Chapter 1

Generalization of the classical integral forumalae and related inequalities

In this chapter we introduce general integral identities using the harmonic sequences of polynomials and $w-$harmonic sequences of functions. Those identities are the main tool for deriving generalizations of some famous quadrature formulas. We deal with quadrature formulas which contain values of the function in nodes, as well as values of higher ordered derivatives in inner nodes. Thereby, the level of exactness of those quadrature formulas is saved. Error estimations with sharp and the best possible constants are developed as well.

In Section 1.1. general integral identities with harmonic polynomials and $w-$harmonic functions are established. Those identities are actually the general quadrature formulas with $m+1$ nodes. For both identites the error estimations for functions whose higher ordered derivatives belong to L_p spaces are given.

In Section 1.2. general one-point quadrature formula is established. Special cases of the well known weights are considered and generalizations of the Gaussian quadrature formulas with one node are obtained.

In Section 1.3. general two-point integral quadrature formula using the concept of harmonic polynomials is established. Improved version of Guessab and Schmeisser's result is given with new integral inequalities involving functions whose derivatives belong to various classes of functions (L_p spaces, convex, concave, bounded functions). Furthermore, several special cases of polynomials are considered, and the generalization of well-known two-point quadrature formulae, such as trapezoid, perturbed trapezoid, two-point Newton-

Cotes formula, two-point Maclaurin formula, midpoint, are obtained. Weighted version of two-point integral quadrature formula is obtained using $w-$harmonic sequences of functions. For special choices of weights w and nodes x and $a+b-x$ the generalization of the well-known two-point quadrature formulas of Gauss type are given.

In Section 1.4. general three-point quadrature formula with nodes $x, \frac{a+b}{2}$ and $a+b-x$ is introduced. From non-weighted version Simpson, dual Simpson and Maclaurin formulas are obtained, while for special weights Gaussian quadrature formulas are given.

The closed four-point quadrature formula is introduced in Section 1.5. Generalization of Lobatto formula is given as special case.

Definition 1.1 *We say that* $\{P_k\}_k \in \mathbb{N}_0$ *is* ***harmonic sequence of the polynomials*** *if* $P_k'(t) = P_{k-1}(t), \forall k \in \mathbb{N}$ *and* $P_0(t) \equiv 1$.

1.1 General integral identities involving $w-$harmonic sequences of functions

Non-weighted integral identity is used for the approximation of an integral of the following form: $\int_a^b f(t)dt$. The next theorem is obtained in [100].

Theorem 1.1 *Let* $\sigma := \{a = x_0 < x_1 < x_2 < \ldots < x_m = b\}$ *be subdivision of the interval* $[a,b]$. *Further, let for each* $j = 1,\ldots,m$, $\{P_{jk}\}_{k\in\mathbb{N}_0}$ *be the harmonic sequences of the polynomials on* $[x_{j-1},x_j]$, *i.e.* $P_{jk}'(t) = P_{j,k-1}(t)$ *i* $P_{j0}(t) \equiv 1$, *for* $j = 1,\ldots,m$ *and* $k \in \mathbb{N}$, *and let*

$$S_n(t,\sigma) = \begin{cases} P_{1n}(t), & t \in [a,x_1] \\ P_{2n}(t), & t \in (x_1,x_2] \\ \vdots & \\ P_{mn}(t), & t \in (x_{m-1},b], \end{cases} \tag{1.1}$$

for some $n \in \mathbb{N}$. *For an arbitrary* $(n-1)-$*times differentiable function* $f : [a,b] \to \mathbb{R}$ *such that* $f^{(n-1)}$ *is bounded, the following identity states*

$$\begin{aligned}(-1)^n \int_a^b S_n(t,\sigma)df^{(n-1)}(t) = \int_a^b f(t)dt + \sum_{k=1}^{n} (-1)^k \Big[P_{mk}(b)f^{(k-1)}(b) \\ + \sum_{j=1}^{m-1} \left[P_{jk}(x_j) - P_{j+1,k}(x_j)\right] f^{(k-1)}(x_j) - P_{1k}(a)f^{(k-1)}(a)\Big],\end{aligned} \tag{1.2}$$

whenever the integrals exist.

