Chapter 1

Preliminaries

1.1 Continuous and Absolutely Continuous Functions

We start with definitions and properties of integrable functions, continuous functions, absolutely continuous functions, and give required notation, terms and overview of some important results (more details could be found in monographs [106, 133]).

L_p spaces

Let $[a,b]$ be a finite interval in $\mathbb{R}$, where $-\infty \leq a < b \leq \infty$. We denote by $L_p[a,b]$, $1 \leq p < \infty$, the space of all Lebesgue measurable functions f for which $\int_a^b |f(t)|^p \, dt < \infty$, with the norm

$$||f||_p = \left(\int_a^b |f(t)|^p \, dt \right)^{\frac{1}{p}}$$

endowed, and by $L_\infty[a,b]$ the set/space of all functions measurable and essentially bounded on $[a,b]$, equipped with the norm

$$||f||_\infty = \operatorname{ess\,sup} \{ |f(x)| : x \in [a,b] \} .$$

Spaces of continuous and absolutely continuous functions

We denote by $C^n[a,b]$, $n \in \mathbb{N}_0$, the space of functions which are n times continuously differentiable on $[a,b]$, that is

$$C^n[a,b] = \left\{ f : [a,b] \to \mathbb{R} \colon f^{(k)} \in C[a,b]\,, k = 0,1,\ldots,n \right\}.$$

In particular, $C^0[a,b] = C[a,b]$ is the space of continuous functions on $[a,b]$ with the norm

$$||f||_{C^n} = \sum_{k=0}^{n} ||f^{(k)}||_C = \sum_{k=0}^{n} \max_{x\in[a,b]} |f^{(k)}(x)|\,,$$

and for $C[a,b]$

$$||f||_C = \max_{x\in[a,b]} |f(x)|\,.$$

Lemma 1.1 *The space $C^n[a,b]$ consists of those and only those functions f which are represented in the form*

$$f(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} \varphi(t)\,dt + \sum_{k=0}^{n-1} c_k (x-a)^k\,, \tag{1.1}$$

where $\varphi \in C[a,b]$ and c_k are arbitrary constants $(k = 0,1,\ldots,n-1)$.
Moreover,

$$\varphi(t) = f^{(n)}(t)\,, \quad c_k = \frac{f^{(k)}(a)}{k!}\ (k = 0,1,\ldots,n-1)\,. \tag{1.2}$$

The space of absolutely continuous functions on a finite interval $[a,b]$ is denoted by $AC[a,b]$. It is known that $AC[a,b]$ coincides with the space of primitives of Lebesgue integrable functions $L_1[a,b]$ (see Kolmogorov and Fomin [93, Chapter 33.2]):

$$f \in AC[a,b] \quad \Leftrightarrow \quad f(x) = f(a) + \int_a^x \varphi(t)\,dt\,, \quad \varphi \in L_1[a,b]\,,$$

and therefore an absolutely continuous function f has an integrable derivative $f'(x) = \varphi(x)$ almost everywhere on $[a,b]$. We denote by $AC^n[a,b]$, $n \in \mathbb{N}$, the space

$$AC^n[a,b] = \left\{ f \in C^{n-1}[a,b] \colon f^{(n-1)} \in AC[a,b] \right\}.$$

In particular, $AC^1[a,b] = AC[a,b]$.

Lemma 1.2 *The space $AC^n[a,b]$ consists of those and only those functions which can be represented in the form* (1.1), *where $\varphi \in L_1[a,b]$ and c_k are arbitrary constants $(k = 0,1,\ldots,n-1)$.*
Moreover, (1.2) *holds.*

