MONOGRAPHS IN INEQUALITIES 2

Euler integral identity, quadrature formulae and error estimations

(from the point of view of inequality theory)

Iva Franjić, Josip Pečarić, Ivan Perić and Ana Vukelić

ELEMENT

Euler integral identity, quadrature formulae and error estimations

(from the point of view of inequality theory)

Iva Franjić
Faculty of Food Technology and Biotechnology
University of Zagreb
Zagreb, Croatia

Josip Pečarić
Faculty of Textile Technology
University of Zagreb
Zagreb, Croatia

Ivan Perić
Faculty of Food Technology and Biotechnology
University of Zagreb
Zagreb, Croatia

Ana Vukelić
Faculty of Food Technology and Biotechnology
University of Zagreb
Zagreb, Croatia

ELEMENT
Zagreb, 2012

MONOGRAPHS IN INEQUALITIES 2

Euler integral identity, quadrature formulae and error estimations

(from the point of view of inequality theory)

Iva Franjić, Josip Pečarić, Ivan Perić and Ana Vukelić

1st edition

www.element.hr/english/

Printed in Croatia.

A CIP catalogue record for this book is available from the National and University Library in Zagreb under 830842.

ISBN 978-953-197-573-5 (soft cover)

Preface

Despite of the title, the main motivation for writing this book (and the papers from which this book has grown) was presenting some aspects of generalizations, refinements, variants of three famous inequalities (actually four). The first inequality is the Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2},$$

which holds for a convex function f on $[a,b] \subset \mathbb{R}$. The second inequality is the Ostrowski inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \frac{1}{(b-a)^2}\left(x - \frac{a+b}{2}\right)^2 \right](b-a)L,$$

which holds for a $L-$Lipschitzian function f on $[a,b] \subset \mathbb{R}$, and the third one is the Iyengar inequality

$$\left| \frac{1}{b-a}\int_a^b f(x)dx - \frac{f(a)+f(b)}{2} \right| \le \left[1 - \left(\frac{f(b)-f(a)}{L(b-a)} \right)^2 \right] \frac{b-a}{4}L,$$

which also holds for a $L-$Lipschitzian function f on $[a,b] \subset \mathbb{R}$.

Generalizations of the Ostrowski inequality are mainly given in Chapter 1, but related results are scattered throughout the book (especially for the case $x = (a+b)/2$). Variants of the Hermite-Hadamard inequality are given for some pairs of quadrature formulae (called dual formulae) and refinements are given in the sense of the Bullen inequalities for higher convex functions. The basic Bullen inequality

$$0 \le \frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx$$

holds for a convex function f on $[a,b] \subset \mathbb{R}$.

The book contains generalizations of many classical quadrature formulae such as Simpson, dual Simpson, Maclaurin, Gauss, Lobatto, Radau. Standard methods in deducing these formulae are very different, spanning from Lagrange, Newton interpolation polynomials to orthogonal polynomials such as Legendre, Chebyshev, Jacobi. The specific feature of this book (regarded nevertheless as a book in numerical integration) is that the unique method is used. This method is based on the, so called, *Euler integral identities* expressing expansion of a function in Bernoulli polynomials proved by V. I. Krylov in [79] as a generalization of the first and the second Euler-Maclaurin sum formula (for details see Chapter 1). The Iyengar inequality is the exception. Although related to the Hermite-Hadamard inequality in the same way as the Ostrowski inequality, Iyengar type inequalities are, it seems, beyond the reach of methods based on the Euler integral identities. This is the reason why generalizations of the Iyengar inequality are given in the Addendum.

Contents

Chapter 1

Euler integral identities

1.1 Introduction

Integral Euler identities extend the well known formula for the expansion of an arbitrary function in Bernoulli polynomials (cf. [79] or Appendix) and were derived in [30]. To prove them, the following lemma is needed:

Lemma 1.1 *Let $a,b \in \mathbb{R}$, $a < b$, $x \in [a,b]$ and $\varphi : \mathbb{R} \to \mathbb{R}$ be defined by*

$$\varphi(t) = B_1^*\left(\frac{x-t}{b-a}\right).$$

Then for every continuous function $F : [a,b] \to \mathbb{R}$ we have

$$\int_{[a,b]} F(t)d\varphi(t) = -\frac{1}{b-a}\int_a^b F(t)dt + F(x), \textit{ for } a < x < b$$

and

$$\int_{[a,b]} F(t)d\varphi(t) = -\frac{1}{b-a}\int_a^b F(t)dt + F(a), \textit{ for } x = a \textit{ or } x = b,$$

with Riemann-Stieltjes integrals on the left hand sides.

Proof. If $a < x < b$ the function φ is differentiable on $[a,b]\backslash\{x\}$ and its derivative is equal to $\frac{-1}{b-a}$, since $B_1(t) = t - 1/2$. It has a jump of $\varphi(x+0) - \varphi(x-0) = 1$ at x, which gives the first formula. For $x = a$ or $x = b$ the function φ is differentiable on (a,b) and its derivative is equal to $\frac{-1}{b-a}$. It has a jump of $\varphi(a+0) - \varphi(a) = 1$ at the point a, while $\varphi(b) - \varphi(b-0) = 0$, which gives the second formula. □

Here, as in the rest of the book, we write $\int_0^1 g(t)d\varphi(t)$ to denote the Riemann-Stieltjes integral with respect to a function $\varphi : [0,1] \to \mathbb{R}$ of bounded variation, and $\int_0^1 g(t)dt$ for the Riemann integral.

Theorem 1.1 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is continuous of bounded variation on $[a,b]$ for some $n \geq 1$. Then for every $x \in [a,b]$ we have*

$$\frac{1}{b-a}\int_a^b f(t)dt = f(x) - T_n(x) + R_n^1(x), \tag{1.1}$$

$$\frac{1}{b-a}\int_a^b f(t)dt = f(x) - T_{n-1}(x) + R_n^2(x) \tag{1.2}$$

where $T_0(x) = 0$, and for $1 \leq m \leq n$

$$T_m(x) = \sum_{k=1}^{m} \frac{(b-a)^{k-1}}{k!} B_k\left(\frac{x-a}{b-a}\right)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right], \tag{1.3}$$

$$R_n^1(x) = \frac{(b-a)^{n-1}}{n!}\int_a^b B_n^*\left(\frac{x-t}{b-a}\right) df^{(n-1)}(t),$$

$$R_n^2(x) = \frac{(b-a)^{n-1}}{n!}\int_a^b \left[B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right)\right] df^{(n-1)}(t).$$

Proof. Using integration by parts we have

$$\begin{aligned} R_k^1(x) &= \frac{(b-a)^{k-1}}{k!} B_k^*\left(\frac{x-t}{b-a}\right) f^{(k-1)}(t)\Bigg|_a^b \\ &\quad - \frac{(b-a)^{k-1}}{k!}\int_{[a,b]} f^{(k-1)}(t) dB_k^*\left(\frac{x-t}{b-a}\right). \end{aligned} \tag{1.4}$$

For every $k \geq 1$ and every $x \in [a,b)$ we have

$$B_k^*\left(\frac{x-b}{b-a}\right) = B_k^*\left(\frac{x-a}{b-a} - 1\right) = B_k^*\left(\frac{x-a}{b-a}\right) = B_k\left(\frac{x-t}{b-a}\right). \tag{1.5}$$

Also, for $k \geq 2$ the above formula is valid for every $x \in [a,b]$. The identity (1.4) for $k = 1$ becomes

$$R_1^1(x) = B_1^*\left(\frac{x-t}{b-a}\right) f(t)\Bigg|_a^b - \int_{[a,b]} f(t) dB_1^*\left(\frac{x-t}{b-a}\right).$$

If $x \in [a,b)$, then using Lemma 1.1 and (1.5) we get

$$\begin{aligned} R_1^1(x) &= B_1\left(\frac{x-a}{b-a}\right)[f(b) - f(a)] + \frac{1}{b-a}\int_a^b f(t)dt - f(x) \\ &= T_1(x) + \frac{1}{b-a}\int_a^b f(t)dt - f(x). \end{aligned}$$

If $x = b$, then using Lemma 1.1 we get

$$\begin{aligned}
R_1^1(b) &= B_1^*(0)f(b) - B_1^*(1)f(a) + \frac{1}{b-a}\int_a^b f(t)dt - f(a) \\
&= -\frac{1}{2}f(b) + \frac{1}{2}f(a) + \frac{1}{b-a}\int_a^b f(t)dt - f(a) \\
&= \frac{1}{2}[f(b) - f(a)] + \frac{1}{b-a}\int_a^b f(t)dt - f(b) \\
&= T_1(b) + \frac{1}{b-a}\int_a^b f(t)dt - f(b).
\end{aligned}$$

So, for every $x \in [a,b]$ we have

$$R_1^1(x) = T_1(x) + \frac{1}{b-a}\int_a^b f(t)dt - f(x), \tag{1.6}$$

which is just the identity (1.1) for $n = 1$. Further, for every $k \geq 2$

$$\frac{d}{dt}B_k^*\left(\frac{x-t}{b-a}\right) = -\frac{k}{b-a}B_{k-1}^*\left(\frac{x-t}{b-a}\right),$$

except for t from discrete set $x + (b-a)\mathbb{Z} \subset \mathbb{R}$, since the Bernoulli polynomials satisfy $\frac{d}{dt}B_k(t) = kB_{k-1}(t)$. Using the above formula and the fact that $B_k^*\left(\frac{x-t}{b-a}\right)$ is continuous for $k \geq 2$, we get

$$\begin{aligned}
&-\frac{(b-a)^{k-1}}{k!}\int_{[a,b]} f^{(k-1)}(t)dB_k^*\left(\frac{x-t}{b-a}\right) \\
&\quad = \frac{(b-a)^{k-2}}{(k-1)!}\int_a^b B_{k-1}^*\left(\frac{x-t}{b-a}\right) f^{(k-1)}(t)dt \\
&\quad = \frac{(b-a)^{k-2}}{(k-1)!}\int_{[a,b]} B_{k-1}^*\left(\frac{x-t}{b-a}\right) df^{(k-2)}(t) \\
&\quad = R_{k-1}^1(x).
\end{aligned}$$

Using this formula and (1.5), from (1.4) we get the identity

$$R_k^1(x) = \frac{(b-a)^{k-1}}{k!}B_k\left(\frac{x-a}{b-a}\right)[f^{(k-1)}(b) - f^{(k-1)}(a)] + R_{k-1}^1(x),$$

which holds for $k = 2,\ldots,n$ and for every $x \in [a,b]$. So, for $n \geq 2$ and for every $x \in [a,b]$ we get

$$R_n^1(x) = \sum_{k=2}^{n}\frac{(b-a)^{k-1}}{k!}B_k\left(\frac{x-a}{b-a}\right)[f^{(k-1)}(b) - f^{(k-1)}(a)] + R_1^1(x),$$

which, in combination with (1.6), yields (1.1).

To obtain the identity (1.2), note that

$$\begin{aligned} R_n^2(x) &= R_n^1(x) - \frac{(b-a)^{n-1}}{n!} B_n\left(\frac{x-a}{b-a}\right) \int_{[a,b]} df^{(n-1)}(t) \\ &= R_n^1(x) - \frac{(b-a)^{n-1}}{n!} B_n\left(\frac{x-a}{b-a}\right) [f^{(n-1)}(b) - f^{(n-1)}(a)] \\ &= R_n^1(x) + T_n(x) - T_{n-1}(x), \end{aligned}$$

and apply (1.1). □

1.2 General Euler-Ostrowski formulae

The main results of this section are the general Euler-Ostrowski formulae which generalize extended Euler identities (1.1) and (1.2), in a sense that the value of the integral is approximated by the values of the function in m equidistant points, instead of by its value in just one point. The results presented in this section were published in [63].

To derive these formulae, we will need an analogue of Multiplication Theorem, stated for periodic functions B_n^*. Multiplication Theorem for Bernoulli polynomials B_n states (cf. [1] or Appendix):

$$B_n(mt) = m^{n-1} \sum_{k=0}^{m-1} B_n\left(t + \frac{k}{m}\right), \quad n \geq 0, \ m \geq 1 \tag{1.7}$$

That (1.7) is true for $B_n^*(t)$ and $t \in [0, 1/m)$ is obvious. For $t \in [j/m, (j+1)/m)$, $1 \leq j \leq m-1$:

$$\begin{aligned} B_n^*(mt) &= B_n^*(m(t - j/m)) = m^{n-1} \sum_{k=0}^{m-1} B_n^*\left(t + \frac{k-j}{m}\right) \\ &= m^{n-1} \sum_{k=0}^{m-1} B_n^*\left(t + \frac{k}{m}\right), \end{aligned}$$

so the statement is true again. Thus, we have

$$B_n^*(mt) = m^{n-1} \sum_{k=0}^{m-1} B_n^*\left(t + \frac{k}{m}\right), \quad n \geq 0, \ m \geq 1. \tag{1.8}$$

Interval $[0,1]$ is used for simplicity and involves no loss in generality.

The following theorem is crucial for our further investigations but is also of independent interest. Namely, the remainder is expressed in terms of $B_n^*(x - mt)$.

Theorem 1.2 *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n-1)}$ is continuous of bounded variation on $[0,1]$ for some $n \geq 1$. Then, for $x \in [0,1]$ and $m \in \mathbb{N}$, we have*

$$\int_0^1 f(t)dt = \frac{1}{m}\sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) - T_n(x) + \frac{1}{n! \cdot m^n}\int_0^1 B_n^*(x-mt)df^{(n-1)}(t), \tag{1.9}$$

where

$$T_n(x) = \sum_{j=1}^{n} \frac{B_j(x)}{j! \cdot m^j}[f^{(j-1)}(1) - f^{(j-1)}(0)]$$

Proof. From (1.8) we get

$$B_n^*(x-mt) = m^{n-1}\sum_{k=0}^{m-1} B_n^*\left(\frac{x+k}{m} - t\right)$$

Multiplying this with $df^{(n-1)}(t)$ and integrating over $[0,1]$ produces formula (1.9) after applying (1.1). □

Formula (1.9) can easily be rewritten as:

$$\begin{aligned}\int_0^1 f(t)dt &= \frac{1}{m}\sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) - T_{n-1}(x) \\ &+ \frac{1}{n! \cdot m^n}\int_0^1 [B_n^*(x-mt) - B_n(x)]df^{(n-1)}(t),\end{aligned} \tag{1.10}$$

with $T_0(x) = 0$.

We call formulae (1.9) and (1.10) the general Euler-Ostrowski formulae.

Theorem 1.3 *Assume (p,q) is a pair of conjugate exponents, that is $1 \leq p,\ q \leq \infty$, $1/p + 1/q = 1$. Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n)} \in L_p[0,1]$ for some $n \geq 1$. Then, for $x \in [0,1]$ and $m \in \mathbb{N}$, we have*

$$\left| \int_0^1 f(t)dt - \frac{1}{m}\sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) + T_n(x) \right| \leq K(n,q) \cdot \|f^{(n)}\|_p, \tag{1.11}$$

$$\left| \int_0^1 f(t)dt - \frac{1}{m}\sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) + T_{n-1}(x) \right| \leq K^*(n,q) \cdot \|f^{(n)}\|_p, \tag{1.12}$$

where

$$K(n,q) = \frac{1}{n! \cdot m^n}\left[\int_0^1 |B_n^*(t)|^q \,\mathrm{d}t\right]^{\frac{1}{q}},$$

$$K^*(n,q) = \frac{1}{n! \cdot m^n}\left[\int_0^1 |B_n^*(t) - B_n(x)|^q \,\mathrm{d}t\right]^{\frac{1}{q}}.$$

These inequalities are sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof. Inequalities (1.11) and (1.12) follow immediately after applying Hölder's inequality to the remainders in formulae (1.9) and (1.10) and using the fact that functions $B_n^*(t)$ are periodic. To prove that the inequalities are sharp, put

$$\begin{aligned}
&f^{(n)}(t) = \operatorname{sgn} B_n^*(x-mt) \cdot |B_n^*(x-mt)|^{1/(p-1)} \quad \text{for} \quad 1<p<\infty \quad \text{and} \\
&f^{(n)}(t) = \operatorname{sgn} B_n^*(x-mt) \quad \text{for} \quad p=\infty \quad \text{in} \quad (1.11), \\
&f^{(n)}(t) = \operatorname{sgn}(B_n^*(x-mt) - B_n(x)) \cdot |B_n^*(x-mt) - B_n(x)|^{1/(p-1)} \\
&\text{for} \quad 1<p<\infty \quad \text{and} \\
&f^{(n)}(t) = \operatorname{sgn}(B_n^*(x-mt) - B_n(x)) \quad \text{for} \quad p=\infty \quad \text{in} \quad (1.12).
\end{aligned}$$

For $p = 1$ it is easy to see that

$$\left| \int_0^1 B_n^*(x-mt) f^{(n)}(t)\,dt \right| \le \max_{t\in[0,1]} |B_n^*(t)| \int_0^1 \left| f^{(n)}(t) \right| dt$$

is the best possible inequality (compare with the proof of Theorem 2.2 in Section 2.3). □

Corollary 1.1 *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n)} \in L_\infty[0,1]$. Let $x \in [0,1]$. If n is odd, then we have*

$$\left| \int_0^1 f(t)\,dt - \frac{1}{m} \sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) + T_n(x) \right| \le \frac{(4-2^{1-n})|B_{n+1}|}{m^n \cdot (n+1)!} \cdot \|f^{(n)}\|_\infty, \tag{1.13}$$

and for $n = 1$

$$\left| \int_0^1 f(t)\,dt - \frac{1}{m} \sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) \right| \le \frac{1}{m} \left[\frac{1}{4} + \left(x - \frac{1}{2}\right)^2 \right] \cdot \|f'\|_\infty, \tag{1.14}$$

while for $n \ge 3$

$$\begin{aligned}
&\left| \int_0^1 f(t)\,dt - \frac{1}{m} \sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) + T_{n-1}(x) \right| \qquad (1.15) \\
&\quad \le \frac{\|f^{(n)}\|_\infty}{m^n \cdot n!} \left((1-2|x-x_1|) \cdot |B_n(x)| + \frac{2}{n+1} |B_{n+1}(x) - B_{n+1}(x_1)| \right),
\end{aligned}$$

where $x_1 \in [0,1]$ is such that $B_n(x_1) = B_n(x)$ and $x_1 \ne x$, except when $B_{n-1}(x) = 0$. If $x = 0$ or $x = 1$, take $x_1 = 1/2$.

If n is even, then we have

$$\begin{aligned}
&\left| \int_0^1 f(t)\,dt - \frac{1}{m} \sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) + T_n(x) \right| \qquad (1.16) \\
&\le \frac{4\|f^{(n)}\|_\infty}{m^n \cdot (n+1)!} \cdot |B_{n+1}(x_1)| = \frac{4\|f^{(n)}\|_\infty}{m^n \cdot (n+1)!} \max_{t\in[0,1]} |B_{n+1}(t)|,
\end{aligned}$$

where $B_n(x_1)=0$*, and*

$$\left|\int_0^1 f(t)dt-\frac{1}{m}\sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right)+T_{n-1}(x)\right| \tag{1.17}$$
$$\le \frac{\|f^{(n)}\|_\infty}{m^n\cdot n!}\left((-1)^{n/2}\left(1-4\left|x-1/2\right|\right)B_n(x)+\frac{4}{n+1}\left|B_{n+1}(x)\right|\right).$$

Proof. Put $p=\infty$ in Theorem 1.3. Inequality (1.13) follows straightforward since it is known that, for an odd n, Bernoulli polynomials have constant sign on $(0,1/2)$ and on $(1/2,1)$. (1.14) also follows by direct calculation.

To prove (1.15), assume first that $0\le x\le 1/2$. For an odd n we have $B_n(1-t)=-B_n(t)$, so we can rewrite $K^*(n,1)$ as

$$\int_0^{1/2}\left|B_n(t)-B_n(x)\right|dt+\int_0^{1/2}\left|B_n(t)+B_n(x)\right|dt.$$

The second integral has no zeros on $(0,1/2)$, so we can calculate it easily. The first integral, however, has two zeros. One is obviously x and the other is x_1, where $x_1\in[0,1/2]$ and $B_n(x_1)=B_n(x)$. When $1/2\le x\le 1$, the statement follows similarly.

Next, assume $0\le x\le 1/2$. Since $B_n(t)$ are symmetric about $t=1/2$ for an even n, we can rewrite $K^*(n,1)$ as $2\int_0^{1/2}\left|B_n(t)-B_n(x)\right|dt$. As Bernoulli polynomials are monotonous on $(0,1/2)$ for an even n, inequality (1.17) follows. For $1/2\le x\le 1$ the statement follows analogously. Using similar arguments we get (1.16). □

Remark 1.1 For $m=1$, formulae (1.9) and (1.10) reduce to (1.1) and (1.2), and thus give all the results from [30] i.e. the generalizations of Ostrowski's inequality; especially, (1.14) produces the classical Ostrowski's inequality for $m=1$.

For $m=1$ and $n=2$, (1.17) gives an improvement of a result obtained in [38]. This was discussed in detail in [30].

Further, taking $m=1$ and $n=3$ in (1.15) produces a result obtained in [4]. These results are therefore a generalization of the results from that paper.

Corollary 1.2 *Let* $f:[0,1]\to\mathbb{R}$ *be such that* $f^{(n)}\in L_1[0,1]$ *and* $x\in[0,1]$*. For* $n=1$*, we have*

$$\left|\int_0^1 f(t)dt-\frac{1}{m}\sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right)+\frac{B_1(x)}{m}[f(1)-f(0)]\right|\le\frac{\|f'\|_1}{2m}, \tag{1.18}$$

$$\left|\int_0^1 f(t)dt-\frac{1}{m}\sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right)\right|\le\frac{\|f'\|_1}{m}\left(\frac{1}{2}+\left|x-\frac{1}{2}\right|\right), \tag{1.19}$$

For an odd n*,* $n\ge 3$*, we have*

$$\left|\int_0^1 f(t)dt-\frac{1}{m}\sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right)+T_n(x)\right|<\frac{2\|f^{(n)}\|_1}{(1-2^{-n-1})(2\pi m)^n}, \tag{1.20}$$

$$\left| \int_0^1 f(t)dt - \frac{1}{m} \sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) + T_{n-1}(x) \right| \tag{1.21}$$
$$< \frac{\|f^{(n)}\|_1}{m^n \cdot n!} \left(\frac{2n!}{(1-2^{-n-1})(2\pi)^n} + |B_n(x)| \right),$$

If n is even, then we have

$$\left| \int_0^1 f(t)dt - \frac{1}{m} \sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) + T_n(x) \right| \le \frac{|B_n|}{m^n \cdot n!} \cdot \|f^{(n)}\|_1, \tag{1.22}$$

$$\left| \int_0^1 f(t)dt - \frac{1}{m} \sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) + T_{n-1}(x) \right| \tag{1.23}$$
$$\le \frac{\|f^{(n)}\|_1}{m^n \cdot n!} \left((1-2^{-n})|B_n| + \left|2^{-n}B_n - B_n(x)\right| \right).$$

Proof. Put $p = 1$ in Theorem 1.3. Inequalities (1.18) and (1.19) follow by direct calculation. Using estimations of the maximal value of Bernoulli polynomials (cf. [1]), we get (1.20), (1.21) and (1.22). Finally, since $B_n(t)$ are symmetric about $t = 1/2$ for an even n, it is enough to consider them on $(0, 1/2)$ and there they are monotonous. So the maximal value of $|B_n(t) - B_n(x)|$ is obtained either for $t = 0$ or for $t = 1/2$. Using formula

$$\max\{|A|, |B|\} = \frac{1}{2}(|A+B| + |A-B|),$$

(1.23) follows. □

Corollary 1.3 *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n)} \in L_2[0,1]$ and $x \in [0,1]$. Then we have*

$$\left| \int_0^1 f(t)dt - \frac{1}{m} \sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) + T_n(x) \right| \le \frac{\|f^{(n)}\|_2}{m^n} \left(\frac{|B_{2n}|}{(2n)!} \right)^{1/2}, \tag{1.24}$$

$$\left| \int_0^1 f(t)dt - \frac{1}{m} \sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) + T_{n-1}(x) \right| \tag{1.25}$$
$$\le \frac{\|f^{(n)}\|_2}{m^n \cdot n!} \left(\frac{(n!)^2}{(2n)!}|B_{2n}| + B_n^2(x) \right)^{1/2},$$

Proof. Both inequalities follow by direct calculation after taking $p = 2$ in Theorem 1.3. □

It is interesting to consider which $x \in [0,1]$ gives the optimal estimation in inequalities (1.15) and (1.17). In (1.14) it is obvious that $x = 1/2$ is such point. Differentiating the function on the right-hand side of (1.17) – this is the case when n is even – it is easy to see that it obtains its minimum for $x = 1/4$ and $x = 3/4$ (for $n \ge 2$) while its maximal value is in $x = 0$ and $x = 1$ (for $n \geq 4$). Of course, the minimal value is of greater interest. In that case, the quadrature formulae take the following form

$$\int_0^1 f(t)dt \approx \frac{1}{4}\left(f(1) + 4f\left(\frac{1}{4}\right) - f(0)\right)$$
$$\int_0^1 f(t)dt \approx \frac{1}{4}\left(f(0) + 4f\left(\frac{3}{4}\right) - f(1)\right)$$

Also, if we take these parameters and put them in (1.10), then add them up and divide by 2, we get a two-point formula where the integral is approximated by values of the function in $x = 1/4$ and $x = 3/4$. The error estimation for this formula can be deduced from the following, more general, estimation. Using triangle inequality, we get

$$\begin{aligned}&\left|\int_0^1 f(t)dt - \frac{1}{2m}\sum_{k=0}^{m-1}\left(f\left(\frac{x+k}{m}\right) + f\left(\frac{1-x+k}{m}\right)\right)\right.\\ &\left.+\sum_{j=1}^{(n-2)/2}\frac{B_{2j}(x)}{(2j)!\cdot m^{2j}}[f^{(2j-1)}(1) - f^{(2j-1)}(0)]\right|\\ &\leq \frac{\|f^{(n)}\|_\infty}{m^n\cdot n!}\left((-1)^{n/2}(1-4|x-1/2|)B_n(x) + \frac{4}{n+1}|B_{n+1}(x)|\right).\end{aligned}$$

Therefore, this formula gives the best error estimate for $x = 1/4$.

On the other hand, inequality (1.15) behaves quite oppositely (this is the case when n is odd and $n \geq 3$). Observe that x_1 is a decreasing function of x and it is differentiable on $(0, 1/2)$. This is sufficient since the function on the right-hand side of that inequality (denote it by $F(x)$) obtains the same value for x and $1-x$. For $0 \leq x \leq 1/2$, we get

$$F'(x) = (-1)^{(n+1)/2}\cdot n(1-2|x-x_1|)B_{n-1}(x).$$

Since $F'(x)$ changes sign from positive to negative when passing through point $\alpha \in (0, 1/2)$ such that $B_{n-1}(\alpha) = 0$, we conclude that $F(x)$ obtains maximal value at α. Note that α is close to $1/4$, but a bit smaller. Minimum is then obtained at the end points of the interval i.e. for $x = 0$ and $x = 1/2$ (the same value is obtained at both of these points).

1.2.1 Trapezoid formula

Choosing $x = 0$ and $x = 1$ in (1.9) and (1.10) when $m = 1$, adding those two formulae up and then dividing the resulting formula by 2, produces the Euler trapezoid formulae - and all the other results - obtained in [25]. Here, we just state the error estimates for this type of quadrature formulae. Namely, for $p = \infty$ and $p = 1$, we have:

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(0) + f(1)]\right| \leq C_T(m,q)\cdot\|f^{(m)}\|_p, \quad m = 1,2$$

where

$$C_T(1,1) = \frac{1}{4}, \quad C_T(1,\infty) = \frac{1}{2}, \quad C_T(2,1) = \frac{1}{12}, \quad C_T(2,\infty) = \frac{1}{8},$$

while for $m = 2, 3, 4$

$$\left| \int_0^1 f(t)dt - \frac{1}{2}[f(0) + f(1)] + \frac{1}{12}[f'(1) - f'(0)] \right| \le C_T(m,q) \cdot \|f^{(m)}\|_p,$$

where

$$C_T(2,1) = \frac{1}{18\sqrt{3}}, \quad C_T(3,1) = \frac{1}{192}, \quad C_T(4,1) = \frac{1}{720},$$
$$C_T(2,\infty) = \frac{1}{12}, \quad C_T(3,\infty) = \frac{1}{72\sqrt{3}}, \quad C_T(4,\infty) = \frac{1}{384}.$$

1.2.2 Midpoint formula

For $m = 1$ and $x = 1/2$ in (1.9) and (1.10), we get the Euler midpoint formulae derived in [23] and of course all other results from that paper follow directly. The error estimates for this type of quadrature formulae, for $p = \infty$ and $p = 1$, are:

$$\left| \int_0^1 f(t)dt - f\left(\frac{1}{2}\right) \right| \le C_M(m,q) \cdot \|f^{(m)}\|_p, \quad m = 1, 2$$

where

$$C_M(1,1) = \frac{1}{4}, \quad C_M(1,\infty) = \frac{1}{2}, \quad C_M(2,1) = \frac{1}{24}, \quad C_M(2,\infty) = \frac{1}{8},$$

while for $m = 2, 3, 4$

$$\left| \int_0^1 f(t)dt - f\left(\frac{1}{2}\right) - \frac{1}{24}[f'(1) - f'(0)] \right| \le C_M(m,q) \cdot \|f^{(m)}\|_p,$$

where

$$C_M(2,1) = \frac{1}{18\sqrt{3}}, \quad C_M(3,1) = \frac{1}{192}, \quad C_M(4,1) = \frac{7}{5760},$$
$$C_M(2,\infty) = \frac{1}{12}, \quad C_M(3,\infty) = \frac{1}{72\sqrt{3}}, \quad C_M(4,\infty) = \frac{1}{384}.$$

Chapter 2

Euler two-point formulae

2.1 Introduction

In this chapter we study, for each real number $x \in [0,1/2]$, the general two-point quadrature formula

$$\int_0^1 f(t)dt = \frac{1}{2}[f(x)+f(1-x)]+E(f;,x) \tag{2.1}$$

with $E(f;x)$ being the remainder. This family of two-point quadrature formulae was considered by Guessab and Schmeisser in [68] and they established sharp estimates for the remainder under various regularity conditions. The aim of this chapter is to establish general two-point formula (2.1) using identities (1.1) and (1.2) and give various error estimates for the quadrature rules based on such generalizations. In Section 2 we use the extended Euler formulae to obtain two new integral identities. We call them the general Euler two-point formulae. In Section 3, we prove a number of inequalities which give error estimates for the general Euler two-point formulae for functions whose derivatives are from the L_p-spaces, thus we extend the results from [68] and we generalize the results from papers [25]-[27], [83] and [84]. These inequalities are generally sharp (in case $p=1$ the best possible). Special attention is devoted to the case where we have some boundary conditions and in some cases we compare our estimates with the Fink's estimates ([68], [45]). In Section 4 we give a variant of the inequality proved in the paper [91] and we use those results to prove some inequalities for the general Euler two-point formula. The general Euler two-point formulae are used in Section 5 with functions possessing various convexity and concavity properties to derive inequalities pertinent to numerical integration. In Section 6 we generalize estimation of two-point formula by using pre-Grüss inequality and in Section 7 we give Hermite-Hadamard's inequalities of Bullen type.

2.2 General Euler two-point formulae

The results from this and next section are published in [98].

For $k \geq 1$ and fixed $x \in [0,1/2]$ define the functions $G_k^x(t)$ and $F_k^x(t)$ as

$$G_k^x(t) = B_k^*(x-t) + B_k^*(1-x-t),\ t \in \mathbb{R}$$

and $F_k^x(t) = G_k^x(t) - \tilde{B}_k(x),\ t \in \mathbb{R}$, where

$$\tilde{B}_k(x) = B_k(x) + B_k(1-x),\ x \in [0,1/2],\ k \geq 1.$$

Especially, we get $\tilde{B}_1(x) = 0,\ \tilde{B}_2(x) = 2x^2 - 2x + 1/3,\ \tilde{B}_3(x) = 0$. Also, for $k \geq 2$ we have $\tilde{B}_k(x) = G_k^x(0)$, that is $F_k^x(t) = G_k^x(t) - G_k^x(0),\ k \geq 2$, and $F_1^x(t) = G_1^x(t),\ t \in \mathbb{R}$. Obviously, $G_k^x(t)$ and $F_k^x(t)$ are periodic functions of period 1 and continuous for $k \geq 2$.

Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n-1)}$ exists on $[0,1]$ for some $n \geq 1$. We introduce the following notation for each $x \in [0,1/2]$

$$D(x) = \frac{1}{2}\left[f(x) + f(1-x)\right].$$

Further, we define $\tilde{T}_0(x) = 0$ and, for $1 \leq m \leq n,\ x \in [0,1/2]$

$$\tilde{T}_m(x) = \frac{1}{2}\left[T_m(x) + T_m(1-x)\right],$$

where $T_m(x)$ is given by (1.3). It is easy to see that

$$\tilde{T}_m(x) = \frac{1}{2}\sum_{k=1}^{m} \frac{\tilde{B}_k(x)}{k!}\left[f^{(k-1)}(1) - f^{(k-1)}(0)\right]. \tag{2.2}$$

In the next theorem we establish two formulae which play the key role in this chapter. We call them the general Euler two-point formulae.

Theorem 2.1 *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$, for some $n \geq 1$. Then for each $x \in [0,1/2]$*

$$\int_0^1 f(t)\mathrm{d}t = D(x) - \tilde{T}_n(x) + \tilde{R}_n^1(f) \tag{2.3}$$

and

$$\int_0^1 f(t)\mathrm{d}t = D(x) - \tilde{T}_{n-1}(x) + \tilde{R}_n^2(f), \tag{2.4}$$

where

$$\tilde{R}_n^1(f) = \frac{1}{2(n!)}\int_0^1 G_n^x(t)\,\mathrm{d}f^{(n-1)}(t),\quad \tilde{R}_n^2(f) = \frac{1}{2(n!)}\int_0^1 F_n^x(t)\,\mathrm{d}f^{(n-1)}(t).$$

Proof. Put $x \equiv x$ and $x \equiv 1-x$ in formula (1.1) to get two new formulae. Then multiply these new formulae by $1/2$ and add them up. The result is formula (2.3). Formula (2.4) is obtained from (1.2) by the same procedure. □

Remark 2.1 If in Theorem 2.1 we choose $x = 0, 1/2, 1/3, 1/4$ we get Euler trapezoid [25], Euler midpoint [23], Euler two-point Newton-Cotes [84] and Euler two-point Maclaurin formulae respectively.

By direct calculations for each $x \in [0,1/2]$, we get

$$F_1^x(t) = G_1^x(t) = \begin{cases} -2t, & 0 \le t \le x \\ -2t+1, & x < t \le 1-x \\ -2t+2, & 1-x < t \le 1 \end{cases}, \tag{2.5}$$

$$G_2^x(t) = \begin{cases} 2t^2+2x^2-2x+1/3, & 0 \le t \le x \\ 2t^2-2t+2x^2+1/3, & x < t \le 1-x \\ 2t^2-4t+2x^2-2x+7/3, & 1-x < t \le 1 \end{cases}, \tag{2.6}$$

$$F_2^x(t) = \begin{cases} 2t^2, & 0 \le t \le x \\ 2t^2-2t+2x, & x < t \le 1-x \\ 2t^2-4t+2, & 1-x < t \le 1 \end{cases} \tag{2.7}$$

and

$$F_3^x(t) = G_3^x(t) = \begin{cases} -2t^3+(-6x^2+6x-1)t, & 0 \le t \le x \\ -2t^3+3t^2+(-6x^2-1)t+3x^2, & x < t \le 1-x \\ -2t^3+6t^2+(-6x^2+6x-7)t & \\ \qquad +6x^2-6x+3, & 1-x < t \le 1. \end{cases} \tag{2.8}$$

Now, we will prove some properties of the functions $G_k^x(t)$ and $F_k^x(t)$ defined above. The Bernoulli polynomials are symmetric with respect to $1/2$, that is

$$B_k(1-x) = (-1)^k B_k(x), \quad k \ge 1. \tag{2.9}$$

Also, we have $B_k(1) = B_k(0) = B_k$, $k \ge 2$, $B_1(1) = -B_1(0) = 1/2$ and $B_{2j-1} = 0$, $j \ge 2$. Therefore, we get $\tilde{B}_{2j-1}(x) = 0,\ j \ge 1$ and $\tilde{B}_{2j}(x) = 2B_{2j}(x),\ x \in [0,1/2]$. Now, we have $F_{2j-1}^x(t) = G_{2j-1}^x(t),\ j \ge 1$, and

$$F_{2j}^x(t) = G_{2j}^x(t) - \tilde{B}_{2j}(x) = G_{2j}^x(t) - 2B_{2j}(x),\ x \in [0,1/2], \quad j \ge 1. \tag{2.10}$$

Further, the points 0 and 1 are zeros of $F_k^x(t) = G_k^x(t) - G_k^x(0),\ k \ge 2$, that is $F_k^x(0) = F_k^x(1) = 0,\ k \ge 1$. As we shall see below, 0 and 1 are the only zeros of $F_{2j}^x(t)$ for $j \ge 2$ and $x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right] \cup \left[\frac{1}{2\sqrt{3}}, \frac{1}{2}\right]$. Next, setting $t = 1/2$ in (2.9) we get $B_k(1/2) = (-1)^k B_k(1/2),\ k \ge 1$, which implies that $B_{2j-1}(1/2) = 0,\ j \ge 1$. Using the above formulae, we get $F_{2j-1}^x(1/2) = G_{2j-1}^x(1/2) = 0,\ j \ge 1$. We shall see that 0, 1/2 and 1 are the only zeros of $F_{2j-1}^x(t) = G_{2j-1}^x(t)$, for $j \ge 2$ and $x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right] \cup \left[\frac{1}{2\sqrt{3}}, \frac{1}{2}\right]$. Also, note that for $x \in [0,1/2],\ j \ge 1$ $G_{2j}^x(1/2) = 2B_{2j}(1/2-x)$ and

$$F_{2j}^x(1/2) = G_{2j}^x(1/2) - \tilde{B}_{2j}(x) = 2B_{2j}(1/2-x) - 2B_{2j}(x). \tag{2.11}$$

Lemma 2.1 *For $k \geq 2$ we have $G_k^x(1-t) = (-1)^k G_k^x(t)$, $0 \leq t \leq 1$ and $F_k^x(1-t) = (-1)^k F_k^x(t)$, $0 \leq t \leq 1$.*

Proof. As the functions $B_k^*(t)$ are periodic with period 1 and continuous for $k \geq 2$. Therefore, for $k \geq 2$ and $0 \leq t \leq 1$ we have

$$\begin{aligned}
G_k^x(1-t) &= B_k^*(x-1+t) + B_k^*(-x+t) \\
&= \begin{cases} B_k(x+t) + B_k(1-x+t), & 0 \leq t \leq x, \\ B_k(x+t) + B_k(-x+t), & x < t \leq 1-x, \\ B_k(-1+x+t) + B_k(-x+t), & 1-x < t \leq 1, \end{cases} \\
&= (-1)^k \times \\
&\quad \begin{cases} B_k(1-x-t) + B_k(x-t), & 0 \leq t \leq x, \\ B_k(1-x-t) + B_k(1+x-t), & x < t \leq 1-x, \\ B_k(2-x-t) + B_k(1+x-t), & 1-x < t \leq 1, \end{cases} \\
&= (-1)^k G_k^x(t),
\end{aligned}$$

which proves the first identity. Further, we have $F_k^x(t) = G_k^x(t) - G_k^x(0)$ and $(-1)^k G_k^x(0) = G_k^x(0)$, since $G_{2j+1}^x(0) = 0$, so that we have

$$F_k^x(1-t) = G_k^x(1-t) - G_k^x(0) = (-1)^k [G_k^x(t) - G_k^x(0)] = (-1)^k F_k^x(t),$$

which proves the second identity. □

Note that the identities established in Lemma 2.1 are valid for $k = 1$, too, except at the points x, and $1-x$ of discontinuity of $F_1^x(t) = G_1^x(t)$.

Lemma 2.2 *For $k \geq 2$ and $x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right] \cup \left[\frac{1}{2\sqrt{3}}, \frac{1}{2}\right]$ the function $G_{2k-1}^x(t)$ has no zeros in the interval $(0, 1/2)$. For $0 < t < 1/2$ the sign of this function is determined by*

$$(-1)^{k-1} G_{2k-1}^x(t) > 0,\ x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right] \text{ and } (-1)^k G_{2k-1}^x(t) > 0,\ x \in \left[\frac{1}{2\sqrt{3}}, \frac{1}{2}\right].$$

Proof. For $k = 2$, $G_3^x(t)$ is given by (2.8) and it is easy to see that for each $x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right]$

$$G_3^x(t) < 0,\ 0 < t < \frac{1}{2}$$

and for each $x \in \left[\frac{1}{2\sqrt{3}}, \frac{1}{2}\right]$

$$G_3^x(t) > 0,\ 0 < t < \frac{1}{2}.$$

Thus, our assertion is true for $k = 2$. Now, assume that $k \geq 3$. Then $2k-1 \geq 5$ and $G_{2k-1}^x(t)$ is continuous and at least twice differentiable function. Using (A-2) we get

$$G^{x\prime}_{2k-1}(t) = -(2k-1) G_{2k-2}^x(t)$$

and

$$G^{x\prime\prime}_{2k-1}(t) = (2k-1)(2k-2)G^{x}_{2k-3}(t).$$

Let us suppose that G^x_{2k-3} has no zeros in the interval $\left(0, \frac{1}{2}\right)$. We know that 0 and $\frac{1}{2}$ are zeros of $G^x_{2k-1}(t)$. Let us suppose that some α, $0 < \alpha < \frac{1}{2}$, is also a zero of $G^x_{2k-1}(t)$. Then inside each of the intervals $(0,\alpha)$ and $\left(\alpha, \frac{1}{2}\right)$ the derivative $G^{x\prime}_{2k-1}(t)$ must have at least one zero, say β_1, $0 < \beta_1 < \alpha$ and β_2, $\alpha < \beta_2 < \frac{1}{2}$. Therefore, the second derivative $G^{x\prime\prime}_{2k-1}(t)$ must have at least one zero inside the interval (β_1, β_2). Thus, from the assumption that $G^x_{2k-1}(t)$ has a zero inside the interval $\left(0, \frac{1}{2}\right)$, it follows that $(2k-1)(2k-2)G^x_{2k-3}(t)$ also has a zero inside this interval. Thus, $G^x_{2k-1}(t)$ can not have a zero inside the interval $\left(0, \frac{1}{2}\right)$. To determine the sign of $G^x_{2k-1}(t)$, note that

$$G^x_{2k-1}(x) = B_{2k-1}(1-2x).$$

We have [1, 23.1.14]

$$(-1)^k B_{2k-1}(t) > 0,\ 0 < t < \frac{1}{2},$$

which implies for $x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right]$

$$(-1)^{k-1} G^x_{2k-1}(x) = (-1)^{k-1} B_{2k-1}(1-2x) = (-1)^k B_{2k-1}(2x) > 0$$

and

$$(-1)^k G^x_{2k-1}(x) = (-1)^k B_{2k-1}(1-2x) > 0 \text{ for } x \in \left[\frac{1}{2\sqrt{3}}, \frac{1}{2}\right].$$

Consequently, we have

$$(-1)^{k-1} G^x_{2k-1}(t) > 0,\ 0 < t < \frac{1}{2} \text{ for } x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right]$$

and

$$(-1)^k G^x_{2k-1}(t) > 0,\ 0 < t < \frac{1}{2} \text{ for } x \in \left[\frac{1}{2\sqrt{3}}, \frac{1}{2}\right].$$

□

Corollary 2.1 *For $k \geq 2$ and $x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right]$ the functions $(-1)^k F^x_{2k}(t)$ and $(-1)^k G^x_{2k}(t)$ are strictly increasing on the interval $(0,1/2)$, and strictly decreasing on the interval $(1/2,1)$. Also, for $x \in \left[\frac{1}{2\sqrt{3}}, \frac{1}{2}\right]$ the functions $(-1)^{k-1} F^x_{2k}(t)$ and $(-1)^{k-1} G^x_{2k}(t)$ are strictly increasing on the interval $(0,1/2)$, and strictly decreasing on the interval $(1/2,1)$. Further, for $k \geq 2$, we have*

$$\max_{t\in[0,1]} |F^x_{2k}(t)| = 2\,|B_{2k}(1/2-x) - B_{2k}(x)|$$

and

$$\max_{t\in[0,1]} |G^x_{2k}(t)| = 2\max\{|B_{2k}(x)|, |B_{2k}(1/2-x)|\}.$$

Proof. Using (A-2) we get

$$\left[(-1)^k F_{2k}^x(t)\right]' = \left[(-1)^k G_{2k}^x(t)\right]' = 2k(-1)^{k-1} G_{2k-1}^x(t)$$

and $(-1)^{k-1} G_{2k-1}^x(t) > 0$ for $0 < t < 1/2$ and $x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right]$, by Lemma 2.2. Thus, $(-1)^k F_{2k}^x(t)$ and $(-1)^k G_{2k}^x(t)$ are strictly increasing on the interval $(0,1/2)$. Also, by Lemma 2.1, we have $F_{2k}^x(1-t) = F_{2k}^x(t)$, $0 \le t \le 1$ and $G_{2k}^x(1-t) = G_{2k}^x(t)$, $0 \le t \le 1$, which implies that $(-1)^k F_{2k}^x(t)$ and $(-1)^k G_{2k}^x(t)$ are strictly decreasing on the interval $(1/2,1)$. The proof of second statement is similar. Further, $F_{2k}^x(0) = F_{2k}^x(1) = 0$, which implies that $\left|F_{2k}^x(t)\right|$ achieves its maximum at $t = 1/2$, that is

$$\max_{t\in[0,1]} \left|F_{2k}^x(t)\right| = \left|F_{2k}^x(1/2)\right| = 2\left|B_{2k}(1/2-x) - B_{2k}(x)\right|.$$

Also

$$\max_{t\in[0,1]} \left|G_{2k}^x(t)\right| = \max\left\{\left|G_{2k}^x(0)\right|, \left|G_{2k}^x(1/2)\right|\right\} = 2\max\left\{\left|B_{2k}(x)\right|, \left|B_{2k}(1/2-x)\right|\right\}, \quad (2.12)$$

which completes the proof. □

Corollary 2.2 *For $k \ge 2$, and $x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right] \cup \left[\frac{1}{2\sqrt{3}}, \frac{1}{2}\right]$ we have*

$$\int_0^1 \left|F_{2k-1}^x(t)\right| \mathrm{d}t = \int_0^1 \left|G_{2k-1}^x(t)\right| \mathrm{d}t = \frac{2}{k}\left|B_{2k}(1/2-x) - B_{2k}(x)\right|.$$

Also, we have

$$\int_0^1 \left|F_{2k}^x(t)\right| \mathrm{d}t = \left|\tilde{B}_{2k}(x)\right| = 2\left|B_{2k}(x)\right| \text{ and } \int_0^1 \left|G_{2k}^x(t)\right| \mathrm{d}t \le 2\left|\tilde{B}_{2k}(x)\right| = 4\left|B_{2k}(x)\right|.$$

Proof. Using Lemma 2.1 and Lemma 2.2 we get

$$\begin{aligned}\int_0^1 \left|G_{2k-1}^x(t)\right| \mathrm{d}t &= 2\left|\int_0^{1/2} G_{2k-1}^x(t)\mathrm{d}t\right| = 2\left|-\frac{1}{2k}\, G_{2k}^x(t)\Big|_0^{1/2}\right| \\ &= \frac{1}{k}\left|G_{2k}^x(1/2) - G_{2k}^x(0)\right| = \frac{2}{k}\left|B_{2k}(1/2-x) - B_{2k}(x)\right|,\end{aligned}$$

which proves the first assertion. By Corollary 2.1 and because $F_{2k}^x(0) = F_{2k}^x(1) = 0$, $F_{2k}^x(t)$ does not change its sign on the interval $(0,1)$. Therefore, using (2.10) we get

$$\begin{aligned}\int_0^1 \left|F_{2k}^x(t)\right| \mathrm{d}t &= \left|\int_0^1 F_{2k}^x(t)\mathrm{d}t\right| = \left|\int_0^1 \left[G_{2k}^x(t) - \tilde{B}_{2k}(x)\right] \mathrm{d}t\right| \\ &= \left|-\frac{1}{2k+1}\, G_{2k+1}^x(t)\Big|_0^1 - \tilde{B}_{2k}(x)\right| = \left|\tilde{B}_{2k}(x)\right|,\end{aligned}$$

which proves the second assertion. Finally, we use (2.10) again and the triangle inequality to obtain the third formula. □

2.3 Inequalities related to the general Euler two-point formulae

In this section we use formulae established in Theorem 2.1 to prove a number of inequalities using L_p norms for $1 \le p \le \infty$. These inequalities are generally sharp (in case $p = 1$ the best possible). Special attention is devoted to the case where we have some boundary conditions and in some cases we compare our constants with the Fink constants ([68], [45]).

Theorem 2.2 *Assume (p,q) is a pair of conjugate exponents, $1 \le p,q \le \infty$ and $f : [0,1] \to \mathbb{R}$ is such that $f^{(n)} \in L_p[0,1]$ for some $n \ge 1$. Then for every $x \in [0,1/2]$, we have*

$$\left| \int_0^1 f(t)\mathrm{d}t - D(x) + \tilde{T}_{n-1}(x) \right| \le K(n,p,x) \cdot \|f^{(n)}\|_p, \tag{2.13}$$

and

$$\left| \int_0^1 f(t)\mathrm{d}t - D(x) + \tilde{T}_n(x) \right| \le K^*(n,p,x) \cdot \|f^{(n)}\|_p, \tag{2.14}$$

where

$$K(n,p,x) = \frac{1}{2(n!)} \left[\int_0^1 |F_n^x(t)|^q \,\mathrm{d}t \right]^{\frac{1}{q}}, K^*(n,p,x) = \frac{1}{2(n!)} \left[\int_0^1 |G_n^x(t)|^q \,\mathrm{d}t \right]^{\frac{1}{q}}.$$

The constants $K(n,p,x)$ and $K^*(n,p,x)$ are sharp for $1 < p \le \infty$ and the best possible for $p = 1$.

Proof. Applying the Hölder inequality we have

$$\left| \frac{1}{2(n!)} \int_0^1 F_n^x(t) f^{(n)}(t)\mathrm{d}t \right| \le \frac{1}{2(n!)} \left[\int_0^1 |F_n^x(t)|^q \,\mathrm{d}t \right]^{\frac{1}{q}} \cdot \left\| f^{(n)} \right\|_p = K(n,p,x) \cdot \|f^{(n)}\|_p.$$

Using the above inequality from (2.4) we get estimate (2.13). In the same manner, from (2.3) we get estimate (2.14). Now, we consider the optimality of $K(n,p,x)$. We shall find a function f such that

$$\left| \int_0^1 F_n^x(t) f^{(n)} dt \right| = \left(\int_0^1 |F_n^x(t)|^q dt \right)^{\frac{1}{q}} \left(\int_0^1 |f^{(n)}(t)|^p dt \right)^{\frac{1}{p}}.$$

For $1 < p < \infty$ take f to be such that

$$f^{(n)}(t) = \mathrm{sgn} F_n^x(t) \cdot |F_n^x(t)|^{\frac{1}{p-1}} \tag{2.15}$$

where for $p = \infty$ we put $f^{(n)}(t) = \mathrm{sgn} F_n^x(t)$. For constant $K^*(n,p,x)$ the proof of sharpness is analogous. For $p = 1$ we shall prove that

$$\left| \int_0^1 F_n^x(t) f^{(n)}(t) dt \right| \le \max_{t \in [0,1]} |F_n^x(t)| \int_0^1 |f^{(n)}(t)| dt \tag{2.16}$$

is the best possible inequality. Suppose that $|F_n^x(t)|$ attains its maximum at $t_0 \in (0,1)$. First, we assume that $F_n^x(t_0) > 0$. For ε small enough define $f_\varepsilon^{(n-1)}(t)$ by

$$f_\varepsilon^{(n-1)}(t) = \begin{cases} 0, & t \le t_0 \\ \frac{1}{\varepsilon}(t-t_0), & t \in [t_0, t_0+\varepsilon] \\ 1, & t \ge t_0+\varepsilon \end{cases}.$$

Then, for ε small enough

$$\left| \int_0^1 F_n^x(t) f_\varepsilon^{(n)}(t)dt \right| = \left| \int_{t_0}^{t_0+\varepsilon} F_n^x(t) \frac{1}{\varepsilon} dt \right| = \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} F_n^x(t)dt.$$

Now, from inequality (2.16) we have

$$\frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} F_n^x(t)dt \le F_n^x(t_0) \int_{t_0}^{t_0+\varepsilon} \frac{1}{\varepsilon} dt = F_n^x(t_0).$$

Since,

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} F_n^x(t)dt = F_n^x(t_0)$$

the statement follows. If $F_n^x(t_0) < 0$, then we take

$$f_\varepsilon^{(n-1)}(t) = \begin{cases} 1, & t \le t_0 \\ -\frac{1}{\varepsilon}(t-t_0-\varepsilon), & t \in [t_0, t_0+\varepsilon] \\ 0, & t \ge t_0+\varepsilon \end{cases}$$

and the rest of proof is the same as above. Proof of the best possibility of the second inequality is similar. □

Remark 2.2 Basically we have three types of estimates

$$\left| \int_0^1 f(t)dt - D(x) + \tilde{T}_{2k}(x) \right| \le \frac{1}{2(2k)!} \left(\int_0^1 |G_{2k}^x(t)|^q dt \right)^{\frac{1}{q}} \left(\int_0^1 |f^{(2k)}(t)|^p dt \right)^{\frac{1}{p}},$$

$$\left| \int_0^1 f(t)dt - D(x) + \tilde{T}_{2k}(x) \right| \le \frac{1}{2(2k+1)!} \left(\int_0^1 |G_{2k+1}^x(t)|^q dt \right)^{\frac{1}{q}} \left(\int_0^1 |f^{(2k+1)}(t)|^p dt \right)^{\frac{1}{p}}$$

and

$$\left| \int_0^1 f(t)dt - D(x) + \tilde{T}_{2k}(x) \right| \le \frac{1}{2(2k+2)!} \left(\int_0^1 |F_{2k+2}^x(t)|^q dt \right)^{\frac{1}{q}} \left(\int_0^1 |f^{(2k+2)}(t)|^p dt \right)^{\frac{1}{p}}.$$

In the following theorem we are interested in sharpness of the above estimates in the presence of boundary conditions.

Theorem 2.3 *Assume that (p,q) is a pair of conjugate exponents, $1 \le p,q \le \infty$ and $k \in \mathbb{N}$. Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(2i-1)}(0) = f^{(2i-1)}(1)$ for $i = 1,\dots,k$ and $x \in [0,1/2]$. If $f^{(2k)} \in L_p[0,1]$, we have*

$$\left| \int_0^1 f(t)dt - D(x) \right| \le \frac{1}{2(2k)!} \left(\int_0^1 |G_{2k}^x(t)|^q dt \right)^{\frac{1}{q}} \left(\int_0^1 |f^{(2k)}(t)|^p dt \right)^{\frac{1}{p}}. \tag{2.17}$$

If $f^{(2k+1)} \in L_p[0,1]$, we have

$$\left| \int_0^1 f(t)dt - D(x) \right| \le \frac{1}{2(2k+1)!} \left(\int_0^1 |G_{2k+1}^x(t)|^q dt \right)^{\frac{1}{q}} \left(\int_0^1 |f^{(2k+1)}(t)|^p dt \right)^{\frac{1}{p}}. \tag{2.18}$$

If $f^{(2k+2)} \in L_p[0,1]$, we have

$$\left| \int_0^1 f(t)dt - D(x) \right| \le \frac{1}{2(2k+2)!} \left(\int_0^1 |F_{2k+2}^x(t)|^q dt \right)^{\frac{1}{q}} \left(\int_0^1 |f^{(2k+2)}(t)|^p dt \right)^{\frac{1}{p}}. \tag{2.19}$$

Inequality (2.17) is sharp for $p = 2$ and inequalities (2.18) and (2.19) are sharp for $1 < p \le \infty$ and best possible for $p = 1$.

Proof. Inequality (2.17) is sharp since a function f for which we have equality in (2.14) in case $p = 2$, $n = 2k$ is defined by $f^{(2k)}(t) = G_{2k}^x(t)$, so we can choose f such that

$$f^{(2k-1)}(t) = -\frac{1}{2k+1} G_{2k+1}^x(t), \quad f^{(2k-3)}(t) = -\frac{1}{(2k+1)(2k+2)(2k+3)} G_{2k+3}^x(t)$$

and generally

$$f^{(2i-1)}(t) = -\frac{1}{(2k+1)(2k+2)\dots(4k-2i+1)} G_{4k-2i+1}^x(t), \; i = 1,\dots,k$$

which give $f^{(2i-1)}(0) = f^{(2i-1)}(1) = 0$, $i = 1,\dots,k$. With regard to the sharpness or the best possibility of inequality (2.18), notice first that approximation $\int_0^1 f(t) \approx D(x) - \tilde{T}_{2k}(x)$ is exact of order $2k+1$. Take any function f which is optimal for inequality (2.13) in case $n = 2k+1$, $1 \le p \le \infty$. Set

$$g(t) = f(t) + \sum_{i=1}^{2k} a_i t^i = f(t) + a_{2k} t^{2k} + a_{2k-1} t^{2k-1} + \dots + a_2 t^2 + a_1 t.$$

We have

$$g^{(2k-1)}(t) = f^{(2k-1)}(t) + (2k)! a_{2k} t + (2k-1)! a_{2k-1}$$

so

$$0 = g^{(2k-1)}(0) = f^{(2k-1)}(0) + (2k-1)! a_{2k-1}$$
$$0 = g^{(2k-1)}(1) = f^{(2k-1)}(1) + (2k)! a_{2k} + (2k-1)! a_{2k-1}$$

which gives a_{2k}, a_{2k-1}. Using $g^{(2k-3)}$ and conditions $g^{(2k-3)}(0) = 0 = g^{(2k-3)}(1)$ we analogously obtain a_{2k-2}, a_{2k-3} and so on. So, the function g is also optimal for (2.13) and satisfies boundary conditions $g^{(2i-1)}(1) = g^{(2i-1)}(0),\ i = 1, \ldots, k$. Inequality (2.19) can be treated in the same way. □

In the following we calculate the optimal constants in cases $p = 1,\ p = \infty$ and $p = 2$.

Corollary 2.3 *Let $f : [0,1] \to \mathbb{R}$ be L-Lipschitzian on $[0,1]$. Then for each $x \in [0,1/2]$*

$$\left| \int_0^1 f(t)\mathrm{d}t - D(x) \right| \leq \frac{4x^2 + (1-2x)^2}{4} \cdot L. \tag{2.20}$$

Proof. Using (2.5) for each $x \in [0,1/2]$ and applying (2.13) with $n = 1$ and $p = \infty$ we get the above inequality. □

Remark 2.3 The inequality (2.20), has been proved by A. Guessab and G. Schmeisser on interval $[a,b]$ in [68] (see also [34]). They also proved that this inequality is sharp for each admissible x. Equality is attained exactly in the case of equality in Theorem 2.2 where we put $f'(t) = \mathrm{sgn} F_1^x(t)$.

Corollary 2.4 *Let $f : [0,1] \to \mathbb{R}$ be such that f' is L-Lipschitzian on $[0,1]$. Then for each $x \in [0,1/4]$*

$$\left| \int_0^1 f(t)\mathrm{d}t - D(x) \right| \leq \left[-\frac{1}{2}\left(x^2 - x - \frac{1}{6} \right) + \frac{1}{6}(1-4x)^{3/2} \right] L \tag{2.21}$$

and for each $x \in [1/4,1/2]$

$$\left| \int_0^1 f(t)\mathrm{d}t - D(x) \right| \leq -\frac{1}{2}\left(x^2 - x + \frac{1}{6} \right) L. \tag{2.22}$$

Proof. Using (2.7) for each $x \in [0,1/4]$ and applying (2.13) with $n = 2$ and $p = \infty$ we get the above inequalities. □

Remark 2.4 The inequalities (2.21) and (2.22) have been proved by A. Guessab and G. Schmeisser on interval $[a,b]$ in [68]. They also proved that these inequalities are sharp for each admissible x.

Corollary 2.5 *Let $f : [0,1] \to \mathbb{R}$ be such that f' is L-Lipschitzian on $[0,1]$. Then for each $x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right]$*

$$\left| \int_0^1 f(t)\mathrm{d}t - D(x) + \frac{B_2(x)}{2}[f'(1) - f'(0)] \right| \leq \frac{1}{18\sqrt{3}}(1-12x^2)^{3/2} L,$$

for each $x \in \left[\frac{1}{2} - \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}\right]$

$$\begin{aligned} &\left| \int_0^1 f(t)\mathrm{d}t - D(x) + \frac{B_2(x)}{2}[f'(1) - f'(0)] \right| \\ &\leq \left[\frac{4}{3}\left(-x^2 + x - \frac{1}{6} \right)^{3/2} + \frac{1}{18\sqrt{3}}(1-12x^2)^{3/2} \right] L \end{aligned}$$

and for each $x \in \left[\frac{1}{2\sqrt{3}}, \frac{1}{2}\right]$

$$\left|\int_0^1 f(t)\mathrm{d}t - D(x) + \frac{B_2(x)}{2}[f'(1)-f'(0)]\right| \leq \frac{4}{3}\left(-x^2+x-\frac{1}{6}\right)^{3/2} L.$$

Proof. Using (2.6) for each $x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right]$ we get

$$\int_0^1 |G_2^x(t)|\,\mathrm{d}t = 2\int_0^{1/2} |G_2^x(t)|dt = 2\left[-\int_0^{1/2} G_2^x(t)dt + 2\int_0^{\frac{1}{2}-\frac{\sqrt{1-12x^2}}{2\sqrt{3}}} G_2^x(t)dt\right]$$

$$= 4\int_0^{\frac{1}{2}-\frac{\sqrt{1-12x^2}}{2\sqrt{3}}} G_2^x(t)dt = 4\left(-\frac{1}{3}G_3^x(t)|_0^{\frac{1}{2}-\frac{\sqrt{1-12x^2}}{2\sqrt{3}}}\right) = -\frac{4}{3}G_3^x\left(\frac{1}{2}-\frac{\sqrt{1-12x^2}}{2\sqrt{3}}\right),$$

for each $x \in \left[\frac{1}{2} - \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}\right]$

$$\begin{aligned}\int_0^1 |G_2^x(t)|\,\mathrm{d}t &= 4\int_{\sqrt{-x^2+x+\frac{1}{6}}}^{\frac{1}{2}-\frac{\sqrt{1-12x^2}}{2\sqrt{3}}} G_2^x(t)dt \\ &= \frac{4}{3}\left[G_3^x\left(\sqrt{-x^2+x-\frac{1}{6}}\right) - G_3^x\left(\frac{1}{2}-\frac{\sqrt{1-12x^2}}{2\sqrt{3}}\right)\right]\end{aligned}$$

and for each $x \in \left[\frac{1}{2\sqrt{3}}, \frac{1}{2}\right]$ we get

$$\int_0^1 |G_2^x(t)|\,\mathrm{d}t = -4\int_0^{\sqrt{-x^2+x-\frac{1}{6}}} G_2^x(t)dt = \frac{4}{3}G_3^x\left(\sqrt{-x^2+x-\frac{1}{6}}\right).$$

Therefore, applying (2.14) with $n=2$ and $p=\infty$ we get the above inequalities. □

Remark 2.5 In Theorem 2.3 it was proved that inequality (2.17) is sharp just for $p=2$. We mention here that comparing the sharp constant from Guessab and Schmeisser in [68] in case $p=\infty$ and our constant, we conclude that inequality (2.17) is not generally sharp. Namely, our constant for boundary conditions $f'(1)=f'(0)$, $n=2$ and $x=0$ is $1/(18\sqrt{3})$, while they have $1/32$ (note that the sharpness of inequality (2.17) under conditions $f'(1) = f'(0)$ implies the sharpness of the same inequality under conditions $f'(1)=f'(0)=0$).

Corollary 2.6 *Let* $f:[0,1]\to\mathbb{R}$ *be such that* f'' *is L-Lipschitzian on* $[0,1]$. *Then for each* $x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right]$

$$\left|\int_0^1 f(t)\mathrm{d}t - D(x) + \frac{B_2(x)}{2}[f'(1)-f'(0)]\right| \leq \left(\frac{x^3}{6}-\frac{x^2}{8}+\frac{1}{192}\right)L,$$

for each $x \in \left[\frac{1}{2} - \frac{1}{2\sqrt{3}}, \frac{1}{4}\right]$

$$\left|\int_0^1 f(t)\mathrm{d}t - D(x) + \frac{B_2(x)}{2}[f'(1) - f'(0)]\right| \le \left[\frac{x^3}{6} - \frac{x^2}{8} + \frac{1}{192} + \frac{1}{6}\left(-3x^2 + 3x - \frac{1}{2}\right)^2\right] L,$$

for each $x \in \left[\frac{1}{4}, \frac{1}{2\sqrt{3}}\right]$

$$\left|\int_0^1 f(t)\mathrm{d}t - D(x) + \frac{B_2(x)}{2}[f'(1) - f'(0)]\right| \le \left[-\frac{x^3}{6} + \frac{x^2}{8} - \frac{1}{192} + \frac{1}{96}(1 - 12x^2)^2\right] L,$$

and for each $x \in \left[\frac{1}{2\sqrt{3}}, \frac{1}{2}\right]$

$$\left|\int_0^1 f(t)\mathrm{d}t - D(x) + \frac{B_2(x)}{2}[f'(1) - f'(0)]\right| \le \left(-\frac{x^3}{6} + \frac{x^2}{8} - \frac{1}{192}\right) L.$$

Proof. Using (2.8) for each $x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right]$ we get

$$\begin{aligned}\int_0^1 |F_3^x(t)|\,\mathrm{d}t &= 2\int_0^{1/2} |F_3^x(t)|dt = -2\int_0^{1/2} F_3^x(t)dt = -2\left(-\frac{1}{4}G_4^x(t)|_0^{1/2}\right) \\ &= \frac{1}{2}\left[G_4^x\left(\frac{1}{2}\right) - \tilde{B}_4(x)\right] = \frac{1}{2}F_4^x\left(\frac{1}{2}\right),\end{aligned}$$

for each $x \in \left[\frac{1}{2} - \frac{1}{2\sqrt{3}}, \frac{1}{4}\right]$

$$\begin{aligned}\int_0^1 |F_3^x(t)|\,\mathrm{d}t &= -2\int_0^{1/2} F_3^x(t)dt + 4\int_0^{\sqrt{-3x^2+3x-\frac{1}{2}}} F_3^x(t)dt \\ &= \frac{1}{2}\left[G_4^x\left(\frac{1}{2}\right) - 2G_4^x\left(\sqrt{-3x^2 + 3x - \frac{1}{2}}\right) + \tilde{B}_4(x)\right] \\ &= \frac{1}{2}\left[F_4^x\left(\frac{1}{2}\right) - 2F_4^x\left(\sqrt{-3x^2 + 3x - \frac{1}{2}}\right)\right],\end{aligned}$$

for each $x \in \left[\frac{1}{4}, \frac{1}{2\sqrt{3}}\right]$

$$\int_0^1 |F_3^x(t)|\,\mathrm{d}t = -2\int_0^{1/2} F_3^x(t)dt + 4\int_0^{\frac{1-\sqrt{1-12x^2}}{2}} F_3^x(t)dt$$

$$= \frac{1}{2}\left[G_4^x\left(\frac{1}{2}\right) - 2G_4^x\left(\frac{1-\sqrt{1-12x^2}}{2}\right) + \tilde{B}_4(x)\right]$$
$$= \frac{1}{2}\left[F_4^x\left(\frac{1}{2}\right) - 2F_4^x\left(\frac{1-\sqrt{1-12x^2}}{2}\right)\right]$$

and for each $x \in \left[\frac{1}{2\sqrt{3}}, \frac{1}{2}\right]$ we get

$$\int_0^1 |F_3^x(t)|\,\mathrm{d}t = 2\int_0^{1/2} F_3^x(t)dt = -\frac{1}{2}\left[G_4^x\left(\frac{1}{2}\right) - \tilde{B}_4(x)\right] = -\frac{1}{2}F_4^x\left(\frac{1}{2}\right).$$

Therefore, applying (2.13) with $n=3$ and $p=\infty$ we get the above inequalities. □

Remark 2.6 Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(n-1)}$ is L-Lipschitzian on $[0,1]$ for some $n\geq 3$. Then for each $x \in \left[0, \frac{1}{2}-\frac{1}{2\sqrt{3}}\right] \cup \left[\frac{1}{2\sqrt{3}}, \frac{1}{2}\right]$, from Corollary 2.2 we get

$$K(2k-1,\infty,x) = \frac{2}{(2k)!}\left|B_{2k}\left(\frac{1}{2}-x\right) - B_{2k}(x)\right|$$

$$K^*(2k,\infty,x) = \frac{1}{(2k)!}\left|B_{2k}(x)\right| \quad \text{and} \quad K(2k,\infty,x) = \frac{2}{(2k)!}\left|B_{2k}(x)\right|.$$

If in the first inequality in Corollary 2.6 we put $k=2$ we get the same inequalities as in Corollary 2.6 when x is from intervals $\left[0, \frac{1}{2}-\frac{1}{2\sqrt{3}}\right]$ and $\left[\frac{1}{2\sqrt{3}}, \frac{1}{2}\right]$.

Remark 2.7 If in Corollaries 2.3, 2.4, 2.5, 2.6 and Remark 2.6 we choose $x = 0, 1/2, 1/3$ we get inequalities related to trapezoid (see [14], [39], and [25]), midpoint (see [15], [40] and [23]) and two-point Newton-Cotes (see [84]), respectively. For $x = 1/4$ in Corollaries 2.3, 2.4, 2.5 and 2.6 we get inequalities related to two-point Maclaurin formulae (see [34]).

Corollary 2.7 *Let $f:[0,1]\to\mathbb{R}$ be continuous of bounded variation on $[0,1]$. Then for $x\in[0,1/2]$*

$$\left|\int_0^1 f(t)\mathrm{d}t - D(x)\right| \leq \frac{1+|4x-1|}{4}\cdot V_0^1(f). \tag{2.23}$$

Proof. From (2.5) we have $\max_{t\in[0,1]}|F_1^x(t)| = \max\{2x,-2x+1\} = \max\{A,B\}$, where $A=2x,\ B=-2x+1$. Also, $\max\{A,B\} = \frac{1}{2}(A+B+|A-B|)$, so using this formula applying (2.13) with $n=1$ and $p=1$ we get the above inequality. □

Remark 2.8 The inequality (2.23) has been proved by Dragomir in [35].

Corollary 2.8 *Let $f:[0,1]\to\mathbb{R}$ be such that f' is continuous of bounded variation on $[0,1]$. Then for each $x\in[0,1/4]$*

$$\left|\int_0^1 f(t)\mathrm{d}t - D(x)\right| \leq \frac{4x^2-4x+1+|4x^2+4x-1|}{16}\cdot V_0^1(f') \tag{2.24}$$

and for each $x \in [1/4, 1/2]$

$$\left| \int_0^1 f(t)\mathrm{d}t - D(x) \right| \le \frac{x^2}{2} \cdot V_0^1(f'). \tag{2.25}$$

Proof. From (2.7) and for each $x \in [0, 1/4]$ we have

$$\max_{t\in[0,1]} |F_2^x(t)| = \max\left\{2x^2, -2x + \frac{1}{2}\right\}$$

and for each $x \in [1/4, 1/2]$, $\max_{t\in[0,1]} |F_2^x(t)| = 2x^2$. So using these two formulae and applying (2.13) with $n = 2$ and $p = 1$ we get the inequalities (2.24) and (2.25). $\square$

Corollary 2.9 *Let* $f : [0,1] \to \mathbb{R}$ *be such that* f' *is continuous of bounded variation on* $[0,1]$*. Then for each* $x \in [0, 1/2]$

$$\left| \int_0^1 f(t)\mathrm{d}t - D(x) + \frac{B_2(x)}{2}[f'(1) - f'(0)] \right| \le \left(x^2 - \frac{x}{2} + \frac{1}{12}\right) V_0^1(f').$$

Proof. Using (2.6) for each $x \in [0, 1/2]$ we get

$$\max_{t\in[0,1]} |G_2^x(t)| = \max\left\{ |G_2(0)|, |G_2(x)|, \left| G_2\left(\frac{1}{2}\right)\right| \right\}.$$

Therefore, applying (2.14) with $n = 2$ and $p = 1$ we get the above inequality. $\square$

Remark 2.9 We mention here that comparing the best possible constant from Guessab and Schmeisser in [68] in case $p = 1$ and our constant, we conclude that inequality (2.17) is not generally the best possible. Namely, our constant for boundary conditions $f'(1) = f'(0)$, $n = 2$ and $x = 0$ is $1/12$, while they have $1/16$.

Corollary 2.10 *Let* $f : [0,1] \to \mathbb{R}$ *be such that* f'' *is continuous of bounded variation on* $[0,1]$*. Then for each* $x \in [0, 1/4]$

$$\left| \int_0^1 f(t)\mathrm{d}t - D(x) + \frac{B_2(x)}{2}[f'(1) - f'(0)] \right| \le \frac{1}{72\sqrt{3}} \left(1 - 12x^2\right)^{3/2} V_0^1(f'')$$

and for each $x \in [1/4, 1/2]$

$$\left| \int_0^1 f(t)\mathrm{d}t - D(x) + \frac{B_2(x)}{2}[f'(1) - f'(0)] \right| \le \frac{1}{3} \left(-x^2 + x - \frac{1}{6}\right)^{3/2} V_0^1(f'').$$

Proof. Using (2.8) for each $x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right]$ we get

$$\max_{t\in[0,1]} |F_3^x(t)| = \left| F_3\left(\frac{1}{2} - \frac{\sqrt{1 - 12x^2}}{2\sqrt{3}}\right)\right|,$$

for each $x \in \left[\frac{1}{2} - \frac{1}{2\sqrt{3}}, \frac{1}{4}\right]$

$$\max_{t\in[0,1]} |F_3^x(t)| = \max\left\{\left|F_3\left(\frac{1}{2} - \frac{\sqrt{1-12x^2}}{2\sqrt{3}}\right)\right|, \left|F_3\left(\sqrt{-x^2+x-\frac{1}{6}}\right)\right|\right\},$$

for each $x \in \left[\frac{1}{4}, \frac{1}{2\sqrt{3}}\right]$

$$\max_{t\in[0,1]} |F_3^x(t)| = \max\left\{\left|F_3\left(\frac{1}{2} - \frac{\sqrt{1-12x^2}}{2\sqrt{3}}\right)\right|, \left|F_3\left(\sqrt{-x^2+x-\frac{1}{6}}\right)\right|\right\}$$

and for each $x \in \left[\frac{1}{2\sqrt{3}}, \frac{1}{2}\right]$ we get

$$\max_{t\in[0,1]} |F_3^x(t)| = \left|F_3\left(\sqrt{-x^2+x-\frac{1}{6}}\right)\right|.$$

Therefore, applying (2.13) with $n = 3$ and $p = 1$ we get the above inequalities. □

Remark 2.10 Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$ for some $n \geq 3$. Then for each $x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right] \cup \left[\frac{1}{2\sqrt{3}}, \frac{1}{2}\right]$, from Corollary 2.1 we get

$$K(2k-1,1,x) = \frac{1}{2(2k-1)!} \max_{t\in[0,1]} \left|F_{2k-1}^x(t)\right|,$$

$$K^*(2k,1,x) = \frac{1}{(2k)!} \left|B_{2k}\left(\frac{1}{2} - x\right) - B_{2k}(x)\right|$$

$$\text{and } K(2k-1,1,x) = \frac{1}{(2k)!} \max\left\{|B_{2k}(x)|, \left|B_{2k}\left(\frac{1}{2} - x\right)\right|\right\}.$$

If in the first inequality in Corollary 2.10 we put $k = 2$ we get the same inequalities as in Corollary 2.10 when x is from intervals $\left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right]$ and $\left[\frac{1}{2\sqrt{3}}, \frac{1}{2}\right]$.

Remark 2.11 If in Corollaries 2.7, 2.8, 2.9, 2.10 and Remark 2.10 we choose $x = 0, 1/2, 1/3$ we get inequalities related to trapezoid (see [14], [39] and [25]), midpoint (see [15], [40] and [23])) and two-point Newton-Cotes (see [84]), respectively. For $x = 1/4$ in Corollaries 2.7, 2.8, 2.9 and 2.10 we get inequalities related to two-point Maclaurin formulae (see [35]).

Now, we calculate the optimal constant for $p = 2$.

Corollary 2.11 *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n)} \in L_2[0,1]$ for some $n \geq 1$. Then for each $x \in [0,1/2]$, we have*

$$\left|\int_0^1 f(t)\mathrm{d}t - D(x) + \tilde{T}_{n-1}(x)\right|$$
$$\leq \frac{1}{2}\left[\frac{2(-1)^{n-1}}{(2n)!}\left[B_{2n} + B_{2n}(1-2x)\right] + \frac{4}{(n!)^2}B_n^2(x)\right]^{1/2} \|f^{(n)}\|_2,$$

and

$$\left|\int_0^1 f(t)\mathrm{d}t - D(x) + \tilde{T}_n(x)\right| \leq \frac{1}{2}\left[\frac{2(-1)^{n-1}}{(2n)!}\left[B_{2n}+B_{2n}(1-2x)\right]\right]^{1/2} \|f^{(n)}\|_2.$$

Proof. Using integration by parts and also using Lemma 1 from [30] we have

$$\begin{aligned}
\int_0^1 G_n^{x2}(t)dt &= (-1)^{n-1}\frac{n(n-1)\dots 2}{(n+1)(n+2)\dots(2n-1)}\left[\frac{1}{2n}\int_0^1 G_{2n}^x(t)dG_1^x(t)\right] \\
&= (-1)^{n-1}\frac{2(n!)^2}{(2n)!}\left[-2\int_0^1 G_{2n}^x(t)dt + G_{2n}^x(x) + G_{2n}^x(1-2x)\right] \\
&= (-1)^{n-1}\frac{2(n!)^2}{(2n)!}\left[B_{2n}+B_{2n}(1-2x)\right].
\end{aligned}$$

Now,

$$\begin{aligned}
\int_0^1 F_n^{x2}(t)dt &= \int_0^1 \left[G_n^x(t)-\tilde{B}_n(x)\right]^2 dt \\
&= \int_0^1 \left[G_n^{x2}(t) - 2G_n^x(t)\tilde{B}_n(x) + \tilde{B}_n^2(x)\right] dt = \int_0^1 G_n^{x2}(t)dt + \tilde{B}_n^2(x) \\
&= (-1)^{n-1}\frac{2(n!)^2}{(2n)!}\left[B_{2n}+B_{2n}(1-2x)\right] + 4B_n^2(x).
\end{aligned}$$

□

Remark 2.12 For $n = 2$ we have boundary conditions $f'(1) = f'(0)$. For $x = 0$ our constant from Theorem 2.3 is $1/(12\sqrt{3})$. Guessab and Schmeisser in [68] also have $1/(12\sqrt{3})$ which confirms the sharpness of our inequality in this case.

Finally, we give the values of optimal constant for $n = 1$ and arbitrary p from Theorem 2.2.

Remark 2.13 Note that $K^*(1,p,x) = K(1,p,x)$, for $1 < p \leq \infty$, since $G_1^x(t) = F_1^x(t)$. Also, for $1 < p \leq \infty$ we can easily calculate $K(1,p,x)$. We get

$$K(1,p,x) = \frac{1}{2}\left[\frac{(2x)^{q+1}+(1-2x)^{q+1}}{q+1}\right]^{\frac{1}{q}}, \ 1 < p \leq \infty. \tag{2.26}$$

Remark 2.14 (2.26) has been proved by S.S. Dragomir on interval $[a,b]$ in [34].

Remark 2.15 If in Remark 2.13 we chose $x = 0, 1/2, 1/3, 1/4$ we get inequality related to trapezoid (see [25]), midpoint (see [23]), two-point Newton-Cotes (see [84]) and two-point Maclaurin formulae (see [34]), respectively.

In the following theorem using the formula (2.3) and one technical result from the recent paper [83] we obtain Grüss type inequalities related to the general Euler two-point formula (see [83]).

Theorem 2.4 *Suppose that $f:[0,1]\to\mathbb{R}$ is such that $f^{(n)}\in L_1[0,1]$ for some $n\geq 1$. Assume that*

$$m_n\leq f^{(n)}(t)\leq M_n,\ 0\leq t\leq 1,$$

for some constants m_n and M_n. Then for $x\in[0,1/2]$

$$\left|\int_0^1 f(t)\mathrm{d}t - D(x) + \tilde{T}_n(x)\right| \leq C_n(M_n-m_n) \tag{2.27}$$

where $C_n=\frac{1}{4(n!)}\int_0^1|G_n^x(t)|dt$.

Remark 2.16 If in Theorem 2.4 we chose $x=0,1/2,1/3$ we get inequalities related to trapezoid, midpoint and two-point Newton-Cotes formulae (see [83]). For $x=1/4$ we get inequalities related to two-point Maclaurin formulae.

Our final results are connected with the series expansion of a function in Bernoulli polynomials.

Theorem 2.5 *If $f:[0,1]\to\mathbb{R}$ is such that $f^{(2k)}$ is a continuous function on $[0,1]$, for some $k\geq 2$ then there exists a point $\eta\in[0,1]$ such that*

$$\tilde{R}_{2k}^2(f) = -\frac{B_{2k}(x)}{[(2k)!]}f^{(2k)}(\eta)\ \text{for}\ x\in\left[0,\frac{1}{2}-\frac{1}{2\sqrt{3}}\right]\cup\left[\frac{1}{2\sqrt{3}},\frac{1}{2}\right] \tag{2.28}$$

Proof. We can rewrite $\tilde{R}_{2k}^2(f)$ for $x\in\left[0,\frac{1}{2}-\frac{1}{2\sqrt{3}}\right]$ as $\tilde{R}_{2k}^2(f)=(-1)^k\frac{J_k}{2[(2k)!]}$, where $J_k=\int_0^1(-1)^kF_{2k}^x(s)f^{(2k)}(s)\mathrm{d}s$. From Corollary 2.1 follows that $(-1)^kF_{2k}^x(s)\geq 0$, $0\leq s\leq 1$ and the claim follows from the mean value theorem for integrals and Corollary 2.2. The proof on interval $\left[\frac{1}{2\sqrt{3}},\frac{1}{2}\right]$ is similar.

□

Remark 2.17 For $k=2$ formulae (2.28) reduces to

$$\tilde{R}_4^2(f) = -\frac{B_4(x)}{24}f^{(4)}(\eta),$$

respectively, which are formulae proved for $x=0$ in [25], for $x=1/2$ in [23] and for $x=1/3$ in [84].

Corollary 2.12 *Let $f\in C^\infty[0,1]$ and $\lambda\in\mathbb{R}$ be such that $0<\lambda<2\pi$ and $|f^{(2k)}(t)|\leq\lambda^{2k}$ for $t\in[0,1]$ and $k\geq k_0$ for some $k_0\geq 2$. Then for $x\in\left[0,\frac{1}{2}-\frac{1}{2\sqrt{3}}\right]\cup\left[\frac{1}{2\sqrt{3}},\frac{1}{2}\right]$ we have*

$$\int_0^1 f(t)dt = D(x)-\frac{1}{2}\sum_{j=1}^{\infty}\frac{B_{2j}(x)}{(2j)!}\left[f^{(2j-1)}(1)-f^{(2j-1)}(0)\right]. \tag{2.29}$$

Proof. From Theorem 2.5 when $k\geq k_0$ we have that

$$|\tilde{R}_{2k}^2(f)|\leq\frac{|B_{2k}(x)|}{(2k)!}\lambda^{2k}\leq\frac{|B_{2k}|}{(2k)!}\lambda^{2k}\approx\frac{1}{(2k)!}\cdot 2\frac{(2k)!}{(2\pi)^{2k}}\lambda^{2k}=2\left(\frac{\lambda}{2\pi}\right)^{2k},$$

so, (2.29) follows. □

2.4 Integration of periodic function and application on the general Euler two-point formula

In the paper [91] the following lemma has been proved:

Lemma 2.3 *Let $\varphi(x) \downarrow 0$. Then*

$$-\int_0^\infty \rho(x)\varphi(x)dx < \frac{1}{8}\varphi(0), \tag{2.30}$$

where $\rho(x) = x - \lfloor x \rfloor - \frac{1}{2}$.

The aim of this section is to give a variant of this inequality, which involves a periodic function ρ, and use those results to prove some inequalities for the general Euler two-point formula. The results from this section are published in [76].

Theorem 2.6 *Let $\varphi : I \to \mathbb{R}$, $I \subset \mathbb{R}$, be a monotone function, and let $\rho : \mathbb{R} \to \mathbb{R}$ be a periodic function with period P such that for some $a \in \mathbb{R}$ and $n \in \mathbb{N}$, $[a, a+nP] \subseteq I$. Suppose that there exists some $x_0 \in (a, a+P)$ such that $\rho(x_0) = 0$, $\rho(x) \geq 0$ for all $x \in [a, x_0)$ and $\rho(x) \leq 0$ for all $x \in (x_0, a+P]$. Suppose also that $\int_a^{a+P} \rho(x)\,dx = 0$. If function φ is increasing on $[a, a+nP]$, then*

$$-\int_a^{a+nP} \rho(x)\,\varphi(x)\,dx \leq \frac{1}{2n}\left(\varphi(a+nP) - \varphi(a)\right)\int_a^{a+nP} |\rho(x)|\,dx, \tag{2.31}$$

and this inequality is sharp. If function φ is decreasing on $[a, a+nP]$, then inequality (2.31) is reversed.

Proof. First we will consider the case of increasing function φ.

Since function ρ is periodic with period P, from the conditions on ρ we can deduce that for all $k \in \{0, \ldots, n-1\}$

$$\begin{aligned}
\int_{a+kP}^{a+(k+1)P} \rho(x)dx &= 0, \\
\rho(x_k) &= 0, \quad x_k = x_0 + kP \\
\rho(x) &\geq 0, \quad x \in [a+kP, x_k) \\
\rho(x) &\leq 0, \quad x \in (x_k, a+(k+1)P].
\end{aligned}$$

Using these properties, we can easily obtain

$$\begin{aligned}
-\int_a^{a+nP} \rho(x)\varphi(x)dx &= -\sum_{k=0}^{n-1} \int_{a+kP}^{a+(k+1)P} \rho(x)\left(\varphi(x) - \varphi(x_k)\right)dx \\
= \sum_{k=0}^{n-1} \Bigg[\int_{a+kP}^{x_k} \rho(x)\left(\varphi(x_k) - \varphi(a+kP)\right)dx
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{x_k}^{a+(k+1)P} \rho(x)\left(\varphi(x_k)-\varphi(a+(k+1)P)\right)dx + a_k\Bigg] \\
&= \sum_{k=0}^{n-1}\Bigg[\left(\varphi(x_k)-\varphi(a+kP)\right)\int_{a+kP}^{x_k}\rho(x)dx \\
&+ \left(\varphi(x_k)-\varphi(a+(k+1)P)\right)\int_{x_k}^{a+(k+1)P}\rho(x)dx + a_k\Bigg] \\
&\le \left(\varphi(a+nP)-\varphi(a)\right)\frac{1}{2n}\int_a^{a+nP}|\rho(x)|dx + \sum_{k=0}^{n-1}a_k,
\end{aligned}
$$

where

$$
\begin{aligned}
a_k &= \int_{a+kP}^{x_k}\rho(x)\left(\varphi(a+kP)-\varphi(x)\right)dx \\
&\quad - \int_{x_k}^{a+(k+1)P}\rho(x)\left(\varphi(x)-\varphi(a+(k+1)P)\right)dx.
\end{aligned}
$$

Due to the fact that φ is increasing function on I, we can deduce that $a_k \le 0$ for all $k \in \{0,...,n-1\}$, i.e. $\sum_{k=1}^{n-1} a_k \le 0$. Immediately it follows that the inequality (2.31) is valid.

In order to prove the sharpness we will define function $\varphi : [a,a+nP] \to \mathbb{R}$ with

$$
\varphi(x) = \begin{cases} a+kP, & x \in [a+kP, x_k] \\ a+(k+1)P, & x \in (x_k, a+(k+1)P] \end{cases}
$$

for all $k \in \{0,...,n-1\}$. It is obvious that function φ is increasing on $[a,a+nP]$, and for any function ρ which fulfils the conditions of this theorem we have:

$$
\begin{aligned}
&-\int_a^{a+nP}\rho(x)\varphi(x)dx = -\sum_{k=0}^{n-1}\int_{a+kP}^{a+(k+1)P}\rho(x)\varphi(x)dx \\
&= -\sum_{k=0}^{n-1}\left[(a+kP)\int_{a+kP}^{x_k}\rho(x)dx + (a+(k+1)P)\int_{x_k}^{a+(k+1)P}\rho(x)dx\right] \\
&= -(a+kP)\sum_{k=0}^{n-1}\int_{a+kP}^{a+(k+1)P}\rho(x)dx - P\sum_{k=0}^{n-1}\int_{x_k}^{a+(k+1)P}\rho(x)dx \\
&= 0 + \frac{P}{2}\sum_{k=0}^{n-1}\int_{a+kP}^{a+(k+1)P}|\rho(x)|dx = \frac{1}{2n}\left(\varphi(a+nP)-\varphi(a)\right)\int_a^{a+nP}|\rho(x)|dx,
\end{aligned}
$$

and this means that inequality (2.31) is sharp.

If function φ is decreasing on I, the reverse of (2.31) can be obtained in the similar way. To prove the sharpness, we can simply choose a decreasing function $\varphi : [a,a+nP] \to \mathbb{R}$ defined with

$$
\varphi(x) = \begin{cases} a+(k+1)P, & x \in [a+kP, x_k] \\ a+kP, & x \in (x_k, a+(k+1)P] \end{cases},
$$

for all $k \in \{0,...,n-1\}$. This completes the proof.

□

Remark 2.18 If we consider inequality (2.31) for a periodic function τ with period P such that $\tau(x) \leq 0$ on $[a+kP, x_0)$, $\tau(x_0) = 0$ and $\tau(x) \geq 0$ on $(x_0, a+(k+1)P]$, then we can use inequality (2.31) with function ρ defined as $\rho(x) = -\tau(x)$, for $x \in \mathbb{R}$.

Now we shall show how can Theorem 2.6 be used in order to obtain some inequalities for general Euler two-point formula. Let $f \in C^{2r-1}([a,b],\mathbb{R})$ for some $r \geq 2$, and let $y \in [a,(a+b)/2]$. We have the general Euler two-point formula

$$\begin{aligned} \int_a^b f(t)dt &= \frac{b-a}{2}[f(y)+f(a+b-y)] - T_{r-1}(y) \\ &+ \frac{(b-a)^{2r-1}}{2(2r-1)!}\int_a^b F_{2r-1}^x\left(\frac{z-a}{b-a}\right) f^{(2r-1)}(z)dz, \end{aligned} \tag{2.32}$$

where $x = \frac{y-a}{b-a}$. Here we define $T_0(y) = T_1(y) = 0$, and for $k \geq 2$

$$T_k(y) = \sum_{j=2}^{k} \frac{(b-a)^{2j}}{(2j)!} B_{2j}\left(\frac{y-a}{b-a}\right)\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right]. \tag{2.33}$$

Theorem 2.7 *Let $f : [a,b] \to \mathbb{R}$ be such that for some $r \geq 2$ derivative $f^{(2r-1)}$ is an increasing function on $[a,b]$. Then for $y \in \left[a, \frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}\right]$ the following inequality holds*

$$\begin{aligned} &(-1)^r \left\{\int_a^b f(t)dt - \frac{b-a}{2}[f(y)+f(a+b-y)] + T_{r-1}(y)\right\} \qquad (2.34)\\ \leq\ &\frac{(b-a)^{2r}}{(2r)!}\left|B_{2r}\left(\frac{1}{2} - \frac{y-a}{b-a}\right) - B_{2r}\left(\frac{y-a}{b-a}\right)\right|\left[f^{(2r-1)}(b) - f^{(2r-1)}(a)\right]. \end{aligned}$$

Also, for $y \in \left[a + \frac{b-a}{2\sqrt{3}}, \frac{a+b}{2}\right]$ we have

$$\begin{aligned} &(-1)^{r-1} \left\{\int_a^b f(t)dt - \frac{b-a}{2}[f(y)+f(a+b-y)] + T_{r-1}(y)\right\} \qquad (2.35)\\ \leq\ &\frac{(b-a)^{2r}}{(2r)!}\left|B_{2r}\left(\frac{1}{2} - \frac{y-a}{b-a}\right) - B_{2r}\left(\frac{y-a}{b-a}\right)\right|\left[f^{(2r-1)}(b) - f^{(2r-1)}(a)\right]. \end{aligned}$$

These two inequalities are sharp.

Proof. We know that function F_{2r-1}^x is periodic with period $P = 1$. It can be easily checked that: $F_{2r-1}^x(0) = F_{2r-1}^x(1/2) = F_{2r-1}^x(1) = 0$, $(-1)^{r-1} F_{2r-1}^x(t) > 0$ for $t \in (0,1/2)$, $(-1)^{r-1} F_{2r-1}^x(t) < 0$ for $t \in (1/2,1)$, and also $\int_0^1 F_{2r-1}^x(t)\,dt = 0$. This means that if in Theorem 2.6 we choose $\rho(t) = (-1)^{r-1} F_{2r-1}^x(t)$, $\varphi(t) = f^{(2r-1)}(t(b-a)+a)$ and $n = 1$, we obtain

$$(-1)^r \int_a^b F_{2r-1}^x\left(\frac{z-a}{b-a}\right) f^{(2r-1)}(z)dz$$

$$
\begin{aligned}
&= (-1)^r (b-a) \int_0^1 F^x_{2r-1}(t) f^{(2r-1)}(t(b-a)+a)dt \\
&\le \left(f^{(2r-1)}(b) - f^{(2r-1)}(a)\right) \frac{b-a}{2} \int_0^1 \left|F^x_{2r-1}(t)\right| dt \\
&= \frac{b-a}{r} \left| B_{2r}\left(\frac{1}{2}-x\right) - B_{2r}(x)\right| \left(f^{(2r-1)}(b) - f^{(2r-1)}(a)\right).
\end{aligned}
$$

and if we combine this with (2.32), we can easily obtain (2.35). The proof of the second statement is similar. □

Remark 2.19 If in (2.35) we let $y = a$, we obtain an inequality for trapezoid formula:

$$
\begin{aligned}
&(-1)^r \left\{ \int_a^b f(t)dt - \frac{b-a}{2}\left[f(a)+f(b)\right] + T_{r-1}(a) \right\} \\
&< \frac{(b-a)^{2r}}{(2r)!}(2-2^{1-2r})\left|B_{2r}\right| \left(f^{(2r-1)}(b) - f^{(2r-1)}(a)\right).
\end{aligned}
$$

If in (2.36) we let $y = (a+b)/2$, we obtain an inequality for mid-point formula:

$$
\begin{aligned}
&(-1)^{r-1} \left\{ \int_a^b f(t)dt - (b-a)f\left(\frac{a+b}{2}\right) + T_{r-1}\left(\frac{a+b}{2}\right) \right\} \\
&< \frac{(b-a)^{2r}}{(2r)!}(2-2^{1-2r})\left|B_{2r}\right| \left(f^{(2r-1)}(b) - f^{(2r-1)}(a)\right)
\end{aligned}
$$

and also for $y = (2a+b)/3$ in inequality (2.36), we get inequality for two-point Newton-Cotes formula:

$$
\begin{aligned}
&(-1)^{r-1} \left\{ \int_a^b f(t)dt - \frac{b-a}{2}\left[f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right] + T_{r-1}\left(\frac{2a+b}{3}\right) \right\} \\
&< \frac{(b-a)^{2r}}{(2r)!}(1-3^{1-2r})(1-2^{-2r})\left|B_{2r}\right| \left(f^{(2r-1)}(b) - f^{(2r-1)}(a)\right),
\end{aligned}
$$

which is an improvement of the Theorem 9 from [84].

2.5 Hermite-Hadamard and Dragomir-Agarwal inequalities and convex functions

One of the cornerstones of nonlinear analysis is the Hermite-Hadamard inequality, which states that if $[a,b]$ $(a < b)$ is a real interval and $f : [a,b] \to \mathbb{R}$ is a convex function, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2}. \tag{2.36}$$

Recently, S.S. Dragomir and R.P. Agarwal [36] considered the trapezoid formula for numerical integration of functions f such that $|f'|^q$ is a convex function for some $q \ge 1$. Their approach was based on estimating the difference between the two sides of the right-hand inequality in (2.36). Improvements of their results were obtained in [97]. In particular, the following result was established.

Suppose $f : [a,b] \to \mathbb{R}$ is differentiable and such that $|f'|^q$ is convex on $[a,b]$ for some $q \ge 1$. Then

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{b-a}{4}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q}. \tag{2.37}$$

Some generalizations to higher-order convexities and applications of these results are given in [32]. Related results for Euler midpoint, Euler-Simpson, Euler two-point, dual Euler-Simpson, Euler-Simpson $3/8$ and Euler-Maclaurin formulae were considered in [105] (see also [33] and [106]). [1]

In this section we consider some related results using the general Euler two-point formulae and these results are published in [103]. We will use interval $[0,1]$ for simplicity and since it involves no loss in generality.

Theorem 2.8 *Suppose $f : [0,1] \to \mathbb{R}$ is $(2r+2)$-convex for $r \ge 2$. Then for $x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right]$ the inequality*

$$\begin{aligned} \frac{|B_{2r}(x)|}{(2r)!} f^{(2r)}\left(\frac{1}{2}\right) &\le (-1)^r \left\{\int_0^1 f(t)dt - \frac{1}{2}[f(x)+f(1-x)] + T_{r-1}(f)\right\} \\ &\le \frac{|B_{2r}(x)|}{(2r)!} \frac{f^{(2r)}(0)+f^{(2r)}(1)}{2} \end{aligned} \tag{2.38}$$

[1] We recall that a function $f : [a,b] \to \mathbb{R}$ is said to be n-convex on $[a,b]$ for some $n \ge 0$ if for any choice of $n+1$ points $x_0, \ldots, x_n$ from $[a,b]$ we have

$$[x_0, \ldots, x_n] f \ge 0,$$

where $[x_0, \ldots, x_n] f$ is n-th order divided difference of f. If f is n-convex, then $f^{(n-2)}$ exists and is convex function in the ordinary sense. Also, if $f^{(n)}$ exists, then f is n-convex if and only if $f^{(n)} \ge 0$. For more details see for example [100].

It should be noted that each continuous n-convex function on $[a,b]$ is the uniform limit of a sequence of the corresponding Bernstein's polynomials (see for example [100, p. 293]). Bernstein's polynomials of any continuous n-convex function are also n-convex functions, so when stating our results for a continuous $(2r)$-convex function f without any loss in generality we may assume that $f^{(2r)}$ exists and is continuous. Actually, our results are valid for any continuous $(2r)$-convex function f.

holds, while for $x \in \left[\frac{1}{2\sqrt{3}}, \frac{1}{2}\right]$ *we have*

$$\begin{aligned}\frac{|B_{2r}(x)|}{(2r)!} f^{(2r)}\left(\frac{1}{2}\right) &\leq (-1)^{r-1}\left\{\int_0^1 f(t)dt - \frac{1}{2}[f(x)+f(1-x)] + T_{r-1}(f)\right\} \\ &\leq \frac{|B_{2r}(x)|}{(2r)!}\frac{f^{(2r)}(0)+f^{(2r)}(1)}{2}. \end{aligned} \tag{2.39}$$

If f *is* $(2r+2)$*-concave, the inequalities are reversed.*

Proof. For $x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right]$ from (2.4) and (2.33) we have

$$\begin{aligned} &(-1)^r \left\{\int_0^1 f(t)dt - \frac{1}{2}[f(x)+f(1-x)] + T_{r-1}(f)\right\} \\ &= (-1)^r \frac{1}{2(2r)!}\int_0^1 f^{(2r)}(t)F_{2r}^x(t)dt = \frac{1}{2(2r)!}\int_0^1 f^{(2r)}(t)\,|F_{2r}^x(t)|\,dt \\ &= \frac{1}{2(2r)!}\int_0^1 f^{(2r)}((1-t)\cdot 0 + t\cdot 1)\,|F_{2r}^x(t)|\,dt. \end{aligned} \tag{2.40}$$

Using the discrete Jensen inequality for the convex function $f^{(2r)}$, we have

$$\begin{aligned} &\int_0^1 f^{(2r)}((1-t)\cdot 0 + t\cdot 1)\,|F_{2r}^x(t)|\,dt \\ &\leq f^{(2r)}(0)\int_0^1 (1-t)\,|F_{2r}^x(t)|\,dt + f^{(2r)}(1)\int_0^1 t\,|F_{2r}^x(t)|\,dt \\ &= |B_{2r}(x)|\left(f^{(2r)}(0) + f^{(2r)}(1)\right), \end{aligned} \tag{2.41}$$

since $\int_0^1 (1-t)\,|F_{2r}^x(t)|\,dt = \int_0^1 t\,|F_{2r}^x(t)|\,dt = \frac{1}{2}\int_0^1 |F_{2r}^x(t)|\,dt$. So, the second inequality in (2.38) follows.

By Jensen's integral inequality we have

$$\begin{aligned} &\int_0^1 f^{(2r)}((1-t)\cdot 0 + t\cdot 1)\,|F_{2r}^x(t)|\,dt \\ &\geq \left(\int_0^1 |F_{2r}^x(t)|\,dt\right) f^{(2r)}\left(\frac{\int_0^1 ((1-t)\cdot 0 + t\cdot 1)\,|F_{2r}^x(t)|\,dt}{\int_0^1 \left|F_{2r}^x(t)\right|dt}\right) \\ &= 2|B_{2k}(x)|f^{(2r)}\left(\frac{1}{2}\right). \end{aligned} \tag{2.42}$$

The first inequality in (2.38) now follows from (2.40).

The proof of inequality (2.39) is similar. □

Remark 2.20 If in Theorem 2.8 we choose $x = 0, 1/2, 1/3$, we get generalizations of Hermite-Hadamard's inequality for Euler trapezoid, Euler midpoint and Euler two-point Newton-Cotes formulae respectively (see [32], [33] and [105]).

Theorem 2.9 *Suppose $f:[0,1]\to\mathbb{R}$ is n-times differentiable and $x\in\left[0,\frac{1}{2}-\frac{1}{2\sqrt{3}}\right]\cup\left[\frac{1}{2\sqrt{3}},\frac{1}{2}\right]$.*
(a) If $\left|f^{(n)}\right|^q$ is convex for some $q\geq 1$, then for $n=2r-1,\ r\geq 2$, we have

$$\begin{aligned}&\left|\int_0^1 f(t)dt-\frac{1}{2}[f(x)+f(1-x)]+T_{r-1}(f)\right|\\ &\leq\frac{2}{(2r)!}\left|B_{2r}\left(\frac{1}{2}-x\right)-B_{2r}(x)\right|\left[\frac{\left|f^{(2r-1)}(0)\right|^q+\left|f^{(2r-1)}(1)\right|^q}{2}\right]^{1/q}.\end{aligned}\tag{2.43}$$

If $n=2r,\ r\geq 2$, then

$$\left|\int_0^1 f(t)dt-\frac{1}{2}[f(x)+f(1-x)]+T_{r-1}(f)\right|\leq\frac{|B_{2r}(x)|}{(2r)!}\left[\frac{\left|f^{(2r)}(0)\right|^q+\left|f^{(2r)}(1)\right|^q}{2}\right]^{1/q}\tag{2.44}$$

and we also have

$$\left|\int_0^1 f(t)dt-\frac{1}{2}[f(x)+f(1-x)]+T_r(f)\right|\leq\frac{2|B_{2r}(x)|}{(2r)!}\left[\frac{\left|f^{(2r)}(0)\right|^q+\left|f^{(2r)}(1)\right|^q}{2}\right]^{1/q}.\tag{2.45}$$

(b) If $\left|f^{(n)}\right|$ is concave, then for $n=2r-1,\ r\geq 2$, we have

$$\begin{aligned}&\left|\int_0^1 f(t)dt-\frac{1}{2}[f(x)+f(1-x)]+T_{r-1}(f)\right|\\ &\leq\frac{2}{(2r)!}\left|\left[B_{2r}\left(\frac{1}{2}-x\right)-B_{2r}(x)\right]f^{(2r-1)}\left(\frac{1}{2}\right)\right|.\end{aligned}\tag{2.46}$$

If $n=2r,\ r\geq 2$, then

$$\left|\int_0^1 f(t)dt-\frac{1}{2}[f(x)+f(1-x)]+T_{r-1}(f)\right|\leq\frac{1}{(2r)!}\left|B_{2r}(x)\cdot f^{(2r)}\left(\frac{1}{2}\right)\right|\tag{2.47}$$

and we also have

$$\left|\int_0^1 f(t)dt-\frac{1}{2}[f(x)+f(1-x)]+T_r(f)\right|\leq\frac{2}{(2r)!}\left|B_{2r}(x)\cdot f^{(2r)}\left(\frac{1}{2}\right)\right|.\tag{2.48}$$

Proof. First, let $n=2r-1$ for some $r\geq 2$. Then for convex function $|f^{(2r)}|^q$ using Hölder's and Jensen's inequality we have

$$\begin{aligned}&\left|\int_0^1 f(t)dt-\frac{1}{2}[f(x)+f(1-x)]+T_{r-1}(f)\right|\\ &\leq\frac{1}{2(2r-1)!}\int_0^1\left|F_{2r-1}^x(t)\right|\cdot\left|f^{(2r-1)}(t\cdot 1+(1-t)\cdot 0)\right|dt\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2(2r-1)!}\left(\int_0^1 \left|F_{2r-1}^x(t)\right| dt\right)^{1-1/q}\left(\int_0^1 \left|F_{2r-1}^x(t)\right|\cdot\left|f^{(2r-1)}(t\cdot 1+(1-t)\cdot 0)\right|^q dt\right)^{1/q}\\
&\leq \frac{1}{2(2r-1)!}\left(\int_0^1 \left|F_{2r-1}^x(t)\right| dt\right)^{1-1/q}\\
&\quad\cdot\left(\int_0^1 \left|F_{2r-1}^x(t)\right|\cdot\left[t\left|f^{(2r-1)}(1)\right|^q+(1-t)\left|f^{(2r-1)}(0)\right|^q\right]dt\right)^{1/q}\\
&=\left(\frac{1}{2(2r-1)!}\int_0^1 \left|F_{2r-1}^x(t)\right| dt\right)^{1-1/q}\cdot\left[\frac{1}{2(2r-1)!}\left|f^{(2r-1)}(1)\right|^q\int_0^1 t\left|F_{2r-1}^x(t)\right| dt\right.\\
&\quad\left.+\frac{1}{2(2r-1)!}\left|f^{(2r-1)}(0)\right|^q\int_0^1 (1-t)\left|F_{2r-1}^x(t)\right| dt\right]^{1/q}\\
&=\left(\frac{2}{(2r)!}\left|B_{2r}\left(\frac{1}{2}-x\right)-B_{2r}(x)\right|\right)^{1-1/q}\cdot\left[\frac{1}{(2r)!}\left|B_{2r}\left(\frac{1}{2}-x\right)-B_{2r}(x)\right|\left|f^{(2r-1)}(1)\right|^q\right.\\
&+\left.\frac{1}{(2r)!}\left|B_{2r}\left(\frac{1}{2}-x\right)-B_{2r}(x)\right|\left|f^{(2r-1)}(0)\right|^q\right]^{1/q}\\
&=\frac{2}{(2r)!}\left|B_{2r}\left(\frac{1}{2}-x\right)-B_{2r}(x)\right|\left[\frac{\left|f^{(2r-1)}(0)\right|^q+\left|f^{(2r-1)}(1)\right|^q}{2}\right]^{1/q}.
\end{aligned}
$$

On the other hand, if $\left|f^{(2r-1)}\right|$ is concave, then

$$
\begin{aligned}
&\left|\int_0^1 f(t)dt-\frac{1}{2}[f(x)+f(1-x)]+T_{r-1}(f)\right|\\
&\leq \frac{1}{2(2r-1)!}\int_0^1 \left|F_{2r-1}^x(t)\right|\cdot\left|f^{(2r-1)}(t)\right| dt\\
&\leq \frac{1}{2(2r-1)!}\left(\int_0^1 \left|F_{2r-1}^x(t)\right| dt\right)\left|f^{(2r-1)}\left(\frac{\int_0^1 \left|F_{2r-1}^x(t)\right|((1-t)\cdot 0+t\cdot 1)dt}{\int_0^1 \left|F_{2r-1}^x(t)\right| dt}\right)\right|\\
&=\frac{2}{(2r)!}\left|\left[B_{2r}\left(\frac{1}{2}-x\right)-B_{2r}(x)\right]f^{(2r-1)}\left(\frac{1}{2}\right)\right|
\end{aligned}
$$

so the inequality (2.43) and (2.46) are proved.

The proofs of the inequalities (2.44), (2.47), (2.45) and (2.48) are similar. □

Remark 2.21 For (2.46) to be satisfied it is enough to suppose that $|f^{(2r-1)}|$ is a concave function. For if $|g|^q$ is concave na $[0,1]$ for some $q \geq 1$, then for $x, y \in [0,1]$ and $\lambda \in [0,1]$

$$|g(\lambda x+(1-\lambda)y)|^q \geq \lambda|g(x)|^q+(1-\lambda)|g(y)|^q \geq (\lambda|g(x)|+(1-\lambda)|g(y)|)^q,$$

therefore $|g|$ is also concave on $[0,1]$.

Remark 2.22 If in Theorem 2.9 we choose $x = 0, 1/2, 1/3$, we get generalizations of Dragomir-Agarwal inequality for Euler trapezoid, Euler midpoint and Euler two-point Newton-Cotes formulae respectively (see [32], [33], [105] and [106]).

The resultant formulae in Theorems 2.8 and 2.9 when $r = 2$ are of special interest, so we isolate them as corollaries.

Corollary 2.13 *If $f : [0,1] \to \mathbb{R}$ is 6-convex, then for $x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right]$ we have*

$$\begin{aligned}
&\frac{1}{24}\left|x^4 - 2x^3 + x^2 - \frac{1}{30}\right| f^{(4)}\left(\frac{1}{2}\right) \\
\leq\ & \int_0^1 f(t)dt - \frac{1}{2}\left[f(x) + f(1-x)\right] + \frac{1}{12}\left[f'(1) - f'(0)\right] \\
\leq\ & \frac{1}{24}\left|x^4 - 2x^3 + x^2 - \frac{1}{30}\right| \frac{f^{(4)}(0) + f^{(4)}(1)}{2}.
\end{aligned}$$

while for $x \in \left[\frac{1}{2\sqrt{3}}, \frac{1}{2}\right]$ the inequalities

$$\begin{aligned}
&\frac{1}{24}\left|x^4 - 2x^3 + x^2 - \frac{1}{30}\right| f^{(4)}\left(\frac{1}{2}\right) \\
\leq\ & \frac{1}{2}\left[f(x) + f(1-x)\right] - \int_0^1 f(t)dt - \frac{1}{12}\left[f'(1) - f'(0)\right] \\
\leq\ & \frac{1}{24}\left|x^4 - 2x^3 + x^2 - \frac{1}{30}\right| \frac{f^{(4)}(0) + f^{(4)}(1)}{2}.
\end{aligned}$$

hold.

If f is 6-concave, the reversed inequalities apply.

Corollary 2.14 *Suppose $f : [0,1] \to \mathbb{R}$ is 4-times differentiable and $x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right] \cup \left[\frac{1}{2\sqrt{3}}, \frac{1}{2}\right]$.*

(a) If $\left|f^{(3)}\right|^q$ is convex for some $q \geq 1$, then

$$\begin{aligned}
&\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f(x) + f(1-x)\right] + \frac{1}{12}\left[f'(1) - f'(0)\right]\right| \\
\leq\ & \frac{1}{12}\left|2x^3 - \frac{3}{2}x^2 + \frac{1}{16}\right| \left[\frac{\left|f^{(3)}(0)\right|^q + \left|f^{(3)}(1)\right|^q}{2}\right]^{1/q}
\end{aligned}$$

and if $\left|f^{(4)}\right|^q$ is convex for some $q \geq 1$, then

$$\begin{aligned}
&\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f(x) + f(1-x)\right] + \frac{1}{12}\left[f'(1) - f'(0)\right]\right| \\
\leq\ & \frac{1}{24}\left|x^4 - 2x^3 + x^2 - \frac{1}{30}\right| \left[\frac{\left|f^{(4)}(0)\right|^q + \left|f^{(4)}(1)\right|^q}{2}\right]^{1/q}.
\end{aligned}$$

(b) If $\left|f^{(3)}\right|$ *is concave, then*

$$\begin{aligned}&\left|\int_0^1 f(t)dt - \frac{1}{2}[f(x)+f(1-x)] + \frac{1}{12}\left[f'(1)-f'(0)\right]\right| \\ &\leq \frac{1}{12}\left|\left[2x^3 - \frac{3}{2}x^2 + \frac{1}{16}\right] f^{(3)}\left(\frac{1}{2}\right)\right|\end{aligned}$$

and if $\left|f^{(4)}\right|$ *is concave, then*

$$\begin{aligned}&\left|\int_0^1 f(t)dt - \frac{1}{2}[f(x)+f(1-x)] + \frac{1}{12}\left[f'(1)-f'(0)\right]\right| \\ &\leq \frac{1}{24}\left|\left[x^4 - 2x^3 + x^2 - \frac{1}{30}\right] f^{(4)}\left(\frac{1}{2}\right)\right|.\end{aligned}$$

Note that inequalities in Theorems 2.8 and 2.9 hold for $r \geq 2$. Now, we will give some results of the same type in the case when $r = 1$.

Theorem 2.10 *Suppose* $f : [0,1] \to \mathbb{R}$ *is* 4*-convex. Then for* $x \in [0,1/4]$ *the following inequalities hold*

$$\begin{aligned}&\left[\frac{-6x^2+6x-1}{24} + \frac{1}{6}(1-4x)^{3/2}\right] f''\left(\frac{1}{2}\right) \leq \frac{1}{2}[f(x)+f(1-x)] - \int_0^1 f(t)dt \\ &\leq \left[\frac{-6x^2+6x-1}{24} + \frac{1}{6}(1-4x)^{3/2}\right] \frac{f''(1)+f''(0)}{2},\end{aligned}$$

while for $x \in [1/4,1/2]$ *we have*

$$\begin{aligned}&\frac{-6x^2+6x-1}{24} f''\left(\frac{1}{2}\right) \leq \int_0^1 f(t)dt - \frac{1}{2}[f(x)+f(1-x)] \\ &\leq \frac{-6x^2+6x-1}{24} \cdot \frac{f''(1)+f''(0)}{2}.\end{aligned}$$

Proof. It was already shown that for $x \in [0,1/4]$ we have

$$\int_0^1 |F_2^x(t)|dt = \frac{-6x^2+6x-1}{6} + \frac{2}{3}(1-4x)^{3/2},$$

while for $x \in [1/4,1/2]$

$$\int_0^1 |F_2^x(t)|dt = \frac{-6x^2+6x-1}{6}.$$

So, using identity (2.4) and following the proof of Theorem 2.8, we get above inequalities. □

Theorem 2.11 *Suppose $f : [0,1] \to \mathbb{R}$ is 2-times differentiable.*
(a) If $|f'|^q$ is convex for some $q \geq 1$, then for $x \in [0,1/2]$ we have

$$\left| \int_0^1 f(t)dt - \frac{1}{2}[f(x) + f(1-x)] \right| \leq \frac{8x^2 - 4x + 1}{4} \left[\frac{|f'(0)|^q + |f'(1)|^q}{2} \right]^{1/q}.$$

If $|f''|^q$ is convex for some $q \geq 1$ and $x \in [0,1/4]$, then

$$\begin{aligned} &\left| \int_0^1 f(t)dt - \frac{1}{2}[f(x) + f(1-x)] \right| \\ &\leq \left[\frac{-6x^2 + 6x - 1}{24} + \frac{1}{6}(1-4x)^{3/2} \right] \left[\frac{|f''(0)|^q + |f''(1)|^q}{2} \right]^{1/q}, \end{aligned}$$

while for $x \in [1/4, 1/2]$ we have

$$\left| \int_0^1 f(t)dt - \frac{1}{2}[f(x) + f(1-x)] \right| \leq \left[\frac{-6x^2 + 6x - 1}{24} \right] \left[\frac{|f''(0)|^q + |f''(1)|^q}{2} \right]^{1/q}.$$

(b) If $|f'|$ is concave for some $q \geq 1$, then for $x \in [0,1/2]$ we have

$$\left| \int_0^1 f(t)dt - \frac{1}{2}[f(x) + f(1-x)] \right| \leq \frac{8x^2 - 4x + 1}{4} \left| f'\left(\frac{1}{2}\right) \right|.$$

If $|f''|$ is concave for some $q \geq 1$ and $x \in [0,1/4]$, then

$$\left| \int_0^1 f(t)dt - \frac{1}{2}[f(x) + f(1-x)] \right| \leq \left[\frac{-6x^2 + 6x - 1}{24} + \frac{1}{6}(1-4x)^{3/2} \right] \left| f''\left(\frac{1}{2}\right) \right|,$$

while for $x \in [1/4, 1/2]$ we have

$$\left| \int_0^1 f(t)dt - \frac{1}{2}[f(x) + f(1-x)] \right| \leq \left[\frac{-6x^2 + 6x - 1}{24} \right] \left| f'\left(\frac{1}{2}\right) \right|.$$

Proof. It was already shown that for $x \in [0,1/2]$

$$\int_0^1 |F_1^x(t)|dt = \frac{8x^2 - 4x + 1}{2},$$

so, using identity (2.4) and following the proof of Theorem 2.12 we get first inequalities in (a) and (b). The second inequality in (a) and (b) we prove similarly. □

Remark 2.23 For $x = 0,\ x = 1/3$ and $x = 1/2$ in above theorems we get the results from [105] and [106].

For $x = 1/4$ we get two-point Maclaurin formula and then we have

$$\frac{1}{192} f^{(2)}\left(\frac{1}{2}\right) \leq \int_0^1 f(t)dt - \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) \right] \leq \frac{1}{192} \frac{f^{(2)}(0) + f^{(2)}(1)}{2}.$$

If $|f'|^q$ is convex for some $q \geq 1$, then

$$\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right]\right| \leq \frac{1}{8}\left[\frac{|f'(0)|^q + |f'(1)|^q}{2}\right]^{1/q}$$

and if $|f^{(2)}|^q$ is convex for some $q \geq 1$, then

$$\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right]\right| \leq \frac{1}{192}\left[\frac{\left|f^{(2)}(0)\right|^q + \left|f^{(2)}(1)\right|^q}{2}\right]^{1/q}.$$

If $|f'|$ is concave, then

$$\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right]\right| \leq \frac{1}{8}\left|f'\left(\frac{1}{2}\right)\right|$$

and if $|f^{(2)}|$ is concave for some $q \geq 1$, then

$$\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right]\right| \leq \frac{1}{192}\left|f^{(2)}\left(\frac{1}{2}\right)\right|.$$

In the paper [73] Dah-Yan Hwang gave some new inequalities of this type and he applied the result to obtain better estimates of the error in the trapezoidal formula.

Here we consider some related results using the general Euler two-point formulae which are published in [118].

By integration by parts, we have that the following identities:

(i) $C_1(x) = \int_0^1 F_{2r-1}^x\left(\frac{y}{2}\right)dy = -\int_0^1 F_{2r-1}^x\left(1-\frac{y}{2}\right)dy = \frac{2}{r}\left[B_{2r}(x) - B_{2r}\left(\frac{1}{2}-x\right)\right],$

(ii) $C_2(x) = \int_0^1 yF_{2r-1}^x\left(\frac{y}{2}\right)dy = -\int_0^1 yF_{2r-1}^x\left(1-\frac{y}{2}\right)dy = -\frac{2}{r}B_{2r}\left(\frac{1}{2}-x\right),$

(iii) $C_3(x) = \int_0^1 (1-y)F_{2r-1}^x\left(\frac{y}{2}\right)dy = -\int_0^1 (1-y)F_{2r-1}^x\left(1-\frac{y}{2}\right)dy = \frac{2}{r}B_{2r}(x),$

(iv) $C_4(x) = \int_0^1 F_{2r}^x\left(\frac{y}{2}\right)dy = \int_0^1 F_{2r}^x\left(1-\frac{y}{2}\right)dy = -2B_{2r}(x),$

(v) $C_5(x) = \int_0^1 yF_{2r}^x\left(\frac{y}{2}\right)dy = \int_0^1 yF_{2r}^x\left(1-\frac{y}{2}\right)dy$
$= \frac{8}{(2r+1)(2r+2)}\left[B_{2r+2}(x) - B_{2r+2}\left(\frac{1}{2}-x\right)\right] - B_{2r}(x),$

(vi) $C_6(x) = \int_0^1 (1-y)F_{2r}^x\left(\frac{y}{2}\right)dy = \int_0^1 (1-y)F_{2r}^x\left(1-\frac{y}{2}\right)dy$
$= \frac{8}{(2r+1)(2r+2)}\left[B_{2r+2}\left(\frac{1}{2}-x\right) - B_{2r+2}(x)\right] - B_{2r}(x),$

(vii) $C_7(x) = \int_0^1 G_{2r}^x\left(\frac{y}{2}\right)dy = \int_0^1 G_{2r}^x\left(1-\frac{y}{2}\right)dy = 0,$

(viii) $C_8(x) = \int_0^1 yG_{2r}^x\left(\frac{y}{2}\right)dy = \int_0^1 yG_{2r}^x\left(1-\frac{y}{2}\right)dy = -\int_0^1 (1-y)G_{2r}^x\left(\frac{y}{2}\right)dy$
$= \int_0^1 (1-y)G_{2r}^x\left(1-\frac{y}{2}\right)dy = \frac{8}{(2r+1)(2r+2)}\left[B_{2r+2}(x) - B_{2r+2}\left(\frac{1}{2}-x\right)\right],$

Theorem 2.12 *Suppose $f : [0,1] \to \mathbb{R}$ is n-times differentiable and $x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right] \cup \left[\frac{1}{2\sqrt{3}}, \frac{1}{2}\right]$.*

(a) If $\left|f^{(n)}\right|^q$ is convex for some $q \geq 1$, then for $n = 2r-1$, $r \geq 2$, we have

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(x)+f(1-x)] + T_{r-1}(f)\right| \leq \frac{2}{(2r)!}\left|B_{2r}\left(\frac{1}{2}-x\right) - B_{2r}(x)\right|^{1-1/q}$$
$$\cdot \left[\left|\frac{r}{2}C_3(x)\right| \cdot \frac{\left|f^{(2r-1)}(0)\right|^q + \left|f^{(2r-1)}(1)\right|^q}{2} + \left|\frac{r}{2}C_2(x)\right| \cdot \left|f^{(2r-1)}\left(\frac{1}{2}\right)\right|^q\right]^{1/q}. \tag{2.49}$$

If $n = 2r$, $r \geq 2$, then

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(x)+f(1-x)] + T_{r-1}(f)\right| \leq \frac{|B_{2r}(x)|^{1-1/q}}{(2r)!}$$
$$\cdot \left[\left|\frac{1}{2}C_6(x)\right| \frac{\left|f^{(2r)}(0)\right|^q + \left|f^{(2r)}(1)\right|^q}{2} + \left|\frac{1}{2}C_5(x)\right|\left|f^{(2r)}\left(\frac{1}{2}\right)\right|^q\right]^{1/q} \tag{2.50}$$

and we also have

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(x)+f(1-x)] + T_r(f)\right| \leq \frac{2|B_{2r}(x)|^{1-1/q}}{(2r)!} \tag{2.51}$$
$$\cdot \left[\left|\frac{1}{8}C_8(x)\right| \left(\left|f^{(2r)}(0)\right|^q + 2\left|f^{(2r)}\left(\frac{1}{2}\right)\right|^q + \left|f^{(2r)}(1)\right|^q\right)\right]^{1/q}.$$

(b) If $\left|f^{(n)}\right|^q$ is concave, then for $n = 2r-1$, $r \geq 2$, we have

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(x)+f(1-x)] + T_{r-1}(f)\right| \tag{2.52}$$
$$\leq \frac{1}{(2r)!}\left|\frac{r}{2}C_1(x)\right| \cdot \left[\left|f^{(2r-1)}\left(\frac{|C_2(x)|}{2|C_1(x)|}\right)\right| + \left|f^{(2r-1)}\left(\frac{\left|C_3(x)+\frac{1}{2}C_2(x)\right|}{|C_1(x)|}\right)\right|\right].$$

If $n = 2r$, $r \geq 2$, then

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(x)+f(1-x)] + T_{r-1}(f)\right| \leq \frac{|C_4(x)|}{4(2r)!}$$
$$\cdot \left[\left|f^{(2r)}\left(\frac{|C_5(x)|}{2|C_4(x)|}\right)\right| + \left|f^{(2r)}\left(\frac{\left|C_6(x)+\frac{1}{2}C_5(x)\right|}{|C_4(x)|}\right)\right|\right]. \tag{2.53}$$

Proof. First, let $n = 2r-1$ for some $r \geq 2$. Then by Hölder inequality

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(x)+f(1-x)] + T_{r-1}(f)\right|$$

$$
\begin{aligned}
&\leq \frac{1}{2(2r-1)!}\int_0^1 \left|F_{2r-1}^x(t)\right|\cdot\left|f^{(2r-1)}(t)\right|dt \\
&\leq \frac{1}{2(2r-1)!}\left(\int_0^1 \left|F_{2r-1}^x(t)\right|dt\right)^{1-1/q}\left(\int_0^1 \left|F_{2r-1}^x(t)\right|\cdot\left|f^{(2r-1)}(t)\right|^q dt\right)^{1/q} \\
&= \frac{1}{2(2r-1)!}\left(\frac{2}{r}\left|B_{2r}\left(\frac{1}{2}-x\right)-B_{2r}(x)\right|\right)^{1-1/q}\left(\int_0^1 \left|F_{2r-1}^x(t)\right|\cdot\left|f^{(2r-1)}(t)\right|^q dt\right)^{1/q}.
\end{aligned}
$$

Now, by the convexity of $|f^{(2r-1)}|^q$ we have

$$
\begin{aligned}
&\int_0^1 \left|F_{2r-1}^x(t)\right|\cdot\left|f^{(2r-1)}(t)\right|^q dt \\
&= \int_0^{1/2} \left|F_{2r-1}^x(t)\right|\cdot\left|f^{(2r-1)}(t)\right|^q dt + \int_{1/2}^1 \left|F_{2r-1}^x(t)\right|\cdot\left|f^{(2r-1)}(t)\right|^q dt \\
&= \frac{1}{2}\int_0^1 \left|F_{2r-1}^x\left(\frac{y}{2}\right)\right|\cdot\left|f^{(2r-1)}\left((1-y)\cdot 0+y\cdot\frac{1}{2}\right)\right|^q dy \\
&+ \frac{1}{2}\int_0^1 \left|F_{2r-1}^x\left(1-\frac{y}{2}\right)\right|\cdot\left|f^{(2r-1)}\left((1-y)\cdot 1+y\cdot\frac{1}{2}\right)\right|^q dy \\
&\leq \frac{1}{2}\left[\left|\int_0^1 (1-y)F_{2r-1}^x\left(\frac{y}{2}\right)dy\right|\cdot\left|f^{(2r-1)}(0)\right|^q + \left|\int_0^1 yF_{2r-1}^x\left(\frac{y}{2}\right)dy\right|\cdot\left|f^{(2r-1)}\left(\frac{1}{2}\right)\right|^q\right. \\
&+ \left|\int_0^1 (1-y)F_{2r-1}^x\left(1-\frac{y}{2}\right)dy\right|\cdot\left|f^{(2r-1)}(1)\right|^q \\
&+ \left.\left|\int_0^1 yF_{2r-1}^x\left(1-\frac{y}{2}\right)dy\right|\cdot\left|f^{(2r-1)}\left(\frac{1}{2}\right)\right|^q\right].
\end{aligned}
$$

On the other hand, if $\left|f^{(2r-1)}\right|^q$ is concave, then

$$
\begin{aligned}
&\left|\int_0^1 f(t)dt-\frac{1}{2}[f(x)+f(1-x)]+T_{r-1}(f)\right| \\
&\leq \frac{1}{2(2r-1)!}\int_0^1 \left|F_{2r-1}^x(t)\right|\cdot\left|f^{(2r-1)}(t)\right|dt \\
&= \frac{1}{2(2r-1)!}\left[\int_0^{1/2} \left|F_{2r-1}^x(t)\right|\cdot\left|f^{(2r-1)}(t)\right|dt+\int_{1/2}^1 \left|F_{2r-1}^x(t)\right|\cdot\left|f^{(2r-1)}(t)\right|dt\right] \\
&= \frac{1}{2(2r-1)!}\left[\int_0^1 \left|F_{2r-1}^x\left(\frac{y}{2}\right)\right|\cdot\left|f^{(2r-1)}\left((1-y)\cdot 0+y\cdot\frac{1}{2}\right)\right|dy\right. \\
&+ \left.\int_0^1 \left|F_{2r-1}^x\left(1-\frac{y}{2}\right)\right|\cdot\left|f^{(2r-1)}\left((1-y)\cdot 1+y\cdot\frac{1}{2}\right)\right|dy\right]
\end{aligned}
$$

$$
\leq \frac{1}{4(2r-1)!}\left[\left|\int_0^1 F_{2r-1}^x\left(\frac{y}{2}\right)dy\right|\cdot\left|f^{(2r-1)}\left(\frac{\left|\int_0^1 F_{2r-1}^x\left(\frac{y}{2}\right)((1-y)\cdot 0+y\cdot\frac{1}{2})dy\right|}{\left|\int_0^1 F_{2r-1}^x\left(\frac{y}{2}\right)dy\right|}\right)\right|\right.
$$

$$+\left|\int_0^1 F_{2r-1}^x\left(1-\frac{y}{2}\right)dy\right|\cdot\left|f^{(2r-1)}\left(\frac{\left|\int_0^1 F_{2r-1}^x\left(1-\frac{y}{2}\right)((1-y)\cdot 1+y\cdot\frac{1}{2})dy\right|}{\left|\int_0^1 F_{2r-1}^x\left(1-\frac{y}{2}\right)dy\right|}\right)\right|\Bigg],$$

so the inequality (2.49) and (2.52) are completely proved.

