Chapter 1

Preliminaries

1.1 Spaces of integrable, continuous and absolutely continuous functions

In this section we listed definitions and properties of integrable functions, continuous functions, absolutely continuous functions and basic properties of the Laplace transform. Also we give required notation, terms and overview of some important results (more details could be found in monographs [57, 59, 70, 74]).

L_p spaces

Let $[a,b]$ be a finite interval in $\mathbb{R}$, where $-\infty \leq a < b \leq \infty$. We denote by $L_p[a,b]$, $1 \leq p < \infty$, the space of all Lebesgue measurable functions f for which $\int_a^b |f(t)|^p \, dt < \infty$, where

$$||f||_p = \left(\int_a^b |f(t)|^p \, dt \right)^{\frac{1}{p}},$$

and by $L_\infty[a,b]$ the set of all functions measurable and essentially bounded on $[a,b]$ with

$$||f||_\infty = \operatorname{ess\,sup} \{|f(x)| \colon x \in [a,b]\}.$$

Theorem 1.1 (INTEGRAL HÖLDER'S INEQUALITY) *Let $p,q \in \mathbb{R}$ such that $1 \leq p,q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $f,g \colon [a,b] \to \mathbb{R}$ be integrable functions such that $f \in L_p[a,b]$ and*

$g \in L_q[a,b]$. Then

$$\int_a^b |f(t)\,g(t)|\,dt \le ||f||_p\,||g||_q\,. \tag{1.1}$$

Equality in (1.1) *holds if and only if $A\,|f(t)|^p = B\,|g(t)|^q$ almost everywhere, where A and B are constants.*

Spaces of continuous and absolutely continuous functions

We denote by $C^n[a,b]$, $n \in \mathbb{N}_0$, the space of functions which are n times continuously differentiable on $[a,b]$, that is

$$C^n[a,b] = \left\{ f : [a,b] \to \mathbb{R} \colon\, f^{(k)} \in C[a,b]\,, k = 0,1,\ldots,n \right\}.$$

In particular, $C^0[a,b] = C[a,b]$ is the space of continuous functions on $[a,b]$ with the norm

$$||f||_{C^n} = \sum_{k=0}^{n} ||f^{(k)}||_C = \sum_{k=0}^{n} \max_{x\in[a,b]} |f^{(k)}(x)|\,,$$

and for $C[a,b]$

$$||f||_C = \max_{x\in[a,b]} |f(x)|\,.$$

Lemma 1.1 *The space $C^n[a,b]$ consists of those and only those functions f which are represented in the form*

$$f(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}\varphi(t)\,dt + \sum_{k=0}^{n-1} c_k (x-a)^k\,, \tag{1.2}$$

where $\varphi \in C[a,b]$ and c_k are arbitrary constants $(k = 0,1,\ldots,n-1)$.
Moreover,

$$\varphi(t) = f^{(n)}(t)\,, \quad c_k = \frac{f^{(k)}(a)}{k!}\ (k = 0,1,\ldots,n-1)\,. \tag{1.3}$$

By $C_a^n[a,b]$ we denote the subspace of the space $C^n[a,b]$ defined by

$$C_a^n[a,b] = \left\{ f \in C^n[a,b] \colon\, f^{(k)}(a) = 0\,, k = 0,1,\ldots,n-1 \right\}.$$

For $f \in C^n[a,b]$ and $0 \le \mu < 1$ we define

$$|f|_{n,\mu} = \sup\left\{ \frac{\left|f^{(n)}(x) - f^{(n)}(y)\right|}{|x-y|^{\mu}} : x,y \in [a,b], x \neq y \right\}.$$

Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, n the integral part of α (notation $n = [\alpha]$) and let $\mu = \alpha - n$. By $\mathscr{D}^{\alpha}[a,b]$ we denote the space

$$\mathscr{D}^{\alpha}[a,b] = \left\{ f \in C^n[a,b] \colon\, |f|_{n,\mu} < \infty \right\},$$

and by $\mathscr{D}_a^\alpha[a,b]$ the subspace of the space $\mathscr{D}^\alpha[a,b]$

$$\mathscr{D}_a^\alpha[a,b] = \left\{ f \in \mathscr{D}^\alpha[a,b] \colon f^{(k)}(a) = 0\,, k = 0,1,\ldots,n \right\}.$$

Specially, for $\alpha = n \in \mathbb{N}$ we have $\mathscr{D}^n[a,b] = C^n[a,b]$ and $\mathscr{D}_a^n[a,b] = C_a^n[a,b]$.

The space of absolutely continuous functions on a finite interval $[a,b]$ is denoted by $AC[a,b]$. It is known that $AC[a,b]$ coincides with the space of primitives of Lebesgue integrable functions $L_1[a,b]$ (see Kolmogorov and Fomin [53, Chapter 33.2]):

$$f \in AC[a,b] \quad \Leftrightarrow \quad f(x) = f(a) + \int_a^x \varphi(t)\,dt\,, \quad \varphi \in L_1[a,b]\,,$$

and therefore an absolutely continuous function f has an integrable derivative $f'(x) = \varphi(x)$ almost everywhere na $[a,b]$. We denote by $AC^n[a,b]$, $n \in \mathbb{N}$, the space

$$AC^n[a,b] = \left\{ f \in C^{n-1}[a,b] \colon f^{(n-1)} \in AC[a,b] \right\}.$$

In particular, $AC^1[a,b] = AC[a,b]$.

Lemma 1.2 *The space $AC^n[a,b]$ consists of those and only those functions which can be represented in the form* (1.2)*, where $\varphi \in L_1[a,b]$ and c_k are arbitrary constants* ($k = 0,1,\ldots,n-1$).
Moreover, (1.3) *holds.*

The next theorem has numerous applications involving multiple integrals.

Theorem 1.2 (FUBINI'S THEOREM) *Let $(X,\mathscr{M},\mu)$ and $(Y,\mathscr{N},\nu)$ be σ-finite measure spaces and f $\mu\times\nu$-measurable function on $X\times Y$. If $f \geq 0$, then next integrals are equal*

$$\int_{X\times Y} f(x,y)\,d(\mu\times\nu)(x,y)\,, \int_X \left(\int_Y f(x,y)\,d\nu(y) \right) d\mu(x) \;\; \text{and} \; \int_Y \left(\int_X f(x,y)\,d\mu(x) \right) d\nu(y).$$

If f is a complex function, then above equalities hold with additional requirement

$$\int_{X\times Y} |f(x,y)|\,d(\mu\times\nu)(x,y) < \infty.$$

Next equalities are consequences of this theorem:

$$\int_a^b dx \int_c^d f(x,y)\,dy = \int_c^d dy \int_a^b f(x,y)\,dx\,;$$

$$\int_a^b dx \int_a^x f(x,y)\,dy = \int_a^b dy \int_y^b f(x,y)\,dx\,. \tag{1.4}$$

The gamma and beta functions

The *gamma function* Γ is the function of complex variable defined by Euler's integral of second kind

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt \,, \quad \Re(z) > 0 \,. \tag{1.5}$$

This integral is convergent for each $z \in \mathbb{C}$ such that $\Re(z) > 0$. It has next property

$$\Gamma(z+1) = z\Gamma(z) \,, \quad \Re(z) > 0 \,,$$

from which follows

$$\Gamma(n+1) = n! \,, \quad n \in \mathbb{N}_0 \,.$$

For domain $\Re(z) \leq 0$ we have

$$\Gamma(z) = \frac{\Gamma(z+n)}{(z)_n} \,, \quad \Re(z) > -n; \; n \in \mathbb{N}; \; z \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\} \,, \tag{1.6}$$

where $(z)_n$ is the *Pochhammer's symbol* defined for $z \in \mathbb{C}$ and $n \in \mathbb{N}_0$ by

$$(z)_0 = 1; \quad (z)_n = z(z+1) \cdots (z+n-1), \; n \in \mathbb{N} \,.$$

The gamma function is analytic in complex plane except in $0, -1, -2, \ldots$ which are simple poles.