Theorem 1.1 (FUBINI'S THEOREM) *Let $(X,\mathscr{M},\mu)$ and $(Y,\mathscr{N},\nu)$ be σ-finite measure spaces and f $\mu\times\nu$-measurable function on $X\times Y$. If $f\geq 0$, then next integrals are equal*

$$\int_{X\times Y} f(x,y)\,d(\mu\times\nu)(x,y),$$

$$\int_X\left(\int_Y f(x,y)\,d\nu(y)\right)d\mu(x)$$

and

$$\int_Y\left(\int_X f(x,y)\,d\mu(x)\right)d\nu(y).$$

If f is a complex function, then above equalities hold with additional requirement

$$\int_{X\times Y} |f(x,y)|\,d(\mu\times\nu)(x,y)<\infty.$$

Fubini's theorem and its consequences below have numerous applications involving multiple integrals:

$$\int_a^b dx\int_c^d f(x,y)\,dy=\int_c^d dy\int_a^b f(x,y)\,dx;$$

$$\int_a^b dx\int_a^x f(x,y)\,dy=\int_a^b dy\int_y^b f(x,y)\,dx. \tag{1.3}$$

1.2 The Gamma and Beta Functions

The *gamma function* Γ is the function of complex variable defined by Euler's integral of second kind

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,dt\,,\quad \Re(z)>0\,. \tag{1.4}$$

This integral is convergent for each $z\in\mathbb{C}$ such that $\Re(z)>0$. It has next property

$$\Gamma(z+1)=z\Gamma(z)\,,\quad \Re(z)>0\,,$$

from which follows

$$\Gamma(n+1)=n!\,,\quad n\in\mathbb{N}_0\,.$$

For domain $\Re(z)\leq 0$ we have

$$\Gamma(z)=\frac{\Gamma(z+n)}{(z)_n}\,,\quad \Re(z)>-n;\ n\in\mathbb{N};\ z\notin\mathbb{Z}_0^-=\{0,-1,-2,\ldots\}\,, \tag{1.5}$$

where $(z)_n$ is the *Pochhammer's symbol* defined for $z\in\mathbb{C}$ and $n\in\mathbb{N}_0$ by

$$(z)_0=1;$$

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)} = z(z+1)\cdots(z+n-1),\ n \in \mathbb{N}. \tag{1.6}$$

The *generalized Pochhammer's symbol* is defined for $z, \nu \in \mathbb{C}$ by

$$(z)_\nu = \frac{\Gamma(z+\nu)}{\Gamma(z)}. \tag{1.7}$$

The gamma function is analytic in complex plane except in $0, -1, -2, \ldots$ which are simple poles. Another interesting equality holds:

$$(z)_{m+n} = (z+m)_n\,(z)_m\,, \quad n, m \in \mathbb{N}.$$

The *beta function* is the function of two complex variables defined by Euler's integral of the first kind

$$\mathrm{B}(z,w) = \int_0^1 t^{z-1}\,(1-t)^{w-1}\,dt\,, \quad \Re(z), \Re(w) > 0. \tag{1.8}$$

It is related to the gamma function with

$$\mathrm{B}(z,w) = \frac{\Gamma(z)\,\Gamma(w)}{\Gamma(z+w)}\,, \quad z, w \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\}\,,$$

which gives

$$\mathrm{B}(z+1,w) = \frac{z}{z+w}\mathrm{B}(z,w).$$

Further, we have an extension of the beta function (for more details see [31])

$$\mathrm{B}_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{-\frac{p}{t(1-t)}}\,dt, \quad \Re(x), \Re(y), \Re(p) > 0. \tag{1.9}$$

Here we emphasize two equalities for the extended beta function:

$$\mathrm{B}_p(x,y+1) + \mathrm{B}_p(x+1,y) = \mathrm{B}_p(x,y)\,,$$

$$\int_0^\infty \mathrm{B}_p(x,y)\,dp = \mathrm{B}(x+1,y+1), \quad \Re(x), \Re(y) > -1.$$

Following examples of integrals will be often used in proofs and calculations in this book.

