Chapter 1

General Linear Inequalities for Sequences

In this chapter we prove several identities for sums $\sum p_k a_k$, $\sum p_{ij} a_i b_j$ involving finite forward or backward differences of higher order. Using these identities we obtain necessary and sufficient conditions under which the above-mentioned sums are nonnegative for different classes of sequences. We consider the classes of convex sequences of higher order, ∇-convex sequences of higher order, starshaped sequences, the class of p,q-convex sequences etc.

1.1 Convex Sequences of Higher Order

This section is devoted to an identity for the sum $\sum p_k a_k$ and to necessary and sufficient conditions under which this sum is nonnegative for the class of convex sequences of higher order. Let us define and discuss some basic concepts. For a real sequence **a** we usually use notation (a_i) or $(a_i)_{i=k}^{\infty}$ when we want to stress that the first element is a_k. Sometimes under the word "sequence" we mean n-tuple also, but it is always clear from the context.

The finite forward difference of a sequence **a** (or, simple, Δ-difference) is defined as

$$\Delta^1 a_i = \Delta a_i := a_{i+1} - a_i,$$

while the difference of order m is defined as

$$\Delta^m a_i := \Delta(\Delta^{m-1} a_i),\ m \in \{2,3,\ldots\}.$$

Similarly, the finite backward difference (∇-difference) is defined as

$$\nabla^1 a_i = \nabla a_i := a_i - a_{i+1},$$

and the ∇-difference of order m as

$$\nabla^m a_i := \nabla(\nabla^{m-1} a_i).$$

For $m = 0$ we put $\Delta^0 a_i = a_i$, and $\nabla^0 a_i = a_i$. It is easy to see that

$$\Delta^m a_i = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} a_{i+k}.$$

We say that a sequence $\mathbf{a}$ is convex of order m or m-convex if

$$\Delta^m a_i \geq 0$$

holds for any $i \in \mathbb{N}$. If $m = 1$, then $\mathbf{a}$ is nondecreasing, while if $m = 2$, then 2-convexity becomes the classical convexity, i.e. the following holds

$$a_{i+2} - 2a_{i+1} + a_i \geq 0,\ \ i \in \mathbb{N}.$$

We say that a sequence $\mathbf{a}$ is ∇-convex of order m if

$$\nabla^m a_i \geq 0$$

holds for any $i \in \mathbb{N}$.

Also, the following notation is frequently used: for some fixed real a and $m \in \mathbb{N}$:

$$a^{(m)} = a(a-1)\cdots(a-m+1), \quad a^{(0)} = 1.$$

In the following Lemma, proved in [61], we give an identity on which all the results of this section are based. It can be observed as a generalization of the well-known Abel identity for an n-tuple $(a_1,\ldots,a_n)$ with weights $(p_1,\ldots,p_n)$, [51, p.334], given by

$$\sum_{i=1}^{n} p_i a_i = a_1 \sum_{i=1}^{n} p_i + \sum_{i=2}^{n} \left(\sum_{k=i}^{n} p_k \right) \Delta a_{i-1}. \tag{1.1}$$

The structure of the Abel identity can be described as following: the sum $\sum_{i=1}^{n} p_i a_i$ is represented as a sum of two sums. In the first sum the difference of the order 0 of element a_1 occures, while in the second sum the differences of the order 1 for the elements $a_1,\ldots,a_{n-m}$ occur. The Abel identity can be looked upon as a discrete analogue of the formula for integration by parts. The new identity has a similar structure: the right-hand side of it consists of two sums, in the first sum differences of order $0,1,\ldots,m-1$ of the first element a_1 appear, while in the second sum only the differences of order m occur but for elements $a_1,\ldots,a_{n-m}$.