Identity (1.2) is used for the approximation of an integral $\int_a^b f(t)dt$ both with the values of the function f and its higher order derivatives in nodes $x_0,x_1,x_2,\ldots,x_m$. With appropriate

choice of polynomials $\{P_{jk}\}$ and nodes x_j we shall get the generalization of the well-known quadrature formulas. In those generalized formulas the integral is approximated not only with the values of the function in certain nodes, but also with values of its derivatives up to $(n-1)^{\text{th}}$ order in inner nodes.
Let us develop an error estimation for the identity (1.2).

Theorem 1.2 *Assume (p,q) is a pair of conjugate exponents, that is $1 \le p,q, \le \infty$, $\frac{1}{p}+\frac{1}{q}=1$. If $f:[a,b]\to\mathbb{R}$ is an arbitrary function such that $f^{(n)}$ is piecewise continuous, for some $n\in\mathbb{N}$, then we have*

$$\begin{aligned}
&\left|\int_a^b f(t)dt+\sum_{k=1}^{n}(-1)^k\left[P_{mk}(b)f^{(k-1)}(b)\right.\right.\\
&\left.\left.+\sum_{j=1}^{m-1}\left[P_{jk}(x_j)-P_{j+1,k}(x_j)\right]f^{(k-1)}(x_j)-P_{1k}(a)f^{(k-1)}(a)\right]\right|\\
&\le C(n,q)\|f^{(n)}\|_p,
\end{aligned} \tag{1.3}$$

where

$$C(n,q)=\|S_n(\cdot,\sigma)\|_q=\begin{cases}\left[\sum_{j=1}^{m}\int_{x_{j-1}}^{x_j}\left|P_{jn}(t)\right|^q dt\right]^{\frac{1}{q}}, & 1\le q<\infty\\ \max_{1\le j\le m}\{\sup_{t\in[x_{j-1},x_j]}\left|P_{jn}(t)\right|\},\ q=\infty.\end{cases}$$

Inequalities are sharp for $1<p\le\infty$ and the best possible for $p=1$. Equality in (1.3) is attained for the functions f of the form:

$$f(t)=Mf_*(t)+r_{n-1}(t), \tag{1.4}$$

where $M\in\mathbb{R}$, r_{n-1} is an arbitrary polynomial of degree $n-1$, and $f_*:[a,b]\to\mathbb{R}$ is function with the following representation:

$$f_*(t):=\int_a^t\frac{(t-s)^{n-1}}{(n-1)!}|S_n(s,\sigma)|^{\frac{1}{p-1}}\operatorname{sgn}S_n(s,\sigma)ds,\quad 1<p<\infty \tag{1.5}$$

and

$$f_*(t):=\int_a^t\frac{(t-s)^{n-1}}{(n-1)!}\operatorname{sgn}S_n(s,\sigma)ds,\quad p=\infty. \tag{1.6}$$

Proof. Applying Hölder inequality to the integral

$$(-1)^n\int_a^b S_n(t,\sigma)df^{(n-1)}(t)=(-1)^n\int_a^b S_n(t,\sigma)f^{(n)}(t)dt$$

an inequality (1.3) is obtained. To prove the inequalities are sharp for $1<p\le\infty$, we have to find function $f:[a,b]\to\mathbb{R}$ such that

$$\left|\int_a^b S_n(t,\sigma)f^{(n)}(t)dt\right|=C(n,q)\cdot\|f^{(n)}\|_p. \tag{1.7}$$

For function f_* defined by (1.5) and (1.6) we have

$$f_*^{(n)}(t) = \begin{cases} \operatorname{sgn} S_n(t,\sigma), & p=\infty, \\ \\ |S_n(t,\sigma)|^{\frac{1}{p-1}} \operatorname{sgn} S_n(t,\sigma), & 1<p<\infty, \end{cases}$$