Example 1.1 Let $\alpha, \beta > 0$ and $x \in [a,b]$. Then by substitution $t = x - s(x-a)$ we have

$$\begin{aligned}\int_a^x (x-t)^{\alpha-1}(t-a)^{\beta-1}\,dt &= \int_0^1 (x-a)^{\alpha+\beta-1} s^{\alpha-1}(1-s)^{\beta-1}\,ds \\ &= B(\alpha,\beta)(x-a)^{\alpha+\beta-1}.\end{aligned}$$

Analogously, by substitution $t = x + s(b-x)$, it follows

$$\int_x^b (t-x)^{\alpha-1}(b-t)^{\beta-1}\,dt = B(\alpha,\beta)\,(b-x)^{\alpha+\beta-1}.$$

Example 1.2 Let $\alpha, \beta > 0$, $f \in L_1[a,b]$ and $x \in [a,b]$. Then interchanging the order of integration and evaluating the inner integral we obtain

$$\begin{aligned}\int_a^x (x-t)^{\alpha-1} \int_a^t (t-s)^{\beta-1} f(s)\, ds\, dt &= \int_{s=a}^x f(s) \int_{t=s}^x (x-t)^{\alpha-1}(t-s)^{\beta-1}\, dt\, ds \\ &= B(\alpha,\beta) \int_a^x (x-s)^{\alpha+\beta-1} f(s)\, ds\,.\end{aligned}$$

Analogously,

$$\int_x^b (t-x)^{\alpha-1} \int_t^b (s-t)^{\beta-1} f(s)\, ds\, dt = B(\alpha,\beta) \int_x^b (s-x)^{\alpha+\beta-1} f(s)\, ds\,.$$

1.3 Convex Functions and Classes of Convexity

Definitions and properties of convex functions, with more details, could be found in monographs [107, 110, 121].

Let I be an interval in $\mathbb{R}$.

Definition 1.1 *A function $f : I \to \mathbb{R}$ is called convex if*

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda) f(x) + \lambda f(y) \tag{1.10}$$

for all points x and y in I and all $\lambda \in [0,1]$. It is called strictly convex if the inequality (1.10) *holds strictly whenever x and y are distinct points and $\lambda \in (0,1)$. If $-f$ is convex (respectively, strictly convex) then we say that f is concave (respectively, strictly concave). If f is both convex and concave, then f is said to be affine.*

Lemma 1.3 (THE DISCRETE CASE OF JENSEN'S INEQUALITY) *A real-valued function f defined on an interval I is convex if and only if for all $x_1, \ldots, x_n$ in I and all scalars $\lambda_1, \ldots, \lambda_n$ in $[0,1]$ with $\sum_{k=1}^n \lambda_k = 1$ we have*

$$f\left(\sum_{k=1}^n \lambda_k x_k\right) \leq \sum_{k=1}^n \lambda_k f(x_k)\,. \tag{1.11}$$

The above inequality is strict if f is strictly convex, all the points x_k are distinct and all scalars λ_k are positive.

Theorem 1.2 (JENSEN) *Let $f : I \to \mathbb{R}$ be a continuous function. Then f is convex if and only if f is midpoint convex, that is,*

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \tag{1.12}$$

for all $x, y \in I$.

Corollary 1.1 *Let $f : I \to \mathbb{R}$ be a continuous function. Then f is convex if and only if*

$$f(x+h)+f(x-h)-2f(x) \geq 0 \tag{1.13}$$

for all $x \in I$ and all $h > 0$ such that both $x+h$ and $x-h$ are in I.

Proposition 1.1 (THE OPERATIONS WITH CONVEX FUNCTIONS)

(i) *The addition of two convex functions (defined on the same interval) is a convex function; if one of them is strictly convex, then the sum is also strictly convex.*

(ii) *The multiplication of a (strictly) convex function with a positive scalar is also a (strictly) convex function.*

(iii) *The restriction of every (strictly) convex function to a subinterval of its domain is also a (strictly) convex function.*

(iv) *If $f : I \to \mathbb{R}$ is a convex (respectively a strictly convex) function and $g : \mathbb{R} \to \mathbb{R}$ is a nondecreasing (respectively an increasing) convex function, then $g \circ f$ is convex (respectively strictly convex)*