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Lemma 1.1 *Let $m,n \in \mathbb{N}$, $m < n$. Let $(p_1,\dots,p_n)$, $(a_1,\dots,a_n)$ be real n-tuples. Then*

$$\sum_{i=1}^{n} p_i a_i = \sum_{k=0}^{m-1}\sum_{i=1}^{n} p_i (i-1)^{(k)} \frac{\Delta^k a_1}{k!} + \sum_{k=m+1}^{n}\left(\sum_{i=k}^{n} p_i (i-k+m-1)^{(m-1)}\right)\frac{\Delta^m a_{k-m}}{(m-1)!}. \tag{1.2}$$

Proof. We prove it by using mathematical induction on m. If $m=1$, then we have

$$\sum_{i=1}^{n} p_i a_i = a_1 \sum_{i=1}^{n} p_i + \sum_{k=2}^{n}\left(\sum_{i=k}^{n} p_i\right)\Delta a_{k-1},$$

which is, in fact, the Abel identity. Suppose that (1.2) is valid. Writting the Abel identity for $(n-m)$-tuple $(\Delta^m a_1, \Delta^m a_2, \dots, \Delta^m a_{n-m})$ with weights $(Q_{m+1}, Q_{m+2}, \dots, Q_n)$, where

$$Q_k = \sum_{i=k}^{n} (i-k+m-1)^{(m-1)} p_i$$

we get

$$\sum_{k=m+1}^{n} Q_k \Delta^m a_{k-m} = \Delta^m a_1 \sum_{j=m+1}^{n} Q_j + \sum_{k=m+2}^{n}\left(\sum_{j=k}^{n} Q_j\right)\Delta^{m+1} a_{k-m-1}.$$

The sum $\sum_{j=k}^{n} Q_j$ is equal to

$$\sum_{j=k}^{n} Q_j = \sum_{j=k}^{n}\sum_{i=j}^{n} (i-j+m-1)^{(m-1)} p_i = \frac{1}{m}\sum_{i=k}^{n} (i-k+m)^{(m)} p_i.$$

For $k=m+1$ we have

$$\sum_{j=m+1}^{n} Q_j = \frac{1}{m}\sum_{j=m+1}^{n} (j-1)^{(m)} p_j = \frac{1}{m}\sum_{j=1}^{n} (j-1)^{(m)} p_j,$$

where we use the fact that for $j=1,\dots,m$ the number $(j-1)^{(m)}$ is equal 0. So, we get

$$\sum_{k=m+1}^{n} Q_k \Delta^m a_{k-m}$$

$$= \frac{\Delta^m a_1}{m}\sum_{j=1}^{n} (j-1)^{(m)} p_j + \sum_{k=m+2}^{n}\left(\sum_{j=k}^{n} (j-k+m)^{(m)} p_j\right)\frac{\Delta^{m+1} a_{k-m-1}}{m}. \tag{1.3}$$

Let us write the right-hand side of identity (1.2) for $m+1$ instead of m:

$$\sum_{k=0}^{m}\sum_{i=1}^{n} p_i (i-1)^{(k)} \frac{\Delta^k a_1}{k!} + \sum_{k=m+2}^{n}\left(\sum_{i=k}^{n} p_i (i-k+m)^{(m)}\right)\frac{\Delta^{m+1} a_{k-m-1}}{m!}$$

$$
\begin{aligned}
&= \left(\sum_{k=0}^{m-1}\sum_{i=1}^{n} p_i(i-1)^{(k)}\frac{\Delta^k a_1}{k!} + \sum_{i=1}^{n} p_i(i-1)^{(m)}\frac{\Delta^m a_1}{m!}\right)\\
&+ \frac{1}{(m-1)!}\left(\sum_{k=m+1}^{n} Q_k\Delta^m a_{k-m} - \Delta^m a_1\frac{1}{m}\sum_{i=1}^{n}(i-1)^{(m)}p_1\right)\\
&= \sum_{k=0}^{m-1}\sum_{i=1}^{n} p_i(i-1)^{(k)}\frac{\Delta^k a_1}{k!} + \sum_{k=m+1}^{n}\left(\sum_{i=k}^{n} p_i(i-k+m-1)^{(m-1)}\right)\frac{\Delta^m a_{k-m}}{(m-1)!}\\
&= \sum_{i=1}^{n} p_i a_i,
\end{aligned}
$$

where we use (1.3) and the assumption of induction. So, by the principle of mathematical induction, identity (1.2) holds. □

Remark 1.1 We use the above identity for $m = n$ also. In that case the second sum vanishes.

