MONOGRAPHS IN INEQUALITIES 5

General Integral Identities and Related Inequalities

Arising from Weighted Montgomery Identity

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ELEMENT

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Dedicated to the memory of Professor Ljuban Dedić
and Professor Nenad Ujević.

Preface

Our objective in writing this book was to expose recent research in the theory of weighted integral and discrete inequalities of Ostrowski type, with emphasis on its applications such as generalizations of classical numerical quadrature rules. We intended to demonstrate varieties and diversities of generalizations and applications of the Ostrowski inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \left\| f' \right\|_\infty$$

which holds for every $x \in [a,b]$ whenever $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on $\langle a,b \rangle$ with derivative $f' : \langle a,b \rangle \to \mathbb{R}$ bounded on $\langle a,b \rangle$.

Since Ostrowski inequality can easily be proved by means of the Montgomery identity

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t + \frac{1}{b-a}\left(\int_a^x (t-a) f'(t)\,\mathrm{d}t + \int_x^b (t-b) f'(t)\,\mathrm{d}t \right)$$

all general identities from which the related inequalities and their applications are deduced in this book are various kinds of different generalizations of weighted Montgomery identity obtained by J. Pečarić in 1980

$$\begin{aligned} & f(x) - \frac{1}{\int_a^b w(t)\,dt}\int_a^b f(t)\,w(t)\,\mathrm{d}t \\ & = \frac{1}{\int_a^b w(t)\,dt}\left(\int_a^x \left(\int_a^t w(x)\,\mathrm{d}x \right) f'(t)\,\mathrm{d}t + \int_x^b \left(\int_b^t w(x)\,\mathrm{d}x \right) f'(t)\,\mathrm{d}t \right) \end{aligned}$$

where $w : [a,b] \to [0,\infty\rangle$ is some integrable weight function.

The book is organized in four chapters.

Chapter 1 introduces general integral identities using the harmonic sequences of polynomials and $w-$harmonic sequences of functions. These identities are used as the main tool for deriving some special cases of quadrature formulas. For derived weighted one-point formula for numerical integration various examples for some famous weight functions are presented. For weighted two-point formula applications are given for trapezoid, midpoint and perturbed trapezoid formulae, Newton-Cotes, Maclaurin, Legendre-Gauss, Chebyshev-Gauss and Hermite-Gauss formula. For derived weighted three-point formula applications are given for Simpson's, Dual Simpson's, Maclaurin, Legendre-Gauss, Chebyshev-Gauss and Gauss-Hermite formula. In case of derived four-point quadrature formula applications for closed quadrature formula with precision 3, and the general Lobatto formula are concerned.

Chapter 2 studies weighted generalizations of the Euler integral identity which express expansion of a function in terms of Bernoulli polynomials. The equivalence of the Euler formula and generalization of weighted Montgomery identity is proven. Several integral and discrete generalizations of Montgomery identity and extensions via Taylor's formula and Fink identity are presented. These identities are utilized to obtain new improvements and generalizations of Ostrowski type inequalities, estimations of difference of two weighted integral and arithmetic means, trapezoid and midpoint inequalities, two-point and three-point Radau, Lobatto and Gauss quadrature rules, error estimates of approximations for the Fourier and Laplace transform and extensions of Landau inequality. Also, integral and discrete weighted generalizations of Montgomery identities for functions of two variables and related Ostrowski type inequalities are given.

Chapter 3 deals with weighted generalizations of classical two-point, three-point and four-point quadrature formulae using generalizations of the weighted Montgomery identity via Taylor's formula. Ostrowski type inequalities and error estimates are derived for trapezoidal and midpoint formula, two-point Gauss-Chebyshev formulae, Newton-Cotes and Maclaurin two-point formula, three-point Gauss-Chebyshev formulae, Simpson's and Maclaurin three-point formula, Simpson's 3/8 formula and Lobatto four-point formula.

Chapter 4 considers generalizations of Euler identities involving μ-harmonic sequences of functions with respect to a real Borel measure μ. Applications for Ostrowski type inequalities, Grüss type and Euler-Grüss type inequalities are given. An integration-by-parts formula, involving finite Borel measures supported by intervals on the real line is obtained. Euler identities for μ-harmonic sequences of functions and moments of a real Borel measure μ are also considered.

Contents

Chapter 1

Generalization of the classical integral forumalae and related inequalities

In this chapter we introduce general integral identities using the harmonic sequences of polynomials and $w-$harmonic sequences of functions. Those identities are the main tool for deriving generalizations of some famous quadrature formulas. We deal with quadrature formulas which contain values of the function in nodes, as well as values of higher ordered derivatives in inner nodes. Thereby, the level of exactness of those quadrature formulas is saved. Error estimations with sharp and the best possible constants are developed as well.

In Section 1.1. general integral identities with harmonic polynomials and $w-$harmonic functions are established. Those identities are actually the general quadrature formulas with $m+1$ nodes. For both identites the error estimations for functions whose higher ordered derivatives belong to L_p spaces are given.

In Section 1.2. general one-point quadrature formula is established. Special cases of the well known weights are considered and generalizations of the Gaussian quadrature formulas with one node are obtained.

In Section 1.3. general two-point integral quadrature formula using the concept of harmonic polynomials is established. Improved version of Guessab and Schmeisser's result is given with new integral inequalities involving functions whose derivatives belong to various classes of functions (L_p spaces, convex, concave, bounded functions). Furthermore, several special cases of polynomials are considered, and the generalization of well-known two-point quadrature formulae, such as trapezoid, perturbed trapezoid, two-point Newton-

Cotes formula, two-point Maclaurin formula, midpoint, are obtained. Weighted version of two-point integral quadrature formula is obtained using $w-$harmonic sequences of functions. For special choices of weights w and nodes x and $a+b-x$ the generalization of the well-known two-point quadrature formulas of Gauss type are given.

In Section 1.4. general three-point quadrature formula with nodes $x, \frac{a+b}{2}$ and $a+b-x$ is introduced. From non-weighted version Simpson, dual Simpson and Maclaurin formulas are obtained, while for special weights Gaussian quadrature formulas are given.

The closed four-point quadrature formula is introduced in Section 1.5. Generalization of Lobatto formula is given as special case.

Definition 1.1 *We say that* $\{P_k\}_k \in \mathbb{N}_0$ *is* ***harmonic sequence of the polynomials*** *if* $P_k'(t) = P_{k-1}(t), \forall k \in \mathbb{N}$ *and* $P_0(t) \equiv 1$.

1.1 General integral identities involving $w-$harmonic sequences of functions

Non-weighted integral identity is used for the approximation of an integral of the following form: $\int_a^b f(t)dt$. The next theorem is obtained in [100].

Theorem 1.1 *Let* $\sigma := \{a = x_0 < x_1 < x_2 < \ldots < x_m = b\}$ *be subdivision of the interval* $[a,b]$. *Further, let for each* $j = 1,\ldots,m$, $\{P_{jk}\}_{k\in\mathbb{N}_0}$ *be the harmonic sequences of the polynomials on* $[x_{j-1},x_j]$, *i.e.* $P_{jk}'(t) = P_{j,k-1}(t)$ *i* $P_{j0}(t) \equiv 1$, *for* $j = 1,\ldots,m$ *and* $k \in \mathbb{N}$, *and let*

$$S_n(t,\sigma) = \begin{cases} P_{1n}(t), & t \in [a,x_1] \\ P_{2n}(t), & t \in (x_1,x_2] \\ \vdots & \\ P_{mn}(t), & t \in (x_{m-1},b], \end{cases} \tag{1.1}$$

for some $n \in \mathbb{N}$. *For an arbitrary* $(n-1)-$*times differentiable function* $f : [a,b] \to \mathbb{R}$ *such that* $f^{(n-1)}$ *is bounded, the following identity states*

$$\begin{aligned}(-1)^n \int_a^b S_n(t,\sigma)df^{(n-1)}(t) = \int_a^b f(t)dt + \sum_{k=1}^{n}(-1)^k \Big[P_{mk}(b)f^{(k-1)}(b) \\ + \sum_{j=1}^{m-1}\left[P_{jk}(x_j) - P_{j+1,k}(x_j)\right] f^{(k-1)}(x_j) - P_{1k}(a)f^{(k-1)}(a)\Big],\end{aligned} \tag{1.2}$$

whenever the integrals exist.

Identity (1.2) is used for the approximation of an integral $\int_a^b f(t)dt$ both with the values of the function f and its higher order derivatives in nodes $x_0,x_1,x_2,\ldots,x_m$. With appropriate

choice of polynomials $\{P_{jk}\}$ and nodes x_j we shall get the generalization of the well-known quadrature formulas. In those generalized formulas the integral is approximated not only with the values of the function in certain nodes, but also with values of its derivatives up to $(n-1)^{\text{th}}$ order in inner nodes.
Let us develop an error estimation for the identity (1.2).

Theorem 1.2 *Assume (p,q) is a pair of conjugate exponents, that is $1 \leq p,q,\leq \infty$, $\frac{1}{p}+\frac{1}{q}=1$. If $f:[a,b]\to\mathbb{R}$ is an arbitrary function such that $f^{(n)}$ is piecewise continuous, for some $n\in\mathbb{N}$, then we have*

$$\begin{aligned}&\Bigg|\int_a^b f(t)dt+\sum_{k=1}^{n}(-1)^k\Big[P_{mk}(b)f^{(k-1)}(b)\\&+\sum_{j=1}^{m-1}\left[P_{jk}(x_j)-P_{j+1,k}(x_j)\right]f^{(k-1)}(x_j)-P_{1k}(a)f^{(k-1)}(a)\Big]\Bigg|\\&\leq C(n,q)\|f^{(n)}\|_p,\end{aligned} \tag{1.3}$$

where

$$C(n,q)=\|S_n(\cdot,\sigma)\|_q=\begin{cases}\left[\sum_{j=1}^m\int_{x_{j-1}}^{x_j}\left|P_{jn}(t)\right|^q dt\right]^{\frac{1}{q}}, & 1\leq q<\infty\\ \max_{1\leq j\leq m}\{\sup_{t\in[x_{j-1},x_j]}\left|P_{jn}(t)\right|\},\ q=\infty.\end{cases}$$

Inequalities are sharp for $1<p\leq\infty$ and the best possible for $p=1$. Equality in (1.3) is attained for the functions f of the form:

$$f(t)=Mf_*(t)+r_{n-1}(t), \tag{1.4}$$

where $M\in\mathbb{R}$, r_{n-1} is an arbitrary polynomial of degree $n-1$, and $f_*:[a,b]\to\mathbb{R}$ is function with the following representation:

$$f_*(t):=\int_a^t\frac{(t-s)^{n-1}}{(n-1)!}|S_n(s,\sigma)|^{\frac{1}{p-1}}\operatorname{sgn}S_n(s,\sigma)ds,\quad 1<p<\infty \tag{1.5}$$

and

$$f_*(t):=\int_a^t\frac{(t-s)^{n-1}}{(n-1)!}\operatorname{sgn}S_n(s,\sigma)ds,\quad p=\infty. \tag{1.6}$$

Proof. Applying Hölder inequality to the integral

$$(-1)^n\int_a^b S_n(t,\sigma)df^{(n-1)}(t)=(-1)^n\int_a^b S_n(t,\sigma)f^{(n)}(t)dt$$

an inequality (1.3) is obtained. To prove the inequalities are sharp for $1<p\leq\infty$, we have to find function $f:[a,b]\to\mathbb{R}$ such that

$$\left|\int_a^b S_n(t,\sigma)f^{(n)}(t)dt\right|=C(n,q)\cdot\|f^{(n)}\|_p. \tag{1.7}$$

For function f_* defined by (1.5) and (1.6) we have

$$f_*^{(n)}(t) = \begin{cases} \operatorname{sgn} S_n(t,\sigma), & p = \infty, \\ \\ |S_n(t,\sigma)|^{\frac{1}{p-1}} \operatorname{sgn} S_n(t,\sigma), & 1 < p < \infty, \end{cases}$$

Function $f : [a,b] \to \mathbb{R}$ defined with (1.4) is $n-$times differentiable also. Further, $f^{(n)}$ is piecewise continuous and $f^{(n)}(t) = Mf_*^{(n)}(t)$ holds.
For $p = \infty$ we have $\|f^{(n)}\|_p = |M|$, so

$$\begin{aligned} \left| \int_a^b S_n(t,\sigma) f^{(n)}(t) dt \right| &= \left| M \int_a^b S_n(t,\sigma) f_*^{(n)}(t) dt \right| \\ &= \left| M \int_a^b S_n(t,\sigma) sgn S_n(t,\sigma) dt \right| \\ &= |M| \int_a^b |S_n(t,\sigma)|\, dt = C(n,1) \|f^{(n)}\|_\infty \end{aligned}$$

holds, while for $1 < p < \infty$ we have

$$\|f^{(n)}\|_p = |M| \left[\int_a^b |S_n(t,\sigma)|^{\frac{p}{p-1}} dt \right]^{\frac{1}{p}} = |M| \left[\int_a^b |S_n(t,\sigma)|^q dt \right]^{\frac{1}{p}},$$

which implies

$$\begin{aligned} \left| \int_a^b S_n(t,\sigma) f^{(n)}(t) dt \right| &= \left| M \int_a^b S_n(t,\sigma) f_*^{(n)}(t) dt \right| \\ &= \left| M \int_a^b S_n(t,\sigma) \, |S_n(t,\sigma)|^{\frac{1}{p-1}} \operatorname{sgn} S_n(t,\sigma) dt \right| \\ &= |M| \int_a^b |S_n(t,\sigma)|^{\frac{p}{p-1}} dt = |M| \int_a^b |S_n(t,\sigma)|^q dt = C(n,q) \|f^{(n)}\|_p, \end{aligned}$$

so the proof of the (1.7) is finished.
Finally, we have to prove that inequality (1.3) is the best possible for $p = 1$. Obviously, because of the continuity of the $P_{jk}(\cdot)$ on $[x_{j-1}, x_j]$, there exists $j \in \{1, \dots, m\}$ and $t_0 \in [x_{j-1}, x_j]$ such that $\sup_{t \in [a,b]} |S_n(t,\sigma)| = \left|P_{jn}(t_0)\right|$. First, let us assume that $P_{jn}(t_0) > 0$. There are two possibilities:

(i) $x_{j-1} < t_0 \le x_j$

(ii) $t_0 = x_{j-1}$

For the case (i) let us define function $f_\varepsilon : [a,b] \to \mathbb{R}$ for $\varepsilon > 0$:

$$f_\varepsilon^{(n-1)}(t) = \begin{cases} 1, & t \le t_0 - \varepsilon, \\ \frac{t_0 - t}{\varepsilon}, & t \in [t_0 - \varepsilon, t_0], \\ 0, & t \ge t_0. \end{cases}$$

When ε is "enough small", we have

$$\left|\int_a^b S_n(t,\sigma)f_\varepsilon^{(n)}(t)dt\right| = \frac{1}{\varepsilon}\left|\int_{t_0-\varepsilon}^{t_0} S_n(t,\sigma)dt\right| = \frac{1}{\varepsilon}\int_{t_0-\varepsilon}^{t_0} P_{jn}(t)dt.$$

Further,

$$\frac{1}{\varepsilon}\int_{t_0-\varepsilon}^{t_0} P_{jn}(t)dt \leq \frac{1}{\varepsilon}P_{jn}(t_0)\int_{t_0-\varepsilon}^{t_0} dt = P_{jn}(t_0).$$

Since $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t_0-\varepsilon}^{t_0} P_{jn}(t)dt = P_{jn}(t_0)$, the assertion follows.
For the case (ii) let us define function $f_\varepsilon : [a,b] \to \mathbb{R}$ for $\varepsilon > 0$:

$$f_\varepsilon^{(n-1)}(t) = \begin{cases} 0, & t \leq t_0, \\ \frac{t-t_0}{\varepsilon}, & t \in [t_0, t_0+\varepsilon], \\ 1, & t \geq t_0+\varepsilon. \end{cases}$$

When ε is "enough small", we have

$$\left|\int_a^b S_n(t,\sigma)f_\varepsilon^{(n)}(t)dt\right| = \frac{1}{\varepsilon}\left|\int_{t_0}^{t_0+\varepsilon} S_n(t,\sigma)dt\right| = \frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} P_{jn}(t)dt.$$

Further,

$$\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} P_{jn}(t)dt \leq \frac{1}{\varepsilon}P_{jn}(t_0)\int_{t_0}^{t_0+\varepsilon} dt = P_{jn}(t_0).$$

Since $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} P_{jn}(t)dt = P_{jn}(t_0)$, the assertion follows.
For the case $P_{jn}(t_0) < 0$, the proof is simmilar. □

Remark 1.1 Inequality (1.3) is obtained in [100], for the case $1 < p \leq \infty$.

In [73] is derived the identity (1.2) with monic polynomials:

Theorem 1.3 *Let $\sigma := \{a = x_0 < x_1 < x_2 < \ldots < x_m = b\}$ be subdivision of the interval $[a,b]$. Further, for $j = 1,\ldots,m$, let M_{jn} be monic polynomials, for some $n \in \mathbb{N}$, with $\deg M_{jn} = n$. Define*

$$V_n(t,\sigma) = \begin{cases} M_{1n}(t), & t \in [a,x_1], \\ M_{2n}(t), & t \in (x_1,x_2], \\ \vdots \\ M_{mn}(t), & t \in (x_{m-1},b]. \end{cases} \tag{1.8}$$

If $f : [a,b] \to \mathbb{R}$ is some $(n-1)-$times differentiable function such that $f^{(n-1)}$ is bounded, then we have

$$\begin{aligned} \int_a^b f(t)dt \; &+ \frac{1}{n!}\sum_{k=0}^{n-1}(-1)^{k+1}\cdot\Big[M_{mn}^{(n-k-1)}(b)f^{(k)}(b) + \sum_{j=1}^{m-1}\Big[M_{jn}^{(n-k-1)}(x_j) \\ &- M_{j+1,n}^{(n-k-1)}(x_j)\Big]f^{(k)}(x_j) - M_{1n}^{(n-k-1)}(a)f^{(k)}(a)\Big] \\ &= \frac{(-1)^n}{n!}\int_a^b V_n(t,\sigma)df^{(n-1)}(t). \end{aligned} \tag{1.9}$$

Proof. The proof follows from the successively integration by parts of the integral

$$\frac{(-1)^n}{n!}\int_a^b V_n(t,\sigma)df^{(n-1)}(t).$$

□

Remark 1.2 Let $\{P_{jk}\}_{k=0,1,\dots,n}$ be harmonic sequences of polynomials such that $P_{j0}(t)=1$. Then we have $P_{jn}^{n-k-1}(t)=P_{j,k+1}(t)$, for $0\le k\le n-1$. Put $M_{jn}=n!P_{jn}$ in (1.9). Now we have $V_n(t,\sigma)=n!S_n(t,\sigma)$, so the identity (1.9) is equivalent to the identity (1.2).

Theorem 1.4 *Assume (p,q) is a pair of conjugate exponents, that is $1\le p,q,\le\infty$, $\frac{1}{p}+\frac{1}{q}=1$. If $f:[a,b]\to\mathbb{R}$ is an arbitrary function such that $f^{(n)}$ is piecewise continuous, for some $n\in\mathbb{N}$, then we have*

$$\begin{aligned}&\left|\int_a^b f(t)dt+\frac{1}{n!}\sum_{k=0}^{n-1}(-1)^{k+1}\cdot\left[M_{mn}^{(n-k-1)}(b)f^{(k)}(b)\right.\right.\\&\left.\left.+\sum_{j=1}^{m-1}\left[M_{jn}^{(n-k-1)}(x_j)-M_{j+1,n}^{(n-k-1)}(x_j)\right]f^{(k)}(x_j)-M_{1n}^{(n-k-1)}(a)f^{(k)}(a)\right]\right|\\&\le\frac{1}{n!}K(n,q)\|f^{(n)}\|_p,\end{aligned}\tag{1.10}$$

where

$$K(n,q)=\|V_n(\cdot,\sigma)\|_q=\begin{cases}\left[\sum_{j=1}^m\int_{x_{j-1}}^{x_j}\left|M_{jn}(t)\right|^q dt\right]^{\frac{1}{q}}, & 1\le q<\infty\\ \max_{1\le j\le m}\{\sup_{t\in[x_{j-1},x_j]}\left|M_{jn}(t)\right|\}, & q=\infty.\end{cases}$$

Inequalities are sharp for $1<p\le\infty$ and the best possible for $p=1$. Equality in (1.10) is attained for the functions f of the form:

$$f(t)=Mf_*(t)+r_{n-1}(t),\tag{1.11}$$

where $M\in\mathbb{R}$, r_{n-1} is an arbitrary polynomial of degree $n-1$, and $f_*:[a,b]\to\mathbb{R}$ is function with the following representation:

$$f_*(t):=\int_a^t\frac{(t-s)^{n-1}}{(n-1)!}|V_n(s,\sigma)|^{\frac{1}{p-1}}\operatorname{sgn}V_n(s,\sigma)ds,\quad 1<p<\infty\tag{1.12}$$

i

$$f_*(t):=\int_a^t\frac{(t-s)^{n-1}}{(n-1)!}\operatorname{sgn}V_n(s,\sigma)ds,\quad p=\infty.\tag{1.13}$$

Proof. The proof is simmilar to the proof of the Theorem 1.2 □

Weighted version of the identity (1.2) and related inequalities are obtained in [72]. In this case the $w-$harmonic sequences of the functions are used.

Lemma 1.1 *Let $w:[a,b]\to\mathbb{R}$ be integrable function on $[a,b]$ and let $\{w_k\}_{k=1,\dots,n}$ be $w-$harmonic sequences of functions, i.e. $w_k:[a,b]\to\mathbb{R}$ are such that $w_k'(t)=w_{k-1}(t)$, for $t\in[a,b]$ and $k=2,3,\dots,n$, and $w_1'(t)=w(t)$. If $g:[a,b]\to\mathbb{R}$ is $n-$times differentiable function such that $g^{(n)}$ is piecewise continuous on $[a,b]$, then we have*

$$\int_a^b w(t)g(t)dt = A_n(w,g;a,b)+R_n(w,g;a,b), \tag{1.14}$$

where

$$A_n(w,g;a,b)=\sum_{k=1}^{n}(-1)^{k-1}\left[w_k(b)g^{(k-1)}(b)-w_k(a)g^{(k-1)}(a)\right]$$

and

$$R_n(w,g;a,b)=(-1)^n\int_a^b w_n(t)g^{(n)}(t).$$

Proof. We prove (1.14) by mathematical induction.
For $n=1$ integration by parts gives

$$\int_a^b w(t)g(t)dt = w_1(b)g(b)-w_1(a)g(a)-\int_a^b w_1(t)g'(t)dt. \tag{1.15}$$

Let us assume that for $l=1,\dots,n-1$ we have

$$\begin{aligned}\int_a^b w(t)g(t)dt &= \sum_{k=1}^{l}(-1)^{k-1}\left[w_k(b)g^{(k-1)}(b)-w_k(a)g^{(k-1)}(a)\right]\\ &+ (-1)^l\int_a^b w_l(t)g^{(l)}(t)dt.\end{aligned} \tag{1.16}$$

Further, integration by parts yields

$$\int_a^b w_l(t)g^{(l)}(t)dt = w_{l+1}(b)g^{(l)}(b)-w_{l+1}(a)g^{(l)}(a)-\int_a^b w_{l+1}(t)g^{(l+1)}(t)dt. \tag{1.17}$$

Finnaly, we impose the identity (1.17) to the relation (1.16) and obtain

$$\begin{aligned}\int_a^b w(t)g(t)dt &= \sum_{k=1}^{l}(-1)^{k-1}\left[w_k(b)g^{(k-1)}(b)-w_k(a)g^{(k-1)}(a)\right]\\ &+ (-1)^l\Big[w_{l+1}(b)g^{(l)}(b)-w_{l+1}(a)g^{(l)}(a)\\ &- \int_a^b w_{l+1}(t)g^{(l+1)}(t)dt\Big]\\ &= \sum_{k=1}^{l+1}(-1)^{k-1}\left[w_k(b)g^{(k-1)}(b)-w_k(a)g^{(k-1)}(a)\right]\\ &+ (-1)^{l+1}\int_a^b w_{l+1}(t)g^{(l+1)}(t)dt,\end{aligned}$$

so the assertion is valid for $l+1$. □

Remark 1.3 Function $w : [a,b] \to \mathbb{R}$ is usually called weight.

Consider subdivision $\sigma = \{a = x_0 < x_1 < \ldots < x_m = b\}$ of the segment $[a,b]$,for some $m \in \mathbb{N}$. Let $w : [a,b] \to \mathbb{R}$ be an arbitrary integrable function. On each interval $[x_{k-1}, x_k]$, $k = 1, \ldots, m$ we consider different $w-$harmonic sequences of functions $\{w_{kj}\}_{j=1,\ldots,n}$, i.e. we have

$$
\begin{aligned}
w'_{k1}(t) &= w(t) && \text{for } t \in [x_{k-1}, x_k] \\
(w_{kj})'(t) &= w_{k,j-1}(t) && \text{for } t \in [x_{k-1}, x_k], \text{ for all } j = 2,3,\ldots,n.
\end{aligned} \tag{1.18}
$$

Further, let us define

$$
W_{n,w}(t,\sigma) = \begin{cases} w_{1n}(t) & \text{for } t \in [a, x_1], \\ w_{2n}(t) & \text{for } t \in (x_1, x_2], \\ \vdots \\ w_{mn}(t) & \text{for } t \in (x_{m-1}, b]. \end{cases} \tag{1.19}
$$

Theorem 1.5 *If $g : [a,b] \to \mathbb{R}$ is such that $g^{(n)}$ is a piecewise continuous on $[a,b]$, then the following identity holds*

$$
\begin{aligned}
\int_a^b w(t)g(t)dt \;=\; & \sum_{j=1}^{n} (-1)^{j-1} \Big[w_{mj}(b) g^{(j-1)}(b) \\
+ & \sum_{k=1}^{m-1} \left[w_{kj}(x_k) - w_{k+1,j}(x_k) \right] g^{(j-1)}(x_k) - w_{1j}(a) g^{(j-1)}(a) \Big] \\
+ & (-1)^n \int_a^b W_{n,w}(t,\sigma) g^{(n)}(t) dt.
\end{aligned} \tag{1.20}
$$

Proof. Using relation (1.14) on each interval $[x_{k-1}, x_k]$ for appropriate $w-$harmonic sequence, we get the following

$$
\int_{x_{k-1}}^{x_k} w(t)g(t)dt = A_n(w,g; x_{k-1}, x_k) + R_n(w,g; x_{k-1}, x_k). \tag{1.21}
$$

By summing relation (1.21) from $k = 1$ to m we obtain

$$
\begin{aligned}
\int_a^b w(t)g(t)dt \;=\; & \sum_{j=1}^{n} (-1)^{j-1} \Big[w_{mj}(b) g^{(j-1)}(b) \\
+ & \sum_{k=1}^{m-1} \left[w_{kj}(x_k) - w_{k+1,j}(x_k) \right] g^{(j-1)}(x_k) - w_{1j}(a) g^{(j-1)}(a) \Big] \\
+ & \sum_{k=1}^{m} R_n(w,g; x_{k-1}, x_k) \\
= & \sum_{j=1}^{n} (-1)^{j-1} \Big[w_{mj}(b) g^{(j-1)}(b)
\end{aligned} \tag{1.22}
$$

$$+\sum_{k=1}^{m-1}\left[w_{kj}(x_k)-w_{k+1,j}(x_k)\right]g^{(j-1)}(x_k)-w_{1j}(a)g^{(j-1)}(a)\Big]$$
$$+(-1)^n\int_a^b W_{n,w}(t,\sigma)g^{(n)}(t)dt.$$

□

Now we shall give the general L_p theorem.

Theorem 1.6 *Assume thet (p,q) is a pair of conjugate exponents, that is $1\le p,q\le\infty$, $\frac{1}{p}+\frac{1}{q}=1$. If $g:[a,b]\to\mathbb{R}$ is some function such that $g^{(n)}$ is piecewise continuous on $[a,b]$ and $g^{(n)}\in L_p[a,b]$, then the following inequality holds*

$$\Bigg|\int_a^b w(t)g(t)dt-\sum_{k=1}^{n}(-1)^{k-1}\Big[w_{mk}(b)g^{(k-1)}(b) \tag{1.23}$$
$$-\sum_{j=1}^{m-1}\left[w_{jk}(x_j)-w_{j+1,k}(x_j)\right]g^{(k-1)}(x_j)-w_{1k}(a)g^{(k-1)}(a)\Big]\Bigg|$$
$$\le C(n,q,w)\cdot\|g^{(n)}\|_p,$$

where

$$C(n,q,w)=\|W_{n,w}(\cdot,\sigma)\|_q=\begin{cases}\left[\sum_{j=1}^{m}\int_{x_{j-1}}^{x_j}\left|w_{jn}(t)\right|^q dt\right]^{\frac{1}{q}},1\le q<\infty,\\ \max_{1\le j\le m}\{\sup_{t\in[x_{j-1},x_j]}\left|w_{jn}(t)\right|\},q=\infty.\end{cases}$$

The inequality is the best possible for $p=1$ and sharp for $1<p\le\infty$. Equality is attained for every function g such that

$$g(t)=M\cdot g_*(t)+p_{n-1}(t),$$

where $M\in\mathbb{R}$, p_{n-1} is an arbitrary polynomial of degree at most $n-1$ and $g_*(t)$ is function on $[a,b]$ defined by

$$g_*(t):=\int_a^t\frac{(t-s)^{n-1}}{(n-1)!}|W_{n,w}(s,\sigma)|^{\frac{1}{p-1}}\operatorname{sgn}W_{n,w}(s,\sigma)ds,\quad 1<p<\infty \tag{1.24}$$

i

$$g_*(t):=\int_a^t\frac{(t-s)^{n-1}}{(n-1)!}\operatorname{sgn}W_{n,w}(s,\sigma)ds,\quad p=\infty. \tag{1.25}$$

1.2 Application to the one-point quadrature formulae

Now we develop the weighted one-point formula for numerical integration. Let $g:[a,b]\to\mathbb{R}$ be some function and $x\in[a,b]$. Let $w:[a,b]\to\mathbb{R}$ be some integrable function. The approximation of the integral $\int_a^b w(t)g(t)dt$ will involve the values of the higher order derivatives of g in the node x. We consider subdivision $\sigma=\{x_0<x_1<x_2\}$ of the interval $[a,b]$, where $x_0=a$, $x_1=x$ and $x_2=b$. Further, let $\{w_{kj}^1\}_{j=1,\dots,n}$ be $w-$harmonic sequences on each subinterval $[x_{k-1},x_k]$, $k=1,2$, defined by the following relations:

$$w_{1j}^1(t):=\frac{1}{(j-1)!}\int_a^t (t-s)^{j-1}w(s)ds,\quad t\in[a,x]$$

$$w_{2j}^1(t):=\frac{1}{(j-1)!}\int_b^t (t-s)^{j-1}w(s)ds,\quad t\in(x,b],$$

for $j=1,\dots,n$. Now we can state the following theorem

Theorem 1.7 *If $g:[a,b]\to\mathbb{R}$ is such that $g^{(n)}$ is a piecewise continuous function, then we have*

$$\int_a^b w(t)g(t)dt = A_1^1(x)g(x)+T_{n,w}^1(x)+(-1)^n\int_a^b W_{n,w}^1(t,x)g^{(n)}(t)\mathrm{d}t, \tag{1.26}$$

where for $j=1,\dots,n$

$$T_{n,w}^1(x)=\sum_{j=2}^{n}A_k^1(x)g^{(k-1)}(x), \tag{1.27}$$

further, for $j=1,\dots,n$

$$A_j^1(x)=\frac{(-1)^{j-1}}{(j-1)!}\int_a^b (x-s)^{j-1}w(s)ds \tag{1.28}$$

and

$$W_{n,w}^1(t,x)=\begin{cases} w_{1n}^1(t)=\frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}w(s)ds & \text{for } t\in[a,x],\\ w_{2n}^1(t)=\frac{1}{(n-1)!}\int_b^t (t-s)^{n-1}w(s)ds & \text{for } t\in(x,b].\end{cases} \tag{1.29}$$

Proof. We apply identity (1.20) for $m=2$ and $x_1=x$ to get

$$\begin{aligned}\int_a^b w(t)g(t)dt &= \sum_{j=1}^{n}(-1)^{j-1}\left[w_{1j}^1(x)-w_{2j}^1(x)\right]g^{(j-1)}(x)\\ &+ (-1)^n\int_a^b W_{n,w}^1(t,x)g^{(n)}(t)dt,\end{aligned}$$

since $w_{1j}^1(a) = 0$ and $w_{2j}^1(b) = 0$, for $j = 1, \ldots, n$. Further, we compute

$$w_{1j}^1(x) - w_{2j}^1(x) = \frac{1}{(j-1)!} \int_a^b (x-s)^{j-1} w(s) ds = (-1)^{j-1} A_j^1(x),$$

so the assertion of the Theorem follows. □

Remark 1.4 The identity in Theorem 1.7 was obtained in [85], so we may call it an integral formula of Matić, Pečarić and Ujević.

Remark 1.5 If we want formula (1.26) to be exact for the polynomials of degree at most 1, such that approximation formula doesn't include the first derivative, the extra condition $A_2(x) = 0$ is required. From this condition we get

$$\int_a^b (x-s) w(s) ds = 0.$$

The solution $x = \frac{\int_a^b s w(s) ds}{\int_a^b w(s) ds}$ of this equation yields the node of the one-point Gaussian quadrature formula.

Theorem 1.8 *Let $w : [a,b] \to [0,\infty)$ be an integrable function and $x \in [a,b]$. Further, let $\{w_{kj}^1\}_{j=1,\ldots,2n+1}$ be $w-$harmonic sequences of functions for $k = 1,2$ and some $n \in \mathbb{N}$, defined by the following relations:*

$$w_{1j}^1(t) := \frac{1}{(j-1)!} \int_a^t (t-s)^{j-1} w(s) ds, \quad t \in [a,x]$$

$$w_{2j}^1(t) := \frac{1}{(j-1)!} \int_b^t (t-s)^{j-1} w(s) ds, \quad t \in (x,b],$$

for $j = 1, \ldots, 2n+1$. If $g : [a,b] \to \mathbb{R}$ is such that $g^{(2n)}$ is continuous function, then there exists $\eta \in [a,b]$ such that

$$\int_a^b w(t) g(t) dt = A_1^1(x) g(x) + T_{n,w}^1(x) + A_{2n+1}^1(x) \cdot g^{(2n)}(\eta). \tag{1.30}$$

Proof. It is easy to check that $W_{2n,w}(t,x) \geq 0$, for $t \in [a,b]$, so we can apply integral mean value theorem to the $\int_a^b W_{2n,w}(t,x) g^{(2n)}(t) dt$ to obtain

$$\int_a^b w(t) g(t) dt - \sum_{j=1}^{2n} A_j^1(x) g^{(j-1)}(x) = g^{(2n)}(\eta) \cdot \int_a^b W_{2n,w}^1(t,x) dt. \tag{1.31}$$

We calculate

$$\begin{aligned} \int_a^b W_{2n}(t,x)^1 dt &= \int_a^x w_{1,2n}^1(t) dt + \int_x^b w_{2,2n}^1(t) dt \\ &= w_{1,2n+1}^1(x) - w_{2,2n+1}^1(x) = A_{2n+1}^1(x), \end{aligned}$$

so we get the assertion. □

Now we can state the L_p-inequality for weighted one-point formula

Theorem 1.9 *Let $g : [a,b] \to \mathbb{R}$ and $\{w_{kj}^1\}_{j=1,\dots,n}$ be as in Theorem 1.7, and let $g^{(n)} \in L_p[a,b]$ for some $1 \le p \le \infty$. Then we have*

$$\left| \int_a^b w(t)g(t)dt - A_1^1(x)g(x) - T_{n,w}^1(x) \right| \le C_1(n,p,x,w) \cdot \|g^{(n)}\|_p,$$

for $\frac{1}{p} + \frac{1}{q} = 1$, where

$$\begin{aligned} C_1(n,p,x,w) &= \frac{1}{(n-1)!} \left[\int_a^x \left| \int_a^t (t-s)^{n-1} w(s)ds \right|^q dt \right. \\ &\quad \left. + \int_x^b \left| \int_b^t (t-s)^{n-1} w(s)ds \right|^q dt \right]^{\frac{1}{q}}, \end{aligned} \tag{1.32}$$

for $1 < p \le \infty$, and

$$\begin{aligned} C_1(n,1,x,w) &= \frac{1}{(n-1)!} \max\left\{ \sup_{t\in[a,x]} \left| \int_a^t (t-s)^{n-1} w(s)ds \right|, \right. \\ &\quad \left. \sup_{t\in[x,b]} \left| \int_b^t (t-s)^{n-1} w(s)ds \right| \right\}. \end{aligned} \tag{1.33}$$

The inequality is the best possible for $p = 1$ and sharp for $1 < p \le \infty$. Equality is attained for some function $g(t) = Mg_*(t) + p_{n-1}(t)$ where $M \in \mathbb{R}$, p_{n-1} is an arbitrary polynomial of degree at most $n-1$ and g_* is function on $[a,b]$ defined by

$$g_*(t) = \int_a^t \frac{(t-\xi)^{n-1}}{(n-1)!} \cdot \operatorname{sgn} W_{n,w}^1(\xi,x) \cdot |W_{n,w}^1(\xi,x)|^{\frac{1}{p-1}} d\xi, \tag{1.34}$$

for $1 < p < \infty$, and

$$g_*(t) = \int_a^t \frac{(t-\xi)^{n-1}}{(n-1)!} \cdot \operatorname{sgn} W_{n,w}^1(\xi,x) d\xi, \tag{1.35}$$

for $p = \infty$.

Proof. This theorem is special case of the Theorem 1.6 □

Now we shall give some special examples of the general one-point quadrature formulae for different choices of weight function.

Example 1.1 (THE CASE $w(t) = 1$) Applying the identity (1.26) to the special case of weight function $w(t) = 1$, the following identity is obtained:

$$\int_a^b g(t)dt = (b-a)g(x) + T_{n,w}^{1,LG}(x) + (-1)^n \int_a^b W_{n,w}^{1,LG}(t,x) g^{(n)}(t)dt, \tag{1.36}$$

where

$$W_{n,w}^{1,LG}(t,x) = \begin{cases} \frac{(t-a)^n}{n!}, & \text{for } t \in [a,x], \\ \frac{(t-b)^n}{n!}, & \text{for } t \in (x,b]. \end{cases} \tag{1.37}$$

Specially, according to Remark 1.5 the condition

$$\int_a^b (x-s)ds = 0,$$

implies $x = \frac{a+b}{2}$, so the generalization of the well-known midpoint quadrature formula is carried out:

$$\begin{aligned}\int_a^b g(t)dt &= (b-a)g\left(\frac{a+b}{2}\right) + T_{n,w}^{1,LG}\left(\frac{a+b}{2}\right) \\ &+ (-1)^n \int_a^b W_{n,w}^{1,LG}\left(t, \frac{a+b}{2}\right) g^{(n)}(t)dt, \end{aligned} \tag{1.38}$$

where

$$T_{n,w}^{1,LG}\left(\frac{a+b}{2}\right) = \sum_{\substack{k=3 \\ \text{even } k}}^{n} A_k^{1,LG}\left(\frac{a+b}{2}\right) g^{(k-1)}\left(\frac{a+b}{2}\right)$$

and

$$A_k^{1,LG}\left(\frac{a+b}{2}\right) = \frac{(b-a)^k}{2^{k-1}k!}.$$

If all the assumptions from Theorem 1.8 hold, then we have

$$\int_a^b g(t)dt = (b-a)g(x) + T_{2n,w}^{1,LG}(x) + A_{2n+1}^{1,LG}(x) \cdot g^{(2n)}(\eta). \tag{1.39}$$

Specially, for $x = \frac{a+b}{2}$ we have

$$\int_a^b g(t)dt = (b-a)g\left(\frac{a+b}{2}\right) + T_{2n,w}^{1,LG}\left(\frac{a+b}{2}\right) + \frac{(b-a)^{2n+1}}{4^n(2n+1)!} \cdot g^{(2n)}(\eta).$$

For $n = 1$ the midpoint formula is established:

$$\int_a^b g(t)dt = (b-a)g\left(\frac{a+b}{2}\right) + \frac{(b-a)^3}{24} g''(\eta).$$

If all the assumptions from Theorem 1.9 hold, then we have

$$\left|\int_a^b g(t)dt - (b-a)g(x) - T_{n,w}^{1,LG}(x)\right| \le C_1^{LG}(n,q,x,w) \cdot \|g^{(n)}\|_p,$$

where

$$C_1^{LG}(n,q,x,w) = \frac{1}{n!}\left[\frac{(x-a)^{nq+1} + (b-x)^{nq+1}}{nq+1}\right]^{\frac{1}{q}}, \tag{1.40}$$

for $1 \le q < \infty$, and

$$C_1^{LG}(n,\infty,x,w) = \frac{1}{n!}\max\{(x-a)^n, (b-x)^n\}. \tag{1.41}$$

Specially, for $x = \frac{a+b}{2}$ we get

$$\left| \int_a^b g(t)dt - (b-a)g\left(\frac{a+b}{2}\right) \right| \le C_1^{LG}\left(n, q, \frac{a+b}{2}, w\right) \|g^{(n)}\|_p, \quad n = 1,2,$$

where

$$\begin{aligned} C_1^{LG}\left(1,1,\frac{a+b}{2},w\right) &= \frac{(b-a)^2}{4}, \quad C_1^{LG}\left(1,\infty,\frac{a+b}{2},w\right) = \frac{b-a}{2}, \\ C_1^{LG}\left(2,1,\frac{a+b}{2},w\right) &= \frac{(b-a)^3}{24}, \quad C_1^{LG}\left(2,\infty,\frac{a+b}{2},w\right) = \frac{(b-a)^2}{8}. \end{aligned}$$

Example 1.2 (THE CASE $w(t) = \frac{1}{\sqrt{1-t^2}}$) Applying the identity (1.26) to the special case of weight function $w(t) = \frac{1}{\sqrt{1-t^2}}$, the following identity is obtained:

$$\int_{-1}^{1} \frac{g(t)}{\sqrt{1-t^2}} dt = \pi g(x) + T_{n,w}^{1,C1}(x) + (-1)^n \int_{-1}^{1} W_{n,w}^{1,C1}(t,x) g^{(n)}(t)dt, \tag{1.42}$$

where

$$W_{n,w}^{1,C1}(t,x) = \begin{cases} w_{1n}^{1,C1}(t) = \frac{2^{-\frac{1}{2}}(t+1)^{n-\frac{1}{2}}}{(n-1)!} B\left(\frac{1}{2},n\right) F\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}+n;\frac{t+1}{2}\right), \\ \text{for } t \in [-1,x], \\ w_{2n}^{1,C1}(t) = (-1)^n \frac{2^{-\frac{1}{2}}(1-t)^{n-\frac{1}{2}}}{(n-1)!} B\left(\frac{1}{2},n\right) F\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}+n;\frac{1-t}{2}\right), \\ \text{for } t \in (x,1]. \end{cases}$$

Here, $B(u,v) = \int_0^1 s^{u-1}(1-s)^{v-1}ds$ is Beta function, and $F(\alpha,\beta,\gamma;z) = \frac{1}{B(\beta,\gamma-\beta)} \int_0^1 t^{\beta-1}$ $(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}dt$, for $\gamma > \beta > 0$ and $z < 1$ is the hypergeometric function. We also use the notation of the hypergeometric function when integral $\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}dt$ converges. Specially, when $\alpha < 0$, then any $z \in \mathbb{R}$ is allowed.

Specially, according to Remark 1.5 the condition

$$\int_a^b \frac{(x-s)}{\sqrt{1-s^2}} ds = 0,$$

implies $x = \frac{a+b}{2}$, so the generalization of the one-point Chebyshev-Gauss quadrature formula is carried out:

$$\int_{-1}^{1} \frac{g(t)}{\sqrt{1-t^2}} dt = \pi g(0) + T_{n,w}^{1,C1}(0) + (-1)^n \int_{-1}^{1} W_{n,w}^{1,C1}(t,0) g^{(n)}(t)dt, \tag{1.43}$$

where

$$T_{n,w}^{1,C1}(0) = \sum_{k=3}^{n} A_k^{1,C1}(0) g^{(k-1)}(0)$$

and

$$A_k^{1,C1}(0) = \frac{1}{(k-1)!}\int_{-1}^{1}\frac{s^{k-1}}{\sqrt{1-s^2}}ds.$$

If all the assumptions from Theorem 1.8 hold, then we have

$$\int_{-1}^{1}\frac{g(t)}{\sqrt{1-t^2}}dt = \pi g(x) + T_{2n,w}^{1,C1}(x) + A_{2n+1}^{1,C1}(x)\cdot g^{(2n)}(\eta). \tag{1.44}$$

Specially, for $x=0$ we have

$$\int_{-1}^{1}\frac{g(t)}{\sqrt{1-t^2}}dt = \pi g(0) + T_{2n,w}^{1,C1}(0) + \frac{1}{(2n)!}\int_{-1}^{1}\frac{s^{2n}}{\sqrt{1-s^2}}ds\cdot g^{(2n)}(\eta).$$

For $n=1$ one-point Chebyshev-Gauss quadrature formula is established:

$$\int_{-1}^{1}\frac{g(t)}{\sqrt{1-t^2}}dt = \pi g(0) + \frac{\pi}{4}g''(\eta).$$

If all the assumptions from Theorem 1.9 hold, then we have

$$\left|\int_{-1}^{1}\frac{g(t)}{\sqrt{1-t^2}}dt - \pi g(x) - T_{n,w}^{1,C1}(x)\right| \le C_1^{C1}(n,q,x,w)\cdot\|g^{(n)}\|_p,$$

where

$$C_1^{C1}(n,q,x,w) = \left[\int_{-1}^{x}|w_{1n}^{1,C1}(t)|^q dt + \int_{x}^{1}|w_{2n}^{1,C1}|^q dt\right]^{\frac{1}{q}}, \tag{1.45}$$

for $1\le q<\infty$, and

$$C_1^{C1}(n,\infty,x,w) = \max\left\{\sup_{t\in[-1,x]}|w_{1n}^{1,C1}(t)|,\ \sup_{t\in[x,1]}|w_{2n}^{1,C1}(t)|\right\}. \tag{1.46}$$

Specially, for $x=0$ we get

$$\left|\int_{-1}^{1}\frac{g(t)}{\sqrt{1-t^2}}dt - \pi g(0)\right| \le C_1^{C1}(n,q,0,w)\|g^{(n)}\|_p,\quad n=1,2,$$

where

$$C_1^{C1}(1,1,0,w) = 2,\quad C_1^{C1}(1,\infty,0,w) = \frac{\pi}{2},\quad C_1^{C1}(2,1,0,w) = \frac{\pi}{4}.$$

Example 1.3 (THE CASE $w(t)=\sqrt{1-t^2}$) Applying the identity (1.26) to the special case of weight function $w(t)=\sqrt{1-t^2}$, the following identity is obtained:

$$\int_{-1}^{1}g(t)\sqrt{1-t^2}dt = \frac{\pi}{2}g(x) + T_{n,w}^{1}(x) + (-1)^n\int_{-1}^{1}W_{n,w}^{1}(t,x)g^{(n)}(t)dt, \tag{1.47}$$

where

$$W_{n,w}^{1,C2}(t,x)=\begin{cases} w_{1n}^{1,C2}(t)=\frac{2^{\frac{1}{2}}(t+1)^{n+\frac{1}{2}}}{(n-1)!}B\left(\frac{3}{2},n\right)F\left(-\frac{1}{2},\frac{3}{2},\frac{3}{2}+n;\frac{t+1}{2}\right), \\ \text{for } t\in[-1,x], \\ w_{2n}^{1,C2}(t)=(-1)^n\frac{2^{\frac{1}{2}}(1-t)^{n+\frac{1}{2}}}{(n-1)!}B\left(\frac{3}{2},n\right)F\left(-\frac{1}{2},\frac{3}{2},\frac{3}{2}+n;\frac{1-t}{2}\right), \\ \text{for } t\in(x,1]. \end{cases}$$

Specially, according to Remark 1.5 the condition

$$\int_a^b (x-s)\sqrt{1-s^2}ds=0,$$

implies $x=0$, so the generalization of the one-point Chebyshev-Gauss quadrature formula of the second kind is carried out:

$$\int_{-1}^1 g(t)\sqrt{1-t^2}dt=\frac{\pi}{2}g(0)+T_{n,w}^{1,C2}(0)+(-1)^n\int_{-1}^1 W_{n,w}^{1,C2}(t,0)g^{(n)}(t)dt, \tag{1.48}$$

where

$$T_{n,w}^{1,C2}(0)=\sum_{k=3}^{n}A_k^{1,C2}(0)g^{(k-1)}(0)$$

and

$$A_k^{1,C2}(0)=\frac{1}{(k-1)!}\int_{-1}^1 s^{k-1}\sqrt{1-s^2}ds.$$

If all the assumptions from Theorem 1.8 hold, then we have

$$\int_{-1}^1 g(t)\sqrt{1-t^2}dt=\frac{\pi}{2}g(x)+T_{2n,w}^{1,C2}(x)+A_{2n+1}^{1,C2}(x)\cdot g^{(2n)}(\eta). \tag{1.49}$$

Specially, for $x=0$ we have

$$\int_{-1}^1 g(t)\sqrt{1-t^2}dt=\frac{\pi}{2}g(0)+T_{2n,w}^{1,C2}(0)+\frac{1}{(2n)!}\int_{-1}^1 s^{2n}\sqrt{1-s^2}ds\cdot g^{(2n)}(\eta).$$

For $n=1$ one-point Chebyshev-Gauss quadrature formula of the second kind is obtained:

$$\int_{-1}^1 g(t)\sqrt{1-t^2}dt=\frac{\pi}{2}g(0)+\frac{\pi}{16}g''(\eta).$$

If all the assumptions from Theorem 1.9 hold, then we have

$$\left|\int_{-1}^1 g(t)\sqrt{1-t^2}dt-\frac{\pi}{2}g(x)-T_{n,w}^{1,C2}(x)\right|\leq C_1^{C2}(n,q,x,w)\cdot\|g^{(n)}\|_p,$$

where

$$C_1^{C2}(n,q,x,w) = \left[\int_{-1}^{x} |w_{1n}^{1,C2}(t)|^q dt + \int_{x}^{1} |w_{2n}^{1,C2}|^q dt \right]^{\frac{1}{q}}, \tag{1.50}$$

for $1 \le q < \infty$, and

$$C_1^{C2}(n,\infty,x,w) = \max \left\{ \sup_{t\in[-1,x]} |w_{1n}^{1,C2}(t)|, \sup_{t\in[x,1]} |w_{2n}^{1,C2}(t)| \right\}. \tag{1.51}$$

Specially, for $x = 0$ we get

$$\left| \int_{-1}^{1} g(t)\sqrt{1-t^2}dt - \frac{\pi}{2}g(0) \right| \le C_1^{C2}(n,q,0,w)\|g^{(n)}\|_p, \quad n = 1,2,$$

where

$$C_1^{C2}(1,1,0,w) = \frac{2}{3}, \quad C_1^{C2}(1,\infty,0,w) = \frac{\pi}{4}, \quad C_1^{C2}(2,1,0,w) = \frac{\pi}{16}.$$

Example 1.4 (THE CASE $w(t) = \sqrt{\frac{1-t}{1+t}}$) Applying the identity (1.26) to the special case of weight function $w(t) = \sqrt{\frac{1-t}{1+t}}$, the following identity is obtained:

$$\int_{-1}^{1} g(t)\sqrt{\frac{1-t}{1+t}}dt = \pi g(x) + T_{n,w}^{1,J3}(x) + (-1)^n \int_{-1}^{1} W_{n,w}^{1,J3}(t,x)g^{(n)}(t)dt, \tag{1.52}$$

where

$$W_{n,w}^{1,J3}(t,x) = \begin{cases} w_{1n}^{1,J3}(t) = \frac{2^{\frac{1}{2}}(t+1)^{n+\frac{1}{2}}}{(n-1)!} B\left(\frac{3}{2},n\right) F\left(-\frac{1}{2},\frac{3}{2},\frac{3}{2}+n;\frac{t+1}{2}\right), \\ \text{for } t \in [-1,x], \\ w_{2n}^{1,J3}(t) = (-1)^n \frac{2^{\frac{1}{2}}(1-t)^{n+\frac{1}{2}}}{(n-1)!} B\left(\frac{3}{2},n\right) F\left(-\frac{1}{2},\frac{3}{2},\frac{3}{2}+n;\frac{1-t}{2}\right), \\ \text{for } t \in (x,1]. \end{cases}$$

Specially, according to Remark 1.5 the condition

$$\int_a^b (x-s)\sqrt{\frac{1-s}{1+s}}ds = 0,$$

implies $x = -\frac{1}{2}$, so the generalization of the one-point Jacobi-Gauss quadrature formula is carried out:

$$\int_{-1}^{1} g(t)\sqrt{\frac{1-t}{1+t}}dt = \pi g\left(-\frac{1}{2}\right) + T_{n,w}^{1,J3}\left(-\frac{1}{2}\right)$$

$$+ (-1)^n \int_{-1}^{1} W_{n,w}^{1,J3}\left(t,-\frac{1}{2}\right) g^{(n)}(t)dt,$$

where

$$T_{n,w}^{1,J3}\left(-\frac{1}{2}\right) = \sum_{k=3}^{n} A_k^{1,J3}\left(-\frac{1}{2}\right) g^{(k-1)}\left(-\frac{1}{2}\right)$$

and

$$A_k^{1,J3}\left(-\frac{1}{2}\right) = \frac{1}{(k-1)!}\int_{-1}^{1}\left(\frac{1}{2}+s\right)^{k-1}\sqrt{\frac{1-s}{1+s}}ds.$$

If all the assumptions from Theorem 1.8 hold, then we have

$$\int_{-1}^{1} g(t)\sqrt{\frac{1-t}{1+t}}dt = \pi g(x) + T_{2n,w}^{1,J3}(x) + A_{2n+1}^{1,J3}(x)\cdot g^{(2n)}(\eta). \tag{1.53}$$

Specially, for $x=-\frac{1}{2}$ we have

$$\int_{-1}^{1} g(t)\sqrt{\frac{1-t}{1+t}}dt = \pi g\left(-\frac{1}{2}\right) + T_{2n,w}^{1,J3}\left(-\frac{1}{2}\right) + \frac{1}{(2n)!}\int_{-1}^{1} s^{2n}\sqrt{\frac{1-s}{1+s}}ds\cdot g^{(2n)}(\eta).$$

For $n=1$ one-point Jacobi-Gauss quadrature formula is obtained:

$$\int_{-1}^{1} g(t)\sqrt{\frac{1-t}{1+t}}dt = \pi g\left(-\frac{1}{2}\right) + \frac{\pi}{8}g''(\eta).$$

If all the assumptions from Theorem 1.9 hold, then we have

$$\left|\int_{-1}^{1} g(t)\sqrt{\frac{1-t}{1+t}}dt - \pi g(x) - T_{n,w}^{1,J3}(x)\right| \le C_1^{J3}(n,q,x,w)\cdot\|g^{(n)}\|_p, \tag{1.54}$$

where

$$C_1^{J3}(n,q,x,w) = \left[\int_{-1}^{x}|w_{1n}^{1,J3}(t)|^q dt + \int_{x}^{1}|w_{2n}^{1,J3}|^q dt\right]^{\frac{1}{q}}, \tag{1.55}$$

for $1\le q<\infty$, and

$$C_1^{J3}(n,\infty,x,w) = \max\left\{\sup_{t\in[-1,x]}|w_{1n}^{1,J3}(t)|,\ \sup_{t\in[x,1]}|w_{2n}^{1,J3}(t)|\right\}. \tag{1.56}$$

Specially, for $x=-\frac{1}{2}$ we get

$$\left|\int_{-1}^{1} g(t)\sqrt{\frac{1-t}{1+t}}dt - \pi g\left(-\frac{1}{2}\right)\right| \le C_1^{J3}\left(n,q,-\frac{1}{2},w\right)\|g^{(n)}\|_p, \quad n=1,2,$$

where

$$C_1^{J3}(1,\infty,0,w) = \frac{\pi}{3}+\frac{\sqrt{3}}{2}\approx 1.91322, \quad C_1^{J3}(2,1,0,w) = \frac{\pi}{8}.$$

Example 1.5 (THE CASE $w(t) = e^{-t^2}$) Applying the identity (1.26) to the special case of weight function $w(t) = e^{-t^2}$, the following identity is obtained:

$$\int_{-\infty}^{\infty} g(t)e^{-t^2}dt = \sqrt{\pi}g(x) + T_{n,w}^{1,HG}(x) + (-1)^n \int_{-\infty}^{\infty} W_{n,w}^{1,HG}(t,x)g^{(n)}(t)dt, \tag{1.57}$$

where

$$W_{n,w}^{1,HG}(t,x) = \begin{cases} w_{1n}^{1,HG}(t) = \frac{1}{(n-1)!}\int_{-\infty}^{t}(t-s)^{n-1}e^{-s^2}ds, & \text{for } t \in (-\infty, x], \\ w_{2n}^{1,HG}(t) = -\frac{1}{(k-1)!}\int_{t}^{\infty}(t-s)^{k-1}e^{-s^2}ds, & \text{for } t \in (x, \infty). \end{cases}$$

Specially, according to Remark 1.5 the condition

$$\int_{-\infty}^{\infty} (x-s)e^{-s^2}ds = 0,$$

implies $x = 0$, so the generalization of the one-point Hermite-Gauss quadrature formula is carried out:

$$\int_{-\infty}^{\infty} g(t)e^{-t^2}dt = \sqrt{\pi}g(0) + T_{n,w}^{1,HG}(0) + (-1)^n \int_{-\infty}^{\infty} W_{n,w}^{1,HG}(t,0)g^{(n)}(t)dt, \tag{1.58}$$

where

$$T_{n,w}^{1,HG}(0) = \sum_{k=3}^{n} A_k^{1,HG}(0)g^{(k-1)}(0)$$

and

$$A_k^{1,HG}(0) = \frac{1}{(k-1)!}\int_{-\infty}^{\infty} s^{k-1}e^{-s^2}ds.$$

If all the assumptions from Theorem 1.8 hold, then we have

$$\int_{-\infty}^{\infty} g(t)e^{-t^2}dt = \sqrt{\pi}g(x) + T_{2n,w}^{1,HG}(x) + A_{2n+1}^{1,HG}(x) \cdot g^{(2n)}(\eta). \tag{1.59}$$

Specially, for $x = 0$ we have

$$\int_{-\infty}^{\infty} g(t)e^{-t^2}dt = \sqrt{\pi}g(0) + T_{2n,w}^{1,HG}(0) + A_{2n+1}^{1,HG}(0) \cdot g^{(2n)}(\eta).$$

For $n = 1$ one-point Hermite-Gauss quadrature formula is obtained:

$$\int_{-\infty}^{\infty} g(t)e^{-t^2}dt = \sqrt{\pi}g(0) + \frac{\sqrt{\pi}}{4}g''(\eta).$$

If all the assumptions from Theorem 1.9 hold, then we have

$$\left|\int_{-\infty}^{\infty} g(t)e^{-t^2}dt - \sqrt{\pi}g(x) - T_{n,w}^{1,HG}(x)\right| \le C_1^{HG}(n,q,x,w) \cdot \|g^{(n)}\|_p,$$

where

$$C_1^{HG}(n,q,x,w) = \left[\int_{-\infty}^{x} |w_{1n}^{1,HG}(t)|^q dt + \int_{x}^{\infty} |w_{2n}^{1,HG}|^q dt\right]^{\frac{1}{q}}, \tag{1.60}$$

for $1 \leq q < \infty$, and

$$C_1(n,\infty,x,w) = \max\left\{\sup_{t\in(-\infty,x]} |w_{1n}^{1,HG}(t)|,\ \sup_{t\in[x,\infty)} |w_{2n}^{1,HG}(t)|\right\}. \tag{1.61}$$

Specially, for $x = 0$ we get

$$\left|\int_{-\infty}^{\infty} g(t)e^{-t^2}dt - \sqrt{\pi}g(0)\right| \leq C_1^{HG}(n,q,0,w)\|g^{(n)}\|_p, \quad n = 1,2,$$

where

$$C_1^{HG}(1,1,0,w) = 1, \quad C_1^{HG}(1,\infty,0,w) = \frac{\sqrt{\pi}}{2}, \quad C_1^{HG}(2,1,0,w) = \frac{\sqrt{\pi}}{4}.$$

1.3 Application to the two-point quadrature formulae

In this section we shall establish general two-point quadrature formulae. The families of the quadrature formulae of the following type are considered:

$$\int_a^b f(t)dt = \frac{b-a}{2}[f(x) + f(a+b-x)] + E(f,x) \tag{1.62}$$

and

$$\int_a^b w(t)f(t)dt = A(x)f(x) + B(x)f(a+b-x) + E(f,x,w), \tag{1.63}$$

where $x \in [a, \frac{a+b}{2}]$, $w : [a,b] \to \mathbb{R}$ is integrable function, $A(x)$ and $B(x)$ are coefficients such that $A(x) + B(x) = \int_a^b w(t)dt$, while $E(f,x)$ and $E(f,x,w)$ are remainders for non-weighted and weighted case. The family of non-weighted formulae (1.62) was considered by Guessab i Schmeisser [65]. They established sharp estimates for the remainder under various regularity conditions. A number of error estimates for the identity (1.62) are obtained, and various examples of the general two-point quadrature formula are given in [73]. Quadrature formulae of the type (1.63) and their error estimates have been developed in [71].

First we shall look at the Guessab and Schmeisser theroem:

Theorem 1.10 *Let f be a function defined on $[a,b]$ and having there a piecewise continuous $n-th$ derivative. Let Q_n be any monic polynomial of degree n such that $Q_n(t) \equiv (-1)^n Q_n(a+b-t)$. Define*

$$K_n(t) = \begin{cases} (t-a)^n, & \text{for } a \le t \le x \\ Q_n(t), & \text{for } x < t \le a+b-x \\ (t-b)^n, & \text{for } a+b-x < t \le b. \end{cases} \tag{1.64}$$

Then, for the remainder in (1.62), we have

$$\begin{aligned} E(f;x) &= \sum_{\nu=1}^{n-1} \left[\frac{(x-a)^{\nu+1}}{(\nu+1)!} - \frac{Q_n^{(n-\nu-1)}(x)}{n!} \right] \frac{f^{(\nu)}(a+b-x) + (-1)^{\nu} f^{(\nu)}(x)}{b-a} \\ &+ \frac{(-1)^n}{n!(b-a)} \int_a^b K_n(t) f^{(n)}(t) \mathrm{d}t. \end{aligned} \tag{1.65}$$

Let us introduce some notes. For $k = 1, 2, \ldots, n$ let us define

$$A_k^2(x) := \frac{(x-a)^k}{k!} - \frac{Q_n^{(n-k)}(x)}{n!}. \tag{1.66}$$

Specially, from the characteristic of the polynomial Q_n we have $A_1^2(x) = \frac{b-a}{2}$. Further, let us define

$$\begin{aligned} T_1^2(x) &:= 0, \quad \text{for } n = 1 \\ T_n^2(x) &:= \sum_{k=2}^{n} A_k^2(x) \cdot \left[f^{(k-1)}(a+b-x) + (-1)^{k-1} f^{(k-1)}(x) \right], \quad \text{for } n \ge 2 \end{aligned} \tag{1.67}$$

Now the identity (1.65) becomes

$$\int_a^b f(t)dt = \frac{b-a}{2} [f(x) + f(a+b-x)] + T_n^2(x) + \frac{(-1)^n}{n!} \int_a^b K_n(t) f^{(n)}(t) dt. \tag{1.68}$$

The identity (1.68) will be used for the derivation of the two-point quadrature formula. Those general formulae will approximate the integral $\int_a^b f(t)dt$ with the values of the integrand f and higher ordered derivatives in nodes x and $a+b-x$. For $x = a, \frac{2a+b}{3}, \frac{3a+b}{4}, \frac{a+b}{2}$ and appropriate polynomials Q_n we shall obtain the generalization of the trapezoid, Newton-Cotes two-point formula, Maclaurin two-point formula and midpoint formula, respectively.

Remark 1.6 In [73] it is shown that the identity (1.65) is a special case of the general integral identity with monic polynimials (1.9).

Now we establish the general inequality using L_p norms for $1 \le p \le \infty$.

Theorem 1.11 *Let $f : [a,b] \to \mathbb{R}$ be a function such that $f^{(n)} \in L_p[a,b]$ for $1 \le p \le \infty$ and some $n \in \mathbb{N}$. Let K_n be defined by (1.64). Then we have for $1 < p \le \infty$*

$$\left| \int_a^b f(t)dt - \frac{b-a}{2}[f(x)+f(a+b-x)] - T_n^2(x) \right| \le C_2(n,q,x) \cdot \|f^{(n)}\|_p, \tag{1.69}$$

where

$$C_2(n,q,x) = \frac{2^{\frac{1}{q}}}{n!} \left[\frac{(x-a)^{nq+1}}{nq+1} + \int_x^{\frac{a+b}{2}} |Q_n(t)|^q dt \right]^{\frac{1}{q}}, \tag{1.70}$$

for $1 \le q < \infty$, and

$$C_2(n,\infty,x) = \frac{1}{n!} \cdot \max \left\{ (x-a)^n, \sup_{t \in [x, \frac{a+b}{2}]} |Q_n(t)| \right\}. \tag{1.71}$$

The inequalities are sharp for $1 < p \le \infty$ and the best possible for $p = 1$. Equality is attained for

$$f(t) = M \cdot f_*(t) + c_0 + c_1 t + \ldots + c_{n-1} t^{n-1},$$

with $M, c_0, c_1, \ldots, c_{n-1} \in \mathbb{R}$, and for $1 < p < \infty$

$$f_*(t) = \begin{cases} \frac{(t-a)^{nq}}{[n(q-1)+1][n(q-1)+2]\ldots nq}, & \text{for } a \le t \le x \\ \int_x^t \frac{(t-\xi)^{n-1}}{(n-1)!} \cdot |Q_n(\xi)|^{\frac{1}{p-1}} \cdot \operatorname{sgn} Q_n(\xi) \mathrm{d}\xi, & \text{for } x < t \le a+b-x \\ \frac{(b-t)^{nq}}{[n(q-1)+1][n(q-1)+2]\ldots nq}, & \text{for } a+b-x < t \le b, \end{cases}$$

while for $p = \infty$

$$f_*(t) = \begin{cases} \frac{(t-x)^n}{n!}, & \text{for } a \le t \le x \\ \int_x^t \frac{(t-\xi)^{n-1}}{(n-1)!} \cdot \operatorname{sgn} Q_n(\xi) \mathrm{d}\xi, & \text{for } x < t \le a+b-x \\ \frac{(a+b-x-t)^n}{n!}, & \text{for } a+b-x < t \le b. \end{cases} \tag{1.72}$$

Proof. This is a special case of the Theorem 1.4. □

Remark 1.7 If we take in previous theorem function f such that $f^{(n-1)} \in Lip_M$, then the proof is equal as for the case $p = \infty$. Specially, for

$$Q_n(t) = \frac{(2x-a-b)^n}{2^{2n}} \cdot U_n \left(\frac{2t-a-b}{2x-a-b} \right),$$

where U_n is the $n-$th Chebyshev polynomial of the second kind, we get the Guessab and Schmeisser's result [65].

Let us introduce a note $R_2(f,n,x)$ for the remainder term in the identity (1.65):

$$R_2(f,n,x) := \frac{(-1)^n}{n!}\int_a^b K_n(t)f^{(n)}(t)dt.$$

Theorem 1.12 *Let $f:[a,b]\to\mathbb{R}$ be a function such that $f^{(2n)}$ is continuous for some $n\geq 1$ and let us assume that Q_{2n} is a monic polynomial of degree $2n$ such that $Q_{2n}(t)\geq 0$ for $t\in[x,\frac{a+b}{2}]$ and $Q_{2n}(t)=Q_{2n}(a+b-t)$. Further, let*

$$K_{2n}(t)=\begin{cases}(t-a)^{2n}, & \text{for } a\leq t\leq x\\ Q_{2n}(t), & \text{for } x<t\leq a+b-x\\ (t-b)^{2n}, & \text{for } a+b-x<t\leq b.\end{cases} \tag{1.73}$$

Then there exists $\eta\in[a,b]$ such that

$$R_2(f,2n,x)=\frac{2}{(2n)!}\cdot\left[\frac{(x-a)^{2n+1}}{2n+1}+\int_x^{\frac{a+b}{2}}Q_{2n}(t)dt\right]\cdot f^{(2n)}(\eta). \tag{1.74}$$

Proof. Since $K_{2n}(t)\geq 0$ for $t\in[a,b]$, we apply mean value theorem so we get

$$\frac{1}{(2n)!}\int_a^b K_{2n}(t)f^{(2n)}(t)dt=f^{(2n)}(\eta)\cdot\frac{1}{(2n)!}\int_a^b K_{2n}(t)dt.$$

Theorem 1.11 implies

$$\int_a^b K_{2n}(t)dt=2\cdot\left[\frac{(x-a)^{2n+1}}{2n+1}+\int_x^{\frac{a+b}{2}}Q_{2n}(t)dt\right],$$

so the assertion follows. □

When apply (1.74) to the remainder in (1.68) for $n=1$, the following identity is obtained:

$$\int_a^b f(t)dt=\frac{b-a}{2}[f(x)+f(a+b-x)]+\left[\frac{(x-a)^3}{3}+\int_x^{\frac{a+b}{2}}Q_2(t)dt\right]\cdot f''(\eta). \tag{1.75}$$

Now we explore the case when $f^{(2n)}$ doesn't change sign on $[a,b]$.

Theorem 1.13 *If $f:[a,b]\to\mathbb{R}$ is such that $f^{(2n)}$ is a continuous function for some $n\in\mathbb{N}$ which doesn't change sign on $[a,b]$, and $Q_{2n}(t)\geq 0$, then there exists $\theta\in[0,1]$ such that*

$$R(f,2n,x)=C_2(2n,\infty,x)\cdot\theta\cdot[f^{(2n-1)}(b)-f^{(2n-1)}(a)]. \tag{1.76}$$

Proof. Suppose that $f^{(2n)}(t)\geq 0$, for $t\in[a,b]$. Then we have

$$0\leq R(f,2n,x)\leq C_2(2n,\infty,x)\cdot\int_a^b f^{(2n)}(t)dt,$$

so there exists a point $\theta\in[0,1]$ such that

$$R(f,2n,x)=C_2(2n,\infty,x)\cdot\theta\cdot[f^{(2n-1)}(b)-f^{(2n-1)}(a)].$$

The case $f^{(2n)}(t)\leq 0$ follows similarly. □

1.3.1 Trapezoid formula

Let the kernel K_n be as in (1.64), where

$$Q_n(t) = \left(t - \frac{a+b}{2}\right)^n - \frac{(b-a)^2 n(n-1)}{8} \cdot \left(t - \frac{a+b}{2}\right)^{n-2}. \tag{1.77}$$

We compute

$$A_k^{2,T}(x) := A_k^2(x) = \frac{(x-a)^k - (x-\frac{a+b}{2})^{k-2} \cdot \left[(x-\frac{a+b}{2})^2 - \frac{(b-a)^2 k(k-1)}{8}\right]}{k!} \tag{1.78}$$

and

$$T_n^{2,T}(x) \;:=\; T_n^2(x) = \sum_{k=2}^{n} A_k^{2,T}(x)\left[f^{(k-1)}(a+b-x) + (-1)^{k-1} f^{(k-1)}(x)\right], \tag{1.79}$$

so the identity (1.68) becomes

$$\int_a^b f(t)dt = \frac{b-a}{2}\left[f(x) + f(a+b-x)\right] + T_n^{2,T}(x) + \frac{(-1)^n}{n!}\int_a^b K_n(t) f^{(n)}(t)dt. \tag{1.80}$$

This formula will be exact for all polynomials of degree ≤ 1 if $A_2^{2,T}(x) = 0$. The solution of this equation is $x = a$ so we have

$$A_k^{2,T}(a) = \frac{(-1)^{k-1}(b-a)^k(k+1)(2-k)}{2^{k+1}k!},$$

and

$$T_n^{2,T}(a) = \sum_{k=3}^{n} A_k^{2,T}(a)\left[f^{(k-1)}(b) + (-1)^{k-1} f^{(k-1)}(a)\right]. \tag{1.81}$$

Therefore we get

$$\int_a^b f(t)dt = \frac{b-a}{2} \cdot \left[f(a) + f(b)\right] + T_n^{2,T}(a) + \frac{(-1)^n}{n!}\int_a^b K_n(t) f^{(n)}(t)dt. \tag{1.82}$$

Corollary 1.1 *Assume $1 \leq p, q \leq \infty$ are such that $\frac{1}{p} + \frac{1}{q} = 1$. If $f : [a,b] \to \mathbb{R}$ is n times differentiable function such that $f^{(n)}$ is piecewise continuous on $[a,b]$ and $f^{(n)} \in L_p[a,b]$, then we have*

$$\left|\int_a^b f(t)dt - \frac{b-a}{2}\left[f(x) + f(a+b-x)\right] - T_n^{2,T}(x)\right| \leq C_2^T(n,q,x) \cdot \|f^{(n)}\|_p, \tag{1.83}$$

where for $1 \leq q < \infty$ and $n > 2$

$$C_2^T(n,q,x) = \frac{1}{n!}\left[\frac{2(x-a)^{nq+1}}{nq+1} + \frac{(b-a)^{2q}(a+b-2x)^{nq-2q+1} n^q (n-1)^q}{2^{nq+q}(nq-2q+1)}\right.$$

$$\cdot F\left(-q,\frac{nq-2q+1}{2},\frac{nq-2q+3}{2};\frac{8}{(b-a)^2n(n-1)}\left(\frac{a+b}{2}-x\right)^2\right)\Bigg]^{\frac{1}{q}},$$

$$C_2^T(2,q,x)=$$

$$\frac{1}{2}\left[\frac{2(x-a)^{2q+1}}{2q+1}+2(b-a)^{2q+1}\cdot\left[\frac{B(q+1,q+1)}{2}-B_{\frac{x-a}{b-a}}(q+1,q+1)\right]\right]^{\frac{1}{q}},$$

$$C_2^T(1,q,x)=\left[\frac{2}{q+1}\cdot\left[(x-a)^{q+1}+\left(\frac{a+b}{2}-x\right)^{q+1}\right]\right]^{\frac{1}{q}}. \tag{1.84}$$

Further, for $q=\infty$ nd $n\neq 2$

$$\begin{aligned} C_2^T(n,\infty,x) &= \frac{|a_n(x)-b_n(x)|+|a_n(x)+b_n(x)|}{2\cdot n!}, \\ C_2^T(2,\infty,x) &= \frac{(b-a)^2}{8}, \end{aligned} \tag{1.85}$$

with

$$a_n(x) = (x-a)^n, \tag{1.86}$$

$$b_n(x) = \left(\frac{a+b}{2}-x\right)^{n-2}\cdot\left[\frac{(b-a)^2n(n-1)}{8}-\left(\frac{a+b}{2}-x\right)^2\right].$$

Inequality is the best possible for $p=1$, and sharp for $1<p\leq\infty$.

Proof. The proof follows from Theorem 1.11 □

Corollary 1.2 *Let $f:[a,b]\to\mathbb{R}$ be a function such that $f^{(n)}$ is continuous for some even $n\geq 2$ and let Q_n be a polynomial defined with (1.77). Then there exists $\eta\in[a,b]$ such that*

$$\int_a^b f(t)dt=\frac{b-a}{2}[f(a)+f(b)]+T_n^{2,T}(a)-C_2^T(n,1,a)\cdot f^{(n)}(\eta), \tag{1.87}$$

where for $n>2$

$$C_2^T(n,1,a)=\frac{1}{(n-1)!}\cdot\left(\frac{b-a}{2}\right)^{n+1}\cdot F\left(-1,\frac{n-1}{2},\frac{n+1}{2};\frac{2}{n(n-1)}\right), \tag{1.88}$$

and

$$C_2^T(2,1,a)=\frac{(b-a)^3}{12}.$$

Proof. Polynomial Q_n satisfies

$$Q_n(t)=\left(t-\frac{a+b}{2}\right)^{n-2}\cdot\left[\left(t-\frac{a+b}{2}\right)^2-\frac{(b-a)^2n(n-1)}{8}\right]. \tag{1.89}$$

For even $n \geq 2$ and for $t \in [a,b]$ we have

$$\left(t-\frac{a+b}{2}\right)^2-\frac{(b-a)^2n(n-1)}{8}\leq 0,$$

and

$$\left(t-\frac{a+b}{2}\right)^{n-2}\geq 0,$$

which implies $K_n(t) \leq 0$, for $t \in [a,b]$. The assertion follows from the mean value theorem. □

Remark 1.8 Formula (1.75) in this case becomes

$$\int_a^b f(t)dt=\frac{b-a}{2}[f(a)+f(b)]-\frac{(b-a)^3}{12}\cdot f''(\eta), \tag{1.90}$$

which is well-known trapezoid formula, so the identity (1.87) represents generalization of the trapezoid quadrature formula.

Remark 1.9 Put $x=a$ and $n=2$ in Corollary 1.1. We obtain the known inequalities related to the trapezoid formula ([56])

$$\left|\int_a^b f(t)dt-\frac{b-a}{2}[f(a)+f(b)]\right|\leq\begin{cases}\frac{(b-a)^{2+\frac{1}{q}}}{2}[B(q+1,q+1)]^{\frac{1}{q}}\|f''\|_p,\\ 1<p\leq\infty\\ \\ \frac{(b-a)^2}{8}\|f''\|_1, p=1.\end{cases}$$

Remark 1.10 Weighted version of the trapezoid formula and related inequalities are established in [74].

1.3.2 Newton-Cotes two-point formula

Let the kernel K_n be defined with (1.64), where

$$Q_n(t)=\left(t-\frac{a+b}{2}\right)^n+\frac{(b-a)^2n(n-1)}{24}\cdot\left(t-\frac{a+b}{2}\right)^{n-2}. \tag{1.91}$$

We compute

$$A_k^{2,NC}(x):=A_k^2(x)=\frac{(x-a)^k-(x-\frac{a+b}{2})^{k-2}\cdot\left[(x-\frac{a+b}{2})^2+\frac{(b-a)^2k(k-1)}{24}\right]}{k!}$$

and

$$T_n^{2,NC}(x)\ :=\ T_n^2(x)=\sum_{k=2}^n A_k^{2,NC}(x)\left[f^{(k-1)}(a+b-x)+(-1)^{k-1}f^{(k-1)}(x)\right],$$

so the identity (1.68) becomes

$$\int_a^b f(t)dt = \frac{b-a}{2}\left[f(x)+f(a+b-x)\right]+T_n^{2,NC}(x)+\frac{(-1)^n}{n!}\int_a^b K_n(t)f^{(n)}(t)dt. \quad (1.92)$$

This formula will be exact for all polynomials of degree ≤ 1 if $A_2^{2,NC}(x)=0$. The solution of this equation is $x=\frac{2a+b}{3}$ so we have

$$A_k^{2,NC}\left(\frac{2a+b}{3}\right)=\frac{(b-a)^k\left[2^{k+1}+(-1)^{k-1}\left(3k^2-3k+2\right)\right]}{2\cdot 6^k k!},$$

and

$$T_n^{2,NC}(a)=\sum_{k=3}^{n}A_k^{2,NC}\left(\frac{2a+b}{3}\right)\left[f^{(k-1)}\left(\frac{2a+b}{3}\right)+(-1)^{k-1}f^{(k-1)}\left(\frac{2a+b}{3}\right)\right].$$

Therefore we get

$$\begin{aligned}\int_a^b f(t)dt &= \frac{b-a}{2}\cdot\left[f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right]\\ &+ T_n^{2,NC}\left(\frac{2a+b}{3}\right)+\frac{(-1)^n}{n!}\int_a^b K_n(t)f^{(n)}(t)dt. \quad (1.93)\end{aligned}$$

Corollary 1.3 *Assume $1\le p,q\le\infty$ are such that $\frac{1}{p}+\frac{1}{q}=1$. If $f:[a,b]\to\mathbb{R}$ is n times differentiable function such that $f^{(n)}$ is piecewise continuous on $[a,b]$ and $f^{(n)}\in L_p[a,b]$, then we have*

$$\left|\int_a^b f(t)dt-\frac{b-a}{2}[f(x)+f(a+b-x)]-T_n^{2,NC}(x)\right|\le C_2^{NC}(n,q,x)\cdot\|f^{(n)}\|_p, \quad (1.94)$$

where for $1\le q<\infty$ and $n\ge 2$

$$\begin{aligned}C_2^{NC}(n,q,x)&=\frac{1}{n!}\left[\frac{2(x-a)^{nq+1}}{nq+1}+\frac{(b-a)^{2q}(a+b-2x)^{nq-2q+1}n^q(n-1)^q}{2^{nq}6^q(nq-2q+1)}\right.\\ &\left.\cdot F\left(-q,\frac{nq-2q+1}{2},\frac{nq-2q+3}{2};-\frac{24}{(b-a)^2n(n-1)}\left(\frac{a+b}{2}-x\right)^2\right)\right]^{\frac{1}{q}},\\ C_2^{NC}(1,q,x)&=\left[\frac{2}{q+1}\left((x-a)^{q+1}+\left(\frac{a+b}{2}-x\right)^{q+1}\right)\right]^{\frac{1}{q}}.\end{aligned}$$

Further, for $q=\infty$ and $n\ne 1$ we have

$$C_2^{NC}(n,\infty,x) = \frac{|a_n(x)-b_n(x)|+|a_n(x)+b_n(x)|}{2\cdot n!},$$

where

$$a_n(x) = (x-a)^n,$$

$$b_n(x) = \left(\frac{a+b}{2} - x\right)^{n-2} \cdot \left[\frac{(b-a)^2 n(n-1)}{24} + \left(\frac{a+b}{2} - x\right)^2\right].$$

Inequality is the best possible for $p = 1$, and sharp for $1 < p \leq \infty$.

Proof. The assertion follows from Theorem 1.11 □

Corollary 1.4 *Let $f : [a,b] \to \mathbb{R}$ be a function such that $f^{(n)}$ is continuous for some even $n \geq 2$ and let Q_n be a polynomial defined with (1.91). Then there exists $\eta \in [a,b]$ such that*

$$\begin{aligned}\int_a^b f(t)dt &= \frac{b-a}{2}\left[f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right] + T_n^{2,NC}\left(\frac{2a+b}{3}\right) \\ &+ C_2^{NC}\left(n, 1, \frac{2a+b}{3}\right) \cdot f^{(n)}(\eta),\end{aligned} \tag{1.95}$$

where

$$\begin{aligned}C_2^{NC}\left(n, 1, \frac{2a+b}{3}\right) &= \frac{(b-a)^{n+1}}{3^n \cdot n!} \cdot \left[\frac{2}{3(n+1)}\right. \\ &+ \left.\frac{n}{2^{n+1}} \cdot F\left(-1, \frac{n-1}{2}, \frac{n+1}{2}; \frac{2}{3n(n+1)}\right)\right].\end{aligned}$$

Proof. It is easy to check that $Q_n(t) \geq 0$, for $t \in \left[\frac{2a+b}{3}, \frac{a+2b}{3}\right]$, and $n \geq 2$, so the assertion follows from Theorem 1.12 □

Remark 1.11 Formula (1.75) in this case becomes

$$\int_a^b f(t)dt = \frac{b-a}{2}\left[f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right] + \frac{(b-a)^3}{36} \cdot f''(\eta), \tag{1.96}$$

which is the well-known Newton-Cotes two-point formula. Therefore the identity (1.95) represents the generalization of the Newton-Cotes two-point quadrature formula.

Remark 1.12 Put $x = \frac{2a+b}{3}$ into the Corollary 1.3 so the best possible and sharp inequalities for Newton-Cotes two-point formula are obtained.

Remark 1.13 More on Newton-Cotes twopoint quadrature formula can be found in [82].

1.3.3 Maclaurin two-point formula. Perturbed trapezoid formula. Midpoint formula

Let the kernel K_n be defined with (1.64), where

$$Q_n(t) = \left(t - \frac{a+b}{2}\right)^n. \tag{1.97}$$

We compute

$$A_k^{2,PT}(x) := A_k^2(x) = \frac{(x-a)^k - (x - \frac{a+b}{2})^k}{k!} \tag{1.98}$$

and

$$T_n^{2,PT}(x) := T_n^2(x) = \sum_{k=2}^{n} A_k^{2,PT}(x)\left[f^{(k-1)}(a+b-x)+(-1)^{k-1}f^{(k-1)}(x)\right],$$

so the identity (1.68) becomes

$$\int_a^b f(t)dt = \frac{b-a}{2}[f(x)+f(a+b-x)]+T_n^{2,PT}(x)+\frac{(-1)^n}{n!}\int_a^b K_n(t)f^{(n)}(t)dt. \quad (1.99)$$

The solution of this equation is is $x=\frac{3a+b}{4}$ so we have

$$A_k^{2,PT}\left(\frac{3a+b}{4}\right)=\begin{cases}\frac{2(b-a)^k}{4^k k!}, & k \text{ odd,}\\ 0, & k \text{ even.}\end{cases}$$

and

$$\begin{aligned} T_n^{2,PT}\left(\frac{3a+b}{4}\right) &= \sum_{k=3}^{n} A_k^{2,PT}\left(\frac{3a+b}{4}\right)\left[f^{(k-1)}\left(\frac{3a+b}{4}\right)\right.\\ &\quad \left. +\ (-1)^{k-1}f^{(k-1)}\left(\frac{3a+b}{4}\right)\right]. \end{aligned}$$

Therefore we have

$$\begin{aligned} \int_a^b f(t)dt &= \frac{b-a}{2}\cdot\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\\ &\quad + T_n^{2,PT}\left(\frac{3a+b}{4}\right)+\frac{(-1)^n}{n!}\int_a^b K_n(t)f^{(n)}(t)dt. \end{aligned}$$

Corollary 1.5 *Assume $1 \le p,q \le \infty$ are such that $\frac{1}{p}+\frac{1}{q}=1$. If $f:[a,b]\to\mathbb{R}$ is n times differentiable function such that $f^{(n)}$ is piecewise continuous on $[a,b]$ and $f^{(n)}\in L_p[a,b]$, then we have*

$$\left|\int_a^b f(t)dt-\frac{b-a}{2}[f(x)+f(a+b-x)]-T_n^{2,PT}(x)\right| \le C_2^{PT}(n,q,x)\cdot\|f^{(n)}\|_p, \quad (1.100)$$

where for $1 \le q < \infty$

$$C_2^{PT}(n,q,x)=\frac{2^{1/q}}{(nq+1)^{1/q}n!}\left[(x-a)^{nq+1}+\left(\frac{a+b}{2}-x\right)^{nq+1}\right]^{\frac{1}{q}}. \quad (1.101)$$

Further, for $q=\infty$ we have

$$C_2^{PT}(n,\infty,x)=\begin{cases}\frac{(\frac{a+b}{2}-x)^2}{n!}, & \text{for } a\le x\le \frac{3a+b}{4},\\ \frac{(x-a)^n}{n!}, & \text{for } \frac{3a+b}{4}\le x\le\frac{a+b}{2}.\end{cases} \quad (1.102)$$

Inequality is the best possible for $p=1$, and sharp for $1<p\le\infty$.

Proof. Apply Theorem 1.11 □

Corollary 1.6 *Let $f : [a,b] \to \mathbb{R}$ be a function such that $f^{(n)}$ is continuous for some even $n \geq 2$ and let Q_n be a polynomial defined with (1.97). Then there exists $\eta \in [a,b]$ such that*

$$\int_a^b f(t)dt = \frac{b-a}{2}[f(x)+f(a+b-x)] + T_n^{2,PT}(x) + C_2^{PT}(n,1,x)\cdot f^{(n)}(\eta). \tag{1.103}$$

Proof. The proof follows from 1.12 □

Remark 1.14 In this case identity (1.75) becomes

$$\int_a^b f(t)dt = \frac{b-a}{2}[f(x)+f(a+b-x)] + \frac{1}{3}\cdot\left[(x-a)^3 + \left(\frac{a+b}{2}-x\right)^3\right]\cdot f''(\eta). \tag{1.104}$$

Specially, for $x = \frac{3a+b}{4}$ we get

$$\int_a^b f(t)dt = \frac{b-a}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] + \frac{(b-a)^3}{96}\cdot f''(\eta),$$

which is the well-known Maclaurin two-point formula. Therefore, the identity (1.103) for $x = \frac{3a+b}{4}$, represents the generalisation of the Maclaurin two-point quadrature formula.

Remark 1.15 Put $x = \frac{3a+b}{4}$ into the Corollary 1.5 so the best possible and sharp inequalities for Maclaurin two-point formula are obtained.

For $x = a$, we get

$$A_k^{2,PT}(a) = \frac{(-1)^{k-1}(b-a)^k}{2^k k!}.$$

Specially, $A_2^{2,PT}(a) = -\frac{(b-a)^2}{8} \neq 0$, so the approximation for $\int_a^b f(t)dt$ will contain values of the first derivatives in nodes a and b. Specially, identity (1.75) becomes

$$\int_a^b f(t)dt = \frac{b-a}{2}[f(a)+f(b)] - \frac{(b-a)^2}{8}[f'(b)-f'(a)] + \frac{(b-a)^3}{24}f''(\eta),$$

which is known as perturbed trapezoid formula.Therefore, the identity (1.103), for $x = a$ is generalization of the perturbed trapezoid formula.

Remark 1.16 Put $x = a$ into the Corollary 1.5 so the best possible and sharp inequalities for perturbed trapezoid formula are obtained.

Remark 1.17 Some inequalities related to the perturbed trapezoid formula are obtained in [37].

Remark 1.18 For $x = \frac{a+b}{2}$, we get the midpoint quadrature formula which is developed in previous section.

Now we shall consider the weighted version of the two-point quadrature formula. Let $w : [a,b] \to \mathbb{R}$ be some integrable function and $x \in [a, \frac{a+b}{2}]$. Consider a subdivision

$$\sigma := \{x_0 = a, x_1 = x, x_2 = a+b-x < x_3 = b\}$$

of $[a,b]$. Let $\{Q_{k,x}\}_{k\in\mathbb{N}}$ be sequence of polynomials such that $\deg Q_{k,x} \leq k-1$, $Q'_{k,x}(t) = Q_{k-1,x}(t)$, $k \in \mathbb{N}$ and $Q_{0,x} \equiv 0$. Define functions $w^2_{jk}(t)$ on $[x_{j-1}, x_j]$, for $j = 1,2,3$ and $k \in \mathbb{N}$:

$$\begin{aligned} w^2_{1k}(t) &= \frac{1}{(k-1)!}\int_a^t (t-s)^{k-1} w(s)\mathrm{d}s \\ w^2_{2k}(t) &= \frac{1}{(k-1)!}\int_x^t (t-s)^{k-1} w(s)\mathrm{d}s + Q_{k,x}(t) \\ w^2_{3k}(t) &= -\frac{1}{(k-1)!}\int_t^b (t-s)^{k-1} w(s)\mathrm{d}s. \end{aligned} \tag{1.105}$$

Obviously, $\{w_{jk}\}_{k\in\mathbb{N}}$ are sequences of $w-$harmonic functions on $[x_{j-1}, x_j]$, for every $j = 1,2,3$. Let us define

$$A^2_{k,w}(x) = (-1)^{k-1}\left[\frac{1}{(k-1)!}\int_a^x (x-s)^{k-1} w(s)ds - Q_{k,x}(x)\right], \tag{1.106}$$

and

$$B^2_{k,w}(x) = (-1)^{k-1}\left[\frac{1}{(k-1)!}\int_x^b (a+b-x-s)^{k-1} w(s)ds + Q_{k,x}(a+b-x)\right]. \tag{1.107}$$

Let $g : [a,b] \to \mathbb{R}$ be such that $g^{(n-1)}$ exists on $[a,b]$ for some $n \in \mathbb{N}$. We introduce the following notation:

$$\begin{aligned} T^2_{n,w}(x) &= 0, \quad \text{for } n = 1 \\ T^2_{n,w}(x) &:= \sum_{k=2}^{n}\left[A^2_{k,w}(x)g^{(k-1)}(x) + B^2_{k,w}(x)g^{(k-1)}(a+b-x)\right], \text{ for } n \geq 2. \end{aligned}$$

Theorem 1.14 *Let $g : [a,b] \to \mathbb{R}$ be such that $g^{(n)}$ is piecewise continuous on $[a,b]$, for some $n \in \mathbb{N}$. Then*

$$\begin{aligned} \int_a^b w(t)g(t)dt &= A^2_1(x)g(x) + B^2_1(x)g(a+b-x) + T^2_{n,w}(x) \\ &+ (-1)^n \int_a^b W^2_{n,w}(t,x)g^{(n)}(t)dt, \end{aligned} \tag{1.108}$$

where

$$W^2_{n,w}(t,x) = \begin{cases} w^2_{1n}(t) & \mathit{for}\ t \in [a,x], \\ w^2_{2n}(t) & \mathit{for}\ t \in (x, a+b-x], \\ w^2_{3n}(t) & \mathit{for}\ t \in (a+b-x, b]. \end{cases}$$

Proof. Put (1.105) in identity (1.20). It follows

$$\begin{aligned}\int_a^b w(t)g(t)dt &= \sum_{k=1}^{n}(-1)^{k-1}\Big[\big[w_{1k}^2(x)-w_{2k}^2(x)\big]g^{(k-1)}(x)\\ &+ \big[w_{2k}^2(a+b-x)-w_{3k}^2(a+b-x)\big]g^{(k-1)}(a+b-x)\Big]\\ &+ (-1)^n\int_a^b W_{n,w}^2(t,x)g^{(n)}(t)dt,\end{aligned}$$

since $w_{1k}^2(a)=0$ and $w_{3k}^2(b)=0$. Further,

$$w_{1k}^2(x)-w_{2k}^2(x)=(-1)^{k-1}A_{k,w}^2(x)$$

and

$$w_{2k}^2(a+b-x)-w_{3k}^2(a+b-x)=(-1)^{k-1}B_{k,w}^2(x),$$

so the proof is finished. □

Remark 1.19 Let R_n be monic polynomial of degree n such that $R_n(t)=(-1)^nR_n(a+b-t)$. For $w(t)\equiv\frac{1}{b-a}$ and polynomials

$$Q_{k,x}(t):=\frac{R_n^{(n-k)}(t)}{n!(b-a)}-\frac{(t-x)^k}{k!(b-a)},\quad k=0,1,\dots,n,$$

Guessab-Schmeisser's identity (1.65) is recovered from (1.108). Therefore, we can say that (1.108) is generalization of Guessab-Schmeisser's integral identity.

Remark 1.20 The polynomials $Q_{k,x}$ satisfy

$$Q_{k,x}(t)=\sum_{j=0}^{k-1}Q_{k-j,x}(x)\frac{(t-x)^j}{j!},$$

so the polynomial $Q_{k,x}$ is uniquely determined by values $Q_{j,x}(x)$, for $j=0,1,\dots,k$.

Theorem 1.15 *Let $w:[a,b]\to[0,\infty)$ be continuous function on (a,b) and let*

$$Q_{2n,x}(t)\ge-\frac{1}{(2n-1)!}\int_x^t(t-s)^{2n-1}w(s)ds,\quad\forall t\in[x,a+b-x],$$

for some $n\in\mathbb{N}$. If $g:[a,b]\to\mathbb{R}$ is function such that $g^{(2n)}$ is continuous on $[a,b]$, then there exists $\eta\in(a,b)$ such that

$$\begin{aligned}\int_a^b w(t)g(t)dt &= A_1^2(x)g(x)+B_1^2(x)g(a+b-x)+T_{2n,w}^2(x)\\ &+ \left(A_{2n+1}^2(x)+B_{2n+1}^2(x)\right)\cdot g^{(2n)}(\eta).\end{aligned}\tag{1.109}$$

Proof. According to the relation (1.108), we have to prove the identity

$$\int_a^b W_{2n,w}^2(t,x)g^{(2n)}(t)dt = \left(A_{2n+1}^2(x)+B_{2n+1}^2(x)\right)\cdot g^{(2n)}(\eta).$$

Observe that $W_{2n,w}^2(\cdot,x)$ is an even function. Since $W_{2n,w}^2(\cdot,x)$ does not change the sign, then by the mean value theorem there exists $\eta \in (a,b)$ such that

$$\begin{aligned}
&\int_a^b W_{2n,w}^2(t,x)g^{(2n)}(t)dt = \\
&= g^{(2n)}(\eta)\cdot\left(\int_a^x w_{1,2n}^2(t)dt + \int_x^{a+b-x} w_{2,2n}^2(t)dt + \int_{a+b-x}^b w_{3,2n}^2(t)dt\right) \\
&= g^{(2n)}(\eta)\cdot\left(w_{1,2n+1}^2(x) - w_{2,2n+1}^2(x) + w_{2,2n+1}^2(a+b-x)\right. \\
&- \left. w_{3,2n+1}^2(a+b-x)\right) \\
&= g^{(2n)}(\eta)\left(A_{2n+1}^2(x)+B_{2n+1}^2(x)\right).
\end{aligned}$$

□

Theorem 1.16 *Assume (p,q) is a pair of conjugate exponents, that is $1 \le p,q \le \infty$, $\frac{1}{p}+\frac{1}{q}=1$. Let $g:[a,b]\to\mathbb{R}$ be such that $g^{(n)} \in L_p[a,b]$. Then we have*

$$\begin{aligned}
&\left|\int_a^b w(t)g(t)dt - A_1^2(x)g(x) - B_1^2(x)g(a+b-x) - T_{n,w}^2(x)\right| \\
&\le C_2(n,q,x,w)\cdot\|g^{(n)}\|_p, \qquad (1.110)
\end{aligned}$$

where for $1 \le q < \infty$

$$\begin{aligned}
C_2(n,q,x,w) &= \frac{1}{(n-1)!}\left[\int_a^x \left|w_{1n}^2(t)\right|^q dt + \int_x^{a+b-x}\left|w_{2n}^2(t)\right|^q dt\right. \\
&+ \left.\int_{a+b-x}^b \left|w_{3n}^2(t)\right|^q dt\right]^{\frac{1}{q}},
\end{aligned}$$

and for $q=\infty$

$$\begin{aligned}
C_2(n,\infty,x,w) &= \frac{1}{(n-1)!}\max\left\{\sup_{t\in[a,x]}\left|w_{1n}^2(t)\right|, \sup_{t\in[x,a+b-x]}\left|w_{2n}^2(t)\right|,\right. \\
&\left.\sup_{t\in[a+b-x,b]}\left|w_{3n}^2(t)\right|\right\}.
\end{aligned}$$

The inequality is the best possible for $p=1$, and sharp for $1<p\le\infty$. The equality is attained for every function $g(t)=Mg_*(t)+p_{n-1}(t), t\in[a,b]$ where $M\in\mathbb{R}$, p_{n-1} is an arbitrary polynomial of degree at most $n-1$, and g_* is function on $[a,b]$ defined by

$$g_*(t) = \int_a^t \frac{(t-\xi)^{n-1}}{(n-1)!}\cdot \operatorname{sgn} W_{n,w}^2(\xi,x)\cdot|W_{n,w}^2(\xi,x)|^{\frac{1}{p-1}}d\xi, \qquad (1.111)$$

for $1 < p < \infty$, *and*

$$g_*(t) = \int_a^t \frac{(t-\xi)^{n-1}}{(n-1)!} \cdot \operatorname{sgn} W_{n,w}^2(\xi,x)\,d\xi, \tag{1.112}$$

for $p = \infty$.

Proof. This is a special case of Theorem 1.6. □

Guessab and Schmeisser's identity (1.65) has symmetric coefficients, while coefficients $A_{k,w}^2(x)$ and $B_{k,w}^2(x)$ in (1.108) are not symmetric. The next result describes conditions which lead to symmetry.

Theorem 1.17 *If*

$$w(t) = w(a+b-t), \quad t \in [a,b] \tag{1.113}$$

and

$$(-1)^k Q_{k,x}(x) - Q_{k,x}(a+b-x) = \frac{1}{(k-1)!}\int_x^{a+b-x} (s-x)^{k-1} w(s)\,ds, \tag{1.114}$$

then $A_{k,w}^2(x) = (-1)^{k-1} B_{k,w}^2(x)$.

Proof. Assume (1.113) and (1.114) for some k. Then we have

$$\begin{aligned}
B_{k,w}^2(x) &= (\text{def.}) = (-1)^{k-1}\left[\frac{(-1)^{k-1}}{(k-1)!}\int_x^b (a+b-x-s)^{k-1} w(s)\,ds\right.\\
&+ \left. Q_{k,x}(a+b-x)\right]\\
&= (-1)^{k-1}\left[\frac{(-1)^{k-1}}{(k-1)!}\int_a^{a+b-x} (s-x)^{k-1} w(a+b-s)\,ds\right.\\
&+ \left.(-1)^k Q_{k,x}(x) - \frac{1}{(k-1)!}\int_x^{a+b-x}(s-x)^{k-1} w(s)\,ds\right]\\
&= (-1)^{k-1}\left[\frac{(-1)^{k-1}}{(k-1)!}\int_a^x (s-x)^{k-1} w(s)\,ds + (-1)^k Q_{k,x}(x)\right]\\
&= (-1)^{k-1} A_{k,w}^2(x).
\end{aligned}$$

□

What about the degree of exactness of quadrature formula (1.108)? We would like to have as great degree of exactness as possible. For fixed x we choose polynomials $Q_{k,x}(t)$ which are uniquely determined by the following (according to the remark 1.20):

$$Q_{1,x}(x) = \frac{1}{2x-a-b}\left(\int_a^x (x-s)w(s)\,ds + \int_x^b (a+b-x-s)w(s)\,ds\right),$$

$$Q_{k,x}(x) = \frac{1}{(k-1)!}\int_a^x (x-s)^{k-1} w(s)\,ds, \quad k = 2,3,4$$

$$Q_{k,x}(x)=0, \quad k\geq 5.$$

Now we have

$$A_1^2(x)=\frac{1}{a+b-2x}\int_a^b (a+b-x-s)w(s)ds$$

and

$$B_1^2(x)=\frac{1}{a+b-2x}\int_a^b (s-x)w(s)ds.$$

Further,

$$A_{k,w}^2(x)=B_{k,w}^2(x)=0, \quad k=2,3,4. \tag{1.115}$$

Now, assume (1.113) holds. So we have

$$A_1^2(x)=B_1^2(x)=\frac{1}{2}\int_a^b w(t)dt.$$

From the condition

$$\int_a^b t^l(t)dt=A_1^2(x)g(x)+B_1^2(x)g(a+b-x), \quad l=2,3 \tag{1.116}$$

we get the equation

$$\int_a^b (t+x-a-b)(t-x)w(t)dt=0, \tag{1.117}$$

which has exactly one solution $x\in[a,\frac{a+b}{2}]$. For that x we get the generalization of the well-known quadrature formulas of Gauss type. Now identity (1.108) becomes

$$\begin{aligned}\int_a^b g(t)w(t)dt &= A_1^2(x)\left[g(x)+g(a+b-x)\right]+T_{n,w}^2(x)\\ &+ (-1)^n\int_a^b W_{n,w}^2(t,x)g^{(n)}(t)dt,\end{aligned}$$

where

$$T_{n,w}^2(x)=\sum_{k=5}^{n}\left[A_{k,w}^2(x)g^{(k-1)}(x)+B_{k,w}^2(x)g^{(k-1)}(a+b-x)\right].$$

In particular, for $n=2$ from the identity (1.109) we get

$$\int_a^b g(t)w(t)dt=A_1^2(x)\left[g(x)+g(a+b-x)\right]+\left[A_5^2(x)+B_5^2(x)\right]g^{(4)}(\eta). \tag{1.118}$$

1.3.4 Legendre-Gauss two-point quadrature formula

Let $w(t)=1, t\in[a,b]$ and $x\in[a,\frac{a+b}{2}]$ an arbitrary and fixed node. Define $\{w_{jk}^{2,LG}\}_{k\in\mathbb{N}}$

$$w_{1k}^{2,LG}(t)=\frac{(t-a)^k}{k!}, \quad t\in[a,x],$$

$$w_{2k}^{2,LG}(t) = \frac{(t-x)^k}{k!} + Q_{k,x}(t), \quad t \in (x, a+b-x]$$

and

$$w_{3k}^{2,LG}(t) = \frac{(t-b)^k}{k!}, \quad t \in (a+b-x, b].$$

Define kernel

$$W_{n,w}^{2,LG}(t,x) = \begin{cases} w_{1n}^{2,LG}(t), & \text{for } t \in [a,x], \\ w_{2n}^{2,LG}(t), & \text{for } t \in (x, a+b-x], \\ w_{3n}^{2,LG}(t), & \text{for } t \in (a+b-x, b]. \end{cases} \tag{1.119}$$

For $k \geq 1$ define

$$A_{k,w}^{2,LG}(x) = (-1)^{k-1}\left[\frac{(x-a)^k}{k!} - Q_{k,x}(x)\right],$$

$$B_{k,w}^{2,LG}(x) = (-1)^{k-1}\left[\frac{(a+b-2x)^k}{k!} - \frac{(a-x)^k}{k!} + Q_{k,x}(a+b-x)\right].$$

In particular, $A_1^{2,LG}(x) = B_1^{2,LG}(x) = \frac{b-a}{2}$. Let $f : [a,b] \to \mathbb{R}$ be such that for $n \in \mathbb{N}$, $f^{(n-1)}$ exists on $[a,b]$. Define $T_{n,w}^{2,LG}(x)$ by

$$T_{n,w}^{2,LG}(x) = \sum_{k=2}^{n}\left[A_{k,w}^{2,LG}(x) f^{(k-1)}(x) + B_{k,w}^{2,LG}(x) f^{(k-1)}(a+b-x)\right].$$

Corollary 1.7 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ is piecewise continuous on $[a,b]$. Then we have*

$$\begin{aligned} \int_a^b f(t)dt &= \frac{b-a}{2}[f(x) + f(a+b-x)] + T_{n,w}^{2,LG}(x) \\ &+ (-1)^n \int_a^b W_{n,w}^{2,LG}(t,x) f^{(n)}(t)dt. \end{aligned} \tag{1.120}$$

Proof. Apply Theorem 1.14 for the case $w(t) = 1$. □

The solution of the equation (1.117) is $x = \frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}$, so we get the generalization of Legendre-Gauss two-point quadrature formula. Further, for the polynomials $Q_{k,x}(t)$ such that $Q_{k,x}(x) = \frac{(x-a)^k}{k!}$ for $k = 2,3,4$, we have

$$\begin{aligned} \int_a^b f(t)dt &= \frac{b-a}{2}\left[f\left(\frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}\right) + f\left(\frac{a+b}{2} + \frac{b-a}{2\sqrt{3}}\right)\right] \\ &+ T_{n,w}^{2,LG}\left(\frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}\right) \\ &+ (-1)^n \int_a^b W_{n,w}^{2,LG}\left(t, \frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}\right) f^{(n)}(t)dt. \end{aligned} \tag{1.121}$$

Corollary 1.8 *If $f : [a,b] \to \mathbb{R}$ is such that $f^{(2n)}$ is continuous on $[a,b]$, and if $w_{2,2n}^{2,LG}(t) \geq 0$, for $t \in [x, a+b-x]$ then there exists $\eta \in [a,b]$ such that*

$$\begin{aligned}\int_a^b f(t)dt &= \frac{b-a}{2}[f(x)+f(a+b-x)] + T_{2n,w}^{2,LG}(x) \\ &+ \left[A_{2n+1}^{2,LG}(x) + B_{2n+1}^{2,LG}(x)\right] \cdot f^{(2n)}(\eta). \qquad (1.122)\end{aligned}$$

For $x = \frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}$, the identity (1.118) becomes Legendre-Gauss quadrature

$$\begin{aligned}\int_a^b f(t)dt &= \frac{b-a}{2}\left[f\left(\frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}\right) + f\left(\frac{a+b}{2} + \frac{b-a}{2\sqrt{3}}\right)\right] \\ &+ \frac{(b-a)^5}{4320} f^{(4)}(\eta).\end{aligned}$$

In particular, from the inequality (1.110) it follows

$$\begin{aligned}&\left|\int_a^b f(t)dt - \frac{b-a}{2}\left[f\left(\frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}\right) + f\left(\frac{a+b}{2} + \frac{b-a}{2\sqrt{3}}\right)\right]\right| \\ &\leq C_2^{LG}\left(n, q, \frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}, w\right) \|f^{(n)}\|_p, \quad n = 1,2,3,4,\end{aligned}$$

where

$$\begin{aligned}
C_2^{LG}\left(1, \infty, \frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}, w\right) &= \frac{b-a}{2\sqrt{3}}, \\
C_2^{LG}\left(1, 1, \frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}, w\right) &= \frac{(5-2\sqrt{3})(b-a)^2}{12}, \\
C_2^{LG}\left(2, \infty, \frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}, w\right) &= \frac{(2-\sqrt{3})(b-a)^2}{12}, \\
C_2^{LG}\left(2, 1, \frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}, w\right) &= \frac{\sqrt{26\sqrt{3}-45}(b-a)^3}{18}, \\
C_2^{LG}\left(3, \infty, \frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}, w\right) &= \frac{(2-\sqrt{3})\sqrt{2\sqrt{3}-3}(b-a)^3}{72}, \\
C_2^{LG}\left(3, 1, \frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}, w\right) &= \frac{(9-4\sqrt{3})(b-a)^4}{1728}, \\
C_2^{LG}\left(4, \infty, \frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}, 1\right) &= \frac{(9-4\sqrt{3})(b-a)^4}{3456}, \\
C_2^{LG}\left(4, 1, \frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}, w\right) &= \frac{(b-a)^5}{4320}.
\end{aligned}$$

1.3.5 Chebyshev-Gauss two-point quadrature formula

Let $w(t) = \frac{1}{\sqrt{1-t^2}}, t \in [-1,1]$ and let $x \in [-1,0]$ be fixed node. Define $\{w_{jk}^{2,C1}\}_{k\in\mathbb{N}}$

$$w_{1k}^{2,C1}(t) = \frac{1}{(k-1)!}\int_{-1}^{t} \frac{(t-s)^{k-1}}{\sqrt{1-s^2}}ds, \quad t \in [-1,x],$$

$$w_{2k}^{2,C1}(t) = \frac{1}{(k-1)!}\int_{x}^{t} \frac{(t-s)^{k-1}}{\sqrt{1-s^2}}ds + Q_{k,x}(t), \quad t \in (x,-x]$$

and

$$w_{3k}^{2,C1}(t) = -\frac{1}{(k-1)!}\int_{t}^{1} \frac{(t-s)^{k-1}}{\sqrt{1-s^2}}ds, \quad t \in (-x,1].$$

Define kernel

$$W_{n,w}^{2,C1}(t,x) = \begin{cases} w_{1n}^{2,C1}(t), & \text{for } t \in [-1,x], \\ \\ w_{2n}^{2,C1}(t), & \text{for } t \in (x,-x], \\ \\ w_{3n}^{2,C1}(t), & \text{for } t \in (-x,1]. \end{cases} \tag{1.123}$$

For $k \geq 1$ define

$$A_{k,w}^{2,C1}(x) = (-1)^{k-1}\left[\frac{2^{k-1/2}(x+1)^{k-1/2}}{(2k-1)!!}F\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}+k,\frac{x+1}{2}\right) - Q_{k,x}(x)\right],$$

$$B_{k,w}^{2,C1}(x) = (-1)^{k-1}\left[\frac{1}{(k-1)!}\int_{x}^{1} \frac{(-x-s)^{k-1}}{\sqrt{1-s^2}}ds + Q_{k,x}(-x)\right].$$

Specially, $A_{1}^{2,C1}(x) = B_{1}^{2,C1}(x) = \frac{\pi}{2}$.

Let $f : [-1,1] \to \mathbb{R}$ be such that for $n \in \mathbb{N}$, $f^{(n-1)}$ exists on $[-1,1]$. Define

$$T_{n,w}^{2,C1}(x) = \sum_{k=2}^{n}\left[A_{k,w}^{2,C1}(x)f^{(k-1)}(x) + B_{k,w}^{2,C1}(x)f^{(k-1)}(-x)\right].$$

Corollary 1.9 *Let $f : [-1,1] \to \mathbb{R}$ be such that $f^{(n)}$ is piecewise continuous. Then we have*

$$\begin{aligned} \int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}}dt &= \frac{\pi}{2}\left[f(x)+f(-x)\right] + T_{n,w}^{2,C1}(x) \\ &+ (-1)^n \int_{-1}^{1} W_{n,w}^{2,C1}(t,x)f^{(n)}(t)dt. \end{aligned} \tag{1.124}$$

Proof. Apply Theorem 1.14 for the case $w(t) = \frac{1}{\sqrt{1-t^2}}$. □

The solution of the equation (1.117) is $x=-\frac{\sqrt{2}}{2}$, so we get the generalization of the Chebysev-Gauss two-point quadrature formula. Further, for the polynomials $Q_{k,x}(t)$ such that $Q_{k,x}(x)=\frac{1}{(k-1)!}\int_{-1}^{x}\frac{(x-s)^{k-1}}{\sqrt{1-s^2}}ds$, for $k=2,3,4$, we have

$$\begin{aligned}\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}dt &= \frac{\pi}{2}\left[f\left(-\frac{\sqrt{2}}{2}\right)+f\left(\frac{\sqrt{2}}{2}\right)\right] \\ &+ T_{n,w}^{2,C1}\left(-\frac{\sqrt{2}}{2}\right)+(-1)^n\int_{-1}^{1}W_{n,w}^{2,C1}\left(t,-\frac{\sqrt{2}}{2}\right)f^{(n)}(t)dt.\end{aligned} \tag{1.125}$$

Corollary 1.10 *If $f:[-1,1]\to\mathbb{R}$ is such that $f^{(2n)}$ is continuous on $[-1,1]$, and if $w_{2,2n}^{2,C1}(t)\geq 0$, for $t\in[x,-x]$, then there exists $\eta\in[-1,1]$ such that*

$$\begin{aligned}\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}dt &= \frac{\pi}{2}[f(x)+f(-x)]+T_{2n,w}^{2,C1}(x) \\ &+ \left[A_{2n+1}^{2,C1}(x)+B_{2n+1}^{2,C1}(x)\right]\cdot f^{(2n)}(\eta).\end{aligned} \tag{1.126}$$

For $x=-\frac{\sqrt{2}}{2}$, from the identity (1.118) we get Chebyshev-Gauss quadrature formula

$$\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}dt=\frac{\pi}{2}\left[f\left(-\frac{\sqrt{2}}{2}\right)+f\left(-\frac{\sqrt{2}}{2}\right)\right]+\frac{\pi}{192}f^{(4)}(\eta).$$

Specially, the inequality (1.110) implies

$$\begin{aligned}&\left|\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}dt-\frac{\pi}{2}\left[f\left(-\frac{\sqrt{2}}{2}\right)+f\left(\frac{\sqrt{2}}{2}\right)\right]\right| \\ &\leq C_2^{C1}\left(n,q,-\frac{\sqrt{2}}{2},w\right)\|f^{(n)}\|_p, \quad n=1,2,3,4,\end{aligned}$$

where

$$\begin{aligned}C_2^{C1}\left(1,\infty,-\frac{\sqrt{2}}{2},w\right)&=\frac{\pi}{4}\approx 0,785398,\\ C_2^{C1}\left(1,1,-\frac{\sqrt{2}}{2},w\right)&=2\sqrt{2}-2\approx 0.828427,\\ C_2^{C1}\left(2,\infty,-\frac{\sqrt{2}}{2},w\right)&\approx 0,151746,\\ C_2^{C1}\left(2,1,-\frac{\sqrt{2}}{2},w\right)&\approx 0,138151\end{aligned}$$

$$C_2^{C1}\left(3,\infty,-\frac{\sqrt{2}}{2},w\right)\approx 0,034537,$$

$$C_2^{C1}\left(3,1,-\frac{\sqrt{2}}{2},w\right)\approx 0,037102,$$

$$C_2^{C1}\left(4,1,-\frac{\sqrt{2}}{2},w\right)=\frac{\pi}{192}\approx 0,0163624.$$

1.3.6 Chebyshev-Gauss formula of the second kind

Let $w(t)=\sqrt{1-t^2}, t\in[-1,1]$ and let $x\in[-1,0]$ be fixed node. Define $\{w_{jk}^{2,C2}\}_{k\in\mathbb{N}}$

$$w_{1k}^{2,C2}(t)=\frac{1}{(k-1)!}\int_{-1}^{t}(t-s)^{k-1}\sqrt{1-s^2}ds,\quad t\in[-1,x],$$

$$w_{2k}^{2,C2}(t)=\frac{1}{(k-1)!}\int_{x}^{t}(t-s)^{k-1}\sqrt{1-s^2}ds+Q_{k,x}(t),\quad t\in(x,-x]$$

and

$$w_{3k}^{2,C2}(t)=-\frac{1}{(k-1)!}\int_{t}^{1}(t-s)^{k-1}\sqrt{1-s^2}ds,\quad t\in(-x,1].$$

Define kernel

$$W_{n,w}^{2,C2}(t,x)=\begin{cases} w_{1n}^{2,C2}(t), & \text{for } t\in[-1,x],\\ w_{2n}^{2,C2}(t), & \text{for } t\in(x,-x],\\ w_{3n}^{2,C2}(t), & \text{for } t\in(-x,1].\end{cases}\tag{1.127}$$

For $k\geq 1$ define

$$A_{k,w}^{2,C2}(x)=(-1)^{k-1}\left[\frac{2^{j+1/2}(x+1)^{j+1/2}}{(2k+1)!!}F\left(-\frac{1}{2},\frac{3}{2},\frac{3}{2}+j,\frac{x+1}{2}\right)-Q_{k,x}(x)\right],$$

$$B_{k,w}^{2,C2}(x)=(-1)^{k-1}\left[\frac{1}{(k-1)!}\int_{x}^{1}(-x-s)^{j-1}\sqrt{1-s^2}ds+Q_{k,x}(-x)\right].$$

Specially, $A_1^{2,C2}(x)=B_1^{2,C2}(x)=\frac{\pi}{4}$.

Let $f:[-1,1]\to\mathbb{R}$ be such that for $n\in\mathbb{N}$, $f^{(n-1)}$ exists on $[-1,1]$. Define

$$T_{n,w}^{2,C2}(x)=\sum_{k=2}^{n}\left[A_{k,w}^{2,C2}(x)f^{(k-1)}(x)+B_{k,w}^{2,C2}(x)f^{(k-1)}(-x)\right].$$

Corollary 1.11 *Let $f:[-1,1]\to\mathbb{R}$ be such that $f^{(n)}$ is piecewise continuous. Then we have*

$$\begin{aligned}\int_{-1}^{1} f(t)\sqrt{1-t^2}dt &= \frac{\pi}{4}[f(x)+f(-x)]+T_{n,w}^{2,C2}(x)\\ &+ (-1)^n\int_{-1}^{1} W_{n,w}^{2,C2}(t,x)f^{(n)}(t)dt. \end{aligned} \tag{1.128}$$

Proof. Apply Theorem 1.14 for the case $w(t)=\sqrt{1-t^2}$. □

The solution of the equation (1.117) is $x=-\frac{1}{2}$, so we get the generalization of the Chebysev-Gauss two-point quadrature formula of the second kind. Further, for the polynomials $Q_{k,x}(t)$ such that $Q_{k,x}(x)=\frac{1}{(k-1)!}\int_{-1}^{x}(x-s)^{k-1}\sqrt{1-s^2}ds$, for $k=2,3,4$, we have

$$\begin{aligned}\int_{-1}^{1} f(t)\sqrt{1-t^2}dt &= \frac{\pi}{4}\left[f\left(-\frac{1}{2}\right)+f\left(\frac{1}{2}\right)\right] \\ &+ T_{n,w}^{2,C2}\left(-\frac{1}{2}\right)+(-1)^n\int_{-1}^{1} W_{n,w}^{2,C2}\left(t,-\frac{1}{2}\right)f^{(n)}(t)dt. \end{aligned} \tag{1.129}$$

Corollary 1.12 *If $f:[-1,1]\to\mathbb{R}$ is such that $g^{(2n)}$ is continuous on $[-1,1]$, and if $w_{2,2n}^{2,C1}(t)\geq 0$, for $t\in[x,-x]$, then there exists $\eta\in[-1,1]$ such that*

$$\begin{aligned}\int_{-1}^{1} f(t)\sqrt{1-t^2}dt &= \frac{\pi}{4}[f(x)+f(-x)]+T_{2n,w}^{2,C2}(x)\\ &+ \left[A_{2n+1}^{2,C2}(x)+B_{2n+1}^{2,C2}(x)\right]\cdot f^{(2n)}(\eta). \end{aligned} \tag{1.130}$$

For $x=-\frac{1}{2}$, from the identity (1.118) we get Chebyshev-Gauss quadrature formula of the second kind

$$\int_{-1}^{1} f(t)\sqrt{1-t^2}dt = \frac{\pi}{4}\left[f\left(-\frac{1}{2}\right)+f\left(-\frac{1}{2}\right)\right]+\frac{\pi}{768}f^{(4)}(\eta).$$

Specially, the inequality (1.110) for this case looks like

$$\begin{aligned}&\left|\int_{-1}^{1} f(t)\sqrt{1-t^2}dt-\frac{\pi}{4}\left[f\left(-\frac{1}{2}\right)+f\left(\frac{1}{2}\right)\right]\right|\\ &\leq C_2^{C2}\left(n,q,-\frac{1}{2},w\right)\|f^{(n)}\|_p, \quad n=1,2,3,4,\end{aligned}$$

where

$$\begin{aligned}C_2^{C2}\left(1,\infty,-\frac{1}{2},w\right) &\approx 0,478305,\\ C_2^{C2}\left(1,1,-\frac{1}{2},w\right) &\approx 0.370572,\end{aligned}$$

$$C_2^{C2}\left(2,\infty,-\frac{1}{2},w\right) \approx 0,062960,$$
$$C_2^{C2}\left(2,1,-\frac{1}{2},w\right) \approx 0.0547145,$$
$$C_2^{C2}\left(3,\infty,-\frac{1}{2},w\right) \approx 0,012251,$$
$$C_2^{C2}\left(3,1,-\frac{1}{2},w\right) \approx 0,0117195,$$
$$C_2^{C2}\left(4,1,-\frac{1}{2},w\right) = \frac{\pi}{768}.$$

1.3.7 Hermite-Gauss two-point formula

Let us consider $w(t) = e^{-t^2}, t \in \mathbb{R}$ and let $x \le 0$. Since this weight function is defined on infinity interval, at first we shall consider it on some finite interval $[-L,L]$, for some $L \in \mathbb{R}_+$ such that $|x| \le L$.

Define
$$w_{1k}^{2,HG,L}(t) = \frac{1}{(k-1)!}\int_{-L}^{t}(t-s)^{k-1}e^{-s^2}ds, \quad t \in [-L,x],$$
$$w_{2k}^{2,HG,L}(t) = \frac{1}{(k-1)!}\int_{x}^{t}(t-s)^{k-1}e^{-s^2}ds + Q_{k,x}(t), \quad t \in (x,-x]$$
and
$$w_{3k}^{2,HG,L}(t) = -\frac{1}{(k-1)!}\int_{t}^{L}(t-s)^{k-1}e^{-s^2}ds, \quad t \in (-x,L].$$
Define kernel
$$W_{n,w}^{2,HG,L}(t,x) = \begin{cases} w_{1n}^{2,HG,L}(t), & \text{for } t \in [-L,x], \\ w_{2n}^{2,HG,L}(t), & \text{for } t \in (x,-x], \\ w_{3n}^{2,HG,L}(t), & \text{for } t \in (-x,L]. \end{cases} \tag{1.131}$$
For $k \ge 1$ define
$$A_{k,w}^{2,HG,L}(x) = (-1)^{k-1}\left[\frac{1}{(k-1)!}\int_{-L}^{x}(x-s)^{k-1}e^{-s^2}ds - Q_{k,x}(x)\right],$$
$$B_{k,w}^{2,HG,L}(x) = (-1)^{k-1}\left[\frac{1}{(k-1)!}\int_{x}^{L}(-x-s)^{k-1}e^{-s^2}ds + Q_{k,x}(-x)\right].$$
Specially, $A_1^{2,HG,L}(x) = B_1^{2,HG,L}(x) = \frac{1}{2}\int_{-L}^{L}e^{-t^2}dt$. Let $f : [-1,1] \to \mathbb{R}$ be such that for $n \in \mathbb{N}$, $f^{(n-1)}$ exists on $[-1,1]$. Define
$$T_{n,w}^{2,HG,L}(x) = \sum_{k=2}^{n}\left[A_{k,w}^{2,HG,L}(x)f^{(k-1)}(x) + B_{k,w}^{2,HG,L}(x)f^{(k-1)}(-x)\right].$$

Corollary 1.13 *Let $f:[-L,L]\to\mathbb{R}$ be such that $g^{(n)}$ is piecewise continuous. Then we have*

$$\begin{aligned}\int_{-L}^{L} f(t)e^{-t^2}dt &= A_1^{2,HG,L}(x)\,[f(x)+f(-x)]+T_{n,w}^{2,HG,L}(x)\\ &+ (-1)^n\int_{-L}^{L} W_{n,w}^{2,HG,L}(t,x)f^{(n)}(t)dt.\end{aligned} \tag{1.132}$$

Proof. Apply Theorem 1.14 forthe case $w(t)=e^{-t^2}$. □

Now, assume f has all the necessary higher ordered derivatives on $\mathbb{R}$. Let us define

$$\begin{aligned}w_{1k}^{2,HG}(t) &= \frac{1}{(k-1)!}\int_{-\infty}^{t}(t-s)^{k-1}e^{-s^2}ds, \quad t\in(-\infty,x],\\ w_{2k}^{2,HG}(t) &= \frac{1}{(k-1)!}\int_{x}^{t}(t-s)^{k-1}e^{-s^2}ds+Q_{k,x}(t), \quad t\in(x,-x]\\ w_{3k}^{2,HG}(t) &= -\frac{1}{(k-1)!}\int_{t}^{\infty}(t-s)^{k-1}e^{-s^2}ds, \quad t\in(-x,\infty),\end{aligned}$$

$$W_{n,w}^{2,HG}(t,x)=\begin{cases} w_{1n}^{2,HG}(t), \ \text{for } t\in(-\infty,x],\\ w_{2n}^{2,HG}(t), \ \text{for } t\in(x,-x],\\ w_{3n}^{2,HG}(t), \ \text{for } t\in(-x,\infty),\end{cases}$$

$$\begin{aligned}A_{k,w}^{2,HG}(x) &= (-1)^{k-1}\left[\frac{1}{(k-1)!}\int_{-\infty}^{x}(x-s)^{k-1}e^{-s^2}ds-Q_{k,x}(x)\right],\\ B_{k,w}^{2,HG}(x) &= (-1)^{k-1}\left[\frac{1}{(k-1)!}\int_{x}^{\infty}(-x-s)^{k-1}e^{-s^2}ds+Q_{k,x}(-x)\right],\\ T_{n,w}^{2,HG}(x) &= \sum_{k=2}^{n}\left[A_{k,w}^{2,HG}(x)f^{(k-1)}(x)+B_{k,w}^{2,HG}(x)f^{(k-1)}(-x)\right].\end{aligned}$$

Specially, $A_1^{2,HG}(x)=B_1^{2,HG}(x)=\frac{\sqrt{\pi}}{2}$. Obviously,

$$\begin{aligned}\lim_{L\to\infty} w_{jk}^{2,HG,L}(t) &= w_{jk}^{2,HG}(t)\\ \lim_{L\to\infty} A_k^{2,HG,L}(x) &= A_k^{2,HG}(x)\\ \lim_{L\to\infty} B_k^{2,HG,L}(x) &= B_k^{2,HG}(x)\\ \lim_{L\to\infty} T_{n,w}^{2,HG,L}(x) &= A_k^{2,HG}(x),\end{aligned}$$

so in (1.132) put $L\to\infty$, and we obtain

$$\int_{-\infty}^{\infty} f(t)e^{-t^2}dt = \frac{\sqrt{\pi}}{2}[f(x)+f(-x)]+T_{n,w}^{2,HG}(x)$$

$$+\ (-1)^n \int_{-\infty}^{\infty} W_{n,w}^{2,HG}(t,x) f^{(n)}(t)dt. \tag{1.133}$$

The solution of the equation (1.117) is $x = -\frac{\sqrt{2}}{2}$, so we get the generalization of the Gauss-Hermite two-point quadrature formula. Further, for the polynomials $Q_{k,x}(t)$ such that $Q_{k,x}(x) = \frac{1}{(k-1)!}\int_{-\infty}^{x}(x-s)^{k-1}e^{-s^2}ds$, for $k = 2,3,4$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)e^{-t^2}dt &= \frac{\sqrt{\pi}}{2}\left[f\left(-\frac{\sqrt{2}}{2}\right)+f\left(\frac{\sqrt{2}}{2}\right)\right]+T_{n,w}^{2,HG}\left(-\frac{\sqrt{2}}{2}\right) \\ &+\ (-1)^n \int_{-\infty}^{\infty} W_{n,w}^{2,HG}\left(t,-\frac{\sqrt{2}}{2}\right) f^{(n)}(t)dt. \end{aligned} \tag{1.134}$$

Corollary 1.14 *If $f : \mathbb{R} \to \mathbb{R}$ is such that $f^{(2n)}$ is continuous on $\mathbb{R}$,and if $w_{2,2n}^{2,HG}(t) \geq 0$, for $t \in [x,-x]$, then there exists $\eta \in \mathbb{R}$ such that*

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)e^{-t^2}dt &= \frac{\sqrt{\pi}}{2}(f(x)+f(-x))+T_{2n,w}^{2,HG}(x) \\ &+\ \left[A_{2n+1}^{2,HG}(x)+B_{2n+1}^{2,HG}(x)\right]\cdot f^{(2n)}(\eta). \end{aligned} \tag{1.135}$$

For $x = -\frac{\sqrt{2}}{2}$ from identity (1.118) we get Hermite-Gauss quadrature formula

$$\int_{-\infty}^{\infty} f(t)e^{-t^2}dt = \frac{\sqrt{\pi}}{2}\left[f\left(-\frac{\sqrt{2}}{2}\right)+f\left(-\frac{\sqrt{2}}{2}\right)\right]+\frac{\sqrt{\pi}}{48}f^{(4)}(\eta).$$

Specially, the inequality (1.110) implies

$$\begin{aligned}&\left|\int_{-\infty}^{\infty} f(t)e^{-t^2}dt - \frac{\sqrt{\pi}}{2}\left[f\left(-\frac{\sqrt{2}}{2}\right)+f\left(\frac{\sqrt{2}}{2}\right)\right]\right| \\ &\leq C_2^{HG}\left(n,q,-\frac{\sqrt{2}}{2},w\right)\|f^{(n)}\|_p, \quad n = 1,2,3,4,\end{aligned}$$

where

$$\begin{aligned}C_2^{2,HG}\left(1,\infty,-\frac{\sqrt{2}}{2},w\right) &\approx 0,605018, \\ C_2^{2,HG}\left(1,1,-\frac{\sqrt{2}}{2},w\right) &\approx 0,670996, \\ C_2^{2,HG}\left(2,\infty,-\frac{\sqrt{2}}{2},w\right) &\approx 0,16266, \\ C_2^{2,HG}\left(2,1,-\frac{\sqrt{2}}{2},w\right) &\approx 0,10442,\end{aligned}$$

$$C_2^{2,HG}\left(3,\infty,-\frac{\sqrt{2}}{2},w\right) \approx 0,061041,$$
$$C_2^{2,HG}\left(4,1,-\frac{\sqrt{2}}{2},w\right) = \frac{\sqrt{\pi}}{48}.$$

1.4 Three-point quadrature formulae

For $x \in [a, \frac{a+b}{2})$ let us consider the following subdivision of the segment $[a,b]$:

$$\sigma := \{x_0 < x_1 < x_2 < x_3 < x_4\}$$

where $x_0 = a$, $x_1 = x$, $x_2 = \frac{a+b}{2}$, $x_3 = a+b-x$ and $x_4 = b$. Let $Q_n(t)$ be some monic polynomial of degree n, for some $n \in \mathbb{N}$. Set

$$S_n^3(t,x) = \begin{cases} P_{1n}(t) = \frac{(t-a)^n}{n!}, & t \in [a,x] \\ P_{2n}(t) = \frac{Q_n(t)}{n!}, & t \in (x, \frac{a+b}{2}] \\ P_{3n}(t) = (-1)^n \frac{Q_n(a+b-t)}{n!}, & t \in (\frac{a+b}{2}, a+b-x] \\ P_{4n}(t) = \frac{(t-b)^n}{n!}, & t \in (a+b-x, b]. \end{cases} \tag{1.136}$$

Further, for $k = 0,1,\ldots,n-1$ we define

$$P_{1k}(t) = \frac{(t-a)^k}{k!}, \quad P_{2k}(t) = \frac{Q_n^{(n-k)}(t)}{n!}$$
$$P_{3k}(t) = \frac{(-1)^k Q_n^{(n-k)}(a+b-t)}{n!}, \quad P_{4k}(t) = \frac{(t-b)^k}{k!}.$$

Remark 1.21 The sequences of the polynomials $\{P_{jk}\}_{k=0,1,\ldots,n}$ are harmonic, for $j = 1,2,3,4$, i.e. $P'_{jk}(t) = P_{j,k-1}(t)$, for $k = 1,\ldots,n$ and $P_{j0}(t) = 1$, for $j = 1,2,3,4$.

Remark 1.22 If we put

$$Q_k(t) := \frac{k!Q_n^{(n-k)}(t)}{n!}, \quad k = 0,1,\ldots,n-1,$$

then we have

$$P_{2k}(t) = \frac{Q_k(t)}{k!}, \quad \text{and} \quad P_{3k}(t) = \frac{(-1)^k Q_k(a+b-t)}{k!}.$$

Further, polynomials Q_k satisfy $Q'_k(t) = Q_{k-1}(t)$.

Remark 1.23 The following symmetry conditions are valid:

$$P_{1k}(t) = (-1)^k P_{4k}(a+b-t), \forall t \in [a,x]$$

and

$$P_{2k}(t) = (-1)^k P_{3k}(a+b-t), \forall t \in \left(x, \frac{a+b}{2}\right).$$

Now we can state the general three point formula:

Theorem 1.18 *Let $f : [a,b] \to \mathbb{R}$ be a function with a piecewise continuous $n-$th derivative, for some $n \in \mathbb{N}$ and $x \in [a, \frac{a+b}{2})$. Further, let $Q_n(t)$ be some monic polynomial of degree n and $S_n(^3t,x)$ be defined by relation (1.136). Then the following formula holds*

$$\begin{aligned}\int_a^b f(t)dt &= \sum_{k=1}^{n} A_k^3(x)\left(f^{(k-1)}(x) + (-1)^{k-1} f^{(k-1)}(a+b-x)\right) \\ &+ \sum_{\substack{k=1 \\ odd}}^{n} B_k^3(x) f^{(k-1)}\left(\frac{a+b}{2}\right) + (-1)^n \int_a^b S_n^3(t,x) f^{(n)}(t)dt,\end{aligned} \tag{1.137}$$

where

$$A_k^3(x) = \frac{(-1)^{k-1}}{k!}\left((x-a)^k - Q_k(x)\right), \quad k \geq 1,$$

$$B_k^3(x) = \frac{2Q_k(\frac{a+b}{2})}{k!}, \quad \textit{for odd } k \geq 1$$

and

$$B_k^3(x) = P_{2k}(x) - P_{3k}(x) = 0, \quad \textit{for even } k \geq 1.$$

Proof. We consider subdivision $x_0 = a$, $x_1 = x$, $x_2 = \frac{a+b}{2}$, $x_3 = a+b-x$ and $x_4 = b$ of the interval $[a,b]$ and apply formula (1.2) with $m = 4$.We have

$$P_{1k}(a) = P_{4k}(b) = 0, \quad \forall k = 1, \ldots, n.$$

Further, imposing polynomials (1.136) in (1.2) we get the coefficient by $f^{(k-1)}(x)$ and $(-1)^{k-1} f^{(k-1)}(a+b-x)$ equals to

$$A_k^3(x) = (-1)^{k-1}\left[P_{1k}(x) - P_{2k}(x)\right] = \frac{(-1)^{k-1}}{k!}\left((x-a)^k - Q_k(x)\right)$$

and coefficient by $f^{(k-1)}(\frac{a+b}{2})$ for odd k equals to

$$B_k^3(x) = (-1)^{k-1}\left[P_{2k}\left(\frac{a+b}{2}\right) - P_{3k}\left(\frac{a+b}{2}\right)\right] = \frac{2Q_k(\frac{a+b}{2})}{k!}.$$

□

Remark 1.24 Analogue results for the general two-point formula with nodes x and $a+b-x$ were considered in [65] and [73].

Now we will state L_p inequalities for the general three point integral formula.

