MONOGRAPHS IN INEQUALITIES 8

Combinatorial Improvements of Jensen's Inequality

Classical and New Refinements of Jensen's Inequality with Applications

László Horváth, Khuram Ali Khan and Josip Pečarić

ELEMENT

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Preface

The study of convex functions as an original mathematical discipline began around the turn of the last century with the work of Jensen [47] (the french translation is [48]), Hermite [28], and Hölder [45], to mention just some of decisive figures. In [48], Jensen obtained his famous inequality for convex functions. One of his principal motivation was to extend the arithmetic and geometric mean inequality. Jensen's inequality is the basic of important inequalities in mathematics (for example, Hölder's and Minkowski's inequalities), and it has many applications. Among such disciplines is the theory of means, which uses Jensen's inequality as an indispensable tool. A great deal of attention has been devoted to Jensen's inequality, its various generalizations, extensions and variants have appeared in the literature, and the subject is growing rapidly.

Specifically, a number of attempts have been made to refine Jensen's inequality, namely to solve the problem of determining expressions between the left hand side and the right hand side of Jensen's inequality. Motivated by these investigations, our book aims to collect results about refinements of Jensen's inequality; to provide methods of constructing refinements of Jensen's inequality, with emphasis on the combinatorial improvements; conformation of old results from new points of view and insights; to define quasi-arithmetic and mixed symmetric means corresponding to the introduced refinements; to generate Cauchy means by using the refinements and the notion of exponential convexity; to study the monotonicity all of these means. It was not our intention to collect all known results in the topic, we wanted to give such an overview which would open the way and inspire for further exploration.

To help the reader, basic facts and conditions are occasionally repeated.

The book consists of nine chapters. In the first four chapters we essentially deal with the refinements of the discrete Jensen's inequality. In the fifth chapter refinements for the integral Jensen's inequality are given. The sixth chapter contains mean value theorems and their applications to Cauchy means via nontrivial classes of exponentially convex functions. Refinements of Hölder's and Minkowski's inequalities are found in the seventh chapter. Refinements for operator convex functions are considered in the eighth chapter. Finally, the ninth chapter is about refinements of determinental inequalities of Jensen's type.

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Notation

$x := y$	x is defined by this equation
$\mathbb{N} := \{0, 1, \ldots\}$	set of natural numbers
$\mathbb{N}_+ := \{1, \ldots\}$	set of positive integers
$\mathbb{R}$	set of real numbers

Intervals in $\mathbb{R}$ are denoted by $[a,b]$, $[a,b)$, $(a,b]$ and (a,b).

$[a]$	the largest natural number that does not exceed $a \in \mathbb{R}$
$\mathbb{R}^n$	n-dimensional Euclidean space
$\subset, \cup, \cap, \setminus$	set theoretic symbols
$\emptyset$	the empty set
$P(X)$	power set of a set X
$\lvert X \rvert$	number of elements of a set X

Functions from a set X into a set Y are denoted by $f : X \to Y$ or by $x \to f(x)$ $(x \in X)$.

$f \circ g$	composition of the function g with the function f
id	the indentity function of a proper set
$C(I)$	space of continuous functions on an interval I
$C^2(I)$	space of two times continuously differentiable functions on an interval I
A°	interior of the set A
$S(I)$	class of all self-adjoint bounded operators on a complex Hilbert space whose spectra are contained in an interval $I \subset \mathbb{R}$
$\mathrm{Sp}(A)$	spectrum of a bounded operator A on on a complex Hilbert space
$\mathscr{M}_m$	set of positive definite matrices of order m
$\lvert M \rvert$	determinant of a square matrix M
$M(j)$	submatrix of a square matrix M obtained by deleting the j^{th} row and column of M
$M[k]$	principal submatrix of a square matrix M formed by taking the first k rows and columns of M
BBF	class of Bellman-Bergstrom-Fan functionals

Contents

Preface v

Notation vii

1 Introduction **1**
1.1 Convex Functions . . . 1
1.2 Interpolations of Jensen's Inequality . . . 4
1.3 Quotients for samples without repetitions . . . 8
1.3.1 New Proof of Theorem . . . 13
1.4 Applications of Quotient inequalities . . . 14
1.5 Application to Mixed Symmetric Means . . . 19

2 Refinements of Jensen's Inequality **25**
2.1 A Refinement of the Discrete Jensen's Inequality . . . 25
2.1.1 Examples and Mixed Symmetric Means Related to Theorem 2.1 . . . 34
2.1.2 Examples and Mixed Symmetric Means Related to Theorem 2.2 . . . 38
2.2 A New Treatment of Discrete Jensen's Inequality . . . 42
2.2.1 Examples and Mixed Symmetric Means Related to Theorem 2.3 . . . 48
2.3 Parameter Dependent Refinement of Discrete Jensen's inequality . . . 55
2.3.1 Some lemmas and the proofs of Theorem 2.5-2.7 . . . 57
2.3.2 Applications to Mixed Symmetric Means . . . 64

3 Further Refinements of Jensen's Inequality **67**
3.1 Preliminary results and the proofs . . . 70
3.2 Applications to Mixed means . . . 78

4 Popoviciu Type Inequalities **81**
4.1 Generalization of Popoviciu's Inequality . . . 82
4.2 Popoviciu Inequality for $2D$-Convex Functions . . . 84

5 Refinements Including Integral Jensen's Inequality **87**
5.0.1 Mixed Symmetric Means Related to Theorems (5.3-5.5) 97
5.1 New Refinement of Classical Jensen's Inequality 102
5.1.1 Mixed Symmetric Means Related to Theorem 5.7 104
5.2 Another Refinement of integral form of Jensen's Inequality 105

6 Mean Value Theorems and Exponential Convexity **125**
6.1 Mean Value Theorems 125
6.2 Exponential Convexity 131
6.3 Applications to Cauchy Means 137

7 Refinements of Hölder's and Minkowski's inequalities **157**
7.1 Generalizations of Beck's result 162
7.2 Generalizations of the Consequences of Beck's Result 173
7.3 Generalization of Minkowski's inequality 182

8 Refinements of Jensen's Inequality for Operator Convex Functions **187**
8.1 Introduction and Preliminary Results 187
8.2 Refinement of Jensen's Operator Inequality 190
8.2.1 Applications of Theorem 8.4 to some special cases 191
8.2.2 Symmetric Means related to Theorem 8.4 193
8.3 Further Refinement of Jensen's Operator Inequality 198
8.3.1 Applications of Theorem 8.4 to some Special Cases 199
8.4 Parameter Dependent Refinement of Jensen's Operator Inequality 203
8.4.1 Applications 208
8.4.2 Parameter Dependent Operator Means 208

9 Refinements of Determinantal Inequalities of Jensen's type **211**
9.1 Introduction and Preliminary Results 211
9.2 Refinement Results 214
9.2.1 Discussion and Applications Related to Theorem 9.1 215
9.3 Generalization of Theorem 9.1 218
9.3.1 Discussion and Applications related to Theorem 9.2 219
9.4 Parameter Dependent Refinements 222

Bibliography **223**

Index **229**

Chapter 1

Introduction

1.1 Convex Functions

Convex functions are very important in the theory of inequalities. The foundations of the theory of convex functions are due to the Danish mathematician and engineer J. L. W. V. Jensen (1859 – 1925).

The natural domain of the different type of convex functions is a convex set in a real vector space V: we say that the subset $C \subset V$ is convex if the segment

$$\{\lambda x_1 + (1-\lambda)x_2 \mid \lambda \in [0,1]\}$$

is a subset of C for every $x_1, x_2 \in C$.

The convex sets in $\mathbb{R}$ exactly the intervals.

Investigation of means under the action of functions is an interesting task. The simplest case which deals with the arithmetic mean leads to the mid-convex (or the J-convex) functions.

***J*-convex function** [69, p.5]: Let V be a real vector space, and $C \subset V$ be a convex set. A function $f : C \to \mathbb{R}$ is called convex in the or mid-convex if

$$f\left(\frac{x_1+x_2}{2}\right) \le \frac{f(x_1)+f(x_2)}{2} \tag{1.1}$$

for all $x_1, x_2 \in C$.

A J-convex function f is called strictly J-convex if for all pairs of points $(x_1, x_2) \in C \times C$, $x_1 \neq x_2$, strict inequality holds in (1.1).

Convex function [69, p.1]: Let V be a real vector space, and $C \subset V$ be a convex set. A function $f : C \to \mathbb{R}$ is called convex if

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2) \tag{1.2}$$

holds for all $x_1, x_2 \in C$ and $\lambda \in [0,1]$.
f is called strictly convex if strict inequality holds in (1.2) for $x_1 \neq x_2$ and $\lambda \in (0,1)$. If the inequality in (1.2) is reversed, then f is called concave function. If it is strict for all $x_1 \neq x_2$ and $\lambda \in (0,1)$, then f is called strictly concave.

Some characterization of convex functions of a real variable can be found in the following three results.

Theorem 1.1 *[63]Let $I \subset \mathbb{R}$ be an interval. Then $f : I \to \mathbb{R}$ is convex, if and only if*

$$(x_3 - x_2)f(x_1) + (x_1 - x_3)f(x_2) + (x_2 - x_1)f(x_3) \geq 0 \tag{1.3}$$

holds for every $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$. Further, f is strictly convex if and only if $\geq$ is replaced by $>$ in (1.3).

A relation between convex and J-convex functions is as follows.

Theorem 1.2 (J. L. W. V. JENSEN [63, P.10]) *If $f : I \to \mathbb{R}$ is continuous on the interval $I \subset \mathbb{R}$, then f is convex if and only if f is convex in the Jensen sense.*

Next, we give the second derivative test for convexity of a function.

Theorem 1.3 *Let $I \subset \mathbb{R}$ be an open interval, and $f : I \to \mathbb{R}$ be a function such that f'' exits on I. Then f is convex if and only if $f''(x) \geq 0$ $(x \in I)$. If $f''(x) > 0$ $(x \in I)$, then f is strictly convex on the interval.*

***J*-log-convex function** [46]: Let V be a real vector space, and $C \subset V$ be a convex set. A function $f : C \to (0,\infty)$ is called log-convex in the Jensen sense if $\log \circ f$ is J-convex, that is

$$f^2\left(\frac{x_1 + x_2}{2}\right) \leq f(x_1)f(x_2)$$

for all $x_1, x_2 \in C$.
Log-convex function [69, p.7]: Let V be a real vector space, and $C \subset V$ be a convex set. A function $f : C \to (0,\infty)$ is called log-convex if $\log \circ f$ is convex, that is

$$f(\lambda x_1 + (1-\lambda)x_2) \leq f^{\lambda}(x_1) f^{1-\lambda}(x_2),$$

holds for all $x_1, x_2 \in C$ and all $\lambda \in [0,1]$.

Lemma 1.1 ([70]) *Let V be a real vector space, and $C \subset V$ be a convex set. Then a function $f : C \to (0,\infty)$ is log-convex in the Jensen sense if and only if the relation*

$$v^2 f(x_1) + 2vw f\left(\frac{x_1 + x_2}{2}\right) + w^2 f(x_2) \geq 0$$

holds for each real v, w and $x_1, x_2 \in C$.

We denote $x_{[1]} \geq \ldots \geq x_{[n]}$ the components of a vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ arranged in decreasing order. We say that a vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is majorized by a vector $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ ($\mathbf{x} \prec \mathbf{y}$) if

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \quad 1 \leq k \leq n$$

with equality for $k = n$ (see [56]). Then the binary relation $\prec$ over $\mathbb{R}^n$ is reflexive and transitive, i.e. a preorder.

Schur-convex function [56]: Let $D \subset \mathbb{R}^n$. A function $f : D \to \mathbb{R}$ is called Schur-convex if $\mathbf{x} \prec \mathbf{y}$ implies $f(\mathbf{x}) \leq f(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in D$.

The following known result is proved in [56].

Theorem 1.4 *Let $D \subset \mathbb{R}^n$ be a symmetric convex set with nonempty interior D°, and $f : D \to \mathbb{R}$ be a continuous function. If f is differentiable on D°, then f is Schur convex (Schur concave) on D if and only if f is symmetric and*

$$(x_2 - x_1)\left(\frac{\partial f(\mathbf{x})}{\partial x_1} - \frac{\partial f(\mathbf{x})}{\partial x_2}\right) \geq 0 \ (\leq 0)$$

for all $\mathbf{x} = (x_1, \ldots, x_n) \in D^\circ$.

In view of applications in different parts of mathematics the Jensen's inequalities are especially noteworthy, as well as useful.

We begin with the discrete version of the Jensen's inequality:

Theorem 1.5 ***Discrete Jensen's inequality****[69, p.43]: (a) Let V be a real vector space, and $C \subset V$ be a convex set, and $f : C \mapsto \mathbb{R}$ be a convex function. Then*

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \tag{1.4}$$

holds, where $x_i \in C$ $(i = 1, \ldots, n)$ and p_i $(i = 1, \ldots, n)$ are nonnegative real numbers, with $P_n = \sum_{i=1}^{n} p_i > 0$. If f is strictly convex and the p_i's are positive, then inequality (1.4) is strict unless $x_1 = x_2 = \ldots = x_n$.

(b) If $f : C \to \mathbb{R}$ is a J-convex function, and the p_i's are rational numbers $(i = 1, \ldots, n)$, then (1.4) also holds.

The integral version of the Jensen's inequality is as follows:

Theorem 1.6 ***Integral Jensen's inequality[26]:*** *Let $(\Omega, \mathscr{A}, \mu)$ be a finite measure space with $\mu(\Omega) > 0$, and $g : \Omega \to \mathbb{R}$ is a μ-integrable function taking values in an interval $I \subset \mathbb{R}$. Then $\frac{1}{\mu(\Omega)}\int_\Omega g d\mu$ lies in I, and for every convex function $f : I \to \mathbb{R}$ the composition $f \circ g$ is measurable. Further, if $f \circ g$ is μ-integrable, then*

$$f\left(\frac{1}{\mu(\Omega)}\int_\Omega g d\mu\right) \leq \frac{1}{\mu(\Omega)}\int_\Omega f \circ g d\mu. \tag{1.5}$$

In case when f is strictly convex on I equality is satisfied in (1.5) if and only if g is constant μ-almost everywhere on Ω.

1.2 Interpolations of Jensen's Inequality

We start with the following interpolation of the discrete Jensen's inequality based on samples without repetitions given by Pečarić and Volenec in 1988 (see [73]).

Theorem 1.7 *Let C be a convex subset of a real vector space V, and let $f : C \to \mathbb{R}$ be a mid-convex function. If $\mathbf{x} = (x_1, ..., x_n) \in C^n$, and*

$$f_{k,n} = f_{k,n}(\mathbf{x}) := \frac{1}{\binom{n}{k}} \sum_{1 \le i_1 < \ldots < i_k \le n} f\left(\frac{x_{i_1} + \ldots + x_{i_k}}{k}\right), \quad 1 \le k \le n, \tag{1.6}$$

then

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = f_{n,n} \le \ldots \le f_{k+1,n} \le f_{k,n} \le \ldots \le f_{1,n} = \frac{1}{n}\sum_{i=1}^{n} f(x_i), \quad 1 \le k \le n-1. \tag{1.7}$$

The weighted version of the above theorem is given by Pečarić.

Theorem 1.8 ([66]) *Let C be a convex subset of a real vector space V, and let $f : C \to \mathbb{R}$ be a convex function. Suppose $\mathbf{x} = (x_1, ..., x_n) \in C^n$, and $\mathbf{p} = (p_1, ..., p_n)$ is a positive n-tuple such that $\sum_{i=1}^{n} p_i = 1$. For $k = 1, \ldots, n$ define*

$$f^1_{k,n} = f^1_{k,n}(\mathbf{x}, \mathbf{p}) := \frac{1}{\binom{n-1}{k-1}} \sum_{1 \le i_1 < \ldots < i_k \le n} \left(\sum_{j=1}^{k} p_{i_j}\right) f\left(\frac{\sum_{j=1}^{k} p_{i_j} x_{i_j}}{\sum_{j=1}^{k} p_{i_j}}\right). \tag{1.8}$$

Then for $1 \le k \le n-1$

$$f(\sum_{i=1}^{n} p_i x_i) = f^1_{n,n} \le ... \le f^1_{k+1,n} \le f^1_{k,n} \le ... \le f^1_{1,n} = \sum_{i=1}^{n} p_i f(x_i). \tag{1.9}$$

The following interpolation of the discrete Jensen's inequality based on samples with repetitions is given by Pečarić and Svrtan in 1998 (see [71]).

Theorem 1.9 *[71] Let C be a convex subset of a real vector space V, and let $f : C \to \mathbb{R}$ be a mid-convex function. If $\mathbf{x} = (x_1, ..., x_n) \in C^n$, and*

$$\overline{f}_{k,n} = \overline{f}_{k,n}(\mathbf{x}) := \frac{1}{\binom{n+k-1}{k}} \sum_{1 \le i_1 \le \ldots \le i_k \le n} f\left(\frac{x_{i_1} + \ldots + x_{i_k}}{k}\right), \quad k \ge 1,$$

then

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \le \ldots \le \overline{f}_{k+1,n} \le \overline{f}_{k,n} \le \ldots \le \overline{f}_{1,n} = \frac{1}{n}\sum_{i=1}^{n} f(x_i). \tag{1.10}$$

The weighted version of the above theorem causes motivation for many authors and it can be found in [60, p.8].

Theorem 1.10 *Let C be a convex subset of a real vector space V, and let $f : C \to \mathbb{R}$ be a convex function. Suppose $\mathbf{x} = (x_1,...,x_n) \in C^n$, and $\mathbf{p} = (p_1,..., p_n)$ is a positive n-tuple such that $\sum_{i=1}^n p_i = 1$. Then*

$$f(\sum_{i=1}^{n} p_i x_i) \leq ... \leq f_{k+1,n}^2 \leq f_{k,n}^2 \leq ... \leq f_{1,n}^2 = \sum_{i=1}^{n} p_i f(x_i), \tag{1.11}$$

where

$$f_{k,n}^2 = f_{k,n}^2(\mathbf{x},\mathbf{p}) = \frac{1}{\binom{n+k-1}{k-1}} \sum_{1\leq i_1\leq \ldots \leq i_k \leq n} \left(\sum_{j=1}^{k} p_{i_j}\right) f\left(\frac{\sum_{j=1}^{k} p_{i_j} x_{i_j}}{\sum_{j=1}^{k} p_{i_j}}\right), \quad k \geq 1. \tag{1.12}$$

Remark 1.1 If f is a concave function then the inequalities (1.9) and (1.11) are reversed.

If p_i $(i = 1,\ldots,n)$ are rational numbers, then (1.9) and (1.11) are also valid for mid-convex functions.

An important consequence of the discrete Jensen's inequality for mid-convex functions is the following Key Lemma from [71].

Lemma 1.2 *Let C be a convex subset of real linear space V, $f : C \to \mathbb{R}$ be a mid-convex function, and $\mathbf{x} = (x_1,...,x_n) \in C^n$. Then*

$$f(\frac{1}{n}\sum_{j=1}^{n} x_j) \leq \frac{1}{n}\sum_{j=1}^{n} f\left(\frac{x_1+\ldots+\hat{x}_j+\ldots+x_n}{n-1}\right), \tag{1.13}$$

where $\hat{x}_j$ means that x_j is omitted.

Proof. Apply the discrete Jensen's inequality for mid-convex functions to

$$x^{(i)} := \big(1/(n-1)\big)\,(x_1+...+\hat{x}_i+...+x_n),$$

and use the identity $\sum\limits_{i=1}^{n} x_i = \sum\limits_{i=1}^{n} x^{(i)}$. □

Unified treatment for samples with and without repetitions: Assume $f : C \to \mathbb{R}$ is a mid-convex function defined on a convex set C in a real linear space V, and $\mathbf{x} = (x_1,...,x_n) \in C^n$. Let $M = \{1^{m_1}, 2^{m_2}, \ldots, n^{m_n}\}$ be a fixed multiset having $m_j =: \nu_j(M) \geq 1$ elements equal to j, for $1 \leq j \leq n$. $N_k(M)$ denotes the k-th rank number of M (the number of subsets of M containing exactly k elements). For every nonempty submultiset $I \subset M$, $x_I := \sum\limits_{i\in I} x_i$, and $|I|$ means the number of elements in I. Now, define the M-dominated k-sample mean of f by

$$f_{k,n}^M = f_{k,n}^M(\mathbf{x}) := \frac{1}{N_k(M)} \sum_{\substack{I\subset M\\ |I|=k}} f(\frac{1}{k}x_I), \quad 1 \leq k \leq m_1+\ldots+m_n.$$

The following Proposition makes a unified treatment of Theorems 1.7 and 1.9.

Proposition 1.1 *Under the previous assumptions, we have*

$$N_{k+1}(M)f_{k+1,n}^{M} = \sum_{J\subset M,|J|=k+1} f\left(\frac{1}{k+1}x_J\right) \leq \frac{1}{k+1}\sum_{I\subset M,|I|=k} c_I f\left(\frac{1}{k}x_I\right) \tag{1.14}$$

for every $1\leq k<m_1+\ldots+m_n$*, where* $c_I := \sum\limits_{\substack{1\leq j\leq n\\ \nu_j(I)<m_j}} (\nu_j(I)+1)$.

Proof. By applying Lemma 1.2 to the terms of the middle sum in (1.14), we have

$$\sum_{J\subset M,|J|=k+1} f\left(\frac{1}{k+1}x_J\right) \leq \frac{1}{k+1}\sum_{J\subset M,|J|=k+1}\sum_{j\in J} f\left(\frac{1}{k}x_{J\setminus\{j\}}\right).$$

Then, the right hand side can be rewritten as

$$\frac{1}{k+1}\sum_{I\subset M,|I|=k} c_I f\left(\frac{1}{k}x_I\right),$$

where c_I can be calculated in the following way: let

$$A_I := \{J\subset M \mid J = I\uplus\{j\} \text{ for some } 1\leq j\leq n\},$$

where $\uplus$ means the multiset sum, and for $J\in A_I$ let $c_I(J)$ be the number of all elements j of J such that $I = J\setminus\{j\}$; then

$$c_I = \sum_{J\in A_I} c_I(J) = \sum_{\substack{1\leq j\leq n\\ \nu_j(I)<m_j}} (\nu_j(I)+1).$$

The proof is complete. □

Now we show that Theorems 1.7 and 1.9 are special cases of Proposition 1.1.

Corollary 1.1 *Let C be a convex subset of a real vector space V, and let $f: C\to\mathbb{R}$ be a mid-convex function. If $\mathbf{x} = (x_1,...,x_n)\in C^n$, then the following refinements of the Jensen's inequality hold:*

a)

$$f\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right) = f_{n,n}\leq\ldots\leq f_{k+1,n}\leq f_{k,n}\leq\ldots\leq f_{1,n} = \frac{1}{n}\sum_{i=1}^{n}f(x_i),$$

b)

$$f\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right) = \ldots\leq \overline{f}_{k+1,n}\leq\overline{f}_{k,n}\leq\ldots\leq\overline{f}_{1,n} = \frac{1}{n}\sum_{i=1}^{n}f(x_i).$$

Proof. (a) We take M to be the following multiset (actually a set): $M := \{1,\ldots,n\}$. In this case $\nu_j(M) = 1$ $(1 \le j \le n)$, and

$$\sum_{I\subset M,|I|=k} f\left(\frac{1}{k}x_I\right) = \binom{n}{k} f_{k,n}, \quad k = 1,\ldots,n-1.$$

By (1.14), this implies that

$$\binom{n}{k+1} f_{k+1,n} \le \frac{1}{k+1} \sum_{I\subset M,|I|=k} c_I f\left(\frac{1}{k}x_I\right)$$

$$= \frac{1}{k+1}(n-k)\binom{n}{k} f_{k,n} = \binom{n}{k+1} f_{k,n}, \quad k = 1,\ldots,n-1,$$

finishes the proof of the first claim.

(b) Let the integers $k \ge 1$ and $l \ge k+1$ be fixed, and let M be the following multiset: $M := \{1^l,\ldots,n^l\}$ (the multiplicity of j is l for $1 \le j \le n$). Then

$$\sum_{I\subset M,|I|=k} f\left(\frac{1}{k}x_I\right) = \binom{n+k-1}{k} \overline{f}_{k,n}, \quad k = 1,\ldots,l.$$

This yields by (1.14)

$$\binom{n+k}{k+1} \overline{f}_{k+1,n} \le \frac{1}{k+1} \sum_{I\subset M,|I|=k} c_I f\left(\frac{1}{k}x_I\right)$$

$$= \frac{1}{k+1}(k+n)\binom{n+k-1}{k} \overline{f}_{k,n} = \binom{n+k}{k+1} \overline{f}_{k,n},$$

and therefore

$$\overline{f}_{k+1,n} \le \overline{f}_{k,n}, \quad k \ge 1.$$

□

The following result is given in [20]:

Theorem 1.11 *Let C be a convex subset of a real vector space V, and let $f : C \to \mathbb{R}$ be a convex function. Suppose $\mathbf{x} = (x_1,...,x_n) \in C^n$, and $\mathbf{p} = (p_1,...,p_n)$ is a nonnegative n-tuple such that $\sum_{i=1}^n p_i = 1$. If*

$$f_{k,n}^3 = f_{k,n}^3(\mathbf{x},\mathbf{p}) := \sum_{i_1,...,i_k=1}^{n} p_{i_1}...p_{i_k} f\left(\frac{1}{k}\sum_{j=1}^{k} x_{i_j}\right), \quad k \ge 1, \tag{1.15}$$

then

$$f\Big(\sum_{i=1}^n p_i x_i\Big) \le ... \le f_{k+1,n}^3 \le f_{k,n}^3 \le ... \le f_{1,n}^3 = \sum_{i=1}^n p_i f(x_i), \quad k \ge 1. \tag{1.16}$$

The next result comes from [19] and [74] (see also Theorem 3.36 in [69, p.97]).

Theorem 1.12 *Let $I \subset \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ be a convex function, σ be an increasing function on $[0,1]$ such that $\int_0^1 d\sigma(x) = 1$, and $u : [0,1] \to I$ be σ-integrable. If $f \circ u$ is also σ-integrable, then*

$$\begin{aligned} f\left(\int_0^1 u(x)d\sigma(x)\right) &\leq \int_{[0,1]^{k+1}} f\left(\frac{1}{k+1}\sum_{i=1}^{k+1} u(x_i)\right)\prod_{i=1}^{k+1} d\sigma(x_i) \\ &\leq \int_{[0,1]^k} f\left(\frac{1}{k}\sum_{i=1}^{k} u(x_i)\right)\prod_{i=1}^{k} d\sigma(x_i) \leq \ldots \\ &\leq \int_{[0,1]^2} f\left(\frac{1}{2}\sum_{i=1}^{2} u(x_i)\right)\prod_{i=1}^{2} d\sigma(x_i) \\ &\leq \int_0^1 f(u(x))\, d\sigma(x), \end{aligned} \tag{1.17}$$

for all positive integers k.

1.3 Quotients for samples without repetitions

Let $I \subset \mathbb{R}$ be an interval, and $f : I \to \mathbb{R}$. Consider the following notations: for $x_i \in I$ $(1 \leq i \leq n)$

$$\begin{aligned} &\mathbf{x} := (x_1, \ldots, x_n);\ f(\mathbf{x}) := (f(x_1), \ldots, f(x_n)); \\ &\text{arithmetic mean: } A(\mathbf{x}) := \tfrac{1}{n}(x_1 + \cdots + x_n); \\ &\text{geometric mean: } G(\mathbf{x}) := \sqrt[n]{x_1 \cdots x_n} \quad (I \subset [0,\infty)). \end{aligned}$$

Then the discrete Jensen's inequality for equal weights is

$$f(A(\mathbf{x})) \leq A(f(\mathbf{x})), \tag{1.18}$$

where $f : I \to \mathbb{R}$ is a convex function, and $\mathbf{x} \in I^n$. The inequality is clearly reversed if $f : I \to \mathbb{R}$ is concave function.

In this context, (1.7) can be written as

$$f(A(\mathbf{x})) = f_{n,n} \leq \ldots \leq f_{k+1,n} \leq f_{k,n} \ldots \leq f_{1,n} = A(f(\mathbf{x})), \quad 1 \leq k \leq n-1. \tag{1.19}$$

In 2003, Tang and Wen [76] obtained the following inequalities which contain (1.19): For all $1 \leq r \leq j \leq s \leq i \leq n$, the following refinement holds:

$$f_{r,s,n} \geq \cdots \geq f_{r,s,i} \geq \cdots \geq f_{r,s,s} \geq \cdots \geq f_{r,j,j} \geq \cdots \geq f_{r,r,r} = 0, \tag{1.20}$$

where

$$f_{r,s,n} := \binom{n}{r}\binom{n}{s}(f_{r,n} - f_{s,n}).$$

Equality conditions are also considered.

In 2008, Gao and Wen [22] obtained the following results in this direction:

Theorem 1.13 *Let $I \subset \mathbb{R}$ be an interval. If $f : I \to \mathbb{R}$, $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n) \in I^n$ $(n \geq 2)$ and*

(i) $a_1 \leq \cdots \leq a_n \leq b_n \leq \ldots \leq b_1, a_1 + b_1 \leq \cdots \leq a_n + b_n$,

(ii) $f(t) > 0, f'(t) > 0$, $f''(t) > 0$, $f'''(t) < 0$ for every $t \in I$,

then

$$\frac{f(A(\mathbf{a}))}{f(A(\mathbf{b}))} = \frac{f_{n,n}(\mathbf{a})}{f_{n,n}(\mathbf{b})} \leq \cdots \leq \frac{f_{k+1,n}(\mathbf{a})}{f_{k+1,n}(\mathbf{b})} \leq \frac{f_{k,n}(\mathbf{a})}{f_{k,n}(\mathbf{b})} \leq \cdots \leq \frac{f_{1,n}(\mathbf{a})}{f_{1,n}(\mathbf{b})} = \frac{A(f(\mathbf{a}))}{A(f(\mathbf{b}))}, \quad 1 \leq k \leq n-1. \tag{1.21}$$

The inequalities are reversed for $f''(t) < 0$, $f'''(t) > 0$ $(t \in I)$. Equality signs hold if and only if $a_1 = \cdots = a_n$ and $b_1 = \cdots = b_n$.

Moreover, Wen and Wang [82] considered some inequalities for linear combinations involving $f_{k,n}$.

Another type of generalization is due to Wen [80]: Let $I \subset \mathbb{R}$ be an interval, and let $f : I \to \mathbb{R}$ be twice continuously differentiable such that f'' is convex. Then

$$f''(D_3(\mathbf{x})) \leq \frac{2J[f(\mathbf{x})]}{J[\mathbf{x}^2]} \leq \frac{1}{3}\left[\max_{1\leq i\leq n}\{f''(x_i)\} + A\left(f''(\mathbf{x})\right) + f''(A(\mathbf{x}))\right], \tag{1.22}$$

where

$$D_3(\mathbf{x}) := \frac{1}{3}\frac{A(\mathbf{x}^3) - A^3(\mathbf{x})}{A(\mathbf{x}^2) - A^2(\mathbf{x})},$$
$$J[f(\mathbf{x})] := A(f(\mathbf{x})) - f(A(\mathbf{x})), \quad J[\mathbf{x}^2] := A(\mathbf{x}^2) - A^2(\mathbf{x}).$$

In [81] an other kind of interesting inequalities, centering about the topic of refinements involving quotients of two functions, are given.

Theorem 1.14 *Let the functions*

$$f : [a,b] \to (0,\infty), g : [a,b] \to (0,\infty)$$

satisfying

$$\sup_{t\in[a,b]}\left\{\left|\frac{g''(t)}{f''(t)}\right|\right\} < \inf_{t\in[a,b]}\left\{\frac{g(t)}{f(t)}\right\}.$$

If $f''(t) > 0$ for each $t \in [a,b]$, then for any $\mathbf{x} \in [a,b]^n$, we have the following inequalities of Jensen-Pečarić-Svrtan-Fan (Abbreviated as J-P-S-F) type:

$$\frac{f(A(\mathbf{x}))}{g(A(\mathbf{x}))} = \frac{f_{n,n}(A(\mathbf{x}))}{g_{n,n}(A(\mathbf{x}))} \leq \cdots \leq \frac{f_{k+1,n}(A(\mathbf{x}))}{g_{k+1,n}(A(\mathbf{x}))} \tag{1.23}$$

$$\le \frac{f_{k,n}(A(\mathbf{x}))}{g_{k,n}(A(\mathbf{x}))} \le \cdots \le \frac{f_{1,n}(A(\mathbf{x}))}{g_{1,n}(A(\mathbf{x}))} = \frac{A(f(\mathbf{x}))}{A(g(\mathbf{x}))}, \quad 1 \le k \le n-1.$$

If $f''(t) < 0$ for each $t \in [a,b]$, then the above inequalities are reversed. In each case, the sign of the equality holding throughout if and only if $x_1 = \cdots = x_n$.

Proof of Theorem 1.14: To prove Theorem 1.14, we set

$$\alpha := (\alpha_1, \ldots, \alpha_n); \quad \Omega_n := \{\alpha \in [0,1]^n | \alpha_1 + \cdots + \alpha_n = 1\},$$

$$S_f(\alpha, \mathbf{x}) := \frac{1}{n!} \sum_{i_1 \cdots i_n} f(\alpha_1 x_{i_1} + \cdots + \alpha_n x_{i_n}); \quad F(\alpha) := \log \frac{S_f(\alpha, \mathbf{x})}{S_g(\alpha, \mathbf{x})};$$

$$u_{\mathbf{i}}(\mathbf{x}) := \alpha_1 x_{i_1} + \alpha_2 x_{i_2} + \sum_{j=3}^{n} \alpha_j x_{i_j}; \quad v_{\mathbf{i}}(\mathbf{x}) := \alpha_1 x_{i_2} + \alpha_2 x_{i_1} + \sum_{j=3}^{n} \alpha_j x_{i_j}. \tag{1.24}$$

Here and in the sequel $\mathbf{x} \in [a,b]^n$, $\alpha \in \Omega_n$, $\mathbf{i} = (i_1, \ldots, i_n)$, and let $i_1 \cdots i_n$ and $i_3 \cdots i_n$ denote the possible permutations of $N_n = \{1, \ldots, n\}$ and the possible permutations of $N_n \setminus \{i_1, i_2\}$, respectively.

We start with two lemmas.

Lemma 1.3 *Under the hypotheses of Theorem 1.14, there exist $\xi_{\mathbf{i}}$ and $\xi_{\mathbf{i}}^*$ between $u_{\mathbf{i}}(\mathbf{x})$ and $v_{\mathbf{i}}(\mathbf{x})$ such that*

$$\begin{aligned}(\alpha_1 - \alpha_2)\left(\frac{\partial F(\alpha)}{\partial \alpha_1} - \frac{\partial F(\alpha)}{\partial \alpha_2}\right) &= \frac{1}{n!} \sum_{i_3 \cdots i_n} \sum_{1 \le i_1 < i_2 \le n} \frac{f''(\xi_{\mathbf{i}})(u_{\mathbf{i}}(\mathbf{x}) - v_{\mathbf{i}}(\mathbf{x}))^2}{S_f(\alpha, \mathbf{x})} \\ &\quad \times \left(1 - \frac{g''(\xi_{\mathbf{i}}^*)}{f''(\xi_{\mathbf{i}}^*)} \frac{S_f(\alpha, \mathbf{x})}{S_g(\alpha, \mathbf{x})}\right).\end{aligned} \tag{1.25}$$

Proof. Note the following identities:

$$\begin{aligned} S_f(\alpha, \mathbf{x}) &:= \frac{1}{n!} \sum_{i_3 \cdots i_n} \sum_{1 \le i_1 \ne i_2 \le n} f(\alpha_1 x_{i_1} + \cdots + \alpha_n x_{i_n}) \\ &= \frac{1}{n!} \sum_{i_3 \cdots i_n} \sum_{1 \le i_1 < i_2 \le n} [f(u_{\mathbf{i}}(\mathbf{x})) - f(v_{\mathbf{i}}(\mathbf{x}))]; \end{aligned}$$

similarly,

$$S_g(\alpha, \mathbf{x}) = \frac{1}{n!} \sum_{i_3 \cdots i_n} \sum_{1 \le i_1 < i_2 \le n} [g(u_{\mathbf{i}}(\mathbf{x})) - g(v_{\mathbf{i}}(\mathbf{x}))];$$

$$\begin{aligned} &\frac{\partial}{\partial \alpha_1} [f(u_{\mathbf{i}}(\mathbf{x})) + f(v_{\mathbf{i}}(\mathbf{x}))] - \frac{\partial}{\partial \alpha_2} [f(u_{\mathbf{i}}(\mathbf{x})) + f(v_{\mathbf{i}}(\mathbf{x}))] \\ &= [x_{i_1} f'(u_{\mathbf{i}}(\mathbf{x})) + x_{i_2} f'(v_{\mathbf{i}}(\mathbf{x}))] - [x_{i_2} f'(u_{\mathbf{i}}(\mathbf{x})) + x_{i_1} f'(v_{\mathbf{i}}(\mathbf{x}))] \\ &= [f'(u_{\mathbf{i}}(\mathbf{x})) - f'(v_{\mathbf{i}}(\mathbf{x}))](x_{i_1} - x_{i_2}); \end{aligned}$$

similarly,

$$\begin{aligned} &\frac{\partial}{\partial \alpha_1} [g(u_{\mathbf{i}}(\mathbf{x})) + f(v_{\mathbf{i}}(\mathbf{x}))] - \frac{\partial}{\partial \alpha_2} [g(u_{\mathbf{i}}(\mathbf{x})) + g(v_{\mathbf{i}}(\mathbf{x}))] \\ &= [g'(u_{\mathbf{i}}(\mathbf{x})) - g'(v_{\mathbf{i}}(\mathbf{x}))](x_{i_1} - x_{i_2}). \end{aligned}$$

Thus

$$\begin{aligned}
&(\alpha_1-\alpha_2)\left(\frac{\partial S_f(\alpha,\mathbf{x})}{\partial\alpha_1}-\frac{\partial S_f(\alpha,\mathbf{x})}{\partial\alpha_2}\right)\\
&=\frac{1}{n!}\sum_{i_3\cdots i_n}\sum_{1\le i_1<i_2\le n}\left[f'\left(u_{\mathbf{i}}(\mathbf{x})\right)-f'\left(v_{\mathbf{i}}(\mathbf{x})\right)\right](\alpha_1-\alpha_2)(x_{i_1}-x_{i_2})\\
&=\frac{1}{n!}\sum_{i_3\cdots i_n}\sum_{1\le i_1<i_2\le n}\left[f'\left(u_{\mathbf{i}}(\mathbf{x})\right)-f'\left(v_{\mathbf{i}}(\mathbf{x})\right)\right](u_{\mathbf{i}}(\mathbf{x})-v_{\mathbf{i}}(\mathbf{x}));
\end{aligned}$$

similarly,

$$\begin{aligned}
&(\alpha_1-\alpha_2)\left(\frac{\partial S_g(\alpha,\mathbf{x})}{\partial\alpha_1}-\frac{\partial S_g(\alpha,\mathbf{x})}{\partial\alpha_2}\right)\\
&=\frac{1}{n!}\sum_{i_3\cdots i_n}\sum_{1\le i_1<i_2\le n}\left[g'\left(u_{\mathbf{i}}(\mathbf{x})\right)-g'\left(v_{\mathbf{i}}(\mathbf{x})\right)\right](u_{\mathbf{i}}(\mathbf{x})-v_{\mathbf{i}}(\mathbf{x})).
\end{aligned}$$

Based on the above facts, we have

$$\begin{aligned}
&(\alpha_1-\alpha_2)\left(\frac{\partial F(\alpha)}{\partial\alpha_1}-\frac{\partial F(\alpha)}{\partial\alpha_2}\right)\\
&=(\alpha_1-\alpha_2)\left(\frac{\frac{S_f(\alpha,\mathbf{x})}{\partial\alpha_1}-\frac{\partial S_f(\alpha,\mathbf{x})}{\partial\alpha_2}}{S_f(\alpha,\mathbf{x})}-\frac{\frac{\partial S_g(\alpha,\mathbf{x})}{\partial\alpha_1}-\frac{\partial S_g(\alpha,\mathbf{x})}{\partial\alpha_2}}{S_g(\alpha,\mathbf{x})}\right)\\
&=\frac{1}{n!}\sum_{i_3\cdots i_n}\sum_{1\le i_1<i_2\le n}\left\{\frac{\left[f'(u_i(\mathbf{x}))-f'(v_i(\mathbf{x}))\right](u_i(\mathbf{x})-v_i(\mathbf{x}))}{S_f(\alpha,\mathbf{x})}\right.\\
&\left.-\frac{\left[g'(u_i(\mathbf{x}))-g'(v_i(\mathbf{x}))\right](u_i(\mathbf{x})-v_i(\mathbf{x}))}{S_g(\alpha,\mathbf{x})}\right\}\\
&=\frac{1}{n!}\sum_{i_3\cdots i_n}\sum_{1\le i_1<i_2\le n}\frac{\left[f'(u_{\mathbf{i}}(\mathbf{x}))-f'(v_{\mathbf{i}}(\mathbf{x}))\right](u_{\mathbf{i}}(\mathbf{x})-v_{\mathbf{i}}(\mathbf{x}))}{S_f(\alpha,\mathbf{x})}\\
&\times\left(1-\frac{g'(u_{\mathbf{i}}(\mathbf{x}))-g'(v_{\mathbf{i}}(\mathbf{x}))}{f'(u_{\mathbf{i}}(\mathbf{x}))-f'(v_{\mathbf{i}}(\mathbf{x}))}\frac{S_f(\alpha,\mathbf{x})}{S_g(\alpha,\mathbf{x})}\right).
\end{aligned}$$

By Lagrange's mean-value theorem, there exists $\xi_{\mathbf{i}}$ between $u_{\mathbf{i}}(\mathbf{x})$ and $v_{\mathbf{i}}(\mathbf{x})$ such that

$$f'\left(u_{\mathbf{i}}(\mathbf{x})\right)-f'\left(v_{\mathbf{i}}(\mathbf{x})\right)=f''(\xi_{\mathbf{i}})(u_{\mathbf{i}}(\mathbf{x})-v_{\mathbf{i}}(\mathbf{x})).$$

By Cauchy's mean-value theorem, there exists $\xi_{\mathbf{i}}^*$ between $u_{\mathbf{i}}(\mathbf{x})$ and $v_{\mathbf{i}}(\mathbf{x})$ such that

$$\frac{g'\left(u_{\mathbf{i}}(\mathbf{x})\right)-g'\left(v_{\mathbf{i}}(\mathbf{x})\right)}{f'\left(u_{\mathbf{i}}(\mathbf{x})\right)-f'\left(v_{\mathbf{i}}(\mathbf{x})\right)}=\frac{g''(\xi_{\mathbf{i}}{}^*)}{f''(\xi_{\mathbf{i}}{}^*)}.$$

Finally

$$\begin{aligned}
&(\alpha_1-\alpha_2)\left(\frac{\partial F(\alpha)}{\partial\alpha_1}-\frac{\partial F(\alpha)}{\partial\alpha_2}\right)\\
&=\frac{1}{n!}\sum_{i_3\cdots i_n}\sum_{1\le i_1<i_2\le n}\frac{\left[f'(u_{\mathbf{i}}(\mathbf{x}))-f'(v_{\mathbf{i}}(\mathbf{x}))\right](u_{\mathbf{i}}(\mathbf{x})-v_{\mathbf{i}}(\mathbf{x}))}{S_f(\alpha,\mathbf{x})}\\
&\times\left(1-\frac{g'(u_{\mathbf{i}}(\mathbf{x}))-g'(v_{\mathbf{i}}(\mathbf{x}))}{f'(u_{\mathbf{i}}(\mathbf{x}))-f'(v_{\mathbf{i}}(\mathbf{x}))}\frac{S_f(\alpha,\mathbf{x})}{S_g(\alpha,\mathbf{x})}\right)\\
&=\frac{1}{n!}\sum_{i_3\cdots i_n}\sum_{1\le i_1<i_2\le n}\frac{f''(\xi_{\mathbf{i}})(u_{\mathbf{i}}(\mathbf{x})-v_{\mathbf{i}}(\mathbf{x}))^2}{S_f(\alpha,\mathbf{x})}\times\left(1-\frac{g''(\xi_{\mathbf{i}}{}^*)}{f''(\xi_{\mathbf{i}}{}^*)}\frac{S_f(\alpha,\mathbf{x})}{S_g(\alpha,\mathbf{x})}\right).
\end{aligned}$$

The proof of Lemma 1.3 has been finished. □

Lemma 1.4 *Let the conditions of Theorem 1.14 be satisfied.*
(I) If $f''(t) > 0$ for each $t \in [a,b]$, then F is a Schur-convex function on Ω_n.
(II) If $f''(t) < 0$ for each $t \in [a,b]$, then F is a Schur-concave function on Ω_n.

Proof. We first affirm that Case (I) is true as follow. One can easily see that Ω_n is a symmetric convex set, and $F(\alpha)$ is a symmetric function on Ω_n and it has continuous partial derivatives. By Theorem 1.4, we need to prove that F satisfies

$$(\alpha_1 - \alpha_2)\left(\frac{\partial F(\alpha)}{\partial \alpha_1} - \frac{\partial F(\alpha)}{\partial \alpha_2}\right) \geq 0, \quad \alpha \in \Omega_n. \tag{1.26}$$

Equality is valid if and only if $\alpha_1 = \alpha_2$ or $x_1 = \cdots = x_n$. In the following, we shall apply the identity (1.25) in Lemma 1.3. Note that $\mathbf{x} \in [a,b]^n$, $\alpha \in \Omega_n$, for any $\mathbf{i} = (i_1, \ldots, i_n)$, we have

$$u_{\mathbf{i}}(\mathbf{x}) = \alpha_1 x_{i_1} + \cdots + \alpha_n x_{i_n} \in [a,b];$$
$$\frac{f(\alpha_1 x_{i_1} + \cdots + \alpha_n x_{i_n})}{g(\alpha_1 x_{i_1} + \cdots + \alpha_n x_{i_n})} = \frac{f(u_{\mathbf{i}}(\mathbf{x}))}{g(u_{\mathbf{i}}(\mathbf{x}))} \leq \sup_{t\in[a,b]}\left\{\frac{f(t)}{g(t)}\right\}$$

$$\begin{aligned}
S_f(\alpha,\mathbf{x}) &= \tfrac{1}{n!}\sum_{i_1\cdots i_n} f(\alpha_1 x_{i_1} + \cdots + \alpha_n x_{i_n})\\
&= \tfrac{1}{n!}\sum_{i_1\cdots i_n} \tfrac{f(\alpha_1 x_{i_1} + \cdots + \alpha_n x_{i_n})}{g(\alpha_1 x_{i_1} + \cdots + \alpha_n x_{i_n})} g(\alpha_1 x_{i_1} + \cdots + \alpha_n x_{i_n})\\
&\leq \tfrac{1}{n!}\sum_{i_1\cdots i_n} \sup_{t\in[a,b]}\left\{\tfrac{f(t)}{g(t)}\right\} g(\alpha_1 x_{i_1} + \cdots + \alpha_n x_{i_n})\\
&= \sup_{t\in[a,b]}\left\{\tfrac{f(t)}{g(t)}\right\} S_g(\alpha,\mathbf{x}),
\end{aligned}$$

or, equivalently,

$$\frac{S_g(\alpha,\mathbf{x})}{S_f(\alpha,\mathbf{x})} \geq \left[\sup_{t\in[a,b]}\left\{\frac{f(t)}{g(t)}\right\}\right]^{-1} = \inf_{t\in[a,b]}\left\{\frac{g(t)}{f(t)}\right\}. \tag{1.27}$$

Combining (1.27) with the following inequality

$$0 \leq \left|\frac{g_i''(\xi_{\mathbf{i}}^*)}{f_i''(\xi_{\mathbf{i}}^*)}\right| \leq \sup_{t\in[a,b]}\left\{\left|\frac{g_i''(t)}{f_i''(t)}\right|\right\} \tag{1.28}$$

and the hypotheses of Theorem 1.14, we obtain that

$$\begin{aligned}
1 - \tfrac{g_i''(\xi_{\mathbf{i}}^*)}{f_i''(\xi_{\mathbf{i}}^*)}\tfrac{S_f(\alpha,\mathbf{x})}{S_g(\alpha,\mathbf{x})} &\geq 1 - \left|\tfrac{g_i''(\xi_{\mathbf{i}}^*)}{f_i''(\xi_{\mathbf{i}}^*)}\right|\tfrac{S_f(\alpha,\mathbf{x})}{S_g(\alpha,\mathbf{x})}\\
&\geq 1 - \sup_{t\in[a,b]}\left\{\left|\tfrac{g_i''(t)}{f_i''(t)}\right|\right\} \Big/ \tfrac{S_g(\alpha,x)}{S_f(\alpha,x)}\\
&\geq 1 - \sup_{t\in[a,b]}\left\{\left|\tfrac{g_i''(t)}{f_i''(t)}\right|\right\} \Big/ \inf_{t\in[a,b]}\left\{\tfrac{g(t)}{f(t)}\right\}\\
&> 0.
\end{aligned}$$

This implies by the identity (1.25), and by the nonnegativity of f'', that (1.26) is satisfied. So F is a Schur-convex function on Ω_n. Let us now turn to the conclusion (II) of our lemma. From the above argument for (I) we know that the inequalities (1.27-1.28) still hold. Using (1.25), and $f''(x) < 0$, the converse of (1.26) can be obtained. Thus, F is a Schur-concave function on Ω_n. From the argument, we obtain that equality is valid if and only if $\alpha_1 = \alpha_2$ or $x_1 = \cdots = x_n$. This completes the proof of Lemma 1.4. □

Proof of Theorem 1.14 We only prove the first assertion, that is, the inequalities (1.23) hold if $f''(t) > 0$ for each $t \in [a,b]$, the second assertion can be proved by an analogous procedure. Define

$$\alpha[k] := \Bigg(\underbrace{k^{-1},\ldots,k^{-1}}_{k},\underbrace{0,\ldots,0}_{n-k}\Bigg), k = 1,\ldots,n$$

Clearly, $\alpha[k] \in \Omega_n, k = 1,2,\ldots,n$, and $\alpha[k+1] \prec \alpha[k]$, $k = 1,2,\ldots,n-1$. By Lemma 1.4, for any $\mathbf{x} \in [a,b]^n, F$ is a Schur-convex function on Ω_n. Using the definition of Schur-convex functions, we have

$$F(\alpha[k+1]) \leq F(\alpha[k]), k = 1,2,\ldots,n-1.$$

Combining this result with the definition of $F(\alpha)$, it follows that the inequalities (1.23) hold. By the argument of Lemma 1.4 and the fact of which $\alpha[k]$ strictly majorizes $\alpha[k+1]$, the sign of equality holding throughout if and only if $x_1 = \cdots = x_n$. So the proof of Theorem 1.14 is complete.

1.3.1 New Proof of Theorem 1.13

We can prove Theorem 1.13 in a similar way by introducing for $\alpha := (\alpha_1,\ldots,\alpha_n) \in \Omega_n$ and $\mathbf{x} \in I^n$.

$$\begin{aligned} S_f(\alpha,\mathbf{x}) &:= \tfrac{1}{n!} \sum_{i_1\cdots i_n} f(\alpha_1 x_{i_1} + \cdots + \alpha_n x_{i_n}) = \tfrac{1}{n!} \sum_{i_3\cdots i_n} \sum_{1\leq i_1 \neq i_2 \leq n} f(\alpha_1 x_{i_1} + \cdots + \alpha_n x_{i_n}) \\ &= \tfrac{1}{n!} \sum_{i_3\cdots i_n} \sum_{1\leq i_1 < i_2 \leq n} [f(u_{\mathbf{i}}(\mathbf{x})) - f(v_{\mathbf{i}}(\mathbf{x}))] \end{aligned}$$

and

$$F(\alpha) := \log \frac{S_f(\alpha,\mathbf{a})}{S_f(\alpha,\mathbf{b})}.$$

To prove that $F(\alpha)$ is Schur-convex it is enough to show that (see Theorem 1.4)

$$\begin{aligned} &(\alpha_1 - \alpha_2)\left(\frac{\partial F(\alpha)}{\partial \alpha_1} - \frac{\partial F(\alpha)}{\partial \alpha_2}\right) \\ &= \tfrac{1}{n!} \sum_{i_3\cdots i_n} \sum_{1\leq i_1 < i_2 \leq n} \left\{ \frac{[f'(u_{\mathbf{i}}(\mathbf{a})) - f'(v_{\mathbf{i}}(\mathbf{a}))](u_{\mathbf{i}}(\mathbf{a}) - v_{\mathbf{i}}(\mathbf{a}))}{S_f(\alpha,\mathbf{a})} - \frac{[f'(u_{\mathbf{i}}(\mathbf{b})) - f'(v_{\mathbf{i}}(\mathbf{b}))](u_{\mathbf{i}}(\mathbf{b}) - v_{\mathbf{i}}(\mathbf{b}))}{S_f(\alpha,\mathbf{b})} \right\} \geq 0 \end{aligned} \tag{1.29}$$

for every $\alpha \in \Omega_n$. This proof is different from the original one, we don't use the mean value theorem.

Since $f(t) > 0$ and $f'(t) > 0$ for every $t \in I$

$$\frac{1}{S_f(\alpha,\mathbf{a})} \geq \frac{1}{S_f(\alpha,\mathbf{b})} > 0. \tag{1.30}$$

From (1.24), we have

$$u_{\mathbf{i}}(\mathbf{a}) - v_{\mathbf{i}}(\mathbf{a}) = \alpha_1(a_{i_1} - a_{i_2}) + \alpha_2(a_{i_2} - a_{i_1})$$

and

$$u_{\mathbf{i}}(\mathbf{b}) - v_{\mathbf{i}}(\mathbf{b}) = \alpha_1(b_{i_1} - b_{i_2}) + \alpha_2(b_{i_2} - b_{i_1}).$$

Therefore, by (i) in Theorem 1.13

$$(u_{\mathbf{i}}(\mathbf{a}) - v_{\mathbf{i}}(\mathbf{a}))^2 \geq (u_{\mathbf{i}}(\mathbf{b}) - v_{\mathbf{i}}(\mathbf{b}))^2. \tag{1.31}$$

It is easy to check by using (i) that for $1 \leq i_1 < i_2 \leq n$

(a) $u_{\mathbf{i}}(\mathbf{a}) = v_{\mathbf{i}}(\mathbf{a})$ implies $u_{\mathbf{i}}(\mathbf{b}) = v_{\mathbf{i}}(\mathbf{b})$,

(b) $u_{\mathbf{i}}(\mathbf{a}) < v_{\mathbf{i}}(\mathbf{a})$ implies $u_{\mathbf{i}}(\mathbf{a}) < v_{\mathbf{i}}(\mathbf{a}) \leq v_{\mathbf{i}}(\mathbf{b}) \leq u_{\mathbf{i}}(\mathbf{b})$,

(c) $v_{\mathbf{i}}(\mathbf{a}) < u_{\mathbf{i}}(\mathbf{a})$ implies $v_{\mathbf{i}}(\mathbf{a}) < u_{\mathbf{i}}(\mathbf{a}) \leq u_{\mathbf{i}}(\mathbf{b}) \leq v_{\mathbf{i}}(\mathbf{b})$.

If either (b) or (c) holds, and $u_{\mathbf{i}}(\mathbf{b}) \neq v_{\mathbf{i}}(\mathbf{b})$, then

$$\frac{f'(u_1(a)) - f'(v_1(a))}{u_1(a) - v_1(a)} \geq \frac{f'(u_1(b)) - f'(v_1(b))}{u_1(b) - v_1(b)} \tag{1.32}$$

since f' is a concave function. Combining (1.30), (1.31) and (1.32) we get (1.29).

Now, we can continue as in the proof of Theorem 1.14.

1.4 Applications of Quotient inequalities

In this section some applications of Theorem 1.14 are given from [81].

Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a positive n-tuples. The Dresher mean of order k $(k = 1, 2, \ldots, n)$ of $\mathbf{x}$ is defined by

$$[D_{p,q}(\mathbf{x})]_{k,n} := \left\{ \begin{array}{ll} \dfrac{1}{k}\left[\dfrac{\sum\limits_{1\leq i_1<\cdots<i_k\leq n}\left(\sum\limits_{j=1}^{n} x_{i_j}\right)^p}{\sum\limits_{1\leq i_1<\cdots<i_k\leq n}\left(\sum\limits_{j=1}^{n} x_{i_j}\right)^q}\right]^{\frac{1}{p-q}} & p \neq q \\[4ex] \dfrac{1}{k}\exp\left[\dfrac{\sum\limits_{1\leq i_1<\cdots<i_k\leq n}\left(\sum\limits_{j=1}^{n} x_{i_j}\right)^p \log\left(\sum\limits_{j=1}^{n} x_{i_j}\right)}{\sum\limits_{1\leq i_1<\cdots<i_k\leq n}\left(\sum\limits_{j=1}^{n} x_{i_j}\right)^p}\right] & p = q \end{array} \right\}$$

Especially, $D_{p,q}(\mathbf{x}) := [D_{p,q}(\mathbf{x})]_{1,n}$ is the Dresher mean of $\mathbf{x}$ (see [11]), and

$$[D_{0,0}(\mathbf{x})]_{k,n} = \left(\prod_{1\le i_1<\cdots<i_k\le n} \frac{x_{i_1}+\cdots+x_{i_n}}{k}\right)^{\frac{1}{\binom{n}{k}}} =: [GA;\mathbf{x}]_{k,n},$$

$$[D_{1,1}(\mathbf{x})]_{k,n} = \left(\prod_{1\le i_1<\cdots<i_k\le n} \left(\frac{x_{i_1}+\cdots+x_{i_n}}{k}\right)^{\frac{x_{i_1}+\cdots+x_{i_n}}{k}}\right)^{\frac{1}{\binom{n}{k}A(x)}}$$

$$\begin{array}{l} [D_{1,1}(\mathbf{x})]_{1,n} = (x_1{}^{x_1}\cdots x_n{}^{x_n})^{\frac{1}{x_1+\cdots+x_n}} \\ [D_{0,0}(\mathbf{x})]_{1,n} = G(\mathbf{x}) \\ [D_{p,q}(\mathbf{x})]_{n,n} = A(\mathbf{x}) \end{array}.$$

Write

$$D(p,q) := \begin{cases} \left(\frac{p(1-p)}{q(1-q)}\right)^{\frac{1}{p-q}} & p\ne q \\ \exp\frac{1-2p}{p(1-p)} & p=q \end{cases}.$$

As a corollary of Theorem 1.14, we have

Corollary 1.2 *Let $\mathbf{x}\in(0,\infty)^n$, and $\frac{\max\{\mathbf{x}\}}{\min\{\mathbf{x}\}} < D(p,q)$.*
(I) If $p>0, q>0$, and $p+q<1$, then

$$\begin{aligned} A(\mathbf{x}) &= [D_{p,q}(\mathbf{x})]_{n,n} \ge \cdots \ge [D_{p,q}(\mathbf{x})]_{k+1,n} \ge [D_{p,q}(\mathbf{x})]_{k,n} \\ &\ge \cdots \ge [D_{p,q}(\mathbf{x})]_{1,n} = [D_{p,q}(\mathbf{x})] \ge G(\mathbf{x}), \quad 1\le k\le n-1. \end{aligned} \tag{1.33}$$

(II) If $p>1$ and $q>1$, then

$$\begin{aligned} A(\mathbf{x}) &= [D_{p,q}(\mathbf{x})]_{n,n} \le \cdots \le [D_{p,q}(\mathbf{x})]_{k+1,n} \le [D_{p,q}(\mathbf{x})]_{k,n} \\ &\le \cdots \le [D_{p,q}(\mathbf{x})]_{1,n} = [D_{p,q}(\mathbf{x})], \quad 1\le k\le n-1. \end{aligned} \tag{1.34}$$

In each case, the sign of equality holds throughout if and only if $x_1=\cdots=x_n$.

Proof. We only prove case (I), that is inequalities (1.33) hold, because case (II) can be proved with the same method. Since $[D_{p,q}(\mathbf{x})]_{k,n} = [D_{q,p}(\mathbf{x})]_{k,n}$ is continuous of (p,q), we can assume that $0<q<p<1$. Now we take $[a,b] = [min\{\mathbf{x}\}, max\{\mathbf{x}\}]$, $f:[a,b]\to(0,\infty)$, $f(t)=t^p$, and $g:[a,b]\to(0,\infty)$, $g(t)=t^q$. We verify that the conditions of Theorem 1.14 are satisfied. First, we notice that

$$\begin{aligned} \sup_{t\in[a,b]}\left\{\left|\frac{g''(t)}{f''(t)}\right|\right\} &= \sup_{t\in[a,b]}\left\{\left|\frac{q(q-1)t^{q-2}}{p(p-1)t^{p-2}}\right|\right\} \\ &= \sup_{t\in[a,b]}\left\{\frac{q(1-q)}{p(1-p)}t^{q-p}\right\} \\ &= \frac{q(1-q)}{p(1-p)}a^{q-p} \end{aligned}$$

$$\inf_{t\in[a,b]}\left\{\frac{g(t)}{f(t)}\right\}=\inf_{t\in[a,b]}\left\{t^{q-p}\right\}=b^{q-p}$$

By $0<q<p<1,\ p+q<1, p(1-p)-q(1-q)=(p-q)(1-p-q)>0$, we have

$$D(p,q)=\left[\frac{p(1-p)}{q(1-q)}\right]^{\frac{1}{p-q}}>1,$$

$$\sup_{t\in[a,b]}\left\{\left|\frac{g''(t)}{f''(t)}\right|\right\}<\inf_{t\in[a,b]}\left\{\frac{g(t)}{f(t)}\right\}\Leftrightarrow\frac{q(1-q)}{p(1-p)}a^{q-p}<b^{q-p}\Leftrightarrow\frac{\max\{\mathbf{x}\}}{\min\{\mathbf{x}\}}<D\{p,q\},$$

$$f''(t)=p(p-1)t^{p-2}<0,\ t\in[a,b].$$

Thus, by Theorem 1.14, the reverse (1.23) holds. In other words, we have

$$\begin{aligned}A(\mathbf{x})&=[D_{p,q}(\mathbf{x})]_{n,n}\geq\cdots\geq[D_{p,q}(\mathbf{x})]_{k+1,n}\geq[D_{p,q}(\mathbf{x})]_{k,n}\\&\geq\cdots\geq[D_{p,q}(\mathbf{x})]_{1,n}=[D_{p,q}(\mathbf{x})].\end{aligned}$$

Second, by using results of [64]:

$$D_{p,q}(\mathbf{x})\geq D_{r,s}(\mathbf{x})\Leftrightarrow max\{p,q\}\geq max\{r,s\}\text{, and } min\{p,q\}\geq min\{r,s\}$$

and $p>0,q>0$, therefore $D_{p,q}(\mathbf{x})\geq G(\mathbf{x})$. This completes the proof of (1.33). □

Remark 1.2 From Corollary 1.2 and

$$\lim_{p\to0^+,q\to0^+}D(p,q)=\lim_{p\to1^+,q\to1^+}D(p,q)=\infty,$$

we can obtain some interesting inequalities (see [71]) : If $\mathbf{x}\in(0,\infty)^n$, then for $1\leq k\leq n-1$

$$A(\mathbf{x})\geq\cdots\geq[GA;\mathbf{x}]_{k+1,n}\geq[GA;\mathbf{x}]_{k,n}\geq\cdots\geq G(\mathbf{x}),\tag{1.35}$$

$$A(\mathbf{x})\leq\cdots\leq[D_{1,1}(\mathbf{x})]_{k+1,n}\leq[D_{1,1}(\mathbf{x})]_{k,n}\leq\cdots\leq[D_{1,1}(\mathbf{x})]_{1,n}\tag{1.36}$$

and the sign of equality holds throughout if and only if $x_1=\cdots=x_n$.

Remark 1.3 Since (1.33) implies the following inequality

$$A(\mathbf{x})\geq\left[\frac{A(\mathbf{x}^p)}{A(\mathbf{x}^q)}\right]^{\frac{1}{p-q}}\geq G(\mathbf{x}),\quad p,q>0,\quad p+q<1,$$

by Corollary 1.2 and the definition of the Riemann integral, we have: If $p>0,q>0$, and $p+q<1$, the function $f:[\alpha,\beta]\to(0,\infty)$ is continuous, and it satisfies the condition

$$\frac{\max\limits_{t\in[\alpha,\beta]}\{f(t)\}}{\min\limits_{t\in[\alpha,\beta]}\{f(t)\}}<D(p,q),$$

then

$$\frac{\int_{\alpha}^{\beta} f\,dt}{\beta-\alpha} \geq \left(\frac{\int_{\alpha}^{\beta} f^p\,dt}{\int_{\alpha}^{\beta} f^q\,dt}\right)^{\frac{1}{p-q}} \geq \exp\left(\frac{\int_{\alpha}^{\beta} \ln f\,dt}{\beta-\alpha}\right), \tag{1.37}$$

where we define

$$\left(\frac{\int_{\alpha}^{\beta} f^p\,dt}{\int_{\alpha}^{\beta} f^q\,dt}\right)^{\frac{1}{p-q}} := \lim_{q\to p}\left(\frac{\int_{\alpha}^{\beta} f^p\,dt}{\int_{\alpha}^{\beta} f^q\,dt}\right)^{\frac{1}{p-q}} = \exp\left(\frac{\int_{\alpha}^{\beta} f^p \ln f\,dt}{\int_{\alpha}^{\beta} f^p\,dt}\right)$$

when $p = q$.

One of the integral analogues of (1.23) is the following inequality (1.38).

Corollary 1.3 *Under the hypotheses of Theorem 1.14, let $E \subset \mathbb{R}^m$ be a bounded closed domain with m-dimensional volume $|E| = 1$, and let $\phi : E \to [a,b]$ be a Riemann integrable function. If $f''(t) > 0$ for every $t \in [a,b]$, then*

$$\frac{f\left(\int_E \phi\right)}{g\left(\int_E \phi\right)} \leq \frac{\int_E f\circ\phi}{\int_E g\circ\phi}. \tag{1.38}$$

If $f''(t) < 0$ for every $t \in [a,b]$, then inequality (1.38) is reversed.

Proof. In fact, the hypotheses of Corollary 1.3 imply that the functions $\phi : E \to \mathbb{R}, f\circ\phi : E \to \mathbb{R}, g\circ\phi : E \to \mathbb{R}$ are integrable, on the other hand, Theorem 1.14 implies the inequality

$$\frac{f\left(A(\mathbf{x},\mathbf{w})\right)}{g\left(A(\mathbf{x},\mathbf{w})\right)} \leq \frac{A\left(f(\mathbf{x}),\mathbf{w}\right)}{A\left(g(\mathbf{x}),\mathbf{w}\right)}, \quad \forall \mathbf{x} \in [a,b]^n \tag{1.39}$$

where

$$\mathbf{w} \in (0,1)^n, \quad \sum_{i=1}^{n} w_i = 1, \quad A(\mathbf{x},\mathbf{w}) := \sum_{i=1}^{n} w_i x_i.$$

Let $T = \{\Delta E_1, ..., \Delta E_n\}$ be a partition of E, and let

$$\|\mathrm{T}\| = \max_{1\leq i\leq n}\max_{u,v\in\Delta E_i}\{\|u-v\|\}$$

be the 'norm' of the partition T, where $\|u-v\|$ is the length of the vector $u-v$. Pick any $\xi \in \Delta E_1 \times\cdots\times \Delta E_n$, then by (1.39) we get

$$\frac{f(\int_E \phi)}{g(\int_E \phi)} = \lim_{\|T\|\to 0}\frac{f(A(\phi(\xi),\mathbf{w}))}{g(A(\phi(\xi),\mathbf{w}))} \leq \lim_{\|T\|\to 0}\frac{A(f(\phi(\xi)),\mathbf{w})}{A(g(\phi(\xi)),\mathbf{w})} = \frac{\int_E f\circ\phi}{\int_E g\circ\phi}, \tag{1.40}$$

where

$$\mathbf{w} = (|\Delta E_1|, ..., |\Delta E_n|) \in (0,1)^n, \sum_{i=1}^{n} |\Delta E_i| = 1, \phi(\xi) \in [a,b]^n.$$

Therefore (1.38) holds from (1.40). This ends the proof. □

Corollary 1.4 *(see [79]) If* $\mathbf{x} \in (0, \frac{1}{2}]^n$, *then*

$$\frac{A(\mathbf{x})}{A(\mathbf{1}-\mathbf{x})} \geq \cdots \geq \frac{[GA;\mathbf{x}]_{k+1,n}}{[GA;\mathbf{1}-\mathbf{x}]_{k+1,n}} \geq \frac{[GA;\mathbf{x}]_{k,n}}{[GA;\mathbf{1}-\mathbf{x}]_{k,n}} \geq \cdots \geq \frac{G(\mathbf{x})}{G(\mathbf{1}-\mathbf{x})}, \quad 1 \leq k \leq n-1, \tag{1.41}$$

where $\mathbf{1}-\mathbf{x} := (1-x_1, \ldots, 1-x_n)$, *and the sign of equality holds throughout if and only if* $x_1 = \cdots = x_n$.

Proof. It goes without saying that, for each $x \in (0,1/2]^n$, we can always find $a \in (0,1/2)$ such that $x \in [a,1/2]^n$. In Theorem 1.14, we take $f : [a,1/2] \to (0,\infty), f(t) = t^r, 0 < r < 1; g : [a,1/2] \to (0,\infty), g(t) = (1-t)^r, 0 < r < 1$. We verify that the conditions of Theorem 1.14 are satisfied as follows:

$$\begin{aligned}\sup_{t\in[a,\frac{1}{2}]} \left\{ \left| \frac{g''(t)}{f''(t)} \right| \right\} &= \sup_{t\in[a,\frac{1}{2}]} \left\{ \left| \frac{r(r-1)(1-t)^{r-2}}{r(r-1)t^{r-2}} \right| \right\} \\ &= \sup_{t\in[a,\frac{1}{2}]} \left\{ \left(\frac{1}{t}-1\right)^{r-2} \right\} = 1\end{aligned}$$

$$\inf_{t\in[a,\frac{1}{2}]} \left\{ \frac{g(t)}{f(t)} \right\} = \inf_{t\in[a,\frac{1}{2}]} \left\{ \left(\frac{1}{t}-1\right)^{r} \right\} = 1$$

From the above we have

$$\sup_{t\in[a,\frac{1}{2}]} \left\{ \left| \frac{g''(t)}{f''(t)} \right| \right\} \leq \inf_{t\in[a,\frac{1}{2}]} \left\{ \frac{g(t)}{f(t)} \right\}$$

It is easy to see that $f''(t) = r(r-1)t^{r-2} < 0$ for all $t \in [a, \frac{1}{2}]$. By now, our verification procedure has been finished. Thus the inverse inequalities (1.23) are true, that is, we have

$$\left[\frac{f_{k+1,n}(\mathbf{x})}{f_{k+1,n}(1-\mathbf{x})}\right]^{1/r} \geq \left[\frac{f_{k,n}(\mathbf{x})}{f_{k,n}(1-\mathbf{x})}\right]^{1/r}, \ k = 1, \ldots, n-1. \tag{1.42}$$

Passing the limit as $r \to 0$ in (1.42), we can obtain (1.41). By the same argument as in Theorem 1.14, we can derive the sign of equality in (1.41) holding throughout if and only if $x_1 = \cdots = x_n$. This ends the proof. □

1.5 Application to Mixed Symmetric Means

Let $I \subset \mathbb{R}$ be an interval, and $n \in \mathbb{N}_+$. The function $M : I^n \mapsto \mathbb{R}$ is called a mean if

$$\inf\{x_1,...,x_n\} \leq M(x_1,...,x_n) \leq \sup\{x_1,...,x_n\},$$

for all n-tuples $(x_1,...,x_n) \in I^n$.

The mean $M(x_1,...,x_n)$ is called a strict mean if these inequalities are strict unless $x_1 = ... = x_n$.
The mean $M(x_1,...,x_n)$ is called symmetric mean if

$$M(x_1,...,x_n) = M(x_{i_1},...,x_{i_n})$$

for any permutation $(i_1,...,i_n)$ of $(1,\ldots,n)$.
Examples of means and symmetric means for positive real numbers are given in [12]. The following famous notion is given in the fascinating and ground-breaking book [26, p. 13].

Quasi-arithmetic Means: Let $I \subset \mathbb{R}$ be an interval, $\mathbf{x} = (x_1,...,x_n) \in I^n$, $\mathbf{p} = (p_1,...,p_n)$ be a positive n-tuples such that $P := \sum_{i=1}^n p_i$, and let $\varphi : I \to \mathbb{R}$ be a continuous and strictly monotone function. The quasi-arithmetic means associated to φ are defined by

$$M_\varphi(\mathbf{x},\mathbf{p}) = M_\varphi(x_1,...,x_n;p_1,...,p_n) := \varphi^{-1}\left(\frac{1}{P}\sum_{i=1}^n p_i\varphi(x_i)\right). \tag{1.43}$$

Particularly, by choosing $\varphi(x) = x^r$ if $r \neq 0$, and $\varphi(x) = \log(x)$ if $r = 0$, we have

Power means: For $n \in \mathbb{N}_+$, let $\mathbf{x} = (x_1,...,x_n)$ and $\mathbf{p} = (p_1,...,p_n)$ be positive n-tuples such that $P := \sum_{i=1}^n p_i$. The well known power means of order $r \in \mathbb{R}$ are defined by

$$M_r(\mathbf{x},\mathbf{p}) = M_r(x_1,...,x_n;p_1,...,p_n) := \begin{cases} \left(\frac{1}{P}\sum\limits_{i=1}^n p_i x_i^r\right)^{\frac{1}{r}}, & r \neq 0, \\ \left(\prod\limits_{i=1}^n x_i^{p_i}\right)^{\frac{1}{P}}, & r = 0. \end{cases} \tag{1.44}$$

If $\mathbf{p} = \left(\frac{1}{n},...,\frac{1}{n}\right)$, $M_\varphi(\mathbf{x},\mathbf{p})$ and $M_r(\mathbf{x},\mathbf{p})$ will be written as $M_\varphi(\mathbf{x})$ and $M_r(\mathbf{x})$, respectively.

Now we give some means defined by integrals.

Integral means: Let $(X,\mathscr{A},\mu)$ be a measure space with $0 < \mu(X) < \infty$, $I \subset \mathbb{R}$ be an interval, $\varphi : I \to \mathbb{R}$ be a continuous and strictly monotone function, and $u : X \to I$ be a measurable function such that $\varphi \circ u$ is μ-integrable. The integral φ-means are defined by

$$\widetilde{M}_\varphi(u,\mu) := \varphi^{-1}\left(\frac{1}{\mu(X)}\int_X \varphi(u(x))d\mu(x)\right).$$

Integral power means: Let $(X, \mathscr{A}, \mu)$ be a measure space with $0 < \mu(X) < \infty$, $r \in \mathbb{R}$, and $u : X \to \mathbb{R}$ be a positive measurable function such that u^r is μ-integrable, if $r \neq 0$, and $\log \circ u$ is μ-integrable, if $r = 0$. Then the integral power means of order r are defined by (see [5]):

$$\widetilde{M}_r(u,\mu) := \begin{cases} \left(\frac{1}{\mu(X)} \int_X (u(x))^r d\mu(x) \right)^{\frac{1}{r}}, & r \neq 0, \\ \exp\left(\frac{1}{\mu(X)} \int_X \log(u(x))\, d\mu(x) \right), & r = 0. \end{cases} \tag{1.45}$$

The power means are monotone in nature: if $s, r \in \mathbb{R}$ and $s < r$, then

$$M_s(\mathbf{x},\mathbf{p}) \leq M_r(\mathbf{x},\mathbf{p}).$$

The same property holds for integral power means.

In cases $s = -1$, $s = 0$ and $s = 1$ the power means are well-known as weighted harmonic, geometric and arithmetic means $H(\mathbf{x},\mathbf{p})$, $G(\mathbf{x},\mathbf{p})$ and $A(\mathbf{x},\mathbf{p})$ respectively, satisfying the order as follows

$$H(\mathbf{x},\mathbf{p}) \leq G(\mathbf{x},\mathbf{p}) \leq A(\mathbf{x},\mathbf{p}).$$

About means see [13].

The history of mixed means is as old as the great C. F. Gauss (1777 - 1855) who represented the limit in the algorithm of the arithmetic-geometric mean by an elliptic integral [14]. The Jensen's inequality is much fertile to study about mixed means thats why our aim in this work is to emphasis on the refinements of Jensen's inequality.

Now, leaning on Theorems 1.8, 1.10, and 1.11, we introduce some new quasi-arithmetic and mixed symmetric means, and study their monotonicity.

First, we define quasi-arithmetic means with respect to (1.8) as follows: Let $I \subset \mathbb{R}$ be an interval, $\mathbf{x} = (x_1, ..., x_n) \in I^n$, $\mathbf{p} = (p_1, ..., p_n)$ be a positive n-tuples such that $\sum_{i=1}^n p_i = 1$, and let $h, g : I \to \mathbb{R}$ be continuous and strictly monotone functions. For $1 \leq k \leq n$, let

$$M^1_{h,g}(\mathbf{x},\mathbf{p};k) := h^{-1}\left(\frac{1}{\binom{n-1}{k-1}} \sum_{1 \leq i_1 < ... < i_k \leq n} \left(\sum_{j=1}^k p_{i_j} \right) h \circ g^{-1} \left(\frac{\sum_{j=1}^k p_{i_j} g(x_{i_j})}{\sum_{j=1}^k p_{i_j}} \right) \right). \tag{1.46}$$

From Theorem 1.8 we get the following result.

Corollary 1.5 ([52]) *Monotonicity properties of means (1.46):*

$$\begin{aligned} M_h(\mathbf{x},\mathbf{p}) = M^1_{h,g}(\mathbf{x},\mathbf{p},1) \geq ... \geq M^1_{h,g}(\mathbf{x},\mathbf{p},k) \geq M^1_{h,g}(\mathbf{x},\mathbf{p},k+1) \\ \geq ... \geq M^1_{h,g}(\mathbf{x},\mathbf{p},n) = M_g(\mathbf{x},\mathbf{p}), \end{aligned} \tag{1.47}$$

if either $f = h \circ g^{-1}$ is convex and h is increasing, or $f = h \circ g^{-1}$ is concave and h is decreasing;

$$M_g(\mathbf{x},\mathbf{p}) = M^1_{g,h}(\mathbf{x},\mathbf{p},1) \leq ... \leq M^1_{g,h}(\mathbf{x},\mathbf{p},k) \leq M^1_{g,h}(\mathbf{x},\mathbf{p},k+1)$$

$$\leq \ldots \leq M^1_{g,h}(\mathbf{x},\mathbf{p},n) = M_h(\mathbf{x},\mathbf{p})$$

if either $f = g \circ h^{-1}$ is convex and g is decreasing, or $f = g \circ h^{-1}$ is concave and g is increasing.

Proof. First, we can apply Theorem 1.8 to the function $h \circ g^{-1}$ and the n-tuples $(g(x_1), \ldots, g(x_n))$, then we can apply h^{-1} to the inequality coming from (1.9). This gives (1.47). A similar argument can be apply to prove the second inequality. □

We introduce the following mixed symmetric means: Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{p} = (p_1, \ldots, p_n)$ be a positive n-tuples such that $\sum_{i=1}^n p_i = 1$. Define for $1 \leq k \leq n$

$$M^1_{s,t}(\mathbf{x},\mathbf{p},k) = \begin{cases} \left(\frac{1}{\binom{n-1}{k-1}} \sum\limits_{1 \leq i_1 < \ldots < i_k \leq n} \left(\sum\limits_{j=1}^k p_{i_j} \right) M_t^s(x_{i_1}, \ldots x_{i_k}; p_{i_1}, \ldots p_{i_k}) \right)^{\frac{1}{s}}, & s \neq 0, \\ \left(\prod\limits_{1 \leq i_1 < \ldots < i_k \leq n} \left(M_t(x_{i_1}, \ldots x_{i_k}; p_{i_1}, \ldots p_{i_k}) \right)^{\left(\sum\limits_{j=1}^k p_{i_j} \right)} \right)^{\frac{1}{\binom{n-1}{k-1}}}, & s = 0. \end{cases} \tag{1.48}$$

The monotonicity of these means is also a consequence of Theorem 1.8.

Corollary 1.6 ([52]) *Let $s,t \in \mathbb{R}$ such that $s \leq t$, and let $\mathbf{x}$ and $\mathbf{p}$ be positive n-tuples such that $\sum_{i=1}^n p_i = 1$. Then we have*

$$M_t(\mathbf{x},\mathbf{p}) = M^1_{t,s}(\mathbf{x},\mathbf{p},1) \geq \ldots \geq M^1_{t,s}(\mathbf{x},\mathbf{p},k) \geq M^1_{t,s}(\mathbf{x},\mathbf{p},k+1) \tag{1.49}$$
$$\geq \ldots \geq M^1_{t,s}(\mathbf{x},\mathbf{p},n) = M_s(\mathbf{x},\mathbf{p}),$$

and

$$M_s(\mathbf{x},\mathbf{p}) = M^1_{s,t}(\mathbf{x},\mathbf{p},1) \leq \ldots \leq M^1_{s,t}(\mathbf{x},\mathbf{p},k) \leq M^1_{s,t}(\mathbf{x},\mathbf{p},k+1) \tag{1.50}$$
$$\leq \ldots \leq M^1_{s,t}(\mathbf{x},\mathbf{p},n) = M_t(\mathbf{x},\mathbf{p}).$$

Proof. Let $s,t \in \mathbb{R}$ such that $s \leq t$, if $s,t \neq 0$, then we set $f(x) = x^{\frac{t}{s}}$, $x_{i_j} = x^s_{i_j}$ in (1.9) and raising the power $\frac{1}{t}$, we get (1.49). Similarly we set $f(x) = x^{\frac{s}{t}}$, $x_{i_j} = x^t_{i_j}$ in (1.9) and raising the power $\frac{1}{s}$, we get (1.50).
When $s = 0$ or $t = 0$, we get the required results by taking limit. □

Next, we define the quasi-arithmetic means with respect to (1.12) as follows: Let $I \subset \mathbb{R}$ be an interval, $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$, $\mathbf{p} = (p_1, \ldots, p_n)$ be a positive n-tuples such that $\sum_{i=1}^n p_i = 1$, and let $h, g : I \to \mathbb{R}$ be continuous and strictly monotone functions. For $k \geq 1$, let

$$M^2_{h,g}(\mathbf{x},\mathbf{p},k) := h^{-1} \left(\frac{1}{\binom{n+k-1}{k-1}} \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq n} \left(\sum_{j=1}^k p_{i_j} \right) h \circ g^{-1} \left(\frac{\sum\limits_{j=1}^k p_{i_j} g(x_{i_j})}{\sum\limits_{j=1}^k p_{i_j}} \right) \right). \tag{1.51}$$

By applying Theorem 1.10, we have the following corollary.

Corollary 1.7 ([52]) *Monotonicity properties of means (1.51):*

$$M_h(\mathbf{x},\mathbf{p}) = M^2_{h,g}(\mathbf{x},\mathbf{p},1) \geq ... \geq M^2_{h,g}(\mathbf{x},\mathbf{p},k) \geq M^2_{h,g}(\mathbf{x},\mathbf{p},k+1) \geq ... \geq M_g(\mathbf{x},\mathbf{p})$$

if either $f = h \circ g^{-1}$ is convex and h is increasing, or $f = h \circ g^{-1}$ is concave and h is decreasing;

$$M_g(\mathbf{x},\mathbf{p}) = M^2_{g,h}(\mathbf{x},\mathbf{p},1) \leq ... \leq M^2_{g,h}(\mathbf{x},\mathbf{p},k) \leq M^2_{h,g}(\mathbf{x},\mathbf{p},k+1) \leq ... \leq M_h(\mathbf{x},\mathbf{p})$$

if either $f = g \circ h^{-1}$ is convex and g is decreasing, or $f = g \circ h^{-1}$ is concave and g is increasing.

We introduce the mixed symmetric means related to (1.12) as follows: For $k \geq 1$,define

$$\begin{aligned} &M^2_{s,t}(\mathbf{x},\mathbf{p},k) \\ &= \begin{cases} \left(\frac{1}{\binom{n+k-1}{k-1}} \sum\limits_{1 \leq i_1 \leq ... \leq i_k \leq n} \left(\sum\limits_{j=1}^{k} p_{i_j} \right) M_t^s(x_{i_1},...x_{i_k};p_{i_1},...p_{i_k}) \right)^{\frac{1}{s}}, & s \neq 0, \\ \left(\prod\limits_{1 \leq i_1 \leq ... \leq i_k \leq n} \left(M_t(x_{i_1},...x_{i_k};p_{i_1},...p_{i_k}) \right)^{\left(\sum\limits_{j=1}^{k} p_{i_j} \right)} \right)^{\frac{1}{\binom{n+k-1}{k-1}}}, & s = 0. \end{cases} \end{aligned} \tag{1.52}$$

Corollary 1.8 ([52]) *Let $s,t \in \mathbb{R}$ such that $s \leq t$, and let $\mathbf{x}$ and $\mathbf{p}$ be positive n-tuples such that $\sum_{i=1}^{n} p_i = 1$. Then we have*

$$M_t(\mathbf{x},\mathbf{p}) = M^2_{t,s}(\mathbf{x},\mathbf{p},1) \geq ... \geq M^2_{t,s}(\mathbf{x},\mathbf{p},k) \geq M^2_{t,s}(\mathbf{x},\mathbf{p},k+1) \geq ... \geq M_s(\mathbf{x},\mathbf{p}),$$

$$M_s(\mathbf{x},\mathbf{p}) = M^2_{s,t}(\mathbf{x},\mathbf{p},1) \leq ... \leq M^2_{s,t}(\mathbf{x},\mathbf{p},k) \leq M^2_{s,t}(\mathbf{x},\mathbf{p},k+1) \leq ... \leq M_t(\mathbf{x},\mathbf{p}).$$

Further quasi-arithmetic means are coming from (1.15): Let $I \subset \mathbb{R}$ be an interval, $\mathbf{x} = (x_1,...,x_n) \in I^n$, $\mathbf{p} = (p_1,...,p_n)$ be a positive n-tuples such that $\sum_{i=1}^{n} p_i = 1$, and let $h,g : I \to \mathbb{R}$ be continuous and strictly monotone functions. For $k \geq 1$, define

$$M^3_{h,g}(\mathbf{x},\mathbf{p},k) := h^{-1} \left(\sum_{i_1,...,i_k=1}^{n} p_{i_1}...p_{i_k} h \circ g^{-1} \left(\frac{1}{k} \sum_{j=1}^{k} g(x_{i_j}) \right) \right). \tag{1.53}$$

Theorem 1.11 implies the following result.

Corollary 1.9 ([52]) *Monotonicity properties of means (1.53):*

$$M_h(\mathbf{x},\mathbf{p}) \geq M^3_{h,g}(\mathbf{x},\mathbf{p},1) \geq ... \geq M^3_{h,g}(\mathbf{x},\mathbf{p},k) \geq M^3_{h,g}(\mathbf{x},\mathbf{p},k+1) \geq ... \geq M_g(\mathbf{x},\mathbf{p})$$

if either $f = h \circ g^{-1}$ is convex and h is increasing, or $f = h \circ g^{-1}$ is concave and h is decreasing;

$$M_g(\mathbf{x},\mathbf{p}) \leq M^3_{g,h}(\mathbf{x},\mathbf{p},1) \leq ... \leq M^3_{g,h}(\mathbf{x},\mathbf{p},k) \leq M^3_{g,h}(\mathbf{x},\mathbf{p},k+1) \leq ... \leq M_h(\mathbf{x},\mathbf{p})$$

if either $f = g \circ h^{-1}$ is convex and g is decreasing, or $f = g \circ h^{-1}$ is concave and g is increasing.

