MONOGRAPHS IN INEQUALITIES 19

Some New Improvements of Jensen's Inequality

Jensen's Type Inequalities in Information Theory

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ELEMENT

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Preface

The classical Jensen inequality is a famous tool to construct new results in the theory of inequalities. It has numerous applications in abstract and applied sciences. In this monograph, some recent developments in theory of inequalities with respect to Jensen's inequality are collected and presented. Applications are given in information theory by evaluating the estimates for different entropies and divergences via some recent and existing refinements of Jensen's inequality. The results for convex functions in this context are also generalized for higher order convex functions by means of interpolating polynomials.

In the first chapter, some basic notions and preliminary results are recalled which are used in the sequel.

In the second chapter, we focus on the refinements of integral Jensen's as well as discrete Jensen's inequalities. First we derive a refinement of integral Jensen's inequality associated to two functions whose sum is equal to unity. As applications of the refinement of integral Jensen's inequality we obtain refinements of Hölder, integral power means and Hermite-Hadamard inequalities. We also give applications in information theory and provide a more general refinement of integral Jensen's inequality. We establish a refinement of discrete Jensen's inequality concerning certain tuples and give applications to different means. Finally, we give applications of discrete main result in information theory and provide a more general refinement of discrete Jensen's inequality.

In the third chapter, we derive refinements of discrete as well as integral Jensen-Steffensen's inequalities associated to certain tuples and functions respectively and also present application to the Zipf Mandelbrot law. Some more general refinements are also presented for Jensen-Steffensen's inequality.

In the fourth chapter, we propose new refinements for the Jensen-Mercer as well as for variant of the Jensen-Mercer inequalities associated to certain positive tuples. We give some related integral versions and present applications for different means. Further generalizations are given which are associated to m finite sequences.

In the fifth chapter, we give a refinement of Jensen's inequality for convex functions of several variables associated to certain tuples. As an application, we deduce refinements of Beck's inequality. At the end, further generalization has been presented for n finite sequences.

In the sixth chapter, we present refinements of generalized Jensen's inequalities given by Jessen and McShane. As applications of the refinement of Jessen's inequality, we deduce refinements of generalized means and Hölder inequalities. Also, as applications of the refinement of McShane's inequality, we obtain refinements of generalized Beck's inequality and discuss their particular cases. At the end of this chapter, we give further generalizations of Jessen's and McShane's inequalities pertaining n certain functions.

In the seventh chapter, we obtain a refinement of Jensen's inequality for operator convex functions. Some applications are presented for different means and also, deduced refinement of operator inequality connected to the operator concavity of operator entropy $A\log A^{-1}$. Further generalization is also given for operator Jensen's inequality.

In the eighth chapter, estimation of f-divergence, Shannon entropy, Rényi divergence and Rényi entropy are studied via refinements of Jensen's inequality. The results show the applications of Jensen's inequality in information theory.

In the ninth chapter, Montgomery identity, Hermite interpolation, Lidstone polynomial, Fink identity and Abel-Gontscharoff Green function, Taylor one point and Taylor two point formula are used to generalize the refinements of Jensen, Rényi and Shannon type inequalities for the class of higher order convex functions.

In the tenth chapter, an integral form of Popoviciu's inequality involving samples with repetitions and without repetitions is given with the refinement of the integral Jensen inequality. Applications to power means are studied with respect to monotonicity property.

In the eleventh chapter, the Jensen differences involving two types of data points are refined for the class of 3-convex functions.

Authors

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Chapter 1

Introduction

Convex functions have a significant role while dealing with optimization problems. Geometry of convex functions leads to many important inequalities which are frequently used to estimate and compare the values related to many physical problems in different branches of mathematics and physics. Entropies and divergences are widely studied in information theory. While dealing with many physical problems physicist have to deal with the structure involving higher dimension convexity. Therefore there are two important gaps first to estimate the entropies and divergences and second one is to study the inequalities for higher dimension problems. Therefore we estimate different entropies and divergences and secondly we generalize the related results for higher order convex functions.
It is of great interest for researchers to study inequalities of continuous data and arbitrary weights. For example integral version of Popoviciu's inequality are studied in the sequel.

1.1 Some Inequalities Involving Convex Functions

The first chapter contains: introduction to convex functions, various inequalities involving convex functions, refinement of these inequalities given by various researchers in recent years, the weighted version of Popoviciu's inequality, some notions from information theory containing entropies and divergences. These will be used frequently in the following chapters while obtaining main results.

1.1.1 J-Convex Functions

In 1905–1906, J. L. W. V. Jensen began the systematic study of convex functions (see [75, p.3]).

A function $\eta : I \to \mathbb{R}$ is said to be J-convex or mid-convex function or convex in Jensen sense on I if

$$\eta\left(\frac{u+v}{2}\right) \leq \frac{\eta(u)+\eta(v)}{2} \tag{1.1}$$

holds for all $u, v \in I$.

The $\eta(x) = x^2$ and $\eta(x) = |x|$ for all $x \in \mathbb{R}$ are the examples of J-convex functions.

1.1.2 Convex Functions

The notion of convex function is the generalization of J-convex function for the arbitrary weight $t \in [0,1]$. In [87, p. 1] the formal definition is given as follows.
Suppose X is a real vector space, $C \subset X$ is a convex set. A function $\eta : C \to \mathbb{R}$ is said to be convex if

$$\eta(\sigma u + (1-\sigma)v) \leq \sigma\eta(u) + (1-\sigma)\eta(v),$$

holds for all $u, v \in C$ and $\sigma \in [0,1]$.

The $\eta(x) = x^2$, $\eta(x) = |x|$, $-\log x$ and e^x for all $x \in \mathbb{R}$ are the examples of convex functions.

1.1.3 Operator Convex Functions

Let I be an interval of real numbers and $S(I)$ denotes the class of all self-adjoint bounded operators defined on complex Hilbert space H whose spectra are in I. Also, assume that $Sp(A)$ denotes the spectrum of a bounded operator A defined on H. An operator $A \in S(I)$ is said to be strictly positive if it is positive and invertible, or equivalently, $Sp(A) \subset [d_1, d_2]$ for $0 < d_1 < d_2$.

Let $\psi : I \to \mathbb{R}$ be a function defined on the interval I. Then ψ is said to be operator convex if ψ is continuous and

$$\psi(\zeta A_1 + (1-\zeta)A_2) \leq \zeta\psi(A_1) + (1-\zeta)\psi(A_2) \tag{1.2}$$

for all $A_1, A_2 \in S(I)$ and $\zeta \in [0,1]$. If the function $-\psi$ is operator convex on I, then ψ is said to be operator concave. The function ψ is said to be operator monotone on I if ψ is continuous on I and $A_1, A_2 \in S(I)$, $A_1 \leq A_2$ (i.e. $A_2 - A_1$ is positive operator), then $\psi(A_1) \leq \psi(A_2)$.

1.1.4 Discrete Jensen's Inequality

The Jensen inequality in discrete version [87, p. 43] generalizes the notion of convex function. Here the function operates on the convex combination of any finite number of points.

Suppose X is a real vector space, $C \subset X$ is a convex set, let $\psi : C \to \mathbb{R}$ be a convex function, $\zeta_1, \ldots, \zeta_n \in [0,1]$ are such that $\sum_{i=1}^{n} \zeta_i = 1$, and $y_1, \ldots, y_n \in C$, then

$$\psi\left(\sum_{\gamma=1}^{n} \zeta_\gamma y_\gamma\right) \leq \sum_{\gamma=1}^{n} \zeta_\gamma \psi(y_\gamma). \tag{1.3}$$

In the Jensen inequality, it is natural to ask the question that is it possible to relax the condition of nonnegative of ζ_γ $(\gamma = 1,2,\ldots,n)$ at the expense of restricting y_γ $(\gamma = 1,2,\ldots,n)$ more severely. The answer of this question was given by Steffensen [96]:

1.1.5 Discrete Jensen-Steffensen's Inequality

Let $\psi : I \to \mathbb{R}$ be a convex function defined on the interval I. Let $y_\gamma \in I$, $\zeta_\gamma \in \mathbb{R}$ $(\gamma = 1,2,\ldots,n)$ with $\overline{\zeta} = \sum_{\gamma=1}^{n} \zeta_\gamma$. If $y_1 \leq y_2 \leq \cdots \leq y_n$ or $y_1 \geq y_2 \geq \cdots \geq y_n$ and

$$0 \leq \sum_{\gamma=1}^{k} \zeta_\gamma \leq \sum_{\gamma=1}^{n} \zeta_\gamma \;\; k = 1,2,\ldots,n, \quad \sum_{\gamma=1}^{n} \zeta_\gamma > 0, \tag{1.4}$$

then

$$\psi\left(\frac{1}{\overline{\zeta}} \sum_{\gamma=1}^{n} \zeta_\gamma y_\gamma\right) \leq \frac{1}{\overline{\zeta}} \sum_{\gamma=1}^{n} \zeta_\gamma \psi(y_\gamma). \tag{1.5}$$

1.1.6 Integral Form of Jensen's Inequality

The integral form of Jensen's inequality [47] is defined as follows.
Let (X, A, μ) be a probability space, consider an integrable function $h : X \to I$. Also let $\psi : I \to \mathbb{R}$ be a convex function. Then

$$\psi\left(\int_X h d\mu\right) \leq \int_X \psi \circ h d\mu. \tag{1.6}$$

1.1.7 Integral Version of Jesnen-Steffensen's Inequality

Integral version of Jensen-Steffensen's inequality is given by:
Let I be an interval in $\mathbb{R}$ and $g, h : [a,b] \to \mathbb{R}$ are integrable functions such that $g(\rho) \in I$ for all $\rho \in [a,b]$. Also, assume that $\psi : I \to \mathbb{R}$ is convex function and $h(\psi \circ g)$ is integrable on $[a,b]$. If g is monotonic on $[a,b]$ and h satisfies

$$0 \leq \int_a^\lambda h(\rho) d\rho \leq \int_a^b h(\rho) d\rho, \;\lambda \in [a,b], \quad \int_a^b h(\rho) d\rho > 0, \tag{1.7}$$

then

$$\psi\left(\frac{\int_a^b g(\rho) h(\rho) d\rho}{\int_a^b h(\rho) d\rho}\right) \leq \frac{\int_a^b h(\rho)(\psi \circ g)(\rho) d\rho}{\int_a^b h(\rho) d\rho}. \tag{1.8}$$

This means convex function and Jensen-type inequalities are linked to each other. In fact definition of convex function involves inequality sign. Until now, inequalities have played a major role in convex function development. In mathematics the role of inequalities is very important, specially in approximation theory and analysis. The linear programming is based on inequalities. A number of mathematicians have a keen interest in the study of mathematical inequalities.

Jensen's inequality is the fundamental inequality for convex function. Many classical inequalities (for instance Minkowski's inequality, Hölder's inequality *etc.*) and other inequalities are the consequences of Jensen's inequality.

L. Horváth and J. Pečarić in [49] used a refinement of discrete Jensen's inequality to construct a new refinement of (1.6), which is a generalization of a result given in [25]. They also gave new monotone quasi arithmetic means.

In a last few decades, many researcher papers have appeared in literature concerning the refinement of discrete Jensen's inequality (see [47]). However the refinement of discrete Jensen's inequality has been studied more compared to the refinement of its integral version. The researchers used the refinements of (1.3) to construct new refinements of (1.6). For instance we can see the following results [88].
Suppose that f is a J-convex function on an interval J, $c_j \in J$, $j = 1,\ldots,n$. Then

$$\eta_{r,n} \geq \eta_{r-1,n} \quad r = 1,\ldots,n-1, \tag{1.9}$$

where

$$\eta_{r,n} = \eta_{k,n}(c_1,\ldots,c_n) := \frac{1}{\binom{n}{r}} \sum_{1\leq j_1<\ldots<j_r\leq n} \eta\left(\frac{1}{r}(c_{j_1}+\ldots+c_{j_r})\right).$$

For positive weights the above results are given in [84].
Suppose $\overline{\eta}$ is convex function defined on an interval J, $c_j \in J$ $(j = 1,...,n)$.

$$\overline{\eta}_{r,n}(c_1,\ldots,c_n,\sigma_1,\ldots,\sigma_n) := \frac{1}{\binom{n-1}{r-1}P_n} \sum_{1\leq j_1<\ldots<j_r\leq n} (\sigma_{j_1}+\ldots+\sigma_{j_r})\overline{\eta}\left(\frac{\sigma_{j_1}c_{j_1}+\ldots+\sigma_{j_r}c_{j_r}}{\sigma_{j_1}+\ldots+\sigma_{j_k}}\right)$$

where $(\sigma_1,\ldots,\sigma_n)$ is suppose to be a positive n-tuple with $\sum_{j=1}^{r}\sigma_j = P_r$, then

$$\overline{\eta}_{r,n}(c_1,\ldots,c_n,\sigma_1,\ldots,\sigma_n) \geq \overline{\eta}_{r+1,n}(c_1,\ldots,c_n,\sigma_1,\ldots,\sigma_n) \quad n = 1,\ldots,r-1. \tag{1.10}$$

J. Pečarić and D. Svrtan noted that by considering the expresssion

$$\tilde{\eta}_{r,n} = \frac{1}{\binom{n+r-1}{r-1}P_n} \sum_{1\leq j_1\leq\ldots\leq j_k\leq n} (\sigma_{j_1}+\ldots+\sigma_{j_r})\left(\frac{\sigma_{j_1}c_{j_1}+\ldots+\sigma_{j_r}c_{j_r}}{\sigma_{j_1}+\ldots+\sigma_{j_r}}\right)$$

we have the same results

$$\eta\left(\frac{1}{P_n}\sum_{i=1}^{n}\sigma_i c_i\right) \leq \ldots \leq \tilde{\eta}_{r+1,n} \leq \tilde{\eta}_{r,n} \leq \ldots \leq \tilde{\eta}_{1,n} = \frac{1}{P_n}\sum_{i=1}^{n}\sigma_i\eta(c_i). \tag{1.11}$$

The researchers have given the refinements of (1.3) by using different indexing sets (see [50, 48]). Like many other researchers L. Horváth and J. Pečarić gave a refinement of

(1.3) for convex functions (see [50]). They defined some essential notions to prove the refinement given as follows:

Let X be a set, let $P(X)$ and $|X|$ represent the power set and number of elements of set X respectively. Let $\mathbb{N} := \{0\} \bigcup \{1,2,\ldots\}$.
Suppose $q \geq 1$ and $r \geq 2$ are two fixed integers. Suppose

$$\nabla_r(q) := \{1,\ldots,q\}^r$$

Now let

$$F_{r,s} : \nabla_r(q) \to \nabla_{r-1}(q) \quad 1 \leq s \leq r,$$

$$F_r : \nabla_r(q) \to P\left(\nabla_{r-1}(q)\right),$$

and

$$T_r : P\left(\nabla_r(q)\right) \to P\left(\nabla_{r-1}(q)\right),$$

are functions defined by

$$F_{r,s}(i_1,\ldots,i_r) := (i_1,i_2,\ldots,i_{s-1},i_{s+1},\ldots,i_r) \quad 1 \leq s \leq r,$$

$$F_r(i_1,\ldots,i_r) := \bigcup_{s=1}^{r} \{F_{r,s}(i_1,\ldots,i_r)\},$$

and

$$T_r(I) = \begin{cases} \emptyset, & I = \emptyset; \\ \bigcup\limits_{(i_1,\ldots,i_r)\in I} F_r(i_1,\ldots,i_r), & I \neq \emptyset. \end{cases}$$

Next for all $i \in \{1,\ldots,q\}$ consider

$$\alpha_{r,i} : \{1,\ldots,q\}^r \to \mathbb{N},$$

defined by

$$\alpha_{r,i}(i_1,\ldots,i_r) \text{ is the number of occurences of } i_j \text{ in } (i_1,\ldots,i_r).$$

For each $I \in P\left(\nabla_r(q)\right)$ let

$$\alpha_{I,i} := \sum_{(i_1,\ldots,i_r)\in I} \alpha_{r,i}(i_1,\ldots,i_r) \quad 1 \leq i \leq q.$$

$(\mathbf{H_1})$: Let $n \in \{1,2,\ldots\}$ and $m \in \{2,3,\ldots\}$, suppose $I_m \subset \nabla_m(n)$ such that for all $i \in \{1,\ldots,n\}$

$$\alpha_{I_m,i} \geq 1. \tag{1.12}$$

Introduce the set $I_l \subset \nabla_l(n)(1 \leq l \leq m-1)$ inductively by

$$I_{l-1} := T_l(I_l) \quad m \geq l \geq 2.$$

Obviously the set $I_1$is $\{1,\ldots,n\}$, by (H_1) and this make certain that $\alpha_{I_1,i} = 1(1 \leq i \leq n)$.
From (H_1) we have $\alpha_{I_l,i} \geq 1(1 \leq i \leq n, m-1 \geq l \geq 1)$.
For $m \geq l \geq 2$, and for any $(j_1,\ldots,j_{l-1}) \in I_{l-1}$, let

$$\mathscr{H}_{I_l}(j_1,\ldots,j_{l-1}) := \{((i_1,\ldots,i_l),k) \times \{1,\ldots,l\} | F_{l,k}(i_1,\ldots,i_l) = (j_1,\ldots,j_{l-1})\}.$$

With the help of these sets they defined the functions $\eta_{I_m,l} : I_l \to \mathbb{N}(m \geq l \geq 1)$ inductively by

$$\eta_{I_m,m}(i_1,\ldots,i_m) := 1 \quad (i_1,\ldots,i_m) \in I_m;$$
$$\eta_{I_m,l-1}(j_1,\ldots,j_{l-1}) := \sum_{((i_1,\ldots,i_l),k)\in\mathscr{H}_{I_l}(j_1,\ldots,j_{l-1})} \eta_{I_m,l}(i_1,\ldots,i_l).$$

They defined some special expressions for $1 \leq l \leq m$, as follows

$$\mathscr{A}_{m,l} = \mathscr{A}_{m,l}(I_m,x_1,\ldots,x_n,p_1,\ldots,p_n;f) := \frac{(m-1)!}{(l-1)!} \sum_{(i_1,\ldots,i_l)\in I_l} \eta_{I_m,l}(i_1,\ldots,i_l)$$
$$\left(\sum_{j=1}^{l} \frac{p_{i_j}}{\alpha_{I_m,i_j}}\right) f\left(\frac{\sum_{j=1}^{l} \frac{p_{i_j}}{\alpha_{I_m,i_j}} x_{i_j}}{\sum_{j=1}^{l} \frac{p_{i_j}}{\alpha_{I_m,i_j}}}\right) \tag{1.13}$$

and constructed the following new refinement of (1.3).

Theorem 1.1 *Assume (H_1), consider a convex function $f : I \to \mathbb{R}$. If $c_1,\ldots,c_n \in I$, $\sigma_1,\ldots,\sigma_n \in \mathbb{R}^+$ such that $\sum_{s=1}^{n} \sigma_s = 1$, then*

$$f\left(\sum_{s=1}^{n} \sigma_s c_s\right) \leq \mathscr{A}_{m,m} \leq \mathscr{A}_{m,m-1} \leq \ldots \leq \mathscr{A}_{m,2}$$
$$\leq \mathscr{A}_{m,1} = \sum_{s=1}^{n} \sigma_s f(c_s). \tag{1.14}$$

L. Horváth and J. Pečarić proved that (1.10) is the special case of Theorem 1.1.

In [25], I. Brnetić *et al*. gave the improvement of (1.6) as follows.
Suppose $\eta : I \to \mathbb{R}$ is a convex function, let $\mu : [a_1,a_2] \to I$ and $\chi : [a_1,a_2] \to \mathbb{R}^+$ be functions. Suppose $\sigma_1,\ldots,\sigma_n \in \mathbb{R}^+$ with $\sum_{i=1}^{n} \sigma_i = 1$, and

$$\bar{\chi} = \int_{a_1}^{a_2} \chi(t)dt,$$

then

$$\eta\left(\frac{1}{\bar{\chi}} \int_{a_1}^{a_2} \chi(t)\mu(t)dt\right) \leq \Delta_{n,n} \leq \ldots \leq \Delta_{k+1,n} \leq \Delta_{k,n} \ldots \leq \Delta_{1,n}$$

$$= \frac{1}{\bar{\chi}} \int_{a_1}^{a_2} \chi(t)\eta(\mu(t))dt,$$

where

$$\Delta_{k,n} = \frac{1}{\binom{n-1}{k-1}\bar{\chi}^k} \sum_{1 \le i_1 < \ldots < i_k \le n} \sum_{r=1}^{k} \sigma_{i_r} \int_{a_1}^{a_2} \cdots \int_{a_1}^{a_2} \left(\prod_{s=1}^{k} \chi(c_{i_s}) \right)$$
$$\times \eta \left(\frac{\sum_{j=1}^{k} \sigma_{i_j} \mu\left(c_{i_j}\right)}{\sum_{j=1}^{k} \sigma_{i_j}} \right) dc_{i_1} dc_{i_2} \ldots dc_{i_k}.$$

1.1.8 Jensen-Mercer's Inequality

In 2003 Mercer proved the following variant of Jensen's inequality, which is known as Jensen-Mercer's inequality.

Theorem 1.2 ([77]) *Let $\psi : [a,b] \to \mathbb{R}$ be a convex function and let $y_\gamma \in [a,b]$, $\zeta_\gamma \in \mathbb{R}^+$ be such that $\zeta = \sum_{\gamma=1}^{n} \zeta_\gamma$. Then*

$$\psi\left(a+b-\frac{1}{\zeta}\sum_{\gamma=1}^{n}\zeta_\gamma y_\gamma\right) \le \psi(a)+\psi(b)-\frac{1}{\zeta}\sum_{\gamma=1}^{n}\zeta_\gamma \psi(y_\gamma). \tag{1.15}$$

1.1.9 Variant of Jensen-Steffensen's Inequality

The following variant of Jensen-Steffensen's inequality has been given in [1].

Theorem 1.3 *Let $\psi : [a,b] \to \mathbb{R}$ be a convex function and $y_\gamma \in [a,b]$, $\zeta_\gamma \in \mathbb{R}, \zeta_\gamma \neq 0$ for $\gamma = 1,2,\ldots,n$ with $\zeta = \sum_{\gamma=1}^{n} \zeta_\gamma$. If $y_1 \le y_2 \le \cdots \le y_n$ or $y_1 \ge y_2 \ge \cdots \ge y_n$ and*

$$0 \le \sum_{\gamma=1}^{k} \zeta_\gamma \le \sum_{\gamma=1}^{n} \zeta_\gamma, \; k = 1,2,\ldots,n, \quad \sum_{\gamma=1}^{n} \zeta_\gamma > 0, \tag{1.16}$$

then

$$\psi\left(a+b-\frac{1}{\zeta}\sum_{\gamma=1}^{n}\zeta_\gamma y_\gamma\right) \le \psi(a)+\psi(b)-\frac{1}{\zeta}\sum_{\gamma=1}^{n}\zeta_\gamma \psi(y_\gamma). \tag{1.17}$$

1.1.10 Jensen's Inequality for Operator Convex Functions

The following Jensen's inequality for operator convex function has been given in [43].

Theorem 1.4 (JENSEN'S OPERATOR INEQUALITY) *Let $\psi : I \to \mathbb{R}$ be an operator convex function defined on the interval I. If $A_p \in S(I)$ and $\zeta_p > 0$ $(p = 1,\ldots,n)$ such that $\sum_{p=1}^{n} \zeta_p = 1$, then*

$$\psi\left(\sum_{p=1}^{n} \zeta_p A_p\right) \le \sum_{p=1}^{n} \zeta_p \psi(A_p). \tag{1.18}$$

1.1.11 Hermite-Hadamard Inequality

The following inequality proved by Hermite and Hadamard for convex function [40]. This inequality says that if the function $\psi : [a,b] \to \mathbb{R}$ is convex function then

$$\psi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \psi(x)dx \leq \frac{\psi(a)+\psi(b)}{2}. \tag{1.19}$$

If ψ is concave function then the inequalities in (1.19) will hold in reverse directions.

1.1.12 Hölder Inequality

The discrete form of well-known Hölder inequality is given below:

Let $p,q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$ and $(a_1,a_2,\cdots,a_n)$, $(b_1,b_2,\cdots,b_n)$ be positive n-tuple. Then

$$\sum_{\mathtt{j}=1}^{n} a_{\mathtt{j}} b_{\mathtt{j}} \leq \Big(\sum_{\mathtt{j}=1}^{n} a_{\mathtt{j}}^{p}\Big)^{\frac{1}{p}} \Big(\sum_{\mathtt{j}=1}^{n} b_{\mathtt{j}}^{q}\Big)^{\frac{1}{q}} \tag{1.20}$$

The integral form of Hölder inequality is given below:

Let $\mathtt{p},\mathtt{q} > 1$ such that $\frac{1}{\mathtt{p}}+\frac{1}{\mathtt{q}} = 1$ and $\mathtt{A},\mathtt{B} : [a,b] \to \mathbb{R}$ be integrable functions functions such that $|\mathtt{A(z)}|^{\mathtt{p}}$, $|\mathtt{B(z)}|^{\mathtt{q}}$ are also integrable on $[a,b]$. Then

$$\int_a^b |\mathtt{A(z)B(z)}|\,d\mathtt{z} \leq \Big(\int_a^b |\mathtt{A(z)}|^{\mathtt{p}}\,d\mathtt{z}\Big)^{\frac{1}{\mathtt{p}}} \Big(\int_a^b |\mathtt{B(z)}|^{\mathtt{q}}\,d\mathtt{z}\Big)^{\frac{1}{\mathtt{q}}}.$$

1.2 Power Means

In [75, p. 14] the power means are given as follows.

Suppose n is a natural number, let $(c_1,\ldots,c_n)$ and $(\sigma_1,\ldots,\sigma_n)$ belong to $(0,\infty)^n$ such that $P_n := \sum_{i=1}^n \sigma_i = 1$. The power mean (of order $s \in \mathbb{R}$) is defined by

$$P(c_1,\ldots,c_n;\sigma_1,\ldots,\sigma_n) = \begin{cases} \left(\frac{1}{P_n}\sum_{i=1}^n c_i^s\right)^{\frac{1}{s}}, & s \neq 0; \\ \left(\prod_{i=1}^n c^{\sigma_i}\right)^{\frac{1}{P_n}}, & s = 0. \end{cases} \tag{1.21}$$

For $c_i = \frac{1}{n}$, $i = 1,\ldots,n$, the power mean (1.21) is arithmetic mean, geometric mean and harmonic mean for $s = 1$, $s \to 0$ and $s = -1$ respectively.

The power means for the n-tuples of strictly positive operators $\mathbf{A} = (A_1,\ldots,A_n)$ with positive weights $\boldsymbol{\zeta} := (\zeta_1,\ldots,\zeta_n)$ of order $r \in \mathbb{R}\setminus\{0\}$ is defined by:

$$M_r(\mathbf{A};\boldsymbol{\zeta}) = \left(\frac{1}{\overline{\zeta}}\sum_{p=1}^n \zeta_p A_p^r\right)^{\frac{1}{r}}, \tag{1.22}$$

where $\overline{\zeta} := \sum_{p=1}^n \zeta_p$.

1.2.1 Quasi-Arithmetic Means

The importance of quasi-arithmetic means has been well understood at least since the 1930's and a number of writers have since then contributed to the characterization and to the study of their properties.

Consider a continuous function $\eta : I \to \mathbb{R}$ such that for all $u, v \in I$ if $u < v$ then $\eta(u) < \eta(v)$ (or if $u > v$ then $\eta(u) > \eta(v)$). Let $(\lambda_1, \ldots, \lambda_n) \in I^n$, also let $(\sigma_1, \ldots, \sigma_n) \in [0, \infty)^n$. Suppose $P_n := \sum_{i=1}^{n} p_i$. Then the quasi-arithmetic mean [75, p. 15] is

$$M_{\eta}^{[n]}(\boldsymbol{\lambda};\boldsymbol{\sigma}) = M_{\eta}(\lambda_1, \ldots, \lambda_n; \sigma_a \ldots, \sigma_n) = \eta^{-1}\left(\frac{1}{P_n}\sum_{i=1}^{n} \sigma_i \eta(\lambda_i)\right). \tag{1.23}$$

If $I = \mathbb{R}^+$ and $\eta(t) = t^p$, then (1.23) is a power mean.

In the current century, the Popoviciu inequality is studied by many authors (see [31, 29, 32, 30]).

The Popoviciu inequality for arbitrary non-negative weights given as follows (see [85]).

Let r and m are positive integers such that $m \geq 3$, $2 \leq r \leq m-1$, let $\eta : [a_1, a_2] \to \mathbb{R}$ be convex function, $(c_1, \ldots, c_m) \in [a_1, a_2]^m$ and $(\sigma_1, \ldots, \sigma_m)$ be non-negative m-tuple such that $\sum_{j=1}^{n} \sigma_j = 1$, then

$$\begin{aligned} \eta_{r,m}(c_1, \ldots, c_m; \sigma_1, \ldots, \sigma_m) \leq \frac{m-r}{m-1}\eta_{1,m}(c_1, \ldots, c_m; \sigma_1, \ldots, \sigma_m) \\ + \frac{r-1}{m-1}\eta_{m,m}(c_1, \ldots, c_m; \sigma_1, \ldots, \sigma_m), \end{aligned} \tag{1.24}$$

where

$$g_{r,m}(c_1, \ldots, c_m; \sigma_1, \ldots, \sigma_m) := \frac{1}{C_{r-1}^{m-1}} \sum_{1 \leq i_1 < \ldots < i_r \leq m} \left(\sum_{j=1}^{m} \sigma_{i_j}\right) \eta\left(\frac{\sum_{j=1}^{m} \sigma_{i_j} c_{i_j}}{\sum_{j=1}^{m} \sigma_{i_j}}\right).$$

Higher order convex function was introduced by T. Popoviciu (see [87, p. 15]). The inequalities involving higher order convex functions are used by physicists in higher dimensional problems. Many of the results that are true for convex functions are not true for higher order convex functions, this fact convince us to study the results involving higher order convexity (see [31]).

Let $\eta : I \to \mathbb{R}$ be a continuous strictly monotone function. Then the quasi arithmetic mean for operators is defined by

$$\tilde{M}_{\eta}(\mathbf{A}; \boldsymbol{\zeta}) = \eta^{-1}\left(\frac{1}{\sum_{p=1}^{n} \zeta_p} \sum_{p=1}^{n} \zeta_p \eta(A_p)\right), \tag{1.25}$$

where $A_p \in S(I)$ and $\zeta_p > 0$ for $p = 1, 2, \ldots, n$.

1.3 Divided Differences

The tools of divided difference are used to define the higher order convex functions. Divided difference is given in [87, p. 14] as follows.
Consider the function $\eta : [a_1, a_2] \to \mathbb{R}$. The r-th order divided difference for $r+1$ distinct points $u_0, u_1, \ldots, u_r \in [a_1, a_2]$ is defined by the following recursive formula

$$[u_i; \eta] = \eta(u_i) \;\; i = 0, 1, \ldots, r,$$

and

$$[u_0, u_1, \ldots, u_r; \eta] = \frac{[u_1, u_2, \ldots, u_r; \eta] - [u_0, u_1, \ldots, u_{r-1}; \eta]}{u_r - u_0}. \tag{1.26}$$

This is equivalent to

$$[u_0, u_1, \ldots, u_r; \eta] = \sum_{j=0}^{k} \frac{\eta(u_j)}{w'(u_j)},$$

where $w(u) = \prod_{j=0}^{k}(u - u_j)$. This definition may be extended to include the case in which some or all the points coincide. Namely, if all the points are same, then by taking limits in (1.26) we obtain

$$[\underbrace{u, u, \ldots, u}_{l-times}; \eta] = \frac{\eta^{(l-1)}(u)}{(l-1)!}, \tag{1.27}$$

where $\eta^{(l-1)}$ is supposed to exist.

1.4 Higher Order Convex Functions

A function $\eta : [a_1, a_2] \to \mathbb{R}$ is called r-convex function ($r \geq 0$) on $[a_1, a_2]$ if and only if

$$[u_0, u_1, \ldots, u_r; \eta] \geq 0 \tag{1.28}$$

for all $(r+1)$ distinct choices in $[a_1, a_2]$ (see [87, p. 14]).

The function η is r-concave on $[a_1, a_2]$ if inequality sign in (1.28) is reverse. The next result is useful to examine the convexity of a function [87, p. 16].

Theorem 1.5 *Suppose the $\eta^{(n)}$ exists where η is a real valued function. Then η is n-convex if and only if $\eta^{(n)}$ is non-negative.*

In recent years many researchers have generalized the inequalities for m-convex functions; like S. I. Butt *et al*. generalized the Popoviciu inequality for m-convex function using Taylor's formula, Lidstone polynomial, Montgomery identity, Fink's identity, Abel-Gonstcharoff interpolation and Hermite interpolating polynomial (see [31, 29, 32, 30, 33]). S. I. Butt *et al*. constructed the linear functional from these generalized Popoviciu type identities and using the inequalities for Cebysev functional and found some bounds for the generalized identities. Also they constructed Grüss and Ostrowski type inequalities. By using these new generalized Popoviciu type functionals they constructed new class of m-exponentially convex functions.

1.5 Information Divergence Measures and Entropies

Information theory is the science of information, which scientifically deals with the storage, quantification and communication of the information. Being an abstract entity information cannot be quantified easily. In 1948 Claude shannon in [93], presented the concept of information theory and introduced entropy as the fundamental measure of information in his first of the two fundamental and important theorems. The information can also be measured with the help of probability density function. Divergences are some important tools for measuring the difference between two probability density functions. A class of information divergence measures, which is one of the important divergence measures due to its compact behavior, is the Csiszár divergence [36, 37].

Let η be a positive function defined on $(0,\infty)$, suppose $\mathbf{p} := (p_1,\ldots,p_n)$ and $\mathbf{q} := (q_1,\ldots,q_n)$ are positive probability distributions. The Csiszár divergence (f-divergence) is defined as

$$I_f(\mathbf{p},\mathbf{q}) := \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right). \tag{1.29}$$

In [52], L. Horv́ath, et al. gave the following generalization of (1.29):

Let $\eta : I \to \mathbb{R}$ be a function. Suppose $\mathbf{p} := (p_1,\ldots,p_n)$ is real and $\mathbf{q} := (q_1,\ldots,q_n)$ is positive n-tuple such that

$$\frac{p_j}{q_j} \in \mathbb{R}, \;\; j = 1,\ldots,n. \tag{1.30}$$

Then

$$\hat{I}_\eta(\mathbf{p},\mathbf{q}) := \sum_{j=1}^{n} q_j \eta\left(\frac{p_j}{q_j}\right). \tag{1.31}$$

They applied the cyclic refinement of Jensen's inequality [52] to $\hat{I}_f(\mathbf{p},\mathbf{q})$ in order to investigate the bounds for (1.31).

Many well-known distance functions or divergences can be obtained for a suitable choice of function f in (1.29), and which are frequently used in mathematical statistics, signal processing, and information theory. One of the divergences is Kullback-Leibler which is defined by

Let $\mathbf{p} := (p_1, \ldots, p_n)$ and $\mathbf{q} := (q_1, \ldots, q_n)$ be two positive probability distributions. Then the Kullback-Leibler divergence between $\mathbf{p}$ and $\mathbf{q}$ is defined as

$$D(\mathbf{p},\mathbf{q}) := \sum_{j=1}^{n} p_j \log\left(\frac{p_j}{q_j}\right). \tag{1.32}$$

One of the another important divergences associated to Csiszár divergence is Rényi divergence which is defined as follows: Suppose $\lambda \in [0,\infty)$ with $\lambda \neq 1$. Let $\mathbf{r} := (r_1, \ldots, r_n)$ and $\mathbf{q} := (q_1, \ldots, q_n)$ be two positive probability distributions. Then the Rényi divergence [89] of order λ is

$$D_\lambda(\mathbf{r},\mathbf{q}) := \frac{1}{\lambda - 1}\log\left(\sum_{i=1}^{n} q_i \left(\frac{r_i}{q_i}\right)^{\lambda}\right). \tag{1.33}$$

The idea of the Shannon entropy [93] plays a key role in information theory, while in some cases, it is denoted as measure of uncertainty. There are basically two methods for understanding the Shannon entropy. Under one point of view, the Shannon entropy quantifies the amount of information in regard to the value of X (after measurement). Under another point of view, the Shannon entropy tells us the amount of uncertainty about the variable of X before we learn its value (before measurement). The random variable, entropy, is characterized regarding its probability distribution and it can appear as a better measure of predictability or uncertainty. Shannon entropy permits the appraisal of the normal least number of bits expected to encode a series of symbols based on the letters in order of estimation and the recurrence of the symbols. The formula for Shannon entropy given in [93], is as follows:

$$S = H(\mathbf{r}) := -\sum_{j=1}^{n} r_j \log r_j. \tag{1.34}$$

where $r_1, r_2, \ldots, r_n \in \mathbb{R}^+$ with $\sum_{i=1}^{n} r_i = 1$.

In the literature there are several generalizations of Shannon entropy. One of the important generalization is Rényi entropy. Rényi entropy is given as follows:

Suppose $\lambda \in [0,\infty)$ such that $\lambda \neq 1$. Let $\mathbf{r} := (r_1, \ldots, r_n)$ is positive probability distributions. Then the Rényi entropy [89] of order λ of $\mathbf{r}$ is given by

$$H_\lambda(\mathbf{r}) := \frac{1}{1-\lambda}\log\left(\sum_{i=1}^{n} r_i^{\lambda}\right). \tag{1.35}$$

If $\lambda \to 1$ in (1.33), we have the (1.32), and if $\lambda \to 1$ in (1.35), then we have the Shannon entropy.

1.5.1 Zipf-Mandelbrot Law

The most commonly used words, the largest cities of countries income of billionare can be described in term of Zipf's law. Many natural phenomena like distribution of wealth and income in a society, distribution of face book likes, distribution of football goals follow power law distribution (Zipf's law). Like above phenomena, distribution of city sizes also follows power law distribution. Auerbach [19] was the first to explore that the Pareto distribution can be used to approximate the distribution of city size. Later, many researchers refined this idea but the work of Zipf [100] is remarkable regarding this field. Black and Henderson [23], Anderson and Ge [18], Rosen and Resnick [90], Ioannides and Overman [55], Bosker et al. [24] and Soo [95] also examined the distribution of city sizes. Zipf's law states that: "The rank of cities with a certain number of inhabitants varies proportional to the city sizes with some negative exponent, say that is close to unit". In other words, Zipf's law says that the product appears to be approximately constant in city sizes and their rank. It gives the idea that the population of the n-th city is $\frac{1}{n}$ of the population of the city with largest population. Let N denotes the number of elements, t denotes the exponent value that characterizes the distribution, suppose s is the rank of N elements. Zipf's law [100] then predicts that normalized frequency of rank element from a population of N elements is $f(s,N,t)$ defined as

$$f(s,N,t) = \frac{\frac{1}{s^t}}{\sum_{j=1}^{N} \frac{1}{j^t}}. \tag{1.36}$$

Let $N \in \{1,2,\dots,\}$, q be a non-negative and t be a positive real number. Zipf-Mandelbrot law [70] is defined as

$$f(s;N,q,t) := \frac{1}{(s+q)^t H_{N,q,t}}, \quad s = 1,\dots,N, \tag{1.37}$$

where

$$H_{N,q,t} = \sum_{j=1}^{N} \frac{1}{(j+q)^t}. \tag{1.38}$$

If the total mass of the law is taken over all $\mathbb{N}$, then for $q \geq 0$, $t > 1$, $s \in \mathbb{N}$, density function of Zipf-Mandelbrot law becomes

$$f(s;q,t) = \frac{1}{(s+q)^t H_{q,t}}, \tag{1.39}$$

where

$$H_{q,t} = \sum_{j=1}^{\infty} \frac{1}{(j+q)^t}. \tag{1.40}$$

For $q = 0$, (1.37) becomes (1.36).

In [52], L. Horváth *et al.* introduced some new functionals based on the (1.30), and estimated these new functionals. They obtained (1.30) and (1.33) divergence by applying a cyclic refinement of Jensen's inequality. They also constructed some new inequalities for (1.35) and (1.34) and used (1.37) to illustrate the results. Also in [13], M. A. Khan *et al*. considered two refinements of (1.3) and investigated bounds for (1.34) and (1.37). N. Lovričević *et al*. [70], applied (1.37) to different types of f-divergence and distances in view of the monotonicity property of the Jensen functional and the deduced comparative inequalities.

Chapter 2

Refinements of Jensen's Inequality

In this chapter we present new refinements of the Jensen inequality in integral as well as in discrete form. This chapter is also devoted to achieve numerous applications in information theory. Some refinements have been obtained for quasi arithmetic means, Hölder and Hermite-Hadamard inequalities. More general refinements of Jensen inequality are presented. The results of this chapter are given in [2, 3].

2.1 Refinement of Integral Jensen's Inequality

We start to derive new refinement of Jensen inequality associated to two functions whose sum is equal to unity.

Theorem 2.1 *Let $\psi : I \to \mathbb{R}$ be a convex function defined on the interval I. Let $p,u,v,g : [a,b] \to \mathbb{R}$ be integrable functions such that $g(\rho) \in I, u(\rho), v(\rho), p(\rho) \in \mathbb{R}^+$ for all $\rho \in [a,b]$ and $v(\rho)+u(\rho)=1$, $P=\int_a^b p(\rho)d\rho$. Then*

$$\frac{1}{P}\int_a^b p(\rho)\psi(g(\rho))d\rho \geq \frac{1}{P}\int_a^b u(\rho)p(\rho)d\rho\,\psi\left(\frac{\int_a^b p(\rho)u(\rho)g(\rho)d\rho}{\int_a^b p(\rho)u(\rho)d\rho}\right)$$

$$+\frac{1}{P}\int_a^b p(\rho)v(\rho)d\rho\,\psi\left(\frac{\int_a^b p(\rho)v(\rho)g(\rho)d\rho}{\int_a^b p(\rho)v(\rho)d\rho}\right) \geq \psi\left(\frac{1}{P}\int_a^b p(\rho)g(\rho)d\rho\right). \quad (2.1)$$

If the function ψ is concave then the reverse inequalities hold in (2.1).

Proof. Since $u(\rho)+v(\rho)=1$, therefore we have

$$\int_a^b p(\rho)\psi(g(\rho))d\rho = \int_a^b u(\rho)p(\rho)\psi(g(\rho))d\rho + \int_a^b v(\rho)p(\rho)\psi(g(\rho))d\rho. \quad (2.2)$$

Applying integral Jensen's inequality on both terms on the right side of (2.2) we obtain

$$\begin{aligned}
&\frac{1}{P}\int_a^b p(\rho)\psi(g(\rho))d\rho \\
&\geq \frac{1}{P}\int_a^b u(\rho)p(\rho)d\rho\,\psi\left(\frac{\int_a^b u(\rho)p(\rho)g(\rho)d\rho}{\int_a^b u(\rho)p(\rho)d\rho}\right) \qquad (2.3)\\
&+\frac{1}{P}\int_a^b v(\rho)p(\rho)d\rho\,\psi\left(\frac{\int_a^b v(\rho)p(\rho)g(\rho)d\rho}{\int_a^b v(\rho)p(\rho)d\rho}\right) \\
&\geq \psi\left(\frac{1}{P}\int_a^b u(\rho)p(\rho)g(\rho)d\rho + \frac{1}{P}\int_a^b v(\rho)p(\rho)g(\rho)d\rho\right) \text{(By the convexity of } \psi) \\
&= \psi\left(\frac{1}{P}\int_a^b p(\rho)g(\rho)d\rho\right) \qquad (2.4)
\end{aligned}$$

As a consequence of the above theorem we deduce the following refinement of Hölder inequality.

Corollary 2.1 *Let $r_1, r_2 > 1$ be such that $\frac{1}{r_1}+\frac{1}{r_2}=1$. If u, v, τ, g_1 and g_2 are non-negative functions defined on $[a,b]$ such that $\tau g_1^{r_1}, \tau g_2^{r_2}, u\tau g_2^{r_2}, v\tau g_2^{r_2}, u\tau g_1 g_2, v\tau g_1 g_2, \tau g_1 g_2 \in L^1([a,b])$ and $u(\rho)+v(\rho)=1$ for all $\rho \in [a,b]$, then*

$$\begin{aligned}
&\left(\int_a^b \tau(\rho)g_1^{r_1}(\rho)d\rho\right)^{\frac{1}{r_1}}\left(\int_a^b \tau(\rho)g_2^{r_2}(\rho)d\rho\right)^{\frac{1}{r_2}} \\
&\geq \left(\int_a^b \tau(\rho)g_2^{r_2}(\rho)d\rho\right)^{\frac{1}{r_2}}\left\{\left(\int_a^b u(\rho)\tau(\rho)g_2^{r_2}(\rho)d\rho\right)^{1-r_1}\left(\int_a^b u(\rho)\tau(\rho)g_1(\rho)g_2(\rho)d\rho\right)^{r_1}\right. \\
&\left.+\left(\int_a^b v(\rho)\tau(\rho)g_2^{r_2}(\rho)d\rho\right)^{1-r_1}\left(\int_a^b v(\rho)\tau(\rho)g_1(\rho)g_2(\rho)d\rho\right)^{r_1}\right\}^{\frac{1}{r_1}} \\
&\geq \int_a^b \tau(\rho)g_1(\rho)g_2(\rho)d\rho. \qquad (2.5)
\end{aligned}$$

In the case when $0 < r_1 < 1$ and $r_2 = \frac{r_1}{r_1-1}$ with $\int_a^b \tau(\rho)g_2^{r_2}(\rho)d\rho > 0$ or $r_1 < 0$ and $\int_a^b \tau(\rho)g_1^{r_1}(\rho)d\rho > 0$, then

$$\begin{aligned}
&\int_a^b \tau(\rho)g_1(\rho)g_2(\rho)d\rho \\
&\geq \left(\int_a^b u(\rho)\tau(\rho)g_2^{r_2}(\rho)d\rho\right)^{\frac{1}{r_2}} \left(\int_a^b u(\rho)\tau(\rho)g_1^{r_1}(\rho)d\rho\right)^{\frac{1}{r_1}} \\
&+\left(\int_a^b v(\rho)\tau(\rho)g_2^{r_2}(\rho)d\rho\right)^{\frac{1}{r_2}} \left(\int_a^b v(\rho)\tau(\rho)g_1^{r_1}(\rho)d\rho\right)^{\frac{1}{r_1}} \\
&\geq \left(\int_a^b \tau(\rho)g_1^{r_1}(\rho)d\rho\right)^{\frac{1}{r_1}} \left(\int_a^b \tau(\rho)g_2^{r_2}(\rho)d\rho\right)^{\frac{1}{r_2}}.
\end{aligned} \tag{2.6}$$

Proof. Let $\int_a^b \tau(\rho)g_2^{r_2}(\rho)d\rho > 0$. Then by using Theorem 2.1 for $\psi(\rho) = \rho^{r_1}, \rho > 0, r_1 > 1$, $p(\rho) = \tau(\rho)g_2^{r_2}(\rho), g(\rho) = g_1(\rho)g_2^{\frac{-r_2}{r_1}}(\rho)$, we obtain (2.5). Let $\int_a^b \tau(\rho)g_1^{r_1}(\rho)d\rho > 0$. Then applying the same procedure but taking r_1, r_2, g_1, g_2 instead of r_2, r_1, g_2, g_1, we obtain (2.5).

Now we prove the inequality for the case when $\int_a^b \tau(\rho)g_2^{r_2}(\rho)d\rho = 0$ and $\int_a^b \tau(\rho)g_1^{r_1}(\rho) d\rho = 0$.
Since we know that

$$0 \leq \tau(\rho)g_1(\rho)g_2(\rho) \leq \frac{1}{r_1}\tau(\rho)g_1^{r_1}(\rho) + \frac{1}{r_2}\tau(\rho)g_2^{r_2}(\rho). \tag{2.7}$$

Therefore taking integral and then using the given conditions we have $\int_a^b \tau(\rho)g_1(\rho)g_2(\rho)d\rho = 0$.
For the case $r_1 > 1$, the proof is completed .

The case when $0<r_1<1$, then $M=\frac{1}{r_1}>1$ and applying (2.5) for M and $N=(1-r_1)^{-1}$, $\overline{g}_1 = (g_1g_2)^{r_1}$, $\overline{g}_2 = g_2^{-r_1}$ instead of r_1, r_2, g_1, g_2.

Finally, if $r_1 < 0$ then $0 < r_2 < 1$ and we may apply similar arguments with r_1, r_2, g_1, g_2 replaced by r_2, r_1, g_2, g_1 provided that $\int_a^b \tau(\rho)g_1^{r_1}(\rho)d\rho > 0$. Another refinement of the Hölder inequality is presented in the following corollary.

Corollary 2.2 *Let $r_1 > 1$, $r_2 = \frac{r_1}{r_1-1}$. Let u, v, τ, g_1 and g_2 be non-negative functions defined on $[a,b]$ such that $\tau g_1^{r_1}, \tau g_2^{r_2}, u\tau g_2^{r_2}, v\tau g_2^{r_2}, \tau g_1 g_2 \in L^1([a,b])$ and $u(\rho)+v(\rho) = 1$ for all $\rho \in [a,b]$. Also assume that $\int_a^b \tau(\rho)g_2^{r_2}(\rho) > 0$. Then*

$$\begin{aligned}
&\left(\int_a^b \tau(\rho)g_1^{r_1}(\rho)d\rho\right)^{\frac{1}{r_1}} \left(\int_a^b \tau(\rho)g_2^{r_2}(\rho)d\rho\right)^{\frac{1}{r_2}} \\
&\geq \left(\int_a^b u(\rho)\tau(\rho)g_1^{r_1}(\rho)d\rho\right)^{\frac{1}{r_1}} \left(\int_a^b u(\rho)\tau(\rho)g_2^{r_2}(\rho)d\rho\right)^{\frac{1}{r_2}} \\
&+\left(\int_a^b v(\rho)\tau(\rho)g_1^{r_1}(\rho)d\rho\right)^{\frac{1}{r_1}} \left(\int_a^b v(\rho)\tau(\rho)g_2^{r_2}(\rho)d\rho\right)^{\frac{1}{r_2}} \\
&\geq \int_a^b \tau(\rho)g_1(\rho)g_2(\rho)d\rho.
\end{aligned} \tag{2.8}$$

In the case when $0 < r_1 < 1$ *and* $r_2 = \frac{r_1}{r_1-1}$ *with* $\int_a^b \tau(\rho)g_2^{r_2}(\rho)d\rho > 0$ *or* $r_1 < 0$ *and* $\int_a^b \tau(\rho)g_1^{r_1}(\rho)d\rho > 0$, *then*

$$\left(\int_a^b \tau(\rho)g_1^{r_1}(\rho)d\rho\right)^{\frac{1}{r_1}}\left(\int_a^b \tau(\rho)g_2^{r_2}(\rho)d\rho\right)^{\frac{1}{r_2}}$$
$$\leq \left(\int_a^b \tau(\rho)g_2^{r_2}(\rho)d\rho\right)^{\frac{1}{r_2}}\left\{\left(\int_a^b u(\rho)\tau(\rho)g_2^{r_2}(\rho)d\rho\right)^{1-r_1}\left(\int_a^b u(\rho)\tau(\rho)g_1(\rho)g_2(\rho)d\rho\right)^{r_1}\right.$$
$$\left.+\left(\int_a^b v(\rho)\tau(\rho)g_2^{r_2}(\rho)d\rho\right)^{1-r_1}\left(\int_a^b v(\rho)\tau(\rho)g_1(\rho)g_2(\rho)d\rho\right)^{r_1}\right\}^{\frac{1}{r_1}}$$
$$\leq \int_a^b \tau(\rho)g_1(\rho)g_2(\rho)d\rho. \tag{2.9}$$

Proof. Assume that $\int_a^b \tau(\rho)g_2^{r_2}(\rho)d\rho > 0$. Let $\psi(\rho) = \rho^{\frac{1}{r_1}}$, $\rho > 0, r_1 > 1$. Then clearly the function ψ is concave. Therefore applying Theorem 2.1 for $\psi(\rho) = \rho^{\frac{1}{r_1}}, p = \tau g_2^{r_2}, g = g_1^{r_1}g_2^{-r_2}$, we obtain (2.8). If $\int_a^b \tau(\rho)g_1^{r_1}(\rho)d\rho > 0$. Then applying the same procedure but taking r_1, r_2, g_1, g_2 instead of r_2, r_1, g_2, g_1, we obtain (2.8).
If $\int_a^b \tau(\rho)g_2^{r_2}(\rho)d\rho = 0$ and $\int_a^b \tau(\rho)g_1^{r_1}(\rho)d\rho = 0$. Since we know that

$$0 \leq \tau(\rho)g_1(\rho)g_2(\rho) \leq \frac{1}{r_1}\tau(\rho)g_1^{r_1}(\rho) + \frac{1}{r_2}\tau(\rho)g_2^{r_2}(\rho), \tag{2.10}$$

therefore taking integral and then using the given conditions we have $\int_a^b \tau(\rho)g_1(\rho)g_2(\rho) d\rho = 0$.
When $0 < r_1 < 1$, then $M = \frac{1}{r_1} > 1$ and applying (2.8) for M and $N = (1-r_1)^{-1}$, $\overline{g}_1 = (g_1g_2)^{r_1}$, $\overline{g}_2 = g_2^{-r_1}$ instead of r_1, r_2, g_1, g_2, we get (2.9).

Finally, if $r_1 < 0$ then $0 < r_2 < 1$ and we may apply similar arguments with r_1, r_2, g_1, g_2 replaced by r_2, r_1, g_2, g_1 provided that $\int_a^b \tau(\rho)g_1^{r_1}(\rho)d\rho > 0$.

Remark 2.1 If we put $u(\rho) = \frac{b-\rho}{b-a}, v(\rho) = \frac{\rho-a}{b-a}$ in (2.8), then we deduce the inequalities which have been obtained by **I**cscan in [56].

Let p and g be positive integrable functions defined on $[a,b]$. Then the integral power means of order $r \in \mathbb{R}$ are defined as follows:

$$M_r(p;g) = \begin{cases} \left(\frac{1}{\int_a^b p(\rho)d\rho}\int_a^b p(\rho)g^r(\rho)d\rho\right)^{\frac{1}{r}}, & \text{if } r \neq 0, \\ \exp\left(\frac{\int_a^b p(\rho)\log g(\rho)d\rho}{\int_a^b p(\rho)d\rho}\right), & \text{if } r = 0. \end{cases} \tag{2.11}$$

In the following corollary we deduce inequalities for power means.

Corollary 2.3 *Let p,u,v and g be positive integrable functions defined on $[a,b]$ with $u(\rho)+v(\rho)=1$ for all $\rho \in [a,b]$. Let $s,t \in \mathbb{R}$ be such that $s \leq t$. Then*

$$M_t(p;g) \geq \left[M_1(u;p)M_s^t(u.p;g)+M_1(v;p)M_s^t(v.p;g)\right]^{\frac{1}{t}} \geq M_s(p;g),\ t \neq 0. \tag{2.12}$$

$$M_t(p;g) \geq \exp\left(M_1(u;p)\log M_s(u.p;g)+M_1(v;p)\log M_s(v.p;g)\right) \geq M_s(p;g),\ t=0. \tag{2.13}$$

$$M_s(p;g) \leq \left[M_1(u;p)M_t^s(u.p;g)+M_1(v;p)M_t^s(v.p;g)\right]^{\frac{1}{s}} \leq M_t(p;g),\ s \neq 0. \tag{2.14}$$

$$M_s(p;g) \leq \exp\left(M_1(u;p)\log M_t(u.p;g)+M_1(v;p)\log M_t(v.p;g)\right) \leq M_t(p;g),\ s=0. \tag{2.15}$$

Proof. If $s,t \in \mathbb{R}$ and $s,t \neq 0$, then using (2.1) for $\psi(\rho)=\rho^{\frac{t}{s}}$, $\rho>0$, $g \to g^s$ and then taking power $\frac{1}{t}$ we get (2.12). For the case $t=0$, taking limit $t \to 0$ in (2.12) we obtain (2.13).