The proofs of the inequalities (2.50), (2.53) and (2.51) are similar. □

Remark 2.24 If in Theorem 2.12 we chose $x=0,1/2,1/3$, we get generalizations of Dragomir-Agarwal inequality for Euler trapezoid (see [73]) , Euler midpoint and Euler two-point Newton-Cotes formulae respectively.

The resultant formulae in Theorem 2.12 when $r=2$ are of special interest, so we isolate it as corollary.

Corollary 2.15 *Suppose $f:[0,1]\to\mathbb{R}$ is 4-times differentiable and $x\in\left[0,\frac{1}{2}-\frac{1}{2\sqrt{3}}\right]\cup\left[\frac{1}{2\sqrt{3}},\frac{1}{2}\right]$.*

(a)If $\left|f^{(3)}\right|^q$ is convex for some $q\geq 1$, then

$$\left|\int_0^1 f(t)dt-\frac{1}{2}\left[f(x)+f(1-x)\right]+\frac{1}{12}\left[f'(1)-f'(0)\right]\right|\leq\frac{1}{12}\left|2x^3-\frac{3}{2}x^2+\frac{1}{16}\right|^{1-1/q}$$

$$\cdot\left[\left|x^4-2x^3+x^2-\frac{1}{30}\right|\frac{\left|f^{(3)}(0)\right|^q+\left|f^{(3)}(1)\right|^q}{2}+\left|-x^4+\frac{x^2}{2}-\frac{7}{240}\right|\left|f^{(3)}\left(\frac{1}{2}\right)\right|^q\right]^{1/q}$$

and if $\left|f^{(4)}\right|^q$ is convex for some $q\geq 1$, then

$$\left|\int_0^1 f(t)dt-\frac{1}{2}\left[f(x)+f(1-x)\right]+\frac{1}{12}\left[f'(1)-f'(0)\right]\right|\leq\frac{1}{24}\left|x^4-2x^3+x^2-\frac{1}{30}\right|^{1-1/q}$$

$$\cdot\left[\left|\frac{2x^5}{5}-x^4+x^3-\frac{3x^2}{8}+\frac{1}{96}\right|\frac{\left|f^{(4)}(0)\right|^q+\left|f^{(4)}(1)\right|^q}{2}\right.$$

$$\left.+\left|-\frac{2x^5}{5}+x^3-\frac{5x^2}{8}+\frac{11}{480}\right|\left|f^{(4)}\left(\frac{1}{2}\right)\right|^q\right]^{1/q}.$$

(b) If $\left|f^{(3)}\right|$ is concave, then

$$\left|\int_0^1 f(t)dt-\frac{1}{2}\left[f(x)+f(1-x)\right]+\frac{1}{12}\left[f'(1)-f'(0)\right]\right|$$

$$\leq\frac{1}{24}\left|2x^3-\frac{3}{2}x^2+\frac{1}{16}\right|\left[\left|f^{(3)}\left(\frac{\left|-x^4+\frac{x^2}{2}-\frac{7}{240}\right|}{\left|-4x^3+3x^2-\frac{1}{8}\right|}\right)\right|+\left|f^{(3)}\left(\frac{\left|\frac{x^4}{2}-2x^3+\frac{5x^2}{4}-\frac{23}{480}\right|}{\left|-2x^3+\frac{3x^2}{2}-\frac{1}{16}\right|}\right)\right|\right]$$

and if $\left|f^{(4)}\right|$ *is concave, then*

$$\begin{aligned}&\left|\int_0^1 f(t)dt-\frac{1}{2}\left[f(x)+f(1-x)\right]+\frac{1}{12}\left[f'(1)-f'(0)\right]\right|\\&\le\frac{1}{48}\left|x^4-2x^3+x^2-\frac{1}{30}\right|\left[\left|f^{(4)}\left(\frac{\left|-\frac{4x^5}{5}+2x^3-\frac{5x^2}{4}+\frac{11}{240}\right|}{\left|-4x^4+8x^3-4x^2+\frac{2}{15}\right|}\right)\right|\right.\\&\left.+\left|f^{(4)}\left(\frac{\left|\frac{2x^5}{5}-2x^4+3x^3-\frac{11x^2}{8}+\frac{7}{160}\right|}{\left|-2x^4+4x^3-2x^2+\frac{1}{15}\right|}\right)\right|\right].\end{aligned}$$

Now, we will give some results of the same type in the case when $r=1$.

Theorem 2.13 *Suppose* $f:[0,1]\to\mathbb{R}$ *is 2-times differentiable.*
(a) If $|f'|^q$ *is convex for some* $q\ge 1$*, then for* $x\in[0,1/2]$ *we have*

$$\begin{aligned}&\left|\int_0^1 f(t)dt-\frac{1}{2}[f(x)+f(1-x)]\right|\le\frac{|8x^2-4x+1|^{1-1/q}}{4}\\&\cdot\left[\left|2x^2-2x+\frac{2}{3}\right|\frac{|f'(0)|^q+|f'(1)|^q}{2}+\left|-2x^2+2x+\frac{1}{3}\right|\left|f'\left(\frac{1}{2}\right)\right|^q\right]^{1/q}.\end{aligned}$$

If $|f''|^q$ *is convex for some* $q\ge 1$ *and* $x\in[0,1/4]$*, then*

$$\begin{aligned}&\left|\int_0^1 f(t)dt-\frac{1}{2}[f(x)+f(1-x)]\right|\le\frac{\left|\frac{-6x^2+6x-1}{3}+\frac{2}{3}(1-4x)^{3/2}\right|^{1-1/q}}{4}\\&\cdot\left[\left|-x^2+x-\frac{1}{8}\right|\frac{|f''(0)|^q+|f''(1)|^q}{2}+\left|-2x^2+2x-\frac{5}{24}\right|\left|f''\left(\frac{1}{2}\right)\right|^q\right]^{1/q},\end{aligned}$$

while for $x\in[1/4,1/2]$ *we have*

$$\begin{aligned}&\left|\int_0^1 f(t)dt-\frac{1}{2}[f(x)+f(1-x)]\right|\le\frac{1}{4}\left|\frac{-6x^2+6x-1}{3}\right|^{1-1/q}\\&\cdot\left[\left|-x^2+x-\frac{1}{8}\right|\frac{|f''(0)|^q+|f''(1)|^q}{2}+\left|-2x^2+2x-\frac{5}{24}\right|\left|f''\left(\frac{1}{2}\right)\right|^q\right]^{1/q}.\end{aligned}$$

(b) If $|f'|$ *is concave for some* $q\ge 1$*, then for* $x\in[0,1/2]$ *we have*

$$\left|\int_0^1 f(t)dt-\frac{1}{2}[f(x)+f(1-x)]\right|\le\frac{1}{8}\left[\left|f'\left(\left|-x^2+x+\frac{1}{6}\right|\right)\right|+\left|f'\left(\left|x^2-x+\frac{5}{6}\right|\right)\right|\right].$$

If $|f''|$ *is concave for some* $q\ge 1$ *and* $x\in[0,1/2]$*, then*

$$\begin{aligned}&\left|\int_0^1 f(t)dt-\frac{1}{2}[f(x)+f(1-x)]\right|\le\frac{1}{8}\left|-3x^2+3x-\frac{1}{3}\right|\\&\cdot\left[\left|f''\left(\frac{\left|-2x^2+2x-\frac{5}{24}\right|}{\left|-6x^2+6x-\frac{2}{3}\right|}\right)\right|+\left|f''\left(\frac{\left|-2x^2+2x-\frac{11}{48}\right|}{\left|-3x^2+3x-\frac{1}{3}\right|}\right)\right|\right].\end{aligned}$$

Proof. Using identities the (2.3) and (2.4) with calculation of $C_i(x), i = 1, \ldots, 6$, similar as in Theorem 2.12, we get the inequalities in (a) and (b). □

Remark 2.25 For $x = 0$ in the above theorem we have the trapezoid formula and for $|f''|^q$ convex function and $|f''|$ concave function we get the results from [73].
If $|f'|^q$ is convex for some $q \geq 1$, then

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(0)+f(1)]\right| \leq \frac{1}{4}\left[\frac{|f'(0)|^q + |f'\left(\frac{1}{2}\right)|^q + |f'(1)|^q}{3}\right]^{1/q}$$

and if $|f'|$ is concave, then

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(0)+f(1)]\right| \leq \frac{1}{8}\left[\left|f'\left(\frac{1}{6}\right)\right| + \left|f'\left(\frac{5}{6}\right)\right|\right].$$

For $x = 1/4$ we get two-point Maclaurin formula and then if $|f'|^q$ is convex for some $q \geq 1$, then

$$\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right]\right| \leq \frac{1}{8}\left[\frac{7|f'(0)|^q + 34|f'\left(\frac{1}{2}\right)|^q + 7|f'(1)|^q}{24}\right]^{1/q}$$

and if $|f''|^q$ is convex for some $q \geq 1$, then

$$\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right]\right| \leq \frac{1}{96}\left[\frac{3|f''(0)|^q + 16|f''\left(\frac{1}{2}\right)|^q + 3|f''(1)|^q}{4}\right]^{1/q}.$$

If $|f'|$ is concave, then

$$\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right]\right| \leq \frac{1}{8}\left[\left|f'\left(\frac{17}{48}\right)\right| + \left|f'\left(\frac{31}{48}\right)\right|\right]$$

and if $|f''|$ is concave for some $q \geq 1$, then

$$\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right]\right| \leq \frac{11}{384}\left[\left|f''\left(\frac{4}{11}\right)\right| + \left|f''\left(\frac{7}{11}\right)\right|\right].$$

For $x = 1/3$ we get two-point Newton-Cotes formula and then if $|f'|^q$ is convex for some $q \geq 1$, then

$$\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right)\right]\right| \leq \frac{5}{36}\left[\frac{|f'(0)|^q + 7|f'\left(\frac{1}{2}\right)|^q + |f'(1)|^q}{5}\right]^{1/q}$$

and if $|f''|^q$ is convex for some $q \geq 1$, then

$$\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right)\right]\right| \leq \frac{1}{36}\left[\frac{7|f''(0)|^q + 34|f''\left(\frac{1}{2}\right)|^q + 7|f''(1)|^q}{16}\right]^{1/q}.$$

If $|f'|$ is concave, then

$$\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f\left(\frac{1}{3}\right)+f\left(\frac{2}{3}\right)\right]\right| \le \frac{1}{8}\left[\left|f'\left(\frac{7}{18}\right)\right|+\left|f'\left(\frac{11}{18}\right)\right|\right]$$

and if $|f''|$ is concave for some $q \ge 1$, then

$$\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f\left(\frac{1}{3}\right)+f\left(\frac{2}{3}\right)\right]\right| \le \frac{1}{24}\left[\left|f''\left(\frac{17}{48}\right)\right|+\left|f''\left(\frac{31}{48}\right)\right|\right].$$

For $x = 1/2$ we get midpoint formula and then if $|f'|^q$ is convex for some $q \ge 1$, then

$$\left|\int_0^1 f(t)dt - f\left(\frac{1}{2}\right)\right| \le \frac{1}{4}\left[\frac{|f'(0)|^q + 10|f'\left(\frac{1}{2}\right)|^q + |f'(1)|^q}{12}\right]^{1/q}$$

and if $|f''|^q$ is convex for some $q \ge 1$, then

$$\left|\int_0^1 f(t)dt - f\left(\frac{1}{2}\right)\right| \le \frac{1}{24}\left[\frac{3|f''(0)|^q + 14|f''\left(\frac{1}{2}\right)|^q + 3|f''(1)|^q}{8}\right]^{1/q}.$$

If $|f'|$ is concave, then

$$\left|\int_0^1 f(t)dt - f\left(\frac{1}{2}\right)\right| \le \frac{1}{8}\left[\left|f'\left(\frac{5}{12}\right)\right|+\left|f'\left(\frac{7}{12}\right)\right|\right]$$

and if $|f''|$ is concave for some $q \ge 1$, then

$$\left|\int_0^1 f(t)dt - f\left(\frac{1}{2}\right)\right| \le \frac{5}{96}\left[\left|f''\left(\frac{7}{20}\right)\right|+\left|f''\left(\frac{13}{20}\right)\right|\right].$$

2.6 Estimations of the error for two-point formula via pre-Grüss inequality

In the paper [116] N. Ujević used the generalization of pre-Grüss inequality to derive some better estimations of the error for Simpson's quadrature rule. In fact, he proved the next as his main result:

Theorem 2.14 *If $g,h,\Psi \in L_2[0,1]$ and $\int_0^1 \Psi(t)dt = 0$ then we have*

$$|S_\Psi(g,h)| \le S_\Psi(g,g)^{1/2} S_\Psi(h,h)^{1/2}, \tag{2.54}$$

where

$$S_\Psi(g,h) = \int_0^1 g(t)h(t)dt - \int_0^1 g(t)dt \int_0^1 h(t)dt - \int_0^1 g(t)\Psi_0(t)dt \int_0^1 h(t)\Psi_0(t)dt$$

and $\Psi_0(t) = \Psi(t)/\|\Psi\|_2$.

Further, he gave some improvements of the Simpson's inequality.

Theorem 2.15 *Let $I \subset \mathbb{R}$ be a closed interval and $a, b \in IntI$, $a < b$. If $f : I \to \mathbb{R}$ is continuous of bounded variation with $f' \in L_2[a,b]$, then we have*

$$\left| \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \int_a^b f(t)dt \right| \leq \frac{(b-a)^{3/2}}{6} K_1, \tag{2.55}$$

where

$$K_1^2 = \|f'\|_2^2 - \frac{1}{b-a} \left(\int_a^b f'(t)dt \right)^2 - \left(\int_a^b f'(t)\Psi_0(t)dt \right)^2 \tag{2.56}$$

and $\Psi(t) = t - \frac{a+b}{2}, \Psi_0(t) = \Psi(t)/\|\Psi\|_2$.

In this section, using Theorem 2.14, we will give similar result for Euler two-point formula and for functions whose derivative of order n, $n \geq 1$, is from $L_2[0,1]$ space. We will use interval $[0,1]$ because of simplicity and since it involves no loss in generality.

The results from this section are published in [104].

Theorem 2.16 *If $f : [0,1] \to \mathbb{R}$ is such that $f^{(n-1)}$ is continuous of bounded variation function with $f^{(n)} \in L_2[0,1]$ then we have*

$$\left| \int_0^1 f(t)\mathrm{d}t - D(x) + T_n(x) \right| \leq \frac{1}{2} \left[\frac{2(-1)^{n-1}}{(2n)!} \left[B_{2n} + B_{2n}(1-2x) \right] \right]^{1/2} K, \tag{2.57}$$

where

$$K^2 = \|f^{(n)}\|_2^2 - \left(\int_0^1 f^{(n)}(t)dt \right)^2 - \left(\int_0^1 f^{(n)}(t)\Psi_0(t)dt \right)^2. \tag{2.58}$$

For n even

$$\Psi(t) = \begin{cases} 1, & t \in \left[0, \frac{1}{2}\right], \\ -1, & t \in \left(\frac{1}{2}, 1\right], \end{cases}$$

while for n odd we have

$$\Psi(t) = \begin{cases} t + \frac{B_{n+1}\left(\frac{1}{2}+x\right)}{2\left(B_{n+1}(x) - B_{n+1}\left(\frac{1}{2}+x\right)\right)}, & t \in \left[0, \frac{1}{2}\right], \\ t + \frac{B_{n+1}\left(\frac{1}{2}+x\right) - 2B_{n+1}(x)}{2\left(B_{n+1}(x) - B_{n+1}\left(\frac{1}{2}+x\right)\right)}, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Proof. It is not difficult to verify that

$$\int_0^1 G_n(t)dt = 0, \tag{2.59}$$

$$\int_0^1 \Psi(t)dt = 0, \tag{2.60}$$

$$\int_0^1 G_n(t)\Psi(t)dt = 0. \tag{2.61}$$

From (2.3), (2.59) and (2.61) it follows that

$$\begin{aligned}
\int_0^1 f(t)\mathrm{d}t - D(x) + T_n(x) &= \frac{1}{2(n!)}\int_0^1 G_n^x(t) f^{(n)}(t)dt \\
&- \frac{1}{2(n!)}\int_0^1 G_n^x(t)dt \int_0^1 f^{(n)}(t)dt \\
&- \frac{1}{2(n!)}\int_0^1 G_n^x(t)\Psi_0(t)dt \int_0^1 f^{(n)}(t)\Psi_0(t)dt \\
&= \frac{1}{2(n!)} S_\Psi(G_n^x, f^{(n)}). \qquad (2.62)
\end{aligned}$$

Using (2.62) and (2.54) we get

$$\left|\int_0^1 f(t)\mathrm{d}t - D(x) + T_n(x)\right| \le \frac{1}{2(n!)} S_\Psi(G_n^x, G_n^x)^{1/2} S_\Psi(f^{(n)}, f^{(n)})^{1/2}. \qquad (2.63)$$

We also have (see Section 3 of this chapter)

$$\begin{aligned}
S_\Psi(G_n^x, G_n^x) &= \|G_n^x\|_2^2 - \left(\int_0^1 G_n^x(t)dt\right)^2 - \left(\int_0^1 G_n^x(t)\Psi_0(t)dt\right)^2 \\
&= (-1)^{n-1}\frac{2(n!)^2}{(2n)!}\left[B_{2n} + B_{2n}(1-2x)\right] \qquad (2.64)
\end{aligned}$$

and

$$S_\Psi(f^{(n)}, f^{(n)}) = \|f^{(n)}\|_2^2 - \left(\int_0^1 f^{(n)}(t)dt\right)^2 - \left(\int_0^1 f^{(n)}(t)\Psi_0(t)dt\right)^2 = K^2. \qquad (2.65)$$

From (2.63)-(2.65) we easily get (2.57). □

Remark 2.26 Function $\Psi(t)$ can be any function which satisfies conditions $\int_0^1 \Psi(t)dt = 0$ and $\int_0^1 G_n^x(t)\Psi(t)dt = 0$. Because $G_n^x(1-t) = (-1)^n G_n^x(t)$ (see Section 3 of this chapter), for n even, we can take function $\Psi(t)$ such that $\Psi(1-t) = -\Psi(t)$. For n odd, we have to calculate $\Psi(t)$ and with no lost in generality in our theorem we take the form $\Psi(t) = \begin{cases} t+a, \ t \in \left[0, \frac{1}{2}\right], \\ t+b, \ t \in \left(\frac{1}{2}, 1\right]. \end{cases}$

Remark 2.27 For $n = 1$ in Theorem 2.16 we have

$$\left|\int_0^1 f(t)\mathrm{d}t - D(x)\right| \le \frac{1}{2}\left[\frac{1}{3} - 2x + 4x^2\right]^{1/2} K, \qquad (2.66)$$

while

$$\Psi(t) = \begin{cases} t + \frac{1-12x^2}{24x-6}, & t \in \left[0, \frac{1}{2}\right], \\ t + \frac{12x^2-24x+5}{24x-6}, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Also, for $n=2$ we have

$$\left|\int_0^1 f(t)\mathrm{d}t - D(x)\right| \leq \frac{1}{2}\left[\frac{1}{180} - \frac{x^2}{3} + \frac{4x^3}{3} - \frac{4x^4}{3}\right]^{1/2} K, \tag{2.67}$$

while

$$\Psi(t) = \begin{cases} 1, & t \in \left[0, \frac{1}{2}\right], \\ -1, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

If in Theorem 2.16 we choose $x = 0, 1/2, 1/3, 1/4$ we get inequality related to the trapezoid, the midpoint, the two-point Newton-Cotes and the two-point Maclaurin formula:

Corollary 2.16 *If $f:[0,1]\to\mathbb{R}$ is such that $f^{(n-1)}$ is continuous of bounded variation with $f^{(n)} \in L_2[0,1]$, then we have*

$$\left|\int_0^1 f(t)\mathrm{d}t - \frac{1}{2}[f(0)+f(1)] + T_n(0)\right| \leq \left[\frac{(-1)^{n-1}}{(2n)!}B_{2n}\right]^{1/2} K, \tag{2.68}$$

where $T_0(0)=0$,

$$T_n(0) = \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right]$$

and

$$K^2 = \|f^{(n)}\|_2^2 - \left(\int_0^1 f^{(n)}(t)dt\right)^2 - \left(\int_0^1 f^{(n)}(t)\Psi_0(t)dt\right)^2.$$

For n even

$$\Psi(t) = \begin{cases} 1, & t \in \left[0, \frac{1}{2}\right], \\ -1, & t \in \left(\frac{1}{2}, 1\right], \end{cases}$$

while for n odd we have

$$\Psi(t) = \begin{cases} t + \frac{2^{-n}-1}{4-2^{1-n}}, & t \in \left[0, \frac{1}{2}\right], \\ t + \frac{2^{-n}-3}{4-2^{1-n}}, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Remark 2.28 For $n=1$ in Corollary 2.16 we have

$$\left|\int_0^1 f(t)\mathrm{d}t - \frac{1}{2}[f(0)+f(1)]\right| \leq \frac{K}{2\sqrt{3}},$$

while

$$\Psi(t) = \begin{cases} t - \frac{1}{6}, & t \in \left[0, \frac{1}{2}\right], \\ t - \frac{5}{6}, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Corollary 2.17 *If $f:[0,1]\to\mathbb{R}$ is such that $f^{(n-1)}$ is continuous of bounded variation with $f^{(n)} \in L_2[0,1]$, then we have*

$$\left|\int_0^1 f(t)\mathrm{d}t - f\left(\frac{1}{2}\right) + T_n\left(\frac{1}{2}\right)\right| \leq \left[\frac{(-1)^{n-1}}{(2n)!}B_{2n}\right]^{1/2} K, \tag{2.69}$$

where $T_0\left(\frac{1}{2}\right)=0$,

$$T_n\left(\frac{1}{2}\right)=\sum_{k=1}^{\lfloor n/2\rfloor}\frac{(2^{1-2k}-1)B_{2k}}{(2k)!}\left[f^{(2k-1)}(1)-f^{(2k-1)}(0)\right]$$

and

$$K^2=\|f^{(n)}\|_2^2-\left(\int_0^1 f^{(n)}(t)dt\right)^2-\left(\int_0^1 f^{(n)}(t)\Psi_0(t)dt\right)^2.$$

For n even

$$\Psi(t)=\begin{cases}1, & t\in\left[0,\frac{1}{2}\right],\\ -1, & t\in\left(\frac{1}{2},1\right],\end{cases}$$

while for n odd we have

$$\Psi(t)=\begin{cases}t+\frac{1}{2^{1-n}-4}, & t\in\left[0,\frac{1}{2}\right],\\ t+\frac{3-2^{1-n}}{2^{1-n}-4}, & t\in\left(\frac{1}{2},1\right].\end{cases}$$

Remark 2.29 For $n=1$ in Corollary 2.17 we have

$$\left|\int_0^1 f(t)\mathrm{d}t-f\left(\frac{1}{2}\right)\right|\le\frac{K}{2\sqrt{3}},$$

while

$$\Psi(t)=\begin{cases}t-\frac{1}{3}, & t\in\left[0,\frac{1}{2}\right],\\ t-\frac{2}{3}, & t\in\left(\frac{1}{2},1\right].\end{cases}$$

Corollary 2.18 *If* $f:[0,1]\to\mathbb{R}$ *is such that* $f^{(n-1)}$ *is continuous of bounded variation with* $f^{(n)}\in L_2[0,1]$, *then we have*

$$\left|\int_0^1 f(t)\mathrm{d}t-\frac{1}{2}\left[f\left(\frac{1}{3}\right)+f\left(\frac{2}{3}\right)\right]+T_n\left(\frac{1}{3}\right)\right|\le\frac{1}{2}\left[\frac{(-1)^{n-1}}{(2n)!}(1+3^{1-2n})B_{2n}\right]^{1/2}K, \tag{2.70}$$

where $T_0\left(\frac{1}{3}\right)=0$,

$$T_n\left(\frac{1}{3}\right)=\frac{1}{2}\sum_{k=1}^{\lfloor n/2\rfloor}\frac{(3^{1-2k}-1)B_{2k}}{(2k)!}\left[f^{(2k-1)}(1)-f^{(2k-1)}(0)\right]$$

and

$$K^2=\|f^{(n)}\|_2^2-\left(\int_0^1 f^{(n)}(t)dt\right)^2-\left(\int_0^1 f^{(n)}(t)\Psi_0(t)dt\right)^2.$$

For n even

$$\Psi(t)=\begin{cases}1, & t\in\left[0,\frac{1}{2}\right],\\ -1, & t\in\left(\frac{1}{2},1\right],\end{cases}$$

while for n odd we have

$$\Psi(t)=\begin{cases}t+\frac{1-2^n}{2^{2+n}-2}, & t\in\left[0,\frac{1}{2}\right],\\ t+\frac{1-3\cdot 2^n}{2^{2+n}-2n}, & t\in\left(\frac{1}{2},1\right].\end{cases}$$

Remark 2.30 For $n = 1$ in Corollary 2.18 we have

$$\left|\int_0^1 f(t)\mathrm{d}t - \frac{1}{2}\left[f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right)\right]\right| \leq \frac{K}{6},$$

while

$$\Psi(t) = \begin{cases} t - \frac{1}{6}, & t \in \left[0, \frac{1}{2}\right], \\ t - \frac{5}{6}, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Corollary 2.19 *If $f : [0,1] \to \mathbb{R}$ is such that $f^{(2m-1)}$ is continuous of bounded variation with $f^{(2m)} \in L_2[0,1]$, then we have*

$$\left|\int_0^1 f(t)\mathrm{d}t - \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right] + T_{2m}\left(\frac{1}{4}\right)\right| \leq \left[\frac{-2^{-4m}}{(4m)!}B_{4m}\right]^{1/2} K, \tag{2.71}$$

where $T_0\left(\frac{1}{4}\right) = 0$,

$$T_{2m}\left(\frac{1}{4}\right) = \sum_{k=1}^{m} \frac{2^{-2k}(2^{1-2k}-1)B_{2k}}{(2k)!}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right]$$

and

$$K^2 = \|f^{(2m)}\|_2^2 - \left(\int_0^1 f^{(2m)}(t)dt\right)^2 - \left(\int_0^1 f^{(2m)}(t)\Psi_0(t)dt\right)^2,$$

while

$$\Psi(t) = \begin{cases} 1, & t \in \left[0, \frac{1}{2}\right], \\ -1, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

2.7 Hermite-Hadamard's inequalities of Bullen type

Hermite-Hadamard's inequality can be generalized in the following way.

Theorem 2.17 *Let $f : [a,b] \to \mathbb{R}$ be a convex function. Then for every $x \in \left[a, \frac{a+b}{2}\right]$*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \geq \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(x)+f(a+b-x)}{2}, \tag{2.72}$$

and for every $x \in \left[\frac{3a+b}{4}, \frac{a+b}{2}\right]$

$$\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(x)+f(a+b-x)}{2} \geq 0. \tag{2.73}$$

Proof. Let $x \in \left[a, \frac{a+b}{2}\right]$. Since f is convex on $[a,b]$, the right hand side of (2.36) gives

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(t)\,dt \\
&= \frac{1}{b-a}\left[\int_a^x f(t)\,dt + \int_x^{a+b-x} f(t)\,dt + \int_{a+b-x}^b f(t)\,dt\right] \\
&\le \frac{1}{b-a}\left[(x-a)\frac{f(a)+f(x)}{2} + (a+b-2x)\frac{f(x)+f(a+b-x)}{2} + \right. \\
&\quad \left. +(x-a)\frac{f(a+b-x)+f(b)}{2}\right] \\
&= \frac{1}{2}\left[\frac{x-a}{b-a}(f(a)+f(b)) + \frac{b-x}{b-a}(f(x)+f(a+b-x))\right].
\end{aligned}$$

Since f is convex on $[a,b]$, for any $h>0$ and $x_1,x_2 \in [a,b]$ such that $x_1 \le x_2$ we have (see for example [115, p. 5,6])

$$f(x_1+h) - f(x_1) \le f(x_2+h) - f(x_2). \tag{2.74}$$

Consider now $x \in \left[a, \frac{a+b}{2}\right]$. If we apply (2.74) on $h = x-a$, $x_1 = a$ and $x_2 = a+b-x$, we obtain

$$f(x) - f(a) \le f(b) - f(a+b-x), \tag{2.75}$$

For $x \in \left[a, \frac{a+b}{2}\right]$ we have $a+b-2x \ge 0$, so for such x the inequality (2.75) can be rewritten as

$$(a+b-2x)\frac{f(x)-f(a)}{b-a} \le (a+b-2x)\frac{f(b)-f(a+b-x)}{b-a},$$

i.e.,

$$(a+b-2x)\frac{f(x)-f(a)}{b-a} + (2x-a-b)\frac{f(b)-f(a+b-x)}{b-a} \le 0.$$

From this, a simple calculation gives us

$$\begin{aligned}
&\tfrac{2(x-a)}{b-a}\left[f(a)+f(b)\right] + \tfrac{2(b-x)}{b-a}\left[f(x)+f(a+b-x)\right] \\
&\quad \le f(a)+f(b)+f(x)+f(a+b-x).
\end{aligned} \tag{2.76}$$

Combining (2.74) and (2.76) we obtain

$$\frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)+f(x)+f(a+b-x)}{4},$$

from which we get

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \ge \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(x)+f(a+b-x)}{2},$$

and this completes the proof of (2.72).

Now let $x \in \left[\frac{3a+b}{4}, \frac{a+b}{2}\right]$. Since f is convex on $[a,b]$, the left hand side of (2.36) gives

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(t)\,dt \\
&= \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(t)\,dt + \int_{\frac{a+b}{2}}^b f(t)\,dt\right] \\
&\geq \frac{1}{b-a}\left[\frac{b-a}{2} f\left(\frac{3a+b}{4}\right) + \frac{b-a}{2} f\left(\frac{a+3b}{4}\right)\right] \\
&= \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right]. \qquad (2.77)
\end{aligned}$$

If we apply again (2.74) on $h = \frac{4x-3a-b}{4}$, $x_1 = \frac{3a+b}{4}$ and $x_2 = a+b-x$, we obtain

$$f(x) - f\left(\frac{3a+b}{4}\right) \leq f\left(\frac{a+3b}{4}\right) - f(a+b-x),$$

i.e.,

$$f(x) + f(a+b-x) \leq f\left(\frac{a+3b}{4}\right) + f\left(\frac{3a+b}{4}\right). \qquad (2.78)$$

Combining (2.78) with (2.77) we obtain

$$\frac{1}{b-a}\int_a^b f(t)\,dt \geq \frac{f(x)+f(a+b-x)}{2},$$

so the inequality (2.73) is proved.

□

Remark 2.31 If in (2.72) and (2.73) we let $x = \frac{a+b}{2}$, we obtain

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \geq \frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right) \geq 0,$$

which is Bullen's result from [11]. His result was generalized for $(2r)$-convex functions $(r \in \mathbb{N})$ in [28].

The next goal is to obtain a variant of inequalities (2.72) and (2.73) for $(2r)$-convex functions $(r \in \mathbb{N})$. To achieve this goal we will construct a general closed 4-point rule based on Euler-type identities (1.1) and (1.2).

For $k \geq 1$ and fixed $x \in \left[a, \frac{a+b}{2}\right]$ we define functions G_k^x and F_k^x as

$$\begin{aligned}
G_k^x(t) &= B_k^*\left(\frac{x-t}{b-a}\right) + B_k^*\left(\frac{a+b-x-t}{b-a}\right) + B_k^*\left(\frac{a-t}{b-a}\right) + B_k^*\left(\frac{b-t}{b-a}\right) \\
&= B_k^*\left(\frac{x-t}{b-a}\right) + B_k^*\left(\frac{a+b-x-t}{b-a}\right) + 2B_k^*\left(\frac{a-t}{b-a}\right)
\end{aligned}$$

and

$$F_k^x(t) = G_k^x(t) - \widetilde{B}_k^x,$$

for all $t \in \mathbb{R}$, where

$$\begin{aligned}
\widetilde{B}_k^x &= B_k\left(\tfrac{x-a}{b-a}\right) + B_k\left(\tfrac{b-x}{b-a}\right) + B_k(0) + B_k(1) \\
&= \left[1+(-1)^k\right]\left[B_k\left(\tfrac{x-a}{b-a}\right) + B_k\right].
\end{aligned}$$

Of course, if $k \geq 2$ we have

$$\widetilde{B}_k^x = \left[1+(-1)^k\right] B_k\left(\frac{x-a}{b-a}\right) + 2B_k.$$

Using the properties of the Bernoulli polynomials we can easily see that for any $x \in \left[a, \frac{a+b}{2}\right]$

$$\begin{aligned}
\widetilde{B}_k^x &= G_k^x(a), \quad k \geq 2 \\
\widetilde{B}_{2r-1}^x &= 0, \quad r \geq 1 \\
\widetilde{B}_{2r}^x &= 2\left[B_{2r}\left(\frac{x-a}{b-a}\right) + B_{2r}\right], \quad r \geq 1 \\
F_{2i-1}^x(t) &= G_{2i-1}^x(t), \quad i \geq 1 \\
F_{2r}^x(t) &= G_{2r}^x(t) - 2\left[B_{2r}\left(\frac{x-a}{b-a}\right) + B_{2r}\right], \; r \geq 1 \\
F_k^x(a) &= F_k^x(b) = 0, \quad k \geq 1 \\
G_k^x(a) &= G_k^x(b) = \left[1+(-1)^k\right] B_k\left(\frac{x-a}{b-a}\right) + 2B_k, \quad k \geq 1.
\end{aligned}$$

We can also easily check that for all $r \geq 1$

$$F_{2r-1}^x\left(\frac{a+b}{2}\right) = G_{2r-1}^x\left(\frac{a+b}{2}\right) = 0$$

and

$$\begin{aligned}
G_{2r}^x\left(\frac{a+b}{2}\right) &= 2B_{2r}\left(\frac{1}{2} - \frac{x-a}{b-a}\right) + 2B_{2r}\left(\frac{1}{2}\right) \\
F_{2r}^x\left(\frac{a+b}{2}\right) &= G_{2r}^x\left(\frac{a+b}{2}\right) - \widetilde{B}_{2r}^x \\
&= 2\left[B_{2r}\left(\frac{1}{2} - \frac{x-a}{b-a}\right) - B_{2r}\left(\frac{x-a}{b-a}\right) + B_{2r}\left(\frac{1}{2}\right) - B_{2r}\right] \\
&= 2\left[B_{2r}\left(\frac{1}{2} - \frac{x-a}{b-a}\right) - B_{2r}\left(\frac{x-a}{b-a}\right) + 2\left(2^{-2r} - 1\right) B_{2r}\right]
\end{aligned}$$

Now let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ exists on $[a,b]$ for some $n \geq 1$. We introduce the following notation for each $x \in \left[a, \frac{a+b}{2}\right]$

$$D(x) = \frac{1}{4}\left[f(x) + f(a+b-x) + f(a) + f(b)\right].$$

Furthermore, we define

$$\begin{aligned}\widetilde{T}_0(x) &= 0\\ \widetilde{T}_m(x) &= \frac{1}{4}\left[T_m(x)+T_m(a+b-x)+T_m(a)+T_m(b)\right],\ 1\le m\le n,\end{aligned}$$

where T_m is given by (1.3). It can be easily checked that

$$\widetilde{T}_m(x)=\frac{1}{4}\sum_{k=1}^{m}\frac{(b-a)^{k-1}}{k!}\widetilde{B}_k^x\left[f^{(k-1)}(b)-f^{(k-1)}(a)\right].$$

For the further use we will denote

$$\begin{aligned}\widetilde{T}_m^V(x) &= \frac{T_m(x)+T_m(a+b-x)}{2},\\ \widetilde{T}_m^F &= \frac{T_m(a)+T_m(b)}{2}.\end{aligned}$$

Obviously,

$$\widetilde{T}_m(x)=\frac{\widetilde{T}_m^V(x)+\widetilde{T}_m^F}{2}.$$

Theorem 2.18 *Let $f:[a,b]\to\mathbb{R}$, $a<b$, be such that for some $n\in\mathbb{N}$, $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$. Then for every $x\in[a,b]$*

$$\frac{1}{b-a}\int_a^b f(t)\,dt=D(x)-\widetilde{T}_n(x)+\widetilde{R}_n^1(x) \tag{2.79}$$

and

$$\frac{1}{b-a}\int_a^b f(t)\,dt=D(x)-\widetilde{T}_{n-1}(x)+\widetilde{R}_n^2(x), \tag{2.80}$$

where

$$\widetilde{R}_n^1(x)=\frac{(b-a)^{n-1}}{4n!}\int_{[a,b]}G_n^x(t)\,df^{(n-1)}(t)$$

and

$$\widetilde{R}_n^2(x)=\frac{(b-a)^{n-1}}{4n!}\int_{[a,b]}F_n^x(t)\,df^{(n-1)}(t).$$

Proof. Put $x\equiv x,a+b-x,a,b$ in the formula (1.1) to get four new formulae. Then multiply these formulae by $\frac{1}{4}$ and add them up. The result is (2.79), and (2.80) is obtained from (1.2) by the same procedure. □

Remark 2.32 If in Theorem 2.18 we choose $x=a$ we obtain Euler trapezoid rule [25], and if we choose $x=\frac{a+b}{2}$ we obtain Euler bitrapezoid rule [28].

Our next goal is to give an estimation of the remainder $\widetilde{R}_n^2(x)$. For the sake of simplicity we will temporarily introduce two new variables:

$$\xi = \frac{x-a}{b-a}, \quad s = \frac{t-a}{b-a}.$$

It can be easily seen that for $x,t \in [a,b]$ we have $\xi, s \in [0,1]$. Using direct calculations, for each $\xi \in \left[0, \frac{1}{2}\right]$ we obtain

$$G_1^{\xi}(s) = F_1^{\xi}(s) = \begin{cases} -4s+1 & 0 \le s \le \xi \\ -4s+2 & \xi < s \le 1-\xi \\ -4s+3 & 1-\xi < s < 1 \end{cases},$$

$$G_2^{\xi}(s) = \begin{cases} 4s^2 - 2s + 2\xi^2 - 2\xi + \frac{2}{3} & 0 \le s \le \xi \\ 4s^2 - 4s + 2\xi^2 + \frac{2}{3} & \xi < s \le 1-\xi \\ 4s^2 - 6s + 2\xi^2 - 2\xi + \frac{8}{3} & 1-\xi < s < 1 \end{cases},$$

$$F_2^{\xi}(s) = \begin{cases} 4s^2 - 2s & 0 \le s \le \xi \\ 4s^2 - 4s + 2\xi & \xi < s \le 1-\xi \\ 4s^2 - 6s + 2 & 1-\xi < s < 1 \end{cases},$$

$$\begin{aligned} G_3^{\xi}(s) &= \begin{cases} -4s^3 + 3s^2 - 2s\left(3\xi^2 - 3\xi + 1\right) & 0 \le s \le \xi \\ -4s^3 + 6s^2 - 2s\left(3\xi^2 + 1\right) + 3\xi^2 & \xi < s \le 1-\xi \\ -4s^3 + 9s^2 - 2s\left(3\xi^2 - 3\xi + 4\right) + 6\xi^2 - 6\xi + 3 & 1-\xi < s < 1 \end{cases} \\ &= F_3^{\xi}(s). \end{aligned}$$

Next we present some properties of the functions G_k^{ξ} and F_k^{ξ} defined as above. First we prove that the functions G_k^{ξ} and F_k^{ξ} are symmetric for even k and skew-symmetric for odd k with respect to $\frac{1}{2}$.

Lemma 2.4 *Let $\xi \in \left[0, \frac{1}{2}\right]$ be fixed. For $k \ge 2$ and $s \in [0,1]$ we have*

$$\begin{aligned} G_k^{\xi}(1-s) &= (-1)^k G_k^{\xi}(s), \\ F_k^{\xi}(1-s) &= (-1)^k F_k^{\xi}(s). \end{aligned}$$

Proof. As it is stated in the beginning of this section, for $k \ge 2$ and $s \in [0,1]$ we have

$$\begin{aligned} G_k^{\xi}(1-s) &= B_k^*(\xi - 1 + s) + B_k^*(-\xi + s) + 2B_k^*(s) \\ &= \begin{cases} B_k(\xi + s) + B_k(1 - \xi + s) + 2B_k(s) & 0 \le s \le \xi \\ B_k(\xi + s) + B_k(-\xi + s) + 2B_k(s) & \xi < s \le 1-\xi \\ B_k(\xi - 1 + s) + B_k(-\xi + s) + 2B_k(s) & 1-\xi < s \le 1 \end{cases} \\ &= (-1)^k \times \\ &\quad \begin{cases} B_k(1 - \xi - s) + B_k(\xi - s) + 2B_k(1-s) & 0 \le s \le \xi \\ B_k(1 - \xi - s) + B_k(1 + \xi - s) + 2B_k(1-s) & \xi < s \le 1-\xi \\ B_k(2 - \xi - s) + B_k(1 + \xi - s) + 2B_k(1-s) & 1-\xi < s \le 1 \end{cases} \\ &= (-1)^k G_k^{\xi}(s), \end{aligned}$$

which proves the first identity. Further, we know that $F_k^{\xi}(s) = G_k^{\xi}(s) - G_k^{\xi}(0)$. If $k = 2i-1$, $i \geq 2$, then $G_{2i-1}^{\xi}(0) = G_{2i-1}^{\xi}(1) = 0$, so we immediately have

$$\begin{aligned} F_{2i-1}^{\xi}(1-s) &= G_{2i-1}^{\xi}(1-s) \\ &= (-1)^{2i-1} G_{2i-1}^{\xi}(s) = (-1)^{2i-1} F_{2i-1}^{\xi}(s). \end{aligned}$$

On the other hand, if $k = 2i$, $i \geq 1$, then $(-1)^{2i} = 1$, so we obtain

$$\begin{aligned} F_{2i}^{\xi}(1-s) &= G_{2i}^{\xi}(1-s) + G_{2i}^{\xi}(0) \\ &= (-1)^{2i} G_{2i}^{\xi}(s) + (-1)^{2i} G_{2i}^{\xi}(0) \\ &= (-1)^{2i} F_{2i}^{\xi}(s), \end{aligned}$$

and this proves the second identity. □

Remark 2.33 It is obvious that analogous assertions hold true for the functions G_k^x and F_k^x, $k \geq 2$. In other words, if $x \in \left[a, \frac{a+b}{2}\right]$ and $t \in [a,b]$ we have

$$\begin{aligned} G_k^x(b-t) &= (-1)^k G_k^x(t), \\ F_k^x(b-t) &= (-1)^k F_k^x(t). \end{aligned}$$

Lemma 2.5 *If* $\xi \in \left[0, \frac{1}{2} - \frac{1}{4\sqrt{6}}\right)$, *than for all* $s \in \left(0, \frac{1}{2}\right)$

$$G_3^{\xi}(s) < 0.$$

Also

$$G_3^{\frac{1}{2}-\frac{1}{4\sqrt{6}}}(s) < 0,\ s \in \left(0, \frac{1}{2}\right) \setminus \left\{\frac{3}{8}\right\},$$

$$\begin{aligned} G_3^{\frac{1}{2}}(s) &< 0,\ s \in \left(0, \tfrac{1}{4}\right), \\ G_3^{\frac{1}{2}}(s) &> 0,\ s \in \left(\tfrac{1}{4}, \tfrac{1}{2}\right). \end{aligned}$$

Proof. For the sake of the simplicity we will denote

$$\begin{aligned} G_3^{\xi}(s) &= \begin{cases} -4s^3 + 3s^2 - 2s\left(3\xi^2 - 3\xi + 1\right), & 0 \leq s \leq \xi \\ -4s^3 + 6s^2 - 2s\left(3\xi^2 + 1\right) + 3\xi^2 & \xi < s \leq 1-\xi \\ -4s^3 + 9s^2 - 2s\left(3\xi^2 - 3\xi + 4\right) + 6\xi^2 - 6\xi + 3 & 1-\xi < s < 1 \end{cases} \\ &= \begin{cases} H_1^{\xi}(s) & 0 \leq s \leq \xi \\ H_2^{\xi}(s) & \xi < s \leq 1-\xi \\ H_3^{\xi}(s) & 1-\xi < s \leq 1 \end{cases}. \end{aligned}$$

If we write $H_1^{\xi}(s)$ as

$$H_1^{\xi}(s) = s\left[-4s^2 + 3s - 2\left(3\xi^2 - 3\xi + 1\right)\right]$$

we can see that $H_1^{\xi}(0) = 0$ and that $H_1^{\xi}(\xi) = \xi\left(-10\xi^2 + 9\xi - 2\right)$, so if for a given $\xi \in \left[0, \frac{1}{2}\right]$ number $-10\xi^2 + 9\xi - 2$ is negative it means that the joining point $\left(\xi, H_1^{\xi}(\xi)\right) = \left(\xi, H_2^{\xi}(\xi)\right)$ is under the x axis. This will be true for $\xi \in \left[0, \frac{2}{5}\right)$. The sign of $H_1^{\xi}(s)$ is determined by the sign of the function $y(s) = -4s^2 + 3s - 2\left(3\xi^2 - 3\xi + 1\right)$. This function will have zeros $s_1 = \frac{3}{8} - \frac{1}{8}\sqrt{D}$ and $s_2 = \frac{3}{8} + \frac{1}{8}\sqrt{D}$ if $D = -96\xi^2 + 96\xi - 23 \geq 0$, i.e., if $\xi \in \left[\frac{1}{2} - \frac{1}{4\sqrt{6}}, \frac{1}{2}\right]$. Furthermore, $y(0) = -2\left(3\xi^2 - 3\xi + 1\right) < 0$ which means that (if they exist) both zeros s_1 and s_2 are positive. Of course, if $\xi = \frac{1}{2} - \frac{1}{4\sqrt{6}}$ the function y has only one zero $s = \frac{3}{8}$. We want to know is it possible for $\xi \in \left(\frac{1}{2} - \frac{1}{4\sqrt{6}}, \frac{2}{5}\right)$ to have $\xi < s_1$ (because this will imply that $H_1^{\xi}(s) < 0$ for all $0 \leq s \leq \xi$). This in fact is not possible because if $\xi < s_1$ than we have $\xi < \frac{3}{8}$, and $\frac{3}{8} < \frac{1}{2} - \frac{1}{4\sqrt{6}}$. This means that $H_1^{\xi}(s) \leq 0$ for all $s \in (0, \xi)$ can be true only if $D \leq 0$, and this will be true for $\xi \in \left[0, \frac{1}{2} - \frac{1}{4\sqrt{6}}\right] \subset \left[0, \frac{2}{5}\right)$.
Now we must check H_2^{ξ} for such ξ. If $\xi < s \leq \frac{1}{2}$ we have

$$\begin{aligned} H_2^{\xi\prime}(s) &= -12s^2 + 12s - 2\left(3\xi^2 + 1\right), \\ H_2^{\xi\prime\prime}(s) &= -24s + 12 = 12(1 - 2s) > 0, \end{aligned}$$

which means that H_2^{ξ} is convex for any choice of such ξ. Since $H_2^{\xi}(\xi) < 0$ and $H_2^{\xi}\left(\frac{1}{2}\right) = 0$ we can deduce that $H_2^{\xi}(s) < 0$ for all $s \in \left(\xi, \frac{1}{2}\right)$. This means that if $\xi \in \left[0, \frac{1}{2} - \frac{1}{4\sqrt{6}}\right)$, then

$$G_3^{\xi}(s) < 0, \ s \in \left(0, \frac{1}{2}\right),$$

and for $\xi = \frac{1}{2} - \frac{1}{4\sqrt{6}}$ we have

$$G_3^{\xi}(s) < 0, \ s \in \left(0, \frac{1}{2}\right) \setminus \left\{\frac{3}{8}\right\}.$$

On the other hand, if $\xi \in \left(\frac{2}{5}, \frac{1}{2}\right]$ the joining point $\left(\xi, H_1^{\xi}(\xi)\right) = \left(\xi, H_2^{\xi}(\xi)\right)$ is above the x axis, and we want $H_1^{\xi}(s)$ to be positive for all $s \in (0, \xi)$. This, of course, can not be true because $\left(\frac{2}{5}, \frac{1}{2}\right] \subset \left(\frac{1}{2} - \frac{1}{4\sqrt{6}}, \frac{1}{2}\right]$, which means that H_1^{ξ} surely has a zero $s_1 < \frac{3}{8} < \frac{2}{5} < \xi$.
And in the end, we must separately investigate $G_3^{\frac{1}{2}}$ because in this special point $\xi = \frac{1}{2}$ function G_3^{ξ} has only one branch for $s \in \left[0, \frac{1}{2}\right]$, i.e., we have

$$G_3^{\frac{1}{2}}(s) = s\left(-4s^2 + 3s - \frac{1}{2}\right), \ s \in \left[0, \frac{1}{2}\right].$$

We can easily see that

$$\begin{aligned} G_3^{\frac{1}{2}}(s) &< 0,\ s \in \left(0, \tfrac{1}{4}\right), \\ G_3^{\frac{1}{2}}(s) &> 0,\ s \in \left(\tfrac{1}{4}, \tfrac{1}{2}\right). \end{aligned}$$

Of course, from the above results we have

$$G_3^x(t) < 0,\ \ t \in \left(a, \frac{a+b}{2}\right)$$

for any $x \in \left[a, \frac{a+b}{2} - \frac{b-a}{4\sqrt{6}}\right)$, and also

$$G_3^{\frac{a+b}{2} - \frac{b-a}{4\sqrt{6}}}(s) < 0,\ s \in \left(a, \frac{a+b}{2}\right) \setminus \left\{\frac{5a+3b}{8}\right\},$$

$$\begin{aligned} G_3^{\frac{a+b}{2}}(t) &< 0,\ t \in \left(a, \tfrac{a+b}{4}\right), \\ G_3^{\frac{a+b}{2}}(t) &> 0,\ t \in \left(\tfrac{3a+b}{4}, \tfrac{a+b}{2}\right). \end{aligned}$$

Lemma 2.6 *For $r \geq 2$ and $x \in \left[a, \frac{a+b}{2} - \frac{b-a}{4\sqrt{6}}\right)$ the function G_{2r-1}^x has no zeros in the interval $\left(a, \frac{a+b}{2}\right)$. The sign of this function is determined by*

$$(-1)^{r-1} G_{2r-1}^x(t) > 0,\ \ t \in \left(a, \frac{a+b}{2}\right).$$

Also,

$$(-1)^{r-1} G_{2r-1}^{\frac{a+b}{2} - \frac{b-a}{4\sqrt{6}}}(t) > 0,\ \ t \in \left(a, \frac{a+b}{2}\right) \setminus \left\{\frac{5a+3b}{8}\right\},$$

$$\begin{aligned} (-1)^{r-1} G_{2r-1}^{\frac{a+b}{2}}(t) &> 0,\ t \in \left(a, \frac{3a+b}{4}\right), \\ (-1)^{r-1} G_{2r-1}^{\frac{a+b}{2}}(s) &< 0,\ s \in \left(\frac{3a+b}{4}, \frac{a+b}{2}\right). \end{aligned}$$

Proof. Let $x \in \left[a, \frac{a+b}{2} - \frac{b-a}{4\sqrt{6}}\right)$. If $r = 2$ assertion follows from Lemma 2.5. Assume now that $r \geq 3$. In that case we have $2r-1 \geq 5$ and the function G_{2r-1}^x is continuous and at least twice differentiable. We know that

$$\begin{aligned} \left(G_{2r-1}^x\right)'(t) &= -\frac{2r-1}{b-a} G_{2r-2}^x(t), \\ \left(G_{2r-1}^x\right)''(t) &= \frac{(2r-1)(2r-2)}{(b-a)^2} G_{2r-3}^x(t), \end{aligned} \tag{2.81}$$

and that

$$G_{2r-1}^{x}(a) = G_{2r-1}^{x}\left(\frac{a+b}{2}\right) = 0.$$

Suppose that G_{2r-1}^{x} has another zero $\alpha \in \left(a, \frac{a+b}{2}\right)$. Then inside each of the intervals (a, α) and $\left(\alpha, \frac{a+b}{2}\right)$ the derivative $\left(G_{2r-1}^{x}\right)'$ must have at least one zero, say $\beta_1 \in (a, \alpha)$ and $\beta_2 \in \left(\alpha, \frac{a+b}{2}\right)$. Therefore, the second derivative $\left(G_{2r-1}^{x}\right)''$ must have at least one zero inside the interval $(\beta_1, \beta_2) \subset \left(a, \frac{a+b}{2}\right)$. Thus, from the assumption that G_{2r-1}^{x} has a zero inside the interval $\left(a, \frac{a+b}{2}\right)$ it follows that G_{2r-3}^{x} also has a zero inside the interval $\left(a, \frac{a+b}{2}\right)$. From this we could deduce that the function G_3^x also has a zero inside of the interval $\left(a, \frac{a+b}{2}\right)$ which is not true. Thus, G_{2r-1}^{x} can not have a zero inside the interval $\left(a, \frac{a+b}{2}\right)$. Furthermore, if $G_{2r-3}^{x}(t) > 0$ for $t \in \left(a, \frac{a+b}{2}\right)$, then from (2.81) follows that G_{2r-1}^{x} is convex on $\left(a, \frac{a+b}{2}\right)$, and hence $G_{2r-1}^{x}(t) < 0$ for $t \in \left(a, \frac{a+b}{2}\right)$. Similarly, if $G_{2r-3}^{x}(t) < 0$ for $t \in \left(a, \frac{a+b}{2}\right)$, then from (2.81) follows that G_{2r-1}^{x} is concave on $\left(a, \frac{a+b}{2}\right)$, and hence $G_{2r-1}^{x}(t) > 0$ for $t \in \left(a, \frac{a+b}{2}\right)$. Since $G_3^x(t) < 0$ for $t \in \left(a, \frac{a+b}{2}\right)$ we can conclude that

$$(-1)^{r-1} G_{2r-1}^{x}(t) > 0,\ t \in \left(a, \frac{a+b}{2}\right).$$

For the special cases $x = \frac{a+b}{2} - \frac{b-a}{4\sqrt{6}}$ and $x = \frac{a+b}{2}$ proof is similar so we skip the details.

□

Corollary 2.20 *For $r \geq 2$ and $x \in \left[a, \frac{a+b}{2} - \frac{b-a}{4\sqrt{6}}\right]$ the functions $(-1)^r F_{2r}^x(t)$ and $(-1)^r G_{2r}^x(t)$ are strictly increasing on the interval $\left(a, \frac{a+b}{2}\right)$ and strictly decreasing on the interval $\left(\frac{a+b}{2}, b\right)$. Consequently, a and b are the only zeros of F_{2r}^x in the interval $[a,b]$ and*

$$\max_{t\in[a,b]} |F_{2r}^x(t)| = 2\left|B_{2r}\left(\frac{1}{2} - \frac{x-a}{b-a}\right) - B_{2r}\left(\frac{x-a}{b-a}\right) + 2\left(2^{-2r} - 1\right) B_{2r}\right|,$$

$$\max_{t\in[a,b]} |G_{2r}^x(t)| = \left\{2\left|B_{2r}\left(\frac{x-a}{b-a}\right) + B_{2r}\right|, 2\left|B_{2r}\left(\frac{1}{2} - \frac{x-a}{b-a}\right) + B_{2r}\left(\frac{1}{2}\right)\right|\right\}.$$

Proof. Let $r \geq 2$ and $x \in \left[a, \frac{a+b}{2} - \frac{b-a}{4\sqrt{6}}\right)$. We know that

$$\left[(-1)^r F_{2r}^x(t)\right]' = \left[(-1)^r G_{2r}^x(t)\right]' = \frac{2r}{b-a}(-1)^{r-1} G_{2r-1}^x(t),$$

and by Lemma 2.6 we also know that $(-1)^{r-1} G_{2r-1}^x(t) > 0$ for all $t \in \left(a, \frac{a+b}{2}\right)$. Thus the functions $(-1)^r F_{2r}^x(t)$ and $(-1)^r G_{2r}^x(t)$ are strictly increasing on the interval $\left(a, \frac{a+b}{2}\right)$. Also, by Lemma 2.4, we have $F_{2r}^x(b-t) = F_{2r}^x(t)$ and $G_{2r}^x(b-t) = G_{2r}^x(t)$ for $t \in [a,b]$, which implies that $(-1)^r F_{2r}^x(t)$ and $(-1)^r G_{2r}^x(t)$ are strictly decreasing on the interval

$\left(\frac{a+b}{2}, b\right)$. Further, $F_{2r}^{x}(a) = F_{2r}^{x}(b) = 0$, which implies that $|F_{2r}^{x}(t)|$ achieves its maximum at $t = \frac{a+b}{2}$, that is

$$\begin{aligned}\max_{t\in[a,b]} |F_{2r}^{x}(t)| &= \left|F_{2r}^{x}\left(\frac{a+b}{2}\right)\right| \\ &= 2\left|B_{2r}\left(\frac{1}{2} - \frac{x-a}{b-a}\right) - B_{2r}\left(\frac{x-a}{b-a}\right) + 2\left(2^{-2r}-1\right)B_{2r}\right|.\end{aligned}$$

Also,

$$\begin{aligned}\max_{t\in[a,b]} |G_{2r}^{x}(t)| &= \max\left\{|G_{2r}^{x}(a)|, \left|G_{2r}^{x}\left(\frac{a+b}{2}\right)\right|\right\} \\ &= \max\left\{2\left|B_{2r}\left(\frac{x-a}{b-a}\right) + B_{2r}\right|, 2\left|B_{2r}\left(\frac{1}{2} - \frac{x-a}{b-a}\right) + B_{2r}\left(\frac{1}{2}\right)\right|\right\}.\end{aligned}$$

The special case $x = \frac{a+b}{2} - \frac{b-a}{4\sqrt{6}}$ can be investigated similarly. □

Corollary 2.21 *For $r \geq 2$ the functions $(-1)^r F_{2r}^{\frac{a+b}{2}}(t)$ and $(-1)^r G_{2r}^{\frac{a+b}{2}}(t)$ are strictly increasing on the intervals $\left(a, \frac{3a+b}{4}\right)$ and $\left(\frac{a+b}{2}, \frac{a+3b}{4}\right)$, and strictly decreasing on the intervals $\left(\frac{3a+b}{4}, \frac{a+b}{2}\right)$ and $\left(\frac{a+3b}{4}, b\right)$. Consequently, a, $\frac{a+b}{2}$ and b are the only zeros of $F_{2r}^{\frac{a+b}{2}}$ in the interval $[a,b]$ and*

$$\max_{t\in[a,b]} \left|F_{2r}^{\frac{a+b}{2}}(t)\right| = \left|F_{2r}^{\frac{a+b}{2}}\left(\frac{3a+b}{4}\right)\right| = 2^{2-2r}\left(2-2^{1-2r}\right)|B_{2r}|,$$

$$\max_{t\in[a,b]} \left|G_{2r}^{\frac{a+b}{2}}(t)\right| = \left|G_{2r}^{\frac{a+b}{2}}\left(\frac{3a+b}{4}\right)\right| = 2^{2-2r}\left(1-2^{1-2r}\right)|B_{2r}|.$$

Proof. Similarly as in the proof of Corollary 2.20 and using the fact

$$F_{2r}^{\frac{a+b}{2}}\left(\frac{a+b}{2}\right) = 2\left[B_{2r} - B_{2r}\left(\frac{1}{2}\right) + 2\left(2^{-2r}-1\right)B_{2r}\right] = 0.$$

□

Corollary 2.22 *For $r \geq 2$ and $x \in \left[a, \frac{a+b}{2} - \frac{b-a}{4\sqrt{6}}\right]$ we have*

$$\begin{aligned}&\tfrac{1}{b-a}\textstyle\int_a^b \left|F_{2r-1}^{x}(t)\right| dt \\ &= \tfrac{1}{b-a}\textstyle\int_a^b \left|G_{2r-1}^{x}(t)\right| dt = \tfrac{1}{r}\left|F_{2r}^{x}\left(\tfrac{a+b}{2}\right)\right| \\ &= \tfrac{2}{r}\left|B_{2r}\left(\tfrac{1}{2} - \tfrac{x-a}{b-a}\right) - B_{2r}\left(\tfrac{x-a}{b-a}\right) + 2\left(2^{-2r}-1\right)B_{2r}\right|.\end{aligned}$$

Also, we have

$$\frac{1}{b-a}\int_a^b |F_{2r}^{x}(t)|\, dt = 2\left|B_{2r}\left(\frac{x-a}{b-a}\right) + B_{2r}\right|$$

and

$$\frac{1}{b-a}\int_a^b |G_{2r}^x(t)|\,dt \le 4\left|B_{2r}\left(\frac{x-a}{b-a}\right)+B_{2r}\right|.$$

Proof. Let $r \ge 2$ and $x \in \left[a, \frac{a+b}{2} - \frac{b-a}{4\sqrt{6}}\right]$. Using Lemma 2.4 and Lemma 2.6 we get

$$\begin{aligned}
\int_a^b \left|G_{2r-1}^x(t)\right| dt &= 2\left|\int_a^{\frac{a+b}{2}} G_{2r-1}^x(t)\,dt\right| \\
&= 2\left|-\frac{b-a}{2r}G_{2r}^x(s)\,|_a^{\frac{a+b}{2}}\right| = \frac{b-a}{r}\left|G_{2r}^x\left(\frac{a+b}{2}\right)-G_{2r}^x(a)\right| \\
&= \frac{b-a}{r}F_{2r}^x\left(\frac{a+b}{2}\right),
\end{aligned}$$

which proves the first assertion. Using Corollary 2.20 and the fact that $F_{2r}^x(a) = F_{2r}^x(b) = 0$, we can deduce that the function F_{2r}^x does not change its sign on the interval (a,b). Therefore we have

$$\begin{aligned}
&\int_a^b |F_{2r}^x(t)|\,dt \\
&= \left|\int_a^b F_{2r}^x(t)\,dt\right| = \left|\int_a^b \left[G_{2r}^x(t)-\widetilde{B}_{2r}^x\right] dt\right| \\
&= \left|-\frac{b-a}{2r+1}G_{2r+1}^x(t)\,|_a^b-(b-a)\widetilde{B}_{2r}^x\right| = (b-a)\left|\widetilde{B}_{2r}^x\right| \\
&= 2(b-a)\left|B_{2r}\left(\frac{x-a}{b-a}\right)+B_{2r}\right|,
\end{aligned}$$

which proves the second assertion. Finally, we use the triangle inequality to obtain the third formula. □

Corollary 2.23 *For $r \ge 2$ we have*

$$\int_a^b \left|F_{2r-1}^{\frac{a+b}{2}}(t)\right| dt = \int_a^b \left|G_{2r-1}^{\frac{a+b}{2}}(t)\right| dt = \frac{b-a}{r}2^{4-2r}\left(1-2^{-2r}\right)|B_{2r}|$$

Also,

$$\frac{1}{b-a}\int_a^b \left|F_{2r}^{\frac{a+b}{2}}(t)\right| dt = 2^{2-2r}|B_{2r}|$$

and

$$\frac{1}{b-a}\int_a^b \left|G_{2r}^{\frac{a+b}{2}}(t)\right| dt \le 2^{3-2r}|B_{2r}|.$$

Proof. Similarly as in the proof of Corollary 2.22. □

Lemma 2.7 *Let $x \in \left[a, \frac{a+b}{2} - \frac{b-a}{4\sqrt{6}}\right]$. If $f : [a,b] \to \mathbb{R}$ is such that for some $r \geq 2$ derivative $f^{(2r)}$ is continuous on $[a,b]$, then there exists a point $\eta \in [a,b]$ such that*

$$\widetilde{R}_{2r}^{2}(x) = -\frac{(b-a)^{2r}}{2(2r)!}\left[B_{2r}\left(\frac{x-a}{b-a}\right) + B_{2r}\right] f^{(2r)}(\eta).$$

Proof. Let $x \in \left[a, \frac{a+b}{2} - \frac{b-a}{4\sqrt{6}}\right]$. For $n = 2r \geq 4$ and f such that $f^{(2r)}$ is continuous on $[a,b]$ we can rewrite $R_{2r}^{2}(f)$ as

$$\begin{aligned}\widetilde{R}_{2r}^{2}(x) &= (-1)^{r}\frac{(b-a)^{2r-1}}{4(2r)!}\int_{a}^{b}(-1)^{r}F_{2r}^{x}(t)f^{(2r)}(t)\,dt \\ &= (-1)^{r}\frac{(b-a)^{2r-1}}{4(2r)!}I_{r},\end{aligned} \tag{2.82}$$

where

$$I_{r} = \int_{a}^{b}(-1)^{r}F_{2r}^{x}(t)f^{(2r)}(t)\,dt. \tag{2.83}$$

If

$$m = \min_{[a,b]} f^{(2r)}(t)\ ,\ M = \max_{[a,b]} f^{(2r)}(t)\ ,$$

then

$$m \leq f^{(2r)}(t) \leq M,\ \ t \in [a,b].$$

From Corollary 2.20 we have

$$(-1)^{r}F_{2r}^{x}(t) \geq 0,\ \ t \in [a,b],$$

so

$$m\int_{a}^{b}(-1)^{r}F_{2r}^{x}(t)\,dt \leq I_{r} \leq M\int_{a}^{b}(-1)^{r}F_{2r}^{x}(t)\,dt.$$

Since

$$\int_{a}^{b}F_{2r}^{x}(t)\,dt = -(b-a)\widetilde{B}_{2r}^{x} = -2(b-a)\left[B_{2r}\left(\frac{x-a}{b-a}\right) + B_{2r}\right],$$

we obtain

$$\begin{aligned}&2m(-1)^{r-1}(b-a)\left[B_{2r}\left(\frac{x-a}{b-a}\right) + B_{2r}\right] \\ &\leq I_{r} \leq 2M(-1)^{r-1}(b-a)\left[B_{2r}\left(\frac{x-a}{b-a}\right) + B_{2r}\right].\end{aligned}$$

By the continuity of $f^{(2r)}$ on $[a,b]$ it follows that there must exist a point $\eta \in [a,b]$ such that

$$I_{r} = 2(-1)^{r-1}(b-a)\left[B_{2r}\left(\frac{x-a}{b-a}\right) + B_{2r}\right] f^{(2r)}(\eta).$$

From that we can easily obtain

$$\widetilde{R}_{2r}^{2}(x) = -\frac{(b-a)^{2r}}{2(2r)!}\left[B_{2r}\left(\frac{x-a}{b-a}\right)+B_{2r}\right]f^{(2r)}(\eta).$$

□

Lemma 2.8 *If $f:[a,b]\to\mathbb{R}$ is such that for some $r\geq 2$ derivative $f^{(2r)}$ is continuous on $[a,b]$, then there exists a point $\eta\in[a,b]$ such that*

$$\widetilde{R}_{2r}^{2}\left(\frac{a+b}{2}\right) = -\frac{(b-a)^{2r}}{(2r)!}2^{-2r}B_{2r}f^{(2r)}(\eta).$$

Proof. Analogously as in the proof of Lemma 2.7. □

Theorem 2.19 *Let $x\in\left[a,\frac{a+b}{2}-\frac{b-a}{4\sqrt{6}}\right]$. Assume that $f:[a,b]\to\mathbb{R}$ is such that for some $r\geq 2$ $f^{(2r)}$ is continuous on $[a,b]$. If f is a $(2r)$-convex or $(2r)$-concave function, then there exists a point $\vartheta\in[0,1]$ such that*

$$\begin{aligned}\widetilde{R}_{2r}^{2}(x) &= \vartheta\left[B_{2r}\left(\frac{1}{2}-\frac{x-a}{b-a}\right)-B_{2r}\left(\frac{x-a}{b-a}\right)+2\left(2^{-2r}-1\right)B_{2r}\right]\\ &\quad\cdot\frac{(b-a)^{2r-1}}{2(2r)!}\left[f^{(2r-1)}(b)-f^{(2r-1)}(a)\right].\end{aligned} \tag{2.84}$$

Proof. By Corollary 2.20 for $t\in[a,b]$ we have

$$0\leq(-1)^{r-1}F_{2r}^{x}(t)\leq(-1)^{r-1}F_{2r}^{x}\left(\frac{a+b}{2}\right).$$

The rest of the proof is similar as in the proof of Lemma 2.7. □

Using the Theorem 2.6 we can improve the above theorem in a way that the derivative $f^{(2r)}$ need not to be continuous on $[a,b]$.

Theorem 2.20 *Assume that the function $f:[a,b]\to\mathbb{R}$ is such that for some $r\geq 2$ the derivative $f^{(2r-1)}$ is continuous and increasing on $[a,b]$. Then for every $x\in\left[a,\frac{a+b}{2}-\frac{b-a}{4\sqrt{6}}\right]$ we have*

$$\begin{aligned}&(-1)^{r}\left\{\frac{1}{b-a}\int_{a}^{b}f(t)\,dt-\frac{f(a)+f(b)+f(x)+f(a+b-x)}{4}+\widetilde{T}_{2r-1}(x)\right\}\\ &\leq\frac{(b-a)^{2r-1}}{2(2r)!}\left[f^{(2r-1)}(b)-f^{(2r-1)}(a)\right]\\ &\quad\cdot\left|B_{2r}\left(\frac{1}{2}-\frac{x-a}{b-a}\right)-B_{2r}\left(\frac{x-a}{b-a}\right)+2\left(2^{-2r}-1\right)B_{2r}\right|,\end{aligned}$$

and this inequality is sharp.

Proof. We know that the function F_{2r-1}^{x} is periodic with the period $P = b-a$. From Theorem 2.19 and Lemma 2.4 for $r \geq 2$ and $x \in \left[a, \frac{a+b}{2} - \frac{b-a}{4\sqrt{6}}\right]$ we have:

$$F_{2r-1}^{x}\left(\frac{a+b}{2}\right) = 0,$$

$$(-1)^{r-1} F_{2r-1}^{x}(t) > 0, \quad t \in \left(a, \frac{a+b}{2}\right)$$

$$(-1)^{r-1} F_{2r-1}^{x}(t) < 0, \quad t \in \left(\frac{a+b}{2}, b\right)$$

and also

$$\int_a^b F_{2r-1}^{x}(t)\,dt = 0.$$

This means that if in Theorem 2.6 we choose $\rho(t) = (-1)^{r-1} F_{2r-1}^{x}(t)$, $\varphi(t) = f^{(2r-1)}(t)$ and $n = 1$, then we obtain

$$\begin{aligned} &- \int_a^b (-1)^{r-1} F_{2r-1}^{x}(t) f^{(2r-1)}(t)\,dt \\ &\leq \frac{1}{2}\left[f^{(2r-1)}(b) - f^{(2r-1)}(a)\right] \int_a^b \left|F_{2r-1}^{x}(t)\right| dt, \end{aligned}$$

and combining this with Corollary 2.22 we obtain

$$\begin{aligned} &(-1)^r \int_a^b F_{2r-1}^{x}(t) f^{(2r-1)}(t)\,dt \\ \leq\ &\frac{b-a}{r}\left[f^{(2r-1)}(b) - f^{(2r-1)}(a)\right] \\ &\cdot \left|B_{2r}\left(\frac{1}{2} - \frac{x-a}{b-a}\right) - B_{2r}\left(\frac{x-a}{b-a}\right) + 2\left(2^{-2r} - 1\right) B_{2r}\right|. \end{aligned}$$

From Theorem 2.18 we know that

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(a) + f(b) + f(x) + f(a+b-x)}{4} + \widetilde{T}_{2r-1}(x) \\ =\ &\frac{(b-a)^{2r-2}}{4(2r-1)!} \int_{[a,b]} F_{2r-1}^{x}(t) f^{(2r-1)}(t)\,dt, \end{aligned}$$

so

$$\begin{aligned} &(-1)^r \left\{\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(a) + f(b) + f(x) + f(a+b-x)}{4} + \widetilde{T}_{2r-1}(x)\right\} \\ =\ &\frac{(b-a)^{2r-2}}{4(2r-1)!} (-1)^r \int_a^b F_{2r-1}^{x}(t) f^{(2r-1)}(t)\,dt \\ \leq\ &\frac{(b-a)^{2r-1}}{2(2r)!}\left[f^{(2r-1)}(b) - f^{(2r-1)}(a)\right] \\ &\cdot \left|B_{2r}\left(\frac{1}{2} - \frac{x-a}{b-a}\right) - B_{2r}\left(\frac{x-a}{b-a}\right) + 2\left(2^{-2r} - 1\right) B_{2r}\right|. \end{aligned}$$

□

Theorem 2.21 *Assume that the function $f:[a,b]\to\mathbb{R}$ is such that for some $r\geq 2$ the derivative $f^{(2r-1)}$ is continuous and increasing on $[a,b]$. Then we have*

$$(-1)^r\left\{\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(a)+f(b)+2f\left(\frac{a+b}{2}\right)}{4}+\widetilde{T}_{2r-1}\left(\frac{a+b}{2}\right)\right\}$$
$$\leq\frac{(b-a)^{2r-1}}{(2r)!}\left[f^{(2r-1)}(b)-f^{(2r-1)}(a)\right]2^{1-2r}\left(1-2^{-2r}\right)|B_{2r}|,$$

and this inequality is sharp.

Proof. Similarly as in the proof of Theorem 2.20. □

Now we can give our main result in this section: a generalization of Hermite-Hadamard's inequalities for $(2r)$-convex functions, $r\geq 2$.

Theorem 2.22 *Assume that $f:[a,b]\to\mathbb{R}$ is such that for some $r\geq 2$ derivative $f^{(2r-1)}$ is continuous on $[a,b]$, and assume that f is $(2r)$-convex on $[a,b]$. If r is odd, then for all $x\in\left[a,\frac{a+b}{2}-\frac{b-a}{4\sqrt{6}}\right]\cup\left\{\frac{a+b}{2}\right\}$*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt-\widetilde{T}^F_{2r-1},\geq$$
$$\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(x)+f(a+b-x)}{2}+\widetilde{T}^V_{2r-1}(x),\tag{2.85}$$

and for all $x\in\left[a+\frac{b-a}{2\sqrt{3}},\frac{a+b}{2}\right]$

$$\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(x)+f(a+b-x)}{2}+\widetilde{T}^V_{2r-1}(x)\geq 0.\tag{2.86}$$

If r is even the above inequalities are reversed.

Proof. Let $x\in\left[a,\frac{a+b}{2}-\frac{b-a}{4\sqrt{6}}\right]$. In case $n=2r\geq 4$ from (2.80) we get

$$\frac{2}{b-a}\int_a^b f(t)\,dt-\frac{f(a)+f(b)+f(x)+f(a+b-x)}{2}+2\widetilde{T}_{2r-1}(x)=2\widetilde{R}^2_{2r}(f),$$

where

$$\widetilde{R}^2_{2r}(x)=\frac{(b-a)^{2r-1}}{4(2r)!}\int_{[a,b]}F^x_{2r}(t)\,df^{(2r-1)}(t).$$

If f is $(2r)$-convex then $df^{(2r-1)}(t)\geq 0$ on $[a,b]$, and since by Corollary 2.20 we know that

$$(-1)^r F^x_{2r}(t)\geq 0,\ t\in[a,b],$$

we obtain $\widetilde{R}_{2r}^{2}(x) \geq 0$ for r even and $\widetilde{R}_{2r}^{2}(x) \leq 0$ for r odd. The same is true if $x = \frac{a+b}{2}$. This means that for r odd we have

$$\frac{2}{b-a}\int_a^b f(t)\,dt - \frac{f(a)+f(b)+f(x)+f(a+b-x)}{2} + 2\widetilde{T}_{2r-1}(x) \leq 0,$$

i.e.,

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt - \widetilde{T}_{2r-1}^{F} \geq \\ &\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(x)+f(a+b-x)}{2} + \widetilde{T}_{2r-1}^{V}(x), \end{aligned}$$

and the above inequality is reversed if r is even. This completes the proof of (2.85).

Now let $x \in \left[a + \frac{b-a}{2\sqrt{3}}, \frac{a+b}{2}\right]$ and suppose that r is odd. We can use the analogous results from Section 3 of this chapter to obtain

$$\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(x)+f(a+b-x)}{2} + \widetilde{T}_{2r-1}^{V}(x) \geq 0,$$

and the reverse if r is even. This completes the proof.

□

The interested reader can find several sharper variants of (2.86) in [105].

Chapter 3

General 3-point quadrature formulae of Euler type

The topic of this chapter are general 3-point quadrature formulae. More precisely, a family of quadrature formulae which approximate the integral over $[0,1]$ by values of the function in nodes $x,\ 1/2,\ 1-x$, where $x \in [0,1/2)$, are studied. The results from the first section were published in [59].

3.1 General approach

Let $x \in [0,1/2)$ and $f : [0,1] \to \mathbb{R}$ be such that $f^{(2n)}$ is continuous of bounded variation on $[0,1]$ for some $n \geq 0$. Put $x \equiv x,\ 1/2,\ 1-x$ in (1.2), multiply by $w(x),\ 1-2w(x),\ w(x)$, respectively, and add up. The following formula is produced:

$$\int_0^1 f(t)dt - w(x)f(x) - (1-2w(x))f\left(\frac{1}{2}\right) - w(x)f(1-x) + T_{2n}(x)$$
$$= \frac{1}{(2n+2)!}\int_0^1 F_{2n+2}(x,t)\,df^{(2n+1)}(t), \tag{3.1}$$

where

$$T_{2n}(x) = \sum_{k=2}^{2n} \frac{1}{k!} G_k(x,0) \left[f^{(k-1)}(1) - f^{(k-1)}(0)\right], \tag{3.2}$$

$$G_k(x,t) = w(x)\left[B_k^*(x-t) + B_k^*(1-x-t)\right] + (1-2w(x))B_k^*\left(\frac{1}{2}-t\right), \tag{3.3}$$

$$F_k(x,t) = G_k(x,t) - G_k(x,0) \tag{3.4}$$

for $k \geq 1$ and $t \in \mathbb{R}$.

Using the properties of Bernoulli polynomials, it is easy to verify that:

$$G_k(x,1-t) = (-1)^k G_k(x,t), \quad t \in [0,1], \tag{3.5}$$

$$\frac{\partial G_k(x,t)}{\partial t} = -kG_{k-1}(x,t). \tag{3.6}$$

Further, notice that $G_{2k-1}(0) = 0$ for $k \geq 2$, and this is not affected by any choice of the weight $w(x)$. On the other hand, in general, $G_{2k}(x,0) \neq 0$.

To obtain the 3-point quadrature formulae with the maximum degree of exactness (which is equal to 3), it is clear from (3.1) that we have to impose a condition: $G_2(x,0) = 0$.

This condition gives:

$$w(x) = \frac{1}{6(2x-1)^2}$$

and formula (3.1) now becomes:

$$\int_0^1 f(t)dt - Q\left(x, \frac{1}{2}, 1-x\right) + T_{2n}^{Q3}(x) = \frac{1}{(2n+2)!}\int_0^1 F_{2n+2}^{Q3}(x,t)df^{(2n+1)}(t), \tag{3.7}$$

where

$$Q\left(x, \frac{1}{2}, 1-x\right) = \frac{1}{6(2x-1)^2}\left[f(x) + 24B_2(x)\cdot f\left(\frac{1}{2}\right) + f(1-x)\right] \tag{3.8}$$

$$T_{2n}^{Q3}(x) = \sum_{k=2}^{n} \frac{1}{(2k)!} G_{2k}^{Q3}(x,0) \left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right], \tag{3.9}$$

$$G_k^{Q3}(x,t) = \frac{1}{6(2x-1)^2}\left[B_k^*(x-t) + 24B_2(x)\cdot B_k^*\left(\frac{1}{2}-t\right) + B_k^*(1-x-t)\right] \tag{3.10}$$

$$F_k^{Q3}(x,t) = G_k^{Q3}(x,t) - G_k^{Q3}(x,0). \tag{3.11}$$

If we assumed $G_{2k}(x,0) = G_{2k+2}(x,0) = 0$ for some $k \geq 2$, it would increase the degree of exactness but the quadrature formulae thus produced would include values of up to $(2k-3)$-th order derivatives at the end points of the interval. When those values are easy to calculate, this is not an obstacle. Furthermore, when $f^{(2k-1)}(1) = f^{(2k-1)}(0)$ for $k \geq 1$, we get a formula with an even higher degree of exactness. This type of quadrature formulae - which include the values of the first derivative at the end points - are sometimes called perturbed or corrected quadrature formulae and will be the topic of the next section.

Changing the assumptions on function f, we can obtain two more identities with the left-hand side equal to that in (3.7): assuming $f^{(2n-1)}$ is continuous of bounded variation on $[0,1]$ for some $n \geq 1$, from (1.1) we get:

$$\int_0^1 f(t)dt - Q\left(x,\, \frac{1}{2},\, 1-x\right) + T_{2n}^{Q3}(x) = \frac{1}{(2n)!}\int_0^1 G_{2n}^{Q3}(x,t)df^{(2n-1)}(t), \tag{3.12}$$

and assuming $f^{(2n)}$ is continuous of bounded variation on $[0,1]$ for some $n \geq 0$, from (1.1) (or (1.2)) we get:

$$\int_0^1 f(t)dt - Q\left(x,\, \frac{1}{2},\, 1-x\right) + T_{2n}^{Q3}(x) = \frac{1}{(2n+1)!}\int_0^1 G_{2n+1}^{Q3}(x,t)df^{(2n)}(t), \tag{3.13}$$

The key step for obtaining the best possible estimates of the error for this type of quadrature formulae is the following lemma.

Lemma 3.1 *For $x \in \{0\} \cup [1/6, 1/2)$ and $k \geq 1$, $G_{2k+1}^{Q3}(x,t)$ has no zeros in variable t on the interval $(0, 1/2)$. The sign of this function is determined by*

$$(-1)^k G_{2k+1}^{Q3}(x,t) > 0 \quad za\;\; x \in [1/6, 1/2),$$
$$(-1)^{k+1} G_{2k+1}^{Q3}(x,t) > 0 \quad za\;\; x = 0.$$

Proof. Observe $G_3^{Q3}(x,t)$. For $0 \leq t \leq x < 1/2$, it takes the form

$$G_3^{Q3}(x,t) = -t^3,$$

so its only zero is obviously $t = 0$. For $0 \leq x \leq t \leq 1/2$, it takes the form

$$G_3^{Q3}(x,t) = -t^3 + \frac{(x-t)^2}{2(2x-1)^2}.$$

Here it has three zeros:

$$t_1 = \frac{1}{2}, \quad t_2 = \frac{x - x^2 - \sqrt{2x^3 - 3x^4}}{(2x-1)^2}, \quad t_3 = \frac{x - x^2 + \sqrt{2x^3 - 3x^4}}{(2x-1)^2}.$$

It is easy to check that $t_2 < x$ which is opposite from our assumption. On the other hand, $t_3 \geq x$ for all $x \in [0, 1/2)$, but $t_3 \geq 1/2$ only for $x \in [1/6, 1/2)$. Thus, our assertion is true for $k = 1$ (for $x = 0$ the assertion is trivial). Assuming the opposite, by induction it follows easily that the assertion is true for all $k \geq 2$.

It is elementary to determine the sign of G_3^{Q3} since we know its form. In order to determine the sign of G_{2k+1}^{Q3} in general, use their second derivatives and the fact that they have no zeros on $(0, 1/2)$. □

Remark 3.1 From Lemma 3.1 it follows immediately that, for $k \geq 1$ and $x \in [1/6, 1/2)$, $(-1)^{k+1} F_{2k+2}^{Q3}(x,t)$ is strictly increasing on $(0, 1/2)$ and strictly decreasing on $(1/2, 1)$. Since $F_{2k+2}^{Q3}(x,0) = F_{2k+2}^{Q3}(x,1) = 0$, it has constant sign on $(0,1)$ and obtains its maximum value at $t = 1/2$. Analogous statement, but with the opposite sign, is valid in the case when $x = 0$.

Denote:

$$R_{2n+2}^{Q3}(x,f)=\frac{1}{(2n+2)!}\int_0^1 F_{2n+2}^{Q3}(x,t)f^{(2n+2)}(t)dt. \tag{3.14}$$

Theorem 3.1 *Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(2n+2)}$ is continuous on $[0,1]$ for some $n\geq 2$ and let $x\in\{0\}\cup[1/6,1/2)$. If $f^{(2n)}$ and $f^{(2n+2)}$ have the same constant sign on $[0,1]$, then the remainder $R_{2n}^{Q3}(x,f)$ has the same sign as the first neglected term $\Delta_{2n}^{Q3}(x,f)$ where*

$$\Delta_{2n}^{Q3}(x,f):=R_{2n}^{Q3}(x,f)-R_{2n+2}^{Q3}(x,f)=-\frac{1}{(2n)!}G_{2n}^{Q3}(x,0)[f^{(2n-1)}(1)-f^{(2n-1)}(0)].$$

Furthermore, $|R_{2n}^{Q3}(x,f)|\leq|\Delta_{2n}^{Q3}(x,f)|$ *and* $|R_{2n+2}^{Q3}(x,f)|\leq|\Delta_{2n}^{Q3}(x,f)|$.

Proof. From Remark 3.1 it follows that $R_{2n}^{Q3}(x,f)$ and $-R_{2n+2}^{Q3}(x,f)$ have the same sign. Therefore, $\Delta_{2n}^{Q3}(x,f)$ has that same sign. Moreover, it follows that $|R_{2n}^{Q3}(x,f)|\leq|\Delta_{2n}^{Q3}(x,f)|$ and $|R_{2n+2}^{Q3}(x,f)|\leq|\Delta_{2n}^{Q3}(x,f)|$. □

Theorem 3.2 *If $f:[0,1]\to\mathbb{R}$ is such that $f^{(2n+2)}$ is continuous on $[0,1]$ for some $n\geq 1$ and $x\in\{0\}\cup[1/6,1/2)$, then there exists $\xi\in[0,1]$ such that*

$$R_{2n+2}^{Q3}(x,f)=-\frac{G_{2n+2}^{Q3}(x,0)}{(2n+2)!}\cdot f^{(2n+2)}(\xi), \tag{3.15}$$

where

$$G_{2n+2}^{Q3}(x,0)=\frac{1}{3(2x-1)^2}\left[B_{2n+2}(x)+\left(1-2^{-2n-1}\right)B_{2n+2}\right]-\left(1-2^{-2n-1}\right)B_{2n+2} \tag{3.16}$$

If, in addition, $f^{(2n+2)}$ does not change sign on $[0,1]$, then there exists $\theta\in[0,1]$ such that

$$R_{2n+2}^{Q3}(x,f)=\frac{\theta}{(2n+2)!}\cdot F_{2n+2}^{Q3}\left(x,\frac{1}{2}\right)\cdot\left[f^{(2n+1)}(1)-f^{(2n+1)}(0)\right], \tag{3.17}$$

where

$$\begin{aligned}F_{2n+2}^{Q3}\left(x,\frac{1}{2}\right)&=\frac{1}{3(2x-1)^2}\left[B_{2n+2}\left(x+\frac{1}{2}\right)-B_{2n+2}(x)-\left(2-2^{-2n-1}\right)B_{2n+2}\right]\\&+\left(2-2^{-2n-1}\right)B_{2n+2}\end{aligned} \tag{3.18}$$

Proof. From Remark 3.1 we know that function $F_{2n+2}^{Q3}(x,t)$ has constant sign on $(0,1)$, so (3.15) follows from the mean value theorem for integrals.

Next, let $x\in[1/6,1/2)$ and suppose $f^{(2n+2)}(t)\geq 0$, $0\leq t\leq 1$. Then we have

$$0\leq\int_0^1(-1)^{n+1}F_{2n+2}^{Q3}(x,t)\,f^{(2n+2)}(t)dt\leq(-1)^{n+1}F_{2n+2}^{Q3}(x,1/2)\cdot\int_0^1 f^{(2n+2)}(t)dt,$$

which means there exists $\theta \in [0,1]$ such that

$$(2n+2)! \cdot R^{Q3}_{2n+2}(x,f) = \theta \cdot F^{Q3}_{2n+2}(x,1/2)\left[f^{(2n+1)}(1) - f^{(2n+1)}(0)\right].$$

When $x=0$ or $f^{(2n+2)}(t) \le 0$, the statement follows similarly. $\square$

When (3.15) is applied to the remainder in formula (3.7) for $n=1$, the following formula is produced:

$$\int_0^1 f(t)dt - \frac{1}{6(2x-1)^2}\left[f(x) + 24B_2(x)\cdot f\left(\frac{1}{2}\right) + f(1-x)\right]$$
$$= \frac{1}{2880}(-10x^2+10x-1)\cdot f^{(4)}(\xi). \tag{3.19}$$

For an adequate choice of node x, formula (3.19) gives classical Simpson's, dual Simpson's and Maclaurin's formula as special cases. Furthermore, for $x=0$, results of this section produce results obtained in [29], where Euler-Simpson formulae were derived. For $x=1/4$, results from [26] are produced, i.e. dual Euler-Simpson formulae and all related results, and finally, for $x=1/6$ Euler-Maclaurin formulae are obtained together with all the results from [24].

Remark 3.2 Formula (3.19) is valid only for $x \in \{0\} \cup [1/6, 1/2)$, so let us consider the limit process when x tends to $1/2$. The following quadrature formula is produced:

$$\int_0^1 f(t)dt - f\left(\frac{1}{2}\right) - \frac{1}{24}f''\left(\frac{1}{2}\right) = \frac{1}{1920}f^{(4)}(\xi).$$

Of course, all other related results can be obtained as well. We have

$$G^{Q3}_k\left(\frac{1}{2},t\right) = B^*_k\left(\frac{1}{2}-t\right) + \frac{k(k-1)}{24}B^*_{k-2}\left(\frac{1}{2}-t\right), \quad k \ge 2.$$

So, when $f^{(m)} \in L_p[-1,1]$ for $p=\infty$ or $p=1$ and $m=2,3,4$ we get the following estimations:

$$\left|\int_0^1 f(t)dt - f\left(\frac{1}{2}\right) - \frac{1}{24}f''\left(\frac{1}{2}\right)\right| \le C(m,q)\cdot \|f^{(m)}\|_p$$

where

$$C(2,1) = \frac{1}{24}, \quad C(3,1) = \frac{1}{192}, \quad C(4,1) = \frac{1}{1920},$$
$$C(2,\infty) = \frac{1}{8}, \quad C(3,\infty) = \frac{1}{48}, \quad C(4,\infty) = \frac{1}{480}.$$

Comparing these estimations with those obtained for the trapezoid formula and the midpoint formula (cf. subsections 1.2.1. and 1.2.2.) shows that for $m=4$, these are better.

The following theorem gives an estimate of error for this type of quadrature formulae.

Theorem 3.3 *Let $p,q \in \mathbb{R}$ be such that $1 \le p,\, q \le \infty$ and $1/p + 1/q = 1$. Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(2n)} \in L_p[0,1]$ for some $n \ge 1$. Then we have*

$$\left| \int_0^1 f(t)dt - Q\left(x,\, \frac{1}{2},\, 1-x\right) + T_{2n}^{Q3}(x) \right| \le K_{Q3}(2n,q) \cdot \|f^{(2n)}\|_p. \tag{3.20}$$

If $f^{(2n+1)} \in L_p[0,1]$ for some $n \ge 0$, then we have

$$\left| \int_0^1 f(t)dt - Q\left(x,\, \frac{1}{2},\, 1-x\right) + T_{2n}^{Q3}(x) \right| \le K_{Q3}(2n+1,q) \cdot \|f^{(2n+1)}\|_p. \tag{3.21}$$

If $f^{(2n+2)} \in L_p[0,1]$ for some $n \ge 0$, then we have

$$\left| \int_0^1 f(t)dt - Q\left(x,\, \frac{1}{2},\, 1-x\right) + T_{2n}^{Q3}(x) \right| \le K_{Q3}^*(2n+2,q) \cdot \|f^{(2n+2)}\|_p, \tag{3.22}$$

where

$$K_{Q3}(m,q) = \frac{1}{m!}\left[\int_0^1 \left|G_m^{Q3}(x,t)\right|^q dt\right]^{\frac{1}{q}}$$
$$K_{Q3}^*(m,q) = \frac{1}{m!}\left[\int_0^1 \left|F_m^{Q3}(x,t)\right|^q dt\right]^{\frac{1}{q}}.$$

These inequalities are sharp for $1 < p \le \infty$ and the best possible for $p = 1$.

Proof. Analogous to the proof of Theorem 2.2. □

For $x \in \{0\} \cup [1/6, 1/2)$ and $n \ge 1$, using Lemma 3.1 and Remark 3.1, we can calculate the following constants as special cases from the previous Theorem:

$$K_{Q3}^*(2n+2,1) = \frac{1}{(2n+2)!}\left|G_{2n+2}^{Q3}(x,0)\right|,$$
$$K_{Q3}^*(2n+2,\infty) = \frac{1}{2}\, K_{Q3}(2n+1,1) = \frac{1}{(2n+2)!}\left|F_{2n+2}^{Q3}\left(x,\frac{1}{2}\right)\right|,$$

where $G_{2n+2}^{Q3}(x,0)$ and $F_{2n+2}^{Q3}(x,1/2)$ are as in (3.16) and (3.18).

Next, we shall consider which x gives the best estimation for $p = \infty$ and $p = 1$. Assume $x \in [1/6, 1/2)$ and define function $H(x) := |G_{2n+2}^{Q3}(x,0)|$, i.e.

$$H(x) = (-1)^n \left[\frac{1}{3(2x-1)^2}\left[B_{2n+2}(x) + \left(1 - 2^{-2n-1}\right)B_{2n+2}\right] - \left(1 - 2^{-2n-1}\right)B_{2n+2}\right]$$

Then

$$H'(x) = \frac{(-1)^n}{3(2x-1)^3}\left[(2x-1)(2n+2)B_{2n+1}(x) + 4\left(B_{2n+2}\left(\frac{1}{2}\right) - B_{2n+2}(x)\right)\right].$$

We claim $H'(x) > 0$. To prove this, it suffices to check that $h(x) < 0$ where

$$h(x) = (-1)^n \left[(2x-1)(2n+2)B_{2n+1}(x) + 4\left(B_{2n+2}\left(\frac{1}{2}\right) - B_{2n+2}(x)\right)\right].$$

Now,

$$h''(x) = (2n+2)(2n+1)(2n)\cdot(2x-1)\cdot(-1)^n B_{2n-1}(x) < 0,$$

so we conclude that h' is decreasing and since $h'(1/2) = 0$, we have $h'(x) > 0$. This means h is increasing and since $h(1/2) = 0$, it follows that $h(x) < 0$ so our claim is true. $H(x)$ is therefore an increasing function and attains its minimal value at $x = 1/6$. Further, it is easy to see (by induction) that $|H(1/6)| < |H(0)|$. This shows that Maclaurin formula, i.e. its generalization - the Euler-Maclaurin formulae, give the best estimation out of all quadrature formulae of the form

$$\int_0^1 f(t)dt \approx Q\left(x,\ \frac{1}{2},\ 1-x\right)$$

where $Q\left(x,\ \frac{1}{2},\ 1-x\right)$ is as in (3.8).

It can be shown analogously that the integrand in $K_{Q3}(2n+1,1)$ is also increasing and attains its minimum value at $x = 1/6$, so the same conclusion is derived again. Furthermore, the same conclusion follows if $K^*_{Q3}(2n+2,\infty)$ is considered.

We will finish this section by considering Hermite-Hadamard's and Dragomir-Agarwal's type inequality for this type of quadrature formulae (cf. Section 2.5.).

Theorem 3.4 *Let $f : [0,1] \to \mathbb{R}$ be $(2n+4)$-convex for $n \geq 1$. Then for $x \in [1/6, 1/2)$, we have*

$$\begin{aligned} &\frac{1}{(2n+2)!}|G^{Q3}_{2n+2}(x,0)|\cdot f^{(2n+2)}\left(\frac{1}{2}\right) \\ &\leq (-1)^{n+1}\left(\int_0^1 f(t)dt - Q\left(x,\ \frac{1}{2},\ 1-x\right) + T^{Q3}_{2n}(x)\right) \\ &\leq \frac{1}{(2n+2)!}|G^{Q3}_{2n+2}(x,0)|\cdot \frac{f^{(2n+2)}(0)+f^{(2n+2)}(1)}{2}, \end{aligned} \tag{3.23}$$

where $G^{Q3}_{2n+2}(x,0)$ is as in (3.16), while

$$\begin{aligned} &\frac{1}{3(2n+2)!}(1-2^{-2n})|B_{2n+2}|\, f^{(2n+2)}\left(\frac{1}{2}\right) \\ &\leq (-1)^n\left(\int_0^1 f(t)dt - Q\left(0,\ \frac{1}{2},\ 1\right) + T^{Q3}_{2n}(0)\right) \\ &\leq \frac{1}{3(2n+2)!}(1-2^{-2n})|B_{2n+2}|\, \frac{f^{(2n+2)}(0)+f^{(2n+2)}(1)}{2}. \end{aligned} \tag{3.24}$$

If f is $(2n+4)$-concave, the inequalities are reversed.

Proof. Analogous to the proof of Theorem 2.8. □

Theorem 3.5 *Let $x \in \{0\} \cup [1/6, 1/2)$ and $f : [0,1] \to \mathbb{R}$ be m-times differentiable for $m \geq 3$. If $|f^{(m)}|^q$ is convex for some $q \geq 1$, then*

$$\left| \int_0^1 f(t)dt - Q\left(x,\, \frac{1}{2},\, 1-x\right) + T_{2n}^{Q3}(x) \right| \leq L_{Q3}(m,x) \left(\frac{|f^{(m)}(0)|^q + |f^{(m)}(1)|^q}{2} \right)^{1/q} \tag{3.25}$$

while if $|f^{(m)}|$ is concave, then

$$\left| \int_0^1 f(t)dt - Q\left(x,\, \frac{1}{2},\, 1-x\right) + T_{2n}^{Q3}(x) \right| \leq L_{Q3}(m,x) \left| f^{(m)}\left(\frac{1}{2}\right) \right|, \tag{3.26}$$

where

$$\begin{aligned} &\text{for } m = 2n+1 \quad L_{Q3}(2n+1,x) = \frac{2}{(2n+2)!} \, |F_{2n+2}^{Q3}(x,1/2)| \\ &\text{and for } m = 2n+2 \quad L_{Q3}(2n+2,x) = \frac{1}{(2n+2)!} \, |G_{2n+2}^{Q3}(x,0)| \end{aligned}$$

with $G_{2n+2}^{Q3}(x,0)$ and $F_{2n+2}^{Q3}(x,1/2)$ as in (3.16) and (3.18), respectively.

Proof. Starting from (3.13) and applying Hölder's inequality and then Jensen's inequality for the convex function $|f^{(2n+1)}|^q$, we get

$$\begin{aligned} &\left| \int_0^1 f(t)dt - Q\left(x,\, \frac{1}{2},\, 1-x\right) + T_{2n}^{Q3}(x) \right| \leq \frac{1}{(2n+1)!} \int_0^1 |G_{2n+1}^{Q3}(x,t)||f^{(2n+1)}(t)|dt \\ &\leq \frac{1}{(2n+1)!} \left(\int_0^1 |G_{2n+1}^{Q3}(x,t)|dt \right)^{1-1/q} \\ &\quad \cdot \left(\int_0^1 |f^{(2n+1)}((1-t)\cdot 0 + t\cdot 1)|^q |G_{2n+1}^{Q3}(x,t)|dt \right)^{1/q} \\ &\leq \frac{1}{(2n+1)!} \left(\int_0^1 |G_{2n+1}^{Q3}(x,t)|dt \right)^{1-1/q} \\ &\quad \cdot \left(|f^{(2n+1)}(0)|^q \int_0^1 (1-t)|G_{2n+1}^{Q3}(x,t)|dt + |f^{(2n+1)}(1)|^q \int_0^1 t|G_{2n+1}^{Q3}(x,t)|dt \right)^{1/q} \end{aligned}$$

Further, it is not difficult to prove that

$$\int_0^1 |F_{2k+2}^{Q3}(x,t)|dt = 2\int_0^1 t|F_{2k+2}^{Q3}(x,t)|dt = |G_{2k+2}^{Q3}(x,0)|, \tag{3.27}$$

$$\int_0^1 |G_{2k+1}^{Q3}(x,t)|dt = 2\int_0^1 t|G_{2k+1}^{Q3}(x,t)|dt = \frac{2}{2k+2} \left| F_{2k+2}^{Q3}\left(x, \frac{1}{2}\right) \right|. \tag{3.28}$$

Applying (3.28), inequality (3.25) for an odd m easily follows. The assertion for an even m follows similarly, starting from (3.7) and applying (3.27).

To prove (3.26), apply Jensen's integral inequality:

$$\begin{aligned}
&\left|\int_0^1 f(t)dt - Q\left(x,\, \frac{1}{2},\, 1-x\right) + T_{2n}^{Q3}(x)\right| \\
&\leq \frac{1}{(2n+1)!}\int_0^1 |G_{2n+1}^{Q3}(x,t)|\cdot|f^{(2n+1)}((1-t)\cdot 0 + t\cdot 1)|dt \\
&\leq \frac{1}{(2n+1)!}\int_0^1 |G_{2n+1}^{Q3}(x,t)|dt\cdot\left|f^{(2n+1)}\left(\frac{\int_0^1((1-t)\cdot 0+t\cdot 1)|G_{2n+1}^{Q3}(x,t)|dt}{\int_0^1 |G_{2n+1}^{Q3}(x,t)|dt}\right)\right|
\end{aligned}$$

Recalling (3.28), (3.26) is proved for an odd m. For an even m, the statement follows similarly. □

Remark 3.3 Inequality (3.24) is in fact Hermite-Hadamard type estimate for the classical Simpson's formula and it was derived in [33]. For $x = 1/4$, (3.23) becomes a Hermite-Hadamard type inequality for the dual Simpson's formula, and for $x = 1/6$ for Maclaurin's formula; these two cases were covered in [26] and [24], respectively.

Theorem 3.5 gives Dragomir-Agarwal type inequalities for Simpson's formula ($x = 0$), dual Simpson's formula ($x = 1/4$) and Maclaurin's formula ($x = 1/6$). These results were already obtained in [33], [26] and [24], respectively.

3.1.1 Gauss 2-point formula

There is an interesting special case of the results from the previous section. Namely, if we choose $w(x) = 1/2$, where $w(x)$ is as in (3.7), we will get

$$x = \frac{1}{2} - \frac{1}{2\sqrt{3}}.$$

Since $\frac{1}{6} < \frac{1}{2} - \frac{1}{2\sqrt{3}} < \frac{1}{2}$, we can apply all the results in this case.

For this choice of the node x, formula (3.19) becomes the classical Gauss 2-point formula stated on the interval $[0,1]$. Since it is customary to study Gauss formulae on the interval $[-1,1]$, in order to make use of the symmetry of the nodes and coefficients, by a simple linear transformation we transform the interval $[0,1]$ we have so far worked with, into $[-1,1]$.

Formulae (3.12), (3.13) and (3.7) now become:

$$\int_{-1}^1 f(t)dt - Q_{G2} + T_{2n}^{G2} = \frac{2^{2n-1}}{(2n)!}\int_{-1}^1 G_{2n}^{G2}(t)df^{(2n-1)}(t), \tag{3.29}$$

$$\int_{-1}^1 f(t)dt - Q_{G2} + T_{2n}^{G2} = \frac{2^{2n}}{(2n+1)!}\int_{-1}^1 G_{2n+1}^{G2}(t)df^{(2n)}(t), \tag{3.30}$$

$$\int_{-1}^1 f(t)dt - Q_{G2} + T_{2n}^{G2} = \frac{2^{2n+1}}{(2n+2)!}\int_{-1}^1 F_{2n+2}^{G2}(t)df^{(2n+1)}(t), \tag{3.31}$$

where

$$Q_{G2} = f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$$

$$T_{2n}^{G2} = \sum_{k=2}^{n} \frac{2^{2k}}{(2k)!} \cdot B_{2k}\left(\frac{3-\sqrt{3}}{6}\right)\left[f^{(2k-1)}(1) - f^{(2k-1)}(-1)\right],$$

$$G_k^{G2}(t) = B_k^*\left(-\frac{\sqrt{3}}{6} - \frac{t}{2}\right) + B_k^*\left(\frac{\sqrt{3}}{6} - \frac{t}{2}\right),$$

$$F_k^{G2}(t) = G_k^{G2}(t) - G_k^{G2}(-1), \quad k \geq 1, \quad t \in \mathbb{R}.$$

Theorem 3.2 takes the form:

Corollary 3.1 *If $f : [-1,1] \to \mathbb{R}$ is such that $f^{(2n+2)}$ is continuous on $[-1,1]$ for some $n \geq 1$, then there exists $\xi \in [-1,1]$ such that*

$$R_{2n+2}^{G2}(f) = -\frac{2^{2n+3}}{(2n+2)!} \cdot B_{2n+2}\left(\frac{3-\sqrt{3}}{6}\right) \cdot f^{(2n+2)}(\xi). \tag{3.32}$$

If, in addition, $f^{(2n+2)}$ does not change sign on $[-1,1]$, then there exists $\theta \in [0,1]$ such that

$$\begin{aligned} R_{2n+2}^{G2}(f) \;=\; & \theta \cdot \frac{2^{2n+2}}{(2n+2)!}\left[B_{2n+2}\left(\frac{\sqrt{3}}{6}\right) - B_{2n+2}\left(\frac{3-\sqrt{3}}{6}\right)\right] \\ & \cdot \left[f^{(2n+1)}(1) - f^{(2n+1)}(-1)\right]. \end{aligned} \tag{3.33}$$

Applying (3.32) to the remainder in (3.31) for $n = 1$, produces the classical Gauss 2-point formula:

$$\int_{-1}^{1} f(t)dt = f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) + \frac{1}{135} \cdot f^{(4)}(\xi), \quad \xi \in [-1,1].$$

Estimates of error from Theorem 3.3 are in this case:

Corollary 3.2 *Let $p, q \in \mathbb{R}$ be such that $1 \leq p,\ q \leq \infty$ and $1/p + 1/q = 1$. Let $f : [-1,1] \to \mathbb{R}$ be such that $f^{(2n)} \in L_p[-1,1]$ for some $n \geq 1$. Then we have*

$$\left|\int_{-1}^{1} f(t)dt - Q_{G2} + T_{2n}^{G2}\right| \leq \frac{2^{2n-1}}{(2n)!}\left[\int_{-1}^{1} \left|G_{2n}^{G2}(t)\right|^q dt\right]^{\frac{1}{q}} \|f^{(2n)}\|_p. \tag{3.34}$$

If $f^{(2n+1)} \in L_p[-1,1]$ for some $n \geq 0$, then

$$\left|\int_{-1}^{1} f(t)dt - Q_{G2} + T_{2n}^{G2}\right| \leq \frac{2^{2n}}{(2n+1)!}\left[\int_{-1}^{1} \left|G_{2n+1}^{G2}(t)\right|^q dt\right]^{\frac{1}{q}} \|f^{(2n+1)}\|_p. \tag{3.35}$$

If $f^{(2n+2)} \in L_p[-1,1]$ for some $n \geq 0$, then

$$\left| \int_{-1}^{1} f(t)dt - Q_{G2} + T_{2n}^{G2} \right| \leq \frac{2^{2n+1}}{(2n+2)!} \left[\int_{-1}^{1} \left| F_{2n+2}^{G2}(t) \right|^q dt \right]^{\frac{1}{q}} \| f^{(2n+2)} \|_p. \tag{3.36}$$

These inequalities are sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

It is easy to see that:

$$\int_{-1}^{1} \left| G_{2n+1}^{G2}(t) \right| dt = \frac{4}{2n+2} \left| F_{2n+2}^{G2}(0) \right| = \frac{8}{2n+2} \left| B_{2n+2}\left(\frac{\sqrt{3}}{6} \right) - B_{2n+2}\left(\frac{3-\sqrt{3}}{6} \right) \right|$$

$$\int_{-1}^{1} \left| F_{2n+2}^{G2}(t) \right| dt = 2 \left| G_{2n+2}^{G2}(-1) \right| = 4 \left| B_{2n+2}\left(\frac{3-\sqrt{3}}{6} \right) \right|.$$

As direct consequences of this and Corollary 3.2, the following estimates of error can be obtained for $p = \infty$ and $p = 1$:

$$\left| \int_{-1}^{1} f(t)dt - f\left(-\frac{\sqrt{3}}{3} \right) - f\left(\frac{\sqrt{3}}{3} \right) \right| \leq C_{G2}(m,q) \cdot \| f^{(m)} \|_p, \quad m = 1,2,3,4,$$

where

$$\begin{aligned}
C_{G2}(1,1) &= \frac{5-2\sqrt{3}}{3} \approx 0.511966, \quad C_{G2}(1,\infty) = \frac{\sqrt{3}}{3} \approx 0.57735, \\
C_{G2}(2,1) &= \frac{4}{9}\sqrt{26\sqrt{3}-45} \approx 0.0811291, \\
C_{G2}(2,\infty) &= \frac{2-\sqrt{3}}{3} \approx 0.0893164, \\
C_{G2}(3,1) &= \frac{9-4\sqrt{3}}{108} \approx 0.0191833, \\
C_{G2}(3,\infty) &= \frac{2-\sqrt{3}}{9}\sqrt{2\sqrt{3}-3} \approx 0.0202823, \\
C_{G2}(4,1) &= \frac{1}{135} \approx 0.00740741, \quad C_{G2}(4,\infty) = \frac{9-4\sqrt{3}}{216} \approx 0.00959165.
\end{aligned}$$

Remark 3.4 The constant $C_{G2}(1,\infty)$ was obtained in Theorem 1.1. in [47].

Remark 3.5 Gauss 2-point formulae of Euler type (3.29), (3.30) and (3.31) were derived also in [98], as a special case of the general 2-point formulae that were studied there, but this was not explicitly mentioned since a small mistake was made in the proof.

Finally, Theorem 3.4 gives the Hermite-Hadamard type inequality for the Gauss 2-point formula:

$$\begin{aligned}
\frac{1}{4320} f^{(4)}\left(\frac{1}{2} \right) &\leq \int_{0}^{1} f(t)dt - \frac{1}{2} f\left(\frac{3-\sqrt{3}}{6} \right) - \frac{1}{2} f\left(\frac{3+\sqrt{3}}{6} \right) \\
&\leq \frac{1}{4320} \frac{f^{(4)}(0) + f^{(4)}(1)}{2}
\end{aligned}$$

and the constants from Theorem 3.5 in this case are:

$$L_{Q3}\left(3,\ \frac{3-\sqrt{3}}{6}\right)=\frac{9-4\sqrt{3}}{1728},\qquad L_{Q3}\left(4,\ \frac{3-\sqrt{3}}{6}\right)=\frac{1}{4320}.$$

3.1.2 Simpson's formula

One of the special cases of the results from the previous section, obtained for $x=0$, is the classical Simpson's formula. Results of this subsection were published in [29].

The quadrature formula is in this case:

$$Q\left(0,\ \frac{1}{2},\ 1\right)=\frac{1}{6}\left[f(0)+4f\left(\frac{1}{2}\right)+f(1)\right].$$

Further,

$$\begin{aligned}
T_{2n}^{S}&=T_{2n}^{Q3}(0)=\sum_{k=2}^{n}\frac{1}{(2k)!}\,G_{2k}^{S}(0)\,[f^{(2k-1)}(1)-f^{(2k-1)}(0)]\\
G_k^S(t)&=G_k^{Q3}(0,t)=\frac{1}{3}\left[B_k^*(1-t)+2B_k^*\left(\frac{1}{2}-t\right)\right],\quad k\geq 1\\
F_k^S(t)&=F_k^{Q3}(0,t)=G_k^S(t)-G_k^S(0),\quad k\geq 2\ \text{ and }\ t\in\mathbb{R}.
\end{aligned}$$

The remainder on the right-hand side of (3.7) for $x=0$ and $n\geq 2$, can be written, according to Theorem 3.2, as:

$$\begin{aligned}
R_{2n+2}^{S}(f)&=\frac{\theta}{3(2n+2)!}(2-2^{-1-2n})B_{2n+2}\cdot\left[f^{(2n+1)}(1)-f^{(2n+1)}(0)\right],\ \theta\in[0,1]\\
R_{2n+2}^{S}(f)&=\frac{1}{3(2n+2)!}(1-2^{-2n})B_{2n+2}\cdot f^{(2n+2)}(\eta),\qquad \eta\in[0,1]
\end{aligned}$$

Formula (3.19) becomes the classical Simpson's formula:

$$\int_0^1 f(t)dt-\frac{1}{6}\left[f(0)+4f\left(\frac{1}{2}\right)+f(1)\right]=-\frac{1}{2880}\,f^{(4)}(\eta) \tag{3.37}$$

As special cases of Theorem 3.3, for $p=\infty$ and $p=1$ we get the following estimates for $m=1,2,3,4$:

$$\left|\int_0^1 f(t)dt-\frac{1}{6}\left[f(0)+4f\left(\frac{1}{2}\right)+f(1)\right]\right|\leq C_S(m,q)\cdot\|f^{(m)}\|_p,$$

where

$$\begin{aligned}
&C_S(1,1)=\frac{5}{36},\quad C_S(2,1)=\frac{1}{81},\quad C_S(3,1)=\frac{1}{576},\quad C_S(4,1)=\frac{1}{2880},\\
&C_S(1,\infty)=\frac{1}{3},\quad C_S(2,\infty)=\frac{1}{24},\quad C_S(3,\infty)=\frac{1}{324},\quad C_S(4,\infty)=\frac{1}{1152}.
\end{aligned}$$

3.1.3 Dual Simpson's formula

For $x = 1/4$, as a special case dual Euler-Simpson's formulae are obtained. Results of this subsection are published in [26]. We have:

$$Q\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right) = \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right],$$
$$T_{2n}^{DS} = T_{2n}^{Q3}\left(\frac{1}{4}\right) = \sum_{k=2}^{n} \frac{1}{(2k)!} G_{2k}^{DS}(0)\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right]$$
$$G_k^{DS}(t) = G_k^{Q3}\left(\frac{1}{4}, t\right) = \frac{1}{3}\left[2B_k^*\left(\frac{1}{4} - t\right) - B_k^*\left(\frac{1}{2} - t\right) + 2B_k^*\left(\frac{3}{4} - t\right)\right],$$
$$F_k^{DS}(t) = F_k^{Q3}\left(\frac{1}{4}, t\right) = G_k^{DS}(t) - G_k^{DS}(0), \quad k \geq 1 \text{ and } t \in \mathbb{R}.$$

The remainder $R_{2n+2}^{DS}(f) = R_{2n+2}^{Q3}(1/4, f)$, according to Theorem 3.2, can be written as:

$$R_{2n+2}^{DS}(f) = \frac{\theta}{3(2n+2)!}(2^{-1-2n} - 2)B_{2n+2}\left[f^{(2n+1)}(1) - f^{(2n+1)}(0)\right], \ \theta \in [0,1]$$
$$R_{2n+2}^{DS}(f) = -\frac{1}{3(2n+2)!}(1 - 4^{-n})(1 - 2^{-1-2n})B_{2n+2} \cdot f^{(2n+2)}(\eta), \ \ \eta \in [0,1]$$

Formula (3.19) produces classical dual Simpson's formula:

$$\int_0^1 f(t)dt - \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right] = \frac{7}{23040} f^{(4)}(\eta). \tag{3.38}$$

Estimate of error for $p = \infty$ and $p = 1$ are for $m = 1, 2, 3, 4$:

$$\left|\int_0^1 f(t)dt - \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right]\right| \leq C_{DS}(m,q) \cdot \|f^{(m)}\|_p,$$

where

$$C_{DS}(1,1) = \frac{5}{24},\ C_{DS}(2,1) = \frac{5}{324},\ C_{DS}(3,1) = \frac{1}{576},\ C_{DS}(4,1) = \frac{7}{23040},$$
$$C_{DS}(1,\infty) = \frac{5}{12},\ C_{DS}(2,\infty) = \frac{1}{24},\ C_{DS}(3,\infty) = \frac{5}{1296},\ C_{DS}(4,\infty) = \frac{1}{1152}.$$

3.1.4 Maclaurin's formula

The next interesting special case is Maclaurin's formula, obtained for $x = 1/6$. Results of this subsection are published in [24]. We have:

$$Q\left(\frac{1}{6}, \frac{1}{2}, \frac{5}{6}\right) = \frac{1}{8}\left[3f\left(\frac{1}{6}\right) + 2f\left(\frac{1}{2}\right) + 3f\left(\frac{5}{6}\right)\right],$$

$$T_{2n}^{M} = T_{2n}^{Q3}\left(\frac{1}{6}\right) = \sum_{k=2}^{n} \frac{1}{(2k)!} G_{2k}^{M}(0)\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right]$$
$$G_k^M(t) = G_k^{Q3}\left(\frac{1}{6},t\right) = \frac{1}{8}\left[3B_k^*\left(\frac{1}{6}-t\right) + 2B_k^*\left(\frac{1}{2}-t\right) + 3B_k^*\left(\frac{5}{6}-t\right)\right],$$
$$F_k^M(t) = F_k^{Q3}\left(\frac{1}{6},t\right) = G_k^M(t) - G_k^M(0), \quad k \geq 1 \text{ and } t \in \mathbb{R}.$$

For $n \geq 2$, the remainder $R_{2n+2}^{M}(f)$ can be written as:

$$R_{2n+2}^{M}(f) = \theta\, \frac{(2-2^{-1-2n})(9^{-n}-1)B_{2n+2}}{8(2n+2)!}\left[f^{(2n-1)}(1) - f^{(2n-1)}(0)\right],\ \theta \in [0,1]$$
$$R_{2n+2}^{M}(f) = \frac{(1-2^{-1-2n})(9^{-n}-1)B_{2n+2}}{8(2n+2)!} \cdot f^{(2n+2)}(\eta), \qquad \eta \in [0,1]$$

Formula (3.19) becomes Maclaurin's formula:

$$\int_0^1 f(t)dt - \frac{1}{8}\left[3f\left(\frac{1}{6}\right) + 2f\left(\frac{1}{2}\right) + 3f\left(\frac{5}{6}\right)\right] = \frac{7}{51840} f^{(4)}(\eta) \tag{3.39}$$

Estimates of error for $p = 1$ and $p = \infty$ are:

$$\left|\int_0^1 f(t)dt - \frac{1}{8}\left[3f\left(\frac{1}{6}\right) + 2f\left(\frac{1}{2}\right) + 3f\left(\frac{5}{6}\right)\right]\right| \leq C_M(m,q) \cdot \|f^{(m)}\|_p,$$

where

$$C_M(1,1) = \frac{25}{288},\ C_M(2,1) = \frac{1}{192},\ C_M(3,1) = \frac{1}{1728},\ C_M(4,1) = \frac{7}{51840},$$
$$C_M(1,\infty) = \frac{5}{24},\ C_M(2,\infty) = \frac{1}{72},\ C_M(3,\infty) = \frac{1}{768},\ C_M(4,\infty) = \frac{1}{3456}.$$

3.1.5 Hermite-Hadamard-type inequality for the 3-point quadrature formulae

The well-known Hermite-Hadamard-type inequality states that: for any convex function $f : [0,1] \to \mathbb{R}$, the following pair of inequalities holds

$$f\left(\frac{1}{2}\right) \leq \int_0^1 f(t)dt \leq \frac{f(0)+f(1)}{2}.$$

If f is concave, inequalities are reversed. The aim of this subsection is to provide this type of inequality for the general 3-point quadrature formulae. The main result states:

Theorem 3.6 *Let $f : [0,1] \to \mathbb{R}$ be 4-convex and such that $f^{(4)}$ is continuous on $[0,1]$. Then, for $x \in \left[\frac{1}{6}, \frac{1}{2}\right)$*

$$\frac{1}{6(2x-1)^2}\left(f(x) + 24B_2(x)f\left(\frac{1}{2}\right) + f(1-x)\right)$$
$$\leq \int_0^1 f(t)dt \ \leq \ \frac{1}{6}\left(f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right). \tag{3.40}$$

If f is 4-concave, the inequalities are reversed.

Proof. For a 4-convex function f, we have $f^{(4)} \geq 0$, so the statement follows easily from (3.19). □

The following corollaries give comparison between dual Simpson's and Simpson's rule, Maclaurin's and Simpson's rule, and finally, the Gauss 2-point and Simpson's rule.

Corollary 3.3 *Let $f : [0,1] \to \mathbb{R}$ be 4-convex and such that $f^{(4)}$ is continuous on $[0,1]$. Then*

$$\frac{1}{3}\left(2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right) \leq \int_0^1 f(t)dt \ \leq \ \frac{1}{6}\left(f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right).$$

If f is 4-concave, the inequalities are reversed.

Proof. Follows from (3.40) for $x = 1/4$. □

Corollary 3.4 *Let $f : [0,1] \to \mathbb{R}$ be 4-convex and such that $f^{(4)}$ is continuous on $[0,1]$. Then*

$$\frac{1}{8}\left(3f\left(\frac{1}{6}\right) + 2f\left(\frac{1}{2}\right) + 3f\left(\frac{5}{6}\right)\right) \leq \int_0^1 f(t)dt \ \leq \ \frac{1}{6}\left(f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right).$$

If f is 4-concave, the inequalities are reversed.

Proof. Follows from (3.40) for $x = 1/6$. □

Corollary 3.5 *Let $f : [0,1] \to \mathbb{R}$ be 4-convex and such that $f^{(4)}$ is continuous on $[0,1]$. Then*

$$\frac{1}{2}f\left(\frac{3-\sqrt{3}}{6}\right) + \frac{1}{2}f\left(\frac{3+\sqrt{3}}{6}\right) \leq \int_0^1 f(t)dt \ \leq \ \frac{1}{6}\left(f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right).$$

If f is 4-concave, the inequalities are reversed.

Proof. Follows from (3.40) for $x = 1/2 - \sqrt{3}/6 \ \Leftrightarrow \ B_2(x) = 0$. □

3.1.6 Bullen-Simpson's inequality

For any function $f : [0,1] \to \mathbb{R}$, with continuous fourth derivative $f^{(4)}$ on $[0,1]$ and $f^{(4)}(t) \geq 0$, $t \in [0,1]$, we have

$$\begin{aligned} \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right] &\leq \int_0^1 f(t)\mathrm{d}t \\ &\leq \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right]. \end{aligned} \tag{3.41}$$

In the case when $f^{(4)}$ exists, the condition $f^{(4)}(t) \geq 0$, $t \in [0,1]$ is equivalent to the requirement that f is 4-convex function on $[0,1]$. However, a function f may be 4-convex although $f^{(4)}$ does not exist.