Theorem 1.19 *Let $f:[a,b]\to\mathbb{R}$ be a function with a piecewise continuous $n-$th derivative and $f^{(n)}\in L_p[a,b]$ for some $n\in\mathbb{N}$ and some $1\le p\le\infty$. Then we have the following inequality*

$$\begin{aligned}\Big|\int_a^b f(t)dt\ &-\ \sum_{k=1}^{n} A_k^3(x)\left(f^{(k-1)}(x)+(-1)^{k-1}f^{(k-1)}(a+b-x)\right)\\ &-\ \sum_{\substack{k=1\\ odd}}^{n} B_k^3(x)f^{(k-1)}\left(\frac{a+b}{2}\right)\Big|\le C_3(n,p,x)\cdot\|f^{(n)}\|_p,\end{aligned}\tag{1.138}$$

where

$$C_3(n,p,x)= \tag{1.139}$$
$$\begin{cases}\frac{2^{1/q}}{n!}\left[\frac{(x-a)^{nq+1}}{nq+1}+\int_x^{\frac{a+b}{2}}|Q_n(t)|^q\,dt\right]^{\frac{1}{q}},\frac{1}{p}+\frac{1}{q}=1,1<p\le\infty\\ \frac{1}{n!}\max\left\{(x-a)^n,\sup_{t\in[x,\frac{a+b}{2}]}|Q_n(t)|\right\},p=1,\end{cases}$$

The inequality is sharp for $1<p\le\infty$ and the best possible for $p=1$. Equality is attained for the function $f_*:[a,b]\to\mathbb{R}$ defined by

$$f_*(t)=\frac{1}{(n-1)!}\int_a^t(t-s)^{n-1}\left|S_n^3(s,x)\right|^{\frac{1}{p-1}}\operatorname{sgn}S_n^3(s,x)ds \tag{1.140}$$

for $1<p<\infty$, while for $p=\infty$

$$f_*(t)=\frac{1}{(n-1)!}\int_a^t(t-s)^{n-1}\operatorname{sgn}S_n^3(s,x)ds \tag{1.141}$$

Proof. This theorem is special case of the Theorem 1.6 □

Let us apply upper results to the following example of monic polynomial Q_n:

$$\begin{aligned}Q_{n,x}(t)\ &:=\ (t-x)^n+n(x-a-\frac{(b-a)^3}{6(2x-a-b)^2})(t-x)^{n-1}\\ &+\ \binom{n}{2}(x-a)^2(t-x)^{n-2}+\binom{n}{3}(x-a)^3(t-x)^{n-3}\\ &+\ \binom{n}{4}(x-a)^4(t-x)^{n-4},\quad t\in[x,\frac{a+b}{2}].\end{aligned}\tag{1.142}$$

After some calculation from Theorem 1.18 we get

$$A_1^3(x)=\frac{(b-a)^3}{6(2x-a-b)^2}\quad B_1^3(x)=b-a-\frac{(b-a)^3}{3(2x-a-b)^2},$$
$$A_2^3(x)=A_3^3(x)=A_4^3(x)=B_3^3(x)=0.$$

Now we have

Corollary 1.15 *Let $f:[a,b]\to\mathbb{R}$ be a function with a piecewise continuous $n-$th derivative, for some $n\in\mathbb{N}$ and $x\in[a,\frac{a+b}{2})$. Further, let $Q_{n,x}(t)$ be defined by relation (1.142) and $S_n^3(t,x)$ be defined by relation (1.136). Then the following formula holds*

$$\int_a^b f(t)dt = D_3(f,x)+T_n^3(f,x)+(-1)^n\int_a^b S_n^3(t,x)f^{(n)}dt, \tag{1.143}$$

where

$$\begin{aligned} D_3(f,x) &= \frac{(b-a)^3}{6(2x-a-b)^2}(f(x)+f(a+b-x)) \\ &+ \left(b-a-\frac{(b-a)^3}{3(2x-a-b)^2}\right)f\left(\frac{a+b}{2}\right), \end{aligned} \tag{1.144}$$

and

$$\begin{aligned} T_n^3(f,x) &= \sum_{k=5}^{n} A_k^3(x)\left(f^{(k-1)}(x)+(-1)^{k-1}f^{(k-1)}(a+b-x)\right) \\ &+ \sum_{\substack{k=5\\ odd}}^{n} B_k^3(x)f^{(k-1)}\left(\frac{a+b}{2}\right). \end{aligned} \tag{1.145}$$

Proof. The proof follows from the Theorem 1.18 for the special case of the polynomial Q_n. □

Lemma 1.2 *For $x\in[\frac{5a+b}{6},\frac{a+b}{2})$ we have $Q_{n,x}(t)>0$, for $t\in(x,\frac{a+b}{2})$, when $n\geq 3$. Further, we have $Q_{n,a}(t)\leq 0$, for $t\in(a,\frac{a+b}{2})$, when $n\geq 3$.*

Proof. We use mathematical induction by n. For $n=3$ we have

$$Q_{3,x}(t)=(t-a)^3-\frac{(b-a)^3}{2(2x-a-b)^2}(t-x)^2.$$

The zeros are $t_1=\frac{a+b}{2}$,

$$\begin{aligned} t_{2,3} &= \tfrac{a+b}{2}+\tfrac{b-a}{4(2x-a-b)^2}\cdot\Big[(b-a)^2-3(2x-a-b)^2 \\ &\pm \sqrt{-3(2x-a-b)^4-8(b-a)(2x-a-b)^3-6(b-a)^2(2x-a-b)^2+(b-a)^4}\Big], \end{aligned}$$

By some calculation, we check that $t_2<x$, and for $x\in[\frac{5a+b}{6},\frac{a+b}{2})$, $t_3>\frac{a+b}{2}$, i.e. $t_{1,2,3}\notin(x,\frac{a+b}{2})$. So, $Q_{3,x}(t)>0$, for $t\in(x,\frac{a+b}{2})$, since $Q_{3,x}(x)>0$. For $n=4$ we have $Q_{4,x}(x)>0$. Since $Q_3(t,x)>0$ on $(x,\frac{a+b}{2})$ and by remark 1.22 we conclude that $Q_{4,x}(t)$ is monotone increasing on $(x,\frac{a+b}{2})$, for $x\in[\frac{5a+b}{6},\frac{a+b}{2})$ so $Q_{4,x}(t)>0$. For $n>5$ we know from the definition of the $Q_{n,x}$ that $Q_{n,x}(x)=0$. Now, let us assume that $Q_{n,x}(t)>0$ for some $n>3$. Relation $Q'_{n+1,x}(t)=(n+1)Q_{n,x}(t)>0$ implies that $Q_{n+1,x}$ is monotone increasing on $((x,\frac{a+b}{2})$. So, since $Q_{n+1,x}(x)=0$, we conclude $Q_{n+1,x}(t)>0$, for $t\in(x,\frac{a+b}{2})$, when $x\in[\frac{5a+b}{6},\frac{a+b}{2})$.

For the case $x=a$ we have $Q_{n,a}(t)=(t-a)^n-\frac{n(b-a)}{6}(t-a)^{n-1}=(t-a)^{n-1}\left[t-a-\frac{n(b-a)}{6}\right]$, so obviously for $n\geq 3$ we have $Q_{n,a}(t)\leq 0$, when $t\in(a,\frac{a+b}{2})$. □

Theorem 1.20 *For $x \in \{a\} \cup [\frac{5a+b}{6}, \frac{a+b}{2})$ and $f^{(2n)}$ continuous function on $[a,b]$ for some $n \geq 2$, we have*

$$\int_a^b f(t)dt = D_3(f,x) + T_{2n}^3(f,x) + f^{(2n)}(\eta) \cdot C_3(2n,\infty,x), \text{ for some } \eta \in (a,b),$$

where $D_3(f,x)$, $T_{2n}^3(f,x)$ and $C_3(2n,\infty,x)$ are defined by relations (1.144), (1.145) and (1.139) respectively.

Proof. The proof follows from the integral mean value theorem. □

Specially, for $n = 2$ we have

$$\int_a^b f(t)dt = D(f,x) + \frac{(b-a)^3}{384}\left[\frac{(b-a)^2}{5} - \frac{(a+b-2x)^2}{3}\right] f^{(4)}(\eta),$$

so for $x = a, \frac{5a+b}{6}, \frac{3a+b}{4}, \frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}$ we get the Simpson's, Maclaurin's, dual Simpson's nad Gauss-Legendre's two-point quadrature formula, respectively.

1.4.1 Simpson's formula

For $x = a$ we get the generalization of the famous Simpson's formula. Using Theorem 1.15 we get $A_1^3(a) = \frac{b-a}{6}$ and $A_k^3(a) = 0$ for $k > 1$, $B_k^3(a) = \frac{(b-a)^k}{2^{k-1}(k-1)!}\left[\frac{1}{k} - \frac{1}{3}\right]$ for odd k and $B_{2k}^3(a) = 0$, so the generalization of the Simpson's formula states

$$\begin{aligned} \int_a^b f(t)dt &= \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right) \\ &+ \sum_{\substack{k=5 \\ odd}}^{n} B_k^3(a) f^{(k-1)}\left(\frac{a+b}{2}\right) + (-1)^n \int_a^b S_n^3(t,a) f^{(n)}(t)dt. \end{aligned} \tag{1.146}$$

For $f : [a,b] \to \mathbb{R}$ such that $f^{(4)}$ is continuous, we get the well-known Simpson rule.

Remark 1.25 This formula and related inequalities were obtained in [79] and [100].

1.4.2 Dual Simpson's formula

For $x = \frac{3a+b}{4}$ we get the generalization of the dual Simpson's formula. Function $S_n^3(t, \frac{3a+b}{4})$ is determined by (1.136) and polynomial

$$\begin{aligned} Q_{n,\frac{3a+b}{4}}(t) &:= \left(t - \frac{3a+b}{4}\right)^n - \frac{5n(b-a)}{12}\left(t - \frac{3a+b}{4}\right)^{n-1} \\ &+ \binom{n}{2}\frac{(b-a)^2}{4^2}\left(t - \frac{3a+b}{4}\right)^{n-2} + \binom{n}{3}\frac{(b-a)^3}{4^3}\left(t - \frac{3a+b}{4}\right)^{n-3} \\ &+ \binom{n}{4}\frac{(b-a)^4}{4^4}\left(t - \frac{3a+b}{4}\right)^{n-4}, \quad t \in \left[\frac{3a+b}{4}, \frac{a+b}{2}\right]. \end{aligned} \tag{1.147}$$

Further, from Theorem 1.15 we have $A_1^3(\frac{3a+b}{4}) = \frac{2(b-a)}{3}$, $A_k^3(\frac{3a+b}{4}) = 0$, for $k = 2,3,4$ and $A_k^3(\frac{3a+b}{4}) = \frac{(-1)^{k-1}(b-a)^k}{4^k k!}$, for $k \geq 5$. $B_1^3(\frac{5a+b}{6}) = \frac{b-a}{4}$, $B_k^3(\frac{5a+b}{6}) = 0$, for $k = 2,3,4$ $B_k^3(\frac{3a+b}{4}) = \frac{(b-a)^k}{2^{2k-1}k!}\left[1 - \frac{5k}{3} + \binom{k}{2} + \binom{k}{3} + \binom{k}{4}\right]$, for odd $k \geq 5$, and $B_{2k}^3(\frac{3a+b}{4}) = 0$. For $f : [a,b] \to \mathbb{R}$ with a piecewise continuous $n-$th derivative we have by Corollary 1.15

$$\int_a^b f(t)dt = D_3\left(f, \frac{3a+b}{4}\right) + T_n^3\left(f, \frac{3a+b}{4}\right) + (-1)^n \int_a^b S_n^3\left(t, \frac{3a+b}{4}\right) f^{(n)}(t)dt \tag{1.148}$$

where

$$D_3\left(f, \frac{3a+b}{4}\right) = \frac{b-a}{3}\left(2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right).$$

Further, if $f^{(n)} \in L_p[a,b]$, then the following inequality holds:

$$\left|\int_a^b f(t)dt - D_3\left(f, \frac{3a+b}{4}\right) - T_n^3\left(f, \frac{3a+b}{4}\right)\right| \leq C_3\left(n, p, \frac{3a+b}{4}\right) \cdot \|f^{(n)}\|_p. \tag{1.149}$$

Specially,

$$C_3\left(1, \infty, \frac{3a+b}{4}\right) = \frac{5(b-a)^2}{24}, \quad C_3\left(1, 1, \frac{3a+b}{4}\right) = \frac{5(b-a)}{12}$$

$$C_3\left(2, \infty, \frac{3a+b}{4}\right) = \frac{5(b-a)^3}{324}, \quad C_3\left(2, 1, \frac{3a+b}{4}\right) = \frac{(b-a)^2}{24}$$

$$C_3\left(3, \infty, \frac{3a+b}{4}\right) = \frac{(b-a)^4}{576}, \quad C_3(3, 1, \frac{3a+b}{4}) = \frac{5(b-a)^3}{1296}$$

$$C_3\left(4, \infty, \frac{3a+b}{4}\right) = \frac{7(b-a)^5}{23040}, \quad C_3\left(4, 1, \frac{3a+b}{4}\right) = \frac{(b-a)^4}{1152}.$$

Remark 1.26 The same constants were obtained in [61].

If $f : [a,b] \to \mathbb{R}$ is such that $f^{(2n)}$ is continuous for some $n \in \mathbb{N}$, then we have

$$\int_a^b f(t)dt = D_3\left(f, \frac{3a+b}{4}\right) + T_{2n}^3\left(f, \frac{3a+b}{4}\right) + C_3\left(2n, \infty, \frac{3a+b}{4}\right) f^{(2n)}(\eta),$$

for some $\eta \in (a,b)$. (1.150)

Specially, for $n = 2$ we get dual Simpson's rule.

1.4.3 Maclaurin formula

For $x = \frac{5a+b}{6}$ we get the generalization of the Maclaurin formula. Function $S_n^3(t, \frac{5a+b}{6})$ is determined by (1.136) and polynomial

$$\begin{aligned} Q_n\left(t, \frac{5a+b}{6}\right) &:= \left(t - \frac{5a+b}{6}\right)^n - \frac{5n(b-a)}{24}\left(t - \frac{5a+b}{6}\right)^{n-1} \\ &+ \binom{n}{2}\frac{(b-a)^2}{6^2}\left(t - \frac{5a+b}{6}\right)^{n-2} + \binom{n}{3}\frac{(b-a)^3}{6^3}\left(t - \frac{5a+b}{6}\right)^{n-3} \\ &+ \binom{n}{4}\frac{(b-a)^4}{6^4}\left(t - \frac{5a+b}{6}\right)^{n-4}, \quad t \in \left[\frac{5a+b}{6}, \frac{a+b}{2}\right]. \end{aligned} \tag{1.151}$$

Further, from Theorem 1.15 we have $A_1^3(\frac{5a+b}{6}) = \frac{3(b-a)}{8}$, $A_k^3(\frac{5a+b}{6}) = 0$, for $k = 2,3,4$ and $A_k^3(\frac{5a+b}{6}) = \frac{(-1)^{k-1}(b-a)^k}{6^k k!}$, for $k \geq 5$. Further,

$$\begin{aligned} B_k^3\left(\frac{5a+b}{6}\right) &= \frac{2(b-a)^k}{3^k k!}\left[1 - \frac{5k}{8} + \frac{k(k-1)}{8} + \frac{k(k-1)(k-2)}{48}\right. \\ &\left. + \frac{k(k-1)(k-2)(k-3)}{384}\right], \end{aligned}$$

for odd k, and $B_{2k}^3(\frac{5a+b}{6}) = 0$. For $f : [a,b] \to \mathbb{R}$ with a piecewise continuous $n-$th derivative we have by Corollary 1.15

$$\int_a^b f(t)dt = D_3\left(f, \frac{5a+b}{6}\right) + T_n^3\left(f, \frac{5a+b}{6}\right) + (-1)^n \int_a^b S_n^3\left(t, \frac{5a+b}{6}\right) f^{(n)}(t)dt, \tag{1.152}$$

where

$$D_3\left(f, \frac{5a+b}{6}\right) = \frac{(b-a)}{8}\left(3f\left(\frac{5a+b}{6}\right) + 2f\left(\frac{a+b}{2}\right) + 3f\left(\frac{a+5b}{6}\right)\right).$$

Further, if $f^{(n)} \in L_p[a,b]$, then the following inequality holds:

$$\left|\int_a^b f(t)dt - D_3\left(f, \frac{5a+b}{6}\right) - T_n^3\left(f, \frac{5a+b}{6}\right)\right| \leq C_3\left(n, p, \frac{5a+b}{6}\right) \cdot \|f^{(n)}\|_p. \tag{1.153}$$

Specially,

$$C_3\left(1, \infty, \frac{5a+b}{6}\right) = \frac{25(b-a)^2}{288}, \quad C_3\left(1, 1, \frac{5a+b}{6}\right) = \frac{5(b-a)}{24}$$

$$C_3\left(2, \infty, \frac{5a+b}{6}\right) = \frac{(b-a)^3}{192}, \quad C_3\left(2, 1, \frac{5a+b}{6}\right) = \frac{(b-a)^2}{72}$$

$$C_3\left(3,\infty,\frac{5a+b}{6}\right)=\frac{(b-a)^4}{1728},\quad C_3\left(3,1,\frac{5a+b}{6}\right)=\frac{(b-a)^3}{768}$$

$$C_3\left(4,\infty,\frac{5a+b}{6}\right)=\frac{7(b-a)^5}{51840},\quad C_3\left(4,1,\frac{5a+b}{6}\right)=\frac{(b-a)^4}{3456}.$$

Remark 1.27 The same constants are also obtained in [45].

If $f:[a,b]\to\mathbb{R}$ is such that $f^{(2n)}$ is continuous for some $n\in\mathbb{N}$, then we have

$$\int_a^b f(t)dt \;=\; D_3\left(f,\frac{5a+6}{b}\right)+T^3_{2n}\left(f,\frac{5a+b}{6}\right)+C_3\left(2n,\infty,\frac{5a+b}{6}\right)f^{(2n)}(\eta),$$
$$\text{for some } \eta\in(a,b).$$

Specially, for $n=2$ we get Maclaurin rule.

1.4.4 Legendre-Gauss two-point formula

We consider case where the term $f(\frac{a+b}{2})$ doesn't appear. If we put in relation (1.143) condition $B^3_1(x)=0$, then we get $A^3_1(x_G)=\frac{b-a}{2}$ and $x_G=\frac{a+b}{2}-\frac{b-a}{2\sqrt{3}}$. Function $S^3_n(t,x_G)$ is determined by (1.136) and polynomial

$$\begin{aligned}
Q_{n,x_G}(t) \;:=&\; (t-x_G)^n-\frac{n(b-a)}{2\sqrt{3}}(t-x_G)^{n-1} \\
&+\binom{n}{2}\frac{(b-a)^2(2-\sqrt{3})}{6}(t-x_G)^{n-2} \\
&+\binom{n}{3}\left(\frac{(b-a)}{2\sqrt{3}}\right)^3(6\sqrt{3}-10)(t-x_G)^{n-3} \\
&+\binom{n}{4}\left(\frac{(b-a)}{2\sqrt{3}}\right)^4(28-16\sqrt{3})(t-x_G)^{n-4},\quad t\in\left[x_G,\frac{a+b}{2}\right].
\end{aligned}\tag{1.154}$$

Further, from Theorem 1.15 we have $A^3_k(x_G)=0$, for $k=2,3,4$ and $A^3_k(x_G)=\frac{(b-a)^k}{2^k k!}(1-\frac{1}{\sqrt{3}})^k$, for $k\geq 5$. Further,

$$\begin{aligned}
B^3_k(x_G) \;=&\; \frac{2(b-a)^k}{(2\sqrt{3})^k k!}\left[1-k+\binom{k}{2}(4-2\sqrt{3})+\binom{k}{3}(6\sqrt{3}-10)\right. \\
&\left.+\binom{k}{4}(28-16\sqrt{3})\right],
\end{aligned}$$

for odd $k\geq 5$, and $B^3_k(x_G)=0$ otherwise. For $f:[a,b]\to\mathbb{R}$ with a piecewise continuous $n-$th derivative we have by Corollary 1.15 the following formula

$$\int_a^b f(t)dt=D_3(f,x_G)+T^3_n(f,x_G)+(-1)^n\int_a^b S^3_n(t,x_G)f^{(n)}(t)dt,\tag{1.155}$$

where

$$D_3(f,x_G) = \frac{b-a}{2}\left(f(x_G)+f(a+b-x_G)\right).$$

Further, if $f^{(n)} \in L_p[a,b]$, then the following inequality holds:

$$\left|\int_a^b f(t)dt - D_3(f,x_G) - T_n^3(f,x_G)\right| \le C_3(n,p,x_{G2}) \cdot \|f^{(n)}\|_p. \tag{1.156}$$

Specially,

$$\begin{aligned}
C_3(1,\infty,x_G) &= \frac{(5-2\sqrt{3})(b-a)^2}{12}, \quad C_3(1,1,x_G) = \frac{(3-\sqrt{3})(b-a)}{6} \\
C_3(2,\infty,x_G) &= \frac{\sqrt{26\sqrt{3}-45}(b-a)^3}{18}, \quad C_3(2,1,x_G) = \frac{(2-\sqrt{3})(b-a)^2}{12} \\
C_3(3,\infty,x_G) &= \frac{(9-4\sqrt{3})(b-a)^4}{1728}, \\
C_3(3,1,x_G) &= \frac{(2-\sqrt{3})\sqrt{2\sqrt{3}-3}(b-a)^3}{72} \\
C_3(4,\infty,x_G) &= \frac{(b-a)^5}{4320}, \quad C_3(4,1,x_G) = \frac{(9-4\sqrt{3})(b-a)^4}{3456}.
\end{aligned}$$

If $f : [a,b] \to \mathbb{R}$ is such that $f^{(2n)}$ is continuous for some $n \in \mathbb{N}$, then we have

$$\int_a^b f(t)dt = D_3(f,x_G) + T_{2n}^3(f,x_G) + C_3(2n,\infty,x_G)f^{(2n)}(\eta),$$

for some $\eta \in (a,b)$.

Specially, for $n = 2$ we get Legendre-Gauss rule two-point rule.

Now we develop weighted version of three-point quadrature formulae. Let $w : [a,b] \to \mathbb{R}$ be some integrable function and $x \in [a, \frac{a+b}{2})$. Let $n \in \mathbb{N}$ and $\{L_j\}_{j=0,1,\dots,n}$ be some sequence of harmonic polynomials such that $\deg L_j \le j-1$ and $L_0 \equiv 0$. Let us consider subdivision of the segment $[a,b]$:

$$\sigma := \{x_0 < x_1 < x_2 < x_3 < x_4\}$$

where $x_0 = a$, $x_1 = x$, $x_2 = \frac{a+b}{2}$, $x_3 = a+b-x$ and $x_4 = b$. For $k = 1,\dots,n$ we define functions $w_{jk}^3 : [x_{j-1},x_j] \to \mathbb{R}$, for $j = 1,2,3,4$, in the following way:

$$\begin{aligned}
w_{1k}^3(t) &:= \frac{1}{(k-1)!}\int_a^t (t-s)^{k-1}w(s)ds, \\
w_{2k}^3(t) &:= \frac{1}{(k-1)!}\int_x^t (t-s)^{k-1}w(s)ds + L_k(t), \\
w_{3k}^3(t) &:= -\frac{1}{(k-1)!}\int_t^{a+b-x} (t-s)^{k-1}w(s)ds + (-1)^k L_k(a+b-t), \\
w_{4k}^3(t) &:= -\frac{1}{(k-1)!}\int_t^b (t-s)^{k-1}w(s)ds,
\end{aligned}$$

and $w_{j0}^3(t) := w(t)$. Further, let us define

$$W_n^3(t,x) = \begin{cases} w_{1n}^3(t), & \text{for } t \in [a,x], \\ w_{2n}^3(t), & \text{for } t \in (x, \frac{a+b}{2}], \\ w_{3n}^3(t), & \text{for } t \in (\frac{a+b}{2}, a+b-x] \\ w_{4n}^3(t), & \text{for } t \in (a+b-x, b]. \end{cases} \tag{1.157}$$

Remark 1.28 Sequences $\{w_{jk}^3\}_{k=0,1,\ldots,n}$ are $w-$harmonic sequences of functions on $[x_{j-1}, x_j]$, for every $j = 1,2,3,4$.

Remark 1.29 If, in addition, we have $w(t) = w(a+b-t)$, for each $t \in [a,b]$, then the following symmetry conditions hold for $k = 1,\ldots,n$:

$$w_{1k}^3(t) = (-1)^k w_{4k}^3(a+b-t), \quad \text{for } t \in [a,x]$$

and

$$w_{2k}^3(t) = (-1)^k w_{3k}^3(a+b-t), \quad \text{for } t \in \left(x, \frac{a+b}{2}\right].$$

Theorem 1.21 *Let $f : [a,b] \to \mathbb{R}$ be a function with piecewise continuous $n-$th derivative, for some $n \in \mathbb{N}$, and $x \in [a, \frac{a+b}{2})$. Further, let $\{L_k\}_{k=0,1,\ldots,n}$ be some sequence of harmonic polynomials such that* $\deg L_j \le j-1$ *and $L_0(t) = 0$, and $W_n^3(t,x)$ be defined by (1.157). Then the following identity holds:*

$$\begin{aligned} \int_a^b w(t)f(t)dt &= \sum_{k=1}^n A_{k,w}^3(x)\left(f^{(k-1)}(x) + (-1)^{k-1}f^{(k-1)}(a+b-x)\right) \\ &+ \sum_{\substack{k=1 \\ odd\ k}} B_{k,w}^3(x) f^{(k-1)}\left(\frac{a+b}{2}\right) + (-1)^n \int_a^b W_n^3(t,x) f^{(n)}(t)dt, \end{aligned} \tag{1.158}$$

where

$$A_{k,w}^3(x) = (-1)^{k-1}\left[\frac{1}{(k-1)!}\int_a^x (x-s)^{k-1}w(s)ds - L_k(x)\right], \quad k \ge 1,$$

$$\begin{aligned} B_{k,w}^3(x) &= 2\left[\frac{1}{(k-1)!}\int_x^{\frac{a+b}{2}} \left(\frac{a+b}{2} - s\right)^{k-1} w(s)ds \right. \\ &\left. + L_k\left(\frac{a+b}{2}\right)\right], \quad \textit{for odd } k \ge 1 \end{aligned}$$

and

$$B_{k,w}^3(x) = w_{2k}^3(x) - w_{3k}^3(x) = 0, \quad \textit{for even } k \ge 1.$$

Proof. We consider subdivision $x_0 = a$, $x_1 = x$, $x_2 = \frac{a+b}{2}$, $x_3 = a+b-x$ and $x_4 = b$ of the interval $[a,b]$ and apply formula (1.20) with $m = 4$. $\square$

Theorem 1.22 *Let $f : [a,b] \to \mathbb{R}$ be a function with a piecewise continuous $n-$th derivative and $f^{(n)} \in L_p[a,b]$ for some $n \in \mathbb{N}$ and some $1 \le p \le \infty$. Then we have the following inequality*

$$\begin{aligned} \Big| \int_a^b w(t)f(t)dt \; &- \sum_{k=1}^{n} A_{k,w}^3(x)\left(f^{(k-1)}(x) + (-1)^{k-1} f^{(k-1)}(a+b-x)\right) \\ &- \sum_{\substack{k=1 \\ \text{odd } k}}^{n} B_{k,w}^3(x) f^{(k-1)}\left(\frac{a+b}{2}\right) \Big| \\ &\le C_3(n,p,x,w) \cdot \|f^{(n)}\|_p, \end{aligned} \tag{1.159}$$

where

$$C_3(n,p,x,w) = \tag{1.160}$$
$$\begin{cases} 2^{1/q}\left[\int_a^{\frac{a+b}{2}} \left|W_n^3(t,x)\right|^q dt\right]^{1/q}, & \frac{1}{p}+\frac{1}{q}=1, \quad 1<p\le\infty \\ \sup\{\left|W_n^3(t,x)\right|, t \in [a, \frac{a+b}{2}]\}, & p=1, \end{cases}$$

The inequality is best possible for $p = 1$ and sharp for $1 < p \le \infty$. Equality is attained for the function $f_* : [a,b] \to \mathbb{R}$ defined by

$$f_*(t) = \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}\left|W_n^3(s,x)\right|^{\frac{1}{p-1}} \operatorname{sgn} W_n^3(s,x)ds \tag{1.161}$$

for $1 < p < \infty$, while for $p = \infty$

$$f_*(t) = \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1} \operatorname{sgn} W_n^3(s,x)ds \tag{1.162}$$

Proof. This theorem is a special case of the general L_p theorem obtained in [72]. □

Theorem 1.23 *If $f : [a,b] \to \mathbb{R}$ is such that $f^{(2n)}$ is continuous and if $w_{2,2n}^3(t) \ge 0$, for each $t \in [x, \frac{a+b}{2}]$, then the following identity holds*

$$\begin{aligned} \int_a^b w(t)f(t)dt \; &= \sum_{k=1}^{2n} A_{k,w}^3(x)\left(f^{(k-1)}(x) + (-1)^{k-1} f^{(k-1)}(a+b-x)\right) \\ &+ \sum_{\substack{k=1 \\ \text{odd } k}}^{2n} B_{k,w}^3(x) f^{(k-1)}\left(\frac{a+b}{2}\right) + C_3(2n,\infty,x,w) f^{(2n)}(\eta), \end{aligned}$$

for some $\eta \in [a,b]$.

Proof. The proof follows from the [72] for the special case $m = 4$. □

Let $w : [a,b] \to \mathbb{R}$ be some integrable function and let $x \in [a, \frac{a+b}{2})$. Let us consider special sequence of polynomials $\{L_{j,x}(t)\}_{j=0,1,\dots,n}$ defined as follows

$$L_{0,x}(t) := 0, \quad \text{for } t \in \left[x, \frac{a+b}{2}\right]$$

$$
\begin{aligned}
L_{1,x}(x) &:= \int_a^x w(s)ds - \frac{2}{(a+b-2x)^2}\int_a^b \left(s^2 - \left(\frac{a+b}{2}\right)^2\right) w(s)ds \\
L_{j,x}(x) &:= \frac{1}{(j-1)!}\int_a^x (x-s)^{j-1} w(s)ds, \quad \text{for } j = 2,3,4,5,6 \\
&\text{and} \\
L_{j,x}(t) &:= \sum_{k=1}^{6\wedge j} L_{k,x}(x)\frac{(t-x)^{j-k}}{(j-k)!}, \quad \text{for } t \in \left[x, \frac{a+b}{2}\right], \quad j = 1,\dots,n.
\end{aligned} \tag{1.163}
$$

Remark 1.30 The polynomials $\{L_{j,x}\}_{j=0,1,\dots,n}$ are harmonic and $\deg L_{j,x} \le j-1$.

Now we can state the general three point weighted integral formula

Corollary 1.16 *Let $f : [a,b] \to \mathbb{R}$ be a function with piecewise continuous $n-$th derivative, for some $n \in \mathbb{N}$. Let $w : [a,b] \to \mathbb{R}$ be an integrable function such that $w(t) = w(a+b-t)$, for each $t \in [a,b]$ and let $x \in [a, \frac{a+b}{2})$. Further, let $L_{j,x}(t)$ and $W_n^3(\cdot,x)$ be defined by (1.163) and (1.157). Then the following identity holds:*

$$
\int_a^b w(t)f(t)dt = D_{w,3}(f,x) + T_{n,w}^3(f,x) + (-1)^n \int_a^b W_n^3(t,x)f^{(n)}(t)dt \tag{1.164}
$$

where

$$
\begin{aligned}
D_{w,3}(f,x) &= A_{1,w}^3(x)\left(f(x) + f(a+b-x)\right) \\
&+ \left(\int_a^b w(s)ds - 2A_{1,w}^3(x)\right) f\left(\frac{a+b}{2}\right),
\end{aligned} \tag{1.165}
$$

$$
A_{1,w}^3(x) = \frac{2}{(2x-a-b)^2}\int_a^b \left(s^2 - \left(\frac{a+b}{2}\right)^2\right) w(s)ds
$$

and

$$
\begin{aligned}
T_{n,w}^3(f,x) &= \sum_{k=7}^{n} A_{k,w}^3(x)\left(f^{(k-1)}(x) + (-1)^{k-1} f^{(k-1)}(a+b-x)\right) \\
&+ \sum_{\substack{k=5 \\ \text{odd } k}} B_{k,w}^3(x) f^{(k-1)}\left(\frac{a+b}{2}\right).
\end{aligned} \tag{1.166}
$$

Proof. The proof follows from the theorem 1.21 for the special choice of the polynomials L_j. □

Remark 1.31 If, in addition, we demand that $B_{5,w}^3(x) = 0$, then we get

$$
x = \frac{a+b}{2} - \sqrt{\frac{\int_a^b \left(s - \frac{a+b}{2}\right)^4 w(s)ds}{\int_a^b (s^2 - (\frac{a+b}{2})^2) w(s)ds}}.
$$

Therefore, for such choice of x we will get the quadrature formula with three nodes which is accurate for the polynomials of degree at most 5, and the approximation formula includes derivatives of order 6 and more.

1.4.5 Legendre-Gauss three-point formula

For $w(t) = 1, \quad t \in [a,b]$ we have

$$\begin{aligned} L_{0,x}(t) &:= 0, \quad \text{for } t \in \left[x, \frac{a+b}{2}\right] \\ L_{1,x}(x) &:= x-a-\frac{(b-a)^3}{6(a+b-2x)^2} \\ L_{j,x}(x) &:= \frac{(x-a)^j}{j!}, \quad \text{for } j=2,3,4,5,6 \\ &\text{and} \\ L_{j,x}(t) &:= \sum_{k=1}^{6\wedge j} L_{k,x}(x)\frac{(t-x)^{j-k}}{(j-k)!}, \quad \text{for } t \in \left[x, \frac{a+b}{2}\right], \quad j=1,\dots,n. \end{aligned} \tag{1.167}$$

and $W^3_{n,w}(t,x)$ is defined by (1.157). Further, from Theorem 1.21 we get

$$A^3_{k,w}(x) = (-1)^{k-1}\left[\frac{(x-a)^k}{k!} - L_{k,x}(x)\right], \quad \text{for } k=1,\dots,n \tag{1.168}$$

and

$$B^3_{k,w}(x) := \begin{cases} 2\cdot\left[\frac{\left(\frac{a+b}{2}-x\right)^k}{k!} + L_{k,x}\left(\frac{a+b}{2}\right)\right] & \text{odd } k=1,\dots,n \\ 0 & \text{even } k=1,\dots,n, \end{cases} \tag{1.169}$$

Specially, we have $A^3_{1,w}(x) = \frac{(b-a)^3}{6(a+b-2x)^2}$, $A^3_{k,w}(x) = 0$, for $k=2,\dots,6$ and $B^3_{3,w}(x) = 0$. If $f:[a,b]\to\mathbb{R}$ is such that $f^{(n)}$ is piecewise continuous, then the following identity holds:

$$\int_a^b f(t)dt = D_{w,3}(f,x) + T^3_{n,w}(f,x) + (-1)^n\int_a^b W^3_n(t,x)f^{(n)}(t)dt, \tag{1.170}$$

where $D_{w,3}(f,x)$ and $T^3_{n,w}(f,x)$ are defined by (1.165) and (1.166) respectively. Specially, if we request $B^3_{5,w}(x) = 0$, and for $[a,b] = [-1,1]$, we get $x = -\frac{\sqrt{15}}{5}$, so we have the generalization of the Legendre-Gauss three-point formula [44]. Further, if the assumptions of the Theorem 1.23 hold, we have

$$\int_{-1}^1 f(t)dt = D_{w,3}\left(f,-\frac{\sqrt{15}}{5}\right) + T^3_{2n,w}\left(f,-\frac{\sqrt{15}}{5}\right) + C\left(2n,\infty,-\frac{\sqrt{15}}{5},w\right)f^{(2n)}(\eta), \tag{1.171}$$

where

$$D_{w,3}\left(f,-\frac{\sqrt{15}}{5}\right) = \frac{1}{9}\left[5f\left(-\frac{\sqrt{15}}{5}\right) + 8f(0) + 5f\left(\frac{\sqrt{15}}{5}\right)\right].$$

Specially, for $n = 3$ we get famous Legendre-Gauss threepoint quadrature formula. If the assumptions of the Theorem 1.22 hold we get

$$\Big|\int_{-1}^{1} f(t)dt - D_{w,3}\left(f, -\frac{\sqrt{15}}{5}\right) - T_{n,w}^{3}\left(f, -\frac{\sqrt{15}}{5}\right)\Big| \leq C_3\left(n, p, -\frac{\sqrt{15}}{5}, w\right) \|f^{(n)}\|_p.$$

Specially, for $n \leq 6$ is $T_{n,w}(f, -\frac{\sqrt{15}}{5}) = 0$ and we compute

$$C_3\left(1, \infty, -\frac{\sqrt{15}}{5}, w\right) \approx 0.357338, \quad C_3\left(1, 1, -\frac{\sqrt{15}}{5}, w\right) \approx 0.444444$$

$$C_3\left(2, \infty, -\frac{\sqrt{15}}{5}, w\right) \approx 0.0374355, \quad C_3\left(2, 1, -\frac{\sqrt{15}}{5}, w\right) \approx 0.0696685$$

$$C_3\left(3, \infty, -\frac{\sqrt{15}}{5}, w\right) \approx 0.00548184, \quad C_3\left(3, 1, -\frac{\sqrt{15}}{5}, w\right) \approx 0.0063794$$

$$C_3\left(4, \infty, -\frac{\sqrt{15}}{5}, w\right) \approx 0.000908828, \quad C_3\left(4, 1, -\frac{\sqrt{15}}{5}, w\right) \approx 0.00136648$$

$$C_3\left(5, \infty, -\frac{\sqrt{15}}{5}, w\right) \approx 0.000195789,$$

$$C_3\left(5, 1, -\frac{\sqrt{15}}{5}, w\right) \approx 0.000227207$$

$$C_3\left(6, \infty, -\frac{\sqrt{15}}{5}, w\right) \approx 0.0000634921,$$

$$C_3\left(6, 1, -\frac{\sqrt{15}}{5}, w\right) \approx 0.0000978944.$$

1.4.6 Chebyshev-Gauss three-point formula

For $w(t) = \frac{1}{\sqrt{1-t^2}}, \quad t \in (-1,1)$ we have

$$\begin{aligned}
L_{0,x}(t) &:= 0, \quad \text{for } t \in [x,0] \\
L_{1,x}(x) &:= \arcsin x + \frac{\pi}{2} - \frac{\pi}{4x^2} \\
L_{j,x}(x) &:= \frac{(x+1)^{j-\frac{1}{2}}\sqrt{\pi}}{\sqrt{2}\Gamma(\frac{1}{2}+j)} F\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}+j, \frac{x+1}{2}\right), \quad \text{for } j = 2,3,4,5,6 \\
&\text{and} \\
L_{j,x}(t) &:= \sum_{k=1}^{6\wedge j} L_{k,x}(x)\frac{(t-x)^{j-k}}{(j-k)!}, \quad \text{for } t \in [x,0], \quad j = 1, \ldots, n.
\end{aligned} \tag{1.172}$$

and $W^3_{n,w}(t,x)$ is defined by (1.157). Further, from Theorem 1.21 we get

$$A^3_{k,w}(x) = (-1)^{k-1}\left[\frac{(x+1)^{j-\frac{1}{2}}\sqrt{\pi}}{\sqrt{2}\Gamma(\frac{1}{2}+j)}F\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}+j,\frac{x+1}{2}\right)-L_{k,x}(x)\right],$$
$$\text{for } k=1,\dots,n$$

and

$$B^3_{k,w}(x) := \begin{cases} 2\cdot\left[\frac{(-1)^{k-1}}{(k-1)!}\int_x^0 \frac{s^{k-1}}{\sqrt{1-s^2}}ds + L_{k,x}(0)\right] & \text{odd } k=1,\dots,n \\ 0 & \text{even } k=1,\dots,n, \end{cases} \tag{1.173}$$

Specially, we have $A^3_{1,w}(x) = \frac{\pi}{4x^2}$, $A^3_{k,w}(x) = 0$, for $k = 2,\dots,6$ and $B^3_{3,w}(x) = 0$. If $f : [-1,1] \to \mathbb{R}$ is such that $f^{(n)}$ is piecewise continuous, then the following identity holds:

$$\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}dt = D_{w,3}(f,x)+T^3_{n,w}(f,x)+(-1)^n\int_{-1}^{1}W^3_{n,w}(t,x)f^{(n)}(t)dt,$$

where $D_{w,3}(f,x)$ and $T^3_{n,w}(f,x)$ are defined by (1.165) and (1.166) respectively. Specially, if we request $B^3_{5,w}(x) = 0$, we get $x = -\frac{\sqrt{3}}{2}$, so we have the generalization of the Chebyshev-Gauss three-point formula of the first kind [44]. Further, if the assumptions of the Theorem 1.23 hold, we have

$$\begin{aligned}\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}dt &= D_{w,3}\left(f,-\frac{\sqrt{3}}{2}\right)+T^3_{2n,w}\left(f,-\frac{\sqrt{3}}{2}\right)\\ &+ C_3(2n,\infty,-\frac{\sqrt{3}}{2},w)f^{(2n)}(\eta),\end{aligned}$$

where

$$D_{w,3}\left(f,-\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}\left[f\left(-\frac{\sqrt{3}}{2}\right)+f(0)+f\left(\frac{\sqrt{3}}{2}\right)\right].$$

Specially, for $n = 3$ we get famous Chebyshev-Gauss threepoint quadrature formula of the first kind. If the assumptions of the Theorem 1.22 hold we get

$$\left|\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}dt - D_{w,3}\left(f,-\frac{\sqrt{3}}{2}\right)-T^3_{n,w}\left(f,-\frac{\sqrt{3}}{2}\right)\right| \le C_3\left(n,p,-\frac{\sqrt{3}}{2},w\right)\|f^{(n)}\|_p.$$

Specially, for $n \le 6$ is $T^3_{n,w}(f,-\frac{\sqrt{3}}{2}) = 0$ and we compute

$$C_3\left(1,\infty,-\frac{\sqrt{3}}{2},w\right) \approx 0.535898, \quad C_3\left(1,1,-\frac{\sqrt{3}}{2},w\right) = \frac{\pi}{6} \approx 0.523598$$
$$C_3\left(2,\infty,-\frac{\sqrt{3}}{2},w\right) \approx 0.0578, \quad C_3\left(2,1,-\frac{\sqrt{3}}{2},w\right) = 1-\frac{\pi\sqrt{3}}{6} \approx 0.0931$$

$$C_3\left(3,\infty,-\frac{\sqrt{3}}{2},w\right) \approx 0.009162, \quad C_3\left(3,1,-\frac{\sqrt{3}}{2},w\right) \approx 0.00959813$$

$$C_3\left(4,\infty,-\frac{\sqrt{3}}{2},w\right) \approx 0.000165293, \quad C_3\left(4,1,-\frac{\sqrt{3}}{2},w\right) \approx 0.002251$$

$$C_3\left(5,\infty,-\frac{\sqrt{3}}{2},w\right) \approx 0.0003867, \quad C_3\left(5,1,-\frac{\sqrt{3}}{2},w\right) \approx 0.000413232$$

$$C_3\left(6,\infty,-\frac{\sqrt{3}}{2},w\right) = \frac{\pi}{23040} \approx 0.00013635,$$

$$C_3\left(6,1,-\frac{\sqrt{3}}{2},w\right) \approx 0.000193352.$$

1.4.7 Chebyshev-Gauss three-point formula of the second kind

For $w(t)=\sqrt{1-t^2}, \quad t\in[-1,1]$ we have

$$\begin{aligned}
L_{0,x}(t) &:= 0, \quad \text{for } t\in[x,0]\\
L_{1,x}(x) &:= \frac{1}{2}\left(\arcsin x+\frac{\pi}{2}-\frac{\pi}{8x^2}+\frac{x\sqrt{1-x^2}}{2}\right)\\
L_{j,x}(x) &:= \frac{(x+1)^{j+\frac{1}{2}}\sqrt{2\pi}}{\Gamma(\frac{3}{2}+j)}F\left(-\frac{1}{2},\frac{3}{2},\frac{3}{2}+j,\frac{x+1}{2}\right), \text{ for } j=2,3,4,5,6\\
&\text{and}\\
L_{j,x}(t) &:= \sum_{k=1}^{6\wedge j} L_{k,x}(x)\frac{(t-x)^{j-k}}{(j-k)!}, \quad \text{for } t\in[x,0], \quad j=1,\dots,n.
\end{aligned}$$

and $W^3_{n,w}(t,x)$ is defined by (1.157). Further, from Theorem 1.21 we get

$$A^3_{k,w}(x)=(-1)^{k-1}\left[\frac{(x+1)^{j+\frac{1}{2}}\sqrt{2\pi}}{\Gamma(\frac{3}{2}+j)}F\left(-\frac{1}{2},\frac{3}{2},\frac{3}{2}+j,\frac{x+1}{2}\right)-L_{k,x}(x)\right], \tag{1.174}$$

for $k=1,\dots,n$ and

$$B^3_{k,w}(x):=\begin{cases} 2\cdot\left[\frac{(-1)^{k-1}}{(k-1)!}\int_x^0 s^{k-1}\sqrt{1-s^2}ds+L_{k,x}(0)\right] & \text{odd } k=1,\dots,n\\ 0 & \text{even } k=1,\dots,n,\end{cases}$$

Specially, we have $A^3_{1,w}(x)=\frac{x\sqrt{1-x^2}}{4}-\frac{\pi}{16x^2}$, $A^3_{k,w}(x)=0$, for $k=2,\dots,6$ and $B^3_{3,w}(x)=0$. If $f:[a,b]\to\mathbb{R}$ is such that $f^{(n)}$ is piecewise continuous, then the following identity holds:

$$\int_{-1}^{1} f(t)\sqrt{1-t^2}dt = D_{w,3}(f,x)+T^3_{n,w}(f,x)+(-1)^n\int_{-1}^{1} W^3_{n,w}(t,x)f^{(n)}(t)dt,$$

where $D_{w,3}(f,x)$ and $T^3_{n,w}(f,x)$ are defined by (1.165) and (1.166) respectively. Specially, if we request $B^3_{5,w}(x)=0$, we get $x=-\frac{\sqrt{2}}{2}$, so we have the generalization of the Chebyshev-Gauss three-point formula of the second kind [44], for the special case of the weight function $w(t)=\sqrt{1-t^2}$. Further, if the assumptions of the Theorem 1.23 hold, we have

$$\begin{aligned}\int_{-1}^{1} f(t)\sqrt{1-t^2}dt &= D_{w,3}\left(f,-\frac{\sqrt{2}}{2}\right)+T^3_{2n,w}\left(f,-\frac{\sqrt{2}}{2}\right)\\ &+ C_3\left(2n,\infty,-\frac{\sqrt{2}}{2},w\right)f^{(2n)}(\eta),\end{aligned}$$

where

$$D_{w,3}\left(f,-\frac{\sqrt{2}}{2}\right)=\frac{\pi}{8}\left[f\left(-\frac{\sqrt{2}}{2}\right)+2f(0)+f\left(\frac{\sqrt{2}}{2}\right)\right].$$

Specially, for $n=3$ we get famous Chebyshev three-point quadrature formula of the second kind. If the assumptions of the Theorem 1.22 hold we get

$$\begin{aligned}&\left|\int_{-1}^{1} f(t)\sqrt{1-t^2}dt-D_{w,3}\left(f,-\frac{\sqrt{2}}{2}\right)-T^3_{n,w}\left(f,-\frac{\sqrt{2}}{2}\right)\right|\\ &\leq C_3\left(n,p,-\frac{\sqrt{2}}{2},w\right)\|f^{(n)}\|_p.\end{aligned}$$

Specially, for $n\leq 6$ is $T^3_{n,w}(f,-\frac{\sqrt{2}}{2})=0$ and we compute

$$C_3\left(1,\infty,-\frac{\sqrt{2}}{2},w\right)\approx 0.2691696884,\quad C_3\left(1,1,-\frac{\sqrt{2}}{2},w\right)=\frac{\pi}{8}\approx 0.392699$$

$$C_3\left(2,\infty,-\frac{\sqrt{2}}{2},w\right)\approx 0.002670866417,\quad C_3\left(2,1,-\frac{\sqrt{2}}{2},w\right)\approx 0.0556531497$$

$$C_3\left(3,\infty,-\frac{\sqrt{2}}{2},w\right)\approx 0.003644471212,\quad C_3\left(3,1,-\frac{\sqrt{2}}{2},w\right)\approx 0.00462924$$

$$C_3\left(4,\infty,-\frac{\sqrt{2}}{2},w\right)\approx 0.0005619519067,$$

$$C_3\left(4,1,-\frac{\sqrt{2}}{2},w\right)\approx 0.0009177930807$$

$$C_3\left(5,\infty,-\frac{\sqrt{2}}{2},w\right)\approx 0.0001128405047,$$

$$
\begin{aligned}
C_3\left(5,1,-\frac{\sqrt{2}}{2},w\right) &\approx 0.000140488\\
C_3\left(6,\infty,-\frac{\sqrt{2}}{2},w\right) &= \frac{\pi}{92160} \approx 0.00003408846195,\\
C_3\left(6,1,-\frac{\sqrt{2}}{2},w\right) &\approx 0.00005642.
\end{aligned}
$$

1.4.8 Gauss-Hermite three-point formula

Let $w(t) = e^{-t^2}, \quad t \in \mathbb{R}$. Since this function is defined on the infinite interval, we consider it on $[-M,M]$, for some $M \in \mathbb{R}_+$ and then let $M \to \infty$. So we have

$$
\begin{aligned}
L_{0,x}(t) &:= 0, \quad \text{for } t \in [x,0]\\
L_{1,x}(x) &:= \int_{\infty}^{x} e^{-s^2} ds - \frac{1}{2x^2}\int_{-\infty}^{\infty} s^2 e^{-s^2} ds\\
L_{j,x}(x) &:= \frac{1}{(j-1)!}\int_{-\infty}^{x} (x-s)^{j-1} e^{-s^2} ds, \quad \text{for } j = 2,3,4,5,6\\
&\text{and}\\
L_{j,x}(t) &:= \sum_{k=1}^{6\wedge j} L_{k,x}(x)\frac{(t-x)^{j-k}}{(j-k)!}, \quad \text{for } t \in [x,0], \quad j = 1,\ldots,n.
\end{aligned}
\tag{1.175}
$$

and $W_{n,w}^3(t,x)$ is defined by (1.157). Further, from Theorem 1.21 we get

$$
A_{k,w}^3(x) = (-1)^{k-1}\left[\frac{1}{(k-1)!}\int_{-\infty}^{x}(x-s)^{k-1}e^{-s^2}ds - L_{k,x}(x)\right], \text{ for } k = 1,\ldots,n
$$

and

$$
B_{k,w}^3(x) := \begin{cases} 2\cdot\left[\frac{(-1)^{k-1}}{(k-1)!}\int_x^0 s^{k-1}e^{-s^2}ds + L_{k,x}(0)\right] & \text{odd } k = 1,\ldots,n\\ 0 & \text{even } k = 1,\ldots,n, \end{cases}
$$

Specially, we have $A_{1,w}^3(x) = \frac{\sqrt{\pi}}{4x^2}$, $A_{k,w}^3(x) = 0$, for $k = 2,\ldots,6$ and $B_{3,w}^3(x) = 0$. If $f:\mathbb{R}\to\mathbb{R}$ is such that $f^{(n)}$ is piecewise continuous, then the following identity holds:

$$
\int_{-\infty}^{\infty} f(t)e^{-t^2}dt = D_{w,3}(f,x) + T_{n,w}^3(f,x) + (-1)^n\int_{-\infty}^{\infty} W_{n,w}^3(t,x)f^{(n)}(t)dt,
$$

where $D_{w,3}(f,x)$ and $T_{n,w}^3(f,x)$ are defined by (1.165) and (1.166) respectively. Specially, if we request $B_{5,w}^3(x) = 0$, we get $x = -\frac{\sqrt{6}}{2}$, so we have the generalization of the Gauss-Hermite three-point formula [44]. Further, if the assumptions of the Theorem 1.23 hold, we have

$$\int_{-\infty}^{\infty} f(t)e^{-t^2}dt = D_{w,3}\left(f,-\frac{\sqrt{6}}{2}\right)+T^3_{2n,w}\left(f,-\frac{\sqrt{6}}{2}\right) + C_3\left(2n,\infty,-\frac{\sqrt{6}}{2},w\right)f^{(2n)}(\eta),$$

where

$$D_{w,3}\left(f,-\frac{\sqrt{2}}{2}\right)=\frac{\sqrt{\pi}}{6}\left[f\left(-\frac{\sqrt{6}}{2}\right)+4f(0)+f\left(\frac{\sqrt{6}}{2}\right)\right].$$

Specially, for $n=3$ we get famous Hermite-Gauss three-point quadrature formula. If the assumptions of the Theorem 1.22 hold we get

$$\left|\int_{-\infty}^{\infty} f(t)e^{-t^2}dt - D_{w,3}\left(f,-\frac{\sqrt{6}}{2}\right)-T^3_{n,w}\left(f,-\frac{\sqrt{6}}{2}\right)\right| \le C_3\left(n,p,-\frac{\sqrt{6}}{2},w\right)\|f^{(n)}\|_p.$$

Specially, for $n\le 6$ is $T^3_{n,w}(f,-\frac{\sqrt{6}}{2})=0$ and we compute

$$C_3\left(1,\infty,-\frac{\sqrt{6}}{2},w\right)\approx 1.808359723,\quad C_3\left(1,1,-\frac{\sqrt{6}}{2},w\right)\approx 0.5908179503$$

$$C_3\left(2,\infty,-\frac{\sqrt{6}}{2},w\right)\approx 0.1036123774,\quad C_3\left(2,1,-\frac{\sqrt{6}}{2},w\right)\approx 0.1381993727$$

$$C_3\left(3,\infty,-\frac{\sqrt{6}}{2},w\right)\approx 0.02668915776,\quad C_3\left(3,1,-\frac{\sqrt{6}}{2},w\right)\approx 0.0193806$$

$$C_3\left(4,\infty,-\frac{\sqrt{6}}{2},w\right)\approx 0.007863163927,\quad C_3\left(4,1,-\frac{\sqrt{6}}{2},w\right)\approx 0.007116823486$$

$$C_3\left(5,\infty,-\frac{\sqrt{6}}{2},w\right)\approx 0.003099143144,\quad C_3\left(5,1,-\frac{\sqrt{6}}{2},w\right)\approx 0.00196579$$

$$C_3\left(6,\infty,-\frac{\sqrt{6}}{2},w\right)=\frac{\sqrt{\pi}}{960}\approx 0.001846306095,$$

$$C_3\left(6,1,-\frac{\sqrt{6}}{2},w\right)\approx 0.001549571572.$$

1.5 Four-point quadrature formulae

In this section we consider closed fourpoint quadrature formulae of the following type:

$$\int_a^b f(t)dt = A_1^4(x)\,[f(a)+f(b)] + B_1^4(x)\,[f(x)+f(a+b-x)] + E(f,x), \tag{1.176}$$

where $x \in [a, \frac{a+b}{2}]$, $A_1^4(x)$ and $B_1^4(x)$ are such that $2A_1^4(x)+2B_1^4(x) = b-a$, and $E(f,x)$ is remainder. Let $x \in [a, \frac{a+b}{2}]$ be a fixed node. For $n \in \mathbb{N}$, let $\{P_{jk}\}_{k=0,1,\ldots,n}$ be sequences of harmonic polynomials for $j = 1,2,3$. In addition, let us assume P_{jk} satisfy following symmetry conditions:

$$\begin{aligned} P_{1k}(t) &= (-1)^k P_{3k}(a+b-t), \quad t \in [a,x], \\ P_{2k}(t) &= (-1)^k P_{2k}(a+b-t), \quad t \in [x, a+b-x]. \end{aligned} \tag{1.177}$$

Define

$$S_n^4(t,x) := \begin{cases} P_{1n}(t), \ t \in [a,x], \\ \\ P_{2n}(t), \ t \in (x, a+b-x], \\ \\ P_{3n}(t), \ t \in (a+b-x, b]. \end{cases} \tag{1.178}$$

Now we introduce some notes. For $k = 1,\ldots,n$ define

$$A_k^4(x) = (-1)^k P_{1k}(a) \tag{1.179}$$

and

$$B_k^4(x) = (-1)^{k-1}\left(P_{1k}(x) - P_{2k}(x)\right). \tag{1.180}$$

Further, for function $f : [a,b] \to \mathbb{R}$ which is $n-1$ times differentiable we define

$$\begin{aligned} D_4(x) &:= A_1^4(x)\,[f(a)+f(b)] + B_1^4(x)\,[f(x)+f(a+b-x)], \\ T_n^4(x) &:= \sum_{k=2}^{n} A_k^4(x)\left[f^{(k-1)}(a) + (-1)^{k-1} f^{(k-1)}(b)\right] \\ &+ \sum_{k=2}^{n} B_k^4(x)\left[f^{(k-1)}(x) + (-1)^{k-1} f^{(k-1)}(a+b-x)\right]. \end{aligned}$$

Theorem 1.24 *Let $f : [a,b] \to \mathbb{R}$ be a function with continuaus $n-$th derivation, for some $n \in \mathbb{N}$ and $x \in [a, \frac{a+b}{2}]$. Further, let $S_n^4(t,x)$ be define with (1.178). Then we have*

$$\int_a^b f(t)dt = D_4(x) + T_n^4(x) + (-1)^n \int_a^b S_n^4(t,x) f^{(n)}(t)dt. \tag{1.181}$$

Proof. For a subdivision $x_0 = a$, $x_1 = x$, $x_2 = a+b-x$, and $x_3 = b$ of the interval $[a,b]$ we apply identity (1.2). □

Theorem 1.25 *Assume $1 \le p,q \le \infty$ are conjugate exponents. If $f:[a,b] \to \mathbb{R}$ is $n-$times differentiable function such that $f^{(n)}$ is piecewise continuous on $[a,b]$ and $f^{(n)} \in L_p[a,b]$, then we have*

$$\left| \int_a^b f(t)dt - D_4(x) - T_n^4(x) \right| \le C_4(n,q,x) \cdot \|f^{(n)}\|_p, \tag{1.182}$$

where

$$C_4(n,q,x) = \tag{1.183}$$
$$\begin{cases} 2^{1/q} \left[\int_a^x |P_{1n}(t)|^q \, dt + \int_x^{\frac{a+b}{2}} |P_{2n}(t)|^q \, dt \right]^{\frac{1}{q}}, & 1 \le q < \infty, \\ \max \left\{ \sup_{t \in [a,x]} |P_{1n}(t)|, \sup_{t \in [x, \frac{a+b}{2}]} |P_{2n}(t)| \right\} & q = \infty. \end{cases}$$

The inequality is the best possible for $p = 1$, and sharp for $1 < p \le \infty$.

Proof. The assertion follows from Theorem 1.2 □

Theorem 1.26 *If polynomials P_{jk} satisfy $P_{1n}(t) \ge 0$, for $t \in [a,x]$ and $P_{2n}(t) \ge 0$, for $t \in (x, \frac{a+b}{2}]$, then for function $f:[a,b] \to \mathbb{R}$ such that $f^{(n)}$ is continuous on $[a,b]$ we have*

$$\int_a^b f(t)dt = D_4(x) + T_n^4(x) + C_4(n,q,x) \cdot f^{(n)}(\eta), \tag{1.184}$$

for some $\eta \in [a,b]$.

Proof. The assertion follows from the mean value theorem. □

1.5.1 The closed four-point quadrature formula with precision 3

In this section we observe the closed four-point formula which doesn't include the values of the first three derivatives in the inner nodes. Such formula will be exact for all polynomials of the degree at most 3. Let $x \in (a, \frac{a+b}{2})$ be fixed.

First, let us define the following constants:

$$\alpha_1(x) := \frac{b-a}{2} \left[1 - \frac{(b-a)^2}{6(x-a)(b-x)} \right], \tag{1.185}$$

$$\beta_2(x) := \frac{(x - \frac{a+b}{2})^2}{2} - \frac{(x-a)^2}{2} + \alpha_1(x) \cdot (x-a) \tag{1.186}$$

and

$$\beta_4(x) := \frac{(x - \frac{a+b}{2})^4}{4!} - \frac{(x-a)^4}{4!} - \beta_2(x) \frac{(x - \frac{a+b}{2})^2}{2} + \alpha_1(x) \frac{(x-a)^3}{3!}. \tag{1.187}$$

Now we consider the following polynomials $P_{jk}(\cdot,x)$:

$$\begin{aligned} P_{1k}(t,x) &:= \frac{(t-a)^k}{k!} - \alpha_1(x)\frac{(t-a)^{k-1}}{(k-1)!}, \\ P_{3k}(t,x) &:= (-1)^k P_{1k}(a+b-t,x) \end{aligned} \tag{1.188}$$

and

$$P_{2k}(t,x) := \tag{1.189}$$
$$\begin{cases} t-\frac{a+b}{2}, & k=1 \\ \frac{(t-\frac{a+b}{2})^2}{2} - \beta_2(x), & k=2 \\ \frac{(t-\frac{a+b}{2})^3}{3!} - \beta_2(x)(t-\frac{a+b}{2}) & k=3 \\ \frac{(t-\frac{a+b}{2})^k}{k!} - \beta_2(x)\frac{(t-\frac{a+b}{2})^{k-2}}{(k-2)!} - \beta_4(x)\frac{(t-\frac{a+b}{2})^{k-4}}{(k-4)!} & k\geq 4. \end{cases}$$

Remark 1.32 The polynomials P_{jk} are harmonic and satisfy the symmetry conditions (1.17).

Now we have:

$$\begin{aligned} A_1^4(x) &= \alpha_1(x) = \frac{b-a}{2}\left(1-\frac{(b-a)^2}{6(x-a)(b-x)}\right), \\ B_1^4(x) &= \frac{(b-a)^3}{12(x-a)(b-x)}, \\ A_k^4(x) &= 0, \quad k\geq 2, \\ B_k^4(x) &= 0, \quad k=2,3,4, \\ B_k^4(x) &= (-1)^{k-1}\left[\frac{(x-a)^k}{k!} - \alpha_1(x)\frac{(x-a)^{k-1}}{(k-1)!}\right. \\ &- \frac{(x-\frac{a+b}{2})^k}{k!} + \beta_2(x)\frac{(x-\frac{a+b}{2})^{k-2}}{(k-2)!} \\ &+ \left.\beta_4(x)\frac{(x-\frac{a+b}{2})^{k-4}}{(k-4)!}\right], \quad k\geq 5. \end{aligned} \tag{1.190}$$

Further, we have

$$D_4(x) = A_1^4(x)\,[f(a)+f(b)] + B_1^4(x)\,[f(x)+f(a+b-x)]$$

and

$$T_n^4(x) = \sum_{k=5}^{n} B_k^4(x)\left[f^{(k-1)}(x) + (-1)^{k-1}f^{(k-1)}(a+b-x)\right].$$

For the choice of $x=\frac{2a+b}{3}$ we can get the generalization of the Simpson's $3/8$ formula.

1.5.2 The general Lobatto four-point formula

We consider four-point quadrature formulae where term $T_n^4(x)$ does not contain derivatives of order less then 5. These formulae are exact for polynomials of degree less then 5. Define:

$$\beta_{4L}(x) := -\frac{1}{x}\left[\frac{(x+1)^5}{5!} - \alpha_1(x)\frac{(x+1)^4}{4!} - \frac{x^5}{5!} + \beta_2(x)\frac{x^3}{3!}\right]$$

and

$$\beta_{6L}(x) := \frac{x^6}{6!} - \frac{(x+1)^6}{6!} + \alpha_1(x)\frac{(x+1)^5}{5!} - \beta_2(x)\frac{x^4}{4!} - \beta_{4L}(x)\frac{x^2}{2}.$$

Further, define polynomials

$$P_{2k,L}^x(t) := \begin{cases} t, & k=1, \\ \frac{t^2}{2} - \beta_2(x), & k=2, \\ \frac{t^3}{3!} - \beta_2(x)t & k=3, \\ \frac{t^4}{4!} - \beta_2(x)\frac{t^2}{2} - \beta_{4L}(x) & k=4, \\ \frac{t^5}{5!} - \beta_2(x)\frac{t^3}{3!} - \beta_{4L}(x)t, & k=5, \\ \frac{t^k}{k!} - \beta_2(x)\frac{t^{k-2}}{(k-2)!} - \beta_{4L}(x)\frac{k-4}{(k-4)!} - \beta_{6L}(x)\frac{t^{k-6}}{(k-6)!} & k \geq 6. \end{cases} \tag{1.191}$$

Polynomials $P_{1k,x}$ and $P_{3k,x}$ are defined with (1.188). Further, define

$$S_n^{4,L}(t,x) = \begin{cases} P_{1n}^x(t), & t \in [-1,x], \\ P_{2n,L}^x(t), & t \in (x,-x], \\ P_{3n}^x(t), & t \in (-x,1]. \end{cases} \tag{1.192}$$

For $x = -\frac{\sqrt{5}}{5}$ we obtain

$$\begin{aligned} \alpha_1\left(-\frac{\sqrt{5}}{5}\right) &= \frac{1}{6}, \\ \beta_2\left(-\frac{\sqrt{5}}{5}\right) &= \frac{\sqrt{5}}{6} - \frac{1}{3}, \\ \beta_{4L}\left(-\frac{\sqrt{5}}{5}\right) &= \frac{\sqrt{5}}{180} - \frac{1}{72}, \\ \beta_{6L}\left(-\frac{\sqrt{5}}{5}\right) &= \frac{1}{3600\sqrt{5}} \end{aligned}$$

and

$$
\begin{aligned}
A_1^4\left(-\frac{\sqrt{5}}{5}\right) &= \frac{1}{6},\\
B_1^4\left(-\frac{\sqrt{5}}{5}\right) &= \frac{5}{6},\\
A_k^4\left(-\frac{\sqrt{5}}{5}\right) &= 0, \quad k \geq 2,\\
B_k^4\left(-\frac{\sqrt{5}}{5}\right) &= 0, \quad k = 2,3,4,5,6,\\
B_k^4\left(-\frac{\sqrt{5}}{5}\right) &= (-1)^{k-1}\left[P_{1k}^{-\frac{\sqrt{5}}{5}}\left(-\frac{\sqrt{5}}{5}\right) - P_{2k,L}^{-\frac{\sqrt{5}}{5}}\left(-\frac{\sqrt{5}}{5}\right)\right], \quad k \geq 5.
\end{aligned}
\tag{1.193}
$$

Then we have

$$
D_4\left(-\frac{\sqrt{5}}{5}\right) = \frac{1}{6}\left[f(-1) + 5f\left(-\frac{\sqrt{5}}{5}\right) + 5f\left(\frac{\sqrt{5}}{5}\right) + f(1)\right]
$$

and

$$
T_n^4\left(-\frac{\sqrt{5}}{5}\right) = \sum_{k=7}^{n} B_k^4\left(-\frac{\sqrt{5}}{5}\right)\left[f^{(k-1)}\left(-\frac{\sqrt{5}}{5}\right) + (-1)^{k-1} f^{(k-1)}\left(\frac{\sqrt{5}}{5}\right)\right].
$$

If $f : [-1,1] \to \mathbb{R}$ is such that $f^{(n)}$ is piecewise continuous, then according to Theorem 1.24 we have

$$
\int_{-1}^{1} f(t)dt = D_4\left(-\frac{\sqrt{5}}{5}\right) + T_n^4\left(-\frac{\sqrt{5}}{5}\right) + (-1)^n \int_{-1}^{1} S_n^4\left(t, -\frac{\sqrt{5}}{5}\right) f^{(n)}(t)dt. \tag{1.194}
$$

Remark 1.33 The identity (1.194) is called the generalization of the Lobatto four-point quadrature formula.

Lemma 1.3 *For $n \geq 6$ we have*

$$
P_{1n}^{-\frac{\sqrt{5}}{5}}(t) \leq 0, \quad t \in \left[-1, -\frac{\sqrt{5}}{5}\right].
$$

Further, for odd $n \geq 7$, we have $P_{2n,L}^{-\frac{\sqrt{5}}{5}}(t) > 0$ on $\left(-\frac{\sqrt{5}}{5}, 0\right)$. For even $n \geq 6$ we have

$$
P_{2n,L}^{-\frac{\sqrt{5}}{5}}(t) < 0, \quad t \in \left(-\frac{\sqrt{5}}{5}, 0\right).
$$

Proof. We have

$$P_{1n}^{-\frac{\sqrt{5}}{5}}(t) = \frac{t^{n-1}}{n!}\cdot\left(t+1-\frac{n}{6}\right).$$

For $t \in \left[-1,-\frac{\sqrt{5}}{5}\right]$, it is easy to check that $t+1-\frac{n}{6} \le 0$, for $n \ge 6$, so $P_{1n}^{-\frac{\sqrt{5}}{5}}(t) \le 0$. After simple computation we conclude that $P_{25,L}^{-\frac{\sqrt{5}}{5}}$ doesn't have zero point on $\left(-\frac{\sqrt{5}}{5},0\right)$, and $P_{25,L}^{-\frac{\sqrt{5}}{5}}(t) < 0$. Therefore, $P_{26,L}^{-\frac{\sqrt{5}}{5}}$ is decreasing, and since $P_{26,L}^{-\frac{\sqrt{5}}{5}}\left(-\frac{\sqrt{5}}{5}\right) < 0$ and $P_{26,L}^{-\frac{\sqrt{5}}{5}}(0) < 0$, we have $P_{26,L}^{-\frac{\sqrt{5}}{5}}(t) < 0$ on $\left(-\frac{\sqrt{5}}{5},0\right)$.

For $n \ge 7$ we get $P_{2n,L}^{-\frac{\sqrt{5}}{5}}(0) = 0$. For $n = 7$ is $\frac{\partial P_{27,L}^{-\frac{\sqrt{5}}{5}}}{\partial t}(t) < 0$, so $P_{27,L}^{-\frac{\sqrt{5}}{5}}(t) > 0$, na $\left(-\frac{\sqrt{5}}{5},0\right)$. Analogusly, $P_{28,L}^{-\frac{\sqrt{5}}{5}}(t)$ is nondecreasing on $\left(-\frac{\sqrt{5}}{5},0\right)$ so $P_{28,L}^{-\frac{\sqrt{5}}{5}}(t) < 0$. Let us assume that for $n \ge 9$ we have

$$P_{2n,L}^{-\frac{\sqrt{5}}{5}}(t) > 0, \quad t \in \left(-\frac{\sqrt{5}}{5},0\right). \tag{1.195}$$

Now, let us assume that there exists $t_0 \in \left(-\frac{\sqrt{5}}{5},0\right)$ such that $P_{2,n+2,L}^{-\frac{\sqrt{5}}{5}}(t_0) = 0$. Then there exists $t_1 \in (t_0,0)$ such that $\frac{\partial P_{2,n+2,L}^{-\frac{\sqrt{5}}{5}}}{\partial t}(t_1) = P_{2,n+1,L}^{-\frac{\sqrt{5}}{5}}(t_1) = 0$. Therefore, there must exist $t_2 \in (t_1,0)$ such that $\frac{\partial P_{2,n+1,L}^{-\frac{\sqrt{5}}{5}}}{\partial t}(t_2) = P_{2,n,L}^{-\frac{\sqrt{5}}{5}}(t_2) = 0$, which is in contradiction with (1.195). Therefore, we proved that for $n \ge 7$ we have (1.195). Now it is easy to check that for $n \ge 8$ we have $P_{2n,L}^{-\frac{\sqrt{5}}{5}}(t) < 0$ on $\left(-\frac{\sqrt{5}}{5},0\right)$. □

If in addition, f satisfies the conditions of Theorem 1.25, then we have

$$\left|\int_{-1}^{1} f(t)dt - D_4\left(-\frac{\sqrt{5}}{5}\right) - T_n^4\left(-\frac{\sqrt{5}}{5}\right)\right| \le C_4\left(n,q,-\frac{\sqrt{5}}{5}\right)\|f^{(n)}\|_p. \tag{1.196}$$

Specially,

$$C_4\left(1,1,-\frac{\sqrt{5}}{5}\right) = 0.376866, \quad C_4\left(1,\infty,-\frac{\sqrt{5}}{5}\right) = 0.447214,$$

$$C_4\left(2,1,-\frac{\sqrt{5}}{5}\right) = 0.041777, \quad C_4\left(2,\infty,-\frac{\sqrt{5}}{5}\right) = 0.060655,$$

$$C_4\left(3,1,-\frac{\sqrt{5}}{5}\right) = 0.006404, \quad C_4\left(3,\infty,-\frac{\sqrt{5}}{5}\right) = 0.007357,$$

$$C_4\left(4,1,-\frac{\sqrt{5}}{5}\right)=0.001132,\quad C_4\left(4,\infty,-\frac{\sqrt{5}}{5}\right)=0.001466,$$

$$C_4\left(5,1,-\frac{\sqrt{5}}{5}\right)=0.000248,\quad C_4\left(5,\infty,-\frac{\sqrt{5}}{5}\right)=0.000283,$$

$$C_4\left(6,1,-\frac{\sqrt{5}}{5}\right)=0.000084,\quad C_4\left(6,\infty,-\frac{\sqrt{5}}{5}\right)=0.000124.$$

If $f:[-1,1]\to\mathbb{R}$ is such that $f^{(n)}$ is continuous for some even $n\geq 6$, then according to Lemma 1.3 and Theorem 1.184 we get

$$\int_{-1}^{1} f(t)dt = D_4\left(-\frac{\sqrt{5}}{5}\right)+T_n^4\left(-\frac{\sqrt{5}}{5}\right)-C_4\left(n,1,-\frac{\sqrt{5}}{5}\right)f^{(n)}(\eta), \tag{1.197}$$

for some $\eta\in[-1,1]$.
Specially, for $n=6$ the famous Lobatto four-point formula is obtained:

$$\int_{-1}^{1} f(t)dt \;=\; D_4\left(-\frac{\sqrt{5}}{5}\right)-\frac{2}{23625}f^{(6)}(\eta). \tag{1.198}$$

Chapter 2

Weighted generalizations of the Montgomery identity

In this chapter weighted generalizations of the Euler integral formula are given, as well as its discrete analogues. They are used to improve some Ostrowski type inequalities, and to obtain new Ostrowski type inequalities, estimations of the difference of two integral means, and generalized trapezoid and midpoint inequalities. For all these inequalities, its discrete analogues are given.

In Section 2.1 a connection between generalized Montgomery identity, Bernoulli polynomials and Euler identity is established. Using this connection, certain strict improvements of some Ostrowski type inequalities are obtained. Some new generalizations of estimations of difference of two weighted integral means are given, by using Euler-type identities and weighted Montgomery identity. Also, a generalization of the discrete weighted Montgomery identity for the infinite sequences is presented as well as discrete analogue of Ostrowski type inequalities and estimations of difference of two arithmetic means.

In Section 2.2 four new weighted generalizations of Euler-type identities are given, and used to obtain new Ostrowski type inequalities, generalized trapezoid and midpoint inequalities, and estimations of the difference of two integral means. Also a discrete analogue of the weighted Montgomery identity (i.e. Euler identity) for finite sequences is given as well as discrete analogue of Ostrowski type inequalities, trapezoid and midpoint inequalities, and estimations of difference of two arithmetic means.

In Section 2.3 new extensions of the weighted Montgomery identity by using Taylor's formula in two ways and one extension by using Fink identity are presented. These identities are used to obtain some Ostrowski type inequalities and estimations of the difference of two integral means. Also, applications for α-L-Hölder type functions are made via

Taylor's extensions.

In Section 2.4 weighted generalization of Montgomery identity for Riemann-Stieltjes integral is given and used to obtain weighted 2-point and 3-point Radau, Lobatto and Gauss quadrature rules for functions of bounded variation and for functions whose first derivatives belongs to Lp spaces.

In Section 2.5 error estimates of approximations in real domain for the Fourier transform and in complex domain for the Laplace transform are given for functions which vanish beyond a finite domain $[a,b] \subset [0,\infty\rangle$ and such that $f' \in L_p[a,b]$. New inequalities involving Fourier and Laplace transform of f, integral mean of f and exponential mean of the endpoints of the domain of f are presented and used to obtain two associated numerical rules and error bounds of their remainders in each case.

In Section 2.6 integral and discrete weighted generalizations of Montgomery identities and Ostrowski type inequalities are presented for functions of two variables.

In Section 2.7 several extensions of Landau inequality are presented using various identities: an extension of Montgomery identity via Taylor's formula with respect to a and b; an extension of Montgomery identity applied to functions f such that $f^{(n)}$ are α-L-Hölder type; Euler type identities; the Fink identity. Also, new Landau type inequalities via Ostrowski type inequalities are given.

2.1 Euler integral formula and Montgomery identity

In this section a connection between generalized Montgomery identity, Bernoulli polynomials and Euler identity is established. This connection enables us to obtain certain strict improvements of some Ostrowski type inequalities. Some new generalizations of estimations of difference of two weighted integral means are given, by using Euler-type identities and weighted Montgomery identity. Also, a generalization of the discrete weighted Montgomery identity for the infinite sequences is presented as well as discrete analogue of Ostrowski type inequalities and estimations of difference of two arithmetic means.

The following Ostrowski inequality gives us an estimate for the deviation of the values of a smooth function from its mean value [92]:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)\left\| f' \right\|_\infty . \tag{2.1}$$

It holds for every $x \in [a,b]$ whenever $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b) with derivative $f' : (a,b) \to \mathbb{R}$ bounded on (a,b) i.e.

$$\left\| f' \right\|_\infty := \sup_{t \in (a,b)} \left| f'(t) \right| < +\infty.$$

The constant $\frac{1}{4}$ is the best possible. Ostrowski proved this inequality in 1938 and since then it has been generalized in a number of ways. Also over the last few years some new inequalities of this type have been intensively considered together with their applications in Numerical analysis and Probability.

Ostrowski inequality can easily be proved by means of the **Montgomery identity** (see [90]):

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t + \int_a^b P(x,t)\,f'(t)\,\mathrm{d}t, \tag{2.2}$$

where $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ and $P(x,t)$ the Peano kernel defined by

$$P(x,t) = \begin{cases} \frac{t-a}{b-a}, & a \le t \le x, \\ \frac{t-b}{b-a} & x < t \le b. \end{cases} \tag{2.3}$$

If $w : [a,b] \to [0,\infty\rangle$ is some nonnegative integrable weight function the **weighted Montgomery identity** (obtained by J. Pečarić in [93]) states

$$f(x) - \frac{1}{\int_a^b w(t)\,dt}\int_a^b f(t)\,w(t)\,\mathrm{d}t = \int_a^b P_w(x,t)\,f'(t)\,\mathrm{d}t \tag{2.4}$$

where $P_w(t,s)$ is the weighted Peano kernel defined by

$$P_w(x,t) = \begin{cases} \frac{W(t)}{W(b)}, & a \le t \le x, \\ \frac{W(t)}{W(b)} - 1, & x < t \le b. \end{cases}$$

and $W(t) = \int_a^t w(x)\,\mathrm{d}x$ for $t \in [a,b]$, $W(t) = 0$ for $t < a$ and $W(t) = W(b)$ for $t > b$. For the nonnegative normalized weight function $w : [a,b] \to [0,\infty\rangle$ i.e. integrable function satisfying $\int_a^b w(t)\,\mathrm{d}t = 1$ it reduces to

$$f(x) = \int_a^b w(t)\,f(t)\,\mathrm{d}t + \int_a^b P_w(x,t)\,f'(t)\,\mathrm{d}t, \tag{2.5}$$

$$P_w(x,t) = \begin{cases} W(t), & a \le t \le x, \\ W(t) - 1 & x < t \le b. \end{cases} \tag{2.6}$$

Further for the uniform weight function $w(t) = 1$, $t \in [a,b]$ it reduces to (2.2).

Here, as in the rest of the book, we write $\int_a^b \varphi(t)\,\mathrm{d}g(t)$ to denote the Riemann-Stieltjes integral with respect to a function $g : [a,b] \to \mathbb{R}$ of bounded variation, and $\int_a^b \varphi(t)\,\mathrm{d}t$ for the Riemann integral.

2.1.1 Generalized Montgomery identity and Bernoulli polynomials

In this subsection a connection between generalized Montgomery identity, Bernoulli polynomials and Euler identity is established. Using it, certain strict improvements of some Ostrowski type inequalities are obtained.

Consider the sequence $(B_k(t), k \geq 0)$ of Bernoulli polynomials which is uniquely determined by the following identities:

$$B_k'(t) = kB_{k-1}(t), \quad k \geq 1, \;\; B_0(t) = 1$$

and

$$B_k(t+1) - B_k(t) = kt^{k-1}, \quad k \geq 0.$$

The values $B_k = B_k(0)$, $k \geq 0$ are known as Bernoulli numbers. For our purposes, the first five Bernoulli polynomials are

$$\begin{aligned} &B_0(t) = 1,\; B_1(t) = t - \frac{1}{2},\; B_2(t) = t^2 - t + \frac{1}{6}, \\ &B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t,\; B_4(t) = t^4 - 2t^3 + t^2 - \frac{1}{30}. \end{aligned} \tag{2.7}$$

For some further details on the Bernoulli polynomials and the Bernoulli numbers see for example [1] or [29]. Let $\left(B_k^*(t), k \geq 0\right)$ be a sequence of periodic functions of period 1, related to Bernoulli polynomials as

$$B_k^*(t) = B_k(t), \quad 0 \leq t < 1, \qquad B_k^*(t+1) = B_k^*(t), \quad t \in \mathbb{R}.$$

From the properties of Bernoulli polynomials it easily follows that $B_0^*(t) = 1, B_1^*$ is discontinuous function with a jump of -1 at each integer, while B_k^* , $k \geq 2$, are continuous functions.

For every function $f : [a,b] \to \mathbb{R}$ with continuous n-th derivative, $n \geq 1$, and for every $x \in [a,b]$ the following formula is valid (see [75]):

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t + T_{n-1}(x) + R_n(x)$$

where

$$T_m(x) = \sum_{k=1}^{m} \frac{(b-a)^{k-1}}{k!} B_k\left(\frac{x-a}{b-a}\right)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right],$$

with convention $T_0(x) = 0$, and

$$R_n(x) = -\frac{(b-a)^{n-1}}{n!}\int_a^b \left[B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right)\right] f^{(n)}(t)\,\mathrm{d}t.$$

This formula can be rewritten as

$$\begin{aligned} &f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \sum_{k=0}^{n-2}\frac{(b-a)^{k+1}}{(k+1)!}B_{k+1}\left(\frac{x-a}{b-a}\right)\frac{f^{(k)}(b) - f^{(k)}(a)}{b-a} \\ &= \frac{(b-a)^{n-1}}{n!}\int_a^b \left[B_n\left(\frac{x-a}{b-a}\right) - B_n^*\left(\frac{x-t}{b-a}\right)\right] f^{(n)}(t)\,\mathrm{d}t. \end{aligned} \tag{2.8}$$

We claim that the formula (2.8) coincide with the generalized Montgomery identity stated in the next theorem obtained by G.A. Anastassiou in [22]:

Let $f:[a,b]\to\mathbb{R}$ *be n-times differentiable on* $[a,b]$, $n\in\mathbb{N}$. *The n-th derivative* $f^{(n)}:[a,b]\to\mathbb{R}$ *is integrable on* $[a,b]$. *Let* $x\in[a,b]$. *Define the kernel*

$$P(r,s):=\begin{cases}\frac{s-a}{b-a}, & a\le s\le r,\\ \frac{s-b}{b-a}, & r<s\le b,\end{cases}$$

where $r,s\in[a,b]$. *Then it holds*

$$\begin{aligned}&f(x)-\frac{1}{b-a}\int_a^b f(s_1)\,\mathrm{d}s_1\\&-\sum_{k=0}^{n-2}\frac{f^{(k)}(b)-f^{(k)}(a)}{b-a}\int_a^b\cdots\int_a^b P(x,s_1)\left(\prod_{i=1}^{k}P(s_i,s_{i+1})\right)\mathrm{d}s_1\cdots\mathrm{d}s_{k+1}\\&=\int_a^b\cdots\int_a^b P(x,s_1)\left(\prod_{i=1}^{n-1}P(s_i,s_{i+1})\right)f^{(n)}(s_n)\,\mathrm{d}s_1\cdots\mathrm{d}s_n\end{aligned}\tag{2.9}$$

We make conventions that $\prod_{k=1}^{0}\bullet:=1$, $\sum_{k=0}^{-1}\bullet:=0$.

We prove this claim by the following two lemmas.

Lemma 2.1 *For all* $k\in\{0,1,2,3,..\}$ *we have*

$$\begin{aligned}&\underbrace{\int_a^b\cdots\int_a^b}_{(k+1)th\ integral}P(x,s_1)P(s_1,s_2)\cdots P(s_k,s_{k+1})\,\mathrm{d}s_1\cdots\mathrm{d}s_{k+1}\\&=\frac{(b-a)^{k+1}}{(k+1)!}B_{k+1}\left(\frac{x-a}{b-a}\right).\end{aligned}\tag{2.10}$$

Proof. We prove our assertion by induction with respect to k. For $k=0$ and $x\in[a,b\rangle$ we have for the left side of (2.10) by integrating by parts

$$\begin{aligned}\int_a^b P(x,s_1)\,\mathrm{d}s_1&=P(x,b)\cdot b-P(x,a)\cdot a-\int_a^b s_1\mathrm{d}P(x,s_1)\\&=-\left(\int_a^b\frac{s_1}{b-a}\mathrm{d}s_1\right)-(-1)x=x-\frac{a+b}{2}.\end{aligned}$$

Since $P(x,b)=P(x,a)=0$ and $P(x,s_1)$ is differentiable on $[a,b]\setminus\{x\}$, its derivative is equal to $\frac{1}{b-a}$, and at x it has a jump of -1. Since $B_1(t)=t-\frac{1}{2}$, the right side of (2.10) is $(b-a)B_1\left(\frac{x-a}{b-a}\right)=x-\frac{a+b}{2}$. For $x=b$ we have $P(b,s_1)=\frac{s_1-a}{b-a}$, $s_1\in[a,b]$ and equality (2.10) obviously holds. Thus we have proved the base of induction. Let's denote

$$Q_{k+1}(x)=\underbrace{\int_a^b\cdots\int_a^b}_{(k+1)th\ integral}P(x,s_1)P(s_1,s_2)\cdots P(s_k,s_{k+1})\,\mathrm{d}s_1\cdots\mathrm{d}s_{k+1}$$

We can suppose now for $k > 0$ that

$$Q_k(y) = \frac{(b-a)^k}{k!} B_k\left(\frac{y-a}{b-a}\right), \quad y \in [a,b].$$

Then we have for $k+1$ and for $x \in [a,b\rangle$

$$Q_{k+1}(x) = \int_a^b P(x,s_1) Q_k(s_1)\,\mathrm{d}s_1 = \int_a^b P(x,s_1) \frac{(b-a)^k}{k!} B_k\left(\frac{s_1-a}{b-a}\right)\mathrm{d}s_1.$$

Since $B_n'(t) = nB_{n-1}(t)$ we have

$$\frac{d}{ds_1}\left[\frac{b-a}{k+1} B_{k+1}\left(\frac{s_1-a}{b-a}\right)\right] = B_k\left(\frac{s_1-a}{b-a}\right).$$

Integrating by parts yields to

$$\begin{aligned}
Q_{k+1}(x) &= \frac{(b-a)^{k+1}}{(k+1)!} \int_a^b P(x,s_1)\,\mathrm{d}B_{k+1}\left(\frac{s_1-a}{b-a}\right) \\
&= \frac{(b-a)^{k+1}}{(k+1)!}\left[-\int_a^b B_{k+1}\left(\frac{s_1-a}{b-a}\right)\mathrm{d}P(x,s_1)\right] \\
&= \frac{(b-a)^{k+1}}{(k+1)!}\left[-\int_a^b \frac{1}{b-a} B_{k+1}\left(\frac{s_1-a}{b-a}\right)\mathrm{d}s_1 - B_{k+1}\left(\frac{x-a}{b-a}\right)(-1)\right] \\
&= \frac{(b-a)^{k+1}}{(k+1)!}\left[-\left[\frac{1}{k+2} B_{k+2}\left(\frac{s_1-a}{b-a}\right)\Bigg|_a^b\right] - B_{k+1}\left(\frac{x-a}{b-a}\right)(-1)\right] \\
&= \frac{(b-a)^{k+1}}{(k+1)!} B_{k+1}\left(\frac{x-a}{b-a}\right),
\end{aligned}$$

since $B_k(1) = B_k(0)$ for all $k \geq 2$. It is easy to see that in case $x = b$ we have

$$Q_{k+1}(b) = \int_a^b \frac{s_1-a}{b-a} \cdot \frac{(b-a)^k}{k!} B_k\left(\frac{s_1-a}{b-a}\right)\mathrm{d}s_1 = \frac{(b-a)^{k+1}}{(k+1)!} B_{k+1}\left(\frac{b-a}{b-a}\right).$$

Thus the statement is valid also for $k+1$, so it is valid for all $k \in \{0,1,2,3,...\}$. □

Lemma 2.2 *For all $n \in \mathbb{N}$ we have*
a)

$$\begin{aligned}
&\int_a^b \cdots \int_a^b P(x,s_1)\left(\prod_{i=1}^{n-1} P(s_i,s_{i+1})\right) f^{(n)}(s_n)\,\mathrm{d}s_1 \cdots \mathrm{d}s_n \\
&= \frac{(b-a)^{n-1}}{n!} \int_a^b \left[B_n\left(\frac{x-a}{b-a}\right) - B_n^*\left(\frac{x-t}{b-a}\right)\right] f^{(n)}(t)\,\mathrm{d}t.
\end{aligned}$$

b)

$$\int_a^b \cdots \int_a^b P(x,s_1)\left(\prod_{i=1}^{n-1} P(s_i,s_{i+1})\right) \mathrm{d}s_1 \cdots \mathrm{d}s_{n-1}$$
$$= \frac{(b-a)^{n-1}}{n!}\left[B_n\left(\frac{x-a}{b-a}\right) - B_n^*\left(\frac{x-s_n}{b-a}\right)\right].$$

Here we make convention that $\prod_{i=1}^{0} \bullet := 1$.

Proof. It is obvious that **a)** follows from **b)** so that we only have to prove **b)**. We do this again by induction. Let us denote

$$q_k(x,t) = \int_a^b \cdots \int_a^b P(x,s_1)P(s_1,s_2)\cdots P(s_k,t)\,\mathrm{d}s_1\cdots\mathrm{d}s_{k.} \tag{2.11}$$

Then our claim is that

$$q_k(x,t) = \frac{(b-a)^k}{(k+1)!}\left[B_{k+1}\left(\frac{x-a}{b-a}\right) - B_{k+1}^*\left(\frac{x-t}{b-a}\right)\right] \tag{2.12}$$

is true for all $k \in \{0,1,2,3,...\}$. Since $B_1(t) = t - \frac{1}{2}$, we have $B_1\left(\frac{x-a}{b-a}\right) - B_1^*\left(\frac{x-t}{b-a}\right) = P(x,t) = q_0(x,t)$ and it is clear that our assertion is true for $k = 0$. Further suppose that (2.12) is true for some $k \geq 0$ and for $x \in [a,b\rangle$ we have

$$\begin{aligned}
q_{k+1}(x,t) &= \int_a^b P(x,s_1)\,q_k(s_1,t)\,\mathrm{d}s_1 \\
&= \int_a^b P(x,s_1)\frac{(b-a)^k}{(k+1)!}\left[B_{k+1}\left(\frac{s_1-a}{b-a}\right) - B_{k+1}^*\left(\frac{s_1-t}{b-a}\right)\right]\mathrm{d}s_1 \\
&= \int_a^t P(x,s_1)\frac{(b-a)^k}{(k+1)!}\left[B_{k+1}\left(\frac{s_1-a}{b-a}\right) - B_{k+1}\left(\frac{s_1-t}{b-a}+1\right)\right]\mathrm{d}s_1 \\
&+ \int_t^b P(x,s_1)\frac{(b-a)^k}{(k+1)!}\left[B_{k+1}\left(\frac{s_1-a}{b-a}\right) - B_{k+1}\left(\frac{s_1-t}{b-a}\right)\right]\mathrm{d}s_1
\end{aligned}$$

Using partial integration and properties of Bernoulli polynomials we get

$$\begin{aligned}
&q_{k+1}(x,t) = \frac{(b-a)^{k+1}}{(k+2)!}\left[\int_a^t P(x,s_1)\,\mathrm{d}\left(B_{k+2}\left(\frac{s_1-a}{b-a}\right) - B_{k+2}\left(\frac{s_1-t}{b-a}+1\right)\right)\right. \\
&\left.+\int_t^b P(x,s_1)\,\mathrm{d}\left(B_{k+2}\left(\frac{s_1-a}{b-a}\right) - B_{k+2}\left(\frac{s_1-t}{b-a}\right)\right)\right] \\
&= -\frac{(b-a)^{k+1}}{(k+2)!}\left[\int_a^t \left(B_{k+2}\left(\frac{s_1-a}{b-a}\right) - B_{k+2}\left(\frac{s_1-t}{b-a}+1\right)\right)\mathrm{d}P(x,s_1)\right. \\
&+P(x,t)\left(B_{k+2}(0) - B_{k+2}(1)\right)
\end{aligned}$$

$$
+\int_t^b \left(B_{k+2}\left(\frac{s_1-a}{b-a}\right)-B_{k+2}\left(\frac{s_1-t}{b-a}\right)\right)\mathrm{d}P(x,s_1)\Bigg]
$$

$$
\begin{aligned}
&= -\frac{(b-a)^{k+1}}{(k+2)!}\left[\frac{b-a}{k+3}\left(B_{k+3}\left(\frac{s_1-a}{b-a}\right)-B_{k+3}\left(\frac{s_1-t}{b-a}+1\right)\right)\Bigg|_a^t\right.\\
&+\frac{b-a}{k+3}\left(B_{k+3}\left(\frac{s_1-a}{b-a}\right)-B_{k+3}\left(\frac{s_1-t}{b-a}\right)\right)\Bigg|_t^b\\
&\left.+(-1)\left(B_{k+2}\left(\frac{x-a}{b-a}\right)-B^*_{k+2}\left(\frac{x-t}{b-a}\right)\right)\right]\\
&= \frac{(b-a)^{k+1}}{(k+2)!}\left[B_{k+2}\left(\frac{x-a}{b-a}\right)-B^*_{k+2}\left(\frac{x-t}{b-a}\right)\right]
\end{aligned}
$$

Since $P(b,s_1)=\frac{s_1-a}{b-a}$, $s_1\in[a,b]$ it's easy to check that (2.12) is also valid for $x=b$. Thus, the formula (2.12) holds when k is replaced by $k+1$, which proves our assertion. □

2.1.2 Improvements of Ostrowski type inequalities

Connection between generalized Montgomery identity, Bernoulli polynomials and Euler identity enables us to improve all Ostrowski type results from [22]. Those results are published in [7] and the following is the main idea.

Let us denote the left hand side in (2.9), that is the left hand side in (2.8), by $R_n(x)$. Then by Lemma **2.2**. we have

$$
\begin{aligned}
R_n(x) &= \int_a^b\cdots\int_a^b P(x,s_1)P(s_1,s_2)\cdots P(s_{n-1},s_n)f^{(n)}(s_n)\,\mathrm{d}s_1\cdots\mathrm{d}s_n\\
&= \int_a^b q_{n-1}(x,s_n)f^{(n)}(s_n)\,\mathrm{d}s_n,
\end{aligned}
$$

where $q_{n-1}(\cdot,\cdot)$ is defined by (2.11). Now, for any function f such $f^{(n)}\in L_\infty[a,b]$, the following inequality obviously holds

$$
|R_n(x)|\le\left\|f^{(n)}\right\|_\infty\int_a^b|q_{n-1}(x,s_n)|\,\mathrm{d}s_n, \tag{2.13}
$$

for all $x\in[a,b]$. The crucial point in this approach is that we can exactly evaluate the integral $\int_a^b|q_{n-1}(x,s_n)|\,\mathrm{d}s_n$, using the formula (2.12).

If we do this for $n=3$ we get strictly better estimates of the same inequality obtained by Anastassiou in [22] (see [7]):

Theorem 2.1 *Let $f:[a,b]\to\mathbb{R}$ be 3-times differentiable on $[a,b]$. Assume that f''' is bounded on $[a,b]$. Then, for all $x\in[a,b]$ we have*

$$
\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right.
$$

$$-\frac{f'(b)-f'(a)}{2(b-a)}\left[x^2-(a+b)x+\frac{a^2+b^2+4ab}{6}\right]\Bigg| \tag{2.14}$$
$$\leq \|f'''\|_\infty \cdot \frac{(b-a)^3}{6} I\left(\frac{x-a}{b-a}\right),$$

where

$$I(\lambda)=\begin{cases} -\frac{3}{2}(t_1)^4+2(t_1)^3-\frac{1}{2}(t_1)^2+\frac{3}{2}\lambda^4-\lambda^3-\lambda^2+\frac{1}{2}\lambda, & \lambda\in[0,\frac{3-\sqrt{3}}{6}] \\ \frac{3}{2}(t_1)^4-2(t_1)^3+\frac{1}{2}(t_1)^2-\frac{3}{2}\lambda^4+3\lambda^3-2\lambda^2+\frac{1}{2}\lambda, & \lambda\in(\frac{3-\sqrt{3}}{6},\frac{1}{2}] \\ \frac{3}{2}(t_2)^4-2(t_2)^3+\frac{1}{2}(t_2)^2-\frac{3}{2}\lambda^4+\lambda^3+\lambda^2-\frac{1}{2}\lambda, & \lambda\in(\frac{1}{2},\frac{3+\sqrt{3}}{6}] \\ -\frac{3}{2}(t_2)^4+2(t_2)^3-\frac{1}{2}(t_2)^2+\frac{3}{2}\lambda^4-3\lambda^3+2\lambda^2-\frac{1}{2}\lambda, & \lambda\in(\frac{3+\sqrt{3}}{6},1] \end{cases}$$

and

$$t_1=\frac{3}{4}-\frac{1}{2}\lambda-\frac{1}{2}\sqrt{\frac{1}{4}+3\lambda-3\lambda^2},\; t_2=\frac{3}{4}-\frac{1}{2}\lambda+\frac{1}{2}\sqrt{\frac{1}{4}+3\lambda-3\lambda^2}. \tag{2.15}$$

Also, the best estimates are obtained for $\lambda\in\{0,\frac{1}{2},1\}$, i.e. in corresponding trapezoid and midpoint inequalities:

Corollary 2.1 *Under the assumptions of Theorem **2.1**. we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{b-a}{12}\left[f'(b)-f'(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right|$$
$$\leq \|f'''\|_\infty\cdot\frac{(b-a)^3}{192} \tag{2.16}$$

and

$$\left|f\left(\frac{a+b}{2}\right)+\frac{b-a}{24}\left[f'(b)-f'(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right|$$
$$\leq \|f'''\|_\infty\cdot\frac{(b-a)^3}{192}. \tag{2.17}$$

Further generalization of the inequality (2.14) are obtained by I. Franjić, J. Pečarić and I. Perić in [62] (see also monograph [61]).

2.1.3 Estimations of the difference between two weighted integral means

In this subsection we estimate the difference between two weighted integral means, each having it's own weight, on two different intervals $[a,b]$ and $[c,d]$. This will be done for the functions whose derivatives f' are from L_p spaces, $1\leq p\leq\infty$ and for both possible cases $[c,d]\subseteq[a,b]$ i.e. $a\leq c<d\leq b$ and $[a,b]\cap[c,d]=[c,b]$ i.e. $a\leq c<b\leq d$ (other two cases, when $[a,b]\cap[c,d]\neq\emptyset$ we simply get by change $a\leftrightarrow c$, $b\leftrightarrow d$). These results are published in [14] and are generalizations of the results from [95], [36] and [33]. In [95]

the same was done for the case $[c,d] \subseteq [a,b]$ without weight function, and in [36] for the same case with only one weight function.

First we give the results for the functions whose derivatives f' belongs to L_p spaces, $1 \leq p \leq \infty$.

Theorem 2.2 *Let $f : [a,b] \cup [c,d] \to \mathbb{R}$ be differentiable on $[a,b] \cup [c,d]$, $w : [a,b] \to [0,\infty\rangle$ and $u : [c,d] \to [0,\infty\rangle$ some normalized weight functions,*

$$W(t) = \begin{cases} 0, & t < a, \\ \int_a^t w(t)\,\mathrm{d}t, & a \leq t \leq b, \\ 1, & t > b, \end{cases}$$

$$U(t) = \begin{cases} 0, & t < c, \\ \int_c^t u(t)\,\mathrm{d}t & c \leq t \leq d, \\ 1, & t > d, \end{cases} \tag{2.18}$$

and $[a,b] \cap [c,d] \neq \emptyset$. Then, for both cases $[c,d] \subseteq [a,b]$ and $[a,b] \cap [c,d] = [c,b]$, (and also for $[a,b] \subseteq [c,d]$ and $[a,b] \cap [c,d] = [a,d]$) the next formula is valid

$$\int_a^b w(t) f(t)\,\mathrm{d}t - \int_c^d u(t) f(t)\,\mathrm{d}t = \int_{\min\{a,c\}}^{\max\{b,d\}} K(t) f'(t)\,\mathrm{d}t \tag{2.19}$$

where

$$K(t) = U(t) - W(t), \; t \in [\min\{a,c\}, \max\{b,d\}].$$

Proof. For $x \in [a,b] \cap [c,d]$, we subtract two weighted Montgomery identities

$$f(x) = \int_a^b w(t) f(t)\,\mathrm{d}t + \int_a^b P(W(x), W(t)) f'(t)\,\mathrm{d}t,$$

and

$$f(x) = \int_c^d u(t) f(t)\,\mathrm{d}t + \int_c^d P(U(x), U(t)) f'(t)\,\mathrm{d}t.$$

Then put

$$K(x,t) = P(U(x), U(t)) - P(W(x), W(t)).$$

$K(x,t)$ doesn't depend on x, so we put $K(t)$ instead:

$$K(t) = \begin{cases} -W(t), & t \in [a,c], \\ -W(t) + U(t), & t \in \langle c,d \rangle, \\ 1 - W(t), & t \in [d,b], \end{cases} \quad \text{if} \quad [c,d] \subseteq [a,b]$$

$$K(t) = \begin{cases} -W(t), & t \in [a,c], \\ -W(t) + U(t), & t \in \langle c,b \rangle, \\ U(t) - 1, & t \in [b,d]. \end{cases} \quad \text{if} \quad [a,b] \cap [c,d] = [c,b]$$

□

Definition 2.1 *We say (p,q) is a pair of conjugate exponents if $1<p,q<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$; or if $p=1$ and $q=\infty$; or if $p=\infty$ and $q=1$.*

Theorem 2.3 *Assume (p,q) is a pair of conjugate exponents, $1\le p,q\le\infty$. Let $f'\in L_p[a,b]\cup[c,d]$, then we have*

$$\left|\int_a^b w(t)f(t)\,\mathrm{d}t-\int_c^d u(t)f(t)\,\mathrm{d}t\right|\le\|K(t)\|_q\left\|f'\right\|_p.$$

This inequality is sharp for $1<p\le\infty$ and the best possible for $p=1$.

Proof. Use the identity (2.19) and apply the Hölder inequality to obtain

$$\left|\int_a^b w(t)f(t)\,\mathrm{d}t-\int_c^d u(t)f(t)\,\mathrm{d}t\right|=\left|\int_{\min\{a,c\}}^{\max\{b,d\}}K(t)f'(t)\,\mathrm{d}t\right|\le\|K(t)\|_q\left\|f'\right\|_p.$$

To prove that inequality is sharp we will find a function f such that

$$\left|\int_{\min\{a,c\}}^{\max\{b,d\}}K(t)f'(t)\,\mathrm{d}t\right|=\left(\int_{\min\{a,c\}}^{\max\{b,d\}}|K(t)|^q\,\mathrm{d}t\right)^{\frac{1}{q}}\left\|f'\right\|_p.$$

For $1<p<\infty$ take f to be such that

$$f'(t)=sgn\,K(t)\cdot|K(t)|^{\frac{1}{p-1}}.$$

For $p=\infty$ take

$$f'(t)=sgn\,K(t).$$

For $p=1$ we shall prove that

$$\left|\int_{\min\{a,c\}}^{\max\{b,d\}}K(t)f'(t)\,\mathrm{d}t\right|\le\max_{t\in[a,b]\cup[c,d]}|K(t)|\left(\int_{\min\{a,c\}}^{\max\{b,d\}}\left|f'(t)\right|\mathrm{d}t\right)\tag{2.20}$$

is the best possible inequality. Suppose that $|K(t)|$ attains its maximum at $t_0\in[a,b]\cup[c,d]$. First we assume that $K(t_0)>0$. For ε small enough define $f_\varepsilon(t)$ by

$$f_\varepsilon(t)=\begin{cases}0, & \min\{a,c\}\le t\le t_0,\\ \frac{1}{\varepsilon}(t-t_0), & t_0\le t\le t_0+\varepsilon,\\ 1, & t_0+\varepsilon\le t\le\max\{b,d\}.\end{cases}$$

Then, for ε small enough

$$\left|\int_{\min\{a,c\}}^{\max\{b,d\}}K(t)f'(t)\,\mathrm{d}t\right|=\left|\int_{t_0}^{t_0+\varepsilon}K(t)\frac{1}{\varepsilon}\mathrm{d}t\right|=\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}K(t)\,\mathrm{d}t.$$

Now, from inequality (2.20) we have

$$\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}K(t)\,\mathrm{d}t\le K(t_0)\int_{t_0}^{t_0+\varepsilon}\frac{1}{\varepsilon}\mathrm{d}t=K(t_0).$$

Since,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}K(t)\,\mathrm{d}t=K(t_0)$$

the statement follows. In case $K(t_0)<0$, we take

$$f_\varepsilon(t)=\begin{cases}1, & \min\{a,c\}\le t\le t_0,\\ -\frac{1}{\varepsilon}(t-t_0-\varepsilon), & t_0\le t\le t_0+\varepsilon,\\ 0, & t_0+\varepsilon\le t\le \max\{b,d\}.\end{cases}$$

and the rest of proof is the same as above. □

Remark 2.1 In the special case, if we take $w(t)=\frac{1}{b-a}$, $t\in[a,b]$ and $u(t)=\frac{1}{d-c}$, $t\in[c,d]$ the inequality from the Theorem 2.3, for $a\le c<d\le b$ reduces to the inequality obtained in [95]:

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t-\frac{1}{d-c}\int_c^d f(t)\,\mathrm{d}t\right|$$
$$\le\begin{cases}\frac{(c-a)^2+(b-d)^2}{2(b-a-d+c)}\cdot\left\|f'\right\|_\infty, & f'\in L_\infty[a,b],\\ \left[\frac{(c-a)^{q+1}+(b-d)^{q+1}}{(q+1)(b-a)^{q-1}(b-a-d+c)}\right]^{\frac{1}{q}}\cdot\|f'\|_p, & f'\in L_p[a,b],\\ \frac{1}{2}\left(\frac{c-a+b-d}{b-a}+\left|\frac{c-a-b+d}{b-a}\right|\right)\cdot\|f'\|_1, & f'\in L_1[a,b].\end{cases}$$

In case $a\le c<b\le d$ we have the following result:

Corollary 2.2 *Let $f:[a,d]\to\mathbb{R}$ be such that f' exists and is bounded on $[a,d]$. Then for $a\le c<b\le d$ we have inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t-\frac{1}{d-c}\int_c^d f(t)\,\mathrm{d}t\right|$$
$$\le\begin{cases}\left[\frac{c-a+d-b}{2}\right]\cdot\left\|f'\right\|_\infty, & f'\in L_\infty[a,b],\\ \left[\left[\frac{(d-b)^{q+1}}{(d-c)^{q-1}}-\frac{(c-a)^{q+1}}{(b-a)^{q-1}}\right]\frac{1}{(q+1)(a-b+d-c)}\right]^{\frac{1}{q}}\cdot\|f'\|_p, & f'\in L_p[a,b],\\ \frac{1}{2}\left(\frac{c-a}{b-a}+\frac{d-b}{d-c}+\left|\frac{c-a}{b-a}-\frac{d-b}{d-c}\right|\right)\cdot\|f'\|_1, & f'\in L_1[a,b].\end{cases}$$

Proof. If we take $p=\infty$, $q=1$, $w(t)=\frac{1}{b-a}$, $t\in[a,b]$ and $u(t)=\frac{1}{d-c}$, $t\in[c,d]$ in the inequality from the Theorem 2.3 we have

$$\int_a^d|K(t)|\,\mathrm{d}t=\int_a^c|-W(t)|\,\mathrm{d}t+\int_c^b|-W(t)+U(t)|\,\mathrm{d}t+\int_b^d|U(t)-1|\,\mathrm{d}t.$$

Then

$$\int_a^c |-W(t)|\,\mathrm{d}t = \frac{(c-a)^2}{2(b-a)} \quad \text{and} \quad \int_b^d |U(t)-1|\,\mathrm{d}t = \frac{(d-b)^2}{2(d-c)},$$

and

$$\begin{aligned}\int_c^b |-W(t)+U(t)|\,\mathrm{d}t &= \frac{1}{(b-a)(d-c)}\int_c^b |(b-a+c-d)t-bc+ad|\,\mathrm{d}t\\ &= \frac{1}{(b-a)(d-c)}\int_c^b ((a-b+d-c)t+bc-ad)\,\mathrm{d}t\\ &= \left[\frac{(b-a)(d-b)^2}{2(d-c)} - \frac{(d-c)(c-a)^2}{2(b-a)}\right]\frac{1}{a-b+d-c}.\end{aligned}$$

Consequently $\int_a^d |K(x,t)|\,\mathrm{d}t = \frac{1}{2}(c-a+d-b)$ and the first inequality is proved. In order to prove the third inequality we put $p=1$, $q=\infty$, $w(t)=\frac{1}{b-a}$, $t\in[a,b]$ and $u(t)=\frac{1}{d-c}$, $t\in[c,d]$ in the inequality from the Theorem 2.3. We have

$$\sup_{t\in[a,c]}|K(t)| = \frac{c-a}{b-a}, \quad \sup_{t\in[b,d]}|K(t)| = \frac{d-b}{d-c}$$

$$\sup_{t\in[c,b]}|K(t)| = \sup_{t\in[c,b]}\left|\frac{t-c}{d-c}-\frac{t-a}{b-a}\right| = \max\left\{\frac{c-a}{b-a},\frac{d-b}{d-c}\right\}$$

and

$$\|K(t)\|_\infty = \sup_{t\in[a,d]}|K(t)| = \max\left\{\frac{c-a}{b-a},\frac{d-b}{d-c}\right\}.$$

Since $0\le\frac{c-a}{b-a}\le\frac{d-b}{d-c}$ or $0\le\frac{d-b}{d-c}\le\frac{c-a}{b-a}$ we get

$$\max\left\{\frac{c-a}{b-a},\frac{d-b}{d-c}\right\} = \frac{1}{2}\left(\frac{c-a}{b-a}+\frac{d-b}{d-c}+\left|\frac{c-a}{b-a}-\frac{d-b}{d-c}\right|\right).$$

In order to prove the second inequality take $p,q\neq 1,\infty$; $w(t)=\frac{1}{b-a}$, $t\in[a,b]$ and $u(t)=\frac{1}{d-c}$, $t\in[c,d]$ in the inequality from the Theorem 2.3. Thus

$$\int_a^c |-W(t)|^q\,\mathrm{d}t = \frac{(c-a)^{q+1}}{(q+1)(b-a)^q}, \quad \int_b^d |U(t)-1|^q\,\mathrm{d}t = \frac{(d-b)^{q+1}}{(q+1)(d-c)^q}$$

and

$$\begin{aligned}&\int_c^b |-W(t)+U(t)|^q\,\mathrm{d}t = \int_c^b \left|\frac{a-t}{b-a}+\frac{t-c}{d-c}\right|^q\mathrm{d}t\\ &= \left[\frac{(b-a)(d-b)^{q+1}}{(d-c)^q} - \frac{(d-c)(c-a)^{q+1}}{(b-a)^q}\right]\frac{1}{(q+1)(a-b+d-c)}.\end{aligned}$$

Consequently,

$$\int_a^d |K(t)|^q \, dt = \left[\frac{(d-b)^{q+1}}{(d-c)^{q-1}} - \frac{(c-a)^{q+1}}{(b-a)^{q-1}} \right] \frac{1}{(q+1)(a-b+d-c)}$$

which completes the proof. □

Corollary 2.3 *Assume (p,q) is a pair of conjugate exponents, $1 \le p,q \le \infty$. Let $f' \in L_p[a,d]$, then for $x \in [a,d]$ we have*

$$\left| \frac{1}{x-a} \int_a^x f(t)\,dt - \frac{1}{d-x} \int_x^d f(t)\,dt \right| \le \begin{cases} \frac{d-a}{2} \cdot \|f'\|_\infty, & f' \in L_\infty[a,b], \\ \left[\frac{d-a}{q+1}\right]^{\frac{1}{q}} \cdot \|f'\|_p, & f' \in L_p[a,b], \\ \|f'\|_1, & f' \in L_1[a,b]. \end{cases}$$

Proof. By setting $b = c = x$ in the previous Corollary. □

Remark 2.2 The incorrect version of the inequality from Corollary 2.3 was first proved by P. Cerone in [33] with $\left[\frac{(x-a)^q+(d-x)^q}{q+1}\right]^{\frac{1}{q}}$ instead of $\left[\frac{d-a}{q+1}\right]^{\frac{1}{q}}$.

Corollary 2.4 *Assume (p,q) is a pair of conjugate exponents, $1 \le p,q \le \infty$. Then for $a \le c < d \le b$, and we have*

$$\left| \int_a^b w(t) f(t)\,dt - \int_c^d u(t) f(t)\,dt \right|$$

$$\le \begin{cases} \|f'\|_\infty \left[\int_a^c W(t)\,dt + \int_d^b [1-W(t)]\,dt + \int_c^d |W(t)-U(t)|\,dt \right], & f' \in L_\infty[a,b], \\ \|f'\|_p \left[\int_a^c W(t)^q\,dt + \int_d^b [1-W(t)]^q\,dt + \int_c^d |W(t)-U(t)|^q\,dt \right]^{\frac{1}{q}}, & f' \in L_p[a,b], \\ \|f'\|_1 \max\left\{ W(c), 1-W(d), \sup_{t\in[c,d]} |W(t)-U(t)| \right\}, & f' \in L_1[a,b], \end{cases}$$

and for $a \le c < b \le d$

$$\left| \int_c^b w(t) f(t)\,dt - \int_c^b u(t) f(t)\,dt \right|$$

$$\leq \begin{cases} \|f'\|_\infty \left[\int\limits_a^c W(t)\,\mathrm{d}t + \int\limits_b^d [1-U(t)]\,\mathrm{d}t + \int\limits_c^b |W(t)-U(t)|\,\mathrm{d}t \right], & f' \in L_\infty[a,d], \\ \|f'\|_p \left[\int\limits_a^c W(t)^q\,\mathrm{d}t + \int\limits_b^d [1-U(t)]^q\,\mathrm{d}t + \int\limits_c^b |W(t)-U(t)|^q\,\mathrm{d}t \right]^{\frac{1}{q}}, & f' \in L_p[a,d], \\ \|f'\|_1 \max\left\{ W(c), 1-U(b), \sup_{t\in[c,b]} |W(t)-U(t)| \right\}, & f' \in L_1[a,d]. \end{cases}$$

Proof. Directly from the Theorem **2.3**. □

The first inequality from the next theorem may be regarded as the weighted Ostrowski inequality.

Corollary 2.5 *Assume (p,q) is a pair of conjugate exponents, $1 \leq p,q \leq \infty$. Then for $x \in [a,b]$ we have*

$$\left| \int_a^b w(t) f(t)\,\mathrm{d}t - f(x) \right|$$
$$\leq \begin{cases} \|f'\|_\infty \left[\int_a^x W(t)\,\mathrm{d}t + \int_x^b [1-W(t)]\,\mathrm{d}t \right], & f' \in L_\infty[a,b], \\ \|f'\|_p \left[\int_a^x W(t)^q\,\mathrm{d}t + \int_x^b [1-W(t)]^q\,\mathrm{d}t \right]^{\frac{1}{q}}, & f' \in L_p[a,b], \\ \|f'\|_1 \max\{W(x), 1-W(x)\}. & f' \in L_1[a,b]. \end{cases}$$

Proof. By setting $c = d = x$ and $u(t) = \frac{1}{d-c}$, $t \in [c,d]$ in the first inequality from the Corollary 2.4, and assuming $\frac{1}{d-c}\int_c^d f(t)\,\mathrm{d}t = f(x)$ as a limit case. □

Corollary 2.6 *Assume (p,q) is a pair of conjugate exponents, $1 \leq p,q \leq \infty$. Then for $x \in [a,d]$ we have*

$$\left| \int_a^x w(t) f(t)\,\mathrm{d}t - \int_x^d u(t) f(t)\,\mathrm{d}t \right|$$
$$\leq \begin{cases} \|f'\|_\infty \left[\int_a^x W(t)\,\mathrm{d}t + \int_x^d [1-U(t)]\,\mathrm{d}t \right], & f' \in L_\infty[a,d], \\ \|f'\|_p \left[\int_a^x W(t)^q\,\mathrm{d}t + \int_x^d [1-U(t)]^q\,\mathrm{d}t \right]^{\frac{1}{q}}, & f' \in L_p[a,d], \\ \|f'\|_1 \max\{W(x), 1-U(x), |W(x)-U(x)|\}. & f' \in L_1[a,d]. \end{cases}$$

Proof. By setting $b = c = x$ in the second inequality from the Corollary **2.4**. □

Now, we generalize the result for the functions whose n-th derivatives $f^{(n)}$ are from L_p spaces, $1 \leq p \leq \infty$.

Theorem 2.4 *Assume (p,q) is a pair of conjugate exponents, $1 \le p,q \le \infty$. Let $f : [a,b] \cup [c,d] \to \mathbb{R}$ be n-times differentiable on $[a,b] \cup [c,d]$,, $n \in \mathbb{N}$ and $f^{(n)} \in L_p [a,b] \cup [c,d]$, $w : [a,b] \to [0,\infty\rangle$ and $u : [c,d] \to [0,\infty\rangle$ are some normalized weight functions, $W(t)$, $U(t)$ as in (2.18). Then for every $x \in [a,b] \cap [c,d]$*

$$\Bigg| \int_a^b w(t) f(t)\,\mathrm{d}t - \int_c^d u(t) f(t)\,\mathrm{d}t + \sum_{k=0}^{n-2} \left(\int_a^b f^{(k+1)}(s_1)\, w(s_1)\,\mathrm{d}s_1 \right)$$

$$\cdot \left(\underbrace{\int_a^b \cdots \int_a^b}_{(k+1)th\ integral} P_w(x,s_1) \prod_{i=1}^{k} P_w(s_i,s_{i+1})\,\mathrm{d}s_1 \cdots \mathrm{d}s_{k+1} \right)$$

$$- \sum_{k=0}^{n-2} \left(\int_c^d f^{(k+1)}(s_1)\, u(s_1)\,\mathrm{d}s_1 \right)$$

$$\cdot \left(\underbrace{\int_c^d \cdot\cdot \int_c^d}_{(k+1)th\ integral} P_u(x,s_1) \prod_{i=1}^{k} P_u(s_i,s_{i+1})\,\mathrm{d}s_1 \cdot\cdot \mathrm{d}s_{k+1} \right) \Bigg| \le \left\| f^{(n)} \right\|_p \left\| \mathbf{K}(x,\cdot) \right\|_q ,$$

where

$$\mathbf{K}(x,s_n) = \underbrace{\int_a^b \cdot\cdot \int_a^b}_{(n-1)th\ integral} P_w(x,s_1) \prod_{i=1}^{n-1} P_w(s_i,s_{i+1})\,\mathrm{d}s_1 \cdot\cdot \mathrm{d}s_{n-1}$$

$$- \underbrace{\int_c^d \cdot\cdot \int_c^d}_{(n-1)th\ integral} P_u(x,s_1) \prod_{i=1}^{n-1} P_u(s_i,s_{i+1})\,\mathrm{d}s_1 \cdot\cdot \mathrm{d}s_{n-1}$$

and we suppose that

$$\underbrace{\int_a^b \cdot\cdot \int_a^b}_{(n-1)th\ integral} P_w(x,s_1) \prod_{i=1}^{n-1} P_w(s_i,s_{i+1})\,\mathrm{d}s_1 \cdot\cdot \mathrm{d}s_{n-1}$$

equals zero for $s_n \notin [a,b]$ and

$$\underbrace{\int_c^d \cdot\cdot \int_c^d}_{(n-1)th\ integral} P_u(x,s_1) \prod_{i=1}^{n-1} P_u(s_i,s_{i+1})\,\mathrm{d}s_1 \cdot\cdot \mathrm{d}s_{n-1}$$

equals zero for $s_n \notin [c,d]$.

Proof. First we take the generalized weighted Montgomery identity (see [22]):

$$f(x) - \int_a^b f(s_1)\, w(s_1)\,\mathrm{d}s_1 - \sum_{k=0}^{n-2} \left(\int_a^b f^{(k+1)}(s_1)\, w(s_1)\,\mathrm{d}s_1 \right)$$

$$\cdot\left(\underbrace{\int_a^b\cdot\cdot\int_a^b}_{(k+1)th\ integral} P_w(x,s_1)\prod_{i=1}^{k}P_w(s_i,s_{i+1})\,\mathrm{d}s_1\cdot\cdot\mathrm{d}s_{k+1}\right)$$
$$=\underbrace{\int_a^b\cdot\cdot\int_a^b}_{n-th\ integral} P_w(x,s_1)\prod_{i=1}^{n-1}P_w(s_i,s_{i+1})\,f^{(n)}(s_n)\,\mathrm{d}s_1\cdot\cdot\mathrm{d}s_n. \tag{2.21}$$

Now, as in Theorem 2.2, we subtract two weighted Montgomery identities, one for interval $[a,b]$ and the other for $[c,d]$, apply the Hölder inequality and the statement follows. □

Since in previous section we have proved the equivalence of Montgomery identity and Euler identity, now we estimate integral means via Euler type identities for case $[a,b]\cap[c,d]=[c,b]$ (generalization of results from [95]) and also we give the proof of the sharpness of given inequalities from [95].

Let us recall: for every function $f:[a,b]\to\mathbb{R}$ such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n\geq 1$ and for every $x\in[a,b]$, the following two formulae have been proved (see [47]):

$$f(x)=\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t+T_n^{[a,b]}(x)+P_n^{[a,b]}(x), \tag{2.22}$$

and

$$f(x)=\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t+T_{n-1}^{[a,b]}(x)+R_n^{[a,b]}(x), \tag{2.23a}$$

where

$$T_m^{[a,b]}(x)=\sum_{k=1}^{m}\frac{(b-a)^{k-1}}{k!}B_k\left(\frac{x-a}{b-a}\right)\left[f^{(k-1)}(b)-f^{(k-1)}(a)\right]$$

with convention $T_0^{[a,b]}(x)=0$, and

$$P_n^{[a,b]}(x)=-\frac{(b-a)^{n-1}}{n!}\int_a^b\left[B_n^*\left(\frac{x-t}{b-a}\right)\right]\mathrm{d}f^{(n-1)}(t),$$

$$R_n^{[a,b]}(x)=-\frac{(b-a)^{n-1}}{n!}\int_a^b\left[B_n^*\left(\frac{x-t}{b-a}\right)-B_n\left(\frac{x-a}{b-a}\right)\right]\mathrm{d}f^{(n-1)}(t).$$

Here $B_k(x)$, $k\geq 0$, are the Bernoulli polynomials, $B_k=B_k(0)$, $k\geq 0$, the Bernoulli numbers, and $B_k^*(x)$, $k\geq 0$, are periodic functions of period 1, related to Bernoulli polynomials as

$$B_k^*(x)=B_k(x),\quad 0\leq x<1,\qquad B_k^*(x+1)=B_k^*(x),\quad x\in\mathbb{R}.$$

From the properties of Bernoulli polynomials it follows $B_0^*(t)=1, B_1^*$ is a discontinuous function with the jump of -1 at each integer, and B_k^*, $k\geq 2$, is a continuous function (see

[1]). The formulae (2.22) and (2.23a) are extensions of the Euler integral formula (see [75]).

As we have proved in previous subsection that in the special case, for $w(t) = \frac{1}{b-a}$, $t \in [a,b]$ and $u(t) = \frac{1}{d-c}$, $t \in [c,d]$ the generalized weighted Montgomery identity (2.21) reduces to the Euler identity (2.23a).

In the recent paper [95] using formulae (2.22) and (2.23a), in case $[c,d] \subseteq [a,b]$ (i.e. $a \leq c < d \leq b$) following two theorems were obtained:

Theorem 2.5 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n \geq 1$. Then for $x \in [c,d]$, if $[c,d] \subset [a,b]$ we have*

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{1}{d-c}\int_c^d f(t)\,\mathrm{d}t + T_n^{[a,b]}(x) - T_n^{[c,d]}(x) = \int_a^b K_n^1(x,t)\,\mathrm{d}f^{(n-1)}(t)$$

and

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{1}{d-c}\int_c^d f(t)\,\mathrm{d}t + T_{n-1}^{[a,b]}(x) - T_{n-1}^{[c,d]}(x) = \int_a^b K_n^2(x,t)\,\mathrm{d}f^{(n-1)}(t),$$

where

$$K_n^1(x,t) = \begin{cases} \frac{(b-a)^{n-1}}{n!}B_n\left(\frac{x-t}{b-a}\right), & t \in [a,c], \\ \frac{(b-a)^{n-1}}{n!}B_n^*\left(\frac{x-t}{b-a}\right) - \frac{(d-c)^{n-1}}{n!}B_n^*\left(\frac{x-t}{d-c}\right), & t \in \langle c,d\rangle, \\ \frac{(b-a)^{n-1}}{n!}B_n\left(\frac{x-t}{b-a}+1\right), & t \in [d,b], \end{cases}$$

and

$$K_n^2(x,t) = \begin{cases} \frac{(b-a)^{n-1}}{n!}\left[B_n\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right)\right], & t \in [a,c], \\ \frac{(b-a)^{n-1}}{n!}\left[B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right)\right] \\ \quad -\frac{(d-c)^{n-1}}{n!}\left[B_n^*\left(\frac{x-t}{d-c}\right) - B_n\left(\frac{x-c}{d-c}\right)\right], & t \in \langle c,d\rangle, \\ \frac{(b-a)^{n-1}}{n!}\left[B_n\left(\frac{x-t}{b-a}+1\right) - B_n\left(\frac{x-a}{b-a}\right)\right], & t \in [d,b]. \end{cases}$$

Theorem 2.6 *Assume (p,q) is a pair of conjugate exponents, $1 \leq p,q \leq \infty$. Let $f^{(n)} \in L_p[a,b]$ for some $n \geq 1$. Then for $a \leq c < d \leq b$, we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{1}{d-c}\int_c^d f(t)\,\mathrm{d}t + T_n^{[a,b]}(x) - T_n^{[c,d]}(x)\right| \leq \left\|K_n^1(x,\cdot)\right\|_q \left\|f^{(n)}\right\|_p \tag{2.24}$$

and

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{1}{d-c}\int_c^d f(t)\,\mathrm{d}t + T_{n-1}^{[a,b]}(x) - T_{n-1}^{[c,d]}(x)\right| \leq \left\|K_n^2(x,\cdot)\right\|_q \left\|f^{(n)}\right\|_p, \tag{2.25}$$

for every $x \in [c,d]$. The inequalities (2.24) and (2.25) are sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Now, we establish result from the Theorems 2.5 and 2.6 (which are valid for $[c,d] \subset [a,b]$, i.e. $a \le c < d \le b$), for the other case $[a,b] \cap [c,d] = [c,b]$, i.e. $a \le c < b \le d$.

Theorem 2.7 *Let $f : [a,d] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,d]$ for some $n \ge 1$. Then if $[a,b] \cap [c,d] = [c,b]$, i.e. $a \le c < b \le d$ for $x \in [c,b]$ we have*

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{1}{d-c}\int_c^d f(t)\,\mathrm{d}t + T_n^{[a,b]}(x) - T_n^{[c,d]}(x) = \int_a^d \widetilde{K}_n^1(x,t)\,\mathrm{d}f^{(n-1)}(t) \tag{2.26}$$

and

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{1}{d-c}\int_c^d f(t)\,\mathrm{d}t + T_{n-1}^{[a,b]}(x) - T_{n-1}^{[c,d]}(x) = \int_a^d \widetilde{K}_n^2(x,t)\,\mathrm{d}f^{(n-1)}(t), \tag{2.27}$$

where

$$\widetilde{K}_n^1(x,t) = \begin{cases} \frac{(b-a)^{n-1}}{n!}B_n\left(\frac{x-t}{b-a}\right), & t \in [a,c], \\ \frac{(b-a)^{n-1}}{n!}B_n^*\left(\frac{x-t}{b-a}\right) - \frac{(d-c)^{n-1}}{n!}B_n^*\left(\frac{x-t}{d-c}\right), & t \in \langle c,b\rangle, \\ -\frac{(d-c)^{n-1}}{n!}B_n\left(\frac{x-t}{d-c}+1\right), & t \in [b,d], \end{cases}$$

and

$$\widetilde{K}_n^2(x,t) = \begin{cases} \frac{(b-a)^{n-1}}{n!}\left[B_n\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right)\right], & t \in [a,c], \\ \frac{(b-a)^{n-1}}{n!}\left[B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right)\right] & \\ -\frac{(d-c)^{n-1}}{n!}\left[B_n^*\left(\frac{x-t}{d-c}\right) - B_n\left(\frac{x-c}{d-c}\right)\right], & t \in \langle c,b\rangle, \\ -\frac{(d-c)^{n-1}}{n!}\left[B_n\left(\frac{x-t}{d-c}+1\right) - B_n\left(\frac{x-c}{d-c}\right)\right], & t \in [b,d]. \end{cases}$$

Proof. We subtract identities (2.22) for interval $[a,b]$ and $[c,d]$, and then using the properties of B_n^*, we get the first formula. By doing the same with identity (2.23a),we get the second formula. □

Theorem 2.8 *Assume (p,q) is a pair of conjugate exponents, $1 \le p,q \le \infty$. Let $f^{(n)} \in L_p[a,b]$ for some $n \ge 1$. Then for $a \le c < b \le d$, we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{1}{d-c}\int_c^d f(t)\,\mathrm{d}t + T_n^{[a,b]}(x) - T_n^{[c,d]}(x)\right| \le \left\|\widetilde{K}_n^1(x,\cdot)\right\|_q \left\|f^{(n)}\right\|_p$$

and

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{1}{d-c}\int_c^d f(t)\,\mathrm{d}t + T_{n-1}^{[a,b]}(x) - T_{n-1}^{[c,d]}(x)\right| \le \left\|\widetilde{K}_n^2(x,\cdot)\right\|_q \left\|f^{(n)}\right\|_p,$$

for every $x \in [c,b]$. The inequalities are sharp for $1 < p \le \infty$ and the best possible for $p = 1$.

Proof. Use the identities (2.26) and (2.27) and apply the Hölder inequality. The proof for sharpness and the best possibility are similar as in Theorem 2.3. □

Remark 2.3 If we take $n = 1$ we get the same result as in the Corollary 2.2.

Remark 2.4 We have showed that in the special case, if we take $w(t) = \frac{1}{b-a}$, $t \in [a,b]$ and $u(t) = \frac{1}{d-c}$, $t \in [c,d]$, in (2.21) the weighted Montgomery identity reduces to the Euler identity. Consequently, the inequality from the Theorem 2.4, for $[c,d] \subset [a,b]$ reduces to the second inequality from the Theorem 2.6; and for $[a,b] \cap [c,d] = [c,b]$ to the second inequality from the Theorem 2.8.

2.1.4 Discrete weighted Montgomery identity for infinite series

In this subsection we present the discrete analogue of weighted Montgomery identity which is the discrete analogue of the identity (2.9) or (2.21) and use it to obtain some new discrete Ostrowski type inequalities as well as the estimations of difference of two (weighted) arithmetic means. All these results are published in [9].

Let $a_1, a_2, .., a_n$ be the finite sequence of real numbers as well as $w_1, w_2, .., w_n$. If for $1 \leq k \leq n$

$$W_k = \sum_{i=1}^{k} w_i, \quad \overline{W_k} = \sum_{i=k+1}^{n} w_i = W_n - W_k$$

then we have (see [89])

$$\sum_{i=1}^{n} w_i a_i = a_k W_n + \sum_{i=1}^{k-1} W_i (a_i - a_{i+1}) + \sum_{i=k}^{n-1} \overline{W_i} (a_{i+1} - a_i), \quad 1 \leq k \leq n. \tag{2.28}$$

Now, let $f : \mathbb{R} \to \mathbb{R}$ be any real-to-real function. The **difference operator** Δ is defined by

$$\Delta f(x) = f(x+1) - f(x). \tag{2.29}$$

This is the finite analogue of the derivative and if f is any real-to-real function, so is Δf.

If we apply formula (2.28) with $a_i = f(i)$, $1 \leq i \leq n$, we get

$$\sum_{i=1}^{n} w_i f(i) = f(k) W_n - \sum_{i=1}^{k-1} W_i \Delta f(i) + \sum_{i=k}^{n-1} \overline{W_i} \Delta f(i).$$

So, the discrete analogue of weighted Montgomery identity is

$$f(k) = \frac{1}{W_n} \sum_{i=1}^{n} w_i f(i) + \sum_{i=1}^{n} D_w(k,i) \Delta f(i), \tag{2.30}$$

where the discrete Peano kernel is defined by

$$D_w(k,i) = \frac{1}{W_n} \cdot \begin{cases} W_i, & 1 \leq i \leq k-1, \\ \left(-\overline{W_i}\right), & k \leq i \leq n. \end{cases} \tag{2.31}$$

Remark 2.5 More generally we could take $a_i = f(x+i),\ 1 \le i \le n$ where $x \in \mathbb{R}$, and in this way obtain

$$f(x+k) = \frac{1}{W_n}\sum_{i=1}^{n} w_i f(x+i) + \sum_{i=1}^{n} D_w(k,i)\,\Delta f(x+i).$$

But, from now on we will assume that $x = 0$ without loss of generality.

If $n \in \mathbb{N}$, Δ^n is inductively defined by

$$\Delta^n f = \Delta^{n-1}(\Delta f).$$

Then, it is easy to prove by induction or directly using the elementary theory of operators (see [63])

$$\Delta^n f(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x+k).$$

In the next Theorem we give the generalization of the identity (2.30).

Theorem 2.9 *Let $f : \mathbb{R} \to \mathbb{R}$ be any real-to-real function and Δ the difference operator, $n, m, k \in \mathbb{N}$, $m \ge 2$ and $1 \le k \le n$. Then it holds*

$$\begin{aligned} f(k) &= \frac{1}{W_n}\sum_{i=1}^{n} w_i f(i) + \frac{1}{W_n}\sum_{r=1}^{m-1}\left(\sum_{i=1}^{n} w_i \Delta^r f(i)\right) \\ &\quad \cdot \left(\sum_{i_1=1}^{n}\cdots\sum_{i_r=1}^{n} D_w(k,i_1)\,D_w(i_1,i_2)\cdots D_w(i_{r-1},i_r)\right) \\ &\quad + \sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n} D_w(k,i_1)\,D_w(i_1,i_2)\cdots D_w(i_{m-1},i_m)\,\Delta^m f(i_m) \end{aligned} \tag{2.32}$$

Proof. We prove our assertion by induction with respect to m. To prove the identity for $m = 2$ we apply (2.30) for the real-to-real function Δf

$$\Delta f(i) = \frac{1}{W_n}\sum_{i=1}^{n} w_i \Delta f(i) + \sum_{j=1}^{n} D_w(i,j)\,\Delta^2 f(j).$$

Again, by (2.30) we have

$$\begin{aligned} f(k) &= \frac{1}{W_n}\sum_{i=1}^{n} w_i f(i) + \sum_{i=1}^{n} D_w(k,i)\left(\frac{1}{W_n}\sum_{j=1}^{n} w_j \Delta f(j) + \sum_{j=1}^{n} D_w(i,j)\,\Delta^2 f(j)\right) \\ &= \frac{1}{W_n}\sum_{i=1}^{n} w_i f(i) + \frac{1}{W_n}\left(\sum_{i=1}^{n} w_i \Delta f(i)\right)\left(\sum_{i=1}^{n} D_w(k,i)\right) \\ &\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} D_w(k,i)\,D_w(i,j)\,\Delta^2 f(j). \end{aligned}$$

Hence the identity (2.32) holds for $m=2$. Now, we assume that it holds for a natural number m. Applying the identity (2.30) for the function $\Delta^m f$

$$\Delta^m f(i_m) = \frac{1}{W_n}\sum_{i=1}^{n} w_i \Delta^m f(i) + \sum_{i_{m+1}=1}^{n} D_w(i_m, i_{m+1})\,\Delta^{m+1} f(i_{m+1})$$

and using the induction hypothesis, we get

$$\begin{aligned}
f(k) &= \frac{1}{W_n}\sum_{i=1}^{n} w_i f(i) + \frac{1}{W_n}\sum_{r=1}^{m-1}\left(\sum_{i=1}^{n} w_i \Delta^r f(i)\right) \\
&\quad \cdot \left(\sum_{i_1=1}^{n}\cdots\sum_{i_r=1}^{n} D_w(k,i_1)\,D_w(i_1,i_2)\cdots D_w(i_{r-1},i_r)\right) \\
&\quad + \left(\sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n} D_w(k,i_1)\,D_w(i_1,i_2)\cdots D_w(i_{m-1},i_m)\right. \\
&\quad \left.\cdot\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i \Delta^m f(i_m) + \sum_{i_{m+1}=1}^{n} D_w(i_m,i_{m+1})\,\Delta^{m+1} f(i_{m+1})\right)\right) \\
&= \frac{1}{W_n}\sum_{i=1}^{n} w_i f(i) + \frac{1}{W_n}\sum_{r=1}^{m}\left(\sum_{i=1}^{n} w_i \Delta^r f(i)\right) \\
&\quad \cdot \left(\sum_{i_1=1}^{n}\cdots\sum_{i_r=1}^{n} D_w(k,i_1)\,D_w(i_1,i_2)\cdots D_w(i_{r-1},i_r)\right) \\
&\quad + \sum_{i_1=1}^{n}\cdots\sum_{i_{m+1}=1}^{n} D_w(k,i_1)\,D_w(i_1,i_2)\cdots D_w(i_m,i_{m+1})\,\Delta^{m+1} f(i_{m+1}).
\end{aligned}$$

We see that (2.32) is valid for $m+1$ and our assertion is proved. □

2.1.5 Discrete Ostrowski type inequalities

Next, we use discrete analogue of weighted Montgomery identity (2.32) to obtain some new discrete Ostrowski type inequalities. These are the discrete analogues of some results obtained in [22], [47] and [7].