Similarly taking (2.1) for $\psi(\rho)=\rho^{\frac{s}{t}}$, $\rho>0, s,t \neq 0$, $g \to g^t$ and then taking power $\frac{1}{s}$ we get (2.14). For $s=0$ taking the limit as above. Let p be positive integrable function defined on $[a,b]$ and g be any integrable function defined on $[a,b]$. Then for a strictly monotone continuous function h whose domain belongs to the image of g, the quasi arithmetic mean is defined as follows:

$$M_h(p;g)=h^{-1}\left(\frac{1}{\int_a^b p(\rho)d\rho}\int_a^b p(\rho)h(g(\rho))d\rho\right). \tag{2.16}$$

We give inequalities for quasi arithmetic mean.

Corollary 2.4 *Let u,v,p be positive integrable functions defined on $[a,b]$ such that $u(\rho)+v(\rho)=1$ for all $\rho \in [a,b]$ and g be any integrable function defined on $[a,b]$. Also assume that h is a strictly monotone continuous function whose domain belongs to the image of g. If $f \circ h^{-1}$ is convex function then*

$$\frac{1}{\int_a^b p(\rho)d\rho}\int_a^b p(\rho)f(g(\rho))d\rho \geq M_1(u;p)f(M_h(p.u;g)) + M_1(v;p)f(M_h(p.v;g)) \geq f\left(M_h(p;g)\right). \tag{2.17}$$

If the function $f \circ h^{-1}$ is concave then the reverse inequalities hold in (2.17).

Proof. The required inequalities may be deduced by using (2.1) for $g \to h \circ g$ and $\psi \to f \circ h^{-1}$.

The following refinement of Hermite-Hadamard inequality may be given:

Corollary 2.5 *Let $\psi:[a,b] \to \mathbb{R}$ be a convex function defined on the interval $[a,b]$. Let $u,v:[a,b] \to \mathbb{R}$ be integrable functions such that $u(\rho),v(\rho) \in \mathbb{R}^+$ for all $\rho \in [a,b]$ and $u(\rho)+v(\rho)=1$. Then*

$$\frac{1}{b-a}\int_a^b \psi(\rho)d\rho \geq \frac{1}{b-a}\int_a^b u(\rho)d\rho\,\psi\left(\frac{\int_a^b \rho u(\rho)d\rho}{\int_a^b u(\rho)d\rho}\right) + \frac{1}{b-a}\int_a^b v(\rho)d\rho\,\psi\left(\frac{\int_a^b \rho v(\rho)d\rho}{\int_a^b v(\rho)d\rho}\right) \geq \psi\left(\frac{a+b}{2}\right). \tag{2.18}$$

For the concave function ψ the reverse inequalities hold in (2.18).

Proof. Using Theorem 2.1 for $p(\rho)=1, g(\rho)=\rho$ for all $\rho \in [a,b]$, we obtain (2.18).

2.1.1 Applications in Information Theory

In this section, we present some important applications of our main result for different divergences and distances to information theory [67].

Definition 2.1 (CSISZÁR-DIVERGENCE) *Let $T : I \to \mathbb{R}$ be a function defined on the positive interval I. Also let $u_1, v_1 : [a,b] \to (0,\infty)$ be two integrable functions such that $\frac{u_1(\rho)}{v_1(\rho)} \in I$ for all $\rho \in [a,b]$, then the integral form of Csiszár-divergence is defined as*

$$C_d(u_1, v_1) = \int_a^b v_1(\rho) T\left(\frac{u_1(\rho)}{v_1(\rho)}\right) d\rho.$$

Theorem 2.2 *Let $T : I \to \mathbb{R}$ be a convex function defined on the positive interval I. Let $u, v, u_1, v_1 : [a,b] \to \mathbb{R}^+$ be integrable functions such that $\frac{u_1(\rho)}{v_1(\rho)} \in I$ and $u(\rho) + v(\rho) = 1$ for all $\rho \in [a,b]$. Then*

$$C_d \geq \int_a^b u(\rho) v_1(\rho) d\rho \, T\left(\frac{\int_a^b u(\rho) u_1(\rho) d\rho}{\int_a^b u(\rho) v_1(\rho) d\rho}\right) + \int_a^b v(\rho) v_1(\rho) d\rho \, T\left(\frac{\int_a^b v(\rho) u_1(\rho) d\rho}{\int_a^b v(\rho) v_1(\rho) d\rho}\right) \geq T\left(\frac{\int_a^b u_1(\rho) d\rho}{\int_a^b v_1(\rho) d\rho}\right) \int_a^b v_1(\rho) d\rho. \tag{2.19}$$

Proof. Using Theorem 2.1 for $\psi = T$, $g = \frac{u_1}{v_1}$ and $p = v_1$, we obtain (2.19).

Definition 2.2 (SHANNON ENTROPY) *If $v_1(\rho)$ is positive probability density function defined on $[a,b]$, then the Shannon-entropy is defined by*

$$SE(v_1) = -\int_a^b v_1(\rho) \log v_1(\rho) d\rho.$$

Corollary 2.6 *Let $u, v, v_1 : [a,b] \to \mathbb{R}^+$ be integrable functions such that v_1 is probability density function and $u(\rho) + v(\rho) = 1$ for all $\rho \in [a,b]$. Then*

$$\int_a^b v_1(\rho) \log(u_1(\rho)) d\rho + SE(v_1) \leq \int_a^b u(\rho) v_1(\rho) d\rho \log\left(\frac{\int_a^b u(\rho) u_1(\rho) d\rho}{\int_a^b u(\rho) v_1(\rho) d\rho}\right) + \int_a^b v(\rho) v_1(\rho) d\rho \log\left(\frac{\int_a^b v(\rho) u_1(\rho) d\rho}{\int_a^b v(\rho) v_1(\rho) d\rho}\right) \leq \log\left(\int_a^b u_1(\rho) d\rho\right). \tag{2.20}$$

Proof. Taking $T(\rho) = -\log \rho, \rho \in \mathbb{R}^+$, in (2.19), we obtain (2.20).

Definition 2.3 (KULLBACK-LEIBLER DIVERGENCE) *If u_1 and v_1 are two positive probability densities defined on $[a,b]$, then the Kullback-Leibler divergence is defined by:*

$$KL_d(u_1, v_1) = \int_a^b u_1(\rho) \log\left(\frac{u_1(\rho)}{v_1(\rho)}\right) d\rho.$$

Corollary 2.7 *Let $u, v, u_1, v_1 : [a,b] \to \mathbb{R}^+$ be integrable functions such that u_1 and v_1 are probability density functions and $u(\rho) + v(\rho) = 1$ for all $\rho \in [a,b]$. Then*

$$\begin{aligned} KL_d(u_1, v_1) &\geq \int_a^b u(\rho)u_1(\rho)d\rho \log\left(\frac{\int_a^b u(\rho)u_1(\rho)d\rho}{\int_a^b u(\rho)v_1(\rho)d\rho}\right) \\ &+ \int_a^b v(\rho)u_1(\rho)d\rho \log\left(\frac{\int_a^b v(\rho)u_1(\rho)d\rho}{\int_a^b v(\rho)v_1(\rho)d\rho}\right) \geq 0. \end{aligned} \tag{2.21}$$

Proof. Taking $T(\rho) = \rho \log \rho, \rho \in \mathbb{R}^+$, in (2.19), we obtain (2.20).

Definition 2.4 (Variational distance) *If two u_1 and v_1 are positive probability density functions defined on $[a,b]$, then the variational distance is defined by*

$$V_d(u_1, v_1) = \int_a^b |u_1(\rho) - v_1(\rho)|\, d\rho.$$

Corollary 2.8 *Let u, v, u_1, v_1 be as stated in Corollary 2.7. Then*

$$\begin{aligned} V_d(u_1, v_1) \geq &\left|\int_a^b u(\rho)(u_1(\rho) - v_1(\rho))d\rho\right| \\ &+ \left|\int_a^b v(\rho)(u_1(\rho) - v_1(\rho))d\rho\right|. \end{aligned} \tag{2.22}$$

Proof. Using the function $T(\rho) = |\rho - 1|, \rho \in \mathbb{R}^+$, in (2.19), we obtain (2.43).

Definition 2.5 (Jeffrey's distance) *If u_1 and v_1 are two positive probability density functions defined on $[a,b]$, then the Jeffrey distance is defined by*

$$J_d(u_1, v_1) = \int_a^b (u_1(\rho) - v_1(\rho)) \log\left(\frac{u_1(\rho)}{v_1(\rho)}\right) d\rho.$$

Corollary 2.9 *Let u, v, u_1, v_1 be as stated in Corollary 2.7. Then*

$$\begin{aligned} J_d(u_1, v_1) &\geq \int_a^b u(\rho)(u_1(\rho) - v_1(\rho))d\rho \log\left(\frac{\int_a^b u(\rho)u_1(\rho)d\rho}{\int_a^b u(\rho)v_1(\rho)d\rho}\right) \\ &+ \int_a^b v(\rho)(u_1(\rho) - v_1(\rho))d\rho \log\left(\frac{\int_a^b v(\rho)u_1(\rho)d\rho}{\int_a^b v(\rho)v_1(\rho)d\rho}\right) \geq 0. \end{aligned} \tag{2.23}$$

Proof. Using the function $T(\rho) = (\rho - 1)\log \rho, \rho \in \mathbb{R}^+$, in (2.19), we obtain (2.23).

Definition 2.6 (Bhattacharyya coefficient) *If u_1 and v_1 are two positive probability density functions defined on $[a,b]$, then the Bhattacharyya coefficient is defined by*

$$B_d(u_1, v_1) = \int_a^b \sqrt{u_1(\rho)v_1(\rho)}\, d\rho.$$

Corollary 2.10 *Let u, v, u_1, v_1 be as stated in Corollary 2.7. Then*

$$B_d(u_1,v_1) \leq \sqrt{\int_a^b u(\rho)v_1(\rho)d\rho \int_a^b u(\rho)u_1(\rho)d\rho} + \sqrt{\int_a^b v(\rho)v_1(\rho)d\rho \int_a^b v(\rho)u_1(\rho)d\rho}. \tag{2.24}$$

Proof. Using the function $T(\rho) = -\sqrt{\rho}, \rho \in \mathbb{R}^+$, in (2.19), we obtain (2.24).

Definition 2.7 (Hellinger distance) *If u_1 and v_1 are two positive probability density functions defined on $[a,b]$, then the Hellinger distance is defined by*

$$H_d(u_1,v_1) = \int_a^b \left(\sqrt{u_1(\rho)} - \sqrt{v_1(\rho)}\right)^2 d\rho.$$

Corollary 2.11 *Let u, v, u_1, v_1 be as stated in Corollary 2.7. Then $u(\rho)+v(\rho) = 1$ for all $\rho \in [a,b]$. Then*

$$H_d(u_1,v_1) \geq \left(\sqrt{\int_a^b u(\rho)u_1(\rho)d\rho} - \sqrt{\int_a^b u(\rho)v_1(\rho)d\rho}\right)^2 + \left(\sqrt{\int_a^b v(\rho)u_1(\rho)d\rho} - \sqrt{\int_a^b v(\rho)v_1(\rho)d\rho}\right)^2 \geq 0. \tag{2.25}$$

Proof. Using the function $T(\rho) = (\sqrt{\rho}-1)^2, \rho \in \mathbb{R}^+$, in (2.19), we obtain (2.25).

Definition 2.8 (Triangular discrimination) *If u_1 and v_1 are two positive probability density functions defined on $[a,b]$, then the triangular discrimination between u_1 and v_1 is defined by*

$$T_d(u_1,v_1) = \int_a^b \frac{(u_1(\rho)-v_1(\rho))^2}{u_1(\rho)+v_1(\rho)} d\rho.$$

Corollary 2.12 *Let u, v, u_1, v_1 be as stated in Corollary 2.7. Then*

$$T_d(u_1,v_1) \geq \frac{\left(\int_a^b u(\rho)(u_1(\rho)-v_1(\rho))d\rho\right)^2}{\int_a^b u(\rho)(u_1(\rho)+v_1(\rho))d\rho} + \frac{\left(\int_a^b (u_1(\rho)-v_1(\rho))v(\rho)d\rho\right)^2}{\int_a^b (u_1(\rho)+v_1(\rho))v(\rho)d\rho} \geq 0. \tag{2.26}$$

Proof. Since the function $\phi(\rho) = \frac{(\rho-1)^2}{\rho+1}, \rho \in \mathbb{R}^+$ is convex. Therefore using the function $T(\rho) = \phi(\rho)$, in (2.19), we obtain (2.26).

2.1.2 Further Generalization

In the following theorem we present further refinement of the Jensen inequality concerning n functions whose sum is equal to unity.

Theorem 2.3 *Let $\psi : I \to \mathbb{R}$ be a convex function defined on the interval I. Let $p, g, u_l \in L[a,b]$ such that $g(\rho) \in I, p(\rho), u_l(\rho) \in \mathbb{R}^+$ for all $\rho \in [a,b]$ $(l = 1,2,...,n)$ and $\sum_{l=1}^{n} u_l(\rho) = 1$, $P = \int_a^b p(\rho)d\rho$. Assume that L_1 and L_2 are non empty disjoint subsets of $\{1,2,...,n\}$ such that $L_1 \cup L_2 = \{1,2,...,n\}$. Then*

$$\begin{aligned}
&\frac{1}{P}\int_a^b p(\rho)\psi(g(\rho))d\rho \\
&\geq \frac{1}{P}\int_a^b \sum_{l\in L_1} u_l(\rho)p(\rho)d\rho\,\psi\left(\frac{\int_a^b \sum_{l\in L_1} u_l(\rho)p(\rho)g(\rho)d\rho}{\int_a^b \sum_{l\in L_1} u_l(\rho)p(\rho)d\rho}\right) \\
&+\frac{1}{P}\int_a^b \sum_{l\in L_2} u_l(\rho)p(\rho)d\rho\,\psi\left(\frac{\int_a^b \sum_{l\in L_2} u_l(\rho)p(\rho)g(\rho)d\rho}{\int_a^b \sum_{l\in L_2} u_l(\rho)p(\rho)d\rho}\right) \\
&\geq \psi\left(\frac{1}{P}\int_a^b p(\rho)g(\rho)d\rho\right).
\end{aligned} \tag{2.27}$$

If the function ψ is concave then the reverse inequalities hold in (2.27).

Proof. Since $\sum_{i=1}^{n} u_l(\rho) = 1$, therefore we may write

$$\int_a^b p(\rho)\psi(g(\rho))d\rho = \int_a^b \sum_{l\in L_1} u_l(\rho)p(\rho)\psi(g(\rho))d\rho + \int_a^b \sum_{l\in L_2} u_l(\rho)p(\rho)\psi(g(\rho))d\rho. \tag{2.28}$$

Applying integral Jensen's inequality on both terms on the right hand side of (2.28) we obtain

$$\begin{aligned}
&\frac{1}{P}\int_a^b p(\rho)\psi(g(\rho))d\rho \\
&\geq \frac{1}{P}\int_a^b \sum_{l\in L_1} u_l(\rho)p(\rho)d\rho\,\psi\left(\frac{\int_a^b \sum_{l\in L_1} u_l(\rho)p(\rho)g(\rho)d\rho}{\int_a^b \sum_{l\in L_1} u_l(\rho)p(\rho)d\rho}\right) \\
&+\frac{1}{P}\int_a^b \sum_{l\in L_2} u_l(\rho)p(\rho)d\rho\,\psi\left(\frac{\int_a^b \sum_{l\in L_2} u_l(\rho)p(\rho)g(\rho)d\rho}{\int_a^b \sum_{l\in L_2} u_l(\rho)p(\rho)d\rho}\right) \\
&\geq \psi\left(\frac{1}{P}\int_a^b \sum_{l\in L_1} u_l(\rho)p(\rho)g(\rho)d\rho + \frac{1}{P}\int_a^b \sum_{l\in L_2} u_l(\rho)p(\rho)g(\rho)d\rho\right) \\
&\qquad\qquad \text{(By the convexity of } \psi\text{)} \\
&= \psi\left(\frac{1}{P}\int_a^b p(\rho)g(\rho)d\rho\right)
\end{aligned} \tag{2.29}$$

Remark 2.2 If we take $n = 2$, in Theorem 2.3, we deduce Theorem 2.1. Also, analogously as in the previous sections we may give applications of Theorem 2.3 for different means, Hölder inequality and information theory.

2.2 Refinement of Discrete Jensen's Inequality

We start to derive new refinement of discrete Jensen's inequality associated to two particular finite sequences.

Theorem 2.4 *Let $\psi : I \to \mathbb{R}$ be a convex function defined on the interval I. Let $y_j \in I$, $\zeta_j, \eta_j, \theta_j \in \mathbb{R}^+$ $(j = 1,2,\dots,n)$ such that $\eta_j + \theta_j = 1$ for all $j \in \{1,2,\dots,n\}$, $\overline{\zeta} = \sum_{j=1}^n \zeta_j$. Then*

$$\psi\left(\frac{1}{\zeta}\sum_{j=1}^n \zeta_j y_j\right) \le \frac{1}{\zeta}\sum_{j=1}^n \eta_j\zeta_j\psi\left(\frac{\sum_{j=1}^n \zeta_j\eta_j y_j}{\sum_{j=1}^n \zeta_j\eta_j}\right) + \frac{1}{\zeta}\sum_{j=1}^n \zeta_j\theta_j\psi\left(\frac{\sum_{j=1}^n \zeta_j\theta_j y_j}{\sum_{j=1}^n \zeta_j\theta_j}\right) \le \frac{1}{\zeta}\sum_{j=1}^n \zeta_j\psi(y_j). \tag{2.30}$$

If the function ψ is concave then the reverse inequalities hold in (2.30).

Proof. Since $\eta_j + \theta_j = 1$ for all $j \in \{1,2,\dots,n\}$, therefore we have

$$\begin{aligned}\psi\left(\frac{1}{\zeta}\sum_{j=1}^n \zeta_j y_j\right) &= \psi\left(\frac{1}{\zeta}\sum_{j=1}^n \zeta_j\eta_j y_j + \frac{1}{\zeta}\sum_{j=1}^n \zeta_j\theta_j y_j\right)\\ &= \psi\left(\frac{\sum_{j=1}^n \zeta_j\eta_j}{\zeta}\frac{\sum_{j=1}^n \zeta_j\eta_j y_j}{\sum_{j=1}^n \zeta_j\eta_j} + \frac{\sum_{j=1}^n \zeta_j\theta_j}{\zeta}\frac{\sum_{j=1}^n \zeta_j\theta_j y_j}{\sum_{j=1}^n \zeta_j\theta_j}\right).\end{aligned} \tag{2.31}$$

Applying convexity of ψ on the right side of (3.4) we obtain

$$\begin{aligned}\psi\left(\frac{1}{\zeta}\sum_{j=1}^n \zeta_j y_j\right) &\le \frac{\sum_{j=1}^n \zeta_j\eta_j}{\zeta}\psi\left(\frac{\sum_{j=1}^n \zeta_j\eta_j y_j}{\sum_{j=1}^n \zeta_j\eta_j}\right) + \frac{\sum_{j=1}^n \zeta_j\theta_j}{\zeta}\psi\left(\frac{\sum_{j=1}^n \zeta_j\theta_j y_j}{\sum_{j=1}^n \zeta_j\theta_j}\right)\\ &\le \frac{1}{\zeta}\sum_{j=1}^n \zeta_j\eta_j\psi(y_j) + \frac{1}{\zeta}\sum_{j=1}^n \zeta_j\theta_j\psi(y_j) \text{ (By Jensen inequality)}\\ &= \frac{1}{\zeta}\sum_{j=1}^n \zeta_j\psi(y_j).\end{aligned} \tag{2.32}$$

This proves the required result.

Let $\mathbf{y} = (y_1, \ldots, y_n)$ and $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n)$ be two positive n-tuples such that $\overline{\zeta} = \sum_{j=1}^{n} \zeta_j$. Then the well known power means of order $r \in \mathbb{R}$ is defined as:

$$M_r(\boldsymbol{\zeta};\mathbf{y}) = \begin{cases} \left(\frac{1}{\zeta}\sum_{j=1}^{n} \zeta_j y_j^r\right)^{\frac{1}{r}}, & \text{if } r \neq 0, \\ \left(\prod_{j=1}^{n} y_j^{\zeta_j}\right)^{\frac{1}{\zeta}}, & \text{if } r = 0. \end{cases} \tag{2.33}$$

In the following corollary we deduce inequalities for power means.

Corollary 2.13 *Let $y_j, \zeta_j, \eta_j, \theta_j \in \mathbb{R}^+$ $(j = 1,2,\ldots,n)$ such that $\eta_j + \theta_j = 1$ for all $j \in \{1,2,\ldots,n\}$, $\zeta = \sum_{j=1}^{n} \zeta_j$. Let $s,t \in \mathbb{R}$ such that $s \leq t$. Then*

$$M_t(\boldsymbol{\zeta};\mathbf{y}) \geq \left[M_1(\boldsymbol{\zeta};\boldsymbol{\eta})M_s^t(\boldsymbol{\zeta}.\boldsymbol{\eta};\mathbf{y}) + M_1(\boldsymbol{\zeta};\boldsymbol{\theta})M_s^t(\boldsymbol{\zeta}.\boldsymbol{\theta};\mathbf{y})\right]^{\frac{1}{t}} \geq M_s(\boldsymbol{\zeta};\mathbf{y}),\ t \neq 0. \tag{2.34}$$

$$M_t(\boldsymbol{\zeta};\mathbf{y}) \geq \exp\left(M_1(\boldsymbol{\zeta};\boldsymbol{\eta})\log M_s(\boldsymbol{\zeta}.\boldsymbol{\eta};\mathbf{y}) + M_1(\boldsymbol{\zeta};\boldsymbol{\theta})\log M_s(\boldsymbol{\zeta}.\boldsymbol{\theta};\mathbf{y})\right) \geq M_s(\boldsymbol{\zeta};\mathbf{y}),\ t = 0. \tag{2.35}$$

$$M_s(\boldsymbol{\zeta};\mathbf{y}) \leq [M_1(\boldsymbol{\zeta};\boldsymbol{\eta})M_t^s(\boldsymbol{\zeta}.\boldsymbol{\eta};\mathbf{y}) + M_1(\boldsymbol{\zeta};\boldsymbol{\theta})M_t^s(\boldsymbol{\zeta}.\boldsymbol{\theta};\mathbf{y})]^{\frac{1}{s}} \leq M_t(\boldsymbol{\zeta};\mathbf{y}),\ s \neq 0. \tag{2.36}$$

$$M_s(\boldsymbol{\zeta};\mathbf{y}) \leq \exp\left(M_1(\boldsymbol{\zeta};\boldsymbol{\eta})\log M_t(\boldsymbol{\zeta}.\boldsymbol{\eta};\mathbf{y}) + M_1(\boldsymbol{\zeta};\boldsymbol{\theta})\log M_t(\boldsymbol{\zeta}.\boldsymbol{\theta};\mathbf{y})\right) \leq M_t(\boldsymbol{\zeta};\mathbf{y}),\ s = 0. \tag{2.37}$$

Proof. If $s,t \in \mathbb{R}$ and $s,t \neq 0$, then using (2.30) for $\psi(\omega) = \omega^{\frac{t}{s}}$, $\omega > 0$, $y_j \to y_j^s$ and then taking power $\frac{1}{t}$ we get (2.34). For the case $t = 0$, taking limit $t \to 0$ in (2.34) we obtain (2.35).

Similarly taking (2.30) for $\psi(\omega) = \omega^{\frac{s}{t}}$, $\omega > 0, s,t \neq 0$, $y_j \to y_j^t$ and then taking power $\frac{1}{s}$ we get (2.36). For $s = 0$ we take limit as above. Let $\mathbf{y} = (y_1, \ldots, y_n)$ and $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n)$ be two positive n-tuples such that $\zeta = \sum_{j=1}^{n} \zeta_j$. If h is a continuous as well as strictly monotone function, then the quasi arithmetic mean is defined by:

$$\tilde{M}_h(\boldsymbol{\zeta};\mathbf{y}) = h^{-1}\left(\frac{1}{\zeta}\sum_{j=1}^{n} \zeta_j h(y_j)\right). \tag{2.38}$$

We give inequalities for quasi arithmetic mean.

Corollary 2.14 *Let $y_j, \zeta_j, \eta_j, \theta_j \in \mathbb{R}^+$ $(j = 1,2,\ldots,n)$ such that $\eta_j + \theta_j = 1$ for all $j \in \{1,2,\ldots,n\}$, $\zeta = \sum_{j=1}^{n} \zeta_j$. Also assume that h is a strictly monotone continuous function. If $f \circ h^{-1}$ is convex function, then*

$$f\left(\tilde{M}_h(\boldsymbol{\zeta};\mathbf{y})\right) \leq M_1(\boldsymbol{\zeta};\boldsymbol{\eta})f(\tilde{M}_h(\boldsymbol{\zeta}.\boldsymbol{\eta};\mathbf{y})) + M_1(\boldsymbol{\zeta};\boldsymbol{\theta})f(\tilde{M}_h(\boldsymbol{\zeta}.\boldsymbol{\theta};\mathbf{y})) \leq \frac{1}{\zeta}\sum_{j=1}^{n} \zeta_j f(y_j). \tag{2.39}$$

If the function $f \circ h^{-1}$ is concave then the reverse inequalities hold in (2.39).

Proof. The required inequalities may be deduced by using (2.30) for $y_j \to h(y_j)$ and $\psi \to f \circ h^{-1}$.

2.2.1 Applications in Information Theory

Definition 2.9 (CSISZÁR F-DIVERGENCE) *Let $f : \subsetneq 0,\infty) \to \mathbb{R}$ be a convex function, also suppose that $\mathbf{c} = (c_1, c_2, \dots, c_n)$ and $\mathbf{d} = (d_1, d_2, \dots, d_n)$ are positive n-tuples, then the Csiszár f-divergence functional is defined by*

$$M_f(\mathbf{c},\mathbf{d}) = \sum_{j=1}^{n} d_j f\left(\frac{c_j}{d_j}\right).$$

Theorem 2.5 *Let $f : (0,\infty) \to \mathbb{R}$ be a convex function and $\mathbf{c} = (c_1, c_2, \dots, c_n)$, $\mathbf{d} = (d_1, d_2, \dots, d_n)$, $\boldsymbol{\eta} = (\eta_1, \eta_2, \dots, \eta_n)$ and $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_n)$ be positive n-tuples such that $\eta_j + \theta_j = 1$ for $j \in \{1, 2, \dots, n\}$. Then*

$$\begin{aligned} M_f(\mathbf{c},\mathbf{d}) &\geq \sum_{j=1}^{n} \eta_j d_j f\left(\frac{\sum_{j=1}^{n} c_j \eta_j}{\sum_{j=1}^{n} d_j \eta_j}\right) \\ &+ \sum_{j=1}^{n} d_j \theta_j f\left(\frac{\sum_{j=1}^{n} c_j \theta_j}{\sum_{j=1}^{n} d_j \theta_j}\right) \geq f\left(\frac{\sum_{j=1}^{n} c_j}{\sum_{j=1}^{n} d_j}\right) \sum_{j=1}^{n} d_j. \end{aligned} \tag{2.40}$$

Proof. Using Theorem 2.4 for $\psi = f$, $y_j = \frac{c_j}{d_j}$ and $\zeta_j = d_j$ for $j \in \{1, 2, \dots, n\}$, we obtain (2.40).

Definition 2.10 (SHANNON ENTROPY) *Let $\mathbf{q} = (q_1, q_2, \dots, q_n)$ be positive probability distribution then the Shannon entropy is defined as*

$$S(\mathbf{q}) = -\sum_{j=1}^{n} q_j \log q_j.$$

Corollary 2.15 *Let $\mathbf{d} = (d_1, d_2, \dots, d_n)$ be positive probability distribution. Also assume that $\boldsymbol{\eta} = (\eta_1, \eta_2, \dots, \eta_n)$ and $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_n)$ be two positive n-tuples such that $\eta_j + \theta_j = 1$ for $j \in \{1, 2, \dots, n\}$. Then*

$$S(\mathbf{d}) \leq \sum_{j=1}^{n} \eta_j d_j \log\left(\frac{\sum_{j=1}^{n} \eta_j}{\sum_{j=1}^{n} d_j \eta_j}\right) + \sum_{j=1}^{n} d_j \theta_j \log\left(\frac{\sum_{j=1}^{n} \theta_j}{\sum_{j=1}^{n} d_j \theta_j}\right) \leq \log n. \tag{2.41}$$

Proof. Taking $f(\omega) = -\log \omega, \omega \in \mathbb{R}^+$, $c_j = 1$, for each $j \in \{1, 2, \dots, n\}$, in (2.40), we obtain (2.41).

Definition 2.11 (KULLBACK-LEIBLER DIVERGENCE) *Let $\mathbf{c} = (c_1, c_2, \dots, c_n)$ and $\mathbf{d} = (d_1, d_2, \dots, d_n)$ be positive probability distributions then the Kullback-Leibler divergence is defined by*

$$K_d(\mathbf{c},\mathbf{d}) = \sum_{i=1}^{n} c_i \log\left(\frac{c_i}{d_i}\right).$$

Corollary 2.16 *Let* $\mathbf{c} = (c_1, c_2, \ldots, c_n)$, $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ *be positive probability distributions and* $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_n)$, $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$ *be positive n-tuples such that* $\eta_j + \theta_j = 1$ *for* $j \in \{1, 2, \ldots, n\}$. *Then*

$$K_d(\mathbf{c}, \mathbf{d}) \geq \sum_{j=1}^{n} \eta_j c_j \log\left(\frac{\sum_{j=1}^{n} c_j \eta_j}{\sum_{j=1}^{n} d_j \eta_j}\right) + \sum_{j=1}^{n} c_j \theta_j \log\left(\frac{\sum_{j=1}^{n} c_j \theta_j}{\sum_{j=1}^{n} d_j \theta_j}\right) \geq 0. \tag{2.42}$$

Proof. Taking $f(\omega) = \omega \log \omega, \omega \in \mathbb{R}^+$, in (2.40), we obtain (2.42).

Definition 2.12 (VARIATIONAL DISTANCE) *For two positive probability distributions* $\mathbf{c} = (c_1, c_2, \cdots, c_n)$ *and* $\mathbf{d} = (d_1, d_2, \cdots, d_n)$ *the variational distance is defined by*

$$V_d(\mathbf{c}, \mathbf{d}) = \sum_{j=1}^{n} \left|c_j - d_j\right|.$$

Corollary 2.17 *Under the assumptions of Corollary 2.16, the following inequality holds:*

$$V_d(\mathbf{c}, \mathbf{d}) \geq \left|\sum_{j=1}^{n} \eta_j (c_j - d_j)\right| + \left|\sum_{j=1}^{n} \theta_j (c_j - d_j)\right|. \tag{2.43}$$

Proof. Using the function $f(\omega) = |\omega - 1|, \omega \in \mathbb{R}^+$, in (2.40), we obtain (2.43).

Definition 2.13 (JEFFREY'S DISTANCE) *For two positive probability distributions* $\mathbf{c} = (c_1, c_2, \cdots, c_n)$ *and* $\mathbf{d} = (d_1, d_2, \cdots, d_n)$, *the Jeffrey distance is defined by*

$$J_d(\mathbf{c}, \mathbf{d}) = \sum_{j=1}^{n} (c_j - d_j) \log\left(\frac{c_j}{d_j}\right).$$

Corollary 2.18 *Under the assumptions of Corollary 2.16, the following inequality holds:*

$$J_d(\mathbf{c}, \mathbf{d}) \geq \sum_{j=1}^{n} \eta_j (c_j - d_j) \log\left(\frac{\sum_{j=1}^{n} \eta_j c_j}{\sum_{j=1}^{n} \eta_j d_j}\right) + \sum_{j=1}^{n} \theta_j (c_j - d_j) \log\left(\frac{\sum_{j=1}^{n} \theta_j c_j}{\sum_{j=1}^{n} \theta_j d_j}\right) \geq 0. \tag{2.44}$$

Proof. Using the function $f(\omega) = (\omega - 1) \log \omega, \omega \in \mathbb{R}^+$, in (2.40), we obtain (2.44).

Definition 2.14 (BHATTACHARYYA COEFFICIENT) *For two positive probability distributions* $\mathbf{c} = (c_1, c_2, \cdots, c_n)$ *and* $\mathbf{d} = (d_1, d_2, \cdots, d_n)$, *the Bhattacharyya coefficient is defined by*

$$B_d(\mathbf{c}, \mathbf{d}) = \sum_{j=1}^{n} \sqrt{c_j d_j}.$$

Corollary 2.19 *Under the assumptions of Corollary 2.16, the following inequality holds:*

$$B_d(u_1,v_1) \leq \sqrt{\sum_{j=1}^{n} \eta_j c_j \sum_{j=1}^{n} \eta_j d_j} + \sqrt{\sum_{j=1}^{n} \theta_j c_j \sum_{j=1}^{n} \theta_j d_j}. \tag{2.45}$$

Proof. Using the function $f(\omega) = -\sqrt{\omega}, \omega \in \mathbb{R}^+$, in (2.40), we obtain (2.45).

Definition 2.15 (HELLINGER DISTANCE) *For two positive probability distributions* $\mathbf{c} = (c_1, c_2, \cdots, c_n)$ *and* $\mathbf{d} = (d_1, d_2, \cdots, d_n)$*, the Hellinger distance is defined by*

$$H_d(\mathbf{c},\mathbf{d}) = \sum_{j=1}^{n} \left(\sqrt{c_j} - \sqrt{d_j}\right)^2.$$

Corollary 2.20 *Under the assumptions of Corollary 2.16, the following inequality holds:*

$$H_d(\mathbf{c},\mathbf{d}) \geq \left(\sqrt{\sum_{j=1}^{n} \eta_j c_j} - \sqrt{\sum_{j=1}^{n} \eta_j d_j}\right)^2 + \left(\sqrt{\sum_{j=1}^{n} \theta_j c_j} - \sqrt{\sum_{j=1}^{n} \theta_j d_j}\right)^2 \geq 0. \tag{2.46}$$

Proof. Using the function $f(\omega) = (\sqrt{\omega} - 1)^2, \omega \in \mathbb{R}^+$, in (2.40), we obtain (2.46).

Definition 2.16 (TRIANGULAR DISCRIMINATION) *For two positive probability distributions* $\mathbf{c} = (c_1, c_2, \cdots, c_n)$ *and* $\mathbf{d} = (d_1, d_2, \cdots, d_n)$*, the triangular discrimination is defined by*

$$T_d(\mathbf{c},\mathbf{d}) = \sum_{j=1}^{n} \frac{(c_j - d_j)^2}{c_j + d_j}.$$

Corollary 2.21 *Under the assumptions of Corollary 2.16, the following inequality holds:*

$$T_d(\mathbf{c},\mathbf{d}) \geq \frac{\left(\sum_{j=1}^{n} \eta_j (c_j - d_j)\right)^2}{\sum_{j=1}^{n} \eta_j (c_j + d_j)} + \frac{\left(\sum_{j=1}^{n} (c_j - d_j)\theta_j\right)^2}{\sum_{j=1}^{n} (c_j + d_j)\theta_j} \geq 0. \tag{2.47}$$

Proof. Since the function $\phi(\omega) = \frac{(\omega-1)^2}{\omega+1}, \omega \in \mathbb{R}^+$ is convex. Therefore using the function $f(\omega) = \phi(\omega)$, in (2.40), we obtain (2.47).

Now we begin to derive bounds for Zipf-Mandelbrot entropy.
In the following corollary we present another bound for Zipf-Mandelbrot entropy.

Corollary 2.22 *Let* $\vartheta \geq 0$, $s, d_j > 0$, $j = 1,2,..,n$ *with* $\sum_{j=1}^{n} d_j = 1$. *Also assume that* $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_n)$, $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$ *be positive n-tuples such that* $\eta_j + \theta_j = 1$ *for* $j \in \{1,2,\ldots,n\}$. *Then*

$$-Z(H,\vartheta,s) - \sum_{j=1}^{n} \frac{\log d_j}{(j+\vartheta)^s H_{n,\vartheta,s}} \geq \sum_{j=1}^{n} \frac{\eta_j}{(j+\vartheta)^s H_{n,\vartheta,s}} \log\left(\frac{\sum_{j=1}^{n} \frac{\eta_j}{(j+\vartheta)^s H_{n,\vartheta,s}}}{\sum_{j=1}^{n} d_j \eta_j}\right)$$

$$+\sum_{j=1}^{n} \frac{\theta_j}{(j+\vartheta)^s H_{n,\vartheta,s}} \log\left(\frac{\sum_{j=1}^{n} \frac{\theta_j}{(j+\vartheta)^s H_{n,\vartheta,s}}}{\sum_{j=1}^{n} d_j\theta_j}\right) \geq 0. \tag{2.48}$$

Proof. Let $c_j = \frac{1}{(j+\vartheta)^s H_{n,\vartheta,s}}$, $j \in \{1,2,...,n\}$, then

$$\begin{aligned}
\sum_{j=1}^{n} c_j \log c_j &= \sum_{j=1}^{n} \frac{1}{(j+\vartheta)^s H_{n,\vartheta,s}} \log \frac{1}{(j+\vartheta)^s H_{n,\vartheta,s}} \\
&= -\sum_{j=1}^{n} \frac{1}{(j+\vartheta)^s H_{n,\vartheta,s}} \log((j+\vartheta)^s H_{n,\vartheta,s}) \\
&= -\sum_{j=1}^{n} \frac{s}{(j+\vartheta)^s H_{n,\vartheta,s}} \log(j+\vartheta) - \sum_{j=1}^{n} \frac{\log H_{n,\vartheta,s}}{(j+\vartheta)^s H_{n,\vartheta,s}} \\
&= -\frac{s}{(j+\vartheta)^s H_{n,\vartheta,s}} \sum_{j=1}^{n} \frac{\log(j+\vartheta)}{(j+\vartheta)^s} - \frac{\log H_{n,\vartheta,s}}{H_{n,\vartheta,s}} \sum_{j=1}^{n} \frac{1}{(j+\vartheta)^s} \\
&= -Z(H,\vartheta,s).
\end{aligned}$$

Since $H_{n,\vartheta,s} = \sum_{j=1}^{n} \frac{1}{(\vartheta+i)^s}$, therefore $\sum_{j=1}^{n} \frac{1}{(j+\vartheta)^s H_{n,\vartheta,s}} = 1$. Hence using (2.42) for $c_j = \frac{1}{(j+\vartheta)^s H_{n,\vartheta,s}}$, $j = 1,2,..,n$, we obtain (2.48). In the following corollary, we derive estimation for Zipf-Mandelbrot entropy by using two Zipf's law corresponding to different parameters.

Corollary 2.23 *Let $\delta_1, \delta_2 \geq 0$, $s_1, s_2 > 0$. Also assume that $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_n)$, $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$ be positive n-tuples such that $\eta_j + \theta_j = 1$ for $j \in \{1,2,\ldots,n\}$. Then*

$$\begin{aligned}
&-Z(H,\delta_1,s_1) + \sum_{j=1}^{n} \frac{\log((j+\delta_2)^{s_2} H_{n,\delta_2,s_2})}{(j+\delta_1)^{s_1} H_{n,\delta_1,s_1}} \\
&\geq \sum_{j=1}^{n} \frac{\eta_j}{(j+\delta_1)^{s_1} H_{n,\delta_1,s_1}} \log\left(\frac{\sum_{j=1}^{n} \frac{\eta_j}{(j+\delta_1)^{s_1} H_{n,\delta_1,s_1}}}{\sum_{j=1}^{n} \frac{\eta_j}{(j+\delta_2)^{s_2} H_{n,\delta_2,s_2}}}\right) \\
&+ \sum_{j=1}^{n} \frac{\theta_j}{(j+\delta_1)^{s_1} H_{n,\delta_1,s_1}} \log\left(\frac{\sum_{j=1}^{n} \frac{\theta_j}{(j+\delta_1)^{s_1} H_{n,\delta_1,s_1}}}{\sum_{j=1}^{n} \frac{\theta_j}{(j+\delta_2)^{s_2} H_{n,\delta_2,s_2}}}\right) \geq 0.
\end{aligned} \tag{2.49}$$

Proof. Let $c_j = \frac{1}{(j+\delta_1)^{s_1} H_{n,\delta_1,s_1}}$ and $d_j = \frac{1}{(j+\delta_2)^{s_2} H_{n,\delta_2,s_2}}$, $j = 1,2,...,n$, then as in the proof of Corollary 2.22, we have

$$\sum_{j=1}^{n} c_j \log c_j = \sum_{j=1}^{n} \frac{1}{(j+\delta_1)^{s_1} H_{n,\delta_1,s_1}} \log \frac{1}{(j+\delta_1)^{s_1} H_{n,\delta_1,s_1}} = -Z(H,\delta_1,s_1).$$

$$\sum_{j=1}^{n} c_j \log d_j = \sum_{j=1}^{n} \frac{\log \frac{1}{(j+\delta_2)^{s_2} H_{n,\delta_2,s_2}}}{(j+\delta_1)^{s_1} H_{n,\delta_1,s_1}} = -\sum_{j=1}^{n} \frac{\log((j+\delta_2)^{s_1} H_{n,\delta_2,s_2})}{(j+\delta_1)^{s_1} H_{n,\delta_1,s_1}}.$$

Also $\sum_{j=1}^{n} c_j = \sum_{j=1}^{n} \frac{1}{(j+\delta_1)^{s_1} H_{n,\delta_1,s_1}} = 1$ and $\sum_{j=1}^{n} \frac{1}{(j+\delta_1)^{s_1} H_{n,\delta_1,s_1}} = 1.$

Therefore using (2.42) for $c_j = \frac{1}{(j+\delta_1)^{s_1} H_{n,\delta_1,s_1}}$ and $d_j = \frac{1}{(j+\delta_2)^{s_2} H_{n,\delta_2,s_2}}$, $j = 1,2,..,n$, we obtain (2.49).

2.2.2 Further Generalization

In the following theorem, we present further refinement of the Jensen inequality concerning m sequences whose sum is equal to unity.

Theorem 2.6 *Let $\psi : I \to \mathbb{R}$ be a convex function defined on the interval I. Let $y_j \in I$, $\zeta_j, \theta_j^l \in \mathbb{R}^+$ $(j = 1,2,\ldots,n, l = 1,2,\ldots,m)$ such that $\sum_{l=1}^{m} \theta_j^l = 1$ for each $j \in \{1,2,\ldots,n\}$, $\zeta = \sum_{j=1}^{n} \zeta_j$. Assume that L_1 and L_2 are non empty disjoint subsets of $\{1,2,...,m\}$ such that $L_1 \cup L_2 = \{1,2,...,m\}$. Then*

$$\begin{aligned}\frac{1}{\zeta}\sum_{j=1}^{n} \zeta_j \psi(y_j) &\geq \frac{1}{\zeta}\sum_{j=1}^{n}\sum_{l\in L_1} \theta_j^l \zeta_j \psi\left(\frac{\sum_{j=1}^{n}\sum_{l\in L_1}\theta_j^l\zeta_j y_j}{\sum_{j=1}^{n}\sum_{l\in L_1}\theta_j^l\zeta_j}\right)\\ &+\frac{1}{\zeta}\sum_{j=1}^{n}\sum_{l\in L_2} \theta_j^l \zeta_j \psi\left(\frac{\sum_{j=1}^{n}\sum_{l\in L_2}\theta_j^l\zeta_j y_j}{\sum_{j=1}^{n}\sum_{l\in L_2}\theta_j^l\zeta_j}\right) \geq \psi\left(\frac{1}{\zeta}\sum_{j=1}^{n}\zeta_j y_j\right). \end{aligned} \tag{2.50}$$

If the function ψ is concave then the reverse inequalities hold in (2.50).

Proof. Since $\sum_{l=1}^{m} \theta_j^l = 1$ for each $j \in \{1,2,\ldots,n\}$, therefore we may write

$$\frac{1}{\zeta}\sum_{j=1}^{n} \zeta_j \psi(y_j) = \frac{1}{\zeta}\sum_{j=1}^{n}\sum_{l\in L_1} \theta_j^l \zeta_j \psi(y_j) + \frac{1}{\zeta}\sum_{j=1}^{n}\sum_{l\in L_2} \theta_j^l \zeta_j \psi(y_j). \tag{2.51}$$

Applying Jensen's inequality on both terms on the right hand side of (2.51) we obtain

$$\begin{aligned}&\frac{1}{\zeta}\sum_{j=1}^{n} \zeta_j \psi(y_j)\\ &\geq \frac{1}{\zeta}\sum_{j=1}^{n}\sum_{l\in L_1} \theta_j^l \zeta_j \psi\left(\frac{\sum_{j=1}^{n}\sum_{l\in L_1}\theta_j^l\zeta_j y_j}{\sum_{j=1}^{n}\sum_{l\in L_1}\theta_j^l\zeta_j}\right) + \frac{1}{\zeta}\sum_{j=1}^{n}\sum_{l\in L_2} \theta_j^l \zeta_j \psi\left(\frac{\sum_{j=1}^{n}\sum_{l\in L_2}\theta_j^l\zeta_j y_j}{\sum_{j=1}^{n}\sum_{l\in L_2}\theta_j^l\zeta_j}\right)\\ &\geq \psi\left(\frac{1}{\zeta}\sum_{j=1}^{n}\sum_{l\in L_1}\theta_j^l\zeta_j\frac{\sum_{j=1}^{n}\sum_{l\in L_1}\theta_j^l\zeta_j y_j}{\sum_{j=1}^{n}\sum_{l\in L_1}\theta_j^l\zeta_j} + \frac{1}{\zeta}\sum_{j=1}^{n}\sum_{l\in L_2}\theta_j^l\zeta_j\frac{\sum_{j=1}^{n}\sum_{l\in L_2}\theta_j^l\zeta_j y_j}{\sum_{j=1}^{n}\sum_{l\in L_2}\theta_j^l\zeta_j}\right)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(By the convexity of } \psi)\\ &= \psi\left(\frac{1}{\zeta}\sum_{j=1}^{n}\zeta_j y_j\right). \end{aligned} \tag{2.52}$$

Remark 2.3 If we take $m = 2$, in Theorem 2.6, we deduce Theorem 2.4. Also, analogously as in the previous sections we may give applications of Theorem 2.6 for means and in information theory.

Chapter 3

Refinements of Jensen-Steffensen's Inequality

The purpose of this chapter is to derive refinements of discrete Jensen-Steffensen's as well as integral Jensen-Steffensen's inequality associated to certain tuples and functions respectively. We present application to Zipf Mandelbrot law. Some more general refinements have also been presented for Jensen-Steffensen's inequality. The results of this chapter are given in [4].

3.1 Refinements of Jensen-Steffensen's Inequality

We begin to derive new refinement of discrete Jensen-Steffensen's inequality associated to two particular finite sequences.

Theorem 3.1 *Let* $\psi : I \to \mathbb{R}$ *be a convex function defined on the interval* I*. Let* $y_\gamma \in I$*,* $\zeta_\gamma, \eta_\gamma, \theta_\gamma \in \mathbb{R}$ $(\gamma = 1,2,\ldots,n)$ *be such that* $\eta_\gamma + \theta_\gamma = 1$ *for all* $j \in \{1,2,\ldots,n\}$ *and*

$\overline{\zeta} = \sum_{\gamma=1}^{n} \zeta_\gamma$. *If* $y_1 \leq y_2 \leq \cdots \leq y_n$ *or* $y_1 \geq y_2 \geq \cdots \geq y_n$ *and*

$$0 \leq \sum_{\gamma=1}^{k} \zeta_\gamma \eta_\gamma \leq \sum_{\gamma=1}^{n} \zeta_\gamma \eta_\gamma, \; k = 1,2,\ldots,n, \; \sum_{\gamma=1}^{n} \zeta_\gamma \eta_\gamma > 0 \; \text{ and } \tag{3.1}$$

$$0 \leq \sum_{\gamma=1}^{k} \zeta_\gamma \theta_\gamma \leq \sum_{\gamma=1}^{n} \zeta_\gamma \theta_\gamma, \; k = 1,2,\ldots,n, \; \sum_{\gamma=1}^{n} \zeta_\gamma \theta_\gamma > 0, \tag{3.2}$$

then

$$\psi\left(\frac{1}{\zeta}\sum_{\gamma=1}^{n} \zeta_\gamma y_\gamma\right) \leq \frac{1}{\zeta}\sum_{\gamma=1}^{n} \eta_\gamma \zeta_\gamma \psi\left(\frac{\sum_{\gamma=1}^{n} \zeta_\gamma \eta_\gamma y_\gamma}{\sum_{\gamma=1}^{n} \zeta_\gamma \eta_\gamma}\right) + \frac{1}{\zeta}\sum_{\gamma=1}^{n} \zeta_\gamma \theta_\gamma \psi\left(\frac{\sum_{\gamma=1}^{n} \zeta_\gamma \theta_\gamma y_\gamma}{\sum_{\gamma=1}^{n} \zeta_\gamma \theta_\gamma}\right) \leq \frac{1}{\zeta}\sum_{\gamma=1}^{n} \zeta_\gamma \psi(y_\gamma). \tag{3.3}$$

If the function ψ *is concave then the reverse inequalities hold in (3.3).*

Proof. Since $\eta_\gamma + \theta_\gamma = 1$ for all $j \in \{1,2,\ldots,n\}$, therefore we have

$$\begin{aligned}\psi\left(\frac{1}{\zeta}\sum_{\gamma=1}^{n} \zeta_\gamma y_\gamma\right) &= \psi\left(\frac{1}{\zeta}\sum_{\gamma=1}^{n} \zeta_\gamma \eta_\gamma y_\gamma + \frac{1}{\zeta}\sum_{\gamma=1}^{n} \zeta_\gamma \theta_\gamma y_\gamma\right) \\ &= \psi\left(\frac{\sum_{\gamma=1}^{n} \zeta_\gamma \eta_\gamma}{\zeta}\frac{\sum_{\gamma=1}^{n} \zeta_\gamma \eta_\gamma y_\gamma}{\sum_{\gamma=1}^{n} \zeta_\gamma \eta_\gamma} + \frac{\sum_{\gamma=1}^{n} \zeta_\gamma \theta_\gamma}{\zeta}\frac{\sum_{\gamma=1}^{n} \zeta_\gamma \theta_\gamma y_\gamma}{\sum_{\gamma=1}^{n} \zeta_\gamma \theta_\gamma}\right).\end{aligned} \tag{3.4}$$

Since $\sum_{\gamma=1}^{n} \zeta_\gamma \eta_\gamma > 0$ and $\sum_{\gamma=1}^{n} \zeta_\gamma \theta_\gamma > 0$, therefore $\sum_{\gamma=1}^{n} \zeta_\gamma \eta_\gamma + \sum_{\gamma=1}^{n} \zeta_\gamma \theta_\gamma > 0$. Which gives that $\zeta > 0$.
Now by applying convexity of ψ to the right side of (3.4) one may obtain

$$\begin{aligned}\psi\left(\frac{1}{\zeta}\sum_{\gamma=1}^{n} \zeta_\gamma y_\gamma\right) &\leq \frac{\sum_{\gamma=1}^{n} \zeta_\gamma \eta_\gamma}{\zeta}\psi\left(\frac{\sum_{\gamma=1}^{n} \zeta_\gamma \eta_\gamma y_\gamma}{\sum_{\gamma=1}^{n} \zeta_\gamma \eta_\gamma}\right) + \frac{\sum_{\gamma=1}^{n} \zeta_\gamma \theta_\gamma}{\zeta}\psi\left(\frac{\sum_{\gamma=1}^{n} \zeta_\gamma \theta_\gamma y_\gamma}{\sum_{\gamma=1}^{n} \zeta_\gamma \theta_\gamma}\right) \\ &\leq \frac{1}{\zeta}\sum_{\gamma=1}^{n} \zeta_\gamma \eta_\gamma \psi(y_\gamma) + \frac{1}{\zeta}\sum_{\gamma=1}^{n} \zeta_\gamma \theta_\gamma \psi(y_\gamma) \\ &\quad \text{(By Jensen-Steffensen's inequality)} \\ &= \frac{1}{\zeta}\sum_{\gamma=1}^{n} \zeta_\gamma \psi(y_\gamma).\end{aligned} \tag{3.5}$$

This proves the required result.

Remark 3.1 If we add (3.1) and (3.2), then the Jensen-Steffensen inequality conditions (1.16) will be obtained.

The integral version of the above theorem may be stated as:

Theorem 3.2 *Let $\psi : I \to \mathbb{R}$ be a convex function defined on the interval I. Let $h,u,v,g : [a,b] \to \mathbb{R}$ be integrable functions such that $g(\rho) \in I, u(\rho), v(\rho), h(\rho) \in \mathbb{R}$ for all $\rho \in [a,b]$ and $v(\rho)+u(\rho) = 1$, $H = \int_a^b h(\rho)d\rho$. If g is monotonic on $[a,b]$ and*

$$0 \leq \int_a^\lambda h(\rho)u(\rho)d\rho \leq \int_a^b h(\rho)u(\rho)d\rho,\ \lambda \in [a,b],\ \int_a^b u(\rho)h(\rho)d\rho > 0 \text{ and} \tag{3.6}$$

$$0 \leq \int_a^\lambda h(\rho)v(\rho)d\rho \leq \int_a^b h(\rho)v(\rho)d\rho,\ \lambda \in [a,b],\ \int_a^b v(\rho)h(\rho)d\rho > 0, \tag{3.7}$$

then

$$\psi\left(\frac{1}{H}\int_a^b h(\rho)g(\rho)d\rho\right) \leq \frac{1}{H}\int_a^b u(\rho)h(\rho)d\rho\,\psi\left(\frac{\int_a^b h(\rho)u(\rho)g(\rho)d\rho}{\int_a^b h(\rho)u(\rho)d\rho}\right)$$
$$+\frac{1}{H}\int_a^b h(\rho)v(\rho)d\rho\,\psi\left(\frac{\int_a^b h(\rho)v(\rho)g(\rho)d\rho}{\int_a^b h(\rho)v(\rho)d\rho}\right) \leq \frac{1}{H}\int_a^b h(\rho)\psi(g(\rho))d\rho. \tag{3.8}$$

If the function ψ is concave then the reverse inequalities hold in (3.8).

Remark 3.2 By adding (3.6) and (3.7), one may obtain conditions (1.7).

In the following corollary we present result for Zipf-Mandelbrot law.

Corollary 3.1 *Let $\vartheta \geq 0$, $s > 0$ and $\psi : \mathbb{R}^+ \to \mathbb{R}$ be a convex function defined on the interval $\underline{I}$. Let $\zeta_\gamma, \eta_\gamma, \theta_\gamma \in \mathbb{R}$ $(j = 1,2,\ldots,n)$ be such that $\eta_\gamma + \theta_\gamma = 1$ for all $\gamma \in \{1,2,\ldots,n\}$ and $\overline{\zeta} = \sum_{\gamma=1}^n \zeta_\gamma$. If*

$$0 \leq \sum_{\gamma=1}^k \zeta_\gamma\eta_\gamma \leq \sum_{\gamma=1}^n \zeta_\gamma\eta_\gamma,\ k = 1,2,\ldots,n,\ \sum_{\gamma=1}^n \zeta_\gamma\eta_\gamma > 0 \text{ and} \tag{3.9}$$

$$0 \leq \sum_{\gamma=1}^k \zeta_\gamma\theta_\gamma \leq \sum_{\gamma=1}^n \zeta_\gamma\theta_\gamma,\ k = 1,2,\ldots,n,\ \sum_{\gamma=1}^n \zeta_\gamma\theta_\gamma > 0, \tag{3.10}$$

then

$$\psi\left(\frac{1}{\zeta}\sum_{\gamma=1}^n \zeta_\gamma f(\gamma,n,\vartheta,s)\right) \leq \frac{1}{\zeta}\sum_{\gamma=1}^n \eta_\gamma\zeta_\gamma\psi\left(\frac{\sum_{\gamma=1}^n \zeta_\gamma\eta_\gamma f(\gamma,n,\vartheta,s)}{\sum_{\gamma=1}^n \zeta_\gamma\eta_\gamma}\right)$$
$$+\frac{1}{\zeta}\sum_{\gamma=1}^n \zeta_\gamma\theta_\gamma\psi\left(\frac{\sum_{\gamma=1}^n \zeta_\gamma\theta_\gamma f(\gamma,n,\vartheta,s)}{\sum_{\gamma=1}^n \zeta_\gamma\theta_\gamma}\right) \leq \frac{1}{\zeta}\sum_{\gamma=1}^n \zeta_\gamma\psi\left(f(\gamma,n,\vartheta,s)\right). \tag{3.11}$$

If the function ψ is concave then the reverse inequalities hold in (3.11).