The following theorem about m-convex sequences is given in [61] by J. Pečarić (see also [77, p. 253]):

Theorem 1.1 *Let $(p_1,\ldots,p_n)$ be a real n-tuple and $m \in \mathbb{N}$, $m < n$. The inequality*

$$\sum_{i=1}^{n} p_i a_i \geq 0 \tag{1.4}$$

holds for every m-convex n-tuple (a_i) if and only if

$$\sum_{i=1}^{n}(i-1)^{(k)}p_i = 0 \tag{1.5}$$

holds for every $k \in \{0,1,\ldots,m-1\}$ and

$$\sum_{i=k}^{n}(i-k+m-1)^{(m-1)}p_i \geq 0 \tag{1.6}$$

holds for every $k \in \{m+1,\ldots,n\}$.

Proof. If equalities (1.5) and inequalities (1.6) are satisfied, then the first sum in identity (1.2) is equal to 0, the second sum is nonnegative and the inequality $\sum_{i=1}^{n} p_i a_i \geq 0$ holds.

Conversely, let us suppose that $\sum_{i=1}^{n} p_i a_i \geq 0$ holds for any m-convex sequence (a_i). Since the sequence $a_i = (i-1)^{(k)}$, $i \in \{1,\ldots,n\}$ is m-convex for every $k \in \{0,\ldots,m-1\}$, we get $\sum_{i=1}^{n} p_i(i-1)^{(k)} \geq 0$. Convexity of the mentioned sequences are proved in Chapter 2 in detail. Similarly, since the sequence $a_i = -(i-1)^{(k)}$, $i \in \{1,\ldots,n\}$ is m-convex for every $k \in \{0,\ldots,m-1\}$, using (1.4) we get $-\sum_{i=1}^{n} p_i(i-1)^{(k)} \geq 0$. Hence, $\sum_{i=1}^{n} p_i(i-1)^{(k)} = 0$.

Also the sequence

$$a_i = \begin{cases} 0, & i \in \{1,\ldots,k-1\},\\ (i-k+m-1)^{(m-1)}, & i \in \{k,\ldots,n\}, \end{cases} \tag{1.7}$$

is m-convex for every $k \in \{m+1,\dots,n\}$. Thus, by (1.4), we get (1.6). □

Remark 1.2 It is easy to see that condition (1.5) is equivalent to the following conditions:

$$\sum_{i=1}^{n}(i-1)^k p_i = 0, \quad k \in \{0,1,\dots,m-1\} \text{ with } 0^0 = 1 \tag{1.8}$$

or

$$\sum_{i=1}^{n} i^k p_i = 0, \quad k \in \{0,1,\dots,m-1\}. \tag{1.9}$$

Also, it is instructive to observe that

$$\frac{(i-1)^{(k)}}{k!} = \binom{i-1}{k}, \quad \frac{(i-k+m-1)^{(m-1)}}{(m-1)!} = \binom{i-k+m-1}{m-1}.$$

In the first sum of (1.2) the numbers $(i-1)^{(k)}$ are equal 0 for $i = 1,\dots,k$, so sometimes as a range for i we use i from $k+1$ till n.

If an n-tuple (a_i) is convex of several consecutive orders we have the following theorem which is a consequence of Theorem 1.1. This result can be found in [71].