Bernoulli numbers B_i, $i \geq 0$, are defined by an implicit recurrence relation

$$\sum_{i=0}^{m}\binom{m+1}{i} B_i = \begin{cases} 1, & \text{if } m=0, \\ 0, & \text{if } m \neq 0. \end{cases}$$

If we denote (for $n \in \mathbb{N}$ and $m \in \mathbb{R}$)

$$S_m(n) = 1^m + 2^m + 3^m + \cdots + (n-1)^m,$$

it is well known that if $m \in \mathbb{N}$ (see [63])

$$S_m(n) = \frac{1}{m+1}\sum_{i=0}^{m}\binom{m+1}{i} B_i\, n^{m+1-i}.$$

Theorem 2.10 *Let $f:\mathbb{R}\to\mathbb{R}$ be any real-to-real function, $n,k\in\mathbb{N}$, and $1\le k\le n$. Let also (p,q) be a pair of conjugate exponents, $1<p,q<\infty$. Then the following inequalities hold:*

$$\left|f(k)-\frac{1}{n}\sum_{i=1}^{n}f(i)\right|\le\begin{cases}\frac{1}{n}\left(\frac{n^2-1}{4}+\left(k-\frac{n+1}{2}\right)^2\right)\cdot\max_{1\le i\le n}\{|\Delta f(i)|\},\\[2ex] \frac{1}{n}\left(S_q(k)+S_q(n-k+1)\right)^{\frac{1}{q}}\cdot\left(\sum_{i=1}^{n}|\Delta f(i)|^p\right)^{\frac{1}{p}},\\[2ex] \frac{1}{n}\max\{k-1,n-k\}\cdot\sum_{i=1}^{n}|\Delta f(i)|.\end{cases}$$

Proof. If we take $w_i=1,\ \ i=1,\ldots,n$, then $W_i=i$ and $\overline{W_i}=n-i$, and discrete Montgomery identity (2.30) reduces to

$$f(k)=\frac{1}{n}\sum_{i=1}^{n}f(i)+\sum_{i=1}^{n}D(k,i)\,\Delta f(i),$$

where

$$D(k,i)=\begin{cases}\frac{i}{n}, & 1\le i\le k-1,\\ \frac{i}{n}-1, & k\le i\le n.\end{cases}$$

We have

$$\left|f(k)-\frac{1}{n}\sum_{i=1}^{n}f(i)\right|=\left|\sum_{i=1}^{n}D(k,i)\,\Delta f(i)\right|\le\max_{1\le i\le n}\{|\Delta f(i)|\}\cdot\sum_{i=1}^{n}|D(k,i)|.$$

Since

$$\sum_{i=1}^{n}|D(k,i)|=\frac{1}{n}\left(\frac{n^2-1}{4}+\left(k-\frac{n+1}{2}\right)^2\right)$$

the first inequality follows. For the second we apply the Hölder inequality

$$\left|\sum_{i=1}^{n}D(k,i)\,\Delta f(i)\right|\le\left(\sum_{i=1}^{n}|D(k,i)|^q\right)^{\frac{1}{q}}\left(\sum_{i=1}^{n}|\Delta f(i)|^p\right)^{\frac{1}{p}}.$$

Since

$$\sum_{i=1}^{n}|D(k,i)|^q=\frac{1}{n^q}\left(\sum_{i=1}^{k-1}i^q+\sum_{i=k}^{n}(n-i)^q\right)=\frac{1}{n^q}\left(S_q(k)+S_q(n-k+1)\right)$$

the second inequality follows. Finally

$$\left|\sum_{i=1}^{n}D(k,i)\,\Delta f(i)\right|\le\max_{1\le i\le n}\{|D(k,i)|\}\cdot\sum_{i=1}^{n}|\Delta f(i)|$$

and

$$\max_{1\le i\le n}\{|D(k,i)|\}=\frac{1}{n}\max\{k-1,n-k\}$$

implies the last inequality. □

The next Theorem is the generalization of the previous one:

Theorem 2.11 *Assume (p,q) is a pair of conjugate exponents, $1 \leq p,q \leq \infty$ and $f : \mathbb{R} \to \mathbb{R}$ any real-to-real function, $n,k,m \in \mathbb{N}$, $m \geq 2$, and $1 \leq k \leq n$. Then the following inequalities hold:*

$$\begin{aligned}&\Bigg| f(k) - \frac{1}{W_n}\sum_{i=1}^{n} w_i f(i) \\ &- \frac{1}{W_n}\sum_{r=1}^{m-1}\left(\sum_{i=1}^{n} w_i \Delta^m f(i)\right)\left(\sum_{i_1=1}^{n}\cdots\sum_{i_r=1}^{n} D_w(k,i_1)D_w(i_1,i_2)\cdots D_w(i_{r-1},i_r)\right)\Bigg| \\ &\leq \left\|\sum_{i_1=1}^{n}\cdots\sum_{i_{m-1}=1}^{n} D_w(k,i_1)D_w(i_1,i_2)\cdots D_w(i_{m-1},\bullet)\right\|_q \left\|\Delta^m f\right\|_p\end{aligned}$$

where

$$\|g\|_p = \begin{cases} \left(\sum_i |g(i)|^p\right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty, \\ \max_i |g(i)| & \text{if } p = \infty. \end{cases}$$

Proof. By using the (2.32) and the Hölder inequality. □

2.1.6 Estimations of the difference of two weighted arithmetic means

In this subsection we will give the estimations of the difference of two weighted arithmetic means using the discrete weighted Montgomery identity. These are the discrete analogues of results obtained in [14], [24], [53], [36] and [95].

From now on we suppose $a,b,c,d \in \mathbb{N}$. and as before consider both cases $a \leq c < d \leq b$ i.e. $[c,d] \subseteq [a,b]$ and $a \leq c < b \leq d$ i.e .$[a,b] \cap [c,d] = [c,b]$. In fact, when $[a,b] \cap [c,d] \neq \emptyset$ we have four possible cases: $[c,d] \subseteq [a,b]$ and $[a,b] \cap [c,d] = [c,b]$ and $[a,b] \subseteq [c,d]$ and $[a,b] \cap [c,d] = [a,d]$, but the last two we can simply obtain from the first two by change $a \leftrightarrow c$, $b \leftrightarrow d$.

Theorem 2.12 *Let $f : \mathbb{R} \to \mathbb{R}$ be any real-to-real function, $a,b,c,d \in \mathbb{N}$, $w_a, w_{a+1}, \ldots, w_b$ finite sequence of real numbers as well as $u_c, u_{c+1}, \ldots, u_d$. Let also $W = \sum_{i=a}^{b} w_i$, $U = \sum_{i=c}^{d} u_i$ and for $k \in \mathbb{N}$*

$$W_k = \begin{cases} 0, & k < a, \\ \sum_{i=a}^{k} w_i, & a \leq k \leq b, \\ W, & k > b, \end{cases}$$

$$U_k = \begin{cases} 0, & k < c, \\ \sum_{i=c}^{k} u_i & c \leq k \leq d, \\ U, & k > d. \end{cases} \tag{2.33}$$

If $[a,b] \cap [c,d] \neq \emptyset$, then, for both cases $[c,d] \subseteq [a,b]$ and $[a,b] \cap [c,d] = [c,b]$, (and also for $[a,b] \subseteq [c,d]$ and $[a,b] \cap [c,d] = [a,d]$), the next formula is valid

$$\frac{1}{W}\sum_{i=a}^{b} w_i f(i) - \frac{1}{U}\sum_{i=c}^{d} u_i f(i) = \sum_{i=\min\{a,c\}}^{\max\{b,d\}} K(i)\Delta f(i) \tag{2.34}$$

where

$$K(i) = \frac{U_i}{U} - \frac{W_i}{W}, \quad \min\{a,c\} \le i \le \max\{b,d\}.$$

Proof. For $k \in ([a,b] \cap [c,d]) \cap \mathbb{N}$, we subtract identities

$$f(k) = \frac{1}{W}\sum_{i=a}^{b} w_i f(i) + \sum_{i=a}^{b} D_w(k,i)\,\Delta f(i),$$

and

$$f(k) = \frac{1}{U}\sum_{i=c}^{d} u_i f(i) + \sum_{i=c}^{d} D_u(k,i)\,\Delta f(i).$$

Then put

$$K(k,i) = D_u(k,i) - D_w(k,i).$$

Since $K(k,i)$ doesn't depend on k, we put $K(i)$ instead:

$$K(i) = \begin{cases} -\frac{W_i}{W}, & a \le i < c, \\ \frac{U_i}{U} - \frac{W_i}{W}, & c \le i \le d, \quad \text{if} \quad [c,d] \subseteq [a,b] \\ 1 - \frac{W_i}{W}, & d < i \le b, \end{cases} \tag{2.35}$$

$$K(i) = \begin{cases} -\frac{W_i}{W}, & a \le i < c, \\ \frac{U_i}{U} - \frac{W_i}{W}, & c \le i \le b, \quad \text{if} \quad [a,b] \cap [c,d] = [c,b] \\ \frac{U_i}{U} - 1, & b < i \le d. \end{cases} \tag{2.36}$$

□

Theorem 2.13 *Assume (p,q) is a pair of conjugate exponents, $1 \le p,q \le \infty$. Let $f : \mathbb{R} \to \mathbb{R}$ be any real-to-real function. Then we have*

$$\left| \frac{1}{W}\sum_{i=a}^{b} w_i f(i) - \frac{1}{U}\sum_{i=c}^{d} u_i f(i) \right| \le \|K\|_q \|\Delta f\|_p.$$

This inequality is sharp for $1 \le p \le \infty$.

Proof. We use the identity (2.19) and apply the Hölder inequality. For the proof of the sharpness we will find a function f such that

$$\left| \sum_{i=\min\{a,c\}}^{\max\{b,d\}} K(i)\,\Delta f(i) \right| = \left(\sum_{i=\min\{a,c\}}^{\max\{b,d\}} |K(i)|^q \right)^{\frac{1}{q}} \|\Delta f\|_p.$$

For $1 < p < \infty$ take f to be such that

$$\Delta f(i) = sgn\, K(i) \cdot |K(i)|^{\frac{1}{p-1}}.$$

For $p=\infty$ take

$$\Delta f(i) = sgn\, K(i)\,.$$

For $p=1$ we will find a function f such that

$$\left|\sum_{i=\min\{a,c\}}^{\max\{b,d\}} K(i)\,\Delta f(i)\right| = \max_{\min\{a,c\}\le i\le \max\{b,d\}} |K(i)| \left(\sum_{i=\min\{a,c\}}^{\max\{b,d\}} |\Delta f(i)|\right)$$

Suppose that $|K(i)|$ attains its maximum at $i_0 \in ([a,b]\cup[c,d])\cap\mathbb{N}$. First we assume that $K(i_0)>0$. Define f such that $\Delta f(i_0)=1$ and $\Delta f(i)=0$, $i\neq i_0$, i.e.

$$f(i) = \begin{cases} 0, & \min\{a,c\} \le i \le i_0, \\ 1, & i_0 < i \le \max\{b,d\}\,. \end{cases}$$

Then,

$$\left|\sum_{i=\min\{a,c\}}^{\max\{b,d\}} K(i)\,\Delta f(i)\right| = |K(i_0)| = \max_{\min\{a,c\}\le i\le \max\{b,d\}} |K(i)| \left(\sum_{i=\min\{a,c\}}^{\max\{b,d\}} |\Delta f(i)|\right)$$

and the statement follows. In case $K(i_0)<0$, we take f such that $\Delta f(i_0)=-1$ and $\Delta f(i)=0$, $i\neq i_0$, i.e.

$$f(i) = \begin{cases} 1, & \min\{a,c\} \le i \le i_0, \\ 0, & i_0 < i \le \max\{b,d\}\,. \end{cases}$$

and the rest of proof is the same as above. □

Case $a\le c<d\le b$

Corollary 2.7 *Let $f:\mathbb{R}\to\mathbb{R}$ be any real-to-real function. Then for $a\le c<d\le b$ we have inequality*

$$\left|\frac{1}{b-a+1}\sum_{i=a}^{b} f(i) - \frac{1}{d-c+1}\sum_{i=c}^{d} f(i)\right| \le C_1\cdot\|\Delta f\|_\infty$$

where

$$\begin{aligned} C_1 = {} & \frac{(a-c)(d-c+1-2\lfloor X\rfloor)+\frac{1}{2}(c-a)(c-a+1)+\frac{1}{2}(b-d-1)(b-d)}{b-a+1} \\ & + \frac{-\frac{1}{2}(d-c+1)(d-c+2)+\lfloor X\rfloor(\lfloor X\rfloor+1)}{b-a+1} - \frac{\lfloor X\rfloor(\lfloor X\rfloor+1)}{d-c+1} + \frac{d-c+2}{2} \end{aligned}$$

and $X=\frac{(d-c+1)(c-a)}{b-a+c-d}$.

Proof. If we put $p=\infty$, $q=1$, and $w_i=1$, $a\le i\le b$ and $u_i=1$, $c\le i\le d$ in the inequality from the Theorem 2.13. By (2.35) we have

$$C_1 = \sum_{i=a}^{b}|K(i)| = \sum_{i=a}^{c-1}\left|-\frac{W_i}{W}\right| + \sum_{i=c}^{d}\left|\frac{U_i}{U}-\frac{W_i}{W}\right| + \sum_{i=d+1}^{b}\left|1-\frac{W_i}{W}\right|$$

$$= \sum_{i=a}^{c-1} \frac{i-a+1}{b-a+1} + \sum_{i=c}^{d} \left| \frac{i-c+1}{d-c+1} - \frac{i-a+1}{b-a+1} \right| + \sum_{i=d+1}^{b} \frac{b-i}{b-a+1}$$

Then

$$\sum_{i=a}^{c-1} \frac{i-a+1}{b-a+1} = \frac{(c-a)(c-a+1)}{2(b-a+1)}, \quad \sum_{i=d+1}^{b} \frac{b-i}{b-a+1} = \frac{(b-d-1)(b-d)}{2(b-a+1)},$$

and

$$\sum_{i=c}^{d} \left| \frac{i-c+1}{d-c+1} - \frac{i-a+1}{b-a+1} \right| = \frac{\sum_{i=1}^{d-c+1} |i(b-a+c-d)+(d-c+1)(a-c)|}{(d-c+1)(b-a+1)}.$$

Since we have

$$\begin{aligned}
&\sum_{i=1}^{d-c+1} |i(b-a+c-d)+(d-c+1)(a-c)| \\
&= \sum_{i=1}^{\lfloor X \rfloor} -i(b-a+c-d)-(d-c+1)(a-c) \\
&+ \sum_{i=\lfloor X \rfloor+1}^{d-c+1} i(b-a+c-d)+(d-c+1)(a-c) \\
&= \frac{(a-c)(d-c+1-2\lfloor X \rfloor)-\frac{1}{2}(d-c+1)(d-c+2)+\lfloor X \rfloor(\lfloor X \rfloor+1)}{b-a+1} \\
&- \frac{\lfloor X \rfloor(\lfloor X \rfloor+1)}{d-c+1} + \frac{d-c+2}{2}.
\end{aligned}$$

the proof follows. □

Corollary 2.8 *Let $f : \mathbb{R} \to \mathbb{R}$ be any real-to-real function. Then for $a \leq c < d \leq b$ we have inequality*

$$\begin{aligned}
&\left| \frac{1}{b-a+1} \sum_{i=a}^{b} f(i) - \frac{1}{d-c+1} \sum_{i=c}^{d} f(i) \right| \\
&\leq \max \left\{ \frac{b-d}{b-a+1}, \frac{c-a}{b-a+1}, \left| \frac{1}{d-c+1} - \frac{c-a+1}{b-a+1} \right| \right\} \cdot \|\Delta f\|_1 .
\end{aligned}$$

Proof. If we put $p=1$, $q=\infty$, and $w_i=1$, $a \leq i \leq b$ and $u_i=1$, $c \leq i \leq d$ in the inequality from the Theorem 2.13. By (2.35) we have

$$\max_{a \leq i \leq c-1} |K(i)| = \max_{a \leq i \leq c-1} \left| \frac{i-a+1}{b-a+1} \right| = \frac{c-a}{b-a+1},$$

$$\max_{d+1 \leq i \leq b} |K(i)| = \max_{d+1 \leq i \leq b} \left| \frac{b-i}{b-a+1} \right| = \frac{b-d-1}{b-a+1}, \quad \text{for } d < b$$

and

$$\max_{c\le i\le d}|K(i)| = \max_{c\le i\le d}\left|\frac{i-c+1}{d-c+1}-\frac{i-a+1}{b-a+1}\right|$$
$$= \max\left\{\frac{b-d}{b-a+1}, \left|\frac{1}{d-c+1}-\frac{c-a+1}{b-a+1}\right|\right\}.$$

So

$$\max_{a\le i\le b}|K(i)| = \max\left\{\frac{b-d}{b-a+1}, \frac{c-a}{b-a+1}, \left|\frac{1}{d-c+1}-\frac{c-a+1}{b-a+1}\right|\right\}.$$

and it is easy to see that this formula is also valid if $b=d$. □

Remark 2.6 In case $a\le c=d\le b$ i.e. if $c=d=k$, we have

$$\left|\frac{1}{b-a+1}\sum_{i=a}^{b} f(i)-f(k)\right| \le \begin{cases} \frac{(k-a)^2+(b-k)^2+b-a}{2(b-a+1)}\cdot\|\Delta f\|_\infty \\ \frac{1}{b-a+1}\left(S_q(k-a+1)+S_q(b-k+1)\right)^{\frac{1}{q}}\cdot\|\Delta f\|_p \\ \max\left\{\frac{b-k}{b-a+1}, \frac{k-a}{b-a+1}\right\}\cdot\|\Delta f\|_1 \end{cases}$$

where (p,q) is a pair of conjugate exponents, $1<p,q<\infty$. For $a=1$ and $b=n$, these inequalities coincides with inequalities from the Theorem 2.10. Indeed, for the first inequality we set $c=d=k$ in the proof of the Corollary 2.7, so

$$\sum_{i=a}^{b}|K(i)| = \sum_{i=a}^{k-1}\frac{i-a+1}{b-a+1}+\left|\frac{k-k+1}{k-k+1}-\frac{k-a+1}{b-a+1}\right|+\sum_{i=k+1}^{b}\frac{b-i}{b-a+1}$$
$$= \frac{(k-a)(k-a+1)}{2(b-a+1)}+\frac{b-k}{b-a+1}+\frac{(b-k-1)(b-k)}{2(b-a+1)}$$
$$= \frac{(k-a)^2+(b-k)^2+b-a}{2(b-a+1)}.$$

For the last inequality, similarly, we set $c=d=k$ in the Corollary 2.8. For the second inequality, we put $w_i=1$, $a\le i\le b$ and $c=d=k$, $u_k=1$, (thus $W_i=i-a+1$, $\overline{W_i}=b-i$, $U=1$) in the inequality from the Theorem 2.13. By (2.35) we have

$$\sum_{i=a}^{b}|K(i)|^q = \sum_{i=a}^{k-1}\left(\frac{i-a+1}{b-a+1}\right)^q+\left|\frac{k-k+1}{k-k+1}-\frac{k-a+1}{b-a+1}\right|^q+\sum_{i=k+1}^{b}\left(\frac{b-i}{b-a+1}\right)^q$$
$$= \frac{1}{(b-a+1)^q}\left(S_q(k-a+1)+(b-k)^q+S_q(b-k)\right)$$
$$= \frac{1}{(b-a+1)^q}\left(S_q(k-a+1)+S_q(b-k+1)\right).$$

Case $a \leq c < b \leq d$

Corollary 2.9 *Let $f : \mathbb{R} \to \mathbb{R}$ be any real-to-real function. Then for $a \leq c < b \leq d$ we have inequality*

$$\left| \frac{1}{b-a+1} \sum_{i=a}^{b} f(i) - \frac{1}{d-c+1} \sum_{i=c}^{d} f(i) \right| \leq C_2 \cdot \|\Delta f\|_\infty$$

where

$$C_2 = \frac{(b-c+1)(2-2a+b+c)+(c-a)(c-a+1)}{2(b-a+1)} + \frac{(d-b-1)(d-b)-(b-c+1)(b-c+2)}{2(d-c+1)}$$

Proof. We put $p = \infty$, $q = 1$, and $w_i = 1$, $a \leq i \leq b$ and $u_i = 1$, $c \leq i \leq d$ in the inequality from the Theorem 2.13. By (2.36) we have

$$\begin{aligned} C_2 &= \sum_{i=a}^{d} |K(i)| = \sum_{i=a}^{c-1} \left| -\frac{W_i}{W} \right| + \sum_{i=c}^{b} \left| \frac{U_i}{U} - \frac{W_i}{W} \right| + \sum_{i=d+1}^{d} \left| \frac{U_i}{U} - 1 \right| \\ &= \sum_{i=a}^{c-1} \frac{i-a+1}{b-a+1} + \sum_{i=c}^{b} \left| \frac{i-c+1}{d-c+1} - \frac{i-a+1}{b-a+1} \right| + \sum_{i=b+1}^{d} \frac{d-i}{d-c+1} \end{aligned}$$

Then

$$\sum_{i=a}^{c-1} \frac{i-a+1}{b-a+1} = \frac{(c-a)(c-a+1)}{2(b-a+1)}, \quad \sum_{i=b+1}^{d} \frac{d-i}{d-c+1} = \frac{(d-b-1)(d-b)}{2(d-c+1)},$$

and

$$\sum_{i=c}^{b} \left| \frac{i-c+1}{d-c+1} - \frac{i-a+1}{b-a+1} \right| = \frac{\sum_{i=1}^{b-c+1} |i(b-a+c-d)+(d-c+1)(a-c)|}{(d-c+1)(b-a+1)}.$$

Since we have

$$\begin{aligned} &\sum_{i=1}^{b-c+1} |i(b-a+c-d)+(d-c+1)(a-c)| \\ &= \sum_{i=1}^{b-c+1} -i(b-a+c-d)-(d-c+1)(a-c) \\ &= \frac{(b-c+1)(2-2a+b+c)}{2(b-a+1)} - \frac{(b-c+1)(b-c+2)}{2(d-c+1)} \end{aligned}$$

the proof follows. $\square$

Corollary 2.10 *Let $f:\mathbb{R}\to\mathbb{R}$ be any real-to-real function. Then for $a\le c<b\le d$ we have inequality*

$$\begin{aligned}&\left|\frac{1}{b-a+1}\sum_{i=a}^{b}f(i)-\frac{1}{d-c+1}\sum_{i=c}^{d}f(i)\right|\\&\le\max\left\{\frac{d-b}{d-c+1},\frac{c-a}{b-a+1},\left|\frac{c-a+1}{b-a+1}-\frac{1}{d-c+1}\right|\right\}\cdot\|\Delta f\|_1.\end{aligned}$$

Proof. If we put $p=1$, $q=\infty$, and $w_i=1$, $a\le i\le b$ and $u_i=1$, $c\le i\le d$ in the inequality from the Theorem 2.13. By (2.36) we have

$$\max_{a\le i\le c-1}|K(i)|=\max_{a\le i\le c-1}\left|\frac{i-a+1}{b-a+1}\right|=\frac{c-a}{b-a+1},$$

$$\max_{b+1\le i\le d}|K(i)|=\max_{b+1\le i\le d}\left|\frac{d-i}{d-c+1}\right|=\frac{d-b-1}{d-c+1},\ \text{if } b<d$$

and

$$\begin{aligned}\max_{c\le i\le b}|K(i)|&=\max_{c\le i\le b}\left|\frac{i-c+1}{d-c+1}-\frac{i-a+1}{b-a+1}\right|\\&=\max\left\{\frac{d-b}{d-c+1},\left|\frac{1}{d-c+1}-\frac{c-a+1}{b-a+1}\right|\right\}.\end{aligned}$$

So

$$\max_{a\le i\le d}|K(i)|=\max\left\{\frac{d-b}{d-c+1},\frac{c-a}{b-a+1},\left|\frac{c-a+1}{b-a+1}-\frac{1}{d-c+1}\right|\right\}.$$

and the formula is also valid if $b=d$. □

Corollary 2.11 *Let $f:\mathbb{R}\to\mathbb{R}$ be any real-to-real function, $k\in\mathbb{N}$, and $a\le k\le d$. Let also (p,q) be a pair of conjugate exponents, $1<p,q<\infty$. Then the following inequalities hold:*

$$\begin{aligned}&\left|\frac{1}{k-a+1}\sum_{i=a}^{k}f(i)-\frac{1}{d-k+1}\sum_{i=k}^{d}f(i)\right|\\&\le\begin{cases}\frac{d-a}{2}\cdot\|\Delta f\|_\infty,\\[2ex]\left(\frac{1}{(k-a+1)^q}S_q(k-a+1)+\frac{1}{(d-k+1)^q}S_q(d-k+1)\right)^{\frac{1}{q}}\cdot\|\Delta f\|_p,\\[2ex]\max\left\{\frac{d-k}{d-k+1},\frac{k-a}{k-a+1}\right\}\cdot\|\Delta f\|_1.\end{cases}\end{aligned}$$

Proof. For the first inequality we set $b=c=k$ in the proof of the Corollary 2.9:

$$\sum_{i=a}^{d}|K(i)|=\sum_{i=a}^{k-1}\frac{i-a+1}{k-a+1}+\left|\frac{k-k+1}{d-k+1}-\frac{k-a+1}{k-a+1}\right|+\sum_{i=k+1}^{d}\frac{d-i}{d-k+1}$$

$$= \frac{k-a}{2} + \frac{d-k}{d-k+1} + \frac{(d-k-1)(d-k)}{2(d-k+1)} = \frac{d-a}{2}.$$

For the last inequality, similarly, we set $b = c = k$ in the Corollary 2.10. And for the second inequality, we put $w_i = 1$, $a \le i \le k$ and $u_i = 1$, $k \le i \le d$, in the inequality from the Theorem 2.13. By (2.36) we have

$$\begin{aligned} \sum_{i=a}^{d} |K(i)|^q &= \sum_{i=a}^{k-1} \left(\frac{i-a+1}{k-a+1} \right)^q + \left| \frac{k-k+1}{d-k+1} - \frac{k-a+1}{k-a+1} \right|^q + \sum_{i=k+1}^{d} \left(\frac{d-i}{d-k+1} \right)^q \\ &= \frac{1}{(k-a+1)^q} S_q(k-a+1) + \left(\frac{d-k}{d-k+1} \right)^q + \frac{1}{(d-k+1)^q} S_q(d-k) \\ &= \frac{1}{(k-a+1)^q} S_q(k-a+1) + \frac{1}{(d-k+1)^q} S_q(d-k+1). \end{aligned}$$

And the proof follows. □

Remark 2.7 If we suppose $b = d$ in both cases $a \le c < d \le b$ and $a \le c < b \le d$, the analogous results coincide.

The "Weighted" case

Corollary 2.12 *Assume $w_a, w_{a+1}, \dots, w_b$ and $u_c, u_{c+1}, \dots, u_d$ are the finite sequences of real, positive numbers and (p,q) is a pair of conjugate exponents, $1 < p, q < \infty$. Let $f : \mathbb{R} \to \mathbb{R}$ be any real-to-real function. Then for $a \le c < d \le b$, we have*

$$\left| \frac{1}{W} \sum_{i=a}^{b} w_i f(i) - \frac{1}{U} \sum_{i=c}^{d} u_i f(i) \right|$$
$$\le \begin{cases} \left[\sum_{i=a}^{c-1} \left| \frac{W_i}{W} \right| + \sum_{i=c}^{d} \left| \frac{U_i}{U} - \frac{W_i}{W} \right| + \sum_{i=d+1}^{b} \left| 1 - \frac{W_i}{W} \right| \right] \|\Delta f\|_\infty, \\ \left[\sum_{i=a}^{c-1} \left| \frac{W_i}{W} \right|^q + \sum_{i=c}^{d} \left| \frac{U_i}{U} - \frac{W_i}{W} \right|^q + \sum_{i=d+1}^{b} \left| 1 - \frac{W_i}{W} \right|^q \right]^{\frac{1}{q}} \|\Delta f\|_p, \\ \max\left\{ \frac{W_{c-1}}{W}, 1 - \frac{W_{d+1}}{W}, \max_{c \le i \le d} \left| \frac{U_i}{U} - \frac{W_i}{W} \right| \right\} \|\Delta f\|_1, \end{cases}$$

and for $a \le c < b \le d$

$$\left| \frac{1}{W} \sum_{i=a}^{b} w_i f(i) - \frac{1}{U} \sum_{i=c}^{d} u_i f(i) \right|$$
$$\le \begin{cases} \left[\sum_{i=a}^{c-1} \left| \frac{W_i}{W} \right| + \sum_{i=c}^{b} \left| \frac{U_i}{U} - \frac{W_i}{W} \right| + \sum_{i=b+1}^{d} \left| \frac{U_i}{U} - 1 \right| \right] \|\Delta f\|_\infty, \\ \left[\sum_{i=a}^{c-1} \left| \frac{W_i}{W} \right|^q + \sum_{i=c}^{b} \left| \frac{U_i}{U} - \frac{W_i}{W} \right|^q + \sum_{i=b+1}^{d} \left| \frac{U_i}{U} - 1 \right|^q \right]^{\frac{1}{q}} \|\Delta f\|_p, \\ \max\left\{ \frac{W_{c-1}}{W}, 1 - \frac{U_{b+1}}{U}, \max_{c \le i \le b} \left| \frac{U_i}{U} - \frac{W_i}{W} \right| \right\} \|\Delta f\|_1. \end{cases}$$

Proof. Directly from the Theorem 2.13. □

The first inequality from the next Theorem may be regarded as the weighted Ostrowski inequality.

Corollary 2.13 *Assume $w_a, w_{a+1}, \ldots, w_b$ is a finite sequence of real, positive numbers and (p,q) is a pair of conjugate exponents, $1 < p,q < \infty$. Let $f : \mathbb{R} \to \mathbb{R}$ be any real-to-real function and $a \le k \le b$. Then*

$$\left| \frac{1}{W} \sum_{i=a}^{b} w_i f(i) - f(k) \right| \le \begin{cases} \left[\sum_{i=a}^{k-1} \left| \frac{W_i}{W} \right| + \sum_{i=k}^{b} \left| 1 - \frac{W_i}{W} \right| \right] \|\Delta f\|_\infty, \\ \left[\sum_{i=a}^{k-1} \left| \frac{W_i}{W} \right|^q + \sum_{i=k}^{b} \left| 1 - \frac{W_i}{W} \right|^q \right]^{\frac{1}{q}} \|\Delta f\|_p, \\ \max\left\{ \frac{W_{k-1}}{W}, 1 - \frac{W_k}{W} \right\} \|\Delta f\|_1. \end{cases}$$

Proof. By setting $c = d = k$ and $u_k = 1$ in the first inequality from the Corollary 2.12. □

Corollary 2.14 *Assume $w_a, w_{a+1}, \ldots, w_b$ and $u_c, u_{c+1}, \ldots, u_d$ are the finite sequences of real, positive numbers and (p,q) is a pair of conjugate exponents, $1 < p,q < \infty$. Let $f : \mathbb{R} \to \mathbb{R}$ be any real-to-real function and $a \le k \le d$. Then*

$$\left| \frac{1}{W} \sum_{i=a}^{k} w_i f(i) - \frac{1}{U} \sum_{i=k}^{d} u_i f(i) \right| \le \begin{cases} \left[\sum_{i=a}^{k-1} \left| \frac{W_i}{W} \right| + \left| \frac{U_k}{U} - \frac{W_k}{W} \right| + \sum_{i=k+1}^{d} \left| \frac{U_i}{U} - 1 \right| \right] \|\Delta f\|_\infty, \\ \left[\sum_{i=a}^{k-1} \left| \frac{W_i}{W} \right|^q + \left| \frac{U_k}{U} - \frac{W_k}{W} \right|^q + \sum_{i=k+1}^{d} \left| \frac{U_i}{U} - 1 \right|^q \right]^{\frac{1}{q}} \|\Delta f\|_p, \\ \max\left\{ \frac{W_{k-1}}{W}, 1 - \frac{U_{k+1}}{U}, \left| \frac{U_k}{U} - \frac{W_k}{W} \right| \right\} \|\Delta f\|_1. \end{cases}$$

Proof. By setting $b = c = k$ in the second inequality from the Corollary 2.12. □

The next Theorem is the generalization of the Theorem 2.13.

Theorem 2.14 *Assume (p,q) is a pair of conjugate exponents, $1 \le p,q \le \infty$. Let $f : \mathbb{R} \to \mathbb{R}$ be any real-to-real function, $a,b,c,d,k,m \in \mathbb{N}$, $m \ge 2$, and $k \in ([a,b] \cap [c,d]) \cap \mathbb{N}$. Then the following inequality hold*

$$\left| \frac{1}{W} \sum_{i=a}^{b} w_i f(i) - \frac{1}{U} \sum_{i=c}^{d} w_i f(i) \right.$$

$$
\begin{aligned}
&+\frac{1}{W}\sum_{r=1}^{m-1}\left(\sum_{i=a}^{b} w_i \Delta^m f(i)\right)\left(\sum_{i_1=a}^{b}\cdots\sum_{i_r=a}^{b} D_w(k,i_1)D_w(i_1,i_2)\cdots D_w(i_{r-1},i_r)\right)\\
&-\frac{1}{U}\sum_{r=1}^{m-1}\left(\sum_{i=c}^{d} u_i \Delta^m f(i)\right)\left(\sum_{i_1=c}^{d}\cdots\sum_{i_r=c}^{d} D_u(k,i_1)D_u(i_1,i_2)\cdots D_u(i_{r-1},i_r)\right)\Bigg|\\
&\leq \|\mathbf{K}(k,\bullet)\|_q \|\Delta^m f\|_p
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{K}(k,i_m) = &\sum_{i_1=a}^{b}\cdots\sum_{i_{m-1}=a}^{b} D_w(k,i_1)D_w(i_1,i_2)\cdots D_w(i_{m-1},i_m)\\
&-\sum_{i_1=c}^{d}\cdots\sum_{i_{m-1}=c}^{d} D_u(k,i_1)D_u(i_1,i_2)\cdots D_u(i_{m-1},i_m)
\end{aligned}
$$

and we suppose that

$$
\sum_{i_1=a}^{b}\cdots\sum_{i_{m-1}=a}^{b} D_w(k,i_1)D_w(i_1,i_2)\cdots D_w(i_{m-1},i_m)=0, \quad \textit{for } i_m \notin [a,b]\cap\mathbb{N}
$$

and

$$
\sum_{i_1=c}^{d}\cdots\sum_{i_{m-1}=c}^{d} D_u(k,i_1)D_u(i_1,i_2)\cdots D_u(i_{m-1},i_m)=0, \quad \textit{for } i_m \notin [c,d]\cap\mathbb{N}.
$$

This inequality is sharp for $1 \leq p \leq \infty$

Proof. As in Theorem 2.13, we subtract two weighted Montgomery identities, one for interval $[a,b]\cap\mathbb{N}$ and the other for $[c,d]\cap\mathbb{N}$. After that, our inequality follows by applying the Hölder inequality. The proof for the sharpness is similar as the proof of the Theorem 2.13 (with $\mathbf{K}(k,\bullet)$ instead of K and $\Delta^m f$ instead of Δf). □

Remark 2.8 Montgomery identity has been generalized in numerous ways over the last decade. For instance, [30], [67], [77], [78], [104] deal with the generalizations of Montgomery identity on time scale, [105] deals with generalization of Montgomery identity for q-integrals, [20], [23] with generalizations for fractal integrals and [19] with generalizations on $\mathbb{R}^{\mathbb{N}}$ over spherical shells and balls.

2.2 Weighted Euler identities

In this section four new weighted generalizations of Euler-type identities are given, and used to obtain new Ostrowski type inequalities, generalized trapezoid and midpoint inequalities, and estimations of the difference of two integral means. Also a discrete analogue of the weighted Montgomery identity (i.e. Euler identity) for finite sequences is given as well as discrete analogue of Ostrowski type inequalities, trapezoid and midpoint inequalities, and estimations of difference of two arithmetic means.

2.2.1 Integral weighted Euler type identities

In this subsection we give four new weighted generalizations of the Euler identity, which can be obtain by using weighted Montgomery identity (2.5). These results are published in [13].

Theorem 2.15 *Let's suppose $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n \geq 1$. Then for $w : [a,b] \to [0,\infty\rangle$ some nonnegative normalized weight function and for $P_w(x,t)$ the weighted Peano kernel given by (2.6), the following identities hold*

$$
\begin{aligned}
f(x) &= \int_a^b w(t) f(t)\,\mathrm{d}t + \sum_{k=1}^{n} \frac{(b-a)^{k-1}}{k!} \\
&\cdot \left(B_k\left(\frac{x-a}{b-a}\right) - \int_a^b w(t) B_k\left(\frac{t-a}{b-a}\right) \mathrm{d}t \right) \left[f^{(k-1)}(b) - f^{(k-1)}(a) \right] \\
&- \frac{(b-a)^{n-1}}{n!} \int_a^b \left(B_n^*\left(\frac{x-t}{b-a}\right) - \int_a^b w(s) B_n^*\left(\frac{s-t}{b-a}\right) \mathrm{d}s \right) df^{(n-1)}(t)
\end{aligned}
\tag{2.37}
$$

and

$$
\begin{aligned}
f(x) &= \int_a^b w(t) f(t)\,\mathrm{d}t + \sum_{k=1}^{n-1} \frac{(b-a)^{k-1}}{k!} \\
&\cdot \left(B_k\left(\frac{x-a}{b-a}\right) - \int_a^b w(t) B_k\left(\frac{t-a}{b-a}\right) \mathrm{d}t \right) \left[f^{(k-1)}(b) - f^{(k-1)}(a) \right] \\
&- \frac{(b-a)^{n-1}}{n!} \int_a^b \left(B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right) \right. \\
&\left. - \int_a^b w(s) \left(B_n^*\left(\frac{s-t}{b-a}\right) - B_n\left(\frac{s-a}{b-a}\right) \right) \mathrm{d}s \right) df^{(n-1)}(t)
\end{aligned}
\tag{2.38}
$$

and for $n > 1$

$$
\begin{aligned}
f(x) &= \int_a^b w(t) f(t)\,\mathrm{d}t \\
&+ \sum_{k=1}^{n} \frac{(b-a)^{k-2}}{(k-1)!} \left(\int_a^b P_w(x,t) B_{k-1}\left(\frac{t-a}{b-a}\right) \mathrm{d}t \right) \left[f^{(k-1)}(b) - f^{(k-1)}(a) \right] \\
&- \frac{(b-a)^{n-2}}{(n-1)!} \int_a^b \left(\int_a^b P_w(x,s) B_{n-1}^*\left(\frac{s-t}{b-a}\right) \mathrm{d}s \right) df^{(n-1)}(t)
\end{aligned}
\tag{2.39}
$$

and

$$\begin{aligned}
f(x) &= \int_a^b w(t) f(t)\,\mathrm{d}t \\
&+ \sum_{k=1}^{n-1} \frac{(b-a)^{k-2}}{(k-1)!} \left(\int_a^b P_w(x,t) B_{k-1}\left(\frac{t-a}{b-a}\right) \mathrm{d}t \right) \left[f^{(k-1)}(b) - f^{(k-1)}(a) \right] \\
&- \frac{(b-a)^{n-2}}{(n-1)!} \int_a^b \left(\int_a^b P_w(x,s) \left[B_{n-1}^*\left(\frac{s-t}{b-a}\right) - B_{n-1}\left(\frac{s-a}{b-a}\right) \right] \mathrm{d}s \right) d f^{(n-1)}(t).
\end{aligned} \tag{2.40}$$

Proof. We multiply identity (2.22) by $w(x)$ and than integrate it to obtain

$$\begin{aligned}
\int_a^b w(x) f(x)\,\mathrm{d}x = \left(\int_a^b w(x)\,\mathrm{d}x \right) \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t + \sum_{k=1}^{n} \frac{(b-a)^{k-1}}{k!} \\
\cdot \left(\int_a^b w(t) B_k\left(\frac{t-a}{b-a}\right) \mathrm{d}t \right) \left[f^{(k-1)}(b) - f^{(k-1)}(a) \right] \\
- \frac{(b-a)^{n-1}}{n!} \int_a^b \left(\int_a^b w(s) B_n^*\left(\frac{s-t}{b-a}\right) \mathrm{d}s \right) d f^{(n-1)}(t).
\end{aligned}$$

If we subtract this identity from (2.22) we obtain (2.37) Further, for $k > 1$ we have

$$\begin{aligned}
&B_k\left(\frac{x-a}{b-a}\right) - \int_a^b w(t) B_k\left(\frac{t-a}{b-a}\right) \mathrm{d}t \\
&= B_k\left(\frac{x-a}{b-a}\right) - W(t) B_k\left(\frac{t-a}{b-a}\right)\Big|_a^b + \frac{k}{b-a} \int_a^b W(t) B_{k-1}\left(\frac{t-a}{b-a}\right) \mathrm{d}t \\
&= B_k\left(\frac{x-a}{b-a}\right) - B_k\left(\frac{b-a}{b-a}\right) + \frac{k}{b-a} \int_a^b W(t) B_{k-1}\left(\frac{t-a}{b-a}\right) \mathrm{d}t \\
&= -\frac{k}{b-a} \int_x^b B_{k-1}\left(\frac{t-a}{b-a}\right) \mathrm{d}t + \frac{k}{b-a} \int_a^b W(t) B_{k-1}\left(\frac{t-a}{b-a}\right) \mathrm{d}t \\
&= \frac{k}{b-a} \int_a^b P_w(x,t) B_{k-1}\left(\frac{t-a}{b-a}\right) \mathrm{d}t.
\end{aligned}$$

Similarly for $n > 1$ we get

$$B_n^*\left(\frac{x-t}{b-a}\right) - \int_a^b w(s) B_n^*\left(\frac{s-t}{b-a}\right) \mathrm{d}s = \frac{n}{b-a} \int_a^b P_w(x,s) B_{n-1}^*\left(\frac{s-t}{b-a}\right) \mathrm{d}s$$

since B_n^*, $n > 1$ is a continuous function, while for $n = 1$ we have

$$B_1^*\left(\frac{x-t}{b-a}\right) - \int_a^b w(s) B_1^*\left(\frac{s-t}{b-a}\right) \mathrm{d}s = -P_w(x,t) + \frac{1}{b-a} \int_a^b P_w(x,s) B_0^*\left(\frac{s-t}{b-a}\right) \mathrm{d}s.$$

Thus, the identity (2.39) follows. The proof for the identities (2.40) and (2.38) is similar (by using the identity (2.23a)). □

Remark 2.9 We could also obtain identities (2.39) and (2.40) in the following way: applying identities (2.22) and (2.23a) with $f'(x)$, then putting these two formulae in the weighted Montgomery identity and finally interchanging the order of integration and replacing n with $n-1$.

Remark 2.10 In the special case, if we take $w(t) = \frac{1}{b-a}$, $t \in [a,b]$ we have

$$\begin{aligned}
&\left(B_k\left(\frac{x-a}{b-a}\right) - \int_a^b w(t) B_k\left(\frac{t-a}{b-a}\right) \mathrm{d}t\right) \\
&= \left(B_k\left(\frac{x-a}{b-a}\right) - \frac{1}{b-a}\int_a^b B_k\left(\frac{t-a}{b-a}\right) \mathrm{d}t\right) \\
&= B_k\left(\frac{x-a}{b-a}\right) - \frac{1}{(b-a)(k+1)}\left(B_{k+1}(1) - B_{k+1}(0)\right) \\
&= B_k\left(\frac{x-a}{b-a}\right),
\end{aligned}$$

and similarly

$$B_n^*\left(\frac{x-t}{b-a}\right) - \int_a^b w(s) B_n^*\left(\frac{s-t}{b-a}\right) \mathrm{d}s = B_n^*\left(\frac{x-t}{b-a}\right),$$

$$\begin{aligned}
&B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right) - \int_a^b w(s)\left[B_n^*\left(\frac{s-t}{b-a}\right) - B_n\left(\frac{s-a}{b-a}\right)\right] \mathrm{d}s \\
&= B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right).
\end{aligned}$$

Consequently, the identities (2.37) and (2.39) reduce to the Euler identity (2.22) and the identities (2.38) and (2.40) reduce to the identity (2.23a). So we may regard them as weighted Euler identities.

Corollary 2.15 *Suppose that all assumptions of Theorem 2.15 hold. Additionally assume that $w : [a,b] \to [0,\infty\rangle$ is symmetric on $[a,b]$, i.e. $w(t) = w(b-a-t)$, for $t \in [a,b]$. Then*

$$\begin{aligned}
f(x) &= \int_a^b w(t) f(t)\,\mathrm{d}t + \sum_{k=1}^{n} \frac{(b-a)^{k-1}}{k!} B_k\left(\frac{x-a}{b-a}\right)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right] \\
&- \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{(b-a)^{2k-1}}{(2k)!}\left(2\int_a^{\frac{a+b}{2}} w(t) B_{2k}\left(\frac{t-a}{b-a}\right) \mathrm{d}t\right)\left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right] \\
&- \frac{(b-a)^{n-1}}{n!}\int_a^b \left(B_n^*\left(\frac{x-t}{b-a}\right) - \int_a^b w(s) B_n^*\left(\frac{s-t}{b-a}\right) \mathrm{d}s\right) df^{(n-1)}(t)
\end{aligned} \tag{2.41}$$

and

$$f(x) = \int_a^b w(t) f(t)\,\mathrm{d}t + \sum_{k=1}^{n-1} \frac{(b-a)^{k-1}}{k!} B_k\left(\frac{x-a}{b-a}\right)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right]$$

$$\sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor}\frac{(b-a)^{2k-1}}{(2k)!}\left(2\int_a^{\frac{a+b}{2}}w(t)\,B_{2k}\left(\frac{t-a}{b-a}\right)\mathrm{d}t\right)\left[f^{(2k-1)}(b)-f^{(2k-1)}(a)\right]$$
$$-\frac{(b-a)^{n-1}}{n!}\int_a^b\left(B_n^*\left(\frac{x-t}{b-a}\right)-B_n\left(\frac{x-a}{b-a}\right)\right.$$
$$\left.-\int_a^b w(s)\left[B_n^*\left(\frac{s-t}{b-a}\right)-B_n\left(\frac{s-a}{b-a}\right)\right]\mathrm{d}s\right)df^{(n-1)}(t).$$
(2.42)

Proof. We have

$$B_k\left(\frac{t-a}{b-a}\right)-\int_a^b w(t)\,B_k\left(\frac{t-a}{b-a}\right)\mathrm{d}t$$
$$=\begin{cases}B_k\left(\frac{t-a}{b-a}\right)-2\int_a^{\frac{a+b}{2}}w(t)\,B_k\left(\frac{t-a}{b-a}\right)\mathrm{d}t, & \text{if } k \text{ is even,}\\ B_k\left(\frac{t-a}{b-a}\right) & \text{if } k \text{ is odd,}\end{cases}$$

since for Bernoulli polynomials hold $B_n(1-x)=(-1)^n B_n(x)$, $x\in[0,1]$. If we apply (2.37) and (2.38) with this the proof follows. □

Remark 2.11 Applying identity (2.41) with $x=b$ we get

$$f(b)=\int_a^b w(t)\,f(t)\,\mathrm{d}t+\sum_{k=1}^{n}\frac{(b-a)^{k-1}}{k!}B_k(1)\left[f^{(k-1)}(b)-f^{(k-1)}(a)\right]$$
$$-\sum_{k=1}^{\lfloor\frac{n}{2}\rfloor}\frac{(b-a)^{2k-1}}{(2k)!}\left(2\int_a^{\frac{a+b}{2}}w(t)\,B_{2k}\left(\frac{t-a}{b-a}\right)\mathrm{d}t\right)\left[f^{(2k-1)}(b)-f^{(2k-1)}(a)\right]$$
$$-\frac{(b-a)^{n-1}}{n!}\int_a^b\left(B_n^*\left(\frac{b-t}{b-a}\right)-\int_a^b w(s)\,B_n^*\left(\frac{s-t}{b-a}\right)\mathrm{d}s\right)df^{(n-1)}(t).$$

Since we have $B_n(1)=(-1)^n B_n(0)=(-1)^n B_n$ for $n\geq 0$ and $B_{2n+1}=0$ for $n\geq 1$ (see [1]) and also for $k=1$, $\frac{(b-a)^{k-1}}{k!}B_k(1)\left[f^{(k-1)}(b)-f^{(k-1)}(a)\right]=\frac{1}{2}\left[f(b)-f(a)\right]$ the last identity reduces to

$$\frac{f(a)+f(b)}{2}=\int_a^b w(t)\,f(t)\,\mathrm{d}t+\sum_{k=1}^{\lfloor\frac{n}{2}\rfloor}\frac{(b-a)^{2k-1}}{(2k)!}$$
$$\cdot\left(B_{2k}-2\int_a^{\frac{a+b}{2}}w(t)\,B_{2k}\left(\frac{t-a}{b-a}\right)\mathrm{d}t\right)\left[f^{(2k-1)}(b)-f^{(2k-1)}(a)\right]$$
$$-\frac{(b-a)^{n-1}}{n!}\int_a^b\left(B_n^*\left(\frac{b-t}{b-a}\right)-\int_a^b w(s)\,B_n^*\left(\frac{s-t}{b-a}\right)\mathrm{d}s\right)df^{(n-1)}(t).\qquad(2.43)$$

So we can regard this as the first Euler-Maclaurin formula (the generalized trapezoid identity). Similarly, applying identity (2.41) with $x = \frac{a+b}{2}$ we get

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &= \int_a^b w(t) f(t)\,\mathrm{d}t + \sum_{k=1}^{n} \frac{(b-a)^{k-1}}{k!} B_k\left(\frac{1}{2}\right)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right] \\
&- \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{(b-a)^{2k-1}}{(2k)!}\left(2\int_a^{\frac{a+b}{2}} w(t) B_{2k}\left(\frac{t-a}{b-a}\right)\mathrm{d}t\right)\left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right] \\
&- \frac{(b-a)^{n-1}}{n!}\int_a^b \left(B_n^*\left(\frac{a+b-2t}{2(b-a)}\right) - \int_a^b w(s) B_n^*\left(\frac{s-t}{b-a}\right)\mathrm{d}s\right) df^{(n-1)}(t).
\end{aligned}
$$

Since $B_n\left(\frac{1}{2}\right) = -\left(1-2^{1-n}\right) B_n$ for $n \geq 0$ (see [1]) the last identity reduces to

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &= \int_a^b w(t) f(t)\,\mathrm{d}t - \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{(b-a)^{2k-1}}{(2k)!} \\
&\cdot \left(\left(1-2^{1-2k}\right) B_{2k} + 2\int_a^{\frac{a+b}{2}} w(t) B_{2k}\left(\frac{t-a}{b-a}\right)\mathrm{d}t\right)\left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right] \\
&- \frac{(b-a)^{n-1}}{n!}\int_a^b \left(B_n^*\left(\frac{a+b-2t}{2(b-a)}\right) - \int_a^b w(s) B_n^*\left(\frac{s-t}{b-a}\right)\mathrm{d}s\right) df^{(n-1)}(t). \qquad (2.44)
\end{aligned}
$$

We can regard this as the second Euler-Maclaurin formula (the generalized midpoint identity).

In the next two subsections identities (2.37), (2.39), (2.38) and (2.40) are used to obtain some Ostrowski type inequalities (weighted generalizations of the results from [47] and [7]), as well as the generalizations of the estimations of the difference of two weighted integral means (generalizations of the results from [14], [24], [35], [36], [95]).

Ostrowski type inequalities

Let's denote for $n \geq 1$

$$
\begin{aligned}
&T_{w,n}(x) \\
&= \sum_{k=1}^{n} \frac{(b-a)^{k-1}}{k!}\left(B_k\left(\frac{x-a}{b-a}\right) - \int_a^b w(t) B_k\left(\frac{t-a}{b-a}\right)\mathrm{d}t\right)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right] \\
&= \sum_{k=1}^{n} \frac{(b-a)^{k-2}}{(k-1)!}\left(\int_a^b P_w(x,t) B_{k-1}\left(\frac{t-a}{b-a}\right)\mathrm{d}t\right)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right].
\end{aligned}
$$

For all the results in this subsection we will use identities (2.39) and (2.40). These results can also be obtained from identities (2.37) and (2.38).

Theorem 2.16 *Assume (p,q) is a pair of conjugate exponents, $1 \le p,q \le \infty$. Let $f^{(n)} \in L_p[a,b]$ for some $n > 1$. Then for $x \in [a,b]$ the following inequalities hold*

$$\left| f(x) - \int_a^b w(t) f(t)\,\mathrm{d}t - T_{w,n}(x) \right|$$
$$\le \frac{(b-a)^{n-2}}{(n-1)!} \left(\int_a^b \left| \int_a^b P_w(x,s) B_{n-1}^{*}\left(\frac{s-t}{b-a}\right) \mathrm{d}s \right|^q \mathrm{d}t \right)^{\frac{1}{q}} \left\| f^{(n)} \right\|_p \tag{2.45}$$

and

$$\left| f(x) - \int_a^b w(t) f(t)\,\mathrm{d}t - T_{w,n-1}(x) \right| \le \frac{(b-a)^{n-2}}{(n-1)!}$$
$$\cdot \left(\int_a^b \left| \int_a^b P_w(x,s) \left[B_{n-1}^{*}\left(\frac{s-t}{b-a}\right) - B_{n-1}\left(\frac{s-a}{b-a}\right) \right] \mathrm{d}s \right|^q \mathrm{d}t \right)^{\frac{1}{q}} \left\| f^{(n)} \right\|_p . \tag{2.46}$$

These inequalities (2.45) and (2.46) are sharp for $1 < p \le \infty$ and the best possible for $p = 1$.

Proof. Let's denote $C_1(t) = \frac{(b-a)^{n-2}}{(n-1)!} \int_a^b P_w(x,s) B_{n-1}^{*}\left(\frac{s-t}{b-a}\right) \mathrm{d}s$. We use the identity (2.39) and apply the Hölder inequality to obtain

$$\left| f(x) - \int_a^b w(t) f(t)\,\mathrm{d}t - T_{w,n}(x) \right|$$
$$= \left| \int_a^b C_1(t) f^{(n)}(t)\,\mathrm{d}t \right| \le \left(\int_a^b |C_1(t)|^q \mathrm{d}t \right)^{\frac{1}{q}} \left\| f^{(n)} \right\|_p .$$

For the proof of the sharpness we will find a function f for which the equality in (2.45) is obtained.

For $1 < p < \infty$ take f to be such that

$$f^{(n)}(t) = sgn\, C_1(t) \cdot |C_1(t)|^{\frac{1}{p-1}} .$$

For $p = \infty$ take

$$f^{(n)}(t) = sgn\, C_1(t) .$$

For $p = 1$ we shall prove that

$$\left| \int_a^b C_1(t) f^{(n)}(t)\,\mathrm{d}t \right| \le \max_{t \in [a,b]} |C_1(t)| \left(\int_a^b \left| f^{(n)}(t) \right| \mathrm{d}t \right) \tag{2.47}$$

is the best possible inequality. Suppose that $|C_1(t)|$ attains its maximum at $t_0 \in [a,b]$. First we assume that $C_1(t_0) > 0$. For $\varepsilon > 0$ define $f_\varepsilon(t)$ by

$$f_\varepsilon(t) = \begin{cases} 0, & a \le t \le t_0, \\ \frac{1}{\varepsilon n!}(t-t_0)^n, & t_0 \le t \le t_0 + \varepsilon, \\ \frac{1}{n!}(t-t_0)^{n-1}, & t_0 + \varepsilon \le t \le b. \end{cases}$$

Then, for ε small enough

$$\left|\int_a^b C_1(t) f^{(n)}(t)\,\mathrm{d}t\right| = \left|\int_{t_0}^{t_0+\varepsilon} C_1(t)\frac{1}{\varepsilon}\mathrm{d}t\right| = \frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} C_1(t)\,\mathrm{d}t.$$

Now, from inequality (2.47) we have

$$\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} C_1(t)\,\mathrm{d}t \le C_1(t_0)\int_{t_0}^{t_0+\varepsilon}\frac{1}{\varepsilon}\mathrm{d}t = C_1(t_0).$$

Since,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} C_1(t)\,\mathrm{d}t = C_1(t_0)$$

the statement follows. In case $C_1(t_0) < 0$, we take

$$f_\varepsilon(t) = \begin{cases} \frac{1}{n!}(t-t_0-\varepsilon)^{n-1}, & a \le t \le t_0, \\ -\frac{1}{\varepsilon n!}(t-t_0-\varepsilon)^n, & t_0 \le t \le t_0+\varepsilon, \\ 0, & t_0+\varepsilon \le t \le b, \end{cases}$$

and the rest of proof is the same as above. For the inequality (2.46) the proof is similar. □

Corollary 2.16 *Suppose that all assumptions of Theorem 2.16 hold. Then the following inequality holds*

$$\left|f(x) - \int_a^b w(t) f(t)\,\mathrm{d}t - T_{w,n}(x)\right| \le \frac{(b-a)^{n-1+\frac{1}{q}}}{(n-1)!}\left(\int_0^1 |B_{n-1}(s)|\,\mathrm{d}s\right)\left\|f^{(n)}\right\|_p. \quad (2.48)$$

Proof. Since $0 \le W(t) \le 1$, $t \in [a,b]$, so $|P_w(x,s)| \le 1$ and B_{n-1}^* is a periodic function with period 1 and $\int_0^1 |B_n^*(y+s)|\,\mathrm{d}s = \int_0^1 |B_n^*(s)|\,\mathrm{d}s = \int_0^1 |B_n(s)|\,\mathrm{d}s$ for every $y \in \mathbb{R}$, we have

$$\begin{aligned} &\left|\int_a^b P_w(x,s) B_{n-1}^*\left(\frac{s-t}{b-a}\right)\mathrm{d}s\right| \le \int_a^b |P_w(x,s)|\left|B_{n-1}^*\left(\frac{s-t}{b-a}\right)\right|\mathrm{d}s \\ &\le \int_a^b \left|B_{n-1}^*\left(\frac{s-t}{b-a}\right)\right|\mathrm{d}s = (b-a)\int_0^1 |B_{n-1}(s)|\,\mathrm{d}s \end{aligned}$$

and by applying (2.45) the inequality (2.48) is proved. □

Remark 2.12 For $n=2$, $n=3$ and $n=4$ inequality (2.48) reduces to:

$$\left|f(x) - \int_a^b w(t) f(t)\,\mathrm{d}t - T_{w,2}(x)\right| \le \frac{1}{4}(b-a)^{1+\frac{1}{q}}\left\|f''\right\|_p,$$

$$\left|f(x) - \int_a^b w(t) f(t)\,\mathrm{d}t - T_{w,3}(x)\right| \le \frac{\sqrt{3}}{34}(b-a)^{2+\frac{1}{q}}\left\|f'''\right\|_p,$$

$$\left|f(x) - \int_a^b w(t) f(t)\,\mathrm{d}t - T_{w,4}(x)\right| \le \frac{1}{192}(b-a)^{3+\frac{1}{q}}\left\|f^{(4)}\right\|_p.$$

Remark 2.13 If we use the identities (2.43) and (2.44) for $n=2$ and $w(t)=\frac{1}{b-a}$, $t\in[a,b]$ and then apply the Hölder inequality with $p=\infty$, $q=1$, we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t-\frac{b-a}{12}\left[f'(b)-f'(a)\right]\right|\leq\left\|f''\right\|_\infty\cdot\frac{\sqrt{3}}{54}(b-a)^2$$

and

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t+\frac{b-a}{24}\left[f'(b)-f'(a)\right]\right|\leq\left\|f''\right\|_\infty\cdot\frac{\sqrt{3}}{54}(b-a)^2.$$

By doing the same for $n=3$ we obtain (2.16) and (2.17) from the Corollary 2.1. Also inequality (2.46) from the Theorem 2.16 applied with $n=3$ is weighted generalization of Theorem 2.1 (from [7]) and applied with $n=2$ is weighted generalization of Corollary 1 from [47].

Estimations of the difference of two weighted integral means

In this section we will denote for $n\geq 1$

$$T_{w,n}^{[a,b]}(x)=\sum_{k=1}^{n}\frac{(b-a)^{k-1}}{k!}$$
$$\cdot\left(B_k\left(\frac{x-a}{b-a}\right)-\int_a^b w(t)B_k\left(\frac{t-a}{b-a}\right)\mathrm{d}t\right)\left[f^{(k-1)}(b)-f^{(k-1)}(a)\right],$$

$$S_{w,n}^{[a,b]}(x)=-\frac{(b-a)^{n-1}}{n!}$$
$$\cdot\int_a^b\left(B_{n-1}^*\left(\frac{x-t}{b-a}\right)-\int_a^b w(s)B_{n-1}^*\left(\frac{s-t}{b-a}\right)\mathrm{d}s\right)df^{(n-1)}(t),$$

$$R_{w,n}^{[a,b]}(x)=-\frac{(b-a)^{n-1}}{n!}\int_a^b\left(B_{n-1}^*\left(\frac{x-t}{b-a}\right)-B_{n-1}\left(\frac{x-a}{b-a}\right)\right.$$
$$\left.-\int_a^b w(s)\left[B_{n-1}^*\left(\frac{s-t}{b-a}\right)-B_{n-1}\left(\frac{s-a}{b-a}\right)\right]\mathrm{d}s\right)df^{(n-1)}(t)$$

for function $f:[a,b]\to\mathbb{R}$ such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$.

Following results are generalizations of the results from the paper [95] (in case $[c,d]\subset[a,b]$) and [14] (in case $[a,b]\cap[c,d]=[c,b]$).

Theorem 2.17 *Let $f:[a,b]\cup[c,d]\to\mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n\geq 1$, $w:[a,b]\to[0,\infty\rangle$ and $u:[c,d]\to[0,\infty\rangle$ some nonnegative normalized weight functions. Then if $[a,b]\cap[c,d]\neq\emptyset$ and $x\in[a,b]\cap[c,d]$, we have*

$$\int_a^b w(t)f(t)\,\mathrm{d}t-\int_c^d u(t)f(t)\,\mathrm{d}t+T_{w.n}^{[a,b]}(x)-T_{u,n}^{[c,d]}(x)$$

$$= \int_{\min\{a,c\}}^{\max\{b,d\}} K_n^1(x,t)\, df^{(n-1)}(t)$$

and

$$\int_a^b w(t) f(t)\,\mathrm{d}t - \int_c^d u(t) f(t)\,\mathrm{d}t + T_{w,n-1}^{[a,b]}(x) - T_{u,n-1}^{[c,d]}(x)$$
$$= \int_{\min\{a,c\}}^{\max\{b,d\}} K_n^2(x,t)\, df^{(n-1)}(t),$$

where in case $[c,d] \subset [a,b]$

$$K_n^1(x,t) = \begin{cases} \frac{(b-a)^{n-1}}{n!}\left(B_n^*\left(\frac{x-t}{b-a}\right) - \int_a^b w(s) B_n^*\left(\frac{s-t}{b-a}\right)\mathrm{d}s\right), & t \in [a,c\rangle, \\[2ex] \frac{(b-a)^{n-1}}{n!}\left(B_n^*\left(\frac{x-t}{b-a}\right) - \int_a^b w(s) B_n^*\left(\frac{s-t}{b-a}\right)\mathrm{d}s\right) \\ -\frac{(d-c)^{n-1}}{n!}\left(B_n^*\left(\frac{x-t}{d-c}\right) - \int_c^d u(s) B_n^*\left(\frac{s-t}{d-c}\right)\mathrm{d}s\right), & t \in [c,d], \\[2ex] \frac{(b-a)^{n-1}}{n!}\left(B_n^*\left(\frac{x-t}{b-a}\right) - \int_a^b w(s) B_n^*\left(\frac{s-t}{b-a}\right)\mathrm{d}s\right), & t \in \langle d,b], \end{cases}$$

and

$$K_n^2(x,t) = \begin{cases} \frac{(b-a)^{n-1}}{n!}\left(B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right)\right. \\ \left. - \int_a^b w(s)\left[B_n^*\left(\frac{s-t}{b-a}\right) - B_n\left(\frac{s-a}{b-a}\right)\right]\mathrm{d}s\right) & t \in [a,c\rangle, \\[2ex] \frac{(b-a)^{n-1}}{n!}\left(B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right)\right. \\ \left. - \int_a^b w(s)\left[B_n^*\left(\frac{s-t}{b-a}\right) - B_n\left(\frac{s-a}{b-a}\right)\right]\mathrm{d}s\right) \\ -\frac{(d-c)^{n-1}}{n!}\left(B_n^*\left(\frac{x-t}{d-c}\right) - B_n\left(\frac{x-c}{d-c}\right)\right. \\ \left. - \int_c^d u(s)\left[B_n^*\left(\frac{s-t}{d-c}\right) - B_n\left(\frac{s-c}{d-c}\right)\right]\mathrm{d}s\right) & t \in [c,d], \\[2ex] \frac{(b-a)^{n-1}}{n!}\left(B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right)\right. \\ \left. - \int_a^b w(s)\left[B_n^*\left(\frac{s-t}{b-a}\right) - B_n\left(\frac{s-a}{b-a}\right)\right]\mathrm{d}s\right) & t \in \langle d,b], \end{cases}$$

while in case $[a,b] \cap [c,d] = [c,b]$

$$K_n^1(x,t) = \begin{cases} \frac{(b-a)^{n-1}}{n!}\left(B_n^*\left(\frac{x-t}{b-a}\right) - \int_a^b w(s) B_n^*\left(\frac{s-t}{b-a}\right)\mathrm{d}s\right), & t \in [a,c\rangle, \\[2ex] \frac{(b-a)^{n-1}}{n!}\left(B_n^*\left(\frac{x-t}{b-a}\right) - \int_a^b w(s) B_n^*\left(\frac{s-t}{b-a}\right)\mathrm{d}s\right) \\ -\frac{(d-c)^{n-1}}{n!}\left(B_n^*\left(\frac{x-t}{d-c}\right) - \int_c^d u(s) B_n^*\left(\frac{s-t}{d-c}\right)\mathrm{d}s\right), & t \in [c,b], \\[2ex] -\frac{(d-c)^{n-1}}{n!}\left(B_n^*\left(\frac{x-t}{d-c}\right) - \int_c^d u(s) B_n^*\left(\frac{s-t}{d-c}\right)\mathrm{d}s\right), & t \in \langle b,d], \end{cases}$$

and

$$K_n^2(x,t)=\begin{cases}\frac{(b-a)^{n-1}}{n!}\left(B_n^*\left(\frac{x-t}{b-a}\right)-B_n\left(\frac{x-a}{b-a}\right)\right.\\ \left.-\int_a^b w(s)\left[B_n^*\left(\frac{s-t}{b-a}\right)-B_n\left(\frac{s-a}{b-a}\right)\right]\mathrm{d}s\right) & t\in[a,c\rangle,\\ \\ \frac{(b-a)^{n-1}}{n!}\left(B_n^*\left(\frac{x-t}{b-a}\right)-B_n\left(\frac{x-a}{b-a}\right)\right.\\ \left.-\int_a^b w(s)\left[B_n^*\left(\frac{s-t}{b-a}\right)-B_n\left(\frac{s-a}{b-a}\right)\right]\mathrm{d}s\right)\\ -\frac{(d-c)^{n-1}}{n!}\left(B_n^*\left(\frac{x-t}{d-c}\right)-B_n\left(\frac{x-c}{d-c}\right)\right.\\ \left.-\int_c^d u(s)\left[B_n^*\left(\frac{s-t}{d-c}\right)-B_n\left(\frac{s-c}{d-c}\right)\right]\mathrm{d}s\right) & t\in[c,b],\\ \\ -\frac{(d-c)^{n-1}}{n!}\left(B_n^*\left(\frac{x-t}{d-c}\right)-B_n\left(\frac{x-c}{d-c}\right)\right.\\ \left.-\int_c^d u(s)\left[B_n^*\left(\frac{s-t}{d-c}\right)-B_n\left(\frac{s-c}{d-c}\right)\right]\mathrm{d}s\right) & t\in\langle b,d].\end{cases}$$

Proof. We subtract identities (2.37) for interval $[a,b]$ and $[c,d]$, to get the first formula. By doing the same with identity (2.38), we get the second formula. □

Theorem 2.18 *Assume (p,q) is a pair of conjugate exponents, $1\le p,q\le\infty$. Let $f^{(n)}\in L_p[a,b]$ for some $n\ge 1$. Then we have*

$$\left|\int_a^b w(t)f(t)\,\mathrm{d}t-\int_c^d u(t)f(t)\,\mathrm{d}t+T_{w,n}^{[a,b]}(x)-T_{u,n}^{[c,d]}(x)\right|$$
$$\le\left(\int_{\min\{a,c\}}^{\max\{b,d\}}\left|K_n^1(x,t)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\left\|f^{(n)}\right\|_p$$

and

$$\left|\int_a^b w(t)f(t)\,\mathrm{d}t-\int_c^d u(t)f(t)\,\mathrm{d}t+T_{w,n-1}^{[a,b]}(x)-T_{u,n-1}^{[c,d]}(x)\right|$$
$$\le\left(\int_{\min\{a,c\}}^{\max\{b,d\}}\left|K_n^2(x,t)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\left\|f^{(n)}\right\|_p,$$

for every $x\in[a,b]\cap[c,d]$. These inequalities are sharp for $1<p\le\infty$ and the best possible for $p=1$.

Proof. Use the identities from the Theorem 2.17 and apply the Hölder inequality. The proof for sharpness and the best possibility are similar as in Theorem 2.16. □

Remark 2.14 Similar results to those in two last theorems could be obtained using the identities (2.39) and (2.40) instead of (2.37) and (2.38).

2.2.2 Discrete weighted Euler identity

In this subsection the discrete analogue of weighted Euler identity for finite sequences of vectors in normed linear spaces is presented and used to obtain some new discrete Ostrowski type inequalities as well as the estimations of difference of two weighted arithmetic means. These results are published in [8] and they are discrete analogues of results from [13].

Let $x_1, x_2, .., x_n$ be the finite sequence of vectors in normed linear space $(X, \|\cdot\|)$ and $w_1, w_2, .., w_n$ finite sequence of positive real numbers. If for $1 \le k \le n$

$$W_k = \sum_{i=1}^{k} w_i, \quad \overline{W_k} = \sum_{i=k+1}^{n} w_i = W_n - W_k$$

then it holds (see [89])

$$\sum_{i=1}^{n} w_i x_i = x_k W_n + \sum_{i=1}^{k-1} W_i (x_i - x_{i+1}) + \sum_{i=k}^{n-1} \overline{W_i} (x_{i+1} - x_i), \quad 1 \le k \le n. \tag{2.49}$$

The **difference operator** Δ is defined by

$$\Delta x_i = x_{i+1} - x_i.$$

So using formula (2.49), we get the discrete analogue of weighted Montgomery identity

$$x_k = \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i + \sum_{i=1}^{n-1} D_w(k,i) \Delta x_i, \tag{2.50}$$

where the discrete Peano kernel is defined by

$$D_w(k,i) = \frac{1}{W_n} \cdot \begin{cases} W_i, & 1 \le i \le k-1, \\ \left(-\overline{W_i}\right), & k \le i \le n. \end{cases} \tag{2.51}$$

In special case, if we take $w_i = 1, \ i = 1, \ldots, n$, then $W_i = i$ and $\overline{W_i} = n - i$, and (2.30) reduces to discrete Montgomery identity

$$x_k = \frac{1}{n} \sum_{i=1}^{n} x_i + \sum_{i=1}^{n-1} D_n(k,i) \Delta x_i, \tag{2.52}$$

where

$$D_n(k,i) = \begin{cases} \frac{i}{n}, & 1 \le i \le k-1, \\ \frac{i}{n} - 1, & k \le i \le n. \end{cases}$$

If $n \in \mathbb{N}$, Δ^n is inductively defined by

$$\Delta^n x_i = \Delta^{n-1} (\Delta x_i).$$

Then, it is easy to prove by induction or directly using the elementary theory of operators (see [63])

$$\Delta^n x_i = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} x_{i+k}.$$

In the next Theorem the generalization of the identity (2.50) is given.

Theorem 2.19 *Let $(X,\|\cdot\|)$ be a normed linear space, $x_1,x_2,..,x_n$ a finite sequence of vectors in X, $w_1,w_2,..,w_n$ finite sequence of positive real numbers. Then for all $m \in \{2,3,..,n-1\}$ and $k \in \{1,2,..,n\}$ the following identity is valid*

$$
\begin{aligned}
x_k = {} & \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i + \sum_{r=1}^{m-1}\frac{1}{n-r}\left(\sum_{i=1}^{n-r}\Delta^r x_i\right) \\
& \cdot\left(\sum_{i_1=1}^{n-1}\sum_{i_2=1}^{n-2}\cdots\sum_{i_r=1}^{n-r} D_w(k,i_1)\, D_{n-1}(i_1,i_2)\cdots D_{n-r+1}(i_{r-1},i_r)\right) \\
& + \sum_{i_1=1}^{n-1}\sum_{i_2=1}^{n-2}\cdots\sum_{i_m=1}^{n-m} D_w(k,i_1)\, D_{n-1}(i_1,i_2)\cdots D_{n-m+1}(i_{m-1},i_m)\,\Delta^m x_{i_m}.
\end{aligned}
\tag{2.53}
$$

Proof. We prove our assertion by induction with respect to m. For $m=2$ we apply the identity (2.52) for the finite sequence of vectors Δx_i, $i=1,2,..,n-1$

$$
\Delta x_i = \frac{1}{n-1}\sum_{i=1}^{n-1}\Delta x_i + \sum_{j=1}^{n-2} D_{n-1}(i,j)\,\Delta^2 x_j,
$$

then use (2.50) to obtain

$$
\begin{aligned}
x_k &= \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i + \sum_{i=1}^{n-1} D_w(k,i)\left(\frac{1}{n-1}\sum_{i=1}^{n-1}\Delta x_i + \sum_{j=1}^{n-2} D_{n-1}(i,j)\,\Delta^2 x_j\right) \\
&= \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i + \frac{1}{n-1}\left(\sum_{i=1}^{n-1}\Delta x_i\right)\left(\sum_{i=1}^{n-1} D_w(k,i)\right) \\
&\quad + \sum_{i=1}^{n-1}\sum_{j=1}^{n-2} D_w(k,i)\, D_{n-1}(i,j)\,\Delta^2 x_j.
\end{aligned}
$$

Hence the identity (2.53) holds for $m=2$. Now, we assume that it holds for a natural number $m \in \{2,3,..,n-2\}$. Applying the identity (2.52) for the $\Delta^m x_{i_m}$

$$
\Delta^m x_{i_m} = \frac{1}{n-m}\sum_{i=1}^{n-m}\Delta^m x_i + \sum_{i_{m+1}=1}^{n-m-1} D_{n-m}(i_m,i_{m+1})\,\Delta^{m+1} x_{i_{m+1}}
$$

and using the induction hypothesis, we get

$$
\begin{aligned}
x_k = {} & \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i + \sum_{r=1}^{m-1}\frac{1}{n-r}\left(\sum_{i=1}^{n-r}\Delta^r x_i\right) \\
& \cdot\left(\sum_{i_1=1}^{n-1}\sum_{i_2=1}^{n-2}\cdots\sum_{i_r=1}^{n-r} D_w(k,i_1)\, D_{n-1}(i_1,i_2)\cdots D_{n-r+1}(i_{r-1},i_r)\right) \\
& + \left(\sum_{i_1=1}^{n-1}\sum_{i_2=1}^{n-2}\cdots\sum_{i_m=1}^{n-m} D_w(k,i_1)\, D_{n-1}(i_1,i_2)\cdots D_{n-m+1}(i_{m-1},i_m)\right.
\end{aligned}
$$

$$
\begin{aligned}
&\cdot\left(\frac{1}{n-m}\sum_{i=1}^{n-m}\Delta^m x_i+\sum_{i_{m+1}=1}^{n-m-1}D_{n-m}\left(i_m,i_{m+1}\right)\Delta^{m+1}x_{i_{m+1}}\right)\Bigg)\\
&=\frac{1}{W_n}\sum_{i=1}^{n}w_ix_i+\sum_{r=1}^{m}\frac{1}{n-r}\left(\sum_{i=1}^{n-r}\Delta^r x_i\right)\\
&\cdot\left(\sum_{i_1=1}^{n-1}\sum_{i_2=1}^{n-2}\cdots\sum_{i_r=1}^{n-r}D_w\left(k,i_1\right)D_{n-1}\left(i_1,i_2\right)\cdots D_{n-r+1}\left(i_{r-1},i_r\right)\right)\\
&+\sum_{i_1=1}^{n-1}\sum_{i_2=1}^{n-2}\cdots\sum_{i_{m+1}=1}^{n-(m+1)}D_w\left(k,i_1\right)D_{n-1}\left(i_1,i_2\right)\cdots D_{n-m}\left(i_m,i_{m+1}\right)\Delta^{m+1}x_{i_{m+1}}.
\end{aligned}
$$

We see that (2.53) is also valid for $m+1$ and our assertion is proved. □

Remark 2.15 For $m=n-1$ (2.53) becomes

$$
\begin{aligned}
x_k=&\frac{1}{W_n}\sum_{i=1}^{n}w_ix_i+\sum_{r=1}^{n-2}\frac{1}{n-r}\left(\sum_{i=1}^{n-r}\Delta^r x_i\right)\\
&\cdot\left(\sum_{i_1=1}^{n-1}\sum_{i_2=1}^{n-2}\cdots\sum_{i_r=1}^{n-r}D_w\left(k,i_1\right)D_{n-1}\left(i_1,i_2\right)\cdots D_{n-r+1}\left(i_{r-1},i_r\right)\right)\\
&+\sum_{i_1=1}^{n-1}\sum_{i_2=1}^{n-2}\cdots\sum_{i_{n-1}=1}^{1}D_w\left(k,i_1\right)D_{n-1}\left(i_1,i_2\right)\cdots D_2\left(i_{n-2},i_{n-1}\right)\Delta^{n-1}x_{i_{n-1}}.
\end{aligned}
$$

Corollary 2.17 *Let $(X,\|\cdot\|)$ be a normed linear space, $x_1,x_2,..,x_n$ a finite sequence of vectors in X. Then for all $m\in\{2,3,..,n-1\}$ and $k\in\{1,2,..,n\}$ the following identity is valid*

$$
\begin{aligned}
x_k=&\frac{1}{n}\sum_{i=1}^{n}x_i+\sum_{r=1}^{m-1}\frac{1}{n-r}\left(\sum_{i=1}^{n-r}\Delta^r x_i\right)\\
&\cdot\left(\sum_{i_1=1}^{n-1}\sum_{i_2=1}^{n-2}\cdots\sum_{i_r=1}^{n-r}D_n\left(k,i_1\right)D_{n-1}\left(i_1,i_2\right)\cdots D_{n-r+1}\left(i_{r-1},i_r\right)\right)\\
&+\sum_{i_1=1}^{n-1}\sum_{i_2=1}^{n-2}\cdots\sum_{i_m=1}^{n-m}D_n\left(k,i_1\right)D_{n-1}\left(i_1,i_2\right)\cdots D_{n-m+1}\left(i_{m-1},i_m\right)\Delta^m x_{i_m}.
\end{aligned}
$$

Proof. We apply Theorem 2.19 with $w_i=1$, $i=1,..,n$. □

Remark 2.16 If we apply (2.53) with $n=2l-1$ and $k=l$ we get

$$
\begin{aligned}
x_l=&\frac{1}{W_{2l-1}}\sum_{i=1}^{2l-1}w_ix_i+\sum_{r=1}^{m-1}\frac{1}{2l-1-r}\left(\sum_{i=1}^{2l-1-r}\Delta^r x_i\right)\\
&\cdot\left(\sum_{i_1=1}^{2l-2}\sum_{i_2=1}^{2l-3}\cdots\sum_{i_r=1}^{2l-1-r}D_w\left(l,i_1\right)D_{2l-2}\left(i_1,i_2\right)\cdots D_{2l-r}\left(i_{r-1},i_r\right)\right)
\end{aligned}
$$

$$+\sum_{i_1=1}^{2l-2}\sum_{i_2=1}^{2l-3}\cdots\sum_{i_m=1}^{2l-1-m} D_w(l,i_1)\,D_{2l-2}(i_1,i_2)\cdots D_{2l-m}(i_{m-1},i_m)\,\Delta^m x_{i_m}.$$

We may regard this identity as generalized midpoint identity since for $m=1$ it reduces to

$$x_l=\frac{1}{W_{2l-1}}\sum_{i=1}^{2l-1} w_i x_i+\sum_{i=1}^{2l-2} D_w(l,i)\,\Delta x_i \tag{2.54}$$

and further for $w_i=1, i=1,2,..,2l-1$ to

$$x_l=\frac{1}{2l-1}\sum_{i=1}^{2l-1} x_i+\frac{1}{2l-1}\sum_{i=1}^{l-1} i\,(\Delta x_i-\Delta x_{2l-1-i}). \tag{2.55}$$

Similarly, if we apply (2.32) with $k=1$ and then with $k=n$, then sum these two equalities and divide them by 2, we get

$$\begin{aligned}\frac{x_1+x_n}{2}&=\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i+\sum_{r=1}^{m-1}\frac{1}{n-r}\left(\sum_{i=1}^{n-r}\Delta^r x_i\right)\\&\cdot\left(\sum_{i_1=1}^{n-1}\cdots\sum_{i_r=1}^{n-r}\frac{D_w(1,i_1)+D_w(n,i_1)}{2}D_{n-1}(i_1,i_2)\cdots D_{n-r+1}(i_{r-1},i_r)\right)\\&+\sum_{i_1=1}^{n-1}\cdots\sum_{i_m=1}^{n-m}\frac{D_w(1,i_1)+D_w(n,i_1)}{2}D_{n-1}(i_1,i_2)\cdots D_{n-m+1}(i_{m-1},i_m)\,\Delta^m x_{i_m}.\end{aligned} \tag{2.56}$$

We may regard this identity as generalized trapezoid identity since for $m=1$ it reduces to

$$\frac{x_1+x_n}{2}=\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i+\sum_{i=1}^{n-1}\frac{D_w(1,i)+D_w(n,i)}{2}\Delta x_i, \tag{2.57}$$

and further for $w_i=1, i=1,2,..,2l-1$ to

$$\frac{x_1+x_n}{2}=\frac{1}{n}\sum_{i=1}^{n} x_i+\frac{1}{n}\sum_{i=1}^{n-1}\left(i-\frac{n}{2}\right)\Delta x_i. \tag{2.58}$$

(2.55) and (2.58) were obtained by Dragomir in [52].

Discrete Ostrowski type inequalities

Here discrete Ostrowski inequality for finite sequences of vectors in normed linear spaces and it's generalization are proved. These are the discrete analogues of some results from [47], [53].

Theorem 2.20 *Let $(X,\|\cdot\|)$ be a normed linear space, $x_1,x_2,..,x_n$ a finite sequence of vectors in X, $w_1,w_2,..,w_n$ finite sequence of positive real numbers. Let also (p,q) be a pair* *of conjugate exponents, $1 < p,q < \infty$. Then for all $m \in \{2,3,..,n-1\}$ and $k \in \{1,2,..,n\}$ the following inequality hold:*

$$\begin{aligned}
&\left\| x_k - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i - \sum_{r=1}^{m-1}\frac{1}{n-r}\left(\sum_{i=1}^{n-r}\Delta^r x_i\right)\right.\\
&\left.\cdot\left(\sum_{i_1=1}^{n-1}\sum_{i_2=1}^{n-2}\cdots\sum_{i_r=1}^{n-r} D_w(k,i_1)D_{n-1}(i_1,i_2)\cdots D_{n-r+1}(i_{r-1},i_r)\right)\right\|\\
&\le \left\|\sum_{i_1=1}^{n-1}\sum_{i_2=1}^{n-2}\cdots\sum_{i_{m-1}=1}^{n-m+1} D_w(k,i_1)D_{n-1}(i_1,i_2)\cdots D_{n-m+1}(i_{m-1},\bullet)\right\|_q \|\Delta^m x\|_p
\end{aligned} \tag{2.59}$$

where

$$\|\Delta^m x\|_p = \begin{cases} \left(\sum_{i=1}^{n-m}\|\Delta^m x_i\|^p\right)^{\frac{1}{p}}, & \text{if } 1 \le p < \infty,\\ \max_{1\le i\le n-m}\|\Delta^m x_i\| & \text{if } p = \infty.\end{cases}$$

Proof. By using the (2.53) and applying the Hölder inequality. □

Corollary 2.18 *Let $(X,\|\cdot\|)$ be a normed linear space, $x_1,x_2,..,x_n$ a finite sequence of vectors in X, $w_1,w_2,..,w_n$ finite sequence of positive real numbers. Let also (p,q) be a pair of conjugate exponents. Then for all $k \in \{1,2,..,n\}$ the following inequalities hold:*

$$\left\| x_k - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right\| \le \begin{cases} \frac{1}{W_n}\left(\sum_{i=1}^{n}|k-i|\,w_i\right)\cdot\|\Delta x\|_\infty, \\[2ex] \frac{1}{W_n}\left(\sum_{i=1}^{k-1}\left(\sum_{j=1}^{i}\sum_{j=1}^{i} w_j\right)^q + \sum_{i=k}^{n-1}\left(\sum_{j=i+1}^{n} w_i\right)^q\right)^{\frac{1}{q}}\cdot\|\Delta x\|_p, \\[2ex] \frac{1}{W_n}\max\{W_{k-1}, W_n - W_k\}\cdot\|\Delta x\|_1 .\end{cases}$$

Proof. By using the discrete analogue of weighted Montgomery identity (2.50) and applying the Hölder inequality we get

$$\left\| x_k - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right\| \le \|D_w(k,\bullet)\|_q\,\|\Delta x\|_p .$$

Since

$$\|D_w(k,\bullet)\|_q = \frac{1}{W_n}\left(\sum_{i=1}^{k-1}|W_i|^q + \sum_{i=k}^{n-1}\left|-\overline{W}_i\right|^q\right)^{\frac{1}{q}}$$

the second inequality is proved. For the first we have

$$\|D_w(k,\bullet)\|_1 = \frac{1}{W_n}\left(\sum_{i=1}^{k-1}(k-i)\,w_i + \sum_{i=1}^{n-k} i w_{k+i}\right) = \frac{1}{W_n}\sum_{i=1}^{n}|k-i|\,w_i$$

and for the third

$$\|D_w(k,\bullet)\|_\infty = \frac{1}{W_n}\max\{W_{k-1}, W_n - W_k\}$$

so the proof is done. □

The first and the third inequality from the Corollary 2.18 and also the next Corollary 2.19 was proved by Dragomir in [52].

For $n \in \mathbb{N}$ and $m \in \mathbb{R}$ we denote

$$S_m(n) = 1^m + 2^m + 3^m + \cdots + (n-1)^m.$$

Corollary 2.19 *Let $(X, \|\cdot\|)$ be a normed linear space, $x_1, x_2, .., x_n$ a finite sequence of vectors in X, $w_1, w_2, .., w_n$ finite sequence of positive real numbers. Let also (p,q) be a pair of conjugate exponents, $1 < p, q < \infty$. Then for all $k \in \{1,2,..,n\}$ the following inequalities hold:*

$$\left\| x_k - \frac{1}{n}\sum_{i=1}^{n} x_i \right\| \le \begin{cases} \frac{1}{n}\left(\frac{n^2-1}{4} + \left(k - \frac{n+1}{2}\right)^2\right) \cdot \|\Delta x\|_\infty, \\ \frac{1}{n}\left(S_q(k) + S_q(n-k+1)\right)^{\frac{1}{q}} \cdot \|\Delta x\|_p, \\ \frac{1}{n}\max\{k-1, n-k\} \cdot \|\Delta x\|_1. \end{cases} \tag{2.60}$$

Proof. If we apply Corollary 2.18 with $w_i = 1$, $i = 1,2,..,n$ (or use discrete Montgomery identity (2.52)), we have

$$\left\| x_k - \frac{1}{n}\sum_{i=1}^{n} x_i \right\| = \left\| \sum_{i=1}^{n-1} D_n(k,i)\,\Delta x_i \right\| \le \|D_n(k,\bullet)\|_q \, \|\Delta x\|_p.$$

Since for $q = 1$

$$\sum_{i=1}^{n-1} |D_n(k,i)| = \frac{1}{n}\left(\frac{n^2-1}{4} + \left(k - \frac{n+1}{2}\right)^2\right)$$

the first inequality follows. For the second let $1 < q < \infty$

$$\sum_{i=1}^{n-1} |D_n(k,i)|^q = \frac{1}{n^q}\left(\sum_{i=1}^{k-1} i^q + \sum_{i=k}^{n-1}(n-i)^q\right) = \frac{1}{n^q}\left(S_q(k) + S_q(n-k+1)\right)$$

the second inequality follows. Finally for $q = \infty$ and

$$\max_{1\le i\le n-1}\{|D(k,i)|\} = \frac{1}{n}\max\{k-1, n-k\}$$

implies the last inequality. □

Corollary 2.20 *Assume all assumptions from the Theorem 2.20 hold. Then following inequality*

$$\left\| x_l - \frac{1}{W_{2l-1}}\sum_{i=1}^{2l-1} w_i x_i - \sum_{r=1}^{m-1}\frac{1}{2l-1-r}\left(\sum_{i=1}^{2l-1-r}\Delta^r x_i\right)\right.$$

$$\cdot \left(\sum_{i_1=1}^{2l-2} \sum_{i_2=1}^{2l-3} \cdots \sum_{i_r=1}^{2l-1-r} D_w(l,i_1) D_{2l-2}(i_1,i_2) \cdots D_{2l-r}(i_{r-1},i_r) \right) \Bigg\|$$

$$\leq \left\| \sum_{i_1=1}^{2l-2} \sum_{i_2=1}^{2l-3} \cdots \sum_{i_{m-1}=1}^{2l-m} D_w(l,i_1) D_{2l-2}(i_1,i_2) \cdots D_{2l-m}(i_{m-1},\bullet) \right\|_q \|\Delta^m x\|_p$$

we may regard as generalized midpoint inequality since for $m = 1$ it reduces to

$$\left\| x_l - \frac{1}{W_{2l-1}} \sum_{i=1}^{2l-1} w_i x_i \right\| \leq \begin{cases} \frac{1}{W_{2l-1}} \left(\sum_{i=1}^{2l-1} |l-i| \, w_i \right) \cdot \|\Delta x\|_\infty , \\ \frac{1}{W_{2l-1}} \left(\sum_{i=1}^{l-1} \left(\sum_{j=1}^{i} w_j \right)^q + \sum_{i=l}^{2l-2} \left(\sum_{j=i+1}^{n} w_i \right)^q \right)^{\frac{1}{q}} \cdot \|\Delta x\|_p , \\ \frac{1}{W_{2l-1}} \max\{W_{l-1}, W_{2l-1} - W_l\} \cdot \|\Delta x\|_1 , \end{cases}$$

and further for $w_i = 1, i = 1,2,..,2l-1$ to

$$\left\| x_l - \frac{1}{2l-1} \sum_{i=1}^{2l-1} x_i \right\| \leq \begin{cases} \frac{l(l-1)}{2l-1} \cdot \|\Delta x\|_\infty , \\ \frac{1}{2l-1} \left(2S_q(l)\right)^{\frac{1}{q}} \cdot \|\Delta x\|_p , \\ \frac{l-1}{2l-1} \cdot \|\Delta x\|_1 . \end{cases} \tag{2.61}$$

Proof. We apply (2.59) with $n = 2l-1$ and $k = l$ we get the first inequality. For the second we take $m = 1$ or apply Hölder inequality on (2.54)

$$\left\| x_l - \frac{1}{W_{2l-1}} \sum_{i=1}^{2l-1} w_i x_i \right\| = \left\| \sum_{i=1}^{2l-2} D_w(l,i) \Delta x_i \right\| \leq \|D_w(l,\bullet)\|_q \|\Delta x\|_p$$

Now,

$$\|D_w(l,\bullet)\|_1 = \frac{1}{W_{2l-1}} \left(\sum_{i=1}^{l-1} |W_i| + \sum_{i=l}^{2l-1} \left|-\overline{W_i}\right| \right) = \frac{1}{W_{2l-1}} \left(\sum_{i=1}^{2l-1} |l-i| \, w_i \right),$$

$$\|D_w(l,\bullet)\|_q = \frac{1}{W_{2l-1}} \left(\sum_{i=1}^{l-1} \left(\sum_{j=1}^{i} w_j \right)^q + \sum_{i=l}^{2l-2} \left(\sum_{j=i+1}^{n} w_i \right)^q \right)^{\frac{1}{q}},$$

$$\|D_w(l,\bullet)\|_\infty = \frac{1}{W_{2l-1}} \max\{W_{l-1}, W_{2l-1} - W_l\}$$

and the second inequality is proved. Now if we take $w_i = 1, i = 1,2,..,2l-1$ (or apply inequality (2.60) with $n = 2l-1$ and $k = l$)

$$\|D_{2l-1}(l,\bullet)\|_1 = \frac{1}{2l-1} \sum_{i=1}^{2l-1} |l-i| = \frac{l(l-1)}{2l-1},$$

$$\|D_{2l-1}(l,\bullet)\|_q = \frac{1}{2l-1}\left(\sum_{i=1}^{2l-1}|l-i|^q\right)^{\frac{1}{q}} = \frac{1}{2l-1}\left(2S_q(l)\right)^{\frac{1}{q}},$$

$$\|D_{2l-1}(l,\bullet)\|_\infty = \frac{1}{2l-1}\max\{l-1,2l-1-l\} = \frac{l-1}{2l-1},$$

so the third inequality is proved. □

Corollary 2.21 *Assume all assumptions from the Theorem 2.20 hold. Then following inequality*

$$\begin{aligned}&\Bigg\|\frac{x_1+x_n}{2} - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i - \sum_{r=1}^{m-1}\frac{1}{n-r}\left(\sum_{i=1}^{n-r}\Delta^r x_i\right)\\ &\cdot\left(\sum_{i_1=1}^{n-1}\cdots\sum_{i_r=1}^{n-r}\frac{D_w(1,i_1)+D_w(n,i_1)}{2}D_{n-1}(i_1,i_2)\cdots D_{n-r+1}(i_{r-1},i_r)\right)\Bigg\|\\ &\le \left\|\sum_{i_1=1}^{n-1}\cdots\sum_{i_{m-1}=1}^{n-m+1}\frac{D_w(1,i_1)+D_w(n,i_1)}{2}D_{n-1}(i_1,i_2)\cdots D_{n-m+1}(i_{m-1},\bullet)\right\|_q \|\Delta^m x\|_p.\end{aligned}$$

We may regard this as generalized trapezoid inequality since for $m=1$ it reduces to

$$\left\|\frac{x_1+x_n}{2} - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right\| \le \begin{cases} \sum_{i=1}^{n-1}\left|\frac{W_i}{W_n}-\frac{1}{2}\right|\cdot\|\Delta x\|_\infty, \\ \left(\sum_{i=1}^{n-1}\left|\frac{W_i}{W_n}-\frac{1}{2}\right|^q\right)^{\frac{1}{q}}\cdot\|\Delta x\|_p, \\ \max\left\{\left|\frac{w_1}{W_n}-\frac{1}{2}\right|,\left|\frac{w_n}{W_n}-\frac{1}{2}\right|\right\}\cdot\|\Delta x\|_1. \end{cases}$$

and further for $w_i = 1, i = 1,2,..,n$ to

$$\begin{aligned}&\left\|\frac{x_1+x_n}{2} - \frac{1}{n}\sum_{i=1}^{n} x_i\right\|\\ &\le \begin{cases} \frac{1}{n}\left(n-1-\left\lfloor\frac{n}{2}\right\rfloor\right)\left(\left\lfloor\frac{n}{2}\right\rfloor\right)\cdot\|\Delta x\|_\infty, \\ \begin{cases} \frac{1}{n}\left(2S_q\left(\frac{n}{2}\right)\right)^{\frac{1}{q}}\cdot\|\Delta x\|_p, & \text{if } n \text{ is even}, \\ \frac{1}{n}\left(\frac{S_q(n-1)}{2^{q-1}} - 2S_q\left(\frac{n-1}{2}\right)\right)^{\frac{1}{q}}\cdot\|\Delta x\|_p, & \text{if } n \text{ is odd}, \end{cases} \\ \frac{n-2}{2n}\cdot\|\Delta x\|_1. \end{cases}\end{aligned} \tag{2.62}$$

Proof. We take (2.56) and apply Hölder inequality to get the first inequality. For the second we take $m=1$ or apply Hölder inequality on (2.57)

$$\left\|\frac{x_1+x_n}{2} - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right\| = \left\|\sum_{i=1}^{n-1}\frac{D_w(1,i)+D_w(n,i)}{2}\Delta x_i\right\|$$

$$\leq \left\| \frac{D_w(1,\bullet)+D_w(n,\bullet)}{2} \right\|_q \|\Delta x\|_p .$$

Now,

$$\left\| \frac{D_w(1,\bullet)+D_w(n,\bullet)}{2} \right\|_1 = \sum_{i=1}^{n-1} \left| \frac{W_i - \overline{W_i}}{2W_n} \right| = \sum_{i=1}^{n-1} \left| \frac{W_i}{W_n} - \frac{1}{2} \right|,$$

$$\left\| \frac{D_w(1,\bullet)+D_w(n,\bullet)}{2} \right\|_q = \left(\sum_{i=1}^{n-1} \left| \frac{W_i}{W_n} - \frac{1}{2} \right|^q \right)^{\frac{1}{q}},$$

$$\left\| \frac{D_w(1,\bullet)+D_w(n,\bullet)}{2} \right\|_\infty = \max_{1\leq i\leq n-1} \left\{ \left| \frac{W_i}{W_n} - \frac{1}{2} \right| \right\}$$
$$= \max\left\{ \left| \frac{W_1}{W_n} - \frac{1}{2} \right|, \left| \frac{W_{n-1}}{W_n} - \frac{1}{2} \right| \right\} = \max\left\{ \left| \frac{w_1}{W_n} - \frac{1}{2} \right|, \left| \frac{w_n}{W_n} - \frac{1}{2} \right| \right\}$$

and the second inequality is proved. Now if we take $w_i = 1, i = 1,2,..,n$ (or use (2.58) and apply Hölder inequality) we get

$$\left\| \frac{x_1+x_n}{2} - \frac{1}{n}\sum_{i=1}^{n} x_i \right\| \leq \left\| \frac{i}{n} - \frac{1}{2} \right\|_q \|\Delta x\|_p .$$

For $q = 1$

$$\left\| \frac{i}{n} - \frac{1}{2} \right\|_1 = \frac{1}{n}\sum_{i=1}^{n-1} \left| i - \frac{n}{2} \right| = \frac{1}{n}\left(n-1-\left\lfloor \frac{n}{2} \right\rfloor \right)\left(\left\lfloor \frac{n}{2} \right\rfloor \right),$$

for $1 < q < \infty$

$$\left\| \frac{i}{n} - \frac{1}{2} \right\|_q = \frac{1}{n}\left(\sum_{i=1}^{n-1} \left| i - \frac{n}{2} \right|^q \right)^{\frac{1}{q}} = \begin{cases} \frac{1}{n}\left(2S_q\left(\frac{n}{2}\right) \right)^{\frac{1}{q}}, & \text{if } n \text{ is even,} \\ \frac{1}{n}\left(\frac{S_q(n-1)}{2^{q-1}} - 2S_q\left(\frac{n-1}{2}\right) \right)^{\frac{1}{q}}, & \text{if } n \text{ is odd,} \end{cases}$$

and for $q = \infty$

$$\left\| \frac{i}{n} - \frac{1}{2} \right\|_\infty = \max_{1\leq i\leq n-1} \left\{ \frac{i}{n} - \frac{1}{2} \right\} = \frac{n-2}{2n}.$$

□

Remark 2.17 The first inequality from (2.61) was obtained by Dragomir in [52] and also the incorrect version of the first inequality from (2.62)

$$\left\| \frac{x_1+x_n}{2} - \frac{1}{n}\sum_{i=1}^{n} x_i \right\| \leq \begin{cases} \frac{k-1}{2}\|\Delta x\|_\infty, & \text{if } n = 2k, \\ \frac{2k^2+2k+1}{2(2k+1)}\|\Delta x\|_\infty, & \text{if } n = 2k+1. \end{cases}$$

Instead of term $\frac{2k^2+2k+1}{2(2k+1)}$ should be $\frac{k^2}{2k+1}$ since

$$\frac{1}{2k+1}\left((2k+1)-1-\left\lfloor \frac{2k+1}{2} \right\rfloor \right)\left(\left\lfloor \frac{2k+1}{2} \right\rfloor \right) = \frac{k^2}{2k+1}.$$

Estimations of the difference of two weighted arithmetic means

In this subsection the estimations of difference of two weighted arithmetic means are given using the discrete weighted Montgomery identity. These are the discrete analogues of some results from [14], [24], [53], [36] and [95].

Here we suppose $l,m,n \in \mathbb{N}$. The first method is by subtracting two weighted Montgomery identities. The second is by summing the discrete weighted Montgomery identity. Both methods are possible for both cases $1 \le l \le m \le n$, i.e. $[l,m] \subseteq [1,n]$ and $1 \le l \le n \le m$, i.e. $[1,n] \cap [l,m] = [l,n]$.

Theorem 2.21 *Let $(X, \|\cdot\|)$ be a normed linear space, $x_1, x_2, .., x_{\max\{m,n\}}$ a finite sequence of vectors in X, $l,m,n \in \mathbb{N}$, $w_1, w_2, \ldots, w_n$ finite sequence of positive real numbers as well as $u_l, u_{l+1}, \ldots, u_m$. Let also $W = \sum_{i=1}^{n} w_i$, $U = \sum_{i=l}^{m} u_i$ and for $k \in \mathbb{N}$*

$$W_k = \begin{cases} \sum_{i=1}^{k} w_i, & 1 \le k \le n, \\ W, & k > n, \end{cases}$$

$$U_k = \begin{cases} 0, & k < l, \\ \sum_{i=l}^{k} u_i & l \le k \le m, \\ U, & k > m. \end{cases} \tag{2.63}$$

If $[1,n] \cap [l,m] \neq \emptyset$, then, for both cases $[l,m] \subseteq [1,n]$ and $[1,n] \cap [l,m] = [l,n]$, the next formula is valid

$$\frac{1}{W}\sum_{i=1}^{n} w_i x_i - \frac{1}{U}\sum_{i=l}^{m} u_i x_i = \sum_{i=1}^{\max\{m,n\}} K(i)\,\Delta x_i \tag{2.64}$$

where

$$K(i) = \frac{U_i}{U} - \frac{W_i}{W}, \quad 1 \le i \le \max\{m,n\}.$$

Proof. For $k \in ([1,n] \cap [l,m]) \cap \mathbb{N}$, we subtract identities

$$x_k = \frac{1}{W}\sum_{i=1}^{n} w_i x_i + \sum_{i=1}^{n-1} D_w(k,i)\,\Delta x_i,$$

and

$$x_k = \frac{1}{U}\sum_{i=l}^{m} u_i x_i + \sum_{i=l}^{m-1} D_u(k,i)\,\Delta x_i.$$

Then put

$$K(k,i) = D_u(k,i) - D_w(k,i).$$

Since $K(k,i)$ doesn't depend on k, we put $K(i)$ instead:

$$K(i) = \begin{cases} -\frac{W_i}{W}, & 1 \le i \le l-1, \\ \frac{U_i}{U} - \frac{W_i}{W}, & l \le i \le m, \qquad \text{if} \quad [l,m] \subseteq [1,n] \\ 1 - \frac{W_i}{W}, & m+1 \le i \le n, \end{cases} \tag{2.65}$$

$$K(i)=\begin{cases}-\frac{W_i}{W}, & 1\le i\le l-1,\\ \frac{U_i}{U}-\frac{W_i}{W}, & l\le i\le n, \qquad \text{if} \quad [1,n]\cap[l,m]=[l,n]\\ \frac{U_i}{U}-1, & n+1\le i\le m.\end{cases} \tag{2.66}$$

□

Theorem 2.22 *Assume all assumptions from the Theorem 2.21 hold and (p,q) is a pair of conjugate exponents, $1\le p,q\le\infty$. Then we have*

$$\left\|\frac{1}{W}\sum_{i=1}^{n}w_ix_i-\frac{1}{U}\sum_{i=l}^{m}u_ix_i\right\|\le\|K\|_q\,\|\Delta x\|_p\,.$$

This inequality is sharp for $1\le p\le\infty$.

Proof. For the proof of the inequality we use the identity (2.64) and apply the Hölder inequality. For the proof of the sharpness, we will find x, a finite sequence of vectors in X such that

$$\left|\sum_{i=1}^{\max\{m,n\}}K(i)\,\Delta x_i\right|=\left(\sum_{i=1}^{\max\{m,n\}}|K(i)|^q\right)^{\frac{1}{q}}\|\Delta x\|_p\,.$$

For $1<p<\infty$ take x to be such that

$$\Delta x_i=sgn\;K(i)\cdot|K(i)|^{\frac{1}{p-1}}\,.$$

For $p=\infty$ take

$$\Delta x_i=sgn\;K(i)\,.$$

For $p=1$ we will find a finite sequence of vectors x such that

$$\left|\sum_{i=1}^{\max\{m,n\}}K(i)\,\Delta x_i\right|=\max_{1\le i\le\max\{m,n\}}|K(i)|\left(\sum_{i=1}^{\max\{m,n\}}|\Delta x_i|\right).$$

Suppose that $|K(i)|$ attains its maximum at $i_0\in([1,n]\cup[l,m])\cap\mathbb{N}$. First we assume that $K(i_0)>0$. Define x such that $\Delta x_{i_0}=1$ and $\Delta x_i=0$, $i\ne i_0$, i.e.

$$x_i=\begin{cases}0, & 1\le i\le i_0,\\ 1, & i_0+1<i\le\max\{m,n\}\,.\end{cases}$$

Then,

$$\left|\sum_{i=1}^{\max\{m,n\}}K(i)\,\Delta x_i\right|=|K(i_0)|=\max_{1\le i\le\max\{m,n\}}|K(i)|\left(\sum_{i=1}^{\max\{m,n\}}|\Delta x_i|\right)$$

and the statement follows. In case $K(i_0)<0$, we take x such that $\Delta x_{i_0}=-1$ and $\Delta x_i=0$, $i\ne i_0$, i.e.

$$x_i=\begin{cases}1, & 1\le i\le i_0,\\ 0, & i_0+1\le i\le\max\{m,n\}\end{cases}$$

and the rest of proof is the same as above. □

Corollary 2.22 *Assume all assumptions from the Theorem 2.22 hold and additionally assume* $1 \leq l < m \leq n$*. Then we have*

$$\left\| \frac{1}{W}\sum_{i=1}^{n} w_i x_i - \frac{1}{U}\sum_{i=l}^{m} u_i x_i \right\|$$

$$\leq \begin{cases} \left[\sum_{i=1}^{l-1}\left|\frac{W_i}{W}\right| + \sum_{i=l}^{m}\left|\frac{Ui}{U}-\frac{W_i}{W}\right| + \sum_{i=m+1}^{n}\left|1-\frac{W_i}{W}\right|\right]\|\Delta x\|_\infty , \\ \left[\sum_{i=1}^{l-1}\left|\frac{W_i}{W}\right|^q + \sum_{i=l}^{m}\left|\frac{Ui}{U}-\frac{W_i}{W}\right|^q + \sum_{i=m+1}^{n}\left|1-\frac{W_i}{W}\right|^q\right]^{\frac{1}{q}}\|\Delta x\|_p , \\ \max\left\{\frac{W_{l-1}}{W}, 1-\frac{W_{m+1}}{W}, \max_{l\leq i\leq m}\left|\frac{Ui}{U}-\frac{W_i}{W}\right|\right\}\|\Delta x\|_1 , \end{cases}$$

and for $1 \leq l < n \leq m$

$$\left| \frac{1}{W}\sum_{i=1}^{n} w_i x_i - \frac{1}{U}\sum_{i=l}^{m} u_i x_i \right|$$

$$\leq \begin{cases} \left[\sum_{i=1}^{l-1}\left|\frac{W_i}{W}\right| + \sum_{i=l}^{n}\left|\frac{Ui}{U}-\frac{W_i}{W}\right| + \sum_{i=n+1}^{m}\left|\frac{U_i}{U}-1\right|\right]\|\Delta x\|_\infty , \\ \left[\sum_{i=1}^{l-1}\left|\frac{W_i}{W}\right|^q + \sum_{i=l}^{n}\left|\frac{Ui}{U}-\frac{W_i}{W}\right|^q + \sum_{i=n+1}^{m}\left|\frac{U_i}{U}-1\right|^q\right]^{\frac{1}{q}}\|\Delta x\|_p , \\ \max\left\{\frac{W_{l-1}}{W}, 1-\frac{U_{n+1}}{U}, \max_{l\leq i\leq n}\left|\frac{Ui}{U}-\frac{W_i}{W}\right|\right\}\|\Delta x\|_1 . \end{cases}$$

Proof. Directly from the Theorem 2.22. □

Remark 2.18 If we suppose $n = m$ in both cases $1 \leq l < m \leq n$ and $1 \leq l < n \leq m$, the analogous results coincides.

Remark 2.19 By setting $l = m = k$ and $u_k = 1$ in the first inequality from the Corollary 2.22 we get the weighted Ostrowski inequality from the Corollary 2.18.

Corollary 2.23 *Assume all assumptions from the Theorem 2.22 hold and additionally assume* $1 \leq k \leq m$*. Then we have*

$$\left| \frac{1}{W}\sum_{i=1}^{k} w_i x_i - \frac{1}{U}\sum_{i=k}^{m} u_i x_i \right|$$

$$\leq \begin{cases} \left[\sum_{i=1}^{k-1}\left|\frac{W_i}{W}\right| + \left|\frac{U_k}{U}-\frac{W_k}{W}\right| + \sum_{i=k+1}^{m}\left|\frac{U_i}{U}-1\right|\right]\|\Delta x\|_\infty , \\ \left[\sum_{i=1}^{k-1}\left|\frac{W_i}{W}\right|^q + \left|\frac{U_k}{U}-\frac{W_k}{W}\right|^q + \sum_{i=k+1}^{m}\left|\frac{U_i}{U}-1\right|^q\right]^{\frac{1}{q}}\|\Delta x\|_p , \\ \max\left\{\frac{W_{k-1}}{W}, 1-\frac{U_{k+1}}{U}, \left|\frac{U_k}{U}-\frac{W_k}{W}\right|\right\}\|\Delta x\|_1 . \end{cases}$$

Proof. By setting $n = l = k$ in the second inequality from the Corollary 2.22. □

Remark 2.20 The second method of giving the estimation of the difference of the two weighted arithmetic means is by summing weighted Montgomery identity and than interchanging the order of summation In this way we also get formula (2.64). For details see [8].

The next Theorem is the generalization of the Theorem 2.22.

Theorem 2.23 *Let* $(X, \|\cdot\|)$ *be a normed linear space,* $x_1, x_2, .., x_{\max\{m,n\}}$ *a finite sequence of vectors in* X, $w_1, w_2, .., w_n$ *and* $u_l, u_{l+1}, .., u_m$ *finite sequences of positive real numbers and* (p, q) *a pair of conjugate exponents. Then for all* $s \in \{2, 3, .., n-1\}$ *and* $k \in ([1,n] \cap [l,m]) \cap \mathbb{N}$ *the following inequality is valid*

$$\left\| \frac{1}{W}\sum_{i=1}^{n} w_i x_i - \frac{1}{U}\sum_{i=l}^{m} w_i x_i \right.$$
$$+ \sum_{r=1}^{s-1} \frac{\left(\sum_{i=1}^{n-r} \Delta^r x_i\right)}{n-r} \left(\sum_{i_1=1}^{n-1} \cdots \sum_{i_r=1}^{n-r} D_w(k, i_1) D_{n-1}(i_1, i_2) \cdots D_{n-r+1}(i_{r-1}, i_r)\right)$$
$$\left. - \sum_{r=1}^{s-1} \frac{\left(\sum_{i=l}^{m-r} \Delta^r x_i\right)}{m-r} \left(\sum_{i_1=l}^{m-1} \cdots \sum_{i_r=l}^{m-r} D_u(k, i_1) D_{m-l}(i_1, i_2) \cdots D_{m-l-r+2}(i_{r-1}, i_r)\right) \right\|$$
$$\leq \|\mathbf{K}(k, \bullet)\|_q \|\Delta^s x\|_p$$

where

$$\mathbf{K}(k, i_m) = \sum_{i_1=1}^{n-1} \sum_{i_2=1}^{n-2} \cdots \sum_{i_{s-1}=1}^{n-s+1} D_w(k, i_1) D_{n-1}(i_1, i_2) \cdots D_{n-s+1}(i_{s-1}, i_s)$$
$$- \sum_{i_1=l}^{m-1} \sum_{i_2=l}^{m-2} \cdots \sum_{i_{s-1}=l}^{m-s+1} D_u(k, i_1) D_{m-l}(i_1, i_2) \cdots D_{m-l-s+2}(i_{s-1}, i_s)$$

and we suppose that for $i_s \notin [1,n] \cap \mathbb{N}$

$$\sum_{i_1=1}^{n-1} \sum_{i_2=1}^{n-2} \cdots \sum_{i_{s-1}=1}^{n-s+1} D_w(k, i_1) D_{n-1}(i_1, i_2) \cdots D_{n-s+1}(i_{s-1}, i_s) = 0,$$

and for $i_s \notin [l,m] \cap \mathbb{N}$

$$\sum_{i_1=l}^{m-1} \sum_{i_2=l}^{m-2} \cdots \sum_{i_{s-1}=l}^{m-s+1} D_u(k, i_1) D_{m-l}(i_1, i_2) \cdots D_{m-l-s+2}(i_{s-1}, i_s) = 0.$$

The inequality is sharp for $1 \leq p \leq \infty$

Proof. As in Theorem 2.22, we subtract two weighted Montgomery identities, one for interval $[1,n] \cap \mathbb{N}$ and the other for $[l,m] \cap \mathbb{N}$. After that, our inequality follows by applying the Hölder inequality. The proof for the sharpness is similar as the proof of the Theorem 2.22 (with $\mathbf{K}(k, \bullet)$ instead of K and $\Delta^s x$ instead of Δx). □

2.2.3 Generalization for n normalized weight functions

In the paper [22] G. A. Anastassiou presented the generalized weighted Montgomery identity that is weighted generalization of the identity (2.9). We state that identity with somewhat changed notation:

Theorem 2.24 *Let $f : [a,b] \to \mathbb{R}$ be n-times differentiable on $[a,b], n \in \mathbb{N}$ with $f^{(n)} : [a,b] \to \mathbb{R}$ integrable on $[a,b]$. Let $g : [a,b] \to \mathbb{R}$ be a function of bounded variation, such that $g(a) \neq g(b)$. For any $x \in [a,b]$ define weighted Peano kernel:*

$$P_g(x,t) = \frac{1}{g(b)-g(a)} \cdot \begin{cases} (g(t)-g(a)), & a \le t \le x, \\ (g(t)-g(b)), & x < t \le b. \end{cases}$$

Then

$$\begin{aligned} & f(x) - \frac{1}{g(b)-g(a)} \int_a^b f(s_1)\, dg(s_1) - \frac{1}{g(b)-g(a)} \sum_{k=0}^{n-2} \left(\int_a^b f^{(k+1)}(s_1)\, dg(s_1) \right) \\ & \cdot \left(\int_a^b \cdots \int_a^b P_g(x,s_1) \prod_{i=1}^{k} P_g(s_i,s_{i+1})\, \mathrm{d}s_1 \cdots \mathrm{d}s_{k+1} \right) \\ & = \int_a^b \cdots \int_a^b P_g(x,s_1) \prod_{i=1}^{n-1} P_g(s_i,s_{i+1})\, f^{(n)}(s_n)\, \mathrm{d}s_1 \cdots \mathrm{d}s_n. \end{aligned}$$

In this section we prove by means of n weight functions the generalizations of weighted Euler identity and use it to obtain some new Ostrowski type inequalities.

Fist, generalization of integral weighted Euler identity and related Ostrowski type inequality are presented. These are the generalizations of the identity (2.40) and identity from the Theorem 2.24 from [13] and [22].

Second, generalization of discrete weighted Euler identity for finite sequences and related Ostrowski type inequality are obtained. These are the generalizations of the results from Subsection 2.1.4 and Subsection 2.1.5. or [9].

Third, generalization of discrete weighted Euler identity for infinite sequences and related Ostrowski type inequality are obtained. These are the generalizations of the results from Subsection 2.2.2. or [8].

Generalization of integral weighted Euler identity and Ostrowski type inequality

Theorem 2.25 *Let $f : [a,b] \to \mathbb{R}$ be n-times differentiable on $[a,b], n \in \mathbb{N}$ with $f^{(n)} : [a,b] \to \mathbb{R}$ integrable on $[a,b]$. Let $w_i : [a,b] \to [0,\infty\rangle$, $i = 1,..,n$ be a sequence of n integrable functions function satisfying $\int_a^b w_i(t)\,\mathrm{d}t = 1$ and $W_i(t) = \int_a^t w_i(x)\,\mathrm{d}x$ for $t \in [a,b]$, $W_i(t) = 0$ for $t < a$ and $W_i(t) = 1$ for $t > b$,for all $i = 1,..,n$. For any $x \in [a,b]$ define weighted Peano kernel:*

$$P_{w_i}(x,t) = \begin{cases} W_i(t), & a \le t \le x, \\ W_i(t) - 1 & x < t \le b. \end{cases}$$

Then

$$
\begin{aligned}
& f(x)-\int_a^b w_1(t) f(t)\,\mathrm{d}t-\sum_{k=0}^{n-2}\left(\int_a^b w_{k+2}(t) f^{(k+1)}(t)\,\mathrm{d}t\right) \\
& \cdot\left(\int_a^b \cdots \int_a^b P_{w_1}(x,t_1)\prod_{i=1}^{k} P_{w_{i+1}}(t_i,t_{i+1})\,\mathrm{d}t_1\cdots\mathrm{d}t_{k+1}\right) \\
& =\int_a^b \cdots \int_a^b P_{w_1}(x,t_1)\prod_{i=1}^{n-1} P_{w_{i+1}}(t_i,t_{i+1}) f^{(n)}(t_n)\,\mathrm{d}t_1\cdots\mathrm{d}t_n .
\end{aligned} \tag{2.67}
$$

Proof. We prove our assertion by induction with respect to n. For $n=1$ we have basic weighted Montgomery identity. Next, we assume that formula (2.67) holds for a natural number $n-1$ i.e.

$$
\begin{aligned}
& f(x)-\int_a^b w_1(t) f(t)\,\mathrm{d}t-\sum_{k=0}^{n-3}\left(\int_a^b w_{k+2}(t) f^{(k+1)}(t)\,\mathrm{d}t\right) \\
& \cdot\left(\int_a^b \cdots \int_a^b P_{w_1}(x,t_1)\prod_{i=1}^{k} P_{w_{i+1}}(t_i,t_{i+1})\,\mathrm{d}t_1\cdots\mathrm{d}t_{k+1}\right) \\
& =\int_a^b \cdots \int_a^b P_{w_1}(x,t_1)\prod_{i=1}^{n-2} P_{w_{i+1}}(t_i,t_{i+1}) f^{(n-1)}(t_{n-1})\,\mathrm{d}t_1\cdots\mathrm{d}t_{n-1} .
\end{aligned}
$$

Applying the basic weighted Montgomery identity for the function $f^{(n-1)}$

$$
f^{(n-1)}(t_{n-1})=\int_a^b w_n(t) f^{(n-1)}(t)\,\mathrm{d}t+\int_a^b P_{w_n}(t_{n-1},t_n) f^{(n)}(t_n)\,\mathrm{d}t_n
$$

so

$$
\begin{aligned}
& \int_a^b \cdots \int_a^b P_{w_1}(x,t_1)\prod_{i=1}^{n-2} P_{w_{i+1}}(t_i,t_{i+1}) f^{(n-1)}(t_{n-1})\,\mathrm{d}t_1\cdots\mathrm{d}t_{n-1} \\
& =\left(\int_a^b w_n(t) f^{(n-1)}(t)\,\mathrm{d}t\right)\left(\int_a^b \cdots \int_a^b P_{w_1}(x,t_1)\prod_{i=1}^{n-2} P_{w_{i+1}}(t_i,t_{i+1})\,\mathrm{d}t_1\cdots\mathrm{d}t_{n-1}\right) \\
& +\int_a^b \cdots \int_a^b P_{w_1}(x,t_1)\prod_{i=1}^{n-1} P_{w_{i+1}}(t_i,t_{i+1}) f^{(n)}(t_n)\,\mathrm{d}t_1\cdots\mathrm{d}t_n
\end{aligned}
$$

and using the induction hypothesis, we get identity (2.67). So, our assertion is proved. □

Remark 2.21 If we take $w_1\equiv w$, $w_i\equiv\frac{1}{b-a}$, $i=2,..,n$ identity (2.67) reduces to identity (2.40). Also, if we take $w_i\equiv w$, $i=1,..,n$ identity (2.67) reduces to identity from the Theorem 2.24 (with $g(t)=W(t)=\int_a^t w(x)\,\mathrm{d}x$ for $t\in[a,b]$).

Corollary 2.24 *Suppose all the assumptions from the Theorem 2.25 hold. Additionally assume (p,q) is a pair of conjugate exponents, $1\le p,q\le\infty$. Then the following inequality* *holds:*

$$\left| f(x) - \int_a^b w_1(t) f(t)\,\mathrm{d}t - \sum_{k=0}^{n-2} \left(\int_a^b w_{k+2}(t) f^{(k+1)}(t)\,\mathrm{d}t \right) \right.$$
$$\left. \cdot \left(\int_a^b \cdots \int_a^b P_{w_1}(x,t_1) \prod_{i=1}^{k} P_{w_{i+1}}(t_i,t_{i+1})\,\mathrm{d}t_1 \cdots \mathrm{d}t_{k+1} \right) \right|$$
$$\leq \left\| \int_a^b \cdots \int_a^b P_{w_1}(x,t_1) \prod_{i=1}^{n-2} P_{w_{i+1}}(t_i,t_{i+1}) P_{w_n}(t_{n-1},\bullet)\,\mathrm{d}t_1 \cdots \mathrm{d}t_{n-1} \right\|_q \left\| f^{(n)} \right\|_p .$$

Proof. The proof follows after applying modulus on the (2.67) and then applying the Hölder inequality. □

Discrete weighted Euler identity for finite sequences and Ostrowski type inequality

Next we give the generalization of the identities (2.30) and (2.32) from Subsection 2.1.4.

Theorem 2.26 *Let $f:\mathbb{R}\to\mathbb{R}$ be any real-to-real function and Δ the difference operator, $n,m,k \in \mathbb{N}$, $m \geq 2$ and $1 \leq k \leq n$. Let also $w_i^{(1)}$, $w_i^{(2)}$,..,$w_i^{(m)}$ $i = 1,..,n$, m finite sequences of real numbers. Then it holds*

$$f(k) = \frac{1}{W_n^{(1)}} \sum_{i=1}^{n} w_i^{(1)} f(i) + \sum_{r=1}^{m-1} \frac{1}{W_n^{(r+1)}} \left(\sum_{i=1}^{n} w_i^{(r+1)} \Delta^r f(i) \right)$$
$$\cdot \left(\sum_{i_1=1}^{n} \cdots \sum_{i_r=1}^{n} D_{w^{(1)}}(k,i_1) D_{w^{(2)}}(i_1,i_2) \cdots D_{w^{(r)}}(i_{r-1},i_r) \right) \tag{2.68}$$
$$+ \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} D_{w^{(1)}}(k,i_1) D_{w^{(2)}}(i_1,i_2) \cdots D_{w^{(m)}}(i_{m-1},i_m) \Delta^m f(i_m).$$

Proof. The proof is similar to the proof of the Theorem 2.9 (induction with respect to m). □

Remark 2.22 If we take $w_i^{(1)} = w_i^{(2)} = ... = w_i^{(m)}$ $i = 1,..,n$ identity (2.68) reduces to identity (2.32) from the Theorem 2.9.

Corollary 2.25 *Assume (p,q) is a pair of conjugate exponents, $1 \leq p,q \leq \infty$, $f:\mathbb{R}\to\mathbb{R}$ any real-to-real function, $n,k,m \in \mathbb{N}$, $m \geq 2$, and $1 \leq k \leq n$. Then the following inequalities hold:*

$$\left| f(k) - \frac{1}{W_n^{(1)}} \sum_{i=1}^{n} w_i^{(1)} f(i) - \sum_{r=1}^{m-1} \frac{1}{W_n^{(r+1)}} \left(\sum_{i=1}^{n} w_i^{(r+1)} \Delta^r f(i) \right) \right.$$
$$\left. \cdot \left(\sum_{i_1=1}^{n} \cdots \sum_{i_r=1}^{n} D_{w^{(1)}}(k,i_1) D_{w^{(2)}}(i_1,i_2) \cdots D_{w^{(r)}}(i_{r-1},i_r) \right) \right|$$

$$\leq \left\| \sum_{i_1=1}^{n} \cdots \sum_{i_{m-1}=1}^{n} D_{w^{(1)}}(k,i_1) D_{w^{(2)}}(i_1,i_2) \cdots D_{w^{(m)}}(i_{m-1},\bullet) \right\|_q \|\Delta^m f\|_p$$

where

$$\|g\|_p = \begin{cases} \left(\sum_i |g(i)|^p\right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty, \\ \max_i |g(i)| & \text{if } p = \infty. \end{cases}$$

Proof. The proof follows after applying modulus on the (2.68) and then applying the Hölder inequality. □

Discrete weighted Euler identity for finite sequences and Ostrowski type inequality

Next we give the generalization of the identities (2.50) and (2.53).

Theorem 2.27 *Let X be a linear space, $x_1, x_2, .., x_n$ a finite sequence of vectors in X, $w_i^{(1)}$, $i = 1,..,n$; $w_i^{(2)}, i = 1,..,n-1;..,w_i^{(m)}, i = 1,..,n-m+1$, m finite sequences of positive real numbers. Then for all $m \in \{2,3,..,n-1\}$ and $k \in \{1,2,..,n\}$ the following identity is valid*

$$\begin{aligned} x_k = {} & \frac{1}{W_n^{(1)}} \sum_{i=1}^{n} w_i^{(1)} x_i + \sum_{r=1}^{m-1} \frac{1}{W_{n-r}^{(r+1)}} \left(\sum_{i=1}^{n-r} w_i^{(r+1)} \Delta^r x_i \right) \\ & \cdot \left(\sum_{i_1=1}^{n-1} \sum_{i_2=1}^{n-2} \cdots \sum_{i_r=1}^{n-r} D_{w^{(1)}}(k,i_1) D_{w^{(2)}}(i_1,i_2) \cdots D_{w^{(r)}}(i_{r-1},i_r) \right) \\ & + \sum_{i_1=1}^{n-1} \sum_{i_2=1}^{n-2} \cdots \sum_{i_m=1}^{n-m} D_{w^{(1)}}(k,i_1) D_{w^{(2)}}(i_1,i_2) \cdots D_{w^{(m)}}(i_{m-1},i_m) \Delta^m x_{i_m}. \end{aligned} \tag{2.69}$$

Proof. The proof is similar to the proof of the Theorem 2.19 (induction with respect to m). □

Remark 2.23 If we take $w_i^{(2)} = 1$, $i = 1,..,n-1;..,w_i^{(m)} = 1$, $i = 1,..,n-m+1$ identity (2.69) reduces to identity (2.53) from the Theorem 2.19.

Corollary 2.26 *Let $(X, \|\cdot\|)$ be a normed linear space, $x_1, x_2, .., x_n$ a finite sequence of vectors in X, $w_i^{(1)}$, $i = 1,..,n$; $w_i^{(2)}, i = 1,..,n-1;..,w_i^{(m)}, i = 1,..,n-m+1$, m finite sequences of positive real numbers. Let also (p,q), $1 \leq p,q \leq \infty$ be a pair of conjugate exponents. Then for all $m \in \{2,3,..,n-1\}$ and $k \in \{1,2,..,n\}$ the following inequality hold:*

$$\begin{aligned} & \left\| x_k - \frac{1}{W_n^{(1)}} \sum_{i=1}^{n} w_i^{(1)} x_i + \sum_{r=1}^{m-1} \frac{1}{W_{n-r}^{(r+1)}} \left(\sum_{i=1}^{n-r} w_i^{(r+1)} \Delta^r x_i \right) \right. \\ & \left. \cdot \left(\sum_{i_1=1}^{n-1} \sum_{i_2=1}^{n-2} \cdots \sum_{i_r=1}^{n-r} D_{w^{(1)}}(k,i_1) D_{w^{(2)}}(i_1,i_2) \cdots D_{w^{(r)}}(i_{r-1},i_r) \right) \right\| \end{aligned}$$

$$\leq \left\| \sum_{i_1=1}^{n-1} \sum_{i_2=1}^{n-2} \cdots \sum_{i_{m-1}=1}^{n-m+1} D_{w^{(1)}}(k,i_1) D_{w^{(2)}}(i_1,i_2) \cdots D_{w^{(m)}}(i_{m-1},\bullet) \right\|_q \|\Delta^m x\|_p$$

where

$$\|\Delta^m x\|_p = \begin{cases} \left(\sum_{i=1}^{n-m} \|\Delta^m x_i\|^p\right)^{\frac{1}{p}}, & \textit{if} \;\; 1 \leq p < \infty, \\ \max_{1\leq i\leq n-m} \|\Delta^m x_i\| & \textit{if} \;\; p = \infty. \end{cases}$$

Proof. The proof follows after applying modulus on the (2.69) and then applying the Hölder inequality. □

2.3 Extensions of the Montgomery identity

In this section we present extensions of the weighted Montgomery identity by using Taylor's formula in two different ways and one extension by using Fink identity. These identities are used to obtain some Ostrowski type inequalities and estimations of the difference of two integral means. Also, applications for α-L-Hölder type functions are made via Taylor's extensions.

2.3.1 Extension via Taylor's formula (centered at the interval endpoints) with applications

Here, an extension of the weighted Montgomery identity is given, by using Taylor's series centered at the interval endpoints. Results from this subsection are published in [16].

Theorem 2.28 *Let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$, $I \subset \mathbb{R}$ an open interval, $a,b \in I$, $a < b$, $w : [a,b] \to [0,\infty\rangle$ is some nonnegative normalized weighted function. Then the following identity holds*

$$\begin{aligned} f(x) = &\int_a^b w(t) f(t)\,\mathrm{d}t + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{(i+1)!} \int_a^x w(s) \left((x-a)^{i+1} - (s-a)^{i+1}\right) \mathrm{d}s \\ &+ \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{(i+1)!} \int_x^b w(s) \left((x-b)^{i+1} - (s-b)^{i+1}\right) \mathrm{d}s \\ &+ \frac{1}{(n-1)!} \int_a^b T^1_{w,n}(x,s) f^{(n)}(s)\,\mathrm{d}s \end{aligned} \tag{2.70}$$

where

$$T^1_{w,n}(x,s) = \begin{cases} \int_x^s w(u)(u-s)^{n-1}\,\mathrm{d}u + W(x)(x-s)^{n-1}, & a \leq s \leq x, \\ \int_x^s w(u)(u-s)^{n-1}\,\mathrm{d}u + (W(x)-1)(x-s)^{n-1} & x < s \leq b. \end{cases}$$

Proof. If we apply Taylor's formula with $f'(t)$, and replace n with $n-1$ ($n \geq 2$) we have

$$\begin{aligned} f'(t) &= \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!}(t-a)^i + \int_a^t f^{(n)}(s)\frac{(t-s)^{n-2}}{(n-2)!}\mathrm{d}s \\ &= \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!}(t-b)^i - \int_t^b f^{(n)}(s)\frac{(t-s)^{n-2}}{(n-2)!}\mathrm{d}s. \end{aligned}$$

By putting these two formulae in the weighted Montgomery identity (2.5) we obtain

$$\begin{aligned} f(x) &= \int_a^b w(t) f(t)\,\mathrm{d}t \\ &+ \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!}\int_a^x (t-a)^i W(t)\,\mathrm{d}t + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!}\int_x^b (t-b)^i (W(t)-1)\,\mathrm{d}t \\ &+ \int_a^x W(t)\left(\int_a^t f^{(n)}(s)\frac{(t-s)^{n-2}}{(n-2)!}\mathrm{d}s\right)\mathrm{d}t \\ &- \int_x^b (W(t)-1)\left(\int_t^b f^{(n)}(s)\frac{(t-s)^{n-2}}{(n-2)!}\mathrm{d}s\right)\mathrm{d}t. \end{aligned}$$

Now, the interchange the order of integration leads us to

$$\begin{aligned} \int_a^x (t-a)^i W(t)\,\mathrm{d}t &= \int_a^x (t-a)^i\left(\int_a^t w(s)\,\mathrm{d}s\right)\mathrm{d}t = \int_a^x w(s)\left(\int_s^x (t-a)^i\,\mathrm{d}t\right)\mathrm{d}s \\ &= \frac{1}{i+1}\int_a^x w(s)\left((x-a)^{i+1}-(s-a)^{i+1}\right)\mathrm{d}s \end{aligned}$$

and similarly

$$\int_x^b (t-b)^i (W(t)-1) = \frac{1}{i+1}\int_x^b w(s)\left((x-b)^{i+1}-(s-b)^{i+1}\right)\mathrm{d}s.$$

Further,

$$\int_a^x W(t)\left(\int_a^t f^{(n)}(s)(t-s)^{n-2}\,\mathrm{d}s\right)\mathrm{d}t = \int_a^x f^{(n)}(s)\left(\int_s^x W(t)(t-s)^{n-2}\,\mathrm{d}t\right)\mathrm{d}s$$

and

$$\begin{aligned} &\int_s^x W(t)(t-s)^{n-2}\,\mathrm{d}t = \int_s^x\left(\int_a^t w(u)\,\mathrm{d}u\right)(t-s)^{n-2}\,\mathrm{d}t \\ &= \int_a^s w(u)\left(\int_s^x (t-s)^{n-2}\,\mathrm{d}t\right)\mathrm{d}u + \int_s^x w(u)\left(\int_u^x (t-s)^{n-2}\,\mathrm{d}t\right)\mathrm{d}u \\ &= \frac{(x-s)^{n-1}}{n-1}W(x) - \int_s^x w(u)\frac{(u-s)^{n-1}}{n-1}\mathrm{d}u. \end{aligned}$$

Similarly

$$
\begin{aligned}
&-\int_x^b (W(t)-1)\left(\int_t^b f^{(n)}(s)(t-s)^{n-2}\mathrm{d}s\right)\mathrm{d}t \\
&=\int_x^b f^{(n)}(s)\left(\int_x^s (1-W(t))(t-s)^{n-2}\mathrm{d}t\right)\mathrm{d}s
\end{aligned}
$$

and

$$
\int_x^s (1-W(t))(t-s)^{n-2}\mathrm{d}t=(W(x)-1)\frac{(x-s)^{n-1}}{n-1}+\int_x^s w(u)\frac{(u-s)^{n-1}}{n-1}\mathrm{d}u.
$$

So the reminder in the weighted Taylor formula is

$$
\begin{aligned}
&\frac{1}{(n-1)!}\left[\int_a^b f^{(n)}(s)\left(\int_x^s w(u)(u-s)^{n-1}\mathrm{d}u\right)\mathrm{d}s\right. \\
&\left.+W(x)\int_a^x f^{(n)}(s)(x-s)^{n-1}\mathrm{d}s+(W(x)-1)\int_x^b f^{(n)}(s)(x-s)^{n-1}\mathrm{d}s\right].
\end{aligned}
$$

□

Remark 2.24 In the special case, if we take $w(t)=\frac{1}{b-a}$, $t\in[a,b]$ the identity (2.70) reduces to

$$
\begin{aligned}
f(x)=&\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t+\sum_{i=0}^{n-2}\frac{f^{(i+1)}(a)}{i!(i+2)}\frac{(x-a)^{i+2}}{b-a}-\sum_{i=0}^{n-2}\frac{f^{(i+1)}(b)}{i!(i+2)}\frac{(x-b)^{i+2}}{b-a} \\
&+\frac{1}{(n-1)!}\int_a^b T_n(x,s)f^{(n)}(s)\mathrm{d}s,
\end{aligned}
\tag{2.71}
$$

where

$$
T_n^1(x,s)=\begin{cases}-\frac{(x-s)^n}{n(b-a)}+\frac{x-a}{b-a}(x-s)^{n-1}, & a\le s\le x,\\[2mm] -\frac{(x-s)^n}{n(b-a)}+\frac{x-b}{b-a}(x-s)^{n-1} & x<s\le b.\end{cases}
$$

The Ostrowski type inequalities

The results in this subsection generalize the results from [86], [87], [21].

Theorem 2.29 *Suppose that all the assumptions of Theorem 2.28 hold. Additionally assume that (p,q) is a pair of conjugate exponents, $1\le p,q\le\infty$. Then we have*

$$
\begin{aligned}
&\left|f(x)-\int_a^b w(t)f(t)\mathrm{d}t-\sum_{i=0}^{n-2}\frac{f^{(i+1)}(a)}{(i+1)!}\int_a^x w(s)\left((x-a)^{i+1}-(s-a)^{i+1}\right)\mathrm{d}s\right. \\
&\left.-\sum_{i=0}^{n-2}\frac{f^{(i+1)}(b)}{(i+1)!}\int_x^b w(s)\left((x-b)^{i+1}-(s-b)^{i+1}\right)\mathrm{d}s\right|
\end{aligned}
$$

$$\leq \frac{1}{(n-1)!} \left\| T^1_{w,n}(x,\cdot) \right\|_q \left\| f^{(n)} \right\|_p . \tag{2.72}$$

The inequality (2.72) is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof. The proof is similar to the proof of the Theorem 2.16. □

The Beta and the incomplete Beta function of Euler type are defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt, \qquad B_r(x,y) = \int_0^r t^{x-1}(1-t)^{y-1}\,dt, \qquad x,y > 0$$

while

$$\Psi_r(x,y) = \int_0^r t^{x-1}(1+t)^{y-1}\,dt$$

is a real positive valued integral.