Proof. Substituting $y_\gamma = f(\gamma,n,\vartheta,s)$, $\gamma \in \{1,2,...,n\}$, in (3.3) we obtain (3.11).

3.2 Further Generalization

In the following theorem, we present further refinement of discrete Jensen-Steffensen's inequality associated to m sequences whose sum is equal to unity.

Theorem 3.3 *Let $\psi : I \to \mathbb{R}$ be a convex function defined on the interval I. Let $y_\gamma \in I$, $\zeta_\gamma, \theta^l_\gamma \in \mathbb{R}$ $(\gamma = 1,2,\ldots,n, l = 1,2,\ldots,m)$ be such that $\sum_{l=1}^m \theta^l_\gamma = 1$ for each $j \in \{1,2,\ldots,n\}$ and $\zeta = \sum_{\gamma=1}^n \zeta_\gamma$. Assume that L_1 and L_2 are non empty disjoint subsets of $\{1,2,...,m\}$ such that $L_1 \cup L_2 = \{1,2,...,m\}$. If $y_1 \le y_2 \le \cdots \le y_n$ or $y_1 \ge y_2 \ge \cdots \ge y_n$ and*

$$0 \le \sum_{\gamma=1}^{k} \zeta_\gamma \theta^l_\gamma \le \sum_{\gamma=1}^{n} \zeta_\gamma \theta^l_\gamma, \ k = 1,2,\ldots,n, \ \sum_{\gamma=1}^{n} \zeta_\gamma \theta^l_\gamma > 0, \quad \text{for each } l \in \{1,2,\ldots,m\}. \tag{3.12}$$

Then

$$\frac{1}{\zeta}\sum_{\gamma=1}^{n} \zeta_\gamma \psi(y_\gamma) \ge \frac{1}{\zeta}\sum_{\gamma=1}^{n}\sum_{l\in L_1} \theta^l_\gamma \zeta_\gamma \psi\left(\frac{\sum_{\gamma=1}^n \sum_{l\in L_1} \theta^l_\gamma \zeta_\gamma y_\gamma}{\sum_{\gamma=1}^n \sum_{l\in L_1} \theta^l_\gamma \zeta_\gamma}\right) + \frac{1}{\zeta}\sum_{\gamma=1}^{n}\sum_{l\in L_2} \theta^l_\gamma \zeta_\gamma \psi\left(\frac{\sum_{\gamma=1}^n \sum_{l\in L_2} \theta^l_\gamma \zeta_\gamma y_\gamma}{\sum_{\gamma=1}^n \sum_{l\in L_2} \theta^l_\gamma \zeta_\gamma}\right) \ge \psi\left(\frac{1}{\zeta}\sum_{\gamma=1}^{n} \zeta_\gamma y_\gamma\right). \tag{3.13}$$

If the function ψ is concave then the reverse inequalities hold in (3.13).

Proof. Since $\sum_{l=1}^m \theta^l_\gamma = 1$ for each $j \in \{1,2,\ldots,n\}$, therefore we may write

$$\sum_{\gamma=1}^{n} \zeta_\gamma \psi(y_\gamma) = \sum_{\gamma=1}^{n}\sum_{l\in L_1} \theta^l_\gamma \zeta_\gamma \psi(y_\gamma) + \sum_{\gamma=1}^{n}\sum_{l\in L_2} \theta^l_\gamma \zeta_\gamma \psi(y_\gamma). \tag{3.14}$$

Since $\sum_{\gamma=1}^n \zeta_\gamma \theta^l_\gamma > 0$ for each $l \in \{1,2,\ldots,m\}$, therefore $\sum_{l=1}^m \sum_{\gamma=1}^n \zeta_\gamma \theta^l_\gamma > 0$. Also, $\sum_{l=1}^m \theta^l_\gamma = 1$. Hence we conclude that $\zeta > 0$.
Now applying Jensen-Steffensen's inequality to both terms on the right hand side of (3.14) we obtain

$$\begin{aligned}
&\frac{1}{\zeta}\sum_{\gamma=1}^{n} \zeta_\gamma \psi(y_\gamma) \\
&\ge \frac{1}{\zeta}\sum_{\gamma=1}^{n}\sum_{l\in L_1} \theta^l_\gamma \zeta_\gamma \psi\left(\frac{\sum_{\gamma=1}^n \sum_{l\in L_1} \theta^l_\gamma \zeta_\gamma y_\gamma}{\sum_{\gamma=1}^n \sum_{l\in L_1} \theta^l_\gamma \zeta_\gamma}\right) + \frac{1}{\zeta}\sum_{\gamma=1}^{n}\sum_{l\in L_2} \theta^l_\gamma \zeta_\gamma \psi\left(\frac{\sum_{\gamma=1}^n \sum_{l\in L_2} \theta^l_\gamma \zeta_\gamma y_\gamma}{\sum_{\gamma=1}^n \sum_{l\in L_2} \theta^l_\gamma \zeta_\gamma}\right) \\
&\ge \psi\left(\frac{1}{\zeta}\sum_{\gamma=1}^{n}\sum_{l\in L_1} \theta^l_\gamma \zeta_\gamma \frac{\sum_{\gamma=1}^n \sum_{l\in L_1} \theta^l_\gamma \zeta_\gamma y_\gamma}{\sum_{\gamma=1}^n \sum_{l\in L_1} \theta^l_\gamma \zeta_\gamma} + \frac{1}{\zeta}\sum_{\gamma=1}^{n}\sum_{l\in L_2} \theta^l_\gamma \zeta_\gamma \frac{\sum_{\gamma=1}^n \sum_{l\in L_2} \theta^l_\gamma \zeta_\gamma y_\gamma}{\sum_{\gamma=1}^n \sum_{l\in L_2} \theta^l_\gamma \zeta_\gamma}\right) \\
&\qquad \text{(By the convexity of } \psi)
\end{aligned}$$

$$= \psi\left(\frac{1}{\zeta}\sum_{\gamma=1}^{n}\zeta_\gamma y_\gamma\right). \tag{3.15}$$

In the following theorem, we present further refinement of integral Jensen-Steffensen's inequality.

Theorem 3.4 *Let $\psi : I \to \mathbb{R}$ be a convex function defined on the interval I. Let $h, g, u_l \in L[a,b]$ be such that $g(\rho) \in I, h(\rho), u_l(\rho) \in \mathbb{R}$ for all $\rho \in [a,b]$ $(l = 1,2,...,n)$ and $\sum_{l=1}^{n} u_l(\rho) = 1$, $H = \int_a^b h(\rho)d\rho$. Assume that L_1 and L_2 are non empty disjoint subsets of $\{1,2,...,n\}$ such that $L_1 \cup L_2 = \{1,2,...,n\}$. If g is monotonic on $[a,b]$ and*

$$0 \le \int_a^\lambda h(\rho)u_l(\rho)d\rho \le \int_a^b h(\rho)u_l(\rho)d\rho,\ \lambda \in [a,b],\ \int_a^b u_l(\rho)h(\rho)d\rho > 0, \tag{3.16}$$
$$\text{for each } l \in \{1,2,\ldots,n\},$$

then

$$\begin{aligned}
&\frac{1}{H}\int_a^b h(\rho)\psi(g(\rho))d\rho \\
&\ge \frac{1}{H}\int_a^b \sum_{l\in L_1} u_l(\rho)h(\rho)d\rho\,\psi\left(\frac{\int_a^b \sum_{l\in L_1} u_l(\rho)h(\rho)g(\rho)d\rho}{\int_a^b \sum_{l\in L_1} u_l(\rho)h(\rho)d\rho}\right) \\
&+\frac{1}{H}\int_a^b \sum_{l\in L_2} u_l(\rho)h(\rho)d\rho\,\psi\left(\frac{\int_a^b \sum_{l\in L_2} u_l(\rho)h(\rho)g(\rho)d\rho}{\int_a^b \sum_{l\in L_2} u_l(\rho)h(\rho)d\rho}\right) \\
&\ge \psi\left(\frac{1}{H}\int_a^b h(\rho)g(\rho)d\rho\right).
\end{aligned} \tag{3.17}$$

If the function ψ is concave then the reverse inequalities hold in (3.17).

Corollary 3.2 *Let $\psi : I \to \mathbb{R}$ be a convex function defined on the interval I. Let $h, g : [a,b] \to \mathbb{R}$ be integrable functions such that $g(\rho) \in I, h(\rho) \in \mathbb{R}$ for all $\rho \in [a,b]$ and $H = \int_a^b h(\rho)d\rho$. If g is monotonic on $[a,b]$ and*

$$0 \le \int_a^\lambda h(\rho)(\rho - a)d\rho \le \int_a^b h(\rho)(\rho - a)d\rho,\ \lambda \in [a,b],\ \int_a^b (\rho - a)h(\rho)d\rho > 0 \text{ and} \tag{3.18}$$

$$0 \le \int_a^\lambda h(\rho)(b-\rho)d\rho \le \int_a^b h(\rho)(b-\rho)d\rho,\ \lambda \in [a,b],\ \int_a^b (b-\rho)h(\rho)d\rho > 0, \tag{3.19}$$

then

$$\psi\left(\frac{1}{H}\int_a^b h(\rho)g(\rho)d\rho\right) \le \frac{1}{H(b-a)}\int_a^b (\rho - a)h(\rho)d\rho\,\psi\left(\frac{\int_a^b h(\rho)(\rho - a)g(\rho)d\rho}{\int_a^b h(\rho)(\rho - a)d\rho}\right)$$

$$+\frac{1}{H(b-a)}\int_a^b h(\rho)(b-\rho)d\rho\,\psi\left(\frac{\int_a^b h(\rho)(b-\rho)g(\rho)d\rho}{\int_a^b h(\rho)(b-\rho)d\rho}\right) \leq \frac{1}{H}\int_a^b h(\rho)\psi(g(\rho))d\rho. \tag{3.20}$$

If the function ψ is concave then the reverse inequalities hold in (3.20).

Proof. The proof follows from Theorem 3.2 by using $u(\rho) = \frac{\rho-a}{b-a}$ and $v(\rho) = \frac{b-\rho}{b-a}$.

Remark 3.3 As in Corollary 3.1 we can give application of Theorem 3.3 for Zipf-Mandelbrot law.

Chapter 4

Refinements of Jensen-Mercer's and Variant of Jensen-Steffensen's Inequalities

In this chapter, we propose new refinements for the Jensen-Mercer as well as variant of the Jensen-Mercer inequalities associated to certain positive tuples. We give some related integral version and present applications to different means. At the end of this chapter, further generalizations have been given which are associated to m finite sequences. The results of this chapter are given in [5].

4.1 Refinements

In the following theorem we present refinement of Jensen-Mercer's inequality.

Theorem 4.1 *Let* $\psi : [a,b] \to \mathbb{R}$ *be a convex function and let* $y_\gamma \in [a,b]$, $\zeta_\gamma, \eta_\gamma, \theta_\gamma \in \mathbb{R}^+$ *be such that* $\zeta = \sum_{\gamma=1}^{n} \zeta_\gamma$ *and* $\eta_\gamma + \theta_\gamma = 1$ *for each* $\gamma \in \{1,2,\ldots,n\}$. *Then*

$$\psi\left(a+b-\frac{1}{\zeta}\sum_{\gamma=1}^{n}\zeta_\gamma y_\gamma\right)$$
$$\leq \sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma\psi\left(\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma}\right)+\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma\psi\left(\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma}\right)$$
$$\leq \psi(a)+\psi(b)-\frac{1}{\zeta}\sum_{\gamma=1}^{n}\zeta_\gamma\psi(y_\gamma). \tag{4.1}$$

Proof. Since $\eta_\gamma+\theta_\gamma=1$ for $\gamma\in\{1,2,\ldots,n\}$, therefore

$$\psi\left(a+b-\frac{1}{\zeta}\sum_{\gamma=1}^{n}\zeta_\gamma y_\gamma\right)=\psi\left(\frac{1}{\zeta}\sum_{\gamma=1}^{n}\zeta_\gamma(a+b-y_\gamma)\right)$$
$$=\psi\left(\frac{1}{\zeta}\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma(a+b-y_\gamma)+\frac{1}{\zeta}\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma(a+b-y_\gamma)\right)$$
$$=\psi\left(\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma}{\zeta}\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma}+\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma}{\zeta}\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma}\right)$$
$$\leq\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma}{\zeta}\psi\left(\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma}\right)$$
$$+\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma}{\zeta}\psi\left(\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma}\right)$$
$$\leq\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma}{\zeta}\left(\psi(a)+\psi(b)-\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma\psi(y_\gamma)}{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma}\right)$$
$$+\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma}{\zeta}\left(\psi(a)+\psi(b)-\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma\psi(y_\gamma)}{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma}\right)$$
$$\leq\psi(a)+\psi(b)-\frac{1}{\zeta}\sum_{\gamma=1}^{n}\zeta_\gamma\psi(y_\gamma).$$

The first inequality holds by using definition of convexity while the second inequality is due to Jensen-Mercer's inequality.

In the following theorem we present integral version of the above theorem.

Theorem 4.2 *Let $\psi:[a,b]\to\mathbb{R}$ be a convex function defined on the interval $[a,b]$. Let $p,u,v,g:[\alpha,\beta]\to\mathbb{R}$ be integrable functions such that $g(\omega)\in[a,b],u(\omega),v(\omega),p(\omega)\in\mathbb{R}^+$ for all $\omega\in[\alpha,\beta]$ and $v(\omega)+u(\omega)=1$, $P=\int_\alpha^\beta p(\omega)d\omega$. Then*

$$\psi\left(a+b-\frac{1}{P}\int_\alpha^\beta p(\omega)g(\omega)d\omega\right)$$
$$\leq\frac{1}{P}\int_\alpha^\beta u(\omega)p(\omega)d\omega\,\psi\left(\frac{\int_\alpha^\beta p(\omega)u(\omega)(a+b-g(\omega))d\omega}{\int_\alpha^\beta p(\omega)u(\omega)d\omega}\right)$$

$$+\frac{1}{P}\int_{\alpha}^{\beta} p(\omega)v(\omega)d\omega\psi\left(\frac{\int_{\alpha}^{\beta} p(\omega)v(\omega)(a+b-g(\omega))d\omega}{\int_{\alpha}^{\beta} p(\omega)v(\omega)d\omega}\right)$$
$$\leq \psi(a)+\psi(b)-\frac{1}{P}\int_{\alpha}^{\beta} p(\omega)\psi(g(\omega))d\omega. \tag{4.2}$$

If the function ψ is concave then the reverse inequalities hold in (4.2).

The following variant of Jensen-Steffensen's inequality has been given in [1].

Theorem 4.3 *Let $\psi : [a,b] \to \mathbb{R}$ be a convex function and $y_\gamma \in [a,b]$, $\zeta_\gamma \in \mathbb{R}, \zeta_\gamma \neq 0$ for $\gamma = 1,2,\ldots,n$ with $\zeta = \sum_{\gamma=1}^{n} \zeta_\gamma$. If $y_1 \leq y_2 \leq \cdots \leq y_n$ or $y_1 \geq y_2 \geq \cdots \geq y_n$ and*

$$0 \leq \sum_{\gamma=1}^{k} \zeta_\gamma \leq \sum_{\gamma=1}^{n} \zeta_\gamma,\ k = 1,2,\ldots,n,\quad \sum_{\gamma=1}^{n} \zeta_\gamma > 0, \tag{4.3}$$

then

$$\psi\left(a+b-\frac{1}{\zeta}\sum_{\gamma=1}^{n} \zeta_\gamma y_\gamma\right) \leq \psi(a)+\psi(b)-\frac{1}{\zeta}\sum_{\gamma=1}^{n} \zeta_\gamma \psi(y_\gamma). \tag{4.4}$$

In the following theorem we present a refinement of variant of Jensen-Steffensen's inequality:

Theorem 4.4 *Let $\psi : [a,b] \to \mathbb{R}$ be a convex function. Let $y_\gamma \in [a,b]$, $\zeta_\gamma, \eta_\gamma, \theta_\gamma \in \mathbb{R}$, $\zeta_\gamma\eta_\gamma, \zeta_\gamma\theta_\gamma \neq 0$ for $\gamma = 1,2,\ldots,n$, such that $\eta_\gamma + \theta_\gamma = 1$ for all $\gamma \in \{1,2,\ldots,n\}$ and $\zeta = \sum_{\gamma=1}^{n} \zeta_\gamma$. If $y_1 \leq y_2 \leq \cdots \leq y_n$ or $y_1 \geq y_2 \geq \cdots \geq y_n$ and*

$$0 \leq \sum_{\gamma=1}^{k} \zeta_\gamma\eta_\gamma \leq \sum_{\gamma=1}^{n} \zeta_\gamma\eta_\gamma,\ k = 1,2,\ldots,n,\quad \sum_{\gamma=1}^{n} \zeta_\gamma\eta_\gamma > 0 \ \text{ and} \tag{4.5}$$

$$0 \leq \sum_{\gamma=1}^{k} \zeta_\gamma\theta_\gamma \leq \sum_{\gamma=1}^{n} \zeta_\gamma\theta_\gamma,\ k = 1,2,\ldots,n,\quad \sum_{\gamma=1}^{n} \zeta_\gamma\theta_\gamma > 0, \tag{4.6}$$

then

$$\psi\left(a+b-\frac{1}{\zeta}\sum_{\gamma=1}^{n} \zeta_\gamma y_\gamma\right) \leq \frac{1}{\zeta}\sum_{\gamma=1}^{n} \eta_\gamma\zeta_\gamma\psi\left(a+b-\frac{\sum_{\gamma=1}^{n} \zeta_\gamma\eta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^{n} \zeta_\gamma\eta_\gamma}\right)$$
$$+\frac{1}{\zeta}\sum_{\gamma=1}^{n} \zeta_\gamma\theta_\gamma\psi\left(\frac{\sum_{\gamma=1}^{n} \zeta_\gamma\theta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^{n} \zeta_\gamma\theta_\gamma}\right) \leq \psi(a)+\psi(b)-\frac{1}{\zeta}\sum_{\gamma=1}^{n} \zeta_\gamma\psi(y_\gamma). \tag{4.7}$$

If the function ψ is concave then the reverse inequalities hold in (4.7).

Proof. Since $\eta_\gamma + \theta_\gamma = 1$ for all $\gamma \in \{1,2,\ldots,n\}$, therefore we have

$$\psi\left(a+b\ -\ \frac{1}{\zeta}\sum_{\gamma=1}^{n} \zeta_\gamma y_\gamma\right) = \psi\left(\frac{1}{\zeta}\sum_{\gamma=1}^{n} \zeta_\gamma(a+b-y_\gamma)\right)$$

$$= \psi\left(\frac{1}{\zeta}\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma(a+b-y_\gamma)+\frac{1}{\zeta}\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma(a+b-y_\gamma)\right)$$

$$= \psi\left(\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma}{\zeta}\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma}+\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma}{\zeta}\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma}\right). \tag{4.8}$$

Since $\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma > 0$ and $\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma > 0$, therefore $\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma + \sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma > 0$. Which gives that $\zeta > 0$.
Now by applying convexity of ψ to the right side of (4.8) one may obtain

$$\begin{aligned}\psi\left(\frac{1}{\zeta}\sum_{\gamma=1}^{n}\zeta_\gamma(a+b-y_\gamma)\right) &\le \frac{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma}{\zeta}\psi\left(\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma}\right)\\ &\quad+\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma}{\zeta}\psi\left(\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma}\right)\\ &\le \frac{1}{\zeta}\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma\left(\psi(a)+\psi(b)-\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma\psi(y_\gamma)}{\sum_{\gamma=1}^{n}\zeta_\gamma\eta_\gamma}\right)\\ &\quad+\frac{1}{\zeta}\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma\left(\psi(a)+\psi(b)-\frac{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma\psi(y_\gamma)}{\sum_{\gamma=1}^{n}\zeta_\gamma\theta_\gamma}\right)\\ &= \psi(a)+\psi(b)-\frac{1}{\zeta}\sum_{\gamma=1}^{n}\zeta_\gamma\psi(y_\gamma).\end{aligned}$$

The above inequality holds by using variant of Jensen-Steffensen's inequality.

Remark 4.1 If we add (4.5) and (4.6), then the variant of Jensen-Steffensen inequality conditions (4.3) will be obtained.

4.2 Applications to Means

Let $y_\gamma \in [a,b], (\gamma = 1,2,..,n)$, where $0 < a < b$, $\zeta_1,...,\zeta_n > 0$ with $\bar{\zeta} := \sum_{\gamma=1}^{n}\zeta_\gamma$. Let $A_n(\mathbf{y};\boldsymbol{\zeta}),\tilde{A}_n(\mathbf{y};\boldsymbol{\zeta}),G_n(\mathbf{y};\boldsymbol{\zeta}),\tilde{G}_n(\mathbf{y};\boldsymbol{\zeta})$, $H_n(\mathbf{y};\boldsymbol{\zeta}),\tilde{H}_n(\mathbf{y};\boldsymbol{\zeta}),M_n^{[r]}(\mathbf{y};\boldsymbol{\zeta})$ and $\tilde{M}_n^{[r]}(\mathbf{y};\boldsymbol{\zeta})$ denote the weighted arithmetic, geometric, harmonic and power means defined as:

$$A_n(\mathbf{y};\boldsymbol{\zeta}) := \frac{1}{\zeta}\sum_{\gamma=1}^{n}\zeta_\gamma y_\gamma, \quad \tilde{A}_n := a+b-\frac{1}{\zeta}\sum_{\gamma=1}^{n}\zeta_\gamma y_\gamma = a+b-A_n(\mathbf{y};\boldsymbol{\zeta}),$$

$$G_n(\mathbf{y};\boldsymbol{\zeta}) := \left(\prod_{\gamma=1}^{n} y_\gamma^{\zeta_\gamma}\right)^{\frac{1}{\zeta}}, \quad \tilde{G}_n(\mathbf{y};\boldsymbol{\zeta}) := \frac{ab}{\left(\prod_{\gamma=1}^{n} y_\gamma^{\zeta_\gamma}\right)^{\frac{1}{\zeta}}} = \frac{ab}{G_n(\mathbf{y};\boldsymbol{\zeta})},$$

$$H_n(\mathbf{y};\boldsymbol{\zeta}) := \frac{1}{\zeta}\sum_{\gamma=1}^{n} \zeta_\gamma y_\gamma^{-1}, \quad \tilde{H}_n(\mathbf{y};\boldsymbol{\zeta}) := \left(a^{-1}+b^{-1}-H_n^{-1}(\mathbf{y};\boldsymbol{\zeta})\right)^{-1},$$

$$M_n^{[s]}(\mathbf{y};\boldsymbol{\zeta}) := \begin{cases} \left(\frac{1}{\zeta}\sum_{\gamma=1}^{n} \zeta_\gamma y_\gamma^s\right)^{\frac{1}{s}}, & s \neq 0, \\ G_n(\mathbf{y};\boldsymbol{\zeta}), & s = 0, \end{cases}$$

$$\tilde{M}_n^{[s]}(\mathbf{y};\boldsymbol{\zeta}) := \begin{cases} \left(a^s+b^s-\left(M_n^{[s]}(\mathbf{y};\boldsymbol{\zeta})\right)^s\right)^{\frac{1}{s}}, & s \neq 0, \\ \tilde{G}_n(\mathbf{y};\boldsymbol{\zeta}), & s = 0. \end{cases}$$

Also, assume that $\eta_\gamma, \theta_\gamma \in \mathbb{R}^+$ are such that $\eta_\gamma + \theta_\gamma = 1$ for each $\gamma \in \{1,2,\ldots,n\}$.

Under the above assumptions we give the following corollaries.

Corollary 4.1 *The following inequality is valid*

$$\tilde{G}_n(\boldsymbol{y};\boldsymbol{\zeta}) \leq \sum_{\gamma=1}^{n} \zeta_\gamma \eta_\gamma \tilde{G}_n(\boldsymbol{y};\boldsymbol{\zeta}.\boldsymbol{\eta}) + \sum_{\gamma=1}^{n} \zeta_\gamma \theta_\gamma \tilde{G}_n(\boldsymbol{y};\boldsymbol{\zeta}.\boldsymbol{\theta}) \leq \tilde{A}_n(\boldsymbol{y};\boldsymbol{\zeta}). \tag{4.9}$$

Proof. Use the function $\psi(x) = \exp(x)$ and replacing a, b, and y_γ by $\ln a, \ln b$, and $\ln y_\gamma$ in(4.1) we will get (4.9).

Corollary 4.2 *By taking $a \to \frac{1}{a}, b \to \frac{1}{b}, y_\gamma \to \frac{1}{y_\gamma}$, in (4.9) we have*

$$\frac{1}{\tilde{G}_n(\boldsymbol{y};\boldsymbol{\zeta})} \leq \frac{\sum_{\gamma=1}^{n} \zeta_\gamma \eta_\gamma}{\tilde{G}_n(\boldsymbol{y};\boldsymbol{\zeta}.\boldsymbol{\eta})} + \frac{\sum_{\gamma=1}^{n} \zeta_\gamma \theta_\gamma}{\tilde{G}_n(\boldsymbol{y};\boldsymbol{\zeta}.\boldsymbol{\theta})} \leq \frac{1}{\tilde{H}_n(\boldsymbol{y};\boldsymbol{\zeta})}. \tag{4.10}$$

Corollary 4.3 *The following inequality holds*

$$\tilde{G}_n(\boldsymbol{y};\boldsymbol{\zeta}) \leq \left(\tilde{A}_n(\boldsymbol{y};\boldsymbol{\zeta}.\boldsymbol{\eta})\right)^{\sum_{\gamma=1}^{n} \zeta_\gamma \eta_\gamma} \left(\tilde{A}_n(\boldsymbol{y};\boldsymbol{\zeta}.\boldsymbol{\theta})\right)^{\sum_{\gamma=1}^{n} \zeta_\gamma \theta_\gamma} \leq \tilde{A}_n(\boldsymbol{y};\boldsymbol{\zeta}). \tag{4.11}$$

Proof. Using the function $f(x) = -\ln x$ in(4.1) we will get (4.11).

Corollary 4.4 *For $s \neq 0$ and $s \leq 1$, we have*

$$\tilde{M}_n^{[s]}(\boldsymbol{y};\boldsymbol{\zeta}) \leq \sum_{\gamma=1}^{n} \zeta_\gamma \eta_\gamma \tilde{M}_n^{[s]}(\boldsymbol{y};\boldsymbol{\zeta}.\boldsymbol{\eta}) + \sum_{\gamma=1}^{n} \zeta_\gamma \theta_\gamma \tilde{M}_n^{[s]}(\boldsymbol{y};\boldsymbol{\zeta}.\boldsymbol{\theta}) \leq \tilde{A}_n(\boldsymbol{y};\boldsymbol{\zeta}). \tag{4.12}$$

Proof. Use the function $\psi(x) = x^{\frac{1}{s}}$ and replace a, b, and x_i by a^s, b^s, and x_γ^s respectively in (4.1) to get (4.12).

Corollary 4.5 *For $t, s \in \mathbb{R}$ with $0 < t \leq s$, we have*

$$\left(\tilde{M}_n^{[t]}(\boldsymbol{y};\boldsymbol{\zeta})\right)^s \leq \sum_{\gamma=1}^{n} \zeta_\gamma \eta_\gamma \left(\tilde{M}_n^{[t]}(\boldsymbol{y};\boldsymbol{\zeta}.\boldsymbol{\eta})\right)^s + \sum_{\gamma=1}^{n} \zeta_\gamma \theta_\gamma \left(\tilde{M}_n^{[t]}(\boldsymbol{y};\boldsymbol{\zeta}.\boldsymbol{\theta})\right)^s \leq \tilde{M}_n^{[s]}(\boldsymbol{y};\boldsymbol{\zeta}). \tag{4.13}$$

Proof. Use the function $\psi(x) = x^{\frac{s}{t}}$ and replace a, b, and x_γ by a^t, b^t, and x_γ^t respectively in (4.1) to get (4.13). Now, we generalize the given applications for quasi-arithmetic mean.

Let $\phi : [a,b] \to \mathbb{R}$ be a strictly monotonic and continuous function. Then for a given n- tuple $\mathbf{y} = (y_1, ..., y_n) \in [a,b]^n$ and positive n- tuple $\boldsymbol{\zeta} = (\zeta_1, ..., \zeta_n)$ with $\zeta = \sum_{\gamma=1}^{n} \zeta_\gamma$, the value $M_\phi^{[n]}(\mathbf{y}; \boldsymbol{\zeta})$ is defined in (1.23) as quasi-arithmetic mean of $\mathbf{y}$ with wight $\boldsymbol{\zeta}$. If we define

$$\tilde{M}_\phi^{[n]}(\mathbf{y}; \boldsymbol{\zeta}) = \phi^{-1}\left(\phi(a) + \phi(b) - \sum_{\gamma=1}^{n} \zeta_\gamma \phi(y_\gamma)\right),$$

then we have the following results.

Corollary 4.6 *The following inequality holds*

$$\psi\left(\tilde{M_\phi}^{[n]}(\boldsymbol{y}; \boldsymbol{\zeta})\right) \leq \sum_{\gamma=1}^{n} \zeta_\gamma \eta_\gamma \psi\left(\tilde{M_\phi}^{[n]}(\boldsymbol{y}; \boldsymbol{\zeta}.\boldsymbol{\eta})\right)$$
$$+ \sum_{\gamma=1}^{n} \zeta_\gamma \theta_\gamma \psi\left(\tilde{M_\phi}^{[n]}(\boldsymbol{y}; \boldsymbol{\zeta}.\boldsymbol{\theta})\right) \leq \psi\left(\tilde{M_\psi}^{[n]}(\boldsymbol{y}; \boldsymbol{\zeta})\right). \tag{4.14}$$

provided that $\psi \circ \phi^{-1}$ is convex and ψ is strictly increasing.

Proof. Replacing $\psi(x)$ by $\psi \circ \phi^{-1}(x)$ and a, b, y_γ by $\phi(a), \phi(b), \phi(y_\gamma)$ respectively in (4.1), then apply ψ^{-1} to get (4.14).

4.3 Further Generalizations

In this section, we present further refinement of the Jensen-Mercer as well as variant of the Jensen-Mercer inequalities concerning to m sequences whose sum is equal to unity.

Theorem 4.5 *Let $\psi : [a,b] \to \mathbb{R}$ be a convex function defined on the interval $[a,b]$. Let $y_\gamma \in [a,b]$, $\zeta_\gamma, \theta_\gamma^l \in \mathbb{R}^+$ $(\gamma = 1,2,\ldots,n, l = 1,2,\ldots,m)$ be such that $\sum_{l=1}^{m} \theta_\gamma^l = 1$ for each $\gamma \in \{1,2,\ldots,n\}$, $\zeta = \sum_{\gamma=1}^{n} \zeta_\gamma$. Assume that L_1 and L_2 are non empty disjoint subsets of $\{1,2,...,m\}$ such that $L_1 \cup L_2 = \{1,2,...,m\}$. Then*

$$\psi\left(a + b - \frac{1}{\zeta}\sum_{\gamma=1}^{n} \zeta_\gamma y_\gamma\right) \leq \frac{1}{\zeta}\sum_{\gamma=1}^{n}\sum_{l \in L_1} \theta_\gamma^l \zeta_\gamma \psi\left(\frac{\sum_{\gamma=1}^{n}\sum_{l \in L_1} \theta_\gamma^l \zeta_\gamma (a + b - y_\gamma)}{\sum_{\gamma=1}^{n}\sum_{l \in L_1} \theta_\gamma^l \zeta_\gamma}\right)$$
$$+ \frac{1}{\zeta}\sum_{\gamma=1}^{n}\sum_{l \in L_2} \theta_\gamma^l \zeta_\gamma \psi\left(\frac{\sum_{\gamma=1}^{n}\sum_{l \in L_2} \theta_\gamma^l \zeta_\gamma (a + b - y_\gamma)}{\sum_{\gamma=1}^{n}\sum_{l \in L_2} \theta_\gamma^l \zeta_\gamma}\right)$$
$$\leq \psi(a) + \psi(b) - \frac{1}{\zeta}\sum_{\gamma=1}^{n} \zeta_\gamma \psi(y_\gamma). \tag{4.15}$$

If the function ψ is concave then the reverse inequalities hold in (4.15).

Proof. Since $\sum_{l=1}^{m}\theta_\gamma^l = 1$ for each $\gamma \in \{1,2,\ldots,n\}$, therefore we may write

$$a+b-\frac{1}{\zeta}\sum_{\gamma=1}^{n}\zeta_\gamma y_\gamma = \frac{1}{\zeta}\sum_{\gamma=1}^{n}\sum_{l\in L_1}\theta_\gamma^l\zeta_\gamma(a+b-y_\gamma)+\frac{1}{\zeta}\sum_{\gamma=1}^{n}\sum_{l\in L_2}\theta_\gamma^l\zeta_\gamma(a+b-y_\gamma).$$

Therefore, we have

$$\begin{aligned}
&\psi\left(a+b-\frac{1}{\zeta}\sum_{\gamma=1}^{n}\zeta_\gamma y_\gamma\right)\\
&=\psi\left(\frac{1}{\zeta}\sum_{\gamma=1}^{n}\sum_{l\in L_1}\theta_\gamma^l\zeta_\gamma(a+b-y_\gamma)+\frac{1}{\zeta}\sum_{\gamma=1}^{n}\sum_{l\in L_2}\theta_\gamma^l\zeta_\gamma(a+b-y_\gamma)\right)\\
&=\psi\left(\frac{1}{\zeta}\sum_{\gamma=1}^{n}\sum_{l\in L_1}\theta_\gamma^l\zeta_\gamma\frac{\sum_{\gamma=1}^{n}\sum_{l\in L_1}\theta_\gamma^l\zeta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^{n}\sum_{l\in L_1}\theta_\gamma^l\zeta_\gamma}\right.\\
&\qquad\left.+\frac{1}{\zeta}\sum_{\gamma=1}^{n}\sum_{l\in L_2}\theta_\gamma^l\zeta_\gamma\frac{\sum_{\gamma=1}^{n}\sum_{l\in L_2}\theta_\gamma^l\zeta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^{n}\sum_{l\in L_2}\theta_\gamma^l\zeta_\gamma}\right)\\
&\leq\frac{1}{\zeta}\sum_{\gamma=1}^{n}\sum_{l\in L_1}\theta_\gamma^l\zeta_\gamma\psi\left(\frac{\sum_{\gamma=1}^{n}\sum_{l\in L_1}\theta_\gamma^l\zeta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^{n}\sum_{l\in L_1}\theta_\gamma^l\zeta_\gamma}\right)\\
&\qquad+\frac{1}{\zeta}\sum_{\gamma=1}^{n}\sum_{l\in L_2}\theta_\gamma^l\zeta_\gamma\psi\left(\frac{\sum_{\gamma=1}^{n}\sum_{l\in L_2}\theta_\gamma^l\zeta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^{n}\sum_{l\in L_2}\theta_\gamma^l\zeta_\gamma}\right)\\
&=\psi(a)+\psi(b)-\frac{1}{\zeta}\sum_{\gamma=1}^{n}\zeta_\gamma\psi(y_\gamma).
\end{aligned}\tag{4.16}$$

The first inequality holds by using definition of convexity while the second holds by using Jensen-Mercer's inequality.

Remark 4.2 We can give applications of Theorem 4.5 for means as given in Section 4.2.

The following theorem is the integral analogue of Theorem 4.5.

Theorem 4.6 *Let $\psi : G \to \mathbb{R}$ be a convex function defined on the interval G. Let $p,g,u_l \in L[a,b]$ such that $g(\omega) \in G, p(\omega),u_l(\omega) \in \mathbb{R}^+$ for all $\omega \in [a,b]$ $(l = 1,2,...,n)$ and $\sum_{l=1}^{n} u_l(\omega) = 1$, $P = \int_a^b p(\omega)d\omega$. Assume that L_1 and L_2 are non empty disjoint subsets of $\{1,2,...,n\}$ such that $L_1 \cup L_2 = \{1,2,...,n\}$. Then*

$$\begin{aligned}
&\frac{1}{P}\int_a^b p(\omega)\psi(g(\omega))d\omega\\
&\geq\frac{1}{P}\int_a^b\sum_{l\in L_1}u_l(\omega)p(\omega)d\omega\,\psi\left(\frac{\int_a^b\sum_{l\in L_1}u_l(\omega)p(\omega)g(\omega)d\omega}{\int_a^b\sum_{l\in L_1}u_l(\omega)p(\omega)d\omega}\right)\\
&+\frac{1}{P}\int_a^b\sum_{l\in L_2}u_l(\omega)p(\omega)d\omega\,\psi\left(\frac{\int_a^b\sum_{l\in L_2}u_l(\omega)p(\omega)g(\omega)d\omega}{\int_a^b\sum_{l\in L_2}u_l(\omega)p(\omega)d\omega}\right)
\end{aligned}$$

$$\geq \psi\left(\frac{1}{P}\int_a^b p(\omega)g(\omega)d\omega\right). \tag{4.17}$$

If the function ψ is concave then the reverse inequalities hold in (4.17).

In the following theorem, we present further refinement of the variant of Jensen-Steffensen's inequality associated to m certain sequences.

Theorem 4.7 *Let $\psi : [a,b] \to \mathbb{R}$ be a convex function defined on the interval $[a,b]$. Let $y_\gamma \in I$, $\zeta_\gamma, \theta_\gamma^l \in \mathbb{R}, \zeta_\gamma\eta_\gamma, \zeta_\gamma\theta_\gamma^l \neq 0$ $(\gamma = 1,2,\ldots,n, l = 1,2,\ldots,m)$ such that $\sum_{l=1}^m \theta_\gamma^l = 1$ for each $\gamma \in \{1,2,\ldots,n\}$ and $\zeta = \sum_{\gamma=1}^n \zeta_\gamma$. Assume that L_1 and L_2 are non empty disjoint subsets of $\{1,2,...,m\}$ such that $L_1 \cup L_2 = \{1,2,...,m\}$. If $y_1 \leq y_2 \leq \cdots \leq y_n$ or $y_1 \geq y_2 \geq \cdots \geq y_n$ and*

$$0 \leq \sum_{\gamma=1}^{k} \zeta_\gamma\theta_\gamma^l \leq \sum_{\gamma=1}^{n} \zeta_\gamma\theta_\gamma^l, \ k = 1,2,\ldots,n, \ \sum_{\gamma=1}^{n} \zeta_\gamma\theta_\gamma^l > 0, \quad \text{for each } l \in \{1,2,\ldots,m\}, \tag{4.18}$$

then

$$\psi\left(a+b-\frac{1}{\zeta}\sum_{\gamma=1}^{n}\zeta_\gamma y_\gamma\right) \leq \frac{1}{\zeta}\sum_{\gamma=1}^{n}\sum_{l\in L_1}\theta_\gamma^l\zeta_\gamma\psi\left(\frac{\sum_{\gamma=1}^n\sum_{l\in L_1}\theta_\gamma^l\zeta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^n\sum_{l\in L_1}\theta_\gamma^l\zeta_\gamma}\right)$$
$$+\frac{1}{\zeta}\sum_{\gamma=1}^{n}\sum_{l\in L_2}\theta_\gamma^l\zeta_\gamma\psi\left(\frac{\sum_{\gamma=1}^n\sum_{l\in L_2}\theta_\gamma^l\zeta_\gamma(a+b-y_\gamma)}{\sum_{\gamma=1}^n\sum_{l\in L_2}\theta_\gamma^l\zeta_\gamma}\right) \leq \psi(a)+\psi(b)-\frac{1}{\zeta}\sum_{\gamma=1}^{n}\zeta_\gamma\psi(y_\gamma). \tag{4.19}$$

If the function ψ is concave then the reverse inequalities hold in (4.19).

Proof. Since $\sum_{\gamma=1}^n \zeta_\gamma\theta_\gamma^l > 0$ for each $l \in \{1,2,\ldots,m\}$, therefore $\sum_{l=1}^m\sum_{\gamma=1}^n \zeta_\gamma\theta_\gamma^l > 0$. Also, $\sum_{l=1}^m \theta_\gamma^l = 1$. Hence we conclude that $\zeta > 0$.
Now proceeding in the same way as in the proof of Theorem 4.5 but use variant of Jensen-Steffensen's inequality instead of Jensen-Steffensen's inequality, we will obtain (4.19).

Chapter 5

Refinement of Jensen's Inequality for Convex Functions of Several Variables

In this chapter we give a refinement of Jensen's inequality for convex functions of several variables associated to certain tuples. As applications we deduce refinements of Beck's inequality. At the end, further generalization has been presented for certain n finite sequences. The results of this chapter are given in [6].

5.1 Refinement of Jensen's Inequality for Convex Functions of Several Variables with Applications

The following Jensen inequality for convex functions of several variables has been given in [87].

Theorem 5.1 *Let $I_1, I_2, \ldots, I_m$ be intervals in $\mathbb{R}$ and $y_j^i \in I_i$, $\zeta_j \in \mathbb{R}^+$ for $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$. If $\psi : I_1 \times I_2 \times \cdots \times I_m \to \mathbb{R}$ is a convex function, then*

$$\psi\left(\frac{\sum_{j=1}^n \zeta_j y_j^1}{\sum_{j=1}^n \zeta_j}, \frac{\sum_{j=1}^n \zeta_j y_j^2}{\sum_{j=1}^n \zeta_j}, \ldots, \frac{\sum_{j=1}^n \zeta_j y_j^m}{\sum_{j=1}^n \zeta_j}\right) \leq \frac{1}{\sum_{j=1}^n \zeta_j} \sum_{j=1}^n \zeta_j \psi(y_j^1, y_j^2, \ldots, y_j^m). \tag{5.1}$$

Let $q : I \to \mathbb{R}$ be a continuous and strictly monotone function defined on the interval I, then the quasi-Arithmetic mean (q-mean) of vector $\mathbf{z} = (z_1, z_2, \ldots, z_n) \in I^n$ with positive weights $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \ldots, \zeta_n)$ is defined in (1.23). Similarly, integral quasi-arithmetic mean $q(g;p)$ for function g is defined in (2.16). The following weighted version of Beck's inequality has been given in [46].

Theorem 5.2 *Let $q_i : I_i \to \mathbb{R}$ $(i = 1, 2, \ldots, m)$ be strictly monotone and $P : I_P \to \mathbb{R}$ be continuous and strictly increasing functions whose domains are intervals in $\mathbb{R}$, and $\psi : I_1 \times I_2 \times \cdots \times I_m \to I_P$ be a continuous function. Let $\mathbf{y}^i = (y_1^i, y_2^i, \ldots, y_n^i) \in I_1 \times I_2 \times \cdots \times I_n$, $i = 1, 2, \ldots, m$ and $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \ldots, \zeta_n)$ be a nonnegative n-tuple such that $\sum_{j=1}^n \zeta_j = 1$, then*

$$\psi\left(q_1(\mathbf{y}^1;\boldsymbol{\zeta})q_2(\mathbf{y}^2;\boldsymbol{\zeta}), \ldots, q_m(\mathbf{y}^m;\boldsymbol{\zeta})\right) \geq P^{-1}\left(\sum_{j=1}^{n} \zeta_j P(\psi(y_j^1, y_j^2, \ldots, y_j^m))\right). \quad (5.2)$$

holds for all possible $\mathbf{y}^i (i = 1, 2, \ldots, n)$ and $\boldsymbol{\zeta}$, if and only if the function $\mathfrak{D}$ defined on $q_1(I_1) \times q_2(I_2) \times \cdots q_m(I_m)$ by

$$\mathfrak{D}(z_1, z_2, \ldots, z_n) = P(\psi(q_1^{-1}(z_1), q_2^{-1}(z_2), \ldots, q_m^{-1}(z_m))$$

is concave.

The inequality in (5.2) is reversed for all possible $\mathbf{y}^i (i = 1, 2, \ldots, m)$ and $\boldsymbol{\zeta}$, if and only if $\mathfrak{D}$ is convex.

Beck's original result (see [27, p. 249], [26, p. 300], [21],[75, p. 194]) was Theorem 5.2 for the case $m = 2$ which is stated as:

Theorem 5.3 *Let $K : I_K \to \mathbb{R}$, $L : I_L \to \mathbb{R}$ be strictly monotone and $N : I_N \to \mathbb{R}$ be continuous and strictly increasing functions whose domains are intervals in $\mathbb{R}$, and let $\psi : I_K \times I_L \to I_N$ be a continuous function. Let $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in I_K^n$, $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in I_L^n$ and $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \ldots, \zeta_n)$ be a nonnegative n-tuple such that $\sum_{j=1}^n \zeta_j = 1$. Then the following inequality holds*

$$\psi(K(\mathbf{a};\boldsymbol{\zeta}), L(\mathbf{b};\boldsymbol{\zeta})) \geq M(\psi(\mathbf{a},\mathbf{b});\boldsymbol{\zeta}), \quad (5.3)$$

$$\text{where } \psi(\mathbf{a},\mathbf{b}) = (\psi(a_1,b_1), \psi(a_2,b_2), \ldots, \psi(a_n,b_n)),$$

if and only if the function $H(s,t) = M(\psi(K^{-1}(s), L^{-1}(t)))$, is concave.

The inequality (5.3) holds in reverse direction if and only if H is convex.

Corollary 5.1 *[75, p. 194] If $\psi(x,y) = x + y$ and $H(s,t) = M(K^{-1}(s) + L^{-1}(t))$, and if $E := \frac{K'}{K''}$, $F := \frac{L'}{L''}$, $G := \frac{N'}{N''}$, where all $K', L', N', K'', L'', N''$ are all positive, then (5.3) holds for all possible tuples $\mathbf{a}$ and $\mathbf{b}$ if and only if*

$$E(x) + F(y) \leq G(x+y). \quad (5.4)$$

Corollary 5.2 *[75, p. 194] Let $\psi(x,y) = xy$ and $H(s,t) = M(K^{-1}(s)L^{-1}(t))$. If $A(x) := \frac{K'(x)}{K'(x)+xK''(x)}$, $B(x) := \frac{L'(x)}{L'(x)+xL''(x)}$, $C(x) := \frac{M'(x)}{M'(x)+xN''(x)}$, and if the functions K', L', N', A, B, C are all positive, then (5.3) holds for all possible tuples $\mathbf{a}$ and $\mathbf{b}$ if and only if*

$$A(x) + B(y) \leq C(xy). \quad (5.5)$$

Now we give refinement of Jensen's inequality for convex functions of several variables.

Theorem 5.4 *Let $I_1, I_2, \ldots, I_m$ be intervals in $\mathbb{R}$, $\psi : I_1 \times I_2 \times \cdots \times I_m \to \mathbb{R}$ be a convex function and $y_j^i \in I_i, \zeta_j, \eta_j, \theta_j \in \mathbb{R}^+$ for $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$. If $\eta_j + \theta_j = 1$ for all $j \in \{1,2,\ldots,n\}$, then*

$$\begin{aligned}
\psi\left(\frac{\sum_{j=1}^n \zeta_j y_j^1}{\sum_{j=1}^n \zeta_j}, \frac{\sum_{j=1}^n \zeta_j y_j^2}{\sum_{j=1}^n \zeta_j}, \ldots, \frac{\sum_{j=1}^n \zeta_j y_j^m}{\sum_{j=1}^n \zeta_j}\right) \\
\leq \frac{\sum_{j=1}^n \zeta_j \eta_j}{\sum_{j=1}^n \zeta_j} \psi\left(\frac{\sum_{j=1}^n \zeta_j \eta_j y_j^1}{\sum_{j=1}^n \zeta_j \eta_j}, \ldots, \frac{\sum_{j=1}^n \zeta_j \eta_j y_j^m}{\sum_{j=1}^n \zeta_j \eta_j}\right) \\
+ \frac{\sum_{j=1}^n \zeta_j \theta_j}{\sum_{j=1}^n \zeta_j} \psi\left(\frac{\sum_{j=1}^n \zeta_j \theta_j y_j^1}{\sum_{j=1}^n \zeta_j \theta_j}, \ldots, \frac{\sum_{j=1}^n \zeta_j \theta_j y_j^m}{\sum_{j=1}^n \zeta_j \theta_j}\right) \\
\leq \frac{1}{\sum_{j=1}^n \zeta_j} \sum_{j=1}^n \zeta_j \psi(y_j^1, y_j^2, \ldots, y_j^m). \qquad (5.6)
\end{aligned}$$

If the function ψ is concave then the reverse inequalities hold in (5.6).

Proof. Since $\eta_j + \theta_j = 1$ for all $j \in \{1,2,\ldots,n\}$, therefore we have

$$\begin{aligned}
&\psi\left(\frac{\sum_{j=1}^n \zeta_j y_j^1}{\sum_{j=1}^n \zeta_j}, \frac{\sum_{j=1}^n \zeta_j y_j^2}{\sum_{j=1}^n \zeta_j}, \ldots, \frac{\sum_{j=1}^n \zeta_j y_j^m}{\sum_{j=1}^n \zeta_j}\right) \\
&= \psi\left(\frac{\sum_{j=1}^n \zeta_j \eta_j y_j^1}{\sum_{j=1}^n \zeta_j} + \frac{\sum_{j=1}^n \zeta_j \theta_j y_j^1}{\sum_{j=1}^n \zeta_j}, \ldots, \frac{\sum_{j=1}^n \zeta_j \eta_j y_j^m}{\sum_{j=1}^n \zeta_j} + \frac{\sum_{j=1}^n \zeta_j \theta_j y_j^m}{\sum_{j=1}^n \zeta_j}\right) \\
&= \psi\left(\frac{\sum_{j=1}^n \zeta_j \eta_j}{\sum_{j=1}^n \zeta_j} \frac{\sum_{j=1}^n \zeta_j \eta_j y_j^1}{\sum_{j=1}^n \zeta_j \eta_j} + \frac{\sum_{j=1}^n \zeta_j \theta_j}{\sum_{j=1}^n \zeta_j} \frac{\sum_{j=1}^n \zeta_j \theta_j y_j^1}{\sum_{j=1}^n \zeta_j \theta_j}, \ldots, \right. \\
&\qquad \left. \frac{\sum_{j=1}^n \zeta_j \eta_j}{\sum_{j=1}^n \zeta_j} \frac{\sum_{j=1}^n \zeta_j \eta_j y_j^m}{\sum_{j=1}^n \zeta_j \eta_j} + \frac{\sum_{j=1}^n \zeta_j \theta_j}{\sum_{j=1}^n \zeta_j} \frac{\sum_{j=1}^n \zeta_j \theta_j y_j^m}{\sum_{j=1}^n \zeta_j \theta_j}\right) \\
&\leq \frac{\sum_{j=1}^n \zeta_j \eta_j}{\sum_{j=1}^n \zeta_j} \psi\left(\frac{\sum_{j=1}^n \zeta_j \eta_j y_j^1}{\sum_{j=1}^n \zeta_j \eta_j}, \ldots, \frac{\sum_{j=1}^n \zeta_j \eta_j y_j^m}{\sum_{j=1}^n \zeta_j \eta_j}\right) \\
&\quad + \frac{\sum_{j=1}^n \zeta_j \theta_j}{\sum_{j=1}^n \zeta_j} \psi\left(\frac{\sum_{j=1}^n \zeta_j \theta_j y_j^1}{\sum_{j=1}^n \zeta_j \theta_j}, \ldots, \frac{\sum_{j=1}^n \zeta_j \theta_j y_j^m}{\sum_{j=1}^n \zeta_j \theta_j}\right) \\
&\leq \frac{1}{\sum_{j=1}^n \zeta_j} \sum_{j=1}^n \zeta_j \psi(y_j^1, y_j^2, \ldots, y_j^m).
\end{aligned}$$

The first inequality has been obtained by using (5.1) for the case $n = 2$, while the second inequality has been obtained by using (5.1) on both the terms. The integral version of the above theorem can be stated as:

Theorem 5.5 *Let $I_1, I_2, \ldots, I_m$ be intervals in $\mathbb{R}$, $\psi : I_1 \times I_2 \times \cdots \times I_m \to \mathbb{R}$ be a convex function. Let $u, v, p, g_i : [a,b] \to \mathbb{R}$ be integrable functions such that $g_i(\omega) \in I_i$, $u(\omega)$, $v(\omega)$, $p(\omega) \in \mathbb{R}^+$ for all $\omega \in [a,b], i = 1, 2, \ldots, m$, and $v(\omega) + u(\omega) = 1$, $P = \int_a^b p(\omega)d\omega$. Then*

$$\psi\left(\frac{1}{P}\int_a^b p(\omega)g_1(\omega)d\omega, \ldots, \frac{1}{P}\int_a^b p(\omega)g_m(\omega)d\omega\right)$$
$$\leq \frac{1}{P}\int_a^b u(\omega)p(\omega)d\omega\, \psi\left(\frac{\int_a^b p(\omega)u(\omega)g_1(\omega)d\omega}{\int_a^b p(\omega)u(\omega)d\omega}, \ldots, \frac{\int_a^b p(\omega)u(\omega)g_m(\omega)d\omega}{\int_a^b p(\omega)u(\omega)d\omega}\right)$$
$$+ \frac{1}{P}\int_a^b u(\omega)p(\omega)d\omega\, \psi\left(\frac{\int_a^b p(\omega)v(\omega)g_1(\omega)d\omega}{\int_a^b p(\omega)v(\omega)d\omega}, \ldots, \frac{\int_a^b p(\omega)v(\omega)g_m(\omega)d\omega}{\int_a^b p(\omega)v(\omega)d\omega}\right)$$
$$\leq \frac{1}{P}\int_a^b p(\omega)\psi(g_1(\omega), g_2(\omega), \ldots, g_m(\omega))d\omega. \tag{5.7}$$

If the function ψ is concave then the reverse inequalities hold in (5.7).

Remark 5.1 Analogously, related refinement can be given for Jensen's inequality (2.8) as given in [87].

In the following theorem we present refinement of the inequality (5.2).

Theorem 5.6 *Let $q_i : I_i \to \mathbb{R}$ $(i = 1, 2, \ldots, m)$ be strictly monotone and $P : I_P \to \mathbb{R}$ be continuous and strictly increasing functions whose domains are intervals in $\mathbb{R}$, and $\psi : I_1 \times I_2 \times \cdots \times I_m \to I_P$ be a continuous function. Let $\mathbf{y}^i = (y_1^i, y_2^i, \ldots, y_n^i) \in I_1 \times I_2 \times \cdots \times I_n$, $i = 1, 2, \ldots, m$. If $\zeta_j, \eta_j, \theta_j \in \mathbb{R}^+$ such that $\eta_j + \theta_j = 1$ for $j = 1, 2, \ldots, n$, then*

$$\psi\Big(q_1(\mathbf{y}^1;\boldsymbol{\zeta}), q_2(\mathbf{y}^2;\boldsymbol{\zeta}), \ldots, q_m(\mathbf{y}^m;\boldsymbol{\zeta})\Big)$$
$$\geq P^{-1}\left[\frac{\sum_{j=1}^n \zeta_j\eta_j}{\sum_{j=1}^n \zeta_j} P\Big(\psi\Big(q_1(\mathbf{y}^1;\boldsymbol{\zeta}.\boldsymbol{\eta}), \cdots, q_m(\mathbf{y}^m;\boldsymbol{\zeta}.\boldsymbol{\eta})\Big)\Big)\right.$$
$$\left. + \frac{\sum_{j=1}^n \zeta_j\theta_j}{\sum_{j=1}^n \zeta_j} P\Big(\psi\Big(q_1(\mathbf{y}^1;\boldsymbol{\zeta}.\boldsymbol{\theta}), \cdots, q_m(\mathbf{y}^m;\boldsymbol{\zeta}.\boldsymbol{\theta})\Big)\Big)\right]$$
$$\geq P^{-1}\left(\frac{1}{\sum_{j=1}^n \zeta_j}\sum_{j=1}^n \zeta_j P(\psi(y_j^1, y_j^2, \ldots, y_j^m))\right). \tag{5.8}$$

if and only if the function $\mathfrak{D}$ defined on $q_1(I_1) \times q_2(I_2) \times \cdots q_m(I_m)$ by

$$\mathfrak{D}(z_1, z_2, \ldots, z_m) = P(\psi(q_1^{-1}(z_1), q_2^{-1}(z_2), \ldots, q_m^{-1}(z_m)) \tag{5.9}$$

is concave.