Theorem 1.2 *Let $(p_1,\dots,p_n)$ be a real n-tuple and $m \in \mathbb{N}$, $m < n$, $j \in \{1,\dots,m\}$. Then inequality (1.4) holds for every n-tuple $(a_1,\dots,a_n)$ that is convex of order $j, j+1,\dots,m$ if and only if*

$$\sum_{i=1}^{n}(i-1)^{(k)} p_i = 0 \tag{1.10}$$

holds for $k \in \{0,1,\dots,j-1\}$,

$$\sum_{i=1}^{n}(i-1)^{(k)} p_i \geq 0 \tag{1.11}$$

holds for $k \in \{j, j+1,\dots,m-1\}$ and

$$\sum_{i=k}^{n}(i-k+m-1)^{(m-1)} p_i \geq 0 \tag{1.12}$$

holds for $k \in \{m+1,\dots,n\}$.

Proof. If $k \in \{0,1,\dots,j-1\}$, then the sequences $((i-1)^{(k)})_i$ and $(-(i-1)^{(k)})_i$ are convex of order $j, j+1,\dots,m$. So, for such k, $\sum_{i=1}^{n}(i-1)^{(k)}p_i = 0$ holds. If $k \in \{j, j+1,\dots,m-1\}$, then the sequence $((i-1)^{(k)})_i$ is convex of order $j, j+1,\dots,m$ and $\sum_{i=1}^{n}(i-1)^{(k)}p_i \geq 0$ for such k.

Since the sequence (a_n) defined as in (1.7) is convex of order $j, j+1,\dots,m$, so (1.12) holds. This proves one implication of the theorem while the other follows from Lemma 1.1. □

A sequence (a_i) is called absolutely monotonic of order m if all the lower order differences of that sequence are nonnegative, i.e. if

$$\Delta^k a_i \geq 0 \quad \text{for } k \in \{1,2,\ldots,m\}.$$

As a consequence of the previous Theorem 1.2 we get the following necessary and sufficient conditions for positivity of sum $\sum p_i a_i$ for an absolutely monotonic sequence of order m. Namely, we obtain the following theorem.

Corollary 1.1 *Let $(p_1,\ldots,p_n)$ be a real n-tuple and $m \in \mathbb{N}$, $m < n$. Then inequality* (1.4) *holds for every n-tuple $(a_1,\ldots,a_n)$ that is absolutely monotonic of order m if and only if*

$$\sum_{i=1}^{n} p_i = 0, \quad \sum_{i=1}^{n} (i-1)^{(k)} p_i \geq 0$$

holds for $k \in \{1,\ldots,m-1\}$, and

$$\sum_{i=k}^{n} (i-k+m-1)^{(m-1)} p_i \geq 0 \ \ \textit{for} \ \ k \in \{m+1,\ldots,n\}.$$

The following theorem describes how bounds for the sum $\sum p_i a_i$ depend on bounds of $\Delta^m a_k$, (see [71]). In fact, using that result we can strengthen the initial inequality.

Theorem 1.3 *Let $m \in \mathbb{N}$, $m < n$ and $(a_1,\ldots,a_n)$, $(p_1,\ldots,p_n)$ be real n-tuples such that*

$$\sum_{i=1}^{n} (i-1)^{(k)} p_i = 0 \ \ \textit{for} \ \ k \in \{0,1,\ldots,m-1\} \tag{1.13}$$

and

$$\sum_{i=k}^{n} (i-k+m-1)^{(m-1)} p_i \geq 0 \ \ \textit{for} \ \ k \in \{m+1,\ldots,n\}. \tag{1.14}$$

If

$$a \leq \Delta^m a_k \leq A \ \textit{for} \ \ k \in \{1,2,\ldots,n-m\}, \tag{1.15}$$

then

$$\frac{a}{m!} \sum_{i=1}^{n} p_i i^{(m)} \leq \sum_{i=1}^{n} p_i a_i \leq \frac{A}{m!} \sum_{i=1}^{n} p_i i^{(m)}.$$

Proof. The sequences

$$b_k = a_k - \frac{a}{m!} k^{(m)} \ \text{ and } \ c_k = \frac{A}{m!} k^{(m)} - a_k$$

have the following properties

$$\Delta^m b_k = \Delta^m a_k - a \ \text{ and } \ \Delta^m c_k = A - \Delta^m a_k.$$