Corollary 2.27 *Let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$, $I \subset \mathbb{R}$ an open interval, $a,b \in I$, $a < b$ and (p,q) a pair of conjugate exponents, $1 < p \leq \infty$. Then we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!\,(i+2)} \frac{(x-a)^{i+2}}{b-a} + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!\,(i+2)} \frac{(x-b)^{i+2}}{b-a} \right|$$
$$\leq \frac{1}{(b-a)(n-1)!} \left\{ \left((x-a)^{qn+1} + (b-x)^{qn+1} \right) n^{q(n-1)+1} \right.$$
$$\left. \left(B(q+1, q(n-1)+1) - B_{\frac{n-1}{n}}(q+1, q(n-1)+1) \right) \right\}^{\frac{1}{q}} \left\| f^{(n)} \right\|_p$$

and this inequality is sharp. For $p = 1$ we have

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!\,(i+2)} \frac{(x-a)^{i+2}}{b-a} + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!\,(i+2)} \frac{(x-b)^{i+2}}{b-a} \right|$$
$$\leq \frac{n-1}{(b-a)n!} \max\left\{ (x-a)^n, (b-x)^n \right\} \left\| f^{(n)} \right\|_1$$

and this inequality is the best possible.

Proof. We apply the inequality (2.72) with $w(t) = \frac{1}{b-a}$, $t \in [a,b]$ and use (2.71)

$$\int_a^b \left| T^1_{w,n}(x,s) \right|^q ds = \int_a^x \left| -\frac{(x-s)^n}{n(b-a)} + \frac{x-a}{b-a}(x-s)^{n-1} \right|^q ds$$
$$+ \int_x^b \left| -\frac{(x-s)^n}{n(b-a)} + \frac{x-b}{b-a}(x-s)^{n-1} \right|^q ds.$$

First with substitution $(x-a) - (x-s) = nt(x-a)$ we have

$$\int_a^x \left| -\frac{(x-s)^n}{n} + (x-a)(x-s)^{n-1} \right|^q ds$$

$$= \int_a^x (x-s)^{q(n-1)} \left((x-a) - \frac{(x-s)}{n} \right)^q \mathrm{d}s$$
$$= (x-a)^{qn+1} n^{q(n-1)+1} \left(B(q+1, q(n-1)+1) - B_{\frac{n-1}{n}}(q+1, q(n-1)+1) \right).$$

Similar with $n(b-x)-(s-x) = nt(b-x)$ we get

$$\int_x^b \left| -\frac{(x-s)^n}{n(b-a)} + \frac{x-b}{b-a}(x-s)^{n-1} \right|^q \mathrm{d}s$$
$$= (b-x)^{qn+1} n^{q(n-1)+1} \left(B(q+1, q(n-1)+1) - B_{\frac{n-1}{n}}(q+1, q(n-1)+1) \right)$$

and the first inequality follows from the Theorem 2.29.

For $p=1$

$$\sup_{s\in[a,b]} \left| T_{w,n}^1(x,s) \right| = \max \left\{ \sup_{s\in[a,x]} \left| -\frac{(x-s)^n}{n(b-a)} + \frac{x-a}{b-a}(x-s)^{n-1} \right|, \right.$$
$$\left. \sup_{s\in[x,b]} \left| -\frac{(x-s)^n}{n(b-a)} + \frac{x-b}{b-a}(x-s)^{n-1} \right| \right\}.$$

By an elementary calculation we get

$$\sup_{s\in[a,x]} \left| -\frac{(x-s)^n}{n(b-a)} + \frac{x-a}{b-a}(x-s)^{n-1} \right| = \frac{(n-1)(x-a)^n}{n(b-a)},$$

$$\sup_{s\in[x,b]} \left| -\frac{(x-s)^n}{n(b-a)} + \frac{x-b}{b-a}(x-s)^{n-1} \right| = \frac{(n-1)(b-x)^n}{n(b-a)}.$$

and the second inequality follows from the Theorem 2.29. □

Remark 2.25 Inequalities from the Corollary 2.27 $n=2$ reduces to

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t + \frac{f'(b)(x-b)^2 - f'(a)(x-a)^2}{2(b-a)} \right|$$
$$\le \frac{1}{(b-a)} \left\{ \left((x-a)^{2q+1} + (b-x)^{2q+1} \right) 2^q (B(q+1, q+1)) \right\}^{\frac{1}{q}} \left\| f'' \right\|_p$$

and (for $p=1$)

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t + \frac{f'(b)(x-b)^2 - f'(a)(x-a)^2}{2(b-a)} \right|$$
$$\le \frac{1}{2(b-a)} \max\left\{ (x-a)^2, (b-x)^2 \right\} \left\| f'' \right\|_1.$$

Also, if $q=1$ we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t + \frac{f'(b)(x-b)^2 - f'(a)(x-a)^2}{2(b-a)} \right| \\
&\le \frac{1}{3(b-a)}\left[(x-a)^3 + (b-x)^3\right] \|f''\|_\infty .
\end{aligned}$$

Remark 2.26 If we apply (2.72) with $x = \frac{a+b}{2}$ we get the generalized midpoint inequality

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \int_a^b w(t) f(t)\,\mathrm{d}t \right. \\
&- \sum_{i=0}^{n-2} \frac{\left(\frac{b-a}{2}\right)^{i+1}}{(i+1)!}\left(f^{(i+1)}(a) W\left(\frac{a+b}{2}\right) + (-1)^{i+1} f^{(i+1)}(b)\left(1 - W\left(\frac{a+b}{2}\right)\right)\right) \\
&\left. + \sum_{i=0}^{n-2} \frac{1}{(i+1)!}\left(f^{(i+1)}(a)\int_a^{\frac{a+b}{2}} w(s)(s-a)^{i+1}\,\mathrm{d}s + f^{(i+1)}(b)\int_{\frac{a+b}{2}}^b w(s)(s-b)^{i+1}\,\mathrm{d}s \right) \right| \\
&\le \frac{1}{(n-1)!}\left(\int_a^b \left| T_{w,n}^1\left(\frac{a+b}{2}, s\right)\right|^q \mathrm{d}s\right)^{\frac{1}{q}} \left\| f^{(n)} \right\|_p .
\end{aligned}$$

If we additionally assume that $w(t)$ is symmetric on $[a,b]$ i.e. $w(t) = w(b-a-t)$ for every $t \in [a,b]$ this inequality reduces to

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \int_a^b w(t) f(t)\,\mathrm{d}t \right. \\
&\left. + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a) + (-1)^{i+1} f^{(i+1)}(b)}{(i+1)!}\left(\int_a^{\frac{a+b}{2}} w(s)(s-a)^{i+1}\,\mathrm{d}s - \frac{1}{2}\left(\frac{b-a}{2}\right)^{i+1} \right) \right| \\
&\le \frac{1}{(n-1)!}\left(\int_a^b \left| T_{w,n}^1\left(\frac{a+b}{2}, s\right)\right|^q \mathrm{d}s\right)^{\frac{1}{q}} \left\| f^{(n)} \right\|_p
\end{aligned}$$

and

$$T_{w,n}^1\left(\frac{a+b}{2}, s\right) = \int_{\frac{a+b}{2}}^s w(u)(u-s)^{n-1}\,\mathrm{d}u + \frac{1}{2}\left(\frac{a+b}{2} - s\right)^{n-1} sgn\left(\frac{a+b}{2} - s\right).$$

For $n=2$ and $w(t) = \frac{1}{b-a}$ we get

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t + \frac{b-a}{8}\left(f'(b) - f'(a)\right) \right|$$

$$\leq \frac{(b-a)^{1+\frac{1}{q}}}{2} \left(B(q+1,q+1)\right)^{\frac{1}{q}} \left\|f''\right\|_p$$

and for $q=1$

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t + \frac{b-a}{8}\left(f'(b)-f'(a)\right) \right| \leq \frac{(b-a)^2}{12}\left\|f''\right\|_\infty .$$

For the generalized trapezoid inequality we apply equality (2.70) first with $x=a$, then with $x=b$ then add them up and divide by 2. After applying the Hölder inequality we get

$$\left| \frac{f(a)+f(b)}{2} - \int_a^b w(t) f(t)\,\mathrm{d}t - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{2(i+1)!}\left((b-a)^{i+1} - \int_a^b w(s)(s-a)^{i+1}\,\mathrm{d}s\right) \right.$$
$$\left. - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{2(i+1)!}\left((-1)^{i+1}(b-a)^{i+1} - \int_a^b w(s)(s-b)^{i+1}\,\mathrm{d}s\right) \right|$$
$$\leq \frac{1}{2(n-1)!}\left(\int_a^b \left|T^1_{w,n}(a,s)+T^1_{w,n}(b,s)\right|^q \mathrm{d}s\right)^{\frac{1}{q}} \left\|f^{(n)}\right\|_p$$

and

$$T^1_{w,n}(a,s)+T^1_{w,n}(b,s) = \int_a^b sgn(s-u)\,w(u)(u-s)^{n-1}\,\mathrm{d}u + (b-s)^{n-1} - (a-s)^{n-1}.$$

Again, if we additionally assume that $w(t)$ is symmetric on $[a,b]$ this inequality reduces to

$$\left| \frac{f(a)+f(b)}{2} - \int_a^b w(t) f(t)\,\mathrm{d}t - \sum_{i=0}^{n-2} \frac{1}{2(i+1)!}\left(f^{(i+1)}(a) + (-1)^{i+1} f^{(i+1)}(b)\right) \right.$$
$$\left. \cdot \left((b-a)^{i+1} - \int_a^b w(s)(s-a)^{i+1}\,\mathrm{d}s\right) \right|$$
$$\leq \frac{1}{2(n-1)!}\left(\int_a^b \left|T^1_{w,n}(a,s)+T^1_{w,n}(b,s)\right|^q \mathrm{d}s\right)^{\frac{1}{q}} \left\|f^{(n)}\right\|_p .$$

For $n=2$, $w(t)=\frac{1}{b-a}$ and $q=1$ we get

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t + \frac{b-a}{4}\left(f'(b)-f'(a)\right) \right| \leq \frac{(b-a)^2}{3}\left\|f''\right\|_\infty .$$

The estimation of the difference of the two weighted integral means

In this section we generalize the results from [24], [83]. We denote

$$t_{w,n}^{[a,b],1}(x) = \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{(i+1)!} \int_a^x w(s)\left((x-a)^{i+1} - (s-a)^{i+1}\right) \mathrm{d}s + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{(i+1)!} \int_x^b w(s)\left((x-b)^{i+1} - (s-b)^{i+1}\right) \mathrm{d}s$$

and

$$D^1(w,u;x) = \int_a^b w(t) f(t)\,\mathrm{d}t - \int_c^d u(t) f(t)\,\mathrm{d}t + t_{w.n}^{[a,b],1}(x) - t_{u,n}^{[c,d],1}(x),$$

for function $f:[a,b]\to\mathbb{R}$ such that $f^{(n)}$ is absolutely continuous function for some $n\geq 2$.

For the two intervals $[a,b]$ and $[c,d]$ we have four possible cases if $[a,b]\cap[c,d]\neq\emptyset$. The first case is $[c,d]\subseteq[a,b]$ and the second $[a,b]\cap[c,d]=[c,b]$. Other two possible cases we simply get by change $a\leftrightarrow c$, $b\leftrightarrow d$.

Theorem 2.30 *Let $f:[a,b]\cup[c,d]\to\mathbb{R}$ be such that $f^{(n-1)}$ is an absolutely continuous function for some $n\geq 2$, $w:[a,b]\to[0,\infty\rangle$ and $u:[c,d]\to[0,\infty\rangle$ some nonnegative normalized weight functions, $W(t)=\int_a^t w(x)\,\mathrm{d}x$ for $t\in[a,b]$, $W(t)=0$ for $t<a$ and $W(t)=1$ for $t>b$, $U(t)=\int_c^t u(x)\,\mathrm{d}x$ for $t\in[c,d]$, $U(t)=0$ for $t<c$ and $U(t)=1$ for $t>d$. Then if $[a,b]\cap[c,d]\neq\emptyset$ and $x\in[a,b]\cap[c,d]$, we have*

$$D^1(w,u;x) = \int_{\min\{a,c\}}^{\max\{b,d\}} K_n(x,s) f^{(n)}(s)\,\mathrm{d}s \tag{2.73}$$

where in case $[c,d]\subseteq[a,b]$

$$K_n^1(x,s) = \begin{cases} -\frac{1}{(n-1)!}\left(\int\limits_x^s w(t)(t-s)^{n-1}\,\mathrm{d}t + W(x)(x-s)^{n-1}\right), & s\in[a,c], \\ -\frac{1}{(n-1)!}\left(\int\limits_x^s (w(t)-u(t))(t-s)^{n-1}\,\mathrm{d}t \right. \\ \left. \quad + (W(x)-U(x))(x-s)^{n-1}\right), & s\in\langle c,d], \\ -\frac{1}{(n-1)!}\left(\int\limits_x^s w(t)(t-s)^{n-1}\,\mathrm{d}t + (W(x)-1)(x-s)^{n-1}\right), & s\in\langle d,b], \end{cases}$$

and in case $[a,b] \cap [c,d] = [c,b]$

$$K_n^1(x,s) = \begin{cases} -\frac{1}{(n-1)!}\left(\int\limits_x^s w(t)(t-s)^{n-1}\,\mathrm{d}t + W(x)(x-s)^{n-1}\right), & s \in [a,c], \\ -\frac{1}{(n-1)!}\left(\int\limits_x^s (w(t)-u(t))(t-s)^{n-1}\,\mathrm{d}t \right. \\ \qquad \left. + (W(x)-U(x))(x-s)^{n-1}\right), & s \in \langle c,b], \\ \frac{1}{(n-1)!}\left(\int\limits_x^s u(t)(t-s)^{n-1}\,\mathrm{d}t + (U(x)-1)(x-s)^{n-1}\right), & s \in \langle b,d]. \end{cases}$$

Proof. We subtract identities (2.70) for interval $[a,b]$ and $[c,d]$, to get the formula (2.73). □

Theorem 2.31 *Assume* (p,q) *is a pair of conjugate exponents,* $1 \le p,q \le \infty$*. Let* $f^{(n)} \in L_p([a,b] \cup [c,d])$ *for some* $n \ge 2$*. Then we have*

$$\left|D^1(w,u;x)\right| \le \left(\int_{\min\{a,c\}}^{\max\{b,d\}} |K_n(x,s)|^q\,\mathrm{d}s\right)^{\frac{1}{q}} \left\|f^{(n)}\right\|_p$$

for every $x \in [a,b] \cap [c,d]$*. This inequality is sharp for* $1 < p \le \infty$ *and the best possible for* $p = 1$*.*

Proof. The proof is similar to the proof of the Theorem 2.16. □

Case $[c,d] \subseteq [a,b]$

Here we denote

$$t_n^{[a,b],1}(x) = \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!(i+2)} \frac{(x-a)^{i+2}}{b-a} - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!(i+2)} \frac{(x-b)^{i+2}}{b-a}$$

and

$$D^1\left(\frac{1}{b-a}, \frac{1}{d-c}; x\right) = \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{1}{d-c}\int_c^d f(t)\,\mathrm{d}t + t_n^{[a,b],1}(x) - t_n^{[c,d],1}(x).$$

Corollary 2.28 *Assume* (p,q) *is a pair of conjugate exponents. Let* $f^{(n)} \in L_p[a,b]$ *for some* $n \ge 2$*,* $[c,d] \subseteq [a,b]$*,* $x \in [c,d]$ *and* $s_0 = \frac{bc-ad}{c-a+b-d}$*. Then for* $1 < p \le \infty$ *and* $x + n(s_0 - x) \notin [c,d]$ *we have*

$$\left|D^1\left(\frac{1}{b-a}, \frac{1}{d-c}; x\right)\right| \le \frac{n^{n-1+\frac{1}{q}}}{(n-1)!(b-a)}$$
$$\cdot \left[(x-a)^{nq+1}\left(B_{1-\frac{x-c}{n(x-a)}}(q+1,(n-1)q+1) - B_{1-\frac{1}{n}}(q+1,(n-1)q+1)\right)\right.$$

$$+\frac{(c-a+b-d)^q}{(d-c)^q}|x-s_o|^{nq+1}\left(B(q+1,(n-1)q+1)\right.$$
$$-B_{r_1}(q+1,(n-1)q+1)+\Psi_{r_2}((n-1)q+1,q+1))+(b-x)^{nq+1}$$
$$\left.\cdot\left(B_{1-\frac{d-x}{n(b-x)}}(q+1,(n-1)q+1)-B_{1-\frac{1}{n}}(q+1,(n-1)q+1)\right)\right]^{\frac{1}{q}}\left\|f^{(n)}\right\|_p$$

where for $x+n(s_0-x)<c$, $r_1=1-\frac{x-c}{n(x-s_0)}$, $r_2=\frac{d-x}{n(x-s_0)}$, *while for* $x+n(s_0-x)>d$, $r_1=1-\frac{d-x}{n(s_0-x)}$, $r_2=\frac{x-c}{n(s_0-x)}$.
If $x+n(s_0-x)\in[c,d]$ *we have*

$$\left|D^1\left(\frac{1}{b-a},\frac{1}{d-c};x\right)\right|\leq\frac{n^{n-1+\frac{1}{q}}}{(n-1)!(b-a)}$$
$$\cdot\left[(x-a)^{nq+1}\left(B_{1-\frac{x-c}{n(x-a)}}(q+1,(n-1)q+1)-B_{1-\frac{1}{n}}(q+1,(n-1)q+1)\right)\right.$$
$$+\frac{(c-a+b-d)^q}{(d-c)^q}|x-s_o|^{nq+1}\left(\Psi_{r_3}((n-1)q+1,q+1)\right.$$
$$+B(q+1,(n-1)q+1)+\Psi_{r_4}(q+1,(n-1)q+1))+(b-x)^{nq+1}$$
$$\left.\cdot\left(B_{1-\frac{d-x}{n(b-x)}}(q+1,(n-1)q+1)-B_{1-\frac{1}{n}}(q+1,(n-1)q+1)\right)\right]^{\frac{1}{q}}\left\|f^{(n)}\right\|_p$$

where for $x<s_0$, $r_3=\frac{x-c}{n(s_0-x)}$, $r_4=\frac{d-x}{n(s_0-x)}-1$, *while for* $s_0<x$, $r_3=\frac{d-x}{n(x-s_0)}$, $r_4=\frac{x-c}{n(x-s_0)}-1$.

For $p=1$ *we have*

$$\left|D\left(\frac{1}{b-a},\frac{1}{d-c};x\right)\right|\leq\frac{1}{(b-a)n!}\max\left\{(n-1)(x-a)^n,(n-1)(b-x)^n,\right.$$
$$\frac{c-a+b-d}{d-c}(x-c)^{n-1}|x-c+n(s_0-x)|,\frac{c-a+b-d}{d-c}|s_0-x|^n(n-1)^{n-1},$$
$$\left.\frac{c-a+b-d}{d-c}(d-x)^{n-1}|d-x+n(x-s_0)|\right\}\left\|f^{(n)}\right\|_1.$$

Proof. We put $w(t)=\frac{1}{b-a}$, $t\in[a,b]$, and $u(t)=\frac{1}{d-c}$, $t\in[c,d]$ in the Theorem 2.31. Thus we have $t_n^{[a,b],1}(x)$ and $t_n^{[c,d],1}(x)$ instead of $t_{w,n}^{[a,b],1}(x)$ and $t_{u,n}^{[c,d],1}(x)$ and

$$\left(\int_{\min\{a,c\}}^{\max\{b,d\}}\left|K_n^1(x,s)\right|^q ds\right)^{\frac{1}{q}}=\frac{1}{(n-1)!}\left(\int_a^c\left|-\frac{(x-s)^n}{n(b-a)}+\frac{x-a}{b-a}(x-s)^{n-1}\right|^q ds\right.$$
$$+\int_c^d\left|-\frac{(x-s)^n}{n(b-a)}+\frac{(x-s)^n}{n(d-c)}+\left(\frac{x-a}{b-a}-\frac{x-c}{d-c}\right)(x-s)^{n-1}\right|^q ds$$
$$\left.+\int_d^b\left|-\frac{(x-s)^n}{n(b-a)}+\frac{x-b}{b-a}(x-s)^{n-1}\right|^q ds\right)^{\frac{1}{q}}.$$

For the first integral let $x-a-\frac{x-s}{n}=t(x-a)$ so

$$I_1=\int_a^c\left|-\frac{(x-s)^n}{n(b-a)}+\frac{x-a}{b-a}(x-s)^{n-1}\right|^q ds=\frac{(x-a)^{nq+1}n^{(n-1)q+1}}{(b-a)^q}$$
$$\cdot\left(B_{1-\frac{x-c}{n(x-a)}}(q+1,(n-1)q+1)-B_{1-\frac{1}{n}}(q+1,(n-1)q+1)\right).$$

For the third integral let $b-x-\frac{s-x}{n}=t(b-x)$ and similar obtain

$$I_3=\int_d^b\left|-\frac{(x-s)^n}{n(b-a)}+\frac{x-b}{b-a}(x-s)^{n-1}\right|^q ds=\frac{(b-x)^{nq+1}n^{(n-1)q+1}}{(b-a)^q}$$
$$\cdot\left(B_{1-\frac{d-x}{n(b-x)}}(q+1,(n-1)q+1)-B_{1-\frac{1}{n}}(q+1,(n-1)q+1)\right).$$

As $c-a+b-d>0$, the second integral is

$$I_2=\int_c^d\left|-\frac{(x-s)^n}{n(b-a)}+\frac{(x-s)^n}{n(d-c)}+\left(\frac{x-a}{b-a}-\frac{x-c}{d-c}\right)(x-s)^{n-1}\right|^q ds$$
$$=\frac{(c-a+b-d)^q}{n^q(b-a)^q(d-c)^q}\int_c^d\left|(x-s)^{n-1}(x-s+n(s_0-x))\right|^q ds.$$

Since $s_0-c=\frac{(d-c)(c-a)}{c-a+b-d}\geq 0$ and $d-s_0=\frac{(d-c)(b-d)}{c-a+b-d}\geq 0$, then $s_0\in[c,d]$. So we have four possible cases:

1. If $x+n(s_0-x)<c$, (then also $x>s_0$) we have

$$\int_c^d\left|(x-s)^{n-1}(x-s+n(s_0-x))\right|^q ds$$
$$=\int_c^x(x-s)^{(n-1)q}(s-x+n(x-s_0))^q ds$$
$$+\int_x^d(s-x)^{(n-1)q}(s-x+n(x-s_0))^q ds.$$

Now, using the substitution $s-x+n(x-s_0)=nt(x-s_0)$ we get

$$\int_c^x(x-s)^{(n-1)q}(s-x+n(x-s_0))^q ds=(x-s_0)^{qn+1}n^{qn+1}$$
$$\cdot\left(B(q+1,(n-1)q+1)-B_{1-\frac{x-c}{n(x-s_0)}}(q+1,(n-1)q+1)\right).$$

Similar, using the substitution $s-x=nt(x-s_0)$ we get

$$\int_x^d(s-x)^{(n-1)q}(s-x+n(x-s_0))^q ds$$
$$=(x-s_0)^{qn+1}n^{qn+1}\Psi_{\frac{d-x}{n(x-s_0)}}((n-1)q+1,q+1). \tag{2.74}$$

2. If $x+n(s_0-x)>d$, (then also $s_0>x$) we have

$$\begin{aligned}&\int_c^d \left|(x-s)^{n-1}(x-s+n(s_0-x))\right|^q \mathrm{d}s\\&=\int_c^x (x-s)^{(n-1)q}(x-s+n(s_0-x))^q\,\mathrm{d}s\\&+\int_x^d (s-x)^{(n-1)q}(x-s+n(s_0-x))^q\,\mathrm{d}s.\end{aligned}$$

Using the substitution $x-s=nt(s_0-x)$ similar we get

$$\begin{aligned}&\int_c^x (x-s)^{(n-1)q}(x-s+n(s_0-x))^q\,\mathrm{d}s\\&=(s_0-x)^{qn+1}n^{qn+1}\Psi_{\frac{x-c}{n(s_0-x)}}((n-1)q+1,q+1)\end{aligned} \tag{2.75}$$

and using the substitution $x-s+n(s_0-x)=nt(s_0-x)$

$$\begin{aligned}&\int_x^d (s-x)^{(n-1)q}(x-s+n(s_0-x))^q\,\mathrm{d}s=(s_0-x)^{qn+1}n^{qn+1}\\&\cdot\left(B(q+1,(n-1)q+1)-B_{1-\frac{d-x}{n(s_0-x)}}(q+1,(n-1)q+1)\right).\end{aligned}$$

3. If $c\le x+n(s_0-x)\le d$ and $x\le x+n(s_0-x)$, (so $x<s_0$) then

$$\begin{aligned}&\int_c^d \left|(x-s)^{n-1}(x-s+n(s_0-x))\right|^q \mathrm{d}s=\int_c^x (x-s)^{(n-1)q}(x-s+n(s_0-x))^q\,\mathrm{d}s\\&+\int_x^{x+n(s_0-x)} (s-x)^{(n-1)q}(x-s+n(s_0-x))^q\,\mathrm{d}s\\&+\int_{x+n(s_0-x)}^d (s-x)^{(n-1)q}(s-x+n(x-s_0))^q\,\mathrm{d}s.\end{aligned}$$

We already had the first integral (see (2.75)), for second let $x-s+n(s_0-x)=nt(s_0-x)$ and then

$$\begin{aligned}&\int_x^{x+n(s_0-x)} (s-x)^{(n-1)q}(x-s+n(s_0-x))^q\,\mathrm{d}s\\&=(s_0-x)^{qn+1}n^{qn+1}B(q+1,(n-1)q+1),\end{aligned}$$

and for third let $s-x+n(x-s_0)=tn(s_0-x)$ so

$$\begin{aligned}&\int_{x+n(s_0-x)}^d (s-x)^{(n-1)q}(s-x+n(x-s_0))^q\,\mathrm{d}s\\&=(s_0-x)^{qn+1}n^{qn+1}\Psi_{\frac{d-x}{n(s_0-x)}-1}(q+1,(n-1)q+1).\end{aligned}$$

4. If $c\le x+n(s_0-x)\le d$ and $x+n(s_0-x)<x$, (so $s_0<x$) then

$$\int_c^d \left|(x-s)^{n-1}(x-s+n(s_0-x))\right|^q \mathrm{d}s$$

$$
\begin{aligned}
&= \int_{c}^{x+n(s_0-x)} (x-s)^{(n-1)q} (x-s+n(s_0-x))^q \, ds \\
&+ \int_{x+n(s_0-x)}^{x} (x-s)^{(n-1)q} (s-x+n(x-s_0))^q \, ds \\
&+ \int_{x}^{d} (s-x)^{(n-1)q} (s-x+n(x-s_0))^q \, ds.
\end{aligned}
$$

Now, let $x-s+n(s_0-x) = tn(x-s_0)$ so

$$
\begin{aligned}
&\int_{c}^{x+n(s_0-x)} (x-s)^{(n-1)q} (x-s+n(s_0-x))^q \, ds \\
&= (x-s_0)^{qn+1} n^{qn+1} \Psi_{\frac{x-c}{n(x-s_0)}-1} (q+1, (n-1)q+1),
\end{aligned}
$$

and with $s-x+n(x-s_0) = nt(x-s_0)$ we get

$$
\begin{aligned}
&\int_{x+n(s_0-x)}^{x} (x-s)^{(n-1)q} (s-x+n(x-s_0))^q \, ds \\
&= (x-s_0)^{qn+1} n^{qn+1} B(q+1, (n-1)q+1),
\end{aligned}
$$

and we already had the third integral before (2.74). Finally, by summing I_1, I_2, and I_3, the statement for $1 < p \le \infty$ follows. For $p = 1$,we have

$$
\begin{aligned}
\left\| K_n^1(x,s) \right\|_\infty = \sup_{s\in[a,b]} \left| K_n^1(x,s) \right| = \max \Bigg\{ &\max_{s\in[a,c]} \left| -\frac{(x-s)^n}{n(b-a)} + \frac{x-a}{b-a}(x-s)^{n-1} \right|, \\
&\max_{s\in[c,d]} \left| -\frac{(x-s)^n}{n(b-a)} + \frac{(x-s)^n}{n(d-c)} + \left(\frac{x-a}{b-a} - \frac{x-c}{d-c} \right)(x-s)^{n-1} \right|, \\
&\max_{s\in[d,b]} \left| -\frac{(x-s)^n}{n(b-a)} + \frac{x-b}{b-a}(x-s)^{n-1} \right| \Bigg\}.
\end{aligned}
$$

By an elementary calculation we get

$$
\max_{s\in[a,c]} \left| -\frac{(x-s)^n}{n(b-a)} + \frac{x-a}{b-a}(x-s)^{n-1} \right| = \frac{(n-1)(x-a)^n}{n(b-a)},
$$

$$
\max_{s\in[d,b]} \left| -\frac{(x-s)^n}{n(b-a)} + \frac{x-b}{b-a}(x-s)^{n-1} \right| = \frac{(n-1)(b-x)^n}{n(b-a)},
$$

$$
\begin{aligned}
&\max_{s\in[c,d]} \left| -\frac{(x-s)^n}{n(b-a)} + \frac{(x-s)^n}{n(d-c)} + \left(\frac{x-a}{b-a} - \frac{x-c}{d-c} \right)(x-s)^{n-1} \right| \\
&= \frac{c-a+b-d}{n(b-a)(d-c)} \max_{s\in[c,d]} \left| (x-s)^{n-1} (x-s+n(s_0-x)) \right|.
\end{aligned}
$$

Again, for calculating $\max_{s\in[c,d]} \left| (x-s)^{n-1} (x-s+n(s_0-x)) \right|$ we have four possible cases:

1. If $x+n(s_0-x)<c$, (then also $x>s_0$) we have

$$\max_{s\in[c,x]}(x-s)^{n-1}(s-x+n(x-s_0))=(x-M)^{n-1}(M-x+n(x-s_0)),$$

where $M=\max\{c,x+(n-1)(s_0-x)\}$

$$\max_{s\in[x,d]}(s-x)^{n-1}(s-x+n(x-s_0))=(d-x)^{n-1}(d-x+n(x-s_0)).$$

2. If $x+n(s_0-x)>d$, (then also $s_0>x$)

$$\max_{s\in[c,x]}(x-s)^{n-1}(x-s+n(s_0-x))=(x-c)^{n-1}(x-c+n(s_0-x)),$$

$$\max_{s\in[x,d]}(s-x)^{n-1}(x-s+n(s_0-x))=(m-x)^{n-1}(x-m+n(s_0-x)).$$

where $m=\min\{b,x+(n-1)(s_0-x)\}$

3. If $c\le x+n(s_0-x)\le d$ and $x\le x+n(s_0-x)$, so $x<s_0$ then

$$\max_{s\in[c,x]}(x-s)^{n-1}(x-s+n(s_0-x))=(x-c)^{n-1}(x-c+n(s_0-x)),$$

$$\max_{s\in[x,x+n(s_0-x)]}(s-x)^{n-1}(x-s+n(s_0-x))=(s_0-x)^{n}(n-1)^{n-1},$$

$$\max_{s\in[x+n(s_0-x),d]}(s-x)^{n-1}(s-x+n(x-s_0))=(d-x)^{n-1}(d-x+n(x-s_0)).$$

4. If $c\le x+n(s_0-x)\le d$ and $x+n(s_0-x)<x$, so $s_0<x$ then

$$\max_{s\in[c,x+n(s_0-x)]}(x-s)^{n-1}(x-s+n(s_0-x))=(x-c)^{n-1}(x-c+n(s_0-x)),$$

$$\max_{s\in[x+n(s_0-x),x]}(x-s)^{n-1}(s-x+n(x-s_0))=(x-s_0)^{n}(n-1)^{n-1},$$

$$\max_{s\in[x,d]}(s-x)^{n-1}(s-x+n(x-s_0))=(d-x)^{n-1}(d-x+n(x-s_0)).$$

Thus, the proof is done. □

Remark 2.27 If we put $c=d=x$ as a limit case, the inequalities from Corollary 2.28 reduce to Ostrowski type inequalities from Corollary 2.27.

Case $[a,b]\cap[c,d]=[c,b]$

Corollary 2.29 *Assume (p,q) is a pair of conjugate exponents. Let $f^{(n)}\in L_p[a,d]$ for some $n\ge 2$, $[a,b]\cap[c,d]=[c,b]$, $x\in[c,b]$. If $c-a+b-d=0$ then for $1<p\le\infty$*

$$\left|D^1\left(\frac{1}{b-a},\frac{1}{d-c};x\right)\right|\le\frac{1}{(n-1)!}\left[\frac{(x-a)^{qn+1}n^{(n-1)q+1}}{(b-a)^q}\right.$$

$$\cdot\left(B_{1-\frac{x-c}{n(x-a)}}\left(q+1,(n-1)q+1\right)-B_{1-\frac{1}{n}}\left(q+1,(n-1)q+1\right)\right)$$
$$+\frac{c-a}{b-a}\left(\frac{(x-c)^{(n-1)q+1}+(b-x)^{(n-1)q+1}}{(n-1)q+1}\right)+\frac{(d-x)^{nq+1}n^{(n-1)q+1}}{(d-c)^{q}}$$
$$\cdot\left(B_{1-\frac{b-x}{n(d-x)}}\left(q+1,(n-1)q+1\right)-B_{1-\frac{1}{n}}\left(q+1,(n-1)q+1\right)\right)\Bigg]^{\frac{1}{q}}\left\|f^{(n)}\right\|_{p}$$

and if $c-a+b-d\neq 0$*, let* $s_0=\frac{bc-ad}{c-a+b-d}$*. Then for* $1<p\leq\infty$ *and* $x+n(s_0-x)\in[c,b]$ *we have*

$$\left|D^{1}\left(\frac{1}{b-a},\frac{1}{d-c};x\right)\right|\leq\frac{n^{n-1+\frac{1}{q}}}{(n-1)!}$$
$$\cdot\left[\frac{(x-a)^{qn+1}}{(b-a)^{q}}\left(B_{1-\frac{x-c}{n(x-a)}}\left(q+1,(n-1)q+1\right)-B_{1-\frac{1}{n}}\left(q+1,(n-1)q+1\right)\right)\right.$$
$$+\frac{|c-a+b-d|^{q}}{(b-a)^{q}(d-c)^{q}}\left|x-s_o\right|^{nq+1}\left(\Psi_{r_1}\left(q+1,(n-1)q+1\right)\right.$$
$$+B\left(q+1,(n-1)q+1\right)+\Psi_{r_2}\left((n-1)q+1,q+1\right)\Big)+\frac{(d-x)^{nq+1}}{(d-c)^{q}}$$
$$\cdot\left(B_{1-\frac{b-x}{n(d-x)}}\left(q+1,(n-1)q+1\right)-B_{1-\frac{1}{n}}\left(q+1,(n-1)q+1\right)\right)\Bigg]^{\frac{1}{q}}\left\|f^{(n)}\right\|_{p}$$

where for $s_0>b$*,* $r_1=\frac{b-x}{n(s_0-x)}-1$*,* $r_2=\frac{x-c}{n(s_0-x)}$*, while for* $s_0<c$*,* $r_1=\frac{x-c}{n(x-s_o)}-1$*,* $r_2=\frac{b-x}{n(x-s_0)}$*.*
If $x+n(s_0-x)\notin[c,b]$ *we have*

$$\left|D^{1}\left(\frac{1}{b-a},\frac{1}{d-c};x\right)\right|\leq\frac{n^{n-1+\frac{1}{q}}}{(n-1)!}$$
$$\cdot\left[\frac{(x-a)^{qn+1}}{(b-a)^{q}}\left(B_{1-\frac{x-c}{n(x-a)}}\left(q+1,(n-1)q+1\right)-B_{1-\frac{1}{n}}\left(q+1,(n-1)q+1\right)\right)\right.$$
$$+\frac{|c-a+b-d|^{q}}{(b-a)^{q}(d-c)^{q}}\left|x-s_o\right|^{nq+1}\left(\Psi_{r_3}\left((n-1)q+1,q+1\right)\right.$$
$$+B\left(q+1,(n-1)q+1\right)-B_{r_4}\left(q+1,(n-1)q+1\right)\Big)+\frac{(d-x)^{nq+1}}{(d-c)^{q}}$$
$$\cdot\left(B_{1-\frac{b-x}{n(d-x)}}\left(q+1,(n-1)q+1\right)-B_{1-\frac{1}{n}}\left(q+1,(n-1)q+1\right)\right)\Bigg]^{\frac{1}{q}}\left\|f^{(n)}\right\|_{p}$$

where for $x+n(s_0-x)>b$*,* $r_3=\frac{x-c}{n(s_o-x)}$*,* $r_4=1-\frac{b-x}{n(s_0-x)}$*, while for* $x+n(s_0-x)<c$*,* $r_3=\frac{b-x}{n(x-s_0)}$*,* $r_4=1-\frac{x-c}{n(x-s_0)}$*.*

For $p=1$ and $c-a+b-d=0$ we have

$$\left|D^1\left(\frac{1}{b-a},\frac{1}{d-c};x\right)\right| \leq \frac{1}{n!}\max\left\{(n-1)\frac{(x-a)^n}{(b-a)},(n-1)\frac{(d-x)^n}{(d-c)},\right.$$
$$\left.\frac{c-a}{b-a}(x-c)^{n-1},\frac{c-a}{b-a}(b-x)^{n-1}\right\}\left\|f^{(n)}\right\|_1$$

and for $p=1$ and $c-a+b-d\neq 0$

$$\left|D^1\left(\frac{1}{b-a},\frac{1}{d-c};x\right)\right| \leq \frac{1}{n!}\max\left\{(n-1)\frac{(x-a)^n}{(b-a)},(n-1)\frac{(d-x)^n}{(d-c)},\right.$$
$$\frac{|c-a+b-d|}{(b-a)(d-c)}(x-c)^{n-1}|x-c+n(s_0-x)|,\frac{|c-a+b-d|}{(b-a)(d-c)}|s_0-x|^n(n-1)^{n-1},$$
$$\left.\frac{|c-a+b-d|}{(b-a)(d-c)}(d-x)^{n-1}|d-x+n(x-s_0)|\right\}\left\|f^{(n)}\right\|_1.$$

Proof. The proof is similar to the proof of the Theorem 2.28. □

Remark 2.28 If we put $b=c=x$ as a limit case, the inequalities from Corollary 2.29 reduce to

$$\left|\frac{1}{x-a}\int_a^x f(t)\,\mathrm{d}t-\frac{1}{d-x}\int_x^d f(t)\,\mathrm{d}t+t_n^{[a,x]}(x)-t_n^{[x,d]}(x)\right|$$
$$\leq \frac{1}{(n-1)!}\left[n^{(n-1)q+1}\left((x-a)^{(n-1)q+1}+(d-x)^{(n-1)q+1}\right)\right.$$
$$\left.\cdot\left(B(q+1,(n-1)q+1)-B_{1-\frac{1}{n}}(q+1,(n-1)q+1)\right)\right]^{\frac{1}{q}}\left\|f^{(n)}\right\|_p$$

for $1<p\leq\infty$ and for $p=1$

$$\left|\frac{1}{x-a}\int_a^x f(t)\,\mathrm{d}t-\frac{1}{d-x}\int_x^d f(t)\,\mathrm{d}t+t_n^{[a,x]}(x)-t_n^{[x,d]}(x)\right|$$
$$\leq \frac{1}{n!}\max\left\{(n-1)(x-a)^{n-1},(n-1)(d-x)^{n-1}\right\}\left\|f^{(n)}\right\|_1.$$

Remark 2.29 If we suppose $b=d$ in both cases $[c,d]\subseteq[a,b]$ and $[a,b]\cap[c,d]=[c,b]$ the analogues results in Corollary 2.28 and Corollary 2.29 coincides.

2.3.2 Extension via Taylor's formula (centered at the point x) with applications

Here, an extension of the weighted Montgomery identity is given, by using Taylor's series centered at the point $x\in[a,b]$. Results from this subsection are published in [12].

Theorem 2.32 *Let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$. If $w : [a,b] \to [0,\infty\rangle$ is some probability density function. Then the following identity hold*

$$f(x) = \int_a^b w(t) f(t)\,\mathrm{d}t - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(x)}{(i+1)!} \int_a^b w(s)(s-x)^{i+1}\,\mathrm{d}s$$
$$+ \frac{1}{(n-1)!} \int_a^b T_{w,n}^2(x,s) f^{(n)}(s)\,\mathrm{d}s \tag{2.76}$$

where

$$T_{w,n}^2(x,s) = \begin{cases} \int_a^s w(u)(u-s)^{n-1}\,\mathrm{d}u, & a \leq s \leq x, \\ -\int_s^b w(u)(u-s)^{n-1}\,\mathrm{d}u, & x < s \leq b. \end{cases}$$

Proof. If we apply Taylor's formula with $f'(t)$, $(n \geq 2)$ we have

$$f'(t) = \sum_{i=0}^{n-2} \frac{f^{(i+1)}(x)}{i!}(t-x)^i + \int_x^t f^{(n)}(s) \frac{(t-s)^{n-2}}{(n-2)!}\,\mathrm{d}s.$$

By putting these two formulae in the weighted Montgomery identity (2.5) we obtain

$$f(x) = \int_a^b w(t) f(t)\,\mathrm{d}t + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(x)}{i!} \int_a^b P_w(x,t)(t-x)^i\,\mathrm{d}t$$
$$+ \int_a^b P_w(x,t) \left(\int_x^t f^{(n)}(s) \frac{(t-s)^{n-2}}{(n-2)!}\,\mathrm{d}s \right) \mathrm{d}t.$$

Now, the interchange the order of integration leads us to

$$\int_a^x (t-x)^i W(t)\,\mathrm{d}t = \int_a^x (t-x)^i \left(\int_a^t w(s)\,\mathrm{d}s \right) \mathrm{d}t$$
$$= \int_a^x w(s) \left(\int_s^x (t-x)^i\,\mathrm{d}t \right) \mathrm{d}s = -\frac{1}{i+1} \int_a^x w(s)(s-x)^{i+1}\,\mathrm{d}s$$

and similarly

$$\int_x^b (t-x)^i (W(t) - 1) = -\int_x^b (t-x)^i \left(\int_t^b w(s)\,\mathrm{d}s \right) \mathrm{d}t$$
$$= -\int_x^b w(s) \left(\int_x^s (t-x)^i\,\mathrm{d}t \right) \mathrm{d}s = -\frac{1}{i+1} \int_x^b w(s)(s-x)^{i+1}\,\mathrm{d}s.$$

Further,

$$\int_a^x W(t) \left(\int_x^t f^{(n)}(s)(t-s)^{n-2}\,\mathrm{d}s \right) \mathrm{d}t = -\int_a^x f^{(n)}(s) \left(\int_a^s W(t)(t-s)^{n-2}\,\mathrm{d}t \right) \mathrm{d}s$$

and

$$\int_a^s W(t)(t-s)^{n-2}\,\mathrm{d}t = \int_a^s \left(\int_a^t w(u)\,\mathrm{d}u\right)(t-s)^{n-2}\,\mathrm{d}t$$
$$= \int_a^s w(u)\left(\int_u^s (t-s)^{n-2}\,\mathrm{d}t\right)\mathrm{d}u = -\int_a^s w(u)\,\frac{(u-s)^{n-1}}{n-1}\,\mathrm{d}u.$$

Similarly we have

$$\int_x^b (W(t)-1)\left(\int_x^t f^{(n)}(s)(t-s)^{n-2}\,\mathrm{d}s\right)\mathrm{d}t$$
$$= -\int_x^b f^{(n)}(s)\left(\int_s^b (1-W(t))(t-s)^{n-2}\,\mathrm{d}t\right)\mathrm{d}s$$

and

$$\int_s^b (1-W(t))(t-s)^{n-2}\,\mathrm{d}t = \int_s^b \left(\int_t^b w(u)\,\mathrm{d}u\right)(t-s)^{n-2}\,\mathrm{d}t$$
$$= \int_s^b w(u)\left(\int_s^u (t-s)^{n-2}\,\mathrm{d}t\right)\mathrm{d}u = \int_s^b w(u)\,\frac{(u-s)^{n-1}}{n-1}\,\mathrm{d}u.$$

So the reminder in the weighted Taylor formula is

$$\frac{1}{(n-1)!}\left[\int_a^x f^{(n)}(s)\left(\int_a^s w(u)(u-s)^{n-1}\,\mathrm{d}u\right)\mathrm{d}s\right.$$
$$\left.+\int_x^b f^{(n)}(s)\left(-\int_s^b w(u)(u-s)^{n-1}\,\mathrm{d}u\right)\mathrm{d}s\right].$$

□

Remark 2.30 *In the special case, for $n=1$, (2.76) reduces to weighted Montgomery identity (2.5). Also, if we take $w(t)=\frac{1}{b-a}$, $t\in[a,b]$ the equality (2.76) reduces to*

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \sum_{i=0}^{n-2} f^{(i+1)}(x)\,\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!\,(b-a)}$$
$$+\frac{1}{(n-1)!}\int_a^b T_n^2(x,s)\,f^{(n)}(s)\,\mathrm{d}s \tag{2.77}$$

where

$$T_n^2(x,s) = \begin{cases} \frac{-1}{n(b-a)}(a-s)^n, & a\le s\le x,\\[2ex] \frac{-1}{n(b-a)}(b-s)^n, & x< s\le b.\end{cases}$$

Identity (2.76) coincides with the identity from the Theorem 1.7 obtained in [85].

The Ostrowski type inequalities

The results in this subsection generalize the results from [86] and [87].

Theorem 2.33 *Suppose that all the assumptions of Theorem 2.32 hold. Additionally assume that (p,q) is a pair of conjugate exponents, $1 \leq p,q \leq \infty$ and $f^{(n)} \in L_p[a,b]$ for some $n \geq 2$. Then we have*

$$\left| f(x) - \int_a^b w(t) f(t)\,\mathrm{d}t + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(x)}{(i+1)!} \int_a^b w(s)(s-x)^{i+1}\,\mathrm{d}s \right| \leq \frac{1}{(n-1)!} \left\| T_{w,n}^2(x,\cdot) \right\|_q \left\| f^{(n)} \right\|_p . \tag{2.78}$$

This inequality is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof. The proof is similar to the proof of the Theorem 3.2. □

Corollary 2.30 *Let $f : I \to \mathbb{R}$ be such that $I \subset \mathbb{R}$ is a open interval, $a,b \in I$, $a < b$ and (p,q) a pair of conjugate exponents, $1 < p \leq \infty$ and $f^{(n)} \in L_p[a,b]$ for some $n \geq 2$. Then we have*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t + \sum_{i=0}^{n-2} f^{(i+1)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!(b-a)} \right| \leq \frac{1}{n!(b-a)} \left(\frac{(x-a)^{qn+1} + (b-x)^{qn+1}}{(nq+1)} \right)^{\frac{1}{q}} \left\| f^{(n)} \right\|_p$$

and this inequality is sharp. For $p = 1$ we have

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t + \sum_{i=0}^{n-2} f^{(i+1)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!(b-a)} \right| \leq \frac{1}{(b-a)n!} \max\left\{ (x-a)^n, (b-x)^n \right\} \left\| f^{(n)} \right\|_1$$

and this inequality is the best possible.

Proof. We apply the inequality (2.78) with $w(t) = \frac{1}{b-a}$, $t \in [a,b]$ and use (2.77)

$$\int_a^b \left| T_{w,n}^2(x,s) \right|^q \mathrm{d}s = \int_a^x \left| \frac{-(a-s)^n}{n(b-a)} \right|^q \mathrm{d}s + \int_x^b \left| \frac{-(b-s)^n}{n(b-a)} \right|^q \mathrm{d}s = \frac{(x-a)^{qn+1} + (b-x)^{qn+1}}{n^q (b-a)^q (nq+1)}$$

and the first inequality follows. For $p = 1$

$$\sup_{s \in [a,b]} \left| T_{w,n}^2(x,s) \right| = \max\left\{ \sup_{s \in [a,x]} \left| \frac{-(a-s)^n}{n(b-a)} \right|, \sup_{s \in [x,b]} \left| \frac{-(b-s)^n}{n(b-a)} \right| \right\}.$$

By an elementary calculation we get

$$\sup_{s\in[a,x]}\left|\frac{-(a-s)^n}{n(b-a)}\right| = \frac{(x-a)^n}{n(b-a)}, \qquad \sup_{s\in[x,b]}\left|\frac{-(b-s)^n}{n(b-a)}\right| = \frac{(b-x)^n}{n(b-a)}$$

and the proof is completed. □

Remark 2.31 If we apply (2.78) with $x=\frac{a+b}{2}$ we get the generalized midpoint inequality

$$\left|f\left(\frac{a+b}{2}\right)-\int_a^b w(t)f(t)\,\mathrm{d}t+\sum_{i=0}^{n-2}\frac{f^{(i+1)}\left(\frac{a+b}{2}\right)}{(i+1)!}\int_a^b w(s)\left(s-\frac{a+b}{2}\right)^{i+1}\mathrm{d}s\right|$$
$$\leq \frac{1}{(n-1)!}\left\|T^2_{w,n}\left(\frac{a+b}{2},\cdot\right)\right\|_q\left\|f^{(n)}\right\|_p.$$

If we additionally assume that $w(t)$ is symmetric on $[a,b]$ i.e. $w(t)=w(b-a-t)$ for every $t\in[a,b]$ this inequality reduces to

$$\left|f\left(\frac{a+b}{2}\right)-\int_a^b w(t)f(t)\,\mathrm{d}t+\sum_{i=0}^{\left\lfloor\frac{n}{2}-1\right\rfloor}\frac{f^{(2i)}\left(\frac{a+b}{2}\right)}{(2i)!}2\int_a^{\frac{a+b}{2}} w(s)\left(s-\frac{a+b}{2}\right)^{2i}\mathrm{d}s\right|$$
$$\leq \frac{1}{(n-1)!}\left\|T^2_{w,n}\left(\frac{a+b}{2},\cdot\right)\right\|_q\left\|f^{(n)}\right\|_p.$$

For the generalized trapezoid inequality we apply equality (2.76) first with $x=a$, then with $x=b$ then add them up and divide by 2. After applying the Hölder inequality we get

$$\left|\frac{f(a)+f(b)}{2}-\int_a^b w(t)f(t)\,\mathrm{d}t\right.$$
$$\left.+\sum_{i=0}^{n-2}\frac{f^{(i+1)}(a)}{2(i+1)!}\int_a^b w(s)(s-a)^{i+1}\mathrm{d}s+\frac{f^{(i+1)}(b)}{2(i+1)!}\int_a^b w(s)(s-b)^{i+1}\mathrm{d}s\right|$$
$$\leq \frac{1}{2(n-1)!}\left\|T^2_{w,n}(a,\cdot)+T^2_{w,n}(b,\cdot)\right\|_q\left\|f^{(n)}\right\|_p$$

and

$$T^2_{w,n}(a,s)+T^2_{w,n}(b,s)=\int_a^s w(u)(u-s)^{n-1}\mathrm{d}u-\int_s^b w(u)(u-s)^{n-1}\mathrm{d}u$$
$$=-\int_a^b w(u)|u-s|^{n-1}\mathrm{d}u$$

Again, if we additionally assume that $w(t)$ is symmetric on $[a,b]$ this inequality reduces to

$$\left|\frac{f(a)+f(b)}{2}-\int_a^b w(t)f(t)\,\mathrm{d}t\right.$$
$$\left.+\sum_{i=0}^{n-2}\frac{f^{(i+1)}(a)+(-1)^{i+1}f^{(i+1)}(b)}{2(i+1)!}\int_a^b w(s)(s-a)^{i+1}\mathrm{d}s\right|$$
$$\leq \frac{1}{2(n-1)!}\left\|T^2_{w,n}(a,\cdot)+T^2_{w,n}(b,\cdot)\right\|_q\left\|f^{(n)}\right\|_p.$$

The estimation of the difference of the two weighted integral means

In this subsection we generalize the results from [24] and [83].We denote

$$t_{w,n}^{[a,b],2}(x)=\sum_{i=0}^{n-2}\frac{f^{(i+1)}(x)}{(i+1)!}\int_a^b w(s)(s-x)^{i+1}\,\mathrm{d}s$$

and

$$D^2(w,u;x)=\int_a^b w(t)f(t)\,\mathrm{d}t-\int_c^d u(t)f(t)\,\mathrm{d}t+t_{w.n}^{[a,b],2}(x)-t_{u,n}^{[c,d],2}(x),$$

for function $f:[a,b]\to\mathbb{R}$ such that $f^{(n-1)}$ is absolutely continuous function for some $n\geq 2$.

Theorem 2.34 *Let $f:[a,b]\cup[c,d]\to\mathbb{R}$ be such that $f^{(n-1)}$ is an absolutely continuous function for some $n\geq 2$, $w:[a,b]\to[0,\infty\rangle$ and $u:[c,d]\to[0,\infty\rangle$ some nonnegative normalized weight functions, $W(t)=\int_a^t w(x)\,\mathrm{d}x$ for $t\in[a,b]$, $W(t)=0$ for $t<a$ and $W(t)=1$ for $t>b$, $U(t)=\int_c^t u(x)\,\mathrm{d}x$ for $t\in[c,d]$, $U(t)=0$ for $t<c$ and $U(t)=1$ for $t>d$. Then if $[a,b]\cap[c,d]\neq\emptyset$ and $x\in[a,b]\cap[c,d]$, we have*

$$D^2(w,u;x)=\int_{\min\{a,c\}}^{\max\{b,d\}}K_n^2(x,s)f^{(n)}(s)\,\mathrm{d}s \tag{2.79}$$

where in case $[c,d]\subset[a,b]$

$$K_n^2(x,s)=\begin{cases}\frac{-1}{(n-1)!}\left(\int_a^s w(t)(t-s)^{n-1}\,\mathrm{d}t\right), & s\in[a,c\rangle,\\ \frac{-1}{(n-1)!}\left(\int_a^s w(t)(t-s)^{n-1}\,\mathrm{d}t-\int_c^s u(t)(t-s)^{n-1}\,\mathrm{d}t\right), & s\in[c,x],\\ \frac{1}{(n-1)!}\left(\int_s^b w(t)(t-s)^{n-1}\,\mathrm{d}t-\int_s^d u(t)(t-s)^{n-1}\,\mathrm{d}t\right), & s\in\langle x,d],\\ \frac{1}{(n-1)!}\left(\int_s^b w(t)(t-s)^{n-1}\,\mathrm{d}t\right), & s\in\langle d,b],\end{cases}$$

and in case $[a,b]\cap[c,d]=[c,b]$

$$K_n^2(x,s)=\begin{cases}\frac{-1}{(n-1)!}\left(\int_a^s w(t)(t-s)^{n-1}\,\mathrm{d}t\right), & s\in[a,c\rangle,\\ \frac{-1}{(n-1)!}\left(\int_a^s w(t)(t-s)^{n-1}\,\mathrm{d}t-\int_c^s u(t)(t-s)^{n-1}\,\mathrm{d}t\right), & s\in[c,x],\\ \frac{1}{(n-1)!}\left(\int_s^b w(t)(t-s)^{n-1}\,\mathrm{d}t-\int_s^d u(t)(t-s)^{n-1}\,\mathrm{d}t\right), & s\in\langle x,b],\\ \frac{-1}{(n-1)!}\left(\int_s^d u(t)(t-s)^{n-1}\,\mathrm{d}t\right) & s\in\langle b,d],\end{cases}$$

Proof. We subtract identities (2.76) for interval $[a,b]$ and $[c,d]$, to get the formula (2.79). □

Theorem 2.35 *Assume (p,q) is a pair of conjugate exponents, $1\leq p,q\leq\infty$. Let $f^{(n)}\in L_p[a,b]$ for some $n\geq 2$. Then we have*

$$\left|D^2(w,u;x)\right|\leq\left(\int_{\min\{a,c\}}^{\max\{b,d\}}\left|K_n^2(x,s)\right|^q\,\mathrm{d}s\right)^{\frac{1}{q}}\left\|f^{(n)}\right\|_p$$

for every $x \in [a,b] \cap [c,d]$. *This inequality is sharp for* $1 < p \le \infty$ *and the best possible for* $p = 1$.

Proof. The proof is similar to the proof of the Theorem 2.33. □

Case $[c,d] \subset [a,b]$

Here we denote

$$t_n^{[a,b],2}(x) = \sum_{i=0}^{n-2} f^{(i+1)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!\,(b-a)}$$

and

$$D^2\left(\frac{1}{b-a}, \frac{1}{d-c}; x\right) = \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{1}{d-c}\int_c^d f(t)\,\mathrm{d}t + t_n^{[a,b],2}(x) - t_n^{[c,d],2}(x).$$

Corollary 2.31 *Let* $f^{(n)} \in L_p[a,b]$ *for some* $n \ge 2$, $[c,d] \subset [a,b]$, $x \in [c,d]$, $s_1 = a + \frac{c-a}{1-\sqrt[n]{\frac{d-c}{b-a}}}$, $s_2 = b + \frac{d-b}{1-\sqrt[n]{\frac{d-c}{b-a}}}$ *Then if* $s_1 \notin [c,x]$, *and* $s_2 \notin [x,d]$

$$\left|D^2\left(\frac{1}{b-a}, \frac{1}{d-c}; x\right)\right| \le \frac{1}{(n+1)!\,(b-a)}\left[(x-a)^{n+1} + (b-x)^{n+1} - \frac{b-a}{d-c}(x-c)^{n+1} - \frac{b-a}{d-c}(d-x)^{n+1}\right]\left\|f^{(n)}\right\|_\infty,$$

if $s_1 \notin [c,x]$, *and* $s_2 \in [x,d]$

$$\left|D^2\left(\frac{1}{b-a}, \frac{1}{d-c}; x\right)\right| \le \frac{1}{(n+1)!\,(b-a)}\left[(x-a)^{n+1} - (b-x)^{n+1} + 2(b-s_2)^{n+1} + \frac{b-a}{d-c}\left((d-x)^{n+1} - (x-c)^{n+1} - 2(d-s_2)^{n+1}\right)\right]\left\|f^{(n)}\right\|_\infty,$$

if $s_1 \in [c,x]$, *and* $s_2 \notin [x,d]$

$$\left|D^2\left(\frac{1}{b-a}, \frac{1}{d-c}; x\right)\right| \le \frac{1}{(n+1)!\,(b-a)}\left[-(x-a)^{n+1} + (b-x)^{n+1} + 2(s_1-a)^{n+1} + \frac{b-a}{d-c}\left((x-c)^{n+1} - (d-x)^{n+1} - 2(s_1-c)^{n+1}\right)\right]\left\|f^{(n)}\right\|_\infty,$$

if $s_1 \in [c,x]$, *and* $s_2 \in [x,d]$

$$\left|D^2\left(\frac{1}{b-a}, \frac{1}{d-c}; x\right)\right| \le \frac{1}{(n+1)!\,(b-a)}\left[2\left((s_1-a)^{n+1} + (b-s_2)^{n+1}\right) - (x-a)^{n+1} - (b-x)^{n+1} + \frac{b-a}{d-c}\left((x-c)^{n+1} + (d-x)^{n+1} - 2\left((s_1-c)^{n+1} + (d-s_2)^{n+1}\right)\right)\right]\left\|f^{(n)}\right\|_\infty,$$

and

$$\left|D^2\left(\frac{1}{b-a},\frac{1}{d-c};x\right)\right| \leq \left\|f^{(n)}\right\|_1$$
$$\cdot\frac{1}{n!}\max\left\{\frac{(c-a)^n}{b-a},\frac{(b-d)^n}{b-a},\left|\frac{(x-c)^n}{d-c}-\frac{(x-a)^n}{b-a}\right|,\left|\frac{(d-x)^n}{d-c}-\frac{(b-x)^n}{b-a}\right|\right\}.$$

Proof. We put $w(t)=\frac{1}{b-a}$, $t\in[a,b]$; $u(t)=\frac{1}{d-c}$, $t\in[c,d]$ and $q=1$ in the Theorem 2.35. Thus we have $t_n^{[a,b],2}(x)$ and $t_n^{[c,d],2}(x)$ instead of $t_{w,n}^{[a,b],2}(x)$ and $t_{u,n}^{[c,d],2}(x)$ and

$$\int_{\min\{a,c\}}^{\max\{b,d\}}\left|K_n^2(x,s)\right|\mathrm{d}s$$
$$=\frac{1}{(n-1)!}\left(\int_a^c\left|\frac{(a-s)^n}{n(b-a)}\right|\mathrm{d}s+\int_c^x\left|\frac{(a-s)^n}{n(b-a)}-\frac{(c-s)^n}{n(d-c)}\right|\mathrm{d}s\right.$$
$$\left.\int_x^d\left|\frac{(b-s)^n}{n(b-a)}-\frac{(d-s)^n}{n(d-c)}\right|\mathrm{d}s+\int_d^b\left|\frac{(b-s)^n}{n(b-a)}\right|\mathrm{d}s\right)$$

The first and the last integrals are

$$I_1=\int_a^c\left|\frac{(a-s)^n}{n(b-a)}\right|\mathrm{d}s=\frac{1}{n(b-a)}\int_a^c(s-a)^n\,\mathrm{d}s=\frac{(c-a)^{n+1}}{n(b-a)(n+1)},$$

$$I_4=\int_d^b\left|\frac{(b-s)^n}{n(b-a)}\right|\mathrm{d}s=\frac{1}{n(b-a)}\int_d^b(b-s)^n\,\mathrm{d}s=\frac{(b-d)^{n+1}}{n(b-a)(n+1)},$$

Now, we suppose n is odd. The second integral is

$$I_2=\int_c^x\left|\frac{(a-s)^n}{n(b-a)}-\frac{(c-s)^n}{n(d-c)}\right|\mathrm{d}s=\frac{1}{n(b-a)(d-c)}\int_c^x|f(s)|\,\mathrm{d}s,$$

where $f(s)=(d-c)(s-a)^n-(b-a)(s-c)^n$. Since $s_1=a+\frac{c-a}{1-\sqrt[n]{\frac{d-c}{b-a}}}$, $f(s_1)=0$ and $f'(s_1)\neq 0$, there are two possible cases:

1. If $s_1>x$, i.e. $s_1\notin[c,x]$ we have

$$\int_c^x|f(s)|\,\mathrm{d}s=\frac{1}{n+1}\left((d-c)\left((x-a)^{n+1}-(c-a)^{n+1}\right)-(b-a)(x-c)^{n+1}\right).$$

2. If $s_1<x$, i.e. $s_1\in[c,x]$ (since $\frac{d-c}{b-a}\leq 1$ so $s_1>c$)

$$\int_c^x|f(s)|\,\mathrm{d}s=\int_c^{s_1}f(s)\,\mathrm{d}s+\int_{s_1}^x(-f(s))\,\mathrm{d}s$$
$$=\frac{1}{n+1}\left((d-c)\left(2(s_1-a)^{n+1}-(c-a)^{n+1}-(x-a)^{n+1}\right)\right.$$
$$\left.+(b-a)\left((x-c)^{n+1}-2(s_1-c)^{n+1}\right)\right).$$

The third integral is

$$I_3 = \int_x^d \left| \frac{(b-s)^n}{n(b-a)} - \frac{(d-s)^n}{n(d-c)} \right| \mathrm{d}s = \frac{1}{n(b-a)(d-c)} \int_x^d |g(s)|\,\mathrm{d}s,$$

where $g(s) = (d-c)(b-s)^n - (b-a)(d-s)^n$. Since $s_2 = b - \frac{b-d}{1-\sqrt[n]{\frac{d-c}{b-a}}}$, $g(s_2) = 0$ and $g'(s_2) \neq 0$, again there are two possible cases:

1. If $s_2 < x$, i.e. $s_2 \notin [x,d]$ (since $s_2 < d$) we have

$$\int_x^d |g(s)|\,\mathrm{d}s = \frac{1}{n+1}\left((d-c)\left((b-x)^{n+1} - (b-d)^{n+1}\right) - (b-a)(d-x)^{n+1}\right).$$

2. If $s_2 > x$, i.e. $s_2 \in [x,d]$ (since $s_2 < d$)

$$\begin{aligned}
\int_x^d |g(s)|\,\mathrm{d}s &= \int_x^{s_2} (-g(s))\,\mathrm{d}s + \int_{s_2}^d g(s)\,\mathrm{d}s \\
&= \frac{1}{n+1}\left((d-c)\left(-(b-x)^{n+1} + 2(b-s_2)^{n+1} - (b-d)^{n+1}\right)\right. \\
&\quad \left. + (b-a)\left((d-x)^{n+1} - 2(d-s_2)^{n+1}\right)\right).
\end{aligned}$$

Now, we suppose n is even. The second integral is

$$I_2 = \frac{1}{n(b-a)(d-c)} \int_c^x |f(s)|\,\mathrm{d}s.$$

Since $s_1 = a + \frac{c-a}{1-\sqrt[n]{\frac{d-c}{b-a}}}$ and $s_3 = a + \frac{c-a}{1+\sqrt[n]{\frac{d-c}{b-a}}}$, we have $f(s_1) = f(s_3) = 0$ and $f'(s_1) \neq 0$, $f'(s_3) \neq 0$. By an elementary calculation we also have $s_3 < c < s_1$ so there are two possible cases:

1. If $s_3 < c < s_1 < x$

$$\begin{aligned}
\int_c^x |f(s)|\,\mathrm{d}s &= \int_c^{s_1} f(s)\,\mathrm{d}s + \int_{s_1}^x (-f(s))\,\mathrm{d}s \\
&= \frac{1}{n+1}\left((d-c)\left(2(s_1-a)^{n+1} - (c-a)^{n+1} - (x-a)^{n+1}\right)\right. \\
&\quad \left. + (b-a)\left((x-c)^{n+1} - 2(s_1-c)^{n+1}\right)\right).
\end{aligned}$$

2. If $s_3 < c < x < s_1$

$$\int_c^x |f(s)|\,\mathrm{d}s = \frac{1}{n+1}\left((d-c)\left((x-a)^{n+1} - (c-a)^{n+1}\right) - (b-a)(x-c)^{n+1}\right).$$

The third integral is

$$I_3 = \frac{1}{n(b-a)(d-c)} \int_x^d |g(s)|\,\mathrm{d}s.$$

Since $s_4 = b - \frac{b-d}{1+\sqrt[n]{\frac{d-c}{b-a}}}$ and $s_2 = b - \frac{b-d}{1-\sqrt[n]{\frac{d-c}{b-a}}}$, we have $g(s_2) = g(s_4) = 0$ and $g'(s_2) \neq 0$, $g'(s_4) \neq 0$. By an elementary calculation we also have $s_2 < s_4$ and $d < s_4$ so there are two possible cases:

1. If $s_2 < x < d < s_4$

$$\int_x^d |g(s)|\, ds = \frac{1}{n+1}\left((d-c)\left((b-x)^{n+1} - (b-d)^{n+1}\right) - (b-a)(d-x)^{n+1}\right).$$

2. If $x < s_2 < d < s_4$

$$\begin{aligned}\int_x^d |g(s)|\, ds &= \int_x^{s_2} (-g(s))\, ds + \int_{s_2}^d g(s)\, ds \\ &= \frac{1}{n+1}\left((d-c)\left(-(b-x)^{n+1} + 2(b-s_2)^{n+1} - (b-d)^{n+1}\right)\right. \\ &\quad \left. + (b-a)\left((d-x)^{n+1} - 2(d-s_2)^{n+1}\right)\right).\end{aligned}$$

Finally, by summing I_1, I_2, I_3 and I_4, the statement for $1 < p \leq \infty$ follows.

For $p = 1$, by putting $w(t) = \frac{1}{b-a}$, $t \in [a,b]$, and $u(t) = \frac{1}{d-c}$, $t \in [c,d]$ in the Theorem 2.35 again, we have

$$\begin{aligned}\left\|K_n^2(x,s)\right\|_\infty &= \frac{1}{(n-1)!}\max\left\{\max_{s\in[a,c]}\left|\frac{(a-s)^n}{n(b-a)}\right|, \max_{s\in[c,x]}\left|\frac{(a-s)^n}{n(b-a)} - \frac{(c-s)^n}{n(d-c)}\right|\right. \\ &\quad \left. \max_{s\in[x,d]}\left|\frac{(b-s)^n}{n(b-a)} - \frac{(d-s)^n}{n(d-c)}\right|, \max_{s\in[d,b]}\left|\frac{(b-s)^n}{n(b-a)}\right|\right\}.\end{aligned}$$

By an elementary calculation we get

$$\max_{s\in[a,c]}\left|\frac{(a-s)^n}{n(b-a)}\right| = \frac{(c-a)^n}{n(b-a)}, \quad \max_{s\in[d,b]}\left|\frac{(b-s)^n}{n(b-a)}\right| = \frac{(b-d)^n}{n(b-a)}$$

and in both cases $s_1 \in [c,x]$ and $s_1 \notin [c,x]$

$$\max_{s\in[c,x]}\left|\frac{(a-s)^n}{n(b-a)} - \frac{(c-s)^n}{n(d-c)}\right| = \max\left\{\frac{(c-a)^n}{n(b-a)}, \left|\frac{(x-a)^n}{n(b-a)} - \frac{(x-c)^n}{n(d-c)}\right|\right\}$$

and similarly for $s_2 \in [x,d]$ and $s_2 \notin [x,d]$

$$\max_{s\in[x,d]}\left|\frac{(b-s)^n}{n(b-a)} - \frac{(d-s)^n}{n(d-c)}\right| = \max\left\{\frac{(b-d)^n}{n(b-a)}, \left|\frac{(b-x)^n}{n(b-a)} - \frac{(d-x)^n}{n(d-c)}\right|\right\}.$$

Thus, the proof is done. □

Remark 2.32 If we put $c = d = x$ as a limit case, the inequalities from Corollary 2.31 reduce to Ostrowski type inequalities from Corollary 2.30.

Case $[a,b]\cap[c,d]=[c,b]$

Corollary 2.32 *Let $f^{(n)}\in L_p[a,d]$ for some $n\geq 2$, $[a,b]\cap[c,d]\subset[c,b]$, $x\in[c,b]$, $s_1=a+\frac{c-a}{1-\sqrt[n]{\frac{d-c}{b-a}}}$, $s_2=b+\frac{d-b}{1-\sqrt[n]{\frac{d-c}{b-a}}}$ Then if $s_1\notin[c,x]$, and $s_2\notin[x,b]$*

$$\left|D^2\left(\frac{1}{b-a},\frac{1}{d-c};x\right)\right| \leq \frac{1}{(n+1)!}\left[\frac{(x-a)^{n+1}-(b-x)^{n+1}}{b-a}+\frac{(d-x)^{n+1}-(x-c)^{n+1}}{d-c}\right]\left\|f^{(n)}\right\|_\infty$$

if $s_1\notin[c,x]$, and $s_2\in[x,b]$

$$\left|D^2\left(\frac{1}{b-a},\frac{1}{d-c};x\right)\right|\leq\frac{1}{(n+1)!}\left[\frac{(x-a)^{n+1}+(b-x)^{n+1}-2(b-s_2)^{n+1}}{(b-a)}\right.$$
$$\left.+\frac{-(d-x)^{n+1}-(x-c)^{n+1}+2(d-s_2)^{n+1}}{d-c}\right]\left\|f^{(n)}\right\|_\infty$$

if $s_1\in[c,x]$, and $s_2\notin[x,b]$

$$\left|D^2\left(\frac{1}{b-a},\frac{1}{d-c};x\right)\right|\leq\frac{1}{(n+1)!}\left[\frac{2(s_1-a)^{n+1}-(x-a)^{n+1}-(b-x)^{n+1}}{b-a}\right.$$
$$\left.+\frac{(x-c)^{n+1}+(d-x)^{n+1}-2(s_1-c)^{n+1}}{d-c}\right]\left\|f^{(n)}\right\|_\infty$$

if $s_1\in[c,x]$, and $s_2\in[x,b]$

$$\left|D^2\left(\frac{1}{b-a},\frac{1}{d-c};x\right)\right| \leq\frac{1}{(n+1)!}\left[\frac{(b-x)^{n+1}-(x-a)^{n+1}+2(s_1-a)^{n+1}-2(b-s_2)^{n+1}}{b-a}\right.$$
$$\left.+\frac{(x-c)^{n+1}-(d-x)^{n+1}+2(d-s_2)^{n+1}-2(s_1-c)^{n+1}}{d-c}\right]\left\|f^{(n)}\right\|_\infty$$

and

$$\left|D^2\left(\frac{1}{b-a},\frac{1}{d-c};x\right)\right|\leq\frac{1}{n!}\max\left\{\frac{(c-a)^n}{b-a},\frac{(d-b)^n}{d-c},\right.$$
$$\left.\left|\frac{(x-a)^n}{b-a}-\frac{(x-c)^n}{d-c}\right|,\left|\frac{(b-x)^n}{b-a}-\frac{(d-x)^n}{d-c}\right|\right\}\left\|f^{(n)}\right\|_1.$$

Proof. The proof is similar to the proof of the Theorem 2.31. □

Remark 2.33 If we put $b = c = x$ as a limit case, the inequalities from Corollary 2.32 reduce to

$$\begin{aligned}&\left|\frac{1}{x-a}\int_a^x f(t)\,\mathrm{d}t - \frac{1}{d-x}\int_x^d f(t)\,\mathrm{d}t - t_n^{[a,x]}(x) + t_n^{[x,d]}(x)\right|\\&\le \frac{1}{(n+1)!}\left[(x-a)^n + (d-x)^n\right]\left\|f^{(n)}\right\|_\infty\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{1}{x-a}\int_a^x f(t)\,\mathrm{d}t - \frac{1}{d-x}\int_x^d f(t)\,\mathrm{d}t - t_n^{[a,x]}(x) + t_n^{[x,d]}(x)\right|\\&\le \frac{1}{n!}\max\left\{(x-a)^{n-1}, (d-x)^{n-1}\right\}\left\|f^{(n)}\right\|_1.\end{aligned}$$

Remark 2.34 If we suppose $b = d$ in both cases $[c,d] \subset [a,b]$ and $[a,b] \cap [c,d] = [c,b]$ the analogues results in Corollary 2.31 and Corollary 2.32 coincides.

2.3.3 Extensions with applications for α-L-Hölder type functions

In the paper [21] G. A. Anastassiou proved the following equality

$$\begin{aligned}&g(y) - g(x) - \sum_{j=1}^{n}\frac{g^{(j)}(x)}{j!}(y-x)^j\\&= \frac{1}{(n-1)!}\int_x^y \left(g^{(n)}(t) - g^{(n)}(x)\right)(y-t)^{n-1}\,\mathrm{d}t \qquad (2.80)\end{aligned}$$

where $g : I \to \mathbb{R}$ is such that $g^{(n)}$ is exists for all $t \in [a,b]$, for some $n \in \mathbb{N}$ and $I \subset \mathbb{R}$ is an open interval, $a, b \in I$, $a < b$ and $x, y \in [a,b]$.

In this subsection we use the formula (2.80) to obtain two extensions of weighted Montgomery identity for α-L-Hölder type functions and further to obtain some new Ostrowski type inequalities, as well as some generalizations of the estimations of the difference of two weighted integral means. Results from this subsection are published in [11].

Two extensions of Montgomery identity

Theorem 2.36 *Let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \ge 2$, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$ and $w : [a,b] \to [0,\infty\rangle$ some probability density function. Then the following identity hold*

$$\begin{aligned}f(x) &= \int_a^b w(t) f(t)\,\mathrm{d}t + \sum_{j=0}^{n-1}\frac{f^{(j+1)}(a)}{j!}\int_a^x W(t)(t-a)^j\,\mathrm{d}t\\&+\sum_{j=0}^{n-1}\frac{f^{(j+1)}(b)}{j!}\int_x^b (W(t)-1)(t-b)^j\,\mathrm{d}t\end{aligned}$$

$$+\frac{1}{(n-2)!}\int_a^x W(t)\left[\int_a^t \left(f^{(n)}(s)-f^{(n)}(a)\right)(t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t$$
$$+\frac{1}{(n-2)!}\int_x^b (1-W(t))\left[\int_t^b \left(f^{(n)}(s)-f^{(n)}(b)\right)(t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t. \tag{2.81}$$

Proof. We apply formula (2.80) with f' instead of g, and first with $x=a$ second with $x=b$, then replace n with $n-1$ (thus $n\geq 2$). By putting these two formulae in the weighted Montgomery identity (2.5) we obtain (2.81). □

Theorem 2.37 *Let $f: I\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n\geq 2$, $I\subset\mathbb{R}$ an open interval, $a,b\in I$, $a<b$ and $w:[a,b]\to[0,\infty\rangle$ some probability density function. Then the following identity hold*

$$f(x)=\int_a^b w(t)f(t)\,\mathrm{d}t+\sum_{j=0}^{n-1}\frac{f^{(j+1)}(x)}{j!}\int_a^b P_w(x,t)(t-x)^j\,\mathrm{d}t$$
$$+\frac{1}{(n-2)!}\int_a^b P_w(x,t)\left[\int_x^t \left(f^{(n)}(s)-f^{(n)}(x)\right)(t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t. \tag{2.82}$$

Proof. If we apply formula (2.80) with f' instead of g, then replace n with $n-1$ (thus $n\geq 2$). By putting this formula in the weighted Montgomery identity (2.5) we obtain (2.82). □

Remark 2.35 In the special case, if we take $w(t)=\frac{1}{b-a}$, $t\in[a,b]$ the equality (2.81) reduces to

$$f(x)=\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t+\sum_{j=0}^{n-1}\frac{f^{(j+1)}(a)}{j!\,(j+2)}(x-a)^{j+2}-\sum_{j=0}^{n-1}\frac{f^{(j+1)}(b)}{j!\,(j+2)}(x-b)^{j+2}$$
$$+\frac{1}{(n-2)!}\int_a^x \frac{t-a}{b-a}\left[\int_a^t \left(f^{(n)}(s)-f^{(n)}(a)\right)(t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t$$
$$+\frac{1}{(n-2)!}\int_x^b \frac{b-t}{b-a}\left[\int_t^b \left(f^{(n)}(s)-f^{(n)}(b)\right)(t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t \tag{2.83}$$

and the equality (2.82) reduces to

$$f(x)=\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t+\sum_{j=0}^{n-1}\frac{f^{(j+1)}(x)}{(b-a)(j+2)!}\left((a-x)^{j+2}-(b-x)^{j+2}\right)$$
$$+\frac{1}{(n-2)!}\int_a^b P(x,t)\left[\int_x^t \left(f^{(n)}(s)-f^{(n)}(x)\right)(t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t \tag{2.84}$$

where $P(x,t)$ is given by (2.3).

The Ostrowski type inequalities

In this section we generalize the results from [21], [86] and [87]. We denote

$$v_{w,n}^{[a,b]}(x) = \sum_{j=0}^{n-1} \frac{f^{(j+1)}(a)}{j!} \int_a^x W(t)(t-a)^j \,\mathrm{d}t + \sum_{j=0}^{n-1} \frac{f^{(j+1)}(b)}{j!} \int_x^b (W(t)-1)(t-b)^j \,\mathrm{d}t,$$

$$s_{w,n}^{[a,b]}(x) = \sum_{j=0}^{n-1} \frac{f^{(j+1)}(x)}{j!} \int_a^b P_w(x,t)(t-x)^j \,\mathrm{d}t.$$

Theorem 2.38 *Suppose that all the assumptions of Theorem 2.36 hold. Additionally assume that* $f^{(n)} : [a,b] \to \mathbb{R}$ *is an* α*-L-Hölder type function, i.e.* $\left|f^{(n)}(x) - f^{(n)}(y)\right| \leq L|x-y|^{\alpha}$ *for every* $x,y \in [a,b]$, *where* $L > 0$ *and* $\alpha \in \langle 0,1]$. *Then we have*

$$\begin{aligned} &\left| f(x) - \int_a^b w(t) f(t) \,\mathrm{d}t - v_{w,n}^{[a,b]}(x) \right| \leq \frac{B(\alpha+1,n-1)}{(n-2)!} \\ &\cdot L\left(\int_a^x |W(t)|(t-a)^{\alpha+n-1} \,\mathrm{d}t + \int_x^b |W(t)-1|(b-t)^{\alpha+n-1} \,\mathrm{d}t \right) \\ &\leq \frac{B(\alpha+1,n-1)}{(n-2)!(\alpha+n)} L\left[(b-x)^{\alpha+n} + (x-a)^{\alpha+n}\right]. \end{aligned} \tag{2.85}$$

Proof. We use the identity (2.81) and apply the properties of modulus to obtain

$$\begin{aligned} &\left| f(x) - \int_a^b w(t) f(t) \,\mathrm{d}t - v_{w,n}^{[a,b]}(x) \right| \\ &\leq \frac{1}{(n-2)!} \left\{ \left| \int_a^x W(t) \left[\int_a^t \left(f^{(n)}(s) - f^{(n)}(a) \right)(t-s)^{n-2} \,\mathrm{d}s \right] \mathrm{d}t \right| \right. \\ &+ \left. \left| \int_x^b (1-W(t)) \left[\int_t^b \left(f^{(n)}(s) - f^{(n)}(b) \right)(t-s)^{n-2} \,\mathrm{d}s \right] \mathrm{d}t \right| \right\} \\ &\leq \frac{L}{(n-2)!} \left\{ \int_a^x |W(t)| \left| \int_a^t |s-a|^{\alpha} |t-s|^{n-2} \,\mathrm{d}s \right| \mathrm{d}t \right. \\ &+ \left. \int_x^b |1-W(t)| \left| \int_t^b |s-b|^{\alpha} |t-s|^{n-2} \,\mathrm{d}s \right| \mathrm{d}t \right\}. \end{aligned}$$

With substitution $(s-a) = u(t-a)$

$$\int_a^t |s-a|^{\alpha} |(t-s)|^{n-2} \,\mathrm{d}s = (t-a)^{\alpha+n-1} B(\alpha+1,n-1),$$

and with $(b-s) = u(b-t)$

$$\int_t^b |s-b|^{\alpha} |t-s|^{n-2} \,\mathrm{d}s = (b-t)^{\alpha+n-1} B(\alpha+1,n-1)$$

and the first inequality from (2.85) follows. Since $0 \leq W(t) \leq 1$, $t \in [a,b]$ and $0 \leq 1 - W(t) \leq 1$, $t \in [a,b]$, so we have

$$\int_a^x |W(t)|(t-a)^{\alpha+n-1}\,\mathrm{d}t \leq \int_a^x (t-a)^{\alpha+n-1}\,\mathrm{d}t = \frac{(x-a)^{\alpha+n}}{\alpha+n},$$

$$\int_x^b |W(t)-1|(b-t)^{\alpha+n-1}\,\mathrm{d}t \leq \int_x^b (b-t)^{\alpha+n-1}\,\mathrm{d}t = \frac{(b-x)^{\alpha+n}}{\alpha+n}.$$

The second inequality from (2.85) follows. □

Theorem 2.39 *Suppose that all the assumptions of Theorem 2.37 hold. Additionally assume that $f^{(n)} : [a,b] \to \mathbb{R}$ is an α-L-Hölder type function, i.e. $\left|f^{(n)}(x) - f^{(n)}(y)\right| \leq L|x-y|^{\alpha}$ for every $x,y \in [a,b]$, where $L > 0$ and $\alpha \in \langle 0,1]$. Then we have*

$$\begin{aligned}
&\left|f(x) - \int_a^b w(t) f(t)\,\mathrm{d}t - s_{w,n}^{[a,b]}(x)\right| \\
&\leq \frac{B(\alpha+1,n-1)}{(n-2)!} L \int_a^b |P_w(x,t)|\,|t-x|^{\alpha+n-1}\,\mathrm{d}t \\
&\leq \frac{B(\alpha+1,n-1)}{(n-2)!\,(\alpha+n)} L\left[(b-x)^{\alpha+n} + (x-a)^{\alpha+n}\right]
\end{aligned} \tag{2.86}$$

Proof. We use the identity (2.82) and apply the properties of modulus to obtain

$$\begin{aligned}
&\left|f(x) - \int_a^b w(t) f(t)\,\mathrm{d}t - s_{w,n}^{[a,b]}(x)\right| \\
&= \left|\frac{1}{(n-2)!}\int_a^b P_w(x,t)\left[\int_x^t \left(f^{(n)}(s) - f^{(n)}(x)\right)(t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t\right| \\
&\leq \frac{1}{(n-2)!}\int_a^b |P_w(x,t)|\left|\int_x^t \left|\left(f^{(n)}(s) - f^{(n)}(x)\right)(t-s)^{n-2}\right|\mathrm{d}s\right|\mathrm{d}t \\
&\leq \frac{L}{(n-2)!}\int_a^b |P_w(x,t)|\left|\int_x^t |s-x|^{\alpha}\,|(t-s)|^{n-2}\,\mathrm{d}s\right|\mathrm{d}t.
\end{aligned}$$

Now, for $t > x$ let $(s-x) = u(t-a)$

$$\int_x^t |s-x|^{\alpha}\,|(t-s)|^{n-2}\,\mathrm{d}s = (t-x)^{\alpha+n-1} B(\alpha+1,n-1),$$

for $x < t$ let $(x-s) = u(x-t)$

$$\int_x^t |s-x|^{\alpha}\,|(t-s)|^{n-2}\,\mathrm{d}s = -(x-t)^{\alpha+n-1} B(\alpha+1,n-1)$$

so

$$\left|\int_x^t |s-x|^{\alpha}\,|(t-s)|^{n-2}\,\mathrm{d}s\right| = |t-x|^{\alpha+n-1} B(\alpha+1,n-1) \tag{2.87}$$

and the first inequality from (2.86) follows. The second one is obvious since $|P_w(x,t)| \leq 1$, $t \in [a,b]$. □

Corollary 2.33 *Suppose that all the assumptions of Theorem 2.38 hold. Then we have*

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \sum_{j=0}^{n-1} \frac{f^{(j+1)}(a)}{j!\,(j+2)}(x-a)^{j+2} + \sum_{j=0}^{n-1} \frac{f^{(j+1)}(b)}{j!\,(j+2)}(x-b)^{j+2} \right| \\
&\leq B(\alpha+1,n-1)\,L\frac{(x-a)^{\alpha+n+1}+(b-x)^{\alpha+n+1}}{(\alpha+n+1)(b-a)(n-2)!}.
\end{aligned}$$

Proof. We apply the first inequality from (2.85) with $w(t)=\frac{1}{b-a}$, $t\in[a,b]$. □

Corollary 2.34 *Suppose that all the assumptions of Theorem 2.39 hold. Then we have*

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \sum_{j=0}^{n-1} \frac{f^{(j+1)}(x)}{(b-a)(j+2)!}\left((a-x)^{j+2}-(b-x)^{j+2}\right) \right| \\
&\leq \frac{B(\alpha+1,n-1)\,B(2,\alpha+n)}{(b-a)(n-2)!}L\left[(x-a)^{\alpha+n+1}+(b-x)^{\alpha+n+1}\right].
\end{aligned}$$

Proof. We apply the first inequality from (2.86) with $w(t)=\frac{1}{b-a}$, $t\in[a,b]$

$$\begin{aligned}
&\int_a^b |P(x,t)|\,|t-x|^{\alpha+n-1}\,\mathrm{d}t \\
&= \frac{1}{b-a}\left(\int_a^x |(t-a)|\,|t-x|^{\alpha+n-1}\,\mathrm{d}t + \int_x^b |(t-b)|\,|t-x|^{\alpha+n-1}\,\mathrm{d}t\right).
\end{aligned}$$

Using substitution $t-a=u(x-a)$ the first integral is equal to

$$\int_a^x (t-a)(x-t)^{\alpha+n-1}\,\mathrm{d}t = (x-a)^{\alpha+n+1}B(2,\alpha+n)$$

and similarly using $b-t=u(b-x)$ the second one is

$$\int_x^b (b-t)(t-x)^{\alpha+n-1}\,\mathrm{d}t = (b-x)^{\alpha+n+1}B(2,\alpha+n).$$

thus the proof follows. □

Remark 2.36 If we apply (2.85) and (2.86) with $x=\frac{a+b}{2}$ we get the generalized midpoint inequalities

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \int_a^b w(t)\,f(t)\,\mathrm{d}t - v_{w,n}^{[a,b]}\left(\frac{a+b}{2}\right)\right| \leq \frac{B(\alpha+1,n-1)}{(n-2)!} \\
&\cdot L\left(\int_a^{\frac{a+b}{2}} |W(t)|\,(t-a)^{\alpha+n-1}\,\mathrm{d}t + \int_{\frac{a+b}{2}}^b |W(t)-1|\,(b-t)^{\alpha+n-1}\,\mathrm{d}t\right) \\
&\leq \frac{B(\alpha+1,n-1)}{(n-2)!\,(\alpha+n)}L\left[\frac{(b-a)^{\alpha+n}}{2^{\alpha+n-1}}\right]
\end{aligned}$$

and

$$
\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\int_a^b w(t)\,f(t)\,\mathrm{d}t-s_{w,n}^{[a,b]}\left(\frac{a+b}{2}\right)\right| \\
&\le \frac{B(\alpha+1,n-1)}{(n-2)!}L\int_a^b\left|P_w\left(\frac{a+b}{2},t\right)\right|\left|t-\frac{a+b}{2}\right|^{\alpha+n-1}\mathrm{d}t \\
&\le \frac{B(\alpha+1,n-1)}{(n-2)!\,(\alpha+n)}L\left[\frac{(b-a)^{\alpha+n}}{2^{\alpha+n-1}}\right].
\end{aligned}
$$

If we additionally assume that $w(t)$ is symmetric on $[a,b]$ i.e. $w(t)=w(b-a-t)$ for every $t\in[a,b]$ these inequalities reduce to

$$
\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\int_a^b w(t)\,f(t)\,\mathrm{d}t\right. \\
&\left.-\sum_{j=0}^{n-1}\frac{f^{(j+1)}(a)+(-1)^{j+1}f^{(j+1)}(b)}{j!}\int_a^{\frac{a+b}{2}}W(t)\,(t-a)^j\,\mathrm{d}t\right| \\
&\le \frac{B(\alpha+1,n-1)}{(n-2)!}2L\int_a^{\frac{a+b}{2}}W(t)\,(t-a)^{\alpha+n-1}\,\mathrm{d}t \\
&\le \frac{B(\alpha+1,n-1)}{(n-2)!\,(\alpha+n)}L\left[\frac{(b-a)^{\alpha+n}}{2^{\alpha+n-1}}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\int_a^b w(t)\,f(t)\,\mathrm{d}t-\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\frac{f^{(2j+1)}(x)}{(2j)!}2\int_a^{\frac{a+b}{2}}W(t)\left(t-\frac{a+b}{2}\right)^j\mathrm{d}t\right| \\
&\le \frac{B(\alpha+1,n-1)}{(n-2)!}2L\int_a^{\frac{a+b}{2}}W(t)\left(\frac{a+b}{2}-t\right)^{\alpha+n-1}\mathrm{d}t \\
&\le \frac{B(\alpha+1,n-1)}{(n-2)!\,(\alpha+n)}L\left[\frac{(b-a)^{\alpha+n}}{2^{\alpha+n-1}}\right].
\end{aligned}
$$

For the generalized trapezoid inequality we apply equality (2.81) and (2.82) first with $x=a$, then with $x=b$, then add them up and divide by 2. After applying the properties of modulus we obtain

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\int_a^b w(t)\,f(t)\,\mathrm{d}t-\frac{v_{w,n}^{[a,b]}(a)+v_{w,n}^{[a,b]}(b)}{2}\right|\le\frac{B(\alpha+1,n-1)}{(n-2)!} \\
&\cdot L\left(\int_a^b W(t)\,(t-a)^{\alpha+n-1}\,\mathrm{d}t+\int_a^b(1-W(t))\,(b-t)^{\alpha+n-1}\,\mathrm{d}t\right) \\
&\le\frac{B(\alpha+1,n-1)}{(n-2)!\,(\alpha+n)}2L\left[(b-a)^{\alpha+n}\right]
\end{aligned}
$$

and

$$\left|\frac{f(a)+f(b)}{2}-\int_a^b w(t)f(t)\,\mathrm{d}t-\frac{s_{w,n}^{[a,b]}(a)+s_{w,n}^{[a,b]}(b)}{2}\right|\leq\frac{B(\alpha+1,n-1)}{(n-2)!}$$
$$\cdot L\left(\int_a^b(1-W(t))(t-a)^{\alpha+n-1}\,\mathrm{d}t+\int_a^b W(t)(b-t)^{\alpha+n-1}\,\mathrm{d}t\right)$$
$$\leq\frac{B(\alpha+1,n-1)}{(n-2)!(\alpha+n)}2L\left[(b-a)^{\alpha+n}\right].$$

If we additionally assume that $w(t)$ is symmetric on $[a,b]$, these inequalities reduce to

$$\left|\frac{f(a)+f(b)}{2}-\int_a^b w(t)f(t)\,\mathrm{d}t\right.$$
$$\left.-\sum_{j=0}^{n-1}\frac{f^{(j+1)}(a)+(-1)^{j+1}f^{(j+1)}(b)}{2(j!)}\int_a^b W(t)(b-a)^j\,\mathrm{d}t\right|$$
$$\leq\frac{B(\alpha+1,n-1)}{(n-2)!}2L\left(\int_a^b W(t)(t-a)^{\alpha+n-1}\,\mathrm{d}t\right)$$
$$\leq\frac{B(\alpha+1,n-1)}{(n-2)!(\alpha+n)}2L\left[(b-a)^{\alpha+n}\right]$$

and

$$\left|\frac{f(a)+f(b)}{2}-\int_a^b w(t)f(t)\,\mathrm{d}t\right.$$
$$\left.-\sum_{j=0}^{n-1}\frac{f^{(j+1)}(a)}{2(j!)}\int_a^b(W(t)-1)(t-a)^j\,\mathrm{d}t-\sum_{j=0}^{n}\frac{f^{(j+1)}(b)}{2(j!)}\int_a^b W(t)(t-b)^j\,\mathrm{d}t\right|$$
$$\leq\frac{B(\alpha+1,n-1)}{(n-2)!}2L\left(\int_a^b W(t)(t-a)^{\alpha+n-1}\,\mathrm{d}t\right)$$
$$\leq\frac{B(\alpha+1,n-1)}{(n-2)!(\alpha+n)}2L\left[(b-a)^{\alpha+n}\right].$$

The estimation of the difference of the two weighted integral means

In this section we generalize the results from [24], [83]. For the two intervals $[a,b]$ and $[c,d]$ we have four possible cases if $[a,b]\cap[c,d]\neq\emptyset$. The first case is $[c,d]\subset[a,b]$ and the second $[a,b]\cap[c,d]=[c,b]$. Other two possible cases we simply get by change $a\leftrightarrow c$, $b\leftrightarrow d$.

Theorem 2.40 *Let $f:I\to\mathbb{R}$ be such that $[a,b]\cup[c,d]\subset I$, $f^{(n)}:[a,b]\cup[c,d]\to\mathbb{R}$ is an α-L-Hölder type function for some $n\geq 2$, $w:[a,b]\to[0,\infty\rangle$ and $u:[c,d]\to[0,\infty\rangle$ some nonnegative normalized weight functions, $W(t)=\int_a^t w(x)\,\mathrm{d}x$ for $t\in[a,b]$, $W(t)=0$ for $t<a$ and $W(t)=1$ for $t>b$, $U(t)=\int_c^t u(x)\,\mathrm{d}x$ for $t\in[c,d]$, $U(t)=0$ for $t<c$ and*

$U(t) = 1$ for $t > d$. Then if $[a,b] \cap [c,d] \neq \emptyset$ and $x \in [a,b] \cap [c,d]$, we have

$$\int_a^b w(t) f(t)\,\mathrm{d}t - \int_c^d u(t) f(t)\,\mathrm{d}t + s_{w.n}^{[a,b]}(x) - s_{u,n}^{[c,d]}(x) = \int_{\min\{a,c\}}^{\max\{b,d\}} \mathscr{K}_n(x,t)\,\mathrm{d}t \tag{2.88}$$

where in case $[c,d] \subset [a,b]$

$$\mathscr{K}_n(x,t) = \begin{cases} \frac{-W(t)}{(n-2)!}\left[\int_x^t \left(f^{(n)}(s) - f^{(n)}(x)\right)(t-s)^{n-2}\,\mathrm{d}s\right], & t \in [a,c\rangle, \\ \frac{U(t)-W(t)}{(n-2)!}\left[\int_x^t \left(f^{(n)}(s) - f^{(n)}(x)\right)(t-s)^{n-2}\,\mathrm{d}s\right], & t \in [c,d], \\ \frac{1-W(t)}{(n-2)!}\left[\int_x^t \left(f^{(n)}(s) - f^{(n)}(x)\right)(t-s)^{n-2}\,\mathrm{d}s\right], & t \in \langle d,b], \end{cases}$$

and in case $[a,b] \cap [c,d] = [c,b]$

$$\mathscr{K}_n(x,t) = \begin{cases} \frac{-W(t)}{(n-2)!}\left[\int_x^t \left(f^{(n)}(s) - f^{(n)}(x)\right)(t-s)^{n-2}\,\mathrm{d}s\right], & t \in [a,c\rangle, \\ \frac{U(t)-W(t)}{(n-2)!}\left[\int_x^t \left(f^{(n)}(s) - f^{(n)}(x)\right)(t-s)^{n-2}\,\mathrm{d}s\right], & t \in [c,b], \\ \frac{U(t)-1}{(n-2)!}\left[\int_x^t \left(f^{(n)}(s) - f^{(n)}(x)\right)(t-s)^{n-2}\,\mathrm{d}s\right], & t \in \langle b,d]. \end{cases}$$

Proof. We subtract identities (2.82) for interval $[a,b]$ and $[c,d]$, to get the formula (2.88). □

Theorem 2.41 *Suppose that all the assumptions of Theorem 2.40 hold. Then we have*

$$\begin{aligned} &\left|\int_a^b w(t) f(t)\,\mathrm{d}t - \int_c^d u(t) f(t)\,\mathrm{d}t + s_{w,n}^{[a,b]}(x) - s_{u,n}^{[c,d]}(x)\right| \\ &\leq \frac{B(\alpha+1,n-1)}{(n-2)!} L \int_{\min\{a,c\}}^{\max\{b,d\}} \left|P_w(x,t) - P_u(x,t)\right| |t-x|^{\alpha+n-1}\,\mathrm{d}t \\ &\leq \frac{B(\alpha+1,n-1)}{(n-2)!} L \int_{\min\{a,c\}}^{\max\{b,d\}} |t-x|^{\alpha+n-1}\,\mathrm{d}t \end{aligned}$$

Proof. Use the identity (2.88) to obtain

$$\begin{aligned} &\left|\int_a^b w(t) f(t)\,\mathrm{d}t - \int_c^d u(t) f(t)\,\mathrm{d}t + s_{w,n}^{[a,b]}(x) - s_{u,n}^{[c,d]}(x)\right| \leq \int_{\min\{a,c\}}^{\max\{b,d\}} \left|\mathscr{K}_n(x,t)\right|\mathrm{d}t \\ &\leq \frac{1}{(n-2)!}\left(\int_{\min\{a,c\}}^{\max\{b,d\}} \left|P_w(x,t) - P_u(x,t)\right| \left|\int_x^t \left|f^{(n)}(s) - f^{(n)}(x)\right| |t-s|^{n-2}\,\mathrm{d}s\right|\mathrm{d}t\right) \\ &\leq \frac{L}{(n-2)!}\left(\int_{\min\{a,c\}}^{\max\{b,d\}} \left|P_w(x,t) - P_u(x,t)\right| \left|\int_x^t |s-x|^{\alpha} |t-s|^{n-2}\,\mathrm{d}s\right|\mathrm{d}t\right) \end{aligned}$$

and like (2.87) in the proof of Theorem 2.39

$$\left|\int_x^t |s-x|^\alpha |t-s|^{n-2}\,\mathrm{d}s\right| = |t-x|^{\alpha+n-1} B(\alpha+1,n-1)$$

which proves the first inequality. The second one immediately follows since $|P_w(x,t)-P_u(x,t)| \le 1$ for all $t \in [\min\{a,c\},\max\{b,d\}]$. □

Case $[c,d] \subset [a,b]$

Here we denote

$$s_n^{[a,b]}(x) = \sum_{j=0}^{n-1} \frac{f^{(j+1)}(x)}{(b-a)(j+2)!}\left((a-x)^{j+2}-(b-x)^{j+2}\right).$$

Corollary 2.35 *Suppose that all the assumptions of Theorem 2.40 hold. Additionally suppose* $[c,d] \subset [a,b]$. *For* $x \in [c,d]$ *and* $s_0 = \frac{bc-ad}{c-a+b-d}$, *the following inequality holds*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{1}{d-c}\int_c^d f(t)\,\mathrm{d}t + s_n^{[a,b]}(x) - s_n^{[c,d]}(x)\right|$$
$$\le \frac{B(\alpha+1,n-1)}{(b-a)(n-2)!}L\left[(x-a)^{\alpha+n+1}B_{\frac{c-a}{x-a}}(2,\alpha+n) + \frac{(c-a+b-d)}{(d-c)}|x-s_o|^{\alpha+n-1}\right.$$
$$\left.\cdot\left(B(2,\alpha+n)+\Psi_{r_1}(2,\alpha+n)+\Psi_{r_2}(\alpha+n,2)\right)+(b-x)^{\alpha+n+1}B_{\frac{b-d}{b-x}}(2,\alpha+n)\right]$$

where for $s_0 < x$

$$r_1 = \frac{s_0-c}{x-s_0}, \qquad r_2 = \frac{d-x}{x-s_0},$$

while for $s_0 > x$

$$r_1 = \frac{d-s_0}{s_0-x}, \qquad r_2 = \frac{x-c}{s_0-x}.$$

Proof. We put $w(t) = \frac{1}{b-a}$, $t \in [a,b]$, and $u(t) = \frac{1}{d-c}$, $t \in [c,d]$ in the first inequality from the Theorem 2.41. Thus we have $s_n^{[a,b]}(x)$ and $s_n^{[c,d]}(x)$ instead of $s_{w,n}^{[a,b]}(x)$ and $s_{u,n}^{[c,d]}(x)$ and

$$\int_{\min\{a,c\}}^{\max\{b,d\}} |P_w(x,t)-P_u(x,t)|\,|t-x|^{\alpha+n-1}\,\mathrm{d}t = \int_a^c \left|\frac{t-a}{b-a}\right| |x-t|^{\alpha+n-1}\,\mathrm{d}t$$
$$+\int_c^d \left|\frac{t-a}{b-a}-\frac{t-c}{d-c}\right| |x-t|^{\alpha+n-1}\,\mathrm{d}t + \int_d^b \left|\frac{b-t}{b-a}\right| |x-t|^{\alpha+n-1}\,\mathrm{d}t.$$

For the first integral let $t-a = u(x-a)$ so

$$\int_a^c (t-a)(x-t)^{\alpha+n-1}\,\mathrm{d}t = (x-a)^{\alpha+n+1}B_{\frac{c-a}{x-a}}(2,\alpha+n). \tag{2.89}$$

For the third integral let $b-t=u(b-x)$ and

$$\int_d^b (b-t)(t-x)^{\alpha+n-1}\,\mathrm{d}t = (b-x)^{\alpha+n+1} B_{\frac{b-d}{b-x}}(2,\alpha+n).$$

$c-a+b-d>0$ so the second integral is

$$\int_c^d \left|\frac{t-a}{b-a}-\frac{t-c}{d-c}\right| |x-t|^{\alpha+n-1}\,\mathrm{d}t = \frac{c-a+b-d}{(b-a)(d-c)}\int_c^d |s_0-t|\,|x-t|^{\alpha+n-1}\,\mathrm{d}t.$$

Since $s_0-c=\frac{(d-c)(c-a)}{c-a+b-d}\geq 0$ and $d-s_0=\frac{(d-c)(b-d)}{c-a+b-d}\geq 0$, $s_0\in[c,d]$. So we have two possible cases:

1. If $s_0<x$ we have

$$\int_c^d |s_0-t|\,|x-t|^{\alpha+n-1}\,\mathrm{d}t = \int_c^{s_0}(s_0-t)(x-t)^{\alpha+n-1}\,\mathrm{d}t$$
$$+\int_{s_0}^x (t-s_0)(x-t)^{\alpha+n-1}\,\mathrm{d}t+\int_x^d (t-s_0)(t-x)^{\alpha+n-1}\,\mathrm{d}t.$$

Now, using the substitution $s_0-t=u(x-s_0)$ we get

$$\int_c^{s_0}(s_0-t)(x-t)^{\alpha+n-1}\,\mathrm{d}t=(x-s_0)^{\alpha+n-1}\Psi_{\frac{s_0-c}{x-s_0}}(2,\alpha+n),$$

with $t-s_0=u(x-s_0)$ we get

$$\int_{s_0}^x (t-s_0)(x-t)^{\alpha+n-1}\,\mathrm{d}t=(x-s_0)^{\alpha+n-1}B(2,\alpha+n),$$

and with $t-x=u(x-s_0)$

$$\int_x^d (t-s_0)(t-x)^{\alpha+n-1}\,\mathrm{d}t=(x-s_0)^{\alpha+n-1}\Psi_{\frac{d-x}{x-s_0}}(\alpha+n,2).$$

2. If $x<s_0$ then

$$\int_c^d |s_0-t|\,|x-t|^{\alpha+n-1}\,\mathrm{d}t = \int_c^{x}(s_0-t)(x-t)^{\alpha+n-1}\,\mathrm{d}t$$
$$+\int_{x}^{s_0} (s_0-t)(t-x)^{\alpha+n-1}\,\mathrm{d}t+\int_{s_0}^d (t-s_0)(t-x)^{\alpha+n-1}\,\mathrm{d}t.$$

Using the substitution $x-t=u(s_0-x)$ we get

$$\int_c^x (s_0-t)(x-t)^{\alpha+n-1}\,\mathrm{d}t=(s_0-x)^{\alpha+n-1}\Psi_{\frac{x-c}{s_0-x}}(\alpha+n,2), \tag{2.90}$$

with $s_0-t=u(s_0-x)$ we get

$$\int_x^{s_0}(s_0-t)(t-x)^{\alpha+n-1}\,\mathrm{d}t=(s_0-x)^{\alpha+n-1}B(2,\alpha+n),$$

and with $t - s_0 = u(s_0 - x)$

$$\int_{s_0}^{d} (t - s_0)(t - x)^{\alpha+n-1}\,\mathrm{d}t = (s_0 - x)^{\alpha+n-1}\,\Psi_{\frac{d-s_0}{s_0-x}}(2, \alpha+n).$$

Thus the proof is done. □

Remark 2.37 If we put $c = d = x$ as a limit case, the inequalities from the Corollary 2.35 reduce to the inequality from the Corollary 2.34.

Case $[a,b] \cap [c,d] = [c,b]$

Corollary 2.36 *Suppose that all the assumptions of Theorem 2.40 hold. Additionally suppose* $[a,b] \cap [c,d] \subset [c,b]$, $x \in [c,b]$. *If* $c - a + b - d = 0$ *then*

$$\begin{aligned}
&\left| \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{1}{d-c}\int_c^d f(t)\,\mathrm{d}t + s_n^{[a,b]}(x) - s_n^{[c,d]}(x) \right| \\
&\leq \frac{B(\alpha+1, n-1)}{(n-2)!} L \left[\frac{(x-a)^{\alpha+n+1}}{b-a} B_{\frac{c-a}{x-a}}(2, \alpha+n) \right. \\
&\left. + \frac{c-a}{b-a}\left(\frac{(x-c)^{\alpha+n} + (b-x)^{\alpha+n}}{\alpha+n} \right) + \frac{(d-x)^{\alpha+n+1}}{d-c} B_{\frac{d-b}{d-x}}(2, \alpha+n) \right]
\end{aligned}$$

and if $c - a + b - d \neq 0$ *and* $s_0 = \frac{bc-ad}{c-a+b-d}$ *then*

$$\begin{aligned}
&\left| \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{1}{d-c}\int_c^d f(t)\,\mathrm{d}t + s_n^{[a,b]}(x) - s_n^{[c,d]}(x) \right| \\
&\leq \frac{B(\alpha+1, n-1)}{(n-2)!} L \left[\frac{(x-a)^{\alpha+n+1}}{b-a} B_{\frac{c-a}{x-a}}(2, \alpha+n) + \frac{|c-a+b-d|}{(b-a)(d-c)} |x - s_o|^{\alpha+n-1} \right. \\
&\left. \cdot \left(B_{r_1}(\alpha+n, 2) + \Psi_{r_2}(\alpha+n, 2) \right) + \frac{(d-x)^{\alpha+n+1}}{d-c} B_{\frac{d-b}{d-x}}(2, \alpha+n) \right]
\end{aligned}$$

where for $s_0 < b, c$ *i.e.* $s_0 < x$

$$r_1 = \frac{x-c}{x-s_0}, \qquad r_2 = \frac{b-x}{x-s_0},$$

while for $s_0 > b, c$ *i.e.* $s_0 > x$

$$r_1 = \frac{b-x}{s_0-x}, \qquad r_2 = \frac{x-c}{s_0-x}.$$

Proof. The proof is similar to the proof of the Theorem 2.35. □

Remark 2.38 If we put $b = c = x$ as a limit case, the inequalities from the Corollary 2.36 reduce to

$$\begin{aligned}&\left|\frac{1}{x-a}\int_a^x f(t)\,\mathrm{d}t - \frac{1}{d-x}\int_x^d f(t)\,\mathrm{d}t + t_n^{[a,x]}(x) - t_n^{[x,d]}(x)\right| \\ &\leq \frac{B(\alpha+1,n-1)B(2,\alpha+n)}{(n-2)!}L\left[(x-a)^{\alpha+n}+(d-x)^{\alpha+n}\right].\end{aligned}$$

Remark 2.39 If we suppose $b = d$ in both cases $[c,d] \subset [a,b]$ and $[a,b] \cap [c,d] = [c,b]$ the analogues results in Corollary 2.35 and Corollary 2.36 coincides.

2.3.4 Extension via Fink identity with applications

G. V. Milovanović and J. Pečarić [88] and A. M. Fink [59] have considered generalizations of The Ostrowski inequality in the form

$$\left|\frac{1}{n}\left(f(x)+\sum_{k=1}^{n-1}F_k(x)\right)-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right| \leq K(n,p,x)\|f^{(n)}\|_p \tag{2.91}$$

which is obtained from identity

$$\frac{1}{n}\left(f(x)+\sum_{k=1}^{n-1}F_k(x)\right)-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t = \frac{1}{n!(b-a)}\int_a^b (x-t)^{n-1}k(t,x)f^{(n)}(t)\mathrm{d}t, \tag{2.92}$$

where

$$F_k(x) = \frac{n-k}{k!}\,\frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a}$$

and

$$k(t,x) = \begin{cases} t-a, & a \leq t \leq x \leq b; \\ t-b, & a \leq x < t \leq b. \end{cases}$$

In fact, G. V. Milovanović and J. Pečarić have proved that

$$K(n,\infty,x) = \frac{(x-a)^{n+1}+(b-x)^{n+1}}{n(n+1)!(b-a)},$$

while A. M. Fink gave the following generalizations of this result:

Theorem 2.42 *Let $f^{(n-1)}$ be absolutely continuous on $[a,b]$ and let $f^{(n)} \in L_p[a,b]$. Then the inequality (2.91) holds with*

$$K(n,p,x) = \frac{\left[(x-a)^{nq+1}+(b-x)^{nq+1}\right]^{1/q}}{n!(b-a)}B((n-1)q+1,q+1)^{1/q} \tag{2.93}$$

where $1 < p \leq \infty$, $1/p + 1/q = 1$, B is the Beta function, and

$$K(n,1,x) = \frac{(n-1)^{n-1}}{n^n n!(b-a)}\max[(x-a)^n,(b-x)^n]. \tag{2.94}$$

In this subsection we give the extension of weighted Montgomery identity (2.5) using identity (2.91) and further, obtain some new Ostrowski type inequalities, as well as the generalizations of the estimations of the difference of two weighted integral means (generalizations of the results from [14], [24], [83] and [95]). Results from this subsection are published in [17].

Theorem 2.43 *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous function on $[a,b]$ for some $n\geq 1$. If $w:[a,b]\to[0,\infty\rangle$ is some normalized weight function, then the following identity holds*

$$\begin{aligned}
f(x) &= \int_a^b w(t)f(t)\,\mathrm{d}t - \sum_{k=1}^{n-1}F_k(x) + \sum_{k=1}^{n-1}\int_a^b w(t)F_k(t)\mathrm{d}t\\
&+ \frac{1}{(n-1)!(b-a)}\int_a^b (x-y)^{n-1}k(y,x)f^{(n)}(y)\,\mathrm{d}y\\
&- \frac{1}{(n-1)!(b-a)}\int_a^b\left(\int_a^b w(t)(t-y)^{n-1}k(y,t)\mathrm{d}t\right)f^{(n)}(y)\,\mathrm{d}y.
\end{aligned} \tag{2.95}$$

Proof. We apply identity (2.92) with $f'(t)$:

$$\begin{aligned}
f'(t) &= -\sum_{k=0}^{n-1}\frac{n-k}{k!}\frac{f^{(k)}(a)(t-a)^k - f^{(k)}(b)(t-b)^k}{b-a}\\
&\quad + \frac{1}{(n-1)!(b-a)}\int_a^b (t-y)^{n-1}k(y,t)f^{(n+1)}(y)\mathrm{d}y.
\end{aligned}$$

Now by putting this formula in the weighted Montgomery identity (2.5) we obtain

$$\begin{aligned}
f(x) &= \int_a^b w(t)f(t)\,\mathrm{d}t\\
&\quad - \sum_{k=0}^{n-1}\frac{n-k}{k!}\int_a^b P_w(x,t)\frac{f^{(k)}(a)(t-a)^k - f^{(k)}(b)(t-b)^k}{b-a}\mathrm{d}t\\
&\quad + \frac{1}{(n-1)!(b-a)}\int_a^b P_w(x,t)\left(\int_a^b (t-y)^{n-1}k(y,t)f^{(n+1)}(y)\,\mathrm{d}y\right)\mathrm{d}t.
\end{aligned}$$

Further,

$$\begin{aligned}
&\int_a^b P_w(x,t)\frac{f^{(k)}(a)(t-a)^k - f^{(k)}(b)(t-b)^k}{b-a}\mathrm{d}t\\
&= \frac{f^{(k)}(a)(x-a)^{k+1} - f^{(k)}(b)(x-b)^{k+1}}{(b-a)(k+1)}\\
&\quad - \int_a^b w(t)\frac{f^{(k)}(a)(t-a)^{k+1} - f^{(k)}(b)(t-b)^{k+1}}{(b-a)(k+1)}\mathrm{d}t
\end{aligned}$$

and

$$\int_a^b P_w(x,t)(t-y)^{n-1}k(y,t)\mathrm{d}t = \frac{1}{n}(x-y)^n k(y,x) - \frac{1}{n}\int_a^b w(t)(t-y)^n k(y,t)\mathrm{d}t.$$

Now, if we replace n with $n-1$ we get (2.95). This identity is valid for $n-1 \geq 1$, i.e. $n > 1$. □

Remark 2.40 We could also obtain identity (2.95) by applying identity (2.92) in the following way: multiply it by $w(x)$ and than integrate it to obtain

$$\int_a^b w(x) f(x)\,\mathrm{d}x = -\sum_{k=1}^{n-1}\int_a^b w(x)F_k(x)\mathrm{d}x + \left(\int_a^b w(x)\,\mathrm{d}x\right)\frac{n}{b-a}\int_a^b f(t)\,\mathrm{d}t$$
$$+\frac{1}{(n-1)!(b-a)}\int_a^b\left(\int_a^b w(x)(x-t)^{n-1}k(t,x)\mathrm{d}x\right)f^{(n)}(t)\mathrm{d}t.$$

Now, if we subtract this identity from (2.92) we obtain (2.95).

Remark 2.41 In the special case, if we take $w(t) = \frac{1}{b-a}$, $t \in [a,b]$ we have

$$\frac{1}{b-a}\sum_{k=1}^{n-1}\int_a^b F_k(t)\mathrm{d}t = \sum_{k=1}^{n-1}\frac{n-k}{(k+1)!}\left[f^{(k-1)}(a)(b-a)^{k-1} + f^{(k-1)}(b)(a-b)^{k-1}\right]$$

and

$$\frac{1}{b-a}\int_a^b (t-y)^{n-1}k(y,t)\mathrm{d}t = \frac{(y-a)(b-y)^n}{n(b-a)} - \frac{(y-b)(a-y)^n}{n(b-a)}.$$

Let's denote

$$I_n = \frac{1}{n!(b-a)^2}\int_a^b \left[(y-a)(b-y)^n - (y-b)(a-y)^n\right] f^{(n)}(y)\mathrm{d}y.$$

Then we have

$$I_n = \frac{1}{n!(b-a)^2}\int_a^b \left[(a-y)^n - (b-y)^n\right] f^{(n-1)}(y)\mathrm{d}y + I_{n-1} = J_n + I_{n-1},$$

where $I_0 = \frac{1}{b-a}\int_a^b f(y)\mathrm{d}y$. Further,

$$J_n = \frac{1}{n!}\left[f^{(n-2)}(a)(b-a)^{n-2} + f^{(n-2)}(b)(a-b)^{n-2}\right] + J_{n-1}$$

and $J_1 = -\frac{1}{b-a}\int_a^b f(y)\mathrm{d}y$. So,

$$J_n = \sum_{k=1}^{n-1}\frac{1}{(k+1)!}\left[f^{(k-1)}(a)(b-a)^{k-1} + f^{(k-1)}(b)(a-b)^{k-1}\right] + J_1$$

and then

$$I_n = \sum_{m=2}^{n} J_m + nJ_1 + I_0$$

$$= \sum_{k=1}^{n-1} \frac{n-k}{(k+1)!} \left[f^{(k-1)}(a)(b-a)^{k-1} + f^{(k-1)}(b)(a-b)^{k-1} \right] - \frac{n-1}{b-a} \int_a^b f(y)\mathrm{d}y.$$

Consequently, identity (2.95) reduces to the identity (2.92), so we may regard it as weighted Fink identity.

Remark 2.42 Applying identity (2.95) with $x = a$ and then $x = b$, we get the generalized trapezoid identity (the first Euler-Maclaurin formula)

$$\begin{aligned} \frac{1}{2}[f(a)+f(b)] &= \int_a^b w(t) f(t)\,\mathrm{d}t + \sum_{k=1}^{n-1} \int_a^b w(t) F_k(t) \mathrm{d}t \\ &- \frac{1}{2} \sum_{k=1}^{n-1} \frac{n-k}{k!} \left[f^{(k-1)}(a)(b-a)^{k-1} + f^{(k-1)}(b)(a-b)^{k-1} \right] \\ &+ \frac{1}{2(n-1)!(b-a)} \int_a^b \left[(a-y)^{n-1}(y-b) + (b-y)^{n-1}(y-a) \right] f^{(n)}(y)\,\mathrm{d}y \\ &- \frac{1}{(n-1)!(b-a)} \int_a^b \left(\int_a^b w(t)(t-y)^{n-1} k(y,t) \mathrm{d}t \right) f^{(n)}(y)\,\mathrm{d}y. \end{aligned} \tag{2.96}$$

Similarly, applying identity (2.95) with $x = \frac{a+b}{2}$ we get the generalized midpoint identity (the second Euler-Maclaurin formula)

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= \int_a^b w(t) f(t)\,\mathrm{d}t + \sum_{k=1}^{n-1} \int_a^b w(t) F_k(t) \mathrm{d}t \\ &- \sum_{k=1}^{n-1} \frac{n-k}{2^k k!} \left[f^{(k-1)}(a)(b-a)^{k-1} + f^{(k-1)}(b)(a-b)^{k-1} \right] \\ &+ \frac{1}{(n-1)!(b-a)} \int_a^b \left(\frac{a+b}{2} - y \right)^{n-1} k\left(y, \frac{a+b}{2}\right) f^{(n)}(y)\,\mathrm{d}y \\ &- \frac{1}{(n-1)!(b-a)} \int_a^b \left(\int_a^b w(t)(t-y)^{n-1} k(y,t) \mathrm{d}t \right) f^{(n)}(y)\,\mathrm{d}y. \end{aligned} \tag{2.97}$$

Ostrowski type inequalities

Let's denote for $n \geq 2$

$$\mathscr{F}_{w,n}(x) = \sum_{k=1}^{n-1} F_k(x) - \sum_{k=1}^{n-1} \int_a^b w(t) F_k(t) \mathrm{d}t.$$

Theorem 2.44 *Assume (p,q) is a pair of conjugate exponents, that is $1 \leq p,q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Let $f^{(n)} \in L_p[a,b]$ for some $n > 1$. Then for $x \in [a,b]$ the following inequality holds*

$$\left| f(x) - \int_a^b w(t) f(t)\,\mathrm{d}t + \mathscr{F}_{w,n}(x) \right| \tag{2.98}$$

$$\leq \frac{1}{(n-2)!(b-a)} \left(\int_a^b \left| \int_a^b P_w(x,t)(t-y)^{n-2} k(y,t) \mathrm{d}t \right|^q \mathrm{d}y \right)^{\frac{1}{q}} \left\| f^{(n)} \right\|_p .$$

This inequality is sharp for $1 < p \leq \infty$ *and the best possible for* $p = 1$.

Proof. From Theorem 2.43 we have

$$\begin{aligned}
&(x-y)^{n-1} k(y,x) - \int_a^b w(t)(t-y)^{n-1} k(y,t) \mathrm{d}t \\
&= (n-1) \int_a^b P_w(x,t)(t-y)^{n-2} k(y,t) \mathrm{d}t .
\end{aligned}$$

Using the identity (2.95) and applying the Hölder inequality we obtain (2.98). The rest of proof is similar to the proof of the Theorem 2.16. □

Remark 2.43 For $w(t) = \frac{1}{b-a}$, $n = 2$ and $q = 1$ (2.98) reduces to

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \mathrm{d}t - \left(x - \frac{a+b}{2} \right) (f(b) - f(a)) \right| \\
&\leq \frac{1}{2(b-a)} \left(\int_a^x |(y-a)(2x-y-b)| \mathrm{d}y + \int_x^b |(b-y)(-2x+y+a)| \mathrm{d}y \right) \|f''\|_\infty \\
&= \left(\frac{4}{3} \delta^3(x) - \frac{1}{2} \delta^2(x) + \frac{1}{24} \right) \|f''\|_\infty ,
\end{aligned}$$

where $\delta(x) = |x - (a+b)/2|$.

If instead of $q = 1$ ($p = \infty$) we take $p = 1$, then similarly we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \mathrm{d}t - \left(x - \frac{a+b}{2} \right) (f(b) - f(a)) \right| \\
&\leq \frac{1}{2(b-a)} \max \left\{ \max_{y \in [a,x]} |(y-a)(2x-y-b)| , \max_{y \in [x,b]} |(b-y)(-2x+y+a)| \right\} \|f''\|_1 \\
&= \frac{1}{4} \left[\frac{1}{4} + \left| \frac{1}{4} - 2 \left(x - \frac{a+b}{2} \right)^2 \right| \right] \|f''\|_1 .
\end{aligned}$$

These two inequalities were obtained in [50].

Corollary 2.37 *Suppose that all assumptions of Theorem 2.44 hold. Then*

$$\begin{aligned}
&\left| f(x) - \int_a^b w(t) f(t) \mathrm{d}t + \mathscr{F}_{w,n}(x) \right| \qquad (2.99) \\
&\leq \frac{1}{(n-1)!(b-a)} \left(\int_a^b \left[(b-y)(y-a)^{n-1} + (y-a)(b-y)^{n-1} \right]^q \mathrm{d}y \right)^{\frac{1}{q}} \left\| f^{(n)} \right\|_p .
\end{aligned}$$

Proof. Since $|P_w(x,t)| \leq 1$, then for every $y \in [a,b]$ we have

$$\left|\int_a^b P_w(x,t)(t-y)^{n-2}k(y,t)\mathrm{d}t\right| \leq \int_a^b |P_w(x,t)|\left|(t-y)^{n-2}k(y,t)\right|\mathrm{d}t$$
$$\leq \int_a^b \left|(t-y)^{n-2}k(y,t)\right|\mathrm{d}t = \frac{1}{n-1}\left[(b-y)(y-a)^{n-1}+(y-a)(b-y)^{n-1}\right]$$

and by applying (2.98) the inequality is proved. □

Remark 2.44 Inequality (2.99) for $n=2$ reduces to:

$$\left|f(x)-\int_a^b w(t)f(t)\,\mathrm{d}t+\mathscr{F}_{w,2}(x)\right| \leq 2(b-a)^{1+\frac{1}{q}}B(q+1,q+1)^{\frac{1}{q}}\left\|f''\right\|_p$$

and for $n=3$ to

$$\left|f(x)-\int_a^b w(t)f(t)\,\mathrm{d}t+\mathscr{F}_{w,3}(x)\right| \leq \frac{(b-a)^{2+\frac{1}{q}}}{2}B(q+1,q+1)^{\frac{1}{q}}\left\|f'''\right\|_p.$$

Remark 2.45 If we use the identities (2.96) and (2.97) for $n=2$ and $w(t)=\frac{1}{b-a}$, $t \in [a,b]$ and then apply the Hölder inequality with $p=\infty$, $q=1$, we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \leq \frac{(b-a)^2}{12}\left\|f''\right\|_\infty$$

and

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \leq \frac{(b-a)^2}{24}\left\|f''\right\|_\infty.$$

By doing the same for $n=3$ we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t-\frac{b-a}{12}[f'(b)-f'(a)]\right| \leq \frac{(b-a)^3}{192}\left\|f'''\right\|_\infty$$

and

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t+\frac{b-a}{24}[f'(b)-f'(a)\right| \leq \frac{(b-a)^3}{192}\left\|f'''\right\|_\infty.$$

The first two inequalities were obtained in [47] and the last two in [13].

Estimations of the difference of two weighted integral means

In this section we will denote for $n>1$

$$\mathscr{F}_{w,n}^{[a,b]}(x)=\sum_{k=1}^{n-1}F_k^{[a,b]}(x)-\sum_{k=1}^{n-1}\int_a^b w(t)F_k^{[a,b]}(t)\mathrm{d}t,$$

for function $f:[a,b]\to\mathbb{R}$ such that $f^{(n-1)}$ is absolutely function on $[a,b]$.

Following results are generalizations of the results from [24] in two cases. First is when $[c,d]\subseteq[a,b]$ and the second when $[a,b]\cap[c,d]=[c,b]$. Other two possible cases, when $[a,b]\cap[c,d]\neq\emptyset$ we simply get by change $a\leftrightarrow c$, $b\leftrightarrow d$.

Theorem 2.45 *Let $f : [a,b] \cup [c,d] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely function on $[a,b]$ for some $n > 1$, $w : [a,b] \to [0,\infty\rangle$ and $u : [c,d] \to [0,\infty\rangle$ some normalized weight functions. Then if $[a,b] \cap [c,d] \neq \emptyset$ and $x \in [a,b] \cap [c,d]$, we have*

$$\int_a^b w(t) f(t)\,\mathrm{d}t - \int_c^d u(t) f(t)\,\mathrm{d}t - \mathscr{F}_{w,n}^{[a,b]}(x) + \mathscr{F}_{u,n}^{[c,d]}(x) = \int_{\min\{a,c\}}^{\max\{b,d\}} K_n(x,y) f^{(n)}(y)\,\mathrm{d}y, \tag{2.100}$$

where in case $[c,d] \subseteq [a,b]$

$$K_n(x,y) = \begin{cases} \frac{-1}{(n-2)!(b-a)} \left[\int_a^b P_w(x,t)(t-y)^{n-2} k^{[a,b]}(y,t)\mathrm{d}t\right], & y \in [a,c], \\ \frac{-1}{(n-2)!(b-a)} \left[\int_a^b P_w(x,t)(t-y)^{n-2} k^{[a,b]}(y,t)\mathrm{d}t\right] \\ \quad + \frac{1}{(n-2)!(d-c)} \left[\int_c^d P_u(x,t)(t-y)^{n-2} k^{[c,d]}(y,t)\mathrm{d}t\right], & y \in \langle c,d], \\ \frac{-1}{(n-2)!(b-a)} \left[\int_a^b P_w(x,t)(t-y)^{n-2} k^{[a,b]}(y,t)\mathrm{d}t\right], & y \in \langle d,b] \end{cases}$$

and in case $[a,b] \cap [c,d] = [c,b]$

$$K_n(x,y) = \begin{cases} \frac{-1}{(n-2)!(b-a)} \left[\int_a^b P_w(x,t)(t-y)^{n-2} k^{[a,b]}(y,t)\mathrm{d}t\right], & y \in [a,c], \\ \frac{-1}{(n-2)!(b-a)} \left[\int_a^b P_w(t)(t-y)^{n-2} k^{[a,b]}(y,t)\mathrm{d}t\right] \\ \quad + \frac{1}{(n-2)!(d-c)} \left[\int_c^d P_u(t)(t-y)^{n-2} k^{[c,d]}(y,t)\mathrm{d}t\right], & y \in \langle c,b], \\ \frac{1}{(n-2)!(d-c)} \left[\int_c^d P_u(t)(t-y)^{n-2} k^{[c,d]}(y,t)\mathrm{d}t\right], & y \in \langle b,d]. \end{cases}$$

Proof. We subtract identities (2.95) for interval $[a,b]$ and $[c,d]$, to get the formula (2.100). □

Theorem 2.46 *Assume (p,q) is a pair of conjugate exponents, that is $1 \leq p,q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Let $f^{(n)} \in L_p[a,b]$ for some $n > 1$. Then we have*

$$\left| \int_a^b w(t) f(t)\,\mathrm{d}t - \int_c^d u(t) f(t)\,\mathrm{d}t - \mathscr{F}_{w,n}^{[a,b]}(x) + \mathscr{F}_{u,n}^{[c,d]}(x) \right| \leq \left(\int_{\min\{a,c\}}^{\max\{b,d\}} |K_n(x,y)|^q \,\mathrm{d}y \right)^{\frac{1}{q}} \left\| f^{(n)} \right\|_p. \tag{2.101}$$

for every $x \in [a,b] \cap [c,d]$. The inequality (2.101) is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof. Use the identity (2.100) and apply the Hölder inequality. The proof for sharpness and the best possibility are similar as in Theorem 2.16. □

Corollary 2.38 *Suppose that all assumptions of Theorem 2.46 hold. Then for $x \in [a,b] \cap [c,d]$*

$$\left| \int_a^b w(t) f(t)\,\mathrm{d}t - \int_c^d u(t) f(t)\,\mathrm{d}t - \mathscr{F}_{w,n}^{[a,b]}(x) + \mathscr{F}_{u,n}^{[c,d]}(x) \right|$$
$$\leq \frac{2}{(n-1)!} \left(\int_a^{\max\{b,d\}} \left| (y-a)^{n-1} + (\max\{b,d\} - y)^{n-1} \right|^q \mathrm{d}y \right)^{\frac{1}{q}} \left\| f^{(n)} \right\|_p.$$

Proof. We have

$$K_n(x,y) = \frac{-1}{(n-2)!} \int\limits_{\min\{a,c\}}^{\max\{b,d\}} \left[P_w(x,t) \frac{k^{[a,b]}(y,t)}{b-a} - P_u(x,t) \frac{k^{[c,d]}(y,t)}{d-c} \right] (t-y)^{n-2} \mathrm{d}t$$

because $P_w(x,t) = 0$, for $x \notin [a,b]$ and $P_u(x,t) = 0$, for $x \notin [c,d]$. Since

$$P_w(x,t), P_u(x,t), \frac{k^{[a,b]}(y,t)}{b-a}, \frac{k^{[c,d]}(y,t)}{d-c} \in [-1,1],$$

we get

$$\left| P_w(x,t) \frac{k^{[a,b]}(y,t)}{b-a} - P_u(x,t) \frac{k^{[c,d]}(y,t)}{d-c} \right| \leq 2,$$

and then we have

$$|K_n(x,y)| \leq \frac{2}{(n-2)!} \int_a^{\max\{b,d\}} |t-y|^{n-2}\,\mathrm{d}t$$
$$= \frac{2\left((y-a)^{n-1} + (\max\{b,d\} - y)^{n-1} \right)}{(n-1)!}.$$

□

2.4 Applications for weighted 2-point and 3-point quadrature rules

In this chapter we present following weighted generalization of Montgomery identity for Riemann-Stieltjes integral which is a weighted generalization of the identity obtained by S.S Dragomir, J. Pečarić and S. Wang in [57] and use it to obtain weighted 2-point and 3-point Radau, Lobatto and Gauss quadrature rules for functions of bounded variation and for functions whose first derivatives belongs to Lp spaces. These results are published in [3] and [4].

Theorem 2.47 *Let $f:[a,b]\to\mathbb{R}$ be a function of bounded variation and $w:[a,b]\to[0,\infty\rangle$ some integrable weight function. Let also $x_1,x_2,x_3,y\in[a,b]$, $y\le x_2$. If f is continuous at x_2 and y then the following identity holds*

$$\begin{aligned}&W(x_1)f(y)+(W(x_3)-W(x_1))f(x_2)+(W(b)-W(x_3))f(b)\\&=\int_a^b w(t)f(t)\,dt+\int_a^b P_w(x_1,x_2,x_3,y,t)\,df(t)\end{aligned}\tag{2.102}$$

where $P_w(x_1,x_2,x_3,y,t)$ is the generalized weighted Peano kernel, defined by

$$P_w(x_1,x_2,x_3,y,t)=\begin{cases}W(t), & a\le t\le y,\\ W(t)-W(x_1), & y<t\le x_2,\\ W(t)-W(x_3), & x_2<t\le b,\end{cases}$$

and $W(t)=\int_a^t w(x)\,dx$ for $t\in[a,b]$.

Proof. We have

$$\begin{aligned}&\int_a^b P_w(x_1,x_2,x_3,y,t)\,df(t)=\int_a^y W(t)\,df(t)\\&+\int_y^{x_2}(W(t)-W(x_1))\,df(t)+\int_{x_2}^b(W(t)-W(x_3))\,df(t).\end{aligned}$$

Since f and P_w do not have common discontinuity points, we may use the integration by parts formula for Riemann-Stieltjes integral to obtain

$$\int_a^y W(t)\,df(t)=W(y)f(y)-\int_a^y f(t)\,dW(t),$$

$$\begin{aligned}&\int_y^{x_2}(W(t)-W(x_1))\,df(t)=(W(x_2)-W(x_1))f(x_2)\\&-(W(y)-W(x_1))f(y)-\int_y^{x_2}f(t)\,dW(t),\end{aligned}$$

and

$$\begin{aligned}&\int_{x_2}^b(W(t)-W(x_3))\,df(t)=(W(b)-W(x_3))f(b)\\&-(W(x_2)-W(x_3))f(x_2)-\int_{x_2}^b f(t)\,dW(t).\end{aligned}$$

By adding these formulas together and by using $dW(t)=w(t)\,dt$ we obtain (2.102). □

Remark 2.46 Identity (2.102) is a weighted generalization of the identity obtained by S. S. Dragomir, J. Pečarić and S. Wang in [57]. If we first take $y=a$ in (2.102) we obtain

$$W(x_1)f(a)+(W(x_3)-W(x_1))f(x_2)+(W(b)-W(x_3))f(b)$$

$$= \int_a^b w(t) f(t)\,dt + \int_a^b P_w(x_1,x_2,x_3,t)\,df(t) \tag{2.103}$$

where $P_w(x_1,x_2,x_3,t)$ is given by

$$P_w(x_1,x_2,x_3,t) = \begin{cases} W(t) - W(x_1), \ a \le t \le x_2, \\ W(t) - W(x_3) \ \ x_2 < t \le b. \end{cases} \tag{2.104}$$

Further, if we take uniform normalized weight function $w(t) = \frac{1}{b-a}$, $t \in [a,b]$ we obtain identity from [57].

Theorem 2.48 *Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$, $w : [a,b] \to [0,\infty\rangle$ some integrable weight function. Let also $x_1,x_2,x_3,y \in [a,b]$, $y \le x_2$. If f is continuous at x_2 and y then the following identity holds*

$$\begin{aligned}
&\left| W(x_1) f(y) + (W(x_3) - W(x_1)) f(x_2) + (W(b) - W(x_3)) f(b) - \int_a^b w(t) f(t)\,dt \right| \\
&\le \max\{W(y), |W(x_2) - W(x_1)|, |W(y) - W(x_1)|, \\
&|W(x_2) - W(x_3)|, W(b) - W(x_3)\} \bigvee_a^b (f).
\end{aligned}$$

Proof. By using the (2.103) and triangle inequality, we have

$$\begin{aligned}
&\left| W(x_1) f(y) + (W(x_3) - W(x_1)) f(x_2) + (W(b) - W(x_3)) f(b) - \int_a^b w(t) f(t)\,dt \right| \\
&= \left| \int_a^b P_w(x_1,x_2,x_3,y,t)\,df(t) \right| \le \left| \int_a^y P_w(x_1,x_2,x_3,y,t)\,df(t) \right| \\
&+ \left| \int_y^{x_2} P_w(x_1,x_2,x_3,y,t)\,df(t) \right| + \left| \int_{x_2}^b P_w(x_1,x_2,x_3,y,t)\,df(t) \right| \\
&= \int_a^y W(t)\,|df(t)| + \int_y^{x_2} |W(t) - W(x_1)|\,|df(t)| + \int_{x_2}^b |W(t) - W(x_3)|\,|df(t)| \\
&\le W(y) \bigvee_a^y (f) + \max\{|W(x_2) - W(x_1)|, |W(y) - W(x_1)|\} \bigvee_y^{x_2} (f) \\
&+ \max\{|W(x_2) - W(x_3)|, W(b) - W(x_3)\} \bigvee_{x_2}^b (f) \\
&\le \max\Big\{ W(y), |W(x_2) - W(x_1)|, |W(y) - W(x_1)|, \\
&|W(x_2) - W(x_3)|, W(b) - W(x_3) \Big\} \bigvee_a^b (f).
\end{aligned}$$

and the proof follows. □

2.4.1 General weighted 2-point and 3-point Radau quadrature rules for functions of bounded variation

Here we use identity (2.102) to obtain weighted 2-point and 3-point quadrature formulae of semi-closed type for functions of bounded variation.

Weighted 3-point quadrature formulae of semi-closed type

here apply previous results to establish bounds of the remainder $E(f)$ of the general weighted 3-point quadrature formula of semi-closed type:

$$\int_a^b w(t) f(t)\,\mathrm{d}t = A_1 f(y) + A_2 f(x) + A_3 f(b) + E(f) \tag{2.105}$$

where $y, x \in [a,b]$, $y \le x$ and $\sum_{k=1}^3 A_k = 1$.

