Chapter 1

Introduction

1.1 Convex Functions

Convex functions are very important in the theory of inequalities. The foundations of the theory of convex functions are due to the Danish mathematician and engineer J. L. W. V. Jensen (1859 – 1925).

The natural domain of the different type of convex functions is a convex set in a real vector space V: we say that the subset $C \subset V$ is convex if the segment

$$\{\lambda x_1 + (1-\lambda)x_2 \mid \lambda \in [0,1]\}$$

is a subset of C for every $x_1, x_2 \in C$.

The convex sets in $\mathbb{R}$ exactly the intervals.

Investigation of means under the action of functions is an interesting task. The simplest case which deals with the arithmetic mean leads to the mid-convex (or the J-convex) functions.

***J*-convex function** [69, p.5]: Let V be a real vector space, and $C \subset V$ be a convex set. A function $f : C \to \mathbb{R}$ is called convex in the or mid-convex if

$$f\left(\frac{x_1+x_2}{2}\right) \leq \frac{f(x_1)+f(x_2)}{2} \tag{1.1}$$

for all $x_1, x_2 \in C$.

A J-convex function f is called strictly J-convex if for all pairs of points $(x_1, x_2) \in C \times C$, $x_1 \neq x_2$, strict inequality holds in (1.1).

Convex function [69, p.1]: Let V be a real vector space, and $C \subset V$ be a convex set. A function $f : C \to \mathbb{R}$ is called convex if

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2) \tag{1.2}$$

holds for all $x_1, x_2 \in C$ and $\lambda \in [0,1]$.
f is called strictly convex if strict inequality holds in (1.2) for $x_1 \neq x_2$ and $\lambda \in (0,1)$. If the inequality in (1.2) is reversed, then f is called concave function. If it is strict for all $x_1 \neq x_2$ and $\lambda \in (0,1)$, then f is called strictly concave.

Some characterization of convex functions of a real variable can be found in the following three results.

Theorem 1.1 *[63]Let $I \subset \mathbb{R}$ be an interval. Then $f : I \to \mathbb{R}$ is convex, if and only if*

$$(x_3 - x_2) f(x_1) + (x_1 - x_3) f(x_2) + (x_2 - x_1) f(x_3) \geq 0 \tag{1.3}$$

holds for every $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$. Further, f is strictly convex if and only if $\geq$ is replaced by $>$ in (1.3).

A relation between convex and J-convex functions is as follows.

Theorem 1.2 (J. L. W. V. JENSEN [63, P.10]) *If $f : I \to \mathbb{R}$ is continuous on the interval $I \subset \mathbb{R}$, then f is convex if and only if f is convex in the Jensen sense.*

Next, we give the second derivative test for convexity of a function.

Theorem 1.3 *Let $I \subset \mathbb{R}$ be an open interval, and $f : I \to \mathbb{R}$ be a function such that f'' exits on I. Then f is convex if and only if $f''(x) \geq 0$ $(x \in I)$. If $f''(x) > 0$ $(x \in I)$, then f is strictly convex on the interval.*

***J*-log-convex function** [46]: Let V be a real vector space, and $C \subset V$ be a convex set. A function $f : C \to (0, \infty)$ is called log-convex in the Jensen sense if $\log \circ f$ is J-convex, that is

$$f^2\left(\frac{x_1 + x_2}{2}\right) \leq f(x_1) f(x_2)$$

for all $x_1, x_2 \in C$.
Log-convex function [69, p.7]: Let V be a real vector space, and $C \subset V$ be a convex set. A function $f : C \to (0, \infty)$ is called log-convex if $\log \circ f$ is convex, that is

$$f(\lambda x_1 + (1-\lambda)x_2) \leq f^{\lambda}(x_1) f^{1-\lambda}(x_2),$$

holds for all $x_1, x_2 \in C$ and all $\lambda \in [0,1]$.

Lemma 1.1 ([70]) *Let V be a real vector space, and $C \subset V$ be a convex set. Then a function $f : C \to (0, \infty)$ is log-convex in the Jensen sense if and only if the relation*

$$v^2 f(x_1) + 2vw f\left(\frac{x_1 + x_2}{2}\right) + w^2 f(x_2) \geq 0$$

holds for each real v, w and $x_1, x_2 \in C$.

