Chapter 1

Basic results and definitions

1.1 Convex functions

In this section we give definitions and some properties of convex functions. Convex functions are very important in the theory of inequalities. The third chapter of the classical book by Hardy, Littlewood and Pólya [60] is devoted to the theory of convex functions (see also [97]).

Definition 1.1 *Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is called* convex *if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \tag{1.1}$$

for all points $x, y \in I$ and all $\lambda \in [0,1]$. It is called strictly convex *if the inequality in (1.1) holds strictly whenever x and y are distinct points and $\lambda \in (0,1)$.*

If the inequality in (1.1) is reversed, then f is said to be concave. *It is called* strictly concave *if the reversed inequality in (1.1) holds strictly whenever x and y are distinct points and $\lambda \in (0,1)$.*

If f is both convex and concave, f is said to be affine.

Remark 1.1 (a) For $x, y \in I$, $p, q \geq 0$, $p+q > 0$, (1.1) is equivalent to

$$f\left(\frac{px+qy}{p+q}\right) \leq \frac{pf(x)+qf(y)}{p+q}.$$

(b) A simple geometrical interpretation of (1.1) is that the graph of f lies below its chords.

(c) If x_1, x_2, x_3 are three points in I such that $x_1 < x_2 < x_3$, then (1.1) is equivalent to

$$\begin{vmatrix} x_1 & f(x_1) & 1 \\ x_2 & f(x_2) & 1 \\ x_3 & f(x_3) & 1 \end{vmatrix} = (x_3 - x_2) f(x_1) + (x_1 - x_3) f(x_2) + (x_2 - x_1) f(x_3) \geq 0$$

which is equivalent to

$$f(x_2) \leq \frac{x_2 - x_3}{x_1 - x_3} f(x_1) + \frac{x_1 - x_2}{x_1 - x_3} f(x_3),$$

or, more symmetrically and without the condition of monotonicity on x_1, x_2, x_3, to

$$\frac{f(x_1)}{(x_1 - x_2)(x_1 - x_3)} + \frac{f(x_2)}{(x_2 - x_3)(x_2 - x_1)} + \frac{f(x_3)}{(x_3 - x_1)(x_3 - x_2)} \geq 0.$$

Proposition 1.1 *If f is a convex function on I and if $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(y_2) - f(y_1)}{y_2 - y_1}.$$

If the function f is concave, the inequality is reversed.

Definition 1.2 *Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is called* convex in the Jensen sense, *or* J-convex on I (midconvex, midpoint convex) *if for all points $x, y \in I$ the inequality*

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2} \tag{1.2}$$

holds. A J-convex function is said to be strictly J-convex *if for all pairs of points $(x, y), x \neq y$, strict inequality holds in (1.2).*

In the context of continuity the following criterion of equivalence of (1.1) and (1.2) is valid.

Theorem 1.1 *Let $f : I \to \mathbb{R}$ be a continuous function. Then f is a convex function if and only if f is a J-convex function.*

Definition 1.3 *Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is called* Wright convex *function if for each $x \leq y$, $z \geq 0$, $x, y + z \in I$, the inequality*

$$f(x+z) - f(x) \leq f(y+z) - f(y)$$

holds.

Next, we want to define convex functions of higher order, but first we need to define divided differences.

Definition 1.4 *Let f be a function defined on $[a,b]$. The n*-th order divided difference *of f at distinct points $x_0, x_1, \ldots, x_n$ in $[a,b]$ is defined recursively by*

$$[x_j; f] = f(x_j), \quad j = 0, \ldots, n$$

and

$$[x_0, x_1, \ldots, x_n; f] = \frac{[x_1, \ldots, x_n; f] - [x_0, \ldots, x_{n-1}; f]}{x_n - x_0}. \tag{1.3}$$