Note that in our case we have

$$A_1 = W(x_1),\ A_2 = W(x_3) - W(x_1),\ A_3 = W(b) - W(x_3),$$

so if we take a normalized weight function $w(t)$, we have

$$\sum_{k=1}^3 A_k = W(b) = 1.$$

In case of **non-symmetric** weight function we can formulate the following result.

Theorem 2.49 *Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$, $w : [a,b] \to [0,\infty)$ normalized weight function. Further, let $a \le y < \frac{I_1 b - I_2}{b - I_1}$ and $\frac{I_1 b - I_2}{b - I_1} \le x \le \frac{I_1 y - I_2}{y - I_1}$ where $I_j = \int_a^b w(t) t^j \mathrm{d}t$, for $j = 1,2$. If f is continuous at y and x, then the following inequality holds:*

$$\left| \frac{I_2 - I_1(b+x) + xb}{(b-y)(x-y)} f(y) + \frac{-I_2 + I_1(b+y) - by}{(x-y)(b-x)} f(x) + \frac{I_2 - I_1(x+y) + xy}{(b-y)(b-x)} f(b) \right.$$
$$\left. - \int_a^b w(t) f(t)\mathrm{d}t \right| \le K_w(y,x) \cdot \bigvee_a^b (f),$$

where

$$K_w(y,x) = \max\left\{ W(y), \left| W(x) - \frac{I_2 - I_1(b+x) + xb}{(x-y)(b-y)} \right|, \left| W(y) - \frac{I_2 - I_1(b+x) + xb}{(x-y)(b-y)} \right|, \right.$$
$$\left. \left| W(x) - 1 + \frac{I_2 - I_1(x+y) + xy}{(b-y)(b-x)} \right|, \frac{I_2 - I_1(x+y) + xy}{(b-y)(b-x)} \right\}.$$

Proof. Obviously, $\left[a, \frac{I_1 b - I_2}{b - I_1}\right) \ne \emptyset$. For fixed $y \in \left[a, \frac{I_1 b - I_2}{b - I_1}\right)$, it is easy to check that $\left[\frac{I_1 b - I_2}{b - I_1}, \frac{I_1 y - I_2}{y - I_1}\right] \ne \emptyset$, so we choose $x \in \left[\frac{I_1 b - I_2}{b - I_1}, \frac{I_1 y - I_2}{y - I_1}\right]$. The quadrature formula (2.105) will be accurate for all polynomials of degree ≤ 2 when

$$A_1 = \frac{I_2 - I_1(b+x) + xb}{(b-y)(x-y)}$$

$$A_2 = \frac{-I_2 + I_1(b+y) - by}{(x-y)(b-x)}$$
$$A_3 = \frac{I_2 - I_1(x+y) + xy}{(b-y)(b-x)}.$$

Since $x \geq \frac{I_1 b - I_2}{b - I_1}$, $(x-y)(b-y) > 0$ we have $A_1 \geq 0$. Further, since $I_1^2 \leq I_2$ we have $y < \frac{I_1 b - I_2}{b - I_1} \leq \frac{I_1 b - I_1^2}{b - I_1} = I_1$, so we have $A_1 = \frac{bx + I_2 - I_1(x+b)}{(x-y)(b-y)} \leq 1$. Therefore, $A_1 \in [0,1]$, so there exist $x_1 \in [a,b]$ such that $W(x_1) = \frac{I_2 - I_1(b+x) + xb}{(b-y)(x-y)}$.On the other hand, since $y \leq \frac{I_1 b - I_2}{b - I_1}$ and $(b-y)(b-x) > 0$ we have $A_3 \in [0,1]$, so there exist $x_3 \in [a,b]$ such that $W(x_3) = 1 - \frac{I_2 - I_1(x+y) + xy}{(b-y)(b-x)}$.
It is easy to check that $W(x_1) \leq W(x_3)$, which implies $x_1 \leq x_3$.
Apply Theorem 2.48 with $y, x_2 = x, x_1$ and x_3 as above. □

Now, we consider the special case of **symmetric** normalized weight function on $[-1,1]$ Analogues results for interval $[a,b]$ can easily be obtained with linear transformation

$$t = \frac{b-a}{2}x + \frac{b+a}{2}.$$

Corollary 2.39 *Let $f : [-1,1] \to \mathbb{R}$ be a function of bounded variation on $[-1,1]$, $w : [-1,1] \to [0,\infty)$ symmetric normalized weight function and $-1 \leq y < -I_2 \leq x < 1$, where $I_2 = \int_{-1}^{1} w(t)t^2 \mathrm{d}t$. If f is continuous at y and x, then the following inequality holds:*

$$\left| \frac{x+I_2}{(x-y)(1-y)} f(y) + \frac{-y-I_2}{(x-y)(1-x)} f(x) + \frac{xy+I_2}{(1-y)(1-x)} f(1) - \int_{-1}^{1} w(t)f(t)\mathrm{d}t \right|$$
$$\leq K_w(y,x) \cdot \bigvee_{-1}^{1}(f), \tag{2.106}$$

where

$$K_w(y,x) = \max \left\{ W(y), \left| W(x) - \frac{x+I_2}{(x-y)(1-y)} \right|, \left| W(y) - \frac{x+I_2}{(x-y)(1-y)} \right|, \right.$$
$$\left. \left| W(x) - 1 + \frac{xy+I_2}{(1-y)(1-x)} \right|, \frac{xy+I_2}{(1-y)(1-x)} \right\}.$$

Proof. Apply Theorem 2.49 with $a = -1$, $b = 1$, and $I_1 = 0$. □

Remark 2.47 If we put $y = -1$ in (2.106), then $-I_2 \leq x \leq I_2$. Specially, for $x = -I_2$ or $x = I_2$, the Radau 2−point inequalities are obtained. For $x = 0$, the Lobatto 3−point inequality is obtained. If we put $x = -\frac{I_2}{y}$, for some $y \in (-1, -I_2)$, the inequality related to the open 2−point quadrature formula is obtained. Specially, for $x = -y = \sqrt{I_2}$, the Gauss 2−point inequality is obtained:

$$\left| \frac{1}{2} f(-\sqrt{I_2}) + \frac{1}{2} f(\sqrt{I_2}) - \int_{-1}^{1} w(t)f(t)\mathrm{d}t \right| \leq K_w(-\sqrt{I_2}, \sqrt{I_2}) \cdot \bigvee_{-1}^{1}(f),$$

where

$$K_w(-\sqrt{I_2},\sqrt{I_2}) = \max\left\{W(-\sqrt{I_2}), \left|W(\sqrt{I_2})-\frac{1}{2}\right|, \left|W(-\sqrt{I_2})-\frac{1}{2}\right|, 1-W(\sqrt{I_2})\right\}.$$

Weighted 2−point quadrature formulae of semi-closed type

In this section we apply all the results to establish bounds of the remainder $E(f)$ of the general weighted 2-point quadrature formula of semi-closed type.

$$\int_a^b w(t)f(t)\,\mathrm{d}t = A_1 f(a) + A_2 f(x) + E(f), \tag{2.107}$$

where $A_1 + A_2 = 1$. Note that in this case we have $x_3 = b$, $A_1 = W(x_1)$ and $A_2 = W(b) - W(x_1)$, so if $w(t)$ is normalized weight function, then $A_1 + A_2 = W(b) = 1$.

In the case of **non-symmetric** weight function, we formulate the following result concerning quadrature formulae with degree of exactness 1.

Theorem 2.50 *Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$, $w : [a,b] \to [0,\infty\rangle$ is some normalized weight function, $I \le x \le b$, where $I = \int_a^b w(t)t\,\mathrm{d}t$. If f is continuous at x and a, then the following inequality holds*

$$\left|\frac{I-x}{a-x}f(a) + \frac{a-I}{a-x}f(x) - \int_a^b w(t)f(t)\,\mathrm{d}t\right| \le K_w(x)\bigvee_a^b(f),$$

where

$$K_w(x) = \max\left\{\frac{I-x}{a-x}, W(x) - \frac{I-x}{a-x}, 1-W(x)\right\},$$

and $W(t) = \int_a^t w(s)\,\mathrm{d}s$.

Proof. Since $a < I < b$, we have $[I,b] \neq \emptyset$. The quadrature formula (2.107) will be accurate for all polynomials of degree ≤ 1 when

$$A_1 = \frac{I-x}{a-x}, \quad A_2 = \frac{a-I}{a-x}.$$

Since $x \in [I,b]$, we have $A_1 = \frac{I-x}{a-x} \in [0,1]$. Apply Theorem 2.48 with $y = a$, $x_2 = x$, $x_3 = b$ and x_1 such that $\int_a^{x_1} w(t)\,\mathrm{d}t = \frac{I-x}{a-x}$. It is easy to check that $\frac{I-x}{a-x} \le W(x)$, so the assertion follows directly. □

In case of **symmetric** normalized weight function we can formulate the following result.

Corollary 2.40 *Let $f : [-1,1] \to \mathbb{R}$ be a function of bounded variation on $[-1,1]$, $w : [-1,1] \to [0,\infty\rangle$ is some symmetric normalized weight function and $0 \le x \le 1$. If f is continuous at x and -1, then the following inequality holds*

$$\left|\frac{x}{1+x}f(-1) + \frac{1}{1+x}f(x) - \int_{-1}^1 w(t)f(t)\,\mathrm{d}t\right| \le K_w(x)\bigvee_{-1}^1(f), \tag{2.108}$$

where

$$K_w(x) = \max\{\frac{x}{1+x}, W(x) - \frac{x}{1+x}, 1 - W(x)\}$$

and $W(t) = \int_{-1}^{t} w(s)\mathrm{d}s$.

Proof. Apply Theorem 2.50 with $a = -1$, $b = 1$ and $I_1 = 0$. □

Remark 2.48 If we take $x = 1$, then $K_w(1) = \frac{1}{2}$, so the inequality (2.108) reduces to the weighted trapezoid inequality

$$\left| \frac{1}{2} f(-1) + \frac{1}{2} f(1) - \int_{-1}^{1} w(t) f(t) \mathrm{d}t \right| \leq \frac{1}{2} \bigvee_{-1}^{1} (f).$$

If we take $x = 0$, then $K_w(0) = \frac{1}{2}$, so the inequality (2.108) reduces to the weighted midpoint inequality

$$\left| f(0) - \int_{-1}^{1} w(t) f(t) \mathrm{d}t \right| \leq \frac{1}{2} \bigvee_{-1}^{1} (f).$$

Remark 2.49 In order to enlarge degree of exactness, we have to consider quadrature formulae of Radau type which are accurate for all polynomials of degree ≤ 2. Therefore we have to take $x = \int_{-1}^{1} w(t) t^2 \mathrm{d}t$ in (2.108).

The next result establishes the inequality related to the 2-point semi-closed quadrature formula with the minimal constant $K(x)$.

Corollary 2.41 *Let* $f : [-1,1] \to \mathbb{R}$ *be a function of bounded variation on* $[-1,1]$, $w : [-1,1] \to [0,\infty\rangle$ *is some symmetric normalized weight function such that* $\int_{-1}^{\frac{1}{2}} w(t)\mathrm{d}t = \frac{2}{3}$. *If* f *is continuous at* $\frac{1}{2}$ *and* -1, *then the following inequality holds*

$$\left| \frac{1}{3} f(-1) + \frac{2}{3} f\left(\frac{1}{2}\right) - \int_{-1}^{1} w(t) f(t) \mathrm{d}t \right| \leq \frac{1}{3} \bigvee_{-1}^{1} (f).$$

Proof. Obviously, constant $K_w(x)$ is minimal if

$$\frac{x}{1+x} = W(x) - \frac{x}{1+x} = 1 - W(x) = \frac{1}{3},$$

where $W(t) = \int_{-1}^{t} w(s)\mathrm{d}s$. That is $x = \frac{1}{2}$ and $W(x) = \frac{2}{3}$. Therefore the proof follows by applying inequality (2.108) with $x = \frac{1}{2}$. □

Remark 2.50 Applications for some special weight functions such as $w(t) = \frac{1}{2}, t \in [-1,1]$; $w(t) = \frac{1}{\pi\sqrt{1-t^2}}, t \in \langle -1,1 \rangle$; $w(t) = \frac{2}{\pi}\sqrt{1-t^2}, t \in [-1,1]$; $w(t) = \frac{3}{2}\sqrt{t}, t \in [0,1]$ and $w(t) = \frac{1}{2\sqrt{t}}, t \in \langle 0,1]$ are given in [3].

2.4.2 General weighted two-point Radau and Gauss and three-point Lobatto quadrature rules for functions in L_p spaces

Here we use identity (2.102) in special case for differentiable functions f to obtain new sharp weighted generalization of Ostrowski type inequality, as well as weighted 2-point Radau and Gauss and 3-point Lobatto quadrature formulae for functions whose first derivatives belongs to Lp spaces.

Theorem 2.51 *Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$, and $f' : [a,b] \to \mathbb{R}$ integrable on $[a,b]$, $w : [a,b] \to [0,\infty\rangle$ is some integrable weight function, $W(t) = \int_a^t w(x)\,\mathrm{d}x$ for $t \in [a,b]$. Let also $x_1, x_2, x_3, y \in [a,b]$ be such that $y \le x_2$ and (p,q) a pair of conjugate exponents, $\frac{1}{p} + \frac{1}{q} = 1$, $1 \le p,q \le \infty$ and $f' \in L_p[a,b]$. Then the following inequality holds*

$$\left| W(x_1) f(y) + (W(x_3) - W(x_1)) f(x_2) + (W(b) - W(x_3)) f(b) - \int_a^b w(t) f(t)\,\mathrm{d}t \right|$$
$$\le \|P_w(x_1,x_2,x_3,y,\cdot)\|_q \left\|f'\right\|_p .$$

The constant $\|P_w(x_1,x_2,x_3,y,\cdot)\|_q$ is sharp for $1 < p \le \infty$ and the best possible for $p = 1$.

Proof. By applying identity (2.102) for differentiable function f and then taking the modulus and applying the Hölder inequality. The rest of proof is similar to the proof of the Theorem 2.33. □

Weighted 2-point formulae of semi-closed type

We apply the results from the Theorem 2.51 with $y = a$, $x_2 = x$ and $x_3 = b$. Note that in our case for 2-point quadrature formula (2.107) of Radau type we have

$$A_1 = W(x_1), \; A_2 = W(b) - W(x_1),$$

so if $w(t)$ is a normalized weight function then $A_1 + A_2 = W(b) = 1$.

The quadrature formula (2.107) will be accurate for all polynomials of degree ≤ 1 if

$$A_1 = \frac{x-I}{x-a}, \qquad A_2 = \frac{I-a}{x-a}$$

where $I = \int_a^b w(t)\,t\,\mathrm{d}t$, $x \in [I,b]$ and x_1 is such that $W(x_1) = \frac{x-I}{x-a}$.

In case of **symmetric** normalized weight function we can formulate the following result concerning quadrature formulae with degree of exactness 1.

Theorem 2.52 *Assume (p,q) are conjugate exponents, $1 \le p,q \le \infty$. Let $f : [-1,1] \to \mathbb{R}$ be a differentiable function such that $f' \in L_p[-1,1]$ and let $x \in [0,1]$. Further, let $w : [-1,1] \to [0,+\infty\rangle$ be some symmetric normalized weight function such that $W(t) = \int_{-1}^t w(s)\mathrm{d}s$. Then the following inequality holds:*

$$\left| \frac{x}{1+x} f(-1) + \frac{1}{1+x} f(x) - \int_{-1}^1 w(t) f(t)\,\mathrm{d}t \right| \le K_w(x,q) \|f'\|_p,$$

where $K_w(x,q) = \|P_w(x,\cdot)\|_q$ *and*

$$P_w(x,t) = \begin{cases} 0, & t=-1, \\ W(t) - \frac{x}{x+1}, & -1<t\le x, \\ W(t)-1, & x<t\le 1. \end{cases}$$

Proof. Let $x \in [0,1]$ be fixed node. The quadrature formula (2.107) will be accurate for all polynomials of degree ≤ 1 when

$$A_1 = \frac{x}{1+x}, \quad A_2 = \frac{1}{x+1}.$$

Since $x \in [0,1]$ we have $A_1 = \frac{x}{1+x} \in \left[0, \frac{1}{2}\right]$. Apply Theorem 2.51 with $[a,b] = [-1,1]$, $y=-1$, $x_2 = x$, $x_3 = 1$ and x_1 such that $\int_{-1}^{x_1} w(t)\,\mathrm{d}t = \frac{x}{1+x}$. □

In the case of **non-symmetric** weight function, we formulate the following result concerning quadrature formulae with degree of exactness 1.

Theorem 2.53 *Assume* (p,q) *are conjugate exponents,* $1 \le p,q \le \infty$. *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable function such that* $f' \in L_p[a,b]$. *Further, let* $w : [a,b] \to [0,+\infty\rangle$ *be some normalized weight function such that* $W(t) = \int_a^t w(s)\mathrm{d}s$ *and let* $x \in [I,b]$, *where* $I = \int_a^b w(t)\,t\,\mathrm{d}t$. *Then the following inequality holds:*

$$\left| \frac{x-I}{x-a} f(a) + \frac{I-a}{x-a} f(x) - \int_a^b w(t) f(t)\,\mathrm{d}t \right| \le K_w(x,q)\,\|f'\|_p,$$

where

$$K_w(x,q) = \|P_w(x,\cdot)\|_q,$$

and

$$P_w(x,t) = \begin{cases} 0, & t=a, \\ W(t) - \frac{x-I}{x-a}, & a<t\le x, \\ W(t)-1, & x<t\le b. \end{cases}$$

Proof. Since $a < I < b$, we have $[I,b] \neq \emptyset$. The quadrature formula (2.107) will be accurate for all polynomials of degree ≤ 1 when

$$A_1 = \frac{x-I}{x-a}, \quad A_2 = \frac{I-a}{x-a}.$$

Since $x \in [I,b]$, we have $A_1 = \frac{x-I}{x-a} \in [0,1]$. Apply Theorem 2.51 with $y=a$, $x_2 = x$, $x_3 = b$ and x_1 such that $\int_a^{x_1} w(t)\,\mathrm{d}t = \frac{x-I}{x-a}$, so the assertion follows directly. □

Remark 2.51 Applications of these inequalities for special weight functions $w(t) = \frac{1}{2}, t \in [-1,1]$; $w(t) = \frac{1}{\pi\sqrt{1-t^2}}, t \in \langle -1,1\rangle$ and $w(t) = \frac{1}{2\sqrt{t}}, t \in \langle 0,1]$ in case $q = 1,2,\infty$ are given in [4]. Also, in each case the minimal value for the constants $K_w(x,q)$ is obtained.

Weighted 3-point formulae of closed type

Here, we establish bounds of the remainder $E(f)$ of the general weighted 3-point quadrature formula of Lobatto type

$$\int_a^b w(t) f(t) \, \mathrm{d}t = A_1 f(a) + A_2 f(x) + A_3 f(b) + E(f), \tag{2.109}$$

where $A_1 + A_2 + A_3 = 1$.We apply the results from the Theorem 2.51 with $y = a$ and $x_2 = x$. Note that in our case we have

$$A_1 = W(x_1), \; A_2 = W(x_3) - W(x_1), \; A_3 = W(b) - W(x_3),$$

so if $w(t)$ is a normalized weight function then $A_1 + A_2 + A_3 = W(b) = 1$.

The quadrature formula (2.109) will be accurate for all polynomials of degree ≤ 2 if

$$A_1 = \frac{I_2 - I_1(b+x) + xb}{(b-a)(x-a)}, \; A_2 = \frac{-I_2 + I_1(b+a) - ba}{(x-a)(b-x)}, \; A_3 = \frac{I_2 - I_1(x+a) + xa}{(b-a)(b-x)}$$

where $I_j = \int_a^b w(t) t^j \mathrm{d}t$, for $j = 1, 2$, and x_1 is such that $W(x_1) = \frac{I_2 - I_1(b+x) + xb}{(b-a)(x-a)}$, x_3 is such that $1 - W(x_3) = \frac{I_2 - I_1(x+a) + xa}{(b-a)(b-x)}$.

In case of **symmetric** normalized weight function we can formulate the following result concerning quadrature formulae with degree of exactness 2.

Theorem 2.54 *Assume (p,q) are conjugate exponents, $1 \leq p, q \leq \infty$. Let $f : [-1,1] \to \mathbb{R}$ be a differentiable function such that $f' \in L_p[-1,1]$. Further, let $w : [-1,1] \to [0, +\infty\rangle$ be some symmetric normalized weight function such that $W(t) := \int_{-1}^t w(s) \mathrm{d}s$. Then if $x \in [-I_2, I_2]$, where $I_2 = \int_{-1}^1 w(t) t^2 \mathrm{d}t$ the following inequality holds:*

$$\left| \frac{x + I_2}{2(x+1)} f(-1) + \frac{1 - I_2}{(x+1)(1-x)} f(x) + \frac{I_2 - x}{2(1-x)} f(1) - \int_{-1}^1 w(t) f(t) \, \mathrm{d}t \right|$$
$$\leq K_w(x,q) \|f'\|_p,$$

where $K_w(x,q) = \|P_w(x,\cdot)\|_q$ and

$$P_w(x,t) = \begin{cases} 0, & t = -1, \\ W(t) - \frac{x+I_2}{2(x+1)}, & -1 < t \leq x, \\ W(t) + \frac{I_2 + x - 2}{2(1-x)}, & x < t \leq 1. \end{cases}$$

Proof. Let $x \in [-I_2, I_2]$ be fixed node. The quadrature formula (2.109) will be accurate for all polynomials of degree ≤ 2 when

$$A_1 = \frac{x + I_2}{2(x+1)}, \; A_2 = \frac{1 - I_2}{(x+1)(1-x)}, \; A_3 = \frac{I_2 - x}{2(1-x)}.$$

Since $x \geq -I_2$ and $I_2 < 1$, we have $A_1 \geq 0$ and

$$A_1 = \frac{x+I_2}{2(x+1)} \leq \frac{x+1}{2(x+1)} = \frac{1}{2},$$

we have $A_1 \in [0, \frac{1}{2}]$, so there exist $x_1 \in [-1,0]$ such that $W(x_1) = \frac{x+I_2}{2(x+1)}$. On the other hand, since $x \leq I_2 < 1$

$$A_3 = \frac{I_2 - x}{2(1-x)} \geq 0,$$
$$A_3 = \frac{I_2 - x}{2(1-x)} < \frac{1}{2}$$

we have $1 - A_3 \in \left\langle \frac{1}{2}, 1\right]$. So there exist $x_3 \in \langle 0,1]$ such that $W(x_3) = \frac{-I_2-x+2}{2(1-x)}$. Apply Theorem 2.51 with $[a,b] = [-1,1]$, $y = -1$, $x_2 = x$, and x_1 and x_3 as above. $\square$

In the case of **non-symmetric** weight function, we formulate the following result concerning quadrature formulae with degree of exactness 2.

Theorem 2.55 *Assume (p,q) are conjugate exponents, $1 \leq p,q \leq \infty$. Let $f : [a,b] \to \mathbb{R}$ be a differentiable function such that $f' \in L_p[a,b]$. Further, let $w : [a,b] \to [0,+\infty\rangle$ be some normalized weight function such that $W(t) = \int_a^t w(s)\mathrm{d}s$ and let $x \in [\frac{I_1 b - I_2}{b-I_1}, \frac{I_2 - I_1 a}{I_1 - a}]$, where where $I_j = \int_a^b w(t)t^j \mathrm{d}t$, for $j = 1,2$. Then the following inequality holds:*

$$\left| \frac{I_2 - I_1(b+x) + xb}{(b-a)(x-a)} f(a) + \frac{-I_2 + I_1(b+a) - ba}{(x-a)(b-x)} f(x) + \frac{I_2 - I_1(x+a) + xa}{(b-a)(b-x)} f(b) \right.$$
$$\left. - \int_a^b w(t) f(t) \mathrm{d}t \right| \leq K_w(x,q) \|f'\|_p,$$

where

$$K_w(x,q) = \|P_w(x,\cdot)\|_q,$$

and

$$P_w(x,t) = \begin{cases} 0, & t = a, \\ W(t) - \frac{I_2 - I_1(b+x) + xb}{(b-a)(x-a)}, & a < t \leq x, \\ W(t) + \frac{I_2 - I_1(x+a) + xa}{(b-a)(b-x)} - 1, & x < t \leq b. \end{cases}$$

Proof. First, we will prove that $\frac{I_1 b - I_2}{b-I_1} < \frac{I_2 - I_1 a}{I_1 - a}$. Since, $\frac{I_2 - I_1 a}{I_1 - a} - \frac{I_1 b - I_2}{b - I_1} = \frac{(b-a)(I_2 - I_1^2)}{(I_1 - a)(b - I_1)}$, this is equivalent to $I_1^2 \leq I_2$. By using the Cauchy-Schwarz inequality we have

$$I_1^2 = \left(\int_a^b w(t)t\mathrm{d}t\right)^2 \leq \left(\int_a^b \left(\sqrt{w(t)}t\right)^2 \mathrm{d}t\right)\left(\int_a^b \left(\sqrt{w(t)}\right)^2 \mathrm{d}t\right)$$
$$= \left(\int_a^b w(t)t^2\mathrm{d}t\right)\left(\int_a^b w(t)\mathrm{d}t\right) = I_2.$$

Thus $[\frac{I_1 b - I_2}{b - I_1}, \frac{I_2 - I_1 a}{I_1 - a}] \neq \emptyset$. Now, let $x \in [\frac{I_1 b - I_2}{b - I_1}, \frac{I_2 - I_1 a}{I_1 - a}]$ be a fixed node. The quadrature formula (2.109) will be accurate for all polynomials of degree ≤ 2 when

$$A_1 = \frac{I_2 - I_1(b+x) + xb}{(b-a)(x-a)}$$
$$A_2 = \frac{-I_2 + I_1(b+a) - ba}{(x-a)(b-x)}$$
$$A_3 = \frac{I_2 - I_1(x+a) + xa}{(b-a)(b-x)}.$$

Since $x \geq \frac{I_1 b - I_2}{b - I_1}$ we have $I_2 - I_1(b+x) + xb \geq 0$ and $A_1 \geq 0$. Similarly, since $x \leq \frac{I_2 - I_1 a}{I_1 - a}$ we have $I_2 - I_1(x+a) + xa \geq 0$ and $A_3 \geq 0$. Finally, we have

$$(t-a)(t-b) \leq 0$$
$$t^2 + ab \leq (a+b)t$$
$$\int_a^b \left(t^2 + ab\right) w(t)\mathrm{d}t \leq \int_a^b (a+b)tw(t)\mathrm{d}t$$
$$I_2 + ab \leq I_1(a+b)$$

so it follows that $A_2 \geq 0$. Further, since $A_1 + A_2 + A_3 = 1$, we have $A_1, A_2, A_3 \in [0,1]$, so there exist $x_1, x_3 \in [a,b]$ such that $W(x_1) = A_1$ and $W(x_3) = 1 - A_3$. Apply Theorem 2.51 with $y = a$, $x_2 = x$, x_1 and x_3 as above and the assertion follows directly. □

Remark 2.52 Applications of these inequalities for special weight functions $w(t) = \frac{1}{2}, t \in [-1,1]$ and $w(t) = \frac{1}{\pi\sqrt{1-t^2}}, t \in \langle -1,1 \rangle$ in case $q = 1, 2, \infty$ are given in [4]. Also, in each case the minimal value for the constants $K_w(x,q)$ is obtained.

Weighted 2-point formulae of open type

Now we establish bounds of the remainder $E(f)$ of the general weighted 2-point quadrature formula of Gauss type

$$\int_a^b w(t) f(t)\,\mathrm{d}t = A_1 f(y) + A_2 f(x) + E(f), \tag{2.110}$$

where $A_1 + A_2 = 1$. We apply the results from the Theorem 2.51 with $x_2 = x$ and $x_3 = b$. Note that in our case we have

$$A_1 = W(x_1), \; A_2 = W(b) - W(x_1),$$

so if $w(t)$ is a normalized weight function then $A_1 + A_2 = W(b) = 1$.

The quadrature formula (2.110) will be accurate for all polynomials of degree ≤ 1 if

$$A_1 = \frac{x - I}{x - y}, \quad A_2 = \frac{I - y}{x - y}$$

where $I=\int_a^b w(t)t\mathrm{d}t$, $y\in[a,I]$, $x\in[I,b]$ and x_1 is such that $W(x_1)=\frac{x-I}{x-y}$.

In case of **symmetric** normalized weight function we can formulate the following result concerning quadrature formulae with degree of exactness 1.

Theorem 2.56 *Assume (p,q) are conjugate exponents, $1\le p,q\le\infty$. Let $f:[-1,1]\to\mathbb{R}$ be a differentiable function such that $f'\in L_p[-1,1]$ and let $y\in[-1,0]$, $x\in[0,1]$. Further, let $w:[-1,1]\to[0,+\infty\rangle$ be some symmetric normalized weight function such that $W(t)=\int_{-1}^t w(s)\mathrm{d}s$. Then the following inequality holds:*

$$\left|\frac{x}{x-y}f(y)+\frac{-y}{x-y}f(x)-\int_{-1}^1 w(t)f(t)\mathrm{d}t\right|\le K_w(y,x,q)\,\|f'\|_p,$$

where $K_w(y,x,q)=\|P_w(y,x,\cdot)\|_q$ and

$$P_w(y,x,t)=\begin{cases} W(t), & -1\le t\le y,\\ W(t)-\frac{x}{x-y}, & y<t\le x,\\ W(t)-1, & x<t\le 1.\end{cases}$$

Proof. Let $x\in[0,1]$ be fixed node. The quadrature formula (2.110) will be accurate for all polynomials of degree ≤ 1 when

$$A_1=\frac{x}{x-y},\quad A_2=\frac{-y}{x-y}.$$

Since $y\in[-1,0]$, $x\in[0,1]$ we have $A_1=\frac{x}{x-y}\in[0,1]$. Apply Theorem 2.51 with $[a,b]=[-1,1]$, $x_2=x$, $x_3=1$ and x_1 such that $\int_{-1}^{x_1}w(t)\mathrm{d}t=\frac{x}{x-y}$. □

In the case of **non-symmetric** weight function, we formulate the following result concerning quadrature formulae with degree of exactness 1.

Theorem 2.57 *Assume (p,q) are conjugate exponents, $1\le p,q\le\infty$. Let $f:[a,b]\to\mathbb{R}$ be a differentiable function such that $f'\in L_p[a,b]$. Further, let $w:[a,b]\to[0,+\infty\rangle$ be some normalized weight function such that $W(t)=\int_a^t w(s)\mathrm{d}s$ and let $y\in[a,I]$, $x\in[I,b]$, where $I=\int_a^b w(t)t\mathrm{d}t$. Then the following inequality holds:*

$$\left|\frac{x-I}{x-y}f(y)+\frac{I-y}{x-y}f(x)-\int_a^b w(t)f(t)\mathrm{d}t\right|\le K_w(y,x,q)\,\|f'\|_p,$$

where

$$K_w(y,x,q)=\|P_w(y,x,\cdot)\|_q,$$

and

$$P_w(y,x,t)=\begin{cases} W(t), & a\le t\le y,\\ W(t)-\frac{x-I}{x-y}, & y<t\le x,\\ W(t)-1, & x<t\le b.\end{cases}$$

Proof. Since $a < I < b$, we have $[I,b] \neq \emptyset$. The quadrature formula (2.110) will be accurate for all polynomials of degree ≤ 1 when

$$A_1 = \frac{x-I}{x-y}, \quad A_2 = \frac{I-y}{x-y}.$$

Since $y \in [a,I]$, $x \in [I,b]$, we have $A_1 = \frac{x-I}{x-y} \in [0,1]$. Apply Theorem 2.51 with $x_2 = x$, $x_3 = b$ and x_1 such that $\int_a^{x_1} w(t)\,\mathrm{d}t = \frac{x-I}{x-y}$ and the assertion follows directly. □

Remark 2.53 Applications of these inequalities for special weight functions $w(t) = \frac{1}{2}, t \in [-1,1]$; $w(t) = \frac{1}{\pi\sqrt{1-t^2}}, t \in \langle -1,1 \rangle$ and $w(t) = \frac{2}{\pi}\sqrt{1-t^2}$, $t \in [-1,1]$ in case $q = 1,2,\infty$ are given in [4]. Also, in each case the minimal value for the constants $K_w(x,q)$ is obtained.

2.5 Applications for estimates of approximations for the integral transforms

In this chapter error estimates of approximations in real domain for the Fourier transform and in complex domain for the Laplace transform are given for functions which vanish beyond a finite domain $[a,b]$ and such that $f' \in L_p[a,b]$. New inequalities involving Fourier and Laplace transform of f, integral mean of f and exponential mean of the endpoints of the domain of f are presented and used to obtain two associated numerical rules and error bounds of their remainders in each case. These results are published in [10] and [2].

2.5.1 Error estimates of approximations for the Fourier transform

The Fourier transform $\mathscr{F}(g)(x)$ of Lebegue integrable mapping which vanish beyond a finite domain $[a,b] \subset \langle -\infty, \infty \rangle$, $g : [a,b] \to \mathbb{R}$ is defined by

$$\mathscr{F}(g)(x) = \int_a^b g(t)\, e^{-2\pi i x t}\,\mathrm{d}t.$$

In the paper [28] the following theorem was proved:

Theorem 2.58 *Let $g : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$, then for all $x \neq 0$ we have the inequality*

$$\left| \mathscr{F}(g)(x) - E(-2\pi i x a, -2\pi i x b) \int_a^b g(s)\,\mathrm{d}s \right| \leq \frac{3}{4}(b-a) \bigvee_a^b (g).$$

Here $\bigvee_a^b(g)$ *is total variation of* g *on* $[a,b]$, $E(z,w)$ *is exponential mean of* z *and* w

$$E(z,w)=\begin{cases}\frac{e^z-e^w}{z-w}, & \text{if } z\neq w,\\ e^w, & \text{if } z=w.\end{cases}\qquad z,w\in\mathbb{C} \tag{2.111}$$

Also, the next theorem was proved in [25]:

Theorem 2.59 *Let* $g:[a,b]\to\mathbb{R}$ *be an absolutely continuous mapping on* $[a,b]$. *Then for all* $x\neq 0$ *we have the inequality*

$$\left|\mathscr{F}(g)(x)-E(-2\pi ixa,-2\pi ixb)\int_a^b g(s)\,\mathrm{d}s\right|$$
$$\leq\begin{cases}\frac{1}{3}(b-a)^2\|g'\|_\infty, & \text{if } g'\in L_\infty[a,b],\\ \frac{2^{\frac{1}{q}}}{[(q+1)(q+2)]^{\frac{1}{q}}}(b-a)^{1+\frac{1}{q}}\|g'\|_p, & \text{if } g'\in L_p[a,b],\\ (b-a)\|g'\|_1, & \text{if } g'\in L_1[a,b].\end{cases}$$

where $E(z,w)$ *is given by (2.111).*

Here we give another error estimates of the same approximations from these two theorems.

Theorem 2.60 *Let* $g:[a,b]\to\mathbb{R}$ *be a mapping of bounded variation on* $[a,b]$, *then for all* $x\neq 0$ *we have the inequality .*

$$\left|\mathscr{F}(g)(x)-E(-2\pi ixa,-2\pi ixb)\int_a^b g(s)\,\mathrm{d}s\right|\leq\frac{1}{\pi|x|}\bigvee_a^b(g),$$

where $\bigvee_a^b(g)$ *is total variation of* g *on* $[a,b]$ *and* $E(z,w)$ *is given by (2.111).*

Proof. Multiplying Montgomery identity $g(t)=\frac{1}{b-a}\int_a^b g(s)\,\mathrm{d}s+\int_a^b P(t,s)\,\mathrm{d}g(s)$ with the kernel of Fourier transform and then integrating on $[a,b]$ gives us

$$\begin{aligned}\mathscr{F}(g)(x)&=\int_a^b g(t)e^{-2\pi ixt}\,\mathrm{d}t\\ &=\frac{1}{b-a}\int_a^b\left[\int_a^b g(s)\,\mathrm{d}s+\int_a^t(s-a)\,\mathrm{d}g(s)+\int_t^b(s-b)\,\mathrm{d}g(s)\right]e^{-2\pi ixt}\,\mathrm{d}t.\end{aligned}$$

By an interchange of the order of integration we get

$$\int_a^b\left(\int_a^b g(s)\,\mathrm{d}s\right)e^{-2\pi ixt}\,\mathrm{d}t=E(-2\pi ixa,-2\pi ixb)(b-a)\int_a^b g(s)\,\mathrm{d}s,$$

$$\int_a^b\left(\int_a^t(s-a)\,\mathrm{d}g(s)\right)e^{-2\pi ixt}\,\mathrm{d}t=\int_a^b\left(\frac{e^{-2\pi ixb}-e^{-2\pi ixs}}{-2\pi ix}\right)(s-a)\,\mathrm{d}g(s),$$

$$\int_a^b \left(\int_t^b (s-b)\,\mathrm{d}g(s) \right) e^{-2\pi ixt}\mathrm{d}t = \int_a^b \left(\frac{e^{-2\pi ixs} - e^{-2\pi ixa}}{-2\pi ix} \right) (s-b)\,\mathrm{d}g(s).$$

So we have

$$\mathscr{F}(g)(x) - E(-2\pi ixa, -2\pi ixb) \int_a^b g(s)\,\mathrm{d}s = \int_a^b \frac{e^{-2\pi ixs}}{2\pi ix}\mathrm{d}g(s) \tag{2.112}$$
$$+ \left[\int_a^b \frac{e^{-2\pi ixb}}{-2\pi ix} \left(\frac{s-a}{b-a} \right) \mathrm{d}g(s) + \int_a^b \left(\frac{e^{-2\pi ixa}}{-2\pi ix} \right) \left(\frac{b-s}{b-a} \right) \mathrm{d}g(s) \right]$$

and the proof follows since

$$\left| \mathscr{F}(g)(x) - E(-2\pi ixa, -2\pi ixb) \int_a^b g(s)\,\mathrm{d}s \right|$$
$$\leq \sup_{s\in[a,b]} \left| \frac{e^{-2\pi ixs}}{2\pi ix} + \left(\frac{s-a}{b-a} \right) \frac{e^{-2\pi ixb}}{-2\pi ix} + \left(\frac{b-s}{b-a} \right) \frac{e^{-2\pi ixa}}{-2\pi ix} \right| \bigvee_a^b (g)$$
$$\leq \frac{1}{2\pi|x|} \left| 1 + \frac{s-a}{b-a} + \frac{b-s}{b-a} \right| \bigvee_a^b (g) = \frac{2}{2\pi|x|} \bigvee_a^b (g).$$

□

Remark 2.54 Whenever it holds $|x| > \frac{4}{3\pi(b-a)}$ for $x \in [a,b]$, the Theorem 2.60 gives better estimate of the Theorem 2.58.

Next theorem is the result for L_p spaces.

Theorem 2.61 *Assume (p,q) is a pair of conjugate exponents. Let $g : [a,b] \to \mathbb{R}$ be absolutely continuous such that $g' \in L_p[a,b]$. Then for $1 < p \leq \infty$, and for all $x \neq 0$ we have the inequality*

$$\left| \mathscr{F}(g)(x) - E(-2\pi ixa, -2\pi ixb) \int_a^b g(s)\,\mathrm{d}s \right| \leq \frac{(b-a)^{\frac{1}{q}}}{\pi|x|} \left\| g' \right\|_p,$$

while for $p = 1$ we have

$$\left| \mathscr{F}(g)(x) - E(-2\pi ixa, -2\pi ixb) \int_a^b g(s)\,\mathrm{d}s \right| \leq \frac{1}{\pi|x|} \left\| g' \right\|_1.$$

Proof. Using formula (2.112) with $\mathrm{d}g(s) = g'(s)\,\mathrm{d}s$ we have

$$\mathscr{F}(g)(x) - E(-2\pi ixa, -2\pi ixb) \int_a^b g(s)\,\mathrm{d}s = \int_a^b \frac{e^{-2\pi ixs}}{2\pi ix} g'(s)\,\mathrm{d}s$$
$$+ \left[\int_a^b \frac{e^{-2\pi ixb}}{-2\pi ix} \left(\frac{s-a}{b-a} \right) g'(s)\,\mathrm{d}s + \int_a^b \left(\frac{e^{-2\pi ixa}}{-2\pi ix} \right) \left(\frac{b-s}{b-a} \right) g'(s)\,\mathrm{d}s \right].$$

For $1 < p \leq \infty$, by applying Hölder inequality we obtain

$$\begin{aligned}
&\left|\mathscr{F}(g)(x) - E(-2\pi ixa, -2\pi ixb)\int_a^b g(s)\,\mathrm{d}s\right| \\
&\leq \left\| \frac{e^{-2\pi ixs}}{2\pi ix} + \left(\frac{s-a}{b-a}\right)\frac{e^{-2\pi ixb}}{-2\pi ix} + \left(\frac{b-s}{b-a}\right)\frac{e^{-2\pi ixa}}{-2\pi ix} \right\|_q \left\|g'\right\|_p \\
&\leq \left\| \frac{1}{2\pi ix}\left(1 + \frac{s-a}{b-a} + \frac{b-s}{b-a}\right) \right\|_q \left\|g'\right\|_p \leq \left\| \frac{1}{\pi ix} \right\|_q \left\|g'\right\|_p \\
&= \frac{(b-a)^{\frac{1}{q}}}{\pi |x|} \left\|g'\right\|_p .
\end{aligned}$$

Similarly for $p = 1$ we have

$$\begin{aligned}
&\left|\mathscr{F}(g)(x) - E(-2\pi ixa, -2\pi ixb)\int_a^b g(s)\,\mathrm{d}s\right| \\
&\leq \sup_{s\in[a,b]} \left| \frac{e^{-2\pi ixs}}{2\pi ix} + \left(\frac{s-a}{b-a}\right)\frac{e^{-2\pi ixb}}{-2\pi ix} + \left(\frac{b-s}{b-a}\right)\frac{e^{-2\pi ixa}}{-2\pi ix} \right| \left\|g'\right\|_1 \\
&\leq \frac{1}{2\pi |x|}\left|1 + \frac{s-a}{b-a} + \frac{b-s}{b-a}\right| \left\|g'\right\|_1 = \frac{1}{\pi |x|}\left\|g'\right\|_1 ,
\end{aligned}$$

and the proof is done. □

Remark 2.55 Whenever it holds $|x| > \frac{1}{\pi(b-a)}\left[\frac{1}{2}(q+1)(q+2)\right]^{\frac{1}{q}}$(if $1 < p \leq \infty$) or $|x| > \frac{1}{\pi(b-a)}$ (if $p = 1$) for $x \in [a,b]$, the Theorem 2.61 gives better estimate of the Theorem 2.59.

Remark 2.56 We have

$$\mathscr{F}(g)(0) = \int_a^b g(t)\,\mathrm{d}t$$

and for $x = 0$ the left-hand side of the inequalities from the previous two theorems reduces to

$$\left|\mathscr{F}(g)(0) - E(0,0)\int_a^b g(s)\,\mathrm{d}s\right| = 0.$$

Next result is a generalization of Theorem 2.2 for weight functions that do not have to be normalized.

Lemma 2.3 *Let $f : [a,b] \cup [c,d] \to \mathbb{R}$ be an absolutely continuous function on $[a,b] \cup [c,d]$, $w : [a,b] \to \mathbb{R}$ and $u : [c,d] \to \mathbb{R}$ some weight functions, such that $\int_a^b w(t)\,\mathrm{d}t \neq 0$, $\int_c^d u(t)\,\mathrm{d}t \neq 0$ and*

$$W(x) = \begin{cases} 0, & t < a, \\ \int_a^x w(t)\,\mathrm{d}t, & a \leq t \leq b, \\ \int_a^b w(t)\,\mathrm{d}t, & t > b, \end{cases} \quad U(x) = \begin{cases} 0, & t < c, \\ \int_c^x u(t)\,\mathrm{d}t & c \leq t \leq d, \\ \int_c^d u(t)\,\mathrm{d}t, & t > d, \end{cases}$$

and $[a,b]\cap[c,d]\neq\emptyset$. *Then, for both cases* $[c,d]\subseteq[a,b]$ *and* $[a,b]\cap[c,d]=[c,b]$, *(and also for* $[a,b]\subseteq[c,d]$ *and* $[a,b]\cap[c,d]=[a,d]$*) the next formula is valid*

$$\frac{1}{\int_a^b w(t)\,\mathrm{d}t}\int_a^b w(t)f(t)\,\mathrm{d}t-\frac{1}{\int_c^d u(t)\,\mathrm{d}t}\int_c^d u(t)f(t)\,\mathrm{d}t=\int_{\min\{a,c\}}^{\max\{b,d\}}K(t)f'(t)\,\mathrm{d}t \quad (2.113)$$

where

$$K(t)=P_u(x,t)-P_w(x,t),\ t\in[\min\{a,c\},\max\{b,d\}]$$

and $P_u(x,t)$, $P_w(x,t)$ *are given by*

$$P_w(x,t)=\begin{cases}\frac{W(t)}{W(b)}, & a\le s\le x,\\ \frac{W(t)}{W(b)}-1, & x<s\le b,\end{cases}\quad,\ P_u(x,t)=\begin{cases}\frac{U(t)}{U(b)}, & c\le s\le x,\\ \frac{U(t)}{U(b)}-1, & x<s\le d.\end{cases}$$

Proof. For $x\in[a,b]\cap[c,d]$, we subtract identities

$$f(x)-\frac{1}{\int_a^b w(t)\,\mathrm{d}t}\int_a^b f(t)w(t)\,\mathrm{d}t=\int_a^b P_w(x,t)f'(t)\,\mathrm{d}t$$

and

$$f(x)-\frac{1}{\int_c^d u(t)\,\mathrm{d}t}\int_c^d f(t)u(t)\,\mathrm{d}t=\int_c^d P_u(x,t)f'(t)\,\mathrm{d}t.$$

Then put

$$K(x,t)=P_u(x,t)-P_w(x,t),\ t\in[\min\{a,c\},\max\{b,d\}].$$

$K(x,t)$ doesn't depend on x, so we put $K(t)$ instead:

$$K(t)=\begin{cases}-\frac{W(t)}{W(b)}, & t\in[a,c],\\ -\frac{W(t)}{W(b)}+\frac{U(t)}{U(d)}, & t\in\langle c,d\rangle,\quad \text{if}\quad [c,d]\subseteq[a,b],\\ 1-\frac{W(t)}{W(b)}, & t\in[d,b],\end{cases} \quad (2.114)$$

$$K(t)=\begin{cases}-\frac{W(t)}{W(b)}, & t\in[a,c],\\ -\frac{W(t)}{W(b)}+\frac{U(t)}{U(d)}, & t\in\langle c,b\rangle,\quad \text{if}\quad [a,b]\cap[c,d]=[c,b].\\ \frac{U(t)}{U(d)}-1, & t\in[b,d].\end{cases}$$

□

Next, we apply identity for the difference of the two weighted integral means (2.113) with two special weight functions: uniform weight function and kernel of the Fourier transform. In such a way new generalizations of the previous results are obtained.

Theorem 2.62 *Assume (p,q) is a pair of conjugate exponents. Let $g:[a,b]\to\mathbb{R}$ be absolutely continuous function on $[a,b]$ and $c,d\in[a,b]$, $c<d$. Then for $1<p\le\infty$, and for $x\neq 0$ we have the inequality*

$$\begin{aligned}&\left|\frac{d-c}{b-a}\mathscr{F}(g)(x)-E(-2\pi ixa,-2\pi ixb)\int_c^d g(t)\,\mathrm{d}t\right|\\&\le (d-c)\left(\frac{(2^q+1)(b-a)}{(q+1)}\right)^{\frac{1}{q}}\left\|g'\right\|_p,\end{aligned}\tag{2.115}$$

while for $p=1$ and $x\neq 0$ we have

$$\left|\frac{d-c}{b-a}\mathscr{F}(g)(x)-E(-2\pi ixa,-2\pi ixb)\int_c^d g(t)\,\mathrm{d}t\right|\le 2(d-c)\left\|g'\right\|_1,\tag{2.116}$$

where $E(z,w)$ is given by (2.111).

Proof. If we apply identity (2.113) with $f(t)=g(t)$, $w(t)=e^{-2\pi ixt}$, $u(t)=\frac{1}{d-c}$ we have $W(t)=\int_a^t e^{-2\pi ixs}\mathrm{d}s=(t-a)E(-2\pi ixa,-2\pi ixt)$,

$$\frac{1}{(b-a)E(-2\pi ixa,-2\pi ixb)}\mathscr{F}(g)(x)-\frac{1}{d-c}\int_c^d g(t)\,\mathrm{d}t=\int_a^b K(t)g'(t)\,\mathrm{d}t,$$

and, since $[c,d]\subseteq[a,b]$ by using (2.114)

$$K(t)=\begin{cases}-\frac{W(t)}{W(b)}, & t\in[a,c],\\[2ex] -\frac{W(t)}{W(b)}+\frac{t-c}{d-c}, & t\in\langle c,d\rangle,\\[2ex] 1-\frac{W(t)}{W(b)}, & t\in[d,b].\end{cases}$$

Thus

$$\frac{d-c}{b-a}\mathscr{F}(g)(x)-E(-2\pi ixa,-2\pi ixb)\int_c^d g(t)\,\mathrm{d}t=\frac{d-c}{b-a}W(b)\int_a^b K(t)g'(t)\,\mathrm{d}t$$

and by taking the modulus and applying Hölder inequality we obtain

$$\left|\frac{d-c}{b-a}\mathscr{F}(g)(x)-E(-2\pi ixa,-2\pi ixb)\int_c^d g(t)\,\mathrm{d}t\right|\le\left\|\frac{d-c}{b-a}W(b)K(t)\right\|_q\left\|g'\right\|_p.$$

Now, for $1<p\le\infty$ (for $1\le q<\infty$) we have

$$\begin{aligned}&\left\|\frac{d-c}{b-a}W(b)K(t)\right\|_q=\left(\int_a^c\left|\frac{d-c}{b-a}W(t)\right|^q\mathrm{d}t\right.\\&\left.+\int_c^d\left|\frac{d-c}{b-a}W(t)-\frac{t-c}{b-a}W(b)\right|^q\mathrm{d}t+\int_d^b\left|\frac{d-c}{b-a}W(t)-\frac{d-c}{b-a}W(b)\right|^q\mathrm{d}t\right)\end{aligned}$$

and since $|W(t)| = \left|\int_a^t e^{-2\pi ixs}\mathrm{d}s\right| \le \int_a^t \left|e^{-2\pi ixs}\right|\mathrm{d}s = \int_a^t \mathrm{d}s = t-a$ for $t \in [a,b]$, we have

$$\int_a^c \left|\frac{d-c}{b-a}W(t)\right|^q \mathrm{d}t \le \int_a^c \left(\frac{d-c}{b-a}(t-a)\right)^q \mathrm{d}t = \left(\frac{d-c}{b-a}\right)^q \frac{(c-a)^{q+1}}{(q+1)},$$

$$\begin{aligned}
&\int_c^d \left|\frac{d-c}{b-a}W(t) - \frac{t-c}{b-a}W(b)\right|^q \mathrm{d}t \\
&\le \int_c^d \left(\left|\frac{d-c}{b-a}W(t)\right| + \left|\frac{t-c}{b-a}W(b)\right|\right)^q \mathrm{d}t \le \int_c^d \left(\frac{d-c}{b-a}(t-a) + t - c\right)^q \mathrm{d}t \\
&= \frac{1}{(b-a)^q}\int_c^d ((b-a+d-c)t - c(b-a) - a(d-c))^q \mathrm{d}t.
\end{aligned}$$

If we denote

$$\lambda(t) = (b-a+d-c)t - c(b-a) - a(d-c) \tag{2.117}$$

we have $\lambda(c) = (d-c)(c-a)$ and $\lambda(d) = (d-c)(b+d-2a)$ so

$$\begin{aligned}
&\frac{1}{(b-a)^q}\int_c^d ((b-a+d-c)t - c(b-a) - a(d-c))^q \mathrm{d}t \\
&= \frac{\left(\lambda(d)^{q+1} - \lambda(c)^{q+1}\right)}{(b-a)^q (q+1)(b-a+d-c)} \\
&= \frac{(d-c)^{q+1}\left((b+d-2a)^{q+1} - (c-a)^{q+1}\right)}{(b-a)^q (q+1)(b-a+d-c)} \le \frac{2^q (d-c)^q (b-a)}{(q+1)}.
\end{aligned}$$

Also

$$\begin{aligned}
&\int_d^b \left|\frac{d-c}{b-a}W(t) - \frac{d-c}{b-a}W(b)\right|^q \mathrm{d}t \\
&\le \int_d^b \left(\frac{d-c}{b-a}(b-t)\right)^q \mathrm{d}t = \left(\frac{d-c}{b-a}\right)^q \frac{(b-d)^{q+1}}{(q+1)}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\left\|\frac{d-c}{b-a}W(b)K(t)\right\|_q \\
&\le \left(\left(\frac{d-c}{b-a}\right)^q \frac{(c-a)^{q+1}}{(q+1)} + \frac{2^q (d-c)^q (b-a)}{(q+1)} + \left(\frac{d-c}{b-a}\right)^q \frac{(b-d)^{q+1}}{(q+1)}\right)^{\frac{1}{q}} \\
&\le \left(\left(\frac{d-c}{b-a}\right)^q \frac{(b-a)^{q+1}}{(q+1)} + \frac{2^q (d-c)^q (b-a)}{(q+1)}\right)^{\frac{1}{q}} = (d-c)\left(\frac{(2^q+1)(b-a)}{(q+1)}\right)^{\frac{1}{q}}
\end{aligned}$$

and inequality (2.115) is proved. For $p = 1$ we have

$$\left\|\frac{d-c}{b-a}W(b)K(t)\right\|_\infty = \max\left\{\sup_{t\in[a,c]}\left|\frac{d-c}{b-a}W(t)\right|,\right.$$

$$\sup_{t\in[c,d]}\left|\frac{d-c}{b-a}W(t)-\frac{t-c}{b-a}W(b)\right|,\ \sup_{t\in[d,b]}\left|\frac{d-c}{b-a}W(t)-\frac{d-c}{b-a}W(b)\right|\Bigg\}$$

and

$$\sup_{t\in[a,c]}\left|\frac{d-c}{b-a}W(t)\right|\le\frac{(d-c)(c-a)}{(b-a)},$$

$$\sup_{t\in[c,d]}\left|\frac{d-c}{b-a}W(t)-\frac{t-c}{b-a}W(b)\right|\le\sup_{t\in[c,d]}\left\{\left|\frac{d-c}{b-a}W(t)\right|+\left|\frac{t-c}{b-a}W(b)\right|\right\}$$
$$=\frac{d-c}{b-a}(d-a)+d-c=(d-c)\frac{b+d-2a}{b-a},$$
$$\sup_{t\in[d,b]}\left|\frac{d-c}{b-a}W(t)-\frac{d-c}{b-a}W(b)\right|\le\frac{(d-c)(b-d)}{(b-a)}.$$

Thus

$$\|U(d)K(t)\|_\infty\le\frac{d-c}{b-a}\max\{(c-a),(b+d-2a),(b-d)\}\le2(d-c)$$

and the proof is completed. □

Theorem 2.63 *Assume (p,q) is a pair of conjugate exponents. Let $g:[a,b]\to\mathbb{R}$ be absolutely continuous function on $[a,b]$ and $c,d\in[a,b]$, $c<d$. Then for $1<p\le\infty$, and for $x\neq0$ we have the inequality*

$$\left|\frac{d-c}{b-a}E(-2\pi ixc,-2\pi ixd)\int_a^b g(t)\,\mathrm{d}t-\int_c^d e^{-2\pi ixt}g(t)\,\mathrm{d}t\right|$$
$$\le(d-c)\left(\frac{(2^q+1)(b-a)}{(q+1)}\right)^{\frac1q}\left\|g'\right\|_p,\tag{2.118}$$

while for $p=1$ and $x\neq0$ we have

$$\left|\frac{d-c}{b-a}E(-2\pi ixc,-2\pi ixd)\int_a^b g(t)\,\mathrm{d}t-\int_c^d e^{-2\pi ixt}g(t)\,\mathrm{d}t\right|\le2(d-c)\left\|g'\right\|_1,\tag{2.119}$$

where $E(z,w)$ is given by (2.111).

Proof. If we apply identity (2.113) with $f(t)=g(t)$, $w(t)=\frac{1}{b-a}$, $u(t)=e^{-2\pi ixt}$ we have $U(t)=\int_c^t e^{-2\pi ixs}\mathrm{d}s=(t-c)E(-2\pi ixc,-2\pi ixt)$,

$$\frac{1}{(b-a)}\int_a^b g(t)\,\mathrm{d}t-\frac{1}{(d-c)E(-2\pi ixc,-2\pi ixd)}\int_c^d e^{-2\pi ixt}g(t)\,\mathrm{d}t=\int_a^b K(t)g'(t)\,\mathrm{d}t,$$

and, since $[c,d]\subseteq[a,b]$ by using (2.114)

$$K(t)=\begin{cases}-\frac{t-a}{b-a}, & t\in[a,c],\\[2mm] \frac{U(t)}{U(d)}-\frac{t-a}{b-a}, & t\in\langle c,d\rangle,\\[2mm] \frac{b-t}{b-a}, & t\in[d,b].\end{cases}$$

Thus

$$\frac{d-c}{b-a}E\left(-2\pi ixc,-2\pi ixd\right)\int_a^b g\left(t\right)\mathrm{d}t-\int_c^d e^{-2\pi ixt}g\left(t\right)\mathrm{d}t=U\left(d\right)\int_a^b K\left(t\right)g'\left(t\right)\mathrm{d}t$$

and by taking the modulus and applying Hölder inequality we obtain

$$\left|\frac{d-c}{b-a}E\left(-2\pi ixc,-2\pi ixd\right)\int_a^b g\left(t\right)\mathrm{d}t-\int_c^d e^{-2\pi ixt}g\left(t\right)\mathrm{d}t\right|\leq\left\|U\left(d\right)K\left(t\right)\right\|_q\left\|g'\right\|_p.$$

Now, for $1<p\leq\infty$ (for $1\leq q<\infty$) we have

$$\begin{aligned}\left\|U\left(d\right)K\left(t\right)\right\|_q&=\left(\int_a^c\left|\frac{t-a}{b-a}U\left(d\right)\right|^q\mathrm{d}t\right.\\&\left.+\int_c^d\left|U\left(t\right)-\frac{t-a}{b-a}U\left(d\right)\right|^q\mathrm{d}t+\int_d^b\left|\frac{b-t}{b-a}U\left(d\right)\right|^q\mathrm{d}t\right)\end{aligned}$$

and since $\left|U\left(t\right)\right|=\left|\int_c^t e^{-2\pi ixs}\mathrm{d}s\right|\leq\int_c^t\left|e^{-2\pi ixs}\right|\mathrm{d}s=\int_c^t\mathrm{d}s=t-c$ for $t\in[c,d]$, we have

$$\int_a^c\left|\frac{t-a}{b-a}U\left(\mathrm{d}\right)\right|^q\mathrm{d}t\leq\int_a^c\left(\frac{t-a}{b-a}\left(d-c\right)\right)^q dt=\left(\frac{d-c}{b-a}\right)^q\frac{\left(c-a\right)^{q+1}}{\left(q+1\right)},$$

$$\begin{aligned}&\int_c^d\left|U\left(t\right)-\frac{t-a}{b-a}U\left(d\right)\right|^q\mathrm{d}t\leq\int_c^d\left(\left|U\left(t\right)\right|+\left|\frac{t-a}{b-a}U\left(d\right)\right|\right)^q dt\\&\leq\int_c^d\left(t-c+\frac{d-c}{b-a}\left(t-a\right)\right)^q\mathrm{d}t\\&\leq\frac{1}{\left(b-a\right)^q}\int_c^d\left(\left(b-a+d-c\right)t-c\left(b-a\right)-a\left(d-c\right)\right)^q\mathrm{d}t\\&=\frac{\left(\lambda\left(d\right)^{q+1}-\lambda\left(c\right)^{q+1}\right)}{\left(b-a\right)^q\left(q+1\right)\left(b-a+d-c\right)}\\&=\frac{\left(d-c\right)^{q+1}\left(\left(b+d-2a\right)^{q+1}-\left(c-a\right)^{q+1}\right)}{\left(b-a\right)^q\left(q+1\right)\left(b-a+d-c\right)}\leq\frac{2^q\left(d-c\right)^q\left(b-a\right)}{\left(q+1\right)},\end{aligned}$$

where $\lambda\left(t\right)$ is given by (2.117) and

$$\int_d^b\left|\frac{b-t}{b-a}U\left(d\right)\right|^q\mathrm{d}t\leq\int_d^b\left(\frac{b-t}{b-a}\left(d-c\right)\right)^q\mathrm{d}t=\left(\frac{d-c}{b-a}\right)^q\frac{\left(b-d\right)^{q+1}}{\left(q+1\right)}.$$

Thus

$$\begin{aligned}&\left\|U\left(d\right)K\left(t\right)\right\|_q\\&\leq\left(\left(\frac{d-c}{b-a}\right)^q\frac{\left(c-a\right)^{q+1}}{\left(q+1\right)}+\frac{2^q\left(d-c\right)^q\left(b-a\right)}{\left(q+1\right)}+\left(\frac{d-c}{b-a}\right)^q\frac{\left(b-d\right)^{q+1}}{\left(q+1\right)}\right)^{\frac{1}{q}}\end{aligned}$$

$$\leq \left(\left(\frac{d-c}{b-a} \right)^q \frac{(b-a)^{q+1}}{(q+1)} + \frac{2^q (d-c)^q (b-a)}{(q+1)} \right)^{\frac{1}{q}} = (d-c) \left(\frac{(2^q+1)(b-a)}{(q+1)} \right)^{\frac{1}{q}}$$

and inequality (2.118) is proved. For $p = 1$ we have

$$\begin{aligned} &\|U(d) K(t)\|_\infty \\ &= \max \left\{ \sup_{t\in[a,c]} \left| \frac{t-a}{b-a} U(d) \right|, \sup_{t\in[c,d]} \left| U(t) - \frac{t-a}{b-a} U(d) \right|, \sup_{t\in[d,b]} \left| \frac{b-t}{b-a} U(d) \right| \right\} \end{aligned}$$

and

$$\sup_{t\in[a,c]} \left| \frac{t-a}{b-a} U(d) \right| \leq \frac{(c-a)(d-c)}{(b-a)},$$

$$\begin{aligned} \sup_{t\in[c,d]} \left| U(t) - \frac{t-a}{b-a} U(d) \right| &= \sup_{t\in[c,d]} \left\{ |U(t)| + \left| \frac{t-a}{b-a} U(d) \right| \right\} \\ \leq \sup_{t\in[c,d]} \left| d-c + \frac{d-a}{b-a}(d-c) \right| &= (d-c) \frac{b+d-2a}{b-a}, \end{aligned}$$

$$\sup_{t\in[d,b]} \left| \frac{b-t}{b-a} U(d) \right| \leq \frac{(b-d)(d-c)}{(b-a)}.$$

Thus

$$\|U(d) K(t)\|_\infty \leq \frac{d-c}{b-a} \max\{(c-a), (b+d-2a), (b-d)\} \leq 2(d-c)$$

and the proof is completed. □

Corollary 2.42 *Assume (p,q) is a pair of conjugate exponents. Let $g : [a,b] \to \mathbb{R}$ be absolutely continuous function on $[a,b]$. Then for $1 < p \leq \infty$, and for all $x \neq 0$ we have the inequality*

$$\left| E(-2\pi ixa, -2\pi ixb) \int_a^b g(t)\,dt - \int_a^b e^{-2\pi ixt} g(t)\,dt \right| \leq 2(b-a)^{1+\frac{1}{q}} \left(\frac{1}{q+1} \right)^{\frac{1}{q}} \|g'\|_p,$$

while for $p = 1$ we have

$$\left| E(-2\pi ixa, -2\pi ixb) \int_a^b g(t)\,dt - \int_a^b e^{-2\pi ixt} g(t)\,dt \right| \leq 2(b-a) \|g'\|_1.$$

Proof. By applying the results of the Theorems 2.62 or 2.63 with $c = a$ and $d = b$. □

The previous corollary can be utilized to obtain numerical quadrature rule.

Let $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$, $h_k := x_{k+1} - x_k,\ k = 0, 1, \ldots, n-1$ and $\nu(h) := \max_k \{h_k\}$. Define the sum

$$\mathscr{E}(g, I_n, x) = \sum_{k=0}^{n-1} E(-2\pi ixx_k, -2\pi ixx_{k+1}) \int_{x_k}^{x_{k+1}} g(t)\,dt \tag{2.120}$$

where $x \neq 0$.

The following approximation theorem holds.

Theorem 2.64 *Assume (p,q) is a pair of conjugate exponents. Let $g : [a,b] \to \mathbb{R}$ be absolutely continuous function on $[a,b]$. Then we have the quadrature rule*

$$\mathscr{F}(g)(x) = \mathscr{E}(g,I_n,x) + R(g,I_n,x)$$

where $x \neq 0$, $\mathscr{E}(g,I_n,x)$ is given by (2.120) and for $1 < p \leq \infty$ the reminder $R(g,I_n,x)$ satisfies the estimate

$$|R(g,I_n,x)| \leq 2\left(\frac{1}{q+1}\right)^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} h_k^{q+1}\right]^{\frac{1}{q}} \left\|g'\right\|_p,$$

while for $p = 1$ we have

$$|R(g,I_n,x)| \leq 2\nu(h) \left\|g'\right\|_1$$

Proof. For $1 < p \leq \infty$ by applying the Corollary 2.42 with $a = x_k$, $b = x_{k+1}$ we have

$$\begin{aligned}
&\left|E(-2\pi i x x_k, -2\pi i x x_{k+1}) \int_{x_k}^{x_{k+1}} g(t)\,\mathrm{d}t - \int_{x_k}^{x_{k+1}} e^{-2\pi i x t} g(t)\,\mathrm{d}t\right| \\
&\leq 2(x_{k+1} - x_k)^{1+\frac{1}{q}} \left(\frac{1}{q+1}\right)^{\frac{1}{q}} \left(\int_{x_k}^{x_{k+1}} \left|g'(t)\right|^p \mathrm{d}t\right)^{\frac{1}{p}}.
\end{aligned}$$

Summing over k from 0 to $n-1$ and using generalized triangle inequality, we obtain

$$\begin{aligned}
|R(g,I_n,x)| &= |\mathscr{F}(g)(x) - \mathscr{E}(g,I_n,x)| \\
&\leq \sum_{k=0}^{n-1} 2(h_k)^{1+\frac{1}{q}} \left(\frac{1}{q+1}\right)^{\frac{1}{q}} \left(\int_{x_k}^{x_{k+1}} \left|g'(t)\right|^p \mathrm{d}t\right)^{\frac{1}{p}}.
\end{aligned}$$

Using the Hölder discrete inequality, we get

$$\begin{aligned}
&2\left(\frac{1}{q+1}\right)^{\frac{1}{q}} \sum_{k=0}^{n-1} (h_k)^{1+\frac{1}{q}} \left(\int_{x_k}^{x_{k+1}} \left|g'(t)\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \\
&\leq 2\left(\frac{1}{q+1}\right)^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} \left((h_k)^{1+\frac{1}{q}}\right)^q\right]^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} \left(\left(\int_{x_k}^{x_{k+1}} \left|g'(t)\right|^p \mathrm{d}t\right)^{\frac{1}{p}}\right)^p\right]^{\frac{1}{p}} \\
&= 2\left(\frac{1}{q+1}\right)^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} h_k^{q+1}\right]^{\frac{1}{q}} \left\|g'\right\|_p
\end{aligned}$$

and the first inequality is proved. For $p = 1$ we have

$$|R(g,I_n,x)| \leq \sum_{k=0}^{n-1} 2h_k \left(\int_{x_k}^{x_{k+1}} \left|g'(t)\right| \mathrm{d}t\right)$$

$$\leq 2\nu(h)\sum_{k=0}^{n-1}\left(\int_{x_k}^{x_{k+1}}\left|g'(t)\right|\mathrm{d}t\right)=2\nu(h)\left\|g'\right\|_1$$

and the proof is completed. □

Corollary 2.43 *Suppose that all assumptions of Theorem 2.64 hold. Additionally suppose*

$$\mathscr{E}(g,I_n,x)=\int_{a+k\cdot\frac{b-a}{n}}^{a+(k+1)\cdot\frac{b-a}{n}}g(t)\,\mathrm{d}t$$
$$\cdot\sum_{k=0}^{n-1}E\left(-2\pi ix\left(a+k\cdot\frac{b-a}{n}\right),-2\pi ix\left(a+(k+1)\cdot\frac{b-a}{n}\right)\right).$$

Then we have the quadrature rule

$$\mathscr{F}(g)(x)=\mathscr{E}(g,I_n,x)+R(g,I_n,x)$$

where $x\neq 0$ and for $1<p\leq\infty$ the reminder $R(g,I_n,x)$ satisfies the estimate

$$|R(g,I_n,x)|\leq 2\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\frac{(b-a)^{1+\frac{1}{q}}}{n}\left\|g'\right\|_p,$$

while for $p=1$ we have

$$|R(g,I_n,x)|\leq\frac{2(b-a)}{n}\left\|g'\right\|_1.$$

Proof. By applying the Theorem 2.64 with equidistant partition of $[a,b]$ $x_j=a+j\cdot\frac{b-a}{n}$, $j=0,1,\ldots,n$ and $h_k=\frac{b-a}{n}$, $k=0,1,\ldots,n-1$. □

2.5.2 Error estimates of approximations in complex domain for the Laplace transform

The Laplace transform $\mathscr{L}(f)$ of Lebegue integrable mapping which vanish beyond a finite domain $[a,b]\subset[0,\infty\rangle$, $f:[a,b]\to\mathbb{R}$ is defined by

$$\mathscr{L}(f)(z)=\int_a^b f(t)e^{-zt}\,\mathrm{d}t. \tag{2.121}$$

for every $z\in\mathbb{C}$ for which the integral on the right hand side of (2.121) exists, i.e. $\left|\int_a^b f(t)e^{-zt}\,\mathrm{d}t\right|<\infty$.

Using the Montgomery identity following result is obtained.

Theorem 2.65 *Assume (p,q) is a pair of conjugate exponents. Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous such that $f'\in L_p[a,b]$. Then for $z\neq 0$, $1<p\leq\infty$ and for $\mathrm{Re}\,z\geq 0$ we have the inequality*

$$\left|\mathscr{L}(f)(z)-E(-za,-zb)\int_a^b f(s)\,\mathrm{d}s\right|\leq\frac{2e^{-a\mathrm{Re}\,z}(b-a)^{\frac{1}{q}}}{|z|}\left\|f'\right\|_p,$$

and for $\operatorname{Re} z < 0$

$$\left| \mathscr{L}(f)(z) - E(-za, -zb) \int_a^b f(s)\,\mathrm{d}s \right| \leq \frac{2e^{-b\operatorname{Re} z}(b-a)^{\frac{1}{q}}}{|z|} \left\| f' \right\|_p,$$

while for $p = 1$ *and* $\operatorname{Re} z \geq 0$ *we have*

$$\left| \mathscr{L}(f)(z) - E(-za, -zb) \int_a^b f(s)\,\mathrm{d}s \right| \leq \frac{2e^{-a\operatorname{Re} z}}{|z|} \left\| f' \right\|_1,$$

and for $\operatorname{Re} z < 0$

$$\left| \mathscr{L}(f)(z) - E(-za, -zb) \int_a^b f(s)\,\mathrm{d}s \right| \leq \frac{2e^{-b\operatorname{Re} z}}{|z|} \left\| f' \right\|_1.$$

Proof. Multiplying the Montgomery identity (2.2) with the kernel of Laplace transform and then integrating on $[a,b]$ gives us

$$\begin{aligned} \mathscr{L}(f)(z) &= \int_a^b f(t)\,e^{-zt}\,\mathrm{d}t \\ &= \frac{1}{b-a} \int_a^b \left[\int_a^b f(s)\,\mathrm{d}s + \int_a^t (s-a)\,f'(s)\,\mathrm{d}s + \int_t^b (s-b)\,f'(s)\,\mathrm{d}s \right] e^{-zt}\,\mathrm{d}t. \end{aligned}$$

By an interchange of the order of integration, since $z \neq 0$ we get

$$\begin{aligned} \int_a^b \left(\int_a^b f(s)\,\mathrm{d}s \right) e^{-zt}\,\mathrm{d}t &= \int_a^b \left(\frac{e^{-zb} - e^{-za}}{-z} \right) f(s)\,\mathrm{d}s \\ &= E(-za, -zb)(b-a) \int_a^b f(s)\,\mathrm{d}s, \end{aligned}$$

$$\begin{aligned} \int_a^b \left(\int_a^t (s-a)\,f'(s)\,\mathrm{d}s \right) e^{-zt}\,\mathrm{d}t &= \int_a^b \left(\int_s^b e^{-zt}\,\mathrm{d}t \right) (s-a)\,f'(s)\,\mathrm{d}s \\ &= \int_a^b \left(\frac{e^{-zb} - e^{-zs}}{-z} \right) (s-a)\,f'(s)\,\mathrm{d}s, \end{aligned}$$

$$\begin{aligned} \int_a^b \left(\int_t^b (s-b)\,f'(s)\,\mathrm{d}s \right) e^{-zt}\,\mathrm{d}t &= \int_a^b \left(\int_a^s e^{-zt}\,\mathrm{d}t \right) (s-b)\,f'(s)\,\mathrm{d}s \\ &= \int_a^b \left(\frac{e^{-zs} - e^{-za}}{-z} \right) (s-b)\,f'(s)\,\mathrm{d}s. \end{aligned}$$

So we have

$$\mathscr{L}(f)(z) - E(-za, -zb) \int_a^b f(s)\,\mathrm{d}s = \int_a^b \frac{e^{-zs}}{z} f'(s)\,\mathrm{d}s$$

$$+\left[\int_a^b \frac{e^{-zb}}{-z}\left(\frac{s-a}{b-a}\right) f'(s)\,\mathrm{d}s + \int_a^b \left(\frac{e^{-za}}{-z}\right)\left(\frac{b-s}{b-a}\right) f'(s)\,\mathrm{d}s\right].$$

For $1 < p \le \infty$, by applying Hölder inequality we obtain

$$\begin{aligned}&\left|\mathscr{L}(f)(z) - E(-za,-zb)\int_a^b f(s)\,\mathrm{d}s\right| \\ &\le \left\|\frac{e^{-zs}}{z} - \left(\frac{s-a}{b-a}\right)\frac{e^{-zb}}{z} - \left(\frac{b-s}{b-a}\right)\frac{e^{-za}}{z}\right\|_q \|f'\|_p .\end{aligned}$$

Now, if $\mathrm{Re}\,z \ge 0$ we have

$$\begin{aligned}&\left\|\frac{e^{-zs}}{z} - \left(\frac{s-a}{b-a}\right)\frac{e^{-zb}}{z} - \left(\frac{b-s}{b-a}\right)\frac{e^{-za}}{z}\right\|_q \\ &\le \left\|\frac{e^{-zs}}{z}\right\|_q + \left\|\left(\frac{s-a}{b-a}\right)\frac{e^{-zb}}{z} + \left(\frac{b-s}{b-a}\right)\frac{e^{-za}}{z}\right\|_q \\ &\le \left\|\frac{e^{-s\mathrm{Re}\,z}}{z}\right\|_q + \left\|\left(\frac{s-a}{b-a}\right)\frac{e^{-b\mathrm{Re}\,z}}{z} + \left(\frac{b-s}{b-a}\right)\frac{e^{-a\mathrm{Re}\,z}}{z}\right\|_q \\ &\le \frac{e^{-a\mathrm{Re}\,z}}{|z|}\left(\|1\|_q + \left\|\left(\frac{s-a}{b-a} + \frac{b-s}{b-a}\right)\right\|_q\right) = \frac{2e^{-a\mathrm{Re}\,z}(b-a)^{\frac{1}{q}}}{|z|},\end{aligned}$$

and if $\mathrm{Re}\,z < 0$ we have

$$\begin{aligned}&\left\|\frac{e^{-zs}}{z} - \left(\frac{s-a}{b-a}\right)\frac{e^{-zb}}{z} - \left(\frac{b-s}{b-a}\right)\frac{e^{-za}}{z}\right\|_q \\ &\le \frac{e^{-b\mathrm{Re}\,z}}{|z|}\left(\|1\|_q + \left\|\left(\frac{s-a}{b-a} + \frac{b-s}{b-a}\right)\right\|_q\right) = \frac{2e^{-b\mathrm{Re}\,z}(b-a)^{\frac{1}{q}}}{|z|}.\end{aligned}$$

Similarly for $p = 1$ we have

$$\begin{aligned}&\left|\mathscr{L}(g)(z) - E(-za,-zb)\int_a^b f(s)\,\mathrm{d}s\right| \\ &\le \left\|\frac{e^{-zs}}{z} - \left(\frac{s-a}{b-a}\right)\frac{e^{-zb}}{z} - \left(\frac{b-s}{b-a}\right)\frac{e^{-za}}{z}\right\|_\infty \|f'\|_1 .\end{aligned}$$

If $\mathrm{Re}\,z \ge 0$

$$\begin{aligned}&\left\|\frac{e^{-zs}}{z} - \left(\frac{s-a}{b-a}\right)\frac{e^{-zb}}{z} - \left(\frac{b-s}{b-a}\right)\frac{e^{-za}}{z}\right\|_\infty \\ &\le \frac{e^{-a\mathrm{Re}\,z}}{|z|}\left(\|1\|_\infty + \left\|\left(\frac{s-a}{b-a} + \frac{b-s}{b-a}\right)\right\|_\infty\right) = \frac{2e^{-a\mathrm{Re}\,z}}{|z|},\end{aligned}$$

and if $\operatorname{Re} z < 0$ we have

$$\begin{aligned}
&\left\| \frac{e^{-zs}}{z} - \left(\frac{s-a}{b-a}\right)\frac{e^{-zb}}{z} - \left(\frac{b-s}{b-a}\right)\frac{e^{-za}}{z} \right\|_\infty \\
&\leq \frac{e^{-b\operatorname{Re} z}}{|z|}\left(\|1\|_\infty + \left\| \left(\frac{s-a}{b-a} + \frac{b-s}{b-a}\right) \right\|_\infty \right) = \frac{2e^{-b\operatorname{Re} z}}{|z|},
\end{aligned}$$

and the proof is done. □

Remark 2.57 For $z = 0$ we have $\mathscr{L}(f)(0) = \int_a^b f(t)\,\mathrm{d}t$ and the left-hand side of the inequalities from the previous theorem reduces to

$$\left| \mathscr{L}(f)(0) - E(0,0)\int_a^b f(s)\,\mathrm{d}s \right| = \left| \int_a^b f(s)\,\mathrm{d}s - \int_a^b f(s)\,\mathrm{d}s \right| = 0.$$

Remark 2.58 It's not difficult to check that Weighted Montgomery identity 2.4 and the Lemma 2.3 hold also for $w : [a,b] \to \mathbb{C}$ integrable and such that $\int_a^b w(t)\,\mathrm{d}t \neq 0$.

Next, we apply identity for the difference of the two weighted integral means (2.113) with two special weight functions: uniform weight function and kernel of the Fourier transform. In such a way new generalizations of the previous results are obtained.

Theorem 2.66 *Assume (p,q) is a pair of conjugate exponents. Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous, $f' \in L_p[a,b]$ and $c,d \in [a,b]$, $c < d$. Then for $\operatorname{Re} z \geq 0$ and $1 < p \leq \infty$ we have inequalities*

$$\begin{aligned}
&\left| \frac{d-c}{b-a}\mathscr{L}(f)(z) - E(-za,-zb)\int_c^d f(t)\,\mathrm{d}t \right| \\
&\leq e^{-a\operatorname{Re} z}(d-c)\left(\frac{(2^q+1)(b-a)}{(q+1)}\right)^{\frac{1}{q}} \|f'\|_p \\
&\leq (d-c)\left(\frac{(2^q+1)(b-a)}{(q+1)}\right)^{\frac{1}{q}} \|f'\|_p,
\end{aligned}$$

while for $p = 1$ we have

$$\begin{aligned}
&\left| \frac{d-c}{b-a}\mathscr{L}(f)(z) - E(-za,-zb)\int_c^d f(t)\,\mathrm{d}t \right| \\
&\leq 2e^{-a\operatorname{Re} z}(d-c)\|f'\|_1 \leq 2(d-c)\|f'\|_1,
\end{aligned}$$

where $E(z,w)$ is exponential mean of z and w given by (2.111).

Proof. If we apply identity (2.113) with $w(t) = e^{-zt}$, $t \in [a,b]$ and $u(t) = \frac{1}{d-c}$, $t \in [c,d]$, we have $W(t) = (t-a)E(-za,-zt)$, $t \in [a,b]$; $U(t) = \frac{t-c}{d-c}$, $t \in [c,d]$ and

$$\frac{1}{(b-a)E(-za,-zb)}\mathscr{L}(f)(z) - \frac{1}{d-c}\int_c^d f(t)\,\mathrm{d}t = \int_a^b K(t)f'(t)\,\mathrm{d}t.$$

Since $[c,d] \subseteq [a,b]$ we use (2.114) so

$$K(t) = \begin{cases} -\frac{W(t)}{W(b)}, & t \in [a,c], \\ -\frac{W(t)}{W(b)} + \frac{t-c}{d-c}, & t \in \langle c,d \rangle, \\ 1 - \frac{W(t)}{W(b)}, & t \in [d,b]. \end{cases}$$

Thus

$$\frac{d-c}{b-a}\mathscr{L}(f)(z) - E(-za,-zb)\int_c^d f(t)\,\mathrm{d}t = \frac{d-c}{b-a}W(b)\int_a^b K(t) f'(t)\,\mathrm{d}t$$

and by taking the modulus and applying Hölder inequality we obtain

$$\left|\frac{d-c}{b-a}\mathscr{L}(f)(z) - E(-za,-zb)\int_c^d f(t)\,\mathrm{d}t\right| \leq \left\|\frac{d-c}{b-a}W(b)K(t)\right\|_q \|f'\|_p.$$

Now, for $1 < p \leq \infty$ (for $1 \leq q < \infty$) we have

$$\left\|\frac{d-c}{b-a}W(b)K(t)\right\|_q = \left(\int_a^c \left|\frac{d-c}{b-a}W(t)\right|^q \mathrm{d}t\right.$$
$$\left. + \int_c^d \left|\frac{d-c}{b-a}W(t) - \frac{t-c}{b-a}W(b)\right|^q \mathrm{d}t + \int_d^b \left|\frac{d-c}{b-a}W(t) - \frac{d-c}{b-a}W(b)\right|^q \mathrm{d}t\right)$$

and since $\mathrm{Re}\,z \geq 0$ we have $|W(t)| = \left|\int_a^t e^{-zs}\mathrm{d}s\right| \leq \int_a^t |e^{-zs}|\,\mathrm{d}s = \int_a^t \left|e^{-s\mathrm{Re}\,z}\right| \mathrm{d}s \leq (t-a)\,e^{-a\mathrm{Re}\,z}$ for $t \in [a,b]$, thus

$$\int_a^c \left|\frac{d-c}{b-a}W(t)\right|^q \mathrm{d}t \leq \int_a^c \left(e^{-a\mathrm{Re}\,z}\frac{d-c}{b-a}(t-a)\right)^q \mathrm{d}t = e^{-aq\mathrm{Re}\,z}\left(\frac{d-c}{b-a}\right)^q \frac{(c-a)^{q+1}}{(q+1)},$$

$$\int_c^d \left|\frac{d-c}{b-a}W(t) - \frac{t-c}{b-a}W(b)\right|^q \mathrm{d}t \leq \int_c^d \left(\left|\frac{d-c}{b-a}W(t)\right| + \left|\frac{t-c}{b-a}W(b)\right|\right)^q \mathrm{d}t$$
$$\leq e^{-aq\mathrm{Re}\,z}\int_c^d \left(\frac{d-c}{b-a}(t-a) + t - c\right)^q \mathrm{d}t$$
$$= \left(\frac{e^{-a\mathrm{Re}\,z}}{b-a}\right)^q \int_c^d ((b-a+d-c)t - c(b-a) - a(d-c))^q\,\mathrm{d}t.$$

If we denote

$$\lambda(t) = (b-a+d-c)t - c(b-a) - a(d-c)$$

we have $\lambda(c) = (d-c)(c-a)$ and $\lambda(d) = (d-c)(b+d-2a)$ so

$$\left(\frac{e^{-a\mathrm{Re}\,z}}{b-a}\right)^q \int_c^d ((b-a+d-c)t - c(b-a) - a(d-c))^q\,\mathrm{d}t$$

$$
\begin{aligned}
&= \frac{e^{-aq\mathrm{Re}z}\left(\lambda(d)^{q+1}-\lambda(c)^{q+1}\right)}{(b-a)^q(q+1)(b-a+d-c)} \\
&= \frac{e^{-aq\mathrm{Re}z}(d-c)^{q+1}\left((b+d-2a)^{q+1}-(c-a)^{q+1}\right)}{(b-a)^q(q+1)(b-a+d-c)} \leq \frac{e^{-aq\mathrm{Re}z}2^q(d-c)^q(b-a)}{(q+1)}.
\end{aligned}
$$

Also

$$
\begin{aligned}
&\int_d^b \left|\frac{d-c}{b-a}W(t)-\frac{d-c}{b-a}W(b)\right|^q \mathrm{d}t = \int_d^b \left|\frac{d-c}{b-a}\int_t^b e^{-zs}\mathrm{d}s\right|^q \mathrm{d}t \\
&\leq e^{-aq\mathrm{Re}z}\int_d^b \left(\frac{d-c}{b-a}(b-t)\right)^q \mathrm{d}t = e^{-aq\mathrm{Re}z}\left(\frac{d-c}{b-a}\right)^q\frac{(b-d)^{q+1}}{(q+1)}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\left\|\frac{d-c}{b-a}W(b)K(t)\right\|_q \\
&\leq e^{-a\mathrm{Re}z}\left(\left(\frac{d-c}{b-a}\right)^q\frac{(c-a)^{q+1}}{(q+1)}+\frac{2^q(d-c)^q(b-a)}{(q+1)}+\left(\frac{d-c}{b-a}\right)^q\frac{(b-d)^{q+1}}{(q+1)}\right)^{\frac{1}{q}} \\
&\leq e^{-a\mathrm{Re}z}\left(\left(\frac{d-c}{b-a}\right)^q\frac{(b-a)^{q+1}}{(q+1)}+\frac{2^q(d-c)^q(b-a)}{(q+1)}\right)^{\frac{1}{q}} \\
&= e^{-a\mathrm{Re}z}(d-c)\left(\frac{(2^q+1)(b-a)}{(q+1)}\right)^{\frac{1}{q}}
\end{aligned}
$$

and since $e^{-a\mathrm{Re}z} \leq 1$ inequalities in case $1 < p \leq \infty$ are proved. For $p = 1$ we have

$$
\begin{aligned}
&\left\|\frac{d-c}{b-a}W(b)K(t)\right\|_\infty = \max\left\{\sup_{t\in[a,c]}\left|\frac{d-c}{b-a}W(t)\right|,\right. \\
&\left.\sup_{t\in[c,d]}\left|\frac{d-c}{b-a}W(t)-\frac{t-c}{b-a}W(b)\right|,\ \sup_{t\in[d,b]}\left|\frac{d-c}{b-a}W(t)-\frac{d-c}{b-a}W(b)\right|\right\}
\end{aligned}
$$

and

$$
\sup_{t\in[a,c]}\left|\frac{d-c}{b-a}W(t)\right| \leq e^{-a\mathrm{Re}z}\frac{(d-c)(c-a)}{(b-a)},
$$

$$
\begin{aligned}
&\sup_{t\in[c,d]}\left|\frac{d-c}{b-a}W(t)-\frac{t-c}{b-a}W(b)\right| \leq \sup_{t\in[c,d]}\left\{\left|\frac{d-c}{b-a}W(t)\right|+\left|\frac{t-c}{b-a}W(b)\right|\right\} \\
&\leq e^{-a\mathrm{Re}z}\frac{d-c}{b-a}(d-a)+e^{-a\mathrm{Re}z}(d-c) = e^{-a\mathrm{Re}z}(d-c)\frac{b+d-2a}{b-a},
\end{aligned}
$$

$$
\sup_{t\in[d,b]}\left|\frac{d-c}{b-a}W(t)-\frac{d-c}{b-a}W(b)\right| \leq e^{-a\mathrm{Re}z}\frac{(d-c)(b-d)}{(b-a)}.
$$

Thus

$$\left\| \frac{d-c}{b-a} W(b) K(t) \right\|_{\infty} \leq e^{-a \operatorname{Re} z} \frac{d-c}{b-a} \max \{(c-a), (b+d-2a), (b-d)\}$$
$$\leq e^{-a \operatorname{Re} z} 2(d-c)$$

and since $e^{-a \operatorname{Re} z} \leq 1$ the proof is completed. □

Remark 2.59 The inequalities from the previous Theorem hods for $\operatorname{Re} z \geq 0$. Similarly it can be proved that in case $\operatorname{Re} z < 0$ and $1 < p \leq \infty$ we have the inequality

$$\left| \frac{d-c}{b-a} \mathscr{L}(f)(z) - E(-za, -zb) \int_c^d f(t) \, \mathrm{d}t \right|$$
$$\leq e^{-b \operatorname{Re} z} (d-c) \left(\frac{(2^q+1)(b-a)}{(q+1)} \right)^{\frac{1}{q}} \left\| f' \right\|_p,$$

while for $\operatorname{Re} z < 0$ and $p = 1$ we have

$$\left| \frac{d-c}{b-a} \mathscr{L}(f)(z) - E(-za, -zb) \int_c^d f(t) \, \mathrm{d}t \right| \leq e^{-b \operatorname{Re} z} 2(d-c) \left\| f' \right\|_1.$$

Theorem 2.67 *Assume (p,q) is a pair of conjugate exponents. Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous, $f' \in L_p[a,b]$ and $c, d \in [a,b]$, $c < d$. Then for* $\operatorname{Re} z \geq 0$ *and $1 < p \leq \infty$, we have inequalities*

$$\left| \frac{d-c}{b-a} E(-zc, -zd) \int_a^b f(t) \, \mathrm{d}t - \int_c^d e^{-zt} f(t) \, \mathrm{d}t \right|$$
$$\leq e^{-c \operatorname{Re} z} (d-c) \left(\frac{(2^q+1)(b-a)}{(q+1)} \right)^{\frac{1}{q}} \left\| f' \right\|_p$$
$$\leq (d-c) \left(\frac{(2^q+1)(b-a)}{(q+1)} \right)^{\frac{1}{q}} \left\| f' \right\|_p,$$

while for $p = 1$ we have

$$\left| \frac{d-c}{b-a} E(-zc, -zd) \int_a^b f(t) \, \mathrm{d}t - \int_c^d e^{-zt} f(t) \, \mathrm{d}t \right|$$
$$\leq e^{-c \operatorname{Re} z} 2(d-c) \left\| f' \right\|_1$$
$$\leq 2(d-c) \left\| f' \right\|_1,$$

where $E(z,w)$ is exponential mean of z and w given by (2.111).

Proof. By applying identity (2.113) with $w(t) = \frac{1}{b-a}$, $t \in [a,b]$ and $u(t) = e^{-zt}$, $t \in [c,d]$ and proceeding in the similar manner as in the proof of the Theorem 2.66. □

Remark 2.60 The inequalities from the previous Theorem hods for $\operatorname{Re} z \geq 0$. Similarly it can be proved that in case $\operatorname{Re} z < 0$ and $1 < p \leq \infty$ we have the inequality

$$\left| \frac{d-c}{b-a} E(-zc,-zd) \int_a^b f(t)\,\mathrm{d}t - \int_c^d e^{-zt} f(t)\,\mathrm{d}t \right|$$
$$\leq e^{-d\operatorname{Re} z}(d-c)\left(\frac{(2^q+1)(b-a)}{(q+1)}\right)^{\frac{1}{q}} \|f'\|_p,$$

while for $\operatorname{Re} z < 0$ and $p = 1$ we have

$$\left| \frac{d-c}{b-a} E(-zc,-zd) \int_a^b f(t)\,\mathrm{d}t - \int_c^d e^{-zt} f(t)\,\mathrm{d}t \right| \leq e^{-d\operatorname{Re} z} 2(d-c)\|f'\|_1.$$

Corollary 2.44 *Assume (p,q) is a pair of conjugate exponents. Let $f:[a,b] \to \mathbb{R}$ be absolutely continuous and $f' \in L_p[a,b]$. Then for all $\operatorname{Re} z \geq 0$ and $1 < p \leq \infty$, we have the inequality*

$$\left| E(-za,-zb) \int_a^b f(t)\,\mathrm{d}t - \mathscr{L}(f)(z) \right| \leq (b-a)^{1+\frac{1}{q}} \left(\frac{2^q+1}{q+1}\right)^{\frac{1}{q}} \|f'\|_p,$$

while for $p = 1$ we have

$$\left| E(-za,-zb) \int_a^b f(t)\,\mathrm{d}t - \mathscr{L}(f)(z) \right| \leq 2(b-a)\|f'\|_1.$$

Proof. By applying the Theorems 2.66 or 2.67 in the special case when $c = a$ and $d = b$. □

Corollary 2.45 *Assume (p,q) is a pair of conjugate exponents. Let $f:[a,b] \to \mathbb{R}$ be absolutely continuous and $f' \in L_p[a,b]$. Then for all $\operatorname{Re} z \geq 0$, for any $c \in [a,b]$ and $1 < p \leq \infty$, we have the inequality*

$$|\mathscr{L}(f)(z) - (b-a)E(-za,-zb)f(c)|$$
$$\leq (b-a)^{1+\frac{1}{q}} \left(\frac{2^q+1}{q+1}\right)^{\frac{1}{q}} \|f'\|_p,$$

while for $p = 1$ we have

$$|\mathscr{L}(f)(z) - (b-a)E(-za,-zb)f(c)| \leq 2(b-a)\|f'\|_1.$$

Proof. By applying the proof of the Theorem 2.66 in the special case when $c = d$. Since f is absolutely continuous, it is continuous, thus as a limit case we have $\lim_{c\to d} \frac{1}{d-c}\int_c^d f(t)\,\mathrm{d}t = f(c)$. □

Two previous corollaries can be utilized to obtain two numerical quadrature rules.

Let $I_n : a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ be a division of the interval $[a,b]$, $h_k := t_{k+1} - t_k$, $k = 0,1,\ldots,n-1$ and $\nu(h) := \max_k \{h_k\}$. Define the sum

$$\mathscr{E}(f, I_n, z) = \sum_{k=0}^{n-1} E(-zt_k, -zt_{k+1}) \int_{t_k}^{t_{k+1}} f(t)\,\mathrm{d}t \tag{2.122}$$

where $\operatorname{Re} z \geq 0$.

The following approximation theorem holds.

Theorem 2.68 *Assume (p,q) is a pair of conjugate exponents. Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous function on $[a,b]$, $f' \in L_p[a,b]$. Then we have the quadrature rule*

$$\mathscr{L}(f)(z) = \mathscr{E}(f, I_n, z) + R(f, I_n, z)$$

where $\operatorname{Re} z \geq 0$, *$\mathscr{E}(f, I_n, z)$ is given by (2.122) and for $1 < p \leq \infty$ the reminder $R(f, I_n, z)$ satisfies the estimate*

$$|R(f, I_n, z)| \leq \left(\frac{2^q+1}{q+1}\right)^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} h_k^{q+1}\right]^{\frac{1}{q}} \left\|f'\right\|_p,$$

while for $p = 1$

$$|R(f, I_n, z)| \leq 2\nu(h) \left\|f'\right\|_1.$$

Proof. For $1 < p \leq \infty$ by applying the Corollary 2.44 with $a = t_k$, $b = t_{k+1}$ we have

$$\begin{aligned}
&\left| E(-zt_k, -zt_{k+1}) \int_{t_k}^{t_{k+1}} f(t)\,\mathrm{d}t - \int_{t_k}^{t_{k+1}} e^{-zt} f(t)\,\mathrm{d}t \right| \\
&\leq (t_{k+1} - t_k)^{1+\frac{1}{q}} \left(\frac{2^q+1}{q+1}\right)^{\frac{1}{q}} \left(\int_{t_k}^{t_{k+1}} \left|f'(t)\right|^p \mathrm{d}t\right)^{\frac{1}{p}}.
\end{aligned}$$

Summing over k from 0 to $n-1$ and using generalized triangle inequality, we obtain

$$\begin{aligned}
|R(f, I_n, z)| &= |\mathscr{L}(f)(z) - \mathscr{E}(f, I_n, z)| \\
&\leq \sum_{k=0}^{n-1} (h_k)^{1+\frac{1}{q}} \left(\frac{2^q+1}{q+1}\right)^{\frac{1}{q}} \left(\int_{t_k}^{t_{k+1}} \left|f'(t)\right|^p \mathrm{d}t\right)^{\frac{1}{p}}.
\end{aligned}$$

Using the Hölder discrete inequality, we get

$$\begin{aligned}
&\left(\frac{2^q+1}{q+1}\right)^{\frac{1}{q}} \sum_{k=0}^{n-1} (h_k)^{1+\frac{1}{q}} \left(\int_{t_k}^{t_{k+1}} \left|f'(t)\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \\
&\leq \left(\frac{2^q+1}{q+1}\right)^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} \left((h_k)^{1+\frac{1}{q}}\right)^q\right]^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} \left(\left(\int_{t_k}^{t_{k+1}} \left|f'(t)\right|^p \mathrm{d}t\right)^{\frac{1}{p}}\right)^p\right]^{\frac{1}{p}} \\
&= \left(\frac{2^q+1}{q+1}\right)^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} h_k^{q+1}\right]^{\frac{1}{q}} \left\|f'\right\|_p
\end{aligned}$$

and the first inequality is proved. For $p = 1$ we have

$$
\begin{aligned}
|R(f, I_n, z)| &\leq \sum_{k=0}^{n-1} 2h_k \left(\int_{t_k}^{k+1} |f'(t)| \, dt \right) \\
&\leq 2\nu(h) \sum_{k=0}^{n-1} \left(\int_{t_k}^{t_{k+1}} |f'(t)| \, dt \right) = 2\nu(h) \|f'\|_1
\end{aligned}
$$

and the proof is completed. □

Corollary 2.46 *Suppose that all assumptions of Theorem 2.68 hold. Additionally suppose*

$$
\mathscr{E}(f, I_n, z) = \cdot \sum_{k=0}^{n-1} E\left(-z\left(a + k \cdot \frac{b-a}{n} \right), -z\left(a + (k+1) \cdot \frac{b-a}{n} \right) \right) \cdot \int_{a+k\cdot\frac{b-a}{n}}^{a+(k+1)\cdot\frac{b-a}{n}} f(t) \, dt.
$$

Then we have the quadrature rule

$$
\mathscr{L}(f)(z) = \mathscr{E}(f, I_n, z) + R(f, I_n, z)
$$

where $\operatorname{Re} z \geq 0$ *and for* $1 < p \leq \infty$ *the reminder* $R(f, I_n, z)$ *satisfies the estimate*

$$
|R(f, I_n, z)| \leq \left(\frac{2^q + 1}{q+1} \right)^{\frac{1}{q}} \frac{(b-a)^{1+\frac{1}{q}}}{n} \|f'\|_p,
$$

while for $p = 1$ *we have*

$$
|R(g, I_n, z)| \leq \frac{2(b-a)}{n} \|f'\|_1.
$$

Proof. If we apply Theorem 2.68 with equidistant partition of $[a, b]$. □

Now, define the sum

$$
\mathscr{A}(f, I_n, z) = \sum_{k=0}^{n-1} (t_{k+1} - t_k) E(-zt_k, -zt_{k+1}) f\left(\frac{t_{k+1} + t_k}{2} \right) \tag{2.123}
$$

where $\operatorname{Re} z \geq 0$.

Also the following approximation theorem holds.

Theorem 2.69 *Assume* (p, q) *is a pair of conjugate exponents. Let* $f : [a, b] \to \mathbb{R}$ *be absolutely continuous function on* $[a, b]$, $f' \in L_p[a, b]$. *Then we have the quadrature rule*

$$
\mathscr{L}(f)(z) = \mathscr{A}(f, I_n, z) + R(f, I_n, z)
$$

where $\operatorname{Re} z \geq 0$, $\mathscr{A}(f, I_n, z)$ *is given by (2.123) and for* $1 < p \leq \infty$ *the reminder* $R(f, I_n, z)$ *satisfies the estimate*

$$
|R(f, I_n, z)| \leq \left(\frac{2^q + 1}{q+1} \right)^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} h_k^{q+1} \right]^{\frac{1}{q}} \|f'\|_p,
$$

while for $p = 1$

$$
|R(f, I_n, z)| \leq 2\nu(h) \|f'\|_1.
$$

Proof. By applying the Corollary 2.45 with $a = t_k$, $b = t_{k+1}$, $c = \frac{t_{k+1}+t_k}{2}$ and then summing over k from 0 to $n-1$, we obtain results similarly as in the proof of the Theorem 2.68. □

Corollary 2.47 *Suppose that all assumptions of Theorem 2.69 hold. Additionally suppose*

$$\mathscr{A}(f,I_n,z) = \frac{b-a}{n} \cdot \sum_{k=0}^{n-1} f\left(a+\frac{k(k+1)(b-a)}{2n}\right) \cdot E\left(-z\left(a+k\cdot\frac{b-a}{n}\right), -z\left(a+(k+1)\cdot\frac{b-a}{n}\right)\right).$$

Then we have the quadrature rule

$$\mathscr{L}(f)(z) = \mathscr{A}(f,I_n,z) + R(f,I_n,z)$$

where $\operatorname{Re} z \geq 0$ *and for* $1 < p \leq \infty$ *the reminder* $R(f,I_n,z)$ *satisfies the estimate*

$$|R(f,I_n,z)| \leq \left(\frac{2^q+1}{q+1}\right)^{\frac{1}{q}} \frac{(b-a)^{1+\frac{1}{q}}}{n} \left\|f'\right\|_p,$$

while for $p = 1$ *we have*

$$|R(g,I_n,z)| \leq \frac{2(b-a)}{n} \left\|f'\right\|_1.$$

Proof. By applying Theorem 2.69 with equidistant partition of $[a,b]$. □

Remark 2.61 For both numerical quadrature formulae in case $\operatorname{Re} z < 0$, for $1 < p \leq \infty$, the reminder $R(f,I_n,z)$ satisfies the estimate

$$|R(f,I_n,z)| \leq e^{-b\operatorname{Re} z} \left(\frac{2^q+1}{q+1}\right)^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} h_k^{q+1}\right]^{\frac{1}{q}} \left\|f'\right\|_p,$$

while for $p = 1$

$$|R(f,I_n,z)| \leq e^{-b\operatorname{Re} z} 2\nu(h) \left\|f'\right\|_1.$$

For equidistant partition of $[a,b]$ and for $1 < p \leq \infty$ we have

$$|R(f,I_n,z)| \leq e^{-b\operatorname{Re} z} \left(\frac{2^q+1}{q+1}\right)^{\frac{1}{q}} \frac{(b-a)^{1+\frac{1}{q}}}{n} \left\|f'\right\|_p,$$

while for $p = 1$

$$|R(f,I_n,z)| \leq e^{-b\operatorname{Re} z} \frac{2(b-a)}{n} \left\|f'\right\|_1.$$

2.6 Weighted Montgomery identity for Functions of Two Variables

In this chapter, integral and discrete weighted Montgomery identities for functions of two variables are given. Further, some new Ostrowski type inequalities for mappings of two independent variables are obtained.

2.6.1 Integral weighted Montgomery identities and Ostrowski type inequalities for functions of two variables

J. Pečarić and A. Vukelić in [102] obtained weighted Montgomery's identities for functions of two variables The analysis used in the proofs is based on the identity established in [94]:

Lemma 2.4 *Let $p : [a,b]\times[c,d] \to \mathbb{R}$ be an integrable function and $P(x,y)$ defined by,*

$$P(x,y) = \int_x^b \int_y^d p(s,t)dsdt.$$

If $f : [a,b]\times[c,d] \to \mathbb{R}$ has continuous partial derivatives $\frac{\partial f(x,y)}{\partial x}$, $\frac{\partial f(x,y)}{\partial y}$, and $\frac{\partial^2 f(x,y)}{\partial x \partial y}$ on $[a,b]\times[c,d]$ then,

$$\int_a^b \int_c^d p(x,y)f(x,y)dxdy = f(a,c)P(a,c) + \int_a^b P(x,c)\frac{\partial f(x,c)}{\partial x}dx$$
$$+ \int_c^d P(a,y)\frac{\partial f(a,y)}{\partial y}dy + \int_a^b \int_c^d P(x,y)\frac{\partial^2 f(x,y)}{\partial x \partial y}dxdy.$$

By using the result from the last lemma following weighted Montgomery's identities for functions of two variables are obtained:

Theorem 2.70 *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ have continuous partial derivatives $\frac{\partial f(s,t)}{\partial s}$, $\frac{\partial f(s,t)}{\partial t}$, and $\frac{\partial^2 f(s,t)}{\partial s \partial t}$ on $[a,b]\times[c,d]$, then for all $(x,y) \in [a,b]\times[c,d]$,*

$$f(x,y)P(a,c) = \int_a^b \int_c^d p(s,t)f(s,t)dsdt + \int_a^b \hat{P}(x,s)\frac{\partial f(s,y)}{\partial s}ds$$
$$+ \int_c^d \tilde{P}(y,t)\frac{\partial f(x,t)}{\partial t}dt - \int_a^b \int_c^d \bar{P}(x,s,y,t)\frac{\partial^2 f(s,t)}{\partial s \partial t}dsdt, \tag{2.124}$$

where $p(\cdot,\cdot)$ and $P(\cdot,\cdot)$ are defined in Lemma 2.4,

$$\hat{P}(x,s) = \begin{cases} \int_a^s \int_c^d p(r,v)drdv, & a \le s \le x, \\ -P(s,c), & x < s \le b, \end{cases},$$

$$\tilde{P}(y,t) = \begin{cases} \int_a^b \int_c^t p(r,v)drdv, & c \le t \le y, \\ -P(a,t), & y < t \le d. \end{cases}$$

and

$$\bar{P}(x,s,y,t) = \begin{cases} \int_a^s \int_c^t p(r,v)drdv, & a \le s \le x,\ c \le t \le y, \\ -\int_s^b \int_c^t p(r,v)drdv, & x < s \le b,\ c \le t \le y, \\ -\int_a^s \int_t^d p(r,v)drdv, & a \le s \le x,\ y < t \le d, \\ P(s,t), & x < s \le b,\ y < t \le d. \end{cases}$$

Proof. See [102]. □

Theorem 2.71 *Let* $f:[a,b]\times[c,d]\to\mathbb{R}$ *have continuous partial derivatives* $\frac{\partial f(s,t)}{\partial s}$, $\frac{\partial f(s,t)}{\partial t}$, *and* $\frac{\partial^2 f(s,t)}{\partial s \partial t}$ *on* $[a,b]\times[c,d]$, *then for all* $(x,y)\in[a,b]\times[c,d]$ *we have,*

$$\begin{aligned} f(x,y)P(a,c) = &-\int_a^b \int_c^d p(s,t)f(s,t)dsdt + \int_a^b \int_c^d p(s,t)f(s,y)dsdt \\ &+ \int_a^b \int_c^d p(s,t)f(x,t)dsdt + \int_a^b \int_c^d \bar{P}(x,s,y,t)\frac{\partial^2 f(s,t)}{\partial s \partial t}dsdt, \end{aligned} \tag{2.125}$$

where $p(\cdot,\cdot), P(\cdot,\cdot), \hat{P}(x,s), \tilde{P}(y,t)$ *and* $\bar{P}(x,s,y,t)$ *are as in Theorem 2.70.*

Proof. See [102]. □

In the next two theorems weighted Ostrowski type inequalities for double weighted integrals for mappings of two independent variables are obtained using identities (2.124) and (2.125). They are weighted generalizations of the results obtained in [27], [54] and [55].

Theorem 2.72 *Let* $f:[a,b]\times[c,d]\to\mathbb{R}$ *be as in Theorem 2.70, then,*

$$\left| f(x,y) - \frac{1}{P(a,c)} \int_a^b \int_c^d p(s,t)f(s,t)dt \right| \le M_1(x) + M_2(y) + M_3(x,y), \tag{2.126}$$

where

$$M_1(x) = \frac{1}{|P(a,c)|} \left(\int_a^b |\hat{P}(x,s)|^{q_1} ds \right)^{1/q_1} \cdot \left\| \frac{\partial f}{\partial s} \right\|_{p_1},$$

if $\frac{\partial f(s,t)}{\partial s} \in L_{p_1}([a,b]\times[c,d])$, $1/p_1 + 1/q_1 = 1$,

$$M_2(y) = \frac{1}{|P(a,c)|} \left(\int_c^d |\tilde{P}(y,t)|^{q_2} ds \right)^{1/q_2} \cdot \left\| \frac{\partial f}{\partial t} \right\|_{p_2},$$

if $\frac{\partial f(s,t)}{\partial t} \in L_{p_2}([a,b]\times[c,d])$, $1/p_2 + 1/q_2 = 1$,

$$M_3(x,y) = \frac{1}{|P(a,c)|} \left(\int_a^b \int_c^d |\bar{P}(x,s,y,t)|^{q_3} dsdt \right)^{1/q_3} \cdot \left\| \frac{\partial^2 f}{\partial s \partial t} \right\|_{p_3},$$

if $\frac{\partial^2 f(s,t)}{\partial s \partial t} \in L_{p_3}([a,b]\times[c,d])$, $1/p_3 + 1/q_3 = 1$, *for all* $(x,y)\in[a,b]\times[c,d]$.

Proof. Using the identity (2.124) and Hölder's integral inequality for double integrals, we get inequality (2.126). □

Remark 2.62 If $p(\cdot,\cdot) = 1$ in Theorem 2.72 then the results of [55] are retrieved.

Theorem 2.73 *Let $f : [a,b]\times[c,d] \to \mathbb{R}$ be continuous on $[a,b]\times[c,d]$, $\frac{\partial^2 f(s,t)}{\partial s\partial t}$ exist on $(a,b)\times(c,d)$ and $\left|\frac{\partial^2 f(s,t)}{\partial s\partial t}\right|^p$ an integrable function such that*

$$\left\|\frac{\partial^2 f}{\partial s\partial t}\right\|_p := \left(\int_a^b\int_c^d \left|\frac{\partial^2 f(s,t)}{\partial s\partial t}\right|^p dsdt\right)^{1/p} < \infty.$$

It follows that,

$$\begin{aligned}
&\left|\int_a^b\int_c^d p(s,t)f(s,t)dsdt - \left[\int_a^b\int_c^d p(s,t)f(x,t)dt\right.\right.\\
&\left.\left.+\int_a^b\int_c^d p(s,t)f(s,y)ds - f(x,y)P(a,c)\right]\right|\\
&\leq \left(\int_a^b\int_c^d |\overline{P}(x,s,y,t)|^q dsdt\right)^{1/q}\cdot\left\|\frac{\partial^2 f}{\partial s\partial t}\right\|_p, \qquad (2.127)
\end{aligned}$$

for all $(x,y)\in[a,b]\times[c,d]$, where $1/p+1/q = 1$.

Proof. Using the identity (2.125) and Hölder's integral inequality for double integrals, we get inequality (2.127). □

Remark 2.63 If $p(\cdot,\cdot) = 1$ in Theorem 2.73 then the results of [27] and [54] are retrieved. Also if $p(s,t) = w(s)\varphi(t)$ then the results of [66] are retrieved.

2.6.2 Discrete weighted Montgomery identity and Ostrowski type inequalities for functions of two variables

Here a discrete analogue of the result of [102] and, at the same time, a generalization for functions of two variables of the discrete weighted Montgomery identity (2.30) obtained in [89] is given.

Let us recall that the difference operator Δ which is the finite analogue of the derivative is defined by $\Delta f(x) = f(x+1) - f(x)$, where $f : \mathbb{R}\to\mathbb{R}$ is any real-to-real function.

For $f : \mathbb{R}^2 \to \mathbb{R}$, the difference operators Δ_1 and Δ_2 are defined as

$$\begin{aligned}
\Delta_1 f(i,j) &= f(i+1,j) - f(i,j),\\
\Delta_2 f(i,j) &= f(i,j+1) - f(i,j).
\end{aligned}$$

These are the finite analogue of the first partial derivatives.

Theorem 2.74 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be any real function of two variables, $n, m \in \mathbb{N}$ and $w_1, w_2, \ldots, w_n$ and $\varphi_1, \varphi_2, \ldots, \varphi_m$ finite sequences of positive real numbers. Further, let $W_k = \sum_{i=1}^{k} w_i$, $\overline{W_k} = W_n - W_k$ for $1 \le k \le n$ and $\Phi_l = \sum_{j=1}^{l} \varphi_j$, $\overline{\Phi_l} = \Phi_m - \Phi_l$ for $1 \le l \le m$, then,*

$$f(k,l) = \frac{1}{W_n \Phi_m} \sum_{i=1}^{n} \sum_{j=1}^{m} w_i \varphi_j f(i,l) + \frac{1}{W_n} \sum_{i=1}^{n} \sum_{j=1}^{m} w_i D_\varphi(l,j) \Delta_2 f(i,j) \tag{2.128}$$

$$+ \frac{1}{\Phi_m} \sum_{i=1}^{n} \sum_{j=1}^{m} \varphi_j D_w(k,i) \Delta_1 f(i,j) + \sum_{i=1}^{n} \sum_{j=1}^{m} D_w(k,i) D_\varphi(l,j) \Delta_1 \Delta_2 f(i,j)$$

where

$$D_w(k,i) = \frac{1}{W_n} \cdot \begin{cases} W_i, & 1 \le i \le k-1, \\ \left(-\overline{W_i}\right), & k \le i \le n, \end{cases}$$

$$D_\varphi(l,j) = \frac{1}{\Phi_m} \cdot \begin{cases} \Phi_j, & 1 \le j \le l-1, \\ \left(-\overline{\Phi_j}\right), & l \le j \le m. \end{cases}$$

Proof. Using (2.30) on the first variable, we have

$$f(k,l) = \frac{1}{W_n} \sum_{i=1}^{n} w_i f(i,l) + \sum_{i=1}^{n} D_w(k,i) \Delta_1 f(i,l)$$

and further, with regard to the second variable

$$f(i,l) = \frac{1}{\Phi_m} \sum_{j=1}^{m} \varphi_j f(i,j) + \sum_{j=1}^{m} D_\varphi(l,j) \Delta_2 f(i,j).$$

Applying Δ_1 on the latter identity gives

$$\Delta_1 f(i,l) = \frac{1}{\Phi_m} \sum_{j=1}^{m} \varphi_j \Delta_1 f(i,j) + \sum_{j=1}^{m} D_\varphi(l,j) \Delta_1 \Delta_2 f(i,j).$$

Substituting the last two identities into the first leads to (2.128). □

For the special case of uniform weights, we obtain the following generalization for functions of two independent variables of the discrete Montgomery identity

$$f(k) = \frac{1}{n} \sum_{i=1}^{n} f(i) + \sum_{i=1}^{n} D(k,i) \Delta f(i),$$

where

$$D(k,i) = \begin{cases} \frac{i}{n}, & 1 \le i \le k-1, \\ \frac{i}{n} - 1, & k \le i \le n. \end{cases}$$

Corollary 2.48 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be any real function of two variables, $n, m, k, l \in \mathbb{N}$, $k \le n$, $l \le m$. Then it holds*

$$f(k,l) = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} f(i,l) + \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m} D_2(l,j) \Delta_2 f(i,j) \tag{2.129}$$

$$+\frac{1}{m}\sum_{i=1}^{n}\sum_{j=1}^{m}D_1(k,i)\Delta_1 f(i,j)+\sum_{i=1}^{n}\sum_{j=1}^{m}D_1(k,i)D_2(l,j)\Delta_1\Delta_2 f(i,j)$$

where

$$D_1(k,i)=\begin{cases}\frac{i}{n}, & 1\le i\le k-1,\\ \frac{i}{n}-1, & k\le i\le n,\end{cases}$$

$$D_2(l,j)=\begin{cases}\frac{j}{m}, & 1\le j\le l-1,\\ \frac{j}{m}-1, & l\le j\le m.\end{cases}$$

Proof. If we take $w_i=1,\ \ i=1,\dots,n$ and $\varphi_j=1,\ \ j=1,\dots,m$, we have $W_i=i$, $\overline{W_i}=n-i$ and $\Phi_j=j$, $\overline{\Phi_j}=m-j$. In this case (2.128) reduces to (2.129). □

Next, using (2.128) new Ostrowski type inequalities for functions f of two variables are derived.

Theorem 2.75 *Suppose that all the assumptions of Theorem 2.74 hold. Additionally assume that (p,q) is a pair of conjugate exponents, $1\le p,q\le\infty$, then,*

$$\Bigg|f(k,l)-\frac{1}{W_n\Phi_m}\sum_{i=1}^{n}\sum_{j=1}^{m}w_i\varphi_j f(i,l)$$
$$-\frac{1}{W_n}\sum_{i=1}^{n}\sum_{j=1}^{m}w_iD_\varphi(l,j)\Delta_2 f(i,j)-\frac{1}{\Phi_m}\sum_{i=1}^{n}\sum_{j=1}^{m}\varphi_jD_w(k,i)\Delta_1 f(i,j)\Bigg|$$
$$\le\|D_w(k,\cdot)\|_q\left\|D_\varphi(l,\cdot)\right\|_q\|\Delta_1\Delta_2 f\|_p \tag{2.130}$$

where

$$\|g\|_p=\begin{cases}\left(\sum_i|g(i)|^p\right)^{\frac{1}{p}}, & \text{if } 1\le p<\infty\\ \max_i|g(i)|, & \text{if } p=\infty,\end{cases}$$

if g is function of one variable and

$$\|g\|_p=\begin{cases}\left(\sum_{i,j}|g(i,j)|^p\right)^{\frac{1}{p}}, & \text{if } 1\le p<\infty,\\ \max_{i,j}|g(i,j)|, & \text{if } p=\infty,\end{cases}$$

if g is function of two variables.

Proof. By using (2.128)

$$\Bigg|f(k,l)-\frac{1}{W_n\Phi_m}\sum_{i=1}^{n}\sum_{j=1}^{m}w_i\varphi_j f(i,l)$$
$$-\frac{1}{W_n}\sum_{i=1}^{n}\sum_{j=1}^{m}w_iD_\varphi(l,j)\Delta_2 f(i,j)-\frac{1}{\Phi_m}\sum_{i=1}^{n}\sum_{j=1}^{m}\varphi_jD_w(k,i)\Delta_1 f(i,j)\Bigg|$$
$$=\left|\sum_{i=1}^{n}\sum_{j=1}^{m}D_w(k,i)D_\varphi(l,j)\Delta_1\Delta_2 f(i,j)\right|$$

and applying the discrete Hölder inequality for double sums we obtain (2.130)

$$\begin{aligned}
&\left|\sum_{i=1}^{n}\sum_{j=1}^{m} D_w(k,i)\, D_\varphi(l,j)\, \Delta_1\Delta_2 f(i,j)\right| \\
&\le \left(\sum_{i=1}^{n}\sum_{j=1}^{m} \left|D_w(k,i)\, D_\varphi(l,j)\right|^q\right)^{\frac{1}{q}} \cdot \left(\sum_{i=1}^{n}\sum_{j=1}^{m} \left|\Delta_1\Delta_2 f(i,j)\right|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{i=1}^{n} \left|D_w(k,i)\right|^q\right)^{\frac{1}{q}} \left(\sum_{j=1}^{m} \left|D_\varphi(l,j)\right|^q\right)^{\frac{1}{q}} \left(\sum_{i=1}^{n}\sum_{j=1}^{m} \left|\Delta_1\Delta_2 f(i,j)\right|^p\right)^{\frac{1}{p}}
\end{aligned}$$

□

In the next three corollaries we give the special cases for uniform weights. Recall that for $n \in \mathbb{N}$ and $m \in \mathbb{R}$ we denote

$$S_m(n) = 1^m + 2^m + 3^m + \cdots + (n-1)^m.$$

Corollary 2.49 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be any real function of two variables, $n,m,k,l \in \mathbb{N}$, $k \le n$, $l \le m$, then*

$$\begin{aligned}
&\left| f(k,l) - \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} f(i,l) \right. \\
&\left. -\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m} D_2(l,j)\,\Delta_2 f(i,j) - \frac{1}{m}\sum_{i=1}^{n}\sum_{j=1}^{m} D_1(k,i)\,\Delta_1 f(i,j) \right| \\
&\le \frac{1}{nm}\left(\frac{n^2-1}{4} + \left(k - \frac{n+1}{2}\right)^2\right)\left(\frac{m^2-1}{4} + \left(l - \frac{m+1}{2}\right)^2\right) \cdot \left\|\Delta_1\Delta_2 f\right\|_\infty .
\end{aligned}$$

Proof. We take $w_i = 1,\ i = 1,\ldots,n;\ \varphi_j = 1,\ j = 1,\ldots,m$, and $p = \infty$ in (2.130) for which we have $W_i = i$, $\overline{W_i} = n-i$, $\Phi_j = j$, $\overline{\Phi_j} = m-j$, giving,

$$\left\|D_w(k,\cdot)\right\|_1 = \sum_{i=1}^{n} \left|D_1(k,i)\right| = \frac{1}{n}\left(\frac{n^2-1}{4} + \left(k - \frac{n+1}{2}\right)^2\right)$$

and similarly

$$\left\|D_\varphi(l,\cdot)\right\|_1 = \sum_{j=1}^{m} \left|D_2(l,j)\right| = \frac{1}{m}\left(\frac{m^2-1}{4} + \left(l - \frac{m+1}{2}\right)^2\right).$$

□

Corollary 2.50 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be any real function of two variables, $n,m,k,l \in \mathbb{N}$, $k \le n$, $l \le m$, then*

$$\left| f(k,l) - \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} f(i,l) \right.$$

$$
\begin{aligned}
& -\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m} D_2(l,j)\,\Delta_2 f(i,j) - \frac{1}{m}\sum_{i=1}^{n}\sum_{j=1}^{m} D_1(k,i)\,\Delta_1 f(i,j)\Bigg| \\
& \le \frac{1}{nm}\max\{k-1,n-k\}\cdot\max\{l-1,m-l\}\cdot\|\Delta_1\Delta_2 f\|_1 .
\end{aligned}
$$

Proof. We take $w_i = 1,\ \ i = 1,\ldots,n;\ \ \varphi_j = 1,\ \ j = 1,\ldots,m$ and $p = 1$ in (2.130), with again, $W_i = i$, $\overline{W_i} = n-i$, $\Phi_j = j$, $\overline{\Phi_j} = m-j$, giving,

$$
\|D_w(k,\cdot)\|_\infty = \max_{1\le i\le n}\{|D_1(k,i)|\} = \frac{1}{n}\max\{k-1,n-k\}
$$

and

$$
\left\|D_\varphi(l,\cdot)\right\|_\infty = \max_{1\le j\le m}\{|D_2(l,j)|\} = \frac{1}{m}\max\{l-1,m-l\}.
$$

□

Corollary 2.51 *Let $f:\mathbb{R}^2\to\mathbb{R}$ be any real function of two variables, $n,m,k,l\in\mathbb{N}$, $k\le n$, $l\le m$. Let also (p,q) be a pair of conjugate exponents, $1<p,q<\infty$, then*

$$
\begin{aligned}
& \Bigg| f(k,l) - \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} f(i,l) \qquad (2.131)\\
& -\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m} D_2(l,j)\,\Delta_2 f(i,j) - \frac{1}{m}\sum_{i=1}^{n}\sum_{j=1}^{m} D_1(k,i)\,\Delta_1 f(i,j)\Bigg| \\
& \le \frac{1}{nm}\left[\left(S_q(k)+S_q(n-k+1)\right)\left(S_q(l)+S_q(m-l+1)\right)\right]^{\frac{1}{q}}\cdot\|\Delta_1\Delta_2 f\|_p .
\end{aligned}
$$

Proof. We take $w_i = 1,\ \ i = 1,\ldots,n$ and $\varphi_j = 1,\ \ j = 1,\ldots,m$ in (2.130),.$W_i = i$, $\overline{W_i} = n-i$, $\Phi_j = j$, $\overline{\Phi_j} = m-j$, giving,

$$
\|D_w(k,\cdot)\|_q = \left(\sum_{i=1}^{n}|D_1(k,i)|^q\right)^{\frac{1}{q}} = \frac{1}{n}\left(S_q(k)+S_q(n-k+1)\right)^{\frac{1}{q}}
$$

and similarly

$$
\left\|D_\varphi(l,\cdot)\right\|_q = \left(\sum_{j=1}^{m}|D_2(l,j)|^q\right)^{\frac{1}{q}} = \frac{1}{m}\left(S_q(l)+S_q(m-l+1)\right)^{\frac{1}{q}}.
$$

□

Remark 2.64 Since for $m\in\mathbb{N}$ it holds (see [63])

$$
S_m(n) = \frac{1}{m+1}\sum_{i=0}^{m}\binom{m+1}{i}B_i\, n^{m+1-i}
$$

where B_i, $i \geq 0$ are Bernoulli numbers, defined by,

$$\sum_{i=0}^{m} \binom{m+1}{i} B_i = \begin{cases} 1, & \text{if } m = 0, \\ 0, & \text{if } m \neq 0, \end{cases}$$

taking $q \in \mathbb{N}$ in (2.131) is equivalent to having

$$\begin{aligned} &\left| f(k,l) - \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} f(i,l) \right. \\ &\left. - \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m} D_2(l,j) \Delta_2 f(i,j) - \frac{1}{m} \sum_{i=1}^{n} \sum_{j=1}^{m} D_1(k,i) \Delta_1 f(i,j) \right| \\ &\leq \frac{1}{nm} \left(\frac{1}{q+1} \sum_{i=0}^{q} \binom{q+1}{i} B_i \left(k^{q+1-i} + (n-k+1)^{q+1-i} \right) \right)^{\frac{1}{q}} \\ &\cdot \left(\frac{1}{q+1} \sum_{i=0}^{q} \binom{q+1}{i} B_i \left(l^{q+1-i} + (m-l+1)^{q+1-i} \right) \right)^{\frac{1}{q}} \cdot \left\| \Delta_1 \Delta_2 f \right\|_p . \end{aligned}$$

2.7 Applications for Landau type inequalities

The following Landau inequality is well known (see [76]):

$$\left\| f' \right\| \leq 2 \sqrt{\|f\| \, \|f''\|} \tag{2.132}$$

where f is a real function, twice differentiable on interval I of the real line, $\|f\| = \sup_{x \in I} |f(x)|$ and $m(I) \geq 2\sqrt{\|f\| / \|f''\|}$, i.e. length of interval I is not less then $2\sqrt{\|f\| / \|f''\|}$. The constant 2 on the right hand side of inequality (2.132) is the best possible.

E. Landau proved this inequality in 1913 and since then it has been generalized in a number of ways (see [90]).

2.7.1 Extensions of Landau inequality

In this section we will give various extensions of inequality (2.133) using an extension of Montgomery identity via Taylor's formula with respect to a and b. Further, for the same purpose we shall use an extension of Montgomery identity applied to functions f such that $f^{(n)}$ are α-L-Hölder type, Euler type identities and the Fink identity. All these results are published in [5].

Remark 2.65 The Landau inequality can also be easily proved using the Montgomery identity in the following way: if we apply Montgomery identity (2.2) to $f'(x)$, with $a,b \in I$, $a < b$, we have

$$f'(x) = \frac{f(b) - f(a)}{b-a} + \int_a^b P(x,t) f''(t)\,\mathrm{d}t,$$

and therefore

$$|f'(x)| \leq \frac{2\|f\|}{b-a} + \|f''\| \int_a^b |P(x,t)|\,\mathrm{d}t.$$

Now,

$$\int_a^b |P(x,t)|\,\mathrm{d}t = \int_a^x \frac{t-a}{b-a}\mathrm{d}t + \int_x^b \frac{b-t}{b-a}\mathrm{d}t = \frac{(x-a)^2 + (b-x)^2}{2(b-a)} \leq \frac{b-a}{2}$$

and we have

$$\sup_{x\in I} |f'(x)| \leq \frac{2\|f\|}{b-a} + \frac{b-a}{2}\|f''\|.$$

It is easy to see that the function $y(h) = \frac{2}{h}\|f\| + \frac{h}{2}\|f''\|$ attains minimal value only for $h_{\min} = 2\sqrt{\|f\| / \|f''\|}$ and the corresponding minimal value is $y_{\min} = 2\sqrt{\|f\|\,\|f''\|}$. Since we have $m(I) \geq h_{\min}$, the inequality (2.132) follows for $b - a = h_{\min}$. If $m(I) < h_{\min}$ we have

$$\|f'\| \leq \frac{2\|f\|}{m(I)} + \frac{m(I)}{2}\|f''\|. \tag{2.133}$$

Landau type inequality via extension of Montgomery identity

We shall also need the following result.

Lemma 2.5 *Let $a,b \in \mathbb{R}$, $a < b$, $\alpha \in [1,\infty\rangle$. Then, for every $x \in [a,b]$ the following inequality holds:*

$$(x-a)^\alpha + (b-x)^\alpha \leq (b-a)^\alpha.$$

Proof. Consider the function $y : [a,b] \to \mathbb{R}$ given by

$$y(x) = (x-a)^\alpha + (b-x)^\alpha.$$

We observe that the unique solution on $[a,b]$, of the equation

$$y'(x) = (\alpha+1)\left[(x-a)^\alpha + (b-x)^\alpha\right] = 0$$

is $x_0 = \frac{a+b}{2}$. The function $y'(x)$ is decreasing on $\langle a,x_0\rangle$ and increasing on $\langle x_0,b\rangle$. Thus, the maximal values for $y(x)$ are attained on the boundary of $[a,b]$: $y(a) = y(b) = (b-a)^\alpha$. □

Remark 2.66 The same result as in the previous lemma holds for $\alpha \in \langle 0,1]$ and it was proved in [80].

Theorem 2.76 *Let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$, $I \subset \mathbb{R}$ an open interval, $a,b \in I$, $a < b$. Additionally assume that $f''(a) = f'''(a) = \cdots = f^{(n-1)}(a) = 0$ and $f''(b) = f'''(b) = \cdots = f^{(n-1)}(b) = 0$ for $n \in \mathbb{N}$, $n \geq 3$. Then we have*

$$\|f'\| \leq \frac{2\|f\|}{b-a} + \frac{(n-1)^{n-1}}{(n-2)!}(b-a)^{n-1}\left(B(2,n-1) - B_{\frac{n-2}{n-1}}(2,n-1)\right)\left\|f^{(n)}\right\|. \quad (2.134)$$

Proof. We apply formula (2.71) to $f'(x)$, and replace n with $n-1$ (thus $n \geq 3$), so we have

$$f'(x) = \frac{f(b)-f(a)}{b-a} + \sum_{i=0}^{n-3}\frac{f^{(i+2)}(a)}{i!(i+2)}\frac{(x-a)^{i+2}}{b-a} - \sum_{i=0}^{n-3}\frac{f^{(i+2)}(b)}{i!(i+2)}\frac{(x-b)^{i+2}}{b-a}$$
$$+ \frac{1}{(n-2)!}\int_a^b T_{n-1}(x,s) f^{(n)}(s)\,\mathrm{d}s,$$

and using the assumptions in the theorem it follows

$$\left|f'(x)\right| \leq \frac{|f(b)-f(a)|}{b-a} + \frac{1}{(n-2)!}\left|\int_a^b T_{n-1}(x,s) f^{(n)}(s)\,\mathrm{d}s\right|$$
$$\leq \frac{2\|f\|}{b-a} + \frac{\left\|f^{(n)}\right\|}{(n-2)!}\int_a^b |T_{n-1}(x,s)|\,\mathrm{d}s. \quad (2.135)$$

Now, we have

$$\int_a^b |T_{n-1}(x,s)|\,\mathrm{d}s = \int_a^x \left|-\frac{(x-s)^{n-1}}{(n-1)(b-a)} + \frac{x-a}{b-a}(x-s)^{n-2}\right|\mathrm{d}s$$
$$+ \int_x^b \left|-\frac{(x-s)^{n-1}}{(n-1)(b-a)} + \frac{x-b}{b-a}(x-s)^{n-2}\right|\mathrm{d}s$$

Using the substitution $s = x-(n-1)(x-a)(1-t)$ we obtain

$$\int_a^x \left|-\frac{(x-s)^{n-1}}{n-1} + (x-a)(x-s)^{n-2}\right|\mathrm{d}s$$
$$= (x-a)^n (n-1)^{n-1}\left(B(2,n-1) - B_{\frac{n-2}{n-1}}(2,n-1)\right).$$

Similarly using the substitution $s = x+(n-1)(b-x)(1-t)$ we get for I_2:

$$\int_x^b \left|-\frac{(x-s)^{n-1}}{(n-1)(b-a)} + \frac{x-b}{b-a}(x-s)^{n-2}\right|\mathrm{d}s$$
$$= (b-x)^n (n-1)^{n-1}\left(B(2,n-1) - B_{\frac{n-2}{n-1}}(2,n-1)\right).$$

Finally, using Lemma 2.5, we obtain

$$\int_a^b |T_{n-1}(x,s)|\,\mathrm{d}s = \frac{(n-1)^{n-1}}{(b-a)}\left[(x-a)^n + (b-x)^n\right]\left(B(2,n-1) - B_{\frac{n-2}{n-1}}(2,n-1)\right)$$

$$\leq (n-1)^{n-1}(b-a)^{n-1}\left(B(2,n-1)-B_{\frac{n-2}{n-1}}(2,n-1)\right).$$

By taking the supremum over $x \in [a,b]$ on the left-hand side of the inequality (2.135), the proof follows. □

Landau type inequality via extension of Montgomery identity with applications to α-L-Hölder type functions

Theorem 2.77 *Let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$, $I \subset \mathbb{R}$ an open interval, $a,b \in I$, $a < b$. Additionally assume that $f''(a) = f'''(a) = \cdots = f^{(n)}(a) = 0$, $f''(b) = f'''(b) = \cdots = f^{(n)}(b) = 0$ for $n \in \mathbb{N}$, $n \geq 3$, and also that $f^{(n)} : [a,b] \to \mathbb{R}$ is an α-L-Hölder type function, i.e. $\left|f^{(n)}(x) - f^{(n)}(y)\right| \leq L|x-y|^{\alpha}$ for every $x,y \in [a,b]$, where $L > 0$ and $\alpha \in \langle 0,1]$. Then we have*

$$\|f'\| \leq \frac{2\|f\|}{b-a} + \frac{L(b-a)^{\alpha+n-1}}{(n-3)!(\alpha+n)}B(\alpha+1,n-2).$$

Proof. We apply formula (2.83) to $f'(x)$, and replace n with $n-1$ (thus $n \geq 3$). Using the assumptions we have

$$\begin{aligned}
f'(x) &= \frac{f(b)-f(a)}{b-a} \\
&\quad + \frac{1}{(n-3)!}\int_a^x \frac{t-a}{b-a}\left[\int_a^t \left(f^{(n)}(s)-f^{(n)}(a)\right)(t-s)^{n-3}\,\mathrm{d}s\right]\mathrm{d}t \\
&\quad + \frac{1}{(n-3)!}\int_x^b \frac{b-t}{b-a}\left[\int_t^b \left(f^{(n)}(s)-f^{(n)}(b)\right)(t-s)^{n-3}\,\mathrm{d}s\right]\mathrm{d}t.
\end{aligned}$$

Therefore

$$\begin{aligned}
\left|f'(x)\right| &\leq \frac{|f(b)-f(a)|}{b-a} \\
&\quad + \frac{1}{(n-3)!}\left|\int_a^x \frac{t-a}{b-a}\left[\int_a^t \left(f^{(n)}(s)-f^{(n)}(a)\right)(t-s)^{n-3}\,\mathrm{d}s\right]\mathrm{d}t\right| \\
&\quad + \frac{1}{(n-3)!}\left|\int_x^b \frac{b-t}{b-a}\left[\int_t^b \left(f^{(n)}(s)-f^{(n)}(b)\right)(t-s)^{n-3}\,\mathrm{d}s\right]\mathrm{d}t\right| \\
&\leq \frac{|f(b)-f(a)|}{b-a} + \frac{L}{(n-3)!}\left\{\int_a^x \left|\frac{t-a}{b-a}\right|\left|\int_a^t |s-a|^{\alpha}|t-s|^{n-3}\,\mathrm{d}s\right|\mathrm{d}t\right. \\
&\quad \left. + \int_x^b \left|\frac{b-t}{b-a}\right|\left|\int_t^b |s-b|^{\alpha}|t-s|^{n-3}\,\mathrm{d}s\right|\mathrm{d}t\right\}.
\end{aligned}$$

Using the substitution $s = a + u(t-a)$ we have

$$\int_a^t |s-a|^{\alpha}|t-s|^{n-3}\,\mathrm{d}s = (t-a)^{\alpha+n-2}B(\alpha+1,n-2),$$

and with $s = b - u(b-t)$ we obtain

$$\int_t^b |s-b|^{\alpha} |t-s|^{n-3} \, \mathrm{d}s = (b-t)^{\alpha+n-2} B(\alpha+1, n-2).$$

Thus we have

$$\left|f'(x)\right| \leq \frac{2\|f\|}{b-a} + \frac{L}{(n-3)!} B(\alpha+1, n-2) \frac{(x-a)^{\alpha+n} + (b-x)^{\alpha+n}}{(b-a)(\alpha+n)}.$$

Finally, by using Lemma 2.5 and taking the supremum over $x \in [a,b]$ on the left-hand side of the last inequality, the proof follows. □

Landau type inequality via Euler type identity

Theorem 2.78 *Let $f : [a,b] \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ for some $n \geq 2$, and $f'(a) = f'(b)$, $f''(a) = f''(b), \ldots, f^{(n-1)}(a) = f^{(n-1)}(b)$. Then the following inequality holds:*

$$\left\|f'\right\| \leq \frac{2\|f\|}{b-a} + \frac{(b-a)^{n-1} \left\|f^{(n)}\right\|}{(n-1)!} \int_0^1 |B_{n-1}(t)| \, \mathrm{d}t.$$

Proof. We apply formula (2.22) to $f'(x)$, and replace n with $n-1$. Using the assumptions we have that $T_{n-1}^{[a,b]}(x) = 0$ and

$$f'(x) = \frac{f(b) - f(a)}{b-a} - \frac{(b-a)^{n-2}}{(n-1)!} \int_a^b \left[B_{n-1}^*\left(\frac{x-t}{b-a}\right)\right] f^{(n)}(t) \, \mathrm{d}t.$$

Thus

$$\begin{aligned} \left|f'(x)\right| &\leq \frac{|f(b) - f(a)|}{b-a} + \frac{(b-a)^{n-2}}{(n-1)!} \left|\int_a^b \left[B_{n-1}^*\left(\frac{x-t}{b-a}\right)\right] f^{(n)}(t) \, \mathrm{d}t\right| \\ &\leq \frac{2\|f\|}{b-a} + \frac{(b-a)^{n-1} \left\|f^{(n)}\right\|}{(n-1)!} \int_0^1 |B_{n-1}(t)| \, \mathrm{d}t, \end{aligned}$$

since B_{n-1}^* is a periodic function with period 1 and

$$\int_y^{y+1} |B_n^*(s)| \, \mathrm{d}s = \int_0^1 |B_n^*(s)| \, \mathrm{d}s = \int_0^1 |B_n(s)| \, \mathrm{d}s$$

for every $y \in \mathbb{R}$. By taking the supremum over $x \in [a,b]$ on the left-hand side of the last inequality, the proof follows. □

Theorem 2.79 *Let $f : [a,b] \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ for some $n \geq 2$, and $f'(a) = f'(b)$, $f''(a) = f''(b), \ldots, f^{(n-2)}(a) = f^{(n-2)}(b)$. Then the following inequality holds:*

$$\left\|f'\right\| \leq \frac{2\|f\|}{b-a} + \frac{(b-a)^{n-1} M}{(n-1)!} \left\|f^{(n)}\right\|.$$

where

$$M = \max_{x\in[a,b]} \left\{ \int_0^1 \left| B_{n-1}(t) - B_{n-1}\left(\frac{x-a}{b-a}\right) \right| \mathrm{d}t \right\}.$$

Proof. We apply formula (2.23a) to $f'(x)$, and replace n with $n-1$. Using the assumption we have that $T_{n-2}^{[a,b]}(x) = 0$ and

$$f'(x) = \frac{f(b)-f(a)}{b-a} - \frac{(b-a)^{n-2}}{(n-1)!}\int_a^b \left[B_{n-1}^*\left(\frac{x-t}{b-a}\right) - B_{n-1}\left(\frac{x-a}{b-a}\right)\right] f^{(n)}(t)\,\mathrm{d}t.$$

Thus

$$\begin{aligned}
\left|f'(x)\right| &\le \frac{|f(b)-f(a)|}{b-a} + \frac{(b-a)^{n-2}}{(n-1)!} \\
&\quad \cdot \left| \int_a^b \left[B_{n-1}^*\left(\frac{x-t}{b-a}\right) - B_{n-1}\left(\frac{x-a}{b-a}\right)\right] f^{(n)}(t)\,\mathrm{d}t \right| \\
&\le \frac{2\|f\|}{b-a} + \frac{(b-a)^{n-1}\left\|f^{(n)}\right\|}{(n-1)!}\int_0^1 \left| B_{n-1}(t) - B_{n-1}\left(\frac{x-a}{b-a}\right)\right| \mathrm{d}t \\
&\le \frac{2\|f\|}{b-a} + \frac{(b-a)^{n-1}\left\|f^{(n)}\right\|}{(n-1)!} M
\end{aligned}$$

since B_{n-1}^* is a periodic function with period 1 and

$$\int_y^{y+1} |B_n^*(s) - z|\,\mathrm{d}s = \int_0^1 |B_n^*(s) - z|\,\mathrm{d}s = \int_0^1 |B_n(s) - z|\,\mathrm{d}s$$

for every $y, z \in \mathbb{R}$. By taking the supremum over $x \in [a,b]$ on the left-hand side of the last inequality, the proof follows. □

The next result is an extension of the Landau inequality (2.132).