We denote $x_{[1]} \geq \ldots \geq x_{[n]}$ the components of a vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ arranged in decreasing order. We say that a vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is majorized by a vector $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ $(\mathbf{x} \prec \mathbf{y})$ if

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \quad 1 \leq k \leq n$$

with equality for $k = n$ (see [56]). Then the binary relation $\prec$ over $\mathbb{R}^n$ is reflexive and transitive, i.e. a preorder.

Schur-convex function [56]: Let $D \subset \mathbb{R}^n$. A function $f : D \to \mathbb{R}$ is called Schur-convex if $\mathbf{x} \prec \mathbf{y}$ implies $f(\mathbf{x}) \leq f(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in D$.

The following known result is proved in [56].

Theorem 1.4 *Let $D \subset \mathbb{R}^n$ be a symmetric convex set with nonempty interior D°, and $f : D \to \mathbb{R}$ be a continuous function. If f is differentiable on D°, then f is Schur convex (Schur concave) on D if and only if f is symmetric and*

$$(x_2 - x_1)\left(\frac{\partial f(\mathbf{x})}{\partial x_1} - \frac{\partial f(\mathbf{x})}{\partial x_2}\right) \geq 0 \ (\leq 0)$$

for all $\mathbf{x} = (x_1, \ldots, x_n) \in D^\circ$.

In view of applications in different parts of mathematics the Jensen's inequalities are especially noteworthy, as well as useful.

We begin with the discrete version of the Jensen's inequality:

Theorem 1.5 ***Discrete Jensen's inequality****[69, p.43]: (a) Let V be a real vector space, and $C \subset V$ be a convex set, and $f : C \mapsto \mathbb{R}$ be a convex function. Then*

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \tag{1.4}$$

holds, where $x_i \in C$ $(i = 1, \ldots, n)$ and p_i $(i = 1, \ldots, n)$ are nonnegative real numbers, with $P_n = \sum_{i=1}^{n} p_i > 0$. If f is strictly convex and the p_i's are positive, then inequality (1.4) is strict unless $x_1 = x_2 = \ldots = x_n$.

(b) If $f : C \to \mathbb{R}$ is a J-convex function, and the p_i's are rational numbers $(i = 1, \ldots, n)$, then (1.4) also holds.

The integral version of the Jensen's inequality is as follows:

Theorem 1.6 ***Integral Jensen's inequality[26]:*** *Let $(\Omega, \mathscr{A}, \mu)$ be a finite measure space with $\mu(\Omega) > 0$, and $g : \Omega \to \mathbb{R}$ is a μ-integrable function taking values in an interval $I \subset \mathbb{R}$. Then $\frac{1}{\mu(\Omega)}\int_{\Omega} g\,d\mu$ lies in I, and for every convex function $f : I \to \mathbb{R}$ the composition $f \circ g$ is measurable. Further, if $f \circ g$ is μ-integrable, then*

$$f\left(\frac{1}{\mu(\Omega)}\int_{\Omega} g\,d\mu\right) \leq \frac{1}{\mu(\Omega)}\int_{\Omega} f \circ g\,d\mu. \tag{1.5}$$

In case when f is strictly convex on I equality is satisfied in (1.5) if and only if g is constant μ-almost everywhere on Ω.

1.2 Interpolations of Jensen's Inequality

We start with the following interpolation of the discrete Jensen's inequality based on samples without repetitions given by Pečarić and Volenec in 1988 (see [73]).

Theorem 1.7 *Let C be a convex subset of a real vector space V, and let $f: C \to \mathbb{R}$ be a mid-convex function. If $\mathbf{x} = (x_1,...,x_n) \in C^n$, and*

$$f_{k,n} = f_{k,n}(\mathbf{x}) := \frac{1}{\binom{n}{k}} \sum_{1 \le i_1 < ... < i_k \le n} f\left(\frac{x_{i_1} + \ldots + x_{i_k}}{k}\right), \quad 1 \le k \le n, \tag{1.6}$$

then

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = f_{n,n} \le \ldots \le f_{k+1,n} \le f_{k,n} \le \ldots \le f_{1,n} = \frac{1}{n}\sum_{i=1}^{n} f(x_i), \quad 1 \le k \le n-1. \tag{1.7}$$

The weighted version of the above theorem is given by Pečarić.