(v) *Suppose that f is a bijection between two intervals I and J. If f is increasing, then f is (strictly) convex if and only if f^{-1} is (strictly) concave. If f is a decreasing bijection, then f and f^{-1} are of the same type of convexity.*

Definition 1.2 *If g is strictly monotonic, then f is said to be (strictly) convex with respect to g if $f \circ g^{-1}$ is (strictly) convex.*

Proposition 1.2 *If $x_1, x_2, x_3 \in I$ are such that $x_1 < x_2 < x_3$, then the function $f : I \to \mathbb{R}$ is convex if and only if the inequality*

$$(x_3 - x_2)\, f(x_1) + (x_1 - x_3)\, f(x_2) + (x_2 - x_1)\, f(x_3) \;\geq\; 0$$

holds.

Proposition 1.3 *If f is a convex function on an interval I and if $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid*

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \leq \frac{f(y_2)-f(y_1)}{y_2-y_1}.$$

If the function f is concave, then the inequality reverses.

The following theorems concern derivatives of convex functions.

Theorem 1.3 *Let $f : I \to \mathbb{R}$ be convex. Then*

(i) *f is Lipschitz on any closed interval in I (recall, a function f such that $|f(x) - f(y)| \leq C|x-y|$ for all x and y, where C is a constant independent of x and y, is called a Lipshitz function);*

(ii) *f'_+ and f'_- exist and are increasing in I, and $f'_- \leq f'_+$ (if f is strictly convex, then these derivatives are strictly increasing);*

(iii) *f' exists, except possibly on a countable set, and on the complement of which it is continuous.*

Proposition 1.4 *Suppose that $f : I \to \mathbb{R}$ is a twice differentiable function. Then*

(i) *f is convex if and only if $f'' \geq 0$;*

(ii) *f is strictly convex if and only if $f'' \geq 0$ and the set of points where f'' vanishes does not include intervals of positive length.*

Next we need divided differences, commonly used when dealing with functions that have different degree of smoothness.

Definition 1.3 *Let $f : I \to \mathbb{R}$, $n \in \mathbb{N}_0$ and let $x_0, x_1, \ldots, x_n \in I$ be mutually different points. The n-th order divided difference of a function at $x_0, \ldots, x_n$ is defined recursively by*

$$\begin{aligned}
[x_i; f] &= f(x_i), \quad i = 0, 1, \ldots, n, \\
[x_0, x_1; f] &= \frac{[x_0; f] - [x_1; f]}{x_0 - x_1} = \frac{f(x_0) - f(x_1)}{x_0 - x_1}, \\
[x_0, x_1, x_2; f] &= \frac{[x_0, x_1; f] - [x_1, x_2; f]}{x_0 - x_2}, \\
&\vdots \\
[x_0, \ldots, x_n; f] &= \frac{[x_0, \ldots, x_{n-1}; f] - [x_1, \ldots, x_n; f]}{x_0 - x_n}.
\end{aligned} \tag{1.14}$$

Remark 1.1 The value $[x_0, x_1, x_2; f]$ is independent of the order of the points x_0, x_1 and x_2. This definition may be extended to include the case in which some or all the points coincide. Namely, taking the limit $x_1 \to x_0$ in (1.14), we get

$$\lim_{x_1 \to x_0} [x_0, x_1, x_2; f] = [x_0, x_0, x_2; f] = \frac{f(x_0) - f(x_2) - f'(x_0)(x_0 - x_2)}{(x_0 - x_2)^2}, \; x_2 \neq x_0$$

provided that f' exists, and furthermore, taking the limits $x_i \to x_0$, $i = 1, 2$ in (1.14), we get

$$\lim_{x_2 \to x_0} \lim_{x_1 \to x_0} [x_0, x_1, x_2; f] = [x_0, x_0, x_0; f] = \frac{f''(x_0)}{2}$$

provided that f'' exists.