Function $f:[a,b]\to\mathbb{R}$ defined with (1.4) is $n-$times differentiable also. Further, $f^{(n)}$ is piecewise continuous and $f^{(n)}(t) = Mf_*^{(n)}(t)$ holds.
For $p=\infty$ we have $\|f^{(n)}\|_p = |M|$, so

$$\begin{aligned} \left|\int_a^b S_n(t,\sigma) f^{(n)}(t)dt\right| &= \left|M\int_a^b S_n(t,\sigma) f_*^{(n)}(t)dt\right| \\ &= \left|M\int_a^b S_n(t,\sigma) sgnS_n(t,\sigma)dt\right| \\ &= |M|\int_a^b |S_n(t,\sigma)|\,dt = C(n,1)\|f^{(n)}\|_\infty \end{aligned}$$

holds, while for $1<p<\infty$ we have

$$\|f^{(n)}\|_p = |M|\left[\int_a^b |S_n(t,\sigma)|^{\frac{p}{p-1}}\,dt\right]^{\frac{1}{p}} = |M|\left[\int_a^b |S_n(t,\sigma)|^q\,dt\right]^{\frac{1}{p}},$$

which implies

$$\begin{aligned} \left|\int_a^b S_n(t,\sigma) f^{(n)}(t)dt\right| &= \left|M\int_a^b S_n(t,\sigma) f_*^{(n)}(t)dt\right| \\ &= \left|M\int_a^b S_n(t,\sigma)\,|S_n(t,\sigma)|^{\frac{1}{p-1}} \operatorname{sgn} S_n(t,\sigma)dt\right| \\ &= |M|\int_a^b |S_n(t,\sigma)|^{\frac{p}{p-1}}\,dt = |M|\int_a^b |S_n(t,\sigma)|^q\,dt = C(n,q)\|f^{(n)}\|_p, \end{aligned}$$

so the proof of the (1.7) is finished.
Finally, we have to prove that inequality (1.3) is the best possible for $p=1$. Obviously, because of the continuity of the $P_{jk}(\cdot)$ on $[x_{j-1},x_j]$, there exists $j\in\{1,\dots,m\}$ and $t_0\in[x_{j-1},x_j]$ such that $\sup_{t\in[a,b]}|S_n(t,\sigma)| = \left|P_{jn}(t_0)\right|$. First, let us assume that $P_{jn}(t_0)>0$. There are two possibilities:

(i) $x_{j-1}<t_0\le x_j$

(ii) $t_0 = x_{j-1}$

For the case (i) let us define function $f_\varepsilon:[a,b]\to\mathbb{R}$ for $\varepsilon>0$:

$$f_\varepsilon^{(n-1)}(t) = \begin{cases} 1, & t\le t_0-\varepsilon, \\ \frac{t_0-t}{\varepsilon}, & t\in[t_0-\varepsilon,t_0], \\ 0, & t\ge t_0. \end{cases}$$

When ε is "enough small", we have

$$\left|\int_a^b S_n(t,\sigma)f_\varepsilon^{(n)}(t)dt\right| = \frac{1}{\varepsilon}\left|\int_{t_0-\varepsilon}^{t_0} S_n(t,\sigma)dt\right| = \frac{1}{\varepsilon}\int_{t_0-\varepsilon}^{t_0} P_{jn}(t)dt.$$

Further,

$$\frac{1}{\varepsilon}\int_{t_0-\varepsilon}^{t_0} P_{jn}(t)dt \leq \frac{1}{\varepsilon}P_{jn}(t_0)\int_{t_0-\varepsilon}^{t_0} dt = P_{jn}(t_0).$$

Since $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t_0-\varepsilon}^{t_0} P_{jn}(t)dt = P_{jn}(t_0)$, the assertion follows.
For the case (ii) let us define function $f_\varepsilon : [a,b] \to \mathbb{R}$ for $\varepsilon > 0$:

$$f_\varepsilon^{(n-1)}(t) = \begin{cases} 0, & t \leq t_0, \\ \frac{t-t_0}{\varepsilon}, & t \in [t_0, t_0+\varepsilon], \\ 1, & t \geq t_0+\varepsilon. \end{cases}$$