Remark 1.2 The value $[x_0, x_1, \ldots, x_n; f]$ is independent of the order of the points $x_0, \ldots, x_n$. The previous definition can be extended to include the case in which some or all of the points coincide by assuming that $x_0 \leq \cdots \leq x_k$ and letting

$$[\underbrace{x, \ldots, x}_{j+1 \text{ times}}; f] = \frac{f^{(j)}(x)}{j!},$$

provided that $f^{(j)}(x)$ exists. Note that (1.3) is equivalent to

$$[x_0, \ldots, x_n; f] = \sum_{k=0}^{n} \frac{f(x_k)}{\tilde{\omega}(x_k)}, \text{ where } \tilde{\omega}(x_k) = \prod_{\substack{j=0 \\ j \neq k}}^{n} (x_k - x_j).$$

Definition 1.5 *Let $n \in \mathbb{N}_0$. A function $f : [a,b] \to \mathbb{R}$ is said to be n*-convex *on $[a,b]$ if and only if for every choice of $n+1$ distinct points $x_0, x_1, \ldots, x_n$ in $[a,b]$*

$$[x_0, x_1, \ldots, x_n; f] \geq 0. \tag{1.4}$$

If the inequality in (1.4) is reversed, the function f is said to be n-concave *on $[a,b]$. If the inequality is strict, f is said to be a* strictly $n-$convex ($n-$concave) *function.*

Remark 1.3 Particularly, $0-$convex functions are nonnegative functions, $1-$convex functions are nondecreasing functions, $2-$convex functions are convex functions.

Theorem 1.2 *If $f^{(n)}$ exists, then f is $n-$convex if and only if $f^{(n)} \geq 0$.*

Definition 1.6 *A positive function f is said to be* logarithmically convex *on an interval $I \subseteq \mathbb{R}$ if* $\log f$ *is a convex function on I, or equivalently if for all $x, y \in I$ and all $\alpha \in [0,1]$*

$$f(\alpha x + (1-\alpha)y) \leq f^{\alpha}(x) f^{1-\alpha}(y). \tag{1.5}$$

For such a function f, we shortly say that f is log-convex.
It is said to be log-concave *if the inequality in (1.5) is reversed.*

Definition 1.7 *A positive function f is said to be* log-convex in the Jensen sense *if for all $x, y \in I$*

$$f^2\left(\frac{x+y}{2}\right) \leq f(x) f(y)$$

holds, i.e. if $\log f$ *is convex in the Jensen sense.*

As a consequence of results from Remark 1.1 (c) and Proposition 1.1 we get the following inequality for a log-convex function f and $a < b < c$:

$$[f(b)]^{c-a} \leq [f(a)]^{c-b} [f(c)]^{b-a}. \tag{1.6}$$

Corollary 1.1 *For a log-convex function f on an interval I and $p,q,r,s \in I$ such that $p \leq r,\ q \leq s,\ p \neq q,\ r \neq s$, it holds*

$$\left(\frac{f(p)}{f(q)}\right)^{\frac{1}{p-q}} \leq \left(\frac{f(r)}{f(s)}\right)^{\frac{1}{r-s}}. \tag{1.7}$$

Inequality (1.7) is known as Galvani's theorem for log-convex functions $f : I \to \mathbb{R}$.

1.2 Exponentially convex functions

In this section we introduce the definition of exponential convexity as given by Bernstein in [27] (see also [13], [93], [94]). Throughout this section I is an open interval in $\mathbb{R}$.

Definition 1.8 *A function $h : I \to \mathbb{R}$ is said to be* exponentially convex *on I if it is continuous and*

$$\sum_{i,j=1}^{n} \xi_i \xi_j h(x_i + x_j) \geq 0$$

for every $n \in \mathbb{N}$ and all sequences $(\xi_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ of real numbers, such that $x_i + x_j \in I$, $1 \leq i,j \leq n$.

The following Proposition follows directly from the previous definition.

Proposition 1.2 *For a function $h : I \to \mathbb{R}$ the following statements are equivalent:*

(i) h is exponentially convex

(ii) h is continuous and

$$\sum_{i,j=1}^{n} \xi_i \xi_j h\left(\frac{x_i + x_j}{2}\right) \geq 0, \tag{1.8}$$

for all $n \in \mathbb{N}$, all sequences $(\xi_n)_{n\in\mathbb{N}}$ of real numbers, and all sequences $(x_n)_{n\in\mathbb{N}}$ in I.

Note that for $n = 1$, it follows from (1.8) that an exponentially convex function is nonnegative.