Theorem 1.8 ([66]) *Let C be a convex subset of a real vector space V, and let $f: C \to \mathbb{R}$ be a convex function. Suppose $\mathbf{x} = (x_1,...,x_n) \in C^n$, and $\mathbf{p} = (p_1,...,p_n)$ is a positive n-tuple such that $\sum_{i=1}^{n} p_i = 1$. For $k = 1,\ldots,n$ define*

$$f_{k,n}^1 = f_{k,n}^1(\mathbf{x},\mathbf{p}) := \frac{1}{\binom{n-1}{k-1}} \sum_{1 \le i_1 < ... < i_k \le n} \left(\sum_{j=1}^{k} p_{i_j}\right) f\left(\frac{\sum_{j=1}^{k} p_{i_j} x_{i_j}}{\sum_{j=1}^{k} p_{i_j}}\right). \tag{1.8}$$

Then for $1 \le k \le n-1$

$$f\left(\sum_{i=1}^{n} p_i x_i\right) = f_{n,n}^1 \le ... \le f_{k+1,n}^1 \le f_{k,n}^1 \le ... \le f_{1,n}^1 = \sum_{i=1}^{n} p_i f(x_i). \tag{1.9}$$

The following interpolation of the discrete Jensen's inequality based on samples with repetitions is given by Pečarić and Svrtan in 1998 (see [71]).

Theorem 1.9 *[71] Let C be a convex subset of a real vector space V, and let $f: C \to \mathbb{R}$ be a mid-convex function. If $\mathbf{x} = (x_1,...,x_n) \in C^n$, and*

$$\overline{f}_{k,n} = \overline{f}_{k,n}(\mathbf{x}) := \frac{1}{\binom{n+k-1}{k}} \sum_{1 \le i_1 \le ... \le i_k \le n} f\left(\frac{x_{i_1} + \ldots + x_{i_k}}{k}\right), \quad k \ge 1,$$

then

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \le \ldots \le \overline{f}_{k+1,n} \le \overline{f}_{k,n} \le \ldots \le \overline{f}_{1,n} = \frac{1}{n}\sum_{i=1}^{n} f(x_i). \tag{1.10}$$

The weighted version of the above theorem causes motivation for many authors and it can be found in [60, p.8].

Theorem 1.10 *Let C be a convex subset of a real vector space V, and let $f : C \to \mathbb{R}$ be a convex function. Suppose $\mathbf{x} = (x_1, ..., x_n) \in C^n$, and $\mathbf{p} = (p_1, ..., p_n)$ is a positive n-tuple such that $\sum_{i=1}^{n} p_i = 1$. Then*

$$f(\sum_{i=1}^{n} p_i x_i) \leq ... \leq f_{k+1,n}^2 \leq f_{k,n}^2 \leq ... \leq f_{1,n}^2 = \sum_{i=1}^{n} p_i f(x_i), \tag{1.11}$$

where

$$f_{k,n}^2 = f_{k,n}^2(\mathbf{x}, \mathbf{p}) = \frac{1}{\binom{n+k-1}{k-1}} \sum_{1 \leq i_1 \leq ... \leq i_k \leq n} \left(\sum_{j=1}^{k} p_{i_j} \right) f \left(\frac{\sum_{j=1}^{k} p_{i_j} x_{i_j}}{\sum_{j=1}^{k} p_{i_j}} \right), \quad k \geq 1. \tag{1.12}$$

Remark 1.1 If f is a concave function then the inequalities (1.9) and (1.11) are reversed.
If p_i $(i = 1, \ldots, n)$ are rational numbers, then (1.9) and (1.11) are also valid for mid-convex functions.

An important consequence of the discrete Jensen's inequality for mid-convex functions is the following Key Lemma from [71].