By (1.15), we get that the sequences (b_k) and (c_k) are m-convex. Since (p_k) satisfies conditions (1.13) and (1.14), then using Theorem 1.1 we get that

$$\sum_{i=1}^{n} p_i b_i \geq 0 \text{ and } \sum_{i=1}^{n} p_i c_i \geq 0$$

and desired inequalities hold. □

Remark 1.3 For $a = -A$ condition (1.15) becomes $|\Delta^m a_k| \leq A$ and then the statement of the above theorem becomes

$$\left|\sum_{i=1}^{n} p_i a_i\right| \leq \frac{A}{m!} \sum_{i=1}^{n} p_i i^{(m)}.$$

Example 1.1 A nice application of Theorem 1.1 is a proof of the Nanson inequality. In [52] E.J. Nanson proved the following inequality: If a real $(2n+1)$-tuple $(a_1, \ldots, a_{2n+1})$ is convex, then

$$\frac{a_1 + a_3 + \ldots + a_{2n+1}}{n+1} \geq \frac{a_2 + a_4 + \ldots + a_{2n}}{n}. \tag{1.16}$$

The original proof of the Nanson inequality (1.16) and some historical remarks are given in $[49, pp.202-203]$. Here we give a proof of (1.16) based on Theorem 1.1.

Putting

$$N = 2n+1,\ p_1 = p_3 = \ldots = p_{2n+1} = \frac{1}{n+1},\ p_2 = p_4 = \ldots = p_{2n} = -\frac{1}{n}$$

we get

$$\sum_{i=1}^{N} p_i = \frac{1}{n+1} - \frac{1}{n} + \ldots + \frac{1}{n+1} - \frac{1}{n} + \frac{1}{n+1} = n\left(\frac{1}{n+1} - \frac{1}{n}\right) + \frac{1}{n+1} = 0,$$

$$\begin{aligned} \sum_{i=1}^{N}(i-1)p_i &= \frac{0}{n+1} - \frac{1}{n} + \frac{2}{n+1} - \frac{3}{n} \ldots + \frac{2n-2}{n+1} - \frac{2n-1}{n} + \frac{2n}{n+1} \\ &= \frac{2+4+\ldots+2n}{n+1} - \frac{1+3+\ldots+2n-1}{n} = \frac{n(n+1)}{n+1} - \frac{n^2}{n} = 0, \end{aligned}$$

and for $k \geq 3$

$$\begin{aligned} &\sum_{i=k}^{N}(i-k+1)p_i = p_k + 2p_{k+1} + 3p_{k+2} + \ldots + (N-k+1)p_N \\ &= \begin{cases} \frac{1}{n+1} + \left(-\frac{2}{n} + \frac{3}{n+1}\right) + \left(-\frac{4}{n} + \frac{5}{n+1}\right) + \ldots \left(-\frac{N-k}{n} + \frac{N-k+1}{n+1}\right), & k \text{ even} \\ \left(-\frac{1}{n} + \frac{2}{n+1}\right) + \left(-\frac{3}{n} + \frac{4}{n+1}\right) + \ldots + \left(-\frac{N-k}{n} + \frac{3}{N-k+1}\right), & k \text{ odd} \end{cases} \end{aligned}$$

$$= \begin{cases} \dfrac{\left(\frac{N-k}{2}+1\right)\left(n-\frac{N-k}{2}\right)}{n(n+1)} \geq 0, \ k \text{ even} \\ \dfrac{1}{n(n+1)} \dfrac{N-k+1}{2}\left(n-\dfrac{N-k+1}{2}\right) \geq 0, \ k \text{ odd.} \end{cases}$$

Applying Theorem 1.1 for $m=2$ we get that $\sum_{i=1}^{N} p_i a_i \geq 0$, i.e.

$$\frac{a_1}{n+1}-\frac{a_2}{n}+\frac{a_3}{n+1}-\frac{a_4}{n}+\ldots\frac{a_{2n-1}}{n+1}-\frac{a_{2n}}{n}+\frac{a_{2n+1}}{n+1} \geq 0$$

which is the desired inequality (1.16).