The *beta function* is the function of two complex variables defined by Euler's integral of the first kind

$$B(z,w) = \int_0^1 t^{z-1} (1-t)^{w-1} \, dt \,, \quad \Re(z), \Re(w) > 0 \,. \tag{1.7}$$

It is related to the gamma function with

$$B(z,w) = \frac{\Gamma(z)\,\Gamma(w)}{\Gamma(z+w)} \,, \quad z, w \notin \mathbb{Z}_0^- \,,$$

which gives

$$B(z+1,w) = \frac{z}{z+w} B(z,w) \,.$$

Next we proceed with examples of integrals often used in proofs and calculations in this book.

Example 1.1 Let $\alpha, \beta > 0$ and $x \in [a,b]$. Then by substitution $t = x - s(x-a)$ we have

$$\begin{aligned} \int_a^x (x-t)^{\alpha-1} (t-a)^{\beta-1} \, dt &= \int_0^1 (x-a)^{\alpha+\beta-1} s^{\alpha-1} (1-s)^{\beta-1} \, ds \\ &= B(\alpha,\beta)(x-a)^{\alpha+\beta-1} \,. \end{aligned}$$

Analogously, by substitution $t = x + s(b-x)$, it follows

$$\int_x^b (t-x)^{\alpha-1} (b-t)^{\beta-1} \, dt = B(\alpha,\beta)\,(b-x)^{\alpha+\beta-1} \,.$$

Example 1.2 Let $\alpha,\beta > 0$, $f \in L_1[a,b]$ and $x \in [a,b]$. Then interchanging the order of integration and evaluating the inner integral we obtain

$$\begin{aligned}\int_a^x (x-t)^{\alpha-1} \int_a^t (t-s)^{\beta-1} f(s)\,ds\,dt &= \int_{s=a}^x f(s) \int_{t=s}^x (x-t)^{\alpha-1}(t-s)^{\beta-1}\,dt\,ds \\ &= B(\alpha,\beta) \int_a^x (x-s)^{\alpha+\beta-1} f(s)\,ds\,.\end{aligned}$$

Analogously,

$$\int_x^b (t-x)^{\alpha-1} \int_t^b (s-t)^{\beta-1} f(s)\,ds\,dt = B(\alpha,\beta) \int_x^b (s-x)^{\alpha+\beta-1} f(s)\,ds\,.$$

The Laplace transform

Let $f\colon [0,\infty) \to \mathbb{R}$ be a function such that mapping $t \mapsto e^{-\sigma t}\,|f(t)|$, $\sigma > 0$, is integrable on $[0,\infty)$. Then for each $p \geq \sigma$ the Lebesgue integral

$$F(p) = \int_0^\infty e^{-pt} f(t)\,dt \tag{1.8}$$

exists. The mapping $f \mapsto F$ is called the *Laplace transform* and noted with $\mathscr{L}$, that is

$$\mathscr{L}[f](p) = F(p)\,.$$

Sufficient conditions for the Laplace transform existence are that function f is locally integrable and exponentially bounded in ∞, that is $|f(t)| \leq Me^{\sigma t}$ for $t > \varepsilon$, where M, σ and ε are constant. The *abscissa of convergence* σ_0 is the smallest value of σ for which $|f(t)| \leq Me^{\sigma t}$.

Example 1.3 Let $f\colon [0,\infty) \to \mathbb{R}$, $f(t) = t^\alpha$, where $\alpha > -1$. Obviously $|f(t)| = t^\alpha < e^{\alpha t}$ for $t > 0$ and $\alpha \geq 0$. For $-1 < \alpha < 0$, the function f is locally integrable and $t^\alpha \leq 1$ for $t \geq 1$. Therefore, by substitution $pt = x$, the Laplace transform has the form

$$\mathscr{L}[f](p) = \int_0^\infty e^{-pt} t^\alpha\,dt = \frac{1}{p^{\alpha+1}} \int_0^\infty e^{-x} x^\alpha\,dx = \frac{\Gamma(\alpha+1)}{p^{\alpha+1}}\,.$$

We give some properties and rules of the Laplace transform, and important uniqueness theorem ([74, Teorem 6.3]):

$$\textit{convolution:} \quad \mathscr{L}\left[\int_0^t f(t-\tau)\,g(\tau)\,d\tau\right](p) = \mathscr{L}[f](p)\,\mathscr{L}[g](p)$$

$$\textit{differentiation:} \quad \mathscr{L}\left[f^{(n)}\right](p) = p^n \mathscr{L}[f](p) - \sum_{k=1}^n p^{n-k} f^{(k-1)}(0)$$

Theorem 1.3 (UNIQUENESS THEOREM) *Let $f,g\colon [0,\infty) \to \mathbb{R}$ be two functions for which the Laplaceova transform exists. If*

$$\int_0^\infty e^{-pt} f(t)\,dt = \int_0^\infty e^{-pt} g(t)\,dt$$

for each p on common area of convergence, then $f(t) = g(t)$ for almost every $t \in [0,\infty)$.

1.2 Convex functions and Jensen's inequalities

Definitions and properties of convex functions and Jensen's inequality, with more details, could be found in monographs [61, 62, 67].

Let I be an interval in $\mathbb{R}$.

Definition 1.1 *A function $f : I \to \mathbb{R}$ is called convex if*

$$f((1-\lambda)x+\lambda y) \leq (1-\lambda)f(x)+\lambda f(y) \tag{1.9}$$

for all points x and y in I and all $\lambda \in [0,1]$. It is called strictly convex if the inequality (1.9) *holds strictly whenever x and y are distinct points and $\lambda \in (0,1)$. If $-f$ is convex (respectively, strictly convex) then we say that f is concave (respectively, strictly concave). If f is both convex and concave, then f is said to be affine.*

Lemma 1.3 (THE DISCRETE CASE OF JENSEN'S INEQUALITY) *A real-valued function f defined on an interval I is convex if and only if for all $x_1,\dots,x_n$ in I and all scalars $\lambda_1,\dots,\lambda_n$ in $[0,1]$ with $\sum_{k=1}^n \lambda_k = 1$ we have*

$$f\left(\sum_{k=1}^{n} \lambda_k x_k\right) \leq \sum_{k=1}^{n} \lambda_k f(x_k)\,. \tag{1.10}$$

The above inequality is strict if f is strictly convex, all the points x_k are distinct and all scalars λ_k are positive.

Theorem 1.4 (JENSEN) *Let $f : I \to \mathbb{R}$ be a continuous function. Then f is convex if and only if f is midpoint convex, that is,*

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \tag{1.11}$$

for all $x,y \in I$.

Corollary 1.1 *Let $f : I \to \mathbb{R}$ be a continuous function. Then f is convex if and only if*

$$f(x+h)+f(x-h)-2f(x) \geq 0 \tag{1.12}$$

for all $x \in I$ and all $h > 0$ such that both $x+h$ and $x-h$ are in I.