Directly from the definition of positive semi-definite matrix and inequality (1.8) we get the following result.

Corollary 1.2 *If h is exponentially convex on I, then the matrix*

$$\left[h\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{n}$$

is a positive semi-definite matrix. In particular,

$$\det\left[h\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{n} \geq 0, \tag{1.9}$$

for every $n\in\mathbb{N}$ and every choice of $x_i\in I,\ i=1,\ldots,n$.

Remark 1.4 Note that for $n=2$ from (1.9) we obtain

$$h(x_1)h(x_2)-h^2\left(\frac{x_1+x_2}{2}\right)\geq 0.$$

Hence, an exponentially convex function is log-convex in the Jensen sense, and, being continuous, it is also a log-convex function.

We continue with the definition of a n-exponentially convex function.

Definition 1.9 *A function $h:I\to\mathbb{R}$ is n-*exponentially convex in the Jensen sense *on I if*

$$\sum_{i,j=1}^{n}\xi_i\xi_j h\left(\frac{x_i+x_j}{2}\right)\geq 0$$

for all choices of $\xi_i\in\mathbb{R}$ and $x_i\in I,\ i=1,\ldots,n$.

A function $h:I\to\mathbb{R}$ is *n-exponentially convex* on I if it is n-exponentially convex in the Jensen sense and continuous on I.

It is clear from the definition that 1-exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k\in\mathbb{N},\ k\leq n$.

A function $h:I\to\mathbb{R}$ is exponentially convex in the Jensen sense on I if it is $n-$exponentially convex in the Jensen sense for all $n\in\mathbb{N}$.

Remark 1.5 It is known that $h:I\to\mathbb{R}$ is log-convex in the Jensen sense if and only if for every $\alpha,\beta\in\mathbb{R}$ and $x,y\in I$

$$\alpha^2h(x)+2\alpha\beta h\left(\frac{x+y}{2}\right)+\beta^2h(y)\geq 0.$$

It follows that a positive function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense. Similarly, a positive function is log-convex if and only if it is 2-exponentially convex.

1.3 The Gamma function and the Gauss hypergeometric function

In this section we give definitions and basic properties of the Gamma function and the Gauss hypergeometric function. More details about these functions can be found e.g. in [73].

The Gamma function $\Gamma(z)$ is a function of complex variable defined by the Euler integral of the second kind

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re(z) > 0.$$

This integral is convergent for every $z \in \mathbb{C}$ with $\Re(z) > 0$. The Gamma function has a property

$$\Gamma(z+1) = z\Gamma(z), \quad \Re(z) > 0,$$

and a simple consequence of it is the following identity

$$\Gamma(n+1) = n!, \quad n \in \mathbb{N}_0.$$

Extension of the Gamma function to $\Re(z) \leq 0$ is given by

$$\Gamma(z) = \frac{\Gamma(z+n)}{(z)_n}, \quad \Re(z) > -n; \quad n \in \mathbb{N}; \quad z \notin \mathbb{Z}_0^- = \{0, -1, -2, \dots\},$$

where $(z)_n$ is the Pochhammer symbol defined for $z \in \mathbb{C}$ and $n \in \mathbb{N}_0$ by

$$(z)_0 = 1; \quad (z)_n = z(z+1)\cdots(z+n-1), n \in \mathbb{N}.$$

The Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ is defined as the sum of the hypergeometric series

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \tag{1.10}$$

where $|z| < 1$; $a, b \in \mathbb{C}$, $c \in \mathbb{C} \setminus \mathbb{Z}_0^-$. The series in (1.10) is absolutely convergent for $|z| < 1$ and for $|z| = 1$, when $\Re(c-a-b) > 0$.

The Euler integral representation of the hypergeometric function is given by

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt,$$

where $0 < \Re(b) < \Re(c)$; and $|\arg(1-z)| < \pi$.