When ε is "enough small", we have

$$\left|\int_a^b S_n(t,\sigma)f_\varepsilon^{(n)}(t)dt\right| = \frac{1}{\varepsilon}\left|\int_{t_0}^{t_0+\varepsilon} S_n(t,\sigma)dt\right| = \frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} P_{jn}(t)dt.$$

Further,

$$\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} P_{jn}(t)dt \leq \frac{1}{\varepsilon}P_{jn}(t_0)\int_{t_0}^{t_0+\varepsilon} dt = P_{jn}(t_0).$$

Since $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} P_{jn}(t)dt = P_{jn}(t_0)$, the assertion follows.
For the case $P_{jn}(t_0) < 0$, the proof is simmilar. □

Remark 1.1 Inequality (1.3) is obtained in [100], for the case $1 < p \leq \infty$.

In [73] is derived the identity (1.2) with monic polynomials:

Theorem 1.3 *Let $\sigma := \{a = x_0 < x_1 < x_2 < \ldots < x_m = b\}$ be subdivision of the interval $[a,b]$. Further, for $j = 1,\ldots,m$, let M_{jn} be monic polynomials, for some $n \in \mathbb{N}$, with $\deg M_{jn} = n$. Define*

$$V_n(t,\sigma) = \begin{cases} M_{1n}(t), & t \in [a,x_1], \\ M_{2n}(t), & t \in (x_1,x_2], \\ \vdots \\ M_{mn}(t), & t \in (x_{m-1},b]. \end{cases} \tag{1.8}$$

If $f : [a,b] \to \mathbb{R}$ is some $(n-1)-$times differentiable function such that $f^{(n-1)}$ is bounded, then we have

$$\begin{aligned} \int_a^b f(t)dt \ &+ \frac{1}{n!}\sum_{k=0}^{n-1}(-1)^{k+1}\cdot\Big[M_{mn}^{(n-k-1)}(b)f^{(k)}(b) + \sum_{j=1}^{m-1}\Big[M_{jn}^{(n-k-1)}(x_j) \\ &- M_{j+1,n}^{(n-k-1)}(x_j)\Big]f^{(k)}(x_j) - M_{1n}^{(n-k-1)}(a)f^{(k)}(a)\Big] \\ &= \frac{(-1)^n}{n!}\int_a^b V_n(t,\sigma)df^{(n-1)}(t). \end{aligned} \tag{1.9}$$

Proof. The proof follows from the successively integration by parts of the integral

$$\frac{(-1)^n}{n!}\int_a^b V_n(t,\sigma)df^{(n-1)}(t).$$

□

Remark 1.2 Let $\{P_{jk}\}_{k=0,1,\dots,n}$ be harmonic sequences of polynomials such that $P_{j0}(t) = 1$. Then we have $P_{jn}^{n-k-1}(t) = P_{j,k+1}(t)$, for $0 \le k \le n-1$. Put $M_{jn} = n!P_{jn}$ in (1.9). Now we have $V_n(t,\sigma) = n!S_n(t,\sigma)$, so the identity (1.9) is equivalent to the identity (1.2).

Theorem 1.4 *Assume (p,q) is a pair of conjugate exponents, that is $1 \le p,q,\le \infty$, $\frac{1}{p}+\frac{1}{q} = 1$. If $f:[a,b]\to\mathbb{R}$ is an arbitrary function such that $f^{(n)}$ is piecewise continuous, for some $n\in\mathbb{N}$, then we have*

$$\begin{aligned}
&\Bigg|\int_a^b f(t)dt + \frac{1}{n!}\sum_{k=0}^{n-1}(-1)^{k+1}\cdot\Big[M_{mn}^{(n-k-1)}(b)f^{(k)}(b)\\
&+\sum_{j=1}^{m-1}\left[M_{jn}^{(n-k-1)}(x_j) - M_{j+1,n}^{(n-k-1)}(x_j)\right]f^{(k)}(x_j) - M_{1n}^{(n-k-1)}(a)f^{(k)}(a)\Big]\Bigg|\\
&\le \frac{1}{n!}K(n,q)\|f^{(n)}\|_p,
\end{aligned} \tag{1.10}$$

where

$$K(n,q) = \|V_n(\cdot,\sigma)\|_q = \begin{cases}\left[\sum_{j=1}^m\int_{x_{j-1}}^{x_j}\left|M_{jn}(t)\right|^q dt\right]^{\frac{1}{q}}, & 1\le q<\infty\\[2ex] \max_{1\le j\le m}\{\sup_{t\in[x_{j-1},x_j]}\left|M_{jn}(t)\right|\}, & q=\infty.\end{cases}$$