Proposition 1.1 (THE OPERATIONS WITH CONVEX FUNCTIONS) **(i)** *The addition of two convex functions (defined on the same interval) is a convex function; if one of them is strictly convex, then the sum is also strictly convex.*

(ii) *The multiplication of a (strictly) convex function with a positive scalar is also a (strictly) convex function.*

(iii) *The restriction of every (strictly) convex function to a subinterval of its domain is also a (strictly) convex function.*

(iv) *If $f : I \to \mathbb{R}$ is a convex (respectively a strictly convex) function and $g : \mathbb{R} \to \mathbb{R}$ is a nondecreasing (respectively an increasing) convex function, then $g \circ f$ is convex (respectively strictly convex)*

(v) *Suppose that f is a bijection between two intervals I and J. If f is increasing, then f is (strictly) convex if and only if f^{-1} is (strictly) concave. If f is a decreasing bijection, then f and f^{-1} are of the same type of convexity.*

Definition 1.2 *If g is strictly monotonic, then f is said to be (strictly) convex with respect to g if $f \circ g^{-1}$ is (strictly) convex.*

Proposition 1.2 *If $x_1, x_2, x_3 \in I$ are such that $x_1 < x_2 < x_3$, then the function $f : I \to \mathbb{R}$ is convex if and only if the inequality*

$$(x_3 - x_2)\, f(x_1) + (x_1 - x_3)\, f(x_2) + (x_2 - x_1)\, f(x_3) \;\geq\; 0$$

holds.

Proposition 1.3 *If f is a convex function on an interval I and if $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \;\leq\; \frac{f(y_2) - f(y_1)}{y_2 - y_1}\,.$$

If the function f is concave, then the inequality reverses.

The following theorems concern derivatives of convex functions.

Theorem 1.5 *Let $f : I \to \mathbb{R}$ be convex. Then*

(i) *f is Lipschitz on any closed interval in I;*

(ii) *f'_+ and f'_- exist and are increasing in I, and $f'_- \leq f'_+$ (if f is strictly convex, then these derivatives are strictly increasing);*

(iii) *f' exists, except possibly on a countable set, and on the complement of which it is continuous.*

Proposition 1.4 *Suppose that $f : I \to \mathbb{R}$ is a twice differentiable function. Then*

(i) *f is convex if and only if $f'' \geq 0$;*

(ii) *f is strictly convex if and only if $f'' \geq 0$ and the set of points where f'' vanishes does not include intervals of positive length.*

Next we need divided differences, commonly used when dealing with functions that have different degree of smoothness.

Definition 1.3 *Let $f: I \to \mathbb{R}$, $n \in \mathbb{N}_0$ and let $x_0, x_1, \ldots, x_n \in I$ be mutually different points. The n-th order divided difference of a function at $x_0, \ldots, x_n$ is defined recursively by*

$$\begin{aligned}
[x_i; f] &= f(x_i), \quad i = 0, 1, \ldots, n, \\
[x_0, x_1; f] &= \frac{[x_0; f] - [x_1; f]}{x_0 - x_1} = \frac{f(x_0) - f(x_1)}{x_0 - x_1}, \\
[x_0, x_1, x_2; f] &= \frac{[x_0, x_1; f] - [x_1, x_2; f]}{x_0 - x_2}, \\
&\vdots \\
[x_0, \ldots, x_n; f] &= \frac{[x_0, \ldots, x_{n-1}; f] - [x_1, \ldots, x_n; f]}{x_0 - x_n}.
\end{aligned} \tag{1.13}$$

Remark 1.1 The value $[x_0, x_1, x_2; f]$ is independent of the order of the points x_0, x_1 and x_2. This definition may be extended to include the case in which some or all the points coincide. Namely, taking the limit $x_1 \to x_0$ in (1.13), we get

$$\lim_{x_1 \to x_0} [x_0, x_1, x_2; f] = [x_0, x_0, x_2; f] = \frac{f(x_0) - f(x_2) - f'(x_0)(x_0 - x_2)}{(x_0 - x_2)^2}, \; x_2 \neq x_0$$

provided that f' exists, and furthermore, taking the limits $x_i \to x_0$, $i = 1, 2$ in (1.13), we get

$$\lim_{x_2 \to x_0} \lim_{x_1 \to x_0} [x_0, x_1, x_2; f] = [x_0, x_0, x_0; f] = \frac{f''(x_0)}{2}$$

provided that f'' exists.

Definition 1.4 *A function $f: I \to \mathbb{R}$ is said to be n-convex ($n \in \mathbb{N}_0$) if for all choices of $n+1$ distinct points $x_0, \ldots, x_n \in I$, the n-th order divided difference of f satisfies*

$$[x_0, \ldots, x_n; f] \geq 0. \tag{1.14}$$

Thus the 1-convex functions are the nondecreasing functions, while the 2-convex functions are precisely the classical convex functions.

Definition 1.5 *A function $f: I \to (0, \infty)$ is called log-convex if*

$$f((1-\lambda)x + \lambda y) \leq f(x)^{1-\lambda} f(y)^{\lambda} \tag{1.15}$$

for all points x and y in I and all $\lambda \in [0, 1]$.

If a function $f: I \to \mathbb{R}$ is log-convex, then it is also convex, which is a consequence of the weighted AG-inequality.

We end this section with the integral form of Jensen's inequality.

Theorem 1.6 (INTEGRAL JENSEN'S INEQUALITY) *Let $(\Omega, \mathscr{A}, \mu)$ be a finite measure space, $0 < \mu(\Omega) < \infty$ and let $f: \Omega \to I$ be a μ-integrable function. If $\varphi: I \to \mathbb{R}$ is convex function, then next inequality holds*

$$\varphi\left(\frac{1}{\mu(\Omega)} \int_{\Omega} f d\mu\right) \leq \frac{1}{\mu(\Omega)} \int_{\Omega} (\varphi \circ f) d\mu. \tag{1.16}$$

If φ is strictly convex, then in (1.16) *we have equality if and only f is constant μ-almost everywhere on Ω.*

1.3 Exponential convexity

Following definitions and properties of exponentially convex functions comes from [28], also [66]. Let I be an interval in $\mathbb{R}$.

Definition 1.6 *A function $\psi\colon I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if*

$$\sum_{i,j=1}^{n} \xi_i \xi_j \psi\left(\frac{x_i+x_j}{2}\right) \geq 0$$

holds for all choices $\xi_i \in \mathbb{R}$ and $x_i \in I$, $i = 1, \ldots, n$.

A function $\psi\colon I \to \mathbb{R}$ is n-exponentially convex if it is n-exponentially convex in the Jensen sense and continuous on I.

Remark 1.2 It is clear from the definition that $1-$exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also, n-exponentially convex functions in the Jensen sense are $k-$exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $k \leq n$.

By definition of positive semi-definite matrices and some basic linear algebra we have the following proposition.

Proposition 1.5 *If ψ is an n-exponentially convex in the Jensen sense, then the matrix $\left[\psi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k}$ is a positive semi-definite matrix for all $k \in \mathbb{N}, k \leq n$. Particularly, $\det\left[\psi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k} \geq 0$ for all $k \in \mathbb{N}$, $k \leq n$.*

Definition 1.7 *A function $\psi\colon I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.*

A function $\psi\colon I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Remark 1.3 It is known (and easy to show) that $\psi : I \to (0, \infty)$ is log-convex in the Jensen sense if and only if

$$\alpha^2 \psi(x) + 2\alpha\beta\psi\left(\frac{x+y}{2}\right) + \beta^2\psi(y) \geq 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a function is log-convex in the Jensen sense if and only if it is $2-$exponentially convex in the Jensen sense.