Basic properties of the Gauss hypergeometric function are:

$${}_2F_1(b,a;c;z) = {}_2F_1(a,b;c;z),$$

$$_2F_1(a,b;c;0) = {}_2F_1(0,b;c;z) = 1,$$

$$_2F_1(a,b;b;z) = (1-z)^{-a},$$

$$_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \Re(c-a-b) > 0.$$

For the Gauss hypergeometric function, the following Euler transformation formula holds

$$_2F_1(a,b;c;z) = (1-z)^{c-a-b}{}_2F_1(c-a,c-b;c;z).$$

1.4 Fractional integrals and fractional derivatives

In this section we give definitions and properties of fractional integrals and fractional derivatives. More details can be found in [16], [59], [73] and [139].

First, we recall definitions and properties of integrable, continuous and absolutely continuous functions.

By $C^m[a,b]$, $m \in \mathbb{N}_0$, we denote the space of all functions which are m times continuously differentiable on $[a,b]$, i.e.

$$C^m[a,b] = \{f : [a,b] \to \mathbb{R} : f^{(k)} \in C[a,b], k = 0,1,\ldots,m\}.$$

By $AC[a,b]$ we denote the space of all absolutely continuous functions on the finite interval $[a,b]$, i.e. $-\infty < a < b < \infty$. By $AC^m[a,b]$, $m \in \mathbb{N}$, we denote the space

$$AC^m[a,b] = \{f \in C^{m-1}[a,b] : f^{(m-1)} \in AC[a,b]\}.$$

Obviously, $AC^1[a,b] = AC[a,b]$.

Let $[a,b]$ be an interval in $\mathbb{R}$, where $-\infty \le a < b \le \infty$. We denote by $L^p[a,b]$, $1 \le p < \infty$, the space of Lebesgue measurable functions f such that $\int_a^b |f(t)|^p dt < \infty$ with the norm

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}},$$

and by $L^\infty[a,b]$ the space of all measurable and almost everywhere bounded functions on $[a,b]$, with the norm

$$\|f\|_\infty = \operatorname{ess\,sup}\{|f(x)| : x \in [a,b]\}.$$

For any $\alpha \in \mathbb{R}$ we denote by $[\alpha]$ the integral part of α i.e. $[\alpha]$ is the integer k satisfying $k \le \alpha < k+1$.

The Riemann-Liouville fractional integral

Let $[a,b]$ be a finite interval in $\mathbb{R}$, i.e. $-\infty < a < b < \infty$. *The left-sided Riemann-Liouville fractional integral* $I^{\alpha}_{a+}f$ of order $\alpha > 0$ is defined by

$$I^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(y)(x-y)^{\alpha-1}dy, \quad x \in [a,b].$$

For $\alpha = n \in \mathbb{N}$ the definition of the left-sided Riemann-Liouville fractional integral coincides with the $n-$th integral of the form

$$I^{n}_{a+}f(x) = \int_a^x dy_1 \int_a^{y_1} dy_2 \cdots \int_a^{y_{n-1}} f(y_n)dy_n = \frac{1}{(n-1)!} \int_a^x (x-y)^{n-1} f(y)dy.$$

The generalized Riemann-Liouville fractional derivative

The left-sided generalized Riemann-Liouville fractional derivative $D^{\alpha}_{a+}f$ of order $\alpha > 0$ is defined by

$$D^{\alpha}_{a+}f(x) := \frac{d^n}{dx^n} I^{n-\alpha}_{a+}f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-y)^{n-\alpha-1} f(y)dy, x \in [a,b],$$

where $n = [\alpha] + 1$.

For $\alpha = n \in \mathbb{N}$ we have $D^{n}_{a+}f(x) = f^{(n)}(x)$, while for $\alpha = 0$ we put $D^{0}_{a+}f(x) = f(x)$. Also, we use

$$I^{-\alpha}_{a+}f := D^{\alpha}_{a+}f \text{ if } \alpha > 0.$$

Definition 1.10 *Let $\alpha > 0$ and $1 \le p \le \infty$. By $I^{\alpha}_{a+}(L^p)$ we denote the following space of functions*

$$I^{\alpha}_{a+}(L^p) = \{f : f = I^{\alpha}_{a+}\varphi,\ \varphi \in L^p[a,b]\}.$$

A characterization of the space $I^{\alpha}_{a+}(L^1)$ is given in the following theorem.