Theorem 2.80 *Let $f : I \to \mathbb{R}$ be a function such that f' is absolutely continuous on I. Then we have*

$$\left\|f'\right\| \le \begin{cases} \frac{2\|f\|}{b-a} + \frac{b-a}{2}\left\|f''\right\|, & \text{if } 0 < m(I) < 2\sqrt{\|f\| / \|f''\|}, \\[2ex] 2\sqrt{\|f\|\,\|f''\|}, & \text{if } m(I) \ge 2\sqrt{\|f\| / \|f''\|}. \end{cases} \tag{2.136}$$

Proof. Using the inequality from the Theorem 2.79 for the $n = 2$ and for some $a, b \in I$, $a < b$ we have

$$M = \max_{x\in[a,b]} \left\{ \int_0^1 \left| B_1(t) - B_1\left(\frac{x-a}{b-a}\right) \right| \mathrm{d}t \right\} = \frac{1}{2}$$

so

$$\left\|f'\right\| \le \frac{2\|f\|}{b-a} + \frac{b-a}{2}\left\|f''\right\|.$$

By taking the infimum over $b-a \in \langle 0, m(I)]$ on the right-hand side of the last inequality, in the similar manner as in the Remark 2.65 the proof follows. □

Landau type inequality via Fink identity

Theorem 2.81 *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and $f'(a)=f''(a)=\cdots=f^{(n-2)}(a)=0$ and $f'(b)=f''(b)=\cdots=f^{(n-2)}(b)=0$ for some $n\in\mathbb{N}$, $n\geq 2$. Then we have*

$$\left\|f'\right\|\leq\frac{2(n-1)\|f\|}{b-a}+\frac{(b-a)^{n-1}}{n!}\left\|f^{(n)}\right\| \tag{2.137}$$

Proof. We apply formula (2.92) to $f'(x)$, and replace n with $n-1$ (thus $n\geq 2$). Using the assumptions we have that $\sum_{k=1}^{n-2}F_k(x)=0$ and

$$\frac{f'(x)}{n-1}-\frac{f(b)-f(a)}{b-a}=\frac{1}{(n-1)!(b-a)}\int_a^b(x-t)^{n-2}k(t,x)f^{(n)}(t)\mathrm{d}t.$$

Thus

$$\begin{aligned}\left|f'(x)\right|&\leq\frac{(n-1)\left|f(b)-f(a)\right|}{b-a}+\frac{1}{(n-2)!(b-a)}\left|\int_a^b(x-t)^{n-2}k(t,x)f^{(n)}(t)\mathrm{d}t\right|\\&\leq\frac{2(n-1)\|f\|}{b-a}+\frac{\left\|f^{(n)}\right\|}{(n-2)!(b-a)}\int_a^b\left|(x-t)^{n-2}k(t,x)\right|\mathrm{d}t.\end{aligned}$$

Since $\int_a^b\left|(x-t)^{n-2}k(t,x)\right|\mathrm{d}t=\frac{(x-a)^n+(b-x)^n}{n(n-1)}$, by using Lemma 2.5 and taking the supremum over $x\in[a,b]$ on the left-hand side of the last inequality, the proof follows. □

Remark 2.67 If we use the inequality from the last theorem for the $n=2$ and repeat the procedure from the proof of the Theorem 2.80, we obtain (2.136).

2.7.2 On Landau type inequalities via Ostrowski inequalities

In this subsection few different versions of Landau type inequalities are given, each using the Ostrowski inequality or Ostrowski type inequalities. These results are published in [6].

Remark 2.68 Note that Ostrowski inequality (2.1) can be formulated in the equivalent form

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right|\leq\frac{(x-a)^2+(b-x)^2}{2(b-a)}\left\|f'\right\|_\infty.$$

The Landau inequality can also be easily proved using the Ostrowski inequality in the following way: if we apply it to $f'(x)$, for some $a,b\in I$, $a<b$ and $x\in[a,b]$ we have

$$\left|f'(x)-\frac{f(b)-f(a)}{b-a}\right|\leq\frac{(x-a)^2+(b-x)^2}{2(b-a)}\left\|f''\right\|_\infty\leq\frac{b-a}{2}\left\|f''\right\|_\infty.$$

Therefore

$$\left|f'(x)\right|-\left|\frac{f(b)-f(a)}{b-a}\right|\leq\left|f'(x)-\frac{f(b)-f(a)}{b-a}\right|\leq\frac{b-a}{2}\left\|f''\right\|_\infty$$

and

$$\left|f'(x)\right| \leq \left|\frac{f(b)-f(a)}{b-a}\right| + \frac{b-a}{2}\left\|f''\right\|_\infty \leq \frac{2\|f\|_\infty}{b-a} + \frac{b-a}{2}\left\|f''\right\|_\infty.$$

Now we have

$$\sup_{x\in I}\left|f'(x)\right| \leq \frac{2\|f\|_\infty}{b-a} + \frac{b-a}{2}\left\|f''\right\|_\infty.$$

It is easy to see that the function $y(t) = \frac{2}{t}\|f\|_\infty + \frac{t}{2}\|f''\|_\infty$ attains minimal value only for $t_{\min} = 2\sqrt{\|f\|_\infty / \|f''\|_\infty}$ and the corresponding minimal value is $y_{\min} = 2\sqrt{\|f\|_\infty \|f''\|_\infty}$. Since $b-a \leq m(I)$, if we set $b-a = t_{\min}$ the inequality (2.132) follows.

Some further generalizations of Landau inequality

We shall need the following result from [80].

Lemma 2.6 *Let $A,B>0$ and $\alpha \in \langle 0,\infty\rangle$. Consider the function $g_\alpha : \langle 0,\infty\rangle \to \mathbb{R}$ given by $g_\alpha(t) = \frac{A}{t} + Bt^\alpha$. Define $t_0 = \left(\frac{A}{\alpha B}\right)^{\frac{1}{\alpha+1}}$. Then it holds:*

$$\inf_{t\in\langle 0,T]} g_\alpha(t) = \begin{cases} \frac{A}{T} + BT^\alpha, & \text{if } 0 < T < t_0, \\ (\alpha+1)\,\alpha^{-\frac{\alpha}{\alpha+1}} A^{\frac{\alpha}{\alpha+1}} B^{\frac{1}{\alpha+1}}, & \text{if } T \geq t_0. \end{cases}$$

In the next theorem Landau's inequality is improved using the idea from [91].

Theorem 2.82 *Let $f : I \to \mathbb{R}$ be twice differentiable on I. If $m \leq f(x) \leq M$ for $x \in I$ and $m(I) \geq \sqrt{2(M-m)/\|f''\|_\infty}$, then we have*

$$\left\|f'\right\|_\infty \leq \sqrt{2(M-m)\left\|f''\right\|_\infty}. \tag{2.138}$$

Proof. If we apply Ostrowski inequality to $f'(x)$, for some $a,b \in I$, $a<b$ and $x \in [a,b]$ we have

$$\left|f'(x) - \frac{f(b)-f(a)}{b-a}\right| \leq \frac{(x-a)^2+(b-x)^2}{2(b-a)}\left\|f''\right\|_\infty \leq \frac{b-a}{2}\left\|f''\right\|_\infty$$

and

$$\left|f'(x)\right| \leq \frac{|f(b)-f(a)|}{b-a} + \frac{b-a}{2}\left\|f''\right\|_\infty \leq \frac{M-m}{b-a} + \frac{b-a}{2}\left\|f''\right\|_\infty.$$

Now we have

$$\sup_{x\in I}\left|f'(x)\right| \leq \frac{M-m}{b-a} + \frac{b-a}{2}\left\|f''\right\|_\infty.$$

It is easy to see that the function $y(t) = \frac{M-m}{t} + \frac{t}{2}\|f''\|_\infty$ attains minimal value only for $t_{\min} = \sqrt{2(M-m)/\|f''\|_\infty}$ and the corresponding minimal value is $y_{\min} = \sqrt{2(M-m)\|f''\|_\infty}$. The inequality (2.138) follows by taking $b-a = t_{\min}$. □

Remark 2.69 In the special case for $M = -m = \|f\|_\infty$ the inequality (2.138) reduces to the inequality (2.132).

Theorem 2.83 *Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable function on $\mathbb{R}$. If $m \le f(x) \le M$ for $x \in \mathbb{R}$, then we have*

$$\|f'\|_\infty \le \sqrt{(M-m)\|f''\|_\infty}.$$

Proof. If we apply Ostrowski inequality to $f'(x)$, for any $a, b \in I$, $a < b$ and $x \in [a,b]$, we have

$$\left| f'(x) - \frac{f(b)-f(a)}{b-a} \right| \le \frac{(x-a)^2 + (b-x)^2}{2(b-a)} \|f''\|_\infty$$

and

$$\begin{aligned} |f'(x)| &\le \frac{|f(b)-f(a)|}{b-a} + \frac{(x-a)^2+(b-x)^2}{2(b-a)}\|f''\|_\infty \\ &\le \frac{M-m}{b-a} + \frac{(x-a)^2+(b-x)^2}{2(b-a)}\|f''\|_\infty. \end{aligned}$$

Now, if we take $a = x-h$ and $b = x+h$ we have

$$|f'(x)| \le \frac{M-m}{2h} + \frac{h}{2}\|f''\|_\infty$$

and

$$\sup_{x\in\mathbb{R}} |f'(x)| \le \frac{M-m}{2h} + \frac{h}{2}\|f''\|_\infty.$$

Since it is easy to see that the function $y(h) = \frac{M-m}{2h} + \frac{h}{2}\|f''\|_\infty$ attains minimal value only for $h_{\min} = \sqrt{(M-m)/\|f''\|_\infty}$ and the corresponding minimal value is $y_{\min} = \sqrt{(M-m)\|f''\|_\infty}$, the inequality follows by taking $b-a = h_{\min}$. □

Now we shall prove a Landau type inequality for α-L-Hölder type function. For doing that we need the following result (see for instance [85]).

Lemma 2.7 *Let $f : [a,b] \to \mathbb{R}$ be α-L-Hölder type function, that is, $|f(t) - f(s)| \le L|t-s|^\alpha$ for all $t, s \in [a,b]$, where $L > 0$ and $0 < \alpha \le 1$ are some constants. Then for any $x \in [a,b]$ we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \right| \le \frac{L}{b-a}\int_a^b |t-x|^\alpha \,\mathrm{d}t. \tag{2.139}$$

Theorem 2.84 *Let $f : \mathbb{R} \to \mathbb{R}$ be such that $m \le f(x) \le M$ for all $x \in \mathbb{R}$ and f' is α-L-Hölder type function. Then we have*

$$\|f'\|_\infty \le \left(\frac{\alpha+1}{2\alpha}\right)^{\frac{\alpha}{\alpha+1}} (M-m)^{\frac{\alpha}{\alpha+1}} L^{\frac{1}{\alpha+1}}.$$

Proof. From (2.139) it follows that for any $a,b \in I$, $a<b$ and $x \in [a,b]$ we have

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \right| \le \frac{L}{(b-a)(\alpha+1)}\left((x-a)^{\alpha+1} + (b-x)^{\alpha+1} \right). \tag{2.140}$$

If we apply this to $f'(x)$, with $a = x-h$, $b = x+h$ we get

$$\left| f'(x) - \frac{f(x+h)-f(x-h)}{2h} \right| \le \frac{L}{\alpha+1}h^{\alpha},$$

so

$$|f'(x)| \le \frac{M-m}{2h} + \frac{L}{\alpha+1}h^{\alpha},$$

and

$$\sup_{x\in\mathbb{R}} |f'(x)| \le \frac{M-m}{2h} + \frac{L}{\alpha+1}h^{\alpha}.$$

By Lemma 2.6, the function $y(h) = \frac{M-m}{2h} + \frac{L}{\alpha+1}h^{\alpha}$ attains minimal value only for $h_{\min} = \left(\frac{(M-m)(\alpha+1)}{2L\alpha}\right)^{\frac{1}{\alpha+1}}$ and the corresponding minimal value is $y_{\min} = \left(\frac{\alpha+1}{2\alpha}\right)^{\frac{\alpha}{\alpha+1}}(M-m)^{\frac{\alpha}{\alpha+1}}L^{\frac{1}{\alpha+1}}$, so the proof follows. □

Remark 2.70 Now we consider the following result from [80]:

$$\|f'\|_{\infty} \le \begin{cases} \frac{2\|f\|_{\infty}}{m(I)} + \frac{L}{\alpha+1}(m(I))^{\alpha}, & \text{if } \ 0 < m(I) < t_0, \\ \left[2\left(1+\frac{1}{\alpha}\right)\right]^{\frac{\alpha}{\alpha+1}} \|f\|_{\infty}^{\frac{\alpha}{\alpha+1}} L^{\frac{1}{\alpha+1}}, & \text{if } \ m(I) \ge t_0, \end{cases}$$

where $f : I \to \mathbb{R}$ is absolutely continuous and f' α-L-Hölder type function, and

$$t_0 = \left(2\frac{\|f\|_{\infty}}{L}\left(1+\frac{1}{\alpha}\right)\right)^{\frac{1}{\alpha+1}}.$$

We can provide another proof of this inequality using the Ostrowski inequality (2.139). Indeed, if we apply (2.139) to $f'(x)$, for some $a,b \in I$, $a<b$ and $x \in [a,b]$ we have

$$\left| f'(x) - \frac{f(b)-f(a)}{b-a} \right| \le \frac{L}{(b-a)(\alpha+1)}\left((x-a)^{\alpha+1} + (b-x)^{\alpha+1} \right) \le \frac{L(b-a)^{\alpha}}{(\alpha+1)}$$

and

$$|f'(x)| \le \frac{2\|f\|_{\infty}}{b-a} + \frac{L}{\alpha+1}(b-a)^{\alpha}.$$

By Lemma 2.6, the function $y(t) = \frac{2\|f\|_{\infty}}{t} + \frac{L}{\alpha+1}t^{\alpha}$ attains minimal value only for $t_{\min} = \left(\frac{2\|f\|_{\infty}(\alpha+1)}{L\alpha}\right)^{\frac{1}{\alpha+1}}$ and the minimal value is $y_{\min} = \left[2\left(1+\frac{1}{\alpha}\right)\right]^{\frac{\alpha}{\alpha+1}} \|f\|_{\infty}^{\frac{\alpha}{\alpha+1}} L^{\frac{1}{\alpha+1}}$. After taking the supremum over $x \in I$ on the left-hand side of the last inequality the proof follows.

Some new Landau type inequalities

Here, we will consider Landau type inequalities of the form

$$M_1(\infty,I) \le C(p,I)\, M_0(\infty,I)^{\frac{p-1}{2p-1}}\, M_2(p,I)^{\frac{p}{2p-1}}$$

where $C(p,I)$ is a real constant, $M_2(p,I) = \|f''\|_p = \left(\int_I |f''(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}$ for $1 \le p < \infty$ and $\|f''\|_\infty = \sup_{x\in I}|f''(x)|$.

In the next two lemmas we cite three Ostrowski type inequalities from [47] and [59] respectively. The first and the second one are obtained via Euler type identities and the third one via the Fink identity.

Lemma 2.8 *Let $f:[a,b]\to\mathbb{R}$ be a function such that $f^{(n)}\in L_p[a,b]$ for some $n\ge 1$, $1\le p\le\infty$ and q is the conjugate exponent of p. Then for every $x\in[a,b]$, the following inequalities hold*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - T_n^{[a,b]}(x)\right| \le \frac{(b-a)^{n-1+\frac{1}{q}}}{n!}\left(\int_0^1 |B_n(t)|^q\,\mathrm{d}t\right)^{\frac{1}{q}}\left\|f^{(n)}\right\|_p, \tag{2.141}$$

$$\begin{aligned}&\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - T_{n-1}^{[a,b]}(x)\right| \\ &\le \frac{(b-a)^{n-1+\frac{1}{q}}}{n!}\left(\int_0^1 \left|B_n(t) - B_n\left(\frac{x-a}{b-a}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\left\|f^{(n)}\right\|_p,\end{aligned} \tag{2.142}$$

where

$$T_n^{[a,b]}(x) = \sum_{k=1}^{n}\frac{(b-a)^{k-1}}{k!}B_k\left(\frac{x-a}{b-a}\right)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right],$$

and $B_k(x)$, $k\ge 0$, are the Bernoulli polynomials, $B_k = B_k(0)$, $k\ge 0$, are the Bernoulli numbers.

Lemma 2.9 *Let $f^{(n-1)}$ be absolutely continuous on $[a,b]$, $n\in\mathbb{N}$, $f^{(n)}\in L_p[a,b]$, $1\le p\le\infty$ and q is the conjugate exponent of p. Then the following inequality holds*

$$\left|\frac{1}{n}\left(f(x) + \sum_{k=1}^{n-1}F_k(x)\right) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right| \le K(n,p,x)\|f^{(n)}\|_p \tag{2.143}$$

where

$$F_k(x) = \left(\frac{n-k}{k!}\right)\frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a},$$

and for $1<p\le\infty$

$$K(n,p,x) = \frac{\left[(x-a)^{nq+1} + (b-x)^{nq+1}\right]^{1/q}}{n!(b-a)}B((n-1)q+1, q+1)^{1/q},$$

while for $p=1$

$$K(n,1,x) = \frac{(n-1)^{n-1}}{n^n n!(b-a)}\max[(x-a)^n, (b-x)^n].$$

Next, we use these Ostrowski type inequalities together with the inequality from the Corollary 2.27 to obtain new Landau type inequalities.

Theorem 2.85 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 3$, $f^{(n)} \in L_p[a,b]$, $1 \leq p \leq \infty$, q the conjugate exponent of p, and $f''(a) = f'''(a) = \cdots = f^{(n-1)}(a) = 0$, $f''(b) = f'''(b) = \cdots = f^{(n-1)}(b) = 0$. Then for $1 < p \leq \infty$ we have*

$$\left\|f'\right\|_\infty \leq \frac{2\|f\|_\infty}{b-a} + \frac{((b-a)(n-1))^{n-2+\frac{1}{q}}}{(n-2)!}\left(\overline{B}(n-1,q)\right)^{\frac{1}{q}}\left\|f^{(n)}\right\|_p \tag{2.144}$$

where $\overline{B}(n-1,q)$ is given by

$$\overline{B}(n,q) = \left(B(q+1,q(n-1)+1) - B_{\frac{n-1}{n}}(q+1,q(n-1)+1)\right)$$

while for $p = 1$ it holds

$$\left\|f'\right\|_\infty \leq \frac{2\|f\|_\infty}{b-a} + \frac{(n-2)(b-a)^{n-2}}{(n-1)!}\left\|f^{(n)}\right\|_1.$$

Proof. For $1 < p \leq \infty$, we apply Corollary 2.27 to $f'(x)$, and replace n with $n-1$ (thus $n \geq 3$), so we have

$$\begin{aligned}
&\left|f'(x) - \frac{f(b)-f(a)}{b-a} - \sum_{i=0}^{n-3}\frac{f^{(i+2)}(a)}{i!(i+2)}\frac{(x-a)^{i+2}}{b-a} + \sum_{i=0}^{n-3}\frac{f^{(i+2)}(b)}{i!(i+2)}\frac{(x-b)^{i+2}}{b-a}\right| \\
&\leq \frac{1}{(b-a)(n-2)!}\left\{\left((x-a)^{q(n-1)+1} + (b-x)^{q(n-1)+1}\right)\right. \\
&\left.\cdot (n-1)^{q(n-2)+1}\overline{B}(n-1,q)\right\}^{\frac{1}{q}}\left\|f^{(n)}\right\|_p.
\end{aligned}$$

which leads to

$$\begin{aligned}
&\left|f'(x)\right| - \frac{|f(b)-f(a)|}{b-a} \\
&\leq \frac{1}{(b-a)(n-2)!}\left\{(b-a)^{q(n-1)+1}(n-1)^{q(n-2)+1}\cdot\overline{B}(n-1,q)\right\}^{\frac{1}{q}}\left\|f^{(n)}\right\|_p
\end{aligned}$$

and

$$\left|f'(x)\right| \leq \frac{2\|f\|_\infty}{b-a} + \frac{((b-a)(n-1))^{n-2+\frac{1}{q}}}{(n-2)!}\left(\overline{B}(n-1,q)\right)^{\frac{1}{q}}\left\|f^{(n)}\right\|_p$$

. For $p = 1$ we apply Corollary 2.27 in the same way to get

$$\begin{aligned}
\left|f'(x)\right| &\leq \frac{2\|f\|_\infty}{b-a} + \frac{n-2}{(b-a)(n-1)!}\max\left\{(x-a)^{n-1},(b-x)^{n-1}\right\}\left\|f^{(n)}\right\|_1 \\
&\leq \frac{2\|f\|_\infty}{b-a} + \frac{(n-2)(b-a)^{n-2}}{(n-1)!}\left\|f^{(n)}\right\|_1.
\end{aligned}$$

The proof follows by taking the supremum over $x \in [a,b]$ on the left-hand side of the last two inequalities. □

Remark 2.71 In the special case, for $p=\infty$ inequality (2.144) reduces to the (2.134).

Theorem 2.86 *Let $f:[a,b]\to\mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ for some $n\geq 2$, $f^{(n)}\in L_p[a,b]$, $1\leq p\leq\infty$ and $f'(a)=f'(b)$, $f''(a)=f''(b),\ldots,$ $f^{(n-1)}(a)=f^{(n-1)}(b)$ and q the conjugate exponent of p. Then the following inequality holds:*

$$\left\|f'\right\|_\infty \leq \frac{2\|f\|_\infty}{b-a}+\frac{(b-a)^{n-2+\frac{1}{q}}}{(n-1)!}\left(\int_0^1 |B_{n-1}(t)|^q\,\mathrm{d}t\right)^{\frac{1}{q}}\left\|f^{(n)}\right\|_p. \tag{2.145}$$

Proof. We apply (2.141) to $f'(x)$, and replace n with $n-1$. Using the assumptions we have that $T_{n-1}^{[a,b]}(x)=0$ and

$$\left|f'(x)-\frac{f(b)-f(a)}{b-a}\right| \leq \frac{(b-a)^{n-2+\frac{1}{q}}}{(n-1)!}\left(\int_0^1 |B_{n-1}(t)|^q\,\mathrm{d}t\right)^{\frac{1}{q}}\left\|f^{(n)}\right\|_p.$$

Since $|f'(x)|-\left|\frac{f(b)-f(a)}{b-a}\right|\leq\left|f'(x)-\frac{f(b)-f(a)}{b-a}\right|$ we have

$$\left|f'(x)\right| \leq \frac{|f(b)-f(a)|}{b-a}+\frac{(b-a)^{n-2+\frac{1}{q}}}{(n-1)!}\left(\int_0^1 |B_{n-1}(t)|^q\,\mathrm{d}t\right)^{\frac{1}{q}}\left\|f^{(n)}\right\|_p.$$

Thus

$$\left|f'(x)\right| \leq \frac{2\|f\|_\infty}{b-a}+\frac{(b-a)^{n-2+\frac{1}{q}}}{(n-1)!}\left(\int_0^1 |B_{n-1}(t)|^q\,\mathrm{d}t\right)^{\frac{1}{q}}\left\|f^{(n)}\right\|_p.$$

The proof follows by taking the supremum over $x\in[a,b]$ on the left-hand side of the inequality. □

Theorem 2.87 *Let $f:[a,b]\to\mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ for some $n\geq 2$, $f^{(n)}\in L_p[a,b]$, $1\leq p\leq\infty$ and $f'(a)=f'(b)$, $f''(a)=f''(b),\ldots,$ $f^{(n-2)}(a)=f^{(n-2)}(b)$ and q the conjugate exponent of p. Then the following inequality holds:*

$$\left\|f'\right\|_\infty \leq \frac{2\|f\|_\infty}{b-a}+\frac{(b-a)^{n-2+\frac{1}{q}}M}{(n-1)!}\left\|f^{(n)}\right\|_p \tag{2.146}$$

where

$$M=\sup_{x\in[a,b]}\left\{\left(\int_0^1\left|B_{n-1}(t)-B_{n-1}\left(\frac{x-a}{b-a}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right\},$$

Proof. We apply (2.142) to $f'(x)$, and replace n with $n-1$. Using the assumptions we have that $T_{n-2}^{[a,b]}(x)=0$ and

$$\left|f'(x)-\frac{f(b)-f(a)}{b-a}\right|$$

$$\leq \frac{(b-a)^{n-2+\frac{1}{q}}}{(n-1)!}\left(\int_0^1\left|B_{n-1}(t)-B_{n-1}\left(\frac{x-a}{b-a}\right)\right|^q \mathrm{d}t\right)^{\frac{1}{q}}\left\|f^{(n)}\right\|_p.$$

Since $|f'(x)|-\left|\frac{f(b)-f(a)}{b-a}\right|\leq\left|f'(x)-\frac{f(b)-f(a)}{b-a}\right|$ we have

$$\begin{aligned}|f'(x)| &\leq \frac{2\|f\|_\infty}{b-a}+\frac{(b-a)^{n-2+\frac{1}{q}}}{(n-1)!}\left(\int_0^1\left|B_{n-1}(t)-B_{n-1}\left(\frac{x-a}{b-a}\right)\right|^q \mathrm{d}t\right)^{\frac{1}{q}}\left\|f^{(n)}\right\|_p \\ &\leq \frac{2\|f\|_\infty}{b-a}+\frac{(b-a)^{n-2+\frac{1}{q}}M}{(n-1)!}\left\|f^{(n)}\right\|_p.\end{aligned}$$

Now, the proof follows by taking the supremum over $x\in[a,b]$ on the left-hand side of the inequality. □

Remark 2.72 In the special case, for $p=\infty$ the inequalities (2.145) and (2.146) reduce to the results from the Theorem 2.78 and Theorem 2.79.

Corollary 2.52 *Let $f: I\to\mathbb{R}$ be a function such that f' is absolutely continuous on I, $f''\in L_p(I)$, $1\leq p\leq\infty$, and q the conjugate exponent of p. Then the following inequality holds:*

$$\left\|f'\right\|_\infty\leq\begin{cases}\frac{2\|f\|_\infty}{m(I)}+\left(\frac{m(I)}{q+1}\right)^{1/q}\|f''\|_p, & \text{if } 0<m(I)<t_0,\\[2ex] (2(q+1))^{\frac{1}{q+1}}q^{-\frac{q}{q+1}}\|f\|_\infty^{\frac{1}{q+1}}\|f''\|_p^{\frac{q}{q+1}}, & \text{if } m(I)\geq t_0,\end{cases}\tag{2.147}$$

where

$$t_0=\left(\frac{2q(q+1)^{\frac{1}{q}}\|f\|_\infty}{\|f''\|_p}\right)^{\frac{q}{q+1}}.$$

Proof. We set the $n=2$ in the inequality (2.146), for some $a,b\in I$, $a<b$. Then we have

$$M=\sup_{x\in[a,b]}\left\{\left(\int_0^1\left|B_1(t)-B_1\left(\frac{x-a}{b-a}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right\}=\left(\frac{1}{q+1}\right)^{\frac{1}{q}},$$

so

$$\left\|f'\right\|_\infty\leq\frac{2\|f\|_\infty}{b-a}+\left(\frac{b-a}{q+1}\right)^{1/q}\|f''\|_p.$$

By Lemma 2.6, the function $y(t)=\frac{2\|f\|_\infty}{t}+\left(\frac{t}{q+1}\right)^{1/q}\|f''\|_p$ attains minimal value only for $t_{\min}=\left(2q(q+1)^{\frac{1}{q}}\|f\|_\infty\right)^{\frac{q}{q+1}}$ and the corresponding minimal value is

$y_{\min}=(2(q+1))^{\frac{1}{q+1}}q^{-\frac{q}{q+1}}\|f\|_\infty^{\frac{1}{q+1}}\|f''\|_p^{\frac{q}{q+1}}$, so the proof follows. □

Theorem 2.88 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ for some $n \geq 2$, $f^{(n)} \in L_p[a,b]$, $1 \leq p \leq \infty$ and $f'(a) = f''(a) = \cdots = f^{(n-2)}(a) = 0$ and $f'(b) = f''(b) = \cdots = f^{(n-2)}(b) = 0$ and q the conjugate exponent of p. Then we have*

$$\left\| f' \right\|_\infty \leq \frac{2(n-1)\|f\|_\infty}{b-a} + \frac{(b-a)^{n-2+\frac{1}{q}}}{(n-2)!} B((n-2)q+1, q+1)^{1/q} \|f^{(n)}\|_p, \qquad (2.148)$$

if $1 < p \leq \infty$, while if $p = 1$ then

$$\left\| f' \right\|_\infty \leq \frac{2(n-1)\|f\|_\infty}{b-a} + \left(\frac{n-2}{n-1}\right)^{n-2} \frac{(b-a)^{n-2}}{(n-1)!} \|f^{(n)}\|_1. \qquad (2.149)$$

Proof. We apply (2.143) to $f'(x)$, and replace n with $n-1$ (thus $n \geq 2$). Using the assumptions we have that $\sum_{k=1}^{n-2} F_k(x) = 0$ and

$$\left| \frac{f'(x)}{n-1} - \frac{f(b)-f(a)}{b-a} \right| \leq K(n-1,p,x)\|f^{(n)}\|_p.$$

Since $\left|\frac{f'(x)}{n-1}\right| - \left|\frac{f(b)-f(a)}{b-a}\right| \leq \left|\frac{f'(x)}{n-1} - \frac{f(b)-f(a)}{b-a}\right|$ we have

$$\left| f'(x) \right| \leq \frac{(n-1)|f(b)-f(a)|}{b-a} + (n-1)K(n-1,p,x)\|f^{(n)}\|_p.$$

Thus, if $1 < p \leq \infty$ then

$$\begin{aligned} \left| f'(x) \right| &\leq \frac{(n-1)|f(b)-f(a)|}{b-a} \\ &\quad + \frac{\left[(x-a)^{(n-1)q+1} + (b-x)^{(n-1)q+1}\right]^{1/q}}{(n-2)!(b-a)} B((n-2)q+1, q+1)^{1/q} \|f^{(n)}\|_p \\ &\leq \frac{2(n-1)\|f\|_\infty}{b-a} + \frac{(b-a)^{n-2+\frac{1}{q}}}{(n-2)!} B((n-2)q+1, q+1)^{1/q} \|f^{(n)}\|_p, \end{aligned}$$

and if $p = 1$ then

$$\begin{aligned} \left| f'(x) \right| &\leq \frac{(n-1)|f(b)-f(a)|}{b-a} \\ &\quad + \frac{(n-2)^{n-2}}{(n-1)^{n-2}(n-1)!(b-a)} \max[(x-a)^{n-1}, (b-x)^{n-1}] \|f^{(n)}\|_1 \\ &\leq \frac{2(n-1)\|f\|_\infty}{b-a} + \left(\frac{n-2}{n-1}\right)^{n-2} \frac{(b-a)^{n-2}}{(n-1)!} \|f^{(n)}\|_1. \end{aligned}$$

The proof follows by taking the supremum over $x \in [a,b]$ on the left-hand side of the both inequalities. □

Remark 2.73 In the special case, for $p = \infty$ the inequality (2.148) reduces to the result from the Theorem 2.81. Also, if we apply (2.148), (2.149) with $n = 2$ and with $f : I \to \mathbb{R}$ such that f' is absolutely continuous on I, $f'' \in L_p(I)$, $1 \le p \le \infty$ and q the conjugate exponent of p, then we obtain the same result as in the Corollary 2.52.

Theorem 2.89 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that f' is absolutely continuous on $\mathbb{R}$, $f'' \in L_p(\mathbb{R})$, $1 \le p \le \infty$, and q the conjugate exponent of p. Then the following inequality holds:*

$$\left\|f'\right\|_\infty \le \left(\frac{q+1}{q}\right)^{\frac{q}{q+1}} 2^{\frac{1-q}{q+1}} \|f\|_\infty^{\frac{1}{q+1}} \|f''\|_p^{\frac{q}{q+1}} \tag{2.150}$$

Proof. If we apply inequality (2.143) with the $n = 1$ to $f'(x)$, for $a = x-h$, $b = x+h$, we obtain

$$\begin{aligned} \left|f'(x)\right| &\le \frac{|f(b) - f(a)|}{2h} + \frac{\left[2h^{q+1}\right]^{1/q}}{2h} \left(\frac{1}{q+1}\right)^{1/q} \|f''\|_p \\ &\le \frac{\|f\|_\infty}{h} + 2^{\frac{1}{q}-1} h^{1/q} \left(\frac{1}{q+1}\right)^{1/q} \|f''\|_p. \end{aligned}$$

Thus we have

$$\left\|f'\right\|_\infty \le \frac{\|f\|_\infty}{h} + 2^{\frac{1}{q}-1} h^{\frac{1}{q}} \left(\frac{1}{q+1}\right)^{1/q} \|f''\|_p.$$

By Lemma 2.6, the function $y(h) = \frac{\|f\|_\infty}{h} + 2^{\frac{1}{q}-1} h^{\frac{1}{q}} (\frac{1}{q+1})^{1/q} \|f''\|_p$ attains minimal value only for $h_{\min} = \left(\frac{2q\|f\|_\infty}{\|f''\|_p} \left(\frac{q+1}{2}\right)^{\frac{1}{q}}\right)^{\frac{q}{q+1}}$ and the corresponding minimal value is $y_{\min} = \left(\frac{q+1}{q}\right)^{\frac{q}{q+1}} 2^{\frac{1-q}{q+1}} \|f\|_\infty^{\frac{1}{q+1}} \|f''\|_p^{\frac{q}{q+1}}$, so inequality (2.150) follows. Inequality (2.150) holds for $1 < p \le \infty$ as well as for $p = 1$. □

Remark 2.74 For $q = 1$ (2.147) and (2.150) reduces to Landau inequality (2.132) and (2.138).

Remark 2.75 If we additionally assume that $m \le f(x) \le M$ for every $x \in [a,b]$, all the inequalities from the Theorems 2.85, 2.86, 2.87, and 2.88 hold with $M - m$ instead of $2\|f\|_\infty$.

Chapter 3

Montgomery identity, quadrature formulae and error estimations

The aim of this chapter is to give generalizations of classical quadrature formulae with two, three and four nodes using some generalizations of the weighted Montgomery identity. Thereby families of weighted and non-weighted quadrature formulae are considered, some error estimates are derived, and sharp and the best possible inequalities as well as Ostrowski type inequalities are proved.

In Section 3.1 classes of weighted and non-weighted two-point quadrature formulae are studied and corresponding error estimates are calculated. Two-point Gauss-Chebyshev formulae of the first and of the second kind as well as generalizations of the trapezoidal formula, Newton-Cotes two-point formula, Maclaurin two-point formula and midpoint formula are obtained as special cases of these formulae.

Section 3.2 deals with three-point quadrature formulae, generalizations of Simpson's, dual Simpson's and Maclaurin's formula, three-point Gauss-Chebyshev formulae of the first kind and of the second kind that follow from a general formula, as well as corresponding error estimates.

Section 3.3 is dedicated to closed four-point quadrature formulae from which a weighted and non-weighted generalization of Bullen type inequalities for $(2n)-$ convex functions is obtained. As a special case, Simpson's $3/8$ formula and Lobatto four-point formula with related inequalities are considered.

3.1 Two-point quadrature formulae

3.1.1 Introduction

In this section, for each number $x \in \left[a, \frac{a+b}{2}\right]$, we study a general weighted two-point quadrature formula

$$\int_a^b w(t)\, f(t)\, dt = \frac{1}{2}\left[f(x) + f(a+b-x)\right] + E(f, w; x), \tag{3.1}$$

where $w : [a,b] \to [0,\infty)$ is a probability density function that is integrable function satisfying $\int_a^b w(t)dt = 1$, and $W(t) = \int_a^t w(u)du$ for $t \in [a,b]$, $W(t) = 0$ for $t < a$ and $W(t) = 1$, for $t > b$ and $E(f,w;x)$ is the remainder. In the special case, for $w(t) = \frac{1}{b-a}$, $t \in [a,b]$, (3.1) reduces to the family of two-point quadrature formulae

$$\int_a^b f(t)\, dt = \frac{b-a}{2}\left[f(x) + f(a+b-x)\right] + E(f,x),$$

which was considered by Guessab and Schmeisser in [65], where they established sharp estimates for the remainder under various regularity conditions.

3.1.2 Quadrature formulae obtained by a generalization of the Montgomery identity

The aim of this subsection is to establish a general weighted and non-weighted version of two-point formula (3.1) using the identities (2.76) and (2.77) and to give various error estimates for quadrature rules based on such generalizations. We prove a number of inequalities which give error estimates for the general two-point formula for functions whose derivatives are from the L_p- spaces, $1 \le p \le \infty$. These inequalities are generally sharp (in case $p = 1$ the best possible). Also, we give some examples of the general two-point formula for the well-known weight functions, that is two-point Gauss-Chebyshev formulae of the first and of the second kind. The results from this subsection are published in [15].

General weighted two-point formula

Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ exists on $[a,b]$ for some $n \ge 2$. We introduce the following notation for each $x \in \left[a, \frac{a+b}{2}\right]$

$$D_2(x) = \frac{1}{2}\left[f(x) + f(a+b-x)\right], \tag{3.2}$$

$$t_{w,n}^2(x) = \frac{1}{2}\left[\sum_{i=0}^{n-2} \frac{f^{(i+1)}(x)}{(i+1)!} \int_a^b w(s)(s-x)^{i+1} \mathrm{d}s\right.$$

$$+\sum_{i=0}^{n-2}\frac{f^{(i+1)}(a+b-x)}{(i+1)!}\int_a^b w(s)(s-a-b+x)^{i+1}\mathrm{d}s\Bigg],\tag{3.3}$$

$$T_{w,n}^2(x,s)=\begin{cases}-\int_a^s w(u)(u-s)^{n-1}\,\mathrm{d}u, & a\le s\le x,\\ -\frac{1}{2}\int_a^s w(u)(u-s)^{n-1}\,\mathrm{d}u & \\ +\frac{1}{2}\int_s^b w(u)(u-s)^{n-1}\,\mathrm{d}u, & x<s\le a+b-x,\\ \int_s^b w(u)(u-s)^{n-1}\,\mathrm{d}u, & a+b-x<s\le b.\end{cases}$$

In the next theorem we establish a general weighted two-point formula.

Theorem 3.1 *Let I be an open interval in $\mathbb{R}$, $[a,b]\subset I$, and let $w:[a,b]\to[0,\infty\rangle$ be some probability density function. Let $f:I\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n\ge 2$. Then for each $x\in\left[a,\frac{a+b}{2}\right]$ the following identity holds*

$$\int_a^b w(t)f(t)\,\mathrm{d}t=D_2(x)+t_{w,n}^2(x)+\frac{1}{(n-1)!}\int_a^b T_{w,n}^2(x,s)\,f^{(n)}(s)\,\mathrm{d}s.\tag{3.4}$$

Proof. We put $x\equiv x$ and $x\equiv a+b-x$ in (2.76) to obtain two new formulae. After adding these two formulae and multiplying by $1/2$, we obtain (3.4). □

Remark 3.1 Identity (3.4) holds true in the case $n=1$. In this special case we have

$$\int_a^b w(t)f(t)\,\mathrm{d}t=D_2(x)+\int_a^b T_{w,1}^2(x,s)\,f'(s)\,\mathrm{d}s,\tag{3.5}$$

where

$$\begin{aligned}T_{w,1}^2(x,s)&=-\frac{1}{2}\left[P_w(x,s)+P_w(a+b-x,s)\right]\\ &=\begin{cases}-W(s), & a\le s\le x,\\ \frac{1}{2}-W(s), & x<s\le a+b-x,\\ 1-W(s), & a+b-x<s\le b.\end{cases}\end{aligned}$$

Now, we use formula (3.4) to prove a number of inequalities using L_p norms for $1\le p\le\infty$. These inequalities are generally sharp (in case $p=1$ the best possible).

Theorem 3.2 *Suppose that all the assumptions of Theorem 3.1 hold. Additionally, assume that (p,q) is a pair of conjugate exponents, $1\le p,q\le\infty$; let $f^{(n)}\in L^p[a,b]$ for some $n\ge 1$. Then for each $x\in\left[a,\frac{a+b}{2}\right]$ the following inequality holds*

$$\left|\int_a^b w(t)f(t)\mathrm{d}t-D_2(x)-t_{w,n}^2(x)\right|\le\frac{1}{(n-1)!}\left\|T_{w,n}^2(x,\cdot)\right\|_q\left\|f^{(n)}\right\|_p.\tag{3.6}$$

Inequality (3.6) is sharp for $1<p\le\infty$ and the best possible for $p=1$.

Proof. Applying the Hölder inequality we have

$$\left|\frac{1}{(n-1)!}\int_a^b T_{w,n}^2(x,s)\,f^{(n)}(s)\,\mathrm{d}s\right| \le \frac{1}{(n-1)!}\left\|T_{w,n}^2(x,\cdot)\right\|_q\left\|f^{(n)}\right\|_p.$$

Using the above inequality from (3.4) we get estimate (3.6). Let us denote $C_n^x(s) = T_{w,n}^2(x,s)$. For the proof of sharpness, we need to find function f such that

$$\left|\int_a^b C_n^x(s)\,f^{(n)}(s)\,\mathrm{d}s\right| = \|C_n^x\|_q\left\|f^{(n)}\right\|_p.$$

For $1 < p < \infty$, take f to be such that

$$f^{(n)}(s) = \mathrm{sign}C_n^x(s)\cdot|C_n^x(s)|^{\frac{1}{p-1}}.$$

For $p = \infty$, take

$$f^{(n)}(s) = \mathrm{sign}C_n^x(s).$$

For $p = 1$, we shall prove that

$$\left|\int_a^b C_n^x(s)\,f^{(n)}(s)\,\mathrm{d}s\right| \le \max_{s\in[a,b]}|C_n^x(s)|\left(\int_a^b |f^{(n)}(s)|\,\mathrm{d}s\right) \tag{3.7}$$

is the best possible inequality.

Function C_n^x is left continuous and has finite jump at x and $a+b-x$. Thus we have four possibilities:

1. $|C_n^x|$ attains its maximum at $s_0 \in [a,b]$ and $C_n^x(s_0) > 0$. Then for $\varepsilon > 0$ small enough define $f_\varepsilon(s)$ by

$$f_\varepsilon(s) = \begin{cases} 0, & a \le s \le s_0-\varepsilon, \\ \frac{1}{\varepsilon n!}(s-s_0+\varepsilon)^n, & s_0-\varepsilon \le s \le s_0, \\ \frac{1}{n!}(s-s_0+\varepsilon)^{n-1}, & s_0 \le s \le b. \end{cases}$$

Thus

$$\left|\int_a^b C_n^x(s)f_\varepsilon^{(n)}(s)\mathrm{d}s\right| = \left|\int_{s_0-\varepsilon}^{s_0} C_n^x(s)\frac{1}{\varepsilon}\mathrm{d}s\right| = \frac{1}{\varepsilon}\int_{s_0-\varepsilon}^{s_0} C_n^x(s)\mathrm{d}s.$$

Now, from inequality (3.7) we have

$$\frac{1}{\varepsilon}\int_{s_0-\varepsilon}^{s_0} C_n^x(s)\mathrm{d}s \le \frac{1}{\varepsilon}C_n^x(s_0)\int_{s_0-\varepsilon}^{s_0}\mathrm{d}s = C_n^x(s_0).$$

Since

$$\lim_{\varepsilon\to 0,\varepsilon>0}\frac{1}{\varepsilon}\int_{s_0-\varepsilon}^{s_0} C_n^x(s)\mathrm{d}s = C_n^x(s_0),$$

the statement follows.

2. $|C_n^x|$ attains its maximum at $s_0 \in [a,b]$ and $C_n^x(s_0) < 0$. Then for $\varepsilon > 0$ small enough define $f_\varepsilon(s)$ by

$$f_\varepsilon(s) = \begin{cases} \frac{1}{n!}(s_0-s)^{n-1}, & a \le s \le s_0-\varepsilon, \\ -\frac{1}{\varepsilon n!}(s_0-s)^n, & s_0-\varepsilon \le s \le s_0, \\ 0, & s_0 \le s \le b, \end{cases}$$

and the rest of the proof is similar to the above.

3. $|C_n^x|$ does not attain a maximum on the $[a,b]$ and let $s_0 \in [a,b]$ be such that

$$\sup_{s\in[a,b]} |C_n^x(s)| = \lim_{\varepsilon\to 0,\varepsilon>0} |f(s_0+\varepsilon)|.$$

If $\lim_{\varepsilon\to 0,\varepsilon>0} f(s_0+\varepsilon) > 0$, we take

$$f_\varepsilon(s) = \begin{cases} 0, & a \le s \le s_0, \\ \frac{1}{\varepsilon n!}(s-s_0)^n, & s_0 \le s \le s_0+\varepsilon, \\ \frac{1}{n!}(s-s_0)^{n-1}, & s_0+\varepsilon \le s \le b, \end{cases}$$

and similarly to before we have

$$\begin{aligned} \left|\int_a^b C_n^x(s) f_\varepsilon^{(n)}(s)\mathrm{d}s\right| &= \left|\int_{s_0}^{s_0+\varepsilon} C_n^x(s)\frac{1}{\varepsilon}\mathrm{d}s\right| = \frac{1}{\varepsilon}\int_{s_0}^{s_0+\varepsilon} C_n^x(s)\,\mathrm{d}s, \\ \frac{1}{\varepsilon}\int_{s_0}^{s_0+\varepsilon} C_n^x(s)\,\mathrm{d}s &\le \frac{1}{\varepsilon}C_n^x(s_0)\int_{s_0}^{s_0+\varepsilon}\mathrm{d}s = C_n^x(s_0), \\ \lim_{\varepsilon\to 0,\varepsilon>0}\frac{1}{\varepsilon}\int_{s_0}^{s_0+\varepsilon} C_n^x(s)\,\mathrm{d}s &= C_n^x(s_0), \end{aligned}$$

and the statement follows.

4. $|C_n^x|$ does not attain a maximum on the $[a,b]$ and let $s_0 \in [a,b]$ be such that

$$\sup_{s\in[a,b]} |C_n^x(s)| = \lim_{\varepsilon\to 0,\varepsilon>0} |f(s_0+\varepsilon)|.$$

If $\lim_{\varepsilon\to 0,\varepsilon>0} f(s_0+\varepsilon) < 0$, we take

$$f_\varepsilon(s) = \begin{cases} \frac{1}{n!}(s-s_0-\varepsilon)^{n-1}, & a \le s \le s_0, \\ -\frac{1}{\varepsilon n!}(s-s_0-\varepsilon)^n, & s_0 \le s \le s_0+\varepsilon, \\ 0, & s_0+\varepsilon \le s \le b, \end{cases}$$

and the rest of the proof is similar to the above. □

Theorem 3.3 *Suppose that all the assumptions of Theorem 3.1 hold. Additionally, assume that $f^{(2n)}$ is a continuous function on $[a,b]$. Then for every $x \in \left[a, \frac{a+b}{2}\right]$ there exists $\eta \in (a,b)$ such that*

$$\int_a^b w(t)f(t)\mathrm{d}t - D_2(x) - t_{w,2n}^2(x) = \frac{f^{(2n)}(\eta)}{(2n-1)!}\int_a^b T_{w,2n}^2(x,s)\mathrm{d}s. \tag{3.8}$$

Proof. We apply (3.4) with $2n$ instead of n. Since

$$-\int_a^s w(u)(u-s)^{2n-1}\,\mathrm{d}u \ge 0, \quad \text{for every } s \in [a,x],$$

$$\int_s^b w(u)(u-s)^{2n-1}\,\mathrm{d}u \ge 0, \quad \text{for every } s \in (a+b-x,b]$$

and

$$\frac{1}{2}\left[-\int_a^s w(u)(u-s)^{2n-1}\,\mathrm{d}u+\int_s^b w(u)(u-s)^{2n-1}\,\mathrm{d}u\right]\geq 0,$$

for every $s\in(x,a+b-x]$, we have $T_{w,2n}^2(x,s)\geq 0$ for every $s\in[a,b]$. By applying the integral mean value theorem to $\int_a^b T_{w,2n}^2(x,s)f^{(2n)}(s)\mathrm{d}s$ we obtain (3.8). □

Theorem 3.4 *Suppose that all the assumptions of Theorem 3.1 hold for $2n$, $n\in\mathbb{N}$. If f is $(2n)$-convex, then for each $x\in\left[a,\frac{a+b}{2}\right]$ the following inequality holds*

$$\int_a^b w(t)f(t)\mathrm{d}t-\frac{f(x)+f(a+b-x)}{2}-t_{w,2n}^2(x)\geq 0. \tag{3.9}$$

If f is $(2n)$-concave, then inequality (3.9) is reversed.

Proof. First note that if $f^{(k)}$ exists, then f is k-convex (k-concave) iff $f^{(k)}\geq 0$ ($f^{(k)}\leq 0$). From (3.4) we have that

$$\int_a^b w(t)f(t)\mathrm{d}t-D_2(x)-t_{w,2n}^2(x)=\frac{1}{(n-1)!}\int_a^b T_{w,2n}^2(x,s)f^{(2n)}(s)\mathrm{d}s.$$

Let us consider the sign of the integral $\int_a^b T_{w,2n}^2(x,s)f^{(2n)}(s)\,\mathrm{d}s$, when f is $2n$-convex. We have $f^{(2n)}\geq 0$ and from the proof of Theorem 3.3 $T_{w,2n}^2(x,s)\geq 0$. Hence,

$$\int_a^b T_{w,2n}^2(x,s)\,f^{(2n)}(s)\,\mathrm{d}s\geq 0,$$

and (3.9) follows.

The reversed (3.9) can be obtained analogously. □

Gauss-Chebyshev two-point formulae of the first kind

Gaussian quadrature formulae are formulae of the following type

$$\int_a^b \rho(t)f(t)\,\mathrm{d}t=\sum_{i=1}^k A_i f(x_i)+E_k(f). \tag{3.10}$$

Without loss of generality, we shall restrict ourselves to $[a,b]=[-1,1]$.

In case $\rho(t)=\frac{1}{\sqrt{1-t^2}}, t\in[-1,1]$ we have Gauss-Chebyshev formulae

$$\int_{-1}^1 \frac{1}{\sqrt{1-t^2}}f(t)\,\mathrm{d}t=\sum_{i=1}^k A_i f(x_i)+E_k(f), \tag{3.11}$$

where A_i are given by

$$A_i=\frac{\pi}{k},\quad i=1,\ldots,k$$

and x_i are zeros of the Chebyshev polynomials of the first kind defined as

$$T_k(x) = \cos(k \arccos(x)), \quad x \in \mathbb{R}.$$

T_k has exactly k distinct zeros, all of which lie in the interval $[-1,1]$ (see for instance [103])

$$x_i = \cos\left(\frac{(2i-1)\pi}{2k}\right), \quad i = 1, \ldots, k.$$

Error of the approximation formula (3.11) is given by

$$E_k(f) = \frac{\pi}{2^{2k-1}(2k)!} f^{(2k)}(\xi), \quad \xi \in (-1,1).$$

In case $k = 2$, (3.11) reduces to

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} f(t)\,dt = \frac{\pi}{2}\left[f\left(-\frac{\sqrt{2}}{2}\right) + f\left(\frac{\sqrt{2}}{2}\right)\right] + \frac{\pi}{192} f^{(4)}(\xi), \quad \xi \in (-1,1).$$

Now, using the results of the previous subsection we establish Gauss-Chebyshev two-point formulae of the first kind and give some sharp and best possible inequalities.

Remark 3.2 If we apply (3.5) with $a = -1$, $b = 1$, $x = -\frac{\sqrt{2}}{2}$ and $w(t) = \frac{1}{\pi\sqrt{1-t^2}}$, $t \in [-1,1]$, we get

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} f(t)\,\mathrm{d}t = \frac{\pi}{2}\left[f\left(-\frac{\sqrt{2}}{2}\right) + f\left(\frac{\sqrt{2}}{2}\right)\right] + \pi \int_{-1}^{1} R_1(s) f'(s)\,\mathrm{d}s,$$

where

$$R_1(s) = \begin{cases} -\frac{1}{2} - \frac{1}{\pi}\arcsin s, & -1 \le s \le -\frac{\sqrt{2}}{2}, \\ -\frac{1}{\pi}\arcsin s, & -\frac{\sqrt{2}}{2} < s \le \frac{\sqrt{2}}{2}, \\ \frac{1}{2} - \frac{1}{\pi}\arcsin s, & \frac{\sqrt{2}}{2} < s \le 1. \end{cases}$$

Corollary 3.1 *Let I be an open interval in $\mathbb{R}$, $[-1,1] \subset I$, (p,q) is a pair of conjugate exponents, $1 \le p,q \le \infty$. Let $f : I \to \mathbb{R}$ be an absolutely continuous function and $f' \in L^p[-1,1]$. Then we have*

$$\left|\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} f(t)\,\mathrm{d}t - \frac{\pi}{2}\left[f\left(-\frac{\sqrt{2}}{2}\right) + f\left(\frac{\sqrt{2}}{2}\right)\right]\right| \le \pi \|R_1\|_q \|f'\|_p. \tag{3.12}$$

Inequality (3.12) is sharp for $1 < p \le \infty$ and best possible for $p = 1$.

Proof. This is a special case of Theorem 3.2 for $a = -1$, $b = 1$, $x = -\frac{\sqrt{2}}{2}$ and $w(t) = \frac{1}{\pi\sqrt{1-t^2}}$, $t \in [-1,1]$. □

Corollary 3.2 *Let I be an open interval in $\mathbb{R}$, $[-1,1] \subset I$, (p,q) a pair of conjugate exponents, $1 \le p,q \le \infty$ and let $f : I \to \mathbb{R}$ be an absolutely continuous function. Then*

$$\left| \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} f(t)\,\mathrm{d}t - \frac{\pi}{2}\left[f\left(-\frac{\sqrt{2}}{2}\right) + f\left(\frac{\sqrt{2}}{2}\right)\right]\right| \le \begin{cases} \left(2\sqrt{2}-2\right)\|f'\|_\infty, & f' \in L^\infty[-1,1] \\ \left(\pi\cdot\sqrt{2}-4\right)^{\frac{1}{2}}\|f'\|_2, & f' \in L^2[-1,1] \\ \frac{1}{4}\pi\|f'\|_1, & f' \in L^1[-1,1]. \end{cases}$$

The first and the second inequality are sharp and the third inequality is the best possible.

Proof. We apply (3.12) with $p = \infty$

$$\begin{aligned} \int_{-1}^{1} |R_1(s)|\,\mathrm{d}s &= \int_{-1}^{-\frac{\sqrt{2}}{2}} \left| -\frac{1}{2} - \frac{1}{\pi}\arcsin s \right| \mathrm{d}s \\ &+ \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \left| -\frac{1}{\pi}\arcsin s \right| \mathrm{d}s + \int_{\frac{\sqrt{2}}{2}}^{1} \left| \frac{1}{2} - \frac{1}{\pi}\arcsin s \right| \mathrm{d}s \\ &= \frac{2\sqrt{2}-2}{\pi} \end{aligned}$$

and the first inequality is obtained. To prove the second inequality we take $p = 2$

$$\begin{aligned} \int_{-1}^{1} |R_1(s)|^2\,\mathrm{d}s &= \int_{-1}^{-\frac{\sqrt{2}}{2}} \left| -\frac{1}{2} - \frac{1}{\pi}\arcsin s \right|^2 \mathrm{d}s \\ &+ \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \left| -\frac{1}{\pi}\arcsin s \right|^2 \mathrm{d}s + \int_{\frac{\sqrt{2}}{2}}^{1} \left| \frac{1}{2} - \frac{1}{\pi}\arcsin s \right|^2 \mathrm{d}s \\ &= \frac{\pi\sqrt{2}-4}{\pi^2}. \end{aligned}$$

If $p = 1$, we have

$$\begin{aligned} \sup_{s\in[-1,1]} |R_1(s)| &= \max\left\{ \sup_{s\in\left[-1,-\frac{\sqrt{2}}{2}\right]} \left| -\frac{1}{2} - \frac{1}{\pi}\arcsin s \right|, \right. \\ &\left. \sup_{s\in\left[-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right]} \left| -\frac{1}{\pi}\arcsin s \right|, \sup_{s\in\left[\frac{\sqrt{2}}{2},1\right]} \left| \frac{1}{2} - \frac{1}{\pi}\arcsin s \right| \right\}. \end{aligned}$$

By elementary calculation we get

$$\sup_{s\in\left[-1,-\frac{\sqrt{2}}{2}\right]} \left| -\frac{1}{2} - \frac{1}{\pi}\arcsin s \right| = \sup_{s\in\left[-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right]} \left| -\frac{1}{\pi}\arcsin s \right|$$

$$= \sup_{s\in\left[\frac{\sqrt{2}}{2},1\right]} \left| \frac{1}{2} - \frac{1}{\pi}\arcsin s \right| = \frac{1}{4},$$

and the third inequality is proved. □

Remark 3.3 The first and the third inequality from Corollary 3.2 were proved by S. Kovač and J. Pečarić in [71].

Remark 3.4 If we apply Theorem 3.1 with $n = 2$, $a = -1$, $b = 1$, $x = -\frac{\sqrt{2}}{2}$ and $w(t) = \frac{1}{\pi\sqrt{1-t^2}}$, $t \in [-1,1]$, we get

$$\begin{aligned}\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} f(t)\,\mathrm{d}t &= \frac{\pi}{2}\left[f\left(-\frac{\sqrt{2}}{2}\right) + f\left(\frac{\sqrt{2}}{2}\right)\right] \\ &+ \frac{\pi\sqrt{2}}{4}\left[f'\left(-\frac{\sqrt{2}}{2}\right) - f'\left(\frac{\sqrt{2}}{2}\right)\right] + \pi\int_{-1}^{1} R_2(s) f''(s)\,\mathrm{d}s,\end{aligned}$$

where

$$R_2(s) = \begin{cases} \frac{1}{2}s + \frac{1}{\pi}\left(s\arcsin s + \sqrt{1-s^2}\right), & -1 \le s \le -\frac{\sqrt{2}}{2}, \\ \frac{1}{\pi}\left(s\arcsin s + \sqrt{1-s^2}\right), & -\frac{\sqrt{2}}{2} < s \le \frac{\sqrt{2}}{2}, \\ -\frac{1}{2}s + \frac{1}{\pi}\left(s\arcsin s + \sqrt{1-s^2}\right), & \frac{\sqrt{2}}{2} < s \le 1. \end{cases}$$

Corollary 3.3 *Let I be an open interval in $\mathbb{R}$, $[-1,1] \subset I$, (p,q) a pair of conjugate exponents, $1 \le p,q \le \infty$ and let $f : I \to \mathbb{R}$ be such that f' is an absolutely continuous function. Then we have*

$$\begin{aligned}&\left| \int_{-1}^{1} \frac{f(t)\,\mathrm{d}t}{\sqrt{1-t^2}} \right. \\ &\left. - \frac{\pi}{2}\left[f\left(-\frac{\sqrt{2}}{2}\right) + f\left(\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2} f'\left(-\frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}}{2} f'\left(\frac{\sqrt{2}}{2}\right)\right]\right| \\ &\le \begin{cases} \frac{1}{2}\pi \left\| f'' \right\|_{\infty}, & f'' \in L^{\infty}[-1,1] \\ \left(\frac{32+3\sqrt{2}\pi}{27}\right)^{1/2} \left\| f'' \right\|_{2}, & f'' \in L^{2}[-1,1] \\ \left(\frac{\sqrt{2}\pi}{8} + \frac{\sqrt{2}}{2}\right) \left\| f'' \right\|_{1}, & f'' \in L^{1}[-1,1]. \end{cases}\end{aligned}$$

The first and the second inequality are sharp and the third inequality is the best possible.

Proof. Similar to the proof of Corollary 3.2. □

Remark 3.5 If we suppose that f'' is a continuous function on $[-1,1]$, by Theorem 3.3 there exists $\eta \in (-1,1)$ such that

$$\int_{-1}^{1} \frac{f(t)\,\mathrm{d}t}{\sqrt{1-t^2}} - \frac{\pi}{2}\left[f\left(-\frac{\sqrt{2}}{2}\right) + f\left(\frac{\sqrt{2}}{2}\right)\right]$$

$$-\frac{\pi\sqrt{2}}{4}\left[f'\left(-\frac{\sqrt{2}}{2}\right)-f'\left(\frac{\sqrt{2}}{2}\right)\right]=\frac{\pi}{2}f''(\eta).$$

Remark 3.6 If we apply Theorem 3.1 with $n=3$, $a=-1$, $b=1$, $x=-\frac{\sqrt{2}}{2}$ and $w(t)=\frac{1}{\pi\sqrt{1-t^2}}$, $t\in[-1,1]$, we get

$$\begin{aligned}&\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}f(t)\,\mathrm{d}t\\&=\frac{\pi}{2}\left[f\left(-\frac{\sqrt{2}}{2}\right)+f\left(\frac{\sqrt{2}}{2}\right)\right]+\frac{\pi\sqrt{2}}{4}\left[f'\left(-\frac{\sqrt{2}}{2}\right)-f'\left(\frac{\sqrt{2}}{2}\right)\right]\\&+\frac{\pi}{4}\left[f''\left(-\frac{\sqrt{2}}{2}\right)+f''\left(\frac{\sqrt{2}}{2}\right)\right]+\frac{\pi}{2}\int_{-1}^{1}R_3(s)f'''(s)\,\mathrm{d}s,\end{aligned}$$

where

$$R_3(s)=\begin{cases}-\frac{3}{2\pi}s\sqrt{1-s^2}\\-\frac{1}{\pi}\left(\frac{1}{2}+s^2\right)\arcsin s-\frac{1}{2}\left(\frac{1}{2}+s^2\right), & -1\le s\le-\frac{\sqrt{2}}{2},\\-\frac{3}{2\pi}s\sqrt{1-s^2}-\frac{1}{\pi}\left(\frac{1}{2}+s^2\right)\arcsin s, & -\frac{\sqrt{2}}{2}<s\le\frac{\sqrt{2}}{2},\\-\frac{3}{2\pi}s\sqrt{1-s^2}\\-\frac{1}{\pi}\left(\frac{1}{2}+s^2\right)\arcsin s+\frac{1}{2}\left(\frac{1}{2}+s^2\right), & \frac{\sqrt{2}}{2}<s\le 1.\end{cases}$$

Corollary 3.4 *Let I be an open interval in $\mathbb{R}$, $[-1,1]\subset I$, (p,q) a pair of conjugate exponents, $1\le p,q\le\infty$ and let $f:I\to\mathbb{R}$ be such that f'' is an absolutely continuous function. Then we have*

$$\begin{aligned}&\left|\int_{-1}^{1}\frac{f(t)\,\mathrm{d}t}{\sqrt{1-t^2}}-\frac{\pi}{2}\left[f\left(-\frac{\sqrt{2}}{2}\right)+f\left(\frac{\sqrt{2}}{2}\right)\right.\right.\\&\left.\left.+\frac{\sqrt{2}}{2}f'\left(-\frac{\sqrt{2}}{2}\right)-\frac{\sqrt{2}}{2}f'\left(\frac{\sqrt{2}}{2}\right)+\frac{1}{2}f''\left(-\frac{\sqrt{2}}{2}\right)+\frac{1}{2}f''\left(\frac{\sqrt{2}}{2}\right)\right]\right|\\&\le\begin{cases}\frac{1}{36}\left(-8+19\sqrt{2}\right)\|f'''\|_\infty, & f'''\in L^\infty[-1,1]\\\frac{1}{2}\left(\frac{-4096+2505\sqrt{2}\pi}{6750}\right)^{1/2}\|f'''\|_2, & f'''\in L^2[-1,1]\\\frac{1}{8}(3+\pi)\|f'''\|_1, & f'''\in L^1[-1,1].\end{cases}\end{aligned}$$

The first and the second inequality are sharp and the third inequality is the best possible.

Proof. Similar to the proof of Corollary 3.2. □

Gauss–Chebyshev two-point formulae of the second kind

If we put $\rho(t)=\sqrt{1-t^2}, t\in[-1,1]$ in (3.10), we have Gauss–Chebyshev formulae

$$\int_{-1}^{1}\sqrt{1-t^2}f(t)\,\mathrm{d}t=\sum_{i=1}^{k}A_if(x_i)+E_k(f),\tag{3.13}$$

where A_i are given by

$$A_i = \frac{\pi}{k+1}\sin^2\frac{i\pi}{k+1}, \quad i = 1,\dots,k,$$

and x_i are zeros of the Chebyshev polynomials of the second kind defined as

$$U_k(x) = \frac{\sin[(k+1)\arccos(x)]}{\sin[\arccos(x)]}.$$

$U_k(x)$ has exactly k distinct zeros, all of which lie in the interval $[-1,1]$ (see for instance [103])

$$x_i = \cos\left(\frac{i\pi}{k+1}\right) \quad i = 1,\dots,k,.$$

Error of the approximation formula (3.13) is given by

$$E_k(f) = \frac{\pi}{2^{2k+1}(2k)!}f^{(2k)}(\xi), \quad \xi \in (-1,1).$$

In case $k = 2$, (3.13) reduces to

$$\int_{-1}^{1}\sqrt{1-t^2}f(t)\,\mathrm{d}t = \frac{\pi}{4}\left[f\left(-\frac{1}{2}\right)+f\left(\frac{1}{2}\right)\right]+\frac{\pi}{768}f^{(4)}(\xi), \quad \xi \in (-1,1).$$

Now, using Theorem 3.1 and Theorem 3.2 we establish Gauss-Chebyshev two-point formulae of the second kind and give some sharp and best possible inequalities.

Remark 3.7 If we apply (3.5) with $a = -1, b = 1, x = -\frac{1}{2}$ and $w(t) = \frac{2\sqrt{1-t^2}}{\pi}, t \in [-1,1]$, we get

$$\int_{-1}^{1}\sqrt{1-t^2}f(t)\,\mathrm{d}t = \frac{\pi}{4}\left[f\left(-\frac{1}{2}\right)+f\left(\frac{1}{2}\right)\right]+\frac{\pi}{2}\int_{-1}^{1}Q_1(s)f'(s)\,\mathrm{d}s,$$

where

$$Q_1(s) = \begin{cases} -\frac{1}{2}-\frac{1}{\pi}\left(s\sqrt{1-s^2}+\arcsin s\right), & -1 \le s \le -\frac{1}{2}, \\ -\frac{1}{\pi}\left(s\sqrt{1-s^2}+\arcsin s\right), & -\frac{1}{2} < s \le \frac{1}{2}, \\ \frac{1}{2}-\frac{1}{\pi}\left(s\sqrt{1-s^2}+\arcsin s\right), & \frac{1}{2} < s \le 1. \end{cases}$$

Corollary 3.5 *Let I be an open interval in $\mathbb{R}$, $[-1,1] \subset I$, (p,q) is a pair of conjugate exponents, $1 \le p,q \le \infty$. Let $f : I \to \mathbb{R}$ be an absolutely continuous function and $f' \in L^p[-1,1]$. Then we have*

$$\left|\int_{-1}^{1}\sqrt{1-t^2}f(t)\,\mathrm{d}t - \frac{\pi}{4}\left[f\left(-\frac{1}{2}\right)+f\left(\frac{1}{2}\right)\right]\right| \le \frac{\pi}{2}\left\|Q_1\right\|_q\left\|f'\right\|_p. \tag{3.14}$$

Inequality (3.14) is sharp for $1 < p \le \infty$ and best possible for $p = 1$.

Proof. This is a special case of Theorem 3.2 for $a=-1, b=1, x=-\frac{1}{2}$ and $w(t)=\frac{2\sqrt{1-t^2}}{\pi}$, $t\in[-1,1]$. □

Corollary 3.6 *Let I be an open interval in $\mathbb{R}$, $[-1,1]\subset I$, (p,q) a pair of conjugate exponents, $1\leq p,q\leq\infty$ and let $f:I\to\mathbb{R}$ be an absolutely continuous function. Then we have*

$$\left|\int_{-1}^{1}\sqrt{1-t^2}f(t)\,\mathrm{d}t-\frac{\pi}{4}\left[f\left(-\frac{1}{2}\right)+f\left(\frac{1}{2}\right)\right]\right|$$
$$\leq\begin{cases}\frac{1}{12}\left(-8+9\sqrt{3}-\pi\right)\|f'\|_{\infty}, & f'\in L^{\infty}[-1,1]\\ \frac{1}{6}\left(\frac{-512+135\sqrt{3}\pi-15\pi^2}{20}\right)^{1/2}\|f'\|_2, & f'\in L^2[-1,1]\\ \frac{1}{24}\left(3\sqrt{3}+2\pi\right)\|f'\|_1, & f'\in L^1[-1,1].\end{cases}$$

The first and the second inequality are sharp and the third inequality is the best possible.

Proof. To prove the first inequality we apply (3.14) with $p=\infty$

$$\begin{aligned}\int_{-1}^{1}|Q_1(s)|\,\mathrm{d}s &= \int_{-1}^{-\frac{1}{2}}\left|-\frac{1}{2}-\frac{1}{\pi}\left(s\sqrt{1-s^2}+\arcsin s\right)\right|\mathrm{d}s\\ &+\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|-\frac{1}{\pi}\left(s\sqrt{1-s^2}+\arcsin s\right)\right|\mathrm{d}s\\ &+\int_{\frac{1}{2}}^{1}\left|\frac{1}{2}-\frac{1}{\pi}\left(s\sqrt{1-s^2}+\arcsin s\right)\right|\mathrm{d}s\\ &=\frac{-16+18\sqrt{3}-2\pi}{12\pi}.\end{aligned}$$

For the second inequality we take $p=2$

$$\begin{aligned}\int_{-1}^{1}|Q_1(s)|^2\,\mathrm{d}s &= \int_{-1}^{-\frac{1}{2}}\left|-\frac{1}{2}-\frac{1}{\pi}\left(s\sqrt{1-s^2}+\arcsin s\right)\right|^2\mathrm{d}s\\ &+\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|-\frac{1}{\pi}\left(s\sqrt{1-s^2}+\arcsin s\right)\right|^2\mathrm{d}s\\ &+\int_{\frac{1}{2}}^{1}\left|\frac{1}{2}-\frac{1}{\pi}\left(s\sqrt{1-s^2}+\arcsin s\right)\right|^2\mathrm{d}s\\ &=\frac{-512+135\sqrt{3}\pi-15\pi^2}{180\pi^2}.\end{aligned}$$

If $p=1$, we have

$$\sup_{s\in[-1,1]}|Q_1(s)| = \max\left\{\sup_{s\in\left[-1,-\frac{1}{2}\right]}\left|-\frac{1}{2}-\frac{1}{\pi}\left(s\sqrt{1-s^2}+\arcsin s\right)\right|,\right.$$

$$\sup_{s\in\left[-\frac{1}{2},\frac{1}{2}\right]}\left|-\frac{1}{\pi}\left(s\sqrt{1-s^2}+\arcsin s\right)\right|,$$

$$\left.\sup_{s\in\left[\frac{1}{2},1\right]}\left|\frac{1}{2}-\frac{1}{\pi}\left(s\sqrt{1-s^2}+\arcsin s\right)\right|\right\}.$$

Since

$$\sup_{s\in\left[-1,-\frac{1}{2}\right]}\left|-\frac{1}{2}-\frac{1}{\pi}\left(s\sqrt{1-s^2}+\arcsin s\right)\right|=\frac{1}{3}-\frac{\sqrt{3}}{4\pi},$$

$$\sup_{s\in\left[-\frac{1}{2},\frac{1}{2}\right]}\left|-\frac{1}{\pi}\left(s\sqrt{1-s^2}+\arcsin s\right)\right|=\frac{1}{6}+\frac{\sqrt{3}}{4\pi},$$

$$\sup_{s\in\left[\frac{1}{2},1\right]}\left|\frac{1}{2}-\frac{1}{\pi}\left(s\sqrt{1-s^2}+\arcsin s\right)\right|=\frac{1}{3}-\frac{\sqrt{3}}{4\pi},$$

the third inequality is proved. □

Remark 3.8 The first and the third inequality from Corollary 3.6 were proved by S. Kovač and J. Pečarić in [71].

Remark 3.9 If we apply Theorem 3.1 with $n=2$, $a=-1$, $b=1$, $x=-\frac{1}{2}$ and $w(t)=\frac{2\sqrt{1-t^2}}{\pi}$, $t\in[-1,1]$, we get

$$\begin{aligned}\int_{-1}^{1}\sqrt{1-t^2}f(t)\,\mathrm{d}t &= \frac{\pi}{4}\left[f\left(-\frac{1}{2}\right)+f\left(\frac{1}{2}\right)\right]\\ &+\frac{\pi}{8}\left[f'\left(-\frac{1}{2}\right)-f'\left(\frac{1}{2}\right)\right]+\frac{\pi}{2}\int_{-1}^{1}Q_2(s)f''(s)\,\mathrm{d}s,\end{aligned}$$

where

$$Q_2(s)=\begin{cases}\frac{1}{3\pi}\left(2+s^2\right)\sqrt{1-s^2}+\frac{1}{\pi}s\arcsin s+\frac{s}{2}, & -1\le s\le -\frac{1}{2},\\ \frac{1}{3\pi}\left(2+s^2\right)\sqrt{1-s^2}+\frac{1}{\pi}s\arcsin s, & -\frac{1}{2}<s\le\frac{1}{2},\\ \frac{1}{3\pi}\left(2+s^2\right)\sqrt{1-s^2}+\frac{1}{\pi}s\arcsin s-\frac{s}{2}, & \frac{1}{2}<s\le 1.\end{cases}$$

Corollary 3.7 *Let I be an open interval in $\mathbb{R}$, $[-1,1]\subset I$, (p,q) a pair of conjugate exponents, $1\le p,q\le\infty$ and let $f:I\to\mathbb{R}$ be such that f' is an absolutely continuous function. Then we have*

$$\left|\int_{-1}^{1}\sqrt{1-t^2}f(t)\,\mathrm{d}t-\frac{\pi}{4}\left[f\left(-\frac{1}{2}\right)+f\left(\frac{1}{2}\right)+\frac{1}{2}f'\left(-\frac{1}{2}\right)-\frac{1}{2}f'\left(\frac{1}{2}\right)\right]\right|$$

$$\le\begin{cases}\frac{1}{8}\pi\left\|f''\right\|_\infty, & f''\in L^\infty[-1,1]\\ \frac{\pi}{2}\left(-\frac{1}{144}+\frac{3\sqrt{3}}{80\pi}+\frac{2048}{4725\pi^2}\right)^{1/2}\left\|f''\right\|_2, & f''\in L^2[-1,1]\\ \left(\frac{\pi}{24}+\frac{3\sqrt{3}}{16}\right)\left\|f''\right\|_1, & f''\in L^1[-1,1].\end{cases}$$

The first and the second inequality are sharp and the third inequality is the best possible.

Proof. The proof is similar to the proof of Corollary 3.6. □

Remark 3.10 If we suppose that f'' is a continuous function on $[-1,1]$ by Theorem 3.3 there exists $\eta \in (-1,1)$ such that

$$\int_{-1}^{1}\sqrt{1-t^2}f(t)\,\mathrm{d}t-\frac{\pi}{4}\left[f\left(-\frac{1}{2}\right)+f\left(\frac{1}{2}\right)\right]-\frac{\pi}{8}\left[f'\left(-\frac{1}{2}\right)-f'\left(\frac{1}{2}\right)\right]$$
$$=\frac{\pi}{8}f''(\eta).$$

Remark 3.11 If we apply Theorem 3.1 with $n=3$, $a=-1$, $b=1$, $x=-\frac{1}{2}$ and $w(t)=\frac{2\sqrt{1-t^2}}{\pi}$, $t\in[-1,1]$, we get

$$\begin{aligned}\int_{-1}^{1}\sqrt{1-t^2}f(t)\,\mathrm{d}t &= \frac{\pi}{4}\left[f\left(-\frac{1}{2}\right)+f\left(\frac{1}{2}\right)\right]+\frac{\pi}{8}\left[f'\left(-\frac{1}{2}\right)-f'\left(\frac{1}{2}\right)\right]\\ &+\frac{\pi}{16}\left[f''\left(-\frac{1}{2}\right)+f''\left(\frac{1}{2}\right)\right]+\frac{\pi}{4}\int_{-1}^{1}Q_3(s)f'''(s)\,\mathrm{d}s,\end{aligned}$$

where

$$Q_3(s)=\begin{cases}\begin{array}{c}-\frac{1}{12\pi}\left(13s+2s^3\right)\sqrt{1-s^2}\\ -\frac{1}{4\pi}\left(1+4s^2\right)\arcsin s-\frac{1}{8}\left(1+4s^2\right),\end{array} & -1\le s\le-\frac{1}{2},\\ -\frac{1}{12\pi}\left(13s+2s^3\right)\sqrt{1-s^2}-\frac{1}{4\pi}\left(1+4s^2\right)\arcsin s, & -\frac{1}{2}<s\le\frac{1}{2},\\ \begin{array}{c}-\frac{1}{12\pi}\left(13s+2s^3\right)\sqrt{1-s^2}\\ -\frac{1}{4\pi}\left(1+4s^2\right)\arcsin s+\frac{1}{8}\left(1+4s^2\right),\end{array} & \frac{1}{2}<s\le 1.\end{cases}$$

Corollary 3.8 *Let I be an open interval in $\mathbb{R}$, $[-1,1]\subset I$, (p,q) a pair of conjugate exponents, $1\le p,q\le\infty$ and let $f:I\to\mathbb{R}$ be such that f'' is an absolutely continuous function. Then we have*

$$\begin{aligned}&\left|\int_{-1}^{1}\sqrt{1-t^2}f(t)\,\mathrm{d}t-\frac{\pi}{4}\left[f\left(-\frac{1}{2}\right)+f\left(\frac{1}{2}\right)\right.\right.\\ &\left.\left.+\frac{1}{2}f'\left(-\frac{1}{2}\right)-\frac{1}{2}f'\left(\frac{1}{2}\right)+\frac{1}{4}f''\left(-\frac{1}{2}\right)+\frac{1}{4}f''\left(\frac{1}{2}\right)\right]\right|\\ &\le\begin{cases}\frac{1}{2880}\left(-128+297\sqrt{3}-40\pi\right)\|f'''\|_\infty, & f'''\in L^\infty[-1,1]\\ \frac{\pi}{4}\left(-\frac{7}{720}+\frac{411\sqrt{3}}{5600\pi}-\frac{65536}{496125\pi^2}\right)^{1/2}\|f'''\|_2, & f'''\in L^2[-1,1]\\ \left(\frac{\pi}{48}+\frac{9\sqrt{3}}{128}\right)\|f'''\|_1, & f'''\in L^1[-1,1].\end{cases}\end{aligned}$$

The first and the second inequality are sharp and the third inequality is the best possible.

Proof. The proof is similar to the proof of Corollary 3.6. □

Non-weighted case of two-point formula and applications

Now, we use formula (3.4) to establish a non-weighted two-point formula and to prove some sharp and best possible inequalities for the functions whose higher order derivatives belong to L^p spaces, $1 \leq p \leq \infty$.

Here we define

$$t_n^2(x) = \frac{1}{2}\sum_{i=0}^{n-2}\left[f^{(i+1)}(x) + (-1)^{i+1} f^{(i+1)}(a+b-x)\right]\frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!\,(b-a)},$$

$$T_n^2(x,s) = \begin{cases} \frac{1}{(b-a)}(a-s)^n, & a \leq s \leq x, \\ \frac{1}{2(b-a)}\left[(a-s)^n + (b-s)^n\right], & x < s \leq a+b-x, \\ \frac{1}{(b-a)}(b-s)^n, & a+b-x < s \leq b. \end{cases}$$

We will use the Beta function and the incomplete Beta function of Euler type defined as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t, \quad B_r(x,y) = \int_0^r t^{x-1}(1-t)^{y-1}\,\mathrm{d}t, \quad x,y > 0.$$

The following theorem gives a non-weighted case of two-point formula.

Theorem 3.5 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$. Then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following identity holds*

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t = D_2(x) + t_n^2(x) + \frac{1}{n!}\int_a^b T_n^2(x,s) f^{(n)}(s)\,\mathrm{d}s. \tag{3.15}$$

Proof. This is a special case of Theorem 3.1 for $w(t) = \frac{1}{b-a}$, $t \in [a,b]$. □

Remark 3.12 Identity (3.15) holds true if $n = 1$.

Theorem 3.6 *Suppose that all the assumptions of Theorem 3.5 hold. Additionally, assume that (p,q) is a pair of conjugate exponents, that is $1 \leq p,q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $f^{(n)} \in L^p[a,b]$ for some $n \geq 1$. Then for each $x \in \left[a, \frac{a+b}{2}\right]$ we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D_2(x) - t_n^2(x)\right| \leq \frac{1}{n!}\left\|T_n^2(x,\cdot)\right\|_q \left\|f^{(n)}\right\|_p. \tag{3.16}$$

Inequality (3.16) is sharp for $1 < p \leq \infty$ and best possible for $p = 1$.

Proof. This is a special case of Theorem 3.2 for $w(t) = \frac{1}{b-a}$, $t \in [a,b]$. □

In the sequel we calculate the optimal constants in cases $p = 1$, $p = 2$ and $p = \infty$.

Corollary 3.9 *Suppose that all the assumptions of Theorem 3.5 hold. Additionally, assume that (p,q) is a pair of conjugate exponents, that is $1 \leq p,q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $x \in \left[a, \frac{a+b}{2}\right]$.*

(a) If $f^{(n)} \in L^{\infty}[a,b]$, then

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D_2(x) - t_n^2(x)\right|$$
$$\leq \frac{1}{(n+1)!}\left(\frac{(x-a)^{n+1}\left[2+(-1)^{n+1}\right]+(b-x)^{n+1}}{(b-a)}\right.$$
$$\left.-\left(\frac{b-a}{2}\right)^n\left[\frac{(-1)^{n+1}+1}{2}\right]\right)\left\|f^{(n)}\right\|_\infty.$$

(b) If $f^{(n)} \in L^2[a,b]$, then

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D_2(x) - t_n^2(x)\right|$$
$$\leq \frac{1}{n!}\left(\frac{3(x-a)^{2n+1}+(b-x)^{2n+1}}{2(2n+1)(b-a)^2}+\frac{(-1)^n(b-a)^{2n-1}}{2}\right.$$
$$\left.\cdot\left[B_{\frac{b-x}{b-a}}(n+1,n+1)-B_{\frac{x-a}{b-a}}(n+1,n+1)\right]\right)^{\frac{1}{2}}\left\|f^{(n)}\right\|_2.$$

(c) If $f^{(n)} \in L^1[a,b]$, then

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D_2(x) - t_n^2(x)\right|$$
$$\leq \frac{1}{n!(b-a)}\max\left\{(x-a)^n, \frac{(a-x)^n+(b-x)^n}{2}\right\}\left\|f^{(n)}\right\|_1.$$

The first and the second inequality are sharp and the third inequality is the best possible.

Proof. Applying (3.16) with $p=\infty$, we have

$$\int_a^b \left|T_n^2(x,s)\right|\mathrm{d}s = \int_a^x \left|\frac{(a-s)^n}{b-a}\right|\mathrm{d}s$$
$$+\int_x^{a+b-x}\left|\frac{(a-s)^n+(b-s)^n}{2(b-a)}\right|\mathrm{d}s + \int_{a+b-x}^b\left|\frac{(b-s)^n}{b-a}\right|\mathrm{d}s$$
$$=2\frac{(x-a)^{n+1}}{(n+1)(b-a)}+\frac{(a-x)^{n+1}+(b-x)^{n+1}-\left(\frac{b-a}{2}\right)^{n+1}\left[(-1)^{n+1}+1\right]}{(n+1)(b-a)}$$
$$=\frac{(x-a)^{n+1}\left[2+(-1)^{n+1}\right]+(b-x)^{n+1}}{(n+1)(b-a)}-\left(\frac{b-a}{2}\right)^n\left[\frac{(-1)^{n+1}+1}{2(n+1)}\right]$$

and the first inequality is obtained. To prove the second inequality we take $p=2$

$$\int_a^b \left|T_n^2(x,s)\right|^2\mathrm{d}s = \int_a^x\left|\frac{(a-s)^n}{b-a}\right|^2\mathrm{d}s$$

$$
\begin{aligned}
&+\int_{x}^{a+b-x}\left|\frac{(a-s)^{n}+(b-s)^{n}}{2(b-a)}\right|^{2}\mathrm{d}s+\int_{a+b-x}^{b}\left|\frac{(b-s)^{n}}{b-a}\right|^{2}\mathrm{d}s\\
&=\frac{3(x-a)^{2n+1}+(b-x)^{2n+1}}{2(2n+1)(b-a)^{2}}\\
&+\frac{(-1)^{n}(b-a)^{2n-1}}{2}\left[B_{\frac{b-x}{b-a}}(n+1,n+1)-B_{\frac{x-a}{b-a}}(n+1,n+1)\right].
\end{aligned}
$$

Finally, for $p=1$, we have

$$
\begin{aligned}
\sup_{s\in[a,b]}\left|T_{n}^{2}(x,s)\right| = \max\Bigg\{&\sup_{s\in[a,x]}\left|\frac{(a-s)^{n}}{b-a}\right|,\\
&\sup_{s\in[x,a+b-x]}\left|\frac{(a-s)^{n}+(b-s)^{n}}{2(b-a)}\right|,\ \sup_{s\in[a+b-x,b]}\left|\frac{(b-s)^{n}}{b-a}\right|\Bigg\}.
\end{aligned}
$$

Now, by elementary calculation we get

$$
\sup_{s\in[a,x]}\left|\frac{(a-s)^{n}}{b-a}\right|=\frac{(x-a)^{n}}{(b-a)},\quad \sup_{s\in[a+b-x,b]}\left|\frac{(b-s)^{n}}{b-a}\right|=\frac{(x-a)^{n}}{(b-a)}.
$$

The function $y:[a,b]\to\mathbb{R}$, $y(x)=(a-x)^{n}+(b-x)^{n}$, is decreasing on $\left(a,\frac{a+b}{2}\right)$ and increasing on $\left(\frac{a+b}{2},b\right)$ if n is even, and decreasing on (a,b) if n is odd. Thus

$$
\sup_{s\in[x,a+b-x]}\left|\frac{(a-s)^{n}+(b-s)^{n}}{2(b-a)}\right|=\frac{(a-x)^{n}+(b-x)^{n}}{2(b-a)}. \tag{3.17}
$$

Since $x\in\left[a,\frac{a+b}{2}\right]$, we have

$$
\sup_{s\in[a,b]}\left|T_{n}^{2}(x,s)\right|=\max\left\{\frac{(x-a)^{n}}{(b-a)},\frac{(a-x)^{n}+(b-x)^{n}}{2(b-a)}\right\},
$$

and the third inequality is proved. □

Corollary 3.10 *Let $f:[a,b]\to\mathbb{R}$ be an L-Lipschitzian function on $[a,b]$, $L\in\mathbb{R}^{+}$. Then for each $x\in\left[a,\frac{a+b}{2}\right]$*

$$
\left|\frac{1}{b-a}\int_{a}^{b}f(t)\,\mathrm{d}t-D_{2}(x)\right|\le\left(\frac{3(x-a)^{2}+(b-x)^{2}}{2(b-a)}-\frac{b-a}{4}\right)L. \tag{3.18}
$$

Proof. We apply the first inequality from Corollary 3.9 with $n=1$. □

Remark 3.13 Inequality (3.18) was proved and generalized for α-L-Lipschitzian functions by A. Guessab and G. Schmeisser in [65]. They also proved that this inequality is sharp for each admissible x.

Corollary 3.11 *Let $f : [a,b] \to \mathbb{R}$ be such that f' is an L-Lipschitzian function on $[a,b]$, $L \in \mathbb{R}^+$. Then for each $x \in \left[a, \frac{a+b}{2}\right]$*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t - D_2(x) - \left[f'(x) - f'(a+b-x)\right] \frac{(b-x)^2-(a-x)^2}{4(b-a)} \right|$$
$$\leq \frac{(x-a)^3+(b-x)^3}{6(b-a)} L.$$

Proof. We apply the first inequality from Corollary 3.9 with $n = 2$. □

Corollary 3.12 *Let $f : [a,b] \to \mathbb{R}$ be a continuous function of bounded variation on $[a,b]$. Then for each $x \in \left[a, \frac{a+b}{2}\right]$*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t - D_2(x) \right| \leq \left(\frac{1}{4} + \frac{|3a+b-4x|}{4(b-a)} \right) V_a^b(f). \tag{3.19}$$

More precisely, if $x \in \left[a, \frac{3a+b}{4}\right]$

$$\left| \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t - D_2(x) \right| \leq \frac{a+b-2x}{2(b-a)} V_a^b(f),$$

and if $x \in \left[\frac{3a+b}{4}, \frac{a+b}{2}\right]$

$$\left| \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t - D_2(x) \right| \leq \frac{x-a}{b-a} V_a^b(f).$$

Proof. We apply the third inequality from Corollary 3.9 with $n = 1$ to get

$$\left| \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t - D_2(x) \right| \leq \frac{1}{(b-a)} \max\left\{ x-a, \frac{a+b}{2} - x \right\} V_a^b(f).$$

The proof for the first inequality follows by the formula
$\max\{A,B\} = \frac{1}{2}(A+B+|A-B|)$. Since

$$\max\left\{ x-a, \frac{a+b}{2} - x \right\} = \begin{cases} \frac{a+b}{2} - x, & \text{za } x \in \left[a, \frac{3a+b}{4}\right], \\ x-a, & \text{za } x \in \left[\frac{3a+b}{4}, \frac{a+b}{2}\right], \end{cases}$$

the proof of the second and the third inequality follows. □

Remark 3.14 Inequalities (3.18) and (3.19) and their generalizations based on extended Euler formulae via Bernoulli polynomials have been proved by I.Franjić, J.Pečarić, I. Perić and A. Vukelić on interval $[0,1]$ in book [61] (see 20.,23).

Corollary 3.13 *Let $f : [a,b] \to \mathbb{R}$ be such that f' is a continuous function of bounded variation on $[a,b]$. Then for each $x \in \left[a, \frac{a+b}{2}\right]$*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t - D_2(x) - \left[f'(x) - f'(a+b-x)\right] \frac{(b-x)^2-(a-x)^2}{4(b-a)} \right|$$
$$\leq \frac{(x-a)^2+(b-x)^2}{4(b-a)} V_a^b(f').$$

Proof. We apply the third inequality from Corollary 3.9 with $n=2$ to get

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D_2(x) - \left[f'(x)-f'(a+b-x)\right]\frac{(b-x)^2-(a-x)^2}{4(b-a)}\right| \\
&\leq \frac{1}{2(b-a)}\max\left\{(x-a)^2, \frac{(a-x)^2+(b-x)^2}{2}\right\} V_a^b(f') \\
&= \frac{(x-a)^2+(b-x)^2}{4(b-a)} V_a^b(f')
\end{aligned}$$

and the proof follows. □

Corollary 3.14 *Suppose that all the assumptions of Theorem 3.6 hold. Then for each $x \in \left[a, \frac{a+b}{2}\right]$ we have*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D_2(x) - t_n^2(x)\right| \leq \frac{1}{n!}\left(\frac{2(x-a)^{nq+1}}{(nq+1)(b-a)^q}\right. \\
&\left. + \frac{((a-x)^n+(b-x)^n)^q}{2^q(b-a)^q}(a+b-2x)\right)^{\frac{1}{q}} \left\|f^{(n)}\right\|_p . \qquad (3.20)
\end{aligned}$$

Proof. We have

$$\begin{aligned}
\int_a^b \left|T_n^2(x,s)\right|^q \mathrm{d}s &= \int_a^x \left|\frac{(a-s)^n}{b-a}\right|^q \mathrm{d}s \\
&+ \int_x^{a+b-x}\left|\frac{(a-s)^n+(b-s)^n}{2(b-a)}\right|^q \mathrm{d}s + \int_{a+b-x}^b \left|\frac{(b-s)^n}{b-a}\right|^q \mathrm{d}s.
\end{aligned}$$

Since

$$\int_a^x \left|\frac{(a-s)^n}{b-a}\right|^q \mathrm{d}s = \int_{a+b-x}^b \left|\frac{(b-s)^n}{b-a}\right|^q \mathrm{d}s = \frac{(x-a)^{nq+1}}{(nq+1)(b-a)^q},$$

and by applying (3.17)

$$\begin{aligned}
\int_x^{a+b-x}\left|\frac{(a-s)^n+(b-s)^n}{2(b-a)}\right|^q \mathrm{d}s &\leq \int_x^{a+b-x}\left(\frac{(a-x)^n+(b-x)^n}{2(b-a)}\right)^q \mathrm{d}s \\
&= \frac{((a-x)^n+(b-x)^n)^q}{2^q(b-a)^q}(a+b-2x)
\end{aligned}$$

we obtain

$$\int_a^b \left|T_n^2(x,s)\right|^q \mathrm{d}s \leq \frac{2(x-a)^{nq+1}}{(nq+1)(b-a)^q} + \frac{((a-x)^n+(b-x)^n)^q}{2^q(b-a)^q}(a+b-2x).$$

□

Remark 3.15 If in Theorem 3.6 and Corollaries 3.9, 3.10, 3.11, 3.12, 3.13, 3.14 we choose $x = a, \frac{2a+b}{3}, \frac{3a+b}{4}, \frac{a+b}{2}$ we obtain a generalized trapezoid, two-point Newton-Cotes, two-point Maclaurin and midpoint inequality.

Remark 3.16 For some related results see [46], [82], [96].

3.1.3 General two-point quadrature formulae with applications for α-L-Hölder type functions

The aim of this subsection is to establish general two-point quadrature formulae (3.1) using the identities (2.81) and (2.82) and to prove several Ostrowski-type inequalities for α-L-Hölder functions. We also show how these results can be applied to obtain some error estimates for Gauss-Chebyshev two-point quadrature rules.

Variant I of general two-point formula

We use (2.81) to study for each number $x \in \left[a, \frac{a+b}{2}\right]$ a general two-point quadrature formula of the type (3.1). In addition, this quadrature formula will be used to obtain Ostrowski-type inequalities for α-L-Hölder functions.

Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ exists on $[a,b]$ for some $n \geq 2$. We introduce the following notation for each $x \in \left[a, \frac{a+b}{2}\right]$

$$\begin{aligned}
s_{w,n}^{2}(x) = \frac{1}{2} \Bigg\{ & \sum_{i=0}^{n-1} \frac{f^{(i+1)}(b)}{i!} \left[\int_x^b (1 - W(t))(t-b)^i \, \mathrm{d}t \right. \\
& \left. + \int_{a+b-x}^{b} (1 - W(t))(t-b)^i \, \mathrm{d}t \right] \\
& - \sum_{i=0}^{n-1} \frac{f^{(i+1)}(a)}{i!} \left[\int_a^x W(t)(t-a)^i \, \mathrm{d}t + \int_a^{a+b-x} W(t)(t-a)^i \, \mathrm{d}t \right] \Bigg\}
\end{aligned} \tag{3.21}$$

and

$$S_{w,n}^{2}(x) = \frac{1}{2} \left[S_{w,n}^{a}(x) + S_{w,n}^{b}(x) + S_{w,n}^{a}(a+b-x) + S_{w,n}^{b}(a+b-x) \right],$$

where

$$\begin{aligned}
S_{w,n}^{a}(x) &= \frac{1}{(n-2)!} \int_a^x W(t) \left[\int_a^t \left(f^{(n)}(a) - f^{(n)}(s) \right) (t-s)^{n-2} \, \mathrm{d}s \right] \mathrm{d}t, \\
S_{w,n}^{b}(x) & \\
&= \frac{1}{(n-2)!} \int_x^b (1 - W(t)) \left[\int_t^b \left(f^{(n)}(b) - f^{(n)}(s) \right) (t-s)^{n-2} \, \mathrm{d}s \right] \mathrm{d}t.
\end{aligned} \tag{3.22}$$

In the next theorem we establish the first variant of a generalized two-point quadrature formula based on the generalized Montgomery identity (2.81).

Theorem 3.7 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$ and let $w : [a,b] \to [0,\infty)$ be some probability density function. Let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$. Then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following identity holds*

$$\int_a^b w(t) f(t) \, \mathrm{d}t = D_2(x) + s_{w,n}^{2}(x) + S_{w,n}^{2}(x). \tag{3.23}$$

Proof. We put $x \equiv x$ and then $x \equiv a+b-x$ in (2.81) to obtain two new formulae. After adding these two formulae and multiplying by $1/2$, we get (3.23). $\square$

Remark 3.17 If in Theorem 3.7 we choose $x = a, \frac{2a+b}{3}, \frac{3a+b}{4}, \frac{a+b}{2}$, we obtain a trapezoid, two-point Newton-Cotes, two-point MacLaurin and midpoint rule, respectively.

Before giving an estimation of the term

$$\left| \int_a^b w(t) f(t) dt - D_2(x) - s_{w,n}^2(x) \right|,$$

let us recall that a function $\varphi : [a,b] \to \mathbb{R}$ is said to be of α-L-Hölder type if $|\varphi(x) - \varphi(y)| \leq L|x-y|^\alpha$ for every $x, y \in [a,b]$, where $L > 0$ and $\alpha \in (0,1]$.

For each $x \in \left[a, \frac{a+b}{2}\right]$ we define

$$W(x,t) = \begin{cases} W(t), & a \leq t \leq x, \\ 1 - W(t), & x < t \leq b, \end{cases}$$

$$U_n(x,t) = \begin{cases} (t-a)^{\alpha+n-1}, & a \leq t \leq x, \\ (b-t)^{\alpha+n-1}, & x < t \leq b. \end{cases} \tag{3.24}$$

Theorem 3.8 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$ and $w : [a,b] \to [0,\infty)$ some probability density function. Let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous and that $f^{(n)} : [a,b] \to \mathbb{R}$ is an α-L-Hölder type function for some $n \geq 2$, $\alpha \in (0,1]$ and $L \in \mathbb{R}^+$. Then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following inequalities hold*

$$\begin{aligned}
&\left| \int_a^b w(t) f(t) \mathrm{d}t - D_2(x) - s_{w,n}^2(x) \right| \\
&\leq \frac{B(\alpha+1, n-1)}{2(n-2)!} L \left[\int_a^b W(x,t) U_n(x,t) \mathrm{d}t \right. \\
&\left. + \int_a^b W(a+b-x,t) U_n(a+b-x,t) \mathrm{d}t \right] \\
&\leq \frac{B(\alpha+1, n-1)}{(\alpha+n)(n-2)!} L \left[(x-a)^{\alpha+n} + (b-x)^{\alpha+n} \right].
\end{aligned}$$

Proof. From (3.23) we have that

$$\begin{aligned}
&\left| \int_a^b w(t) f(t) \mathrm{d}t - D_2(x) - s_{w,n}^2(x) \right| \\
&= \frac{1}{2} \left| S_{w,n}^a(x) + S_{w,n}^a(a+b-x) + S_{w,n}^b(x) + S_{w,n}^b(a+b-x) \right| \\
&\leq \frac{1}{2} \left(\left| S_{w,n}^a(x) \right| + \left| S_{w,n}^a(a+b-x) \right| + \left| S_{w,n}^b(x) \right| + \left| S_{w,n}^b(a+b-x) \right| \right).
\end{aligned} \tag{3.25}$$

Since $f^{(n)}$ is an α-L-Hölder type function, from (3.25) we obtain

$$\begin{aligned}
&\left|\int_a^b w(t) f(t)\,\mathrm{d}t - D_2(x) - s_{w,n}^2(x)\right| \\
&\leq \frac{L}{2(n-2)!}\left\{\int_a^x W(t)\left[\int_a^t (s-a)^\alpha (t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t\right. \\
&+\int_a^{a+b-x} W(t)\left[\int_a^t (s-a)^\alpha (t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t \\
&+\int_x^b (1-W(t))\left[\int_t^b (b-s)^\alpha (s-t)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t \\
&\left.+\int_{a+b-x}^b (1-W(t))\left[\int_t^b (b-s)^\alpha (s-t)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t\right\}.
\end{aligned} \tag{3.26}$$

The first integral over ds in (3.26) can be written as

$$\begin{aligned}
&\int_a^t (s-a)^\alpha (t-s)^{n-2}\,\mathrm{d}s \\
&=(t-a)^{\alpha+n-2}\int_a^t \left(\frac{s-a}{t-a}\right)^\alpha \left(\frac{t-s}{t-a}\right)^{n-2}\mathrm{d}s \\
&=\left[u=\frac{s-a}{t-a}\right]=(t-a)^{\alpha+n-1}\int_0^1 u^\alpha (1-u)^{n-2}\,\mathrm{d}u \\
&=(t-a)^{\alpha+n-1} B(\alpha+1,n-1).
\end{aligned}$$

Similar can be done with other integrals in (3.26), so we obtain

$$\begin{aligned}
&\left|\int_a^b w(t) f(t)\,\mathrm{d}t - D_2(x) - s_{w,n}^2(x)\right| \leq \frac{B(\alpha+1,n-1)}{2(n-2)!}L \\
&\cdot\left[\int_a^x W(t)(t-a)^{\alpha+n-1}\,\mathrm{d}t + \int_a^{a+b-x} W(t)(t-a)^{\alpha+n-1}\,\mathrm{d}t\right. \\
&\left.+\int_x^b (1-W(t))(b-t)^{\alpha+n-1}\,\mathrm{d}t + \int_{a+b-x}^b (1-W(t))(b-t)^{\alpha+n-1}\,\mathrm{d}t\right] \\
&=\frac{B(\alpha+1,n-1)}{2(n-2)!}L\left[\int_a^b W(x,t)\,U_n(x,t)\,\mathrm{d}t\right. \\
&\left.+\int_a^b W(a+b-x,t)\,U_n(a+b-x,t)\,\mathrm{d}t\right].
\end{aligned} \tag{3.27}$$

Since we have $0 \leq W(t) \leq 1$, $t \in [a,b]$, from (3.27) we obtain

$$\begin{aligned}
&\frac{B(\alpha+1,n-1)}{2(n-2)!}L\left[\int_a^x W(t)(t-a)^{\alpha+n-1}\,\mathrm{d}t\right. \\
&+\int_a^{a+b-x} W(t)(t-a)^{\alpha+n-1}\,\mathrm{d}t
\end{aligned}$$

$$+\int_x^b (1-W(t))(b-t)^{\alpha+n-1}\,\mathrm{d}t+\int_{a+b-x}^b (1-W(t))(b-t)^{\alpha+n-1}\,\mathrm{d}t\Bigg]$$
$$\leq \frac{B(\alpha+1,n-1)}{(\alpha+n)(n-2)!}L\left[(x-a)^{\alpha+n}+(b-x)^{\alpha+n}\right],$$

which completes the proof. □

Corollary 3.15 *Let I be an open interval in $\mathbb{R}$, $[a,b]\subset I$ and let $f:I\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous and that $f^{(n)}:[a,b]\to\mathbb{R}$ is an α-L-Hölder type function for some $n\geq 2$, $\alpha\in(0,1]$ and $L\in\mathbb{R}^+$. Then for each $x\in\left[a,\frac{a+b}{2}\right]$ the following inequality holds*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t-D_2(x)-s_n^2(x)\right|$$
$$\leq \frac{B(\alpha+1,n-1)}{(b-a)(\alpha+n+1)(n-2)!}L\left[(x-a)^{\alpha+n+1}+(b-x)^{\alpha+n+1}\right],$$

where

$$s_n^2(x)=\frac{1}{2}\sum_{i=0}^{n-1}\left[(-1)^i f^{(i+1)}(b)-f^{(i+1)}(a)\right]\frac{(x-a)^{i+2}+(b-x)^{i+2}}{i!\,(i+2)(b-a)}.$$

Proof. This is a special case of Theorem 3.8 for $w(t)=\frac{1}{b-a}$, $t\in[a,b]$. □

Corollary 3.16 *Let I be an open interval in $\mathbb{R}$, $[a,b]\subset I$ and let $f:I\to\mathbb{R}$ be such that f' is absolutely continuous and that $f'':[a,b]\to\mathbb{R}$ is an α-L-Hölder type function, $\alpha\in(0,1]$ and $L\in\mathbb{R}^+$. Then for each $x\in\left[a,\frac{a+b}{2}\right]$ the following inequality holds*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t-D_2(x)-s_2^2(x)\right|$$
$$\leq \frac{1}{(b-a)(\alpha+1)(\alpha+3)}L\left[(x-a)^{\alpha+3}+(b-x)^{\alpha+3}\right],$$

where

$$\begin{aligned} s_2^2(x) &= \left(f'(b)-f'(a)\right)\frac{(x-a)^2+(b-x)^2}{4(b-a)} \\ &- \left(f''(b)+f''(a)\right)\frac{(x-a)^3+(b-x)^3}{6(b-a)}. \end{aligned}$$

Proof. This is a special case of Corollary 3.15 for $n=2$. □

Theorem 3.9 *Suppose that all the assumptions of Theorem 3.7 hold for some $n=2k-1$, $k\geq 2$. If f is $(2k)$-convex, then for each $x\in\left[a,\frac{a+b}{2}\right]$ the following inequality holds*

$$\int_a^b w(t)f(t)\,\mathrm{d}t-\frac{f(x)+f(a+b-x)}{2}-s_{w,n}^2(x)\leq 0. \tag{3.28}$$

If f is $(2k)$-concave, then inequality (3.28) is reversed.

Proof. First note that if f is $(2k)$-convex, the derivative $f^{(2k-1)} = f^{(n)}$ is nondecreasing, and if f is $(2k)$-concave, the derivative $f^{(n)}$ is nonincreasing (see [97]).

From (3.23) we have that

$$\begin{aligned}&\int_a^b w(t)\, f(t)\, \mathrm{d}t - D_2(x) - s_{w,n}^2(x)\\&= \frac{1}{2}\left[S_{w,n}^a(x) + S_{w,n}^a(a+b-x) + S_{w,n}^b(x) + S_{w,n}^b(a+b-x)\right].\end{aligned}$$

Let us consider the sign of the integral

$$S_{w,n}^a(x) = \frac{1}{(n-2)!}\int_a^x W(t)\left[\int_a^t \left(f^{(n)}(a) - f^{(n)}(s)\right)(t-s)^{n-2}\, \mathrm{d}s\right]\mathrm{d}t$$

when $f^{(n)}$ is nondecreasing. We have

$$\int_a^t \left(f^{(n)}(a) - f^{(n)}(s)\right)(t-s)^{n-2}\, \mathrm{d}s \leq 0,$$

hence we may conclude that $S_{w,n}^a(x) \leq 0$. Analogously, we obtain $S_{w,n}^a(a+b-x) \leq 0$. On the other hand, the sign of the integral

$$S_{w,n}^b(x) = \frac{1}{(n-2)!}\int_x^b (1 - W(t))\left[\int_t^b \left(f^{(n)}(b) - f^{(n)}(s)\right)(t-s)^{n-2}\, \mathrm{d}s\right]\mathrm{d}t$$

depends on the parity of n : if n is odd, that is $n = 2k-1$, then $S_{w,n}^b(x) \leq 0$ and analogously $S_{w,n}^b(a+b-x) \leq 0$. Hence, if $n = 2k-1$ and $f^{(n)}$ is nondecreasing, we have that

$$\int_a^b w(t)\, f(t)\, \mathrm{d}t - D_2(x) - s_{w,n}^2(x) \leq 0.$$

The reversed (3.28) can be obtained analogously. □

Corollary 3.17 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is an absolutely continuous for some $n = 2k-1$, $k \geq 2$. If f is $(2k)$-convex, then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following inequality holds*

$$\frac{1}{b-a}\int_a^b f(t)\, \mathrm{d}t - \frac{f(x) + f(a+b-x)}{2} - s_n^2(x) \leq 0. \tag{3.29}$$

If f is $(2k)$-concave, then inequality (3.29) is reversed.

Proof. This is a special case of Theorem 3.9 for $w(t) = \frac{1}{b-a}$, $t \in [a,b]$. □

Corollary 3.18 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and let $f : I \to \mathbb{R}$ be such that f'' is an absolutely continuous function. If f is 4-convex, then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following inequality holds*

$$\frac{1}{b-a}\int_a^b f(t)\, \mathrm{d}t - \frac{f(x) + f(a+b-x)}{2}$$

$$\begin{aligned}
&\leq \frac{1}{2(b-a)}\left[\left(f'(b)-f'(a)\right)\frac{(x-a)^2+(b-x)^2}{2}\right. \\
&-\left(f''(b)+f''(a)\right)\frac{(x-a)^3+(b-x)^3}{3} \\
&\left.+\left(f'''(b)-f'''(a)\right)\frac{(x-a)^4+(b-x)^4}{8}\right].
\end{aligned} \tag{3.30}$$

If f is 4-concave, then inequality (3.30) *is reversed.*

Proof. This is a special case of Corollary 3.17 for $n=3$. □

Variant II of general two-point formula

Now, we establish a general two-point quadrature formula based on the generalized Montgomery identity (2.82) and use this formula to prove some Ostrowski-type inequalities for α-L-Hölder functions.

Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ exists on $[a,b]$ for some $n\geq 2$. We introduce the following notation for each $x\in\left[a,\frac{a+b}{2}\right]$

$$\begin{aligned}
r^2_{w,n}(x) &= -\frac{1}{2}\sum_{i=0}^{n-1}\left[\frac{f^{(i+1)}(x)}{i!}\int_a^b P_w(x,t)(t-x)^i\,\mathrm{d}t\right. \\
&+\left.\frac{f^{(i+1)}(a+b-x)}{i!}\int_a^b P_w(a+b-x,t)(t-a-b+x)^i\,\mathrm{d}t\right]
\end{aligned} \tag{3.31}$$

and

$$R^2_{w,n}(x)=\frac{1}{2}\left[R_{w,n}(x)+R_{w,n}(a+b-x)\right],$$

where

$$\begin{aligned}
&R_{w,n}(x) \\
&=\frac{1}{(n-2)!}\int_a^b P_w(x,t)\left[\int_x^t\left(f^{(n)}(x)-f^{(n)}(s)\right)(t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t,
\end{aligned} \tag{3.32}$$

and $P_w(x,t)$ is the weighted Peano kernel defined by

$$P_w(x,t)=\begin{cases} W(t), & a\leq t\leq x,\\ W(t)-1, & x<t\leq b.\end{cases}$$

In the next theorem we establish the second variant of a generalized two-point quadrature formula based on the generalized Montgomery identity (2.82).

Theorem 3.10 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$ and let $w : [a,b] \to [0,\infty)$ be some probability density function. Let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$. Then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following identity holds*

$$\int_a^b w(t) f(t) \, \mathrm{d}t = D_2(x) + r_{w,n}^2(x) + R_{w,n}^2(x). \tag{3.33}$$

Proof. We put $x \equiv x$ and then $x \equiv a+b-x$ in (2.82) to obtain two new formulae. After adding these two formulae and multiplying by $1/2$, we get (3.33). □

Now, we obtain some error estimates for two-point quadrature formula (3.33).

Theorem 3.11 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$ and $w : [a,b] \to [0,\infty)$ some probability density function. Let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous and that $f^{(n)} : [a,b] \to \mathbb{R}$ is an α-L-Hölder type function for some $n \geq 2$, $\alpha \in (0,1]$ and $L \in \mathbb{R}^+$. Then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following inequalities hold*

$$\begin{aligned}
&\left| \int_a^b w(t) f(t) \, \mathrm{d}t - D_2(x) - r_{w,n}^2(x) \right| \\
&\leq \frac{B(\alpha+1, n-1)}{2(n-2)!} L \left[\int_a^b \left| P_w(x,t) (x-t)^{\alpha+n-1} \right| \mathrm{d}t \right. \\
&\left. + \int_a^b \left| P_w(a+b-x,t) (a+b-x-t)^{\alpha+n-1} \right| \mathrm{d}t \right] \\
&\leq \frac{B(\alpha+1, n-1)}{(n-2)! (\alpha+n)} L \left[(x-a)^{\alpha+n} + (b-x)^{\alpha+n} \right].
\end{aligned}$$

Proof. From (3.33) we have that

$$\left| \int_a^b w(t) f(t) \, \mathrm{d}t - D_2(x) - r_{w,n}^2(x) \right| = \left| R_{w,n}^2(x) \right|. \tag{3.34}$$

Since $f^{(n)}$ is an α-L-Hölder type function, from (3.34) we obtain

$$\begin{aligned}
&\left| \int_a^b w(t) f(t) \, \mathrm{d}t - D_2(x) - r_{w,n}^2(x) \right| \\
&\leq \frac{L}{2(n-2)!} \left[\int_a^b |P_w(x,t)| \left| \int_x^t \left| (s-x)^\alpha (t-s)^{n-2} \right| \mathrm{d}s \right| \mathrm{d}t \right. \\
&\left. + \int_a^b |P_w(a+b-x,t)| \left| \int_{a+b-x}^t \left| (s-a-b+x)^\alpha (t-s)^{n-2} \right| \mathrm{d}s \right| \mathrm{d}t \right].
\end{aligned} \tag{3.35}$$

From (3.35) similarly to Theorem 3.8 we get

$$\left| \frac{1}{b-a} \int_a^b f(t) \, \mathrm{d}t - D_2(x) - r_{w,n}^2(x) \right|$$

$$
\begin{aligned}
&\leq \frac{B(\alpha+1,n-1)}{2(n-2)!} L \left[\int_a^x W(t)(x-t)^{\alpha+n-1}\,\mathrm{d}t \right. \\
&+ \int_a^{a+b-x} W(t)(a+b-x-t)^{\alpha+n-1}\,\mathrm{d}t \\
&+ \int_x^b (1-W(t))(t-x)^{\alpha+n-1}\,\mathrm{d}t \\
&\left. + \int_{a+b-x}^b (1-W(t))(t-a-b+x)^{\alpha+n-1}\,\mathrm{d}t \right] \\
&\leq \frac{B(\alpha+1,n-1)}{(\alpha+n)(n-2)!} L \left[(x-a)^{\alpha+n} + (b-x)^{\alpha+n} \right],
\end{aligned}
$$

and this completes the proof. □

Corollary 3.19 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$. Let $f: I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous and that $f^{(n)}: [a,b] \to \mathbb{R}$ is an α-L-Hölder type function for some $n \geq 2$, $\alpha \in (0,1]$ and $L \in \mathbb{R}^+$. Then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following inequality holds*

$$
\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t - D_2(x) - r_n^2(x) \right| \\
&\leq \frac{B(\alpha+1,n-1)B(2,\alpha+n)}{(b-a)(n-2)!} L \left[(x-a)^{\alpha+n+1} + (b-x)^{\alpha+n+1} \right],
\end{aligned}
$$

where

$$
r_n^2(x) = \frac{1}{2} \sum_{i=0}^{n-1} \left[f^{(i+1)}(x) + (-1)^{i+1} f^{(i+1)}(a+b-x) \right] \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!(b-a)}.
$$

Proof. This is a special case of Theorem 3.11 for $w(t) = \frac{1}{b-a}$, $t \in [a,b]$. □

Corollary 3.20 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$. Let $f: I \to \mathbb{R}$ be such that f' is absolutely continuous and that $f'': [a,b] \to \mathbb{R}$ is an α-L-Hölder type function for some $\alpha \in (0,1]$ and $L \in \mathbb{R}^+$. Then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following inequality holds*

$$
\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t - D_2(x) - r_2^2(x) \right| \\
&\leq \frac{1}{(b-a)(\alpha+1)(\alpha+2)(\alpha+3)} L \left[(x-a)^{\alpha+3} + (b-x)^{\alpha+3} \right],
\end{aligned}
$$

where

$$
\begin{aligned}
r_2^2(x) &= \left(f'(x) - f'(a+b-x) \right) \frac{a+b-2x}{4} \\
&+ \left(f''(x) + f''(a+b-x) \right) \frac{(a-x)^2 + (a-x)(b-x) + (b-x)^2}{12}.
\end{aligned}
$$

Proof. This is a special case of Corollary 3.19 for $n = 2$. □

Theorem 3.12 *Suppose that all the assumptions of Theorem 3.10 hold for some $n = 2k-1$, $k \geq 2$. If f is $(2k)$-convex, then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following inequality holds*

$$\int_a^b w(t) f(t)\,\mathrm{d}t - \frac{f(x) + f(a+b-x)}{2} - r^2_{w,n}(x) \geq 0. \tag{3.36}$$

If f is $(2k)$-concave, then inequality (3.36) is reversed.

Proof. The proof is analogous to the proof of Theorem 3.9. We have that

$$\int_a^b w(t) f(t)\,\mathrm{d}t - D_2(x) - r^2_{w,n}(x) = R^2_{w,n}(x).$$

Let us consider the sign of the integral

$$\begin{aligned} R^a_{w,n}(x) &= \int_a^x W(t) \left[\int_x^t \left(f^{(n)}(x) - f^{(n)}(s)\right)(t-s)^{n-2}\,\mathrm{d}s\right] \mathrm{d}t \\ &= \int_a^x W(t) \left[\int_t^x \left(f^{(n)}(s) - f^{(n)}(x)\right)(t-s)^{n-2}\,\mathrm{d}s\right] \mathrm{d}t \end{aligned}$$

when $f^{(n)}$ is nondecreasing. For $s \in [t,x]$ we have

$$f^{(n)}(s) - f^{(n)}(x) \leq 0, \quad t - s \leq 0,$$

hence for n odd we obtain

$$\int_a^t \left(f^{(n)}(a) - f^{(n)}(s)\right)(t-s)^{n-2}\,\mathrm{d}s \geq 0$$

and we may conclude that under such assumptions $R^a_{w,n}(x) \geq 0$. Analogously, we obtain $R^a_{w,n}(a+b-x) \geq 0$. On the other hand, the sign of the integral

$$\begin{aligned} R^b_{w,n}(x) &= \int_x^b (W(t) - 1) \left[\int_x^t \left(f^{(n)}(x) - f^{(n)}(s)\right)(t-s)^{n-2}\,\mathrm{d}s\right] \mathrm{d}t \\ &= \int_x^b (1 - W(t)) \left[\int_x^t \left(f^{(n)}(s) - f^{(n)}(x)\right)(t-s)^{n-2}\,\mathrm{d}s\right] \mathrm{d}t \end{aligned}$$

(and analogously to $R^b_{w,n}(a+b-x)$) does not depend on the parity of n, and for $f^{(n)}$ nondecreasing we obtain $R^b_{w,n}(x) \geq 0$ and $R^b_{w,n}(a+b-x) \geq 0$.

Since we can write

$$\begin{aligned} &R^2_{w,n}(x) \\ &= \frac{1}{2(n-2)!} \left[R^a_{w,n}(x) + R^b_{w,n}(x) + R^a_{w,n}(a+b-x) + R^b_{w,n}(a+b-x)\right] \end{aligned}$$

the assertion follows immediately.

For $f^{(n)}$ nonincreasing we obtain reversed (3.36) in a similar way. □

Corollary 3.21 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n = 2k-1$, $k \geq 2$. If f is $(2k)$-convex, then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following inequality holds*

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{f(x)+f(a+b-x)}{2} - r_n^2(x) \geq 0. \tag{3.37}$$

If f is $(2k)$-concave, then inequality (3.37) is reversed.

Proof. This is a special case of Theorem 3.12 for $w(t) = \frac{1}{b-a}$, $t \in [a,b]$. □

Corollary 3.22 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and let $f : I \to \mathbb{R}$ be such that f'' is an absolutely continuous function. If f is 4-convex, then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following inequality holds*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{f(x)+f(a+b-x)}{2} \\ &\geq \left(f'(x) - f'(a+b-x)\right)\frac{a+b-2x}{4} \\ &+ \left(f''(x) + f''(a+b-x)\right)\frac{(a-x)^2+(a-x)(b-x)+(b-x)^2}{12} \\ &+ \left(f'''(x) - f'''(a+b-x)\right)(a+b-2x)\frac{(a-x)^2+(b-x)^2}{48}. \end{aligned} \tag{3.38}$$

If f is 4-concave, then inequality (3.38) is reversed.

Proof. This is a special case of Corollary 3.21 for $n = 3$. □

Gauss-Chebyshev two-point formulae

Now, we show how to apply the results of previous subsections to obtain some error estimates for Gauss-Chebyshev quadrature rules involving α-L-Hölder type functions.

Theorem 3.13 *Let I be an open interval in $\mathbb{R}$, $[-1,1] \subset I$, and let $f : I \to \mathbb{R}$ be such that the derivative $f^{(n-1)}$ is absolutely continuous and $f^{(n)} : [-1,1] \to \mathbb{R}$ is an α-L-Hölder function for some $n \geq 2$, $\alpha \in (0,1]$ and $L \in \mathbb{R}^+$. Then*

$$\begin{aligned} &\left|\int_{-1}^1 \frac{1}{\pi\sqrt{1-t^2}} f(t)\,\mathrm{d}t - \frac{1}{2}\left[f\left(-\frac{\sqrt{2}}{2}\right) + f\left(\frac{\sqrt{2}}{2}\right)\right] - s_{w,n}^2\left(-\frac{\sqrt{2}}{2}\right)\right| \\ &\leq \frac{B(\alpha+1,n-1)}{(\alpha+n)(n-2)!}L\left[\left(1-\frac{\sqrt{2}}{2}\right)^{\alpha+n} + \left(1+\frac{\sqrt{2}}{2}\right)^{\alpha+n}\right], \end{aligned}$$

where $s_{w,n}^2$ is defined as in (3.21) and $W(t) = \frac{1}{\pi}\left(\arcsin t + \frac{\pi}{2}\right)$.

Proof. This is a special case of Theorem 3.8 for $[a,b] = [-1,1]$, $x = -\frac{\sqrt{2}}{2}$ and $w(t) = \frac{1}{\pi\sqrt{1-t^2}}$, $t \in [-1,1]$. □

Theorem 3.14 *Let I be an open interval in $\mathbb{R}$, $[-1,1] \subset I$, and let $f : I \to \mathbb{R}$ be such that the derivative $f^{(n-1)}$ is absolutely continuous and $f^{(n)} : [-1,1] \to \mathbb{R}$ is an α-L-Hölder function for some $n \geq 2$, $\alpha \in (0,1]$ and $L \in \mathbb{R}^+$. Then*

$$\left| \int_{-1}^{1} \frac{2}{\pi}\sqrt{1-t^2} f(t)\,\mathrm{d}t - \frac{1}{2}\left[f\left(-\frac{1}{2}\right) + f\left(\frac{1}{2}\right)\right] - s_{w,n}^{2}\left(-\frac{1}{2}\right)\right|$$
$$\leq \frac{B(\alpha+1,n-1)}{(\alpha+n)(n-2)!} L \left[\left(1-\frac{1}{2}\right)^{\alpha+n} + \left(1+\frac{1}{2}\right)^{\alpha+n}\right],$$

where $s_{w,n}^{2}$ is defined as in (3.21) and $W(t) = \frac{1}{\pi}\left(t\sqrt{1-t^2} + \arcsin t + \frac{\pi}{2}\right)$.

Proof. This is a special case of Theorem 3.8 for $[a,b] = [-1,1]$, $x = -\frac{1}{2}$ and $w(t) = \frac{2}{\pi}\sqrt{1-t^2}$, $t \in [-1,1]$. □

3.2 Three-point quadrature formulae

In this section, for each number $x \in \left[a, \frac{a+b}{2}\right)$, we study the general weighted three-point quadrature formula

$$\int_a^b w(t) f(t)\,\mathrm{d}t \tag{3.39}$$
$$= A(x)\left[f(x) + f(a+b-x)\right] + (1-2A(x)) f\left(\frac{a+b}{2}\right) + E(f,w;x),$$

where $A : \left[a, \frac{a+b}{2}\right) \to \mathbb{R}$ and $E(f,w;x)$ being the remainder. Further, $w : [a,b] \to [0,\infty)$ is a probability density function, that is, integrable function satisfying $\int_a^b w(t)dt = 1$, and $W(t) = \int_a^t w(u)du$ for $t \in [a,b]$, $W(t) = 0$ for $t < a$ and $W(t) = 1$ for $t > b$. Some results from this section are published in [69].

3.2.1 Quadrature formulae obtained by a generalization of the Montgomery identity

The aim of this subsection is to consider a general weighted and non-weighted three-point quadrature formula using identity (2.76) and to calculate corresponding error estimates. We obtain three-point Gauss-Chebyshev formulae of the first and of the second kind as special cases of the general weighted three-point quadrature formula and prove some sharp and best possible inequalities. As special cases of general non-weighted three-point quadrature formula, we obtain generalizations of the well-known Simpson's formula ($x = a$), dual Simpson's formula ($x = (3a+b)/4$) and Maclaurin's formula ($x = (5a+b)/6$).

General weighted three-point formula and related inequalities

Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ exists on $[a,b]$ for some $n\geq 2$. We introduce the following notation for each $x\in\left[a,\frac{a+b}{2}\right)$

$$D_3(x)=A(x)[f(x)+f(a+b-x)]+(1-2A(x))f\left(\frac{a+b}{2}\right), \tag{3.40}$$

$$\begin{aligned} t_{w,n}^3(x) &= A(x)\left[\sum_{i=0}^{n-2}\frac{f^{(i+1)}(x)}{(i+1)!}\int_a^b w(s)(s-x)^{i+1}\mathrm{d}s\right. \\ &+ \left.\sum_{i=0}^{n-2}\frac{f^{(i+1)}(a+b-x)}{(i+1)!}\int_a^b w(s)(s-a-b+x)^{i+1}\mathrm{d}s\right] \\ &+ (1-2A(x))\sum_{i=0}^{n-2}\frac{f^{(i+1)}\left(\frac{a+b}{2}\right)}{(i+1)!}\int_a^b w(s)\left(s-\frac{a+b}{2}\right)^{i+1}\mathrm{d}s \end{aligned} \tag{3.41}$$

and

$$\begin{aligned} &T_{w,n}^3(x,s) \\ &= -A(x)[T_{w,n}(x,s)+T_{w,n}(a+b-x,s)]-(1-2A(x))T_{w,n}\left(\frac{a+b}{2},s\right) \end{aligned}$$

where $T_{w,n}(x,s)$ is defined by

$$T_{w,n}(x,s) = \begin{cases} \int_a^s w(u)(u-s)^{n-1}\mathrm{d}u, & a\leq s\leq x, \\ -\int_s^b w(u)(u-s)^{n-1}\mathrm{d}u, & x<s\leq b. \end{cases}$$

The following is a general weighted three-point formula.

Theorem 3.15 *Let I be an open interval in $\mathbb{R}$, $[a,b]\subset I$, and let $w:[a,b]\to[0,\infty\rangle$ be some probability density function. Let $f:I\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n\geq 2$. Then for each $x\in\left[a,\frac{a+b}{2}\right)$ the following identity holds*

$$\int_a^b w(t)f(t)\mathrm{d}t=D_3(x)+t_{w,n}^3(x)+\frac{1}{(n-1)!}\int_a^b T_{w,n}^3(x,s)f^{(n)}(s)\mathrm{d}s. \tag{3.42}$$

Proof. We put $x\equiv x, x\equiv\frac{a+b}{2}$ and $x\equiv a+b-x$ in (2.76) to obtain three new formulae. After multiplying these three formulae by $A(x), 1-2A(x)$ and $A(x)$ and adding, we get (3.42). □

Remark 3.18 Identity (3.42) holds true in the case $n=1$. In this special case we have

$$\int_a^b w(t)f(t)\mathrm{d}t=D_3(x)+\int_a^b T_{w,1}^3(x,s)f'(s)\mathrm{d}s, \tag{3.43}$$

where

$$
\begin{aligned}
&T^3_{w,1}(x,s)\\
&= -A(x)\left[T_{w,1}(x,s)+T_{w,1}(a+b-x,s)\right]-(1-2A(x))\,T_{w,1}\left(\frac{a+b}{2},s\right)\\
&= -A(x)\left[P_w(x,s)+P_w(a+b-x,s)\right]-(1-2A(x))\,P_w\left(\frac{a+b}{2},s\right)
\end{aligned}
$$

and $P_w(x,s)$ is the weighted Peano kernel defined by

$$
P_w(x,s) = \begin{cases} W(s), & a \le s \le x,\\ W(s)-1, & x < s \le b. \end{cases}
$$

There follows an error estimate for general formula (3.42).

Theorem 3.16 *Suppose that all the assumptions of Theorem 3.15 hold. Additionally, assume that (p,q) is a pair of conjugate exponents, $1 \le p,q \le \infty$; let $f^{(n)} \in L^p[a,b]$ for some $n \ge 1$. Then for each $x \in \left[a, \frac{a+b}{2}\right)$ we have*

$$
\left|\int_a^b w(t)f(t)\mathrm{d}t - D_3(x) - t^3_{w,n}(x)\right| \le \frac{1}{(n-1)!}\left\|T^3_{w,n}(x,\cdot)\right\|_q \left\|f^{(n)}\right\|_p. \tag{3.44}
$$

Inequality (3.44) *is sharp for* $1 < p \le \infty$.

Proof. Applying the Hölder inequality on (3.42) we get estimate (3.44). The proof of sharpness is analogous to Theorem 3.2. □

Gauss-Chebyshev three-point formulae of the first kind

In case $k=3$, (3.11) reduces to Gauss-Chebyshev three-point formulae of the first kind

$$
\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}f(t)\,dt = \frac{\pi}{3}\left[f\left(-\frac{\sqrt{3}}{2}\right)+f(0)+f\left(\frac{\sqrt{3}}{2}\right)\right]+\frac{\pi}{23040}f^{(6)}(\xi),
$$

for some $\xi \in (-1,1)$.

Remark 3.19 If we apply (3.43) with $a=-1, b=1, x=-\frac{\sqrt{3}}{2}, A\left(-\frac{\sqrt{3}}{2}\right)=\frac{1}{3}$ and $w(t) = \frac{1}{\pi\sqrt{1-t^2}}$, $t \in [-1,1]$, we get

$$
\begin{aligned}
&\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}f(t)\,\mathrm{d}t\\
&= \frac{\pi}{3}\left[f\left(-\frac{\sqrt{3}}{2}\right)+f(0)+f\left(\frac{\sqrt{3}}{2}\right)\right]+\pi\int_{-1}^{1}R_1(s)f'(s)\,\mathrm{d}s,
\end{aligned}
$$

where

$$R_1(s) = \begin{cases} -\frac{1}{2} - \frac{1}{\pi}\arcsin s, & -1 \le s \le -\frac{\sqrt{3}}{2}, \\ -\frac{1}{6} - \frac{1}{\pi}\arcsin s, & -\frac{\sqrt{3}}{2} < s \le 0, \\ \frac{1}{6} - \frac{1}{\pi}\arcsin s, & 0 < s \le \frac{\sqrt{3}}{2}, \\ \frac{1}{2} - \frac{1}{\pi}\arcsin s, & \frac{\sqrt{3}}{2} < s \le 1. \end{cases}$$

Corollary 3.23 *Let I be an open interval in $\mathbb{R}$, $[-1,1] \subset I$, (p,q) a pair of conjugate exponents, $1 \le p,q \le \infty$. Let $f : I \to \mathbb{R}$ be an absolutely continuous function and $f' \in L^p[-1,1]$. Then we have*

$$\left| \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} f(t)\,\mathrm{d}t - \frac{\pi}{3}\left[f\left(-\frac{\sqrt{3}}{2}\right) + f(0) + f\left(\frac{\sqrt{3}}{2}\right) \right] \right| \le \pi \|R_1\|_q \|f'\|_p . \quad (3.45)$$

Inequality (3.45) *is sharp for* $1 < p \le \infty$.

Proof. This is a special case of Theorem 3.16 for $a=-1$, $b=1$, $x=-\frac{\sqrt{3}}{2}$, $A\left(-\frac{\sqrt{3}}{2}\right) = \frac{1}{3}$ and $w(t) = \frac{1}{\pi\sqrt{1-t^2}}$, $t \in [-1,1]$. □

Corollary 3.24 *Let I be an open interval in $\mathbb{R}$, $[-1,1] \subset I$, (p,q) a pair of conjugate exponents, $1 \le p,q \le \infty$ and $f : I \to \mathbb{R}$ absolutely continuous function. Then we have*

$$\left| \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} f(t)\,\mathrm{d}t - \frac{\pi}{3}\left[f\left(-\frac{\sqrt{3}}{2}\right) + f(0) + f\left(\frac{\sqrt{3}}{2}\right) \right] \right|$$

$$\le \begin{cases} \left(4-2\sqrt{3}\right)\|f'\|_\infty, & f' \in L^\infty[-1,1], \\ 2\sqrt{\frac{1}{3}\pi - 1}\,\|f'\|_2, & f' \in L^2[-1,1], \\ \frac{1}{6}\pi\|f'\|_1, & f' \in L^1[-1,1]. \end{cases}$$

The first and the second inequality are sharp and the third inequality is the best possible.

Proof. Applying (3.45) with $p=\infty$, $p=2$ and $p=1$, respectively, we get these inequalities. Since function R_1 is left continuous and has finite jump at $0, \pm\frac{\sqrt{3}}{2}$, the proof of the best possibility of the third inequality is similar to the proof given in Theorem 3.2. □

Remark 3.20 Inequalities from Corollary 3.24 were proved by J. Pečarić et al. in [99].