Inequalities are sharp for $1 < p \le \infty$ and the best possible for $p = 1$. Equality in (1.10) is attained for the functions f of the form:

$$f(t) = Mf_*(t) + r_{n-1}(t), \tag{1.11}$$

where $M\in\mathbb{R}$, r_{n-1} is an arbitrary polynomial of degree $n-1$, and $f_*:[a,b]\to\mathbb{R}$ is function with the following representation:

$$f_*(t) := \int_a^t \frac{(t-s)^{n-1}}{(n-1)!}\left|V_n(s,\sigma)\right|^{\frac{1}{p-1}}\operatorname{sgn}V_n(s,\sigma)ds, \quad 1<p<\infty \tag{1.12}$$

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$$f_*(t) := \int_a^t \frac{(t-s)^{n-1}}{(n-1)!}\operatorname{sgn}V_n(s,\sigma)ds, \quad p=\infty. \tag{1.13}$$

Proof. The proof is simmilar to the proof of the Theorem 1.2 □

Weighted version of the identity (1.2) and related inequalities are obtained in [72]. In this case the $w-$harmonic sequences of the functions are used.

Lemma 1.1 *Let $w:[a,b]\to\mathbb{R}$ be integrable function on $[a,b]$ and let $\{w_k\}_{k=1,\dots,n}$ be $w-$harmonic sequences of functions, i.e. $w_k:[a,b]\to\mathbb{R}$ are such that $w_k'(t)=w_{k-1}(t)$, for $t\in[a,b]$ and $k=2,3,\dots,n$, and $w_1'(t)=w(t)$. If $g:[a,b]\to\mathbb{R}$ is $n-$times differentiable function such that $g^{(n)}$ is piecewise continuous on $[a,b]$, then we have*

$$\int_a^b w(t)g(t)dt = A_n(w,g;a,b)+R_n(w,g;a,b), \tag{1.14}$$

where

$$A_n(w,g;a,b)=\sum_{k=1}^{n}(-1)^{k-1}\left[w_k(b)g^{(k-1)}(b)-w_k(a)g^{(k-1)}(a)\right]$$

and

$$R_n(w,g;a,b)=(-1)^n\int_a^b w_n(t)g^{(n)}(t).$$

Proof. We prove (1.14) by mathematical induction.
For $n=1$ integration by parts gives

$$\int_a^b w(t)g(t)dt = w_1(b)g(b)-w_1(a)g(a)-\int_a^b w_1(t)g'(t)dt. \tag{1.15}$$

Let us assume that for $l=1,\dots,n-1$ we have

$$\begin{aligned}\int_a^b w(t)g(t)dt &= \sum_{k=1}^{l}(-1)^{k-1}\left[w_k(b)g^{(k-1)}(b)-w_k(a)g^{(k-1)}(a)\right]\\ &+ (-1)^l\int_a^b w_l(t)g^{(l)}(t)dt. \end{aligned}\tag{1.16}$$

Further, integration by parts yields

$$\int_a^b w_l(t)g^{(l)}(t)dt = w_{l+1}(b)g^{(l)}(b)-w_{l+1}(a)g^{(l)}(a)-\int_a^b w_{l+1}(t)g^{(l+1)}(t)dt. \tag{1.17}$$