Theorem 1.3 *Let $\alpha > 0$ and $n = [\alpha] + 1$. A function f belongs to $I^{\alpha}_{a+}(L^1)$ if and only if*

$$I^{n-\alpha}_{a+}f \in AC^n[a,b],$$

$$\frac{d^j}{dx^j} I^{n-\alpha}_{a+}f(a) = 0, \quad j = 0,1,\ldots,n-1.$$

Composition identity for the left-sided generalized Riemann-Liouville fractional derivative is given by Handley, Koliha and Pečarić in [59]. We use the following lemma which summarizes conditions in composition identity for the left-sided generalized Riemann-Liouville fractional derivatives given in [20].

Lemma 1.1 *Let $\beta > \alpha \ge 0$, $n = [\beta] + 1$, $m = [\alpha] + 1$. Composition identity*

$$D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\beta-\alpha)} \int_a^x (x-y)^{\beta-\alpha-1} D^{\beta}_{a+}f(y)\,dy, \quad x \in [a,b]$$

is valid if one of the following conditions holds:

(i) $f \in I^{\beta}_{a+}\left(L^1\right)$.

(ii) $I^{n-\beta}_{a+} f \in AC^n[a,b]$ *and* $D^{\beta-k}_{a+} f(a) = 0$ *for* $k = 1, \ldots n$.

(iii) $D^{\beta-1}_{a+} f \in AC[a,b]$, $D^{\beta-k}_{a+} f \in C[a,b]$ *and* $D^{\beta-k}_{a+} f(a) = 0$ *for* $k = 1, \ldots n$.

(iv) $f \in AC^n[a,b]$, $D^{\beta}_{a+} f, D^{\alpha}_{a+} f \in L^1[a,b]$, $\beta - \alpha \notin \mathbb{N}$, $D^{\beta-k}_{a+} f(a) = 0$ *for* $k = 1, \ldots, n$ *and* $D^{\alpha-k}_{a+} f(a) = 0$ *for* $k = 1, \ldots, m$.

(v) $f \in AC^n[a,b]$, $D^{\beta}_{a+} f, D^{\alpha}_{a+} f \in L^1[a,b]$, $\beta - \alpha = l \in \mathbb{N}$, $D^{\beta-k}_{a+} f(a) = 0$ *for* $k = 1, \ldots, l$.

(vi) $f \in AC^n[a,b]$, $D^{\beta}_{a+} f, D^{\alpha}_{a+} f \in L^1[a,b]$ *and* $f^{(k)}(a) = 0$ *for* $k = 0, \ldots, n-2$.

(vii) $f \in AC^n[a,b]$, $D^{\beta}_{a+} f, D^{\alpha}_{a+} f \in L^1[a,b]$, $\beta \notin \mathbb{N}$ *and* $D^{\beta-1}_{a+} f$ *is bounded in a neighborhood of* $t = a$.

The Caputo fractional derivative

The following type of fractional derivative which we use is the Caputo fractional derivative. We give the definition from [16].

Definition 1.11 *Let* $\alpha > 0$, $n = [\alpha] + 1$, $f \in AC^n[a,b]$. The Caputo fractional derivative $D^{\alpha}_{*a} f$ *is defined by*

$$D^{\alpha}_{*a} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

for every $t \in [a,b]$.

For $\alpha = n \in \mathbb{N}$ we have $D^{n}_{*a} f(x) = f^{(n)}(x)$, while for $\alpha = 0$ we put $D^{0}_{*a} f(x) = f(x)$.

The Canavati fractional derivative

A definition of the Canavati fractional derivative is given in [16], but we use it with some new conditions given in [19].

Let $\alpha > 0$ and $n = [\alpha] + 1$. By $C^{\alpha}_{a+}[a,b]$ we denote the space defined by

$$C^{\alpha}_{a+}[a,b] = \{f \in C^{n-1}[a,b] : I^{n-\alpha}_{a+} f^{(n-1)} \in C^1[a,b]\}.$$

Definition 1.12 *Let* $\alpha > 0$, $n = [\alpha] + 1$. The left-sided Canavati fractional derivative *of* $f \in C^{\alpha}_{a+}[a,b]$*, denoted by* ${}^{C_1}D^{\alpha}_{a+} f$*, is defined by*

$${}^{C_1}D^{\alpha}_{a+} f(x) = \frac{d}{dx} I^{n-\alpha}_{a+} f^{(n-1)}(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{n-\alpha-1} f^{(n-1)}(t) dt.$$

For $\alpha = n \in \mathbb{N}$ we have ${}^{C_1}D^{n}_{a+} f(x) = f^{(n)}(x)$, while for $\alpha = 0$ we put ${}^{C_1}D^{0}_{a+} f(x) = f(x)$.