The mixed symmetric means with positive weights related to (1.15) are

$$M^3_{s,t}(\mathbf{x},\mathbf{p},k) := \begin{cases} \left(\sum\limits_{i_1,\dots,i_k=1}^{n} \left(\prod\limits_{j=1}^{k} p_{i_j} \right) M_t^s\left(x_{i_1},\dots,x_{i_k}\right) \right)^{\frac{1}{s}}, & s \neq 0, \\ \prod\limits_{i_1,\dots,i_k=1}^{n} \left(M_t\left(x_{i_1},\dots,x_{i_k}\right)\right)^{\left(\prod\limits_{j=1}^{k} p_{i_j}\right)}, & s = 0, \end{cases} \quad , \quad k \geq 1.$$

Corollary 1.10 ([52]) *Let $s,t \in \mathbb{R}$ such that $s \leq t$, and let $\mathbf{x}$ and $\mathbf{p}$ be positive n-tuples such that $\sum_{i=1}^{n} p_i = 1$. Then we have*

$$M_t(\mathbf{x},\mathbf{p}) \geq M^3_{t,s}(\mathbf{x},\mathbf{p},1) \geq \dots \geq M^3_{t,s}(\mathbf{x},\mathbf{p},k) \geq M^3_{t,s}(\mathbf{x},\mathbf{p},k+1) \geq \dots \geq M_s(\mathbf{x},\mathbf{p}),$$

$$M_s(\mathbf{x},\mathbf{p}) \leq M^3_{s,t}(\mathbf{x},\mathbf{p},1) \leq \dots \leq M^3_{s,t}(\mathbf{x},\mathbf{p},k) \leq M^3_{s,t}(\mathbf{x},\mathbf{p},k+1) \leq \dots \leq M_t(\mathbf{x},\mathbf{p}).$$

Finally, we define some integral means with respect to (1.17) as follows: Let $I \subset \mathbb{R}$ be an interval, $h,g : I \to \mathbb{R}$ be continuous and strictly monotone functions, and σ be an increasing function on $[0,1]$ such that $\int\limits_0^1 d\sigma(x) = 1$. Further, let $u : [0,1] \to I$ be a measurable function such that $h \circ u$ and $g \circ u$ are σ-integrable. For $k \geq 1$, introduce

$$M^4_{h,g}(u,\sigma,k) := h^{-1}\left(\int\limits_0^1 \dots . \int\limits_0^1 h \circ g^{-1}\left(\frac{1}{k} \sum_{i=1}^{k} g\left(u(x_i)\right) \right) \prod_{i=1}^{k} d\sigma(x_i) \right). \tag{1.54}$$

Theorem 1.12 implies the following result.

Corollary 1.11 ([52]) *Monotonicity properties of means (1.54):*

$$\widetilde{M}_h\left(u,\sigma\right) = M^4_{h,g}(u,\sigma,1) \geq \dots \geq M^4_{h,g}(u,\sigma,k) \geq M^4_{h,g}(u,\sigma,k+1) \geq \dots \geq \widetilde{M}_g\left(u,\sigma\right)$$

where $f = h \circ g^{-1}$ is convex and h is increasing, or $f = h \circ g^{-1}$ is concave and h is decreasing;

$$\widetilde{M}_g\left(u,\sigma\right) = M^4_{g,h}(u,\sigma,1) \leq \dots \leq M^4_{g,h}(u,\sigma,k) \leq M^4_{g,h}(u,\sigma,k+1) \leq \dots \leq \widetilde{M}_h\left(u,\sigma\right)$$

where $f = g \circ h^{-1}$ is convex and g is decreasing, or $f = g \circ h^{-1}$ is concave and g is increasing.

The mixed symmetric means with positive weights related to

$$\int\limits_0^1 \dots \int\limits_0^1 f\left(\frac{1}{k} \sum_{i=1}^{k} u(x_i) \right) \prod_{i=1}^{k} d\sigma(x_i), \quad k \geq 1 \tag{1.55}$$

are defined as:

$$M^4_{s,t}(u,\sigma,k) := \begin{cases} \left(\int\limits_0^1 \dots . \int\limits_0^1 M_t^s\left(u(x_1),\dots,u(x_k)\right) \prod\limits_{i=1}^{k} d\sigma(x_i) \right)^{\frac{1}{s}}, & s \neq 0, \\ \exp\left(\left(\int\limits_0^1 \dots . \int\limits_0^1 \log M_t\left(u(x_1),\dots,u(x_k)\right) \prod\limits_{i=1}^{k} d\sigma(x_i) \right) \right), & s = 0. \end{cases}$$

Corollary 1.12 ([52]) *Let $s,t \in \mathbb{R}$ such that $s \le t$. Then we have*

$$\widetilde{M}_t\,(u,\sigma) = M^4_{t,s}(u,\sigma,1) \ge \ldots \ge M^4_{t,s}(u,\sigma,k) \ge M^4_{t,s}(u,\sigma,k{+}1) \ge \ldots \ge \widetilde{M}_s\,(u,\sigma)\,, \quad (1.56)$$

$$\widetilde{M}_s\,(u,\sigma) = M^4_{s,t}(u,\sigma,1) \le \ldots \le M^4_{s,t}(u,\sigma,k) \le M^4_{s,t}(u,\sigma,k{+}1) \le \ldots \le \widetilde{M}_t\,(u,\sigma)\,. \quad (1.57)$$

Remark 1.4 In fact unweighted version of these results were proved in [6], but it has been mentioned in Remark 2.14 of [6] that the same is valid for the weighted case. In [52], not only results for weighted mixed symmetric means are given, but the exponential convexity of some expressions coming from (1.8), (1.12), (1.15) and (1.17) is also proved by using the convex functions $\varphi_s : (0,\infty) \to \mathbb{R}$ and $\phi_s : \mathbb{R} \to [0,\infty)$, defined by

$$\varphi_s(x) := \begin{cases} \frac{x^s}{s(s-1)}\,, & s \ne 0,1; \\ -\log(x)\,, & s = 0; \\ x\log(x)\,, & s = 1, \end{cases}$$

$$\phi_s(x) := \begin{cases} \frac{1}{s^2}e^{sx}, & s \ne 0; \\ \frac{1}{2}x^2, & s = 0. \end{cases}$$

Mean value theorems are also given together with the corresponding monotone means of Cauchy type. In [6] these classes are used to prove the log-convexity (not the exponential convexity) of positive linear functionals.

Remark 1.5 In 1998, T. Hara et al. introduced some unweighted mixed means and proved their monotonicity property (see [25]). The Hamy's symmetric function (see [25, 54]) has interesting properties and it has been studied by many authors (see [9, 15, 16, 23, 49, 54]). It generates some mixed symmetric means without weights, named as Hamy's means. For a positive integer r, the r-th order Hamy's Mean is a special case of the mixed symmetric (arithmetic-geometric or geometric-arithmetic) mean given by Mitrinović and Pečarić in 1988 [58].

Chapter 2

Refinements of Jensen's Inequality

A method to refine the well known discrete Jensen's inequality is developed in [32], and a parameter dependent refinement of the discrete Jensen's inequality is proved in [33]. Mixed symmetric means are constructed with respect to these refinements and the monotonicity of them is studied. We also apply the new exponential convexity method as illustrated in [68], to the functionals obtained from the refinement results of [32] and [33]. In this way we are able to generalize the results given in [37] as well as given in [6]. The results of this chapter are given in [40], [51] and [53].

Throughout the text $P(X)$ denotes the power set of a set X, $|X|$ means the number of elements in X, and for any nonnegative integer d let

$$P_d(X) := \{Y \subset X \mid |Y| = d\}.$$

2.1 A Refinement of the Discrete Jensen's Inequality

A refinement of the discrete Jensen's inequality is given in [44]. The following notations are also introduced in [44]:

($\mathscr{N}_1$): Let $u \geq 1$ and $v \geq 2$ be fixed integers. Define the functions

$$S_{v,w} : \{1,...,u\}^v \to \{1,...,u\}^{v-1}, \quad 1 \leq w \leq v,$$

$$S_v : \{1,...,u\}^v \to P\left(\{1,...,u\}^{v-1}\right),$$

and

$$T_v : P(\{1,...,u\}^v) \to P\left(\{1,...,u\}^{v-1}\right)$$

by

$$S_{v,w}(i_1,...,i_v) := (i_1,...,i_{w-1},i_{w+1},...,i_v), \quad 1 \leq w \leq v,$$

$$S_v(i_1,...,i_v) := \bigcup_{w=1}^{v} \{S_{v,w}(i_1,...,i_v)\},$$

and

$$T_v(I) := \begin{cases} \bigcup\limits_{(i_1,...,i_v)\in I} S_v(i_1,...,i_v), \; I \neq \emptyset, \\ \phi, \qquad\qquad\qquad , \; I = \emptyset, \end{cases}$$

where $\emptyset$ means the empty set.

Further, introduce the functions

$$\alpha_{v,i} : \{1,...,u\}^v \to \mathbb{N}, \quad 1 \leq i \leq u,$$

defined by

$$\alpha_{v,i}(i_1,...,i_v) := \text{Number of occurrences of } i \text{ in the sequence } (i_1,...,i_v).$$

For each $I \in P(\{1,...,u\}^v)$, let

$$\alpha_{I,i} := \sum_{(i_1,...,i_v)\in I} \alpha_{v,i}(i_1,...,i_v), \qquad 1 \leq i \leq u.$$

It is easy to observe from the construction of the functions $S_{v,w}$, S_v, T_v and $\alpha_{v,i}$ that they do not depend essentially on u, so u is not marked in the notations.

($\mathscr{H}_0$) The following considerations concern a subset I_k of $\{1,...,n\}^k$ satisfying

$$\alpha_{I_k,i} \geq 1, \quad 1 \leq i \leq n, \tag{2.1}$$

where $n \geq 1$ and $k \geq 2$ are fixed integers.

Next, we proceed inductively to define the sets $I_l \subset \{1,...,n\}^l$ $(k-1 \geq l \geq 1)$ by

$$I_{l-1} := T_l(I_l), \quad k \geq l \geq 2.$$

By (2.1), $I_1 = \{1,...,n\}$ and this implies that $\alpha_{I_1,i} = 1$ for $1 \leq i \leq n$. From (2.1) again, we have $\alpha_{I_l,i} \geq 1$ $(k-1 \geq l \geq 1, 1 \leq i \leq n)$. It is evident that

$$\alpha_{1,i}(j) = \begin{cases} 1, & \text{if} \quad j = i \\ 0, & \text{if} \quad j \neq i \end{cases}, \quad 1 \leq i \leq n. \tag{2.2}$$

For every $k \geq l \geq 2$ and for any $(j_1,...,j_{l-1}) \in I_{l-1}$ let

$$H_{I_l}(j_1,...,j_{l-1}) \\ := \{((i_1,...,i_l),m) \in I_l \times \{1,...,l\} \mid S_{l,m}(i_1,...,i_l) = (j_1,...,j_{l-1})\}.$$

Using these sets we define the functions $t_{I_k,l} : I_l \to \mathbb{N}$ $(k \geq l \geq 1)$ inductively by

$$t_{I_k,k}(i_1,...,i_k) := 1,\ (i_1,...,i_k) \in I_k; \tag{2.3}$$

$$t_{I_k,l-1}(j_1,...,j_{l-1}) := \sum_{((i_1,...,i_l),m) \in H_{I_l}(j_1,...,j_{l-1})} t_{I_k,l}(i_1,...,i_l). \tag{2.4}$$

In the sequel we need the following hypotheses:

($\mathscr{H}_1$) Let V be a real vector space, $C \subset V$ be a convex set, $\mathbf{x} := (x_1,...,x_n) \in C^n$, and let $\mathbf{p} := (p_1,...,p_n)$ be a positive n-tuples such that $\sum\limits_{i=1}^{n} p_i = 1$.

($\mathscr{H}_2$) Let $f : C \to \mathbb{R}$ be a convex function.

($\tilde{\mathscr{H}_2}$) Let $f : C \to \mathbb{R}$ be a mid-convex function, and $p_1,\ldots,p_n$ be rational numbers.

We introduce some special expressions, which will be important in our results.

For any $k \geq l \geq 1$ set

$$A_{l,l} = A_{l,l}(I_k,\mathbf{x},\mathbf{p}) := \sum_{(i_1,...,i_l) \in I_l} \left(\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_l,i_s}} \right) f \left(\frac{\sum\limits_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_l,i_s}} x_{i_s}}{\sum\limits_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_l,i_s}}} \right), \tag{2.5}$$

and associate to each $k-1 \geq l \geq 1$ the number

$$A_{k,l} = A_{k,l}(I_k,\mathbf{x},\mathbf{p}) \\ := \frac{1}{(k-1)...l} \sum_{(i_1,...,i_l) \in I_l} t_{I_k,l}(i_1,\ldots,i_l) \left(\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}} \right) f \left(\frac{\sum\limits_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}} x_{i_s}}{\sum\limits_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}}} \right). \tag{2.6}$$

Now we are in a position to formulate the following interpolatory result by Horváth and Pečarić:

Theorem 2.1 ([44]) *Assume that ($\mathscr{H}_0$), ($\mathscr{H}_1$) and either ($\mathscr{H}_2$) or ($\tilde{\mathscr{H}_2}$) are satisfied. Then*

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq A_{k,k} \leq A_{k,k-1} \leq \ldots \leq A_{k,2} \leq A_{k,1} = \sum_{i=1}^{n} p_i f(x_i). \tag{2.7}$$

If f is a concave function then the inequalities in (2.7) are reversed.

Under the conditions of Theorem 2.1, we have

$$\Upsilon_1(f) = \Upsilon_1(f,m,l,I_k,\mathbf{x},\mathbf{p}) := A_{k,m} - A_{k,l} \geq 0, \quad k-1 \geq l > m \geq 1,$$
$$\Upsilon_2(f) = \Upsilon_2(f,l,I_k,\mathbf{x},\mathbf{p}) := A_{k,l} - f\left(\sum_{i=1}^{n} p_i x_i\right) \geq 0, \quad k-1 \geq l \geq 1.$$

The following result is also given in [44].

Theorem 2.2 ([44]) *Assume that* $(\mathscr{H}_0)$, $(\mathscr{H}_1)$ *and either* $(\mathscr{H}_2)$ *or* $(\tilde{\mathscr{H}}_2)$ *are satisfied. Also suppose* $|H_{I_l}(j_1,...,j_{l-1})| = \beta_{l-1}$ *for any* $(j_1,...,j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$. *Then*

$$A_{k,l} = A_{l,l} = \frac{n}{l\,|I_l|} \sum_{(i_1,...,i_l)\in I_l} \left(\sum_{s=1}^{l} p_{i_s}\right) f\left(\frac{\sum\limits_{s=1}^{l} p_{i_s} x_{i_s}}{\sum\limits_{s=1}^{l} p_{i_s}}\right), \quad k \geq l \geq 1, \tag{2.8}$$

and thus

$$f\left(\sum_{r=1}^{n} p_r x_r\right) \leq A_{k,k} \leq A_{k-1,k-1} \leq ... \leq A_{2,2} \leq A_{1,1} = \sum_{r=1}^{n} p_r f(x_r). \tag{2.9}$$

If f is a concave function then the inequalities in (2.9) are reversed.

Under the conditions of the previous theorem, we have from (2.9) that

$$\begin{aligned}
\Upsilon_3(f) = \Upsilon_3(f, I_k, \mathbf{x}, \mathbf{p}) &:= A_{m,m} - A_{l,l} \geq 0, \quad k \geq l > m \geq 1, \\
\Upsilon_4(f) = \Upsilon_4(f, I_k, \mathbf{x}, \mathbf{p}) &:= A_{l,l} - f\left(\sum_{r=1}^{n} p_r x_r\right) \geq 0, \quad k \geq l \geq 1.
\end{aligned}$$

To prove the previous results we begin with a deeper property of the function $t_{I_k,1}$.

Lemma 2.1 *If* $(\mathscr{H}_0)$ *is satisfied, then*

$$t_{I_k,1}(i) = \alpha_{I_k,i}\,(k-1)!, \quad 1 \leq i \leq n. \tag{2.10}$$

Proof. For a fixed $1 \leq i \leq n$ we first prove by induction on l that

$$\begin{aligned}
&\sum_{(i_1,...,i_l)\in I_l} \alpha_{l,i}(i_1,\ldots,i_l) t_{I_k,l}(i_1,\ldots,i_l) \qquad (2.11)\\
&= \begin{cases} \alpha_{I_k,i}, & \text{if} \quad l = k \\ \alpha_{I_k,i}(k-1)(k-2)\ldots l, & \text{if} \quad k-1 \geq l \geq 1 \end{cases}.
\end{aligned}$$

If $l = k$, then (2.1) and (2.3) give (2.11). Suppose then that l $(k \geq l \geq 2)$ is an integer for which (2.11) holds. By (2.4)

$$\begin{aligned}
&\sum_{(j_1,...,j_{l-1})\in I_{l-1}} \alpha_{l-1,i}(j_1,\ldots,j_{l-1}) t_{I_k,l-1}(j_1,\ldots,j_{l-1}) \\
&= \sum_{(j_1,...,j_{l-1})\in I_{l-1}} \alpha_{l-1,i}(j_1,\ldots,j_{l-1}) \\
&\cdot \left(\sum_{((i_1,...,i_l),m)\in H_{I_l}(j_1,...,j_{l-1})} t_{I_k,l}(i_1,\ldots,i_l) \right).
\end{aligned}$$

From this and the definition of $S_{l,m}$ $(1 \leq m \leq l)$ it follows

$$\sum_{(j_1,\ldots,j_{l-1})\in I_{l-1}} \alpha_{l-1,i}(j_1,\ldots,j_{l-1})t_{I_k,l-1}(j_1,\ldots,j_{l-1}) = \sum_{(j_1,\ldots,j_{l-1})\in I_{l-1}}$$

$$\left(\sum_{\{((i_1,\ldots,i_l),m)\in H_{I_l}(j_1,\ldots,j_{l-1})|i_m\neq i\}} \alpha_{l,i}(i_1,\ldots,i_l)t_{I_k,l}(i_1,\ldots,i_l) \right.$$

$$\left. + \sum_{\{((i_1,\ldots,i_l),m)\in H_{I_l}(j_1,\ldots,j_{l-1})|i_m=i\}} \left(\alpha_{l,i}(i_1,\ldots,i_l)-1\right) t_{I_k,l}(i_1,\ldots,i_l) \right)$$

$$= \sum_{(i_1,\ldots,i_l)\in I_l} \left(\left(l-\alpha_{l,i}(i_1,\ldots,i_l)\right) \alpha_{l,i}(i_1,\ldots,i_l)t_{I_k,l}(i_1,\ldots,i_l) \right.$$

$$\left. +\alpha_{l,i}(i_1,\ldots,i_l)\left(\alpha_{l,i}(i_1,\ldots,i_l)-1\right) t_{I_k,l}(i_1,\ldots,i_l)\right)$$

$$= (l-1) \sum_{(i_1,\ldots,i_l)\in I_l} \alpha_{l,i}(i_1,\ldots,i_l)t_{I_k,l}(i_1,\ldots,i_l),$$

and therefore the induction hypothesis shows that

$$\sum_{(j_1,\ldots,j_{l-1})\in I_{l-1}} \alpha_{l-1,i}(j_1,\ldots,j_{l-1})t_{I_k,l-1}(j_1,\ldots,j_{l-1})$$

$$= \alpha_{I_k,i}(k-1)\ldots l(l-1).$$

(2.11) for $l = 1$, taking into consideration (2.2), implies (2.10). The proof is complete. □

In Theorem 2.2 our arguments depend on the following lemma.

Lemma 2.2 *Assume* $(\mathscr{H}_0)$. *If*

$$\left|H_{I_l}(j_1,\ldots,j_{l-1})\right| = \beta_{l-1}, \text{ for all } (j_1,\ldots,j_{l-1}) \in I_{l-1}, \quad k \geq l \geq 2, \tag{2.12}$$

then

(a) $\beta_{l-1} = l\frac{|I_l|}{|I_{l-1}|}$ $(k \geq l \geq 2)$.

(b) $t_{I_k,l}(j_1,\ldots,j_l) = \beta_{k-1}\ldots\beta_l = k\ldots(l+1)\frac{|I_k|}{|I_l|}$ $((j_1,\ldots,j_l) \in I_l,\ k-1 \geq l \geq 1)$.

(c) $\alpha_l := \alpha_{I_l,n} = \ldots = \alpha_{I_l,1}$ $(k \geq l \geq 1)$.

(d) $\alpha_l = \frac{l|I_l|}{n}$ $(k \geq l \geq 1)$.

(e)

$$A_{k.l} = A_{l,l} = \frac{n}{l\,|I_l|} \sum_{(i_1,\ldots,i_l)\in I_l} \left(\sum_{s=1}^{l} p_{i_s}\right) f\left(\frac{\sum\limits_{s=1}^{l} p_{i_s}x_s}{\sum\limits_{s=1}^{l} p_{i_s}}\right), \quad k \geq l \geq 1.$$

(f) If $p_1 = \ldots = p_n = \frac{1}{n}$, *then*

$$A_{k.l} = A_{l,l} = \frac{1}{|I_l|} \sum_{(i_1,\ldots,i_l)\in I_l} f\left(\frac{x_{i_1}+\ldots+x_{i_l}}{l}\right), \quad k \geq l \geq 1.$$

Proof. (a) By the definition of $H_{I_l}(j_1,\ldots,j_{l-1})$

$$\sum_{(j_1,\ldots,j_{l-1})\in I_{l-1}} \left|H_{I_l}(j_1,\ldots,j_{l-1})\right| = l\,|I_l|, \quad k\geq l\geq 2.$$

Consequently, (2.12) yields (a).

(b) We prove this by induction on l, the case $l=k$ being

$$\begin{aligned} t_{I_k,k-1}(j_1,\ldots,j_{k-1}) &:= \sum_{((i_1,\ldots,i_k),m)\in H_{I_k}(j_1,\ldots,j_{k-1})} t_{I_k,k}(i_1,\ldots,i_k) \\ &= \beta_{k-1}, \quad (j_1,\ldots,j_{k-1})\in I_{k-1}. \end{aligned}$$

Let l $(k-1\geq l\geq 2)$ be an integer such that the result holds. Then

$$\begin{aligned} t_{I_k,l-1}(j_1,\ldots,j_{l-1}) &:= \sum_{((i_1,\ldots,i_l),m)\in H_{I_l}(j_1,\ldots,j_{l-1})} t_{I_k,l}(i_1,\ldots,i_l) \\ = \sum_{((i_1,\ldots,i_l),m)\in H_{I_l}(j_1,\ldots,j_{l-1})} \beta_{k-1}\ldots\beta_l &= \left|H_{I_l}(j_1,\ldots,j_{l-1})\right|\beta_{k-1}\ldots\beta_l \\ &= \beta_{k-1}\ldots\beta_l\beta_{l-1}. \end{aligned}$$

The second equality in (b) comes from (a).

(c) Part (b) and (2.10) show that

$$t_{I_k,1}(i) = \beta_{k-1}\ldots\beta_1 = \alpha_{I_k,i}(k-1)!, \quad 1\leq i\leq n,$$

and thus $\alpha_{I_k,n}=\ldots=\alpha_{I_k,1}$. It follows from (b) and (2.11) that

$$\begin{aligned} &\sum_{(i_1,\ldots,i_l)\in I_l} \alpha_{l,i}(i_1,\ldots,i_l)t_{I_k,l}(i_1,\ldots,i_l) = \beta_{k-1}\ldots\beta_l \sum_{(i_1,\ldots,i_l)\in I_l} \alpha_{l,i}(i_1,\ldots,i_l) \\ &= \beta_{k-1}\ldots\beta_l\alpha_{I_l,i} = \alpha_{I_k,i}(k-1)(k-2)\ldots l, \quad k-1\geq l\geq 1, \quad 1\leq i\leq n, \end{aligned}$$

giving

$$\alpha_{I_l,i} = \frac{\alpha_k(k-1)(k-2)\ldots l}{\beta_{k-1}\ldots\beta_l}, \quad k-1\geq l\geq 1, \quad 1\leq i\leq n,$$

and this implies the result for $k-1\geq l\geq 1$.

(d) It is an easy consequence of (c).

(e) Using the definition of $A_{k,l}$ $(k-1\geq l\geq 1)$, then (b), (c) and (d), we get

$$\begin{aligned} A_{k,l} &:= \frac{1}{(k-1)\ldots l} \sum_{(i_1,\ldots,i_l)\in I_l} t_{I_k,l}(i_1,\ldots,i_l)\left(\sum_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right) f\left(\frac{\sum\limits_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_k,i_s}}x_s}{\sum\limits_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_k,i_s}}}\right) \\ &= \frac{1}{\alpha_k(k-1)\ldots l}k\ldots(l+1)\frac{|I_k|}{|I_l|} \sum_{(i_1,\ldots,i_l)\in I_l}\left(\sum_{s=1}^{l}p_{i_s}\right) f\left(\frac{\sum\limits_{s=1}^{l}p_{i_s}x_s}{\sum\limits_{s=1}^{l}p_{i_s}}\right) \end{aligned}$$

$$= \frac{k}{l\alpha_k}\frac{|I_k|}{|I_l|}\sum_{(i_1,\ldots,i_l)\in I_l}\left(\sum_{s=1}^{l} p_{i_s}\right) f\left(\frac{\sum\limits_{s=1}^{l} p_{i_s}x_s}{\sum\limits_{s=1}^{l} p_{i_s}}\right)$$

$$= \frac{n}{l\,|I_l|}\sum_{(i_1,\ldots,i_l)\in I_l}\left(\sum_{s=1}^{l} p_{i_s}\right) f\left(\frac{\sum\limits_{s=1}^{l} p_{i_s}x_s}{\sum\limits_{s=1}^{l} p_{i_s}}\right), \quad (k-1\geq l\geq 1).$$

Similarly, the definition of $A_{l,l}$ $(k\geq l\geq 1)$, (c) and (d) insures that

$$A_{l,l} := \sum_{(i_1,\ldots,i_l)\in I_l}\left(\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_l,i_s}}\right) f\left(\frac{\sum\limits_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_l,i_s}}x_s}{\sum\limits_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_l,i_s}}}\right)$$

$$= \frac{1}{\alpha_l}\sum_{(i_1,\ldots,i_l)\in I_l}\left(\sum_{s=1}^{l} p_{i_s}\right) f\left(\frac{\sum\limits_{s=1}^{l} p_{i_s}x_s}{\sum\limits_{s=1}^{l} p_{i_s}}\right)$$

$$= \frac{n}{l\,|I_l|}\sum_{(i_1,\ldots,i_l)\in I_l}\left(\sum_{s=1}^{l} p_{i_s}\right) f\left(\frac{\sum\limits_{s=1}^{l} p_{i_s}x_s}{\sum\limits_{s=1}^{l} p_{i_s}}\right), \quad (k\geq l\geq 1).$$

(f) This is a special case of (e).

The proof is now complete. □

Remark 2.1 Assume $(\mathscr{H}_0)$. Lemma 2.2 shows that (2.12) implies $\alpha_l := \alpha_{I_l,n} = \ldots = \alpha_{I_l,1}$ $(k\geq l\geq 1)$. The converse of this is not true in general, as it is seen by easy examples.

The following lemma will be fundamental.

Lemma 2.3 *Assume that $(\mathscr{H}_0)$, $(\mathscr{H}_1)$ and either $(\mathscr{H}_2)$ or $(\tilde{\mathscr{H}}_2)$ are satisfied. Then*

$$A_{k,l} \leq A_{k,l-1}, \quad k\geq l\geq 2.$$

Proof. Assume $(\mathscr{H}_0)$, $(\mathscr{H}_1)$ and $(\mathscr{H}_2)$. We prove first that $A_{k,k} \le A_{k,k-1}$. Since

$$A_{k,k} = \sum_{(i_1,\ldots,i_k)\in I_k} \left(\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}}\right) f\left(\frac{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} x_{i_s}}{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}}}\right)$$

$$= \sum_{(i_1,\ldots,i_k)\in I_k} \left(\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}}\right)$$

$$\cdot f\left(\sum_{m=1}^{k}\left(\frac{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} - \frac{p_{i_m}}{\alpha_{I_k,i_m}}}{(k-1)\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}}} \cdot \frac{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} x_{i_s} - \frac{p_{i_m}}{\alpha_{I_k,i_m}} x_{i_m}}{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} - \frac{p_{i_m}}{\alpha_{I_k,i_m}}}\right)\right),$$

and

$$\frac{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} - \frac{p_{i_m}}{\alpha_{I_k,i_m}}}{(k-1)\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}}} \ge 0, \quad 1 \le m \le k,$$

and

$$\sum_{m=1}^{k}\left(\frac{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} - \frac{p_{i_m}}{\alpha_{I_k,i_m}}}{(k-1)\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}}}\right) = 1, \quad (i_1,\ldots,i_k) \in I_k,$$

the discrete Jensen's inequality for convex functions (see Theorem 1.5) implies

$$A_{k,k} \le \sum_{(i_1,\ldots,i_k)\in I_k} \left(\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}}\right) \sum_{m=1}^{k}\left(\frac{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_j}} - \frac{p_{i_m}}{\alpha_{I_k,i_m}}}{(k-1)\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}}} \tag{2.13}$$

$$\cdot f\left(\frac{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} x_{i_s} - \frac{p_{i_m}}{\alpha_{I_k,i_m}} x_{i_m}}{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} - \frac{p_{i_m}}{\alpha_{I_k,i_m}}}\right)\right) = \frac{1}{k-1} \sum_{(i_1,\ldots,i_k)\in I_k}$$

$$\left(\sum_{m=1}^{k}\left(\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} - \frac{p_{i_m}}{\alpha_{I_k,i_m}}\right) f\left(\frac{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} x_{i_s} - \frac{p_{i_m}}{\alpha_{I_k,i_m}} x_{i_m}}{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} - \frac{p_{i_m}}{\alpha_{I_k,i_m}}}\right)\right).$$

In light of the meaning of $t_{I_k,k-1}$, this yields

$$A_{k,k} \le \frac{1}{k-1} \sum_{(j_1,\ldots,j_{k-1})\in I_{k-1}} t_{I_k,k-1}(j_1,\ldots,j_{k-1})$$

$$\cdot \left(\sum_{s=1}^{k-1} \frac{p_{j_s}}{\alpha_{I_k,j_s}} \right) f \left(\frac{\sum\limits_{s=1}^{k-1} \frac{p_{j_s}}{\alpha_{I_k,j_s}} x_{j_s}}{\sum\limits_{s=1}^{k-1} \frac{p_{j_s}}{\alpha_{I_k,j_s}}} \right) = A_{k,k-1}.$$

Suppose now that $k-1 \ge l \ge 2$. By an argument analogous to that employed in the first part we have that

$$A_{k,l} \le \frac{1}{(k-1)\ldots l\,(l-1)} \sum_{(i_1,\ldots,i_l)\in I_l} \Bigg(t_{I_k,l}(i_1,\ldots i_l)$$

$$\cdot \sum_{m=1}^{l} \left(\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}} - \frac{p_{i_m}}{\alpha_{I_k,i_m}} \right) f \left(\frac{\sum\limits_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}} x_{i_s} - \frac{p_{i_m}}{\alpha_{I_k,i_m}} x_{i_m}}{\sum\limits_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}} - \frac{p_{i_m}}{\alpha_{I_k,i_m}}} \right) \Bigg),$$

and therefore the definitions of the set $H_{I_l}(j_1,\ldots,j_{l-1})$ and the function $t_{I_k,l-1}$ give

$$A_{k,l} \le \frac{1}{(k-1)\ldots l\,(l-1)}$$

$$\cdot \sum_{(j_1,\ldots,j_{l-1})\in I_{l-1}} \Bigg(\Bigg(\sum_{((i_1,\ldots,i_l),m)\in H_{I_l}(j_1,\ldots,j_{l-1})} t_{I_k,l}(i_1,\ldots,i_l) \Bigg)$$

$$\left(\sum_{s=1}^{l-1} \frac{p_{j_s}}{\alpha_{I_k,j_s}} \right) f \left(\frac{\sum\limits_{s=1}^{l-1} \frac{p_{j_s}}{\alpha_{I_k,j_s}} x_{j_s}}{\sum\limits_{s=1}^{l-1} \frac{p_{j_s}}{\alpha_{I_k,j_s}}} \right) \Bigg) = A_{k,l-1},$$

and this completes the proof in the considered case.

We turn now to the other case: assume $(\mathscr{H}_0)$, $(\mathscr{H}_1)$ and $(\tilde{\mathscr{H}}_2)$. Since the numbers $\alpha_{I_k,i}$ $(1 \le i \le n)$ are integers the proof is entirely similar as above (the discrete Jensen's inequality for mid-convex functions can be applied in (2.13)).

The proof is now complete. $\square$

After these preliminaries we arrive to the proof of Theorem 2.1.

Proof. Assume $(\mathscr{H}_0)$, $(\mathscr{H}_1)$ and $(\mathscr{H}_2)$.

Since

$$\sum_{r=1}^{n} p_r x_r = \sum_{(i_1,\ldots,i_k)\in I_k} \left(\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} \frac{\sum\limits_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} x_{i_s}}{\sum\limits_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}}} \right),$$

and

$$\sum_{(i_1,\ldots,i_k)\in I_k}\left(\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right)=1,$$

it follows from the discrete Jensen's inequality for convex functions that

$$f\left(\sum_{r=1}^{n}p_r x_r\right)\leq\sum_{(i_1,\ldots,i_k)\in I_k}\left(\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right)f\left(\frac{\sum\limits_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_j}}x_{i_s}}{\sum\limits_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}}\right)=A_{k,k},\tag{2.14}$$

which proves the first inequality in (2.7).

The inequalities

$$A_{k,k}\leq A_{k,k-1}\leq\ldots\leq A_{k,2}\leq A_{k,1}$$

can be obtained from Lemma 2.3.

It remains only to show that

$$A_{k,1}=\sum_{r=1}^{n}p_r f(x_r)\tag{2.15}$$

By the definition of $A_{k,1}$

$$A_{k,1}=\frac{1}{(k-1)!}\sum_{s=1}^{n}t_{I_k,1}(s)\frac{p_s}{\alpha_{I_k,s}}f(x_s),$$

and therefore Lemma 2.1 insures (2.15).

If $(\mathscr{H}_0)$, $(\mathscr{H}_1)$ and $(\tilde{\mathscr{H}}_2)$ are satisfied, then we can prove as before, since the numbers $\alpha_{I_k,i}$ $(1\leq i\leq n)$ are integers, and since the discrete Jensen's inequality for mid-convex functions can be used in (2.14).

The proof of the theorem is complete. □

Proof of Theorem 2.2: It follows from Theorem 2.1 by applying Lemma 2.2 (e).

2.1.1 Examples and Mixed Symmetric Means Related to Theorem 2.1

In the following two examples $(\mathscr{H}_1)$ and either $(\mathscr{H}_2)$ or $(\tilde{\mathscr{H}}_2)$ will be assumed. They originated from [44].

Example 2.1 Let

$$I_2:=\left\{(i_1,i_2)\in\{1,\ldots,n\}^2\mid i_1|i_2\right\}.\tag{2.16}$$

The notation $i_1|i_2$ means that i_1 divides i_2. Since $i|i$ $(i=1,\ldots,n)$, $(\mathscr{H}_0)$ holds. In this case

$$\alpha_{I_2,i}=\left[\frac{n}{i}\right]+d(i),\quad i=1,\ldots,n,$$

where $\left[\frac{n}{i}\right]$ is the largest natural number that does not exceed $\frac{n}{i}$, and $d(i)$ denotes the number of positive divisors of i. By Theorem 2.1 , we have

$$f\left(\sum_{r=1}^{n} p_r x_r\right) \leq \sum_{(i_1,i_2)\in I_2} \left(\frac{p_{i_1}}{\left[\frac{n}{i_1}\right]+d(i_1)} + \frac{p_{i_2}}{\left[\frac{n}{i_2}\right]+d(i_2)}\right)$$
$$\cdot f\left(\frac{\frac{p_{i_1}}{\left[\frac{n}{i_1}\right]+d(i_1)}x_{i_1} + \frac{p_{i_2}}{\left[\frac{n}{i_2}\right]+d(i_2)}x_{i_2}}{\frac{p_{i_1}}{\left[\frac{n}{i_1}\right]+d(i_1)} + \frac{p_{i_2}}{\left[\frac{n}{i_2}\right]+d(i_2)}}\right) \leq \sum_{r=1}^{n} p_r f(x_r).$$

Example 2.2 Let $c_i \geq 1$ be an integer $(i = 1,\ldots,n)$, let $k := \sum_{i=1}^{n} c_i$, and let $I_k = P^{c_1,\ldots,c_n}$ consist of all sequences $(i_1,\ldots,i_k)$ in which the number of occurrences of $i \in \{1,\ldots,n\}$ is c_i $(i = 1,\ldots,n)$. Evidently, $(\mathscr{H}_0)$ is satisfied. A simple calculation shows that

$$I_{k-1} = \bigcup_{i=1}^{n} P^{c_1,\ldots,c_{i-1},c_i-1,c_{i+1},\ldots,c_n}, \quad \alpha_{I_k,i} = \frac{k!}{c_1!\ldots c_n!}c_i, \quad i = 1,\ldots,n,$$

and

$$t_{I_k,k-1}(i_1,\ldots,i_{k-1}) = k,$$
$$\text{if} \quad (i_1,\ldots,i_{k-1}) \in P^{c_1,\ldots,c_{i-1},c_i-1,c_{i+1},\ldots,c_n}, \quad i = 1,\ldots,n,$$

and

$$f\left(\sum_{r=1}^{n} p_r x_r\right) = A_{k,k}$$

$$= \frac{c_1!\ldots c_n!}{k!} \sum_{(i_1,\ldots,i_k)\in I_k} \left(\sum_{s=1}^{k} \frac{p_{i_s}}{c_{i_s}}\right) f\left(\frac{\sum_{s=1}^{k} \frac{p_{i_s}}{c_{i_s}} x_{i_s}}{\sum_{s=1}^{k} \frac{p_{i_s}}{c_{i_s}}}\right).$$

According to Theorem 2.1

$$f\left(\sum_{r=1}^{n} p_r x_r\right) \leq A_{k,k-1} \leq \sum_{r=1}^{n} p_r f(x_r),$$

where

$$A_{k,k-1} = \frac{1}{k-1} \sum_{i=1}^{n} (c_i - p_i) f\left(\frac{\sum_{r=1}^{n} p_r x_r - \frac{p_i}{c_i} x_i}{1 - \frac{p_i}{c_i}}\right).$$

To introduce some new means corresponding to the expressions (2.5) and (2.6), we need the following two additional hypotheses:

$(\mathscr{H}_3)$ Let $\mathbf{x} := (x_1, ..., x_n)$ and $\mathbf{p} := (p_1, ..., p_n)$ be positive n-tuples such that $\sum_{i=1}^{n} p_i = 1$.

$(\tilde{\mathscr{H}_3})$ Let $J \subset \mathbb{R}$ be an interval, $\mathbf{x} := (x_1, ..., x_n) \in J^n$, let $\mathbf{p} := (p_1, ..., p_n)$ be a positive n-tuples such that $\sum_{i=1}^{n} p_i = 1$, and let $h,\ g : J \to \mathbb{R}$ be continuous and strictly monotone functions.

Assume $(\mathscr{H}_0)$ and $(\mathscr{H}_3)$. The power means of order $r \in \mathbb{R}$ corresponding to $\mathbf{i}^l := (i_1, \ldots, i_l) \in I_l$ $(l = 1, \ldots, k)$ are given as:

$$M_r(I_k, \mathbf{i}^l) = M_r(I_k, \mathbf{i}^l, \mathbf{x}, \mathbf{p}) := \begin{cases} \left(\dfrac{\sum\limits_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}} x_{i_s}^r}{\sum\limits_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}}} \right)^{\frac{1}{r}}, & r \neq 0, \\ \left(\prod\limits_{s=1}^{l} x_{i_s}^{\frac{p_{i_s}}{\alpha_{I_k,i_s}}} \right)^{\frac{1}{\sum\limits_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}}}}, & r = 0. \end{cases}$$

For $\eta, \gamma \in \mathbb{R}$ we introduce the mixed symmetric means with positive weights as follows:

$$M^1_{\eta,\gamma}(I_k, k, \mathbf{x}, \mathbf{p}) := \begin{cases} \left[\sum\limits_{\mathbf{i}^k = (i_1,...,i_k) \in I_k} \left(\sum\limits_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} \right) \left(M_\gamma(I_k, \mathbf{i}^k) \right)^\eta \right]^{\frac{1}{\eta}}, & \eta \neq 0, \\ \prod\limits_{\mathbf{i}^k = (i_1,...,i_k) \in I_k} \left(M_\gamma(I_k, \mathbf{i}^k) \right)^{\left(\sum\limits_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} \right)}, & \eta = 0, \end{cases} \tag{2.17}$$

and for $k-1 \geq l \geq 1$

$$\begin{aligned} &M^1_{\eta,\gamma}(I_k, l, \mathbf{x}, \mathbf{p}) \\ &:= \begin{cases} \left[\frac{1}{(k-1)\ldots l} \sum\limits_{\mathbf{i}^l = (i_1,...,i_l) \in I_l} t_{I_k,l}(\mathbf{i}^l) \left(\sum\limits_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}} \right) \left(M_\gamma(I_k, \mathbf{i}^l, \mathbf{p}) \right)^\eta \right]^{\frac{1}{\eta}}, & \eta \neq 0, \\ \left[\prod\limits_{\mathbf{i}^l = (i_1,...,i_l) \in I_l} \left(M_\gamma(I_k, \mathbf{i}^l, \mathbf{p}) \right)^{t_{I_k,l}(\mathbf{i}^l) \left(\sum\limits_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}} \right)} \right]^{\frac{1}{(k-1)\ldots l}}, & \eta = 0. \end{cases} \end{aligned} \tag{2.18}$$

We deduce the monotonicity of these means from Theorem 2.1 as follows.

Corollary 2.1 ([37]) *Assume $(\mathscr{H}_0)$ and $(\mathscr{H}_3)$. Let $\eta,\ \gamma \in \mathbb{R}$ such that $\eta \leq \gamma$. Then*

$$M_\gamma(\mathbf{x}, \mathbf{p}) = M^1_{\gamma,\eta}(I_k, 1, \mathbf{x}, \mathbf{p}) \geq \ldots \geq M^1_{\gamma,\eta}(I_k, k, \mathbf{x}, \mathbf{p}) \geq M_\eta(\mathbf{x}, \mathbf{p}), \tag{2.19}$$

and

$$M_\eta(\mathbf{x}, \mathbf{p}) = M^1_{\eta,\gamma}(I_k, 1, \mathbf{x}, \mathbf{p}) \leq .... \leq M^1_{\eta,\gamma}(I_k, k, \mathbf{x}, \mathbf{p}) \leq M_\gamma(\mathbf{x}, \mathbf{p}), \tag{2.20}$$

where $M_r(\mathbf{x}, \mathbf{p})$ is the power mean of order $r \in \mathbb{R}$ (see 1.44).

Proof. Assume η, $\gamma \neq 0$. To obtain (2.19), we can apply Theorem 2.1 to the function $f(x) = x^{\frac{\gamma}{\eta}}$ $(x > 0)$ and the n-tuples $(x_1^{\eta}, \ldots, x_n^{\eta})$ to get the analogue of (2.7) and raising the power $\frac{1}{\gamma}$. (2.20) can be proved in a similar way by using $f(x) = x^{\frac{\eta}{\gamma}}$ $(x > 0)$ and $(x_1^{\gamma}, \ldots, x_n^{\gamma})$ and raising the power $\frac{1}{\eta}$.
When $\eta = 0$ or $\gamma = 0$, we get the required results by taking limit. □

Assume $(\mathscr{H}_0)$ and $(\tilde{\mathscr{H}}_3)$. Then we define the quasi-arithmetic means with respect to (2.5) and (2.6) as follows:

$$M_{h,g}^{1}(I_k, k, \mathbf{x}, \mathbf{p}) := h^{-1}\left(\sum_{(i_1,\ldots,i_k)\in I_k}\left(\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right) h\circ g^{-1}\left(\frac{\sum\limits_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}g(x_{i_s})}{\sum\limits_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}}\right)\right), \tag{2.21}$$

and for $k-1 \geq l \geq 1$

$$\begin{aligned} &M_{h,g}^{1}(I_k, k, \mathbf{x}, \mathbf{p}) \\ &:= h^{-1}\left(\frac{1}{(k-1)\ldots l}\sum_{\mathbf{i}^l=(i_1,\ldots,i_l)\in I_l} t_{I_k,l}(\mathbf{i}^l)\left(\sum_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right) h\circ g^{-1}\left(\frac{\sum\limits_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_k,i_s}}g(x_{i_s})}{\sum\limits_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_k,i_s}}}\right)\right). \end{aligned} \tag{2.22}$$

The monotonicity of these generalized means is obtained in the next corollary.

Corollary 2.2 ([37]) *Assume $(\mathscr{H}_0)$ and $(\tilde{\mathscr{H}}_3)$. Then*

$$M_h(\mathbf{x}, \mathbf{p}) = M_{h,g}^{1}(I_k, 1, \mathbf{x}, \mathbf{p}) \geq \ldots \geq M_{h,g}^{1}(I_k, k, \mathbf{x}, \mathbf{p}) \geq M_g(\mathbf{x}, \mathbf{p}), \tag{2.23}$$

if either $h\circ g^{-1}$ is convex and h is increasing or $h\circ g^{-1}$ is concave and h is decreasing;

$$M_g(\mathbf{x}, \mathbf{p}) = M_{g,h}^{1}(I_k, 1, \mathbf{x}, \mathbf{p}) \leq \ldots \leq M_{g,h}^{1}(I_k, k, \mathbf{x}, \mathbf{p}) \leq M_h(\mathbf{x}, \mathbf{p}), \tag{2.24}$$

if either $g\circ h^{-1}$ is convex and g is decreasing or $g\circ h^{-1}$ is concave and g is increasing. $M_h(\mathbf{x}, \mathbf{p})$ and $M_g(\mathbf{x}, \mathbf{p})$ are the quasi-arithmetic means associated to h and g, respectively (see (1.43).

Proof. First, we can apply Theorem 2.1 to the function $h\circ g^{-1}$ and the n-tuples $(g(x_1), \ldots, g(x_n))$, then we can apply h^{-1} to the inequality coming from (2.7). This gives (2.23). A similar argument gives (2.24): $g\circ h^{-1}$, $(h(x_1), \ldots, h(x_n))$ and g^{-1}can be used. □

Based on Examples 2.1 and 2.2, we can generate concrete means (see [37]). We just consider Example 2.1.

Example 2.3 ([37]) Let I_2 be the set defined in (2.16).

If $(\mathscr{H}_3)$ holds, then (2.17) gives for $\eta, \gamma \in \mathbb{R}$

$$M^1_{\eta,\gamma}(I_2, 2, \mathbf{x}, \mathbf{p}) = \begin{cases} \left(\sum\limits_{\mathbf{i}^2=(i_1,i_2)\in I_2} \left(\sum\limits_{s=1}^{2} \frac{p_{i_s}}{\left[\frac{n}{i_s}\right]+d(i_s)} \right) \left(M_\gamma(I_2, \mathbf{i}^2)\right)^\eta \right)^{\frac{1}{\eta}}, & \eta \neq 0 \\ \prod\limits_{\mathbf{i}^2=(i_1,i_2)\in I_2} \left(M_\gamma(I_2, \mathbf{i}^2)\right)^{\left(\sum\limits_{s=1}^{2} \frac{p_{i_s}}{\left[\frac{n}{i_s}\right]+d(i_s)} \right)}, & \eta = 0 \end{cases},$$

while if $(\tilde{\mathscr{H}}_3)$ is satisfied, then (2.21) gives

$$M^1_{h,g}(I_2, 2, \mathbf{x}, \mathbf{p})$$

$$= h^{-1} \left(\sum_{(i_1,i_2)\in I_2} \left(\sum_{s=1}^{2} \frac{p_{i_s}}{\left[\frac{n}{i_s}\right]+d(i_s)} \right) h \circ g^{-1} \left(\frac{\sum\limits_{s=1}^{2} \frac{p_{i_s}}{\left[\frac{n}{i_s}\right]+d(i_s)} g(x_{i_s})}{\sum\limits_{s=1}^{2} \frac{p_{i_s}}{\left[\frac{n}{i_s}\right]+d(i_s)}} \right) \right).$$

2.1.2 Examples and Mixed Symmetric Means Related to Theorem 2.2

Throughout the following four examples $(\mathscr{H}_1)$ and either $(\mathscr{H}_2)$ or $(\tilde{\mathscr{H}}_2)$ will be assumed. They come from [44].

The first example shows that Theorem 2.2 contains Theorem 1.8.

Example 2.4 Let

$$I_k := \left\{ (i_1, \ldots, i_k) \in \{1, \ldots, n\}^k \mid i_1 < \ldots < i_k \right\}, \quad 1 \le k \le n.$$

Then $\alpha_{I_n,i} = 1$ $(i = 1, \ldots, n)$ ensuring $(\mathscr{H}_0)$ with $k = n$. It is easy to check that $T_k(I_k) = I_{k-1}$ $(k = 2, \ldots, n)$, $|I_k| = \binom{n}{k}$ $(k = 1, \ldots, n)$, and for every $k = 2, \ldots, n$

$$\left| H_{I_k}(j_1, \ldots, j_{k-1}) \right| = n - (k-1), \quad (j_1, \ldots, j_{k-1}) \in I_{k-1},$$

and therefore, thanks to Theorem 2.2,

$$A_{k,k} = \frac{1}{\binom{n-1}{k-1}} \sum_{1\le i_1<\ldots<i_k\le n} \left(\sum_{s=1}^{k} p_{i_s} \right) f\left(\frac{\sum\limits_{s=1}^{k} p_{i_s} x_{i_s}}{\sum\limits_{s=1}^{k} p_{i_s}} \right), \quad k = 1, \ldots, n.$$

and

$$f\left(\sum_{r=1}^{n} p_r x_r \right) \le A_{k,k} \le A_{k-1,k-1} \le \ldots \le A_{2,2} \le A_{1,1} = \sum_{r=1}^{n} p_r f(x_r).$$

The next example illustrates that Theorem 1.10 is a special case of Theorem 2.2.

Example 2.5 Let

$$I_k := \left\{(i_1,\ldots,i_k) \in \{1,\ldots,n\}^k \mid i_1 \le \ldots \le i_k\right\}, \quad k \ge 1.$$

Obviously, $\alpha_{I_k,i} \ge 1$ $(i = 1,\ldots,n)$, and therefore $(\mathscr{H}_0)$ is satisfied. It is not hard to see that $T_k(I_k) = I_{k-1}$ $(k = 2,\ldots)$, $|I_k| = \binom{n+k-1}{k}$ $(k = 1,\ldots)$, and for each $l = 2,\ldots,k$

$$\left|H_{I_l}(j_1,\ldots,j_{l-1})\right| = n, \quad (j_1,\ldots,j_{l-1}) \in I_{l-1}.$$

Consequently, by applying Theorem 2.2, we deduce that

$$A_{k,k} = \frac{1}{\binom{n+k-1}{k-1}} \sum_{1\le i_1\le\ldots\le i_k\le n} \left(\sum_{s=1}^{k} p_{i_s}\right) f\left(\frac{\sum\limits_{s=1}^{k} p_{i_s}x_{i_s}}{\sum\limits_{s=1}^{k} p_{i_s}}\right), \quad k \ge 1,$$

and

$$f\left(\sum_{r=1}^{n} p_r x_r\right) \le \ldots \le A_{k,k} \le \ldots \le A_{k,1} = \sum_{r=1}^{n} p_r f(x_r).$$

For $p_1 = \ldots = p_n = \frac{1}{n}$ Theorem 1.11 is contained in the next example.

Example 2.6 Let

$$I_k := \{1,\ldots,n\}^k, \quad k \ge 1.$$

Trivially, $\alpha_{I_k,i} \ge 1$ $(i = 1,\ldots,n)$, hence $(\mathscr{H}_0)$ holds. It is evident that $T_k(I_k) = I_{k-1}$ $(k = 2,\ldots)$, $|I_k| = n^k$ $(k = 1,\ldots)$, and for every $l = 2,\ldots,k$

$$\left|H_{I_l}(j_1,\ldots,j_{l-1})\right| = n^l, \quad (j_1,\ldots,j_{l-1}) \in I_{l-1},$$

and so Theorem 2.2 leads to

$$A_{k,k} = \frac{1}{kn^{k-1}} \sum_{(i_1,\ldots,i_k)\in I_k} \left(\sum_{s=1}^{k} p_{i_s}\right) f\left(\frac{\sum\limits_{s=1}^{k} p_{i_s}x_{i_s}}{\sum\limits_{s=1}^{k} p_{i_s}}\right), \quad k \ge 1,$$

and

$$f\left(\sum_{r=1}^{n} p_r x_r\right) \le \ldots \le A_{k,k} \le \ldots \le A_{1,1} = \sum_{r=1}^{n} p_r f(x_r), \quad k \ge 1.$$

Especially, for $p_1 = \ldots = p_n = \frac{1}{n}$ we find from Lemma 2.2 (f) that

$$A_{k,k} = \frac{1}{n^k} \sum_{(i_1,\ldots,i_k)\in I_k} f\left(\frac{x_{i_1} + \ldots x_{i_k}}{k}\right), \quad k = 1,\ldots,n.$$

The final example is the next:

Example 2.7 For $1 \leq k \leq n$ let I_k consist of all sequences $(i_1, \ldots, i_k)$ of k distinct numbers from $\{1, \ldots, n\}$. Then $\alpha_{I_n,i} \geq 1$ $(i = 1, \ldots, n)$, hence $(\mathscr{H}_0)$ is valid. It is immediate that $T_k(I_k) = I_{k-1}$ $(k = 2, \ldots, n)$, $|I_k| = n(n-1)\ldots(n-k+1)$ $(k = 1, \ldots, n)$, and for each $k = 2, \ldots, n$

$$\left|H_{I_k}(j_1, \ldots, j_{k-1})\right| = (n-(k-1))\,k, \quad (j_1, \ldots, j_{k-1}) \in I_{k-1}.$$

and from them, on account of Theorem 2.2, follows

$$A_{k,k} = \frac{n}{kn(n-1)\ldots(n-k+1)}$$
$$\cdot \sum_{(i_1,\ldots,i_k)\in I_k} \left(\sum_{s=1}^{k} p_{i_s}\right) f\left(\frac{\sum\limits_{s=1}^{k} p_{i_s} x_{i_s}}{\sum\limits_{s=1}^{k} p_{i_s}}\right), \quad k = 1, \ldots, n$$

and

$$f\left(\sum_{r=1}^{n} p_r x_r\right) \leq A_{n,n} \leq \ldots \leq A_{k,k} \leq \ldots \leq A_{1,1} = \sum_{r=1}^{n} p_r f(x_r).$$

If we set $p_1 = \ldots = p_n = \frac{1}{n}$, then

$$A_{k,k} = \frac{1}{n(n-1)\ldots(n-k+1)} \sum_{(i_1,\ldots,i_k)\in I_k} f\left(\frac{x_{i_1} + \ldots x_{i_k}}{k}\right), \quad k = 1, \ldots, n.$$

Now we introduce some means corresponding to (2.8).

Assume $(\mathscr{H}_0)$, $(\mathscr{H}_3)$, and suppose also that $|H_{I_l}(j_1, ..., j_{l-1})| = \beta_{l-1}$ for any $(j_1, ..., j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$.

The power means of order $r \in \mathbb{R}$ corresponding to $\mathbf{i}^l := (i_1, \ldots, i_l) \in I_l$ $(l = 1, \ldots, k)$ has the form

$$M_r(I_l, \mathbf{i}^l) = M_r(I_k, \mathbf{i}^l) = M_r(I_l, \mathbf{i}^l, \mathbf{x}, \mathbf{p}) := \begin{cases} \left(\dfrac{\sum\limits_{s=1}^{l} p_{i_s} x_{i_s}^r}{\sum\limits_{s=1}^{l} p_{i_s}}\right)^{\frac{1}{r}}, & r \neq 0, \\ \left(\prod\limits_{s=1}^{l} x_{i_s}{}^{p_{i_s}}\right)^{\frac{1}{\sum_{s=1}^{l} p_{i_s}}}, & r = 0. \end{cases}$$

Now, for $\eta, \gamma \in \mathbb{R}$ and $k \geq l \geq 1$ we introduce the mixed symmetric means with positive weights related to (2.8) as follows:

$$M^2_{\eta,\gamma}(I_l, \mathbf{x}, \mathbf{p}) := \begin{cases} \left[\frac{n}{l|I_l|} \sum\limits_{\mathbf{i}^l=(i_1,\ldots,i_l)\in I_l} \left(\sum\limits_{s=1}^{l} p_{i_s}\right) \left(M_\gamma\left(I_l, \mathbf{i}^l\right)\right)^{\eta}\right]^{\frac{1}{\eta}}, & \eta \neq 0, \\ \left[\prod\limits_{\mathbf{i}^l=(i_1,\ldots,i_l)\in I_l} \left(M_\gamma(I_l, \mathbf{i}^l)\right)^{\left(\sum_{s=1}^{l} p_{i_s}\right)}\right]^{\frac{n}{l|I_l|}}, & \eta = 0. \end{cases} \tag{2.25}$$

Corollary 2.3 ([37]) *Assume* ($\mathscr{H}_0$) *and* ($\mathscr{H}_3$). *Suppose further that* $|H_{I_l}(j_1,\ldots,j_{l-1})| = \beta_{l-1}$ *for any* $(j_1,\ldots,j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$. *Let* η, $\gamma \in \mathbb{R}$ *such that* $\eta \leq \gamma$. *Then*

$$M_\gamma(\mathbf{x},\mathbf{p}) = M^2_{\gamma,\eta}(I_1,\mathbf{x},\mathbf{p}) \geq \ldots \geq M^2_{\gamma,\eta}(I_k,\mathbf{x},\mathbf{p}) \geq M_\eta(\mathbf{x},\mathbf{p}),$$

and

$$M_\eta(\mathbf{x},\mathbf{p}) = M^2_{\eta,\gamma}(I_1,\mathbf{x},\mathbf{p}) \leq \ldots \leq M^2_{\eta,\gamma}(I_k,\mathbf{x},\mathbf{p}) \leq M_\gamma(\mathbf{x},\mathbf{p}),$$

where $M_r(\mathbf{x},\mathbf{p})$ *is the power mean of order* $r \in \mathbb{R}$ *(see 1.44).*

Proof. Similar to the proof of Corollary 2.1. □

Assume ($\mathscr{H}_0$) and ($\tilde{\mathscr{H}}_3$). Suppose further that $|H_{I_l}(j_1,\ldots,j_{l-1})| = \beta_{l-1}$ for any $(j_1,\ldots,j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$. We define for $k \geq l \geq 1$ the quasi-arithmetic means with respect to (2.8) as follows:

$$M^2_{h,g}(I_l,\mathbf{x},\mathbf{p}) := h^{-1}\left(\frac{n}{l\,|I_l|}\sum_{(i_1,\ldots,i_l)\in I_l}\left(\sum_{s=1}^{l} p_{i_s}\right) h\circ g^{-1}\left(\frac{\sum\limits_{s=1}^{l} p_{i_s} g(x_{i_s})}{\sum\limits_{s=1}^{l} p_{i_s}}\right)\right). \tag{2.26}$$

Corollary 2.4 ([37]) *Assume* ($\mathscr{H}_0$) *and* ($\tilde{\mathscr{H}}_3$). *Suppose further that* $|H_{I_l}(j_1,\ldots,j_{l-1})| = \beta_{l-1}$ *for any* $(j_1,\ldots,j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$. *Then*

$$M_h(\mathbf{x},\mathbf{p}) = M^2_{h,g}(I_1,\mathbf{x},\mathbf{p}) \geq \ldots \geq M^2_{h,g}(I_k,\mathbf{x},\mathbf{p}) \geq M_g(\mathbf{x},\mathbf{p}),$$

where either $h\circ g^{-1}$ *is convex and* h *is increasing or* $h\circ g^{-1}$ *is concave and* h *is decreasing;*

$$M_g(\mathbf{x},\mathbf{p}) = M^2_{g,h}(I_1,\mathbf{x},\mathbf{p}) \leq \ldots \leq M^2_{g,h}(I_k,\mathbf{x},\mathbf{p}) \leq M_h(\mathbf{x},\mathbf{p}),$$

where either $g\circ h^{-1}$ *is convex and* g *is decreasing or* $g\circ h^{-1}$ *is concave and* g *is increasing.* $M_h(\mathbf{x},\mathbf{p})$ *and* $M_g(\mathbf{x},\mathbf{p})$ *are the quasi-arithmetic means associated to* h *and* g*, respectively (see (1.43).*

Proof. Similar to the proof of Corollary 2.2. □

We illustrate these means with an example coming from Example 2.6.

Example 2.8 ([37]) Consider the set I_k defined in Example 2.6.

If ($\mathscr{H}_3$) holds, then (2.25) leads to

$$M^2_{\eta,\gamma}(I_k,\mathbf{x},\mathbf{p}) = \begin{cases} \left[\frac{1}{kn^{k-1}}\sum\limits_{\mathbf{i}^k=(i_1,\ldots,i_k)\in I_k}\left(\sum\limits_{s=1}^{k} p_{i_s}\right)\left(M_\gamma(I_k,\mathbf{i}^k)\right)^\eta\right]^{\frac{1}{\eta}}, & \eta \neq 0, \\ \left[\prod\limits_{\mathbf{i}^k=(i_1,\ldots,i_k)\in I_k}\left(M_\gamma(I_k,\mathbf{i}^k)\right)^{\left(\sum\limits_{s=1}^{k} p_{i_s}\right)}\right]^{\frac{1}{kn^{k-1}}}, & \eta = 0, \end{cases}$$

and if ($\tilde{\mathscr{H}}_3$) is satisfied, then (2.26) gives

$$M^2_{h,g}(I_k,\mathbf{x},\mathbf{p}) = h^{-1}\left(\frac{1}{kn^{k-1}}\sum_{\mathbf{ik}=(i_1,\ldots,i_k)\in I_k}\left(\sum_{s=1}^{k} p_{i_s}\right) h\circ g^{-1}\left(\frac{\sum\limits_{s=1}^{k} p_{i_s} g(x_{i_s})}{\sum\limits_{s=1}^{k} p_{i_s}}\right)\right).$$

2.2 A New Treatment of Discrete Jensen's Inequality

The aim of this chapter is to give such a generalization of Theorem 2.1 and Theorem 2.2 which shows the essence the methods employed in a lot of known results and unifies them.

In this chapter we also use the conditions $(\mathscr{H}_1)$, $(\mathscr{H}_2)$ and $(\tilde{\mathscr{H}_2})$:

$(\mathscr{H}_1)$ Let V be a real vector space, $C \subset V$ be a convex set, $\mathbf{x} := (x_1, ..., x_n) \in C^n$, and let $\mathbf{p} := (p_1, ..., p_n)$ be a positive n-tuples such that $\sum\limits_{i=1}^{n} p_i = 1$.

$(\mathscr{H}_2)$ Let $f : C \to \mathbb{R}$ be a convex function.

$(\tilde{\mathscr{H}_2})$ Let $f : C \to \mathbb{R}$ be a mid-convex function, and $p_1, \ldots, p_n$ be rational numbers.

We need the following two additional hypotheses:

$(\mathscr{H}_4)$ Let $S_1, \ldots, S_n$ be finite, pairwise disjoint and nonempty sets, let

$$S := \bigcup_{j=1}^{n} S_j,$$

and let c be a function from S into $\mathbb{R}$ such that

$$c(s) > 0, \quad s \in S, \quad \text{and} \quad \sum_{s \in S_j} c(s) = 1, \quad j = 1, \ldots, n. \tag{2.27}$$

Let the function $\tau : S \to \{1, \ldots, n\}$ be defined by

$$\tau(s) := j, \quad \text{if} \quad s \in S_j.$$

$(\mathscr{H}_5)$ Suppose $\mathscr{A} \subset P(S)$ is a partition of S into pairwise disjoint and nonempty sets. Let

$$k := \max\{|A| \mid A \in \mathscr{A}\},$$

and let

$$\mathscr{A}_l := \{A \in \mathscr{A} \mid |A| = l\}, \quad l = 1, \ldots, k.$$

Then $\mathscr{A}_l$ $(l = 1, \ldots, k-1)$ may be the empty set, and $|S| = \sum\limits_{l=1}^{k} l\,|\mathscr{A}_l|$.

Hereinafter, the empty sum is taken to be zero.

Theorem 2.3 ([32]) *(a) Assume $(\mathscr{H}_1)$, $(\mathscr{H}_2)$, $(\mathscr{H}_4)$ and $(\mathscr{H}_5)$. Then*

$$f\left(\sum_{j=1}^{n} p_j x_j\right) \leq N_k \leq N_{k-1} \leq \ldots \leq N_2 \leq N_1 = \sum_{j=1}^{n} p_j f(x_j), \tag{2.28}$$

where

$$N_k = N_k(S, c, \mathscr{A}, \mathbf{x}, \mathbf{p})$$

$$:= \sum_{l=1}^{k} \left(\sum_{A \in \mathscr{A}_l} \left(\left(\sum_{s \in A} c(s) p_{\tau(s)} \right) f\left(\frac{\sum\limits_{s \in A} c(s) p_{\tau(s)} x_{\tau(s)}}{\sum\limits_{s \in A} c(s) p_{\tau(s)}} \right) \right) \right), \tag{2.29}$$

and for every $1 \leq m \leq k-1$ *the number* N_{k-m} *is given by*

$$N_{k-m} = N_{k-m}(S, c, \mathscr{A}, \mathbf{x}, \mathbf{p})$$
$$:= \sum_{l=1}^{m}\left(\sum_{A \in \mathscr{A}_l}\left(\sum_{s \in A} c(s) p_{\tau(s)} f(x_{\tau(s)})\right)\right) + \sum_{l=m+1}^{k}\left(\frac{m!}{(l-1)\dots(l-m)}\right.$$
$$\left.\cdot \sum_{A \in \mathscr{A}_l}\left(\sum_{B \in P_{l-m}(A)}\left(\left(\sum_{s \in B} c(s) p_{\tau(s)}\right) f\left(\frac{\sum_{s \in B} c(s) p_{\tau(s)} x_{\tau(s)}}{\sum_{s \in B} c(s) p_{\tau(s)}}\right)\right)\right)\right). \tag{2.30}$$

(b) Suppose $(\mathscr{H}_1)$, $(\tilde{\mathscr{H}}_2)$, $(\mathscr{H}_4)$ *and* $(\mathscr{H}_5)$*. If the numbers* $c(s)$ $(s \in S)$ *are rational, then the inequality (2.28) remains true.*

Under the conditions of Theorem 2.3

$$\Upsilon_5(f) = \Upsilon_5(f, m, l, \mathbf{x}, \mathbf{p}) := N_m - N_l \geq 0, \quad 1 \leq m < l \leq k,$$
$$\Upsilon_6(f) = \Upsilon_6(f, l, \mathbf{x}, \mathbf{p}) := N_l - f\left(\sum_{j=1}^{n} p_j x_j\right) \geq 0, \quad 1 \leq l \leq k.$$

The following application of Theorem 2.3 leads to a generalization of Theorem 2.1.

Theorem 2.4 *Let* $n \geq 1$ *and* $k \geq 1$ *be fixed integers, and let* $I_k \subset \{1, \dots, n\}^k$ *such that*

$$\alpha_{I_k,i} \geq 1, \quad 1 \leq i \leq n,$$

where $\alpha_{I_k,i}$ *means the number of occurrences of* i *in the sequences* $(i_1, \dots, i_k) \in I_k$*. For* $j = 1, \dots, n$ *we introduce the sets*

$$S_j := \{((i_1, \dots, i_k), l) \mid (i_1, \dots, i_k) \in I_k, \quad 1 \leq l \leq k, \quad i_l = j\}. \tag{2.31}$$

Let c *be a positive function on* $S := \bigcup_{j=1}^{n} S_j$ *such that*

$$\sum_{((i_1, \dots, i_k), l) \in S_j} c((i_1, \dots, i_k), l) = 1, \quad j = 1, \dots, n. \tag{2.32}$$

(a) Assume that $(\mathscr{H}_1)$ *and* $(\mathscr{H}_2)$ *are satisfied. Then*

$$f\left(\sum_{j=1}^{n} p_j x_j\right) \leq N_k \leq N_{k-1} \leq \dots \leq N_2 \leq N_1 = \sum_{j=1}^{n} p_j f(x_j), \tag{2.33}$$

where the numbers N_{k-m} $(0 \leq m \leq k-1)$ *can be written in the following forms:*

$$N_k := \sum_{(i_1, \dots, i_k) \in I_k}\left(\left(\sum_{l=1}^{k} c((i_1, \dots, i_k), l) p_{i_l}\right) f\left(\frac{\sum_{l=1}^{k} c((i_1, \dots, i_k), l) p_{i_l} x_{i_l}}{\sum_{l=1}^{k} c((i_1, \dots, i_k), l) p_{i_l}}\right)\right), \tag{2.34}$$

and for every $1 \le m \le k-1$

$$N_{k-m} := \frac{m!}{(k-1)\dots(k-m)} \sum_{(i_1,\dots,i_k)\in I_k} \left(\sum_{1\le l_1<\dots<l_{k-m}\le k} \right.$$

$$\left. \left(\left(\sum_{j=1}^{k-m} c\left((i_1,\dots,i_k),l_j\right) p_{i_{l_j}} \right) f \left(\frac{\sum\limits_{j=1}^{k-m} c\left((i_1,\dots,i_k),l_j\right) p_{i_{l_j}} x_{i_{l_j}}}{\sum\limits_{j=1}^{k-m} c\left((i_1,\dots,i_k),l_j\right) p_{i_{l_j}}} \right) \right) \right). \tag{2.35}$$

(b) If ($\mathscr{H}_1$) *and* ($\tilde{\mathscr{H}}_2$) *are satisfied and the numbers* $c\left((i_1,\dots,i_k),l\right)$ $\left(\left((i_1,\dots,i_k),l\right)\in S\right)$ *are rational, then the inequality (2.33) remains true.*

An immediate consequence of the previous result is Theorem 2.1: choosing

$$c\left((i_1,\dots,i_k),l\right) = \frac{1}{|S_j|} = \frac{1}{\alpha_{I_k,j}} \quad \text{if} \quad \left((i_1,\dots,i_k),l\right)\in S_j,$$

we can check easily that the inequalities (2.33) corresponds to the inequalities (2.7).

By using Theorem 2.2, some extensions of Theorem 1.8 and Theorem 1.10 have been obtained in Example 2.4 and in Example 2.5. Theorem 2.4 generalizes all these results: apply it to either

$$I_k := \left\{ (i_1,\dots,i_k)\in\{1,\dots,n\}^k \mid i_1<\dots<i_k \right\}, \quad 1\le k\le n,$$

or

$$I_k := \left\{ (i_1,\dots,i_k)\in\{1,\dots,n\}^k \mid i_1\le\dots\le i_k \right\}, \quad 1\le k.$$

We confine here our attention to the proof of Theorem 2.3 (a), so we shall suppose the conditions ($\mathscr{H}_1$), ($\mathscr{H}_2$) and ($\mathscr{H}_4$), ($\mathscr{H}_5$). It is easy to verify that the following results and their proofs remain valid under the hypotheses of Theorem 2.3 (b) (Theorem 1.5 (b) can be applied in place of Theorem 1.5 (a)).

Lemma 2.4 *If* ($\mathscr{H}_1$), ($\mathscr{H}_2$), ($\mathscr{H}_4$) *and* ($\mathscr{H}_5$) *are satisfied, then*

$$f\left(\sum_{j=1}^{n} p_j x_j\right) \le N_k.$$

Proof. Since $\{S_1,\dots,S_n\}$ and $\mathscr{A}$ are partitions of S into pairwise disjoint and nonempty sets, it comes from the definition of the function τ that

$$\sum_{j=1}^{n} p_j x_j = \sum_{j=1}^{n}\left(\sum_{s\in S_j} c(s)\right) p_j x_j = \sum_{A\in\mathscr{A}}\left(\sum_{s\in A} c(s) p_{\tau(s)} x_{\tau(s)}\right)$$

$$= \sum_{l=1}^{k} \left(\sum_{A \in \mathscr{A}_l} \left(\sum_{s \in A} c(s) p_{\tau(s)} x_{\tau(s)} \right) \right). \tag{2.36}$$

Therefore, recalling that the numbers p_j $(j = 1, \ldots, n)$ and $c(s)$ $(s \in S)$ are positive, we have

$$\sum_{j=1}^{n} p_j x_j = \sum_{l=1}^{k} \left(\sum_{A \in \mathscr{A}_l} \left(\left(\sum_{s \in A} c(s) p_{\tau(s)} \right) \frac{\sum\limits_{s \in A} c(s) p_{\tau(s)} x_{\tau(s)}}{\sum\limits_{s \in A} c(s) p_{\tau(s)}} \right) \right). \tag{2.37}$$

Similar reasoning as above leads to

$$\sum_{l=1}^{k} \left(\sum_{A \in \mathscr{A}_l} \left(\sum_{s \in A} c(s) p_{\tau(s)} \right) \right) = \sum_{A \in \mathscr{A}} \left(\sum_{s \in A} c(s) p_{\tau(s)} \right) = \sum_{j=1}^{n} \left(\sum_{s \in S_j} c(s) \right) p_j.$$

Then, using $\sum\limits_{j=1}^{n} p_j = 1$ and (2.27) we get

$$\sum_{l=1}^{k} \left(\sum_{A \in \mathscr{A}_l} \left(\sum_{s \in A} c(s) p_{\tau(s)} \right) \right) = 1. \tag{2.38}$$

It now follows from (2.37), (2.38) and Theorem 1.5 (a) that

$$f\left(\sum_{j=1}^{n} p_j x_j \right) \le \sum_{l=1}^{k} \left(\sum_{A \in \mathscr{A}_l} \left(\left(\sum_{s \in A} c(s) p_{\tau(s)} \right) f\left(\frac{\sum\limits_{s \in A} c(s) p_{\tau(s)} x_{\tau(s)}}{\sum\limits_{s \in A} c(s) p_{\tau(s)}} \right) \right) \right),$$

as was being claimed. □

Lemma 2.5 *If* $(\mathscr{H}_1)$, $(\mathscr{H}_2)$, $(\mathscr{H}_4)$ *and* $(\mathscr{H}_5)$ *are satisfied, and* $A \in \mathscr{A}_l$, *where* $2 \le l \le k$, *then*

$$f\left(\frac{\sum\limits_{s \in A} c(s) p_{\tau(s)} x_{\tau(s)}}{\sum\limits_{s \in A} c(s) p_{\tau(s)}} \right) \le \frac{1}{(l-1) \sum\limits_{s \in A} c(s) p_{\tau(s)}}$$

$$\cdot \sum_{B \in P_{l-1}(A)} \left(\left(\sum_{s \in B} c(s) p_{\tau(s)} \right) f\left(\frac{\sum\limits_{s \in B} c(s) p_{\tau(s)} x_{\tau(s)}}{\sum\limits_{s \in B} c(s) p_{\tau(s)}} \right) \right).$$

Proof. A simple calculation confirms that

$$\frac{\sum\limits_{s \in A} c(s) p_{\tau(s)} x_{\tau(s)}}{\sum\limits_{s \in A} c(s) p_{\tau(s)}} = \sum_{B \in P_{l-1}(A)} \left(\frac{\sum\limits_{s \in B} c(s) p_{\tau(s)} x_{\tau(s)}}{(l-1) \sum\limits_{s \in A} c(s) p_{\tau(s)}} \right)$$

$$= \sum_{B \in P_{l-1}(A)} \left(\frac{\sum\limits_{s \in B} c(s) p_{\tau(s)}}{(l-1) \sum\limits_{s \in A} c(s) p_{\tau(s)}} \cdot \frac{\sum\limits_{s \in B} c(s) p_{\tau(s)} x_{\tau(s)}}{\sum\limits_{s \in B} c(s) p_{\tau(s)}} \right). \tag{2.39}$$

Since

$$\sum_{B\in P_{l-1}(A)}\left(\frac{\sum\limits_{s\in B}c(s)p_{\tau(s)}}{(l-1)\sum\limits_{s\in A}c(s)p_{\tau(s)}}\right)=1,$$

the result follows from (2.39) and Theorem 1.5 (a).

The proof is complete. □

Proof of Theorem 2.3 (a).

Proof. The definition of the number N_1 shows that

$$N_1=\sum_{l=1}^{k-1}\left(\sum_{A\in\mathscr{A}_l}\left(\sum_{s\in A}c(s)p_{\tau(s)}f(x_{\tau(s)})\right)\right)+\frac{(k-1)!}{(k-1)\dots 1}$$

$$\cdot\sum_{A\in\mathscr{A}_k}\left(\sum_{B\in P_1(A)}\left(\left(\sum_{s\in B}c(s)p_{\tau(s)}\right)f\left(\frac{\sum\limits_{s\in B}c(s)p_{\tau(s)}x_{\tau(s)}}{\sum\limits_{s\in B}c(s)p_{\tau(s)}}\right)\right)\right)$$

$$=\sum_{l=1}^{k-1}\left(\sum_{A\in\mathscr{A}_l}\left(\sum_{s\in A}c(s)p_{\tau(s)}f(x_{\tau(s)})\right)\right)+\sum_{A\in\mathscr{A}_k}\left(\sum_{s\in A}c(s)p_{\tau(s)}f(x_{\tau(s)})\right)$$

$$=\sum_{l=1}^{k}\left(\sum_{A\in\mathscr{A}_l}\left(\sum_{s\in A}c(s)p_{\tau(s)}f(x_{\tau(s)})\right)\right).$$

Therefore

$$N_1=\sum_{j=1}^{n}p_jf(x_j)$$

follows by an argument entirely similar to that for (2.36).

So according to Lemma 2.4 only the task of confirming the inequalities

$$N_k\le N_{k-1}\le\dots\le N_{k-m}\le\dots\le N_2\le N_1 \tag{2.40}$$

remains.

To this end, we suppose first that $\mathscr{A}_k=\mathscr{A}$ and thus $\mathscr{A}_1=\dots=\mathscr{A}_{k-1}=\emptyset$. By Lemma 2.5

$$N_k=\sum_{A\in\mathscr{A}_k}\left(\left(\sum_{s\in A}c(s)p_{\tau(s)}\right)f\left(\frac{\sum\limits_{s\in A}c(s)p_{\tau(s)}x_{\tau(s)}}{\sum\limits_{s\in A}c(s)p_{\tau(s)}}\right)\right) \tag{2.41}$$

$$\le\sum_{A\in\mathscr{A}_k}\left(\left(\sum_{s\in A}c(s)p_{\tau(s)}\right)\frac{1}{(k-1)\sum\limits_{s\in A}c(s)p_{\tau(s)}}\right.$$

$$\left.\cdot\sum_{B\in P_{k-1}(A)}\left(\left(\sum_{s\in B}c(s)p_{\tau(s)}\right)f\left(\frac{\sum\limits_{s\in B}c(s)p_{\tau(s)}x_{\tau(s)}}{\sum\limits_{s\in B}c(s)p_{\tau(s)}}\right)\right)\right)$$

$$= \frac{1}{k-1} \sum_{A \in \mathscr{A}_k} \left(\sum_{B \in P_{k-1}(A)} \left(\left(\sum_{s \in B} c(s) p_{\tau(s)} \right) f \left(\frac{\sum_{s \in B} c(s) p_{\tau(s)} x_{\tau(s)}}{\sum_{s \in B} c(s) p_{\tau(s)}} \right) \right) \right).$$

$$= N_{k-1},$$

Suppose then that $1 \le m \le k-2$. By applying Lemma 2.5 again, we have

$$N_{k-m} = \frac{m!}{(k-1)\dots(k-m)} \tag{2.42}$$

$$\cdot \sum_{A \in \mathscr{A}_k} \left(\sum_{B \in P_{k-m}(A)} \left(\left(\sum_{s \in B} c(s) p_{\tau(s)} \right) f \left(\frac{\sum_{s \in B} c(s) p_{\tau(s)} x_{\tau(s)}}{\sum_{s \in B} c(s) p_{\tau(s)}} \right) \right) \right)$$

$$\le \frac{m!}{(k-1)\dots(k-m)}$$

$$\cdot \sum_{A \in \mathscr{A}_k} \left(\sum_{B \in P_{k-m}(A)} \left(\left(\sum_{s \in B} c(s) p_{\tau(s)} \right) \frac{1}{(k-m-1) \sum_{s \in B} c(s) p_{\tau(s)}} \right.\right.$$

$$\left.\left. \cdot \sum_{C \in P_{k-m-1}(B)} \left(\left(\sum_{s \in C} c(s) p_{\tau(s)} \right) f \left(\frac{\sum_{s \in C} c(s) p_{\tau(s)} x_{\tau(s)}}{\sum_{s \in C} c(s) p_{\tau(s)}} \right) \right) \right) \right)$$

$$= \frac{m!}{(k-1)\dots(k-m)(k-m-1)} \sum_{A \in \mathscr{A}_k} \left(\sum_{B \in P_{k-m}(A)} \right.$$

$$\left. \left(\sum_{C \in P_{k-m-1}(B)} \left(\left(\sum_{s \in C} c(s) p_{\tau(s)} \right) f \left(\frac{\sum_{s \in C} c(s) p_{\tau(s)} x_{\tau(s)}}{\sum_{s \in C} c(s) p_{\tau(s)}} \right) \right) \right) \right)$$

$$= \frac{(m+1)!}{(k-1)\dots(k-m)(k-m-1)}$$

$$\cdot \sum_{A \in \mathscr{A}_k} \left(\sum_{C \in P_{k-m-1}(A)} \left(\left(\sum_{s \in C} c(s) p_{\tau(s)} \right) f \left(\frac{\sum_{s \in C} c(s) p_{\tau(s)} x_{\tau(s)}}{\sum_{s \in C} c(s) p_{\tau(s)}} \right) \right) \right)$$

$$= N_{k-m-1}.$$

Together with (2.41) this gives (2.40) in the considered special case.

In the general case we can majorize the members

$$\sum_{A \in \mathscr{A}_l} \left(\left(\sum_{s \in A} c(s) p_{\tau(s)} \right) f \left(\frac{\sum_{s \in A} c(s) p_{\tau(s)} x_{\tau(s)}}{\sum_{s \in A} c(s) p_{\tau(s)}} \right) \right), \quad 2 \le l \le k$$

in (2.29) exactly as N_k in (2.41). Similarly, the argument employed in the proof of the inequality $N_{k-m} \le N_{k-m-1}$ in (2.42) can be extended to estimate the members

$$\frac{m!}{(l-1)\ldots(l-m)}$$

$$\sum_{A\in\mathscr{A}_l}\left(\sum_{B\in P_{l-m}(A)}\left(\left(\sum_{s\in B}c(s)p_{\tau(s)}\right)f\left(\frac{\sum\limits_{s\in B}c(s)p_{\tau(s)}x_{\tau(s)}}{\sum\limits_{s\in B}c(s)p_{\tau(s)}}\right)\right)\right),$$

$$3\le l\le k,\quad 1\le m\le l-2$$

in (2.30). Now (2.40) follow from these facts.

The proof is complete. □

Proof of Theorem 2.4.

Proof. It is obvious that the sets $S_1,\ldots,S_n$ and the function c defined in the theorem satisfy the condition $(\mathscr{H}_4)$. In this case

$$\tau((i_1,\ldots,i_k),l) = i_l,\quad ((i_1,\ldots,i_k),l)\in S.$$

The condition $(\mathscr{H}_5)$ is also fulfilled if

$$\mathscr{A} := \{\{((i_1,\ldots,i_k),l)\mid l=1,\ldots,k\}\mid (i_1,\ldots,i_k)\in I_k\}.$$

Then $\mathscr{A}_k=\mathscr{A}$ and $\mathscr{A}_l=\emptyset$ $(l=1,\ldots,k-1)$.

The result can be obtained by an application of Theorem 2.3 in this environment. □

2.2.1 Examples and Mixed Symmetric Means Related to Theorem 2.3

Now we apply Theorem 2.3 to some special situations which correspond to some recent results.

Example 2.9 Let n, r be fixed integers, where $n\ge 3$, and $1\le r\le n-2$. In this example, for every $i=1,2,\ldots,n$ and for every $l=0,1,\ldots,r$ the integer $i+l$ will be identified with the uniquely determined integer j from $\{1,\ldots,n\}$ for which

$$l+i\equiv j \pmod n. \tag{2.43}$$

Introducing the notation

$$D:=\{1,\ldots,n\}\times\{0,\ldots,r\},$$

let for every $j\in\{1,\ldots,n\}$

$$S_j:=\{(i,l)\in D\mid i+l\equiv j \pmod n\}\bigcup\{j\},$$

and let $\mathscr{A} \subset P(S)$ $(S := \bigcup\limits_{j=1}^{n} S_j)$ contain the following sets:

$$A_i := \{(i,l) \in D \mid l = 0, \ldots, r\}, \quad i = 1, \ldots, n$$

and

$$A := \{1, \ldots, n\}.$$

Let c be a positive function on S such that

$$\sum_{(i,l) \in S_j} c(i,l) + c(j) = 1, \quad j = 1, \ldots, n.$$

A patient verification shows that the sets $S_1, \ldots, S_n$, the partition $\mathscr{A}$ and the function c defined above satisfy the conditions $(\mathscr{H}_4)$ and $(\mathscr{H}_5)$,

$$\tau(i,l) = i + l, \quad (i,l) \in D,$$

(by the agreement (see (2.43)), $i + l$ is identified with j)

$$\tau(j) = j, \quad j = 1, \ldots, n,$$

$$\left|S_j\right| = r + 2, \quad j = 1, \ldots, n,$$

and

$$|A_i| = r + 1, \quad i = 1, \ldots, n, \quad |A| = n.$$

Now we suppose that $(\mathscr{H}_1)$ and either $(\mathscr{H}_2)$ or $(\tilde{\mathscr{H}}_2)$ are satisfied and in the latter case the numbers $c(i,l)$ $((i,l) \in D)$ and $c(j)$ $(j = 1, \ldots, n)$ are rational. Then by Theorem 2.3

$$f\left(\sum_{j=1}^{n} p_j x_j\right) \le N_n = \sum_{i=1}^{n} \left(\left(\sum_{l=0}^{r} c(i,l)\, p_{i+l}\right) f\left(\frac{\sum\limits_{l=0}^{r} c(i,l)\, p_{i+l} x_{i+l}}{\sum\limits_{l=0}^{r} c(i,l)\, p_{i+l}}\right)\right)$$

$$+ \left(\sum_{j=1}^{n} c(j) p_j\right) f\left(\frac{\sum\limits_{j=1}^{n} c(j) p_j x_j}{\sum\limits_{j=1}^{n} c(j) p_j}\right) \le \sum_{j=1}^{n} p_j f(x_j). \tag{2.44}$$

Let $m \ge 2$ be an integer. In case

$$p_j := \frac{1}{n}, \quad j = 1, \ldots, n,$$

$$c(i,l) := \frac{1}{m(r+1)}, \quad (i,l) \in D, \quad c(j) := \frac{m-1}{m} \quad j = 1, \ldots, n,$$

it follows from (2.44) that

$$f\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right) \le \frac{1}{mn}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \ldots + x_{i+r}}{r+1}\right)$$

$$+\frac{m-1}{m}f\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\leq\frac{1}{n}\sum_{j=1}^{n}f(x_j), \tag{2.45}$$

which is an essential part of Theorem 2.1 in [85]. Really, in that theorem a sequence of inequalities (similar to (1.10)) has been proved. On the one hand (2.44) generalizes (2.45), on the other hand the sequence of inequalities in (2.28) is different from that in (2.45).

Example 2.10 Let n and k be fixed positive integers. Let

$$D:=\{(i_1,\ldots,i_n)\in\{1,\ldots,k\}^n\mid i_1+\ldots+i_n=n+k-1\},$$

and for each $j=1,\ldots,n$, denote S_j the set

$$S_j:=D\times\{j\}.$$

For every $(i_1,\ldots,i_n)\in D$ designate by $A_{(i_1,\ldots,i_n)}$ the set

$$A_{(i_1,\ldots,i_n)}:=\{((i_1,\ldots,i_n),l)\mid l=1,\ldots,n\}.$$

It is obvious that S_j $(j=1,\ldots,n)$ and $A_{(i_1,\ldots,i_n)}$ $((i_1,\ldots,i_n)\in D)$ are decompositions of $S:=\bigcup\limits_{j=1}^{n}S_j$ into pairwise disjoint and nonempty sets, respectively. Let c be a function on S such that

$$c((i_1,\ldots,i_n),j)>0,\quad ((i_1,\ldots,i_n),j)\in S$$

and

$$\sum_{(i_1,\ldots,i_n)\in D}c((i_1,\ldots,i_n),j)=1,\quad j=1,\ldots,n. \tag{2.46}$$

In summary we have that the conditions $(\mathscr{H}_4)$ and $(\mathscr{H}_5)$ are valid, and

$$\tau((i_1,\ldots,i_n),j)=j,\quad ((i_1,\ldots,i_n),j)\in S.$$

Now we suppose that $(\mathscr{H}_1)$ and either $(\mathscr{H}_2)$ or $(\tilde{\mathscr{H}_2})$ are satisfied and in the latter case the numbers $c((i_1,\ldots,i_n),j)$ $(((i_1,\ldots,i_n),j)\in S)$ are rational. Then by Theorem 2.3

$$f\left(\sum_{j=1}^{n}p_jx_j\right)\leq N_n=\sum_{(i_1,\ldots,i_n)\in D}\left(\left(\sum_{l=1}^{n}c((i_1,\ldots,i_n),l)\,p_l\right)\right.$$

$$\left.f\left(\frac{\sum\limits_{l=1}^{n}c((i_1,\ldots,i_n),l)\,p_lx_l}{\sum\limits_{l=1}^{n}c((i_1,\ldots,i_n),l)\,p_l}\right)\right)\leq\sum_{j=1}^{n}p_jf(x_j). \tag{2.47}$$

If we set

$$p_j:=\frac{1}{n},\quad j=1,\ldots,n,$$

and

$$c((i_1,\ldots,i_n),j):=\frac{i_j}{\binom{n+k-1}{k-1}},$$

then (2.46) holds, since by some combinatorial considerations

$$|D| = \binom{n+k-2}{n-1},$$

and

$$\sum_{(i_1,\ldots,i_n)\in D} i_j = \frac{n+k-1}{n}\binom{n+k-2}{n-1} = \binom{n+k-1}{k-1}, \quad j = 1,\ldots,n.$$

In this situation (2.47) can therefore be expressed thus

$$f\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right) \le \frac{1}{\binom{n+k-2}{k-1}} \sum_{(i_1,\ldots,i_n)\in D} f\left(\frac{1}{n+k-1}\sum_{l=1}^{n} i_l x_l\right) \le \frac{1}{n}\sum_{j=1}^{n} f(x_j), \tag{2.48}$$

which inequality is contained in Theorem 1 of [86]. The inequality (2.48) is placed in a more general framework in [86], but the treatment of (2.48) is different from our approach. Theorem 2.3 generalizes (2.48) and (2.28) gives a new sequence of inequalities even in the considered special case.