Also, using basic convexity theory it follows that a function is log-convex if and only if it is $2-$exponentially convex.

One of the main features of exponentially convex functions is its integral representation given by Bernstein ([32]) in the following theorem.

Theorem 1.7 *The function $\psi\colon I \to \mathbb{R}$ is exponentially convex on I if and only if*

$$\psi(x) = \int_{-\infty}^{\infty} e^{tx} d\sigma(t), \quad x \in I$$

for some non-decreasing function $\sigma : \mathbb{R} \to \mathbb{R}$.

1.4 Opial-type inequalities

In 1960. Opial published an inequality involving integrals of a function and its derivative, which now bear his name ([64]). Over the last five decades, an enormous amount of work has been done on Opial's inequality: several simplifications of the original proof, various extensions, generalizations and discrete analogues. More details can be found in the monograph by Agarwal and Pang [5] which is dedicated to the theory of Opial-type inequalities and its applications in theory of differential and difference equations. We observe Beesack's, Wirtinger's, Willett's, Godunova-Levin's, Rozanova's, Fink's, Agarwal-Pang's and Alzer's versions of Opial's inequality.

Theorem 1.8 (OPIAL'S INEQUALITY) *Let $f \in C^1[0,h]$ be such that $f(0) = f(h) = 0$ and $f(x) > 0$ for $x \in (0,h)$. then*

$$\int_0^h \left| f(x) f'(x) \right| dx \le \frac{h}{4} \int_0^h \left[f'(x) \right]^2 dx, \tag{1.17}$$

where constant $h/4$ is the best possible.

The novelty of Opial's result is thus in establishing the best possible constant $h/4$.

Example 1.4 It is easy to construct the function which satisfy equality in (1.17). For instance, let f be defined by

$$f(x) = \begin{cases} cx, & 0 \le x \le \frac{h}{2} \\ c(h-x), & \frac{h}{2} \le x \le h \end{cases}$$

where $c > 0$ is arbitrary constant. Although this function is not derivable in $t = h/2$, it could be approximated by the function belonging to $C^1[0,h]$ that satisfy (1.17). Then constant $h/4$ is the best possible.

Opial's inequality (1.17) holds even if function f' has discontinuity at $t = h/2$, provided that f is absolutely continuous on both of the subintervals $[0, \frac{h}{2}]$ and $[\frac{h}{2}, h]$, with $f(0) = f(h) = 0$. Also, the positivity requirement of f on $(0,h)$ is unnecessary, that is, next Beesack's inequality holds ([31]).

Theorem 1.9 (BEESACK'S INEQUALITY) *Let $f \in AC[0,h]$ be such that $f(0)=0$. Then*

$$\int_0^h \left|f(x)f'(x)\right| dx \leq \frac{h}{2}\int_0^h \left[f'(x)\right]^2 dx. \tag{1.18}$$

Equality in (1.18) *holds if and only if $f(x)=cx$, where c is a constant.*

Theorem 1.10 (WIRTINGER'S INEQUALITY) *Let $f\colon [0,h]\to\mathbb{R}$ be such that $f' \in L_2[0,h]$. If $f(0)=f(h)=0$, then*

$$\int_0^h [f(x)]^2 dx \leq \left(\frac{h}{\pi}\right)^2 \int_0^h \left[f'(x)\right]^2 dx. \tag{1.19}$$

Equality in (1.19) *holds if and only if $f(x)=c\sin\frac{\pi x}{h}$, where c is a constant.*

Remark 1.4 A weaker form of Opial's inequality can be obtained by combining Cauchy-Schwarz-Buniakowski's inequality and Wirtinger's inequality:

$$\int_0^h \left|f(x)f'(x)\right| dx \leq \left(\int_0^h |f(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_0^h \left|f'(x)\right|^2 dx\right)^{\frac{1}{2}} \leq \frac{h}{\pi}\int_0^h \left[f'(x)\right]^2 dx.$$

Next inequality involving $x^{(n)}$, $n \geq 1$, is given by Willett [75] (see also [5, p. 128]).

Theorem 1.11 (WILLETT'S INEQUALITY) *Let $x \in C^n[0,h]$ be such that $x^{(i)}(0)=0$, $i=0,\ldots,n-1$, $n\geq 1$. Then*

$$\int_0^h \left|x(t)x^{(n)}(t)\right| dt \ \leq\ \frac{h^n}{2}\int_0^h \left|x^{(n)}(t)\right|^2 dt. \tag{1.20}$$

More generalizations and extensions of Willett's inequality are done by Boyd in [33].

Following generalization of Opial's inequality is due to Godunova and Levin [46] (see also [5, p. 74]).

Theorem 1.12 (GODUNOVA-LEVIN'S INEQUALITY) *Let f be a convex and increasing function on $[0,\infty)$ with $f(0)=0$. Further, let x be absolutely continuous on $[a,\tau]$ and $x(a)=0$. Then, the following inequality holds*

$$\int_a^\tau f'\left(|x(t)|\right)|x'(t)|\, dt \leq f\left(\int_a^\tau |x'(t)|\, dt\right). \tag{1.21}$$

An extension of the inequality (1.21) is embodied in the following inequality by Rozanova [69] (see also [5, p. 82]).

Theorem 1.13 (ROZANOVA'S INEQUALITY) *Let f, g be convex and increasing functions on $[0,\infty)$ with $f(0)=0$, and let $p(t)\geq 0$, $p'(t)>0$, $t\in[a,\tau]$ with $p(a)=0$. Further, let x be absolutely continuous on $[a,\tau]$ and $x(a)=0$. Then, the following inequality holds*

$$\int_a^\tau p'(t)\,g\left(\frac{|x'(t)|}{p'(t)}\right) f'\left(p(t)\,g\left(\frac{|x(t)|}{p(t)}\right)\right) dt \leq f\left(\int_a^\tau p'(t)\,g\left(\frac{|x'(t)|}{p'(t)}\right) dt\right). \tag{1.22}$$

Moreover, equality holds in (1.22) *for the function $x(t)=c\,p(t)$.*

Remark 1.5 The condition in the two previous theorems that function f is to be increasing is actually unneeded, and also, the condition $g \geq 0$ is missing in Theorem 1.13 (it can be easily seen from proofs of the theorems).

Among inequalities of Opial-type, there is a class of inequality involving higher order derivatives. First we have Fink's inequality ([45]).