Remark 3.21 If we apply Theorem 3.15 with $n=2$, $a=-1$, $b=1$, $x=-\frac{\sqrt{3}}{2}$, $A\left(-\frac{\sqrt{3}}{2}\right) = \frac{1}{3}$ and $w(t) = \frac{1}{\pi\sqrt{1-t^2}}$, $t \in [-1,1]$, we get

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} f(t)\,\mathrm{d}t = \frac{\pi}{3}\left[f\left(-\frac{\sqrt{3}}{2}\right) + f(0) + f\left(\frac{\sqrt{3}}{2}\right) \right]$$

$$+\frac{\pi\sqrt{3}}{6}\left[f'\left(-\frac{\sqrt{3}}{2}\right)-f'\left(\frac{\sqrt{3}}{2}\right)\right]+\pi\int_{-1}^{1}R_2(s)f''(s)\,\mathrm{d}s,$$

where

$$R_2(s)=\begin{cases}\frac{1}{2}s+\frac{1}{\pi}\left(s\arcsin s+\sqrt{1-s^2}\right), & -1\le s\le-\frac{\sqrt{3}}{2},\\ \frac{1}{6}s+\frac{1}{\pi}\left(s\arcsin s+\sqrt{1-s^2}\right), & -\frac{\sqrt{3}}{2}<s\le 0,\\ -\frac{1}{6}s+\frac{1}{\pi}\left(s\arcsin s+\sqrt{1-s^2}\right), & 0<s\le\frac{\sqrt{3}}{2},\\ -\frac{1}{2}s+\frac{1}{\pi}\left(s\arcsin s+\sqrt{1-s^2}\right), & \frac{\sqrt{3}}{2}<s\le 1.\end{cases}$$

Corollary 3.25 *Let I be an open interval in $\mathbb{R}$, $[-1,1]\subset I$, (p,q) a pair of conjugate exponents, $1\le p,q\le\infty$ and let $f:I\to\mathbb{R}$ be such that f' is an absolutely continuous function. Then we have*

$$\left|\int_{-1}^{1}\frac{f(t)\,\mathrm{d}t}{\sqrt{1-t^2}}-\frac{\pi}{3}\left[f\left(-\frac{\sqrt{3}}{2}\right)+f(0)\right.\right.$$
$$\left.\left.+f\left(\frac{\sqrt{3}}{2}\right)+\frac{\sqrt{3}}{2}f'\left(-\frac{\sqrt{3}}{2}\right)-\frac{\sqrt{3}}{2}f'\left(\frac{\sqrt{3}}{2}\right)\right]\right|$$
$$\le\begin{cases}\frac{1}{2}\pi\|f''\|_\infty, & f''\in L^\infty[-1,1],\\ \frac{1}{3}\sqrt{\frac{32+2\pi}{3}}\|f''\|_2, & f''\in L^2[-1,1],\\ \|f''\|_1, & f''\in L^1[-1,1].\end{cases}$$

The first and the second inequality are sharp and the third inequality is the best possible.

Proof. Similar to the proof of Corollary 3.24 □

Remark 3.22 If we apply Theorem 3.15 with $n=3, a=-1, b=1, x=-\frac{\sqrt{3}}{2}, A\left(-\frac{\sqrt{3}}{2}\right)=\frac{1}{3}$ and $w(t)=\frac{1}{\pi\sqrt{1-t^2}}$, $t\in[-1,1]$, we get

$$\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}f(t)\,\mathrm{d}t$$
$$=\frac{\pi}{3}\left[f\left(-\frac{\sqrt{3}}{2}\right)+f(0)+f\left(\frac{\sqrt{3}}{2}\right)\right]+\frac{\pi\sqrt{3}}{6}\left[f'\left(-\frac{\sqrt{3}}{2}\right)-f'\left(\frac{\sqrt{3}}{2}\right)\right]$$
$$+\frac{\pi}{12}\left[\frac{5}{2}f''\left(-\frac{\sqrt{3}}{2}\right)+f''(0)+\frac{5}{2}f''\left(\frac{\sqrt{3}}{2}\right)\right]+\frac{\pi}{2}\int_{-1}^{1}R_3(s)f'''(s)\,\mathrm{d}s,$$

where

$$R_3(s)=\begin{cases}-\frac{1}{2}\left(\frac{1}{2}+s^2\right)-\frac{3}{2\pi}s\sqrt{1-s^2}-\frac{1}{\pi}\left(\frac{1}{2}+s^2\right)\arcsin s, & -1\le s\le-\frac{\sqrt{3}}{2},\\ -\frac{1}{6}\left(\frac{1}{2}+s^2\right)-\frac{3}{2\pi}s\sqrt{1-s^2}-\frac{1}{\pi}\left(\frac{1}{2}+s^2\right)\arcsin s, & -\frac{\sqrt{3}}{2}<s\le 0,\\ \frac{1}{6}\left(\frac{1}{2}+s^2\right)-\frac{3}{2\pi}s\sqrt{1-s^2}-\frac{1}{\pi}\left(\frac{1}{2}+s^2\right)\arcsin s, & 0<s\le\frac{\sqrt{3}}{2},\\ \frac{1}{2}\left(\frac{1}{2}+s^2\right)-\frac{3}{2\pi}s\sqrt{1-s^2}-\frac{1}{\pi}\left(\frac{1}{2}+s^2\right)\arcsin s, & \frac{\sqrt{3}}{2}<s\le 1.\end{cases}$$

Corollary 3.26 *Let I be an open interval in $\mathbb{R}$, $[-1,1]\subset I$, (p,q) a pair of conjugate exponents, $1\le p,q\le\infty$ and let $f:I\to\mathbb{R}$ be such that f'' is an absolutely continuous function. Then we have*

$$\begin{aligned}&\left|\int_{-1}^{1}\frac{f(t)\,\mathrm{d}t}{\sqrt{1-t^2}}-\frac{\pi}{3}\left[f\left(-\frac{\sqrt{3}}{2}\right)+f(0)+f\left(\frac{\sqrt{3}}{2}\right)+\frac{\sqrt{3}}{2}f'\left(-\frac{\sqrt{3}}{2}\right)\right.\right.\\&\left.\left.-\frac{\sqrt{3}}{2}f'\left(\frac{\sqrt{3}}{2}\right)+\frac{5}{8}f''\left(-\frac{\sqrt{3}}{2}\right)+\frac{1}{4}f''(0)+\frac{5}{8}f''\left(\frac{\sqrt{3}}{2}\right)\right]\right|\\&\le\begin{cases}0.493373\,\|f'''\|_\infty, & f'''\in L^\infty[-1,1],\\ 0.45485\,\|f'''\|_2, & f'''\in L^2[-1,1],\\ \frac{1}{48}\left(9\sqrt{3}+5\pi\right)\|f'''\|_1, & f'''\in L^1[-1,1].\end{cases}\end{aligned}$$

Proof. The proof is similar to the proof of Corollary 3.24. □

Gauss-Chebyshev three-point formulae of the second kind

In case $k=3$, (3.13) reduces to Gauss-Chebyshev three-point formulae of the second kind

$$\int_{-1}^{1}\sqrt{1-t^2}f(t)\,\mathrm{d}t=\frac{\pi}{8}\left[f\left(-\frac{\sqrt{2}}{2}\right)+2f(0)+f\left(\frac{\sqrt{2}}{2}\right)\right]+\frac{\pi}{92160}f^{(6)}(\xi),$$

for some $\xi\in(-1,1)$.

Remark 3.23 If we apply (3.43) with $a=-1$, $b=1$, $x=-\frac{\sqrt{2}}{2}$, $A\left(-\frac{\sqrt{2}}{2}\right)=\frac{1}{4}$ and $w(t)=\frac{2\sqrt{1-t^2}}{\pi}$, $t\in[-1,1]$, we get

$$\begin{aligned}&\int_{-1}^{1}\sqrt{1-t^2}f(t)\,\mathrm{d}t\\&=\frac{\pi}{8}\left[f\left(-\frac{\sqrt{2}}{2}\right)+2f(0)+f\left(\frac{\sqrt{2}}{2}\right)\right]+\frac{\pi}{2}\int_{-1}^{1}Q_1(s)f'(s)\,\mathrm{d}s,\end{aligned}$$

where

$$Q_1(s)=\begin{cases}-\frac{1}{2}-\frac{1}{\pi}\left(s\sqrt{1-s^2}+\arcsin s\right), & -1\le s\le-\frac{\sqrt{2}}{2},\\ -\frac{1}{4}-\frac{1}{\pi}\left(s\sqrt{1-s^2}+\arcsin s\right), & -\frac{\sqrt{2}}{2}<s\le 0,\\ \frac{1}{4}-\frac{1}{\pi}\left(s\sqrt{1-s^2}+\arcsin s\right), & 0<s\le\frac{\sqrt{2}}{2},\\ \frac{1}{2}-\frac{1}{\pi}\left(s\sqrt{1-s^2}+\arcsin s\right), & \frac{\sqrt{2}}{2}<s\le 1.\end{cases}$$

Corollary 3.27 *Let I be an open interval in $\mathbb{R}$, $[-1,1]\subset I$, (p,q) a pair of conjugate exponents, $1\le p,q\le\infty$. Let $f:I\to\mathbb{R}$ be an absolutely continuous function and $f'\in$*

$L^p[-1,1]$. *Then we have*

$$\left|\int_{-1}^{1}\sqrt{1-t^2}f(t)\,\mathrm{d}t-\frac{\pi}{8}\left[f\left(-\frac{\sqrt{2}}{2}\right)+2f(0)+f\left(\frac{\sqrt{2}}{2}\right)\right]\right|\leq\frac{\pi}{2}\|Q_1\|_q\|f'\|_p. \quad (3.46)$$

Inequality (3.46) *is sharp for* $1<p\leq\infty$.

Proof. This is a special case of Theorem 3.16 for $a=-1$, $b=1$, $x=-\frac{\sqrt{2}}{2}$, $A\left(-\frac{\sqrt{2}}{2}\right)=\frac{1}{4}$ and $w(t)=\frac{2\sqrt{1-t^2}}{\pi}$, $t\in[-1,1]$. □

Corollary 3.28 *Let I be an open interval in $\mathbb{R}$, $[-1,1]\subset I$, (p,q) a pair of conjugate exponents, $1\leq p,q\leq\infty$ and $f:I\to\mathbb{R}$ an absolutely continuous function. Then we have*

$$\left|\int_{-1}^{1}\sqrt{1-t^2}f(t)\,\mathrm{d}t-\frac{\pi}{8}\left[f\left(-\frac{\sqrt{2}}{2}\right)+2f(0)+f\left(\frac{\sqrt{2}}{2}\right)\right]\right|$$
$$\leq\begin{cases}0.26917\,\|f'\|_\infty, & f'\in L^\infty[-1,1],\\ 0.239162\,\|f'\|_2, & f'\in L^2[-1,1],\\ \frac{1}{8}\pi\,\|f'\|_1, & f'\in L^1[-1,1].\end{cases}$$

The first and the second inequality are sharp and the third inequality is the best possible.

Proof. These inequalities follow by direct calculation after taking $p=\infty$, $p=2$ and $p=1$ in (3.46). Since function Q_1 is left continuous and has finite jump at $0,\pm\frac{\sqrt{2}}{2}$, the proof of the best possibility of the third inequality is similar to Theorem 3.2. □

Remark 3.24 Inequalities from Corollary 3.28 were proved by J. Pečarić et al. in [99].

Remark 3.25 If we apply Theorem 3.15 with $n=2$, $a=-1$, $b=1$, $x=-\frac{\sqrt{2}}{2}$, $A\left(-\frac{\sqrt{2}}{2}\right)=\frac{1}{4}$ and $w(t)=\frac{2\sqrt{1-t^2}}{\pi}$, $t\in[-1,1]$, we get

$$\int_{-1}^{1}\sqrt{1-t^2}f(t)\,\mathrm{d}t=\frac{\pi}{8}\left[f\left(-\frac{\sqrt{2}}{2}\right)+2f(0)+f\left(\frac{\sqrt{2}}{2}\right)\right]$$
$$+\frac{\pi\sqrt{2}}{16}\left[f'\left(-\frac{\sqrt{2}}{2}\right)-f'\left(\frac{\sqrt{2}}{2}\right)\right]+\frac{\pi}{2}\int_{-1}^{1}Q_2(s)f''(s)\,\mathrm{d}s,$$

where

$$Q_2(s)=\begin{cases}\frac{s}{2}+\frac{1}{3\pi}\left(2+s^2\right)\sqrt{1-s^2}+\frac{1}{\pi}s\arcsin s, & -1\leq s\leq-\frac{\sqrt{2}}{2},\\ \frac{s}{4}+\frac{1}{3\pi}\left(2+s^2\right)\sqrt{1-s^2}+\frac{1}{\pi}s\arcsin s, & -\frac{\sqrt{2}}{2}<s\leq 0,\\ -\frac{s}{4}+\frac{1}{3\pi}\left(2+s^2\right)\sqrt{1-s^2}+\frac{1}{\pi}s\arcsin s, & 0<s\leq\frac{\sqrt{2}}{2},\\ -\frac{s}{2}+\frac{1}{3\pi}\left(2+s^2\right)\sqrt{1-s^2}+\frac{1}{\pi}s\arcsin s, & \frac{\sqrt{2}}{2}<s\leq 1.\end{cases}$$

Corollary 3.29 *Let I be an open interval in $\mathbb{R}$, $[-1,1] \subset I$, (p,q) a pair of conjugate exponents, $1 \le p,q \le \infty$ and let $f : I \to \mathbb{R}$ be such that f' is an absolutely continuous function. Then we have*

$$\left| \int_{-1}^{1} \sqrt{1-t^2} f(t)\,\mathrm{d}t - \frac{\pi}{8} \left[f\left(-\frac{\sqrt{2}}{2}\right) + 2f(0) + f\left(\frac{\sqrt{2}}{2}\right) \right.\right.$$
$$\left.\left. + \frac{\sqrt{2}}{2} f'\left(-\frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}}{2} f'\left(\frac{\sqrt{2}}{2}\right) \right] \right|$$
$$\le \begin{cases} \frac{1}{8}\pi \left\|f''\right\|_{\infty}, & f'' \in L^{\infty}[-1,1], \\ 0.3287364 \left\|f''\right\|_{2}, & f'' \in L^{2}[-1,1], \\ \frac{1}{3}\left\|f''\right\|_{1}, & f'' \in L^{1}[-1,1]. \end{cases}$$

The first and the second inequality are sharp and the third inequality is the best possible.

Proof. The proof is similar to the proof of Corollary 3.28. □

Remark 3.26 If we apply Theorem 3.15 with $n = 3$, $a = -1$, $b = 1$, $x = -\frac{\sqrt{2}}{2}$, $A\left(-\frac{\sqrt{2}}{2}\right) = \frac{1}{4}$ and $w(t) = \frac{2\sqrt{1-t^2}}{\pi}$, $t \in [-1,1]$, we get

$$\int_{-1}^{1} \sqrt{1-t^2} f(t)\,\mathrm{d}t$$
$$= \frac{\pi}{8}\left[f\left(-\frac{\sqrt{2}}{2}\right) + 2f(0) + f\left(\frac{\sqrt{2}}{2}\right)\right] + \frac{\pi\sqrt{2}}{16}\left[f'\left(-\frac{\sqrt{2}}{2}\right) - f'\left(\frac{\sqrt{2}}{2}\right)\right]$$
$$+ \frac{\pi}{64}\left[3f''\left(-\frac{\sqrt{2}}{2}\right) + 2f''(0) + 3f''\left(\frac{\sqrt{2}}{2}\right)\right] + \frac{\pi}{4}\int_{-1}^{1} Q_3(s) f'''(s)\,\mathrm{d}s,$$

where

$$Q_3(s) = \begin{cases} -\frac{1}{8}\left(1+4s^2\right) - \frac{1}{12\pi}\left(13s+2s^3\right)\sqrt{1-s^2} & \\ \qquad -\frac{1}{4\pi}\left(1+4s^2\right)\arcsin s, & -1 \le s \le -\frac{\sqrt{2}}{2}, \\ -\frac{1}{16}\left(1+4s^2\right) - \frac{1}{12\pi}\left(13s+2s^3\right)\sqrt{1-s^2} & \\ \qquad -\frac{1}{4\pi}\left(1+4s^2\right)\arcsin s, & -\frac{\sqrt{2}}{2} < s \le 0, \\ \frac{1}{16}\left(1+4s^2\right) - \frac{1}{12\pi}\left(13s+2s^3\right)\sqrt{1-s^2} & \\ \qquad -\frac{1}{4\pi}\left(1+4s^2\right)\arcsin s, & 0 < s \le \frac{\sqrt{2}}{2}, \\ \frac{1}{8}\left(1+4s^2\right) - \frac{1}{12\pi}\left(13s+2s^3\right)\sqrt{1-s^2} & \\ \qquad -\frac{1}{4\pi}\left(1+4s^2\right)\arcsin s, & \frac{\sqrt{2}}{2} < s \le 1. \end{cases}$$

Corollary 3.30 *Let I be an open interval in $\mathbb{R}$, $[-1,1] \subset I$, (p,q) a pair of conjugate exponents, $1 \le p,q \le \infty$ and let $f : I \to \mathbb{R}$ be such that f'' is an absolutely continuous* *function. Then we have*

$$\left|\int_{-1}^{1}\sqrt{1-t^2}f(t)\,\mathrm{d}t-\frac{\pi}{8}\left[f\left(-\frac{\sqrt{2}}{2}\right)+2f(0)+f\left(\frac{\sqrt{2}}{2}\right)+\frac{\sqrt{2}}{2}f'\left(-\frac{\sqrt{2}}{2}\right)\right.\right.$$
$$\left.\left.-\frac{\sqrt{2}}{2}f'\left(\frac{\sqrt{2}}{2}\right)+\frac{3}{8}f''\left(-\frac{\sqrt{2}}{2}\right)+\frac{1}{4}f''(0)+\frac{3}{8}f''\left(\frac{\sqrt{2}}{2}\right)\right]\right|$$
$$\leq\begin{cases}0.0869419\left\|f'''\right\|_{\infty}, & f'''\in L^{\infty}[-1,1],\\ 0.0885601\left\|f'''\right\|_{2}, & f'''\in L^{2}[-1,1],\\ \frac{7}{48}\left\|f'''\right\|_{1}, & f'''\in L^{1}[-1,1].\end{cases}$$

The first and the second inequality are sharp and the third inequality is the best possible.

Proof. Similar to the proof of Corollary 3.28. □

Non-weighted three-point formula

Here we establish a general non-weighted three-point quadrature formula as a special case of formula (3.42).

We define

$$t_n^3(x)=A(x)\sum_{i=0}^{n-2}\left[f^{(i+1)}(x)+(-1)^{i+1}f^{(i+1)}(a+b-x)\right]$$
$$\cdot\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!\,(b-a)}$$
$$+(1-2A(x))\sum_{i=0}^{n-2}f^{(i+1)}\left(\frac{a+b}{2}\right)\frac{\left(1-(-1)^{i}\right)(b-a)^{i+1}}{2^{i+2}(i+2)!}$$

and

$$T_n^3(x,s)$$
$$=-nA(x)\left[T_n(x,s)+T_n(a+b-x,s)\right]-n(1-2A(x))\,T_n\left(\frac{a+b}{2},s\right),$$

where

$$T_n(x,s)=\begin{cases}\frac{-1}{n(b-a)}(a-s)^n, & a\leq s\leq x,\\ \frac{-1}{n(b-a)}(b-s)^n, & x<s\leq b.\end{cases}$$

Theorem 3.17 *Let I be an open interval in $\mathbb{R}$, $[a,b]\subset I$, and let $f:I\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n\geq 2$. Then for each $x\in\left[a,\frac{a+b}{2}\right)$ the following identity holds*

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t=D_3(x)+t_n^3(x)+\frac{1}{n!}\int_a^b T_n^3(x,s)\,f^{(n)}(s)\,\mathrm{d}s. \tag{3.47}$$

Proof. This is a special case of Theorem 3.15 for $w(t)=\frac{1}{b-a}$, $t\in[a,b]$. □

Remark 3.27 Identity (3.47) holds true in the case $n = 1$.

Theorem 3.18 *Suppose that all the assumptions of Theorem 3.17 hold. Additionally, assume that (p,q) is a pair of conjugate exponents, that is $1 \le p,q \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $f^{(n)} \in L^p[a,b]$ for some $n \ge 1$. Then for each $x \in \left[a, \frac{a+b}{2}\right)$ we have*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t - D_3(x) - t_n^3(x) \right| \le \frac{1}{n!} \left\| T_n^3(x,\cdot) \right\|_q \left\| f^{(n)} \right\|_p. \tag{3.48}$$

Inequality (3.48) *is sharp for* $1 < p \le \infty$.

Proof. This is a special case of Theorem 3.16 for $w(t) = \frac{1}{b-a}$, $t \in [a,b]$. □

Simpson's formula

Now, we set

$$A(x) = \frac{(b-a)^2}{6(a+b-2x)^2},\ x \in \left[a, \frac{a+b}{2}\right).$$

This special choice of the function A enables us to establish our generalizations of the well-known Simpson's formula $(x = a)$

Suppose that all the assumptions of Theorem 3.17 hold. Then the generalization of the Simpson's formula reads

$$\frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t = D_3(a) + t_n^3(a) + \frac{1}{n!} \int_a^b T_n^3(a,s) f^{(n)}(s)\,\mathrm{d}s, \tag{3.49}$$

where

$$D_3(a) = \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right],$$

$$\begin{aligned} t_n^3(a) &= \frac{1}{6} \sum_{i=0}^{n-2} \left[f^{(i+1)}(a) + (-1)^{i+1} f^{(i+1)}(b) \right] \frac{(b-a)^{i+1}}{(i+2)!} \\ &+ \frac{4}{6} \sum_{i=1}^{n-2} f^{(i+1)}\left(\frac{a+b}{2}\right) \frac{\left(1-(-1)^i\right)(b-a)^{i+1}}{2^{i+2}(i+2)!} \end{aligned}$$

and

$$\begin{aligned} T_n^3(a,s) &= -\frac{n}{6}\left[T_n(a,s) + 4T_n\left(\frac{a+b}{2}, s\right) + T_n(b,s) \right] \\ &= \begin{cases} \dfrac{1}{6(b-a)} \left[5(a-s)^n + (b-s)^n\right],\ a \le s \le \dfrac{a+b}{2}, \\ \dfrac{1}{6(b-a)} \left[(a-s)^n + 5(b-s)^n\right],\ \dfrac{a+b}{2} < s \le b. \end{cases} \end{aligned}$$

Corollary 3.31 *Suppose that all the assumptions of Theorem 3.17 hold. Additionally, assume that (p,q) is a pair of conjugate exponents, that is $1 \leq p,q \leq \infty$, $\frac{1}{p}+\frac{1}{q}=1$ and $n \in \mathbb{N}$.*

(a) If $f^{(n)} \in L^{\infty}[a,b]$, then

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D_3(a) - t_n^3(a)\right| \leq \frac{1}{(n+1)!}$$
$$\cdot\left(\frac{\left[5-(-1)^n+2^{n+1}\right](b-a)^n}{3\cdot 2^{n+1}} - \frac{5\left(1-(-1)^n\right)(b-a)^n}{3\left(1+\sqrt[n]{5}\right)^n}\right)\left\|f^{(n)}\right\|_{\infty}.$$

(b) If $f^{(n)} \in L^2[a,b]$, then

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D_3(a) - t_n^3(a)\right| \leq \frac{1}{n!}\left(\frac{\left(2^{2n-2}+3\right)(b-a)^{2n-1}}{9\cdot 2^{2n-1}(2n+1)}\right.$$
$$\left.+\frac{5(-1)^n(b-a)^{2n-1}}{18}B(n+1,n+1)\right)^{\frac{1}{2}}\left\|f^{(n)}\right\|_2.$$

(c) If $f^{(n)} \in L^1[a,b]$, then

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D_3(a) - t_n^3(a)\right|$$
$$\leq \frac{1}{n!}K_n(a)\left\|f^{(n)}\right\|_1,$$

where $K_1(a)=\frac{1}{3}$, $K_2(a)=\frac{1}{4}(b-a)$ and $K_n(a)=\frac{1}{6}(b-a)^{n-1}$, for $n \geq 3$.

The first and the second inequality are sharp and the third inequality is the best possible.

Proof. Applying (3.48) with $x=a$ and $p=\infty$ we get

$$\int_a^b \left|T_n^3(a,s)\right|\mathrm{d}s$$
$$=\int_a^{\frac{a+b}{2}}\left|\frac{5(a-s)^n+(b-s)^n}{6(b-a)}\right|\mathrm{d}s+\int_{\frac{a+b}{2}}^b\left|\frac{(a-s)^n+5(b-s)^n}{6(b-a)}\right|\mathrm{d}s$$
$$=2\cdot\frac{\left[5-(-1)^n+2^{n+1}\right](b-a)^n}{6\cdot 2^{n+1}(n+1)} - 2\cdot\frac{5\left(1-(-1)^n\right)(b-a)^n}{6\left(1+\sqrt[n]{5}\right)^n(n+1)}$$

and the first inequality is obtained. To prove the second inequality we take $p=2$

$$\int_a^b \left|T_n^3(a,s)\right|^2\mathrm{d}s$$

$$= \int_a^{\frac{a+b}{2}} \left| \frac{5(a-s)^n + (b-s)^n}{6(b-a)} \right|^2 \mathrm{d}s + \int_{\frac{a+b}{2}}^b \left| \frac{(a-s)^n + 5(b-s)^n}{6(b-a)} \right|^2 \mathrm{d}s$$

$$= \frac{(b-a)^{2n-1}}{36} \left[\frac{24 + 2^{2n+1}}{2^{2n+1}(2n+1)} + 10 \cdot (-1)^n B_{\frac{1}{2}}(n+1, n+1) \right]$$

$$+ \frac{(b-a)^{2n-1}}{36} \left[\frac{24 + 2^{2n+1}}{2^{2n+1}(2n+1)} + 10 \cdot (-1)^n \Big(B(n+1, n+1) - B_{\frac{1}{2}}(n+1, n+1) \Big) \right].$$

For $p = 1$, we have

$$\sup_{s \in [a,b]} \left| T_n^3(a,s) \right| = \max \left\{ \sup_{s \in \left[a, \frac{a+b}{2}\right]} \left| \frac{5(a-s)^n + (b-s)^n}{6(b-a)} \right|, \sup_{s \in \left[\frac{a+b}{2}, b\right]} \left| \frac{(a-s)^n + 5(b-s)^n}{6(b-a)} \right| \right\}.$$

By elementary calculations we get

$$\sup_{s \in \left[a, \frac{a+b}{2}\right]} \left| \frac{5(a-s)^n + (b-s)^n}{6(b-a)} \right| = \max \left\{ \frac{(b-a)^{n-1}}{2^n}, \frac{(b-a)^{n-1}}{6} \right\},$$

$$\sup_{s \in \left[\frac{a+b}{2}, b\right]} \left| \frac{(a-s)^n + 5(b-s)^n}{6(b-a)} \right| = \max \left\{ \frac{(b-a)^{n-1}}{2^n}, \frac{(b-a)^{n-1}}{6} \right\},$$

if n is even, and

$$\sup_{s \in \left[a, \frac{a+b}{2}\right]} \left| \frac{5(a-s)^n + (b-s)^n}{6(b-a)} \right| = \max \left\{ \frac{(b-a)^{n-1}}{3 \cdot 2^{n-1}}, \frac{(b-a)^{n-1}}{6} \right\},$$

$$\sup_{s \in \left[\frac{a+b}{2}, b\right]} \left| \frac{(a-s)^n + 5(b-s)^n}{6(b-a)} \right| = \max \left\{ \frac{(b-a)^{n-1}}{3 \cdot 2^{n-1}}, \frac{(b-a)^{n-1}}{6} \right\},$$

if n is odd. Since function $T_n^3(a, \cdot)$ is left continuous and has finite jump at $\frac{a+b}{2}$, the proof of the best possibility of the third inequality is similar to the proof of Theorem 3.2. □

Dual Simpson's formula

Suppose that all the assumptions of Theorem 3.17 hold for

$$A(x) = \frac{(b-a)^2}{6(a+b-2x)^2}.$$

and $x = \frac{3a+b}{4}$. Then the generalization of the famous dual Simpson's formula states

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\\&= D_3\left(\frac{3a+b}{4}\right) + t_n^3\left(\frac{3a+b}{4}\right) + \frac{1}{n!}\int_a^b T_n^3\left(\frac{3a+b}{4},s\right) f^{(n)}(s)\,\mathrm{d}s,\end{aligned} \tag{3.50}$$

where

$$D_3\left(\frac{3a+b}{4}\right) = \frac{1}{3}\left(2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right),$$

$$\begin{aligned}t_n^3\left(\frac{3a+b}{4}\right) &= \frac{2}{3}\sum_{i=0}^{n-2}\left[f^{(i+1)}\left(\frac{3a+b}{4}\right) + (-1)^{i+1} f^{(i+1)}\left(\frac{a+3b}{4}\right)\right]\\&\quad\cdot\frac{\left[3^{i+2}-(-1)^{i+2}\right](b-a)^{i+1}}{4^{i+2}(i+2)!}\\&\quad-\frac{1}{3}\sum_{i=1}^{n-2} f^{(i+1)}\left(\frac{a+b}{2}\right)\frac{\left(1-(-1)^i\right)(b-a)^{i+1}}{2^{i+2}(i+2)!}\end{aligned}$$

and

$$\begin{aligned}&T_n^3\left(\frac{3a+b}{4},s\right)\\&= -\frac{n}{3}\left[2T_n\left(\frac{3a+b}{4},s\right) - T_n\left(\frac{a+b}{2},s\right) + 2T_n\left(\frac{a+3b}{4},s\right)\right]\\&= \begin{cases}\dfrac{(a-s)^n}{b-a}, & a \le s \le \dfrac{3a+b}{4},\\ \dfrac{(a-s)^n + 2(b-s)^n}{3(b-a)}, & \dfrac{3a+b}{4} < s \le \dfrac{a+b}{2},\\ \dfrac{2(a-s)^n + (b-s)^n}{3(b-a)}, & \dfrac{a+b}{2} < s \le \dfrac{a+3b}{4},\\ \dfrac{(b-s)^n}{b-a}, & \dfrac{a+3b}{4} < s \le b.\end{cases}\end{aligned}$$

Corollary 3.32 *Suppose that all the assumptions of Theorem 3.17 hold. Additionally, assume that (p,q) is a pair of conjugate exponents, that is $1 \le p,q \le \infty$, $\frac{1}{p}+\frac{1}{q} = 1$ and $n \in \mathbb{N}$.*

(a) If $f^{(n)} \in L^\infty[a,b]$, then

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D_3\left(\frac{3a+b}{4}\right) - t_n^3\left(\frac{3a+b}{4}\right)\right| \le \frac{1}{(n+1)!}\\&\cdot\left(\frac{\left[3\left(2\cdot 3^n+1\right) - 2^{n+1}\left(2+(-1)^{n+1}\right) + (-1)^{n+1}\right](b-a)^n}{3\cdot 2^{2n+1}}\right.\end{aligned}$$

$$\cdot \left\| f^{(n)} \right\|_{\infty}.$$

(b) If $f^{(n)} \in L^2[a,b]$, then

$$\left| \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t - D_3\left(\frac{3a+b}{4}\right) - t_n^3\left(\frac{3a+b}{4}\right) \right|$$
$$\leq \frac{1}{n!} \left(\frac{\left[9\left(4\cdot 3^{2n-1}+1\right) - 3\cdot 2^{2n+1} - 1\right](b-a)^{2n-1}}{9\cdot 2^{4n+1}(2n+1)} \right.$$
$$\left. + \frac{4(-1)^n (b-a)^{2n-1}}{9} \left[B_{\frac{3}{4}}(n+1,n+1) - B_{\frac{1}{4}}(n+1,n+1)\right] \right)^{\frac{1}{2}}$$
$$\cdot \left\| f^{(n)} \right\|_2.$$

(c) If $f^{(n)} \in L^1[a,b]$, then

$$\left| \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t - D_3\left(\frac{3a+b}{4}\right) - t_n^3\left(\frac{3a+b}{4}\right) \right|$$
$$\leq \frac{\left(2\cdot 3^n + (-1)^n\right)(b-a)^{n-1}}{3\cdot 2^{2n}\cdot n!} \left\| f^{(n)} \right\|_1.$$

The first and the second inequality are sharp and the third inequality is the best possible.

Proof. Applying (3.48) with $x = \frac{3a+b}{4}$ and $p = \infty$, $p = 2$, $p = 1$,respectively, we get the above inequalities. Since function $T_n^3\left(\frac{3a+b}{4},\cdot\right)$ is left continuous and has finite jump at $\frac{3a+b}{4}$, $\frac{a+b}{2}$, $\frac{a+3b}{4}$ the proof of the best possibility of the third inequality is similar to the proof of Theorem 3.2 □

Maclaurin's formula

If all the assumptions of Theorem 3.17 hold for

$$A(x) = \frac{(b-a)^2}{6(a+b-2x)^2}.$$

and $x = \frac{5a+b}{6}$ then the generalization of Maclaurin's formula reads

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t$$
$$= D_3\left(\frac{5a+b}{6}\right) + t_n^3\left(\frac{5a+b}{6}\right) + \frac{1}{n!}\int_a^b T_n^3\left(\frac{5a+b}{6},s\right) f^{(n)}(s)\,\mathrm{d}s, \tag{3.51}$$

where

$$D_3\left(\frac{5a+b}{6}\right)=\frac{1}{8}\left(3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right),$$

$$\begin{aligned}t_n^3\left(\frac{5a+b}{6}\right) &= \frac{3}{8}\sum_{i=0}^{n-2}\left[f^{(i+1)}\left(\frac{5a+b}{6}\right)+(-1)^{i+1}f^{(i+1)}\left(\frac{a+5b}{6}\right)\right]\\ &\quad\cdot\frac{\left[5^{i+2}-(-1)^{i+2}\right](b-a)^{i+1}}{6^{i+2}(i+2)!}\\ &\quad+\frac{1}{4}\sum_{i=1}^{n-2}f^{(i+1)}\left(\frac{a+b}{2}\right)\frac{\left(1-(-1)^i\right)(b-a)^{i+1}}{2^{i+2}(i+2)!}\end{aligned}$$

and

$$\begin{aligned}&T_n^3\left(\frac{5a+b}{6},s\right)\\ &=-\frac{n}{8}\left[3T_n\left(\frac{5a+b}{6},s\right)+2T_n\left(\frac{a+b}{2},s\right)+3T_n\left(\frac{a+5b}{6},s\right)\right]\\ &=\begin{cases}\dfrac{(a-s)^n}{b-a}, & a\le s\le \dfrac{5a+b}{6},\\ \dfrac{5(a-s)^n+3(b-s)^n}{8(b-a)}, & \dfrac{5a+b}{6}<s\le\dfrac{a+b}{2},\\ \dfrac{3(a-s)^n+5(b-s)^n}{8(b-a)}, & \dfrac{a+b}{2}<s\le\dfrac{a+5b}{6},\\ \dfrac{(b-s)^n}{b-a}, & \dfrac{a+5b}{6}<s\le b.\end{cases}\end{aligned}$$

Corollary 3.33 *Suppose that all the assumptions of Theorem 3.17 hold. Additionally, assume that (p,q) is a pair of conjugate exponents, that is $1\le p,q\le\infty$, $\frac{1}{p}+\frac{1}{q}=1$ and $n\in\mathbb{N}$.*

(a) If $f^{(n)}\in L^\infty[a,b]$, then

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t-D_3\left(\frac{5a+b}{6}\right)-t_n^3\left(\frac{5a+b}{6}\right)\right|\le\frac{(b-a)^n}{(n+1)!}\\ &\cdot\left(\frac{5\left(3\cdot5^n-(-1)^n\right)+2\left(4+3^{n+1}\right)+\left(1+(-1)^{n+1}\right)3^{n+2}}{2^{n+3}\cdot3^{n+1}}\right.\\ &\left.-\frac{15\left(1+(-1)^{n+1}\right)}{4\left(\sqrt[n]{3}+\sqrt[n]{5}\right)^n}\right)\left\|f^{(n)}\right\|_\infty.\end{aligned}$$

(b) If $f^{(n)} \in L^2[a,b]$, then

$$\left| \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t - D_3\left(\frac{5a+b}{6}\right) - t_n^3\left(\frac{5a+b}{6}\right) \right|$$
$$\le \frac{1}{n!}\left(\frac{\left[25\left(9\cdot 5^{2n-1}-1\right)+16\cdot 3^{2n+1}+64\right](b-a)^{2n-1}}{2^{2n+6}\cdot 3^{2n+1}(2n+1)} \right.$$
$$\left. + \frac{15(-1)^n (b-a)^{2n-1}}{32}\left[B_{\frac{5}{6}}(n+1,n+1) - B_{\frac{1}{6}}(n+1,n+1)\right]\right)^{\frac{1}{2}}$$
$$\cdot \left\| f^{(n)} \right\|_2 .$$

(c) If $f^{(n)} \in L^1[a,b]$, then

$$\left| \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t - D_3\left(\frac{5a+b}{6}\right) - t_n^3\left(\frac{5a+b}{6}\right) \right|$$
$$\le \frac{5\left(3\cdot 5^{n-1}+(-1)^n\right)(b-a)^{n-1}}{2^{n+3}\cdot 3^n\cdot n!}\left\| f^{(n)} \right\|_1 .$$

The first and the second inequality are sharp and the third inequality is the best possible.

Proof. Applying (3.48) with $x = \frac{5a+b}{6}$ and $p = \infty$, $p = 2$, $p = 1$ we obtain the above inequalities. Since function $T_n^3\left(\frac{5a+b}{6},\cdot\right)$ is left continuous and has finite jump at $\frac{5a+b}{6}$, $\frac{a+b}{2}$, $\frac{a+5b}{6}$ the proof of the best possibility of the third inequality is similar to Theorem 3.2 □

3.2.2 General three-point quadrature formulae with applications for α-L-Hölder type functions

In this subsection, using identities (2.81) and (2.82), we consider a general three-point quadrature formula (3.39) and give various error estimates for α-L-Hölder type functions. From the general non-weighted formula we get generalizations of the well-known Simpson's formula ($x = a$), dual Simpson's formula ($x = (3a+b)/4$) and Maclaurin's formula ($x = (5a+b)/6$).

The results from this subsection are published in [69].

Variant I of general three-point formula

Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ exists on $[a,b]$ for some $n \ge 2$. We introduce the following notation for each $x \in \left[a, \frac{a+b}{2}\right)$

$$s_{w,n}^3(x) = A(x)\left\{ \sum_{i=0}^{n-1} \frac{f^{(i+1)}(b)}{i!} \left[\int_x^b (1-W(t))(t-b)^i\,\mathrm{d}t \right.\right.$$
$$\left. + \int_{a+b-x}^b (1-W(t))(t-b)^i\,\mathrm{d}t \right]$$

$$
\begin{aligned}
&-\sum_{i=0}^{n-1}\frac{f^{(i+1)}(a)}{i!}\left[\int_a^x W(t)(t-a)^i\,\mathrm{d}t+\int_a^{a+b-x}W(t)(t-a)^i\,\mathrm{d}t\right]\Bigg\}\\
&+(1-2A(x))\left\{\sum_{i=0}^{n-1}\frac{f^{(i+1)}(b)}{i!}\int_{\frac{a+b}{2}}^{b}(1-W(t))(t-b)^i\,\mathrm{d}t\right.\\
&\left.-\sum_{i=0}^{n-1}\frac{f^{(i+1)}(a)}{i!}\int_a^{\frac{a+b}{2}}W(t)(t-a)^i\,\mathrm{d}t\right\} \qquad (3.52)
\end{aligned}
$$

and

$$
\begin{aligned}
S_{w,n}^3(x) &= A(x)\left[S_{w,n}^a(x)+S_{w,n}^b(x)+S_{w,n}^a(a+b-x)+S_{w,n}^b(a+b-x)\right]\\
&+ (1-2A(x))\left[S_{w,n}^a\left(\frac{a+b}{2}\right)+S_{w,n}^b\left(\frac{a+b}{2}\right)\right],
\end{aligned}
$$

where $S_{w,n}^a$ and $S_{w,n}^b$ are defined by (3.22), and $D_3(x)$ are as in (3.40).

In the next theorem we establish the first variant of the generalized three-point quadrature formula based on the generalized Montgomery identity (2.81).

Theorem 3.19 *Let I be an open interval in $\mathbb{R}$, $[a,b]\subset I$, and $w:[a,b]\to[0,\infty\rangle$ some probability density function. Let $f:I\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n\geq 2$. Then for each $x\in\left[a,\frac{a+b}{2}\right)$ the following identity holds*

$$
\int_a^b w(t)f(t)\,\mathrm{d}t = D_3(x)+s_{w,n}^3(x)+S_{w,n}^3(x). \qquad (3.53)
$$

Proof. We put $x\equiv x, x\equiv\frac{a+b}{2}$ and $x\equiv a+b-x$ in (2.81) to obtain three new formulae. After multiplying these three formulae by $A(x), 1-2A(x), A(x)$, respectively, and adding, we obtain (3.53). □

Theorem 3.20 *Let I be an open interval in $\mathbb{R}$, $[a,b]\subset I$ and $w:[a,b]\to[0,\infty)$ some probability density function. Let $f:I\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous and that $f^{(n)}:[a,b]\to\mathbb{R}$ is an α-L-Hölder type function for some $n\geq 2$, $\alpha\in(0,1]$ and $L\in\mathbb{R}^+$. Then for each $x\in\left[a,\frac{a+b}{2}\right)$ the following inequalities hold*

$$
\begin{aligned}
&\left|\int_a^b w(t)f(t)\,\mathrm{d}t-D_3(x)-s_{w,n}^3(x)\right|\\
&\leq\frac{B(\alpha+1,n-1)L}{(n-2)!}\left\{|A(x)|\int_a^b W(x,t)U_n(x,t)\,\mathrm{d}t\right.\\
&+|A(x)|\int_a^b W(a+b-x,t)U_n(a+b-x,t)\,\mathrm{d}t\\
&\left.+|1-2A(x)|\int_a^b W\left(\frac{a+b}{2},t\right)U_n\left(\frac{a+b}{2},t\right)\mathrm{d}t\right\}\\
&\leq\frac{2B(\alpha+1,n-1)L}{(\alpha+n)(n-2)!}\left\{|A(x)|\left[(x-a)^{\alpha+n}+(b-x)^{\alpha+n}\right]\right.
\end{aligned}
$$

$$+|1-2A(x)|\left(\frac{b-a}{2}\right)^{\alpha+n}\Bigg\},$$

where $W(x,t)$ and $U_n(x,t)$ are as in (3.24).

Proof. From (3.53) we have

$$\begin{aligned}
&\left|\int_a^b w(t)f(t)\,\mathrm{d}t - D_3(x) - s_{w,n}^3(x)\right| \\
&= \Bigg|A(x)\left[S_{w,n}^a(x)+S_{w,n}^b(x)+S_{w,n}^a(a+b-x)+S_{w,n}^b(a+b-x)\right] \\
&+(1-2A(x))\left[S_{w,n}^a\left(\frac{a+b}{2}\right)+S_{w,n}^b\left(\frac{a+b}{2}\right)\right]\Bigg| \\
&\le |A(x)|\left[\left|S_{w,n}^a(x)\right|+\left|S_{w,n}^b(x)\right|+\left|S_{w,n}^a(a+b-x)\right|+\left|S_{w,n}^b(a+b-x)\right|\right] \\
&+|1-2A(x)|\left[\left|S_{w,n}^a\left(\frac{a+b}{2}\right)\right|+\left|S_{w,n}^b\left(\frac{a+b}{2}\right)\right|\right].
\end{aligned} \tag{3.54}$$

Since $f^{(n)}$ is an α-L-Hölder type function, from (3.54) we obtain

$$\begin{aligned}
&\left|\int_a^b w(t)f(t)\,\mathrm{d}t - D_3(x) - s_{w,n}^3(x)\right| \\
&\le \frac{|A(x)|}{(n-2)!}L\Bigg\{\int_a^x W(t)\left[\int_a^t (s-a)^\alpha (t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t \\
&+\int_a^{a+b-x} W(t)\left[\int_a^t (s-a)^\alpha (t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t \\
&+\int_x^b (1-W(t))\left[\int_t^b (b-s)^\alpha (s-t)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t \\
&+\int_{a+b-x}^b (1-W(t))\left[\int_t^b (b-s)^\alpha (s-t)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t\Bigg\} \\
&+\frac{|1-2A(x)|}{(n-2)!}L\Bigg\{\int_a^{\frac{a+b}{2}} W(t)\left[\int_a^t (s-a)^\alpha (t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t \\
&+\int_{\frac{a+b}{2}}^b (1-W(t))\left[\int_t^b (b-s)^\alpha (s-t)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t\Bigg\}.
\end{aligned} \tag{3.55}$$

The first integral over ds in (3.55) can be written as

$$\begin{aligned}
&\int_a^t (s-a)^\alpha (t-s)^{n-2}\,\mathrm{d}s \\
&= (t-a)^{\alpha+n-2}\int_a^t \left(\frac{s-a}{t-a}\right)^\alpha \left(\frac{t-s}{t-a}\right)^{n-2}\mathrm{d}s \\
&= \left[u=\frac{s-a}{t-a}\right] = (t-a)^{\alpha+n-1}\int_0^1 u^\alpha (1-u)^{n-2}\,\mathrm{d}u
\end{aligned}$$

$$= (t-a)^{\alpha+n-1} B(\alpha+1, n-1).$$

Similar can be done with other integrals in (3.55). Hence we obtain

$$\begin{aligned}
&\left| \int_a^b w(t) f(t)\,\mathrm{d}t - D_3(x) - s^3_{w,n}(x) \right| \\
&\le \frac{B(\alpha+1,n-1)}{(n-2)!} L \left\{ |A(x)| \int_a^b W(x,t) U_n(x,t)\,\mathrm{d}t \right. \\
&+ |A(x)| \int_a^b W(a+b-x,t) U_n(a+b-x,t)\,\mathrm{d}t \\
&\left. + |1-2A(x)| \int_a^b W\left(\frac{a+b}{2},t\right) U_n\left(\frac{a+b}{2},t\right)\mathrm{d}t \right\}.
\end{aligned} \tag{3.56}$$

Since we have $0 \le W(t) \le 1$, $t \in [a,b]$, from (3.56) we obtain

$$\begin{aligned}
&\frac{B(\alpha+1,n-1)}{(n-2)!} L \left\{ |A(x)| \int_a^b W(x,t) U_n(x,t)\,\mathrm{d}t \right. \\
&+ |A(x)| \int_a^b W(a+b-x,t) U_n(a+b-x,t)\,\mathrm{d}t \\
&\left. + |1-2A(x)| \int_a^b W\left(\frac{a+b}{2},t\right) U_n\left(\frac{a+b}{2},t\right)\mathrm{d}t \right\} \\
&\le \frac{2B(\alpha+1,n-1)}{(\alpha+n)(n-2)!} L \left\{ |A(x)| \left[(x-a)^{\alpha+n} + (b-x)^{\alpha+n}\right] \right. \\
&\left. + |1-2A(x)| \left(\frac{b-a}{2}\right)^{\alpha+n} \right\},
\end{aligned}$$

which completes the proof. □

Variant I of non-weighted three-point formula

Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ exists on $[a,b]$ for some $n \ge 2$. Then for each $x \in \left[a, \frac{a+b}{2}\right)$ we define

$$\begin{aligned}
s^3_n(x) \;&=\; A(x) \sum_{i=0}^{n-1} \left[(-1)^i f^{(i+1)}(b) - f^{(i+1)}(a)\right] \frac{(x-a)^{i+2} + (b-x)^{i+2}}{i!\,(i+2)(b-a)} \\
&+\; (1-2A(x)) \sum_{i=0}^{n-1} \left[(-1)^i f^{(i+1)}(b) - f^{(i+1)}(a)\right] \frac{(b-a)^{i+1}}{i!\,(i+2)2^{i+2}}
\end{aligned}$$

and

$$\begin{aligned}
S^3_n(x) \;&=\; A(x) \left[S^a_n(x) + S^b_n(x) + S^a_n(a+b-x) + S^b_n(a+b-x)\right] \\
&+\; (1-2A(x)) \left[S^a_n\left(\frac{a+b}{2}\right) + S^b_n\left(\frac{a+b}{2}\right)\right],
\end{aligned}$$

where

$$\begin{aligned}
&S_n^a(x) \\
&= \frac{1}{(n-2)!\,(b-a)} \int_a^x (t-a) \left[\int_a^t \left(f^{(n)}(a) - f^{(n)}(s) \right) (t-s)^{n-2} \mathrm{d}s \right] \mathrm{d}t, \\
&S_n^b(x) \\
&= \frac{1}{(n-2)!\,(b-a)} \int_x^b (b-t) \left[\int_t^b \left(f^{(n)}(b) - f^{(n)}(s) \right) (t-s)^{n-2} \mathrm{d}s \right] \mathrm{d}t.
\end{aligned} \tag{3.57}$$

Corollary 3.34 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$. Then for each $x \in \left[a, \frac{a+b}{2}\right)$ the following identity holds*

$$\frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t = D_3(x) + s_n^3(x) + S_n^3(x). \tag{3.58}$$

Proof. This is a special case of Theorem 3.19 for $w(t) = \frac{1}{b-a}$, $t \in [a,b]$. □

Corollary 3.35 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$ and let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous and that $f^{(n)} : [a,b] \to \mathbb{R}$ is an α-L-Hölder type function for some $n \geq 2$, $\alpha \in (0,1]$ and $L \in \mathbb{R}^+$. Then for each $x \in \left[a, \frac{a+b}{2}\right)$ the following inequality holds*

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t - D_3(x) - s_n^3(x) \right| \\
&\leq \frac{2B(\alpha+1, n-1)}{(b-a)(\alpha+n+1)(n-2)!} L \left\{ |A(x)| \left[(x-a)^{\alpha+n+1} + (b-x)^{\alpha+n+1} \right] \right. \\
&\left. + |1-2A(x)| \left(\frac{b-a}{2} \right)^{\alpha+n+1} \right\}.
\end{aligned}$$

Proof. This is a special case of Theorem 3.20 for $w(t) = \frac{1}{b-a}$, $t \in [a,b]$. □

Variant II of general three-point formula

Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ exists on $[a,b]$ for some $n \geq 2$. We introduce the following notation for each $x \in \left[a, \frac{a+b}{2}\right)$

$$\begin{aligned}
r_{w,n}^3(x) = &-A(x) \sum_{i=0}^{n-1} \left[\frac{f^{(i+1)}(x)}{i!} \int_a^b P_w(x,t)(t-x)^i \,\mathrm{d}t \right. \\
&\left. + \frac{f^{(i+1)}(a+b-x)}{i!} \int_a^b P_w(a+b-x,t)(t-a-b+x)^i \,\mathrm{d}t \right] \\
&- (1-2A(x)) \sum_{i=0}^{n-1} \frac{f^{(i+1)}\left(\frac{a+b}{2}\right)}{i!} \int_a^b P_w\left(\frac{a+b}{2}, t\right) \left(t - \frac{a+b}{2}\right)^i \mathrm{d}t
\end{aligned}$$

and

$$R_{w,n}^{3}(x)=A(x)\left[R_{w,n}(x)+R_{w,n}(a+b-x)\right]+(1-2A(x))R_{w,n}\left(\frac{a+b}{2}\right),$$

where $R_{w,n}$ is defined by (3.32) and $D_3(x)$ are as in (3.40).

Theorem 3.21 *Let I be an open interval in $\mathbb{R}$, $[a,b]\subset I$, and $w:[a,b]\to[0,\infty\rangle$ some probability density function. Let $f:I\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n\geq 2$. Then for each $x\in\left[a,\frac{a+b}{2}\right)$ the following identity holds*

$$\int_a^b w(t)f(t)\,\mathrm{d}t=D_3(x)+r_{w,n}^{3}(x)+R_{w,n}^{3}(x). \tag{3.59}$$

Proof. We put $x\equiv x, x\equiv\frac{a+b}{2}$ and $x\equiv a+b-x$ in (2.82) to obtain three new formulae. After multiplying these three formulae by $A(x), 1-2A(x)$, $A(x)$, respectively, and adding, we get (3.59). □

Theorem 3.22 *Let I be an open interval in $\mathbb{R}$, $[a,b]\subset I$ and $w:[a,b]\to[0,\infty)$ some probability density function. Let $f:I\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous and that $f^{(n)}:[a,b]\to\mathbb{R}$ is an α-L-Hölder type function for some $n\geq 2$, $\alpha\in(0,1]$ and $L\in\mathbb{R}^{+}$. Then for each $x\in\left[a,\frac{a+b}{2}\right)$ the following inequalities hold*

$$\begin{aligned}&\left|\int_a^b w(t)f(t)\,\mathrm{d}t-D_3(x)-r_{w,n}^{3}(x)\right|\\&\leq\frac{B(\alpha+1,n-1)}{(n-2)!}L\left\{|A(x)|\left[\int_a^b|P_w(x,t)|\,|x-t|^{\alpha+n-1}\,\mathrm{d}t\right.\right.\\&\left.+\int_a^b|P_w(a+b-x,t)|\,|a+b-x-t|^{\alpha+n-1}\,\mathrm{d}t\right]\\&\left.+|1-2A(x)|\int_a^b\left|P_w\left(\frac{a+b}{2},t\right)\right|\left|\frac{a+b}{2}-t\right|^{\alpha+n-1}\mathrm{d}t\right\}\\&\leq\frac{2B(\alpha+1,n-1)}{(\alpha+n)(n-2)!}L\left\{|A(x)|\left[(x-a)^{\alpha+n}+(b-x)^{\alpha+n}\right]\right.\\&\left.+|1-2A(x)|\left(\frac{b-a}{2}\right)^{\alpha+n}\right\}.\end{aligned}$$

Proof. From (3.59) we have that

$$\left|\int_a^b w(t)f(t)\,\mathrm{d}t-D_3(x)-r_{w,n}^{3}(x)\right|=\left|R_{w,n}^{3}(x)\right|. \tag{3.60}$$

Since $f^{(n)}$ is an α-L-Hölder type function, from (3.60) we obtain

$$\left|\int_a^b w(t)f(t)\,\mathrm{d}t-D_3(x)-r_{w,n}^{3}(x)\right|$$

$$
\begin{aligned}
&\leq \frac{L}{(n-2)!}\left\{|A(x)|\left[\int_a^b |P_w(x,t)|\left|\int_x^t \left|(s-x)^{\alpha}(t-s)^{n-2}\right|\mathrm{d}s\right|\mathrm{d}t\right.\right.\\
&+\left.\int_a^b |P_w(a+b-x,t)|\left|\int_{a+b-x}^t \left|(s-a-b+x)^{\alpha}(t-s)^{n-2}\right|\mathrm{d}s\right|\mathrm{d}t\right]\\
&+\left.|1-2A(x)|\int_a^b \left|P_w\left(\frac{a+b}{2},t\right)\right|\left|\int_{\frac{a+b}{2}}^t \left|\left(s-\frac{a+b}{2}\right)^{\alpha}(t-s)^{n-2}\right|\mathrm{d}s\right|\mathrm{d}t\right\}.
\end{aligned} \tag{3.61}
$$

From (3.61), similarly to Theorem 3.20 we get

$$
\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D_3(x) - r_{w,n}^3(x)\right|\\
&\leq \frac{B(\alpha+1,n-1)}{(n-2)!}L\left\{|A(x)|\left[\int_a^x W(t)(x-t)^{\alpha+n-1}\,\mathrm{d}t\right.\right.\\
&+\int_a^{a+b-x} W(t)(a+b-x-t)^{\alpha+n-1}\,\mathrm{d}t+\int_x^b (1-W(t))(t-x)^{\alpha+n-1}\,\mathrm{d}t\\
&+\left.\int_{a+b-x}^b (1-W(t))(t-a-b+x)^{\alpha+n-1}\,\mathrm{d}t\right]\\
&+|1-2A(x)|\left[\int_a^{\frac{a+b}{2}} W(t)\left(\frac{a+b}{2}-t\right)^{\alpha+n-1}\mathrm{d}t\right.\\
&+\left.\left.\int_{\frac{a+b}{2}}^b (1-W(t))\left(t-\frac{a+b}{2}\right)^{\alpha+n-1}\mathrm{d}t\right]\right\}\\
&\leq \frac{2B(\alpha+1,n-1)}{(\alpha+n)(n-2)!}L\left\{|A(x)|\left[(x-a)^{\alpha+n}+(b-x)^{\alpha+n}\right]\right.\\
&+\left.|1-2A(x)|\left(\frac{b-a}{2}\right)^{\alpha+n}\right\}
\end{aligned}
$$

and this completes the proof. □

Variant II of non-weighted three-point formula

Here we define

$$
\begin{aligned}
r_n^3(x) &= A(x)\sum_{i=0}^{n-1}\left[f^{(i+1)}(x)+(-1)^{i+1}f^{(i+1)}(a+b-x)\right]\\
&\quad\cdot\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!\,(b-a)}\\
&+\ (1-2A(x))\sum_{i=0}^{n-1} f^{(i+1)}\left(\frac{a+b}{2}\right)\frac{\left(1+(-1)^{i+1}\right)(b-a)^{i+1}}{2^{i+2}(i+2)!}
\end{aligned}
$$

and

$$R_n^3(x) = A(x)[R_n(x) + R_n(a+b-x)] + (1-2A(x))R_n\left(\frac{a+b}{2}\right),$$

where

$$R_n(x) = \frac{1}{(n-2)!}\int_a^b P(x,t)\left(\int_x^t \left(f^{(n)}(x) - f^{(n)}(s)\right)(t-s)^{n-2}\,\mathrm{d}s\right)\mathrm{d}t, \tag{3.62}$$

and

$$P(x,t) = \begin{cases} \frac{t-a}{b-a}, & a \le t \le x, \\ \frac{t-b}{b-a}, & x < t \le b. \end{cases}$$

Corollary 3.36 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \ge 2$. Then for each $x \in \left[a, \frac{a+b}{2}\right)$ the following identity holds*

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t = D_3(x) + r_n^3(x) + R_n^3(x).$$

Proof. This is a special case of Theorem 3.21 for $w(t) = \frac{1}{b-a}$, $t \in [a,b]$. □

Corollary 3.37 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$ and let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous and that $f^{(n)} : [a,b] \to \mathbb{R}$ is an α-L-Hölder type function for some $n \ge 2$, $\alpha \in (0,1]$ and $L \in \mathbb{R}^+$. Then for each $x \in \left[a, \frac{a+b}{2}\right)$ the following inequality holds*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D_3(x) - r_n^3(x)\right| \\ &\le \frac{2B(\alpha+1, n-1)L}{(b-a)(\alpha+n)(\alpha+n+1)(n-2)!} \\ &\cdot \left\{|A(x)|\left[(x-a)^{\alpha+n+1} + (b-x)^{\alpha+n+1}\right] + |1-2A(x)|\left(\frac{b-a}{2}\right)^{\alpha+n+1}\right\}. \end{aligned}$$

Proof. This is a special case of Theorem 3.22 for $w(t) = \frac{1}{b-a}$, $t \in [a,b]$. □

Simpson's formula

Now, we set

$$A(x) = \frac{(b-a)^2}{6(a+b-2x)^2}, \; x \in \left[a, \frac{a+b}{2}\right).$$

This special choice of the function A enables us to establish our generalizations of the well-known Simpson's formula $(x = a)$. We will also show how to apply the results of the previous subsections to obtain some error estimates for these quadrature rules if they involve α-L-Hölder type functions.

Suppose that all the assumptions of Corollary 3.34 and Corollary 3.36 hold. Then the following two generalizations of Simpson's formula read

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t = D_3(a) + s_n^3(a) + S_n^3(a),$$

and

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t = D_3(a) + r_n^3(a) + R_n^3(a),$$

where

$$D_3(a) = \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right],$$

$$s_n^3(a) = \frac{1}{6}\sum_{i=0}^{n-1}\left[(-1)^i f^{(i+1)}(b) - f^{(i+1)}(a)\right]\frac{\left(2^i+1\right)(b-a)^{i+1}}{2^i i!\,(i+2)}$$

and

$$\begin{aligned} r_n^3(a) &= \frac{1}{6}\sum_{i=0}^{n-1}\left[f^{(i+1)}(a) + (-1)^{i+1} f^{(i+1)}(b)\right]\frac{(b-a)^{i+1}}{(i+2)!} \\ &+ \frac{2}{3}\sum_{i=0}^{n-1} f^{(i+1)}\left(\frac{a+b}{2}\right)\frac{\left(1+(-1)^{i+1}\right)(b-a)^{i+1}}{2^{i+2}(i+2)!}. \end{aligned}$$

Corollary 3.38 *Suppose that all the assumptions of Corollary 3.35 hold. Then the following inequality holds*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D_3(a) - s_n^3(a)\right| \\ &\leq \frac{B(\alpha+1, n-1)\left(2^{\alpha+n-1}+1\right)(b-a)^{\alpha+n}}{3\cdot 2^{\alpha+n-1}(\alpha+n+1)(n-2)!}L. \end{aligned}$$

Proof. This is a special case of Corollary 3.35 for $x = a$. □

For example, if in Corollary 3.38 we have $n = 2$, we obtain this estimation

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - s_2^3(a)\right| \\ &\leq \frac{\left(2^{\alpha+1}+1\right)(b-a)^{\alpha+2}}{3\cdot 2^{\alpha+1}(\alpha+1)(\alpha+3)}L, \end{aligned}$$

where

$$s_2^3(a) = \left(f'(b) - f'(a)\right)\frac{b-a}{6} - \left(f''(b) + f''(a)\right)\frac{(b-a)^2}{12}.$$

Corollary 3.39 *Suppose that all the assumptions of Corollary 3.37 hold. Then we have*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D_3(a) - r_n^3(a)\right| \\
&\le \frac{B(\alpha+1,n-1)\left(2^{\alpha+n-1}+1\right)(b-a)^{\alpha+n}}{3\cdot 2^{\alpha+n-1}(\alpha+n)(\alpha+n+1)(n-2)!}L.
\end{aligned}$$

Proof. This is a special case of Corollary 3.37 for $x=a$. □

For example, if in Corollary 3.39 we have $n=2$ we obtain the following estimation

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] - r_2^3(a)\right| \\
&\le \frac{\left(2^{\alpha+1}+1\right)(b-a)^{\alpha+2}L}{3\cdot 2^{\alpha+1}(\alpha+1)(\alpha+2)(\alpha+3)},
\end{aligned}$$

where

$$r_2^3(a) = \left(f'(a)-f'(b)\right)\frac{b-a}{12} + \left(f''(a)+f''\left(\frac{a+b}{2}\right)+f''(b)\right)\frac{(b-a)^2}{36}.$$

Dual Simpson's formula

As special cases of Corollary 3.34 and Corollary 3.36 for $A(x) = \frac{(b-a)^2}{6(a+b-2x)^2}$ and $x = \frac{3a+b}{4}$ we have two generalizations of the dual Simpson's formula

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t = D_3\left(\frac{3a+b}{4}\right) + s_n^3\left(\frac{3a+b}{4}\right) + S_n^3\left(\frac{3a+b}{4}\right),$$

and

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t = D_3\left(\frac{3a+b}{4}\right) + r_n^3\left(\frac{3a+b}{4}\right) + R_n^3\left(\frac{3a+b}{4}\right),$$

where

$$D_3\left(\frac{3a+b}{4}\right) = \frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right],$$

$$\begin{aligned}
&s_n^3\left(\frac{3a+b}{4}\right) \\
&= \frac{1}{3}\sum_{i=0}^{n-1}\left[(-1)^i f^{(i+1)}(b) - f^{(i+1)}(a)\right]\frac{\left[2\left(3^{i+2}+1\right)-2^{i+2}\right](b-a)^{i+1}}{4^{i+2}i!(i+2)}
\end{aligned}$$

and

$$r_n^3\left(\frac{3a+b}{4}\right) = \frac{2}{3}\sum_{i=0}^{n-1}\left[f^{(i+1)}\left(\frac{3a+b}{4}\right) + (-1)^{i+1} f^{(i+1)}\left(\frac{a+3b}{4}\right)\right]$$

$$\cdot \frac{\left(3^{i+2}+(-1)^{i+1}\right)(b-a)^{i+1}}{4^{i+2}(i+2)!}$$
$$-\frac{1}{3}\sum_{i=0}^{n-1} f^{(i+1)}\left(\frac{a+b}{2}\right)\frac{\left(1+(-1)^{i+1}\right)(b-a)^{i+1}}{2^{i+2}(i+2)!}.$$

Corollary 3.40 *Suppose that all the assumptions of Corollary 3.35 hold. Then we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D_3\left(\frac{3a+b}{4}\right) - s_n^3\left(\frac{3a+b}{4}\right)\right|$$
$$\le \frac{B(\alpha+1,n-1)\left(3^{\alpha+n+1}+2^{\alpha+n}+1\right)(b-a)^{\alpha+n}}{3\cdot 4^{\alpha+n}(\alpha+n+1)(n-2)!}L.$$

Proof. This is a special case of Corollary 3.35 for $x=\frac{3a+b}{4}$. □

Corollary 3.41 *Suppose that all the assumptions of Corollary 3.37 hold. Then we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D_3\left(\frac{3a+b}{4}\right) - r_n^3\left(\frac{3a+b}{4}\right)\right|$$
$$\le \frac{B(\alpha+1,n-1)\left(3^{\alpha+n+1}+2^{\alpha+n}+1\right)(b-a)^{\alpha+n}}{3\cdot 4^{\alpha+n}(\alpha+n)(\alpha+n+1)(n-2)!}L.$$

Proof. This is a special case of Corollary 3.37 for $x=\frac{3a+b}{4}$. □

Maclaurin's formula

If we put $A(x)=\frac{(b-a)^2}{6(a+b-2x)^2}$ and $x=\frac{5a+b}{6}$ in Corollary 3.34 and Corollary 3.36 we get the following generalizations of Maclaurin's formula

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t = D_3\left(\frac{5a+b}{6}\right)+s_n^3\left(\frac{5a+b}{6}\right)+S_n^3\left(\frac{5a+b}{6}\right),$$

and

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t = D_3\left(\frac{5a+b}{6}\right)+r_n^3\left(\frac{5a+b}{6}\right)+R_n^3\left(\frac{5a+b}{6}\right),$$

where

$$D_3\left(\frac{5a+b}{6}\right)=\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right],$$

$$s_n^3\left(\frac{5a+b}{6}\right)$$
$$=\frac{3}{8}\sum_{i=0}^{n-1}\left[(-1)^i f^{(i+1)}(b)-f^{(i+1)}(a)\right]\frac{\left[2\cdot 3^{i+1}+5^{i+2}+1\right](b-a)^{i+1}}{6^{i+2}\cdot i!\,(i+2)}$$

and

$$\begin{aligned}
r_n^3\left(\frac{5a+b}{6}\right) &= \frac{3}{8}\sum_{i=0}^{n-1}\left[f^{(i+1)}\left(\frac{5a+b}{6}\right)+(-1)^{i+1}f^{(i+1)}\left(\frac{a+5b}{6}\right)\right] \\
&\quad\cdot\frac{\left(5^{i+2}+(-1)^{i+1}\right)(b-a)^{i+1}}{6^{i+2}(i+2)!} \\
&\quad+\frac{1}{4}\sum_{i=0}^{n-1}f^{(i+1)}\left(\frac{a+b}{2}\right)\frac{\left(1+(-1)^{i+1}\right)(b-a)^{i+1}}{2^{i+2}(i+2)!}.
\end{aligned}$$

Corollary 3.42 *Suppose that all the assumptions of Corollary 3.35 hold. Then we have*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t-D_3\left(\frac{5a+b}{6}\right)-s_n^3\left(\frac{5a+b}{6}\right)\right| \\
&\le\frac{B(\alpha+1,n-1)\left(5^{\alpha+n+1}+2\cdot 3^{\alpha+n}+1\right)(b-a)^{\alpha+n}}{8\cdot 6^{\alpha+n}(\alpha+n+1)(n-2)!}L.
\end{aligned}$$

Proof. This is a special case of Corollary 3.35 for $x=\frac{5a+b}{6}$. □

Corollary 3.43 *Suppose that all the assumptions of Corollary 3.37 hold. Then we have*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t-D_3\left(\frac{5a+b}{6}\right)-r_n^3\left(\frac{5a+b}{6}\right)\right| \\
&\le\frac{B(\alpha+1,n-1)\left(5^{\alpha+n+1}+2\cdot 3^{\alpha+n}+1\right)(b-a)^{\alpha+n}}{8\cdot 6^{\alpha+n}(\alpha+n)(\alpha+n+1)(n-2)!}L.
\end{aligned}$$

Proof. This is a special case of Corollary 3.37 for $x=\frac{5a+b}{6}$. □

Gauss-Chebyshev three-point formulae

Now we show how to apply the previous results to obtain some error estimates for Gauss-Chebyshev quadrature rules involving α-L-Hölder type functions.

Theorem 3.23 *Let I be an open interval in $\mathbb{R}$, $[-1,1]\subset I$, and let $f:I\to\mathbb{R}$ be such that the derivative $f^{(n-1)}$ is absolutely continuous and $f^{(n)}:[-1,1]\to\mathbb{R}$ is an α-L-Hölder function for some $n\ge 2$, $\alpha\in(0,1]$ and $L\in\mathbb{R}^+$. Then*

$$\begin{aligned}
&\left|\int_{-1}^{1}\frac{1}{\pi\sqrt{1-t^2}}f(t)\,\mathrm{d}t\right. \\
&\left.-\frac{1}{3}\left[f\left(-\frac{\sqrt{3}}{2}\right)+f(0)+f\left(\frac{\sqrt{3}}{2}\right)\right]-s_{w,n}^3\left(-\frac{\sqrt{3}}{2}\right)\right| \\
&\le\frac{2B(\alpha+1,n-1)}{3(\alpha+n)(n-2)!}L\left[\left(1-\frac{\sqrt{3}}{2}\right)^{\alpha+n}+\left(1+\frac{\sqrt{3}}{2}\right)^{\alpha+n}+1\right],
\end{aligned}$$

where $s_{w,n}^3$ is defined as (3.52) *and* $W(t)=\frac{1}{\pi}\left(\arcsin t+\frac{\pi}{2}\right)$.

Proof. This is a special case of Theorem 3.20 for $[a,b]=[-1,1]$, $x=-\frac{\sqrt{3}}{2}$, $A\left(-\frac{\sqrt{3}}{2}\right)=\frac{1}{3}$ and $w(t)=\frac{1}{\pi\sqrt{1-t^2}}$, $t\in(-1,1)$. □

Theorem 3.24 *Let I be an open interval in $\mathbb{R}$, $[-1,1]\subset I$, and let $f:I\to\mathbb{R}$ be such that the derivative $f^{(n-1)}$ is absolutely continuous and $f^{(n)}:[-1,1]\to\mathbb{R}$ is an α-L-Hölder function for some $n\geq 2$, $\alpha\in(0,1]$ and $L\in\mathbb{R}^+$. Then*

$$\begin{aligned}&\left|\int_{-1}^{1}\frac{2}{\pi}\sqrt{1-t^2}f(t)\,\mathrm{d}t\right.\\&\left.-\frac{1}{4}\left[f\left(-\frac{\sqrt{2}}{2}\right)+2f(0)+f\left(\frac{\sqrt{2}}{2}\right)\right]-s_{w,n}^{3}\left(-\frac{\sqrt{2}}{2}\right)\right|\\&\leq\frac{B(\alpha+1,n-1)}{2(\alpha+n)(n-2)!}L\left[\left(1-\frac{\sqrt{2}}{2}\right)^{\alpha+n}+\left(1+\frac{\sqrt{2}}{2}\right)^{\alpha+n}+2\right],\end{aligned}$$

where $s_{w,n}^{3}$ is defined as (3.52) *and* $W(t)=\frac{1}{\pi}\left(t\sqrt{1-t^2}+\arcsin t+\frac{\pi}{2}\right)$.

Proof. This is a special case of Theorem 3.20 for $[a,b]=[-1,1]$, $x=-\frac{\sqrt{2}}{2}$, $A\left(-\frac{\sqrt{2}}{2}\right)=\frac{1}{4}$ and $w(t)=\frac{2}{\pi}\sqrt{1-t^2}$, $t\in[-1,1]$. □

3.3 Four-point quadrature formulae. Bullen type inequalities

In this section we study closed four-point weighted quadrature formulae

$$\begin{aligned}&\int_a^b w(t)f(t)\,\mathrm{d}t\\&=\left(\frac{1}{2}-A(x)\right)[f(a)+f(b)]+A(x)[f(x)+f(a+b-x)]+E(f,w;x),\end{aligned}\tag{3.63}$$

where $A:\left(a,\frac{a+b}{2}\right]\to\mathbb{R}$, $w:[a,b]\to[0,\infty)$ is some probability density function, that is, integrable function, satisfying $\int_a^b w(t)\mathrm{d}t=1$, and $W(t)=\int_a^t w(u)\mathrm{d}u$ for $t\in[a,b]$, $W(t)=0$, for $t<a$ and $W(t)=1$ for $t>b$ and $E(f,w;x)$ is the remainder. Some results from this section are published in [15], [70] and [98].

3.3.1 Quadrature formulae obtained by a generalization of the Montgomery identity

Here we use identities (2.76) and (2.77) for the purpose of studying weighted and non-weighted quadrature four-point formula (3.63) by means of which we derive weighted and non-weighted generalizations of Bullen type inequalities for $(2n)-$ convex functions $(n \in \mathbb{N})$.

General weighted four-point formula and related inequalities

Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ exists on $[a,b]$ for some $n \geq 2$. We introduce the following notation for each $x \in \left(a, \frac{a+b}{2}\right]$

$$D_4(x) = \left(\frac{1}{2} - A(x)\right)[f(a) + f(b)] + A(x)[f(x) + f(a+b-x)], \tag{3.64}$$

$$\begin{aligned} t_{w,n}^4(x) &= A(x)\left[\sum_{i=0}^{n-2} \frac{f^{(i+1)}(x)}{(i+1)!} \int_a^b w(s)(s-x)^{i+1} \mathrm{d}s \right. \\ &\quad \left. + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a+b-x)}{(i+1)!} \int_a^b w(s)(s-a-b+x)^{i+1} \mathrm{d}s \right] \\ &\quad + \left(\frac{1}{2} - A(x)\right)\left[\sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{(i+1)!} \int_a^b w(s)(s-a)^{i+1} \mathrm{d}s \right. \\ &\quad \left. + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{(i+1)!} \int_a^b w(s)(s-b)^{i+1} \mathrm{d}s \right] \end{aligned}$$

and

$$T_{w,n}^4(x,s) = \begin{cases} -\left(\frac{1}{2} + A(x)\right) \int_a^s w(u)(u-s)^{n-1} du \\ +\left(\frac{1}{2} - A(x)\right) \int_s^b w(u)(u-s)^{n-1} du, & a \leq s \leq x, \\ \quad -\frac{1}{2} \int_a^s w(u)(u-s)^{n-1} du \\ \quad +\frac{1}{2} \int_s^b w(u)(u-s)^{n-1} du, & x < s \leq a+b-x, \\ -\left(\frac{1}{2} - A(x)\right) \int_a^s w(u)(u-s)^{n-1} du \\ +\left(\frac{1}{2} + A(x)\right) \int_s^b w(u)(u-s)^{n-1} du, & a+b-x < s \leq b. \end{cases}$$

In the next theorem we give a general weighted four-point formula.

Theorem 3.25 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and $w : [a,b] \to [0,\infty\rangle$ some probability density function. Let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$. Then for each $x \in \left(a, \frac{a+b}{2}\right]$ the following identity holds*

$$\int_a^b w(t) f(t) \mathrm{d}t = D_4(x) + t_{w,n}^4(x) + \frac{1}{(n-1)!} \int_a^b T_{w,n}^4(x,s) f^{(n)}(s) \mathrm{d}s. \tag{3.65}$$

Proof. We put $x \equiv a, x \equiv x, x \equiv a+b-x$ and $x \equiv b$ in (2.76) to obtain four new formulae. After multiplying these four formulae by $1/2 - A(x)$, $A(x)$, $A(x)$ and $1/2 - A(x)$, respectively, and adding, we get (3.65). □

Remark 3.28 Identity (3.65) holds true in the case $n = 1$.

Theorem 3.26 *Suppose that all the assumptions of Theorem 3.25 hold. Additionally, assume that (p,q) is a pair of conjugate exponents, $1 \le p,q \le \infty$; let $f^{(n)} \in L^p[a,b]$ for some $n \ge 1$. Then for each $x \in \left(a, \frac{a+b}{2}\right]$ we have*

$$\left| \int_a^b w(t) f(t) \mathrm{d}t - D_4(x) - t_{w,n}^4(x) \right| \le \frac{1}{(n-1)!} \left\| T_{w,n}^4(x,\cdot) \right\|_q \left\| f^{(n)} \right\|_p . \tag{3.66}$$

Inequality (3.66) is sharp for $1 < p \le \infty$.

Proof. Applying the Hölder inequality on (3.65) we obtain estimate (3.66). The proof of sharpness is analogous to Theorem 3.2. □

Non-weighted case of four-point formula

For $x \in \left(a, \frac{a+b}{2}\right]$ we define

$$\begin{aligned} & t_n^4(x) \\ & = A(x) \sum_{i=0}^{n-2} \left[f^{(i+1)}(x) + (-1)^{i+1} f^{(i+1)}(a+b-x) \right] \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!\,(b-a)} \\ & + \left(\frac{1}{2} - A(x) \right) \sum_{i=0}^{n-2} \left[f^{(i+1)}(a) + (-1)^{i+1} f^{(i+1)}(b) \right] \frac{(b-a)^{i+1}}{(i+2)!} \end{aligned}$$

and

$$\begin{aligned} & T_n^4(x,s) \\ & = \begin{cases} \left(\frac{1}{2} + A(x) \right) \frac{(a-s)^n}{(b-a)} + \left(\frac{1}{2} - A(x) \right) \frac{(b-s)^n}{(b-a)}, & a \le s \le x, \\ \frac{(a-s)^n + (b-s)^n}{2(b-a)}, & x < s \le a+b-x, \\ \left(\frac{1}{2} - A(x) \right) \frac{(a-s)^n}{(b-a)} + \left(\frac{1}{2} + A(x) \right) \frac{(b-s)^n}{(b-a)}, & a+b-x < s \le b. \end{cases} \end{aligned}$$

Theorem 3.27 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \ge 1$. Then for each $x \in \left(a, \frac{a+b}{2}\right]$ the following identity holds*

$$\frac{1}{b-a} \int_a^b f(t) \mathrm{d}t = D_4(x) + t_n^4(x) + \frac{1}{n!} \int_a^b T_n^4(x,s) f^{(n)}(s) \mathrm{d}s. \tag{3.67}$$

Proof. This is a special case of Theorem 3.25 for $w(t) = \frac{1}{b-a}$, $t \in [a,b]$. □

Theorem 3.28 *Suppose that all the assumptions of Theorem 3.27 hold. Additionally, assume that (p,q) is a pair of conjugate exponents, that is $1 \le p,q \le \infty$, $\frac{1}{p}+\frac{1}{q}=1$ and $f^{(n)} \in L^p[a,b]$ for some $n \ge 1$. Then for each $x \in \left(a, \frac{a+b}{2}\right]$ we have*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t - D_4(x) - t_n^4(x) \right| \le \frac{1}{n!} \left\| T_n^4(x,\cdot) \right\|_q \left\| f^{(n)} \right\|_p. \tag{3.68}$$

Inequality (3.68) *is sharp for* $1 < p \le \infty$.

Proof. This is a special case of Theorem 3.26 for $w(t) = \frac{1}{b-a}$, $t \in [a,b]$. □

Simpson's 3/8 formula

Now, we set

$$A(x) = \frac{(b-a)^2}{12(x-a)(b-x)}, \quad x \in \left(a, \frac{a+b}{2}\right].$$

This special choice of the function A enables us to establish a generalization of the well-known Simpson's 3/8 formula.

Suppose that all the assumptions of Theorem 3.27 hold. Then the following generalization of Simpson's 3/8 formula states

$$\begin{aligned} &\frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t \\ &= D_4\left(\frac{2a+b}{3}\right) + t_n^4\left(\frac{2a+b}{3}\right) + \frac{1}{n!} \int_a^b T_n^4\left(\frac{2a+b}{3}, s\right) f^{(n)}(s)\,\mathrm{d}s, \end{aligned} \tag{3.69}$$

where

$$D_4\left(\frac{2a+b}{3}\right) = \frac{1}{8}\left(f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right),$$

$$\begin{aligned} t_n^4\left(\frac{2a+b}{3}\right) &= \frac{1}{8} \sum_{i=0}^{n-2} \left[f^{(i+1)}\left(\frac{2a+b}{3}\right) + (-1)^{i+1} f^{(i+1)}\left(\frac{a+2b}{3}\right) \right] \\ &\quad \cdot \frac{\left[2^{i+2} + (-1)^{i+1}\right](b-a)^{i+1}}{3^{i+1}(i+2)!} \\ &\quad + \frac{1}{8} \sum_{i=0}^{n-2} \left[f^{(i+1)}(a) + (-1)^{i+1} f^{(i+1)}(b) \right] \frac{(b-a)^{i+1}}{(i+2)!} \end{aligned}$$

and

$$T_n^4\left(\frac{2a+b}{3}, s\right) = \begin{cases} \dfrac{7(a-s)^n + (b-s)^n}{8(b-a)}, & a \le s \le \dfrac{2a+b}{3}, \\ \dfrac{(a-s)^n + (b-s)^n}{2(b-a)}, & \dfrac{2a+b}{3} < s \le \dfrac{a+2b}{3}, \\ \dfrac{(a-s)^n + 7(b-s)^n}{8(b-a)}, & \dfrac{a+2b}{3} < s \le b. \end{cases}$$

Corollary 3.44 *Suppose that all the assumptions of Theorem 3.27 hold. Additionally, assume that (p,q) is a pair of conjugate exponents and $n \in \mathbb{N}$.*

(a) If $f^{(n)} \in L^{\infty}[a,b]$, then

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D_4\left(\frac{2a+b}{3}\right)\right| \le \frac{25}{288}(b-a)\left\|f'\right\|_{\infty}$$

and

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D_4\left(\frac{2a+b}{3}\right) - t_n^4\left(\frac{2a+b}{3}\right)\right|$$
$$\le \frac{1}{(n+1)!}\left(\frac{\left[3^{n+1}+3\cdot 2^{n+1}+3(-1)^n\right](b-a)^n}{4\cdot 3^{n+1}}\right.$$
$$\left.-\left(\frac{b-a}{2}\right)^n\left[\frac{(-1)^{n+1}+1}{2}\right]\right)\left\|f^{(n)}\right\|_{\infty}, \quad n \ge 2.$$

(b) If $f^{(n)} \in L^2[a,b]$, then

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D_4\left(\frac{2a+b}{3}\right) - t_n^4\left(\frac{2a+b}{3}\right)\right|$$
$$\le \frac{1}{n!}\left(\frac{\left[3^{2n}+5\cdot 2^{2n+1}+11\right](b-a)^{2n-1}}{32\cdot 3^{2n}(2n+1)}\right.$$
$$+\frac{(-1)^n(b-a)^{2n-1}}{32}\left[7B(n+1,n+1)+9B_{\frac{2}{3}}(n+1,n+1)\right.$$
$$\left.\left.-9B_{\frac{1}{3}}(n+1,n+1)\right]\right)^{\frac{1}{2}}\left\|f^{(n)}\right\|_2.$$

(c) If $f^{(n)} \in L^1[a,b]$, then

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D_4\left(\frac{2a+b}{3}\right) - t_n^4\left(\frac{2a+b}{3}\right)\right|$$
$$\le \frac{1}{n!}K_n\left(\frac{2a+b}{3}\right)\left\|f^{(n)}\right\|_1,$$

where $K_1\left(\frac{2a+b}{3}\right) = \frac{5}{24}$, $K_2\left(\frac{2a+b}{3}\right) = \frac{5}{18}(b-a)$, $K_3\left(\frac{2a+b}{3}\right) = \frac{7}{54}(b-a)^2$ and $K_n\left(\frac{2a+b}{3}\right) = \frac{1}{8}(b-a)^{n-1}$, for $n \ge 4$.

The first and the second inequality are sharp and the third inequality is the best possible.

Proof. Using (3.68) and carrying out the same analysis as in Corollary 3.31, we obtain these three inequalities. Function $T_n^4\left(\frac{2a+b}{3},\cdot\right)$ is left continuous and has finite jump at $\frac{2a+b}{3}$ and $\frac{a+2b}{3}$, so the proof of the best possibility of the third inequality is similar to the proof given in Theorem 3.2. $\square$

Generalizations of Bullen type inequalities

We recall that for a convex function $f : [a,b] \to \mathbb{R}$ double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2}$$

is known in the literature as Hermite-Hadamard inequalities for convex functions. Inequalities

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \geq \frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right) \geq 0, \tag{3.70}$$

for any convex function f defined on $[a,b]$, were first proved by Bullen in [31]. His results were generalized for $(2n)$-convex functions $(n \in \mathbb{N})$ in [49].

Now we use previous results to prove weighted and non-weighted generalizations of Bullen type inequalities (3.70) for $(2n)$-convex functions $(n \in \mathbb{N})$.

Again, let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ exists on $[a,b]$ for some $n \geq 2$. We introduce the following notation for each $x \in \left[a, \frac{a+b}{2}\right]$:

$$\begin{aligned}
\widetilde{D}_4(x) &= \frac{f(a)+f(x)+f(a+b-x)+f(b)}{4}, &(3.71)\\
\widetilde{T}^4_{w,n}(x,s) &= \frac{T^2_{w,n}(a,s)+T^2_{w,n}(x,s)}{2},\\
\widetilde{t}^4_{w,n}(x) &= \frac{t^2_{w,n}(x)+t^2_{w,n}(a)}{2},
\end{aligned}$$

where $T^2_{w,n}$ and $t^2_{w,n}$ are defined as (3.3).

Theorem 3.29 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and $w : [a,b] \to [0,\infty\rangle$ some probability density function. Let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$. Then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following identity holds*

$$\int_a^b w(t)\,f(t)\,\mathrm{d}t = \widetilde{D}_4(x) + \widetilde{t}^4_{w,n}(x) + \frac{1}{(n-1)!}\int_a^b \widetilde{T}^4_{w,n}(x,s)\,f^{(n)}(s)\,\mathrm{d}s. \tag{3.72}$$

Proof. We put $x \equiv x$, $x \equiv a+b-x$, $x \equiv a$ and $x \equiv b$ in (2.76) to obtain four new formulae. After adding these four formulae and multiplying by $1/4$, we obtain (3.72). □

Remark 3.29 For instance, if in Theorem 3.29 we choose $x = \frac{2a+b}{3}, \frac{a+b}{2}$, we obtain closed Newton-Cotes formulae with the same nodes as Simpson's 3/8 rule and Simpson's rule, respectively.

Theorem 3.30 *Suppose that all the assumptions of Theorem 3.29 hold. Additionally, assume that (p,q) is a pair of conjugate exponents, that is $1 \leq p,q \leq \infty$, $\frac{1}{p}+\frac{1}{q} = 1$ and $f^{(n)} \in L^p[a,b]$ for some $n \geq 1$. Then for each $x \in \left[a, \frac{a+b}{2}\right]$ we have*

$$\left|\int_a^b w(t)\,f(t)\,\mathrm{d}t - \widetilde{D}_4(x) - \widetilde{t}^4_{w,n}(x)\right| \leq \frac{1}{(n-1)!}\left\|\widetilde{T}^4_{w,n}(x,\cdot)\right\|_q \left\|f^{(n)}\right\|_p. \tag{3.73}$$

Inequality (3.73) *is sharp for $1 < p \leq \infty$ and best possible for $p = 1$.*

Proof. The proof is analogous to the proof of Theorem 3.2 □

Theorem 3.31 *Suppose that all the assumptions of Theorem 3.29 hold. Additionally, assume that $f^{(2n)}$ is a continuous function on $[a,b]$. Then for each $x \in \left[a, \frac{a+b}{2}\right]$ there exists $\eta \in (a,b)$ such that*

$$\int_a^b w(t) f(t) \, dt - \widetilde{D}_4(x) - \widetilde{t}_{w,2n}^4(x) = \frac{f^{(2n)}(\eta)}{(2n-1)!} \int_a^b \widetilde{T}_{w,2n}^4(x,s) \, ds.$$

Proof. Similarly to Theorem 3.3, we have $\widetilde{T}_{w,2n}^4(x,s) \geq 0$ for $s \in [a,b]$. Thus we can apply the integral mean value theorem to $\int_a^b \widetilde{T}_{w,2n}^4(x,s) f^{(2n)}(s) ds$. □

Theorem 3.32 *Suppose that all the assumptions of Theorem 3.29 hold for $2n$, $n \in \mathbb{N}$. If f is $(2n)$-convex, then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following inequality holds*

$$\begin{aligned} &\int_a^b w(t) f(t) \, dt - \frac{f(x) + f(a+b-x)}{2} - t_{w,2n}^2(x) \\ &\geq \frac{f(a) + f(b)}{2} - \int_a^b w(t) f(t) \, dt + t_{w,2n}^2(a). \end{aligned} \tag{3.74}$$

If f is $(2n)$-concave, then inequality (3.74) is reversed.

Proof. From (3.72) we have that

$$\begin{aligned} &2\int_a^b w(t) f(t) \, dt - \frac{f(a) + f(x) + f(a+b-x) + f(b)}{2} \\ &-t_{w,2n}^2(x) - t_{w,2n}^2(a) = \frac{1}{(2n-1)!} \int_a^b \widetilde{T}_{w,2n}^4(x,s) f^{(2n)}(s) \, ds. \end{aligned}$$

Similarly to Theorem 3.4 we have

$$\widetilde{T}_{w,2n}^4(x,s) \geq 0 \text{ and } \int_a^b \widetilde{T}_{w,2n}^4(x,s) f^{(2n)}(s) \, ds \geq 0,$$

from which (3.74) follows immediately. □

Corollary 3.45 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and let $f : I \to \mathbb{R}$ be such that $f^{(2n-1)}$ is absolutely continuous for $n \geq 1$. If f is $(2n)$-convex, then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following inequality holds*

$$\begin{aligned} &\frac{1}{b-a} \int_a^b f(t) \, dt - \frac{f(x) + f(a+b-x)}{2} - t_{2n}^2(x) \\ &\geq \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) \, dt + t_{2n}^2(a). \end{aligned} \tag{3.75}$$

If f is $(2n)$-concave, then inequality (3.75) is reversed.

Proof. This is a special case of Theorem 3.32 for $w(t) = \frac{1}{b-a}$, $t \in [a,b]$. □

Remark 3.30 Generalizations of Bullen type inequalities (3.70) for $(2n)$-convex functions ($n \in \mathbb{N}$) and $x \in \left[a, \frac{a+b}{2} - \frac{b-a}{4\sqrt{6}}\right] \cup \left\{\frac{a+b}{2}\right\}$ (of the same type as in Theorem 3.32) were first proved by M. Klaričić Bakula and J.Pečarić in [68].

Corollary 3.46 *Suppose that all the assumptions of Corollary 3.45 hold. If f is 2-convex, then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following inequality holds*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{f(x)+f(a+b-x)}{2} - r(x) \\ &\geq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t, \end{aligned} \tag{3.76}$$

where

$$r(x) = \left(f'(x) - f'(a+b-x)\right)\frac{a+b-2x}{4} + \left(f'(a) - f'(b)\right)\frac{b-a}{4}.$$

If f is 2-concave, then inequality (3.76) is reversed.

Proof. This is a special case of Corollary 3.45 for $n = 1$. □

Corollary 3.47 *Suppose that all the assumptions of Corollary 3.45 hold. If f is 4-convex, then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following inequality holds*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{f(x)+f(a+b-x)}{2} - r(x) \\ &\geq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t, \end{aligned} \tag{3.77}$$

where

$$\begin{aligned} r(x) &= \left(f'(x) - f'(a+b-x)\right)\frac{a+b-2x}{4} + \left(f'(a) - f'(b)\right)\frac{b-a}{4} \\ &+ \left(f''(x) + f''(a+b-x)\right)\frac{(a-x)^2 + (a-x)(b-x) + (b-x)^2}{12} \\ &+ \left(f''(a) + f''(b)\right)\frac{(b-a)^2}{12} + \left(f'''(a) - f'''(b)\right)\frac{(b-a)^3}{48} \\ &+ \left(f'''(x) - f'''(a+b-x)\right)(a+b-2x)\frac{(a-x)^2 + (b-x)^2}{48}. \end{aligned}$$

If f is 4-concave, then inequality (3.77) is reversed.

Proof. This is a special case of Corollary 3.45 for $n = 2$. □

3.3.2 General four-point quadrature formulae with applications for α-L-Hölder type functions

Variant I of general four-point formula

Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ exists on $[a,b]$ for some $n \geq 2$. We introduce the following notation for each $x \in \left(a, \frac{a+b}{2}\right]$

$$
\begin{aligned}
s_{w,n}^{4}(x) &= \left(\frac{1}{2} - A(x)\right)\left\{\sum_{i=0}^{n-1} \frac{f^{(i+1)}(b)}{i!}\left[\int_a^b (1-W(t))(t-b)^i \,\mathrm{d}t\right]\right. \\
&\left. - \sum_{i=0}^{n-1} \frac{f^{(i+1)}(a)}{i!}\left[\int_a^b W(t)(t-a)^i \,\mathrm{d}t\right]\right\} + A(x)\left\{\sum_{i=0}^{n-1} \frac{f^{(i+1)}(b)}{i!}\right. \\
&\cdot \left[\int_x^b (1-W(t))(t-b)^i \,\mathrm{d}t + \int_{a+b-x}^b (1-W(t))(t-b)^i \,\mathrm{d}t\right] \\
&\left. - \sum_{i=0}^{n-1} \frac{f^{(i+1)}(a)}{i!}\left[\int_a^x W(t)(t-a)^i \,\mathrm{d}t + \int_a^{a+b-x} W(t)(t-a)^i \,\mathrm{d}t\right]\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
S_{w,n}^{4}(x) &= \left(\frac{1}{2} - A(x)\right)\left[S_{w,n}^{a}(b) + S_{w,n}^{b}(a)\right] \\
&+ A(x)\left[S_{w,n}^{a}(x) + S_{w,n}^{b}(x) + S_{w,n}^{a}(a+b-x) + S_{w,n}^{b}(a+b-x)\right],
\end{aligned}
$$

where $S_{w,n}^{a}$ and $S_{w,n}^{b}$ are defined as (3.22).

In the next theorem we establish the first variant of the generalized four-point quadrature formula based on the generalized Montgomery identity (2.81).

Theorem 3.33 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and $w : [a,b] \to [0,\infty\rangle$ some probability density function. Let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$. Then for each $x \in \left(a, \frac{a+b}{2}\right]$ the following identity holds*

$$
\int_a^b w(t) f(t) \,\mathrm{d}t = D_4(x) + s_{w,n}^{4}(x) + S_{w,n}^{4}(x). \tag{3.78}
$$

Proof. We put $x \equiv a, x \equiv x, x \equiv a+b-x$ and $x \equiv b$ in (2.81) to obtain four new formulae. After multiplying these four formulae by $1/2 - A(x)$, $A(x)$, $A(x)$ and $1/2 - A(x)$, respectively, and adding, we get (3.78). □

Now, we will give an estimation of the term

$$
\left|\int_a^b w(t) f(t) \,\mathrm{d}t - D_4(x) - s_{w,n}^{4}(x)\right|.
$$

Theorem 3.34 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$ and $w : [a,b] \to [0,\infty)$ some probability density function. Let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous and that $f^{(n)} : [a,b] \to \mathbb{R}$ is an α-L-Hölder type function for some $n \geq 2$, $\alpha \in (0,1]$ and $L \in \mathbb{R}^+$. Then for each $x \in \left(a, \frac{a+b}{2}\right]$ the following inequalities hold*

$$\begin{aligned}
&\left| \int_a^b w(t) f(t)\,\mathrm{d}t - D_4(x) - s_{w,n}^4(x) \right| \leq \frac{B(\alpha+1,n-1)}{(n-2)!} L \\
&\cdot \left\{ \left| \frac{1}{2} - A(x) \right| \left[\int_a^b W(a,t) U_n(a,t)\,\mathrm{d}t + \int_a^b W(b,t) U_n(b,t)\,\mathrm{d}t \right] \right. \\
&+ |A(x)| \left[\int_a^b W(x,t) U_n(x,t)\,\mathrm{d}t \right. \\
&\left. \left. + \int_a^b W(a+b-x,t) U_n(a+b-x,t)\,\mathrm{d}t \right] \right\} \\
&\leq \frac{2B(\alpha+1,n-1)L}{(\alpha+n)(n-2)!} \\
&\cdot \left\{ \left| \frac{1}{2} - A(x) \right| (b-a)^{\alpha+n} + |A(x)| \left[(x-a)^{\alpha+n} + (b-x)^{\alpha+n} \right] \right\},
\end{aligned}$$

where $W(x,t)$ and $U_n(x,t)$ are defined as in (3.24).

Proof. The proof is similar to the proof of Theorem 3.20. □

Variant I of non-weighted four-point formula

For each $x \in \left(a, \frac{a+b}{2}\right]$ we define

$$\begin{aligned}
s_n^4(x) &= \left(\frac{1}{2} - A(x) \right) \sum_{i=0}^{n-1} \left[(-1)^i f^{(i+1)}(b) - f^{(i+1)}(a) \right] \frac{(b-a)^{i+1}}{i!\,(i+2)} \\
&+ A(x) \sum_{i=0}^{n-1} \left[(-1)^i f^{(i+1)}(b) - f^{(i+1)}(a) \right] \frac{(x-a)^{i+2} + (b-x)^{i+2}}{i!\,(i+2)(b-a)}
\end{aligned}$$

and

$$\begin{aligned}
S_n^4(x) &= \left(\frac{1}{2} - A(x) \right) \left[S_n^a(b) + S_n^b(a) \right] \\
&+ A(x) \left[S_n^a(x) + S_n^b(x) + S_n^a(a+b-x) + S_n^b(a+b-x) \right],
\end{aligned}$$

where S_n^a and S_n^b are defined as (3.57).

Corollary 3.48 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$. Then for each $x \in \left(a, \frac{a+b}{2}\right]$ the following identity holds*

$$\frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t = D_4(x) + s_n^4(x) + S_n^4(x). \tag{3.79}$$

Proof. This is a special case of Theorem 3.33 for $w(t) = \frac{1}{b-a}$, $t \in [a,b]$. □

Corollary 3.49 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous and that $f^{(n)} : [a,b] \to \mathbb{R}$ is an α-L-Hölder type function for some $n \geq 2$. Then for each $x \in \left(a, \frac{a+b}{2}\right]$ the following inequality holds*

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t - D_4(x) - s_n^4(x) \right| \\
&\leq \frac{2B(\alpha+1, n-1)L}{(b-a)(\alpha+n+1)(n-2)!} \\
&\cdot \left\{ \left| \frac{1}{2} - A(x) \right| (b-a)^{\alpha+n+1} + |A(x)| \left[(x-a)^{\alpha+n+1} + (b-x)^{\alpha+n+1} \right] \right\}.
\end{aligned}$$

Proof. This is a special case of Theorem 3.34 for $w(t) = \frac{1}{b-a}$, $t \in [a,b]$. □

Generalizations of Bullen type inequalities I

Now, by means of quadrature formulae obtained in previous two subsections we prove weighted and non-weighted generalizations of Bullen type inequalities (3.70) for $(2k)$-convex functions ($k \in \mathbb{N}$).

Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ exists on $[a,b]$ for some $n \geq 2$. We introduce the following notation for each $x \in \left[a, \frac{a+b}{2}\right]$

$$\begin{aligned}
\widetilde{S}_{w,n}^4(x) &= \frac{S_{w,n}^2(a) + S_{w,n}^2(x)}{2}, \\
\widetilde{s}_{w,n}^4(x) &= \frac{s_{w,n}^2(x) + s_{w,n}^2(a)}{2},
\end{aligned}$$

where $S_{w,n}^2$ and $s_{w,n}^2$ are defined as (3.21) and $\widetilde{D}_4(x)$ are as in (3.71).

Theorem 3.35 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and $w : [a,b] \to [0,\infty\rangle$ some probability density function. Let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$. Then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following identity holds*

$$\int_a^b w(t) f(t)\,\mathrm{d}t = \widetilde{D}_4(x) + \widetilde{s}_{w,n}^4(x) + \widetilde{S}_{w,n}^4(x). \tag{3.80}$$

Proof. We put $x \equiv x$, $x \equiv a+b-x$, $x \equiv a$ and then $x \equiv b$ in (2.81) to obtain four new formulae. After adding these four formulae and multiplying by $1/4$, we get (3.80). □

Remark 3.31 For instance, if in Theorem 3.35 we choose $x = \frac{2a+b}{3}, \frac{a+b}{2}$, we obtain closed Newton-Cotes formulae with the same nodes as Simpson's 3/8 rule and Simpson's rule, respectively.

Theorem 3.36 *Suppose that all the assumptions of Theorem 3.35 hold. Additionally, assume that $f^{(n)}:[a,b]\to\mathbb{R}$ is an α-L-Hölder type function for some $n\geq 2$, $\alpha\in(0,1]$ and $L\in\mathbb{R}^{+}$. Then for each $x\in\left[a,\frac{a+b}{2}\right]$ the following inequalities hold*

$$\begin{aligned}
&\left|\int_a^b w(t)f(t)\,dt-\widetilde{D}_4(x)-\widetilde{s}^{4}_{w,n}(x)\right| \\
&\leq\frac{B(\alpha+1,n-1)}{4(n-2)!}L\left[\int_a^b W(x,t)U_n(x,t)\,dt+\int_a^b W(a,t)U_n(a,t)\,dt\right. \\
&\left.+\int_a^b W(a+b-x,t)U_n(a+b-x,t)\,dt+\int_a^b W(b,t)U_n(b,t)\,dt\right] \\
&\leq\frac{B(\alpha+1,n-1)}{2(\alpha+n)(n-2)!}L\left[(x-a)^{\alpha+n}+(b-x)^{\alpha+n}+(b-a)^{\alpha+n}\right].
\end{aligned}$$

Proof. This is a special case of Theorem 3.34 for $A(x)=1/4$. □

Corollary 3.50 *Let I be an open interval in $\mathbb{R}$, $[a,b]\subset I$ and let $f:I\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous and $f^{(n)}:[a,b]\to\mathbb{R}$ is an α-L-Hölder type function for some $n\geq 2$, $\alpha\in(0,1]$ and $L\in\mathbb{R}^{+}$. Then for each $x\in\left[a,\frac{a+b}{2}\right]$ the following inequality holds*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\widetilde{D}_4(x)-\frac{s_n^2(x)+s_n^2(a)}{2}\right| \\
&\leq\frac{B(\alpha+1,n-1)}{2(b-a)(\alpha+n+1)(n-2)!}L \\
&\cdot\left[(x-a)^{\alpha+n+1}+(b-x)^{\alpha+n+1}+(b-a)^{\alpha+n+1}\right].
\end{aligned}$$

Proof. This is a special case of Theorem 3.36 for $w(t)=\frac{1}{b-a}$, $t\in[a,b]$. □

Theorem 3.37 *Suppose that all the assumptions of Theorem 3.35 hold for some $n=2k-1$, $k\geq 2$. If f is $(2k)$-convex, then for each $x\in\left[a,\frac{a+b}{2}\right]$ the following inequality holds*

$$\begin{aligned}
&\int_a^b w(t)f(t)\,dt-\frac{f(x)+f(a+b-x)}{2}-s^2_{w,n}(x) \\
&\leq\frac{f(a)+f(b)}{2}-\int_a^b w(t)f(t)\,dt+s^2_{w,n}(a).
\end{aligned}\tag{3.81}$$

If f is $(2k)$-concave, then inequality (3.81) is reversed.

Proof. From (3.80) we have that

$$\begin{aligned}
&2\int_a^b w(t)f(t)\,dt-\frac{f(a)+f(x)+f(a+b-x)+f(b)}{2} \\
&-s^2_{w,n}(x)-s^2_{w,n}(a)=\widetilde{S}^4_{w,n}(x).
\end{aligned}$$

Carrying out the same analysis as in Theorem 3.9 we obtain $\widetilde{S}^4_{w,n}(x)\leq 0$, from which (3.81) follows immediately. □

Corollary 3.51 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n = 2k-1$, $k \geq 2$. If f is $(2k)$-convex, then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following inequality holds*

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(x)+f(a+b-x)}{2} - s_n^2(x) \\
&\leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt + s_n^2(a). \qquad (3.82)
\end{aligned}$$

If f is $(2k)$-concave, then inequality (3.82) is reversed.

Proof. This is a special case of Theorem 3.37 for $w(t) = \frac{1}{b-a}$, $t \in [a,b]$. □

Corollary 3.52 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and let $f : I \to \mathbb{R}$ be such that f'' is an absolutely continuous function. If f is 4-convex, then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following inequality holds*

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(x)+f(a+b-x)}{2} \\
&\leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt + \rho(x), \qquad (3.83)
\end{aligned}$$

where

$$\begin{aligned}
\rho(x) = & \frac{1}{2(b-a)}\left[\left(f'(b)-f'(a)\right)\frac{(x-a)^2+(b-a)^2+(b-x)^2}{2}\right. \\
& - \left(f''(b)+f''(a)\right)\frac{(x-a)^3+(b-a)^3+(b-x)^3}{3} \\
& \left. + \left(f'''(b)-f'''(a)\right)\frac{(x-a)^4+(b-a)^4+(b-x)^4}{8}\right].
\end{aligned}$$

If f is 4-concave, then inequality (3.83) is reversed.

Proof. This is a special case of Corollary 3.51 for $n = 3$. □

Variant II of general four-point formula

For each $x \in \left(a, \frac{a+b}{2}\right]$ we define

$$\begin{aligned}
r_{w,n}^4(x) = & -\left(\frac{1}{2}-A(x)\right)\sum_{i=0}^{n-1}\left[\frac{f^{(i+1)}(a)}{i!}\int_a^b P_w(a,t)(t-a)^i\,\mathrm{d}t\right. \\
& \left. + \frac{f^{(i+1)}(b)}{i!}\int_a^b P_w(b,t)(t-b)^i\,\mathrm{d}t\right] \\
& - A(x)\sum_{i=0}^{n-1}\left[\frac{f^{(i+1)}(x)}{i!}\int_a^b P_w(x,t)(t-x)^i\,\mathrm{d}t\right.
\end{aligned}$$

$$+\quad \frac{f^{(i+1)}(a+b-x)}{i!}\int_a^b P_w(a+b-x,t)(t-a-b+x)^i\,\mathrm{d}t\Bigg]$$

and

$$\begin{aligned} R^4_{w,n}(x) &= \left(\frac{1}{2}-A(x)\right)[R_{w,n}(a)+R_{w,n}(b)] \\ &\quad + A(x)[R_{w,n}(x)+R_{w,n}(a+b-x)], \end{aligned}$$

where $R_{w,n}$ is defined by (3.32).

Theorem 3.38 *Let I be an open interval in $\mathbb{R}$, $[a,b]\subset I$, and $w:[a,b]\to[0,\infty\rangle$ some probability density function. Let $f:I\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n\geq 2$. Then for each $x\in\left(a,\frac{a+b}{2}\right]$ the following identity holds*

$$\int_a^b w(t)f(t)\,\mathrm{d}t = D_4(x)+r^4_{w,n}(x)+R^4_{w,n}(x). \tag{3.84}$$

Proof. We put $x\equiv a, x\equiv x, x\equiv a+b-x$ and $x\equiv b$ in (2.82) to obtain four new formulae. After multiplying these four formulae by $1/2-A(x)$, $A(x)$, $A(x)$ and $1/2-A(x)$, respectively, and adding, we get (3.84). □

Theorem 3.39 *Let I be an open interval in $\mathbb{R}$, $[a,b]\subset I$ and let $w:[a,b]\to[0,\infty)$ be some probability density function. Let $f:I\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous and that $f^{(n)}:[a,b]\to\mathbb{R}$ is an α-L-Hölder type function for some $n\geq 2$, $\alpha\in(0,1]$ and $L\in\mathbb{R}^+$. Then for each $x\in\left(a,\frac{a+b}{2}\right]$ the following inequalities hold*

$$\begin{aligned} &\left|\int_a^b w(t)f(t)\,\mathrm{d}t - D_4(x) - r^4_{w,n}(x)\right| \\ &\leq \frac{B(\alpha+1,n-1)}{(n-2)!}L\left\{\left|\frac{1}{2}-A(x)\right|\left[\int_a^b W(t)(b-t)^{\alpha+n-1}\,\mathrm{d}t\right.\right. \\ &\left.+\int_a^b (1-W(t))(t-a)^{\alpha+n-1}\,\mathrm{d}t\right] \\ &+|A(x)|\left[\int_a^b |P_w(x,t)|\,|(x-t)|^{\alpha+n-1}\,\mathrm{d}t\right. \\ &\left.\left.+\int_a^b |P_w(a+b-x,t)|\,|(a+b-x-t)|^{\alpha+n-1}\,\mathrm{d}t\right]\right\} \\ &\leq \frac{2B(\alpha+1,n-1)L}{(\alpha+n)(n-2)!} \\ &\cdot\left\{\left|\frac{1}{2}-A(x)\right|(b-a)^{\alpha+n}+|A(x)|\left[(x-a)^{\alpha+n}+(b-x)^{\alpha+n}\right]\right\}. \end{aligned}$$

Proof. The proof is similar to the proof of Theorem 3.22. □

Variant II of non-weighted four-point formula

For each $x \in \left(a, \frac{a+b}{2}\right]$ we define

$$\begin{aligned}
r_n^4(x) &= \left(\frac{1}{2} - A(x)\right) \sum_{i=0}^{n-1} \left[f^{(i+1)}(a) + (-1)^{i+1} f^{(i+1)}(b)\right] \frac{(b-a)^{i+1}}{(i+2)!} \\
&+ A(x) \sum_{i=0}^{n-1} \left[f^{(i+1)}(x) + (-1)^{i+1} f^{(i+1)}(a+b-x)\right] \\
&\cdot \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!(b-a)}
\end{aligned}$$

and

$$R_n^4(x) = \left(\frac{1}{2} - A(x)\right) [R_n(a) + R_n(b)] + A(x) [R_n(x) + R_n(a+b-x)],$$

where R_n is determined by (3.62).

Corollary 3.53 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$. Then for each $x \in \left(a, \frac{a+b}{2}\right]$ the following identity holds*

$$\frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t = D_4(x) + r_n^4(x) + R_n^4(x).$$

Proof. This is a special case of Theorem 3.38 for $w(t) = \frac{1}{b-a}$, $t \in [a,b]$. □

Corollary 3.54 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous and $f^{(n)} : [a,b] \to \mathbb{R}$ is an α-L-Hölder type function for some $n \geq 2$. Then for each $x \in \left(a, \frac{a+b}{2}\right]$ the following inequality holds*

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t - D_4(x) - r_n^4(x) \right| \\
&\leq \frac{2B(\alpha+1, n-1) B(2, \alpha+n) L}{(b-a)(n-2)!} \\
&\cdot \left\{ |A(x)| \left[(x-a)^{\alpha+n+1} + (b-x)^{\alpha+n+1}\right] + \left|\frac{1}{2} - A(x)\right| (b-a)^{\alpha+n+1} \right\}.
\end{aligned}$$

Proof. This is a special case of Theorem 3.39 for $w(t) = \frac{1}{b-a}$, $t \in [a,b]$. □

Generalizations of Bullen type inequalities II

Here we use previous results to prove weighted and non-weighted generalizations of Bullen type inequalities (3.70) for $(2k)$-convex functions ($k \in \mathbb{N}$).

Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ exists on $[a,b]$ for some $n \geq 2$. We introduce the following notation for each $x \in \left[a, \frac{a+b}{2}\right]$

$$\widetilde{R}_{w,n}^4(x) = \frac{R_{w,n}^2(a) + R_{w,n}^2(x)}{2},$$

$$\widetilde{r}_{w,n}^{4}(x) = \frac{r_{w,n}^{2}(x) + r_{w,n}^{2}(a)}{2},$$

where $R_{w,n}^2$ and $r_{w,n}^2$ are defined as (3.31), and $\widetilde{D}_4(x)$ are as in (3.71).

Theorem 3.40 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and $w : [a,b] \to [0,\infty\rangle$ some probability density function. Let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$. Then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following identity holds*

$$\int_a^b w(t) f(t) \mathrm{d}t = \widetilde{D}_4(x) + \widetilde{r}_{w,n}^{4}(x) + \widetilde{R}_{w,n}^{4}(x). \tag{3.85}$$

Proof. We put $x \equiv x$, $x \equiv a+b-x$, $x \equiv a$ and then $x \equiv b$ in (2.82) to obtain four new formulae. After adding these four formulae and multiplying by $1/4$, we get (3.85). □

Theorem 3.41 *Suppose that all the assumptions of Theorem 3.40 hold. Additionally, assume that $f^{(n)} : [a,b] \to \mathbb{R}$ is an α-L-Hölder type function for some $n \geq 2$, $\alpha \in (0,1]$ and $L \in \mathbb{R}^+$. Then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following inequalities hold*

$$\begin{aligned}
&\left| \int_a^b w(t) f(t) dt - \widetilde{D}_4(x) - \widetilde{r}_{w,n}^{4}(x) \right| \\
&\leq \frac{B(\alpha+1, n-1)}{4(n-2)!} L \left[\int_a^b \left| P_w(x,t)(x-t)^{\alpha+n-1} \right| dt \right. \\
&+ \int_a^b \left| P_w(a+b-x,t)(a+b-x-t)^{\alpha+n-1} \right| dt \\
&\left. + \int_a^b W(t)(b-t)^{\alpha+n-1} dt + \int_a^b (1-W(t))(t-a)^{\alpha+n-1} dt \right] \\
&\leq \frac{B(\alpha+1, n-1)}{2(n-2)!(\alpha+n)} L \left[(x-a)^{\alpha+n} + (b-x)^{\alpha+n} + (b-a)^{\alpha+n} \right].
\end{aligned}$$

Proof. This is a special case of Theorem 3.39 for $A(x) = 1/4$. □

Corollary 3.55 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$ and let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous and that $f^{(n)} : [a,b] \to \mathbb{R}$ is an α-L-Hölder type function for some $n \geq 2$, $\alpha \in (0,1]$ and $L \in \mathbb{R}^+$. Then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following inequality holds*

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(t) dt - \widetilde{D}_4(x) - \frac{r_n^2(x) + r_n^2(a)}{2} \right| \\
&\leq \frac{B(\alpha+1, n-1)}{2(b-a)(\alpha+n+1)(n-2)!} L \\
&\cdot \left[(x-a)^{\alpha+n+1} + (b-x)^{\alpha+n+1} + (b-a)^{\alpha+n+1} \right].
\end{aligned}$$

Proof. This is a special case of Theorem 3.41 for $w(t) = \frac{1}{b-a}$, $t \in [a,b]$. □

Theorem 3.42 *Suppose that all the assumptions of Theorem 3.40 hold for some $n = 2k-1$, $k \geq 2$. If f is $(2k)$-convex, then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following inequality holds*

$$\begin{aligned} &\int_a^b w(t) f(t)\,dt - \frac{f(x)+f(a+b-x)}{2} - r_{w,n}^2(x) \\ &\geq \frac{f(a)+f(b)}{2} - \int_a^b w(t) f(t)\,dt + r_{w,n}^2(a). \end{aligned} \tag{3.86}$$

If f is $(2k)$-concave, then inequality (3.86) *is reversed.*

Proof. From (3.85) we have that

$$\begin{aligned} &2\int_a^b w(t) f(t)\,dt - \frac{f(a)+f(x)+f(a+b-x)+f(b)}{2} \\ &-r_{w,n}^2(x) - r_{w,n}^2(a) = \widetilde{R}_{w,n}^4(x). \end{aligned}$$

Carrying out the same analysis as in Theorem 3.12 we obtain $\widetilde{R}_{w,n}^4(x) \geq 0$, from which (3.86) follows immediately. □

Corollary 3.56 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n = 2k-1$, $k \geq 2$. If f is $(2k)$-convex, then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following inequality holds*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(x)+f(a+b-x)}{2} - r_n^2(x) \\ &\geq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt + r_n^2(a). \end{aligned} \tag{3.87}$$

If f is $(2k)$-concave, then inequality (3.87) *is reversed.*

Proof. This is a special case of Theorem 3.42 for $w(t) = \frac{1}{b-a}$, $t \in [a,b]$. □

Corollary 3.57 *Let I be an open interval in $\mathbb{R}$, $[a,b] \subset I$, and let $f : I \to \mathbb{R}$ be such that f'' is absolutely continuous function. If f is 4-convex, then for each $x \in \left[a, \frac{a+b}{2}\right]$ the following inequality holds*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(x)+f(a+b-x)}{2} \\ &\geq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt + \rho(x), \end{aligned} \tag{3.88}$$

where

$$\begin{aligned} \rho(x) &= \left(f'(x) - f'(a+b-x)\right)\frac{a+b-2x}{4} + \left(f'(a) - f'(b)\right)\frac{b-a}{4} \\ &+ \left(f''(x) + f''(a+b-x)\right)\frac{(a-x)^2 + (a-x)(b-x) + (b-x)^2}{12} \end{aligned}$$

$$+\ \left(f''(a)+f''(b)\right)\frac{(b-a)^2}{12}+\left(f'''(a)-f'''(b)\right)\frac{(b-a)^3}{48}$$
$$+\ \left(f'''(x)-f'''(a+b-x)\right)(a+b-2x)\frac{(a-x)^2+(b-x)^2}{48}.$$

If f is 4-concave, then inequality (3.88) is reversed.

Proof. This is a special case of Corollary 3.56 for $n=3$. □

Simpson's $3/8$ formula

If all the assumptions of Corollary 3.48 and Corollary 3.53 hold for $A(x)=\frac{(b-a)^2}{12(x-a)(b-x)}$ and $x=\frac{2a+b}{3}$ then the following generalizations of Simpson's $3/8$ formula read

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t = D_4\left(\frac{2a+b}{3}\right)+s_n^4\left(\frac{2a+b}{3}\right)+S_n^4\left(\frac{2a+b}{3}\right)$$

and

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t = D_4\left(\frac{2a+b}{3}\right)+r_n^4\left(\frac{2a+b}{3}\right)+R_n^4\left(\frac{2a+b}{3}\right),$$

where

$$D_4\left(\frac{2a+b}{3}\right)=\frac{1}{8}\left(f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right),$$

$$s_n^4\left(\frac{2a+b}{3}\right)$$
$$=\frac{1}{8}\sum_{i=0}^{n-1}\left[(-1)^i f^{(i+1)}(b)-f^{(i+1)}(a)\right]\frac{\left[3^{i+1}+2^{i+2}+1\right](b-a)^{i+1}}{3^{i+1}i!\,(i+2)}$$

and

$$r_n^4\left(\frac{2a+b}{3}\right) = \frac{1}{8}\sum_{i=0}^{n-1}\left[f^{(i+1)}(a)+(-1)^{i+1}f^{(i+1)}(b)\right]\frac{(b-a)^{i+1}}{(i+2)!}$$
$$+\ \frac{3}{8}\sum_{i=0}^{n-1}\left[f^{(i+1)}\left(\frac{2a+b}{3}\right)+(-1)^{i+1}f^{(i+1)}\left(\frac{a+2b}{3}\right)\right]$$
$$\cdot\ \frac{\left(2^{i+2}+(-1)^{i+1}\right)(b-a)^{i+1}}{3^{i+2}(i+2)!}.$$

Corollary 3.58 *Suppose that all the assumptions of Corollary 3.49 hold. Then we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t-D_4\left(\frac{2a+b}{3}\right)-s_n^4\left(\frac{2a+b}{3}\right)\right|$$
$$\leq\frac{B(\alpha+1,n-1)\left(3^{\alpha+n}+2^{\alpha+n+1}+1\right)(b-a)^{\alpha+n}}{4\cdot 3^{\alpha+n}(\alpha+n+1)(n-2)!}L.$$

Proof. This is a special case of Corollary 3.49 for $x = \frac{2a+b}{3}$. □

Corollary 3.59 *Suppose that all the assumptions of Corollary 3.54 hold. Then we have*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D_4\left(\frac{2a+b}{3}\right) - r_n^4\left(\frac{2a+b}{3}\right)\right| \\ &\leq \frac{B(\alpha+1,n-1)B(2,\alpha+n)\left(3^{\alpha+n}+2^{\alpha+n+1}+1\right)(b-a)^{\alpha+n}}{4\cdot 3^{\alpha+n}(n-2)!}L.\end{aligned}$$

Proof. This is a special case of Corollary 3.54 for $x = \frac{2a+b}{3}$. □

Lobatto four-point formula

As special cases of Corollary 3.48 and Corollary 3.53 we get two generalizations of Lobatto four-point formula

$$\frac{1}{2}\int_{-1}^{1} f(t)\,\mathrm{d}t = D_4\left(-\frac{\sqrt{5}}{5}\right) + s_n^4\left(-\frac{\sqrt{5}}{5}\right) + S_n^4\left(-\frac{\sqrt{5}}{5}\right)$$

and

$$\frac{1}{2}\int_{-1}^{1} f(t)\,\mathrm{d}t = D_4\left(-\frac{\sqrt{5}}{5}\right) + r_n^4\left(-\frac{\sqrt{5}}{5}\right) + R_n^4\left(-\frac{\sqrt{5}}{5}\right),$$

where

$$D_4\left(-\frac{\sqrt{5}}{5}\right) = \frac{1}{12}\left(f(-1) + 5f\left(-\frac{\sqrt{5}}{5}\right) + 5f\left(\frac{\sqrt{5}}{5}\right) + f(1)\right),$$

$$\begin{aligned} s_n^4\left(-\frac{\sqrt{5}}{5}\right) &= \frac{1}{12}\sum_{i=0}^{n-1}\left[(-1)^i f^{(i+1)}(1) - f^{(i+1)}(-1)\right] \\ &\quad \cdot \frac{\left[2^{i+2}\cdot 5^{i+1} + \left(5-\sqrt{5}\right)^{i+2} + \left(5+\sqrt{5}\right)^{i+2}\right]}{2\cdot 5^{i+1} i!\,(i+2)} \end{aligned}$$

and

$$\begin{aligned} r_n^4\left(-\frac{\sqrt{5}}{5}\right) &= \frac{1}{12}\left\{\sum_{i=0}^{n-1}\left[f^{(i+1)}(-1) + (-1)^{i+1} f^{(i+1)}(1)\right]\frac{2^{i+1}}{(i+2)!}\right. \\ &\quad + \sum_{i=0}^{n-1}\left[f^{(i+1)}\left(-\frac{\sqrt{5}}{5}\right) + (-1)^{i+1} f^{(i+1)}\left(\frac{\sqrt{5}}{5}\right)\right] \\ &\quad \left.\cdot \frac{\left(\sqrt{5}+5\right)^{i+2} - \left(\sqrt{5}-5\right)^{i+2}}{2\cdot 5^{i+1}(i+2)!}\right\}. \end{aligned}$$

Corollary 3.60 *Suppose that all the assumptions of Corollary 3.49 hold. Then we have*

$$\left|\frac{1}{2}\int_{-1}^{1} f(t)\,\mathrm{d}t - D_4\left(-\frac{\sqrt{5}}{5}\right) - s_n^4\left(-\frac{\sqrt{5}}{5}\right)\right|$$
$$\leq \frac{B(\alpha+1,n-1)\left(2^{\alpha+n+1}\cdot 5^{\alpha+n} + \left(5-\sqrt{5}\right)^{\alpha+n+1} + \left(5+\sqrt{5}\right)^{\alpha+n+1}\right)}{12\cdot 5^{\alpha+n}(\alpha+n+1)(n-2)!}L.$$

Proof. This is a special case of Corollary 3.49 for $[a,b]=[-1,1]$ and $x=-\frac{\sqrt{5}}{5}$. □

Corollary 3.61 *Suppose that all the assumptions of Corollary 3.54 hold. Then we have*

$$\left|\frac{1}{2}\int_{-1}^{1} f(t)\,\mathrm{d}t - D_4\left(-\frac{\sqrt{5}}{5}\right) - r_n^4\left(-\frac{\sqrt{5}}{5}\right)\right|$$
$$\leq \frac{B(\alpha+1,n-1)B(2,\alpha+n)}{(n-2)!}$$
$$\cdot\frac{\left(2^{\alpha+n+1}\cdot 5^{\alpha+n} + \left(5-\sqrt{5}\right)^{\alpha+n+1} + \left(5+\sqrt{5}\right)^{\alpha+n+1}\right)}{12\cdot 5^{\alpha+n}}L.$$

Proof. This is a special case of Corollary 3.54 for $[a,b]=[-1,1]$ i $x=-\frac{\sqrt{5}}{5}$. □

Chapter 4

Euler harmonic identities for real Borel measures

In this chapter we consider Euler harmonic identities for real Borel measures and some of their applications to general Ostrowski and Euler-Grüss type inequalities.

Generalizations of Euler identities involving μ-harmonic sequences of functions with respect to a real Borel measure μ are established in the first section. Some Ostrowski type inequalities for functions of various classes are proved.

An inequality of Grüss type for a real Borel measure is proved in the second section. Some Euler-Grüss type inequalities are given by using general Euler identities derived in the previous section.

An integration-by-parts formula, involving finite Borel measures supported by intervals on the real line, is proved. Some applications to Ostrowski type and Grüss type inequalities are presented in the third section.

In the last section Euler identities for μ-harmonic sequences of functions and moments of a real Borel measure μ are considered. Also, some Ostrowski and Euler-Grüss type inequalities for functions of various classes are obtained.

4.1 Euler harmonic identities for measures

4.1.1 Introduction

Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a function of bounded variation on $[a,b]$ for some $n \geq 1$. In the recent paper [47] the following two identities have been proved:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t + T_n(x) + R_n^1(x) \tag{4.1}$$

and

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t + T_{n-1}(x) + R_n^2(x), \tag{4.2}$$

where $T_0(x) = 0$ and

$$T_m(x) = \sum_{k=1}^{m} \frac{(b-a)^{k-1}}{k!} B_k\left(\frac{x-a}{b-a}\right)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right],$$

for $1 \leq m \leq n$, while

$$R_n^1(x) = -\frac{(b-a)^{n-1}}{n!}\int_{[a,b]} B_n^*\left(\frac{x-t}{b-a}\right)\mathrm{d}f^{(n-1)}(t)$$

and

$$R_n^2(x) = -\frac{(b-a)^{n-1}}{n!}\int_{[a,b]}\left[B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right)\right]\mathrm{d}f^{(n-1)}(t).$$

Here, $\int_{[a,b]} g(t)d\varphi(t)$ denotes the Riemann–Stieltjes integral with respect to a function $\varphi : [a,b] \to \mathbb{R}$ of bounded variation, and $\int_a^b g(t)dt$ is the Riemann integral. The formulae (4.1) and (4.2) hold for every $x \in [a,b]$. They are extensions of the well known formula for the expansion of an arbitrary function f with a continuous n^{th} derivative $f^{(n)}$ in Bernoulli polynomials [75]:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t + T_{n-1}(x) + R_n(x),$$

where

$$R_n(x) = -\frac{(b-a)^{n-1}}{n!}\int_a^b\left[B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right)\right]f^{(n)}(t)\mathrm{d}t.$$

The formulae (4.1) and (4.2) have been used in [47] to prove some generalized Ostrowski inequalities.

Further natural generalization of such results arises by replacing the Bernoulli polynomials with an arbitrary harmonic sequence of polynomials.

(2) Assume that $(P_k,\ k \geq 0)$ is a **harmonic sequence of polynomials** i.e. the sequence of polynomials satisfying

$$P_k'(t) = P_{k-1}(t),\ k \geq 1;\ \ P_0(t) = 1.$$

Define $P_k^*(t)$, $k \geq 0$, to be a periodic functions of period 1, related to $P_k(t)$, $k \geq 0$, as

$$P_k^*(t) = P_k(t),\ 0 \leq t < 1,$$

$$P_k^*(t+1) = P_k^*(t),\ t \in \mathbb{R}.$$

Thus, $P_0^*(t) = 1$, while for $k \geq 1$, $P_k^*(t)$ is continuous on $\mathbb{R} \setminus \mathbb{Z}$ and has a jump of

$$\alpha_k = P_k(0) - P_k(1)$$

at every integer t, whenever $\alpha_k \neq 0$. Note that $\alpha_1 = -1$, since $P_1(t) = t + c$, for some $c \in \mathbb{R}$. Also, note that from the definition it follows

$$P_k^{*\prime}(t) = P_{k-1}^*(t),\ \ k \geq 1,\ \ t \in \mathbb{R} \setminus \mathbb{Z}.$$

Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n \geq 1$. In the recent paper [50] the following two identities have been proved:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t + \widetilde{T}_n(x) + \tau_n(x) + \widetilde{R}_n^1(x) \tag{4.3}$$

and

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t + \widetilde{T}_{n-1}(x) + \tau_n(x) + \widetilde{R}_n^2(x), \tag{4.4}$$

where

$$\widetilde{T}_m(x) = \sum_{k=1}^{m} (b-a)^{k-1} P_k\left(\frac{x-a}{b-a}\right)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right],$$

for $1 \leq m \leq n$, and

$$\tau_m(x) = \sum_{k=2}^{m} (b-a)^{k-1} \alpha_k f^{(k-1)}(x),$$

with convention $\widetilde{T}_0(x) = 0$, $\tau_1(x) = 0$, while

$$\widetilde{R}_n^1(x) = -(b-a)^{n-1}\int_{[a,b]} P_n^*\left(\frac{x-t}{b-a}\right)\mathrm{d}f^{(n-1)}(t)$$

and

$$\widetilde{R}_n^2(x) = -(b-a)^{n-1}\int_{[a,b]} \left[P_n^*\left(\frac{x-t}{b-a}\right) - P_n\left(\frac{x-a}{b-a}\right)\right]\mathrm{d}f^{(n-1)}(t).$$

The formulae (4.3) and (4.4) hold for every $x \in [a,b]$. They have been used in [50] to prove some generalized Ostrowski inequalities.

Further natural generalization of such results arises by replacing harmonic sequence of polynomials with a harmonic sequence of functions generated by some weight function. Some results of this type involving integration by parts formula have been obtained recently by Dragomir [51]. For some other weighted generalizations of Euler identity, Ostrowski type inequalities and its discrete analogues the reader is referred to papers [13], [8], and [9].

(3) For $a,b \in \mathbb{R}$, $a < b$, let $w : [a,b] \to [\mathbf{0},\infty)$ be a probability density function i.e. integrable function satisfying

$$\int_a^b w(t)dt = 1.$$

For $n \geq 0$ and $t \in [a,b]$ let

$$w_n(t) = \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}w(s)ds, \;\; n \geq 1; \qquad w_0(t) = w(t).$$

It is well known that w_n is equal to the n-th indefinite integral of w, being equal to zero at a, i.e. $w_n^{(n)}(t) = w(t)$ and $w_n(a) = 0$, for every $n \geq 1$.

A sequence of functions $H_n : [a,b] \to \mathbb{R}$, $n \geq 0$, is called ***w*-harmonic sequence of functions** on $[a,b]$ if

$$H_n'(t) = H_{n-1}(t), \; n \geq 1; \qquad H_0(t) = w(t), \quad t \in [a,b].$$

The sequence $(w_n,\; n \geq 0)$ is an example of w-harmonic sequence of functions on $[a,b]$.

Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n \geq 1$. In a recent paper [48] two identities named the weighted Euler harmonic identities were proved. We state here those two identities with notations which are somewhat different from those used in [48]. The first weighted Euler harmonic identity is

$$f(x) = \int_a^b f_x(t)w(t)dt + \hat{S}_n(x) + \hat{R}_n^1(x) \tag{4.5}$$

and the second one is

$$\begin{aligned} f(x) &= \int_a^b f_x(t)w(t)dt + \hat{S}_{n-1}(x) \\ &\quad + [H_n(a) - H_n(b)]\, f^{(n-1)}(x) + \hat{R}_n^2(x), \end{aligned} \tag{4.6}$$

where

$$\hat{S}_m(x) = \sum_{k=1}^{m} H_k(x)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right] + \sum_{k=2}^{m}\left[H_k(a) - H_k(b)\right] f^{(k-1)}(x)$$

for $1 \leq m \leq n$, with convention

$$\hat{S}_1(x) = H_1(x)\left[f(b) - f(a)\right],$$

and

$$f_x(t) = \begin{cases} f(a+x-t), & a \leq t \leq x \\ f(b+x-t), & x < t \leq b \end{cases}, \tag{4.7}$$

while

$$\hat{R}_n^1(x) = -(b-a)^n \int_{[a,b]} H_n^*\left(\frac{x-t}{b-a}\right) df^{(n-1)}(t)$$

and

$$\hat{R}_n^2(x) = -(b-a)^n \int_{[a,b]} \left[H_n^*\left(\frac{x-t}{b-a}\right) - \frac{1}{(b-a)^n}H_n(x)\right] df^{(n-1)}(t).$$

Here H_n^* denotes a periodic function of period 1 defined by H_n as

$$H_n^*(t) = \frac{1}{(b-a)^n} H_n(a+(b-a)t),\ 0 \le t < 1,\ \text{and}\ H_n^*(t+1) = H_n^*(t),\ t \in \mathbb{R}.$$

Identities (4.5) and (4.6) hold for every $x \in [a,b]$. They were used in [48] to prove some generalized Ostrowski inequalities. The reader can find further references to some recent results on generalizations and applications of Euler identities in the monograph [61], [18] or paper [38]. In this section we generalize formulae (4.5) and (4.6), by replacing the w-harmonic sequence of functions by more general harmonic sequence of functions, and using them we prove some generalizations of Ostrowski's inequality. This section is based on [40].

4.1.2 Euler identities for measures

For $a,b \in \mathbb{R}$, $a<b$, let $C[a,b]$ be the Banach space of all continuous functions $f : [a,b] \to \mathbb{R}$ with the max norm,

$$\|f\| = \max_{a \le t \le b} |f(t)|,$$

and $M[a,b]$ the Banach space of all real Borel measures on $[a,b]$ with the total variation norm,

$$\|\mu\| = |\mu|([a,b]) = \sup_{E_i} \sum_i |\mu(E_i)|,$$

where the sup being taken over all partitions $\{E_i\}$ of $[a,b]$. In the rest of the paper we use the notation

$$\int_{[a,b]} F(s)\mathrm{d}\mu(s)$$

to denote the Lebesgue integral of F over $[a,b]$ with respect to the measure μ, while for a given function $\varphi : [a,b] \to \mathbb{R}$ of bounded variation

$$\int_{[a,b]} F(s)\mathrm{d}\varphi(s)$$

denotes the Lebesgue-Stieltjes integral of F over $[a,b]$ with respect to φ. Also, by

$$\int_a^b F(s)\mathrm{d}s$$

we denote the usual Lebesgue integral of F over $[a,b]$.

For $\mu \in M[a,b]$ define the function $\check{\mu}_n : [a,b] \to \mathbb{R}$, $n \ge 1$, by

$$\check{\mu}_n(t) = \frac{1}{(n-1)!} \int_{[a,t]} (t-s)^{n-1} d\mu(s). \tag{4.8}$$

For $n = 1$,

$$\check{\mu}_1(t) = \int_{[a,t]} d\mu(s) = \mu([a,t]),\ a \le t \le b,$$

which means that $\check{\mu}_1$ is equal to the distribution function of μ.

Substituting

$$\check{\mu}_1(s) = \int_{[a,s]} d\mu(u)$$

in

$$\int_a^t (t-s)^{n-2}\check{\mu}_1(s)ds$$

and using the Fubini theorem we easily get the formula

$$\check{\mu}_n(t) = \frac{1}{(n-2)!}\int_a^t (t-s)^{n-2}\check{\mu}_1(s)ds, \ \ n \geq 2. \tag{4.9}$$

From this formula we see immediately that $\check{\mu}_n(a) = 0, \ n \geq 2$.

Substituting

$$\check{\mu}_n(s) = \frac{1}{(n-1)!}\int_{[a,s]} (s-u)^{n-1} d\mu(u), \ a \leq s \leq b$$

in

$$\int_a^t \check{\mu}_n(s)ds, \ a \leq s \leq t \leq b$$

and using the Fubini theorem once again we easily get the following formula

$$\check{\mu}_{n+1}(t) = \int_a^t \check{\mu}_n(s)ds, \ a \leq t \leq b, \ n \geq 1. \tag{4.10}$$

This means that for $n \geq 1, \ \check{\mu}_{n+1}$ is differentiable at almost all points of $[a,b]$ and

$$\check{\mu}_{n+1}' = \check{\mu}_n$$

almost everywhere on $[a,b]$ with respect to Lebesgue measure.

The function $s \to g(s) = (t-s)^{n-1}$ is nonincreasing on $[a,t]$ so that from (4.8) we get the estimate

$$|\check{\mu}_n(t)| \leq \frac{1}{(n-1)!}(t-a)^{n-1}\|\mu\|, \ t \in [a,b], \ n \geq 1. \tag{4.11}$$

Definition 4.1 *Let $\mu \in M[a,b]$. A sequence of functions $P_n : [a,b] \to \mathbb{R}$, $n \geq 1$, is called a **μ-harmonic sequence of functions** on $[a,b]$ if*

$$P_1(t) = c + \check{\mu}_1(t), \ a \leq t \leq b,$$

for some $c \in \mathbb{R}$, and

$$P_{n+1}(t) = P_{n+1}(a) + \int_a^t P_n(s)ds, \ a \leq t \leq b, \ n \geq 1.$$

Since $P_{n+1},\ n \geq 1$ is defined as an indefinite Lebesgue integral of P_n, it is well known that $P_{n+1},\ n \geq 1$ is an absolutely continuous function,

$$P_{n+1}^{'} = P_n, \text{ a.e. on } [a,b]$$

with respect to Lebesgue measure, and for every $f \in C[a,b]$ we have

$$\int_{[a,b]} f(t)dP_{n+1}(t) = \int_a^b f(t)P_n(t)dt,\ n \geq 1.$$

The sequence $(\check{\mu}_n,\ n \geq 1)$ is an example of a μ-harmonic sequence of functions on $[a,b]$.

Remark 4.1 In the special case, when the measure μ is a probability measure with the density w, the μ-harmonic sequence of functions $(P_n,\ n \geq 1)$ on $[a,b]$ becomes w-harmonic sequence of functions from Introduction. In this case P_1 is differentiable a.e. and $P_1'(t) = w(t)$, a.e..

Assume that $(P_n,\ n \geq 1)$ is a μ-harmonic sequence of functions on $[a,b]$. Define P_n^*, for $n \geq 1$, to be a periodic function of period 1, related to P_n as

$$P_n^*(t) = \frac{1}{(b-a)^n} P_n(a+(b-a)t),\ 0 \leq t < 1,$$

and

$$P_n^*(t+1) = P_n^*(t),\ t \in \mathbb{R}.$$

Thus, for $n \geq 2$, P_n^* is continuous on $\mathbb{R}\backslash\mathbb{Z}$ and has a jump of

$$\alpha_n = [P_n(a) - P_n(b)]/(b-a)^n$$

at every $t \in \mathbb{Z}$, whenever $\alpha_n \neq 0$. Note that for $n \geq 1$,

$$\left(P_{n+1}^*\right)' = P_n^* \text{ a.e. on } \mathbb{R}.$$

For a given μ-harmonic sequence of functions $(P_n,\ n \geq 1)$ and any fixed $x \in [a,b]$ define functions $\varphi_n(x,\cdot) : [a,b] \to \mathbb{R}$, $n \geq 1$, as

$$\varphi_n(x,t) = P_n^*\left(\frac{x-t}{b-a}\right), \quad a \leq t \leq b. \tag{4.12}$$

Note that by the definition of P_n^* we have

$$P_n^*\left(\frac{b-t}{b-a}\right) = P_n^*\left(\frac{b-t}{b-a} - 1\right) = P_n^*\left(\frac{a-t}{b-a}\right),$$

which means that

$$\varphi_n(b,t) = \varphi_n(a,t),\ a \leq t \leq b,$$

for any $n \geq 1$.