The inequalities in (5.8) hold in reverse direction for all possible $\mathbf{y}^i (i = 1, 2, \ldots, n)$ and $\boldsymbol{\zeta}$, if and only if $\mathfrak{D}$ is convex.

Proof. Replace ψ by $\mathfrak{D}$ and y^i_j by $q_j(y^i_j)$ $(i = 1,2,\ldots,m, j = 1,2,\ldots,n)$ and then applying the increasing function P^{-1} in the reverse inequality in (5.6), we obtain (5.8). The integral version of the above theorem can be stated as:

Theorem 5.7 *Let $q_i : I_i \to \mathbb{R}$ $(i = 1,2,\ldots,m)$ be strictly monotone and $T : I_T \to \mathbb{R}$ be continuous and strictly increasing functions whose domains are intervals in $\mathbb{R}$, and $\psi : I_1 \times I_2 \times \cdots \times I_m \to I_T$ be a continuous function. Let $u,v,p,g_i : [a,b] \to \mathbb{R}$ be integrable functions such that $g_i(\omega) \in I_i, u(\omega), v(\omega), p(\omega) \in \mathbb{R}^+$ for all $\omega \in [a,b], i = 1,2,\ldots,m$, and $v(\omega)+u(\omega) = 1$, $P = \int_a^b p(\omega)d\omega$. Then*

$$\begin{aligned}
&\psi\bigg(q_1(g_1;p),q_2(g_2;p),\ldots,q_m(g_m;p)\bigg)\\
&\qquad\geq T^{-1}\Bigg[\frac{1}{P}\int_a^b u(\omega)p(\omega)d\omega T\Bigg(\psi\bigg(q_1(g_1;p.u),\cdots,q_m(g_m;p.u)\bigg)\Bigg)\\
&\qquad\quad+\frac{1}{P}\int_a^b v(\omega)p(\omega)d\omega T\Bigg(\psi\bigg(q_1(g_1;p.v),\cdots,q_m(g_m;p.v)\bigg)\Bigg)\Bigg]\\
&\qquad\geq T^{-1}\left(\frac{1}{P}\int_a^b p(\omega)T(\psi(g_1(\omega),g_2(\omega),\ldots,g_m(\omega))d\omega\right). \qquad (5.10)
\end{aligned}$$

if and only if the function $\mathfrak{D}$ defined on $q_1(I_1)\times q_2(I_2)\times\cdots q_m(I_m)$ by

$$\mathfrak{D}(z_1,z_2,\ldots,z_m) = T(\psi(q_1^{-1}(z_1),q_2^{-1}(z_2),\ldots,q_m^{-1}(z_m))$$

is concave.

The inequalities in (5.10) hold in reverse direction for all possible $\mathbf{y}^i (i = 1,2,\ldots,n)$ and $\boldsymbol{\zeta}$, if and only if $\mathfrak{D}$ is convex.

As a consequence of the above theorem for the case $m = 2$, the following refinement of Beck's inequality holds:

Corollary 5.3 *Let all the assumptions of Theorem 5.3 hold. If $\eta_j, \theta_j \in \mathbb{R}^+$ such that $\eta_j + \theta_j = 1$ for $j = 1,2,\ldots,n$, then*

$$\begin{aligned}
\psi(K(\mathbf{a};\boldsymbol{\zeta}),L(\mathbf{b};\boldsymbol{\zeta})) &\geq M^{-1}\left[\frac{\sum_{j=1}^n \zeta_j\eta_j}{\sum_{j=1}^n \zeta_j}M\Bigg(\psi\bigg(K(\mathbf{a};\boldsymbol{\zeta}.\boldsymbol{\eta}),L(\mathbf{b};\boldsymbol{\zeta}.\boldsymbol{\eta})\bigg)\Bigg)\right.\\
&\left.+\frac{\sum_{j=1}^n \zeta_j\theta_j}{\sum_{j=1}^n \zeta_j}M\Bigg(\psi\bigg(K(\mathbf{a};\boldsymbol{\zeta}.\boldsymbol{\theta}),(L(\mathbf{a};\boldsymbol{\zeta}.\boldsymbol{\theta}))\bigg)\Bigg)\right] \geq M(\psi(\mathbf{a},\mathbf{b});\boldsymbol{\zeta}). \qquad (5.11)
\end{aligned}$$

if and only if the function $H(s,t) = M(\psi(K^{-1}(s),L^{-1}(t)))$, is concave.

The inequality (5.11) holds in reverse direction if and only if H is convex.

Corollary 5.4 *Let K,L,M be twice continuously differentiable and strictly monotone functions such that K',L',M',K'',L'',M'' are all positive. If $\eta_j,\theta_j \in \mathbb{R}^+$ such that $\eta_j + \theta_j =$*

1 for $j = 1,2,\ldots,n$*, then*

$$K(\mathbf{a};\boldsymbol{\zeta}) + L(\mathbf{b};\boldsymbol{\zeta}) \geq M^{-1}\left[\frac{\sum_{j=1}^{n}\zeta_j\eta_j}{\sum_{j=1}^{n}\zeta_j}M\Big(K(\mathbf{a};\boldsymbol{\zeta}.\boldsymbol{\eta}) + L(\mathbf{b};\boldsymbol{\zeta}.\boldsymbol{\eta})\Big)\right.$$
$$\left.+\frac{\sum_{j=1}^{n}\zeta_j\theta_j}{\sum_{j=1}^{n}\zeta_j}M\Big(K(\mathbf{a};\boldsymbol{\zeta}.\boldsymbol{\theta}) + L(\mathbf{a};\boldsymbol{\zeta}.\boldsymbol{\theta}))\Big)\right] \geq M(\mathbf{a}+\mathbf{b};\boldsymbol{\zeta}). \tag{5.12}$$

holds for all possible tuples $\mathbf{a},\mathbf{b}$ *and positive tuple* $\boldsymbol{\zeta}$ *if and only if*

$$E(x) + F(y) \leq G(x+y), \tag{5.13}$$

where $E := \frac{K'}{K''}, F := \frac{L'}{L''}, G := \frac{M'}{M''}$*,*

Proof. Let $H(s,t) = M(K^{-1}(s) + L^{-1}(t))$. We prove that the function H is concave. Since H is twice continuously differentiable function, therefore for the concavity of H, we show that

$$a_1^2\frac{\partial^2 H}{\partial s^2} + 2a_1a_2\frac{\partial^2 H}{\partial s\partial t} + a_2^2\frac{\partial^2 H}{\partial t^2} \leq 0, \text{ for all } a_1, a_2 \in \mathbb{R}. \tag{5.14}$$

But by taking partial derivatives of H of order 2 and using (5.13), we obtain (5.14).

Finally, using H and $\psi(x,y) = x+y$ in (5.11), we deduce (5.12).

In the following corollary we present refinement of the inequality given in Corollary 5.2. The idea of the proof is similar to the proof of Corollary 5.2.

Corollary 5.5 *Let* K,L,M *be twice continuously differentiable and strictly monotone functions and let* $A(x) := \frac{K'(x)}{K'(x)+xK''(x)}$, $B(x) := \frac{L'(x)}{L'(x)+xL''(x)}, C(x) := \frac{M'(x)}{M'(x)+xN''(x)}$*. Also, assume that the functions* K', L', N', A, B, C *are all positive. If* $\eta_j, \theta_j \in \mathbb{R}^+$ *such that* $\eta_j + \theta_j = 1$ *for* $j = 1,2,\ldots,n$*, then*

$$K(\mathbf{a};\boldsymbol{\zeta})L(\mathbf{b};\boldsymbol{\zeta}) \geq M^{-1}\left[\frac{\sum_{j=1}^{n}\zeta_j\eta_j}{\sum_{j=1}^{n}\zeta_j}M\Big(K(\mathbf{a};\boldsymbol{\zeta}.\boldsymbol{\eta})L(\mathbf{b};\boldsymbol{\zeta}.\boldsymbol{\eta})\Big)\right.$$
$$\left.+\frac{\sum_{j=1}^{n}\zeta_j\theta_j}{\sum_{j=1}^{n}\zeta_j}M\Big(K(\mathbf{a};\boldsymbol{\zeta}.\boldsymbol{\theta})(L(\mathbf{a};\boldsymbol{\zeta}.\boldsymbol{\theta}))\Big)\right]$$
$$\geq M(\mathbf{a}.\mathbf{b};\boldsymbol{\zeta}). \tag{5.15}$$

holds for all possible tuples $\mathbf{a},\mathbf{b}$ *and positive tuple* $\boldsymbol{\zeta}$ *if and only if*

$$A(x) + B(y) \leq C(xy). \tag{5.16}$$

We give a refinement of the Minkowski inequality.

Corollary 5.6 *Let* I *be an interval in* $\mathbb{R}$, $\mathbf{y}_j = (y_j^1, y_j^2, \ldots, y_j^m) \in I^m$, $\zeta_j, \eta_j, \theta_j \in \mathbb{R}^+ (j = 1,2,\ldots,n)$ *such that* $\eta_j + \theta_j = 1$ *for all* $j \in \{1,2,\ldots,n\}$*, and* $M : I \to \mathbb{R}$ *be a continuous* *and strictly monotone function. Consider the quasi-arithmetic mean function* $M_n : I^n \to \mathbb{R}$ *defined by*

$$M_n(\boldsymbol{y};\boldsymbol{\zeta}) = M^{-1}\left(\frac{1}{\sum_{j=1}^{n}\zeta_j}\sum_{j=1}^{n}\zeta_j M(y_j)\right)$$

is convex, then

$$\begin{aligned} M_m\left(\frac{1}{n}\sum_{j=1}^{n}\boldsymbol{y}_j;\boldsymbol{\zeta}\right) &\le \frac{1}{n}\sum_{j=1}^{n}\eta_j M_m\left(\boldsymbol{y}_\eta;\boldsymbol{\zeta}\right)+\frac{1}{n}\sum_{j=1}^{n}\theta_j M_m\left(\boldsymbol{y}_\theta;\boldsymbol{\zeta}\right) \\ &\le \frac{1}{n}\sum_{j=1}^{n}M_m(\boldsymbol{y}_j;\boldsymbol{\zeta}), \end{aligned} \tag{5.17}$$

where $\boldsymbol{y}_\eta = \left(\frac{\sum_{j=1}^{n}\eta_j y_j^1}{\sum_{j=1}^{n}\eta_j},\cdots,\frac{\sum_{j=1}^{n}\eta_j y_j^m}{\sum_{j=1}^{n}\eta_j}\right)$, $\boldsymbol{y}_\theta = \left(\frac{\sum_{j=1}^{n}\theta_j y_j^1}{\sum_{j=1}^{n}\theta_j},\cdots,\frac{\sum_{j=1}^{n}\theta_j y_j^m}{\sum_{j=1}^{n}\theta_j}\right)$.

Proof. The proof follows by using Theorem 5.4 for $\zeta_j = 1$ and then taking the function $M_m(;\boldsymbol{\zeta})$ instead of ψ.

Remark 5.2 Analogously as above we can give the integral version of Corollaries 5.3–5.6.

5.2 Further Generalization

In the following theorem, we present further refinement of the Jensen inequality related to n sequences.

Theorem 5.8 *Let* $I_1, I_2, \ldots, I_m$ *be intervals in* $\mathbb{R}$, $\psi : I_1 \times I_2 \times \cdots \times I_m \to \mathbb{R}$ *be a convex function and* $y_j^i \in I_i$, $\zeta_j, \theta_j^l \in \mathbb{R}^+$ $(i = 1,2,\ldots,m, j = 1,2,\ldots,n, l = 1,2,\ldots,t)$ *such that* $\sum_{l=1}^{t}\theta_j^l = 1$ *for each* $j \in \{1,2,\ldots,n\}$, $\overline{\zeta} = \sum_{j=1}^{n}\zeta_j$. *Assume that* L_1 *and* L_2 *are non empty disjoint subsets of* $\{1,2,...,m\}$ *such that* $L_1 \cup L_2 = \{1,2,...,t\}$. *Then*

$$\begin{aligned} &\psi\left(\frac{\sum_{j=1}^{n}\zeta_j y_j^1}{\overline{\zeta}},\frac{\sum_{j=1}^{n}\zeta_j y_j^2}{\overline{\zeta}},\cdots,\frac{\sum_{j=1}^{n}\zeta_j y_j^m}{\overline{\zeta}}\right) \\ &\le \frac{\sum_{j=1}^{n}\sum_{l\in L_1}\zeta_j\theta_j^l}{\overline{\zeta}}\psi\left(\frac{\sum_{j=1}^{n}\sum_{l\in L_1}\zeta_j\theta_j^l y_j^1}{\sum_{j=1}^{n}\sum_{l\in L_1}\zeta_j\theta_j^l},\cdots,\frac{\sum_{j=1}^{n}\sum_{l\in L_1}\zeta_j\theta_j^l y_j^m}{\sum_{j=1}^{n}\sum_{l\in L_1}\zeta_j\theta_j^l}\right) \\ &+\frac{\sum_{j=1}^{n}\sum_{l\in L_2}\zeta_j\theta_j^l}{\overline{\zeta}}\psi\left(\frac{\sum_{j=1}^{n}\sum_{l\in L_2}\zeta_j\theta_j^l y_j^1}{\sum_{j=1}^{n}\sum_{l\in L_2}\zeta_j\theta_j^l},\cdots,\frac{\sum_{j=1}^{n}\sum_{l\in L_2}\zeta_j\theta_j^l y_j^m}{\sum_{j=1}^{n}\sum_{l\in L_2}\zeta_j\theta_j^l}\right) \\ &\le \frac{1}{\overline{\zeta}}\sum_{j=1}^{n}\zeta_j\psi(y_j^1,y_j^2,\ldots,y_j^m). \end{aligned} \tag{5.18}$$

If the function ψ *is concave then the reverse inequalities hold in (5.18).*

Proof. Since $\sum_{l=1}^{t}\theta_j^l=1$ for each $j\in\{1,2,\ldots,n\}$, therefore we may write

$$\frac{1}{\zeta}\sum_{j=1}^{n}\zeta_j\psi(y_j^1,y_j^2,\ldots,y_j^m)=\frac{1}{\zeta}\sum_{j=1}^{n}\sum_{l\in L_1}\theta_j^l\zeta_j\psi(y_j^1,y_j^2,\ldots,y_j^m)$$
$$+\frac{1}{\zeta}\sum_{j=1}^{n}\sum_{l\in L_2}\theta_j^l\zeta_j\psi(y_j^1,y_j^2,\ldots,y_j^m). \tag{5.19}$$

Applying Jensen's inequality (5.1) to both terms on the right hand side of (5.19) we obtain

$$\frac{1}{\zeta}\sum_{j=1}^{n}\zeta_j\psi(y_j^1,y_j^2,\ldots,y_j^m)$$
$$\geq\frac{\sum_{j=1}^{n}\sum_{l\in L_1}\zeta_j\theta_j^l}{\zeta}\psi\left(\frac{\sum_{j=1}^{n}\sum_{l\in L_1}\zeta_j\theta_j^l y_j^1}{\sum_{j=1}^{n}\sum_{l\in L_1}\zeta_j\theta_j^l},\cdots,\frac{\sum_{j=1}^{n}\sum_{l\in L_1}\zeta_j\theta_j^l y_j^m}{\sum_{j=1}^{n}\sum_{l\in L_1}\zeta_j\theta_j^l}\right)$$
$$+\frac{\sum_{j=1}^{n}\sum_{l\in L_2}\zeta_j\theta_j^l}{\zeta}\psi\left(\frac{\sum_{j=1}^{n}\sum_{l\in L_2}\zeta_j\theta_j^l y_j^1}{\sum_{j=1}^{n}\sum_{l\in L_2}\zeta_j\theta_j^l},\cdots,\frac{\sum_{j=1}^{n}\sum_{l\in L_2}\zeta_j\theta_j^l y_j^m}{\sum_{j=1}^{n}\sum_{l\in L_2}\zeta_j\theta_j^l}\right)$$
$$\geq\psi\left[\frac{\sum_{j=1}^{n}\sum_{l\in L_1}\zeta_j\theta_j^l}{\zeta}\left(\frac{\sum_{j=1}^{n}\sum_{l\in L_1}\zeta_j\theta_j^l y_j^1}{\sum_{j=1}^{n}\sum_{l\in L_1}\zeta_j\theta_j^l},\cdots,\frac{\sum_{j=1}^{n}\sum_{l\in L_1}\zeta_j\theta_j^l y_j^m}{\sum_{j=1}^{n}\sum_{l\in L_1}\zeta_j\theta_j^l}\right)\right.$$
$$\left.+\frac{\sum_{j=1}^{n}\sum_{l\in L_2}\zeta_j\theta_j^l}{\zeta}\left(\frac{\sum_{j=1}^{n}\sum_{l\in L_2}\zeta_j\theta_j^l y_j^1}{\sum_{j=1}^{n}\sum_{l\in L_2}\zeta_j\theta_j^l},\cdots,\frac{\sum_{j=1}^{n}\sum_{l\in L_2}\zeta_j\theta_j^l y_j^m}{\sum_{j=1}^{n}\sum_{l\in L_2}\zeta_j\theta_j^l}\right)\right]$$
(By the convexity of ψ)
$$=\psi\left(\frac{\sum_{j=1}^{n}\zeta_j y_j^1}{\zeta},\frac{\sum_{j=1}^{n}\zeta_j y_j^2}{\zeta},\cdots,\frac{\sum_{j=1}^{n}\zeta_j y_j^m}{\zeta}\right).$$

The integral version of the above theorem can be stated as:

Theorem 5.9 *Let $I_1,I_2,\ldots,I_m$ be intervals in $\mathbb{R}$, $\psi:I_1\times I_2\times\cdots\times I_m\to\mathbb{R}$ be a convex function. Let $p,g_i,u_l\in L[a,b]$ such that $g_i(\omega)\in I_i, p(\omega),u_l(\omega)\in\mathbb{R}^+$ for all $\omega\in[a,b]$ $(i=1,2,...,n,l=1,2,...,t)$ and $\sum\limits_{l=1}^{t}u_l(\omega)=1$, $P=\int_a^b p(\omega)d\omega$. Assume that L_1 and L_2 are non empty disjoint subsets of $\{1,2,...,t\}$ such that $L_1\cup L_2=\{1,2,...,t\}$. Then*

$$\psi\left(\frac{1}{P}\int_a^b p(\omega)g_1(\omega)d\omega,\ldots,\frac{1}{P}\int_a^b p(\omega)g_m(\omega)d\omega\right)$$
$$\leq\frac{1}{P}\int_a^b\sum_{l\in L_1}u_l(\omega)p(\omega)d\omega\,\psi\left(\frac{\int_a^b\sum_{l\in L_1}u_l(\omega)p(\omega)g_1(\omega)d\omega}{\int_a^b\sum_{l\in L_1}u_l(\omega)p(\omega)d\omega},\ldots\right.$$
$$\left.\ldots\frac{\int_a^b\sum_{l\in L_1}u_l(\omega)p(\omega)g_n(\omega)d\omega}{\int_a^b\sum_{l\in L_1}u_l(\omega)p(\omega)d\omega}\right)$$

$$+\frac{1}{P}\int_a^b \sum_{l\in L_2} u_l(\omega)p(\omega)d\omega\,\psi\left(\frac{\int_a^b \sum_{l\in L_2} u_l(\omega)p(\omega)g_1(\omega)d\omega}{\int_a^b \sum_{l\in L_2} u_l(\omega)p(\omega)d\omega},\ldots\right.$$
$$\left.\ldots,\frac{\int_a^b \sum_{l\in L_1} u_l(\omega)p(\omega)g_n(\omega)d\omega}{\int_a^b \sum_{l\in L_1} u_l(\omega)p(\omega)d\omega}\right)$$
$$\leq \frac{1}{P}\int_a^b p(\omega)\psi(g_1(\omega),g_2(\omega),\ldots,g_m(\omega))d\omega. \tag{5.20}$$

If the function ψ is concave then the reverse inequalities hold in (5.20).

Remark 5.3 All the results presented in this paper may also be generalized using Theorem 5.8.

Chapter 6

Refinements of Jessen's and McShane's Inequalities

In this chapter we consider generalized forms of Jensen's inequality for isotonic linear functionals given by Jessen and McShane. We derive refinements of Jessen's and McShane's inequalities connected to the certain functions from the linear space. As applications of the refinement of Jessen's inequality, we deduce refinements of generalized means and Hölder inequalities. Also, as applications of refinement of McShane's inequality, we obtain refinement of generalized Beck's inequality and discuss their particular cases. At the end of this chapter, we give further generalizations of Jessen's and McShane's inequalities pertaining n certain functions. The results of this chapter are given in [7].

6.1 Refinement of Jessen's Inequality with Applications

The main focus of this section is to present refinements of generalized Jensen's inequality given by Jessen in 1931 [57] and McShane in 1937 [78] for isotonic linear functionals for convex functions of single and multiple variables respectively. Before giving Jessen's and McShane's results, we consider the following hypothesis and recall a definition.

Hypothesis H: For a non empty set $\mathscr{M}$, let $\mathscr{L}$ be a class of real valued functions $f : \mathscr{M} \to \mathbb{R}$ which satisfies the following properties:

(i) : if $g_1, g_2 \in \mathscr{L}$ and $\alpha_1, \alpha_2 \in \mathbb{R}$, then $\alpha_1 g_1 + \alpha_2 g_2 \in \mathscr{L}$,

(ii) : $\mathbf{1} \in \mathscr{L}$, that is if $g(z) = 1$ for all $z \in \mathscr{M}$, then $g \in \mathscr{L}$.

Definition 6.1 *If $\mathscr{G} : \mathscr{L} \to \mathbb{R}$ is a functional which satisfies the following conditions*

(i) $\mathscr{G}(\alpha_1 g_1 + \alpha_2 g_2) = \alpha_1 \mathscr{G}(g_1) + \alpha_2 \mathscr{G}(g_2)$ *for* $g_1, g_2 \in \mathscr{L}, \alpha_1, \alpha_2 \in \mathbb{R}$,

(ii) $g \in \mathscr{L}, g(t) \geq 0$ *on* $\mathscr{M} \Rightarrow \mathscr{G}(g) \geq 0$,

then $\mathscr{G}$ is said to be isotonic linear functional.

In 1931, Jessen [57](also see [87, p-47]) constructed the functional version of Jensen's inequality for convex functions of one variable. In the following theorem we present weighted version of Jessen's inequality.

Theorem 6.1 *Let the hypothesis H holds and $\mathscr{G} : \mathscr{L} \to \mathbb{R}$ be an isotonic linear functional and $\psi : [a,b] \to \mathbb{R}$ be a continuous convex function. Then for all $f, \vartheta \in \mathscr{L}$ such that $\vartheta\psi(f), \vartheta f \in \mathscr{L}$ and $\mathscr{G}(\vartheta) > 0$, we have $\frac{\mathscr{G}(\vartheta f)}{\mathscr{G}(\vartheta)} \in [a,b]$ and*

$$\psi\left(\frac{\mathscr{G}(\vartheta f)}{\mathscr{G}(\vartheta)}\right) \leq \frac{\mathscr{G}(\vartheta\psi(f))}{\mathscr{G}(\vartheta)}. \tag{6.1}$$

In 1937, McShane [78] further extended the above functional version of Jensen's inequality from convex functions of one variable to the convex functions of several variables. The following theorem is weighted version of McShane's result.

Theorem 6.2 *Let the hypothesis H holds and $\mathscr{G} : \mathscr{L} \to \mathbb{R}$ be an isotonic linear functional. Also let C be a convex closed subset of $\mathbb{R}^n$ and ψ be convex and continuous function defined on C. Let $\phi_1(x), \phi_2(x), \ldots, \phi_n(x), \vartheta(x)$ be functions from $\mathscr{L}$ such that $\boldsymbol{\phi}(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_n(x)) \in C$ for all $x \in \mathscr{M}$, $\vartheta\psi(\boldsymbol{\phi}(x)), \vartheta\phi_i \in \mathscr{L}$ $(i = 1, 2, \ldots, n)$ and $\mathscr{G}(\vartheta) > 0$. Then*

$$\psi\left(\frac{\mathscr{G}(\vartheta\phi_1)}{\mathscr{G}(\vartheta)}, \frac{\mathscr{G}(\vartheta\phi_2)}{\mathscr{G}(\vartheta)}, \ldots, \frac{\mathscr{G}(\vartheta\phi_n)}{\mathscr{G}(\vartheta)}\right) \leq \frac{1}{\mathscr{G}(\vartheta)}\mathscr{G}(\vartheta\psi(\phi_1, \phi_2, \ldots, \phi_n)). \tag{6.2}$$

In the following fundamental result we present a refinement of Jessen's inequity.

Theorem 6.3 *Under the assumptions of Theorem 6.1, if $u, v \in \mathscr{L}$ such that $u(t) + v(t) = 1$ for $t \in \mathscr{M}$ and $u\vartheta f, v\vartheta f, u\vartheta, v\vartheta \in \mathscr{L}$ with $\mathscr{G}(u\vartheta), \mathscr{G}(v\vartheta) > 0$, then*

$$\psi\left(\frac{\mathscr{G}(\vartheta f)}{\mathscr{G}(\vartheta)}\right) \leq \frac{\mathscr{G}(u\vartheta)}{\mathscr{G}(\vartheta)}\psi\left(\frac{\mathscr{G}(u\vartheta f)}{\mathscr{G}(u\vartheta)}\right) + \frac{\mathscr{G}(v\vartheta)}{\mathscr{G}(\vartheta)}\psi\left(\frac{\mathscr{G}(v\vartheta f)}{\mathscr{G}(v\vartheta)}\right) \leq \frac{\mathscr{G}(\vartheta\psi(f))}{\mathscr{G}(\vartheta)}. \tag{6.3}$$

Proof. Since $u(t) + v(t) = 1$ for $t \in \mathscr{M}$, therefore we have

$$\begin{aligned}
\psi\left(\frac{\mathscr{G}(\vartheta f)}{\mathscr{G}(\vartheta)}\right) &= \psi\left(\frac{\mathscr{G}((u+v)\vartheta f)}{\mathscr{G}(\vartheta)}\right) \\
&= \psi\left(\frac{\mathscr{G}(u\vartheta f)}{\mathscr{G}(\vartheta)} + \frac{\mathscr{G}(v\vartheta f)}{\mathscr{G}(\vartheta)}\right) \text{(As } \mathscr{G} \text{ is linear)}
\end{aligned}$$

$$= \psi\left(\frac{\mathscr{G}(u\vartheta)}{\mathscr{G}(\vartheta)}\frac{\mathscr{G}(u\vartheta f)}{\mathscr{G}(u\vartheta)}+\frac{\mathscr{G}(v\vartheta)}{\mathscr{G}(\vartheta)}\frac{\mathscr{G}(v\vartheta f)}{\mathscr{G}(v\vartheta)}\right). \tag{6.4}$$

Due to linearity of $\mathscr{G}$, we have

$$\frac{\mathscr{G}(u\vartheta)}{\mathscr{G}(\vartheta)}+\frac{\mathscr{G}(v\vartheta)}{\mathscr{G}(\vartheta)}=\frac{\mathscr{G}(u\vartheta+v\vartheta)}{\mathscr{G}(\vartheta)}=\frac{\mathscr{G}(\vartheta)}{\mathscr{G}(\vartheta)}=1.$$

Therefore using convexity of ψ on the right hand side of (6.4) we obtain

$$\psi\left(\frac{\mathscr{G}(\vartheta f)}{\mathscr{G}(\vartheta)}\right)\le\frac{\mathscr{G}(u\vartheta)}{\mathscr{G}(\vartheta)}\psi\left(\frac{\mathscr{G}(u\vartheta f)}{\mathscr{G}(u\vartheta)}\right)+\frac{\mathscr{G}(v\vartheta)}{\mathscr{G}(\vartheta)}\psi\left(\frac{\mathscr{G}(v\vartheta f)}{\mathscr{G}(v\vartheta)}\right) \tag{6.5}$$

Applying Jessen's inequality (6.1) on both terms in (6.5), we deduce

$$\begin{aligned}\frac{\mathscr{G}(u\vartheta)}{\mathscr{G}(\vartheta)}\psi\left(\frac{\mathscr{G}(u\vartheta f)}{\mathscr{G}(u\vartheta)}\right)&+\frac{\mathscr{G}(v\vartheta)}{\mathscr{G}(\vartheta)}\psi\left(\frac{\mathscr{G}(v\vartheta f)}{\mathscr{G}(v\vartheta)}\right)\\
&\le\frac{\mathscr{G}(u\vartheta)}{\mathscr{G}(\vartheta)}\frac{\mathscr{G}(u\vartheta\psi(f))}{\mathscr{G}(u\vartheta)}+\frac{\mathscr{G}(v\vartheta)}{\mathscr{G}(\vartheta)}\frac{\mathscr{G}(v\vartheta\psi(f))}{\mathscr{G}(v\vartheta)}\\
&=\frac{\mathscr{G}(u\vartheta\psi(f))}{\mathscr{G}(\vartheta)}+\frac{\mathscr{G}(v\vartheta\psi(f))}{\mathscr{G}(\vartheta)}\\
&=\frac{\mathscr{G}(\vartheta\psi(f))}{\mathscr{G}(\vartheta)}.\end{aligned} \tag{6.6}$$

From (6.5) and (6.6), we derive (6.3). Now we demonstrate applications of the above theorem to means.

Consider the generalization of classical power mean $M_r(\vartheta,f;\mathscr{G})$ for isotonic functionals $\mathscr{G}$, defined for $r\in\mathbb{R}$ by

$$M_r(\vartheta,f;\mathscr{G})=\begin{cases}\left(\frac{\mathscr{G}(\vartheta f^r)}{\mathscr{G}(\vartheta)}\right)^{\frac{1}{r}}, & \text{if } r\neq 0,\\ \exp\left(\frac{\mathscr{G}(\vartheta\log f)}{\mathscr{G}(\vartheta)}\right), & \text{if } r=0,\end{cases} \tag{6.7}$$

where $f(x)>0$ for $x\in\mathscr{M}$, $\vartheta,\vartheta f^r\in\mathscr{L}$ for $r\in\mathbb{R}$, $\vartheta\log f\in\mathscr{L}$ and $\mathscr{G}(\vartheta)>0$.

Corollary 6.1 *Let the hypothesis H hold and the functions f,ϑ,u,v be defined on $\mathscr{M}$ such that $f,\vartheta,u,v,u\vartheta,v\vartheta,u\vartheta f^r,v\vartheta f^r\ \vartheta f^r\in\mathscr{L}$ $(r\in\mathbb{R})$ and $f(x)>0$ for $x\in\mathscr{M}$. Let $\mathscr{G}$ be an isotonic linear functional on $\mathscr{L}$ such that $\mathscr{G}(\vartheta),\mathscr{G}(u\vartheta),\mathscr{G}(v\vartheta)>0$. Also assume that $u(x)+v(x)=1$ for $x\in\mathscr{M}$. If $p,l\in\mathbb{R}$ such that $p\le l$, then*

$$\begin{aligned}M_l(\vartheta,f;\mathscr{G})\ge\Big[M_1(\vartheta,u;\mathscr{G})M_p^l(u\vartheta,f;\mathscr{G})+M_1(\vartheta,v;\mathscr{G})M_p^l(v\vartheta,f;\mathscr{G})\Big]^{\frac{1}{l}}\\
\ge M_p(\vartheta,f;\mathscr{G});l\neq 0.\end{aligned} \tag{6.8}$$

$$M_l(\vartheta,f;\mathscr{G}) \geq \exp\left(M_1(\vartheta,u;\mathscr{G})\log M_p(u\vartheta,f;\mathscr{G}) + M_1(\vartheta,v;\mathscr{G})\log M_p(v\vartheta,f;\mathscr{G})\right)$$
$$\geq M_p(\vartheta,f;\mathscr{G}); l=0. \quad (6.9)$$

$$M_p(\vartheta,f;\mathscr{G}) \leq \left[M_1(\vartheta,u;\mathscr{G})M_l^p(u\vartheta,f;\mathscr{G}) + M_1(\vartheta,v;\mathscr{G})M_l^p(v\vartheta,f;\mathscr{G})\right]^{\frac{1}{p}}$$
$$\leq M_l(\vartheta,f;\mathscr{G}); p \neq 0. \quad (6.10)$$

$$M_p(\vartheta,f;\mathscr{G}) \leq \exp\left(M_1(\vartheta,u;\mathscr{G})\log M_l(u\vartheta,f;\mathscr{G}) + M_1(\vartheta,v;\mathscr{G})\log M_l(v\vartheta,f;\mathscr{G})\right)$$
$$\leq M_l(\vartheta,f;\mathscr{G}); p=0. \quad (6.11)$$

Proof. Let $p,l \in \mathbb{R}$ such that $p,l \neq 0$, then using (6.3) for $\psi(z) = z^{\frac{l}{p}}$, $z>0$, $f \to f^p$ and taking the power $\frac{1}{l}$ we get (6.8). For the case $l=0$, taking limit $l \to 0$ in (6.8) we obtain (6.9).

Similarly utilizing (6.3) for $\psi(z) = z^{\frac{p}{l}}$, $z>0, p,l \neq 0$, $f \to f^l$ and then taking power $\frac{1}{p}$ we get (6.10). For $p=0$ taking the limit in (6.10) we deduce (6.11). Let $\mathscr{L}$ satisfy $\mathscr{L}_1$ and $\mathscr{L}_2$ on a nonempty set $\mathscr{M}$ and $\mathscr{G}$ be an isotonic linear functional on $\mathscr{L}$. Let $\vartheta, h : [a,b] \to \mathbb{R}$ be functions such that h is strictly monotone continuous function with $\vartheta h(f) \in \mathscr{L}$ for $f \in \mathscr{L}$ with $f(x) \in [a,b]$ and $\mathscr{G}(\vartheta) > 0$, then the generalized quasi arithmetic mean ([87, p-47]) is defined as:

$$M_h(\vartheta,f;\mathscr{G}) = h^{-1}\left(\frac{\mathscr{G}(\vartheta h(f))}{\mathscr{G}(\vartheta)}\right). \quad (6.12)$$

We give inequalities for generalized quasi arithmetic mean.

Corollary 6.2 *Let the above hypotheses hold and $g : [a,b] \to \mathbb{R}$ be strictly monotone continuous function such that $\vartheta g(f) \in \mathscr{L}$ for $f \in \mathscr{L}$ with $f(x) \in [a,b]$ and let $u,v \in \mathscr{L}$ such that $u(x)+v(x) = 1$ for $x \in \mathscr{M}$ and $\mathscr{G}(\vartheta), \mathscr{G}(u\vartheta), \mathscr{G}(v\vartheta) > 0$. If $g \circ h^{-1}$ is convex function then*

$$\frac{\mathscr{G}(\vartheta g(f))}{\mathscr{G}(\vartheta)} \geq M_1(\vartheta,u;\mathscr{G})g(M_h(\vartheta u,f;\mathscr{G}))$$
$$+M_1(\vartheta,v;\mathscr{G})g(M_h(\vartheta v,f;\mathscr{G})) \geq g\left(M_h(\vartheta,f;\mathscr{G})\right). \quad (6.13)$$

Proof. Using (6.3) for $f \to h \circ f$ and $\psi \to g \circ h^{-1}$. In the following corollaries, we present refinements of Hölder inequality as applications of Theorem 6.3.

Corollary 6.3 *Let the hypothesis H hold and $\mathscr{G} : \mathscr{L} \to \mathbb{R}$ be an isotonic linear functional. Suppose $r_1 > 1$, $r_2 = \frac{r_1}{r_1 - 1}$. If u,v,w,g_1 and g_2 are non-negative functions defined on $\mathscr{M}$ such that $wg_1^{r_1}$, $wg_2^{r_2}$, $uwg_2^{r_2}$, $vwg_2^{r_2}$, uwg_1g_2, vwg_1g_2, $wg_1g_2 \in \mathscr{L}$ and $u(x)+v(x) = 1$ for $x \in \mathscr{M}$, then*

$$\mathscr{G}(wg_1g_2)$$
$$\leq \mathscr{G}^{\frac{1}{r_2}}(wg_2^{r_2})\left\{\left(\mathscr{G}(uwg_2^{r_2})\right)^{1-r_1}\left(\mathscr{G}(uwg_1g_2)\right)^{r_1}\right.$$

$$+\left(\mathscr{G}(vwg_2^{r_2})\right)^{1-r_1}(\mathscr{G}(vwg_1g_2))^{r_1}\Big\}^{\frac{1}{r_1}}$$
$$\leq \mathscr{G}^{\frac{1}{r_1}}(wg_1^{r_1})\mathscr{G}^{\frac{1}{r_2}}(wg_2^{r_2}). \tag{6.14}$$

In the case when $0<r_1<1$ *and* $r_2=\frac{r_1}{r_1-1}$ *with* $\mathscr{G}(wg_2^{r_2})>0$ *or* $r_1<0$ *and* $\mathscr{G}(wg_1^{r_1})>0$, *then we have*

$$\mathscr{G}(wg_1g_2)\geq \mathscr{G}^{\frac{1}{r_2}}(uwg_2^{r_2})\mathscr{G}^{\frac{1}{r_1}}(uwg_1^{r_1})+\mathscr{G}^{\frac{1}{r_2}}(vwg_2^{r_2})\mathscr{G}^{\frac{1}{r_1}}(vwg_1^{r_1})$$
$$\geq \mathscr{G}^{\frac{1}{r_1}}(wg_1^{r_1})\mathscr{G}^{\frac{1}{r_2}}(wg_2^{r_2}). \tag{6.15}$$

Proof. Assume that $\mathscr{G}(wg_2^{r_2})>0$. Since $wg_2^{r_2}g_1g_2^{\frac{-r_2}{r_1}}=wg_1g_2\in\mathscr{L}$ and $wg_2^{r_2}g_1^{r_1}g_2^{-r_2}=wg_1^{r_1}\in\mathscr{L}$, therefore by using Theorem 6.3 for $\psi(z)=z^{r_1}, z>0, r_1>1$, $\vartheta=wg_2^{r_2}$, $f=g_1g_2^{\frac{-r_2}{r_1}}$, we obtain (6.14). If $\mathscr{G}(wg_1^{r_1})>0$. Then applying the same procedure but taking r_1,r_2,g_1,g_2 instead of r_2,r_1,g_2,g_1, we obtain (6.14).

If $\mathscr{G}(wg_2^{r_2})=0$ and $\mathscr{G}(wg_1^{r_1})=0$ then, as we know that

$$0\leq wg_1g_2\leq \frac{1}{r_1}wg_1^{r_1}+\frac{1}{r_2}wg_2^{r_2}, \tag{6.16}$$

it gives that $\mathscr{G}(wg_1g_2)=0$. The proof for the case $r_1>1$ is completed.

The case when $0<r_1<1$, then $M=\frac{1}{r_1}>1$ and utilizing (6.14) for M and $N=(1-r_1)^{-1}$, $\overline{g}_1=(g_1g_2)^{r_1}$, $\overline{g}_2=g_2^{-r_1}$ instead of r_1,r_2,g_1,g_2.

Finally, if $r_1<0$ then $0<r_2<1$ and we may apply similar arguments with r_1,r_2,g_1,g_2 replaced by r_2,r_1,g_2,g_1 provided that $\mathscr{G}(wg_1^{r_1})>0$.

Corollary 6.4 *Let the hypothesis H hold and* $\mathscr{G}:\mathscr{L}\to\mathbb{R}$ *be an isotonic linear functional. Suppose* $r_1>1$, $r_2=\frac{r_1}{r_1-1}$. *If* u,v,w,g_1 *and* g_2 *are non-negative functions defined on* $\mathscr{M}$ *such that* $wg_1^{r_1}, wg_2^{r_2}, uwg_2^{r_2}, vwg_2^{r_2}, uwg_1g_2, vwg_1g_2, wg_1g_2\in\mathscr{L}$ *and* $u(x)+v(x)=1$ *for* $x\in\mathscr{M}$, *then*

$$\mathscr{G}(wg_1g_2)\leq \mathscr{G}^{\frac{1}{r_1}}(uwg_1^{r_1})\mathscr{G}^{\frac{1}{r_2}}(uwg_2^{r_2})+\mathscr{G}^{\frac{1}{r_1}}(vwg_1^{r_1})\mathscr{G}^{\frac{1}{r_2}}(vwg_2^{r_2})$$
$$\leq \mathscr{G}^{\frac{1}{r_1}}(wg_1^{r_1})\mathscr{G}^{\frac{1}{r_2}}(wg_2^{r_2}). \tag{6.17}$$

In the case when $0<r_1<1$ *and* $r_2=\frac{r_1}{r_1-1}$ *with* $\mathscr{G}(wg_2^{r_2})>0$ *or* $r_1<0$ *and* $\mathscr{G}(wg_1^{r_1})>0$, *then*

$$\mathscr{G}^{\frac{1}{r_1}}(wg_1^{r_1})\mathscr{G}^{\frac{1}{r_2}}(wg_2^{r_2})$$
$$\leq \mathscr{G}^{\frac{1}{r_2}}(wg_2^{r_2})\Big\{\left(\mathscr{G}(uwg_2^{r_2})\right)^{1-r_1}(\mathscr{G}(uwg_1g_2))^{r_1}$$
$$+\left(\mathscr{G}(vwg_2^{r_2})\right)^{1-r_1}(vwg_1g_2))^{r_1}\Big\}^{\frac{1}{r_1}}$$
$$\leq \mathscr{G}(wg_1g_2). \tag{6.18}$$

Proof. If $\mathscr{G}(wg_2^{r_2}) > 0$. Let $\psi(z) = -z^{\frac{1}{r_1}}$, $z > 0, r_1 > 1$. Then obviously the function ψ is convex. Therefore using Theorem 6.3 for $\psi(z) = -z^{\frac{1}{r_1}}, \vartheta = wg_2^{r_2}, f = g_1^{r_1}g_2^{-r_2}$, we obtain (6.17). If $\mathscr{G}(wg_1^{r_1}) > 0$, then applying the same procedure but taking r_1, r_2, g_1, g_2 instead of r_2, r_1, g_2, g_1, we obtain (6.17).
For the case when $\mathscr{G}(wg_2^{r_2}) = 0$ and $\mathscr{G}(wg_1^{r_1}) = 0$, we proceed in a similar fashion as in the proof of Corollary 6.3.

When $0 < r_1 < 1$, then $M = \frac{1}{r_1} > 1$ and applying (6.17) for M and $N = (1-r_1)^{-1}$, $\overline{g}_1 = (g_1g_2)^{r_1}$, $\overline{g}_2 = g_2^{-r_1}$ instead of r_1, r_2, g_1, g_2, we get (6.18).

Finally, if $r_1 < 0$ then $0 < r_2 < 1$, then we may proceed as above but instead r_1, r_2, g_1, g_2, use r_2, r_1, g_2, g_1 respectively, provided that $\int_a^b w(\rho)g_1^{r_1}(\rho)d\rho > 0$.

6.2 Refinement of McShane's Inequality with Applications

We begin this section by giving a refinement of McShane's inequality.

Theorem 6.4 *Under the assumptions of Theorem 6.2, if $u, v \in \mathscr{L}$ are such that $u(x) + v(x) = 1$ for $x \in \mathscr{M}$ and $\vartheta\psi(\boldsymbol{\phi}(x)), u\vartheta\phi_i, v\vartheta\phi_i, u\vartheta, v\vartheta \in \mathscr{L}$ $(i = 1, 2, \ldots, n)$ with $\mathscr{G}(u\vartheta)$, $\mathscr{G}(v\vartheta) > 0$, then*

$$\begin{aligned}
&\psi\left(\frac{\mathscr{G}(\vartheta\phi_1)}{\mathscr{G}(\vartheta)}, \frac{\mathscr{G}(\vartheta\phi_2)}{\mathscr{G}(\vartheta)}, \ldots, \frac{\mathscr{G}(\vartheta\phi_n)}{\mathscr{G}(\vartheta)}\right) \\
&\quad\leq \frac{\mathscr{G}(u\vartheta)}{\mathscr{G}(\vartheta)}\psi\left(\frac{\mathscr{G}(u\vartheta\phi_1)}{\mathscr{G}(u\vartheta)}, \ldots, \frac{\mathscr{G}(u\vartheta\phi_n)}{\mathscr{G}(u\vartheta)}\right) + \frac{\mathscr{G}(v\vartheta)}{\mathscr{G}(\vartheta)}\psi\left(\frac{\mathscr{G}(v\vartheta\phi_1)}{\mathscr{G}(v\vartheta)}, \ldots, \frac{\mathscr{G}(v\vartheta\phi_n)}{\mathscr{G}(v\vartheta)}\right) \\
&\quad\leq \frac{1}{\mathscr{G}(\vartheta)}\mathscr{G}(\vartheta\psi(\phi_1, \phi_2, \ldots, \phi_n). \qquad (6.19)
\end{aligned}$$

Proof. Since $u(x) + v(x) = 1$ for $x \in \mathscr{M}$ and $\mathscr{G}$ is linear, therefore we have

$$\begin{aligned}
&\psi\left(\frac{\mathscr{G}(\vartheta\phi_1)}{\mathscr{G}(\vartheta)}, \frac{\mathscr{G}(\vartheta\phi_2)}{\mathscr{G}(\vartheta)}, \ldots, \frac{\mathscr{G}(\vartheta\phi_n)}{\mathscr{G}(\vartheta)}\right) \\
&= \psi\left(\frac{\mathscr{G}(u\vartheta\phi_1)}{\mathscr{G}(\vartheta)} + \frac{\mathscr{G}(v\vartheta\phi_1)}{\mathscr{G}(\vartheta)}, \ldots, \frac{\mathscr{G}(u\vartheta\phi_n)}{\mathscr{G}(\vartheta)} + \frac{\mathscr{G}(v\vartheta\phi_n)}{\mathscr{G}(\vartheta)}\right) \\
&= \psi\left(\frac{\mathscr{G}(u\vartheta)}{\mathscr{G}(\vartheta)}\frac{\mathscr{G}(u\vartheta\phi_1)}{\mathscr{G}(u\vartheta)} + \frac{\mathscr{G}(v\vartheta)}{\mathscr{G}(\vartheta)}\frac{\mathscr{G}(v\vartheta\phi_1)}{\mathscr{G}(v\vartheta)}, \ldots, \frac{\mathscr{G}(u\vartheta)}{\mathscr{G}(\vartheta)}\frac{\mathscr{G}(u\vartheta\phi_n)}{\mathscr{G}(u\vartheta)} + \frac{\mathscr{G}(v\vartheta)}{\mathscr{G}(\vartheta)}\frac{\mathscr{G}(v\vartheta\phi_n)}{\mathscr{G}(v\vartheta)}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \psi\left(\frac{\mathscr{G}(u\vartheta)}{\mathscr{G}(\vartheta)}\left(\frac{\mathscr{G}(u\vartheta\phi_1)}{\mathscr{G}(u\vartheta)},\dots,\frac{\mathscr{G}(u\vartheta\phi_n)}{\mathscr{G}(u\vartheta)}\right)+\frac{\mathscr{G}(v\vartheta)}{\mathscr{G}(\vartheta)}\left(\frac{\mathscr{G}(v\vartheta\phi_1)}{\mathscr{G}(v\vartheta)},\dots,\frac{\mathscr{G}(v\vartheta\phi_n)}{\mathscr{G}(v\vartheta)}\right)\right)\\
&\le \frac{\mathscr{G}(u\vartheta)}{\mathscr{G}(\vartheta)}\psi\left(\frac{\mathscr{G}(u\vartheta\phi_1)}{\mathscr{G}(u\vartheta)},\dots,\frac{\mathscr{G}(u\vartheta\phi_n)}{\mathscr{G}(u\vartheta)}\right)+\frac{\mathscr{G}(v\vartheta)}{\mathscr{G}(\vartheta)}\psi\left(\frac{\mathscr{G}(v\vartheta\phi_1)}{\mathscr{G}(v\vartheta)},\dots,\frac{\mathscr{G}(v\vartheta\phi_n)}{\mathscr{G}(v\vartheta)}\right)\\
&\le \frac{1}{\mathscr{G}(\vartheta)}\mathscr{G}(u\vartheta\psi(\phi_1,\phi_1,\dots,\phi_n)+\frac{1}{\mathscr{G}(\vartheta)}\mathscr{G}(v\vartheta\psi(\phi_1,\phi_1,\dots,\phi_n)\\
&= \frac{1}{\mathscr{G}(\vartheta)}\mathscr{G}(\vartheta\psi(\phi_1,\phi_2,\dots,\phi_n).
\end{aligned}
$$

The first inequality has been obtained by using definition of convex function while the second inequality has been obtained by using (6.2) on both terms and at the end linearity of $\mathscr{G}$ is utilized. The following theorem provides a refinement of the generalized Beck's inequality (4.50) given in [87, p-127].

Theorem 6.5 *Let the hypothesis H hold, $\mathscr{G}:\mathscr{L}\to\mathbb{R}$ be an isotonic linear functional and $\psi_i : I_i \to \mathbb{R}$ $(i=1,2,\dots,n)$ be continuous and strictly monotonic, $\tau : I\to\mathbb{R}$ be continuous and increasing functions. Also, let $g_1,g_2,\dots,g_n:\mathscr{M}\to\mathbb{R}$ and $\psi : I_1\times I_2\times\cdots\times I_n\to\mathbb{R}$ be real valued functions such that $g_1(\mathscr{M})\subset I_1, g_2(\mathscr{M})\subset I_2,\dots,g_n(\mathscr{M})\subset I_n$, $\psi_1(g_1),\psi_2(g_2),\dots,\psi_n(g_n)$, $\tau(\psi(g_1,g_2,\dots,g_n))$, u, v, ϑ, $u\vartheta$, $v\vartheta\in\mathscr{L}$ with $u(x)+v(x)=1$ for $x\in\mathscr{M}$ and $\mathscr{G}(\vartheta)$, $\mathscr{G}(u\vartheta)$, $\mathscr{G}(v\vartheta)>0$. Then the following inequalities hold*

$$
\begin{aligned}
&\psi\Big(M_{\psi_1}(\vartheta,g_1;\mathscr{G}),M_{\psi_2}(\vartheta,g_2;\mathscr{G}),\dots,M_{\psi_n}(\vartheta,g_n;\mathscr{G})\Big)\\
&\qquad\ge \tau^{-1}\left[\frac{\mathscr{G}(u\vartheta)}{\mathscr{G}(\vartheta)}\tau\left(\psi\Big(M_{\psi_1}(u\vartheta,g_1;\mathscr{G}),\dots,M_{\psi_n}(u\vartheta,g_n;\mathscr{G})\Big)\right)\right.\\
&\qquad\quad\left.+\frac{\mathscr{G}(v\vartheta)}{\mathscr{G}(\vartheta)}\tau\left(\psi\Big(M_{\psi_1}(v\vartheta,g_1;\mathscr{G}),\dots,M_{\psi_n}(v\vartheta,g_n;\mathscr{G})\Big)\right)\right]\\
&\qquad\ge M_\tau(\vartheta,\psi(g_1,g_2,\dots,g_n);\mathscr{G}),
\end{aligned}
\tag{6.20}
$$

if the function H defined by

$$H(s_1,s_2,\dots,s_n) = -\tau(\psi(\psi_1^{-1}(s_1),\psi_2^{-1}(s_2),\dots,\psi_n^{-1}(s_n))$$

is convex.

Proof. Applying Theorem 6.4 for the function H instead of ψ, we obtain

$$
\begin{aligned}
&\tau\left(\psi\left(\psi_1^{-1}\left(\frac{\mathscr{G}(\vartheta\phi_1)}{\mathscr{G}(\vartheta)}\right),\psi_2^{-1}\left(\frac{\mathscr{G}(\vartheta\phi_2)}{\mathscr{G}(\vartheta)}\right),\dots,\psi_n^{-1}\left(\frac{\mathscr{G}(\vartheta\phi_n)}{\mathscr{G}(\vartheta)}\right)\right)\right)\\
&\ge \frac{\mathscr{G}(u\vartheta)}{\mathscr{G}(\vartheta)}\tau\left(\psi\left(\psi_1^{-1}\left(\frac{\mathscr{G}(u\vartheta\phi_1)}{\mathscr{G}(u\vartheta)}\right),\dots,\psi_n^{-1}\left(\frac{\mathscr{G}(u\vartheta\phi_n)}{\mathscr{G}(u\vartheta)}\right)\right)\right)
\end{aligned}
$$

$$+\frac{\mathscr{G}(v\vartheta)}{\mathscr{G}(\vartheta)}\tau\left(\psi\left(\psi_1^{-1}\left(\frac{\mathscr{G}(v\vartheta\phi_1)}{\mathscr{G}(v\vartheta)}\right),\ldots,\psi_n^{-1}\left(\frac{\mathscr{G}(v\vartheta\phi_n)}{\mathscr{G}(v\vartheta)}\right)\right)\right)$$
$$\geq\frac{1}{\mathscr{G}(\vartheta)}\mathscr{G}(\vartheta\tau(\psi(\psi_1^{-1}(\phi_1),\psi_2^{-1}(\phi_2),\ldots,\psi_n^{-1}(\phi_n))). \tag{6.21}$$

Let $\phi_i=\psi_i(g_i)$ $(i=1,2,\ldots,n)$. Then (6.21) becomes

$$\tau\left(\psi\left(M_{\psi_1}(\vartheta,g_1;\mathscr{G}),M_{\psi_2}(\vartheta,g_2;\mathscr{G}),\ldots,M_{\psi_n}(\vartheta,g_n;\mathscr{G})\right)\right)$$
$$\geq\frac{\mathscr{G}(u\vartheta)}{\mathscr{G}(\vartheta)}\tau\left(\psi\left(M_{\psi_1}(u\vartheta,g_1;\mathscr{G}),\ldots,M_{\psi_n}(u\vartheta,g_n;\mathscr{G})\right)\right)$$
$$+\frac{\mathscr{G}(v\vartheta)}{\mathscr{G}(\vartheta)}\tau\left(\psi\left(M_{\psi_1}(v\vartheta,g_1;\mathscr{G}),\ldots,M_{\psi_n}(v\vartheta,g_n;\mathscr{G})\right)\right)$$
$$\geq\frac{1}{\mathscr{G}(\vartheta)}\mathscr{G}(\vartheta\tau(\psi(g_1,g_2,\ldots,g_n))). \tag{6.22}$$

which is equivalent to (6.20). A refinement of Beck's inequality ([27, p. 249]) is given in the following corollary.

Corollary 6.5 *Under the assumptions of Theorem 6.5 for $n=2$, the following inequalities hold*

$$\psi\left(M_{\psi_1}(\vartheta,g_1;\mathscr{G}),M_{\psi_2}(\vartheta,g_2;\mathscr{G})\right)$$
$$\geq\tau^{-1}\left[\frac{\mathscr{G}(u\vartheta)}{\mathscr{G}(\vartheta)}\tau\left(\psi\left(M_{\psi_1}(u\vartheta,g_1;\mathscr{G}),M_{\psi_2}(u\vartheta,g_2;\mathscr{G})\right)\right)\right.$$
$$\left.+\frac{\mathscr{G}(v\vartheta)}{\mathscr{G}(\vartheta)}\tau\left(\psi\left(M_{\psi_1}(v\vartheta,g_1;\mathscr{G}),M_{\psi_2}(v\vartheta,g_2;\mathscr{G})\right)\right)\right]$$
$$\geq M_\tau(\vartheta,\psi(g_1,g_2);\mathscr{G}), \tag{6.23}$$

if the function H defined by $H(s_1,s_2)=-\tau(\psi(\psi_1^{-1}(s_1),\psi_2^{-1}(s_2))$ is convex.