We remind the conditions $(\mathscr{H}_3)$ and $(\tilde{\mathscr{H}_3})$.

$(\mathscr{H}_3)$ Let $\mathbf{x} := (x_1,...,x_n)$ and $\mathbf{p} := (p_1,...,p_n)$ be positive n-tuples such that $\sum\limits_{i=1}^{n} p_i = 1$.

$(\tilde{\mathscr{H}_3})$ Let $J \subset \mathbb{R}$ be an interval, $\mathbf{x} := (x_1,...,x_n) \in J^n$, let $\mathbf{p} := (p_1,...,p_n)$ be a positive n-tuples such that $\sum\limits_{i=1}^{n} p_i = 1$, and let $h,\ g : J \to \mathbb{R}$ be continuous and strictly monotone functions.

Assume $(\mathscr{H}_3)$, $(\mathscr{H}_4)$ and $(\mathscr{H}_5)$.

First, we define the power means of order $r \in \mathbb{R}$ corresponding to $A \in \mathscr{A}_l$ $(l = 1,\ldots,k)$ as follows:

$$M_r(A) = M_r(A,S,c,\mathscr{A},\mathbf{x},\mathbf{p}) := \begin{cases} \left(\dfrac{\sum\limits_{s\in A} c(s)p_{\tau(s)}x_{\tau(s)}^{r}}{\sum\limits_{s\in A} c(s)p_{\tau(s)}}\right)^{\frac{1}{r}}, & r \neq 0, \\ \left(\prod\limits_{s\in A} x_{\tau(s)}^{c(s)p_{\tau(s)}}\right)^{\frac{1}{\sum\limits_{s\in A} c(s)p_{\tau(s)}}}, & r = 0. \end{cases} \tag{2.49}$$

Let $\eta,\gamma \in \mathbb{R}$. Now, we define the mixed symmetric means corresponding to (2.29) and (2.30) as follows:

$$M^1_{\eta,\gamma}(S,c,\mathscr{A},k,\mathbf{x},\mathbf{p})$$
$$:= \begin{cases} \left(\sum\limits_{l=1}^{k}\left(\sum\limits_{A\in\mathscr{A}_l}\left(\left(\sum\limits_{s\in A} c(s)p_{\tau(s)}\right)M_{\gamma}^{\eta}(A)\right)\right)\right)^{\frac{1}{\eta}}, & \eta \neq 0, \\ \prod\limits_{l=1}^{k}\left(\prod\limits_{A\in\mathscr{A}_l}\left((M_{\gamma}(A))^{\sum\limits_{s\in A} c(s)p_{\tau(s)}}\right)\right), & \eta = 0, \end{cases}$$

and for $1 \leq m \leq k-1$

$$M^1_{\eta,\gamma}(S,c,\mathscr{A},k-m,\mathbf{x},\mathbf{p})$$
$$:= \left(\begin{array}{l} \sum\limits_{l=1}^{m} \left(\sum\limits_{A\in\mathscr{A}_l} \left(\sum\limits_{s\in A} c(s) p_{\tau(s)} x^{\eta}_{\tau(s)} \right)\right) + \\ \sum\limits_{l=m+1}^{k} \left(\frac{m!}{(l-1)\ldots(l-m)} \sum\limits_{A\in\mathscr{A}_l} \left(\sum\limits_{B\in P_{l-m}(A)} \left(\left(\sum\limits_{s\in B} c(s)p_{\tau(s)} \right) M^{\eta}_{\gamma}(B) \right)\right)\right) \end{array} \right)^{\frac{1}{\eta}},$$

if $\eta \neq 0$ and for $\eta = 0$, we have

$$M^1_{\eta,\gamma}(S,c,\mathscr{A},k-m,\mathbf{x},\mathbf{p}) := \prod_{l=1}^{m} \left(\prod_{A\in\mathscr{A}_l} \left(\prod_{s\in A} x_{\tau(s)}^{c(s)p_{\tau(s)}} \right)\right) \times$$
$$\prod_{l=m+1}^{k} \left(\left(\prod_{A\in\mathscr{A}_l} \left(\prod_{B\in P_{l-m}(A)} (M_\gamma(B))^{\left(\sum\limits_{s\in B} c(s)p_{\tau(s)}\right)} \right)\right)^{\frac{m!}{(l-1)\ldots(l-m)}} \right).$$

The monotonicity of these mixed symmetric means is a consequence of Theorem 2.3.

Corollary 2.5 *Assume $(\mathscr{H}_3)$, $(\mathscr{H}_4)$ and $(\mathscr{H}_5)$. Let η, $\gamma \in \mathbb{R}$ such that $\eta \leq \gamma$. Then*

$$M_\eta(\mathbf{x},\mathbf{p}) \leq M^1_{\eta,\gamma}(S,c,\mathscr{A},k,\mathbf{x},\mathbf{p}) \leq \ldots \leq M^1_{\gamma,\eta}(S,c,\mathscr{A},1,\mathbf{x},\mathbf{p}) = M_\gamma(\mathbf{x},\mathbf{p}),$$

and

$$M_\eta(\mathbf{x},\mathbf{p}) = M^1_{\gamma,\eta}(S,c,\mathscr{A},1,\mathbf{x},\mathbf{p}) \leq \ldots. \leq M^1_{\eta,\gamma}(S,c,\mathscr{A},k,\mathbf{x},\mathbf{p}) \leq M_\gamma(\mathbf{x},\mathbf{p}).$$

Proof. The proof is similar to the proof of Corollary 2.1. We can apply Theorem 2.3 instead of Theorem 2.1. □

Assume $(\tilde{\mathscr{H}_3})$, $(\mathscr{H}_4)$ and $(\mathscr{H}_5)$. Then we define the generalized means with respect to (2.29) and (2.30) as follows:

$$M^1_{h,g}(S,c,\mathscr{A},k,\mathbf{x},\mathbf{p}) := h^{-1}\left(\sum_{l=1}^{k} \left(\sum_{A\in\mathscr{A}_l} \left(\left(\sum_{s\in A} c(s)p_{\tau(s)} \right) h\circ g^{-1} \left(\frac{\sum\limits_{s\in A} c(s)p_{\tau(s)} g(x_{\tau(s)})}{\sum\limits_{s\in A} c(s)p_{\tau(s)}} \right)\right)\right)\right),$$

and for $1 \leq m \leq k-1$

$$M^1_{h,g}(S,c,\mathscr{A},k-m,\mathbf{x},\mathbf{p}) :=$$
$$h^{-1} \left(\begin{array}{l} \sum\limits_{l=1}^{m} \left(\sum\limits_{A\in\mathscr{A}_l} \left(\sum\limits_{s\in A} c(s)p_{\tau(s)} h(x_{\tau(s)}) \right)\right) + \sum\limits_{l=m+1}^{k} \left(\frac{m!}{(l-1)\ldots(l-m)} \right. \\ \left. \times \sum\limits_{A\in\mathscr{A}_l} \left(\sum\limits_{B\in P_{l-m}(A)} \left(\left(\sum\limits_{s\in B} c(s)p_{\tau(s)} \right) h\circ g^{-1} \left(\frac{\sum\limits_{s\in B} c(s)p_{\tau(s)} g(x_{\tau(s)})}{\sum\limits_{s\in B} c(s)p_{\tau(s)}} \right)\right)\right)\right) \end{array} \right).$$

The monotonicity of the generalized means is given in the next corollary.

Corollary 2.6 *Assume ($\tilde{\mathscr{H}}_3$) and ($\mathscr{H}_4$)-($\mathscr{H}_5$). Then*

$$M_g(\mathbf{x},\mathbf{p}) \le M^1_{h,g}(S,c,\mathscr{A},k,\mathbf{x},\mathbf{p}) \le \ldots \le M^1_{h,g}(S,c,\mathscr{A},1,\mathbf{x},\mathbf{p}) = M_h(\mathbf{x},\mathbf{p}),$$

if either $h \circ g^{-1}$ is convex and h is strictly increasing or $h \circ g^{-1}$ is concave and h is strictly decreasing;

$$M_g(\mathbf{x},\mathbf{p}) = M^1_{h,g}(S,c,\mathscr{A},1,\mathbf{x},\mathbf{p}) \le \ldots \le M^1_{h,g}(S,c,\mathscr{A},k,\mathbf{x},\mathbf{p}) \le M_h(\mathbf{x},\mathbf{p}),$$

if either $g \circ h^{-1}$ is convex and g is strictly decreasing or $g \circ h^{-1}$ is concave and g is strictly increasing.

Proof. The proof is similar to the proof of Corollary 2.2. We can apply Theorem 2.3 instead of Theorem 2.1. □

We illustrate the means defined above by a concrete example based on Theorem 2.4. Further interesting means can be derived from Example 2.9 and Example 2.10.

Example 2.11 As in Theorem 2.4 *let $n \ge 1$ and $k \ge 1$ be fixed integers, and let $I_k \subset \{1,\ldots,n\}^k$ such that*

$$\alpha_{I_k,i} \ge 1, \quad 1 \le i \le n,$$

where $\alpha_{I_k,i}$ means the number of occurrences of i in all the sequences $\mathbf{i}^k := (i_1,\ldots,i_k)$ from *I_k. For $j = 1,\ldots,n$* we introduce *the sets*

$$S_j := \{((i_1,\ldots,i_k),l) \mid (i_1,\ldots,i_k) \in I_k, \quad 1 \le l \le k, \quad i_l = j\}.$$

Let c be a positive function on $S := \bigcup_{j=1}^{n} S_j$ such that

$$\sum_{((i_1,\ldots,i_k),l)\in S_j} c((i_1,\ldots,i_k),l) = 1, \quad j = 1,\ldots,n.$$

In the proof of the theorem we have seen that the condition ($\mathscr{H}_5$) is fulfilled if

$$\mathscr{A} := \{\{((i_1,\ldots,i_k),l) \mid l = 1,\ldots,k\} \mid (i_1,\ldots,i_k) \in I_k\}.$$

In this case $\mathscr{A}_k = \mathscr{A}$ and $\mathscr{A}_l = \emptyset$ ($l = 1,\ldots,k-1$).

(a) Assume ($\mathscr{H}_3$). For $1 \le m \le k-1$ let

$$J_{k-m} := \left\{(l_1,\ldots,l_{k-m}) \in \{1,\ldots,k\}^{k-m} \mid 1 \le l_1 < \ldots < l_{k-m} \le k\right\}.$$

We give the analogue of the power means defined in (2.49). For $r \in \mathbb{R}$ and $\mathbf{i}^k \in I_k$

$$M_r(I_k,c,\mathbf{i}^k) = M_r(I_k,c,\mathbf{i}^k,\mathbf{x},\mathbf{p})$$
$$:= \begin{cases} \left(\dfrac{\sum\limits_{l=1}^{k} c((i_1,\ldots,i_k),l)p_{i_l}x^r_{i_l}}{\sum\limits_{l=1}^{k} c((i_1,\ldots,i_k),l)p_{i_l}}\right)^{\frac{1}{r}}, & r \ne 0, \\ \left(\prod\limits_{l=1}^{k} x_{i_l}^{c((i_1,\ldots,i_k),l)p_{i_l}}\right)^{\frac{1}{\sum\limits_{l=1}^{k-m} c((i_1,\ldots,i_k),l)p_{i_l}}}, & r = 0, \end{cases}$$

and for $r \in \mathbb{R}$, $\mathbf{i}^k \in I_k$ and $\mathbf{l}^{k-m} \in J_{k-m}$

$$M_r(I_k, c, \mathbf{i}^k, \mathbf{l}^{k-m}) = M_r(I_k, c, \mathbf{i}^k, \mathbf{l}^{k-m}, \mathbf{x}, \mathbf{p})$$
$$:= \begin{cases} \left(\dfrac{\sum\limits_{j=1}^{k-m} c\big((i_1,\ldots,i_k), l_j\big) p_{i_{l_j}} x_{i_{l_j}}^r}{\sum\limits_{j=1}^{k-m} c\big((i_1,\ldots,i_k), l_j\big) p_{i_{l_j}}} \right)^{\frac{1}{r}}, & r \neq 0, \\ \left(\prod\limits_{j=1}^{k-m} x_{i_{l_j}}^{c\big((i_1,\ldots,i_k),l_j\big) p_{i_{l_j}}} \right)^{\frac{1}{\sum\limits_{j=1}^{k-m} c((i_1,\ldots,i_k),l) p_{i_l}}}, & r = 0, \end{cases}$$

Now let $\eta, \gamma \in \mathbb{R}$. The mixed symmetric means corresponding to (2.34) and (2.35) can be written as

$$M^1_{\eta,\gamma}(I_k, c, k, \mathbf{x}, \mathbf{p})$$
$$= \begin{cases} \left(\sum\limits_{(i_1,\ldots,i_k) \in I_k} \left(\sum\limits_{l=1}^{k} c\left((i_1, \ldots, i_k), l\right) p_{i_l} \right) M_\gamma^\eta\left(I_k, c, \mathbf{i}^k\right) \right)^{\frac{1}{\eta}}, \eta \neq 0, \\ \prod\limits_{(i_1,\ldots,i_k) \in I_k} \left(\left(M_\gamma(I_k, c, \mathbf{i}^k)\right)^{\sum\limits_{l=1}^{k} c((i_1,\ldots,i_k),l) p_{i_l}} \right), & \eta = 0, \end{cases}$$

and for $1 \leq m \leq k-1$

$$M^1_{\eta,\gamma}(I_k, c, k-m, \mathbf{x}, \mathbf{p})$$
$$= \left(\frac{m!}{(k-1)\ldots(k-m)} \sum_{(i_1,\ldots,i_k) \in I_k} \left(\sum_{1 \leq l_1 < \ldots < l_{k-m} \leq k} \left(\left(\sum_{j=1}^{k-m} c\left((i_1, \ldots, i_k), l_j\right) p_{i_{l_j}} \right) M_\gamma^\eta\left(I_k, c, \mathbf{i}^k, \mathbf{l}^{k-m}\right) \right) \right) \right)^{\frac{1}{\eta}},$$

if $\eta \neq 0$, and for $\eta = 0$, we have

$$M^1_{\eta,\gamma}(I_k, c, k-m, \mathbf{x}, \mathbf{p}) :=$$
$$\left(\prod_{(i_1,\ldots,i_k) \in I_k} \left(\prod_{1 \leq l_1 < \ldots < l_{k-m} \leq k} \left(M_\gamma(I_k, c, \mathbf{i}^k, \mathbf{l}^{k-m})\right)^{\left(\sum\limits_{j=1}^{k-m} c\big((i_1,\ldots,i_k), l_j\big) p_{i_{l_j}} \right)} \right) \right)^{\frac{m!}{(k-1)\ldots(k-m)}}.$$

(b) Assume $(\tilde{\mathscr{H}}_3)$. The generalized means with respect to (2.34) and (2.35) can be written as

$$M^1_{h,g}(I_k, c, k) = M^1_{h,g}(I_k, c, k, \mathbf{x}, \mathbf{p}) :=$$
$$h^{-1} \left(\sum_{(i_1,\ldots,i_k) \in I_k} \left(\left(\sum_{l=1}^{k} c\left((i_1, \ldots, i_k), l\right) p_{i_l} \right) h \circ g^{-1} \left(\frac{\sum\limits_{l=1}^{k} c((i_1,\ldots,i_k),l) p_{i_l} g(x_{i_l})}{\sum\limits_{l=1}^{k} c((i_1,\ldots,i_k),l) p_{i_l}} \right) \right) \right),$$

and for $1 \leq m \leq k-1$

$$M_{h,g}^{1}(I_k, c, k-m, \mathbf{x}, \mathbf{p}) :=$$

$$h^{-1}\left(\frac{m!}{(l-1)\ldots(l-m)} \sum_{(i_1,\ldots,i_k)\in I_k} \left(\sum_{1\leq l_1<\ldots<l_{k-m}\leq k} \left(\left(\sum_{j=1}^{k-m} c\left((i_1,\ldots,i_k),l_j\right) p_{i_{l_j}} \right) h\circ g^{-1}\left(\frac{\sum_{j=1}^{k-m} c\left((i_1,\ldots,i_k),l_j\right) p_{i_{l_j}} g(x_{i_{l_j}})}{\sum_{j=1}^{k-m} c\left((i_1,\ldots,i_k),l_j\right) p_{i_{l_j}}} \right)\right)\right)\right).$$

Remark 2.2 By choosing

$$c\left((i_1,\ldots,i_k),l\right) = \frac{1}{|S_j|} = \frac{1}{\alpha_{I_k,j}} \quad \text{if} \quad \left((i_1,\ldots,i_k),l\right) \in S_j$$

in the previous example, we have the means defined in (2.17), (2.18), (2.21) and (2.22) as special cases. Thus we have some extensions of these means.

Remark 2.3 Results about mixed symmetric means and generalized means similar to Corollary 2.5 and Corollary 2.6 can be given for Example 2.11 as a special case.

2.3 Parameter Dependent Refinement of Discrete Jensen's inequality

Now we consider parameter dependent refinement of discrete Jensen's inequality given by L. Horváth recently [33]. Generalizations of these results can be found in [35].

We need the following hypotheses (($\mathscr{H}_2$) and ($\tilde{\mathscr{H}}_2$) have already been introduced earlier):

($\tilde{\mathscr{H}}_1$) Let V be a real vector space, $C \subset V$ be a convex set, $\mathbf{x} := (x_1, \ldots, x_n) \in C^n$, and let $\mathbf{p} := (p_1, \ldots, p_n)$ be a nonnegative n-tuples such that $\sum_{i=1}^{n} p_i = 1$.

($\mathscr{H}_2$) Let $f : C \to \mathbb{R}$ be a convex function.

($\tilde{\mathscr{H}}_2$) Let $f : C \to \mathbb{R}$ be a mid-convex function, and $p_1, \ldots, p_n$ be rational numbers.

Theorem 2.5 *Assume either ($\tilde{\mathscr{H}}_1$-$\mathscr{H}_2$), and in this case let $\lambda \geq 1$, or ($\tilde{\mathscr{H}}_1$-$\tilde{\mathscr{H}}_2$), and in this case let $\lambda \geq 1$ be a rational number. We introduce the sets*

$$S_k := \left\{ (i_1, \ldots, i_n) \in \mathbb{N}^n \mid \sum_{j=1}^{n} i_j = k \right\}, \quad k \in \mathbb{N},$$

and for $k \in \mathbb{N}$ define the numbers

$$\begin{aligned} C_k(\lambda) &= C_k(\lambda, \mathbf{x}, \mathbf{p}) \\ &:= \frac{1}{(n+\lambda-1)^k} \sum_{(i_1,\ldots,i_n)\in S_k} \frac{k!}{i_1!\ldots i_n!} \left(\sum_{j=1}^{n} \lambda^{i_j} p_j \right) f \left(\frac{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j x_j}{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j} \right). \end{aligned} \tag{2.50}$$

Then

$$f\left(\sum_{j=1}^{n} p_j x_j\right) = C_0(\lambda) \le C_1(\lambda) \le \ldots \le C_k(\lambda) \le \ldots \le \sum_{j=1}^{n} p_j f(x_j).$$

Remark 2.4 (a) It follows from the definition of S_k that $S_k \subset \{0,\ldots,k\}^n$ $(k \in \mathbb{N})$.

(b) It is easy to see that

$$C_k(1) = f\left(\sum_{j=1}^{n} p_j x_j\right), \quad k \in \mathbb{N}. \tag{2.51}$$

We establish two convergence theorems.

Theorem 2.6 *Suppose ($\tilde{\mathscr{H}}_1$-$\mathscr{H}_2$), and let $\lambda \ge 1$. If X is a normed space and f is continuous, then*

(a) For every fixed $\lambda > 1$

$$\lim_{k\to\infty} C_k(\lambda) = \sum_{j=1}^{n} p_j f(x_j).$$

(b) The function $\lambda \to C_k(\lambda)$ $(\lambda \ge 1)$ is continuous for every $k \in \mathbb{N}$.

The proof of Theorem 2.6 (a) requires a lemma (see Lemma 2.7) which is interesting in its own right. Probability theoretical technique will be used to handle this problem.

Remark 2.5 In the previous theorem it suffices to consider the case when ($\tilde{\mathscr{H}}_1$-$\mathscr{H}_2$) and $\lambda \ge 1$ are satisfied. Really, if f is mid-convex and continuous, then convex.

By (2.51)

$$\lim_{k\to\infty} C_k(1) = f\left(\sum_{j=1}^{n} p_j x_j\right).$$

We come now to the second convergence theorem.

Theorem 2.7 *Suppose ($\tilde{\mathscr{H}}_1$-$\mathscr{H}_2$) and $\lambda \ge 1$. For each fixed $k \in \mathbb{N}_+$*

$$\lim_{\lambda\to\infty} C_k(\lambda) = \sum_{j=1}^{n} p_j f(x_j).$$

Suppose ($\tilde{\mathscr{H}}_1$-$\mathscr{H}_2$) and $\lambda \ge 1$. Theorem 2.5 implies

$$\begin{aligned} \Upsilon_7(f) &= \Upsilon_7(f,m,l,\lambda,\mathbf{x},\mathbf{p}) := C_m(\lambda) - C_l(\lambda) \ge 0, \quad 0 \le l < m, \\ \Upsilon_8(f) &= \Upsilon_8(f,k,\lambda,\mathbf{x},\mathbf{p}) := \sum_{j=1}^{n} p_j f(x_j) - C_k(\lambda) \ge 0, \quad 0 \le k. \end{aligned}$$

2.3.1 Some lemmas and the proofs of Theorem 2.5-2.7

Lemma 2.6 *Let $k \in \mathbb{N}$ and $(i_1, \ldots, i_n) \in S_{k+1}$ be fixed. If we set*

$$z(i_1, \ldots, i_n) := \left\{ j \in \{1, \ldots, n\} \mid i_j \neq 0 \right\},$$

then

$$\sum_{j \in z(i_1, \ldots, i_n)} \frac{k!}{i_1! \ldots i_{j-1}!(i_j - 1)! i_{j+1}! \ldots i_n!} = \frac{(k+1)!}{i_1! \ldots i_n!}.$$

Proof. The lowest common denominator is $i_1! \ldots i_n!$. Combined with $\sum\limits_{j=1}^{n} i_j = k+1$ the result follows. □

The proof of Theorem 2.5.

Proof. (a) We separate the proof of this part of the theorem into three steps. Let $\lambda \geq 1$ be fixed.

I. Since $S_0 = \{(0, \ldots, 0)\}$

$$C_0(\lambda) = \left(\sum_{j=1}^{n} \lambda^0 p_j \right) f \left(\frac{\sum\limits_{j=1}^{n} \lambda^0 p_j x_j}{\sum\limits_{j=1}^{n} \lambda^0 p_j} \right) = f \left(\sum_{j=1}^{n} p_j x_j \right).$$

II. Next, we prove that $C_k(\lambda) \leq C_{k+1}(\lambda)$ $(k \in \mathbb{N})$.

It is easy to check that for every $(i_1, \ldots, i_n) \in S_k$

$$\frac{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j x_j}{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j} = \frac{1}{n + \lambda - 1}$$

$$\cdot \sum_{l=1}^{n} \left(\frac{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j x_j + (\lambda - 1) \lambda^{i_l} p_l x_l}{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j + (\lambda - 1) \lambda^{i_l} p_l} \cdot \frac{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j + (\lambda - 1) \lambda^{i_l} p_l}{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j} \right).$$

With the help of discrete Jensen's inquality this yields that

$$f \left(\frac{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j x_j}{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j} \right) \leq \frac{1}{n + \lambda - 1} \sum_{l=1}^{n} \left(\frac{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j + (\lambda - 1) \lambda^{i_l} p_l}{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j} \right.$$

$$\left. \cdot f \left(\frac{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j x_j + (\lambda - 1) \lambda^{i_l} p_l x_l}{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j + (\lambda - 1) \lambda^{i_l} p_l} \right) \right).$$

Consequently,

$$C_k(\lambda) \leq \frac{1}{(n+\lambda-1)^{k+1}} \sum_{(i_1,\ldots,i_n)\in S_k} \frac{k!}{i_1!\ldots i_n!}$$

$$\cdot \sum_{l=1}^{n} \left(\left(\sum_{j=1}^{n} \lambda^{i_j} p_j + (\lambda-1)\lambda^{i_l} p_l \right) f \left(\frac{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j x_j + (\lambda-1)\lambda^{i_l} p_l x_l}{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j + (\lambda-1)\lambda^{i_l} p_l} \right) \right). \tag{2.52}$$

By Lemma 2.6, it is easy to see that the right hand side of (2.52) can be written in the form

$$\frac{1}{(n+\lambda-1)^{k+1}} \sum_{(i_1,\ldots,i_n)\in S_{k+1}} \frac{(k+1)!}{i_1!\ldots i_n!} \left(\sum_{j=1}^{n} \lambda^{i_j} p_j \right) f \left(\frac{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j x_j}{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j} \right)$$

which is just $C_{k+1}(\lambda)$.

III. Finally, we prove that

$$C_k(\lambda) \leq \sum_{j=1}^{n} p_j f(x_j), \quad k \in \mathbb{N}_+. \tag{2.53}$$

It follows from the discrete Jensen's inequality that

$$C_k(\lambda) \leq \frac{1}{(n+\lambda-1)^{k}} \sum_{(i_1,\ldots,i_n)\in S_k} \left(\frac{k!}{i_1!\ldots i_n!} \sum_{j=1}^{n} \lambda^{i_j} p_j f(x_j) \right)$$

$$= \frac{1}{(n+\lambda-1)^{k}} \sum_{j=1}^{n} \left(\sum_{(i_1,\ldots,i_n)\in S_k} \frac{k!}{i_1!\ldots i_n!} \lambda^{i_j} \right) p_j f(x_j), \quad k \in \mathbb{N}_+. \tag{2.54}$$

The multinomial theorem shows that

$$\sum_{(i_1,\ldots,i_n)\in S_k} \frac{k!}{i_1!\ldots i_n!} \lambda^{i_j} = (n+\lambda-1)^k, \quad 1 \leq j \leq n,$$

hence (2.54) implies (2.53). □

The proof of Theorem 2.6 (a) is based on the following interesting result. The σ-algebra of Borel subsets of $\mathbb{R}^n$ is denoted by $\mathscr{B}^n$.

Lemma 2.7 *Let $p_1,\ldots,p_n$ be a discrete distribution with $n \geq 2$, and let $\lambda > 1$. Let $l \in \{1,\ldots,n\}$ be fixed. e_l denotes the vector in $\mathbb{R}^n$ that has 0s in all coordinate positions except the lth, where it has a 1. Let $q_1,\ldots,q_n$ be also a discrete distribution such that $q_j > 0$ $(1 \leq j \leq n)$ and*

$$q_l > \max(q_1,\ldots q_{l-1}, q_{l+1},\ldots,q_n). \tag{2.55}$$

If

$$g:\left\{(t_1,\ldots,t_n)\in\mathbb{R}^n \mid t_j>0\ (1\le j\le n),\ \sum_{j=1}^{n}t_j=1\right\}\to\mathbb{R}$$

is a bounded function for which

$$\tau_l := \lim_{e_l} g$$

exists in $\mathbb{R}$, *and* $p_l>0$, *then*

$$\lim_{k\to\infty}\sum_{(i_1,\ldots,i_n)\in S_k}\frac{k!}{i_1!\ldots i_n!}q_1^{i_1}\ldots q_n^{i_n}g\left(\frac{\lambda^{i_1}p_1}{\sum\limits_{j=1}^{n}\lambda^{i_j}p_j},\ldots,\frac{\lambda^{i_n}p_n}{\sum\limits_{j=1}^{n}\lambda^{i_j}p_j}\right)=\tau_l. \tag{2.56}$$

Proof. To prove the result we can obviously suppose that $l=1$.

For the sake of clarity we shall denote the element $(i_1,\ldots,i_n)$ of S_k by $(i_{1k},\ldots,i_{nk})$ $(k\in\mathbb{N}_+)$.

Let $\xi_k:=(\xi_{1k},\ldots,\xi_{nk})$ $(k\in\mathbb{N}_+)$ be a $(\mathbb{R}^n,\mathscr{B}^n)$-random variable on a probability space $(\Omega,\mathscr{A},P)$ such that ξ_k has multinomial distribution of order k and with parameters $q_1,\ldots,q_n$. A fundamental theorem of the statistics (see [18] Theorem 5.4.13), which is based on the multidimensional Central Limit Theorem and the Cochran-Fisher theorem, implies that

$$\lim_{k\to\infty}P\left(\sum_{j=1}^{n}\frac{\left(\xi_{jk}-kq_j\right)^2}{kq_j}<t\right)=F_{n-1}(t),\quad t\in\mathbb{R}, \tag{2.57}$$

where F_{n-1} means the distribution function of the Chi-squared distribution (χ^2-distribution) with $n-1$ degrees of freedom.

Choose $0<\varepsilon<1$. Since F_{n-1} is continuous, and strictly increasing on $(0,\infty)$, there exists a unique $t_\varepsilon>0$ such that

$$F_{n-1}(t_\varepsilon)=1-\varepsilon.$$

Define

$$S_k^1:=\left\{(i_{1k},\ldots,i_{nk})\in S_k \mid \sum_{j=1}^{n}k\frac{\left(\frac{i_{jk}}{k}-q_j\right)^2}{q_j}<t_\varepsilon\right\}.$$

The definition of the set S_k^1 shows that

$$\sum_{(i_{1k},\ldots,i_{nk})\in S_k^1}\frac{k!}{i_{1k}!\ldots i_{nk}!}q_1^{i_{1k}}\ldots q_n^{i_{nk}}=P\left((\xi_{1k},\ldots,\xi_{nk})\in S_k^1\right) \tag{2.58}$$

$$=P\left(\sum_{j=1}^{n}k\frac{\left(\frac{\xi_{jk}}{k}-q_j\right)^2}{q_j}<t_\varepsilon\right)=P\left(\sum_{j=1}^{n}\frac{\left(\xi_{jk}-kq_j\right)^2}{kq_j}<t_\varepsilon\right)$$

$$= F_{n-1}(t_\varepsilon) + \left(P\left(\sum_{j=1}^{n} \frac{\left(\xi_{jk} - kq_j\right)^2}{kq_j} < t_\varepsilon \right) - F_{n-1}(t_\varepsilon) \right)$$
$$= 1 - \varepsilon + \delta_\varepsilon(k), \quad k \in \mathbb{N}_+, \tag{2.59}$$

where, by (2.57)

$$\lim_{k\to\infty} \delta_\varepsilon(k) = 0. \tag{2.60}$$

For $j = 1,\ldots,n$ construct the sequences $(I_k^j)_{k\geq 1}$ by

$$I_k^j := i_{jk}^*, \text{ if } \left| \frac{i_{jk}^*}{k} - q_j \right| = \max\left\{ \left| \frac{i_{jk}}{k} - q_j \right| \mid (i_{1k},\ldots,i_{nk}) \in S_k^1 \right\}, \quad k \in \mathbb{N}_+. \tag{2.61}$$

We claim that

$$\lim_{k\to\infty} \frac{I_k^j}{k} = q_j, \quad 1 \leq j \leq n. \tag{2.62}$$

Fix $1 \leq j \leq n$. If (2.62) is false, then (2.61) yields that we can find a positive number ρ, a strictly increasing sequence $(k_u)_{u\geq 1}$, and points

$$(i_{1k_u},\ldots,i_{nk_u}) \in S_{k_u}^1, \quad u \in \mathbb{N}_+ \tag{2.63}$$

such that

$$\left| \frac{i_{jk_u}}{k_u} - q_j \right| \geq \rho, \quad u \in \mathbb{N}_+,$$

and therefore

$$k_u \frac{\left(\frac{i_{jk_u}}{k_u} - q_j\right)^2}{q_j} \geq k_u \frac{\rho^2}{q_j} \to \infty \text{ as } u \to \infty,$$

contrary to (2.63).

Let

$$q := \max\left(q_2,\ldots,q_n\right).$$

It follows from (2.55) that

$$\gamma := \frac{1}{3}\left(q_1 - q\right) > 0. \tag{2.64}$$

By (2.61) and (2.62), we can find an integer k_γ such that for each $k > k_\gamma$

$$\left| \frac{i_{jk}}{k} - q_j \right| \leq \left| \frac{I_k^j}{k} - q_j \right| < \gamma, \quad (i_{1k},\ldots,i_{nk}) \in S_k^1, \quad 1 \leq j \leq n.$$

Thus for every $k > k_\gamma$

$$\frac{i_{1k}}{k} > q_1 - \gamma \text{ and } \frac{i_{jk}}{k} < q_j + \gamma, \quad 2 \leq j \leq n, \quad (i_{1k},\ldots,i_{nk}) \in S_k^1,$$

and hence we get from (2.64) that

$$i_{1k} - i_{jk} > k\gamma \quad 2 \leq j \leq n, \quad (i_{1k},\ldots,i_{nk}) \in S_k^1, \quad k > k_\gamma. \tag{2.65}$$

We can see that

$$i_{1k} - i_{jk} \to \infty \text{ as } k \to \infty, \quad 2 \le j \le n, \quad (i_{1k}, \ldots, i_{nk}) \in S_k^1. \tag{2.66}$$

Now, set $S_k^2 := S_k \setminus S_k^1$ $(k \in \mathbb{N}_+)$, and consider the sequences

$$a_k^1 := \sum_{(i_{1k},\ldots,i_{nk}) \in S_k^1} \frac{k!}{i_{1k}! \ldots i_{nk}!} q_1^{i_{1k}} \ldots q_n^{i_{nk}} g\left(\frac{\lambda^{i_{1k}} p_1}{\sum\limits_{j=1}^n \lambda^{i_{jk}} p_j}, \ldots, \frac{\lambda^{i_{nk}} p_n}{\sum\limits_{j=1}^n \lambda^{i_{jk}} p_j}\right),$$

and

$$a_k^2 := \sum_{(i_{1k},\ldots,i_{nk}) \in S_k^2} \frac{k!}{i_{1k}! \ldots i_{nk}!} q_1^{i_{1k}} \ldots q_n^{i_{nk}} g\left(\frac{\lambda^{i_{1k}} p_1}{\sum\limits_{j=1}^n \lambda^{i_{jk}} p_j}, \ldots, \frac{\lambda^{i_{nk}} p_n}{\sum\limits_{j=1}^n \lambda^{i_{jk}} p_j}\right),$$

where $k \in \mathbb{N}_+$. The sum of these sequences is just the studied sequence in (2.56). Since $p_1 > 0$, we obtain from (2.66) that

$$\lim_{k\to\infty} \frac{\lambda^{i_{1k}} p_1}{\sum\limits_{j=1}^n \lambda^{i_{jk}} p_j} = 1, \quad (i_{1k}, \ldots, i_{nk}) \in S_k^1, \tag{2.67}$$

and

$$\lim_{k\to\infty} \frac{\lambda^{i_{lk}} p_1}{\sum\limits_{j=1}^n \lambda^{i_{jk}} p_j} = 0, \quad 2 \le l \le n, \quad (i_{1k}, \ldots, i_{nk}) \in S_k^1. \tag{2.68}$$

According to (2.65), the convergence is uniform for all the possible sequences in (2.67) and (2.68), hence for every $\varepsilon_1 > 0$ we can find an integer $k_{\varepsilon_1} > k_\gamma$ such that for all $k > k_{\varepsilon_1}$

$$\tau_1 - \varepsilon_1 < g\left(\frac{\lambda^{i_{1k}} p_1}{\sum\limits_{j=1}^n \lambda^{i_{jk}} p_j}, \ldots, \frac{\lambda^{i_{nk}} p_n}{\sum\limits_{j=1}^n \lambda^{i_{jk}} p_j}\right) < \tau_1 + \varepsilon_1, \quad (i_{1k}, \ldots, i_{nk}) \in S_k^1. \tag{2.69}$$

Bringing in (2.58-2.59), we find that

$$P\left((\xi_{1k}, \ldots, \xi_{nk}) \in S_k^2\right) = \varepsilon - \delta_\varepsilon(k), \quad k \in \mathbb{N}_+$$

and therefore, thanks to (2.58-2.59), (2.69) and the boundedness of g $(|g| \le m)$

$$(1 - \varepsilon + \delta(k))(\tau_1 - \varepsilon_1) - (\varepsilon - \delta(k))m \le a_k^1 + a_k^2$$

$$\le (1 - \varepsilon + \delta(k))(\tau_1 + \varepsilon_1) + (\varepsilon - \delta(k))m, \quad k > k_{\varepsilon_1}.$$

Consequently, by (2.60)

$$(1 - \varepsilon)(\tau_1 - \varepsilon_1) - \varepsilon m \le \liminf_{k\to\infty}\left(a_k^1 + a_k^2\right) \le \limsup_{k\to\infty}\left(a_k^1 + a_k^2\right)$$

$$\leq (1-\varepsilon)(\tau_1+\varepsilon_1)+\varepsilon m,$$

and this proves the convergence claim (2.56).

The proof is now complete. □

The proof of Theorem 2.6.

Proof. (a) We have only to observe that for every fixed $1 \leq l \leq n$

$$\lim_{k\to\infty} \frac{1}{(n+\lambda-1)^k} \sum_{(i_1,\ldots,i_n)\in S_k} \frac{k!}{i_1!\ldots i_n!} \lambda^{i_l} p_l f\left(\frac{\sum_{j=1}^{n} \lambda^{i_j} p_j x_j}{\sum_{j=1}^{n} \lambda^{i_j} p_j}\right) = p_l f(x_l). \tag{2.70}$$

The case $p_l = 0$ is trivial.

To prove the case $p_l > 0$, define the function

$$g : \left\{ (t_1,\ldots,t_n) \in \mathbb{R}^n \mid t_j > 0 \ (1 \leq j \leq n), \ \sum_{j=1}^{n} t_j = 1 \right\} \to \mathbb{R}$$

by

$$g(t_1,\ldots,t_n) := f\left(\sum_{j=1}^{n} t_j x_j\right).$$

Consequently, the limit in (2.70) can be written in the form

$$\lim_{k\to\infty} p_l \sum_{(i_1,\ldots,i_n)\in S_k} \frac{k!}{i_1!\ldots i_n!} \left(\frac{1}{n+\lambda-1}\right)^{i_1} \cdots \left(\frac{1}{n+\lambda-1}\right)^{i_{l-1}} \left(\frac{\lambda}{n+\lambda-1}\right)^{i_l}$$

$$\cdot \left(\frac{1}{n+\lambda-1}\right)^{i_{l+1}} \cdots \left(\frac{1}{n+\lambda-1}\right)^{i_n} g\left(\frac{\lambda^{i_1} p_1}{\sum_{j=1}^{n} \lambda^{i_j} p_j}, \ldots, \frac{\lambda^{i_n} p_n}{\sum_{j=1}^{n} \lambda^{i_j} p_j}\right).$$

Now we can apply Lemma 2.7 with

$$q_j = \frac{1}{n+\lambda-1}, \quad 1 \leq j \leq n, \quad j \neq l, \text{ and } q_l = \frac{\lambda}{n+\lambda-1}$$

and

$$\lim_{e_l} g = f(x_l), \quad 1 \leq l \leq n.$$

(b) Elementary considerations show this part of the theorem.

The proof is complete. □

The proof of Theorem 2.7.

Proof. The discrete Jensen's inequality confirms that f is bounded on the set

$$G := \left\{ \sum_{j=1}^{n} t_j x_j \in C \mid t_j \geq 0 \quad (1 \leq j \leq n), \quad \sum_{j=1}^{n} t_j = 1 \right\}.$$

It is elementary that for every $(i_1,\ldots,i_n)\in S_k$

$$\lim_{\lambda\to\infty}\frac{\lambda^{i_l}}{(n+\lambda-1)^k}=\begin{cases}1, \text{ if } i_l=k\\ 0, \text{ if } i_l<k\end{cases}, \quad 1\le l\le n.$$

By the definition of the set S_k, $(0,\ldots,0,k,0,\ldots,0)$ (the vector has 0s in all coordinate positions except the lth) is the only element of S_k for which $i_l=k$ $(1\le l\le n)$. By using the boundedness of f on G, the previous assumptions imply the result, bringing the proof to an end. □

Suppose either $(\tilde{\mathscr{H}}_1\text{-}\mathscr{H}_2)$, and in this case let $\lambda\ge 1$, or $(\tilde{\mathscr{H}}_1\text{-}\tilde{\mathscr{H}}_2)$ and in this case let $\lambda\ge 1$ be a rational number. First, we give three special cases of (2.50).

(a) $k=1,\, n\in\mathbb{N}_+$:

$$C_1(\lambda)=\frac{1}{n+\lambda-1}\sum_{i=1}^{n}(1+(\lambda-1)p_i)f\left(\frac{\sum\limits_{j=1}^{n}p_jx_j+(\lambda-1)p_ix_i}{1+(\lambda-1)p_i}\right).$$

(b) $k\in\mathbb{N},\, n=2$:

$$C_k(\lambda)=\frac{1}{(\lambda+1)^k}\sum_{i=0}^{k}\binom{k}{i}\left(\lambda^ip_1+\lambda^{k-i}p_2\right)f\left(\frac{\lambda^ip_1x_1+\lambda^{k-i}p_2x_2}{\lambda^ip_1+\lambda^{k-i}p_2}\right).$$

(c) $p_1=\ldots=p_n:=\frac{1}{n}$:

$$C_k(\lambda)=\frac{1}{n\,(n+\lambda-1)^k}\sum_{(i_1,\ldots,i_n)\in S_k}\frac{k!}{i_1!\ldots i_n!}\left(\sum_{j=1}^{n}\lambda^{i_j}\right)f\left(\frac{\sum\limits_{j=1}^{n}\lambda^{i_j}x_j}{\sum\limits_{j=1}^{n}\lambda^{i_j}}\right).$$

Assume further that f is strictly convex (strictly mid-convex). Then it comes from the third part of the proof of Theorem 2.5 that

$$C_k(\lambda)<\sum_{j=1}^{n}p_jf(x_j), \quad k\in\mathbb{N}, \tag{2.71}$$

if not all x_i are equal.

If $p_1=\ldots=p_n:=\frac{1}{n}$ and f is strictly convex (strictly mid-convex), then the analysis of the proof of Theorem 2.5 shows that

$$f\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)=C_0(\lambda)<C_1(\lambda)<\ldots<C_k(\lambda)<\ldots<\frac{1}{n}\sum_{j=1}^{n}f(x_j), \quad k\in\mathbb{N},$$

whenever not all x_i are equal.

If the inequality (2.71) holds, X is a normed space and f is continuous (see Remark 2.5), then Theorem 2.6 (b) and Theorem 2.7 insure that the range of the function $\lambda \to C_k(\lambda)$ $(k \in \mathbb{N}_+)$ is the interval

$$\left[f\left(\sum_{j=1}^{n} p_j x_j\right), \quad \sum_{j=1}^{n} p_j f(x_j) \right[.$$

Conjecture 2.1 *Suppose either ($\tilde{\mathscr{H}}_1$-$\mathscr{H}_2$), and in this case let $\lambda \geq 1$, or ($\tilde{\mathscr{H}}_1$-$\tilde{\mathscr{H}}_2$) and in this case let $\lambda \geq 1$ be a rational number. The function $\lambda \to C_k(\lambda)$ $(\lambda \geq 1)$ is increasing for every $k \in \mathbb{N}$.*

2.3.2 Applications to Mixed Symmetric Means

We define some new quasi-arithmetic means and study their monotonicity and convergence.

($\overline{\mathscr{H}}_3$) Let $J \subset \mathbb{R}$ be an interval, $\mathbf{x} := (x_1, \ldots, x_n) \in J^n$, let $\mathbf{p} := (p_1, \ldots, p_n)$ be a nonegative n-tuples such that $\sum\limits_{i=1}^{n} p_i = 1$, and let h, $g : J \to \mathbb{R}$ be continuous and strictly monotone functions. Assume $\lambda \geq 1$.

We define the quasi-arithmetic means with respect to (2.50) by

$$M_{h,g}(k,\lambda,\mathbf{x},\mathbf{p}) := h^{-1}\left(\frac{1}{(n+\lambda-1)^k} \sum_{(i_1,\ldots,i_n)\in S_k} \frac{k!}{i_1!\ldots i_n!}\left(\sum_{j=1}^{n} \lambda^{i_j} p_j\right)\right. \tag{2.72}$$

$$\left.\cdot(h \circ g^{-1})\left(\frac{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j g(x_j)}{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j}\right)\right), \quad k \in \mathbb{N}.$$

We now prove the monotonicity of these means and give limit formulas.

Proposition 2.1 *Assume ($\overline{\mathscr{H}}_3$). Then*

(a)

$$M_g(\mathbf{x},\mathbf{p}) = M_{h,g}(0,\lambda,\mathbf{x},\mathbf{p}) \leq \ldots \leq M_{h,g}(k,\lambda,\mathbf{x},\mathbf{p}) \leq \ldots \leq M_h(\mathbf{x},\mathbf{p}), \quad k \in \mathbb{N},$$

if either $h \circ g^{-1}$ is convex and h is increasing or $h \circ g^{-1}$ is concave and h is decreasing.

(b)

$$M_g(\mathbf{x},\mathbf{p}) = M_{h,g}(0,\lambda,\mathbf{x},\mathbf{p}) \geq \ldots \geq M_{h,g}(k,\lambda,\mathbf{x},\mathbf{p}) \geq \ldots \geq M_h(\mathbf{x},\mathbf{p}), \quad k \in \mathbb{N},$$

if either $h \circ g^{-1}$ is convex and h is decreasing or $h \circ g^{-1}$ is concave and h is increasing.

(c) Moreover, in both cases

$$\lim_{k\to\infty} M_{h,g}(k,\lambda,\mathbf{x},\mathbf{p}) = M_h(\mathbf{x},\mathbf{p})$$

for each fixed $\lambda > 1$, and

$$\lim_{\lambda\to\infty} M_{h,g}(k,\lambda,\mathbf{x},\mathbf{p}) = M_h(\mathbf{x},\mathbf{p})$$

for each fixed $k \in \mathbb{N}_+$.

Proof. Theorem 2.5 can be applied to the function $h \circ g^{-1}$, if it is convex ($-h \circ g^{-1}$, if it is concave) and the n-tuples $(g(x_1), \ldots, g(x_n))$, then upon taking h^{-1}, we get (a) and (b). (c) comes from Theorem 2.6 (a) and Theorem 2.7. □

As a special case we consider the following example.

Example 2.12 If $J := (0,\infty)$, $h := \ln$ and $g(x) := x$ $(x \in (0,\infty))$, then by Proposition 2.1 (b), we have the following inequality: for every $x_j > 0$ $(1 \le j \le n)$, $\lambda \ge 1$, and $k \in \mathbb{N}_+$

$$\sum_{j=1}^{n} p_j x_j \ge \prod_{(i_1,\ldots,i_n)\in S_k} \left(\frac{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j x_j}{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j} \right)^{\frac{1}{(n+\lambda-1)^k} \frac{k!}{i_1!\ldots i_n!} \sum\limits_{j=1}^{n} \lambda^{i_j} p_j} \ge \prod_{j=1}^{n} x_j^{p_j},$$

which gives a sharpened version of the arithmetic mean - geometric mean inequality

$$\frac{1}{n} \sum_{j=1}^{n} x_j \ge \prod_{(i_1,\ldots,i_n)\in S_k} \left(\frac{\sum\limits_{j=1}^{n} \lambda^{i_j} x_j}{\sum\limits_{j=1}^{n} \lambda^{i_j}} \right)^{\frac{1}{n(n+\lambda-1)^k} \frac{k!}{i_1!\ldots i_n!} \sum\limits_{j=1}^{n} \lambda^{i_j}} \ge \prod_{j=1}^{n} x_j^{\frac{1}{n}}.$$

Supported by the power means we can introduce mixed symmetric means correspond to (2.50) under the condition

($\mathscr{H}_3$) Let $\mathbf{x} := (x_1, \ldots, x_n)$ and $\mathbf{p} := (p_1, \ldots, p_n)$ be positive n-tuples such that $\sum\limits_{i=1}^{n} p_i = 1$.

Assume ($\mathscr{H}_3$), and let $\lambda \ge 1$, and $k \in \mathbb{N}$. We define the mixed symmetric means with respect to (2.50) by

$$M_{s,t}(k,\lambda,\mathbf{x},\mathbf{p}) := \left(\frac{1}{(n+\lambda-1)^k} \sum_{(i_1,\ldots,i_n)\in S_k} \frac{k!}{i_1!\ldots i_n!} \left(\sum_{j=1}^{n} \lambda^{i_j} p_j \right) \cdot M_t^s \left(x_1, \ldots, x_n; \frac{\lambda^{i_1} p_1}{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j}, \ldots, \frac{\lambda^{i_n} p_n}{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j} \right) \right)^{\frac{1}{s}},$$

if s, $t \in \mathbb{R}$ and $s \ne 0$, and

$$M_{0,t}(k,\lambda,\mathbf{x},\mathbf{p}) := \prod_{(i_1,\ldots,i_n)\in S_k} \left(M_t \left(x_1, \ldots, x_n; \frac{\lambda^{i_1} p_1}{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j}, \ldots, \frac{\lambda^{i_n} p_n}{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j} \right) \right)^{\frac{1}{(n+\lambda-1)^k} \frac{k!}{i_1!\ldots i_n!} \left(\sum\limits_{j=1}^{n} \lambda^{i_j} p_j \right)},$$

where $t \in \mathbb{R}$.

The monotonicity and the convergence of the previous means are studied in the next result.

Proposition 2.2 *Assume ($\mathscr{H}_3$), let $\lambda \geq 1$, and $k \in \mathbb{N}$. Suppose $s,\ t \in \mathbb{R}$ such that $s \leq t$. Then*

(a)

$$M_t(\mathbf{x},\mathbf{p}) = M_{s,t}(0,\lambda) \geq \ldots \geq M_{s,t}(k,\lambda) \geq \ldots \geq M_s(\mathbf{x},\mathbf{p}), \quad k \in \mathbb{N}.$$

(b) In case of $s,\ t \neq 0$

$$\lim_{k\to\infty} M_{s,t}(k,\lambda,\mathbf{x},\mathbf{p}) = M_s(\mathbf{x},\mathbf{p})$$

for each fixed $\lambda > 1$, and

$$\lim_{\lambda\to\infty} M_{s,t}(k,\lambda,\mathbf{x},\mathbf{p}) = M_s(\mathbf{x},\mathbf{p})$$

for each fixed $k \in \mathbb{N}_+$.

Proof. Assume $s,t \neq 0$. Then Proposition 2.1 (b) can be applied with $g,h : (0,\infty) \to \mathbb{R}$, $g(x) := x^t$ and $h(x) := x^s$. If $s = 0$ or $t = 0$, the result follows by taking limit. $\square$

Chapter 3

Further Refinements of Jensen's Inequality

The expression $\overline{f}_{k,n}$ given in Theorem 1.9 can be written as follows

$$\overline{f}_{k,n} = \frac{1}{\binom{n+k-1}{k}} \sum_{\substack{i_1+\ldots+i_n=k \\ i_j\in\mathbb{N};\quad 1\le j\le n}} f\left(\frac{1}{k}\sum_{j=1}^{k} i_j x_j\right).$$

Inspired by this interpretation of $\overline{f}_{k,n}(\mathbf{x})$, Xiao, Srivastava and Zhang have obtained the following result:

Theorem 3.1 *(see [86]) Let C be a convex subset of a real vector space X, and $\{x_1,\ldots,x_n\}$ be a finite subset of C, where $n \ge 1$ is fixed. If $f : C \to \mathbb{R}$ is a mid-convex function, and*

$$F_{k,n} := \frac{1}{\binom{n+k-2}{k-1}} \sum_{\substack{i_1+\ldots+i_n=n+k-1 \\ i_j\in\mathbb{N}_+;\quad 1\le j\le n}} f\left(\frac{1}{n+k-1}\sum_{j=1}^{n} i_j x_j\right), \quad k\in\mathbb{N}_+, \tag{3.1}$$

then

(a)

$$f\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right) = F_{1,n} \le \ldots \le F_{k,n} \le F_{k+1,n} \le \ldots \le \frac{1}{n}\sum_{j=1}^{n} f(x_j).$$

(b)

$$F_{k,n} \le \overline{f}_{k,n}, \quad k\in\mathbb{N}_+.$$

The limit of the constructed increasing sequence is also determined. We recall this result too:

Theorem 3.2 *(see [86]) Let C be a convex subset of a real vector space X, and $\{x_1,\ldots,x_n\}$ be a finite subset of C, where $n \geq 1$ is fixed. Suppose $f : C \to \mathbb{R}$ is a mid-convex function. Define the function g on the set*

$$E_n := \left\{ (t_1,\ldots,t_{n-1}) \in \mathbb{R}^{n-1} \mid \sum_{j=1}^{n-1} t_j \leq 1, \quad t_j \geq 0, \ j = 1,\ldots,n-1 \right\} \tag{3.2}$$

by

$$g(t_1,\ldots,t_{n-1}) := f\left(\sum_{j=1}^{n-1} t_j x_j + \left(1 - \sum_{j=1}^{n-1} t_j\right) x_n \right).$$

If g is integrable over E_n, then

$$\lim_{k\to\infty} F_{k,n} = \lim_{k\to\infty} \overline{f}_{k,n} = (n-1)! \int_{E_n} g(t_1,\ldots,t_{n-1})\, dt_1 \ldots dt_{n-1}.$$

In Theorem 3.1 and 3.2 the discrete uniform distribution is used. Recently, Horváth has discussed in [34] some new weighted versions of Theorem 3.1 and 3.2 for convex and mid-convex functions.

A method has been developed to refine the discrete Jensen's inequality by Horváth given in Section 2.2. The results given in Section 2.2 include those considered in Section 2.1, but the method can not be applied to solve the present problem (details are given in [33]). In Section 2.3, a different approach led to a parameter dependent refinement, whose construction is similar to (3.1) in Theorem 3.1. However, the treatment of the problem in Section 2.3 is totally different from that in [86].

First, we give the generalization of Theorem 3.1. Moreover, we compare the expressions $F_{k,n}$, $f_{k,n}^2$ and $G_{k,n}$ (see 3.3).

The following conditions will be used:

($\mathscr{H}_1$) Let V be a real vector space, $C \subset V$ be a convex set, $\mathbf{x} := (x_1,...,x_n) \in C^n$, and let $\mathbf{p} := (p_1,...,p_n)$ be a positive n-tuples such that $\sum_{i=1}^{n} p_i = 1$.

($\tilde{\mathscr{H}}_1$) Let V be a real vector space, $C \subset V$ be a convex set, $\mathbf{x} := (x_1,...,x_n) \in C^n$, and let $\mathbf{p} := (p_1,...,p_n)$ be a nonnegative n-tuples such that $\sum_{i=1}^{n} p_i = 1$.

($\mathscr{H}_2$) Let $f : C \to \mathbb{R}$ be a convex function.

($\tilde{\mathscr{H}}_2$) Let $f : C \to \mathbb{R}$ be a mid-convex function, and $p_1,\ldots,p_n$ be rational numbers.

Theorem 3.3 *Assume ($\tilde{\mathscr{H}}_1$) and either ($\mathscr{H}_2$) or ($\tilde{\mathscr{H}}_2$). Define*

$$G_{k,n} = G_{k,n}(\mathbf{x},\mathbf{p})$$

$$:= \frac{1}{\binom{n+k-1}{k-1}} \sum_{\substack{i_1+\ldots+i_n=n+k-1 \\ i_j\in\mathbb{N}_+; \quad 1\leq j\leq n}} \left(\sum_{j=1}^{n} i_j p_j \right) f\left(\frac{1}{\sum\limits_{j=1}^{n} i_j p_j} \sum_{j=1}^{n} i_j p_j x_j \right), \quad k \in \mathbb{N}_+. \tag{3.3}$$

Then
(a)

$$f\left(\sum_{j=1}^{n} p_j x_j\right) = G_{1,n} \leq \ldots \leq G_{k,n} \leq G_{k+1,n} \leq \ldots \leq \sum_{j=1}^{n} p_j f(x_j).$$

(b)

$$F_{k,n} \leq G_{k,n}, \quad k \in \mathbb{N}_+.$$

(c) If the numbers $p_1, \ldots, p_n$ are positive, then

$$G_{k,n} \leq f_{k,n}^2, \quad k \in \mathbb{N}_+.$$

Remark 3.1 It is easy to see that in case $p_j = \frac{1}{n}$ $(1 \leq j \leq n)$

$$G_{k,n} = F_{k,n}, \quad k \in \mathbb{N}_+,$$

so $G_{k,n}$ is the weighted form of $F_{k,n}$.

Next, we extend Theorem 3.2.

Theorem 3.4 *Assume $(\mathscr{H}_1)$ and $(\mathscr{H}_2)$, where $n \geq 2$. Define the function h on the set E_n (see 3.2) by*

$$h(t_1, \ldots, t_{n-1}) := \left(\sum_{j=1}^{n-1} t_j p_j + \left(1 - \sum_{j=1}^{n-1} t_j\right) p_n\right) \tag{3.4}$$

$$\cdot f\left(\frac{1}{\sum\limits_{j=1}^{n-1} t_j p_j + \left(1 - \sum\limits_{j=1}^{n-1} t_j\right) p_n} \left(\sum_{j=1}^{n-1} t_j p_j x_j + \left(1 - \sum_{j=1}^{n-1} t_j\right) p_n x_n\right)\right).$$

(a) The function h is convex on E_n, and it is Riemann integrable over E_n.
(b)

$$\lim_{k\to\infty} G_{k,n} = \lim_{k\to\infty} f_{k,n}^2 = n! \int\limits_{E_n} h(t_1, \ldots, t_{n-1})\, dt_1 \ldots dt_{n-1}.$$

Xiao, Srivastava and Zhang seems to have regarded it as evident that the proof of Theorem 3.2 is valid for every integral concept. What does integrable mean in Theorem 3.2? The proof of Theorem 3.4 actually uses the Riemann integrability of h over E_n, but then f is essentially convex as the following result shows.

For a fixed subset $\{x_1, \ldots, x_n\}$ of C, only the restriction of f to the set

$$H := \left\{\sum_{j=1}^{n} \alpha_j x_j \in C \mid \sum_{j=1}^{n} \alpha_j = 1, \quad \alpha_j \geq 0,\ j = 1, \ldots, n\right\}$$

is important in Theorem 3.3 and 3.4.

Lemma 3.1 *Assume $(\mathscr{H}_1)$ and $(\tilde{\mathscr{H}_2})$, where $n \geq 2$. If the function h in (3.4) is Riemann integrable over E_n, then f is convex on the set*

$$\hat{H} := \left\{ \sum_{j=1}^{n} \alpha_j x_j \in C \mid \sum_{j=1}^{n} \alpha_j = 1, \quad \alpha_j > 0,\ j = 1, \ldots, n \right\}.$$

3.1 Preliminary results and the proofs

Lemma 3.2 *Let $p_1, \ldots, p_n$ be a discrete distribution with positive p_j's $(1 \leq j \leq n)$, and let $q_1, \ldots, q_n$ be another discrete distribution. Then there is a discrete distribution $t_1, \ldots, t_n$ such that*

$$\frac{t_i p_i}{\sum\limits_{j=1}^{n} t_j p_j} = q_i, \quad i = 1, \ldots, n. \tag{3.5}$$

Proof. At this proof the Perron-Frobenius theory comes into play (see [57]).

Suppose $q_j > 0$ $(1 \leq j \leq n)$. Consider the $n \times n$ matrix

$$A := \begin{pmatrix} q_1 & q_1 & \ldots & q_1 \\ q_2 & q_2 & \ldots & q_2 \\ \vdots & \vdots & \vdots & \vdots \\ q_n & q_n & \ldots & q_n \end{pmatrix}.$$

Since A is positive and $\sum\limits_{j=1}^{n} q_j = 1$, the Perron-Frobenius eigenvalue of A is 1. Then there exists an eigenvector $(v_1, \ldots, v_n)$ of A corresponding to the eigenvalue 1 such that $v_j > 0$ $(1 \leq j \leq n)$. It follows that $(v_1, \ldots, v_n)$ is a positive solution of the system of equations

$$\frac{x_i}{\sum\limits_{j=1}^{n} x_j} = q_i, \quad i = 1, \ldots, n. \tag{3.6}$$

It is easy to see that we can abandon the supplementary hypothesis on q_j $(1 \leq j \leq n)$: if $q_j \geq 0$ $(1 \leq j \leq n)$, then (3.6) has a nonnegative solution $(v_1, \ldots, v_n)$ different from $(0, \ldots, 0)$. In this case

$$\left(\frac{v_1}{p_1}, \ldots, \frac{v_n}{p_n} \right)$$

is a solution of (3.5). We have from this that

$$t_i = \frac{1}{\sum\limits_{j=1}^{n} \frac{v_j}{p_j}} \frac{v_i}{p_i}, \quad i = 1, \ldots, n$$

is appropriate.

The proof is complete. □

Proof of Theorem 3.3 We introduce the following set:

$$S_{k,n} := \left\{ (i_1, \dots, i_n) \in \mathbb{N}_+^n \mid \sum_{j=1}^{n} i_j = n+k-1 \right\}, \quad k \in \mathbb{N}_+.$$

(a) Since $S_{1,n} = \{(1,\dots,1)\}$

$$f\left(\sum_{v=1}^{n} p_v x_v\right) = G_{1,n}.$$

Next, we prove that

$$G_{k,n} \le G_{k+1,n}, \quad k \in \mathbb{N}_+.$$

Let $k \in \mathbb{N}_+$ be fixed. First we note that

$$\binom{n+k-1}{k-1} = \binom{n+k}{k}\frac{k}{n+k},$$

and therefore

$$k = \sum_{u=1}^{n} (i_u - 1), \quad (i_1, \dots, i_n) \in S_{k+1,n}$$

implies

$$G_{k+1,n} = \frac{1}{\binom{n+k-1}{k-1}} \frac{1}{n+k} \sum_{(i_1,\dots,i_n)\in S_{k+1,n}} \left(\sum_{u=1}^{n} (i_u - 1) \right.$$
$$\left. \cdot \left(\sum_{v=1}^{n} i_v p_v\right) f\left(\frac{1}{\sum_{v=1}^{n} i_v p_v} \sum_{v=1}^{n} i_v p_v x_v \right)\right).$$

By introducing

$$j_u := i_u - 1, \quad u = 1, \dots, n, \quad (i_1, \dots, i_n) \in S_{k+1,n},$$

we have that

$$G_{k+1,n} = \frac{1}{\binom{n+k-1}{k-1}} \frac{1}{n+k} \sum_{(j_1,\dots,j_n)\in S_{k,n}} \left(\sum_{u=1}^{n} j_u \left(\sum_{v=1}^{n} j_v p_v + p_u \right) \right.$$
$$\left. f\left(\frac{1}{\left(\sum_{v=1}^{n} j_v p_v + p_u\right)} \left(\sum_{v=1}^{n} j_v p_v x_v + p_u x_u \right)\right)\right). \tag{3.7}$$

It is easy to observe that

$$\sum_{u=1}^{n} j_u \left(\sum_{v=1}^{n} j_v p_v + p_u \right) = (n+k) \sum_{v=1}^{n} j_v p_v, \quad (j_1,\ldots,j_n) \in S_{k,n}, \tag{3.8}$$

and

$$\sum_{u=1}^{n} j_u \left(\sum_{v=1}^{n} j_v p_v x_v + p_u x_u \right) = (n+k) \sum_{v=1}^{n} j_v p_v x_v, \quad (j_1,\ldots,j_n) \in S_{k,n}. \tag{3.9}$$

With the help of the discrete Jensen's inequality (either for convex or mid convex function) (3.7), (3.8) and (3.9) yield

$$G_{k+1,n} \geq \frac{1}{\binom{n+k-1}{k-1}} \frac{1}{n+k} \sum_{(j_1,\ldots,j_n)\in S_{k,n}} \left((n+k) \sum_{v=1}^{n} j_v p_v \right.$$

$$\left. f\left(\frac{1}{(n+k)\sum\limits_{v=1}^{n} j_v p_v} \sum_{u=1}^{n} j_u \left(\sum_{v=1}^{n} j_v p_v x_v + p_u x_u \right) \right) \right)$$

$$= \frac{1}{\binom{n+k-1}{k-1}} \sum_{(j_1,\ldots,j_n)\in S_{k,n}} \left(\sum_{v=1}^{n} j_v p_v \right) f\left(\frac{1}{\sum\limits_{v=1}^{n} j_v p_v} \sum_{v=1}^{n} j_v p_v x_v \right) = G_{k,n}.$$

It remained to prove that

$$G_{k,n} \leq \sum_{v=1}^{n} p_v f(x_v), \quad k \in \mathbb{N}_+.$$

We can apply the discrete Jensen's inequality (either for convex or mid convex function) again, which insures

$$G_{k,n} = \frac{1}{\binom{n+k-1}{k-1}} \sum_{(i_1,\ldots,i_n)\in S_{k,n}} \left(\sum_{v=1}^{n} i_v p_v \right) f\left(\frac{1}{\sum\limits_{v=1}^{n} i_v p_v} \sum_{v=1}^{n} i_v p_v x_v \right)$$

$$\leq \frac{1}{\binom{n+k-1}{k-1}} \sum_{(i_1,\ldots,i_n)\in S_{k,n}} \sum_{v=1}^{n} i_v p_v f(x_v)$$

$$= \frac{1}{\binom{n+k-1}{k-1}} \sum_{v=1}^{n} \sum_{(i_1,\ldots,i_n)\in S_{k,n}} i_v p_v f(x_v), \quad k \in \mathbb{N}_+.$$

Since the set $S_{k,n}$ has $\binom{n+k-2}{k-1}$ elements

$$\frac{1}{\binom{n+k-1}{k-1}}\sum_{v=1}^{n}\sum_{(i_1,\ldots,i_n)\in S_{k,n}} i_v p_v f(x_v)=\frac{1}{\binom{n+k-1}{k-1}}\sum_{v=1}^{n}\binom{n+k-1}{k-1}p_v f(x_v)$$

$$=\sum_{v=1}^{n} p_v f(x_v), \quad k\in\mathbb{N}_+.$$

(b) Let $\pi_i(j)$ be the unique integer from $\{1,\ldots,n\}$ for which

$$\pi_i(j)\equiv i+j-1 \quad (mod\ n), \quad i,j=1,\ldots,n.$$

Then the functions π_i $(i=1,\ldots,n)$ are permutations of the numbers $1,\ldots,n$. Clearly, $\sum\limits_{j=1}^{n} p_{\pi_i(j)}=1$ $(i=1,\ldots,n)$, and $\pi_i(j)=\pi_j(i)$ $(i,j=1,\ldots,n)$.

Fix $k\in\mathbb{N}_+$. The previous establishments imply

$$F_{k,n}=\frac{1}{\binom{n+k-2}{k-1}}\sum_{(i_1,\ldots,i_n)\in S_{k,n}} f\left(\frac{1}{n+k-1}\sum_{v=1}^{n}\left(\sum_{u=1}^{n} p_{\pi_v(u)} i_v x_v\right)\right)$$

$$=\frac{1}{\binom{n+k-2}{k-1}}\sum_{(i_1,\ldots,i_n)\in S_{k,n}} f\left(\frac{1}{n+k-1}\sum_{u=1}^{n}\left(\sum_{v=1}^{n} p_{\pi_u(v)} i_v x_v\right)\right)=\frac{1}{\binom{n+k-2}{k-1}}$$

$$\sum_{(i_1,\ldots,i_n)\in S_{k,n}} f\left(\frac{1}{n+k-1}\sum_{u=1}^{n}\left(\sum_{w=1}^{n} p_{\pi_u(w)} i_w \sum_{v=1}^{n}\frac{p_{\pi_u(v)} i_v}{\sum\limits_{w=1}^{n} p_{\pi_u(w)} i_w} x_v\right)\right). \tag{3.10}$$

Noting that

$$\sum_{u=1}^{n}\left(\sum_{w=1}^{n} p_{\pi_u(w)} i_w\right)=\sum_{w=1}^{n} i_w=n+k-1,$$

the discrete Jensen's inequality (either for convex or mid convex function) can be applied in (3.10), and we get

$$F_{k,n}\le\frac{1}{\binom{n+k-2}{k-1}(n+k-1)}$$

$$\cdot\sum_{(i_1,\ldots,i_n)\in S_{k,n}}\sum_{u=1}^{n}\left(\sum_{w=1}^{n} p_{\pi_u(w)} i_w f\left(\sum_{v=1}^{n}\frac{p_{\pi_u(v)} i_v}{\sum\limits_{w=1}^{n} p_{\pi_u(w)} i_w} x_v\right)\right)$$

$$=\frac{1}{\binom{n+k-1}{k}}\frac{1}{n}\sum_{u=1}^{n}\left(\sum_{(i_1,\ldots,i_n)\in S_{k,n}}\sum_{w=1}^{n} p_{\pi_u(w)} i_w f\left(\sum_{v=1}^{n}\frac{p_{\pi_u(v)} i_v}{\sum\limits_{w=1}^{n} p_{\pi_u(w)} i_w} x_v\right)\right).$$

Since π_u $(u = 1,\ldots,n)$ is a permutation of the numbers $1,\ldots,n$, and $\pi_u(S_{k,n}) = S_{k,n}$ $(u = 1,\ldots,n)$ we can see that for every fixed $u \in \{1,\ldots,n\}$

$$\sum_{(i_1,\ldots,i_n)\in S_{k,n}} \sum_{w=1}^{n} p_{\pi_u(w)} i_w f\left(\sum_{v=1}^{n} \frac{p_{\pi_u(v)} i_v}{\sum\limits_{w=1}^{n} p_{\pi_u(w)} i_w} x_v\right)$$

$$= \sum_{(i_1,\ldots,i_n)\in S_{k,n}} \left(\sum_{v=1}^{n} i_v p_v\right) f\left(\frac{1}{\sum\limits_{v=1}^{n} i_v p_v} \sum_{v=1}^{n} i_v p_v x_v\right).$$

(c) Fix $k \in \mathbb{N}_+$. By the definition of $G_{k+1,n}$

$$G_{k+1,n} = \frac{1}{\binom{n+k}{k}} \sum_{(i_1,\ldots,i_n)\in S_{k+1,n}} \left(\sum_{v=1}^{n} i_v p_v\right) f\left(\frac{1}{\sum\limits_{v=1}^{n} i_v p_v} \sum_{v=1}^{n} i_v p_v x_v\right)$$

$$= \frac{1}{\binom{n+k}{k}} \sum_{(i_1,\ldots,i_n)\in S_{k+1,n}} \left(\sum_{v=1}^{n} (i_v - 1) p_v + \sum_{v=1}^{n} p_v\right)$$

$$\cdot f\left(\frac{1}{\sum\limits_{v=1}^{n} (i_v - 1) p_v + \sum\limits_{v=1}^{n} p_v} \left(\sum_{v=1}^{n} (i_v - 1) p_v x_v + \sum_{v=1}^{n} p_v x_v\right)\right)$$

$$= \frac{1}{\binom{n+k}{k}} \sum_{\substack{j_1+\ldots+j_n=k \\ j_l\in\mathbb{N};\quad 1\le l\le n}} \left(\sum_{v=1}^{n} j_v p_v + 1\right)$$

$$\cdot f\left(\frac{1}{\sum\limits_{v=1}^{n} j_v p_v + 1} \left(\sum_{v=1}^{n} j_v p_v \frac{\sum\limits_{v=1}^{n} j_v p_v x_v}{\sum\limits_{v=1}^{n} j_v p_v} + \sum_{v=1}^{n} p_v x_v\right)\right).$$

In this situation the discrete Jensen's inequality (either for convex or mid convex function) implies that

$$G_{k+1,n} \le \frac{1}{\binom{n+k}{k}}$$

$$\cdot \sum_{\substack{j_1+\ldots+j_n=k \\ j_l\in\mathbb{N};\quad 1\le l\le n}} \left(\left(\sum_{v=1}^{n} j_v p_v\right) f\left(\frac{1}{\sum\limits_{v=1}^{n} j_v p_v} \sum_{v=1}^{n} j_v p_v x_v\right) + f\left(\sum_{v=1}^{n} p_v x_v\right)\right)$$

$$= \frac{1}{\binom{n+k}{k}} \sum_{\substack{j_1+\ldots+j_n=k \\ j_l \in \mathbb{N};\quad 1\le l\le n}} \left(\sum_{v=1}^{n} j_v p_v\right) f\left(\frac{1}{\sum_{v=1}^{n} j_v p_v} \sum_{v=1}^{n} j_v p_v x_v\right)$$

$$+ \frac{\binom{n+k-1}{k}}{\binom{n+k}{k}} f\left(\sum_{v=1}^{n} p_v x_v\right).$$

From this, by means of Theorem 3.1, we get

$$G_{k+1,n} \le \left(\frac{1}{\binom{n+k}{k}}\left(1+\frac{\binom{n+k-1}{k}}{\binom{n+k-1}{k-1}}\right)\right)$$

$$\cdot \sum_{\substack{j_1+\ldots+j_n=k \\ j_l \in \mathbb{N};\quad 1\le l\le n}} \left(\sum_{v=1}^{n} j_v p_v\right) f\left(\frac{1}{\sum_{v=1}^{n} j_v p_v} \sum_{v=1}^{n} j_v p_v x_v\right) = B_{k,n}.$$

Combining this and (a) yields finally

$$G_{k,n} \le G_{k+1,n} \le B_{k,n}, \quad k \in \mathbb{N}_+.$$

The proof is complete. □

Proof of Theorem 3.4 (a) E_n is obviously a convex set, and by using the convexity of f, some elementary computation shows that h is convex. Since f is bounded on the convex set

$$\left\{\sum_{j=1}^{n} \alpha_j x_j \in X \mid \sum_{j=1}^{n} \alpha_j = 1, \quad \alpha_j \ge 0, \quad j=1,\ldots,n\right\},$$

h is bounded too. The convexity of h implies that it is continuous on the interior of E_n. The previous two establishments, together with the fact that the measure of the boundary of E_n is 0, yield that h is Riemann integrable over E_n.

(b) Fix $k \in \mathbb{N}_+$.

By the definition of $G_{k,n}$, elementary considerations show that

$$G_{k,n} = \frac{1}{\binom{n+k-1}{k-1}} \sum_{\substack{i_1+\ldots+i_n=n+k-1 \\ i_j \in \mathbb{N}_+;\quad 1\le j\le n}} \left(\sum_{j=1}^{n} i_j p_j\right) f\left(\frac{1}{\sum_{j=1}^{n} i_j p_j} \sum_{j=1}^{n} i_j p_j x_j\right)$$

$$= n! \frac{(n+k-2)^{n-2}}{k(k+1)\ldots(n+k-3)}$$

$$\cdot \frac{1}{(n+k-2)^{n-1}} \sum_{i_1=1}^{k} \sum_{i_2=1}^{k+1-i_1} \sum_{i_3=1}^{k+2-(i_1+i_2)} \cdots \sum_{i_{n-1}=1}^{n+k-2-(i_1+\ldots+i_{n-2})}$$

$$\left(\sum_{j=1}^{n-1}\frac{i_j}{n+k-1}p_j+\left(1-\sum_{j=1}^{n-1}\frac{i_j}{n+k-1}\right)p_n\right)$$

$$\cdot f\left(\frac{\sum\limits_{j=1}^{n-1}\frac{i_j}{n+k-1}p_jx_j+\left(1-\sum\limits_{j=1}^{n-1}\frac{i_j}{n+k-1}\right)p_nx_n}{\sum\limits_{j=1}^{n-1}\frac{i_j}{n+k-1}p_j+\left(1-\sum\limits_{j=1}^{n-1}\frac{i_j}{n+k-1}\right)p_n}\right).$$

Since h is Riemann integrable, the result for the sequence $\left(G_{k,n}\right)$ follows from this and from

$$\frac{i-1}{n+k-2}<\frac{i}{n+k-1}<\frac{i}{n+k-2},\quad i=1,\ldots,k.$$

Similarly, according to the definition of $B_{k,n}$, we have

$$B_{k,n}=\frac{1}{\binom{n+k-1}{k-1}}\sum_{\substack{i_1+\ldots+i_n=k\\ i_j\in\mathbb{N};\quad 1\leq j\leq n}}\left(\sum_{j=1}^{n}i_jp_j\right)\left(\frac{1}{\sum\limits_{j=1}^{n}i_jp_j}\sum_{j=1}^{n}i_jp_jx_j\right)$$

$$=n!\frac{(k+1)^{n-1}}{(k+1)\ldots(n+k-1)}\frac{1}{(k+1)^{n-1}}\sum_{i_1=0}^{k}\sum_{i_2=0}^{k-i_1}\sum_{i_3=0}^{k-(i_1+i_2)}\cdots\sum_{i_{n-1}=0}^{k-(i_1+\ldots+i_{n-2})}$$

$$\left(\sum_{j=1}^{n-1}\frac{i_j}{k}p_j+\left(1-\sum_{j=1}^{n-1}\frac{i_j}{k}\right)p_n\right)f\left(\frac{\sum\limits_{j=1}^{n-1}\frac{i_j}{k}p_jx_j+\left(1-\sum\limits_{j=1}^{n-1}\frac{i_j}{k}\right)p_nx_n}{\sum\limits_{j=1}^{n-1}\frac{i_j}{k}p_j+\left(1-\sum\limits_{j=1}^{n-1}\frac{i_j}{k}\right)p_n}\right).$$

By taking into account the Riemann integrability of h and

$$\frac{i}{k+1}<\frac{i}{k}<\frac{i+1}{k+1},\quad i=0,\ldots,k,$$

we have the result for the sequence $\left(B_{k,n}\right)$. □

Proof of Lemma 3.1 Let

$$p:=\min\left\{p_1,\ldots,p_n\right\}.$$

Then $p>0$ and

$$\sum_{j=1}^{n-1}t_jp_j+\left(1-\sum_{j=1}^{n-1}t_j\right)p_n\geq p,\quad (t_1,\ldots,t_{n-1})\in E_n.$$

Therefore, recalling the definition of h

$$f\left(\frac{1}{\sum_{j=1}^{n-1}t_jp_j+\left(1-\sum_{j=1}^{n-1}t_j\right)p_n}\left(\sum_{j=1}^{n-1}t_jp_jx_j+\left(1-\sum_{j=1}^{n-1}t_j\right)p_nx_n\right)\right)$$
$$\leq \frac{1}{p}h(t_1,\ldots,t_{n-1}),\quad (t_1,\ldots,t_{n-1})\in E_n. \tag{3.11}$$

By Lemma 3.2, the function

$$(t_1,\ldots,t_{n-1})\to\left(\frac{t_1p_1}{\sum_{j=1}^{n-1}t_jp_j+\left(1-\sum_{j=1}^{n-1}t_j\right)p_n},\right.$$
$$\left.\ldots,\frac{t_{n-1}p_{n-1}}{\sum_{j=1}^{n-1}t_jp_j+\left(1-\sum_{j=1}^{n-1}t_j\right)p_n},\frac{\left(1-\sum_{j=1}^{n-1}t_j\right)p_n}{\sum_{j=1}^{n-1}t_jp_j+\left(1-\sum_{j=1}^{n-1}t_j\right)p_n}\right)$$

maps E_n onto the set

$$\left\{(\alpha_1,\ldots,\alpha_n)\in\mathbb{R}^n \mid \sum_{j=1}^{n}\alpha_j=1,\quad \alpha_j\geq 0,\ j=1,\ldots,n\right\},$$

and hence (3.11) and the Riemann integrability of h over E_n (h is bounded on E_n) show that f is bounded above on H.

Since f is mid-convex, the function $\overline{h}$ defined on E_n by

$$\overline{h}(t_1,\ldots,t_{n-1}):=f\left(\sum_{j=1}^{n-1}t_jx_j+\left(1-\sum_{j=1}^{n-1}t_j\right)x_n\right)$$

is also mid-convex on E_n. Because f is bounded above on H, $\overline{h}$ is bounded above on E_n. These two properties of $\overline{h}$, together with the Bernstein-Doetsch theorem (see [55]) give that $\overline{h}$ is convex on the interior of E_n, and therefore f is convex on $\hat{H}$.

The proof is complete. □

3.2 Applications to Mixed means

As an application we introduce some new quasi-arithmetic means and study their monotonicity and convergence.

($\tilde{\mathscr{H}}_3$) Let $J \subset \mathbb{R}$ be an interval, $\mathbf{x} := (x_1, ..., x_n) \in J^n$, let $\mathbf{p} := (p_1, ..., p_n)$ be a nonnegative n-tuples such that $\sum_{i=1}^{n} p_i = 1$, and let h, $g : J \to \mathbb{R}$ be continuous and strictly monotone functions.

Definition 3.1 *Assume ($\tilde{\mathscr{H}}_3$). We define the quasi-arithmetic means with respect to (3.3) by*

$$M_{h,g}(k,\mathbf{x},\mathbf{p}) := h^{-1}\left(\frac{1}{\binom{n+k-1}{k-1}} \sum_{\substack{i_1+\ldots+i_n=n+k-1 \\ i_j \in \mathbb{N}_+; \quad 1\le j\le n}} \left(\sum_{j=1}^{n} i_j p_j\right)\right. \tag{3.12}$$

$$\left.\cdot(h \circ g^{-1})\left(\frac{1}{\sum_{j=1}^{n} i_j p_j} \sum_{j=1}^{n} i_j p_j g(x_j)\right)\right), \quad k \in \mathbb{N}_+.$$

We now prove the monotonicity of the means (3.12) and give limit formulas.

Proposition 3.1 *Assume ($\tilde{\mathscr{H}}_3$). Then*

(a)

$$M_g(\mathbf{x},\mathbf{p}) = M_{h,g}(1,\mathbf{x},\mathbf{p}) \le \ldots \le M_{h,g}(k,\mathbf{x},\mathbf{p}) \le \ldots \le M_h(\mathbf{x},\mathbf{p}), \quad k \in \mathbb{N}_+,$$

if either $h \circ g^{-1}$ is convex and h is increasing or $h \circ g^{-1}$ is concave and h is decreasing.

(b)

$$M_g(\mathbf{x},\mathbf{p}) = M_{h,g}(1,\mathbf{x},\mathbf{p}) \ge \ldots \ge M_{h,g}(k,\mathbf{x},\mathbf{p}) \ge \ldots \ge M_h(\mathbf{x},\mathbf{p}), \quad k \in \mathbb{N}_+,$$

if either $h \circ g^{-1}$ is convex and h is decreasing or $h \circ g^{-1}$ is concave and h is increasing.

(c) Moreover, in both cases

$$\lim_{k\to\infty} M_{h,g}(k,\mathbf{x},\mathbf{p}) = \psi^{-1}\left(n! \int_{E_n} h(t_1,\ldots,t_{n-1})\, dt_1 \ldots dt_{n-1}\right),$$

where the function h is defined on the set E_n (see 3.2) by

$$h(t_1,\ldots,t_{n-1}) := \left(\sum_{j=1}^{n-1} t_j p_j + \left(1 - \sum_{j=1}^{n-1} t_j\right) p_n\right) \left(h \circ g^{-1}\right)$$

$$\left(\frac{1}{\sum_{j=1}^{n-1} t_j p_j + \left(1 - \sum_{j=1}^{n-1} t_j\right) p_n} \left(\sum_{j=1}^{n-1} t_j p_j g(x_j) + \left(1 - \sum_{j=1}^{n-1} t_j\right) p_n g(x_n) \right) \right).$$

Proof. Theorem 3.3 (a) can be applied to the function $h \circ g^{-1}$, if it is convex ($-h \circ g^{-1}$, if it is concave) and the n-tuples $(g(x_1), \ldots, g(x_n))$, then upon taking h^{-1}, we get (a) and (b). (c) comes from Theorem 3.4 (b). □

As a special case we consider the following example.

Example 3.1 If $I := (0, \infty)$, $h := \ln$ and $g(x) := x$ $(x \in (0, \infty))$, then by Proposition 3.1 (b), we have the following sharpened version of the weighted arithmetic mean - geometric mean inequality: for every $x_j > 0$ $(1 \le j \le n)$ and $k \in \mathbb{N}_+$

$$\sum_{j=1}^{n} p_j x_j \ge \prod_{\substack{i_1 + \ldots + i_n = n+k-1 \\ i_j \in \mathbb{N}_+; \quad 1 \le j \le n}} \left(\frac{\sum_{j=1}^{n} i_j p_j x_j}{\sum_{j=1}^{n} i_j p_j} \right)^{\frac{1}{\binom{n+k-1}{k-1}} \sum_{j=1}^{n} i_j p_j} \ge \prod_{j=1}^{n} x_j^{p_j}.$$

Moreover, by Proposition 3.1 (c)

$$\lim_{k \to \infty} \prod_{\substack{i_1 + \ldots + i_n = n+k-1 \\ i_j \in \mathbb{N}_+; \quad 1 \le j \le n}} \left(\frac{\sum_{j=1}^{n} i_j p_j x_j}{\sum_{j=1}^{n} i_j p_j} \right)^{\frac{1}{\binom{n+k-1}{k-1}} \sum_{j=1}^{n} i_j p_j}$$

$$= \exp \left(n! \int_{E_n} h(t_1, \ldots, t_{n-1}) \, dt_1 \ldots dt_{n-1} \right),$$

where the function h is defined *on the set* E_n (see 3.2) by

$$h(t_1, \ldots, t_{n-1}) := \left(\sum_{j=1}^{n-1} t_j p_j + \left(1 - \sum_{j=1}^{n-1} t_j\right) p_n \right)$$

$$\cdot \ln \left(\frac{1}{\sum_{j=1}^{n-1} t_j p_j + \left(1 - \sum_{j=1}^{n-1} t_j\right) p_n} \left(\sum_{j=1}^{n-1} t_j p_j x_j + \left(1 - \sum_{j=1}^{n-1} t_j\right) p_n x_n \right) \right).$$

Chapter 4

Popoviciu Type Inequalities

In 1965 T. Popoviciu [67] has introduced a characterization of the convex functions of one real variable, relating the arithmetic mean of its values and the values taken at the barycenters of certain subfamilies of the given family of points. The inequality of Popoviciu as given by Vasić and Stanković in [77] (see also [69, p.173]) can be written in the following form:

Theorem 4.1 *Suppose that the conditions of Theorem 1.8 are satisfied. Then for $n \geq 3$ and $2 \leq k \leq n-1$*

$$f_{k,n}^{1}(\mathbf{x},\mathbf{p}) \leq \frac{n-k}{n-1} f_{1,n}^{1}(\mathbf{x},\mathbf{p}) + \frac{k-1}{n-1} f_{n,n}^{1}(\mathbf{x},\mathbf{p}), \tag{4.1}$$

where $f_{k,n}^{1}(\mathbf{x},\mathbf{p})$ is given by (1.8).

Corollary 4.1 ([52]) *Let $I \subset \mathbb{R}$ be an interval, $\mathbf{x} \in I^n$, $\mathbf{p}$ be a positive n-tuples such that $\sum_{i=1}^{n} p_i = 1$, and let $h,\ g : I \to \mathbb{R}$ be continuous and strictly monotone functions such that $h \circ g^{-1}$ is convex. We set $x_{i_j} = g(x_{i_j})$ and $f = h \circ g^{-1}$ in (4.1) to get*

$$h\left(M_{h,g}^{1}(\mathbf{x},\mathbf{p};k)\right) \leq \frac{n-k}{n-1} h\left(M_h(\mathbf{x},\mathbf{p})\right) + \frac{k-1}{n-1} h\left(M_g(\mathbf{x},\mathbf{p})\right).$$

Corollary 4.2 ([52]) *Let $s,t \in \mathbb{R}$ such that $s \leq t$, and let $\mathbf{x}$ and $\mathbf{p}$ be positive n-tuples such that $\sum_{i=1}^{n} p_i = 1$. Then we have*

$$M_{t,s}^{t}(\mathbf{x},\mathbf{p};k) \leq \frac{n-k}{n-1} M_t^{t}(\mathbf{x},\mathbf{p}) + \frac{k-1}{n-1} M_s^{t}(\mathbf{x},\mathbf{p}), \tag{4.2}$$

$$M_{s,t}^{s}(\mathbf{x},\mathbf{p};k) \geq \frac{n-k}{n-1} M_{s}^{s}(\mathbf{x},\mathbf{p}) + \frac{k-1}{n-1} M_{t}^{s}(\mathbf{x},\mathbf{p}), \tag{4.3}$$

where $M_{s,t}(\mathbf{x},\mathbf{p};k)$ means $M_{s,t}^{1}(\mathbf{x},\mathbf{p};k)$ in (1.48).