Definition 1.4 *A function $f : I \to \mathbb{R}$ is said to be n-convex ($n \in \mathbb{N}_0$) if for all choices of $n+1$ distinct points $x_0, \ldots, x_n \in I$, the n-th order divided difference of f satisfies*

$$[x_0, \ldots, x_n; f] \geq 0. \tag{1.15}$$

Thus the 1-convex functions are the nondecreasing functions, while the 2-convex functions are precisely the classical convex functions.

Definition 1.5 *A function $f : I \to (0,\infty)$ is called log-convex if*

$$f((1-\lambda)x+\lambda y) \leq f(x)^{1-\lambda} f(y)^{\lambda} \tag{1.16}$$

for all points x and y in I and all $\lambda \in [0,1]$.

If a function $f : I \to \mathbb{R}$ is log-convex, then it is also convex, which is a consequence of the weighted AG-inequality.

We continue with definitions and properties of other types of convex functions that will be used in the book.

Definition 1.6 [112] *Let T_g be a set of real numbers. This set T_g is said to be relative convex with respect to an arbitrary function $g : \mathbb{R} \to \mathbb{R}$ if*

$$(1-t)x+tg(y) \in T_g$$

where $x,y \in \mathbb{R}$ such that $x, g(y) \in T_g$, $t \in [0,1]$.

Definition 1.7 [112] *A function $f : T_g \to \mathbb{R}$ is said to be relative convex, if there exists an arbitrary function $g : \mathbb{R} \to \mathbb{R}$ such that*

$$f((1-t)x+tg(y)) \leq (1-t)f(x)+tf(g(y)),$$

holds, where $x,y \in \mathbb{R}$ such that $x, g(y) \in T_g$, $t \in [0,1]$.

Note that if g is identity function, then convex set and convex function are reproduced from relative convex set and relative convex function.

Definition 1.8 [73] *Let I be an interval of real numbers. Then a function $f : I \to \mathbb{R}$ is said to be quasi-convex function, if for all $a,b \in I$ and $0 \leq t \leq 1$ the following inequality holds*

$$f(ta+(1-t)b) \leq max\{f(a), f(b)\}. \tag{1.17}$$

Example 1.3 [80] The function $f : [-2,2] \to \mathbb{R}$, given by

$$f(x) = \begin{cases} 1 & x \in [-2,-1] \\ x^2 & x \in (-1,2] \end{cases}$$

is not a convex function on $[-2,2]$, but it is quasi-convex function on $[-2,2]$.

Definition 1.9 *Let I be an interval of non-zero real numbers. A function $f : I \to \mathbb{R}$ is said to be harmonically convex function, if*

$$f\left(\frac{ab}{ta+(1-t)b}\right) \leq tf(b)+(1-t)f(a) \tag{1.18}$$

holds for all $a,b \in I$ and $t \in [0,1]$. If inequality in (1.18) is reversed, then f is said to be harmonically concave function.

Example 1.4 [76] Let $f:(0,\infty)\to\mathbb{R}$ be defined by $f(t)=t$ and $g:(-\infty,0)\to\mathbb{R}$ defined by $g(t)=t$. Then f is harmonically convex function and g is harmonically concave function.

Following results are obvious from above example.

(i) If $I\subset(0,\infty)$ and f is non-decreasing convex function, then f is harmonically convex.

(ii) If $I\subset(0,\infty)$ and f is non-increasing harmonically convex function, then f is convex.

(iii) If $I\subset(-\infty,0)$ and f is non-decreasing harmonically convex function, then f is convex.

(iv) If $I\subset(-\infty,0)$ and f is non-increasing convex function, then f is harmonically convex.