We discuss some particular cases of Corollary 6.5.

Corollary 6.6 *Let all the assumptions of Theorem 6.5 hold for $n=2$ with $\psi(z_1,z_2)=z_1+z_2$ and ψ_1,ψ_2 and τ be twice continuously differentiable such that $\psi_1',\psi_2',\tau',\psi_1'',\psi_2'',\tau''$ are all positive then the following inequalities hold*

$$M_{\psi_1}(\vartheta,g_1;\mathscr{G})+M_{\psi_2}(\vartheta,g_2;\mathscr{G})$$
$$\geq\tau^{-1}\left[\frac{\mathscr{G}(u\vartheta)}{\mathscr{G}(\vartheta)}\tau\left(M_{\psi_1}(u\vartheta,g_1;\mathscr{G})+M_{\psi_2}(u\vartheta,g_2;\mathscr{G})\right)\right.$$
$$\left.+\frac{\mathscr{G}(v\vartheta)}{\mathscr{G}(\vartheta)}\tau\left(M_{\psi_1}(v\vartheta,g_1;\mathscr{G})+M_{\psi_2}(v\vartheta,g_2;\mathscr{G})\right)\right]$$

$$\geq M_\tau(\vartheta, g_1+g_2;\mathscr{G}), \tag{6.24}$$

if and only if $G(z_1)+H(z_2) \leq K(z_1+z_2)$*, where* $G := \frac{\psi_1'}{\psi_1''}, H := \frac{\psi_2'}{\psi_2''}, K := \frac{\tau'}{\tau''}$.

Proof. The idea of the proof is similar to the proof of Corollary 3.2 given in [38]. Similar to the idea of the proof of Corollary 6.6, we can state the following Corollary.

Corollary 6.7 *Let all the assumptions of Theorem 6.5 hold for* $n=2$ *with* $\psi(z_1,z_2)=z_1z_2$ *and* ψ_1,ψ_2,τ *be twice continuously differentiable and let* $L_1(z) := \frac{\psi_1'(z)}{\psi_1'(z)+z\psi_1''(z)}$, $L_2(z) := \frac{\psi_2'(z)}{\psi_2'(z)+z\psi_2''(z)}, L_3(z) := \frac{\tau'(z)}{\tau'(z)+z\tau''(z)}$*. Also, assume that the functions* $\psi_1', \psi_2', \tau', \tau'', \psi_1'', \psi_2'', L_1, L_2, L_3$ *are all positive, then the inequalities*

$$\begin{aligned}
& M_{\psi_1}(\vartheta, g_1;\mathscr{G})M_{\psi_2}(\vartheta, g_2;\mathscr{G}) \\
& \quad \geq \tau^{-1}\left[\frac{\mathscr{G}(u\vartheta)}{\mathscr{G}(\vartheta)}\tau\left(M_{\psi_1}(u\vartheta, g_1;\mathscr{G})M_{\psi_2}(u\vartheta, g_2;\mathscr{G})\right)\right. \\
& \quad\quad \left.+\frac{\mathscr{G}(v\vartheta)}{\mathscr{G}(\vartheta)}\tau\left(M_{\psi_1}(v\vartheta, g_1;\mathscr{G})M_{\psi_2}(v\vartheta, g_2;\mathscr{G})\right)\right] \\
& \geq M_\tau(\vartheta, g_1g_2;\mathscr{G}),
\end{aligned} \tag{6.25}$$

hold if $L_1(z_1)+L_2(z_2) \leq L_3(z_1z_2)$.

6.3 Further Generalizations

The following theorem provides further generalization of the refinement of Jessen's inequality associated to n certain functions.

Theorem 6.6 *Let all the assumptions of Theorem 6.1 hold. Also, let* $u_l \in \mathscr{L}$ *be such that* $\sum_{l=1}^{n} u_l = 1$ *and* $u_l\vartheta f, u_l\vartheta \in \mathscr{L}$ *with* $\mathscr{G}(u_l\vartheta) > 0$ *for all* $l \in \{1,2,\ldots,n\}$*. Assume that* $\mathscr{S}_1, \mathscr{S}_2 \subset \{1,2,\ldots,n\}$ *are such that* $\mathscr{S}_1$ *and* $\mathscr{S}_2$ *are non empty,* $\mathscr{S}_1 \cap \mathscr{S}_2 = \emptyset$ *and* $\mathscr{S}_1 \cup \mathscr{S}_2 = \{1,2,\ldots,n\}$*. Then*

$$\begin{aligned}
& \psi\left(\frac{\mathscr{G}(\vartheta f)}{\mathscr{G}(\vartheta)}\right) \\
& \quad \leq \frac{\mathscr{G}\left(\sum\limits_{l\in\mathscr{S}_1} u_l\vartheta\right)}{\mathscr{G}(\vartheta)}\psi\left(\frac{\mathscr{G}\left(\sum\limits_{l\in\mathscr{S}_1} u_l\vartheta f\right)}{\mathscr{G}\left(\sum\limits_{l\in\mathscr{S}_1} u_l\vartheta\right)}\right)+\frac{\mathscr{G}\left(\sum\limits_{l\in\mathscr{S}_2} u_l\vartheta\right)}{\mathscr{G}(\vartheta)}\psi\left(\frac{\mathscr{G}\left(\sum\limits_{l\in\mathscr{S}_2} u_l\vartheta f\right)}{\mathscr{G}\left(\sum\limits_{l\in\mathscr{S}_2} u_l\vartheta\right)}\right) \\
& \quad \leq \frac{\mathscr{G}(\vartheta\psi(f))}{\mathscr{G}(\vartheta)}.
\end{aligned} \tag{6.26}$$

Proof. Since $\mathscr{G}$ is linear and $\sum_{l=1}^{n} u_l = 1$, therefore we may write

$$\psi\left(\frac{\mathscr{G}(\vartheta f)}{\mathscr{G}(\vartheta)}\right) = \psi\left(\frac{\mathscr{G}\left(\sum_{l=1}^{n} u_l \vartheta f\right)}{\mathscr{G}(\vartheta)}\right) = \psi\left(\frac{\mathscr{G}\left(\sum_{l\in\mathscr{S}_1} u_l \vartheta f + \sum_{l\in\mathscr{S}_2} u_l \vartheta f\right)}{\mathscr{G}(\vartheta)}\right)$$

$$= \psi\left(\frac{\mathscr{G}\left(\sum_{l\in\mathscr{S}_1} u_l \vartheta f\right) + \mathscr{G}\left(\sum_{l\in\mathscr{S}_2} u_l \vartheta f\right)}{\mathscr{G}(\vartheta)}\right)$$

$$= \psi\left(\frac{\mathscr{G}\left(\sum_{l\in\mathscr{S}_1} u_l \vartheta\right)}{\mathscr{G}(\vartheta)} \frac{\mathscr{G}\left(\sum_{l\in\mathscr{S}_1} u_l \vartheta f\right)}{\mathscr{G}\left(\sum_{l\in\mathscr{S}_1} u_l \vartheta\right)} + \frac{\mathscr{G}\left(\sum_{l\in\mathscr{S}_2} u_l \vartheta\right)}{\mathscr{G}(\vartheta)} \frac{\mathscr{G}\left(\sum_{l\in\mathscr{S}_2} u_l \vartheta f\right)}{\mathscr{G}\left(\sum_{l\in\mathscr{S}_2} u_l \vartheta\right)}\right). \quad (6.27)$$

By using definition of convex function in (6.27) we obtain

$$\psi\left(\frac{\mathscr{G}(\vartheta f)}{\mathscr{G}(\vartheta)}\right) \leq \frac{\mathscr{G}\left(\sum_{l\in\mathscr{S}_1} u_l \vartheta\right)}{\mathscr{G}(\vartheta)} \psi\left(\frac{\mathscr{G}\left(\sum_{l\in\mathscr{S}_1} u_l \vartheta f\right)}{\mathscr{G}\left(\sum_{l\in\mathscr{S}_1} u_l \vartheta\right)}\right)$$

$$+ \frac{\mathscr{G}\left(\sum_{l\in\mathscr{S}_2} u_l \vartheta\right)}{\mathscr{G}(\vartheta)} \psi\left(\frac{\mathscr{G}\left(\sum_{l\in\mathscr{S}_2} u_l \vartheta f\right)}{\mathscr{G}\left(\sum_{l\in\mathscr{S}_2} u_l \vartheta\right)}\right) \quad (6.28)$$

$$\leq \frac{\mathscr{G}\left(\sum_{l\in\mathscr{S}_1} u_l \vartheta\right)}{\mathscr{G}(\vartheta)} \frac{\mathscr{G}\left(\sum_{l\in\mathscr{S}_1} u_l \vartheta \psi(f)\right)}{\mathscr{G}\left(\sum_{l\in\mathscr{S}_1} u_l \vartheta\right)} + \frac{\mathscr{G}\left(\sum_{l\in\mathscr{S}_2} u_l \vartheta\right)}{\mathscr{G}(\vartheta)} \frac{\mathscr{G}\left(\sum_{l\in\mathscr{S}_2} u_l \vartheta \psi(f)\right)}{\mathscr{G}\left(\sum_{l\in\mathscr{S}_2} u_l \vartheta\right)} \quad (6.29)$$

$$= \frac{\mathscr{G}\left(\sum_{l\in\mathscr{S}_1} u_l \vartheta \psi(f)\right)}{\mathscr{G}(\vartheta)} + \frac{\mathscr{G}\left(\sum_{l\in\mathscr{S}_2} u_l \vartheta \psi(f)\right)}{\mathscr{G}(\vartheta)}$$

$$= \frac{\mathscr{G}\left(\sum_{l\in\mathscr{S}_1} u_l \vartheta \psi(f) + \sum_{l\in\mathscr{S}_2} u_l \vartheta \psi(f)\right)}{\mathscr{G}(\vartheta)} = \frac{\mathscr{G}(\vartheta \psi(f))}{\mathscr{G}(\vartheta)}.$$

The inequality (6.29) has been obtained by applying Jessen's inequality to both terms in (6.28) while linearity of $\mathscr{G}$ and $\sum_{l=1}^{n} u_l = 1$ have been further utilized for derivation of required result. Similarly to the above theorem, in the following theorem we present further generalization of McShane's inequality.

Theorem 6.7 *Let all the assumptions of Theorem 6.2 hold and let* $u_l \in \mathscr{L}$ *be such that* $\sum_{l=1}^{n} u_l = 1$ *and* $u_l \vartheta \psi(\boldsymbol{\phi}(x)), v_l \vartheta \psi(\boldsymbol{\phi}(x)), u_l \vartheta \phi_l, u_l \vartheta \in \mathscr{L}$ *with* $\mathscr{G}(u_l \vartheta) > 0$ *for all* $l \in$

$\{1,2,\ldots,n\}$. *If* $\mathscr{S}_1$ *and* $\mathscr{S}_2$ *are non empty and disjoint subsets of* $\{1,2,\ldots,n\}$ *such that* $\mathscr{S}_1 \cup \mathscr{S}_2 = \{1,2,\ldots,n\}$, *then*

$$\begin{aligned}
&\psi\left(\frac{\mathscr{G}(\vartheta\phi_1)}{\mathscr{G}(\vartheta)}, \frac{\mathscr{G}(\vartheta\phi_2)}{\mathscr{G}(\vartheta)}, \ldots, \frac{\mathscr{G}(\vartheta\phi_n)}{\mathscr{G}(\vartheta)}\right)\\
&\qquad \leq \frac{\sum_{l\in\mathscr{S}_1}\mathscr{G}(u_l\vartheta)}{\mathscr{G}(\vartheta)}\psi\left(\frac{\sum_{l\in\mathscr{S}_1}\mathscr{G}(u_l\vartheta\phi_1)}{\sum_{l\in\mathscr{S}_1}\mathscr{G}(u_l\vartheta)}, \ldots, \frac{\sum_{l\in\mathscr{S}_1}\mathscr{G}(u_l\vartheta\phi_n)}{\sum_{l\in\mathscr{S}_1}\mathscr{G}(u_l\vartheta)}\right)\\
&\qquad + \frac{\sum_{l\in\mathscr{S}_2}\mathscr{G}(v_l\vartheta)}{\mathscr{G}(\vartheta)}\psi\left(\frac{\sum_{l\in\mathscr{S}_2}\mathscr{G}(v_l\vartheta\phi_1)}{\sum_{l\in\mathscr{S}_2}\mathscr{G}(v_l\vartheta)}, \ldots, \frac{\sum_{l\in\mathscr{S}_2}\mathscr{G}(v_l\vartheta\phi_n)}{\sum_{l\in\mathscr{S}_2}\mathscr{G}(v_l\vartheta)}\right)\\
&\qquad \leq \frac{1}{\mathscr{G}(\vartheta)}\mathscr{G}(\vartheta\psi(\phi_1,\phi_2,\ldots,\phi_n)).
\end{aligned}$$

Proof. Since $\sum_{l=1}^{n} u_l(x) = 1$ for $x \in \mathscr{M}$ and $\mathscr{G}$ is linear, therefore we have

$$\begin{aligned}
&\psi\left(\frac{\mathscr{G}(\vartheta\phi_1)}{\mathscr{G}(\vartheta)}, \frac{\mathscr{G}(\vartheta\phi_2)}{\mathscr{G}(\vartheta)}, \ldots, \frac{\mathscr{G}(\vartheta\phi_n)}{\mathscr{G}(\vartheta)}\right)\\
&= \psi\left(\frac{\sum_{l\in\mathscr{S}_1}\mathscr{G}(u_l\vartheta\phi_1)}{\mathscr{G}(\vartheta)} + \frac{\sum_{l\in\mathscr{S}_2}\mathscr{G}(v_l\vartheta\phi_1)}{\mathscr{G}(\vartheta)}, \ldots, \frac{\sum_{l\in\mathscr{S}_1}\mathscr{G}(u_l\vartheta\phi_n)}{\mathscr{G}(\vartheta)} + \frac{\sum_{l\in\mathscr{S}_2}\mathscr{G}(v_l\vartheta\phi_n)}{\mathscr{G}(\vartheta)}\right)\\
&= \psi\left(\frac{\sum_{l\in\mathscr{S}_1}\mathscr{G}(u_l\vartheta)}{\mathscr{G}(\vartheta)}\frac{\sum_{l\in\mathscr{S}_1}\mathscr{G}(u_l\vartheta\phi_1)}{\sum_{l\in\mathscr{S}_1}\mathscr{G}(u_l\vartheta)} + \frac{\sum_{l\in\mathscr{S}_2}\mathscr{G}(v_l\vartheta)}{\mathscr{G}(\vartheta)}\frac{\sum_{l\in\mathscr{S}_2}\mathscr{G}(v_l\vartheta\phi_1)}{\sum_{l\in\mathscr{S}_2}\mathscr{G}(v_l\vartheta)}, \ldots\right.\\
&\quad \left.\ldots, \frac{\sum_{l\in\mathscr{S}_1}\mathscr{G}(u_l\vartheta)}{\mathscr{G}(\vartheta)}\frac{\sum_{l\in\mathscr{S}_1}\mathscr{G}(u_l\vartheta\phi_n)}{\sum_{l\in\mathscr{S}_1}\mathscr{G}(u_l\vartheta)} + \frac{\sum_{l\in\mathscr{S}_2}\mathscr{G}(v_l\vartheta)}{\mathscr{G}(\vartheta)}\frac{\sum_{l\in\mathscr{S}_2}\mathscr{G}(v_l\vartheta\phi_n)}{\sum_{l\in\mathscr{S}_2}\mathscr{G}(v_l\vartheta)}\right)\\
&= \psi\left(\frac{\sum_{l\in\mathscr{S}_1}\mathscr{G}(u_l\vartheta)}{\mathscr{G}(\vartheta)}\left(\frac{\sum_{l\in\mathscr{S}_1}\mathscr{G}(u_l\vartheta\phi_1)}{\sum_{l\in\mathscr{S}_1}\mathscr{G}(u_l\vartheta)}, \ldots, \frac{\mathscr{G}(\sum_{l\in\mathscr{S}_1}u_l\vartheta\phi_n)}{\sum_{l\in\mathscr{S}_1}\mathscr{G}(u_l\vartheta)}\right)\right.\\
&\left. + \frac{\sum_{l\in\mathscr{S}_2}\mathscr{G}(v_l\vartheta)}{\mathscr{G}(\vartheta)}\left(\frac{\sum_{l\in\mathscr{S}_2}\mathscr{G}(v_l\vartheta\phi_1)}{\sum_{l\in\mathscr{S}_2}\mathscr{G}(v_l\vartheta)}, \ldots, \frac{\sum_{l\in\mathscr{S}_2}\mathscr{G}(v_l\vartheta\phi_n)}{\sum_{l\in\mathscr{S}_2}\mathscr{G}(v_l\vartheta)}\right)\right)\\
&\quad \leq \frac{\sum_{l\in\mathscr{S}_1}\mathscr{G}(u_l\vartheta)}{\mathscr{G}(\vartheta)}\psi\left(\frac{\sum_{l\in\mathscr{S}_1}\mathscr{G}(u_l\vartheta\phi_1)}{\sum_{l\in\mathscr{S}_1}\mathscr{G}(u_l\vartheta)}, \ldots, \frac{\sum_{l\in\mathscr{S}_1}\mathscr{G}(u_l\vartheta\phi_n)}{\sum_{l\in\mathscr{S}_1}\mathscr{G}(u_l\vartheta)}\right)\\
&\qquad + \frac{\sum_{l\in\mathscr{S}_2}\mathscr{G}(v_l\vartheta)}{\mathscr{G}(\vartheta)}\psi\left(\frac{\sum_{l\in\mathscr{S}_2}\mathscr{G}(v_l\vartheta\phi_1)}{\sum_{l\in\mathscr{S}_2}\mathscr{G}(v_l\vartheta)}, \ldots, \frac{\sum_{l\in\mathscr{S}_2}\mathscr{G}(v_l\vartheta\phi_n)}{\sum_{l\in\mathscr{S}_2}\mathscr{G}(v_l\vartheta)}\right)\\
&\leq \frac{1}{\mathscr{G}(\vartheta)}\mathscr{G}\left(\sum_{l\in\mathscr{S}_1}u_l\vartheta\psi(\phi_1,\phi_1,\ldots,\phi_n)\right) + \frac{1}{\mathscr{G}(\vartheta)}\mathscr{G}\left(\sum_{l\in\mathscr{S}_2}v_l\vartheta\psi(\phi_1,\phi_1,\ldots,\phi_n)\right)\\
&= \frac{1}{\mathscr{G}(\vartheta)}\mathscr{G}(\vartheta\psi(\phi_1,\phi_2,\ldots,\phi_n)).
\end{aligned}$$

The first inequality has been obtained by using definition of convex function while the second inequality has been obtained by using (6.2) on both terms and at the end linearity of $\mathscr{G}$ is utilized.

Remark 6.1 Analogously to the applications of Theorem 6.3 we may give applications of Theorem 6.6. Also, we can give applications of Theorem 6.7 as given for Theorem 6.4.

Chapter 7

Refinement of Jensen's Operator Inequality

In this chapter we present a new refinement of Jensen's inequality for operator convex function associated to certain n-tuples. Some applications are presented for different means and also, deduced refinement of operator inequality connected to the operator concavity of operator entropy $A\log A^{-1}$. At the end, further generalization is given related to certain m finite sequences. The results of this chapter are given in [8].

7.1 Refinement of Jensen's Operator Inequality with Applications

In the following theorem we give a refinement of Jensen's Operator Inequality.

Theorem 7.1 *Let* $\psi : I \to \mathbb{R}$ *be an operator convex function defined on the interval* I*. Let* $A_p \in S(I)$ *and* $\zeta_p, \eta_p, \theta_p \in \mathbb{R}^+$ $(p = 1,2,\ldots,n)$ *such that* $\eta_p + \theta_p = 1$ *for all* $p \in \{1,2,\ldots,n\}$ *and* $\sum_{p=1}^{n} \zeta_p = 1$*. Then*

$$\psi\left(\sum_{p=1}^{n} \zeta_p A_p\right) \leq \sum_{p=1}^{n} \eta_p \zeta_p \psi\left(\frac{\sum_{p=1}^{n} \zeta_p \eta_p A_p}{\sum_{p=1}^{n} \zeta_p \eta_p}\right)$$

$$+\sum_{p=1}^{n}\zeta_p\theta_p\psi\left(\frac{\sum_{p=1}^{n}\zeta_p\theta_pA_p}{\sum_{p=1}^{n}\zeta_p\theta_p}\right)\leq\sum_{p=1}^{n}\zeta_p\psi(A_p). \tag{7.1}$$

Proof. Since $\eta_p+\theta_p=1$ for all $p\in\{1,2,\dots,n\}$, therefore we have

$$\begin{aligned}\psi\left(\sum_{p=1}^{n}\zeta_pA_p\right)&=\psi\left(\sum_{p=1}^{n}\zeta_p\eta_pA_p+\sum_{p=1}^{n}\zeta_p\theta_pA_p\right)\\&=\psi\left(\sum_{p=1}^{n}\zeta_p\eta_p\frac{\sum_{p=1}^{n}\zeta_p\eta_pA_p}{\sum_{p=1}^{n}\zeta_p\eta_p}+\sum_{p=1}^{n}\zeta_p\theta_p\frac{\sum_{p=1}^{n}\zeta_p\theta_pA_p}{\sum_{p=1}^{n}\zeta_p\theta_p}\right).\end{aligned} \tag{7.2}$$

Applying convexity of ψ to the right side of (7.2) we obtain

$$\begin{aligned}\psi\left(\sum_{p=1}^{n}\zeta_pA_p\right)&\leq\sum_{p=1}^{n}\zeta_p\eta_p\psi\left(\frac{\sum_{p=1}^{n}\zeta_p\eta_pA_p}{\sum_{p=1}^{n}\zeta_p\eta_p}\right)+\sum_{p=1}^{n}\zeta_p\theta_p\psi\left(\frac{\sum_{p=1}^{n}\zeta_p\theta_pA_p}{\sum_{p=1}^{n}\zeta_p\theta_p}\right)\\&\leq\sum_{p=1}^{n}\zeta_p\eta_p\psi(A_p)+\sum_{p=1}^{n}\zeta_p\theta_p\psi(A_p)\ \text{(By Jensen operator inequality)}\\&=\sum_{p=1}^{n}\zeta_p\psi(A_p).\end{aligned} \tag{7.3}$$

This proves the required result.

Remark 7.1 If $\mathbf{z}=(z_1,z_2,\dots,z_n)$ and $\mathbf{z}'=(z'_1,z'_2,\dots,z'_n)$ are two tuples, then we define $\mathbf{z}.\mathbf{z}'$ as $\mathbf{z}.\mathbf{z}'=(z_1z'_1,z_2z'_2,\dots,z_nz'_n)$.

As application of Theorem 7.1, in the following corollary we give a refinement of inequality (4.2) given in [42].

Corollary 7.1 *Let* $\mathbf{A}=(A_1,\dots,A_n)$ *be an n-tuple of strictly positive operators and* $\boldsymbol{\zeta}:=(\zeta_1,\dots,\zeta_n),\boldsymbol{\eta}:=(\eta_1,\dots,\eta_n)$ *and* $\boldsymbol{\theta}:=(\theta_1,\dots,\theta_n)$ *be positive n-tuples of real numbers such that* $\sum_{p=1}^{n}\zeta_p=1$ *and* $\eta_p+\theta_p=1$ *for all* $p\in\{1,2,\dots,n\}$. *Then the following inequalities hold:*

(i)

$$\begin{aligned}M_r(\mathbf{A},\boldsymbol{\zeta})&\leq[M_1(\boldsymbol{\eta},\boldsymbol{\zeta})M_s^r(\mathbf{A},\boldsymbol{\eta}.\boldsymbol{\zeta})+M_1(\boldsymbol{\theta},\boldsymbol{\zeta})M_s^r(\mathbf{A},\boldsymbol{\theta}.\boldsymbol{\zeta})]^{\frac{1}{r}}\\&\leq M_s(\mathbf{A},\boldsymbol{\zeta});\ \textit{if}\ 1\leq r\leq s\,.\end{aligned} \tag{7.4}$$

(ii)

$$\begin{aligned}M_r(\mathbf{A},\boldsymbol{\zeta})&\leq[M_1(\boldsymbol{\eta},\boldsymbol{\zeta})M_r^s(\mathbf{A},\boldsymbol{\eta}.\boldsymbol{\zeta})+M_1(\boldsymbol{\theta},\boldsymbol{\zeta})M_r^s(\mathbf{A},\boldsymbol{\theta}.\boldsymbol{\zeta})]^{\frac{1}{s}}\\&\leq M_s(\mathbf{A},\boldsymbol{\zeta}),\ \textit{if}\ r\leq s\leq-1\ \textit{or}\ r\leq-1\leq s\leq\tfrac{-1}{2}.\end{aligned} \tag{7.5}$$

(iii)

$$M_r(\mathbf{A},\boldsymbol{\zeta})\leq\left[M_1(\boldsymbol{\eta},\boldsymbol{\zeta})M_r^{-1}(\mathbf{A},\boldsymbol{\eta}.\boldsymbol{\zeta})+M_1(\boldsymbol{\theta},\boldsymbol{\zeta})M_r^{-1}(\mathbf{A},\boldsymbol{\theta}.\boldsymbol{\zeta})\right]^{-1}$$

$$\leq M_{-1}(\mathbf{A},\boldsymbol{\zeta}) \leq \left(M_1(\boldsymbol{\eta},\boldsymbol{\zeta})\left(M_1(\mathbf{A},\boldsymbol{\eta}.\boldsymbol{\zeta})\right)^{-1} + M_1(\boldsymbol{\theta},\boldsymbol{\zeta})\left(M_1(\mathbf{A},\boldsymbol{\theta}.\boldsymbol{\zeta})\right)^{-1}\right)^{-1}$$
$$\leq M_1(\mathbf{A},\boldsymbol{\zeta}) \leq M_1(\boldsymbol{\eta},\boldsymbol{\zeta})M_s(\mathbf{A},\boldsymbol{\eta}.\boldsymbol{\zeta}) + M_1(\boldsymbol{\theta},\boldsymbol{\zeta})M_s(\mathbf{A},\boldsymbol{\theta}.\boldsymbol{\zeta})$$
$$\leq M_s(\mathbf{A},\boldsymbol{\zeta}), \textit{if } r \leq -1 \textit{ and } s \geq 1\,. \tag{7.6}$$

(iv)

$$M_r(\mathbf{A},\boldsymbol{\zeta}) \leq \left[M_1(\boldsymbol{\eta},\boldsymbol{\zeta})M_r(\mathbf{A},\boldsymbol{\eta}.\boldsymbol{\zeta}) + M_1(\boldsymbol{\theta},\boldsymbol{\zeta})M_r(\mathbf{A},\boldsymbol{\theta}.\boldsymbol{\zeta})\right]^{\frac{1}{r}}$$
$$\leq M_s(\mathbf{A},\boldsymbol{\zeta}),\ \textit{if } \tfrac{1}{2} \leq r \leq 1 \leq s\,. \tag{7.7}$$

Proof. In the proof we use the fact that function $\psi(x) = x^t$ is operator convex on $(0,\infty)$ if either $1 \leq t \leq 2$ or $-1 \leq t \leq 0$, and operator concave on $(0,\infty)$ if $0 \leq t \leq 1$ ([42]).

(i) Suppose $1 \leq r \leq s$ then $0 < \frac{r}{s} \leq 1$. Therefore using Theorem 7.1 for the operator concave function $\psi(x) = x^{\frac{r}{s}}$ and taking $A_p \to A_p^s$ and then raising the power $\frac{1}{r}$ to both sides, we obtain (7.4).

(ii) Suppose $r \leq s < -1$ then $0 < \frac{s}{r} \leq 1$. Analogously as above using Theorem 7.1 for the operator concave function $\psi(x) = x^{\frac{s}{r}}$ and taking $A_p \to A_p^r$ and then raising the power $\frac{1}{s}$ to both sides, we obtain (7.5).

If $r \leq -1 \leq s \leq \frac{-1}{2}$, then $0 < \frac{s}{r} \leq 1$. Using Theorem 7.1 for the operator concave function $\psi(x) = x^{\frac{s}{r}}$, taking $A_p \to A_p^r$ and then raising the power to both sides $\frac{1}{s}$ we obtain (7.5).

(iii) Suppose $r \leq -1$ and $s \geq 1$ then using the operator convexity of the function $\psi(x) = x^{-1}$ in (7.1) we have

$$M_1(\mathbf{A},\boldsymbol{\zeta}) \geq \Big(M_1(\boldsymbol{\eta},\boldsymbol{\zeta})\left(M_1(\mathbf{A},\boldsymbol{\eta}.\boldsymbol{\zeta})\right)^{-1}$$
$$+ M_1(\boldsymbol{\theta},\boldsymbol{\zeta})\left(M_1(\mathbf{A},\boldsymbol{\theta}.\boldsymbol{\zeta})\right)^{-1}\Big)^{-1} \geq M_{-1}(\mathbf{A},\boldsymbol{\zeta}). \tag{7.8}$$

Using (7.4) for $r = 1$ we obtain

$$M_1(\mathbf{A},\boldsymbol{\zeta}) \leq M_1(\boldsymbol{\eta},\boldsymbol{\zeta})M_s(\mathbf{A},\boldsymbol{\eta}.\boldsymbol{\zeta}) + M_1(\boldsymbol{\theta},\boldsymbol{\zeta})M_s(\mathbf{A},\boldsymbol{\theta}.\boldsymbol{\zeta}) \leq M_s(\mathbf{A},\boldsymbol{\zeta}). \tag{7.9}$$

Similarly using (7.5) for $s = -1$ we obtain

$$M_r(\mathbf{A},\boldsymbol{\zeta}) \leq \left[M_1(\boldsymbol{\eta},\boldsymbol{\zeta})M_r^{-1}(\mathbf{A},\boldsymbol{\eta}.\boldsymbol{\zeta}) + M_1(\boldsymbol{\theta},\boldsymbol{\zeta})M_r^{-1}(\mathbf{A},\boldsymbol{\theta}.\boldsymbol{\zeta})\right]^{-1} \leq M_{-1}(\mathbf{A},\boldsymbol{\zeta}). \tag{7.10}$$

Combining (7.8)-(7.10) we obtain (7.6).

(iv) If $\frac{1}{2} \leq r \leq 1 \leq s$, then $1 \leq \frac{1}{r} \leq 2$. Now using Theorem 7.1 for the operator convex function $\psi(x) = x^{\frac{1}{r}}$ and taking $A_p \to A_p^r$ we obtain

$$M_r(\mathbf{A},\boldsymbol{\zeta}) \leq \left[M_1(\boldsymbol{\eta},\boldsymbol{\zeta})M_r(\mathbf{A},\boldsymbol{\eta}.\boldsymbol{\zeta}) + M_1(\boldsymbol{\theta},\boldsymbol{\zeta})M_r(\mathbf{A},\boldsymbol{\theta}.\boldsymbol{\zeta})\right]^{\frac{1}{r}} \leq M_1(\mathbf{A},\boldsymbol{\zeta}), \tag{7.11}$$

But $s \geq 1$, therefore $M_1(\mathbf{A},\boldsymbol{\zeta}) \leq M_s(\mathbf{A},\boldsymbol{\zeta})$. Combining these two inequalities we have (7.7).

In the following corollary we present a refinement of the inequality given ([53, Theorem 2.1]).

Corollary 7.2 *Let $A_p \in S(I)$ and $\zeta_p, \eta_p, \theta_p \in \mathbb{R}^+$ $(p = 1,2,\ldots,n)$ be such that $\eta_p + \theta_p = 1$ for all $p \in \{1,2,\ldots,n\}$ and $\sum_{p=1}^n \zeta_p = 1$ and let $f,h : I \to \mathbb{R}$ be continuous and strictly operator monotone functions. If one of the following conditions*

(i) $f \circ h^{-1} : I \to \mathbb{R}$ is operator convex function and f^{-1} is operator monotone

(ii) $f \circ h^{-1} : I \to \mathbb{R}$ is operator concave function and $-f^{-1}$ is operator monotone,

is satisfied, then

$$M_h(\boldsymbol{\zeta};\mathbf{A}) \le f^{-1}\Big(M_1(\boldsymbol{\zeta};\boldsymbol{\eta}) f(M_h(\boldsymbol{\zeta}.\boldsymbol{\eta};\mathbf{A})) + M_1(\boldsymbol{\zeta};\boldsymbol{\theta}) f(M_h(\boldsymbol{\zeta}.\boldsymbol{\theta};\mathbf{A}))\Big) \le M_f(\boldsymbol{\zeta};\mathbf{A}). \tag{7.12}$$

If one of the following conditions

(i) $f \circ h^{-1} : I \to \mathbb{R}$ is operator concave function and f^{-1} is operator monotone

(ii) $f \circ h^{-1} : I \to \mathbb{R}$ is operator convex function and $-f^{-1}$ is operator monotone

is satisfied, then the reverse inequalities hold in (7.12).

Proof. (i) The required inequalities may be deduced by using (7.1) for $A_p \to h(A_p)$ and $\psi \to f \circ h^{-1}$ and then applying f^{-1}.

Similarly we can prove inequalities for other conditions.

In the following theorem we obtain a refinement of integral operator inequality involving quasi-arithmetic mean as given in [53, Theorem 2.3] in discrete form.

Theorem 7.2 *Let $A_p \in S([m,M])$ $(p = 1,2,\ldots,n)$ and $\phi, \psi \in C[m,M]$, $0 < m < M$, be strictly monotone functions. Also assume that $\zeta_p, \eta_p, \theta_p \in \mathbb{R}^+$ $(p = 1,2,\ldots,n)$ are such that $\eta_p + \theta_p = 1$ for all $p \in \{1,2,\ldots,n\}$ and $\sum_{p=1}^n \zeta_p = 1$.*

(i) If ϕ^{-1} is operator convex and ψ^{-1} is operator concave, then

$$\begin{aligned} M_\phi(\boldsymbol{\zeta};\mathbf{A}) &\le M_1(\boldsymbol{\zeta};\boldsymbol{\eta}) M_\phi(\boldsymbol{\zeta}.\boldsymbol{\eta};\mathbf{A}) + M_1(\boldsymbol{\zeta};\boldsymbol{\theta}) M_\phi(\boldsymbol{\zeta}.\boldsymbol{\theta};\mathbf{A}) \le M_1(\boldsymbol{\zeta};\mathbf{A}) \\ &\le M_1(\boldsymbol{\zeta};\boldsymbol{\eta}) M_\psi(\boldsymbol{\zeta}.\boldsymbol{\eta};\mathbf{A}) + M_1(\boldsymbol{\zeta};\boldsymbol{\theta}) M_\psi(\boldsymbol{\zeta}.\boldsymbol{\theta};\mathbf{A}) \\ &\le M_\psi(\boldsymbol{\zeta};\mathbf{A}). \end{aligned} \tag{7.13}$$

(ii) If ϕ^{-1} is operator concave and ψ^{-1} is operator convex, then the reverse inequalities hold in (7.13).

Proof. (i). Using Theorem 7.1 for the operator convex function ϕ^{-1} on $[\phi_m, \phi_M]$, where $\phi_m = \min\{\phi(m), \phi(M)\}$, $\phi_M = \max\{\phi(m), \phi(M)\}$, we have

$$\phi^{-1}\left(\sum_{p=1}^n \zeta_p \phi(A_p)\right) \le \sum_{p=1}^n \eta_p \zeta_p \phi^{-1}\left(\frac{\sum_{p=1}^n \zeta_p \eta_p \phi(A_p)}{\sum_{p=1}^n \zeta_p \eta_p}\right) + \sum_{p=1}^n \zeta_p \theta_p \phi^{-1}\left(\frac{\sum_{p=1}^n \zeta_p \theta_p \phi(A_p)}{\sum_{p=1}^n \zeta_p \theta_p}\right) \le \sum_{p=1}^n \zeta_p A_p, \tag{7.14}$$

$$\text{i.e. } M_\phi(\boldsymbol{\zeta};\mathbf{A}) \leq M_1(\boldsymbol{\zeta};\boldsymbol{\eta})M_\phi(\boldsymbol{\zeta}.\boldsymbol{\eta};\mathbf{A}) + M_1(\boldsymbol{\zeta};\boldsymbol{\theta})M_\phi(\boldsymbol{\zeta}.\boldsymbol{\theta};\mathbf{A}) \leq M_1(\boldsymbol{\zeta};\mathbf{A}). \tag{7.15}$$

Similarly, since ψ^{-1} is operator concave on $I = [\psi_m, \psi_M]$, therefore we have

$$\psi^{-1}\left(\sum_{p=1}^n \zeta_p \psi(A_p)\right) \geq \sum_{p=1}^n \eta_p \zeta_p \psi^{-1}\left(\frac{\sum_{p=1}^n \zeta_p \eta_p \psi(A_p)}{\sum_{p=1}^n \zeta_p \eta_p}\right) + \sum_{p=1}^n \zeta_p \theta_p \psi^{-1}\left(\frac{\sum_{p=1}^n \zeta_p \theta_p \psi(A_p)}{\sum_{p=1}^n \zeta_p \theta_p}\right) \geq \sum_{p=1}^n \zeta_p A_p. \tag{7.16}$$

$$\text{i.e. } M_\psi(\boldsymbol{\zeta};\mathbf{A}) \geq M_1(\boldsymbol{\zeta};\boldsymbol{\eta})M_\psi(\boldsymbol{\zeta}.\boldsymbol{\eta};\mathbf{A}) + M_1(\boldsymbol{\zeta};\boldsymbol{\theta})M_\psi(\boldsymbol{\zeta}.\boldsymbol{\theta};\mathbf{A}) \geq M_1(\boldsymbol{\zeta};\mathbf{A}). \tag{7.17}$$

Combining (7.15) and (7.17) we obtain (7.13).

In the following corollary we present a refinement of operator inequality given in ([43, Theorem 5]) associated to the operator concavity of operator entropy $A\log A^{-1}$.

Corollary 7.3 *Let A_p be strictly positive operators and $\zeta_p, \eta_p, \theta_p \in \mathbb{R}^+$ $(p = 1,2,\ldots,n)$ such that $\eta_p + \theta_p = 1$ for all $p \in \{1,2,\ldots,n\}$ and $\sum_{p=1}^n \zeta_p = 1$. Then*

$$\sum_{p=1}^n \zeta_p A_p \log\left(\sum_{p=1}^n \zeta_p A_p\right) \leq \sum_{p=1}^n \eta_p \zeta_p A_p \log\left(\frac{\sum_{p=1}^n \zeta_p \eta_p A_p}{\sum_{p=1}^n \zeta_p \eta_p}\right) + \sum_{p=1}^n \zeta_p \theta_p A_p \log\left(\frac{\sum_{p=1}^n \zeta_p \theta_p A_p}{\sum_{p=1}^n \zeta_p \theta_p}\right) \leq \sum_{p=1}^n \zeta_p A_p \log(A_p). \tag{7.18}$$

Proof. Since function $\psi(x) = x\log x$ is operator convex, therefore utilizing (7.1) for the function $\psi(x) = x\log x$, we obtain (7.18). In the following corollary we present refinement of inequality given in ([43, Theorem 4].

Corollary 7.4 *Let A_p be strictly positive operators and $\zeta_p, \eta_p, \theta_p \in \mathbb{R}^+$ $(p = 1,2,\ldots,n)$ such that $\eta_p + \theta_p = 1$ for all $p \in \{1,2,\ldots,n\}$ and $\sum_{p=1}^n \zeta_p = 1$. Then*

$$\sum_{p=1}^n \zeta_p \log(A_p) \leq \sum_{p=1}^n \eta_p \zeta_p \log\left(\frac{\sum_{p=1}^n \zeta_p \eta_p A_p}{\sum_{p=1}^n \zeta_p \eta_p}\right) + \sum_{p=1}^n \zeta_p \theta_p \log\left(\frac{\sum_{p=1}^n \zeta_p \theta_p A_p}{\sum_{p=1}^n \zeta_p \theta_p}\right) \leq \log\left(\sum_{p=1}^n \zeta_p A_p\right). \tag{7.19}$$

Proof. Since $\psi(x) = \log x$ is an operator concave function, therefore utilizing reverse inequalities in (7.1) for the function $\psi(x) = \log x$, we obtain (7.19).

7.2 Further Generalization

In the following theorem, we present further refinement of the Jensen operator inequality concerning m sequences whose sum is equal to unity.

Theorem 7.3 *Let $\psi : I \to \mathbb{R}$ be an operator convex function. Let $A_p \in S(I)$, $\zeta_p, \theta_p^l \in \mathbb{R}^+$ $(l = 1,2,\ldots,m, p = 1,2,\ldots,n)$ be such that $\sum_{l=1}^{m} \theta_p^l = 1$ for each $p \in \{1,2,\ldots,n\}$ and $\sum_{p=1}^{n} \zeta_p = 1$. Assume that L_1 and L_2 are non empty disjoint subsets of $\{1,2,...,m\}$ such that $L_1 \cup L_2 = \{1,2,...,m\}$. Then*

$$\psi\left(\sum_{p=1}^{n} \zeta_p A_p\right) \leq \sum_{p=1}^{n} \sum_{l\in L_1} \theta_p^l \zeta_p \psi\left(\frac{\sum_{p=1}^{n} \sum_{l\in L_1} \theta_p^l \zeta_p A_p}{\sum_{p=1}^{n} \sum_{l\in L_1} \theta_p^l \zeta_p}\right)$$
$$+ \sum_{p=1}^{n} \sum_{l\in L_2} \theta_p^l \zeta_p \psi\left(\frac{\sum_{p=1}^{n} \sum_{l\in L_2} \theta_p^l \zeta_p A_p}{\sum_{p=1}^{n} \sum_{l\in L_2} \theta_p^l \zeta_p}\right) \leq \sum_{p=1}^{n} \zeta_p \psi(A_p). \tag{7.20}$$

If ψ is concave function then the reverse inequalities hold in (7.20).

Proof. Since $\sum_{l=1}^{m} \theta_p^l = 1$ for each $p \in \{1,2,\ldots,n\}$, therefore we may write

$$\sum_{p=1}^{n} \zeta_p \psi(A_p) = \sum_{p=1}^{n} \sum_{l\in L_1} \theta_p^l \zeta_p \psi(A_p) + \sum_{p=1}^{n} \sum_{l\in L_2} \theta_p^l \zeta_p \psi(A_p). \tag{7.21}$$

Apply Jensen's operator inequality to both terms on the right hand side of (7.21) and then using operator convexity of ψ in the obtained result, we have

$$\begin{aligned}
&\sum_{p=1}^{n} \zeta_p \psi(A_p) \\
&\geq \sum_{p=1}^{n} \sum_{l\in L_1} \theta_p^l \zeta_p \psi\left(\frac{\sum_{p=1}^{n} \sum_{l\in L_1} \theta_p^l \zeta_p A_p}{\sum_{p=1}^{n} \sum_{l\in L_1} \theta_p^l \zeta_p}\right) + \sum_{p=1}^{n} \sum_{l\in L_2} \theta_p^l \zeta_p \psi\left(\frac{\sum_{p=1}^{n} \sum_{l\in L_2} \theta_p^l \zeta_p A_p}{\sum_{p=1}^{n} \sum_{l\in L_2} \theta_p^l \zeta_p}\right) \\
&\geq \psi\left(\sum_{p=1}^{n} \sum_{l\in L_1} \theta_p^l \zeta_p \frac{\sum_{p=1}^{n} \sum_{l\in L_1} \theta_p^l \zeta_p A_p}{\sum_{p=1}^{n} \sum_{l\in L_1} \theta_p^l \zeta_p} + \sum_{p=1}^{n} \sum_{l\in L_2} \theta_p^l \zeta_p \frac{\sum_{p=1}^{n} \sum_{l\in L_2} \theta_p^l \zeta_p A_p}{\sum_{p=1}^{n} \sum_{l\in L_2} \theta_p^l \zeta_p}\right) \\
&= \psi\left(\sum_{p=1}^{n} \zeta_p A_p\right).
\end{aligned} \tag{7.22}$$

Remark 7.2 All the results presented in this paper may also be generalized using Theorem 7.3.

Chapter 8

Estimation of Different Entropies and Divergences via Refinement of Jensen's Inequality

Jensen's inequality is important to obtain inequalities for divergences between probability distributions. In this chapter, some suitable substitutions are used in (1.14) to construct new inequalities. These new inequalities actually give: estimation of f-divergence, Shannon entropy, Rényi divergence and Réyi entropy, and some relations among these entropies and divergences. Also the bounds are investigated for the Zipf-Mandelbrot law and hybrid Zipf-Mandelbrot law. The results of this chapter are published in [60].

8.1 Estimation of Csiszár Divergence

The first result gives the estimation for $\hat{I}_f(\mathbf{r},\mathbf{q})$ under different conditions by using some suitable substitution in (1.14).

Theorem 8.1 *Assume* (1.12), *let* $I \subset \mathbb{R}$ *be an interval and let* $\mathbf{r} = (r_1, \ldots, r_n)$ *and* $\mathbf{q} = (q_1, \ldots, q_n)$ *be in* $(0, \infty)^n$ *such that*

$$\frac{r_s}{q_s} \in I, \quad s = 1, \ldots, n.$$

(*i*) *If* $f : I \to \mathbb{R}$ *is convex function, then*

$$\hat{I}_f(\mathbf{r}, \mathbf{q}) = \sum_{s=1}^{n} q_s f\left(\frac{r_s}{q_s}\right) = A_{m,1}^{[1]} \geq A_{m,2}^{[1]} \geq \ldots \geq A_{m,m-1}^{[1]} \geq A_{m,m}^{[1]} \geq f\left(\frac{\sum_{s=1}^{n} r_s}{\sum_{s=1}^{n} q_s}\right) \sum_{s=1}^{n} q_s. \tag{8.1}$$

where

$$A_{m,l}^{[1]} = \frac{(m-1)!}{(l-1)!} \sum_{(i_1,\ldots,i_l) \in I_l} \eta_{I_m,l}(i_1, \ldots, i_l) \left(\sum_{j=1}^{l} \frac{q_{i_j}}{\alpha_{I_m,i_j}}\right) f\left(\frac{\sum_{j=1}^{l} \frac{r_{i_j}}{\alpha_{I_m,i_j}}}{\sum_{j=1}^{l} \frac{q_{i_j}}{\alpha_{I_m,i_j}}}\right). \tag{8.2}$$

The inequalities signs (8.1) *are reverse when* f *is concave function.*
(*ii*) *If* $f : I \to \mathbb{R}$ *be a function such that for all* $x \in I$ *the function* $x \to xf(x)$ *is convex, then*

$$\begin{aligned} \left(\sum_{s=1}^{n} r_s\right) f\left(\sum_{s=1}^{n} \frac{r_s}{\sum_{s=1}^{n} q_s}\right) &\leq A_{m,m}^{[2]} \leq A_{m,m-1}^{[2]} \leq \ldots \leq A_{m,2}^{[2]} \leq A_{m,1}^{[2]} \\ &= \sum_{s=1}^{n} r_s f\left(\frac{r_s}{q_s}\right) = \hat{I}_{idf}(\mathbf{r}, \mathbf{q}), \end{aligned} \tag{8.3}$$

where

$$A_{m,l}^{[2]} = \frac{(m-1)!}{(l-1)!} \sum_{(i_1,\ldots,i_l) \in I_l} \eta_{I_m,l}(i_1, \ldots, i_l) \left(\sum_{j=1}^{l} \frac{q_{i_j}}{\alpha_{I_m,i_j}}\right) \left(\frac{\sum_{j=1}^{l} \frac{r_{i_j}}{\alpha_{I_m,i_j}}}{\sum_{j=1}^{l} \frac{q_{i_j}}{\alpha_{I_m,i_j}}}\right) f\left(\frac{\sum_{j=1}^{l} \frac{r_{i_j}}{\alpha_{I_m,i_j}}}{\sum_{j=1}^{l} \frac{q_{i_j}}{\alpha_{I_m,i_j}}}\right).$$

Proof. (*i*) Considering $p_s = \frac{q_s}{\sum_{s=1}^{n} q_s}$ and $x_s = \frac{r_s}{q_s}$ in Theorem 1.1, we have

$$\begin{aligned} f\left(\sum_{s=1}^{n} \frac{q_s}{\sum_{s=1}^{n} q_s} \frac{r_s}{q_s}\right) \leq \ldots \leq \frac{(m-1)!}{(l-1)!} \sum_{(i_1,\ldots,i_l) \in I_l} \eta_{I_m,l}(i_1, \ldots, i_l) \\ \left(\sum_{j=1}^{l} \frac{\frac{q_{i_j}}{\sum_{s=1}^{n} q_s}}{\alpha_{I_m,i_j}}\right) f\left(\frac{\sum_{j=1}^{l} \frac{\frac{q_{i_j}}{\sum_{i=1}^{n} q_i}}{\alpha_{I_m,i_j}} \frac{r_{i_j}}{q_{i_j}}}{\sum_{j=1}^{l} \frac{\frac{q_{i_j}}{\sum_{i=1}^{n} q_i}}{\alpha_{I_m,i_j}}}\right) \leq \ldots \leq \sum_{s=1}^{n} \frac{q_s}{\sum_{i=1}^{n} q_s} f\left(\frac{r_s}{q_s}\right). \end{aligned} \tag{8.4}$$

Multiplying $\sum_{s=1}^{n} q_i$ we have (8.1).
(*ii*) Using $f := idf$ (where "*id*" is the identity function) in Theorem 1.1, we have

$$\sum_{s=1}^{n} p_s x_s f\left(\sum_{s=1}^{n} p_s x_s\right) \leq \ldots \leq \frac{(m-1)!}{(l-1)!} \sum_{(i_1,\ldots,i_l) \in I_l} \eta_{I_m,l}(i_1, \ldots, i_l)$$

$$\left(\sum_{j=1}^{l}\frac{p_{i_j}}{\alpha_{I_m,i_j}}\right)\left(\frac{\sum\limits_{j=1}^{l}\frac{p_{i_j}}{\alpha_{I_m,i_j}}x_{i_j}}{\sum\limits_{j=1}^{l}\frac{p_{i_j}}{\alpha_{I_m,i_j}}}\right)f\left(\frac{\sum\limits_{j=1}^{l}\frac{p_{i_j}}{\alpha_{I_m,i_j}}x_{i_j}}{\sum\limits_{j=1}^{l}\frac{p_{i_j}}{\alpha_{I_m,i_j}}}\right)\leq\ldots\leq\sum_{s=1}^{n}p_sx_sf(x_s). \tag{8.5}$$

Now on using $p_s=\frac{q_s}{\sum_{s=1}^{n}q_s}$ and $x_s=\frac{r_s}{q_s}$, $s=1,\ldots,n$, we get

$$\sum_{s=1}^{n}\frac{q_s}{\sum_{s=1}^{n}q_s}\frac{r_s}{q_s}f\left(\sum_{s=1}^{n}\frac{q_s}{\sum_{s=1}^{n}q_s}\frac{r_s}{q_s}\right)\leq\ldots\leq\frac{(m-1)!}{(l-1)!}\sum_{(i_1,\ldots,i_l)\in I_l}\eta_{I_m,l}(i_1,\ldots,i_l)$$

$$\left(\sum_{j=1}^{l}\frac{\frac{q_{i_j}}{\sum_{s=1}^{n}q_s}}{\alpha_{I_m,i_j}}\right)\left(\frac{\sum_{j=1}^{l}\frac{\frac{q_{i_j}}{\sum_{s=1}^{n}q_s}}{\alpha_{I_m,i_j}}\frac{r_{i_j}}{q_{i_j}}}{\sum_{j=1}^{l}\frac{\frac{q_{i_j}}{\sum_{s=1}^{n}q_s}}{\alpha_{I_m,i_j}}}\right)f\left(\frac{\sum_{j=1}^{l}\frac{\frac{q_{i_j}}{\sum_{s=1}^{n}q_s}}{\alpha_{I_m,i_j}}\frac{r_{i_j}}{q_{i_j}}}{\sum_{j=1}^{l}\frac{\frac{q_{i_j}}{\sum_{s=1}^{n}q_s}}{\alpha_{I_m,i_j}}}\right)\leq\ldots\leq\sum_{s=1}^{n}\frac{q_s}{\sum_{s=1}^{n}q_s}\frac{r_s}{q_s}f\left(\frac{r_s}{q_s}\right).$$

Multiplying $\sum_{s=1}^{n}q_s$ on both sides, we get (8.3).

8.2 Estimation of Shannon Entropy

The second result estimates the bounds for Shannon entropy by using suitable substitution in Theorem 8.1 for two different conditions.

Corollary 8.1 *Assume* (1.12).
(i) *If* $\mathbf{q}=(q_1,\ldots,q_n)\in(0,\infty)^n$, *and the base of* log *lies in the interval* $(1,\infty)$, *then*

$$S\leq A_{m,m}^{[3]}\leq A_{m,m-1}^{[3]}\leq\ldots\leq A_{m,2}^{[3]}\leq A_{m,1}^{[3]}=\log\left(\frac{n}{\sum_{s=1}^{n}q_s}\right)\sum_{s=1}^{n}q_s, \tag{8.6}$$

where

$$A_{m,l}^{[3]}=-\frac{(m-1)!}{(l-1)!}\sum_{(i_1,\ldots,i_l)\in I_l}\eta_{I_m,l}(i_1,\ldots,i_l)\left(\sum_{j=1}^{l}\frac{q_{i_j}}{\alpha_{I_m,i_j}}\right)\log\left(\sum_{j=1}^{l}\frac{q_{i_j}}{\alpha_{I_m,i_j}}\right). \tag{8.7}$$

The inequalities in (8.6) *are reversed if the base of* log *lies in interval* $(0,1)$.
(ii) *Consider the positive probability distribution* $\mathbf{q}=(q_1,\ldots,q_n)$ *and let the base lies in the interval* $(1,\infty)$, *then estimate for the Shannon entropy is given by*

$$S\leq A_{m,m}^{[4]}\leq A_{m,m-1}^{[4]}\leq\ldots\leq A_{m,2}^{[4]}\leq A_{m,1}^{[4]}=\log(n), \tag{8.8}$$

where

$$A_{m,l}^{[4]}=-\frac{(m-1)!}{(l-1)!}\sum_{(i_1,\ldots,i_l)\in I_l}\eta_{I_m,l}(i_1,\ldots,i_l)\left(\sum_{j=1}^{l}\frac{q_{i_j}}{\alpha_{I_m,i_j}}\right)\log\left(\sum_{j=1}^{l}\frac{q_{i_j}}{\alpha_{I_m,i_j}}\right).$$

Proof. (i) Using $f := \log$ and $\mathbf{r} = (1, \ldots, 1)$ in Theorem 8.1 (i), we get (8.6).
(ii) It is the special case of (i).

8.3 Estimation of Kullback-Leibler Divergence

The following result is the estimation of Kullback-Leibler divergence by using assumption in Theorem 8.1.