Lemma 4.1 *Let $(P_n,\ n \geq 1)$ be a μ-harmonic sequence of functions on $[a,b]$ and $x \in [a,b]$. If $n \geq 2$ and $F \in C[a,b]$, then*

$$\int_{[a,b]} F(t)d\varphi_n(x,t) = -\frac{1}{b-a}\int_a^b F(t)\varphi_{n-1}(x,t)dt - \alpha_n F(x),$$

for $a \leq x < b$, while for $x = b$

$$\int_{[a,b]} F(t)d\varphi_n(b,t) = -\frac{1}{b-a}\int_a^b F(t)\varphi_{n-1}(b,t)dt - \alpha_n F(a).$$

Proof. First assume that $a < x < b$. The function $\varphi_n(x,\cdot)$ is differentiable a.e. on $[a,b] \setminus \{x\}$ and its derivative is equal to $\frac{-1}{b-a}\varphi_{n-1}(x,\cdot)$. Further, it has a jump of $\varphi_n(x,x+0) - \varphi_n(x,x-0) = -\alpha_n$ at x, which gives the first formula in this case. For $x = a$ the function $\varphi_n(a,\cdot)$ is differentiable a.e. on (a,b) and its derivative is equal to $\frac{-1}{b-a}\varphi_{n-1}(a,\cdot)$. Further, it has a jump of $\varphi_n(a,a+0) - \varphi_n(a,a) = -\alpha_n$ at the point a, while $\varphi_n(a,b) - \varphi_n(a,b-0) = 0$, which gives the first formula for $x = a$. The second formula is a consequence of the first one and the fact that $\varphi_n(b,\cdot) = \varphi_n(a,\cdot)$. □

Lemma 4.2 *For every $\mu \in M[a,b]$, $F \in C[a,b]$, and $a \leq x < b$ we have*

$$\int_{[a,b]} F(t)d\varphi_1(x,t) = -\frac{1}{b-a}\int_{[a,b]} F_x(t)d\mu(t) + \frac{1}{b-a}\mu([a,b])F(x) \tag{4.13}$$

while, for $x = b$

$$\int_{[a,b]} F(t)d\varphi_1(b,t) = -\frac{1}{b-a}\int_{[a,b]} F_a(t)d\mu(t) + \frac{1}{b-a}\mu([a,b])F(a), \tag{4.14}$$

where $F_x(t)$ is defined by

$$F_x(t) = \begin{cases} F(a+x-t), & a \leq t \leq x \\ F(b+x-t), & x < t \leq b \end{cases}.$$

Proof. Because of the fact that $\varphi_1(b,t) = \varphi_1(a,t)$, $a \leq t \leq b$, the identity (4.14) follows from (4.13) with $x = a$. Therefore, we may assume that $a \leq x < b$. Since $P_1(t) = c + \check{\mu}_1(t)$, $a \leq t \leq b$, for some constant c, and obviously the integral on the left hand side of (4.13) is independet on the choice of the constant c we may assume that $c = 0$. Therefore, from (4.12) we easily see that for $n = 1$

$$\varphi_1(x,t) = \check{\mu}_1^*\left(\frac{x-t}{b-a}\right) = \frac{1}{b-a} \times \begin{cases} \mu([a,a+x-t]), & \text{for } a \leq t \leq x \\ \mu([a,b+x-t]), & \text{for } x < t \leq b \end{cases}. \tag{4.15}$$

Let us introduce the notations

$$I(F,\mu) = \int_{[a,b]} F(t)d\varphi_1(x,t),$$

$$J(F,\mu) = -\frac{1}{b-a}\int_{[a,b]} F_x(t)d\mu(t) + \frac{1}{b-a}\mu([a,b])F(x).$$

Then $I,J : C[a,b] \times M[a,b] \to \mathbb{R}$ are continuous bilinear functionals with

$$|I(F,\mu)| \leq \frac{1}{b-a} \|F\| \|\mu\|, \qquad |J(F,\mu)| \leq \frac{2}{b-a} \|F\| \|\mu\|.$$

Let us prove that $I(F,\mu) = J(F,\mu)$ for every $F \in C[a,b]$ and every discrete measure $\mu \in M[a,b]$ with finite support. For $y \in [a,b]$ let $\mu = \delta_y$ be the Dirac measure at y, i.e. the measure defined by

$$\int_{[a,b]} F(t) d\delta_y(t) = F(y).$$

If $y > x$, then from (4.15) we get

$$\varphi_1(x,t) = \frac{1}{b-a} \times \begin{cases} 1, & \text{for } x < t \leq b+x-y \\ 0, & \text{for } a \leq t \leq x \text{ or } b+x-y < t \leq b \end{cases},$$

and, by a simple calculation, we see that

$$\begin{aligned} I(F,\delta_y) &= -\frac{1}{b-a} F(b+x-y) + \frac{1}{b-a} F(x) \\ &= -\frac{1}{b-a} F_x(y) + \frac{1}{b-a} \delta_y([a,b]) F(x) = J(F,\delta_y). \end{aligned}$$

Similarly, if $y \leq x$, then from (4.15) we get

$$\varphi_1(x,t) = \frac{1}{b-a} \times \begin{cases} 0, & \text{for } a+x-y < t \leq x \\ 1, & \text{for } a \leq t \leq a+x-y \text{ or } x < t \leq b \end{cases},$$

and by analogous calculation

$$\begin{aligned} I(F,\delta_y) &= -\frac{1}{b-a} F(a+x-y) + \frac{1}{b-a} F(x) \\ &= -\frac{1}{b-a} F_x(y) + \frac{1}{b-a} \delta_y([a,b]) F(x) = J(F,\delta_y). \end{aligned}$$

Therefore, for every $F \in C[a,b]$ and every $y \in [a,b]$ we have $I(F,\delta_y) = J(F,\delta_y)$ which means that (4.13) holds for $\mu = \delta_y$. Every discrete measure $\mu \in M[a,b]$, with finite support, is a linear combination of Dirac measures, i.e. it has the form

$$\mu = \sum_{k=1}^{n} c_k \delta_{x_k},$$

for some real numbers c_k, and $x_k \in [a,b]$. By linearity of I and J we get $I(F,\mu) = J(F,\mu)$, for every $F \in C[a,b]$, which means that (4.13) holds for every discrete measure $\mu \in M[a,b]$ with finite support.

Let $\mathscr{T}$ be the minimal topology on $M[a,b]$ such that linear functionals

$$\mu \mapsto \int F d\mu$$

are continuous, for every bounded Borel function $F : [a,b] \to \mathbb{R}$. By the definition we see that $\mathscr{T}$ contains the weak* topology on $M[a,b]$ and is contained in the weak topology on $M[a,b]$. Further, the curve $x \mapsto \delta_x$ is bounded and $\mathscr{T}$-measurable since

$$x \mapsto \int F d\delta_x = F(x)$$

is measurable by assumption. Therefore, the integral $\int \delta_x d\mu(x)$ exists in the $\mathscr{T}$ topology, for every $\mu \in M[a,b]$. It is easy to see that this integral is equal to μ i.e.

$$\int \delta_x d\mu(x) = \mu,$$

for every measure $\mu \in M[a,b]$, which means that μ is a $\mathscr{T}$-limit of a sequence of discrete measures with finite support. Thus, we conclude that the subspace of all discrete measures with finite support is $\mathscr{T}$-dense in $M[a,b]$, and therefore the functionals $I(F,\cdot)$ and $J(F,\cdot)$ are equal, for every $F \in C[a,b]$, since they are equal on a dense subspace and they are $\mathscr{T}$-continuous. □

Let $(P_n,\ n \geq 1)$ be a μ-harmonic sequence of functions on $[a,b]$ and let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ exists on $[a,b]$, for some $n \geq 1$. For every $x \in [a,b]$ and $1 \leq m \leq n$ we introduce the following notation

$$S_m(x) = \sum_{k=1}^{m} P_k(x) \left[f^{(k-1)}(b) - f^{(k-1)}(a) \right] + \sum_{k=2}^{m} \left[P_k(a) - P_k(b) \right] f^{(k-1)}(x) \tag{4.16}$$

with convention

$$S_1(x) = P_1(x) \left[f(b) - f(a) \right].$$

Theorem 4.1 *For $\mu \in M[a,b]$ let $(P_n,\ n \geq 1)$ be a μ-harmonic sequence of functions on $[a,b]$ and $f : [a,b] \to \mathbb{R}$ such that $f^{(n-1)}$ is a continuous function of bounded variation for some $n \geq 1$. Then for every $x \in [a,b]$*

$$\mu([a,b]) f(x) = \int_{[a,b]} f_x(t) d\mu(t) + S_n(x) + R_n^1(x) \tag{4.17}$$

where $S_n(x)$ is defined by (4.16), $f_x(t)$ by (4.7) and

$$R_n^1(x) = -(b-a)^n \int_{[a,b]} P_n^* \left(\frac{x-t}{b-a} \right) df^{(n-1)}(t).$$

Proof. For $1 \leq k \leq n$ consider the integral

$$I_k(x) = (b-a)^k \int_{[a,b]} P_k^* \left(\frac{x-t}{b-a} \right) df^{(k-1)}(t).$$

Integrating by parts we get

$$I_k(x) \ = \ (b-a)^k P_k^* \left(\frac{x-t}{b-a} \right) f^{(k-1)}(t) \Big|_a^b$$

$$-(b-a)^k \int_{[a,b]} f^{(k-1)}(t)\,d\varphi_k(x,t) \tag{4.18}$$

where $\varphi_k(x,t)$ is defined by (4.12). First, assume that $a \le x < b$. For every $k \ge 1$ we have

$$P_k^*\left(\frac{x-b}{b-a}\right) = P_k^*\left(\frac{x-a}{b-a}-1\right) = P_k^*\left(\frac{x-a}{b-a}\right) = \frac{1}{(b-a)^k}P_k(x).$$

Therefore, using the first formula from Lemma 4.1, we get for $k \ge 2$

$$\begin{aligned} I_k(x) &= P_k(x)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right] + (b-a)^k \alpha_k f^{(k-1)}(x) \\ &\quad + (b-a)^{k-1}\int_a^b f^{(k-1)}(t) P_{k-1}^*\left(\frac{x-t}{b-a}\right) dt. \end{aligned} \tag{4.19}$$

Also, for $k \ge 2$ we have

$$\begin{aligned} &(b-a)^{k-1}\int_a^b f^{(k-1)}(t) P_{k-1}^*\left(\frac{x-t}{b-a}\right) dt \\ &= (b-a)^{k-1}\int_{[a,b]} P_{k-1}^*\left(\frac{x-t}{b-a}\right) df^{(k-2)}(t) = I_{k-1}(x). \end{aligned}$$

By Lemma 4.2, for $k=1$, (4.18) becomes

$$\begin{aligned} I_1(x) &= P_1(x)\left[f(b)-f(a)\right] - (b-a)\int_{[a,b]} f(t)\,d\varphi_1(x,t) \\ &= P_1(x)\left[f(b)-f(a)\right] + \int_{[a,b]} f_x(t)\,d\mu(t) - \mu([a,b])f(x), \end{aligned} \tag{4.20}$$

where $f_x(t)$ is given by (4.7). Therefore, (4.19) can be rewritten as

$$I_k(x) = P_k(x)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right] + (b-a)^k \alpha_k f^{(k-1)}(x) + I_{k-1}(x)$$

which is equal to

$$P_k(x)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right] + \left[P_k(a) - P_k(b)\right] f^{(k-1)}(x) + I_{k-1}(x) \tag{4.21}$$

since $\alpha_k = [P_k(a) - P_k(b)]/(b-a)^k$. From (4.20) and (4.21) we obtain

$$\begin{aligned} I_n(x) &= \sum_{k=1}^n P_k(x)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right] \\ &\quad + \sum_{k=2}^n \left[P_k(a) - P_k(b)\right] f^{(k-1)}(x) + \int_{[a,b]} f_x(t)\,d\mu(t) - \mu([a,b])f(x) \end{aligned}$$

which proves our assertion in this case, since $I_n(x) = -R_n^1(x)$. Thus, (4.17) holds for $a \le x < b$. If $x = b$, then

$$P_k^*\left(\frac{b-b}{b-a}\right) = P_k^*(0) = P_k^*(1) = P_k^*\left(\frac{b-a}{b-a}\right) = \frac{1}{(b-a)^k}P_k(a).$$

Similarly as we did for $a \le x < b$, using the above equalities and the second formula from Lemma 4.1, we get

$$I_k(b) = P_k(a)[f^{(k-1)}(b) - f^{(k-1)}(a)] + (b-a)^k \alpha_k f^{(k-1)}(a) + I_{k-1}(b)$$

for $k \ge 2$, and

$$I_1(b) = P_1(a)\,[f(b) - f(a)] - \mu([a,b])f(a) + \int_{[a,b]} f_a(t)d\mu(t).$$

Applying the above identities, we get

$$\begin{aligned}
I_n(b) &= \sum_{k=1}^{n} P_k(a)[f^{(k-1)}(b) - f^{(k-1)}(a)] \\
&\quad + \sum_{k=2}^{n} [P_k(a) - P_k(b)]\, f^{(k-1)}(a) - \mu([a,b])f(a) + \int_{[a,b]} f_a(t)d\mu(t) \\
&= \sum_{k=1}^{n} P_k(b)[f^{(k-1)}(b) - f^{(k-1)}(a)] \\
&\quad + \sum_{k=2}^{n} [P_k(a) - P_k(b)]\, f^{(k-1)}(b) - \mu([a,b])f(b) + \int_{[a,b]} f_b(t)d\mu(t)
\end{aligned}$$

since

$$f_b(t) - f_a(t) = 0, \; a < t \le b,$$
$$f_b(a) - f_a(a) = f(b) - f(a)$$

and

$$\int_{[a,b]} [f_b(t) - f_a(t)]\, d\mu(t) = [f(b) - f(a)]\, \mu(\{a\}),$$

while

$$\begin{aligned}
P_1(b) - P_1(a) &= \check{\mu}_1(b) - \check{\mu}_1(a) \\
&= \mu([a,b]) - \mu(\{a\}).
\end{aligned}$$

This proves (4.17) for $x = b$, since $I_n(b) = -R_n^1(b)$. □

Theorem 4.2 *For $\mu \in M[a,b]$ let $(P_n,\ n \ge 1)$ be a μ-harmonic sequence of functions on $[a,b]$ and $f : [a,b] \to \mathbb{R}$ such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n \ge 1$. Then for every $x \in [a,b]$ and $n \ge 2$*

$$\begin{aligned}
\mu([a,b])f(x) &= \int_{[a,b]} f_x(t)d\mu(t) + S_{n-1}(x) \\
&\quad + [P_n(a) - P_n(b)]\, f^{(n-1)}(x) + R_n^2(x), \qquad (4.22)
\end{aligned}$$

while for $n = 1$

$$\mu([a,b])f(x) = \int_{[a,b]} f_x(t)d\mu(t) + R_1^2(x),$$

where $S_{n-1}(x)$ is defined by (4.16), and for $n \geq 1$

$$R_n^2(x) = -(b-a)^n \int_{[a,b]} \left[P_n^* \left(\frac{x-t}{b-a} \right) - \frac{1}{(b-a)^n} P_n(x) \right] df^{(n-1)}(t).$$

Proof. Note first that for $n \geq 2$

$$S_n(x) - S_{n-1}(x) = P_n(x) \left[f^{(n-1)}(b) - f^{(n-1)}(a) \right] + \left[P_n(a) - P_n(b) \right] f^{(n-1)}(x).$$

Thus for $n \geq 2$

$$\begin{aligned} R_n^2(x) &= R_n^1(x) + P_n(x) \int_{[a,b]} df^{(n-1)}(t) \\ &= R_n^1(x) + P_n(x) \left[f^{(n-1)}(b) - f^{(n-1)}(a) \right] \\ &= R_n^1(x) + S_n(x) - S_{n-1}(x) - \left[P_n(a) - P_n(b) \right] f^{(n-1)}(x), \end{aligned}$$

while

$$R_1^2(x) = R_1^1(x) + P_1(x) \left[f(b) - f(a) \right].$$

Therefore, our assertion follows from Theorem 4.1. □

Remark 4.2 In the special case, when μ is a probability measure with density w, formulae (4.17) and (4.22) reduce to (4.5) and (4.6), respectively, since in this case $\mu([a,b]) = 1$ and for every $x \in [a,b]$ we have

$$\int_{[a,b]} f_x(t) d\mu(t) = \int_a^b f_x(t) w(t) dt.$$

4.1.3 Some Ostrowski type inequalities

In this section we use the identities obtained in Theorems 4.1 and 4.2 to prove a number of inequalities which hold for a class of functions whose derivatives are either L-Lipschitzian on $[a,b]$ or continuous and of bounded variation on $[a,b]$. Analogous results can be obtained for a class of functions f such that $f^{(n)}$ are in $L_p[a,b]$, $1 \leq p \leq \infty$. Throughout this section we use the same notations as in the previous section.

Lemma 4.3 *For $\mu \in M[a,b]$ let $(P_n,\ n \geq 1)$ be a μ-harmonic sequence of functions on $[a,b]$. Then for every integrable function $f : [a,b] \to \mathbb{R}$, and every continuous function $F : \mathbb{R} \to \mathbb{R}$ we have*

$$\int_a^b f(t) F\left(P_n^* \left(\frac{x-t}{b-a} \right) \right) dt = \int_a^b f_x(t) F\left(\frac{1}{(b-a)^n} P_n(t) \right) dt \tag{4.23}$$

for every $x \in [a,b]$ and $n \geq 1$, where $f_x(t)$ is defined by (4.7).

Proof. For any function $f : [a,b] \to \mathbb{R}$ and any fixed $x \in [a,b]$ let $f_x : [a,b] \to \mathbb{R}$ be defined by (4.7). It is easily seen that for $x \in [a,b)$ we have

$$(f_x)_x(t) = f(t),\ a \leq t < b,$$

and

$$(f_x)_x(b) = f(a),$$

while for $x = b$ we have

$$(f_b)_b(t) = f(t), a \leq t \leq b.$$

Consequently, for any two integrable functions $f,\ g : [a,b] \to \mathbb{R}$ and for any fixed $x \in [a,b]$ by a simple calculation we get

$$\int_a^b f(t)g_x(t)dt = \int_a^b f_x(t)g(t)dt.$$

Applying this identity to $f(t)$ and

$$g(t) = F\left(\frac{1}{(b-a)^n}P_n(t)\right),\ a \leq t \leq b,$$

we obtain (4.23), since the equality

$$F\left(P_n^*\left(\frac{x-t}{b-a}\right)\right) = g_x(t)$$

holds for all $x \in [a,b]$ and $a \leq t \leq b$, except when $x = b$ and $t = a$. □

Theorem 4.3 *For $\mu \in M[a,b]$ let $(P_n,\ n \geq 1)$ be a μ-harmonic sequence of functions on $[a,b]$. Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is L-Lipschitzian function on $[a,b]$ for some $n \geq 1$. If $x \in [a,b]$, then for $n \geq 2$ the absolute value of*

$$\mu([a,b])f(x) - \int_{[a,b]} f_x(t)d\mu(t) - S_{n-1}(x) - [P_n(a) - P_n(b)]\, f^{(n-1)}(x)$$

is less than or equal to

$$L\int_a^b |P_n(t) - P_n(x)|\, dt,$$

while for $n = 1$

$$\left|\mu([a,b])f(x) - \int_{[a,b]} f_x(t)d\mu(t)\right| \leq L\int_a^b |P_1(t) - P_1(x)|\, dt.$$

Proof. If $\varphi : [a,b] \to \mathbb{R}$ is L-Lipschitzian on $[a,b]$, then for any integrable function $g : [a,b] \to \mathbb{R}$

$$\left|\int_{[a,b]} g(t)d\varphi(t)\right| \leq L\int_a^b |g(t)|\, dt. \tag{4.24}$$

Using this estimate and Lemma 4.3 we get

$$\begin{aligned}|R_n^2(x)| &= (b-a)^n\left|\int_{[a,b]}\left[P_n^*\left(\frac{x-t}{b-a}\right) - \frac{1}{(b-a)^n}P_n(x)\right]df^{(n-1)}(t)\right| \\ &\leq (b-a)^n L\int_a^b \left|P_n^*\left(\frac{x-t}{b-a}\right) - \frac{1}{(b-a)^n}P_n(x)\right| dt\end{aligned}$$

$$= (b-a)^n L \int_a^b \left| \frac{1}{(b-a)^n} P_n(t) - \frac{1}{(b-a)^n} P_n(x) \right| dt$$
$$= L \int_a^b |P_n(t) - P_n(x)|\, dt.$$

Therefore, our assertion follows from Theorem 4.2. □

Corollary 4.1 *If f is L-Lipschitzian on $[a,b]$ and $\mu \geq 0$, then for every $x \in [a,b]$*

$$\left| \mu([a,b]) f(x) - \int_{[a,b]} f_x(t) d\mu(t) \right| \leq L [(2x-a-b)\check{\mu}_1(x) - 2\check{\mu}_2(x) + \check{\mu}_2(b)].$$

Proof. Apply Theorem 4.3 to $n=1$. We have $P_1(t) = c + \check{\mu}_1(t)$ for some real constant c, so that $P_1(t) - P_1(x) = \check{\mu}_1(t) - \check{\mu}_1(x)$. Also $\mu \geq 0$ implies that $\check{\mu}_1$ is nondecreasing on $[a,b]$ and

$$\int_a^b |P_1(t) - P_1(x)|\, dt = \int_a^x (\check{\mu}_1(x) - \check{\mu}_1(t))\, dt + \int_x^b (\check{\mu}_1(t) - \check{\mu}_1(x))\, dt$$
$$= (2x-a-b)\check{\mu}_1(x) - 2\check{\mu}_2(x) + \check{\mu}_2(b),$$

which proves the assertion. □

Corollary 4.2 *If $f : [a,b] \to \mathbb{R}$ is such that f' is L-Lipschitzian on $[a,b]$, then for every $x \in [a,b]$ and $c \in \mathbb{R}$ the absolute value of*

$$\mu([a,b]) f(x) - \int_{[a,b]} f_x(t) d\mu(t) - [c + \check{\mu}_1(x)] [f(b) - f(a)] + [c(b-a) + \check{\mu}_2(b)] f'(x)$$

is less than or equal to

$$L \int_a^b |c(t-x) + \check{\mu}_2(t) - \check{\mu}_2(x)|\, dt.$$

Proof. $P_1(t) = c + \check{\mu}_1(t)$ and $P_2(t) = c_1 + c(t-a) + \check{\mu}_2(t)$ are two beginning terms of a μ-harmonic sequence $(P_n,\ n \geq 1)$ of functions on $[a,b]$. Apply Theorem 4.3 to $n=2$ and use the relations

$$S_1(x) = [c + \check{\mu}_1(x)] [f(b) - f(a)],$$
$$P_2(a) - P_2(b) = -c(b-a) - \check{\mu}_2(b)$$

and

$$P_2(t) - P_2(x) = c(t-x) + \check{\mu}_2(t) - \check{\mu}_2(x).$$

□

Remark 4.3 If $\mu \geq 0$, then $\check{\mu}_1 \geq 0$ so that $\check{\mu}_2$ is nondecreasing. Therefore, when $c \geq 0$ and $\mu \geq 0$, we see that

$$\int_a^b |c(t-x) + \check{\mu}_2(t) - \check{\mu}_2(x)|\, dt$$

is equal to

$$\frac{c}{2} \left[(x-a)^2 + (b-x)^2 \right] + (2x-a-b)\check{\mu}_2(x) - 2\check{\mu}_3(x) + \check{\mu}_3(b).$$

Corollary 4.3 *Under the assumptions of Theorem 4.3, if $n \geq 2$ and $\mu \geq 0$, then for every $x \in [a,b]$ the absolute value of*

$$\mu([a,b])f(x) - \int_{[a,b]} f_x(t)d\mu(t) - \check{S}_{n-1}(x) + \check{\mu}_n(b)f^{(n-1)}(x)$$

is less than or equal to

$$L[(2x-a-b)\check{\mu}_n(x) - 2\check{\mu}_{n+1}(x) + \check{\mu}_{n+1}(b)],$$

where

$$\check{S}_1(x) = \check{\mu}_1(x)[f(b) - f(a)]$$

and for $2 \leq m \leq n$

$$\check{S}_m(x) = \sum_{k=1}^{m} \check{\mu}_k(x)[f^{(k-1)}(b) - f^{(k-1)}(a)] - \sum_{k=2}^{m} \check{\mu}_k(b)f^{(k-1)}(x). \tag{4.25}$$

Proof. Apply Theorem 4.3 to the μ-harmonic sequence $(\check{\mu}_n,\ n \geq 1)$. Then $S_{n-1}(x)$ becomes $\check{S}_{n-1}(x)$, while $\mu \geq 0$ implies that $\check{\mu}_n$ is nondecreasing on $[a,b]$ and

$$\begin{aligned}\int_a^b |\check{\mu}_n(t) - \check{\mu}_n(x)|\,dt &= \int_a^x [\check{\mu}_n(x) - \check{\mu}_n(t)]\,dt + \int_x^b [\check{\mu}_n(t) - \check{\mu}_n(x)]\,dt \\ &= (2x-a-b)\check{\mu}_n(x) - 2\check{\mu}_{n+1}(x) + \check{\mu}_{n+1}(b),\end{aligned}$$

which proves our assertion. □

Corollary 4.4 *Let f be as in Theorem 4.3. If $n \geq 2$, then for every $x, y \in [a,b]$ such that $y \leq x$, the absolute value of*

$$f(x) - f(a+x-y) - T_{n-1}(x,y) + \tfrac{(b-y)^{n-1}}{(n-1)!} f^{(n-1)}(x)$$

is less than or equal to

$$L\left[(2x-a-b)\frac{(x-y)^{n-1}}{(n-1)!} - 2\frac{(x-y)^n}{n!} + \frac{(b-y)^n}{n!}\right],$$

where

$$T_1(x,y) = (x-y)[f(b) - f(a)]$$

and

$$\begin{aligned}T_m(x,y) = &\sum_{k=1}^{m} \frac{1}{(k-1)!}(x-y)^{k-1}[f^{(k-1)}(b) - f^{(k-1)}(a)] \\ &- \sum_{k=2}^{m} \frac{1}{(k-1)!}(b-y)^{k-1} f^{(k-1)}(x),\ 2 \leq m \leq n.\end{aligned} \tag{4.26}$$

Proof. Apply Corollary 4.3 to $\mu = \delta_y$, $a \le y \le x$, and note that then $S_{n-1}(x)$ becomes $T_{n-1}(x,y)$, while

$$\check{\mu}_n(t) = 0,\ a \le t < y,$$

and

$$\check{\mu}_n(t) = \frac{1}{(n-1)!}(t-y)^{n-1},\ \ y \le t \le b,$$

for every $n \ge 1$. □

Theorem 4.4 *For $\mu \in M[a,b]$ let $(P_n,\ n \ge 0)$ be a μ-harmonic sequence of functions on $[a,b]$. Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is L-Lipschitzian function on $[a,b]$ for some $n \ge 1$. Then for every $x \in [a,b]$*

$$\left|\mu([a,b])f(x) - \int_{[a,b]} f_x(t)d\mu(t) - S_n(x)\right| \le L\int_a^b |P_n(t)|\,dt. \tag{4.27}$$

Proof. Using estimate (4.24) and arguing similarly as in the proof of Theorem 4.3, we get

$$\begin{aligned}
|R_n^1(x)| &= (b-a)^n \left|\int_{[a,b]} P_n^*\left(\frac{x-t}{b-a}\right) df^{(n-1)}(t)\right| \\
&\le (b-a)^n L \int_a^b \left|P_n^*\left(\frac{x-t}{b-a}\right)\right| dt \\
&= L\int_a^b |P_n(t)|\,dt.
\end{aligned}$$

Therefore, our assertion follows from Theorem 4.1. □

Corollary 4.5 *Let $\mu \in M[a,b]$ and assume f to be L-Lipschitzian on $[a,b]$.*
(i) For every $x \in [a,b]$ and $c \in \mathbb{R}$ we have

$$\left|\mu([a,b])f(x) - \int_{[a,b]} f_x(t)d\mu(t) - [c+\check{\mu}_1(x)]\,[f(b)-f(a)]\right| \le L\int_a^b |c+\check{\mu}_1(t)|dt.$$

(ii) If $\mu \ge 0$ and $x,y \in [a,b]$, then

$$\begin{aligned}
&\left|\mu([a,b])f(x) - \int_{[a,b]} f_x(t)d\mu(t) - [\check{\mu}_1(x)-\check{\mu}_1(y)]\,[f(b)-f(a)]\right| \\
&\le\ L\left[(2y-a-b)\check{\mu}_1(y) - 2\check{\mu}_2(y) + \check{\mu}_2(b)\right].
\end{aligned}$$

Proof. To prove (i) apply theorem above to $n = 1$. If $\mu \ge 0$ then $\check{\mu}_1$ is nondecreasing so that (ii) follows from (i) with $c = -\check{\mu}_1(y)$. □

Corollary 4.6 *Under the assumptions of Theorem 4.4, for $\mu \ge 0$ we have*

$$\begin{aligned}
\left|\mu([a,b])f(x) - \int_{[a,b]} f_x(t)d\mu(t) - \check{S}_n(x)\right| &\le\ L\check{\mu}_{n+1}(b) \\
&\le\ \frac{1}{n!}L(b-a)^n\,\|\mu\|
\end{aligned}$$

for every $x \in [a,b]$, where $\check{S}_n(x)$ is defined by (4.25).

Proof. Apply Theorem 4.4 to the μ-harmonic sequence $(\check{\mu}_n,\ n\geq 1)$. Then $S_n(x)$ becomes $\check{S}_n(x)$, while $\mu\geq 0$ implies that $\check{\mu}_n$ is nondecreasing on $[a,b]$ and we have

$$\int_a^b |\check{\mu}_n(t)|\,dt=\int_a^b \check{\mu}_n(t)dt=\check{\mu}_{n+1}(b)\leq \frac{1}{n!}(b-a)^n\,\|\mu\|$$

for every $n\geq 1$. □

Corollary 4.7 *Under the assumptions of Theorem 4.4 we have*

$$|f(x)-f(a+x-y)-T_n(x,y)|\leq L\frac{1}{n!}(b-y)^n$$

for every $x,y\in[a,b]$, $a\leq y\leq x$, where $T_n(x,y)$ is defined by (4.26).

Proof. Apply Corollary 4.6 to the special case $\mu=\delta_y$, for $a\leq y\leq x$. □

Theorem 4.5 *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n\geq 1$. Then for $n\geq 2$ the absolute value of*

$$\mu([a,b])f(x)-\int_{[a,b]} f_x(t)d\mu(t)-S_{n-1}(x)-[P_n(a)-P_n(b)]\,f^{(n-1)}(x)$$

is less than or equal to

$$\sup_{t\in[a,b]}|P_n(t)-P_n(x)|\cdot V_a^b(f^{(n-1)}),$$

while for $n=1$

$$\left|\mu([a,b])f(x)-\int_{[a,b]} f_x(t)d\mu(t)\right|\leq \sup_{t\in[a,b]}|P_1(t)-P_1(x)|\cdot V_a^b(f),$$

for every $x\in[a,b]$, where $V_a^b(f)$ is the total variation of f on $[a,b]$.

Proof. If $F:[a,b]\to\mathbb{R}$ is bounded and the Stieltjes integral $\int_{[a,b]}F(t)df^{(n-1)}(t)$ exists, then

$$\left|\int_{[a,b]}F(t)df^{(n-1)}(t)\right|\leq \sup_{t\in[a,b]}|F(t)|\cdot V_a^b(f^{(n-1)}).$$

Applying formula (4.22) we get

$$\begin{aligned}
|R_n^2(x)| &= \left|-(b-a)^n\int_a^b\left[P_n^*\left(\frac{x-t}{b-a}\right)-P_n^*\left(\frac{x-a}{b-a}\right)\right]df^{(n-1)}(t)\right|\\
&\leq (b-a)^n\sup_{t\in[a,b]}\left|P_n^*\left(\frac{x-t}{b-a}\right)-P_n^*\left(\frac{x-a}{b-a}\right)\right|\cdot V_a^b(f^{(n-1)})\\
&= \sup_{t\in[a,b]}|P_n(t)-P_n(x)|\cdot V_a^b(f^{(n-1)}),
\end{aligned}$$

which proves our assertion. □

Corollary 4.8 *If f is a continuous function of bounded variation on $[a,b]$ and $\mu \geq 0$, then*

$$\left|\mu([a,b])f(x) - \int_{[a,b]} f_x(t)d\mu(t)\right| \leq \frac{1}{2}\left[\check{\mu}_1(b) - \check{\mu}_1(a) + |\check{\mu}_1(a) + \check{\mu}_1(b) - 2\check{\mu}_1(x)|\right] V_a^b(f).$$

Proof. Apply Theorem 4.5 to $n = 1$ and notice that

$$\begin{aligned}\sup_{t\in[a,b]} |P_1(t) - P_1(x)| &= \sup_{t\in[a,b]} |\check{\mu}_1(t) - \check{\mu}_1(x)| \\ &= \max\{\check{\mu}_1(x) - \check{\mu}_1(a), \check{\mu}_1(b) - \check{\mu}_1(x)\} \\ &= \frac{1}{2}\left[\check{\mu}_1(b) - \check{\mu}_1(a) + |\check{\mu}_1(a) + \check{\mu}_1(b) - 2\check{\mu}_1(x)|\right].\end{aligned}$$

□

Corollary 4.9 *If f' is a continuous function of bounded variation on $[a,b]$, then for every $x \in [a,b]$ and $c \in \mathbb{R}$ the absolute value of*

$$\mu([a,b])f(x) - \int_{[a,b]} f_x(t)d\mu(t) - [c + \check{\mu}_1(x)][f(b) - f(a)] + [c(b-a) + \check{\mu}_2(b)] f'(x)$$

is less than or equal to

$$\sup_{t\in[a,b]} |c(t-x) + \check{\mu}_2(t) - \check{\mu}_2(x)| \cdot V_a^b(f').$$

Proof. Apply Theorem 4.5 to $n = 2$. □

Corollary 4.10 *Under the conditions of Theorem 4.5 if $\mu \geq 0$, then*

$$\begin{aligned}&\left|\mu([a,b])f(x) - \int_{[a,b]} f_x(t)d\mu(t) - \check{S}_{n-1}(x) + \check{\mu}_n(b) f^{(n-1)}(x)\right| \\ &\leq \frac{1}{2}\left[\check{\mu}_n(b) + |\check{\mu}_n(b) - 2\check{\mu}_n(x)|\right] \cdot V_a^b(f^{(n-1)})\end{aligned}$$

for every $x \in [a,b]$ and $n \geq 2$, where $\check{S}_{n-1}(x)$ is defined by (4.25).

Proof. Apply Theorem 4.5 to μ-harmonic sequence $(\check{\mu}_n)$. Then $S_n(x)$ becomes $\check{S}_n(x)$ while $\check{\mu}_n(a) = 0$, for $n \geq 2$ and

$$\max_{t\in[a,b]} |\check{\mu}_n(t) - \check{\mu}_n(x)| = \frac{1}{2}\left[\check{\mu}_n(b) + |\check{\mu}_n(b) - 2\check{\mu}_n(x)|\right]$$

for every $n \geq 2$. □

Corollary 4.11 *Under the conditions of Theorem 4.5 we have*

$$\begin{aligned}&\left|f(x) - f(a+x-y) - T_{n-1}(x,y) + \frac{(b-y)^{n-1}}{(n-1)!} f^{(n-1)}(x)\right| \\ &\leq \frac{(b-y)^{n-1}}{2(n-1)!}\left[1 + \left|1 - 2\frac{(x-y)^{n-1}}{(b-y)^{n-1}}\right|\right] V_a^b(f^{(n-1)})\end{aligned}$$

for every $x, y \in [a,b]$, $a \leq y \leq x$, and $n \geq 2$, where $T_{n-1}(x,y)$ is defined by (4.26).

Proof. Apply Corollary 4.10 to the special case when $\mu = \delta_y$. □

Theorem 4.6 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n \geq 1$. Then*

$$\left| \mu([a,b])f(x) - \int_{[a,b]} f_x(t)d\mu(t) - S_n(x) \right| \leq \sup_{t\in[a,b]} |P_n(t)| \cdot V_a^b(f^{(n-1)})$$

for every $x \in [a,b]$.

Proof. Argue similarly as in the proof of Theorem 4.5, using identity (4.17). □

Corollary 4.12 *For $\mu \in M[a,b]$ let f be a continuous function of bounded variation on $[a,b]$.*
(i) For every $x \in [a,b]$ and $c \in \mathbb{R}$ we have

$$\left| \mu([a,b])f(x) - \int_{[a,b]} f_x(t)d\mu(t) - [c + \check{\mu}_1(x)]\,[f(b) - f(a)] \right| \leq \sup_{t\in[a,b]} |c + \check{\mu}_1(t)| \cdot V_a^b(f).$$

(ii) If $\mu \geq 0$ and $x, y \in [a,b]$, then

$$\begin{aligned} &\left| \mu([a,b])f(x) - \int_{[a,b]} f_x(t)d\mu(t) - [\check{\mu}_1(x) - \check{\mu}_1(y)]\,[f(b) - f(a)] \right| \\ &\leq \frac{1}{2}\left[\check{\mu}_1(b) - \check{\mu}_1(a) + |\check{\mu}_1(a) + \check{\mu}_1(b) - 2\check{\mu}_1(y)|\right] \cdot V_a^b(f). \end{aligned}$$

Proof. To prove (i) apply Theorem 4.6 to $n = 1$. If $\mu \geq 0$ then $\check{\mu}_1$ is nondecreasing so that (ii) follows from (i) with $c = -\check{\mu}_1(y)$, since

$$\begin{aligned} \sup_{t\in[a,b]} |\check{\mu}_1(t) - \check{\mu}_1(y)| &= \max\{\check{\mu}_1(y) - \check{\mu}_1(a), \check{\mu}_1(b) - \check{\mu}_1(y)\} \\ &= \frac{1}{2}\left[\check{\mu}_1(b) - \check{\mu}_1(a) + |\check{\mu}_1(a) + \check{\mu}_1(b) - 2\check{\mu}_1(y)|\right]. \end{aligned}$$

□

Corollary 4.13 *Under the conditions of Theorem 4.6, for $\mu \geq 0$*

$$\begin{aligned} \left| \mu([a,b])f(x) - \int_{[a,b]} f_x(t)d\mu(t) - \check{S}_n(x) \right| &\leq \check{\mu}_n(b)V_a^b(f^{(n-1)}) \\ &\leq \frac{1}{(n-1)!}(b-a)^{n-1}\,\|\mu\|\,V_a^b(f^{(n-1)}) \end{aligned}$$

for every $x \in [a,b]$, where $\check{S}_n(x)$ is defined by (4.25).

Proof. Apply Theorem 4.6 to μ-harmonic sequence $(\check{\mu}_n,\ n \geq 1)$. Then $S_n(x)$ becomes $\check{S}_n(x)$ while

$$\sup_{t\in[a,b]} \check{\mu}_n(t) = \check{\mu}_n(b) \leq \frac{1}{(n-1)!}(b-a)^{n-1}\,\|\mu\|\,.$$

□

Corollary 4.14 *Under the conditions of Theorem 4.6 we have*

$$|f(x) - f(a+x-y) - T_n(x,y)| \le \frac{1}{(n-1)!}(b-a)^{n-1}V_a^b(f^{(n-1)})$$

for every $x,y \in [a,b]$, $a \le y \le x$, where $T_n(x,y)$ is defined by (4.26).

Proof. Apply Corollary 4.13 to $\mu = \delta_y$ and note that

$$\check{\mu}_n(b) = \frac{(b-y)^{n-1}}{(n-1)!}.$$

□

Remark 4.4 If in Theorems 4.3 and 4.4 we additionally assume that $f^{(n)}$ exists and $f^{(n)} \in L_\infty[a,b]$, then we obtain the estimates with $L = \|f^{(n)}\|_\infty$. Clearly, this is true for all corollaries of these theorems. Similarly, if in Theorems 4.5 and 4.6 we additionally assume that $f^{(n)}$ exists and $f^{(n)} \in L_1[a,b]$, then we obtain the estimates with $V_a^b(f^{(n-1)})$ replaced by $\|f^{(n)}\|_1$ and this is true for all corollaries of these theorems. Finally, if we assume that $f^{(n)} \in L_p[a,b]$, $1 < p < \infty$, then we can use the Hölder's inequality to obtain the estimates

$$\left|R_n^1(x)\right| \le \|P_n\|_q \cdot \|f^{(n)}\|_p$$

and

$$\left|R_n^2(x)\right| \le \|P_n - P_n(x)\|_q \cdot \|f^{(n)}\|_p,$$

for every $x \in [a,b]$, where $1/p + 1/q = 1$.

In the next section, obtained Euler identities will be applied to the Grüss type inequality.

4.2 Euler-Grüss type inequalities involving measures

4.2.1 Introduction

Similar to the previous section, this one is also devoted to a generalization of another well known inequality – the **Grüss inequality** (1935):

Let f and g be two functions defined and integrable on $[a,b]$. Further, let

$$\gamma \le f(x) \le \Gamma, \quad \varphi \le g(x) \le \Phi,$$

for all $x \in [a,b]$, where γ, Γ, φ, Φ are given real constants. *H. Grüss stated the hypothesis that*

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x - \frac{1}{(b-a)^2}\int_a^b f(x)\mathrm{d}x\int_a^b g(x)\mathrm{d}x\right| \le \frac{1}{4}(\Gamma-\gamma)(\Phi-\varphi).$$

In [64] G. Grüss proved that previous relation is valid and that the constant $\frac{1}{4}$ *is the best possible.*

A great deal of effort has been invested in developing generalizations, extensions, and refinements of the Grüss inequality. The reader is referred to books [60], [97], and [18] for more extensive exploration.

Recently, the following theorem was proved [81, Theorem 4]:

Theorem A *Let* $f,\ g:[a,b]\to\mathbb{R}$ *be integrable functions such that*

$$m\le f(t)\le M,\ a.e.$$

for some $m,\ M\in\mathbb{R}$ and

$$\int_a^b g(t)dt=0.$$

Then

$$\left|\int_a^b f(t)g(t)dt\right|\le\frac{1}{2}(M-m)\int_a^b|g(t)|\,dt,$$

with equality if and only if either

$$f(t)=M,\ t\in I_+\ \textit{and}\ f(t)=m,\ t\in I_-,\ \ a.e.\ on\ I_+\cup I_-$$

or

$$f(t)=m,\ t\in I_+\ \textit{and}\ f(t)=M,\ t\in I_-,\ \ a.e.\ on\ I_+\cup I_-,$$

where

$$I_+=\{t\in[a,b];\ g(t)>0\},\qquad I_-=\{t\in[a,b];\ g(t)<0\}.$$

Remark 4.5 The assumption

$$\int_a^b g(t)dt=0$$

is not essential since $g(t)$ can be replaced with

$$\tilde{g}(t)=g(t)-\frac{1}{b-a}\int_a^b g(s)ds.$$

The aim of this section is to give generalizations of Theorem A (see Theorem 4.7 and Remark 4.8), and applying them to formulae (4.17) and (4.22) to prove some general Euler-Grüss type inequalities. This section follows [41].

4.2.2 Some inequalities of Grüss type

Let $X\subset\mathbb{R}^m$ be a Borel set in $\mathbb{R}^m$, $m\ge1$, and let $M(X)$ denotes the Banach space of all real Borel measures on X with the total variation norm. For $\mu\in M(X)$ let

$$\mu=\mu_+-\mu_-$$

be the Jordan-Hahn decomposition of μ, where μ_+ and μ_- are orthogonal and positive measures. Then we have

$$|\mu|=\mu_++\mu_-$$

and

$$\|\mu\| = |\mu|(X) = \mu_+(X) + \mu_-(X) = \|\mu_+\| + \|\mu_-\|.$$

Measure $\mu \in M(X)$ is called **balanced** if $\mu(X) = 0$. This is equivalent to

$$\|\mu_+\| = \|\mu_-\| = \frac{1}{2}\|\mu\|.$$

Theorem 4.7 (Grüss inequality for measures) *For a balanced measure $\mu \in M(X)$ let $f \in L_\infty(X,\mu)$ be such that*

$$m \le f(t) \le M,\ t \in X,\ \ \mu - a.e., \tag{4.28}$$

for some $m,\ M \in \mathbb{R}$. Then

$$\left|\int_X f(t)d\mu\right| \le \frac{1}{2}(M-m)\|\mu\|, \tag{4.29}$$

with the equality if and only if either

$$(f(t) = M,\ t \in I_+) \text{ and } (f(t) = m,\ t \in I_-),\ \mu - a.e. \tag{4.30}$$

or

$$(f(t) = m,\ t \in I_+) \text{ and } (f(t) = M,\ t \in I_-),\ \mu - a.e., \tag{4.31}$$

where I_+ and I_- are disjoint Borel sets satisfying

$$\mu_+(I_+) = \|\mu_+\|, \qquad \mu_-(I_-) = \|\mu_-\|, \qquad \mu_+(I_-) = \mu_-(I_+) = 0.$$

Proof. Integrating the relation (4.28) with respect to μ_+ and μ_- we get

$$\frac{1}{2}m\|\mu\| \le \int_X f(t)d\mu_+(t) \le \frac{1}{2}M\|\mu\| \tag{4.32}$$

and

$$-\frac{1}{2}M\|\mu\| \le -\int_X f(t)d\mu_-(t) \le -\frac{1}{2}m\|\mu\|. \tag{4.33}$$

Adding these relations together we get our inequality.

The equality case occurs in (4.29) if and only if we have the equality either in the both right hand sides of (4.32) and (4.33), or in the both left hand sides of (4.32) and (4.33). The former case is equivalent to (4.30), while the later case to (4.31). □

Remark 4.6 Let f and g be from Theorem A and let $\mu \in M([a,b])$ be defined by

$$d\mu(t) = g(t)dt.$$

Then μ_+ and μ_- are given by

$$d\mu_+(t) = g_+(t)dt, \qquad\qquad d\mu_-(t) = g_-(t)dt,$$

where

$$g_+(t) = \frac{1}{2}\left[|g(t)| + g(t)\right], \qquad g_-(t) = \frac{1}{2}\left[|g(t)| - g(t)\right],$$

and we have

$$\mu([a,b]) = \int_a^b g(t)dt = 0,$$

which means that μ is balanced. Now we see that Theorem 4.7 reduces to Theorem A since

$$\|\mu\| = \int_a^b |g(t)|\,dt.$$

Remark 4.7 The inequality (4.29) is obviously sharp. Namely for the function f defined as

$$f(t) = M\chi_{I_+}(t) + m\chi_{I_-}(t), t \in X,$$

we have equality in (4.29). Clearly, the same is true for the function

$$f(t) = m\chi_{I_+}(t) + M\chi_{I_-}(t), t \in X.$$

Corollary 4.15 (Discrete Grüss inequality) *Let $(c_k,\ k \geq 1)$ be a sequence in $\mathbb{R}$ such that*

$$\sum_{k\geq 1} |c_k| < \infty \qquad \text{and} \qquad \sum_{k\geq 1} c_k = 0.$$

Then for every bounded sequence $(d_k,\ k \geq 1)$ in $\mathbb{R}$ we have

$$\left|\sum_{k\geq 1} c_k d_k\right| \leq \frac{1}{2}(M - m)\sum_{k\geq 1} |c_k|,$$

where

$$M = \sup\{d_k : c_k \neq 0\} \quad \text{and} \quad m = \inf\{d_k : c_k \neq 0\}.$$

The equality occurs if and only if either

$$d_k = M,\ k \in I_+ \ \text{ and } \ d_k = m,\ k \in I_-,$$

or

$$d_k = m,\ k \in I_+ \ \text{ and } \ d_k = M,\ k \in I_-,$$

where

$$I_+ = \{k : c_k > 0\} \ \text{ and } \ I_- = \{k : c_k < 0\}.$$

Proof. Choose any sequence $(x_k,\ k \geq 1)$ of distinct points $x_k \in \mathbb{R}$ and set $X = \{x_k : k \geq 1\}$. Then apply the theorem above to discrete measure $\mu = \sum_{k\geq 1} c_k \delta_{x_k}$, and to the function $f : X \to \mathbb{R}$ defined as $f(x_k) = d_k,\ k \geq 1$. In this case

$$\|\mu\| = \sum_{k\geq 1} |c_k| < \infty$$

and

$$\mu(X) = \sum_{k\geq 1} c_k = 0,$$

which means that μ is balanced, while

$$\int_X f(t)d\mu(t) = \sum_{k\geq 1} c_k f(x_k) = \sum_{k\geq 1} c_k d_k.$$

□

Corollary 4.16 *For $\mu \in M(X)$ let $f : X \to \mathbb{R}$ be a Borel function such that $m \leq f(t) \leq M$, $t \in X$ for some m, $M \in \mathbb{R}$. Then for every $x \in X$ we have*

$$\left|\mu(X)f(x) - \int_X f(t)d\mu(t)\right| \leq \frac{1}{2}(M-m)\left[|\mu(X)| + \|\mu\|\right].$$

Proof. For $x \in X$ define measure ν_x by

$$\nu_x = \mu(X)\delta_x - \mu.$$

Then

$$\nu_x(X) = \mu(X) - \mu(X) = 0,$$

and

$$\|\nu_x\| = \|\mu(X)\delta_x - \mu\| \leq |\mu(X)| + \|\mu\|,$$

while

$$\int_X f(t)d\nu_x(t) = \mu(X)f(x) - \int_X f(t)d\mu(t).$$

Apply now the theorem above. □

Corollary 4.17 *Let $\mu, \nu \in M(X)$ be probability measures and let $f : X \to \mathbb{R}$ be a Borel function such that $m \leq f(t) \leq M$, $t \in X$, μ and ν - a.e. for some m, $M \in \mathbb{R}$. Then we have*

$$\left|\int_X f(t)d\mu(t) - \int_X f(t)d\nu(t)\right| \leq M - m.$$

Proof. Apply the theorem above to $\mu - \nu$ and note that $(\mu - \nu)(X) = 1 - 1 = 0$ and $\|\mu - \nu\| \leq \|\mu\| + \|\nu\| = 2$. □

Corollary 4.18 *For a probability measure $\mu \in M(X)$ and a Borel function $f : X \to \mathbb{R}$ such that $m \leq f(t) \leq M$, $t \in X$, for every $x \in X$ we have*

$$\left|f(x) - \int_X f(t)d\mu(t)\right| \leq M - m.$$

Proof. Apply Corollary 4.17 to μ and $\nu = \delta_x$. □

4.2.3 Some Euler-Grüss type inequalities

Throughout this section we use the same notations as in the previous one for the special case $X = [a,b] \subset \mathbb{R}$. Hence μ denotes a real Borel measure on $[a,b]$. Also, whenever $f : [a,b] \to \mathbb{R}$ is such that $f^{(k)}$ exists and is bounded, for some $k \geq 1$, we assume that there are some real constants m_k and M_k such that

$$m_k \leq f^{(k)}(t) \leq M_k,\ t \in [a,b]. \tag{4.34}$$

Theorem 4.8 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation, for some $n \geq 2$. Then for every μ-harmonic sequence $(P_k,\ k \geq 1)$ we have*

$$\begin{aligned}
&\left|\mu([a,b])f(x) - \int_{[a,b]} f_x(t)d\mu(t) - S_{n-1}(x) - [P_n(a) - P_n(b)]\, f^{(n-1)}(x)\right| \\
&\leq \frac{1}{2}(M_{n-1} - m_{n-1})\left[|P_n(b) - P_n(a)| + \int_a^b |P_{n-1}(t)|\, dt\right]
\end{aligned} \tag{4.35}$$

Proof. Note that the expression within the absolute value signs on the left hand side of (4.35) is equal to $R_n^2(x)$ from (4.22), Theorem 4.2. Therefore, the left hand side of (4.35) is equal to $|R_n^2(x)|$ and to get asserted estimate we shall rewrite $R_n^2(x)$ in more suitable form. Integration by parts yields

$$\begin{aligned}
R_n^2(x) = &-(b-a)^n \left[P_n^*\left(\frac{x-t}{b-a}\right) - \frac{1}{(b-a)^n}P_n(x)\right] f^{(n-1)}(t)\Big|_a^b \\
&+ (b-a)^n \int_{[a,b]} f^{(n-1)}(t)dP_n^*\left(\frac{x-t}{b-a}\right).
\end{aligned} \tag{4.36}$$

For $a \leq x < b$ we have

$$P_n^*\left(\frac{x-b}{b-a}\right) = P_n^*\left(\frac{x-a}{b-a}\right) = \frac{1}{(b-a)^n}P_n(x)$$

so that from (4.36) we get

$$R_n^2(x) = (b-a)^n \int_{[a,b]} f^{(n-1)}(t)dP_n^*\left(\frac{x-t}{b-a}\right). \tag{4.37}$$

Using the first formula from Lemma 4.1 relation (4.37) becomes

$$\begin{aligned}
R_n^2(x) \ = \ &-(b-a)^{n-1} \int_a^b f^{(n-1)}(t)P_{n-1}^*\left(\frac{x-t}{b-a}\right) dt \\
&- [P_n(a) - P_n(b)]\, f^{(n-1)}(x).
\end{aligned} \tag{4.38}$$

If $x = b$, then

$$P_n^*\left(\frac{b-b}{b-a}\right) = P_n^*\left(\frac{b-a}{b-a}\right) = \frac{1}{(b-a)^n}P_n(a)$$

so that from (4.36) we get

$$\begin{aligned} R_n^2(b) &= [P_n(b) - P_n(a)]\,[f^{(n-1)}(b) - f^{(n-1)}(a)] \\ &\quad + (b-a)^n \int_{[a,b]} f^{(n-1)}(t)\,dP_n^*\left(\frac{b-t}{b-a}\right). \end{aligned}$$

Now, using the second formula from Lemma 4.1, from the last equality we easily get formula (4.38) for $x = b$. Therefore, formula (4.38) holds for all $x \in [a,b]$.

Define measures ν_n and ξ_n by

$$d\xi_n(t) = -(b-a)^{n-1} P_{n-1}^*\left(\frac{x-t}{b-a}\right) dt$$

and

$$\nu_n = \xi_n - [P_n(a) - P_n(b)]\,\delta_x,$$

and note that by (4.38) we have

$$R_n^2(x) = \int_{[a,b]} f^{(n-1)}(t)\,d\nu_n(t).$$

For all $k \geq 1$ we have

$$P_k^*\left(\frac{x-t}{b-a}\right) = \frac{1}{(b-a)^k} \times \begin{cases} P_k(a+x-t), & \text{for } a \leq t \leq x \\ P_k(b+x-t), & \text{for } x < t \leq b \end{cases} \tag{4.39}$$

so, applying Lemma 4.3 we have

$$\begin{aligned} \nu_n([a,b]) &= -(b-a)^{n-1} \int_a^b P_{n-1}^*\left(\tfrac{x-t}{b-a}\right) dt - [P_n(a) - P_n(b)] \\ &= -\int_a^b P_{n-1}(t)\,dt - [P_n(a) - P_n(b)] \\ &= P_n(a) - P_n(b) - [P_n(a) - P_n(b)] = 0, \end{aligned}$$

which means that ν_n is balanced. Further

$$\begin{aligned} \|\nu_n\| &= (b-a)^{n-1} \int_a^b \left| P_{n-1}^*\left(\frac{x-t}{b-a}\right) \right| dt + |P_n(a) - P_n(b)| \\ &= \int_a^b |P_{n-1}(t)|\,dt + |P_n(a) - P_n(b)|. \end{aligned}$$

Now we can apply Theorem 4.7 to obtain (4.35). □

Corollary 4.19 *Let f and μ be as in Theorem 4.8. Then we have*

$$\begin{aligned} &\left| \mu([a,b]) f(x) - \int_{[a,b]} f_x(t)\,d\mu(t) - \check{S}_{n-1}(x) + \check{\mu}_n(b) f^{(n-1)}(x) \right| \\ &\leq \frac{1}{2}(M_{n-1} - m_{n-1})[|\check{\mu}_n(b)| + \int_a^b |\check{\mu}_{n-1}(t)|\,dt] \end{aligned}$$

$$\leq \frac{1}{(n-1)!}(b-a)^{n-1}\left[M_{n-1}-m_{n-1}\right]\|\mu\|,$$

where $\check{S}_m(x)$ is defined by (4.25).

Proof. The first inequality follows by Theorem 4.8 applied to the μ-harmonic sequence $(\check{\mu}_k,\ k \geq 1)$. In that case $S_{n-1}(x)$ becomes $\check{S}_{n-1}(x)$ since $\check{\mu}_k(a) = 0,\ k \geq 2$. The second inequality follows from inequality (4.11). □

Corollary 4.20 *Let f and μ be as in Theorem 4.8. If $\mu \geq 0$, then we have*

$$\begin{aligned}
&\left|\mu([a,b])f(x) - \int_{[a,b]} f_x(t)d\mu(t) - \check{S}_{n-1}(x) + \check{\mu}_n(b) f^{(n-1)}(x)\right| \\
&\leq \left[M_{n-1}-m_{n-1}\right]\check{\mu}_n(b) \\
&\leq \frac{1}{(n-1)!}(b-a)^{n-1}\left[M_{n-1}-m_{n-1}\right]\|\mu\|.
\end{aligned}$$

Proof. Apply Corollary 4.19. Since $\mu \geq 0$, we have $\check{\mu}_k \geq 0$, for $k \geq 1$ and $\int_a^b \check{\mu}_{n-1}(t)\,dt = \check{\mu}_n(b)$. □

Corollary 4.21 *Let f be as in Theorem 4.8. Then for every $x \in [a,b]$*

$$\begin{aligned}
&\left|(b-a)f(x) - \int_a^b f(t)dt - U_{n-1}(x) + \frac{1}{n!}(b-a)^n f^{(n-1)}(x)\right| \\
&\leq \frac{1}{n!}(b-a)^n\left[M_{n-1}-m_{n-1}\right],
\end{aligned}$$

where

$$U_1(x) = (x-a)[f(b)-f(a)]$$

and

$$U_m(x) = \sum_{k=1}^{m}\frac{1}{k!}(x-a)^k[f^{(k-1)}(b)-f^{(k-1)}(a)] - \sum_{k=2}^{m}\frac{1}{k!}(b-a)^k f^{(k-1)}(x),\ 2 \leq m \leq n.$$

Proof. Apply Corollary 4.20 to the case when μ is the Lebesgue measure on $[a,b]$. For any $x \in [a,b]$ we then have

$$\int_{[a,b]} f_x(t)d\mu(t) = \int_a^b f(t)dt$$

and

$$\check{\mu}_k(t) = \frac{1}{k!}(t-a)^k,\ k \geq 1,$$

so that $\check{S}_{n-1}(x)$ becomes $U_{n-1}(x)$. □

Corollary 4.22 *Let f be as in Theorem 4.8. Then for every $x,y \in [a,b]$, $y \leq x$ we have*

$$\left| f(x) - f(a+x-y) - T_{n-1}(x,y) + \frac{1}{(n-1)!}(b-y)^{n-1} f^{(n-1)}(x) \right|$$
$$\leq \frac{1}{(n-1)!}(b-y)^{n-1}\left[M_{n-1} - m_{n-1}\right],$$

where $T_{n-1}(x,y)$ is defined by (4.26).

Proof. Apply Corollary 4.20 to $\mu = \delta_y$, $a \leq y \leq x$. Then $\check{S}_{n-1}(x)$ becomes $T_{n-1}(x,y)$ since

$$\check{\mu}_n(t) = \frac{1}{(n-1)!}(t-y)^{n-1},\ y \leq t \leq b$$

and $\check{\mu}_n(t) = 0$, $a \leq t < y$. □

Corollary 4.23 *If f is such that f' is a continuous function of bounded variation, then for every $\mu \in M[a,b]$ and every real constant c the absolute value of*

$$\mu([a,b]) f(x) - \int_{[a,b]} f_x(t) d\mu(t) - [c + \check{\mu}_1(x)][f(b) - f(a)] + [c(b-a) + \check{\mu}_2(b)] f'(x)$$

is less than or equal to

$$\frac{1}{2}(M_1 - m_1)[|c(b-a) + \check{\mu}_2(b)| + \int_a^b |c + \check{\mu}_1(t)| dt].$$

Proof. Apply Theorem 4.8 for $n = 2$ and follow the proof of Corollary (4.2). □

Corollary 4.24 *Under assumptions of Corollary 4.23 if $\mu \geq 0$ and $c \geq 0$, then the absolute value of*

$$\mu([a,b]) f(x) - \int_{[a,b]} f_x(t) d\mu(t) - [c + \check{\mu}_1(x)][f(b) - f(a)] + [c(b-a) + \check{\mu}_2(b)] f'(x)$$

is less than or equal to

$$(M_1 - m_1)[c(b-a) + \check{\mu}_2(b)] \leq (M_1 - m_1)(b-a)(c + \|\mu\|).$$

Proof. Apply Corollary 4.23 and note that in this case

$$|c(b-a) + \check{\mu}_2(b)| + \int_a^b |c + \check{\mu}_1(t)| dt = 2[c(b-a) + \check{\mu}_2(b)]$$

and $\check{\mu}_2(b) \leq (b-a)\|\mu\|$. □

Theorem 4.9 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 1$, and let*

$$m_n \leq f^{(n)}(t) \leq M_n,\ a.e.\ on\ [a,b],$$

for some real constants m_n and M_n. If $(P_k,\ k \geq 1)$ is a μ-harmonic sequence such that

$$P_{n+1}(a) = P_{n+1}(b)$$

for that particular n, then for every $x \in [a,b]$ we have

$$\left| \mu([a,b]) f(x) - \int_{[a,b]} f_x(t) d\mu(t) - S_n(x) \right| \leq \frac{1}{2}(M_n - m_n) \int_a^b |P_n(t)|\, dt, \tag{4.40}$$

where $f_x(t)$ is defined by (4.7).

Proof. An expression $R_n^1(x)$ from (4.17), Theorem 4.1, can be rewritten as

$$R_n^1(x) = -(b-a)^n \int_a^b P_n^*\left(\frac{x-t}{b-a}\right) f^{(n)}(t)dt = \int_a^b f^{(n)}(t) d\nu_n(t),$$

where measure ν_n is defined by

$$d\nu_n(t) = -(b-a)^n P_n^*\left(\frac{x-t}{b-a}\right) dt.$$

Using (4.39) and Lemma 4.3 we get

$$\begin{aligned} \nu_n([a,b]) &= -(b-a)^n \int_a^b P_n^*\left(\frac{x-t}{b-a}\right) dt \\ &= -\int_a^b P_n(t)\, dt = P_{n+1}(a) - P_{n+1}(b) = 0, \end{aligned}$$

which means that ν_n is balanced. Further,

$$\|\nu_n\| = (b-a)^n \int_a^b \left| P_n^*\left(\frac{x-t}{b-a}\right) \right| dt = \int_a^b |P_n(t)|\, dt.$$

Now (4.40) follows immediately from Theorem 4.7, since the left hand side of (4.40) is equal to $|R_n^1(x)|$. □

Corollary 4.25 *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function and let $m_1 \leq f'(t) \leq M_1$, a.e. on $[a,b]$, for some real constants m_1 and M_1. If $\mu \in M[a,b]$ and $c \in \mathbb{R}$ are such that*

$$c(b-a) + \check{\mu}_2(b) = 0,$$

then for every $x \in [a,b]$ we have

$$\begin{aligned} &\left| \mu([a,b]) f(x) - \int_{[a,b]} f_x(t) d\mu(t) - [c + \check{\mu}_1(x)]\,[f(b) - f(a)] \right| \\ &\leq \frac{1}{2}(M_1 - m_1) \int_a^b |c + \check{\mu}_1(t)| dt. \end{aligned}$$

Proof. Apply Theorem 4.9 to $n = 1$. Note that $P_1(t) = c + \check{\mu}_1(t)$ and $P_2(t) = c_1 + c(t - a) + \check{\mu}_2(t)$ are two beginning terms of a μ-harmonic sequence of functions on $[a,b]$. The condition $P_2(a) = P_2(b)$ reduces to $c(b-a) + \check{\mu}_2(b) = 0$. □

Measure $\mu \in M[a,b]$ is called k**-balanced** if $\check{\mu}_k(b) = 0$. We see that a 1-balanced measure is the same as a balanced measure. We also define the k-th moment of μ as

$$m_k(\mu) = \int_{[a,b]} t^k d\mu(t), \ k \geq 0.$$

Theorem 4.10 *For any $\mu \in M[a,b]$ the following assertions hold:*
1) $\check{\mu}_n(b) = \sum_{k=0}^{n-1} \frac{(-1)^k}{(n-1)!} \binom{n-1}{k} b^{n-1-k} m_k(\mu)$, $n \geq 1$
2) $m_n(\mu) = \sum_{k=0}^{n} (-1)^k k! \binom{n}{k} b^{n-k} \check{\mu}_{k+1}(b)$, $n \geq 0$
3) *μ is a k-balanced for every $k \in \{1,\ldots,n\}$ if and only if $m_k(\mu) = 0$ for every $k \in \{0,\ldots,n-1\}$.*
4) *μ is uniquely determined by the sequence $(\check{\mu}_k(b),\ k \geq 1)$.*

Proof. 1) By definition of $\check{\mu}_n$ we have

$$\check{\mu}_n(b) = \frac{1}{(n-1)!} \int_{[a,b]} (b-s)^{n-1} d\mu(s).$$

The staded identity follows from the binomial formula applied to $(b-s)^{n-1}$.

2) For every real α, by a simple calculation we have

$$\sum_{k \geq 0} \alpha^k \check{\mu}_{k+1}(b) = \int_{[a,b]} \exp[\alpha(b-s)] d\mu(s)$$

and

$$\sum_{k \geq 0} \alpha^k \check{\mu}_{k+1}(b) \exp(-\alpha b) = \int_{[a,b]} \exp(-\alpha s) d\mu(s).$$

Expand both sides of this identity in Taylor series in variable α and then equate the coefficients to get the formula.

3) Follows immediately from 1) and 2).

4) Every compactly supported real Borel measure μ in $\mathbb{R}$ is uniquely determined by its moments, and therefore $\mu \in M[a,b]$ is uniquely determined by $(\check{\mu}_k(b),\ k \geq 1)$, because of 1) and 2). □

Corollary 4.26 *Let f be as in Theorem 4.9. Then for every $(n+1)$-balanced measure $\mu \in M[a,b]$ and for every $x \in [a,b]$ we have*

$$\begin{aligned}\left| \mu([a,b]) f(x) - \int_{[a,b]} f_x(t) d\mu(t) - \check{S}_n(x) \right| &\leq \frac{1}{2}(M_n - m_n) \int_a^b |\check{\mu}_n(t)| dt \\ &\leq \frac{1}{2}(M_n - m_n) \frac{1}{n!}(b-a)^n \|\mu\|,\end{aligned}$$

where $\check{S}_n(x)$ is defined by (4.25).

Proof. To obtain the first inequality apply Theorem 4.9 to μ-harmonic sequence $(\check{\mu}_k,\ k \geq 1)$ and note that the condition $P_{n+1}(a) = P_{n+1}(b)$ reduces to $\check{\mu}_{n+1}(b) = 0$, which means that μ is $(n+1)$-balanced. The second inequality follows by (4.11). □

Corollary 4.27 *Let f be as in Theorem 4.9. Then for every $\mu \in M[a,b]$, such that all k-moments of μ are zero, for $k = 0,\dots,n$, and for any $x \in [a,b]$ we have*

$$\left|\int f_x(t)d\mu(t) - \sum_{k=1}^{n} \check{\mu}_k(x)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right]\right| \leq \frac{1}{2}(M_n - m_n)\int_a^b |\check{\mu}_n(t)|\,dt$$
$$\leq \frac{1}{2\,(n!)}(M_n - m_n)(b-a)^n \|\mu\|.$$

Proof. By Theorem 4.10 the condition $m_k(\mu) = 0$, $k = 0,\dots,n$, is equivalent to $\check{\mu}_k(b) = 0$, $k = 1,\dots,n+1$. Apply now Corollary 4.26 and note that in this case $\mu([a,b]) = 0$ and

$$\check{S}_n(x) = \sum_{k=1}^{n} \check{\mu}_k(x)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right].$$

□

Corollary 4.28 *Let f be as in Theorem 4.9. Then for every $\mu \in M[a,b]$, such that all k-moments of μ are zero, for $k = 0,\dots,n$, we have*

$$\left|\int_{[a,b]} f(t)d\mu(t)\right| \leq \frac{1}{2}(M_n - m_n)\int_a^b |\check{\mu}_n(t)|\,dt$$
$$\leq \frac{1}{2\,(n!)}(M_n - m_n)(b-a)^n \|\mu\|.$$

Proof. Put $x = b$ in Corollary 4.27. Then $\check{S}_n(b) = 0$, and we can replace $f_b(t) = f(a+b-t)$ by $f(t)$ since the constants M_n and m_n are the same for both $f_x(t)$ and $f(t)$. □

Remark 4.8 The inequality of Corollary 4.28

$$\left|\int_{[a,b]} f(t)d\mu(t)\right| \leq \frac{1}{2\,(n!)}(M_n - m_n)(b-a)^n \|\mu\|$$

can be regarded as an n-th order generalization of inequality (4.29) of Theorem 4.7.

Corollary 4.29 *Let $f : [a,b] \to \mathbb{R}$, m_1 and M_1 be as in Corollary 4.25. Then for every 2-balanced measure μ and for any $x \in [a,b]$ we have*

$$\left|\mu([a,b])f(x) - \int_{[a,b]} f_x(t)d\mu(t) - \check{\mu}_1(x)\left[f(b) - f(a)\right]\right| \leq \frac{1}{2}(M_1 - m_1)\int_a^b |\check{\mu}_1(t)|\,dt$$
$$\leq \frac{1}{2}(M_1 - m_1)(b-a)\|\mu\|.$$

Proof. Put $n = 1$ in Corollary 4.26. □

Corollary 4.30 *Let $f : [a,b] \to \mathbb{R}$, m_1 and M_1 be as in Corollary 4.25. Then for every $\mu \in M[a,b]$ such that*

$$\int_{[a,b]} d\mu(t) = \int_{[a,b]} t d\mu(t) = 0$$

and for any $x \in [a,b]$ we have

$$\begin{aligned}\left| \int_{[a,b]} f_x(t) d\mu(t) - \check{\mu}_1(x)[f(b) - f(a)] \right| &\leq \frac{1}{2}(M_1 - m_1) \int_a^b |\check{\mu}_1(t)| dt \\ &\leq \frac{1}{2}(M_1 - m_1)(b-a) \|\mu\|,\end{aligned}$$

and

$$\begin{aligned}\left| \int_{[a,b]} f(t) d\mu(t) \right| &\leq \frac{1}{2}(M_1 - m_1) \int_a^b |\check{\mu}_1(t)| dt \\ &\leq \frac{1}{2}(M_1 - m_1)(b-a) \|\mu\|.\end{aligned}$$

Proof. Put $n = 1$ in Corollaries 4.27 and 4.28. □

Corollary 4.31 *Let $\{x_k : k \geq 1\}$ be a subset of $[a,b]$ of different points and let $(c_k,\ k \geq 1)$ be a sequence in $\mathbb{R}$ such that*

$$\sum_{k\geq 1} |c_k| < \infty, \qquad \sum_{k\geq 1} c_k(b - x_k) = 0.$$

Then for every $x \in [a,b]$ and $f : [a,b] \to \mathbb{R}$ such that $m_1 \leq f'(x_k) \leq M_1$, $k \geq 1$, we have

$$\begin{aligned}&\left| f(x) \sum_{k\geq 1} c_k - \sum_{k\geq 1} c_k f_x(x_k) - \sum_{k\geq 1} c_k \chi_{[a,x]}(x_k)[f(b) - f(a)] \right| \\ &\leq \frac{1}{2}(M_1 - m_1)(b-a) \sum_{k\geq 1} |c_k|,\end{aligned}$$

where $\chi_{[a,x]}$ is the indicator function of $[a,x]$ and $f_x(t)$ is defined by (4.7).

Proof. Apply Corollary 4.29 to discrete measure $\mu = \sum_{k\geq 1} c_k \delta_{x_k}$. In this case $\|\mu\| = \sum_{k\geq 1} |c_k|$,

$$\check{\mu}_1(x) = \sum_{k\geq 1} c_k \chi_{[x_k,b]}(x) = \sum_{k\geq 1} c_k \chi_{[a,x]}(x_k),$$

$$\check{\mu}_2(b) = \sum_{k\geq 1} c_k(b - x_k) = 0$$

and

$$\int_{[a,b]} f_x(t) d\mu(t) = \sum_{k\geq 1} c_k f_x(x_k).$$

□

Corollary 4.32 *Let $\{x_k;\ k \geq 1\}$ be a subset of $[a,b]$ of different points and let $(c_k,\ k \geq 1)$ be a sequence in $\mathbb{R}$ such that*

$$\sum_{k\geq 1} |c_k| < \infty, \qquad \sum_{k\geq 1} c_k = \sum_{k\geq 1} c_k x_k = 0.$$

Then for every $f : [a,b] \to \mathbb{R}$, such that $m_1 \leq f'(x_k) \leq M_1$, $k \geq 1$, we have

$$\left|\sum_{k\geq 1} c_k f(x_k)\right| \leq \frac{1}{2}(M_1 - m_1)(b-a) \sum_{k\geq 1} |c_k|.$$

Proof. Apply Corollary 4.30 to discrete measure $\mu = \sum_{k\geq 1} c_k \delta_{x_k}$. In this case we have $\|\mu\| = \sum_{k\geq 1} |c_k|$,

$$\check{\mu}_1(b) = \sum_{k\geq 1} c_k = 0$$

and

$$\check{\mu}_2(b) = \sum_{k\geq 1} c_k(b - x_k) = 0,$$

while

$$\int_{[a,b]} f(t) d\mu(t) = \sum_{k\geq 1} c_k f(x_k).$$

Some of the results from the first two sections will be used in the following sections. □

4.3 On an integration-by-parts formula for measures

4.3.1 Introduction

In the paper [51] S. S. Dragomir introduced the notion of w_0-Appell type sequence of functions as a sequence $w_0, w_1, \ldots, w_n$, for $n \geq 1$, of real absolutely continuous functions defined on $[a,b]$, such that

$$w_k' = w_{k-1},\ a.e. \text{ on } [a,b],\ k = 1,..,n.$$

For such a sequence the author proved a generalization of Mitrinović-Pečarić integration-by-parts formula

$$\int_a^b w_0(t) g(t) dt = A_n + B_n \tag{4.41}$$

where

$$A_n = \sum_{k=1}^{n} (-1)^{k-1} [w_k(b) g^{(k-1)}(b) - w_k(a) g^{(k-1)}(a)]$$

and

$$B_n = (-1)^n \int_a^b w_n(t) g^{(n)}(t)dt$$

for every function $g : [a,b] \to \mathbb{R}$ such that $g^{(n-1)}$ is absolutely continuous on $[a,b]$ and $w_n g^{(n)} \in L_1[a,b]$. Using identity (4.41) the author proved following inequality

$$\left| \int_a^b w_0(t)g(t)dt - A_n \right| \le \|w_n\|_p \, \|g^{(n)}\|_q, \tag{4.42}$$

for $w_n \in L_p[a,b]$, $g^{(n)} \in L_q[a,b]$ where $p,q \in [1,\infty]$ and $1/p + 1/q = 1$, giving explicitly some interesting special cases. For some similar inequalities see also [32], [34] and [58]. The aim of this section is to give a generalization of integration-by-parts formula (4.41), by replacing w_0-Appell type sequence of functions by a more general sequence of functions, and to generalize inequality (4.42), as well as to prove some related inequalities. The results from this and following sections are published in [42].

4.3.2 Integration-by-parts formula for measures

Remark 4.9 Let $w_0 : [a,b] \to \mathbb{R}$ be an absolutely integrable function and let $\mu \in M[a,b]$ be defined by $d\mu(t) = w_0(t)dt$. If $(P_n,\ n \ge 1)$ is a μ-harmonic sequence of functions on $[a,b]$, then $w_0, P_1, \ldots, P_n$ is a w_0-Appell type sequence of functions on $[a,b]$.

Lemma 4.4 *For every $f \in C[a,b]$ and $\mu \in M[a,b]$ we have*

$$\int_{[a,b]} f(t)d\check{\mu}_1(t) = \int_{[a,b]} f(t)d\mu(t) - \mu(\{a\})f(a).$$

Proof. Define $I,J : C[a,b] \times M[a,b] \to \mathbb{R}$ by

$$I(f,\mu) = \int_{[a,b]} f(t)d\check{\mu}_1(t)$$

and

$$J(f,\mu) = \int_{[a,b]} f(t)d\mu(t) - \mu(\{a\})f(a).$$

Then I and J are continuous bilinear functionals since

$$|I(f,\mu)| \le \|f\| \, \|\mu\|, \quad |J(f,\mu)| \le 2\|f\| \, \|\mu\|.$$

Let us prove that $I(f,\mu) = J(f,\mu)$ for every $f \in C[a,b]$ and every discrete measure $\mu \in M[a,b]$ with finite support. For $x \in [a,b]$ let $\mu = \delta_x$ be the Dirac measure at x. If $a < x \le b$ then

$$\check{\mu}_1(t) = \delta_x([a,t]) = \begin{cases} 0, & a \le t < x \\ 1, & x \le t \le b \end{cases}$$

and by a simple calculation we have

$$I(f,\delta_x) = \int_{[a,b]} f(t)d\check{\mu}_1(t) = f(x) = \int_{[a,b]} f(t)d\delta_x(t) - 0$$

$$= \int_{[a,b]} f(t) d\delta_x(t) - \delta_x(\{a\}) f(a) = J(f, \delta_x).$$

Similarly, if $x = a$ then $\check{\mu}_1(t) = \delta_a([a,t]) = 1,\ a \le t \le b$, and by a similar calculation we have

$$\begin{aligned} I(f, \delta_a) &= \int_{[a,b]} f(t) d\check{\mu}_1(t) \ = 0 = f(a) - f(a) \\ &= \int_{[a,b]} f(t) d\delta_a(t) - \delta_a(\{a\}) f(a) = J(f, \delta_a). \end{aligned}$$

Therefore, for every $f \in C[a,b]$ and every $x \in [a,b]$ we have $I(f, \delta_x) = J(f, \delta_x)$. By linearity of I and J, for every $f \in C[a,b]$ and every discrete measure $\mu = \sum_{k=1}^{n} c_k \delta_{x_k}$ with finite support we have $I(f, \mu) = J(f, \mu)$.

Apply now the same argument as in the proof of Lemma 4.2 to conclude that $I(f, \mu) = J(f, \mu)$ for every $f \in C[a,b]$ and every $\mu \in M[a,b]$. □

Theorem 4.11 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation, for some $n \ge 1$. Then for every μ-harmonic sequence $(P_n,\ n \ge 1)$ we have*

$$\int_{[a,b]} f(t) d\mu/t) = \mu(\{a\}) f(a) + S_n + R_n, \tag{4.43}$$

where

$$S_n = \sum_{k=1}^{n} (-1)^{k-1} [P_k(b) f^{(k-1)}(b) - P_k(a) f^{(k-1)}(a)] \tag{4.44}$$

and

$$R_n = (-1)^n \int_{[a,b]} P_n(t) d f^{(n-1)}(t). \tag{4.45}$$

Proof. Integrating by parts, for $n \ge 2$, we have

$$\begin{aligned} R_n &= (-1)^n \int_{[a,b]} P_n(t) d f^{(n-1)}(t) \\ &= (-1)^n [P_n(b) f^{(n-1)}(b) - P_n(a) f^{(n-1)}(a)] \\ &- (-1)^n \int_{[a,b]} P_{n-1}(t) f^{(n-1)}(t) dt \\ &= (-1)^n [P_n(b) f^{(n-1)}(b) - P_n(a) f^{(n-1)}(a)] + R_{n-1}. \end{aligned}$$

By Lemma 4.4 we have

$$\begin{aligned} R_1 &= -\int_{[a,b]} P_1(t) d f(t) \\ &= -[P_1(b) f(b) - P_1(a) f(a)] + \int_{[a,b]} f(t) d P_1(t) \\ &= -[P_1(b) f(b) - P_1(a) f(a)] + \int_{[a,b]} f(t) d\check{\mu}_1(t) \end{aligned}$$

$$= -[P_1(b)f(b) - P_1(a)f(a)] + \int_{[a,b]} f(t)d\mu(t) - \mu(\{a\})f(a).$$

Therefore, by iteration, we have

$$\begin{aligned} R_n &= \sum_{k=1}^{n} (-1)^k [P_k(b)f^{(k-1)}(b) - P_k(a)f^{(k-1)}(a)] \\ &\quad + \int_{[a,b]} f(t)d\mu(t) - \mu(\{a\})f(a), \end{aligned}$$

which proves our assertion. □

Remark 4.10 By Remark 4.9 we see that identity (4.43) is a generalization of the integration-by-parts formula (4.41).

Corollary 4.33 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation, for some $n \geq 1$. Then for every $\mu \in M[a,b]$ we have*

$$\int_{[a,b]} f(t)d\mu(t) = \check{S}_n + \check{R}_n,$$

where

$$\check{S}_n = \sum_{k=1}^{n} (-1)^{k-1} \check{\mu}_k(b) f^{(k-1)}(b)$$

and

$$\check{R}_n = (-1)^n \int_{[a,b]} \check{\mu}_n(t) df^{(n-1)}(t).$$

Proof. Apply the theorem above for the μ-harmonic sequence $(\check{\mu}_n,\ n \geq 1)$ and note that $\check{\mu}_n(a) = 0$, for $n \geq 2$. □

Corollary 4.34 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation, for some $n \geq 1$. Then for every $x \in [a,b]$ we have*

$$f(x) = \sum_{k=1}^{n} \tfrac{1}{(k-1)!}(x-b)^{k-1} f^{(k-1)}(b) + R_n(x),$$

where

$$R_n(x) = \frac{(-1)^n}{(n-1)!} \int_{[x,b]} (t-x)^{n-1} df^{(n-1)}(t).$$

Proof. Apply Corollary 4.33 to $\mu = \delta_x$ and note that in this case

$$\check{\mu}_k(t) = \begin{cases} 0, & a \leq t < x \\ \frac{(t-x)^{k-1}}{(k-1)!}, & x \leq t \leq b \end{cases},$$

for $k \geq 1$. □

Corollary 4.35 *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation, for some $n\geq 1$. Further, let $(c_m,\ m\geq 1)$ be a sequence in $\mathbb{R}$ such that $\sum_{m\geq 1}|c_m|<\infty$ and let $\{x_m;\ m\geq 1\}\subset[a,b]$. Then*

$$\sum_{m\geq 1} c_m f(x_m)=\sum_{m\geq 1}\sum_{k=1}^{n} c_m \tfrac{(x_m-b)^{k-1}}{(k-1)!} f^{(k-1)}(b)+\sum_{m\geq 1} c_m R_n(x_m),$$

where $R_n(x_m)$ is from Corollary 4.34.

Proof. Apply Corollary 4.33 to the discrete measure $\mu=\sum_{m\geq 1} c_m\delta_{x_m}$. □

4.3.3 Some Ostrowski-type inequalities

In this section we shall use the same notations as above.

Theorem 4.12 *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is L-Lipschitzian for some $n\geq 1$. Then for every μ-harmonic sequence $(P_n,\ n\geq 1)$ we have*

$$\left|\int_{[a,b]} f(t)d\mu(t)-\mu(\{a\})f(a)-S_n\right|\leq L\int_a^b |P_n(t)|\,dt, \tag{4.46}$$

where S_n is defined by (4.44).

Proof. By Theorem 4.11 we have

$$|R_n|=\left|\int_{[a,b]} P_n(t)df^{(n-1)}(t)\right|\leq L\int_a^b |P_n(t)|\,dt,$$

which proves our assertion. □

Corollary 4.36 *If f is L-Lipschitzian, then for every $c\in\mathbb{R}$ and $\mu\in M[a,b]$ we have*

$$\left|\int_{[a,b]} f(t)d\mu(t)-\mu([a,b])f(b)-c\,[f(b)-f(a)]\right|\leq L\int_a^b |c+\check{\mu}_1(t)|\,dt.$$

Proof. Put $n=1$ in the theorem above and note that $P_1(t)=c+\check{\mu}_1(t)$ for some $c\in\mathbb{R}$. □

Corollary 4.37 *If f is L-Lipschitzian, then for every $c\geq 0$ and $\mu\geq 0$ we have*

$$\begin{aligned}&\left|\int_{[a,b]} f(t)d\mu(t)-\mu([a,b])f(b)-c\,[f(b)-f(a)]\right|\\ &\leq L\,[c(b-a)+\check{\mu}_2(b)]\\ &\leq L(b-a)(c+\|\mu\|).\end{aligned}$$

Proof. Apply Corollary 4.36 and note that in this case

$$\int_a^b |c+\breve{\mu}_1(t)|\,dt = \int_a^b [c+\breve{\mu}_1(t)]\,dt = c(b-a)+\breve{\mu}_2(b)$$
$$\le c(b-a)+(b-a)\,\|\mu\| = (b-a)(c+\|\mu\|),$$

which proves our assertion. □

Corollary 4.38 *Let f be L-Lipschitzian, $(c_m,\ m\ge 1)$ a sequence in $[0,\infty)$ such that $\sum_{m\ge 1} c_m < \infty$, and let $\{x_m;\ m\ge 1\}\subset [a,b]$. Then for every $c\ge 0$ we have*

$$\left|\sum_{m\ge 1} c_m\,[f(b)-f(x_m)] + c\,[f(b)-f(a)]\right| \le L\left[c(b-a)+\sum_{m\ge 1} c_m(b-x_m)\right]$$
$$\le L(b-a)\left[c+\sum_{m\ge 1} c_m\right].$$

Proof. Apply Corollary 4.37 to the discrete measure $\mu=\sum_{m\ge 1} c_m\delta_{x_m}$. □

Corollary 4.39 *If f is L-Lipschitzian and $\mu\ge 0$, then*

$$\left|\int_{[a,b]} f(t)d\mu(t) - \mu([a,x])f(a) - \mu((x,b])f(b)\right|$$
$$\le L\,[(2x-a-b)\breve{\mu}_1(x) - 2\breve{\mu}_2(x) + \breve{\mu}_2(b)]$$

for every $x\in[a,b]$.

Proof. Apply Corollary 4.36 to $c=-\breve{\mu}_1(x)$. Then $c+\breve{\mu}_1(b)=\mu((x,b])$, $\breve{\mu}_1(x)=\mu([a,x])$ and

$$\int_a^b |-\breve{\mu}_1(x)+\breve{\mu}_1(t)|\,dt = \int_a^x (\breve{\mu}_1(x)-\breve{\mu}_1(t))\,dt + \int_x^b (\breve{\mu}_1(t)-\breve{\mu}_1(x))\,dt$$
$$= (2x-a-b)\breve{\mu}_1(x) - 2\breve{\mu}_2(x) + \breve{\mu}_2(b).$$

□

Corollary 4.40 *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is L-Lipschitzian, for some $n\ge 1$. Then for every $\mu\in M[a,b]$ we have*

$$\left|\int_{[a,b]} f(t)d\mu(t) - \breve{S}_n\right| \le L\int_a^b |\breve{\mu}_n(t)|\,dt$$
$$\le \frac{1}{n!}(b-a)^n L\|\mu\|,$$

where $\breve{S}_n$ is from Corollary 4.33.

Proof. Apply the theorem above to the μ-harmonic sequence $(\breve{\mu}_n,\ n\ge 1)$. □

Corollary 4.41 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is L-Lipschitzian, for some $n \geq 1$. Further, let $(c_m,\ m \geq 1)$ be a sequence in $\mathbb{R}$ such that $\sum_{m\geq 1} |c_m| < \infty$ and let $\{x_m;\ m \geq 1\} \subset [a,b]$. Then*

$$\begin{aligned}&\left|\sum_{m\geq 1} c_m f(x_m) - \sum_{m\geq 1}\sum_{k=1}^{n} c_m \tfrac{(x_m-b)^{k-1}}{(k-1)!} f^{(k-1)}(b)\right| \\ &\leq \frac{1}{n!} L \sum_{m\geq 1} |c_m| (b-x_m)^n \\ &\leq \frac{1}{n!} L (b-a)^n \sum_{m\geq 1} |c_m| .\end{aligned}$$

Proof. Apply Corollary 4.40 to the discrete measure $\mu = \sum_{m\geq 1} c_m \delta_{x_m}$. □

Theorem 4.13 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation, for some $n \geq 1$. Then for every μ-harmonic sequence $(P_n,\ n \geq 1)$ we have*

$$\left|\int_{[a,b]} f(t)d\mu(t) - \mu(\{a\})f(a) - S_n\right| \leq \sup_{t\in[a,b]} |P_n(t)| V_a^b(f^{(n-1)}),$$

where $V_a^b(f^{(n-1)})$ is the total variation of $f^{(n-1)}$ on $[a,b]$.

Proof. By Theorem 4.11 we have

$$|R_n| = \left|\int_{[a,b]} P_n(t) d f^{(n-1)}(t)\right| \leq \sup_{t\in[a,b]} |P_n(t)| V_a^b(f^{(n-1)}),$$

which proves our assertion. □

Corollary 4.42 *If f is a continuous function of bounded variation, then for every $c \in \mathbb{R}$ and $\mu \in M[a,b]$ we have*

$$\left|\int_{[a,b]} f(t)d\mu(t) - \mu([a,b])f(b) - c\,[f(b)-f(a)]\right| \leq \sup_{t\in[a,b]} |c + \check{\mu}_1(t)| V_a^b(f).$$

Proof. Put $n = 1$ in the theorem above. □

Corollary 4.43 *If f is a continuous function of bounded variation, then for every $c \geq 0$ and $\mu \geq 0$ we have*

$$\left|\int_{[a,b]} f(t)d\mu(t) - \mu([a,b])f(b) - c\,[f(b)-f(a)]\right| \leq (c + \|\mu\|) V_a^b(f).$$

Proof. In this case we have

$$\sup_{t\in[a,b]} |c + \check{\mu}_1(t)| = c + \check{\mu}_1(b) = c + \|\mu\| .$$

□

Corollary 4.44 *Let f be a continuous function of bounded variation, $(c_m,\ m \geq 1)$ a sequence in $[0,\infty)$ such that $\sum_{m\geq 1} c_m < \infty$ and let $\{x_m;\ m \geq 1\} \subset [a,b]$. Then for every $c \geq 0$ we have*

$$\left| \sum_{m\geq 1} c_m \left[f(b) - f(x_m)\right] + c\left[f(b) - f(a)\right] \right| \leq \left(c + \sum_{m\geq 1} c_m \right) V_a^b(f)$$

Proof. Apply Corollary 4.43 to the discrete measure $\mu = \sum_{m\geq 1} c_m \delta_{x_m}$. □

Corollary 4.45 *If f is a continuous function of bounded variation and $\mu \geq 0$, then we have*

$$\begin{aligned} &\left| \int_{[a,b]} f(t)d\mu(t) - \mu([a,x])f(a) - \mu((x,b])f(b) \right| \\ &\leq \frac{1}{2}\left[\check{\mu}_1(b) - \check{\mu}_1(a) + |\check{\mu}_1(a) + \check{\mu}_1(b) - 2\check{\mu}_1(x)|\right] V_a^b(f). \end{aligned}$$

Proof. Apply Corollary 4.42 to $c = -\check{\mu}_1(x)$. Then

$$\begin{aligned} \sup_{t\in[a,b]} |c + \check{\mu}_1(t)| &= \sup_{t\in[a,b]} |\check{\mu}_1(t) - \check{\mu}_1(x)| \\ &= \max\{\check{\mu}_1(x) - \check{\mu}_1(a),\ \check{\mu}_1(b) - \check{\mu}_1(x)\} \\ &= \frac{1}{2}\left[\check{\mu}_1(b) - \check{\mu}_1(a) + |\check{\mu}_1(a) + \check{\mu}_1(b) - 2\check{\mu}_1(x)|\right], \end{aligned}$$

which proves our assertion. □

Corollary 4.46 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation, for some $n \geq 1$. Then for every $\mu \in M[a,b]$ we have*

$$\begin{aligned} \left| \int_{[a,b]} f(t)d\mu(t) - \check{S}_n \right| &\leq \sup_{t\in[a,b]} |\check{\mu}_n(t)| V_a^b(f^{(n-1)}) \\ &\leq \frac{(b-a)^{n-1}}{(n-1)!} \|\mu\| V_a^b(f^{(n-1)}), \end{aligned}$$

where $\check{S}_n$ is from Corollary 4.33.

Proof. Apply the theorem above to the μ-harmonic sequence $(\check{\mu}_n,\ n \geq 1)$. □

Corollary 4.47 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation, for some $n \geq 1$. Then for every $x \in [a,b]$ we have*

$$\left| f(x) - \sum_{k=1}^{n} \frac{(x-b)^{k-1}}{(k-1)!} f^{(k-1)}(b) \right| \leq \frac{1}{(n-1)!}(b-x)^{n-1} V_a^b(f^{(n-1)}).$$

Proof. Apply Corollary 4.46 for $\mu = \delta_x$ and note that in this case

$$\sup_{t\in[a,b]} |\check{\mu}_n(t)| \leq \frac{1}{(n-1)!}(b-x)^{n-1}.$$

□

Corollary 4.48 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation, for some $n \geq 1$. Further, let $(c_m,\ m \geq 1)$ be a sequence in $\mathbb{R}$ such that $\sum_{m\geq 1} |c_m| < \infty$ and let $\{x_m;\ m \geq 1\} \subset [a,b]$. Then*

$$\begin{aligned}
&\left| \sum_{m\geq 1} c_m f(x_m) - \sum_{m\geq 1} \sum_{k=1}^{n} c_m \tfrac{(x_m-b)^{k-1}}{(k-1)!} f^{(k-1)}(b) \right| \\
&\leq \frac{1}{(n-1)!} V_a^b(f^{(n-1)}) \sum_{m\geq 1} |c_m| (b-x_m)^{n-1} \\
&\leq \frac{(b-a)^{n-1}}{(n-1)!} V_a^b(f^{(n-1)}) \sum_{m\geq 1} |c_m| .
\end{aligned}$$

Proof. Apply Corollary 4.46 to the discrete measure $\mu = \sum_{m\geq 1} c_m \delta_{x_m}$. □

Theorem 4.14 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)} \in L_p[a,b]$, for some $n \geq 1$. Then for every μ-harmonic sequence $(P_n, \geq 1)$ we have*

$$\left| \int_{[a,b]} f(t) d\mu(t) - \mu(\{a\}) f(a) - S_n \right| \leq \|P_n\|_q \|f^{(n)}\|_p,$$

where $p,q \in [1,\infty]$ and $1/p + 1/q = 1$.

Proof. By Theorem 4.11 and the Hölder's inequality we have

$$\begin{aligned}
|R_n| &= \left| \int_{[a,b]} P_n(t) df^{(n-1)}(t) \right| = \left| \int_a^b P_n(t) f^{(n)}(t) dt \right| \\
&\leq \left(\int_a^b |P_n(t)|^q \, dt \right)^{\frac{1}{q}} \left(\int_a^b \left| f^{(n)}(t) \right|^p dt \right)^{\frac{1}{p}} = \|P_n\|_q \|f^{(n)}\|_p.
\end{aligned}$$

□

Remark 4.11 We see that the inequality of the theorem above is a generalization of inequality (4.42).

Corollary 4.49 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)} \in L_p[a,b]$ for some $n \geq 1$, and $\mu \in M[a,b]$. Then*

$$\begin{aligned}
\left| \int_{[a,b]} f(t) d\mu(t) - \check{S}_n \right| &\leq \|\check{\mu}_n\|_q \, \|f^{(n)}\|_p \\
&\leq \frac{(b-a)^{n-1+1/q}}{(n-1)! \, [(n-1)q+1]^{1/q}} \|\mu\| \, \|f^{(n)}\|_p,
\end{aligned}$$

where $p,q \in [1,\infty]$ and $1/p + 1/q = 1$.

Proof. Apply the theorem above to the μ-harmonic sequence $(\check{\mu}_n,\ n \geq 1)$. □

Corollary 4.50 *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}\in L_p[a,b]$, for some $n\geq 1$. Further, let $(c_m,\ m\geq 1)$ be a sequence in $\mathbb{R}$ such that $\sum_{m\geq 1}|c_m|<\infty$ and let $\{x_m;\ m\geq 1\}\subset[a,b]$. Then*

$$\begin{aligned}&\left|\sum_{m\geq 1} c_m f(x_m) - \sum_{m\geq 1}\sum_{k=1}^{n} c_m \frac{(x_m-b)^{k-1}}{(k-1)!} f^{(k-1)}(b)\right|\\ &\leq \frac{1}{(n-1)!\,[(n-1)q+1]^{1/q}}\|f^{(n)}\|_p \sum_{m\geq 1}|c_m|\,(b-x_m)^{n-1+1/q}\\ &\leq \frac{(b-a)^{n-1+1/q}}{(n-1)!\,[(n-1)q+1]^{1/q}}\|f^{(n)}\|_p \sum_{m\geq 1}|c_m|,\end{aligned}$$

where $p,q\in[1,\infty]$ and $1/p+1/q=1$.

Proof. Apply the theorem above to the discrete measure $\mu=\sum_{m\geq 1} c_m\delta_{x_m}$. □

4.3.4 Some Grüss-type inequalities

Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}\in L_\infty[a,b]$, for some $n\geq 1$. Then

$$m_n\leq f^{(n)}(t)\leq M_n,\ t\in[a,b],\ \text{a.e.}$$

for some real constants m_n and M_n.

Theorem 4.15 *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}\in L_\infty[a,b]$, for some $n\geq 1$. Further, let $(P_k,\ k\geq 1)$ be a μ-harmonic sequence such that*

$$P_{n+1}(a)=P_{n+1}(b)$$

for that particular n. Then

$$\left|\int_{[a,b]} f(t)d\mu(t)-\mu(\{a\})f(a)-S_n\right|\leq\frac{1}{2}(M_n-m_n)\int_a^b|P_n(t)|\,dt.$$

Proof. Apply Theorem 4.11 to the special case when $f^{(n-1)}$ is absolutely continuous and its derivative $f^{(n)}$, existing *a.e.*, is bounded *a.e.* Define measure ν_n by

$$d\nu_n(t)=-P_n(t)\,dt.$$

Then

$$\nu_n([a,b])=-\int_a^b P_n(t)\,dt=P_{n+1}(a)-P_{n+1}(b)=0,$$

which means that ν_n is balanced. Further

$$\|\nu_n\|=\int_a^b|P_n(t)|\,dt,$$

and applying Theorem 4.7 we have following estimate

$$
\begin{aligned}
|R_n| &= \left| \int_a^b P_n(t) f^{(n)}(t) dt \right| \\
&\leq \frac{1}{2}(M_n - m_n) \|\nu_n\| \\
&= \frac{1}{2}(M_n - m_n) \int_a^b |P_n(t)| dt,
\end{aligned}
$$

which proves our assertion. □

Corollary 4.51 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)} \in L_\infty[a,b]$, for some $n \geq 1$. Then for every $(n+1)$-balanced measure $\mu \in M[a,b]$ we have*

$$
\begin{aligned}
\left| \int_{[a,b]} f(t) d\mu(t) - \check{S}_n \right| &\leq \frac{1}{2}(M_n - m_n) \int_a^b |\check{\mu}_n(t)| dt \\
&\leq \frac{1}{2(n!)}(M_n - m_n)(b-a)^n \|\mu\|,
\end{aligned}
$$

where $\check{S}_n$ is from Corollary 4.33.

Proof. Apply Theorem 4.15 to the μ-harmonic sequence $(\check{\mu}_k,\ k \geq 1)$ and note that the condition $P_{n+1}(a) = P_{n+1}(b)$ reduces to $\check{\mu}_{n+1}(b) = 0$, which means that μ is $(n+1)$-balanced. □

Corollary 4.52 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)} \in L_\infty[a,b]$, for some $n \geq 1$. Further, let $(c_m,\ m \geq 1)$ be a sequence in $\mathbb{R}$ such that $\sum_{m\geq 1} |c_m| < \infty$ and let $\{x_m;\ m \geq 1\} \subset [a,b]$ satisfies the condition $\sum_{m\geq 1} c_m(b - x_m)^n = 0$. Then*

$$
\begin{aligned}
&\left| \sum_{m\geq 1} c_m f(x_m) - \sum_{m\geq 1} \sum_{k=1}^{n} c_m \frac{(x_m - b)^{k-1}}{(k-1)!} f^{(k-1)}(b) \right| \\
&\leq \frac{1}{2(n!)}(M_n - m_n) \sum_{m\geq 1} |c_m| (b - x_m)^n \\
&\leq \frac{1}{2(n!)}(M_n - m_n)(b-a)^n \sum_{m\geq 1} |c_m|.
\end{aligned}
$$

Proof. Apply Corollary 4.51 to the discrete measure $\mu = \sum_{m\geq 1} c_m \delta_{x_m}$. □

Corollary 4.53 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)} \in L_\infty[a,b]$, for some $n \geq 1$. Then for every $\mu \in M[a,b]$, such that all k-moments of μ are zero, for $k = 0, \ldots, n$, we have*

$$
\begin{aligned}
\left| \int_{[a,b]} f(t) d\mu(t) \right| &\leq \frac{1}{2}(M_n - m_n) \int_a^b |\check{\mu}_n(t)| dt \\
&\leq \frac{1}{2(n!)}(M_n - m_n)(b-a)^n \|\mu\|.
\end{aligned}
$$

Proof. By Theorem 4.10 the condition $m_k(\mu) = 0$, $k = 0, ..., n$ is equivalent to $\check{\mu}_k(b) = 0$, $k = 1, ..., n+1$. Apply Corollary 4.51 and note that $\check{S}_n = 0$. □

Corollary 4.54 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)} \in L_\infty[a,b]$, for some $n \geq 1$. Further, let $(c_m,\ m \geq 1)$ be a sequence in $\mathbb{R}$ such that $\sum_{m\geq 1} |c_m| < \infty$ and let $\{x_m;\ m \geq 1\} \subset [a,b]$. If*

$$\sum_{m\geq 1} c_m = \sum_{m\geq 1} c_m x_m = \ ... \ = \sum_{m\geq 1} c_m x_m^n = 0,$$

then

$$\begin{aligned} \left| \sum_{m\geq 1} c_m f(x_m) \right| &\leq \frac{1}{2\,(n!)} (M_n - m_n) \sum_{m\geq 1} |c_m|\,(b - x_m)^n \\ &\leq \frac{1}{2\,(n!)} (M_n - m_n)(b-a)^n \sum_{m\geq 1} |c_m|\,. \end{aligned}$$

Proof. Apply Corollary 4.53 to the discrete measure $\mu = \sum_{m\geq 1} c_m \delta_{x_m}$. □

4.4 Euler harmonic identities and moments of measures

4.4.1 Introduction

For $a, b \in \mathbb{R}$, $a < b$, let $w : [a,b] \to [0,\infty)$ be an integrable function satisfying

$$\int_a^b w(t)dt > 0.$$

For $n \geq 1$, and $x, t \in [a,b]$ let

$$K_n(x,t) = \begin{cases} \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} w(s)ds, & a \leq t < x \\ 0, & t = x \\ \frac{1}{(n-1)!} \int_b^t (t-s)^{n-1} w(s)ds, & x < t \leq b \end{cases},$$

and $K_0(x,t) = w(t)$. Also let

$$e_n(x,w) = \int_a^b (t-x)^n w(t)dt,\ n \geq 0.$$

It is easy to see that $K_n(x,\cdot)$ is continuous on $[a,b] \backslash \{x\}$ and has a total jump of

$$K_n(x, x+0) - K_n(x, x-0) = \frac{(-1)^n}{(n-1)!} e_{n-1}(x,w)$$

at x. It is differentiable on $[a,b]\backslash\{x\}$ and $K'_{n+1}(x,t)=K_n(x,t)$.

Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n\geq 1$. In a paper [85] the following identity has been proved:

$$\int_a^b f(t)w(t)dt=\sum_{k=0}^{n-1}\frac{1}{k!}f^{(k)}(x)e_k(x,w)+\tilde{R}_n(x) \tag{4.47}$$

where $x\in[a,b]$ and

$$\tilde{R}_n(x)=(-1)^n\int_{[a,b]}K_n(x,t)df^{(n-1)}(t). \tag{4.48}$$

This identity has been used in [85] to prove some generalizations of weighted Ostrowski inequalities. The aim of this section is to generalize formula (4.47), by replacing weight function w by a real Borel measure on $[a,b]$, and using it to prove some further generalizations of weighted Ostrowski inequality. The results presented in this section were published in [43]. It has been generalized in [39].

4.4.2 Some integral identities

For $\mu\in M[a,b]$ we write

$$m_n(\mu)=\int_{[a,b]}s^n d\mu(s),\ n\geq 0$$

for the n-th moment of μ, and

$$e_n(x,\mu)=\int_{[a,b]}(s-x)^n d\mu(s),\ n\geq 0,\ x\in[a,b]$$

for the n-th x-centered moment of μ. We introduce the sequence of functions $P_n:[a,b]\times[a,b]\to\mathbb{R}$, $n\geq 1$, by

$$P_n(x,t)=\begin{cases}\check{\mu}_n(t), & a\leq t\leq x\\ \check{\mu}_n(t)+\frac{(-1)^n}{(n-1)!}e_{n-1}(t,\mu), & x<t\leq b\end{cases}$$

for $a\leq x<b$, while for $x=b$

$$P_n(b,t)=\begin{cases}\check{\mu}_n(t), & a\leq t<b\\ 0, & t=b\end{cases}.$$

It is easy to see that for $n\geq 2$

$$P_n(x,a)=P_n(x,b)=0$$

and

$$P_1(x,a)=\check{\mu}_1(a)=\mu(\{a\}),\ \ P_1(x,b)=0,$$

for every $x\in[a,b]$, and that $P_n(x,\cdot)$, $n\geq 2$, is continuous on $[a,b]\backslash\{x\}$, having a jump of

$$\frac{(-1)^n}{(n-1)!}e_{n-1}(x,\mu)$$

at x. Further, $P_n(x,\cdot)$, $n \geq 2$, is almost everywhere differentiable on $[a,b]\backslash\{x\}$ and $P'_{n+1}(x,t) = P_n(x,t)$ a.e. on $[a,b]$.

Remark 4.12 Note that

$$|P_n(x,t)| \leq \frac{1}{(n-1)!}(t-a)^{n-1}\|\mu\|,\ a \leq t \leq x,\ n \geq 1$$

and

$$|P_n(x,t)| \leq \frac{1}{(n-1)!}(b-t)^{n-1}\|\mu\|,\ x < t \leq b,\ n \geq 1$$

since for $x < t \leq b$ and $n \geq 1$ we have

$$\begin{aligned} P_n(x,t) &= \check{\mu}_n(t) - \frac{1}{(n-1)!}\int_{[a,b]}(t-s)^{n-1}d\mu(s) \\ &= -\frac{1}{(n-1)!}\int_{(t,b]}(t-s)^{n-1}d\mu(s). \end{aligned}$$

Remark 4.13 In the special case, when the measure μ has the density w, with respect to Lebesgue measure on $[a,b]$, the sequence $(P_n(x,t),\ n \geq 1)$ reduces to the sequence $(K_n(x,t),\ n \geq 1)$ from Introduction, except for $t = x$. In this case also $P_1(x,\cdot)$ is differentiable a.e. and $P'_1(x,t) = w(t)$, a.e..

Lemma 4.5 *For $n \geq 2$, $x \in [a,b]$, and $f \in C[a,b]$, we have*

$$\int_{[a,b]} f(t)dP_n(x,t) = \int_a^b f(t)P_{n-1}(x,t)dt + \frac{(-1)^n}{(n-1)!}e_{n-1}(x,\mu)f(x),$$

while for $n = 1$

$$\int_{[a,b]} f(t)dP_1(x,t) = \int_{[a,b]} f(t)d\mu(t) - \mu(\{a\})f(a) - \mu([a,b])f(x).$$

Proof. For $n \geq 2$, the function $P_n(x,\cdot)$ is differentiable a.e. on $[a,b]\setminus\{x\}$ and its derivative is equal to $P_{n-1}(x,\cdot)$ a.e.. Further, it has a jump of $\frac{(-1)^n}{(n-1)!}e_{n-1}(x,\mu)$ at x, which gives the first formula. Further, $P_1(x,\cdot)$ has a jump of $-\check{\mu}_1(b)$ at x, and by Lemma 4.4 we have

$$\begin{aligned} \int_{[a,b]} f(t)dP_1(x;t) &= \int_{[a,b]} f(t)d\check{\mu}_1(t) - \check{\mu}_1(b)f(x) \\ &= \int_{[a,b]} f(t)d\mu(t) - \check{\mu}_1(a)f(a) - \check{\mu}_1(b)f(x) \\ &= \int_{[a,b]} f(t)d\mu(t) - \mu(\{a\})f(a) - \mu([a,b])f(x), \end{aligned}$$

which proves the second formula. □

Theorem 4.16 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation for some $n \geq 1$. Then for every $x \in [a,b]$*

$$\int_{[a,b]} f(t)d\mu(t) = \hat{S}_n(x) + \hat{R}_n(x) \tag{4.49}$$

where

$$\hat{S}_n(x) = \sum_{k=0}^{n-1} \frac{1}{k!} f^{(k)}(x) e_k(x,\mu),$$

and

$$\hat{R}_n(x) = (-1)^n \int_{[a,b]} P_n(x,t)\, df^{(n-1)}(t).$$

Proof. Integrating by parts, for $k \geq 1$, we have

$$\hat{R}_k(x) = (-1)^k P_k(x,t) f^{(k-1)}(t)|_a^b - (-1)^k \int_{[a,b]} f^{(k-1)}(t) dP_k(x,t).$$

Since $P_n(x,a) = P_n(x,b) = 0$, for $k \geq 2$, by the first formula of Lemma 4.5,

$$\begin{aligned}
\hat{R}_k(x) &= (-1)^{k-1} \int_{[a,b]} f^{(k-1)}(t) dP_k(x,t) \\
&= (-1)^{k-1} \int_a^b f^{(k-1)}(t) P_{k-1}(x,t)\, dt + \\
&(-1)^{k-1} \frac{(-1)^k}{(k-1)!} e_{k-1}(x,\mu) f^{(k-1)}(x) \\
&= -\frac{1}{(k-1)!} f^{(k-1)}(x) e_{k-1}(x,\mu) + \hat{R}_{k-1}(x).
\end{aligned} \tag{4.50}$$

By the second formula of Lemma 4.5, for $k = 1$, (4.50) becomes

$$\begin{aligned}
\hat{R}_1(x) &= \check{\mu}_1(a) f(a) + \int_{[a,b]} f(t) dP_1(x,t) \\
&= \check{\mu}_1(a) f(a) + \int_{[a,b]} f(t) d\mu(t) - \mu(\{a\}) f(a) - \mu([a,b]) f(x) \\
&= \int_{[a,b]} f(t) d\mu(t) - \mu([a,b]) f(x).
\end{aligned} \tag{4.51}$$

From (4.50) and (4.51) follows, by iteration

$$\begin{aligned}
\hat{R}_n(x) &= -\sum_{k=2}^{n} \frac{1}{(k-1)!} f^{(k-1)}(x) e_{k-1}(x,\mu) + \hat{R}_1(x) \\
&= -\sum_{k=2}^{n} \frac{1}{(k-1)!} f^{(k-1)}(x) e_{k-1}(x,\mu) - \mu([a,b]) f(x) + \int_{[a,b]} f(t) d\mu(t) \\
&= -\sum_{k=0}^{n-1} \frac{1}{k!} f^{(k)}(x) e_k(x,\mu) + \int_{[a,b]} f(t) d\mu(t),
\end{aligned}$$

which proves our assertion. □

Remark 4.14 Note that $\hat{R}_n(x)$ can be rewritten for $n \geq 2$, by Lemma 4.5, as

$$\begin{aligned}\hat{R}_n(x) &= (-1)^n \int_{[a,b]} P_n(x,t)\, d[f^{(n-1)}(t) - f^{(n-1)}(x)] \\ &= (-1)^{n-1} \int_{[a,b]} [f^{(n-1)}(t) - f^{(n-1)}(x)] dP_n(x,t) \\ &= (-1)^{n-1} \int_{[a,b]} [f^{(n-1)}(t) - f^{(n-1)}(x)] P_{n-1}(x,t)\, dt\end{aligned}$$

It can be easily seen that the theorem above also holds for functions $f : [a,b] \to \mathbb{R}$ such that $f^{(n-1)}$ is integrable on $[a,b]$, for $n \geq 2$, and

$$\hat{R}_n(x) = (-1)^{n-1} \int_{[a,b]} [f^{(n-1)}(t) - f^{(n-1)}(x)] P_{n-1}(x,t)\, dt.$$

Note that formula (4.49) is a generalization of (4.47).

4.4.3 Generalizations of weighted Ostrowski inequality

In this section we shall use the same notations as above.

Theorem 4.17 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is an L-Lipschitzian function on $[a,b]$ for some $n \geq 1$. Then*

$$\begin{aligned}\left| \int_{[a,b]} f(t) d\mu(t) - \hat{S}_n(x) \right| &\leq L \int_a^b |P_n(x,t)|\, dt \\ &\leq \frac{1}{n!} [(x-a)^n + (b-x)^n] L \|\mu\| \end{aligned} \tag{4.52}$$

for every $x \in [a,b]$.

Proof. Using estimate (4.24) and Theorem 4.16 we get

$$|\hat{R}_n(x)| = \left| \int_{[a,b]} P_n(x,t)\, df^{(n-1)}(t) \right| \leq L \int_a^b |P_n(x,t)|\, dt.$$

By Remark 4.12 we have

$$\begin{aligned}\int_a^b |P_n(x,t)|\, dt &= \int_a^x |P_n(x,t)|\, dt + \int_x^b |P_n(x,t)|\, dt \\ &\leq \frac{1}{(n-1)!} \|\mu\| \int_a^x (t-a)^{n-1} dt + \frac{1}{(n-1)!} \|\mu\| \int_x^b (b-t)^{n-1} dt \\ &= \frac{1}{n!} \|\mu\| [(x-a)^n + (b-x)^n],\end{aligned}$$

which proves our assertion. □

Remark 4.15 For positive measure μ we have

$$\begin{aligned}\int_a^b |P_n(x,t)|\,dt &= \int_a^x P_n(x,t)\,dt + (-1)^n \int_x^b P_n(x,t)\,dt \\ &= \frac{1}{n!}\int_{[a,b]} |t-x|^n\,d\mu(t).\end{aligned}$$

Therefore, for every $\mu \in M[a,b]$

$$\int_a^b |P_n(x,t)|\,dt \le \frac{1}{n!}\int_{[a,b]} |t-x|^n\,d|\mu|(t),$$

which gives

$$\begin{aligned}\int_a^b |P_n(x,t)|\,dt &\le \frac{1}{n!}\|\mu\| \sup_{a\le t\le b} |t-x|^n \\ &= \frac{1}{n!}\|\mu\| \max\{(x-a)^n,(b-x)^n\} \\ &= \frac{1}{n!}\|\mu\| [\max\{x-a,b-x\}]^n \\ &= \frac{1}{n!}\|\mu\| \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^n.\end{aligned}$$

Corollary 4.55 *If f is L-Lipschitzian on $[a,b]$, then*

$$\left|\int_{[a,b]} f(t)d\mu(t) - \mu([a,b])f(x)\right| \le L\int_a^b |P_1(x,t)|\,dt \le (b-a)L\|\mu\|$$

for every $x \in [a,b]$.

Proof. Put $n = 1$ in the theorem above. □

Corollary 4.56 *If f' is L-Lipschitzian on $[a,b]$, then*

$$\begin{aligned}\left|\int_{[a,b]} f(t)d\mu(t) - f(x)\mu([a,b]) - f'(x)e_1(x,\mu)\right| &\le L\int_a^b |P_2(x,t)|\,dt \\ &\le \tfrac{1}{2}\left[(x-a)^2 + (b-x)^2\right] L\|\mu\|\end{aligned}$$

for every $x \in [a,b]$.

Proof. Put $n = 2$ in the theorem above. □

Corollary 4.57 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is an L-Lipschitzian function on $[a,b]$ for some $n \ge 1$. Then*

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{(k+1)!}[(b-x)^{k+1} - (a-x)^{k+1}]\right| \le \frac{L}{(n+1)!}[(x-a)^{n+1} + (b-x)^{n+1}]$$

for every $x \in [a,b]$.

Proof. Apply the theorem above to the Lebesgue measure on $[a,b]$. □

Corollary 4.58 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is an L-Lipschitzian function on $[a,b]$ for some $n \geq 1$. Then*

$$\left| f(y) - \sum_{k=0}^{n-1} \frac{1}{k!} f^{(k)}(x)(y-x)^k \right| \leq \frac{1}{n!} L|x-y|^n$$

for every $x, y \in [a,b]$.

Proof. Apply the theorem above to $\mu = \delta_y$. Then

$$e_k(x,\mu) = (y-x)^k, \ k \geq 0$$

and

$$|P_n(x,t)| = \frac{1}{(n-1)!} |t-y|^{n-1}, \ t \in [x,y] \text{ or } t \in [y,x],$$

while $P_n(x,t) = 0$ for other t. □

Corollary 4.59 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is an L-Lipschitzian function on $[a,b]$ for some $n \geq 1$. Further, let $(c_m, \ m \geq 1)$ be a sequence in $\mathbb{R}$ such that $\sum_{m\geq 1} |c_m| < \infty$ and let $\{x_m; \ m \geq 1\}$ be different points in $[a,b]$. Then*

$$\begin{aligned}
&\left| \sum_{m\geq 1} c_m f(x_m) - \sum_{m\geq 1} \sum_{k=0}^{n-1} \frac{1}{k!} f^{(k)}(x) c_m (x_m - x)^k \right| \\
&\leq \frac{1}{n!} L \sum_{m\geq 1} |c_m| \, |x - x_m|^n \\
&\leq \frac{1}{n!} L(b-a)^n \sum_{m\geq 1} |c_m|
\end{aligned}$$

for every $x \in [a,b]$.

Proof. Apply the theorem above for the discrete measure $\mu = \sum_{m\geq 1} c_m \delta_{x_m}$. □

Theorem 4.18 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n \geq 1$. Then*

$$\begin{aligned}
\left| \int_{[a,b]} f(t) d\mu(t) - \hat{S}_n(x) \right| &\leq \sup_{t\in[a,b]} |P_n(x,t)| \, V_a^b(f^{(n-1)}) \\
&\leq \frac{1}{(n-1)!} \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right]^{n-1} \|\mu\| \, V_a^b(f^{(n-1)})
\end{aligned}$$

for every $x \in [a,b]$.

Proof. If $F:[a,b]\to\mathbb{R}$ is bounded and the Stieltjes integral $\int_{[a,b]}F(t)df^{(n-1)}(t)$ exists, then

$$\left|\int_{[a,b]}F(t)df^{(n-1)}(t)\right|\leq\sup_{t\in[a,b]}|F(t)|\cdot V_a^b(f^{(n-1)}).$$

Let us apply this estimation to formula (4.49)

$$\left|\hat{R}_n(x)\right|=\left|\int_a^b P_n(x,t)\,df^{(n-1)}(t)\right|\leq\sup_{t\in[a,b]}|P_n(x,t)|\,V_a^b(f^{(n-1)})$$

Further, by Remark 4.12 we have

$$\begin{aligned}\sup_{t\in[a,b]}|P_n(x,t)|&\leq\max\{\frac{(x-a)^{n-1}}{(n-1)!}\|\mu\|,\frac{(b-x)^{n-1}}{(n-1)!}\|\mu\|\}\\&=\frac{1}{(n-1)!}\|\mu\|\max\{(x-a)^{n-1},(b-x)^{n-1}\}\\&=\frac{1}{(n-1)!}\|\mu\|\,[\max\{(x-a),(b-x)\}]^{n-1}\\&=\frac{1}{(n-1)!}\|\mu\|\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]^{n-1},\end{aligned}$$

which proves our assertion. □

Corollary 4.60 *If f is a continuous function of bounded variation on $[a,b]$, then*

$$\left|\int_{[a,b]}f(t)d\mu(t)-\mu([a,b])f(x)\right|\leq\|\mu\|\,V_a^b(f)$$

for every $x\in[a,b]$.

Proof. Put $n=1$ in the theorem above. □

Corollary 4.61 *If f' is a continuous function of bounded variation on $[a,b]$, then*

$$\begin{aligned}&\left|\int_{[a,b]}f(t)d\mu(t)-f(x)\mu([a,b])-f'(x)e_1(x,\mu)\right|\\&\leq\sup_{t\in[a,b]}|P_2(x,t)|\,V_a^b(f')\\&\leq\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\|\mu\|\,V_a^b(f')\end{aligned}$$

for every $x\in[a,b]$.

Proof. Put $n=2$ in Theorem 4.18. □

Corollary 4.62 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n \geq 1$. Then*

$$\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} f^{(k)}(x) \left[(b-x)^{k+1} - (a-x)^{k+1} \right] \right| \leq \frac{1}{n!} \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] V_a^b(f^{(n-1)})$$

for every $x \in [a,b]$.

Proof. Apply the theorem above to the Lebesgue measure on $[a,b]$. □

Corollary 4.63 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n \geq 1$. Then*

$$\left| f(y) - \sum_{k=0}^{n-1} \frac{1}{k!} f^{(k)}(x)(y-x)^k \right| \leq \frac{1}{(n-1)!} |x-y|^{n-1} V_a^b(f^{(n-1)})$$

for every $x, y \in [a,b]$.

Proof. Apply the theorem above to $\mu = \delta_y$. □

Corollary 4.64 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n \geq 1$. Further, let $(c_m,\ m \geq 1)$ be a sequence in $\mathbb{R}$ such that $\sum_{m \geq 1} |c_m| < \infty$ and let $\{x_m;\ m \geq 1\}$ be different points in $[a,b]$. Then*

$$\begin{aligned} &\left| \sum_{m\geq 1} c_m f(x_m) - \sum_{m\geq 1} \sum_{k=0}^{n-1} \frac{1}{k!} f^{(k)}(x) c_m (x_m - x)^k \right| \\ &\leq \frac{1}{(n-1)!} \sum_{m \geq 1} |c_m| \, |x - x_m|^{n-1} V_a^b(f^{(n-1)}) \\ &\leq \frac{(b-a)^{n-1}}{(n-1)!} V_a^b(f^{(n-1)}) \sum_{m\geq 1} |c_m| \end{aligned}$$

for every $x \in [a,b]$.

Proof. Apply the theorem above to the discrete measure $\mu = \sum_{m \geq 1} c_m \delta_{x_m}$. □

Theorem 4.19 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ is integrable, for some $n \geq 1$. Then*

$$\begin{aligned} \left| \int_{[a,b]} f(t)d\mu(t) - \hat{S}_n(x) \right| &\leq \sup_{t \in [a,b]} |P_n(x,t)| \, \|f^{(n)}\|_1 \\ &\leq \frac{1}{(n-1)!} \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right]^{n-1} \|\mu\| \, \|f^{(n)}\|_1 \end{aligned}$$

for every $x \in [a,b]$.

Proof. Note that in this case

$$V_a^b(f^{(n-1)}) = \int_a^b \left| f^{(n)}(t) \right| dt = \|f^{(n)}\|_1$$

and apply Theorem 4.18. □

Theorem 4.20 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)} \in L_\infty[a,b]$, for some $n \geq 1$. Then*

$$\begin{aligned}\left| \int_{[a,b]} f(t)d\mu(t) - \hat{S}_n(x) \right| &\leq \int_a^b |P_n(x,t)|\, dt \cdot \|f^{(n)}\|_\infty \\ &\leq \frac{1}{n!} \left[(x-a)^n + (b-x)^n \right] \|\mu\| \, \|f^{(n)}\|_\infty\end{aligned}$$

for every $x \in [a,b]$.

Proof. In this case $f^{(n-1)}$ is L-Lipschitzian with $L = \|f^{(n)}\|_\infty$. □

Theorem 4.21 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)} \in L_p[a,b]$, for some $n \geq 1$ and $1 < p < \infty$. Then*

$$\begin{aligned}\left| \int_{[a,b]} f(t)d\mu(t) - \hat{S}_n(x) \right| &\leq \|P_n(x,\cdot)\|_q \|f^{(n)}\|_p \\ &\leq \frac{1}{(n-1)!} \|\mu\| \, \|f^{(n)}\|_p \left[\frac{(x-a)^{(n-1)q+1} + (b-x)^{(n-1)q+1}}{(n-1)q+1} \right]^{1/q}\end{aligned}$$

for every $x \in [a,b]$, where $1/p + 1/q = 1$.

Proof. By applying the Hölder inequality we have

$$\begin{aligned}\left| \int_{[a,b]} f(t)d\mu(t) - \hat{S}_n(x) \right| &\leq \int_a^b |P_n(x,t)| \, |f^{(n)}(t)| dt \\ &\leq \left(\int_a^b |P_n(x,t)|^q \, dt \right)^{1/q} \|f^{(n)}\|_p.\end{aligned}$$

Further, by Remark 4.12 we have

$$\begin{aligned}\int_a^b |P_n(x,t)|^q \, dt &= \int_a^x |P_n(x,t)|^q \, dt + \int_x^b |P_n(x,t)|^q \, dt \\ &\leq \left[\frac{1}{(n-1)!} \|\mu\| \right]^q \left[\int_a^x (t-a)^{(n-1)q} dt + \int_x^b (b-t)^{(n-1)q} dt \right] \\ &= \left[\frac{1}{(n-1)!} \|\mu\| \right]^q \frac{(x-a)^{(n-1)q+1} + (b-x)^{(n-1)q+1}}{(n-1)q+1},\end{aligned}$$

which proves our assertion. □

Corollary 4.65 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)} \in L_p[a,b]$, for some $n \geq 1$ and $1 < p < \infty$. Then*

$$\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} f^{(k)}(x)[(b-x)^{k+1} - (a-x)^{k+1}] \right|$$
$$\leq \frac{1}{n!} \|f^{(n)}\|_p \left[\frac{(x-a)^{nq+1} + (b-x)^{nq+1}}{nq+1} \right]^{1/q}$$

for every $x \in [a,b]$, where $1/p + 1/q = 1$.

Proof. Apply the theorem above to the Lebesgue measure on $[a,b]$. □

Corollary 4.66 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)} \in L_p[a,b]$, for some $n \geq 1$ and $1 < p < \infty$. Then*

$$\left| f(y) - \sum_{k=0}^{n-1} \frac{1}{k!} f^{(k)}(x)(y-x)^k \right| \leq \frac{1}{(n-1)!\,[(n-1)q+1]^{1/q}} |x-y|^{n-1+1/q} \|f^{(n)}\|_p$$

for every $x, y \in [a,b]$, where $1/p + 1/q = 1$.

Proof. Apply the theorem above to $\mu = \delta_y$. □

Corollary 4.67 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)} \in L_p[a,b]$, for some $n \geq 1$ and $1 < p < \infty$. Further, let $(c_m,\ m \geq 1)$ be a sequence in $\mathbb{R}$ such that $\sum_{m\geq 1} |c_m| < \infty$ and let $\{x_m;\ m \geq 1\}$ be different points in $[a,b]$. Then*

$$\left| \sum_{m\geq 1} c_m f(x_m) - \sum_{m\geq 1} \sum_{k=0}^{n-1} \frac{1}{k!} f^{(k)}(x) c_m (x_m - x)^k \right|$$
$$\leq \frac{1}{(n-1)!\,[(n-1)q+1]^{1/q}} \|f^{(n)}\|_p \sum_{m\geq 1} |c_m|\,|x - x_m|^{n-1+1/q}$$
$$\leq \frac{1}{(n-1)!\,[(n-1)q+1]^{1/q}} (b-a)^{n-1+1/q} \|f^{(n)}\|_p \sum_{m\geq 1} |c_m|$$

for every $x \in [a,b]$.

Proof. Apply the theorem above to the discrete measure $\mu = \sum_{m\geq 1} c_m \delta_{x_m}$. □

Let $\alpha \in (0,1]$ and $L \geq 0$. Function $g : [a,b] \to \mathbb{R}$ is called **α-Hölder** function with constant L if

$$|g(t) - g(s)| \leq L|t-s|^{\alpha}, \ s,t \in [a,b].$$

Theorem 4.22 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is α-Hölder function with constant L for some $n \geq 2$. Then*

$$\left| \int_{[a,b]} f(t)d\mu(t) - \hat{S}_n(x) \right| \leq L \int_a^b |t-x|^{\alpha} |P_{n-1}(x,t)|\,dt$$

$$\leq \frac{(x-a)^{\alpha+n-1}+(b-x)^{\alpha+n-1}}{(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)} L\|\mu\|,$$

for every $x \in [a,b]$.

Proof. By Remark 4.14

$$\begin{aligned}\left|\hat{R}_n(x)\right| &\leq \int_{[a,b]} |[f^{(n-1)}(t)-f^{(n-1)}(x)]| \, |P_{n-1}(x,t)| \, dt \\ &\leq L \int_a^b |t-x|^{\alpha} |P_{n-1}(x,t)| \, dt.\end{aligned}$$

Further, by Remark 4.12 we have

$$\begin{aligned}&\int_a^b |t-x|^{\alpha} |P_{n-1}(x,t)| \, dt \\ &\leq \|\mu\| \int_a^x (x-t)^{\alpha} \frac{(t-a)^{n-2}}{(n-2)!} dt + \|\mu\| \int_x^b (t-x)^{\alpha} \frac{(b-t)^{n-2}}{(n-2)!} dt \\ &= \frac{1}{(n-2)!} \|\mu\| \left[\int_a^x (x-t)^{\alpha} (t-a)^{n-2} dt + \int_x^b (t-x)^{\alpha} (b-t)^{n-2} dt \right] \\ &= \frac{1}{(n-2)!} \|\mu\| B(\alpha+1, n-1) \left[(x-a)^{\alpha+n-1} + (b-x)^{\alpha+n-1} \right] \\ &= \frac{(x-a)^{\alpha+n-1}+(b-x)^{\alpha+n-1}}{(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)} \|\mu\|,\end{aligned}$$

which proves our assertion, where B is the beta function. □

Corollary 4.68 *If f' is an α-Hölder function with constant L, then*

$$\begin{aligned}&\left| \int_{[a,b]} f(t) d\mu(t) - f(x)\mu([a,b]) - f'(x) e_1(x,\mu) \right| \\ &\leq L \int_a^b |t-x|^{\alpha} |P_1(x,t)| \, dt \\ &\leq \frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{\alpha+1} L\|\mu\|,\end{aligned}$$

for every $x \in [a,b]$.

Proof. Put $n = 2$ in the theorem above. □

Remark 4.16 Applying calculations as in Remark 4.15 and in the proof of Theorem 4.22, for positive measure μ we have

$$\int_a^b |t-x|^{\alpha} |P_n(x,t)| \, dt = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)} \int_{[a,b]} |t-x|^{\alpha+n} d\mu(t).$$

Therefore, for every $\mu \in M[a,b]$

$$\int_a^b |t-x|^{\alpha} |P_n(x,t)| \, dt \leq \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)} \int_{[a,b]} |t-x|^{\alpha+n} \, d|\mu|(t),$$

which gives

$$\begin{aligned} \int_a^b |t-x|^{\alpha} |P_n(x,t)| \, dt &\leq \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)} \|\mu\| \max_{a\leq t\leq b} |t-x|^{\alpha+n} \\ &= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)} \|\mu\| \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^{\alpha+n}. \end{aligned}$$

Corollary 4.69 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is α-Hölder function with constant L for some $n \geq 2$. Then*

$$\left| \int_{[a,b]} f(t) d\mu(t) - S_n(x) \right| \leq \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)} \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^{\alpha+n-1} L\|\mu\|$$

for every $x \in [a,b]$.

Proof. Follows from Theorem 4.22 and Remark 4.16. □

Corollary 4.70 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is α-Hölder function with constant L for some $n \geq 2$. Then*

$$\left| f(y) - \sum_{k=0}^{n-1} \frac{1}{k!} f^{(k)}(x)(y-x)^k \right| \leq \frac{1}{(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)} |x-y|^{\alpha+n-1} L$$

for every $x, y \in [a,b]$.

Proof. Apply the theorem above to $\mu = \delta_y$. □

Corollary 4.71 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is α-Hölder function with constant L, for some $n \geq 2$. Further, let $(c_m,\ m \geq 1)$ be a sequence in $\mathbb{R}$ such that $\sum |c_m| < \infty$ and let $\{x_m;\ m \geq 1\}$ be different points in $[a,b]$. Then*

$$\begin{aligned} &\left| \sum_{m\geq 1} c_m f(x_m) - \sum_{m\geq 1} \sum_{k=0}^{n-1} \frac{1}{k!} f^{(k)}(x) c_m (x_m - x)^k \right| \\ &\leq \frac{1}{(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)} L \sum_{m\geq 1} |c_m| \, |x-x_m|^{\alpha+n-1} \\ &\leq \frac{1}{(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)} L(b-a)^{\alpha+n-1} \textstyle\sum_{m\geq 1} |c_m| \end{aligned}$$

for every $x \in [a,b]$.

Proof. Apply the theorem above to the discrete measure $\mu = \sum_{m\geq 1} c_m \delta_{x_m}$. □

4.4.4 Some Grüss-type inequalities

Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}\in L_\infty[a,b]$, for some $n\geq 1$. Then

$$m_n\leq f^{(n)}(t)\leq M_n,\ t\in[a,b],\ \text{a.e.}$$

for some real constants m_n and M_n.

Theorem 4.23 *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}\in L_\infty[a,b]$, for some $n\geq 2$. If $x\in[a,b]$ and $\mu\in M[a,b]$ are such that $e_{n-1}(x,\mu)=0$, then*

$$\left|\int_{[a,b]}f(t)d\mu(t)-\hat{S}_n(x)\right|\leq\frac{1}{2(n-1)!}(M_{n-1}-m_{n-1})[(x-a)^{n-1}+(b-x)^{n-1}]\,\|\mu\|\,.$$

Proof. By Remark 4.14 we have

$$\hat{R}_n(x)=(-1)^{n-1}\int_{[a,b]}[f^{(n-1)}(t)-f^{(n-1)}(x)]P_{n-1}(x,t)\,dt.$$

Define measure ν_{n-1} by

$$d\nu_{n-1}(t)=(-1)^{n-1}P_{n-1}(x,t)\,dt.$$

Then

$$\begin{aligned}\nu_{n-1}([a,b])&=(-1)^{n-1}\int_a^b P_{n-1}(x,t)\,dt\\&=(-1)^{n-1}\tfrac{(-1)^{n-1}}{(n-1)!}e_{n-1}(x,\mu)\\&=\tfrac{1}{(n-1)!}e_{n-1}(x,\mu),\end{aligned}$$

so by our condition $\nu_{n-1}([a,b])=0$, which means that ν_{n-1} is balanced measure. Further,

$$\|\nu_{n-1}\|=\int_a^b|P_{n-1}(x,t)|\,dt\leq\frac{1}{(n-1)!}\,\|\mu\|\,[(x-a)^{n-1}+(b-x)^{n-1}].$$

Therefore, applying Theorem 4.7 we have

$$\begin{aligned}\left|\hat{R}_n(x)\right|&\leq\frac{M_{n-1}-m_{n-1}}{2}\,\|\nu_{n-1}\|\\&\leq\frac{M_{n-1}-m_{n-1}}{2}\,\frac{1}{(n-1)!}\,\|\mu\|\left[(x-a)^{n-1}+(b-x)^{n-1}\right]\\&=\frac{M_{n-1}-m_{n-1}}{2(n-1)!}\left[(x-a)^{n-1}+(b-x)^{n-1}\right]\|\mu\|\,,\end{aligned}$$

which proves our assertion. □

Corollary 4.72 *For $f' \in L_\infty[a,b]$ let $x \in [a,b]$ and $\mu \in M[a,b]$ be such that*

$$\int_{[a,b]} (t-x)d\mu(t) = 0.$$

Then

$$\left| \int_{[a,b]} f(t)d\mu(t) - \mu([a,b]) f(x) \right| \leq \frac{1}{2}(M_1 - m_1)(b-a)\|\mu\|$$

Proof. Put $n = 2$ in the theorem above. □

Corollary 4.73 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)} \in L_\infty[a,b]$, for some $n \geq 2$. If $\mu \in M[a,b]$ is such that*

$$m_0(\mu) = m_1(\mu) = \ \dots \ = m_{n-1}(\mu) = 0,$$

then

$$\left| \int_{[a,b]} f(t)d\mu(t) \right| \leq \frac{1}{2(n-1)!}(M_{n-1} - m_{n-1})[(x-a)^{n-1} + (b-x)^{n-1}]\|\mu\|,$$

for every $x \in [a,b]$.

Proof. Apply the theorem above and note that in this case we have $e_k(x,\mu) = 0$, for $k = 0,1,\dots,n-1$, and also $\hat{S}_n(x) = 0$, for every $x \in [a,b]$. □

Corollary 4.74 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)} \in L_\infty[a,b]$, for some $n \geq 2$. If $\mu \in M[a,b]$ is k-balanced, for $k = 1,..,n$, then*

$$\left| \int_{[a,b]} f(t)d\mu(t) \right| \leq \frac{1}{2(n-1)!}(M_{n-1} - m_{n-1})[(x-a)^{n-1} + (b-x)^{n-1}]\|\mu\|$$

for every $x \in [a,b]$.

Proof. Note that in this case $m_0(\mu) = m_1(\mu) = \cdots = m_{n-1}(\mu) = 0$. Apply now Corollary 4.73. □

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