P. S. Bullen in [11] proved that, if f is 4-convex, then (3.41) is valid. Moreover, he proved that the dual Simpson's quadrature rule is more accurate than the Simpson's quadrature rule, that is we have

$$\begin{aligned} 0 &\leq \int_0^1 f(t)\mathrm{d}t - \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right] \\ &\leq \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right] - \int_0^1 f(t)\mathrm{d}t, \end{aligned} \tag{3.42}$$

provided f is 4-convex. We shall call this inequality Bullen-Simpson's inequality.

The aim of this section is to establish a generalization of the inequalities (3.41) and (3.42) for a class of $(2r)$-convex functions and also to obtain some estimates for the absolute value of difference between the absolute value of error in the dual Simpson's quadrature rule and the absolute value of error in the Simpson's quadrature rule. We shall make use of the following five-point quadrature formula

$$\int_0^1 f(t)\mathrm{d}t \approx \frac{1}{12}\left[f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + f(1)\right],$$

obtained by adding the Simpson's and the dual Simpson's quadrature formulae. It is suitable for our purposes to rewrite the inequality (3.41) in the form

$$\int_0^1 f(t)\mathrm{d}t \leq \frac{1}{12}\left[f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + f(1)\right] \tag{3.43}$$

As we mentioned earlier, this inequality is valid for any 4-convex function f and we call it the Bullen-Simpson's inequality. The results from this section are published in [86].

We consider the sequences of functions $(G_k(t))_{k\geq 1}$ and $(F_k(t))_{k\geq 1}$ defined for $t \in \mathbb{R}$ by

$$G_k(t) := G_k^S(t) + G_k^{DS}(t),\ F_k(t) := F_k^S(t) + F_k^{DS}(t),$$

where $G_k^S(t)$, $G_k^{DS}(t)$, $F_k^S(t)$ and $F_k^{DS}(t)$ are defined as in Section 3.1.2. and Section 3.1.3., respectively. So we have

$$G_1(t) = F_1(t) = B_1(1-t) + 2B_1^*\left(\frac{1}{4} - t\right) + B_1^*\left(\frac{1}{2} - t\right) + 2B_1^*\left(\frac{3}{4} - t\right)$$

and, for $k \geq 2$,

$$G_k(t) = B_k(1-t) + 2B_k^*\left(\frac{1}{4} - t\right) + B_k^*\left(\frac{1}{2} - t\right) + 2B_k^*\left(\frac{3}{4} - t\right),$$

$$F_k(t) := G_k(t) - \tilde{B}_k\,,$$

where

$$\tilde{B}_k := G_k(0) = B_k + 2B_k\left(\frac{1}{4}\right) + B_k\left(\frac{1}{2}\right) + 2B_k\left(\frac{3}{4}\right).$$

Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(n-1)}$ exists on $[0,1]$ for some $n\geq 1$. We introduce the following notation

$$D(0,1):=\frac{1}{12}\left[f(0)+4f\left(\frac{1}{4}\right)+2f\left(\frac{1}{2}\right)+4f\left(\frac{3}{4}\right)+f(1)\right]$$

Further, we define $T_0(f)=T_1(f):=0$ and, for $2\leq m\leq[n/2]$,

$$T_m(f):=\frac{1}{2}\left[T_m^S(f)+T_m^{DS}(f)\right],$$

where $T_m^S(f)$ and $T_m^{DS}(f)$ are given in Section 3.1.2 and Section 3.1.3, respectively. It is easy to see that

$$T_m(f)=\frac{1}{3}\sum_{k=2}^{m}\frac{1}{(2k)!}2^{-2k}(1-4\cdot 2^{-2k})B_{2k}\left[f^{(2k-1)}(1)-f^{(2k-1)}(0)\right]. \tag{3.44}$$

In the next lemma we establish two formulae which play the key role here. We call them Bullen-Simpson formulae of Euler type.

Lemma 3.2 *Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$, for some $n\geq 1$. Then we have*

$$\int_0^1 f(t)\mathrm{d}t=D(0,1)+T_r(f)+\tau_n^1(f), \tag{3.45}$$

where $r=[n/2]$ and

$$\tau_n^1(f)=\frac{1}{6(n!)}\int_0^1 G_n(t)\,\mathrm{d}f^{(n-1)}(t).$$

Also,

$$\int_0^1 f(t)\mathrm{d}t=D(0,1)+T_s(f)+\tau_n^2(f), \tag{3.46}$$

where $s=[(n-1)/2]$ and

$$\tau_n^2(f)=\frac{1}{6(n!)}\int_0^1 F_n(t)\,\mathrm{d}f^{(n-1)}(t).$$

Proof. We multiply Euler-Simpson's and dual Euler-Simpson's formulae by the factor $1/2$ and then add them up to obtain the identities (3.45) and (3.46). □

Remark 3.6 The interval $[0,1]$ is used for simplicity and involves no loss in generality. The results which follow will be applied, without comment, to any interval that is convenient.
Namely it is easy to transform the identities (3.45) and (3.46) to the identities which hold for any function $f:[a,b]\to\mathbb{R}$ such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$, for some $n\geq 1$. We get

$$\int_a^b f(t)\mathrm{d}t=D(a,b)+\tilde{T}_r(f)+\frac{(b-a)^n}{6(n!)}\int_a^b G_n\left(\frac{t-a}{b-a}\right)\mathrm{d}f^{(n-1)}(t) \tag{3.47}$$

and

$$\int_a^b f(t)\mathrm{d}t = D(a,b) + \tilde{T}_s(f) + \frac{(b-a)^n}{6(n!)} \int_a^b F_n\left(\frac{t-a}{b-a}\right) \mathrm{d}f^{(n-1)}(t), \tag{3.48}$$

where

$$D(a,b) := \frac{b-a}{12}\left[f(a) + 4\left(\frac{3a+b}{4}\right) + 2\left(\frac{a+b}{2}\right) + 4\left(\frac{a+3b}{4}\right) + f(b)\right],$$

while $\tilde{T}_0(f) = \tilde{T}_1(f) = 0$ and

$$\tilde{T}_m(f) = \frac{1}{3}\sum_{k=2}^{m} \frac{(b-a)^{2k}}{(2k)!} 2^{-2k}(1-4\cdot 2^{-2k})B_{2k}\left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right],$$

for $2 \le m \le [n/2]$.

Now, we use Bullen-Simpson formulae of Euler type established in Lemma 3.2 to obtain a generalization of Bullen-Simpson's inequality for $(2r)$-convex functions. First, we need some properties of the functions $G_k(t)$ and $F_k(t)$.

Since $B_1(t) = t - (1/2)$, we have

$$G_1(t) = F_1(t) = \begin{cases} -6t + 1/2, & t \in [0,1/4] \\ -6t + 5/2, & t \in (1/4,1/2] \\ -6t + 7/2, & t \in (1/2,3/4] \\ -6t + 11/2, & t \in (3/4,1] \end{cases}. \tag{3.49}$$

Further, for $k \ge 2$ the functions $B_k^*(t)$ are periodic with period 1 and continuous. We have

$$G_k(0) = G_k(1/2) = G_k(1) = \tilde{B}_k \text{ and } F_k(0) = F_k(1/2) = F_k(1) = 0.$$

Moreover, it is enough to know the values of the functions $G_k(t)$ and $F_k(t)$, $k \ge 2$ only on the interval $[0,1/2]$ since for $0 \le t \le 1/2$ we have

$$\begin{aligned} G_k\left(t + \frac{1}{2}\right) &= B_k\left(\frac{1}{2} - t\right) + 2B_k^*\left(-\frac{1}{4} - t\right) + B_k^*(-t) + 2B_k^*\left(\frac{1}{4} - t\right) \\ &= B_k^*\left(\frac{1}{2} - t\right) + 2B_k^*\left(\frac{3}{4} - t\right) + B_k(1-t) + 2B_k^*\left(\frac{1}{4} - t\right) \\ &= G_k(t). \end{aligned}$$

For $k = 2$ and $k = 3$ we have $B_2(t) = t^2 - t + (1/6)$ and $B_3(t) = t^3 - (3/2)t^2 + (1/2)t$, so that by direct calculation we get $\tilde{B}_2 = \tilde{B}_3 = 0$ and

$$G_2(t) = F_2(t) = \begin{cases} 6t^2 - t, & t \in [0,1/4] \\ 6t^2 - 5t + 1, & t \in (1/4,1/2] \end{cases}, \tag{3.50}$$

$$G_3(t) = F_3(t) = \begin{cases} -6t^3 + (3/2)t^2, & t \in [0,1/4] \\ -6t^3 + (15/2)t^2 - 3t + (3/8), & t \in (1/4,1/2] \end{cases}. \tag{3.51}$$

The Bernoulli polynomials have the property of symmetry with respect to $\frac{1}{2}$, that is [1, 23.1.8]

$$B_k(1-t) = (-1)^k B_k(t),\ t \in \mathbb{R},\ k \geq 1. \tag{3.52}$$

Also, we have

$$B_k(1) = B_k(0) = B_k,\ k \geq 2,\ B_1(1) = -B_1(0) = \frac{1}{2}$$

and

$$B_{2r-1} = 0,\ r \geq 2.$$

This implies

$$\tilde{B}_{2r-1} = 0,\ r \geq 2 \tag{3.53}$$

and

$$\tilde{B}_{2r} = B_{2r} + 4B_{2r}\left(\frac{1}{4}\right) + B_{2r}\left(\frac{1}{2}\right),\ r \geq 1.$$

Also, we have [1, 23.1.21, 23.1.22]

$$B_{2r}\left(\frac{1}{2}\right) = -\left(1-2^{1-2r}\right)B_{2r},\ B_{2r}\left(\frac{1}{4}\right) = -2^{-2r}\left(1-2^{1-2r}\right)B_{2r},\ r \geq 1,$$

which gives the formula

$$\tilde{B}_{2r} = 2 \cdot 2^{-2r}(4 \cdot 2^{-2r} - 1)B_{2r},\ r \geq 1. \tag{3.54}$$

Now, by (3.53) we have

$$F_{2r-1}(t) = G_{2r-1}(t),\ r \geq 1. \tag{3.55}$$

Also,

$$F_{2r}(t) = G_{2r}(t) - 2 \cdot 2^{-2r}(4 \cdot 2^{-2r} - 1)B_{2r},\ r \geq 1. \tag{3.56}$$

Further, as we pointed out earlier, the points 0 and $\frac{1}{2}$ are the zeros of $F_k(t),\ k \geq 2.$ As we shall see below, 0 and $\frac{1}{2}$ are the only zeros of $F_k(t)$ in $\left[0, \frac{1}{2}\right]$ for $k = 2r$, $r \geq 1$, while for $k = 2r-1$, $r \geq 2$ we have $F_{2r-1}\left(\frac{1}{4}\right) = G_{2r-1}\left(\frac{1}{4}\right) = 0$. We shall see that 0, $\frac{1}{4}$ and $\frac{1}{2}$ are the only zeros of $F_{2r-1}(t) = G_{2r-1}(t)$, in $\left[0, \frac{1}{2}\right]$ for $r \geq 2$. Also, note that for $r \geq 1$ we have

$$G_{2r}(0) = G_{2r}\left(\frac{1}{2}\right) = \tilde{B}_{2r} = 2 \cdot 2^{-2r}(4 \cdot 2^{-2r} - 1)B_{2r}$$

and

$$G_{2r}\left(\frac{1}{4}\right) = 2B_{2r} + 2B_{2r}\left(\frac{1}{4}\right) + 2B_{2r}\left(\frac{1}{2}\right) = 2 \cdot 2^{-2r}(2 \cdot 2^{-2r} + 1)B_{2r},$$

while

$$F_{2r}\left(\frac{1}{4}\right) = G_{2r}\left(\frac{1}{4}\right) - \tilde{B}_{2r} = 4 \cdot 2^{-2r}(1 - 2^{-2r})B_{2r}. \tag{3.57}$$

Lemma 3.3 *For $k \geq 2$ we have*

$$G_k\left(\frac{1}{2}-t\right) = (-1)^k G_k(t),\ 0 \leq t \leq \frac{1}{2},$$

and

$$F_k\left(\frac{1}{2}-t\right) = (-1)^k F_k(t),\ 0 \leq t \leq \frac{1}{2}.$$

Proof. As the functions $B_k^*(t)$ are periodic with period 1 and continuous for $k \geq 2$, we get these two identities. □

Note that the identities established in Lemma 3.3 are valid for $k = 1$, too, except at the points 0, $\frac{1}{4}$ and $\frac{1}{2}$.

Lemma 3.4 *For $r \geq 2$ the function $G_{2r-1}(t)$ has no zeros in the interval $\left(0, \frac{1}{4}\right)$. The sign of this function is determined by*

$$(-1)^r G_{2r-1}(t) > 0,\ 0 < t < \frac{1}{4}.$$

Proof. For $r = 2$, $G_3(t)$ is given by (3.51) and it is easy to see that

$$G_3(t) > 0,\ 0 < t < \frac{1}{4}. \tag{3.58}$$

Thus, our assertion is true for $r = 2$. Now, using a simple induction we prove that can not have a zero inside the interval $\left(0, \frac{1}{4}\right)$. Further, if $G_{2r-3}(t) > 0$, $0 < t < \frac{1}{4}$, then from $G''_{2k-1}(t) = (2k-1)(2k-2)G_{2k-3}(t)$ it follows that $G_{2r-1}(t)$ is convex on $\left(0, \frac{1}{4}\right)$ and hence $G_{2r-1}(t) < 0$, $0 < t < \frac{1}{4}$, while in the case when $G_{2r-3}(t) < 0$, $0 < t < \frac{1}{4}$ we have that $G_{2r-1}(t)$ is concave and hence $G_{2r-1}(t) > 0$, $0 < t < \frac{1}{4}$. Since (3.58) is valid we conclude that

$$(-1)^r G_{2r-1}(t) > 0,\ 0 < t < \frac{1}{4}.$$

□

Corollary 3.6 *For $r \geq 2$ the functions $(-1)^{r-1}F_{2r}(t)$ and $(-1)^{r-1}G_{2r}(t)$ are strictly increasing on the interval $\left(0, \frac{1}{4}\right)$, and strictly decreasing on the interval $\left(\frac{1}{4}, \frac{1}{2}\right)$. Consequently, 0 and $\frac{1}{2}$ are the only zeros of $F_{2r}(t)$ in the interval $\left[0, \frac{1}{2}\right]$ and*

$$\max_{t\in[0,1]} |F_{2r}(t)| = 4 \cdot 2^{-2r}(1 - 2^{-2r})\,|B_{2r}|\,,\ r \geq 1.$$

Also, we have

$$\max_{t\in[0,1]} |G_{2r}(t)| = 2 \cdot 2^{-2r}\left(2 \cdot 2^{-2r} + 1\right)|B_{2r}|\,,\ r \geq 1.$$

Proof. We have

$$\left[(-1)^{r-1}F_{2r}(t)\right]' = \left[(-1)^{r-1}G_{2r}(t)\right]' = 2r(-1)^r G_{2r-1}(t)$$

and $(-1)^r G_{2r-1}(t) > 0$ for $0 < t < \frac{1}{4}$, by Lemma 3.4. Thus, $(-1)^{r-1}F_{2r}(t)$ and $(-1)^{r-1}G_{2r}(t)$ are strictly increasing on the interval $\left(0, \frac{1}{4}\right)$. Also, by Lemma 3.3, we have $F_{2r}\left(\frac{1}{2} - t\right) = F_{2r}(t)$, $0 \le t \le \frac{1}{2}$ and $G_{2r}\left(\frac{1}{2} - t\right) = G_{2r}(t)$, $0 \le t \le \frac{1}{2}$, which implies that $(-1)^{r-1}F_{2r}(t)$ and $(-1)^{r-1}G_{2r}(t)$ are strictly decreasing on the interval $\left(\frac{1}{4}, \frac{1}{2}\right)$. Further, $F_{2r}(0) = F_{2r}\left(\frac{1}{2}\right) = 0$, which implies that $|F_{2r}(t)|$ achieves its maximum at $t = \frac{1}{4}$, that is

$$\max_{t\in[0,1]} |F_{2r}(t)| = \left| F_{2r}\left(\frac{1}{4}\right) \right| = 4 \cdot 2^{-2r}(1 - 2^{-2r})\, |B_{2r}|\,.$$

Also,

$$\max_{t\in[0,1]} |G_{2r}(t)| = \max\left\{ |G_{2r}(0)|\,, \left| G_{2r}\left(\frac{1}{4}\right) \right| \right\} = 2 \cdot 2^{-2r}\left(1 + 2 \cdot 2^{-2r}\right) |B_{2r}|\,,$$

which completes the proof. □

Corollary 3.7 *Assume $r \ge 2$. Then we have*

$$\int_0^1 |G_{2r-1}(t)|\, \mathrm{d}t = \frac{8 \cdot 2^{-2r}\left(1 - 2^{-2r}\right)}{r}\, |B_{2r}|\,.$$

Also, we have

$$\int_0^1 |F_{2r}(t)|\, \mathrm{d}t = \left|\tilde{B}_{2r}\right| = 2 \cdot 2^{-2r}\left(1 - 4 \cdot 2^{-2r}\right) |B_{2r}|$$

and

$$\int_0^1 |G_{2r}(t)|\, \mathrm{d}t \le 2\left|\tilde{B}_{2r}\right| = 4 \cdot 2^{-2r}\left(1 - 4 \cdot 2^{-2r}\right) |B_{2r}|\,.$$

Proof. Using Lemma 3.3, Lemma 3.4 we get

$$\int_0^1 |G_{2r-1}(t)|\, \mathrm{d}t = 4\left| \int_0^{\frac{1}{4}} G_{2r-1}(t)\mathrm{d}t \right| = 4\left| -\frac{1}{2r}\, G_{2r}(t)|_0^{\frac{1}{4}} \right|$$
$$= \frac{2}{r}\left| G_{2r}\left(\frac{1}{4}\right) - G_{2r}(0) \right| = \frac{8 \cdot 2^{-2r}\left(1 - 2^{-2r}\right)}{r}\, |B_{2r}|\,,$$

which proves the first assertion. By the Corollary 3.6, $F_{2r}(t)$ does not change the sign on the interval $\left(0, \frac{1}{2}\right)$. Therefore, using (3.56) we get

$$\int_0^1 |F_{2r}(t)|\, \mathrm{d}t = 2\left| \int_0^{1/2} F_{2r}(t)\mathrm{d}t \right| = 2\left| \int_0^{1/2} \left[G_{2r}(t) - \tilde{B}_{2r}\right] \mathrm{d}t \right|$$
$$= 2\left| -\frac{1}{2r+1}\, G_{2r+1}(t)|_0^{1/2} - \frac{1}{2}\tilde{B}_{2r} \right| = \left|\tilde{B}_{2r}\right| = 2 \cdot 2^{-2r}\left(1 - 4 \cdot 2^{-2r}\right) |B_{2r}|\,.$$

This proves the second assertion. Finally, we use (3.56) again and the triangle inequality to obtain the third formula. □

In the following discussion we assume that $f:[0,1]\to\mathbb{R}$ has a continuous derivative of order n, for some $n\geq 1$. In this case we can use remainders $\tau_n^1(f)$ and $\tau_n^2(f)$ as

$$\tau_n^1(f)=\frac{1}{6(n!)}\int_0^1 G_n(s)f^{(n)}(s)\mathrm{d}s \tag{3.59}$$

and

$$\tau_n^2(f)=\frac{1}{6(n!)}\int_0^1 F_n(s)f^{(n)}(s)\mathrm{d}s. \tag{3.60}$$

Lemma 3.5 *If $f:[0,1]\to\mathbb{R}$ is such that $f^{(2r)}$ is continuous on $[0,1]$, for some $r\geq 2$, then there exists a point $\eta\in[0,1]$ such that*

$$\tau_{2r}^2(f)=\frac{1}{3(2r)!}2^{-2r}(1-4\cdot 2^{-2r})B_{2r}f^{(2r)}(\eta). \tag{3.61}$$

Proof. Using (3.60) with $n=2r$, we can rewrite $\tau_{2r}^2(f)$ as

$$\tau_{2r}^2(f)=(-1)^{r-1}\frac{1}{6(2r)!}J_r, \tag{3.62}$$

where

$$J_r=\int_0^1(-1)^{r-1}F_{2r}(s)f^{(2r)}(s)\mathrm{d}s. \tag{3.63}$$

From Corollary 3.6 it follows that $(-1)^{r-1}F_{2r}(s)\geq 0,\ o\leq s\leq 1$ and the claim follows from the mean value theorem for integrals and Corollary 3.81. □

Now, we prove the main result:

Theorem 3.7 *Assume $f:[0,1]\to\mathbb{R}$ is such that $f^{(2r)}$ is continuous on $[0,1]$, for some $r\geq 2$. If f is $(2r)$-convex function, then for even r we have*

$$\begin{aligned}0 &\leq \int_0^1 f(t)\mathrm{d}t-\frac{1}{3}\left[2f\left(\frac{1}{4}\right)-f\left(\frac{1}{2}\right)+2f\left(\frac{3}{4}\right)\right]-T_{r-1}^D(f)\\ &\leq \frac{1}{6}\left[f(0)+4f\left(\frac{1}{2}\right)+f(1)\right]+T_{r-1}^S-\int_0^1 f(t)\mathrm{d}t,\end{aligned} \tag{3.64}$$

while for odd r we have reversed inequalities in (3.64).

Proof. Let us denote by LHS and RHS respectively the left hand side and the right hand side in the second inequality in (3.64). Then we have

$$LHS=\rho_{2r}^2(f)$$

and

$$RHS-LHS=-2\tau_{2r}^2(f),$$

For dual Euler-Simpson's formula is proved that, under given assumption on f, there exists a point $\xi \in [0,1]$ such that

$$\rho_{2r}^{2}(f) = -\frac{1}{3(2r)!}\left(1-2\cdot 2^{-2r}\right)(1-4\cdot 2^{-2r})B_{2r}f^{(2r)}(\xi). \tag{3.65}$$

Also by Lemma 3.5 we know that

$$-2\tau_{2r}^{2}(f) = -\frac{2}{3(2r)!}2^{-2r}(1-4\cdot 2^{-2r})B_{2r}f^{(2r)}(\eta), \tag{3.66}$$

for some point $\eta \in [0,1]$. Finally, we know that [1, 23.1.15]

$$(-1)^{r-1}B_{2r} > 0,\ r = 1,2,\cdots. \tag{3.67}$$

Now, if f is $(2r)$-convex function, then $f^{(2r)}(\xi) \geq 0$ and $f^{(2r)}(\eta) \geq 0$ so that using (3.65), (3.66) and (3.67) we get the inequalities

$$LHS \geq 0,\ RHS - LHS \geq 0,\ \text{when } r \text{ is even;}$$

$$LHS \leq 0,\ RHS - LHS \leq 0,\ \text{when } r \text{ is odd.}$$

This proves our assertions. □

Remark 3.7 For $r = 2$ formula (3.61) reduces to

$$\tau_4^2(f) = -\frac{1}{46080}f^{(4)}(\eta).$$

Theorem 3.8 *Assume $f : [0,1] \to \mathbb{R}$ is such that $f^{(2r)}$ is continuous on $[0,1]$, for some $r \geq 2$. If f is either $(2r)$-convex or $(2r)$-concave function, then there exists a point $\vartheta \in [0,1]$ such that*

$$\tau_{2r}^{2}(f) = \vartheta\frac{2}{3(2r)!}2^{-2r}(1-2^{-2r})B_{2r}\left[f^{(2r-1)}(1) - f^{(2r-1)}(0)\right]. \tag{3.68}$$

Proof. With (3.62) and using (3.57) we get (3.68). □

Remark 3.8 If we approximate $\int_0^1 f(t)\mathrm{d}t$ by

$$I_{2r}(f) := D(0,1) + T_{r-1}(f),$$

then the next approximation will be $I_{2r+2}(f)$. The difference

$$\Delta_{2r}(f) = I_{2r+2}(f) - I_{2r}(f)$$

is equal to the last term in $I_{2r+2}(f)$, that is

$$\Delta_{2r}(f) = \frac{1}{3(2r)!}2^{-2r}(1-4\cdot 2^{-2r})B_{2r}\left[f^{(2r-1)}(1) - f^{(2r-1)}(0)\right].$$

We see that, under the assumptions of Theorem 3.8,

$$\tau_{2r}^{2}(f) = \frac{2\vartheta\left(1-2^{-2r}\right)}{1-4\cdot 2^{-2r}}\Delta_{2r}(f)$$

Theorem 3.9 *Assume $f:[0,1]\to\mathbb{R}$ is such that $f^{(2r+2)}$ is continuous on $[0,1]$, for some $r\geq 2$. If f is either $(2r)$-convex and $(2r+2)$-convex or $(2r)$-concave and $(2r+2)$-concave function, then the remainder $\tau_{2r}^{2}(f)$ has the same sign as the first neglected term $\Delta_{2r}(f)$ and*

$$\left|\tau_{2r}^{2}(f)\right|\leq\left|\Delta_{2r}(f)\right|.$$

Proof. We have

$$\Delta_{2r}(f)+\tau_{2r+2}^{2}(f)=\tau_{2r}^{2}(f),$$

that is

$$\Delta_{2r}(f)=\tau_{2r}^{2}(f)-\tau_{2r+2}^{2}(f). \tag{3.69}$$

By (3.60) we have

$$\tau_{2r}^{2}(f)=\frac{1}{6(2r)!}\int_{0}^{1}F_{2r}(s)f^{(2r)}(s)\mathrm{d}s$$

and

$$-\tau_{2r+2}^{2}(f)=\frac{1}{6(2r+2)!}\int_{0}^{1}[-F_{2r+2}(s)]f^{(2r+2)}(s)\mathrm{d}s.$$

It follows that

$$\left|\tau_{2r}^{2}(f)\right|\leq\left|\Delta_{2r}(f)\right| \text{ and } \left|-\tau_{2r+2}^{2}(f)\right|\leq\left|\Delta_{2r}(f)\right|.$$

□

Now, we use Bullen-Simpson formulae of Euler type established in Lemma 3.2 to estimate the absolute value of difference between the absolute value of error in the dual Simpson's quadrature rule and the absolute value of error in the Simpson's quadrature rule. We do this by proving a number of inequalities for various classes of functions.

First, let us denote

$$R_S:=\int_0^1 f(t)\mathrm{d}t-\frac{1}{6}\left[f(0)+4f\left(\frac{1}{2}\right)+f(1)\right]$$

and

$$R_D:=\int_0^1 f(t)\mathrm{d}t-\frac{1}{3}\left[2f\left(\frac{1}{4}\right)-f\left(\frac{1}{2}\right)+2f\left(\frac{3}{4}\right)\right].$$

By the triangle inequality we have

$$||R_D|-|R_S||\leq|R_D+R_S|.$$

Now, if we define $R:=R_D+R_S$, then

$$\begin{aligned}\frac{R}{2}&=\int_0^1 f(t)\mathrm{d}t-\frac{1}{12}\left[f(0)+4f\left(\frac{1}{4}\right)+2f\left(\frac{1}{2}\right)+4f\left(\frac{3}{4}\right)+f(1)\right]\\&=\int_0^1 f(t)\mathrm{d}t-D(0,1).\end{aligned} \tag{3.70}$$

Theorem 3.10 *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n-1)}$ is an L-Lipschitzian function on $[0,1]$ for some $n \geq 1$.*
If $n = 2r-1$, $r \geq 2$, then

$$\left| \int_0^1 f(t)\mathrm{d}t - D(0,1) - T_{r-1}(f) \right| \leq \frac{1}{6(2r-1)!} \int_0^1 |G_{2r-1}(t)|\mathrm{d}t \cdot L$$
$$= \frac{8 \cdot 2^{-2r}(1-2^{-2r})}{3(2r)!} |B_{2r}| \cdot L. \tag{3.71}$$

If $n = 2r$, $r \geq 2$, then

$$\left| \int_0^1 f(t)\mathrm{d}t - D(0,1) - T_{r-1}(f) \right| \leq \frac{1}{6(2r)!} \int_0^1 |F_{2r}(t)|\mathrm{d}t \cdot L$$
$$= \frac{2^{-2r}(1-4 \cdot 2^{-2r})}{3(2r)!} |B_{2r}| \cdot L \tag{3.72}$$

and also

$$\left| \int_0^1 f(t)\mathrm{d}t - D(0,1) - T_r(f) \right| \leq \frac{1}{6(2r)!} \int_0^1 |G_{2r}(t)|\mathrm{d}t \cdot L$$
$$\leq \frac{2 \cdot 2^{-2r}(1-4 \cdot 2^{-2r})}{3(2r)!} |B_{2r}| \cdot L. \tag{3.73}$$

Proof. For any integrable function $\Phi : [0,1] \to \mathbb{R}$ we have

$$\left| \int_0^1 \Phi(t)\mathrm{d}f^{(n-1)}(t) \right| \leq \int_0^1 |\Phi(t)|\,\mathrm{d}t \cdot L, \tag{3.74}$$

since $f^{(n-1)}$ is L-Lipschitzian function. Applying (3.74) with $\Phi(t) = G_{2r-1}(t)$, we get

$$\left| \frac{1}{6(2r-1)!} \int_0^1 G_{2r-1}(t)\mathrm{d}f^{(2r-2)}(t) \right| \leq \frac{1}{6(2r-1)!} \int_0^1 |G_{2r-1}(t)|\,\mathrm{d}t \cdot L.$$

Applying the above inequality and the identity (3.46), we get the inequality in (3.71). Similarly, we can apply the inequality (3.74) with $\Phi(t) = F_{2r}(t)$ and again the identity (3.46) to get the inequality in (3.72). Finally, applying (3.74) with $\Phi(t) = G_{2r}(t)$ and the identity (3.45), we get the first inequality in (3.73). The equalities in (3.71) and (3.72) and the second inequality in (3.73) follow from Corollary 3.7. □

Corollary 3.8 *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n-1)}$ is an L-Lipschitzian function on $[0,1]$ for some $n \geq 1$.*
If $n = 2r-1$, $r \geq 2$, then

$$|R - 2T_{r-1}(f)| \leq \frac{1}{3(2r-1)!} \int_0^1 |G_{2r-1}(t)|\mathrm{d}t \cdot L = \frac{16 \cdot 2^{-2r}(1-2^{-2r})}{3(2r)!} |B_{2r}| \cdot L. \tag{3.75}$$

If $n = 2r$, $r \geq 2$, then

$$|R - 2T_{r-1}(f)| \leq \frac{1}{3(2r)!}\int_0^1 |F_{2r}(t)|\mathrm{d}t \cdot L = \frac{2 \cdot 2^{-2r}(1 - 4 \cdot 2^{-2r})}{3(2r)!}|B_{2r}| \cdot L \tag{3.76}$$

and also

$$|R - 2T_r(f)| \leq \frac{1}{3(2r)!}\int_0^1 |G_{2r}(t)|\mathrm{d}t \cdot L \leq \frac{4 \cdot 2^{-2r}(1 - 4 \cdot 2^{-2r})}{3(2r)!}|B_{2r}| \cdot L. \tag{3.77}$$

Proof. Follows from Theorem 3.10 and (3.70). □

Corollary 3.9 *Let $f : [0,1] \to \mathbb{R}$ be such that f'' is L-Lipschitzian on $[0,1]$. Then*

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1)\right| \leq \frac{1}{4608}L, \ |R| \leq \frac{1}{2304}L.$$

If f''' is L-Lipschitzian on $[0,1]$, then

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1)\right| \leq \frac{1}{46080}L, \ |R| \leq \frac{1}{23040}L.$$

Proof. The first pair of inequalities follow from (3.71)and (3.75) with $r = 2$, while the second pair follow from (3.72) and (3.76) with $r = 2$. □

Remark 3.9 If f is L-Lipschitzian on $[0,1]$, then, as above

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1)\right| \leq \frac{1}{6}\int_0^1 |G_1(t)|\mathrm{d}t \cdot L.$$

Since

$$\int_0^1 |G_1(t)|\mathrm{d}t = \frac{5}{12},$$

we get

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1)\right| \leq \frac{5}{72} \cdot L \text{ and } |R| \leq \frac{5}{36} \cdot L.$$

If f' is L-Lipschitzian on $[0,1]$, then

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1)\right| \leq \frac{1}{12}\int_0^1 |F_2(t)|\mathrm{d}t \cdot L.$$

Since

$$\int_0^1 |F_2(t)|\mathrm{d}t = \frac{1}{27},$$

we get

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1)\right| \leq \frac{1}{324} \cdot L \text{ and } |R| \leq \frac{1}{162} \cdot L.$$

Remark 3.10 Suppose that $f:[0,1]\to\mathbb{R}$ is such that $f^{(n)}$ exists and is bounded on $[0,1]$, for some $n\geq 1$ In this case we have for all $t,s\in[0,1]$

$$\left|f^{(n-1)}(t)-f^{(n-1)}(s)\right|\leq\|f^{(n)}\|_\infty\cdot|t-s|\,,$$

which means that $f^{(n-1)}$ is $\|f^{(n)}\|_\infty$-Lipschitzian function on $[0,1]$. Therefore, the inequalities established in Theorem 3.10 hold with $L=\|f^{(n)}\|_\infty$. Consequently, under appropriate assumptions on f, the inequalities from Corollary 3.9 and Remark 3.9 hold with $L=\|f'\|_\infty,\ \|f''\|_\infty,\ \|f'''\|_\infty,\ \|f^{(4)}\|_\infty$.

Theorem 3.11 *Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$ for some $n\geq 1$.*
If $n=2r-1$, $r\geq 2$, then

$$\left|\int_0^1 f(t)\mathrm{d}t-D(0,1)-T_{r-1}(f)\right|\leq\frac{1}{6(2r-1)!}\max_{t\in[0,1]}|G_{2r-1}(t)|\cdot V_0^1(f^{(2r-2)}).\qquad(3.78)$$

If $n=2r$, $r\geq 2$, then

$$\begin{aligned}\left|\int_0^1 f(t)\mathrm{d}t-D(0,1)-T_{r-1}(f)\right|&\leq\frac{1}{6(2r)!}\max_{t\in[0,1]}|F_{2r}(t)|\cdot V_0^1(f^{(2r-1)})\\&=\frac{2\cdot 2^{-2r}(1-2^{-2r})}{3(2r)!}|B_{2r}|\cdot V_0^1(f^{(2r-1)}).\end{aligned}\qquad(3.79)$$

Also, we have

$$\begin{aligned}\left|\int_0^1 f(t)\mathrm{d}t-D(0,1)-T_r(f)\right|&\leq\frac{1}{6(2r)!}\max_{t\in[0,1]}|G_{2r}(t)|\cdot V_0^1(f^{(2r-1)})\\&=\frac{2^{-2r}(2\cdot 2^{-2r}+1)}{3(2r)!}|B_{2r}|\cdot V_0^1(f^{(2r-1)}).\end{aligned}\qquad(3.80)$$

Here $V_0^1(f^{(n-1)})$ denotes the total variation of $f^{(n-1)}$ on $[0,1]$.

Proof. If $\Phi:[0,1]\to\mathbb{R}$ is bounded on $[0,1]$ and the Riemann-Stieltjes integral $\int_0^1\Phi(t)\mathrm{d}f^{(n-1)}(t)$ exists, then

$$\left|\int_0^1\Phi(t)\mathrm{d}f^{(n-1)}(t)\right|\leq\max_{t\in[0,1]}|\Phi(t)|\cdot V_0^1(f^{(n-1)}).\qquad(3.81)$$

We apply this estimate to $\Phi(t)=G_{2r-1}(t)$ to obtain

$$\left|\frac{1}{6(2r-1)!}\int_0^1 G_{2r-1}(t)\,\mathrm{d}f^{(2r-2)}(t)\right|\leq\frac{1}{6(2r-1)!}\max_{t\in[0,1]}|G_{2r-1}(t)|\cdot V_0^1(f^{(2r-2)}),$$

which is just the inequality (3.78), because of the identity (3.46). Similarly, we can apply the estimate (3.81) with $\Phi(t)=F_{2r}(t)$ and use the identity (3.46) and Corollary 3.6 to obtain (3.79). Finally, (3.80) follows from (3.81) with $\Phi(t)=G_{2r}(t)$, the identity (3.45) and Corollary 3.6. □

Corollary 3.10 *Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$ for some $n\geq 1$.*
If $n=2r-1$, $r\geq 2$, then

$$|R-2T_{r-1}(f)|\leq \frac{1}{3(2r-1)!}\max_{t\in[0,1]}|G_{2r-1}(t)|\cdot V_0^1(f^{(2r-2)}). \tag{3.82}$$

If $n=2r$, $r\geq 2$, then

$$\begin{aligned}|R-2T_{r-1}(f)| &\leq \frac{1}{3(2r)!}\max_{t\in[0,1]}|F_{2r}(t)|\cdot V_0^1(f^{(2r-1)})\\ &= \frac{4\cdot 2^{-2r}(1-2^{-2r})}{3(2r)!}|B_{2r}|\cdot V_0^1(f^{(2r-1)}).\end{aligned} \tag{3.83}$$

Also, we have

$$\begin{aligned}|R-2T_{r}(f)| &\leq \frac{1}{3(2r)!}\max_{t\in[0,1]}|G_{2r}(t)|\cdot V_0^1(f^{(2r-1)})\\ &= \frac{2\cdot 2^{-2r}(2\cdot 2^{-2r}+1)}{3(2r)!}|B_{2r}|\cdot V_0^1(f^{(2r-1)}).\end{aligned} \tag{3.84}$$

Proof. Follows from Theorem 3.11 and (3.70). □

Corollary 3.11 *Let $f:[0,1]\to\mathbb{R}$ be such that f'' is a continuous function of bounded variation on $[0,1]$. Then*

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1)\right|\leq \frac{1}{2592}V_0^1(f''),\;\; |R|\leq\frac{1}{1296}V_0^1(f'').$$

If f''' is a continuous function of bounded variation on $[0,1]$, then

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1)\right|\leq \frac{1}{18432}V_0^1(f'''),\;\; |R|\leq\frac{1}{9216}V_0^1(f''').$$

Proof. From (3.51), we get

$$\max_{t\in[0,1]}|G_3(t)|=\frac{1}{72},$$

so that the first pair of inequalities follow from (3.78) and (3.82) with $r=2$. The second pair of inequalities follow from (3.79) and (3.83) with $r=2$. □

Remark 3.11 If f is a continuous function of bounded variation on $[0,1]$, then, as above

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1)\right|\leq \frac{1}{6}\max_{t\in[0,1]}|G_1(t)|\cdot V_0^1(f).$$

Since

$$\max_{t\in[0,1]}|G_1(t)|=1,$$

we get

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1)\right| \le \frac{1}{6}\cdot V_0^1(f) \text{ and } |R| \le \frac{1}{3}\cdot V_0^1(f).$$

If f' is a continuous function of bounded variation on $[0,1]$, then

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1)\right| \le \frac{1}{12}\max_{t\in[0,1]}|F_2(t)|\cdot V_0^1(f').$$

Since

$$\max_{t\in[0,1]}|F_2(t)| = \frac{1}{8},$$

we get

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1)\right| \le \frac{1}{96}\cdot V_0^1(f) \text{ and } |R| \le \frac{1}{48}\cdot V_0^1(f).$$

Remark 3.12 Suppose that $f:[0,1]\to\mathbb{R}$ is such that $f^{(n)}\in L_1[0,1]$ for some $n\ge 1$ In this case $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$ and we have

$$V_0^1(f^{(n-1)}) = \int_0^1 \left|f^{(n)}(t)\right|\mathrm{d}t = \|f^{(n)}\|_1,$$

Therefore, the inequalities established in Theorem 3.11 hold with $\|f^{(n)}\|_1$ in place of $V_0^1(f^{(n-1)})$. However, a similar observation can be made for the results of Corollary 3.11 and Remark 3.11.

Theorem 3.12 *Assume (p,q) is a pair of conjugate exponents, that is $1<p,q<\infty$, $\frac{1}{p}+\frac{1}{q}=1$ or $p=\infty$, $q=1$. Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(n)}\in L_p[0,1]$ for some $n\ge 1$. If $n=2r-1$, $r\ge 1$, then*

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1) - T_{r-1}(f)\right| \le K(2r-1,p)\|f^{(2r-1)}\|_p. \tag{3.85}$$

If $n=2r$, $r\ge 1$, then

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1) - T_{r-1}(f)\right| \le K^*(2r,p)\|f^{(2r)}\|_p. \tag{3.86}$$

Also, we have

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1) - T_r(f)\right| \le K(2r,p)\|f^{(2r)}\|_p. \tag{3.87}$$

Here

$$K(n,p) = \frac{1}{6n!}\left[\int_0^1 |G_n(t)|^q\,\mathrm{d}t\right]^{\frac{1}{q}}$$

and

$$K^*(n,p) = \frac{1}{6n!}\left[\int_0^1 |F_n(t)|^q\,\mathrm{d}t\right]^{\frac{1}{q}}.$$

Proof. Applying the Hölder inequality we have

$$\begin{aligned}&\left|\frac{1}{6(2r-1)!}\int_0^1 G_{2r-1}(t)f^{(2r-1)}(t)\mathrm{d}t\right|\\ &\leq \frac{1}{6(2r-1)!}\left[\int_0^1 |G_{2r-1}(t)|^q \,\mathrm{d}t\right]^{\frac{1}{q}}\cdot\left\|f^{(2r-1)}\right\|_p = K(2r-1,p)\|f^{(2r-1)}\|_p\end{aligned}$$

The above estimate is just (3.85), by the identity (3.47). The inequalities (3.86) and (3.87) are obtained in the same manner from (3.46) and (3.45), respectively. □

Corollary 3.12 *Assume (p,q) is a pair of conjugate exponents, that is $1<p,q<\infty$, $\frac{1}{p}+\frac{1}{q}=1$ or $p=\infty$, $q=1$. Let $f:[0,1]\to\mathbb{R}$ be integrable function such that $f^{(n)}\in L_p[0,1]$ for some $n\geq 1$.*
If $n=2r-1$, $r\geq 1$, then

$$|R-2T_{r-1}(f)|\leq 2K(2r-1,p)\|f^{(2r-1)}\|_p.$$

If $n=2r$, $r\geq 1$, then

$$|R-2T_{r-1}(f)|\leq 2K^*(2r,p)\|f^{(2r)}\|_p.$$

Also, we have

$$|R-2T_r(f)|\leq 2K(2r,p)\|f^{(2r)}\|_p.$$

Proof. Follows from Theorem 3.12 and (3.70). □

Remark 3.13 Note that $K^*(1,p)=K(1,p)$, for $1<p\leq\infty$, since $G_1(t)=F_1(t)$. Also, for $1<p\leq\infty$ we can easily calculate $K(1,p)$. We get

$$K(1,p)=\frac{1}{6}\left[\frac{2+2^{-q}}{3(1+q)}\right]^{\frac{1}{q}},\ 1<p\leq\infty.$$

At the end of this section we prove an interesting Grüss type inequality related to Bullen-Simpson's identity (3.45). To do this we use the following variant of the key technical result from the paper [83]:

Lemma 3.6 *Let $F,G:[0,1]\to\mathbb{R}$ be two integrable functions. If, for some constants $m,M\in\mathbb{R}$*

$$m\leq F(t)\leq M,\ \ 0\leq t\leq 1$$

and

$$\int_0^1 G(t)\mathrm{d}t=0,$$

then

$$\left|\int_0^1 F(t)G(t)\mathrm{d}t\right|\leq\frac{M-m}{2}\int_0^1|G(t)|\,\mathrm{d}t. \tag{3.88}$$

Theorem 3.13 *Suppose that $f : [0,1] \to \mathbb{R}$ is such that $f^{(n)}$ exists and is integrable on $[0,1]$, for some $n \geq 1$. Assume that*

$$m_n \leq f^{(n)}(t) \leq M_n,\ 0 \leq t \leq 1,$$

for some constants m_n and M_n. Then

$$\left| \int_0^1 f(t)\mathrm{d}t - D(0,1) - T_k(f) \right| \leq \frac{1}{12\,(n!)} C_n (M_n - m_n), \tag{3.89}$$

where $k = \left[\frac{n}{2}\right]$ and

$$C_n = \int_0^1 |G_n(t)|\,\mathrm{d}t,\ n \geq 1.$$

Moreover, if $n = 2r-1$, $r \geq 2$, then

$$\left| \int_0^1 f(t)\mathrm{d}t - D(0,1) - T_{r-1}(f) \right| \leq \frac{4 \cdot 2^{-2r}\left(1 - 2^{-2r}\right)}{3\,(2r)!} |B_{2r}|\,(M_{2r-1} - m_{2r-1}). \tag{3.90}$$

Proof. We can rewrite the identity (3.45) in the form

$$\int_0^1 f(t)\mathrm{d}t - D(0,1) - T_k(f) = \frac{1}{6(n!)} \int_0^1 F(t)G(t)\mathrm{d}t, \tag{3.91}$$

where

$$F(t) = f^{(n)}(t),\ G(t) = G_n(t),\ 0 \leq t \leq 1.$$

In [27, Lemma 2 (i)] it was proved that for all $n \geq 1$ and for every $\gamma \in \mathbb{R}$

$$\int_0^1 B_n^*(\gamma - t)\mathrm{d}t = 0,$$

so that we have

$$\begin{aligned} &\int_0^1 G(t)\mathrm{d}t \\ &= \int_0^1 \left[B_n(1-t) + 2B_n^*\left(\frac{1}{4} - t\right) + B_n^*\left(\frac{1}{2} - t\right) + 2B_n^*\left(\frac{3}{4} - t\right) \right] \mathrm{d}t = 0. \end{aligned}$$

Thus, we can apply (3.88) to the integral in the right hand side of (3.91) and (3.89) follows immediately. The inequality (3.90) follows from (3.89) and Corollary 3.7. □

Remark 3.14 For $n = 1$ and $n = 2$ we have already calculated

$$C_1 = \int_0^1 |G_1(t)|\mathrm{d}t = \frac{5}{12},\ C_2 = \int_0^1 |G_2(t)|\mathrm{d}t = \frac{1}{27},$$

so that we have

$$\left| \int_0^1 f(t)\mathrm{d}t - D(0,1) \right| \leq \frac{5}{144}(M_1 - m_1)$$

and

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1)\right| \le \frac{1}{648}(M_2 - m_2).$$

For $n = 3$ we apply (3.90) with $r = 2$ to get the inequality

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1)\right| \le \frac{1}{9216}(M_3 - m_3).$$

3.2 Corrected 3-point quadrature formulae

The aim of this section is to derive general 3-point quadrature formulae with a degree of exactness higher than that which the formulae from the previous section had. Observe identity (3.1) again. Imposing the condition $G_2(x,0) = 0$ led us to a family of quadrature formulae with a degree of exactness equal to 3 (assuming the values of the derivatives of order 3 or higher are not included in the quadrature; otherwise, an arbitrary degree of exactness can be achieved).

Now, impose a condition $G_4(x,0) = 0$ (for function $G_k(x,t)$ defined in (3.3) and $x \in [0,1/2)$). Formulae thus obtained will include the values of the first derivative at the end points of the interval and we will call them corrected quadrature formulae. This name was first introduced in [117]. The weight is now of the form:

$$w_c(x) = \frac{7}{30(2x-1)^2(-4x^2+4x+1)}\,. \tag{3.92}$$

Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(2n-1)}$ is continuous of bounded variation on $[0,1]$ for some $n \ge 1$. Similarly as before, we now obtain:

$$\int_0^1 f(t)dt - Q_C\left(x,\, \frac{1}{2},\, 1-x\right) + T_{2n}^{CQ3}(x) = \frac{1}{(2n)!}\int_0^1 G_{2n}^{CQ3}(x,t)df^{(2n-1)}(t). \tag{3.93}$$

If $f^{(2n)}$ is continuous of bounded variation on $[0,1]$ for some $n \ge 0$, then:

$$\int_0^1 f(t)dt - Q_C\left(x,\, \frac{1}{2},\, 1-x\right) + T_{2n}^{CQ3}(x) = \frac{1}{(2n+1)!}\int_0^1 G_{2n+1}^{CQ3}(x,t)df^{(2n)}(t), \tag{3.94}$$

and finally, if $f^{(2n+1)}$ is continuous of bounded variation on $[0,1]$ for some $n \ge 0$, then:

$$\int_0^1 f(t)dt - Q_C\left(x,\, \frac{1}{2},\, 1-x\right) + T_{2n}^{CQ3}(x) = \frac{1}{(2n+2)!}\int_0^1 F_{2n+2}^{CQ3}(x,t)df^{(2n+1)}(t), \tag{3.95}$$

where

$$
\begin{aligned}
& Q_C\left(x,\, \frac{1}{2},\, 1-x\right) \qquad (3.96)\\
&= \frac{1}{30(2x-1)^2(-4x^2+4x+1)}\left[7f(x) - 480B_4(x)f\left(\frac{1}{2}\right) + 7f(1-x)\right]\\
& T_{2n}^{CQ3}(x) = \sum_{k=1}^{n} \frac{1}{(2k)!}\, G_{2k}^{CQ3}(x,0)\, [f^{(2k-1)}(1) - f^{(2k-1)}(0)]\\
& \qquad = \frac{10x^2-10x+1}{60(-4x^2+4x+1)}[f'(1)-f'(0)] \qquad (3.97)\\
& \qquad + \sum_{k=3}^{n} \frac{1}{(2k)!}\, G_{2k}^{CQ3}(x,0)\, [f^{(2k-1)}(1) - f^{(2k-1)}(0)]\\
& G_k^{CQ3}(x,t) = w_c(x)[B_k^*(x-t) + B_k^*(1-x-t)] + (1-2w_c(x))B_k^*(1/2-t),\\
& F_k^{CQ3}(x,t) = G_k^{CQ3}(x,t) - G_k^{CQ3}(x,0), \quad k \geq 1, \quad t \in \mathbb{R},
\end{aligned}
$$

and $w_c(x)$ as in (3.92).

What follows is a lemma which is, similarly as before, key for obtaining the rest of the results in this section.

Lemma 3.7 *For $x \in \left[0, \frac{1}{2} - \frac{\sqrt{15}}{10}\right] \cup \left[\frac{1}{6}, \frac{1}{2}\right)$ and $k \geq 2$, $G_{2k+1}^{CQ3}(x,t)$ has no zeros in variable t on $(0,1/2)$. The sign of the function is determined by:*

$$
\begin{aligned}
&(-1)^{k+1} G_{2k+1}^{CQ3}(x,t) > 0 \quad \text{for } x \in [0, 1/2 - \sqrt{15}/10],\\
&(-1)^{k} G_{2k+1}^{CQ3}(x,t) > 0 \quad \text{for } x \in [1/6, 1/2).
\end{aligned}
$$

Proof. We start from G_5^{CQ3} and claim that $G_5^{CQ3}(x,t)$ has constant sign for $x \in \left[0, \frac{1}{2} - \frac{\sqrt{15}}{10}\right] \cup \left[\frac{1}{6}, \frac{1}{2}\right)$. We will show that it is increasing in x for $x \in [0,1/2)$ and after considering its behavior at the end points, the claim will follow. For $0 \leq t \leq x < 1/2$, we have

$$
\frac{\partial G_5^{CQ3}}{\partial x}(x,t) = \frac{t^3}{3} \cdot \frac{14(1-2x)}{(4x^2-4x-1)^2} > 0
$$

so our statement is true. For $0 \leq x \leq t \leq 1/2$, we have

$$
\frac{\partial G_5^{CQ3}}{\partial x}(x,t) = \frac{14x \cdot (1-2t)}{3(1-2x)^3(4x^2-4x-1)^2} \cdot g(x,t),
$$

where $g(x,t) = 4t^3(x-1) + t^2(-8x^3+4x^2+3) + tx(8x^2-4x-3) + x^2 + 2x^3 - 4x^4$. The zeros of $\frac{\partial g}{\partial t}(x,t)$ are $t_1 = 1/2$ and $t_2 = \frac{8x^3-4x^2-3x}{6(x-1)}$ and it is elementary to see that $t_2 < x$. Also, it is very simple to check that $g(x,0) > 0$ and $g(x,1/2) > 0$, so we have $g(x,t) > 0$. Thus, it follows that $G_5^{CQ3}(x,t)$ is increasing in x.

Since $G_5^{CQ3}\left(\frac{1}{2}-\frac{\sqrt{15}}{10},t\right)<0$ (cf. [59]) and $G_5^{CQ3}\left(\frac{1}{6},t\right)>0$ (cf. [55]) for $t\in(0,1/2)$, we see that $G_5^{CQ3}(x,t)<0$ for $x\in\left[0,\frac{1}{2}-\frac{\sqrt{15}}{10}\right]$ and $G_5^{CQ3}(x,t)>0$ for $x\in\left[\frac{1}{6},\frac{1}{2}\right)$. So, $G_5^{CQ3}(x,t)$ has constant sign, for an adequate choice of x, and our statement is therefore valid for $k=2$. Assuming the opposite, the assertion follows for all $k\geq 3$ by induction.

Knowing the sign of $G_5^{CQ3}(x,t)$ allows us to see whether $G_7^{CQ3}(x,t)$ is convex or concave on $(0,1/2)$. As it has no zeros there, that is enough to determine its sign. With this procedure we can determine the sign of $G_{2k+1}^{CQ3}(x,t)$ for $k\geq 4$ which completes the proof. □

Remark 3.15 Note that for $x\in\left(\frac{1}{2}-\frac{\sqrt{15}}{10},\frac{1}{6}\right)$, $G_5^{CQ3}(x,t)$ has at least one zero in t on $(0,1/2)$. Namely, we have

$$G_5^{CQ3}(x,0)=\frac{\partial G_5^{CQ3}}{\partial t}(x,0)=\frac{\partial^2 G_5^{CQ3}}{\partial t^2}(x,0)=G_5^{CQ3}(x,1/2)=0 \tag{3.98}$$

and it is not difficult to see that

$$x\in\left(\frac{1}{2}-\frac{\sqrt{15}}{10},\frac{1}{6}\right)\ \Leftrightarrow\ \frac{\partial^3 G_5^{CQ3}}{\partial t^3}(x,0)>0\ \ \&\ \ \frac{\partial G_5^{CQ3}}{\partial t}(x,1/2)>0.$$

From $\frac{\partial G_5^{CQ3}}{\partial t}(x,1/2)>0$ we conclude that $\frac{\partial G_5^{CQ3}}{\partial t}(x,t)>0$ in some neighborhood of $t=1/2$, and then from $G_5^{CQ3}(x,1/2)=0$ it follows that $G_5^{CQ3}(x,t)<0$ in that neighborhood. On the other hand, from $\frac{\partial^3 G_5^{CQ3}}{\partial t^3}(x,0)>0$ we conclude that $\frac{\partial^3 G_5^{CQ3}}{\partial t^3}(x,t)>0$ in some neighborhood of $t=0$. Similarly as above, using (3.98), we conclude that $G_5^{CQ3}(x,t)>0$ in this neighborhood of 0. Therefore, it is now clear that for $x\in\left(\frac{1}{2}-\frac{\sqrt{15}}{10},\frac{1}{6}\right)$, $G_5^{CQ3}(x,t)$ has at least one zero in $(0,1/2)$.

Remark 3.16 Similarly as for the function F_{2k+2}^{Q3}, $k\geq 1$ in the previous section, we conclude that for $k\geq 2$ and $x\in[0,1/2-\sqrt{15}/10]$, the function $(-1)^k F_{2k+2}^{CQ3}(x,t)$ is strictly increasing in variable t on $(0,1/2)$ and strictly decreasing on $(1/2,1)$. Since $F_{2k+2}^{CQ3}(x,0)$ $=F_{2k+2}^{CQ3}(x,1)=0$, it has constant sign on $(0,1)$ and attains its maximal value at $t=1/2$. Analogous statement, but with the opposite sign, is valid in the case when $x\in[1/6,1/2)$.

Denote by $R_{2n+2}^{CQ3}(x,f)$ the right-hand side of (3.95) and let the weight $w_c(x)$ be as in (3.92).

Theorem 3.14 *Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(2n+2)}$ is continuous on $[0,1]$ for some $n\geq 3$ and let $x\in\left[0,\frac{1}{2}-\frac{\sqrt{15}}{10}\right]\cup\left[\frac{1}{6},\frac{1}{2}\right)$. If $f^{(2n)}$ and $f^{(2n+2)}$ have the same constant sign on $[0,1]$, then the remainder $R_{2n}^{CQ3}(x,f)$ has the same sign as the first neglected term $\Delta_{2n}^{CQ3}(x,f)$ where*

$$\Delta_{2n}^{CQ3}(x,f):=R_{2n}^{CQ3}(x,f)-R_{2n+2}^{CQ3}(x,f)=-\frac{1}{(2n)!}G_{2n}^{CQ3}(x,0)[f^{(2n-1)}(1)-f^{(2n-1)}(0)].$$

Furthermore, $|R_{2n}^{CQ3}(x,f)| \leq |\Delta_{2n}^{CQ3}(x,f)|$ *and* $|R_{2n+2}^{CQ3}(x,f)| \leq |\Delta_{2n}^{CQ3}(x,f)|$.

Proof. Analogous to the proof of Theorem 3.1. □

Theorem 3.15 *Let* $f:[0,1]\to\mathbb{R}$ *be such that* $f^{(2n+2)}$ *is continuous on* $[0,1]$ *for some* $n\geq 2$ *and let* $x \in \left[0, \frac{1}{2}-\frac{\sqrt{15}}{10}\right] \cup \left[\frac{1}{6},\frac{1}{2}\right)$. *Then there exists* $\xi \in [0,1]$ *such that*

$$R_{2n+2}^{CQ3}(x,f) = -\frac{G_{2n+2}^{CQ3}(x,0)}{(2n+2)!} f^{(2n+2)}(\xi). \tag{3.99}$$

where

$$G_{2n+2}^{CQ3}(x,0) = 2w_c(x)\left[B_{2n+2}(x) + (1-2^{-2n-1})B_{2n+2}\right] - (1-2^{-2n-1})B_{2n+2}. \tag{3.100}$$

If, in addition, $f^{(2n+2)}$ *has constant sign on* $[0,1]$, *then there exists* $\theta \in [0,1]$ *such that*

$$R_{2n+2}^{CQ3}(x,f) = \frac{\theta}{(2n+2)!} F_{2n+2}^{CQ3}\left(x, \frac{1}{2}\right)\left[f^{(2n+1)}(1) - f^{(2n+1)}(0)\right] \tag{3.101}$$

where

$$\begin{aligned} F_{2n+2}^{CQ3}(x,1/2) \; &= \; 2w_c(x)\left[B_{2n+2}(1/2-x) - B_{2n+2}(x) - \left(2-2^{-2n-1}\right)B_{2n+2}\right] \\ &\quad + \left(2-2^{-2n-1}\right)B_{2n+2}. \end{aligned} \tag{3.102}$$

Proof. Analogous to the proof of Theorem 3.2. □

When (3.99) is applied to the remainder in (3.95) for $n=2$, the following formula is produced:

$$\begin{aligned} \int_0^1 f(t)dt - Q_C\left(x, \frac{1}{2}, 1-x\right) + \frac{10x^2-10x+1}{60(-4x^2+4x+1)}[f'(1)-f'(0)] \\ = \frac{98x^4-196x^3+102x^2-4x-1}{604800(4x^2-4x-1)} f^{(6)}(\xi). \end{aligned} \tag{3.103}$$

For $x=0$, formula (3.103) becomes corrected Simpson's formula, for $x=1/4$ corrected dual Simpson's formula and for $x=1/6$ corrected Maclaurin's formula. Furthermore, for $x=\frac{5-\sqrt{15}}{10}$, (3.103) becomes the classical Gauss 3-point formula (stated on $[0,1]$), while for $x=\frac{15-\sqrt{15(15-2\sqrt{30})}}{30}$ the corrected Gauss 2-point formula is produced. These quadrature formulae will be the main topics of the sections that follow.

Note that $x=\frac{5-\sqrt{15}}{10}$ is the unique solution of the equation $G_2(x,0)=0$. In fact, the nodes and coefficients of the Gauss 3-point formula are the unique solution of the system:

$$G_2(x,0) = G_4(x,0) = 0.$$

This would be the system one would set to obtain from (3.1) the quadrature formula which is not corrected (i.e. does not include values of the first derivatives in the quadrature) and has the highest possible degree of exactness, which is 5 in this case. That this is exactly the classical Gauss 3-point formula is no surprise.

Remark 3.17 Although only $x \in [0,1/2)$ were taken into consideration here, results for $x = 1/2$ can easily be obtained by considering the limit process when x tends to 1/2. Namely,

$$\lim_{x \to 1/2} Q_C\left(x, \frac{1}{2}, 1-x\right) = -f\left(\frac{1}{2}\right) + \frac{7}{240} f''\left(\frac{1}{2}\right)$$

$$\lim_{x \to 1/2} G_k^{CQ3}(x,t) = B_k^*\left(\frac{1}{2} - t\right) + \frac{7}{240} k(k-1) B_{k-2}^*\left(\frac{1}{2} - t\right)$$

Consequently, from (4.14) it follows:

$$\int_0^1 f(t)dt + f\left(\frac{1}{2}\right) - \frac{7}{240} f''\left(\frac{1}{2}\right) - \frac{1}{80}[f'(1) - f'(0)] = -\frac{11}{3225600} f^{(6)}(\xi). \quad (3.104)$$

Next, some sharp estimates of error for this type of quadrature formulae are given.

Theorem 3.16 *Let $p,q \in \mathbb{R}$ be such that $1 \le p,\ q \le \infty$ and $1/p + 1/q = 1$. If $f : [0,1] \to \mathbb{R}$ is such that $f^{(2n)} \in L_p[0,1]$ for some $n \ge 1$, then we have*

$$\left| \int_0^1 f(t)dt - Q_C\left(x,\ \frac{1}{2},\ 1-x\right) + T_{2n}^{CQ3}(x) \right| \le K_{CQ3}(2n,q) \|f^{(2n)}\|_p. \quad (3.105)$$

If $f^{(2n+1)} \in L_p[0,1]$ for some $n \ge 0$, then we have

$$\left| \int_0^1 f(t)dt - Q_C\left(x,\ \frac{1}{2},\ 1-x\right) + T_{2n}^{CQ3}(x) \right| \le K_{CQ3}(2n+1,q) \|f^{(2n+1)}\|_p. \quad (3.106)$$

If $f^{(2n+2)} \in L_p[0,1]$ for some $n \ge 0$, then we have

$$\left| \int_0^1 f(t)dt - Q_C\left(x,\ \frac{1}{2},\ 1-x\right) + T_{2n}^{CQ3}(x) \right| \le K_{CQ3}^*(2n+2,q) \|f^{(2n+2)}\|_p, \quad (3.107)$$

where

$$K_{CQ3}(m,q) = \frac{1}{m!} \left[\int_0^1 \left| G_m^{CQ3}(x,t) \right|^q dt \right]^{\frac{1}{q}}$$

$$K_{CQ3}^*(m,q) = \frac{1}{m!} \left[\int_0^1 \left| F_m^{CQ3}(x,t) \right|^q dt \right]^{\frac{1}{q}}.$$

These inequalities are sharp for $1 < p \le \infty$ and the best possible for $p = 1$.

Proof. Analogous to the proof of Theorem 2.2. □

For $x \in \left[0, \frac{1}{2} - \frac{\sqrt{15}}{10}\right] \cup \left[\frac{1}{6}, \frac{1}{2}\right)$ and $n \ge 2$, using Lemma 3.7 and Remark 3.16, we can calculate the following constants for $p = 1$ and $p = \infty$ from the previous Theorem:

$$K_{CQ3}^*(2n+2,1) = \frac{1}{(2n+2)!} \left| G_{2n+2}^{CQ3}(x,0) \right|, \quad (3.108)$$

$$K_{CQ3}^*(2n+2,\infty) = \frac{1}{2} K_{CQ3}(2n+1,1) = \frac{1}{(2n+2)!} \left| F_{2n+2}^{CQ3}\left(x, \frac{1}{2}\right) \right| \quad ,(3.109)$$

where $G_{2n+2}^{CQ3}(x,0)$ and $F_{2n+2}^{CQ3}(x,1/2)$ are as in (3.100) and (3.102).

The following two theorems give Hermite-Hadamard and Dragomir-Agarwal type inequalities for the general corrected 3-point quadrature formulae (cf. Section 2.5.).

Theorem 3.17 *Let $f:[0,1]\to\mathbb{R}$ be $(2n+4)$-convex for $n\geq 2$. Then for $x\in\left[0,\frac{1}{2}-\frac{\sqrt{15}}{10}\right]$, we have*

$$\begin{aligned}
&\frac{1}{(2n+2)!}\,|G_{2n+2}^{CQ3}(x,0)|\,f^{(2n+2)}\left(\frac{1}{2}\right)\\
&\leq\ (-1)^n\left(\int_0^1 f(t)dt-Q_C\left(x,\ \frac{1}{2},\ 1-x\right)+T_{2n}^{CQ3}(x)\right) \qquad (3.110)\\
&\leq\ \frac{1}{(2n+2)!}\,|G_{2n+2}^{CQ3}(x,0)|\,\frac{f^{(2n+2)}(0)+f^{(2n+2)}(1)}{2},
\end{aligned}$$

while for $x\in\left[\frac{1}{6},\frac{1}{2}\right)$ we have

$$\begin{aligned}
&\frac{1}{(2n+2)!}\,|G_{2n+2}^{CQ3}(x,0)|\,f^{(2n+2)}\left(\frac{1}{2}\right)\\
&\leq\ (-1)^{n+1}\left(\int_0^1 f(t)dt-Q_C\left(x,\ \frac{1}{2},\ 1-x\right)+T_{2n}^{CQ3}(x)\right) \qquad (3.111)\\
&\leq\ \frac{1}{(2n+2)!}\,|G_{2n+2}^{CQ3}(x,0)|\,\frac{f^{(2n+2)}(0)+f^{(2n+2)}(1)}{2},
\end{aligned}$$

where $G_{2n+2}^{CQ3}(x,0)$ is as in (3.100). If f is $(2n+4)$-concave, the inequalities are reversed.

Proof. Analogous to the proof of Theorem 2.8. □

Theorem 3.18 *Let $x\in\left[0,\frac{1}{2}-\frac{\sqrt{15}}{10}\right]\cup\left[\frac{1}{6},\frac{1}{2}\right)$ and $f:[0,1]\to\mathbb{R}$ be m-times differentiable for $m\geq 5$. If $|f^{(m)}|^q$ is convex for some $q\geq 1$, then*

$$\begin{aligned}
&\left|\int_0^1 f(t)dt-Q_C\left(x,\ \frac{1}{2},\ 1-x\right)+T_{2n}^{CQ3}(x)\right|\\
&\leq\ L_{CQ3}(m,x)\left(\frac{|f^{(m)}(0)|^q+|f^{(m)}(1)|^q}{2}\right)^{1/q} \qquad (3.112)
\end{aligned}$$

while if $|f^{(m)}|$ is concave, then

$$\left|\int_0^1 f(t)dt-Q_C\left(x,\ \frac{1}{2},\ 1-x\right)+T_{2n}^{CQ3}(x)\right|\ \leq\ L_{CQ3}(m,x)\left|f^{(m)}\left(\frac{1}{2}\right)\right|, \qquad (3.113)$$

where

$$\text{for } m=2n+1 \quad L_{CQ3}(2n+1,x)=\frac{2}{(2n+2)!}\,|F_{2n+2}^{CQ3}(x,1/2)|$$

$$\text{and for } m=2n+2 \quad L_{CQ3}(2n+2,x)=\frac{1}{(2n+2)!}\,|G_{2n+2}^{CQ3}(x,0)|$$

with $G_{2n+2}^{CQ3}(x,0)$ *and* $F_{2n+2}^{CQ3}(x,1/2)$ *as in (3.100) and (3.102), respectively.*

Proof. Analogous to the proof of Theorem 3.5. □

3.2.1 Corrected Simpson's formula

One of the special cases of the results from the previous section, obtained for $x=0$, is corrected Simpson's formula. It was introduced for the first time in [117] and [80]. Results of this subsection are in fact a generalization of the results from [117] and were published in [50].

The quadrature formula is in this case:

$$Q_C\left(0,\,\frac{1}{2},\,1\right)=\frac{1}{30}\left[7f(0)+16f\left(\frac{1}{2}\right)+7f(1)\right].$$

Further,

$$\begin{aligned}
T_{2n}^{CS} &= T_{2n}^{CQ3}(0)=\frac{1}{60}[f'(1)-f'(0)] \\
&\quad+\sum_{k=3}^{n}\frac{1}{(2k)!}\,G_{2k}^{CS}(0)\,[f^{(2k-1)}(1)-f^{(2k-1)}(0)] \\
G_k^{CS}(t) &= G_k^{CQ3}(0,t)=\frac{1}{30}\left[14B_k^*(1-t)+16B_k^*\left(\frac{1}{2}-t\right)\right],\quad k\geq 1 \qquad (3.114)\\
F_k^{CS}(t) &= F_k^{CQ3}(0,t)=G_k^{CS}(t)-G_k^{CS}(0),\quad k\geq 2\ \text{ and }\ t\in\mathbb{R}.
\end{aligned}$$

The remainder $R_{2n+2}^{CS}(f)$ on the right-hand side of (3.95) for $x=0$ and $n\geq 2$ can be written, according to Theorem 3.15, as:

$$\begin{aligned}
R_{2n+2}^{CS}(f) &= \frac{\theta}{15(2n+2)!}(2-2^{-1-2n})B_{2n+2}\left[f^{(2n+1)}(1)-f^{(2n+1)}(0)\right],\ \theta\in[0,1]\\
R_{2n+2}^{CS}(f) &= \frac{1}{15(2n+2)!}(1-2^{2-2n})B_{2n+2}f^{(2n+2)}(\eta),\qquad \eta\in[0,1]
\end{aligned}$$

Formula (3.103) becomes:

$$\int_0^1 f(t)dt-\frac{1}{30}\left[7f(0)+16f\left(\frac{1}{2}\right)+7f(1)\right]+\frac{1}{60}[f'(1)-f'(0)]=\frac{1}{604800}\,f^{(6)}(\eta) \tag{3.115}$$

Formula (3.115) is called corrected Simpson's formula.

As special cases of Theorem 3.16, (3.108) and (3.109), for $p=\infty$ and $p=1$ we get the following estimates for $m=1,2$:

$$\left|\int_0^1 f(t)dt-\frac{1}{30}\left[7f(0)+16f\left(\frac{1}{2}\right)+7f(1)\right]\right|\leq C_{CS}(m,q)\|f^{(m)}\|_p,$$

where

$$C_{CS}(1,1) = \frac{113}{900}, \qquad C_{CS}(1,\infty) = \left| G_1^{CS}\left(\frac{1}{2}\right) \right| = \frac{4}{15},$$
$$C_{CS}(2,1) = \frac{697}{40500}, \quad C_{CS}(2,\infty) = \frac{1}{2}\left| F_2^{CS}\left(\frac{7}{30}\right) \right| = \frac{49}{1800},$$

while for $m = 2,3,4,5,6$

$$\left| \int_0^1 f(t)dt - \frac{1}{30}\left[7f(0) + 16f\left(\frac{1}{2}\right) + 7f(1) \right] + \frac{1}{60}[f'(1) - f'(0)] \right| \\ \leq C_{CS}(m,q)\|f^{(m)}\|_p,$$

where

$$C_{CS}(2,1) = \frac{19\sqrt{19}}{10125}, \qquad C_{CS}(2,\infty) = \frac{1}{2}\left| G_2^{CS}\left(\frac{1}{2}\right) \right| = \frac{1}{40},$$
$$C_{CS}(3,1) = \frac{253}{360000}, \qquad C_{CS}(3,\infty) = \frac{28 + 19\sqrt{19}}{81000}$$
$$C_{CS}(4,1) = \frac{1}{14580}, \qquad C_{CS}(4,\infty) = \frac{1}{24}\left| G_4^{CS}\left(\frac{1}{2}\right) \right| = \frac{1}{5760}$$
$$C_{CS}(5,1) = \frac{1}{360}\left| F_6^{CS}\left(\frac{1}{2}\right) \right| = \frac{1}{115200},$$
$$C_{CS}(5,\infty) = \frac{1}{120}\left| G_5^{CS}\left(\frac{1}{3}\right) \right| = \frac{1}{58320}$$
$$C_{CS}(6,1) = \frac{1}{720}\left| G_6^{CS}(0) \right| = \frac{1}{604800},$$
$$C_{CS}(6,\infty) = \frac{1}{720}\left| F_6^{CS}\left(\frac{1}{2}\right) \right| = \frac{1}{230400}.$$

From (3.108) and (3.109), estimates can easily be obtained for $m \geq 7$. However, for those m, the values of higher order derivatives at the end points are included in the quadrature.

The Hermite-Hadamard-type inequality for the corrected Simpson's formula is:

$$\begin{aligned} &\frac{1}{604800} f^{(6)}\left(\frac{1}{2}\right) \\ &\leq \int_0^1 f(t)dt - \frac{1}{30}\left[7f(0) + 16f\left(\frac{1}{2}\right) + 7f(1) \right] + \frac{1}{60}[f'(1) - f'(0)] \\ &\leq \frac{1}{604800} \frac{f^{(6)}(0) + f^{(6)}(1)}{2} \end{aligned}$$

while the constants from Theorem 3.18 are:

$$L_{CQ3}(5,0) = \frac{1}{115200}, \qquad L_{CQ3}(6,0) = \frac{1}{604800}.$$

3.2.2 Corrected dual Simpson's formula

For $x = 1/4$, as a special case corrected dual Euler-Simpson's formulae are obtained. Results of this subsection are published in [52]. We have

$$Q_C\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right) = \frac{1}{15}\left[8f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 8f\left(\frac{3}{4}\right)\right],$$

$$T_{2n}^{CDS} = T_{2n}^{CQ3}\left(\frac{1}{4}\right) = -\frac{1}{120}[f'(1) - f'(0)] + \sum_{k=3}^{n} \frac{1}{(2k)!} G_{2k}^{CDS}(0)\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right]$$

$$G_k^{CDS}(t) = G_k^{CQ3}\left(\frac{1}{4}, t\right) = \frac{1}{15}\left[8B_k^*\left(\frac{1}{4} - t\right) - B_k^*\left(\frac{1}{2} - t\right) + 8B_k^*\left(\frac{3}{4} - t\right)\right],$$

$$F_k^{CDS}(t) = F_k^{CQ3}\left(\frac{1}{4}, t\right) = G_k^{CDS}(t) - G_k^{CDS}(0), \quad k \geq 1 \text{ and } t \in \mathbb{R}.$$

The remainder $R_{2n+2}^{CDS}(f) = R_{2n+2}^{CQ3}(1/4, f)$, according to Theorem 3.15, can be written as:

$$R_{2n+2}^{CDS}(f) = \frac{\theta}{15(2n+2)!}(2^{-1-2n} - 2)B_{2n+2}\left[f^{(2n+1)}(1) - f^{(2n+1)}(0)\right], \ \theta \in [0,1]$$

$$R_{2n+2}^{CDS}(f) = \frac{1}{15(2n+2)!}(9 \cdot 2^{-1-2n} - 2^{1-4n} - 1)B_{2n+2} \cdot f^{(2n+2)}(\eta), \ \ \eta \in [0,1] \tag{3.116}$$

Formula (3.103) takes the form:

$$\int_0^1 f(t)dt - \frac{1}{15}\left[8f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 8f\left(\frac{3}{4}\right)\right] - \frac{1}{120}[f'(1) - f'(0)] = -\frac{31}{19353600} f^{(6)}(\eta) \tag{3.117}$$

Formula (3.117) is called corrected dual Simpson's formula.

Estimate of error for $p = \infty$ i $p = 1$ are for $m = 1, 2$:

$$\left|\int_0^1 f(t)dt - \frac{1}{15}\left[8f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 8f\left(\frac{3}{4}\right)\right]\right| \leq C_{CDS}(m,q)\|f^{(m)}\|_p,$$

where

$$C_{CDS}(1,1) = \frac{17}{120}, \qquad C_{CDS}(1,\infty) = \left|G_1^{CDS}\left(\frac{3}{4}\right)\right| = \frac{17}{60},$$

$$C_{CDS}(2,1) = \frac{31}{3000}, \qquad C_{CDS}(2,\infty) = \frac{1}{2}\left|F_2^{CDS}\left(\frac{1}{4}\right)\right| = \frac{1}{32},$$

while for $m = 2,3,4,5,6$

$$\left|\int_0^1 f(t)dt - \frac{1}{15}\left[8f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 8f\left(\frac{3}{4}\right)\right] - \frac{1}{120}[f'(1) - f'(0)]\right| \leq C_{CDS}(m,q)\|f^{(m)}\|_p,$$

where

$$
\begin{aligned}
C_{CDS}(2,1) &= \frac{31\sqrt{31}+15\sqrt{15}-46}{20250}, \\
C_{CDS}(2,\infty) &= \frac{1}{2}\left|G_2^{CDS}\left(\frac{1}{4}\right)\right| = \frac{11}{480}, \\
C_{CDS}(3,1) &= \frac{11}{14400}, \qquad C_{CDS}(3,\infty) = \frac{31\sqrt{31}-46}{81000}, \\
C_{CDS}(4,1) &= \frac{17}{233280}, \qquad C_{CDS}(4,\infty) = \frac{1}{24}\left|G_4^{CDS}\left(\frac{1}{2}\right)\right| = \frac{1}{5760}, \\
C_{CDS}(5,1) &= \frac{1}{360}\left|F_6^{CDS}\left(\frac{1}{2}\right)\right| = \frac{1}{115200}, \\
C_{CDS}(5,\infty) &= \frac{1}{120}\left|G_5^{CDS}\left(\frac{1}{3}\right)\right| = \frac{17}{933120}, \\
C_{CDS}(6,1) &= \frac{1}{720}\left|G_6^{CDS}(0)\right| = \frac{31}{19353600}, \\
C_{CDS}(6,\infty) &= \frac{1}{720}\left|F_6^{CDS}\left(\frac{1}{2}\right)\right| = \frac{1}{230400}.
\end{aligned}
$$

Similary as before, (3.108) and (3.109) give estimates of error for $m \geq 7$.

The Hermite-Hadamard-type inequality for the corrected dual Simpson's formula is:

$$
\begin{aligned}
&\frac{31}{19353600} f^{(6)}\left(\frac{1}{2}\right) \\
&\leq -\left(\int_0^1 f(t)dt - \frac{1}{15}\left[8f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 8f\left(\frac{3}{4}\right)\right] - \frac{1}{120}[f'(1)-f'(0)]\right) \\
&\leq \frac{31}{19353600}\,\frac{f^{(6)}(0)+f^{(6)}(1)}{2}
\end{aligned}
$$

while the constants from Theorem 3.18 are:

$$
L_{CQ3}\left(5,\frac{1}{4}\right) = \frac{1}{115200}, \qquad L_{CQ3}\left(6,\frac{1}{4}\right) = \frac{31}{19353600}.
$$

3.2.3 Corrected Maclaurin's formula

The next interesting special case is corrected Maclaurin's formula, obtained for $x = 1/6$. Results of this subsection were published in [55]. We have:

$$
\begin{aligned}
Q_C\left(\frac{1}{6},\frac{1}{2},\frac{5}{6}\right) &= \frac{1}{80}\left[27f\left(\frac{1}{6}\right) + 26f\left(\frac{1}{2}\right) + 27f\left(\frac{5}{6}\right)\right], \\
T_{2n}^{CM} = T_{2n}^{CQ3}\left(\frac{1}{6}\right) &= -\frac{1}{240}[f'(1)-f'(0)] \\
&\quad + \sum_{k=3}^{n}\frac{1}{(2k)!}\,G_{2k}^{CM}(0)\left[f^{(2k-1)}(1)-f^{(2k-1)}(0)\right]
\end{aligned}
$$

$$G_k^{CM}(t)=G_k^{CQ3}\left(\frac{1}{6},t\right)=\frac{1}{80}\left[27B_k^*\left(\frac{1}{6}-t\right)+26B_k^*\left(\frac{1}{2}-t\right)+27B_k^*\left(\frac{5}{6}-t\right)\right], \quad (3.118)$$
$$F_k^{CM}(t)=F_k^{CQ3}\left(\frac{1}{6},t\right)=G_k^{CM}(t)-G_k^{CM}(0), \quad k\geq 1 \text{ and } t\in\mathbb{R}.$$

For $n\geq 2$, the remainder $R_{2n+2}^{CM}(f)$ can be written as:

$$R_{2n+2}^{CM}(f)=-\theta\frac{\left(2-2^{-1-2n}\right)\left(1-3^{2-2n}\right)B_{2n+2}}{80(2n+2)!}\left[f^{(2n-1)}(1)-f^{(2n-1)}(0)\right], \ \theta\in[0,1]$$
$$R_{2n+2}^{CM}(f)=-\frac{(1-2^{-1-2n})(1-3^{2-2n})B_{2n+2}}{80(2n+2)!}f^{(2n+2)}(\eta), \quad \eta\in[0,1] \quad (3.119)$$

Formula (3.103) becomes corrected Maclaurin's formula and is of the form:

$$\int_0^1 f(t)dt-\frac{1}{80}\left[27f\left(\frac{1}{6}\right)+26f\left(\frac{1}{2}\right)+27f\left(\frac{5}{6}\right)\right]-\frac{1}{240}[f'(1)-f'(0)]$$
$$=-\frac{31}{87091200}f^{(6)}(\eta) \quad (3.120)$$

Estimates of error for $p=1$ and $p=\infty$ are:

$$\left|\int_0^1 f(t)dt-\frac{1}{80}\left[27f\left(\frac{1}{6}\right)+26f\left(\frac{1}{2}\right)+27f\left(\frac{5}{6}\right)\right]\right|\leq C_{CM}(m,q)\|f^{(m)}\|_p,$$

where

$$C_{CM}(1,1)=\frac{2401}{28800}, \qquad C_{CM}(1,\infty)=\left|G_1^{CM}\left(\frac{5}{6}\right)\right|=\frac{41}{240},$$
$$C_{CM}(2,1)=\frac{827}{192000}, \qquad C_{CM}(2,\infty)=\frac{1}{2}\left|F_2^{CM}\left(\frac{1}{6}\right)\right|=\frac{1}{72},$$

while for $m=2,3,4,5,6$

$$\left|\int_0^1 f(t)dt-\frac{1}{80}\left[27f\left(\frac{1}{6}\right)+26f\left(\frac{1}{2}\right)+27f\left(\frac{5}{6}\right)\right]-\frac{1}{240}[f'(1)-f'(0)]\right|$$
$$\leq C_{CM}(m,q)\|f^{(m)}\|_p,$$

where

$$C_{CM}(2,1)=\frac{320\sqrt{30}+187\sqrt{561}}{1728000},$$
$$C_{CM}(2,\infty)=\frac{1}{2}\left|G_2^{CM}\left(\frac{1}{6}\right)\right|=\frac{7}{720},$$
$$C_{CM}(3,1)=\frac{48693+3133\sqrt{241}}{491520000},$$
$$C_{CM}(3,\infty)=\frac{1053+187\sqrt{561}}{13824000},$$

$$C_{CM}(4,1) = \frac{1}{73728}, \qquad C_{CM}(4,\infty) = \frac{1}{38400},$$
$$C_{CM}(5,1) = \frac{1}{360}\left|F_6^{CM}\left(\frac{1}{2}\right)\right| = \frac{1}{691200},$$
$$C_{CM}(5,\infty) = \frac{1}{120}\left|G_5^{CM}\left(\frac{1}{4}\right)\right| = \frac{1}{294912},$$
$$C_{CM}(6,1) = \frac{1}{720}\left|G_6^{CM}(0)\right| = \frac{31}{87091200},$$
$$C_{CM}(6,\infty) = \frac{1}{720}\left|F_6^{CM}\left(\frac{1}{2}\right)\right| = \frac{1}{1382400}.$$

Let us mention again that from (3.108) and (3.109) estimates of error for $m \geq 7$ can also be calculated.

The Hermite-Hadamard-type inequality for the corrected Maclaurin's formula is:

$$\begin{aligned}
&\frac{31}{87091200} f^{(6)}\left(\frac{1}{2}\right) \\
&\leq -\left(\int_0^1 f(t)dt - \frac{1}{80}\left[27f\left(\frac{1}{6}\right) + 26f\left(\frac{1}{2}\right) + 27f\left(\frac{5}{6}\right)\right] - \frac{1}{240}[f'(1) - f'(0)]\right) \\
&\leq \frac{31}{87091200}\,\frac{f^{(6)}(0) + f^{(6)}(1)}{2}
\end{aligned}$$

and the constants from Theorem 3.18 are:

$$L_{CQ3}\left(5, \frac{1}{6}\right) = \frac{1}{691200}, \qquad L_{CQ3}\left(6, \frac{1}{6}\right) = \frac{31}{87091200}.$$

3.2.4 Gauss 3-point formula

Perhaps the most interesting special case is the Gauss 3-point formula. Namely, if we put $x = \frac{5-\sqrt{15}}{10}$ in (3.103), we obtain

$$\int_0^1 f(t)dt - \frac{1}{18}\left[5f\left(\frac{5-\sqrt{15}}{10}\right) + 8f\left(\frac{1}{2}\right) + 5f\left(\frac{5+\sqrt{15}}{10}\right)\right] = \frac{1}{2016000} f^{(6)}(\xi)$$

which is exactly the Gauss 3-point formula on the interval $[0,1]$. Note once more that this node is the unique solution of the system $G_2^{CQ3}(x,0) = G_4^{CQ3}(x,0) = 0$. Again, we switch to interval $[-1,1]$ for reasons mentioned in the subsection on the Gauss 2-point formula.

Let $f : [-1,1] \to \mathbb{R}$ be such that $f^{(2n-1)}$ is continuous of bounded variation on $[-1,1]$ for some $n \geq 1$. Then we have:

$$\int_{-1}^1 f(t)dt - Q_{G3} + T_{2n}^{G3} = \frac{2^{2n-1}}{(2n)!}\int_{-1}^1 G_{2n}^{G3}(t)df^{(2n-1)}(t); \tag{3.121}$$

if $f^{(2n)}$ is continuous of bounded variation on $[-1,1]$ for some $n \geq 0$ then

$$\int_{-1}^1 f(t)dt - Q_{G3} + T_{2n}^{G3} = \frac{2^{2n}}{(2n+1)!}\int_{-1}^1 G_{2n+1}^{G3}(t)df^{(2n)}(t), \tag{3.122}$$

and finally, if $f^{(2n+1)}$ is continuous of bounded variation on $[-1,1]$ for some $n \geq 0$, then

$$\int_{-1}^{1} f(t)dt - Q_{G3} + T_{2n}^{G3} = \frac{2^{2n+1}}{(2n+2)!} \int_{-1}^{1} F_{2n+2}^{G3}(t) d f^{(2n+1)}(t), \tag{3.123}$$

where

$$Q_{G3} = \frac{1}{9}\left[5f\left(-\frac{\sqrt{15}}{5}\right) + 8f(0) + 5f\left(\frac{\sqrt{15}}{5}\right)\right] \tag{3.124}$$

$$T_{2n}^{G3} = \sum_{k=3}^{n} \frac{2^{2k-1}}{(2k)!} G_{2k}^{G3}(-1)[f^{(2k-1)}(1) - f^{(2k-1)}(-1)] \tag{3.125}$$

$$G_k^{G3}(t) = \frac{1}{9}\left[5B_k^*\left(-\frac{\sqrt{15}}{10} - \frac{t}{2}\right) + 8B_k^*\left(-\frac{t}{2}\right) + 5B_k^*\left(\frac{\sqrt{15}}{10} - \frac{t}{2}\right)\right] \tag{3.126}$$

$$F_k^{G3}(t) = G_k^{G3}(t) - G_k^{G3}(-1). \tag{3.127}$$

The remainder in formula (3.123) $R_{2n+2}^{G3}(f)$ can, for $n \geq 2$, be written as:

$$R_{2n+2}^{G3}(f) = -\frac{2^{2n+2}}{(2n+2)!} G_{2n+2}^{G3}(-1) f^{(2n+2)}(\xi), \quad \xi \in [-1,1] \tag{3.128}$$

$$R_{2n+2}^{G3}(f) = \theta \frac{2^{2n+1}}{(2n+2)!} F_{2n+2}^{G3}(0)\left[f^{(2n+1)}(1) - f^{(2n+1)}(-1)\right], \theta \in [0,1] \tag{3.129}$$

where

$$G_{2n+2}^{G3}(-1) = \frac{1}{9}\left[10B_{2n+2}\left(\frac{5-\sqrt{15}}{10}\right) - (8 - 2^{2-2n})B_{2n+2}\right] \tag{3.130}$$

$$F_{2n+2}^{G3}(0) = \frac{1}{9}\left[10B_{2n+2}\left(\frac{\sqrt{15}}{10}\right) - 10B_{2n+2}\left(\frac{5-\sqrt{15}}{10}\right) + (16 - 2^{2-2n})B_{2n+2}\right] \tag{3.131}$$

From (3.128) and (3.123) for $n = 2$ the classical Gauss 3-point formula is produced:

$$\int_{-1}^{1} f(t)dt = \frac{1}{9}\left[5f\left(-\frac{\sqrt{15}}{5}\right) + 8f(0) + 5f\left(\frac{\sqrt{15}}{5}\right)\right] + \frac{1}{15750} f^{(6)}(\xi), \quad \xi \in [-1,1].$$

Applying Hölder's inequality (see Theorem 3.16), sharp, i.e. the best possible, estimates of error for the formulae (3.121)-(3.123) can be obtained. For $p = \infty$ and $p = 1$, and $n \geq 2$, they are:

$$K_{G3}(2n+1,1) = 2K_{G3}^*(2n+2,\infty) = \frac{2^{2n+2}}{(2n+2)!}\left|F_{2n+2}^{G3}(0)\right|$$

$$K_{G3}^*(2n+2,1) = \frac{2^{2n+2}}{(2n+2)!}\left|G_{2n+2}^{G3}(-1)\right|,$$

where $F_{2n+2}^{G3}(0)$ and $G_{2n+2}^{G3}(-1)$ as in (3.131) and (3.130).

Further, for $1 \leq m \leq 6$ we have:

$$\left| \int_{-1}^{1} f(t)dt - \frac{1}{9}\left[5f\left(-\frac{\sqrt{15}}{5}\right) + 8f(0) + 5f\left(\frac{\sqrt{15}}{5}\right)\right]\right| \leq C_{G3}(m,q)\|f^{(m)}\|_p,$$

where

$$\begin{aligned}
C_{G3}(1,1) &= \frac{1051-234\sqrt{15}}{405} \approx 0.357338, \quad C_{G3}(1,\infty) = \frac{4}{9} \approx 0.444444,\\
C_{G3}(2,1) &= \frac{8}{2187}(18\sqrt{15}-65)^{3/2} \approx 0.0374355,\\
C_{G3}(2,\infty) &= \frac{9-2\sqrt{15}}{18} \approx 0.0696685,\\
C_{G3}(3,1) &= \frac{160(6\sqrt{15}-23)^{3/2}+6516\sqrt{15}-25175}{14580} \approx 0.00548184,\\
C_{G3}(3,\infty) &= \frac{(18\sqrt{15}-65)^{3/2}-108\sqrt{15}+422}{2187} \approx 0.0063794,\\
C_{G3}(4,1) &\approx 0.000908828, \quad C_{G3}(4,\infty) = \frac{4\sqrt{15}-15}{360} \approx 0.00136648,\\
C_{G3}(5,1) &= \frac{25-6\sqrt{15}}{9000} \approx 0.000195789, \quad C_{G3}(5,\infty) \approx 0.000227207,\\
C_{G3}(6,1) &= \frac{1}{15750} \approx 0.0000634921,\\
C_{G3}(6,\infty) &= \frac{25-6\sqrt{15}}{18000} \approx 0.0000978944.
\end{aligned}$$

Integral $\int_{-1}^{1}|G_4^{G3}(t)|dt$ is calculated only approximately, with the help of Wolfram's Mathematica, because it is difficult to obtain the exact value of the zero of $G_4^{G3}(t)$ (which is $t \approx 0.280949$). Further, $|G_5^{G3}(t)|$ attains its maximal value in the zero of $G_4^{G3}(t)$, so we have the same problem again. Therefore, $C_{G3}(5,\infty)$ is also calculated with the help of Wolfram's Mathematica.

Remark 3.18 It was shown previously that in the class of all 3-point quadrature formulae where the integral is approximated by (3.8), the best estimation is obtained for $x = 1/6$ (when considering an integral over $[-1,1]$, for $x = 2/3$), i.e. Maclaurin's formula. Comparing estimates $C(m,1)$ and $C(m,\infty)$ obtained for Maclaurin's formula and the Gauss 3-point formula shows that, when $p = \infty$, the Gauss 3-point formula gives better approximation for $m = 2,3,4$, while when $p = 1$, it gives better approximation for $m = 3$ and $m = 4$.

Remark 3.19 The constant $C_{G3}(1,\infty)$ coincides with the constant $\rho_V(R_3^G)$, obtained in Theorem 1.1. in [47].

The Hermite-Hadamard-type inequality for the Gauss 3-point formula is:

$$\frac{1}{2016000} f^{(6)}\left(\frac{1}{2}\right)$$

$$\leq \int_0^1 f(t)dt - \frac{1}{18}\left[5f\left(\frac{5-\sqrt{15}}{10}\right) + 8f\left(\frac{1}{2}\right) + 5f\left(\frac{5+\sqrt{15}}{10}\right)\right]$$
$$\leq \frac{1}{2016000} \frac{f^{(6)}(0) + f^{(6)}(1)}{2}$$

and the constants from Theorem 3.18 are:

$$L_{CQ3}\left(5, \frac{5-\sqrt{15}}{10}\right) = \frac{25-6\sqrt{15}}{576000}, \qquad L_{CQ3}\left(6, \frac{5-\sqrt{15}}{10}\right) = \frac{1}{2016000}.$$

3.2.5 Corrected Gauss 2-point formula

What if in formulae (3.93)-(3.95) a condition $w(x) = 1/2$ is imposed? The unique solution x_0 of this equation, such that $x_0 \in [0, 1/2)$, is

$$x_0 = \frac{1}{2} - \frac{1}{30}\sqrt{15(15-2\sqrt{30})}.$$

At the same time, $B_4(x_0) = 0$, which means the quadrature formula thus obtained has only two nodes. We will call this formula corrected Gauss 2-point formula.

The results are again transformed to the interval $[-1, 1]$. Formulae (3.93)-(3.95) become:

$$\int_{-1}^1 f(t)dt - Q_{CG2} + T_{2n}^{CG2} = \frac{2^{2n-1}}{(2n)!}\int_{-1}^1 G_{2n}^{CG2}(t)df^{(2n-1)}(t), \tag{3.132}$$

$$\int_{-1}^1 f(t)dt - Q_{CG2} + T_{2n}^{CG2} = \frac{2^{2n}}{(2n+1)!}\int_{-1}^1 G_{2n+1}^{CG2}(t)df^{(2n)}(t) \tag{3.133}$$

$$\int_{-1}^1 f(t)dt - Q_{CG2} + T_{2n}^{CG2} = \frac{2^{2n+1}}{(2n+2)!}\int_{-1}^1 F_{2n+2}^{CG2}(t)df^{(2n+1)}(t), \tag{3.134}$$

where

$$Q_{CG2} = f\left(-\sqrt{1-\frac{2\sqrt{30}}{15}}\right) + f\left(\sqrt{1-\frac{2\sqrt{30}}{15}}\right)$$
$$T_{2n}^{CG2} = \sum_{k=1}^{n} \frac{2^{2k-1}}{(2k)!} G_{2k}^{CG2}(-1)[f^{(2k-1)}(1) - f^{(2k-1)}(-1)]$$
$$= \frac{5-\sqrt{30}}{15}[f'(1) - f'(-1)]$$
$$+ \sum_{k=3}^{n} \frac{2^{2k-1}}{(2k)!} G_{2k}^{CG2}(-1)[f^{(2k-1)}(1) - f^{(2k-1)}(-1)]$$
$$G_k^{CG2}(t) = B_k^*\left(-\frac{1}{2}\sqrt{1-\frac{2\sqrt{30}}{15}} - \frac{t}{2}\right) + B_k^*\left(\frac{1}{2}\sqrt{1-\frac{2\sqrt{30}}{15}} - \frac{t}{2}\right)$$
$$F_k^{CG2}(t) = G_k^{CG2}(t) - G_k^{CG2}(-1), \ k \geq 1.$$

The remainder in formula (3.134) $R^{CG2}_{2n+2}(f)$ for $n \geq 2$ can be written similarly as in (3.128) and (3.129), where

$$G^{CG2}_{2n+2}(-1) = 2\,B_{2n+2}\left(\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2\sqrt{30}}{15}}\right) \tag{3.135}$$

$$F^{CG2}_{2n+2}(0) = 2\left[B_{2n+2}\left(\frac{1}{2}\sqrt{1 - \frac{2\sqrt{30}}{15}}\right) - B_{2n+2}\left(\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{2\sqrt{30}}{15}}\right)\right] \tag{3.136}$$

Using this, from (3.134) for $n = 2$, the corrected Gauss 2-point formula is obtained:

$$\int_{-1}^{1} f(t)dt = Q_{CG2} - \frac{5 - \sqrt{30}}{15}[f'(1) - f'(-1)] + \frac{14\sqrt{30} - 90}{70875} f^{(6)}(\xi),\ \xi \in [-1,1] \tag{3.137}$$

Note that

$$\frac{14\sqrt{30} - 90}{70875} \approx -0.00018792.$$

Further, similarly as in the previous subsection where the Gauss 3-point formula was considered, from Theorem 3.16 we get the sharp estimates of error for $n \geq 2$, with the appropriate values of $G^{CG2}_{2n+2}(-1)$ and $F^{CG2}_{2n+2}(0)$ as in (3.135) and (3.136). Specially, let us see what these estimates are for $p = \infty$ and $p = 1$:

$$\left|\int_{-1}^{1} f(t)dt - Q_{CG2}\right| \leq C_{CG2}(m,q)\|f^{(m)}\|_p,\ \ m = 1,2$$

where

$$\begin{aligned}
C_{CG2}(1,1) &= 3 - \frac{4\sqrt{30}}{15} - 2\sqrt{1 - \frac{2\sqrt{30}}{15}} \approx 0.500747,\\
C_{CG2}(1,\infty) &= \sqrt{1 - \frac{2\sqrt{30}}{15}} \approx 0.51933,\\
C_{CG2}(2,1) &= \frac{2}{45}\left[3\sqrt{30} - 15 + 4\sqrt{15\left(2\sqrt{30} - 15 + 2\sqrt{585 - 106\sqrt{30}}\right)}\right.\\
&\quad \left. -2\sqrt{30\sqrt{225 - 30\sqrt{30}} - 225}\right] \approx 0.073765,\\
C_{CG2}(2,\infty) &= 1 - \frac{\sqrt{30}}{15} - \sqrt{1 - \frac{2\sqrt{30}}{15}} \approx 0.115522,
\end{aligned}$$

while for $2 \leq m \leq 6$

$$\left|\int_{-1}^{1} f(t)dt - Q_{CG2} + \frac{5 - \sqrt{30}}{15}[f'(1) - f'(-1)]\right| \leq C_{CG2}(m,q)\|f^{(m)}\|_p,$$

where

$$
\begin{aligned}
C_{CG2}(2,1) &\approx 0.0650208, \\
C_{CG2}(2,\infty) &= \frac{1}{15}\left(20-2\sqrt{30}-\sqrt{225-30\sqrt{30}}\right) \approx 0.083707, \\
C_{CG2}(3,1) &= \frac{1}{180}\left(339-60\sqrt{30}-4\sqrt{585-106\sqrt{30}}\right) \approx 0.0109032, \\
C_{CG2}(3,\infty) &= \frac{1}{45}\sqrt{2\sqrt{25653825-4649430\sqrt{30}}+1226\sqrt{30}-\frac{22745}{3}} \\
&\approx 0.010905, \\
C_{CG2}(4,1) &\approx 0.00209577, \\
C_{CG2}(4,\infty) &= \frac{5}{24}-\frac{1}{\sqrt{30}}-\frac{1}{90}\sqrt{585-106\sqrt{30}} \approx 0.002415, \\
C_{CG2}(5,1) &= \frac{2\sqrt{392625-71670\sqrt{30}}+150\sqrt{30}-825}{27000} \approx 0.000503075, \\
C_{CG2}(5,\infty) &\approx 0.000523942, \quad C_{CG2}(6,1) = \frac{90-14\sqrt{30}}{70875} \approx 0.00018792, \\
C_{CG2}(6,\infty) &= \frac{1}{3600}\left(2\sqrt{1745-\frac{4778\sqrt{30}}{15}}+10\sqrt{30}-55\right) \approx 0.000251537.
\end{aligned}
$$

The exact values of $C_{CG2}(2,1)$ (for the case when the values of the first derivative are included in the quadrature formula), $C_{CG2}(4,1)$ and $C_{CG2}(5,\infty)$ can be calculated (with the help of Wolfram's Mathematica) but obtained expressions are rather cumbersome so we state only their approximate values.

Also, notice that when $p=\infty$ and $m=1,2$, and when $p=1$ and $m=1$, corrected Gauss 2-point formulae give better estimations than Gauss 2-point formulae.

The Hermite-Hadamard-type inequality for the corrected Gauss 2-point formula is:

$$
\begin{aligned}
&\frac{45-7\sqrt{30}}{4536000} f^{(6)}\left(\frac{1}{2}\right) \\
&\leq -\left(\int_0^1 f(t)dt - \frac{1}{2}f\left(\frac{1}{2}-\frac{1}{30}\sqrt{15(15-2\sqrt{30})}\right) - \frac{1}{2}f\left(\frac{1}{2}+\frac{1}{30}\sqrt{15(15-2\sqrt{30})}\right)\right. \\
&\qquad\qquad \left. + \frac{5-\sqrt{30}}{60}[f'(1)-f'(0)]\right) \\
&\leq \frac{45-7\sqrt{30}}{4536000}\frac{f^{(6)}(0)+f^{(6)}(1)}{2}
\end{aligned}
$$

and the constants from Theorem 3.18 are:

$$
L_{CQ3}\left(5,\ \frac{1}{2}-\frac{1}{30}\sqrt{15(15-2\sqrt{30})}\right)
$$

$$= \frac{825 - 150\sqrt{30} - 46\sqrt{225 - 30\sqrt{30}} + 120\sqrt{30 - 4\sqrt{30}}}{1728000}$$

$$L_{CQ3}\left(6,\ \frac{1}{2} - \frac{1}{30}\sqrt{15(15 - 2\sqrt{30})}\right) = \frac{45 - 7\sqrt{30}}{4536000}.$$

3.2.6 Hermite-Hadamard-type inequality for the corrected 3-point quadrature formulae

The main result of this section provides Hermite-Hadamard-type inequality for the corrected 3-point quadrature formulae.

Theorem 3.19 *Let $f : [0,1] \to \mathbb{R}$ be 6-convex and such that $f^{(6)}$ is continuous on $[0,1]$. Then, for $x \in [0, \frac{1}{2} - \frac{\sqrt{15}}{10}]$ and $y \in \left[\frac{1}{6}, \frac{1}{2}\right)$*

$$\begin{aligned}
& Q_C\left(x, \frac{1}{2}, 1 - x\right) - \frac{10x^2 - 10x + 1}{60(-4x^2 + 4x + 1)}[f'(1) - f'(0)] \\
& \leq \int_0^1 f(t)dt \qquad (3.138) \\
& \leq\ Q_C\left(y, \frac{1}{2}, 1 - y\right) - \frac{10y^2 - 10y + 1}{60(-4y^2 + 4y + 1)}[f'(1) - f'(0)],
\end{aligned}$$

where $Q_C\left(x, \frac{1}{2}, 1 - x\right)$ is defined in (3.96). If f is 6-concave, the inequalities are reversed.

Proof. First, note that

$$\begin{aligned}
\frac{1}{98}(98x^4 - 196x^3 + 102x^2 - 4x - 1) &= \left(x - \frac{1}{2}\right)^4 - \frac{45}{98}\left(x - \frac{1}{2}\right)^2 + \frac{33}{784} \\
&= \left(\left(x - \frac{1}{2}\right)^2 - \frac{45}{196}\right)^2 - \frac{51}{4802} \\
&= \left(\left(x - \frac{1}{2}\right)^2 - \frac{45 + 2\sqrt{102}}{196}\right)\left(\left(x - \frac{1}{2}\right)^2 - \frac{45 - 2\sqrt{102}}{196}\right)
\end{aligned}$$

and therefore, we can find the zeros of this polynomial explicitly:

$$x_{1,2,3,4} = \frac{1}{2} \pm \frac{1}{14}\sqrt{45 \pm 2\sqrt{102}}.$$

Now, it is easy to determine the sign of the remainder in the formula (3.103). Finally, for a 6-convex function, we have $f^{(6)} \geq 0$, and thus the statement follows. □

The following corollaries give comparison between corrected Simpson's rule and corrected dual Simpson's rule, corrected Simpson's rule and corrected Maclaurin's rule, corrected Simpson's rule and the corrected Gauss 2-point rule, and then the Gauss 3-point rule and corrected dual Simpson's rule, the Gauss 3-point rule and corrected Maclaurin's rule, the Gauss 3-point rule and the corrected Gauss 2-point rule.

Corollary 3.13 *Let $f : [0,1] \to \mathbb{R}$ be 6-convex and such that $f^{(6)}$ is continuous on $[0,1]$. Then*

$$\begin{aligned}
&\frac{1}{30}\left(7f(0)+16f\left(\frac{1}{2}\right)+7f(1)\right)-\frac{1}{60}[f'(1)-f'(0)] \\
&\leq \int_0^1 f(t)dt \\
&\leq \frac{1}{15}\left(8f\left(\frac{1}{4}\right)-f\left(\frac{1}{2}\right)+8f\left(\frac{3}{4}\right)\right)+\frac{1}{120}[f'(1)-f'(0)].
\end{aligned}$$

If f is 6-concave, the inequalities are reversed.

Proof. Follows from (3.138) for $x=0$ and $y=1/4$. □

Corollary 3.14 *Let $f : [0,1] \to \mathbb{R}$ be 6-convex and such that $f^{(6)}$ is continuous on $[0,1]$. Then*

$$\begin{aligned}
&\frac{1}{30}\left(7f(0)+16f\left(\frac{1}{2}\right)+7f(1)\right)-\frac{1}{60}[f'(1)-f'(0)] \\
&\leq \int_0^1 f(t)dt \\
&\leq \frac{1}{80}\left(27f\left(\frac{1}{6}\right)+26f\left(\frac{1}{2}\right)+27f\left(\frac{5}{6}\right)\right)+\frac{1}{240}[f'(1)-f'(0)].
\end{aligned}$$

If f is 6-concave, the inequalities are reversed.

Proof. Follows from (3.138) for $x=0$ and $y=1/6$. □

Corollary 3.15 *Let $f : [0,1] \to \mathbb{R}$ be 6-convex and such that $f^{(6)}$ is continuous on $[0,1]$. Then*

$$\begin{aligned}
&\frac{1}{30}\left(7f(0)+16f\left(\frac{1}{2}\right)+7f(1)\right)-\frac{1}{60}[f'(1)-f'(0)] \\
&\leq \int_0^1 f(t)dt \\
&\leq \frac{1}{2}f\left(\frac{1}{2}-\frac{1}{30}\sqrt{225-30\sqrt{30}}\right)+\frac{1}{2}f\left(\frac{1}{2}+\frac{1}{30}\sqrt{225-30\sqrt{30}}\right) \\
&\qquad +\frac{7\sqrt{30}-5}{420}[f'(1)-f'(0)].
\end{aligned}$$

If f is 6-concave, the inequalities are reversed.

Proof. Follows from (3.138) for $x=0$ and $y=1/2-\sqrt{225-30\sqrt{30}}\Big/30 \Leftrightarrow B_4(y)=0$. □

Corollary 3.16 *Let $f:[0,1]\to\mathbb{R}$ be 6-convex and such that $f^{(6)}$ is continuous on $[0,1]$. Then*

$$\begin{aligned}&\frac{1}{18}\left(5f\left(\frac{5-\sqrt{15}}{10}\right)+8f\left(\frac{1}{2}\right)+5f\left(\frac{5+\sqrt{15}}{10}\right)\right)\\&\leq\int_0^1 f(t)dt\\&\leq\frac{1}{15}\left(8f\left(\frac{1}{4}\right)-f\left(\frac{1}{2}\right)+8f\left(\frac{3}{4}\right)\right)+\frac{1}{120}[f'(1)-f'(0)].\end{aligned}$$

If f is 6-concave, the inequalities are reversed.

Proof. Follows from (3.138) for $x=1/2-\sqrt{15}/10\Leftrightarrow 10x^2-10x+1=0$ and $y=1/4$. □

Corollary 3.17 *Let $f:[0,1]\to\mathbb{R}$ be 6-convex and such that $f^{(6)}$ is continuous on $[0,1]$. Then*

$$\begin{aligned}&\frac{1}{18}\left(5f\left(\frac{5-\sqrt{15}}{10}\right)+8f\left(\frac{1}{2}\right)+5f\left(\frac{5+\sqrt{15}}{10}\right)\right)\\&\leq\int_0^1 f(t)dt\\&\leq\frac{1}{80}\left(27f\left(\frac{1}{6}\right)+26f\left(\frac{1}{2}\right)+27f\left(\frac{5}{6}\right)\right)+\frac{1}{240}[f'(1)-f'(0)].\end{aligned}$$

If f is 6-concave, the inequalities are reversed.

Proof. Follows from (3.138) for $x=1/2-\sqrt{15}/10\Leftrightarrow 10x^2-10x+1=0$ and $y=1/6$. □

Corollary 3.18 *Let $f:[0,1]\to\mathbb{R}$ be 6-convex and such that $f^{(6)}$ is continuous on $[0,1]$. Then*

$$\begin{aligned}&\frac{1}{18}\left(5f\left(\frac{5-\sqrt{15}}{10}\right)+8f\left(\frac{1}{2}\right)+5f\left(\frac{5+\sqrt{15}}{10}\right)\right)\\&\leq\int_0^1 f(t)dt\\&\leq\frac{1}{2}f\left(\frac{1}{2}-\frac{1}{30}\sqrt{225-30\sqrt{30}}\right)+\frac{1}{2}f\left(\frac{1}{2}+\frac{1}{30}\sqrt{225-30\sqrt{30}}\right)\\&+\frac{7\sqrt{30}-5}{420}[f'(1)-f'(0)].\end{aligned}$$

If f is 6-concave, the inequalities are reversed.

Proof. Follows from (3.138) for $x=1/2-\sqrt{15}/10\Leftrightarrow 10x^2-10x+1=0$ and $y=1/2-\sqrt{225-30\sqrt{30}}\Big/30\Leftrightarrow B_4(y)=0$. □

3.2.7 On corrected Bullen-Simpson's inequality

In [70], it was shown by a simple geometric argument that for a convex function f the following inequality is valid :

$$0 \leq \int_0^1 f(t)dt - f\left(\frac{1}{2}\right) \leq \frac{f(0)+f(1)}{2} - \int_0^1 f(t)\mathrm{d}t. \tag{3.139}$$

An elementary analytic proof of (3.40) and (3.139) was given in [11]. Another interesting result of a similar type was given in that same paper. Namely, provided f is 4-convex, we have:

$$\begin{aligned} 0 \leq & \int_0^1 f(t)dt - \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right] \\ \leq & \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right] - \int_0^1 f(t)dt \end{aligned} \tag{3.140}$$

This implies that dual Simpson's quadrature rule is more accurate than Simpson's quadrature rule. Inequality (3.140) is sometimes called Bullen-Simpson's inequality and was generalized for a class of $(2k)$-convex functions in [86].

The aim here is to derive similar type inequalities, only this time starting from corrected Simpson's and dual corrected Simpson's formula. The results of this subsection were published in [53].

For $k \geq 1$ and $t \in \mathbb{R}$, we define functions

$$G_k(t) = G_k^{CQ3}(0,t) + G_k^{CQ3}(1/4,t) = G_k^{CS}(t) + G_k^{CDS}(t),$$
$$F_k(t) = F_k^{CQ3}(0,t) + F_k^{CQ3}(1/4,t) = F_k^{CS}(t) + F_k^{CDS}(t),$$

where $G_k^{CS}(t)$ and $G_k^{CDS}(t)$ are defined in (3.114) and (3.116), respectively. Thus,

$$G_k(t) = 7B_k^*(1-t) + 8B_k^*\left(\frac{1}{4}-t\right) + 7B_k^*\left(\frac{1}{2}-t\right) + 8B_k^*\left(\frac{3}{4}-t\right), \quad k \geq 1,$$
$$F_1(t) = G_1(t), \quad F_k(t) = G_k(t) - G_k(0) \quad \text{for} \quad k \geq 2.$$

Introduce notation $\tilde{B}_k = G_k(0)$. By direct calculation we get

$$\tilde{B}_2 = 1/4 \qquad \text{and} \qquad \tilde{B}_3 = \tilde{B}_4 = \tilde{B}_5 = 0.$$

Using the properties of Bernoulli polynomials, it is easy to check that $\tilde{B}_{2k-1} = 0, \ k \geq 2$.

Now, let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n-1)}$ exists on $[0,1]$ for some $n \geq 1$. Introduce the following notation

$$D(0,1) = \frac{1}{60}\left[7f(0) + 16f\left(\frac{1}{4}\right) + 14f\left(\frac{1}{2}\right) + 16f\left(\frac{3}{4}\right) + 7f(1)\right].$$

Define $T_0(f) = 0$ and for $1 \leq m \leq n$

$$T_m(f) = \frac{1}{2}\left[T_m^{CQ3}(0) + T_m^{CQ3}(1/4)\right],$$

where $T_m^{CQ3}(x)$ was given by (3.97). So, we have $T_1(f) = 0$,

$$T_2(f) = T_3(f) = T_4(f) = T_5(f) = \frac{1}{240}[f'(1) - f'(0)]$$

and, for $m \geq 6$,

$$\begin{aligned} T_m(f) &= \frac{1}{240}[f'(1) - f'(0)] \\ &+ \frac{1}{15}\sum_{k=3}^{[m/2]} \frac{B_{2k}}{(2k)!}(16 \cdot 2^{-4k} - 2^{-2k})\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right]. \end{aligned}$$

In the next theorem we establish two formulae which play the key role here. We call them corrected Bullen-Simpson's formulae of Euler type.

Theorem 3.20 *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n-1)}$ is continuous of bounded variation on $[0,1]$, for some $n \geq 1$. Then*

$$\int_0^1 f(t)dt = D(0,1) - T_n(f) + \tilde{R}_n^{(1)}(f), \tag{3.141}$$

and

$$\int_0^1 f(t)dt = D(0,1) - T_{n-1}(f) + \tilde{R}_n^{(2)}(f), \tag{3.142}$$

where

$$\tilde{R}_n^{(1)}(f) = \frac{1}{30n!}\int_0^1 G_n(t)\, df^{(n-1)}(t),$$

and

$$\tilde{R}_n^{(2)}(f) = \frac{1}{30n!}\int_0^1 F_n(t)\, df^{(n-1)}(t).$$

Proof. Apply (3.94) for $x = 0$ and $x = 1/4$, add those two formulae, divide by 2 and identity (3.141) is produced. Identity (3.142) is obtained analogously from (3.95). □

Remark 3.20 Interval $[0,1]$ is used for simplicity and involves no loss in generality. In what follows, Theorem 3.20 and others will be applied, without comment, to any interval that is convenient.

It is easy to see that if $f : [a,b] \to \mathbb{R}$ is such that $f^{(n-1)}$ is continuous of bounded variation on $[a,b]$, for some $n \geq 1$, then

$$\int_a^b f(t)dt = D(a,b) - \tilde{T}_n(f) + \frac{(b-a)^n}{30n!}\int_a^b G_n\left(\frac{t-a}{b-a}\right) df^{(n-1)}(t)$$

and

$$\int_a^b f(t)dt = D(a,b) - \tilde{T}_{n-1}(f) + \frac{(b-a)^n}{30n!}\int_a^b F_n\left(\frac{t-a}{b-a}\right) df^{(n-1)}(t),$$

where

$$D(a,b) = \frac{b-a}{60}\left[7f(a) + 16f\left(\frac{3a+b}{4}\right) + 14f\left(\frac{a+b}{2}\right) + 16f\left(\frac{a+3b}{4}\right) + 7f(b)\right],$$

$$\tilde{T}_0(f) = \tilde{T}_1(f) = 0,$$
$$\tilde{T}_2(f) = \tilde{T}_3(f) = \tilde{T}_4(f) = \tilde{T}_5(f) = \frac{(b-a)^2}{240}\left[f'(b) - f'(a)\right]$$

and for $m \geq 6$

$$\begin{aligned}\tilde{T}_m(f) &= \frac{(b-a)^2}{240}\left[f'(b) - f'(a)\right] \\ &+ \frac{1}{15}\sum_{k=3}^{[m/2]} \frac{(b-a)^{2k}}{(2k)!}(16\cdot 2^{-4k} - 2^{-2k})B_{2k}\left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right].\end{aligned}$$

Remark 3.21 Suppose that $f : [0,1] \to \mathbb{R}$ is such that $f^{(n)}$ exists and is integrable on $[0,1]$, for some $n \geq 1$. In this case (3.141) holds with

$$\tilde{R}_n^{(1)}(f) = \frac{1}{30n!}\int_0^1 G_n(t) f^{(n)}(t)dt,$$

while (3.142) holds with

$$\tilde{R}_n^{(2)}(f) = \frac{1}{30n!}\int_0^1 F_n(t) f^{(n)}(t)dt.$$

Remark 3.22 For $n = 6$, (3.142) yields

$$\int_0^1 f(t)dt - D(0,1) + \frac{f'(1) - f'(0)}{240} = \frac{1}{21600}\int_0^1 F_6(t)\, df^{(5)}(t).$$

From this identity it is clear that corrected Bullen-Simpson's formula of Euler type is exact for all polynomials of order ≤ 5.

Before the main result is stated, we will need to prove some properties of functions G_k and F_k. Notice that it is enough to know the values of those functions on the interval $[0,1/2]$, since $G_k(t+1/2) = G_k(t)$.

Lemma 3.8 *For $k \geq 3$, function $G_{2k-1}(t)$ has no zeros in the interval $\left(0,\frac{1}{4}\right)$. The sign of this function is determined by*

$$(-1)^k G_{2k-1}(t) > 0, \qquad 0 < t < \frac{1}{4}. \tag{3.143}$$

Proof. For $k = 3$, we have

$$G_5(t) = \begin{cases} -30t^5 + 35/2\cdot t^4 - 5/2\cdot t^3, & 0 \leq t \leq 1/4 \\ -30t^5 + 115/2\cdot t^4 - 85/2\cdot t^3 + 15t^2 - 5/2\cdot t + 5/32, & 1/4 \leq t \leq 1/2 \end{cases}$$

and it is elementary to see that

$$G_5(t) < 0, \qquad 0 < t < 1/4, \tag{3.144}$$

so our assertion is true for $k = 3$. Assuming the opposite, by induction, it follows easily that the assertion is true for all $k \geq 4$.

Further, if $G_{2k-3}(t) > 0,\ 0 < t < 1/4$, then since

$$G''_{2k-1}(t) = (2k-1)(2k-2)G_{2k-3}(t)$$

it follows that G_{2k-1} is convex and hence $G_{2k-1}(t) < 0$ on $(0,1/4)$. Similarly, we conclude that if $G_{2k-3}(t) < 0$, then $G_{2k-1}(t) > 0$ on $(0,1/4)$. (3.143) now follows from (3.144). □

Corollary 3.19 *For $k \geq 3$, functions $(-1)^{k-1}F_{2k}(t)$ and $(-1)^{k-1}G_{2k}(t)$ are strictly increasing on the interval $(0,1/4)$ and strictly decreasing on the interval $(1/4,1/2)$. Consequently, 0 and $1/2$ are the only zeros of $F_{2k}(t)$ on $[0,1/2]$ and*

$$\max_{t\in[0,1]} |F_{2k}(t)| = 2^{2-2k}(1-2^{-2k})|B_{2k}|,$$
$$\max_{t\in[0,1]} |G_{2k}(t)| = 2^{1-2k}(1+14\cdot 2^{-2k})|B_{2k}|.$$

Proof. Since

$$[(-1)^{k-1}F_{2k}(t)]' = [(-1)^{k-1}G_{2k}(t)]' = (-1)^k\cdot 2k\cdot G_{2k-1}(t),$$

from Lemma 3.8 we conclude that $(-1)^{k-1}F_{2k}(t)$ and $(-1)^{k-1}G_{2k}(t)$ are strictly increasing on $(0,1/4)$. It is easy to check that for $k \geq 2$ and $0 \leq t \leq 1/2$,

$$G_k(1/2-t) = (-1)^k G_k(t) \qquad \text{and} \qquad F_k(1/2-t) = (-1)^k F_k(t).$$

From there we conclude that $(-1)^{k-1}F_{2k}(t)$ and $(-1)^{k-1}G_{2k}(t)$ are strictly decreasing on $(1/4,1/2)$. Further, $F_{2k}(0) = F_{2k}(1/2) = 0$, which implies $|F_{2k}(t)|$ achieves maximum at $t = 1/4$ and thus, the first assertion is proved.

On the other hand,

$$\max_{t\in[0,1]} |G_{2k}(t)| = \max\left\{|G_{2k}(0)|, \left|G_{2k}\left(\frac{1}{4}\right)\right|\right\} = \left|G_{2k}\left(\frac{1}{4}\right)\right|.$$

The proof is now complete. □

Corollary 3.20 *For $k \geq 3$, we have*

$$\int_0^1 |F_{2k-1}(t)|\,dt = \int_0^1 |G_{2k-1}(t)|\,dt = \frac{2^{3-2k}}{k}\left(1-2^{-2k}\right)|B_{2k}|,$$
$$\int_0^1 |F_{2k}(t)|\,dt = |\tilde{B}_{2k}| = 2^{1-2k}\left(1-16\cdot 2^{-2k}\right)|B_{2k}|$$
$$\int_0^1 |G_{2k}(t)|\,dt \leq 2|\tilde{B}_{2k}| = 2^{2-2k}\left(1-16\cdot 2^{-2k}\right)|B_{2k}|.$$

Proof. Using the properties of functions G_k, i.e. properties of Bernoulli polynomials, we get

$$\int_0^1 |G_{2k-1}(t)|\,dt = 4\left|\int_0^{1/4} G_{2k-1}(t)dt\right| = \frac{2}{k}\left|F_{2k}\left(\frac{1}{4}\right)\right|,$$

which proves the first assertion. Since $F_{2k}(0) = F_{2k}(1/2) = 0$, from Corollary 3.19 we conclude that $F_{2k}(t)$ does not change sign on $(0,1/2)$. Therefore,

$$\int_0^1 |F_{2k}(t)|\,dt = 2\left|\int_0^{1/2} G_{2k}(t)dt - \frac{1}{2}\tilde{B}_{2k}\right| = |\tilde{B}_{2k}|,$$

which proves the second assertion. Finally, we use the triangle inequality to obtain

$$\int_0^1 |G_{2k}(t)|\,dt \le \int_0^1 |F_{2k}(t)|\,dt + |\tilde{B}_{2k}| = 2|\tilde{B}_{2k}|,$$

which proves the third assertion. □

Theorem 3.21 *If $f:[0,1]\to\mathbb{R}$ is such that $f^{(2k)}$ is a continuous function on $[0,1]$, for some $k\ge 3$, then there exists $\eta\in[0,1]$ such that*

$$\tilde{R}_{2k}^{(2)}(f) = \frac{1}{15(2k)!}2^{-2k}(1-16\cdot 2^{-2k})B_{2k}\cdot f^{(2k)}(\eta). \tag{3.145}$$

Proof. We can rewrite $\tilde{R}_{2k}^{(2)}(f)$ as

$$\tilde{R}_{2k}^{(2)}(f) = \frac{(-1)^{k-1}}{30(2k)!}J_k, \tag{3.146}$$

where

$$J_k = \int_0^1 (-1)^{k-1}F_{2k}(t)f^{(2k)}(t)dt. \tag{3.147}$$

From Corollary 3.19 we know that $(-1)^{k-1}F_{2k}(t)\ge 0,\ 0\le t\le 1$, so the claim follows from the mean value theorem for integrals and Corollary 3.20 □

Remark 3.23 For $k=3$ formula (3.145) reduces to

$$\tilde{R}_6^{(2)}(f) = \frac{1}{38707200}f^{(6)}(\eta).$$

Now, we prove our main result:

Theorem 3.22 *Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(2k)}$ is continuous on $[0,1]$ for some $k\ge 3$. If f is a $(2k)-$convex function, then for an even k we have*

$$\begin{aligned} 0 &\le \int_0^1 f(t)dt - \frac{1}{15}\left[8f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 8f\left(\frac{3}{4}\right)\right] + T_{2k-1}^D(f) \\ &\le \frac{1}{30}\left[7f(0) + 16f\left(\frac{1}{2}\right) + 7f(1)\right] - T_{2k-1}^S(f) - \int_0^1 f(t)dt \end{aligned} \tag{3.148}$$

while for an odd k the inequalities are reversed.