Proof. Let $s,t \in \mathbb{R}$ such that $s \leq t$, if $s,t \neq 0$, then we set $f(x) = x^{\frac{t}{s}}$, $x_{i_j} = x_{i_j}^{s}$ in (4.1) to obtain (4.2) and we set $f(x) = x^{\frac{s}{t}}$, $x_{i_j} = x_{i_j}^{t}$ in (4.1) to obtain (4.3).
When $s = 0$ or $t = 0$, we get the required results by taking limit. □

Remark 4.1 The unweighted versions of (1.48) and (1.46) were introduced in [58] with their monotone property. Hence Corollary 1.6, Corollary 1.5, Corollary 4.2 and Corollary 4.1 are weighted versions of corresponding results given in [58].

4.1 Generalization of Popoviciu's Inequality

Consider the Green function $G : [\alpha,\beta] \times [\alpha,\beta] \to \mathbb{R}$ defined as

$$G(t,s) = \begin{cases} \frac{(t-\beta)(s-\alpha)}{\beta-\alpha}, & \alpha \leq s \leq t, \\ \frac{(s-\beta)(t-\alpha)}{\beta-\alpha}, & t \leq s \leq \beta. \end{cases} \tag{4.4}$$

The function G is convex and continuous w.r.t s and due to symmetry also w.r.t t.

For any function $h \in C^2([\alpha,\beta])$, we have

$$h(x) = \frac{\beta - x}{\beta - \alpha} h(\alpha) + \frac{x-\alpha}{\beta-\alpha} h(\beta) + \int_{\alpha}^{\beta} G(x,s) h''(s) ds, \tag{4.5}$$

where the function G is defined in (4.4) (see [84]).

It is assumed in Theorem 4.1 that p_i $(i = 1,\ldots,n)$ are positive real numbers. Now we give the generalization of that result for real values of p_i $(i = 1,\ldots,n)$ with $\sum_{i=1}^{n} p_i = 1$ using the Green function defined in (4.4).

Theorem 4.2 ([53]) *Let $n,k \in \mathbb{N}$, $n \geq 3$, $2 \leq k \leq n-1$, $[\alpha,\beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1,\ldots,x_n) \in [\alpha,\beta]^n$, $\mathbf{p} = (p_1,\ldots,p_n)$ be a real n-tuple such that $\sum_{j=1}^{k} p_{i_j} \neq 0$ for any $1 \leq i_1 < \ldots < i_k \leq n$ and $\sum_{i=1}^{n} p_i = 1$. Also let $\frac{\sum_{j=1}^{k} p_{i_j} x_{i_j}}{\sum_{j=1}^{k} p_{i_j}} \in [\alpha,\beta]$ for any $1 \leq i_1 < \ldots < i_k \leq n$. Then the following statements are equivalent:*

(i) For every continuous convex function $f : [\alpha,\beta] \to \mathbb{R}$

$$f_{k,n}(\mathbf{x},\mathbf{p}) \leq \frac{n-k}{n-1} f_{1,n}(\mathbf{x},\mathbf{p}) + \frac{k-1}{n-1} f_{n,n}(\mathbf{x},\mathbf{p}), \tag{4.6}$$

where

$$f_{k,n}(\mathbf{x},\mathbf{p}) := \frac{1}{\binom{n-1}{k-1}} \sum_{1\le i_1<\ldots<i_k\le n} \left(\sum_{j=1}^{k} p_{i_j}\right) f\left(\frac{\sum_{j=1}^{k} p_{i_j}x_{i_j}}{\sum_{j=1}^{k} p_{i_j}}\right).$$

(ii) For all $s \in [\alpha,\beta]$

$$G_{k,n}(\mathbf{x},s,\mathbf{p}) \le \frac{n-k}{n-1} G_{1,n}(\mathbf{x},s,\mathbf{p}) + \frac{k-1}{n-1} G_{n,n}(\mathbf{x},s,\mathbf{p}), \tag{4.7}$$

where

$$G_{k,n}(\mathbf{x},s,\mathbf{p}) := \frac{1}{\binom{n-1}{k-1}} \sum_{1\le i_1<\ldots<i_k\le n} \left(\sum_{j=1}^{k} p_{i_j}\right) G\left(\frac{\sum_{j=1}^{k} p_{i_j}x_{i_j}}{\sum_{j=1}^{k} p_{i_j}}, s\right), \quad 1 \le k \le n,$$

for the function $G : [\alpha,\beta]\times[\alpha,\beta] \to \mathbb{R}$ defined in (4.4).
Moreover, the statements (i) and (ii) are also equivalent if we change the sign of inequality in both (4.6) and (4.7).

Proof. (i)⇒(ii): Let (i) be valid. Since the function $G(\cdot,s)$ ($s \in [\alpha,\beta]$) is also continuous and convex, (4.7) is a special case of (4.6).

(ii)⇒(i): Let $f : [\alpha,\beta] \to \mathbb{R}$ be a convex function such that $f \in C^2([\alpha,\beta])$ and (ii) holds. Then, we can represent f in the form (4.5). Now by means of some simple calculations we can write

$$\begin{aligned} &\tfrac{n-k}{n-1} f_{1,n}(\mathbf{x},\mathbf{p}) + \tfrac{k-1}{n-1} f_{n,n}(\mathbf{x},\mathbf{p}) - f_{k,n}(\mathbf{x},\mathbf{p}) \\ &= \int_{\alpha}^{\beta} \left(\tfrac{n-k}{n-1} G_{1,n}(\mathbf{x},s,\mathbf{p}) + \tfrac{k-1}{n-1} G_{n,n}(\mathbf{x},s,\mathbf{p}) - G_{k,n}(\mathbf{x},s,\mathbf{p})\right) f''(s)ds. \end{aligned} \tag{4.8}$$

By the convexity of f, we have $f''(s) \ge 0$ for all $s \in [\alpha,\beta]$. Hence, if for every $s \in [\alpha,\beta]$, (4.7) is valid, then it follows that for every convex function $f : [\alpha,\beta] \to \mathbb{R}$, with $f \in C^2([\alpha,\beta])$, (4.6) is valid.

Here we can eliminate the differentiability condition due to the fact that it is possible to approximate uniformly a continuous convex function by convex polynomials [69, p.172].

Analogous to the above proof we can give the proof of the last part of our theorem. □

Remark 4.2 Note that in the case when $\mathbf{p}$ is a positive n-tuple, the inequality (4.6) gives (4.1).

Remark 4.3 Consider $n,k \in \mathbb{N}$, $n \ge 3$, $2 \le k \le n-1$, $[\alpha,\beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1,\ldots,x_n) \in [\alpha,\beta]^n$, $\mathbf{p} = (p_1,\ldots,p_n)$ be a real n-tuple such that $\sum_{j=1}^{k} p_{i_j} \ne 0$ for any $1 \le i_1 < \ldots < i_k \le n$ and $\sum_{i=1}^{n} p_i = 1$. Also assume that $\frac{\sum_{j=1}^{k} p_{i_j}x_{i_j}}{\sum_{j=1}^{k} p_{i_j}} \in [\alpha,\beta]$ for any $1 \le i_1 < \ldots < i_k \le n$.

If for all $s \in [\alpha, \beta]$ the inequality (4.7) holds then from the above theorem we have

$$\Upsilon_9(f) = \Upsilon_9(f, \mathbf{x}, \mathbf{p}) := \frac{n-k}{n-1} f_{1,n}(\mathbf{x}, \mathbf{p}) + \frac{k-1}{n-1} f_{n,n}(\mathbf{x}, \mathbf{p}) - f_{k,n}(\mathbf{x}, \mathbf{p}) \geq 0.$$

Further, we give an extension of the inequality (6.4) in [69, p.174] by Popoviciu.

Theorem 4.3 ([53]) *Let $n, k \in \mathbb{N}$, $n \geq 3$, $2 \leq k \leq n-1$, $a > 0$, $\mathbf{x} = (x_1, \ldots, x_n) \in (0, a]^n$ such that $\sum_{i=1}^n x_i \leq a$. If $f : (0, a] \to \mathbb{R}$ is a function such that $x \to \frac{f(x)}{x}$, $x \in (0, a]$ is convex, then*

$$f_{k,n}(\mathbf{x}) \leq \frac{n-k}{n-1} f_{1,n}(\mathbf{x}) + \frac{k-1}{n-1} f_{n,n}(\mathbf{x}), \tag{4.9}$$

where

$$f_{k,n}(\mathbf{x}) := \frac{1}{\binom{n-1}{k-1}} \sum_{1 \leq i_1 < \ldots < i_k \leq n} f\left(\sum_{j=1}^{k} x_{i_j} \right).$$

Proof. For $k = 2$ and $n = 3$ the result follows from inequality (6.4) in [69, p.174]. For $(n, k) \neq (3, 2)$ the result comes from the case $(n, k) = (3, 2)$ and from Theorem 6.9 in [69, p.176]. □

By analyzing the proofs of Theorem 6.5 and Theorem 6.9 in [69, p.174], we can give another version of the previous result.

Theorem 4.4 *Let $n, k \in \mathbb{N}$, $n \geq 3$, $2 \leq k \leq n-1$, $a > 0$, $\mathbf{x} = (x_1, \ldots, x_n) \in (0, a]^n$ such that $\sum_{i=1}^n x_i \leq a$. Let $0 < \alpha \leq \min_{1 \leq i \leq n} x_i$. If $f : (0, a] \to \mathbb{R}$ is a function such that $x \to \frac{f(x)}{x}$, $x \in [\alpha, a]$ is convex, then (4.9) also holds.*

Remark 4.4 Under the conditions of either Theorem 4.3 or Theorem 4.4

$$\Psi(f) = \Psi(k, \mathbf{x}, f) := \frac{n-k}{n-1} f_{1,n}(\mathbf{x}) + \frac{k-1}{n-1} f_{n,n}(\mathbf{x}) - f_{k,n}(\mathbf{x}) \geq 0. \tag{4.10}$$

4.2 Popoviciu Inequality for $2D$-Convex Functions

The refinement of (4.1) is given in [63], while in [62] the integral form has been established. In [53] the generalization of Theorem 4.1 is given for real values of weights p_i's by using the Green function associated to second order differential operator with homogenous boundary conditions. Motivated by Hlawka's inequality (see [30]), in 2010, Bencze et al. extented the Popoviciu's inequality for functions of several variables [8]. For this purpose they introduced a new concept of convex function, namely $2D$-convex function. Every 2D-convex function is convex in the usual sense, but there are convex functions which are not 2D-convex.

Definition 4.1 *Let U be a convex subset of a real linear space V. A function $f : U \to \mathbb{R}$ is called $2D$-convex if it verifies the inequality*

$$(C_{2,3}): \quad \sum_{1 \le i_1 < i_2 \le 3} \left(\sum_{j=1}^{2} p_{i_j} \right) f \left(\frac{\sum_{j=1}^{2} p_{i_j} x_{i_j}}{\sum_{j=1}^{2} p_{i_j}} \right) \le \sum_{i=1}^{3} p_i f(x_i) + \left(\sum_{i=1}^{3} p_i \right) f \left(\frac{\sum_{i=1}^{3} p_i x_i}{\sum_{i=1}^{3} p_i} \right),$$

for all $x_1, x_2, x_3 \in U$ and $p_1, p_2, p_3 \ge 0$, with $p_1 + p_2 + p_3 > 0$.

For more than three points the $2D$ analogue of Jensen's inequality is given in [8] as follows.

Theorem 4.5 *If $f : U \to \mathbb{R}$ is a $2D$-convex function then*

$$\left(C_{k,n}\right): \quad f_{k,n}(\mathbf{x}, \mathbf{p}) \le \frac{n-k}{n-1} f_{1,n}(\mathbf{x}, \mathbf{p}) + \frac{k-1}{n-1} f_{n,n}(\mathbf{x}, \mathbf{p}),$$

where

$$f_{k,n}(\mathbf{x}, \mathbf{p}) := \frac{1}{\binom{n-1}{k-1}} \sum_{1 \le i_1 < \ldots < i_k \le n} \left(\sum_{j=1}^{k} p_{i_j} \right) f \left(\frac{\sum_{j=1}^{k} p_{i_j} x_{i_j}}{\sum_{j=1}^{k} p_{i_j}} \right),$$

and $\mathbf{x} = (x_1, \ldots, x_n) \in U^n$, $n \ge 3$, $k \in \{2, \ldots, n\}$, and $\mathbf{p} = (p_1, \ldots, p_n)$ is a positive tuple.

In [8] the proof is given by mathematical induction. But here we prove Theorem 4.5 as a consequence of a more general result given by Vasić and Adamović [1]. After some modification the result of Vasić and Adamović [67] (see also [69, p.176]) is given as follows:

Consider D as a commutative additive semigroup and let $E \subset D$ be a non-empty set satisfy the condition:

$a_i \in E$ for $i = 1, \ldots, n$ and $\sum_{i=1}^{n} a_i \in E \Rightarrow \sum_{j=1}^{k} a_{i_j} \in E$ for $1 \le i_1 < \ldots < i_k \le n$.

Further, suppose that G be a commutative additive group with total order ($\le$ is a total order, satisfies $a < b \Rightarrow a + c < b + c$; $a, b, c \in G$).

Theorem 4.6 *For any given $g : E \to G$ and $2 \le k \le n$, let*

$$\left(P_{k,n}\right): \quad g_{k,n}(\mathbf{a}) \le \frac{n-k}{n-1} g_{1,n}(\mathbf{a}) + \frac{k-1}{n-1} g_{n,n}(\mathbf{a}),$$

where

$$g_{k,n}(\mathbf{a}) = g_{k,n}(a_1, \ldots, a_n) := \frac{1}{\binom{n-1}{k-1}} \sum_{1 \le i_1 < \ldots < i_k \le n} g \left(\sum_{j=1}^{k} a_{i_j} \right),$$

and $\mathbf{a} \in E^n$, $\sum\limits_{i=1}^{n} a_i \in E$. *Then*
a) $(P_{2,3})$ *implies* $\left(P_{k,n}\right)$,
b) For $(k,n) \neq (2,3)$ $\left(P_{k,n}\right)$ *implies* $(P_{2,3})$ *if* D *contains additive identity '0',* $0 \in E$ *and* $f(0) = 0$.

To apply Theorem 4.6, we make use of scheme introduced by Popoviciu in [67] (see also [69, p.179]).

Let L be a real linear space and U be a convex set in L. Define a semi-group structure on $L \times (0,\infty)\,(= D)$. The operation "+" is defined by

$$X + Y = (x,p) + (y,p) = \left(\frac{px+qy}{p+q}, p+q\right); \quad X,Y \in D.$$

Obviously this operation is commutative and associative, also

$$X_1,\dots,X_n \in E \Rightarrow \sum_{i=1}^{n} X_i \in E,$$

where $E = U \times (0,\infty)$.

If $f : U \to \mathbb{R}$ is a $2D$-convex and $g : E \to \mathbb{R}$ is defined by

$$g(X) = pf(x); \quad X = (x,p) \in E,$$

then $\left(P_{k,n}\right)$ becomes $\left(C_{k,n}\right)$.

Hence by applying Theorem 4.6 (a), we get Theorem 4.5.

Remark 4.5 As $0 \in L$, so if $0 \in U$ and $f(0) = 0$, then from Theorem 4.6 (b) we have the converse of Theorem 4.5.

Chapter 5

Refinements Including Integral Jensen's Inequality

The following refinement of the discrete Jensen's inequality is proved in [20].

Theorem 5.1 *Let C be a convex subset of a real vector space V, and let $f : C \to \mathbb{R}$ be a convex function. If $r_1, \dots, r_k$ are nonnegative numbers with $r_1 + \dots + r_k = 1$, and $v_1, \dots, v_k \in V$, then*

$$f\left(\sum_{i=1}^{k} r_i v_i\right) \le \sum_{i_1,\dots,i_{n+1}=1}^{k} r_{i_1} \dots r_{i_{n+1}} f\left(\frac{v_{i_1} + \dots + v_{i_{n+1}}}{n+1}\right) \tag{5.1}$$

$$\le \sum_{i_1,\dots,i_n=1}^{k} r_{i_1} \dots r_{i_n} f\left(\frac{v_{i_1} + \dots + v_{i_n}}{n}\right) \le \sum_{i=1}^{k} r_i f(v_i), \quad n \ge 1.$$

Next result is taken from [17].

Theorem 5.2 *Let C be a convex subset of a real vector space V, and let $f : C \to \mathbb{R}$ be a convex function. Let $r_1, \dots, r_k$ be nonnegative numbers with $r_1 + \dots + r_k = 1$, and let $v_1, \dots, v_k \in V$. If $p_1, \dots, p_n$ are nonnegative numbers with $p_1 + \dots + p_n = 1$, then*

$$f\left(\sum_{i=1}^{k} r_i v_i\right) \le \sum_{i_1,\dots,i_n=1}^{k} r_{i_1} \dots r_{i_n} f\left(\frac{v_{i_1} + \dots + v_{i_n}}{n}\right) \tag{5.2}$$

$$\le \sum_{i_1,\dots,i_n=1}^{k} r_{i_1} \dots r_{i_n} f\left(p_1 v_{i_1} + \dots p_n v_{i_n}\right) \le \sum_{i=1}^{k} r_i f(v_i), \quad 1 \le n \le k.$$

Inspired by (5.1) and (5.2), Horváth [31] established some new inequalities in a measure theoretical setting. We have some refinements of the classical Jensen's inequality from the results.

We need some facts from measure and integration theory from [31] (see also [29]).

Let the index set T be either $\{1,...,n\}$ with $n \in \mathbb{N}_+$ or $\mathbb{N}_+$.

Suppose we are given a family $\{A_i \mid i \in T\}$ of non-empty sets. If S is a non-empty subset of T, then the projection mapping $pr_S^T : \underset{i\in T}{\times} A_i \to \underset{i\in S}{\times} A_i$ is defined by associating with every point of $\underset{i\in T}{\times} A_i$ its restriction to S. We write for short pr_i^n for $pr_{\{i\}}^T$ $(i \in T = \{1,\ldots,n\})$ and pr_i^∞ for $pr_{\{i\}}^T$ $(i \in T = \mathbb{N}_+)$. Similarly, if $T = \mathbb{N}_+$, then $pr_{1\ldots k}^\infty$ means $pr_{\{1,\ldots,k\}}^T$ $(k \in \mathbb{N}_+)$.

Consider the probability spaces $(Y_i, \mathscr{B}_i, \nu_i)$, $i \in T$. The product of these spaces is denoted by $(Y^T, \mathscr{B}^T, \nu^T)$, i.e $Y^T := \underset{i\in T}{\times} Y_i$ and $\mathscr{B}^T$ is the smallest σ-algebra in Y such that each $pr_{\{i\}}^T$ is $\mathscr{B}^T - \mathscr{B}_i$ measurable $(i \in T)$. If $T = \{1,...,n\}$, then ν^T is the only measure on $\mathscr{B}^T$ which satisfies

$$\nu^T(B_1 \times ... \times B_n) = \nu_1(B_1)...\nu_n(B_n)$$

for every $B_i \in \mathscr{B}_i$. If $T = \mathbb{N}_+$, then ν^T is the unique measure on $\mathscr{B}^T$ such that the image measure of ν^T under the projection mapping $pr_{1\ldots k}^\infty$ is the product of the measures $\nu_1,...,\nu_k$ $(k \in \mathbb{N}_+)$. We observe that $\left(Y^T, \mathscr{B}^T, \nu^T\right)$ is also a probability space. The n-fold $(n \geq 1$ or $n = \infty)$ product of the probability spaces $(X, \mathscr{A}, \mu)$ is denoted by $(X^n, \mathscr{A}^n, \mu^n)$. We suppose that the μ-integrability of a function $g : X \to \mathbb{R}$ over X implies the measurability of g.

Theorem 5.3 ([31]) *Let $I \subset \mathbb{R}$ be an interval, and let $f : I \to \mathbb{R}$ be a convex function. Let (Y_i, B_i, ν_i), $i \in T := \{1,...,n\}$ be probability spaces, and $u_i : Y_i \to I$ be a ν_i-integrable function over Y_i $(i \in T)$. Assume $p_1,...,p_n$ are nonnegative numbers such that $\sum_{i=1}^n p_i = 1$. If $f \circ u_i$ is ν_i-integrable over Y_i $(i \in T)$, then*

$$f\left(\sum_{i=1}^n p_i \int_{Y_i} u_i d\nu_i\right) \leq \int_{Y^T} f\left(\sum_{i=1}^n p_i u_i(y_i)\right) d\nu^T(y_1,...,y_n) \leq \sum_{i=1}^n p_i \int_{Y_i} f \circ u_i d\nu_i. \tag{5.3}$$

The next theorem corresponds to the asymptotic behavior of the sequence

$$\int_{Y^{\{1,\ldots,n\}}} f\left(\frac{1}{n}\sum_{i=1}^n u_i(y_i)\right) d\nu^{\{1,\ldots,n\}}(y_1,\ldots,y_n), \quad n \in \mathbb{N}_+, \tag{5.4}$$

in some cases. (5.4) corresponds to the middle member of (5.3).

Theorem 5.4 ([31]) *Let $I \subset \mathbb{R}$ be an interval, and let $f : I \to \mathbb{R}$ be a convex and bounded function. Let (Y_i, B_i, ν_i), $i \in \mathbb{N}_+$ be probability spaces and u_i: $Y_i \to I$ be a square ν_i-integrable function over $Y_i (i \in \mathbb{N}_+)$ such that*

$$\int_{Y_i} u_i d\nu_i = \int_{Y_1} u_1 d\nu_1, \; i \in \mathbb{N}_+, \tag{5.5}$$

and

$$\sum_{i=1}^{\infty} \frac{1}{i^2} \int_{Y_i} u_i^2 d\nu_i < \infty. \tag{5.6}$$

Then

$$\lim_{n\to\infty} \int_{Y^{\{1,\dots,n\}}} f\left(\frac{1}{n}\sum_{i=1}^{n} u_i(y_i)\right) d\nu^{\{1,\dots,n\}}(y_1,\dots,y_n) = f\left(\int_{Y_1} u_1 d\nu_1\right).$$

The following result corresponds to (5.1) and (5.2).

Theorem 5.5 ([31]) *Let $I \subset \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a convex function. Let (X, A, μ) be probability space, $u : X \to I$ be a μ-integrable function over X such that $f \circ u$ is μ-integrable over X. Assume $p_1, \dots, p_n$ are nonnegative numbers such that $\sum_{i=1}^{n} p_i = 1$. Then*
(a)

$$f\left(\int_X u d\mu\right) \le \int_{X^n} f\left(\sum_{i=1}^{n} p_i u(x_i)\right) d\mu^n(x_1,\dots,x_n) \le \int_X f \circ u d\mu. \tag{5.7}$$

(b)

$$\begin{aligned} &\int_{X^{n+1}} f\left(\frac{1}{n+1}\sum_{i=1}^{n+1} u(x_i)\right) d\mu^{n+1}(x_1,\dots,x_{n+1}) \\ &\le \int_{X^n} f\left(\frac{1}{n}\sum_{i=1}^{n} u(x_i)\right) d\mu^n(x_1,\dots,x_n) \le \int_{X^n} f\left(\sum_{i=1}^{n} p_i u(x_i)\right) d\mu^n(x_1,\dots,x_n). \end{aligned} \tag{5.8}$$

(c) If f is bounded, then

$$\lim_{n\to\infty} \int_{X^n} f\left(\frac{1}{n}\sum_{i=1}^{n} u(x_i)\right) d\mu^n(x_1,\dots,x_n) = f\left(\int_X u d\mu\right).$$

Suppose $V := \mathbb{R}$ and $C := I$ in Theorem 5.1 and Theorem 5.2. Then the inequalities (5.1) and (5.2) can be obtained easily from the inequalities in Theorem 5.3 and Theorem 5.5, but Theorem 5.3 and Theorem 5.5 are in more general settings.

An application: As an application, we give some discrete inequalities.

Let $I \subset \mathbb{R}$ be an interval, and let $f : I \to \mathbb{R}$ be a convex function on I. Suppose $Y_i := \{1, \dots, k_i\}$ $(i \in \mathbb{N}_+)$, $\mathscr{B}_i$ is the power set of Y_i $(i \in \mathbb{N}_+)$, and $\nu_i(\{j\}) := r_{ij} \ge 0$ $(i \in \mathbb{N}_+,$ $j = 1, \dots, k_i)$ such that $\sum_{j=1}^{k_i} r_{ij} = 1$ $(i \in \mathbb{N}_+)$. Let $u_i(j) := v_{ij} \in I$ $(i \in \mathbb{N}_+,\ j = 1, \dots, k_i)$.

Suppose that for a fixed $n \in \mathbb{N}_+$, $p_1, \ldots, p_n$ are nonnegative numbers with $p_1 + \ldots + p_n = 1$. Now inequality (5.3) gives that

$$f\left(\sum_{i=1}^{n} p_i \left(\sum_{j=1}^{k_i} r_{ij} v_{ij}\right)\right) \leq \sum_{(j_1,\ldots,j_n) \in Y^{\{1,\ldots,n\}}} f\left(\sum_{i=1}^{n} p_i v_{ij_i}\right) r_{1j_1} \ldots r_{nj_n} \tag{5.9}$$
$$\leq \sum_{i=1}^{n} p_i \left(\sum_{j=1}^{k_i} r_{ij} f(v_{ij})\right),$$

which generalizes (5.2) (except the second member).

If f is bounded on I and

$$\sum_{j=1}^{k_i} r_{ij} v_{ij} = m, \quad i \in \mathbb{N}_+, \tag{5.10}$$

and

$$\sum_{i=1}^{\infty} \frac{1}{i^2} \left(\sum_{j=1}^{k_i} r_{ij} v_{ij}^2\right) < \infty. \tag{5.11}$$

then it comes from Theorem 5.4 that

$$\lim_{n \to \infty} \sum_{(j_1,\ldots,j_n) \in Y^{\{1,\ldots,n\}}} f\left(\frac{1}{n} \sum_{i=1}^{n} v_{ij_i}\right) r_{1j_1} \ldots r_{nj_n} = f(m).$$

Especially, suppose $f := -\ln$, $p_i > 0$ $(i = 1, \ldots, n)$, $r_{ij} > 0$ and $v_{ij} > 0$ $(i \in \mathbb{N}_+,\ j = 1, \ldots, k_i)$. Then (5.9) yields that

$$-\ln\left(\sum_{i=1}^{n} p_i \left(\sum_{j=1}^{k_i} r_{ij} v_{ij}\right)\right) \leq -\ln\left(\prod_{(j_1,\ldots,j_n) \in Y^{\{1,\ldots,n\}}} \left(\sum_{i=1}^{n} p_i v_{ij_i}\right)^{r_{1j_1} \ldots r_{nj_n}}\right)$$
$$\leq -\ln\left(\prod_{i=1}^{n}\left(\left(\prod_{j=1}^{k_i} v_{ij}^{r_{ij}}\right)^{p_i}\right)\right),$$

which shows the next inequality

$$\sum_{i=1}^{n} p_i \left(\sum_{j=1}^{k_i} r_{ij} v_{ij}\right) \geq \prod_{(j_1,\ldots,j_n) \in Y^{\{1,\ldots,n\}}} \left(\sum_{i=1}^{n} p_i v_{ij_i}\right)^{r_{1j_1} \ldots r_{nj_n}} \tag{5.12}$$
$$\geq \prod_{i=1}^{n}\left(\left(\prod_{j=1}^{k_i} v_{ij}^{r_{ij}}\right)^{p_i}\right).$$

Further, if there are positive numbers a and b such that

$$a \leq v_{ij} \leq b, \quad i \in \mathbb{N}_+, \quad j = 1, \ldots, k_i,$$

and (5.10) holds (obviously, (5.11) satisfies too), then

$$\lim_{n\to\infty}\prod_{(j_1,\ldots,j_n)\in Y^{\{1,\ldots,n\}}}\left(\frac{1}{n}\sum_{i=1}^{n}v_{ij_i}\right)^{r_{1j_1}\ldots r_{nj_n}}=m. \tag{5.13}$$

In order to proof the results as transparent as possible, we begin some preparatory lemmas. First, we investigate the integrability properties of some functions.

Lemma 5.1 *Let $I\subset\mathbb{R}$ be an interval, and let $f:I\to\mathbb{R}$ be a convex function on I. Let $(Y_i,\mathscr{B}_i,\nu_i)$, $i\in T:=\{1,\ldots,n\}$ be probability spaces, and let the function $u_i:Y_i\to I$ be ν_i-integrable over Y_i $(i=1,\ldots,n)$.*

(a) The function $u_i\circ pr_i^n$ is ν^T-integrable over Y^T, and

$$\int_{Y^T}u_i\circ pr_i^n d\nu^T=\int_{Y_i}u_i d\nu_i,\quad i=1,\ldots,n. \tag{5.14}$$

(b) If $f\circ u_i$ is also ν_i-integrable over Y_i $(i=1,\ldots,n)$, then the function

$$f\circ\left(\sum_{i=1}^{n}p_i\left(u_i\circ pr_i^n\right)\right),\quad \text{where}\quad p_i\geq 0\quad (i=1,\ldots,n),\quad \sum_{i=1}^{n}p_i=1 \tag{5.15}$$

is ν^T-integrable over Y^T.

Proof. (a) Since the image measure of ν^T under the mapping pr_i^n is ν_i $(i=1,\ldots,n)$, and u_i is ν_i-integrable over Y_i $(i=1,\ldots,n)$, we therefore get from the connection between ν^T-integrals and ν_i-integrals that $u_i\circ pr_i^n$ is ν^T-integrable over Y^T $(i=1,\ldots,n)$, and (5.14) holds.

(b) Since the range of u_i is a subset of I $(i=1,\ldots,n)$, the properties of the numbers $p_1,\ldots,p_n$ in (5.15) imply that the range of the function

$$\sum_{i=1}^{n}p_i\left(u_i\circ pr_i^n\right)$$

is a subset of I too. Thus the domain of the function (5.15) is Y^T.

The function f, being convex on I, is lower semicontinuous on I, and therefore f is measurable on I.

The measurability of the function (5.15) now follows from the above statements.

Let a be a fixed interior point of I. Now the convexity of f on I insures that

$$f(t)\geq f(a)+f'_+(a)(t-a),\quad t\in I,$$

where $f'_+(a)$ denotes the right-hand derivative of f at a. Using the previous inequality, and the convexity of f again, we have

$$f(a)+f'_+(a)\left(\sum_{i=1}^{n}p_iu_i(y_i)-a\right)\leq f\left(\sum_{i=1}^{n}p_iu_i(y_i)\right) \tag{5.16}$$

$$\leq\sum_{i=1}^{n}p_if(u_i(y_i)),\quad (y_1,\ldots,y_n)\in Y^T.$$

By what has already been proved in (a) the lower bound for the function (5.15) in (5.16) is ν^T-integrable over Y^T. The condition on $f \circ u_i$ ensures that u_i can be replaced by $f \circ u_i$ in (a), and thus the upper bound for the function (5.15) in (5.16) is ν^T-integrable over Y^T too. These and the inequality (5.16), together with the measurability of the function (5.15) on Y^T imply the ν^T-integrability of the function (5.15) on Y^T.

The proof is now complete. □

We derive an analog of Lemma 5.1 for a sequence of probability spaces.

Lemma 5.2 *Let $I \subset \mathbb{R}$ be a bounded interval, and let $f : I \to \mathbb{R}$ be a convex and bounded function on I. Let $(Y_i, \mathscr{B}_i, \nu_i)$, $i \in T := \mathbb{N}_+$ be probability spaces, and let the function $u_i : Y_i \to I$ be ν_i-integrable over Y_i $(i \in \mathbb{N}_+)$.*

(a) The function $u_i \circ pr_i^{\infty}$ is ν^T-integrable over Y^T, and

$$\int_{Y^T} u_i \circ pr_i^{\infty} d\nu^T = \int_{Y_i} u_i d\nu_i, \quad i \in \mathbb{N}_+.$$

(b) If $f \circ u_i$ is also ν_i-integrable over Y_i $(i \in \mathbb{N}_+)$, then for every $n \in \mathbb{N}_+$

$$\int_{Y^T} f \circ \left(\frac{1}{n} \sum_{i=1}^{n} u_i \circ pr_i^{\infty} \right) d\nu^T = \int_{Y^{\{1,\dots,n\}}} f \circ \left(\frac{1}{n} \sum_{i=1}^{n} u_i \circ pr_i^{n} \right) d\nu^{\{1,\dots,n\}}. \tag{5.17}$$

Proof. (a) We argue as in the proof of Lemma 5.1 (a).

(b) According to Lemma 5.1 (b), the function

$$f \circ \left(\frac{1}{n} \sum_{i=1}^{n} u_i \circ pr_i^{n} \right)$$

is $\nu^{\{1,\dots,n\}}$-integrable over $Y^{\{1,\dots,n\}}$ $(n \in \mathbb{N}_+)$. By the definition of ν^T, the image measure of ν^T under the mapping $pr_{1\dots n}^{\infty}$ is $\nu^{\{1,\dots,n\}}$ $(n \in \mathbb{N}_+)$. Therefore the connection between ν^T-integrals and $\nu^{\{1,\dots,n\}}$-integrals gives (5.17).

The result is completely proved. □

The next result is simple to prove but useful.

Lemma 5.3 *Let $(X, \mathscr{A}, \mu)$ be a probability space, and let $u : X^n \to \mathbb{R}$ be a μ^n-integrable function.*

(a) Let π be a permutation of the numbers $1, \dots, n$, and let the mapping $T : X^n \to X^n$ be defined by

$$T(x_1, \dots, x_n) := (x_{\pi(1)}, \dots, x_{\pi(n)}).$$

Then the function $u \circ T$ is μ^n-integrable on X^n and

$$\int_{X^n} u \circ T d\mu^n = \int_{X^n} u d\mu^n.$$

(b) If $1 \leq i < n$, then

$$\int_{X^n} u d\mu^n = \int_{X^{n-1}} \left(\int_X u(x_1,\ldots,x_n) d\mu(x_i) \right) d\mu^{n-1}(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n).$$

Proof. (a) Let $T(\mu^n)$ be the image of μ^n under the mapping T. Since $T(\mu^n) = \mu^n$, the result follows from the connection between μ^n-integrals and $T(\mu^n)$-integrals.

(b) We have only to apply (a) and the Fubini's theorem. □

The last result that we discuss corresponds to the laws of large numbers.

Lemma 5.4 *Let $I \subset \mathbb{R}$ be an interval, and let $p : I \to \mathbb{R}$ be a function on I, which is continuous at every interior point of I, and bounded on I.*

(a) Let $(Y_i, \mathscr{B}_i, \nu_i)$, $i \in T := \mathbb{N}_+$ be probability spaces, and let $u_i : Y_i \to I$ be a square ν_i-integrable function over Y_i $(i \in \mathbb{N}_+)$ such that

$$\int_{Y_i} u_i d\nu_i = \int_{Y_1} u_1 d\nu_1, \quad i \in \mathbb{N}_+,$$

and

$$\sum_{i=1}^{\infty} \frac{1}{i^2} \int_{Y_i} u_i^2 d\nu_i < \infty. \tag{5.18}$$

Then

$$\lim_{n\to\infty} \int_{Y^{\{1,\ldots,n\}}} p\left(\frac{1}{n} \sum_{i=1}^{n} u_i(y_i) \right) d\nu^{\{1,\ldots,n\}}(y_1,\ldots,y_n) = p\left(\int_{Y_1} u_1 d\nu_1 \right).$$

(b) Let $(X, \mathscr{A}, \mu)$ be a probability space, and let $u : X \to I$ be a μ-integrable function over X. Then

$$\lim_{n\to\infty} \int_{X^n} p\left(\frac{1}{n} \sum_{i=1}^{n} u(x_i) \right) d\mu^n(x_1,\ldots,x_n) = p\left(\int_X u d\mu \right).$$

Proof. (a) Since u_i is a square ν_i-integrable function over Y_i, u_i is ν_i-integrable over Y_i $(i \in \mathbb{N}_+)$. From Lemma 5.2 (a) we obtain that $u_i \circ pr_i^\infty$ and $u_i^2 \circ pr_i^\infty$ $(i \in \mathbb{N}_+)$ are ν^T-integrable over Y^T, and

$$\int_{Y^T} u_i \circ pr_i^\infty d\nu^T = \int_{Y_i} u_i d\nu_i = \int_{Y_1} u_1 d\nu_1, \tag{5.19}$$

and

$$\int_{Y^T} u_i^2 \circ pr_i^\infty d\nu^T = \int_{Y_i} u_i^2 d\nu_i. \tag{5.20}$$

If $V(u_i \circ pr_i^\infty)$ means the variance of the random variable $u_i \circ pr_i^\infty$ $(i \in \mathbb{N}_+)$, then by using (5.19), (5.20) and (5.18), we have that

$$\sum_{i=1}^{\infty} \frac{V(u_i \circ pr_i^\infty)}{i^2} = \sum_{i=1}^{\infty} \frac{1}{i^2} \left(\int_{Y_i} u_i^2 d\nu_i - \left(\int_{Y_1} u_1 d\nu_1 \right)^2 \right) < \infty.$$

It is easy to verify that the random variables $u_i \circ pr_i^\infty$ $(i \in \mathbb{N}_+)$ are independent. Now, Kolmogorov's criterion (see [7]) implies that the sequence $(u_i \circ pr_i^\infty)_{i \in \mathbb{N}_+}$ of random variables obeys the strong law of large numbers, that is

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} u_i \circ pr_i^\infty = \int_{Y_1} u_1 d\nu_1 \quad \nu^T\text{-almost everywhere on } Y^T. \tag{5.21}$$

If $\int_{Y_1} u_1 d\nu_1$ is an interior point of I, then the continuity of p at $\int_{Y_1} u_1 d\nu_1$ and (5.21) yield that

$$\lim_{n\to\infty} p \circ \left(\frac{1}{n} \sum_{i=1}^{n} u_i \circ pr_i^\infty \right) = p \left(\int_{Y_1} u_1 d\nu_1 \right) \quad \nu^T\text{-almost everywhere on } Y^T. \tag{5.22}$$

Suppose $\int_{Y_1} u_1 d\nu_1$ is either the left-hand endpoint or the right-hand endpoint of I. In either case

$$u_i = \int_{Y_1} u_1 d\nu_1 \quad \nu_i\text{-almost everywhere on } Y_i \quad i \in \mathbb{N}_+,$$

hence

$$u_i \circ pr_i^\infty = \int_{Y_1} u_1 d\nu_1 \quad \nu^T\text{-almost everywhere on } Y^T, \quad i \in \mathbb{N}_+,$$

and this justifies (5.22).

Since p is bounded on I, it follows from (5.22) and Lebesgue's convergence theorem that

$$\lim_{n\to\infty} \int_{Y^T} p \circ \left(\frac{1}{n} \sum_{i=1}^{n} u_i \circ pr_i^\infty \right) d\nu^T = p \left(\int_{Y_1} u_1 d\nu_1 \right),$$

thus we can apply Lemma 5.2 (b).

(b) In this case the proof of (a) also works if instead of Kolmogorov's criterion Kolmogorov's law of large numbers (see [7]) is used, since the random variables $u \circ pr_i^\infty$ $(i \in \mathbb{N}_+)$ are independent, ν^T-integrable over Y^T, and identically distributed.

The whole theorem is proved. □

Proof of Theorem 5.3 By Lemma 5.1 (a) and (b), the functions

$$\sum_{i=1}^{n} p_i(u_i \circ pr_i^n) \quad \text{and} \quad f \circ \left(\sum_{i=1}^{n} p_i(u_i \circ pr_i^n) \right)$$

are ν^T-integrable over Y^T. An application of the integral form of Jensen's inequality (see Theorem 1.6) yields that

$$f\left(\int_{Y^T} \left(\sum_{i=1}^{n} p_i(u_i \circ pr_i^n) \right) d\nu^T \right) \leq \int_{Y^T} \left(f \circ \left(\sum_{i=1}^{n} p_i(u_i \circ pr_i^n) \right) d\nu^T \right).$$

The first inequality in (5.3) follows from this, by (5.14).

It remains to prove the second inequality in (5.3). By Lemma 5.1 (a) (replaced u_i by $f \circ u_i$), the function $f \circ (u_i \circ pr_i^n)$ is ν^T-integrable over Y^T $(i = 1, \ldots, n)$, and

$$\int_{Y^T} f \circ (u_i \circ pr_i^n) d\nu^T = \int_{Y_i} f \circ u_i d\nu_i.$$

Applying this and taking account of the convexity of f on I, we calculate

$$\begin{aligned} \int_{Y^T} f\left(\sum_{i=1}^{n} p_i u_i(y_i) \right) d\nu^T(y_1, \ldots, y_n) &\leq \int_{Y^T} \sum_{i=1}^{n} p_i f(u_i(y_i)) d\nu^T(y_1, \ldots, y_n) \\ &= \sum_{i=1}^{n} p_i \int_{Y_i} f \circ u_i d\nu_i, \end{aligned}$$

and this completes the proof. □

Proof of Theorem 5.4 This is an immediate consequence of Lemma 5.2 (b), since f is continuous at every interior point of I. □

Proof of Theorem 5.5 (a) This is an immediate consequence of Theorem 5.3.

(b) Since f is convex on I

$$\begin{aligned} &\int_{X^{n+1}} f\left(\frac{1}{n+1} \sum_{i=1}^{n+1} u(x_i) \right) d\mu^{n+1}(x_1, \ldots, x_{n+1}) \\ &= \int_{X^{n+1}} f\left(\frac{1}{n+1} \sum_{i=1}^{n+1} \left(\frac{1}{n} \sum_{j \in \{1,\ldots,n+1\} \setminus \{i\}} u(x_j) \right) \right) d\mu^{n+1}(x_1, \ldots, x_{n+1}) \\ &\leq \int_{X^{n+1}} \frac{1}{n+1} \sum_{i=1}^{n+1} f\left(\frac{1}{n} \sum_{j \in \{1,\ldots,n+1\} \setminus \{i\}} u(x_j) \right) d\mu^{n+1}(x_1, \ldots, x_{n+1}). \end{aligned}$$

By the Fubini's theorem (see Lemma 5.3 (b)), the right-hand side of the previous inequality can be written in the form

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\int_{X^n}\left(\int_X f\left(\frac{1}{n}\sum_{j\in\{1,\dots,n+1\}\setminus\{i\}} u(x_j)\right)d\mu(x_i)\right)d\mu^n(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n+1})$$

$$=\frac{1}{n+1}\sum_{i=1}^{n+1}\int_{X^n} f\left(\frac{1}{n}\sum_{j\in\{1,\dots,n+1\}\setminus\{i\}} u(x_j)\right)d\mu^n(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n+1})$$

$$=\int_{X^n} f\left(\frac{1}{n}\sum_{i=1}^{n} u(x_i)\right)d\mu^n(x_1,\dots,x_n),$$

confirming the first inequality in Theorem 5.5 (b).

Finally, we prove the second inequality in Theorem 5.5 (b). Let $\pi_i(j)$ $(i,j=1,\dots,n)$ be the unique number from $\{1,\dots,n\}$ for which

$$\pi_i(j)\equiv i+j-1 \pmod n,\quad i,j=1,\dots,n.$$

Then the functions π_i $(i=1,\dots,n)$ are permutations of the numbers $1,\dots,n$. Clearly, $\sum\limits_{j=1}^{n} p_{\pi_i(j)}=1$ $(i=1,\dots,n)$, and $\pi_i(j)=\pi_j(i)$, $i,j=1,\dots,n$.

The convexity of f on I implies that

$$\int_{X^n} f\left(\frac{1}{n}\sum_{i=1}^{n} u(x_i)\right)d\mu^n(x_1,\dots,x_n) \tag{5.23}$$

$$=\int_{X^n} f\left(\frac{1}{n}\sum_{i=1}^{n}\left(\sum_{j=1}^{n} p_{\pi_i(j)}u(x_i)\right)\right)d\mu^n(x_1,\dots,x_n)$$

$$\le\int_{X^n}\sum_{j=1}^{n}\frac{1}{n}f\left(\sum_{i=1}^{n} p_{\pi_j(i)}u(x_i)\right)d\mu^n(x_1,\dots,x_n)$$

$$=\sum_{i=1}^{n}\frac{1}{n}\int_{X^n} f\left(\sum_{i=1}^{n} p_{\pi_j(i)}u(x_i)\right)d\mu^n(x_1,\dots,x_n).$$

It follows from Lemma 5.3 (a) that

$$\int_{X^n} f\left(\sum_{i=1}^{n} p_{\pi_j(i)}u(x_i)\right)d\mu^n(x_1,\dots,x_n)$$

$$=\int_{X^n} f\left(\sum_{i=1}^{n} p_i u(x_i)\right)d\mu^n(x_1,\dots,x_n),\quad (j=1,\dots,n).$$

This fact and (5.23) yields the result, bringing the proof to an end.

(c) It comes from Lemma 5.4 (b), since f is continuous at every interior point of I. □

Remark 5.1 From Theorem 5.3, we can write

$$\Upsilon_{10}(f)=\Upsilon_{10}(f,\mathbf{p},u):=\sum_{i=1}^{n}p_i\int_{Y_i}f\circ u_i dv_i-\int_{Y^T}f\left(\sum_{i=1}^{n}p_iu_i(y_i)\right)dv^T(y_1,\dots,y_n)\geq 0,$$

$$\Upsilon_{11}(f)=\Upsilon_{11}(f,\mathbf{p},u):=\int_{Y^T}f\left(\sum_{i=1}^{n}p_iu_i(y_i)\right)dv^T(y_1,\dots,y_n)-f\left(\sum_{i=1}^{n}p_i\int_{Y_i}u_i dv_i\right)\geq 0,$$

$$\Upsilon_{12}(f)=\Upsilon_{12}(f,\mathbf{p},u):=\sum_{i=1}^{n}p_i\int_{Y_i}f\circ u_i dv_i-f\left(\sum_{i=1}^{n}p_i\int_{Y_i}u_i dv_i\right)\geq 0,$$

where $u:=(u_1,\dots,u_n)$.
From Theorem 5.5 (a), we can write

$$\Upsilon_{13}(f)=\Upsilon_{13}(f,\mathbf{p},u):=\int_{X^n}f\left(\sum_{i=1}^{n}p_iu(x_i)\right)d\mu^n(x_1,.,x_n)-f\left(\int_X u d\mu\right)\geq 0,$$

$$\Upsilon_{14}(f)=\Upsilon_{14}(f,\mathbf{p},u):=\int_X f\circ u dv-\int_{X^n}f\left(\sum_{i=1}^{n}p_iu(x_i)\right)d\mu^n(x_1,.,x_n)\geq 0,$$

$$\Upsilon_{15}(f)=\Upsilon_{15}(f,u):=\int_X f\circ u dv-f\left(\int_X u d\mu\right)\geq 0.$$

Remark 5.2 The first inequality in Theorem 5.5 (b) provides the generalization of Theorem 1.12 which is utilized in [6] to give the log-convexity and in [52] to give exponential convexity for two classes of convex functions stated in Remark 1.4.

5.0.1 Mixed Symmetric Means Related to Theorems (5.3-5.5)

We need the following condition:

($\mathscr{H}_6$) Let $(Y_i,\mathscr{B}_i,\nu_i)$ be probability spaces, and let u_i: $Y_i\to\mathbb{R}$ be a measurable function $(i=1,\dots,n)$. Suppose $\mathbf{p}=(p_1,\dots,p_n)$ is a nonnegative n-tuples such that $\sum_{i=1}^{n}p_i=1$.

The product of the probability spaces $(Y_i,\mathscr{B}_i,\nu_i)$ is denoted by $(Y^{T_n},\mathscr{B}^{T_n},\nu^{T_n})$.

Assume ($\mathscr{H}_6$), and let u_i be positive $(i=1,\dots,n)$. By using (1.44) and (1.45), we define weighted power means and mixed symmetric means as follows:

For every $(y_1,\dots,y_n)\in Y^{T_n}$

$$M_s(u_1,\dots,u_n,\mathbf{p})(y_1,\dots,y_n):=M_s(u_1(y_1),\dots,u_n(y_n),\mathbf{p}).$$

Let $r,s\in\mathbb{R}$, and suppose u_i^s is ν_i-integrable if $s\neq 0$, and $\log\circ u_i$ is ν_i-integrable if $s=0$ $(i=1,\dots,n)$. Then define

$$M_{r,s}(u_1,\dots,u_n,\mathbf{p}):=\begin{cases}\left(\sum_{i=1}^{n}p_i\widetilde{M}_s^r(u_i,\nu_i)\right)^{\frac{1}{r}}, & r\neq 0,\\ \prod_{i=1}^{n}\widetilde{M}_s^{p_i}(u_i,\nu_i), & r=0.\end{cases}.$$

Let $r,s\in\mathbb{R}$, and suppose u_i^s and u_i^r are ν_i-integrable if $s,r\neq 0$, and $\log\circ u_i$ is ν_i-

integrable if either $s=0$ or $r=0$ $(i=1,\ldots,n)$. Then define

$$\widetilde{M}_{r,s}(u_1,\ldots,u_n,\mathbf{p}) := \begin{cases} \left(\int\limits_{Y^{T_n}} M_s{}^r(u_1,\ldots,u_n,\mathbf{p})\, d\nu^{T_n} \right)^{\frac{1}{r}}, & r \neq 0, \\ \exp\left(\int\limits_{Y^{T_n}} \log M_s(u_1,\ldots,u_n,\mathbf{p})\, d\nu^{T_n} \right), & r = 0. \end{cases} \tag{5.24}$$

We can establish the relations among these means as an application of Theorem 5.3. It also follows that the integrals in (5.24) are finite.

Corollary 5.1 *Assume ($\mathscr{H}_6$), and let $\mathbf{u} := (u_1,\ldots,u_n)$ be positive. Let $r,s \in \mathbb{R}$, and suppose u_i^s and u_i^r are ν_i-integrable if $s,r \neq 0$, and $\log\circ u_i$ is ν_i-integrable if either $s=0$ or $r=0$ $(i=1,\ldots,n)$. If $s \le r$, then we have*

$$M_{s,s}(\mathbf{u},\mathbf{p}) \le \widetilde{M}_{r,s}(\mathbf{u},\mathbf{p}) \le M_{r,r}(\mathbf{u},\mathbf{p}), \tag{5.25}$$

$$M_{r,r}(\mathbf{u},\mathbf{p}) \ge \widetilde{M}_{s,r}(\mathbf{u},\mathbf{p}) \ge M_{s,s}(\mathbf{u},\mathbf{p}). \tag{5.26}$$

Proof. If $r,s \neq 0$, we set $f(x) = x^{\frac{r}{s}}$ $(x>0)$, $u_i = u_i{}^s$ in (5.3) and raising to the power $\frac{1}{r}$, then we get (5.25). Similarly, we set $f(x) = x^{\frac{s}{r}}$ $(x>0)$, $u_i = u_i{}^r$ in (5.3) and raising to the power $\frac{1}{s}$, then we get (5.26).
When $s=0$ or $r=0$, we get the required results by taking limit. □

Corollary 5.2 *Let $(Y_i,\mathscr{B}_i,\nu_i)$, $i \in \mathbb{N}_+$ be probability spaces, and suppose $u_i : Y_i \to \mathbb{R}$ $(i \in \mathbb{N}_+)$ are positive and measurable functions with a common upper bound. Let $0 < s \le r$.*

If

$$\int\limits_{Y_i} u_i^s d\nu_i = \int\limits_{Y_1} u_1^s d\nu_1, \quad i \in \mathbb{N}_+,$$

then

$$\lim_{n\to\infty} \widetilde{M}_{r,s}\left(u_1,\ldots,u_n,\frac{\mathbf{1}}{\mathbf{n}}\right) = \widetilde{M}_s(u_1,\nu_1),$$

where $\frac{\mathbf{1}}{\mathbf{n}}$ denotes the n-tuples $\left(\frac{1}{n},\ldots,\frac{1}{n}\right)$.

If

$$\int\limits_{Y_i} u_i^r d\nu_i = \int\limits_{Y_1} u_1^r d\nu_1, \quad i \in \mathbb{N}_+,$$

$$\lim_{n\to\infty} \widetilde{M}_{s,r}\left(u_1,\ldots,u_n,\frac{\mathbf{1}}{\mathbf{n}}\right) = \widetilde{M}_r(u_1,\nu_1).$$

Proof. u_i^t is obviously ν_i-integrable for all $t \ge 0$ $(i \in \mathbb{N})$. We can apply Theorem 5.4 taking into account the proof of Corollary 5.1. □

We also need the following hypothesis:

($\mathscr{H}_7$) Let $J \subset \mathbb{R}$ be an interval, and let $h, g : J \to \mathbb{R}$ be continuous and strictly monotone functions.
Assume ($\mathscr{H}_6$) and ($\mathscr{H}_7$), and let u_i: $Y_i \to J$ $(i = 1, \ldots, n)$. Then quasi-arithmetic means can be defined as follows:

For every $(y_1, \ldots, y_n) \in Y^{T_n}$

$$M_g(u_1, ..., u_n, \mathbf{p})(y_1, \ldots, y_n) := M_g(u_1(y_1), ..., u_n(y_n), \mathbf{p}).$$

Let $g \circ u_i$ be ν_i-integrable $(i = 1, \ldots, n)$. Then define

$$\widetilde{M}_g(u_1, ..., u_n, \mathbf{p}) = g^{-1}\left(\sum_{i=1}^{n} p_i \int\limits_{Y_i} g \circ u_i d\nu_i\right).$$

Let $g \circ u_i$ and $h \circ u_i$ be ν_i-integrable $(i = 1, \ldots, n)$. Then define

$$\widetilde{M}_{h,g}(u_1, ..., u_n, \mathbf{p}) = h^{-1}\left(\int\limits_{Y^{T_n}} h(M_g(u_1, ..., u_n, \mathbf{p}))\, d\nu^{T_n}\right). \tag{5.27}$$

The monotonicity of these means are described in the next result. We have that the integral in (5.27) is finite too.

Corollary 5.3 *Assume ($\mathscr{H}_6$) and ($\mathscr{H}_7$), and let u_i: $Y_i \to J$ $(i = 1, \ldots, n)$. Suppose $g \circ u_i$ and $h \circ u_i$ are ν_i-integrable $(i = 1, \ldots, n)$. If either $h \circ g^{-1}$ is convex and h is increasing, or $h \circ g^{-1}$ is concave and h is decreasing, then*

$$\widetilde{M}_g(u_1, ..., u_n, \mathbf{p}) \le \widetilde{M}_{h,g}(u_1, ..., u_n, \mathbf{p}) \le \widetilde{M}_h(u_1, ..., u_n, \mathbf{p}), \tag{5.28}$$

while if either $g \circ h^{-1}$ is convex and g is decreasing, or $g \circ h^{-1}$ is concave and g is increasing, then

$$\widetilde{M}_h(u_1, ..., u_n, \mathbf{p}) \ge \widetilde{M}_{g,h}(u_1, ..., u_n, \mathbf{p}) \ge \widetilde{M}_g(u_1, ..., u_n, \mathbf{p}). \tag{5.29}$$

Proof. By exchanging f for $h \circ g^{-1}$ and u_i for $g \circ u_i$ in (5.3) and applying h^{-1}, we obtain (5.28). We also exchange f for $g \circ h^{-1}$ and u_i for $h \circ u_i$ in (5.3) and apply g^{-1}, (5.29) is obtained. □

Corollary 5.4 *Assume ($\mathscr{H}_7$). Let $(Y_i, \mathscr{B}_i, \nu_i)$, $i \in \mathbb{N}_+$ be probability spaces, and suppose $u_i : Y_i \to J$ $(i \in \mathbb{N}_+)$ are measurable functions.*

(a) Suppose that either $h \circ g^{-1}$ is bounded and convex and h is increasing, or $h \circ g^{-1}$ is bounded and concave and h is decreasing. If $g \circ u_i$ is square ν_i-integrable $(i \in \mathbb{N})$, and

$$\int\limits_{Y_i} g \circ u_i d\nu_i = \int\limits_{Y_1} g \circ u_1 d\nu_1, \quad i \in \mathbb{N}_+,$$

and

$$\sum_{i=1}^{\infty} \frac{1}{i^2} \int\limits_{Y_i} (g \circ u_i)^2 d\nu_i < \infty,$$

then

$$\lim_{n\to\infty} \widetilde{M}_{h,g}\left(u_1,\dots,u_n,\frac{\mathbf{1}}{\mathbf{n}}\right) = \widetilde{M}_g\left(u_1,\nu_1\right).$$

(b) Suppose that either $g\circ h^{-1}$ *is bounded and convex and* g *is increasing, or* $g\circ h^{-1}$ *is bounded and concave and* g *is decreasing. If* $h\circ u_i$ *is square* ν_i*-integrable* $(i\in\mathbb{N}_+)$*, and*

$$\int_{Y_i} h\circ u_i d\nu_i = \int_{Y_1} h\circ u_1 d\nu_1, \quad i\in\mathbb{N}_+,$$

and

$$\sum_{i=1}^{\infty}\frac{1}{i^2}\int_{Y_i}\left(h\circ u_i\right)^2 d\nu_i < \infty,$$

then

$$\lim_{n\to\infty} \widetilde{M}_{g,h}\left(u_1,\dots,u_n,\frac{\mathbf{1}}{\mathbf{n}}\right) = \widetilde{M}_h\left(u_1,\nu_1\right).$$

Proof. Apply Theorem 5.4 taking into account the proof of Corollary 5.3. □

We consider the special case of $(\mathscr{H}_6)$ when the measure spaces $(Y_i,\mathscr{B}_i,\nu_i)$ are equal.

$(\tilde{\mathscr{H}}_6)$ Let $(X,\mathscr{A},\mu)$ be a probability space, and $u: X\to\mathbb{R}$ be a measurable function. Suppose $\mathbf{p}=(p_1,\dots,p_n)$ is a nonnegative n-tuples such that $\sum_{i=1}^n p_i = 1$.

In this case simplified notations will be used to the introduced means: for example, $\widetilde{M}_{r,s}(u,\mathbf{p})$ means $\widetilde{M}_{r,s}(u,\dots,u,\mathbf{p})$.

Corollary 5.5 *Assume* $(\tilde{\mathscr{H}}_6)$*, and let* u *be positive. Let* $r,s\in\mathbb{R}$*, and suppose* u^s *and* u^r *are* μ*-integrable if* $s,r\neq 0$*, and* $\log\circ u$ *is* μ*-integrable if either* $s=0$ *or* $r=0$*. If* $s\le r$*, then we have*

$$\widetilde{M}_s(u,\mu) \le \widetilde{M}_{r,s}(u,\mathbf{p}) \le \widetilde{M}_r(u,\mu),$$

and

$$\widetilde{M}_r(u,\mu) \ge \widetilde{M}_{s,r}(u,\mathbf{p}) \ge \widetilde{M}_s(u,\mu).$$

Proof. Apply Theorem 5.5 (a) and follow the proof of Corollary 5.1. □

Corollary 5.6 *Under the conditions Corollary 5.5, we have*

$$\widetilde{M}_{r,s}\left(u,\frac{\mathbf{1}}{\mathbf{n+1}}\right) \le \widetilde{M}_{r,s}\left(u,\frac{\mathbf{1}}{\mathbf{n}}\right) \le \widetilde{M}_{r,s}(u,\mathbf{p}),$$

and

$$\widetilde{M}_{s,r}\left(u,\frac{\mathbf{1}}{\mathbf{n+1}}\right) \ge \widetilde{M}_{s,r}\left(u,\frac{\mathbf{1}}{\mathbf{n}}\right) \ge \widetilde{M}_{s,r}(u,\mathbf{p}).$$

Proof. Apply Theorem 5.5 (b) and follow the proof of Corollary 5.1. □

Corollary 5.7 *Let $(X, \mathscr{A}, \mu)$ be a probability space, and let $u : X \to \mathbb{R}$ be a positive, measurable and bounded function. If $0 < s \leq r$, then*

$$\lim_{n\to\infty} \widetilde{M}_{r,s}\left(u, \frac{\mathbf{1}}{\mathbf{n}}\right) = \widetilde{M}_s\left(u, \mu\right),$$

$$\lim_{n\to\infty} \widetilde{M}_{s,r}\left(u, \frac{\mathbf{1}}{\mathbf{n}}\right) = \widetilde{M}_r\left(u, \mu\right).$$

Proof. Apply Theorem 5.5 (c) and follow the proof of Corollary 5.1. □

Corollary 5.8 *Assume $(\widetilde{\mathscr{H}_6})$ and $(\mathscr{H}_7)$, and let $u : X \to J$. Suppose $g \circ u$ and $h \circ u$ are μ-integrable. If either $h \circ g^{-1}$ is convex and h is increasing, or $h \circ g^{-1}$ is concave and h is decreasing, then*

$$\widetilde{M}_g\left(u, \mu\right) \leq \widetilde{M}_{h,g}\left(u, \mathbf{p}\right) \leq \widetilde{M}_h\left(u, \mu\right),$$

while if either $g \circ h^{-1}$ is convex and g is decreasing, or $g \circ h^{-1}$ is concave and g is increasing, then

$$\widetilde{M}_h\left(u, \mu\right) \geq \widetilde{M}_{g,h}\left(u, \mathbf{p}\right) \geq \widetilde{M}_g\left(u, \mu\right).$$

Proof. Apply Theorem 5.5 (a) and follow the proof of Corollary 5.3. □

Corollary 5.9 *Assume $(\widetilde{\mathscr{H}_6})$ and $(\mathscr{H}_7)$, and let $u : X \to J$. Suppose $g \circ u$ and $h \circ u$ are μ-integrable. If either $h \circ g^{-1}$ is convex and h is increasing, or $h \circ g^{-1}$ is concave and h is decreasing, then*

$$\widetilde{M}_{h,g}\left(u, \frac{\mathbf{1}}{\mathbf{n+1}}\right) \leq \widetilde{M}_{h,g}\left(u, \frac{\mathbf{1}}{\mathbf{n}}\right) \leq \widetilde{M}_{h,g}\left(u, \mathbf{p}\right),$$

while if either $g \circ h^{-1}$ is convex and g is decreasing, or $g \circ h^{-1}$ is concave and g is increasing, then

$$\widetilde{M}_{g,h}\left(u, \frac{\mathbf{1}}{\mathbf{n+1}}\right) \geq \widetilde{M}_{g,h}\left(u, \frac{\mathbf{1}}{\mathbf{n}}\right) \geq \widetilde{M}_{g,h}\left(u, \mathbf{p}\right).$$

Proof. Apply Theorem 5.5 (b) and follow the proof of Corollary 5.3. □

Corollary 5.10 *Let $(X, \mathscr{A}, \mu)$ be a probability space, and let $u : X \to \mathbb{R}$ be a measurable function. Assume $(\mathscr{H}_7)$. If either $h \circ g^{-1}$ is bounded and convex and h is increasing, or $h \circ g^{-1}$ is bounded and concave and h is decreasing, then*

$$\lim_{n\to\infty} \widetilde{M}_{h,g}\left(u, \frac{\mathbf{1}}{\mathbf{n}}\right) = \widetilde{M}_g\left(u, \mu\right),$$

while if either $g \circ h^{-1}$ is bounded and convex and g is increasing, or $g \circ h^{-1}$ is bounded and concave and g is decreasing, then

$$\lim_{n\to\infty} \widetilde{M}_{g,h}\left(u, \frac{\mathbf{1}}{\mathbf{n}}\right) = \widetilde{M}_h\left(u, \mu\right).$$

Proof. Apply Theorem 5.5 (c) and follow the proof of Corollary 5.3. □

5.1 New Refinement of Classical Jensen's Inequality

Consider $\mathbf{t} := (t_1, \ldots, t_{n-1})$, where $t_i \in [0,1]$ $(n \geq 2)$. Then the following interpolation of the classical discrete Jensen's inequality was proved by Pečarić:

Theorem 5.6 ([65]) *Let C be a convex subset of a real vector space V, and let $f : C \to \mathbb{R}$ be a convex function. Suppose $\mathbf{x} = (x_1, \ldots, x_n) \in C^n$, and $\mathbf{p} = (p_1, \ldots, p_n)$ is a nonnegative n-tuple such that $\sum_{i=1}^n p_i = 1$. Define*

$$f_{n,k} = f_{n,k}(\mathbf{x}, \mathbf{p}, \mathbf{t}, f) :=$$
$$\sum_{i_1=1}^{n} \ldots \sum_{i_k=1}^{n} p_{i_1} \ldots p_{i_k} f\left(x_{i_1}(1-t_1) + \sum_{j=1}^{k-1} x_{i_j}(1-t_{j+1}) t_1 \ldots t_j + \overline{x} t_1 \ldots t_k\right);$$

for $k = 1, \ldots, n-1$, where $\overline{x} = \sum_{i=1}^{n} p_i x_i$. Then

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq f_{n,1} \leq f_{n,2} \leq \ldots \leq f_{n,n-1} \leq$$
$$\leq \sum_{i_1=1}^{n} \ldots \sum_{i_n=1}^{n} p_{i_1} \ldots p_{i_n} f\left(x_{i_1}(1-t_1) + \sum_{j=1}^{n-2} x_{i_j}(1-t_{j+1}) t_1 \ldots t_j + x_{i_n} t_1 \ldots t_{n-1}\right)$$
$$\leq \sum_{i=1}^{n} p_i f(x_i).$$

We generalize this result for integrals.

Theorem 5.7 ([51]) *Let $I \subset \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a convex function. Let (X, A, μ) be probability space, $u : X \to I$ be a μ-integrable function over X such that $f \circ u$ is μ-integrable over X. We define*

$$Q_{n,k} := \int_{X^k} f((1-t_1)u(x_1) + \sum_{j=1}^{k-1} (1-t_{j+1}) t_1 \ldots t_j u(x_j) + t_1 \ldots t_k \overline{u}) d\mu^k(x_1, \ldots x_k),$$

where $\overline{u} := \int_X u \, d\mu$ and $t_i \in [0,1]$ $i = 1, \ldots n-1$. Then

$$f\left(\int_X u \, d\mu\right) \leq Q_{n,1} \leq Q_{n,2} \leq \ldots \leq Q_{n,n-1} \leq$$
$$\leq \int_{X^n} f((1-t_1)u(x_1) + \sum_{j=1}^{n-2} (1-t_{j+1}) t_1 \ldots t_j u(x_j) + t_1 \ldots t_{n-1} u(x_n)) d\mu^n(x_1, \ldots x_n)$$
$$\leq \int_X f \circ u \, d\mu.$$

Proof. By using the integral Jensen's inequality and integrate with respect to μ, we have:

$$\begin{aligned}
&\int_X f\circ u\, d\mu = \\
&\int_{X^n}((1-t_1)f(u(x_1))+\sum_{j=1}^{n-2}(1-t_{j+1})t_1...t_j f(u(x_j))+t_1...t_{n-1}f(u(x_n)))d\mu^n(x_1,...x_n) \\
&\geq \int_{X^n} f((1-t_1)u(x_1)+\sum_{j=1}^{n-2}(1-t_{j+1})t_1...t_j u(x_j)+t_1...t_{n-1}u(x_n))d\mu^n(x_1,...x_n) \\
&\geq \int_{X^{n-1}} f((1-t_1)u(x_1)+\sum_{j=1}^{n-2}(1-t_{j+1})t_1...t_j u(x_j)+t_1...t_{n-1}\overline{u})d\mu^{n-1}(x_1,...x_{n-1}) \\
&\geq \\
&. \\
&. \\
&. \\
&\geq \int_X f((1-t_1)u(x_1)+t_1\overline{u})d\mu(x_1) \geq f(\int_X u\, d\mu).
\end{aligned}$$

□

Remark 5.3 From Theorem 5.6, we write

$$\Upsilon_{16}(f)=\Upsilon_{16}(f,\mathbf{x},\mathbf{p},\mathbf{t}) := \sum_{i=1}^{n} p_i f(x_i) - \sum_{i_1=1}^{n}...\sum_{i_n=1}^{n} p_{i_1}...p_{i_n} f(x_{i_1}(1-t_1)+\sum_{j=1}^{n-2} x_{i_j}(1-t_{j+1})t_1...t_j + x_{i_n}t_1...t_{n-1}) \geq 0,$$

$$\Upsilon_{17}(f)=\Upsilon_{17}(f,\mathbf{x},\mathbf{p},\mathbf{t}) := \sum_{i=1}^{n} p_i f(x_i) - f_{n,k} \geq 0, \quad k=1,...,n-1,$$

$$\Upsilon_{18}(f)=\Upsilon_{18}(f,\mathbf{x},\mathbf{p},\mathbf{t}) := f_{n,k} - f\left(\sum_{i=1}^{n} p_i x_i\right) \geq 0, \quad k=1,...,n-1,$$

$$\Upsilon_{19}(f)=\Upsilon_{19}(f,\mathbf{x},\mathbf{p},\mathbf{t}) := \sum_{i_1=1}^{n}...\sum_{i_n=1}^{n} p_{i_1}...p_{i_n} f(x_{i_1}(1-t_1)+\sum_{j=1}^{n-2} x_{i_j}(1-t_{j+1})t_1...t_j + x_{i_n}t_1...t_{n-1}) - f\left(\sum_{i=1}^{n} p_i x_i\right) \geq 0.$$

From Theorem 5.7 we write

$$\Upsilon_{20}(f)=\Upsilon_{20}(f,\mathbf{t},u) := \int_X f\circ u\, d\mu - \int_{X^n} f((1-t_1)u(x_1)+\sum_{j=1}^{n-2}(1-t_{j+1})t_1...t_j u(x_j)+t_1...t_{n-1}u(x_n))d\mu^n(x_1,...x_n) \geq 0,$$

$$\Upsilon_{21}(f)=\Upsilon_{21}(f,\mathbf{t},u) := \int_X f\circ u\, d\mu - Q_{n,k} \geq 0, \quad k=1,...,n-1,$$

$$\Upsilon_{22}(f)=\Upsilon_{22}(f,\mathbf{t},u) := Q_{n,k} - f(\int_X u\, d\mu) \geq 0, \quad k=1,...,n-1,$$

$\Upsilon_{23}(f) = \Upsilon_{23}(f,\mathbf{t},u) :=$
$\int\limits_{X^n} f((1-t_1)u(x_1) + \sum\limits_{j=1}^{n-2}(1-t_{j+1})t_1 \ldots t_j u(x_j) + t_1 \ldots t_{n-1}u(x_n))d\mu^n(x_1, \ldots x_n)$
$-f(\int\limits_X u\, d\mu) \geq 0.$

5.1.1 Mixed Symmetric Means Related to Theorem 5.7

($\mathscr{H}_8$) Let $(X, \mathscr{A}, \mu)$ be a probability space, and $u : X \to \mathbb{R}$ be a measurable function. Let $\mathbf{t} := (t_1, \ldots, t_{n-1})$, where $t_i \in [0,1]$ $(n \geq 2)$.

Assume ($\mathscr{H}_8$), and let u be positive. Then associated to the core term of Theorem 5.7, we define mixed means as follows:

Let $s \in \mathbb{R}$, and suppose u^s is μ-integrable if $s \neq 0$, and $\log \circ u$ is μ-integrable if $s = 0$. For $k = 1, \ldots, n-1$ and for every $(x_1, \ldots, x_k) \in X^k$

$$M_s(u, \mathbf{t}, k)(x_1, \ldots, x_k) :=$$
$$\begin{cases} \left((1-t_1)u^s(x_1) + \sum\limits_{j=1}^{k-1}(1-t_{j+1})t_1 \ldots t_j u^s(x_j) + t_1 \ldots t_k \widetilde{M}_s^s(u,\mu)\right)^{\frac{1}{s}}, & s \neq 0, \\ \exp\begin{pmatrix} (1-t_1)\log u(x_1) + \\ \sum\limits_{j=1}^{k-1}(1-t_{j+1})t_1 \ldots t_j \log u(x_j) + t_1 \ldots t_k \log \widetilde{M}_0(u,\mu) \end{pmatrix}, & s = 0. \end{cases}$$

Let $r, s \in \mathbb{R}$, and suppose u^s and u^r are μ-integrable if $s, r \neq 0$, and $\log \circ u$ is μ-integrable if either $s = 0$ or $r = 0$. For $k = 1, \ldots, n-1$ define

$$\widetilde{M}_{r,s}(n,k,u,\mathbf{t}) := \begin{cases} \left(\int\limits_{X^k} M_s^r(u,\mathbf{t};k)d\mu^k\right)^{\frac{1}{r}}, & r \neq 0, \\ \exp\left(\int\limits_{X^k} \log M_s(u,\mathbf{t};k)d\mu^k\right), & r = 0. \end{cases}$$

Corollary 5.11 ([51]) *Assume ($\mathscr{H}_8$). Let $r, s \in \mathbb{R}$, and suppose u^s and u^r are μ-integrable if $s, r \neq 0$, and $\log \circ u$ is μ-integrable if either $s = 0$ or $r = 0$. If $s \leq r$, then*

$$\widetilde{M}_s(u,\mu) \leq \widetilde{M}_{r,s}(n,1,u,\mathbf{t}) \leq \ldots \leq \widetilde{M}_{r,s}(n,n-1,u,\mathbf{t}) \leq \widetilde{M}_{r,s}(u,\mathbf{t}) \leq \widetilde{M}_r(u,\mu),$$

and

$$\widetilde{M}_r(u,\mu) \leq \widetilde{M}_{s,r}(n,1,u,\mathbf{t}) \leq \ldots \leq \widetilde{M}_{s,r}(n,n-1,u,\mathbf{t}) \leq \widetilde{M}_{s,r}(u,\mathbf{t}) \leq \widetilde{M}_s(u,\mu).$$

Proof. Apply Theorem 5.7 and follow the proof of Corollary 5.1. □

We need

($\mathscr{H}_7$) Let $J \subset \mathbb{R}$ be an interval, and let $h, g : J \to \mathbb{R}$ be continuous and strictly monotone functions.

Assume ($\mathscr{H}_8$) and ($\mathscr{H}_7$). Then using $Q_{n,k}$ from Theorem 5.7, we define the generalized means as follows:

Let $g \circ u$ be μ-integrable. For $k = 1,...,n-1$ and for every $(x_1,\ldots,x_k) \in X^k$

$$M_g(u,\mathbf{t},k) :=$$
$$g^{-1}\left((1-t_1)g(u(x_1)) + \sum_{j=1}^{k-1}(1-t_{j+1})t_1...t_j g(u(x_j)) + t_1...t_k g(\widetilde{M}_g(u))\right).$$

Let $g \circ u$ and $h \circ u$ be μ-integrable. For $k = 1,...,n-1$

$$\widetilde{M}_{h,g}(n,k,u,\mathbf{t}) := h^{-1}\left(\int_{X^k} h(M_g(u,\mathbf{t},k))\,d\mu^k\right),$$

Corollary 5.12 ([51]) *Assume ($\mathscr{H}_8$) and ($\mathscr{H}_7$). Suppose $g \circ u$ and $h \circ u$ are μ-integrable. If either $h \circ g^{-1}$ is convex and h is increasing, or $h \circ g^{-1}$ is concave and h is decreasing, then*

$$\widetilde{M}_g(u,\mu) \le \widetilde{M}_{h,g}(n,1,u,\mathbf{t}) \le ... \le \widetilde{M}_{h,g}(n,n-1,u,\mathbf{t}) \le \widetilde{M}_{h,g}(u,\mathbf{t}) \le \widetilde{M}_h(u,\mu),$$

while if either $g \circ h^{-1}$ is convex and g is decreasing, or $g \circ h^{-1}$ is concave and g is increasing, then

$$\widetilde{M}_h(u,\mu) \le \widetilde{M}_{g,h}(n,1,u,\mathbf{t}) \le ... \le \widetilde{M}_{g,h}(n,n-1,u,\mathbf{t}) \le \widetilde{M}_{g,h}(u,\mathbf{t}) \le \widetilde{M}_g(u,\mu).$$

Proof. Apply Theorem 5.7 and follow the proof of Corollary 5.3. □

Remark 5.4 Similarly to Corollary 5.11 and Corollary 5.12, we can give the results for Theorem 5.6 and those will be special cases of Corollary 5.11 and Corollary 5.12 with discrete measure.

5.2 Another Refinement of integral form of Jensen's Inequality

For the results from integration theory see [29].

We consider the following conditions:

($\mathbf{C}_1$) Let $(X,\mathscr{A},\mu)$ be a σ-finite measure space such that $\mu(X) > 0$.

The integrable functions are considered to be measurable.

($\mathbf{C}_2$) Let φ be a positive function on X such that $\int_X \varphi\,d\mu = 1$.

In this case the measure P defined on $\mathscr{A}$ by

$$P(A) := \int_A \varphi d\mu$$

is a probability measure having density φ with respect to μ. An $\mathscr{A}$-measurable function $g : X \to \mathbb{R}$ is P-integrable if and only if $g\varphi$ is μ-integrable, and the relationship between the P- and μ-integrals is

$$\int_X g dP = \int_X g\varphi d\mu.$$

(C$_3$) Let $k \geq 2$ be a fixed integer.

The σ-algebra in X^k generated by the projection mappings $pr_m : X^k \to X$ $(m = 1, \ldots, k)$

$$pr_m(x_1, \ldots, x_k) := x_m$$

is denoted by $\mathscr{A}^k$. μ^k means the product measure on $\mathscr{A}^k$: this measure is uniquely (μ is σ-finite) specified by

$$\mu^k(A_1 \times \ldots \times A_k) := \mu(A_1) \ldots \mu(A_k), \quad A_m \in \mathscr{A}, \quad m = 1, \ldots, k.$$

We shall also use the following projection mappings: for $m = 1, \ldots, k$ define $pr^m : X^k \to X^{k-1}$ by

$$pr^m(x_1, \ldots, x_k) := (x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_k).$$

For every $Q \in \mathscr{A}^k$ and for all $m = 1, \ldots, k$ the sets

$$Q_{m,x} := \Big\{(x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_k) \in X^{k-1} \mid$$
$$(x_1, \ldots, x_{m-1}, x, x_{m+1}, \ldots, x_k) \in Q\}$$

and

$$Q_{x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_k} := \{x \in X \mid (x_1, \ldots, x_{m-1}, x, x_{m+1}, \ldots, x_k) \in Q\}$$

are called x-sections of Q $(x \in X)$ and $(x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_k)$-sections of Q $((x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_k) \in X^{k-1})$, respectively. We note that the sets $Q_{m,x}$ lie in $\mathscr{A}^{k-1}$, while $Q_{x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_k} \in \mathscr{A}$.

(C$_4$) Let $S \in \mathscr{A}^k$ such that
(i)

$$pr_m(S) \in \mathscr{A}, \quad m = 1, \ldots, k \quad \text{and} \quad \bigcup_{m=1}^{k} pr_m(S) = X, \tag{5.30}$$

and
(ii)

$$l(x) := \sum_{m=1}^{k} \beta_m \mu^{k-1}(S_{m,x}) \in \,]0, \infty[, \quad x \in X,$$

where $\beta_1,\dots\beta_k$ are fixed positive numbers.

We stress that the first condition in (5.30) is necessary. For example, there exists a Borel set in $\mathbb{R}^2$ whose image under the first projection map is not a Borel set in $\mathbb{R}$ (see [50]). The function l is $\mathscr{A}$-measurable.

Under the conditions (C_1)-(C_4) we introduce the functions $\psi : X^k \to]0,\infty[$

$$\psi(x_1,\dots,x_k) := \sum_{j=1}^{k} \frac{\beta_j \varphi(x_j)}{l(x_j)}$$

and $\psi^i : X^{k-1} \to]0,\infty[$ $(i=1,\dots,k)$

$$\psi^i(x_1,\dots,x_{i-1},x_{i+1},\dots,x_k) := \sum_{\substack{j=1\\ j\neq i}}^{k} \frac{\beta_j \varphi(x_j)}{l(x_j)}.$$

Since

$$\psi = \sum_{j=1}^{k} \frac{\beta_j(\varphi \circ pr_j)}{l \circ pr_j},$$

ψ is $\mathscr{A}^k$-measurable. Similarly, ψ^i $(i=1,\dots,k)$ is $\mathscr{A}^{k-1}$-measurable.

(C_5) Suppose $pr^m(S) \in \mathscr{A}^{k-1}$ $(m=1,\dots,k)$.

Theorem 5.8 *Assume (C_1)-(C_4). Let $f : X \to \mathbb{R}$ be a P-integrable function taking values in an interval $I \subset \mathbb{R}$, and let q be a convex function on I such that $q \circ f$ is P-integrable. Then*

(a)

$$q\left(\int_X f dP\right) = q\left(\int_X f\varphi d\mu\right) \leq N_k$$

$$:= \int_S q\left(\frac{1}{\psi(x_1,\dots,x_k)} \sum_{j=1}^{k} \frac{\beta_j \varphi(x_j)}{l(x_j)} f(x_j)\right) \psi(x_1,\dots,x_k)\, d\mu^k(x_1,\dots,x_k)$$

$$\leq \int_X (q\circ f)\varphi d\mu = \int_X q\circ f dP.$$

(b) If (C_5) is also satisfied, then

$$q\left(\int_X f dP\right) = q\left(\int_X f\varphi d\mu\right) \leq N_k \leq N_{k-1} := \frac{1}{k-1}\sum_{i=1}^{k} \int_{pr^i(S)}$$

$$\left(\mu\left(S_{x_1,\dots,x_{i-1},x_{i+1},\dots,x_k}\right) q\left(\frac{1}{\psi^i(x_1,\dots,x_{i-1},x_{i+1},\dots,x_k)} \sum_{\substack{j=1\\ j\neq i}}^{k} \frac{\beta_j \varphi(x_j)}{l(x_j)} f(x_j)\right)\right.$$

$$\left.\cdot\psi^{i}\left(x_{1},\ldots,x_{i-1},x_{i+1},\ldots,x_{k}\right)\right)d\mu^{k-1}\left(x_{1},\ldots,x_{i-1},x_{i+1},\ldots,x_{k}\right)$$

$$\leq\int_{X}\left(q\circ f\right)\varphi d\mu=\int_{X}q\circ fdP.$$

By applying the method used in the proof of the preceding theorem, it is possible to obtain a chain of refinements of the form

$$q\left(\int_{X}f\varphi d\mu\right)\leq N_{k}\leq N_{k-1}\leq\ldots\leq N_{2}\leq N_{1}=\int_{X}\left(q\circ f\right)\varphi d\mu,$$

but some measurability problems crop up and it is not so easy to construct the expressions N_i $(i=k-2,\ldots,2)$. These difficulties disappear entirely if $S:=X$. In this case we have

Theorem 5.9 *Assume (C_1)-(C_3), and let $\mu\left(X\right)<\infty$. If $f:X\rightarrow\mathbb{R}$ is a P-integrable function taking values in an interval $I\subset\mathbb{R}$, and q is a convex function on I such that $q\circ f$ is P-integrable, then*

$$q\left(\int_{X}fdP\right)=q\left(\int_{X}f\varphi d\mu\right)$$

$$\leq\ldots\leq N_{k}\leq\ldots\leq N_{2}\leq N_{1}=\int_{X}\left(q\circ f\right)\varphi d\mu=\int_{X}q\circ fdP,\quad k\geq1,$$

where

$$N_{k}:=\frac{1}{k\mu(X)^{k-1}}\int_{X^{k}}q\left(\frac{\sum\limits_{j=1}^{k}\varphi\left(x_{j}\right)f(x_{j})}{\sum\limits_{j=1}^{k}\varphi\left(x_{j}\right)}\right)\sum_{j=1}^{k}\varphi\left(x_{j}\right)d\mu^{k}\left(x_{1},\ldots,x_{k}\right),\quad k\geq1.$$

Applications: The following special situations show the force of our results: they extend and generalize some earlier results; new refinements of the discrete Jensen's inequality can be constructed; the integral version of known discrete inequalities can be derived.

1. Suppose $(X,\mathscr{A},\mu)$ is a probability space, $\varphi(x):=1$ $(x\in X)$, $S:=X^k$, and $\sum\limits_{m=1}^{k}\beta_m=1$. Then ($C_1$)-($C_5$) are satisfied. Suppose also that $f:X\rightarrow\mathbb{R}$ is a μ-integrable function taking values in an interval $I\subset\mathbb{R}$, and q is a convex function on I such that $q\circ f$ is μ-integrable. In this case Theorem 5.8 (a) gives Theorem 5.5 (a):

$$q\left(\int_{X}fd\mu\right)\leq\int_{X^{k}}q\left(\sum_{j=1}^{k}\beta_{j}f(x_{j})\right)d\mu^{k}\left(x_{1},\ldots,x_{k}\right)\leq\int_{X}\left(q\circ f\right)d\mu.\tag{5.31}$$

If $\beta_m = \frac{1}{k}$ $(m = 1,\ldots,k)$ also holds, then Theorem 5.5 (b) comes from Theorem 5.8 (b):

$$\int\limits_{X^{k+1}} q\left(\frac{1}{k+1}\sum_{j=1}^{k+1} f(x_j)\right) d\mu^{k+1}(x_1,\ldots,x_{k+1}) \tag{5.32}$$
$$\leq \int\limits_{X^k} q\left(\frac{1}{k}\sum_{j=1}^{k} f(x_j)\right) d\mu^k(x_1,\ldots,x_k), \quad k \geq 1.$$

We can see that Theorem 5.8 much more general than (5.31) even if μ is a probability measure. Moreover, Theorem 5.8 (b) makes it possible to obtain a chain of refinements in (5.31):

$$q\left(\int\limits_X f d\mu\right) \leq \int\limits_{X^k} q\left(\sum_{j=1}^{k} \beta_j f(x_j)\right) d\mu^k(x_1,\ldots,x_k) \leq \frac{1}{(k-1)}$$
$$\cdot \sum_{i=1}^{k}(1-\beta_i) \int\limits_{X^{k-1}} q\left(\frac{1}{1-\beta_i}\sum_{\substack{j=1\\ j\neq i}}^{k} \beta_j f(x_j)\right) d\mu^{k-1}(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_k)$$
$$\leq \int\limits_X (q\circ f)\, d\mu.$$

2. Let $X := [a,b] \subset \mathbb{R}$ $(a < b)$. The set of the Borel subsets of $\mathbb{R}$ $(\mathbb{R}^k)$ is denoted by $\mathscr{B}$ $(\mathscr{B}^k)$. λ means the Lebesgue measure on $\mathscr{B}$. Let $q : [a,b] \to \mathbb{R}$ be a convex function.

The classical Hermite-Hadamard inequality (see [24]) says:

$$q\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b q \leq \frac{q(a)+q(b)}{2}.$$

We can obtain the following refinement of the left hand side of the Hermite-Hadamard inequality:

Corollary 5.13 *Let $S \subset [a,b]^k$ be a Borel set such that*

$$pr_m(S) \in \mathscr{B}, \quad m = 1,\ldots,k \quad \text{and} \quad \bigcup_{m=1}^{k} pr_m(S) = [a,b],$$

and

$$l(x) = \sum_{m=1}^{k} \beta_m \lambda^{k-1}(S_{m,x}) > 0, \quad x \in [a,b],$$

where $\beta_m > 0$ $(m = 1,\ldots,k)$.

Then

$$q\left(\frac{a+b}{2}\right)$$

$$\leq \int_S q\left(\frac{1}{(b-a)\psi(x_1,\ldots,x_k)}\sum_{j=1}^{k}\frac{\beta_j x_j}{l(x_j)}\right)\psi(x_1,\ldots,x_k)\,d\mu^k(x_1,\ldots,x_k) \tag{5.33}$$

$$\leq \frac{1}{b-a}\int_a^b q.$$

Proof. We can apply Theorem 5.8 (a) to the pair of functions φ, $f:[a,b]\to\mathbb{R}$, $\varphi(x)=\frac{1}{b-a}$ and $f(x)=x$. □

If $S=[a,b]^k$ and $\sum\limits_{m=1}^{k}\beta_m=1$, then we have from (5.33) and Theorem 5.8 (b) one of the main results in [87] as a special case:

$$q\left(\frac{a+b}{2}\right)\leq\frac{1}{(b-a)^k}\int_{[a,b]^k}q\left(\sum_{j=1}^{k}\beta_j x_j\right)d\lambda^k(x_1,\ldots,x_k)$$

$$\leq\sum_{i=1}^{k}\frac{1-\beta_i}{(k-1)(b-a)^{k-1}}\int_{[a,b]^{k-1}}q\left(\frac{1}{1-\beta_i}\sum_{\substack{j=1\\ j\neq i}}^{k}\beta_j x_j\right)$$

$$d\lambda^{k-1}(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_k)\leq\frac{1}{b-a}\int_a^b q.$$

Another concrete example can be constructed for (5.33) by using Corollary 5.15.