Finnaly, we impose the identity (1.17) to the relation (1.16) and obtain

$$\begin{aligned}\int_a^b w(t)g(t)dt &= \sum_{k=1}^{l}(-1)^{k-1}\left[w_k(b)g^{(k-1)}(b)-w_k(a)g^{(k-1)}(a)\right]\\ &+ (-1)^l\Big[w_{l+1}(b)g^{(l)}(b)-w_{l+1}(a)g^{(l)}(a)\\ &- \int_a^b w_{l+1}(t)g^{(l+1)}(t)dt\Big]\\ &= \sum_{k=1}^{l+1}(-1)^{k-1}\left[w_k(b)g^{(k-1)}(b)-w_k(a)g^{(k-1)}(a)\right]\\ &+ (-1)^{l+1}\int_a^b w_{l+1}(t)g^{(l+1)}(t)dt,\end{aligned}$$

so the assertion is valid for $l+1$. □

Remark 1.3 Function $w : [a,b] \to \mathbb{R}$ is usually called weight.

Consider subdivision $\sigma = \{a = x_0 < x_1 < \ldots < x_m = b\}$ of the segment $[a,b]$,for some $m \in \mathbb{N}$. Let $w : [a,b] \to \mathbb{R}$ be an arbitrary integrable function. On each interval $[x_{k-1}, x_k]$, $k = 1, \ldots, m$ we consider different $w-$harmonic sequences of functions $\{w_{kj}\}_{j=1,\ldots,n}$, i.e. we have

$$\begin{aligned} w'_{k1}(t) &= w(t) && \text{for } t \in [x_{k-1}, x_k] \\ (w_{kj})'(t) &= w_{k,j-1}(t) && \text{for } t \in [x_{k-1}, x_k], \text{ for all } j = 2, 3, \ldots, n. \end{aligned} \tag{1.18}$$

Further, let us define

$$W_{n,w}(t,\sigma) = \begin{cases} w_{1n}(t) & \text{for } t \in [a, x_1], \\ w_{2n}(t) & \text{for } t \in (x_1, x_2], \\ \vdots \\ w_{mn}(t) & \text{for } t \in (x_{m-1}, b]. \end{cases} \tag{1.19}$$

Theorem 1.5 *If $g : [a,b] \to \mathbb{R}$ is such that $g^{(n)}$ is a piecewise continuous on $[a,b]$, then the following identity holds*

$$\begin{aligned} \int_a^b w(t)g(t)dt &= \sum_{j=1}^{n} (-1)^{j-1} \Big[w_{mj}(b) g^{(j-1)}(b) \\ &+ \sum_{k=1}^{m-1} \left[w_{kj}(x_k) - w_{k+1,j}(x_k) \right] g^{(j-1)}(x_k) - w_{1j}(a) g^{(j-1)}(a) \Big] \\ &+ (-1)^n \int_a^b W_{n,w}(t,\sigma) g^{(n)}(t)dt. \end{aligned} \tag{1.20}$$

Proof. Using relation (1.14) on each interval $[x_{k-1}, x_k]$ for appropriate $w-$harmonic sequence, we get the following

$$\int_{x_{k-1}}^{x_k} w(t)g(t)dt = A_n(w, g; x_{k-1}, x_k) + R_n(w, g; x_{k-1}, x_k). \tag{1.21}$$

By summing relation (1.21) from $k = 1$ to m we obtain

$$\begin{aligned} \int_a^b w(t)g(t)dt &= \sum_{j=1}^{n} (-1)^{j-1} \Big[w_{mj}(b) g^{(j-1)}(b) \\ &+ \sum_{k=1}^{m-1} \left[w_{kj}(x_k) - w_{k+1,j}(x_k) \right] g^{(j-1)}(x_k) - w_{1j}(a) g^{(j-1)}(a) \Big] \\ &+ \sum_{k=1}^{m} R_n(w, g; x_{k-1}, x_k) \\ &= \sum_{j=1}^{n} (-1)^{j-1} \Big[w_{mj}(b) g^{(j-1)}(b) \end{aligned} \tag{1.22}$$

$$+\sum_{k=1}^{m-1}\left[w_{kj}(x_k)-w_{k+1,j}(x_k)\right]g^{(j-1)}(x_k)-w_{1j}(a)g^{(j-1)}(a)\Big]$$
$$+(-1)^n\int_a^b W_{n,w}(t,\sigma)g^{(n)}(t)dt.$$

□

Now we shall give the general L_p theorem.