Let us use Theorem 1.3 to get an estimate for the difference of the left-hand and the right-hand side of the Nanson inequality if the second differences are bounded. This result is proved in [3] using different approach.

Let us suppose that for sequence (a_i) the following holds

$$a \leq \Delta^2 a_k \leq A, \ \ k \in \{1,2,\ldots 2n-1\}$$

for some $a, A \in \mathbf{R}$. Then

$$\frac{2n+1}{6}a \leq \frac{a_1+a_3+\ldots+a_{2n+1}}{n+1}-\frac{a_2+a_4+\ldots+a_{2n}}{n} \leq \frac{2n+1}{6}A. \tag{1.17}$$

From the previous calculation we have that (1.13) holds for $k=0,1$ and (1.14) holds for $k=2$. Let us calculate $\sum_{i=1}^{N} p_i i^{(2)}$.

$$\begin{aligned} \sum_{i=1}^{N} p_i i^{(2)} &= \sum_{i=1}^{N} p_i i^2 - \sum_{i=1}^{N} p_i i = \sum_{i=1}^{N} p_i i^2 \\ &= \frac{1}{n+1}(1^2+3^2+\ldots+(2n+1)^2)+\frac{1}{n}(2^2+4^2+\ldots+(2n)^2) \\ &= \frac{2n+1}{3}. \end{aligned}$$

From that result we get (1.17).

Example 1.2 Let us illustrate an application of Theorem 1.1 to another inequality due to N. Ozeki. In [55], and also in [49, p.199], the following result is given: If $a_{n-1}+a_{n+1} \geq 2a_n$ for $n=2,3,\ldots$, then

$$A_{n-1}+A_{n+1} \geq 2A_n, \ \ n=2,3,\ldots, \tag{1.18}$$

where

$$A_n = \frac{a_1+\ldots+a_n}{n}.$$

In other words, if a sequence (a_i) is convex, then the sequence (A_i) of arithmetic means is also convex.

Putting

$$p_1=p_2=\ldots=p_{n-1}=\frac{1}{n-1}+\frac{1}{n+1}-\frac{2}{n}, \ \ p_n=\frac{1}{n+1}-\frac{2}{n}, \ \ p_{n+1}=\frac{1}{n+1},$$

we get

$$\sum_{i=1}^{n+1} p_i = 0,\ \sum_{i=1}^{n+1}(i-1)p_i = 0,\ \sum_{i=k}^{n+1}(i-k+1)p_i \geq 0.$$

Using Theorem 1.1 for $m=2$ we get that $\sum_{i=1}^{n+1} p_i a_i \geq 0$, i.e.

$$a_1\left(\frac{1}{n-1}+\frac{1}{n+1}-\frac{2}{n}\right)+\ldots+a_{n-1}\left(\frac{1}{n-1}+\frac{1}{n+1}-\frac{2}{n}\right)$$
$$+a_n\left(\frac{1}{n+1}-\frac{2}{n}\right)+\frac{1}{n+1}a_{n+1} \geq 0,$$
$$\frac{a_1+a_2+\ldots+a_{n-1}}{n-1}+\frac{a_1+a_2+\ldots+a_{n+1}}{n+1}-2\frac{a_1+a_2+\ldots+a_n}{n} \geq 0$$

which is the desired inequality (1.18).

Example 1.3 If (a_i) is convex, then for any $n \geq 1$

$$a_1+a_3+\ldots+a_{2n+1} \geq a_2+a_4+\ldots+a_{2n}+\frac{a_1+a_3+\ldots+a_{2n+1}}{n+1}. \tag{1.19}$$

This inequality for $a_i = a$ is due to Steinig ([3, 92]).