3. In the following results we consider noteworthy proper subsets of $\mathbb{R}^k$.
(a) Let z, $w\in\mathbb{R}$, $z<w$, and let $m\geq 1$ be an integer. The simplex $S^m_{z,w}$ is defined by

$$S^m_{z,w}:=\{(x_1,\ldots,x_m)\in\mathbb{R}^m \mid z\leq x_1\leq\ldots\leq x_m\leq w\}.$$

Let $X:=[a,b]\subset\mathbb{R}$ $(a<b)$, and μ be a finite measure on the trace σ-algebra $[a,b]\cap\mathscr{B}$ such that $\mu([a,b])>0$. Suppose $\varphi:[a,b]\to\mathbb{R}$ is a positive function such that $\int\limits_{[a,b]}\varphi d\mu=1$. Fix an integer $k\geq 2$, and let $\beta_m>0$ $(m=1,\ldots,k)$.

Choose $S:=S^k_{a,b}$. Then

$$S_{1,x}=S^{k-1}_{x,b},\quad S_{k,x}=S^{k-1}_{a,x},\quad S_{m,x}=S^{m-1}_{a,x}\times S^{k-m}_{x,b},\quad 2\leq m\leq k-1,$$

once the appropriate identification of $\mathbb{R}^{k-1}$ with $\mathbb{R}^{j-1}\times\mathbb{R}^{k-j}$ $(2\leq j\leq k-1)$ has been made. Therefore

$$l(x)=\sum_{m=1}^{k}\beta_m\mu^{m-1}\left(S^{m-1}_{a,x}\right)\mu^{k-m}\left(S^{k-m}_{x,b}\right)>0,\quad x\in[a,b],$$

where $\mu^0\left(S^0_{a,x}\right) = \mu^0\left(S^0_{x,b}\right) := 1$. Thus

$$\psi(x_1,\ldots,x_k) = \sum_{j=1}^{k} \frac{\beta_j\varphi(x_j)}{\sum\limits_{m=1}^{k} \beta_m \mu^{m-1}\left(S^{m-1}_{a,x_j}\right)\mu^{k-m}\left(S^{k-m}_{x_j,b}\right)}, \quad (x_1,\ldots,x_k) \in [a,b]^k.$$

We can see that under the above assumptions (C_1)-(C_4) are satisfied, so Theorem 5.8 can be applied:

Corollary 5.14 *If $f : [a,b] \to \mathbb{R}$ is a P-integrable function taking values in an interval $I \subset \mathbb{R}$, and q is a convex function on I such that $q \circ f$ is P-integrable, then*

$$q\left(\int\limits_{[a,b]} f dP\right) = q\left(\int\limits_{[a,b]} f\varphi d\mu\right)$$

$$\leq \int\limits_{S} q\left(\frac{1}{\sum\limits_{j=1}^{k} \frac{\beta_j\varphi(x_j)}{\sum\limits_{m=1}^{k} \beta_m\mu^{m-1}\left(S^{m-1}_{a,x_j}\right)\mu^{k-m}\left(S^{k-m}_{x_j,b}\right)}} \sum_{j=1}^{k} \frac{\beta_j\varphi(x_j)f(x_j)}{\sum\limits_{m=1}^{k} \beta_m\mu^{m-1}\left(S^{m-1}_{a,x_j}\right)\mu^{k-m}\left(S^{k-m}_{x_j,b}\right)}\right) \tag{5.34}$$

$$\cdot \sum_{j=1}^{k} \frac{\beta_j\varphi(x_j)}{\sum\limits_{m=1}^{k} \beta_m\mu^{m-1}\left(S^{m-1}_{a,x_j}\right)\mu^{k-m}\left(S^{k-m}_{x_j,b}\right)} d\mu^k(x_1,\ldots,x_k)$$

$$\leq \int\limits_{[a,b]} (q\circ f)\varphi d\mu = \int\limits_{[a,b]} q\circ f dP.$$

Specifically, if $\mu = \lambda$, we have

$$l(x) = \frac{1}{(k-1)!}\sum_{m=1}^{k} \beta_m \binom{k-1}{m-1}(x_j-a)^{m-1}(b-x_j)^{k-m}, \quad x\in[a,b].$$

When $\beta_1 = \ldots = \beta_k = \beta$, this says

$$l(x) = \frac{\beta}{(k-1)!}(b-a)^{k-1}, \quad x\in[a,b],$$

and in this case we have the inequality for the cube $[a,b]^k$

$$q\left(\int\limits_{a}^{b} f\varphi d\lambda\right) \leq \frac{(k-1)!}{(b-a)^{k-1}}\int\limits_{S} q\left(\frac{1}{\sum\limits_{j=1}^{k}\varphi(x_j)}\sum_{j=1}^{k}\varphi(x_j)f(x_j)\right)$$

$$\cdot \sum_{j=1}^{k} \varphi(x_j) d\lambda^k (x_1, \dots, x_k) = \frac{1}{k(b-a)^{k-1}} \int_{[a,b]^k} q \left(\frac{1}{\sum\limits_{j=1}^{k} \varphi(x_j)} \sum_{j=1}^{k} \varphi(x_j) f(x_j) \right)$$

$$\cdot \sum_{j=1}^{k} \varphi(x_j) d\lambda^k (x_1, \dots, x_k) \leq \int_a^b (q \circ f) \varphi d\lambda.$$

Next, we show that inequality (5.34) extends Theorem 1.9, a well known discrete inequality to an integral form. Similar results are quite rare in the literature (see [31]). We give Theorem 1.9 again:

Theorem C. *Let I be an interval in $\mathbb{R}$, and let $q : I \to \mathbb{R}$ be a convex function. If $v_1, \dots, v_n \in I$, then for each $k \geq 1$*

$$q\left(\frac{1}{n}\sum_{i=1}^{n} v_i\right) \leq \frac{1}{\binom{n+k-1}{k}} \sum_{1 \leq i_1 \leq \dots \leq i_k \leq n} q\left(\frac{v_{i_1} + \dots + v_{i_k}}{k}\right) \leq \frac{1}{n}\sum_{i=1}^{n} q(v_i). \tag{5.35}$$

Let $X := [1,n] \subset \mathbb{R}$ ($n \geq 1$ is an integer), and let μ be the measure on the trace σ-algebra $[1,n] \cap \mathscr{B}$ defined by $\mu := \sum\limits_{m=1}^{n} \frac{1}{n} \varepsilon_m$, where ε_m is the unit mass at m ($m = 1, \dots, n$). Suppose $\varphi(x) := 1$ ($x \in [1,n]$), $k \geq 2$ is a fixed integer, and $\beta_m = 1$ ($m = 1, \dots, k$).

Some easy combinatorial considerations yield that for every $x \in [1,n]$ and $m = 1, \dots, k$

$$\mu^{k-1}(S_{m,x}) = \frac{1}{n^{k-1}} \binom{[x]+m-2}{m-1} \binom{n-[x]+k-m}{k-m},$$

where $[x]$ is the largest natural number that does not exceed x. Therefore

$$l(x) = \frac{1}{n^{k-1}} \binom{n+k-1}{k-1}, \quad x \in [1,n].$$

Now, if I is an interval in $\mathbb{R}$, $q : I \to \mathbb{R}$ is a convex function, and $f : [1,n] \to \mathbb{R}$ defined by

$$f(i) := \begin{cases} v_i, & i = 1, \dots, n \\ 0, & \text{elsewhere} \end{cases},$$

then (5.35) follows immediately from (5.34).

(b) Let $m \geq 1$ be an integer, let $z \in \mathbb{R}^m$, and let $r > 0$. The open ball of radius r centered at the point z is denoted by $B_m(z,r)$.

Consider the measure space $(]a,b[, \mathscr{B}, \lambda)$ ($a < b$). Suppose $\varphi :]a,b[\to \mathbb{R}$ is a positive function such that $\int\limits_a^b \varphi d\lambda = 1$. Fix an integer $k \geq 2$, and let $\beta_m > 0$ ($m = 1, \dots, k$). Choose

$$S := B_k\left(\left(\frac{a+b}{2}, \dots, \frac{a+b}{2}\right), \frac{b-a}{2}\right).$$

Then for all $x \in]a,b[$ and $m = 1,\dots,k$

$$S_{m,x} = B_{k-1}\left(\left(\frac{a+b}{2},\dots,\frac{a+b}{2}\right),\sqrt{(b-x)(x-a)}\right).$$

Consequently

$$l(x) = k\frac{2^{k-1}}{(k-1)!!}\left(\frac{\pi}{2}\right)^{\left[\frac{k-1}{2}\right]}((b-x)(x-a))^{\frac{k-1}{2}}, \quad x \in]a,b[,$$

where

$$(k-1)!! := (k-1)(k-3)\dots\varepsilon_{k-1}$$

and

$$\varepsilon_{k-1} := \begin{cases} 2, & k \text{ is odd} \\ 1, & k \text{ is even} \end{cases}.$$

According to this, for all $(x_1,\dots,x_k) \in]a,b[^k$

$$\psi(x_1,\dots,x_k) = \frac{1}{k\frac{2^{k-1}}{(k-1)!!}\left(\frac{\pi}{2}\right)^{\left[\frac{k-1}{2}\right]}}\sum_{j=1}^{k}\frac{\beta_j\varphi(x_j)}{\sum\limits_{m=1}^{k}\beta_m((b-x_j)(x_j-a))^{\frac{k-1}{2}}}.$$

It is not hard to check that (C_1)-(C_4) are satisfied in this situation, and thus Theorem 5.8 says:

Corollary 5.15 *If $f :]a,b[\to \mathbb{R}$ is a P-integrable function taking values in an interval $I \subset \mathbb{R}$, and q is a convex function on I such that $q \circ f$ is P-integrable, then*

$$q\left(\int\limits_{]a,b[} f dP\right) = q\left(\int\limits_a^b f\varphi d\lambda\right) \le \frac{1}{k\frac{2^{k-1}}{(k-1)!!}\left(\frac{\pi}{2}\right)^{\left[\frac{k-1}{2}\right]}}$$

$$\cdot\int\limits_S q\left(\frac{1}{\sum\limits_{j=1}^{k}\frac{\beta_j\varphi(x_j)}{\sum\limits_{m=1}^{k}\beta_m((b-x_j)(x_j-a))^{\frac{k-1}{2}}}}\sum_{j=1}^{k}\frac{\beta_j\varphi(x_j)f(x_j)}{\sum\limits_{m=1}^{k}\beta_m((b-x_j)(x_j-a))^{\frac{k-1}{2}}}\right)$$

$$\cdot\sum_{j=1}^{k}\frac{\beta_j\varphi(x_j)}{\sum\limits_{m=1}^{k}\beta_m((b-x_j)(x_j-a))^{\frac{k-1}{2}}}d\lambda^k(x_1,\dots,x_k)$$

$$\le \int\limits_a^b (q\circ f)\varphi d\lambda = \int\limits_{]a,b[} q\circ f dP.$$

By applying this result (φ, $f : [a,b] \to \mathbb{R}$, $\varphi(x) = \frac{1}{b-a}$ and $f(x) = x$), we can have a special case of the refinement of the left hand side of the Hermite-Hadamard inequality in (5.33).

4. We turn now to the case where X is a countable set.

(C_1^1) Consider the measure space $\left(X, 2^X, \mu\right)$, where either $X := \{1, \ldots, n\}$ for some positive integer n or $X := \{0, 1, \ldots\}$, 2^X denotes the power set of X, and $\mu(\{u\}) := \mu_u$ is a positive number for all $u \in X$.

(C_2^1) Let $(p_u)_{u \in X}$ be a sequence of positive numbers for which $\sum\limits_{u \in X} p_u \mu_u = 1$.

(C_3^1) Let $k \geq 2$ be a fixed integer.

We define the functions α_v^j $(v \in X,\ j = 1, \ldots, k)$ on X^k by

$$\alpha_v^j(u_1, \ldots, u_k) := \begin{cases} 1, & \text{if} \quad u_j = v \\ 0, & \text{if} \quad u_j \neq v \end{cases}.$$

Then $\sum\limits_{j=1}^{k} \alpha_v^j(u_1, \ldots, u_k)$ means the number of occurrences of v in $(u_1, \ldots, u_k) \in X^k$. If $S \subset X^k$, we introduce the following sums

$$\alpha_{S,v}^j := \sum_{(u_1, \ldots, u_k) \in S} \alpha_v^j(u_1, \ldots, u_k), \quad v \in X, \quad j = 1, \ldots, k$$

and

$$\alpha_{S,v} := \sum_{j=1}^{k} \alpha_{S,v}^j, \quad v \in X.$$

Every sum is either a nonnegative integer or ∞.

(C_4^1) Let $S \subset X^k$ such that $\alpha_{S,v} \geq 1$ for all $v \in X$, and

$$l(u) := \sum_{m=1}^{k} \beta_m \mu^{k-1}(S_{m,u}) < \infty, \quad u \in X,$$

where $\beta_m > 0$ $(m = 1, \ldots, k)$.

Since $\mu_u > 0$ for all $u \in X$, $l(u) > 0$ for all $u \in X$. By the definition of the measure μ

$$l(u) = \sum_{m=1}^{k} \beta_m \sum_{\left(u_1, \ldots, u_{j-1}, u, u_{j+1}, \ldots, u_k\right) \in S} \mu_{u_1} \ldots \mu_{u_{j-1}} \mu_{u_{j+1}} \ldots \mu_{u_k}, \quad u \in X.$$

In this case the function ψ has the form

$$\psi : X^k \to]0, \infty[, \quad \psi(u_1, \ldots, u_k) := \sum_{j=1}^{k} \frac{\beta_j p_{u_j}}{l(u_j)}.$$

Now Theorem 5.8 (a) can be formulated in the following way:

Corollary 5.16 *Assume* (C_1^1)-(C_4^1)*, and let* $(f_u)_{u\in X}$ *be a sequence taking values in an interval* $I \subset \mathbb{R}$ *such that* $\sum_{u\in X} |f_u| p_u \mu_u < \infty$. *If* q *is a convex function on* I *such that* $\sum_{u\in X} |q(f_u)| p_u \mu_u < \infty$, *then*

$$q\left(\sum_{u\in X} f_u p_u \mu_u\right)$$

$$\leq \sum_{(u_1,\ldots,u_k)\in S} q\left(\frac{1}{\psi(u_1,\ldots,u_k)} \sum_{j=1}^{k} \frac{\beta_j p_{u_j}}{l(u_j)} f_{u_j}\right) \psi(u_1,\ldots,u_k)\, \mu_{u_1}\ldots\mu_{u_k}$$

$$\leq \sum_{u\in X} q(f_u)\, p_u \mu_u.$$

Assume (C_1^1)-(C_4^1), and suppose μ is the counting measure on $P(X)$, that is $\mu_u := 1$ for all $u \in X$. For a set $A \in 2^X$ let $|A|$ denote the number of elements of A. Then $\sum_{u\in X} p_u = 1$,

$$l(u) = \sum_{m=1}^{k} \beta_m \alpha_{S,u}^m, \quad u \in X,$$

and

$$\psi : X^k \to]0,\infty[, \quad \psi(u_1,\ldots,u_k) := \sum_{j=1}^{k} \frac{\beta_j p_{u_j}}{\sum_{m=1}^{k} \beta_m \alpha_{S,u_j}^m}.$$

We note explicitly this particular case of Corollary 5.16:

Corollary 5.17 *Assume* (C_1^1)-(C_4^1)*, where* μ *is the counting measure on* 2^X*, and let* $(f_u)_{u\in X}$ *be a sequence taking values in an interval* $I \subset \mathbb{R}$ *such that* $\sum_{u\in X} |f_u| p_u < \infty$. *If* q *is a convex function on* I *such that* $\sum_{u\in X} |q(f_u)| p_u < \infty$, *then*

$$q\left(\sum_{u\in X} f_u p_u\right)$$

$$\leq \sum_{(u_1,\ldots,u_k)\in S} \left(\sum_{j=1}^{k} \frac{\beta_j p_{u_j}}{\sum_{m=1}^{k} \beta_m \alpha_{S,u_j}^m}\right) q\left(\frac{1}{\sum_{j=1}^{k} \frac{\beta_j p_{u_j}}{\sum_{m=1}^{k} \beta_m \alpha_{S,u_j}^m}} \sum_{j=1}^{k} \frac{\beta_j p_{u_j}}{\sum_{m=1}^{k} \beta_m \alpha_{S,u_j}^m} f_{u_j}\right)$$

$$\leq \sum_{u\in X} q(f_u)\, p_u.$$

Corollary 5.16 corresponds to Theorem 2.4, but in that result only finite sets are considered. If $X = \{1, \ldots, n\}$ and $\beta_m = 1$ $(m = 1, \ldots, k)$, then Theorem 2.1 contains Corollary 5.17, but Corollary 5.16 makes sense in a lot of other cases (for example, for countably infinite sets).

Next, some examples are given.

The first example deals with a relatively flexile case.

Example 5.1 (a) Assume (C_1)-(C_3), and let $A_m \in \mathscr{A}$ $(m = 1, \ldots, k)$ such that $0 < \mu(A_m) < \infty$ $(m = 1, \ldots, k)$ and $\bigcup\limits_{m=1}^{k} A_m = X$. Define $S := A_1 \times \ldots \times A_k$. Then (5.30) holds and

$$l(x) = \left(\prod_{m=1}^{k} \mu(A_m) \right) \sum_{m=1}^{k} \left(\frac{\beta_m \chi_m(x)}{\mu(A_m)} \right), \quad x \in X,$$

where $\beta_m > 0$ $(m = 1, \ldots, k)$, and $\chi_m : X \to \mathbb{R}$ means the characteristic function of A_m $(m = 1, \ldots, k)$. We can see that (C_4) is satisfied and

$$\psi(x_1, \ldots, x_k) = \frac{1}{\left(\prod\limits_{m=1}^{k} \mu(A_m) \right)} \sum_{j=1}^{k} \frac{\beta_j \varphi(x_j)}{\sum\limits_{m=1}^{k} \left(\frac{\beta_m \chi_m(x_j)}{\mu(A_m)} \right)}, \quad (x_1, \ldots, x_k) \in X^k.$$

The condition (C_5) is also true, since

$$pr^m(S) = A_1 \times \ldots \times A_{m-1} \times A_{m+1} \times \ldots \times A_k, \quad m = 1, \ldots, k.$$

Moreover for $m = 1, \ldots, k$

$$S_{x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_k} = \begin{cases} A_m, & \text{if} \quad (x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_k) \in A_1 \times \ldots \times A_{m-1} \times A_{m+1} \times \ldots \times A_k \\ \emptyset, & \text{otherwise} \end{cases}.$$

It follows that Theorem 5.8 can be applied.

(b) We consider the special case of (a), when the sets A_m $(m = 1, \ldots, k)$ are pairwise disjoint (a special partition of X). Let the function τ be defined on X by

$$\tau(x) := m, \quad \text{if} \quad x \in A_m.$$

Then

$$l(x) = \left(\prod_{m=1}^{k} \mu(A_m) \right) \frac{\beta_{\tau(x)}}{\mu(A_{\tau(x)})}, \quad x \in X,$$

and

$$\psi(x_1, \ldots, x_k) = \frac{1}{\left(\prod\limits_{m=1}^{k} \mu(A_m) \right)} \sum_{j=1}^{k} \frac{\beta_j \varphi(x_j) \mu(A_{\tau(x_j)})}{\beta_{\tau(x_j)}}, \quad (x_1, \ldots, x_k) \in X^k.$$

The second example corresponds to Corollary 5.16.

Example 5.2 Let $X := \{0,1,\ldots\}$, let $(p_u)_{u=0}^{\infty}$ be a sequence of positive numbers for which $\sum\limits_{u=0}^{\infty} p_u = 1$, and *let* $(f_u)_{u=0}^{\infty}$ *be a sequence taking values in an interval* $I \subset \mathbb{R}$ such that $\sum\limits_{u=0}^{\infty} |f_u|\, p_u < \infty$. *Define*

$$S := \left\{ (u_1,u_2) \in X^2 \mid u_1 \le u_2 \le 2u_1 \right\}.$$

An easy calculation shows that $l(u) = \alpha_{S,u} = u + \left[\frac{u}{2}\right] + 2 (\ge 2)$ for all $u \in X$, where $\left[\frac{u}{2}\right]$ denotes the greatest integer that does not exceed $\frac{u}{2}$. *If q is a convex function on I such that* $\sum\limits_{u=0}^{\infty} |q(f_u)|\, p_u < \infty$, *then by Corollary 5.17*

$$q\left(\sum_{u=0}^{\infty} f_u p_u\right) \le \sum_{u=0}^{\infty}\left(\sum_{v=u}^{2u}\left(\frac{p_u}{u+\left[\frac{u}{2}\right]+2} + \frac{p_v}{v+\left[\frac{v}{2}\right]+2}\right.\right.$$

$$\left.\left.\cdot q\left(\frac{\frac{p_u}{u+\left[\frac{u}{2}\right]+2} f_u + \frac{p_v}{v+\left[\frac{v}{2}\right]+2} f_v}{\frac{p_u}{u+\left[\frac{u}{2}\right]+2} + \frac{p_v}{v+\left[\frac{v}{2}\right]+2}}\right)\right)\right) \le \sum_{u=0}^{\infty} q(f_u)\, p_u.$$

The final example illustrates the case $X := \mathbb{R}$.

Example 5.3 Consider the measure space $(\mathbb{R}, \mathscr{B}, \lambda)$. The function $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is the density of the standard normal distribution on $\mathbb{R}$, *and thus* $\int\limits_{-\infty}^{\infty} \varphi = 1$. Let

$$S := \left\{ (x,y) \in \mathbb{R}^2 \mid x-1 \le y \le x+1 \right\}.$$

Then (C_1)-(C_3) are satisfied. Let $f : \mathbb{R} \to \mathbb{R}$ *be a* Borel measurable *function taking values in an interval* $I \subset \mathbb{R}$ *such that* $f\varphi$ is integrable, and let q *be a convex function on* I *such that* $(q \circ f)\,\varphi$ *is integrable. By Theorem 5.8 (a)*

$$q\left(\int_{-\infty}^{\infty} f\varphi\right)$$

$$\le \frac{1}{4\sqrt{2\pi}} \int_{-\infty}^{\infty}\left(\int_{x-1}^{x+1} q\left(\frac{e^{-x^2/2} f(x) + e^{-y^2/2} f(y)}{e^{-x^2/2} + e^{-y^2/2}}\right)\left(e^{-x^2/2} + e^{-y^2/2}\right) dy\right) dx$$

$$\le \int_{-\infty}^{\infty} (q \circ f)\,\varphi.$$

Now, we prove our results.

We first establish a result which will be fundamental to our treatment.

Lemma 5.5 *Assume* (C_1)*-*(C_4)*, and let* $f : X \to \mathbb{R}$ *be a* P*-integrable function. Then*

$$\int_X f dP = \int_X f\varphi d\mu = \int_S \left(\sum_{j=1}^{k} \frac{\beta_j \varphi(x_j)}{l(x_j)} f(x_j) \right) d\mu^k (x_1, \ldots, x_k).$$

Proof. The functions

$$\frac{\beta_j (\varphi \circ pr_j)}{l \circ pr_j} f \circ pr_j, \quad j = 1, \ldots, k$$

are obviously $\mathscr{A}^k$-measurable on S.

Suppose first that the function f is nonnegative. By (C_3), $pr_j(S) \in \mathscr{A}$, and hence the theorem of Fubini implies that

$$\int_S \frac{\beta_j \varphi(x_j)}{l(x_j)} f(x_j) d\mu^k (x_1, \ldots, x_k)$$

$$= \int_{pr_j(S)} \left(\int_{S_{j,x_j}} \frac{\beta_j \varphi(x_j)}{l(x_j)} f(x_j) d\mu^{k-1}((x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_k)) \right) d\mu(x_j)$$

$$= \int_{pr_j(S)} \frac{\beta_j \varphi(x_j)}{l(x_j)} f(x_j) \mu^{k-1}\left(S_{j,x_j}\right) d\mu(x_j), \quad j = 1, \ldots, k. \tag{5.36}$$

It follows from (5.36) that

$$\int_S \left(\sum_{j=1}^{k} \frac{\beta_j \varphi(x_j)}{l(x_j)} f(x_j) \right) d\mu^k (x_1, \ldots, x_k) \tag{5.37}$$

$$= \sum_{j=1}^{k} \int_{pr_j(S)} \frac{\beta_j \varphi(x_j)}{l(x_j)} f(x_j) \mu^{k-1}\left(S_{j,x_j}\right) d\mu(x_j).$$

If $(i_1, \ldots, i_k) \in \{0,1\}^k$, then let

$$A_{i_1,\ldots,i_k} := \bigcap_{m=1}^{k} pr_m(S)^{(i_m)},$$

where

$$pr_m(S)^{(i_m)} := \begin{cases} pr_m(S), & \text{if} \quad i_m = 1 \\ X \setminus pr_m(S), & \text{if} \quad i_m = 0 \end{cases} \qquad m = 1, \ldots, k.$$

The sets $A_{i_1,\ldots,i_k}$ $((i_1,\ldots,i_k) \in \{0,1\}^k)$ are pairwise disjoint and measurable. Moreover, by (5.30)

$$\bigcup_{(i_1,\ldots,i_k)\in\{0,1\}^k} A_{i_1,\ldots,i_k} = \bigcup_{m=1}^{k} pr_m(S) = X.$$

These establishments with (5.37) imply that

$$\int_S \left(\sum_{j=1}^{k} \frac{\beta_j \varphi(x_j)}{l(x_j)} f(x_j) \right) d\mu^k(x_1,\ldots,x_k)$$

$$= \sum_{(i_1,\ldots,i_k)\in\{0,1\}^k} \int_{A_{i_1,\ldots,i_k}} \left(\frac{\varphi(x)}{l(x)} f(x) \sum_{\substack{m\in\{1,\ldots,k\} \\ i_m=1}} \beta_m \mu^{k-1}(S_{m,x}) \right) d\mu(x). \tag{5.38}$$

Choose $(i_1,\ldots,i_k) \in \{0,1\}^k$. It is clear that $S_{m,x} = \emptyset$ if $x \in A_{i_1,\ldots,i_k}$ and $i_m = 0$, and hence

$$\sum_{\substack{m\in\{1,\ldots,k\} \\ i_m=1}} \beta_m \mu^{k-1}(S_{m,x}) = l(x), \quad x \in A_{i_1,\ldots,i_k}.$$

Therefore (5.38) gives

$$\int_S \left(\sum_{j=1}^{k} \frac{\beta_j \varphi(x_j)}{l(x_j)} f(x_j) \right) d\mu^k(x_1,\ldots,x_k)$$

$$= \sum_{(i_1,\ldots,i_k)\in\{0,1\}^k} \int_{A_{i_1,\ldots,i_k}} \varphi f d\mu = \int_X f\varphi d\mu.$$

Having disposed of the nonnegativity of the function f, we have from the first part of the proof that

$$\int_X |f|\, dP = \int_X |f|\, \varphi d\mu = \int_S \left(\sum_{j=1}^{k} \frac{\beta_j \varphi(x_j)}{l(x_j)} |f(x_j)| \right) d\mu^k(x_1,\ldots,x_k),$$

and therefore the functions

$$\frac{\beta_j (\varphi \circ pr_j)}{l \circ pr_j} f \circ pr_j, \quad j = 1,\ldots,k$$

are μ^k-integrable over S. By using this, the result follows by an argument entirely similar to that for the nonnegative case.

The proof is complete. □

Remark 5.5 Under the conditions of Lemma 5.5, we have

(a) The functions

$$\frac{\beta_j(\varphi \circ pr_j)}{l \circ pr_j} f \circ pr_j, \quad j = 1, \ldots, k$$

are μ^k-integrable over S.

(b) The measure P_k defined on $S \cap \mathscr{A}^k$ by

$$P_k(A) := \int_A \psi d\mu^k = \int_A \left(\sum_{j=1}^{k} \frac{\beta_j \varphi(x_j)}{l(x_j)} \right) d\mu^k(x_1, \ldots, x_k)$$

is a probability measure.

Lemma 5.6 *Assume (C_1)-(C_4). Let $f : X \to \mathbb{R}$ be a P-integrable function taking values in an interval $I \subset \mathbb{R}$, and let q be a convex function on I such that $q \circ f$ is P-integrable.*

(a) The function

$$g := \left(q \circ \left(\frac{1}{\psi} \sum_{j=1}^{k} \frac{\beta_j(\varphi \circ pr_j)}{l \circ pr_j} (f \circ pr_j) \right) \right) \psi$$

is μ^k-integrable over S.

(b) The functions

$$h_i := \left(q \circ \left(\frac{1}{\psi^i \circ pr^i} \sum_{\substack{j=1 \\ j \neq i}}^{k} \frac{\beta_j(\varphi \circ pr_j)}{l \circ pr_j} (f \circ pr_j) \right) \right) \left(\psi^i \circ pr^i \right), \quad i = 1, \ldots, k$$

are μ^k-integrable over S.

Proof. (a) It is easy to check that for fixed $(x_1, \ldots, x_k) \in S$

$$\frac{1}{\psi(x_1, \ldots x_k)} \frac{\beta_j \varphi(x_j)}{l(x_j)}, \quad j = 1, \ldots, k$$

are positive numbers with

$$\frac{1}{\psi(x_1, \ldots, x_k)} \sum_{j=1}^{k} \frac{\beta_j \varphi(x_j)}{l(x_j)} = 1.$$

This gives immediately that for every $(x_1, \ldots, x_k) \in S$

$$\frac{1}{\psi(x_1, \ldots x_k)} \sum_{j=1}^{k} \frac{\beta_j \varphi(x_j)}{l(x_j)} f(x_j) \in I,$$

and therefore by the discrete Jensen's inequality,

$$g(x_1, \ldots, x_k) \leq \sum_{j=1}^{k} \frac{\beta_j \varphi(x_j)}{l(x_j)} q(f(x_j)), \quad (x_1, \ldots, x_k) \in S. \tag{5.39}$$

Since the function q is convex on I, it is lower semicontinuous on I, and therefore the function g is $\mathscr{A}^k$-measurable.

Choose an interior point a of I. The convexity of q on I implies that

$$q(t) \geq q(a) + q_+^{'}(a)(t-a), \quad t \in I,$$

where $q_+^{'}(a)$ means the right-hand derivative of q at a. It follows from this and from (5.39) that

$$q(a)\psi(x_1,\ldots x_k) + q_+^{'}(a)\left(\sum_{j=1}^{k}\frac{\beta_j\varphi(x_j)}{l(x_j)}f(x_j) - a\psi(x_1,\ldots x_k)\right)$$

$$\leq g(x_1,\ldots,x_k) \leq \sum_{j=1}^{k}\frac{\beta_j\varphi(x_j)}{l(x_j)}q(f(x_j)), \quad (x_1,\ldots,x_k) \in S.$$

Now we can apply Remark 5.5 (a), by the P-integrability of the functions 1_X, f and $q \circ f$.

(b) Fix i from the set $\{1,\ldots,k\}$. We can prove as in (a) by using the $\mathscr{A}^k$-measurability of h_i and the estimates

$$q(a)\psi^i(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_k)$$

$$+q_+^{'}(a)\left(\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{\beta_j\varphi(x_j)}{l(x_j)}f(x_j) - a\psi^i(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_k)\right)$$

$$\leq h_i(x_1,\ldots,x_k) \leq \sum_{\substack{j=1\\ j\neq i}}^{k}\frac{\beta_j\varphi(x_j)}{l(x_j)}q(f(x_j)), \quad (x_1,\ldots,x_k) \in S.$$

The proof is complete. □

Proof of Theorem 5.8 (a) By Lemma 5.5

$$q\left(\int_X f dP\right) = q\left(\int_X f\varphi d\mu\right) = q\left(\int_S\left(\sum_{j=1}^{k}\frac{\beta_j\varphi(x_j)}{l(x_j)}f(x_j)\right)d\mu^k(x_1,\ldots,x_k)\right)$$

$$= q\left(\int_S\left(\frac{1}{\psi(x_1,\ldots x_k)}\sum_{j=1}^{k}\frac{\beta_j\varphi(x_j)}{l(x_j)}f(x_j)\psi(x_1,\ldots x_k)\right)d\mu^k(x_1,\ldots,x_k)\right)$$

$$= q\left(\int_S\left(\frac{1}{\psi(x_1,\ldots x_k)}\sum_{j=1}^{k}\frac{\beta_j\varphi(x_j)}{l(x_j)}f(x_j)\right)dP_k(x_1,\ldots,x_k)\right).$$

Since P_k is a probability measure on $S \cap \mathscr{A}^k$, it follows from the previous part, the classical Jensen's inequality, Lemma 5.5 and Lemma 5.6 (a) that

$$q\left(\int_X f dP\right) = q\left(\int_X f\varphi d\mu\right)$$

$$\leq \int_S q\left(\frac{1}{\psi(x_1,\ldots x_k)}\sum_{j=1}^k \frac{\beta_j\varphi(x_j)}{l(x_j)}f(x_j)\right)dP_k(x_1,\ldots,x_k)$$

$$= \int_S q\left(\frac{1}{\psi(x_1,\ldots x_k)}\sum_{j=1}^k \frac{\beta_j\varphi(x_j)}{l(x_j)}f(x_j)\right)\psi(x_1,\ldots x_k)\,d\mu^k(x_1,\ldots x_k)$$

$$\leq \int_S \left(\sum_{j=1}^k \frac{\beta_j\varphi(x_j)}{l(x_j)}q(f(x_j))\right)d\mu^k = \int_X (q\circ f)\varphi d\mu = \int_X q\circ f dP.$$

Now (a) has been proven.
(b) By using the convexity of q, an easy manipulation leads to

$$q\left(\frac{1}{\psi(x_1,\ldots,x_k)}\sum_{j=1}^k \frac{\beta_j\varphi(x_j)}{l(x_j)}f(x_j)\right)$$

$$= q\left(\sum_{i=1}^k \frac{\sum_{\substack{j=1\\ j\neq i}}^k \frac{\beta_j\varphi(x_j)}{l(x_j)}f(x_j)}{\psi^i(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_k)}\frac{\psi^i(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_k)}{(k-1)\psi(x_1,\ldots,x_k)}\right)$$

$$\leq \frac{1}{(k-1)\psi(x_1,\ldots,x_k)}\sum_{i=1}^k \psi^i(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_k)$$

$$\cdot q\left(\frac{1}{\psi^i(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_k)}\sum_{\substack{j=1\\ j\neq i}}^k \frac{\beta_j\varphi(x_j)}{l(x_j)}f(x_j)\right)$$

$$\leq \frac{1}{(k-1)\psi(x_1,\ldots,x_k)}\sum_{i=1}^k\left(\sum_{\substack{j=1\\ j\neq i}}^k \frac{\beta_j\varphi(x_j)}{l(x_j)}q(f(x_j))\right)$$

$$= \frac{1}{\psi(x_1,\ldots,x_k)}\sum_{j=1}^k \frac{\beta_j\varphi(x_j)}{l(x_j)}q(f(x_j)), \quad (x_1,\ldots,x_k)\in S.$$

Consequently, by applying (C_5), Lemma 5.6 (b) and the theorem of Fubini, we have

$$N_k = \int_S q\left(\frac{1}{\psi(x_1,\ldots,x_k)}\sum_{j=1}^k \frac{\beta_j\varphi(x_j)}{l(x_j)}f(x_j)\right)\psi(x_1,\ldots,x_k)\,d\mu^k(x_1,\ldots,x_k)$$

$$\leq \frac{1}{k-1}\sum_{i=1}^k\int_S q\left(\frac{1}{\psi^i(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_k)}\sum_{\substack{j=1\\ j\neq i}}^k \frac{\beta_j\varphi(x_j)}{l(x_j)}f(x_j)\right)$$

$$\cdot\psi^{i}\left(x_{1},\ldots,x_{i-1},x_{i+1},\ldots,x_{k}\right)d\mu^{k}\left(x_{1},\ldots,x_{k}\right)=\frac{1}{k-1}\sum_{i=1}^{k}\int\limits_{pr^{i}(S)}$$

$$\left(\int\limits_{S_{x_{1},\ldots,x_{i-1},x_{i+1},\ldots,x_{k}}} q\left(\frac{1}{\psi^{i}\left(x_{1},\ldots,x_{i-1},x_{i+1},\ldots,x_{k}\right)}\sum_{\substack{j=1\\ j\neq i}}^{k}\frac{\beta_{j}\varphi\left(x_{j}\right)}{l\left(x_{j}\right)}f(x_{j})\right)d\mu(x_{i})\right)$$

$$\cdot\psi^{i}\left(x_{1},\ldots,x_{i-1},x_{i+1},\ldots,x_{k}\right)d\mu^{k-1}\left(x_{1},\ldots,x_{i-1},x_{i+1},\ldots,x_{k}\right)$$

$$=N_{k-1}\leq\int\limits_{X}\left(q\circ f\right)\varphi d\mu.$$

The proof is complete. □

Proof of Theorem 5.9 Apply Theorem 5.8 with $S:=X^{k}$ and $\beta_{m}=1$ $(m=1,\ldots,k)$. Then the conditions (C_4) (by using $\mu\left(X\right)<\infty$) and (C_5) are satisfied,

$$\psi\left(x_{1},\ldots,x_{k}\right):=\frac{1}{k\mu(X)^{k-1}}\sum_{j=1}^{k}\varphi\left(x_{j}\right),\quad\left(x_{1},\ldots,x_{k}\right)\in S,$$

and $\psi^{i}:X^{k-1}\to\,]0,\infty[$ $(i=1,\ldots,k)$ has the form

$$\psi^{i}\left(x_{1},\ldots,x_{i-1},x_{i+1},\ldots,x_{k}\right):=\frac{1}{k\mu(X)^{k-1}}\sum_{\substack{j=1\\ j\neq i}}^{k}\varphi\left(x_{j}\right).$$

Therefore

$$N_{k}=\frac{1}{k\mu(X)^{k-1}}\int\limits_{X^{k}}q\left(\frac{\sum\limits_{j=1}^{k}\varphi\left(x_{j}\right)f(x_{j})}{\sum\limits_{j=1}^{k}\varphi\left(x_{j}\right)}\right)\sum_{j=1}^{k}\varphi\left(x_{j}\right)d\mu^{k}\left(x_{1},\ldots,x_{k}\right)$$

and

$$N_{k-1}=\frac{1}{k-1}\sum_{i=1}^{k}\int\limits_{X^{k-1}}\left(\mu(X)q\left(\frac{\sum\limits_{\substack{j=1\\ j\neq i}}^{k}\frac{\varphi(x_{j})}{l(x_{j})}f(x_{j})}{\sum\limits_{\substack{j=1\\ j\neq i}}^{k}\varphi\left(x_{j}\right)}\right)\right)$$

$$\cdot\frac{1}{k\mu(X)^{k-1}}\sum_{\substack{j=1\\ j\neq i}}^{k}\varphi\left(x_{j}\right)d\mu^{k-1}\left(x_{1},\ldots,x_{i-1},x_{i+1},\ldots,x_{k}\right)$$

$$= \frac{1}{(k-1)\mu(X)^{k-2}} \int\limits_{X^{k-1}} q\left(\frac{\sum\limits_{j=1}^{k-1} \varphi(x_j) f(x_j)}{\sum\limits_{j=1}^{k-1} \varphi(x_j)}\right) \sum_{j=1}^{k-1} \varphi(x_j)\, d\mu^{k-1}(x_1, \ldots, x_{k-1}).$$

The proof is complete. □

Chapter 6

Mean Value Theorems and Exponential Convexity

6.1 Mean Value Theorems

The first two mean value theorems are found in [40].

Theorem 6.1 *Let $I \subset \mathbb{R}$ be an interval. Let Υ be a linear functional on a subspace $D(\Upsilon)$ of the vector space of real functions defined on I such that $id^2 \in D(\Upsilon)$ $\left(id^2(x) = x^2,\ x \in I\right)$. Suppose further that $[a,b] \subset I$ is an interval with the following property:*

(i) if $f \in D(\Upsilon)$ such that the restriction of f on $[a,b]$ is convex, then $\Upsilon(f) \geq 0$.

If $g \in C^2(I) \cap D(\Upsilon)$, then there exists $\xi \in [a,b]$ such that

$$\Upsilon(g) = \frac{1}{2} g''(\xi)\, \Upsilon\left(id^2\right).$$

Proof. Since $g \in C^2(I)$, there exist the real numbers $m = \min_{x \in [a,b]} g''(x)$ and $M = \max_{x \in [a,b]} g''(x)$. It is easy to show that the functions ϕ_1 and ϕ_2 defined on I by

$$\phi_1(x) = \frac{M}{2} x^2 - g(x),$$

and

$$\phi_2(x) = g(x) - \frac{m}{2} x^2,$$

belong to $D(\Upsilon)$, and their restrictions on $[a,b]$ are convex.

By applying the functional Υ to the functions ϕ_1 and ϕ_2, we have from the properties of Υ that

$$\Upsilon\left(\frac{M}{2}id^2 - g\right) \geq 0,$$

$$\Rightarrow \Upsilon(g) \leq \frac{M}{2}\Upsilon\left(id^2\right), \tag{6.1}$$

and

$$\Upsilon\left(g - \frac{m}{2}id^2\right) \geqslant 0$$

$$\Rightarrow \frac{m}{2}\Upsilon\left(id^2\right) \leq \Upsilon(g). \tag{6.2}$$

From (6.1) and from (6.2), we get

$$\frac{m}{2}\Upsilon\left(id^2\right) \leq \Upsilon(g) \leq \frac{M}{2}\Upsilon\left(id^2\right).$$

If $\Upsilon\left(id^2\right) = 0$, then nothing to prove. If $\Upsilon\left(id^2\right) \neq 0$, then

$$m \leq \frac{2\Upsilon(g)}{\Upsilon\left(id^2\right)} \leq M.$$

Hence we have

$$\Upsilon(g) = \frac{1}{2}g''(\xi)\Upsilon\left(id^2\right).$$

□

Remark 6.1 (a) Under the conditions of the previous theorem, $\Upsilon(f) \geq 0$ for every convex $f \in D(\Upsilon)$.

(b) Define the linear functional $\Upsilon : C(I) \to \mathbb{R}$ on the vector space of real continuous functions defined on an interval $I \subset \mathbb{R}$ by

$$\Upsilon(f) := \frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right),$$

where $a,b \in I$ and $a < b$. Then Υ is linear, $id^2 \in C(I)$, and $\Upsilon(f) \geq 0$ for every $f \in C(I)$ such that the restriction of f on $[a,b]$ is convex.

Theorem 6.2 *Let $I \subset \mathbb{R}$ be an interval. Let Υ be a linear functional on a subspace $D(\Upsilon)$ of the vector space of real functions defined on I such that $id^2 \in D(\Upsilon)$. Suppose further that $[a,b] \subset I$ is an interval with the following property:*

(i) if $f \in D(\Upsilon)$ such that the restriction of f on $[a,b]$ is convex, then $\Upsilon(f) \geq 0$.

If $g,h \in C^2(I) \cap D(\Upsilon)$, then there exists $\xi \in [a,b]$ such that

$$\frac{\Upsilon(g)}{\Upsilon(h)} = \frac{g''(\xi)}{h''(\xi)},$$

provided that $\Upsilon(h) \neq 0$ and $h''(x) \neq 0$ $(x \in [a,b])$.

Proof. Define $f \in C^2(I) \cap D(\Upsilon)$ by

$$f := c_1 g - c_2 h,$$

where

$$c_1 := \Upsilon(h)$$

and

$$c_2 := \Upsilon(g).$$

Now, by applying Theorem 6.1 for the function f, we have

$$\left(c_1 \frac{g''(\xi)}{2} - c_2 \frac{h''(\xi)}{2}\right) \Upsilon\left(id^2\right) = 0. \tag{6.3}$$

Since $\Upsilon(h) \neq 0$, Theorem 6.1 implies that $\Upsilon\left(id^2\right) \neq 0$, and therefore (6.3) and $h''(x) \neq 0$ $(x \in [a,b])$ give

$$\frac{\Upsilon(g)}{\Upsilon(h)} = \frac{g''(\xi)}{h''(\xi)}.$$

□

Now we give two additional mean value theorems.

Theorem 6.3 *Let $n,k \in \mathbb{N}$, $n \geq 3$, $2 \leq k \leq n-1$, $[\alpha,\beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1,\ldots,x_n) \in [\alpha,\beta]^n$, and $\mathbf{p} = (p_1,\ldots,p_n)$ be a real n-tuple such that $\sum_{j=1}^{k} p_{i_j} \neq 0$ for any $1 \leq i_1 < \ldots < i_k \leq n$, $\sum_{i=1}^{n} p_i = 1$ and $\frac{\sum_{j=1}^{k} p_{i_j} x_{i_j}}{\sum_{j=1}^{k} p_{i_j}} \in [\alpha,\beta]$ for any $1 \leq i_1 < \ldots < i_k \leq n$. Assume $f : [\alpha,\beta] \to \mathbb{R}$ is a function from $C^2([\alpha,\beta])$, and let*

$$f_{k,n}(\mathbf{x},\mathbf{p}) := \frac{1}{\binom{n-1}{k-1}} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \left(\sum_{j=1}^{k} p_{i_j}\right) f\left(\frac{\sum_{j=1}^{k} p_{i_j} x_{i_j}}{\sum_{j=1}^{k} p_{i_j}}\right),$$

and

$$\Upsilon(\mathbf{x},\mathbf{p},f) := \frac{n-k}{n-1} f_{1,n}(\mathbf{x},\mathbf{p}) + \frac{k-1}{n-1} f_{n,n}(\mathbf{x},\mathbf{p}) - f_{k,n}(\mathbf{x},\mathbf{p}). \tag{6.4}$$

If for all $s \in [\alpha,\beta]$ either the inequality (4.7) holds or the reverse inequality holds in (4.7), then there exists $\xi \in [\alpha,\beta]$ such that

$$\Upsilon(\mathbf{x},\mathbf{p},f) = \frac{1}{2} f''(\xi) \Upsilon(\mathbf{x},\mathbf{p},id^2),$$

where $id^2(t) = t^2$ $(t \in [\alpha,\beta])$.

Proof. By the assumption, we have that the function f'' is continuous and

$$\frac{n-k}{n-1}G_{1,n}(\mathbf{x},s,\mathbf{p})+\frac{k-1}{n-1}G_{n,n}(\mathbf{x},s,\mathbf{p})-G_{k,n}(\mathbf{x},s,\mathbf{p})$$

does not change its sign on $[\alpha,\beta]$. By applying the integral mean value theorem, we have from (4.8) that there exists a $\xi\in[\alpha,\beta]$ such that

$$\begin{aligned}&\Upsilon(\mathbf{x},\mathbf{p},f)\\&=f''(\xi)\int_\alpha^\beta\left(\tfrac{n-k}{n-1}G_{1,n}(\mathbf{x},s,\mathbf{p})+\tfrac{k-1}{n-1}G_{n,n}(\mathbf{x},s,\mathbf{p})-G_{k,n}(\mathbf{x},s,\mathbf{p})\right)ds.\end{aligned} \tag{6.5}$$

By the definition of the function G, we can observe that

$$\int_\alpha^\beta G(t,s)ds=\frac{1}{2}(t-\alpha)(t-\beta). \tag{6.6}$$

Now, we calculate the integral on the right hand side of (6.5) with the help of (6.6): $\overline{x}$ means $\sum\limits_{i=1}^n p_ix_i$;

$$\begin{aligned}\Upsilon(\mathbf{x},\mathbf{p},f)&=f''(\xi)\left(\begin{array}{l}\frac{n-k}{n-1}\sum\limits_{i=1}^n p_i\int_\alpha^\beta G(x_i,s)ds+\frac{k-1}{n-1}\int_\alpha^\beta G(\overline{x},s)ds\\ -\frac{1}{\binom{n-1}{k-1}}\sum\limits_{1\le i_1<\ldots<i_k\le n}\left(\sum\limits_{j=1}^k p_{i_j}\right)\displaystyle\int_\alpha^\beta G\left(\frac{\sum_{j=1}^k p_{i_j}x_{i_j}}{\sum_{j=1}^k p_{i_j}},s\right)ds\end{array}\right)\\
&=\frac{f''(\xi)}{2}\left(\begin{array}{l}\frac{n-k}{n-1}\sum\limits_{i=1}^n p_i(x_i-\alpha)(x_i-\beta)+\frac{k-1}{n-1}(\overline{x}-\alpha)(\overline{x}-\beta)\\ -\frac{1}{\binom{n-1}{k-1}}\sum\limits_{1\le i_1<\ldots<i_k\le n}\left(\sum\limits_{j=1}^k p_{i_j}\right)\left(\frac{\sum_{j=1}^k p_{i_j}x_{i_j}}{\sum_{j=1}^k p_{i_j}}-\alpha\right)\left(\frac{\sum_{j=1}^k p_{i_j}x_{i_j}}{\sum_{j=1}^k p_{i_j}}-\beta\right)\end{array}\right)\\
&=\frac{f''(\xi)}{2}\left(\frac{n-k}{n-1}\sum_{i=1}^n p_ix_i^2+\frac{k-1}{n-1}\overline{x}^2-\frac{1}{\binom{n-1}{k-1}}\sum_{1\le i_1<\ldots<i_k\le n}\left(\sum_{j=1}^k p_{i_j}\right)\left(\frac{\sum_{j=1}^k p_{i_j}x_{i_j}}{\sum_{j=1}^k p_{i_j}}\right)^2\right)\\
&=\tfrac{1}{2}f''(\xi)\Upsilon(\mathbf{x},\mathbf{p},id^2),\end{aligned}$$

which completes the proof. □

Theorem 6.4 *Let $n,k\in\mathbb{N}$, $n\ge 3$, $2\le k\le n-1$, $[\alpha,\beta]\subset\mathbb{R}$, $\mathbf{x}=(x_1,\ldots,x_n)\in[\alpha,\beta]^n$, and $\mathbf{p}=(p_1,\ldots,p_n)$ be a real n-tuple such that $\sum_{j=1}^k p_{i_j}\ne 0$ for any $1\le i_1<\ldots<i_k\le n$, $\sum_{i=1}^n p_i=1$ and $\frac{\sum_{j=1}^k p_{i_j}x_{i_j}}{\sum_{j=1}^k p_{i_j}}\in[\alpha,\beta]$ for any $1\le i_1<\ldots<i_k\le n$. Assume $f,g:[\alpha,\beta]\to$*

$\mathbb{R}$ are functions from $C^2([\alpha,\beta])$ such that $g''(s)\neq 0,\ s\in[\alpha,\beta]$. Assume further that $\Upsilon(\mathbf{x},\mathbf{p},id^2)\neq 0$, where Υ is defined in (6.4).

If for all $s\in[\alpha,\beta]$ either the inequality (4.7) holds or the reverse inequality holds in (4.7), then there exists $\xi\in[\alpha,\beta]$ such that

$$\frac{\Upsilon(\mathbf{x},\mathbf{p},f)}{\Upsilon(\mathbf{x},\mathbf{p},g)}=\frac{f''(\xi)}{g''(\xi)}. \tag{6.7}$$

Proof. Consider the function

$$h(t)=\Upsilon(\mathbf{x},\mathbf{p},g)f(t)-\Upsilon(\mathbf{x},\mathbf{p},f)g(t),\quad t\in[\alpha,\beta].$$

Then $h\in C^2([\alpha,\beta])$. Therefore we can apply Theorem 6.3 to the function h which shows that there exists a $\xi\in[\alpha,\beta]$ such that

$$\Upsilon(\mathbf{x},\mathbf{p},h)=\frac{h''(\xi)}{2}\left[\Upsilon(\mathbf{x},\mathbf{p},id^2)\right]. \tag{6.8}$$

Since $\Upsilon(\mathbf{x},\mathbf{p},h)=0$ and $\Upsilon(\mathbf{x},\mathbf{p},id^2)$ is nonzero, hence on the one hand

$$h''(\xi)=0,$$

and on the other hand Theorem 6.3 applied to the function g, shows that $\Upsilon(\mathbf{x},\mathbf{p},g)\neq 0$.
This completes the proof. □

The following lemma is given in [3]:

Lemma 6.1 *Let $h\in C^2(I)$ for a compact interval $I\subset\mathbb{R}\setminus\{0\}$ and consider $m,M\in\mathbb{R}$ such that*

$$m\leq\frac{x^2h''(x)-2xh'(x)+2h(x)}{x^3}\leq M,\quad x\in I.$$

If the functions h_1,h_2 are defined on I by

$$h_1(x)=M\frac{x^3}{2}-h(x)$$

and

$$h_2(x)=h(x)-m\frac{x^3}{2},$$

then the functions $x\to\frac{h_1(x)}{x}$ and $x\to\frac{h_2(x)}{x}$ $(x\in I)$ are convex.

Theorem 6.5 *Let $I\subset\mathbb{R}\setminus\{0\}$ be an interval. Let Ψ be a linear functional on a subspace of the vector space of real functions defined on I such that $f\in D(\Psi)$ implies $\frac{f}{id}\in D(\Psi)$ and $id^3\in D(\Psi)$. Suppose further that $[a,b]\subset I$ is an interval with the following property:*

(i) if $f\in D(\Psi)$ such that the restriction of $\frac{f}{id}$ to $[a,b]$ is convex, then $\Psi(f)\geq 0$.

If $g\in C^2(I)\cap D(\Psi)$, then there exists $\xi\in[a,b]$ such that

$$\Psi(g)=\frac{\xi^2g''(\xi)-2\xi g'(\xi)+2g(\xi)}{2\xi^3}\Psi(id^3).$$

Proof. By $g \in C^2(I)$, there exist

$$m := \min_{x \in [a,b]} \frac{x^2 g''(x) - 2xg'(x) + 2g(x)}{x^3},$$

and

$$M := \max_{x \in [a,b]} \frac{x^2 g''(x) - 2xg'(x) + 2g(x)}{x^3}.$$

Define the functions u and v on I by

$$u(x) = M\frac{x^3}{2} - g(x)$$

and

$$v(x) = g(x) - m\frac{x^3}{2}.$$

Lemma 6.1 implies that the functions $x \longrightarrow \frac{u(x)}{x}$ and $x \longrightarrow \frac{v(x)}{x}$ $(x \in I)$ are convex on $[a,b]$, and therefore (i) shows that $\Psi(u) \geq 0$ and $\Psi(v) \geq 0$. By the linearity of Ψ, we have

$$\frac{m}{2}\Psi(id^3) \leq \Psi(g) \leq \frac{M}{2}\Psi(id^3).$$

If $\Psi\left(id^3\right) = 0$, then nothing to prove. $\Psi\left(id^3\right) \neq 0$ yields

$$m \leq \frac{2\Psi(g)}{\Psi(id^3)} \leq M,$$

which gives the result. □

Remark 6.2 The previous theorem can be applied to the functional Ψ defined in (4.10).

Theorem 6.6 *Let $I \subset \mathbb{R} \setminus \{0\}$ be an interval. Let Ψ be a linear functional on a subspace of the vector space of real functions defined on I such that $f \in D(\Psi)$ implies $\frac{f}{id} \in D(\Psi)$ and $id^3 \in D(\Psi)$. Suppose further that $[a,b] \subset I$ is an interval with the following property:*

(i) if $f \in D(\Psi)$ such that the restriction of $\frac{f}{id}$ to $[a,b]$ is convex, then $\Psi(f) \geq 0$.

If $g,h \in C^2(I) \cap D(\Psi)$, then there exists $\xi \in [a,b]$ such that

$$\frac{\Psi(g)}{\Psi(h)} = \frac{\xi^2 g''(\xi) - 2\xi g'(\xi) + 2g(\xi)}{\xi^2 h''(\xi) - 2\xi h'(\xi) + 2h(\xi)},$$

provided that $\Psi(h) \neq 0$ and $x^2 h''(x) - 2xh'(x) + 2h(x) \neq 0$ $(x \in [a,b])$.

Proof. Define $f \in C^2(I) \cap D(\Psi)$ by

$$f := c_1 g - c_2 h,$$

where

$$c_1 := \Psi(h)$$

and

$$c_2 := \Psi(g).$$

Now, by using Theorem 6.5 for the function f, we have

$$\left(c_1\frac{\xi^2 g''(\xi) - 2\xi g'(\xi) + 2g(\xi)}{2\xi^3} - c_2\frac{\xi^2 h''(\xi) - 2\xi h'(\xi) + 2h(\xi)}{2\xi^3}\right)\Psi(id^3) = 0. \quad (6.9)$$

Since $\Psi(h) \neq 0$, Theorem 6.5 implies that $\Psi(id^3) \neq 0$, and therefore (6.9) and $x^2h''(x) - 2xh'(x) + 2h(x) \neq 0$ $(x \in [a,b])$ give the result. □

6.2 Exponential Convexity

The notion of *n-exponentially convex* functions is initiated in [68], while *exponentially convex* functions are invented by Bernstein in [10].

Definition 6.1 *Let $I \subset \mathbb{R}$ be an interval. A function $g : I \to \mathbb{R}$ is called* n-exponentially convex *in the Jensen sense if*

$$\sum_{i,j=1}^{n} a_i a_j g\left(\frac{x_i + x_j}{2}\right) \geq 0$$

holds for every $a_i \in \mathbb{R}$ and $x_i \in I$, $i = 1,2,\dots,n$.

*A function $g : I \to \mathbb{R}$ is n-*exponentially convex *if it is* n-exponentially convex *in the Jensen sense and continuous on I.*

Remark 6.3 From the definition it is clear that 1-exponentially convex functions on I in the Jensen sense are in fact the nonnegative functions on I. Also, n-exponentially convex functions in the Jensen sense are m-exponentially convex in the Jensen sense for every $m \in \mathbb{N}_+$, $m \leq n$.

Definition 6.2 *Let $I \subset \mathbb{R}$ be an interval.*

A function $g : I \to \mathbb{R}$ is exponentially convex in the Jensen sense, if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}_+$.

A function $g : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Remark 6.4 Let $I \subset \mathbb{R}$ be an interval. Lemma 1.1 shows that a positive function $g : I \to \mathbb{R}$ is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense, that is

$$a_1^2 g(x) + 2a_1 a_2 g\left(\frac{x+y}{2}\right) + a_2^2 g(y) \geq 0$$

holds for every $a_1, a_2 \in \mathbb{R}$ and $x, y \in I$.

Similarly, if g is positive and 2-exponentially convex, then g is log-convex. Conversely, if g is log-convex and continuous, then g is 2-exponentially convex.

The following result will be used later. This is a special case of a much more general result (see [2]).

Theorem 6.7 *Let $I \subset \mathbb{R}$ be an interval. If $g : I \to \mathbb{R}$ is a continuous function which has the form*

$$g(x) = \int_{-\infty}^{\infty} e^{-xt} d\mu(t), \quad x \in I,$$

with a measure μ on the Borel sets of $\mathbb{R}$, then g is exponentially convex.

Proof. Let $a_i \in \mathbb{R}$ and $x_i \in I$ $(i = 1, 2, \ldots, n)$. Then

$$\sum_{i,j=1}^{n} a_i a_j g\left(\frac{x_i + x_j}{2}\right) = \sum_{i,j=1}^{n} a_i a_j \int_{-\infty}^{\infty} e^{-\frac{x_i + x_j}{2} t} d\mu(t)$$

$$= \int_{-\infty}^{\infty} \sum_{i,j=1}^{n} a_i a_j e^{-\frac{x_i}{2} t} e^{-\frac{x_j}{2} t} d\mu(t) = \int_{-\infty}^{\infty} \left(\sum_{i=1}^{n} a_i e^{-\frac{x_i}{2} t}\right)^2 d\mu(t) \geq 0.$$

□

Divided differences are fertile to study functions having different degree of smoothness.

Definition 6.3 *Let $I \subset \mathbb{R}$ be an interval. The second order divided difference of a function $g : I \to \mathbb{R}$ at mutually different points $y_0, y_1, y_2 \in I$ is defined recursively by*

$$[y_i; g] = g(y_i), \quad i = 0, 1, 2$$

$$[y_i, y_{i+1}; g] = \frac{g(y_{i+1}) - g(y_i)}{y_{i+1} - y_i}, \quad i = 0, 1$$

$$[y_0, y_1, y_2; g] = \frac{[y_1, y_2; g] - [y_0, y_1; g]}{y_2 - y_0}. \tag{6.10}$$

Remark 6.5 The value $[y_0,y_1,y_2;g]$ is independent of the order of the points y_0,y_1, and y_2. By taking limits this definition may be extended to include the cases in which any two or all three points coincide as follows: for y_0, y_1, $y_2 \in I$ such that $y_2 \neq y_0$

$$[y_0,y_0,y_2;g] := \lim_{y_1 \to y_0} [y_0,y_1,y_2;g] = \frac{g(y_2) - g(y_0) - g^{'}(y_0)(y_2 - y_0)}{(y_2 - y_0)^2}$$

provided that g' exists, and furthermore, taking the limits $y_i \to y_0$, $i = 1,2$ in (6.10), we get

$$[y_0,y_0,y_0;g] \; := \lim_{y_1 \to y_0,\, y_2 \to y_0} [y_0,y_1,y_2;g] = \frac{g^{''}(y_0)}{2}$$

provided that $g^{''}$ exist on I.

The following lemma is well known.

Lemma 6.2 *Let $I \subset \mathbb{R}$ be an interval, and $g : I \to \mathbb{R}$.*

(a) g is convex if and only if $[y_0,y_1,y_2;g] \geq 0$ for every mutually different points $y_0,y_1,y_2 \in I$.

(b) Suppose g is differentiable. g is convex if and only if $[y_0,y_0,y_2;g] \geq 0$ ($[y_0,y_1,y_1;g] \geq 0$) for every $y_0,y_2 \in I$, $y_0 \neq y_2$ ($y_1,y_2 \in I, y_1 \neq y_2$).

(c) Suppose g is twice differentiable. g is convex if and only if $[y_0,y_0,y_0;g] \geq 0$ for every $y_0 \in I$.

$\mathscr{F}(X)$ will means the space of all real valued functions defined on the set X.

By analyzing the essential properties of the functionals Υ_i $(i = 1,\ldots,23)$, we introduce the following condition:

(F) Let $I \subset \mathbb{R}$ be an interval, and let $\Upsilon : D(\Upsilon) \to \mathbb{R}$ be a linear functional on a subspace of $\mathscr{F}(I)$ such that $\Upsilon(f) \geq 0$ for every convex f from $D(\Upsilon)$.

Theorem 6.8 *Assume (F). Let $J \subset \mathbb{R}$ be an interval, and let $\Lambda := \{\phi_t \mid t \in J\} \subset D(\Upsilon)$ such that the function $t \to [y_0,y_1,y_2;\phi_t]$ $(t \in J)$ is n-exponentially convex in the Jensen sense on J for every three mutually different points $y_0,y_1,y_2 \in I$. Then $t \to \Upsilon(\phi_t)$ $(t \in J)$ is an n-exponentially convex function in the Jensen sense on J. If the function $t \to \Upsilon(\phi_t)$ $(t \in J)$ is continuous, then it is n-exponentially convex on J.*

Proof. Let $t_k,t_l \in J$, $t_{kl} := \frac{t_k+t_l}{2}$ and $b_k \in \mathbb{R}$ for $k,l = 1,2,...,n$, and define the function ω by

$$\omega := \sum_{k,l=1}^{n} b_k b_l \phi_{t_{kl}}.$$

Then $\omega \in D(\Upsilon)$. By hypothesis the function $t \to [y_0,y_1,y_2;\phi_t]$ $(t \in J)$ is n-exponentially convex in the Jensen sense, therefore we have

$$[y_0,y_1,y_2;\omega] = \sum_{k,l=1}^{n} b_k b_l [y_0,y_1,y_2;\phi_{t_{kl}}] \geq 0,$$

which implies that ω is a convex function on I. Therefore we have $\Upsilon(\omega) \geq 0$, which yields by the linearity of Υ, that

$$\sum_{k,l=1}^{n} b_k b_l \Upsilon(\phi_{t_{kl}}) \geq 0.$$

We conclude that the function $t \to \Upsilon(\phi_t)$ $(t \in J)$ is an n-exponentially convex function in the Jensen sense on J.

If the function $t \to \Upsilon(\phi_t)$ $(t \in J)$ is continuous on J, then it is n-exponentially convex on J by definition. □

As a consequence of the above theorem we can give the following corollaries.

Corollary 6.1 *Assume (F). Let $J \subset \mathbb{R}$ be an interval, and let $\Lambda := \{\phi_t \mid t \in J\} \subset D(\Upsilon)$ such that the function $t \to [y_0, y_1, y_2; \phi_t]$ $(t \in J)$ is exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Then $t \to \Upsilon(\phi_t)$ $(t \in J)$ is an exponentially convex function in the Jensen sense on J. If the function $t \to \Upsilon(\phi_t)$ $(t \in J)$ is continuous, then it is exponentially convex on J.*

Corollary 6.2 *Assume (F). Let $J \subset \mathbb{R}$ be an interval, and let $\Lambda := \{\phi_t \mid t \in J\} \subset D(\Upsilon)$ such that the function $t \to [y_0, y_1, y_2; \phi_t]$ $(t \in J)$ is 2-exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Then the following two statements hold:*

(i) If the function $t \to \Upsilon(\phi_t)$ $(t \in J)$ is positive and continuous, then it is log*-convex on J.*

(ii) If the function $t \to \Upsilon(\phi_t)$ $(t \in J)$ is positive and differentiable, then for every $s, t, u, v \in J$, such that $s \leq u$ and $t \leq v$, we have

$$\mathfrak{u}_{s,t}(\Upsilon, \Lambda) \leq \mathfrak{u}_{u,v}(\Upsilon, \Lambda) \tag{6.11}$$

where for $s, t \in J$

$$\mathfrak{u}_{s,t}(\Upsilon, \Lambda) := \begin{cases} \left(\frac{\Upsilon(\phi_s)}{\Upsilon(\phi_t)} \right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(\frac{\frac{d}{ds}\Upsilon(\phi_s)}{\Upsilon(\phi_s)} \right), & t = s. \end{cases} \tag{6.12}$$

Proof.

(i) See Remark 6.4 and Theorem 6.8.

(ii) It is well known that if ψ is a convex function on J, then

$$\frac{\psi(s) - \psi(t)}{s - t} \leq \frac{\psi(u) - \psi(v)}{u - v}, \tag{6.13}$$

for every $s, t, u, v \in J$ such that $s \leq u$, $t \leq v$, $s \neq t$, $u \neq v$.
By (i), $s \to \Upsilon(\phi_s)$, $s \in J$ is log-convex, and hence (6.13) shows with $\psi(s) = \log \Upsilon(\phi_s)$, $s \in J$ that

$$\frac{\log \Upsilon(\phi_s) - \log \Upsilon(\phi_t)}{s - t} \leq \frac{\log \Upsilon(\phi_u) - \log \Upsilon(\phi_v)}{u - v} \tag{6.14}$$

for every $s,t,u,v \in J$ such that $s \leq u$, $t \leq v$, $s \neq t$, $u \neq v$, which is equivalent to (6.11). For $s = t$ or $u = v$ (6.11) follows from (6.14) by taking limit.

□

Remark 6.6 Suppose the function $t \to \Upsilon(\phi_t)$ $(t \in J)$ is positive and continuous in (ii). It can be seen from the proof of (ii) that (6.11) remains true for every $s,t,u,v \in J$ such that $s \leq u$, $t \leq v$, $s \neq t$ and $u \neq v$.

Remark 6.7 Note that the results Theorem 6.8, Corollary 6.1 and Corollary 6.2 are valid when two of the points $y_0, y_1, y_2 \in I$ coincide, say $y_1 = y_0$, for a family of differentiable functions $\Lambda = \{\phi_t \mid t \in J\} \subset D(\Upsilon)$ such that the function $t \to [y_0, y_0, y_2; \phi_t]$ is n-exponentially convex or exponentially convex or 2-exponentially convex in the Jensen sense on J for every two mutually different points $y_0, y_2 \in I$. Moreover, they are also valid when all three points coincide for a family of twice differentiable functions $\Lambda = \{\phi_t \mid t \in J\} \subset D(\Upsilon)$ with the same properties. The proofs can be obtained by recalling Remark 6.5 and Lemma 6.2.

Remark 6.8 A refinement of the inequality of Popoviciu from [67] is given by Niculescu and Popoviciu in [63]. An integral version of Theorem 4.1 is given by Niculescu in [62]. Results, similar to Theorem 4.2, Theorem 6.3, Theorem 6.4, Theorem 6.8, Corollary 6.1 and Corollary 6.2 can also be obtained from the results of the mentioned papers.

Remark 6.9 The functional Ψ defined in (4.10) depends on a real valued function defined on either $(0,a]$ or $[\alpha,a]$ $(\alpha > 0)$ and some other parameters. The parameters will be fixed except the function. The common notation of $(0,a]$ and $[\alpha,a]$ is I. The functional Ψ can be defined on $\mathscr{F}(I)$, and $\Psi(f) \geq 0$ for every function $f \in \mathscr{F}(I)$ for which $x \to \frac{f(x)}{x}$, $x \in I$ is convex. Moreover, Ψ is linear on $\mathscr{F}(I)$.

By taking into account this remark, we introduce the following condition:

(G) Let $I \subset \mathbb{R} \setminus \{0\}$ be an interval, and denote id the identity function on I. Let $\Psi : D(\Psi) \to \mathbb{R}$ be a linear functional which satisfies

(i) $D(\Psi)$ is a subspace of $\mathscr{F}(I)$ such that $f \in D(\Psi)$ implies $\frac{f}{id} \in D(\Psi)$,

(ii) $\Psi(f) \geq 0$ for every $f \in D(\Psi)$ for which $\frac{f}{id}$ is convex.

Theorem 6.9 *Assume (G). Let $J \subset \mathbb{R}$ be an interval, and let $\Phi := \{\phi_t \mid t \in J\} \subset D(\Psi)$ such that the function $t \to [y_0, y_1, y_2; \frac{\phi_t}{id}]$ $(t \in J)$ is n-exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Then $t \to \Psi(\phi_t)$ $(t \in J)$ is an n-exponentially convex function in the Jensen sense on J. If the function $t \to \Psi(\phi_t)$ $(t \in J)$ is continuous, then it is n-exponentially convex on J.*

Proof. Let $t_k, t_l \in J$, $t_{kl} := \frac{t_k + t_l}{2}$ and $b_k \in \mathbb{R}$ for $k,l = 1,2,\ldots,n$, and define the function ω by

$$\omega := \sum_{k,l=1}^{n} b_k b_l \phi_{t_{kl}}.$$

Then $\omega, \frac{\omega}{id} \in D(\Psi)$. By hypothesis the function $t \to [y_0, y_1, y_2; \frac{\phi_t}{id}]$ $(t \in J)$ is n-exponentially convex in the Jensen sense, therefore we have

$$[y_0, y_1, y_2; \frac{\omega}{id}] = \sum_{k,l=1}^{n} b_k b_l [y_0, y_1, y_2; \frac{\phi_{t_{kl}}}{id}] \geq 0,$$

which implies that $\frac{\omega}{id}$ is a convex function on I. Therefore we have $\Psi(\omega) \geq 0$, which yields by the linearity of Ψ, that

$$\sum_{k,l=1}^{n} b_k b_l \Psi(\phi_{t_{kl}}) \geq 0.$$

We conclude that the function $t \to \Psi(\phi_t)$ $(t \in J)$ is an n-exponentially convex function in the Jensen sense on J.

If the function $t \to \Psi(\phi_t)$ $(t \in J)$ is continuous on J, then it is n-exponentially convex on J by definition. □

As a consequence of the above theorem we can give the following corollaries.

Corollary 6.3 *Assume (G). Let $J \subset \mathbb{R}$ be an interval, and let $\Phi := \{\phi_t \mid t \in J\} \subset D(\Psi)$ such that the function $t \to [y_0, y_1, y_2; \frac{\phi_t}{id}]$ $(t \in J)$ is exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Then $t \to \Psi(\phi_t)$ $(t \in J)$ is an exponentially convex function in the Jensen sense on J. If the function $t \to \Psi(\phi_t)$ $(t \in J)$ is continuous, then it is exponentially convex on J.*

Corollary 6.4 *Assume (G). Let $J \subset \mathbb{R}$ be an interval, and let $\Phi := \{\phi_t \mid t \in J\} \subset D(\Psi)$ such that the function $t \to [y_0, y_1, y_2; \frac{\phi_t}{id}]$ $(t \in J)$ is 2-exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Then the following two statements hold:*

(i) If the function $t \to \Psi(\phi_t)$ $(t \in J)$ is positive and continuous, then it is log*-convex.*

(ii) If the function $t \to \Psi(\phi_t)$ $(t \in J)$ is positive and differentiable, then for every $s, t, u, v \in J$, such that $s \leq u$ and $t \leq v$, we have

$$\bar{\mathrm{u}}_{s,t}(\Psi, \Phi) \leq \bar{\mathrm{u}}_{u,v}(\Psi, \Phi) \tag{6.15}$$

where for $s, t \in J$

$$\bar{\mathrm{u}}_{s,t}(\Psi, \Phi) := \begin{cases} \left(\frac{\Psi(\phi_s)}{\Psi(\phi_t)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(\frac{\frac{d}{ds}\Psi(\phi_s)}{\Psi(\phi_s)}\right), & t = s. \end{cases} \tag{6.16}$$

Proof. The proof is similar to the proof of Corollary 6.2. □

Remark 6.10 Suppose the function $t \to \Psi(\phi_t)$ $(t \in J)$ is positive and continuous in (ii). It can be seen that (6.15) remains true for every $s, t, u, v \in J$ such that $s \leq u$, $t \leq v$, $s \neq t$ and $u \neq v$.

Remark 6.11 Note that the results Theorem 6.9, Corollary 6.3 and Corollary 6.4 are valid when two of the points $y_0, y_1, y_2 \in I$ coincide, say $y_1 = y_0$, for a family of differentiable functions $\Phi = \{\phi_t \mid t \in J\} \subset D(\Psi)$ such that the function $t \to [y_0, y_0, y_2; \frac{\phi_t}{id}]$ is n-exponentially convex or exponentially convex or 2-exponentially convex in the Jensen sense on J for every two mutually different points $y_0, y_2 \in I$. Moreover, they are also valid when all three points coincide for a family of twice differentiable functions $\Phi = \{\phi_t \mid t \in J\} \subset D(\Psi)$ with the same properties. The proofs can be obtained by recalling Remark 6.5 and Lemma 6.2.

The next result related to the first condition of Theorem 6.8 is given in [40].

Theorem 6.10 *Assume $J \subset \mathbb{R}$ is an interval, and assume $\Lambda = \{\phi_t \mid t \in J\}$ is a family of twice differentiable functions defined on an interval $I \subset \mathbb{R}$ such that the function $t \mapsto \phi_t''(x)$ $(t \in J)$ is exponentially convex for every fixed $x \in I$. Then the function $t \mapsto [y_0, y_1, y_2; \phi_t]$ $(t \in J)$ is exponentially convex in the Jensen sense for any three points y_0, y_1, $y_2 \in I$.*

Proof. Let n be a positive integer, $t_k, t_l \in J$, $t_{kl} := \frac{t_k + t_l}{2}$ and $b_k \in \mathbb{R}$ for $k, l = 1, 2, ..., n$. Then

$$\left(\sum_{k,l=1}^{n} b_k b_l \phi_{t_{kl}} \right)''(x) = \sum_{k,l=1}^{n} b_k b_l \phi_{t_{kl}}''(x) \geq 0, \quad x \in I.$$

It follows that the function

$$\sum_{k,l=1}^{n} b_k b_l \phi_{t_{kl}}$$

is convex, and hence (see Lemma 6.2)

$$\sum_{k,l=1}^{n} b_k b_l [y_0, y_1, y_2; \phi_{t_{kl}}] = [y_0, y_1, y_2; \sum_{k,l=1}^{n} b_k b_l \phi_{t_{kl}}] \geq 0$$

for every points $y_0, y_1, y_2 \in I$. This implies the exponential convexity of $t \mapsto [y_0, y_1, y_2; \phi_t]$ $(t \in J)$ in the Jensen sense. □

Remark 6.12 It comes from either the conditions of Theorem 6.10 or the proof of this theorem that the functions ϕ_t, $t \in J$ are convex.

6.3 Applications to Cauchy Means

In this section we generate new Cauchy means by applying the results in the previous section to some special classes of functions. We remind the condition

(F) Let $I \subset \mathbb{R}$ be an interval, and let $\Upsilon : D(\Upsilon) \to \mathbb{R}$ be a linear functional on a subspace of $\mathscr{F}(I)$ such that $\Upsilon(f) \geq 0$ for every convex f from $D(\Upsilon)$.

First, we summarize the essential properties of the functionals Υ_i $(i = 1, \ldots, 23)$.

Remark 6.13 (a) Every functional Υ_i $(i = 1,\ldots,23)$ depends on a real valued function defined on a convex set, and some other parameters. In this chapter the convex set will be an interval $I \subset \mathbb{R}$, and the parameters will be fixed except the function. By considering Υ_9, it is supposed that I is compact and the inequality (4.7) holds.

(b) The functionals Υ_i $(i = 1,\ldots,23)$ can be defined on a subspace $D(\Upsilon_i) \subset \mathscr{F}(I)$. It is easy to see that $D(\Upsilon_i) = \mathscr{F}(I)$, when $1 \le i \le 9$ and $16 \le i \le 19$, but in the other cases some integrability conditions are necessary, and hence $D(\Upsilon_i)$ is a proper subspace of $\mathscr{F}(I)$ in general.

(c) Υ_i is linear on $D(\Upsilon_i)$, and $\Upsilon_i(f) \ge 0$ for every convex f from $D(\Upsilon_i)$ $(i = 1,\ldots,23)$. It would be worthwhile to study the following problem: give those convex functions from $D(\Upsilon_i)$ for which the value of Υ_i is positive.

(d) Consider the functionals Υ_i, when $1 \le i \le 9$ and $16 \le i \le 19$. Every such functional depends on a fixed n-tuples $(x_1,\ldots,x_n) \in I^n$. Let

$$a := \min_{1\le i\le n} x_i \quad \text{and} \quad b := \max_{1\le i\le n} x_i.$$

Then $[a,b] \subset I$, and if $f \in D(\Upsilon_i)$ such that the restriction of f on $[a,b]$ is convex, then $\Upsilon_i(f) \ge 0$ $(1 \le i \le 9,\ 16 \le i \le 19)$.

(e) Consider the functionals Υ_i, when $10 \le i \le 12$. These functionals depends on functions u_i $(1 \le i \le n)$, whose ranges are subsets of I. If the ranges of the functions u_i $(1 \le i \le n)$ are subsets of an interval $\hat{I} \subset I$, and if $f \in D(\Upsilon_i)$ such that the restriction of f on $\hat{I}$ is convex, then $\Upsilon_i(f) \ge 0$ $(10 \le i \le 12)$.

(f) Consider the functionals Υ_i, when $13 \le i \le 15$, and $20 \le i \le 23$. These functionals depends on a function u, whose range is a subset of I. If the range of the function u is a subset of an interval $\hat{I} \subset I$, and if $f \in D(\Upsilon_i)$ such that the restriction of f on $\hat{I}$ is convex, then $\Upsilon_i(f) \ge 0$ $(13 \le i \le 15,\ 20 \le i \le 23)$.

Example 6.1 Assume (F) with $I = \mathbb{R}$ and $D(\Upsilon) = C(\mathbb{R})$. Consider the class of continuous convex functions

$$\Lambda_1 := \{\phi_t : \mathbb{R} \to [0,\infty) \mid t \in \mathbb{R}\},$$

where

$$\phi_t(x) := \begin{cases} \frac{1}{t^2}e^{tx}; & t \ne 0, \\ \frac{1}{2}x^2; & t = 0. \end{cases} \tag{6.17}$$

Elementary calculations show that $t \mapsto \phi_t''(x) = e^{tx}$ $(t \in \mathbb{R})$ is exponentially convex for every fixed $x \in \mathbb{R}$, and therefore Theorem 6.10 implies that the function $t \mapsto [y_0, y_1, y_2; \phi_t]$, $t \in \mathbb{R}$ is exponentially convex in the Jensen sense for every three mutually different points $y_0, y_1, y_2 \in \mathbb{R}$.

(a) By applying Corollary 6.1 with $J = \mathbb{R}$ and $\Lambda = \Lambda_1$, we get the exponential convexity of $t \mapsto \Upsilon(\phi_t)$ $(t \in \mathbb{R})$ in the Jensen sense. If the mapping $t \mapsto \Upsilon(\phi_t)$ $(t \in \mathbb{R})$ is also continuous, then it is exponentially convex.

(b) If the mapping $t \mapsto \Upsilon(\phi_t)$ $(t \in \mathbb{R})$ is positive and differentiable, then Corollary 6.2 (ii) gives the monotonicity of the function $\mathfrak{u}_{s,t}(\Upsilon, \Lambda_1)$ $(s,t \in \mathbb{R})$ (defined by (6.12) with $\Lambda = \Lambda_1$) in both parameters.

(c) Suppose the mapping $t \mapsto \Upsilon(\phi_t)$ $(t \in \mathbb{R})$ is positive and differentiable. Suppose further that $[a,b] \subset \mathbb{R}$ is an interval with the following property:

(i) if $f \in D(\Upsilon)$ such that the restriction of f on $[a,b]$ is convex, then $\Upsilon(f) \geq 0$.

Introduce

$$\mathfrak{M}_{s,t}(\Upsilon, \Lambda_1) := \log \mathfrak{u}_{s,t}(\Upsilon, \Lambda_1), \quad s,t \in \mathbb{R}.$$

If $s \neq t$, we have from Theorem 6.2 (the conditions of the theorem are satisfied) that there exists $\xi \in [a,b]$ for which

$$\mathfrak{M}_{s,t}(\Upsilon, \Lambda_1) = \frac{1}{s-t} \log\left(\frac{\Upsilon(\phi_s)}{\Upsilon(\phi_t)}\right) = \frac{1}{s-t} \log\left(\frac{\phi_s''(\xi)}{\phi_t''(\xi)}\right)$$

$$= \frac{1}{s-t} \log \frac{e^{s\xi}}{e^{t\xi}} = \xi \in [a,b].$$

It follows from this by taking limit that

$$\mathfrak{M}_{s,s}(\Upsilon, \Lambda_1) \in [a,b], \quad s \in \mathbb{R}.$$

It can be seen that

$$a \leq \mathfrak{M}_{s,t}(\Upsilon, \Lambda_1) \leq b, \quad s,t \in \mathbb{R}.$$

The monotonicity of the function $\mathfrak{M}_{s,t}(\Upsilon, \Lambda_1)$ in both parameters comes from the similar property of $\mathfrak{u}_{s,t}(\Upsilon, \Lambda_1)$.

(d) Remark 6.13 shows that the functionals Υ_i $(i = 1, \ldots, 23)$ satisfy condition (F). Υ denotes one of the functionals Υ_i $(i = 1, \ldots, 23)$.

Suppose the mapping $t \mapsto \Upsilon(\phi_t)$ $(t \in \mathbb{R})$ is positive (see Remark 6.13 (c)).

If $1 \leq i \leq 9$ or $16 \leq i \leq 19$, then it is easy to check that the mapping $t \mapsto \Upsilon(\phi_t)$ $(t \in \mathbb{R})$ is differentiable and

$$\mathfrak{u}_{s,t}(\Upsilon, \Lambda_1) = \begin{cases} \left(\frac{\Upsilon(\phi_s)}{\Upsilon(\phi_t)}\right)^{\frac{1}{s-t}}, \; s \neq t, \\ \exp\left(\frac{\Upsilon(id\,\phi_s)}{\Upsilon(\phi_s)} - \frac{2}{s}\right), \; s = t \neq 0, \\ \exp\left(\frac{\Upsilon(id\,\phi_0)}{3\Upsilon(\phi_0)}\right), \; s = t = 0. \end{cases} \tag{6.18}$$

In these cases (c) and Remark 6.13 (d) yield that $\mathfrak{M}_{s,t}(\Upsilon, \Lambda_1)$ $(s,t \in \mathbb{R})$ are means on $\mathbb{R}^n$, and they are monotonic in both parameters.

If $10 \leq i \leq 15$ or $20 \leq i \leq 23$, then the differentiability of the mapping $t \mapsto \Upsilon(\phi_t)$ $(t \in \mathbb{R})$ does not follow in general. It is also not trivial whether (6.18) remains true provided $t \mapsto \Upsilon(\phi_t)$ $(t \in \mathbb{R})$ is differentiable (this concerns the behaviour of parameter dependent integrals). But, if the ranges of the functions u_i $(1 \leq i \leq n)$ and u (see Remark 6.13 (e) and (f)) are subsets of an interval $[a,b] \subset I$, and $t \mapsto \Upsilon(\phi_t)$ $(t \in \mathbb{R})$ is differentiable, then as in the previous cases, $\mathfrak{M}_{s,t}(\Upsilon, \Lambda_1)$ $(s,t \in \mathbb{R})$ are means defined by integrals.

Example 6.2 Assume (F) with $I = (0, \infty)$ and $D(\Upsilon) = C(0, \infty)$. Consider the class of continuous convex functions

$$\Lambda_2 := \{\psi_t : (0,\infty) \to \mathbb{R} \mid t \in \mathbb{R}\},$$

where

$$\psi_t(x) := \begin{cases} \frac{x^t}{t(t-1)};\ t \neq 0,1, \\ -\log x;\ t=0, \\ x\log x;\ t=1. \end{cases}$$

As in the previous example, $t \mapsto \psi_t''(x) = x^{t-2} = e^{(t-2)\log x}$ $(t \in \mathbb{R})$ is exponentially convex for every fixed $x \in (0,\infty)$, and therefore Theorem 6.10 implies that the function $t \mapsto [y_0, y_1, y_2; \psi_t]$, $t \in \mathbb{R}$ is exponentially convex in the Jensen sense for every three mutually different points $y_0, y_1, y_2 \in (0,\infty)$.

(a) By applying Corollary 6.1 with $J = \mathbb{R}$ and $\Lambda = \Lambda_2$, we get the exponential convexity of $t \mapsto \Upsilon(\psi_t)$ $(t \in \mathbb{R})$ in the Jensen sense. If the mapping $t \mapsto \Upsilon(\psi_t)$ $(t \in \mathbb{R})$ is also continuous, then it is exponentially convex.

(b) If the mapping $t \mapsto \Upsilon(\psi_t)$ $(t \in \mathbb{R})$ is positive and differentiable, then Corollary 6.2 (ii) gives the monotonicity of the function $\mathfrak{u}_{s,t}(\Upsilon, \Lambda_2)$ (defined by (6.12) with $\Lambda = \Lambda_2$) in both parameters.

(c) Suppose the mapping $t \mapsto \Upsilon(\psi_t)$ $(t \in \mathbb{R})$ is positive and differentiable. Suppose further that $[a,b] \subset (0,\infty)$ is an interval with the following property:

(i) if $f \in D(\Upsilon)$ such that the restriction of f on $[a,b]$ is convex, then $\Upsilon(f) \geq 0$.

If $s \neq t$, we can apply Theorem 6.2 (the conditions of the theorem are satisfied) which shows that there exists $\xi \in [a,b]$ such that

$$\mathfrak{u}_{s,t}(\Upsilon, \Lambda_2) = \left(\frac{\Upsilon(\psi_s)}{\Upsilon(\psi_t)}\right)^{\frac{1}{s-t}} = \left(\frac{\psi_s''(\xi)}{\psi_t''(\xi)}\right)^{\frac{1}{s-t}}$$

$$= \left(e^{(s-t)\log(\xi)}\right)^{\frac{1}{s-t}} = \xi \in [a,b].$$

It follows from this by taking limit that

$$\mathfrak{u}_{s,s}(\Upsilon, \Lambda_2) \in [a,b], \quad s \in \mathbb{R}.$$

It can be seen that

$$a \leq \mathfrak{u}_{s,t}(\Upsilon, \Lambda_2) \leq b, \quad s,t \in \mathbb{R}.$$

(d) Now, we consider functionals Υ_i $(i = 1, \ldots, 23)$ which satisfy condition (F), by Remark 6.13. Υ denotes one of the functionals Υ_i $(i = 1, \ldots, 23)$.

Suppose the mapping $t \mapsto \Upsilon(\psi_t)$ $(t \in \mathbb{R})$ is positive (see Remark 6.13 (c)).