Proof. Denote the middle part and the right-hand side of (3.148) by LHS and DHS, respectively. Then we have

$$LHS = R_{2k}^{CDS}(f) \qquad \text{and} \qquad RHS - LHS = -2\tilde{R}_{2k}^{(2)}(f)$$

where $\tilde{R}_{2k}^{(2)}(f)$ is as in Theorem 3.20 and according to (3.116), $R_{2k}^{CDS}(f)$ can be written in the form:

$$R_{2k}^{CDS}(f) = -\frac{1}{15(2k)!}(1 - 18\cdot 2^{-2k} + 32\cdot 2^{-4k})B_{2k}\cdot f^{(2k)}(\xi),\ \xi \in [0,1] \tag{3.149}$$

Recall that if f is $(2k)-$convex on $[0,1]$, then $f^{(2k)}(x) \geq 0$, $x \in [0,1]$. Now, having in mind that $(-1)^{k-1}B_{2k} > 0$ ($k \in \mathbb{N}$), from (3.149) and (3.145) we get

$$\begin{aligned} LHS \geq 0, \quad & RHS - LHS \geq 0, \quad \text{for even k} \\ LHS \leq 0, \quad & RHS - LHS \leq 0, \quad \text{for odd k} \end{aligned}$$

and thus the proof is complete. □

Remark 3.24 From (3.148) for $k = 3$ we get

$$\begin{aligned} 0 &\leq \int_0^1 f(t)dt - \frac{1}{15}\left[8f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 8f\left(\frac{3}{4}\right)\right] - \frac{1}{120}[f'(1) - f'(0)] \\ &\leq \frac{1}{30}\left[7f(0) + 16f\left(\frac{1}{2}\right) + 7f(1)\right] - \frac{1}{60}[f'(1) - f'(0)] - \int_0^1 f(t)dt \end{aligned}$$

Theorem 3.23 *If $f : [0,1] \to \mathbb{R}$ is such that $f^{(2k)}$ is a continuous function on $[0,1]$ and f is either $(2k)-$convex or $(2k)-$concave, for some $k \geq 3$, then there exists $\theta \in [0,1]$ such that*

$$\tilde{R}_{2k}^{(2)}(f) = \theta\frac{2^{1-2k}}{15(2k)!}(1 - 2^{-2k})B_{2k}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right]. \tag{3.150}$$

Proof. Suppose f is $(2k)-$convex, so $f^{(2k)}(t) \geq 0$, $0 \leq t \leq 1$. If J_k is given by (3.147), using Corollary 3.19, we obtain

$$0 \leq J_k \leq (-1)^{k-1}F_{2k}\left(\frac{1}{4}\right)\cdot\int_0^1 f^{(2k)}(t)dt.$$

which means that there exists $\theta \in [0,1]$ such that

$$J_k = \theta\cdot(-1)^{k-1}\cdot 2^{2-2k}\left(1 - 2^{-2k}\right)B_{2k}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right].$$

When f is $(2k)-$concave, the statement follows similarly. □

Now define

$$\Delta_{2k}(f) = \frac{2^{-2k}}{15(2k)!}(1 - 16\cdot 2^{-2k})B_{2k}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right].$$

Clearly,

$$\tilde{R}_{2k}^{(2)}(f) = \theta\cdot\frac{2 - 2^{1-2k}}{1 - 2^{4-2k}}\cdot\Delta_{2k}(f).$$

Theorem 3.24 *Suppose that $f:[0,1]\to\mathbb{R}$ is such that $f^{(2k+2)}$ is a continuous function on $[0,1]$ for some $k\geq 3$. If f is either $(2k)$-convex and $(2k+2)$-convex or $(2k)$-concave and $(2k+2)$-concave, then the remainder $\tilde{R}^{(2)}_{2k}(f)$ has the same sign as the first neglected term $\Delta_{2k}(f)$ and*

$$|\tilde{R}^{(2)}_{2k}(f)|\leq|\Delta_{2k}(f)|.$$

Proof. We have

$$\Delta_{2k}(f)=\tilde{R}^{(2)}_{2k}(f)-\tilde{R}^{(2)}_{2k+2}(f).$$

From Corollary 3.19 it follows that for all $t\in[0,1]$

$$(-1)^{k-1}F_{2k}(t)\geq 0 \quad\text{and}\quad (-1)^{k-1}[-F_{2k+2}(t)]\geq 0,$$

so we conclude that $\tilde{R}^{(2)}_{2k}(f)$ has the same sign as $-\tilde{R}^{(2)}_{2k+2}(f)$. Therefore, $\Delta_{2k}(f)$ must have the same sign as $\tilde{R}^{(2)}_{2k}(f)$ and $-\tilde{R}^{(2)}_{2k+2}(f)$. Moreover, it follows that

$$|\tilde{R}^{(2)}_{2k}(f)|\leq|\Delta_{2k}(f)| \quad\text{and}\quad |\tilde{R}^{(2)}_{2k+2}(f)|\leq|\Delta_{2k}(f)|.$$

□

In this subsection, using formulae derived in Theorem 3.20, we shall prove a number of inequalities for various classes of functions.

Theorem 3.25 *Assume (p,q) is a pair of conjugate exponents, that is $1\leq p,\ q\leq\infty$, $\frac{1}{p}+\frac{1}{q}=1$. If $f:[0,1]\to\mathbb{R}$ is such that $f^{(n)}\in L_p[0,1]$ for some $n\geq 1$, then we have*

$$\left|\int_0^1 f(t)dt-D(0,1)+T_{n-1}(f)\right|\leq K(n,p)\|f^{(n)}\|_p, \tag{3.151}$$

and

$$\left|\int_0^1 f(t)dt-D(0,1)+T_n(f)\right|\leq K^*(n,p)\|f^{(n)}\|_p, \tag{3.152}$$

where

$$K(n,p)=\frac{1}{30n!}\left[\int_0^1|F_n(t)|^q\,dt\right]^{\frac{1}{q}} \quad\textit{and}\quad K^*(n,p)=\frac{1}{30n!}\left[\int_0^1|G_n(t)|^q\,dt\right]^{\frac{1}{q}}.$$

Proof. Applying the Hölder inequality we get

$$\begin{aligned}\left|\frac{1}{30n!}\int_0^1 F_n(t)f^{(n)}(t)dt\right| &\leq \frac{1}{30n!}\left[\int_0^1|F_n(t)|^q dt\right]^{\frac{1}{q}}\left\|f^{(n)}\right\|_p\\ &= K(n,p)\|f^{(n)}\|_p\end{aligned}$$

Having in mind Remark 3.21, from (3.142) and the above inequality, we obtain (3.151). Similarly, from (3.141) we obtain (3.152). □

Remark 3.25 Taking $p = \infty$ and $n = 1,2$ in Theorem 3.25, i.e. (3.151), we get

$$\left|\int_0^1 f(t)dt - D(0,1)\right| \leq K(n,\infty)\|f^{(n)}\|_\infty,$$

where

$$K(1,\infty) = \frac{113}{1800}, \quad K(2,\infty) = \frac{697}{162000}.$$

Taking $p = 1$ and $n = 1,2$, we get

$$\left|\int_0^1 f(t)dt - D(0,1)\right| \leq K(n,1)\|f^{(n)}\|_1,$$

where

$$K(1,1) = \frac{2}{15}, \quad K(2,1) = \frac{49}{7200}.$$

Comparing these estimates with the analogous ones obtained for Bullen-Simpson's formula shows that these are better in all cases except for $n = 2$ and $p = \infty$.

Moreover, for $p = \infty$ and $n = 3,4,5$ we obtain

$$\left|\int_0^1 f(t)dt - D(0,1) + \frac{1}{240}[f'(1) - f'(0)]\right| \leq K(n,\infty)\|f^{(n)}\|_\infty,$$

where

$$K(3,\infty) = \frac{253}{2880000}, \quad K(4,\infty) = \frac{1}{233280}, \quad K(5,\infty) = \frac{1}{3686400},$$

and for $p = 1$ and $n = 3,4,5$ we get

$$\left|\int_0^1 f(t)dt - D(0,1) + \frac{1}{240}[f'(1) - f'(0)]\right| \leq K(n,1)\|f^{(n)}\|_1,$$

where

$$K(3,1) = \frac{28 + 19\sqrt{19}}{648000}, \quad K(4,1) = \frac{1}{92160}, \quad K(5,1) = \frac{1}{1866240}.$$

Finally, for $p = 2$ we get

$$\left|\int_0^1 f(t)dt - D(0,1)\right| \leq K(n,2)\|f^{(n)}\|_2,$$

where

$$K(1,2) = \frac{\sqrt{19}}{60}, \quad K(2,2) = \frac{\sqrt{3}}{360},$$

and

$$\left|\int_0^1 f(t)dt - D(0,1) + \frac{1}{240}[f'(1) - f'(0)]\right| \leq K(n,2)\|f^{(n)}\|_2,$$

where

$$K(3,2) = \frac{\sqrt{105}}{100800}, \quad K(4,2) = \frac{\sqrt{70}}{1612800}, \quad K(5,2) = \frac{\sqrt{5005}}{212889600}.$$

Remark 3.26 Note that $K^*(1,p) = K(1,p)$, for $1 < p \leq \infty$, since $G_1(t) = F_1(t)$. Also, for $1 < p \leq \infty$, we can easily calculate $K(1,p)$. Namely,

$$K(1,p) = \frac{1}{60}\left[\frac{7^{q+1}+8^{q+1}}{15(q+1)}\right]^{\frac{1}{q}}.$$

In the limit case when $p \to 1$, that is when $q \to \infty$, we have

$$\lim_{p\to 1} K(1,p) = \frac{2}{15} = K(1,1).$$

Now we use formula (3.141) and a Grüss type inequality to obtain estimations of corrected Bullen-Simpson's formulae in terms of oscillation of derivatives of a function. To do this, we need the following two technical lemmas. The first one was proved in [27] and the second one is the key result from [83].

Lemma 3.9 *Let $k \geq 1$ and $\gamma \in \mathbb{R}$. Then*

$$\int_0^1 B_k^*(\gamma - t)dt = 0.$$

Lemma 3.10 *Let $F, G : [0,1] \to \mathbb{R}$ be two integrable functions. If*

$$m \leq F(t) \leq M, \qquad 0 \leq t \leq 1$$

and

$$\int_0^1 G(t)dt = 0,$$

then

$$\left|\int_0^1 F(t)G(t)dt\right| \leq \frac{M-m}{2} \cdot \int_0^1 |G(t)|dt. \tag{3.153}$$

Theorem 3.26 *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n)}$ exists and is integrable on $[0,1]$, for some $n \geq 1$. Suppose*

$$m_n \leq f^{(n)}(t) \leq M_n, \qquad 0 \leq t \leq 1,$$

for some constants m_n and M_n. Then

$$\left|\int_0^1 f(t)dt - D(0,1) + T_n(0,1)\right| \leq C_n(M_n - m_n) \tag{3.154}$$

where

$$C_1 = \frac{113}{3600}, \quad C_2 = \frac{19\sqrt{19}}{81000}, \quad C_3 = \frac{253}{5760000}, \quad C_4 = \frac{1}{466560},$$
$$C_{2k-1} = \frac{2^{2-2k}(1-2^{-2k})}{15(2k)!}|B_{2k}|, \quad C_{2k} = \frac{2^{-2k}(1-16\cdot 2^{-2k})}{15(2k)!}|B_{2k}|, \quad k \geq 3.$$

Proof. Lemma 3.9 ensures that the second condition of Lemma 3.10 is satisfied. Having in mind Remark 3.21, apply inequality (3.153) to obtain the estimate for $|\tilde{R}_n^{(1)}(f)|$. Now our statement follows easily from Corollary 3.20 for $n \geq 5$ and direct calculation for $n = 1,2,3,4$. □

3.3 General Simpson formulae

3.3.1 General Euler-Simpson formulae

The results from this section are published in [65].

Here we study the general Simpson quadrature formula

$$\int_0^1 f(t)dt = \frac{1}{2u+v}\left[uf(0)+vf\left(\frac{1}{2}\right)+uf(1)\right]+E(f;u,v) \tag{3.155}$$

with $E(f;u,v)$ being the remainder, $u,v\in\mathbb{Z}^+$ and the greatest common divisor of u and v is 1. This quadrature formula was considered by V. Čuljak, C. E. M. Pearce and J. Pečarić in [20]. The aim of this section is to establish general Simpson formula (3.155) using identities (1.1) and (1.2) and give various error estimates for the quadrature rules based on such generalizations. We use the extended Euler formulae for $a=0,\ b=1$ to obtain two new integral identities. We call them the general Euler-Simpson formulae. After that we prove a number of inequalities which give error estimates for the general Euler-Simpson formulae for functions whose derivatives are in L_p-spaces.

For $k\geq 1$ define the functions $G_k(t)$ and $F_k(t)$ as

$$G_k(t)=2uB_k^*(1-t)+vB_k^*\left(\frac{1}{2}-t\right),\ t\in\mathbb{R}$$

and

$$F_k(t)=G_k(t)-\tilde{B}_k,\ t\in\mathbb{R},\ k\geq 1,$$

where

$$\tilde{B}_k=uB_k(0)+vB_k\left(\frac{1}{2}\right)+uB_k(1),\ k\geq 1.$$

Using $B_1(t)=t-1/2$ we get $\tilde{B}_1=0$. Also, for $k\geq 2$ we have $\tilde{B}_k=G_k(0)$, that is

$$F_k(t)=G_k(t)-G_k(0),\ k\geq 2,\ \text{ and }\ F_1(t)=G_1(t),\ \ t\in\mathbb{R}.$$

Obviously, $G_k(t)$ and $F_k(t)$ are periodic functions of period 1 and continuous for $k\geq 2$.

Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(n-1)}$ exists on $[0,1]$ for some $n\geq 1$. We introduce the following notation

$$D(u,v)=\frac{1}{2u+v}\left[uf(0)+vf\left(\frac{1}{2}\right)+uf(1)\right].$$

Further, we define $\tilde{T}_0(u,v)=0$ and, for $1\leq m\leq n$,

$$\tilde{T}_m(u,v)=\frac{1}{2u+v}\left[uT_m(0)+vT_m\left(\frac{1}{2}\right)+uT_m(1)\right],$$

where $T_m(x)$ is given by (1.3) (for $a=0,\ b=1$). For $m\geq 1$

$$\tilde{T}_m(u,v)=\frac{1}{2u+v}\sum_{k=1}^{m}\frac{\tilde{B}_k}{k!}\left[f^{(k-1)}(1)-f^{(k-1)}(0)\right]. \tag{3.156}$$

In the next theorem we establish two formulae which play the key role in this paper. We call them the general Euler-Simpson formulae.

Theorem 3.27 *Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$, for some $n\geq 1$. Then*

$$\int_0^1 f(t)\mathrm{d}t=D(u,v)-\tilde{T}_n(u,v)+\tilde{R}_n^1(f), \tag{3.157}$$

and

$$\int_0^1 f(t)\mathrm{d}t=D(u,v)-\tilde{T}_{n-1}(u,v)+\tilde{R}_n^2(f), \tag{3.158}$$

where

$$\tilde{R}_n^1(f)=\frac{1}{(2u+v)(n!)}\int_0^1 G_n(t)\,\mathrm{d}f^{(n-1)}(t),$$

and

$$\tilde{R}_n^2(f)=\frac{1}{(2u+v)(n!)}\int_0^1 F_n(t)\,\mathrm{d}f^{(n-1)}(t).$$

Proof. Put $x=0,\ 1/2,\ 1$ in formula (1.1) to get three new formulae. Then multiply these new formulae by u, v, u respectively, and add them up. The result is formula (3.157). Formula (3.158) is obtained from (1.2) by the same procedure. □

Remark 3.27 If in Theorem 3.27 we chose $u=1$ and $v=4$ we get Euler Simpson formulae [29], for $u=1$ and $v=2$ Euler bitrapezoid formulae [28] and for $u=7$ and $v=16$ we get corrected Euler Simpson formulae [50].

By direct calculations we get

$$F_1(t)=G_1(t)=\begin{cases}-(2u+v)t+u, & 0<t\leq 1/2\\ -(2u+v)t+u+v, & 1/2<t\leq 1\end{cases}, \tag{3.159}$$

$$G_2(t)=\begin{cases}(2u+v)t^2-2ut+(4u-v)/12, & 0\leq t\leq 1/2\\ (2u+v)t^2-2(u+v)t+(4u+11v)/12, & 1/2<t\leq 1\end{cases}, \tag{3.160}$$

$$F_2(t)=\begin{cases}(2u+v)t^2-2ut, & 0\leq t\leq 1/2\\ (2u+v)t^2-2(u+v)t+v, & 1/2<t\leq 1\end{cases}, \tag{3.161}$$

$$F_3(t)=G_3(t)=\begin{cases}-(2u+v)t^3+3ut^2-(4u-v)t/4, & 0\leq t\leq 1/2\\ -(2u+v)t^3+(3u+3v)t^2-(4u+11v)t/4+3v/4, & 1/2<t\leq 1\end{cases}. \tag{3.162}$$

Now, we will prove some properties of the functions $G_k(t)$ and $F_k(t)$ defined above. The Bernoulli polynomials are symmetric with respect to $1/2$, that is $[1, 23.1.8]$

$$B_k(1-t) = (-1)^k B_k(t),\ \forall t \in \mathbb{R},\ k \geq 1. \tag{3.163}$$

Also, we have

$$B_k(1) = B_k(0) = B_k,\ k \geq 2,\ \ B_1(1) = -B_1(0) = \frac{1}{2}$$

and

$$B_{2j-1} = 0,\ j \geq 2.$$

Therefore, using $[1, 23.1.21, 23.1.22]$

$$B_{2j}\left(\frac{1}{2}\right) = -\left(1-2^{1-2j}\right)B_{2j},\ \ B_{2j}\left(\frac{1}{4}\right) = -2^{-2j}\left(1-2^{1-2j}\right)B_{2j}\quad j \geq 1,$$

we get

$$\tilde{B}_{2j-1} = 0,\ j \geq 1 \tag{3.164}$$

and

$$\tilde{B}_{2j} = 2uB_{2j} + vB_{2j}\left(\frac{1}{2}\right) = \left[2u - v(1-2^{1-2j})\right]B_{2j},\ j \geq 1. \tag{3.165}$$

Now, by (3.164) we have

$$F_{2j-1}(t) = G_{2j-1}(t),\ j \geq 1, \tag{3.166}$$

and, by (3.165),

$$F_{2j}(t) = G_{2j}(t) - \tilde{B}_{2j} = G_{2j}(t) - \left[2u - v(1-2^{1-2j})\right]B_{2j},\ j \geq 1. \tag{3.167}$$

Further, the points 0 and 1 are the zeros of $F_k(t) = G_k(t) - G_k(0)$, $k \geq 2$, that is

$$F_k(0) = F_k(1) = 0,\ k \geq 2.$$

As we shall see below, for $j \geq 1$, 0 and 1 are the only zeros of $F_{2j}(t)$ for $v \leq u$ or $v \geq 4u$. Next, setting $t = 1/2$ in (3.163) we get

$$B_k\left(\frac{1}{2}\right) = (-1)^k B_k\left(\frac{1}{2}\right),\ k \geq 1.$$

which implies that

$$B_{2j-1}\left(\frac{1}{2}\right) = 0,\ j \geq 1.$$

Using the above formulae, we get

$$F_{2j-1}\left(\frac{1}{2}\right) = G_{2j-1}\left(\frac{1}{2}\right) = 0,\ j \geq 1.$$

We shall see that for $j \geq 2$, 0, $1/2$ and 1 are the only zeros of $F_{2j-1}(t) = G_{2j-1}(t)$, for $v \leq u$ or $v \geq 4u$. Also, note that

$$G_{2j}\left(\frac{1}{2}\right) = 2uB_{2j}\left(\frac{1}{2}\right) + vB_{2j} = \left[v - 2u(1-2^{1-2j})\right] B_{2j},\ j \geq 1,$$

$$F_{2j}\left(\frac{1}{2}\right) = G_{2j}\left(\frac{1}{2}\right) - \tilde{B}_{2j} = (v-2u)(2-2^{1-2j})B_{2j},\ j \geq 1. \tag{3.168}$$

Lemma 3.11 *For $k \geq 2$ we have*

$$G_k(1-t) = (-1)^k G_k(t),\ 0 \leq t \leq 1,$$

and

$$F_k(1-t) = (-1)^k F_k(t),\ 0 \leq t \leq 1.$$

Proof. As we noted in introduction, the functions $B_k^*(t)$ are periodic with period 1 and continuous for $k \geq 2$. Therefore, for $k \geq 2$ and $0 \leq t \leq 1$ we have

$$G_k(1-t) = 2uB_k^*(t) + vB_k^*\left(-\frac{1}{2}+t\right)$$

$$= \begin{cases} 2uB_k(t) + vB_k\left(\frac{1}{2}+t\right), & 0 \leq t \leq 1/2, \\ 2uB_k(t) + vB_k\left(-\frac{1}{2}+t\right), & 1/2 < t \leq 1, \end{cases}$$

$$= (-1)^k \times \begin{cases} 2uB_k(1-t) + vB_k\left(\frac{1}{2}-t\right), & 0 \leq t \leq 1/2, \\ 2uB_k(1-t) + vB_k\left(\frac{3}{2}-t\right), & 1/2 < t \leq 1, \end{cases} = (-1)^k G_k(t),$$

which proves the first identity. Further, we have $F_k(t) = G_k(t) - G_k(0)$ and $(-1)^k G_k(0) = G_k(0)$, since $G_{2j+1}(0) = 0$, so that we have

$$F_k(1-t) = G_k(1-t) - G_k(0) = (-1)^k \left[G_k(t) - G_k(0)\right] = (-1)^k F_k(t),$$

which proves the second identity. □

Note that the identities established in Lemma 3.11 are valid for $k = 1$, too, except at the points 0, $1/2$ and 1 of discontinuity of $F_1(t) = G_1(t)$.

Lemma 3.12 *For $k \geq 2$ and $v \leq u$ or $v \geq 4u$ the function $G_{2k-1}(t)$ has no zeros in the interval $(0, 1/2)$. The sign of this function is determined by*

$$(-1)^{k-1} G_{2k-1}(t) > 0,\ 0 < t < \frac{1}{2} \text{ for } v \leq u,$$

$$(-1)^k G_{2k-1}(t) > 0,\ 0 < t < \frac{1}{2} \text{ for } v \geq 4u.$$

Proof. For $k = 2$, $G_3(t)$ is given by (3.162) and it is easy to see that

$$-G_3(t) > 0,\ 0 < t < \frac{1}{2},\ v \leq u, \qquad G_3(t) > 0,\ 0 < t < \frac{1}{2},\ v \geq 4u,$$

Thus, our assertion is true for $k=2$. Now, assume that $k \geq 3$. Then $2k-1 \geq 5$ and $G_{2k-1}(t)$ is continuous and at least twice differentiable function. Using (A-2) we get

$$G'_{2k-1}(t) = -(2k-1)G_{2k-2}(t)$$

and

$$G''_{2k-1}(t) = (2k-1)(2k-2)G_{2k-3}(t).$$

Let us suppose that G_{2k-3} has no zeros in the interval $(0,1/2)$. We know that 0 and $1/2$ are the zeros of $G_{2k-1}(t)$. Let us suppose that some α, $0 < \alpha < 1/2$, is also a zero of $G_{2k-1}(t)$. Then inside each of the intervals $(0,\alpha)$ and $(\alpha,1/2)$ the derivative $G'_{2k-1}(t)$ must have at least one zero, say β_1, $0 < \beta_1 < \alpha$ and β_2, $\alpha < \beta_2 < 1/2$. Therefore, the second derivative $G''_{2k-1}(t)$ must have at least one zero inside the interval (β_1,β_2). Thus, from the assumption that $G_{2k-1}(t)$ has a zero inside the interval $(0,1/2)$, it follows that $(2k-1)(2k-2)G_{2k-3}(t)$ also has a zero inside this interval. Thus, $G_{2k-1}(t)$ can not have a zero inside the interval $(0,1/2)$. To determine the sign of $G_{2k-1}(t)$, note that

$$G_{2k-1}\left(\frac{1}{4}\right) = (v-2u)B_{2k-1}\left(\frac{1}{4}\right).$$

We have $[1,23.1.14]$

$$(-1)^k B_{2k-1}(t) > 0,\ 0 < t < \frac{1}{2},$$

which implies

$$(-1)^{k-1}G_{2k-1}\left(\frac{1}{4}\right) = (-1)^k(2u-v)B_{2k-1}\left(\frac{1}{4}\right) > 0 \text{ for } v \leq u$$

and

$$(-1)^k G_{2k-1}\left(\frac{1}{4}\right) > 0 \text{ for } v \geq 4u$$

So, we proved our assertions. □

Corollary 3.21 *For $k \geq 2$ and $v \leq u$, functions $(-1)^k F_{2k}(t)$ and $(-1)^k G_{2k}(t)$ are strictly increasing on the interval $(0,1/2)$, and strictly decreasing on the interval $(1/2,1)$. Also, for $k \geq 2$ and $v \geq 4u$, functions $(-1)^{k-1}F_{2k}(t)$ and $(-1)^{k-1}G_{2k}(t)$ are strictly increasing on the interval $(0,1/2)$, and strictly decreasing on the interval $(1/2,1)$. Further, for $k \geq 2$, $v \leq u$ or $v \geq 4u$ we have*

$$\max_{t\in[0,1]} |F_{2k}(t)| = \left(2-2^{1-2k}\right)|(v-2u)B_{2k}|,$$

and

$$\max_{t\in[0,1]} |G_{2k}(t)| = \begin{cases} \left[2u-v(1-2^{1-2k})\right]|B_{2k}|, & \text{for } v \leq u, \\ \left[v-2u(1-2^{1-2k})\right]|B_{2k}|, & \text{for } v \geq 4u. \end{cases}$$

Proof. Using (A-2) for $v \leq u$ we get

$$\left[(-1)^k F_{2k}(t)\right]' = \left[(-1)^k G_{2k}(t)\right]' = 2k(-1)^{k-1}G_{2k-1}(t)$$

and $(-1)^{k-1}G_{2k-1}(t) > 0$ for $0 < t < 1/2$, by Lemma 3.12. Thus, $(-1)^k F_{2k}(t)$ and $(-1)^k G_{2k}(t)$ are strictly increasing on the interval $(0,1/2)$. Also, by Lemma 3.11, we have $F_{2k}(1-t) = F_{2k}(t)$, $0 \leq t \leq 1$ and $G_{2k}(1-t) = G_{2k}(t)$, $0 \leq t \leq 1$, which implies that $(-1)^k F_{2k}(t)$ and $(-1)^k G_{2k}(t)$ are strictly decreasing on the interval $(1/2,1)$. The proof of the second assertion is similar. Further, $F_{2k}(0) = F_{2k}(1) = 0$, which implies that $|F_{2k}(t)|$ achieves its maximum at $t = 1/2$, that is

$$\max_{t\in[0,1]} |F_{2k}(t)| = \left|F_{2k}\left(\frac{1}{2}\right)\right| = \left(2 - 2^{1-2k}\right)|(v-2u)B_{2k}|.$$

Also

$$\begin{aligned}\max_{t\in[0,1]} |G_{2k}(t)| &= \max\left\{|G_{2k}(0)|, \left|G_{2k}\left(\frac{1}{2}\right)\right|\right\} \\ &= \begin{cases}\left[2u - v(1-2^{1-2k})\right]|B_{2k}|, & \text{for } v \leq u, \\ \left[v - 2u(1-2^{1-2k})\right]|B_{2k}|, & \text{for } v \geq 4u.\end{cases}\end{aligned}$$

which completes the proof. □

Corollary 3.22 *For $k \geq 2$, and $v \leq u$ or $v \geq 4u$ we have*

$$\int_0^1 |F_{2k-1}(t)|\,dt = \int_0^1 |G_{2k-1}(t)|\,dt = \frac{\left(2-2^{1-2k}\right)}{k}|(v-2u)B_{2k}|.$$

Also, we have

$$\int_0^1 |F_{2k}(t)|\,\mathrm{d}t = \left|\tilde{B}_{2k}\right| = \left|[2u - v(1-2^{1-2j})]B_{2k}\right|$$

and

$$\int_0^1 |G_{2k}(t)|\,\mathrm{d}t \leq 2\left|\tilde{B}_{2k}\right| = 2\left|[2u - v(1-2^{1-2j})]B_{2k}\right|.$$

Proof. Using (A-2) it is easy to see that

$$G_m'(t) = -mG_{m-1}(t),\ m \geq 3. \tag{3.169}$$

Now, using Lemma 3.11, Lemma 3.12 and (3.169) we get

$$\begin{aligned}\int_0^1 |G_{2k-1}(t)|\,dt &= 2\left|\int_0^{1/2} G_{2k-1}(t)dt\right| = 2\left|-\frac{1}{2k}\,G_{2k}(t)|_0^{1/2}\right| \\ &= \frac{1}{k}\left|G_{2k}\left(\frac{1}{2}\right) - G_{2k}(0)\right| = \frac{\left(2-2^{1-2k}\right)}{k}|(v-2u)B_{2k}|,\end{aligned}$$

which proves the first assertion. By Corollary 3.21 and because $F_{2k}(0) = F_{2k}(1) = 0$, $F_{2k}(t)$ does not change sign on the interval $(0,1)$. Therefore, using (3.167) and (3.169), we get

$$\begin{aligned}\int_0^1 |F_{2k}(t)|\,dt &= \left|\int_0^1 F_{2k}(t)dt\right| = \left|\int_0^1 \left[G_{2k}(t) - \tilde{B}_{2k}\right] dt\right| \\ &= \left|-\frac{1}{2k+1}\, G_{2k+1}(t)|_0^1 - \tilde{B}_{2k}\right| = \left|\tilde{B}_{2k}\right|,\end{aligned}$$

which proves the second assertion. Finally, we use (3.167) again and the triangle inequality to obtain

$$\int_0^1 |G_{2k}(t)|\,dt = \int_0^1 \left|F_{2k}(t) + \tilde{B}_{2k}\right| dt \leq \int_0^1 |F_{2k}(t)|\,dt + \left|\tilde{B}_{2k}\right| = 2\left|\tilde{B}_{2k}\right|,$$

which proves the third assertion. □

Now, we use formulae established in Theorem 3.27 to prove a number of inequalities using L_p norms for $1 \leq p \leq \infty$. These inequalities are generally sharp (in case $p = 1$ the best possible).

Theorem 3.28 *Assume (p,q) is a pair of conjugate exponents, $1 \leq p,q \leq \infty$. If $f : [0,1] \to \mathbb{R}$ is such that $f^{(n)} \in L_p[0,1]$ for some $n \geq 1$. Then we have*

$$\left|\int_0^1 f(t)\mathrm{d}t - D(u,v) + \tilde{T}_{n-1}(u,v)\right| \leq K(n,p;u,v) \cdot \|f^{(n)}\|_p, \tag{3.170}$$

and

$$\left|\int_0^1 f(t)\mathrm{d}t - D(u,v) + \tilde{T}_n(u,v)\right| \leq K^*(n,p;u,v) \cdot \|f^{(n)}\|_p, \tag{3.171}$$

where

$$K(n,p;u,v) = \frac{1}{(2u+v)(n!)} \left[\int_0^1 |F_n(t)|^q\,\mathrm{d}t\right]^{1/q} \quad \textit{and}$$

$$K^*(n,p;u,v) = \frac{1}{(2u+v)(n!)} \left[\int_0^1 |G_n(t)|^q\,\mathrm{d}t\right]^{1/q}.$$

The constants $K(n,p;u,v)$ and $K^*(n,p;u,v)$ are sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof. The proof is analogous to the proof of Theorem 2.2. □

Corollary 3.23 *Let $f : [0,1] \to \mathbb{R}$ be a L-Lipschitzian function on $[0,1]$. Then*

$$\left|\int_0^1 f(t)\mathrm{d}t - D(u,v)\right| \leq \frac{4u^2+v^2}{4(2u+v)^2} \cdot L.$$

If f' is L-Lipschitzian on $[0,1]$, then

$$\left|\int_0^1 f(t)\mathrm{d}t - D(u,v)\right| \leq \frac{48u^3 - 12u^2v + v^3}{24(2u+v)^3} \cdot L.$$

Proof. Using (3.159) and (3.160) we get

$$\int_0^1 |F_1(t)|\,\mathrm{d}t = \frac{4u^2+v^2}{4(2u+v)} \text{ and } \int_0^1 |F_2(t)|\,\mathrm{d}t = \frac{48u^3-12u^2v+v^3}{12(2u+v)^2}.$$

Therefore, applying (3.170) with $n = 1,2$ and $p = \infty$ we get the above inequalities. □

Remark 3.28 The estimation in the first inequality in Corollary 3.23 achieves minimum of $1/8$ for $u = 1$ and $v = 2$, which is bitrapezoid formula (see [20] and [28]) and the second achieves minimum of $1/81$ for $u = 1$ and $v = 4$, which is Simpson's formula (see [29]).

Remark 3.29 Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n-1)}$ is an L-Lipschitzian function on $[0,1]$ for some $n \geq 3$. Then for $v \leq u$ or $v \geq 4u$, from Corollary 3.22 we get

$$K(2k-1,\infty;u,v) = \frac{2}{(2u+v)\,[(2k)!]}(2-2^{1-2k})\,|(v-2u)B_{2k}|,$$

$$K^*(2k,\infty;u,v) = \frac{1}{(2u+v)\,[(2k)!]}\left|[2u-v(1-2^{1-2k})]B_{2k}\right|$$

and

$$K(2k,\infty;u,v) = \frac{2}{(2u+v)\,[(2k)!]}\left|[2u-v(1-2^{1-2k})]B_{2k}\right|.$$

Corollary 3.24 *Let $f : [0,1] \to \mathbb{R}$ be a continuous function of bounded variation on $[0,1]$. Then*

$$\left|\int_0^1 f(t)\mathrm{d}t - D(u,v)\right| \leq \frac{1}{4}\left[1+\frac{|2u-v|}{2u+v}\right]\cdot V_0^1(f).$$

If f' is a continuous function of bounded variation on $[0,1]$, then

$$\left|\int_0^1 f(t)\mathrm{d}t - D(u,v)\right| \leq \frac{1}{16(2u+v)^2}[v^2+|8u^2-v^2|]\cdot V_0^1(f').$$

Proof. From (3.159) and (3.160), we get

$$\max_{t\in[0,1]}|F_1(t)| = \max\left\{u,\frac{v}{2}\right\} = \frac{1}{4}[2u+v+|2u-v|] \text{ and}$$
$$\max_{t\in[0,1]}|F_2(t)| = \max\left\{\frac{u^2}{2u+v},\frac{v-2u}{4}\right\} = \frac{1}{8(2u+v)}[v^2+|8u^2-v^2|].$$

Therefore, applying (3.170) with $n = 1,2$ and $p = 1$ we get the above inequalities. □

Remark 3.30 The estimations in inequalities in Corollary 3.24 achieve minimuma of $1/4$ and $1/32$ for $u = 1$ and $v = 2$ which is bitrapezoid formula (see [28]).

Remark 3.31 Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$ for some $n \geq 3$. Then for $v \leq u$ and $v \geq 4u$ from Corollary 3.21 we get

$$K(2k-1,1;u,v) = \frac{1}{(2u+v)\,[(2k-1)!]} \max_{t\in[0,1]} |F_{2k-1}(t)|,$$

$$K^*(2k,1;u,v) = \frac{\left(2-2^{1-2k}\right)}{(2u+v)\,[(2k)!]} |(v-2u)B_{2k}|$$

and

$$K(2k,1;u,v) = \frac{1}{(2u+v)\,[(2k)!]} \begin{cases} \left[2u - v(1-2^{1-2k})\right] |B_{2k}|, & \text{for } v \leq u, \\ \left[v - 2u(1-2^{1-2k})\right] |B_{2k}|, & \text{for } v \geq 4u. \end{cases}$$

Now, we calculate the optimal constant for $p = 2$.

Corollary 3.25 *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n)} \in L_2[0,1]$ for some $n \geq 1$. Then, we have*

$$\begin{aligned} &\left| \int_0^1 f(t)\mathrm{d}t - D(u,v) + \tilde{T}_{n-1}(u,v) \right| \\ &\leq \frac{1}{(2u+v)} \left[\frac{(-1)^{n-1}}{(2n)!} \left[4u^2+v^2-4uv(1-2^{1-2n})\right] B_{2n} + \frac{\tilde{B}_n^2}{(n!)^2} \right]^{1/2} \|f^{(n)}\|_2, \end{aligned}$$

and

$$\begin{aligned} &\left| \int_0^1 f(t)\mathrm{d}t - D(u,v) + \tilde{T}_n(u,v) \right| \\ &\leq \frac{1}{(2u+v)} \left[\frac{(-1)^{n-1}}{(2n)!} \left[4u^2+v^2-4uv(1-2^{1-2n})\right] B_{2n} \right]^{1/2} \|f^{(n)}\|_2. \end{aligned}$$

Proof. Using integration by parts and also using Lemma 1 from [30] we have

$$\begin{aligned} \int_0^1 G_n^2(t)dt &= (-1)^{n-1} \frac{n(n-1)\dots 2}{(n+1)(n+2)\dots(2n-1)} \\ &\quad \cdot \left[-\frac{1}{2n} G_{2n}(t)G_1(t)|_0^1 + \frac{1}{2n}\int_0^1 G_{2n}(t)dG_1(t) \right] \\ &= (-1)^{n-1} \frac{(n!)^2}{(2n)!} \left[-(2u+v)\int_0^1 G_{2n}(t)dt + 2uG_{2n}(0) + vG_{2n}\left(\frac{1}{2}\right) \right] \\ &= (-1)^{n-1} \frac{(n!)^2}{(2n)!} \left[4uvB_{2n}\left(\frac{1}{2}\right) + (4u^2+v^2)B_{2n} \right] \\ &= (-1)^{n-1} \frac{(n!)^2}{(2n)!} \left[4u^2+v^2-4uv(1-2^{1-2n})\right] B_{2n}. \end{aligned}$$

Now,

$$\begin{aligned}\int_0^1 F_n^2(t)dt &= \int_0^1 \left[G_n(t)-\tilde{B}_n\right]^2 dt\\ &= \int_0^1 \left[G_n^2(t)-2G_n(t)\tilde{B}_n+\tilde{B}_n^2\right] dt = \int_0^1 G_n^2(t)dt+\tilde{B}_n^2\\ &= (-1)^{n-1}\frac{(n!)^2}{(2n)!}\left[4u^2+v^2-4uv(1-2^{1-2n})\right]B_{2n}+\tilde{B}_n^2.\end{aligned}$$

□

Finally, we give the values of optimal constant for $n=1$ and arbitrary p from Theorem 3.28.

Remark 3.32 Note that $K^*(1,p;u,v)=K(1,p;u,v)$, for $1<p\leq\infty$, since $G_1(t)=F_1(t)$. Also, for $1<p\leq\infty$ we can easily calculate $K(1,p;u,v)$. We get

$$K(1,p;u,v)=\frac{1}{2(2u+v)}\left[\frac{(2u)^{q+1}+v^{q+1}}{(2u+v)(q+1)}\right]^{\frac{1}{q}},\ 1<p\leq\infty.$$

Now we use the formula (3.157) and one technical result from [83] to obtain Grüss type inequality related to that general Euler-Simpson formula:

Theorem 3.29 *Suppose that $f:[0,1]\to\mathbb{R}$ is such that $f^{(n)}$ exists and is integrable on $[0,1]$, for some $n\geq 1$. Assume that*

$$m_n\leq f^{(n)}(t)\leq M_n,\ 0\leq t\leq 1,$$

for some constants m_n and M_n. Then

$$\left|\int_0^1 f(t)\mathrm{d}t-D(u,v)+\tilde{T}_n(u,v)\right|\leq C_n(M_n-m_n), \tag{3.172}$$

where $C_n=\frac{1}{(2u+v)(n!)}\int_0^1|G_n(t)|dt$.

Our final results are connected with the series expansion of a function in Bernoulli polynomials.

Theorem 3.30 *If $f:[0,1]\to\mathbb{R}$ is such that $f^{(2k)}$ is a continuous function on $[0,1]$, for some $k\geq 2$ and $v\leq u$ or $v\geq 4u$, then there exists a point $\eta\in[0,1]$ such that*

$$\tilde{R}_{2k}^2(f)=-\frac{[2u-v(1-2^{1-2k})]B_{2k}}{(2u+v)[(2k)!]}f^{(2k)}(\eta). \tag{3.173}$$

Proof. We can rewrite $\tilde{R}_{2k}^2(f)$ for $v\leq u$ as $\tilde{R}_{2k}^2(f)=(-1)^k\frac{J_k}{(2u+v)[(2k)!]}$, where $J_k=\int_0^1(-1)^kF_{2k}(s)f^{(2k)}(s)ds$. From Corollary 3.21 follows that $(-1)^kF_{2k}(s)\geq 0$, $0\leq s\leq 1$ and the claim from the mean value theorem for integrals and Corollary 3.22. The proof for $v\geq 4u$ is similar. □

Remark 3.33 For $k=2$ formula (3.173) reduces to

$$\tilde{R}_4^2(f)=\frac{16u-7v}{5760(2u+v)}f^{(4)}(\eta).$$

Now we study, the general Simpson quadrature formula

$$\int_0^1 f(t)dt=\frac{1}{u+v+w}\left[uf(0)+vf\left(\frac{1}{2}\right)+wf(1)\right]+E(f;u,v,w) \tag{3.174}$$

with $E(f;u,v,w)$ being the remainder, $u,v,w\in\mathbb{Z}^+$ and we are using identities (2.36) and (2.37) to get two new identities of Euler type (see [20]).

Theorem 3.31 *Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$, for some $n\geq 1$. Then*

$$\int_0^1 f(t)\mathrm{d}t=D(u,v,w)-\overline{T}_n(u,v,w)+\overline{R}_n^1(f), \tag{3.175}$$

and

$$\int_0^1 f(t)\mathrm{d}t=D(u,v,w)-\overline{T}_{n-1}(u,v,w)+\overline{R}_n^2(f), \tag{3.176}$$

where

$$D(u,v,w)=\frac{1}{u+v+w}\left[uf(0)+vf\left(\frac{1}{2}\right)+wf(1)\right],$$

$$\overline{R}_n^1(f)=\frac{1}{(u+v+w)(n!)}\int_0^1 \overline{G}_n(t)\,\mathrm{d}f^{(n-1)}(t),$$

$$\overline{R}_n^2(f)=\frac{1}{(u+v+w)(n!)}\int_0^1 \overline{F}_n(t)\,\mathrm{d}f^{(n-1)}(t),$$

$$\overline{G}_k(t)=(u+w)B_k^*(1-t)+vB_k^*\left(\frac{1}{2}-t\right),\quad t\in\mathbb{R},$$

$$\overline{F}_k(t)=\overline{G}_k(t)-\overline{B}_k,\quad t\in\mathbb{R},\quad k\geq 1,$$

$$\overline{B}_k=uB_k(0)+vB_k\left(\frac{1}{2}\right)+wB_k(1),\quad k\geq 1$$

and

$$\overline{T}_m(u,v,w)=\frac{1}{u+v+w}\left[uT_m(0)+vT_m\left(\frac{1}{2}\right)+wT_m(1)\right].$$

Proof. Put $x=0,\ 1/2,\ 1$ in formula (1.1) to get three new formulae. Then multiply these new formulae by u, v, w respectively, and add them up. The result is formula (3.175). Formula (3.176) is obtained from (1.2) by the same procedure. □

Theorem 3.32 *Assume (p_1,q_1) and (p_2,q_2) are two pairs of conjugate exponents, $1 \le p_1,q_1,p_2,q_2 \le \infty$. Let $x \in [0,1]$ and $f:[0,1] \to \mathbb{R}$ be such that $f^{(n)} \in L_{p_1}[0,x]$ and $f^{(n)} \in L_{p_2}[x,1]$, for some $n \ge 1$. Then, we have*

$$\left| \int_0^1 f(t)\mathrm{d}t - D(u,v,w) + \overline{T}_{n-1}(u,v,w) \right| \tag{3.177}$$
$$\le K(n,p_1;u,v,w,x) \cdot \|f^{(n)}\|_{L_{p_1}[0,x]} + K(n,p_2;u,v,w,x) \cdot \|f^{(n)}\|_{L_{p_2}[x,1]},$$

and

$$\left| \int_0^1 f(t)\mathrm{d}t - D(u,v,w) + \overline{T}_n(u,v,w) \right| \tag{3.178}$$
$$\le K^*(n,p_1;u,v,w,x) \cdot \|f^{(n)}\|_{L_{p_1}[0,x]} + K^*(n,p_2;u,v,w,x) \cdot \|f^{(n)}\|_{L_{p_2}[x,1]},$$

where

$$K(n,p_1;u,v,w,x) = \frac{1}{(u+v+w)(n!)} \left[\int_0^x |\overline{F}_n(t)|^{q_1} \,\mathrm{d}t \right]^{1/q_1},$$
$$K(n,p_2;u,v,w,x) = \frac{1}{(u+v+w)(n!)} \left[\int_x^1 |\overline{F}_n(t)|^{q_2} \,\mathrm{d}t \right]^{1/q_2},$$
$$K^*(n,p_1;u,v,w,x) = \frac{1}{(u+v+w)(n!)} \left[\int_0^x \left|\overline{G}_n(t)\right|^{q_1} \mathrm{d}t \right]^{1/q_1} \quad \textit{and}$$
$$K^*(n,p_2;u,v,w,x) = \frac{1}{(u+v+w)(n!)} \left[\int_x^1 \left|\overline{G}_n(t)\right|^{q_2} \mathrm{d}t \right]^{1/q_2}.$$

The inequalities are sharp for $1 < p_1,p_2 \le \infty$ and the best possible for $p_1 = 1$ or $p_2 = 1$.

Proof. Applying the Hölder inequality we have

$$\begin{aligned}
&\left| \frac{1}{(u+v+w)(n!)} \int_0^1 \overline{F}_n(t) f^{(n)}(t)dt \right| \\
&= \left| \frac{1}{(u+v+w)(n!)} \int_0^x \overline{F}_n(t) f^{(n)}(t)dt + \frac{1}{(u+v+w)(n!)} \int_x^1 \overline{F}_n(t) f^{(n)}(t)dt \right| \\
&\le \frac{1}{(u+v+w)(n!)} \left\{ \left[\int_0^x |\overline{F}_n(t)|^{q_1} dt \right]^{1/q_1} \|f^{(n)}\|_{L_{p_1}[0,x]} \right. \\
&+ \left. \left[\int_x^1 |\overline{F}_n(t)|^{q_2} dt \right]^{1/q_2} \|f^{(n)}\|_{L_{p_2}[x,1]} \right\} \\
&= K(n,p_1;u,v,w,x) \|f^{(n)}\|_{L_{p_1}[0,x]} + K(n,p_2;u,v,w,x) \|f^{(n)}\|_{L_{p_2}[x,1]}.
\end{aligned}$$

Using the above inequality from (3.158) we get estimate (3.177). In the same manner, from (3.157) we get estimate (3.178). The proof of sharpness and best possibility is similar as in the proof of Theorem 2.2. □

Remark 3.34 For $n = 1$, $0 \le u \le \frac{1}{2} \le 1 - w \le 1$ and $u + v + w = 1$ in inequality (3.177) we get inequality

$$\left| \int_0^1 f(t)\mathrm{d}t - D(u,v,w) \right| \le \left[\frac{u^{q_1+1} + \left(\frac{1}{2} - u\right)^{q_1+1}}{q_1+1} \right]^{1/q_1} \|f'\|_{L_{p_1}[0,1/2]} + \left[\frac{w^{q_2+1} + \left(w - \frac{1}{2}\right)^{q_2+1}}{q_2+1} \right]^{1/q_2} \|f'\|_{L_{p_2}[1/2,1]},$$

which is inequality proved in [21] for $t_1 = u$, $t_2 = \frac{1}{2}$, $t_3 = 1 - w$.

Remark 3.35 Using formulae (3.177) and (3.178) we can also get the other inequalities from [21].

3.3.2 General dual Euler-Simpson formulae

Results from this section are published in [102].

Here we study the general dual Simpson quadrature formula

$$\int_0^1 f(t)dt = \frac{1}{2u - v} \left[uf\left(\frac{1}{4}\right) - vf\left(\frac{1}{2}\right) + uf\left(\frac{3}{4}\right) \right] + E(f;u,v) \tag{3.179}$$

with $E(f;u,v)$ being the remainder, $u, v \in \mathbb{Z}^+$, $v < 2u$ and the greatest common divisor of u and v is 1. The aim of this section is to establish general dual Simpson formula (3.179) using identities (1.1) and (1.2) and give various error estimates for the quadrature rules based on such generalizations. We use the extended Euler formulae for $a = 0$, $b = 1$ to obtain two new integral identities. We call them the general dual Euler-Simpson formulae. After that we prove a number of inequalities which give error estimates for the general dual Euler-Simpson formulae for functions whose derivatives are from the L_p-spaces.

For $k \ge 1$ define the functions $G_k^D(t)$ and $F_k^D(t)$ as

$$G_k^D(t) = uB_k^*\left(\frac{1}{4} - t\right) - vB_k^*\left(\frac{1}{2} - t\right) + uB_k^*\left(\frac{3}{4} - t\right),\ t \in \mathbb{R}$$

and

$$F_k^D(t) = G_k^D(t) - \tilde{B}_k^D,\ t \in \mathbb{R},\ k \ge 1,$$

where

$$\tilde{B}_k^D = uB_k\left(\frac{1}{4}\right) - vB_k\left(\frac{1}{2}\right) + uB_k\left(\frac{3}{4}\right),\ k \ge 1.$$

Especially, using $B_1(t) = t - 1/2$ we get $\tilde{B}_1^D = 0$. Also, for $k \ge 2$ we have $\tilde{B}_k^D = G_k^D(0)$, that is

$$F_k^D(t) = G_k^D(t) - G_k^D(0),\ k \ge 2,\ \text{ and }\ F_1^D(t) = G_1^D(t),\ \ t \in \mathbb{R}.$$

Obviously, $G_k^D(t)$ and $F_k^D(t)$ are periodic functions of period 1 and continuous for $k \ge 2$.

Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n-1)}$ exists on $[0,1]$ for some $n \geq 1$. We introduce the following notation

$$F(u,v) = \frac{1}{2u-v}\left[uf\left(\frac{1}{4}\right) - vf\left(\frac{1}{2}\right) + uf\left(\frac{3}{4}\right)\right].$$

Further, we define $\tilde{T}_0^D(u,v) = 0$ and, for $1 \leq m \leq n$,

$$\tilde{T}_m^D(u,v) = \frac{1}{2u-v}\left[uT_m\left(\frac{1}{4}\right) - vT_m\left(\frac{1}{2}\right) + uT_m\left(\frac{3}{4}\right)\right],$$

where $T_m(x)$ is given by (1.3). For $m \geq 1$

$$\tilde{T}_m^D(u,v) = \frac{1}{2u-v}\sum_{k=1}^{m}\frac{\tilde{B}_k}{k!}\left[f^{(k-1)}(1) - f^{(k-1)}(0)\right]. \tag{3.180}$$

In the next theorem we establish two formulae which play the key role in this section. We call them the general dual Euler-Simpson formulae.

Theorem 3.33 *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$, for some $n \geq 1$. Then*

$$\int_0^1 f(t)\mathrm{d}t = F(u,v) - \tilde{T}_n^D(u,v) + \tilde{R}_n^{D1}(f), \tag{3.181}$$

and

$$\int_0^1 f(t)\mathrm{d}t = F(u,v) - \tilde{T}_{n-1}^D(u,v) + \tilde{R}_n^{D2}(f), \tag{3.182}$$

where

$$\tilde{R}_n^{D1}(f) = \frac{1}{(2u-v)(n!)}\int_0^1 G_n(t)\,\mathrm{d}f^{(n-1)}(t),$$

and

$$\tilde{R}_n^{D2}(f) = \frac{1}{(2u-v)(n!)}\int_0^1 F_n(t)\,\mathrm{d}f^{(n-1)}(t).$$

Proof. Put $x = 1/4,\ 1/2,\ 3/4$ in formula (1.1) to get three new formulae. Then multiply these new formulae by $u,\ -v,\ u$ respectively, and add them up. The result is formula (3.181). Formula (3.182) is obtained from (1.2) by the same procedure. □

Remark 3.36 If in Theorem 3.33 we chose $u = 2$ and $v = 1$ we get dual Euler Simpson formulae [26] and for $u = 8$ and $v = 1$ we get corrected dual Euler Simpson formulae [52].

By direct calculations we get

$$F_1^D(t) = G_1^D(t) = \begin{cases} (v-2u)t, & 0 \leq t \leq 1/4 \\ (v-2u)t + u, & 1/4 < t \leq 1/2 \\ (v-2u)t + u - v, & 1/2 < t \leq 3/4 \\ (v-2u)t + 2u - v, & 3/4 < t \leq 1 \end{cases}, \tag{3.183}$$

$$G_2^D(t)=\begin{cases}(2u-v)t^2+(2v-u)/24, & 0\le t\le 1/4\\ (2u-v)t^2-2ut+(11u+2v)/24, & 1/4<t\le 1/2\\ (2u-v)t^2+(2v-2u)t+(11u-22v)/24, & 1/2<t\le 3/4\\ (2u-v)t^2+(2v-4u)t+(47u-22v)/24, & 3/4<t\le 1\end{cases}, \tag{3.184}$$

$$F_2^D(t)=\begin{cases}(2u-v)t^2, & 0\le t\le 1/4\\ (2u-v)t^2-2ut+u/2, & 1/4<t\le 1/2\\ (2u-v)t^2+(2v-2u)t+(u-2v)/2, & 1/2<t\le 3/4\\ (2u-v)t^2+(2v-4u)t+2u-v, & 3/4<t\le 1\end{cases}, \tag{3.185}$$

$$F_3^D(t)=G_3^D(t)=\begin{cases}(v-2u)t^3+(u-2v)t/8, & 0\le t\le 1/4\\ (v-2u)t^3+3ut^2-(11u+2v)t/8+3u/16, & 1/4<t\le 1/2\\ (v-2u)t^3+(3u-3v)t^2 & \\ +(22v-11u)t/8+(3u-12v)/16, & 1/2<t\le 3/4\\ (v-2u)t^3+(6u-3v)t^2 & \\ +(22v-47u)t/8+(15u-6v)/8, & 3/4<t\le 1\end{cases}. \tag{3.186}$$

Now, we will prove some properties of the functions $G_k^D(t)$ and $F_k^D(t)$ defined above. The Bernoulli polynomials are symmetric with respect to $1/2$, that is $[1,23.1.8]$

$$B_k(1-t)=(-1)^k B_k(t),\ \forall t\in\mathbb{R},\ k\ge 1. \tag{3.187}$$

Also, we have

$$B_k(1)=B_k(0)=B_k,\ k\ge 2,\ \ B_1(1)=-B_1(0)=\frac{1}{2}$$

and

$$B_{2j-1}=0,\ j\ge 2.$$

Therefore, using $[1,23.1.21,23.1.22]$

$$B_{2j}\left(\frac{1}{2}\right)=-\left(1-2^{1-2j}\right)B_{2j},\ \ B_{2j}\left(\frac{1}{4}\right)=-2^{-2j}\left(1-2^{1-2j}\right)B_{2j}\quad j\ge 1,$$

we get

$$\tilde{B}_{2j-1}^D=0,\ j\ge 1 \tag{3.188}$$

and for $j\ge 1$

$$\tilde{B}_{2j}^D=uB_{2j}\left(\frac{1}{4}\right)-vB_{2j}\left(\frac{1}{2}\right)+uB_{2j}\left(\frac{3}{4}\right)=(v-u\cdot 2^{1-2j})(1-2^{1-2j})B_{2j}. \tag{3.189}$$

Now, by (3.188) we have

$$F_{2j-1}^D(t)=G_{2j-1}^D(t),\ j\ge 1, \tag{3.190}$$

and, by (3.189),

$$F_{2j}^D(t)=G_{2j}^D(t)-\tilde{B}_{2j}^D=G_{2j}^D(t)-(v-u\cdot 2^{1-2j})(1-2^{1-2j})B_{2j},\ j\ge 1. \tag{3.191}$$

Further, the points 0 and 1 are the zeros of $F_k^D(t)=G_k^D(t)-G_k^D(0)$, $k\ge 2$, that is

$$F_k^D(0)=F_k^D(1)=0,\ k\ge 2.$$

As we shall see below, for $j \geq 1$, 0 and 1 are the only zeros of $F_{2j}^{D}(t)$ for $u/2 \leq v < 2u$. Next, setting $t = 1/2$ in (3.187) we get

$$B_k\left(\frac{1}{2}\right) = (-1)^k B_k\left(\frac{1}{2}\right),\ k \geq 1.$$

which implies that

$$B_{2j-1}\left(\frac{1}{2}\right) = 0,\ j \geq 1.$$

Using the above formulae, we get

$$F_{2j-1}^{D}\left(\frac{1}{2}\right) = G_{2j-1}^{D}\left(\frac{1}{2}\right) = 0,\ j \geq 1.$$

We shall see that for $j \geq 2$, 0, $1/2$ and 1 are the only zeros of $F_{2j-1}^{D}(t) = G_{2j-1}^{D}(t)$ for $u/2 \leq v < 2u$. Also, note that

$$G_{2j}^{D}\left(\frac{1}{2}\right) = uB_{2j}\left(\frac{3}{4}\right) - vB_{2j} + uB_{2j}\left(\frac{1}{4}\right) = \left[-v - u \cdot 2^{1-2j}(1-2^{1-2j})\right] B_{2j},\ j \geq 1,$$

$$F_{2j}^{D}\left(\frac{1}{2}\right) = G_{2j}^{D}\left(\frac{1}{2}\right) - \tilde{B}_{2j}^{D} = -2v(1-2^{-2j})B_{2j},\ j \geq 1. \tag{3.192}$$

Lemma 3.13 *For $k \geq 2$ we have*

$$G_k^{D}(1-t) = (-1)^k G_k^{D}(t),\ 0 \leq t \leq 1,$$

and

$$F_k^{D}(1-t) = (-1)^k F_k^{D}(t),\ 0 \leq t \leq 1.$$

Proof. As we noted in introduction, the functions $B_k^*(t)$ are periodic with period 1 and continuous for $k \geq 2$. Therefore, for $k \geq 2$ and $0 \leq t \leq 1$ we have

$$\begin{aligned}
G_k^{D}(1-t) &= uB_k^*\left(-\frac{3}{4}+t\right) - vB_k^*\left(-\frac{1}{2}+t\right) + uB_k^*\left(-\frac{1}{4}+t\right) \\
&= \begin{cases} uB_k^*\left(\frac{1}{4}+t\right) - vB_k^*\left(\frac{1}{2}+t\right) + uB_k^*\left(\frac{3}{4}+t\right), & 0 \leq t \leq 1/4, \\ uB_k^*\left(\frac{1}{4}+t\right) - vB_k^*\left(\frac{1}{2}+t\right) + uB_k^*\left(-\frac{1}{4}+t\right), & 1/4 < t \leq 1/2, \\ uB_k^*\left(\frac{1}{4}+t\right) - vB_k^*\left(-\frac{1}{2}+t\right) + uB_k^*\left(-\frac{1}{4}+t\right), & 1/2 < t \leq 3/4, \\ uB_k^*\left(-\frac{3}{4}+t\right) - vB_k^*\left(-\frac{1}{2}+t\right) + uB_k^*\left(-\frac{1}{4}+t\right), & 3/4 < t \leq 1, \end{cases} \\
&= (-1)^k \times \begin{cases} uB_k^*\left(\frac{3}{4}-t\right) - vB_k^*\left(\frac{1}{2}-t\right) + uB_k^*\left(\frac{1}{4}-t\right), & 0 \leq t \leq 1/4, \\ uB_k^*\left(\frac{3}{4}-t\right) - vB_k^*\left(\frac{1}{2}-t\right) + uB_k^*\left(\frac{5}{4}-t\right), & 1/4 < t \leq 1/2, \\ uB_k^*\left(\frac{3}{4}-t\right) - vB_k^*\left(\frac{3}{2}-t\right) + uB_k^*\left(\frac{5}{4}-t\right), & 1/2 < t \leq 3/4, \\ uB_k^*\left(\frac{7}{4}-t\right) - vB_k^*\left(\frac{3}{2}-t\right) + uB_k^*\left(\frac{5}{4}-t\right), & 3/4 < t \leq 1, \end{cases} \\
&= (-1)^k G_k^{D}(t),
\end{aligned}$$

which proves the first identity. Further, we have $F_k^D(t) = G_k^D(t) - G_k^D(0)$ and $(-1)^k G_k^D(0) = G_k^D(0)$, since $G_{2j+1}^D(0) = 0$, so that we have

$$F_k^D(1-t) = G_k^D(1-t) - G_k^D(0) = (-1)^k \left[G_k^D(t) - G_k^D(0)\right] = (-1)^k F_k^D(t),$$

which proves the second identity. □

Note that the identities established in Lemma 3.11 are valid for $k = 1$, too, except at the points $1/4$, $1/2$ and $3/4$ of discontinuity of $F_1^D(t) = G_1^D(t)$.

Lemma 3.14 *For $k \geq 2$ and $u/2 \leq v < 2u$ the function $G_{2k-1}^D(t)$ has no zeros in the interval $(0,1/2)$. The sign of this function is determined by*

$$(-1)^{k-1} G_{2k-1}^D(t) > 0,\ 0 < t < \frac{1}{2}.$$

Proof. For $k = 2$, $G_3^D(t)$ is given by (3.186) and it is easy to see that for $u/2 \leq v < 2u$

$$-G_3^D(t) > 0,\ 0 < t < \frac{1}{2},$$

Thus, our assertion is true for $k = 2$. Now, assume that $k \geq 3$. Then $2k-1 \geq 5$ and $G_{2k-1}^D(t)$ is continuous and at least twice differentiable function. Using (A-2) we get

$$G_{2k-1}^{D\prime}(t) = -(2k-1)G_{2k-2}^D(t)$$

and

$$G_{2k-1}^{D\prime\prime}(t) = (2k-1)(2k-2)G_{2k-3}^D(t).$$

Let us suppose that G_{2k-3}^D has no zeros in the interval $(0,1/2)$. We know that 0 and $1/2$ are the zeros of $G_{2k-1}^D(t)$. Let us suppose that some α, $0 < \alpha < 1/2$, is also a zero of $G_{2k-1}^D(t)$. Then inside each of the intervals $(0,\alpha)$ and $(\alpha,1/2)$ the derivative $G_{2k-1}^{D\prime}(t)$ must have at least one zero, say β_1, $0 < \beta_1 < \alpha$ and β_2, $\alpha < \beta_2 < 1/2$. Therefore, the second derivative $G_{2k-1}^{D\prime\prime}(t)$ must have at least one zero inside the interval (β_1, β_2). Thus, from the assumption that $G_{2k-1}^D(t)$ has a zero inside the interval $(0,1/2)$, it follows that $(2k-1)(2k-2)G_{2k-3}^D(t)$ also has a zero inside this interval. Thus, $G_{2k-1}^D(t)$ can not have a zero inside the interval $(0,1/2)$. To determine the sign of $G_{2k-1}^D(t)$, note that

$$G_{2k-1}^D\left(\frac{1}{4}\right) = -vB_{2k-1}\left(\frac{1}{4}\right).$$

We have [1, 23.1.14]

$$(-1)^k B_{2k-1}(t) > 0,\ 0 < t < \frac{1}{2},$$

which implies

$$(-1)^{k-1} G_{2k-1}^D\left(\frac{1}{4}\right) = (-1)^k v B_{2k-1}\left(\frac{1}{4}\right) > 0.$$

So, we proved our assertions. □

Corollary 3.26 *For $k \geq 2$ and $u/2 \leq v < 2u$ the functions $(-1)^k F_{2k}^D(t)$ and $(-1)^k G_{2k}^D(t)$ are strictly increasing on the interval $(0,1/2)$, and strictly decreasing on the interval $(1/2,1)$. Further, for $k \geq 2$ and $u/2 \leq v < 2u$ we have*

$$\max_{t\in[0,1]} \left|F_{2k}^D(t)\right| = 2v\left(1-2^{-2k}\right)|B_{2k}|,$$

and

$$\max_{t\in[0,1]} \left|G_{2k}^D(t)\right| = \left[v + u\cdot 2^{1-2k}(1-2^{1-2k})\right]|B_{2k}|.$$

Proof. Using (A-2) we get

$$\left[(-1)^k F_{2k}^D(t)\right]' = \left[(-1)^k G_{2k}^D(t)\right]' = 2k(-1)^{k-1}G_{2k-1}^D(t)$$

and $(-1)^{k-1}G_{2k-1}^D(t) > 0$ for $0 < t < 1/2$, by Lemma 3.14. Thus, $(-1)^k F_{2k}^D(t)$ and $(-1)^k G_{2k}^D(t)$ are strictly increasing on the interval $(0,1/2)$. Also, by Lemma 3.13, we have $F_{2k}^D(1-t) = F_{2k}^D(t)$, $0 \leq t \leq 1$ and $G_{2k}^D(1-t) = G_{2k}^D(t)$, $0 \leq t \leq 1$, which implies that $(-1)^k F_{2k}^D(t)$ and $(-1)^k G_{2k}^D(t)$ are strictly decreasing on the interval $(1/2,1)$. Further, $F_{2k}^D(0) = F_{2k}^D(1) = 0$, which implies that $\left|F_{2k}^D(t)\right|$ achieves its maximum at $t = 1/2$, that is

$$\max_{t\in[0,1]} \left|F_{2k}^D(t)\right| = \left|F_{2k}^D\left(\frac{1}{2}\right)\right| = 2v\left(1-2^{-2k}\right)|B_{2k}|.$$

Also

$$\max_{t\in[0,1]} \left|G_{2k}^D(t)\right| = \max\left\{\left|G_{2k}^D(0)\right|, \left|G_{2k}^D\left(\frac{1}{2}\right)\right|\right\} = \left[v + u\cdot 2^{1-2k}(1-2^{1-2k})\right]|B_{2k}|,$$

which completes the proof. □

Corollary 3.27 *For $k \geq 2$ and $u/2 \leq v < 2u$ we have*

$$\int_0^1 \left|F_{2k-1}^D(t)\right| \mathrm{d}t = \int_0^1 \left|G_{2k-1}^D(t)\right| \mathrm{d}t = \frac{2v}{k}(1-2^{-2k})|B_{2k}|.$$

Also, we have

$$\int_0^1 \left|F_{2k}^D(t)\right| \mathrm{d}t = \left|\tilde{B}_{2k}^D\right| = (v - u\cdot 2^{1-2j})(1-2^{1-2j})|B_{2k}|$$

and

$$\int_0^1 \left|G_{2k}^D(t)\right| \mathrm{d}t \leq 2\left|\tilde{B}_{2k}^D\right| = 2(v - u\cdot 2^{1-2j})(1-2^{1-2j})|B_{2k}|.$$

Proof. Using (A-2) it is easy to see that

$$G_m^{D\prime}(t) = -mG_{m-1}^D(t),\ m \geq 3. \tag{3.193}$$

Now, using Lemma 3.13, Lemma 3.14 and (3.193) we get

$$\begin{aligned}\int_0^1 \left|G_{2k-1}^D(t)\right| \mathrm{d}t &= 2\left|\int_0^{1/2} G_{2k-1}^D(t)\mathrm{d}t\right| = 2\left|-\frac{1}{2k}\, G_{2k}^D(t)\Big|_0^{1/2}\right| \\ &= \frac{1}{k}\left|G_{2k}^D\left(\frac{1}{2}\right) - G_{2k}^D(0)\right| = \frac{2v}{k}(1-2^{-2k})\,|B_{2k}|,\end{aligned}$$

which proves the first assertion. By Corollary 3.26 and because $F_{2k}^D(0) = F_{2k}^D(1) = 0$, $F_{2k}^D(t)$ does not change its sign on the interval $(0,1)$. Therefore, using (3.191) and (3.193), we get

$$\begin{aligned}\int_0^1 \left|F_{2k}^D(t)\right| \mathrm{d}t &= \left|\int_0^1 F_{2k}^D(t)\mathrm{d}t\right| = \left|\int_0^1 \left[G_{2k}^D(t) - \tilde{B}_{2k}^D\right] \mathrm{d}t\right| \\ &= \left|-\frac{1}{2k+1}\, G_{2k+1}^D(t)\Big|_0^1 - \tilde{B}_{2k}^D\right| = \left|\tilde{B}_{2k}^D\right|,\end{aligned}$$

which proves the second assertion. Finally, we use (3.191) again and the triangle inequality to obtain

$$\int_0^1 \left|G_{2k}^D(t)\right| \mathrm{d}t = \int_0^1 \left|F_{2k}^D(t) + \tilde{B}_{2k}^D\right| \mathrm{d}t \le \int_0^1 \left|F_{2k}^D(t)\right| \mathrm{d}t + \left|\tilde{B}_{2k}^D\right| = 2\left|\tilde{B}_{2k}^D\right|,$$

which proves the third assertion. □

Now we use formulae established in Theorem 3.33 to prove a number of inequalities using L_p norms for $1 \le p \le \infty$. These inequalities are generally sharp (in case $p = 1$ the best possible).

Theorem 3.34 *Assume (p,q) is a pair of conjugate exponents, $1 \le p,q \le \infty$. Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n)} \in L_p[0,1]$ for some $n \ge 1$. Then, we have*

$$\left|\int_0^1 f(t)\mathrm{d}t - F(u,v) + \tilde{T}_{n-1}^D(u,v)\right| \le K^D(n,p;u,v) \cdot \|f^{(n)}\|_p, \tag{3.194}$$

and

$$\left|\int_0^1 f(t)\mathrm{d}t - F(u,v) + \tilde{T}_n^D(u,v)\right| \le K^{D*}(n,p;u,v) \cdot \|f^{(n)}\|_p, \tag{3.195}$$

where

$$K^D(n,p;u,v) = \frac{1}{(2u-v)(n!)}\left[\int_0^1 |F_n(t)|^q \,\mathrm{d}t\right]^{1/q} \quad \textit{and}$$

$$K^{D*}(n,p;u,v) = \frac{1}{(2u-v)(n!)}\left[\int_0^1 |G_n(t)|^q \,\mathrm{d}t\right]^{1/q}.$$

The constants $K^D(n,p;u,v)$ and $K^{D*}(n,p;u,v)$ are sharp for $1 < p \le \infty$ and the best possible for $p = 1$.

Proof. Applying the Hölder inequality we have

$$\begin{aligned}\left|\frac{1}{(2u-v)(n!)}\int_0^1 F_n^D(t)f^{(n)}(t)dt\right| &\le \frac{1}{(2u-v)(n!)}\left[\int_0^1 \left|F_n^D(t)\right|^q \mathrm{d}t\right]^{1/q}\cdot\left\|f^{(n)}\right\|_p\\ &= K^D(n,p;u,v)\cdot\|f^{(n)}\|_p.\end{aligned}$$

Using the above inequality from (3.182) we get estimate (3.194). In the same manner, from (3.181) we get estimate (3.195). The proof of the optimality of $K^D(n,p;u,v)$ is analogous to the proof of Theorem 2.2. □

Corollary 3.28 *Let $f:[0,1]\to\mathbb{R}$ be a L-Lipschitzian function on $[0,1]$. Then*

$$\left|\int_0^1 f(t)\mathrm{d}t - F(u,v)\right| \le \frac{2u+v}{8(2u-v)}\cdot L.$$

If f' is L-Lipschitzian on $[0,1]$, then

$$\left|\int_0^1 f(t)\mathrm{d}t - F(u,v)\right| \le \frac{2u^2(3v+\sqrt{2uv})+uv(5v-\sqrt{2uv})+2v^2(v+3\sqrt{2uv})}{48(2u-v)(v+\sqrt{2uv})(2u+v+2\sqrt{2uv})}\cdot L.$$

Proof. Using (3.183) and (3.184) we get

$$\int_0^1 \left|F_1^D(t)\right|\mathrm{d}t = \frac{2u+v}{8} \quad \text{and}$$

$$\int_0^1 \left|F_2^D(t)\right|\mathrm{d}t = \frac{2u^2(3v+\sqrt{2uv})+uv(5v-\sqrt{2uv})+2v^2(v+3\sqrt{2uv})}{24(v+\sqrt{2uv})(2u+v+2\sqrt{2uv})}.$$

Therefore, applying (3.194) with $n=1,2$ and $p=\infty$ we get the above inequalities. □

Remark 3.37 The estimation in the first inequality in Corollary 3.28 achieves an infimum of $1/24$ and the second inequality an infimum of 0 for $u\to\infty$ and $v=1$.

Remark 3.38 Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(n-1)}$ is an L-Lipschitzian function on $[0,1]$ for some $n\ge 3$. Then from Corollary 3.27 for $u/2\le v<2u$ we get

$$K^D(2k-1,\infty;u,v) = \frac{2v}{(2u-v)\,[(2k)!]}(1-2^{-2k})\,|B_{2k}|,$$

$$K^{D*}(2k,\infty;u,v) = \frac{1}{(2u-v)\,[(2k)!]}(v-u\cdot 2^{1-2k})(1-2^{1-2k})\,|B_{2k}|$$

and

$$K^D(2k,\infty;u,v) = \frac{2}{(2u-v)\,[(2k)!]}(v-u\cdot 2^{1-2k})(1-2^{1-2k})\,|B_{2k}|.$$

Corollary 3.29 *Let $f:[0,1]\to\mathbb{R}$ be a continuous function of bounded variation on $[0,1]$. Then*

$$\left|\int_0^1 f(t)\mathrm{d}t - F(u,v)\right| \le \frac{2u+v}{4(2u-v)}\cdot V_0^1(f).$$

If f' is a continuous function of bounded variation on $[0,1]$, then

$$\left|\int_0^1 f(t)\mathrm{d}t - F(u,v)\right| \le \frac{1}{64(2u-v)}[2u+3v+|2u-5v|]\cdot V_0^1(f').$$

Proof. From (3.183) and (3.184), we get

$$\max_{t\in[0,1]}\left|F_1^D(t)\right| = \max\left\{\frac{2u-v}{4},\frac{2u+v}{4}\right\} = \frac{2u+v}{4} \text{ and}$$
$$\max_{t\in[0,1]}\left|F_2^D(t)\right| = \max\left\{\frac{2u-v}{16},\frac{v}{4}\right\} = \frac{1}{32}[2u+3v+|2u-5v|].$$

Therefore, applying (3.194) with $n=1,2$ and $p=1$ we get the above inequalities. □

Remark 3.39 The estimation in the first inequality in Corollary 3.29 achieves an infimum of $1/4$ and in the second inequality an infimum of 0 for $u\to\infty$ and $v=1$.

Remark 3.40 Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$ for some $n\ge 3$. Then from Corollary 3.26 for $u/2\le v<2u$ we get

$$K^D(2k-1,1;u,v) = \frac{1}{(2u-v)\,[(2k-1)!]}\max_{t\in[0,1]}\left|F_{2k-1}^D(t)\right|,$$

$$K^{D*}(2k,1;u,v) = \frac{2v}{(2u-v)\,[(2k)!]}(1-2^{-2k})\,|B_{2k}|$$

and

$$K^D(2k,1;u,v) = \frac{1}{(2u-v)\,[(2k)!]}\left[v+u\cdot 2^{1-2k}(1-2^{1-2k})\right]|B_{2k}|$$

Now, we calculate the optimal constant for $p=2$.

Corollary 3.30 *Let $f^{(n)}\in L_2[0,1]$ for some $n\ge 1$. Then, we have*

$$\begin{aligned}&\left|\int_0^1 f(t)\mathrm{d}t - F(u,v)+\tilde{T}_{n-1}^D(u,v)\right|\\ \le\;& \frac{1}{(2u-v)}\left[\frac{(-1)^{n-1}}{(2n)!}\left[2u^2+v^2-(2u^2-uv\cdot 2^{2-2n})(1-2^{1-2n})\right]B_{2n}\right.\\ +\;& \left.\frac{\tilde{B}_n^2}{(n!)^2}\right]^{1/2}\|f^{(n)}\|_2,\end{aligned}$$

and

$$
\begin{aligned}
&\left|\int_0^1 f(t)\mathrm{d}t - F(u,v) + \tilde{T}_n^D(u,v)\right| \\
&\le \frac{1}{(2u-v)}\left[\frac{(-1)^{n-1}}{(2n)!}\left[2u^2+v^2-(2u^2-uv\cdot 2^{2-2n})(1-2^{1-2n})\right]B_{2n}\right]^{1/2}\|f^{(n)}\|_2.
\end{aligned}
$$

Proof. Using integration by parts and also using Lemma 1 from [30] we have

$$
\begin{aligned}
\int_0^1 (G_n^D(t))^2 dt &= (-1)^{n-1}\frac{n(n-1)\dots 2}{(n+1)(n+2)\dots(2n-1)} \\
&\quad\cdot \left[-\frac{1}{2n}G_{2n}^D(t)G_1^D(t)|_0^1 + \frac{1}{2n}\int_0^1 G_{2n}^D(t)dG_1^D(t)\right] \\
&= (-1)^{n-1}\frac{(n!)^2}{(2n)!}\left[(v-2u)\int_0^1 G_{2n}^D(t)dt + 2uG_{2n}^D\left(\frac{1}{4}\right) - vG_{2n}^D\left(\frac{1}{2}\right)\right] \\
&= (-1)^{n-1}\frac{(n!)^2}{(2n)!}\left[-4uvB_{2n}\left(\frac{1}{4}\right) + 2u^2B_{2n}\left(\frac{1}{2}\right) + (2u^2+v^2)B_{2n}\right] \\
&= (-1)^{n-1}\frac{(n!)^2}{(2n)!}\left[2u^2+v^2-(2u^2-uv\cdot 2^{2-2n})(1-2^{1-2n})\right]B_{2n}.
\end{aligned}
$$

Now,

$$
\begin{aligned}
\int_0^1 (F_n^D(t))^2 dt &= \int_0^1 \left[G_n^D(t) - \tilde{B}_n^D\right]^2 dt \\
&= \int_0^1 \left[(G_n(t))^2 - 2G_n(t)\tilde{B}_n^D + (\tilde{B}_n^D)^2\right]dt = \int_0^1 (G_n^D)^2(t)dt + (\tilde{B}_n^D)^2 \\
&= (-1)^{n-1}\frac{(n!)^2}{(2n)!}\left[2u^2+v^2-(2u^2-uv\cdot 2^{2-2n})(1-2^{1-2n})\right]B_{2n} + (\tilde{B}_n^D)^2.
\end{aligned}
$$

□

Finally, we give the values of optimal constant for $n=1$ and arbitrary p from Theorem 3.34.

Remark 3.41 Note that $K^{D*}(1,p;u,v) = K^D(1,p;u,v)$, for $1 < p \le \infty$, since $G_1^D(t) = F_1^D(t)$. Also, for $1 < p \le \infty$ we can easily calculate $K^D(1,p;u,v)$. We get

$$
K^D(1,p;u,v) = \frac{1}{(2u-v)}\left[\frac{(2u-v)^{q+1} + (2u+v)^{q+1} - 2^{q+1}v^{q+1}}{(2u-v)(q+1)2^{2q+1}}\right]^{\frac{1}{q}},\ 1 < p \le \infty.
$$

Now we use the formula (3.181) and one technical result from [83] to obtain Grüss type inequality related to the general dual Euler-Simpson formula:

Theorem 3.35 *Suppose that $f:[0,1]\to\mathbb{R}$ is such that $f^{(n)}$ exists and is integrable on $[0,1]$, for some $n \ge 1$. Assume that*

$$
m_n \le f^{(n)}(t) \le M_n,\ 0 \le t \le 1,
$$

for some constants m_n and M_n. Then

$$\left| \int_0^1 f(t)\mathrm{d}t - F(u,v) + \tilde{T}_n^D(u,v) \right| \leq C_n(M_n - m_n), \tag{3.196}$$

where $C_n = \frac{1}{(2u-v)(n!)} \int_0^1 |G_n^D(t)|dt$.

Our final results are connected with the series expansion of a function in Bernoulli polynomials.

Theorem 3.36 *If $f:[0,1] \to \mathbb{R}$ is such that $f^{(2k)}$ is a continuous function on $[0,1]$, for some $k \geq 2$, then for $u/2 \leq v < 2u$ there exists a point $\eta \in [0,1]$ such that*

$$\tilde{R}_{2k}^{D2}(f) = -\frac{(v - u \cdot 2^{1-2k})(1-2^{1-2k})B_{2k}}{(2u-v)[(2k)!]} f^{(2k)}(\eta). \tag{3.197}$$

Proof.
We can rewrite $\tilde{R}_{2k}^{D2}(f)$ as $\tilde{R}_{2k}^{D2}(f) = (-1)^k \frac{J_k}{2[(2k)!]}$, where $J_k = \int_0^1 (-1)^k F_{2k}^D(s) f^{(2k)}(s)ds$. From Corollary 3.26 follows that $(-1)^k F_{2k}^D(s) \geq 0,\ \ 0 \leq s \leq 1$ and the claim follows from the mean value theorem for integrals and Corollary 3.27. □

Remark 3.42 For $k=2$ formula (3.197) reduces to

$$\tilde{R}_4^{D2}(f) = \frac{7(8v-u)}{46080(2u-v)} f^{(4)}(\eta).$$

Now, we study, the general dual Simpson quadrature formula

$$\int_0^1 f(t)dt = \frac{1}{u+-v+w}\left[uf\left(\frac{1}{4}\right) - vf\left(\frac{1}{2}\right) + wf\left(\frac{3}{4}\right)\right] + E(f;u,v,w) \tag{3.198}$$

with $E(f;u,v,w)$ being the remainder, $u,v,w \in \mathbb{Z}^+$ and $u+w>v$. We are using identities (1.1) and (1.2) to get two new identities of Euler type.

Theorem 3.37 *Let $f:[0,1] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$, for some $n \geq 1$. Then*

$$\int_0^1 f(t)\mathrm{d}t = F(u,v,w) - \overline{T}_n^D(u,v,w) + \overline{R}_n^{D1}(f), \tag{3.199}$$

and

$$\int_0^1 f(t)\mathrm{d}t = F(u,v,w) - \overline{T}_{n-1}^D(u,v,w) + \overline{R}_n^{D2}(f), \tag{3.200}$$

where

$$F(u,v,w) = \frac{1}{u-v+w}\left[uf\left(\frac{1}{4}\right) - vf\left(\frac{1}{2}\right) + wf\left(\frac{3}{4}\right)\right],$$

$$\overline{R}_n^{D1}(f) = \frac{1}{(u-v+w)(n!)} \int_0^1 \overline{G}_n^D(t)\,\mathrm{d}f^{(n-1)}(t),$$

$$\bar{R}_n^{D2}(f) = \frac{1}{(u-v+w)(n!)} \int_0^1 \bar{F}_n^D(t)\,\mathrm{d}f^{(n-1)}(t),$$

$$\bar{G}_k^D(t) = uB_k^*\left(\frac{1}{4}-t\right) - vB_k^*\left(\frac{1}{2}-t\right) + wB_k^*\left(\frac{3}{4}-t\right), \quad t \in \mathbb{R},$$

$$\bar{F}_k^D(t) = \bar{G}_k^D(t) - \bar{B}_k^D, \quad t \in \mathbb{R}, \quad k \geq 1,$$

$$\bar{B}_k^D = uB_k\left(\frac{1}{4}\right) - vB_k\left(\frac{1}{2}\right) + wB_k\left(\frac{3}{4}\right), \quad k \geq 1$$

and

$$\bar{T}_m^D(u,v,w) = \frac{1}{u-v+w}\left[uT_m\left(\frac{1}{4}\right) - vT_m\left(\frac{1}{2}\right) + wT_m\left(\frac{3}{4}\right)\right].$$

Proof. Put $x = 1/4,\ 1/2,\ 3/4$ in formula (1.1) to get three new formulae. Then multiply these new formulae by $u,\ -v,\ w$ respectively, and add them up. The result is formula (3.199). Formula (3.200) is obtained from (1.2) by the same procedure. □

Theorem 3.38 *Assume (p_1,q_1) and (p_2,q_2) are two pairs of conjugate exponents, $1 \leq p_1,q_1,p_2,q_2 \leq \infty$. Let $x \in [0,1]$ and $f : [0,1] \to \mathbb{R}$ be such that $f^{(n)} \in L_{p_1}[0,x]$ and $f^{(n)} \in L_{p_2}[x,1]$ for some $n \geq 1$. Then, we have*

$$\left|\int_0^1 f(t)\mathrm{d}t - F(u,v,w) + \bar{T}_{n-1}^D(u,v,w)\right| \tag{3.201}$$
$$\leq K^D(n,p_1;u,v,w,x)\cdot\|f^{(n)}\|_{L_{p_1}[0,x]} + K^D(n,p_2;u,v,w,x)\cdot\|f^{(n)}\|_{L_{p_2}[x,1]},$$

and

$$\left|\int_0^1 f(t)\mathrm{d}t - F(u,v,w) + \bar{T}_n^D(u,v,w)\right| \tag{3.202}$$
$$\leq K^{D*}(n,p_1;u,v,w,x)\cdot\|f^{(n)}\|_{L_{p_1}[0,x]} + K^{D*}(n,p_2;u,v,w,x)\cdot\|f^{(n)}\|_{L_{p_2}[x,1]},$$

where

$$K^D(n,p_1;u,v,w,x) = \frac{1}{(u-v+w)(n!)}\left[\int_0^x \left|\bar{F}_n^D(t)\right|^{q_1}\mathrm{d}t\right]^{1/q_1},$$

$$K^D(n,p_2;u,v,w,x) = \frac{1}{(u-v+w)(n!)}\left[\int_x^1 \left|\bar{F}_n^D(t)\right|^{q_2}\mathrm{d}t\right]^{1/q_2},$$

$$K^{D*}(n,p_1;u,v,w,x) = \frac{1}{(u-v+w)(n!)}\left[\int_0^x \left|\bar{G}_n^D(t)\right|^{q_1}dt\right]^{1/q_1} \quad \text{and}$$

$$K^{D*}(n,p_2;u,v,w,x) = \frac{1}{(u-v+w)(n!)}\left[\int_x^1 \left|\bar{G}_n^D(t)\right|^{q_2}dt\right]^{1/q_2}.$$

The constants $K^D(n,p_1;u,v,w,x)$, $K^D(n,p_2;u,v,w,x)$, $K^{D*}(n,p_1;u,v,w,x)$ and $K^{D*}(n,p_2;u,v,w,x)$ are sharp for $1 < p_1,p_2 \leq \infty$ and the best possible for $p_1 = 1$ or $p_2 = 1$.

Proof. Applying the Hölder inequality we have

$$
\begin{aligned}
&\left|\frac{1}{(u-v+w)(n!)}\int_0^1 \bar{F}_n^D(t)f^{(n)}(t)dt\right| \\
&= \left|\frac{1}{(u-v+w)(n!)}\int_0^x \bar{F}_n^D(t)f^{(n)}(t)dt + \frac{1}{(u-v+w)(n!)}\int_x^1 \bar{F}_n^D(t)f^{(n)}(t)dt\right| \\
&\le \frac{1}{(u-v+w)(n!)}\left\{\left[\int_0^x |\bar{F}_n^D(t)|^{q_1}dt\right]^{1/q_1}\|f^{(n)}\|_{L_{p_1}[0,x]}\right. \\
&+ \left.\left[\int_x^1 |\bar{F}_n^D(t)|^{q_2}dt\right]^{1/q_2}\|f^{(n)}\|_{L_{p_2}[x,1]}\right\} \\
&= K^D(n,p_1;u,v,w,x)\|f^{(n)}\|_{L_{p_1}[0,x]} + K^D(n,p_2;u,v,w,x)\|f^{(n)}\|_{L_{p_2}[x,1]}.
\end{aligned}
$$

Using the above inequality from (3.182) we get estimate (3.201). In the same manner, from (3.181) we get estimate (3.202). The proof of sharpness and best possibility is similar as in the proof of Theorem 2.2. □

Remark 3.43 For $n = 1$, $\frac{1}{4} \le u \le w \le \frac{1}{2}$ and $u - v + w = 1$ in inequality (3.201) we get inequality

$$
\begin{aligned}
&\left|\int_0^1 f(t)\mathrm{d}t - F(u,v,w)\right| \\
&\le \left[\frac{(w-u)^{q_1+1} + (u-w+1)^{q_1+1} + (3u+w-1)^{q_1+1} + (2-3u-w)^{q_1+1}}{4^{q_1+1}(q_1+1)}\right]^{1/q_1} \\
&\cdot \ \|f'\|_{L_{p_1}[0,1/2]} \\
&+ \left[\frac{-(w-u)^{q_2+1} + (w-u+1)^{q_2+1} + (3w+u-1)^{q_2+1} + (2-u-3w)^{q_2+1}}{4^{q_2+1}(q_2+1)}\right]^{1/q_2} \\
&\cdot \ \|f'\|_{L_{p_2}[1/2,1]}.
\end{aligned}
$$

3.3.3 Estimations of the error via pre-Grüss inequality

In [116], N. Ujević used the generalization of pre-Grüss inequality to derive some better estimations of the error for Simpson's quadrature rule. In fact, he proved the next three theorems:

Theorem 3.39 *Let $I \subset \mathbb{R}$ be a closed interval and $a,b \in \mathrm{Int}I$, $a < b$. If $f : I \to \mathbb{R}$ is a continuous function of bounded variation with $f' \in L_2[a,b]$, then we have*

$$
\left|\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] - \int_a^b f(t)dt\right| \le \frac{(b-a)^{3/2}}{6}K_1, \tag{3.203}
$$

where

$$
K_1^2 = \|f'\|_2^2 - \frac{1}{b-a}\left(\int_a^b f'(t)dt\right)^2 - \left(\int_a^b f'(t)\Psi_0(t)dt\right)^2 \tag{3.204}
$$

and $\Psi(t) = t - \frac{a+b}{2}, \Psi_0(t) = \Psi(t)/\|\Psi\|_2$.

Theorem 3.40 *Let $I \subset \mathbb{R}$ be a closed interval and $a,b \in IntI$, $a < b$. If $f : I \to \mathbb{R}$ is such that f' is a continuous function of bounded variation with $f'' \in L_2[a,b]$, then we have*

$$\left| \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \int_a^b f(t)dt \right| \leq \frac{(b-a)^{5/2}}{12\sqrt{30}} K_2, \tag{3.205}$$

where

$$K_2^2 = \|f''\|_2^2 - \frac{1}{b-a}\left(\int_a^b f''(t)dt\right)^2 - \left(\int_a^b f''(t)\Psi_0(t)dt\right)^2, \tag{3.206}$$

$$\Psi(t) = \begin{cases} 1, & t \in \left[a, \frac{a+b}{2}\right] \\ -1, & t \in \left(\frac{a+b}{2}, b\right] \end{cases} \tag{3.207}$$

and $\Psi_0(t) = \Psi(t)/\|\Psi\|_2$.

Theorem 3.41 *Let $I \subset \mathbb{R}$ be a closed interval and $a,b \in IntI$, $a < b$. If $f : I \to \mathbb{R}$ is such that f'' is a continuous function of bounded variation with $f''' \in L_2[a,b]$ then we have*

$$\left| \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \int_a^b f(t)dt \right| \leq \frac{(b-a)^{7/2}}{48\sqrt{105}} K_3, \tag{3.208}$$

where

$$K_3^2 = \|f'''\|_2^2 - \frac{1}{b-a}\left(\int_a^b f'''(t)dt\right)^2 - \left(\int_a^b f'''(t)\Psi_0(t)dt\right)^2, \tag{3.209}$$

$$\Psi(t) = \begin{cases} t - \frac{7a+3b}{10}, & t \in \left[a, \frac{a+b}{2}\right] \\ t - \frac{3a+7b}{10}, & t \in \left(\frac{a+b}{2}, b\right] \end{cases} \tag{3.210}$$

and $\Psi_0(t) = \Psi(t)/\|\Psi\|_2$.

In this section using Theorem 2.14 we will give a similar result for general Euler-Simpson formula and for functions whose derivative of order n, $n \geq 1$, is from $L_2[0,1]$ space. We will also give related results for general dual Euler-Simpson formula. We will use interval $[0,1]$ for simplicity and since it involves no loss in generality.

Theorem 3.42 *If $f : [0,1] \to \mathbb{R}$ is such that $f^{(n-1)}$ is a continuous function of bounded variation with $f^{(n)} \in L_2[0,1]$, then we have*

$$\left| \int_0^1 f(t)\mathrm{d}t - D(u,v) + \tilde{T}_n(u,v) \right| \tag{3.211}$$

$$\leq \frac{1}{2u+v}\left[\frac{(-1)^{n-1}}{(2n)!}\left[4u^2 + v^2 - 4uv(1-2^{1-2n})\right] B_{2n}\right]^{1/2} K,$$

where

$$K^2 = \|f^{(n)}\|_2^2 - \left(\int_0^1 f^{(n)}(t)dt\right)^2 - \left(\int_0^1 f^{(n)}(t)\Psi_0(t)dt\right)^2. \tag{3.212}$$

For n even

$$\Psi(t) = \begin{cases} 1, & t \in \left[0, \frac{1}{2}\right], \\ -1, & t \in \left(\frac{1}{2}, 1\right], \end{cases}$$

while for n odd we have

$$\Psi(t) = \begin{cases} t + \frac{2^{1-n}u-2u+v}{2^{1-n}v-2^{2-n}u+8u-4v}, & t \in \left[0, \frac{1}{2}\right], \\ t + \frac{2^{1-n}(u-v)+3v-6u}{2^{1-n}v-2^{2-n}u+8u-4v}, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Proof. It is not difficult to verify that

$$\int_0^1 G_n(t)dt = 0, \tag{3.213}$$

$$\int_0^1 \Psi(t)dt = 0, \tag{3.214}$$

$$\int_0^1 G_n(t)\Psi(t)dt = 0. \tag{3.215}$$

From (3.157), (3.213) and (3.215) it follows that

$$\begin{aligned} \int_0^1 f(t)\mathrm{d}t - D(u,v) + \tilde{T}_n(u,v) &= \frac{1}{(2u+v)n!}\int_0^1 G_n(t)f^{(n)}(t)dt \\ - & \frac{1}{(2u+v)n!}\int_0^1 G_n(t)dt\int_0^1 f^{(n)}(t)dt \\ - & \frac{1}{(2u+v)n!}\int_0^1 G_n(t)\Psi_0(t)dt\int_0^1 f^{(n)}(t)\Psi_0(t)dt \\ = & \frac{1}{(2u+v)n!}S_\Psi(G_n, f^{(n)}). \end{aligned} \tag{3.216}$$

Using (3.216) and (2.54) we get

$$\left|\int_0^1 f(t)\mathrm{d}t - D(u,v) + \tilde{T}_n(u,v)\right| \le \frac{1}{(2u+v)n!}S_\Psi(G_n,G_n)^{1/2}S_\Psi(f^{(n)},f^{(n)})^{1/2}. \tag{3.217}$$

We also have (see [65])

$$\begin{aligned} S_\Psi(G_n,G_n) &= \|G_n\|_2^2 - \left(\int_0^1 G_n(t)dt\right)^2 - \left(\int_0^1 G_n(t)\Psi_0(t)dt\right)^2 \\ &= (-1)^{n-1}\frac{(n!)^2}{(2n)!}\left[4u^2+v^2-4uv(1-2^{1-2n})\right]B_{2n} \end{aligned} \tag{3.218}$$

and

$$S_\Psi(f^{(n)},f^{(n)}) = \|f^{(n)}\|_2^2 - \left(\int_0^1 f^{(n)}(t)dt\right)^2 - \left(\int_0^1 f^{(n)}(t)\Psi_0(t)dt\right)^2 = K^2. \qquad (3.219)$$

From (3.217)-(3.219) we easily get (3.211). □

Remark 3.44 The estimation in inequality (3.211) achieves minimum of $\left[\frac{(-1)^{n-1}}{(2n)!}2^{-2n}B_{2n}\right]^{1/2}$ for $u=1$ and $v=2$, which is bitrapezoid formula (see [28]). For $n=1$ it is $1/4\sqrt{3}$.

Remark 3.45 For $u=1$ and $v=4$ in Theorem 3.42, we get Euler-Simpson formula (see [29]) and then we have

$$\left|\int_0^1 f(t)\mathrm{d}t - D(1,4) + T_n(1,4)\right| \le \frac{1}{3}\left[\frac{(-1)^{n-1}}{(2n)!}\left(1+2^{3-2n}\right)\right] B_{2n}\Bigg]^{1/2} K, \qquad (3.220)$$

where

$$D(1,4) = \frac{1}{6}\left[f(0)+4f\left(\frac{1}{2}\right)+f(1)\right],$$

and

$$\tilde{T}_n(1,4) = \sum_{k=2}^{\lfloor n/2\rfloor}\frac{1}{3(2k)!}(1-2^{2-2k})B_{2k}\left[f^{(2k-1)}(1)-f^{(2k-1)}(0)\right].$$

For n even

$$\Psi(t) = \begin{cases} 1, & t\in\left[0,\frac{1}{2}\right], \\ -1, & t\in\left(\frac{1}{2},1\right], \end{cases}$$

while for n odd we have

$$\Psi(t) = \begin{cases} t+\frac{2^{-n}+1}{4(2^{-1-n}-1)}, & t\in\left[0,\frac{1}{2}\right], \\ t+\frac{3(1-2^{-n})}{4(2^{-1-n}-1)}, & t\in\left(\frac{1}{2},1\right]. \end{cases}$$

For $n=1,\ 2$ and 3 in the inequality (3.220) we get inequalities (3.203), (3.205) and (3.208) respectively.

Theorem 3.43 *If $f:[0,1]\to\mathbb{R}$ is such that $f^{(n-1)}$ is a continuous function of bounded variation with $f^{(n)}\in L_2[0,1]$, then we have*

$$\left|\int_0^1 f(t)\mathrm{d}t - F(u,v) + \tilde{T}_n^D(u,v)\right| \qquad (3.221)$$

$$\le \frac{1}{2u-v}\left[\frac{(-1)^{n-1}}{(2n)!}\left[2u^2+v^2-(2u^2-uv\cdot 2^{2-2n})(1-2^{1-2n})\right]B_{2n}\right]^{1/2} K,$$

where

$$K^2 = \|f^{(n)}\|_2^2 - \left(\int_0^1 f^{(n)}(t)dt\right)^2 - \left(\int_0^1 f^{(n)}(t)\Psi_0(t)dt\right)^2. \qquad (3.222)$$

For n even

$$\Psi(t) = \begin{cases} 1, & t \in \left[0, \frac{1}{2}\right], \\ -1, & t \in \left(\frac{1}{2}, 1\right], \end{cases}$$

while for n odd we have

$$\Psi(t) = \begin{cases} t + \frac{2^{-n}u(1-2^{-n})+v}{4v(2^{-n-1}-1)}, & t \in \left[0, \frac{1}{2}\right], \\ t + \frac{v(3-2^{-n+1})-2^{-n}u(1-2^{-n})}{4v(2^{-n-1}-1)}, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Proof. Similar as in Theorem 3.42. □

Remark 3.46 For $u = 2$ and $v = 1$ in Theorem 3.43 we get dual Euler-Simpson formula (see [26]) and then we have

$$\left| \int_0^1 f(t)\mathrm{d}t - F(2,1) + \tilde{T}_n^D(2,1) \right| \le \frac{1}{3} \left[\frac{(-1)^{n-1}}{(2n)!} \left[9 - (8 - 2^{3-2n})(1 - 2^{1-2n})\right] B_{2n} \right]^{1/2} K, \tag{3.223}$$

where

$$F(2,1) = \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right],$$

and

$$\tilde{T}_n^D(2,1) = \sum_{k=2}^{\lfloor n/2 \rfloor} \frac{1}{3(2k)!}\left(8 \cdot 2^{-4k} - 6 \cdot 2^{-2k} + 1\right) B_{2k} \left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right].$$

For n even

$$\Psi(t) = \begin{cases} 1, & t \in \left[0, \frac{1}{2}\right], \\ -1, & t \in \left(\frac{1}{2}, 1\right], \end{cases}$$

while for n odd we have

$$\Psi(t) = \begin{cases} t + \frac{2^{1-n}(1-2^{-n})+1}{4(2^{-1-n}-1)}, & t \in \left[0, \frac{1}{2}\right], \\ t + \frac{3-2^{2-n}+2^{1-2n}}{4(2^{-1-n}-1)}, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

For $n = 1,\ 2$ and 3 we get inequalities

$$\left| \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right] - \int_0^1 f(t)dt \right| \le \frac{1}{3\sqrt{2}} K_1,$$

$$\left| \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right] - \int_0^1 f(t)dt \right| \le \frac{\sqrt{13}}{48\sqrt{15}} K_2$$

and

$$\left| \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right] - \int_0^1 f(t)dt \right| \le \frac{\sqrt{13}}{192\sqrt{70}} K_3$$

respectively.

Chapter 4

General 4-point quadrature formulae of Euler type

The object of interest in this chapter are the general 4-point formulae which approximate the integral over $[0,1]$ by values of the function in points $0,\ x,\ 1-x$ and 1, with $x\in(0,1/2]$. The results from this chapter were published in [56].

4.1 General approach

Let $x\in(0,1/2]$ and $f:[0,1]\to\mathbb{R}$ be such that $f^{(2n+1)}$ is continuous of bounded variation on $[0,1]$ for some $n\geq 0$. We proceed similarly as in the previous chapter, since the main idea of the method is the same: put $x\equiv 0,\ x,\ 1-x$ and 1 in (1.2), multiply by $1/2-w(x),\ w(x),\ w(x),\ 1/2-w(x),$ respectively and add up. The following formula is obtained:

$$\begin{aligned}\int_0^1 f(t)dt-(1/2-A(x))[f(0)+f(1)]-A(x)[f(x)+f(1-x)]+T_{2n}(x)\\ =\frac{1}{(2n+2)!}\int_0^1 F_{2n+2}(x,t)df^{(2n+1)}(t),\qquad (4.1)\end{aligned}$$

where, for $t \in \mathbb{R}$,

$$T_{2n}(x) = \sum_{k=2}^{2n} \frac{1}{k!} G_k(x,0) \, [f^{(k-1)}(1) - f^{(k-1)}(0)],$$
$$G_k(x,t) = [1-2A(x)]B_k^*(1-t) + A(x)\,[B_k^*(x-t) + B_k^*(1-x-t)], \ \ k \geq 1$$
$$F_k(x,t) = G_k(x,t) - G_k(x,0), \ \ k \geq 2.$$

Functions G_k have all the properties that functions G_k from Chapter 3 had, including (3.5) and (3.6). If one wants to obtain from (4.1) the quadrature formula with the maximum degree of exactness (if values of derivatives at the end points are not to be included in the quadrature, then it is equal to 3), similarly as before, a condition $G_2(x,0) = 0$ has to be imposed. In this way we get:

$$w(x) = \frac{1}{12x(1-x)}. \tag{4.2}$$

Formula (4.1) now becomes:

$$\int_0^1 f(t)dt - Q(0,x,1-x,1) + T_{2n}^{Q4}(x) = \frac{1}{(2n+2)!} \int_0^1 F_{2n+2}^{Q4}(x,t)df^{(2n+1)}(t), \tag{4.3}$$

where

$$Q(0,x,1-x,1) = \frac{1}{12x(1-x)} \left[-6B_2(x)f(0) + f(x) + f(1-x) - 6B_2(x)f(1)\right], \tag{4.4}$$

$$T_{2n}^{Q4}(x) = \sum_{k=2}^{n} \frac{1}{(2k)!} G_{2k}^{Q4}(x,0) \, [f^{(2k-1)}(1) - f^{(2k-1)}(0)], \tag{4.5}$$

$$G_k^{Q4}(x,t) = \frac{1}{12x(1-x)} \left[B_k^*(x-t) - 12B_2(x) \cdot B_k^*(1-t) + B_k^*(1-x-t)\right], \tag{4.6}$$

$$F_k^{Q4}(x,t) = G_k^{Q4}(x,t) - G_k^{Q4}(x,0), \ \ k \geq 2\,. \tag{4.7}$$

Assuming $f^{(2n-1)}$ is continuous of bounded variation on $[0,1]$ for some $n \geq 1$, we get:

$$\int_0^1 f(t)dt - Q(0,x,1-x,1) + T_{2n}^{Q4}(x) = \frac{1}{(2n)!} \int_0^1 G_{2n}^{Q4}(x,t)df^{(2n-1)}(t), \tag{4.8}$$

while assuming $f^{(2n)}$ fulfills the same condition for some $n \geq 0$, we get:

$$\int_0^1 f(t)dt - Q(0,x,1-x,1) + T_{2n}^{Q4}(x) = \frac{1}{(2n+1)!} \int_0^1 G_{2n+1}^{Q4}(x,t)df^{(2n)}(t). \tag{4.9}$$

The following lemma is the key step for obtaining sharp estimates of error for the formulae (4.3), (4.8) and (4.9).

Lemma 4.1 *For $x \in (0, \frac{1}{2} - \frac{\sqrt{3}}{6}] \cup [\frac{1}{3}, \frac{1}{2}]$ and $k \geq 1$, $G_{2k+1}^{Q4}(x,t)$ has no zeros in variable t on $(0,1/2)$. The sign of this function is determined by*

$$(-1)^k G_{2k+1}^{Q4}(x,t) > 0 \quad \text{for } x \in (0, 1/2 - \sqrt{3}/6],$$
$$(-1)^{k+1} G_{2k+1}^{Q4}(x,t) > 0 \quad \text{for } x \in [1/3, 1/2].$$

Proof. Observe $G_3^{Q4}(x,t)$. For $0 \leq t \leq x$, it takes the form:

$$G_3^{Q4}(x,t) = -t^2\left(t + \frac{3B_2(x)}{2x(1-x)}\right).$$

Its only zero, except 0, is $t_1 = \frac{3B_2(x)}{2x(x-1)}$. It is easy to see that $0 < t_1 \leq x$ iff $x \in (\frac{1}{2} - \frac{\sqrt{3}}{6}, 1 - \frac{\sqrt{2}}{2}]$. Further, for $x \leq t \leq 1/2$, function $G_3^{Q4}(x,t)$ takes the form:

$$G_3^{Q4}(x,t) = -t^3 + \frac{3t^2}{2} - \frac{t}{2(1-x)} + \frac{x}{4(1-x)}.$$

Here it has 3 zeros:

$$t = \frac{1}{2}, \quad t_2 = \frac{1}{2} - \frac{\sqrt{3x^2 - 4x + 1}}{2(1-x)}, \quad t_3 = \frac{1}{2} + \frac{\sqrt{3x^2 - 4x + 1}}{2(1-x)}.$$

It only needs to be checked if t_2 is a zero for $x \in (0,1/3)$ since $t_2, t_3 \in \mathbb{R}$ iff $x \in (0,1/3]$ and it is obvious that $t_3 \geq 1/2$. That $t_2 < 1/2$ is trivial and it is easy to see that $t_2 \geq x$ iff $x \in [1 - \frac{\sqrt{2}}{2}, \frac{1}{3})$. Therefore, our statement is valid for $k = 1$. Assuming the opposite, the statement for $k \geq 2$ follows by induction. The rest of the proof is analogous to the same part of the proof of Lemma 3.1 □

Remark 4.1 From Lemma 4.1 it follows immediately that for $k \geq 1$ and $x \in (0, 1/2 - \sqrt{3}/6]$, function $(-1)^{k+1}F_{2k+2}^{Q4}(x,t)$ is strictly increasing in variable t on $(0,1/2)$ and strictly decreasing on $(1/2,1)$. Since $F_{2k+2}^{Q4}(x,0) = F_{2k+2}^{Q4}(x,1) = 0$, it has constant sign on $(0,1)$ and obtains its maximum at $t = 1/2$. Analogous statement, but with the opposite sign, is valid in the case when $x \in [1/3, 1/2]$.

Denote by $R_{2n+2}^{Q4}(x,f)$ the right-hand side of (4.3).

Theorem 4.1 *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(2n+2)}$ is continuous on $[0,1]$ for some $n \geq 2$ and let $x \in (0, \frac{1}{2} - \frac{\sqrt{3}}{6}] \cup [\frac{1}{3}, \frac{1}{2}]$. If $f^{(2n)}$ and $f^{(2n+2)}$ have the same constant sign on $[0,1]$, then the remainder $R_{2n}^{Q4}(x,f)$ has the same sign as the first neglected term $\Delta_{2n}^{Q4}(x,f)$ where*

$$\Delta_{2n}^{Q4}(x,f) := R_{2n}^{Q4}(x,f) - R_{2n+2}^{Q4}(x,f) = -\frac{1}{(2n)!}G_{2n}^{Q4}(x,0)[f^{(2n-1)}(1) - f^{(2n-1)}(0)].$$

Furthermore, $|R_{2n}^{Q4}(x,f)| \leq |\Delta_{2n}^{Q4}(x,f)|$ *and* $|R_{2n+2}^{Q4}(x,f)| \leq |\Delta_{2n}^{Q4}(x,f)|$.

Proof. Analogous to the proof of Theorem 3.1. □

Theorem 4.2 *If $f : [0,1] \to \mathbb{R}$ is such that $f^{(2n+2)}$ is continuous on $[0,1]$ for some $n \geq 1$ and $x \in (0, \frac{1}{2} - \frac{\sqrt{3}}{6}] \cup [\frac{1}{3}, \frac{1}{2}]$, then there exists $\xi \in [0,1]$ such that*

$$R_{2n+2}^{Q4}(x,f) = -\frac{G_{2n+2}^{Q4}(x,0)}{(2n+2)!} \cdot f^{(2n+2)}(\xi) \qquad (4.10)$$

where

$$G_{2n+2}^{Q4}(x,0) = \frac{1}{6x(1-x)}\left[B_{2n+2}(x) - B_{2n+2}\right] + B_{2n+2}. \tag{4.11}$$

If, in addition, $f^{(2n+2)}$ does not change sign on $[0,1]$, then there exists $\theta \in [0,1]$ such that

$$R_{2n+2}^{Q4}(x,f) = \frac{\theta}{(2n+2)!} \cdot F_{2n+2}^{Q4}\left(x,\frac{1}{2}\right) \cdot \left[f^{(2n+1)}(1) - f^{(2n+1)}(0)\right] \tag{4.12}$$

where

$$\begin{aligned} F_{2n+2}^{Q4}\left(x,\frac{1}{2}\right) &= \frac{1}{6x(1-x)}\left[B_{2n+2}(1/2-x) - B_{2n+2}(x) + \left(2-2^{-2n-1}\right)B_{2n+2}\right] \\ &- \left(2-2^{-2n-1}\right)B_{2n+2}. \end{aligned} \tag{4.13}$$

Proof. Analogous to the proof of Theorem 3.2. □

When (4.10) is applied to the remainder in formula (4.3) for $n=1$, we obtain:

$$\int_0^1 f(t)dt - Q(0,x,1-x,1) = \frac{1}{720}(5x^2-5x+1)\cdot f^{(4)}(\xi). \tag{4.14}$$

For $x = 1/3$, this formula becomes the classical Simpson's 3/8 formula, for $x = 1/2$ it becomes the well-known Simpson's formula, and finally for $x = 1/2 - \sqrt{3}/6 \Leftrightarrow w(x) = 1/2$ it becomes the classical Gauss 2-point formula (stated on $[0,1]$). These three formulae were studied and generalized using a similar technique as here, in [31], [29] and [59], respectively. Of course, all related results from those papers follow as special cases of our results.

Remark 4.2 Although only $x \in (0,1/2]$ were taken into consideration here, results for $x=0$ can easily be obtained by considering the limit process when x tends to 0. Namely,

$$\lim_{x\to 0} Q(0,x,1-x,1) = \frac{1}{2}[f(0)+f(1)] - \frac{1}{12}[f'(1)-f'(0)]$$
$$\lim_{x\to 0} G_k^{Q4}(x,t) = B_k^*(1-t)$$

Consequently, from (4.14) it follows:

$$\int_0^1 f(t)dt - \frac{1}{2}[f(0)+f(1)] + \frac{1}{12}[f'(1)-f'(0)] = \frac{1}{720}f^{(4)}(\xi). \tag{4.15}$$

Theorem 4.3 *Let $p,q \in \mathbb{R}$ be such that $1 \le p,\ q \le \infty$ and $1/p+1/q=1$. If $f:[0,1]\to\mathbb{R}$ is such that $f^{(2n)} \in L_p[0,1]$ for some $n \ge 1$, then we have*

$$\left|\int_0^1 f(t)dt - Q(0,x,1-x,1) + T_{2n}^{Q4}(x)\right| \le K_{Q4}(2n,q)\cdot \|f^{(2n)}\|_p. \tag{4.16}$$

If $f^{(2n+1)} \in L_p[0,1]$ for some $n \geq 0$, then we have

$$\left| \int_0^1 f(t)dt - Q(0,x,1-x,1) + T_{2n}^{Q4}(x) \right| \leq K_{Q4}(2n+1,q) \cdot \|f^{(2n+1)}\|_p. \tag{4.17}$$

If $f^{(2n+2)} \in L_p[0,1]$ for some $n \geq 0$, then we have

$$\left| \int_0^1 f(t)dt - Q(0,x,1-x,1) + T_{2n}^{Q4}(x) \right| \leq K_{Q4}^*(2n+2,q) \cdot \|f^{(2n+2)}\|_p, \tag{4.18}$$

where

$$K_{Q4}(m,q) = \frac{1}{m!} \left[\int_0^1 \left| G_m^{Q4}(x,t) \right|^q dt \right]^{\frac{1}{q}}, \quad K_{Q4}^*(m,q) = \frac{1}{m!} \left[\int_0^1 \left| F_m^{Q4}(x,t) \right|^q dt \right]^{\frac{1}{q}}.$$

These inequalities are sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof. Analogous to the proof of Theorem 2.2. □

For $x \in (0, \frac{1}{2} - \frac{\sqrt{3}}{6}] \cup [\frac{1}{3}, \frac{1}{2}]$ and $n \geq 1$, using Lemma 4.1 and Remark 4.1, we can calculate the following constants as special cases of the previous Theorem:

$$K_{Q4}^*(2n+2,1) = \frac{1}{(2n+2)!} \left| G_{2n+2}^{Q4}(x,0) \right|,$$

$$K_{Q4}^*(2n+2,\infty) = \frac{1}{2} K_{Q4}(2n+1,1) = \frac{1}{(2n+2)!} \left| F_{2n+2}^{Q4}\left(x, \frac{1}{2}\right) \right|,$$

where $G_{2n+2}^{Q4}(x,0)$ is as in (4.11) and $F_{2n+2}^{Q4}(x,1/2)$ as in (4.13). In view of this, let us consider inequalities (4.17) and (4.18) for $n = 1$ and $p = \infty$:

$$\left| \int_0^1 f(t)dt - Q(0,x,1-x,1) \right| \leq \frac{1}{576} \left| \frac{16x^2 - 15x + 3}{1-x} \right| \cdot \|f'''\|_\infty$$

$$\left| \int_0^1 f(t)dt - Q(0,x,1-x,1) \right| \leq \frac{1}{720} \left| 5x^2 - 5x + 1 \right| \cdot \|f^{(4)}\|_\infty$$

In order to find which admissible x gives the least estimate of error, we have to minimize the functions on the right-hand side. It is easy to see that both those functions are decreasing on $(0, \frac{1}{2} - \frac{\sqrt{3}}{6}]$ and increasing on $[\frac{1}{3}, \frac{1}{2}]$ and that they reach their minimal values at $x = 1/3$. In fact, the same is valid in the case when $n = 1$ and $p = 1$, since $K_{Q4}^*(4,\infty) = \frac{1}{2} K_{Q4}(3,1)$.

Therefore, the node that gives the least estimate of error in these three cases is $x = 1/3$, i.e. the optimal closed 4-point quadrature formula is Simpson's 3/8 formula.

The following two theorems give Hermite-Hadamard and Dragomir-Agarwal type inequalities for the general 4-point quadrature formulae:

Theorem 4.4 *Let $f:[0,1]\to\mathbb{R}$ be $(2n+4)$-convex for $n\geq 1$. Then for $x\in\left(0,\frac{1}{2}-\frac{\sqrt{3}}{6}\right]$, we have*

$$\begin{aligned}
&\frac{1}{(2n+2)!}\,|G_{2n+2}^{Q4}(x,0)|\,f^{(2n+2)}\left(\frac{1}{2}\right)\\
&\leq(-1)^{n+1}\left(\int_0^1 f(t)dt-Q(0,x,1-x,1)+T_{2n}^{Q4}(x)\right)\\
&\leq\frac{1}{(2n+2)!}\,|G_{2n+2}^{Q4}(x,0)|\,\frac{f^{(2n+2)}(0)+f^{(2n+2)}(1)}{2},
\end{aligned}\tag{4.19}$$

while for $x\in\left[\frac{1}{3},\frac{1}{2}\right]$ we have

$$\begin{aligned}
&\frac{1}{(2n+2)!}\,|G_{2n+2}^{Q4}(x,0)|\,f^{(2n+2)}\left(\frac{1}{2}\right)\\
&\leq(-1)^{n}\left(\int_0^1 f(t)dt-Q(0,x,1-x,1)+T_{2n}^{Q4}(x)\right)\\
&\leq\frac{1}{(2n+2)!}\,|G_{2n+2}^{Q4}(x,0)|\,\frac{f^{(2n+2)}(0)+f^{(2n+2)}(1)}{2},
\end{aligned}\tag{4.20}$$

where $G_{2n+2}^{Q4}(x,0)$ is as in (4.11).

If f is $(2n+4)$-concave, the inequalities are reversed.

Proof. Analogous to the proof of Theorem 2.8. □

Theorem 4.5 *Let $x\in\left(0,\frac{1}{2}-\frac{\sqrt{3}}{6}\right]\cup\left[\frac{1}{3},\frac{1}{2}\right]$ and $f:[0,1]\to\mathbb{R}$ be m-times differentiable for $m\geq 3$. If $|f^{(m)}|^q$ is convex for some $q\geq 1$, then*

$$\left|\int_0^1 f(t)dt-Q(0,x,1-x,1)+T_{2n}^{Q4}(x)\right|\leq L_{Q4}(m,x)\left(\frac{|f^{(m)}(0)|^q+|f^{(m)}(1)|^q}{2}\right)^{1/q}\tag{4.21}$$

while if $|f^{(m)}|$ is concave, then

$$\left|\int_0^1 f(t)dt-Q(0,x,1-x,1)+T_{2n}^{Q4}(x)\right|\leq L_{Q4}(m,x)\left|f^{(m)}\left(\frac{1}{2}\right)\right|,\tag{4.22}$$

where

$$\text{for } m=2n+1\quad L_{Q4}(2n+1,x)=\frac{2}{(2n+2)!}\,|F_{2n+2}^{Q4}(x,1/2)|$$
$$\text{and for } m=2n+2\quad L_{Q4}(2n+2,x)=\frac{1}{(2n+2)!}\,|G_{2n+2}^{Q4}(x,0)|$$

with $G_{2n+2}^{Q4}(x,0)$ and $F_{2n+2}^{Q4}(x,1/2)$ as in (4.11) and (4.13), respectively.

Proof. Analogous to the proof of Theorem 3.5. □

4.1.1 Simpson's 3/8 formula

For $x = 1/3$, (4.14) becomes classical Simpson's 3/8 formula:

$$\int_0^1 f(t)dt - \frac{1}{8}\left[f(0) + 3f\left(\frac{1}{3}\right) + 3f\left(\frac{2}{3}\right) + f(1)\right] = -\frac{1}{6480}\cdot f^{(4)}(\eta). \tag{4.23}$$

The results from this subsection are published in [31]. We have:

$$Q_{S38} = \frac{1}{8}\left[f(0) + 3f\left(\frac{1}{3}\right) + 3f\left(\frac{2}{3}\right) + f(1)\right],$$

$$T_{2n}^{S38} = T_{2n}^{Q4}\left(\frac{1}{3}\right) = \sum_{k=2}^{n}\frac{1}{(2k)!}\,G_{2k}^{S38}(0)\,[f^{(2k-1)}(1) - f^{(2k-1)}(0)]$$

$$G_k^{S38}(t) = \frac{1}{8}\left[3B_k^*\left(\frac{1}{3} - t\right) + 3B_k^*\left(\frac{2}{3} - t\right) + 2B_k^*(1-t)\right],\ k \geq 1$$

$$F_k^{S38}(t) = G_k^{S38}(t) - G_k^{S38}(0),\ k \geq 2$$

The error $R_{2n+2}^{S38}(f)$ for $n \geq 2$ can be expressed as:

$$R_{2n+2}^{S38}(f) = \frac{1}{8(2n+2)!}(1 - 9^{-n})B_{2n+2}\cdot f^{(2n+2)}(\eta), \quad \eta \in [0,1]$$

$$R_{2n+2}^{S38}(f) = \theta\,\frac{(2 - 2^{-1-2n})(1 - 9^{-n})B_{2n+2}}{8(2n+2)!}\left[f^{(2n+1)}(1) - f^{(2n+1)}(0)\right], \quad \theta \in [0,1]$$

Estimates of error for $p = 1$, $p = \infty$ and $m = 1,2,3,4$ are:

$$\left|\int_0^1 f(t)dt - Q_{CS38}\right| \leq C_{S38}(m,q)\cdot\|f^{(m)}\|_p,$$

where

$$C_{S38}(1,1) = \frac{25}{288},\ C_{S38}(2,1) = \frac{1}{192},\ C_{S38}(3,1) = \frac{1}{1728},\ C_{S38}(4,1) = \frac{1}{6480},$$
$$C_{S38}(1,\infty) = \frac{5}{24},\ C_{S38}(2,\infty) = \frac{1}{72},\ C_{S38}(3,\infty) = \frac{1}{768},\ C_{S38}(4,\infty) = \frac{1}{3456}.$$

4.1.2 Hermite-Hadamard-type inequality for the 4-point quadrature formulae

The main result of this subsection provides Hermite-Hadamard-type inequality for the 4-point quadrature formulae.

Theorem 4.6 *Let $f : [0,1] \to \mathbb{R}$ be 4-convex and such that $f^{(4)}$ is continuous on $[0,1]$. Then, for $x \in (0, \frac{1}{2} - \frac{\sqrt{3}}{6}]$ and $y \in \left[\frac{1}{3}, \frac{1}{2}\right]$*

$$Q(0,x,1-x,1) \leq \int_0^1 f(t)dt \ \leq\ Q(0,y,1-y,1), \tag{4.24}$$

where $Q(0,x,1-x,1)$ is defined in (4.4). If f is 4-concave, the inequalities are reversed.

Proof. Analogous to the proof of Theorem 3.6. □

The following corollary gives comparison between the Gauss 2-point and Simpson's 3/8 rule.

Corollary 4.1 *Let $f:[0,1]\to\mathbb{R}$ be 4-convex and such that $f^{(4)}$ is continuous on $[0,1]$. Then*

$$\frac{1}{2}f\left(\frac{3-\sqrt{3}}{6}\right)+\frac{1}{2}f\left(\frac{3+\sqrt{3}}{6}\right)\le\int_0^1 f(t)dt\ \le\ \frac{1}{8}\left(f(0)+3f\left(\frac{1}{3}\right)+3f\left(\frac{2}{3}\right)+f(1)\right).$$

If f is 4-concave, the inequalities are reversed.

Proof. Follows from (4.24) for $x=1/2-\sqrt{3}/6\Leftrightarrow B_2(x)=0$ and $y=1/3$. □

Remark 4.3 The result of Corollary 3.5 can be recaptured from (4.24) for $x=1/2-\sqrt{3}/6$ and $y=1/2$.

4.1.3 Bullen-Simpson's 3/8 inequality

For function $f:[0,1]\to\mathbb{R}$ such that $f^{(4)}$ is continuous on $[0,1]$ and $f^{(4)}(t)\ge 0$, $t\in[0,1]$, we have

$$\begin{aligned}\frac{1}{8}\left[3f\left(\frac{1}{6}\right)+2f\left(\frac{1}{2}\right)+3f\left(\frac{5}{6}\right)\right]\ &\le\ \int_0^1 f(t)\mathrm{d}t \qquad (4.25)\\ &\le\ \frac{1}{8}\left[f(0)+3f\left(\frac{1}{3}\right)+3f\left(\frac{2}{3}\right)+f(1)\right].\end{aligned}$$

In the case when $f^{(4)}$ exists, the condition $f^{(4)}(t)\ge 0$, $t\in[0,1]$ is equivalent to the requirement that f is 4-convex function on $[0,1]$. However, a function f may be 4-convex although $f^{(4)}$ does not exist.

P. S. Bullen in [11] proved that, if f is 4-convex, then (4.26) is valid. Moreover, he proved that the Maclaurin quadrature rule is more accurate than the Simpson's 3/8 quadrature rule, that is we have

$$\begin{aligned}0\ &\le\ \int_0^1 f(t)\mathrm{d}t-\frac{1}{8}\left[3f\left(\frac{1}{6}\right)+2f\left(\frac{1}{2}\right)+3f\left(\frac{5}{6}\right)\right]\\ &\le\ \frac{1}{8}\left[f(0)+3f\left(\frac{1}{3}\right)+3f\left(\frac{2}{3}\right)+f(1)\right]-\int_0^1 f(t)\mathrm{d}t, \qquad (4.26)\end{aligned}$$

provided f is 4-convex. We shall call this inequality Bullen-Simpson's 3/8 inequality.

The aim of this section is to establish a generalization of the inequalities (4.26) and (4.26) for a class of $(2r)$-convex functions and also to obtain some estimates for the absolute value of difference between the absolute value of error in the Maclaurin quadrature rule and the absolute value of error in the Simpson's 3/8 quadrature rule. Let us define

$$\begin{aligned}&D(0,1)\\ &:=\frac{1}{16}\left[f(0)+3f\left(\frac{1}{6}\right)+3f\left(\frac{1}{3}\right)+2f\left(\frac{1}{2}\right)+3f\left(\frac{2}{3}\right)+3f\left(\frac{5}{6}\right)+f(1)\right].\end{aligned}$$

We shall make use of the following seven-point quadrature formula

$$\int_0^1 f(t)\mathrm{d}t \approx D(0,1),$$

obtained by adding the Simpson 3/8 and the Maclaurin quadrature formulae. It is suitable for our purposes to rewrite the second inequality in (4.26) in the form

$$\int_0^1 f(t)\mathrm{d}t \leq D(0,1). \tag{4.27}$$

As we mentioned earlier, this inequality is valid for any 4-convex function f and we call it the Bullen-Simpson's 3/8 inequality. The results from this section are published in [87].

We consider the sequences of functions $(G_k(t))_{k\geq 1}$ and $(F_k(t))_{k\geq 1}$ defined for $t \in \mathbb{R}$ by

$$G_k(t) := G_k^{S38}(t) + G_k^M(t),\ F_k(t) := F_k^{S38}(t) + F_k^M(t),$$

where $G_k^{S38}(t)$, $G_k^M(t)$, $F_k^{S38}(t)$ and $F_k^M(t)$ are defined as Section 4.1.1 and Section 3.1.4 respectively. So we have

$$\begin{aligned} G_k(t) &= 2B_k^*(1-t) + 3B_k^*\left(\frac{1}{6}-t\right) + 3B_k^*\left(\frac{1}{3}-t\right) \\ &+ 2B_k^*\left(\frac{1}{2}-t\right) + 3B_k^*\left(\frac{2}{3}-t\right) + 3B_k^*\left(\frac{5}{6}-t\right),\ t \in \mathbb{R} \end{aligned}$$

and

$$F_k(t) = G_k(t) - \tilde{B}_k,\ t \in \mathbb{R}$$

where

$$\begin{aligned} \tilde{B}_k &= \tilde{B}_k^{S38} + \tilde{B}_k^M \\ &= B_k(0) + 3B_k\left(\frac{1}{6}\right) + 3B_k\left(\frac{1}{3}\right) + 2B_k\left(\frac{1}{2}\right) + 3B_k\left(\frac{2}{3}\right) + 3B_k\left(\frac{5}{6}\right) + B_k(1). \end{aligned}$$

For any function $f : [0,1] \to \mathbb{R}$ such that $f^{(n-1)}$ exists on $[0,1]$ for some $n \geq 1$ let $D(0,1)$ be defined as in Introduction. Further, we define $T_0(f) = T_1(f) := 0$ and, for $2 \leq m \leq [n/2]$,

$$T_m(f) := \frac{1}{2}\left[T_m^{S38}(f) + T_m^M(f)\right],$$

where $T_m^{S38}(f)$ and $T_m^M(f)$ are given in Section 4.1.1 and Section 3.1.4, respectively. It is easy to see that

$$T_m(f) = \frac{1}{8}\sum_{k=2}^{m} \frac{1}{(2k)!} 2^{-2k}(1-3^{2-2k})B_{2k}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right]. \tag{4.28}$$

In the next lemma we establish two formulae which play the key role in this paper. We call them the Euler Bullen-Simpson 3/8 formulae.