To prove this, we use Theorem 1.1 for $m=2$. Putting

$$N = 2n+1,\ p_1 = p_3 = \ldots = p_{2n+1} = \frac{n}{n+1},\ p_2 = p_4 = \ldots = p_{2n} = -1$$

we get that property (1.13) holds for $k=0,1$ and (1.14) holds for $k=2$. So, by Theorem 1.1 inequality (1.19) holds. Furthermore, if (a_i) satisfies (1.13) for $k=0,1$, (1.14) for $k=2$ and if $a \leq \Delta^2 a_k \leq A$ $(k=1,\ldots,2n-1)$, then

$$\frac{n(2n+1)}{6}a \leq a_1-a_2+a_3-\ldots+a_{2n+1}-\frac{a_1+a_3+\ldots+a_{2n+1}}{n+1} \leq \frac{n(2n+1)}{6}A.$$

Let us again consider a basic identity from Lemma 1.1, with slightly modified indexing in the first sum:

$$\begin{aligned}\sum_{i=1}^{n} p_i a_i &= \sum_{k=1}^{m}\sum_{i=1}^{n} p_i (i-1)^{(k-1)}\frac{\Delta^{k-1}a_1}{(k-1)!}\\ &+ \sum_{k=m+1}^{n}\left(\sum_{i=k}^{n} p_i (i-k+m-1)^{(m-1)}\right)\frac{\Delta^m a_{k-m}}{(m-1)!}.\end{aligned}$$

Putting $p_1 = \ldots = p_{n-1} = 0$ and $p_n = 1$ we obtain the following ([79])

$$a_n = \begin{cases} \displaystyle\sum_{k=1}^{m}(n-1)^{(k-1)}\frac{\Delta^{k-1}a_1}{(k-1)!} \\ \qquad\displaystyle + \sum_{k=m+1}^{n}(n-k+m-1)^{(m-1)}\frac{\Delta^m a_{k-m}}{(m-1)!}, & m<n, \\ \displaystyle\sum_{k=1}^{n}(n-1)^{(k-1)}\frac{\Delta^{k-1}a_1}{(k-1)!}, & m=n. \end{cases}$$

The above-mentioned identity can be considered as the Taylor formula for sequences.

The following theorem was published in [62] and it gives results about preservation of convexity of a sequence which is made from a sequence (a_i).

Let $(a_0,a_1,a_2,\ldots)$ be a real sequence and $[p_{n,i}]$, $i=0,1,\ldots,n$; $n=0,1,2,\ldots$ a lower triangular matrix of real numbers, i.e.

$$\begin{bmatrix} p_{00} & 0 & 0 & 0 & \ldots & 0 & \ldots \\ p_{10} & p_{11} & 0 & 0 & \ldots & 0 & \ldots \\ p_{20} & p_{21} & p_{22} & 0 & \ldots & 0 & \ldots \\ \vdots & \vdots & \vdots & & & & \\ p_{n0} & p_{n1} & p_{n2} & p_{n3} & \cdots & p_{nn} & 0\ldots \\ \vdots & \vdots & \vdots & & & & \end{bmatrix}.$$

Let (σ_n) be a sequence defined as

$$\sigma_n = \sum_{k=0}^{n} p_{n,n-k}a_k, \quad n=0,1,2,\ldots \tag{1.20}$$

Theorem 1.4 *Let σ_n be defined as in (1.20) and $s\in\mathbb{N}$. Then the implication*

$$\Delta^m a_n \geq 0 \Rightarrow \Delta^s \sigma_n \geq 0$$

is valid for every sequence (a_n) if and only if

$$\Delta^s X_n(k+1,k) = 0 \quad \textit{for} \quad k\in\{0,1,\ldots,m-1\}; \quad n\in\{0,1,2,\ldots\}$$

and

$$\Delta^s X_n(m,k) \geq 0 \quad \textit{for} \quad k\in\{m,\ldots,n+s\}; \quad n\in\{0,1,2,\ldots\}$$

where

$$X_n(m,k) = \begin{cases} 0 & \textit{for} \quad n<k \\ \displaystyle\sum_{j=0}^{n-k}\binom{n-k+m-1-j}{m-1}p_{n,j} & \textit{for} \quad n\geq k. \end{cases} \tag{1.21}$$