(d_1) If $1 \leq i \leq 9$ or $16 \leq i \leq 19$, then it is easy to check that the mapping $t \mapsto \Upsilon(\psi_t)$ $(t \in \mathbb{R})$ is differentiable and

$$u_{s,t}(\Upsilon, \Lambda_2) = \begin{cases} \left(\frac{\Upsilon(\psi_s)}{\Upsilon(\psi_t)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(\frac{1-2s}{s(s-1)} - \frac{\Upsilon(\psi_s\psi_0)}{\Upsilon(\psi_s)}\right), & s = t \neq 0,1, \\ \exp\left(1 - \frac{\Upsilon(\psi_0^2)}{2\Upsilon(\psi_0)}\right), & s = t = 0, \\ \exp\left(-1 - \frac{\Upsilon(\psi_0\psi_1)}{2\Upsilon(\psi_1)}\right), & s = t = 1. \end{cases} \tag{6.19}$$

In these cases (c) and Remark 6.13 (d) yield that $\mathfrak{u}_{s,t}(\Upsilon,\Lambda_2)$ $(s,t\in\mathbb{R})$ are means on $(0,\infty)^n$, and they are monotonic in both parameters.

If $10\le i\le 15$ or $20\le i\le 23$, then the differentiability of the mapping $t\mapsto\Upsilon(\psi_t)$ $(t\in\mathbb{R})$ does not follow in general. It is also not trivial whether (6.19) remains true provided $t\mapsto\Upsilon(\psi_t)$ $(t\in\mathbb{R})$ is differentiable (see Example 6.1 (d)). But, if $t\mapsto\Upsilon(\psi_t)$ $(t\in\mathbb{R})$ is differentiable, then as in the previous cases, $\mathfrak{u}_{s,t}(\Upsilon,\Lambda_2)$ $(s,t\in\mathbb{R})$ are means defined by integrals.

(d_2) Let $1\le i\le 9$ or $16\le i\le 19$. As we have mentioned in Remark 6.13 (d), Υ depends on a fixed n-tuples $\mathbf{x}=(x_1,...,x_n)$ which must belong to $(0,\infty)^n$ in the present situation. Henceforth this dependence is important, thus we shall make out it. For $r\in\mathbb{R}$, denote $\mathbf{x}^r:=(x_1^r,...,x_n^r)$. $\log(\mathbf{x})$ means $(\log x_1,\ldots,\log(x_n))$.

Let $s,t,r\in\mathbb{R}$, $r\ne 0$. Since $t\mapsto\Upsilon(\psi_t)$ $(t\in\mathbb{R})$ is differentiable, we can introduce

$$\mathfrak{u}_{s,t,r}(\Upsilon,\mathbf{x},\Lambda_2):=\left(\mathfrak{u}_{s/r,t/r}(\Upsilon,\mathbf{x}^r,\Lambda_2)\right)^{1/r}$$
$$=\begin{cases}\left(\frac{\Upsilon(\psi_{s/r},\mathbf{x}^r)}{\Upsilon(\psi_{t/r},\mathbf{x}^r)}\right)^{\frac{1}{s-t}}, & s\ne t,\\ \exp\left(\frac{\frac{d}{ds}\Upsilon(\psi_{s/r},\mathbf{x}^r)}{\Upsilon(\psi_{s/r},\mathbf{x}^r)}\right), & t=s.\end{cases}$$

It follows from (c) that

$$a^r\le\mathfrak{u}_{s/r,t/r}(\Upsilon,\mathbf{x}^r,\Lambda_2)\le b^r,\quad s,t,r\in\mathbb{R},\quad r\ne 0,$$

and therefore

$$a\le\mathfrak{u}_{s,t,r}(\Upsilon,\mathbf{x},\Lambda_2)\le b,\quad s,t,r\in\mathbb{R},\quad r\ne 0.\tag{6.20}$$

(6.19) yields that

$$\mathfrak{u}_{s,t,r}(\Upsilon,\mathbf{x},\Lambda_2)=\begin{cases}\left(\frac{\Upsilon(\psi_{s/r},\mathbf{x}^r)}{\Upsilon(\psi_{t/r},\mathbf{x}^r)}\right)^{\frac{1}{s-t}}, & s\ne t,\quad r\ne 0,\\ \exp\left(\frac{r-2s}{s(s-r)}-\frac{\Upsilon(\psi_{s/r}\psi_0)}{r\Upsilon(\psi_{s/r})}\right), & s=t\ne 0,r,\quad r\ne 0,\\ \exp\left(\frac{1}{r}-\frac{\Upsilon(\psi_0^2)}{2r\Upsilon(\psi_0)}\right), & s=t=0,\quad r\ne 0,\\ \exp\left(-\frac{1}{r}-\frac{\Upsilon(\psi_0\psi_1)}{2r\Upsilon(\psi_1)}\right), & s=t=r\ne 0.\end{cases}\tag{6.21}$$

By taking limit $r\to 0$, we can extend the meaning of $\mathfrak{u}_{s,t,r}(\Upsilon,\mathbf{x},\Lambda_2)$ for $r=0$ with some tedious calculations:

$$\mathfrak{u}_{s,t,0}(\Upsilon,\mathbf{x},\Lambda_2)=\left(\frac{\Upsilon(\phi_s,\log(\mathbf{x}))}{\Upsilon(\phi_t,\log(\mathbf{x}))}\right)^{\frac{1}{s-t}},\quad s\ne t,$$
$$\mathfrak{u}_{s,s,0}(\Upsilon,\mathbf{x},\Lambda_2)=\exp\left(-\frac{2}{s}+\frac{\Upsilon(id\cdot\phi_s,\log(\mathbf{x}))}{\Upsilon(\phi_s,\log(\mathbf{x}))}\right),\quad s=t\ne 0,$$
$$\mathfrak{u}_{0,0,0}(\Upsilon,\mathbf{x},\Lambda_2)=\exp\left(\frac{\Upsilon(id\cdot\phi_0,\log(\mathbf{x}))}{3\Upsilon(\phi_0,\log(\mathbf{x}))}\right),$$

where the definition of the function ϕ_t $(t\in\mathbb{R})$ can be found in (6.17), and $id(x):=x$ $(x\in\mathbb{R})$.

According to (6.20)

$$a \le \mathfrak{u}_{s,t,r}(\Upsilon,\mathbf{x},\Lambda_2) \le b, \quad s,t,r \in \mathbb{R},$$

thus we have obtained new means on $(0,\infty)^n$ with three parameters.

The monotonicity of the means $\mathfrak{u}_{s,t}(\Upsilon,\Lambda_2)$ $(s,t \in \mathbb{R})$ implies that for $s \le u$, $t \le v$ and $r \in \mathbb{R}$

$$\mathfrak{u}_{s,t,r}(\Upsilon,\mathbf{x},\Lambda_2) \le \mathfrak{u}_{u,v,r}(\Upsilon,\mathbf{x},\Lambda_2).$$

(d_3) Let $10 \le i \le 12$. Then Υ depends on functions $u_1,\ldots,u_n$ which have positive ranges in the present investigation (see Remark 6.13 (e)). Let $\mathbf{u} = (u_1,\ldots,u_n)$. If some integrability conditions are satisfied (see Theorem 5.3), and $t \mapsto \Upsilon(\psi_t)$ $(t \in \mathbb{R})$ is differentiable, then we can introduce

$$\mathfrak{u}_{s,t,r}(\Upsilon,\mathbf{u},\Lambda_2) := \left(\mathfrak{u}_{s/r,t/r}(\Upsilon,\mathbf{u}^r,\Lambda_2)\right)^{1/r}$$
$$= \begin{cases} \left(\dfrac{\Upsilon(\psi_{s/r},\mathbf{u}^r)}{\Upsilon(\psi_{t/r},\mathbf{u}^r)}\right)^{\frac{1}{s-t}}, & s \ne t, \\ \exp\left(\dfrac{\frac{d}{ds}\Upsilon(\psi_{s/r},\mathbf{u}^r)}{\Upsilon(\psi_{s/r},\mathbf{u}^r)}\right), & t = s, \end{cases}$$

for $s,t,r \in \mathbb{R}$, $r \ne 0$. If the ranges of the functions u_i $(1 \le i \le n)$ are subsets of an interval $[a,b] \subset (0,\infty)$, then as in the previous cases, $\mathfrak{u}_{s,t,r}(\Upsilon,\mathbf{u},\Lambda_2)$ $(s,t,r \in \mathbb{R},\ r \ne 0)$ are means defined by integrals. It requires further study whether (6.21) remains true (this concerns the behaviour of parameter dependent integrals), and how can we extended $\mathfrak{u}_{s,t,r}(\Upsilon,\mathbf{u},\Lambda_2)$ for $r = 0$.

(d_4) Let $13 \le i \le 15$ or $20 \le i \le 23$. Then Υ depends on a functions u which has positive range in the present investigation (see Remark 6.13 (f)). If some integrability conditions are satisfied (see Theorem 5.5), and $t \mapsto \Upsilon(\psi_t)$ $(t \in \mathbb{R})$ is differentiable, then we can introduce

$$\mathfrak{u}_{s,t,r}(\Upsilon,u,\Lambda_2) := \left(\mathfrak{u}_{s/r,t/r}(\Upsilon,u^r,\Lambda_2)\right)^{1/r}$$
$$= \begin{cases} \left(\dfrac{\Upsilon(\psi_{s/r},u^r)}{\Upsilon(\psi_{t/r},u^r)}\right)^{\frac{1}{s-t}}, & s \ne t, \\ \exp\left(\dfrac{\frac{d}{ds}\Upsilon(\psi_{s/r},u^r)}{\Upsilon(\psi_{s/r},u^r)}\right), & t = s, \end{cases}$$

for $s,t,r \in \mathbb{R}$, $r \ne 0$. If the range of the functions u is a subset of an interval $[a,b] \subset (0,\infty)$, then as in the previous cases, $\mathfrak{u}_{s,t,r}(\Upsilon,u,\Lambda_2)$ $(s,t,r \in \mathbb{R},\ r \ne 0)$ are means defined by integrals. Like (d_3), it requires further study whether (6.21) remains true (this concerns the behaviour of parameter dependent integrals), and how can we extended $\mathfrak{u}_{s,t,r}(\Upsilon,u,\Lambda_2)$ for $r = 0$.

Example 6.3 Assume (F) with $I = (0,\infty)$ and $D(\Upsilon) = C(0,\infty)$. Consider the class of continuous convex functions

$$\Lambda_3 := \{\eta_t : (0,\infty) \to (0,\infty) \mid t \in (0,\infty)\},$$

where

$$\eta_t(x) := \begin{cases} \frac{t^{-x}}{\log^2 t}; t \neq 1, \\ \frac{x^2}{2}; \quad t = 1. \end{cases}$$

For every fixed $x \in (0,\infty)$, he function $t \mapsto \eta_t''(x) = t^{-x} = e^{-x\log t}$ $(t \in (0,\infty))$ is the restriction of the Laplace transform of the nonnegative function $s \to \frac{s^{x-1}}{\Gamma(x)}$ $(s \in (0,\infty))$ (see [75]), and hence Theorem 6.7 shows that it is exponentially convex. Now, Theorem 6.10 yields that the function $t \mapsto [y_0, y_1, y_2; \eta_t]$, $t \in (0,\infty)$ is exponentially convex in the Jensen sense for every three mutually different points $y_0, y_1, y_2 \in (0,\infty)$.

(a) By applying Corollary 6.1 with $J = (0,\infty)$ and $\Lambda = \Lambda_3$, we get the exponential convexity of $t \mapsto \Upsilon(\eta_t)$ $(t \in (0,\infty))$ in the Jensen sense. If the mapping $t \mapsto \Upsilon(\eta_t)$ $(t \in (0,\infty))$ is also continuous, then it is exponentially convex.

(b) If the mapping $t \mapsto \Upsilon(\eta_t)$ $(t \in (0,\infty))$ is positive and differentiable, then Corollary 6.2 (ii) gives the monotonicity of the function $\mathfrak{u}_{s,t}(\Upsilon, \Lambda_3)$ (defined by (6.12) with $\Lambda = \Lambda_3$) in both parameters.

(c) Suppose the mapping $t \mapsto \Upsilon(\eta_t)$ $(t \in (0,\infty))$ is positive and differentiable. Suppose further that $[a,b] \subset (0,\infty)$ is an interval with the following property:

(i) if $f \in D(\Upsilon)$ such that the restriction of f on $[a,b]$ is convex, then $\Upsilon(f) \geq 0$.

By using the well known logarithmic mean

$$L(s,t) := \begin{cases} \frac{t-s}{\log t - \log s}, & s \neq t, \\ s, & t = s. \end{cases}, \quad s,t > 0, \tag{6.22}$$

we introduce

$$\mathfrak{M}_{s,t}(\Upsilon, \Lambda_3) := -L(s,t) \log \mathfrak{u}_{s,t}(\Upsilon, \Lambda_3), \quad s,t > 0.$$

If $s \neq t$, we can apply Theorem 6.2 (the conditions of the theorem are satisfied) which shows that there exists $\xi \in [a,b]$ such that

$$\mathfrak{M}_{s,t}(\Upsilon, \Lambda_3) = -L(s,t) \frac{1}{s-t} \log\left(\frac{\Upsilon(\eta_s)}{\Upsilon(\eta_t)}\right) = \frac{1}{\log t - \log s} \log\left(\frac{\eta_s''(\xi)}{\eta_t''(\xi)}\right)$$

$$= \frac{-\xi(\log s - \log t)}{\log t - \log s} = \xi \in [a,b].$$

It follows from this by taking limit that

$$\mathfrak{M}_{s,s}(\Upsilon, \Lambda_3) \in [a,b], \quad s \in (0,\infty).$$

It can be seen that

$$a \leq \mathfrak{M}_{s,t}(\Upsilon, \Lambda_3) \leq b, \quad s,t \in (0,\infty).$$

The function $\mathfrak{M}_{s,t}(\Upsilon, \Lambda_3)$ is decreasing in both parameters, because $\mathfrak{u}_{s,t}(\Upsilon, \Lambda_3)$ and $L(s,t)$ are increasing in both parameters: if $s,t,u,v \in (0,\infty)$ such that $s \leq u$ and $t \leq v$, then

$$\mathfrak{M}_{s,t}(\Upsilon, \Lambda_3) \geq \mathfrak{M}_{u,v}(\Upsilon, \Lambda_3).$$

(d) Now, we consider functionals Υ_i $(i = 1, \ldots, 23)$ which satisfy condition (F), by Remark 6.13. Υ denotes one of the functionals Υ_i $(i = 1, \ldots, 23)$.

Suppose the mapping $t \mapsto \Upsilon(\eta_t)$ $(t \in (0, \infty))$ is positive (see Remark 6.13 (c)).

If $1 \leq i \leq 9$ or $16 \leq i \leq 19$, then it is easy to check that the mapping $t \mapsto \Upsilon(\eta_t)$ $(t \in (0, \infty))$ is differentiable and

$$\mathfrak{u}_{s,t}(\Upsilon, \Lambda_3) = \begin{cases} \left(\frac{\Upsilon(\eta_s)}{\Upsilon(\eta_t)}\right)^{\frac{1}{s-t}}, \ s \neq t, \\ \exp\left(-\frac{2}{s \log s} - \frac{\Upsilon(id\eta_s)}{s\Upsilon(\eta_s)}\right), \ s = t \neq 1, \\ \exp\left(-\frac{\Upsilon(id\eta_1)}{3\Upsilon(\eta_1)}\right), \ s = t = 1. \end{cases} \tag{6.23}$$

In these cases (c) and Remark 6.13 (d) yield that $\mathfrak{M}_{s,t}(\Upsilon, \Lambda_3)$ $(s, t \in (0, \infty))$ are means on $(0, \infty)^n$, and they are monotonic in both parameters.

If $10 \leq i \leq 15$ or $20 \leq i \leq 23$, then the differentiability of the mapping $t \mapsto \Upsilon(\eta_t)$ $(t \in (0, \infty))$ does not follow in general. It is also not trivial whether (6.23) remains true provided $t \mapsto \Upsilon(\eta_t)$ $(t \in (0, \infty))$ is differentiable (this concerns the behaviour of parameter dependent integrals). But, if the ranges of the functions u_i $(1 \leq i \leq n)$ and u (see Remark 6.13 (e) and (f)) are subsets of an interval $[a, b] \subset I$, and $t \mapsto \Upsilon(\eta_t)$ $(t \in (0, \infty))$ is differentiable, then as in the previous cases, $\mathfrak{M}_{s,t}(\Upsilon, \Lambda_3)$ $(s, t \in (0, \infty))$ are means defined by integrals.

Example 6.4 Assume (F) with $I = (0, \infty)$ and $D(\Upsilon) = C(0, \infty)$. Consider the class of continuous convex functions

$$\Lambda_4 := \{\gamma_t : (0, \infty) \to (0, \infty) \mid t \in (0, \infty)\},$$

where

$$\gamma_t(x) := \frac{e^{-x\sqrt{t}}}{t}.$$

For every fixed $x \in (0, \infty)$, he function $t \mapsto \gamma_t''(x) = e^{-x\sqrt{t}}$ $(t \in (0, \infty))$ is the restriction of the Laplace transform of the nonnegative function $s \to \frac{x}{2\sqrt{\pi s^3}} e^{-x^2/4s}$ $(s \in (0, \infty))$ (see [75]), and hence Theorem 6.7 shows that it is exponentially convex. Now, Theorem 6.10 yields that the function $t \mapsto [y_0, y_1, y_2; \gamma_t]$, $t \in (0, \infty)$ is exponentially convex in the Jensen sense for every three mutually different points $y_0, y_1, y_2 \in (0, \infty)$.

(a) By applying Corollary 6.1 with $J = (0, \infty)$ and $\Lambda = \Lambda_4$, we get the exponential convexity of $t \mapsto \Upsilon(\gamma_t)$ $(t \in (0, \infty))$ in the Jensen sense. If the mapping $t \mapsto \Upsilon(\gamma_t)$ $(t \in (0, \infty))$ is also continuous, then it is exponentially convex.

(b) If the mapping $t \mapsto \Upsilon(\gamma_t)$ $(t \in (0, \infty))$ is positive and differentiable, then Corollary 6.2 (ii) gives the monotonicity of the function $\mathfrak{u}_{s,t}(\Upsilon, \Lambda_4)$ (defined by (6.12) with $\Lambda = \Lambda_4$) in both parameters.

(c) Suppose the mapping $t \mapsto \Upsilon(\gamma_t)$ $(t \in (0, \infty))$ is positive and differentiable. Suppose further that $[a, b] \subset (0, \infty)$ is an interval with the following property:

(i) if $f \in D(\Upsilon)$ such that the restriction of f on $[a, b]$ is convex, then $\Upsilon(f) \geq 0$.

Introduce

$$\mathfrak{M}_{s,t}(\Upsilon, \Lambda_4) := -(\sqrt{s} + \sqrt{t}) \log \mathfrak{u}_{s,t}(\Upsilon, \Lambda_4), \quad s, t \in (0, \infty).$$

If $s \neq t$, we can apply Theorem 6.2 (the conditions of the theorem are satisfied) which shows that there exists $\xi \in [a,b]$ such that

$$\mathfrak{M}_{s,t}(\Upsilon,\Lambda_4) = -(\sqrt{s}+\sqrt{t})\frac{1}{s-t}\log\left(\frac{\Upsilon(\gamma_s)}{\Upsilon(\gamma_t)}\right) = \frac{1}{\sqrt{t}-\sqrt{s}}\log\left(\frac{\gamma_s''(\xi)}{\gamma_t''(\xi)}\right)$$

$$= \frac{1}{\sqrt{t}-\sqrt{s}}\log e^{-\xi(\sqrt{s}-\sqrt{t})} = \xi \in [a,b].$$

It follows from this by taking limit that

$$\mathfrak{M}_{s,s}(\Upsilon,\Lambda_4) \in [a,b], \quad s \in (0,\infty).$$

It can be seen that

$$a \leq \mathfrak{M}_{s,t}(\Upsilon,\Lambda_4) \leq b, \quad s,t \in (0,\infty).$$

The function $\mathfrak{M}_{s,t}(\Upsilon,\Lambda_4)$ is decreasing in both parameters, because $\mathfrak{u}_{s,t}(\Upsilon,\Lambda_4)$ is increasing in both parameters: if $s,t,u,v \in (0,\infty)$ such that $s \leq u$ and $t \leq v$, then

$$\mathfrak{M}_{s,t}(\Upsilon,\Lambda_4) \geq \mathfrak{M}_{u,v}(\Upsilon,\Lambda_4).$$

(d) Now, we consider functionals Υ_i $(i = 1,\ldots,23)$ which satisfy condition (F), by Remark 6.13. Υ denotes one of the functionals Υ_i $(i = 1,\ldots,23)$.

Suppose the mapping $t \mapsto \Upsilon(\gamma_t)$ $(t \in \mathbb{R})$ is positive (see Remark 6.13 (c)).

If $1 \leq i \leq 9$ or $16 \leq i \leq 19$, then it is easy to check that the mapping $t \mapsto \Upsilon(\gamma_t)$ $(t \in (0,\infty))$ is differentiable and

$$\mathfrak{u}_{s,t}(\Upsilon_2,\Lambda_4) = \begin{cases} \left(\frac{\Upsilon_2(\gamma_s)}{\Upsilon_2(\gamma_t)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(-\frac{1}{t} - \frac{\Upsilon_2(id\gamma_t)}{2\sqrt{t}\Upsilon_2(\gamma_t)}\right), & s = t. \end{cases} \tag{6.24}$$

In these cases (c) and Remark 6.13 (d) yield that $\mathfrak{M}_{s,t}(\Upsilon,\Lambda_4)$ $(s,t \in (0,\infty))$ are means on $(0,\infty)^n$, and they are monotonic in both parameters.

If $10 \leq i \leq 15$ or $20 \leq i \leq 23$, then the differentiability of the mapping $t \mapsto \Upsilon(\gamma_t)$ $(t \in (0,\infty))$ does not follow in general. It is also not trivial whether (6.24) remains true provided $t \mapsto \Upsilon(\gamma_t)$ $(t \in (0,\infty))$ is differentiable (this concerns the behaviour of parameter dependent integrals). But, if the ranges of the functions u_i $(1 \leq i \leq n)$ and u (see Remark 6.13 (e) and (f)) are subsets of an interval $[a,b] \subset I$, and $t \mapsto \Upsilon(\gamma_t)$ $(t \in (0,\infty))$ is differentiable, then as in the previous cases, $\mathfrak{M}_{s,t}(\Upsilon,\Lambda_4)$ $(s,t \in (0,\infty))$ are means defined by integrals.

In the remaining examples (Example 6.5-6.8) we need the following condition which was introduced previously:

(G) Let $I \subset \mathbb{R}\setminus\{0\}$ be an interval, and denote id the identity function on I. Let $\Psi : D(\Psi) \to \mathbb{R}$ be a linear functional which satisfies

(i) $D(\Psi)$ is a subspace of $\mathscr{F}(I)$ such that $f \in D(\Psi)$ implies $\frac{f}{id} \in D(\Psi)$,

(ii) $\Psi(f) \geq 0$ for every $f \in D(\Psi)$ for which $\frac{f}{id}$ is convex.

Example 6.5 Assume (G) with $I=(0,\infty)$ and $D(\Psi)=C(0,\infty)$. Consider the class of continuous convex functions

$$\Phi_1 := \{\tau_t : (0,\infty) \to (0,\infty) \mid t \in \mathbb{R}\},$$

where

$$\tau_t(x) := \begin{cases} \frac{xe^{tx}}{t^2}; \ t \neq 0, \\ \frac{x^3}{2}; \ t = 0. \end{cases}$$

Then the function $\frac{\tau_t}{id}$ is the restriction of ϕ_t (see Example 6.1) to $(0,\infty)$ (which will be denoted by ϕ_t too) for every $t \in \mathbb{R}$. Thus, as we have seen in Example 6.1 the function $t \mapsto [y_0,y_1,y_2;\frac{\tau_t}{id}] = [y_0,y_1,y_2;\phi_t]$, $t \in \mathbb{R}$ is exponentially convex in the Jensen sense for every three mutually different points $y_0,y_1,y_2 \in (0,\infty)$.

(a) By applying Corollary 6.3 with $J=\mathbb{R}$ and $\Phi=\Phi_1$, we get the exponential convexity of $t \mapsto \Psi(\tau_t)$ $(t \in \mathbb{R})$ in the Jensen sense. If the mapping $t \mapsto \Psi(\tau_t)$ $(t \in \mathbb{R})$ is also continuous, then it is exponentially convex.

(b) If the mapping $t \mapsto \Psi(\tau_t)$ $(t \in \mathbb{R})$ is positive and differentiable, then Corollary 6.4 (ii) gives the monotonicity of the function $\overline{\mathfrak{u}}_{s,t}(\Psi,\Phi_1)$ (defined by (6.16) with $\Phi=\Phi_1$) in both parameters.

(c) Suppose the mapping $t \mapsto \Psi(\tau_t)$ $(t \in \mathbb{R})$ is positive and differentiable. Suppose further that $[a,b] \subset (0,\infty)$ is an interval with the following property:

(i) if $f \in D(\Psi)$ such that the restriction of $\frac{f}{id}$ to $[a,b]$ is convex, then $\Psi(f) \geq 0$.

Introduce

$$\overline{\mathfrak{M}}_{s,t}(\Psi,\Phi_1) := \log \overline{\mathfrak{u}}_{s,t}(\Psi,\Phi_1), \quad s,t \in \mathbb{R}.$$

If $s \neq t$, we can apply Theorem 6.6 (the conditions of the theorem are satisfied) which shows that there exists $\xi \in [a,b]$ such that

$$\overline{\mathfrak{M}}_{s,t}(\Psi,\Phi_1) = \log\left(\frac{\Psi(\tau_s)}{\Psi(\tau_t)}\right)^{\frac{1}{s-t}} = \log\left(\frac{\xi^2\tau_s''(\xi)-2\xi\tau_s'(\xi)+2\tau_s(\xi)}{\xi^2\tau_t''(\xi)-2\xi\tau_t'(\xi)+2\tau_t(\xi)}\right)^{\frac{1}{s-t}}$$

$$= \log\left(e^{(s-t)\xi}\right)^{\frac{1}{s-t}} = \xi \in [a,b].$$

It follows from this by taking limit that

$$\overline{\mathfrak{M}}_{s,s}(\Psi,\Phi_1) \in [a,b], \quad s \in \mathbb{R}.$$

It can be seen that

$$a \leq \overline{\mathfrak{M}}_{s,t}(\Psi,\Phi_1) \leq b, \quad s,t \in \mathbb{R}.$$

The monotonicity of the function $\overline{\mathfrak{M}}_{s,t}(\Psi,\Phi_1)$ in both parameters comes from the similar property of $\overline{\mathfrak{u}}_{s,t}(\Psi,\Phi_1)$.

(d) Now, we consider the functional Ψ defined in (4.10): let $n,k \in \mathbb{N}$, $n \geq 3$, $2 \leq k \leq n-1$, $\mathbf{x}=(x_1,\ldots,x_n) \in I^n$ such that $x=\sum_{i=1}^n x_i \in I$; then

$$f_{k,n}(\mathbf{x}) := \frac{1}{\binom{n-1}{k-1}} \sum_{1 \leq i_1 < \ldots < i_k \leq n} f\left(\sum_{j=1}^k x_{i_j}\right),$$

and

$$\Psi(f) = \Psi(k,\mathbf{x},f) := \frac{n-k}{n-1} f_{1,n}(\mathbf{x}) + \frac{k-1}{n-1} f_{n,n}(\mathbf{x}) - f_{k,n}(\mathbf{x}).$$

In this case Ψ is a linear functional defined on $D(\Psi) = \mathscr{F}(I)$, and $\Psi(f) \geq 0$ for every $f \in D(\Psi)$ for which $\frac{f}{id}$ is convex, and therefore Ψ satisfies (G). It should analyse: give all the functions $f \in D(\Psi)$ for which $\frac{f}{id}$ is convex and $\Psi(f) > 0$.

Suppose the mapping $t \mapsto \Psi(\tau_t)$ $(t \in \mathbb{R})$ is positive.

It is easy to check that the mapping $t \mapsto \Psi(\tau_t)$ $(t \in \mathbb{R})$ is differentiable. By applying (c), we have that

$$\min_{1 \leq i \leq n} x_i \leq \overline{\mathfrak{M}}_{s,t}(\Psi,\mathbf{x},\Phi_1) \leq \sum_{i=1}^{n} x_i, \quad s,t \in \mathbb{R}, \tag{6.25}$$

which shows that $\overline{\mathfrak{M}}_{s,t}(\Psi,\mathbf{x},\Phi_1)$ is not a mean of the numbers x_i $(1 \leq i \leq n)$.

To construct means of the numbers x_i $(1 \leq i \leq n)$, we assume that

$$x_1 \leq x_i - x_{i-1}, \quad 2 \leq i \leq n.$$

Introduce

$$\triangle\mathbf{x} := (y_1, y_2, \ldots y_n) = (x_1, x_2 - x_1, \ldots, x_n - x_{n-1}).$$

Then $x_1 < \ldots < x_n$, $\min\limits_{1 \leq i \leq n} y_i = x_1$ and $\sum\limits_{i=1}^{n} y_i = x_n = \max\limits_{1 \leq i \leq n} x_i$. It is not hard to calculate that

$\overline{\mathfrak{u}}_{s,t}(\Psi,\triangle\mathbf{x},\Phi_1)$

$$= \left(\frac{t^2}{s^2} \frac{\frac{n-k}{n-1}\sum\limits_{i=1}^{n} y_i e^{sy_i} + \frac{k-1}{n-1} x_n e^{sx_n} - \frac{1}{\binom{n-1}{k-1}} \sum\limits_{1 \leq i_1 < \ldots < i_k \leq n} \left(\sum\limits_{j=1}^{k} y_{i_j}\right) e^{s\left(\sum\limits_{j=1}^{k} y_{i_j}\right)}}{\frac{n-k}{n-1}\sum\limits_{i=1}^{n} y_i e^{ty_i} + \frac{k-1}{n-1} x_n e^{tx_n} - \frac{1}{\binom{n-1}{k-1}} \sum\limits_{1 \leq i_1 < \ldots < i_k \leq n} \left(\sum\limits_{j=1}^{k} y_{i_j}\right) e^{t\left(\sum\limits_{j=1}^{k} y_{i_j}\right)}} \right)^{\frac{1}{s-t}}, \quad \begin{array}{l} s \neq t, \\ s,t \neq 0, \end{array}$$

$$= \left(\frac{2}{s^2} \frac{\frac{n-k}{n-1}\sum\limits_{i=1}^{n} y_i e^{sx_i} + \frac{k-1}{n-1} x_n e^{sx_n} - \frac{1}{\binom{n-1}{k-1}} \sum\limits_{1 \leq i_1 < \ldots < i_k \leq n} \left(\sum\limits_{j=1}^{k} y_{i_j}\right) e^{s\left(\sum\limits_{j=1}^{k} y_{i_j}\right)}}{\frac{n-k}{n-1}\sum\limits_{i=1}^{n} y_i^3 + \frac{k-1}{n-1} x_n^3 - \frac{1}{\binom{n-1}{k-1}} \sum\limits_{1 \leq i_1 < \ldots < i_k \leq n} \left(\sum\limits_{j=1}^{k} y_{i_j}\right)^3} \right)^{\frac{1}{s}}, \quad \begin{array}{l} s \neq 0, \\ t = 0, \end{array}$$

$$= \left(\frac{t^2}{2} \frac{\frac{n-k}{n-1}\sum\limits_{i=1}^{n} y_i^3 + \frac{k-1}{n-1} x_n^3 - \frac{1}{\binom{n-1}{k-1}} \sum\limits_{1 \leq i_1 < \ldots < i_k \leq n} \left(\sum\limits_{j=1}^{k} y_{i_j}\right)^3}{\frac{n-k}{n-1}\sum\limits_{i=1}^{n} y_i e^{ty_i} + \frac{k-1}{n-1} x_n e^{tx_n} - \frac{1}{\binom{n-1}{k-1}} \sum\limits_{1 \leq i_1 < \ldots < i_k \leq n} \left(\sum\limits_{j=1}^{k} y_{i_j}\right) e^{t\left(\sum\limits_{j=1}^{k} y_{i_j}\right)}} \right)^{-\frac{1}{t}}, \quad \begin{array}{l} s = 0, \\ t \neq 0, \end{array}$$

$$= \exp\left(\frac{\frac{n-k}{n-1}\sum_{i=1}^{n} y_i^2 e^{sy_i} + \frac{k-1}{n-1}x_n^2 e^{sx_n} - \frac{1}{\binom{n-1}{k-1}}\sum_{1\le i_1<\ldots<i_k\le n}\left(\sum_{j=1}^{k} y_{i_j}\right)^2 e^{s\left(\sum_{j=1}^{k} y_{i_j}\right)}}{\frac{n-k}{n-1}\sum_{i=1}^{n} y_i e^{sy_i} + \frac{k-1}{n-1}x_n e^{sx_n} - \frac{1}{\binom{n-1}{k-1}}\sum_{1\le i_1<\ldots<i_k\le n}\left(\sum_{j=1}^{k} y_{i_j}\right) e^{s\left(\sum_{j=1}^{k} y_{i_j}\right)}} - \frac{2}{s}\right),$$

$$s = t \neq 0,$$

$$= \exp\left(\frac{1}{3}\frac{\frac{n-k}{n-1}\sum_{i=1}^{n} y_i^4 + \frac{k-1}{n-1}x_n^4 - \frac{1}{\binom{n-1}{k-1}}\sum_{1\le i_1<\ldots<i_k\le n}\left(\sum_{j=1}^{k} y_{i_j}\right)^4}{\frac{n-k}{n-1}\sum_{i=1}^{n} y_i^3 + \frac{k-1}{n-1}x_n^3 - \frac{1}{\binom{n-1}{k-1}}\sum_{1\le i_1<\ldots<i_k\le n}\left(\sum_{j=1}^{k} y_{i_j}\right)^3}\right), \quad s = t = 0.$$

(6.25) yields that $\overline{\mathfrak{M}}_{s,t}(\Psi, \triangle\mathbf{x}, \Phi_1)$ $(s,t \in \mathbb{R})$ are means (of the numbers $x_1 < \ldots < x_n$) on

$$\{(x_1,\ldots,x_n) \mid x_1 > 0,\ x_1 \le x_i - x_{i-1}, \quad 2 \le i \le n\},$$

and they are monotonic in both parameters.

Example 6.6 Assume (G) with $I = (0,\infty)$ and $D(\Psi) = C(0,\infty)$. Consider the class of continuous convex functions

$$\Phi_2 = \{\mu_t : (0,\infty) \to \mathbb{R} \mid t \in \mathbb{R}\},$$

where

$$\mu_t(x) := \begin{cases} \frac{x^{t+1}}{t(t-1)};\ t \neq 0,1, \\ -x\log x;\ t = 0, \\ x^2 \log x;\ t = 1. \end{cases}$$

Then the function $\frac{\mu_t}{id}$ is ψ_t (see Example 6.2) for every $t \in \mathbb{R}$. Thus, as we have seen in Example 6.2 the function $t \mapsto [y_0, y_1, y_2; \frac{\mu_t}{id}] = [y_0, y_1, y_2; \psi_t]$, $t \in \mathbb{R}$ is exponentially convex in the Jensen sense for every three mutually different points $y_0, y_1, y_2 \in (0,\infty)$.

(a) By applying Corollary 6.3 with $J = \mathbb{R}$ and $\Phi = \Phi_2$, we get the exponential convexity of $t \mapsto \Psi(\mu_t)$ $(t \in \mathbb{R})$ in the Jensen sense. If the mapping $t \mapsto \Psi(\mu_t)$ $(t \in \mathbb{R})$ is also continuous, then it is exponentially convex.

(b) If the mapping $t \mapsto \Psi(\mu_t)$ $(t \in \mathbb{R})$ is positive and differentiable, then Corollary 6.4 (ii) gives the monotonicity of the function $\overline{\mathfrak{u}}_{s,t}(\Psi, \Phi_2)$ (defined by (6.16) with $\Phi = \Phi_2$) in both parameters..

(c) Suppose the mapping $t \mapsto \Psi(\mu_t)$ $(t \in \mathbb{R})$ is positive and differentiable. Suppose further that $[a,b] \subset (0,\infty)$ is an interval with the following property:

(i) if $f \in D(\Psi)$ such that the restriction of $\frac{f}{id}$ to $[a,b]$ is convex, then $\Psi(f) \ge 0$.

If $s \neq t$, we can apply Theorem 6.6 (the conditions of the theorem are satisfied) which shows that there exists $\xi \in [a,b]$ such that

$$\overline{\mathfrak{u}}_{s,t}(\Psi, \Phi_2) = \left(\frac{\Psi(\mu_s)}{\Psi(\mu_t)}\right)^{\frac{1}{s-t}} = \left(\frac{\xi^2 \mu_s''(\xi) - 2\xi\mu_s'(\xi) + 2\mu_s(\xi)}{\xi^2 \mu_t''(\xi) - 2\xi\mu_t'(\xi) + 2\mu_t(\xi)}\right)^{\frac{1}{s-t}}$$

$$= \left(\xi^{s-t}\right)^{\frac{1}{s-t}} = \xi \in [a,b].$$

It follows from this by taking limit that

$$\overline{u}_{s,s}(\Psi,\Phi_2) \in [a,b], \quad s \in \mathbb{R}.$$

It can be seen that

$$a \le \overline{u}_{s,t}(\Psi,\Phi_2) \le b, \quad s,t \in \mathbb{R}.$$

The function $\overline{u}_{s,t}(\Psi,\Phi_2)$ is monotonic in both parameters.

(d) Now, we consider the functional Ψ defined in (4.10) (see (d) in the previous example).

Suppose the mapping $t \mapsto \Psi(\mu_t)$ $(t \in \mathbb{R})$ is positive.

It is easy to check that the mapping $t \mapsto \Psi(\mu_t)$ $(t \in \mathbb{R})$ is differentiable. It follows from (c) that

$$\min_{1\le i\le n} x_i \le \overline{u}_{s,t}(\Psi,\mathbf{x},\Phi_2) \le \sum_{i=1}^{n} x_i, \quad s,t \in \mathbb{R},$$

and therefore $\overline{u}_{s,t}(\Psi,\mathbf{x},\Phi_2)$ is not a mean of the numbers x_i $(1 \le i \le n)$.

To construct means of the numbers x_i $(1 \le i \le n)$, we assume that

$$x_1 \le x_i - x_{i-1}, \quad 2 \le i \le n.$$

As in the previous example we use the notation

$$\triangle\mathbf{x} := (y_1, y_2, \ldots y_n) = (x_1, x_2 - x_1, \ldots, x_n - x_{n-1}).$$

Then $x_1 < \ldots < x_n$, $\min\limits_{1\le i\le n} y_i = x_1$ and $\sum\limits_{i=1}^{n} y_i = x_n = \max\limits_{1\le i\le n} x_i$.

It is not hard to calculate
$\overline{u}_{s,t}(\Psi,\triangle\mathbf{x},\Phi_2)$

$$= \left(\frac{t(t-1)}{s(s-1)} \frac{\frac{n-k}{n-1}\sum\limits_{i=1}^{n} y_i^{s+1} + \frac{k-1}{n-1}x_n^{s+1} - \frac{1}{\binom{n-1}{k-1}} \sum\limits_{1\le i_1<\ldots<i_k\le n} \left(\sum\limits_{j=1}^{k} y_{i_j}\right)^{s+1}}{\frac{n-k}{n-1}\sum\limits_{i=1}^{n} y_i^{t+1} + \frac{k-1}{n-1}x_n^{t+1} - \frac{1}{\binom{n-1}{k-1}} \sum\limits_{1\le i_1<\ldots<i_k\le n} \left(\sum\limits_{j=1}^{k} y_{i_j}\right)^{t+1}} \right)^{\frac{1}{s-t}},$$

$$s \ne t, \quad s,t \ne 0,1,$$

$$= \left(\frac{-1}{s(s-1)} \frac{\frac{n-k}{n-1}\sum\limits_{i=1}^{n} y_i^{s+1} + \frac{k-1}{n-1}x_n^{s+1} - \frac{1}{\binom{n-1}{k-1}} \sum\limits_{1\le i_1<\ldots<i_k\le n} \left(\sum\limits_{j=1}^{k} y_{i_j}\right)^{s+1}}{\frac{n-k}{n-1}\sum\limits_{i=1}^{n} y_i \log y_i + \frac{k-1}{n-1}x_n \log x_n - \frac{1}{\binom{n-1}{k-1}} \sum\limits_{1\le i_1<\ldots<i_k\le n} \left(\sum\limits_{j=1}^{k} y_{i_j}\right) \log\left(\sum\limits_{j=1}^{k} y_{i_j}\right)} \right)^{\frac{1}{s}},$$

$$t = 0, \quad s \ne 0,1,$$

$$= \left(\frac{1}{s(s-1)} \frac{\frac{n-k}{n-1} \sum_{i=1}^{n} y_i^{s+1} + \frac{k-1}{n-1} x_n^{s+1} - \frac{1}{\binom{n-1}{k-1}} \sum_{1 \le i_1 < \ldots < i_k \le n} \left(\sum_{j=1}^{k} y_{i_j} \right)^{s+1}}{\frac{n-k}{n-1} \sum_{i=1}^{n} y_i^2 \log y_i + \frac{k-1}{n-1} x_n^2 \log x_n - \frac{1}{\binom{n-1}{k-1}} \sum_{1 \le i_1 < \ldots < i_k \le n} \left(\sum_{j=1}^{k} y_{i_j} \right)^2 \log \left(\sum_{j=1}^{k} y_{i_j} \right)} \right)^{\frac{1}{s-1}},$$

$$t = 1, \quad s \neq 0, 1,$$

$$= -\frac{\frac{n-k}{n-1} \sum_{i=1}^{n} y_i^2 + \frac{k-1}{n-1} x_n^2 \log x_n - \frac{1}{\binom{n-1}{k-1}} \sum_{1 \le i_1 < \ldots < i_k \le n} \left(\sum_{j=1}^{k} y_{i_j} \right)^2 \log \left(\sum_{j=1}^{k} y_{i_j} \right)}{\frac{n-k}{n-1} \sum_{i=1}^{n} y_i \log y_i + \frac{k-1}{n-1} x_n \log x_n - \frac{1}{\binom{n-1}{k-1}} \sum_{1 \le i_1 < \ldots < i_k \le n} \left(\sum_{j=1}^{k} y_{i_j} \right) \log \left(\sum_{j=1}^{k} y_{i_j} \right)},$$

$$t = 0, \quad s = 1,$$

$$= \exp \left(\frac{1-2s}{s(s-1)} + \frac{\frac{n-k}{n-1} \sum_{i=1}^{n} y_i^{s+1} \log y_i + \frac{k-1}{n-1} x_n^{s+1} \log x_n - \frac{1}{\binom{n-1}{k-1}} \sum_{1 \le i_1 < \ldots < i_k \le n} \left(\sum_{j=1}^{k} y_{i_j} \right)^{s+1} \log \left(\sum_{j=1}^{k} y_{i_j} \right)}{\frac{n-k}{n-1} \sum_{i=1}^{n} y_i^{s+1} + \frac{k-1}{n-1} x_n^{s+1} - \frac{1}{\binom{n-1}{k-1}} \sum_{1 \le i_1 < \ldots < i_k \le n} \left(\sum_{j=1}^{k} y_{i_j} \right)^{s+1}} \right),$$

$$s = t \neq 0, 1,$$

$$= \exp \left(1 + \frac{\frac{n-k}{n-1} \sum_{i=1}^{n} y_i \log^2 y_i + \frac{k-1}{n-1} \log^2 x_n - \frac{1}{\binom{n-1}{k-1}} \sum_{1 \le i_1 < \ldots < i_k \le n} \left(\sum_{j=1}^{k} y_{i_j} \right) \log^2 \left(\sum_{j=1}^{k} y_{i_j} \right)}{2 \left(\frac{n-k}{n-1} \sum_{i=1}^{n} y_i \log y_i + \frac{k-1}{n-1} x_n \log x_n - \frac{1}{\binom{n-1}{k-1}} \sum_{1 \le i_1 < \ldots < i_k \le n} \left(\sum_{j=1}^{k} y_{i_j} \right) \log \left(\sum_{j=1}^{k} y_{i_j} \right) \right)} \right),$$

$$s = t = 0,$$

$$= \exp \left(-1 + \frac{\frac{n-k}{n-1} \sum_{i=1}^{n} y_i^2 \log^2 y_i + \frac{k-1}{n-1} x_n^2 \log^2 x_n - \frac{1}{\binom{n-1}{k-1}} \sum_{1 \le i_1 < \ldots < i_k \le n} \left(\sum_{j=1}^{k} y_{i_j} \right)^2 \log^2 \left(\sum_{j=1}^{k} y_{i_j} \right)}{2 \left(\frac{n-k}{n-1} \sum_{i=1}^{n} y_i^2 \log y_i + \frac{k-1}{n-1} x_n^2 \log x_n - \frac{1}{\binom{n-1}{k-1}} \sum_{1 \le i_1 < \ldots < i_k \le n} \left(\sum_{j=1}^{k} y_{i_j} \right)^2 \log \left(\sum_{j=1}^{k} y_{i_j} \right) \right)} \right),$$

$$s = t = 1.$$

For simplicity, we don't list the cases $s = 0$, $t \neq 0, 1$ and $t = 1$, $s \neq 0, 1$ and $s = 0$, $t = 1$ in the previous table. They can be easily obtained from the similar cases when s and t are reversed.

By applying (c), we have that $\overline{u}_{s,t}(\Psi, \triangle\mathbf{x}, \Phi_2)$ $(s,t \in \mathbb{R})$ are means (of the numbers $x_1 < \ldots < x_n$) on

$$\{(x_1, \ldots, x_n) \mid x_1 > 0,\ x_1 \le x_i - x_{i-1}, \quad 2 \le i \le n\},$$

and they are monotonic in both parameters.

Example 6.7 Assume (G) with $I = (0,\infty)$ and $D(\Psi) = C(0,\infty)$. Consider the class of continuous convex functions

$$\Phi_3 = \{\chi_t : (0,\infty) \to (0,\infty) \mid t \in (0,\infty)\},$$

where

$$\chi_t(x) := \begin{cases} \frac{xt^{-x}}{\log^2 t}, & t \neq 1, \\ \frac{x^3}{2}, & t = 1. \end{cases}$$

Then the function $\frac{\chi_t}{id}$ is η_t (see Example 6.3) for every $t \in (0,\infty)$. Thus, as we have seen in Example 6.3 the function $t \mapsto [y_0, y_1, y_2; \frac{\chi_t}{id}] = [y_0, y_1, y_2; \eta_t]$, $t \in (0,\infty)$ is exponentially convex in the Jensen sense for every three mutually different points $y_0, y_1, y_2 \in (0,\infty)$.

(a) By applying Corollary 6.3 with $J = (0,\infty)$ and $\Phi = \Phi_3$, we get the exponential convexity of $t \mapsto \Psi(\chi_t)$ $(t \in (0,\infty))$ in the Jensen sense. If the mapping $t \mapsto \Psi(\chi_t)$ $(t \in (0,\infty))$ is also continuous, then it is exponentially convex.

(b) If the mapping $t \mapsto \Psi(\chi_t)$ $(t \in (0,\infty))$ is positive and differentiable, then Corollary 6.4 (ii) gives the monotonicity of the function $\overline{\mathfrak{u}}_{s,t}(\Psi, \Phi_3)$ (defined by (6.16) with $\Phi = \Phi_3$) in both parameters..

(c) Suppose the mapping $t \mapsto \Psi(\chi_t)$ $(t \in (0,\infty))$ is positive and differentiable. Suppose further that $[a,b] \subset (0,\infty)$ is an interval with the following property:

(i) if $f \in D(\Psi)$ such that the restriction of $\frac{f}{id}$ to $[a,b]$ is convex, then $\Psi(f) \geq 0$.

By using the logarithmic mean (see (6.22))

$$L(s,t) := \begin{cases} \frac{t-s}{\log t - \log s}, & s \neq t, \\ s, & t = s. \end{cases}, \quad s,t > 0,$$

we introduce

$$\overline{\mathfrak{M}}_{s,t}(\Psi, \Phi_3) := -L(s,t) \log \overline{\mathfrak{u}}_{s,t}(\Psi, \Phi_3), \quad s,t > 0.$$

If $s \neq t$, we can apply Theorem 6.6 (the conditions of the theorem are satisfied) which shows that there exists $\xi \in [a,b]$ such that

$$\begin{aligned} \overline{\mathfrak{M}}_{s,t}(\Psi, \Phi_3) &= -L(s,t) \frac{1}{s-t} \log \left(\frac{\Psi(\chi_s)}{\Psi(\chi_t)} \right) \\ &= \frac{1}{\log t - \log s} \log \left(\frac{\xi^2 \chi_s''(\xi) - 2\xi \chi_s'(\xi) + 2\chi_s(\xi)}{\xi^2 \chi_t''(\xi) - 2\xi \chi_t'(\xi) + 2\chi_t(\xi)} \right) \\ &= \frac{-\xi (\log s - \log t)}{\log t - \log s} = \xi \in [a,b]. \end{aligned}$$

It follows from this by taking limit that

$$\overline{\mathfrak{M}}_{s,s}(\Psi, \Phi_3) \in [a,b], \quad s \in (0,\infty).$$

It can be seen that

$$a \leq \overline{\mathfrak{M}}_{s,t}(\Psi, \Phi_3) \leq b, \quad s,t \in (0,\infty).$$

The function $\bar{\mathfrak{M}}_{s,t}(\Psi,\Phi_3)$ is decreasing in both parameters, because $\mathfrak{u}_{s,t}(\Upsilon,\Lambda_3)$ and $L(s,t)$ are increasing in both parameters: if $s,t,u,v \in (0,\infty)$ such that $s \le u$ and $t \le v$, then

$$\bar{\mathfrak{M}}_{s,t}(\Psi,\Phi_3) \ge \bar{\mathfrak{M}}_{u,v}(\Psi,\Phi_3).$$

(d) Now, we consider the functional Ψ defined in (4.10) (see (d) in Example 6.5).

Suppose the mapping $t \mapsto \Psi(\chi_t)$ $(t \in (0,\infty))$ is positive.

It is easy to check that the mapping $t \mapsto \Psi(\chi_t)$ $(t \in (0,\infty))$ is differentiable. It follows from (c) that

$$\min_{1\le i\le n} x_i \le \bar{\mathfrak{M}}_{s,t}(\Psi,\mathbf{x},\Phi_3) \le \sum_{i=1}^{n} x_i, \quad s,t \in (0,\infty),$$

and therefore $\bar{\mathfrak{M}}_{s,t}(\Psi,\mathbf{x},\Phi_3)$ is not a mean of the numbers x_i $(1 \le i \le n)$.

To construct means of the numbers x_i $(1 \le i \le n)$, we assume as in the previous two examples that

$$x_1 \le x_i - x_{i-1}, \quad 2 \le i \le n.$$

We also use the notation

$$\triangle\mathbf{x} := (y_1, y_2, \ldots y_n) = (x_1, x_2 - x_1, \ldots, x_n - x_{n-1}).$$

Then $x_1 < \ldots < x_n$, $\min\limits_{1\le i\le n} y_i = x_1$ and $\sum\limits_{i=1}^{n} y_i = x_n = \max\limits_{1\le i\le n} x_i$.

It is not hard to calculate

$\bar{\mathfrak{u}}_{s,t}(\Psi,\triangle\mathbf{x},\Phi_3)$

$$= \left(\frac{\log^2 t}{\log^2 s} \frac{\frac{n-k}{n-1}\sum\limits_{i=1}^{n} y_i s^{-y_i} + \frac{k-1}{n-1} x_n s^{-x_n} - \frac{1}{\binom{n-1}{k-1}} \sum\limits_{1\le i_1<\ldots<i_k\le n} \left(\sum\limits_{j=1}^{k} y_{i_j}\right) s^{-\left(\sum\limits_{j=1}^{k} y_{i_j}\right)}}{\frac{n-k}{n-1}\sum\limits_{i=1}^{n} y_i t^{-y_i} + \frac{k-1}{n-1} x_n t^{-x_n} - \frac{1}{\binom{n-1}{k-1}} \sum\limits_{1\le i_1<\ldots<i_k\le n} \left(\sum\limits_{j=1}^{k} y_{i_j}\right) t^{-\left(\sum\limits_{j=1}^{k} y_{i_j}\right)}} \right)^{\frac{1}{s-t}},$$

$$s \ne t, \quad s,t \ne 1,$$

$$= \left(\frac{2}{\log^2 s} \frac{\frac{n-k}{n-1}\sum\limits_{i=1}^{n} y_i s^{-y_i} + \frac{k-1}{n-1} x_n s^{-x_n} - \frac{1}{\binom{n-1}{k-1}} \sum\limits_{1\le i_1<\ldots<i_k\le n} \left(\sum\limits_{j=1}^{k} y_{i_j}\right) s^{-\left(\sum\limits_{j=1}^{k} y_{i_j}\right)}}{\frac{n-k}{n-1}\sum\limits_{i=1}^{n} y_i^3 + \frac{k-1}{n-1} x_n^3 - \frac{1}{\binom{n-1}{k-1}} \sum\limits_{1\le i_1<\ldots<i_k\le n} \left(\sum\limits_{j=1}^{k} y_{i_j}\right)^3} \right)^{\frac{1}{s-1}},$$

$$t = 1, \quad s \ne 1,$$

$$= \exp\left(-\frac{2}{s\log s} - \frac{\frac{n-k}{n-1}\sum\limits_{i=1}^{n} y_i^2 s^{-y_i} + \frac{k-1}{n-1} x_n^2 s^{-x_n} - \frac{1}{\binom{n-1}{k-1}} \sum\limits_{1\le i_1<\ldots<i_k\le n} \left(\sum\limits_{j=1}^{k} y_{i_j}\right)^2 s^{-\left(\sum\limits_{j=1}^{k} y_{i_j}\right)}}{s\left(\frac{n-k}{n-1}\sum\limits_{i=1}^{n} y_i s^{-y_i} + \frac{k-1}{n-1} x_n s^{-x_n} - \frac{1}{\binom{n-1}{k-1}} \sum\limits_{1\le i_1<\ldots<i_k\le n} \left(\sum\limits_{j=1}^{k} y_{i_j}\right) s^{-\left(\sum\limits_{j=1}^{k} y_{i_j}\right)}\right)} \right),$$

$$s = t \ne 1,$$

$$= \exp\left(-\frac{1}{3} \frac{\frac{n-k}{n-1}\sum_{i=1}^{n} y_i^4 + \frac{k-1}{n-1}x_n^4 - \frac{1}{\binom{n-1}{k-1}} \sum_{1\le i_1<\ldots<i_k\le n} \left(\sum_{j=1}^{k} y_{i_j}\right)^4}{\frac{n-k}{n-1}\sum_{i=1}^{n} y_i^3 + \frac{k-1}{n-1}x_n^3 - \frac{1}{\binom{n-1}{k-1}} \sum_{1\le i_1<\ldots<i_k\le n} \left(\sum_{j=1}^{k} y_{i_j}\right)^3} \right), \qquad s=t=1.$$

By applying (c), we have that $\overline{\mathfrak{M}}_{s,t}(\Psi,\mathbf{x},\Phi_3)$ $(s,t\in(0,\infty))$ are means (of the numbers $x_1<\ldots<x_n$) on

$$\{(x_1,\ldots,x_n) \mid x_1>0,\ x_1\le x_i-x_{i-1}, \quad 2\le i\le n\},$$

and they are monotonic in both parameters.

Example 6.8 Assume (G) with $I=(0,\infty)$ and $D(\Psi)=C(0,\infty)$. Consider the class of continuous convex functions

$$\Phi_4 = \{\delta_t : (0,\infty)\to(0,\infty) \mid t\in(0,\infty)\},$$

where

$$\delta_t(x) := \frac{xe^{-x\sqrt{t}}}{t}.$$

Then the function $\frac{\delta_t}{id}$ is γ_t (see Example 6.4) for every $t\in(0,\infty)$. Thus, as we have seen in Example 6.4 the function $t\mapsto[y_0,y_1,y_2;\frac{\delta_t}{id}]=[y_0,y_1,y_2;\gamma_t]$, $t\in(0,\infty)$ is exponentially convex in the Jensen sense for every three mutually different points $y_0,y_1,y_2\in(0,\infty)$.

(a) By applying Corollary 6.3 with $J=(0,\infty)$ and $\Phi=\Phi_4$, we get the exponential convexity of $t\mapsto\Psi(\delta_t)$ $(t\in(0,\infty))$ in the Jensen sense. If the mapping $t\mapsto\Psi(\delta_t)$ $(t\in(0,\infty))$ is also continuous, then it is exponentially convex.

(b) If the mapping $t\mapsto\Psi(\delta_t)$ $(t\in(0,\infty))$ is positive and differentiable, then Corollary 6.4 (ii) gives the monotonicity of the function $\overline{\mathfrak{u}}_{s,t}(\Psi,\Phi_4)$ (defined by (6.16) with $\Phi=\Phi_4$) in both parameters..

(c) Suppose the mapping $t\mapsto\Psi(\delta_t)$ $(t\in(0,\infty))$ is positive and differentiable. Suppose further that $[a,b]\subset(0,\infty)$ is an interval with the following property:

(i) if $f\in D(\Psi)$ such that the restriction of $\frac{f}{id}$ to $[a,b]$ is convex, then $\Psi(f)\ge 0$.

Introduce

$$\overline{\mathfrak{M}}_{s,t}(\Psi,\Phi_4) := -(\sqrt{s}+\sqrt{t})\log\overline{\mathfrak{u}}_{s,t}(\Psi,\Phi_4), \quad s,t\in(0,\infty).$$

If $s\neq t$, we can apply Theorem 6.6 (the conditions of the theorem are satisfied) which shows that there exists $\xi\in[a,b]$ such that

$$\overline{\mathfrak{M}}_{s,t}(\Psi,\Phi_4) = -(\sqrt{s}+\sqrt{t})\frac{1}{s-t}\log\left(\frac{\Psi(\delta_s)}{\Psi(\delta_t)}\right)$$
$$= \frac{1}{\sqrt{t}-\sqrt{s}}\log\left(\frac{\xi^2\delta_s''(\xi)-2\xi\delta_s'(\xi)+2\delta_s(\xi)}{\xi^2\delta_t''(\xi)-2\xi\delta_t'(\xi)+2\delta_t(\xi)}\right)$$

$$= \frac{1}{\sqrt{t}-\sqrt{s}} \log e^{-\xi\left(\sqrt{s}-\sqrt{t}\right)} = \xi \in [a,b].$$

It follows from this by taking limit that

$$\overline{\mathfrak{M}}_{s,s}(\Psi,\Phi_4) \in [a,b], \quad s \in (0,\infty).$$

It can be seen that

$$a \leq \overline{\mathfrak{M}}_{s,t}(\Psi,\Phi_4) \leq b, \quad s,t \in (0,\infty).$$

The function $\overline{\mathfrak{M}}_{s,t}(\Psi,\Phi_4)$ is decreasing in both parameters, because $\overline{\mathfrak{u}}_{s,t}(\Psi,\Phi_4)$ is increasing in both parameters: if $s,t,u,v \in (0,\infty)$ such that $s \leq u$ and $t \leq v$, then

$$\overline{\mathfrak{M}}_{s,t}(\Psi,\Phi_4) \geq \overline{\mathfrak{M}}_{u,v}(\Psi,\Phi_4).$$

(d) Now, we consider the functional Ψ defined in (4.10) (see (d) in Example 6.5).

Suppose the mapping $t \mapsto \Psi(\delta_t)$ $(t \in (0,\infty))$ is positive.

It is easy to check that the mapping $t \mapsto \Psi(\delta_t)$ $(t \in (0,\infty))$ is differentiable. It follows from (c) that

$$\min_{1\leq i\leq n} x_i \leq \overline{\mathfrak{M}}_{s,t}(\Psi,\mathbf{x},\Phi_4) \leq \sum_{i=1}^{n} x_i, \quad s,t \in (0,\infty),$$

and therefore $\overline{\mathfrak{M}}_{s,t}(\Psi,\mathbf{x},\Phi_4)$ is not a mean of the numbers x_i $(1 \leq i \leq n)$.

To construct means of the numbers x_i $(1 \leq i \leq n)$, we assume as in the previous three examples that

$$x_1 \leq x_i - x_{i-1}, \quad 2 \leq i \leq n.$$

We also use the notation

$$\triangle\mathbf{x} := (y_1, y_2, \ldots y_n) = (x_1, x_2 - x_1, \ldots, x_n - x_{n-1}).$$

Then $x_1 < \ldots < x_n$, $\min\limits_{1\leq i\leq n} y_i = x_1$ and $\sum\limits_{i=1}^{n} y_i = x_n = \max\limits_{1\leq i\leq n} x_i$.

It is easy to calculate

$\overline{\mathfrak{u}}_{s,t}(\Psi,\Lambda_4)$

$$= \left(\frac{t}{s} \frac{\frac{n-k}{n-1}\sum\limits_{i=1}^{n} y_i e^{-y_i\sqrt{s}} + \frac{k-1}{n-1} x_n e^{-x_n\sqrt{s}} - \frac{1}{\binom{n-1}{k-1}} \sum\limits_{1\leq i_1<\ldots<i_k\leq n} \left(\sum\limits_{j=1}^{k} y_{i_j}\right) e^{-\left(\sum\limits_{j=1}^{k} y_{i_j}\right)\sqrt{s}}}{\frac{n-k}{n-1}\sum\limits_{i=1}^{n} y_i e^{-y_i\sqrt{t}} + \frac{k-1}{n-1} x_n e^{-x_n\sqrt{t}} - \frac{1}{\binom{n-1}{k-1}} \sum\limits_{1\leq i_1<\ldots<i_k\leq n} \left(\sum\limits_{j=1}^{k} y_{i_j}\right) e^{-\left(\sum\limits_{j=1}^{k} y_{i_j}\right)\sqrt{t}}} \right)^{\frac{1}{s-t}},$$

$$s \neq t,$$

$$= \exp\left(-\frac{1}{s} - \frac{1}{2\sqrt{s}} \frac{\frac{n-k}{n-1}\sum\limits_{i=1}^{n} y_i^2 e^{-y_i\sqrt{s}} + \frac{k-1}{n-1} x_n^2 e^{-x_n\sqrt{s}} - \frac{1}{\binom{n-1}{k-1}} \sum\limits_{1\leq i_1<\ldots<i_k\leq n} \left(\sum\limits_{j=1}^{k} y_{i_j}\right)^2 e^{-\left(\sum\limits_{j=1}^{k} y_{i_j}\right)\sqrt{s}}}{\frac{n-k}{n-1}\sum\limits_{i=1}^{n} y_i e^{-y_i\sqrt{s}} + \frac{k-1}{n-1} x_n e^{-x_n\sqrt{s}} - \frac{1}{\binom{n-1}{k-1}} \sum\limits_{1\leq i_1<\ldots<i_k\leq n} \left(\sum\limits_{j=1}^{k} y_{i_j}\right) e^{-\left(\sum\limits_{j=1}^{k} y_{i_j}\right)\sqrt{s}}} \right),$$

$$s = t.$$

By applying (c), we have that $\overline{\mathfrak{M}}_{s,t}(\Psi, \mathbf{x}, \Phi_4)$ $(s,t \in (0,\infty))$ are means (of the numbers $x_1 < \ldots < x_n$) on

$$\{(x_1, \ldots, x_n) \mid x_1 > 0,\ x_1 \leq x_i - x_{i-1}, \quad 2 \leq i \leq n\},$$

and they are monotonic in both parameters.

Chapter 7

Refinements of Hölder's and Minkowski's inequalities

The results about interpolation of Mixed Means given in [58] are without weights. But in [38], we have given results with weights and improved the results given in [58] by using a refinement of the discrete Jensen's inequality from [44]. Further, in [39] we work on the refinement given in [32] to establish the generalizations of corresponding results given in [38] and we presents some parameter dependent refinements of Hölder's and Minkowski's inequalities with the help of [33].

The results of this chapter are given in [38] and [39].

We start with the extensions of Beck's results [9], given in [38]. The following hypothesis is assumed:

(A_1) Let $L_t : I_t \to \mathbb{R}$ $(t = 1,...,m)$ and $N : I_N \to \mathbb{R}$ be continuous and strictly monotone functions whose domains are intervals in $\mathbb{R}$, and let $f : I_1 \times ... \times I_m \to I_N$ be a continuous function. Let $\mathbf{x}^{(1)},...,\mathbf{x}^{(m)} \in \mathbb{R}^n$ $(n \geq 2)$ such that $\mathbf{x}^{(t)} \in I_t^n$ for each $t = 1,\ldots,m$, and let $\mathbf{p} = (p_1,...,p_n)$ be a nonnegative n-tuple such that $\sum_{i=1}^n p_i = 1$.

The following result is a simple consequence of the discrete Jensen's inequality (Theorem 1.5).

Theorem 7.1 *[38] Assume (A_1). If N is an increasing function, then the inequality*

$$f\left(L_1(\mathbf{x}^{(1)};\mathbf{p};n),...,L_m(\mathbf{x}^{(m)};\mathbf{p};n)\right) \geq N^{-1}\left(\sum_{i=1}^{n} p_i N(f(x_i^{(1)},...,x_i^{(m)}))\right), \tag{7.1}$$

holds for all possible $\mathbf{x}^{(t)}$ $(t = 1,\ldots,m)$ and $\mathbf{p}$, if and only if the function H defined on

$L_1(I_1)\times ... \times L_m(I_m)$ by

$$H(t_1,...,t_m) := N\left(f\left(L_1^{-1}(t_1),...,L_m^{-1}(t_m)\right)\right)$$

is concave. The inequality in (7.1) is reversed for all possible $\mathbf{x}^{(t)}$ $(t = 1,\dots,m)$ and $\mathbf{p}$, if and only if H is convex.

Beck's original result was the special case of Theorem 7.1, where $m = 2$ and $I_1 = [k_1,k_2]$, $I_2 = [l_1,l_2]$ and $I_N = [n_1,n_2]$ (see [12], p. 249).

For simplicity, in the case $m = 2$ we use the following form of (A_1):

(A_2) Let $K : I_K \to \mathbb{R}$, $L : I_L \to \mathbb{R}$ and $N : I_N \to \mathbb{R}$ be continuous and strictly monotone functions whose domains are intervals in $\mathbb{R}$, and let $f : I_K \times I_L \to I_N$ be a continuous function. Let $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^n$ $(n \geq 2)$ such that $\mathbf{a} \in I_K^n$ and $\mathbf{b} \in I_L^n$, and let $\mathbf{p} = (p_1,...,p_n)$ be a nonnegative n-tuple such that $\sum_{i=1}^n p_i = 1$.

Then (7.1) has the form

$$f\left(K_n(\mathbf{a};\mathbf{p}), L_n(\mathbf{b};\mathbf{p})\right) \geq N_n(f(\mathbf{a},\mathbf{b});\mathbf{p}), \tag{7.2}$$

where $f(\mathbf{a},\mathbf{b})$ means $(f(a_1,b_1),...,f(a_n,b_n))$.

The following results are important special cases of Theorem 7.1, and generalize the corresponding results of Beck. The next hypothesis will be used:

(A_3) Let $K : I_K \to \mathbb{R}$, $L : I_L \to \mathbb{R}$ and $N : I_N \to \mathbb{R}$ be continuous and strictly monotone functions whose domains are intervals in $\mathbb{R}$ such that either $I_K + I_L \subset I_N$ and $f(x,y) = x+y$ $((x,y) \in I_K \times I_L)$ or $I_K, I_L \subset (0,\infty)$, $I_K \cdot I_L \subset I_N$ and $f(x,y) = xy$ $((x,y) \in I_K \times I_L)$. Assume further that the functions K, L and N are twice continuously differentiable on the interior of their domains, respectively. Let $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^n$ $(n \geq 2)$ such that $\mathbf{a} \in I_K^n$ and $\mathbf{b} \in I_L^n$, and let $\mathbf{p} = (p_1,...,p_n)$ be a nonnegative n-tuple such that $\sum_{i=1}^n p_i = 1$.

The interior of a subset A of $\mathbb{R}$ is denoted by A°.

Corollary 7.1 *[38] Assume (A_3) with $f(x,y) = x+y$ $((x,y) \in I_K \times I_L)$, and assume that K', L', N', K'', L'' and N'' are all positive. Introducing $E := \frac{K'}{K''}$, $F := \frac{L'}{L''}$, $G := \frac{N'}{N''}$, (7.2) holds for all possible $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{p}$ if and only if*

$$E(x) + F(y) \leq G(x+y), \quad (x,y) \in I_K^\circ \times I_L^\circ. \tag{7.3}$$

Corollary 7.2 *[38] Assume (A_3) with $f(x,y) = xy$ $((x,y) \in I_K \times I_L)$. Suppose the functions $A(x) := \frac{K'(x)}{K'(x)+xK''(x)}$, $B(x) := \frac{L'(x)}{L'(x)+xL''(x)}$ and $C(x) := \frac{N'(x)}{N'(x)+xN''(x)}$ are defined on I_K°, I_L° and I_N°, respectively. Assume further that K', L', N', A, B and C are all positive. Then (7.2) holds for all possible $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{p}$ if and only if*

$$A(x) + B(y) \leq C(xy), \quad (x,y) \in I_K^\circ \times I_L^\circ.$$

To prove these corollaries, similar arguments can be applied as in the analogous results of Beck. We just sketch the proof of Corollary 7.1.

Proof. By Theorem 7.1, it is enough to prove that the function

$$H : K(I_K) \times L(I_L), \quad H(t,s) := N\left(K^{-1}(t) + L^{-1}(s)\right)$$

is concave. Since H is continuous, and twice continuously differentiable on the interior $K(I_K^\circ) \times L(I_L^\circ)$ of its domain, we have to show that

$$D_{11}H(t,s)h_1^2 + 2D_{12}H(t,s)h_1h_2 + D_{22}H(t,s)h_2^2 \le 0$$

for all $(t,s) \in K(I_K^\circ) \times L(I_L^\circ)$ and $(h_1,h_2) \in \mathbb{R}^2$. By computing the partial derivatives of H of order 2 at the points of $K(I_K^\circ) \times L(I_L^\circ)$, we have that this condition follows from (7.3). □

In [58], Mitrinović and Pečarić obtained a new inequality like (7.2), which is based on Theorem 1.7.

Assume (A_2). We denote by α_i^k $(1 \le i \le v)$ and β_i^k $(1 \le i \le v)$ the k-tuples of **a** and **b** respectively, where $v = \begin{pmatrix} n \\ k \end{pmatrix}$. Following [58], we introduce the mixed N-K-L means of **a** and **b**:

$$M(N,K,L;k) := N_v(f(K_k(\alpha_i^k), L_k(\beta_i^k)); 1 \le i \le v), \quad 1 < k < n, \tag{7.4}$$

and

$$M(N,K,L;1) := N_n(f(\mathbf{a},\mathbf{b})),$$

$$M(N,K,L;n) := f\left(K_n(\mathbf{a}), L_n(\mathbf{b})\right).$$

The promised theorem from [58] is the next:

Theorem A. *Assume (A_2). Let N be an increasing (decreasing) function, and let*

$$H : K(I_K) \times L(I_L) \to \mathbb{R}, \quad H(s,t) := N\left(f\left(K^{-1}(s), L^{-1}(t)\right)\right)$$

be a convex (concave) function. Then

$$M(N,K,L;k+1) \le M(N,K,L;k), \quad k = 1,\ldots,n-1. \tag{7.5}$$

If N is increasing (decreasing) but H is concave (convex) then the inequalities in (7.5) are reversed.

Here we can apply Theorem 1.7 to the function H and to the points $(K(a_i), L(b_i))$ $(1 \le i \le n)$.

On the analogy of Corollary 7.1 and Corollary 7.2, we have the following consequences of Theorem A.

Corollary A. *Assume (A_3) with $f(x,y) = x + y$ $((x,y) \in I_K \times I_L)$. Assume further that K', L', N', K'', L'' and N'' are all positive and $E(x) + F(y) \le G(x+y)$ $((x,y) \in I_K^\circ \times I_L^\circ)$, where $E := \frac{K'}{K''}$, $F := \frac{L'}{L''}$, $G := \frac{N'}{N''}$. Then (7.5) with reverse inequality is valid.*

Corollary B. *Assume (A_3) with $f(x,y) = xy$ $((x,y) \in I_K \times I_L)$. Suppose the functions $A(x) := \frac{K'(x)}{K'(x)+xK''(x)}$, $B(x) := \frac{L'(x)}{L'(x)+xL''(x)}$ and $C(x) := \frac{N'(x)}{N'(x)+xN''(x)}$ are defined on I_K°, I_L°*

and I_N°, respectively. If K', L', M', A, B and C are all positive and $A(x)+B(y) \le C(xy)$ $((x,y) \in I_K^\circ \times I_L^\circ)$, then (7.5) with reverse inequality is valid.

Next, we collect four results which are special cases of earlier results. We need the following hypothesis:

($\mathscr{G}_1$) Let U be a convex set in $\mathbb{R}^m$, $\mathbf{x}_1,\ldots,\mathbf{x}_n \in U$, and let $\mathbf{p} := (p_1,...,p_n)$ be a positive n-tuples such that $\sum_{i=1}^n p_i = 1$. Further, let $f : U \to \mathbb{R}$ be a convex function.

By using the notations introduced in Section 2.1 (see $\mathscr{N}_1$), we remind:

($\mathscr{H}_0$) For fixed integers $n \ge 1$ and $k \ge 2$ consider a subset I_k of $\{1,...,n\}^k$ such that

$$\alpha_{I_k,i} \ge 1, \quad 1 \le i \le n.$$

For any $k \ge l \ge 1$ set

$$A_{l,l} = A_{l,l}(I_l;\mathbf{x}_1,\ldots,\mathbf{x}_n;\mathbf{p}) := \sum_{(i_1,...,i_l)\in I_l} \left(\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_l,i_s}}\right) f\left(\frac{\sum\limits_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_l,i_s}}\mathbf{x}_{i_s}}{\sum\limits_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_l,i_s}}}\right), \tag{7.6}$$

and associate to each $k-1 \ge l \ge 1$ the number

$$\begin{aligned} &A_{k,l} = A_{k,l}(I_k;\mathbf{x}_1,\ldots,\mathbf{x}_n;\mathbf{p}) \\ &:= \frac{1}{(k-1)\ldots l} \sum_{(i_1,...,i_l)\in I_l} t_{I_k,l}(i_1,...,i_l) \left(\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}}\right) f\left(\frac{\sum\limits_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}}\mathbf{x}_{i_s}}{\sum\limits_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}}}\right). \end{aligned} \tag{7.7}$$

The following refinement of the discrete Jensen's inequality is coming from Theorem 2.1.

Theorem B. *Assume ($\mathscr{H}_0$) and ($\mathscr{G}_1$). Then*

$$f\left(\sum_{i=1}^n p_i\mathbf{x}_i\right) \le A_{k,k} \le A_{k,k-1} \le \ldots \le A_{k,2} \le A_{k,1} = \sum_{i=1}^n p_i f(\mathbf{x}_i), \tag{7.8}$$

where the numbers $A_{k,l}$ $(k \ge l \ge 1)$ are defined in (7.6) and (7.7). If f is a concave function then the inequalities in (7.8) are reversed.

The following result is a special case of Theorem 2.2.

Theorem C. *Assume ($\mathscr{H}_0$) and ($\mathscr{G}_1$), and suppose $\left|H_{I_l}(j_1,...,j_{l-1})\right| = \beta_{l-1}$ for any $(j_1,...,j_{l-1}) \in I_{l-1}$ $(k \ge l \ge 2)$. Then*

$$A_{k,l} = A_{l,l} = \frac{n}{l\,|I_l|} \sum_{(i_1,...,i_l)\in I_l} \left(\sum_{s=1}^{l} p_{i_s}\right) f\left(\frac{\sum\limits_{s=1}^{l} p_{i_s}\mathbf{x}_{i_s}}{\sum\limits_{s=1}^{l} p_{i_s}}\right), \quad k \ge l \ge 1,$$

and thus

$$f\left(\sum_{r=1}^n p_r\mathbf{x}_r\right) \le A_{k,k} \le A_{k-1,k-1} \le \ldots \le A_{2,2} \le A_{1,1} = \sum_{r=1}^n p_r f(\mathbf{x}_r). \tag{7.9}$$

If f is a concave function then the inequalities in (7.9) are reversed.

By the virtue of hypotheses (they are introduced in Section 2.2)

($\mathscr{H}_4$) Let $S_1,\ldots,S_n$ be finite, pairwise disjoint and nonempty sets, let

$$S := \bigcup_{j=1}^{n} S_j,$$

and let c be a function from S into $\mathbb{R}$ such that

$$c(s) > 0, \quad s \in S, \quad \text{and} \quad \sum_{s\in S_j} c(s) = 1, \quad j = 1,\ldots,n.$$

Let the function $\tau : S \to \{1,\ldots,n\}$ be defined by

$$\tau(s) := j, \quad \text{if} \quad s \in S_j.$$

and

($\mathscr{H}_5$) Suppose $\mathscr{A} \subset P(S)$ is a partition of S into pairwise disjoint and nonempty sets. Let

$$k := \max\{|A| \mid A \in \mathscr{A}\},$$

and let

$$\mathscr{A}_l := \{A \in \mathscr{A} \mid |A| = l\}, \quad l = 1,\ldots,k.$$

we state the following refinement of the discrete Jensen's inequality which is contained in Theorem 2.3:

Theorem D. *Assume* ($\mathscr{H}_4$), ($\mathscr{H}_5$) *and* ($\mathscr{G}_1$). *Then*

$$f\left(\sum_{j=1}^{n} p_j \mathbf{x}_j\right) \leq A_k \leq A_{k-1} \leq \ldots \leq A_2 \leq A_1 = \sum_{j=1}^{n} p_j f(\mathbf{x}_j),$$

where

$$A_k := \sum_{l=1}^{k}\left(\sum_{A\in\mathscr{A}_l}\left(\left(\sum_{s\in A} c(s)p_{\tau(s)}\right) f\left(\frac{\sum_{s\in A} c(s)p_{\tau(s)}\mathbf{x}_{\tau(s)}}{\sum_{s\in A} c(s)p_{\tau(s)}}\right)\right)\right), \tag{7.10}$$

and for every $1 \leq d \leq k-1$ the number A_{k-d} is given by

$$A_{k-d} := \sum_{l=1}^{d}\left(\sum_{A\in\mathscr{A}_l}\left(\sum_{s\in A} c(s)p_{\tau(s)} f(\mathbf{x}_{\tau(s)})\right)\right) + \sum_{l=d+1}^{k}\left(\frac{d!}{(l-1)\ldots(l-d)}\right. \tag{7.11}$$

$$\left.\cdot \sum_{A\in\mathscr{A}_l}\left(\sum_{B\in P_{l-d}(A)}\left(\left(\sum_{s\in B} c(s)p_{\tau(s)}\right) f\left(\frac{\sum_{s\in B} c(s)p_{\tau(s)}\mathbf{x}_{\tau(s)}}{\sum_{s\in B} c(s)p_{\tau(s)}}\right)\right)\right)\right).$$

From the parameter dependent refinement of the discrete Jensen's inequality (see Theorem 2.5), we have

Theorem E. *For any real number $\lambda \geq 1$, we suppose (G_1) and consider the sets*

$$S_k := \left\{ (i_1, \ldots, i_n) \in \mathbb{N}^n \mid \sum_{j=1}^{n} i_j = k \right\}, \quad k \in \mathbb{N}. \tag{7.12}$$

Let

$$C_k(\lambda) = C_k(\mathbf{x}_1, \ldots, \mathbf{x}_n; p_1, \ldots, p_n; \lambda)$$

$$:= \frac{1}{(n+\lambda-1)^k} \sum_{(i_1,\ldots,i_n)\in S_k} \frac{k!}{i_1!\ldots i_n!} \left(\sum_{j=1}^{n} \lambda^{i_j} p_j \right) f \left(\frac{\sum_{j=1}^{n} \lambda^{i_j} p_j \mathbf{x}_j}{\sum_{j=1}^{n} \lambda^{i_j} p_j} \right), \tag{7.13}$$

for any $k \in N$. Then

$$f\left(\sum_{j=1}^{n} p_j \mathbf{x}_j \right) = C_0(\lambda) \leq C_1(\lambda) \leq \ldots \leq C_k(\lambda) \leq \ldots \leq \sum_{j=1}^{n} p_j f(\mathbf{x}_j), \quad k \in \mathbb{N}.$$

7.1 Generalizations of Beck's result

In what follows (A_1) and $(\mathscr{H}_0)$ are assumed. The weighted mixed means relative to (7.6) and (7.7) are defined in the following ways:

$$M^1_{k,k}(L_1,\ldots,L_m;\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(m)};\mathbf{p}) :=$$
$$N^{-1}\left(\sum_{\mathbf{i}^k\in I_k} \left(\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} \right) N\left(f\left(L_1(\mathbf{x}^{(1)}; \tfrac{\mathbf{p}}{\alpha_{I_k}}; k), \ldots, L_m(\mathbf{x}^{(m)}; \tfrac{\mathbf{p}}{\alpha_{I_k}}; k) \right) \right) \right)$$

and for $k-1 \geq l \geq 1$

$$M^1_{k,l}(L_1,\ldots,L_m;\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(m)};\mathbf{p}) :=$$
$$N^{-1}\left(\tfrac{1}{(k-1)\ldots l} \sum_{\mathbf{i}^l\in I_l} t_{I_k,l}(\mathbf{i}^l) \left(\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}} \right) N\left(f\left(L_1(\mathbf{x}^{(1)}; \tfrac{\mathbf{p}}{\alpha_{I_k}}; l), \ldots, L_m(\mathbf{x}^{(m)}; \tfrac{\mathbf{p}}{\alpha_{I_k}}; l) \right) \right) \right)$$

where for $k \geq l \geq 1$

$$L_t(\mathbf{x}^{(t)}; \frac{\mathbf{p}}{\alpha_{I_k}}; l) := L_t^{-1} \left(\frac{\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}} L_t(x^{(t)}_{i_s})}{\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}}} \right), \quad t = 1, \ldots, m,$$

respectively, and $\mathbf{i}^l := (i_1, \ldots, i_l)$.

Now, we get an interpolation of (7.1) by the direct application of Theorem B as follows.

Theorem 7.2 *Assume (A_1) and $(\mathscr{H}_0)$. If N is an increasing (decreasing) function, then the inequalities*

$$\begin{aligned}
f\left(L_1(\mathbf{x}^{(1)};\mathbf{p};n),...,L_m(\mathbf{x}^{(m)};\mathbf{p};n)\right) &\leq M^1_{k,k}(L_1,...,L_m;\mathbf{x}^{(1)},...,\mathbf{x}^{(m)};\mathbf{p})\\
&\leq M^1_{k,k-1}(L_1,...,L_m;\mathbf{x}^{(1)},...,\mathbf{x}^{(m)};\mathbf{p})\\
&\leq\\
&\quad\vdots\\
&\quad\leq\\
&\quad\leq M^1_{k,2}(L_1,\ldots,L_m;\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(m)};\mathbf{p})\\
&\leq M^1_{k,1}(L_1,...,L_m;\mathbf{x}^{(1)},...,\mathbf{x}^{(m)};\mathbf{p})\\
&= N^{-1}\left(\sum_{i=1}^{n} p_i N(f(x_i^{(1)},...,x_i^{(m)}))\right)
\end{aligned}\tag{7.14}$$

hold for all possible $\mathbf{x}^{(t)}$ $(t=1,\ldots,m)$ and $\mathbf{p}$, if and only if the function H defined in Theorem 7.1 is convex (concave). If N is an increasing (decreasing) function, then the inequalities in (7.14) are reversed for all possible $\mathbf{x}^{(t)}$ $(t=1,\ldots,m)$ and $\mathbf{p}$, if and only if H is concave (convex).

Proof. Suppose N is increasing and the function $H : L_1(I_1)\times\ldots\times L_m(I_m)\to\mathbb{R}$,

$$H(t_1,...,t_m) = N\left(f\left(L_1^{-1}(t_1),...,L_m^{-1}(t_m)\right)\right)$$

is convex. We apply Theorem B to the function H and to the vectors $(L_1(x_i^1),\ldots,L_m(x_i^m))$, $i=1,\ldots,n$. Then the first term in (7.8) gives

$$\begin{aligned}
&H\left(\sum_{i=1}^{n} p_i(L_1(x_i^1),\ldots,L_m(x_i^m))\right)\\
&= N\left(f\left(L_1^{-1}\left(\sum_{i=1}^{n} p_iL_1(x_i^1)\right),...,L_m^{-1}\left(\sum_{i=1}^{n} p_iL_m(x_i^m)\right)\right)\right)\\
&= N\left(f\left(L_1(\mathbf{x}^{(1)};\mathbf{p};n),...,L_m(\mathbf{x}^{(m)};\mathbf{p};n)\right)\right).
\end{aligned}$$

The last term in (7.8) will be

$$\sum_{i=1}^{n} p_iH(L_1(x_i^1),\ldots,L_m(x_i^m)) = \sum_{i=1}^{n} p_iN\left(f\left(x_i^1,\ldots,x_i^m\right)\right).$$

$A_{k,k}$ in (7.8) has the form

$$\begin{aligned}
&\sum_{(i_1,...,i_k)\in I_k}\left(\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right) H\left(\frac{\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}(L_1(x_{i_s}^1),...,L_m(x_{i_s}^m))}{\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}}\right)\\
&= \sum_{(i_1,...,i_k)\in I_k}\left(\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right) N\left(f\left(L_1^{-1}\left(\frac{\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}L_1(x_{i_s}^1)}{\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}}\right),\ldots,L_m^{-1}\left(\frac{\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}L_m(x_{i_s}^m)}{\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}}\right)\right)\right)\\
&= \sum_{\mathbf{i}^k\in I_k}\left(\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right) N\left(f\left(L_1(\mathbf{x}^{(1)};\frac{\mathbf{p}}{\alpha_{I_k}};k),...,L_m(\mathbf{x}^{(m)};\frac{\mathbf{p}}{\alpha_{I_k}};k)\right)\right)
\end{aligned}$$

$= M^1_{k,k}(L_1,...,L_m;\mathbf{x}^{(1)},...,\mathbf{x}^{(m)};\mathbf{p})$.

A similar argument shows that for $k-1 \geq l \geq 1$ $A_{k,l}$ in (7.8) can be written as

$$M^1_{k,l}(L_1,...,L_m;\mathbf{x}^{(1)},...,\mathbf{x}^{(m)};\mathbf{p}).$$

The inequality (7.14) follows from these observations and Theorem B since N^{-1} is increasing.

The converse is obtained by Theorem 7.1. □

The following applications of Theorem 7.2 are motivated by Example 2.1 and Example 2.2 corresponding to Theorem B.