Lemma 4.2 *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$, for some $n \geq 1$. Then we have*

$$\int_0^1 f(t)\mathrm{d}t = D(0,1) + T_r(f) + \tau_n^1(f), \tag{4.29}$$

where $r = [n/2]$ and

$$\tau_n^1(f) = \frac{1}{16\,(n!)} \int_0^1 G_n\,(t)\,\mathrm{d}f^{(n-1)}(t).$$

Also,

$$\int_0^1 f(t)\mathrm{d}t = D(0,1) + T_s(f) + \tau_n^2(f), \tag{4.30}$$

where $s = [(n-1)/2]$ and

$$\tau_n^2(f) = \frac{1}{16\,(n!)} \int_0^1 F_n\,(t)\,\mathrm{d}f^{(n-1)}(t).$$

Proof. We multiply Euler-Simpson's 3/8 and Euler-Maclaurin's formulae by the factor $1/2$ and then add them up to obtain the identities (4.29) and (4.30). □

Remark 4.4 In the case when $f : [0,1] \to \mathbb{R}$ is such that $f^{(n)}$ exists and is integrable on $[0,1]$, then the Riemann-Stieltjes integral $\int_0^1 H(t)\mathrm{d}f^{(n-1)}(t)$ is equal to the Riemann integral $\int_0^1 H(t) f^{(n)}(t)\mathrm{d}t$. Therefore, if $f^{(n)}$ exists for some $n \geq 1$ and is integrable on $[0,1]$, then (4.29) and (4.30) reduce to

$$\int_0^1 f(t)\mathrm{d}t = D(0,1) + T_r(f) + \frac{1}{16\,(n!)} \int_0^1 G_n\,(t)\,f^{(n)}(t)\mathrm{d}t. \tag{4.31}$$

and

$$\int_0^1 f(t)\mathrm{d}t = D(0,1) + T_s(f) + \frac{1}{16\,(n!)} \int_0^1 F_n\,(t)\,f^{(n)}(t)\mathrm{d}t. \tag{4.32}$$

Remark 4.5 The interval $[0,1]$ is used for simplicity and involves no loss in generality. The results which follow will apply, without comment, to any interval that is convenient. Namely it is easy to transform the identities (4.29) and (4.30) to the identities which hold for any function $f : [a,b] \to \mathbb{R}$ such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$, for some $n \geq 1$. We get

$$\int_a^b f(t)\mathrm{d}t = D(a,b) + \tilde{T}_r(f) + \frac{(b-a)^n}{16\,(n!)} \int_a^b G_n\left(\frac{t-a}{b-a}\right)\mathrm{d}f^{(n-1)}(t) \tag{4.33}$$

and

$$\int_a^b f(t)\mathrm{d}t = D(a,b) + \tilde{T}_s(f) + \frac{(b-a)^n}{16\,(n!)} \int_a^b F_n\left(\frac{t-a}{b-a}\right)\mathrm{d}f^{(n-1)}(t), \tag{4.34}$$

where

$$
\begin{aligned}
D(a,b) := \quad & \frac{b-a}{16}\left[f(a)+3\left(\frac{5a+b}{6}\right)+3\left(\frac{2a+b}{3}\right)\right. \\
& + \left. 2\left(\frac{a+b}{2}\right)+3\left(\frac{a+5b}{6}\right)+3\left(\frac{a+2b}{3}\right)+f(b)\right],
\end{aligned}
$$

while $\tilde{T}_0(f)=\tilde{T}_1(f)=0$ and

$$
\tilde{T}_m(f)=\frac{1}{8}\sum_{k=2}^{m}\frac{(b-a)^{2k}}{(2k)!}2^{-2k}(1-3^{2-2k})B_{2k}\left[f^{(2k-1)}(b)-f^{(2k-1)}(a)\right],
$$

for $2\leq m\leq[n/2]$.

Now, we use the Euler Bullen-Simpson 3/8 formulae established in Lemma 4.2 to extend the Bullen-Simpson's 3/8 inequality for $(2r)$-convex functions. First, we need some properties of the functions $G_k(t)$ and $F_k(t)$.

First note that it is enough to know the values of the functions $G_k(t)$ and $F_k(t)$, $k\geq 1$ only on the interval $[0,1/2]$. Namely, the functions $B_k^*(t)$ are periodic with period 1 so that for $0\leq t\leq 1/2$ we have

$$
\begin{aligned}
G_k\left(t+\frac{1}{2}\right) &= 2B_k^*\left(\frac{1}{2}-t\right)+3B_k^*\left(-\frac{1}{3}-t\right)+3B_k^*\left(-\frac{1}{6}-t\right) \\
&\quad +2B_k^*(-t)+3B_k^*\left(\frac{1}{6}-t\right)+3B_k^*\left(\frac{1}{3}-t\right) \\
&= 2B_k^*\left(\frac{1}{2}-t\right)+3B_k^*\left(\frac{2}{3}-t\right)+3B_k^*\left(\frac{5}{6}-t\right) \\
&\quad +2B_k^*(1-t)+3B_k^*\left(\frac{1}{6}-t\right)+3B_k^*\left(\frac{1}{3}-t\right) \\
&= G_k(t)
\end{aligned}
$$

and

$$
F_k\left(t+\frac{1}{2}\right)=G_k\left(t+\frac{1}{2}\right)-\tilde{B}_k=G_k(t)-\tilde{B}_k=F_k(t).
$$

Since $B_1(t)=t-(1/2)$, we get $\tilde{B}_1=0$ and

$$
G_1(t)=F_1(t)=\begin{cases}-1, & t=0\\ -16t+1, & t\in(0,1/6]\\ -16t+4, & t\in(1/6,1/3]\\ -6t+11/2, & t\in(1/3,1/2]\end{cases}. \tag{4.35}
$$

Further, for $k\geq 2$ the functions $B_k^*(t)$ are continuous and the same is true for $G_k(t)$ and $F_k(t)$, $k\geq 2$. Also we have for $k\geq 2$

$$
G_k(0)=G_k(1/2)=G_k(1)=\tilde{B}_k
$$

and

$$F_k(0) = F_k(1/2) = F_k(1) = 0.$$

For example, for $k = 2$ and $k = 3$ we have $B_2(t) = t^2 - t + (1/6)$ and $B_3(t) = t^3 - (3/2)t^2 + (1/2)t$, so that by direct calculation we get $\tilde{B}_2 = \tilde{B}_3 = 0$ and

$$G_2(t) = F_2(t) = \begin{cases} 16t^2 - 2t, & t \in [0, 1/6] \\ 16t^2 - 8t + 1, & t \in (1/6, 1/3] \\ 16t^2 - 14t + 3, & t \in (1/3, 1/2] \end{cases}, \tag{4.36}$$

$$G_3(t) = F_3(t) = \begin{cases} -16t^3 + 3t^2, & t \in [0, 1/6] \\ -16t^3 + 12t^2 - 3t + (1/4), & t \in (1/6, 1/3] \\ -16t^3 + 21t^2 - 9t + (5/4), & t \in (1/3, 1/2] \end{cases}. \tag{4.37}$$

The Bernoulli polynomials of even order are symmetric and those of odd order skew-symmetric about $1/2$, that is [1, 23.1.8]

$$B_k(1-t) = (-1)^k B_k(t), \ 0 \le t \le 1, \ k \ge 1. \tag{4.38}$$

Also, we have

$$B_k(1) = B_k(0) = B_k, \ k \ge 2, \ B_1(1) = -B_1(0) = \frac{1}{2}$$

and

$$B_{2r-1} = 0, \ r \ge 2.$$

Therefore, for $r \ge 1$ we have

$$\tilde{B}_{2r-1} = 0 \tag{4.39}$$

and

$$\tilde{B}_{2r} = 2B_{2r} + 3B_{2r}\left(\frac{1}{6}\right) + 3B_{2r}\left(\frac{1}{3}\right) + 2B_{2r}\left(\frac{1}{2}\right) + 3B_{2r}\left(\frac{2}{3}\right) + 3B_{2r}\left(\frac{5}{6}\right).$$

Also, we have [1, 23.1.21, 23.1.23, 23.1.24]

$$B_{2r}\left(\frac{1}{2}\right) = -\left(1 - 2^{1-2r}\right) B_{2r}, \ B_{2r}\left(\frac{1}{3}\right) = -\frac{1}{2}\left(1 - 3^{1-2r}\right) B_{2r},$$

and

$$B_{2r}\left(\frac{1}{6}\right) = \frac{1}{2}\left(1 - 2^{1-2r}\right)\left(1 - 3^{1-2r}\right) B_{2r},$$

which gives the formula

$$\tilde{B}_{2r} = -2^{1-2r}(1 - 3^{2-2r}) B_{2r}, \ r \ge 1. \tag{4.40}$$

Now, using (4.39) and (4.40) we get

$$F_{2r-1}(t) = G_{2r-1}(t), \ r \ge 1 \tag{4.41}$$

and

$$F_{2r}(t) = G_{2r}(t) + 2^{1-2r}(1-3^{2-2r})B_{2r},\ r \geq 1. \tag{4.42}$$

Further, as we pointed out earlier, the points 0 and $1/2$ are the zeros of $F_k(t)$, $k \geq 2$. As we shall see below, 0 and $1/2$ are the only zeros of $F_k(t)$ in $[0,1/2]$ for $k = 2r$, $r \geq 2$, while for $k = 2r-1$, $r \geq 2$, using (4.38) we easily get

$$G_{2r-1}\left(\frac{1}{4}\right) = F_{2r-1}\left(\frac{1}{4}\right) = 0.$$

We shall see that 0, $1/4$ and $1/2$ are the only zeros of $F_{2r-1}(t) = G_{2r-1}(t)$, in $[0,1/2]$ for $r \geq 2$. Also, note that for $r \geq 1$ we have

$$G_{2r}(0) = G_{2r}\left(\frac{1}{2}\right) = \tilde{B}_{2r} = -2^{1-2r}(1-3^{2-2r})B_{2r}.$$

The values $G_{2r}(1/4)$ and $F_{2r}(1/4)$ can also be evaluated exactly.

Lemma 4.3 *For $r \geq 1$ we have*

$$G_{2r}\left(\frac{1}{4}\right) = 2^{1-2r}(1-2^{1-2r})(1-3^{2-2r})B_{2r},$$

and

$$F_{2r}\left(\frac{1}{4}\right) = 2^{2-2r}(1-2^{-2r})(1-3^{2-2r})B_{2r}. \tag{4.43}$$

Proof. We use the formula [1, 23.1.10]

$$B_{2r}(mx) = m^{2r-1}\sum_{i=0}^{m-1} B_{2r}\left(x+\frac{i}{m}\right),\ r \geq 0,\ m \geq 1.$$

Setting $m = 3$, $x = 1/12$ we get

$$B_{2r}\left(\frac{1}{4}\right) = B_{2r}\left(\frac{3}{12}\right) = 3^{2r-1}\left[B_{2r}\left(\frac{1}{12}\right) + B_{2r}\left(\frac{5}{12}\right) + B_{2r}\left(\frac{3}{4}\right)\right],$$

which can be rewritten as

$$B_{2r}\left(\frac{1}{12}\right) + B_{2r}\left(\frac{5}{12}\right) = (3^{1-2r}-1)B_{2r}\left(\frac{1}{4}\right)$$

since $B_{2r}(3/4) = B_{2r}(1/4)$, by (4.38). Now we have (see [1, 23.1.22])

$$B_{2r}\left(\frac{1}{4}\right) = -2^{-2r}(1-2^{1-2r})B_{2r}$$

and using again (4.38) we get

$$\begin{aligned} G_{2r}\left(\frac{1}{4}\right) &= 4B_{2r}\left(\frac{1}{4}\right) + 6\left[B_{2r}\left(\frac{1}{12}\right) + B_{2r}\left(\frac{5}{12}\right)\right] \\ &= 2(3^{2-2r}-1)B_{2r}\left(\frac{1}{4}\right) \\ &= 2^{1-2r}(1-2^{1-2r})(1-3^{2-2r})B_{2r}. \end{aligned}$$

Also using (4.40) we get (4.43)

$$F_{2r}\left(\frac{1}{4}\right) = G_{2r}\left(\frac{1}{4}\right) - \tilde{B}_{2r} = 2^{2-2r}(1-2^{-2r})(1-3^{2-2r})B_{2r}.$$

□

Now we prove that the functions $G_k(t)$ and $F_k(t)$ are symmetric for even k and skew-symmetric for odd k, about $1/4$.

Lemma 4.4 *For $k \geq 2$ we have*

$$G_k\left(\frac{1}{2}-t\right) = (-1)^k G_k(t),\ 0 \leq t \leq \frac{1}{2},$$

and

$$F_k\left(\frac{1}{2}-t\right) = (-1)^k F_k(t),\ 0 \leq t \leq \frac{1}{2}.$$

Proof. As we already noted, the functions $B_k^*(t)$ are periodic with period 1 and continuous for $k \geq 2$. Also, from (4.38) we get

$$B_k^*(1-t) = (-1)^k B_k^*(t),\ t \in \mathbb{R},\ k \geq 2.$$

Therefore, for $k \geq 2$ and $0 \leq t \leq \frac{1}{2}$ we have

$$\begin{aligned} G_k\left(\frac{1}{2}-t\right) &= 2B_k^*\left(\frac{1}{2}+t\right) + 3B_k^*\left(-\frac{1}{3}+t\right) + 3B_k^*\left(-\frac{1}{6}+t\right) \\ &+ 2B_k^*(t) + 3B_k^*\left(\frac{1}{6}+t\right) + 3B_k^*\left(\frac{1}{3}+t\right) \\ &= (-1)^k \left[2B_k^*\left(\frac{1}{2}-t\right) + 3B_k^*\left(\frac{4}{3}-t\right) + 3B_k^*\left(\frac{7}{6}-t\right)\right. \\ &+ \left. 2B_k^*(1-t) + 3B_k^*\left(\frac{5}{6}-t\right) + 3B_k^*\left(\frac{2}{3}-t\right)\right] \\ &= (-1)^k G_k(t), \end{aligned}$$

which proves the first identity. Further, we have $\tilde{B}_k = (-1)^k \tilde{B}_k$, $k \geq 2$, since (4.39) holds, so that

$$F_k\left(\frac{1}{2}-t\right) = G_k\left(\frac{1}{2}-t\right) - \tilde{B}_k = (-1)^k \left[G_k(t) - \tilde{B}_k\right] = (-1)^k F_k(t),$$

which proves the second identity. □

Note that the identities established in Lemma 4.4 are valid for $k = 1$, too, except at the points 0, $1/6$, $1/3$ and $1/2$.

Lemma 4.5 *For $r \geq 2$ the function $G_{2r-1}(t)$ has no zeros in the interval $\left(0, \frac{1}{4}\right)$. The sign of this function is determined by*

$$(-1)^r G_{2r-1}(t) > 0,\ 0 < t < \frac{1}{4}.$$

Proof. For $r=2$, $G_3(t)$ is given by (4.37) and it is easy to see that

$$G_3(t) > 0,\ 0 < t < \frac{1}{4}. \tag{4.44}$$

Thus, our assertion is true for $r=2$. Now, assume that $r \geq 3$. Then $2r-1 \geq 5$ and $G_{2r-1}(t)$ is continuous and twice differentiable function. We get

$$G'_{2r-1}(t) = -(2r-1)G_{2r-2}(t)$$

and

$$G''_{2r-1}(t) = (2r-1)(2r-2)G_{2r-3}(t). \tag{4.45}$$

We know that 0 and $\frac{1}{4}$ are the zeros of $G_{2r-1}(t)$. Let us suppose that some α, $0<\alpha<\frac{1}{4}$, is also a zero of $G_{2r-1}(t)$. Then inside each of the intervals $(0,\alpha)$ and $\left(\alpha,\frac{1}{4}\right)$ the derivative $G'_{2r-1}(t)$ must have at least one zero, say β_1, $0<\beta_1<\alpha$ and β_2, $\alpha<\beta_2<\frac{1}{4}$. Therefore, the second derivative $G''_{2r-1}(t)$ must have at least one zero inside the interval (β_1,β_2). Thus, from the assumption that $G_{2r-1}(t)$ has a zero inside the interval $\left(0,\frac{1}{4}\right)$, it follows that $(2r-1)(2r-2)G_{2r-3}(t)$ also has a zero inside this interval. From this it follows that $G_3(t)$ would have a zero inside the interval $\left(0,\frac{1}{4}\right)$, which is not true. Thus, $G_{2r-1}(t)$ can not have a zero inside the interval $\left(0,\frac{1}{4}\right)$. Further, if $G_{2r-3}(t)>0$, $0<t<\frac{1}{4}$, then from (4.45) it follows that $G_{2r-1}(t)$ is convex on $\left(0,\frac{1}{4}\right)$ and hence $G_{2r-1}(t)<0$, $0<t<\frac{1}{4}$, while in the case when $G_{2r-3}(t)<0$, $0<t<\frac{1}{4}$ we have that $G_{2r-1}(t)$ is concave and hence $G_{2r-1}(t)>0$, $0<t<\frac{1}{4}$. Since (4.44) is valid we conclude that

$$(-1)^r G_{2r-1}(t) > 0,\ 0<t<\frac{1}{4}.$$

□

Corollary 4.2 *For $r \geq 2$ the functions $(-1)^{r-1}F_{2r}(t)$ and $(-1)^{r-1}G_{2r}(t)$ are strictly increasing on the interval $\left(0,\frac{1}{4}\right)$, and strictly decreasing on the interval $\left(\frac{1}{4},\frac{1}{2}\right)$. Consequently, 0 and $\frac{1}{2}$ are the only zeros of $F_{2r}(t)$ in the interval $\left[0,\frac{1}{2}\right]$ and*

$$\max_{t\in[0,1]} |F_{2r}(t)| = 2^{2-2r}(1-2^{-2r})(1-3^{2-2r})|B_{2r}|,\ r\geq 2.$$

Also, we have

$$\max_{t\in[0,1]} |G_{2r}(t)| = 2^{1-2r}(1-3^{2-2r})|B_{2r}|,\ r\geq 2.$$

Proof. We have

$$\left[(-1)^{r-1}F_{2r}(t)\right]' = \left[(-1)^{r-1}G_{2r}(t)\right]' = 2r(-1)^r G_{2r-1}(t)$$

and $(-1)^r G_{2r-1}(t) > 0$ for $0<t<\frac{1}{4}$, by the Lemma 4.5. Thus, $(-1)^{r-1}F_{2r}(t)$ and $(-1)^{r-1}G_{2r}(t)$ are strictly increasing on the interval $\left(0,\frac{1}{4}\right)$. Also, by the Lemma 4.4, we have $F_{2r}\left(\frac{1}{2}-t\right) = F_{2r}(t)$, $0\leq t\leq\frac{1}{2}$ and $G_{2r}\left(\frac{1}{2}-t\right) = G_{2r}(t)$, $0\leq t\leq \frac{1}{2}$, which implies that $(-1)^{r-1}F_{2r}(t)$ and $(-1)^{r-1}G_{2r}(t)$ are strictly decreasing on the interval $\left(\frac{1}{4},\frac{1}{2}\right)$. Further, $F_{2r}(0)=F_{2r}\left(\frac{1}{2}\right)=0$, which implies that $|F_{2r}(t)|$ achieves its maximum at $t=\frac{1}{4}$, that is

$$\max_{t\in[0,1]}|F_{2r}(t)|=\left|F_{2r}\left(\frac{1}{4}\right)\right|=2^{2-2r}(1-2^{-2r})(1-3^{2-2r})|B_{2r}|.$$

Also,

$$\max_{t\in[0,1]}|G_{2r}(t)|=\max\left\{|G_{2r}(0)|,\left|G_{2r}\left(\frac{1}{4}\right)\right|\right\}=2^{1-2r}(1-3^{2-2r})|B_{2r}|,$$

which completes the proof. □

Corollary 4.3 *Assume $r\geq 2$. Then we have*

$$\int_0^1|G_{2r-1}(t)|\,\mathrm{d}t=\frac{2^{3-2r}(1-2^{-2r})(1-3^{2-2r})}{r}|B_{2r}|.$$

Also, we have

$$\int_0^1|F_{2r}(t)|\,\mathrm{d}t=\left|\tilde{B}_{2r}\right|=2^{1-2r}\left(1-3^{2-2r}\right)|B_{2r}|$$

and

$$\int_0^1|G_{2r}(t)|\,\mathrm{d}t\leq 2\left|\tilde{B}_{2r}\right|=2^{2-2r}\left(1-3^{2-2r}\right)|B_{2r}|.$$

Proof. It is easy to see that

$$G_m'(t)=-mG_{m-1}(t),\ m\geq 3. \tag{4.46}$$

By (4.41) we have $\int_0^1|F_{2r-1}(t)|dt=\int_0^1|G_{2r-1}(t)|dt$. Now, using Lemma 4.4, Lemma 4.5 and (4.46) we get

$$\begin{aligned}\int_0^1|G_{2r-1}(t)|\,\mathrm{d}t&=4\left|\int_0^{\frac{1}{4}}G_{2r-1}(t)\mathrm{d}t\right|=4\left|-\frac{1}{2r}\,G_{2r}(t)|_0^{\frac{1}{4}}\right|\\&=\frac{2}{r}\left|G_{2r}\left(\frac{1}{4}\right)-G_{2r}(0)\right|=\frac{2^{3-2r}(1-2^{-2r})(1-3^{2-2r})}{r}|B_{2r}|,\end{aligned}$$

which proves the first assertion. By the Corollary 4.2, $F_{2r}(t)$ does not change the sign on the interval $\left(0,\frac{1}{2}\right)$. Therefore, using (4.42) and (4.46), we get

$$\begin{aligned}\int_0^1|F_{2r}(t)|\,\mathrm{d}t&=2\left|\int_0^{1/2}F_{2r}(t)\mathrm{d}t\right|=2\left|\int_0^{1/2}\left[G_{2r}(t)-\tilde{B}_{2r}\right]\mathrm{d}t\right|\\&=2\left|-\frac{1}{2r+1}\,G_{2r+1}(t)|_0^{1/2}-\frac{1}{2}\tilde{B}_{2r}\right|=\left|\tilde{B}_{2r}\right|=2^{1-2r}\left(1-3^{2-2r}\right)|B_{2r}|.\end{aligned}$$

This proves the second assertion. Finally, we use (4.42) again and the triangle inequality to obtain

$$\begin{aligned}\int_0^1 |G_{2r}(t)|\,\mathrm{d}t &= \int_0^1 \left|F_{2r}(t)+\tilde{B}_{2r}\right|\mathrm{d}t\\ &\le \int_0^1 |F_{2r}(t)|\,\mathrm{d}t + \left|\tilde{B}_{2r}\right| = 2\left|\tilde{B}_{2r}\right| = 2^{2-2r}\left(1-3^{2-2r}\right)|B_{2r}|,\end{aligned}$$

which proves the third assertion. □

In the following discussion we assume that $f:[0,1]\to\mathbb{R}$ has a continuous derivative of order n, for some $n\ge 1$. In this case we can use formulae (4.31) and (4.32) from Remark 4.4 so that remainders $\tau_n^1(f)$ and $\tau_n^2(f)$ are given as

$$\tau_n^1(f) = \frac{1}{16(n!)}\int_0^1 G_n(s)f^{(n)}(s)\mathrm{d}s \tag{4.47}$$

and

$$\tau_n^2(f) = \frac{1}{16(n!)}\int_0^1 F_n(s)f^{(n)}(s)\mathrm{d}s. \tag{4.48}$$

Lemma 4.6 *If $f:[0,1]\to\mathbb{R}$ is such that $f^{(2r)}$ is continuous on $[0,1]$, for some $r\ge 2$, then there exists a point $\eta\in[0,1]$ such that*

$$\tau_{2r}^2(f) = \frac{1}{8(2r)!}2^{-2r}(1-3^{2-2r})B_{2r}f^{(2r)}(\eta). \tag{4.49}$$

Proof. Using (4.48) with $n=2r$, we can rewrite $\tau_{2r}^2(f)$ as

$$\tau_{2r}^2(f) = (-1)^{r-1}\frac{1}{16(2r)!}J_r, \tag{4.50}$$

where

$$J_r = \int_0^1 (-1)^{r-1}F_{2r}(s)f^{(2r)}(s)\mathrm{d}s. \tag{4.51}$$

If

$$m=\min_{t\in[0,1]} f^{(2r)}(t),\;\; M=\max_{t\in[0,1]} f^{(2r)}(t),$$

then

$$m\le f^{(2r)}(s)\le M,\;\; 0\le s\le 1.$$

On the other side, from Corollary 4.2 it follows that

$$(-1)^{r-1}F_{2r}(s)\ge 0,\;\; 0\le s\le 1,$$

which implies

$$m\int_0^1 (-1)^{r-1}F_{2r}(s)ds \le J_r \le M\int_0^1 (-1)^{r-1}F_{2r}(s)ds.$$

Similarly as we have already calculated in the proof of Corollary 4.3, we get

$$\int_0^1 F_{2r}(s)ds = -\tilde{B}_{2r} = 2^{1-2r}(1-3^{2-2r})B_{2r},$$

so that

$$m(-1)^{r-1}2^{1-2r}(1-3^{2-2r})B_{2r} \leq J_r \leq M(-1)^{r-1}2^{1-2r}(1-3^{2-2r})B_{2r}.$$

By the continuity of $f^{(2r)}(s)$ on $[0,1]$, it follows that there must exist a point $\eta \in [0,1]$ such that

$$J_r = (-1)^{r-1}2^{1-2r}(1-3^{2-2r})B_{2r}f^{(2r)}(\eta).$$

Combining this with (4.50) we get (4.49). □

Now, we prove the main result:

Theorem 4.7 *Assume $f:[0,1]\to\mathbb{R}$ is such that $f^{(2r)}$ is continuous on $[0,1]$, for some $r\geq 2$. If f is $(2r)$-convex function, then for even r we have*

$$\begin{aligned} 0 \ \leq\ & \int_0^1 f(t)\mathrm{d}t - \frac{1}{8}\left[3f\left(\frac{1}{6}\right)+2f\left(\frac{1}{2}\right)+3f\left(\frac{5}{6}\right)\right] - T_{r-1}^M(f) \\ \leq\ & \frac{1}{8}\left[f(0)+3f\left(\frac{1}{3}\right)+3f\left(\frac{2}{3}\right)+f(1)\right] + T_{r-1}^S - \int_0^1 f(t)\mathrm{d}t, \end{aligned} \tag{4.52}$$

while for odd r we have reversed inequalities in (4.52).

Proof. Let us denote by *LHS* and *RHS* respectively the left hand side and the right hand side in the second inequality in (4.52). Then we have

$$LHS = \rho_{2r}^2(f)$$

and

$$RHS - LHS = -2\tau_{2r}^2(f),$$

For Euler-Maclaurin formula, under given assumption on f, there exists a point $\xi \in [0,1]$ such that

$$\rho_{2r}^2(f) = -\frac{1}{8(2r)!}\left(1-2^{1-2r}\right)(1-3^{2-2r})B_{2r}f^{(2r)}(\xi). \tag{4.53}$$

Also by Lemma 4.6 we know that

$$-2\tau_{2r}^2(f) = -\frac{1}{8(2r)!}2^{1-2r}(1-3^{2-2r})B_{2r}f^{(2r)}(\eta), \tag{4.54}$$

for some point $\eta \in [0,1]$. Finally, we know that [1, 23.1.15]

$$(-1)^{r-1}B_{2r} > 0,\ r = 1,2,\cdots. \tag{4.55}$$

Now, if f is $(2r)$-convex function, then $f^{(2r)}(\xi) \geq 0$ and $f^{(2r)}(\eta) \geq 0$ so that using (4.53), (4.54) and (4.55) we get the inequalities

$$LHS \geq 0,\ RHS - LHS \geq 0, \text{ when } r \text{ is even;}$$

$$LHS \leq 0,\ RHS - LHS \leq 0, \text{ when } r \text{ is odd.}$$

This proves our assertions. □

Remark 4.6 In the case when $r = 2$ we have $B_4 = -1/30$ and formula (4.49) reduces to

$$\tau_4^2(f) = -\frac{1}{103680} f^{(4)}(\eta).$$

Note that in this case the result stated in Theorem 4.7 reduces to Bullen's result that we mentioned in Introduction.

Theorem 4.8 *Assume $f : [0,1] \to \mathbb{R}$ is such that $f^{(2r)}$ is continuous on $[0,1]$, for some $r \geq 2$. If f is either $(2r)$-convex or $(2r)$-concave function, then there exists a point $\vartheta \in [0,1]$ such that*

$$\tau_{2r}^2(f) = \vartheta \frac{1}{4(2r)!} 2^{-2r}(1-2^{-2r})(1-3^{2-2r})B_{2r}\left[f^{(2r-1)}(1) - f^{(2r-1)}(0)\right]. \tag{4.56}$$

Proof. First, consider the case when f is $(2r)$-convex, that is $f^{(2r)}(t) \geq 0$, $0 \leq t \leq 1$. By Corollary 4.2 we get

$$0 \leq (-1)^{r-1} F_{2r}(s) \leq (-1)^{r-1} F_{2r}\left(\frac{1}{4}\right), \quad 0 \leq s \leq 1.$$

Therefore, if J_r is given by (4.51), then

$$\begin{aligned} 0 \ &\leq\ J_r \leq (-1)^{r-1} F_{2r}\left(\frac{1}{4}\right) \int_0^1 f^{(2r)}(s)\mathrm{d}s \\ &=\ (-1)^{r-1} F_{2r}\left(\frac{1}{4}\right)\left[f^{(2r-1)}(1) - f^{(2r-1)}(0)\right]. \end{aligned}$$

So, there must exist a point $\vartheta \in [0,1]$ such that

$$J_r = \vartheta (-1)^{r-1} F_{2r}\left(\frac{1}{4}\right)\left[f^{(2r-1)}(1) - f^{(2r-1)}(0)\right].$$

Combining this with (4.50) and using (4.43) we get (4.56). The argument is the same when f is $(2r)$-concave since in that case $f^{(2r)}(t) \leq 0$, $0 \leq t \leq 1$ and we get

$$(-1)^{r-1} F_{2r}\left(\frac{1}{4}\right)\left[f^{(2r-1)}(1) - f^{(2r-1)}(0)\right] \leq J_r \leq 0.$$

□

Remark 4.7 If we approximate $\int_0^1 f(t)\mathrm{d}t$ by

$$I_{2r}(f) := D(0,1) + T_{r-1}(f),$$

then the next approximation will be $I_{2r+2}(f)$. The difference

$$\Delta_{2r}(f) = I_{2r+2}(f) - I_{2r}(f)$$

is equal to the last term in $I_{2r+2}(f)$, that is

$$\Delta_{2r}(f) = \frac{1}{8(2r)!} 2^{-2r}(1-3^{2-2r})B_{2r}\left[f^{(2r-1)}(1) - f^{(2r-1)}(0)\right].$$

We see that, under the assumptions of Theorem 4.8,

$$\tau_{2r}^2(f) = 2\vartheta\left(1-2^{-2r}\right)\Delta_{2r}(f).$$

Theorem 4.9 *Assume $f:[0,1]\to\mathbb{R}$ is such that $f^{(2r+2)}$ is continuous on $[0,1]$, for some $r\geq 2$. If f is either $(2r)$-convex and $(2r+2)$-convex or $(2r)$-concave and $(2r+2)$-concave function, then the remainder $\tau_{2r}^2(f)$ has the same sign as the first neglected term $\Delta_{2r}(f)$ and*

$$\left|\tau_{2r}^2(f)\right| \leq \left|\Delta_{2r}(f)\right|.$$

Proof. We have

$$\Delta_{2r}(f) + \tau_{2r+2}^2(f) = \tau_{2r}^2(f),$$

that is

$$\Delta_{2r}(f) = \tau_{2r}^2(f) - \tau_{2r+2}^2(f). \tag{4.57}$$

By (4.48) we have

$$\tau_{2r}^2(f) = \frac{1}{16(2r)!}\int_0^1 F_{2r}(s)f^{(2r)}(s)\mathrm{d}s$$

and

$$-\tau_{2r+2}^2(f) = \frac{1}{16(2r+2)!}\int_0^1 [-F_{2r+2}(s)]f^{(2r+2)}(s)\mathrm{d}s.$$

Under the assumptions made on f, we have for all $s\in[0,1]$ either

$$f^{(2r)}(s)\geq 0 \text{ and } f^{(2r+2)}(s)\geq 0$$

or

$$f^{(2r)}(s)\leq 0 \text{ and } f^{(2r+2)}(s)\leq 0.$$

Also, from Corollary 4.2 it follows that for all $s\in[0,1]$

$$(-1)^{r-1}F_{2r}(s)\geq 0 \text{ and } (-1)^{r-1}[-F_{2r+2}(s)]\geq 0.$$

We conclude that $\tau_{2r}^2(f)$ has the same sign as $-\tau_{2r+2}^2(f)$. Therefore, because of (4.57), $\Delta_{2r}(f)$ must have the same sign as $\tau_{2r}^2(f)$ and $-\tau_{2r+2}^2(f)$. Moreover, it follows that

$$\left|\tau_{2r}^2(f)\right| \leq \left|\Delta_{2r}(f)\right| \text{ and } \left|-\tau_{2r+2}^2(f)\right| \leq \left|\Delta_{2r}(f)\right|.$$

□

Now, we use the Euler Bullen-Simpson 3/8 formulae established in Lemma 4.2 to determine the absolute value of difference between the absolute value of error in the Maclaurin quadrature rule and the absolute value of error in the Simpson's 3/8 quadrature rule. We do this by proving a number of inequalities for various classes of functions.

First, let us denote

$$R_S := \int_0^1 f(t)\mathrm{d}t - \frac{1}{8}\left[f(0) + 3f\left(\frac{1}{3}\right) + 3f\left(\frac{2}{3}\right) + f(1)\right]$$

and

$$R_M := \int_0^1 f(t)\mathrm{d}t - \frac{1}{8}\left[3f\left(\frac{1}{6}\right) + 2f\left(\frac{1}{2}\right) + 3f\left(\frac{5}{6}\right)\right].$$

By the triangle inequality we have

$$||R_M| - |R_S|| \leq |R_M + R_S|.$$

Now, if we define $R := R_M + R_S$, then

$$\frac{R}{2} = \int_0^1 f(t)\mathrm{d}t - D(0,1) \tag{4.58}$$

where $D(0,1)$ denotes the same expression as in Introduction.

Theorem 4.10 *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n-1)}$ is an L-Lipschitzian function on $[0,1]$ for some $n \geq 2$.*
If $n = 2r-1$, $r \geq 2$, then

$$\begin{aligned} \left|\int_0^1 f(t)\mathrm{d}t - D(0,1) - T_{r-1}(f)\right| &\leq \frac{1}{16(2r-1)!}\int_0^1 |G_{2r-1}(t)|\mathrm{d}t \cdot L \\ &= \frac{2^{-2r}(1-2^{-2r})(1-3^{2-2r})}{(2r)!}|B_{2r}| \cdot L. \end{aligned} \tag{4.59}$$

If $n = 2r$, $r \geq 2$, then

$$\begin{aligned} \left|\int_0^1 f(t)\mathrm{d}t - D(0,1) - T_{r-1}(f)\right| &\leq \frac{1}{16(2r)!}\int_0^1 |F_{2r}(t)|\mathrm{d}t \cdot L \\ &= \frac{2^{-2r}(1-3^{2-2r})}{8(2r)!}|B_{2r}| \cdot L. \end{aligned} \tag{4.60}$$

and also

$$\begin{aligned} \left|\int_0^1 f(t)\mathrm{d}t - D(0,1) - T_r(f)\right| &\leq \frac{1}{16(2r)!}\int_0^1 |G_{2r}(t)|\mathrm{d}t \cdot L \\ &\leq \frac{2^{-2r}(1-3^{2-2r})}{4(2r)!}|B_{2r}| \cdot L. \end{aligned} \tag{4.61}$$

Proof. For any integrable function $\Phi : [0,1] \to \mathbb{R}$ we have

$$\left| \int_0^1 \Phi(t) \mathrm{d} f^{(n-1)}(t) \right| \leq \int_0^1 |\Phi(t)| \, \mathrm{d}t \cdot L, \tag{4.62}$$

since $f^{(n-1)}$ is L-Lipschitzian function. Applying (4.62) with $\Phi(t) = G_{2r-1}(t)$, we get

$$\left| \frac{1}{16(2r-1)!} \int_0^1 G_{2r-1}(t) \mathrm{d} f^{(2r-2)}(t) \right| \leq \frac{1}{16(2r-1)!} \int_0^1 |G_{2r-1}(t)| \, \mathrm{d}t \cdot L.$$

Applying the above inequality and the identity (4.30), we get the inequality in (4.59). Similarly, we can apply the inequality (4.62) with $\Phi(t) = F_{2r}(t)$ and again the identity (4.30) to get the inequality in (4.60). Finally, applying (4.62) with $\Phi(t) = G_{2r}(t)$ and the identity (4.29), we get the first inequality in (4.61). The equalities in (4.59) and (4.60) and the second inequality in (4.61) follow from Corollary 4.3. □

Corollary 4.4 *Let $f : [0,1] \to \mathbb{R}$ be such that f'' is L-Lipschitzian on $[0,1]$, then*

$$\left| \int_0^1 f(t) \mathrm{d}t - D(0,1) \right| \leq \frac{1}{13824} L, \;\; |R| \leq \frac{1}{6912} L.$$

If f''' is L-Lipschitzian on $[0,1]$, then

$$\left| \int_0^1 f(t) \mathrm{d}t - D(0,1) \right| \leq \frac{1}{103680} L, \;\; |R| \leq \frac{1}{51840} L.$$

Proof. The first pair of inequalities follows from (4.59) with $r = 2$, while the second pair follows from (4.60) with $r = 2$. □

Remark 4.8 If f is L-Lipschitzian on $[0,1]$, then, as above

$$\left| \int_0^1 f(t) \mathrm{d}t - D(0,1) \right| \leq \frac{1}{16} \int_0^1 |G_1(t)| \mathrm{d}t \cdot L.$$

Since

$$\int_0^1 |G_1(t)| \mathrm{d}t = \frac{25}{36},$$

we get

$$\left| \int_0^1 f(t) \mathrm{d}t - D(0,1) \right| \leq \frac{25}{576} \cdot L \text{ and } |R| \leq \frac{25}{288} \cdot L.$$

If f' is L-Lipschitzian on $[0,1]$, then

$$\left| \int_0^1 f(t) \mathrm{d}t - D(0,1) \right| \leq \frac{1}{32} \int_0^1 |F_2(t)| \mathrm{d}t \cdot L.$$

Since

$$\int_0^1 |F_2(t)| \mathrm{d}t = \frac{1}{24},$$

we get

$$\left| \int_0^1 f(t)\mathrm{d}t - D(0,1) \right| \le \frac{1}{768} \cdot L \text{ and } |R| \le \frac{1}{384} \cdot L.$$

Theorem 4.11 *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$ for some $n \ge 2$.*
If $n = 2r-1, r \ge 2$, then

$$\begin{aligned} &\left| \int_0^1 f(t)\mathrm{d}t - D(0,1) - T_{r-1}(f) \right| \\ &\le \frac{1}{16(2r-1)!} \max_{t\in[0,1]} |G_{2r-1}(t)| \cdot V_0^1(f^{(2r-2)}). \end{aligned} \tag{4.63}$$

If $n = 2r, r \ge 2$, then

$$\begin{aligned} \left| \int_0^1 f(t)\mathrm{d}t - D(0,1) - T_{r-1}(f) \right| &\le \frac{1}{16(2r)!} \max_{t\in[0,1]} |F_{2r}(t)| \cdot V_0^1(f^{(2r-1)}) \\ &= \frac{2^{-2r}(1-2^{-2r})(1-3^{2-2r})}{4(2r)!} |B_{2r}| \cdot V_0^1(f^{(2r-1)}). \end{aligned} \tag{4.64}$$

Also, we have

$$\begin{aligned} \left| \int_0^1 f(t)\mathrm{d}t - D(0,1) - T_r(f) \right| &\le \frac{1}{16(2r)!} \max_{t\in[0,1]} |G_{2r}(t)| \cdot V_0^1(f^{(2r-1)}) \\ &= \frac{2^{-2r}(1-3^{2-2r})}{8(2r)!} |B_{2r}| \cdot V_0^1(f^{(2r-1)}). \end{aligned} \tag{4.65}$$

Here $V_0^1(f^{(n-1)})$ denotes the total variation of $f^{(n-1)}$ on $[0,1]$.

Proof. If $\Phi : [0,1] \to \mathbb{R}$ is bounded on $[0,1]$ and the Riemann-Stieltjes integral $\int_0^1 \Phi(t)\mathrm{d}f^{(n-1)}(t)$ exists, then

$$\left| \int_0^1 \Phi(t)\mathrm{d}f^{(n-1)}(t) \right| \le \max_{t\in[0,1]} |\Phi(t)| \cdot V_0^1(f^{(n-1)}). \tag{4.66}$$

We apply this estimate to $\Phi(t) = G_{2r-1}(t)$ to obtain

$$\left| \frac{1}{16(2r-1)!} \int_0^1 G_{2r-1}(t)\,\mathrm{d}f^{(2r-2)}(t) \right| \le \frac{1}{16(2r-1)!} \max_{t\in[0,1]} |G_{2r-1}(t)| \cdot V_0^1(f^{(2r-2)})$$

which is just the inequality (4.63), because of the identity (4.30). Similarly, we can apply the estimate (4.66) with $\Phi(t) = F_{2r}(t)$ and use the identity (4.30) and Corollary 4.2 to obtain (4.64). Finally, (4.65) follows from (4.66) with $\Phi(t) = G_{2r}(t)$, the identity (4.29) and Corollary 4.2. □

Corollary 4.5 *Let $f:[0,1]\to\mathbb{R}$ be such that f'' is a continuous function of bounded variation on $[0,1]$, then*

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1)\right| \le \frac{1}{6144}V_0^1(f''),\ \ |R| \le \frac{1}{3072}V_0^1(f'').$$

If f''' is a continuous function of bounded variation on $[0,1]$, then

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1)\right| \le \frac{1}{55296}V_0^1(f'''),\ \ |R| \le \frac{1}{27648}V_0^1(f''').$$

Proof. From (4.37), we get

$$\max_{t\in[0,1]}|G_3(t)| = \frac{1}{64}$$

so that the first pair of inequalities follow from (4.63) with $r=2$. The second pair of inequalities follow from (4.64) with $r=2$. □

Remark 4.9 If f is a continuous function of bounded variation on $[0,1]$, then, as above

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1)\right| \le \frac{1}{16}\max_{t\in[0,1]}|G_1(t)|\cdot V_0^1(f).$$

Since

$$\max_{t\in[0,1]}|G_1(t)| = \frac{5}{3},$$

we get

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1)\right| \le \frac{5}{48}\cdot V_0^1(f) \text{ and } |R| \le \frac{5}{24}\cdot V_0^1(f).$$

If f' is a continuous function of bounded variation on $[0,1]$, then

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1)\right| \le \frac{1}{32}\max_{t\in[0,1]}|F_2(t)|\cdot V_0^1(f').$$

Since

$$\max_{t\in[0,1]}|F_2(t)| = \frac{1}{9},$$

we get

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1)\right| \le \frac{1}{288}\cdot V_0^1(f') \text{ and } |R| \le \frac{1}{144}\cdot V_0^1(f').$$

Theorem 4.12 *Assume (p,q) is a pair of conjugate exponents, that is $1<p,q<\infty$, $\frac{1}{p}+\frac{1}{q}=1$ or $p=\infty, q=1$. Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(n)}\in L_p[0,1]$ for some $n\ge 1$. If $n=2r-1, r\ge 1$, then*

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1) - T_{r-1}(f)\right| \le K(2r-1,p)\|f^{(2r-1)}\|_p. \tag{4.67}$$

If $n = 2r, r \geq 1$, *then*

$$\left| \int_0^1 f(t)\mathrm{d}t - D(0,1) - T_{r-1}(f) \right| \leq K^*(2r,p)\|f^{(2r)}\|_p. \tag{4.68}$$

Also, we have

$$\left| \int_0^1 f(t)\mathrm{d}t - D(0,1) - T_r(f) \right| \leq K(2r,p)\|f^{(2r)}\|_p. \tag{4.69}$$

Here

$$K(n,p) = \frac{1}{16n!}\left[\int_0^1 |G_n(t)|^q \,\mathrm{d}t\right]^{\frac{1}{q}},$$

and

$$K^*(n,p) = \frac{1}{16n!}\left[\int_0^1 |F_n(t)|^q \,\mathrm{d}t\right]^{\frac{1}{q}}.$$

Proof. Applying the Hölder inequality we have

$$\begin{aligned} &\left| \frac{1}{16(2r-1)!}\int_0^1 G_{2r-1}(t) f^{(2r-1)}(t)\mathrm{d}t \right| \\ \leq\ &\frac{1}{16(2r-1)!}\left[\int_0^1 |G_{2r-1}(t)|^q \,\mathrm{d}t\right]^{\frac{1}{q}} \cdot \left\| f^{(2r-1)} \right\|_p = K(2r-1,p)\|f^{(2r-1)}\|_p \end{aligned}$$

The above estimate is just (4.67), by the identity (4.33). The inequalities (4.68) and (4.69) are obtained in the same manner from (4.32) and (4.31), respectively. □

Remark 4.10 For $p = \infty$ we have

$$K(n,\infty) = \frac{1}{16n!}\int_0^1 |G_n(t)|\,\mathrm{d}t \ \text{ and } \ K^*(n,\infty) = \frac{1}{16n!}\int_0^1 |F_n(t)|\,\mathrm{d}t.$$

The results established in Theorem 4.12 for $p = \infty$ coincide with the results of Theorem 4.10 with $L = \|f^{(n)}\|_\infty$. Moreover, by Remark 4.8 and Corollary 4.4, we have

$$\left| \int_0^1 f(t)\mathrm{d}t - D(0,1) - T_{r-1}(f) \right| \leq K(2r-1,\infty)\|f^{(2r-1)}\|_\infty, \ \ r = 1,2,$$

where

$$K(1,\infty) = \frac{25}{576},\ K(3,\infty) = \frac{1}{13824}.$$

Also, we have

$$\left| \int_0^1 f(t)\mathrm{d}t - D(0,1) - T_{r-1}(f) \right| \leq K^*(2r,\infty)\|f^{(2r)}\|_\infty, \ \ r = 1,2,$$

where

$$K^*(2,\infty) = \frac{1}{768},\ K^*(4,\infty) = \frac{1}{103680}.$$

Remark 4.11 Let us define for $p = 1$

$$K(n,1) = \frac{1}{16n!} \max_{t\in[0,1]} |G_n(t)| \quad \text{and} \quad K^*(n,1) = \frac{1}{16n!} \max_{t\in[0,1]} |F_n(t)|.$$

Then, using Corollary 4.5, Remark 4.9 and Theorem 4.11, we can extend the results established in Theorem 4.12 to the pair $p = 1$, $q = \infty$. This means that if we set $\frac{1}{q} = 0$, then (4.67) and (4.68) hold for $p = 1$. Also, by Corollary 4.5, we have

$$\left| \int_0^1 f(t)\mathrm{d}t - D(0,1) - T_{r-1}(f) \right| \leq K(2r-1,1) \|f^{(2r-1)}\|_1, \quad r = 1,2,$$

where

$$K(1,1) = \frac{5}{48}, \; K(3,1) = \frac{1}{6144}.$$

Also, we have

$$\left| \int_0^1 f(t)\mathrm{d}t - D(0,1) - T_{r-1}(f) \right| \leq K^*(2r,1) \|f^{(2r)}\|_1, \quad r = 1,2,$$

where

$$K^*(2,1) = \frac{1}{288}, \; K^*(4,1) = \frac{1}{55296}.$$

Remark 4.12 Note that $K^*(1,p) = K(1,p)$, for $1 < p \leq \infty$, since $G_1(t) = F_1(t)$. Also, for $1 < p \leq \infty$ we can easily calculate $K(1,p)$. We get

$$K(1,p) = \frac{1}{16} \left[\frac{3^{q+1} + 4^{q+1} + 5^{q+1}}{4(q+1)3^{q+1}} \right]^{\frac{1}{q}}, \; 1 < p \leq \infty.$$

So, from (4.67) with $r = 1$ we get the following inequality

$$\left| \int_0^1 f(t)\mathrm{d}t - D(0,1) \right| \leq \frac{1}{16} \left[\frac{3^{q+1} + 4^{q+1} + 5^{q+1}}{4(q+1)3^{q+1}} \right]^{\frac{1}{q}} \cdot \|f'\|_p.$$

In the limit case when $p \to 1$, that is when $q \to \infty$, we have

$$\lim_{q\to\infty} \frac{1}{16} \left[\frac{3^{q+1} + 4^{q+1} + 5^{q+1}}{4(q+1)3^{q+1}} \right]^{\frac{1}{q}} = \frac{5}{48} = K(1,1).$$

At the end of this section we prove an interesting Grüss type inequality related to to Euler Bullen-Simpson's 3/8 identity (4.31). To do this we use the following variant of the key technical result from the paper [83]:

Lemma 4.7 *Let $F, G : [0,1] \to \mathbb{R}$ be two integrable functions. If, for some constants $m, M \in \mathbb{R}$*

$$m \leq F(t) \leq M, \quad 0 \leq t \leq 1$$

and

$$\int_0^1 G(t)\mathrm{d}t = 0,$$

then

$$\left|\int_0^1 F(t)G(t)\mathrm{d}t\right| \le \frac{M-m}{2}\int_0^1 |G(t)|\mathrm{d}t. \tag{4.70}$$

Theorem 4.13 *Suppose that $f:[0,1]\to\mathbb{R}$ is such that $f^{(n)}$ exists and is integrable on $[0,1]$, for some $n\ge 1$. Assume that*

$$m_n \le f^{(n)}(t) \le M_n,\ 0\le t\le 1,$$

for some constants m_n and M_n. Then

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1) - T_k(f)\right| \le \frac{1}{32\,(n!)}C_n(M_n-m_n), \tag{4.71}$$

where $k=\left[\frac{n}{2}\right]$ and

$$C_n = \int_0^1 |G_n(t)|\,\mathrm{d}t,\ n\ge 1.$$

Moreover, if $n=2r-1$, $r\ge 2$, then

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1) - T_{r-1}(f)\right|$$
$$\le \frac{2^{-2r}\left(1-2^{-2r}\right)\left(1-3^{2-2r}\right)}{2\left[(2r)!\right]}|B_{2r}|\left(M_{2r-1}-m_{2r-1}\right). \tag{4.72}$$

Proof. We can rewrite the identity (4.31) in the form

$$\int_0^1 f(t)\mathrm{d}t - D(0,1) - T_k(f) = \frac{1}{16(n!)}\int_0^1 F(t)G(t)\mathrm{d}t, \tag{4.73}$$

where

$$F(t)=f^{(n)}(t),\ G(t)=G_n(t),\ 0\le t\le 1.$$

In [27, Lemma 2 (i)] it was proved that for all $n\ge 1$ and for every $\gamma\in\mathbb{R}$

$$\int_0^1 B_n^*(\gamma - t)\mathrm{d}t = 0,$$

so that we have

$$\int_0^1 G(t)\mathrm{d}t = \int_0^1 \left[2B_n^*(1-s) + 3B_n^*\left(\frac{1}{6}-s\right) + 3B_n^*\left(\frac{1}{3}-s\right)\right.$$
$$\left. +2B_n^*\left(\frac{1}{2}-s\right) + 3B_n^*\left(\frac{2}{3}-s\right) + 3B_n^*\left(\frac{5}{6}-s\right)\right]\mathrm{d}s = 0.$$

Thus, we can apply (4.70) to the integral in the right hand side of (4.73) and (4.71) follows immediately. The inequality (4.72) follows from (4.71) and Corollary 4.3. □

Remark 4.13 For $n = 1$ and $n = 2$ we have already evaluated

$$C_1 = \int_0^1 |G_1(t)|\mathrm{d}t = \frac{25}{36},\ C_2 = \int_0^1 |G_2(t)|\mathrm{d}t = \frac{1}{24},$$

so that we have

$$\left| \int_0^1 f(t)\mathrm{d}t - D(0,1) \right| \le \frac{25}{1152}(M_1 - m_1)$$

and

$$\left| \int_0^1 f(t)\mathrm{d}t - D(0,1) \right| \le \frac{1}{1536}(M_2 - m_2).$$

For $n = 3$ we apply (4.72) with $r = 2$ to get the inequality

$$\left| \int_0^1 f(t)\mathrm{d}t - D(0,1) \right| \le \frac{1}{27648}(M_3 - m_3).$$

4.2 Closed corrected 4-point quadrature formulae

In this section, we follow the same idea as in section on corrected 3-point quadrature formulae. Let us observe formula (4.1) again. Instead of the condition $G_2(x,0) = 0$, we impose condition $G_4(x,0) = 0$, thus leaving the values of the first derivative in the quadrature formula and removing the values of the third. This new condition produces the following weight:

$$w_c(x) := w(x) = -\frac{B_4}{2(B_4(x) - B_4)} = \frac{1}{60x^2(1-x)^2}. \tag{4.74}$$

Now, assuming $f^{(2n-1)}$ is continuous of bounded variation on $[0,1]$ for some $n \ge 1$, we have:

$$\int_0^1 f(t)dt - Q_C(0,x,1-x,1) + T_{2n}^{CQ4}(x) = \frac{1}{(2n)!}\int_0^1 G_{2n}^{CQ4}(x,t)df^{(2n-1)}(t), \tag{4.75}$$

assuming $f^{(2n)}$ is continuous of bounded variation on $[0,1]$ for some $n \ge 0$, we have:

$$\int_0^1 f(t)dt - Q_C(0,x,1-x,1) + T_{2n}^{CQ4}(x) = \frac{1}{(2n+1)!}\int_0^1 G_{2n+1}^{CQ4}(x,t)df^{(2n)}(t), \tag{4.76}$$

and finally, assuming $f^{(2n+1)}$ is continuous of bounded variation on $[0,1]$ for some $n \ge 0$, we have:

$$\int_0^1 f(t)dt - Q_C(0,x,1-x,1) + T_{2n}^{CQ4}(x) = \frac{1}{(2n+2)!}\int_0^1 F_{2n+2}^{CQ4}(x,t)df^{(2n+1)}(t), \tag{4.77}$$

where

$$Q_C(0,x,1-x,1) \tag{4.78}$$
$$= \frac{1}{60x^2(1-x)^2}[30B_4(x)f(0)+f(x)+f(1-x)+30B_4(x)f(1)],$$

$$T_{2n}^{CQ4}(x) = \sum_{k=2}^{2n} \frac{1}{k!}\, G_k^{CQ4}(x,0)\; [f^{(k-1)}(1)-f^{(k-1)}(0)] \tag{4.79}$$
$$= \frac{5x^2-5x+1}{60x(x-1)}[f'(1)-f'(0)] + \sum_{k=3}^{n} \frac{G_{2k}^{CQ4}(x,0)}{(2k)!}\; [f^{(2k-1)}(1)-f^{(2k-1)}(0)]$$

$$G_k^{CQ4}(x,t) = \frac{1}{60x^2(1-x)^2}\left[60B_4(x)\cdot B_k^*(1-t)+B_k^*(x-t)+B_k^*(1-x-t)\right], \quad k \geq 1$$

$$F_k^{CQ4}(x,t) = G_k^{CQ4}(x,t) - G_k^{CQ4}(x,0), \;\; k \geq 2.$$

What follows is the key lemma.

Lemma 4.8 *For $x \in (0, \frac{1}{2}-\frac{\sqrt{5}}{10}] \cup [\frac{1}{3}, \frac{1}{2}]$ and $k \geq 2$, $G_{2k+1}^{CQ4}(x,t)$ has no zeros in variable t in the interval $(0,1/2)$. The sign of the function is determined by:*

$$(-1)^k G_{2k+1}^{CQ4}(x,t) > 0 \quad \text{for } x \in (0,1/2-\sqrt{5}/10],$$
$$(-1)^{k+1} G_{2k+1}^{CQ4}(x,t) > 0 \quad \text{for } x \in [1/3,1/2].$$

Proof. We start from $G_5^{CQ4}(x,t)$ and claim that for $x \in (1/2-\sqrt{5}/10, 1/3)$, $G_5^{CQ4}(x,t)$ has at least one zero in variable t in $(0,1/2)$. To prove this, first notice that $G_5^{CQ4}(x,0) = \frac{\partial G_5^{CQ4}}{\partial t}(x,0) = \frac{\partial^2 G_5^{CQ4}}{\partial t^2}(x,0) = G_5^{CQ4}(x,1/2) = 0$ and that $x \in (1/2-\sqrt{5}/10, 1/3)$ is equivalent to $\frac{\partial^3 G_5^{CQ4}}{\partial t^3}(x,0) < 0$ and $\frac{\partial G_5^{CQ4}}{\partial t}(x,\frac{1}{2}) < 0$. From $\frac{\partial^3 G_5^{CQ4}}{\partial t^3}(x,0) < 0$ we conclude $\frac{\partial^3 G_5^{CQ4}}{\partial t^3}(x,t) < 0$ in some neighborhood of $t=0$. Therefore, $\frac{\partial^2 G_5^{CQ4}}{\partial t^2}(x,t)$ is decreasing in some neighborhood of $t=0$ and since $\frac{\partial^2 G_5^{CQ4}}{\partial t^2}(x,0) = 0$, it follows that there we have $\frac{\partial^2 G_5^{CQ4}}{\partial t^2}(x,t) < 0$. Further, $\frac{\partial G_5^{CQ4}}{\partial t}(x,t)$ is then also decreasing and since $\frac{\partial G_5^{CQ4}}{\partial t}(x,0) = 0$, we conclude $\frac{\partial G_5^{CQ4}}{\partial t}(x,t) < 0$ in some neighborhood of $t=0$. Finally, from here we see that $G_5^{CQ4}(x,t)$ is decreasing and since $G_5^{CQ4}(x,0) = 0$ we have $G_5^{CQ4}(x,t) < 0$ in some neighborhood of 0. On the other hand, from $\frac{\partial G_5^{CQ4}}{\partial t}(x,\frac{1}{2}) < 0$ we conclude that $\frac{\partial G_5^{CQ4}}{\partial t}(x,t) < 0$ in some neighborhood of $t = 1/2$. Then $G_5^{CQ4}(x,t)$ is decreasing and since $G_5^{CQ4}(x,1/2) = 0$ we see that $G_5^{CQ4}(x,t) > 0$ in that neighborhood. Now it is clear that when $\frac{\partial^3 G_5^{CQ4}}{\partial t^3}(x,0) < 0$ and $\frac{\partial G_5^{CQ4}}{\partial t}(x,\frac{1}{2}) < 0$, i.e. when $x \in (1/2-\sqrt{5}/10, 1/3)$, $G_5^{CQ4}(x,t)$ has at least one zero on $(0,1/2)$.

It is left to prove that for $x \in (0, \frac{1}{2}-\frac{\sqrt{5}}{10}] \cup [\frac{1}{3}, \frac{1}{2}]$, $G_5^{CQ4}(x,t)$ has constant sign. This will be done by showing that $G_5^{CQ4}(x,t)$ is decreasing in the variable x and after checking its behavior at the end points, our statement will follow. First assume $0 < t \le x \le 1/2$. Then

$$\frac{\partial G_5^{CQ4}}{\partial x}(x,t) = t^3 \cdot \frac{1-2x}{6x^3(1-x)^3}\,[t - 2x(1-x)].$$

Since $t \le x \le 2x(1-x)$, it follows that $\frac{\partial G_5^{CQ4}}{\partial x}(x,t) < 0$ on this interval. When $0 < x \le t < 1/2$, we have

$$\frac{\partial G_5^{CQ4}}{\partial x}(x,t) = \frac{(1-2t)}{6(1-x)^3}\,[x - 2t(1-t)].$$

Similarly as before, now $x \le t \le 2t(1-t)$. Therefore $G_5^{CQ4}(x,t)$ is decreasing in x. To complete our proof, we need to consider the sign of $G_5^{CQ4}\left(\frac{5-\sqrt{5}}{10},t\right)$ and $G_5^{CQ4}\left(\frac{1}{3},t\right)$. Assume $0 < t \le x \le 1/2$. Then $G_5^{CQ4}(x,t) = \frac{-t^3}{12x^2(1-x)^2} \cdot g(x,t)$ where $g(x,t) = 12t^2(1-x)^2 \cdot x^2 + t(-30x^4 + 60x^3 - 30x^2 + 1) + 4x(5x^3 - 10x^2 + 6x - 1)$. Now, it is trivial to see that $G_5^{CQ4}\left(\frac{5-\sqrt{5}}{10},t\right) > 0$ and that $G_5^{CQ4}\left(\frac{1}{3},t\right) < 0$. Similarly, when $0 < x \le t \le 1/2$, we have $G_5^{CQ4}(x,t) = \frac{1-2t}{12(1-x)^2} \cdot h(x,t)$ where $h(x,t) = 6t^4(1-x)^2 - 12t^3(1-x)^3 + t^2(4x^2 - 8x + 6) + 2t \cdot x(x-2) + x^2$ and again $G_5^{CQ4}\left(\frac{5-\sqrt{5}}{10},t\right) > 0$ and $G_5^{CQ4}\left(\frac{1}{3},t\right) < 0$. Therefore, since $G_5^{CQ4}(x,t)$ is decreasing in x, it follows that $G_5^{CQ4}(x,t) > 0$ for $x \in (0, \frac{1}{2} - \frac{\sqrt{5}}{10}]$ and that $G_5^{CQ4}(x,t) < 0$ for $x \in [\frac{1}{3}, \frac{1}{2}]$.

Thus, the assertion is true for $k = 2$. For $k \ge 3$ it follows by induction. As for the sign of functions $G_{2k+1}^{CQ4}(x,t)$, the proof is analogous to the same part of the proof of Lemma 3.1 □

Denote by $R_{2n+2}^{CQ4}(x,f)$ the right-hand side of (4.77).

Theorem 4.14 *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(2n+2)}$ is continuous on $[0,1]$ for some $n \ge 3$ and let $x \in (0, \frac{1}{2} - \frac{\sqrt{5}}{10}] \cup [\frac{1}{3}, \frac{1}{2}]$. If $f^{(2n)}$ and $f^{(2n+2)}$ have the same constant sign on $[0,1]$, then the remainder $R_{2n}^{CQ4}(x,f)$ has the same sign as the first neglected term $\Delta_{2n}^{CQ4}(x,f)$ where*

$$\Delta_{2n}^{CQ4}(x,f) := R_{2n}^{CQ4}(x,f) - R_{2n+2}^{CQ4}(x,f) = -\frac{1}{(2n)!} G_{2n}^{CQ4}(x,0)[f^{(2n-1)}(1) - f^{(2n-1)}(0)].$$

Furthermore, $|R_{2n}^{CQ4}(x,f)| \le |\Delta_{2n}^{CQ4}(x,f)|$ and $|R_{2n+2}^{CQ4}(x,f)| \le |\Delta_{2n}^{CQ4}(x,f)|$.

Proof. Analogous to the proof of Theorem 3.1. □

Theorem 4.15 *If $f : [0,1] \to \mathbb{R}$ is such that $f^{(2n+2)}$ is continuous on $[0,1]$ for some $n \ge 2$ and $x \in (0, \frac{1}{2} - \frac{\sqrt{5}}{10}] \cup [\frac{1}{3}, \frac{1}{2}]$, then there exists $\xi \in [0,1]$ such that*

$$R_{2n+2}^{CQ4}(x,f) = -\frac{G_{2n+2}^{CQ4}(x,0)}{(2n+2)!} \cdot f^{(2n+2)}(\xi) \tag{4.80}$$

where

$$G_{2n+2}^{CQ4}(x,0) = \frac{1}{30x^2(1-x)^2}\left[B_{2n+2}(x) - B_{2n+2}\right] + B_{2n+2}. \tag{4.81}$$

If, in addition, $f^{(2n+2)}$ *does not change sign on [0,1], then there exists* $\theta \in [0,1]$ *such that*

$$R_{2n+2}^{CQ4}(x,f) = \frac{\theta}{(2n+2)!} \cdot F_{2n+2}^{CQ4}\left(x,\frac{1}{2}\right)\left[f^{(2n+1)}(1) - f^{(2n+1)}(0)\right] \tag{4.82}$$

where

$$\begin{aligned} F_{2n+2}^{CQ4}(x,1/2) \;=\; & \frac{1}{30x^2(1-x)^2}\left[B_{2n+2}(1/2-x) - B_{2n+2}(x) + \left(2-2^{-2n-1}\right)B_{2n+2}\right] \\ & - \left(2-2^{-2n-1}\right)B_{2n+2}. \end{aligned} \tag{4.83}$$

Proof. Analogous to the proof of Theorem 3.2. □

When we apply (4.80) to the remainder in formula (4.77) for $n = 2$, we obtain:

$$\begin{aligned} \int_0^1 f(t)dt - Q_C(0,x,1-x,1) + \frac{5x^2-5x+1}{60x(x-1)}[f'(1) - f'(0)] \\ = -\frac{14x^2-14x+3}{302400} \cdot f^{(6)}(\xi). \end{aligned} \tag{4.84}$$

For $x = 1/2$, formula (4.84) produces corrected Simpson's formula which was already studied some sections earlier. For $x = 1/3$ corrected Simpson's 3/8 formula is produced. Imposing condition $w_c(x) = 1/2$, where $w_c(x)$ is as in (4.74), gives a 2-point quadrature formula (since $w_c(x) = 1/2 \Leftrightarrow B_4(x) = 0$), and the corrected Gauss 2-point formula, which was also already studied here. Possible the most interesting special case is for the $x = 1/2 - \sqrt{5}/10$ which produces the Lobatto 4-point formula. The following subsections will be dedicated to these special cases.

Remark 4.14 Similarly as in Remark 4.2, one might wonder if similar results can be obtained for $x = 0$. By considering the limit process we get:

$$\begin{aligned} &\lim_{x\to 0}\left(Q_C(0,x,1-x,1) - \frac{5x^2-5x+1}{60x(x-1)}[f'(1) - f'(0)]\right) \\ &\qquad = \frac{1}{2}[f(0)+f(1)] - \frac{1}{10}[f'(1)-f'(0)] + \frac{1}{120}[f''(0)+f''(1)] \\ &\lim_{x\to 0} G_k^{CQ4}(x,t) = B_k^*(1-t) + \frac{k(k-1)}{60}B_{k-2}^*(1-t) \end{aligned}$$

Consequently, from (4.84) it follows:

$$\int_0^1 f(t)dt - \frac{1}{2}[f(0)+f(1)] + \frac{1}{10}[f'(1)-f'(0)] - \frac{1}{120}[f''(0)+f''(1)] = -\frac{1}{100800}f^{(6)}(\xi). \tag{4.85}$$

Note that quadrature formulae (4.15) and (4.85) were derived in [22], by integrating the two-point Taylor interpolation formula.

Theorem 4.16 *Let $p,q \in \mathbb{R}$ be such that $1 \le p,\ q \le \infty$ and $1/p + 1/q = 1$. If $f : [0,1] \to \mathbb{R}$ is such that $f^{(2n)} \in L_p[0,1]$ for some $n \ge 1$, then we have*

$$\left| \int_0^1 f(t)dt - Q_C(0,x,1-x,1) + T_{2n}^{CQ4}(x) \right| \le K_{CQ4}(2n,q) \cdot \|f^{(2n)}\|_p. \tag{4.86}$$

If $f^{(2n+1)} \in L_p[0,1]$ for some $n \ge 0$, then we have

$$\left| \int_0^1 f(t)dt - Q_C(0,x,1-x,1) + T_{2n}^{CQ4}(x) \right| \le K_{CQ4}(2n+1,q) \cdot \|f^{(2n+1)}\|_p \tag{4.87}$$

and if $f^{(2n+2)} \in L_p[0,1]$ for some $n \ge 0$, then we have

$$\left| \int_0^1 f(t)dt - Q_C(0,x,1-x,1) + T_{2n}^{CQ4}(x) \right| \le K^*_{CQ4}(2n+2,q) \cdot \|f^{(2n+2)}\|_p, \tag{4.88}$$

where

$$K_{CQ4}(m,q) = \frac{1}{m!} \left[\int_0^1 \left| G_m^{CQ4}(x,t) \right|^q dt \right]^{\frac{1}{q}}$$

$$\text{and} \quad K^*_{CQ4}(m,q) = \frac{1}{m!} \left[\int_0^1 \left| F_m^{CQ4}(x,t) \right|^q dt \right]^{\frac{1}{q}}.$$

These inequalities are sharp for $1 < p \le \infty$ and the best possible for $p = 1$.

Proof. Analogous to the proof of Theorem 2.2. □

Similarly as in the previous section, for $x \in (0, \frac{1}{2} - \frac{\sqrt{5}}{10}] \cup [\frac{1}{3}, \frac{1}{2}]$ and $n \ge 2$, we can calculate the following constants as special cases of the previous Theorem:

$$K^*_{CQ4}(2n+2,1) = \frac{1}{(2n+2)!} \left| G_{2n+2}^{CQ4}(x,0) \right|,$$

$$K^*_{CQ4}(2n+2,\infty) = \frac{1}{2} K_{CQ4}(2n+1,1) = \frac{1}{(2n+2)!} \left| F_{2n+2}^{CQ4}\left(x, \frac{1}{2}\right) \right|,$$

where $G_{2n+2}^{CQ4}(x,0)$ and $F_{2n+2}^{CQ4}(x,1/2)$ are as in (4.81) and (4.83), respectively.

We now seek for the optimal corrected closed 4-point quadrature formula for $x \in (0, \frac{1}{2} - \frac{\sqrt{5}}{10}] \cup [\frac{1}{3}, \frac{1}{2}]$, $n = 2$ and $p = \infty$. Theorem 4.16 gives:

$$\left| \int_0^1 f(t)dt - Q_C(0,x,1-x,1) + \frac{5x^2-5x+1}{60x(x-1)} [f'(1) - f'(0)] \right|$$
$$\le \frac{|32x^3 - 55x^2 + 30x - 5|}{115200(1-x)^2} \cdot \|f^{(5)}\|_\infty$$

$$\left| \int_0^1 f(t)dt - Q_C(0,x,1-x,1) + \frac{5x^2-5x+1}{60x(x-1)} [f'(1) - f'(0)] \right|$$
$$\le \frac{|14x^2 - 14x + 3|}{302400} \cdot \|f^{(6)}\|_\infty$$

It is not hard to see that functions on the right-hand sides of both of these inequalities are decreasing on $(0, \frac{1}{2} - \frac{\sqrt{5}}{10}]$ and increasing on $[\frac{1}{3}, \frac{1}{2}]$ and they reach their minimum at $x = 1/3$. The same goes for the case when $n = 2$ and $p = 1$.

Thus, once again, we conclude that the node which gives the best estimation of error in these three cases is $x = 1/3$, i.e. the optimal corrected closed 4-point quadrature formula is corrected Simpson's 3/8 formula.

The following two theorems give Hermite-Hadamard and Dragomir-Agarwal type inequalities for the general corrected 4-point quadrature formulae:

Theorem 4.17 *Let $f : [0,1] \to \mathbb{R}$ be $(2n+4)$-convex for $n \geq 2$. Then for $x \in \left(0, \frac{1}{2} - \frac{\sqrt{5}}{10}\right]$, we have*

$$\begin{aligned}
&\frac{1}{(2n+2)!} \left|G_{2n+2}^{CQ4}(x,0)\right| f^{(2n+2)}\left(\frac{1}{2}\right) \\
&\leq (-1)^{n+1} \left(\int_0^1 f(t)dt - Q_C(0,x,1-x,1) + T_{2n}^{CQ4}(x) \right) \qquad (4.89) \\
&\leq \frac{1}{(2n+2)!} \left|G_{2n+2}^{CQ4}(x,0)\right| \frac{f^{(2n+2)}(0) + f^{(2n+2)}(1)}{2},
\end{aligned}$$

while for $x \in \left[\frac{1}{3}, \frac{1}{2}\right]$ we have

$$\begin{aligned}
&\frac{1}{(2n+2)!} \left|G_{2n+2}^{CQ4}(x,0)\right| f^{(2n+2)}\left(\frac{1}{2}\right) \\
&\leq (-1)^{n} \left(\int_0^1 f(t)dt - Q_C(0,x,1-x,1) + T_{2n}^{CQ4}(x) \right) \qquad (4.90) \\
&\leq \frac{1}{(2n+2)!} \left|G_{2n+2}^{CQ4}(x,0)\right| \frac{f^{(2n+2)}(0) + f^{(2n+2)}(1)}{2},
\end{aligned}$$

where $G_{2n+2}^{CQ4}(x,0)$ is as in (4.81).

If f is $(2n+4)$-concave, the inequalities are reversed.

Proof. Analogous to the proof of Theorem 2.8. □

Theorem 4.18 *Let $x \in \left(0, \frac{1}{2} - \frac{\sqrt{5}}{10}\right] \cup \left[\frac{1}{3}, \frac{1}{2}\right]$ and $f : [0,1] \to \mathbb{R}$ be m-times differentiable for $m \geq 5$. If $|f^{(m)}|^q$ is convex for some $q \geq 1$, then*

$$\begin{aligned}
&\left| \int_0^1 f(t)dt - Q_C(0,x,1-x,1) + T_{2n}^{CQ4}(x) \right| \\
&\leq L_{CQ4}(m,x) \cdot \left(\frac{|f^{(m)}(0)|^q + |f^{(m)}(1)|^q}{2} \right)^{1/q} \qquad (4.91)
\end{aligned}$$

while if $|f^{(m)}|$ is concave, then

$$\left| \int_0^1 f(t)dt - Q_C(0,x,1-x,1) + T_{2n}^{CQ4}(x) \right| \leq L_{CQ4}(m,x) \cdot \left| f^{(m)}\left(\frac{1}{2}\right) \right|, \qquad (4.92)$$

where

$$\text{for } m=2n+1 \quad L_{CQ4}(2n+1,x)=\frac{2}{(2n+2)!}\,|F_{2n+2}^{CQ4}(x,1/2)|$$
$$\text{and for } m=2n+2 \quad L_{CQ4}(2n+2,x)=\frac{1}{(2n+2)!}\,|G_{2n+2}^{CQ4}(x,0)|$$

with $G_{2n+2}^{CQ4}(x,0)$ *and* $F_{2n+2}^{CQ4}(x,1/2)$ *as in (4.81) and (4.83), respectively.*

Proof. Analogous to the proof of Theorem 3.5. □

4.2.1 Lobatto 4-point formula

If one wants to obtain from (4.77) the quadrature formula with the maximum degree of exactness, and for that formula not to be "corrected" at the same time, one has to impose the condition:

$$G_2^{CQ4}(x,0)=G_4^{CQ4}(x,0)=0.$$

The unique solution to this system are exactly the nodes ($x_0=1/2-\sqrt{5}/10$) and the weights of the Lobatto 4-point formula (on the interval $[0,1]$), which was to be expected. The same is obtained from (4.3) when considering the system: $G_2^{Q4}(x,0)=G_4^{Q4}(x,0)=0$, where the functions G_k^{Q4} are as in (4.6).

Let us now see the results and the estimates for the Lobatto 4-point formula. We move to $[-1,1]$. Formulae (4.75)-(4.77) now become:

$$\int_{-1}^{1} f(t)dt - Q_{L4} + T_{2n}^{L4} = \frac{2^{2n-1}}{(2n)!}\int_{-1}^{1} G_{2n}^{L4}(t)df^{(2n-1)}(t), \tag{4.93}$$
$$\int_{-1}^{1} f(t)dt - Q_{L4} + T_{2n}^{L4} = \frac{2^{2n}}{(2n+1)!}\int_{-1}^{1} G_{2n+1}^{L4}(t)df^{(2n)}(t), \tag{4.94}$$
$$\int_{-1}^{1} f(t)dt - Q_{L4} + T_{2n}^{L4} = \frac{2^{2n+1}}{(2n+2)!}\int_{-1}^{1} F_{2n+2}^{L4}(t)df^{(2n+1)}(t), \tag{4.95}$$

where

$$Q_{L4}=\frac{1}{6}\left[f(-1)+5f\left(-\frac{\sqrt{5}}{5}\right)+5f\left(\frac{\sqrt{5}}{5}\right)+f(1)\right],$$
$$T_{2n}^{L4}=\sum_{k=3}^{n}\frac{2^{2k-1}}{(2k)!}\,G_{2k}^{L4}(-1)\,[f^{(2k-1)}(1)-f^{(2k-1)}(-1)],$$
$$G_k^{L4}(t)=\frac{1}{3}B_k^*\left(\frac{1}{2}-\frac{t}{2}\right)+\frac{5}{6}\left[B_k^*\left(\frac{\sqrt{5}}{10}-\frac{t}{2}\right)+B_k^*\left(-\frac{\sqrt{5}}{10}-\frac{t}{2}\right)\right],\ k\ge 1$$
$$F_k^{L4}(t)=G_k^{L4}(t)-G_k^{L4}(-1),\ k\ge 2.$$

Theorem 4.15 becomes:

Corollary 4.6 *If $f:[-1,1]\to\mathbb{R}$ is such that $f^{(2n+2)}$ is continuous on $[-1,1]$ for some $n\geq 2$, then there exists $\xi\in[-1,1]$ such that*

$$R_{2n+2}^{L4}(f)=-\frac{2^{2n+2}}{(2n+2)!}\,G_{2n+2}^{L4}(-1)\cdot f^{(2n+2)}(\xi) \tag{4.96}$$

where

$$G_{2n+2}^{L4}(-1)=\frac{1}{3}\left[B_{2n+2}+5B_{2n+2}\left(\frac{1}{2}-\frac{\sqrt{5}}{10}\right)\right]. \tag{4.97}$$

Applying (4.96) for $n=2$ to the remainder in (4.95) produces Lobatto 4-point formula:

$$\int_{-1}^{1} f(t)dt-Q_{L4}=-\frac{2}{23625}f^{(6)}(\xi) \tag{4.98}$$

Using Hölder's inequality one can easily obtain the analogue of Theorem 4.16 for this quadrature formula. As a direct consequence, for $p=1$ and $p=\infty$ the following estimations are obtained:

$$\left|\int_{-1}^{1} f(t)dt-Q_{L4}\right|\leq C_{L4}(m,q)\cdot\|f^{(m)}\|_p,\ \ m=1,\dots,6$$

where

$$\begin{aligned}
&C_{L4}(1,1)\approx 0.376866, \qquad C_{L4}(1,\infty)=\left|G_1^{L4}\left(1/\sqrt{5}\right)\right|=1/\sqrt{5}\approx 0.447214,\\
&C_{L4}(2,1)\approx 0.0417772, \qquad C_{L4}(2,\infty)=G_2^{L4}\left(1/\sqrt{5}\right)\approx 0.0606553,\\
&C_{L4}(3,1)\approx 0.0064048, \qquad C_{L4}(3,\infty)\approx 0.00735788\\
&C_{L4}(4,1)\approx 0.00113265, \qquad C_{L4}(4,\infty)=G_4^{L4}(0)/3\approx 0.00146629,\\
&C_{L4}(5,1)=4|F_6^{L4}(0)|/45\approx 0.000248452, \qquad C_{L4}(5,\infty)\approx 0.000283162,\\
&C_{L4}(6,1)=4|G_6^{L4}(-1)|/45\approx 0.0000846561,\\
&C_{L4}(6,\infty)=2|F_6^{L4}(0)|/45=\sqrt{5}/18000\approx 0.000124226.
\end{aligned}$$

The Hermite-Hadamard type inequality for the Lobatto 4-point formula is:

$$\begin{aligned}
&\frac{1}{1512000}\,f^{(6)}\left(\frac{1}{2}\right)\\
&\leq-\left(\int_0^1 f(t)dt-\frac{1}{12}\left[f(0)+5f\left(\frac{5-\sqrt{5}}{10}\right)+5f\left(\frac{5+\sqrt{5}}{10}\right)+f(1)\right]\right)\\
&\leq\frac{1}{1512000}\,\frac{f^{(6)}(0)+f^{(6)}(1)}{2}.
\end{aligned}$$

The constants from Theorem 4.18 are:

$$L_{CQ4}\left(5,\,\frac{5-\sqrt{5}}{10}\right)=\frac{\sqrt{5}}{576000},\qquad L_{CQ4}\left(6,\,\frac{5-\sqrt{5}}{10}\right)=\frac{1}{1512000}.$$

4.2.2 Corrected Simpson's 3/8 formula

For $x = 1/3$, (4.84) becomes corrected Simpson's 3/8 formula:

$$\int_0^1 f(t)dt = \frac{1}{80}\left[13f(0)+27f\left(\frac{1}{3}\right)+27f\left(\frac{2}{3}\right)+13f(1)\right] - \frac{1}{120}[f'(1)-f'(0)] + \frac{1}{2721600}\cdot f^{(6)}(\eta). \tag{4.99}$$

The results from this subsection are published in [51]. We have:

$$\begin{aligned}
Q_{CS38} &= \frac{1}{80}\left[13f(0)+27f\left(\frac{1}{3}\right)+27f\left(\frac{2}{3}\right)+13f(1)\right],\\
T_{2n}^{CS38} &= T_{2n}^{CQ4}\left(\frac{1}{3}\right) = \sum_{k=2}^{2n}\frac{1}{k!}\,G_k^{CS38}(0)\,[f^{(k-1)}(1)-f^{(k-1)}(0)]\\
&= \frac{5x^2-5x+1}{60x(x-1)}[f'(1)-f'(0)] + \sum_{k=3}^{n}\frac{G_{2k}^{CS38}(0)}{(2k)!}\,[f^{(2k-1)}(1)-f^{(2k-1)}(0)]\\
G_k^{CS38}(t) &= \frac{1}{80}\left[27B_k^*\left(\frac{1}{3}-t\right)+27B_k^*\left(\frac{2}{3}-t\right)+26B_k^*(1-t)\right],\ k\geq 1 \qquad (4.100)\\
F_k^{CS38}(t) &= G_k^{CS38}(t)-G_k^{CS38}(0),\ k\geq 2
\end{aligned}$$

The error $R_{2n+2}^{CS38}(f)$ for $n \geq 2$ can be expressed as:

$$R_{2n+2}^{CS38}(f) = \frac{1}{80(2n+2)!}(1-3^{2-2n})B_{2n+2}\cdot f^{(2n+2)}(\eta), \quad \eta\in[0,1]$$

$$R_{2n+2}^{CS38}(f) = \theta\,\frac{(2-2^{-1-2n})\left(1-3^{2-2n}\right)B_{2n+2}}{80(2n+2)!}\left[f^{(2n+1)}(1)-f^{(2n+1)}(0)\right], \quad \theta\in[0,1]$$

Estimates of error for $p=1$, $p=\infty$ and $m=1,2$ are:

$$\left|\int_0^1 f(t)dt - Q_{CS38}\right| \leq C_{CS38}(m,q)\cdot\|f^{(m)}\|_p,$$

where

$$\begin{aligned}
C_{CS38}(1,1) &= \frac{2401}{28800}, \quad C_{CS38}(1,\infty) = \left|G_1^{CS38}\left(\frac{1}{3}\right)\right| = \frac{41}{240},\\
C_{CS38}(2,1) &= \frac{597+320\sqrt{10}}{192000}, \quad C_{CS38}(2,\infty) = \frac{1}{2}\left|F_2^{CS38}\left(\frac{13}{80}\right)\right| = \frac{169}{12800}.
\end{aligned}$$

Comparing these estimates with the ones the classical Simpson's 3/8 formula provides, shows that the corrected formula gives better estimates for $m=1$.

Furthermore, for $p=1,\infty$ and $m=2,3,4,5,6$ we get:

$$\left|\int_0^1 f(t)dt - Q_{CS38} + \frac{1}{120}[f'(1)-f'(0)]\right| \leq C_{CS38}(m,q)\cdot\|f^{(m)}\|_p,$$

where

$$C_{CS38}(2,1) = \frac{320\sqrt{30}+187\sqrt{561}}{1728000}, \quad C_{CS38}(2,\infty) = \frac{1}{2}\left|G_2^{CS38}\left(\frac{1}{3}\right)\right| = \frac{7}{720},$$
$$C_{CS38}(3,1) = \frac{48693+3133\sqrt{241}}{491520000}, \quad C_{CS38}(3,\infty) = \frac{1053+187\sqrt{561}}{13824000},$$
$$C_{CS38}(4,1) = \frac{1}{73728}, \quad C_{CS38}(4,\infty) = \frac{1}{38400}$$
$$C_{CS38}(5,1) = \frac{1}{691200}, \quad C_{CS38}(5,\infty) = \frac{1}{5!}\left|G_5^{CS38}\left(\frac{1}{4}\right)\right| = \frac{1}{294912}$$
$$C_{CS38}(6,1) = \frac{1}{2721600}, \quad C_{CS38}(6,\infty) = \frac{1}{6!}\left|F_6^{CS38}\left(\frac{1}{2}\right)\right| = \frac{1}{1382400}.$$

The Hermite-Hadamard type inequality for the corrected Simpson's 3/8 formula is:

$$\begin{aligned}
&\frac{1}{2721600} f^{(6)}\left(\frac{1}{2}\right) \\
&\leq \int_0^1 f(t)dt - \frac{1}{80}\left[13f(0)+27f\left(\frac{1}{3}\right)+27f\left(\frac{2}{3}\right)+13f(1)\right] + \frac{1}{120}[f'(1)-f'(0)] \\
&\leq \frac{1}{2721600}\,\frac{f^{(6)}(0)+f^{(6)}(1)}{2}
\end{aligned}$$

and the constants from Theorem 4.18 are:

$$L_{CQ4}\left(5,\,\frac{1}{3}\right) = \frac{1}{691200}, \qquad L_{CQ4}\left(6,\,\frac{1}{3}\right) = \frac{1}{2721600}.$$

4.2.3 Hermite-Hadamard-type inequality for the corrected 4-point quadrature formulae

The main result of this section provides Hermite-Hadamard-type inequality for the corrected 4-point quadrature formulae.

Theorem 4.19 *Let $f : [0,1] \to \mathbb{R}$ be 6-convex and such that $f^{(6)}$ is continuous on $[0,1]$. Then, for $x \in (0, \frac{1}{2} - \frac{\sqrt{5}}{10}]$ and $y \in \left[\frac{1}{3}, \frac{1}{2}\right]$*

$$\begin{aligned}
&Q_C(0,y,1-y,1) - \frac{5y^2-5y+1}{60y(y-1)}[f'(1)-f'(0)] \\
&\leq \int_0^1 f(t)dt \qquad (4.101)\\
&\leq Q_C(0,x,1-x,1) - \frac{5x^2-5x+1}{60x(x-1)}[f'(1)-f'(0)],
\end{aligned}$$

where $Q_C(0,x,1-x,1)$ is defined in (4.78). If f is 6-concave, the inequalities are reversed.

Proof. Analogous to the proof of Theorem 3.6. □

The following corollaries give comparison between corrected Simpson's 3/8 and the corrected Gauss 2-point rule, corrected Simpson's 3/8 and the Lobatto 4-point, and finally corrected Simpson's and the Lobatto 4-point.

Corollary 4.7 *Let $f:[0,1]\to\mathbb{R}$ be 6-convex and such that $f^{(6)}$ is continuous on $[0,1]$. Then*

$$\begin{aligned}
&\frac{1}{80}\left(13f(0)+27f\left(\frac{1}{3}\right)+27f\left(\frac{2}{3}\right)+13f(1)\right)-\frac{1}{120}[f'(1)-f'(0)]\\
&\le \int_0^1 f(t)dt\\
&\le \frac{1}{2}f\left(\frac{1}{2}-\frac{1}{30}\sqrt{225-30\sqrt{30}}\right)+\frac{1}{2}f\left(\frac{1}{2}+\frac{1}{30}\sqrt{225-30\sqrt{30}}\right)\\
&+\frac{7\sqrt{30}-5}{420}[f'(1)-f'(0)].
\end{aligned}$$

If f is 6-concave, the inequalities are reversed.

Proof. Follows from (4.101) for $x=1/2-\sqrt{225-30\sqrt{30}}\Big/30 \Leftrightarrow B_4(y)=0$ and $y=1/3$. □

Corollary 4.8 *Let $f:[0,1]\to\mathbb{R}$ be 6-convex and such that $f^{(6)}$ is continuous on $[0,1]$. Then*

$$\begin{aligned}
&\frac{1}{80}\left(13f(0)+27f\left(\frac{1}{3}\right)+27f\left(\frac{2}{3}\right)+13f(1)\right)-\frac{1}{120}[f'(1)-f'(0)]\\
&\le \int_0^1 f(t)dt\\
&\le \frac{1}{12}\left(f(0)+5f\left(\frac{5-\sqrt{5}}{10}\right)+5f\left(\frac{5+\sqrt{5}}{10}\right)+f(1)\right).
\end{aligned}$$

If f is 6-concave, the inequalities are reversed.

Proof. Follows from (4.101) for $x=1/2-\sqrt{5}/10 \Leftrightarrow 5x^2-5x+1=0$ and $y=1/3$. □

Corollary 4.9 *Let $f:[0,1]\to\mathbb{R}$ be 6-convex and such that $f^{(6)}$ is continuous on $[0,1]$. Then*

$$\begin{aligned}
&\frac{1}{30}\left(7f(0)+16f\left(\frac{1}{2}\right)+7f(1)\right)-\frac{1}{60}[f'(1)-f'(0)]\\
&\le \int_0^1 f(t)dt\\
&\le \frac{1}{12}\left(f(0)+5f\left(\frac{5-\sqrt{5}}{10}\right)+5f\left(\frac{5+\sqrt{5}}{10}\right)+f(1)\right).
\end{aligned}$$

If f is 6-concave, the inequalities are reversed.

Proof. Follows from (4.101) for $x = 1/2 - \sqrt{5}/10 \Leftrightarrow 5x^2 - 5x + 1 = 0$ and $y = 1/2$. □

Remark 4.15 The result of Corollary 3.15 can be recaptured from (4.101) for $x = 1/2 - \sqrt{225 - 30\sqrt{30}}\Big/30$ and $y = 1/2$.

4.2.4 On corrected Bullen-Simpson's 3/8 inequality

In [11], an elementary analytic proof of the following inequality was given: provided f is 4-convex, we have

$$\begin{aligned} 0 &\le \int_0^1 f(t)dt - \frac{1}{8}\left[3f\left(\frac{1}{6}\right) + 2f\left(\frac{1}{2}\right) + 3f\left(\frac{5}{6}\right)\right] \\ &\le \frac{1}{8}\left[f(0) + 3f\left(\frac{1}{3}\right) + 3f\left(\frac{2}{3}\right) + f(1)\right] - \int_0^1 f(t)dt \end{aligned} \tag{4.102}$$

This implies that, for a 4-convex function, Maclaurin's quadrature rule is more accurate than Simpson's 3/8 quadrature rule. Inequality (4.102) is sometimes called Bullen-Simpson's 3/8 inequality and was generalized for a class of $(2k)$-convex functions in [87].

The aim is to derive an inequality of similar type, only this time starting from corrected Simpson's 3/8 and corrected Maclaurin's formula. The results of this subsection were published in [54].

For $k \ge 1$ and $t \in \mathbb{R}$, we define functions

$$G_k(t) = G_k^{CQ4}(1/3,t) + G_k^{CQ3}(1/6,t) = G_k^{CS38}(t) + G_k^{CM}(t),$$
$$F_k(t) = F_k^{CQ4}(1/3,t) + F_k^{CQ3}(1/6,t) = F_k^{CS38}(t) + F_k^{CM}(t),$$

where $G_k^{CS38}(t)$ and $G_k^{CM}(t)$ where defined as in (4.100) and (3.118), respectively. So,

$$\begin{aligned} G_k(t) &= 27B_k^*\left(\frac{1}{6} - t\right) + 27B_k^*\left(\frac{1}{3} - t\right) + 26B_k^*\left(\frac{1}{2} - t\right) \\ &+ 27B_k^*\left(\frac{2}{3} - t\right) + 27B_k^*\left(\frac{5}{6} - t\right) + 26B_k^*(1-t), \qquad k \ge 1, \\ F_1(t) &= G_1(t), \qquad F_k(t) = G_k(t) - G_k(0), \qquad k \ge 2. \end{aligned}$$

Introduce notation $\tilde{B}_k = G_k(0)$. By direct calculation we get

$$\tilde{B}_2 = 2/3 \qquad \text{and} \qquad \tilde{B}_3 = \tilde{B}_4 = \tilde{B}_5 = 0$$

Using the properties of Bernoulli polynomials, it is easy to check that $\tilde{B}_{2k-1} = 0,\ k \ge 2$.

Now, let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n-1)}$ exists on $[0,1]$ for some $n \ge 1$. Introduce the following notation

$$\begin{aligned} &D(0,1) \\ &= \frac{1}{160}\left[13f(0) + 27f\left(\frac{1}{6}\right) + 27f\left(\frac{1}{3}\right) + 26f\left(\frac{1}{2}\right) + 27f\left(\frac{2}{3}\right) + 27f\left(\frac{5}{6}\right) + 13f(1)\right]. \end{aligned}$$

Define $T_0(f) = 0$ and for $1 \leq m \leq n$

$$T_m(f) = \frac{1}{2}\left[T_m^{CQ4}(1/3, f) + T_m^{CQ3}(1/6, f)\right],$$

where $T_m^{CQ4}(x, f)$ and $T_m^{CQ3}(x, f)$ are given by (4.79) and (3.97), respectively. So, we have $T_1(f) = 0$,

$$T_2(f) = T_3(f) = T_4(f) = T_5(f) = \frac{1}{480}[f'(1) - f'(0)]$$

and for $m \geq 6$,

$$T_m(f) = \frac{1}{480}[f'(1) - f'(0)] + \frac{1}{80}\sum_{k=3}^{[m/2]} \frac{B_{2k}}{(2k)!} 2^{-2k}(3^{4-2k} - 1)\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right].$$

In the next theorem we establish two formulae which we call corrected Bullen-Simpson's 3/8 formulae of Euler type.

Theorem 4.20 *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$, for some $n \geq 1$. Then*

$$\int_0^1 f(t)dt = D(0,1) - T_n(f) + \tilde{R}_n^{(1)}(f), \tag{4.103}$$

and

$$\int_0^1 f(t)dt = D(0,1) - T_{n-1}(f) + \tilde{R}_n^{(2)}(f), \tag{4.104}$$

where

$$\tilde{R}_n^{(1)}(f) = \frac{1}{160n!}\int_0^1 G_n(t)\, df^{(n-1)}(t),$$

and

$$\tilde{R}_n^{(2)}(f) = \frac{1}{160n!}\int_0^1 F_n(t)\, df^{(n-1)}(t).$$

Proof. Apply (3.94) for $x = 1/6$ and (4.76) for $x = 1/3$, add them and divide by 2. Identity (4.103) is produced. Identity (4.104) is obtained similarly from (3.95) and (4.77). □

Remark 4.16 Interval $[0,1]$ is used for simplicity and involves no loss in generality. In what follows, Theorem 4.20 and others will be applied, without comment, to any interval that is convenient.

It is easy to see that if $f : [a,b] \to \mathbb{R}$ is such that $f^{(n-1)}$ is continuous of bounded variation on $[a,b]$, for some $n \geq 1$, then

$$\int_a^b f(t)dt = D(a,b) - \tilde{T}_n(f) + \frac{(b-a)^n}{160n!}\int_a^b G_n\left(\frac{t-a}{b-a}\right) df^{(n-1)}(t)$$

and

$$\int_a^b f(t)dt = D(a,b) - \tilde{T}_{n-1}(f) + \frac{(b-a)^n}{160n!}\int_a^b F_n\left(\frac{t-a}{b-a}\right) df^{(n-1)}(t),$$

where

$$D(a,b) = \frac{b-a}{160}\left[13f(a)+27f\left(\frac{5a+b}{6}\right)+27f\left(\frac{2a+b}{3}\right)+26f\left(\frac{a+b}{2}\right)\right.$$
$$\left.+27f\left(\frac{a+2b}{3}\right)+27f\left(\frac{a+5b}{6}\right)+13f(b)\right],$$

$$\tilde{T}_0(f) = \tilde{T}_1(f) = 0,$$
$$\tilde{T}_2(f) = \tilde{T}_3(f) = \tilde{T}_4(f) = \tilde{T}_5(f) = \frac{(b-a)^2}{480}\left[f'(b)-f'(a)\right]$$

and for $m \geq 6$

$$\begin{aligned}\tilde{T}_m(f) &= \frac{(b-a)^2}{480}\left[f'(b)-f'(a)\right]\\ &+ \frac{1}{80}\sum_{k=3}^{[m/2]}\frac{(b-a)^{2k}}{(2k)!}\cdot 2^{-2k}(3^{4-2k}-1)B_{2k}\left[f^{(2k-1)}(b)-f^{(2k-1)}(a)\right].\end{aligned}$$

Remark 4.17 Suppose that $f:[0,1]\to\mathbb{R}$ is such that $f^{(n)}$ exists and is integrable on $[0,1]$, for some $n \geq 1$. In this case (4.103) holds with

$$\tilde{R}_n^{(1)}(f) = \frac{1}{160n!}\int_0^1 G_n(t)f^{(n)}(t)dt,$$

while (4.104) holds with

$$\tilde{R}_n^{(2)}(f) = \frac{1}{160n!}\int_0^1 F_n(t)f^{(n)}(t)dt.$$

Remark 4.18 For $n=6$, (4.104) yields

$$\int_0^1 f(t)dt - D(0,1) + \frac{1}{480}[f'(1)-f'(0)] = \frac{1}{115200}\int_0^1 F_6(t)\,df^{(5)}(t).$$

From this identity it is clear that corrected Bullen-Simpson's 3/8 formula of Euler type is exact for all polynomials of order ≤ 5.

Before we state our main result, we will need to prove some properties of functions G_k and F_k. Notice that it is enough to know the values of those functions on the interval $\left[0,\frac{1}{2}\right]$, since $G_k\left(t+\frac{1}{2}\right) = G_k(t)$.

Lemma 4.9 *For $k \geq 3$, function $G_{2k-1}(t)$ has no zeros in the interval $(0,1/4)$. The sign of this function is determined by*

$$(-1)^k G_{2k-1}(t) > 0, \qquad 0 < t < 1/4. \tag{4.105}$$

Proof. For $k=3$ we have

$$G_5(t)=\begin{cases}-160t^5+65t^4-20/3\cdot t^3, & 0\le t\le 1/6\\ -160t^5+200t^4-290/3\cdot t^3+45/2\cdot t^2-5/2\cdot t+5/48, & 1/6\le t\le 1/3\\ -160t^5+335t^4-830/3\cdot t^3+225/2\cdot t^2-45/2\cdot t+85/48, & 1/3\le t\le 1/2\end{cases}$$

and it is elementary to see that

$$G_5(t)<0, \qquad 0<t<1/4, \tag{4.106}$$

so our first assertion is true for $k=3$. Assuming the opposite, by induction, it follows easily that the assertion is true for all $k\ge 4$.

Further, if $G_{2k-3}(t)>0$, $0<t<1/4$, then since

$$G''_{2k-1}(t)=(2k-1)(2k-2)G_{2k-3}(t)$$

it follows that G_{2k-1} is convex and hence $G_{2k-1}(t)<0$ on $(0,1/4)$. Similarly, we conclude that if $G_{2k-3}(t)<0$, then $G_{2k-1}(t)>0$ on $(0,1/4)$. (4.105) now follows from (4.106). □

Corollary 4.10 *For $k\ge 3$, functions $(-1)^{k-1}F_{2k}(t)$ and $(-1)^{k-1}G_{2k}(t)$ are strictly increasing on the interval $(0,1/4)$ and strictly decreasing on the interval $(1/4,1/2)$. Consequently, 0 and $1/2$ are the only zeros of $F_{2k}(t)$ on $[0,1/2]$ and*

$$\max_{t\in[0,1]}|F_{2k}(t)|=2^{2-2k}(1-2^{-2k})(1-3^{4-2k})|B_{2k}|,$$
$$\max_{t\in[0,1]}|G_{2k}(t)|=2^{1-2k}(1-3^{4-2k})|B_{2k}|.$$

Proof. Since

$$[(-1)^{k-1}F_{2k}(t)]'=[(-1)^{k-1}G_{2k}(t)]'=(-1)^k\cdot 2k\cdot G_{2k-1}(t),$$

from Lemma 4.9 we conclude that $(-1)^{k-1}F_{2k}(t)$ and $(-1)^{k-1}G_{2k}(t)$ are strictly increasing on $(0,1/4)$. It is easy to check that for $k\ge 2$ and $0\le t\le 1/2$,

$$G_k(1/2-t)=(-1)^kG_k(t) \qquad \text{and} \qquad F_k(1/2-t)=(-1)^kF_k(t).$$

From there we conclude that $(-1)^{k-1}F_{2k}(t)$ and $(-1)^{k-1}G_{2k}(t)$ are strictly decreasing on $(1/4,1/2)$. Further, $F_{2k}(0)=F_{2k}(1/2)=0$, which implies $|F_{2k}(t)|$ achieves maximum at $t=1/4$ and thus, the first assertion is proved.

On the other hand,

$$\max_{t\in[0,1]}|G_{2k}(t)|=\max\left\{|G_{2k}(0)|,\left|G_{2k}\left(\frac{1}{4}\right)\right|\right\}=|G_{2k}(0)|.$$

The proof is now complete. □

Corollary 4.11 *For $k \geq 3$, we have*

$$\int_0^1 |F_{2k-1}(t)|\,dt = \int_0^1 |G_{2k-1}(t)|\,dt = \frac{2^{3-2k}}{k}\left(1-2^{-2k}\right)\left(1-3^{4-2k}\right)|B_{2k}|,$$
$$\int_0^1 |F_{2k}(t)|\,dt = |\tilde{B}_{2k}| = 2^{1-2k}\left(1-3^{4-2k}\right)|B_{2k}|$$
$$\int_0^1 |G_{2k}(t)|\,dt \leq 2|\tilde{B}_{2k}| = 2^{2-2k}\left(1-3^{4-2k}\right)|B_{2k}|.$$

Proof. Using the properties of functions G_k, i.e. properties of Bernoulli polynomials, we get

$$\int_0^1 |G_{2k-1}(t)|\,dt \;=\; 4\left|\int_0^{1/4} G_{2k-1}(t)dt\right| = \frac{2}{k}\left|F_{2k}\left(\frac{1}{4}\right)\right|,$$

which proves the first assertion. Since $F_{2k}(0) = F_{2k}(1/2) = 0$, from Corollary 4.10 we conclude that $F_{2k}(t)$ does not change sign on $(0,1/2)$. Therefore,

$$\int_0^1 |F_{2k}(t)|\,dt = 2\left|\int_0^{1/2} G_{2k}(t)dt - \frac{1}{2}\tilde{B}_{2k}\right| = |\tilde{B}_{2k}|,$$

which proves the second assertion. Finally, we use the triangle inequality to obtain

$$\int_0^1 |G_{2k}(t)|\,dt \leq \int_0^1 |F_{2k}(t)|\,dt + |\tilde{B}_{2k}| = 2|\tilde{B}_{2k}|,$$

which proves the third assertion. □

Theorem 4.21 *If $f:[0,1]\to\mathbb{R}$ is such that $f^{(2k)}$ is a continuous function on $[0,1]$, for some $k \geq 3$, then there exists $\eta \in [0,1]$ such that*

$$\tilde{R}_{2k}^{(2)}(f) = \frac{2^{-2k}}{80(2k)!}(1-3^{4-2k})B_{2k}\cdot f^{(2k)}(\eta). \tag{4.107}$$

Proof. We can rewrite $\tilde{R}_{2k}^{(2)}(f)$ as

$$\tilde{R}_{2k}^{(2)}(f) = \frac{(-1)^{k-1}}{160(2k)!}J_k, \tag{4.108}$$

where

$$J_k = \int_0^1 (-1)^{k-1}F_{2k}(t)f^{(2k)}(t)dt. \tag{4.109}$$

From Corollary 4.10 we know that $(-1)^{k-1}F_{2k}(t) \geq 0$, $0 \leq t \leq 1$, so the claim follows from the mean value theorem for integrals and Corollary 4.11 □

Remark 4.19 For $k=3$ formula (4.107) reduces to

$$\tilde{R}_6^{(2)}(f)=\frac{1}{174182400}\cdot f^{(6)}(\eta).$$

Now, we prove our main result:

Theorem 4.22 *Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(2k)}$ is a continuous function on $[0,1]$ for some $k\geq 3$. If f is a $(2k)-$convex function, then for even k we have*

$$\begin{aligned} 0 &\leq \int_0^1 f(t)dt-\frac{1}{80}\left[27f\left(\frac{1}{6}\right)+26f\left(\frac{1}{2}\right)+27f\left(\frac{5}{6}\right)\right]+T_{2k-1}^D(f) \qquad (4.110)\\ &\leq \frac{1}{80}\left[13f(0)+27f\left(\frac{1}{3}\right)+27f\left(\frac{2}{3}\right)+13f(1)\right]-T_{2k-1}^S(f)-\int_0^1 f(t)dt \end{aligned}$$

while for odd k inequalities are reversed.

Proof. Denote the middle part and the right-hand side of (4.110) by LHS and DHS, respectively. Then we have

$$LHS=R_{2k}^{CM}(f)\quad\text{and}\quad RHS-LHS=-2\tilde{R}_{2k}^{(2)}(f)$$

where $\tilde{R}_{2k}^{(2)}(f)$ is defined in Theorem 4.20 and according to (3.119), $R_{2k}^{CM}(f)$ can be written in a form

$$R_{2k}^{CM}(f)=-\frac{1}{80(2k)!}(1-2^{1-2k})(1-3^{4-2k})B_{2k}\cdot f^{(2k)}(\xi),\ \xi\in[0,1] \qquad (4.111)$$

Recall that if f is $(2k)-$convex on $[0,1]$, then $f^{(2k)}(x)\geq 0$, $x\in[0,1]$. Now, having in mind that $(-1)^{k-1}B_{2k}>0$ $(k\in\mathbb{N})$, from (4.111) and (4.107), it follows

$$\begin{aligned} &LHS\geq 0,\quad RHS-LHS\geq 0,\quad \text{for even k}\\ &LHS\leq 0,\quad RHS-LHS\leq 0,\quad \text{for odd k} \end{aligned}$$

and thus the proof is complete. □

Remark 4.20 From (4.110) for $k=3$ it follows

$$\begin{aligned} 0 &\leq \int_0^1 f(t)dt-\frac{1}{80}\left[27f\left(\frac{1}{6}\right)+26f\left(\frac{1}{2}\right)+27f\left(\frac{5}{6}\right)\right]-\frac{1}{240}[f'(1)-f'(0)]\\ &\leq \frac{1}{80}\left[13f(0)+27f\left(\frac{1}{3}\right)+27f\left(\frac{2}{3}\right)+13f(1)\right]-\frac{1}{120}[f'(1)-f'(0)]-\int_0^1 f(t)dt \end{aligned}$$

Theorem 4.23 *If $f:[0,1]\to\mathbb{R}$ is such that $f^{(2k)}$ is a continuous function on $[0,1]$ and f is either $(2k)-$convex or $(2k)-$concave, for some $k\geq 3$, then there exists $\theta\in[0,1]$ such that*

$$\tilde{R}_{2k}^{(2)}(f)=\theta\cdot\frac{2^{-2k}}{40(2k)!}(1-2^{-2k})\left(1-3^{4-2k}\right)B_{2k}\left[f^{(2k-1)}(1)-f^{(2k-1)}(0)\right]. \qquad (4.112)$$

Proof. Suppose f is $(2k)-$convex, so $f^{(2k)}(t) \geq 0,\ 0 \leq t \leq 1$. If J_k is given by (4.109), using Corollary 4.10, we obtain

$$0 \leq J_k \leq (-1)^{k-1} F_{2k}\left(\frac{1}{4}\right) \cdot \int_0^1 f^{(2k)}(t)dt.$$

which means there exists $\theta \in [0,1]$ such that

$$J_k = \theta \cdot (-1)^{k-1} \cdot 2^{2-2k}\left(1-2^{-2k}\right)\left(1-3^{4-2k}\right) B_{2k}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right].$$

When f is $(2k)-$concave, the statement follows similarly. □

Now define

$$\Delta_{2k}(f) = \frac{2^{-2k}}{80(2k)!} \cdot (1-3^{4-2k}) B_{2k}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right].$$

Clearly,

$$\tilde{R}_{2k}^{(2)}(f) = \theta \cdot (2-2^{1-2k}) \cdot \Delta_{2k}(f).$$

Theorem 4.24 *Suppose that $f:[0,1] \to \mathbb{R}$ is such that $f^{(2k+2)}$ is a continuous function on $[0,1]$ for some $k \geq 3$. If f is either $(2k)$-convex and $(2k+2)$-convex or $(2k)$-concave and $(2k+2)$-concave, then the remainder $\tilde{R}_{2k}^{(2)}(f)$ has the same sign as the first neglected term $\Delta_{2k}(f)$ and*

$$|\tilde{R}_{2k}^{(2)}(f)| \leq |\Delta_{2k}(f)|.$$

Proof. We have

$$\Delta_{2k}(f) = \tilde{R}_{2k}^{(2)}(f) - \tilde{R}_{2k+2}^{(2)}(f).$$

From Corollary 4.10 it follows that for all $t \in [0,1]$

$$(-1)^{k-1} F_{2k}(t) \geq 0 \quad \text{and} \quad (-1)^{k-1}(-F_{2k+2}(t)) \geq 0,$$

so we conclude $\tilde{R}_{2k}^{(2)}(f)$ has the same sign as $-\tilde{R}_{2k+2}^{(2)}(f)$. Therefore, $\Delta_{2k}(f)$ must have the same sign as $\tilde{R}_{2k}^{(2)}(f)$ and $-\tilde{R}_{2k+2}^{(2)}(f)$. Moreover, it follows that

$$|\tilde{R}_{2k}^{(2)}(f)| \leq |\Delta_{2k}(f)| \quad \text{and} \quad |\tilde{R}_{2k+2}^{(2)}(f)| \leq |\Delta_{2k}(f)|.$$

□

Using formulae derived in Theorem 4.20, we shall prove a number of inequalities for various classes of functions.

Theorem 4.25 *Assume (p,q) is a pair of conjugate exponents, that is $1 \leq p,\ q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Let $f:[0,1] \to \mathbb{R}$ be such that $f^{(n)} \in L_p[0,1]$ for some $n \geq 1$. Then we have*

$$\left|\int_0^1 f(t)dt - D(0,1) + T_{n-1}(f)\right| \leq K(n,p) \cdot \|f^{(n)}\|_p, \tag{4.113}$$

and

$$\left|\int_0^1 f(t)dt - D(0,1) + T_n(f)\right| \le K^*(n,p)\cdot \|f^{(n)}\|_p, \tag{4.114}$$

where

$$K(n,p) = \frac{1}{160n!}\left[\int_0^1 |F_n(t)|^q\, dt\right]^{\frac{1}{q}} \quad \textit{and} \quad K^*(n,p) = \frac{1}{160n!}\left[\int_0^1 |G_n(t)|^q\, dt\right]^{\frac{1}{q}}.$$

Proof. Applying the Hölder inequality we get

$$\left|\frac{1}{160n!}\int_0^1 F_n(t)f^{(n)}(t)dt\right| \le \frac{1}{160n!}\left[\int_0^1 |F_n(t)|^q dt\right]^{\frac{1}{q}}\cdot \left\|f^{(n)}\right\|_p$$

Having in mind Remark 4.17, from (4.104) and the above inequality, we obtain (4.113). Similarly, from (4.103) we obtain (4.114). □

Remark 4.21 Taking $p=\infty$ and $n=1,2$ in Theorem 4.25, i.e. (4.113), we get

$$\left|\int_0^1 f(t)dt - D(0,1)\right| \le K(n,\infty)\cdot \|f^{(n)}\|_\infty,$$

where

$$K(1,\infty) = \frac{2401}{57600}, \quad K(2,\infty) = \frac{597+320\sqrt{10}}{768000}.$$

Taking $p=1$ and $n=1,2$, we get

$$\left|\int_0^1 f(t)dt - D(0,1)\right| \le K(n,1)\cdot \|f^{(n)}\|_1,$$

where

$$K(1,1) = \frac{41}{480}, \quad K(2,1) = \frac{169}{51200}.$$

Comparing these estimates with the analogous ones obtained for the Bullen-Simpson's 3/8 formula shows that these are better in all cases except for $n=2$ and $p=\infty$.

Moreover, for $p=\infty$ and $n=3,4,5$ we obtain

$$\left|\int_0^1 f(t)dt - D(0,1) + \frac{1}{480}[f'(1)-f'(0)]\right| \le K(n,\infty)\cdot \|f^{(n)}\|_\infty,$$

where

$$K(3,\infty) = \frac{48693+3133\sqrt{241}}{3932160000}, \quad K(4,\infty) = \frac{1}{1179648}, \quad K(5,\infty) = \frac{1}{22118400},$$

and for $p=1$ and $n=3,4,5$ we get

$$\left|\int_0^1 f(t)dt - D(0,1) + \frac{1}{480}[f'(1)-f'(0)]\right| \le K(n,1)\cdot \|f^{(n)}\|_1,$$

where

$$K(3,1) = \frac{1053+187\sqrt{561}}{110592000}, \quad K(4,1) = \frac{1}{614400}, \quad K(5,1) = \frac{1}{9437184}.$$

Finally, for $p = 2$ we get

$$\left| \int_0^1 f(t)dt - D(0,1) \right| \leq K(n,2) \cdot \|f^{(n)}\|_2,$$

where

$$K(1,2) = \frac{\sqrt{534}}{480}, \quad K(2,2) = \frac{\sqrt{5}}{960},$$

and

$$\left| \int_0^1 f(t)dt - D(0,1) + \frac{1}{480}[f'(1) - f'(0)] \right| \leq K(n,2) \cdot \|f^{(n)}\|_2,$$

where

$$K(3,2) = \frac{\sqrt{1155}}{1209600}, \quad K(4,2) = \frac{\sqrt{210}}{14515200}, \quad K(5,2) = \frac{\sqrt{116655}}{5748019200}.$$

Remark 4.22 Note that $K^*(1,p) = K(1,p)$, for $1 < p \leq \infty$, since $G_1(t) = F_1(t)$. Also, for $1 < p \leq \infty$, we can easily calculate $K(1,p)$. Namely,

$$K(1,p) = \frac{1}{480}\left[\frac{39^{q+1}+40^{q+1}+41^{q+1}}{120(q+1)}\right]^{\frac{1}{q}}.$$

In the limit case when $p \to 1$, that is when $q \to \infty$, we have

$$\lim_{p\to 1} K(1,p) = \frac{41}{480} = K(1,1).$$

Now we use formula (4.103) and a Grüss type inequality to obtain estimations of corrected Bullen-Simpson's 3/8 formulae in terms of oscillation of derivatives of a function.

Theorem 4.26 *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n)}$ exists and is integrable on $[0,1]$, for some $n \geq 1$. Suppose*

$$m_n \leq f^{(n)}(t) \leq M_n, \qquad 0 \leq t \leq 1,$$

for some constants m_n and M_n. Then

$$\left| \int_0^1 f(t)dt - D(0,1) + T_n(0,1) \right| \leq C_n(M_n - m_n) \tag{4.115}$$

where

$$C_1 = \frac{2401}{115200}, \quad C_2 = \frac{320\sqrt{30}+187\sqrt{561}}{27648000},$$

$$C_3 = \frac{48693 + 3133\sqrt{241}}{7864320000}, \quad C_4 = \frac{1}{2359296},$$
$$C_{2k-1} = \frac{2^{-2k}}{20(2k)!}(1-2^{-2k})(1-3^{4-2k})|B_{2k}|, \quad k \geq 3,$$
$$C_{2k} = \frac{2^{-2k}}{80(2k)!}(1-3^{4-2k})|B_{2k}|, \quad k \geq 3.$$

Proof. Similarly as in the proof of Theorem 3.26, Lemma 3.9 ensures that the second condition of Lemma 3.10 is satisfied. Having in mind Remark 4.17, apply inequality (3.153) to obtain the estimate for $|\tilde{R}_n^{(1)}(f)|$. Now our statement follows easily from Corollary 4.11 for $n \geq 5$ and direct calculation for $n = 1,2,3,4$. □

4.3 Gauss 4-point formula

One of the most interesting properties to consider when studying the quadrature formulae is the maximum degree of exactness. It is well-known that the formulae that have property are the Gauss formulae.

The Gauss 4-point formula is an open quadrature formula and is furthermore exact for all polynomials of order ≤ 7, so it is clear why it has not appeared as a special case of the two families of closed 4-point quadrature formulae considered in this chapter. Nevertheless, we are going to consider it separately, using the same technique as before. The results from this section are published in [58].

Let $f : [-1,1] \to \mathbb{R}$ be such that $f^{(2n)}$ is continuous of bounded variation on $[-1,1]$ for some $n \geq 0$. Assume $0 < x < y < 1$. Put $x \equiv -y,\ -x,\ x,\ y$ in (1.2), multiply by $w(x,y),\ 1-w(x,y),\ 1-w(x,y),\ w(x,y)$, respectively, and add. The following formula is obtained:

$$\begin{aligned}\int_{-1}^{1} f(t)dt - w(x,y)[f(-y)+f(y)] - (1-w(x,y))[f(-x)+f(x)] + T_{2n}(x,y)\\ = \frac{2^{2n+1}}{(2n+2)!}\int_{-1}^{1} F_{2n+2}(x,y,t)\,df^{(2n+1)}(t),\end{aligned} \tag{4.116}$$

where, for $k \geq 1$ and $t \in \mathbb{R}$,

$$\begin{aligned}
T_{2n}(x,y) &= \sum_{k=1}^{2n} \frac{2^{k-1}}{k!}\, G_k(x,y,-1)\,[f^{(k-1)}(1) - f^{(k-1)}(-1)]\\
G_k(x,y,t) &= w(x,y)\left[B_k^*\left(\frac{-y-t}{2}\right) + B_k^*\left(\frac{y-t}{2}\right)\right]\\
&+ (1-w(x,y))\left[B_k^*\left(\frac{-x-t}{2}\right) + B_k^*\left(\frac{x-t}{2}\right)\right],\\
F_k(x,y,t) &= G_k(x,y,t) - G_k(x,y,-1).
\end{aligned}$$

Functions G_k have the same properties again as the analogous functions from previous sections. In order to produce the quadrature formula with the maximum degree of exactness, impose conditions:

$$G_2(x,y,-1) = G_4(x,y,-1) = G_6(x,y-1) = 0.$$

The solution of this system is:

$$x_0 = \sqrt{\frac{15-2\sqrt{30}}{35}}, \; y_0 = \sqrt{\frac{15+2\sqrt{30}}{35}}, \; w(x_0,y_0) = \frac{18-\sqrt{30}}{36}$$

and from there

$$\begin{aligned}\int_{-1}^{1} f(t)dt \approx \; & \frac{18-\sqrt{30}}{36}\left[f\left(-\sqrt{\frac{15+2\sqrt{30}}{35}}\right)+f\left(\sqrt{\frac{15+2\sqrt{30}}{35}}\right)\right] \\ & + \frac{18+\sqrt{30}}{36}\left[f\left(-\sqrt{\frac{15-2\sqrt{30}}{35}}\right)+f\left(\sqrt{\frac{15-2\sqrt{30}}{35}}\right)\right],\end{aligned}$$

which is exactly the classical Gauss 4-point formula. To shorten notation, denote the right-hand side of the upper expression with Q_{G4}. Now we have:

$$T_{2n}^{G4} = T_{2n}(x_0,y_0) = \sum_{k=4}^{n} \frac{2^{2k-1}}{(2k)!} G_{2k}^{G4}(-1)[f^{(2k-1)}(1) - f^{(2k-1)}(-1)] \tag{4.117}$$

$$G_k^{G4}(t) = G_k(x_0,y_0,t) \tag{4.118}$$

$$F_k^{G4}(t) = F_k(x_0,y_0,t) - F_k(x_0,y_0,-1), \tag{4.119}$$

so formula (4.116) becomes

$$\int_{-1}^{1} f(t)dt - Q_{G4} + T_{2n}^{G4} = \frac{2^{2n+1}}{(2n+2)!}\int_{-1}^{1} F_{2n+2}^{G4}(t)df^{(2n+1)}(t). \tag{4.120}$$

Assuming $f^{(2n-1)}$ is continuous of bounded variation on $[-1,1]$ for some $n \geq 1$ we obtain analogously:

$$\int_{-1}^{1} f(t)dt - Q_{G4} + T_{2n}^{G4} = \frac{2^{2n-1}}{(2n)!}\int_{-1}^{1} G_{2n}^{G4}(t)df^{(2n-1)}(t), \tag{4.121}$$

and if $f^{(2n)}$ satisfies the same property for some $n \geq 0$, then

$$\int_{-1}^{1} f(t)dt - Q_{G4} + T_{2n}^{G4} = \frac{2^{2n}}{(2n+1)!}\int_{-1}^{1} G_{2n+1}^{G4}(t)df^{(2n)}(t). \tag{4.122}$$

Lemma 4.10 *For $k \geq 3$, $G_{2k+1}^{G4}(t)$ has no zeros in $(0,1)$. The sign of this function is determined by*

$$(-1)^{k+1} G_{2k+1}^{G4}(t) > 0, \quad 0 < t < 1.$$

Proof. We start from $G_7^{G4}(t)$. For $t \in [y_0, 1)$, $G_7^{G4}(t) = \frac{1}{64}(1-t)^7$, so obviously it has no zeros there. It is easy to check that $(G^{G4})_7'(0) > 0$. Now, assume G_7^{G4} has at least one zero in $(0,1)$. Then G_7^{G4} has at least 3 local extrema in $(0,1)$. This means G_6^{G4} has at least 3 zeros and therefore at least 3 local extrema, since $G_6^{G4}(1) = 0$. This implies G_5^{G4} has at least 3 zeros and since $G_5^{G4}(0) = G_5(1) = 0$, it has at least 4 local extrema. From there we conclude G_4^{G4} has at least 4 zeros and as $G_4^{G4}(1) = 0$, G_4^{G4} has at least 4 local extrema as well. Therefore, G_3^{G4} has at least 4 zeros and since $G_3^{G4}(0) = G_3^{G4}(1) = 0$, it has at least 5 local extrema in this interval. Finally, this implies G_2^{G4} has at least 5 zeros which is obviously impossible - we know it has none on $[y_0, 1)$, so it could have 4 zeros at most. Thus, we conclude G_7^{G4} has no zeros and $G_7^{G4}(t) > 0$ on $(0,1)$. Assuming the opposite, by induction, it follows easily that the assertion is true for all $k \geq 4$. The sign of the functions $G_{2k+1}^{G4}(t)$, $k \geq 5$ can be determined analogously as in Lemma 3.1 □

Remark 4.23 From Lemma 4.10 it follows immediately that for $k \geq 3$, $(-1)^{k+1} F_{2k+2}^{G4}(t)$ is strictly increasing on $(-1,0)$ and strictly decreasing on $(0,1)$. Since $F_{2k+2}^{G4}(-1) = F_{2k+2}^{G4}(1) = 0$, it has constant sign on $(-1,1)$ and attains maximum at $t = 0$.

Using Hölder's inequality, estimates of error for this type of quadrature formulae are obtained:

Theorem 4.27 *Let $p, q \in \mathbb{R}$ be such that $1 \leq p,\ q \leq \infty$, $1/p + 1/q = 1$. Let $f : [-1,1] \to \mathbb{R}$ be such that $f^{(2n)} \in L_p[-1,1]$ for some $n \geq 1$. Then we have*

$$\left| \int_{-1}^{1} f(t)dt - Q_{G4} + T_{2n}^{G4} \right| \leq K_{G4}(2n, q) \cdot \|f^{(2n)}\|_p. \tag{4.123}$$

If $f^{(2n+1)} \in L_p[-1,1]$ for some $n \geq 0$, then we have

$$\left| \int_{-1}^{1} f(t)dt - Q_{G4} + T_{2n}^{G4} \right| \leq K_{G4}(2n+1, q) \cdot \|f^{(2n+1)}\|_p. \tag{4.124}$$

If $f^{(2n+2)} \in L_p[-1,1]$ for some $n \geq 0$, then we have

$$\left| \int_{-1}^{1} f(t)dt - Q_{G4} + T_{2n}^{G4} \right| \leq K_{G4}^{*}(2n+2, q) \cdot \|f^{(2n+2)}\|_p, \tag{4.125}$$

where

$$K_{G4}(m,q) = \frac{2^{m-1}}{m!} \left[\int_{-1}^{1} \left| G_m^{G4}(t) \right|^q dt \right]^{\frac{1}{q}}, \qquad K_{G4}^{*}(m,q) = \frac{2^{m-1}}{m!} \left[\int_{-1}^{1} \left| F_m^{G4}(t) \right|^q dt \right]^{\frac{1}{q}}.$$

These inequalities are sharp for $1 < p \leq \infty$ and the the best possible for $p = 1$.

Proof. Analogous to the proof of Theorem 2.2. □

Using Lemma 4.10 and Remark 4.23, for $p = 1$ and $p = \infty$ we get:

$$K_{G4}^{*}(2n+2,1) = \frac{2^{2n+2}}{(2n+2)!} \left| G_{2n+2}^{G4}(-1) \right|, \tag{4.126}$$

$$K_{G4}^{*}(2n+2,\infty) = \frac{1}{2} K_{G4}(2n+1,1) = \frac{2^{2n+1}}{(2n+2)!} \left| F_{2n+2}^{G4}(0) \right|, \tag{4.127}$$

where

$$G_{2n+2}^{G4}(-1) = \frac{18-\sqrt{30}}{18} B_{2n+2}\left(\frac{1-y_0}{2}\right) + \frac{18+\sqrt{30}}{18} B_{2n+2}\left(\frac{1-x_0}{2}\right) \tag{4.128}$$

$$\begin{aligned} F_{2n+2}^{G4}(0) &= \frac{18-\sqrt{30}}{18}\left[B_{2n+2}\left(\frac{y_0}{2}\right) - B_{2n+2}\left(\frac{1-y_0}{2}\right)\right] \\ &+ \frac{18+\sqrt{30}}{18}\left[B_{2n+2}\left(\frac{x_0}{2}\right) - B_{2n+2}\left(\frac{1-x_0}{2}\right)\right]. \end{aligned} \tag{4.129}$$

The following theorem shows how the remainder $R_{2n+2}^{G4}(f)$ in formula (4.120) can be expressed.