A theorem on composition identity for the left-sided Canavati fractional derivative is proved by Anastassiou in [16]. We use an improvement of that theorem with weaker conditions given in [19].

Lemma 1.2 *Let $\beta > \alpha \geq 0$, $n = [\beta] + 1$, $m = [\alpha] + 1$. Let $f \in C^{\beta}_{a+}[a,b]$ such that $f^{(i)}(a) = 0$, $i = m-1, \ldots, n-2$. Then $f \in C^{\alpha}_{a+}[a,b]$ and*

$$^{C_1}D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\beta-\alpha)} \int_a^x (x-t)^{\beta-\alpha-1}\, {}^{C_1}D^{\beta}_{a+}f(t)dt, \quad x \in [a,b].$$

The fractional integral of a function f with respect to a given function g

Let (a,b) $(-\infty \leq a < b \leq \infty)$ be a finite or infinite interval in $\mathbb{R}$ and let $\alpha > 0$. Let g be an increasing function on (a,b) such that g' is continuous on (a,b). *The left-sided fractional integral of a function f with respect to a given function g* on $[a,b]$ is defined by

$$I^{\alpha}_{a+;g}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(y)f(y)dy}{[g(x)-g(y)]^{1-\alpha}}, \quad x > a.$$

Remark 1.6 If $g(x) = x$, then $I^{\alpha}_{a+;x}f$ coincides with the left-sided Riemann-Liouville fractional integral $I^{\alpha}_{a+}f$.

The Hadamard fractional integral

Let (a,b) $(0 \leq a < b \leq \infty)$ be a finite or infinite interval in $\mathbb{R}^+$ and let $\alpha > 0$. *The left-sided Hadamard fractional integral* of order $\alpha > 0$ is defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log\frac{x}{y}\right)^{\alpha-1} \frac{f(y)dy}{y}, \quad a < x < b.$$

Note that the left-sided Hadamard fractional integral of order α is a special case of the left-sided fractional integral of a function f with respect to the given function g, where $g(x) = \log x$ on $[a,b]$ where $0 < a < b \leq \infty$.

The Erdélyi-Kober fractional integral

Let (a,b) $(0 \leq a < b \leq \infty)$ be a finite or infinite interval in $\mathbb{R}^+$. Let $\alpha > 0, \sigma > 0$ and $\eta \in \mathbb{R}$. *The left-sided Erdélyi-Kober fractional integral* of order $\alpha > 0$ is defined by

$$I^{\alpha}_{a+;\sigma;\eta}f(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^x \frac{y^{\sigma\eta+\sigma-1}f(y)dy}{(x^{\sigma}-y^{\sigma})^{1-\alpha}}, \quad a < x < b.$$

The mixed Riemann-Liouville fractional integral

Multidimensional fractional integrals are natural generalizations of corresponding one-dimensional fractional integrals.
For $\mathbf{x} = (x_1,...,x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1,...,\alpha_n) \in \mathbb{R}^n$, we use the following notation:

$$\Gamma(\alpha) = \Gamma(\alpha_1)\cdot \ldots \cdot \Gamma(\alpha_n); \quad \mathbf{x}^{\alpha} = x_1^{\alpha_1} \ldots x_n^{\alpha_n}; \quad \frac{\partial}{\partial \mathbf{x}} = \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n};$$

$$[\mathbf{a},\mathbf{b}] = [a_1,b_1] \times \cdots \times [a_n,b_n]; \mathbf{a} = (a_1,\ldots,a_n) \in \mathbb{R}^n, \mathbf{b} = (b_1,\ldots,b_n) \in \mathbb{R}^n$$