Example 7.1 Assume (A_1). Consider

$$I_2 := \{(i_1,i_2) \in \{1,...,n\}^2 \mid i_1|i_2\},$$

where $i_1|i_2$ means that i_1 divides i_2. Since $i|i$ $(i=1,...,n)$, therefore $(\mathscr{H}_0)$ holds and

$$\alpha_{I_2,i} = \left[\frac{n}{i}\right] + d(i), \quad i = 1,...,n,$$

where $[\frac{n}{i}]$ is the largest positive integer not greater than $\frac{n}{i}$, and $d(i)$ means the number of positive divisors of i. Then a corresponding weighted mixed mean is

$$\begin{aligned}
&M^1_{2,2}(L_1,...,L_m;\mathbf{x}^{(1)},...,\mathbf{x}^{(m)};\mathbf{p})\\
&= N^{-1}\left(\sum_{(i_1,i_2)\in I_2}\left(\sum_{s=1}^{2}\frac{p_{i_s}}{[\frac{n}{i_s}]+d(i_s)}\right)N\left(f\left(L_1(\mathbf{x}^{(1)};\frac{\mathbf{p}}{\alpha_{I_2}}),...,L_m(\mathbf{x}^{(m)};\frac{\mathbf{p}}{\alpha_{I_2}})\right)\right)\right),
\end{aligned}$$

where

$$L_t(\mathbf{x}^{(t)};\frac{\mathbf{p}}{\alpha_{I_2}}) := L_t^{-1}\left(\frac{\sum_{s=1}^{2}\frac{p_{i_s}}{[\frac{n}{i_s}]+d(i_s)}L_t(x^{(t)}_{i_s})}{\sum_{s=1}^{2}\frac{p_{i_s}}{[\frac{n}{i_s}]+d(i_s)}}\right), \quad t = 1,...,m.$$

If N is increasing and the function H defined in Theorem 7.1 is convex, then Theorem 7.2 gives

$$\begin{aligned}
&f\left(L_1(\mathbf{x}^{(1)};\mathbf{p};n),...,L_m(\mathbf{x}^{(m)};\mathbf{p};n)\right)\\
&\leq M^1_{2,2}(L_1,...,L_m;\mathbf{x}^{(1)},...,\mathbf{x}^{(m)};\mathbf{p}) \leq N^{-1}\left(\sum_{i=1}^{n}p_iN(f(x_i^{(1)},...,x_i^{(m)}))\right).
\end{aligned}$$

Example 7.2 Assume (A_1). Let $c_i \geq 1$ $(i=1,...,n)$ be integers, let $k := \sum_{i=1}^{n} c_i$, and also let $I_k = P_k^{c_1,...,c_n}$ consist of all sequences $(i_1,...,i_k)$ in which the number of occurrences of $i \in \{1,...,n\}$ is c_i $(i=1,...,n)$. Then $(\mathscr{H}_0)$ is satisfied, and

$$I_{k-1} = \bigcup_{i=1}^{n} P_{k-1}^{c_1,...,c_{i-1},c_i-1,c_{i+1},...,c_n}, \quad \alpha_{I_k,i} = \frac{k!}{c_1!...c_n!}c_i, \ i=1,...,n,$$

Moreover, $t_{I_k,k-1}(i_1,....,i_{k-1}) = k$ for

$$(i_1,....,i_{k-1}) \in P_{k-1}^{c_1,...,c_{i-1},c_i-1,c_{i+1},...,c_n}, \quad i = 1,...,n.$$

Then we can write a corresponding mixed mean as follows:

$$\begin{aligned}&M_{k,k-1}^1(L_1,...,L_m;\mathbf{x}^{(1)},...,\mathbf{x}^{(m)};\mathbf{p}) \\ &= N^{-1}\left(\frac{1}{k-1}\sum_{i=1}^n (c_i - p_i)N\left(f\left(L_1(\mathbf{x};\tfrac{\mathbf{p}}{\mathbf{c}_i}),...,L_m(\mathbf{x};\tfrac{\mathbf{p}}{\mathbf{c}_i})\right)\right)\right),\end{aligned}$$

where

$$L_t(\mathbf{x};\frac{\mathbf{p}}{\mathbf{c}_i}) := L_t^{-1}\left(\frac{\sum_{r=1}^n p_r L_t(x_r^{(t)}) - \frac{p_i}{c_i}L_t(x_i^{(t)})}{1-\frac{p_i}{c_i}}\right), \quad t = 1,...,m.$$

If M is increasing and the function H defined in Theorem 7.1 is convex, then Theorem 7.2 gives

$$\begin{aligned}&f\left(L_1(\mathbf{x}^{(1)};\mathbf{p};n),...,L_m(\mathbf{x}^{(m)};\mathbf{p};n)\right) \\ &\leq M_{k,k-1}^1(L_1,...,L_m;\mathbf{x}^{(1)},...,\mathbf{x}^{(m)};\mathbf{p}) \leq N^{-1}\left(\sum_{i=1}^n p_i N(f(x_i^{(1)},...,x_i^{(m)}))\right).\end{aligned}$$

Now, we assume (A_1), $(\mathscr{H}_0)$ and suppose $\left|H_{I_l}(j_1,...,j_{l-1})\right| = \beta_{l-1}$ for any $(j_1,...,j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$. Then corresponding to the core term of Theorem C, we define for $k \geq l \geq 1$

$$\begin{aligned}&M_{l,l}^2(L_1,...,L_m;\mathbf{x}^{(1)},...,\mathbf{x}^{(m)};\mathbf{p}) \\ &= N^{-1}\left(\frac{n}{l|I_l|}\sum_{\mathbf{i}^l\in I_l}\left(\sum_{s=1}^l p_{i_s}\right)N\left(f\left(L_1(\mathbf{x}^{(1)};\mathbf{p}_{I_l}),...,L_m(\mathbf{x}^{(m)};\mathbf{p}_{I_l})\right)\right)\right),\end{aligned} \tag{7.15}$$

where

$$L_t(\mathbf{x}^{(t)};\mathbf{p}_{I_l}) := L_t^{-1}\left(\frac{\sum_{s=1}^l p_{i_s}L_t(x_{i_s}^{(t)})}{\sum_{s=1}^l p_{i_s}}\right), \quad t = 1,\ldots,m.$$

In this case Theorem C gives another interpolation of (7.1) as follows:

Theorem 7.3 *Assume* (A_1), $(\mathscr{H}_0)$ *and suppose* $\left|H_{I_l}(j_1,...,j_{l-1})\right| = \beta_{l-1}$ *for any* $(j_1,...,j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$. *If* N *is an increasing (decreasing) function, then inequalities*

$$\begin{aligned}&f\left(L_1(\mathbf{x}^{(1)};\mathbf{p};n),...,L_m(\mathbf{x}^{(m)};\mathbf{p};n)\right) \leq M_{k,k}^2(L_1,...,L_m;\mathbf{x}^{(1)},...,\mathbf{x}^{(m)};\mathbf{p}) \leq \\ &\leq M_{k-1,k-1}^2(L_1,...,L_m;\mathbf{x}^{(1)},...,\mathbf{x}^{(m)};\mathbf{p}) \leq ... \leq M_{2,2}^2(L_1,...,L_m;\mathbf{x}^{(1)},...,\mathbf{x}^{(m)};\mathbf{p}) \leq \\ &\leq M_{1,1}^2(L_1,...,L_m;\mathbf{x}^{(1)},...,\mathbf{x}^{(m)};\mathbf{p}) = N^{-1}\left(\sum_{i=1}^n p_i N(f(x_i^{(1)},...,x_i^{(m)}))\right).\end{aligned} \tag{7.16}$$

hold for all possible $\mathbf{x}^{(t)}$ $(t = 1, \ldots, m)$ *and* $\mathbf{p}$*, if and only if the function* H *defined in Theorem 7.1 is convex (concave). If* N *is an increasing (decreasing) function, then the inequalities in (7.16) are reversed for all possible* $\mathbf{x}^{(t)}$ $(t = 1, \ldots, m)$ *and* $\mathbf{p}$*, if and only if* H *is concave (convex).*

Proof. The proof is similar to the proof of Theorem 7.2. □

Now, we give some applications of Theorem 7.3 with the help of Examples 2.4-2.7.

Example 7.3 Assume (A_1). If we set

$$I_k := \{(i_1, ..., i_k) \in \{1, ..., n\}^k \mid i_1 < ... < i_k\}, \quad 1 \le k \le n,$$

then $\alpha_{I_n,i} = 1$ $(i = 1, ..., n)$ i.e. $(\mathscr{H}_0)$ is satisfied for $k = n$. It comes easily that $T_k(I_k) = I_{k-1}$ $(k = 2, ..., n)$, $|I_k| = \binom{n}{k}$ $(k = 1, ..., n)$, and for each $k = 2, ..., n$

$$\left|H_{I_k}(j_1, ..., j_{k-1})\right| = n - (k-1), \quad (j_1, ..., j_{k-1}) \in I_{k-1}.$$

In this case (7.15) becomes for $n \ge k \ge 1$

$$\begin{aligned} &M^2_{k,k}(L_1, ..., L_m; \mathbf{x}^{(1)}, ..., \mathbf{x}^{(m)}; \mathbf{p}) \\ &= N^{-1}\left(\frac{1}{\binom{n-1}{k-1}} \sum_{1 \le i_1 < ... < i_k \le n} \left(\sum_{s=1}^{k} p_{i_s}\right) N\left(f\left(L_1(\mathbf{x}^{(1)}; \mathbf{p}_{I_k}), ..., L_m(\mathbf{x}^{(m)}; \mathbf{p}_{I_k})\right)\right)\right). \end{aligned} \tag{7.17}$$

If N is increasing and the function H defined in Theorem 7.1 is convex, then Theorem 7.3 gives

$$\begin{aligned} &f\left(L_1(\mathbf{x}^{(1)}; \mathbf{p}; n), ..., L_m(\mathbf{x}^{(m)}; \mathbf{p}; n)\right) \le M^2_{n,n}(L_1, ..., L_m; \mathbf{x}^{(1)}, ..., \mathbf{x}^{(m)}; \mathbf{p}) \le \\ &\le M^2_{n-1,n-1}(L_1, ..., L_m; \mathbf{x}^{(1)}, ..., \mathbf{x}^{(m)}; \mathbf{p}) \le ... \le M^2_{2,2}(L_1, \ldots, L_m; \mathbf{x}^{(1)}, ..., \mathbf{x}^{(m)}; \mathbf{p}) \le \\ &\le M^2_{1,1}(L_1, \ldots, L_m; \mathbf{x}^{(1)}, ..., \mathbf{x}^{(m)}; \mathbf{p}) = N^{-1}\left(\sum_{i=1}^{n} p_i N(f(x_i^{(1)}, ..., x_i^{(m)}))\right). \end{aligned} \tag{7.18}$$

Remark 7.1 If we take $p_1 = ... = p_n = \frac{1}{n}$ and $m = 2$ in (7.17), then we get (7.4). Hence the interpolation given in (7.18) is a generalization of (7.5).

Example 7.4 Assume (A_1). If we set

$$I_k := \{(i_1, ..., i_k) \in \{1, ..., n\}^k \mid i_1 \le ... \le i_k\}, \quad k \ge 1,$$

then $\alpha_{I_k,i} \ge 1$ $(i = 1, ..., n)$ and thus $(\mathscr{H}_0)$ is satisfied. It is easy to see that $T_k(I_k) = I_{k-1}$ $(k = 2, ...)$, $|I_k| = \binom{n+k-1}{k}$ $(k = 1, ...)$, and for each $l = 2, ..., k$

$$\left|H_{I_l}(j_1, ..., j_{l-1})\right| = n, \quad (j_1, ..., j_{l-1}) \in I_{l-1}.$$

Under these settings (7.15) becomes

$$M^2_{k,k}(L_1,...,L_m;\mathbf{x}^{(1)},...,\mathbf{x}^{(m)};\mathbf{p})$$
$$= N^{-1}\left(\frac{1}{\binom{n+k-1}{k}}\sum_{1\le i_1\le...\le i_k\le n}\left(\sum_{s=1}^{k} p_{i_s}\right)N\left(f\left(L_1(\mathbf{x}^{(1)};\mathbf{p}_{I_k}),...,L_m(\mathbf{x}^{(m)};\mathbf{p}_{I_k})\right)\right)\right).$$

If N is increasing and the function H defined in Theorem 7.1 is convex, then Theorem 7.3 gives

$$f\left(L_1(\mathbf{x}^{(1)};\mathbf{p};n),...,L_m(\mathbf{x}^{(m)};\mathbf{p};n)\right) \le ... \le M^2_{k,k}(L_1,\dots,L_m;\mathbf{x}^{(1)},\dots,\mathbf{x}^{(m)};\mathbf{p}) \le$$
$$\le ... \le M^2_{k,1}(L_1,...,L_m;\mathbf{x}^{(1)},...,\mathbf{x}^{(m)};\mathbf{p}) = N^{-1}\left(\sum_{i=1}^{n} p_i N(f(x_i^{(1)},...,x_i^{(m)}))\right).$$

Example 7.5 Assume (A_1). Let

$$I_k := \{1,...,n\}^k, \quad k \ge 1.$$

Then $\alpha_{I_k,i} \ge 1$ $(i = 1,...,n)$, hence $(\mathscr{H}_0)$ holds and $T_k(I_k) = I_{k-1}$ $(k = 2,...)$, $|I_k| = n^k$ $(k = 1,...)$, also for $l = 2,...,k$

$$\left|H_{I_l}(j_1,...,j_{l-1})\right| = n^l, \ (j_1,...,j_{l-1}) \in I_{l-1}.$$

Therefore under these settings, for $k \ge 1$, (7.15) leads to

$$M^2_{k,k}(L_1,L_2;\mathbf{x}^{(1)},\mathbf{x}^{(2)};\mathbf{p})$$
$$= N^{-1}\left(\frac{1}{kn^{k-1}}\sum_{(i_1,...,i_k)\in I_k}\left(\sum_{s=1}^{k} p_{i_s}\right)N\left(f\left(L_1(\mathbf{x}^{(1)};\mathbf{p}_{I_k}),...,L_m(\mathbf{x}^{(m)};\mathbf{p}_{I_k})\right)\right)\right).$$

If N is increasing and the function H defined in Theorem 7.1 is convex, then for $k \ge 1$ Theorem 7.3 gives

$$f\left(L_1(\mathbf{x}^{(1)};\mathbf{p};n),...,L_m(\mathbf{x}^{(m)};\mathbf{p};n)\right) \le ... \le M^2_{k,k}(L_1,.,L_m;\mathbf{x}^{(1)},\dots,\mathbf{x}^{(m)};\mathbf{p}) \le$$
$$\le ... \le M^2_{1,1}(L_1,...,L_m;\mathbf{x}^{(1)},...,\mathbf{x}^{(m)};\mathbf{p}) = N^{-1}\left(\sum_{i=1}^{n} p_i N(f(x_i^{(1)},...,x_i^{(m)}))\right).$$

Example 7.6 Assume (A_1). Let $1 \le k \le n$ and let I_k consist of all sequences $(i_1,...,i_k)$ of k distinct numbers from $\{1,...,n\}$. Then $\alpha_{I_n,i} \ge 1$ $(i = 1,...,n)$, and $(\mathscr{H}_0)$ is satisfied. It is immediate that $T_k(I_k) = I_{k-1}$ $(k = 2,...)$, $|I_k| = n(n-1)...(n-k+1)$ $(k = 1,...,n)$, and for every $k = 2,...,n$

$$\left|H_{I_k}(j_1,...,j_{k-1})\right| = (n-k+1)k, \quad (j_1,...,j_{k-1}) \in I_{k-1}.$$

Therefore under these settings, for $k = 1,...,n$, (7.15) gives

$$M^2_{k,k}(L_1,.,L_m;\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(m)};\mathbf{p})$$
$$= N^{-1}\left(\frac{n}{kn(n-1)(n-k+1)}\sum_{(i_1,.,i_k)\in I_k}\left(\sum_{s=1}^{k} p_{i_s}\right)N\left(f\left(L_1(\mathbf{x}^{(1)};\mathbf{p}_{I_k}),...,L_m(\mathbf{x}^{(m)};\mathbf{p}_{I_k})\right)\right)\right).$$

If N is increasing and the function H defined in Theorem 7.1 is convex, then Theorem 7.3 gives

$$f\left(L_1(\mathbf{x}^{(1)};\mathbf{p};n),...,L_m(\mathbf{x}^{(m)};\mathbf{p};n)\right) \le M^2_{n,n}(L_1,...,L_m;\mathbf{x}^{(1)},...,\mathbf{x}^{(m)};\mathbf{p}) \le$$
$$\le ... \le M^2_{k,k}(L_1,\ldots,L_m;\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(m)};\mathbf{p}) \le ... \le$$
$$\le M^2_{1,1}(L_1,...,L_m;\mathbf{x}^{(1)},...,\mathbf{x}^{(m)};\mathbf{p}) = N^{-1}\left(\sum_{i=1}^{n} p_i N(f(x_i^{(1)},...,x_i^{(m)}))\right).$$

Assume (A_1) with positive n-tuple $\mathbf{p}$, ($\mathscr{H}_4$) and ($\mathscr{H}_5$). Let

$$L_t(\mathbf{x}^{(t)};c\mathbf{p};B) = L_t^{-1}\left(\frac{\sum_{s\in B} c(s)p_{\tau(s)}L_t(x^{(t)}_{\tau(s)})}{\sum_{s\in B} c(s)p_{\tau(s)}}\right), \quad t = 1,\ldots,m, \quad B \subset S,$$

and let

$$\mathbf{x}_i := \left(x_i^{(1)},\ldots,x_i^{(m)}\right), \quad i = 1,\ldots,n.$$

Then weighted mixed means corresponding to (7.10) and (7.11) are defined in the following ways:

$$M^1_k := M^1_k(L_1,...,L_m;\mathbf{x}^{(1)},...,\mathbf{x}^{(m)};c\mathbf{p}) :=$$
$$N^{-1}\left(\sum_{l=1}^{k}\left(\sum_{A\in\mathscr{A}_l}\left(\left(\sum_{s\in A} c(s)p_{\tau(s)}\right)N\left(f\left(L_1(\mathbf{x}^{(1)};c\mathbf{p};A),...,L_m(\mathbf{x}^{(m)};c\mathbf{p};A)\right)\right)\right)\right)\right),$$

and for $1 \le d \le k-1$

$$M^1_{k-d} := M^1_{k-d}(L_1,...,L_m;\mathbf{x}^{(1)},...,\mathbf{x}^{(m)};c\mathbf{p}) :=$$
$$N^{-1}\left(\begin{array}{l}\sum_{l=1}^{d}\left(\sum_{A\in\mathscr{A}_l}\left(\sum_{s\in A} c(s)p_{\tau(s)}N(f(\mathbf{x}_{\tau(s)}))\right)\right) + \sum_{l=d+1}^{k}\left(\frac{d!}{(l-1)...(l-d)}\right.\\ \left.\sum_{A\in\mathscr{A}_l}\left(\sum_{B\in P_{l-d}(A)}\left(\left(\sum_{s\in B} c(s)p_{\tau(s)}\right)N(f(L_1(\mathbf{x}^{(1)};c\mathbf{p};B),...,L_m(\mathbf{x}^{(m)};c\mathbf{p};B)))\right)\right)\right)\end{array}\right).$$

Now, we get an interpolation of (7.1) by the direct application of Theorem D as follows.

Theorem 7.4 *Assume (A_1) with a positive n-tuple* $\mathbf{p}$, *($\mathscr{H}_4$) and ($\mathscr{H}_5$). If N is a strictly increasing (decreasing) function, then the inequalities*

$$f\left(L_1(\mathbf{x}^{(1)};\mathbf{p};n),...,L_m(\mathbf{x}^{(m)};\mathbf{p};n)\right) \le M^1_k \le M^1_{k-1} \le ... \le$$
$$\le M^1_2 \le M^1_1 = N^{-1}\left(\sum_{i=1}^{n} p_i N(f(\mathbf{x}_i))\right), \tag{7.19}$$

hold for all possible $\mathbf{x}^{(t)}$ $(t = 1,\ldots,m)$ *and* $\mathbf{p}$*, if and only if the function* H *defined in Theorem 7.1 is convex (concave). If* N *is a strictly increasing (decreasing) function, then the inequalities in (7.19) are reversed for all possible* $\mathbf{x}^{(t)}$ $(t = 1,\ldots,m)$ *and* $\mathbf{p}$*, if and only if* H *is concave (convex).*

Proof. It comes from Theorem D and Theorem 7.1. We can apply Theorem D to the vectors

$$\left(L_1\left(x_i^{(1)}\right),\ldots,L_1\left(x_i^{(m)}\right)\right), \quad i = 1,\ldots,n,$$

and the function H if either H is convex and N is strictly increasing or H is concave and N is strictly decreasing. $-H$ is used if either H is convex and N is strictly decreasing or H is concave and N is strictly increasing. □

The following applications of Theorem 7.4 are based on Theorem 2.4, Example 2.9 and Example 2.10.

Example 7.7 Let $n \geq 1$ and $k \geq 1$ be fixed integers, and let $I_k \subset \{1,\ldots,n\}^k$ such that

$$\alpha_{I_k,i} \geq 1, \quad 1 \leq i \leq n,$$

where $\alpha_{I_k,i}$ means the number of occurrences of i in the sequences $\mathbf{i}_k := (i_1,\ldots,i_k) \in I_k$. For $j = 1,\ldots,n$ we introduce the sets

$$S_j := \{((i_1,\ldots,i_k),l) \mid (i_1,\ldots,i_k) \in I_k, \quad 1 \leq l \leq k, \quad i_l = j\}.$$

Let c be a positive function on $S := \bigcup_{j=1}^{n} S_j$ such that

$$\sum_{((i_1,\ldots,i_k),l)\in S_j} c((i_1,\ldots,i_k),l) = 1, \quad j = 1,\ldots,n.$$

Assume (A_1) with positive n-tuple $\mathbf{p}$. Then the corresponding weighted mixed means are

$$M_k^1 := N^{-1}\left(\sum_{(i_1,\ldots,i_k)\in I_k}\left(\begin{array}{l}\left(\sum_{l=1}^{k} c((i_1,\ldots,i_k),l)\,p_{i_l}\right)\\ \times N(f(L_1(\mathbf{x}^{(1)};c\mathbf{p};\mathbf{i}_k),\ldots,L_m(\mathbf{x}^{(m)};c\mathbf{p};\mathbf{i}_k)))\end{array}\right)\right),$$

where

$$L_t(\mathbf{x}^{(t)};c\mathbf{p};\mathbf{i}_k) = L_t^{-1}\left(\frac{\sum_{l=1}^{k} c((i_1,\ldots,i_k),l)\,p_{i_l}L_t(x_{i_l}^{(t)})}{\sum_{l=1}^{k} c((i_1,\ldots,i_k),l)\,p_{i_l}}\right); \mathbf{i}_k \in I_k, \quad 1 \leq t \leq m,$$

while for $1 \leq d \leq k-1$,

$$M_{k-d}^1 :=$$

$$N^{-1}\left\{\begin{array}{l}\left\{\frac{d!}{(k-1)\ldots(k-d)}\sum_{(i_1,\ldots,i_k)\in I_k}\left(\sum_{1\leq l_1<\ldots<l_{k-d}\leq k}\left(\left(\sum_{j=1}^{k-m} c((i_1,\ldots,i_k),l_j)\,p_{i_{l_j}}\right)\right.\right.\right.\\ \left.\left.\times N(f(L_1(\mathbf{x}^{(1)};c\mathbf{p};\mathbf{i}_k;\mathbf{l}_{k-d}),\ldots,L_m(\mathbf{x}^{(m)};c\mathbf{p};\mathbf{i}_k;\mathbf{l}_{k-d})))\right)\right)\end{array}\right\},$$

where

$$L_t(\mathbf{x}^{(t)};c\mathbf{p};\mathbf{i}_k;\mathbf{l}_{k-d}) = L_t^{-1}\left(\frac{\sum\limits_{j=1}^{k-d} c\left((i_1,\ldots,i_k),l_j\right)p_{i_{l_j}}L_t(x_{i_{l_j}}^{(t)})}{\sum\limits_{j=1}^{k-d} c\left((i_1,\ldots,i_k),l_j\right)p_{i_{l_j}}}\right), \quad \begin{array}{c} 1\le l_1<\ldots<l_{k-d}\le k, \\ 1\le t\le m. \end{array} \tag{7.20}$$

If N is strictly increasing and the function H defined in Theorem 7.1 is convex, then Theorem 7.2 gives

$$\begin{aligned} f\left(L_1(\mathbf{x}^{(1)};\mathbf{p};n),\ldots,L_m(\mathbf{x}^{(m)};\mathbf{p};n)\right) &\le M_k^1 \le M_{k-1}^1 \le \ldots \le \\ \le M_2^1 \le M_1^1 = N^{-1}&\left(\sum_{i=1}^n p_i N(f(x_i^{(1)},\ldots,x_i^{(m)}))\right). \end{aligned} \tag{7.21}$$

Taking

$$c((i_1,\ldots,i_k),l)=\frac{1}{|S_j|}=\frac{1}{\alpha_{I_k,j}};\quad ((i_1,\ldots,i_k),l)\in S_j,$$

in (7.21) we get Theorem 2.1 of [38].

Example 7.8 We summarize the essence of Example 2.9.

Let n, r be fixed integers, where $n\ge 3$, and $1\le r\le n-2$. In this example, for every $i=1,2,\ldots,n$ and for every $l=0,1,\ldots,r$ the integer $i+l$ will be identified with the uniquely determined integer j from $\{1,\ldots,n\}$ for which

$$l+i\equiv j \quad (mod\ n).$$

Introducing the notation

$$D:=\{1,\ldots,n\}\times\{0,\ldots,r\},$$

let for every $j\in\{1,\ldots,n\}$

$$S_j:=\{(i,l)\in D \mid i+l\equiv j \quad (mod\ n)\}\bigcup\{j\},$$

and let $\mathscr{A}\subset P(S)$ $(S:=\bigcup\limits_{j=1}^n S_j)$ contain the following sets:

$$A_i:=\{(i,l)\in D \mid l=0,\ldots,r\},\quad i=1,\ldots,n$$

and

$$A:=\{1,\ldots,n\}.$$

Let c be a positive function on S such that

$$\sum_{(i,l)\in S_j} c(i,l)+c(j)=1,\quad j=1,\ldots,n.$$

Then the sets $S_1,\dots,S_n$, the partition $\mathscr{A}$, and the function c defined above satisfy the conditions $(\mathscr{H}_4)$ and $(\mathscr{H}_5)$.

Assume (A_1) with positive n-tuple $\mathbf{p}$. If N is increasing and the function H defined in Theorem 7.1 is convex, then from Theorem 7.2 we get

$$\begin{aligned}
& f\left(L_1(\mathbf{x}^{(1)};\mathbf{p};n),\dots,L_m(\mathbf{x}^{(m)};\mathbf{p};n)\right) \\
& \leq N^{-1}\left\{\begin{array}{l} \sum\limits_{i=1}^{n}\left(\sum\limits_{l=0}^{r} c(i,l)\,p_{i+l}\right) N\left(f\left(L_1(\mathbf{x}^{(1)},c\mathbf{p};i),\dots,L_m(\mathbf{x}^{(m)},c\mathbf{p};i)\right)\right) \\ +\left(\sum\limits_{j=1}^{n} c(j)p_j\right) N\left(f\left(L_1(\mathbf{x}^{(1)},c\mathbf{p}),\dots,L_m(\mathbf{x}^{(m)},c\mathbf{p})\right)\right) \end{array}\right\} \\
& \leq N^{-1}\left(\sum_{i=1}^{n} p_i N(f(x_i{}^{(1)},\dots,x_i^{(m)}))\right),
\end{aligned}$$

where

$$L_t(\mathbf{x}^{(t)},c\mathbf{p};i) = L_t^{-1}\left(\frac{\sum\limits_{l=0}^{r} c(i,l)\,p_{i+l}L_t(x_{i+l}^{(t)})}{\sum\limits_{l=0}^{r} c(i,l)\,p_{i+l}}\right), \quad 1\leq i\leq n, \quad 1\leq t\leq m,$$

and

$$L_t(\mathbf{x}^{(t)},c\mathbf{p}) = L_t^{-1}\left(\frac{\sum\limits_{j=1}^{n} c(j)p_jL_t(x_j^{(t)})}{\sum\limits_{j=1}^{n} c(j)p_j}\right), \quad 1\leq t\leq m.$$

Example 7.9 We describe the basic situation in Example 2.10.

Let n and k be fixed positive integers. Let

$$D := \{(i_1,\dots,i_n)\in\{1,\dots,k\}^n \mid i_1+\dots+i_n = n+k-1\},$$

and for each $j=1,\dots,n$, denote S_j the set

$$S_j := D\times\{j\}.$$

For every $\mathbf{i}_n := (i_1,\dots,i_n)\in D$ designate by $A_{(i_1,\dots,i_n)}$ the set

$$A_{(i_1,\dots,i_n)} := \{((i_1,\dots,i_n),l) \mid l=1,\dots,n\}.$$

It is obvious that S_j $(j=1,\dots,n)$ and $A_{(i_1,\dots,i_n)}$ $((i_1,\dots,i_n)\in D)$ are decompositions of $S := \bigcup\limits_{j=1}^{n} S_j$ into pairwise disjoint and nonempty sets, respectively. Let c be a function on S such that

$$c((i_1,\dots,i_n),j) > 0, \quad ((i_1,\dots,i_n),j)\in S$$

and

$$\sum_{(i_1,\dots,i_n)\in D} c((i_1,\dots,i_n),j) = 1, \quad j=1,\dots,n.$$

Then we have that the conditions ($\mathscr{H}_4$) and ($\mathscr{H}_5$) are satisfied.

Assume (A$_1$) with positive n-tuple $\mathbf{p}$. If N is strictly increasing and the function H defined in Theorem 7.1 is convex, then from Theorem 7.2 we get

$$\begin{aligned}
&f\left(L_1(\mathbf{x}^{(1)};\mathbf{p};n),\ldots,L_m(\mathbf{x}^{(m)};\mathbf{p};n)\right)\\
&\leq N^{-1}\left(\sum_{(i_1,\ldots,i_n)\in D}\left(\left(\sum_{l=1}^{n} c\left((i_1,\ldots,i_n),l\right)p_l\right)N\left(f\left(L_1(\mathbf{x}^{(1)},c\mathbf{p};\mathbf{i}_n),\ldots,L_m(\mathbf{x}^{(m)},c\mathbf{p};\mathbf{i}_n)\right)\right)\right)\right)\\
&\leq N^{-1}\left(\sum_{i=1}^{n} p_i N(f(x_i{}^{(1)},\ldots,x_i^{(m)}))\right),
\end{aligned}$$

where

$$L_t(\mathbf{x}^{(t)},c\mathbf{p};\mathbf{i}_n)=L_t^{-1}\left(\frac{\sum_{l=1}^{n} c\left((i_1,\ldots,i_n),l\right)p_l L_t(x_l^{(t)})}{\sum_{l=1}^{n} c\left((i_1,\ldots,i_n),l\right)p_l}\right);\quad \mathbf{i}_n\in D,\quad 1\leq t\leq m.$$

Now assume (A$_1$), consider a real number $\lambda\geq 1$, and let S_k be the set defined in (7.12). Then the mixed means corresponding to (7.13) are

$$\begin{aligned}
&M_k^2(\lambda):=M_k^2(L_1,\ldots,L_m;\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(m)};\mathbf{p};\lambda):=\\
&N^{-1}\left(\begin{array}{l}\frac{1}{(n+\lambda-1)^k}\sum_{(i_1,\ldots,i_n)\in S_k}\left(\frac{k!}{i_1!\ldots i_n!}\left(\sum_{j=1}^{n}\lambda^{i_j}p_j\right)\right.\\
\left.\times N\left(f\left(L_1(\mathbf{x}^{(1)};\mathbf{p};\mathbf{i}_{n,k};\lambda),\ldots,L_m(\mathbf{x}^{(m)};\mathbf{p};\mathbf{i}_{n,k};\lambda)\right)\right)\right)\end{array}\right),
\end{aligned}$$

where

$$L_t(\mathbf{x}^{(t)};\mathbf{p};\mathbf{i}_{n,k};\lambda)=L_t^{-1}\left(\frac{\sum_{j=1}^{n}\lambda^{i_j}p_j L_t(x_j^{(t)})}{\sum_{j=1}^{n}\lambda^{i_j}p_j}\right),\quad \mathbf{i}_{n,k}\in S_k,\quad 1\leq t\leq m.$$

In this case Theorem E gives another interpolation of (7.1) as follows:

Theorem 7.5 *Assume (A$_1$), let $\lambda\geq 1$ be a real number, and let S_k be the set defined in (7.12). If N is a strictly increasing (decreasing) function, then the inequalities*

$$\begin{aligned}
&f\left(L_1(\mathbf{x}^{(1)};\mathbf{p};n),\ldots,L_m(\mathbf{x}^{(m)};\mathbf{p};n)\right)=M_0^2(\lambda)\leq M_1^2(\lambda)\leq\ldots\leq\\
&\leq M_k^2(\lambda)\leq\ldots\leq N^{-1}\left(\sum_{i=1}^{n} p_i N(f(x_i^{(1)},\ldots,x_i^{(m)}))\right),\quad k\in\mathbb{N},
\end{aligned}\tag{7.22}$$

hold for all possible $\mathbf{x}^{(t)}$ $(t=1,\ldots,m)$ and $\mathbf{p}$, if and only if the function H defined in Theorem 7.1 is convex (concave). If N is an increasing (decreasing) function, then the inequalities in (7.22) are reversed for all possible $\mathbf{x}^{(t)}$ $(t=1,\ldots,m)$ and $\mathbf{p}$, if and only if H is concave (convex).

Proof. Similar to the proof of Theorem 7.4. □

7.2 Generalizations of the Consequences of Beck's Result

Assume (A_2) and ($\mathscr{H}_0$). Then, for $m = 2$, the reverse of (7.14) can be written as

$$\begin{aligned} & f\left(K_n(\mathbf{a};\mathbf{p}), L_n(\mathbf{b};\mathbf{p})\right) \geq M^1_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq M^1_{k,k-1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq \cdots \geq \\ & \geq M^1_{k,2}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq M^1_{k,1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = N^{-1}\left(\sum_{i=1}^{n} p_i N(f(a_i,b_i))\right). \end{aligned} \tag{7.23}$$

Analogous to the results of Corollary A and Corollary B (see [58] and also [60, p.195]), we have immediately from Theorem 7.2 and Corollaries 7.1, 7.2 that

Corollary 7.3 *Assume (A_3) with $f(x,y) = x+y$ ($(x,y) \in I_K \times I_L$), assume ($\mathscr{H}_0$), and assume that K', L', N', K'', L'' and N'' are all positive. Introducing $E := \frac{K'}{K''}$, $F := \frac{L'}{L''}$, $G := \frac{N'}{N''}$, (7.23) holds for all possible* **a**, **b** *and* **p** *if and only if*

$$E(x) + F(y) \leq G(x+y), \quad (x,y) \in I_K^\circ \times I_L^\circ.$$

In this case

$$\begin{aligned} & M^1_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = \\ & N^{-1}\left(\sum_{\mathbf{i}^k \in I_k}\left(\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right) N\left(K(\mathbf{a};\frac{\mathbf{p}}{\alpha_{I_k}};k) + L(\mathbf{b};\frac{\mathbf{p}}{\alpha_{I_k}};k)\right)\right), \end{aligned} \tag{7.24}$$

and for $k-1 \geq l \geq 1$

$$\begin{aligned} & M_{k,l}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = \\ & N^{-1}\left(\frac{1}{(k-1)\ldots l}\sum_{\mathbf{i}^l \in I_l} t_{I_k,l}(\mathbf{i}^l)\left(\sum_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right) N\left(K(\mathbf{a};\frac{\mathbf{p}}{\alpha_{I_l}};l) + L(\mathbf{b};\frac{\mathbf{p}}{\alpha_{I_l}};l)\right)\right), \end{aligned} \tag{7.25}$$

respectively, where $\mathbf{i}^l := (i_1,\ldots,i_l)$.

Corollary 7.4 *Assume (A_3) with $f(x,y) = xy$ ($(x,y) \in I_K \times I_L$) and assume ($\mathscr{H}_0$). Suppose the functions $A(x) := \frac{K'(x)}{K'(x)+xK''(x)}$, $B(x) := \frac{L'(x)}{L'(x)+xL''(x)}$ and $C(x) := \frac{N'(x)}{N'(x)+xN''(x)}$ are defined on I_K°, I_L° and I_N° respectively. Assume further that K', L', M', A, B and C are all positive. Then (7.23) holds for all possible* **a**, **b** *and* **p** *if and only if*

$$A(x) + B(y) \leq C(xy), \quad (x,y) \in I_K^\circ \times I_L^\circ.$$

In this case

$$\begin{aligned} & M^1_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = \\ & N^{-1}\left(\sum_{\mathbf{i}^k \in I_k}\left(\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right) N\left(K(\mathbf{a};\frac{\mathbf{p}}{\alpha_{I_k}};k)L(\mathbf{b};\frac{\mathbf{p}}{\alpha_{I_k}};k)\right)\right), \end{aligned} \tag{7.26}$$

and for $k-1 \geq l \geq 1$

$$M_{k,l}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = \\ N^{-1}\left(\frac{1}{(k-1)\dots l}\sum_{\mathbf{i}^l\in I_l} t_{I_k,l}(\mathbf{i}^l)\left(\sum_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right)N\left(K(\mathbf{a};\tfrac{\mathbf{p}}{\alpha_{I_l}};l)L(\mathbf{b};\tfrac{\mathbf{p}}{\alpha_{I_l}};l)\right)\right), \tag{7.27}$$

respectively, where $\mathbf{i}^l := (i_1,\dots,i_l)$.

We also give some special cases of the Corollaries 7.3 and 7.4 as illustrations.

Remark 7.2 Under the settings of Example 7.1, if $f(x_1,x_2) = x_1 + x_2$, then (7.24) becomes

$$M^1_{2,2}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \\ = N^{-1}\left(\sum_{(i_1,i_2)\in I_2}\left(\sum_{s=1}^{2}\frac{p_{i_s}}{\left[\frac{n}{i_s}\right]+d(i_s)}\right)N\left(K(\mathbf{a};\tfrac{\mathbf{p}}{\alpha_{I_2}})+L(\mathbf{b};\tfrac{\mathbf{p}}{\alpha_{I_2}})\right)\right).$$

Under the conditions of Corollary 7.3

$$K_n(\mathbf{a};\mathbf{p})+L_n(\mathbf{a};\mathbf{p}) \geq M^1_{2,2}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq N^{-1}\left(\sum_{i=1}^{n}p_iN(a_i+b_i)\right).$$

Similarly, if $f(x_1,x_2) = x_1x_2$, then from (7.26) we have

$$M^1_{2,2}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \\ = N^{-1}\left(\sum_{(i_1,i_2)\in I_2}\left(\sum_{s=1}^{2}\frac{p_{i_s}}{\left[\frac{n}{i_s}\right]+d(i_s)}\right)N\left(K(\mathbf{a};\tfrac{\mathbf{p}}{\alpha_{I_2}})L(\mathbf{b};\tfrac{\mathbf{p}}{\alpha_{I_2}})\right)\right).$$

Under the conditions of Corollary 7.4

$$K_n(\mathbf{a};\mathbf{p})L_n(\mathbf{a};\mathbf{p}) \geq M^1_{2,2}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq N^{-1}\left(\sum_{i=1}^{n}p_iN(a_ib_i)\right).$$

Remark 7.3 Under the settings of Example 7.2, if $f(x_1,x_2) = x_1 + x_2$ then (7.25) becomes

$$M^1_{k,k-1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = N^{-1}\left(\frac{1}{k-1}\sum_{i=1}^{n}(c_i-p_i)N\left(K(\mathbf{a};\tfrac{\mathbf{p}}{\mathbf{c}_i})+L(\mathbf{b};\tfrac{\mathbf{p}}{\mathbf{c}_i})\right)\right),$$

Under the conditions of Corollary 7.3

$$K_n(\mathbf{a};\mathbf{p})+L_n(\mathbf{a};\mathbf{p}) \geq M^1_{k,k-1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq N^{-1}\left(\sum_{i=1}^{n}p_iN(a_i+b_i)\right).$$

Similarly if $f(x_1,x_2) = x_1x_2$ then from (7.27) we have

$$M^1_{k,k-1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \\ = N^{-1}\left(\frac{1}{k-1}\sum_{i=1}^{n}(c_i-p_i)N\left(K(\mathbf{a};\tfrac{\mathbf{p}}{\mathbf{c}_i})L(\mathbf{b};\tfrac{\mathbf{p}}{\mathbf{c}_i})\right)\right),$$

Under the conditions of Corollary 7.4

$$K_n(\mathbf{a};\mathbf{p})L_n(\mathbf{a};\mathbf{p}) \geq M^1_{k,k-1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq N^{-1}\left(\sum_{i=1}^{n} p_i N(a_i b_i)\right).$$

Next, assume (A_2), ($\mathscr{H}_0$) and suppose $|H_{I_l}(j_1,...,j_{l-1})| = \beta_{l-1}$ for any $(j_1,...,j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$. For $m=2$, the reverse of (7.16) becomes

$$\begin{aligned} & f\left(K_n(\mathbf{a};\mathbf{p}),L_n(\mathbf{b};\mathbf{p})\right) \geq M^2_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq M^2_{k-1,k-1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \\ & \geq ... \geq M^2_{2,2}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq M^2_{1,1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = N^{-1}\left(\sum_{i=1}^{n} p_i N(f(a_i,b_i))\right), \end{aligned} \tag{7.28}$$

where

$$\begin{aligned} & M^2_{l,l}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \\ & = N^{-1}\left(\frac{n}{l|I_l|}\sum_{\mathbf{i}^l \in I_l}^{n}\left(\sum_{s=1}^{l} p_{i_s}\right) N\left(f(K(\mathbf{a};\mathbf{p}_{I_l}),L(\mathbf{b};\mathbf{p}_{I_l}))\right)\right) \end{aligned}$$

for $k \geq l \geq 1$.
Now using Theorem 7.3 (for $m=2$) and Corollaries 7.1, 7.2, we get generalizations of Beck's results in [9] (see [58] and also [60, p.195]).

Corollary 7.5 *Assume (A_3) with $f(x,y) = x+y$ $((x,y) \in I_K \times I_L)$,assume ($\mathscr{H}_0$), and suppose $|H_{I_l}(j_1,...,j_{l-1})| = \beta_{l-1}$ for any $(j_1,...,j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$. Assume further that K', L', N', K'', L'' and N'' are all positive. Introducing $E := \frac{K'}{K''}$, $F := \frac{L'}{L''}$, $G := \frac{N'}{N''}$, (7.28) holds for all possible* **a**, **b** *and* **p** *if and only if*

$$E(x) + F(y) \leq G(x+y), \quad (x,y) \in I_K^\circ \times I_L^\circ.$$

In this case for $k \geq l \geq 1$

$$\begin{aligned} & M^2_{l,l}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \\ & = N^{-1}\left(\frac{n}{l|I_l|}\sum_{\mathbf{i}^l \in I_l}^{n}\left(\sum_{s=1}^{l} p_{i_s}\right) N\left(K(\mathbf{a};\mathbf{p}_{I_l}) + L(\mathbf{b};\mathbf{p}_{I_l})\right)\right), \end{aligned} \tag{7.29}$$

where $\mathbf{i}^l := (i_1,...,i_l)$.

Corollary 7.6 *Assume (A_3) with $f(x,y) = xy$ $((x,y) \in I_K \times I_L)$, assume ($\mathscr{H}_0$), and suppose $|H_{I_l}(j_1,...,j_{l-1})| = \beta_{l-1}$ for any $(j_1,...,j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$. Suppose the functions $A(x) := \frac{K'(x)}{K'(x)+xK''(x)}$, $B(x) := \frac{L'(x)}{L'(x)+xL''(x)}$ and $C(x) := \frac{N'(x)}{N'(x)+xN''(x)}$ are defined on I_K°, I_L° and I_N° respectively. Assume further that K', L', M', A, B and C are all positive. Then (7.28) holds for all possible* **a**, **b** *and* **p** *if and only if*

$$A(x) + B(y) \leq C(xy), \quad (x,y) \in I_K^\circ \times I_L^\circ.$$

In this case for $k \geq l \geq 1$

$$M_{l,l}^2(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = N^{-1}\left(\frac{n}{l|I_l|}\sum_{\mathbf{i}^l \in I_l}^{n}\left(\sum_{s=1}^{l} p_{i_s}\right) N\left(K(\mathbf{a};\mathbf{p}_{I_l}) + L(\mathbf{b};\mathbf{p}_{I_l})\right)\right), \tag{7.30}$$

where $\mathbf{i}^l := (i_1,...,i_l)$.

The special cases correspond to Examples 7.3, 7.4, 7.5 and 7.6 are as follows:

Remark 7.4 Under the settings of Example 7.3, for $n \geq k \geq 1$, (7.29) becomes

$$N_{k,k}^2(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = N^{-1}\left(\frac{1}{\binom{n-1}{k-1}}\sum_{1\leq i_1<...<i_k\leq n}\left(\sum_{s=1}^{k} p_{i_s}\right) N\left(K(\mathbf{a};\mathbf{p}_{I_k}) + L(\mathbf{b};\mathbf{p}_{I_k})\right)\right).$$

Under the conditions of Corollary 7.5

$$K_n(\mathbf{a};\mathbf{p}) + L_n(\mathbf{b};\mathbf{p}) \geq M_{k,k}^2(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq M_{k-1,k-1}^2(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq ... \geq M_{2,2}^2(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq M_{1,1}^2(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = N^{-1}\left(\sum_{i=1}^{n} p_i N(a_i+b_i)\right).$$

Similarly if $f(x_1,x_2) = x_1x_2$ then for $n \geq k \geq 1$, (7.30) can be written as

$$M_{k,k}^2(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = N^{-1}\left(\frac{1}{\binom{n-1}{k-1}}\sum_{1\leq i_1<...<i_k\leq n}\left(\sum_{s=1}^{k} p_{i_s}\right) N\left(K(\mathbf{a};\mathbf{p}_{I_k})L(\mathbf{b};\mathbf{p}_{I_k})\right)\right).$$

Under the conditions of Corollary 7.6

$$K_n(\mathbf{a};\mathbf{p})L_n(\mathbf{b};\mathbf{p}) \geq M_{k,k}^2(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq M_{k-1,k-1}^2(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq ... \geq M_{2,2}^2(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq M_{1,1}^2(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = N^{-1}\left(\sum_{i=1}^{n} p_i N(a_ib_i)\right).$$

Remark 7.5 Under the settings of Example 7.4, for $k \geq 1$, (7.29) becomes

$$M_{k,k}^2(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = N^{-1}\left(\frac{1}{\binom{n+k-1}{k}}\sum_{1\leq i_1\leq...\leq i_k\leq n}\left(\sum_{s=1}^{k} p_{i_s}\right) N\left(K(\mathbf{a};\mathbf{p}_{I_k}) + L(\mathbf{b};\mathbf{p}_{I_k})\right)\right),$$

Under the conditions of Corollary 7.5

$$K_n(\mathbf{a};\mathbf{p})+L_n(\mathbf{b};\mathbf{p}) \geq M^2_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq M^2_{k-1,k-1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p})$$
$$\geq \ldots \geq M^2_{2,2}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq M^2_{1,1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = N^{-1}\left(\sum_{i=1}^{n} p_i N(a_i+b_i)\right).$$

Similarly if $f(x_1,x_2)=x_1x_2$ then for $k\geq 1$, (7.30) can be written as

$$M^2_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p})$$
$$= N^{-1}\left(\frac{1}{\binom{n+k-1}{k}} \sum_{1\leq i_1\leq\ldots\leq i_k\leq n}\left(\sum_{s=1}^{k} p_{i_s}\right) N\left(K(\mathbf{a};\mathbf{p}_{I_k})L(\mathbf{b};\mathbf{p}_{I_k})\right)\right),$$

Under the conditions of Corollary 7.6

$$K_n(\mathbf{a};\mathbf{p})L_n(\mathbf{b};\mathbf{p}) \geq M^2_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p})$$
$$\geq \ldots \geq M^2_{k,1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = N^{-1}\left(\sum_{i=1}^{n} p_i N(a_i b_i)\right).$$

Remark 7.6 Under the settings of Example 7.5, for $k\geq 1$, (7.29) becomes

$$M^2_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p})$$
$$= N^{-1}\left(\frac{1}{kn^{k-1}} \sum_{(i_1,\ldots,i_k)\in I_k}\left(\sum_{s=1}^{k} p_{i_s}\right) N\left(K(\mathbf{a};\mathbf{p}_{I_k})+L(\mathbf{b};\mathbf{p}_{I_k})\right)\right).$$

Under the conditions of Corollary 7.5

$$K_n(\mathbf{a};\mathbf{p})+L_n(\mathbf{b};\mathbf{p}) \geq \ldots \geq M^2_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p})$$
$$\geq \ldots \geq M^2_{1,1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = N^{-1}\left(\sum_{i=1}^{n} p_i N(a_i+b_i)\right).$$

Similarly if $f(x_1,x_2)=x_1x_2$ then for $k\geq 1$, (7.30) can be written as

$$M^2_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p})$$
$$= N^{-1}\left(\frac{1}{kn^{k-1}} \sum_{(i_1,\ldots,i_k)\in I_k}\left(\sum_{s=1}^{k} p_{i_s}\right) N\left(K(\mathbf{a};\mathbf{p}_{I_k})L(\mathbf{b};\mathbf{p}_{I_k})\right)\right).$$

Under the conditions of Corollary 7.6 gives

$$K_n(\mathbf{a};\mathbf{p})L_n(\mathbf{b};\mathbf{p}) \geq \ldots \geq M^2_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p})$$
$$\geq \ldots \geq M^2_{1,1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = N^{-1}\left(\sum_{i=1}^{n} p_i N(a_i b_i)\right).$$

Remark 7.7 Under the settings of Example 7.6, if $f(x_1,x_2) = x_1 + x_2$ then for $1 \leq k \leq n$, (7.29) becomes

$$\begin{aligned}
&M^2_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \\
&= N^{-1}\left(\frac{n}{kn(n-1)\ldots(n-k+1)}\sum_{(i_1,\ldots,i_k)\in I_k}\left(\sum_{s=1}^{k} p_{i_s}\right)N\left(K(\mathbf{a};\mathbf{p}_{I_k}) + L(\mathbf{b};\mathbf{p}_{I_k})\right)\right).
\end{aligned}$$

Under the conditions of Corollary 7.5

$$\begin{aligned}
&K_n(\mathbf{a};\mathbf{p}) + L_n(\mathbf{b};\mathbf{p}) \geq M^2_{n,n}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq \ldots \geq M^2_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq \\
&\geq \ldots \geq M^2_{1,1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = N^{-1}\left(\sum_{i=1}^{n} p_i N(a_i + b_i)\right).
\end{aligned}$$

Similarly if $f(x_1,x_2) = x_1x_2$ then for $1 \leq k \leq n$, (7.30) can be written as

$$\begin{aligned}
&M^2_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \\
&= N^{-1}\left(\frac{n}{kn(n-1)\ldots(n-k+1)}\sum_{(i_1,\ldots,i_k)\in I_k}\left(\sum_{s=1}^{k} p_{i_s}\right)N\left(K(\mathbf{a};\mathbf{p}_{I_k})L(\mathbf{b};\mathbf{p}_{I_k})\right)\right).
\end{aligned}$$

Under the conditions of Corollary 7.6

$$\begin{aligned}
&K_n(\mathbf{a};\mathbf{p})L_n(\mathbf{b};\mathbf{p}) \geq M^2_{n,n}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq \ldots \geq M^2_{k,k}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) \geq \\
&\geq \ldots \geq M^2_{1,1}(K,L;\mathbf{a},\mathbf{b};\mathbf{p}) = N^{-1}\left(\sum_{i=1}^{n} p_i N(a_i b_i)\right).
\end{aligned}$$

Assume (A_2) with positive n-tuple $\mathbf{p}$, ($\mathscr{H}_4$) and ($\mathscr{H}_5$). Then for $m = 2$, the reverse of (7.19) can be written as

$$f\left(K_n(\mathbf{a};\mathbf{p}), L_n(\mathbf{b};\mathbf{p})\right) \geq M^1_k \geq M^1_{k-1} \geq \ldots \geq M^1_1 = N^{-1}\left(\sum_{j=1}^{n} p_j N(f(a_j,b_j))\right). \tag{7.31}$$

Analogous to the results of Corollary A and Corollary B (see [59] and also [60, p.195]), we have immediately from Theorem 7.4 and Corollaries 7.1, 7.2 that

Corollary 7.7 *Assume (A_3) with $f(x,y) = x + y$ ($(x,y) \in I_K \times I_L$) and with positive n-tuple $\mathbf{p}$, assume ($\mathscr{H}_4$)-($\mathscr{H}_5$), and assume that K', L', N', K'', L'' and N'' are all positive. Introducing $E := \frac{K'}{K''}$, $F := \frac{L'}{L''}$, $G := \frac{N'}{N''}$, (7.31) holds for all possible $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{p}$ if and only if*

$$E(x) + F(y) \leq G(x+y), \quad (x,y) \in I^\circ_K \times I^\circ_L.$$

In this case

$$\begin{aligned}
&M^1_k := M^1_k(K,L;\mathbf{a},\mathbf{b};c\mathbf{p}) := \\
&N^{-1}\left(\sum_{l=1}^{k}\left(\sum_{A\in\mathscr{A}_l}\left(\left(\sum_{s\in A} c(s)p_{\tau(s)}\right)N\left((K(\mathbf{a};c\mathbf{p};A) + L(\mathbf{b};c\mathbf{p};A))\right)\right)\right)\right),
\end{aligned} \tag{7.32}$$

and for $1 \le d \le k-1$

$$M_{k-d}^1 := M_{k-d}^1(K,L;\mathbf{a},\mathbf{b};c\mathbf{p}) :=$$
$$N^{-1}\left\{\begin{array}{l}\sum\limits_{l=1}^{d}\left(\sum\limits_{A\in\mathscr{A}_l}\left(\sum\limits_{s\in A}c(s)p_{\tau(s)}N(a_{\tau(s)}+b_{\tau(s)})\right)\right)+\sum\limits_{l=d+1}^{k}\left(\frac{d!}{(l-1)\dots(l-d)}\right.\\ \left.\sum\limits_{A\in\mathscr{A}_l}\left(\sum\limits_{B\in P_{l-d}(A)}\left(\left(\sum\limits_{s\in B}c(s)p_{\tau(s)}\right)N(K(\mathbf{a};c\mathbf{p};B)+L(\mathbf{b};c\mathbf{p};B))\right)\right)\right)\end{array}\right\}. \tag{7.33}$$

Corollary 7.8 *Assume* ($\mathscr{H}_4$), ($\mathscr{H}_5$) *and consider* (A_3) *with* $f(x,y) = xy$ $((x,y) \in I_K \times I_L)$ *and with positive n-tuple* $\mathbf{p}$*. Suppose the functions* $A(x) := \frac{K'(x)}{K'(x)+xK''(x)}$, $B(x) := \frac{L'(x)}{L'(x)+xL''(x)}$ *and* $C(x) := \frac{N'(x)}{N'(x)+xN''(x)}$ *are defined on* I_K°, I_L° *and* I_N° *respectively. Assume further that* K', L', M', *A, B and C are all positive. Then (7.31) holds for all possible* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{p}$ *if and only if*

$$A(x)+B(y) \le C(xy), \quad (x,y) \in I_K^\circ \times I_L^\circ.$$

In this case

$$M_k^1 := M_k^1(K,L;\mathbf{a},\mathbf{b};c\mathbf{p}) :=$$
$$N^{-1}\left(\sum_{l=1}^{k}\left(\sum_{A\in\mathscr{A}_l}\left(\left(\sum_{s\in A}c(s)p_{\tau(s)}\right)N\left((K(\mathbf{a};c\mathbf{p};A)L(\mathbf{b};c\mathbf{p};A))\right)\right)\right)\right), \tag{7.34}$$

and for $1 \le d \le k-1$,

$$M_{k-d}^1 := M_{k-d}^1(K,L;\mathbf{a},\mathbf{b};c\mathbf{p}) :=$$
$$N^{-1}\left\{\begin{array}{l}\sum\limits_{l=1}^{d}\left(\sum\limits_{A\in\mathscr{A}_l}\left(\sum\limits_{s\in A}c(s)p_{\tau(s)}N(a_{\tau(s)}b_{\tau(s)})\right)\right)+\sum\limits_{l=d+1}^{k}\left(\frac{d!}{(l-1)\dots(l-d)}\right.\\ \left.\sum\limits_{A\in\mathscr{A}_l}\left(\sum\limits_{B\in P_{l-d}(A)}\left(\left(\sum\limits_{s\in B}c(s)p_{\tau(s)}\right)N(K(\mathbf{a};c\mathbf{p};B)L(\mathbf{b};c\mathbf{p};B))\right)\right)\right)\end{array}\right\}. \tag{7.35}$$

Under the considerations of examples in Section 2, we show some special cases of the Corollaries 7.7 and 7.8.

Remark 7.8 Under the settings of Example 7.7, if $f(x_1,x_2) = x_1+x_2$, then (7.32) becomes

$$M_k^1 := M_k^1(K,L;\mathbf{a},\mathbf{b};c\mathbf{p}) :=$$
$$N^{-1}\left(\sum_{(i_1,\dots,i_k)\in I_k}\left(\left(\sum_{l=1}^{k}c\left((i_1,\dots,i_k),l\right)p_{i_l}\right)N(K(\mathbf{a};c\mathbf{p};\mathbf{i}_k)+L(\mathbf{b};c\mathbf{p};\mathbf{i}_k))\right)\right),$$

and for $1 \le d \le k-1$ (7.33) becomes

$$M_{k-d}^1 := M_k^1(K,L;\mathbf{a},\mathbf{b};c\mathbf{p}) :=$$
$$N^{-1}\left\{\begin{array}{l}\dfrac{d!}{(k-1)\dots(k-d)}\sum\limits_{(i_1,\dots,i_k)\in I_k}\left(\sum\limits_{1\le l_1<\dots<l_{k-d}\le k}\left(\left(\sum\limits_{j=1}^{k-m}c((i_1,\dots,i_k),l_j)p_{i_{l_j}}\right)\right.\right.\\ \left.\left.\times N(K(\mathbf{a};c\mathbf{p};\mathbf{i}_k;\mathbf{l}_{k-d})+L(\mathbf{b};c\mathbf{p};\mathbf{i}_k;\mathbf{l}_{k-d}))\right)\right)\end{array}\right\}$$

Under the conditions of Corollary 7.7, we have

$$K_n(\mathbf{a};\mathbf{p})+L_n(\mathbf{a};\mathbf{p}) \geq M_k^1 \geq M_{k-1}^1 \geq \ldots \geq M_1^1 = N^{-1}\left(\sum_{i=1}^{n} p_i N(a_i+b_i)\right). \tag{7.36}$$

Similarly, if $f(x_1,x_2)=x_1x_2$, then from (7.34) we have

$$\begin{aligned} &M_k^1 := M_k^1(K,L;\mathbf{a},\mathbf{b};c\mathbf{p}) := \\ &N^{-1}\left(\sum_{(i_1,\ldots,i_k)\in I_k}\left(\left(\sum_{l=1}^{k} c\left((i_1,\ldots,i_k),l\right)p_{i_l}\right)N(K(\mathbf{a};c\mathbf{p};\mathbf{i}_k)L(\mathbf{b};c\mathbf{p};\mathbf{i}_k))\right)\right), \end{aligned}$$

and for $1\leq d\leq k-1$, we have from (7.35)

$$\begin{aligned} &M_{k-d}^1 := M_k^1(K,L;\mathbf{a},\mathbf{b};c\mathbf{p}) := \\ &N^{-1}\left\{\begin{array}{l} \dfrac{d!}{(k-1)\ldots(k-d)}\displaystyle\sum_{(i_1,\ldots,i_k)\in I_k}\left(\sum_{1\leq l_1<\ldots<l_{k-d}\leq k}\left(\left(\sum_{j=1}^{k-m} c((i_1,\ldots,i_k),l_j)p_{i_{l_j}}\right)\right.\right. \\ \left.\left.\times N(K(\mathbf{a};c\mathbf{p};\mathbf{i}_k;\mathbf{l}_{k-d})L(\mathbf{b};c\mathbf{p};\mathbf{i}_k;\mathbf{l}_{k-d}))\right)\right) \end{array}\right\} \end{aligned}$$

Under the conditions of Corollary 7.8, we have

$$K_n(\mathbf{a};\mathbf{p})L_n(\mathbf{a};\mathbf{p}) \geq M_k^1 \geq M_{k-1}^1 \geq \ldots \geq M_1^1 = N^{-1}\left(\sum_{i=1}^{n} p_i N(a_ib_i)\right). \tag{7.37}$$

Taking

$$c((i_1,\ldots,i_k),l) = \frac{1}{|S_j|} = \frac{1}{\alpha_{I_k,j}}; \quad ((i_1,\ldots,i_k),l)\in S_j,$$

in (7.36) and (7.37), we get Corollary 7.3 and Corollary 7.4, respectively.

Remark 7.9 We consider Example 7.8. If $f(x_1,x_2)=x_1+x_2$, then under the conditions of Corollary 7.7 we have

$$\begin{aligned} &K_n(\mathbf{a};\mathbf{p})+L_n(\mathbf{b};\mathbf{p}) \\ &\geq N^{-1}\left\{\begin{array}{l} \displaystyle\sum_{i=1}^{n}\left(\sum_{l=0}^{r} c(i,l)\,p_{i+l}\right)N\left(K_r(\mathbf{a},c\mathbf{p};i)+L_r(\mathbf{b},c\mathbf{p};i)\right) \\ +\left(\displaystyle\sum_{j=1}^{n} c(j)p_j\right)N\left(K_n(\mathbf{a};c\mathbf{p})+L_n(\mathbf{b};c\mathbf{p})\right) \end{array}\right\} \\ &\geq N^{-1}\left(\sum_{i=1}^{n} p_i N(a_ib_i)\right). \end{aligned}$$

Similarly, if $f(x_1,x_2)=x_1x_2$, then under the conditions of Corollary 7.8 we have

$$
\begin{aligned}
&K_n(\mathbf{a};\mathbf{p})L_n(\mathbf{b};\mathbf{p}) \\
&\geq N^{-1}\left\{\begin{array}{l}\sum\limits_{i=1}^{n}\left(\sum\limits_{l=0}^{r}c\,(i,l)\,p_{i+l}\right)N\left(K_r(\mathbf{a};c\mathbf{p};i)L_r(\mathbf{b};c\mathbf{p};i)\right) \\ +\left(\sum\limits_{j=1}^{n}c(j)p_j\right)N\left(K_n(\mathbf{a};c\mathbf{p})L_n(\mathbf{b};c\mathbf{p})\right)\end{array}\right\} \\
&\geq N^{-1}\left(\sum_{i=1}^{n}p_iN(a_ib_i)\right).
\end{aligned}
$$

Remark 7.10 We now consider Example 7.9. If $f(x_1,x_2)=x_1+x_2$, then under the conditions of Corollary 7.7 we have

$$
\begin{aligned}
&K_n(\mathbf{a};\mathbf{p})+L_n(\mathbf{b};\mathbf{p}) \\
&\geq N^{-1}\left(\sum_{(i_1,\ldots,i_n)\in D}\left(\left(\sum_{l=1}^{n}c\left((i_1,\ldots,i_n),l\right)p_l\right)N\left(K_n(\mathbf{a};c\mathbf{p},\mathbf{i_n})+L_n(\mathbf{b};c\mathbf{p},\mathbf{i_n})\right)\right)\right) \\
&\geq N^{-1}\left(\sum_{i=1}^{n}p_iN(a_i+b_i)\right).
\end{aligned}
$$

Similarly, if $f(x_1,x_2)=x_1x_2$, then under the conditions of Corollary 7.8 we have

$$
\begin{aligned}
&K_n(\mathbf{a};\mathbf{p})L_n(\mathbf{b};\mathbf{p}) \\
&\geq N^{-1}\left(\sum_{(i_1,\ldots,i_n)\in D}\left(\left(\sum_{l=1}^{n}c\left((i_1,\ldots,i_n),l\right)p_l\right)N\left(K_n(\mathbf{a},c\mathbf{p},\mathbf{i_n})L_n(\mathbf{b},c\mathbf{p},\mathbf{i_n})\right)\right)\right) \\
&\geq N^{-1}\left(\sum_{i=1}^{n}p_iN(a_ib_i)\right).
\end{aligned}
$$

Next, assume (A_2), let $\lambda\geq 1$, and let S_k be the set defined in (7.12). Then for $m=2$, the reverse of (7.22) becomes

$$
\begin{aligned}
&f\left(K_n(\mathbf{a};\mathbf{p}),L_n(\mathbf{b};\mathbf{p})\right)=M_0^2(\lambda)\geq M_1^2(\lambda)\geq\ldots\geq \\
&\geq M_k^2(\lambda)\geq\ldots\geq N^{-1}\left(\sum_{i=1}^{n}p_iN(f(a_i,b_i))\right);\quad k\in\mathbb{N},
\end{aligned}
\tag{7.38}
$$

where

$$
\begin{aligned}
&M_k^2(\lambda):=M_k^2(K,L;\mathbf{a},\mathbf{b};\mathbf{p};\lambda):= \\
&N^{-1}\left(\begin{array}{l}\frac{1}{(n+\lambda-1)^k}\sum\limits_{(i_1,\ldots,i_n)\in S_k}\left(\frac{k!}{i_1!\ldots i_n!}\left(\sum\limits_{j=1}^{n}\lambda^{i_j}p_j\right)\right. \\ \left.\times N\left(f\left(K_n(\mathbf{a};\mathbf{p};\mathbf{i}_{n,k};\lambda),L_n(\mathbf{x}^{(m)};\mathbf{p};\mathbf{i}_{n,k};\lambda)\right)\right)\right)\end{array}\right),
\end{aligned}
$$

By using Theorem 7.5 (for $m=2$) and Corollaries 7.1, 7.2, we get parameter dependent generalizations of Beck's results.

Corollary 7.9 *Assume (A_3) with $f(x,y)=x+y$ ($(x,y)\in I_K\times I_L$), let $\lambda\geq 1$, and let T_k be the set defined in (7.12). Assume further that K', L', N', K'', L'' and N'' are all positive.*

Introducing $E := \frac{K'}{K''}$, $F := \frac{L'}{L''}$, $G := \frac{N'}{N''}$, *(7.38) holds for all possible* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{p}$ *if and only if*

$$E(x) + F(y) \leq G(x+y), \quad (x,y) \in I_K^\circ \times I_L^\circ.$$

In this case for $k \in \mathbb{N}$, *we have*

$$M_k^2(\lambda) := M_k^2(K,L;\mathbf{a},\mathbf{b};\mathbf{p};\lambda) :=$$
$$N^{-1}\left(\begin{array}{l} \frac{1}{(n+\lambda-1)^k} \sum\limits_{(i_1,\ldots,i_n)\in S_k} \left(\frac{k!}{i_1!\ldots i_n!}\left(\sum\limits_{j=1}^{n} \lambda^{i_j} p_j\right)\right. \\ \left.\times N\left(K_n(\mathbf{a};\mathbf{p};\mathbf{i}_{n,k};\lambda) + L_n(\mathbf{x}^{(m)};\mathbf{p};\mathbf{i}_{n,k};\lambda)\right)\right) \end{array}\right).$$

Corollary 7.10 *Assume* (A_3) *with* $f(x,y) = xy$ $((x,y) \in I_K \times I_L)$, *let* $\lambda \geq 1$, *and let* T_k *be the set defined in (7.12). Suppose the functions* $A(x) := \frac{K'(x)}{K'(x)+xK''(x)}$, $B(x) := \frac{L'(x)}{L'(x)+xL''(x)}$ *and* $C(x) := \frac{N'(x)}{N'(x)+xN''(x)}$ *are defined on* I_K°, I_L° *and* I_N° *respectively. Assume further that* K', L', M', A, B *and* C *are all positive. Then (7.38) holds for all possible* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{p}$ *if and only if*

$$A(x) + B(y) \leq C(xy), \quad (x,y) \in I_K^\circ \times I_L^\circ.$$

In this case for $k \in \mathbb{N}$, *we have*

$$M_k^2(\lambda) := M_k^2(K,L;\mathbf{a},\mathbf{b};\mathbf{p};\lambda) := N^{-1}\left(\begin{array}{l} \frac{1}{(n+\lambda-1)^k} \sum\limits_{(i_1,\ldots,i_n)\in S_k} \left(\frac{k!}{i_1!\ldots i_n!}\left(\sum\limits_{j=1}^{n} \lambda^{i_j} p_j\right)\right. \\ \left.\times N\left(K_n(\mathbf{a};\mathbf{p};\mathbf{i}_{n,k};\lambda) L_n(\mathbf{x}^{(m)};\mathbf{p};\mathbf{i}_{n,k};\lambda)\right)\right) \end{array}\right).$$

7.3 Generalization of Minkowski's inequality

(A_4) Let I be an interval in $\mathbb{R}$, and let $\varphi : I \to \mathbb{R}$ be a continuous and strictly monotone function. Let $\mathbf{x}_i \in I^m$ $(i = 1,\ldots,n)$, let $\mathbf{p} = (p_1,\ldots,p_n)$ be a positive n-tuple such that $\sum\limits_{i=1}^{n} p_i = 1$, and let $\mathbf{w} = (w_1,\ldots,w_m)$ be a nonnegative m-tuple such that $\sum\limits_{i=1}^{m} w_i = 1$.

We give a generalization of the Minkowski's inequality by using Theorem B.

Theorem 7.6 *Assume* (A_4) *and* ($\mathscr{H}_0$), *and assume that the quasi-arithmetic mean function*

$$\mathbf{x} \to M_\varphi(\mathbf{x},\mathbf{w}), \quad \mathbf{x} \in I^m$$

is convex. Then

$$M_\varphi\Big(\sum_{r=1}^{n} p_r \mathbf{x}_r,\mathbf{w}\Big) \leq A_{k,k} \leq A_{k,k-1} \leq \ldots \leq A_{k,2} \leq A_{k,1} = \sum_{r=1}^{n} p_r M_\varphi(\mathbf{x}_r,\mathbf{w}), \tag{7.39}$$

where

$$A_{k,k} := \sum_{(i_1,\ldots,i_l)\in I_k} \left(\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}}\right) M_\varphi\left(\frac{\sum\limits_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}}\mathbf{x}_{i_s}}{\sum\limits_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}}}, \mathbf{w}\right), \tag{7.40}$$

and

$$A_{k,l} := \frac{1}{(k-1)\ldots l} \sum_{(i_1,\ldots,i_l)\in I_l} t_{I_k,l}(i_1,\ldots,i_l) \left(\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}}\right) M_\varphi\left(\frac{\sum\limits_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}}\mathbf{x}_{i_s}}{\sum\limits_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}}}, \mathbf{w}\right), \tag{7.41}$$

for $k-1 \geq l \geq 1$.

Proof. This is obtained by applying Theorem B to the function $M_\varphi(\cdot,\mathbf{w})$ and to the vectors $\mathbf{x}_i$ $(i = 1,\ldots,n)$. It is enough to show that $A_{k,l}$ in (7.8) has the form (7.40) and (7.41) depending on l, but this is easy to check. □

Similarly, by using Theorem C we get

Theorem 7.7 *Assume* (A_4), $(\mathscr{H}_0)$, *and suppose* $|H_{I_l}(j_1,\ldots,j_{l-1})| = \beta_{l-1}$ *for any* $(j_1,\ldots,j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$. *Then*

$$M_\varphi\Big(\sum_{r=1}^{n} p_r\mathbf{x}_r,\mathbf{w}\Big) \leq A_{k,k} \leq A_{k-1,k-1} \leq \ldots \leq A_{2,2} \leq A_{1,1} = \sum_{r=1}^{n} p_r M_\varphi(\mathbf{x}_r,\mathbf{w}),$$

where

$$A_{l,l} := \frac{n}{l\,|I_l|} \sum_{(i_1,\ldots,i_l)\in I_l} \left(\sum_{s=1}^{l} p_{i_s}\right) M_\varphi\left(\frac{\sum\limits_{s=1}^{l} p_{i_s}\mathbf{x}_{i_s}}{\sum\limits_{s=1}^{l} p_{i_s}}, \mathbf{w}\right), \quad k \geq l \geq 1.$$

We give a generalization of the Minkowski's inequality by using Theorem D.

Theorem 7.8 *Assume* (A_4), $(\mathscr{H}_4)$ *and* $(\mathscr{H}_5)$. *Further, assume that the quasi-arithmetic mean function*

$$\mathbf{x} \to M_\varphi(\mathbf{x},\mathbf{w}), \quad \mathbf{x} \in I^m$$

is convex. Then

$$M_\varphi\Big(\sum_{r=1}^{n} p_r\mathbf{x}_r,\mathbf{w}\Big) \leq A_k \leq A_{k-1} \leq \ldots \leq A_2 \leq A_1 = \sum_{r=1}^{n} p_r M_\varphi(\mathbf{x}_r,\mathbf{w}),$$

where

$$A_k := \sum_{l=1}^{k} \left(\sum_{A\in\mathscr{A}_l} \left(\left(\sum_{s\in A} c(s)p_{\tau(s)}\right) M_\varphi\left(\frac{\sum\limits_{s\in A} c(s)p_{\tau(s)}\mathbf{x}_{\tau(s)}}{\sum\limits_{s\in A} c(s)p_{\tau(s)}}, \mathbf{w}\right)\right)\right), \tag{7.42}$$

and for $1 \le d \le k-1$

$$\begin{aligned} A_{k-d} := \sum_{l=1}^{d} \left(\sum_{A \in \mathscr{A}_l} \left(\sum_{s \in A} c(s) p_{\tau(s)} M_\varphi(\mathbf{x}_{\tau(s)}, \mathbf{w}) \right) \right) + \sum_{l=d+1}^{k} \left(\frac{d!}{(l-1)\ldots(l-d)} \right. \\ \left. \cdot \sum_{A \in \mathscr{A}_l} \left(\sum_{B \in P_{l-d}(A)} \left(\left(\sum_{s \in B} c(s) p_{\tau(s)} \right) M_\varphi \left(\frac{\sum_{s \in B} c(s) p_{\tau(s)} \mathbf{x}_{\tau(s)}}{\sum_{s \in B} c(s) p_{\tau(s)}}, \mathbf{w} \right) \right) \right) \right). \end{aligned} \tag{7.43}$$

Proof. We apply Theorem D to the convex function $M_\varphi(\cdot, \mathbf{w})$ and the vectors $\mathbf{x}_i$ $(i = 1, \ldots, n)$. We get A_d $(k \ge d \ge 1)$ in (7.42) and (7.43) from (7.10) and (7.11) respectively. □

Similarly, by using Theorem E we get

Theorem 7.9 *Let* $\lambda \ge 1$ *be a real number, assume* (A_4) *and suppose* S_k $(k \in \mathbb{N})$ *is the set given in (7.12). If the quasi-arithmetic mean function*

$$\mathbf{x} \to M_\varphi(\mathbf{x}, \mathbf{w}), \quad \mathbf{x} \in I^m$$

is convex, then

$$\begin{aligned} M_\varphi \left(\sum_{r=1}^{n} p_r \mathbf{x}_r, \mathbf{w} \right) = C_0(\lambda) \le C_1(\lambda) \le \ldots \le \\ \le C_k(\lambda) \le \ldots \le \sum_{r=1}^{n} p_r M_\varphi(\mathbf{x}_r, \mathbf{w}), \quad k \in \mathbb{N}, \end{aligned}$$

where

$$\begin{aligned} & C_k(\lambda) = C_k(\mathbf{x}_1, \ldots, \mathbf{x}_n; p_1, \ldots, p_n; \lambda) \\ & := \frac{1}{(n+\lambda-1)^k} \sum_{(i_1, \ldots, i_n) \in S_k} \frac{k!}{i_1! \ldots i_n!} \left(\sum_{j=1}^{n} \lambda^{i_j} p_j \right) M_\varphi \left(\frac{\sum_{j=1}^{n} \lambda^{i_j} p_j \mathbf{x}_j}{\sum_{j=1}^{n} \lambda^{i_j} p_j}, \mathbf{w} \right), \quad k \in \mathbb{N}. \end{aligned}$$

The following special case a necessary and sufficient condition for the quasi-arithmetic mean function to be convex is given in [60], p. 197:

Theorem F. *If* $\varphi : [m_1, m_2] \to \mathbb{R}$ *has continuous derivatives of second order and it is strictly increasing and strictly convex, then the quasi-arithmetic mean function* $M_\varphi(\cdot, w)$ *is convex if and only if* φ' / φ'' *is a concave function.*

(A_5) Let $\varphi : (0, \infty) \to (0, \infty)$ be a continuous and strictly monotone function such that $\lim_{x \to 0} \varphi(x) = \infty$ or $\lim_{x \to \infty} \varphi(x) = \infty$. Let $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{w} = (w_1, \ldots, w_m)$ be positive m-tuples such that $w_i \ge 1$ $(i = 1, \ldots, m)$. Let $\mathbf{p} = (p_1, \ldots, p_n)$ be a positive n-tuple such that $\sum_{i=1}^{n} p_i = 1$.

Then we define

$$\widetilde{M}_\varphi(\mathbf{x}; \mathbf{w}) = \varphi^{-1} \left(\sum_{i=1}^{m} w_i \varphi(x_i) \right). \tag{7.44}$$

The following result is also given in [60, p.197]:

Theorem G. *If* $\varphi : (0,\infty) \to (0,\infty)$ *has continuous derivatives of second order and it is strictly increasing and strictly convex, then* $\widetilde{M}_\varphi(\cdot, w)$ *is a convex function if* φ/φ' *is a convex function.*

By using (7.44) we have

Theorem 7.10 *Assume* (A_5) *and* $(\mathscr{H}_0)$. *If the function*

$$\mathbf{x} \to \widetilde{M}_\varphi(\mathbf{x}, \mathbf{w}), \quad \mathbf{x} \in (0,\infty)^m$$

is convex, then Theorem 7.6 and Theorem 7.7 (in this case we suppose $|H_{I_l}(j_1, \dots, j_{l-1})| = \beta_{l-1}$ *for any* $(j_1, \dots, j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$*) remain valid for* $\widetilde{M}_\varphi(\mathbf{x}, \mathbf{w})$ *instead of* $M_\varphi(\mathbf{x}, \mathbf{w})$.

Remark 7.11 All special cases (as given in Section 2) can be considered for Theorem 7.6, Theorem 7.7 and Theorem 7.10.

Again by using (7.44) we have

Theorem 7.11 *Assume* (A_5) *and let*

$$\mathbf{x} \to \widetilde{M}_\varphi(\mathbf{x}, \mathbf{w}), \quad \mathbf{x} \in (0,\infty)^m$$

be a convex function.

(a) Consider $(\mathscr{H}_4)$ *and* $(\mathscr{H}_5)$. *Then Theorem 7.8 remains valid for* $\widetilde{M}_\varphi(\mathbf{x}, \mathbf{w})$ *instead of* $M_\varphi(\mathbf{x}, \mathbf{w})$.

(b) Consider $\lambda \in \mathbb{R}$ *such that* $\lambda \geq 1$ *and suppose* S_k $(k \in \mathbb{N})$ *is the set defined in (7.12). Then Theorem 7.9 also remains valid for* $\widetilde{M}_\varphi(\mathbf{x}, \mathbf{w})$ *instead of* $M_\varphi(\mathbf{x}, \mathbf{w})$.

Remark 7.12 All special cases (as given in Section 2) can also be considered for Theorem 7.8, Theorem 7.9 and Theorem 7.11.

Chapter 8

Refinements of Jensen's Inequality for Operator Convex Functions

In this chapter, we consider the class of self-adjoint operators defined on a complex Hilbert space, whose spectra are contained in an interval. We give several refinements of the well known discrete Jensen's inequality in this class. The corresponding mixed symmetric means are defined for a subclass of positive self-adjoint operators which insure the refinements of inequality between power means of strictly positive operators.

8.1 Introduction and Preliminary Results

Let H denote a complex Hilbert space. $S(I)$ means the class of all self-adjoint bounded operators on H whose spectra are contained in an interval $I \subset \mathbb{R}$. The spectrum of a bounded operator A on H is denoted by $\mathrm{Sp}(A)$.

Let $f : D_f (\subset \mathbb{R}) \to \mathbb{R}$ be a function and let $I \subset D_f$ be an interval. f is said to be operator monotone on I if f is continuous on I and $A, B \in S(I)$, $A \leq B$ (i.e. $A - B$ is a positive operator) imply $f(A) \leq f(B)$. The function f is said to be operator convex on I if f is continuous on I and

$$f(sA + tB) \leq sf(A) + tf(B)$$

for all $A, B \in S(I)$ and for all positive numbers s and t such that $s+t=1$. The function f is called operator concave on I if $-f$ is operator convex on I.

Jensen's Operator Inequality: Let $I \subset \mathbb{R}$ be an interval, and let $f : I \to \mathbb{R}$ be an operator convex function on I. If $T_i \in S(I)$ $(i = 1,\dots,n)$, and $w_i > 0$ $(i = 1,...,n)$ such that $\sum_{i=1}^{n} w_i = 1$, then

$$f\left(\sum_{i=1}^{n} w_i T_i\right) \leq \sum_{i=1}^{n} w_i f(T_i). \tag{8.1}$$

If f is an operator concave function on I, then the inequality in (8.1) is reversed.

Some interpolations of (8.1) are given in [61] as follows.

Theorem 8.1 *Under the conditions of the Jensen's operator inequality*

$$f(\sum_{i=1}^{n} w_i T_i) = f_{n,n} \leq ... \leq f_{k,n} \leq ... \leq f_{1,n} = \sum_{i=1}^{n} w_i f(T_i),$$

where for $1 \leq k \leq n$

$$f_{k,n} := \frac{1}{\binom{n-1}{k-1}} \sum_{1 \leq i_1 < ... < i_k \leq n} \left(\sum_{j=1}^{k} w_{i_j}\right) f\left(\frac{\sum_{j=1}^{k} w_{i_j} T_{i_j}}{\sum_{j=1}^{k} w_{i_j}}\right). \tag{8.2}$$

Theorem 8.2 *If the conditions of the Jensen's operator inequality are satisfied, then*

$$f(\sum_{i=1}^{n} w_i T_i) \leq ... \leq \overline{f}_{k+1,n} \leq \overline{f}_{k,n} \leq ... \leq \overline{f}_{1,n} = \sum_{i=1}^{n} w_i f(T_i),$$

where for $k \geq 1$

$$\overline{f}_{k,n} = \frac{1}{\binom{n+k-1}{k-1}} \sum_{1 \leq i_1 \leq ... \leq i_k \leq n} \left(\sum_{j=1}^{k} w_{i_j}\right) f\left(\frac{\sum_{j=1}^{k} w_{i_j} T_{i_j}}{\sum_{j=1}^{k} w_{i_j}}\right). \tag{8.3}$$

A self-adjoint bounded operator A on H is called strictly positive if it is positive and invertible, or equivalently, $\mathrm{Sp}(A) \subset [m, M]$ for some $0 < m < M$.

The power means for strictly positive operators $\mathbf{T} := (T_1, ..., T_n)$ with positive weights $\mathbf{w} := (w_1, ..., w_n)$ are defined in [61] as follows:

$$M_r(\mathbf{T}, \mathbf{w}) = M_r(T_1, ..., T_n; w_1, ..., w_n) := \left(\frac{1}{W_n} \sum_{i=1}^{n} w_i T_i^r\right)^{\frac{1}{r}},$$

where $r \in \mathbb{R} \setminus \{0\}$ and $W_n := \sum_{i=1}^{n} w_i$. The following result about the monotonicity of power means is also given in [61]:

$$M_s(\mathbf{T}, \mathbf{w}) \leq M_r(\mathbf{T}, \mathbf{w}) \tag{8.4}$$

holds if either $s \le r$, $s \notin (-1,1)$, $r \notin (-1,1)$ or $1/2 \le s \le 1 \le r$ or $s \le -1 \le r \le -1/2$.

Some symmetric mixed means, corresponding to the expressions (8.2) and (8.3) are introduced in [61]: for $r,s \in \mathbb{R}\setminus\{0\}$ and for $W_n = 1$, define

$$M_n(s,r;k) := \left(\frac{1}{\binom{n-1}{k-1}} \sum_{1\le i_1<\ldots<i_k\le n} \left(\sum_{j=1}^{k} w_{i_j}\right) M_r^s(T_{i_1},\ldots,T_{i_k}; w_{i_1},\ldots,w_{i_k})\right)^{\frac{1}{s}},$$

where $1 \le k \le n$, and

$$\overline{M}_n(s,r;k) := \left(\frac{1}{\binom{n+k-1}{k-1}} \sum_{1\le i_1\le\ldots\le i_k\le n} \left(\sum_{j=1}^{k} w_{i_j}\right) M_r^s(T_{i_1},\ldots,T_{i_k}; w_{i_1},\ldots,w_{i_k})\right)^{\frac{1}{s}},$$

where $k \ge 1$.

The following result from [61] gives some refinements of (8.4).

Theorem 8.3 *Let $\mathbf{T}$ be an n-tuple of strictly positive operators, and let $w_i > 0$ $(i = 1,\ldots,n)$ such that $W_n = 1$. Then the following inequalities are valid*

$$M_s(\mathbf{T},\mathbf{w}) = M_n(s,r;1) \le \ldots \le M_n(s,r;k) \le \ldots \le M_n(s,r;n) = M_r(\mathbf{T},\mathbf{w}),$$

and

$$M_s(\mathbf{T},\mathbf{w}) = \overline{M}_n(s,r;1) \le \ldots \le \overline{M}_n(s,r;k) \le \ldots \le M_r(\mathbf{T},\mathbf{w}),$$

if either

(i) $1 \le s \le r$ *or*

(ii) $-r \le s \le -1$ *or*

(iii) $s \le -1$, $r \ge s \ge 2r$;
while the reverse inequalities are valid if either

(iv) $r \le s \le -1$ *or*

(v) $1 \le s \le -r$ *or*

(vi) $s \ge 1$, $r \le s \le 2r$.

In [41], we generalize the above results of [61] by using a refinement of the Jensen's inequality from [44].

We use the notations from [44] (see $\mathscr{N}_1$ of Section 2.1).

The following hypotheses will give the basic context of our results.

$(\mathscr{O}_1)$ Let $n \ge 1$ and $k \ge 2$ be fixed integers, and let I_k be a subset of $\{1,\ldots,n\}^k$ such that

$$\alpha_{I_k,i} \ge 1, \quad 1 \le i \le n.$$

($\mathscr{O}_2$) Let $I \subset \mathbb{R}$ be an interval, and let $T_i \in S(I)$ $(1 \leq i \leq n)$.

($\mathscr{O}_3$) Let $w_1, \ldots, w_n$ be positive numbers such that $\sum\limits_{j=1}^{n} w_j = 1$.

($\mathscr{O}_4$) Let the function $f : I \to \mathbb{R}$ be operator convex.

($\mathscr{O}_5$) Let h, $g : I \to \mathbb{R}$ be continuous and strictly operator monotone functions.

Here ($\mathscr{O}_1$) is the same as ($\mathscr{H}_0$) stated in Section 2.1, in seek of symmetry we change the symbol.

8.2 Refinement of Jensen's Operator Inequality

The main results of this section involve some special expressions, which we now describe. Suppose ($\mathscr{O}_1$)-($\mathscr{O}_4$). For any $k \geq l \geq 1$ let

$$A_{l,l} = A_{l,l}\left(I_k, T_1, \ldots, T_n, w_1, \ldots, w_n\right) \tag{8.5}$$
$$:= \sum_{(i_1,\ldots,i_l)\in I_l} \left(\sum_{s=1}^{l} \frac{w_{i_s}}{\alpha_{I_l,i_s}}\right) f\left(\frac{\sum\limits_{s=1}^{l} \frac{w_{i_s}}{\alpha_{I_l,i_s}} T_{i_s}}{\sum\limits_{s=1}^{l} \frac{w_{i_s}}{\alpha_{I_l,i_s}}}\right),$$

and associate to each $k-1 \geq l \geq 1$ the operator

$$A_{k,l} = A_{k,l}\left(I_k, T_1, \ldots, T_n, w_1, \ldots, w_n\right) \tag{8.6}$$
$$:= \frac{1}{(k-1)\ldots l} \sum_{(i_1,\ldots,i_l)\in I_l} t_{I_k,l}\left(i_1, \ldots, i_l\right) \left(\sum_{s=1}^{l} \frac{w_{i_s}}{\alpha_{I_k,i_s}}\right) f\left(\frac{\sum\limits_{s=1}^{l} \frac{w_{i_s}}{\alpha_{I_k,i_s}} T_{i_s}}{\sum\limits_{s=1}^{l} \frac{w_{i_s}}{\alpha_{I_k,i_s}}}\right).$$

With these preparations out of the way we come to

Theorem 8.4 *Assume ($\mathscr{O}_1$)-($\mathscr{O}_4$). Then*

(a)

$$f\left(\sum_{r=1}^{n} w_r T_r\right) \leq A_{k,k} \leq A_{k,k-1} \leq \ldots \leq A_{k,2} \leq A_{k,1} = \sum_{r=1}^{n} w_r f(T_r). \tag{8.7}$$

(b) Suppose $\left|H_{I_l}\left(j_1, \ldots, j_{l-1}\right)\right| = \beta_{l-1}$ *for any* $\left(j_1, \ldots, j_{l-1}\right) \in I_{l-1}$ $(k \geq l \geq 2)$. *Then*

$$A_{k,l} = A_{l,l} = \frac{n}{l\left|I_l\right|} \sum_{(i_1,\ldots,i_l)\in I_l} \left(\sum_{s=1}^{l} w_{i_s}\right) f\left(\frac{\sum\limits_{s=1}^{l} w_{i_s} T_{i_s}}{\sum\limits_{s=1}^{l} w_{i_s}}\right), \quad (k \geq l \geq 1), \tag{8.8}$$

and thus

$$f\left(\sum_{r=1}^{n} w_r T_r\right) \le A_{k,k} \le A_{k-1,k-1} \le \ldots \le A_{2,2} \le A_{1,1} = \sum_{r=1}^{n} w_r f(T_r).$$

To prove these results we can use the same method as in the proofs of Theorem 2.1 and Theorem 2.2, so we omit the proofs.

8.2.1 Applications of Theorem 8.4 to some special cases

Throughout Examples 8.1-8.6 (based on Examples 2.4-2.7, Example 2.2 and Example 2.1) the conditions $(\mathscr{O}_2)$-$(\mathscr{O}_4)$ will be assumed.

Theorem 8.4 contains Theorem 8.1, as the first example shows.