Theorem 4.28 *If $f : [-1,1] \to \mathbb{R}$ is such that $f^{(2n+2)}$ is continuous on $[-1,1]$ for some $n \geq 3$, then there exists $\xi \in [-1,1]$ such that*

$$R_{2n+2}^{G4}(f) = -\frac{2^{2n+2}}{(2n+2)!} G_{2n+2}^{G4}(-1) \cdot f^{(2n+2)}(\xi), \tag{4.130}$$

where $G_{2n+2}^{G4}(-1)$ is as in (4.128).

If, in addition, $f^{(2n+2)}$ does not change sign on $[-1,1]$, then there exists $\theta \in [0,1]$ such that

$$R_{2n+2}^{G4}(f) = \theta \cdot \frac{2^{2n+1}}{(2n+2)!} F_{2n+2}^{G4}(0) \left[f^{(2n+1)}(1) - f^{(2n+1)}(-1) \right], \tag{4.131}$$

where $F_{2n+2}^{G4}(0)$ is as in (4.129).

Proof. Analogous to the proof of Theorem 3.2. □

Applying (4.130) to the remainder in (4.120) for $n = 3$ produces the classical Gauss 4-point formula:

$$\int_{-1}^{1} f(t)dt - Q_{G4} = \frac{1}{3472875} f^{(8)}(\xi), \quad \xi \in [-1,1].$$

As direct consequences of Theorem 4.27, i.e. (4.126) and (4.127), the following estimations are obtained for $p = \infty$ and $p = 1$:

$$\left| \int_{-1}^{1} f(t)dt - Q_{G4} \right| \leq C_{G4}(m,q) \cdot \|f^{(m)}\|_p, \; 1 \leq m \leq 8$$

where

$$
\begin{aligned}
&C_{G4}(1,1) \approx 2.75994 \cdot 10^{-1}, \quad C_{G4}(1,\infty) = x_0 \approx 3.39981 \cdot 10^{-1},\\
&C_{G4}(2,1) \approx 2.18566 \cdot 10^{-2}, \quad C_{G4}(2,\infty) = |G_2^{G4}(x_0)| \approx 3.65261 \cdot 10^{-2},\\
&C_{G4}(3,1) \approx 2.3501 \cdot 10^{-3}, \quad C_{G4}(3,\infty) \approx 2.92413 \cdot 10^{-3},\\
&C_{G4}(4,1) \approx 2.69484 \cdot 10^{-4}, \quad C_{G4}(4,\infty) = |G_4^{G4}(0)|/3 \approx 3.73171 \cdot 10^{-4},\\
&C_{G4}(5,1) \approx 3.48131 \cdot 10^{-5}, \quad C_{G4}(5,\infty) \approx 4.89802 \cdot 10^{-5},\\
&C_{G4}(6,1) \approx 5.25522 \cdot 10^{-6}, \quad C_{G4}(6,\infty) = 2|G_6^{G4}(0)|/45 \approx 8.498485 \cdot 10^{-6},\\
&C_{G4}(7,1) = 2|F_8^{G4}(0)|/315 \approx 9.941993 \cdot 10^{-7}, \quad C_{G4}(7,\infty) \approx 1.313805 \cdot 10^{-6},\\
&C_{G4}(8,1) = 2|G_8^{G4}(-1)|/315 = 1/3472875 \approx 2.8794587 \cdot 10^{-7},\\
&C_{G4}(8,\infty) = |F_8^{G4}(0)|/315 \approx 4.971 \cdot 10^{-7}.
\end{aligned}
$$

Above constants are obtained with the help of Wolfram's Mathematica, as the expressions involved are rather cumbersome. Similar estimations can be obtained for $m \geq 9$ from (4.126) and (4.127). However, the values of derivatives (starting from the 7th) in the end points of the interval are then also included in the quadrature formula. In cases when those values are easy to calculate, this is not an obstacle. Furthermore, in cases when $f^{(k)}(1) = f^{(k)}(-1)$ for $k \geq 7$, we get a formula with an even higher degree of exactness.

For $m = 1$ and $1 < p \leq \infty$, we get:

$$
C_{G4}(1,q) = \left[\frac{2}{q+1} \left(x_0^{q+1} + \left(\frac{18+\sqrt{30}}{36} - x_0 \right)^{q+1} + \left(y_0 - \frac{18+\sqrt{30}}{36} \right)^{q+1} + (1-y_0)^{q+1} \right) \right]^{1/q}
$$

When $p = 1$, i.e. when $q = \infty$, we obtain $C_{G4}(1,\infty) = x_0$, as we did before by calculating directly.

Remark 4.24 The constant $C_{G4}(1,\infty)$ coincides with the constant $\rho_V(R_4^G)$ from Theorem 1.1. in [47].

Estimations for $m = 2$ and $1 < p \leq \infty$ are expressed in terms of hypergeometric functions. Namely,

$$
\begin{aligned}
C_{G4}(2,q) = \frac{1}{2} \Bigg[& \frac{2}{2q+1}(1-y_0)^{2q+1} + 2(\gamma-\beta)^{2q+1} B(q+1,q+1) \\
& + \frac{2}{q+1} \left(\alpha^{2q+1} 2^q F\left(-q, q+1; q+2; \frac{1}{2}\right) \right. \\
& + (x_0-\alpha)^{q+1} (2\alpha)^q F\left(-q, q+1; q+2; -\frac{x_0-\alpha}{2\alpha}\right) \\
& + (\beta-x_0)^{q+1} (\gamma-\beta)^q F\left(-q, q+1; q+2; -\frac{\beta-x_0}{\gamma-\beta}\right) \\
& \left. + (y_0-\gamma)^{q+1} (\gamma-\beta)^q F\left(-q, q+1; q+2; -\frac{y_0-\gamma}{\gamma-\beta}\right) \right) \Bigg]^{1/q},
\end{aligned}
$$

where α, β and γ are zeros of $G_2(t)$ such that $0 < \alpha < x_0 < \beta < \gamma < y_0$. Here, the integral representation of a hypergeometric function was used. It is given by:

$$F(a,b;c;x) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}dt,$$

if $c > b > 0$ and $|x| < 1$. B is the well-known Beta function.

Similar estimations could be obtained for higher derivatives but it is difficult to calculate the zeros of the integrand for $m \geq 3$.

Chapter 5

General 5-point quadrature formulae of Euler type

This chapter is dedicated to the closed 5-point quadrature formulae, i.e. quadratures which estimate the integral over $[0,1]$ with the values of the function at nodes $0,\ x,\ 1/2, 1-x$ and 1, where $x \in (0,1/2)$. The results from this chapter can be found in [60].

5.1 General approach

Let $x \in (0,1/2)$ and $f:[0,1] \to \mathbb{R}$ be such that $f^{(2n+1)}$ is continuous of bounded variation on $[0,1]$ for some $n \geq 0$. Analogously as before: put $x \equiv 0,\ x,\ 1/2, 1-x$ and 1 in (1.2), multiply by $w_1(x),\ w_2(x),\ w_3(x),\ w_2(x),\ w_1(x)$ respectively, where $2w_1(x)+2w_2(x) + w_3(x) = 1$, and add up. The following formula is obtained:

$$\int_0^1 f(t)dt - Q\left(0,x,\frac{1}{2},1-x,1\right) + T_{2n}(x) = \frac{1}{(2n+2)!}\int_0^1 F_{2n+2}(x,t)df^{(2n+1)}(t), \tag{5.1}$$

where

$$Q\left(0,x,\frac{1}{2},1-x,1\right) \tag{5.2}$$

$$= w_1(x)[f(0)+f(1)] + w_2(x)[f(x)+f(1-x)] + w_3(x) f\left(\frac{1}{2}\right),$$

$$T_{2n}(x) = \sum_{k=1}^{n} \frac{1}{(2k)!}\, G_{2k}(x,0)\, [f^{(2k-1)}(1) - f^{(2k-1)}(0)], \tag{5.3}$$

$$G_k(x,t) = 2w_1(x) B_k^*(1-t) + w_2(x)\,[B_k^*(x-t) + B_k^*(1-x-t)] + w_3(x) B_k^*(1/2-t), \tag{5.4}$$

$$F_k(x,t) = G_k(x,t) - G_k(x,0). \tag{5.5}$$

To obtain the formula with the maximum degree of exactness from (5.1), impose the condition:

$$G_2(x,0) = G_4(x,0) = 0.$$

Then:

$$w_1(x) = \frac{10x^2-10x+1}{60x(x-1)}, \qquad w_2(x) = \frac{1}{60x(1-x)(2x-1)^2},$$
$$w_3(x) = 1 - 2w_1(x) - 2w_2(x) = \frac{8(5x^2-5x+1)}{15(2x-1)^2}\ ; \tag{5.6}$$

these are the weights we shall work with in this section.

Changing the assumptions on function f, we can obtain two more identities with the left-hand side equal to that in (5.1). Namely, assuming $f^{(2n-1)}$ is continuous of bounded variation on $[0,1]$ for some $n \geq 1$, from (1.1) we get:

$$\int_0^1 f(t)dt - Q\left(0, x, \frac{1}{2}, 1-x, 1\right) + T_{2n}^{Q5}(x) = \frac{1}{(2n)!} \int_0^1 G_{2n}^{Q5}(x,t)\, df^{(2n-1)}(t), \tag{5.7}$$

and assuming $f^{(2n)}$ is continuous of bounded variation on $[0,1]$ for some $n \geq 0$, from (1.1) (or (1.2)) we get:

$$\int_0^1 f(t)dt - Q\left(0, x, \frac{1}{2}, 1-x, 1\right) + T_{2n}^{Q5}(x) = \frac{1}{(2n+1)!} \int_0^1 G_{2n+1}^{Q5}(x,t)\, df^{(2n)}(t), \tag{5.8}$$

where $T_{2n}^{Q5}(x), G_{2n}^{Q5}(x,t), F_{2n}^{Q5}(x,t)$ are as in (5.3)-(5.5) with weights as in (5.6).

Lemma 5.1 *For $x \in (0, \frac{5-\sqrt{15}}{10}] \cup [\frac{1}{5}, \frac{1}{2})$ and $k \geq 2$, $G_{2k+1}^{Q5}(x,t)$ has no zeros in the variable t on $(0,1/2)$. The sign of the function is determined by:*

$$(-1)^{k+1} G_{2k+1}^{Q5}(x,t) > 0 \quad za\ \ x \in \left(0,\ 1/2 - \sqrt{15}/10\right],$$
$$(-1)^{k} G_{2k+1}^{Q5}(x,t) > 0 \quad za\ \ x \in [1/5,\ 1/2).$$

Proof. First, note that

$$x \in \left(\frac{5-\sqrt{15}}{10}, \frac{1}{5}\right) \ \Leftrightarrow\ \frac{\partial^4 G_5^{Q5}}{\partial t^4}(x,0) > 0 \quad \& \quad \frac{\partial G_5^{Q5}}{\partial t}\left(x, \frac{1}{2}\right) > 0.$$

Further, $G_k^{Q5}(x,0)=0$ for $2\le k\le 5$ and $G_5^{Q5}\left(x,\frac{1}{2}\right)=0$. The claim is that for $x\in\left(\frac{5-\sqrt{15}}{10},\ \frac{1}{5}\right)$, $G_5^{Q5}(x,t)$ has at least one zero in variable t on $(0,1/2)$. Assume first $\frac{\partial^4 G_5^{Q5}}{\partial t^4}(x,0)>0$. This means $\frac{\partial^4 G_5^{Q5}}{\partial t^4}>0$ in some neighborhood of zero so we conclude $\frac{\partial^3 G_5^{Q5}}{\partial t^3}$ is increasing in this neighborhood. Now, similarly as in the proof of Lemma 3.7, knowing the values of the k-th derivative in 0 and the sign of the $(k+1)$-th derivative of G_5^{Q5}, we conclude on the sign of the k-th derivative (for $k=0,1,2,3$). Thus it follows $G_5^{Q5}(x,t)>0$ in this neighborhood of zero. From the assumption $\frac{\partial G_5^{Q5}}{\partial t}\left(x,\frac{1}{2}\right)>0$, analogously follows that $G_5^{Q5}(x,t)<0$ in some neighborhood of $1/2$. Now it is clear that for $x\in\left(\frac{5-\sqrt{15}}{10},\ \frac{1}{5}\right)$, G_5^{Q5} has at least one zero.

The next step is to prove $G_5^{Q5}(x,t)$ is monotonous (in x). Assume first $0\le t\le x<1/2$. Then

$$\frac{\partial G_5^{Q5}(x,t)}{\partial x}=\frac{t^4(1-2x)}{12x^2(x-1)^2},$$

so obviously G_5^{Q5} is strictly increasing. Next, let $0<x\le t\le 1/2$. Then

$$\frac{\partial G_5^{Q5}(x,t)}{\partial x}=\frac{1-2t}{12(x-1)^2(2x-1)^3}\cdot h(x,t),$$

where $h(x,t)=8t^3(x-1)^2+t^2(4x^2+4x-6)+2t(4x-5x^2)+x^2(4x-3)$. We claim $h(x,t)<0$. It is easy to see that

$$\frac{\partial h(x,t)}{\partial t}=0\ \Leftrightarrow\ t_1=\frac{1}{2},\ t_2=\frac{4x-5x^2}{6(x-1)^2}$$

but we also have $t_2<x$, so $h(x,t)$ is strictly decreasing for $t\in[x,1/2]$. Since $h(x,x)<0$, our claim follows, and from there it is clear that G_5^{Q5} is strictly increasing in x on this interval.

Further, since $G_5^{Q5}(\frac{5-\sqrt{15}}{10},t)<0$ for $t\in(0,1/2)$ (see [59]), $G_5^{Q5}(\frac{1}{5},t)=t^4(5/16-t)>0$ for $t\in(0,1/5]$ and $G_5^{Q5}(\frac{1}{5},t)=\frac{1}{432}(1-2t)^3(54t^2-14t+1)>0$ for $t\in[1/5,1/2)$, we conclude $G_5^{Q5}(x,t)<0$ for $x\in\left(0,\ \frac{5-\sqrt{15}}{10}\right]$ and $G_5^{Q5}(x,t)>0$ for $x\in\left[\frac{1}{5},\ \frac{1}{2}\right)$.

The rest of the proof is analogous to the proof of Lemma 3.1 □

Denote by $R_{2n+2}^{Q5}(x,f)$ the right-hand side of (5.1) with weights as in (5.6).

Theorem 5.1 *Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(2n+2)}$ is continuous on $[0,1]$ for some $n\ge 3$ and let $x\in(0,\frac{1}{2}-\frac{\sqrt{15}}{10}]\cup[\frac{1}{5},\frac{1}{2})$. If $f^{(2n)}$ and $f^{(2n+2)}$ have the same constant sign on $[0,1]$, then the remainder $R_{2n}^{Q5}(x,f)$ has the same sign as the first neglected term $\Delta_{2n}^{Q5}(x,f)$ where*

$$\Delta_{2n}^{Q5}(x,f):=R_{2n}^{Q5}(x,f)-R_{2n+2}^{Q5}(x,f)=-\frac{1}{(2n)!}G_{2n}^{Q5}(x,0)[f^{(2n-1)}(1)-f^{(2n-1)}(0)].$$

Furthermore, $|R_{2n}^{Q5}(x,f)|\le|\Delta_{2n}^{Q5}(x,f)|$ and $|R_{2n+2}^{Q5}(x,f)|\le|\Delta_{2n}^{Q5}(x,f)|$.

Proof. Analogous to the proof of Theorem 3.1. □

Theorem 5.2 *If $f:[0,1]\to\mathbb{R}$ is such that $f^{(2n+2)}$ is continuous on $[0,1]$ for some $n\geq 2$ and $x\in(0,\frac{1}{2}-\frac{\sqrt{15}}{10}]\cup[\frac{1}{5},\frac{1}{2})$, then there exists $\xi\in[0,1]$ such that*

$$R_{2n+2}^{Q5}(x,f)=-\frac{G_{2n+2}^{Q5}(x,0)}{(2n+2)!}\cdot f^{(2n+2)}(\xi) \tag{5.9}$$

where

$$G_{2n+2}^{Q5}(x,0)=2w_2(x)\cdot B_{2n+2}(x)+[2w_1(x)-w_3(x)(1-2^{-2n-1})]B_{2n+2}. \tag{5.10}$$

If, in addition, $f^{(2n+2)}$ has constant sign on $[0,1]$, then there exists a point $\theta\in[0,1]$ such that

$$R_{2n+2}^{Q5}(x,f)=\frac{\theta}{(2n+2)!}\cdot F_{2n+2}^{Q5}\left(x,\frac{1}{2}\right)\cdot\left[f^{(2n+1)}(1)-f^{(2n+1)}(0)\right] \tag{5.11}$$

where

$$\begin{aligned}F_{2n+2}^{Q5}(x,1/2) &= 2w_2(x)[B_{2n+2}(1/2-x)-B_{2n+2}(x)]\\ &+ [w_3(x)-2w_1(x)](2-2^{-2n-1})B_{2n+2}.\end{aligned} \tag{5.12}$$

Proof. Analogous to the proof of Theorem 3.2. □

When (5.9) is applied to (5.1) for $n=2$, the following formula is produced:

$$\int_0^1 f(t)dt-Q\left(0,x,\frac{1}{2},1-x,1\right)=\frac{1}{604800}(7x^2-7x+1)\cdot f^{(6)}(\xi). \tag{5.13}$$

For $x=1/4$, (5.13) produces the classical Boole's formula. When $w_1(x)=0$, which is equivalent to $x=\frac{5-\sqrt{15}}{10}$, formula (5.13) becomes the Gauss 3-point formula (stated on $[0,1]$). When $w_3(x)=0$, i.e. when $x=\frac{5-\sqrt{5}}{10}$, (5.13) gives the Lobatto 4-point formula (on $[0,1]$ again). Note that $w_2(x)\neq 0$ for every x what could be expected since if x such that $w_2(x)=0$ existed, that x would generate a closed 3-point quadrature formula with a degree of exactness equal to 5, and we know that such a formula does not exist - unless of course it is corrected, but we do not deal with that kind of quadrature formulae just yet.

Remark 5.1 Although only $x\in(0,1/2)$ were taken into consideration here, results for $x=0$ and $x=1/2$ can be obtained by considering the limit processes. Namely,

$$\lim_{x\to 0}Q\left(0,x,\frac{1}{2},1-x,1\right)=\frac{1}{30}\left[7f(0)+16f\left(\frac{1}{2}\right)+7f(1)\right]-\frac{1}{60}[f'(1)-f'(0)]$$
$$\lim_{x\to 0}G_k^{Q5}(x,t)=\frac{7}{15}B_k^*(1-t)+\frac{8}{15}B_k^*\left(\frac{1}{2}-t\right)$$

Consequently, from (5.13) it follows:

$$\int_0^1 f(t)dt - \frac{1}{30}[7f(0)+16f\left(\frac{1}{2}\right)+7f(1)] + \frac{1}{60}[f'(1)-f'(0)] \\ = \frac{1}{604800}f^{(6)}(\xi)$$

which is exactly corrected Simpson's formula (cf. (3.115)).

Furthermore,

$$\lim_{x\to 1/2} Q\left(0,x,\frac{1}{2},1-x,1\right) = \frac{1}{10}[f(0)+8f\left(\frac{1}{2}\right)+f(1)] + \frac{1}{60}f''\left(\frac{1}{2}\right)$$

$$\lim_{x\to 1/2} G_k^{Q5}(x,t) = \frac{1}{5}B_k^*(1-t) + \frac{4}{5}B_k^*\left(\frac{1}{2}-t\right) + \frac{k(k-1)}{60}B_{k-2}^*\left(\frac{1}{2}-t\right)$$

and then from (5.13):

$$\int_0^1 f(t)dt - \frac{1}{10}[f(0)+8f\left(\frac{1}{2}\right)+f(1)] - \frac{1}{60}f''\left(\frac{1}{2}\right) = -\frac{1}{806400}f^{(6)}(\xi).$$

The next theorem gives some sharp estimates of error for this type of quadrature formulae.

Theorem 5.3 *Let $p,q\in\mathbb{R}$ be such that $1\le p,\ q\le\infty$ and $1/p+1/q=1$. If $f:[0,1]\to\mathbb{R}$ is such that $f^{(2n)}\in L_p[0,1]$ for some $n\ge 1$, then*

$$\left|\int_0^1 f(t)dt - Q\left(0,x,\frac{1}{2},1-x,1\right) + T_{2n}^{Q5}(x)\right| \le K_{Q5}(2n,q,x)\cdot\|f^{(2n)}\|_p, \tag{5.14}$$

if $f^{(2n+1)}\in L_p[0,1]$ for some $n\ge 0$, then

$$\left|\int_0^1 f(t)dt - Q\left(0,x,\frac{1}{2},1-x,1\right) + T_{2n}^{Q5}(x)\right| \le K_{Q5}(2n+1,q,x)\cdot\|f^{(2n+1)}\|_p, \tag{5.15}$$

and finally, if $f^{(2n+2)}\in L_p[0,1]$ for some $n\ge 0$, then

$$\left|\int_0^1 f(t)dt - Q\left(0,x,\frac{1}{2},1-x,1\right) + T_{2n}^{Q5}(x)\right| \le K_{Q5}^*(2n+2,q,x)\cdot\|f^{(2n+2)}\|_p, \tag{5.16}$$

where

$$K_{Q5}(m,q,x) = \frac{1}{m!}\left[\int_0^1 \left|G_m^{Q5}(x,t)\right|^q dt\right]^{\frac{1}{q}}, \quad K_{Q5}^*(m,q,x) = \frac{1}{m!}\left[\int_0^1 \left|F_m^{Q5}(x,t)\right|^q dt\right]^{\frac{1}{q}}.$$

These inequalities are sharp for $1<p\le\infty$ and best possible for $p=1$.

Proof. Analogous to the proof of Theorem 2.2. □

For $x \in (0, \frac{1}{2} - \frac{\sqrt{15}}{10}] \cup [\frac{1}{5}, \frac{1}{2})$ and $n \geq 2$, using the previous theorem, the following constants can be calculated:

$$K_{Q5}^{*}(2n+2,1,x) = \frac{1}{(2n+2)!}\left|G_{2n+2}^{Q5}(x,0)\right|,$$
$$K_{Q5}^{*}(2n+2,\infty,x) = \frac{1}{2}\,K_{Q5}(2n+1,1,x) = \frac{1}{(2n+2)!}\left|F_{2n+2}^{Q5}\left(x,\frac{1}{2}\right)\right|,$$

where $G_{2n+2}^{Q5}(x,0)$ and $F_{2n+2}^{Q5}\left(x,\frac{1}{2}\right)$ are as in (5.10) and (5.12), respectively. In view of this, let us consider inequalities (5.15) and (5.16) for $n = 2$ and $p = \infty$:

$$\left|\int_0^1 f(t)dt - Q\left(0,x,\frac{1}{2},1-x,1\right)\right| \leq \frac{|8x^2-7x+1|}{115200(1-x)} \cdot \|f^{(5)}\|_\infty$$
$$\left|\int_0^1 f(t)dt - Q\left(0,x,\frac{1}{2},1-x,1\right)\right| \leq \frac{|7x^2-7x+1|}{604800} \cdot \|f^{(6)}\|_\infty$$

In order to find which admissible x gives the least estimate of error, the functions of the right-hand sides have to be minimized. It is easy to verify that both functions are decreasing on $(0, 1/2 - \sqrt{15}/10]$, increasing on $[1/5, 1/2)$ and reach their minimal value at $x = 1/5$. Since $K_{Q5}(5,1) = 2K_{Q5}^{*}(6,\infty)$, the same conclusion applies for $n = 2$ and $p = 1$. For $x = 1/5$, (5.13) readily gives:

$$\int_0^1 f(t)dt - \frac{1}{432}\left(27f(0) + 125f\left(\frac{1}{5}\right) + 128f\left(\frac{1}{2}\right) + 125f\left(\frac{4}{5}\right) + 27f(1)\right)$$
$$= -\frac{1}{5040000}f^{(6)}(\xi).$$

For functions with the degree of smoothness lower than 6, this formula gives the following error estimates (cf. Theorem 5.3):

$$\left|\int_0^1 f(t)dt - \frac{1}{432}\left(27f(0) + 125f\left(\frac{1}{5}\right) + 128f\left(\frac{1}{2}\right) + 125f\left(\frac{4}{5}\right) + 27f(1)\right)\right|$$
$$\leq C(m,q,1/5) \cdot \|f^{(m)}\|_p, \quad m = 1,\ldots,6$$

where

$$C(1,1,1/5) \approx 6.78194 \cdot 10^{-2},$$
$$C(1,\infty,1/5) = |G_1^{Q5}(1/5,4/5)| \approx 0.151852,$$
$$C(2,1,1/5) \approx 2.58029 \cdot 10^{-3},$$
$$C(2,\infty,1/5) = \frac{1}{2}|G_2^{Q5}(1/5,1/5)| \approx 7.5 \cdot 10^{-3},$$
$$C(3,1,1/5) \approx 1.34151 \cdot 10^{-4}, \quad C(3,\infty,1/5) \approx 2.96195 \cdot 10^{-4}$$
$$C(4,1,1/5) \approx 8.68677 \cdot 10^{-6}, \quad C(4,\infty,1/5) \approx 1.71661 \cdot 10^{-5},$$

$$
\begin{aligned}
&C(5,1,1/5) \approx 8.68056\cdot 10^{-7}, \quad C(5,\infty,1/5) \approx 2.17169\cdot 10^{-6},\\
&C(6,1,1/5) \approx 1.98413\cdot 10^{-7},\\
&C(6,\infty,1/5) = \frac{1}{6!}|F_6^{Q5}(1/5,1/2)| \approx 4.34028\cdot 10^{-7}.
\end{aligned}
$$

The following two theorems give the Hermite-Hadamard and Dragomir-Agawal type inequalities for the general 5-point quadrature formulae:

Theorem 5.4 *Let $f:[0,1]\to\mathbb{R}$ be $(2n+4)$-convex for $n\geq 2$. Then for $x\in\left(0,\frac{1}{2}-\frac{\sqrt{15}}{10}\right]$, we have*

$$
\begin{aligned}
&\frac{1}{(2n+2)!}\,|G_{2n+2}^{Q5}(x,0)|\, f^{(2n+2)}\left(\frac{1}{2}\right) \qquad (5.17)\\
&\quad\leq (-1)^n\left(\int_0^1 f(t)dt - Q\left(0,x,\frac{1}{2},1-x,1\right)+T_{2n}^{Q5}(x)\right)\\
&\quad\leq \frac{1}{(2n+2)!}\,|G_{2n+2}^{Q5}(x,0)|\,\frac{f^{(2n+2)}(0)+f^{(2n+2)}(1)}{2},
\end{aligned}
$$

while for $x\in\left[\frac{1}{5},\frac{1}{2}\right)$ we have

$$
\begin{aligned}
&\frac{1}{(2n+2)!}\,|G_{2n+2}^{Q5}(x,0)|\, f^{(2n+2)}\left(\frac{1}{2}\right) \qquad (5.18)\\
&\quad\leq (-1)^{n+1}\left(\int_0^1 f(t)dt - Q\left(0,x,\frac{1}{2},1-x,1\right)+T_{2n}^{Q5}(x)\right)\\
&\quad\leq \frac{1}{(2n+2)!}\,|G_{2n+2}^{Q5}(x,0)|\,\frac{f^{(2n+2)}(0)+f^{(2n+2)}(1)}{2},
\end{aligned}
$$

with $G_{2n+2}^{Q5}(x,0)$ as in (5.10) and $w_1(x),w_2(x)$ and $w_3(x)$ as in (5.6).

If f is $(2n+4)$-concave, the inequalities are reversed.

Proof. Analogous to the proof of Theorem 2.8. □

Theorem 5.5 *Let $x\in\left(0,\frac{1}{2}-\frac{\sqrt{15}}{10}\right]\cup\left[\frac{1}{5},\frac{1}{2}\right)$ and $f:[0,1]\to\mathbb{R}$ be m-times differentiable for $m\geq 5$. If $|f^{(m)}|^q$ is convex for some $q\geq 1$, then*

$$
\begin{aligned}
&\left|\int_0^1 f(t)dt - Q\left(0,x,\frac{1}{2},1-x,1\right)+T_{2n}^{Q5}(x)\right|\\
&\quad\leq L_{Q5}(m,x)\cdot\left(\frac{|f^{(m)}(0)|^q+|f^{(m)}(1)|^q}{2}\right)^{1/q} \qquad (5.19)
\end{aligned}
$$

while if $|f^{(m)}|$ is concave, then

$$\left|\int_0^1 f(t)dt - Q\left(0,x,\frac{1}{2},1-x,1\right) + T_{2n}^{Q5}(x)\right| \le L_{Q5}(m,x)\cdot\left|f^{(m)}\left(\frac{1}{2}\right)\right|, \tag{5.20}$$

where

$$\textit{for } m=2n+1 \quad L_{Q5}(2n+1,x) = \frac{2}{(2n+2)!}\,|F_{2n+2}^{Q5}(x,1/2)|$$
$$\textit{and for } m=2n+2 \quad L_{Q5}(2n+2,x) = \frac{1}{(2n+2)!}\,|G_{2n+2}^{Q5}(x,0)|$$

with $G_{2n+2}^{Q5}(x,0)$ *as in (5.10) and* $F_{2n+2}^{Q5}(x,1/2)$ *as in (5.12).*

5.1.1 Boole's formula

The special case which is considered in this subsection is the case when $x = 1/4$. These results are published in [99].

For $x = 1/4$, formula (5.13) becomes the classical Boole's formula:

$$\int_0^1 f(t)dt - \frac{1}{90}\left[7f(0)+32f\left(\frac{1}{4}\right)+12f\left(\frac{1}{2}\right)+32f\left(\frac{3}{4}\right)+7f(1)\right]$$
$$= -\frac{1}{1935360}\cdot f^{(6)}(\xi), \quad \xi\in[0,1] \tag{5.21}$$

We now have:

$$Q_B = \frac{1}{90}\left[7f(0)+32f\left(\frac{1}{4}\right)+12f\left(\frac{1}{2}\right)+32f\left(\frac{3}{4}\right)+7f(1)\right]$$
$$T_{2n}^B = T_{2n}^{Q5}\left(\frac{1}{4}\right) = \sum_{k=3}^{n}\frac{1}{(2k)!}\,G_{2k}^B(0)\,[f^{(2k-1)}(1)-f^{(2k-1)}(0)]$$
$$G_k^B(t) = G_k^{Q5}\left(\frac{1}{4},t\right) = \frac{14}{90}B_k^*(1-t)+\frac{12}{90}B_k^*\left(\frac{1}{2}-t\right)$$
$$+\frac{32}{90}\left[B_k^*\left(\frac{1}{4}-t\right)+B_k^*\left(\frac{3}{4}-t\right)\right],$$
$$F_k^B(t) = F^{Q5}\left(\frac{1}{4},t\right) = G_k^B(t)-G_k^B(0).$$

Note that

$$G_{2n+2}^B(0) = G_{2n+2}^{Q5}(1/4,0) = \frac{1}{45}(1-5\cdot 4^{-n}+4\cdot 16^{-n})B_{2n+2},$$
$$F_{2n+2}^B(1/2) = F_{2n+2}^{Q5}(1/4,1/2) = -\frac{1}{45}(2-2^{-2n-1})B_{2n+2}.$$

Finally, let us see what the estimates of error for functions with a low degree of smoothness for this type of formula are:

$$\left|\int_0^1 f(t)dt - Q_{CB}\right| \le C_B(m,q)\cdot\|f^{(m)}\|_p,$$

where

$$C_B(1,1)=\frac{239}{3240},\quad C_B(1,\infty)=\frac{11}{60},$$
$$C_B(2,1)=\frac{1018}{273375},\quad C_B(2,\infty)=\frac{17}{1440}.$$

For $x=1/4$ and $n=2$, (5.18) gives Hermite-Hadamard type estimate for Boole's formula:

$$\begin{aligned}
&\frac{1}{1935360}f^{(6)}\left(\frac{1}{2}\right)\\
&\leq -\left(\int_0^1 f(t)dt-\frac{1}{90}\left[7f(0)+32f\left(\frac{1}{4}\right)+12f\left(\frac{1}{2}\right)+32f\left(\frac{3}{4}\right)+7f(1)\right]\right)\\
&\leq \frac{1}{1935360}\,\frac{f^{(6)}(0)+f^{(6)}(1)}{2}
\end{aligned}$$

The Dragomir-Agarwal estimates for Boole's formula are:

$$L_{Q5}\left(5,\frac{1}{4}\right)=\frac{1}{345600},\qquad L_{Q5}\left(6,\frac{1}{4}\right)=\frac{1}{1935360}.$$

5.1.2 Hermite-Hadamard-type inequality for the 5-point quadrature formulae

The main result of this section provides Hermite-Hadamard-type inequality for the 5-point quadrature formulae.

Theorem 5.6 *Let $f:[0,1]\to\mathbb{R}$ be 6-convex and such that $f^{(6)}$ is continuous on $[0,1]$. Then, for $x\in(0,\frac{1}{2}-\frac{\sqrt{15}}{10}]$ and $y\in\left[\frac{1}{5},\frac{1}{2}\right)$*

$$Q\left(0,x,\frac{1}{2},1-x,1\right)\ \leq \int_0^1 f(t)dt\ \leq\ Q\left(0,y,\frac{1}{2},1-y,1\right), \tag{5.22}$$

where $Q\left(0,x,\frac{1}{2},1-x,1\right)$ is defined in (5.2). If f is 6-concave, the inequalities are reversed.

Proof. Analogous to the proof of Theorem 3.6. □

The following corollaries give comparison between the Gauss 3-point and the Lobatto 4-point, and the Gauss 3-point and Boole's rule.

Corollary 5.1 *Let $f:[0,1]\to\mathbb{R}$ be 6-convex and such that $f^{(6)}$ is continuous on $[0,1]$. Then*

$$\begin{aligned}
&\frac{1}{18}\left(5f\left(\frac{5-\sqrt{15}}{10}\right)+8f\left(\frac{1}{2}\right)+5f\left(\frac{5+\sqrt{15}}{10}\right)\right)\\
&\leq \int_0^1 f(t)dt\\
&\leq \frac{1}{12}\left(f(0)+5f\left(\frac{5-\sqrt{5}}{10}\right)+5f\left(\frac{5+\sqrt{5}}{10}\right)+f(1)\right).
\end{aligned}$$

If f is 6-concave, the inequalities are reversed.

Proof. Follows from (5.22) for $x = 1/2 - \sqrt{15}/10 \Leftrightarrow 10x^2 - 10x + 1 = 0$ and $y = 1/2 - \sqrt{5}/10 \Leftrightarrow 5x^2 - 5x + 1 = 0$. □

Corollary 5.2 *Let $f : [0,1] \to \mathbb{R}$ be 6-convex and such that $f^{(6)}$ is continuous on $[0,1]$. Then*

$$\begin{aligned}
&\frac{1}{18}\left(5f\left(\frac{5-\sqrt{15}}{10}\right)+8f\left(\frac{1}{2}\right)+5f\left(\frac{5+\sqrt{15}}{10}\right)\right)\\
&\leq \int_0^1 f(t)dt\\
&\leq \frac{1}{90}\left(7f(0)+32f\left(\frac{1}{4}\right)+12f\left(\frac{1}{2}\right)+32f\left(\frac{3}{4}\right)+7f(1)\right).
\end{aligned}$$

If f is 6-concave, the inequalities are reversed.

Proof. Follows from (5.22) for $x = 1/2 - \sqrt{15}/10 \Leftrightarrow 10x^2 - 10x + 1 = 0$ and $y = 1/4$. □

5.2 Closed corrected 5-point quadrature formulae

The idea is now to derive a family of closed 5-point quadrature formulae with a degree of exactness higher than what the family from the previous section had. Observe formula (5.1) again. If we choose weights such that

$$G_4(x,0) = G_6(x,0) = 0,$$

it will increase the degree of exactness but it will also include values of the first derivative of the integrand in the quadrature - i.e. ”corrected” quadrature formulae will be produced.

The weights thus produced are:

$$\begin{aligned}
w_1(x) &= \frac{98x^4 - 196x^3 + 102x^2 - 4x - 1}{420x^2(1-x)^2}, \qquad (5.23)\\
w_2(x) &= \frac{1}{420x^2(1-x)^2(1-2x)^2}, \qquad w_3(x) = \frac{16(14x^2 - 14x + 3)}{105(2x-1)^2}
\end{aligned}$$

All related results from the previous section can now be obtained completely analogously, just having in mind that we are now working with the weights from (5.23). These results are therefore not going to be stated explicitly, except for the key lemma. To emphasize the weights we are using, denote the notions from (5.3)-(5.5) by $T_{2n}^{CQ5}(x)$, $G_{2n}^{CQ5}(x,t)$ and $F_{2n}^{CQ5}(x,t)$, and the quadrature itself by $Q_C\left(0,x,\frac{1}{2},1-x,1\right)$. Note that

$$\begin{aligned} T_{2n}^{CQ5}(x) &= \sum_{k=1}^{n} \frac{1}{(2k)!}\, G_{2k}^{CQ5}(x,0)\, [f^{(2k-1)}(1)-f^{(2k-1)}(0)] \\ &= \frac{7x^2-7x+1}{420x(x-1)}[f'(1)-f'(0)] \\ &\quad + \sum_{k=4}^{n} \frac{1}{(2k)!}\, G_{2k}^{CQ5}(x,0)\, [f^{(2k-1)}(1)-f^{(2k-1)}(0)]. \end{aligned}$$

Lemma 5.2 *For $x \in (0,\frac{7-\sqrt{21}}{14}] \cup [\frac{3-\sqrt{2}}{7},\frac{1}{2})$ and $k \geq 3$, $G_{2k+1}^{CQ5}(x,t)$ has no zeros in the variable t on the interval $(0,1/2)$. The sign of this function is determined by:*

$$\begin{aligned} (-1)^{k+1} G_{2k+1}^{CQ5}(x,t) > 0 \quad &\textit{for } x \in \left(0,\ 1/2-\sqrt{21}/14\right], \\ (-1)^{k} G_{2k+1}^{CQ5}(x,t) > 0 \quad &\textit{for } x \in \left[3/7-\sqrt{2}/7,\ 1/2\right). \end{aligned}$$

Proof. The concept is the same as in the proof of Lemma 5.1: first show that for $x \in (\frac{7-\sqrt{21}}{14},\frac{3-\sqrt{2}}{7})$, G_7^{CQ5} has at least one zero on $(0,1/2)$. It is easy to check that

$$x \in \left(\frac{7-\sqrt{21}}{14},\ \frac{3-\sqrt{2}}{7}\right) \quad \Leftrightarrow \quad \frac{\partial^5 G_7^{CQ5}}{\partial t^5}(x,0) < 0 \quad \& \quad \frac{\partial G_7^{CQ5}}{\partial t}\left(x,\frac{1}{2}\right) < 0.$$

From the assumption $\frac{\partial^5 G_7^{CQ5}}{\partial t^5}(x,0) < 0$, it follows that $\frac{\partial^5 G_7^{CQ5}}{\partial t^5} < 0$ on some neighborhood of zero. Recall $\frac{\partial^k G_7^{CQ5}}{\partial t^k}(x,0) = 0$ for $0 \leq k \leq 4$. Knowing the sign of the $(k+1)$-th derivative, we can conclude on the monotonicity of the k-th derivative, and from its behavior in zero we can conclude what its sign is. In short, all derivatives (for $0 \leq k \leq 4$) are strictly decreasing and have negative sign. Therefore, $G_7^{CQ5}(x,t) < 0$ on some neighborhood of zero. Similarly, from $\frac{\partial G_7^{CQ5}}{\partial t}\left(x,\frac{1}{2}\right) < 0$, we conclude $G_7^{CQ5}(x,t) > 0$ on some neighborhood of $1/2$, and it is now obvious that for $x \in \left(\frac{7-\sqrt{21}}{14},\ \frac{3-\sqrt{2}}{7}\right)$, G_7^{CQ5} has at least one zero.

It is left to prove that G_7^{CQ5} is monotonous (more precisely: strictly decreasing) in x and to check its behavior at the end points in order to determine its sign. Let $0 \leq t \leq x < 1/2$. Then

$$\begin{aligned} \frac{\partial G_7^{CQ5}(x,t)}{\partial x} = 0 \ \Leftrightarrow\ x_1 = \frac{1}{2},\ x_2 &= \frac{1}{2} - \frac{\sqrt{3(4t^2-8t+3)}}{2(2t-3)}, \\ x_3 &= \frac{1}{2} + \frac{\sqrt{3(4t^2-8t+3)}}{2(2t-3)}. \end{aligned}$$

Obviously, $x_2 > 1/2$, and it is not hard to see $x_3 < t$. So, $\frac{\partial G_7^{CQ5}(x,t)}{\partial x}$ has constant sign and

$G_7^{CQ5}(x,t)$ is therefore strictly decreasing in x. Next, let $0 < x \leq t \leq 1/2$. Then

$$\frac{\partial G_7^{CQ5}(x,t)}{\partial x} = \frac{1-2t}{30(2x^2-3x+1)^3} \cdot \mu(x,t),$$

where $\mu(x,t) = 16t^5(x-1)^3 + t^4(-16x^3+48x^2-60x+25) - 2t^3(4x^3-12x^2+5) + t^2(-4x^3-18x^2+15x) + t(13x^3-9x^2) + (2-3x)x^3$. Further,

$$\frac{\partial \mu(x,t)}{\partial t} = (1-2t)\cdot \mu_1(x,t),$$
$$\frac{\partial \mu_1(x,t)}{\partial t} = 0 \Leftrightarrow t_1 = \frac{1}{2},\ t_2 = \frac{3x^3-9x^2+5x}{10(1-x)^3}.$$

But, $t_2 < x$, so $\mu_1(x,t)$ is strictly decreasing in t on this interval, and since $\mu_1(x,x) < 0$, we conclude $\mu_1(x,t) < 0$. From here it follows that $\mu(x,t)$ is strictly decreasing in t and since $\mu(x,x) < 0$, we see that G_7^{CQ5} is strictly decreasing on this interval as well. Finally, it is not difficult to verify that $G_7^{CQ5}(\frac{7-\sqrt{21}}{14},t) > 0$ and $G_7^{CQ5}(\frac{3-\sqrt{2}}{7},t) < 0$ for $t \in (0,1/2)$, so we conclude $G_7^{CQ5}(x,t) > 0$ for $x \in \left(0, \frac{7-\sqrt{21}}{14}\right]$ and $G_7^{CQ5}(x,t) < 0$ for $x \in \left[\frac{3-\sqrt{2}}{7}, \frac{1}{2}\right)$. This completes the proof of the statement for $k = 3$. The rest of the proof is analogous to the proof of Lemma 3.1 □

Applying the analogue of (5.9) for $n = 3$ and $x \in (0, \frac{7-\sqrt{21}}{14}] \cup [\frac{3-\sqrt{2}}{7}, \frac{1}{2})$ the following formula is produced:

$$\int_0^1 f(t)dt - Q_C\left(0,x,\frac{1}{2},1-x,1\right) + \frac{7x^2-7x+1}{420x(x-1)}[f'(1)-f'(0)]$$
$$= \frac{1}{203212800}(-6x^2+6x-1)\cdot f^{(8)}(\xi), \quad \xi \in [0,1] \tag{5.24}$$

This formula has several interesting special cases that are worth studying: for $x = 1/4$ it gives corrected Boole's formula; when $w_1(x) = 0$ it becomes the corrected Gauss 3-point formula while when $w_3(x) = 0$, it produces the corrected Lobatto's 4-point formula. As an especially interesting case, the classical Lobatto 5-point formula is obtained (upon taking $G_2^{CQ5}(x,0) = 0$). Once again, as it could be expected, $w_2(x) \neq 0$ for every x. If x such that $w_2(x) = 0$ existed, it would generate a corrected closed 3-point quadrature formula with a degree of exactness equal to 7, and such a formula does not exist. The only 3-point quadrature formula that has a degree of exactness that high is the corrected Gauss 3-point formula (and that is an open quadrature formula). We will investigate these formulae further in the following subsections.

Remark 5.2 Here, only $x \in (0,1/2)$ were considered, but results for $x = 0$ and $x = 1/2$ can be obtained by considering the limit processes. Namely,

$$\lim_{x\to 0}\left(Q_C\left(0,x,\frac{1}{2},1-x,1\right) + \frac{7x^2-7x+1}{420x(1-x)}[f'(1)-f'(0)]\right)$$

$$= \frac{1}{70}[19f(0)+32f\left(\frac{1}{2}\right)+19f(1)] - \frac{1}{35}[f'(1)-f'(0)] + \frac{1}{840}[f''(0)+f''(1)]$$

$$\lim_{x\to 0} G_k^{CQ5}(x,t) = \frac{19}{35}B_k^*(1-t) + \frac{16}{35}B_k^*\left(\frac{1}{2}-t\right) + \frac{k(k-1)}{420}B_{k-2}^*(1-t)$$

Consequently, from (5.24) it follows:

$$\int_0^1 f(t)dt - \frac{1}{70}[19f(0)+32f\left(\frac{1}{2}\right)+19f(1)] + \frac{1}{35}[f'(1)-f'(0)] - \frac{1}{840}[f''(0)+f''(1)] = -\frac{1}{203212800}f^{(8)}(\xi).$$

Furthermore,

$$\lim_{x\to 1/2}\left(Q_C\left(0,x,\frac{1}{2},1-x,1\right) + \frac{7x^2-7x+1}{420x(1-x)}[f'(1)-f'(0)]\right)$$
$$= \frac{1}{70}[11f(0)+48f\left(\frac{1}{2}\right)+11f(1)] - \frac{1}{140}[f'(1)-f'(0)] + \frac{1}{105}f''\left(\frac{1}{2}\right)$$

$$\lim_{x\to 1/2} G_k^{CQ5}(x,t) = \frac{11}{35}B_k^*(1-t) + \frac{24}{35}B_k^*\left(\frac{1}{2}-t\right) + \frac{k(k-1)}{105}B_{k-2}^*(1-t)$$

and then from (5.24):

$$\int_0^1 f(t)dt - \frac{1}{70}[11f(0)+48f\left(\frac{1}{2}\right)+11f(1)] + \frac{1}{140}[f'(1)-f'(0)] - \frac{1}{105}f''\left(\frac{1}{2}\right) = \frac{1}{406425600}f^{(8)}(\xi).$$

Next, for $x \in (0, \frac{7-\sqrt{21}}{14}] \cup [\frac{3-\sqrt{2}}{7}, \frac{1}{2})$ and $n \geq 3$, using Lemma 5.2, from the analogue of Theorem 5.3 we get:

$$K_{CQ5}^*(2n+2,1,x) = \frac{1}{(2n+2)!}\left|G_{2n+2}^{CQ5}(x,0)\right|,$$
$$K_{CQ5}^*(2n+2,\infty,x) = \frac{1}{2}K_{CQ5}(2n+1,1,x) = \frac{1}{(2n+2)!}\left|F_{2n+2}^{CQ5}\left(x,\frac{1}{2}\right)\right|,$$

where $G_{2n+2}^{CQ5}(x,0)$ and $F_{2n+2}^{CQ5}(x,1/2)$ are similar as in (5.10) and (5.12) but with weights $w_1(x)$, $w_2(x)$, $w_3(x)$ as in (5.23). For $n=3$ and $p=\infty$ we obtain:

$$\left|\int_0^1 f(t)dt - Q_C\left(0,x,\frac{1}{2},1-x,1\right) + \frac{7x^2-7x+1}{420x(x-1)}[f'(1)-f'(0)]\right|$$
$$\leq \frac{|128x^3-215x^2+110x-15|}{541900800(1-x)^2}\cdot\|f^{(7)}\|_\infty$$
$$\left|\int_0^1 f(t)dt - Q_C\left(0,x,\frac{1}{2},1-x,1\right) + \frac{7x^2-7x+1}{420x(x-1)}[f'(1)-f'(0)]\right|$$
$$\leq \frac{|6x^2-6x+1|}{203212800}\cdot\|f^{(8)}\|_\infty$$

In order to find which admissible x gives the least estimate of error for this kind of formulae, the functions on the right-hand sides have to be minimized. It is elementary to establish that both functions are decreasing on $(0, \frac{7-\sqrt{21}}{14}]$, increasing on $[\frac{3-\sqrt{2}}{7}, \frac{1}{2})$ and reach their minimal value at $x_{\text{op}} := \frac{3-\sqrt{2}}{7}$. Thus, the nodes of the quadrature formula that gives the least estimate of error are

$$0,\ x_{\text{op}} \approx 0.226541,\ 1/2,\ 1 - x_{\text{op}} \approx 0.773459,\ 1$$

and the weights are

$$w_1(x_{\text{op}}) \approx 0.10143,\ \ w_2(x_{\text{op}}) \approx 0.259261,\ \ w_3(x_{\text{op}}) \approx 0.278617.$$

The formula itself is

$$\int_0^1 f(t)dt - \left(w_1(x_{\text{op}})[f(0)+f(1)] + w_2(x_{\text{op}})[f(x_{\text{op}}) + f(1-x_{\text{op}})] + w_3(x_{\text{op}}) f\left(\frac{1}{2}\right)\right)$$
$$+3.07832 \cdot 10^{-3} \cdot [f'(1) - f'(0)] = 2.52547 \cdot 10^{-10} \cdot f^{(8)}(\xi)$$

Sharp error estimates for functions with low degree of smoothness are as follows:

$$\left| \int_0^1 f(t)dt - \left(w_1(x_{\text{op}})[f(0)+f(1)] + w_2(x_{\text{op}})[f(x_{\text{op}}) + f(1-x_{\text{op}})]\right.\right.$$
$$\left.\left. + w_3(x_{\text{op}}) f\left(\frac{1}{2}\right)\right)\right| \le C(m,q,x_{\text{op}}) \cdot \|f^{(m)}\|_p, \quad m = 1,2$$

where

$$\begin{aligned} C(1,1,x_{\text{op}}) &\approx 6.3344 \cdot 10^{-2}, & C(1,\infty,x_{\text{op}}) &= |G_1^{CQ5}(x_{\text{op}},1/2)| \approx 0.139309, \\ C(2,1,x_{\text{op}}) &\approx 3.49012 \cdot 10^{-3}, & C(2,\infty,x_{\text{op}}) &\approx 6.31576 \cdot 10^{-3} \end{aligned}$$

and

$$\left| \int_0^1 f(t)dt - (w_1(x_{op})[f(0)+f(1)] + w_2(x_{op})[f(x_{op}) + f(1-x_{op})]\right.$$
$$\left. + w_3(x_{op}) f\left(\frac{1}{2}\right)\right) + 3.07832 \cdot 10^{-3} \cdot [f'(1) - f'(0)]\bigg| \le C(m,q,x_{\text{op}}) \cdot \|f^{(m)}\|_p,$$
$$m = 2, \ldots, 8$$

where

$$\begin{aligned} C(2,1,x_{\text{op}}) &\approx 2.09749 \cdot 10^{-3}, \\ C(2,\infty,x_{\text{op}}) &= \frac{1}{2}|G_2^{CQ5}(x_{op},1/2)| \approx 6.46602 \cdot 10^{-3}, \\ C(3,1,x_{\text{op}}) &\approx 9.0368 \cdot 10^{-5}, & C(3,\infty,x_{\text{op}}) &\approx 1.74082 \cdot 10^{-4} \\ C(4,1,x_{\text{op}}) &\approx 3.99829 \cdot 10^{-6}, & C(4,\infty,x_{\text{op}}) &\approx 7.90283 \cdot 10^{-6}, \\ C(5,1,x_{\text{op}}) &\approx 1.95347 \cdot 10^{-7}, & C(5,\infty,x_{\text{op}}) &\approx 4.63008 \cdot 10^{-7}, \\ C(6,1,x_{\text{op}}) &\approx 1.21228 \cdot 10^{-8}, & C(6,\infty,x_{\text{op}}) &\approx 2.52003 \cdot 10^{-8}, \\ C(7,1,x_{\text{op}}) &\approx 1.15271 \cdot 10^{-9}, & C(7,\infty,x_{\text{op}}) &\approx 3.03071 \cdot 10^{-9}, \\ C(8,1,x_{\text{op}}) &\approx 2.52547 \cdot 10^{-10}, & C(8,\infty,x_{\text{op}}) &\approx 5.76355 \cdot 10^{-10}. \end{aligned}$$

5.2.1 Lobatto 5-point formula

To obtain from (5.1) the node x which generates the quadrature formula with the maximum degree of exactness, clearly conditions

$$G_2(x,0) = G_4(x,0) = G_6(x,0) = 0$$

have to be imposed. As a matter of fact, the same node will be produced if the condition:

$$G_2^{CQ5}(x,0) = 0 \tag{5.25}$$

is imposed (thus, here the weights are as in (5.23)). From (5.25) we get:

$$x_0 = \frac{7-\sqrt{21}}{14}, \;\; w_1(x_0) = \frac{1}{20}, \;\; w_2(x_0) = \frac{49}{180}, \;\; w_3(x_0) = \frac{16}{45}.$$

With this node and these weights, (5.24) becomes the classical Lobatto 5-point formula stated on $[0,1]$. Once again, we switch to $[-1,1]$.

Analogues of the formulae (5.7), (5.8) and (5.1) in this case are:

$$\int_{-1}^{1} f(t)dt - Q_{L5} + T_{2n}^{L5} = \frac{2^{2n-1}}{(2n)!}\int_{-1}^{1} G_{2n}^{L5}(t)df^{(2n-1)}(t), \tag{5.26}$$

$$\int_{-1}^{1} f(t)dt - Q_{L5} + T_{2n}^{L5} = \frac{2^{2n}}{(2n+1)!}\int_{-1}^{1} G_{2n+1}^{L5}(t)df^{(2n)}(t), \tag{5.27}$$

$$\int_{-1}^{1} f(t)dt - Q_{L5} + T_{2n}^{L5} = \frac{2^{2n+1}}{(2n+2)!}\int_{-1}^{1} F_{2n+2}^{L5}(t)df^{(2n+1)}(t), \tag{5.28}$$

where

$$Q_{L5} = \frac{1}{90}\left[9f(-1) + 49f\left(-\sqrt{\frac{3}{7}}\right) + 64f(0) + 49f\left(\sqrt{\frac{3}{7}}\right) + 9f(1)\right],$$

$$T_{2n}^{L5} = \sum_{k=4}^{n} \frac{2^{2k-1}}{(2k)!}\, G_{2k}^{L5}(-1)\,[f^{(2k-1)}(1) - f^{(2k-1)}(-1)],$$

$$G_k^{L5}(t) = \frac{1}{5}B_k^*\left(\frac{1-t}{2}\right) + \frac{32}{45}B_k^*\left(1-\frac{t}{2}\right) + \frac{49}{90}\left[B_k^*\left(\frac{\sqrt{21}}{14} - \frac{t}{2}\right) + B_k^*\left(-\frac{\sqrt{21}}{14} - \frac{t}{2}\right)\right],$$

$$F_k^{L5}(t) = G_k^{L5}(t) - G_k^{L5}(-1).$$

Using the analogue of Theorem 5.2, for $n \geq 3$ the remainder $R_{2n+2}^{L5}(f)$ in (5.28) can be written as:

$$R_{2n+2}^{L5}(f) = -\frac{2^{2n+2}}{(2n+2)!}\, G_{2n+2}^{L5}(-1)\cdot f^{(2n+2)}(\xi), \quad \xi \in [-1,1], \tag{5.29}$$

$$R_{2n+2}^{L5}(f) = \theta\cdot\frac{2^{2n+1}}{(2n+2)!}\, F_{2n+2}^{L5}(0)\left[f^{(2n+1)}(1) - f^{(2n+1)}(-1)\right], \quad \theta \in [0,1] \tag{5.30}$$

where

$$G_{2n+2}^{L5}(-1) = \frac{1}{45}\left[49B_{2n+2}\left(\frac{1}{2}-\frac{\sqrt{21}}{14}\right)+(4^{2-n}-23)B_{2n+2}\right], \tag{5.31}$$

$$\begin{aligned} F_{2n+2}^{L5}(0) = \frac{49}{45}\left[B_{2n+2}\left(\frac{\sqrt{21}}{14}\right)-B_{2n+2}\left(\frac{1}{2}-\frac{\sqrt{21}}{14}\right)\right] \\ +\frac{23}{45}\left(2-2^{-2n-1}\right)B_{2n+2}. \end{aligned} \tag{5.32}$$

Finally, from (5.28) and (5.29) for $n = 3$ the Lobatto 5-point formula is produced:

$$\int_{-1}^{1} f(t)dt - Q_{L5} = -\frac{1}{2778300}\, f^{(8)}(\xi). \tag{5.33}$$

Applying Hölder's inequality, sharp estimates for the formulae (5.26)-(5.28) can easily be obtained (cf.Theorem 5.3); especially, for $n \geq 3$ and $p = \infty$, i.e. $p = 1$, we have

$$\begin{aligned} K_{L5}^{*}(2n+2,1) &= \frac{2^{2n+2}}{(2n+2)!}\left|G_{2n+2}^{L5}(-1)\right|, \\ K_{L5}^{*}(2n+2,\infty) &= \frac{1}{2}\,K_{L5}(2n+1,1) = \frac{2^{2n+1}}{(2n+2)!}\left|F_{2n+2}^{L5}(0)\right|, \end{aligned}$$

where $G_{2n+2}^{L5}(-1)$ and $F_{2n+2}^{L5}(0)$ are as in (5.31) and (5.32). Further,

$$\left|\int_{-1}^{1} f(t)dt - Q_{L5}\right| \leq C_{L5}(m,q)\cdot\|f^{(m)}\|_p, \quad m = 1,\ldots,8$$

where

$$\begin{aligned} C_{L5}(1,1) &= \frac{10943-2034\sqrt{21}}{5670} \approx 0.286074, \\ C_{L5}(1,\infty) &= \left|G_1^{L5}(0)\right| = \frac{16}{45} \approx 0.355556, \\ C_{L5}(2,1) &\approx 0.0234146, \\ C_{L5}(2,\infty) &= |G_2^{L5}(0)| = \frac{36-7\sqrt{21}}{90} \approx 0.0435774, \\ C_{L5}(3,1) &\approx 2.58631\cdot 10^{-3}, \qquad C_{L5}(3,\infty) \approx 3.10461\cdot 10^{-3}, \\ C_{L5}(4,1) &\approx 3.05134\cdot 10^{-4}, \\ C_{L5}(4,\infty) &= |G_4^{L5}(0)|/3 = \frac{2\sqrt{21}-9}{360} \approx 4.58754\cdot 10^{-4}, \\ C_{L5}(5,1) &\approx 4.0538\cdot 10^{-5}, \qquad C_{L5}(5,\infty) \approx 5.46895\cdot 10^{-5}, \\ C_{L5}(6,1) &\approx 6.21866\cdot 10^{-6}, \\ C_{L5}(6,\infty) &= 2|G_6^{L5}(0)|/45 = \frac{14-3\sqrt{21}}{25200} \approx 1.00108\cdot 10^{-5}, \end{aligned}$$

$$
\begin{aligned}
C_{L5}(7,1) &= 2|F_8^{L5}(0)|/315 = \frac{12\sqrt{21}-49}{4939200} \approx 1.21293 \cdot 10^{-6},\\
C_{L5}(7,\infty) &\approx 1.55466 \cdot 10^{-6},\\
C_{L5}(8,1) &= 2|G_8^{L5}(-1)|/315 = \frac{1}{2778300} \approx 3.59932 \cdot 10^{-7},\\
C_{L5}(8,\infty) &= |F_8^{L5}(0)|/315 = \frac{12\sqrt{21}-49}{9878400} \approx 6.06465 \cdot 10^{-7}.
\end{aligned}
$$

The Hermite-Hadamard type estimate for the Lobatto 5-point formula is:

$$
\begin{aligned}
&\frac{1}{1422489600} f^{(8)}\left(\frac{1}{2}\right)\\
&\le -\int_0^1 f(t)dt\\
&+\frac{1}{180}\left[9f(0)+49f\left(\frac{7-\sqrt{21}}{14}\right)+64f\left(\frac{1}{2}\right)+49f\left(\frac{7+\sqrt{21}}{14}\right)+9f(1)\right]\\
&\le \frac{1}{1422489600}\,\frac{f^{(8)}(0)+f^{(8)}(1)}{2}
\end{aligned}
$$

and the Dragomir-Agarwal type estimates for this formula are:

$$
C_{CQ5}\left(7,\frac{7-\sqrt{21}}{14}\right) = \frac{12\sqrt{21}-49}{1264435200}, \qquad C_{CQ5}\left(8,\frac{7-\sqrt{21}}{14}\right) = \frac{1}{1422489600}.
$$

5.2.2 Corrected Lobatto 4-point formula

What happens if the condition $w_3(x)=0$ is imposed, where $w_3(x)$ is as in (5.23)? The node $x_0 = \frac{7-\sqrt{7}}{14}$ is produced. This node generates from (5.24) the corrected closed 4-point quadrature formula with the maximum degree of exactness for this type of formula (i.e. with this number of nodes and the necessity of the end points being included), so we call it the corrected Lobatto 4-point formula. The results are transformed to the interval $[-1,1]$ again. Analogues of the formulae (5.7), (5.8) and (5.1) are in this case:

$$
\int_{-1}^{1} f(t)dt - Q_{CL4} + T_{2n}^{CL4} = \frac{2^{2n-1}}{(2n)!}\int_{-1}^{1} G_{2n}^{CL4}(t)df^{(2n-1)}(t), \tag{5.34}
$$

$$
\int_{-1}^{1} f(t)dt - Q_{CL4} + T_{2n}^{CL4} = \frac{2^{2n}}{(2n+1)!}\int_{-1}^{1} G_{2n+1}^{CL4}(t)df^{(2n)}(t), \tag{5.35}
$$

$$
\int_{-1}^{1} f(t)dt - Q_{CL4} + T_{2n}^{CL4} = \frac{2^{2n+1}}{(2n+2)!}\int_{-1}^{1} F_{2n+2}^{CL4}(t)df^{(2n+1)}(t), \tag{5.36}
$$

where

$$
Q_{CL4} = \frac{1}{135}\left[37f(-1)+98f\left(-\frac{\sqrt{7}}{7}\right)+98f\left(\frac{\sqrt{7}}{7}\right)+37f(1)\right],
$$

$$T_{2n}^{CL4} = \sum_{k=2}^{2n} \frac{2^{k-1}}{k!} G_k^{CL4}(-1)\,[f^{(k-1)}(1) - f^{(k-1)}(-1)]$$
$$= \frac{1}{45}[f'(1) - f'(-1)] + \sum_{k=4}^{n} \frac{2^{2k-1}}{(2k)!} G_{2k}^{CL4}(-1)\,[f^{(2k-1)}(1) - f^{(2k-1)}(-1)],$$
$$G_k^{CL4}(t) = \frac{74}{135} B_k^*\left(\frac{1-t}{2}\right) + \frac{98}{135}\left[B_k^*\left(\frac{\sqrt{7}}{14} - \frac{t}{2}\right) + B_k^*\left(-\frac{\sqrt{7}}{14} - \frac{t}{2}\right)\right],$$
$$F_k^{CL4}(t) = G_k^{CL4}(t) - G_k^{CL4}(-1).$$

The remainder $R_{2n+2}^{CL4}(f)$ for $n \geq 3$ in (5.36) can be written as:

$$R_{2n+2}^{CL4}(f) = -\frac{2^{2n+2}}{(2n+2)!} G_{2n+2}^{CL4}(-1) \cdot f^{(2n+2)}(\xi), \quad \xi \in [-1,1], \tag{5.37}$$
$$R_{2n+2}^{CL4}(f) = \theta \cdot \frac{2^{2n+1}}{(2n+2)!} F_{2n+2}^{CL4}(0)\left[f^{(2n+1)}(1) - f^{(2n+1)}(-1)\right], \tag{5.38}$$
$$\theta \in [0,1]$$

where

$$G_{2n+2}^{CL4}(-1) = \frac{1}{135}\left[196 B_{2n+2}\left(\frac{1}{2} - \frac{\sqrt{7}}{14}\right) + 74 B_{2n+2}\right], \tag{5.39}$$
$$F_{2n+2}^{CL4}(0) = \frac{196}{135}\left[B_{2n+2}\left(\frac{\sqrt{7}}{14}\right) - B_{2n+2}\left(\frac{1}{2} - \frac{\sqrt{7}}{14}\right)\right] - \frac{74}{135}\left(2 - 2^{-2n-1}\right) B_{2n+2}. \tag{5.40}$$

From (5.36) and (5.37) we get the corrected Lobatto 4-point formula:

$$\int_{-1}^{1} f(t)dt - Q_{CL4} + \frac{1}{45}[f'(1) - f'(-1)] = \frac{1}{1389150} f^{(8)}(\xi). \tag{5.41}$$

Estimates of error for this formula for $n \geq 3$ and $p = \infty$, i.e. $p = 1$, are:

$$K_{CL4}^*(2n+2,1) = \frac{2^{2n+2}}{(2n+2)!}\left|G_{2n+2}^{CL4}(-1)\right|,$$
$$K_{CL4}^*(2n+2,\infty) = \frac{1}{2} K_{CL4}(2n+1,1) = \frac{2^{2n+1}}{(2n+2)!}\left|F_{2n+2}^{CL4}(0)\right|.$$

Further, we have

$$\left|\int_{-1}^{1} f(t)dt - Q_{CL4}\right| \leq C_{CL4}(m,q) \cdot \|f^{(m)}\|_p, \quad m = 1,2$$

where

$$C_{CL4}(1,1) \approx 0.339051, \quad C_{CL4}(1,\infty) = \left|G_1^{CL4}\left(1/\sqrt{7}\right)\right| = 1/\sqrt{7} \approx 0.377964,$$
$$C_{CL4}(2,1) \approx 0.0506718, \quad C_{CL4}(2,\infty) = |F_2^{CL4}(0)| \approx 0.0484483$$

and

$$\left|\int_{-1}^{1} f(t)dt - Q_{CL4} + \frac{1}{45}[f'(1) - f'(-1)]\right| \le C_{CL4}(m,q)\cdot\|f^{(m)}\|_p, \quad m = 2,\dots,8$$

where

$$\begin{aligned}
&C_{CL4}(2,1) \approx 3.03418\cdot 10^{-2},\\
&C_{CL4}(2,\infty) = |G_2^{CL4}\left(1/\sqrt{7}\right)| \approx 4.52025\cdot 10^{-2},\\
&C_{CL4}(3,1) \approx 3.59487\cdot 10^{-3}, \qquad C_{CL4}(3,\infty) \approx 4.00427\cdot 10^{-3},\\
&C_{CL4}(4,1) \approx 4.55146\cdot 10^{-4},\\
&C_{CL4}(4,\infty) = |G_4^{CL4}(0)|/3 \approx 5.66046\cdot 10^{-4},\\
&C_{CL4}(5,1) \approx 6.49236\cdot 10^{-5}, \qquad C_{CL4}(5,\infty) \approx 8.23873\cdot 10^{-5},\\
&C_{CL4}(6,1) \approx 1.08105\cdot 10^{-5},\\
&C_{CL4}(6,\infty) = 2|G_6^{CL4}(0)|/45 \approx 1.57981\cdot 10^{-5},\\
&C_{CL4}(7,1) = 2|F_8^{CL4}(0)|/315 \approx 2.254586\cdot 10^{-6},\\
&C_{CL4}(7,\infty) \approx 2.70262\cdot 10^{-6},\\
&C_{CL4}(8,1) = 2|G_8^{CL4}(-1)|/315 = 1/1389150 \approx 7.19865\cdot 10^{-7},\\
&C_{CL4}(8,\infty) = |F_8^{CL4}(0)|/315 \approx 1.12729\cdot 10^{-6}.
\end{aligned}$$

The Hermite-Hadamard type estimate for the corrected Lobatto 4-point formula is:

$$\begin{aligned}
&\frac{1}{711244800} f^{(8)}\left(\frac{1}{2}\right)\\
&\le \int_0^1 f(t)dt - \frac{1}{270}\left[37f(0) + 98f\left(\frac{7-\sqrt{7}}{14}\right) + 98f\left(\frac{7+\sqrt{7}}{14}\right) + 37f(1)\right]\\
&\qquad + \frac{1}{180}[f'(1) - f'(0)]\\
&\le \frac{1}{711244800}\,\frac{f^{(8)}(0) + f^{(8)}(1)}{2}
\end{aligned}$$

and the Dragomir-Agarwal type estimates for this formula are:

$$C_{CQ5}\left(7, \frac{7-\sqrt{7}}{14}\right) = \frac{343 - 16\sqrt{7}}{34139750400}, \qquad C_{CQ5}\left(8, \frac{7-\sqrt{7}}{14}\right) = \frac{1}{711244800}.$$

5.2.3 Corrected Gauss 3-point formula

Consider the case when $w_1(x) = 0$, with $w_1(x)$ as in (5.23). Then $x = \frac{7-\sqrt{45-2\sqrt{102}}}{14}$. For this choice of x, we obtain from (5.24) the corrected open 3-point quadrature formula with the maximum degree of exactness, so we call it the corrected Gauss 3-point formula. It should be mentioned that this formula was also consider in [47] but not in as much detail. We have:

$$\int_{-1}^{1} f(t)dt - Q_{CG3} + T_{2n}^{CG3} = \frac{2^{2n-1}}{(2n)!}\int_{-1}^{1} G_{2n}^{CG3}(t)df^{(2n-1)}(t), \tag{5.42}$$

$$\int_{-1}^{1} f(t)dt - Q_{CG3} + T_{2n}^{CG3} = \frac{2^{2n}}{(2n+1)!}\int_{-1}^{1} G_{2n+1}^{CG3}(t)df^{(2n)}(t), \tag{5.43}$$

$$\int_{-1}^{1} f(t)dt - Q_{CG3} + T_{2n}^{CG3} = \frac{2^{2n+1}}{(2n+2)!}\int_{-1}^{1} F_{2n+2}^{CG3}(t)df^{(2n+1)}(t), \tag{5.44}$$

where

$$\begin{aligned}
x_0 &= \frac{1}{7}\sqrt{45-2\sqrt{102}},\\
Q_{CG3} &= \frac{1977+16\sqrt{102}}{3465}\left[f(-x_0)+f(x_0)\right] + \frac{2976-32\sqrt{102}}{3465}f(0),\\
T_{2n}^{CG3} &= \sum_{k=2}^{2n}\frac{2^{k-1}}{k!}\,G_k^{CG3}(-1)\,\left[f^{(k-1)}(1)-f^{(k-1)}(-1)\right]\\
&= \frac{9-\sqrt{102}}{105}\left[f'(1)-f'(-1)\right] + \sum_{k=4}^{n}\frac{2^{2k-1}}{(2k)!}\,G_{2k}^{CG3}(-1)\,\left[f^{(2k-1)}(1)-f^{(2k-1)}(-1)\right],\\
G_k^{CG3}(t) &= \frac{2976-32\sqrt{102}}{3465}B_k^*\left(1-\frac{t}{2}\right) + \frac{1977+16\sqrt{102}}{3465}\left[B_k^*\left(\frac{x_0-t}{2}\right)+B_k^*\left(\frac{-x_0-t}{2}\right)\right],\\
F_k^{CG3}(t) &= G_k^{CG3}(t) - G_k^{CG3}(-1).
\end{aligned}$$

The remainder $R_{2n+2}^{CG3}(f)$ for $n \geq 3$ can be written as:

$$R_{2n+2}^{CG3}(f) = -\frac{2^{2n+2}}{(2n+2)!}\,G_{2n+2}^{CG3}(-1)\cdot f^{(2n+2)}(\xi), \quad \xi\in[-1,1], \tag{5.45}$$

$$R_{2n+2}^{CG3}(f) = \theta\cdot\frac{2^{2n+1}}{(2n+2)!}\,F_{2n+2}^{CG3}(0)\left[f^{(2n+1)}(1)-f^{(2n+1)}(-1)\right], \quad \theta\in[0,1] \tag{5.46}$$

where

$$\begin{aligned}
G_{2n+2}^{CG3}(-1) = {}& \frac{3954+32\sqrt{102}}{3465}\,B_{2n+2}\left(\frac{1-x_0}{2}\right)\\
&-\frac{2976-32\sqrt{102}}{3465}(1-2^{-2n-1})B_{2n+2},
\end{aligned} \tag{5.47}$$

$$\begin{aligned}
F_{2n+2}^{CG3}(0) = {}& \frac{3954+32\sqrt{102}}{3465}\left[B_{2n+2}\left(\frac{x_0}{2}\right)-B_{2n+2}\left(\frac{1-x_0}{2}\right)\right]\\
&+\frac{2976-32\sqrt{102}}{3465}\left(2-2^{-2n-1}\right)B_{2n+2}.
\end{aligned} \tag{5.48}$$

Now, from (5.44) and (5.45) for $n=3$ the corrected Gauss 3-point formula is produced:

$$\int_{-1}^{1} f(t)dt - Q_{CG3} + \frac{9-\sqrt{102}}{105}[f'(1) - f'(-1)] = \frac{3\sqrt{102}-43}{19448100} f^{(8)}(\xi). \tag{5.49}$$

Similarly as before, error estimates can be obtained which, for $n \geq 3$ and $p = \infty$ or $p = 1$, can be expressed through the values of $G_{2n+2}^{CG3}(-1)$ and $F_{2n+2}^{CG3}(0)$. For functions with lower degree of smoothness, the appropriate estimates can also be explicitly calculated. We have

$$\left| \int_{-1}^{1} f(t)dt - Q_{CG3} \right| \leq C_{CG3}(m,q) \cdot \|f^{(m)}\|_p, \quad m = 1,2$$

where

$$\begin{aligned} C_{CG3}(1,1) &\approx 0.337807, \quad C_{CG3}(1,\infty) \approx 0.382802, \\ C_{CG3}(2,1) &\approx 0.0313153, \quad C_{CG3}(2,\infty) \approx 0.0609023, \end{aligned}$$

and

$$\left| \int_{-1}^{1} f(t)dt - Q_{CG3} + \frac{9-\sqrt{102}}{105}[f'(1) - f'(-1)] \right| \leq C_{CG3}(m,q) \cdot \|f^{(m)}\|_p, \quad 2 \leq m \leq 8$$

where

$$\begin{aligned} C_{CG3}(2,1) &\approx 0.0300722, \quad C_{CG3}(2,\infty) \approx 0.0504308, \\ C_{CG3}(3,1) &\approx 0.00351347, \quad C_{CG3}(3,\infty) \approx 0.00386067 \\ C_{CG3}(4,1) &\approx 0.000438656, \quad C_{CG3}(4,\infty) \approx 0.000610086, \\ C_{CG3}(5,1) &\approx 0.0000618111, \quad C_{CG3}(5,\infty) \approx 0.000077903, \\ C_{CG3}(6,1) &\approx 1.00553 \cdot 10^{-5}, \quad C_{CG3}(6,\infty) \approx 0.0000151755, \\ C_{CG3}(7,1) &\approx 2.077547 \cdot 10^{-6}, \quad C_{CG3}(7,\infty) \approx 2.513835 \cdot 10^{-6}, \\ C_{CG3}(8,1) &\approx 6.530965 \cdot 10^{-7}, \quad C_{CG3}(8,\infty) \approx 1.038773 \cdot 10^{-6}. \end{aligned}$$

The Hermite-Hadamard type estimate for the corrected Gauss 3-point formula (on $[0,1]$) is:

$$\begin{aligned} &\frac{43-3\sqrt{102}}{9957427200} f^{(8)}\left(\frac{1}{2}\right) \\ &\leq -\left(\int_0^1 f(t)dt - \frac{1977+16\sqrt{102}}{6930}\left[f\left(\frac{7-\sqrt{45-2\sqrt{102}}}{14}\right) + f\left(\frac{7+\sqrt{45-2\sqrt{102}}}{14}\right)\right] \right. \\ &\qquad \left. - \frac{1488-16\sqrt{102}}{3465} f\left(\frac{1}{2}\right) + \frac{9-\sqrt{102}}{420}[f'(1) - f'(0)] \right) \\ &\leq \frac{43-3\sqrt{102}}{9957427200} \frac{f^{(8)}(0) + f^{(8)}(1)}{2} \end{aligned}$$

and the Dragomir-Agarwal type estimates are:

$$C_{CQ5}\left(7, \frac{7-\sqrt{45-2\sqrt{102}}}{14}\right) = \frac{24\sqrt{60933-6014\sqrt{102}}-49(87-8\sqrt{102})}{3793305600},$$

$$C_{CQ5}\left(8, \frac{7-\sqrt{45-2\sqrt{102}}}{14}\right) = \frac{43-3\sqrt{102}}{9957427200}.$$

5.2.4 Corrected Boole's formula

The last special case which is to be considered is the case when $x = 1/4$. For this x, formula (5.24) becomes:

$$\int_0^1 f(t)dt - \frac{1}{1890}\left[217f(0)+512f\left(\frac{1}{4}\right)+432f\left(\frac{1}{2}\right)+512f\left(\frac{3}{4}\right)+217f(1)\right]$$
$$+\frac{1}{252}[f'(1)-f'(0)] = \frac{1}{1625702400}\cdot f^{(8)}(\xi), \quad \xi\in[0,1] \quad (5.50)$$

We call formula (5.50) corrected Boole's formula. We now have:

$$\int_0^1 f(t)dt - Q_{CB} + T_{2n}^{CB} = \frac{1}{(2n)!}\int_0^1 G_{2n}^{CB}(t)df^{(2n-1)}(t), \quad (5.51)$$

$$\int_0^1 f(t)dt - Q_{CB} + T_{2n}^{CB} = \frac{1}{(2n+1)!}\int_0^1 G_{2n+1}^{CB}(t)df^{(2n)}(t), \quad (5.52)$$

$$\int_0^1 f(t)dt - Q_{CB} + T_{2n}^{CB} = \frac{1}{(2n+2)!}\int_0^1 F_{2n+2}^{CB}(t)df^{(2n+1)}(t), \quad (5.53)$$

where

$$Q_{CB} = \frac{1}{1890}\left[217f(0)+512f\left(\frac{1}{4}\right)+432f\left(\frac{1}{2}\right)+512f\left(\frac{3}{4}\right)+217f(1)\right]$$

$$T_{2n}^{CB} = T_{2n}^{CQ5}\left(\frac{1}{4}\right) = \sum_{k=2}^{2n}\frac{1}{k!}\,G_k^{CB}(0)\,[f^{(k-1)}(1)-f^{(k-1)}(0)]$$
$$= \frac{1}{252}[f'(1)-f'(0)] + \sum_{k=4}^{n}\frac{1}{(2k)!}\,G_{2k}^{CB}(0)\,[f^{(2k-1)}(1)-f^{(2k-1)}(0)],$$

$$G_k^{CB}(t) = G_k^{CQ5}\left(\frac{1}{4},t\right) = \frac{31}{135}B_k^*(1-t)$$
$$+\frac{256}{945}\left[B_k^*\left(\frac{1}{4}-t\right)+B_k^*\left(\frac{3}{4}-t\right)\right]+\frac{8}{35}B_k^*\left(\frac{1}{2}-t\right),$$

$$F_k^{CB}(t) = F^{CQ5}\left(\frac{1}{4},t\right) = G_k^{CB}(t)-G_k^{CB}(0).$$

Note that

$$G_{2n+2}^{CB}(0) = G_{2n+2}^{CQ5}(1/4,0) = \frac{1}{945}(1-20\cdot4^{-n}+64\cdot16^{-n})B_{2n+2},$$

$$F_{2n+2}^{CB}(1/2) = F_{2n+2}^{CQ5}(1/4, 1/2) = \frac{1}{945}(2^{-2n-1} - 2)B_{2n+2}.$$

Finally, let us see what the estimates of error for functions with a low degree of smoothness for this type of formula are.

$$\left| \int_0^1 f(t)dt - Q_{CB} \right| \leq C_{CB}(m,q) \cdot \|f^{(m)}\|_p, \quad m = 1,2$$

where

$$\begin{aligned}
C_{CB}(1,1) &= \frac{89927}{1428840} \approx 0.0629371, \\
C_{CB}(1,\infty) &= |G_1^{CB}(3/4)| = \frac{19}{140} \approx 0.135714, \\
C_{CB}(2,1) &= \frac{21454879 + 285606\sqrt{5289}}{10126903500} \approx 0.00416966, \\
C_{CB}(2,\infty) &= \frac{1}{2}\left| F_2^{CB}\left(\frac{27}{70}\right) \right| = \frac{1763}{264600} \approx 0.00666289
\end{aligned}$$

and further

$$\left| \int_0^1 f(t)dt - Q_{CB} + \frac{1}{252}[f'(1) - f'(0)] \right| \leq C_{CB}(m,q) \cdot \|f^{(m)}\|_p, \quad 2 \leq m \leq 8$$

where

$$\begin{aligned}
C_{CB}(2,1) &= \frac{57753\sqrt{2139} + 18739\sqrt{18739}}{2531725875} \approx 0.00206824, \\
C_{CB}(2,\infty) &= \frac{1}{2}\left| G_2^{CB}\left(\frac{1}{4}\right) \right| = \frac{197}{30240} \approx 0.00651455, \\
C_{CB}(3,1) &\approx 9.15728 \cdot 10^{-5}, \quad C_{CB}(3,\infty) \approx 1.91051 \cdot 10^{-4} \\
C_{CB}(4,1) &\approx 5.05748 \cdot 10^{-6}, \quad C_{CB}(4,\infty) \approx 1.22339 \cdot 10^{-5}, \\
C_{CB}(5,1) &\approx 3.47811 \cdot 10^{-7}, \\
C_{CB}(5,\infty) &= \frac{1}{5!}\left| G_5^{CB}\left(\frac{1}{3}\right) \right| = \frac{1}{1224720} \approx 8.16513 \cdot 10^{-7}, \\
C_{CB}(6,1) &= \frac{1}{31492800} \approx 3.17533 \cdot 10^{-8}, \\
C_{CB}(6,\infty) &= \frac{1}{6!}\left| G_6^{CB}\left(\frac{1}{2}\right) \right| = \frac{1}{14515200} \approx 6.88933 \cdot 10^{-8}, \\
C_{CB}(7,1) &= \frac{17}{4877107200} \approx 3.48567 \cdot 10^{-9} \\
C_{CB}(7,\infty) &= \frac{1}{7!}\left| G_7^{CB}\left(\frac{1}{3}\right) \right| = \frac{1}{125971200} \approx 7.93832 \cdot 10^{-9}, \\
C_{CB}(8,1) &= \frac{1}{1625702400} \approx 6.15119 \cdot 10^{-10}, \\
C_{CB}(8,\infty) &= \frac{1}{8!}\left| F_8^{CB}\left(\frac{1}{2}\right) \right| = \frac{17}{9754214400} \approx 1.74284 \cdot 10^{-9}.
\end{aligned}$$

The Hermite-Hadamard type estimate for the corrected Boole's formula is:

$$
\begin{aligned}
&\frac{1}{1625702400} f^{(8)}\left(\frac{1}{2}\right) \\
&\leq \int_0^1 f(t)dt - \frac{1}{1890}\left[217f(0)+512f\left(\frac{1}{4}\right)+432f\left(\frac{1}{2}\right)+512f\left(\frac{3}{4}\right)+217f(1)\right] \\
&\qquad +\frac{1}{252}[f'(1)-f'(0)] \\
&\leq \frac{1}{1625702400}\,\frac{f^{(8)}(0)+f^{(8)}(1)}{2}
\end{aligned}
$$

and the Dragomir-Agarwal type estimates for this formula are:

$$
C_{CQ5}\left(7,\frac{1}{4}\right)=\frac{17}{4877107200}, \qquad C_{CQ5}\left(8,\frac{1}{4}\right)=\frac{1}{1625702400}.
$$

5.2.5 Hermite-Hadamard-type inequality for the corrected 5-point quadrature formulae

The main result of this subsection provides Hermite-Hadamard-type inequality for the corrected 5-point quadrature formulae.

Theorem 5.7 *Let $f:[0,1]\to\mathbb{R}$ be 8-convex and such that $f^{(8)}$ is continuous on $[0,1]$. Then, for $x\in(0,\frac{1}{2}-\frac{\sqrt{21}}{14}]$ and $y\in\left[\frac{3-\sqrt{2}}{7},\frac{1}{2}\right)$*

$$
\begin{aligned}
&Q_C\left(0,y,\frac{1}{2},1-y,1\right)-\frac{7y^2-7y+1}{420y(y-1)}[f'(1)-f'(0)] \\
&\leq \int_0^1 f(t)dt \qquad (5.54)\\
&\leq Q_C\left(0,x,\frac{1}{2},1-x,1\right)-\frac{7x^2-7x+1}{420x(x-1)}[f'(1)-f'(0)],
\end{aligned}
$$

where

$$
\begin{aligned}
&Q_C\left(0,x,\frac{1}{2},1-x,1\right) \\
&=\frac{1}{420x^2(1-x)^2(1-2x)^2}\big((98x^4-196x^3+102x^2-4x-1)(1-2x)^2[f(0)+f(1)] \\
&\quad +f(x)+f(1-x)+64x^2(1-x)^2(14x^2-14x+3)f\left(\frac{1}{2}\right)\big).
\end{aligned}
$$

If f is 8-concave, the inequalities are reversed.

Proof. Analogous to the proof of Theorem 3.6. □

The following corollaries give comparison between the corrected Lobatto 4-point and the Lobatto 5-point rule, corrected Boole's and the Lobatto 5-point rule, the corrected Lobatto 4-point and the corrected Gauss 3-point rule, corrected Boole's and the corrected Gauss 3-point rule.

Corollary 5.3 *Let $f : [0,1] \to \mathbb{R}$ be 8-convex and such that $f^{(8)}$ is continuous on $[0,1]$. Then*

$$\begin{aligned}
&\frac{1}{270}\left(37f(0)+98f\left(\frac{7-\sqrt{7}}{14}\right)+98f\left(\frac{7+\sqrt{7}}{14}\right)+37f(1)\right)-\frac{1}{180}[f'(1)-f'(0)]\\
&\leq \int_0^1 f(t)dt\\
&\leq \frac{1}{180}\left(9f(0)+49f\left(\frac{7-\sqrt{21}}{14}\right)+64f\left(\frac{1}{2}\right)+49f\left(\frac{7+\sqrt{21}}{14}\right)+9f(1)\right).
\end{aligned}$$

If f is 8-concave, the inequalities are reversed.

Proof. Follows from (5.22) for $x = 1/2-\sqrt{21}/14 \Leftrightarrow 7x^2-7x+1=0$ and $y = 1/2-\sqrt{7}/14 \Leftrightarrow 5y^2-5y+1=0$. □

Corollary 5.4 *Let $f : [0,1] \to \mathbb{R}$ be 8-convex and such that $f^{(8)}$ is continuous on $[0,1]$. Then*

$$\begin{aligned}
&\frac{1}{1890}\left(217f(0)+512f\left(\frac{1}{4}\right)+432f\left(\frac{1}{2}\right)+512f\left(\frac{3}{4}\right)+217f(1)\right)-\frac{1}{252}[f'(1)-f'(0)]\\
&\leq \int_0^1 f(t)dt\\
&\leq \frac{1}{180}\left(9f(0)+49f\left(\frac{7-\sqrt{21}}{14}\right)+64f\left(\frac{1}{2}\right)+49f\left(\frac{7+\sqrt{21}}{14}\right)+9f(1)\right).
\end{aligned}$$

If f is 8-concave, the inequalities are reversed.

Proof. Follows from (5.22) for $x = 1/2-\sqrt{21}/14 \Leftrightarrow 7x^2-7x+1=0$ and $y = 1/4$. □

Corollary 5.5 *Let $f : [0,1] \to \mathbb{R}$ be 8-convex and such that $f^{(8)}$ is continuous on $[0,1]$. Then*

$$\begin{aligned}
&\frac{1}{270}\left(37f(0)+98f\left(\frac{7-\sqrt{7}}{14}\right)+98f\left(\frac{7+\sqrt{7}}{14}\right)+37f(1)\right)-\frac{1}{180}[f'(1)-f'(0)]\\
&\leq \int_0^1 f(t)dt\\
&\leq \frac{1977+16\sqrt{102}}{6930}\left(f\left(\frac{7-\sqrt{45-2\sqrt{102}}}{14}\right)+f\left(\frac{7+\sqrt{45-2\sqrt{102}}}{14}\right)\right)\\
&\quad+\frac{2976-32\sqrt{102}}{6930}f\left(\frac{1}{2}\right)-\frac{9-\sqrt{102}}{420}[f'(1)-f'(0)].
\end{aligned}$$

If f is 8-concave, the inequalities are reversed.

Proof. Follows from (5.22) for $x = 1/2 - \sqrt{45-2\sqrt{102}}\Big/14 \Leftrightarrow 98x^4 - 196x^3 + 102x^2 - 4x - 1 = 0$ and $y = 1/2 - \sqrt{7}/14 \Leftrightarrow 5y^2 - 5y + 1 = 0$. $\square$

Corollary 5.6 *Let $f : [0,1] \to \mathbb{R}$ be 8-convex and such that $f^{(8)}$ is continuous on $[0,1]$. Then*

$$\begin{aligned}
&\frac{1}{1890}\left(217f(0)+512f\left(\frac{1}{4}\right)+432f\left(\frac{1}{2}\right)+512f\left(\frac{3}{4}\right)+217f(1)\right)-\frac{1}{252}[f'(1)-f'(0)]\\
&\quad\le \int_0^1 f(t)dt\\
&\quad\le \frac{1977+16\sqrt{102}}{6930}\left(f\left(\frac{7-\sqrt{45-2\sqrt{102}}}{14}\right)+f\left(\frac{7+\sqrt{45-2\sqrt{102}}}{14}\right)\right)\\
&\qquad+\frac{2976-32\sqrt{102}}{6930}f\left(\frac{1}{2}\right)-\frac{9-\sqrt{102}}{420}[f'(1)-f'(0)].
\end{aligned}$$

If f is 8-concave, the inequalities are reversed.

Proof. Follows from (5.22) for $x = 1/2 - \sqrt{45-2\sqrt{102}}\Big/14 \Leftrightarrow 98x^4 - 196x^3 + 102x^2 - 4x - 1 = 0$ and $y = 1/4$. $\square$

5.3 Corrected Lobatto 5-point formula

One of the concepts of this book was to adjoin the classical quadrature formulae with the corresponding corrected one. This was done for all closed 5-point formulae except for the Lobatto 5-point formula.

Recall how the classical Lobatto 5-point formula was derived: starting from (5.1) and setting the system

$$G_2(x,0) = G_4(x,0) = G_6(x,0) = 0.$$

The classical Lobatto 5-point formula belongs to the family of the corrected closed 5-point formulae, studied in the previous section, which are exact for all polynomials of order ≤ 7. What we want to do now is derive the closed 5-point quadrature formula that has a degree of exactness higher than the classical Lobatto 5-point formula but on the other hand includes in the quadrature the values of the first derivative. To do this, we impose the conditions:

$$G_4(x,0) = G_6(x,0) = G_8(x,0) = 0.$$

Formula thus obtained will have the maximum degree of exactness for this type of formula (which is 9) so we call it the corrected Lobatto 5-point formula.

Instead of on the interval $[0,1]$, we shall work on $[-1,1]$. Having this transformation in mind, we get

$$x_0 = \frac{1}{\sqrt{3}}, \quad w_1(x_0) = \frac{19}{105}, \quad w_2(x_0) = \frac{18}{35}, \quad w_3(x_0) = \frac{64}{105} ;$$

these are the node and the weights of the corrected Lobatto 5-point formula.

So, if $f^{(2n-1)}$ is continuous of bounded variation on $[-1,1]$ for some $n \geq 1$, then:

$$\int_{-1}^{1} f(t)dt - Q_{CL5} + T_{2n}^{CL5} = \frac{2^{2n-1}}{(2n)!} \int_{-1}^{1} G_{2n}^{CL5}(t)df^{(2n-1)}(t); \tag{5.55}$$

if $f^{(2n)}$ satisfies the same condition for some $n \geq 0$, then:

$$\int_{-1}^{1} f(t)dt - Q_{CL5} + T_{2n}^{CL5} = \frac{2^{2n}}{(2n+1)!} \int_{-1}^{1} G_{2n+1}^{CL5}(t)df^{(2n)}(t), \tag{5.56}$$

and, finally, if $f^{(2n+1)}$ satisfies the same condition for some $n \geq 0$, then:

$$\int_{-1}^{1} f(t)dt - Q_{CL5} + T_{2n}^{CL5} = \frac{2^{2n+1}}{(2n+2)!} \int_{-1}^{1} F_{2n+2}^{CL5}(t)df^{(2n+1)}(t), \tag{5.57}$$

where

$$Q_{CL5} = \frac{1}{105}\left[19f(-1) + 54f\left(-\frac{1}{\sqrt{3}}\right) + 64f(0) + 54\left(\frac{1}{\sqrt{3}}\right) + 19f(1)\right],$$

$$T_{2n}^{CL5} = \frac{1}{105}[f'(1) - f'(-1)] + \sum_{k=5}^{n} \frac{2^{2k-1}}{(2k)!} G_{2k}^{CL5}(-1)\,[f^{(2k-1)}(1) - f^{(2k-1)}(-1)],$$

$$G_k^{CL5}(t) = \frac{38}{105}B_k^*\left(\frac{1-t}{2}\right) + \frac{18}{35}\left[B_k^*\left(\frac{\sqrt{3}-3t}{6}\right) + B_k^*\left(\frac{-\sqrt{3}-3t}{6}\right)\right]$$
$$+\frac{64}{105}B_k^*\left(1-\frac{t}{2}\right),$$

$$F_k^{CL5}(t) = G_k^{CL5}(t) - G_k^{CL5}(-1).$$

To prove the rest of the results for these formulae, we need the following lemma:

Lemma 5.3 *For $k \geq 4$, $G_{2k+1}^{CL5}(t)$ has no zeros on $(0,1)$. The sign of this function is determined by $(-1)^k G_{2k+1}^{CL5}(t) > 0$.*

Proof. We start from G_9^{CL5}. For $1/\sqrt{3} \leq t < 1$,

$$G_9^{CL5}(t) = \frac{1}{8960}(1-t)^7(35t^2 - 13t + 2),$$

so it is trivial to see that here $G_9^{CL5}(t) > 0$. For $0 < t \leq 1/\sqrt{3}$, it is a bit more complicated:

$$G_9^{CL5}(t) = \frac{t}{8960} \cdot k(t),$$

where
$k(t) = -35t^8 + 96t^7 + 36(12\sqrt{3} - 23)t^6 + 42(24\sqrt{3} - 41)t^4 + 84(4\sqrt{3} - 7)t^2 + 16\sqrt{3} - 27.$
We have:

$$k'(t) = -8t \cdot k_1(t) \quad \text{and} \quad k_1'(t) = 6t \cdot k_2(t),$$

where $k_2(t) = 35t^4 - 70t^3 + (414 - 216\sqrt{3})t^2 + 287 - 168\sqrt{3}$. It is easy to check that $k_2'(t) > 0$ which, together with $k_2(1/\sqrt{3}) < 0$, leads to a conclusion that $k_2 < 0$. From $k_1(1/\sqrt{3}) > 0$ it follows that $k_1 > 0$. Finally, we conclude that $k > 0$ since $k(1/\sqrt{3}) > 0$. Therefore, it is now clear that $G_9^{CL5}(t) > 0$ on this interval. This proves the assertion for $k = 4$. The rest of the proof is analogous to the same part of the proof of Lemma 5.1. □

Denote by $R_{2n+2}^{CL5}(f)$ the remainder in (5.57).

Theorem 5.8 *Let $f : [-1,1] \to \mathbb{R}$ be such that $f^{(2n+2)}$ is continuous on $[-1,1]$ for some $n \geq 4$. Then there exists $\xi \in [-1,1]$ such that*

$$R_{2n+2}^{CL5}(f) = -\frac{2^{2n+2}}{(2n+2)!} G_{2n+2}^{CL5}(-1) \cdot f^{(2n+2)}(\xi) \tag{5.58}$$

where

$$G_{2n+2}^{CL5}(-1) = \frac{36}{35} B_{2n+2}\left(\frac{3-\sqrt{3}}{6}\right) + \frac{1}{105}(2^{5-2n} - 26)B_{2n+2}. \tag{5.59}$$

If, in addition, $f^{(2n+2)}$ has constant sign on $[-1,1]$, then there exists $\theta \in [0,1]$ such that

$$R_{2n+2}^{CL5}(f) = \theta \cdot \frac{2^{2n+1}}{(2n+2)!} \cdot F_{2n+2}^{CL5}(0) \cdot \left[f^{(2n+1)}(1) - f^{(2n+1)}(-1)\right], \tag{5.60}$$

where

$$F_{2n+2}^{CL5}(0) = \frac{36}{35}\left[B_{2n+2}\left(\frac{\sqrt{3}}{6}\right) - B_{2n+2}\left(\frac{3-\sqrt{3}}{6}\right)\right] + \frac{1}{105}(52 - 13 \cdot 2^{-2n})B_{2n+2} \tag{5.61}$$

Proof. Analogous to the proof of Theorem 3.2. □

The corrected Lobatto 5-point formula is produced after applying (5.58) for $n = 4$:

$$\int_{-1}^{1} f(t)dt - Q_{CL5} + \frac{1}{105}[f'(1) - f'(-1)] = \frac{1}{589396500} \cdot f^{(10)}(\xi), \quad \xi \in [-1,1]. \tag{5.62}$$

Using Hölder's inequality, sharp error estimates can easily be obtained for these formulae (cf. Theorem 5.3). Especially, we have

$$\left|\int_{-1}^{1} f(t)dt - Q_{CL5}\right| \leq C_{CL5}(m,q) \cdot \|f^{(m)}\|_p, \quad m = 1,2$$

where

$$
\begin{aligned}
C_{CL5}(1,1) &= \frac{3431-1652\sqrt{3}}{2205} \approx 0.258346, \quad C_{CL5}(2,1) \approx 0.0253504,\\
C_{CL5}(1,\infty) &= \left| G_1^{CL5}\left(\frac{1}{\sqrt{3}}\right)\right| = \frac{35\sqrt{3}-32}{105} \approx 0.272588,\\
C_{CL5}(2,\infty) &= \left| F_2^{CL5}\left(\frac{32}{105}\right)\right| = \frac{3780\sqrt{3}-6011}{22050} \approx 0.0243153
\end{aligned}
$$

and

$$
\left|\int_{-1}^{1} f(t)dt - Q_{CL5} + \frac{1}{105}[f'(1)-f'(-1)]\right| \le C_{CL5}(m,q)\cdot\|f^{(m)}\|_p, \quad m=2,\ldots,10
$$

where

$$
\begin{aligned}
&C_{CL5}(2,1) \approx 1.78428\cdot 10^{-2},\\
&C_{CL5}(2,\infty) = \left|G_2^{CL5}(0)\right| = \frac{23-12\sqrt{3}}{70} \approx 3.16484\cdot 10^{-2},\\
&C_{CL5}(3,1) \approx 1.63266\cdot 10^{-3}, \quad C_{CL5}(3,\infty) \approx 1.90589\cdot 10^{-3},\\
&C_{CL5}(4,1) \approx 1.55231\cdot 10^{-4},\\
&C_{CL5}(4,\infty) = |G_4^{CL5}(0)|/3 = \frac{24\sqrt{3}-41}{2520} \approx 2.25881\cdot 10^{-4},\\
&C_{CL5}(5,1) \approx 1.60278\cdot 10^{-5}, \quad C_{CL5}(5,\infty) \approx 2.19844\cdot 10^{-5},\\
&C_{CL5}(6,1) \approx 1.79762\cdot 10^{-6},\\
&C_{CL5}(6,\infty) = 2|G_6^{CL5}(0)|/45 = \frac{7-4\sqrt{3}}{25200} \approx 2.84908\cdot 10^{-6},\\
&C_{CL5}(7,1) \approx 2.25172\cdot 10^{-7}, \quad C_{CL5}(7,\infty) \approx 3.30395\cdot 10^{-7},\\
&C_{CL5}(8,1) \approx 3.27708\cdot 10^{-8},\\
&C_{CL5}(8,\infty) = |G_8^{CL5}(0)|/315 = \frac{16\sqrt{3}-27}{12700800} \approx 5.61235\cdot 10^{-8}\\
&C_{CL5}(9,1) = \frac{9-4\sqrt{3}}{342921600} \approx 6.0416\cdot 10^{-9}, \quad C_{CL5}(9,\infty) \approx 8.19271\cdot 10^{-9},\\
&C_{CL5}(10,1) = \frac{1}{589396500} \approx 1.69665\cdot 10^{-9},\\
&C_{CL5}(10,\infty) = 2|F_{10}^{CL5}(0)|/14175 = \frac{9-4\sqrt{3}}{685843200} \approx 3.0208\cdot 10^{-9}.
\end{aligned}
$$

5.4 General Euler-Boole's and dual Euler-Boole's formulae

The aim of this section is to derive general Euler-Boole's and dual Euler-Boole's formulae. The idea is to obtain Boole type quadrature formulae such that $G_{2n-2}(1/4,0) = G_{2n}(1/4,0) = 0,$ where G_k are as in (5.4), thus achieving an arbitrary degree of exactness. Dual Euler-Boole's formulae are derived by analogy with Simpson's and dual Simpson's formula, and Simpson's 3/8 and its dual formula - Maclaurin's formula. Results from this section are published in [64].

5.4.1 General Euler-Boole's formulae

To start with, a quadrature formula with the nodes: $0,\ 1/4,\ 1/2,\ 3/4,\ 1$ and general weights, needs to be derived. The technique is the same as in deriving (5.1), only with $x = 1/4$. Then, for $f:[0,1]\to\mathbb{R}$ such that $f^{(2n-1)}$ is continuous of bounded variation on $[0,1]$ for some $n\geq 1$, we have

$$\int_0^1 f(t)dt - Q_{GB} + T_{2n}^{GB} = \frac{1}{(2n)!}\int_0^1 G_{2n}^{GB}(t)df^{(2n-1)}(t); \tag{5.63}$$

if $f^{(2n)}$ has that same property for some $n\geq 0$, then:

$$\int_0^1 f(t)dt - Q_{GB} + T_{2n}^{GB} = \frac{1}{(2n+1)!}\int_0^1 G_{2n+1}^{GB}(t)df^{(2n)}(t), \tag{5.64}$$

and finally, if $f^{(2n+1)}$ has it for some $n\geq 0$, then:

$$\int_0^1 f(t)dt - Q_{GB} + T_{2n}^{GB} = \frac{1}{(2n+2)!}\int_0^1 F_{2n+2}^{GB}(t)df^{(2n+1)}(t), \tag{5.65}$$

where, for $t\in\mathbb{R}$,

$$\begin{aligned}
Q_{GB} &= \lambda_1[f(0)+f(1)] + \lambda_2[f(1/4)+f(3/4)] + \lambda_3 f(1/2),\\
T_{2n}^{GB} &= \sum_{k=1}^{n}\frac{1}{(2k)!}\,G_{2k}^{GB}(0)\,[f^{(2k-1)}(1) - f^{(2k-1)}(0)],\\
G_k^{GB}(t) &= 2\lambda_1 B_k^*(1-t) + \lambda_2[B_k^*(1/4-t) + B_k^*(3/4-t)] + \lambda_3 B_k^*(1/2-t),\\
F_k^{GB}(t) &= G_k^{GB}(t) - G_k^{GB}(0),
\end{aligned}$$

and $2\lambda_1 + 2\lambda_2 + \lambda_3 = 1$.

Formulae (5.63), (5.64) and (5.65) shall be called general Euler-Boole's formulae. Now, for $n\geq 2$, set the following linear system:

$$G_{2n-2}^{GB}(0) = G_{2n}^{GB}(0) = 0.$$

These conditions will provide that the formula thus obtained has the maximum degree of exactness and includes the values of up to $(2n-5)$-th derivative at the endpoints (which is why $n \geq 2$). Using properties of Bernoulli polynomials and numbers, it is not difficult to find the solutions of this system:

$$\lambda_1 = \frac{16 - 10 \cdot 4^n + 4^{2n}}{8(4^n - 1)(4^n - 4)}, \quad \lambda_2 = \frac{4^{2n-1}}{(4^n - 1)(4^n - 4)}, \quad \lambda_3 = \frac{(4^n - 10) \cdot 4^{n-1}}{(4^n - 1)(4^n - 4)}$$

and these are the weights we will work with.

What follows is a lemma that is a key step for all the results in this section.

Lemma 5.4 *For $n \geq 2$, $G_{2n+1}^{GB}(t)$ has no zeros in the interval $(0, 1/2)$. The sign of the function is determined by*

$$(-1)^n G_{2n+1}^{GB}(t) > 0, \qquad 0 < t < 1/2.$$

Proof. Applying (1.8), we can rewrite $G_{2n+1}^{GB}(t)$ as

$$G_{2n+1}^{GB}(t) = \frac{-1}{4(4^n - 1)(4^n - 4)} [B_{2n+1}^*(4t) - 10B_{2n+1}^*(2t) + 16B_{2n+1}^*(t)]. \tag{5.66}$$

There cannot exist $t \in (1/4, 3/8)$ such that $G_{2n+1}^{GB}(t) = 0$ because $B_{2n+1}^*(t)$, $-B_{2n+1}^*(2t)$ and $B_{2n+1}^*(4t)$ have the same sign on $(1/4, 3/8)$.

Let us assume there exists $t_1 \in (0, 1/4]$ such that $G_{2n+1}^{GB}(t_1) = 0$. Since $G_{2n+1}^{GB}(0) = 0$, we conclude there must exist $t_2 \in (0, t_1)$ such that $(G^{GB})'_{2n+1}(t_2) = 0$. So, we must have

$$B_{2n}^*(4t_2) - 5B_{2n}^*(2t_2) + 4B_{2n}^*(t_2) = 0,$$

which is equivalent to

$$\frac{B_{2n}^*(4t_2) - B_{2n}^*(2t_2)}{B_{2n}^*(2t_2) - B_{2n}^*(t_2)} = 4$$

since for $z \in (0, 1/2)$, $B_{2n}^*(2z) = B_{2n}^*(z)$ iff $z = 1/3$ and that cannot be the case. Define functions

$$f(x) = B_{2n}^*(2xt_2), \quad g(x) = B_{2n}^*(xt_2), \quad x \in [1, 2].$$

Note that $g'(x) \neq 0$ for $x \in [1, 2]$, since $0 < xt_2 < 1/2$. From Cauchy's mean value theorem we know there exists $x_1 \in (1, 2)$ such that

$$\frac{B_{2n}^*(4t_2) - B_{2n}^*(2t_2)}{B_{2n}^*(2t_2) - B_{2n}^*(t_2)} = \frac{f'(x_1)}{g'(x_1)} = 4,$$

and from there

$$\frac{B_{2n-1}^*(2x_1t_2)}{B_{2n-1}^*(x_1t_2)} = 2, \quad \text{for some} \quad 0 < x_1t_2 < 1/2. \tag{5.67}$$

Next, define a function

$$h(t) = 2B_{2n-1}^*(t) - B_{2n-1}^*(2t).$$

From (5.67) it follows that $h(x_1t_2) = 0$. To obtain a contradiction, we will prove $h(t) \neq 0$ for $t \in (0, 1/2)$. First, assume $t \in (0, 1/4]$. Suppose there exists $t_3 \in (0, 1/4]$ such that $h(t_3) = 0$. Since $h(0) = 0$, we conclude there must exist $t_4 \in (0, t_3)$ such that $h'(t_4) = 0$. But from there it would follow that $B^*_{2n-2}(t_4) = B^*_{2n-2}(2t_4)$ which cannot be the case. When $t \in (1/4, 1/2)$, $B^*_{2n-1}(t)$ and $-B^*_{2n-1}(2t)$ have the same sign, so our statement follows easily.

Finally, consider the case $t \in [3/8, 1/2)$. We have

$$B^*_{2n+1}(4t) - 10B^*_{2n+1}(2t) + 16B^*_{2n+1}(t) = k(t) - 8B^*_{2n+1}(2t) + 16B^*_{2n+1}(t),$$

where

$$k(t) = B^*_{2n+1}(4t) - 2B^*_{2n+1}(2t) = 2B^*_{2n+1}(1-2t) - B^*_{2n+1}[2(1-2t)].$$

It follows from the previous proof for the function $h(t)$, that $k(t)$ has no zeros on $[3/8, 1/2)$. Furthermore, $k(t)$, $-B^*_{2n+1}(2t)$ and $B^*_{2n+1}(t)$ have the same sign on this interval. So, in conclusion, the function $G^{GB}_{2n+1}(t)$ has no zeros on $(0, 1/2)$.

It is clear now that $G^{GB}_{2n+1}(t)$ has constant sign on $(0, 1/2)$. To determine the sign, it is enough to calculate the value of that function in any point from the interval $(0, 1/2)$, e.g. $t = 1/4$. □

The proof of the previous Lemma, compared to the proof of Lemma 2 in [99], is much more difficult, since we cannot reduce it to the case where we can explicitly calculate zeros of the function.

Remark 5.3 It follows immediately from the previous Lemma that for $n \geq 2$, $(-1)^{n+1}F^{GB}_{2n+2}(t)$ is strictly increasing on $(0, 1/2)$ and strictly decreasing on $(1/2, 1)$ and since $F^{GB}_{2n+2}(0) = F^{GB}_{2n+2}(1) = 0$, we have:

$$\max_{t \in [0,1]} \left| F^{GB}_{2n+2}(t) \right| = \left| F^{GB}_{2n+2}(1/2) \right| = \frac{2(4 - 4^{-n})}{(4^n - 1)(4^n - 4)} |B_{2n+2}|.$$

Furthermore,

$$\int_0^1 \left| G^{GB}_{2n+1}(t) \right| dt = \int_0^1 \left| F^{GB}_{2n+1}(t) \right| dt = \frac{1}{n+1} \left| F^{GB}_{2n+2}\left(\frac{1}{2}\right) \right|$$
$$= \frac{2(4 - 4^{-n})|B_{2n+2}|}{(n+1)(4^n - 1)(4^n - 4)},$$
$$\int_0^1 \left| F^{GB}_{2n+2}(t) \right| dt = |G^{GB}_{2n+2}(0)| = \frac{45 \cdot |B_{2n+2}|}{16(4^n - 1)(4^n - 4)}.$$

Theorem 5.9 *Let $p, q \in \mathbb{R}$ be such that $1 \leq p, q \leq \infty$ and $1/p + 1/q = 1$. If $f : [0,1] \to \mathbb{R}$ is such that $f^{(2n)} \in L_p[0,1]$ for some $n \geq 2$, then:*

$$\left| \int_0^1 f(t)dt - Q_{GB} + T^{GB}_{2n} \right| \leq K_{GB}(2n, q) \cdot \|f^{(2n)}\|_p, \tag{5.68}$$

if $f^{(2n+1)} \in L_p[0,1]$ *for some* $n \geq 2$*, then*

$$\left| \int_0^1 f(t)dt - Q_{GB} + T_{2n}^{GB} \right| \leq K_{GB}(2n+1,q) \cdot \|f^{(2n+1)}\|_p, \tag{5.69}$$

if $f^{(2n+2)} \in L_p[0,1]$ *for some* $n \geq 2$*, then*

$$\left| \int_0^1 f(t)dt - Q_{GB} + T_{2n}^{GB} \right| \leq K_{GB}^*(2n+2,q) \cdot \|f^{(2n+2)}\|_p, \tag{5.70}$$

where

$$K_{GB}(m,q) = \frac{1}{m!} \left[\int_0^1 \left| G_m^{GB}(t) \right|^q dt \right]^{\frac{1}{q}} \quad \text{and} \quad K_{GB}^*(m,q) = \frac{1}{m!} \left[\int_0^1 \left| F_m^{GB}(t) \right|^q dt \right]^{\frac{1}{q}}.$$

These inequalities are sharp for $1 < p \leq \infty$ *and best possible for* $p = 1$.

Proof. Analogous to the proof of Theorem 2.2. □

Using Remark 5.3 it is easy to calculate $K_{GB}(2n+1,1)$, $K_{GB}^*(2n+2,1)$, and $K_{GB}^*(2n+2,\infty)$. Further, using Lemma 1 from [30] for $p = 2$ we obtain using integration by parts:

$$K_{GB}(2n,2) = \frac{1}{(4^n-1)(4^n-4)} \left[\frac{|B_{4n}|}{(4n)!} (2^{3-4n} - 25 \cdot 2^{1-2n} + 42) \right]^{1/2},$$

$$K_{GB}(2n+1,2) = \frac{1}{4(4^n-1)(4^n-4)} \left[\frac{B_{4n+2}}{(4n+2)!} \left(2^{3-4n} - 85 \cdot 2^{1-2n} + 357\right) \right]^{1/2},$$

$$K_{GB}^*(2n+2,2) = \frac{1}{16(2n+2)!(4^n-1)(4^n-4)} \Big[2025 B_{2n+2}^2 + \frac{[(2n+2)!]^2}{(4n+4)!} \left(2^{3-4n} - 325 \cdot 2^{1-2n} + 4497\right) |B_{4n+4}| \Big]^{1/2}.$$

Theorem 5.10 *Let* $f : [0,1] \to \mathbb{R}$ *be such that* $f^{(2n+2)}$ *is continuous on* $[0,1]$ *for some* $n \geq 2$*, then there exists* $\xi \in [0,1]$ *such that*

$$R_{2n+2}^{GB}(f) = -\frac{45 \cdot B_{2n+2}}{16(2n+2)!(4^n-1)(4^n-4)} \cdot f^{(2n+2)}(\xi), \tag{5.71}$$

where $R_{2n+2}^{GB}(f)$ *is the remainder in (5.65).*

If, in addition, $f^{(2n+2)}$ *does not change sign on* $[0,1]$*, then there exists* $\theta \in [0,1]$ *such that*

$$R_{2n+2}^{GB}(f) = -\frac{\theta \cdot 2(4-4^{-n}) \cdot B_{2n+2}}{(2n+2)!(4^n-1)(4^n-4)} \left[f^{(2n+1)}(1) - f^{(2n+1)}(0) \right]. \tag{5.72}$$

Proof. Analogous to the proof of Theorem 3.2. □

Remark 5.4 Applying (5.71) for $n = 2$, from (5.65) classical Boole's formula is produced, while for $n = 3$ corrected Boole's formula (5.50) is obtained.

5.4.2 General dual Euler-Boole's formulae

Boole's formula is a quadrature formula of closed type, and so are the general Euler-Boole's formulae. When the value of the function at the end point of the interval cannot be computed, formulae of closed type cannot be applied. For such functions, open formulae are much more effective. That is why quadrature formulae are usually considered in pairs: a closed and a corresponding open one, both with the same degree of exactness. For example, the well-known Simpson's rule (cf. subsection 3.1.2.) is sometimes studied in pair with the dual Simpson's formula (cf. subsection 3.1.3.). Another such pair of formulae is Simpson's 3/8 formula (cf. subsection 4.1.1.) and Maclaurin's formula (cf. subsection 3.1.4.).

Similar reasoning can be applied for corrected quadrature formulae: corrected Simpson's (3.115) and corrected dual Simpson's (3.117) can be considered dual quadrature formulae, as well as corrected Simpson's 3/8 (4.99) and corrected Maclaurin's (3.120).

So, now the idea is to derive a formula of open type that will be dual to Boole's formula in this sense, or, more generally, open formulae dual to general Euler-Boole's formulae. We shall call those formulae general dual Euler-Boole's formulae.

It can easily be checked that in all of these cases we have

$$G_k^D(t) = 2^{1-k} G_k(2t) - G_k(t), \tag{5.73}$$

where G_k is obtained in case when a closed quadrature formula is considered and G_k^D in case of the corresponding dual quadrature formula. We will use this identity as a definition of a dual formula, since from the function G_k we can deduce the quadrature formula itself. Especially, using (1.8) and (5.73) gives

$$\begin{aligned} G_k^{GDB}(t) &= \frac{1}{4(4^n-1)(4^n-4)} \left[4^{2n} B_k^*(1/8-t) - 10 \cdot 4^n B_k^*(1/4-t)\right. \\ &+ 4^{2n} B_k^*(3/8-t) + 16 B_k^*(1/2-t) + 4^{2n} B_k^*(5/8-t) \\ &- \left. 10 \cdot 4^n B_k^*(3/4-t) + 4^{2n} B_k^*(7/8-t)\right], \end{aligned} \tag{5.74}$$

for $k \geq 1$ and $t \in \mathbb{R}$.

The procedure is from now on similar as before: take $f : [0,1] \to \mathbb{R}$ such that $f^{(2n)}$ is continuous of bounded variation on $[0,1]$ for some $n \geq 2$; put $x = 1/8,\ 1/4,\ 3/8,\ 1/2,\ 5/8,\ 3/4,\ 7/8$ in (1.1), multiply by 4^{2n}, $-10 \cdot 4^n$, 4^{2n}, 16, 4^{2n}, $-10 \cdot 4^n$, 4^{2n}, respectively; add those formulae up and divide by $4(4^n-1)(4^n-4)$. We obtain:

$$\int_0^1 f(t)dt - Q_{GDB} + T_{2n}^{GDB} = \frac{1}{(2n+1)!} \int_0^1 G_{2n+1}^{GDB}(t) d f^{(2n)}(t), \tag{5.75}$$

where

$$\begin{aligned} Q_{GDB} &= \frac{1}{4(4^n-1)(4^n-4)} \left[4^{2n} f(1/8) - 10 \cdot 4^n f(1/4) + 4^{2n} f(3/8)\right. \\ &+ \left. 16 f(1/2) + 4^{2n} f(5/8) - 10 \cdot 4^n f(3/4) + 4^{2n} f(7/8)\right] \\ T_{2n}^{GDB} &= \sum_{k=1}^{n} \frac{1}{(2k)!} G_{2k}^{GDB}(0) \left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right]. \end{aligned}$$

Assuming $f^{(2n-1)}$ is continuous of bounded variation on $[0,1]$ for some $n \geq 2$, we get:

$$\int_0^1 f(t)dt - Q_{GDB} + T_{2n}^{GDB} = \frac{1}{(2n)!}\int_0^1 G_{2n}^{GDB}(t)df^{(2n-1)}(t), \tag{5.76}$$

and if $f^{(2n+1)}$ satisfies the same property, we get:

$$\int_0^1 f(t)dt - Q_{GDB} + T_{2n}^{GDB} = \frac{1}{(2n+2)!}\int_0^1 F_{2n+2}^{GDB}(t)df^{(2n+1)}(t), \tag{5.77}$$

where $F_k^{GDB}(t) = G_k^{GDB}(t) - G_k^{GDB}(0),\ k \geq 2$. Formulae (5.75)-(5.77) are general dual Euler-Boole's formulae.

Lemma 5.5 *For $n \geq 2$, $G_{2n+1}^{GDB}(t)$ has no zeros in $(0,1/2)$. The sign of the function is determined by*

$$(-1)^{n-1}G_{2n+1}^{GDB}(t) > 0, \qquad 0 < t < 1/2.$$

Proof. We have $G_{2n+1}^{GB}(1-t) = -G_{2n+1}^{GB}(t)$, so from Lemma 5.4 it follows that $G_{2n+1}^{GB}(2t)$ and $-G_{2n+1}^{GB}(t)$ have the same sign on $(1/4,1/2)$ so from (5.73) we conclude $G_{2n+1}^{GDB}(t)$ cannot have any zeros here.

Next, we can rewrite $G_{2n+1}^{GDB}(t)$ as

$$\begin{aligned} G_{2n+1}^{GDB}(t) \ &= \ \frac{-1}{4(4^n-1)(4^n-4)}\left[B_{2n+1}^*(4t-1/2)\right. \\ &- \ \left. 10B_{2n+1}^*(2t-1/2) + 16B_{2n+1}^*(t-1/2)\right]. \end{aligned} \tag{5.78}$$

Using this in the case when $t \in (0,1/4]$, the proof is completely analogous to the same part of the proof of Lemma 5.4. As for the sign of the function, again it is enough to calculate the value of the function in any point of the interval $(0,1/2)$, e.g. $t = 1/4$. □

Notice the analogy of the form of the dual function G_{2n+1}^{GDB} in (5.78) with the form of the function G_{2n+1}^{GB} in (5.66). One can easily deduce one from the other having this connection in mind. Therefore, (5.78) can also be used as a definition of the dual function G_{2n+1}^{GDB}.

Theorem 5.11 *Let $p,q \in \mathbb{R}$ be such that $1 \leq p,\ q \leq \infty,\ 1/p + 1/q = 1$. If $f : [0,1] \to \mathbb{R}$ is such that $f^{(2n)} \in L_p[0,1]$ for some $n \geq 2$, then*

$$\left|\int_0^1 f(t)dt - Q_{GDB} + T_{2n}^{GDB}\right| \leq K_{GDB}(2n,q)\cdot\|f^{(2n)}\|_p, \tag{5.79}$$

if $f^{(2n+1)} \in L_p[0,1]$ for some $n \geq 2$, then

$$\left|\int_0^1 f(t)dt - Q_{GDB} + T_{2n}^{GDB}\right| \leq K_{GDB}(2n+1,q)\cdot\|f^{(2n+1)}\|_p, \tag{5.80}$$

and finally, if $f^{(2n+2)} \in L_p[0,1]$ for some $n \geq 2$, then

$$\left|\int_0^1 f(t)dt - Q_{GDB} + T_{2n}^{GDB}\right| \leq K_{GDB}^*(2n+2,q)\cdot\|f^{(2n+2)}\|_p, \tag{5.81}$$

where

$$K_{GDB}(m,q) = \frac{1}{m!}\left[\int_0^1 \left|G_m^{GDB}(t)\right|^q dt\right]^{\frac{1}{q}}$$
$$K_{GDB}^*(m,q) = \frac{1}{m!}\left[\int_0^1 \left|F_m^{GDB}(t)\right|^q dt\right]^{\frac{1}{q}}.$$

These inequalities are sharp for $1 < p \leq \infty$ and best possible for $p = 1$.

Proof. Analogous to the proof of Theorem 2.2. □

One can easily find that:

$$K_{GDB}^*(2n+2,1) = \frac{45(1-2^{-2n-1})}{16(2n+2)!(4^n-1)(4^n-4)}|B_{2n+2}|,$$
$$K_{GDB}^*(2n+2,\infty) = \frac{1}{2}\,K_{GDB}(2n+1,1) = \frac{2(4-4^{-n})|B_{2n+2}|}{(2n+2)!(4^n-1)(4^n-4)},$$
$$K_{GDB}(2n,2)$$
$$= \frac{1}{(4^n-1)(4^n-4)}\left[\frac{|B_{4n}|}{(4n)!}(2^{4-8n}-25\cdot 2^{2-6n}-2^{3-4n}+25\cdot 2^{1-2n}+42)\right]^{1/2},$$
$$K_{GDB}(2n+1,2)$$
$$= \frac{1}{4(4^n-1)(4^n-4)}\left[\frac{B_{4n+2}}{(4n+2)!}\left(2^{2-8n}-85\cdot 2^{-6n}-2^{3-4n}+85\cdot 2^{1-2n}+357\right)\right]^{1/2},$$
$$K_{GDB}^*(2n+2,2) = \frac{1}{16(2n+2)!(4^n-1)(4^n-4)}\left[2025(1-2^{-2n-1})^2 B_{2n+2}^2\right.$$
$$\left.+\frac{[(2n+2)!]^2}{(4n+4)!}\left(2^{-8n}-325\cdot 2^{-2-6n}-2^{3-4n}+325\cdot 2^{1-2n}+4497\right)|B_{4n+4}|\right]^{1/2}.$$

Theorem 5.12 *Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(2n+2)}$ is continuous on $[0,1]$ for some $n \geq 2$. Then there exists $\xi \in [0,1]$ such that*

$$R_{2n+2}^{GDB}(f) = \frac{45(1-2^{-2n-1})\cdot B_{2n+2}}{16(2n+2)!(4^n-1)(4^n-4)}\cdot f^{(2n+2)}(\xi), \tag{5.82}$$

where $R_{2n+2}^{GDB}(f)$ is the remainder in (5.77).

If, in addition, $f^{(2n+2)}$ does not change sign on $[0,1]$ for some $n \geq 2$, then there exists $\theta \in [0,1]$ such that

$$R_{2n+2}^{GDB}(f) = \theta\frac{2(4-4^{-n})\cdot B_{2n+2}}{(2n+2)!(4^n-1)(4^n-4)}\left[f^{(2n+1)}(1)-f^{(2n+1)}(0)\right]. \tag{5.83}$$

Proof. Analogous to the proof of Theorem 3.2. □

From (5.82) for $n = 2$ dual Boole's formula is obtained:

$$\begin{aligned}\int_0^1 f(t)dt - \frac{1}{45}\Big[16\left(f\left(\frac{1}{8}\right)+f\left(\frac{3}{8}\right)+f\left(\frac{5}{8}\right)+f\left(\frac{7}{8}\right)\right) \\ -10\left(f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)\right)+f\left(\frac{1}{2}\right)\Big] = \frac{31}{61931520}f^{(6)}(\xi), \quad 0<\xi<1\end{aligned} \tag{5.84}$$

and for $n = 3$, corrected dual Boole's formula:

$$\begin{aligned}\int_0^1 f(t)dt - \frac{1}{945}\Big[256\left(f\left(\frac{1}{8}\right)+f\left(\frac{3}{8}\right)+f\left(\frac{5}{8}\right)+f\left(\frac{7}{8}\right)\right) \\ -40\left(f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)\right)+f\left(\frac{1}{2}\right)\Big] - \frac{1}{504}[f'(1)-f'(0)] \\ = -\frac{127}{208089907200}f^{(8)}(\xi), \qquad 0<\xi<1\end{aligned} \tag{5.85}$$

Note that

$$\frac{31}{61931520} \approx 5.00553\cdot 10^{-7} \quad \text{and} \quad \frac{127}{208089907200} \approx 6.10313\cdot 10^{-10}.$$

5.4.3 General Bullen-Boole's inequality

Several prior subsections were devoted to the generalizations and variants of the Bullen's inequality. In this subsection we derive an inequality of similar type, only this time starting from general Boole's formula and its dual formula. We call it general Bullen-Boole's inequality.