Proof. Let us write the difference $\Delta^s\sigma_n$ as a linear combination of the elements a_j. Using the notation:

$$q_n(j) = \begin{cases} 0 & \text{for} \quad n<j \\ p_{n,n-j} & \text{for} \quad n\geq j \end{cases}$$

we get the following

$$\begin{aligned} \Delta\sigma_n &= \sigma_{n+1}-\sigma_n = \sum_{j=0}^{n+1} p_{n+1,n+1-j}a_j - \sum_{j=0}^{n} p_{n,n-j}a_j \\ &= \sum_{j=0}^{n}(p_{n+1,n+1-j}-p_{n,n-j})a_j + p_{n+1,0}a_{n+1} \end{aligned}$$

$$= \sum_{j=0}^{n} \Delta q_n(j) a_j + \Delta q_n(n+1) a_{n+1} = \sum_{j=0}^{n+1} \Delta q_n(j) a_j,$$

$$\begin{aligned}
\Delta^2 \sigma_n &= \Delta\sigma_{n+1} - \Delta\sigma_n = \sum_{j=0}^{n+2} \Delta q_{n+1}(j) a_j - \sum_{j=0}^{n+1} \Delta q_n(j) a_j \\
&= \sum_{j=0}^{n+1} \Delta(q_{n+1}(j) - q_n(j)) a_j + \Delta q_{n+1}(n+2) a_{n+1} \\
&= \sum_{j=0}^{n+2} \Delta^2 q_n(j) a_j.
\end{aligned}$$

Similarly, we get

$$\Delta^s \sigma_n = \sum_{j=0}^{n+s} \Delta^s q_n(j)\, a_j \quad \text{for every } s \tag{1.22}$$

and

$$\Delta^s X_n(m,k) = \sum_{i=k}^{n+s} \binom{i-k+m-1}{m-1} \Delta^s q_n(i).$$

Writting identity (1.2) for $n+s+1$-tuples $(a_0, a_1, \ldots, a_{n+s})$ and $(\Delta^s q_n(0), \Delta^s q_n(1), \ldots, \Delta^s q_n(n+s))$ and using the above results we get the identity

$$\Delta^s \sigma_n = \sum_{k=0}^{m-1} \Delta^k a_0\, \Delta^s X_n(k+1,k) + \sum_{k=m}^{n+s} \Delta^m a_{k-m}\, \Delta^s X_n(m,k). \tag{1.23}$$

Hence, the statement follows from Theorem 1.1. □

Theorem 1.4 is a generalization of several previously published results. Firstly, in [56] N. Ozeki obtained conditions on a matrix $[p_{n,i}]$ implying that for each convex sequence (a_n) the sequence (σ_n) is also convex, i.e. it is a particular case of Theorem 1.4 for $m = s = 2$. One decade later a particular case of Theorem 1.4 for $m = s$ was published in [34] and [41].

A result which is based on identity (1.23) is given as the following theorem, [62].

Theorem 1.5 *Let (a_n) be a real sequence and let σ_n be defined as in (1.20). If $|\Delta^m a_n| \le N$ for $n \in \{0,1,2,\ldots\}$, and*

$$\Delta^s X_n(k+1,k) = 0 \quad \text{for} \quad k \in \{0,1,\ldots,m-1\}; \quad n \in \{0,1,2,\ldots\} \tag{1.24}$$

where $X_n(m,k)$ is given in (1.21), then

$$|\Delta^s \sigma_n| \le N \sum_{k=m}^{n+s} |\Delta^s X_n(m,k)|.$$

Proof. This is an immediate consequence of (1.23). □

The following theorem also gives a bounds for $\Delta^s \sigma_n$, (see [71]).