Example 8.1 Let

$$I_k := \left\{(i_1, \ldots, i_k) \in \{1, \ldots, n\}^k \mid i_1 < \ldots < i_k\right\}, \quad 1 \le k \le n.$$

Then, by taking into account Examples 2.4, Theorem 8.4 (b) can be applied: we have

$$A_{k,k} = \frac{1}{\binom{n-1}{k-1}} \sum_{1 \le i_1 < \ldots < i_k \le n} \left(\sum_{s=1}^{k} w_{i_s}\right) f\left(\frac{\sum\limits_{s=1}^{k} w_{i_s} T_{i_s}}{\sum\limits_{s=1}^{k} w_{i_s}}\right), \quad k = 1, \ldots, n.$$

and

$$f\left(\sum_{r=1}^{n} w_r T_r\right) \le A_{k,k} \le A_{k-1,k-1} \le \ldots \le A_{2,2} \le A_{1,1} = \sum_{r=1}^{n} w_r f(T_r). \tag{8.9}$$

If $w_1 = \ldots = w_n = \frac{1}{n}$, then

$$A_{k,k} = \frac{1}{\binom{n}{k}} \sum_{1 \le i_1 < \ldots < i_k \le n} f\left(\frac{T_{i_1} + \ldots + T_{i_k}}{k}\right), \quad k = 1, \ldots, n,$$

and thus (8.9) gives Theorem 8.1.

The next example illustrates that Theorem 8.2 is a also special case of Theorem 8.4.

Example 8.2 Let

$$I_k := \left\{(i_1, \ldots, i_k) \in \{1, \ldots, n\}^k \mid i_1 \le \ldots \le i_k\right\}, \quad k \ge 1.$$

Then, by taking into account Examples 2.5, Theorem 8.4 (b) can be applied: we can deduce

$$A_{k,k} = \frac{1}{\binom{n+k-1}{k-1}} \sum_{1 \le i_1 \le \ldots \le i_k \le n} \left(\sum_{s=1}^{k} w_{i_s}\right) f\left(\frac{\sum\limits_{s=1}^{k} w_{i_s} T_{i_s}}{\sum\limits_{s=1}^{k} w_{i_s}}\right), \quad k \ge 1,$$

and

$$f\left(\sum_{r=1}^{n} w_r T_r\right) \leq \ldots \leq A_{k,k} \leq \ldots \leq A_{k,1} = \sum_{r=1}^{n} w_r f(T_r). \tag{8.10}$$

By taking $w_1 = \ldots = w_n = \frac{1}{n}$, we obtain that

$$A_{k,k} = \frac{1}{\binom{n+k-1}{k}} \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq n} f\left(\frac{T_{i_1} + \ldots + T_{i_k}}{k}\right), \quad k \geq 1,$$

and thus (8.10) gives Theorem 8.2.

The following two examples are particular cases of Theorem 8.4 (b).

Example 8.3 Let

$$I_k := \{1, \ldots, n\}^k, \quad k \geq 1.$$

Then, by taking into account Examples 2.6, Theorem 8.4 (b) can be applied: this leads to

$$A_{k,k} = \frac{1}{kn^{k-1}} \sum_{(i_1,\ldots,i_k) \in I_k} \left(\sum_{s=1}^{k} w_{i_s}\right) f\left(\frac{\sum\limits_{s=1}^{k} w_{i_s} T_{i_s}}{\sum\limits_{s=1}^{k} w_{i_s}}\right), \quad k \geq 1,$$

and

$$f\left(\sum_{r=1}^{n} w_r T_r\right) \leq \ldots \leq A_{k,k} \leq \ldots \leq A_{1,1} = \sum_{r=1}^{n} w_r f(T_r), \quad k \geq 1.$$

Especially, for $w_1 = \ldots w_n = \frac{1}{n}$ we find that

$$A_{k,k} = \frac{1}{n^k} \sum_{(i_1,\ldots,i_k) \in I_k} f\left(\frac{T_{i_1} + \ldots + T_{i_k}}{k}\right), \quad k = 1, \ldots, n.$$

Example 8.4 For $1 \leq k \leq n$ let I_k consist of all sequences $(i_1, \ldots, i_k)$ of k distinct numbers from $\{1, \ldots, n\}$.

Then, by taking into account Examples 2.7, Theorem 8.4 (b) can be applied: it follows that

$$A_{k,k} = \frac{n}{kn(n-1)\ldots(n-k+1)}$$
$$\cdot \sum_{(i_1,\ldots,i_k) \in I_k} \left(\sum_{s=1}^{k} w_{i_s}\right) f\left(\frac{\sum\limits_{s=1}^{k} w_{i_s} T_{i_s}}{\sum\limits_{s=1}^{k} w_{i_s}}\right), \quad k = 1, \ldots, n$$

and

$$f\left(\sum_{r=1}^{n} w_r T_r\right) \leq A_{n,n} \leq \ldots \leq A_{k,k} \leq \ldots \leq A_{1,1} = \sum_{r=1}^{n} w_r f(T_r).$$

If we set $w_1 = \ldots = w_n = \frac{1}{n}$, then

$$A_{k,k} = \frac{1}{n(n-1)\ldots(n-k+1)} \sum_{(i_1,\ldots,i_k)\in I_k} f\left(\frac{T_{i_1}+\ldots+T_{i_k}}{k}\right), \quad k=1,\ldots,n.$$

In the sequel two interesting consequences of Theorem 8.4 (a) are given.

Example 8.5 Let $c_i \geq 1$ be an integer $(i=1,\ldots,n)$, let $k := \sum\limits_{i=1}^{n} c_i$, and let $I_k = P^{c_1,\ldots,c_n}$ consist of all sequences $(i_1,\ldots,i_k)$ in which the number of occurrences of $i \in \{1,\ldots,n\}$ is c_i $(i=1,\ldots,n)$.

By taking into account Examples 2.2, Theorem 8.4 (a) can be applied. According to the result

$$f\left(\sum_{r=1}^{n} w_r T_r\right) \leq A_{k,k-1} \leq \sum_{r=1}^{n} w_r f(T_r),$$

where

$$A_{k,k-1} = \frac{1}{k-1}\sum_{i=1}^{n}(c_i - w_i) f\left(\frac{\sum\limits_{r=1}^{n} w_r T_r - \frac{w_i}{c_i}T_i}{1-\frac{w_i}{c_i}}\right).$$

Example 8.6 Let

$$I_2 := \left\{(i_1,i_2) \in \{1,\ldots,n\}^2 \mid i_1|i_2\right\}.$$

The notation $i_1|i_2$ means that i_1 divides i_2. $\left[\frac{n}{i}\right]$ is the largest natural number that does not exceed $\frac{n}{i}$, and $d(i)$ denotes the number of positive divisors of i.

By taking into account Examples 2.1, Theorem 8.4 (a) can be applied. We have

$$f\left(\sum_{r=1}^{n} w_r T_r\right) \leq \sum_{(i_1,i_2)\in I_2}\left(\frac{w_{i_1}}{\left[\frac{n}{i_1}\right]+d(i_1)} + \frac{w_{i_2}}{\left[\frac{n}{i_2}\right]+d(i_2)}\right)$$
$$\cdot f\left(\frac{\frac{w_{i_1}}{\left[\frac{n}{i_1}\right]+d(i_1)}T_{i_1} + \frac{w_{i_2}}{\left[\frac{n}{i_2}\right]+d(i_2)}T_{i_2}}{\frac{w_{i_1}}{\left[\frac{n}{i_1}\right]+d(i_1)} + \frac{w_{i_2}}{\left[\frac{n}{i_2}\right]+d(i_2)}}\right) \leq \sum_{r=1}^{n} w_r f(T_r).$$

8.2.2 Symmetric Means related to Theorem 8.4

Assume $(\mathscr{O}_1)$-$(\mathscr{O}_3)$. The power means corresponding to $\mathbf{i}^l := (i_1,\ldots,i_l) \in I_l$ $(l=1,\ldots,k)$ are given as:

$$M_r(I_k,\mathbf{i}^l) := \left(\frac{\sum\limits_{s=1}^{l}\frac{w_{i_s}}{\alpha_{I_k,i_s}}T_{i_s}^r}{\sum\limits_{s=1}^{l}\frac{w_{i_s}}{\alpha_{I_k,i_s}}}\right)^{\frac{1}{r}}, \quad r \neq 0.$$

Next, we introduce the mixed symmetric means corresponding to the expressions (8.5) and (8.6) as follows:

$$M^1_{s,r}(I_k,k) := \left(\sum_{\mathbf{i}^k=(i_1,\ldots,i_k)\in I_k} \left(\sum_{j=1}^{k} \frac{w_{i_j}}{\alpha_{I_k,i_j}} \right) \left(M_r(I_k,\mathbf{i}^k) \right)^s \right)^{\frac{1}{s}}, \quad s \neq 0,$$

and for $k-1 \geq l \geq 1$

$$M^1_{s,r}(I_k,l) := \left(\frac{1}{(k-1)\ldots l} \sum_{\mathbf{i}^l=(i_1,\ldots,i_l)\in I_l} t_{I_k,l}(\mathbf{i}^l) \left(\sum_{j=1}^{l} \frac{w_{i_j}}{\alpha_{I_k,i_j}} \right) \left(M_r(I_k,\mathbf{i}^l) \right)^s \right)^{\frac{1}{s}}, \, s \neq 0. \tag{8.11}$$

The following result is a comprehensive generalization of Theorem 8.3.

Theorem 8.5 *Assume ($\mathscr{O}_1$)-($\mathscr{O}_3$) for an n-tuple* $\mathbf{T}$ *of strictly positive operators. Then*

$$M_s(\mathbf{T},\mathbf{w}) = M^1_{s,r}(I_k,1) \leq \ldots. \leq M^1_{s,r}(I_k,k) \leq M_r(\mathbf{T},\mathbf{w}). \tag{8.12}$$

holds if either

(i) $1 \leq s \leq r$ *or*
(ii) $-r \leq s \leq -1$ *or*
(iii) $s \leq -1$, $r \geq s \geq 2r$;
while the reverse inequalities hold in (8.12) if either
(iv) $r \leq s \leq -1$ *or*
(v) $1 \leq s \leq -r$ *or*
(vi) $s \geq 1$, $r \leq s \leq 2r$.

Proof. It is well known (see [21]) that the function $f : D_f(\subset \mathbb{R}) \to \mathbb{R}$, $f(x) = x^p$ is operator convex on $(0,\infty)$ if either $1 \leq p \leq 2$ or $-1 \leq p \leq 0$, and operator concave on $(0,\infty)$ if $0 \leq p \leq 1$, while f is operator monotone on $(0,\infty)$ if $0 \leq p \leq 1$. It is also true that $-f$ is operator monotone on $(0,\infty)$ if $-1 \leq p \leq 0$. By using these facts, we can apply Theorem 8.4 (a) to the function $f(x) = x^{\frac{s}{r}}$, and the operators T_i^r $(i = 1,\ldots,n)$. □

Assume ($\mathscr{O}_1$)-($\mathscr{O}_3$) and ($\mathscr{O}_5$). Then we define the quasi-arithmetic means with respect to (8.5) and (8.6) as follows:

$$M^1_{h,g}(I_k,k) := h^{-1}\left(\sum_{(i_1,\ldots,i_k)\in I_k} \left(\sum_{s=1}^{k} \frac{w_{i_s}}{\alpha_{I_k,i_s}} \right) h \circ g^{-1} \left(\frac{\sum\limits_{s=1}^{k} \frac{w_{i_s}}{\alpha_{I_k,i_s}} g(T_{i_s})}{\sum\limits_{s=1}^{k} \frac{w_{i_s}}{\alpha_{I_k,i_s}}} \right) \right), \tag{8.13}$$

and for $k-1 \geq l \geq 1$

$$M^1_{h,g}(I_k,l) := h^{-1}\left(\frac{1}{(k-1)\ldots l} \sum_{\mathbf{i}^l=(i_1,\ldots,i_l)\in I_l} t_{I_k,l}(\mathbf{i}^l) \left(\sum_{s=1}^{l} \frac{w_{i_s}}{\alpha_{I_k,i_s}} \right) h \circ g^{-1} \left(\frac{\sum\limits_{s=1}^{l} \frac{w_{i_s}}{\alpha_{I_k,i_s}} g(T_{i_s})}{\sum\limits_{s=1}^{l} \frac{w_{i_s}}{\alpha_{I_k,i_s}}} \right) \right). \tag{8.14}$$

The monotonicity of these generalized means is obtained in the next corollary.

Corollary 8.1 *Assume $(\mathscr{O}_1)$-$(\mathscr{O}_3)$ and $(\mathscr{O}_5)$. For a continuous and strictly operator monotone function $q: I \to \mathbb{R}$ we define*

$$M_q := q^{-1}\left(\sum_{i=1}^{n} w_i q(T_i)\right).$$

Then

$$M_h = M^1_{h,g}(I_k,1) \geq \ldots \geq M^1_{h,g}(I_k,k) \geq M_g, \tag{8.15}$$

if either $h \circ g^{-1}$ is operator convex and h^{-1} is operator monotone or $h \circ g^{-1}$ is operator concave and $-h^{-1}$ is operator monotone;

$$M_g = M^1_{g,h}(I_k,1) \leq \ldots \leq M^1_{g,h}(I_k,k) \leq M_h, \tag{8.16}$$

if either $g \circ h^{-1}$ is operator convex and $-g^{-1}$ is operator monotone or $g \circ h^{-1}$ is operator concave and g^{-1} is operator monotone.

Proof. First, we apply Theorem 8.4 (a) to the function $h \circ g^{-1}$ and replace T_i to $g(T_i)$, then we apply h^{-1} to the inequality coming from (8.7). This gives (8.15). A similar argument gives (8.16): $g \circ h^{-1}$, $T_i = h(T_i)$ and g^{-1}can be used. □

Assume $(\mathscr{O}_1)$-$(\mathscr{O}_3)$, and suppose $|H_{I_l}(j_1,...,j_{l-1})| = \beta_{l-1}$ for any $(j_1,...,j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$. In this case the power means corresponding to $\mathbf{i}^l := (i_1,\ldots,i_l) \in I_l$ $(l = 1,\ldots,k)$ has the form

$$M_r(I_l,\mathbf{i}^l) = M_r(I_k,\mathbf{i}^l) = \left(\frac{\sum\limits_{s=1}^{l} w_{i_j} T^r_{i_j}}{\sum\limits_{s=1}^{l} w_{i_j}}\right)^{\frac{1}{r}}, \quad r \neq 0.$$

Now, for $k \geq l \geq 1$ we introduce the mixed symmetric means related to (8.8) as follows:

$$M^2_{s,r}(I_l) := \left[\frac{n}{l\,|I_l|} \sum_{\mathbf{i}^l=(i_1,\ldots,i_l)\in I_l} \left(\sum_{j=1}^{l} w_{i_j}\right)\left(M_r\left(I_l,\mathbf{i}^l\right)\right)^s\right]^{\frac{1}{s}}, \quad s \neq 0. \tag{8.17}$$

Corollary 8.2 *Assume $(\mathscr{O}_1)$-$(\mathscr{O}_3)$, and suppose $|H_{I_l}(j_1,...,j_{l-1})| = \beta_{l-1}$ for any $(j_1,...,j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$. Then*

$$M_s(\mathbf{T},\mathbf{w}) = M^2_{s,r}(I_1) \leq \ldots \leq M^2_{s,r}(I_k) \leq M_r(\mathbf{T},\mathbf{w}). \tag{8.18}$$

holds if either

(i) $1 \leq s \leq r$ *or*

(ii) $-r \leq s \leq -1$ *or*

(iii) $s \leq -1,\ r \geq s \geq 2r$;
while the reverse inequalities hold in (8.18) if either

(iv) $r \leq s \leq -1$ *or*

(v) $1 \leq s \leq -r$ *or*

(vi) $s \geq 1,\ r \leq s \leq 2r$.

Proof. It comes from Theorem 8.5. □

Assume $(\mathscr{O}_1)$-$(\mathscr{O}_3)$ and $(\mathscr{O}_5)$, and suppose $|H_{I_l}(j_1,...,j_{l-1})| = \beta_{l-1}$ for any $(j_1,...,j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$. We define for $k \geq l \geq 1$ the quasi-arithmetic means with respect to (8.8) as follows:

$$M^2_{h,g}(I_l) := h^{-1}\left(\frac{n}{l|I_l|}\sum_{(i_1,...,i_l)\in I_l}\left(\sum_{s=1}^{l} w_{i_s}\right) h\circ g^{-1}\left(\frac{\sum_{s=1}^{l} w_{i_s} g(T_{i_s})}{\sum_{s=1}^{l} w_{i_s}}\right)\right). \tag{8.19}$$

Corollary 8.3 *Assume $(\mathscr{O}_1)$-$(\mathscr{O}_3)$ and $(\mathscr{O}_5)$, and suppose $|H_{I_l}(j_1,...,j_{l-1})| = \beta_{l-1}$ for any $(j_1,...,j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$. Then*

$$M_h = M^2_{h,g}(I_1) \geq \ldots \geq M^2_{h,g}(I_k) \geq M_g,$$

where either $h\circ g^{-1}$ is operator convex and h^{-1} is operator monotone or $h\circ g^{-1}$ is operator concave and $-h^{-1}$ is operator monotone;

$$M_g = M^2_{g,h}(I_1) \leq \ldots \leq M^2_{g,h}(I_k) \leq M_h,$$

where either $g\circ h^{-1}$ is operator convex and $-g^{-1}$ is operator monotone or $g\circ h^{-1}$ is operator concave and g^{-1} is operator monotone.

Proof. Similar to the proof of Corollary 8.1. □

Finally, we apply the results of this section in some special cases. Throughout Remarks 8.1-8.4 and 8.5-8.6, which are based on Examples 2.4-2.7, Example 2.2 and Example 2.1, the conditions $(\mathscr{O}_2)$-$(\mathscr{O}_3)$ (in the mixed symmetric means) and $(\mathscr{O}_5)$ (in the quasi-arithmetic means) will be assumed.

Remark 8.1 In the case of Example 8.1, for $n \geq k \geq 1$ (8.17) becomes

$$M^2_{s,r}(I_k) = \left(\frac{1}{\binom{n-1}{k-1}}\sum_{1\leq i_1<\ldots<i_k\leq n}\left(\sum_{j=1}^{k} w_{i_j}\right)\left(M_r(I_k,\mathbf{i}^k)\right)^s\right)^{\frac{1}{s}}, \quad s \neq 0. \tag{8.20}$$

and (8.19) has the form

$$M^2_{h,g}(I_k) = h^{-1}\left(\frac{1}{\binom{n-1}{k-1}}\sum_{1\leq i_1<\ldots<i_k\leq n}\left(\sum_{s=1}^{k} w_{i_s}\right) h\circ g^{-1}\left(\frac{\sum_{s=1}^{k} w_{i_s} g(T_{i_s})}{\sum_{s=1}^{k} w_{i_s}}\right)\right). \tag{8.21}$$

Remark 8.2 Under the setting of Example 8.2, for $k \geq 1$ (8.17) becomes

$$M_{s,r}^2(I_k) = \left(\frac{1}{\binom{n+k-1}{k-1}} \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq n} \left(\sum_{j=1}^{k} w_{i_j} \right) \left(M_r(I_k, \mathbf{i}^k) \right)^s \right)^{\frac{1}{s}}, \quad s \neq 0.$$

and (8.19) has the form

$$M_{h,g}^2(I_k) = h^{-1} \left(\frac{1}{\binom{n+k-1}{k-1}} \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq n} \left(\sum_{s=1}^{k} w_{i_s} \right) h \circ g^{-1} \left(\frac{\sum_{s=1}^{k} w_{i_s} g(T_{i_s})}{\sum_{s=1}^{k} w_{i_s}} \right) \right).$$

(8.20) and (8.2) represents mixed symmetric means as given in [61]. Therefore Corollary 8.2 is a generalization of results given in [61].

Remark 8.3 Under the setting of Example 8.3, for $k \geq 1$, (8.17) leads to

$$M_{s,r}^2(I_k) = \left(\frac{1}{kn^{k-1}} \sum_{\mathbf{i}^k = (i_1, \ldots, i_k) \in I_k} \left(\sum_{j=1}^{k} w_{i_j} \right) \left(M_r(I_k, \mathbf{i}^k) \right)^s \right)^{\frac{1}{s}}, \quad s \neq 0.$$

and (8.19) gives

$$M_{h,g}^2(I_k) = h^{-1} \left(\frac{1}{kn^{k-1}} \sum_{\mathbf{i}^k = (i_1, \ldots, i_k) \in I_k} \left(\sum_{s=1}^{k} w_{i_s} \right) h \circ g^{-1} \left(\frac{\sum_{s=1}^{k} w_{i_s} g(T_{i_s})}{\sum_{s=1}^{k} w_{i_s}} \right) \right),$$

respectively.

Remark 8.4 Under the setting of Example 8.4, for $k = 1, \ldots, n$, (8.17) gives

$$M_{s,r}^2(I_k) = \left(\frac{n}{kn(n-1)\ldots(n-k+1)} \sum_{\mathbf{i}^k = (i_1, \ldots, i_k) \in I_k} \left(\sum_{j=1}^{k} w_{i_j} \right) \left(M_r(I_k, \mathbf{i}^k) \right)^s \right)^{\frac{1}{s}}, \quad s \neq 0.$$

and (8.19) has the form

$$\begin{aligned} &M_{h,g}^2(I_k) \\ &= h^{-1} \left(\frac{n}{kn(n-1)\ldots(n-k+1)} \sum_{\mathbf{i}^k = (i_1, \ldots, i_k) \in I_k} \left(\sum_{s=1}^{k} w_{i_s} \right) h \circ g^{-1} \left(\frac{\sum_{s=1}^{k} w_{i_s} g(T_{i_s})}{\sum_{s=1}^{k} w_{i_s}} \right) \right), \end{aligned}$$

respectively.

Remark 8.5 Under the construction of Example 8.5, (8.11) is written as

$$M^1_{s,r}(I_k,k-1)=\left(\frac{1}{k-1}\sum_{i=1}^{n}(c_i-w_i)\left(\frac{\sum\limits_{j=1}^{n} w_jT_j^r-\frac{w_i}{c_i}T_i^r}{1-\frac{w_i}{c_i}}\right)^{\frac{s}{r}}\right)^{\frac{1}{s}},\quad s\neq 0, r\neq 0,$$

while (8.14) becomes

$$M^1_{h,g}(I_k,k-1)=h^{-1}\left(\frac{1}{k-1}\sum_{i=1}^{n}(c_i-w_i)h\circ g^{-1}\left(\frac{\sum\limits_{r=1}^{n} w_rg(T_r)-\frac{w_i}{c_i}g(T_i)}{1-\frac{w_i}{c_i}}\right)\right).$$

Remark 8.6 Under the construction of Example 8.6, (8.2.2) gives

$$M^1_{s,r}(I_2,2)=\left(\sum_{\mathbf{i}^2=(i_1,i_2)\in I_2}\left(\sum_{j=1}^{2}\frac{w_{i_j}}{\left[\frac{n}{i_j}\right]+d(i_j)}\right)\left(M_r(I_2,\mathbf{i}^2)\right)^s\right)^{\frac{1}{s}},\quad s\neq 0.$$

while (8.13) gives

$$\begin{aligned}&M^1_{h,g}(I_2,2)\\&=h^{-1}\left(\sum_{(i_1,i_2)\in I_2}\left(\sum_{s=1}^{2}\frac{w_{i_s}}{\left[\frac{n}{i_s}\right]+d(i_s)}\right)h\circ g^{-1}\left(\frac{\sum\limits_{s=1}^{2}\frac{w_{i_s}}{\left[\frac{n}{i_s}\right]+d(i_s)}g(T_{i_s})}{\sum\limits_{s=1}^{2}\frac{w_{i_s}}{\left[\frac{n}{i_s}\right]+d(i_s)}}\right)\right).\end{aligned}$$

8.3 Further Refinement of Jensen's Operator Inequality

In this section, we first use the method of Horváth adopted in [32] (see section 2.2) to construct a new refinement of Jensen's inequality for operator convex functions. In this way we are able to generalize the refinement results given in [41] as well as the results of Mond and Pečarić in [61]. The results of this section are published in [42].

Now, we give the generalization of Theorem 8.4. For this we use two further hypotheses $(\mathscr{H}_4)$ and $(\mathscr{H}_5)$ given in Section 2.2:

$(\mathscr{H}_4)$ Let $S_1,\dots,S_n$ be finite, pairwise disjoint and nonempty sets, let

$$S:=\bigcup_{j=1}^{n}S_j,$$

and let c be a function from S into $\mathbb{R}$ such that

$$c(s) > 0, \quad s \in S, \quad \text{and} \quad \sum_{s \in S_j} c(s) = 1, \quad j = 1, \ldots, n.$$

Let the function $\tau : S \to \{1, \ldots, n\}$ be defined by

$$\tau(s) := j, \quad \text{if} \quad s \in S_j.$$

($\mathscr{H}_5$) Suppose $\mathscr{A} \subset P(S)$ is a partition of S into pairwise disjoint and nonempty sets. Let

$$k := \max\{|A| \mid A \in \mathscr{A}\},$$

and let

$$\mathscr{A}_l := \{A \in \mathscr{A} \mid |A| = l\}, \quad l = 1, \ldots, k.$$

Then $\mathscr{A}_l$ $(l = 1, \ldots, k-1)$ may be the empty set, and $|S| = \sum\limits_{l=1}^{k} l\,|\mathscr{A}_l|$.

Theorem 8.6 *[42] If ($\mathscr{O}_2$)-($\mathscr{O}_4$) and ($\mathscr{H}_4$)-($\mathscr{H}_5$) are satisfied, then*

$$f\left(\sum_{j=1}^{n} w_j T_j\right) \leq N_k \leq N_{k-1} \leq \ldots \leq N_2 \leq N_1 = \sum_{j=1}^{n} w_j f(T_j),$$

where

$$N_k := \sum_{l=1}^{k}\left(\sum_{A \in \mathscr{A}_l}\left(\left(\sum_{s \in A} c(s) w_{\tau(s)}\right) f\left(\frac{\sum\limits_{s \in A} c(s) w_{\tau(s)} T_{\tau(s)}}{\sum\limits_{s \in A} c(s) w_{\tau(s)}}\right)\right)\right),$$

and for every $1 \leq m \leq k-1$ the operator N_{k-m} is given by

$$N_{k-m} := \sum_{l=1}^{m}\left(\sum_{A \in \mathscr{A}_l}\left(\sum_{s \in A} c(s) w_{\tau(s)} f(T_{\tau(s)})\right)\right) + \sum_{l=m+1}^{k}\left(\frac{m!}{(l-1)\ldots(l-m)}\right.$$

$$\left.\cdot \sum_{A \in \mathscr{A}_l}\left(\sum_{B \in P_{l-m}(A)}\left(\left(\sum_{s \in B} c(s) w_{\tau(s)}\right) f\left(\frac{\sum\limits_{s \in B} c(s) w_{\tau(s)} T_{\tau(s)}}{\sum\limits_{s \in B} c(s) w_{\tau(s)}}\right)\right)\right)\right).$$

Proof. The proof is entirely similar to the proof of Theorem 2.3, so we omit it. □

8.3.1 Applications of Theorem 8.6 to some Special Cases

The first application of Theorem 8.6 leads to a generalization of Theorem 8.4.

Theorem 8.7 *Assume that ($\mathscr{O}_2$)-($\mathscr{O}_4$) are satisfied, let $k \geq 1$ be a fixed integer, and let $I_k \subset \{1, \ldots, n\}^k$ such that*

$$\alpha_{I_k, i} \geq 1, \quad 1 \leq i \leq n,$$

where $\alpha_{I_k,i}$ *means the number of occurrences of* i *in the sequences* $(i_1,\ldots,i_k) \in I_k$*. For* $j = 1,\ldots,n$ *we consider the sets*

$$S_j := \{((i_1,\ldots,i_k),l) \mid (i_1,\ldots,i_k) \in I_k, \quad 1 \le l \le k, \quad i_l = j\}.$$

Let c *be a positive function on* $S := \bigcup_{j=1}^{n} S_j$ *such that*

$$\sum_{((i_1,\ldots,i_k),l)\in S_j} c((i_1,\ldots,i_k),l) = 1, \quad j = 1,\ldots,n.$$

Then

$$f\left(\sum_{j=1}^{n} w_j T_j\right) \le N_k \le N_{k-1} \le \ldots \le N_2 \le N_1 = \sum_{j=1}^{n} w_j f(T_j), \tag{8.22}$$

where

$$N_k := \sum_{(i_1,\ldots,i_k)\in I_k} \left(\left(\sum_{l=1}^{k} c((i_1,\ldots,i_k),l)\, w_{i_l}\right) f\left(\frac{\sum_{l=1}^{k} c((i_1,\ldots,i_k),l)\, w_{i_l} T_{i_l}}{\sum_{l=1}^{k} c((i_1,\ldots,i_k),l)\, w_{i_l}}\right)\right),$$

and for every $1 \le m \le k-1$

$$N_{k-m} := \frac{m!}{(k-1)\ldots(k-m)} \sum_{(i_1,\ldots,i_k)\in I_k} \left(\sum_{1\le l_1<\ldots<l_{k-m}\le k}\right.$$

$$\left.\left(\left(\sum_{j=1}^{k-m} c((i_1,\ldots,i_k),l_j)\, w_{i_{l_j}}\right) f\left(\frac{\sum_{l=1}^{k-m} c((i_1,\ldots,i_k),l_j)\, w_{i_{l_j}} T_{i_{l_j}}}{\sum_{l=1}^{k-m} c((i_1,\ldots,i_k),l_j)\, w_{i_{l_j}}}\right)\right)\right).$$

An immediate consequence of the previous result is Theorem 8.4: choosing

$$c((i_1,\ldots,i_k),l) = \frac{1}{|S_j|} = \frac{1}{\alpha_{I_k,j}} \quad \text{if} \quad ((i_1,\ldots,i_k),l) \in S_j,$$

it can be checked easily that inequality (8.22) corresponds to inequality (8.7).

Theorem 8.4 has the interestng special cases Example 8.1 and Example 8.2. Theorem 8.7 generalizes these results: apply it to either

$$I_k := \left\{(i_1,\ldots,i_k) \in \{1,\ldots,n\}^k \mid i_1 < \ldots < i_k\right\}, \quad 1 \le k \le n,$$

or

$$I_k := \left\{(i_1,\ldots,i_k) \in \{1,\ldots,n\}^k \mid i_1 \le \ldots \le i_k\right\}, \quad 1 \le k.$$

Now we apply Theorem 8.6 to some special situations which correspond to some results about operator convexity. The next examples based on Examples 2.9-2.10 and Example 5 in [32].

Example 8.7 Let n, m, r be fixed integers, where $n \geq 3$, $m \geq 2$ and $1 \leq r \leq n-2$. In this example, for every $i = 1,2,\ldots,n$ and for every $l = 0,1,\ldots,r$ the integer $i+l$ will be identified with the uniquely determined integer j from $\{1,\ldots,n\}$ for which

$$l+i \equiv j \pmod n.$$

Introducing the notation

$$D := \{1,\ldots,n\} \times \{0,\ldots,r\},$$

let for every $j \in \{1,\ldots,n\}$

$$S_j := \{(i,l) \in D \mid i+l \equiv j \pmod n\} \bigcup \{j\},$$

and let $\mathscr{A} \subset P(S)$ $(S := \bigcup_{j=1}^{n} S_j)$ contain the following sets:

$$A_i := \{(i,l) \in D \mid l = 0,\ldots,r\}, \quad i = 1,\ldots,n$$

and

$$A := \{1,\ldots,n\}.$$

Let c be a positive function on S such that

$$\sum_{(i,l)\in S_j} c(i,l) + c(j) = 1, \quad j = 1,\ldots,n.$$

As we have seen in Example 2.9, the sets $S_1,\ldots,S_n$, the partition $\mathscr{A}$ and the function c defined above satisfy the conditions $(\mathscr{H}_4)$ and $(\mathscr{H}_5)$.

Now we suppose $(\mathscr{O}_2)$-$(\mathscr{O}_4)$ are satisfied. Then by Theorem 8.6

$$f\left(\sum_{j=1}^{n} w_j T_j\right) \leq N_k = \sum_{i=1}^{n}\left(\left(\sum_{l=0}^{r} c(i,l)\, w_{i+l}\right) f\left(\frac{\sum_{l=0}^{r} c(i,l)\, w_{i+l} T_{i+l}}{\sum_{l=0}^{r} c(i,l)\, w_{i+l}}\right)\right)$$

$$+\left(\sum_{j=1}^{n} c(j) w_j\right) f\left(\frac{\sum_{j=1}^{n} c(j) w_j T_j}{\sum_{j=1}^{n} c(j) w_j}\right) \leq \sum_{j=1}^{n} w_j f(T_j). \tag{8.23}$$

In case

$$w_j := \frac{1}{n}, \quad j = 1,\ldots,n,$$

$$c(i,l) := \frac{1}{m(r+1)}, \quad (i,l) \in D, \quad c(j) := \frac{m-1}{m} \quad j = 1,\ldots,n,$$

it follows from (8.23) that

$$f\left(\frac{1}{n}\sum_{j=1}^{n} T_j\right) \leq \frac{1}{mn}\sum_{i=1}^{n} f\left(\frac{T_i + T_{i+1} + \ldots + T_{i+r}}{r+1}\right)$$

$$+\frac{m-1}{m}f\left(\frac{1}{n}\sum_{j=1}^{n}T_j\right)\leq\frac{1}{n}\sum_{j=1}^{n}f(T_j).$$

Example 8.8 Let n and k be fixed positive integers. Let

$$D:=\{(i_1,\ldots,i_n)\in\{1,\ldots,k\}^n \mid i_1+\ldots+i_n=n+k-1\},$$

and for each $j=1,\ldots,n$, denote S_j the set

$$S_j:=D\times\{j\}.$$

For every $(i_1,\ldots,i_n)\in D$ designate by $A_{(i_1,\ldots,i_n)}$ the set

$$A_{(i_1,\ldots,i_n)}:=\{((i_1,\ldots,i_n),l)\mid l=1,\ldots,n\}.$$

It is obvious that S_j $(j=1,\ldots,n)$ and $A_{(i_1,\ldots,i_n)}$ $((i_1,\ldots,i_n)\in D)$ are decompositions of $S:=\bigcup\limits_{j=1}^{n}S_j$ into pairwise disjoint and nonempty sets, respectively. Let c be a function on S such that

$$c((i_1,\ldots,i_n),j)>0,\quad ((i_1,\ldots,i_n),j)\in S$$

and

$$\sum_{(i_1,\ldots,i_n)\in D}c((i_1,\ldots,i_n),j)=1,\quad j=1,\ldots,n. \tag{8.24}$$

As in Example 2.10 we have that the conditions $(\mathscr{H}_4)$ and $(\mathscr{H}_5)$ are valid.

Suppose $(\mathscr{O}_2)$-$(\mathscr{O}_4)$ are satisfied. Then by Theorem 8.6

$$f\left(\sum_{j=1}^{n}w_jT_j\right)\leq N_k=\sum_{(i_1,\ldots,i_n)\in D}\left(\left(\sum_{l=1}^{n}c((i_1,\ldots,i_n),l)w_l\right)\right.$$

$$\left.f\left(\frac{\sum\limits_{l=1}^{n}c((i_1,\ldots,i_n),l)w_lT_l}{\sum\limits_{l=1}^{n}c((i_1,\ldots,i_n),l)w_l}\right)\right)\leq\sum_{j=1}^{n}w_jf(T_j). \tag{8.25}$$

If we set

$$w_j:=\frac{1}{n},\quad j=1,\ldots,n,$$

and

$$c((i_1,\ldots,i_n),j):=\frac{i_j}{\binom{n+k-1}{k-1}},$$

then (8.24) holds, since by some combinatorial considerations

$$|D|=\binom{n+k-2}{n-1},$$

and

$$\sum_{(i_1,\ldots,i_n)\in D} i_j = \frac{n+k-1}{n}\binom{n+k-2}{n-1} = \binom{n+k-1}{k-1}, \quad j=1,\ldots,n.$$

In this situation (8.25) can therefore be expressed as

$$f\left(\frac{1}{n}\sum_{j=1}^{n} T_j\right) \le \frac{1}{\binom{n+k-2}{k-1}} \sum_{(i_1,\ldots,i_n)\in D} f\left(\frac{1}{n+k-1}\sum_{l=1}^{n} i_l T_l\right) \le \frac{1}{n}\sum_{j=1}^{n} f(T_j).$$

Let us close this section by deriving a sharpened version of the arithmetic mean - geometric mean inequality. We note that ln is operator concave in $(0,\infty)$.

Example 8.9 Let $n \ge 2$ be a fixed positive integer, let

$$S_j := \left\{(i,j) \in \{1,\ldots,n\}^2 \mid i=1,\ldots,j\right\}, \quad j=1,\ldots,n,$$

and let

$$A_i := \left\{(i,j) \in \{1,\ldots,n\}^2 \mid j=i,\ldots,n\right\}, \quad i=1,\ldots,n.$$

If $T_1,\ldots,T_n$ are strictly positive operators, then it follows from Theorem 8.6 that

$$-\ln\left(\frac{T_1+\ldots+T_n}{n}\right) \le \sum_{i=1}^{n}\left(-\left(\frac{1}{n}\sum_{j=i}^{n}\frac{1}{j}\right)\ln\left(\frac{\sum_{j=i}^{n}\frac{T_j}{j}}{\sum_{j=i}^{n}\frac{1}{j}}\right)\right)$$

$$\le -\frac{\ln(T_1)+\ldots+\ln(T_n)}{n},$$

and therefore

$$(T_1\ldots T_n)^{\frac{1}{n}} \le \prod_{i=1}^{n}\left(\frac{\sum_{j=i}^{n}\frac{T_j}{j}}{\sum_{j=i}^{n}\frac{1}{j}}\right)^{\frac{1}{n}\sum_{j=i}^{n}\frac{1}{j}} \le \frac{T_1+\ldots+T_n}{n}.$$

8.4 Parameter Dependent Refinement of Jensen's Operator Inequality

In this section, we introduce a parameter dependent refinement of (8.1) by using the method given in Section 2.3. With the help of this new refinement, we construct the parameter dependent mixed symmetric means for a subclass of $S(I)$ and also give the monotonicity property of these operator means. The results of this section are published in [42].

Now, we give a parameter dependent refinement of the discrete Jensen's operator inequality (8.1).

we also need the following hypothesis:

($\mathscr{O}_6$) Consider a real number λ such that $\lambda \geq 1$.

Theorem 8.8 *[42] Suppose ($\mathscr{O}_2$)-($\mathscr{O}_4$) and ($\mathscr{O}_6$). For $k \in \mathbb{N}$, we introduce the sets*

$$S_k := \left\{ (i_1, \ldots, i_n) \in \mathbb{N}^n \mid \sum_{j=1}^{n} i_j = k \right\}, \quad k \in \mathbb{N},$$

and define the operators

$$C_k(\lambda) = C_k(T_1, \ldots, T_n; w_1, \ldots, w_n; \lambda)$$

$$:= \frac{1}{(n+\lambda-1)^k} \sum_{(i_1,\ldots,i_n)\in S_k} \frac{k!}{i_1! \ldots i_n!} \left(\sum_{j=1}^{n} \lambda^{i_j} w_j \right) f \left(\frac{\sum_{j=1}^{n} \lambda^{i_j} w_j T_j}{\sum_{j=1}^{n} \lambda^{i_j} w_j} \right). \tag{8.26}$$

Then

(a)

$$f\left(\sum_{j=1}^{n} w_j T_j \right) = C_0(\lambda) \leq C_1(\lambda) \leq \ldots \leq C_k(\lambda) \leq \ldots \leq \sum_{j=1}^{n} w_j f(T_j), \quad k \in \mathbb{N}.$$

(b) For every fixed $\lambda > 1$

$$\lim_{k\to\infty} C_k(\lambda) = \sum_{j=1}^{n} w_j f(T_j).$$

It follows from the definition of S_k that $S_k \subset \{0, \ldots, k\}^n$ ($k \in \mathbb{N}$), and it is obvious that

$$C_k(1) = f\left(\sum_{j=1}^{n} w_j T_j \right), \quad k \in \mathbb{N}.$$

The proof of Theorem 8.8 is essentially the same as the proofs of Theorem 2.5 and Theorem 2.6, so it is omitted. But to prove the second part of the theorem we need the following two results. First, we generalize Lemma 2.7.

Lemma 8.1 *[42] Let $(X, \|\cdot\|)$ be a normed space. Let $p_1, \ldots, p_n$ be a discrete distribution with $n \geq 2$, and let $\lambda > 1$. Let $l \in \{1, \ldots, n\}$ be fixed. e_l denotes the vector in $\mathbb{R}^n$ that has 0s in all coordinate positions except the lth, where it has a 1. Let $q_1, \ldots, q_n$ be also a discrete distribution such that $q_j > 0$ $(1 \leq j \leq n)$ and*

$$q_l > \max(q_1, \ldots q_{l-1}, q_{l+1}, \ldots, q_n).$$

If

$$g : \left\{ (t_1, \ldots, t_n) \in \mathbb{R}^n \mid t_j > 0 \ (1 \leq j \leq n), \ \sum_{j=1}^{n} t_j = 1 \right\} \to X$$

is a bounded function for which

$$\tau_l := \lim_{e_l} g$$

exists, and $p_l > 0$, then

$$\lim_{k \to \infty} \sum_{(i_1, \ldots, i_n) \in S_k} \frac{k!}{i_1! \ldots i_n!} q_1^{i_1} \ldots q_n^{i_n} g \left(\frac{\lambda^{i_1} p_1}{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j}, \ldots, \frac{\lambda^{i_n} p_n}{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j} \right) = \tau_l.$$

Proof. We have to modify just the final part of the proof of Lemma 2.7. We can suppose that $l = 1$.

Choose $0 < \varepsilon < 1$. Since the distribution function F_{n-1} of the Chi-squared distribution (χ^2-distribution) with $n-1$ degrees of freedom is continuous, and strictly increasing on $(0, \infty)$, there exists a unique $t_\varepsilon > 0$ such that

$$F_{n-1}(t_\varepsilon) = 1 - \varepsilon.$$

Define

$$S_k^1 := \left\{ (i_{1k}, \ldots, i_{nk}) \in S_k \mid \sum_{j=1}^{n} k \frac{\left(\frac{i_{jk}}{k} - q_j \right)^2}{q_j} < t_\varepsilon \right\},$$

let $S_k^2 := S_k \setminus S_k^1$ $(k \in \mathbb{N}_+)$, and consider the sequences

$$a_k^1 := \sum_{(i_{1k}, \ldots, i_{nk}) \in S_k^1} \frac{k!}{i_{1k}! \ldots i_{nk}!} q_1^{i_{1k}} \ldots q_n^{i_{nk}} g \left(\frac{\lambda^{i_{1k}} p_1}{\sum\limits_{j=1}^{n} \lambda^{i_{jk}} p_j}, \ldots, \frac{\lambda^{i_{nk}} p_n}{\sum\limits_{j=1}^{n} \lambda^{i_{jk}} p_j} \right),$$

and

$$a_k^2 := \sum_{(i_{1k}, \ldots, i_{nk}) \in S_k^2} \frac{k!}{i_{1k}! \ldots i_{nk}!} q_1^{i_{1k}} \ldots q_n^{i_{nk}} g \left(\frac{\lambda^{i_{1k}} p_1}{\sum\limits_{j=1}^{n} \lambda^{i_{jk}} p_j}, \ldots, \frac{\lambda^{i_{nk}} p_n}{\sum\limits_{j=1}^{n} \lambda^{i_{jk}} p_j} \right),$$

where $k \in \mathbb{N}_+$.

By using the first part of the proof of Lemma 2.7, we have that

(i)

$$\sum_{(i_{1k}, \ldots, i_{nk}) \in S_k^1} \frac{k!}{i_{1k}! \ldots i_{nk}!} q_1^{i_{1k}} \ldots q_n^{i_{nk}} = 1 - \varepsilon + \delta_\varepsilon(k), \quad k \in \mathbb{N}_+,$$

where $\lim\limits_{k \to \infty} \delta_\varepsilon(k) = 0$ (let $k_\varepsilon \in \mathbb{N}_+$ such that $\delta_\varepsilon(k) < \varepsilon$ for all $k > k_\varepsilon$),

(ii) for every $\varepsilon_1 > 0$ we can find an integer $k_{\varepsilon_1} > k_\varepsilon$ such that for all $k > k_{\varepsilon_1}$

$$\left\| g\left(\frac{\lambda^{i_{1k}} p_1}{\sum\limits_{j=1}^{n} \lambda^{i_{jk}} p_j}, \ldots, \frac{\lambda^{i_{nk}} p_n}{\sum\limits_{j=1}^{n} \lambda^{i_{jk}} p_j}\right) - \tau_1 \right\| < \varepsilon_1, \quad (i_{1k}, \ldots, i_{nk}) \in S_k^1.$$

Since g bounded on its domain ($\|g - \tau_1\| \leq m$), it follows from (i) and (ii) that

$$\left\| \sum_{(i_1,\ldots,i_n)\in S_k} \frac{k!}{i_1! \ldots i_n!} q_1^{i_1} \ldots q_n^{i_n} g\left(\frac{\lambda^{i_1} p_1}{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j}, \ldots, \frac{\lambda^{i_n} p_n}{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j}\right) - \tau_1 \right\|$$

$$\leq \sum_{(i_1,\ldots,i_n)\in S_k^1} \frac{k!}{i_1! \ldots i_n!} q_1^{i_1} \ldots q_n^{i_n} \left\| g\left(\frac{\lambda^{i_1} p_1}{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j}, \ldots, \frac{\lambda^{i_n} p_n}{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j}\right) - \tau_1 \right\|$$

$$+ \sum_{(i_1,\ldots,i_n)\in S_k^2} \frac{k!}{i_1! \ldots i_n!} q_1^{i_1} \ldots q_n^{i_n} \left\| g\left(\frac{\lambda^{i_1} p_1}{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j}, \ldots, \frac{\lambda^{i_n} p_n}{\sum\limits_{j=1}^{n} \lambda^{i_j} p_j}\right) - \tau_1 \right\|$$

$$\leq \varepsilon_1 \left(1 - \varepsilon + \delta_\varepsilon(k)\right) + m\left(\varepsilon - \delta_\varepsilon(k)\right), \quad k > k_{\varepsilon_1},$$

and this gives the result. □

The second lemma corresponds to the symbolic calculus for self-adjoint operators.

Lemma 8.2 *[42] Assume ($\mathscr{O}_2$) and let $f : I \to \mathbb{R}$ be continuous. Let the function*

$$g : \left\{(t_1, \ldots, t_n) \in \mathbb{R}^n \mid t_j > 0 \ (1 \leq j \leq n), \ \sum_{j=1}^{n} t_j = 1\right\} \to B(H)$$

defined by

$$g(t_1, \ldots, t_n) := f\left(\sum_{j=1}^{n} t_j T_j\right).$$

Then

$$\lim_{e_l} g = f(T_l), \quad 1 \leq l \leq n.$$

Proof. Let

$$\alpha := \min_{1\leq j\leq n} \left(\min Sp(T_j)\right) \quad \text{and} \quad \beta := \max_{1\leq j\leq n} \left(\max Sp(T_j)\right),$$

where $Sp(T)$ denotes the spectrum of T. Then

$$Sp\left(\sum_{j=1}^{n} t_j T_j\right) \subset [\alpha,\beta] \subset I$$

for all $t_j \geq 0$ $(1 \leq j \leq n)$ with $\sum\limits_{j=1}^{n} t_j = 1$.

It is enough to prove that f is continuous on $S([\alpha,\beta])$.

To prove this let $\varepsilon > 0$ be fixed, and let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $S([\alpha,\beta])$ such that $A_n \to A \in S([\alpha,\beta])$.

Since f is continuous on $[\alpha,\beta]$, the Stone-Weierstrass theorem implies the existence of a sequence of real polynomial functions $(f_k)_{k\in\mathbb{N}}$ which converges uniformly on $[\alpha,\beta]$ to f. It follows that there exists $k_0 \in \mathbb{N}$ such that

$$\left|f_{k_0}(t) - f(t)\right| < \frac{\varepsilon}{3}, \quad t \in [\alpha,\beta].$$

The fundamental result for continuous functional calculus (see for example [27]) yields that

$$\left\|f(A_n) - f_{k_0}(A_n)\right\| = \left\|\left(f - f_{k_0}\right)(A_n)\right\| = \sup_{t\in Sp(A_n)} \left|f(t) - f_{k_0}(t)\right| \tag{8.27}$$

$$\leq \sup_{t\in[\alpha,\beta]} \left|f(t) - f_{k_0}(t)\right| < \frac{\varepsilon}{3}, \quad n \in \mathbb{N},$$

where $\|\cdot\|$ means the norm on H. Similarly, we have

$$\left\|f_{k_0}(A) - f(A)\right\| < \frac{\varepsilon}{3}. \tag{8.28}$$

Since $A_n \to A$, we obtain $A_n^i \to A^i$ for every $i \in \mathbb{N}$, and therefore there is $n_0 \in \mathbb{N}$ such that

$$\left\|f_{k_0}(A_n) - f_{k_0}(A)\right\| < \frac{\varepsilon}{3} \tag{8.29}$$

for all $n > n_0$.

Now the inequalities (8.27-8.29) give that

$$\|f(A_n) - f(A)\| \leq \left\|f(A_n) - f_{k_0}(A_n)\right\| + \left\|f_{k_0}(A_n) - f_{k_0}(A)\right\|$$

$$+ \left\|f_{k_0}(A) - f(A)\right\| < \varepsilon$$

for all $n > n_0$, and hence $f(A_n) \to f(A)$.

The proof is complete. $\square$

8.4.1 Applications

Suppose ($\mathcal{O}_2$)-($\mathcal{O}_4$) and ($\mathcal{O}_6$). We consider three special cases of (8.26).

(a) $k=1$, $n\in\mathbb{N}_+$:

$$C_1(\lambda)=\frac{1}{n+\lambda-1}\sum_{i=1}^{n}\left(1+(\lambda-1)w_i\right)f\left(\frac{\sum\limits_{j=1}^{n}w_jT_j+(\lambda-1)w_iT_i}{1+(\lambda-1)w_i}\right).$$

(b) $k\in\mathbb{N}$, $n=2$:

$$C_k(\lambda)=\frac{1}{(\lambda+1)^k}\sum_{i=0}^{k}\binom{k}{i}\left(\lambda^i w_1+\lambda^{k-i}w_2\right)f\left(\frac{\lambda^i w_1T_1+\lambda^{k-i}w_2T_2}{\lambda^i w_1+\lambda^{k-i}w_2}\right).$$

(c) $w_1=\ldots=w_n:=\frac{1}{n}$:

$$C_k(\lambda)=\frac{1}{n\,(n+\lambda-1)^k}\sum_{(i_1,\ldots,i_n)\in S_k}\frac{k!}{i_1!\ldots i_n!}\left(\sum_{j=1}^{n}\lambda^{i_j}\right)f\left(\frac{\sum\limits_{j=1}^{n}\lambda^{i_j}T_j}{\sum\limits_{j=1}^{n}\lambda^{i_j}}\right).$$

8.4.2 Parameter Dependent Operator Means

Next, we define some further operator means with parameter and study their monotonicity and convergence.

Definition 8.1 *[42] We assume that ($\mathcal{O}_2$), ($\mathcal{O}_3$) and ($\mathcal{O}_5$) are satisfied and $\lambda\geq 1$. Then we define the operator means with respect to (8.26) by*

$$M_{h,g}(k,\lambda):=h^{-1}\left(\frac{1}{(n+\lambda-1)^k}\sum_{(i_1,\ldots,i_n)\in S_k}\frac{k!}{i_1!\ldots i_n!}\left(\sum_{j=1}^{n}\lambda^{i_j}w_j\right)\right.$$
$$\left.\cdot(h\circ g^{-1})\left(\frac{\sum\limits_{j=1}^{n}\lambda^{i_j}w_jg(T_j)}{\sum\limits_{j=1}^{n}\lambda^{i_j}w_j}\right)\right),\quad k\in\mathbb{N}. \tag{8.30}$$

We now give the monotonicity of the means (8.30) by the virtue of Theorem 8.8.

Proposition 8.1 *[42] For $\lambda\geq 1$, we assume ($\mathcal{O}_2$), ($\mathcal{O}_3$) and ($\mathcal{O}_5$). Then*

(a)

$$M_g=M_{h,g}(0,\lambda)\leq\ldots\leq M_{h,g}(k,\lambda)\leq\ldots\leq M_h,\quad k\in\mathbb{N},$$

if either $h\circ g^{-1}$ is operator convex and h^{-1} is operator monotone or $h\circ g^{-1}$ is operator concave and $-h^{-1}$ is operator monotone.

(b)

$$M_g = M_{h,g}(0,\lambda) \geq \ldots \geq M_{h,g}(k,\lambda) \geq \ldots \geq M_h, \quad k \in \mathbb{N},$$

if either $h \circ g^{-1}$ is operator convex and $-h^{-1}$ is operator monotone or $h \circ g^{-1}$ is operator concave and h^{-1} is operator monotone.

(c) In both cases

$$\lim_{k \to \infty} M_{h,g}(k,\lambda) = M_h.$$

Proof. The idea of the proof is the same as given in Corollary 8.1. □

As a special case we consider the following example.

Example 8.10 [42] If $I := (0,\infty)$, $h := \ln$ and $g(x) := x$ $(x \in (0,\infty))$, then by Proposition 8.1 (b), we have the following inequality: for every $T_j > 0$ $(1 \leq j \leq n)$, $\lambda \geq 1$, and $k \in \mathbb{N}_+$

$$\prod_{j=1}^{n} T_j^{w_j} \leq \prod_{(i_1,\ldots,i_n) \in S_k} \left(\frac{\sum\limits_{j=1}^{n} \lambda^{i_j} w_j T_j}{\sum\limits_{j=1}^{n} \lambda^{i_j} w_j} \right)^{\frac{1}{(n+\lambda-1)^k} \frac{k!}{i_1!\ldots i_n!} \sum\limits_{j=1}^{n} \lambda^{i_j} w_j} \leq \sum_{j=1}^{n} w_j T_j,$$

which gives a sharpened version of the arithmetic mean - geometric mean inequality

$$\prod_{j=1}^{n} T_j^{\frac{1}{n}} \leq \prod_{(i_1,\ldots,i_n) \in S_k} \left(\frac{\sum\limits_{j=1}^{n} \lambda^{i_j} T_j}{\sum\limits_{j=1}^{n} \lambda^{i_j}} \right)^{\frac{1}{n(n+\lambda-1)^k} \frac{k!}{i_1!\ldots i_n!} \sum\limits_{j=1}^{n} \lambda^{i_j}} \leq \frac{1}{n} \sum_{j=1}^{n} T_j.$$

Supported by the power means we can introduce mixed symmetric operator means corresponding to (8.26):

Definition 8.2 *[42] Assume $(\mathcal{O}_2)$ with $I := (0,\infty)$ and $(\mathcal{O}_3)$. We define the mixed symmetric means with respect to (8.26) by*

$$M_{s,r}(k,\lambda) := \left(\frac{1}{(n+\lambda-1)^k} \sum_{(i_1,\ldots,i_n) \in S_k} \frac{k!}{i_1!\ldots i_n!} \left(\sum_{j=1}^{n} \lambda^{i_j} w_j \right) \right.$$

$$\left. \cdot M_r^s \left(T_1,\ldots,T_n; \frac{\lambda^{i_1} w_1}{\sum\limits_{j=1}^{n} \lambda^{i_j} w_j}, \ldots, \frac{\lambda^{i_n} w_n}{\sum\limits_{j=1}^{n} \lambda^{i_j} w_j} \right) \right)^{\frac{1}{s}},$$

if s, $r \in \mathbb{R}$ and $s \neq 0$.

The monotonicity and the convergence of the previous means is studied in the next result.

Proposition 8.2 *[42] Assume ($\mathscr{O}_2$) with $I := (0,\infty)$ and ($\mathscr{O}_3$). Then*

(a)

$$M_s \leq \ldots \leq M_{s,r}(k,\lambda) \leq \ldots \leq M_{s,r}(0,\lambda) = M_r, \tag{8.31}$$

if either

(i) $1 \leq s \leq r$ *or*

(ii) $-r \leq s \leq -1$ *or*

(iii) $s \leq -1$, $r \geq s \geq 2r$;

while the reverse inequalities hold in (8.31) if either

(iv) $r \leq s \leq -1$ *or*

(v) $1 \leq s \leq -r$ *or*

(vi) $s \geq 1$, $r \leq s \leq 2r$.

(b) All of these cases

$$\lim_{k\to\infty} M_{s,r}(k,\lambda) = M_s$$

for each fixed $\lambda > 1$.

Proof. We apply Proposition 8.1 (b). □

Chapter 9

Refinements of Determinantal Inequalities of Jensen's type

In this chapter, some new refinements are given for Jensen's type inequalities involving the determinants of positive definite matrices. Bellman-Bergstrom-Fan functionals are considered. These functionals are not only concave, but superlinear which is a stronger condition. The results take advantage of this property.

The results of this chapter are given in [43].

9.1 Introduction and Preliminary Results

We start this section with the following notations introduced in [59] (see also [60]):
$\mathscr{M}_m$ denotes the set of positive definite matrices of order m. It is evident that $\mathscr{M}_m$ is closed under addition and multiplication with a positive number, i.e. if $M_1, M_2 \in \mathscr{M}_m$, $a > 0$, then $M_1 + M_2$, $aM_1 \in \mathscr{M}_m$ ($\mathscr{M}_m$ is a convex cone).
If $M \in \mathscr{M}_m$, let
$|M|$:= the determinant of M,
$|M|_k = \Pi_{j=1}^{k} \lambda_j$, $k = 1, \ldots, m$, where $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of M with $\lambda_1 \leq \ldots \leq \lambda_m$ (here $|M|_m = |M|$),
$M(j)$:= the submatrix of M obtained by deleting the j^{th} row and column of M,
$M[k]$:= the principal submatrix of M formed by taking the first k rows and columns of M;

then $M[m] = M$, $M[m-1] = M(m)$ and $M[0]:=$ the identity matrix.
BBF means the class of Bellman-Bergstrom-Fan functionals σ_i, δ_j and ν_k defined on $\mathscr{M}_m$ by

$$\sigma_i(M) = |M|_i^{\frac{1}{i}}, \quad i = 1,...,m,$$

$$\delta_j(M) = \frac{|M|}{|M(j)|}, \quad j = 1,...,m,$$

and

$$\nu_k(M) = \left(\frac{|M|}{|M[k]|}\right)^{\frac{1}{(m-k)}}, \quad k = 1,...,m,$$

respectively.

The *BBF* functionals are superlinear (see [59]), i.e. $f \in BBF$ is both superadditive

$$f(M_1 + M_2) \geq f(M_1) + f(M_2), \quad M_1, M_2 \in \mathscr{M}_m$$

and positive homogeneous

$$f(pM) = pf(M), \quad M_1, M_2 \in \mathscr{M}_m, \quad p > 0.$$

More generally, for $f \in BBF$, $M_i \in \mathscr{M}_m$, $p_i > 0$ $(i = 1,...,n)$, and $P_k = \sum_{i=1}^k p_i$ $(k = 1,...,n)$, we have (see also [59]):

$$f\left(\sum_{i=1}^n p_i M_i\right) \geq \sum_{i=1}^n p_i f(M_i) \geq P_n \prod_{i=1}^n f(M_i)^{\frac{p_i}{P_n}}, \tag{9.1}$$

which is an interpolating inequality for

$$f\left(\frac{1}{P_n}\sum_{i=1}^n p_i M_i\right) \geq \prod_{i=1}^n f(M_i)^{\frac{p_i}{P_n}}. \tag{9.2}$$

Remark 9.1 (a) Since a functional $f \in BBF$ is superlinear, it is also concave. The inequality (9.2) comes from the second inequality in (9.1), which is just an arithmetic-geometric mean inequality, by using only the concavity of f.

For $P_n = 1$, interpolations corresponding to the second inequality in (9.1) can be found in [37] and [40]. In [33] parameter dependent interpolations are given.

(b) A concave functional on $\mathscr{M}_m$ is not superlinear in general, hence the interpolations of the first inequality in (9.1) are the most interesting (of course, in the case $P_n \neq 1$).

Unweighted versions of (9.1) and (9.2) are given by

$$f\left(\frac{1}{n}\sum_{i=1}^n M_i\right) \geq \frac{1}{n}\sum_{i=1}^n f(M_i) \geq \prod_{i=1}^n f(M_i)^{\frac{1}{n}}, \tag{9.3}$$

and

$$f\left(\frac{1}{n}\sum_{i=1}^{n} M_i\right) \geq \prod_{i=1}^{n} f(M_i)^{\frac{1}{n}}, \tag{9.4}$$

respectively.

The following interpolations of the first inequality in (9.3) are given in [59]:

$$f\left(\frac{1}{n}\sum_{i=1}^{n} M_i\right) = f_{n,n} \geq \ldots \geq f_{k+1,n} \geq f_{k,n} \geq \ldots \geq f_{1,n} = \frac{1}{n}\sum_{i=1}^{n} f(M_i), \tag{9.5}$$

where

$$f_{k,n} = \frac{1}{\binom{n}{k}} \sum_{1\leq i_1<\ldots<i_k\leq n} f\left(\frac{1}{k}\left(M_{i_1}+\ldots+M_{i_k}\right)\right).$$

[59] contains interpolations for the second inequality in (9.3) too:

$$\frac{1}{n}\sum_{i=1}^{n} f(M_i) = g_{n,n} \geq \ldots \geq g_{k+1,n} \geq g_{k,n} \geq \ldots \geq g_{1,n} = \prod_{i=1}^{n} f(M_i)^{\frac{1}{n}}, \tag{9.6}$$

where

$$g_{k,n} = \prod_{1\leq i_1<\ldots<i_k\leq n} \left(\frac{1}{k}\left(f(M_{i_1})+\ldots+f(M_{i_k})\right)\right)^{\frac{1}{\binom{n}{k}}},$$

and

$$\frac{1}{n}\sum_{i=1}^{n} f(M_i) = h_{1,n} \geq \ldots \geq h_{k,n} \geq h_{k+1,n} \geq \ldots \geq h_{n,n} = \prod_{i=1}^{n} f(M_i)^{\frac{1}{n}}, \tag{9.7}$$

where

$$h_{k,n} = \frac{1}{\binom{n}{k}} \sum_{1\leq i_1<\ldots<i_k\leq n} \left(f(M_{i_1})\ldots f(M_{i_k})\right)^{\frac{1}{k}}.$$

There are similar interpolations for (9.4) in [59]:

$$f\left(\frac{1}{n}\sum_{i=1}^{n} M_i\right) = r_{n,n} \geq \ldots \geq r_{k+1,n} \geq r_{k,n} \geq \ldots \geq r_{1,n} = \prod_{i=1}^{n} f(M_i)^{\frac{1}{n}}, \tag{9.8}$$

where

$$r_{k,n} = \prod_{1\leq i_1<\ldots<i_k\leq n} f\left(\frac{1}{k}\left(M_{i_1}+\ldots+M_{i_k}\right)\right)^{\frac{1}{\binom{n}{k}}}.$$

The above interpolations from [59] based on the concavity of f.

We give interpolations of the first inequality in (9.1) (see Remark 9.1 (b)), which insure generalizations of (9.5). By using the results in the papers [37], [40] and [33], we can also generalize the second inequality in (9.3) and the inequality (9.4), and thus inequalities (9.6-9.8), but these interpolations are just concrete examples of the inequalities in the papers [37], [40] and [33] (see Remark 9.1 (a)).

We consider the notations introduced in ($\mathscr{N}_1$); see Section 2.1 of Chapter 2 (see also [44, 43]).

The following hypotheses are required to give the basic context of our results.

($\mathscr{D}_1$) Let $n \geq 1$ and $k \geq 2$ be fixed integers, and let I_k be a subset of $\{1,\ldots,n\}^k$ such that

$$\alpha_{I_k,i} \geq 1, \quad 1 \leq i \leq n.$$

($\mathscr{D}_2$) Let $M_1,\ldots,M_n \in \mathscr{M}_m$.

($\mathscr{D}_3$) Let $p_1,\ldots,p_n$ be positive real numbers. Let $P_n := \sum_{i=1}^{n} p_i$.

($\mathscr{D}_4$) Let the function $f : \mathscr{M}_m \to \mathbb{R}$ be a Bellman-Bergström-Fan (BBF) functional.

($\mathscr{D}_5$) Let $\left|H_{I_l}(j_1,\ldots,j_{l-1})\right| = \beta_{l-1}$ for any $(j_1,\ldots,j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$.

($\mathscr{D}_1$) is the same as ($\mathscr{H}_0$) given in Section 2.1 of Chapter 2, in seek of symmetry we use this.

9.2 Refinement Results

The refinement results of this section involve some special expressions, which we now describe. Assume ($\mathscr{D}_1$)-($\mathscr{D}_4$). We shall use that $f \in BBF$ is positive homogeneous. For any $k \geq l \geq 1$ let

$$\begin{aligned} A_{l,l} &= A_{l,l}(I_k, M_1,\ldots,M_n,p_1,\ldots,p_n) \\ &:= \sum_{(i_1,\ldots,i_l)\in I_l} \left(\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_l,i_s}}\right) f\left(\frac{\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_l,i_s}} M_{i_s}}{\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_l,i_s}}}\right) \\ &= \sum_{(i_1,\ldots,i_l)\in I_l} f\left(\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_l,i_s}} M_{i_s}\right), \end{aligned} \tag{9.9}$$

and associate to each $k-1 \geq l \geq 1$ the number

$$\begin{aligned} A_{k,l} &= A_{k,l}(I_k, M_1,\ldots,M_n,p_1,\ldots,p_n) \\ &:= \frac{1}{(k-1)\ldots l} \sum_{(i_1,\ldots,i_l)\in I_l} t_{I_k,l}(i_1,\ldots,i_l) \left(\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}}\right) f\left(\frac{\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}} M_{i_s}}{\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}}}\right) \\ &= \frac{1}{(k-1)\ldots l} \sum_{(i_1,\ldots,i_l)\in I_l} t_{I_k,l}(i_1,\ldots,i_l) f\left(\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}} M_{i_s}\right). \end{aligned}$$

Under the above constructions we come to

Theorem 9.1 *Assume $(\mathscr{D}_1)$-$(\mathscr{D}_4)$. Then*
(a)

$$f\left(\sum_{r=1}^{n} p_r M_r\right) \geq A_{k,k} \geq A_{k,k-1} \geq \ldots \geq A_{k,2} \geq A_{k,1} = \sum_{r=1}^{n} p_r f(M_r). \tag{9.10}$$

(b) Assume $(\mathscr{D}_5)$ is also satisfied. Then

$$A_{k,l} = A_{l,l} = \frac{n}{l\,|I_l|} \sum_{(i_1,\ldots,i_l)\in I_l} f\left(\sum_{s=1}^{l} p_{i_s} M_{i_s}\right), \quad (k \geq l \geq 1),$$

and thus

$$f\left(\sum_{r=1}^{n} p_r M_r\right) \geq A_{k,k} \geq A_{k-1,k-1} \geq \ldots \geq A_{2,2} \geq A_{1,1} = \sum_{r=1}^{n} p_r f(M_r).$$

Proof. We prove (a), (b) can be proved similarly. Since f is a Bellman-Bergström-Fan functional, it is concave. Therefore Theorem 2.1 implies that

$$f\left(\frac{1}{P_n}\sum_{r=1}^{n} p_r M_r\right) \geq \overline{A}_{k,k} \geq \overline{A}_{k,k-1} \geq \ldots \geq \overline{A}_{k,2} \geq \overline{A}_{k,1} = \frac{1}{P_n}\sum_{r=1}^{n} p_r f(M_r), \tag{9.11}$$

where

$$\overline{A}_{l,l} := A_{l,l}\left(I_k, M_1, \ldots, M_n, \frac{p_1}{P_n}, \ldots, \frac{p_n}{P_n}\right), \quad k \geq l \geq 1$$

and

$$\overline{A}_{k,l} := A_{k,l}\left(I_k, M_1, \ldots, M_n, \frac{p_1}{P_n}, \ldots, \frac{p_n}{P_n}\right)$$

for $k-1 \geq l \geq 1$. The result now follows from (9.11), since f is positive homogeneous. □

9.2.1 Discussion and Applications Related to Theorem 9.1

Throughout Examples (9.1-9.6) the conditions $(\mathscr{D}_2)$-$(\mathscr{D}_4)$ will be assumed. These examples based on Examples 2.4-2.7, Example 2.2 and Example 2.1.

First, we generalize (9.5).

Example 9.1 Let

$$I_k := \left\{(i_1,\ldots,i_k) \in \{1,\ldots,n\}^k \mid i_1 < \ldots < i_k\right\}, \quad 1 \leq k \leq n.$$

Then, by right of Examples 2.4, Theorem 9.1 (b) can be applied: we have

$$A_{k,k} = \frac{1}{\binom{n-1}{k-1}} \sum_{1\leq i_1<\ldots<i_k\leq n} f\left(\sum_{s=1}^{k} p_{i_s} M_{i_s}\right), \quad k = 1,\ldots,n.$$

and

$$f\left(\sum_{r=1}^{n} p_r M_r\right) \geq A_{k,k} \geq A_{k-1,k-1} \geq \ldots \geq A_{2,2} \geq A_{1,1} = \sum_{r=1}^{n} p_r f(M_r). \tag{9.12}$$

If $p_1 = \ldots = p_n = \frac{1}{n}$, then (see (9.9))

$$A_{k,k} = \frac{1}{\binom{n}{k}} \sum_{1 \leq i_1 < \ldots < i_k \leq n} f\left(\frac{M_{i_1} + \ldots + M_{i_k}}{k}\right), \quad k = 1, \ldots, n,$$

and thus (9.12) gives the generalization of (9.5).

The structure of the second example is similar to the previous one.

Example 9.2 Let

$$I_k := \left\{(i_1, \ldots, i_k) \in \{1, \ldots, n\}^k \mid i_1 \leq \ldots \leq i_k\right\}, \quad k \geq 1.$$

Then, by right of Examples 2.5, Theorem 9.1 (b) can be applied: we can deduce

$$A_{k,k} = \frac{1}{\binom{n+k-1}{k-1}} \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq n} f\left(\sum_{s=1}^{k} p_{i_s} M_{i_s}\right), \quad k \geq 1,$$

and

$$f\left(\sum_{r=1}^{n} p_r M_r\right) \geq \ldots \geq A_{k,k} \geq \ldots \geq A_{k,1} = \sum_{r=1}^{n} p_r f(M_r).$$

By taking $p_1 = \ldots = p_n = \frac{1}{n}$ we obtain (see (9.9))

$$A_{k,k} = \frac{1}{\binom{n+k-1}{k}} \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq n} f\left(\frac{M_{i_1} + \ldots + M_{i_k}}{k}\right), \quad k \geq 1.$$

The following two examples are particular cases of Theorem 9.1 (b).

Example 9.3 Let

$$I_k := \{1, \ldots, n\}^k, \quad k \geq 1.$$

Then, by right of Examples 2.6, Theorem 9.1 (b) can be applied: this leads to

$$A_{k,k} = \frac{1}{kn^{k-1}} \sum_{(i_1, \ldots, i_k) \in I_k} f\left(\sum_{s=1}^{k} p_{i_s} M_{i_s}\right), \quad k \geq 1,$$

and

$$f\left(\sum_{r=1}^{n} p_r M_r\right) \geq \ldots \geq A_{k,k} \geq \ldots \geq A_{1,1} = \sum_{r=1}^{n} p_r f(M_r), \quad k \geq 1.$$

Especially, for $p_1 = \ldots = p_n = \frac{1}{n}$ we find (see (9.9)) that

$$A_{k,k} = \frac{1}{n^k} \sum_{(i_1, \ldots, i_k) \in I_k} f\left(\frac{M_{i_1} + \ldots + M_{i_k}}{k}\right), \quad k = 1, \ldots, n.$$

Example 9.4 For $1 \le k \le n$ let I_k consist of all sequences $(i_1, \ldots, i_k)$ of k distinct numbers from $\{1, \ldots, n\}$.

Then, by right of Examples 2.7, Theorem 9.1 (b) can be applied: it follows that

$$A_{k,k} = \frac{n}{kn(n-1)\ldots(n-k+1)} \sum_{(i_1,\ldots,i_k)\in I_k} f\left(\sum_{s=1}^{k} p_{i_s} M_{i_s}\right), \quad k = 1, \ldots, n$$

and

$$f\left(\sum_{r=1}^{n} p_r M_r\right) \ge A_{n,n} \ge \ldots \ge A_{k,k} \ge \ldots \ge A_{1,1} = \sum_{r=1}^{n} p_r f(M_r).$$

If we set $p_1 = \ldots = p_n = \frac{1}{n}$, then by (9.9)

$$A_{k,k} = \frac{1}{n(n-1)\ldots(n-k+1)} \sum_{(i_1,\ldots,i_k)\in I_k} f\left(\frac{M_{i_1} + \ldots + M_{i_k}}{k}\right), \quad k = 1, \ldots, n.$$

In the sequel two interesting consequences of Theorem 9.1 (a) are given.

Example 9.5 Let $c_i \ge 1$ be an integer $(i = 1, \ldots, n)$, let $k := \sum\limits_{i=1}^{n} c_i$, and let $I_k = P^{c_1,\ldots,c_n}$ consist of all sequences $(i_1, \ldots, i_k)$ in which the number of occurrences of $i \in \{1, \ldots, n\}$ is c_i $(i = 1, \ldots, n)$.

Then, by right of Examples 2.2, Theorem 9.1 (a) can be applied. According to the result

$$f\left(\sum_{r=1}^{n} p_r M_r\right) = A_{k,k}$$

$$= \frac{c_1! \ldots c_n!}{k!} \sum_{(i_1,\ldots,i_k)\in I_k} f\left(\sum_{s=1}^{k} \frac{p_{i_s}}{c_{i_s}} M_{i_s}\right) \ge A_{k,k-1}$$

$$= \frac{1}{k-1} \sum_{i=1}^{n} c_i f\left(\sum_{r=1}^{n} p_r M_r - \frac{p_i}{c_i} M_i\right) \ge \sum_{r=1}^{n} p_r f(M_r).$$

Example 9.6 Let

$$I_2 := \left\{(i_1, i_2) \in \{1, \ldots, n\}^2 \mid i_1 | i_2\right\}.$$

The notation $i_1 | i_2$ means that i_1 divides i_2. $\left[\frac{n}{i}\right]$ is the largest natural number that does not exceed $\frac{n}{i}$, and $d(i)$ denotes the number of positive divisors of i.

Then, by right of Examples 2.1, Theorem 9.1 (a) can be applied. We have

$$f\left(\sum_{r=1}^{n} p_r M_r\right) \ge$$

$$= \sum_{(i_1,i_2)\in I_2} f\left(\frac{p_{i_1}}{\left[\frac{n}{i_1}\right] + d(i_1)} M_{i_1} + \frac{p_{i_2}}{\left[\frac{n}{i_2}\right] + d(i_2)} M_{i_2}\right) \ge \sum_{r=1}^{n} p_r f(M_r).$$

9.3 Generalization of Theorem 9.1

In this section, we give the generalization of some refinements given in Section 9.2. Here we consider the hypotheses ($\mathscr{H}_4$) and ($\mathscr{H}_5$) from Section 2.2 of Chapter 2 (see also [32, 43]).

By virtue of the above considerations we give another refinement of the first inequality in (9.1).

Theorem 9.2 *If ($\mathscr{D}_2$-$\mathscr{D}_4$) and ($\mathscr{H}_4$-$\mathscr{H}_5$) are satisfied, then*

$$f\left(\sum_{j=1}^{n} p_j M_j\right) \geq N_k \geq N_{k-1} \geq \ldots \geq N_2 \geq N_1 = \sum_{j=1}^{n} p_j f(M_j),$$

where

$$\begin{aligned} N_k &:= \sum_{l=1}^{k}\left(\sum_{A\in\mathscr{A}_l}\left(\left(\sum_{s\in A} c(s)p_{\tau(s)}\right) f\left(\frac{\sum_{s\in A} c(s)p_{\tau(s)}M_{\tau(s)}}{\sum_{s\in A} c(s)p_{\tau(s)}}\right)\right)\right) \\ &= \sum_{l=1}^{k}\left(\sum_{A\in\mathscr{A}_l}\left(f\left(\sum_{s\in A} c(s)p_{\tau(s)}M_{\tau(s)}\right)\right)\right), \end{aligned} \tag{9.13}$$

and for every $1 \leq m \leq k-1$ the number N_{k-m} is given by

$$\begin{aligned} N_{k-m} &:= \sum_{l=1}^{m}\left(\sum_{A\in\mathscr{A}_l}\left(\sum_{s\in A} c(s)p_{\tau(s)}f(M_{\tau(s)})\right)\right) + \sum_{l=m+1}^{k}\left(\frac{m!}{(l-1)\ldots(l-m)}\right. \\ &\quad \left.\cdot \sum_{A\in\mathscr{A}_l}\left(\sum_{B\in P_{l-m}(A)}\left(\left(\sum_{s\in B} c(s)p_{\tau(s)}\right) f\left(\frac{\sum_{s\in B} c(s)p_{\tau(s)}M_{\tau(s)}}{\sum_{s\in B} c(s)p_{\tau(s)}}\right)\right)\right)\right) \\ &= \sum_{l=1}^{m}\left(\sum_{A\in\mathscr{A}_l}\left(\sum_{s\in A} c(s)p_{\tau(s)}f(M_{\tau(s)})\right)\right) + \sum_{l=m+1}^{k}\left(\frac{m!}{(l-1)\ldots(l-m)}\right. \\ &\quad \left.\cdot \sum_{A\in\mathscr{A}_l}\left(\sum_{B\in P_{l-m}(A)}\left(f\left(\sum_{s\in B} c(s)p_{\tau(s)}M_{\tau(s)}\right)\right)\right)\right). \end{aligned}$$

Proof. We can prove as in Theorem 9.1, by applying Theorem 2.3. □

9.3.1 Discussion and Applications related to Theorem 9.2

The first application of Theorem 9.2 leads to a generalization of Theorem 9.1.

Theorem 9.3 *Assume ($\mathscr{D}_2$-$\mathscr{D}_4$). For $j = 1,\ldots,n$, we introduce the sets*

$$S_j := \{((i_1,\ldots,i_k),l) \mid (i_1,\ldots,i_k) \in I_k, \quad 1 \le l \le k, \quad i_l = j\}.$$

Let c be a positive function on $S := \bigcup_{j=1}^{n} S_j$ such that

$$\sum_{((i_1,\ldots,i_k),l)\in S_j} c((i_1,\ldots,i_k),l) = 1, \quad j = 1,\ldots,n.$$

Then we have

$$f\left(\sum_{j=1}^{n} p_j M_j\right) \ge N_k \ge N_{k-1} \ge \ldots \ge N_2 \ge N_1 = \sum_{j=1}^{n} p_j f(M_j), \tag{9.14}$$

where the numbers N_{k-m} $(0 \le m \le k-1)$ can be written in the following forms:

$$N_k = \sum_{(i_1,\ldots,i_k)\in I_k} \left(f\left(\sum_{l=1}^{k} c((i_1,\ldots,i_k),l)\, p_{i_l} M_{i_l}\right)\right),$$

and for every $1 \le m \le k-1$

$$N_{k-m} := \frac{m!}{(k-1)\ldots(k-m)} \sum_{(i_1,\ldots,i_k)\in I_k} \left(\sum_{1\le l_1<\ldots<l_{k-m}\le k} \left(f\left(\sum_{l=1}^{k-m} c((i_1,\ldots,i_k),l_j)\, p_{i_{l_j}} M_{i_{l_j}}\right)\right)\right).$$

An immediate consequence of the previous result is Theorem 9.1: by choosing

$$c((i_1,\ldots,i_k),l) := \frac{1}{|S_j|} = \frac{1}{\alpha_{I_k,j}} \quad \text{if} \quad ((i_1,\ldots,i_k),l) \in S_j,$$

we can see that the inequality (9.14) corresponds to the inequality (9.10).

By applying Theorem 9.3 to either the set

$$I_k := \left\{(i_1,\ldots,i_k) \in \{1,\ldots,n\}^k \mid i_1 < \ldots < i_k\right\}, \quad 1 \le k \le n,$$

or the set

$$I_k := \left\{(i_1,\ldots,i_k) \in \{1,\ldots,n\}^k \mid i_1 \le \ldots \le i_k\right\}, \quad 1 \le k,$$

generalizations of Example 9.1 and Example 9.2 are obtained. Therefore Theorem 9.2 also provides the generalizations of the corresponding results given in [59].

Now we apply Theorem 9.2 to some special situations based on Examples 2.9-2.10.

Example 9.7 Let n, m, r be fixed integers, where $n \geq 3$, $m \geq 2$ and $1 \leq r \leq n-2$. In this example, for every $i = 1,2,\ldots,n$ and for every $l = 0,1,\ldots,r$ the integer $i+l$ will be identified with the uniquely determined integer j from $\{1,\ldots,n\}$ for which

$$l+i \equiv j \pmod n. \tag{9.15}$$

Introducing the notation

$$D := \{1,\ldots,n\} \times \{0,\ldots,r\},$$

let for every $j \in \{1,\ldots,n\}$

$$S_j := \{(i,l) \in D \mid i+l \equiv j \pmod n\} \bigcup \{j\},$$

and let $\mathscr{A} \subset P(S)$ $(S := \bigcup_{j=1}^{n} S_j)$ contain the following sets:

$$A_i := \{(i,l) \in D \mid l = 0,\ldots,r\}, \quad i = 1,\ldots,n$$

and

$$A := \{1,\ldots,n\}.$$

Let c be a positive function on S such that

$$\sum_{(i,l)\in S_j} c(i,l) + c(j) = 1, \quad j = 1,\ldots,n.$$

As we have seen in Example 2.9, the sets $S_1,\ldots,S_n$, the partition $\mathscr{A}$ and the function c defined above satisfy the conditions $(\mathscr{H}_4)$ and $(\mathscr{H}_5)$.

Now we suppose that $(\mathscr{D}_2)$-$(\mathscr{D}_4)$ are satisfied. Then by Theorem 9.2, we have

$$f\left(\sum_{j=1}^{n} p_j M_j\right) \geq N_n = \sum_{i=1}^{n}\left(f\left(\sum_{l=0}^{r} c(i,l)\, p_{i+l} M_{i+l}\right)\right) \tag{9.16}$$

$$+ f\left(\sum_{j=1}^{n} c(j) p_j M_j\right) \geq \sum_{j=1}^{n} p_j f(M_j).$$

In case

$$p_j := \frac{1}{n}, \quad j = 1,\ldots,n,$$

$$c(i,l) := \frac{1}{m(r+1)}, \quad (i,l) \in D, \quad c(j) := \frac{m-1}{m} \quad j = 1,\ldots,n,$$

it follows from (9.16) and (9.13) that

$$f\left(\frac{1}{n}\sum_{j=1}^{n} M_j\right) \geq \frac{1}{mn}\sum_{i=1}^{n} f\left(\frac{M_i + M_{i+1} + \ldots + M_{i+r}}{r+1}\right)$$

$$+ \frac{m-1}{m} f\left(\frac{1}{n}\sum_{j=1}^{n} M_j\right) \geq \frac{1}{n}\sum_{j=1}^{n} f(M_j).$$

Example 9.8 Let n and k be fixed positive integers. Let

$$D := \{(i_1,\ldots,i_n) \in \{1,\ldots,k\}^n \mid i_1+\ldots+i_n = n+k-1\},$$

and for each $j=1,\ldots,n$, denote S_j the set

$$S_j := D \times \{j\}.$$

For every $(i_1,\ldots,i_n) \in D$ designate by $A_{(i_1,\ldots,i_n)}$ the set

$$A_{(i_1,\ldots,i_n)} := \{((i_1,\ldots,i_n),l) \mid l = 1,\ldots,n\}.$$

It is obvious that S_j $(j=1,\ldots,n)$ and $A_{(i_1,\ldots,i_n)}$ $((i_1,\ldots,i_n) \in D)$ are decompositions of $S := \bigcup\limits_{j=1}^{n} S_j$ into pairwise disjoint and nonempty sets, respectively. Let c be a function on S such that

$$c((i_1,\ldots,i_n),j) > 0, \quad ((i_1,\ldots,i_n),j) \in S$$

and

$$\sum_{(i_1,\ldots,i_n)\in D} c((i_1,\ldots,i_n),j) = 1, \quad j=1,\ldots,n. \tag{9.17}$$

As in Example 2.10 we have that the conditions $(\mathscr{H}_4)$ and $(\mathscr{H}_5)$ are valid.
Now we suppose that $(\mathscr{D}_2)$-$(\mathscr{D}_4)$ are satisfied. Then, by Theorem 9.2, we have

$$\begin{aligned} f\left(\sum_{j=1}^{n} p_j M_j\right) \geq N_n &= \sum_{(i_1,\ldots,i_n)\in D} f\left(\sum_{l=1}^{n} c((i_1,\ldots,i_n),l)\, p_l M_l\right) \\ &\geq \sum_{j=1}^{n} p_j f(M_j). \end{aligned} \tag{9.18}$$

If we set

$$p_j := \frac{1}{n}, \quad j=1,\ldots,n,$$

and

$$c((i_1,\ldots,i_n),j) := \frac{i_j}{\binom{n+k-1}{k-1}},$$

then (9.17) holds, since by some combinatorial considerations

$$|D| = \binom{n+k-2}{n-1},$$

and

$$\sum_{(i_1,\ldots,i_n)\in D} i_j = \frac{n+k-1}{n}\binom{n+k-2}{n-1} = \binom{n+k-1}{k-1}, \quad j=1,\ldots,n.$$

In this situation (9.18) can therefore be expressed thus

$$f\left(\frac{1}{n}\sum_{j=1}^{n} M_j\right) \geq \frac{1}{\binom{n+k-2}{k-1}} \sum_{(i_1,\ldots,i_n)\in D} f\left(\frac{1}{n+k-1}\sum_{l=1}^{n} i_l M_l\right) \geq \frac{1}{n}\sum_{j=1}^{n} f(M_j).$$

9.4 Parameter Dependent Refinements

Now, we give parameter dependent refinements for determinantal inequalities of Jensen's type. We use the constructions introduced in Section 2.3.

Theorem 9.4 *Let $\lambda \geq 1$ be a real number. Suppose $(\mathscr{D}_2)$-$(\mathscr{D}_4)$ are satisfied, consider the sets*

$$S_k := \left\{ (i_1, \ldots, i_n) \in \mathbb{N}^n \mid \sum_{j=1}^{n} i_j = k \right\}, \quad k \in \mathbb{N},$$

and for $k \in \mathbb{N}$ define the numbers

$$C_k(\lambda) = C_k(M_1, \ldots, M_n; p_1, \ldots, p_n; \lambda)$$

$$:= \frac{1}{(n+\lambda-1)^k} \sum_{(i_1,\ldots,i_n)\in S_k} \frac{k!}{i_1! \ldots i_n!} \left(\sum_{j=1}^{n} \lambda^{i_j} p_j \right) f \left(\frac{\sum_{j=1}^{n} \lambda^{i_j} p_j M_j}{\sum_{j=1}^{n} \lambda^{i_j} p_j} \right)$$

$$= \frac{1}{(n+\lambda-1)^k} \sum_{(i_1,\ldots,i_n)\in S_k} \frac{k!}{i_1! \ldots i_n!} f \left(\sum_{j=1}^{n} \lambda^{i_j} p_j M_j \right).$$

Then

$$f\left(\sum_{j=1}^{n} p_j M_j\right) = C_0(\lambda) \geq C_1(\lambda) \geq \ldots \geq C_k(\lambda) \geq \ldots \geq \sum_{j=1}^{n} p_j f(M_j), \quad k \in \mathbb{N}.$$

Proof. It is similar to the proof of Theorem 9.1, by applying Theorem 2.5. □

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Index

Arithmetic-geometric mean inequality, 79
  for operators, 203
    parameter dependent, 209

BBF functional, 212

Concave function, 2
  operator concave function, 188
Convex function, 1
  J-convex, 1
  $2D$-convex function, 84
  log-convex, 132
  n-exponentially convex, 131
  exponentially convex, 131
    integral form, 132
  operator convex function, 187
  Jensen sense, 1
  mid convex, 1

Divided difference, 132

Green function, 82

Hermite-Hadamard inequality, 109

Jensen's inequality
  $2D$ analogue, 85
  discrete, 3
    parameter dependent refinement, 55
  for operator convex function, 188
  integral analogue, 3

Linear functional, 126
  for convex function, 133

Mean, 19
  Cauchy mean, 137
  Dresher mean, 14
  Hamy's mean, 24
  arithmetic mean, 8
  geometric mean, 8
  integral mean, 19
    integral power mean, 20
  logarithmic mean, 143
  power mean, 19
    for operators, 188
  symmetric mean, 19
  quasi-arithmetic mean, 19
    for operators, 194
      parameter dependent, 208
Mean value theorem, 125
Minkowski's inequality, 182

Operator monotone function, 187

Popoviciu's inequality, 84
  integral version, 135
  of several variables, 84
  weighted form, 81

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