First, add (5.65) and (5.77) then divide by 2. We get:

$$\int_0^1 f(t)dt - \hat{D}(0,1) + \hat{T}_{2n}(f) = \hat{R}_{2n+2}(f), \tag{5.86}$$

where

$$\begin{aligned}\hat{D}(0,1) &= \frac{1}{8(4^n-1)(4^n-4)}\Big[(2^{4n-1}-5\cdot 4^n+8)f(0)+4^{2n}f\left(\frac{1}{8}\right) \\ &+ 4^n(4^n-10)f\left(\frac{1}{4}\right)+4^{2n}f\left(\frac{3}{8}\right)+(4^{2n}-10\cdot 4^n+16)f\left(\frac{1}{2}\right) \\ &+ 4^{2n}f\left(\frac{5}{8}\right)+4^n(4^n-10)f\left(\frac{3}{4}\right)+4^{2n}f\left(\frac{7}{8}\right)+(2^{4n-1}-5\cdot 4^n+8)f(1)\Big] \\ \hat{T}_m(f) &= \sum_{k=1}^{m}\frac{\hat{B}_k}{k!}[f^{(k-1)}(1)-f^{(k-1)}(0)], \qquad 1\le m\le 2n \\ \hat{G}_k(t) &= \frac{1}{8(4^n-1)(4^n-4)}\left[(4^{2n}-10\cdot 4^n+16)B_k^*(1-t)+4^{2n}B_k^*(1/8-t)\right. \\ &+ 4^n(4^n-10)B_k^*(1/4-t)+4^{2n}B_k^*(3/8-t)\end{aligned}$$

$$\begin{aligned}
&+\ (4^{2n}-10\cdot 4^n+16)B_k^*(1/2-t)\\
&+\ 4^{2n}B_k^*(5/8-t)+4^n(4^n-10)B_k^*(3/4-t)+4^{2n}B_k^*(7/8-t)\big]\\
\hat{B}_1 &= 0, \quad \hat{B}_k=\hat{G}_k(0), \quad k\geq 2\\
\hat{F}_k(t) &= \hat{G}_k(t)-\hat{B}_k, \quad k\geq 1\\
\hat{R}_{2n+2}(f) &= \frac{1}{(2n+2)!}\int_0^1 \hat{F}_{2n+2}(t)df^{(2n+1)}(t)
\end{aligned}$$

The function $\hat{G}_k$ has the property $\hat{G}_k(t+1/2)=\hat{G}_k(t)$ so it is enough to study it on the interval $(0,1/4)$.

Lemma 5.6 *For $n\geq 2$, $\hat{G}_{2n+1}(t)$ has no zeros in the interval $(0,1/4)$. The sign of the function is determined by*

$$(-1)^n\hat{G}_{2n+1}(t)>0, \qquad 0<t<1/4.$$

Proof. As (5.73) implies that $\hat{G}_{2n+1}(t)=2^{-2n-1}G_{2n+1}(2t)$, the statement follows immediately from Lemma 5.4. □

Theorem 5.13 *If $f:[0,1]\to\mathbf{R}$ is such that $f^{(2n+2)}$ is continuous on $[0,1]$ for some $n\geq 2$, then there exists a point $\eta\in[0,1]$ such that*

$$\hat{R}_{2n+2}(f)=-\frac{45\cdot 4^{-n-3}\cdot B_{2n+2}}{(2n+2)!(4^n-1)(4^n-4)}\cdot f^{(2n+2)}(\eta). \tag{5.87}$$

Proof. Analogous to the proof of Theorem 3.2. □

Theorem 5.14 *Let $f:[0,1]\to\mathbf{R}$ be such that $f^{(2n+2)}$ is continuous on $[0,1]$ for some $n\geq 2$. If f is a $(2n+2)-$convex function, then for an even n we have*

$$0\leq\int_0^1 f(t)\mathrm{d}t-\tilde{D}(0,1)+T_{2n}^D(f)\leq D(0,1)-T_{2n}(f)-\int_0^1 f(t)\mathrm{d}t. \tag{5.88}$$

For an odd n inequalities are reversed.

Proof. Denote the middle part of (5.88) by LHS and the right-hand side by RHS. Then

$$LHS=R_{2n+2}^D(f) \quad \text{and} \quad RHS-LHS=-2\hat{R}_{2n+2}(f).$$

Now, applying (5.82) and (5.87), we conclude

$$\begin{aligned}
&LHS\geq 0, \quad RHS-LHS\geq 0, \quad \text{for even } n\\
&LHS\leq 0, \quad RHS-LHS\leq 0, \quad \text{for odd } n
\end{aligned}$$

and thus the proof is complete. □

Remark 5.5 For $n = 2$, (5.88) becomes

$$
\begin{aligned}
0 \leq & \int_0^1 f(t)\mathrm{d}t - \frac{1}{45}[16(f(1/8)+f(3/8)+f(5/8)+f(7/8)) \\
& - 10(f(1/4)+f(3/4))+f(1/2)] \\
\leq & \frac{1}{90}[7f(0)+32f(1/4)+12f(1/2)+32f(3/4)+7f(1)] - \int_0^1 f(t)\mathrm{d}t
\end{aligned}
$$

which implies dual Boole's formula is more accurate than classical Boole's formula. For $n = 3$, (5.88) becomes

$$
\begin{aligned}
0 \leq & \frac{1}{945}[256(f(1/8)+f(3/8)+f(5/8)+f(7/8)) - 40(f(1/4)+f(3/4)) \\
& + f(1/2)] + \frac{1}{504}[f'(1)-f'(0)] - \int_0^1 f(t)\mathrm{d}t \\
\leq & \int_0^1 f(t)\mathrm{d}t - \frac{1}{1890}[217f(0)+512f(1/4)+432f(1/2)+512f(3/4)+217f(1)] \\
& + \frac{1}{252}[f'(1)-f'(0)].
\end{aligned}
$$

Therefore, dual corrected Boole's formula is more accurate than corrected Boole's formula.

For this new quadrature formula (5.86), similar results as those obtained for general Euler-Boole's and general dual Euler-Boole's formulae can be derived analogously.

Chapter 6

Radau-type quadrature formulae

In the previous chapters, Gauss, Lobatto and Newton-Cotes quadrature formulae were obtained, using the extended Euler formulae. It is natural to wonder if Radau quadrature formulae can be obtained using the same technique. The results from this chapter were published in [48].

Radau-type quadrature formulae involve one end of the interval as a node (cf. [22]):

$$\int_{-1}^{1} f(t)dt \approx (2-w(x))\,f(-1)+w(x)f(x)$$

and

$$\int_{-1}^{1} f(t)dt \approx w(x)f(x)+(2-w(x))\,f(1).$$

Thus, let $x \in (-1,1]$ and $f:[-1,1] \to \mathbb{R}$ be such that $f^{(n-1)}$ is continuous of bounded variation on $[-1,1]$ for some $n \geq 1$. Put $x \equiv -1,\ x$ in (1.2), multiply by $2-w(x),\ w(x)$ respectively and add up. The following formula is produced:

$$\int_{-1}^{1} f(t)dt - Q(-1,x) + T_{n-1}(x) = \frac{2^{n-1}}{n!}\int_{-1}^{1} F_n(x,t)\,df^{(n-1)}(t), \tag{6.1}$$

where

$$Q(-1,x) = (2-w(x))\,f(-1)+w(x)f(x) \tag{6.2}$$

$$T_{n-1}(x) = \sum_{k=1}^{n-1} \frac{2^{k-1}}{k!}\,G_k(x,1)\,[f^{(k-1)}(1)-f^{(k-1)}(-1)],\ T_0(x)=0 \tag{6.3}$$

$$G_n(x,t) = [2 - w(x)]B_n^*\left(\frac{1-t}{2}\right) + w(x)B_n^*\left(\frac{x-t}{2}\right), \tag{6.4}$$

$$F_n(x,t) = G_n(x,t) - G_n(x,1). \tag{6.5}$$

Note that

$$\frac{\partial^k G_n(x,t)}{\partial t^k} = \frac{n!}{(-2)^k (n-k)!}\, G_{n-k}(x,t).$$

The following theorem gives the best possible estimate of error for this type of quadrature formulae.

Theorem 6.1 *Let $p,q \in \mathbb{R}$ be such that $1 \le p,\ q \le \infty$ and $1/p + 1/q = 1$. If $f : [-1,1] \to \mathbb{R}$ is such that $f^{(n)} \in L_p[-1,1]$ for some $n \ge 1$, then*

$$\left| \int_{-1}^{1} f(t)dt - Q(-1,x) + T_{n-1}(x) \right| \le \frac{2^{n-1}}{n!} \left[\int_{-1}^{1} |F_n(x,t)|^q \, dt \right]^{\frac{1}{q}} \|f^{(n)}\|_p\ . \tag{6.6}$$

The inequality is sharp for $1 < p \le \infty$ and the best possible for $p = 1$.

Proof. Analogous to the proof of Theorem 2.2. □

6.1 The first family of Radau-type quadratures

As the coefficient $w(x)$ is arbitrary, it can be chosen so that:

$$G_1(x,1) = 0 \quad \Leftrightarrow \quad w(x) = \frac{2}{x+1}\ . \tag{6.7}$$

This coefficient removes the values of the function at the end points of the interval out of $T_{n-1}(x)$ and thus provides the highest possible degree of exactness (namely, such a quadrature rule is exact for all first degree polynomials), without the values of the derivatives being included in the quadrature. To emphasize the coefficient we are working with, we denote notions (6.2)-(6.5) by $Q_{R1}(-1,x)$, $T_{n-1}^{R1}(x)$, $G_n^{R1}(x,t)$ and $F_n^{R1}(x,t)$.

Lemma 6.1 *For $x \in (-1,0] \cup \{1\}$, $F_2^{R1}(x,t)$ has no zeros in the variable t on $(-1,1)$. The sign of the function is determined by:*

$$F_2^{R1}(x,t) > 0 \ \textit{for}\ x \in (-1,0] \quad \textit{and} \quad F_2^{R1}(1,t) < 0.$$

Proof. We have:

$$F_2^{R1}(x,t) = \frac{2x}{x+1}\left[B_2\left(\frac{1-t}{2}\right) - \frac{1}{6}\right] + \frac{2}{x+1}\left[B_2^*\left(\frac{x-t}{2}\right) - B_2\left(\frac{x+1}{2}\right)\right].$$

It is obvious that $F_2^{R1}(x,-1) = F_2^{R1}(x,1) = 0$. Assume: $-1 < t \le x \le 1$. Then:

$$F_2^{R1}(x,t) = \frac{1+t}{2(1+x)}(t(1+x) - 3x + 1) = 0 \Leftrightarrow t^* = \frac{3x-1}{x+1}.$$

It is elementary to see that $t^* \le x$, but $t^* > -1$ iff $x > 0$. Also, $x = 1 \Rightarrow t^* = 1$. If $-1 < x \le t < 1$, $F_2^{R1}(x,t) = \frac{1}{2}(1-t)^2 > 0$, so the assertion is proved. □

Theorem 6.2 *Let $f : [-1,1] \to \mathbb{R}$ be such that f'' is continuous on $[-1,1]$ and let $x \in (-1,0] \cup \{1\}$. Then there exists $\xi \in [-1,1]$ such that*

$$\int_{-1}^{1} f(t)dt - \frac{2x}{x+1}f(-1) - \frac{2}{x+1}f(x) = \frac{1}{3}(1-3x)f''(\xi) \tag{6.8}$$

and

$$\int_{-1}^{1} f(t)dt - \frac{2}{x+1}f(-x) - \frac{2x}{x+1}f(1) = \frac{1}{3}(1-3x)f''(-\xi). \tag{6.9}$$

Proof. (6.8) follows after applying the Mean Value Theorem for integrals and Lemma 6.1 to the remainder in (6.1) for $n = 2$ and coefficients from (6.7). Note that $\int_{-1}^{1} F_n^{R1}(x,t)dt = -2G_n^{R1}(x,1)$. (6.9) follows analogously for $f(-x)$. □

Remark 6.1 When considering the limit process $x \to -1$, we obtain the following quadrature rules:

$$\int_{-1}^{1} f(t)dt - 2f(-1) - 2f'(-1) = \frac{4}{3}f''(\xi)$$

and

$$\int_{-1}^{1} f(t)dt - 2f(1) + 2f'(1) = \frac{4}{3}f''(-\xi).$$

Theorem 6.3 *If $f : [-1,1] \to \mathbb{R}$ is such that $f' \in L_\infty[-1,1]$, then for $x \in (-1,0]$*

$$\left| \int_{-1}^{1} f(t)dt - \frac{2x}{x+1}f(-1) - \frac{2}{x+1}f(x) \right| \le (1-x)^2 \|f'\|_\infty \tag{6.10}$$

while for $x \in [0,1]$

$$\left| \int_{-1}^{1} f(t)dt - \frac{2x}{x+1}f(-1) - \frac{2}{x+1}f(x) \right| \le \left(\frac{1+x^2}{1+x} \right)^2 \|f'\|_\infty \tag{6.11}$$

The node which provides the smallest error here is $x = \sqrt{2} - 1 \approx 0.4142$ and we have

$$\left| \int_{-1}^{1} f(t)dt - (2-\sqrt{2})\, f(-1) - \sqrt{2}\, f(\sqrt{2}-1) \right| \le (12 - 8\sqrt{2})\, \|f'\|_\infty$$

$(12 - 8\sqrt{2} \approx 0.6863)$.

Furthermore, if $f:[-1,1]\to\mathbb{R}$ *is such that* $f''\in L_\infty[-1,1]$*, then for* $x\in(-1,0]\cup\{1\}$ *we have*

$$\left|\int_{-1}^{1} f(t)dt-\frac{2x}{x+1}f(-1)-\frac{2}{x+1}f(x)\right|\le\frac{1}{3}|1-3x|\cdot\|f''\|_\infty \tag{6.12}$$

while for $x\in(0,1)$

$$\left|\int_{-1}^{1} f(t)dt-\frac{2x}{x+1}f(-1)-\frac{2}{x+1}f(x)\right|\le\frac{1-6x^2+24x^3-3x^4}{3(1+x)^3}\,\|f''\|_\infty \tag{6.13}$$

The node which provides the smallest error in this case is $x^*:=2\sqrt{2}-1-2\sqrt{2-\sqrt{2}}\approx 0.2977$ *and we have:*

$$\left|\int_{-1}^{1} f(t)dt-0.4588\cdot f(-1)-1.5412\cdot f(x^*)\right|\le 0.1644\cdot\|f''\|_\infty$$

Proof. (6.10) and (6.11) follow after taking $p=\infty$ and $n=1$ in (6.6) with coefficients from (6.7). (6.12) and (6.13) follow similarly, for $n=2$.

In order to find the nodes which provide the smallest error, the functions on the right-hand sides of all four inequalities have to be minimized. Routine calculation confirms the claims. When trying to minimize the function on the right-hand side of (6.13), note that $x^4+4x^3-26x^2+4x+1=(x+1)^4-32x^2$, so the zeros can be found analytically. □

Theorem 6.4 *Let* $f:[-1,1]\to\mathbb{R}$ *be* 2*-convex and such that* f'' *is continuous on* $[-1,1]$*, and let* $x\in(-1,0]$*. Then*

$$\frac{x}{x+1}f(-1)+\frac{1}{x+1}f(x)\le\frac{1}{2}\int_{-1}^{1} f(t)dt\le\frac{f(-1)+f(1)}{2}. \tag{6.14}$$

If f *is* 2*-concave, the inequalities are reversed.*

Proof. For a 2-convex function f, we have $f''\ge 0$, so the statement follows easily from (6.8). □

As a special case, we now obtain the classical Hermite-Hadamard inequality.

Corollary 6.1 *If* $f:[-1,1]\to\mathbb{R}$ *is* 2*-convex and such that* f'' *is continuous on* $[-1,1]$*, then*

$$f(0)\le\frac{1}{2}\int_{-1}^{1} f(t)dt\le\frac{f(-1)+f(1)}{2}.$$

If f *is* 2*-concave, the inequalities are reversed.*

Proof. Take $x=0$ in (6.14). □

Remark 6.2 All the results obtained here easily follow for the quadrature rule with the right-end of the interval as the preassigned node, therefore we will not state them explicitly.

6.2 The second family of Radau-type quadratures

Suppose we want to obtain a quadrature rule exact for all polynomials of order ≤ 2, instead of ≤ 1, as were (6.8) and (6.9). Observe (6.1) again. We considered the case when $G_1(x,1) = 0$. Now, impose another condition and choose the coefficient so that $G_2(x,1) = 0$.

$$G_2(x,1) = 0 \Leftrightarrow w(x) = \frac{4}{3(1-x^2)} \tag{6.15}$$

This will produce a quadrature rule with the desired degree of exactness, however, as a downside, the value of the function at the right end of the interval will now also be included in the quadrature. To emphasize the coefficient we are working with, we denote notions (6.2)-(6.5) by $Q_{R2}(-1,x)$, $T_{n-1}^{R2}(x)$, $G_n^{R2}(x,t)$ and $F_n^{R2}(x,t)$ for this specific coefficient.

Lemma 6.2 *For $x \in (-1,-1/3] \cup [1/3,1)$, $F_3^{R2}(x,t)$ has no zeros in t on $(-1,1)$. The sign of this function is determined by:*

$$F_3^{R2}(x,t) > 0 \text{ for } x \in [1/3,1)$$
$$F_3^{R2}(x,t) < 0 \text{ for } x \in (-1,-1/3].$$

Proof. For $-1 < t \leq x < 1$, we have

$$F_3^{R2}(x,t) = (1+t)^2 \left(\frac{2x}{1+x} - t \right) = 0 \Leftrightarrow t^* = \frac{2x}{1+x},$$

and $-1 < t^* \leq x$ iff $-1/3 < x \leq 0$. If $-1 < x \leq t < 1$,

$$F_3^{R2}(x,t) = \frac{(1-t)^2}{4} \left(\frac{2x}{1-x} - t \right) = 0 \Leftrightarrow t^{**} = \frac{2x}{1-x}.$$

Now, $x \leq t^{**} < 1$ iff $0 \leq x < 1/3$. Therefore, the claim follows. □

Theorem 6.5 *Let $f : [-1,1] \to \mathbb{R}$ be such that f''' is continuous on $[-1,1]$ and let $x \in (-1,-1/3] \cup [1/3,1)$. Then there exists $\xi \in [-1,1]$ such that*

$$\int_{-1}^{1} f(t)dt - \frac{1+3x}{3(1+x)} f(-1) - \frac{4}{3(1-x^2)} f(x) - \frac{1-3x}{3(1-x)} f(1) = \frac{2x}{9} f'''(\xi). \tag{6.16}$$

Proof. Analogous to the proof of Theorem 6.2. □

Remark 6.3 For $x = 1/3$ and $x = -1/3$, from (6.16) we get the Radau 2-point formulae:

$$\int_{-1}^{1} f(t)dt - \frac{1}{2} f(-1) - \frac{3}{2} f\left(\frac{1}{3}\right) = \frac{2}{27} f'''(\xi)$$

and

$$\int_{-1}^{1} f(t)dt - \frac{3}{2} f\left(-\frac{1}{3}\right) - \frac{1}{2} f(1) = -\frac{2}{27} f'''(\xi)$$

Remark 6.4 When considering the limit processes $x \to 1$ and $x \to -1$, the following quadrature rules are produced:

$$\int_{-1}^{1} f(t)dt - \frac{2}{3}f(-1) - \frac{4}{3}f(1) + \frac{2}{3}f'(1) = \frac{2}{9}f'''(\xi)$$

and

$$\int_{-1}^{1} f(t)dt - \frac{4}{3}f(-1) - \frac{2}{3}f(1) - \frac{2}{3}f'(-1) = -\frac{2}{9}f'''(\xi).$$

Next, we consider the error estimates for this type of quadrature rules.

Theorem 6.6 *If $f : [-1,1] \to \mathbb{R}$ is such that $f'' \in L_\infty[-1,1]$, then for $x \in (-1,-1/3] \cup [1/3,1)$*

$$\left|\int_{-1}^{1} f(t)dt - \frac{1+3x}{3(1+x)}f(-1) - \frac{4}{3(1-x^2)}f(x) - \frac{1-3x}{3(1-x)}f(1)\right| \le \frac{4}{81}\left(\frac{1+3|x|}{1+|x|}\right)^3 \|f''\|_\infty \tag{6.17}$$

while for $x \in (-1/3,1/3)$

$$\left|\int_{-1}^{1} f(t)dt - \frac{1+3x}{3(1+x)}f(-1) - \frac{4}{3(1-x^2)}f(x) - \frac{1-3x}{3(1-x)}f(1)\right| \le \frac{8(1-3x^2)(1+3x^2)^2}{81(1-x^2)^3} \|f''\|_\infty \tag{6.18}$$

Further, if $f : [-1,1] \to \mathbb{R}$ is such that $f''' \in L_\infty[-1,1]$, then for $x \in (-1,-1/3] \cup [1/3,1)$,

$$\left|\int_{-1}^{1} f(t)dt - \frac{1+3x}{3(1+x)}f(-1) - \frac{4}{3(1-x^2)}f(x) - \frac{1-3x}{3(1-x)}f(1)\right| \le \frac{2|x|}{9} \|f'''\|_\infty \tag{6.19}$$

while for $x \in (-1/3,1/3)$

$$\left|\int_{-1}^{1} f(t)dt - \frac{1+3x}{3(1+x)}f(-1) - \frac{4}{3(1-x^2)}f(x) - \frac{1-3x}{3(1-x)}f(1)\right| \le \frac{8|x|^5+49x^4-60|x|^3+22x^2-4|x|+1}{36(1-|x|)^4} \|f'''\|_\infty \tag{6.20}$$

In both cases, the node which provides the smallest error is $x = 0$. The quadrature rule thus obtained is the classical Simpson's rule. More precisely, we have:

$$\left|\int_{-1}^{1} f(t)dt - \frac{1}{3}f(-1) - \frac{4}{3}f(0) - \frac{1}{3}f(1)\right| \le \frac{8}{81}\|f''\|_\infty$$

and

$$\left|\int_{-1}^{1} f(t)dt - \frac{1}{3}f(-1) - \frac{4}{3}f(0) - \frac{1}{3}f(1)\right| \le \frac{1}{36}\|f'''\|_\infty$$

Proof. (6.17) and (6.18) follow after taking $p = \infty$ and $n = 2$ in (6.6) with coefficients from (6.15). (6.19) and (6.20) follow similarly, for $n = 3$.

As for finding the node which provides the smallest error, the functions on the right-hand sides of all four inequalities have to be minimized. The claim follows after somewhat lengthy but routine calculation. □

Corollary 6.2 *Let $f : [-1,1] \to \mathbb{R}$ be such that $f^{(n)} \in L_\infty[-1,1]$ for $n = 1,2$ or 3. Then we have:*

$$\left| \int_{-1}^{1} f(t)dt - \frac{1}{2}f(-1) - \frac{3}{2}f\left(\frac{1}{3}\right) \right| \leq C_n^\infty \, \|f^{(n)}\|_\infty, \quad n = 1,2,3 \tag{6.21}$$

where

$$C_1^\infty = \frac{25}{36}, \quad C_2^\infty = \frac{1}{6}, \quad C_3^\infty = \frac{2}{27}.$$

Proof. For $n = 2$ and $n = 3$ the assertions follow directly after taking $x = 1/3$ in (6.17) and (6.19). As for $n = 1$, take $n = 1$ and $p = \infty$ in (6.6). □

Corollary 6.3 *Let $f : [-1,1] \to \mathbb{R}$ be such that $f^{(n)} \in L_1[-1,1]$ for $n = 1,2$ or 3. Then we have:*

$$\left| \int_{-1}^{1} f(t)dt - \frac{1}{2}f(-1) - \frac{3}{2}f\left(\frac{1}{3}\right) \right| \leq C_n^1 \, \|f^{(n)}\|_1, \quad n = 1,2,3 \tag{6.22}$$

where

$$C_1^1 = \frac{5}{6}, \quad C_2^1 = \frac{2}{9}, \quad C_3^1 = \frac{1}{12}.$$

Proof. Take $p = 1$ and $n = 1,2,3$, respectively, in (6.6) and then find $\sup_{t \in [-1,1]} |F_n(1/3,t)|$. □

Theorem 6.7 *Let $f : [-1,1] \to \mathbb{R}$ be 3-convex and such that f''' is continuous on $[-1,1]$, and let $x \in (-1,1/3]$ and $y \in [1/3,1)$. Then*

$$\begin{aligned} &\frac{1+3y}{3(1+y)}f(-1) + \frac{4}{3(1-y^2)}f(y) + \frac{1-3y}{3(1-y)}f(1) \\ &\leq \int_{-1}^{1} f(t)dt \\ &\leq \frac{1+3x}{3(1+x)}f(-1) + \frac{4}{3(1-x^2)}f(x) + \frac{1-3x}{3(1-x)}f(1) \end{aligned} \tag{6.23}$$

If f is 3-concave, the inequalities are reversed.

Proof. Analogous to the proof of Theorem 6.4. □

Corollary 6.4 *If $f : [-1,1] \to \mathbb{R}$ is 3-convex and such that f''' is continuous on $[-1,1]$, then*

$$\frac{1}{2}f(-1)+\frac{3}{2}f\left(\frac{1}{3}\right) \le \int_{-1}^{1} f(t)dt \le \frac{3}{2}f\left(-\frac{1}{3}\right)+\frac{1}{2}f(1) \tag{6.24}$$

If f is 3-concave, the inequalities are reversed.

Proof. Take $x=-1/3$ and $y=1/3$ in (6.23). □

Remark 6.5 Using another, more general approach, the inequality (6.24) was also obtained in [10], i.e. [9].

6.3 Radau 3-point formulae

Let $x,y \in (-1,1],\ x<y$, and let $f : [-1,1] \to \mathbb{R}$ be such that $f^{(n-1)}$ is continuous and of bounded variation on $[-1,1]$ for some $n \ge 1$. Put $x \equiv -1,\ x,\ y$ in (1.2), multiply by $2-w_1(x,y)-w_2(x,y),\ w_1(x,y),\ w_2(x,y)$ respectively and add up. The following formula is produced:

$$\int_{-1}^{1} f(t)dt - Q(-1,x,y)+T_{n-1}(x,y) = \frac{2^{n-1}}{n!}\int_{-1}^{1} F_n(x,y,t)df^{(n-1)}(t), \tag{6.25}$$

where

$$Q(-1,x,y) = [2-w_1(x,y)-w_2(x,y)]\,f(-1)+w_1(x,y)f(x)+w_2(x,y)f(y) \tag{6.26}$$

$$T_{n-1}(x,y) = \sum_{k=1}^{n-1}\frac{2^{k-1}}{k!}\,G_k(x,y,1)\,[f^{(k-1)}(1)-f^{(k-1)}(-1)], \quad T_0(x)=0 \tag{6.27}$$

$$G_n(x,y,t) = [2-w_1(x,y)-w_2(x,y)]\,B_n^*\left(\frac{1-t}{2}\right)+w_1(x,y)B_n^*\left(\frac{x-t}{2}\right) \tag{6.28}$$

$$+w_2(x,y)B_n^*\left(\frac{y-t}{2}\right) \tag{6.29}$$

$$F_n(x,y,t) = G_n(x,y,t)-G_n(x,y,1). \tag{6.30}$$

Now, impose conditions:

$$G_1(x,y,1)=G_2(x,y,1)=G_3(x,y,1)=G_4(x,y,1)=0.$$

The unique solution of this system

$$x=\frac{1-\sqrt{6}}{5}, \quad y=\frac{1+\sqrt{6}}{5}, \quad w_1(x,y)=\frac{16+\sqrt{6}}{18}, \quad w_2(x,y)=\frac{16-\sqrt{6}}{18} \tag{6.31}$$

are the nodes and the coefficients of the Radau 3-point formula.

To emphasize the nodes and the coefficients we are going to be working with in this subsection, denote notions (6.27)- (6.30) by T_{n-1}^{R3}, $G_n^{R3}(t)$, $F_n^{R3}(t)$ and

$$Q_{R3} = \frac{2}{9}f(-1) + \frac{16+\sqrt{6}}{18}f\left(\frac{1-\sqrt{6}}{5}\right) + \frac{16-\sqrt{6}}{18}f\left(\frac{1+\sqrt{6}}{5}\right).$$

Lemma 6.3 *$F_5^{R3}(t)$ has no zeros in $(-1,1)$ and its sign is determined by $F_5^{R3}(t) > 0$.*

Proof. For $-1 \le t \le (1-\sqrt{6})/5$, we have $F_5^{R3}(t) = \frac{1}{144}(1+t)^4(1-9t)$ so the claim is obvious. So it is for $(1+\sqrt{6})/5 \le t < 1$, since there $F_5^{R3}(t) = \frac{1}{16}(1-t)^5$. For $(1-\sqrt{6})/5 \le t \le (1+\sqrt{6})/5$, the function is a bit more complicated:

$$F_5^{R3}(t) = \frac{1}{288}\, k(t)$$

where

$$k(t) = -18t^5 + 5(\sqrt{6}+2)t^4 + 20(3\sqrt{6}-7)t^3 - 30(\sqrt{6}-2)t^2 + 10(2\sqrt{6}-5)t + 10 - 3\sqrt{6}.$$

We have to prove that $k(t) > 0$. From

$$k'''(t) = -1080t^2 + 120(\sqrt{6}+2)t + 120(3\sqrt{6}-7)$$

we conclude that k'' increases on (t_1,t_2) and decreases on $[\frac{1-\sqrt{6}}{5}, t_1) \cup (t_2, \frac{1+\sqrt{6}}{5}]$, where $t_1 \approx -0.068755$ and $t_2 \approx 0.563143$. This, together with the fact that $k''(\frac{1-\sqrt{6}}{5}) < 0$, $k''(t_1) < 0$, $k''(t_2) > 0$, $k''(\frac{1+\sqrt{6}}{5}) > 0$, shows that k'' has only one zero $t^{**} \in (t_1,t_2)$. This means k' is decreasing on $[\frac{1-\sqrt{6}}{5}, t^{**})$ and increasing on $(t^{**}, \frac{1+\sqrt{6}}{5}]$. Since $k'(\frac{1-\sqrt{6}}{5}) > 0$ and $k'(\frac{1+\sqrt{6}}{5}) < 0$, it follows that k' has only one zero $t^* \in (t_1,t_2)$. From there we conclude that k increases on $[\frac{1-\sqrt{6}}{5}, t^*)$ and decreases on $(t^*, \frac{1+\sqrt{6}}{5}]$. Since $k(\frac{1-\sqrt{6}}{5}) > 0$ and $k(\frac{1+\sqrt{6}}{5}) > 0$, the claim follows. □

Theorem 6.8 *If $f : [-1,1] \to \mathbb{R}$ is such that $f^{(5)}$ is continuous on $[-1,1]$, then there exists $\xi \in [-1,1]$ such that*

$$\int_{-1}^{1} f(t)dt - Q_{R3} = \frac{1}{1125}\, f^{(5)}(\xi)$$

and

$$\int_{-1}^{1} f(t)dt - \frac{16-\sqrt{6}}{18}f\left(-\frac{1+\sqrt{6}}{5}\right) - \frac{16+\sqrt{6}}{18}f\left(-\frac{1-\sqrt{6}}{5}\right) - \frac{2}{9}f(1)$$
$$= -\frac{1}{1125}\, f^{(5)}(-\xi).$$

Proof. Analogous to the proof of Theorem 6.2. □

Theorem 6.9 *If $f : [-1,1] \to \mathbb{R}$ is 5-convex and such that $f^{(5)}$ is continuous on $[-1,1]$, then*

$$\begin{aligned}
&\frac{2}{9}f(-1) + \frac{16+\sqrt{6}}{18} f\left(\frac{1-\sqrt{6}}{5}\right) + \frac{16-\sqrt{6}}{18} f\left(\frac{1+\sqrt{6}}{5}\right) \\
&\leq \int_{-1}^{1} f(t)dt \\
&\leq \frac{16-\sqrt{6}}{18} f\left(-\frac{1+\sqrt{6}}{5}\right) + \frac{16+\sqrt{6}}{18} f\left(-\frac{1-\sqrt{6}}{5}\right) + \frac{2}{9}f(1)
\end{aligned}$$

Proof. Follows trivially from Theorem 6.8 □

Theorem 6.10 *Let $p,q \in \mathbb{R}$ be such that $1 \leq p,\ q \leq \infty$ and $1/p + 1/q = 1$. If $f : [-1,1] \to \mathbb{R}$ is such that $f^{(n)} \in L_p[-1,1]$ for some $n \geq 1$, then*

$$\left| \int_{-1}^{1} f(t)dt - Q_{R3} + T_{n-1}^{R3} \right| \leq \frac{2^{n-1}}{n!} \left[\int_{-1}^{1} \left| F_n^{R3}(t) \right|^q dt \right]^{\frac{1}{q}} \|f^{(n)}\|_p \,. \tag{6.32}$$

The inequality is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof. Analogous to the proof of Theorem 2.2. □

Corollary 6.5 *Let $f : [-1,1] \to \mathbb{R}$ be such that $f^{(k)} \in L_\infty[-1,1]$ for $k = 1,2,3,4$ or 5. Then we have*

$$\left| \int_{-1}^{1} f(t)dt - Q_{R3} \right| \leq C_k^\infty \, \|f^{(k)}\|_\infty$$

where

$$C_1^\infty \approx 0.434014, \quad C_2^\infty \approx 0.0566841, \quad C_3^\infty \approx 0.0106218,$$
$$C_4^\infty \approx 0.00247235, \quad C_5^\infty = \frac{1}{1125} \approx 0.000888889.$$

Proof. Take $p = \infty$ and $n = 1,2,3,4,5$ in (6.32). □

Corollary 6.6 *Let $f : [-1,1] \to \mathbb{R}$ be such that $f^{(k)} \in L_1[-1,1]$ for $k = 1,2,3,4$ or 5. Then we have*

$$\left| \int_{-1}^{1} f(t)dt - Q_{R3} \right| \leq C_k^1 \, \|f^{(k)}\|_1$$

where

$$C_1^1 = \left| F_1^{R3}\left(\frac{1-\sqrt{6}}{5}\right) \right| \approx 0.537092, \quad C_2^1 = \left| F_2^{R3}\left(\frac{1-\sqrt{6}}{5}\right) \right| \approx 0.094322,$$
$$C_3^1 \approx 0.0131784, \quad C_4^1 = \left| F_4^{R3}\left(-\frac{1}{3}\right) \right| \approx 0.00274348, \quad C_5^1 \approx 0.00123618.$$

Proof. Take $p = 1$ and $n = 1,2,3,4,5$ in (6.32). □

Chapter 7

A general problem of non-vanishing of the kernel in the quadrature formulae

7.1 Introduction

From the previous chapters it is clear that the procedure of deducing the quadrature formulae can be summarized as follows. Using symmetric (with respect to $1/2$) nodes $0 \le x_1 < x_2 < \cdots x_k \le 1/2 \le x_{k+1} < \cdots < x_{2k} \le 1$ and affine combinations of (1.1) it follows

$$\int_0^1 f(t)dt = \sum_{i=1}^{2k} \lambda_i f(x_i) - \widetilde{T}_n + \frac{1}{n!}\int_0^1 \left(\sum_{i=1}^{2k} \lambda_i B_n^*(x_i - t)\right) df^{(n-1)}(t), \tag{7.1}$$

where $\widetilde{T}_n = \sum_{i=1}^{n} \frac{\widetilde{B}_i}{i!}\left[f^{(i-1)}(1) - f^{(2i-1)}(0)\right]$, $\widetilde{B}_i = \sum_{j=1}^{2k} \lambda_j B_i(x_j)$, $\sum_{i=1}^{2k} \lambda_i = 1$ and $\lambda_j = \lambda_{2k+1-j}$, $j = 1,\ldots,k$. Notice that chosen symmetry implies $\widetilde{B}_{2i-1} = 0$, $i \in \mathbb{N}$, consequently (7.1) is usually written as

$$\int_0^1 f(t)dt = \sum_{i=1}^{2k} \lambda_i f(x_i) - \widetilde{T}_{2n} + \frac{1}{(2n)!}\int_0^1 G_{2n+1}(t)df^{(2n)}(t), \tag{7.2}$$

$$\int_0^1 f(t)dt = \sum_{i=1}^{2k} \lambda_i f(x_i) - \widetilde{T}_{2n} + \frac{1}{(2n+2)!}\int_0^1 F_{2n+2}(t)df^{(2n+1)}(t), \tag{7.3}$$

where $\widetilde{T}_{2n} = \sum_{i=1}^{n} \frac{\widetilde{B}_{2i}}{(2i)!}\left[f^{(2i-1)}(1) - f^{(2i-1)}(0)\right]$, $G_{2n+1}(t) = \sum_{i}^{2k} \lambda_i B^*_{2n+1}(x_i - t)$, $F_{2n+2}(t) = \sum_{i=1}^{2k} \lambda_i \left[B^*_{2n+2}(x_i - t) - B^*_{2n+2}(x_i)\right]$.

To produce quadrature formulae for preassigned nodes the following conditions are usually imposed:

$$\widetilde{B}_{2n} = \widetilde{B}_{2n-2} = \cdots = \widetilde{B}_{2(n-k+2)} = 0,\ n \geq k-1. \tag{7.4}$$

Unperturbed (uncorrected) quadrature formulae are obtained for $n = k-1$, i.e. formulae which do not involve derivatives at boundary points. Notice that (7.4) is equivalent to

$$G^{(1)}_{2n+1}(0) = G^{(3)}_{2n+1}(0) = \cdots = G^{(2k-3)}_{2n+1}(0) = 0. \tag{7.5}$$

The main step in obtaining the best possible error estimates is to prove that

$$G_{2n+1}(t) = \sum_{i=1}^{2k} \lambda_i B^*_{2n+1}(x_i - t)$$

has some "nice" zeros in $(0,1/2)$ (usually G_{2n+1} has no zeros at all in $(0,1/2)$). We formulate the following problem which seems to be interesting independently of the present context.

Problem 7.1 *Find the distribution of nodes $0 \leq x_1 < x_2 < \cdots < x_k \leq 1/2$, such that $G_{2n+1}(t) = \sum_{i=1}^{2k} \lambda_i B^*_{2n+1}(x_i - t)$ has no zeros in $(0,1/2)$, if $\sum_{i=1}^{2k} \lambda_i = 1$, $x_{2k-j+1} = 1 - x_k$, $j = 1,...,k$, $G^{(1)}_{2n+1}(0) = G^{(3)}_{2n+1}(0) = \cdots = G^{(2k-3)}_{2n+1}(0) = 0$, where $n \geq k-1$.*

Some partial results can be found in previous chapters, where nodes and weights are explicitly calculated or a priori given, thus allowing explicit expression of G_{2n+1} for some small n. An exception is Section 5.4, where some elementary motivations for the present chapter can be found.

To prove some special cases of Problem 7.1 (but of a general nature as stated above), we found the "frequency" variant of identities (1.1) and (1.2) more tractable. An easy consequence of Multiplication Theorem for periodic Bernoulli functions B^*_n in the form

$$B^*_n(x - mt) = m^{n-1} \sum_{k=0}^{m-1} B^*_n\left(\frac{x+k}{m} - t\right),\ n \geq 0,\ m \geq 1,$$

is the following theorem (see Section 1.2):

Theorem 7.1 *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n-1)}$ is continuous of bounded variation on $[0,1]$ for some $n \geq 1$. Then, for $x \in [0,1]$ and $m \in \mathbb{N}$, we have*

$$\int_0^1 f(t)dt = \frac{1}{m} \sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) - T_n(x) + \frac{1}{n! \cdot m^n} \int_0^1 B^*_n(x - mt)\,df^{(n-1)}(t), \tag{7.6}$$

where

$$T_n(x) = \sum_{j=1}^{n} \frac{B_j(x)}{j! \cdot m^j}\left[f^{(j-1)}(1) - f^{(j-1)}(0)\right].$$

Setting $x=0$ in (7.6) and using that $B_1=-1/2$, $B_{2i-1}=0$, $i\geq 2$, we write (7.6), with appropriate assumptions, in a more convenient form:

$$\int_0^1 f(t)dt = \frac{1}{m}\frac{f(0)+f(1)}{2} + \frac{1}{m}\sum_{i=1}^{m-1} f\left(\frac{i}{m}\right)$$
$$+\sum_{i=1}^{n}\frac{B_{2i}}{(2i)!m^{2i}}\left[f^{(2i-1)}(1)-f^{(2i-1)}(0)\right]$$
$$-\frac{1}{(2n+1)!m^{2n+1}}\int_0^1 B^*_{2n+1}(mt)df^{(2n)}(t). \tag{7.7}$$

Affine combinations of (7.7) with frequencies $m_0=1<m_1<\cdots m_s$, $m_i\in\mathbb{N}$, $s\in\mathbb{N}$, and weights $\lambda_0,\ldots,\lambda_s$, $\sum_{i=0}^{s}\lambda_i=1$ give:

$$\begin{aligned}\int_0^1 f(t)dt &= \frac{f(0)+f(1)}{2}\sum_{j=0}^{s}\frac{\lambda_j}{m_j}+\sum_{j=0}^{s}\frac{\lambda_j}{m_j}\sum_{i=1}^{m_j-1} f\left(\frac{i}{m_j}\right)\\ &+\sum_{i=1}^{n}\frac{B_{2i}}{(2i)!}\left[f^{(2i-1)}(1)-f^{(2i-1)}(0)\right]\sum_{j=0}^{s}\frac{\lambda_j}{m_j^{2i}}\\ &-\frac{1}{(2n+1)!}\int_0^1\left(\sum_{j=0}^{s}\frac{\lambda_j}{m_j^{2n+1}}B^*_{2n+1}(m_jt)\right)df^{(2n)}(t).\end{aligned} \tag{7.8}$$

Analogously,

$$\begin{aligned}\int_0^1 f(t)dt &= \frac{f(0)+f(1)}{2}\sum_{j=0}^{s}\frac{\lambda_j}{m_j}+\sum_{j=0}^{s}\frac{\lambda_j}{m_j}\sum_{i=1}^{m_j-1} f\left(\frac{i}{m_j}\right)\\ &+\sum_{i=1}^{n}\frac{B_{2i}}{(2i)!}\left[f^{(2i-1)}(1)-f^{(2i-1)}(0)\right]\sum_{j=0}^{s}\frac{\lambda_j}{m_j^{2i}}\\ &-\frac{1}{(2n+2)!}\int_0^1\left(\sum_{j=0}^{s}\frac{\lambda_j}{m_j^{2n+2}}\left(B^*_{2n+2}(m_jt)-B_{2n+2}\right)\right)df^{(2n+1)}(t).\end{aligned} \tag{7.9}$$

It is clear that identities (7.8) and (7.9) can be written in the form of identities (7.2) and (7.3), respectively. Also, it is easy to see that there are identities of the type (7.2) and (7.3) which cannot be of a type (7.8) and (7.9), respectively.

Again, as in (7.4) and (7.5), to produce quadrature formulae it is natural to impose the following conditions:

$$\sum_{j=0}^{s}\frac{\lambda_j}{m_j^{2n}}=\sum_{j=0}^{s}\frac{\lambda_j}{m_j^{2(n-1)}}=\cdots=\sum_{j=0}^{s}\frac{\lambda_j}{m_j^{2(n-s+1)}}=0,\ n\geq s, \tag{7.10}$$

or equivalently:

$$G^{(1)}_{2n+1}(0)=G^{(3)}_{2n+1}(0)=\cdots=G^{(2s-1)}_{2n+1}(0)=0, \tag{7.11}$$

where

$$G_{2n+1}(t) = \sum_{j=0}^{s} \frac{\lambda_j}{m_j^{2n+1}} B_{2n+1}^*(m_j t). \tag{7.12}$$

Now, we can state the following special case of Problem 7.1.

Problem 7.2 *Find the distribution of frequencies $m_0 = 1 < m_1 < m_2 < \cdots < m_s$, $m_i \in \mathbb{N}$, such that $G_{2n+1}(t) = \sum_{j=0}^{s} \frac{\lambda_j}{m_j^{2n+1}} B_{2n+1}^*(m_j t)$ has no zeros in $(0,1/2)$, if $\sum_{j=0}^{s} \lambda_j = 1$, $G_{2n+1}^{(1)}(0) = G_{2n+1}^{(3)}(0) = \cdots = G_{2n+1}^{(2s-1)}(0) = 0$, where $n \geq s$.*

7.2 Some preliminary considerations

To obtain quadrature formulae based on identities (7.8) and (7.9), we determine weights $\lambda_0, \lambda_1, \ldots, \lambda_s$ from the linear system

$$M\lambda = b, \tag{7.13}$$

where

$$M = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \frac{1}{m_1^{2n}} & \cdots & \frac{1}{m_s^{2n}} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \frac{1}{m_1^{2(n-s+1)}} & \cdots & \frac{1}{m_s^{2(n-s+1)}} \end{pmatrix}, \tag{7.14}$$

$\lambda = (\lambda_0 \ \ \lambda_1 \ \ \cdots \ \ \lambda_s)^T$ and $b = (1 \ \ 0 \ \ \cdots \ \ 0)^T$. It is easy to see that $\mathrm{Det}M \neq 0$ (see also [108]), so the system (7.13) has a unique solution. Cramer's rule and (7.12) immediately imply:

$$\begin{aligned} G_{2n+1}(t) &= \sum_{j=0}^{s} \frac{\lambda_j}{m_j^{2n+1}} B_{2n+1}^*(m_j t) \\ &= \frac{1}{\mathrm{Det}M} \begin{vmatrix} B_{2n+1}^*(t) & \frac{B_{2n+1}^*(m_1 t)}{m_1^{2n+1}} & \cdots & \frac{B_{2n+1}^*(m_s t)}{m_s^{2n+1}} \\ 1 & \frac{1}{m_1^{2n}} & \cdots & \frac{1}{m_s^{2n}} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \frac{1}{m_1^{2(n-s+1)}} & \cdots & \frac{1}{m_s^{2(n-s+1)}} \end{vmatrix}, \end{aligned} \tag{7.15}$$

which gives

$$G_{2n+1}(t)=\frac{(-1)^s}{(m_1\cdots m_s)^{2n+1}\mathrm{Det}M}\begin{vmatrix} 1 & 1 & \cdots & 1 & B^*_{2n+1}(t)\\ m_1 & m_1^3 & \cdots & m_1^{2s-1} & B^*_{2n+1}(m_1t)\\ m_2 & m_2^3 & \cdots & m_2^{2s-1} & B^*_{2n+1}(m_2t)\\ \vdots & \vdots & \cdots & \vdots & \vdots\\ m_s & m_s^3 & \cdots & m_s^{2s-1} & B^*_{2n+1}(m_st)\end{vmatrix}. \tag{7.16}$$

Define

$$H_{2n+1}(t)=\begin{vmatrix} 1 & 1 & \cdots & 1 & B^*_{2n+1}(t)\\ m_1 & m_1^3 & \cdots & m_1^{2s-1} & B^*_{2n+1}(m_1t)\\ m_2 & m_2^3 & \cdots & m_2^{2s-1} & B^*_{2n+1}(m_2t)\\ \vdots & \vdots & \cdots & \vdots & \vdots\\ m_s & m_s^3 & \cdots & m_s^{2s-1} & B^*_{2n+1}(m_st)\end{vmatrix}. \tag{7.17}$$

In this way Problem 7.2 is equivalent to the following problem.

Problem 7.3 *Find the distribution of frequencies such that $H_{2n+1}(t)$ has no zeros in $(0,1/2)$ for $n\geq s$.*

Example 7.1 *Suppose that $m_0=1<m_1=3<m_2=4$, $n=s=2$. Using Wolfram's Mathematica, for $H_5(t)=\begin{vmatrix} 1 & 1 & B_5^*(t)\\ 3 & 3^3 & B_5^*(3t)\\ 4 & 4^3 & B_5(4t)\end{vmatrix}$, $H_5(0.45)=1.11285$ and $H_5(0.3)=-3.3996$, so H_5 has zeros in $(0,1/2)$.*

For a given sequence of functions $a_0,\ldots,a_n$ defined on some real interval I and given sequence $x_0,\ldots,x_n$ in I, we introduce

$$D\begin{pmatrix} a_0 & \cdots & a_{n-1} & a_n\\ x_0 & \cdots & x_{n-1} & x_n\end{pmatrix}=\begin{vmatrix} a_0(x_0) & a_1(x_0) & \cdots & a_n(x_0)\\ a_0(x_1) & a_1(x_1) & \cdots & a_n(x_1)\\ \vdots & \vdots & \cdots & \vdots\\ a_0(x_n) & a_1(x_n) & \cdots & a_n(x_n)\end{vmatrix},$$

and, if $a_0,\ldots,a_n$ are sufficiently smooth, we denote by $W(a_0,\ldots,a_n)(x)$ the Wronskian of the sequence $a_0,\ldots,a_n$ at $x\in I$.

To transform the functions H_{2n+1} in a more suitable form, the following General Mean Value theorem from [66] appears to be useful.

Theorem 7.2 *Let $a_0,\ldots,a_n$ be a sequence of real functions of a real variable x, possessing derivatives up to the order n, and further such that the Wronskians $W(a_0,\ldots,a_k)$, $k=0,1,\ldots,n$, do not vanish on a certain interval I. Let $f(x)$ be a function possessing derivatives up to the order n in I. Finally let $x_0,x_1,\ldots,x_n$ be a system of $(n+1)$ values of x in I. There exists at least one value ξ in I such that*

$$\frac{D\begin{pmatrix} a_0 & \cdots & a_{n-1} & f\\ x_0 & \cdots & x_{n-1} & x_n\end{pmatrix}}{D\begin{pmatrix} a_0 & \cdots & a_{n-1} & a_n\\ x_0 & \cdots & x_{n-1} & x_n\end{pmatrix}}=\frac{W(a_0,\ldots,a_{n-1},f)(\xi)}{W(a_0,\ldots,a_{n-1},a_n)(\xi)}. \tag{7.18}$$

To apply Theorem 7.2 we set:

$$a_0(x) = x, a_1(x) = x^3, \ldots, a_s(x) = x^{2s+1},$$

$$f(x) = B^*_{2n+1}(xt) = g(xt),$$

$$x_0 = 1, x_1 = m_1, \ldots, x_s = m_s,\ I = [1, m_s].$$

Assumptions of Theorem 7.2 are obviously satisfied, so there is an $\xi \in [1, m_s]$ such that

$$H_{2n+1}(t) = D\begin{pmatrix} x & x^3 & \cdots & x^{2s-1} & g(xt) \\ 1 & m_1 & \cdots & m_{s-1} & m_s \end{pmatrix}$$
$$= \frac{D\begin{pmatrix} x & x^3 & \cdots & x^{2s-1} & x^{2s+1} \\ 1 & m_1 & \cdots & m_{s-1} & m_s \end{pmatrix}}{W\left(x, x^3, \ldots, x^{2s+1}\right)(\xi)}$$
$$\cdot \begin{vmatrix} \xi & \xi^3 & \cdots & \xi^{2s-1} & g(\xi t) \\ 1 & 3\xi^2 & \cdots & (2s-1)\xi^{2s-2} & tg'(\xi t) \\ 0 & 3!\xi & \cdots & (2s-1)(2s-2)\xi^{2s-3} & t^2 g''(\xi t) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \frac{(2s-1)!}{(s-1)!}\xi^{s-1} & t^s g^{(s)}(\xi t) \end{vmatrix}$$

Denote the last determinant in (7.19) by $\widetilde{H}_{2n+1}(t, \xi)$. Multiplying the kth row of this determinant by ξ^{k-1}, $k = 2, \ldots, s$, then extracting from the lth column ξ^{2l-1}, $l = 1, \ldots, s$, we have

$$\widetilde{H}_{2n+1}(t, \xi) = \xi^{\frac{s(s+1)}{2}} \begin{vmatrix} 1 & 1 & \cdots & 1 & f(\xi t) \\ 1 & 3 & \cdots & 2s-1 & \xi t f'(\xi t) \\ 0 & 3! & \cdots & (2s-1)(2s-2) & (\xi t)^2 f''(\xi t) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \frac{(2s-1)!}{(s-1)!} & (\xi t)^s f^{(s)}(\xi t) \end{vmatrix}$$
$$= \xi^{\frac{s(s+1)}{2}} \begin{vmatrix} a_0(1) & a_1(1) & \cdots & a_{s-1}(1) & f(u) \\ a_0'(1) & a_1'(1) & \cdots & a_{s-1}'(1) & uf'(u) \\ a_0''(1) & a_1''(1) & \cdots & a_{s-1}''(1) & u^2 f''(u) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_0^{(s)}(1) & a_1^{(s)}(1) & \cdots & a_{s-1}^{(s)}(1) & u^s f^{(s)}(u) \end{vmatrix}, \tag{7.19}$$

where $u = \xi t$. Note that $0 < u < m_s/2$ for $0 < u < 1/2$. It remains to investigate the sign of the function given by the last determinant in (7.19). Using the Laplace expansion of determinants, this function is up to the sign equal to the function

$$F(u) = \sum_{j=0}^{s} (-i)^j \mathrm{Ch}^{(j)} u^j f^{(j)}(u), \tag{7.20}$$

where $\mathrm{Ch}^{(j)}$ (Ch stands for Chebyshev) is the determinant of the matrix obtained from the $(s+1)\times s$ matrix

$$\mathrm{Ch} = \begin{pmatrix} a_0(1) & a_1(1) & \cdots & a_{s-1}(1) \\ a_0'(1) & a_1'(1) & \cdots & a_{s-1}'(1) \\ \vdots & \vdots & \cdots & \vdots \\ a_0^{(s)}(1) & a_1^{(s)}(1) & \cdots & a_{s-1}^{(s)}(1) \end{pmatrix}$$

by deleting the $(j+1)$th row. The sequence of functions $a_0, a_1, \ldots, a_s$ can be obtained by using the universal construction of Chebyshev systems (S. Karlin, W. J. Studden, Tchebyscheff systems with applications in analysis and statistics, Interscience, New York, 1966.). Take $\omega_0(x) = x$, $\omega_1(x) = 2x, \ldots$, $\omega_s(x) = 2sx$. Then

$$a_0(x) = \omega_0(x),\ a_1(x) = \omega_0(x)\int_0^x \omega_1(t_1)dt_1,$$

$$a_2(x) = \omega_0(x)\int_0^x \omega_1(t_1)\int_0^{t_1} \omega_2(t_2)dt_2dt_1, \ldots,$$

$$a_s(x) = \omega_0(x)\int_0^x \omega_1(t_1)\int_0^{t_1} \omega_2(t_2)\cdots\int_0^{t_{s-1}} \omega_s(t_s)dt_s\cdots dt_2dt_1.$$

Using this and properties of determinants (manipulating with columns of $\mathrm{Ch}^{(j)}$), straightforward calculation reveals that

$$\mathrm{Ch}^{(j)} = \mathrm{Ch}^{(s)}\cdot\frac{(2s-j)!}{j!(2s-2j)!!} = 2^{s-1}\cdot 4^{s-2}\cdots(2s-4)^2\cdot(2s-2)\cdot\frac{(2s-j)!}{j!(2s-2j)!!}$$

(for the case $j = s$ see [75]).

Lemma 7.1 *Let $(F_k)_{k=0}^s$ be the sequence of functions defined by:*

$$F_k(u) = \sum_{j=k}^{s-1}(-1)^j\mathrm{Ch}_k^{(j)}u^{j-k}f^{(j+k)}(u) + (-1)^s\mathrm{Ch}^{(s)}u^{s-k}f^{(s+k)}(u),\ u\in\mathbb{R}, \tag{7.21}$$

where

$$\mathrm{Ch}_k^{(j)} = \frac{j!}{(j-k)!}\frac{(2s-j-k)!}{(2s-j)!}\mathrm{Ch}^{(j)},\ j = k,\ldots,s. \tag{7.22}$$

Then

(i) $F_0(u) = F(u)$ *where* $F(u)$ *is given by (7.20)*

(ii) $F_s(u) = (-1)^s\mathrm{Ch}^{(s)}f^{(2s)}(u)$,

(iii) $F_k'(u) = uF_{k+1}(u)$, $k = 0,\ldots s-1$.

Proof. The first two properties are obvious. Let us prove the third property. Simple rearranging gives:

$$\begin{aligned}
F_k'(u) &= (-1)^k \left[\mathrm{Ch}_k^{(k)} - \mathrm{Ch}_k^{(k+1)}\right] f^{(2k+1)}(u) \\
&+ \sum_{j=k+1}^{s-1} (-1)^j \left[\mathrm{Ch}_k^{(j)} - (j+1-k)\mathrm{Ch}_k^{(j+1)}\right] u^{j-k} f^{(j+k+1)}(u) \\
&+(-1)^s \mathrm{Ch}^{(s)} u^{s-k} f^{(s+k+1)}(u).
\end{aligned} \tag{7.23}$$

It is obvious that $\mathrm{Ch}_k^{(k)} = \mathrm{Ch}_k^{(k+1)} = (2s-2k-1)!!$. It remains to show that $\mathrm{Ch}_k^{(j)} - (j+1-k)\mathrm{Ch}_k^{(j+1)} = \mathrm{Ch}_{k+1}^{(j)}$. Using (7.22) and that $\mathrm{Ch}^{(j+1)} = \frac{2s-2j}{(j+1)(2s-j)}\mathrm{Ch}^{(j)}$ we have:

$$\begin{aligned}
&\mathrm{Ch}_k^{(j)} - (j+1-k)\mathrm{Ch}_k^{(j+1)} \\
&= \frac{j!}{(j-k)!}\frac{(2s-j-k)!}{(2s-j)!}\mathrm{Ch}^{(j)} - \frac{(j+1)!}{(j-k)!}\frac{(2s-j-k-1)!}{(2s-j-1)!}\mathrm{Ch}^{(j+1)} \\
&= \mathrm{Ch}^{(j)}\left[\frac{j!}{(j-k)!}\frac{(2s-j-k)!}{(2s-j)!} - \frac{(j+1)!}{(j-k)!}\frac{(2s-j-k-1)!}{(2s-j-1)!}\frac{2s-2j}{(j+1)(2s-j)}\right] \\
&= \mathrm{Ch}^{(j)}\frac{j!}{(j-k-1)!}\frac{(2s-j-k-1)!}{(2s-j)!} = \mathrm{Ch}_{k+1}^{(j)}.
\end{aligned}$$

We can write

$$\begin{aligned}
F_k'(u) = u&\left[\sum_{j=k+1}^{s-1} (-1)^j \mathrm{Ch}_{k+1}^{(j)} u^{j-k-1} f^{(j+k+1)}(u)\right. \\
&\left. +(-1)^s \mathrm{Ch}^{(s)} u^{s-k-1} f^{(s+k+1)}(u)\right] \\
&= uF_{k+1}(u).
\end{aligned}$$

□

Theorem 7.3 *Suppose that $m_0 = 1 < m_1 < m_2 < \cdots < m_s$, $m_i \in \mathbb{N}$ and $\sum_{j=0}^{s} \lambda_j = 1$, $\lambda_j \in \mathbb{R}$. Then the function $G_{2n+1}(t) = \sum_{j=0}^{s} \frac{\lambda_j}{m_j^{2n+1}} B_{2n+1}^*(m_j t)$ such that $G_{2n+1}^{(1)}(0) = G_{2n+1}^{(3)}(0) = \cdots = G_{2n+1}^{(2s-1)}(0) = 0$, $s \le n$, has no zeros in $\left(0, \frac{1}{2m_s}\right]$.*

Proof. Suppose that $0 < t \le \frac{1}{2m_s}$. Then $0 < u = \xi t \le \frac{1}{2}$ (see the discussion and notation below Theorem 7.2). The claim follows from previous reductions and because Lemma 7.23 implies

$$F(u) = F_0(u) = \int_0^u t_1 \int_0^{t_1} t_2 \cdots t_{s-1} \int_0^{t_{s-1}} t_s (-1)^s \mathrm{Ch}^{(s)} B_{2n+1-2s}^*(t_s) dt_s \cdots dt_2 dt_1. \tag{7.24}$$

□

Remark 7.1 Notice that $F_k(u) = (-1)^k \mathrm{Ch}_k^{(k)} f^{(2k)}(u) + u \cdot [\cdots]$. Consequently $F_k(0) = 0$ for functions for which $f^{(2k)}(0) = 0$, $k = 0, \ldots, s$. From Lemma 7.23 follows:

$$F(u) = F_0(u) = \int_0^u t_1 \int_0^{t_1} t_2 \cdots \int_0^{t_{s-1}} t_s (-1)^s \mathrm{Ch}^{(s)} f^{(2s)}(t_s) dt_s \cdots dt_2 dt_1. \tag{7.25}$$

7.3 Case $m_i = m^i$

In the previous section we proved that the function $G_{2n+1}(t)$, defined in (7.12) such that conditions (7.11) hold and $\sum_{j=0}^{s} \lambda_j \neq 0$, has no zeros on $(0, \frac{1}{2m_s}]$.

In the present section we give the complete answer for the case $m_i = m^i$, $i = 0, \ldots s$, $m \geq 2$, $m \in \mathbb{N}$, in the sense that we prove that the function H_{2n+1} defined in (7.17), using frequencies $1, m, m^2, \ldots, m^s$, has no zeros on $(0, 1/2)$.

Theorem 7.4 *Let $s \in \mathbb{N}$ and $m \in \mathbb{N}$, $m \geq 2$. Then the function*

$$K_s(t; f) = \begin{vmatrix} 1 & 1 & \cdots & 1 & f(t) \\ m & m^3 & \cdots & m^{2s-1} & f(mt) \\ m^2 & (m^2)^3 & \cdots & (m^2)^{2s-1} & f(m^2 t) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ m^s & (m^s)^3 & \cdots & (m^s)^{2s-1} & f(m^s t) \end{vmatrix} \tag{7.26}$$

has no zeros on $(0, 1/2)$, for any odd function $f : \mathbb{R} \to \mathbb{R}$ which is periodic with period $T = 1$, such that $f^{(2s-2)}$ is continuous on $\mathbb{R}$ and strictly concave (convex) on $(0, 1/2)$.

Proof. The proof is by induction. Suppose that f is continuous on $\mathbb{R}$ and strictly concave on $(0, 1/2)$. We shall prove that $K_1(t; f) = f(mt) - mf(t)$ is strictly negative for $t \in (0, 1/2)$. Using strict concavity $f(mt) < mf(t)$ for $0 < t \leq \frac{1}{2m}$. Now, we split the proof into two cases.

Case $m = 2k+1$: Suppose that $\frac{1}{2} - \frac{1}{2m} \leq t < \frac{1}{2}$. Set $g(x) = f(\frac{1}{2} - x)$. Obviously g is strictly concave on $(0, 1/2)$ and $g(0) = 0$. This implies $g(mx) < mg(x)$ for $x = \frac{1}{2} - t$, which gives $f(-k + mt) = f(mt) < mf(t)$. In this way we conclude that $f(mt) < mf(t)$ for $t \in (0, \frac{1}{2m}] \cup [\frac{1}{2} - \frac{1}{2m}, \frac{1}{2})$.

Set $M = \max_{t \in [0,1/2]} f(t)$. There is $t_1 \in (0, \frac{1}{2m})$ such that $f(mt_1) = M$, and there is $t_2 \in \left(\frac{1}{2} - \frac{1}{2m}, \frac{1}{2}\right)$ such that $f(mt_2) = M$. Suppose that $t \in \left(\frac{1}{2m}, \frac{1}{2} - \frac{1}{2m}\right)$ is arbitrary. Then there is $\lambda \in (0, 1)$ such that $t = \lambda t_1 + (1 - \lambda) t_2$. Finally:

$$f(mt) \leq M = \lambda f(mt_1) + (1 - \lambda) f(mt_2) < f(m(\lambda t_1 + (1 - \lambda) t_2)) = f(mt).$$

Case $m = 2k$: Notice that $f(mt) < 0$ for $\frac{1}{2} - \frac{1}{2m} < t < \frac{1}{2}$. Arguing as in the final step of the proof for the case $m = 2k+1$, it is enough to prove that $f\left(m\left(t - \frac{1}{2m}\right)\right) < mf(t)$ for

$\frac{1}{2}-\frac{1}{2m}\le t<\frac{1}{2}$. Set again $g(x)=f(\frac{1}{2}-x)$. Obviously g is strictly concave on $(0,1/2)$ and $g(0)=0$, so $g(mx)<mg(x)$ for $0<x\le\frac{1}{2m}$. This implies for $x=\frac{1}{2}-t$ that $g(\frac{m}{2}-mt)<mg(\frac{1}{2}-t)$, so $f(mt+\frac{1}{2}-k)=f(mt-\frac{1}{2})<mf(t)$.

The proof when f is convex is analogous.

To prove the inductive step we use the Sylvester identity for determinants with the first and the last row and with the two last columns. It follows (where we denote by $\mathrm{V}[\alpha_1,\ldots,\alpha_n]$ the Vandermonde determinant):

$$K_s(t;f)\cdot\begin{vmatrix} m & m^3 & \cdots & m^{2s-3}\\ m^2 & (m^2)^3 & \cdots & (m^2)^{2s-3}\\ \vdots & \vdots & \cdots & \vdots\\ m^{s-1} & \left(m^{s-1}\right)^3 & \cdots & \left(m^{s-1}\right)^{2s-3}\end{vmatrix}$$

$$=\begin{vmatrix} \begin{vmatrix} 1 & 1 & \cdots & 1\\ m & & \cdots & m^{2s-1}\\ \vdots & \vdots & \cdots & \vdots\\ m^{s-1} & & \cdots & \left(m^{s-1}\right)^{2s-1}\end{vmatrix} & \begin{vmatrix} 1 & 1 & \cdots & 1 & f(t)\\ m & m^3 & \cdots & m^{2s-3} & f(mt)\\ \vdots & \vdots & \cdots & \vdots & \vdots\\ m^{s-1} & \left(m^{s-1}\right)^3 & \cdots & \left(m^{s-1}\right)^{2s-3} & f(m^{s-1}t)\end{vmatrix}\\ \begin{vmatrix} m & & \cdots & m^{2s-1}\\ m^2 & & \cdots & (m^2)^{2s-1}\\ \vdots & \vdots & \cdots & \vdots\\ m^s & & \cdots & (m^s)^{2s-1}\end{vmatrix} & \begin{vmatrix} m & m^3 & \cdots & m^{2s-3} & f(mt)\\ m^2 & (m^2)^3 & \cdots & (m^2)^{2s-3} & f(m^2t)\\ \vdots & \vdots & \cdots & \vdots & \vdots\\ m^s & (m^s)^3 & \cdots & (m^s)^{2s-3} & f(m^st)\end{vmatrix}\end{vmatrix}$$

$$=\begin{vmatrix} m\cdot m^2\cdots m^{s-1}\mathrm{V}[1,m^2,\ldots,m^{2(s-1)}] & K_{s-1}(t;f)\\ m\cdot m^2\cdots m^s\mathrm{V}[m^2,m^4,\ldots m^{2s}] & m\cdot m^3\cdots m^{2s-3}K_{s-1}(mt;f)\end{vmatrix}$$

$$=\begin{vmatrix} m^{\frac{(s-1)s}{2}}\mathrm{V}[1,m^2,\ldots,m^{2(s-1)}] & K_{s-1}(t;f)\\ m^{\frac{s(s+1)}{2}}\cdot m^{(s-1)s}\mathrm{V}\left[1,m^2,\ldots,m^{2(s-1)}\right] & m^{(s-1)^2}K_{s-1}(mt;f)\end{vmatrix}$$

$$=m^{\frac{(s-1)(3s-2)}{2}}\mathrm{V}\left[1,m^2,\ldots,m^{2(s-1)}\right]\begin{vmatrix} 1 & K_{s-1}(t;f)\\ m^{2s-1} & K_{s-1}(mt;f)\end{vmatrix},$$

which gives

$$\begin{aligned} K_s(t;f)&=m^{\frac{(s-1)(s+2)}{2}}\frac{\mathrm{V}\left[1,m^2,\ldots,m^{2(s-1)}\right]}{\mathrm{V}\left[1,m^2,\ldots,m^{2(s-2)}\right]}\begin{vmatrix} 1 & K_{s-1}(t;f)\\ m^{2s-1} & K_{s-1}(mt;f)\end{vmatrix}\\ &=C\begin{vmatrix} 1 & 1 & \cdots & 1 & f(mt)-m^{2s-1}f(t)\\ m & m^3 & \cdots & m^{2s-3} & f(m^2t)-m^{2s-1}f(mt)\\ \vdots & \vdots & \cdots & \vdots & \vdots\\ m^{s-1} & \left(m^{s-1}\right)^3 & \cdots & \left(m^{s-1}\right)^{2s-3} & f(m^st)-m^{2s-1}f(m^{s-1}t)\end{vmatrix}\\ &=CK_{s-1}(t;g), \end{aligned}\tag{7.27}$$

where $g(t) = f(mt) - m^{2s-1}f(t)$. To use the inductive assumption we only have to prove that $g^{(2s-4)}$ is strictly concave or strictly convex on $(0,1/2)$. We have

$$g^{(2s-4)}(t) = m^{2s-4}\left[f^{(2s-4)}(mt) - m^3 f^{(2s-4)}(t)\right].$$

Set $h(t) = f^{(2s-4)}(mt) - m^3 f^{(2s-4)}(t)$. Since $h''(t) = m^2\left[f^{(2s-2)}(mt) - mf^{(2s-2)}(t)\right]$, using assumption ($f^{(2s-2)}$ is strictly concave or strictly convex on $(0,1/2)$) and the basis of induction, we conclude that h'' has no zeros on $(0,1/2)$. Since $f^{(2s-2)}$ is continuous, h'' has constant sign on $(0,1/2)$. It follows that h, and consequently $g^{(2s-4)}$, is strictly concave or strictly convex on $(0,1/2)$. Using inductive assumption, $K_{s-1}(t;g)$ has no zeros on $(0,1/2)$, so by (7.27), $K_s(t;f)$ has no zeros on $(0,1/2)$. The proof is complete. □

Obvious examples of the functions which satisfy conditions in the previous theorem are $f(t) = B^*_{2n+1}(t)$ and $f(t) = \sin 2\pi t$.

Example 7.2 *The Boole and Simpson formula can be easily deduced using above procedure.*

7.4 Using the Fourier expansion of the periodic Bernoulli functions

In this section we present yet another method to study zeros of the functions defined as the function H_{2n+1}. This method is motivated by the Fourier expansion of the periodic Bernoulli functions given by

$$B^*_{2n+1}(t) = \frac{(-1)^{n+1}(2n+1)!}{(2\pi)^{2n+1}}\sum_{k=1}^{\infty}\frac{\sin(2k\pi t)}{k^{2n+1}},\ n \geq 1, x \in \mathbf{R},\ n = 0,\ x \neq k. \tag{7.28}$$

Recall that we reduced the problem of zeros of the function G_{2n+1} to the one of the function H_{2n+1}. Using (7.28) we can write

$$H_{2n+1}(t) = C_n \sum_{k=1}^{\infty}\frac{1}{k^{2n+1}}\begin{vmatrix} 1 & 1 & \cdots & 1 & \sin(2k\pi t) \\ m_1 & m_1^3 & \cdots & m_1^{2s-1} & \sin(2km_1\pi t) \\ m_2 & m_2^3 & \cdots & m_2^{2s-1} & \sin(2km_2\pi t) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ m_s & m_s^3 & \cdots & m_s^{2s-1} & \sin(2km_s\pi t) \end{vmatrix}. \tag{7.29}$$

We consider the case with no gaps in frequencies i.e. case with $(s-1)$-nontrivial frequencies $m_1 = 2 < m_2 = 3 < \cdots < m_{s-1} = s$. In that case we have

$$H_{2n+1}(t) = C_n \sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} \begin{vmatrix} 1 & 1 & \cdots & 1 & \sin(2k\pi t) \\ 2 & 2^3 & \cdots & 2^{2s-3} & \sin(4k\pi t) \\ 3 & 3^3 & \cdots & 3^{2s-3} & \sin(6k\pi t) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ s & s^3 & \cdots & s^{2s-3} & \sin(2sk\pi t) \end{vmatrix}. \tag{7.30}$$

To simplify terms in the above expansion set:

$$S(\alpha) = \begin{vmatrix} 1 & 1 & \cdots & 1 & \sin\alpha \\ 2 & 2^3 & \cdots & 2^{2s-3} & \sin 2\alpha \\ 3 & 3^3 & \cdots & 3^{2s-3} & \sin 3\alpha \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ s & s^3 & \cdots & s^{2s-3} & \sin s\alpha \end{vmatrix} = \sin\alpha \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ 2 & 2^3 & \cdots & 2^{2s-3} & \frac{\sin 2\alpha}{\sin\alpha} \\ 3 & 3^3 & \cdots & 3^{2s-3} & \frac{\sin 3\alpha}{\sin\alpha} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ s & s^3 & \cdots & s^{2s-3} & \frac{\sin s\alpha}{\sin\alpha} \end{vmatrix}.$$

Recall that the Chebyshev polynomials of the second kind are defined by

$$U_n(x) = \frac{\sin(n+1)\alpha}{\sin\alpha}, \ \alpha = \arccos x, \ n = 0, 1, \ldots.$$

We will need the following properties of the Chebyshev polynomials U_n:

(i)

$$U_n(1) = n+1$$

(ii)

$$U_n^{(k)}(1) = \frac{(n+k+1)!}{(2k+1)!!(n-k)!} \Leftarrow U_n^{(k)}(x) = xU_{n-1}^{(k)}(x) + (k+n)U_{n-1}^{(k-1)}(x)$$

(iii)

$$|U_n(x)| \le n+1$$

Set:

$$\overline{S}(x) = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ 2 & 2^3 & \cdots & 2^{2s-3} & U_1(x) \\ 3 & 3^3 & \cdots & 3^{2s-3} & U_2(x) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ s & s^3 & \cdots & s^{2s-3} & U_{s-1}(x) \end{vmatrix}.$$

Obviously $\overline{S}(1) = 0$, $\overline{S}^{(s-1)}(1) = (s-1)!\mathrm{V}[1, 2^2, \ldots, (s-1)^2]$. We want to prove that $\overline{S}^{(k)}(1) = 0$ for $k = 1, \ldots, s-2$. We have:

$$\overline{S}'(x) = \begin{vmatrix} 1 & 1 & \cdots & 1 & 0 \\ 2 & 2^3 & \cdots & 2^{2s-3} & U_1'(x) \\ 3 & 3^3 & \cdots & 3^{2s-3} & U_2'(x) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ s & s^3 & \cdots & s^{2s-3} & U_{s-1}'(x) \end{vmatrix}.$$

We have $U_l'(1) = \frac{(l+2)!}{3!!(l-1)!}$. Multiplying the first column with -1 and adding to the second column, we have in the $(l+1)$th row:

$$(l+1)^3 - (l+1) = (l+1)(l+2)l = \frac{(l+2)!}{(l-1)!},$$

which obviously implies that $\overline{S}'(1) = 0$. To make a general argument we compare $U_{l-1}^{(k)}(1)$, $k \le l-1$, with the lth row in the $(k+1)$th column after reducing the first $k+1$ columns on the lower trapezoid form. Using properties of the polynomials U_n we have:

$$U_{l-1}^{(k)}(1) = \frac{(l+k)!}{(2k+1)!!(l-k-1)!}.$$

After reducing the first $k+1$ columns on the lower trapezoid form in the lth row and the $(k+1)$th column, using inductive property of the Vandermonde determinant applied on determinant

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^3 & \cdots & 2^{2k+1} \\ \vdots & \vdots & \cdots & \vdots \\ k & k^3 & \cdots & k^{2k+1} \\ l & l^3 & \cdots & l^{2k+1} \end{vmatrix},$$

we obtain

$$\begin{aligned} & l(l^2-k^2)(l^2-(k-1)^2 \cdots (l^2-2^2)(l^2-1) \\ = {} & l(l-k)(l+k)(l+k-1)(l-k+1)\cdots(l+2)(l-2))(l+1)(l-1) = \frac{(l+k)!}{(l-k-1)!}. \end{aligned}$$

This finishes the proof that $\overline{S}^{(k)}(1) = 0$ for $k = 0, 1, \ldots, s-2$.

Since $\overline{S}$ is the polynomial of degree $s-1$, we conclude

$$\overline{S}(x) = \mathrm{V}[1, 2^2, \ldots, (s-1)^2](x-1)^{s-1}.$$

It follows

$$\begin{aligned} S(\alpha) &= \sin\alpha \mathrm{V}[1, 2^2, \ldots, (s-1)^2](x-1)^{s-1} \\ &= (-1)^{s-1}\mathrm{V}[1, 2^2, \ldots, (s-1)^2]\sin\alpha(1-\cos\alpha)^{s-1}. \end{aligned}$$

Finally, we can write

$$H_{2n+1}(t) = D_n \sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} \sin(2k\pi t)(1-\cos(2k\pi t))^{s-1}.$$

To illustrate how this expression helps in proving that H_{2n+1} has no zeros in $(0, 1/2)$, we will prove that

$$\sin(2\pi t)(1-\cos(2\pi t))^{s-1} > -\sum_{k=2}^{\infty} \frac{1}{k^{2n+1}} \sin(2k\pi t)(1-\cos(2k\pi t))^{s-1}, \ s \le n.$$

Rearranging this is obviously equivalent to inequality

$$-\sum_{k=2}^{\infty}\frac{1}{k^{2n+1}}\frac{\sin(2k\pi t)}{\sin(2\pi t)}\left(\frac{\sin(k\pi t)}{\sin\pi t}\right)^{2s-2}<1.$$

Since $\left|\frac{\sin k\alpha}{\sin\alpha}\right|\le k$, it is enough to prove that

$$\sum_{k=2}^{\infty}\frac{1}{k^{2n+1}}k^{2s-1}=\sum_{k=2}^{\infty}\frac{1}{k^{2(n-s)+2}}<1.$$

Recall that $s\le n$, so it is enough to prove that

$$\sum_{k=2}^{\infty}\frac{1}{k^2}<1,$$

and this is obvious since the LHS is equal to $\pi^2/6-1$.

The method of this section can give more information in negative direction. Let us consider Example 1 from Section 2 i.e.

$$H_5(t)=\begin{vmatrix}1 & 1 & B_5^*(t)\\ 3 & 3^3 & B_5^*(3t)\\ 4 & 4^3 & B_5^*(4t)\end{vmatrix}=C_n\sum_{k=1}^{\infty}\frac{1}{k^5}\begin{vmatrix}1 & 1 & \sin(2k\pi t)\\ 3 & 3^3 & \sin(6k\pi t)\\ 4 & 4^3 & \sin(8k\pi t)\end{vmatrix},$$

and consider the first term

$$\begin{vmatrix}1 & 1 & \sin(2\pi t)\\ 3 & 3^3 & \sin(6\pi t)\\ 4 & 4^3 & \sin(8\pi t)\end{vmatrix}=\sin(2\pi t)\begin{vmatrix}1 & 1 & 1\\ 3 & 3^3 & U_2(x)\\ 4 & 4^3 & U_3(x)\end{vmatrix}=\sin(2\pi t)\overline{S}(x).$$

It can be shown easily that

$$\overline{S}(x)=(x-1)^2(144+192x),$$

which implies

$$\begin{vmatrix}1 & 1 & \sin(2\pi t)\\ 3 & 3^3 & \sin(6\pi t)\\ 4 & 4^3 & \sin(8\pi t)\end{vmatrix}=\sin(2\pi t)(1-\cos(2\pi t))^2(144+192\cos(2\pi t)).$$

It can be shown easily that terms with higher frequencies (and small amplitudes) cannot remove the zeros in the basic term.

Chapter 8

Euler's method for weighted integral formulae

8.1 Introduction

In this chapter we will consider the remainder of the quadrature formula

$$\int_a^b w(x)f(x)dx = \sum_{k=1}^{n} A_k f(x_k) + E_n(f), \tag{8.1}$$

where $\sum_{k=1}^{n} A_k = 1$.

In [79] V.I.Krylov assumed that this formula is exact for all polynomials of degree $m-1$, and using the representation by the Taylor series, transformed this formula into an equation of Euler's form:

$$\begin{aligned}\int_a^b w(x)f(x)dx \;&=\; \sum_{k=1}^{n} A_k f(x_k) + C_0\left[f^{(m-1)}(b) - f^{(m-1)}(a)\right] + \cdots \\ &+\; C_{s-1}\left[f^{(m+s-2)}(b) - f^{(m+s-2)}(a)\right] + E_n^s(f),\end{aligned} \tag{8.2}$$

where for $E(x) = \begin{cases} 1, & x > 0 \\ \frac{1}{2}, & x = 0 \\ 0, & x < 0 \end{cases}$, we have

$$\begin{aligned}
C_i &= (b-a)^{-1} \int_a^b L_i(t)dt, \\
L_0(t) &= K(t) = \int_t^b w(x)\frac{(x-t)^{m-1}}{(m-1)!}dx - \sum_{k=1}^n A_k E(x_k - t)\frac{(x_k-t)^{m-1}}{(m-1)!}, \\
L_{i+1}(t) &= \int_a^t [C_i - L_i(x)]\,dx, \\
E_n^s(f) &= \int_a^b f^{(m+s)}(t)L_s(t)dt.
\end{aligned} \tag{8.3}$$

Equations (8.3) give a method for sequentially calculating the C_i and $L_i(t)$. However, V.I.Krylov found a representation for C_i and $L_i(t)$ directly from the kernel $K(t)$. To do this he returned to the initial quadrature formula (8.1) with the integral representation for the remainder

$$E_n(f) = \int_a^b f^{(m)}(t)K(t)dt,$$

with replacing $f^{(m)}(t)$ by its expansion in terms of Bernoulli polynomials. Then

$$\begin{aligned}
C_i &= \frac{(b-a)^{m+i-1}}{(m+i)!} E_n\left[B_{m+i}\left(\frac{t-a}{b-a}\right)\right] \\
&= \frac{(b-a)^{m+i-1}}{(m+i)!}\left\{\int_a^b w(t)B_{m+i}\left(\frac{t-a}{b-a}\right)dt - \sum_{k=1}^n A_k B_{m+i}\left(\frac{x_k-a}{b-a}\right)\right\}.
\end{aligned} \tag{8.4}$$

Similarly he obtained for $L_s(t)$ the expression

$$L_s(t) = -\frac{(b-a)^{m+s-1}}{(m+s)!} E_{n,x}\left[B^*_{m+s}\left(\frac{x-t}{b-a}\right) - B^*_{m+s}\left(\frac{x-a}{b-a}\right)\right], \tag{8.5}$$

where $E_{n,x}$ indicates the remainder when the quadrature formula is applied with respect to the variable x.

He also gave the series in (8.3) for increasing the precision of quadrature formula

$$\begin{aligned}
&\int_{-1}^{-1} (1-x)^{\alpha}(1+x)^{\beta} f(x)dx \approx \sum_{k=1}^n A_k f(x_k) \\
&+ C_0\left[f^{(2n-1)}(1) - f^{(2n-1)}(-1)\right] + C_1\left[f^{(2n)}(1) - f^{(2n)}(-1)\right] + \dots
\end{aligned}$$

where $\alpha, \beta > -1$ and the nodes are the zeros of the Jacobi polynomial. So,

$$C_0 = \frac{2^{\alpha+\beta+2n} n!\Gamma(\alpha+\beta+n+1)\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{(2n)!(\alpha+\beta+2n+1)[\Gamma(\alpha+\beta+2n+1)]^2},$$

$$
\begin{aligned}
C_1 &= \frac{\beta-\alpha}{\alpha+\beta+2n}\left[\frac{\alpha+\beta}{\alpha+\beta+2n+2}+2n\right] \\
&\cdot \frac{n!2^{\alpha+\beta+2n}\Gamma(\alpha+\beta+n+1)\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{(2n+1)!\Gamma(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+2n+2)}.
\end{aligned}
$$

For the special ultraspherical case, $\alpha=\beta$, the C_i with odd subscripts are zero:

$$
C_0 = \frac{2^{2\alpha}n!\Gamma(2\alpha+n+1)}{(2n)!(2\alpha+2n+1)}\left[\frac{2^n\Gamma(\alpha+n+1)}{\Gamma(2\alpha+2n+1)}\right]^2,
$$

$$
\begin{aligned}
C_2 &= \frac{2^{2\alpha}n!\Gamma(2\alpha+n+1)}{(2n+2)!}\left[\frac{2^n\Gamma(\alpha+n+1)}{\Gamma(2\alpha+2n+1)}\right]^2 \\
&\cdot\left[\frac{2n^2+2(2\alpha+1)n+2\alpha-1}{(2\alpha+2n-1)(2\alpha+2n+1)(2\alpha+2n+3)}+\frac{n(n-1)}{(2\alpha+2n-1)(2\alpha+2n+1)}-\frac{(n+1)(2n+1)}{3(2\alpha+2n+1)}\right].
\end{aligned}
$$

8.2 Main results

Let us suppose $f^{(r-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $r\geq 1$ and let $w:[a,b]\to[0,\infty)$ be some probability density function, that is, an integrable function satisfying $\int_a^b w(t)dt=1$.

A. Aglić Aljinović and J. Pečarić (in [5]) have proved the following two identities

$$
\begin{aligned}
f(x) &= \int_a^b w(t)f(t)\mathrm{d}t+\sum_{k=1}^{r}\frac{(b-a)^{k-1}}{k!} \\
&\cdot\left(B_k\left(\frac{x-a}{b-a}\right)-\int_a^b w(t)B_k\left(\frac{t-a}{b-a}\right)\mathrm{d}t\right)\left[f^{(k-1)}(b)-f^{(k-1)}(a)\right] \\
&- \frac{(b-a)^{r-1}}{r!}\int_a^b\left(B_r^*\left(\frac{x-t}{b-a}\right)-\int_a^b w(u)B_r^*\left(\frac{u-t}{b-a}\right)\mathrm{d}u\right)\mathrm{d}f^{(r-1)}(t)
\end{aligned}
\tag{8.6}
$$

and

$$
\begin{aligned}
f(x) &= \int_a^b w(t)f(t)\mathrm{d}t+\sum_{k=1}^{r-1}\frac{(b-a)^{k-1}}{k!} \\
&\cdot\left(B_k\left(\frac{x-a}{b-a}\right)-\int_a^b w(t)B_k\left(\frac{t-a}{b-a}\right)\mathrm{d}t\right)\left[f^{(k-1)}(b)-f^{(k-1)}(a)\right] \\
&- \frac{(b-a)^{r-1}}{r!}\int_a^b\left(B_r^*\left(\frac{x-t}{b-a}\right)-B_r\left(\frac{x-a}{b-a}\right)\right. \\
&- \left.\int_a^b w(u)\left(B_r^*\left(\frac{u-t}{b-a}\right)-B_r\left(\frac{u-a}{b-a}\right)\right)\mathrm{d}u\right)\mathrm{d}f^{(r-1)}(t).
\end{aligned}
\tag{8.7}
$$

Now, using the identities (8.6) i (8.7), we will consider the remainder $E_n(f)$ of the quadrature formula (8.1).

Theorem 8.1 *Let us suppose $f^{(r-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $r \geq 1$. If $w:[a,b] \to [0,\infty)$ is some probability density function, that is, an integrable function satisfying $\int_a^b w(t)\mathrm{d}t = 1$, then the following formulae hold*

$$\begin{aligned}
\int_a^b w(t)f(t)dt &= \sum_{k=1}^n A_k f(x_k) + \sum_{i=1}^r \frac{(b-a)^{i-1}}{i!} \\
&\cdot \left(\int_a^b w(t)B_i\left(\frac{t-a}{b-a}\right)dt - \sum_{k=1}^n A_k B_i\left(\frac{x_k-a}{b-a}\right)\right)\left[f^{(i-1)}(b) - f^{(i-1)}(a)\right] \\
&- \frac{(b-a)^{r-1}}{r!}\int_a^b \left(\int_a^b w(u)B_r^*\left(\frac{u-t}{b-a}\right)\mathrm{d}u - \sum_{k=1}^n A_k B_r^*\left(\frac{x_k-t}{b-a}\right)\right)\mathrm{d}f^{(r-1)}(t)
\end{aligned} \tag{8.8}$$

and

$$\begin{aligned}
\int_a^b w(t)f(t)dt &= \sum_{k=1}^n A_k f(x_k) + \sum_{i=1}^{r-1} \frac{(b-a)^{i-1}}{i!} \\
&\cdot \left(\int_a^b w(t)B_i\left(\frac{t-a}{b-a}\right)dt - \sum_{k=1}^n A_k B_i\left(\frac{x_k-a}{b-a}\right)\right)\left[f^{(i-1)}(b) - f^{(i-1)}(a)\right] \\
&- \frac{(b-a)^{r-1}}{r!}\int_a^b \left(\int_a^b w(u)\left(B_r^*\left(\frac{u-t}{b-a}\right) - B_r\left(\frac{u-a}{b-a}\right)\right)\mathrm{d}u\right. \\
&- \left.\sum_{k=1}^n A_k\left(B_r^*\left(\frac{x_k-t}{b-a}\right) - B_r\left(\frac{x_k-a}{b-a}\right)\right)\right)\mathrm{d}f^{(r-1)}(t).
\end{aligned} \tag{8.9}$$

Proof. First, we put $x = x_k$ in the identity (8.6). Then multiplying it by A_k and summing up from 1 to n, we obtain identity (8.8).

The proof of formula (8.9) is similar. □

Corollary 8.1 *Let us suppose $f^{(m+s-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $(m+s) \geq 1$. If $w:[a,b] \to [0,\infty)$ is some probability density function, that is, an integrable function satisfying $\int_a^b w(t)\mathrm{d}t = 1$, then the following formulae hold*

$$\begin{aligned}
\int_a^b w(t)f(t)dt &= \sum_{k=1}^n A_k f(x_k) + \sum_{j=0}^s \frac{(b-a)^{j+m-1}}{(m+j)!} \\
&\cdot \left(\int_a^b w(t)B_{j+m}\left(\frac{t-a}{b-a}\right)dt - \sum_{k=1}^n A_k B_{j+m}\left(\frac{x_k-a}{b-a}\right)\right)\left[f^{(m+j-1)}(b) - f^{(m+j-1)}(a)\right] \\
&- \frac{(b-a)^{m+s-1}}{(m+s)!}\int_a^b \left(\int_a^b w(u)B_{m+s}^*\left(\frac{u-t}{b-a}\right)\mathrm{d}u - \sum_{k=1}^n A_k B_{m+s}^*\left(\frac{x_k-t}{b-a}\right)\right)\mathrm{d}f^{(m+s-1)}(t)
\end{aligned} \tag{8.10}$$

and

$$\int_a^b w(t)f(t)dt = \sum_{k=1}^{n} A_k f(x_k) + \sum_{j=0}^{s-1} \frac{(b-a)^{j+m-1}}{(m+j)!} \tag{8.11}$$

$$\cdot \left(\int_a^b w(t) B_{j+m}\left(\frac{t-a}{b-a}\right) dt - \sum_{k=1}^{n} A_k B_{j+m}\left(\frac{x_k-a}{b-a}\right) \right) \left[f^{(m+j-1)}(b) - f^{(m+j-1)}(a) \right]$$

$$- \frac{(b-a)^{m+s-1}}{(m+s)!} \int_a^b \left(\int_a^b w(u) \left(B^*_{m+s}\left(\frac{u-t}{b-a}\right) - B_{m+s}\left(\frac{u-a}{b-a}\right) \right) \mathrm{d}u \right.$$

$$\left. - \sum_{k=1}^{n} A_k \left(B^*_{m+s}\left(\frac{x_k-t}{b-a}\right) - B_{m+s}\left(\frac{x_k-a}{b-a}\right) \right) \right) \mathrm{d}f^{(m+s-1)}(t).$$

These formulae are exact for all polynomials of degree $\leq m-1$.

Proof. First, we put $r = m+s$ in the identity (8.8). Now, if we put $f(t) = P_l(t)$ $(l \leq m-1)$ in the identity (8.8) we get

$$\int_a^b w(t)P_l(t)dt = \sum_{k=1}^{n} A_k P_l(x_k) + \sum_{i=1}^{l} C_i \left[f^{(i-1)}(b) - f^{(i-1)}(a) \right],$$

where $C_i = \frac{(b-a)^{i-1}}{i!} \left(\int_a^b w(t) B_i\left(\frac{t-a}{b-a}\right) dt - \sum_{k=1}^{n} A_k B_i\left(\frac{x_k-a}{b-a}\right) \right)$.

If formula (8.8) has to be exact for polynomial $P_l(t)$, $l = 1,2,\ldots,m-1$, by induction we get $C_1 = C_2 = \cdots = C_{m-1} = 0$.

So, with substitution $i = j+m$ in formula (8.8) we obtain formula (8.10).

The proof of formula (8.11) is similar. □

Remark 8.1 The formula (8.11) was proved by V.I. Krylov in [79].

Remark 8.2 If we put $s = 0$ in the identity (8.11), then for $m_1 < m$ we get

$$\int_a^b w(t)f(t)dt = \sum_{k=1}^{n} A_k f(x_k) \tag{8.12}$$

$$- \frac{(b-a)^{m_1-1}}{m_1!} \int_a^b \left(\int_a^b w(u) \left(B^*_{m_1}\left(\frac{u-t}{b-a}\right) - B_{m_1}\left(\frac{u-a}{b-a}\right) \right) \mathrm{d}u \right.$$

$$\left. - \sum_{k=1}^{n} A_k \left(B^*_{m_1}\left(\frac{x_k-t}{b-a}\right) - B_{m_1}\left(\frac{x_k-a}{b-a}\right) \right) \right) \mathrm{d}f^{(m_1-1)}(t).$$

In the following discussion we assume that $f : [a,b] \to \mathbb{R}$ has a continuous derivative of order $m+s$, for some $m+s \geq 1$. In this case we will use (8.11) and we define

$$F_{m+s}(t) = \int_a^b w(u) \left(B^*_{m+s}\left(\frac{u-t}{b-a}\right) - B_{m+s}\left(\frac{u-a}{b-a}\right) \right) \mathrm{d}u$$

$$- \sum_{k=1}^{n} A_k \left(B^*_{m+s}\left(\frac{x_k-t}{b-a}\right) - B_{m+s}\left(\frac{x_k-a}{b-a}\right) \right). \tag{8.13}$$

Theorem 8.2 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(m+s)}$ is a continuous function on $[a,b]$ and $F_{m+s}(t) > 0, t \in [a,b]$. Then there exists a point $\eta \in (a,b)$ such that*

$$\int_a^b w(t)f(t)dt = \sum_{k=1}^{n} A_k f(x_k) + \sum_{j=0}^{s-1} \frac{(b-a)^{j+m-1}}{(m+j)!} \tag{8.14}$$
$$\cdot \left(\int_a^b w(t) B_{j+m}\left(\frac{t-a}{b-a}\right) dt - \sum_{k=1}^{n} A_k B_{j+m}\left(\frac{x_k-a}{b-a}\right)\right) \left[f^{(m+j-1)}(b) - f^{(m+j-1)}(a) \right]$$
$$+ \frac{(b-a)^{m+s}}{(m+s)!} f^{(m+s)}(\eta) \left(\int_a^b w(u) B_{m+s}\left(\frac{u-a}{b-a}\right) du - \sum_{k=1}^{n} A_k B_{m+s}\left(\frac{x_k-a}{b-a}\right)\right).$$

Proof. From the mean value theorem for integrals we have

$$\int_a^b w(t)f(t)dt = \sum_{k=1}^{n} A_k f(x_k) + \sum_{j=0}^{s-1} \frac{(b-a)^{j+m-1}}{(m+j)!}$$
$$\cdot \left(\int_a^b w(t) B_{j+m}\left(\frac{t-a}{b-a}\right) dt - \sum_{k=1}^{n} A_k B_{j+m}\left(\frac{x_k-a}{b-a}\right)\right) \left[f^{(m+j-1)}(b) - f^{(m+j-1)}(a) \right]$$
$$- \frac{(b-a)^{m+s-1}}{(m+s)!} f^{(m+s)}(\eta) \int_a^b \left(\int_a^b w(u) \left(B^*_{m+s}\left(\frac{u-t}{b-a}\right) - B_{m+s}\left(\frac{u-a}{b-a}\right)\right) \mathrm{d}u \right.$$
$$\left. - \sum_{k=1}^{n} A_k \left(B^*_{m+s}\left(\frac{x_k-t}{b-a}\right) - B_{m+s}\left(\frac{x_k-a}{b-a}\right)\right)\right) \mathrm{d}t.$$

Because

$$\int_a^b B^*_{m+s}\left(\frac{y-t}{b-a}\right) dt = \int_a^y B_{m+s}\left(\frac{y-t}{b-a}\right) dt + \int_y^b B_{m+s}\left(\frac{y-t}{b-a}+1\right) dt = 0, \quad y \in [a,b]$$

we get identity (8.14). □

Remark 8.3 We can rewrite the identity (8.14) as

$$\int_a^b w(t)f(t)dt = \sum_{k=1}^{n} A_k f(x_k) + \sum_{j=0}^{s-1} C_j \left[f^{(m+j-1)}(b) - f^{(m+j-1)}(a) \right] + C_s f^{(m+s)}(\eta)\,(b-a),$$

where

$$C_j = \frac{(b-a)^{j+m-1}}{(m+j)!} \left(\int_a^b w(t) B_{j+m}\left(\frac{t-a}{b-a}\right) dt - \sum_{k=1}^{n} A_k B_{j+m}\left(\frac{x_k-a}{b-a}\right)\right).$$

For $f(t) = P_m(t) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$ we get

$$\int_a^b w(t) P_m(t) dt = \sum_{k=1}^{n} A_k P_m(x_k) + C_0 m! a_m\,(b-a).$$

8.3 Related inequalities

In this section we use formulae established in Corollary 8.1 to prove a number of inequalities using L_p norms for $1 \leq p \leq \infty$. These inequalities are generally sharp (in case $p = 1$ the best possible).

Theorem 8.3 *Assume (p,q) is a pair of conjugate exponents, $1 \leq p,q \leq \infty$. Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(m+s)} \in L_p[a,b]$ for some $m+s \geq 1$. Then we have*

$$\begin{aligned}
&\Bigg| \int_a^b w(t)f(t)dt - \sum_{k=1}^{n} A_k f(x_k) - \sum_{j=0}^{s} \frac{(b-a)^{j+m-1}}{(m+j)!} \\
&\cdot \left(\int_a^b w(t) B_{j+m}\left(\frac{t-a}{b-a}\right) dt - \sum_{k=1}^{n} A_k B_{j+m}\left(\frac{x_k-a}{b-a}\right) \right) \left[f^{(m+j-1)}(b) - f^{(m+j-1)}(a) \right] \Bigg| \\
&\leq K_n(m,s,p,w) \cdot \| f^{(m+s)} \|_p, \qquad (8.15)
\end{aligned}$$

and

$$\begin{aligned}
&\Bigg| \int_a^b w(t)f(t)dt - \sum_{k=1}^{n} A_k f(x_k) - \sum_{j=0}^{s-1} \frac{(b-a)^{j+m-1}}{(m+j)!} \\
&\cdot \left(\int_a^b w(t) B_{j+m}\left(\frac{t-a}{b-a}\right) dt - \sum_{k=1}^{n} A_k B_{j+m}\left(\frac{x_k-a}{b-a}\right) \right) \left[f^{(m+j-1)}(b) - f^{(m+j-1)}(a) \right] \Bigg| \\
&\leq K_n^*(m,s,p,w) \cdot \| f^{(m+s)} \|_p, \qquad (8.16)
\end{aligned}$$

where

$$K_n(m,s,p,w) = \frac{(b-a)^{m+s-1}}{(m+s)!} \left[\int_a^b \left| \int_a^b w(u) B_{m+s}^*\left(\frac{u-t}{b-a}\right) \mathrm{d}u - \sum_{k=1}^{n} A_k B_{m+s}^*\left(\frac{x_k-t}{b-a}\right) \right|^q \mathrm{d}t \right]^{\frac{1}{q}},$$

$$\begin{aligned}
K_n^*(m,s,p,w) \; = \; & \frac{(b-a)^{m+s-1}}{(m+s)!} \left[\int_a^b \left| \int_a^b w(u) \left(B_{m+s}^*\left(\frac{u-t}{b-a}\right) - B_{m+s}\left(\frac{u-a}{b-a}\right) \right) \mathrm{d}u \right. \right. \\
& \left. \left. - \sum_{k=1}^{n} A_k \left(B_{m+s}^*\left(\frac{x_k-t}{b-a}\right) - B_{m+s}\left(\frac{x_k-a}{b-a}\right) \right) \right|^q \mathrm{d}t \right]^{\frac{1}{q}}.
\end{aligned}$$

The constants $K_n(m,s,p,w)$ and $K_n^*(m,s,p,w)$ are sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof. The proof is analogous to the proof of Theorem 2.2. □

8.4 Chebyshev-Gauss formulae of the first kind of Euler type

If the weight function is $w(t) = \frac{1}{\pi\sqrt{1-t^2}}$, $t \in [-1,1]$, we have Chebyshev-Gauss formulae given by

$$\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} dt = \pi \sum_{k=1}^{n} A_k f(x_k) + E_n(f), \tag{8.17}$$

where

$$A_k = \frac{1}{n}, \quad k = 1, \dots, n,$$

and x_k are zeros of Chebyshev polynomials of the first kind defined as

$$T_n(x) = \cos(n \arccos(x)).$$

$T_n(x)$ has exactly n distinct zeros, all of which lie in the interval $[-1,1]$ and

$$x_k = \cos\left(\frac{(2k-1)\pi}{2n}\right).$$

Error of approximation formula (8.17) is given by

$$E_n(f) = \frac{\pi}{2^{2n-1}(2n)!} f^{(2n)}(\eta), \quad \eta \in (-1,1).$$

In the next theorem we establish Chebyshev-Gauss formulae of the first kind of Euler type which are exact for all polynomials of degree $\leq m-1$.

Theorem 8.4 *Let us suppose $f^{(m+s-1)}$ is a continuous function of bounded variation on $[-1,1]$ for some $(m+s) \geq 1$. Then the following formulae hold*

$$\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} dt = \frac{\pi}{n} \sum_{k=1}^{n} f(x_k) + T_{m+s}^{CG1}(f,n) + \frac{2^{m+s-1}}{(m+s)!} \int_{-1}^{1} G_{m+s}^{CG1}(t,n) \mathrm{d}f^{(m+s-1)}(t) \tag{8.18}$$

and

$$\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} dt = \frac{\pi}{n} \sum_{k=1}^{n} f(x_k) + T_{m+s-1}^{CG1}(f,n) + \frac{2^{m+s-1}}{(m+s)!} \int_{-1}^{1} F_{m+s}^{CG1}(t,n) \mathrm{d}f^{(m+s-1)}(t), \tag{8.19}$$

where

$$T_{m+s}^{CG1}(f,n) = \sum_{j=0}^{s} \frac{2^{j+m-1}}{(m+j)!} \cdot \left(\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} B_{j+m}\left(\frac{t+1}{2}\right) dt \right.$$
$$\left. - \frac{\pi}{n} \sum_{k=1}^{n} B_{j+m}\left(\frac{x_k+1}{2}\right) \right) \left[f^{(m+j-1)}(1) - f^{(m+j-1)}(-1) \right],$$

$$G_{m+s}^{CG1}(t,n) = \frac{\pi}{n}\sum_{k=1}^{n} B_{m+s}^{*}\left(\frac{x_k-t}{2}\right) - \int_{-1}^{1}\frac{1}{\sqrt{1-u^2}} B_{m+s}^{*}\left(\frac{u-t}{2}\right)\mathrm{d}u$$

and

$$\begin{aligned} F_{m+s}^{CG1}(t,n) &= \frac{\pi}{n}\sum_{k=1}^{n}\left(B_{m+s}^{*}\left(\frac{x_k-t}{2}\right) - B_{m+s}\left(\frac{x_k+1}{2}\right)\right) \\ &\quad - \int_{-1}^{1}\frac{1}{\sqrt{1-u^2}}\left(B_{m+s}^{*}\left(\frac{u-t}{2}\right) - B_{m+s}\left(\frac{u+1}{2}\right)\right)\mathrm{d}u. \end{aligned}$$

These formulae are exact for all polynomials of degree $\leq m-1$.

Proof. This is a special case of Corollary 8.1 for $a=-1,\ b=1,\ w(t)=\frac{1}{\pi\sqrt{1-t^2}}$ and $A_k=\frac{1}{n}$. □

Theorem 8.5 *Assume (p,q) is a pair of conjugate exponents, $1\leq p,q\leq\infty$. Let $f:[-1,1]\to\mathbb{R}$ be such that $f^{(m+s)}\in L_p[-1,1]$ for some $m+s\geq 1$. Then we have*

$$\left|\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}dt - \frac{\pi}{n}\sum_{k=1}^{n} f(x_k) - T_{m+s}^{CG1}(f,n)\right| \leq \pi K_n\left(m,s,p,\frac{1}{\pi\sqrt{1-t^2}}\right)\cdot\|f^{(m+s)}\|_p, \tag{8.20}$$

and

$$\left|\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}dt - \frac{\pi}{n}\sum_{k=1}^{n} f(x_k) - T_{m+s-1}^{CG1}(f,n)\right| \leq \pi K_n^{*}\left(m,s,p,\frac{1}{\pi\sqrt{1-t^2}}\right)\cdot\|f^{(m+s)}\|_p. \tag{8.21}$$

The constants $K_n\left(m,s,p,\frac{1}{\pi\sqrt{1-t^2}}\right)$ and $K_n^{*}\left(m,s,p,\frac{1}{\pi\sqrt{1-t^2}}\right)$ are sharp for $1<p\leq\infty$ and the best possible for $p=1$.

Proof. This is a special case of Theorem 8.3 for $a=-1,\ b=1,\ w(t)=\frac{1}{\pi\sqrt{1-t^2}}$ and $A_k=\frac{1}{n}$.
□

Remark 8.4 For $n=1$ and $x_1=0$ we get one-point Chebyshev-Gauss formulae of the first kind of Euler type

$$\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}dt = \pi f(0) + T_{m+s}^{CG1}(f,1) + \frac{2^{m+s-1}}{(m+s)!}\int_{-1}^{1} G_{m+s}^{CG1}(t,1)\mathrm{d}f^{(m+s-1)}(t)$$

and

$$\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}dt = \pi f(0) + T_{m+s-1}^{CG1}(f,1) + \frac{2^{m+s-1}}{(m+s)!}\int_{-1}^{1} F_{m+s}^{CG1}(t,1)\mathrm{d}f^{(m+s-1)}(t).$$

Especially for $m=1$ and $s=0$ we get

$$G_1^{CG1}(t,1) = F_1^{CG1}(t,1) = \begin{cases} -\frac{\pi}{2} - \arcsin t, & -1\leq t\leq 0, \\ \frac{\pi}{2} - \arcsin t, & 0<t\leq 1. \end{cases}$$

Now, inequalities (8.20) and (8.21) reduce to

$$\left|\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}\mathrm{d}t-\pi f(0)\right|\leq \pi K_1\left(1,0,p,\frac{1}{\pi\sqrt{1-t^2}}\right)\cdot\|f'\|_p$$

and

$$\left|\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}\mathrm{d}t-\pi f(0)\right|\leq \pi K_1^*\left(1,0,p,\frac{1}{\pi\sqrt{1-t^2}}\right)\cdot\|f'\|_p,$$

where $K_1\left(1,0,\infty,\frac{1}{\pi\sqrt{1-t^2}}\right) = K_1^*\left(1,0,\infty,\frac{1}{\pi\sqrt{1-t^2}}\right) = \frac{2}{\pi}$, $K_1\left(1,0,2,\frac{1}{\pi\sqrt{1-t^2}}\right) = K_1^*\left(1,0,2,\frac{1}{\pi\sqrt{1-t^2}}\right) = \frac{\sqrt{2\pi-4}}{\pi} \approx \frac{1.51102}{\pi}$ and $K_1\left(1,0,1,\frac{1}{\pi\sqrt{1-t^2}}\right) = K_1^*\left(1,0,1,\frac{1}{\pi\sqrt{1-t^2}}\right) = \frac{1}{2}$.

The first and third constant have also been obtained in [77].

If the presumptions of the Theorem 8.2 hold, for $m = 2$ and $s = 0$ we get

$$\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}dt=\pi f(0)+\frac{\pi}{4}f''(\eta),\quad \eta\in(-1,1), \tag{8.22}$$

which is the well known one-point Chebyshev-Gauss formula of the first kind.

Remark 8.5 For $n = 2$, $x_1 = -\frac{\sqrt{2}}{2}$ and $x_2 = \frac{\sqrt{2}}{2}$ we get two-point Chebyshev-Gauss formulae of the first kind of Euler type

$$\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}dt=\frac{\pi}{2}\left[f\left(-\frac{\sqrt{2}}{2}\right)+f\left(\frac{\sqrt{2}}{2}\right)\right]+T_{m+s}^{CG1}(f,2)$$
$$+\frac{2^{m+s-1}}{(m+s)!}\int_{-1}^{1}G_{m+s}^{CG1}(t,2)\mathrm{d}f^{(m+s-1)}(t)$$

and

$$\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}dt=\frac{\pi}{2}\left[f\left(-\frac{\sqrt{2}}{2}\right)+f\left(\frac{\sqrt{2}}{2}\right)\right]+T_{m+s-1}^{CG1}(f,2)$$
$$+\frac{2^{m+s-1}}{(m+s)!}\int_{-1}^{1}F_{m+s}^{CG1}(t,2)\mathrm{d}f^{(m+s-1)}(t).$$

Especially, for $m_1 < 4$ and $s = 0$ inequalities (8.20) and (8.21) reduce to

$$\left|\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}\mathrm{d}t-\frac{\pi}{2}\left[f\left(-\frac{\sqrt{2}}{2}\right)+f\left(\frac{\sqrt{2}}{2}\right)\right]\right|$$
$$\leq \pi K_2\left(m_1,0,p,\frac{1}{\pi\sqrt{1-t^2}}\right)\cdot\|f^{(m_1)}\|_p,$$

$$\left|\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}\mathrm{d}t-\frac{\pi}{2}\left[f\left(-\frac{\sqrt{2}}{2}\right)+f\left(\frac{\sqrt{2}}{2}\right)\right]\right|$$
$$\leq \pi K_2^*\left(m_1,0,p,\frac{1}{\pi\sqrt{1-t^2}}\right)\cdot\|f^{(m_1)}\|_p,$$

where

$K_2\left(1,0,\infty,\frac{1}{\pi\sqrt{1-t^2}}\right) = K_2^*\left(1,0,\infty,\frac{1}{\pi\sqrt{1-t^2}}\right) = \frac{2\sqrt{2}-2}{\pi} \approx \frac{0.828427}{\pi}$,

$K_2\left(1,0,2,\frac{1}{\pi\sqrt{1-t^2}}\right) = K_2^*\left(1,0,2,\frac{1}{\pi\sqrt{1-t^2}}\right) = \frac{\sqrt{\pi\sqrt{2}-4}}{\pi} \approx \frac{0.665495}{\pi}$,

$K_2\left(1,0,1,\frac{1}{\pi\sqrt{1-t^2}}\right) = K_2^*\left(1,0,1,\frac{1}{\pi\sqrt{1-t^2}}\right) = \frac{1}{4}$,

$K_2\left(2,0,\infty,\frac{1}{\pi\sqrt{1-t^2}}\right) = K_2^*\left(2,0,\infty,\frac{1}{\pi\sqrt{1-t^2}}\right) \approx \frac{0.138151}{\pi}$,

$K_2\left(2,0,2,\frac{1}{\pi\sqrt{1-t^2}}\right) = K_2^*\left(2,0,2,\frac{1}{\pi\sqrt{1-t^2}}\right) \approx \frac{0.11295}{\pi}$,

$K_2\left(2,0,1,\frac{1}{\pi\sqrt{1-t^2}}\right) = K_2^*\left(2,0,1,\frac{1}{\pi\sqrt{1-t^2}}\right) \approx \frac{0.151746}{\pi}$,

$K_2\left(3,0,\infty,\frac{1}{\pi\sqrt{1-t^2}}\right) = K_2^*\left(3,0,\infty,\frac{1}{\pi\sqrt{1-t^2}}\right) \approx \frac{0.0371021}{\pi}$,

$K_2\left(3,0,2,\frac{1}{\pi\sqrt{1-t^2}}\right) = K_2^*\left(3,0,2,\frac{1}{\pi\sqrt{1-t^2}}\right) \approx \frac{0.0472213}{\pi}$,

$K_2\left(3,0,1,\frac{1}{\pi\sqrt{1-t^2}}\right) = K_2^*\left(3,0,1,\frac{1}{\pi\sqrt{1-t^2}}\right) \approx \frac{0.0345377}{\pi}$.

The constants $K_2\left(2,0,\infty,\frac{1}{\pi\sqrt{1-t^2}}\right)$ and $K_2\left(2,0,1,\frac{1}{\pi\sqrt{1-t^2}}\right)$ are better than the constants obtained in [77].

If the presumptions of the Theorem 8.2 hold, for $m = 4$ and $s = 0$ we get

$$\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}}dt = \frac{\pi}{2}\left[f\left(-\frac{\sqrt{2}}{2}\right)+f\left(\frac{\sqrt{2}}{2}\right)\right] + \frac{\pi}{192}f^{(4)}(\eta), \quad \eta\in(-1,1), \tag{8.23}$$

which is the well known two-point Chebyshev-Gauss formula of the first kind.

Remark 8.6 For $n = 3,\ x_1 = -\frac{\sqrt{3}}{2}, x_2 = 0$ and $x_3 = \frac{\sqrt{3}}{2}$ we get three-point Chebyshev-Gauss formulae of the first kind of Euler type

$$\begin{aligned}\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}}dt = \frac{\pi}{3}\left[f\left(-\frac{\sqrt{3}}{2}\right)+f(0)+f\left(\frac{\sqrt{3}}{2}\right)\right] + T_{m+s}^{CG1}(f,3)\\ +\frac{2^{m+s-1}}{(m+s)!}\int_{-1}^{1} G_{m+s}^{CG1}(t,3)\mathrm{d}f^{(m+s-1)}(t)\end{aligned}$$

and

$$\begin{aligned}\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}}dt = \frac{\pi}{3}\left[f\left(-\frac{\sqrt{3}}{2}\right)+f(0)+f\left(\frac{\sqrt{3}}{2}\right)\right] + T_{m+s-1}^{CG1}(f,3)\\ +\frac{2^{m+s-1}}{(m+s)!}\int_{-1}^{1} F_{m+s}^{CG1}(t,3)\mathrm{d}f^{(m+s-1)}(t).\end{aligned}$$

Especially, for $m_1 < 6$ and $s = 0$ inequalities (8.20) and (8.21) reduce to

$$\left| \int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} \mathrm{d}t - \frac{\pi}{3} \left[f\left(-\frac{\sqrt{3}}{2}\right) + f(0) + f\left(\frac{\sqrt{3}}{2}\right) \right] \right| \le \pi K_3 \left(m_1, 0, p, \frac{1}{\pi\sqrt{1-t^2}} \right) \cdot \|f^{(m_1)}\|_p,$$

$$\left| \int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} \mathrm{d}t - \frac{\pi}{3} \left[f\left(-\frac{\sqrt{3}}{2}\right) + f(0) + f\left(\frac{\sqrt{3}}{2}\right) \right] \right| \le \pi K_3^* \left(m_1, 0, p, \frac{1}{\pi\sqrt{1-t^2}} \right) \cdot \|f^{(m_1)}\|_p,$$

where

$K_3\left(1,0,\infty,\frac{1}{\pi\sqrt{1-t^2}}\right) = K_3^*\left(1,0,\infty,\frac{1}{\pi\sqrt{1-t^2}}\right) \approx \frac{0.535898}{\pi}$,

$K_3\left(1,0,2,\frac{1}{\pi\sqrt{1-t^2}}\right) = K_3^*\left(1,0,2,\frac{1}{\pi\sqrt{1-t^2}}\right) \approx \frac{0.4345}{\pi}$,

$K_3\left(1,0,1,\frac{1}{\pi\sqrt{1-t^2}}\right) = K_3^*\left(1,0,1,\frac{1}{\pi\sqrt{1-t^2}}\right) = \frac{1}{6}$,

$K_3\left(2,0,\infty,\frac{1}{\pi\sqrt{1-t^2}}\right) = K_3^*\left(2,0,\infty,\frac{1}{\pi\sqrt{1-t^2}}\right) \approx \frac{0.0578}{\pi}$,

$K_3\left(2,0,2,\frac{1}{\pi\sqrt{1-t^2}}\right) = K_3^*\left(2,0,2,\frac{1}{\pi\sqrt{1-t^2}}\right) \approx \frac{0.0487106}{\pi}$,

$K_3\left(2,0,1,\frac{1}{\pi\sqrt{1-t^2}}\right) = K_3^*\left(2,0,1,\frac{1}{\pi\sqrt{1-t^2}}\right) \approx \frac{0.0931}{\pi}$,

$K_3\left(3,0,\infty,\frac{1}{\pi\sqrt{1-t^2}}\right) = K_3^*\left(3,0,\infty,\frac{1}{\pi\sqrt{1-t^2}}\right) \approx \frac{0.009162}{\pi}$,

$K_3\left(3,0,2,\frac{1}{\pi\sqrt{1-t^2}}\right) = K_3^*\left(3,0,2,\frac{1}{\pi\sqrt{1-t^2}}\right) \approx \frac{0.0077668}{\pi}$,

$K_3\left(3,0,1,\frac{1}{\pi\sqrt{1-t^2}}\right) = K_3^*\left(3,0,1,\frac{1}{\pi\sqrt{1-t^2}}\right) \approx \frac{0.00959813}{\pi}$,

$K_3\left(4,0,\infty,\frac{1}{\pi\sqrt{1-t^2}}\right) = K_3^*\left(4,0,\infty,\frac{1}{\pi\sqrt{1-t^2}}\right) \approx \frac{0.00165293}{\pi}$,

$K_3\left(4,0,2,\frac{1}{\pi\sqrt{1-t^2}}\right) = K_3^*\left(4,0,2,\frac{1}{\pi\sqrt{1-t^2}}\right) \approx \frac{0.00146947}{\pi}$,

$K_3\left(4,0,1,\frac{1}{\pi\sqrt{1-t^2}}\right) = K_3^*\left(4,0,1,\frac{1}{\pi\sqrt{1-t^2}}\right) \approx \frac{0.002251}{\pi}$,

$K_3\left(5,0,\infty,\frac{1}{\pi\sqrt{1-t^2}}\right) = K_3^*\left(5,0,\infty,\frac{1}{\pi\sqrt{1-t^2}}\right) \approx \frac{0.0003867}{\pi}$,

$K_3\left(5,0,2,\frac{1}{\pi\sqrt{1-t^2}}\right) = K_3^*\left(5,0,2,\frac{1}{\pi\sqrt{1-t^2}}\right) \approx \frac{0.000348327}{\pi}$,

$K_3\left(5,0,1,\frac{1}{\pi\sqrt{1-t^2}}\right) = K_3^*\left(5,0,1,\frac{1}{\pi\sqrt{1-t^2}}\right) \approx \frac{0.000413232}{\pi}$.

The constants for $p=\infty$ and $p=1$ are obtained in [78].

If the presumptions of Theorem 8.2 hold, for $m=6$ and $s=0$ we get

$$\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}}dt = \frac{\pi}{3}\left[f\left(-\frac{\sqrt{3}}{2}\right)+f(0)+f\left(\frac{\sqrt{3}}{2}\right)\right]+\frac{\pi}{23040}f^{(6)}(\eta), \tag{8.24}$$

which is the well known three-point Chebyshev-Gauss formula of the first kind.

8.5 Chebyshev-Gauss formulae of the second kind of Euler type

If the weight function is $w(t)=\frac{2\sqrt{1-t^2}}{\pi}$, $t\in[-1,1]$, we have Chebyshev-Gauss formulae given by

$$\int_{-1}^{1}\sqrt{1-t^2}f(t)dt = \frac{\pi}{2}\sum_{k=1}^{n}A_k f(x_k)+E_n(f), \tag{8.25}$$

where

$$A_k=\frac{2}{n+1}\sin^2\left(\frac{k\pi}{n+1}\right), \quad k=1,\dots,n,$$

x_k are zeros of Chebyshev polynomials of the second kind defined as

$$U_n(x)=\frac{\sin[(n+1)\arccos(x)]}{\sin[\arccos(x)]}.$$

$U_n(x)$ has exactly n distinct zeros, all of which lie in the interval $[-1,1]$ and

$$x_k=\cos\left(\frac{k\pi}{n+1}\right).$$

Error of approximation formula (8.25) is given by

$$E_n(f)=\frac{\pi}{2^{2n+1}(2n)!}f^{(2n)}(\eta), \quad \eta\in(-1,1).$$

In the next theorem we establish Chebyshev-Gauss formulae of the second kind of Euler type which are exact for all polynomials of degree $\leq m-1$.

Theorem 8.6 *Let us suppose $f^{(m+s-1)}$ is a continuous function of bounded variation on $[-1,1]$ for some $(m+s)\geq 1$. Then the following formulae hold*

$$\begin{aligned}\int_{-1}^{1}\sqrt{1-t^2}f(t)dt &= \frac{\pi}{n+1}\sum_{k=1}^{n}\sin^2\left(\frac{k\pi}{n+1}\right)f(x_k)+T_{m+s}^{CG2}(f,n)\\ &+\frac{2^{m+s-1}}{(m+s)!}\int_{-1}^{1}G_{m+s}^{CG2}(t,n)\mathrm{d}f^{(m+s-1)}(t)\end{aligned} \tag{8.26}$$

and

$$\begin{aligned}\int_{-1}^{1}\sqrt{1-t^2}f(t)dt &= \frac{\pi}{n+1}\sum_{k=1}^{n}\sin^2\left(\frac{k\pi}{n+1}\right)f(x_k)+T_{m+s-1}^{CG2}(f,n)\\ &+\frac{2^{m+s-1}}{(m+s)!}\int_{-1}^{1}F_{m+s}^{CG2}(t,n)\mathrm{d}f^{(m+s-1)}(t),\end{aligned} \tag{8.27}$$

where

$$T_{m+s}^{CG2}(f,n)=\sum_{j=0}^{s}\frac{2^{j+m-1}}{(m+j)!}\cdot\left(\int_{-1}^{1}\sqrt{1-t^2}B_{j+m}\left(\frac{t+1}{2}\right)dt\right.$$

$$\left.-\frac{\pi}{n+1}\sum_{k=1}^{n}\sin^2\left(\frac{k\pi}{n+1}\right)B_{j+m}\left(\frac{x_k+1}{2}\right)\right)\left[f^{(m+j-1)}(1)-f^{(m+j-1)}(-1)\right],$$

$$G_{m+s}^{CG2}(t,n)=\frac{\pi}{n+1}\sum_{k=1}^{n}\sin^2\left(\frac{k\pi}{n+1}\right)B_{m+s}^{*}\left(\frac{x_k-t}{2}\right)-\int_{-1}^{1}\sqrt{1-u^2}B_{m+s}^{*}\left(\frac{u-t}{2}\right)\mathrm{d}u$$

and

$$\begin{aligned}F_{m+s}^{CG2}(t,n) &= \frac{\pi}{n+1}\sum_{k=1}^{n}\sin^2\left(\frac{k\pi}{n+1}\right)\left(B_{m+s}^{*}\left(\frac{x_k-t}{2}\right)-B_{m+s}\left(\frac{x_k+1}{2}\right)\right)\\ &-\int_{-1}^{1}\sqrt{1-u^2}\left(B_{m+s}^{*}\left(\frac{u-t}{2}\right)-B_{m+s}\left(\frac{u+1}{2}\right)\right)\mathrm{d}u.\end{aligned}$$

These formulae are exact for all polynomials of degree $\le m-1$.

Proof. This is a special case of Corollary 8.1 for $a=-1,\ b=1,\ w(t)=\frac{2\sqrt{1-t^2}}{\pi}$ and $A_k=\frac{2}{n+1}\sin^2\left(\frac{k\pi}{n+1}\right)$. □

Theorem 8.7 *Assume (p,q) is a pair of conjugate exponents, $1\le p,q\le\infty$. Let $f:[-1,1]\to\mathbb{R}$ be such that $f^{(m+s)}\in L_p[-1,1]$ for some $m+s\ge 1$. Then we have*

$$\begin{aligned}&\left|\int_{-1}^{1}\sqrt{1-t^2}f(t)dt-\frac{\pi}{n+1}\sum_{k=1}^{n}\sin^2\left(\frac{k\pi}{n+1}\right)f(x_k)-T_{m+s}^{CG2}(f,n)\right|\\ &\qquad\le\frac{\pi}{2}K_n\left(m,s,p,\frac{2\sqrt{1-t^2}}{\pi}\right)\cdot\|f^{(m+s)}\|_p\end{aligned} \tag{8.28}$$

and

$$\begin{aligned}&\left|\int_{-1}^{1}\sqrt{1-t^2}f(t)dt-\frac{\pi}{n+1}\sum_{k=1}^{n}\sin^2\left(\frac{k\pi}{n+1}\right)f(x_k)-T_{m+s-1}^{CG2}(f,n)\right|\\ &\qquad\le\frac{\pi}{2}K_n^{*}\left(m,s,p,\frac{2\sqrt{1-t^2}}{\pi}\right)\cdot\|f^{(m+s)}\|_p.\end{aligned} \tag{8.29}$$

The constants $K_n\left(m,s,p,\frac{2\sqrt{1-t^2}}{\pi}\right)$ *and* $K_n^*\left(m,s,p,\frac{2\sqrt{1-t^2}}{\pi}\right)$ *are sharp for* $1<p\le\infty$ *and the best possible for* $p=1$.

Proof. This is a special case of Theorem 8.3 for $a=-1,\ b=1,\ w(t)=\frac{2\sqrt{1-t^2}}{\pi}$ and $A_k=\frac{2}{n+1}\sin^2\left(\frac{k\pi}{n+1}\right)$. □

Remark 8.7 For $n=1$ and $x_1=0$ we get one-point Chebyshev-Gauss formulae of second kind of Euler type

$$\int_{-1}^{1}\sqrt{1-t^2}f(t)dt=\frac{\pi}{2}f(0)+T_{m+s}^{CG2}(f,1)+\frac{2^{m+s-1}}{(m+s)!}\int_{-1}^{1}G_{m+s}^{CG2}(t,1)\mathrm{d}f^{(m+s-1)}(t)$$

and

$$\int_{-1}^{1}\sqrt{1-t^2}f(t)dt=\frac{\pi}{2}f(0)+T_{m+s-1}^{CG2}(f,1)+\frac{2^{m+s-1}}{(m+s)!}\int_{-1}^{1}F_{m+s}^{CG2}(t,1)\mathrm{d}f^{(m+s-1)}(t).$$

Specially for $m=1$ and $s=0$ we get

$$G_1^{CG2}(t,1)=F_1^{CG2}(t,1)=\begin{cases}-\frac{\pi}{4}-\frac{1}{2}(t\sqrt{1-t^2}+\arcsin t), & -1\le t\le 0,\\ \frac{\pi}{4}-\frac{1}{2}(t\sqrt{1-t^2}+\arcsin t), & 0<t\le 1.\end{cases}$$

Now, inequalities (8.28) and (8.29) reduce to

$$\left|\int_{-1}^{1}\sqrt{1-t^2}f(t)\,\mathrm{d}t-\frac{\pi}{2}f(0)\right|\le\frac{\pi}{2}K_1\left(1,0,p,\frac{2\sqrt{1-t^2}}{\pi}\right)\cdot\|f'\|_p$$

and

$$\left|\int_{-1}^{1}\sqrt{1-t^2}f(t)\,\mathrm{d}t-\frac{\pi}{2}f(0)\right|\le\frac{\pi}{2}K_1^*\left(1,0,p,\frac{2\sqrt{1-t^2}}{\pi}\right)\cdot\|f'\|_p,$$

where $K_1\left(1,0,\infty,\frac{2\sqrt{1-t^2}}{\pi}\right)=K_1^*\left(1,0,\infty,\frac{2\sqrt{1-t^2}}{\pi}\right)=\frac{4}{3\pi}$, $K_1\left(1,0,2,\frac{2\sqrt{1-t^2}}{\pi}\right)=K_1^*\left(1,0,2,\frac{2\sqrt{1-t^2}}{\pi}\right)\approx\frac{1.15946}{\pi}$ and $K_1\left(1,0,1,\frac{2\sqrt{1-t^2}}{\pi}\right)=K_1^*\left(1,0,1,\frac{2\sqrt{1-t^2}}{\pi}\right)=\frac{1}{2}$.