Theorem 1.6 *Assume thet (p,q) is a pair of conjugate exponents, that is $1\le p,q\le\infty$, $\frac{1}{p}+\frac{1}{q}=1$. If $g:[a,b]\to\mathbb{R}$ is some function such that $g^{(n)}$ is piecewise continuous on $[a,b]$ and $g^{(n)}\in L_p[a,b]$, then the following inequality holds*

$$\Bigg|\int_a^b w(t)g(t)dt-\sum_{k=1}^{n}(-1)^{k-1}\Big[w_{mk}(b)g^{(k-1)}(b) \tag{1.23}$$
$$-\sum_{j=1}^{m-1}\left[w_{jk}(x_j)-w_{j+1,k}(x_j)\right]g^{(k-1)}(x_j)-w_{1k}(a)g^{(k-1)}(a)\Big]\Bigg|$$
$$\le C(n,q,w)\cdot\|g^{(n)}\|_p,$$

where

$$C(n,q,w)=\|W_{n,w}(\cdot,\sigma)\|_q=\begin{cases}\left[\sum_{j=1}^{m}\int_{x_{j-1}}^{x_j}\left|w_{jn}(t)\right|^q dt\right]^{\frac{1}{q}},1\le q<\infty,\\ \max_{1\le j\le m}\{\sup_{t\in[x_{j-1},x_j]}\left|w_{jn}(t)\right|\},q=\infty.\end{cases}$$

The inequality is the best possible for $p=1$ and sharp for $1<p\le\infty$. Equality is attained for every function g such that

$$g(t)=M\cdot g_*(t)+p_{n-1}(t),$$

where $M\in\mathbb{R}$, p_{n-1} is an arbitrary polynomial of degree at most $n-1$ and $g_*(t)$ is function on $[a,b]$ defined by

$$g_*(t):=\int_a^t\frac{(t-s)^{n-1}}{(n-1)!}|W_{n,w}(s,\sigma)|^{\frac{1}{p-1}}\operatorname{sgn}W_{n,w}(s,\sigma)ds,\quad 1<p<\infty \tag{1.24}$$

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$$g_*(t):=\int_a^t\frac{(t-s)^{n-1}}{(n-1)!}\operatorname{sgn}W_{n,w}(s,\sigma)ds,\quad p=\infty. \tag{1.25}$$

1.2 Application to the one-point quadrature formulae

Now we develop the weighted one-point formula for numerical integration. Let $g : [a,b] \to \mathbb{R}$ be some function and $x \in [a,b]$. Let $w : [a,b] \to \mathbb{R}$ be some integrable function. The approximation of the integral $\int_a^b w(t)g(t)dt$ will involve the values of the higher order derivatives of g in the node x. We consider subdivision $\sigma = \{x_0 < x_1 < x_2\}$ of the interval $[a,b]$, where $x_0 = a$, $x_1 = x$ and $x_2 = b$. Further, let $\{w_{kj}^1\}_{j=1,\dots,n}$ be $w-$harmonic sequences on each subinterval $[x_{k-1}, x_k]$, $k = 1,2$, defined by the following relations:

$$w_{1j}^1(t) := \frac{1}{(j-1)!}\int_a^t (t-s)^{j-1}w(s)ds, \quad t \in [a,x]$$

$$w_{2j}^1(t) := \frac{1}{(j-1)!}\int_b^t (t-s)^{j-1}w(s)ds, \quad t \in (x,b],$$

for $j = 1,\dots,n$. Now we can state the following theorem

Theorem 1.7 *If $g : [a,b] \to \mathbb{R}$ is such that $g^{(n)}$ is a piecewise continuous function, then we have*

$$\int_a^b w(t)g(t)dt = A_1^1(x)g(x) + T_{n,w}^1(x) + (-1)^n \int_a^b W_{n,w}^1(t,x)g^{(n)}(t)\mathrm{d}t, \tag{1.26}$$