Corollary 8.2 *Assume* (1.12).
(i) *Let* $\mathbf{r} = (r_1, \ldots, r_n) \in (0, \infty)^n$ *and* $\mathbf{q} := (q_1, \ldots, q_n) \in (0, \infty)^n$. *If the base of* log *is in interval* $(1, \infty)$, *then*

$$\sum_{s=1}^{n} r_s \log\left(\sum_{s=1}^{n} \frac{r_s}{\sum_{s=1}^{n} q_s}\right) \leq A_{m,m}^{[5]} \leq A_{m,m-1}^{[5]} \leq \ldots \leq A_{m,2}^{[5]} \leq A_{m,1}^{[5]} = \sum_{s=1}^{n} r_s \log\left(\frac{r_s}{q_s}\right) = D(\mathbf{r}, \mathbf{q}), \tag{8.9}$$

where

$$A_{m,l}^{[5]} = \frac{(m-1)!}{(l-1)!} \sum_{(i_1, \ldots, i_l) \in I_l} \eta_{I_m,l}(i_1, \ldots, i_l) \left(\sum_{j=1}^{l} \frac{q_{i_j}}{\alpha_{I_m,i_j}}\right) \left(\frac{\sum_{j=1}^{l} \frac{r_{i_j}}{\alpha_{I_m,i_j}}}{\sum_{j=1}^{l} \frac{q_{i_j}}{\alpha_{I_m,i_j}}}\right) \log\left(\frac{\sum_{j=1}^{l} \frac{r_{i_j}}{\alpha_{I_m,i_j}}}{\sum_{j=1}^{l} \frac{q_{i_j}}{\alpha_{I_m,i_j}}}\right).$$

The inequalities in (8.9) *are reversed, when the base of* log *lies in the interval* $(0, 1)$.
(ii) *Suppose two positive probability distributions* $\boldsymbol{r} := (r_1, \ldots, r_n)$ *and* $\boldsymbol{q} := (q_1, \ldots, q_n)$ *and also let the base of* log *being in the interval* $(1, \infty)$, *then*

$$D(\boldsymbol{r}, \boldsymbol{q}) = A_{m,1}^{[6]} \geq A_{m,2}^{[6]} \geq \ldots \geq A_{m,m-1}^{[6]} \geq A_{m,m}^{[6]} \geq 0, \tag{8.10}$$

where

$$A_{m,l}^{[6]} = \frac{(m-1)!}{(l-1)!} \sum_{(i_1, \ldots, i_l) \in I_l} \eta_{I_m,l}(i_1, \ldots, i_l) \left(\sum_{j=1}^{l} \frac{q_{i_j}}{\alpha_{I_m,i_j}}\right) \left(\frac{\sum_{j=1}^{l} \frac{r_{i_j}}{\alpha_{I_m,i_j}}}{\sum_{j=1}^{l} \frac{q_{i_j}}{\alpha_{I_m,i_j}}}\right) \log\left(\frac{\sum_{j=1}^{l} \frac{r_{i_j}}{\alpha_{I_m,i_j}}}{\sum_{j=1}^{l} \frac{q_{i_j}}{\alpha_{I_m,i_j}}}\right).$$

The inequalities in (8.10) *are reversed when the base of* log *being in the inteval* $(0, 1)$.

Proof. (i) On taking $f := \log$ in Theorem 8.1 (ii), we get (8.9).
(ii) It is a special case of (i).

8.4 Inequalities for Rényi Divergence and Entropy

In this section we investigate the bounds for Rényi divergence and Rényi entropy by using some suitable substitution in Theorem 1.14.

Theorem 8.2 *Assume* (1.12)*, let* $\mathbf{r} = (r_1, \ldots, r_n)$ *and* $\mathbf{q} = (q_1, \ldots, q_n)$ *be probability distributions.*
(i) If $0 \leq \lambda \leq \mu$ *such that* $\lambda, \mu \neq 1$*, and suppose that the base of* log *lies in* $(1, \infty)$*, then*

$$D_\lambda(\mathbf{r}, \mathbf{q}) \leq A^{[7]}_{m,m} \leq A^{[7]}_{m,m-1} \leq \ldots \leq A^{[7]}_{m,2} \leq A^{[7]}_{m,1} = D_\mu(\mathbf{r}, \mathbf{q}), \tag{8.11}$$

where

$$A^{[7]}_{m,l} = \frac{1}{\mu - 1} \log\left(\frac{(m-1)!}{(l-1)!} \sum_{(i_1,\ldots,i_l)\in I_l} \eta_{I_m,l}(i_1, \ldots, i_l)\left(\sum_{j=1}^{l} \frac{r_{i_j}}{\alpha_{I_m,i_j}}\right)\left(\frac{\sum\limits_{j=1}^{l} \frac{r_{i_j}}{\alpha_{I_m,i_j}}\left(\frac{r_{i_j}}{q_{i_j}}\right)^{\lambda-1}}{\sum\limits_{j=1}^{l} \frac{r_{i_j}}{\alpha_{I_m,i_j}}}\right)^{\frac{\mu-1}{\lambda-1}}\right)$$

The reverse inequalities hold in (8.11) *if the base of* log *lies in* $(0, 1)$.
(ii) Suppose $1 < \mu$ *and the base of* log *lies in* $(1, \infty)$*, then*

$$D_1(\mathbf{r}, \mathbf{q}) = D(\mathbf{r}, \mathbf{q}) = \sum_{s=1}^{n} r_s \log\left(\frac{r_s}{q_s}\right) \leq A^{[8]}_{m,m} \leq A^{[8]}_{m,m-1} \leq \ldots \leq A^{[8]}_{m,2} \leq A^{[8]}_{m,1} = D_\mu(\mathbf{r}, \mathbf{q}), \tag{8.12}$$

where

$$A^{[8]}_{m,l} = \leq \frac{1}{\mu - 1} \log\left(\frac{(m-1)!}{(l-1)!} \sum_{(i_1,\ldots,i_l)\in I_l} \eta_{I_m,l}(i_1, \ldots, i_l)\left(\sum_{j=1}^{l} \frac{r_{i_j}}{\alpha_{I_m,i_j}}\right) \exp\left(\frac{(\mu-1)\sum\limits_{j=1}^{l} \frac{r_{i_j}}{\alpha_{I_m,i_j}} \log\left(\frac{r_{i_j}}{q_{i_j}}\right)}{\sum\limits_{j=1}^{l} \frac{r_{i_j}}{\alpha_{I_m,i_j}}}\right)\right),$$

here the bases of exp *and* log *are same, and if the base of* log *lies in* $(0, 1)$ *then the reverse inequalities hold.*
(iii) If $\lambda \in [0, 1)$*, and the base of* log *lies in* $(1, \infty)$*, then*

$$D_\lambda(\mathbf{r}, \mathbf{q}) \leq A^{[9]}_{m,m} \leq A^{[9]}_{m,m-1} \leq \ldots \leq A^{[9]}_{m,2} \leq A^{[9]}_{m,1} = D_1(\mathbf{r}, \mathbf{q}), \tag{8.13}$$

where

$$A^{[9]}_{m,l} = \frac{1}{\lambda-1}\frac{(m-1)!}{(l-1)!}\sum_{(i_1,\ldots,i_l)\in I_l}\eta_{I_m,l}(i_1,\ldots,i_l)\left(\sum_{j=1}^{l}\frac{r_{i_j}}{\alpha_{I_m,i_j}}\right)\log\left(\frac{\sum\limits_{j=1}^{l}\frac{r_{i_j}}{\alpha_{I_m,i_j}}\left(\frac{r_{i_j}}{q_{i_j}}\right)^{\lambda-1}}{\sum\limits_{j=1}^{l}\frac{r_{i_j}}{\alpha_{I_m,i_j}}}\right). \tag{8.14}$$

Proof. Suppose $f:(0,\infty)\to\mathbb{R}$ is function defined as $f(t):=t^{\frac{\mu-1}{\lambda-1}}$. On using this function in Theorem 1.1 together with substitution

$$p_s := r_s, \quad x_s := \left(\frac{r_s}{q_s}\right)^{\lambda-1}, \; s=1,\ldots,n,$$

we get

$$\left(\sum_{s=1}^{n} q_s\left(\frac{r_s}{q_s}\right)^{\lambda}\right)^{\frac{\mu-1}{\lambda-1}} = \left(\sum_{s=1}^{n} r_s\left(\frac{r_s}{q_s}\right)^{\lambda-1}\right)^{\frac{\mu-1}{\lambda-1}} \le$$

$$\cdots \le \frac{(m-1)!}{(l-1)!}\sum_{(i_1,\ldots,i_l)\in I_l}\eta_{I_m,l}(i_1,\ldots,i_l)\left(\sum_{j=1}^{l}\frac{r_{i_j}}{\alpha_{I_m,i_j}}\right)\left(\frac{\sum\limits_{j=1}^{l}\frac{r_{i_j}}{\alpha_{I_m,i_j}}\left(\frac{r_{i_j}}{q_{i_j}}\right)^{\lambda-1}}{\sum\limits_{j=1}^{l}\frac{r_{i_j}}{\alpha_{I_m,i_j}}}\right)^{\frac{\mu-1}{\lambda-1}}$$

$$\le \cdots \le \sum_{s=1}^{n} r_s\left(\left(\frac{r_s}{q_s}\right)^{\lambda-1}\right)^{\frac{\mu-1}{\lambda-1}}, \tag{8.15}$$

if either $0\le\lambda<1<\beta$ or $1<\lambda\le\mu$, and the reverse inequality in (8.15) holds if $0\le\lambda\le\beta<1$. On taking power $\frac{1}{\mu-1}$, we get

$$\left(\sum_{s=1}^{n} q_s\left(\frac{r_s}{q_s}\right)^{\lambda}\right)^{\frac{1}{\lambda-1}} \le$$

$$\cdots \le \left(\frac{(m-1)!}{(l-1)!}\sum_{(i_1,\ldots,i_l)\in I_l}\eta_{I_m,l}(i_1,\ldots,i_l)\left(\sum_{j=1}^{l}\frac{r_{i_j}}{\alpha_{I_m,i_j}}\right)\left(\frac{\sum\limits_{j=1}^{l}\frac{r_{i_j}}{\alpha_{I_m,i_j}}\left(\frac{r_{i_j}}{q_{i_j}}\right)^{\lambda-1}}{\sum\limits_{j=1}^{l}\frac{r_{i_j}}{\alpha_{I_m,i_j}}}\right)^{\frac{\mu-1}{\lambda-1}}\right)^{\frac{1}{\mu-1}}$$

$$\le \cdots \le \left(\sum_{s=1}^{n} r_s\left(\left(\frac{r_s}{q_s}\right)^{\lambda-1}\right)^{\frac{\mu-1}{\lambda-1}}\right)^{\frac{1}{\mu-1}} = \left(\sum_{s=1}^{n} q_s\left(\frac{r_s}{q_s}\right)^{\mu}\right)^{\frac{1}{\mu-1}}. \tag{8.16}$$

If the base of log function lies in $(1,\infty)$, then log is increasing function, therefore (8.11) is valid, and log is decreasing function if its base lies in $(0,1)$. Thus reverse inequality holds in (8.11). If $\lambda = 1$ and $\beta = 1$, we have (ii) and (iii) respectively by taking limit.

Theorem 8.3 *Assume* (1.12)*, let* $\mathbf{r} = (r_1,\ldots,r_n)$ *and* $\mathbf{q} = (q_1,\ldots,q_n)$ *be probability distributions. If either* $0 \le \lambda < 1$ *and the base of* log *is greater than 1, or* $1 < \lambda$ *and the base of* log *is between 0 and 1, then*

$$\frac{1}{\sum_{s=1}^{n} q_s \left(\frac{r_s}{q_s}\right)^{\lambda}} \sum_{s=1}^{n} q_s \left(\frac{r_s}{q_s}\right)^{\lambda} \log\left(\frac{r_s}{q_s}\right) = A_{m,1}^{[10]} \le A_{m,2}^{[10]} \le \ldots \le A_{m,m-1}^{[10]} \le A_{m,m}^{[10]}$$
$$\le D_\lambda(\mathbf{r},\mathbf{q}) \le A_{m,m}^{[11]} \le A_{m,m}^{[11]} \le \ldots \le A_{m,2}^{[11]} \le A_{m,1}^{[11]} = D_1(\mathbf{r},\mathbf{q}) \tag{8.17}$$

where

$$A_{m,m}^{[10]} = \frac{1}{(\lambda-1)\sum_{s=1}^{n} q_s \left(\frac{r_s}{q_s}\right)^{\lambda}} \frac{(m-1)!}{(l-1)!} \sum_{(i_1,\ldots,i_l)\in I_l} \eta_{I_m,l}(i_1,\ldots,i_l) \left(\sum_{j=1}^{l} \frac{r_{i_j}}{\alpha_{I_m,i_j}} \left(\frac{r_{i_j}}{q_{i_j}}\right)^{\lambda-1}\right) \times \log\left(\frac{\sum_{j=1}^{l} \frac{r_{i_j}}{\alpha_{I_m,i_j}} \left(\frac{r_{i_j}}{q_{i_j}}\right)^{\lambda-1}}{\sum_{j=1}^{l} \frac{r_{i_j}}{\alpha_{I_m,i_j}}}\right) \tag{8.18}$$

and

$$A_{m,m}^{[11]} = \frac{1}{\lambda-1} \frac{(m-1)!}{(l-1)!} \sum_{(i_1,\ldots,i_l)\in I_l} \eta_{I_m,l}(i_1,\ldots,i_l) \left(\sum_{j=1}^{l} \frac{r_{i_j}}{\alpha_{I_m,i_j}}\right) \log\left(\frac{\sum_{j=1}^{l} \frac{r_{i_j}}{\alpha_{I_m,i_j}} \left(\frac{r_{i_j}}{q_{i_j}}\right)^{\lambda-1}}{\sum_{j=1}^{l} \frac{r_{i_j}}{\alpha_{I_m,i_j}}}\right). \tag{8.19}$$

The inequalities in (8.17) *are reversed if either* $0 \le \lambda < 1$ *and the base of* log *lies in interval* $(0,1)$*, or* $1 < \lambda$ *and the base of* log *lies in interval* $(1,\infty)$.

Proof. We only give the proof for the case when $0 \le \lambda < 1$ and base of log lies in the interval $(1,\infty)$. Following the similar technique one can prove the other cases. Since $\frac{1}{\lambda-1} < 0$ and the function log is concave then choosing $I = (0,\infty)$, $f := \log$, $p_s = r_s$, $x_s := \left(\frac{r_s}{q_s}\right)^{\lambda-1}$ in Theorem 1.1, we have

$$D_\lambda(\mathbf{r},\mathbf{q}) = \frac{1}{\lambda-1} \log\left(\sum_{s=1}^{n} q_s \left(\frac{r_s}{q_s}\right)^{\lambda}\right) = \frac{1}{\lambda-1} \log\left(\sum_{s=1}^{n} r_s \left(\frac{r_s}{q_s}\right)^{\lambda-1}\right)$$

$$\leq \cdots \leq \frac{1}{\lambda-1}\frac{(m-1)!}{(l-1)!}\sum_{(i_1,\ldots,i_l)\in I_l}\eta_{I_m,l}(i_1,\ldots,i_l)\left(\sum_{j=1}^{l}\frac{r_{i_j}}{\alpha_{I_m,i_j}}\right)\log\left(\frac{\sum_{j=1}^{l}\frac{r_{i_j}}{\alpha_{I_m,i_j}}\left(\frac{r_{i_j}}{q_{i_j}}\right)^{\lambda-1}}{\sum_{j=1}^{l}\frac{r_{i_j}}{\alpha_{I_m,i_j}}}\right)$$

$$\leq \cdots \leq \frac{1}{\lambda-1}\sum_{s=1}^{n} r_s\log\left(\left(\frac{r_s}{q_s}\right)^{\lambda-1}\right)=\sum_{s=1}^{n} r_s\log\left(\frac{r_s}{q_s}\right)=D_1(\mathbf{r},\mathbf{q}) \tag{8.20}$$

and it is giving the upper bound for the Rényi divergence $D_\lambda(\mathbf{r},\mathbf{q})$.
Since for all $x\in(0,\infty)$ the function $x\mapsto xf(x)$ is convex, also base of log lies in interval $(0,\infty)$ and $\frac{1}{1-\lambda}<0$, therefore Theorem 1.1 provides

$$D_\lambda(\mathbf{r},\mathbf{q})=\frac{1}{\lambda-1}\log\left(\sum_{s=1}^{n} q_s\left(\frac{r_s}{q_s}\right)^{\lambda}\right) \tag{8.21}$$

$$=\frac{1}{\lambda-1\left(\sum_{s=1}^{n} q_s\left(\frac{r_s}{q_s}\right)^{\lambda}\right)}\left(\sum_{s=1}^{n} q_s\left(\frac{r_s}{q_s}\right)^{\lambda}\right)\log\left(\sum_{s=1}^{n} q_s\left(\frac{r_s}{q_s}\right)^{\lambda}\right)$$

$$\geq \cdots \geq \frac{1}{\lambda-1\left(\sum_{s=1}^{n} q_s\left(\frac{r_s}{q_s}\right)^{\lambda}\right)}\frac{(m-1)!}{(l-1)!}\sum_{(i_1,\ldots,i_l)\in I_l}\eta_{I_m,l}(i_1,\ldots,i_l)\left(\sum_{j=1}^{l}\frac{r_{i_j}}{\alpha_{I_m,i_j}}\right)$$

$$\left(\frac{\sum_{j=1}^{l}\frac{r_{i_j}}{\alpha_{I_m,i_j}}\left(\frac{r_{i_j}}{q_{i_j}}\right)^{\lambda-1}}{\sum_{j=1}^{l}\frac{r_{i_j}}{\alpha_{I_m,i_j}}}\right)\log\left(\frac{\sum_{j=1}^{l}\frac{r_{i_j}}{\alpha_{I_m,i_j}}\left(\frac{r_{i_j}}{q_{i_j}}\right)^{\lambda-1}}{\sum_{j=1}^{l}\frac{r_{i_j}}{\alpha_{I_m,i_j}}}\right)=$$

$$\frac{1}{\lambda-1\left(\sum_{s=1}^{n} q_s\left(\frac{r_s}{q_s}\right)^{\lambda}\right)}\frac{(m-1)!}{(l-1)!}\sum_{(i_1,\ldots,i_l)\in I_l}\eta_{I_m,l}(i_1,\ldots,i_l)$$

$$\left(\sum_{j=1}^{l}\frac{r_{i_j}}{\alpha_{I_m,i_j}}\left(\frac{r_{i_j}}{q_{i_j}}\right)^{\lambda-1}\right)\log\left(\frac{\sum_{j=1}^{l}\frac{r_{i_j}}{\alpha_{I_m,i_j}}\left(\frac{r_{i_j}}{q_{i_j}}\right)^{\lambda-1}}{\sum_{j=1}^{l}\frac{r_{i_j}}{\alpha_{I_m,i_j}}}\right)\geq\cdots\geq$$

$$\frac{1}{\lambda-1}\sum_{s=1}^{n} r_s\left(\frac{r_s}{q_s}\right)^{\lambda-1}\log\left(\frac{r_s}{q_s}\right)^{\lambda-1}\frac{1}{\sum_{s=1}^{n} r_s\left(\frac{r_s}{q_s}\right)^{\lambda-1}}$$

$$=\frac{1}{\sum_{s=1}^{n} q_s\left(\frac{r_s}{q_s}\right)^{\lambda}}\sum_{s=1}^{n} q_s\left(\frac{r_s}{q_s}\right)^{\lambda}\log\left(\frac{r_s}{q_s}\right). \tag{8.22}$$

This completes the proof.

8.5 Relation Between Shannon Entropy and Divergence

The next result gives the relation between Shannon entropy and divergence by using suitable substitution in Theorem 8.2 and Theorem 8.3. First consider the discrete probability distribution $\frac{\mathbf{1}}{\mathbf{n}} = (\frac{1}{n}, \ldots, \frac{1}{n})$.

Corollary 8.3 *Assume* (1.12)*, let* $\boldsymbol{r} = (r_1, \ldots, r_n)$ *and* $\boldsymbol{q} = (q_1, \ldots, q_n)$ *be positive probability distributions.*
(i) If $0 \le \lambda \le \mu$, $\lambda, \mu \ne 1$, *and the base of* log *lies in the interval* $(1, \infty)$*, then*

$$H_\lambda(\boldsymbol{r}) = \log(n) - D_\lambda\left(\boldsymbol{r}, \frac{1}{\boldsymbol{n}}\right) \ge A^{[12]}_{m,m} \ge A^{[12]}_{m,m} \ge \ldots A^{[12]}_{m,2} \ge A^{[12]}_{m,1} = H_\mu(\boldsymbol{r}), \tag{8.23}$$

where

$$A^{[12]}_{m,l} = \frac{1}{1-\mu} \log\left(\frac{(m-1)!}{(l-1)!} \sum_{(i_1,\ldots,i_l)\in I_l} \eta_{I_m,l}(i_1,\ldots,i_l) \times \left(\sum_{j=1}^{l} \frac{r_{i_j}}{\alpha_{I_m,i_j}}\right) \left(\frac{\sum_{j=1}^{l} \frac{r^{\lambda}_{i_j}}{\alpha_{I_m,i_j}}}{\sum_{j=1}^{l} \frac{r_{i_j}}{\alpha_{I_m,i_j}}}\right)^{\frac{\mu-1}{\lambda-1}}\right).$$

The reverse inequalities hold in (8.23) *if the base of* log *lies in* $(0,1)$.
(ii) If $1 < \mu$ *and base of* log *lies in* $(1, \infty)$*, then*

$$S = -\sum_{s=1}^{n} p_i \log(p_i) \ge A^{[13]}_{m,m} \ge A^{[13]}_{m,m-1} \ge \ldots \ge A^{[13]}_{m,2} \ge A^{[13]}_{m,1} = H_\mu(\boldsymbol{r}) \tag{8.24}$$

where

$$A^{[13]}_{m,l} = \log(n) + \frac{1}{1-\mu} \log\left(\frac{(m-1)!}{(l-1)!} \sum_{(i_1,\ldots,i_l)\in I_l} \eta_{I_m,l}(i_1,\ldots,i_l) \left(\sum_{j=1}^{l} \frac{r_{i_j}}{\alpha_{I_m,i_j}}\right) \exp\left(\frac{(\mu-1)\sum_{j=1}^{l} \frac{r_{i_j}}{\alpha_{I_m,i_j}} \log\left(n r_{i_j}\right)}{\sum_{j=1}^{l} \frac{r_{i_j}}{\alpha_{I_m,i_j}}}\right)\right),$$

the bases of exp *and* log *are same. The inequalities in* (8.24) *are reversed if the base of* log *lies in* $(0,1)$.
(iii) If $0 \le \lambda < 1$, *and the base of* log *lies in interval* $(1, \infty)$*, then*

$$H_\lambda(\boldsymbol{r}) \ge A^{[14]}_{m,m} \ge A^{[14]}_{m,m-1} \ge \ldots \ge A^{[14]}_{m,2} \le A^{[14]}_{m,1} = S, \tag{8.25}$$

where

$$A^{[14]}_{m,m} = \frac{1}{1-\lambda}\frac{(m-1)!}{(l-1)!}\sum_{(i_1,\ldots,i_l)\in I_l}\eta_{I_m,l}(i_1,\ldots,i_l)\left(\sum_{j=1}^{l}\frac{r_{i_j}}{\alpha_{I_m,i_j}}\right)\log\left(\frac{\sum_{j=1}^{l}\frac{r^{\lambda}_{i_j}}{\alpha_{I_m,i_j}}}{\sum_{j=1}^{l}\frac{r_{i_j}}{\alpha_{I_m,i_j}}}\right). \tag{8.26}$$

The inequalities in (8.25) *are reversed if the base of* log *lies in interval* $(0,1)$.

Proof. (i) Suppose $\mathbf{q}=\frac{\mathbf{1}}{\mathbf{n}}$ then from (1.33), we have

$$D_{\lambda}(\mathbf{r},\mathbf{q}) = \frac{1}{\lambda-1}\log\left(\sum_{s=1}^{n} n^{\lambda-1}r_s^{\lambda}\right) = \log(n)+\frac{1}{\lambda-1}\log\left(\sum_{s=1}^{n} r_s^{\lambda}\right), \tag{8.27}$$

therefore we have

$$H_{\lambda}(\mathbf{r}) = \log(n) - D_{\lambda}(\mathbf{r},\frac{\mathbf{1}}{\mathbf{n}}). \tag{8.28}$$

Now using Theorem 8.2 (i) and (8.28), we get

$$\begin{aligned} H_{\lambda}(\mathbf{r}) = \log(n) - D_{\lambda}\left(\mathbf{r},\frac{\mathbf{1}}{\mathbf{n}}\right) \geq \ldots \geq \log(n) - \frac{1}{\mu-1}\log\Bigg(n^{\mu-1}\frac{(m-1)!}{(l-1)!} \\ \sum_{(i_1,\ldots,i_l)\in I_l}\eta_{I_m,l}(i_1,\ldots,i_l)\times\left(\sum_{j=1}^{l}\frac{r_{i_j}}{\alpha_{I_m,i_j}}\right)\left(\frac{\sum_{j=1}^{l}\frac{r^{\lambda}_{i_j}}{\alpha_{I_m,i_j}}}{\sum_{j=1}^{l}\frac{r_{i_j}}{\alpha_{I_m,i_j}}}\right)^{\frac{\mu-1}{\lambda-1}}\Bigg) \geq \ldots \geq \\ \log(n) - D_{\mu}(\mathbf{r},\mathbf{q}) = H_{\mu}(\mathbf{r}). \end{aligned} \tag{8.29}$$

Relations (ii) and (iii) can be proved similarly.

Corollary 8.4 *Assume* (1.12) *and let* $\boldsymbol{r}=(r_1,\ldots,r_n)$ *and* $\boldsymbol{q}=(q_1,\ldots,q_n)$ *be positive probability distributions.*
If either $0\leq\lambda<1$ *and the base of* log *is greater than 1, or* $1<\lambda$ *and the base of* log *is between 0 and 1, then*

$$\begin{aligned} -\frac{1}{\sum_{s=1}^{n} r_s^{\lambda}}\sum_{s=1}^{n} r_s^{\lambda}\log(r_s) = A^{[15]}_{m,1} \geq A^{[15]}_{m,2} \geq \ldots \geq A^{[15]}_{m,m-1} \geq A^{[15]}_{m,m} \geq H_{\lambda}(\boldsymbol{r}) \\ \geq A^{[16]}_{m,m} \geq A^{[16]}_{m,m-1} \geq \ldots A^{[16]}_{m,2} \geq A^{[16]}_{m,1} = H(\boldsymbol{r}), \end{aligned} \tag{8.30}$$

where

$$A^{[15]}_{m,l} = \frac{1}{(\lambda-1)\sum_{s=1}^{n} r_s^{\lambda}}\frac{(m-1)!}{(l-1)!}\sum_{(i_1,\ldots,i_l)\in I_l}\eta_{I_m,l}(i_1,\ldots,i_l)\left(\sum_{j=1}^{l}\frac{r^{\lambda}_{i_j}}{\alpha_{I_m,i_j}}\right) \log\left(n^{\lambda-1}\frac{\sum_{j=1}^{l}\frac{r^{\lambda}_{i_j}}{\alpha_{I_m,i_j}}}{\sum_{j=1}^{l}\frac{r_{i_j}}{\alpha_{I_m,i_j}}}\right)$$

and

$$A_{m,1}^{[16]} = \frac{1}{1-\lambda}\frac{(m-1)!}{(l-1)!}\sum_{(i_1,\ldots,i_l)\in I_l}\eta_{I_m,l}(i_1,\ldots,i_l)\left(\sum_{j=1}^{l}\frac{r_{i_j}}{\alpha_{I_m,i_j}}\right)\log\left(\frac{\sum_{j=1}^{l}\frac{r_{i_j}^{\lambda}}{\alpha_{I_m,i_j}}}{\sum_{j=1}^{l}\frac{r_{i_j}}{\alpha_{I_m,i_j}}}\right). \tag{8.31}$$

The inequalities in (8.30) *are reversed if either* $0 \le \lambda < 1$ *and the base of* log *lies in interval* $(0,1)$, *or* $1 < \lambda$ *and the base of* log *lies in* $(1,\infty)$.

Proof. It can be proved similarly by following similar steps of Corollary 8.3 and using Theorem 8.3.

8.6 Inequalities by Using Zipf-Mandelbrot Law

The following results give the estimates for Rényi entropy and Shannon entropy and also the relation between them.

Conclusion 8.1 *Assume* $\alpha_{I_m,i_j} \ge 1$, *let* $\mathbf{r}$ *be a Zipf-Mandelbrot law and if* $0 \le \lambda < 1$, *and the base of* log *lies in interval* $(1,\infty)$. *Then by Corollary 8.3* (*iii*), *we get*

$$\begin{aligned} H_\lambda(\mathbf{r}) = \frac{1}{1-\lambda}\log\left(\frac{1}{H_{N,q,t}^{\lambda}}\sum_{s=1}^{n}\frac{1}{(s+q)^{\lambda s}}\right) \ge \ldots \ge \\ \frac{1}{1-\lambda}\frac{(m-1)!}{(l-1)!}\sum_{(i_1,\ldots,i_l)\in I_l}\eta_{I_m,l}(i_1,\ldots,i_l)\left(\sum_{j=1}^{l}\frac{1}{\alpha_{I_m,i_j}(i_j+q)H_{N.q,t}}\right) \\ \log\left(\frac{1}{H_{N,q,t}^{\lambda-1}}\frac{\sum_{j=1}^{l}\frac{1}{\alpha_{I_m,i_j}(i_j-q)^{\lambda s}}}{\sum_{j=1}^{l}\frac{1}{\alpha_{I_m,i_j}(i_j-q)^{s}}}\right) \ge \ldots \ge \\ \frac{t}{H_{N,q,t}}\sum_{s=1}^{N}\frac{\log(s+q)}{(s+q)^t} + \log(H_{N,q,t}) = S. \end{aligned} \tag{8.32}$$

The inequalities in (8.32) *are reversed if the base of* log *is between 0 and 1.*

Conclusion 8.2 *Assume* $\alpha_{I_m,i_j} \ge 1$, *let* $\mathbf{r}_1$ *and* $\mathbf{r}_2$ *be the Zipf-Mandelbort laws with parameters* $N \in \{1,2,\ldots\}$, $q_1, q_2 \in [0,\infty)$ *and* $s_1, s_2 > 0$, *respectively. Also if the base of* log *lies in interval* $(1,\infty)$, *then from Corollary 8.2* (ii), *we have*

$$D(\boldsymbol{r}_1,\boldsymbol{r}_2) = \sum_{s=1}^{n} \frac{1}{(s+q_1)^{t_1} H_{N,q_1,t_1}} \log\left(\frac{(s+q_2)^{t_2} H_{N,q_2,t_2}}{(s+q_1)^{t_1} H_{N,q_2,t_1}}\right) \geq \ldots \geq$$

$$\frac{(m-1)!}{(l-1)!} \sum_{(i_1,\ldots,i_l)\in I_l} \eta_{I_m,l}(i_1,\ldots,i_l) \left(\sum_{j=1}^{l} \frac{\frac{1}{(i_j+q_2)^{t_2} H_{N,q_2,t_2}}}{\alpha_{I_m,i_j}}\right) \left(\frac{\sum_{j=1}^{l} \frac{\frac{1}{(i_j+q_1)^{t_1} H_{N,q_1,t_1}}}{\alpha_{I_m,i_j}}}{\sum_{j=1}^{l} \frac{\frac{1}{(i_j+q_2)^{t_2} H_{N,q_2,t_2}}}{\alpha_{I_m,i_j}}}\right) \tag{8.33}$$

$$\log\left(\frac{\sum_{j=1}^{l} \frac{\frac{1}{(i_j+q_1)^{t_1} H_{N,q_1,t_1}}}{\alpha_{I_m,i_j}}}{\sum_{j=1}^{l} \frac{\frac{1}{(i_j+q_2)^{t_2} H_{N,q_2,t_2}}}{\alpha_{I_m,i_j}}}\right) \geq \ldots \geq 0.$$

The inequalities in (8.33) *are reversed if base of* log *is between 0 and 1.*

8.7 Relation Between Shannon Entropy and Zipf-Mandelbrot Law

Here we maximize the Shannon entropy using method of Lagrange multiplier under some equations constraints and get the Zipf-Mandelbrot law.

Theorem 8.4 *If* $J = \{1,2,\ldots,N\}$, *for a given* $q \geq 0$ *a probability distribution that maximizes the Shannon entropy under the constraints*

$$\sum_{s\in J} r_s = 1, \quad \sum_{s\in J} r_s\,(\ln(s+q)) := \Psi,$$

is Zipf-Mandelbrot law.

Proof. If $J = \{1,2,\ldots,N\}$. Set the Lagrange multipliers λ and t and consider the expression

$$\widetilde{S} = -\sum_{s=1}^{N} r_s \ln r_s - \lambda\left(\sum_{s=1}^{N} r_s - 1\right) - t\left(\sum_{s=1}^{N} r_s \ln(s+q) - \Psi\right).$$

Just for the sake of convenience, replace λ by $\ln\lambda - 1$, thus the last expression gives

$$\widetilde{S} = -\sum_{s=1}^{N} r_s \ln r_s - (\ln\lambda - 1)\left(\sum_{s=1}^{N} r_s - 1\right) - t\left(\sum_{s=1}^{N} r_s \ln(s+q) - \Psi\right).$$

From $\widetilde{S}_{r_s} = 0$, for $s = 1, 2, \ldots, N$, we get

$$r_s = \frac{1}{\lambda\,(s+q)^t},$$

and on using the constraint $\sum_{s=1}^{N} r_s = 1$, we have

$$\lambda = \sum_{s=1}^{N} \left(\frac{1}{(s+1)^t} \right),$$

where $t > 0$, concluding that

$$r_s = \frac{1}{(s+q)^t H_{N,q,t}}, \qquad s = 1, 2, \ldots, N.$$

Remark 8.1 Observe that the Zipf-Mandelbrot law and Shannon Entroy can be bounded from above (see [74]).

$$S = -\sum_{s=1}^{N} f(s,N,q,t) \ln f(s,N,q,t) \le -\sum_{s=1}^{N} f(s,N,q,t) \ln q_s$$

where $(q_1, \ldots, q_N)$ is a positive N-tuple such that $\sum_{s=1}^{N} q_s = 1$.

8.8 Relation Between Shannon Entropy and Hybrid Zipf-Mandelbrot Law

Here we maximize the Shannon entropy using method of Lagrange multiplier under some equations constraints and get the hybrid Zipf-Mandelbrot law.

Theorem 8.5 *If $J = \{1, \ldots, N\}$, then probability distribution that maximizes Shannon entropy under constraints*

$$\sum_{s \in J} r_s = 1, \quad \sum_{s \in J} r_s \ln(s+q) := \Psi, \quad \sum_{s \in J} s r_s := \eta$$

is hybrid Zipf-Mandelbrot law given as

$$r_s = \frac{w^s}{(s+q)^k\,\Phi^J(k,q,w)}, \qquad s \in J,$$

where

$$\Phi_J(k,q,w) = \sum_{s \in J} \frac{w^s}{(s+q)^k}.$$

Proof. First consider $J = \{1,\ldots,N\}$, we set the Lagrange multiplier and consider the expression

$$\tilde{S} = -\sum_{s=1}^{N} r_s \ln r_s + \ln w \left(\sum_{s=1}^{N} s r_s - \eta \right) - (\ln\lambda - 1) \left(\sum_{s=1}^{N} r_s - 1 \right) - k \left(\sum_{s=1}^{N} r_s \ln(s+q) - \Psi \right).$$

On setting $\tilde{S}_{r_s} = 0$, for $s = 1,\ldots,N$, we get

$$-\ln r_s + s\ln w - \ln\lambda - k\ln(s+q) = 0,$$

after solving for r_s, we get

$$\lambda = \sum_{s=1}^{N} \frac{w^s}{(s+q)^k},$$

and we recognize this as the partial sum of Lerch's transcendent that we denote with

$$\Phi_N^*(k,q,w) = \sum_{s=1}^{N} \frac{w^s}{(s+q)^k}$$

with $w \geq 0, k > 0$.

Remark 8.2 Observe that for Zipf-Mandelbrot law, Shannon entropy can be bounded from above (see [74]).

$$S = -\sum_{s=1}^{N} f_h(s,N,q,k) \ln f_h(s,N,q,k) \leq -\sum_{s=1}^{N} f_h(s,N,q,k) \ln q_s$$

where $(q_1,\ldots,q_N)$ is any positive N-tuple such that $\sum_{s=1}^{N} q_s = 1$.

Chapter 9

Divergence and Entropy Results via Interpolating Polynomials for m-convex Function

This chapter contains the generalization of refinement of Jensen's, Rényi and Shannon type inequalities via different interpolation for m-convex function. For this we construct the non-negative functionals from these inequalities and use various interpolation like: Abel-Gontscharoff Green function, Montgomery identity, Hermite interpolation, Lidstone polynomial, Fink's identity and Abel-Gontscharoff Green function, Taylor one point and Taylor two point formulas. The results of this chapter can be found in [60, 83, 64, 59, 82, 63].

9.1 New Generalized Functionals

We define following functionals by taking non-negative differences from the inequalities given in (1.14).

$$\Theta_1(f) = \mathscr{A}_{m,r} - f\left(\sum_{s=1}^{n} p_s x_s\right), \quad r = 1,\ldots,m, \tag{9.1}$$

$$\Theta_2(f) = \mathscr{A}_{m,r} - \mathscr{A}_{m,k}, \quad 1 \leq r < k \leq m. \tag{9.2}$$

If the suppositions of Theorem 1.1 hold, we have

$$\Theta_i(f) \geq 0, \quad i = 1,2. \tag{9.3}$$

Inequalities (9.3) are reversed if f is concave on I.

By using the conditions of Theorem 8.1 (i), define the non-negative functionals as follows.

$$\Theta_3(f) = \mathscr{A}^{[1]}_{m,r} - f\left(\frac{\sum_{s=1}^{n} r_s}{\sum_{s=1}^{n} q_s}\right)\sum_{s=1}^{n} q_s, \quad r = 1,\ldots,m, \tag{9.4}$$

$$\Theta_4(f) = \mathscr{A}^{[1]}_{m,r} - \mathscr{A}^{[1]}_{m,k}, \quad 1 \leq r < k \leq m. \tag{9.5}$$

By using the conditions of Theorem 8.1 (ii), define the non-negative functionals as follows.

$$\Theta_5(f) = \mathscr{A}^{[2]}_{m,r} - \left(\sum_{s=1}^{n} r_s\right) f\left(\frac{\sum_{s=1}^{n} r_s}{\sum_{s=1}^{n} q_s}\right), \quad r = 1,\ldots,m, \tag{9.6}$$

$$\Theta_6(f) = \mathscr{A}^{[2]}_{m,r} - \mathscr{A}^{[2]}_{m,k}, \quad 1 \leq r < k \leq m. \tag{9.7}$$

Under the assumption of Corollary 8.1 (i), define the following non-negative functionals.

$$\Theta_7(f) = A^{[3]}_{m,r} + \sum_{i=1}^{n} q_i \log(q_i), \ r = 1,\ldots,m, \tag{9.8}$$

$$\Theta_8(f) = A^{[3]}_{m,r} - A^{[3]}_{m,k}, \ 1 \leq r < k \leq m. \tag{9.9}$$

Under the assumption of Corollary 8.1 (ii), define the following non-negative functionals given as.

$$\Theta_9(f) = A^{[4]}_{m,r} - S, \ r = 1,\ldots,m, \tag{9.10}$$

$$\Theta_{10}(f) = A^{[4]}_{m,r} - A^{[4]}_{m,k}, \ 1 \leq r < k \leq m. \tag{9.11}$$

Under the assumption of Corollary 8.2 (i), let us define the non-negative functionals as follows.

$$\Theta_{11}(f) = A^{[5]}_{m,r} - \sum_{s=1}^{n} r_s \log\left(\sum_{s=1}^{n} \log\frac{r_n}{\sum_{s=1}^{n} q_s}\right), \ r = 1,\ldots,m, \tag{9.12}$$

$$\Theta_{12}(f) = A^{[5]}_{m,r} - A^{[5]}_{m,k}, \ 1 \leq r < k \leq m. \tag{9.13}$$

By using the conditions of Corollary 8.2 (ii), define the non-negative functionals as follows.

$$\Theta_{13}(f) = A^{[6]}_{m,r} - A^{[6]}_{m,k}, \ 1 \le r < k \le m. \tag{9.14}$$

Under the assumption of Theorem 8.2 (i), consider the following functionals.

$$\Theta_{14}(f) \ = \ A^{[7]}_{m,r} - D_\lambda(\mathbf{r},\mathbf{q}), \ r = 1,\ldots,m, \tag{9.15}$$
$$\Theta_{15}(f) \ = \ A^{[7]}_{m,r} - A^{[7]}_{m,k}, \ 1 \le r < k \le m. \tag{9.16}$$

Under the assumption of Theorem 8.2 (ii), consider the following functionals.

$$\Theta_{16}(f) = A^{[8]}_{m,r} - D_1(\mathbf{r},\mathbf{q}), \ r = 1,\ldots,m, \tag{9.17}$$
$$\Theta_{17}(f) = A^{[8]}_{m,r} - A^{[8]}_{m,k}, \ 1 \le r < k \le m. \tag{9.18}$$

Under the assumption of Theorem 8.2 (iii), consider the following functionals.

$$\Theta_{18}(f) = A^{[9]}_{m,r} - D_\lambda(\mathbf{r},\mathbf{q}), \ r = 1,\ldots,m, \tag{9.19}$$
$$\Theta_{19}(f) = A^{[9]}_{m,r} - A^{[9]}_{m,k}, \ 1 \le r < k \le m. \tag{9.20}$$

Under the assumption of Theorem 8.3 consider the following non-negative functionals.

$$\Theta_{20}(f) \ = \ D_\lambda(\mathbf{r},\mathbf{q}) - A^{[10]}_{m,r}, \quad r = 1,\ldots,m, \tag{9.21}$$
$$\Theta_{21}(f) \ = \ A^{[10]}_{m,k} - A^{[10]}_{m,r}, \ 1 \le r < k \le m, \tag{9.22}$$
$$\Theta_{22}(f) \ = \ A^{[11]}_{m,r} - D_\lambda(\mathbf{r},\mathbf{q}), \ r = 1,\ldots,m, \tag{9.23}$$
$$\Theta_{23}(f) \ = \ A^{[11]}_{m,r} - A^{[11]}_{m,r}, \ 1 \le r < k \le m, \tag{9.24}$$
$$\Theta_{24}(f) \ = \ A^{[11]}_{m,r} - A^{[10]}_{m,k}, \ r = 1,\ldots,m, \ k = 1,\ldots,m. \tag{9.25}$$
$$\tag{9.26}$$

Under the assumption of Corollary 8.3 (i), consider the following non-negative functionals.

$$\Theta_{25}(f) \ = \ H_\lambda(\mathbf{r}) - A^{[12]}_{m,r}, \ r = 1,\ldots,m, \tag{9.27}$$
$$\Theta_{26}(f) \ = \ A^{[12]}_{m,k} - A^{[12]}_{m,r}, \ 1 \le r < k \le m. \tag{9.28}$$
$$\tag{9.29}$$

Under the assumption of Corollary 8.3 (ii), consider the following functionals

$$\Theta_{27}(f) \ = \ S - A^{[13]}_{m,r}, \ r = 1,\ldots,m, \tag{9.30}$$
$$\Theta_{28}(f) \ = \ A^{[13]}_{m,k} - A^{[13]}_{m,r}, \ 1 \le r < k \le m. \tag{9.31}$$

Under the assumption of Corollary 8.3 (iii), consider the following functionals.

$$\Theta_{29}(f) \ = \ H_\lambda(\mathbf{r}) - A^{[14]}_{m,r}, \ r = 1,\ldots,m, \tag{9.32}$$

$$\Theta_{30}(f) = A^{[14]}_{m,k} - A^{[14]}_{m,r}, \ 1 \leq r < k \leq m. \tag{9.33}$$

Under the assumption of Corollary 8.4, defined the following functionals.

$$\Theta_{31} = A^{[15]}_{m,r} - H_\lambda(\mathbf{r}), \ r = 1,\dots,m, \tag{9.34}$$

$$\Theta_{32} = A^{[15]}_{m,r} - A^{[15]}_{m,k}, \ 1 \leq r < k \leq m, \tag{9.35}$$

$$\Theta_{33} = H_\lambda(\mathbf{r}) - A^{[16]}_{m,r}, \ r = 1,\dots,m, \tag{9.36}$$

$$\Theta_{34} = A^{[16]}_{m,k} - A^{[16]}_{m,r}, \ 1 \leq r < k \leq m, \tag{9.37}$$

$$\Theta_{35} = A^{[15]}_{m,r} - A^{[16]}_{m,k}, \ r = 1,\dots,m, \ k = 1,\dots,m. \tag{9.38}$$

9.2 Generalization of Refinement of Jensen's, Rényi and Shannon Type Inequalities via Montgomery Identity

The following two results contain the Montgomery identity using Taylor's formula [16, 17].

Theorem 9.1 *Let $m \in \mathbb{N}$, $f : (a,b) \to \mathbb{R}$ be such that $f^{(m-1)}$ is absolutely continuous $\alpha_1, \alpha_2 \in (a,b)$, $\alpha_1 < \alpha_2$. Then*

$$f(x) = \frac{1}{\alpha_2 - \alpha_1}\int_{\alpha_1}^{\alpha_2} f(u)du + \sum_{k=0}^{m-2} \frac{f^{(k+1)}(\alpha_1)(x-\alpha_1)^{k+2}}{k!(k+2)(\alpha_2-\alpha_1)} - \sum_{k=0}^{m-2} \frac{f^{(k+1)}(\alpha_2)(x-\alpha_2)^{k+2}}{k!(k+2)(\alpha_2-\alpha_1)}$$
$$+ \frac{1}{(m-1)!}\int_{\alpha_1}^{\alpha_2} R_m(x,u) f^{(m)}(u)du \tag{9.39}$$

where

$$R_m(x,u) = \begin{cases} -\frac{(x-u)^m}{m(\alpha_2-\alpha_1)} + \frac{x-\alpha_1}{\alpha_2-\alpha_1}(x-u)^{m-1}, & \alpha_1 \leq u \leq x; \\ -\frac{(x-u)^m}{m(\alpha_2-\alpha_1)} + \frac{x-\alpha_2}{\alpha_2-\alpha_1}(x-u)^{m-1}, & x \leq u \leq \alpha_2. \end{cases} \tag{9.40}$$

Theorem 9.2 *Let $m \in \mathbb{N}$, $f : I \to \mathbb{R}$ be such that $f^{(m-1)}$ is absolutely continuous, $\alpha_1, \alpha_2 \in I$, $\alpha_1 < \alpha_2$. Then*

$$f(x) = \frac{1}{\alpha_2 - \alpha_1}\int_{\alpha_1}^{\alpha_2} f(u)du + \sum_{k=0}^{m-2} f^{(k+1)}(x)\frac{(\alpha_1-x)^{k+2} - (\alpha_2-x)^{k+2}}{(k+2)!(\alpha_2-\alpha_1)}$$
$$+ \frac{1}{(m-1)!}\int_{\alpha_1}^{\alpha_2} \widehat{R}(x,u) f^{(m)}(u)du \tag{9.41}$$

where

$$\widehat{R}(x,u) = \begin{cases} -\frac{1}{m(\alpha_2-\alpha_1)}(\alpha_1-u), & \alpha_1 \le u \le x; \\ -\frac{1}{m(\alpha_2-\alpha_1)}(\alpha_2-u), & x \le u \le \alpha_2. \end{cases} \tag{9.42}$$

In case $m=1$, the sum $\sum_{k=0}^{m-2}\ldots$ is empty, so (9.39) and (9.41) reduce to well-known Montgomery identity (see [75])

$$f(x) = \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2} f(t)dt + \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2} p(x,u)f'(u)du,$$

where $p(x,u)$ denotes the Peano kernel which is defined as

$$p(x,u) = \begin{cases} \frac{u-\alpha_1}{\alpha_2-\alpha_1}, & \alpha_1 \le u \le x; \\ \frac{u-\alpha_2}{\alpha_2-\alpha_1}, & x \le u \le \alpha_2. \end{cases}$$

We construct some new identities the with the help of generalized Montgomery identity (9.39).

Theorem 9.3 *Let $f:[\alpha_1,\alpha_2]\to\mathbb{R}$ be a function. Also let $x_1,\ldots,x_n\in[\alpha_1,\alpha_2]$ and $p_1,\ldots,p_n$ are positive real numbers such that $\sum\limits_{i=1}^{n} p_i = 1$, and $R_m(x,u)$ be the same as defined in* (9.40)*, then the following identity holds.*

$$\Theta_i(f) = \frac{1}{\alpha_2-\alpha_1}\sum_{k=0}^{m-2}\left(\frac{1}{k!(k+2)}\right)\Big(f^{(k+1)}(\alpha_1)\Theta_i((x-\alpha_1)^{k+1}) - f^{(k+1)}(\alpha_2)$$
$$\times\Theta_2((x-\alpha_2)^{k+1})\Big)\frac{1}{(m-1)!}\int_{\alpha_1}^{\alpha_2}\Theta_i(R_m(x,u))f^{(m)}(u)du,,\; i=1,\ldots,35. \tag{9.43}$$

Proof. Using (9.39) in (9.1), (9.2) and (9.4)-(9.38), we get the result.

Theorem 9.4 *Let $f:[\alpha_1,\alpha_2]\to\mathbb{R}$ be a function. Also let $x_1,\ldots,x_n\in[\alpha_1,\alpha_2]$ and $p_1,\ldots,p_n$ are positive real numbers such that $\sum\limits_{i=1}^{n} p_i = 1$, and $R_m(x,u)$ be the same as defined in* (9.40)*. Let for $m\ge 2$*

$$\Theta_i(R_m(x,u)) \ge 0 \quad \textit{for all } u\in[\alpha_1,\alpha_2]\;\; i=1,\ldots,35.$$

If f is m-convex such that $f^{(m-1)}$ is absolutely continuous, then

$$\Theta_i(f) \ge \frac{1}{\alpha_2-\alpha_1}\sum_{k=0}^{m-2}\frac{1}{k!(k+2)}\Big(f^{(k+1)}(\alpha_1)\Theta_i((x-\alpha_1)^{k+1})$$
$$-f^{(k+1)}(\alpha_2)\Theta_i((x-\alpha_2)^{k+1}\Big),\; i=1,\ldots,35. \tag{9.44}$$

Proof. As $f^{(m-1)}$ is absolutely continuous on $[\alpha_1,\alpha_2]$, therefore $f^{(m)}$ exists almost everywhere. As f is m-convex, so $f^{(m)}(u)\ge 0$ for all $u\in[\alpha_1,\alpha_2]$(see [87, p.16]). Hence using Theorem 9.3, we get (9.44).

Theorem 9.5 *Let $f:[\alpha_1,\alpha_2]\to\mathbb{R}$ be a function. Also let $x_1,\ldots,x_n\in[\alpha_1,\alpha_2]$ and $p_1,\ldots,p_n$ are positive real numbers such that $\sum_{i=1}^{n} p_i=1$, let $f:[\alpha_1,\alpha_2]\to\mathbb{R}$ be a convex function.*
(i) If $m\geq 2$ is even, then (9.44) *holds.*
(ii) Let (9.44) *is valid. If the function*

$$\lambda(x)=\frac{1}{\alpha_2-\alpha_1}\sum_{l=0}^{m-2}\left(\frac{f^{(l+1)}(\alpha_1)(x-\alpha_1)^{l+2}-f^{(l+1)}(\alpha_2)(x-\alpha_2)^{l+2}}{l!(l+2)}\right)$$

is convex, then the smaller side of (9.44) *is non-negative and*

$$\Theta_i(f)\geq 0 \;\; i=1,\ldots,35.$$

Proof. (i) The function $R_m(\cdot,v)$ is convex (see [33]). Hence for even integers $m\geq 2$

$$\Theta_i(R_m(u,v))\geq 0,$$

therefore from Theorem 9.4, we have (9.44).
(ii) By using the linearity of $\Theta_i(f)$ we can write the smaller side of (9.44) in the form $\Theta_i(\lambda)$. As λ is supposed to be convex therefore the smaller side of (9.44) is non-negative, so $\Theta_i(f)\geq 0$.

Theorem 9.6 *Let $f:[\alpha_1,\alpha_2]\to\mathbb{R}$ be a function. Also let $x_1,\ldots,x_n\in[\alpha_1,\alpha_2]$ and $p_1,\ldots,p_n$ are positive real numbers such that $\sum_{i=1}^{n} p_i=1$, and $\widehat{R}_m(x,u)$ be the same as defined in* (9.42)*, then the following identity holds.*

$$\begin{aligned}\Theta_i(f) &= \frac{1}{\alpha_2-\alpha_1}\sum_{k=0}^{m-2}\frac{1}{k!(k+2)}\left(\Theta_i(f^{(k+1)}(x)(\alpha_1-x)^{k+1})-\Theta_i(f^{(k+1)}(x)(\alpha_2-x)^{k+1}\right)\\ &\quad+\frac{1}{(m-1)!}\int_{\alpha_1}^{\alpha_2}\Theta_i(\widehat{R}_m(x,u))f^{(m)}(u)du \;\; i=1,\ldots,35. \qquad (9.45)\end{aligned}$$

Proof. Using (9.41) in (9.1), (9.2) and (9.4)-(9.38), we get the identity (9.45).

Theorem 9.7 *Let $f:[\alpha_1,\alpha_2]\to\mathbb{R}$ be a function. Also let $x_1,\ldots,x_n\in[\alpha_1,\alpha_2]$ and $p_1,\ldots,p_n$ are positive real numbers such that $\sum_{i=1}^{n} p_i=1$, and $R_m(x,u)$ be the same as defined in* (9.42)*. Let for $m\geq 2$*

$$\Theta_i(\widehat{R}_m(x,u))\geq 0 \;\; \textit{for all } u\in[\alpha_1,\alpha_2] \;\;\; i=1,\ldots,35.$$

Suppose f is m-convex and let $f^{(m-1)}$ be an absolutely continuous function, then for $i=1,\ldots,35$

$$\begin{aligned}\Theta_i(f)\geq \frac{1}{\alpha_2-\alpha_1}\sum_{k=0}^{m-2}\left(\frac{1}{k!(k+2)}\right)&\left(\Theta_i(f^{(k+1)}(x)(\alpha_1-x)^{k+1})\right.\\ &\left.-\Theta_i(f^{(k+1)}(x)(\alpha_2-x)^{k+1}\right). \qquad (9.46)\end{aligned}$$

Proof. As $f^{(m-1)}$ is absolutely continuous on $[\alpha_1,\alpha_2]$, therefore $f^{(m)}$ exists almost everywhere. As f is m-convex, so $f^{(m)}(u) \geq 0$ for all $u \in [\alpha_1,\alpha_2]$(see [87, p.16]). Hence using Theorem 9.6, we get (9.46).

Remark 9.1 We can give related mean value theorems, also construct the new families of m-exponentialy convex functions and Cauchy means related to the functionals Θ_i, $i = 1,\ldots,35$ as given in [29].

9.3 Generalization of Refinement of Jensen's, f-divergence, Shannon and Rényi type Inequalities via Hermite Interpolating Polynomial

In [15], the Hermite interpolating polynomial is given as follows.
Let α_1,α_2 be two real numbers such that $\alpha_1 = c_1 < c_2 < \ldots < c_l = \alpha_2 (l \geq 2)$ be the points. For $f \in C^{2m}[\alpha_1,\alpha_2]$, a unique polynomial $\sigma_H^{(i)}(s)$ of degree $(m-1)$ exists and satisfies any of the following conditions:
Hermite Conditions

$$\sigma_H^{(i)}(c_j) = f^{(i)}(c_j);\ 0 \leq i \leq k_j,\ 1 \leq j \leq l,\ \sum_{j=1}^{l} k_j + l = m.$$

It is noted that Hermite conditions include the following particular cases.
Lagrange Conditions $(l = m,\ k_j = 0 \text{ for all } i)$

$$\sigma_L(c_j) = f(c_j),\ 1 \leq j \leq m.$$

Type $(q, m-q)$ **Conditions**$(l = 2,\ 1 \leq q \leq m-1,\ k_1 = q-1,\ k_2 = m-q-1)$

$$\sigma_{(q,m)}^{(i)}(\alpha_1) = f^{(i)}(\alpha_1),\ 0 \leq i \leq q-1$$
$$\sigma_{(q,m)}^{(i)}(\alpha_2) = f^{(i)}(\alpha_2),\ 0 \leq i \leq m-q-1.$$

Two Point Taylor Conditions $(m = 2q,\ l = 2,\ k_1 = k_2 = q-1)$

$$\sigma_{2T}^{(i)}(\alpha_1) = f^{(i)}(\alpha_1),\ f_{2T}^{(i)}(\alpha_2) \ = f^{(i)}(\alpha_2).\ \ 0 \leq i \leq q-1$$

In [15], the following result is given.