The first and the third constant have also been obtained in [77].

If the presumptions of the Theorem 8.2 hold, for $m=2$ and $s=0$ we get

$$\int_{-1}^{1}\sqrt{1-t^2}f(t)\,dt=\frac{\pi}{2}f(0)+\frac{\pi}{16}f''(\eta),\quad \eta\in(-1,1), \tag{8.30}$$

which is the well known one-point Chebyshev-Gauss formula of the second kind.

Remark 8.8 For $n=2$, $x_1=-\frac{1}{2}$ and $x_2=\frac{1}{2}$ we get two-point Chebyshev-Gauss formulae of the second kind of Euler type

$$\begin{aligned}\int_{-1}^{1}\sqrt{1-t^2}f(t)dt &=\frac{\pi}{4}\left[f\left(-\frac{1}{2}\right)+f\left(\frac{1}{2}\right)\right]\\ &+T_{m+s}^{CG2}(f,2)+\frac{2^{m+s-1}}{(m+s)!}\int_{-1}^{1}G_{m+s}^{CG2}(t,2)\mathrm{d}f^{(m+s-1)}(t)\end{aligned}$$

and

$$\int_{-1}^{1}\sqrt{1-t^2}f(t)dt = \frac{\pi}{4}\left[f\left(-\frac{1}{2}\right)+f\left(\frac{1}{2}\right)\right]+T^{CG2}_{m+s-1}(f,2)+\frac{2^{m+s-1}}{(m+s)!}\int_{-1}^{1}F^{CG2}_{m+s}(t,2)\mathrm{d}f^{(m+s-1)}(t).$$

Specially for $m_1 < 4$ and $s = 0$ inequalities (8.28) and (8.29) reduce to

$$\left|\int_{-1}^{1}\sqrt{1-t^2}f(t)\,\mathrm{d}t-\frac{\pi}{4}\left[f\left(-\frac{1}{2}\right)+f\left(\frac{1}{2}\right)\right]\right| \le \frac{\pi}{2}K_2\left(m_1,0,p,\frac{2\sqrt{1-t^2}}{\pi}\right)\cdot\|f^{(m_1)}\|_p,$$

$$\left|\int_{-1}^{1}\sqrt{1-t^2}f(t)\,\mathrm{d}t-\frac{\pi}{4}\left[f\left(-\frac{1}{2}\right)+f\left(\frac{1}{2}\right)\right]\right| \le \frac{\pi}{2}K_2^*\left(m_1,0,p,\frac{2\sqrt{1-t^2}}{\pi}\right)\cdot\|f^{(m_1)}\|_p,$$

where

$K_2\left(1,0,\infty,\frac{2\sqrt{1-t^2}}{\pi}\right) = K_2^*\left(1,0,\infty,\frac{2\sqrt{1-t^2}}{\pi}\right) \approx \frac{0.741144}{\pi}$,
$K_2\left(1,0,2,\frac{2\sqrt{1-t^2}}{\pi}\right) = K_2^*\left(1,0,2,\frac{2\sqrt{1-t^2}}{\pi}\right) \approx \frac{0.643534}{\pi}$,
$K_2\left(1,0,1,\frac{2\sqrt{1-t^2}}{\pi}\right) = K_2^*\left(1,0,1,\frac{2\sqrt{1-t^2}}{\pi}\right) \approx \frac{0.95661}{\pi}$,
$K_2\left(2,0,\infty,\frac{2\sqrt{1-t^2}}{\pi}\right) = K_2^*\left(2,0,\infty,\frac{2\sqrt{1-t^2}}{\pi}\right) \approx \frac{0.109429}{\pi}$,
$K_2\left(2,0,2,\frac{2\sqrt{1-t^2}}{\pi}\right) = K_2^*\left(2,0,2,\frac{2\sqrt{1-t^2}}{\pi}\right) \approx \frac{0.0968132}{\pi}$,
$K_2\left(2,0,1,\frac{2\sqrt{1-t^2}}{\pi}\right) = K_2^*\left(2,0,1,\frac{2\sqrt{1-t^2}}{\pi}\right) \approx \frac{0.1259202}{\pi}$,
$K_2\left(3,0,\infty,\frac{2\sqrt{1-t^2}}{\pi}\right) = K_2^*\left(3,0,\infty,\frac{2\sqrt{1-t^2}}{\pi}\right) \approx \frac{0.023439}{\pi}$,
$K_2\left(3,0,2,\frac{2\sqrt{1-t^2}}{\pi}\right) = K_2^*\left(3,0,2,\frac{2\sqrt{1-t^2}}{\pi}\right) \approx \frac{0.0218954}{\pi}$,
$K_2\left(3,0,1,\frac{2\sqrt{1-t^2}}{\pi}\right) = K_2^*\left(3,0,1,\frac{2\sqrt{1-t^2}}{\pi}\right) \approx \frac{0.0273572}{\pi}$.

The constants $K_2\left(1,0,\infty,\frac{2\sqrt{1-t^2}}{\pi}\right)$, $K_2\left(1,0,1,\frac{2\sqrt{1-t^2}}{\pi}\right)$ and $K_2\left(2,0,1,\frac{2\sqrt{1-t^2}}{\pi}\right)$ have also been obtained in [77].

If the presumptions of the Theorem 8.2 hold, for $m = 4$ and $s = 0$ we get

$$\int_{-1}^{1}\sqrt{1-t^2}f(t)\,dt = \frac{\pi}{4}\left[f\left(-\frac{1}{2}\right)+f\left(\frac{1}{2}\right)\right]+\frac{\pi}{768}f^{(4)}(\eta),\quad \eta\in(-1,1), \tag{8.31}$$

which is the well known two-point Chebyshev-Gauss formula of the second kind.

Remark 8.9 For $n = 3,\ x_1 = -\frac{\sqrt{2}}{2}, x_2 = 0$ and $x_3 = \frac{\sqrt{2}}{2}$ we get three-point Chebyshev-Gauss formulae of second kind of Euler type

$$\int_{-1}^{1}\sqrt{1-t^2}f(t)dt \;=\; \frac{\pi}{8}\left[f\left(-\frac{\sqrt{2}}{2}\right)+2f(0)+f\left(\frac{\sqrt{2}}{2}\right)\right]+T^{CG2}_{m+s}(f,3)$$

$$+\ \frac{2^{m+s-1}}{(m+s)!}\int_{-1}^{1} G_{m+s}^{CG2}(t,3)\mathrm{d}f^{(m+s-1)}(t)$$

and

$$\begin{aligned}\int_{-1}^{1}\sqrt{1-t^2}f(t)dt &= \frac{\pi}{8}\left[f\left(-\frac{\sqrt{2}}{2}\right)+2f(0)+f\left(\frac{\sqrt{2}}{2}\right)\right]+T_{m+s-1}^{CG2}(f,3)\\ &+\ \frac{2^{m+s-1}}{(m+s)!}\int_{-1}^{1} F_{m+s}^{CG2}(t,3)\mathrm{d}f^{(m+s-1)}(t).\end{aligned}$$

Specially for $m_1 < 6$ and $s = 0$ inequalities (8.28) and (8.29) reduce to

$$\begin{aligned}&\left|\int_{-1}^{1}\sqrt{1-t^2}f(t)\,\mathrm{d}t-\frac{\pi}{8}\left[f\left(-\frac{\sqrt{2}}{2}\right)+2f(0)+f\left(\frac{\sqrt{2}}{2}\right)\right]\right|\\ &\qquad\le \frac{\pi}{2}K_3\left(m_1,0,p,\frac{2\sqrt{1-t^2}}{\pi}\right)\cdot\|f^{(m_1)}\|_p,\end{aligned}$$

$$\begin{aligned}&\left|\int_{-1}^{1}\sqrt{1-t^2}f(t)\,\mathrm{d}t-\frac{\pi}{8}\left[f\left(-\frac{\sqrt{2}}{2}\right)+2f(0)+f\left(\frac{\sqrt{2}}{2}\right)\right]\right|\\ &\qquad\le \frac{\pi}{2}K_3^*\left(m_1,0,p,\frac{2\sqrt{1-t^2}}{\pi}\right)\cdot\|f^{(m_1)}\|_p,\end{aligned}$$

where

$K_3\left(1,0,\infty,\frac{2\sqrt{1-t^2}}{\pi}\right)=K_3^*\left(1,0,\infty,\frac{2\sqrt{1-t^2}}{\pi}\right)\approx\frac{0.53833976}{\pi}$,

$K_3\left(1,0,2,\frac{2\sqrt{1-t^2}}{\pi}\right)=K_3^*\left(1,0,2,\frac{2\sqrt{1-t^2}}{\pi}\right)\approx\frac{0.478324}{\pi}$,

$K_3\left(1,0,1,\frac{2\sqrt{1-t^2}}{\pi}\right)=K_3^*\left(1,0,1,\frac{2\sqrt{1-t^2}}{\pi}\right)=\frac{1}{4}$,

$K_3\left(2,0,\infty,\frac{2\sqrt{1-t^2}}{\pi}\right)=K_3^*\left(2,0,\infty,\frac{2\sqrt{1-t^2}}{\pi}\right)\approx\frac{0.053417328}{\pi}$,

$K_3\left(2,0,2,\frac{2\sqrt{1-t^2}}{\pi}\right)=K_3^*\left(2,0,2,\frac{2\sqrt{1-t^2}}{\pi}\right)\approx\frac{0.0493938}{\pi}$,

$K_3\left(2,0,1,\frac{2\sqrt{1-t^2}}{\pi}\right)=K_3^*\left(2,0,1,\frac{2\sqrt{1-t^2}}{\pi}\right)\approx\frac{0.111306298}{\pi}$,

$K_3\left(3,0,\infty,\frac{2\sqrt{1-t^2}}{\pi}\right)=K_3^*\left(3,0,\infty,\frac{2\sqrt{1-t^2}}{\pi}\right)\approx\frac{0.007288942}{\pi}$,

$K_3\left(3,0,2,\frac{2\sqrt{1-t^2}}{\pi}\right)=K_3^*\left(3,0,2,\frac{2\sqrt{1-t^2}}{\pi}\right)\approx\frac{0.0067384}{\pi}$,

$K_3\left(3,0,1,\frac{2\sqrt{1-t^2}}{\pi}\right)=K_3^*\left(3,0,1,\frac{2\sqrt{1-t^2}}{\pi}\right)\approx\frac{0.00925848}{\pi}$,

$K_3\left(4,0,\infty,\frac{2\sqrt{1-t^2}}{\pi}\right)=K_3^*\left(4,0,\infty,\frac{2\sqrt{1-t^2}}{\pi}\right)\approx\frac{0.001123902}{\pi}$,

$K_3\left(4,0,2,\frac{2\sqrt{1-t^2}}{\pi}\right) = K_3^*\left(4,0,2,\frac{2\sqrt{1-t^2}}{\pi}\right) \approx \frac{0.001081414}{\pi},$
$K_3\left(4,0,1,\frac{2\sqrt{1-t^2}}{\pi}\right) = K_3^*\left(4,0,1,\frac{2\sqrt{1-t^2}}{\pi}\right) \approx \frac{0.001835586}{\pi},$
$K_3\left(5,0,\infty,\frac{2\sqrt{1-t^2}}{\pi}\right) = K_3^*\left(5,0,\infty,\frac{2\sqrt{1-t^2}}{\pi}\right) \approx \frac{0.00022568}{\pi},$
$K_3\left(5,0,2,\frac{2\sqrt{1-t^2}}{\pi}\right) = K_3^*\left(5,0,2,\frac{2\sqrt{1-t^2}}{\pi}\right) \approx \frac{0.000218644}{\pi},$
$K_3\left(5,0,1,\frac{2\sqrt{1-t^2}}{\pi}\right) = K_3^*\left(5,0,1,\frac{2\sqrt{1-t^2}}{\pi}\right) \approx \frac{0.000280976}{\pi}.$

The constants for $p = \infty$ and $p = 1$ were obtained in [78].

If the presumptions of the Theorem 8.2 hold, for $m = 6$ and $s = 0$ we get

$$\int_{-1}^{1} \sqrt{1-t^2} f(t)\,dt = \frac{\pi}{8}\left[f\left(-\frac{\sqrt{2}}{2}\right) + 2f(0) + f\left(\frac{\sqrt{2}}{2}\right)\right] + \frac{\pi}{92160} f^{(6)}(\eta), \quad (8.32)$$

which is the well known three-point Chebyshev-Gauss formula of the second kind.

Addendum

Iyengar's inequality

8.5.1 Weighted generalizations of Iyengar type inequalities

In 1938. K.S.K.Iyengar proved the following inequality (see [74]):

Theorem 8.8 *Let f be a differentiable function on $[a,b]$ and $|f'(x)| \leq M$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}\right| \leq \frac{M(b-a)}{4} - \frac{(f(b)-f(a))^2}{4M(b-a)}. \tag{1}$$

Inequality (1) can be written in a form:

$$\left|A(f;1) - \frac{f(a)+f(b)}{2}\right| \leq \frac{M(b-a)}{4}(1-q^2), \tag{2}$$

where

$$A(f;w) = \frac{\int_a^b w(x)f(x)dx}{\int_a^b w(x)dx} \tag{3}$$

and

$$q = \frac{|f(b)-f(a)|}{M(b-a)}. \tag{4}$$

In [89], G.Milovanović generalized Theorem 8.8. He proved the following:

Theorem 8.9 *Let f be such that $f \in Lip_M$ [1], let w be an integrable function on (a,b) and let there exist $\lambda \geq 1$ such that $0 < c \leq w(x) \leq \lambda c$ for each $x \in [a,b]$. Then*

$$\left|A(f;w) - \frac{1}{2}(f(a)+f(b))\right| \leq \frac{M(b-a)}{2} \cdot \frac{(\lambda+q)(1-q^2)+2(\lambda-1)q}{2\lambda(1+q)-(\lambda-1)(1+q^2)} \tag{5}$$

where $A(f;w)$ and q are defined by (3) and (4), respectively.

[1] Recall that for a function f defined on an interval $[a,b]$, we write $f \in Lip_M(\alpha)$ with $M > 0$ and $0 < \alpha \leq 1$ and say that f satisfies a *Lipschitz condition of order α with the Lipschitz constant M*, if

$$|f(t_2)-f(t_1)| \leq M|t_2-t_1|^{\alpha}, \quad \textit{for each } t_1,t_2 \in [a,b].$$

For notational convenience, the class $Lip_M(1)$ is denoted simply Lip_M.

Upon taking $w(x) = 1 (\Rightarrow \lambda = 1)$, (5) reduces to (2).

In [90], G.V.Milovanović and J.Pečarić proved another generalization of Iyengar's inequality.

Theorem 8.10 *Let function $f : [a,b] \to \mathbb{R}$ satisfy the following conditions:*

$$
\begin{aligned}
&1^\circ \quad f^{(n-1)} \in Lip_M(\alpha)\\
&2^\circ \quad f^{(k)}(a) = f^{(k)}(b) = 0, \ \ k = 1,2,\ldots,n-1 \quad (n \in \mathbb{N}).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(x)dx - \frac{1}{2}(f(a)+f(b)) \right| \\
&\leq \frac{M(b-a)^{\alpha+n-1}}{(\alpha+n)^{(n)}} \left\{ \zeta^{\alpha+n-1} - \frac{q}{2}[1+(\alpha+n-1)(2\zeta-1)] \right\}
\end{aligned}
\tag{6}
$$

where ζ is the real root of the equation

$$
\zeta^{\alpha+n-1} - (1-\zeta)^{\alpha+n-1} = q,
$$

$$
q = \frac{(\alpha+n-1)^{(n-1)}}{M(b-a)^{\alpha+n-1}} |f(b)-f(a)|, \quad p^{(n)} = p(p-1)\ldots(p-n+1). \tag{7}
$$

Taking $\alpha = 1$ and $n = 1$ in (6), produces Iyengar's inequality (1).

For $\alpha = 1$ and $n = 2$, inequality (6) reduces to

$$
\left| \frac{1}{b-a} \int_a^b f(x)dx - \frac{1}{2}(f(a)+f(b)) \right| \leq \frac{M(b-a)^2}{24} - \frac{1}{2M}\left(\frac{f(b)-f(a)}{b-a} \right)^2. \tag{8}
$$

This inequality is called the Milovanović - Pečarić inequality.

The weighted version of Theorem 8.10 was given in [57] by Franjić, Pečarić and Perić.

Theorem 8.11 *Let function $f : [a,b] \to \mathbb{R}$ satisfy the following conditions:*

$$
\begin{aligned}
&1^\circ \quad f^{(n-1)} \in Lip_M(\alpha)\\
&2^\circ \quad f^{(k)}(a) = f^{(k)}(b) = 0, \ \ k = 1,2,\ldots,n-1 \quad (n \in \mathbb{N}).
\end{aligned}
$$

Let w be a non-negative and integrable function on $[a,b]$. Then we have

$$
\begin{aligned}
&\Bigg| \int_a^b f(x)w(x)dx - \frac{1}{2}(f(a)+f(b)) \int_a^b w(x)dx \\
&\quad + \frac{1}{4}\int_a^b (|f(b)-f(a)-G(x)+F(x)| - |f(b)-f(a)+G(x)-F(x)|)\, w(x)dx \Bigg| \\
&\leq \frac{1}{2}\Bigg(\int_a^b \Big(F(x)+G(x)\Big) w(x)dx - \frac{1}{2}\int_a^b \Big(|f(b)-f(a)-G(x)+F(x)| \\
&\quad + |f(b)-f(a)+G(x)-F(x)| \Big) w(x)dx \Bigg)
\end{aligned}
\tag{9}
$$

where

$$F(x) = M\frac{(x-a)^{\alpha+n-1}}{(\alpha+n-1)^{(n-1)}}\ ,\ G(x) = M\frac{(b-x)^{\alpha+n-1}}{(\alpha+n-1)^{(n-1)}}. \tag{10}$$

Proof. From condition 1° it follows:

$$-M(x-a)^{\alpha} \le f^{(n-1)}(x) - f^{(n-1)}(a) \le M(x-a)^{\alpha},$$
$$-M(b-x)^{\alpha} \le f^{(n-1)}(b) - f^{(n-1)}(x) \le M(b-x)^{\alpha}.$$

Using condition 2° and $(n-1)$-times successive integration of these two inequalities on (a,x) and (x,b), respectively, we get

$$f(a) - M\frac{(x-a)^{\alpha+n-1}}{(\alpha+n-1)^{(n-1)}} \le f(x) \le f(a) + M\frac{(x-a)^{\alpha+n-1}}{(\alpha+n-1)^{(n-1)}},$$
$$f(b) - M\frac{(b-x)^{\alpha+n-1}}{(\alpha+n-1)^{(n-1)}} \le f(x) \le f(b) + M\frac{(b-x)^{\alpha+n-1}}{(\alpha+n-1)^{(n-1)}}. \tag{11}$$

Introducing notation from (10), we conclude

$$\max\{f(a) - F(x),\ f(b) - G(x)\} \le f(x) \le \min\{f(a) + F(x),\ f(b) + G(x)\}. \tag{12}$$

It is easy to check that for each $\alpha, \beta \in \mathbb{R}$ we have:

$$\min\{\alpha,\beta\} = \frac{1}{2}(\alpha+\beta-|\beta-\alpha|),\quad \max\{\alpha,\beta\} = \frac{1}{2}(\alpha+\beta+|\beta-\alpha|). \tag{13}$$

Applying (13) to (12) gives

$$\frac{1}{2}\Big(-F(x) - G(x) + |f(b) - G(x) - f(a) + F(x)|\Big) \le f(x) - \frac{1}{2}\Big(f(a) + f(b)\Big)$$
$$\le \frac{1}{2}\Big(F(x) + G(x) - |f(b) + G(x) - f(a) - F(x)|\Big). \tag{14}$$

Now, multiply (14) by $w(x)$ and then integrate over (a,b). We get:

$$-\frac{1}{2}\int_a^b \Big(F(x)+G(x)\Big)w(x)dx + \frac{1}{2}\int_a^b |f(b) - f(a) - G(x) + F(x)|w(x)dx$$
$$\le \int_a^b f(x)w(x)dx - \frac{1}{2}\Big(f(a)+f(b)\Big)\int_a^b w(x)dx \tag{15}$$
$$\le \frac{1}{2}\int_a^b \Big(F(x)+G(x)\Big)w(x)dx - \frac{1}{2}\int_a^b |f(b) - f(a) + G(x) - F(x)|w(x)dx.$$

Applying

$$A \le B \le C \Leftrightarrow \left|B - \frac{C+A}{2}\right| \le \frac{C-A}{2} \tag{16}$$

to (15) produces (9). Thus, the proof is complete. □

Corollary 8.2 *Let the assumptions of Theorem 8.11 be valid. Let function w be symmetric about the mid-point $\frac{a+b}{2}$, i.e. let $w(x) = w(a+b-x)$ for every $x \in [a, \frac{a+b}{2}]$. Then we have*

$$\left| \int_a^b f(x)w(x)dx - \frac{f(a)+f(b)}{2} \int_a^b w(x)dx \right| \tag{17}$$
$$\leq \int_a^b F(x)w(x)dx - \frac{1}{2} \int_a^b |f(b) - f(a) + G(x) - F(x)| w(x)dx$$

where $F(x)$ and $G(x)$ are defined by (10).

Proof. First, note that

$$F(a+b-x) = M \frac{(a+b-x-a)^{\alpha+n-1}}{(\alpha+n-1)^{(n-1)}} = G(x),$$
$$G(a+b-x) = M \frac{(b-a-b+x)^{\alpha+n-1}}{(\alpha+n-1)^{(n-1)}} = F(x). \tag{18}$$

Now, using the substitution $x = a+b-t$, the symmetric-property of function w, together with (18), from the left-hand side of (15) we get

$$-\frac{1}{2} \int_a^b \Big(F(x) + G(x)\Big) w(x)dx + \frac{1}{2} \int_a^b |f(b) - f(a) - G(x) + F(x)| w(x)dx$$
$$= -\int_a^b F(x)w(x)dx + \frac{1}{2} \int_a^b |f(b) - f(a) - F(t) + G(t)| w(t)dt$$

which is equal to the negative value of the right-hand side of (15). Therefore, (17) is proved. □

Remark 8.10 Taking $w(x) = 1$ in Corollary 8.2, produces Theorem 8.10.

For the proof of the next corollary (which generalizes Theorem 8.9), we need the following result (cf. [101]).

Theorem 8.12 *Let f be an integrable function on (a,b) and $m \leq f(x) \leq M$ for each $x \in (a,b)$. Let w be an integrable function on (a,b) and let there exist $\lambda \geq 1$ such that $0 < c \leq w(x) \leq \lambda c$ for each $x \in [a,b]$. Then*

$$\frac{\lambda m(M-\mu) + M(\mu - m)}{\lambda(M-\mu) + (\mu - m)} \leq A(f;w) \leq \frac{m(M-\mu) + \lambda M(\mu - m)}{(M-\mu) + \lambda(\mu - m)} \tag{19}$$

where $A(f,w)$ is defined by (3) and $\mu = \frac{1}{b-a} \int_a^b f(t)dt$.

Corollary 8.3 *Let the assumptions of Corollary 8.2 be valid and let there exist $\lambda \geq 1$ such that $0 < c \leq w(x) \leq \lambda c$ for each $x \in [a,b]$. Then*

$$\left| A(f;w) - \frac{1}{2}(f(a) + f(b)) \right| \tag{20}$$
$$\leq \frac{F(b)}{2} \left[\frac{(\alpha+n)(1-\lambda) + \lambda 2^{\alpha+n-1} - 2}{(\alpha+n)(2^{\alpha+n-2} - \lambda) + 2^{\alpha+n-1}(\lambda - 1)} - \frac{(q+1)\mu}{F(b)\lambda(q+1) - (\lambda-1)\mu} \right]$$

where $F(x)$ and $G(x)$ are defined in (10), $\quad q = \dfrac{|f(b)-f(a)|}{F(b)}$,

ζ is the real root of the equation $\quad \zeta^{\alpha+n-1} - (1-\zeta)^{\alpha+n-1} = q$,

and $\quad \mu = \dfrac{2F(b)}{\alpha+n}\Big(1-\zeta^{\alpha+n-1}+\dfrac{q}{2}[1+(\alpha+n-1)(2\zeta-1)]\Big)$.

Proof. We start from (17). Using the symmetric property of function w and (18), write it in a form

$$\begin{aligned}
&\left|\int_a^b f(x)w(x)dx - \frac{f(a)+f(b)}{2}\int_a^b w(x)dx\right| \\
&\le \frac{1}{2}\int_a^b [F(x)+G(x)]w(x)dx - \frac{1}{2}\int_a^b |f(b)-f(a)+G(x)-F(x)|w(x)dx
\end{aligned}$$

and then divide it by $\int_a^b w(x)dx$. It follows

$$\begin{aligned}
&\left|A(f;w) - \frac{f(a)+f(b)}{2}\right| \\
&\le \frac{1}{2}\frac{\int_a^b [F(x)+G(x)]w(x)dx}{\int_a^b w(x)dx} - \frac{1}{2}\frac{\int_a^b |f(b)-f(a)+G(x)-F(x)|w(x)dx}{\int_a^b w(x)dx}.
\end{aligned}$$

Set

$$B = \frac{\int_a^b [F(x)+G(x)]w(x)dx}{\int_a^b w(x)dx}.$$

We have

$$\frac{4F(b)}{2^{\alpha+n}} \le F(x)+G(x) \le F(b), \quad x \in [a,b]$$

and

$$\mu = \frac{1}{b-a}\int_a^b (F(x)+G(x))dx = \frac{2F(b)}{\alpha+n}.$$

Now we can apply (the right side of) (19). We get:

$$\begin{aligned}
B &\le \frac{F(b)\left[\frac{4}{2^{\alpha+n}}\left(1-\frac{2}{\alpha+n}\right)+\lambda\left(\frac{2}{\alpha+n}-\frac{4}{2^{\alpha+n}}\right)\right]}{\left(1-\frac{2}{\alpha+n}\right)+\lambda\left(\frac{2}{\alpha+n}-\frac{4}{2^{\alpha+n}}\right)} \\
&= F(b)\frac{(\alpha+n)(1-\lambda)+\lambda 2^{\alpha+n-1}-2}{(\alpha+n)(2^{\alpha+n-2}-\lambda)+2^{\alpha+n-1}(\lambda-1)}.
\end{aligned} \tag{21}$$

Similarly, for

$$C = \frac{\int_a^b |f(b)-f(a)+G(x)-F(x)|w(x)dx}{\int_a^b w(x)dx},$$

applying the left side of (19) we get

$$C \ge \frac{(|f(b)-f(a)|+F(b))\,\mu}{\lambda\,(|f(b)-f(a)|+F(b))-(\lambda-1)\mu} = \frac{F(b)(q+1)\mu}{F(b)\lambda(q+1)-(\lambda-1)\mu}. \tag{22}$$

Namely, we have

$$0 \le |f(b)-f(a)+G(x)-F(x)| \le |f(b)-f(a)|+F(b) = F(b)(1+q)$$

and

$$\begin{aligned}\mu &= \frac{1}{b-a}\int_a^b |f(b)-f(a)+G(x)-F(x)|dx \\ &= F(b)\int_0^1 \left|q+(1-t)^{\alpha+n+1}-t^{\alpha+n+1}\right| dt.\end{aligned}$$

From (11) it follows that $|q| \le 1$. On the other hand, function

$$h(t) = t^{\alpha+n+1}-(1-t)^{\alpha+n+1}$$

is strictly increasing and $h(0)=-1$, $h(1)=1$. Therefrom we conclude there exists a real zero $\zeta \in [0,1]$ of the integrand. Simple calculation now gives

$$\mu = \frac{2F(b)}{\alpha+n}\left(1-\zeta^{\alpha+n-1}+\frac{q}{2}[1+(\alpha+n-1)(2\zeta-1)]\right).$$

Our statement now follows from (21) and (22). □

Remark 8.11 Taking $w(x)=1$ in Corollary 8.3, produces Theorem 8.10 again.

Remark 8.12 It should be noted that the first expression on the right side of (20) is of the indeterminate form $(\frac{0}{0})$ for $n=1$ and $\alpha=1$, but the limit of that expression, as $\alpha+n$ approaches 2, is equal to 1, so this Corollary really is a generalization of Theorem 8.9.

8.5.2 Improvements of the Milovanović - Pečarić inequality

Yet another generalization of Iyengar's inequality was given in [68] by A.Guessab and G.Schmeisser. They studied, for each real number $x \in [a, \frac{1}{2}(a+b)]$, the more general quadrature formula

$$\frac{1}{b-a}\int_a^b f(t)dt = \frac{1}{2}(f(x)+f(a+b-x))+E(f;x), \tag{23}$$

with $E(f;x)$ being the remainder. Before their main result is stated, a remark is needed.

Remark 8.13 Let $f \in Lip_M$ and suppose that the graph of f passes through the point (ξ,η). Then from

$$|f(t)-f(\xi)| \le M|t-\xi|$$

it follows

$$\varphi(\xi,\eta;t) := \eta - M|t-\xi| \le f(t) \le \eta + M|t-\xi| =: \psi(\xi,\eta;t). \tag{24}$$

The functions $\varphi(\xi,\eta;t)$ and $\psi(\xi,\eta;t)$ themselves belong to Lip_M. Moreover, if we know k points $(\xi_1,\eta_1),(\xi_2,\eta_2),\dots,(\xi_k,\eta_k)$ on the graph of f, then the estimate (24) can be refined. In fact, defining

$$\varphi(t) := \max_{1\le j\le k}\varphi(\xi_j,\eta_j;t) \qquad \textit{and} \qquad \psi(t) := \min_{1\le j\le k}\psi(\xi_j,\eta_j;t)$$

we have

$$\varphi(t)\le f(t)\le\psi(t)$$

and again $\varphi,\psi\in Lip_M$.

Theorem 8.13 *Let f be a function defined on $[a,b]$ and belonging to Lip_M. Then, for each $x\in[a,\frac{1}{2}(a+b)]$, the remainder in (23) satisfies:*

$$|E(f;x)|\le\frac{M}{4}\cdot\frac{(2x-2a)^2+(a+b-2x)^2}{b-a}-\frac{(f(a+b-x)-f(x))^2}{4M(b-a)}. \tag{25}$$

This inequality is sharp for each admissible x. Equality is attained if and only if $f=\pm Mf_*(\delta;\cdot)+c$ $(c\in\mathbb{R})$ and

$$f_*(\delta;t):=\begin{cases} x-t & a\le t\le x\\ t-x, & x\le t\le\frac{1}{2}(a+b+\delta)\\ a+b-x-t+\delta, & \frac{1}{2}(a+b+\delta)\le t\le a+b-x\\ t-a-b+x+\delta, & a+b-x\le t\le b,\end{cases} \tag{26}$$

where $\delta\in\mathbb{R}$ is any real number satisfying $|\delta|\le a+b-2x$.

Proof. Let $u,v\in\mathbb{R}$. Denote by $\mathscr{F}_M(u,v)$ the class of all functions which belong to Lip_M on $[a,b]$ and satisfy $f(x)=u$ and $f(a+b-x)=v$. In view of Remark 8.13 with $k=2$, we have

$$\begin{aligned}&\max\{u-M|t-x|,v-M|t-a-b+x|\}\\ &\le\ f(t)\le\min\{u+M|t-x|,v+M|t-a-b+x|\}.\end{aligned}$$

To shorten notation put

$$\max\{\varphi_1(t),\varphi_2(t)\}\le f(t)\le\min\{\psi_1(t),\psi_2(t)\}.$$

The assumption $f\in Lip_M$ yields

$$|u-v|\le M|2x-a-b|=M(a+b-2x)$$

and therefrom

$$\begin{aligned}&-M(a+b-2x)-M|t-x|+M|t-a-b+x|\\ &\le\ u-M|t-x|-v+M|t-a-b+x|\\ &\le\ M(a+b-2x)-M|t-x|+M|t-a-b+x|\end{aligned}$$

The middle part is equal to $\varphi_1(t) - \varphi_2(t)$, so

$$\varphi_1(t) \geq \varphi_2(t), \; t \in \left[a, \frac{1}{2}\left(a+b+\frac{u-v}{M}\right)\right],$$

while

$$\varphi_1(t) \leq \varphi_2(t), \; t \in \left[\frac{1}{2}\left(a+b+\frac{u-v}{M}\right), b\right].$$

Further, note that

$$v - M|t-a-b+x| = u - Mf_*\left(\frac{u-v}{M};t\right), \; x \in \left[\frac{1}{2}\left(a+b+\frac{u-v}{M}\right), b\right]$$

where f_* is as in (26).

We conclude that

$$\varphi(t) = \max\{\varphi_1(t), \varphi_2(t)\} = u - Mf_*\left(\frac{u-v}{M};t\right), \; t \in [a,b]$$

and analogously,

$$\psi(t) = \min\{\psi_1(t), \psi_2(t)\} = u + Mf_*\left(\frac{v-u}{M};t\right), \; t \in [a,b].$$

Thus, we have proved that for every $f \in \mathscr{F}_M(u,v)$ we have $\varphi(t) \leq f(t) \leq \psi(t)$. Functions φ and ψ are both in $\mathscr{F}_M(u,v)$.

Now

$$\begin{aligned} |E(f;x)| &= \left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{u+v}{2}\right| \\ &\leq \sup_{g \in \mathscr{F}_M(u,v)} \left|\frac{1}{b-a}\int_a^b g(t)dt - \frac{u+v}{2}\right| \\ &= \max\{|E(\varphi;x), |E(\psi;x)|\}. \end{aligned}$$

and also

$$\begin{aligned} |E(\varphi;x)| &= |E(\psi;x)| \\ &= \frac{M[(2x-2a)^2 + (a+b-2x)^2]}{4(b-a)} - \frac{(u-v)^2}{4M(b-a)} \end{aligned}$$

which proves (25), as well as the sharpness. □

Remark 8.14 Note that from Theorem 8.13 for $x = a$, we obtain the conclusion of Theorem 8.8 under a weaker hypothesis.

Next, we give an alternative proof of Theorem 8.13 which was published in [57] by I.Franjić, J.Pečarić and I.Perić. First, notice that

$$\begin{aligned}
&\frac{b-a}{2}(f(x)+f(a+b-x)) \qquad (27)\\
&= (x-a)f(x)+\frac{a+b-2x}{2}(f(x)+f(a+b-x))+(x-a)f(a+b-x).
\end{aligned}$$

The idea is to apply Ostrowski's inequality to the first and the last expression on the right-hand side and Iyengar's inequality to the middle one.

The well-known Ostrowski's inequality (cf. [96]) states that for a differentiable function f on (a,b) such that $|f'(x)| \le M$ for each $x \in (a,b)$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(x)\right| \le M(b-a)\left(\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right), \quad x \in (a,b) \qquad (28)$$

Now,

$$\begin{aligned}
&|(b-a)E(f;x)| = \left|\int_a^b f(t)dt - \frac{b-a}{2}(f(x)+f(a+b-x))\right|\\
&\le \left|(x-a)f(x)-\int_a^x f(t)dt\right|\\
&+ \left|\frac{a+b-2x}{2}(f(x)+f(a+b-x))-\int_x^{a+b-x} f(t)dt\right|\\
&+ \left|(x-a)f(a+b-x)-\int_{a+b-x}^b f(t)dt\right|\\
&\le M(x-a)^2\left(\frac{1}{4}+\frac{\left(x-\frac{a+x}{2}\right)^2}{(x-a)^2}\right)\\
&+ \frac{M}{4}(a+b-2x)^2 - \frac{1}{4M}(f(a+b-x)-f(x))^2\\
&+ M(b-a-b+x)^2\left(\frac{1}{4}+\frac{\left(a+b-x-\frac{a+b-x+b}{2}\right)^2}{(b-a-b+x)^2}\right)\\
&= \frac{M}{4}\cdot\frac{(2x-2a)^2+(a+b-2x)^2}{b-a} - \frac{(f(a+b-x)-f(x))^2}{4M(b-a)}
\end{aligned}$$

which is exactly (25).

Applying this technique of proof, the same authors gave the weighted generalization of the inequality (25) for the class of functions whose derivatives are in $Lip_M(\alpha)$. Of course, the weighted generalization of Ostrowski's inequality is needed in this case. It was given by Matić, Pečarić and Ujević in [85]:

Theorem 8.14 *Assume $f^{(n)}$ exists for each $t \in [a,b]$ while $n \in \mathbb{N} \cup \{0\}$. Let $f^{(n)} \in Lip_M(\alpha)$ and let w be a non-negative and integrable function on $[a,b]$. Then, for $x \in [a,b]$,* *we have*

$$\left|\int_a^b f(t)w(t)dt - \sum_{j=0}^{n} \frac{f^{(j)}(x)}{j!}\int_a^b (t-x)^j w(t)dt\right|$$
$$\leq M\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)}\int_a^b |t-x|^{\alpha+n}w(t)dt, \tag{29}$$

where Γ *is the standard Gamma function.*

For $w(x)=1$, (29) reduces to

$$\left|\int_a^b f(t)dt - \sum_{j=0}^{n} \frac{f^{(j)}(x)}{(j+1)!}\left[(b-x)^{j+1}-(a-x)^{j+1}\right]\right|$$
$$\leq \frac{M\Gamma(\alpha+1)}{\Gamma(\alpha+n+2)}\left[(b-x)^{j+1}+(x-a)^{j+1}\right]. \tag{30}$$

Now everything is set for:

Theorem 8.15 *Let function* $f:[a,b]\to\mathbb{R}$ *satisfy the following conditions:*

1° $f^{(n-1)}\in Lip_M(\alpha)$

2° $f^{(k)}(x)=f^{(k)}(a+b-x)=0,\ x\in\left[a,\frac{a+b}{2}\right],\ k=1,2,\ldots,n-1.$

Let w *be a non-negative and integrable function on* $[a,b]$ *and such that* $w(x)=w(a+b-x)$*, for each* $x\in\left[a,\frac{a+b}{2}\right]$*. Then we have*

$$\left|\int_a^b f(t)w(t)dt-\frac{1}{2}(f(x)+f(a+b-x))\int_a^b w(t)dt\right|$$
$$\leq \int_x^{a+b-x} F(t)w(t)dt-\frac{1}{2}\int_x^{a+b-x}|f(a+b-x)-f(x)+G(t)-F(t)|w(t)dt$$
$$+\ \frac{2M}{(\alpha+n-1)^{(n-1)}}\int_a^x |t-x|^{\alpha+n-1}w(t)dt \tag{31}$$

where

$$F(t)=M\frac{(t-x)^{\alpha+n-1}}{(\alpha+n-1)^{(n-1)}}\ ,\ G(t)=M\frac{(a+b-x-t)^{\alpha+n-1}}{(\alpha+n-1)^{(n-1)}}.$$

Proof. We start from the left side of (31)

$$I=\left|\int_a^b f(t)w(t)dt-\frac{1}{2}(f(x)+f(a+b-x))\int_a^b w(t)dt\right|$$
$$=\left|\int_x^{a+b-x} f(t)w(t)dt-\frac{1}{2}(f(x)+f(a+b-x))\int_x^{a+b-x} w(t)dt\right.$$
$$+\ \int_a^x f(t)w(t)dt+\int_{a+b-x}^b f(t)w(t)dt$$
$$\left.-\ \frac{1}{2}(f(x)+f(a+b-x))\left(\int_a^x w(t)dt+\int_{a+b-x}^b w(t)dt\right)\right|.$$

Function w is symmetric about the midpoint so we have

$$\int_a^x w(t)dt = \int_{a+b-x}^b w(t)dt.$$

Therefore

$$\begin{aligned}
I &= \left| \int_x^{a+b-x} f(t)w(t)dt - \frac{1}{2}(f(x)+f(a+b-x)) \int_x^{a+b-x} w(t)dt \right. \\
&+ \left. \int_a^x f(t)w(t)dt + \int_{a+b-x}^b f(t)w(t)dt - (f(x)+f(a+b-x)) \int_a^x w(t)dt \right| \\
&\leq \left| \int_x^{a+b-x} f(t)w(t)dt - \frac{1}{2}(f(x)+f(a+b-x)) \int_x^{a+b-x} w(t)dt \right| \\
&+ \left| \int_a^x f(t)w(t)dt - f(x) \int_a^x w(t)dt \right| \\
&+ \left| \int_{a+b-x}^b f(t)w(t)dt - f(a+b-x) \int_{a+b-x}^b w(t)dt \right|.
\end{aligned}$$

Now, apply the weighted generalization of Iyengar's inequality (17) to the first expression on the right side and the weighted generalization of Ostrowski's inequality (29) to the second and the third. It follows

$$\begin{aligned}
I &\leq \int_x^{a+b-x} F(t)w(t)dt - \frac{1}{2} \int_x^{a+b-x} |f(a+b-x) - f(x) + G(t) - F(t)| w(t)dt \\
&+ M \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n)} \int_a^x |t-x|^{\alpha+n-1} w(t)dt \\
&+ M \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n)} \int_{a+b-x}^b |t-(a+b-x)|^{\alpha+n-1} w(t)dt.
\end{aligned}$$

Using the symmetric property of function w, it is easy to check that the second and the third expression on the right side are equal and thus inequality (31) is proved. □

Corollary 8.4 *Let function f satisfy the assumptions of Theorem 8.15. Then we have*

$$\begin{aligned}
|E(f;x)| &\leq \frac{M}{(b-a)(\alpha+n)^{(n)}} \left[2(x-a)^{\alpha+n} \right. \\
&+ \left. (a+b-2x)^{\alpha+n} \left(\zeta^{\alpha+n-1} - \frac{q}{2} [1+(\alpha+n-1)(2\zeta-1)] \right) \right]
\end{aligned} \tag{32}$$

where ζ is defined as in Theorem 8.10.

Proof. Statement follows from Theorem 8.15 by taking $w(x) = 1$. It can, of course, be proved directly, using the same idea and applying inequalities (6) and (30). □

Remark 8.15 Taking $n = 1$ and $\alpha = 1$ in Corollary 8.4 produces Theorem 8.13.

If we weaken the condition $f^{(n-1)} \in Lip_M(\alpha)$ to $f^{(n-1)}$ being continuous and satisfying

$$\begin{aligned} |f^{(n-1)}(t_1) - f^{(n-1)}(t_2)| &\leq M_1|t_1 - t_2|^\alpha, \qquad t_1,t_2 \in [a,x] \\ |f^{(n-1)}(t_1) - f^{(n-1)}(t_2)| &\leq M_2|t_1 - t_2|^\alpha, \qquad t_1,t_2 \in [x,a+b-x] \\ |f^{(n-1)}(t_1) - f^{(n-1)}(t_2)| &\leq M_3|t_1 - t_2|^\alpha, \qquad t_1,t_2 \in [a+b-x,b] \end{aligned} \tag{33}$$

by an analogous proof we would get:

$$\begin{aligned} &\left| \int_a^b f(t)w(t)dt - \frac{1}{2}(f(x) + f(a+b-x)) \int_a^b w(t)dt \right| \\ &\leq \int_x^{a+b-x} F(t)w(t)dt - \frac{1}{2} \int_x^{a+b-x} |f(a+b-x) - f(x) + G(t) - F(t)| w(t)dt \\ &+ \frac{M_1 + M_3}{(\alpha + n - 1)^{(n-1)}} \int_a^x |t-x|^{\alpha+n-1} w(t)dt \end{aligned} \tag{34}$$

where

$$F(t) = M_2 \frac{(t-x)^{\alpha+n-1}}{(\alpha+n-1)^{(n-1)}} \ , \ G(t) = M_2 \frac{(a+b-x-t)^{\alpha+n-1}}{(\alpha+n-1)^{(n-1)}}. \tag{35}$$

Inequality (31) follows upon taking $M_1 = M_2 = M_3$ and using concavity of function t^α for $\alpha \in (0,1]$.

Another interesting result from [68] is the following theorem.

Theorem 8.16 *Let f be a differentiable function defined on $[a,b]$ with $f' \in Lip_M$. Let $x \in \left[a, \frac{a+b}{2}\right)$, and suppose that $f'(x) = f'(a+b-x) = 0$. Then the remainder in (23) satisfies*

$$|E(f;x)| \leq \frac{1}{b-a} \left[\frac{M}{3}(x-a)^3 + \frac{M}{32}(a+b-2x)^3 - \frac{(f(a+b-x)-f(x))^2}{2M(a+b-2x)} \right]. \tag{36}$$

The inequality is sharp for each $x \in \left[a, \frac{a+b}{2}\right)$. Equality is attained for $f(t) = \pm M \int f'_*(t)dt + c$ with $c \in \mathbb{R}$ and

$$f'_*(t) := \begin{cases} x - t, & a \leq t \leq \frac{1}{4}(a+b+2x) - \delta =: x_1 \\ t - \frac{1}{2}(a+b) + 2\delta, & x_1 \leq t \leq \frac{1}{4}(3a+3b-2x) - \delta =: x_2 \\ a+b-x-t, & x_2 \leq t \leq b, \end{cases}$$

where $\delta \in \mathbb{R}$ is any real number satisfying $|\delta| \leq \frac{1}{4}(a+b-2x)$.

Proof. Denote by $\mathscr{F}'_M(\Delta)$ the class of all functions which are differentiable on $[a,b]$ with f' belonging Lip_M and which satsify

$$f(a+b-x) - f(x) = \Delta \qquad \text{and} \qquad f'(x) = f'(a+b-x) = 0.$$

We want to determine for each $x \in \left[a, \frac{a+b}{2}\right)$ the supremum od $|E(f;x)|$ over all $f \in \mathscr{F}'_M(\Delta)$. Using integration by parts, it is easy to check the following formula

$$E(f;x) = \frac{1}{b-a} \int_a^b K(t) f'(t)dt,$$

where

$$K(t) := \begin{cases} a-t, & a \le t \le x \\ \frac{1}{2}(a+b)-t, & x < t \le a+b-x \\ b-t, & a+b-x < t \le b. \end{cases}$$

Applying this yields

$$\sup_{f \in \mathscr{F}'_M(\Delta)} |E(f;x)| = S_1 + S_2 + S_3, \tag{37}$$

where

$$\begin{aligned} S_1 &= \sup_{f \in \mathscr{F}'_M(\Delta)} \left| \frac{1}{b-a} \int_a^x (a-t) f'(t)\,dt \right|, \\ S_2 &= \sup_{f \in \mathscr{F}'_M(\Delta)} \left| \frac{1}{b-a} \int_x^{a+b-x} \left(\frac{1}{2}(a+b) - t \right) f'(t)\,dt \right|, \\ S_3 &= \sup_{f \in \mathscr{F}'_M(\Delta)} \left| \frac{1}{b-a} \int_{a+b-x}^{b} (b-t) f'(t)\,dt \right|. \end{aligned}$$

In view of Remark 8.13, it follows

$$S_1 = S_3 = \frac{M}{b-a} \int_a^x (t-a)(x-t)\,dt = \frac{M(x-a)^3}{6(b-a)}. \tag{38}$$

What is left is to calculate S_2. Use a substitution

$$t \mapsto x + \frac{a+b-2x}{2}(t+1)$$

and introduce the function

$$g(t) := \frac{2}{M(a+b-2x)} \cdot f'\left(x + \frac{a+b-2x}{2}(t+1) \right).$$

Now

$$\int_x^{a+b-x} \left(\frac{1}{2}(a+b) - t \right) f'(t)\,dt = -M \left(\frac{a+b-2x}{2} \right)^3 \int_{-1}^{1} t g(t)\,dt.$$

The condition $f \in \mathscr{F}'_M(\Delta)$ implies that the function g is defined on $[-1,1]$ and satisfies

$$g \in Lip_1, \qquad g(-1) = g(1) = 0 \qquad \text{and} \qquad \int_{-1}^{1} g(t)\,dt = D \tag{39}$$

where

$$D := \frac{\Delta}{M} \left(\frac{2}{a+b-2x} \right)^2 \tag{40}$$

We can assume D is non-negative; otherwise, take $-g$ instead of g. Furthermore, assume $\int_{-1}^{1} t g(t)\,dt$ is non-negative; otherwise, replace g by $g(-.)$, which is again a function satisfying (39). Now

$$S_2 = \frac{M}{b-a} \left(\frac{a+b-2x}{2} \right)^3 \Omega, \tag{41}$$

where Ω is the solution of the following optimization problem:

$$\text{Maximize} \qquad \Phi(g) := \int_{-1}^{1} t g(t)dt \qquad \text{under the constraints (39).}$$

The solution of this problem is the function (for details see [68])

$$G^*(t) = \begin{cases} -1-t, & -1 \le t \le -\frac{1}{2}(1+D) \\ t+D, & -\frac{1}{2}(1+D) \le t \le \frac{1}{2}(1-D) \\ 1-t, & \frac{1}{2}(1-D) \le t \le 1. \end{cases}$$

and the maximal value of this functional Φ is

$$\Omega := \Phi(G^*) = \frac{1-D^2}{4}.$$

Now, combining (37)-(41), we readily obtain (36). Functions f for which equality is attained are easily deduced from G^*. The proof is thus completed. □

Remark 8.16 For $x = a$, inequality (36) is obviously an improvement of inequality (8).

Though A.Guessab and G.Schmeisser proved Theorem 8.16 for $x \in [a, \frac{1}{2}(a+b))$, it is in fact enough to prove their statement for $x = a$. From that, the more general case when $x \in [a, \frac{1}{2}(a+b))$ follows. So, suppose we have

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{2}(f(a)+f(b)) \right| \le \frac{M}{32}(b-a)^2 - \frac{(f(b)-f(a))^2}{2M(b-a)^2}. \tag{42}$$

We will use, again, the same idea as in proof of Theorem 31, or more precisely, we will now start from (29). To the middle part we apply (42):

$$\begin{aligned} &\left| \int_x^{a+b-x} f(t)dt - \frac{a+b-2x}{2}(f(x)+f(a+b-x)) \right| \\ &\le \frac{M}{32}(a+b-2x)^3 - \frac{(f(a+b-x)-f(x))^2}{2M(a+b-2x)}. \end{aligned} \tag{43}$$

To the first and the last part, we apply (30) for $n = 1$ and $\alpha = 1$. In that case (30) reduces to

$$\left| \int_a^b f(t)dt - f(x)(b-a) - f'(x)(b-a)(b+a-2x) \right| \le \frac{M}{2} \int_a^b |t-x|^2 dt \tag{44}$$

By assumption, function f satisfies $f'(x) = f'(a+b-x) = 0$, so from (44) follows

$$\left| \int_a^x f(t)dt - (x-a)f(x) \right| \le \frac{M}{6}(x-a)^3, \tag{45}$$

$$\left| \int_{a+b-x}^b f(t)dt - (x-a)f(a+b-x) \right| \le \frac{M}{6}(x-a)^3. \tag{46}$$

Addition of estimations (43), (45) and (46) produces (36).

Once again, replacing condition $f^{(n-1)} \in Lip_M(\alpha)$ with condition (33), gives us

$$|E(f;x)| \leq \frac{1}{b-a}\left[\frac{M_1+M_3}{6}(x-a)^3+\frac{M_2}{32}(a+b-2x)^3-\frac{(f(a+b-x)-f(x))^2}{2M_2(a+b-2x)}\right].$$

Again, taking $M_1 = M_2 = M_3$ produces (36).

We will finish this subsection with another generalization of Iyengar's inequality, obtained by X.L.Cheng and J.Sun in [18].

Theorem 8.17 *Let $f : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$, be twice differentiable in the interior I° of I and let $a,b \in I^\circ$, $a < b$. If $|f''(x)| \leq M$ for every $x \in [a,b]$, then*

$$\left|\int_a^b f(x)dx-\frac{1}{2}(b-a)(f(a)+f(b))+\frac{1}{8}(b-a)^2(f'(b)-f'(a))\right| \leq \frac{M}{24}(b-a)^3-\sqrt{\frac{|\Delta|^3(b-a)^3}{72M}}, \tag{47}$$

where

$$\Delta = f'(a)-2\frac{f(b)-f(a)}{b-a}+f'(b). \tag{48}$$

Proof. Denote

$$J_f = \int_a^b f(x)dx-\frac{1}{2}(b-a)(f(a)+f(b))+\frac{1}{8}(b-a)^2(f'(b)-f'(a)).$$

It is easy to see that

$$J_f = \frac{1}{2}\int_a^b \left(x-\frac{a+b}{2}\right)^2 f''(x)dx$$

and

$$\Delta = \frac{2}{b-a}\int_a^b \left(x-\frac{a+b}{2}\right) f''(x)dx. \tag{49}$$

Now, for any ε such that $|\varepsilon| \leq \frac{1}{8}$ we have

$$\begin{aligned} J_f+\varepsilon(b-a)^2\Delta &= \int_a^b \left[\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2+2\varepsilon(b-a)\left(x-\frac{a+b}{2}\right)\right] f''(x)dx \\ &\leq M(b-a)^3F(\varepsilon), \end{aligned} \tag{50}$$

where

$$\begin{aligned} F(\varepsilon) &= \frac{1}{(b-a)^3}\int_a^b \left|\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2+2\varepsilon(b-a)\left(x-\frac{a+b}{2}\right)\right| dx \\ &= \int_0^1 \left|\frac{1}{2}\left(x-\frac{1}{2}\right)^2+2\varepsilon\left(x-\frac{1}{2}\right)\right| dx. \end{aligned}$$

The zeros of the integrands are $x_1 = 1/2$ and $x_2 = 1/2 - 4\varepsilon$, so $0 \le x_2 \le 1/2$ when $0 \le \varepsilon \le 1/8$, while $1/2 \le x_2 \le 1$ when $-1/8 \le \varepsilon \le 0$. Thus, for $0 \le \varepsilon \le \frac{1}{8}$:

$$\begin{aligned}F(\varepsilon) &= \int_0^{\frac{1}{2}-4\varepsilon}\left(\frac{1}{2}\left(x-\frac{1}{2}\right)^2+2\varepsilon\left(x-\frac{1}{2}\right)\right)dx\\ &- \int_{\frac{1}{2}-4\varepsilon}^{\frac{1}{2}}\left(\frac{1}{2}\left(x-\frac{1}{2}\right)^2+2\varepsilon\left(x-\frac{1}{2}\right)\right)dx+\int_{\frac{1}{2}}^{1}\left(\frac{1}{2}\left(x-\frac{1}{2}\right)^2+2\varepsilon\left(x-\frac{1}{2}\right)\right)dx\\ &= \frac{1}{24}+\frac{32}{3}\varepsilon^3.\end{aligned}$$

Analogously, for $-\frac{1}{8} \le \varepsilon \le 0$, it follows

$$F(\varepsilon) = \frac{1}{24} - \frac{32}{3}\varepsilon^3.$$

It is not difficult to check that

$$\varepsilon_* = \operatorname{sgn}(\Delta)\sqrt{\frac{|\Delta|}{32M(b-a)}}$$

is the point in which the function

$$f(\varepsilon) = M(b-a)^3F(\varepsilon) - \varepsilon(b-a)^2\Delta$$

achieves its minimal value. (49) implies

$$|\Delta| \le \frac{2M}{b-a}\int_a^b\left|x-\frac{a+b}{2}\right|dx = \frac{M}{2}(b-a),$$

and therefrom $|\varepsilon_*| \le 1/8$.

From (50) it follows

$$J_f \le f(\varepsilon_*) = \frac{M}{24}(b-a)^3 - \sqrt{\frac{|\Delta|^3(b-a)^3}{72M}}.$$

Replacing f with $-f$, analogously we get

$$J_{-f} = -J_f \le \frac{M}{24}(b-a)^3 - \sqrt{\frac{|\Delta|^3(b-a)^3}{72M}}$$

and thus

$$|J_f| \le \frac{M}{24}(b-a)^3 - \sqrt{\frac{|\Delta|^3(b-a)^3}{72M}}.$$

□

Remark 8.17 We have shown that $\frac{|\Delta|}{M} \leq \frac{b-a}{2}$. If we use this in estimating the right-hand side of (47), it easily follows

$$\begin{aligned}
&\left|\int_a^b f(x)dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{1}{8}(b-a)^2(f'(b)-f'(a))\right| \\
&\leq \frac{M}{24}(b-a)^3 - \frac{|\Delta|(b-a)}{6}\sqrt{\frac{b-a}{2}}\sqrt{\frac{|\Delta|}{M}} \\
&\leq \frac{M}{24}(b-a)^3 - \frac{b-a}{6M}\Delta^2.
\end{aligned}$$

Assume now that we have $f'(a) = f'(b) = 0$. Then

$$\Delta = -2\frac{f(b)-f(a)}{b-a},$$

and (47) reduces to

$$\left|\int_a^b f(x)dx - \frac{1}{2}(b-a)(f(a)+f(b))\right| \leq \frac{M}{24}(b-a)^3 - \frac{2}{3}\frac{(f(b)-f(a))^2}{M(b-a)} \tag{51}$$

This inequality is sharper than the inequality (8).

Remark 8.18 With the additional assumption $f'(a) = f'(b) = 0$, the estimate of the trapezoid formula in (47) is weaker than the one in (36) with $x = a$, i.e. (42). We claim:

$$\begin{aligned}
\frac{M}{32}(b-a)^2 - \frac{(f(b)-f(a))^2}{2M(b-a)^2} &\leq \frac{M}{24}(b-a)^2 - \frac{|\Delta|}{6}\sqrt{\frac{|\Delta|(b-a)}{2M}}, \\
&= \frac{M}{24}(b-a)^2 - \frac{|f(b)-f(a)|}{3(b-a)}\sqrt{\frac{|f(b)-f(a)|}{M}}
\end{aligned}$$

since now $\Delta = -2\frac{f(b)-f(a)}{b-a}$. The claim is equivalent to

$$\frac{|f(b)-f(a)|}{3(b-a)}\sqrt{\frac{|f(b)-f(a)|}{M}} - \frac{(f(b)-f(a))^2}{2M(b-a)^2} \leq \frac{M}{96}(b-a)^2$$

Introduce the function $g(x) = \dfrac{f(x)}{M}$. Now

$$\frac{|g(b)-g(a)|}{3(b-a)}\sqrt{|g(b)-g(a)|} - \frac{(g(b)-g(a))^2}{2(b-a)^2} \leq \frac{(b-a)^2}{96}. \tag{52}$$

Denote $t = |g(b)-g(a)|$ and

$$h(t) = \frac{t^{3/2}}{3(b-a)} - \frac{t^2}{2(b-a)^2}.$$

It is easy to check that $t = (b-a)^2/4$ is the point in which the function $h(t)$ attains its maximal value which is $(b-a)^2/96$. This proves (52), and our claim.

An estimate of a similar type, i.e. an estimate of the functional J_f, was given by X.L.Cheng in [17].

Theorem 8.18 *Let $f \in C^2([a,b])$ and $|f''(x)| \le M$. Then*

$$\left| \int_a^b f(x)dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{1}{8}(b-a)^2(f'(b)-f'(a)) \right| \le \frac{M}{24}(b-a)^3 - \frac{b-a}{16M}\Delta^2, \tag{53}$$

where Δ is as in (48).

Proof. By Taylor's expansion formula, for $x \in [a,b]$ we get

$$f(x) \le f(a) + f'(a)(x-a) + \frac{M}{2}(x-a)^2$$

and

$$f(x) \le f(b) - f'(b)(b-x) + \frac{M}{2}(b-x)^2.$$

Using this, for $c = (a+b)/2$ and $|\delta| \le (b-a)/2$, we obtain

$$\int_a^{c+\delta} f(x)dx \le f(a)\left(\frac{b-a}{2}+\delta\right) + \frac{1}{2}f'(a)\left(\frac{b-a}{2}+\delta\right)^2 + \frac{M}{6}\left(\frac{b-a}{2}+\delta\right)^3$$

$$\int_{c+\delta}^b f(x)dx \le f(b)\left(\frac{b-a}{2}-\delta\right) - \frac{1}{2}f'(b)\left(\frac{b-a}{2}-\delta\right)^2 + \frac{M}{6}\left(\frac{b-a}{2}-\delta\right)^3.$$

Then

$$\int_a^b f(x)dx = \int_a^{c+\delta} f(x)dx + \int_{c+\delta}^b f(x)dx \le F_0 + \delta F_1 + \delta^2 F_2,$$

where

$$F_0 = \frac{b-a}{2}(f(a)+f(b)) + \frac{(b-a)^2}{8}(f'(a)-f'(b)) + \frac{M}{24}(b-a)^3,$$

$$F_1 = f(a) - f(b) + \frac{b-a}{2}(f'(a)+f'(b)),$$

$$F_2 = \frac{1}{2}(f'(a)-f'(b)+M(b-a)).$$

Introduce $\widetilde{F}_2 = M(b-a)$. Then $F_2 \le \widetilde{F}_2$. Now, we find the minimal value of $F(\delta) = F_0 + \delta F_1 + \delta^2 \widetilde{F}_2$ for $|\delta| \le (b-a)/2$:

$$F'(\delta) = f(a) - f(b) + \frac{b-a}{2}(f'(a)+f'(b)) + 2M(b-a)\delta = 0,$$

$$F''(\delta) = 2M(b-a) > 0.$$

Thus, the point in which $F(\delta)$ attains minimal value is

$$\delta_0 = \frac{2(f(b)-f(a))/(b-a) - f'(a) - f'(b)}{4M} = -\frac{\Delta}{4M},$$

where Δ is as in (48). Using Taylor's expansion formula it is not difficult to verify that $|\delta_0| \le (b-a)/2$. Thus,

$$\begin{aligned}\int_a^b f(x)dx &\le F(\delta_0) \\ &= \frac{b-a}{2}(f(a)+f(b)) + \frac{(b-a)^2}{8}(f'(a)-f'(b)) + \frac{M}{24}(b-a)^3 - \frac{b-a}{16M}\Delta^2.\end{aligned}$$

To obtain the lower bound, apply Taylor's expansion formula again:

$$f(x) \ge f(a) + f'(a)(x-a) - \frac{M}{2}(x-a)^2$$

and

$$f(x) \ge f(b) - f'(b)(b-x) - \frac{M}{2}(b-x)^2.$$

Now, analogously as before we obtain

$$\int_a^b f(x)dx \ge \frac{b-a}{2}(f(a)+f(b)) + \frac{(b-a)^2}{8}(f'(a)-f'(b)) - \frac{M}{24}(b-a)^3 + \frac{b-a}{16M}\Delta^2$$

which in conclusion proves (53). □

Remark 8.19 Minimizing the polynomial $F(\delta) = F_2\delta^2 + F_1\delta + F_0$ which estimates the right-hand side and the appropriate polynomial which estimates the left-hand side, would produce

$$\begin{aligned}&\left|\int_a^b f(x)dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{1}{8}(b-a)^2(f'(b)-f'(a))\right| \\ &\le \frac{M}{24}(b-a)^3 - \frac{(b-a)^2\Delta^2}{8[M(b-a)+f'(a)-f'(b)]} \qquad (54)\end{aligned}$$

which is exactly inequality (65) from Corollary 8.6 for $m = -M$, only written in somewhat different form. Obviously, the estimate (54) is better than the one in (53).

8.5.3 Weighted generalizations of Iyengar's inequality through Taylor's formula

The results of this subsection were obtained by F.Qi in [111]. Introduce notation:

$$h_{s,k}(t) = \int_s^t (x-s)^k w(x)dx, \quad s,t \in [a,b], \quad k \in \mathbb{N}. \qquad (55)$$

where the function w in non-negative and integrable on $[a,b]$.

Theorem 8.19 *Let f be continuous on $[a,b]$ and differentiable on (a,b). Suppose $f(a) = f(b) = 0$ and $m \le f'(x) \le M$ for every $x \in (a,b)$. Let $w(x) > 0$ for every $x \in [a,b]$. If $f \neq 0$, then $m < 0 < M$ and*

$$mh_{a,1}(t_1) - Mh_{b,1}(t_1) \le \int_a^b w(x)f(x)dx \le Mh_{a,1}(t_0) - mh_{b,1}(t_0), \tag{56}$$

where $t_0 = \frac{Ma-mb}{M-m} \in (a,b)$, $\quad t_1 = \frac{Mb-ma}{M-m} \in (a,b)$.

Proof. $m < 0 < M$ is a direct consequence of the Rolle's Mean Value Theorem. The idea is to apply Lagrange's Mean Value Theorem in order to estimate the weighted integral.

Let $\Theta \in (a,b)$. Now,

$$\begin{aligned}\int_a^b w(x)f(x)dx &= \int_a^\Theta w(x)[f(x)-f(a)]dx + \int_\Theta^b w(x)[f(x)-f(b)]dx \\ &= \int_a^\Theta w(x)(x-a)f'(\xi_1)dx + \int_\Theta^b w(x)(x-b)f'(\xi_2)dx,\end{aligned}$$

where $a < \xi_1 < \Theta < \xi_2 < b$. Using the fact that the first derivative is bounded we get

$$\begin{aligned}\int_a^b w(x)f(x)dx &\le M\int_a^\Theta (x-a)w(x)dx + m\int_\Theta^b (x-b)w(x)dx \\ &= Mh_{a,1}(\Theta) - mh_{b,1}(\Theta).\end{aligned} \tag{57}$$

We wish to determine the minimal value of the upper bound. From (55) it follows

$$\frac{dh_{s,k}(t)}{dt} = (t-s)^k w(t),$$

$$\text{i.e.} \quad \frac{d(Mh_{a,1}(\Theta) - mh_{b,1}(\Theta))}{d\Theta} = [(M-m)\Theta + (bm-aM)]w(\Theta).$$

It is easy to check that the minimal value is attained for $\Theta = \frac{Ma-mb}{M-m} = t_0 \in (a,b)$.

Similarly,

$$\begin{aligned}\int_a^b w(x)f(x)dx &\ge m\int_a^\Theta (x-a)w(x)dx + M\int_\Theta^b (x-b)w(x)dx \\ &= mh_{a,1}(\Theta) - Mh_{b,1}(\Theta).\end{aligned}$$

The lower bound attains its maximal value for $\Theta = \frac{Mb-ma}{M-m} = t_1 \in (a,b)$. This completes the proof. □

Theorem 8.20 *Let f be continuous on $[a,b]$ and differentiable on (a,b). Assume f is not a constant function and that $m \le f'(x) \le M$ for every $x \in (a,b)$. Let $w(x) > 0$ for every $x \in [a,b]$. Then*

$$\begin{aligned}&\left[\frac{f(b)-f(a)}{b-a} - M\right]h_{b,1}(t_3) - \left[\frac{f(b)-f(a)}{b-a} - m\right]h_{a,1}(t_3) \\ &\le \int_a^b w(x)f(x)dx - f(a)h_{a,0}(b) - \frac{f(b)-f(a)}{b-a}h_{a,1}(b) \\ &\le \left[\frac{f(b)-f(a)}{b-a} - m\right]h_{b,1}(t_2) - \left[\frac{f(b)-f(a)}{b-a} - M\right]h_{a,1}(t_2)\end{aligned} \tag{58}$$

where $t_2 = \frac{Ma-mb+f(b)-f(a)}{M-m} \in (a,b)$, $\quad t_3 = \frac{Mb-ma-f(b)+f(a)}{M-m} \in (a,b)$.

Proof. For $x \in [a,b]$ define function

$$\Phi(x) = [f(x)-f(a)](b-a) - [f(b)-f(a)](x-a).$$

Obviously, $\Phi(a) = \Phi(b) = 0$. Furthermore,

$$\Phi'(x) = (b-a)f'(x) - f(b) + f(a),$$

and therefrom

$$(b-a)m - f(b) + f(a) \le \Phi'(x) \le (b-a)M - f(b) + f(a).$$

Now, notice that

$$\begin{aligned} \int_a^b w(x)\Phi(x)dx &= (b-a)\int_a^b w(x)[f(x)-f(a)]dx - [f(b)-f(a)]\int_a^b (x-a)w(x)dx \\ &= (b-a)\left[\int_a^b w(x)f(x)dx - f(a)h_{a,0}(b)\right] - [f(b)-f(a)]h_{a,1}(b). \end{aligned}$$

Applying Theorem 8.19 to the function $\Phi(x)$ now yields the statement. □

Further results are given without proof. For details see [111].

Theorem 8.21 *Let f be differentiable, $f \in C^{n-1}([a,b])$ and such that $m \le f^{(n)}(x) \le M$ for every $x \in (a,b)$. Let $w(x) > 0$ for $x \in [a,b]$. If n is odd, then for every $t \in (a,b)$*

$$\begin{aligned} &\frac{mh_{a,n}(t) - Mh_{b,n}(t)}{n!} \\ &\quad \le \int_a^b w(x)f(x)dx + \sum_{i=0}^{n-1} \frac{f^{(i)}(b)h_{b,i}(t) - f^{(i)}(a)h_{a,i}(t)}{i!} \\ &\quad \le \frac{Mh_{a,n}(t) - mh_{b,n}(t)}{n!}; \end{aligned} \tag{59}$$

while for an even n

$$\begin{aligned} &\frac{m(h_{a,n}(t) - h_{b,n}(t))}{n!} \\ &\quad \le \int_a^b w(x)f(x)dx + \sum_{i=0}^{n-1} \frac{f^{(i)}(b)h_{b,i}(t) - f^{(i)}(a)h_{a,i}(t)}{i!} \\ &\quad \le \frac{M(h_{a,n}(t) - h_{b,n}(t))}{n!}. \end{aligned} \tag{60}$$

Corollary 8.5 *Let $f \in C^n([a,b])$ and $m \le f^{(n)} \le M$ for $x \in [a,b]$. Denote*

$$S_n(u,v,w) = \sum_{k=1}^{n-1} \frac{(-1)^k}{k!} \cdot u^k f^{(k-1)}(v) + \frac{w}{n!} \cdot (-1)^n u^n, \tag{61}$$

$$\frac{\partial^k S_n}{\partial u^k} = S_n^{(k)}(u,v,w). \tag{62}$$

Then, for every $t \in [a,b]$, when n is even we have

$$\sum_{i=0}^{n+1} \frac{(-1)^i}{i!} \left(S_{n+1}^{(i)}(a,a,m) - S_{n+1}^{(i)}(b,b,m) \right) t^i \leq \int_a^b f(x)dx$$
$$\leq \sum_{i=0}^{n+1} \frac{(-1)^i}{i!} \left(S_{n+1}^{(i)}(a,a,M) - S_{n+1}^{(i)}(b,b,M) \right) t^i; \tag{63}$$

while when n is odd

$$\sum_{i=0}^{n+1} \frac{(-1)^i}{i!} \left(S_{n+1}^{(i)}(a,a,m) - S_{n+1}^{(i)}(b,b,M) \right) t^i \leq \int_a^b f(x)dx$$
$$\leq \sum_{i=0}^{n+1} \frac{(-1)^i}{i!} \left(S_{n+1}^{(i)}(a,a,M) - S_{n+1}^{(i)}(b,b,m) \right) t^i. \tag{64}$$

Corollary 8.6 *Let $f \in C^2([a,b])$ and $m \leq f''(x) \leq M$. Then*

$$\begin{aligned} &\frac{m(b^3-a^3)}{6} + \frac{\left[f(b)-f(a)-bf'(b)+af'(a)+\frac{m}{2}(b^2-a^2)\right]^2}{2\left[f'(b)-f'(a)+m(a-b)\right]} \\ &\leq \int_a^b f(x)dx - bf(b) + af(a) + \frac{b^2 f'(b) - a^2 f'(a)}{2} \\ &\leq \frac{M(b^3-a^3)}{6} + \frac{\left[f(b)-f(a)-bf'(b)+af'(a)+\frac{M}{2}(b^2-a^2)\right]^2}{2\left[f'(b)-f'(a)+M(a-b)\right]}. \end{aligned} \tag{65}$$

Corollary 8.7 *Let f be continuous on $[a,b]$ and differentiable on (a,b). Suppose f is not a constant function and $m \leq f'(x) \leq M$ for every $x \in (a,b)$. Then*

$$\begin{aligned} &\frac{mM(b-a)^2 + 2(b-a)[Mf(a)-mf(b)] + [f(b)-f(a)]^2}{2(M-m)} \\ &\leq \int_a^b f(x)dx \\ &\leq -\frac{mM(b-a)^2 + 2(b-a)[mf(a)-Mf(b)] + [f(b)-f(a)]^2}{2(M-m)} \end{aligned} \tag{66}$$

Remark 8.20 The inequality (66) was also derived by R.P.Agarwal and S.S. Dragomir in [3]. Note that for $m = -M$, (66) recaptures Iyengar's inequality (1).

Remark 8.21 In [114], F.Qi, P.Cerone and S.S.Dragomir gave a generalization of Iyengar's inequality using a generalized Taylor's formula with an integral remainder. The main tool used is a harmonic sequence of polynomials. A sequence of polynomials $\{P_i(x)\}_{i=0}^{\infty}$ is called harmonic if

$$P_i'(x) = P_{i-1}(x), \quad i \in \mathbb{N}; \qquad P_0(x) = 1.$$

8.5.4 Weighted generalizations of Iyengar's inequality through Steffensen's inequality

Through the years, Iyengar's inequality has been generalized in various ways. Generalizations that are of interest in this subsection are the ones obtained using Hayashi's modification of the well-known Steffensen's inequality. For example, in [3] R.P.Agarwal and S.S.Dragomir first proved:

Theorem 8.22 *Let function f be differentiable on $[a,b]$ and $m \le f'(x) \le M$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}\right| \le \frac{[f(b)-f(a)-m(b-a)][M(b-a)-f(b)+f(a)]}{2(M-m)(b-a)}. \tag{67}$$

Inequality (67) is in fact inequality (66), only written in somewhat different form.

In [12], P.Cerone proved the following result for the trapezoidal rule:

Theorem 8.23 *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous on I° (I° being the interior od I) and $[a,b] \subset I^\circ$. Assume $m = \inf_{x\in[a,b]} f^{(n)}(x) > -\infty$ and $M = \sup_{x\in[a,b]} f^{(n)}(x) < \infty$. Then*

$$\left|\int_a^b f(x)dx - \sum_{k=1}^{n} E_k(\Theta;a,b) + R - \frac{M-m}{2(n+1)!}(U+L)\right| \le \frac{M-m}{2(n+1)!}(U-L) \tag{68}$$

where

$$E_k(\Theta;a,b) = \frac{1}{k!}[(\Theta-a)^k f^{(k-1)}(a) - (\Theta-b)^k f^{(k-1)}(b)] \tag{69}$$

$$R = \frac{m}{(n+1)!}\left[(\Theta-b)^{n+1} - (\Theta-a)^{n+1}\right] \tag{70}$$

$$L = \begin{cases} (\lambda_n^a)^{n+1} + (\lambda_n^b)^{n+1}, & n \text{ even} \\ (\Theta-b+\lambda_n^0)^{n+1} - (\Theta-b)^{n+1}, & n \text{ odd} \end{cases} \tag{71}$$

$$U = \begin{cases} (\Theta-b+\lambda_n^b)^{n+1} - (\Theta-a-\lambda_n^a)^{n+1} + (\Theta-a)^{n+1} - (\Theta-b)^{n+1}, & n \text{ even} \\ (\Theta-a)^{n+1} - (\Theta-a-\lambda_n^0)^{n+1}, & n \text{ odd} \end{cases} \tag{72}$$

$$\lambda_n^0 = \frac{1}{M-m}\left[f^{(n-1)}(b) - f^{(n-1)}(a) - m(b-a)\right], \tag{73}$$

$$\lambda_n^a = \frac{1}{M-m}\left[f^{(n-1)}(\Theta) - f^{(n-1)}(a) - m(\Theta-a)\right], \tag{74}$$

$$\lambda_n^b = \frac{1}{M-m}\left[f^{(n-1)}(b) - f^{(n-1)}(\Theta) - m(b-\Theta)\right]. \tag{75}$$

Taking $n = 1$ and $\Theta = (a+b)/2$ in (68), produces (67).

In [67], H.Gauchman proved two inequalities involving Taylor's remainder. He denotes by $R_{n,f}(c,x)$ the nth Taylor's remainder of function $f(x)$ with center c:

$$R_{n,f}(c,x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k.$$

Theorem 8.24 *Let $f : I \to \mathbb{R}$ and $w : I \to \mathbb{R}$ be two functions, $a,b \in I^\circ$, $a < b$ and let $f \in C^{n+1}([a,b])$ and $w \in C([a,b])$. Assume that $m \le f^{(n+1)}(x) \le M$, $m \ne M$ and $w(x) \ge 0$ for each $x \in [a,b]$. Then*

$$\begin{aligned}
(i)\quad & \frac{1}{(n+1)!}\int_{b-\lambda_n^0}^{b}(x-b+\lambda_n^0)^{n+1}w(x)dx \qquad (76)\\
&\le \frac{1}{M-m}\int_a^b \left[R_{n,f}(a,x) - m\frac{(x-a)^{n+1}}{(n+1)!}\right]w(x)dx\\
&\le \frac{1}{(n+1)!}\int_a^b[(x-a)^{n+1} - (x-a-\lambda_n^0)^{n+1}]w(x)dx\\
&+\frac{(-1)^{n+1}}{(n+1)!}\int_a^{a+\lambda_n^0}(a+\lambda_n^0-x)^{n+1}w(x)dx;\\
(ii)\quad & \frac{1}{(n+1)!}\int_a^{a+\lambda_n^0}(a+\lambda_n^0-x)^{n+1}w(x)dx \qquad (77)\\
&\le \frac{(-1)^{n+1}}{M-m}\int_a^b \left[R_{n,f}(b,x) - m\frac{(x-b)^{n+1}}{(n+1)!}\right]w(x)dx\\
&\le \frac{1}{(n+1)!}\int_a^b[(b-x)^{n+1} - (b-\lambda_n^0-x)^{n+1}]w(x)dx\\
&+\frac{(-1)^{n+1}}{(n+1)!}\int_{b-\lambda_n^0}^{b}(x-b+\lambda_n^0)^{n+1}w(x)dx;
\end{aligned}$$

where λ_n^0 is defined by (73).

Addition of (76) and (77) upon taking $n = 0$ and $w(x) = 1$ followed by division by 2, produces (67) again. Of course, as a special case we get Iyengar's inequality once more.

Now, we give a generalization of both Theorem 8.23 and Theorem 8.24 in a sense that an inequality involving both the weight $w(x)$ and the parameter Θ is given. This was published in [61].

Before we proceed, it should be mentioned that using the same technique similar inequalities were proved in a number of papers. In [2], R.P.Agarwal, V.Čuljak and J.Pečarić derived inequality (68) for an odd n. For an even n, using a somewhat different technique, they obtained a result which involves only the midpoint. In [44], only the case $n = 2$ was considered. In [19], an even more special case was considered. The results obtained there follow from (68) by taking $\Theta = (a+b)/2$ again and assuming function f satisfies $f^{(k)}(a) = (-1)^{k+1}f^{(k)}(b)$, for $1 < k < n$. Results obtained in [13] by P.Cerone and S.S.Dragomir are special cases of Theorem 8.24 produced after taking $n = 0$.

For the proof of our main result we use the Hayashi modification of the well-known Steffensen's inequality, so we state it first (cf. [93]).

Theorem 8.25 *Let $F : [a,b] \to \mathbf{R}$ be a nonincreasing function and $G : [a,b] \to \mathbf{R}$ an integrable function such that $0 \le G(x) \le A$ for each $x \in [a,b]$. Then*

$$A\int_{b-\lambda}^{b} F(x)dx \le \int_{a}^{b} F(x)G(x)dx \le A\int_{a}^{a+\lambda} F(x)dx, \tag{78}$$

where $\lambda = \frac{1}{A}\int_a^b G(x)dx$.

We introduce:

$$h_k(s,t) = \frac{1}{k!}\int_s^t (x-s)^k w(x)dx \tag{79}$$

for $s,t \in [a,b]$ and $k \in \mathbb{N}$.

Now we state our main result:

Theorem 8.26 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$. Assume that $m \le f^{(n)}(x) \le M$ for each $x \in [a,b]$. Let $w : I \to \mathbb{R}$ be integrable and such that $w(x) \ge 0$ for each $x \in [a,b]$. Let $\Theta \in [a,b]$. Then, when n is odd we have*

$$\begin{aligned}
&(M-m)h_n(b-\lambda_n^0,\Theta) - Mh_n(b,\Theta) + mh_n(a,\Theta)\\
&\le \int_a^b f(x)w(x)dx + \sum_{k=0}^{n-1}\left[f^{(k)}(b)h_k(b,\Theta) - f^{(k)}(a)h_k(a,\Theta)\right]\\
&\le Mh_n(a,\Theta) - mh_n(b,\Theta) - (M-m)h_n(a+\lambda_n^0,\Theta)
\end{aligned} \tag{80}$$

and when n is even we have

$$\begin{aligned}
&(M-m)[h_n(\Theta-\lambda_n^a,\Theta) - h_n(\Theta+\lambda_n^b,\Theta)] + m[h_n(a,\Theta) - h_n(b,\Theta)]\\
&\le \int_a^b f(x)w(x)dx + \sum_{k=0}^{n-1}\left[f^{(k)}(b)h_k(b,\Theta) - f^{(k)}(a)h_k(a,\Theta)\right]\\
&\le M[h_n(a,\Theta) - h_n(b,\Theta)] + (M-m)[h_n(b-\lambda_n^b,\Theta) - h_n(a+\lambda_n^a,\Theta)],
\end{aligned} \tag{81}$$

where λ_n^0, λ_n^a and λ_n^b are defined by (73), (74) and (75), respectively.

Proof. For $\Theta \in [a,b]$, set

$$\begin{aligned}
G_k(x) &= f^{(k)}(x) - m, && k = 0,1,\ldots n\\
F_k(x) &= \frac{1}{k!}\int_x^\Theta (t-x)^k w(t)dt && k = 0,1,\ldots n-1
\end{aligned}$$

for each $x \in [a,b]$. Now we have: $0 \le G_n(x) \le M-m$, for each $x \in [a,b]$, so $G_n(x)$ satisfies the conditions of Theorem 8.25. It is easy to prove that

$$F_k'(x) = -F_{k-1}(x)$$

and from there we conclude that for $x \le \Theta$, function $F_{n-1}(x)$ is nonincreasing. For $x \ge \Theta$ and odd n, $F_{n-1}(x)$ is again nonincreasing. However, for $x \ge \Theta$ and even n, $F_{n-1}(x)$ is nondecreasing. Therefore, inequality (78) is in that case reversed.

Let us assume first that n is odd. From (78) we get

$$(M-m)\int_{b-\lambda_n^0}^{b} F_{n-1}(x)dx \le \int_a^b F_{n-1}(x)G_n(x)dx \le (M-m)\int_a^{a+\lambda_n^0} F_{n-1}(x)dx.$$

where

$$\lambda_n^0 = \frac{1}{M-m}\int_a^b (f^{(n)}(x)-m)dx$$

as defined in (73). Using integration by parts and the fact that $F'_{n-1}(x) = -F_{n-2}(x)$, we easily obtain

$$\begin{aligned} I_n &= \int_a^b F_{n-1}(x)G_n(x)dx \qquad (82)\\ &= \int_a^b f(x)w(x)dx + \sum_{k=0}^{n-1}[f^{(k)}(b)h_k(b,\Theta) - f^{(k)}(a)h_k(a,\Theta)]\\ &\quad -mh_n(a,\Theta)+mh_n(b,\Theta). \end{aligned}$$

The upper bound is

$$U_o = \frac{M-m}{(n-1)!}\int_a^{a+\lambda_n^0}\left[\int_x^{\Theta}(t-x)^{n-1}w(t)dt\right]dx.$$

Assume first that $\Theta \le a+\lambda_n^0$. Changing the order of integration, we obtain

$$U_o = (M-m)[h_n(a,\Theta) - h_n(a+\lambda_n^0,\Theta)]. \qquad (83)$$

Assuming $\Theta \ge a+\lambda_n^0$, we get the same expression for the upper bound again.

Analogously, after changing the order of integration in the case when $\Theta \ge b-\lambda_n^0$, the lower bound equals

$$\begin{aligned} L_o &= \frac{M-m}{(n-1)!}\int_{b-\lambda_n^0}^{b}\left[\int_x^{\Theta}(t-x)^{n-1}w(t)dt\right]dx\\ &= (M-m)[h_n(b-\lambda_n^0,\Theta) - h_n(b,\Theta)]. \qquad (84) \end{aligned}$$

For $\Theta \le b-\lambda_n^0$, we get the same expression and thus, once again, obtain the same expression in both cases. Inequality (80) is produced by combining (82), (83) and (84), so the statement is proved for the case when n is odd.

Assume now n is even. $F_{n-1}(x)$ is nonincreasing on $[a,\Theta]$ so inequality (78) gives us:

$$L_e^a \le \int_a^{\Theta} F_{n-1}(x)G_n(x)dx \le U_e^a, \qquad (85)$$

It is easy to check that $a+\lambda_n^a \le \Theta$. We calculate both lower and upper bound by changing the order of integration:

$$U_e^a = (M-m)\int_a^{a+\lambda_n^a} F_{n-1}(x)dx = (M-m)[h_n(a,\Theta) - h_n(a+\lambda_n^a,\Theta)], \qquad (86)$$

$$L_e^a = (M-m)\int_{\Theta-\lambda_n^a}^{\Theta} F_{n-1}(x)dx = (M-m)h_n(\Theta-\lambda_n^a,\Theta), \qquad (87)$$

where

$$\lambda_n^a = \frac{1}{M-m}\int_a^{\Theta}(f^{(n)}(x)-m)dx$$

as defined in (74).

On $[\Theta, b]$, $F_{n-1}(x)$ is nondecreasing so inequality (78) is reversed. We have:

$$L_e^b \leq \int_{\Theta}^{b} F_{n-1}(x)G_n(x)dx \leq U_e^b. \tag{88}$$

This time $b-\lambda_n^b \geq \Theta$, so it follows

$$U_e^b = (M-m)\int_{b-\lambda_n^b}^{b} F_{n-1}(x)dx = (M-m)[h_n(b-\lambda_n^b,\Theta)-h_n(b,\Theta)], \tag{89}$$

$$L_e^b = (M-m)\int_{\Theta}^{\Theta+\lambda_n^b} F_{n-1}(x)dx = -(M-m)h_n(\Theta+\lambda_n^b,\Theta), \tag{90}$$

where

$$\lambda_n^b = \frac{1}{M-m}\int_{\Theta}^{b}(f^{(n)}(x)-m)dx$$

as defined in (75).

Addition of (85) and (88) gives:

$$L_e \leq I_n \leq U_e$$

where

$$U_e = U_e^a + U_e^b \qquad \text{and} \qquad L_e = L_e^a + L_e^b,$$

and thus inequality (81) is produced. The proof of this theorem is now complete. □

Remark 8.22 Taking $w(x)=1$ in Theorem 8.26 recaptures Theorem 8.23. Taking $\Theta=b$ produces inequality (76) and $\Theta=a$ produces inequality (77). Of course, for $w(x)=1$, $n=1$ and $\Theta=(a+b)/2$, we get inequality (67) again.

Next, we prove an alternative inequality for an even n and thus generalize results from [2]. Taking $\Theta=(a+b)/2$ and $w(x)=1$ will produce results from there.

Theorem 8.27 *Assume assumptions of Theorem 8.26 are valid. Then, for $\Theta\in[a,b]$ and even n, we have*

$$\begin{aligned}
&m(h_n(a,\Theta)-h_n(b,\Theta))+(M-m)|h_n(b-\lambda_n,\Theta)| \\
&\leq \int_a^b f(x)w(x)dx+\sum_{k=0}^{n-1}\left[f^{(k)}(b)h_k(b,\Theta)-f^{(k)}(a)h_k(a,\Theta)\right] \\
&\leq M(h_n(a,\Theta)-h_n(b,\Theta))-(M-m)|h_n(a+\lambda_n,\Theta)|
\end{aligned} \tag{91}$$

where $\lambda_n=\lambda_n^a-\lambda_n^b+b-\Theta,\ 0\leq\lambda_n\leq b-a.$

Proof. We use Hayashi's modification of Steffensen's inequality again. Set

$$F_{n-1}(x) = \begin{cases} \frac{1}{(n-1)!}\int_x^{\Theta}(t-x)^{n-1}w(t)dt, & a \le x \le \Theta, \\ \frac{1}{(n-1)!}\int_{\Theta}^{x}(t-x)^{n-1}w(t)dt, & \Theta \le x \le b. \end{cases} \tag{92}$$

From the proof of Theorem 8.26 it is clear that F_{n-1} is decreasing on $[a,b]$. Taking

$$G_n(x) = \begin{cases} f^{(n)}(x) - m, & a \le x \le \Theta, \\ M - f^{(n)}(x), & \Theta \le x \le b. \end{cases} \tag{93}$$

produces our statement. □

Remark 8.23 Estimates for an even n from Theorem 8.26 are better than the ones from Theorem 8.27. To prove this, we have to check that

$$|h_n(a+\lambda_n,\Theta)| \le h_n(a+\lambda_n^a,\Theta) - h_n(b-\lambda_n^b,\Theta), \tag{94}$$

$$|h_n(b-\lambda_n,\Theta)| \le h_n(\Theta-\lambda_n^a,\Theta) - h_n(\Theta+\lambda_n^b,\Theta). \tag{95}$$

After introducing notation

$$c_1 = a+\lambda_n^a, \qquad c_2 = b-\lambda_n^b, \qquad d_1 = \Theta-\lambda_n^a, \qquad d_2 = \Theta+\lambda_n^b,$$

(94) and (95) become

$$|h_n(c_1+c_2-\Theta,\Theta)| \le h_n(c_1,\Theta) - h_n(c_2,\Theta), \tag{96}$$

$$|h_n(d_1+d_2-\Theta,\Theta)| \le h_n(d_1,\Theta) - h_n(d_2,\Theta). \tag{97}$$

We already know that $c_1 \le \Theta$ and $c_2 \ge \Theta$ and it is clear that $d_1 \le \Theta$ and $d_2 \ge \Theta$, so we have $c_1 \le c_1+c_2-\Theta \le c_2$ and $d_1 \le d_1+d_2-\Theta \le d_2$. For an even n, function $h_n(x,\Theta)$ is decreasing. Also, $h_n(\Theta,\Theta) = 0$. Let us consider (96). First assume $c_1+c_2-\Theta \le \Theta$. Then $h_n(c_1+c_2-\Theta,\Theta) \ge 0$ and

$$h_n(c_1+c_2-\Theta,\Theta) \le h_n(c_1,\Theta) \le h_n(c_1,\Theta) - h_n(c_2,\Theta)$$

since $h_n(c_2,\Theta) \le 0$. Next, suppose $c_1+c_2-\Theta \ge \Theta$. Then $h_n(c_1+c_2-\Theta,\Theta) \le 0$ and

$$h_n(c_1+c_2-\Theta,\Theta) \ge h_n(c_2,\Theta) \ge h_n(c_2,\Theta) - h_n(c_1,\Theta)$$

since $h_n(c_1,\Theta) \ge 0$. Proof of (97) is analogous.

Finally, we give a comparison between Theorem 8.26 and Theorem 8.21. The connection is obvious - under the same assumptions, the same expression is estimated. The claim is that the estimation given in Theorem 8.26 is better than the one in Theorem 8.21.

First, consider the case when n is odd. The upper bound in (80) is

$$U_S = Mh_n(a,\Theta) - mh_n(b,\Theta) - (M-m)h_n(a+\lambda_n^0,\Theta)$$

and in (59)

$$U_T = Mh_n(a,\Theta) - mh_n(b,\Theta),$$

so obviously $U_S \leq U_T$. Similarly, the lower bound in (80) is

$$L_S = (M-m)h_n(b-\lambda_n^0,\Theta) - Mh_n(b,\Theta) + mh_n(a,\Theta)$$

and in (59)

$$L_T = mh_n(a,\Theta) - Mh_n(b,\Theta),$$

so $L_S \geq L_T$.

Next, consider the case when n is even. The upper bound in (81) is

$$U_S = M[h_n(a,\Theta) - h_n(b,\Theta)] + (M-m)[h_n(b-\lambda_n^b,\Theta) - h_n(a+\lambda_n^a,\Theta)]$$

and in (60)

$$U_T = M[h_n(a,\Theta) - h_n(b,\Theta)].$$

Again, $U_S \leq U_T$. Finally, the lower bound in (81) is

$$L_S = (M-m)[h_n(\Theta-\lambda_n^a,\Theta) - h_n(\Theta+\lambda_n^b,\Theta)] + m[h_n(a,\Theta) - h_n(b,\Theta)]$$

and in (60)

$$L_T = m[h_n(a,\Theta) - h_n(b,\Theta)],$$

so $L_S \geq L_T$.

This completes the proof of the claim.

8.5.5 Comparison between different generalizations of Iyengar's inequality

We give yet another comparison between generalizations of Iyengar's inequality obtained through different methods, for a function f such that $f \in C^2[a,b]$ and $|f''(x)| \leq M$. The results given here were published in [62].

For $w(x) = 1$, $m = -M$, $n = 2$ and $\Theta = (a+b)/2$, (81) from Theorem 8.26 yields:

$$\begin{aligned}
&-\frac{M(b-a)^3}{24} + \frac{M}{3}\left(\lambda_a^3 + \lambda_b^3\right) \\
&\leq \int_a^b f(x)dx - \frac{b-a}{2}(f(a)+f(b)) + \frac{(b-a)^2}{8}(f'(b)-f'(a)) \qquad (98)\\
&\leq \frac{M(b-a)^3}{24} - \frac{M}{3}\left[\left(\frac{b-a}{2}-\lambda_a\right)^3 + \left(\frac{b-a}{2}-\lambda_b\right)^3\right],
\end{aligned}$$

where

$$\lambda_a = \frac{1}{2M}\left(f'\left(\frac{a+b}{2}\right) - f'(a)\right) + \frac{b-a}{4}, \qquad (99)$$

$$\lambda_b = \frac{1}{2M}\left(f'(b) - f'\left(\frac{a+b}{2}\right)\right) + \frac{b-a}{4}. \qquad (100)$$

For the same parameters, (91) from Theorem 8.27 yields:

$$\left|\int_a^b f(x)dx - \frac{b-a}{2}(f(a)+f(b)) + \frac{(b-a)^2}{8}(f'(b)-f'(a))\right| \\ \leq \frac{M}{24}(b-a)^3 - \frac{|\Delta_1|^3}{24M^2}, \tag{101}$$

where $\Delta_1 = f'(a) - 2f'(\frac{a+b}{2}) + f'(b)$.

Another result of a similar type was given by Cheng in [17] (cf. Theorem 8.18 and Remark 8.19):

$$\left|\int_a^b f(x)dx - \frac{b-a}{2}(f(a)+f(b)) + \frac{(b-a)^2}{8}(f'(b)-f'(a))\right| \\ \leq \frac{M}{24}(b-a)^3 - \frac{(b-a)^2\Delta_2^2}{8[M(b-a)+f'(a)-f'(b)]} \tag{102}$$

where $\Delta_2 = f'(a) - 2\frac{f(b)-f(a)}{b-a} + f'(b)$.

In the same paper, Cheng showed that, for some classes of functions, inequality (102) gives better estimations than inequality (101). Now, we prove that (98) is always better than (101) and better than (102) for the same class of functions for which (102) is better than (101).

Define:

$$H(x) = \frac{M}{3}\left(\frac{a+b}{2} - x\right)^3, \qquad \text{for } \; x \in [a,b].$$

Now we can write (101) and (98) in a following form:

$$\left|\int_a^b f(x)dx - \frac{b-a}{2}(f(a)+f(b)) + \frac{(b-a)^2}{8}(f'(b)-f'(a))\right| \\ \leq \frac{M(b-a)^3}{24} - |H(a+\lambda)|, \tag{103}$$

and

$$-\frac{M(b-a)^3}{24} + H(\Theta-\lambda_a) - H(\Theta+\lambda_b) \\ \leq \int_a^b f(x)dx - \frac{b-a}{2}(f(a)+f(b)) + \frac{(b-a)^2}{8}(f'(b)-f'(a)) \\ \leq \frac{M(b-a)^3}{24} - H(a+\lambda_a) + H(b-\lambda_b), \tag{104}$$

where $\lambda = \lambda_a - \lambda_b + \frac{b-a}{2}$.

$H(x)$ is decreasing, $H(\Theta) = 0$ and $a+\lambda_a \leq \Theta \leq b-\lambda_b$, $\;0 \leq \lambda_a, \lambda_b \leq \frac{b-a}{2}$. Also, $a+\lambda_a \leq a+\lambda \leq b-\lambda_b$. Assume first $\lambda_a \leq \lambda_b$. Then $H(a+\lambda) \geq 0$ and

$$H(a+\lambda) \leq H(a+\lambda_a) \leq H(a+\lambda_a) - H(b-\lambda_b),$$

since $H(b-\lambda_b) \leq 0$. Suppose $\lambda_a \geq \lambda_b$. Then $H(a+\lambda) \leq 0$ and

$$H(a+\lambda) \geq H(b-\lambda_b) \geq H(b-\lambda_b) - H(a+\lambda_a),$$

since $H(a+\lambda_a) \geq 0$. The proof that the lower bound in (104) is also better is analogous: just note $|H(b-\lambda)| = |H(a+\lambda)|$.

Finally, we give some classes of functions for which (98) gives better estimates than (102). We claim that

$$\frac{(b-a)^2\Delta_1^2}{8[M(b-a)+f'(a)-f'(b)]} \leq \frac{M}{3}\left[\left(\frac{b-a}{2}-\lambda_a\right)^3 + \left(\frac{b-a}{2}-\lambda_b\right)^3\right] \tag{105}$$

$$\frac{(b-a)^2\Delta_1^2}{8[M(b-a)+f'(a)-f'(b)]} \leq \frac{M}{3}\left(\lambda_a^3+\lambda_b^3\right), \tag{106}$$

for $f(x) = x^n$, $n \geq 5$ on $[0,1]$. Inequalities (105) and (106) in this case reduce to

$$\frac{n-2}{8n} \leq \frac{n(n-1)}{3}\left[\left(\frac{1}{4}-\frac{1-2^{1-n}}{2(n-1)}\right)^3 + \left(\frac{1}{4}-\frac{2^{1-n}}{2(n-1)}\right)^3\right], \tag{107}$$

$$\frac{n-2}{8n} \leq \frac{n(n-1)}{3}\left[\left(\frac{1}{4}+\frac{1-2^{1-n}}{2(n-1)}\right)^3 + \left(\frac{1}{4}+\frac{2^{1-n}}{2(n-1)}\right)^3\right]. \tag{108}$$

Routine calculation shows that (107) is valid for $n \geq 5$ (for $n = 2$ we get equality) and (108) is valid for $n \geq 2$. Thus, we have shown that (98) is better than (102) for the same class of functions for which (102) is better than (101).

Further, with no loss in generality, we can consider functions on $[0,1]$ such that $f(0) = f'(0) = 0$ and $|f''(x)| \leq 1$. Inequalities (105) and (106) turn to:

$$\frac{[2f(1)-f'(1)]^2}{1-f'(1)} \leq \frac{1}{24}\left[\left(1-2f'\left(\frac{1}{2}\right)\right)^3 + \left(1-2f'(1)+2f'\left(\frac{1}{2}\right)\right)^3\right], \tag{109}$$

$$\frac{[2f(1)-f'(1)]^2}{1-f'(1)} \leq \frac{1}{24}\left[\left(1+2f'\left(\frac{1}{2}\right)\right)^3 + \left(1+2f'(1)-2f'\left(\frac{1}{2}\right)\right)^3\right]. \tag{110}$$

When $f(1) = f'(1) = 0$, or, more generally, when $2f(1) = f'(1)$ and $f'(1) \neq 1$, (98) gives better estimates than (102), since the right-hand sides of (109) and (110) are obviously positive. If we take $f'(1/2) = t$, $0 \leq t \leq 1/2$, when $f'(1) = 0$, (109) and (110) reduce to

$$4f^2(1) \leq t^2 + 1/12. \tag{111}$$

Maximizing the left-hand side of (111) using continuous piecewise linear function with $|f''(x)| = 1$ (where f'' exists), (111) will follow if

$$(t^2 - t - 1/4)^2 \leq t^2 + 1/12. \tag{112}$$

Using Wolfram's Mathematica 5.0, we see that the approximate solutions on $[0,1/2]$ of the equation in (112) are $t_1 = 0.044$ and $t_2 = 0.395$, so for $0 \le t \le t_1$ or $t_2 \le t \le 1/2$, (98) is better than (102). For $t_1 < t < t_2$, (102) may give better estimates.

Remark 8.24 Related results on Iyengar type inequalities can be found in: [16, 43, 69, 81, 82, 88, 95, 109, 110, 112, 113, 119, 120].

Appendix

Bernoulli polynomials and Bernoulli numbers

The Bernoulli polynomial $B_k(x)$ **of the** k**th degree** is defined as the coefficient of $t^k/k!$ in the expansion

$$\frac{t\,e^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} \frac{t^k}{k!} B_k(x). \tag{A-1}$$

The first ten Bernoulli polynomials are

$$\begin{aligned}
&B_0(x) = 1\\
&B_1(x) = x - 1/2\\
&B_2(x) = x^2 - x + 1/6\\
&B_3(x) = x^3 - 3/2\,x^2 + 1/2\,x\\
&B_4(x) = x^4 - 2x^3 + x^2 - 1/30\\
&B_5(x) = x^5 - 5/2\,x^4 + 5/3\,x^3 - 1/6\,x\\
&B_6(x) = x^6 - 3x^5 + 5/2\,x^4 - 1/2\,x^2 + 1/42\\
&B_7(x) = x^7 - 7/2\,x^6 + 7/2\,x^5 - 7/6\,x^3 + 1/6\,x\\
&B_8(x) = x^8 - 4x^7 + 14/3\,x^6 - 7/3\,x^4 + 2/3\,x^2 - 1/30\\
&B_9(x) = x^9 - 9/2\,x^8 + 6x^7 - 21/5\,x^5 + 2x^3 - 3/10\,x\\
&B_{10}(x) = x^{10} - 5x^9 + 15/2\,x^8 - 7x^6 + 5x^4 - 3/2\,x^2 + 5/66.
\end{aligned}$$

Differentiating (A-1) with respect to x gives

$$t\,\frac{t\,e^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} \frac{t^k}{k!} B_k'(x).$$

From (A-1) it follows

$$t\,\frac{t\,e^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} \frac{t^{k+1}}{k!} B_k(x).$$

Therefore

$$\frac{B_k'(x)}{k!} = \frac{B_{k-1}(x)}{(k-1)!}$$

and thus

$$B_k'(x) = kB_{k-1}(x). \tag{A-2}$$

Bernoulli polynomials $B_k(t)$ are uniquely determined by (A-2) and

$$B_0(t) = 1, \qquad B_k(t+1) - B_k(t) = kt^{k-1},\ k \geq 0.$$

Let $m \in \mathbb{N}$. Then

$$e^{mxt}\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{t^k}{k!} B_k(mx)$$

and furthermore

$$\begin{aligned} e^{mxt}\frac{t}{e^t-1} &= \frac{e^{mxt}}{m}\left[\frac{mt(1+e^t+\ldots+e^{(m-1)t})}{e^{mt}-1}\right] \\ &= \frac{1}{m}\sum_{j=0}^{m-1}\frac{mt\ e^{\left(x+\frac{j}{m}\right)mt}}{e^{mt}-1} = \frac{1}{m}\sum_{j=0}^{m-1}\sum_{k=0}^{\infty}\frac{(mt)^k}{k!}B_k\left(x+\frac{j}{m}\right). \end{aligned}$$

Now, similarly as before

$$B_k(mx) = m^{k-1}\sum_{j=0}^{m-1} B_k\left(x+\frac{j}{m}\right). \tag{A-3}$$

The (A-3) is called **the Multiplication Theorem for Bernoulli polynomials**.

Some properties of Bernoulli polynomials

- For the kth Bernoulli polynomial we have

  $$B_k(1-x) = (-1)^k B_k(x), \quad x \in \mathbb{R}, \quad k \geq 1 \tag{A-4}$$

  so the graph of $B_{2k}(x)$ is symmetric with respect to the line $x = 1/2$, while the graph of $B_{2k-1}(x)$ is centrally symmetric with respect to the point $x = 1/2$.

- The k**th Bernoulli number** B_k is defined by the relation $B_k = B_k(0)$. From (A-4) it follows that

  $$B_{2k}(1) = B_{2k}$$

  and

  $$B_{2k-1}(1) = -B_{2k-1},$$

  and therefore for $k \geq 2$, we have

  $$B_k(1) = B_k(0) = B_k.$$

Note that

$$B_{2k-1} = 0, \quad k \geq 2$$

and

$$B_1(1) = -B_1(0) = 1/2.$$

- For $k \in \mathbb{N}$

$$B_k\left(\frac{1}{2}\right) = -(1-2^{1-k})B_k.$$

- For $0 < x < 1/2$, we have

$$(-1)^k[B_{2k}(x) - B_{2k}] > 0$$

and

$$(-1)^k B_{2k-1}(x) > 0$$

Periodic functions related to Bernoulli polynomials

$B_k^*(x)$ are periodic functions of period 1 defined by the condition

$$B_k^*(x+1) = B_k^*(x), \quad x \in \mathbb{R},$$

and related to Bernoulli polynomials as

$$B_k^*(x) = B_k(x), \quad 0 \leq x < 1.$$

$B_0^*(x)$ is a constant equal to 1, while $B_1^*(x)$ is a discontinuous function with a jump of -1 at each integer. For $k \geq 2$, $B_k^*(t)$ is a continuous function.

Direct calculations give the following results:

$$B_{2k}^*(x) = \frac{(-1)^{k-1}(2k)!}{2^{2k-1}\,\pi^{2k}} \sum_{m=1}^{\infty} \frac{\cos 2\pi mx}{m^{2k}} \tag{A-5}$$

$$B_{2k+1}^*(x) = \frac{(-1)^{k-1}(2k+1)!}{2^{2k}\,\pi^{2k+1}} \sum_{m=1}^{\infty} \frac{\sin 2\pi mx}{m^{2k+1}}. \tag{A-6}$$

For further details on Bernoulli polynomials and Bernoulli numbers see [1] or [79].

Bibliography

[1] M. Abramowitz and I. A. Stegun (Eds), *Handbook of mathematical functions with formulae,graphs and mathematical tables*, National Bureau of Standards, Applied Math. Series 55, 4th printing, Washington, 1965.

[2] R. P. Agarwal, V. Čuljak, J. Pečarić, *Some integral inequalities involving bounded higher order derivatives*, Math. Comput. Modelling. **28**(3) (1998), 51–57.

[3] R. P. Agarwal, S. S. Dragomir, *An application of Hayashi's inequality for differentiable functions*, Comput. Math. Appl. **32**(6) (1996) 95–99,

[4] A. Aglić Aljinović, M. Matić, J. Pečarić, *Improvements of some Ostrowski type inequalities*, J. Comput. Anal. Appl. **7**(3) (2005), 289–304.

[5] A. Aglić Aljinović, J. Pečarić, *The weighted Euler identity*, Math. Inequal. Appl. **8**(2) (2005), 207–221.

[6] G. A. Anastassiou, *Univariate Ostrowski inequalities, Revisited*, Monatsh. Math. **135** (2002), 175–189.

[7] E. F. Beckenbach, R. Bellman, *Inequalities*, Springer - Verlag, Berlin and New York, 1961, 1965.

[8] I. S. Berezin and N. P. Zhidkov, *Computing methods.* Vol. I, Pergamon Press, Oxford, 1965.

[9] M. Bessenyei, *Hermite-Hadamard-type inequalities for generalized convex functions*, Ph.D. dissertation, available on RGMIA

[10] M. Bessenyei, Z. Páles, *Higher-order generalizations of Hadamard's inequality*, Publ. Math. Debrecen **61**(3-4) (2002), 623-643.

[11] P. S. Bullen, *Error estimates for some elementary quadrature rules*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz. **602-633** (1978), 97–103.

[12] P. Cerone, *Generalised trapezoidal rules with error involving bounds of the n-th derivative*, Math. Inequal. Appl. **5**(3) (2002), 451–462.

[13] P. Cerone, S. S. Dragomir, *On a weighted generalizations of Iyengar type inequalities involving bounded first derivative*, Math. Inequal. Appl. **3**(1) (2000), 35–44.

[14] P. Cerone, S. S. Dragomir, *Trapezoidal type rules from an inequalities point of view*, Handbook of Analytic-Computational Methods in Applied Mathematics, Chapman and Halle / CRC, Boca Raton, 2000.

[15] P. Cerone, S. S. Dragomir, *Midpoint type rules from an inequalities point of view*, Handbook of Analytic-Computational Methods in Applied Mathematics, Chapman and Halle / CRC, Boca Raton, 2000.

[16] P. Cerone, S. S. Dragomir, *Lobatto type quadrature rules for functions with bounded derivative*, Math. Inequal. Appl. **3**(2) (2000), 197–209.

[17] X. L. Cheng, *The Iyengar type inequality*, Appl. Math. Lett. **14** (2001), 975–978.

[18] X. L. Cheng, J. Sun, *A note on the perturbed trapezoid inequality*, JIPAM. J. Inequal. Pure Appl. Math. **3**(2), Article 29, 2002.

[19] V. Čuljak, N. Elezović, *A note on Steffensen's and Iyengar's inequality*, Glas. Mat. **33**(53) (1998), 167–171.

[20] V. Čuljak, C. E. M. Pearce, J. Pečarić, The unified treatment of some inequalities of Ostrowski and Simpson types, Soochow J. Math. **26**(4) (2000), 377–390.

[21] V. Čuljak, J. Pečarić, L. E. Persson, A note on Simpson type numerical integration, Soochow J. Math. **29**(2) (2003), 191–200.

[22] P. J. Davis, P. Rabinowitz, *Methods of numerical integration*, Academic Press, New York, San Francisco, London, 1975.

[23] Lj. Dedić, M. Matić, J. Pečarić, *On Euler midpoint formulae*, ANZIAM J **46** (2005), 417–438.

[24] Lj. Dedić, M. Matić, J. Pečarić, *Euler-Maclaurin formulae*, Math. Inequal. Appl. **6**(2) (2003), 247–275.

[25] Lj. Dedić, M. Matić, J. Pečarić, *On Euler trapezoid formulae*, Appl. Math. Comput. **123** (2001), 37–62.

[26] Lj. Dedić, M. Matić, J. Pečarić, *On dual Euler-Simpson formulae*, Bull. Belg. Math. Soc. **8** (2001), 479–504.

[27] Lj. Dedić, M. Matić, J. Pečarić, *Some inequalities of Euler-Grüss type*, Comput. Math. Appl. **41** (2001), 843–856.

[28] Lj. Dedić, M. Matić, J. Pečarić, A.Vukelić, *Hadamard type inequalities via some Euler type identities - Euler bitrapezoid formulae*, Nonlinear Stud. **8**(3) (2001), 343–372.

[29] Lj. Dedić, M. Matić, J. Pečarić, *On Euler-Simpson formulae*, PanAmer. Math. J. **11**(2) (2001), 47–64.

[30] Lj. Dedić, M. Matić, J. Pečarić, *On generalizations of Ostrowski inequality via some Euler-type identities*, Math. Inequal. Appl., **3**(3) (2000), 337–353.

[31] Lj. Dedić, M. Matić, J. Pečarić, A. Vukelić, *On Euler-Simpson 3/8 formulae* (to appear in Nonlinear Studies)

[32] Lj. Dedić, C. E. M. Pearce and J. Pečarić, *Hadamard and Dragomir-Agarwal inequalities, higher-order convexity and the Euler formula*, J. Korean Math. Soc. **38** (2001), 1235–1243.

[33] Lj. Dedić, C. E. M. Pearce and J. Pečarić, *The Euler formulae and convex functions*, Math. Inequal. Appl. **3**(2) (2000), 211–221.

[34] S. S. Dragomir, *Some companions of Ostrowski's inequality for absolutely continuous functions and applications*, Bull. Korean Math. Soc. **42**(2) (2005), 213–230.

[35] S. S. Dragomir, *A companion of Ostrowski's inequality for functions of bounded variation and applications*, RGMIA 5 (2002), supplement.

[36] S. S. Dragomir and R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapesoidal formula*, Appl. Math. Lett. **11**(5) (1998), 91–95.

[37] S. S. Dragomir, R. P. Agarwal, P. Cerone, *On Simpson's inequality and applications*, J. Inequal. Appl. **5**(6) (2000), 533–579.

[38] S. S. Dragomir, N. S. Barnett, *An Ostrowski type inequality for mappings whose second derivatives are bounded and applications*, J. Indian Math. Soc.(N.S.) **66** (1-4) (1999), 237–245.

[39] S. S. Dragomir, P. Cerone, A. Sofo, *Some remarks on the trapezoid rule in numerical integration*, Indian J. Pure Appl. Math. **31**(5) (2000), 475–494.

[40] S. S. Dragomir, P. Cerone, A. Sofo, *Some remarks on the midpoint rule in numerical integration*, Studia Math. Babes-Bolyai Uni XLV (2000), 63–73.

[41] S. S. Dragomir, C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, monography, RGMIA

[42] S. S. Dragomir, T. M. Rassias, *Ostrowski Type Inequalities and Applications in Numerical Integration*, monography, RGMIA

[43] S. S. Dragomir, S. Wang, *Applications of Iyengar's type inequalities to the estimation of error bounds for the trapezoidal quadrature rule*, Tamkang J. Math. **29**(1) (1998), 55–58.

[44] N. Elezović, J. Pečarić, *Steffensen's inequality and estimates of error in trapezoidal rule*, Appl. Math. Lett. **11**(6) (1998), 63–69.

[45] A. M. Fink, *Bounds on the deviation of a function from its averages*, Czech. Math. J. **42** (1992), 289–310.

[46] G. B. Folland, *Real Analysis, Modern Techniques and Their Applications*, A Wiley-Interscience publication, 1984.

[47] K. J. Förster, K. Petras, *Error estimates in Gaussian quadrature for functions of bounded variation*, SIAM J. Numer. Anal. **28** (1991), 880–889.

[48] I. Franjić, *Hermite-Hadamard-type inequalities for Radau-type quadrature rules*, J. Math. Inequal. **3**(3) (2009), 395–407.

[49] I. Franjić, J. Pečarić, *Dragomir-Agarwal type inequalities for several families of quadratures*, JIPAM. J. Inequal. Pure Appl. Math. **10**(3) (2009), Article 65, 11 pp.

[50] I. Franjić, J. Pečarić, *Generalisation of corrected Simpson's formula*, ANZIAM J. **47** (2006), 367–385.

[51] I. Franjić, J. Pečarić, *On corrected Euler-Simpson's 3/8 formulae*, Nonlinear Stud. **13**(4) (2006), 309–319.

[52] I. Franjić, J. Pečarić, *On corrected dual Euler-Simpson formulae*, Soochow J. Math. **32**(4) (2006), 575–587.

[53] I. Franjić, J. Pečarić, *On corrected Bullen-Simpson's inequality*, Bull. Allahabad Math. Soc. **21** (2006), 105–123.

[54] I. Franjić, J. Pečarić, *On corrected Bullen-Simpson's 3/8 inequality*, Tamkang J. Math. **37**(2) (2006), 135–148.

[55] I. Franjić, J. Pečarić, *Corrected Euler-Maclaurin's formulae*, Rend. Circ. Mat. Palermo (2) **54**(2) (2005), 259–272.

[56] I. Franjić, J. Pečarić, I. Perić, *General closed 4-point quadrature formulae of Euler type*, Math. Inequal. Appl. **12**(3) (2009), 573–586.

[57] I. Franjić, J. Pečarić , I. Perić, *Weighted generalizations of Iyengar type inequalities*, Indian J.Pure Appl. Math. **38**(4) (2007), 291–304.

[58] I. Franjić, J. Pečarić, I. Perić, *Estimates for the Gauss four-point formula for functions with low degree of smoothness*, Appl. Math. Lett. **20**(1) (2007), 1–6.

[59] I. Franjić, J. Pečarić, I. Perić, *Quadrature formulae of Gauss type based on Euler identitites* Math. Comput. Modelling **45**(3-4) (2007), 355–370.

[60] I. Franjić, J. Pečarić, I. Perić, *On families of quadrature formulas based on Euler identities*, Appl. Math. Comput. **217**(9) (2011), 4516–4528.

[61] I. Franjić, J. Pečarić, I. Perić, *Iyengar type estimate of error in trapezoidal rule*, Int. J. Pure Appl. Math. **26**(3) (2006), 285–295.

[62] I. Franjić, J. Pečarić, I. Perić, *Note on an Iyengar type inequality*, Appl. Math. Lett. **19** (2006), 657–660.

[63] I. Franjić, J. Pečarić, I. Perić, *General Euler-Ostrowski formulae and applications to quadratures*, Appl. Math. Comput. **177** (2006), 92–98.

[64] I. Franjić, J. Pečarić, I. Perić, *General Euler-Boole's and dual Euler-Boole's formulae*, Math. Inequal. Appl. **8**(2) (2005), 287–303.

[65] I. Franjić, J. Pečarić, A. Vukelić, *General Euler-Simpson formulae*, Indian J. Math. **47**(2-3) (2005), 139–157.

[66] R. Frisch, *On approximation to a certain type of integrals*, Skandinavisk Aktuarietidskrift **12** (1929), 129–181.

[67] H. Gauchman, *Some integral inequalities involving Taylor's remainder.I*, JIPAM. J. Inequal. Pure Appl. Math. **3**(2) (2002), Article 26.

[68] A. Guessab, G. Schmeisser, *Sharp integral inequalities of the Hermite-Hadamard type*, J. Approx. Theory **115** (2002), 260–288.

[69] B. N. Guo, F. Qi, *Some estimates of an integral in terms of the L^p-norm of the $(n+1)$st derivative of its integrand*, Anal. Math. **29** (1) (2003), 1-6.

[70] P. C. Hammer, *The mid-point method of numerical integration*, Math. Mag. **31**, (1957-58), 193–195.

[71] G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, Cambridge, 1934.

[72] F. B. Hildebrand, *Introduction to Numerical Analysis*, McGraw-Hill Book Company Inc., New York, 1956.

[73] D. Y. Hwang, *Improvements of Euler-trapezodial type inequalities with higher-order convexity and applications*, JIPAM. J. Inequal. Pure Appl. Math. **5**(3) (2004), Article 66.

[74] K. S. K. Iyengar, *Note on an inequality*, Math. Student **6** (1938), 75–76.

[75] S. Karlin, W. J. Studden, Tchebyscheff systems with applications in analysis and statistics, Interscience, New York, 1966.

[76] M. Klaričić Bakula, J. Pečarić, A. Vukelić, *Integration of periodic function and applications on integration formulae*, Bull. Math. Soc. Sci. Math. Roumanie. **48**(96)(3) (2005), 261–275.

[77] S. Kovač, J. E. Pečarić, *Generalization of an integral formula of Guessab and Schmeisser*, Banach J. Math. Anal., **5**(1) (2011), 1–18.

[78] S. Kovač, J. E. Pečarić, *The general three-point quadrature formula*, manuscript

[79] V. I. Krylov, *Approximate Calculation of Integrals*, The Macmillan Company, New York, 1962.

[80] C. Lanczos, *Applied Analysis*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1956.

[81] Z. Liu, *Note on Iyengar's inequality*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. **16** (2005), 29-35.

[82] G. S. Mahajani, *A note on an inequality*, Math. Student **6** (1938), 125–128.

[83] M. Matić, *Improvement of some inequalities of Euler-Grüss type*, Comput. Math. Appl. **46** (2003), 1325–1336.

[84] M. Matić, C. E. M. Pearce, J. Pečarić, *Two-point formulae of Euler type*, ANZIAM J. **44** (2002), 221–245.

[85] M. Matić, J. Pečarić, N. Ujević, *Generalizations of weighted version of Ostrowski's inequality and some related results*, J. Inequal. Appl. **5** (2000), 639–666.

[86] M. Matić, J. Pečarić, A. Vukelić, *On generalization of Bullen-Simpson's inequality*, Rocky Mountain J. Math. **35** (2005), 1727–1754.

[87] M. Matić, J. Pečarić, A. Vukelić, *Generalization of Bullen-Simpson's 3/8 inequality*, Math. Comput. Modelling **41**(4-5) (2005), 463–483.

[88] Y. Miao, *A note on Iyengar type inequality*, East J. Approx. **15**(4) (2009), 451-457.

[89] G. V. Milovanović, *O nekim funkcionalnim nejednakostima*, Univ. Beograd. Publ. Elektrotehn. Fak., Ser. Mat. Fiz. **599** (1977), 1–59.

[90] G. V. Milovanović, J. Pečarić, *Some considerations on Iyengar's inequality and some related applications*, Univ.Beograd. Publ. Elektrotehn. Fak., Ser. Mat. Fiz. **544 - 576** (1976), 166–170.

[91] G. Mingzhe, G. Xuemei, *On the generalized Hardy-Hilbert inequality and its applications*, Math. Ineq. Appl. **7**(1) (2004), 19–26.

[92] D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, Berlin - Heildelberg - New York, 1970.

[93] D. S. Mitrinović, J. Pečarić, A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer, Dordrecht, 1993.

[94] D. S. Mitrinović, J. Pečarić, A. M. Fink, *Inequalities Involving Functions and Their Integrals and Derivatives*, Kluwer, Dordrecht, 1991.

[95] V. Nhat Huy, Q. A. Ngô, *On an Iyengar-type inequality involving quadratures in n knots*, Appl. Math. Comput. **217**(1) (2010), 289-294.

[96] A. Ostrowski, *Über die Absolutabweichung einer differentierbaren Funktion von ihrem Integralmittelwert*, Comment. Math. Helv. **10** (1938), 226–227.

[97] C. E. M. Pearce, J. Pečarić, *Inequalities for differentiable mappings and applications to special means and quadrature formulas*, Appl. Math. Lett. **13** (2000), 51–55.

[98] J. Pečarić, I. Perić, A. Vukelić, *Sharp integral inequalities based on general Euler two-point formulae*, ANZIAM J **46** (2005), 1–20.

[99] J. Pečarić, I. Perić, A. Vukelić, *On Euler-Boole formulae*, Math. Inequal. Appl. **7**(1) (2004), 27–46.

[100] J. Pečarić, F. Proschan, Y. L. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Mathematics in Science and Engineering Volume 187, Academic Press inc., 1992.

[101] J. Pečarić, B. Savić, *O generalizacijama nekih rezultata J.Karamate i nekim primjenama*, Zbor. Rad. AKoV (Beograd) (1984), 245–268.

[102] J. Pečarić, A. Vukelić, *General dual Euler-Simpson formulae*, J. Math. Inequal. **2**(4) (2008), 511–526.

[103] J. Pečarić, A. Vukelić, *Hadamard and Dragomir-Agarwal inequalities, the general Euler two point formulae and convex functions*, Rad HAZU 491. Matematičke znanosti 15 (2005), 139–151.

[104] J. Pečarić, A. Vukelić, *Estimations of the error for two-point formula via pre-Grüss inequality*, Gen. Math. **13**(2) (2005), 95–104.

[105] J. Pečarić, A. Vukelić, *Hadamard and Dragomir-Agarwal inequalities, the Euler formulae and convex functions*, Functional Equations, Inequalities and Applications / Rassias, Themistocles M. (ed.), Dordrecht, Kluwer Academic Publishers, 2003.

[106] J. Pečarić, A. Vukelić, *On generalizations of Dragomir-Agarwal inequality via some Euler-type identities*, Bulletin de la Société des Mathématiciens de R. Macédoine 26, (LII)(2002), 17–42.

[107] G. Pôlya, *On the mean value theorem corresponding to a given linear homogeneous differential equation*, Trans. Am. Math. Soc. **24** (1922) 312–324.

[108] G. Pôlya, G. Szegö, Aufgaben und Lehrsätze aus der Analysis II, Berlin, Verlag von Julius Springer, 1925.

[109] F. Qi, *Inequalities for an integral*, Math. Gaz. **80**(488) (1996), 376-377.

[110] F. Qi, *Further generalizations of inequalities for an integral*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. **8** (1997), 79–83.

[111] F. Qi, Y. J. Zhang, *Inequalities for a weighted integral*, Adv. Stud. Contemp. Math. (Kyungshang), **4**(2) (2002), 93–101.

[112] F. Qi, *Inequalities for a multiple integral*, Acta Math. Hungar. **84**(1-2) (1999), 19-26.

[113] F. Qi, *Inequalities for a weighted multiple integral*, J. Math. Anal. Appl. **253**(2) (2001), 381-388.

[114] F. Qi, P. Cerone, S. S. Dragomir, *Some new Iyengar type inequalities*, Rocky Mountain J. Math., **35**(3) (2005), 997–1015.

[115] A. W. Roberts, D. E. Varberg, *Convex Functions*, Academic Press, New York, 1973.

[116] N. Ujević, *New bounds for Simpson's inequality*, Tamkang J. Math. **33**(2) (2002), 129–138.

[117] N. Ujević, A. J. Roberts, *A corrected quadrature formula and applications*, ANZIAM J. **45**(E) (2004), ppE41-E56.

[118] A. Vukelić, *Note on Dragomir-Agarwal inequalities, the general Euler two-point formulae and convex functions*, JIPAM. J. Inequal. Pure Appl. Math. **6**(1) (2005), Article 19.

[119] X. H. Wang, C. C. Xie, S. J. Yang, *The Iyengar inequality, revisited*, BIT **47**(4) (2007), 839-852.

[120] X. H. Wang, S. J. Yang, *The Iyengar type inequalities with exact estimations and the Chebyshev central algorithms of integrals*, Acta Math. Sin. (Engl. Ser.) **21**(6) (2005), 1361-1376.

Zagreb, March 2013