Theorem 1.14 (FINK'S INEQUALITY) *Let $q \geq 1$, $\frac{1}{p}+\frac{1}{q}=1$, $n \geq 2$ and $0 \leq i \leq j \leq n-1$. Let $f \in AC^n[0,h]$ be such that $f(0)=f'(0)=\cdots=f^{(n-1)}(0)=0$ and $f^{(n)} \in L_q[0,h]$. Then*

$$\int_0^h \left|f^{(i)}(x)f^{(j)}(x)\right| dx \leq C h^{2n-i-j+1-\frac{2}{q}} \left(\int_0^h \left|f^{(n)}(x)\right|^q dx\right)^{\frac{2}{q}}, \tag{1.23}$$

where $C = C(n,i,j,q)$ is given by

$$C = \left[2^{\frac{1}{q}}(n-i-1)!\,(n-j)!\,[p(n-j)+1]^{\frac{1}{p}}\,[p(2n-i-j-1)+2]^{\frac{1}{p}}\right]^{-1}. \tag{1.24}$$

Inequality (1.23) is sharp for $j=i+1$, where equality in this case is achieved for $q>1$ and function f such that

$$f(x) = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}(h-t)^{\frac{p}{q}(n-i-1)}\,dt\,.$$

Remark 1.6 Agarwal and Pang proved in [65] that Fink's inequality does not hold for $i=j$, and that is not necessary to assume that $f^{(k)}(0)=0$ for $k<i$.

Next inequality is due to Agarwal and Pang ([65]).

Theorem 1.15 (AGARWAL-PANG'S INEQUALITY) *Let $n \in \mathbb{N}$ and $f \in AC^n[0,h]$ be such that $f(0)=f'(0)=\cdots=f^{(n-1)}(0)=0$. Let w_1 and w_2 be positive, measurable functions on $[0,h]$. Let $r_i>0$, $i=0,\ldots,n-1$, and let $r=\sum_{i=0}^{n-1} r_i$. Let $s_k>1$ and $\frac{1}{s_k}+\frac{1}{s_k'}=1$ for $k=1,2$, and $q \in \mathbb{R}$ such that $q>s_2$. Further, let*

$$P = \left(\int_0^h [w_2(x)]^{-\frac{s_2'}{q}}\,dx\right)^{\frac{r}{s_2'}} < \infty,$$

$$Q = \left(\int_0^h [w_1(x)]^{s_1'}\,dx\right)^{\frac{1}{s_1'}} < \infty.$$

Then

$$\int_0^h w_1(x)\prod_{i=0}^{n-1}\left|f^{(i)}(x)\right|^{r_i} dx \leq C h^{\rho+\frac{1}{s_1}} \left(\int_0^h w_2(x)\left|f^{(n)}(x)\right|^q dx\right)^{\frac{r}{q}}, \tag{1.25}$$

where $\rho = \sum\limits_{i=0}^{n-1} I r_i + \sigma r$, $I=n-i-1$, $\sigma = \frac{1}{s_2}-\frac{1}{q}$, and $C = C(n,\{r_i\},w_1,w_2,s_1,s_2,q)$ is given by

$$C \leq QP\prod_{i=0}^{n-1}(I!)^{-r_i}\left[\frac{I}{\sigma}+1\right]^{-r_i\sigma}\left[\sum_{i=0}^{n-1} I r_i s_1 + \sigma r s_1 + 1\right]^{-\frac{1}{s_1}},$$

provided that integral on the right side in (1.25) exists.

Alzer's inequalities are given in [10, 11], where second one includes higher order derivatives of two functions.

Theorem 1.16 (ALZER'S INEQUALITY 1) *Let $n \in \mathbb{N}$ and $f \in C^n[a,b]$ be such that $f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0$. Let w be continuous, positive, decreasing function on $[a,b]$. Let $r_i \geq 0$, $i = 0, \ldots, n-1$, and $\sum_{i=0}^{n-1} r_i = 1$. Let $p \geq 1$, $q > 0$ and $\sigma = 1/(p+q)$. Then*

$$\int_a^b w(x) \left(\prod_{i=0}^{n-1} \left| f^{(i)}(x) \right|^{r_i} \right)^p \left| f^{(n)}(x) \right|^q dx \leq A_1 \int_a^b w(x) \left| f^{(n)}(x) \right|^{p+q} dx, \tag{1.26}$$

where

$$A_1 = \sigma q^{\sigma q} \left[n - \sum_{i=1}^{n-1} i r_i \right]^{-\sigma p} (b-a)^{(n - \sum_{i=1}^{n-1} i r_i) p} \prod_{i=0}^{n-1} \left[\left(\frac{1-\sigma}{n-i-\sigma} \right)^{1-\sigma} \frac{1}{(n-i-1)!} \right]^{r_i p}.$$

Theorem 1.17 (ALZER'S INEQUALITY 2) *Let $p \geq 0$, $q > 0$, $r > 1$ and $r > q$. Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$, $0 \leq k \leq n-1$. Let $w_1 \geq 0$ and $w_2 > 0$ be measurable functions on $[a,b]$. Further, let $f, g \in AC^n[a,b]$ be such that $f^{(i)}(a) = g^{(i)}(a) = 0$ for $i = 0, \ldots, n-1$ and let integrals $\int_a^b w_2(x)|f^{(n)}(x)|^r\, dx$ and $\int_a^b w_2(x)|g^{(n)}(x)|^r\, dx$ exist. Then*

$$\begin{aligned} &\int_a^b w_1(x) \left[\left| g^{(k)}(x) \right|^p \left| f^{(n)}(x) \right|^q + \left| f^{(k)}(x) \right|^p \left| g^{(n)}(x) \right|^q \right] dx \\ &\leq A_2 \left(\int_a^b w_2(x) \left[\left| f^{(n)}(x) \right|^r + \left| g^{(n)}(x) \right|^r \right] dx \right)^{\frac{p+q}{r}}, \end{aligned} \tag{1.27}$$

where

$$A_2 = \frac{2M}{[(n-k-1)!]^p} \left[\frac{q}{2(p+q)} \right]^{\frac{q}{r}} \left[\int_a^b [w_1(x)]^{\frac{r}{r-q}} [w_2(x)]^{\frac{q}{q-r}} [s(x)]^{\frac{p(r-1)}{r-q}} dx \right]^{\frac{r-q}{r}},$$

$$s(x) = \int_a^x (x-u)^{\frac{r(n-k-1)}{r-1}} [w_2(u)]^{\frac{1}{1-r}} du,$$

$$M = \begin{cases} \left(1 - 2^{-\frac{p}{q}}\right)^{\frac{q}{r}}, & p \geq q, \\ 2^{-\frac{p}{r}}, & p \leq q. \end{cases}$$

Chapter 2

Fractional integrals and fractional derivatives

Fractional calculus is a theory of differential and integral operators of non-integer order. This chapter contains definitions and basic properties of the Riemann-Liouville fractional integral and three main types of fractional derivatives (more detailed information may be found in [38, 51, 68, 72]). The last part of the chapter is based on our results involving composition identities for fractional derivatives: Andrić, Pečarić and Perić [23, 25, 26]. At the same time we investigate the role of the initial conditions on functions included in composition identities, and also relations between the order of the Riemann-Liouville fractional integrals and mentioned fractional derivatives.

Fractional integrals and fractional derivatives will be observed in the real domain. Let $[a,b] \subset \mathbb{R}$ be a finite interval, that is $-\infty < a < b < \infty$. For the integral part of a real number α we use notation $[\alpha]$. Also, Γ is the gamma function defined by (1.5) on $\mathbb{R}^+$, and by (1.6) on $\mathbb{R}_0^- \backslash \mathbb{Z}_0^-$. Throughout this chapter let $x \in [a,b]$.

2.1 The Riemann-Liouville fractional integrals

In [48] G. H. Hardy showed that the Riemann-Liouville fractional integrals are defined for a function $f \in L_1[a,b]$, existing almost everywhere on $[a,b]$. Also, which is in accordance with the classical theorem of Vallée-Poussin and the Young convolution theorem, he proved $J^{\alpha}_{a+}f, J^{\alpha}_{b-}f \in L_1[a,b]$.