Theorem 9.8 *Let $-\infty < \alpha_1 < \alpha_2 < \infty$ and $\alpha_1 \leq c_1 < c_2 < \ldots < c_l \leq \alpha_2$ $(l \geq 2)$ are the given points and $f \in C^m([\alpha_1, \alpha_2])$. Then we have*

$$f(u) = \sigma_H(u) + R_H(f,u), \tag{9.47}$$

where $\sigma_H(u)$ is the Hermite interpolation polynomial that is

$$\sigma_H(u) = \sum_{j=1}^{l} \sum_{i=0}^{k_j} H_{i_j}(u) f^{(i)}(c_j);$$

the H_{i_j} are the fundamental polynomials of the Hermite basis given as

$$H_{i_j}(u) = \frac{1}{i!} \frac{\omega(u)}{(u-c_j)^{k_j+1-i}} \sum_{k=0}^{k_j-i} \frac{1}{k!} \frac{d^k}{du^k} \left(\frac{(u-c_j)^{k_j+1}}{\omega(u)} \right) \Bigg|_{u=c_j} (u-c_j)^k, \tag{9.48}$$

with

$$\omega(u) = \Pi_{j=1}^{l} (u-c_j)^{k_j+1},$$

and the remainder is given by

$$R_H(f,u) = \int_{\alpha_1}^{\alpha_2} G_{H,m}(u,s) f^{(m)}(s) ds,$$

where $G_{H,m}(u,s)$ is defined by

$$G_{H,m}(u,s) = \begin{cases} \sum_{j=1}^{l} \sum_{i=0}^{k_j} \frac{(c_j-s)^{m-i-1}}{(m-i-1)!} H_{i_j}(u), & s \leq u; \\ -\sum_{j=r+1}^{l} \sum_{i=0}^{k_j} \frac{(c_j-s)^{m-i-1}}{(m-i-1)!} H_{i_j}(u), & s \geq u. \end{cases}, \tag{9.49}$$

for all $c_r \leq s \leq c_{r+1}$; $r = 0, 1, \ldots, l$, with $c_0 = \alpha_1$ and $c_{l+1} = \alpha_2$.

Remark 9.2 In particular cases, for Lagrange condition from Theorem 9.8, we have

$$f(u) = \sigma_L(u) + R_L(f,u),$$

where $\sigma_L(u)$ is the Lagrange interpolating polynomial that is

$$\sigma_L(u) = \sum_{j=1}^{m} \sum_{k=1, k\neq j}^{m} \left(\frac{u-c_k}{c_j-c_k} \right) f(c_j),$$

and the remainder $R_L(f,u)$ is given by

$$R_L(f,u) = \int_{\alpha_1}^{\alpha_2} G_L(u,s) f^{(m)}(s) ds,$$

with

$$G_L(u,s) = \frac{1}{(m-1)!} \begin{cases} \sum_{j=1}^{r} (c_j-s)^{m-1} \prod_{k=1, k\neq j}^{m} \left(\frac{u-c_k}{c_j-c_k} \right), & s \leq u; \\ -\sum_{j=r+1}^{m} (c_j-s)^{m-1} \prod_{k=1, k\neq j}^{m} \left(\frac{u-c_k}{c_j-c_k} \right), & s \geq u. \end{cases}, \tag{9.50}$$

$c_r \leq s \leq c_{r+1}$ $r = 1, 2, \ldots, m-1$, with $c_1 = \alpha_1$ and $c_m = \alpha_2$, for type $(q, m-q)$ condition, from Theorem 9.8, we have

$$f(u) = \sigma_{(q,m)}(u) + R_{q,m}(f,u),$$

where $\sigma_{(q,m)}(u)$ is $(q, m-q)$ interpolating that is

$$\sigma_{(q,m)}(u) = \sum_{i=0}^{q-1} \tau_i(u) f^{(i)}(\alpha_1) + \sum_{i=0}^{m-q-1} \eta_i(u) f^{(i)}(\alpha_2),$$

with

$$\tau_i(u) = \frac{1}{i!}(u-\alpha_1)^i \left(\frac{u-\alpha_1}{\alpha_1-\alpha_2}\right)^{m-q} \sum_{k=0}^{q-1-i} \binom{m-q+k-1}{k} \left(\frac{u-\alpha_1}{\alpha_2-\alpha_1}\right)^k \tag{9.51}$$

and

$$\eta_i(u) = \frac{1}{i!}(u-\alpha_1)^i \left(\frac{u-\alpha_1}{\alpha_2-\alpha_1}\right)^{q} \sum_{k=0}^{m-q-1-i} \binom{q+k-1}{k} \left(\frac{u-\alpha_2}{\alpha_2-\alpha_1}\right)^k, \tag{9.52}$$

and the remainder $R_{(q,m)}(f,u)$ is defined as

$$R_{(q,m)}(f,u) = \int_{\alpha_1}^{\alpha_2} G_{q,m}(u,s) f^{(m)}(s) ds,$$

with

$$G_{(q,m)}(u,s) = \begin{cases} \sum_{j=0}^{q-1} \left[\sum_{p=0}^{q-1-j} \binom{m-q+p-1}{p} \left(\frac{u-\alpha_1}{\alpha_2-\alpha_1}\right)^p\right] \\ \times \frac{(u-\alpha_1)^j (\alpha_1-s)^{m-j-1}}{j!(m-j-1)!} \left(\frac{\alpha_2-u}{\alpha_2-\alpha_1}\right)^{m-q}, & \alpha_1 \le s \le u \le \alpha_2; \\ -\sum_{j=0}^{m-q-1} \left[\sum_{\lambda=0}^{m-q-j-1} \binom{q+\lambda-1}{\lambda} \left(\frac{\alpha_2-u}{\alpha_2-\alpha_1}\right)^\lambda\right] \\ \times \frac{(u-\alpha_2)^j (\alpha_2-s)^{m-j-1}}{j!(m-j-1)!} \left(\frac{u-\alpha_1}{\alpha_2-\alpha_1}\right)^{q}, & \alpha_1 \le u \le s \le \alpha_2. \end{cases} \tag{9.53}$$

From type Two-point Taylor condition from Theorem 9.8, we have

$$f(u) = \sigma_{2T}(u) + R_{2T}(f,u),$$

where

$$\begin{aligned} \sigma_{2T}(u) = & \sum_{i=0}^{q-1} \sum_{k=0}^{q-1-i} \binom{q+k-1}{k} \left[\frac{(u-\alpha_1)^i}{i!} \left(\frac{u-\alpha_2}{\alpha_1-\alpha_2}\right)^q \left(\frac{u-\alpha_1}{\alpha_2-\alpha_1}\right)^k f^{(i)}(\alpha_1) \right. \\ & \left. - \frac{(u-\alpha_2)^i}{i!} \left(\frac{u-\alpha_1}{\alpha_2-\alpha_1}\right)^q \left(\frac{u-\alpha_1}{\alpha_1-\alpha_2}\right)^k f^{(i)}(\alpha_2)\right] \end{aligned}$$

and the remainder $R_{2T}(f,u)$ is given by

$$R_{2T}(f,u) = \int_{\alpha_1}^{\alpha_2} G_{2T}(u,s) f^{(m)}(s) ds$$

with

$$G_{2T}(u,s)=\begin{cases}\frac{(-1)^q}{(2q-1)!}p^m(u,s)\sum_{j=0}^{q-1}\binom{q-1+j}{j}(u-s)^{q-1-j}\delta^j(u,s), & \alpha_1\le s\le u\le\alpha_2;\\ \frac{(-1)^q}{(2q-1)!}\delta^m(u,s)\sum_{j=0}^{q-1}\binom{q-1+j}{j}(s-u)^{q-1-j}p^j(u,s), & \alpha_1\le u\le s\le\alpha_2.\end{cases} \tag{9.54}$$

where $p(u,s)=\frac{(s-\alpha_1)(\alpha_2-u)}{\alpha_2-\alpha_1}$, $\delta(u,s)=p(u,s)$ for all $u,s\in[\alpha_1,\alpha_2]$.
In [22] and [68] the positivity of Green's functions is given as follows.

Lemma 9.1 *For the Green function $G_{H,m}(u,s)$ as defined in* (9.49)*, the following results hold.* (i)

$$\frac{G_{H,m}(u,s)}{\omega(u)}>0 \quad c_1\le u\le c_l,\ c_1\le s\le c_l.$$

(ii)

$$G_{H,m}(u,s)\le\frac{1}{(m-1)!(\alpha_2-\alpha_1)}|\omega(u)|.$$

(iii)

$$\int_{\alpha_1}^{\alpha_2}G_{H,m}(u,s)ds=\frac{\omega(u)}{m!}.$$

Theorem 9.9 *Let $f:[\alpha_1,\alpha_2]\to\mathbb{R}$ be a function. Also let $x_1,\ldots,x_n\in[\alpha_1,\alpha_2]$ and $p_1,\ldots,p_n\in(0,\infty)$ be such that $\sum_{i=1}^{n}p_i=1$. Also let $\alpha_1=c_1<c_2<\ldots<c_l=\alpha_2$ $(l\ge 2)$ be the points and $f\in C^m([\alpha_1,\alpha_2])$. Moreover H_{i_j} and $G_{H,m}$ are as defined by* (9.48) *and* (9.49) *respectively. Then for $i=1,\ldots,35$ we have*

$$\Theta_i(f(u))=\sum_{j=1}^{l}\sum_{i=0}^{k_j}f^{(i)}(c_j)\Theta_i(H_{i_j}(u))+\int_{\alpha_1}^{\alpha_2}\Theta_i\left(G_{H,m}(u,s)\right)f^{(m)}(s)ds. \tag{9.55}$$

Proof. Using (9.47) and (9.2) and by the linearity of $\Theta_i(f)$ we get (9.55).

Theorem 9.10 *Let $f:[\alpha_1,\alpha_2]\to\mathbb{R}$ be a function. Also let $x_1,\ldots,x_n\in[\alpha_1,\alpha_2]$ and $p_1,\ldots,p_n\in(0,\infty)$ be such that $\sum_{i=1}^{n}p_i=1$. Also let $\alpha_1=c_1<c_2\le\ldots<c_l=\alpha_2$ $(l\ge 2)$ be the points and $f\in C^m([\alpha_1,\alpha_2])$. Moreover H_{i_j} and $G_{H,m}$ are as defined by* (9.48) *and* (9.49) *respectively. Assume f is m-convex function and*

$$\Theta_i\left(G_{H,m}(u,s)\right)\ge 0 \quad \textit{for all } s\in[\alpha_1,\alpha_2],\quad i=1,\ldots,35.,$$

then

$$\Theta_i(f(u))\ge\sum_{j=1}^{l}\sum_{i=0}^{k_j}f^{(i)}(c_j)\Theta_i(H_{i_j}(u)),\quad i=1,\ldots,35. \tag{9.56}$$

Proof. As it is given that f is m-convex therefore $f^{(m)}(u) \geq 0$ for all $u \in [\alpha_1, \alpha_2]$. Hence by applying Theorem 9.9 we get (9.56).

Remark 9.3 If (9.56) is reversed then (9.56) is reversed under the assumption of Theorem 9.10.

Lagrange conditions give following results.

Corollary 9.1 *Let all the assumptions of Theorem 9.9 hold. Let G_L be as defined in* (9.50)*, f be m-convex function and*

$$\Theta_i\left(G_L(u,s)\right) \geq 0 \quad \textit{for all } s \in [\alpha_1, \alpha_2], \quad i = 1, \ldots, 35.,$$

then

$$\Theta_i(f(u)) \geq \sum_{j=1}^{m} f^{(i)}(c_j)\Theta_i\left(\prod_{k=1,k\neq j}^{m}\left(\frac{u-c_j}{c_j-c_k}\right)\right), \quad i = 1,2,\ldots,35.$$

On using the type $(q, m-q)$ conditions we have the following result.

Corollary 9.2 *Let all the assumptions of Theorem 9.9 hold, $G_{(q,m)}$ be a Green function as defined in* (9.53) *and τ_i and η_i as defined in* (9.51) *and* (9.52) *respectively. Also let f be m-convex function and*

$$\Theta_i\left(G_{(q,m)}(u,s)\right) \geq 0 \quad \textit{for all } s \in [\alpha_1, \alpha_2], \quad i = 1, \ldots, 35.,$$

then

$$\Theta_i(f(u)) \geq \sum_{i=0}^{q-1} f^{(i)}(\alpha_1)\Theta_i(\tau_i(u)) + \sum_{i=0}^{m-q-1} f^{(i)}(\alpha_2)\Theta_i(\eta_i(u)), \quad i = 1, \ldots, 35.$$

Two-point Taylor condition is used in order to obtain the following result.

Corollary 9.3 *Let all the assumptions of Theorem 9.9 hold, G_{2T} be a Green function as defined in* (9.54)*. Also let f be m-convex function and*

$$\Theta_i\left(G_{2T}(u,s)\right) \geq 0 \quad \textit{for all } s \in [\alpha_1, \alpha_2], \quad i = 1, \ldots, 35.,$$

then

$$\begin{aligned}\Theta_i(f(u)) \geq \sum_{i=0}^{q-1}\sum_{k=0}^{q-1-i}\binom{q+k-1}{k}\Bigg[&f^{(i)}(\alpha_1)\Theta_i\left(\frac{(u-\alpha_1)^i}{i!}\left(\frac{u-\alpha_2}{\alpha_1-\alpha_2}\right)^q\left(\frac{u-\alpha_1}{\alpha_2-\alpha_1}\right)^k\right)\\ &+ f^{(i)}(\alpha_2)\Theta_i\left(\frac{(u-\alpha_2)^i}{i!}\left(\frac{u-\alpha_1}{\alpha_2-\alpha_1}\right)^q\left(\frac{u-\alpha_2}{\alpha_1-\alpha_2}\right)^k\right)\Bigg], \quad i = 1, \ldots, 35.\end{aligned}$$

Theorem 9.11 *Let all the assumptions of Theorem* (9.9) *hold,* $f : [\alpha_1, \alpha_2] \to \mathbb{R}$ *be m-convex function.*
(i) *If* k_j *is odd for each* $j = 2, \ldots, l$ *then* (9.56) *holds.*
(ii) *Let* (9.56) *be satisfied and the function*

$$F(u) = \sum_{j=1}^{l} \sum_{i=1}^{k_j} f^{(i)}(c_j) H_{i_j}(u)$$

is supposed to be convex. Then the smaller side of (9.56) *is non-negative and we have*

$$\Theta_i(f(u)) \geq 0, \quad i = 1, \ldots, 35.$$

Proof. (i) Since k_j is odd for all $j = 2, \ldots, l$ so we have $\omega(u) \geq 0$, we have $G_{H,m-2}(u,s) \geq 0$, so $G_{H,m}$ is convex, therefore $\Theta_i(G_{H,m}(u,s)) \geq 0$, using Theorem 9.10, we get (9.56).
(ii) By using the linearity of $\Theta_i(f)$ we can write the smaller side of (9.56) in the form $\Theta_i(\lambda)$. As λ is supposed to be convex therefore the smaller side of (9.56) is non-negative, so $\Theta_i(f) \geq 0$.

Theorem 9.12 *Let* $f : [\alpha_1, \alpha_2] \to \mathbb{R}$ *be a function. Also let* $x_1, \ldots, x_n \in [\alpha_1, \alpha_2]$ *and* $p_1, \ldots, p_n \in (0, \infty)$ *be such that* $\sum_{i=1}^{n} p_i = 1$. *Also let* $\alpha_1 = c_1 < c_2 < \ldots < c_l = \alpha_2$ $(l \geq 2)$ *be the points and* $f \in C^m([\alpha_1, \alpha_2])$. *Furthermore let* H_{i_j}, $G_{H,m}$ *and* G *be as defined in* (9.48), (9.49) *and* (9.93) *respectively. Then we have*

$$\Theta_i(f(u)) = \int_{\alpha_1}^{\alpha_2} \Theta_i(G(u,t)) \sum_{j=1}^{l} \sum_{i=0}^{k_j} f^{(i+2)}(c_j) H_{i_j}(t) dt + \int_{\alpha_1}^{\alpha_2} \int_{\alpha_1}^{\alpha_2} \Theta_i(G(u,t)) G_{H,m-2}(t,s) f^{(m)}(s) ds dt, \quad i = 1, 2, \ldots, 35. \tag{9.57}$$

Proof. Using (9.94) and (9.2) and following the linearity of $\Theta_i(.)$, we have

$$\Theta_i(f(u)) = \int_{\alpha_1}^{\alpha_2} \Theta_i(G(u,t)) f''(t) dt. \tag{9.58}$$

By Theorem 9.8, $f''(t)$ can be expressed as

$$f''(t) = \sum_{j=1}^{l} \sum_{i=0}^{k_j} H_{i_j}(t) f^{(i+2)}(c_j) + \int_{\alpha_1}^{\alpha_2} G_{H,m-2}(t,s) f^{(m)}(s) ds. \tag{9.59}$$

Using (9.59) in (9.58), we get (9.57).

Theorem 9.13 *Let* $f : [\alpha_1, \alpha_2] \to \mathbb{R}$ *be a function. Also let* $x_1, \ldots, x_n \in [\alpha_1, \alpha_2]$ *and* $p_1, \ldots, p_n$ *be positive real numbers such that* $\sum_{i=1}^{n} p_i = 1$. *Also let* $\alpha_1 = c_1 < c_2 < \ldots < c_l = \alpha_2$ $(l \geq 2)$ *be the points and* $f \in C^m([\alpha_1, \alpha_2])$. *Furthermore let* H_{i_j}, $G_{H,m}$ *and* G *be as* *defined in* (9.48), (9.49) *and* (9.93) *respectively. Let* $f : [\alpha_1, \alpha_2] \to \mathbb{R}$ *be m-convex function and*

$$\int_{\alpha_1}^{\alpha_2} \Theta_i(G(u,t))G_{H,m-2}(t,s)dt \geq 0 \quad t \in [\alpha_1, \alpha_2], \quad i = 1,2,\ldots,35. \tag{9.60}$$

Then

$$\Theta_i(f(u)) \geq \int_{\alpha_1}^{\alpha_2} \Theta_i\left(G(u,t)\right) \sum_{j=1}^{l} \sum_{i=0}^{k_j} f^{(i+2)}(c_j)H_{i_j}(u)du, \quad i = 1,2,\ldots,35. \tag{9.61}$$

Proof. Since the function f is m-convex therefore $f^{(m)}(u) \geq 0$ for all $u \in [\alpha_1, \alpha_2]$. Hence by applying Theorem 9.12 we obtain (9.61).

Theorem 9.14 *Let* $f : [\alpha_1, \alpha_2] \to \mathbb{R}$ *be a function. Also let* $x_1, \ldots, x_n \in [\alpha_1, \alpha_2]$ *and* $p_1, \ldots, p_n \in (0, \infty)$ *be such that* $\sum_{i=1}^{n} p_i = 1$. *Also let* $\alpha_1 = c_1 < c_2 < \ldots < c_l = \alpha_2$ $(l \geq 2)$ *be the points and* $f \in C^m([\alpha_1, \alpha_2])$. *Let* $f : [\alpha_1, \alpha_2] \to \mathbb{R}$ *be m-convex function. Then the following holds.*
(i) *If* k_j *is odd for each* $j = 2, \ldots, l$ *then* (9.61) *holds.*
(ii) *Let the inequality* (9.61) *be satisfied*

$$F(.) = \sum_{j=1}^{l} \sum_{i=0}^{k_j} f^{(i+2)}(c_j)H_{i_j}(.) \tag{9.62}$$

is non-negative. Then $\Theta_i(f(u)) \geq 0$, *i = 1, 2, ..., 35.*

Proof. (i) Since $G(u,t)$ is convex and weights are positive, so $\Theta_i\left(G(u,t)\right) \geq 0$. Also as k_j is odd for all $j = 2, \ldots, l$, therefore $\omega(t) \geq 0$ and by using Lemma 9.1 (i), we have $G_{H,m-2}(u,s) \geq 0$ so (9.60) holds. Now using Theorem 9.13 we have (9.61).
(ii) Using (9.62) in (9.61), we get $\Theta_i(f(u)) \geq 0$. For the particular case of Hermite conditions, we can give the following corollaries to above Theorem 9.14. By using type $(q, m-q)$ conditions we give the following results.

Corollary 9.4 *Let* τ_i, η_i *be as defined in* (9.51) *and* (9.52) *respectively. Let* $f : [\alpha_1, \alpha_2] \to \mathbb{R}$ *be m-convex function.*
(i) *If* $m-q$ *is even, then the inequality*

$$\Theta_i(f(u)) \geq \int_{\alpha_1}^{\alpha_2} \Theta_i\left(G(u,t)\right) \left(\sum_{i=0}^{q-1} \tau_i(t) f^{(i+2)}(\alpha_1) + \sum_{i=0}^{m-q-1} \eta_i(t) f^{(i+2)}(\alpha_2) \right) dt,$$

holds for $i = 1,2,\ldots,35$.
(ii) *Let the inequality* (9.63) *be satisfied*

$$F(\cdot) = \sum_{i=0}^{q-1} \tau_i(\cdot) f^{(i+2)}(\alpha_1) + \sum_{i=0}^{m-q-1} \eta_i(\cdot) f^{(i+2)}(\alpha_2)$$

is non-negative. Then $\Theta_i(f(u)) \geq 0$, *i = 1, 2, ..., 35.*

Two points Taylor conditions give help to derive following results.

Corollary 9.5 *Let $f:[\alpha_1,\alpha_2]\to\mathbb{R}$ be m-convex function.*
(i) If m is even, then

$$\Theta_i(f(u)) \geq \int_{\alpha_1}^{\alpha_2}\Theta_i(G(u,t))\sum_{i=0}^{q-1}\sum_{k=0}^{q-i-1}\binom{q+k-1}{k}\left[\frac{(t-\alpha_1)^i}{i!}\left(\frac{t-\alpha_2}{\alpha_1-\alpha_2}\right)^q\left(\frac{t-\alpha_1}{\alpha_2-\alpha_1}\right)^k f^{(i+2)}(\alpha_1)\right.$$
$$\left.+\frac{(t-\alpha_2)^i}{i!}\left(\frac{t-\alpha_1}{\alpha_2-\alpha_1}\right)^q\left(\frac{t-\alpha_2}{\alpha_1-\alpha_2}\right)^k f^{(i+2)}(\alpha_2)\right]dt,\quad i=1,2,\ldots,35.$$

(ii) Let the inequality (9.63) *be satisfied and*

$$F(t)=\sum_{i=0}^{q-1}\sum_{k=0}^{q-i-1}\binom{q+k-1}{k}\left[\frac{(t-\alpha_1)^i}{i!}\left(\frac{t-\alpha_2}{\alpha_1-\alpha_2}\right)^q\left(\frac{t-\alpha_1}{\alpha_2-\alpha_1}\right)^k f^{(i+2)}(\alpha_1)\right.$$
$$\left.+\frac{(t-\alpha_2)^i}{i!}\left(\frac{t-\alpha_1}{\alpha_2-\alpha_1}\right)^q\left(\frac{t-\alpha_2}{\alpha_1-\alpha_2}\right)^k f^{(i+2)}(\alpha_2)\right]$$

is non-negative. Then $\Theta_i(f(u))\geq 0$, i = 1, 2, ..., 35.

9.4 Generalization of Refinement of Jensen's, Rényi and Shannon Type Inequalities via Lidstone Polynomial

In this section refinement of Jensen's inequality is generalized for higher order convex functions using Lidstone interpolating polynomial.

Lemma 9.2 *If $g\in C^\infty([0,1])$, then*

$$g(u)=\sum_{l=0}^{m-1}\left[g^{(2l)}(0)\mathfrak{F}_l(1-u)+g^{(2l)}(0)\mathfrak{F}_l(t)\right]+\int_0^1 G_m(u,s)g^{(2m)}(s)ds$$

where $\mathfrak{F}_l$ is a polynomial of degree $2l+1$ defined by the relation

$$\mathfrak{F}_0(u)=u,\ \mathfrak{F}''_m(u)=\mathfrak{F}_{m-1}(u),\ \mathfrak{F}_m(0)=\mathfrak{F}_m(1)=0,\ m\geq 1, \tag{9.63}$$

and

$$G_1(u,s)=G(u,s)=\begin{cases}(u-1)s, & \alpha_1\leq s\leq u\leq\alpha_2;\\ (s-1)u, & \alpha_1\leq u\leq s\leq\alpha_2,\end{cases}$$

is a homogeneous Green function of the differential operator $\frac{d^2}{d^2s}$ *on* $[0,1]$*, and with the iterates of* $G(u,s)$

$$G_m(u,s) = \int_0^1 G_1(u,p)G_{m-1}(p,s)dp, \quad m \geq 2.$$

The Lidstone polynomial can be expressed in terms of $G_m(u,s)$ as

$$\mathfrak{F}_m(u) = \int_0^1 G_m(u,s)sds.$$

Lidstone series representation of $g \in C^{2m}[\alpha_1,\alpha_2]$ is given by

$$\begin{aligned} g(u) &= \sum_{l=0}^{m-1}(\alpha_2-\alpha_1)^{2l}g^{(2l)}(\alpha_1)\mathfrak{F}_l\left(\frac{\alpha_2-u}{\alpha_2-\alpha_1}\right)+\sum_{l=0}^{m-1}(\alpha_2-\alpha_1)^{2l}g^{(2l)}(\alpha_2)\mathfrak{F}_l\left(\frac{u-\alpha_1}{\alpha_2-\alpha_1}\right)\\ &+ (\alpha_2-\alpha_1)^{2l-1}\int_{\alpha_1}^{\alpha_2}G_m\left(\frac{u-\alpha_1}{\alpha_2-\alpha_1},\frac{t-\alpha_1}{\alpha_2-\alpha_1}\right)g^{(2l)}(t)dt. \end{aligned} \tag{9.64}$$

We construct some new identities with the help of generalized Lidstone polynomial (9.64).

Theorem 9.15 *Let* $f:[\alpha_1,\alpha_2]\to\mathbb{R}$ *be a function be such that* $f \in C^{2m}[\alpha_1,\alpha_2]$ *for* $m \geq 1$*. Also let* $x_1,\ldots,x_n \in [\alpha_1,\alpha_2]$ *and* $p_1,\ldots,p_n \in (0,\infty)$ *such that* $\sum_{i=1}^{n} p_i = 1$*, and* $\mathfrak{F}_m(t)$ *be the same as defined in* (9.63)*, then*

$$\begin{aligned} \Theta_i(f) &= \sum_{k=1}^{m-1}(\alpha_2-\alpha_1)^{2k}f^{(2k)}(\alpha_1)\Theta_i\left(\mathfrak{F}_l\left(\frac{\alpha_2-x}{\alpha_2-\alpha_1}\right)\right)\\ &+\sum_{k=1}^{m-1}(\alpha_2-\alpha_1)^{2k}f^{(2k)}(\alpha_2)\Theta_i\left(\mathfrak{F}_l\left(\frac{x-\alpha_1}{\alpha_2-\alpha_1}\right)\right)\\ &+(\alpha_2-\alpha_1)^{2k-1}\int_{\alpha_1}^{\alpha_2}\Theta_i\left(G_m\left(\frac{x-\alpha_1}{\alpha_2-\alpha_1},\frac{t-\alpha_1}{\alpha_2-\alpha_1}\right)\right)f^{(2m)}(t)dt, \quad i=1,2,\ldots,35. \end{aligned} \tag{9.65}$$

Proof. Using (9.64) in place of f in $\Theta_i(f)$, $i=1,2,\ldots,35$, we get (9.65).

Theorem 9.16 *Let* $f:[\alpha_1,\alpha_2]\to\mathbb{R}$ *be a function such that* $f \in C^{2m}[\alpha_1,\alpha_2]$ *for* $m \geq 1$*. Also let* $x_1,\ldots,x_n \in [\alpha_1,\alpha_2]$ *and* $p_1,\ldots,p_n \in (0,\infty)$ *be such that* $\sum_{i=1}^{n} p_i = 1$*, and* $\mathfrak{F}_m(t)$ *be the same as defined in* (9.63)*, let for* $m \geq 1$

$$\Theta_i\left(G_m\left(\frac{x-\alpha_1}{\alpha_2-\alpha_1},\frac{t-\alpha_1}{\alpha_2-\alpha_1}\right)\right) \geq 0, \text{ for all } t \in [\alpha_1,\alpha_2]. \tag{9.66}$$

If f *is* $2m$*-convex function then we have*

$$\begin{aligned} \Theta_i(f) &\geq \sum_{k=1}^{m-1}(\alpha_2-\alpha_1)^{2k}f^{(2k)}(\alpha_1)\Theta_i\left(\mathfrak{F}_l\left(\frac{\alpha_2-x}{\alpha_2-\alpha_1}\right)\right)\\ &+\sum_{k=1}^{m-1}(\alpha_2-\alpha_1)^{2k}f^{(2k)}(\alpha_2)\Theta_i\left(\mathfrak{F}_l\left(\frac{x-\alpha_1}{\alpha_2-\alpha_1}\right)\right), \quad i=1,2,\ldots,35. \end{aligned} \tag{9.67}$$

Proof. Since f is $2m$-convex therefore $f^{(2m)} \geq 0$ for all $x \in [\alpha_1, \alpha_2]$, then by using (9.66) in (9.65) we get (9.67).

Theorem 9.17 *Let $f : [\alpha_1, \alpha_2] \to \mathbb{R}$ be a function. Also let $x_1, \ldots, x_n \in [\alpha_1, \alpha_2]$ and $p_1, \ldots, p_n \in (0, \infty)$ be such that $\sum_{i=1}^{n} p_i = 1$, also let $f : [\alpha_1, \alpha_2] \to \mathbb{R}$ be $2m$-convex function then following results are valid.*
(i) If m is odd integer, then for every $2m$-convex function (9.67) *holds.*
(ii) Suppose (9.67) *holds, if the function*

$$\lambda(u) = \sum_{l=0}^{m-1} (\alpha_2 - \alpha_1)^{2l} g^{(2l)}(\alpha_1) \mathfrak{F}_l \left(\frac{\alpha_2 - u}{\alpha_2 - \alpha_1} \right) + \sum_{l=0}^{m-1} (\alpha_2 - \alpha_1)^{2l} g^{(2l)}(\alpha_2) \mathfrak{F}_l \left(\frac{u - \alpha_1}{\alpha_2 - \alpha_1} \right)$$

is convex, then the smaller side of (9.67) *is non-negative and we have*

$$\Theta_i(f) \geq 0, \quad i = 1, 2, \ldots, 35. \tag{9.68}$$

Proof. (i) Note that $G_1(u,s) \leq 0$ for $1 \leq u, s, \leq 1$ and also note that $G_m(u,s) \leq 0$ for odd integer m and $G_m(u,s) \geq 0$ for even integer m. As G_1 is convex function and G_{m-1} is positive for odd integer m, therefore

$$\frac{d^2}{d^2u}(G_m(u,s)) = \int_0^1 \frac{d^2}{d^2u} G_1(u,p) G_{m-1}(p,s) dp \geq 0, \; m \geq 2.$$

This shows that G_m is convex in the first variable u if m is convex. Similarly G_m is concave in the first variable if m is even. Hence if m is odd then

$$\Theta_i \left(G_m \left(\frac{x - \alpha_1}{\alpha_2 - \alpha_1}, \frac{t - \alpha_1}{\alpha_2 - \alpha_1} \right) \right) \geq 0,$$

therefore (9.68) is valid.
(ii) By using linearity of $\Theta_i(f)$ we can write the smaller side of (9.67) in the form $\Theta_i(\lambda)$. As λ is supposed to be convex therefore the smaller side of (9.67) is non-negative, so $\Theta_i(f) \geq 0$.

9.5 Generalization of Refinement of Jensen's, Rényi and Shannon Type Inequalities via Fink Identity and Abel-Gontscharoff Green Fuction

In [41], A. M. Fink gave the following result.

Let $f : [\alpha_1, \alpha_2] \to \mathrm{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous then we have the following identity:

$$f(z) = \frac{n}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} f(\zeta) d\zeta$$

$$+\sum_{\lambda=1}^{n-1}\frac{n-\lambda}{\lambda!}\left(\frac{f^{(\lambda-1)}(\alpha_2)(z-\alpha_2)^{\lambda}-f^{(\lambda-1)}(\alpha_1)(z-\alpha_1)^{\lambda}}{\alpha_2-\alpha_1}\right)$$
$$+\frac{1}{(n-1)!(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2}(z-\zeta)^{n-1}F_{\alpha_1}^{\alpha_2}(\zeta,z)f^{(n)}(\zeta)\,d\zeta, \tag{9.69}$$

where

$$F_{\alpha_1}^{\alpha_2}(\zeta,z)=\begin{cases}\zeta-\alpha_1, & \alpha_1\le\zeta\le z\le\alpha_2;\\ \zeta-\alpha_2, & \alpha_1\le z<\zeta\le\alpha_2.\end{cases} \tag{9.70}$$

The complete reference about Abel-Gontscharoff polynomial and theorem for 'two-point right focal' problem is given in [15].

The Abel-Gontscharoff polynomial for 'two-point right focal' interpolating polynomial for $n=2$ can be given as

$$f(z)=f(\alpha_1)+(z-\alpha_1)f'(\alpha_2)+\int_{\alpha_1}^{\alpha_2}G_1(z,w)f''(w)\,dw, \tag{9.71}$$

where

$$G_1(z,w)=\begin{cases}\alpha_1-w, & \alpha_1\le w\le z;\\ \alpha_1-z, & z\le w\le\alpha_2.\end{cases} \tag{9.72}$$

In [31], S. I. Butt et al. gave some new types of Green functions defined as

$$G_2(z,w)=\begin{cases}\alpha_2-z, & \alpha_1\le w\le z;\\ \alpha_2-w, & z\le w\le\alpha_2,\end{cases} \tag{9.73}$$

$$G_3(z,w)=\begin{cases}z-\alpha_1, & \alpha_1\le w\le z;\\ w-\alpha_1, & z\le w\le\alpha_2,\end{cases} \tag{9.74}$$

$$G_4(z,w)=\begin{cases}\alpha_2-w, & \alpha_1\le w\le z;\\ \alpha_2-z, & z\le w\le\alpha_2.\end{cases} \tag{9.75}$$

They also introduced some new Abel-Gontscharoff type identities by using these new Green functions in the next result.

Lemma 9.3 *Let $f:[\alpha_1,\alpha_2]$ be a function such that f'' exists and G_k ($k=2,3,4$) be the two-point right focal problem-type Green functions defined by (9.73)-(9.75). Then the following identities hold:*

$$f(z)=f(\alpha_2)-(\alpha_2-z)f'(\alpha_1)-\int_{\alpha_1}^{\alpha_2}G_2(z,w)f''(w)\,dw, \tag{9.76}$$

$$f(z)=f(\alpha_2)-(\alpha_2-\alpha_1)f'(\alpha_2)+(z-\alpha_1)f'(\alpha_1)+\int_{\alpha_1}^{\alpha_2}G_3(z,w)f''(w)\,dw, \tag{9.77}$$

$$f(z)=f(\alpha_1)+(\alpha_2-\alpha_1)f'(\alpha_1)-(\alpha_2-z)f'(\alpha_2)+\int_{\alpha_1}^{\alpha_2}G_4(z,w)f''(w)\,dw. \tag{9.78}$$

Theorem 9.18 *Let $f : [\alpha_1, \alpha_2] \to \mathrm{R}$ be a function such that for $m \geq 3$ (an integer) $f^{(m-1)}$ is absolutely continuous. Also, let $x_1, \ldots, x_n \in [\alpha_1, \alpha_2]$, $p_1, \ldots, p_n \in (0, \infty)$ be such that $\sum_{i=1}^{n} p_i = 1$. Assume that $F_{\alpha_1}^{\alpha_2}$, G_k $(k = 1,2,3,4)$ and Θ_i $(i = 1, \ldots, 35)$ are the same as defined in (9.70), (9.72)-(9.75), (9.1), (9.2),* (9.4)-(9.38) *respectively. Then:*

(i) For $k = 1,3,4$ we have the following identities:

$$\begin{aligned}\Theta_i(f) &= (m-2)\left(\frac{f'(\alpha_2) - f'(\alpha_1)}{\alpha_2 - \alpha_1}\right)\int_{\alpha_1}^{\alpha_2}\Theta_i\big(G_k(\cdot,w)\big)\,dw \\ &+ \frac{1}{\alpha_2 - \alpha_1}\int_{\alpha_1}^{\alpha_2}\Theta_i\big(G_k(\cdot,w)\big) \\ &\times \sum_{\lambda=1}^{m-3}\left(\frac{m-2-\lambda}{\lambda!}\right)\big(f^{(\lambda+1)}(\alpha_2)(w-\alpha_2)^{\lambda} - f^{(\lambda+1)}(\alpha_1)(w-\alpha_1)^{\lambda}\big)\,dw \\ &+ \frac{1}{(m-3)!(\alpha_2 - \alpha_1)}\int_{\alpha_1}^{\alpha_2} f^{(m)}(\zeta) \\ &\times \left(\int_{\alpha_1}^{\alpha_2}\Theta_i\big(G_k(\cdot,w)\big)(w-\zeta)^{m-3}F_{\alpha_1}^{\alpha_2}(\zeta,w)\,dw\right)d\zeta, \quad i = 1,\ldots,35.\end{aligned} \tag{9.79}$$

(ii) For $k = 2$ we have

$$\begin{aligned}\Theta_i(f) &= (-1)(m-2)\left(\frac{f'(\alpha_2) - f'(\alpha_1)}{\alpha_2 - \alpha_1}\right)\int_{\alpha_1}^{\alpha_2}\Theta_i\big(G_2(\cdot,w)\big)\,dw \\ &+ \frac{(-1)}{\alpha_2 - \alpha_1}\int_{\alpha_1}^{\alpha_2}\Theta_i\big(G_2(\cdot,w)\big) \\ &\times \sum_{\lambda=1}^{m-3}\left(\frac{m-2-\lambda}{\lambda!}\right)\big(f^{(\lambda+1)}(\alpha_2)(w-\alpha_2)^{\lambda} - f^{(\lambda+1)}(\alpha_1)(w-\alpha_1)^{\lambda}\big)\,dw \\ &+ \frac{(-1)}{(m-3)!(\alpha_2 - \alpha_1)}\int_{\alpha_1}^{\alpha_2} f^{(m)}(\zeta) \\ &\times \left(\int_{\alpha_1}^{\alpha_2}\Theta_i\big(G_2(\cdot,w)\big)(w-\zeta)^{m-3}F_{\alpha_1}^{\alpha_2}(\zeta,w)\,dw\right)d\zeta, \quad i = 1,\ldots,35.\end{aligned} \tag{9.80}$$

Proof. (i) Using Abel-Gontsharoff-type identities (9.71), (9.77), (9.78) in $\Theta_i(f)$, $i = 1, \ldots, 35$, and using properties of $\Theta_i(f)$, we get

$$\Theta_i(f) = \int_{\alpha_1}^{\alpha_2}\Theta_i\big(G_k(\cdot,w)\big)f''(w)\,dw, \quad i = 1,2. \tag{9.81}$$

From identity (9.69), we get

$$f'(w) = (m-2)\left(\frac{f'(\alpha_2) - f'(\alpha_1)}{\alpha_2 - \alpha_1}\right)$$

$$+\sum_{\lambda=1}^{m-3}\left(\frac{m-2-\lambda}{\lambda!}\right)\left(\frac{f^{(\lambda)}(\alpha_2)(w-\alpha_2)^{\lambda-1}-f^{(\lambda)}(\alpha_2)(w-\alpha_2)^{\lambda-1}}{\alpha_2-\alpha_1}\right)$$
$$+\frac{1}{(m-3)!(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2}(w-\zeta)^{m-3}F_{\alpha_1}^{\alpha_2}(\zeta,w)f^{(m)}(\zeta)\,d\zeta. \tag{9.82}$$

Using (9.81) and (9.82) and applying Fubini's theorem we get the result (9.79) for $k=1,3,4$.

(ii) Substituting Abel-Gontsharoff-type inequality (9.76) in $\Theta_i(f)$, $i=1,\ldots,35$, and following similar steps to (i), we get (9.80).

Theorem 9.19 *Let $f: I=[\alpha_1,\alpha_2]\to\mathbb{R}$ be a function such that for $m\geq 3$ (an integer) $f^{(m-1)}$ is absolutely continuous. Also, let $x_1,\ldots,x_n\in I$, $p_1,\ldots,p_n$ be positive real numbers such that $\sum_{i=1}^{n}p_i=1$. Assume that $F_{\alpha_1}^{\alpha_2}$, G_k ($k=1,2,3,4$) and Θ_i ($i=1,2$) are the same as defined in (9.70), (9.72)-(9.75) (9.1), (9.2), (9.4)-(9.38) respectively. For $m\geq 3$ assume that*

$$\int_{\alpha_1}^{\alpha_2}\Theta_i\big(G_k(\cdot,\zeta)\big)(w-\zeta)^{m-3}F_{\alpha_1}^{\alpha_2}(\zeta,w)\,dw\geq 0,\quad \zeta\in[\alpha_1,\alpha_2],\quad i=1,\ldots,35. \tag{9.83}$$

If f is an m-convex function, then

(i) For $k=1,3,4$, the following holds:

$$\begin{aligned}\Theta_i(f)\geq{}&(m-2)\left(\frac{f'(\alpha_2)-f'(\alpha_1)}{\alpha_2-\alpha_1}\right)\int_{\alpha_1}^{\alpha_2}\Theta_i\big(G_k(\cdot,w)\big)\,dw\\&+\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\Theta_i\big(G_k(\cdot,w)\big)\\&\times\sum_{\lambda=1}^{m-3}\left(\frac{m-2-\lambda}{\lambda!}\right)\big(f^{(\lambda+1)}(\alpha_2)(w-\alpha_2)^{\lambda}\\&-f^{(\lambda+1)}(\alpha_1)(w-\alpha_1)^{\lambda}\big)\,dw,\quad i=1,\ldots,35.\end{aligned} \tag{9.84}$$

(ii) For $k=2$, we have

$$\begin{aligned}\Theta_i(f)\leq{}&(-1)(m-2)\left(\frac{f'(\alpha_2)-f'(\alpha_1)}{\alpha_2-\alpha_1}\right)\int_{\alpha_1}^{\alpha_2}\Theta_i\big(G_2(\cdot,w)\big)\,dw\\&+\frac{(-1)}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\Theta_i\big(G_2(\cdot,w)\big)\\&\times\sum_{\lambda=1}^{m-3}\left(\frac{m-2-\lambda}{\lambda!}\right)\big(f^{(\lambda+1)}(\alpha_2)(w-\alpha_2)^{\lambda}\\&-f^{(\lambda+1)}(\alpha_1)(w-\alpha_1)^{\lambda}\big)\,dw,\quad i=1,\ldots,35.\end{aligned} \tag{9.85}$$

Proof. (i) Since $f^{(m-1)}$ is absolutely continuous on $[\alpha_1,\alpha_2]$, $f^{(m)}$ exists almost everywhere. Also, since f is m-convex therefore we have $f^{(m)}(\zeta)\geq 0$ for a.e. on $[\alpha_1,\alpha_2]$. So, applying Theorem 9.18, we obtain (9.84).

(ii) Similar to (i).

9.6 Generalization of Refinement of Jensen's, Rényi and Shannon type Inequalities via Taylor's one and two point Polynomials

In [29], the well known Taylor formula is given as follows:
Let m be a positive integer and $f:[\alpha_1,\alpha_2]\to\mathbb{R}$ be such that $f^{(m-1)}$ is absolutely continuous, then for all $u\in[\alpha_1,\alpha_2]$ the Taylor formula at point $c\in[\alpha_1,\alpha_2]$ is

$$f(u)=T_{m-1}(f;c;u)+R_{m-1}(f;c;u), \tag{9.86}$$

where

$$T_{m-1}(f;c;u)=\sum_{l=0}^{m-1}\frac{f^{(l)}(c)}{l!}(u-c)^l,$$

and the remainder is given by

$$R_{m-1}(f;c;u)=\frac{1}{(m-1)!}\int_c^u f^{(m)}(t)(u-t)_+^{m-1}dt,$$

for

$$(u-v)_+:=\begin{cases}(u-v), & v\le u;\\ 0, & v>u.\end{cases}$$

The Taylor formula at points α_1 and α_2 is given by:

$$f(u)=\sum_{l=0}^{m-1}\frac{f^{(l)}(\alpha_1)}{l!}(u-\alpha_1)^l+\frac{1}{(m-1)!}\int_{\alpha_1}^{\alpha_2}f^{(m)}(t)\left((u-t)_+^{m-1}\right)dt. \tag{9.87}$$

$$f(u)=\sum_{l=0}^{m-1}\frac{(-1)^l f^{(l)}(\alpha_2)}{l!}(\alpha_2-u)^l+\frac{(-1)^{m-1}}{(m-1)!}\int_{\alpha_1}^{\alpha_2}f^{(m)}(t)\left((t-u)_+^{m-1}\right)dt. \tag{9.88}$$

We construct some new identities with the help of Taylor polynomial (9.86).

Theorem 9.20 *Let $f:[\alpha_1,\alpha_2]\to\mathbb{R}$ be a function. Also let $x_1,\ldots,x_n\in[\alpha_1,\alpha_2]$ and $p_1,\ldots,p_n\in(0,\infty)$ be such that $\sum_{i=1}^{n}p_i=1$. Then for $i=1,\ldots,35$ we have the following identities:*
(i)

$$\Theta_i(f)=\sum_{l=2}^{m-1}\frac{f^{(l)}(\alpha_1)}{l!}\Theta_i\left((u-\alpha_1)^l\right)+\frac{1}{(m-1)!}\int_{\alpha_1}^{\alpha_2}f^{(m)}(t)\Theta_i\left((u-t)_+^{m-1}\right)dt. \tag{9.89}$$

(ii)

$$\Theta_i(f)=\sum_{l=2}^{m-1}\frac{(-1)^l f^{(l)}(\alpha_2)}{l!}\Theta_i\left((\alpha_2-u)^l\right)+\frac{(-1)^{m-1}}{(m-1)!}\int_{\alpha_1}^{\alpha_2}f^{(m)}(t)\Theta_i\left((t-u)_+^{m-1}\right)dt. \tag{9.90}$$

Proof. Using (9.87) and (9.88) in (9.2), we get (9.89) and (9.90).

Theorem 9.21 *Assume* (H_1)*, let* $f:[\alpha_1,\alpha_2]\to\mathbb{R}$ *be a function. Also let* $x_1,\ldots,x_n\in[\alpha_1,\alpha_2]$ *and* $p_1,\ldots,p_n\in(0,\infty)$ *be such that* $\sum_{i=1}^{n} p_i=1$*. Let* f *be a m-convex function such that* $f^{(m-1)}$ *is absolutely continuous. Then we have the following results:*
(i) *If*

$$\Theta_i\left((u-t)_+^{m-1}\right)\geq 0 \quad t\in[\alpha_1,\alpha_2],\ i=1,2,\ldots,35,$$

then

$$\Theta_i(f(u))\geq\sum_{l=2}^{m-1}\frac{f^{(l)}(\alpha_1)}{l!}\Theta_i\left((u-\alpha_1)^l\right),\ i=1,2,\ldots,35. \tag{9.91}$$

(ii) *If*

$$(-1)^{m-1}\Theta_i\left((t-u)_+^{m-1}\right)\leq 0 \quad t\in[\alpha_1,\alpha_2],\ i=1,2,\ldots,35,$$

then

$$\Theta_i(f(u))\geq\sum_{l=2}^{m-1}\frac{(-1)^l f^{(l)}(\alpha_2)}{l!}\Theta_i\left((\alpha_2-u)^l\right),\ i=1,2,\ldots,35. \tag{9.92}$$

Proof. Since $f^{(m-1)}$ is absolutely continuous on $[\alpha_1,\alpha_2]$, $f^{(m)}$ exists almost everywhere. As f is m-convex therefore $f^{(m)}(u)\geq 0$ for all $u\in[\alpha_1,\alpha_2]$. Hence using Theorem 9.20 we obtain (9.91) and (9.92).

Theorem 9.22 *Let* $f:[\alpha_1,\alpha_2]\to\mathbb{R}$ *be a function. Also let* $x_1,\ldots,x_n\in[\alpha_1,\alpha_2]$ *and* $p_1,\ldots,p_n\in(0,\infty)$ *be such that* $\sum_{i=1}^{n} p_i=1$*. Then we have the following three results.*

(i) *If* f *is m-convex, then* (9.91) *holds. Also if* $f^{(l)}(\alpha_1)\geq 0$ *for* $l=2,\ldots,m-1$*, then the smaller side of* (9.91) *will be non-negative.*
(ii) *If* m *is even and* f *is m-convex, then* (9.92) *holds. Also if* $f^{(l)}(\alpha_1)\leq 0$ *for* $l=2,\ldots,m-1$ *and* $f^{(l)}\geq 0$ *for* $l=3,\ldots,m-1$*, then smaller side of* (9.92) *will be non-negative.*
(iii)*If* m *is odd and* f *is m-convex function then* (9.92) *is valid. Also if* $f^{(l)}(\alpha_2)\geq 0$ *for* $l=2,\ldots,m-1$ *and* $f^{(l)}(\alpha_2)\leq 0$ *for* $l=2,\ldots,m-2$*, then smaller side of* (9.92) *will be non positive.*

In [32, p.20] the Green function $G:[\alpha_1,\alpha_2]\times[\alpha_1,\alpha_2]\to\mathbb{R}$ is defined as

$$G(u,v)=\begin{cases}\frac{(u-\alpha_2)(v-\alpha_1)}{\alpha_2-\alpha_1}, & \alpha_1\leq v\leq u;\\ \frac{(v-\alpha_2)(u-\alpha_1)}{\alpha_2-\alpha_1}, & u\leq v\leq\alpha_2.\end{cases} \tag{9.93}$$

We can check that G is convex as well as continuous function with respect to both variables u and v. The function G is convex and continuous with respect to v, since G is symmetric

therefore it is also convex and continuous with respect to variable u.
Let $\psi \in C^2([\alpha_1,\alpha_2])$, then

$$\psi(t) = \frac{\alpha_2 - t}{\alpha_2 - \alpha_1}\psi(\alpha_1) + \frac{t-\alpha_1}{\alpha_2-\alpha_1}\psi(\alpha_2) + \int_{\alpha_1}^{\alpha_2} G(t,v)\,\psi''(v)dv. \tag{9.94}$$

Theorem 9.23 *Let $f : [\alpha_1,\alpha_2] \to \mathbb{R}$ be a function. Also let $x_1,\ldots,x_n \in [\alpha_1,\alpha_2]$ and $p_1,\ldots,p_n \in (0,\infty)$ be such that $\sum_{i=1}^{n} p_i = 1$. Then we have the following results:*
(i) For $i = 1,2,\ldots,35$,

$$\Theta_i(f) = \int_{\alpha_1}^{\alpha_2} \Theta_i(G(t,v)) \left(\sum_{l=1}^{n-1} \frac{f^{(l)}(\alpha_1)(v-\alpha_1)^{l-2}}{(l-2)!}\right) dv + \frac{1}{(n-3)!}\int_{\alpha_1}^{\alpha_2} f^{(m)}(s)\left(\int_{\alpha_1}^{\alpha_2} \Theta_i(G(t,v))(v-s)^{n-3}dv\right) ds. \tag{9.95}$$

(ii) For $i = 1,2,\ldots,35$,

$$\Theta_i(f) = \int_{\alpha_1}^{\alpha_2} \Theta_i(G(t,v)) \left(\sum_{l=1}^{n-1} \frac{f^{l}(\alpha_2)(v-\alpha_2)^{l-2}}{(l-2)!}\right) dv - \frac{1}{(n-3)!}\int_{\alpha_1}^{\alpha_2} f^{(m)}(s)\left(\int_{\alpha_1}^{\alpha_2} \Theta_i(G(t,v))(v-s)^{n-3}dv\right) ds. \tag{9.96}$$

Proof. Using (9.94) in $\Theta_i, i = 1,2,\ldots,35$, we get

$$\Theta_i(f) = \int_{\alpha_1}^{\alpha_2} \Theta_i(G(t,v))\, f''(v)dv. \tag{9.97}$$

Differentiating (9.87) twice, we get

$$f''(v) = \sum_{l=2}^{n-1} \frac{f^{(l)}(\alpha_1)}{(l-2)!}(v-\alpha_1)^{l-2} + \frac{1}{(m-3)!}\int_{\alpha_1}^{\alpha_2} f^{(m)}(v-u)^{m-3}du. \tag{9.98}$$

Using (9.98) in (9.97) and using Fubini's theorem, we get (9.95). Similarly using second derivative of (9.88) in (9.97) and applying Fubini's theorem, we get (9.96). The next result contains the generalization of refinement of Jensen's inequality for higher order convex function. Now we obtain generalization of refinement of Jensen's inequality for n-convex function.

Theorem 9.24 *Let $f : [\alpha_1, \alpha_2] \to \mathbb{R}$ be a function. Also let $x_1, \dots, x_n \in [\alpha_1, \alpha_2]$ and $p_1, \dots, p_n \in (0, \infty)$ be such that $\sum_{i=1}^{n} p_i = 1$. Let f is m-convex function such that $f^{(m-1)}$ is absolutely continuous. Then we have the following results:*
(i) If

$$\int_{u}^{\alpha_2} \Theta_i \left(G(t,v)\right) (v-u)^{n-3} dv \geq 0 \;\; u \in [\alpha_1, \alpha_2], \;\; i = 1,2,\dots,35, \tag{9.99}$$

then

$$\Theta_i(f) \geq \int_{\alpha_1}^{\alpha_2} \Theta_i \left(G(t,v)\right) \left(\sum_{l=2}^{n-2} \frac{f^{(l)}(\alpha_1)(v-\alpha_1)^{l-2}}{(l-2)!} \right) dv, \;\; i = 1,2,\dots,35, \tag{9.100}$$

and if

$$\int_{\alpha_1}^{u} \Theta_i \left(G(t,v)\right) (v-u)^{n-3} dv \leq 0 \;\; u \in [\alpha_1, \alpha_2], \;\; i = 1,2,\dots,35, \tag{9.101}$$

then

$$\Theta_i(f) \geq \int_{\alpha_1}^{\alpha_2} \Theta i \left(G(t,v)\right) \left(\sum_{l=2}^{n-2} \frac{f^{(l)}(\alpha_2)(v-\alpha_2)^{l-2}}{(l-2)!} \right) dv \;\; i = 1,2,\dots,35. \tag{9.102}$$

Proof. It can be proved in similar manner as Theorem 9.21.

Corollary 9.6 *Let $f : [\alpha_1, \alpha_2] \to \mathbb{R}$ be a function. Also let $x_1, \dots, x_n \in [\alpha_1, \alpha_2]$ and $p_1, \dots, p_n \in (0, \infty)$ be such that $\sum_{i=1}^{n} p_i = 1$. Then the following two results are valid.*

(i) Suppose f is m-convex function, then (9.100) *holds. Also if*

$$\sum_{l=2}^{n-1} \frac{f^{(l)}(\alpha_1)(v-\alpha_1)^{l-2}}{(l-2)!} \geq 0, \tag{9.103}$$

then

$$\Theta_i(f) \geq 0, \quad i = 1,2,\dots,35. \tag{9.104}$$

(ii) If m is even and f is m-convex, then (9.102) *holds. Also if*

$$\sum_{l=2}^{n-1} \frac{f^{(l)}(\alpha_2)(v-\alpha_2)^{l-2}}{(l-2)!} \geq 0, \tag{9.105}$$

then (9.104) *holds.*

Remark 9.4 We can discover the bounds for the identities given in (9.43), (9.45), (9.55), (9.57), (9.65), (9.79), (9.80), (9.89) and (9.90). By using and some new results related to the Grüss and Ostrowski type inequalities can be constructed by using inequalities for the Čebyšev functional as given in Section 3 of [29]. Also we can construct the non-negative functionals using inequalities (9.44), (9.46), (9.56), (9.61), (9.67), (9.84), (9.85), (9.91) and (9.92) and give related mean value theorems and we can construct the new families of m-exponentially convex functions and Cauchy means related to these functionals as given in Section 4 of [29].