where for $j = 1,\dots,n$

$$T_{n,w}^1(x) = \sum_{j=2}^{n} A_k^1(x)g^{(k-1)}(x), \tag{1.27}$$

further, for $j = 1,\dots,n$

$$A_j^1(x) = \frac{(-1)^{j-1}}{(j-1)!}\int_a^b (x-s)^{j-1}w(s)ds \tag{1.28}$$

and

$$W_{n,w}^1(t,x) = \begin{cases} w_{1n}^1(t) = \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}w(s)ds & \text{for } t \in [a,x], \\ w_{2n}^1(t) = \frac{1}{(n-1)!}\int_b^t (t-s)^{n-1}w(s)ds & \text{for } t \in (x,b]. \end{cases} \tag{1.29}$$

Proof. We apply identity (1.20) for $m = 2$ and $x_1 = x$ to get

$$\int_a^b w(t)g(t)dt = \sum_{j=1}^{n}(-1)^{j-1}\left[w_{1j}^1(x) - w_{2j}^1(x)\right]g^{(j-1)}(x) + (-1)^n \int_a^b W_{n,w}^1(t,x)g^{(n)}(t)dt,$$

since $w_{1j}^1(a)=0$ and $w_{2j}^1(b)=0$, for $j=1,\dots,n$. Further, we compute

$$w_{1j}^1(x)-w_{2j}^1(x)=\frac{1}{(j-1)!}\int_a^b (x-s)^{j-1}w(s)ds=(-1)^{j-1}A_j^1(x),$$

so the assertion of the Theorem follows. □

Remark 1.4 The identity in Theorem 1.7 was obtained in [85], so we may call it an integral formula of Matić, Pečarić and Ujević.

Remark 1.5 If we want formula (1.26) to be exact for the polynomials of degree at most 1, such that approximation formula doesn't include the first derivative, the extra condition $A_2(x)=0$ is required. From this condition we get

$$\int_a^b (x-s)w(s)ds=0.$$

The solution $x=\frac{\int_a^b sw(s)ds}{\int_a^b w(s)ds}$ of this equation yields the node of the one-point Gaussian quadrature formula.

Theorem 1.8 *Let $w:[a,b]\to[0,\infty)$ be an integrable function and $x\in[a,b]$. Further, let $\{w_{kj}^1\}_{j=1,\dots,2n+1}$ be $w-$harmonic sequences of functions for $k=1,2$ and some $n\in\mathbb{N}$, defined by the following relations:*

$$w_{1j}^1(t):=\frac{1}{(j-1)!}\int_a^t (t-s)^{j-1}w(s)ds, \quad t\in[a,x]$$

$$w_{2j}^1(t):=\frac{1}{(j-1)!}\int_b^t (t-s)^{j-1}w(s)ds, \quad t\in(x,b],$$

for $j=1,\dots,2n+1$. If $g:[a,b]\to\mathbb{R}$ is such that $g^{(2n)}$ is continuous function, then there exists $\eta\in[a,b]$ such that

$$\int_a^b w(t)g(t)dt=A_1^1(x)g(x)+T_{n,w}^1(x)+A_{2n+1}^1(x)\cdot g^{(2n)}(\eta). \tag{1.30}$$

Proof. It is easy to check that $W_{2n,w}(t,x)\geq 0$, for $t\in[a,b]$, so we can apply integral mean value theorem to the $\int_a^b W_{2n,w}(t,x)g^{(2n)}(t)dt$ to obtain

$$\int_a^b w(t)g(t)dt-\sum_{j=1}^{2n}A_j^1(x)g^{(j-1)}(x)=g^{(2n)}(\eta)\cdot\int_a^b W_{2n,w}^1(t,x)dt. \tag{1.31}$$

We calculate

$$\begin{aligned}\int_a^b W_{2n}(t,x)^1dt &= \int_a^x w_{1,2n}^1(t)dt+\int_x^b w_{2,2n}^1(t)dt\\ &= w_{1,2n+1}^1(x)-w_{2,2n+1}^1(x)=A_{2n+1}^1(x),\end{aligned}$$

so we get the assertion. □

Now we can state the L_p-inequality for weighted one-point formula