Definition 2.1 *Let $\alpha > 0$ and $f \in L_1[a,b]$. The left-sided and the right-sided Riemann-Liouville fractional integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order α are defined by*

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt\,, \quad x\in[a,b]\,, \tag{2.1}$$

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\,dt\,, \quad x\in[a,b]\,. \tag{2.2}$$

For $\alpha = n \in \mathbb{N}$ fractional integrals are actually n-fold integrals, that is

$$\begin{aligned} J^{n}_{a+}f(x) &= \int_a^x dt_1 \int_a^{t_1} dt_2 \cdots \int_a^{t_{n-1}} f(t_n)\,dt_n \\ &= \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)\,dt\,, \end{aligned} \tag{2.3}$$

$$\begin{aligned} J^{n}_{b-}f(x) &= \int_x^b dt_1 \int_{t_1}^{b} dt_2 \cdots \int_{t_{n-1}}^{b} f(t_n)\,dt_n \\ &= \frac{1}{(n-1)!}\int_x^b (t-x)^{n-1} f(t)\,dt\,. \end{aligned} \tag{2.4}$$

Example 2.1 Let $\alpha,\beta > 0$, $f(x) = (x-a)^{\beta-1}$ and $g(x) = (b-x)^{\beta-1}$. By Example 1.1, for the left-sided Riemann-Liouville fractional integral of a function f we have

$$\begin{aligned} J^{\alpha}_{a+}(x-a)^{\beta-1} &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}(t-a)^{\beta-1}\,dt \\ &= \frac{(x-a)^{\alpha+\beta-1}}{\Gamma(\alpha)}B(\alpha,\beta) \\ &= \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1}\,. \end{aligned}$$

Analogously, the right-sided Riemann-Liouville fractional integral of a function g is

$$J^{\alpha}_{b-}(b-x)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(b-x)^{\alpha+\beta-1}\,.$$

Example 2.2 Let $\alpha > 0$ and $\lambda \in \mathbb{R}$. By using Taylor series for the exponential function we have

$$\begin{aligned} J^{\alpha}_{a+}e^{\lambda x} &= J^{\alpha}_{a+}\left(e^{\lambda a}e^{\lambda(x-a)}\right) \\ &= J^{\alpha}_{a+}\left[e^{\lambda a}\sum_{n=0}^{\infty}\frac{\lambda^n(x-a)^n}{n!}\right] \\ &= e^{\lambda a}\sum_{n=0}^{\infty}\frac{\lambda^n}{\Gamma(n+1)}J^{\alpha}_{a+}(x-a)^n \end{aligned}$$

$$= e^{\lambda a}(x-a)^{\alpha}\sum_{n=0}^{\infty}\frac{\lambda^n(x-a)^n}{\Gamma(\alpha+n+1)},$$

$$J^{\alpha}_{b-}e^{\lambda x} = e^{\lambda b}(b-x)^{\alpha}\sum_{n=0}^{\infty}\frac{(-\lambda)^n(b-x)^n}{\Gamma(\alpha+n+1)}.$$

Next we give some properties of the Riemann-Liouville fractional integral, basically presented by Samko et al. in [72] and by Canavati in [38]. Those result we will unify and give complete proofs. We start with a following lemma by Canavati ([38]): the Riemann-Liouville fractional integral of a continuous function is also continuous function.

Lemma 2.1 *Let $\alpha > 0$ and $f \in C[a,b]$. Then for each $x, y \in [a,b]$ we have*

$$\left|J^{\alpha}_{a+}f(x) - J^{\alpha}_{a+}f(y)\right| \le \frac{||f||_C}{\Gamma(\alpha+1)}\left(2\,|x-y|^{\alpha} + |(x-a)^{\alpha}-(y-a)^{\alpha}|\right). \tag{2.5}$$

In particular, if $0 < \alpha < 1$, then

$$\left|J^{\alpha}_{a+}f(x) - J^{\alpha}_{a+}f(y)\right| \le \frac{3\,||f||_C}{\Gamma(\alpha+1)}\,|x-y|^{\alpha}. \tag{2.6}$$

Proof. Let $x < y$. Then

$$\begin{aligned}
&J^{\alpha}_{a+}f(x) - J^{\alpha}_{a+}f(y)\\
&= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt \;-\; \frac{1}{\Gamma(\alpha)}\int_a^y (y-t)^{\alpha-1} f(t)\,dt\\
&= \frac{1}{\Gamma(\alpha)}\int_a^x \left[(x-t)^{\alpha-1}-(y-t)^{\alpha-1}\right] f(t)\,dt \;-\; \frac{1}{\Gamma(\alpha)}\int_x^y (y-t)^{\alpha-1} f(t)\,dt.
\end{aligned}$$

$$\begin{aligned}
&\left|J^{\alpha}_{a+}f(x) - J^{\alpha}_{a+}f(y)\right|\\
&\le \frac{1}{\Gamma(\alpha)}\int_a^x \left|(x-t)^{\alpha-1}-(y-t)^{\alpha-1}\right| |f(t)|\,dt \;+\; \frac{1}{\Gamma(\alpha)}\left|\int_x^y (y-t)^{\alpha-1} f(t)\,dt\right|\\
&\le \frac{||f||_C}{\Gamma(\alpha)}\int_a^x \left[(x-t)^{\alpha-1}-(y-t)^{\alpha-1}\right] dt \;+\; \frac{||f||_C}{\Gamma(\alpha)}\int_x^y (y-t)^{\alpha-1}\,dt\\
&= \frac{||f||_C}{\Gamma(\alpha+1)}\left(2(y-x)^{\alpha} \;+\; (x-a)^{\alpha}-(y-a)^{\alpha}\right)\\
&\le \frac{||f||_C}{\Gamma(\alpha+1)}\left(2\,|x-y|^{\alpha} \;+\; |(x-a)^{\alpha}-(y-a)^{\alpha}|\right).
\end{aligned}$$

The same inequality follows for $x > y$, that is (2.5) holds. If $0 < \alpha < 1$, then $\left||a|^{\alpha}-|b|^{\alpha}\right| \le |a-b|^{\alpha}$, and

$$\begin{aligned}
\left|J^{\alpha}_{a+}f(x) - J^{\alpha}_{a+}f(y)\right| &\le \frac{||f||_C}{\Gamma(\alpha+1)}\left(2\,|x-y|^{\alpha} + |(x-a)^{\alpha}-(y-a)^{\alpha}|\right)\\
&\le \frac{3\,||f||_C}{\Gamma(\alpha+1)}\,|x-y|^{\alpha}.
\end{aligned}$$

□

We give lemma for the right-sided Riemann-Liouville fractional integrals. The proof is analogous to the previous one, and is omitted.

Lemma 2.2 *Let $\alpha > 0$ and $f \in C[a,b]$. Then for each $x, y \in [a,b]$ we have*

$$\left|J_{b-}^{\alpha} f(x) - J_{b-}^{\alpha} f(y)\right| \le \frac{||f||_C}{\Gamma(\alpha+1)} \left(2\,|x-y|^{\alpha} + |(b-x)^{\alpha} - (b-y)^{\alpha}|\right). \tag{2.7}$$

In particular, if $0 < \alpha < 1$, then

$$\left|J_{b-}^{\alpha} f(x) - J_{b-}^{\alpha} f(y)\right| \le \frac{3\,||f||_C}{\Gamma(\alpha+1)} |x-y|^{\alpha}. \tag{2.8}$$

Corollary 2.1 *Let $\alpha > 0$ and $f \in C[a,b]$. Then $J_{a+}^{\alpha} f, J_{b-}^{\alpha} f \in C[a,b]$.*

Next we observe the composition of fractional integrals (see Samko et al. [72], Section 2).