9.7 Bounds for the Identities Related to Generalization of Refinement of Jensen's Inequality

For two Lebesgue integrable functions $f_1, f_2 : [\alpha_1, \alpha_2] \to \mathbb{R}$, the Čebyšev functional [29] is defined as

$$\Omega(f_1, f_2)$$

$$= \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2} f_1(t)f_2(t)dt - \frac{1}{(\alpha_2-\alpha_1)^2}\int_{\alpha_1}^{\alpha_2} f_1(t)dt \cdot \int_{\alpha_1}^{\alpha_2} f_2(t)dt, \tag{9.106}$$

where the integrals are assumed to exist.

In [34], the following theorems are given.

Theorem 9.25 *Suppose functions $f_1, f_2 : [\alpha_1, \alpha_2] \to \mathbb{R}$, where f_1 is a Lebesgue integrable function and f_2 is an absolutely continuous function such that $(.-\alpha_1)(.-\alpha_2)[f_2']^2 \in L[\alpha_1, \alpha_2]$. Then the following inequality holds:*

$$|\Omega(f_1, f_2)| \le \frac{1}{\sqrt{2}}[\Omega(f_1, f_1)]^{\frac{1}{2}} \frac{1}{\sqrt{\beta-\alpha}}\left(\int_{\alpha_1}^{\alpha_2}(x-\alpha_1)(\alpha_2-x)[f_2'(x)]^2 dx\right)^{\frac{1}{2}}. \tag{9.107}$$

The constant $\frac{1}{\sqrt{2}}$ in (9.107) *is the best possible.*

Theorem 9.26 *Suppose functions $f_1, f_2 : [\alpha_1, \alpha_2] \to \mathbb{R}$, such that f_1 is absolutely continuous with $f_1' \in L_\infty[\alpha_1, \alpha_2]$ and f_2 is a monotonic non-decreasing on interval $[\alpha_1, \alpha_2]$. Then the following inequality holds:*

$$|\Omega(f_1, f_2)| \le \frac{1}{2(\alpha_2-\alpha_1)}||f_1'||_\infty \int_{\alpha_1}^{\alpha_2}(x-\alpha_1)(\alpha_2-x)[f_2'(x)]^2 df_2(x), \tag{9.108}$$

where the constant $\frac{1}{2}$ is best possible in (9.108).

Now we consider Theorem 9.25 and Theorem 9.26 to generalize results given in previous section. Let us first denote for $\zeta \in [\alpha_1, \alpha_2]$

$$\mathscr{K}(\zeta) = \int_{\alpha_1}^{\alpha_2} \Theta_i(G_k(\cdot,w)(w-\zeta)^{n-3} F_{\alpha_1}^{\alpha_2}(\zeta,w)dw \quad k=1,3,4., \tag{9.109}$$

$$\hat{\mathscr{K}}(\zeta) = (-1)\int_{\alpha_1}^{\alpha_2} \Theta_i(G_2(\cdot,w))(w-\zeta)^{n-3} F_{\alpha_1}^{\alpha_2}(\zeta,w)dw, \; i=1,2. \tag{9.110}$$

In the next two results bounds are investigated for the generalized identities related to the refinement of Jensen's inequality using Theorem 9.25 and Theorem 9.26.

The following results investigate the bounds for the identities related to generalization of refinement of Jensen's inequality using inequalities for Čebyšev function given in Theorem 9.25 and Theorem 9.26.

Theorem 9.27 *Let $m \geq 3$ be an integer, and $f : [\alpha_1, \alpha_2] \to \mathbb{R}$ be a function such that $f^{(m)}$ is absolutely continuous with $(.-\alpha_1)(\alpha_2-.)[f^{(m+1)}]^2 \in L[\alpha_1, \alpha_2]$. Let $p_1, \ldots, p_n \in (0,\infty)$ be such that $\sum_{i=1}^n p_i = 1$. Also, assume $F_{\alpha_1}^{\alpha_2}$ and $\Theta_i (i=1,2)$ are the same as defined in* (9.70) *and* (9.1)-(9.2) *respectively. Then*
(i) for $G_k(\cdot,w)(k=1,3,4)$ as defined in (9.72), (9.74) *and* (9.75) *respectively, we have*

$$\begin{aligned}\Theta_i(f) &= (m-2)\left(\frac{f'(\alpha_2)-f'(\alpha_1)}{\alpha_2-\alpha_1}\right)\int_{\alpha_1}^{\alpha_2}\Theta_i(G_k(\cdot,w))dw + \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\Theta_i(G_k(\cdot,w)) \\ &\times \sum_{\lambda=1}^{m-3}\left(\frac{m-2-\lambda}{\lambda!}\right)\left(f^{(\lambda+1)}(\alpha_2)(w-\alpha_2)^{\lambda} - f^{(\lambda+1)}(\alpha_1)(w-\alpha_1)^{\lambda}\right)dw \\ &+ \frac{f^{(m-1)}(\alpha_2)-f^{(m-1)}(\alpha_1)}{(m-3)!(\alpha_2-\alpha_1)^2}\int_{\alpha_1}^{\alpha_2}\mathscr{K}(\zeta)d\zeta + \mathscr{R}_m^1(\alpha_1,\alpha_2;f), \; i=1,2,\end{aligned} \tag{9.111}$$

where the remainder $\mathscr{R}_m^1(\alpha_1,\alpha_2;f)$ satisfies the bound

$$\begin{aligned}|\mathscr{R}_m^1(\alpha_1,\alpha_2;f)| \leq \frac{1}{\sqrt{2}(m-3)!}[\Omega(\mathscr{K},\mathscr{K})]^{\frac{1}{2}}\frac{1}{\sqrt{\alpha_2-\alpha_1}} \\ \left(\int_{\alpha_1}^{\alpha_2}(\zeta-\alpha_1)(\alpha_2-\zeta)[f^{(m+1)}(\zeta)]^2 d\zeta\right)^{\frac{1}{2}}.\end{aligned} \tag{9.112}$$

(ii) for $G_2(z,w)$ as defined in (9.73)*, we have*

$$\begin{aligned}\Theta_i(f) &= (-1)(m-2)\left(\frac{f'(\alpha_2)-f'(\alpha_1)}{\alpha_2-\alpha_1}\right)\int_{\alpha_1}^{\alpha_2}\Theta_i(G_2(\cdot,w))dw + \frac{(-1)}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\Theta_i(G_2(\cdot,w)) \\ &\times \sum_{\lambda=1}^{m-3}\left(\frac{m-2-\lambda}{\lambda!}\right)\left(f^{(\lambda+1)}(\alpha_2)(w-\alpha_2)^{\lambda} - f^{(\lambda+1)}(\alpha_1)(w-\alpha_1)^{\lambda}\right)dw \\ &+ \frac{f^{(m-1)}(\alpha_2)-f^{(m-1)}(\alpha_1)}{(m-3)!(\alpha_2-\alpha_1)^2}\int_{\alpha_1}^{\alpha_2}\mathscr{K}(\zeta)d\zeta + \mathscr{R}_m^2(\alpha_1,\alpha_2;f), \; i=1,2,\end{aligned} \tag{9.113}$$

where the remainder $\mathscr{R}_m^2(\alpha_1,\alpha_2;f)$ satisfies the bound

$$|\mathscr{R}_m^2(\alpha_1,\alpha_2;f)| \leq \frac{1}{\sqrt{2}(m-3)!}[\Omega(\hat{\mathscr{K}},\mathscr{K})]^{\frac{1}{2}}\frac{1}{\sqrt{\alpha_2-\alpha_1}}$$
$$\left(\int_{\alpha_1}^{\alpha_2}(\zeta-\alpha_1)(\alpha_2-\zeta)[f^{(m+1)}(\zeta)]^2 d\zeta\right)^{\frac{1}{2}}.$$

Proof. (i) Setting $f_1 \mapsto \mathscr{K}$ and $f_2 \mapsto f^{(m)}$ in Theorem 9.25, we get

$$\left|\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\mathscr{K}(\zeta)f^{(m)}(\zeta)d\zeta - \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\mathscr{K}(\zeta)d\zeta\cdot\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}f^{(m)}(\zeta)d\zeta\right|$$
$$\leq \frac{1}{\sqrt{2}}[\Omega(\mathscr{K},\mathscr{K})]^{\frac{1}{2}}\frac{1}{\sqrt{\alpha_2-\alpha_1}}\left(\int_{\alpha_1}^{\alpha_2}(\zeta-\alpha_1)(\alpha_2-\zeta)[f^{(m+1)}(\zeta)]^2 d\zeta\right)^{\frac{1}{2}}.$$

Hence, we have

$$\frac{1}{(m-3)!(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2}\mathscr{K}(\zeta)f^{(m)}d\zeta$$
$$= \frac{f^{(m-1)}(\alpha_2)-f^{(m-1)}(\alpha_1)}{(m-3)!(\alpha_2-\alpha_1)^2}\int_{\alpha_1}^{\alpha_2}\mathscr{K}(\zeta)d\zeta + \mathscr{R}_m^1(\alpha_1,\alpha_2;f)$$

where the remainder satisfies the estimation (9.112). Using identity (9.79) we get (9.111).
(ii) Similar to the above part. The Grüss type inequalities can be obtained by using Theorem 9.26.

Theorem 9.28 *Let $m \geq 3$ be an integer, $f : [\alpha_1,\alpha_2] \to \mathbb{R}$ be a function such that $f^{(m)}$ is absolutely continuous function and $f^{(m+1)} \geq 0$ a.e on $[\alpha_1,\alpha_2]$ and let the function $\mathscr{K}$ and $\hat{\mathscr{K}}$ be defined as in* (9.109) *and* (9.110)*. Then we have*
(i) *identity* (9.111) *where the remainder satisfies the estimation*

$$|\mathscr{R}_m^1(\alpha_1,\alpha_2;f)| \leq \frac{1}{(m-3)!}||\mathscr{K}'||_\infty$$
$$\left[\frac{f^{(m-1)}(\alpha_2)+f^{(m-1)}(\alpha_1)}{2} - \frac{f^{(m-1)}(\alpha_2)-f^{8(m-1)}(\alpha_1)}{\alpha_2-\alpha_1}\right].$$

(ii) *identity* (9.113) *where the remainder satisfies the estimation*

$$|\mathscr{R}_m^1(\alpha_1,\alpha_2;f)| \leq \frac{1}{(m-3)!}||\hat{\mathscr{K}}||_\infty$$
$$\left[\frac{f^{(m-1)}(\alpha_2)+f^{(m-1)}(\alpha_1)}{2} - \frac{f^{(m-1)}(\alpha_2)-f^{(m-1)}(\alpha_1)}{\alpha_2-\alpha_1}\right].$$

Proof. (i) Setting $f_1 \mapsto \mathscr{K}$ and $f_2 \mapsto f^{(m)}$ in Theorem 9.26, we get

$$\left|\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\mathscr{K}(\zeta)f^{(m)}(\zeta)d\zeta - \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\mathscr{K}(\zeta)d\zeta\cdot\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}f^{(m)}(\zeta)d\zeta\right|$$

$$\leq \frac{1}{2}||\mathscr{K}'||_\infty \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}(\zeta-\alpha_1)(\alpha_2-\zeta)[f^{(m+1)}(\zeta)]^2 d\zeta. \tag{9.114}$$

Since

$$\begin{aligned}&\textstyle\int_{\alpha_1}^{\alpha_2}(\zeta-\alpha_1)(\alpha_2-\zeta)[f^{(m+1)}(\zeta)]^2 d\zeta = \int_{\alpha_1}^{\alpha_2}[2\zeta-\alpha_1-\alpha_2]f^m(\zeta)d\zeta\\ &= (\alpha_2-\alpha_1)[f^{(m-1)}(\alpha_2)+f^{(m-1)}(\alpha_1)]-2(f^{(m-1)}(\alpha_2)-f^{(m-1)}(\alpha_1)),\end{aligned} \tag{9.115}$$

using (9.79), (9.114) and (9.115), we have (9.114).
(ii) Similar to above part.

Theorem 9.29 *Let $f:[\alpha_1,\alpha_2]\to\mathbb{R}$ be a function such that $f^{(m-1)}$ is absolutely continuous, let $x_1,\ldots,x_n\in[\alpha_1,\alpha_2]$, $p_1,\ldots,p_n\in(0,\infty)$ be such that $\sum_{i=1}^n p_i=1$. Also, let $F_{\alpha_1}^{\alpha_2}$, $G_k(k=1,2,3,4)$ and $\Theta_i(i=1,2)$ are the same as defined in* (9.70), (9.72)-(9.75) *and* (9.1)-(9.2) *respectively. Moreover, assume (p,q) such that $1\leq p,q,\leq\infty$, $\frac{1}{p}+\frac{1}{q}=1$ (pair of conjugate exponent). Let $|f^{(m)}|^p:[\alpha_1,\alpha_2]\to\mathbb{R}$ be Riemann integrable function. Then*
(i) for $k=1,3,4$, we have

$$\begin{aligned}&\left|\Theta_i(f)-\int_{\alpha_1}^{\alpha_2}\left[(m-2)\left(\frac{f'(\alpha_2)-f'(\alpha_2)}{\alpha_2-\alpha_1}\right)+\frac{1}{\alpha_2-\alpha_1}\sum_{\lambda=1}^{m-3}\left(\frac{m-2-\lambda}{\lambda}\right)\right.\right.\\ &\qquad\left.\left.\left(f^{(\lambda+1)}(\alpha_2)(w-\alpha_2)-f^{(\lambda+1)}(\alpha_2)(w-\alpha_2)\right)\right]\Theta_i(G_k(\cdot,w))dw\right|\\ &\leq\frac{1}{(\alpha_2-\alpha_1)(m-3)!}||f^{(m)}||_p\left(\int_{\alpha_1}^{\alpha_2}\left|\int_{\alpha_1}^{\alpha_2}\Theta_i(G_k(\cdot,w))(w-\zeta)^{m-3}F_{\alpha_1}^{\alpha_2}(\zeta,w)dw\right|^q\right)^{\frac{1}{q}},\\ &\qquad\qquad i=1,2.\end{aligned} \tag{9.116}$$

(ii) for $k=2$, we have

$$\begin{aligned}&\left|\Theta_i(f)-\int_{\alpha_1}^{\alpha_2}\left[(-1)(m-2)\left(\frac{f'(\alpha_2)-f'(\alpha_2)}{\alpha_2-\alpha_1}\right)+\frac{1}{\alpha_2-\alpha_1}\sum_{\lambda=1}^{m-3}\left(\frac{m-2-\lambda}{\lambda}\right)\right.\right.\\ &\qquad\left.\left.(f^{(\lambda+1)}(\alpha_2)(w-\alpha_2)-f^{(\lambda+1)}(\alpha_2)(w-\alpha_2))\right]\Theta_i(G_2(\cdot,w))dw\right|\\ &\leq\frac{(-1)}{(\alpha_2-\alpha_1)(m-3)!}||f^{(m)}||_p\left(\int_{\alpha_1}^{\alpha_2}\left|\int_{\alpha_1}^{\alpha_2}\Theta_i(G_2(\cdot,w))(w-\zeta)^{m-3}F_{\alpha_1}^{\alpha_2}(\zeta,w)dw\right|^q\right)^{\frac{1}{q}},\\ &\qquad\qquad i=1,2.\end{aligned} \tag{9.117}$$

Proof. It can be proved similarly as Theorem 3.5 in [29].

Remark 9.5 We can construct the non-negative functional by taking the differences of (9.79) and (9.80). By using the idea of Section 4 and Section 5 of [28], we can construct new class of n-exponential convexity. A new class of monotonic Cauchy means can be constructed by using the related mean value theorem.

Remark 9.6 Following the similar way as above we can find the bounds related to the generalized f-divergence, Shannon and Rényi type inequalities. Further by using the generalized f-divergence, Shannon and Rényi we can construct the results related to Grüss and Owstrowski type inequalities.

Remark 9.7 Analogous to the results presented in this chapter by using different interpolating polynomials we may use inequalities (2.1), (2.27), (2.30) and (2.50) to present related results.

Chapter 10

Integral form of Popoviciu's Inequality for Convex Functions

This chapter contains a new integral form of Popoviciu's inequality. We construct new refinement for the integral Jensen's inequality. Also a new class of quasi-arithmetic means along with their monotonicity property is given. These results are published in paper (see [61]).

10.1 Integral form of Popoviciu's Inequality

First consider some hypotheses which we use in our work given as follow.
(H_1) Let $(X,\mathscr{E},\mu)$ be a probability space, suppose $p_1,\ldots,p_n \in (0,\infty)$ such that $\sum_{i=1}^{n} p_i = 1$.
(H_2) Let $h : X \to I \subset \mathbb{R}$ be an integrable function.
(H_3) Let g be a convex function on interval I, suppose such that the composition $g \circ h$ is integrable.
Let $m \geq 2$ be a fixed integer. The σ-algebra in X^k generated by the projection mapping $pr_l : X^m \to X$ $(l = 1,\ldots,m)$

$$pr_l(x_1,\ldots,x_m) := x_l \tag{10.1}$$

is denoted by $\mathscr{E}^k$. And μ^m is defined as the product measure on $\mathscr{E}$, this measure is uniquely (μ is σ-finite) specified by

$$\mu^m(B_1 \times \ldots \times B_m) := \mu(B_1)\ldots\mu(B_k), \quad B_l \in \mathscr{E}, \quad l = 1, \ldots, m. \tag{10.2}$$

Theorem 10.1 *Assume (H_1)-(H_3), then the following inequalities hold.*

(i)

$$\frac{1}{C_{m-1}^{n-1}} \sum_{1\le i_1<\ldots<i_m\le n} \left(\sum_{j=1}^{m} p_{i_j}\right) \int_{X^m} g\left(\frac{\sum_{j=1}^{m} p_{i_j} h(x_{i_j})}{\sum_{j=1}^{m} p_{i_j}}\right) d\mu^m(x_{i_1},\ldots,x_{i_m})$$
$$\le \frac{n-m}{n-1} \sum_{i=1}^{n} p_i \int_X g(h(x_i))d\mu(x_i)$$
$$+ \frac{m-1}{n-1} \int_{X^n} g\left(\sum_{i=1}^{n} p_i h(x_i)\right) d\mu^n(x_{i_1},\ldots,x_{i_n}).$$

(ii)

$$\frac{1}{C_{m-1}^{n-1}} \sum_{1\le i_1<\ldots<i_m\le n} \left(\sum_{j=1}^{m} p_{i_j}\right) \int_{X^m} g\left(\frac{\sum_{j=1}^{m} p_{i_j} h(x_{i_j})}{\sum_{j=1}^{m} p_{i_j}}\right) d\mu^m(x_{i_1},\ldots,x_{i_m})$$
$$\le \sum_{i=1}^{n} p_i \int_X g(h(x_i))d\mu(x_i).$$

Proof. (i) On integrating the inequality (1.24) over X^n and replacing x_{i_j} by $h(x_{i_j})$, we have

$$\frac{1}{C_{m-1}^{n-1}} \sum_{1\le i_1<\ldots<i_m\le n} \left(\sum_{j=1}^{m} p_{i_j}\right) \int_{X^n} g\left(\frac{\sum_{j=1}^{m} p_{i_j} h(x_{i_j})}{\sum_{j=1}^{m} p_{i_j}}\right) d\mu^n(x_{i_1},\ldots,x_{i_n})$$
$$\le \frac{n-m}{n-1} \sum_{i=1}^{n} p_i \int_{X^n} g(h(x_i))d\mu^n(x_{i_1},\ldots,x_{i_n})$$
$$+ \frac{m-1}{n-1} \int_{X^n} g\left(\sum_{i=1}^{n} p_i h(x_i)\right) d\mu^n(x_{i_1},\ldots,x_{i_n}).$$

On simplification we have

$$\frac{1}{C_{m-1}^{n-1}} \sum_{1\le i_1<\ldots<i_m\le n} \left(\sum_{j=1}^{m} p_{i_j}\right) \int_{X^m} g\left(\frac{\sum_{j=1}^{m} p_{i_j} h(x_{i_j})}{\sum_{j=1}^{m} p_{i_j}}\right) d\mu^m(x_{i_1},\ldots,x_{i_m})$$
$$\times \int_X d\mu(x_{i_{m+1}}) \ldots \int_X d\mu(x_{i_n})$$
$$\le \frac{n-m}{n-1} \sum_{i=1}^{n} p_i \int_X g\circ h(x_i) d\mu(x_i) \times \int_X d\mu(x_{i_1}) \ldots \int_X d\mu(x_{i_m}) \int_X d\mu(x_{i_{m+1}}) \ldots \int_X d\mu(x_{i_n})$$
$$+ \frac{m-1}{n-1} \int_{X^n} g\left(\sum_{i=1}^{n} p_i h(x_i)\right) d\mu^n(x_{i_1},\ldots,x_{i_n}).$$

This gives

$$\frac{1}{C_{m-1}^{n-1}} \sum_{1\le i_1<\ldots<i_m\le n} \left(\sum_{j=1}^{m} p_{i_j}\right) \int_{X^m} g\left(\frac{\sum_{j=1}^{m} p_{i_j} h(x_{i_j})}{\sum_{j=1}^{m} p_{i_j}}\right) d\mu^m(x_{i_1},\ldots,x_{i_m})$$
$$\le \frac{n-m}{n-1} \sum_{i=1}^{n} p_i \int_X g(h(x_i)) d\mu(x_i)$$
$$+ \frac{m-1}{n-1} \int_{X^n} g\left(\sum_{i=1}^{n} p_i h(x_i)\right) d\mu^n(x_{i_1},\ldots,x_{i_n})$$

(ii) Applying discrete Jensen's inequality to the last term of inequality given in (i) and on solving, we have

$$\frac{1}{C_{m-1}^{n-1}} \sum_{1\le i_1<\ldots<i_m\le n} \left(\sum_{j=1}^{m} p_{i_j}\right) \int_{X^m} g\left(\frac{\sum_{j=1}^{m} p_{i_j} h(x_{i_j})}{\sum_{j=1}^{m} p_{i_j}}\right) d\mu^m(x_{i_1},\ldots,x_{i_m})$$
$$\le \frac{n-m}{n-1} \sum_{i=1}^{n} p_i \int_X g(h(x_i)) d\mu(x_i)$$
$$+ \frac{m-1}{n-1} \left(p_1 \int_{X^n} g(h(x_1)) d\mu^n(x_{i_1},\ldots,x_{i_n}) + \ldots + p_n \int_{X^n} g(h(x_n)) d\mu^n(x_{i_1},\ldots,x_{i_n}) \right),$$

this gives

$$\frac{1}{C_{m-1}^{n-1}} \sum_{1 \le i_1 < \ldots < i_m \le n} \left(\sum_{j=1}^{m} p_{i_j} \right) \int_{X^m} g \left(\frac{\sum_{j=1}^{m} p_{i_j} h(x_{i_j})}{\sum_{j=1}^{m} p_{i_j}} \right) d\mu^m(x_{i_1}, \ldots, x_{i_m})$$

$$\le \sum_{i=1}^{n} p_i \int_X g(h(x_i)) d\mu(x_i).$$

10.2 New Refinement of the Integral form of Jensen's Inequality

Under the hypothesis (H_1), (H_2) and (H_3), define the function $H_m(t)$ on $[0,1]$ given by

$$H_m(t) = \frac{1}{C_{m-1}^{n-1}} \sum_{1 \le i_1 < \ldots < i_m \le n} \left(\sum_{j=1}^{m} p_{i_j} \right) \times \int_{X^m} g \left(t \frac{\sum_{j=1}^{m} p_{i_j} h(x_{i_j})}{\sum_{j=1}^{m} p_{i_j}} + (1-t) \int_X h d\mu \right) d\mu^m(x_{i_1}, \ldots, x_{i_m}) \quad (10.3)$$

Theorem 10.2 *Assume* (H_1)-(H_3), *then*

(i) H_m *is convex.*

(ii) $\min_{t \in [0,1]} H_m(t) = H_m(0) = g\left(\int_X h d\mu \right)$

(iii) $\max_{t \in [0,1]} H_m(t) = H_m(1)$

(iv) H_m *is increasing.*

Proof. (i) Suppose $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$ and $u, v \in [0,1]$, then from (10.3) we have

$$H_m(\alpha u + \beta v) = \frac{1}{C_{m-1}^{n-1}} \sum_{1 \le i_1 < \ldots < i_m \le n} \left(\sum_{j=1}^{m} p_{i_j} \right) \times \int_{X^m} g \left((\alpha u + \beta v) \frac{\sum_{j=1}^{m} p_{i_j} h(x_{i_j})}{\sum_{j=1}^{m} p_{i_j}} + (\alpha + \beta - \alpha u - \beta v) \int_X h d\mu \right) d\mu^m(x_{i_1}, \ldots, x_{i_m}).$$

On simplification we have

$$H_m(\alpha u+\beta v)=\frac{1}{C_{m-1}^{n-1}}\sum_{1\le i_1<\ldots<i_m\le n}\left(\sum_{j=1}^{m}p_{i_j}\right)$$

$$\times\int_{X^m} g\left(\alpha\left(u\frac{\sum_{j=1}^{m}p_{i_j}h(x_{i_j})}{\sum_{j=1}^{m}p_{i_j}}+(1-u)\int_X hd\mu\right)+\right.$$

$$\left.\beta\left(v\frac{\sum_{j=1}^{m}p_{i_j}h(x_{i_j})}{\sum_{j=1}^{m}p_{i_j}}+(1-v)\int_X hd\mu\right)\right)d\mu^m(x_{i_1},\ldots,x_{i_m}).$$

By convexity of g, we have

$$H_m(\alpha u+\beta v)\le\alpha\frac{1}{C_{m-1}^{n-1}}\sum_{1\le i_1<\ldots<i_m\le n}\left(\sum_{j=1}^{m}p_{i_j}\right)$$

$$\times\int_{X^m} g\left(u\frac{\sum_{j=1}^{m}p_{i_j}h(x_{i_j})}{\sum_{j=1}^{m}p_{i_j}}+(1-u)\int_X gd\mu\right)d\mu^m(x_{i_1},\ldots,x_{i_m})+$$

$$\beta\frac{1}{C_{m-1}^{n-1}}\sum_{1\le i_1<\ldots<i_m\le n}\left(\sum_{j=1}^{m}p_{i_j}\right)\int_{X^m} g\left(v\frac{\sum_{j=1}^{m}p_{i_j}h(x_{i_j})}{\sum_{j=1}^{m}p_{i_j}}+(1-v)\int_X hd\mu\right)d\mu^m(x_{i_1},\ldots,x_{i_m}),$$

that is

$$H_m(\alpha u+\beta v)\le\alpha H_m(u)+\beta H_m(v).$$

Therefore H_m is convex function.
(ii) Integral from of Jensen's inequality yields

$$H_m(t)\ge\frac{1}{C_{m-1}^{n-1}}\sum_{1\le i_1<\ldots<i_m\le n}\left(\sum_{j=1}^{m}p_{i_j}\right)$$

$$\times g\left(\int_{X^m}\left(t\frac{\sum_{j=1}^{m}p_{i_j}h(x_{i_j})}{\sum_{j=1}^{m}p_{i_j}}+(1-t)\int_X gd\mu\right)d\mu^m(x_{i_1},\ldots,x_{i_m})\right)$$

or

$$H_m(t)\ge\frac{1}{C_{m-1}^{n-1}}\sum_{1\le i_1<\ldots<i_m\le n}\left(\sum_{j=1}^{m}p_{i_j}\right)g\left(I\right) \tag{10.4}$$

where

$$\begin{aligned}
I &= \int\limits_{X^m} \left(t \frac{\sum\limits_{j=1}^{m} p_{i_j} h(x_{i_j})}{\sum\limits_{j=1}^{m} p_{i_j}} + (1-t) \int\limits_X g d\mu \right) d\mu^m(x_{i_1}, \ldots, x_{i_m}) \\
&= t \int\limits_{X^m} \frac{\sum\limits_{j=1}^{m} p_{i_j} h(x_{i_j})}{\sum\limits_{j=1}^{m} p_{i_j}} d\mu^m(x_{i_1}, \ldots, x_{i_m}) + (1-t) \int\limits_{X^m} \left(\int\limits_X g d\mu \right) d\mu^m(x_{i_1}, \ldots, x_{i_m}) \\
&= \frac{t}{\sum\limits_{j=1}^{m} p_{i_j}} \sum_{i=1}^{m} p_i \int\limits_X g d\mu + (1-t) \int\limits_X g d\mu \\
&= \int\limits_X g d\mu
\end{aligned}$$

so from (10.4), we have

$$H_m(t) \geq \frac{1}{C_{m-1}^{n-1}} \sum_{1 \leq i_1 < \ldots < i_m \leq n} \left(\sum_{j=1}^{m} p_{i_j} \right) g \left(\int\limits_X g d\mu \right) = H_m(0), \qquad \forall\, t \in [0,1].$$

(iii)

$$\begin{aligned}
H_m(t) = H_m(1.t + (1-t)0) &\leq tH_m(1) + (1-t)H_m(0) \\
&\leq tH_m(1) + (1-t)H_m(1) \\
&= H_m(1), \quad \forall\, t \in [0,1].
\end{aligned}$$

(iv) Since $H_m(t)$ is convex and $H_m(t) \geq H_m(0) (t \in [0,1])$, therefore for $0 \leq t_1 < t_2 \leq 1$, we have

$$\frac{H_m(t_2) - H_m(t_1)}{t_2 - t_1} \geq \frac{H_m(t_2) - H_m(0)}{t_2} \geq 0,$$

so

$$H_m(t_2) \geq H_m(t_1).$$

Theorem 10.3 *Assume (H_1), (H_2) and (H_3), then*

$$g\left(\int\limits_X h d\mu \right) \leq H_m(t) \leq H_m(1) \leq \int\limits_X g \circ h d\mu. \tag{10.5}$$

Proof. Using (ii) and (iii) of Theorem 10.2 we get first two inequalities, and for the last inequality

$$H_m(1) = \frac{1}{C_{m-1}^{n-1}} \sum_{1\le i_1<\ldots<i_m\le n} \left(\sum_{j=1}^{m} p_{i_j}\right) \times \int_{X^m} g\left(\frac{\sum_{j=1}^{m} p_{i_j} h(x_{i_j})}{\sum_{j=1}^{m} p_{i_j}}\right) d\mu^m(x_{i_1},\ldots,x_{i_m}).$$

Using discrete Jensen's inequality, we have

$$H_m(1) \le \frac{1}{C_{m-1}^{n-1}} \sum_{1\le i_1<\ldots<i_m\le n} \left(\sum_{j=1}^{m} p_{i_j}\right) \times \frac{\sum_{j=1}^{m} p_{i_j}}{\sum_{j=1}^{m} p_{i_j}} \int_{X^m} g\left(h(x_{i_j})\right) d\mu^m(x_{i_1},\ldots,x_{i_m}),$$

this gives

$$H_m(1) \le \int_X g\circ h d\mu.$$

Remark 10.1 A refinement similar to (10.5) of integral form of Jensen's inequality is proved in Proposition 7 of [49].

10.3 New Quasi-Arithmetic Means

Now we introduce some new quasi arithmetic means. First we assume the following conditions:
(H_4) Let $h : X \to I$, where $I \subset \mathbb{R}$ be an interval, is measurable.
(H_5) Let α and β be two real valued, continuous and strictly monotone functions defined on interval I.

Definition 10.1 *Assume (H_1), (H_4) and (H_5).*
For $t \in [0,1]$ we define the class of quasi-arithmetic means given by

$$M_{\lambda,\chi}(t,g,\mu) := \chi^{-1}\left(\frac{1}{C_{m-1}^{n-1}} \sum_{1\le i_1<\ldots<i_m\le n} \left(\sum_{j=1}^{m} p_{i_j}\right)\right.$$
$$\left.\times \int_{X^m} \chi\circ\lambda^{-1}\left(t\frac{\sum_{j=1}^{m} p_{i_j}\lambda(g(x_{i_j}))}{\sum_{j=1}^{m} p_{i_j}} + (1-t)\int_X \lambda(g)d\mu\right) d\mu^m(x_{i_1},\ldots,x_{i_m})\right) \quad (10.6)$$

where the integrals are supposed to exist.

Assume (H_6), let η be a real valued, continuous and strictly monotone function defined on interval I such that the composition $\eta \circ h$ is integrable on X, and $M_\eta(h,\mu)$ is the quasi-arithmetic mean defined in (2.16).

Theorem 10.4 *Assume (H_1),(H_4), (H_5) and assume that $\lambda \circ h$ and $\chi \circ h$ are integrable on X.*
(i) If $\chi \circ \lambda^{-1}$ is convex with χ is increasing or $\chi \circ \lambda^{-1}$ is concave with χ is decreasing, then

$$M_\lambda(h,\mu) \leq M_{\chi,\lambda}(t,h,\mu) \leq M_\chi(h,\mu), \tag{10.7}$$

holds for all $t \in [0,1]$.
(ii) If $\chi \circ \lambda^{-1}$ is convex with χ is decreasing or $\chi \circ \lambda^{-1}$ is concave with χ is increasing, then

$$M_\lambda(h,\mu) \geq M_{\chi,\lambda}(t,h,\mu) \geq M_\chi(h,\mu), \tag{10.8}$$

holds for all $t \in [0,1]$.

Proof. (i) Using pair of functions $\chi \circ \lambda^{-1}$ and $\lambda(h)$ ($\lambda(I)$ is an interval) in Theorem 10.3, we have

$$\chi \circ \lambda^{-1}\left(\int_X \lambda(h)d\mu\right) \leq \frac{1}{C_{m-1}^{n-1}} \sum_{1 \leq i_1 < \ldots < i_m \leq n} \left(\sum_{j=1}^m p_{i_j}\right)$$
$$\times \int_{X^m} \chi \circ \lambda^{-1}\left(t \frac{\sum_{j=1}^m p_{i_j}\lambda(h(x_{i_j}))}{\sum_{j=1}^m p_{i_j}} + (1-t)\int_X \lambda(h)d\mu\right) d\mu^m(x_{i_1},\ldots,x_{i_m})$$
$$\leq \frac{1}{C_{m-1}^{n-1}} \sum_{1 \leq i_1 < \ldots < i_m \leq n} \left(\sum_{j=1}^m p_{i_j}\right) \int_{X^m} \chi \circ \lambda^{-1}\left(\frac{\sum_{j=1}^m p_{i_j}\lambda(h)}{\sum_{j=1}^m p_{i_j}}\right) d\mu^m(x_{i_1},\ldots,x_{i_m})$$

Using the discrete Jensen inequality on the larger side of last inequality we get

$$\chi \circ \lambda^{-1}\left(\int_X \lambda(h)d\mu\right) \leq \frac{1}{C_{m-1}^{n-1}} \sum_{1 \leq i_1 < \ldots < i_m \leq n} \left(\sum_{j=1}^m p_{i_j}\right)$$
$$\times \int_{X^m} \chi \circ \lambda^{-1}\left(t \frac{\sum_{j=1}^m p_{i_j}\lambda(h(x_{i_j}))}{\sum_{j=1}^m p_{i_j}} + (1-t)\int_X \lambda(h)d\mu\right) d\mu^m(x_{i_1},\ldots,x_{i_m})$$
$$\leq \int_X \chi(h)d\mu$$

On taking χ^{-1} on both sides we have (10.7).
(ii) Similarly using the pair of functions $-\chi \circ \lambda^{-1}$ and $\lambda(h)$ in Theorem 1.14, where $\chi \circ \lambda^{-1}$ is concave. On taking χ^{-1} we have (10.8).

Chapter 11

Refinement of Jensen's Inequality for 3-convex Functions

We use (1.14) and establish the inequalities for classes of functions $\mathcal{K}_1^a(I)$ and $\mathcal{K}_2^a(I)$, as it was done for convex functions. We also improve these inequalities. These results can be found in [62].

11.1 Refinement of Jensen's Inequality for 3-convex Functions at a Point

In [25], I. A. Baloch *et al*. introduced the new classes of functions that are $\mathcal{K}_1^a(I)$ and $\mathcal{K}_2^a(I)$ given in the following definition.

Definition 11.1 *Let $f : I \to \mathbb{R}$ and $a \in I^\circ$ (I° denotes the interior of I). Consider the classes*

$$\mathcal{K}_1^a(I) := \Big\{ f : \text{there exists a real number } B \text{ such that} f(x) - \frac{B}{2}x^2 \text{is concave on } I \cap (-\infty, a] \text{and convex on} I \cap [a, \infty) \} \tag{11.1}$$

and

$$\mathscr{K}_2^a(I) := \Big\{ f : \text{there exists a real number } B \text{ such that} f(x) - \frac{B}{2}x^2 \text{is convex on } I \cap (-\infty, a] \text{and concave on} I \cap [a, \infty) \Big\}. \tag{11.2}$$

The function $f \in \mathscr{K}_1^a(I)$ is called 3-convex function at the point a. The function $f \in \mathscr{K}_2^a(I))$ is called 3-concave function at the point a.

They also showed that the $\mathscr{K}_1^a(I)(\mathscr{K}_2^a(I))$ is larger class of functions than the class of all 3-convex (3-concave) functions in the following result (see [25], Theorem 2.4).

Theorem 11.1 *If $g \in \mathscr{K}_1^a(I)(g \in \mathscr{K}_2^a(I))$ for every $a \in I$, then g is 3-convex (3-concave).*

In [25], I. A. Baloch *et al*. gave the Levinson inequality for the classes $\mathscr{K}_1^a(I)$ and $\mathscr{K}_2^a(I)$.

Theorem 11.2 *Let $I = [\alpha, \beta]$ be an interval. Consider $\boldsymbol{x} = (x_1, \ldots, x_n) \in [\alpha, \beta]^n$ and $\boldsymbol{y} = (y_1, \ldots, y_s) \in [\alpha, \beta]^s$. Also let there exist $a \in I$ such that*

$$\max_i x_i \leq a \leq \min_j y_j.$$

Suppose $\boldsymbol{p} = (p_1, \ldots, p_n) \in (0, \infty)^n$, $\boldsymbol{q} = (q_1, \ldots, q_s)^s \in (0, \infty)^s$ such that $\sum_{j=1}^{n} p_j = \sum_{i=1}^{s} q_i = 1$ and

$$\mathscr{A}_{m,r}(I_m, \boldsymbol{x}, \boldsymbol{p}, id^2) - \mathscr{A}_{m,k}(I_m, \boldsymbol{x}, \boldsymbol{p}, id^2) = \mathscr{A}_{m,r}(I_m, \boldsymbol{y}, \boldsymbol{q}, id^2) - \mathscr{A}_{m,k}(I_m, \boldsymbol{y}, \boldsymbol{q}, id^2). \tag{11.3}$$

If $f \in \mathscr{K}_1^a(I)$, then

$$\mathscr{A}_{m,r}(I_m, \boldsymbol{x}, \boldsymbol{p}, f) - \mathscr{A}_{m,k}(\boldsymbol{x}, \boldsymbol{p}, f) \leq \mathscr{A}_{m,r}(\boldsymbol{y}, \boldsymbol{q}, f) - \mathscr{A}_{m,k}(\boldsymbol{y}, \boldsymbol{q}, f) \tag{11.4}$$

holds for $\mathscr{A}_{m,r}$ defined in (1.13).

Proof. Since $H_1(x) := f(x) - \frac{B}{2}x^2$ is concave on $I \cap [\alpha, a]$, therefore from Remark 9.3, we have

$$\begin{aligned} 0 &\geq \mathscr{A}_{m,r}(\mathbf{x}, \mathbf{p}, H_1) - \mathscr{A}_{m,k}(\mathbf{x}, \mathbf{p}, H_1) \\ &= \mathscr{A}_{m,r}(\mathbf{x}, \mathbf{p}, f) - \mathscr{A}_{m,k}(\mathbf{x}, \mathbf{p}, f) - \frac{B}{2}\Big[\mathscr{A}_{m,r}(\mathbf{x}, \mathbf{p}, id^2) \\ &- \mathscr{A}_{m,k}(\mathbf{x}, \mathbf{p}, id^2)\Big]. \end{aligned} \tag{11.5}$$

As $H_2(y) := f(y) - \frac{B}{2}y^2$ is convex on $[a, \beta]$, therefore from Remark 9.3, we get

$$0 \leq \mathscr{A}_{m,r}(\mathbf{y}, \mathbf{p}, H_2) - \mathscr{A}_{m,k}(\mathbf{y}, \mathbf{p}, H_2)$$

$$= \mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},f) - \mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},f) - \frac{B}{2}\Big[\mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},id^2) - \mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},id^2)\Big]. \tag{11.6}$$

From (11.5) and (11.6), we have

$$\mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},f) - \mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},f) - \frac{B}{2}\Big[\mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},id^2) - \mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},id^2)\Big]$$
$$\le \mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},f) - \mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},f) - \frac{B}{2}\Big[\mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},id^2) - \mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},id^2)\Big].$$

Using the assumption (11.3), we get (11.4).

Corollary 11.1 *Let $I = [0,2a]$ be an interval, $\mathbf{x} = (x_1,\dots,x_n) \in [0,a]^n$, $\mathbf{y} = (y_1,\dots,y_m) \in [a,2a]^m$ and $\mathbf{p} = (p_1,\dots,p_n)$ be positive n-tuple such that $\sum_{j=1}^{n} p_j = 1$, also let*

$$x_1 + y_1 = \ldots = x_n + y_n = 2a.$$

If $f \in \mathscr{K}_1^a(I)$, then the inequality (11.4) *holds for $n = m$ and $\mathbf{p} = \mathbf{q}$.*

Proof. Note that

$$id^2\left(\frac{\sum_{j=1}^{k} p_{i_j} y_{i_j}}{\sum_{j=1}^{k} p_{i_j}}\right) = id^2\left(\frac{\sum_{j=1}^{k} p_{i_j}(c - x_{i_j})}{\sum_{j=1}^{k} p_{i_j}}\right) \tag{11.7}$$

$$= c^2 - 2c\frac{\sum_{j=1}^{k} p_{i_j} x_{i_j}}{\sum_{j=1}^{k} p_{i_j}} + \left(\frac{\sum_{j=1}^{k} p_{i_j} x_{i_j}}{\sum_{j=1}^{k} p_{i_j}}\right)^2. \tag{11.8}$$

We can observe that

$$\mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},id^2) - \mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},id^2) = \mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},id^2) - \mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},id^2).$$

Following the same steps as in Theorem 11.2, we get (11.4).

Remark 11.1 Using (11.5) and (11.6) from proof of Theorem 11.2, we have

$$\mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},f) - \mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},f) \le \frac{B}{2}\Big[\mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},id^2) - \mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},id^2)\Big] \tag{11.9}$$

and

$$\frac{B}{2}\left[\mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},id^2) - \mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},id^2)\right] \leq \mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},f) - \mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},f). \quad (11.10)$$

Using (11.9) and (11.10), we have the refinement of (11.4) given by

$$\mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},f) - \mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},f) \leq \frac{B}{2}\left[\mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},id^2) - \mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},id^2)\right]$$
$$\left(= \frac{B}{2}\left[\mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},id^2) - \mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},id^2)\right]\right) \leq \mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},f) - \mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},f).$$

The next result is the generalization of Theorem 11.2, with weaker assumptions on (11.3).

Theorem 11.3 *Let $I = [\alpha,\beta]$ be an interval, $\boldsymbol{x} = (x_1,\ldots,x_n) \in [\alpha,\beta]^n$, $\boldsymbol{y} = (y_1,\ldots,y_s) \in [\alpha,\beta]^s$ with*

$$\max_i x_i \leq \min_j y_j. \quad (11.11)$$

Also let $\boldsymbol{p} = (p_1,\ldots,p_n) \in (0,\infty)^n$, $\boldsymbol{q} = (q_1,\ldots,q_s) \in (0,\infty)^s$ such that $\sum_{j=1}^{n} p_j = \sum_{i=1}^{s} q_i = 1$ and $f \in \mathscr{K}_1^a(I)$ for some $a \in [\max x_i, \min y_j]$. Then if

(i)

$$f''_-(\max x_i) \geq 0$$

and

$$\mathscr{A}_{m,r}(\boldsymbol{x},\boldsymbol{p},id^2) - \mathscr{A}_{m,k}(\boldsymbol{x},\boldsymbol{p},id^2) \leq \mathscr{A}_{m,r}(\boldsymbol{y},\boldsymbol{q},id^2) - \mathscr{A}_{m,k}(\boldsymbol{y},\boldsymbol{q},id^2)$$

(ii)

$$f''_+(\min y_j) \leq 0$$

and

$$\mathscr{A}_{m,r}(\boldsymbol{x},\boldsymbol{p},id^2) - \mathscr{A}_{m,k}(\boldsymbol{x},\boldsymbol{p},id^2) \geq \mathscr{A}_{m,r}(\boldsymbol{y},\boldsymbol{q},id^2) - \mathscr{A}_{m,k}(\boldsymbol{y},\boldsymbol{q},id^2)$$

(iii) *$f''_-(\max x_i) < 0 < f''_+(\min y_j)$ and f is 3-convex,*

then (11.4) *holds.*

Proof. Since $f \in \mathscr{K}_1^a[\alpha,\beta]$ for some $a \in [\max x_i, \max y_j]$, therefore there exists a constant B such that $H_1(x) := f(x) - \frac{B}{2}x^2$, is concave on $[\alpha,a]$, such that for $x_1,\ldots,x_n \in I \cap [\alpha,a]$, we have

$$0 \geq \mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},H_1) - \mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},H_1),$$

that is

$$0 \geq \mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},f) - \mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},f) - \frac{B}{2}\left[\mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},id^2) - \mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},id^2)\right]. \quad (11.12)$$

Also $H_2(y) := f(y) - \frac{B}{2}y^2$ is convex on $[a,\beta]$, for $y_1,\dots,y_s \in [a,\beta]$, we have

$$0 \le \mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},H_2) - \mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},H_2),$$

that is

$$\begin{aligned} 0 \ge\ & \mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},f) - \mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},f) - \frac{B}{2}\Big[\mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},id^2) \\ -\ & \mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},id^2)\Big]. \end{aligned} \tag{11.13}$$

From (11.12) and (11.13), we have

$$\begin{aligned} &\mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},f) - \mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},f) - \frac{B}{2}\left[\mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},id^2) - \mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},id^2)\right] \\ &\le \mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},f) - \mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},f) - \frac{B}{2}\left[\mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},id^2) - \mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},id^2)\right]. \end{aligned}$$

So

$$\begin{aligned} &\frac{B}{2}\left[\mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},id^2) - \mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},id^2) - \mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},id^2) - \mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},id^2)\right] \\ &\qquad \le \mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},f) - \mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},f) - \mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},f) - \mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},f). \end{aligned} \tag{11.14}$$

Now due to concavity of H_1 and convexity of H_2 for every distinct point $\tilde{x}_j \in [\alpha, \max x_i]$ and $\tilde{y}_j \in [\min y_i, \beta]$, $j = 1,2,3$, we have

$$[\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,f] \le B \le [\tilde{y}_1,\tilde{y}_2,\tilde{y}_3,f]. \tag{11.15}$$

Letting $\tilde{x}_j \nearrow \max x_i$ and $\tilde{y}_j \searrow \min y_j$, we get the inequalities if derivatives exist

$$f''_{-}(\max x_i) \le B \le f''_{+}(\min y_i). \tag{11.16}$$

Since from assumption (a), $f''(\max x_i) \ge 0$, therefore $B \ge 0$, so using the assumption

$$\left[\mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},id^2) - \mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},id^2) - \mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},id^2) - \mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},id^2)\right] \ge 0,$$

the expression

$$\frac{B}{2}\left[\mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},id^2) - \mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},id^2) - \mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},id^2) - \mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},id^2)\right]$$

is non-negative and on using it on left side of (11.14) we have the result (11.4). And similarly for assumption (b), the inequality $f''_{+}(\min y_j) \le 0$ gives $B \le 0$, so the expression with assumption of (b) is also non-negative, this gives the result (11.4). Under the assumption of (c), f''_{-} and f''_{+} are both left and right continuous respectively and both are nondecreasing with $f''_{-} \le f''_{+}$, so their exists a point $\tilde{a} \in [\max x_i, \min y_j]$ such that $f \in \mathscr{K}_1^{\tilde{a}}[\alpha,\beta]$ with constant $\tilde{B} = 0$, and thus we have the inequality (11.4).

Remark 11.2 From the proof of Theorem 11.3, we have

$$\mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},f)-\mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},f)\leq \frac{B}{2}\left[\mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},id^2)-\mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},id^2)\right]$$

and

$$\mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},f)-\mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},f)\geq \frac{B}{2}\left[\mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},id^2)-\mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},id^2)\right].$$

In Theorem 11.3, B is positive, negative and zero for the assumptions (a), (b) and (c) respectively as discussed in proof. Therefore, we have the better improvement of (11.4) than (11.11) given as

$$\begin{aligned}\mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},f)-\mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},f)&\leq \frac{B}{2}\left[\mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},id^2)-\mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},id^2)\right]\\ \leq \frac{B}{2}\left[\mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},id^2)-\mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},id^2)\right]&\leq \mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},f)-\mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},f).\end{aligned}$$

If the assumptions of Theorem 11.2 with $f\in\mathscr{K}_2^a[\alpha,\beta]$, the reverse of inequality (11.4) holds. The generalization of this result is proven in the following result.

Theorem 11.4 *Let $I=[\alpha,\beta]\subset\mathbb{R}$ be an interval, $\boldsymbol{x}=(x_1,\ldots,x_n)\in[\alpha,\beta]^n$, $\boldsymbol{y}=(y_1,\ldots,y_s)\in[\alpha,\beta]^s$ with*

$$\max_i x_i\leq \min_j y_j. \tag{11.17}$$

Also let $\boldsymbol{p}=(p_1,\ldots,p_n)\in(0,\infty)^n$, $\boldsymbol{q}=(q_1,\ldots,q_s)\in(0,\infty)^s$ such that $\sum_{j=1}^{n}p_j=1=\sum_{i=1}^{s}q_i$ and $f\in\mathscr{K}_2^a(I)$ for some $a\in[\max x_i,\min y_j]$. Then if

(i)

$$f''_-(\max x_i)\leq 0$$

and

$$\mathscr{A}_{m,r}(\boldsymbol{x},\boldsymbol{p},id^2)-\mathscr{A}_{m,k}(\boldsymbol{x},\boldsymbol{p},id^2)\leq \mathscr{A}_{m,r}(\boldsymbol{y},\boldsymbol{q},id^2)-\mathscr{A}_{m,k}(\boldsymbol{y},\boldsymbol{q},id^2)$$

(ii)

$$f''_+(\min y_j)\geq 0$$

and

$$\mathscr{A}_{m,r}(\boldsymbol{x},\boldsymbol{p},id^2)-\mathscr{A}_{m,k}(\boldsymbol{x},\boldsymbol{p},id^2)\geq \mathscr{A}_{m,r}(\boldsymbol{y},\boldsymbol{q},id^2)-\mathscr{A}_{m,k}(\boldsymbol{y},\boldsymbol{q},id^2)$$

(iii) $f''_-(\max x_i)<0<f''_+(\min y_j)$ *and f is 3-concave,*

then the inequality

$$\mathscr{A}_{m,r}(\boldsymbol{x},\boldsymbol{p},f)-\mathscr{A}_{m,k}(\boldsymbol{x},\boldsymbol{p},f)\geq \mathscr{A}_{m,r}(\boldsymbol{y},\boldsymbol{q},f)-\mathscr{A}_{m,k}(\boldsymbol{y},\boldsymbol{q},f) \tag{11.18}$$

holds.

Proof. Since $f \in \mathscr{K}_2^a[\alpha,\beta]$ for some $a \in [\max x_i, \max y_j]$, therefore there exists a constant B such that $H_1(x) = f(x) - \frac{B}{2}x^2$, is convex on $I \cap (-\infty, a]$, such that for $x_1, \dots, x_n \in I \cap (-\infty, a]$, we have

$$0 \le \mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},H_1) - \mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},H_1),$$

that is

$$\begin{aligned} 0 \le\ & \mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},\ f) - \mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},f) - \frac{C}{2}\Big[\mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},id^2) \\ -\ & \mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},id^2)\Big]. \end{aligned} \tag{11.19}$$

Also $H_2(y) = f(y) - \frac{B}{2}y^2$ is concave on $I \cap [a,\infty)$, for $y_1, \dots, y_s \in [a,\infty)$, we have

$$0 \ge \mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},H_2) - \mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},H_2),$$

that is

$$\begin{aligned} 0 \ge\ & \mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},f) - \mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},f) - \frac{B}{2}\Big[\mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},id^2) \\ -\ & \mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},id^2)\Big]. \end{aligned} \tag{11.20}$$

From (11.19) and (11.20), we have

$$\begin{aligned} &\mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},f) - \mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},f) - \frac{B}{2}\left[\mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},id^2) - \mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},id^2)\right] \\ &\ge \mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},f) - \mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},f) - \frac{B}{2}\left[\mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},id^2) - \mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},id^2)\right]. \end{aligned}$$

So

$$\begin{aligned} &\frac{B}{2}\left[\mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},id^2) - \mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},id^2) - \mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},id^2) - \mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},id^2)\right] \\ &\qquad \ge \mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},f) - \mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},f) - \mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},f) - \mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},f). \end{aligned} \tag{11.21}$$

Now due to convexity of H_1 and concavity of H_2 for every distinct point $\tilde{x}_j \in [\alpha, \max x_i]$ and $\tilde{y}_j \in [\min y_i, \beta]$, $j = 1,2,3$, we have

$$[\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,f] \ge B \ge [\tilde{y}_1,\tilde{y}_2,\tilde{y}_3,f]. \tag{11.22}$$

Letting $\tilde{x}_j \nearrow \max x_i$ and $\tilde{y}_j \searrow \min y_j$, we get the inequalities if derivatives exist

$$f''_-(\max x_i) \ge B \ge f''_+(\min y_i). \tag{11.23}$$

Since from assumption (a), $f''(\max x_i) \le 0$, therefore $B \ge 0$, using the assumption

$$\left[\mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},id^2) - \mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},id^2) - \mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},id^2) - \mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},id^2)\right] \ge 0$$

we have

$$\frac{B}{2}\left[\mathscr{A}_{m,r}(\mathbf{y},\mathbf{p},id^2)-\mathscr{A}_{m,k}(\mathbf{y},\mathbf{p},id^2)-\mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},id^2)-\mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},id^2)\right]$$

is negative and on using it on left side of (11.21) we have the result (11.4). And similarly for assumption (b), the inequality $f''_+(\min y_j)\geq 0$ gives $B>0$, so the expression with assumption of (b) is also positive, this gives the result (11.4). Under the assumption of (c), f''_- and f''_+ are left and right continuous respectively and both are decreasing with $f''_-\geq f''_+$, so there exists a point $\tilde{a}\in[\max x_i,\min y_j]$ such that $f\in\mathscr{K}_1^{\tilde{a}}[\alpha,\beta]$ with constant $\tilde{B}=0$, and thus we have the inequality (11.18).

Remark 11.3 In Theorem 11.4, B is negative or positive or zero under the assumption $(i),(ii)$ and (iii) respectively as discussed earlier in the proof of the Theorem 11.4. Therefore we get the improvement of (11.18) as follows.

$$\begin{aligned}\mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},f)-\mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},f)&\geq\mathscr{A}_{m,r}(\mathbf{x},\mathbf{p},id^2)-\mathscr{A}_{m,k}(\mathbf{x},\mathbf{p},id^2)\\ \geq\mathscr{A}_{m,r}(\mathbf{y},\mathbf{q},id^2)-\mathscr{A}_{m,k}(\mathbf{y},\mathbf{q},id^2)&\geq\mathscr{A}_{m,r}(\mathbf{y},\mathbf{q},f)-\mathscr{A}_{m,k}(\mathbf{y},\mathbf{q},f).\end{aligned}$$

Remark 11.4 Theorem 11.2, Remark 11.1, Theorem 11.3, Remark 11.2 and Theorem 11.4 are also valid for the differences given in (9.1) and (9.2) for $r=1,\ldots,m$ and $1\leq r<k\leq m$ respectively.

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