Definition 1.10 [95] *A function $h:[a,b]\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ is said to be harmonically symmetric about $\frac{a+b}{2ab}$ if for all $x\in[a,b]$*

$$h\left(\frac{1}{x}\right)=h\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right).$$

We give another new notion of harmonically $(h-m)$-convex function by setting $\alpha=1$ as follows:

Definition 1.11 *Let $h:[0,1]\subseteq J\to\mathbb{R}$ be a nonnegative function. A function $f:J\subseteq\mathbb{R}_+\to\mathbb{R}$ is said to be harmonically $(h-m)$-convex if*

$$f\left(\frac{mxy}{mty+(1-t)x}\right)\leq h(t)f(x)+mh(1-t)f(y)$$

holds for all $x,y\in J$, $t\in[0,1]$ and $m\in(0,1]$.

Next is a harmonically $(\alpha,h-m)$-convex function:

Definition 1.12 *Let $h:[0,1]\subseteq J\to\mathbb{R}$ be a nonnegative function. A function $f:J\subseteq\mathbb{R}_+\to\mathbb{R}$ is said to be harmonically $(\alpha,h-m)$-convex if*

$$f\left(\frac{mxy}{mty+(1-t)x}\right)\leq h(t^{\alpha})f(x)+mh(1-t^{\alpha})f(y),$$

holds for all $x,y\in J$, t, $\alpha\in[0,1]$ and $m\in(0,1]$.

This unifies the definitions of harmonically (α,m)-convexity and harmonically h-convexity of functions. For different specific choices of α,h,m, almost all kinds of well-known harmonically convex functions can be obtained:

Remark 1.2 (i) If $h(t) = t$, then harmonically (α,m)-convex function can be obtained [74].

(ii) If $\alpha = 1$ and $h(t) = t^s$, then harmonically (s,m)-convex function can be obtained [27].

(iii) If $\alpha = 1$ and $h(t) = t$, then harmonically m-convex function can be obtained [27].

(iv) If $\alpha = h(t) = m = 1$, then harmonically P-function can be obtained [113].

(v) If $\alpha = 1$, $h(t) = t^s$ and $m = 1$, then harmonically s-convex function can be obtained [113].

(vi) If $\alpha = 1$, $h(t) = \frac{1}{t}$ and $m = 1$, then harmonically Godunova-Levin function can be obtained [113].

(vii) If $\alpha = 1$, $h(t) = \frac{1}{t^s}$ and $m = 1$, then harmonically s-Godunova-Levin function can be obtained [113].

(viii) If we set $m = 1$ and $\alpha = 1$, then harmonically h-convex function can be achieved [113].

(ix) By putting $\alpha = 1$, $h(t) = t$ and $m = 1$, then harmonically-convex function can be obtained [76].

Definition 1.13 [147] *A function $f : [0,b] \to \mathbb{R}$, $b > 0$ is said to be m-convex function if for all $x, y \in [0,b]$ and $t \in [0,1]$*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

holds where $m \in [0,1]$.

Example 1.5 [105] A function $f : [0,\infty] \to \mathbb{R}$ given by

$$f(x) = \frac{1}{12}(x^3 - 5x^2 + 9x - 5x)$$

is $\frac{16}{17}$-convex function. If $m \in (\frac{16}{17}, 1]$, then f is not m-convex.

For $m = 1$ the m-convex function reduces to convex function and for $m = 0$ it gives star-shaped function. If set of m-convex functions on $[0,b]$ for which $f(0) < 0$ is denoted by $K_m(b)$, then we have

$$K_1(b) \subset K_m(b) \subset K_0(b)$$

whenever $m \in (0,1)$. In the class $K_1(b)$ there are convex functions $f : [0,b] \to \mathbb{R}$ for which $f(0) \leq 0$ (see, [40]).

Definition 1.14 [130] *A function $f : [a,b] \to \mathbb{R}$ is said to be exponentially m-convex if*

$$e^{f(zx+m(1-z)y)} \leq ze^{f(x)} + m(1-z)e^{f(y)}. \tag{1.19}$$

for all $x, y \in [a,b]$ and $z \in [0,1]$ where $m \in (0,1]$.