Lemma 2.3 *Let $\alpha, \beta > 0$ and $f \in L_p[a,b]$, $1 \le p \le \infty$. Then for almost every $x \in [a,b]$ we have*

$$J_{a+}^{\alpha} J_{a+}^{\beta} f(x) = J_{a+}^{\alpha+\beta} f(x), \qquad J_{b-}^{\alpha} J_{b-}^{\beta} f(x) = J_{b-}^{\alpha+\beta} f(x). \tag{2.9}$$

If $f \in C[a,b]$ or $\alpha + \beta > 1$, then equalities (2.9) hold for each x in $[a,b]$.

Proof. Straightforward calculations with Example 1.2 gives us

$$\begin{aligned}
J_{a+}^{\alpha} J_{a+}^{\beta} f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} J_{a+}^{\beta} f(t)\, dt \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x (x-t)^{\alpha-1} \int_a^t (t-s)^{\beta-1} f(s)\, ds\, dt \\
&= \frac{1}{\Gamma(\alpha+\beta)} \int_a^x (x-s)^{\alpha+\beta-1} f(s)\, ds \\
&= J_{a+}^{\alpha+\beta} f(x).
\end{aligned}$$

Analogously for the right-sided fractional integrals follows

$$J_{b-}^{\alpha} J_{b-}^{\beta} f(x) = \frac{1}{\Gamma(\alpha+\beta)} \int_x^b (s-x)^{\alpha+\beta-1} f(s)\, ds = J_{b-}^{\alpha+\beta} f(x).$$

If $f \in C[a,b]$, then $J_{a+}^{\beta} f \in C[a,b]$ by Lemma 2.1, and also $J_{a+}^{\alpha} J_{a+}^{\beta} f \in C[a,b]$, $J_{a+}^{\alpha+\beta} f \in C[a,b]$. Hence, two function $J_{a+}^{\alpha} J_{a+}^{\beta} f$ and $J_{a+}^{\alpha+\beta} f$ coincide almost everywhere on $[a,b]$, and by continuity follows that they coincide on whole interval $[a,b]$. If $f \in L_p[a,b]$ and $\alpha + \beta > 1$, then

$$J_{a+}^{\alpha} J_{a+}^{\beta} f = J_{a+}^{\alpha+\beta} f = J_{a+}^{\alpha+\beta-1} J_{a+}^{1} f$$

almost everywhere on $[a,b]$. Since $J_{a+}^{1} f$ is continuous function, then $J_{a+}^{\alpha+\beta} = J_{a+}^{\alpha+\beta-1} J_{a+}^{1} f \in C[a,b]$, that is once again they coincide on whole interval $[a,b]$ due to continuity.

The same goes for the right-sided Riemann-Liouville fractional integral, so we conclude that equalities (2.9) hold for each x in $[a,b]$. $\square$

The homogeneous Abel integral equation has only trivial solution (see Samko et al. [72], Section 2.4).

Lemma 2.4 *Let $\alpha > 0$ and $f \in L_1[a,b]$. Then integral equations $J^{\alpha}_{a+}f = 0$ and $J^{\alpha}_{b-}f = 0$ have only trivial solution $f = 0$ (almost everywhere).*

Proof. Let $J^{\alpha}_{a+}f = 0$. If $0 < \alpha < 1$, then by Lemma 2.3 follows $J^{1}_{a+}f = J^{1-\alpha}_{a+}J^{\alpha}_{a+}f = 0$. Now we have

$$f = \frac{d}{dx}\int_a^x f(t)\,dt = \frac{d}{dx}J^{1}_{a+}f = 0\,.$$

Let $\alpha \geq 1$, $m = [\alpha]$, $\alpha = m + \beta$, $0 \leq \beta < 1$. If $\beta = 0$, then $\alpha = m \in \mathbb{N}$, and by (2.3) follows $f = \frac{d^m}{dx^m} J^{m}_{a+}f = 0$. Let $\beta > 0$. Again by Lemma 2.3 follows

$$J^{\beta}_{a+}J^{m}_{a+}f = J^{\alpha}_{a+}f = 0,$$

and by just proven, for $0 < \beta < 1$ we have $J^{m}_{a+}f = 0$ and also $f = 0$. The proof is analogous for the right-sided Riemann-Liouville fractional integral. $\square$

Lemma 2.1 and Lemma 2.2 showed that the Riemann-Liouville fractional integral of continuous function is also continuous function. Moreover, for the image of the Riemann-Liouville fractional integral of continuous function we have next result by Canavati ([38]).

Lemma 2.5 *Let $\alpha > 0$ and $f \in C[a,b]$. Then $J^{\alpha}_{a+}f \in \mathscr{D}^{\alpha}_{a}[a,b]$ and $J^{\alpha}_{b-}f \in \mathscr{D}^{\alpha}_{b}[a,b]$.*

Proof. Let $m = [\alpha]$ and $\mu = \alpha - m$. For $\mu = 0$, that is $\alpha = m \in \mathbb{N}$ ($m \geq 1$), we use (2.3) and Lemma 2.3

$$\frac{d^k}{dx^k}J^{m}_{a+}f(x) = \frac{d^k}{dx^k}J^{k}_{a+}J^{m-k}_{a+}f(x) = J^{m-k}_{a+}f(x)\,, \quad k = 0,1,\ldots,m-1\,,$$

that is $\frac{d^k}{dx^k}J^{m}_{a+}f(a) = 0$, for $k = 0,1,\ldots,m-1$ (since f is continuous at a), and then $J^{m}_{a+}f \in C^{m}_{a}[a,b] = \mathscr{D}^{m}_{a}[a,b]$.
Let $0 < \alpha < 1$. Then by Lemma 2.1 we have (2.6),

$$\left|J^{\alpha}_{a+}f(x) - J^{\alpha}_{a+}f(y)\right| \leq \frac{3\,||f||_C}{\Gamma(\alpha+1)}\,|x-y|^{\alpha}\,,$$

that is $J^{\alpha}_{a+}f \in C[a,b]$. Since $0 < \alpha < 1$, then $m = 0$ and

$$\left|J^{\alpha}_{a+}f\right|_{m,\mu} = \sup\left\{\frac{\left|J^{\alpha}_{a+}f(x) - J^{\alpha}_{a+}f(y)\right|}{|x-y|^{\mu}}\right\} \leq \frac{3\,||f||_C}{\Gamma(\alpha+1)} < \infty,$$

that is $J^{\alpha}_{a+}f \in \mathscr{D}^{\alpha}[a,b]$. Further, $J^{\alpha}_{a+}f(a) = 0$, $m = 0$, and then $J^{\alpha}_{a+}f \in \mathscr{D}^{\alpha}_{a}[a,b]$.
Let $\alpha > 1$ ($m \geq 1$) and $0 < \mu < 1$. Then

$$\left|\frac{d^m}{dx^m}J^{\alpha}_{a+}f(x) - \frac{d^m}{dx^m}J^{\alpha}_{a+}f(y)\right| = \left|J^{\mu}_{a+}f(x) - J^{\mu}_{a+}f(y)\right| \leq \frac{3\,||f||_C}{\Gamma(\alpha+1)}\,|x-y|^{\mu}\,,$$

that is $\left|J^{\alpha}_{a+}f\right|_{m,\mu} \leq \frac{3||f||_C}{\Gamma(\alpha+1)} < \infty$ and $J^{\alpha}_{a+}f \in \mathscr{D}^{\alpha}[a,b]$. Again, using Lemma 2.3, (2.3) and continuity of f at a, we have $\frac{d^k}{dx^k}J^{\alpha}_{a+}f(a) = J^{\alpha-k}_{a+}f(a) = 0$ za $k = 0,1,\ldots,m$, that is $J^{\alpha}_{a+}f \in \mathscr{D}^{\alpha}_{a}[a,b]$.
The proof is analogous for the right-sided Riemann-Liouville fractional integral. $\square$

Since $\mathscr{D}^{\alpha}_{a}[a,b], \mathscr{D}^{\alpha}_{b}[a,b] \subseteq \mathscr{D}^{\alpha}[a,b] \subseteq C^{[\alpha]}[a,b]$, next corollary is valid.

Corollary 2.2 *Let $\alpha > 0$, $m = [\alpha]$ and $f \in C[a,b]$. Then $J^{\alpha}_{a+}f, J^{\alpha}_{b-}f \in C^m[a,b]$.*

Next result by Samko et al. ([72]) shows that the Riemann-Liouville fractional integral is bounded operator on $L_p[a,b]$.

Lemma 2.6 *Let $\alpha > 0$ and $1 \leq p \leq \infty$. Then the Riemann-Liouville fractional integrals are bounded on $L_p[a,b]$, that is*

$$||J^{\alpha}_{a+}f||_p \leq K||f||_p\,, \quad ||J^{\alpha}_{b-}f||_p \leq K||f||_p\,, \tag{2.10}$$

where

$$K = \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}.$$

For $C[a,b]$ we have

$$||J^{\alpha}_{a+}f||_C \leq K||f||_C\,, \quad ||J^{\alpha}_{b-}f||_C \leq K||f||_C\,. \tag{2.11}$$

Proof. We give a proof for the left-sided fractional integrals in spaces $L_p[a,b]$ and $C[a,b]$. The proof for the right-sided fractional integrals is analogous. By Jensen's inequality (1.16) and Fubini's theorem follows

$$\int_a^b (x-a)^{\alpha}\left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}\left|J^{\alpha}_{a+}f(x)\right|\right)^p dx \tag{2.12}$$
$$= \int_a^b (x-a)^{\alpha}\left(\frac{\alpha}{(x-a)^{\alpha}}\int_a^x (x-t)^{\alpha-1}|f(t)|\,dt\right)^p dx$$
$$= \int_a^b (x-a)^{\alpha}\left(\int_a^x (x-t)^{\alpha-1}|f(t)|\,dt\Big/\int_a^x (x-t)^{\alpha-1}\,dt\right)^p dx$$
$$\leq \int_a^b (x-a)^{\alpha}\left(\int_a^x (x-t)^{\alpha-1}|f(t)|^p\,dt\Big/\int_a^x (x-t)^{\alpha-1}\,dt\right) dx$$
$$= \int_a^b \alpha\int_a^x (x-t)^{\alpha-1}\,|f(t)|^p\,dt\,dx$$
$$= \int_a^b \alpha|f(t)|^p\int_t^b (x-t)^{\alpha-1}\,dx\,dt$$
$$= \int_a^b |f(t)|^p\,(b-t)^{\alpha}\,dt\,. \tag{2.13}$$

Since $x \in [a,b]$ and $\alpha(1-p) < 0$, for (2.12) we have

$$\int_a^b (x-a)^{\alpha}\left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}\left|J^{\alpha}_{a+}f(x)\right|\right)^p dx$$

$$
\begin{aligned}
&= \int_a^b (x-a)^{\alpha(1-p)} [\Gamma(\alpha+1)]^p \left|J_{a+}^{\alpha} f(x)\right|^p dx \\
&\geq (b-a)^{\alpha(1-p)} [\Gamma(\alpha+1)]^p \int_a^b \left|J_{a+}^{\alpha} f(x)\right|^p dx, \qquad (2.14)
\end{aligned}
$$

and for (2.13)

$$
\int_a^b (b-t)^{\alpha} |f(t)|^p dt \leq (b-a)^{\alpha} \int_a^b |f(t)|^p dt. \qquad (2.15)
$$

Now from (2.14) and (2.15) follows

$$
(b-a)^{\alpha(1-p)} [\Gamma(\alpha+1)]^p \int_a^b \left|J_{a+}^{\alpha} f(x)\right|^p dx \leq (b-a)^{\alpha} \int_a^b |f(t)|^p dt,
$$

that is

$$
\int_a^b \left|J_{a+}^{\alpha} f(x)\right|^p dx \leq \left[\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right]^p \int_a^b |f(t)|^p dt. \qquad (2.16)
$$

If we use exponent $1/p$ for both sides of inequality (2.16), we get that the left-sided Riemann-Liouville fractional integrals are bounded on $L_p[a,b]$.
For $C[a,b]$ we have inequality

$$
||J_{a+}^{\alpha} f||_C = \max_{x\in[a,b]} |J_{a+}^{\alpha} f(x)| \leq \max_{x\in[a,b]} |J_{a+}^{\alpha} 1| \cdot ||f||_C
$$

and by Example 2.1 for $\beta=1$ we have $J_{a+}^{\alpha} 1 = (x-a)^{\alpha}/\Gamma(\alpha+1)$, that is $\max\limits_{x\in[a,b]} |J_{a+}^{\alpha} 1| = K$. □

At the end of this section, we give our result showing that for $\alpha \in (0,1]$ the Riemann-Liouville fractional integral of an absolutely continuous function is also absolutely continuous.

Proposition 2.1 *Let $n \in \mathbb{N}$, $0 < \alpha \leq 1$ and $f \in AC^n[a,b]$. Then $J_{a+}^{\alpha} f \in AC^n[a,b]$.*

Proof. Let $f \in AC^n[a,b]$, that is $f \in C^{n-1}[a,b]$ and $f^{(n-1)} \in AC[a,b]$. Let

$$
g(x) = f(x) - \sum_{and=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k.
$$

The statement $J_{a+}^{\alpha} f \in AC^n[a,b]$ will follow if we prove that $J_{a+}^{\alpha} g \in AC^n[a,b]$.

First we prove that $J_{a+}^{\alpha} g \in C^{n-1}[a,b]$, that is $\frac{d^k}{dx^k} J_{a+}^{\alpha} g \in C[a,b]$ for $k = 0, \ldots, n-1$. Notice that $g(a) = g'(a) = \cdots = g^{(n-1)}(a) = 0$. Using integration by parts we get

$$
\begin{aligned}
J_{a+}^{\alpha} g(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} g(t)\, dt \\
&= \frac{1}{\Gamma(\alpha)} \left[-\frac{(x-t)^{\alpha}}{\alpha} g(t) \Big|_{t=a}^{x} + \int_a^x \frac{(x-t)^{\alpha}}{\alpha} g'(t)\, dt \right] \\
&= \frac{1}{\Gamma(\alpha+1)} \int_a^x (x-t)^{\alpha} g'(t)\, dt.
\end{aligned}
$$