Lemma 1.2 *Let C be a convex subset of real linear space V, $f : C \to \mathbb{R}$ be a mid-convex function, and $\mathbf{x} = (x_1, ..., x_n) \in C^n$. Then*

$$f(\frac{1}{n} \sum_{j=1}^{n} x_j) \leq \frac{1}{n} \sum_{j=1}^{n} f \left(\frac{x_1 + \ldots + \hat{x}_j + \ldots + x_n}{n-1} \right), \tag{1.13}$$

where $\hat{x}_j$ means that x_j is omitted.

Proof. Apply the discrete Jensen's inequality for mid-convex functions to

$$x^{(i)} := \big(1/(n-1)\big) (x_1 + ... + \hat{x}_i + ... + x_n),$$

and use the identity $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x^{(i)}$. □

Unified treatment for samples with and without repetitions: Assume $f : C \to \mathbb{R}$ is a mid-convex function defined on a convex set C in a real linear space V, and $\mathbf{x} = (x_1, ..., x_n) \in C^n$. Let $M = \{1^{m_1}, 2^{m_2}, \ldots, n^{m_n}\}$ be a fixed multiset having $m_j =: \nu_j(M) \geq 1$ elements equal to j, for $1 \leq j \leq n$. $N_k(M)$ denotes the k-th rank number of M (the number of subsets of M containing exactly k elements). For every nonempty submultiset $I \subset M$, $x_I := \sum_{i \in I} x_i$, and $|I|$ means the number of elements in I. Now, define the M-dominated k-sample mean of f by

$$f_{k,n}^M = f_{k,n}^M(\mathbf{x}) := \frac{1}{N_k(M)} \sum_{\substack{I \subset M \\ |I| = k}} f(\frac{1}{k} x_I), \quad 1 \leq k \leq m_1 + \ldots + m_n.$$

The following Proposition makes a unified treatment of Theorems 1.7 and 1.9.

Proposition 1.1 *Under the previous assumptions, we have*

$$N_{k+1}(M) f^M_{k+1,n} = \sum_{J \subset M, |J|=k+1} f\left(\frac{1}{k+1} x_J\right) \leq \frac{1}{k+1} \sum_{I \subset M, |I|=k} c_I f\left(\frac{1}{k} x_I\right) \tag{1.14}$$

for every $1 \leq k < m_1 + \ldots + m_n$, *where* $c_I := \sum_{\substack{1 \leq j \leq n \\ \nu_j(I) < m_j}} (\nu_j(I) + 1)$.

Proof. By applying Lemma 1.2 to the terms of the middle sum in (1.14), we have

$$\sum_{J \subset M, |J|=k+1} f\left(\frac{1}{k+1} x_J\right) \leq \frac{1}{k+1} \sum_{J \subset M, |J|=k+1} \sum_{j \in J} f\left(\frac{1}{k} x_{J \setminus \{j\}}\right).$$

Then, the right hand side can be rewritten as

$$\frac{1}{k+1} \sum_{I \subset M, |I|=k} c_I f\left(\frac{1}{k} x_I\right),$$

where c_I can be calculated in the following way: let

$$A_I := \{J \subset M \mid J = I \uplus \{j\} \text{ for some } 1 \leq j \leq n\},$$

where $\uplus$ means the multiset sum, and for $J \in A_I$ let $c_I(J)$ be the number of all elements j of J such that $I = J \setminus \{j\}$; then

$$c_I = \sum_{J \in A_I} c_I(J) = \sum_{\substack{1 \leq j \leq n \\ \nu_j(I) < m_j}} (\nu_j(I) + 1).$$

The proof is complete. □

Now we show that Theorems 1.7 and 1.9 are special cases of Proposition 1.1.

Corollary 1.1 *Let C be a convex subset of a real vector space V, and let $f : C \to \mathbb{R}$ be a mid-convex function. If $\mathbf{x} = (x_1, \ldots, x_n) \in C^n$, then the following refinements of the Jensen's inequality hold:*

a)

$$f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) = f_{n,n} \leq \ldots \leq f_{k+1,n} \leq f_{k,n} \leq \ldots \leq f_{1,n} = \frac{1}{n} \sum_{i=1}^n f(x_i),$$

b)

$$f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) = \ldots \leq \bar{f}_{k+1,n} \leq \bar{f}_{k,n} \leq \ldots \leq \bar{f}_{1,n} = \frac{1}{n} \sum_{i=1}^n f(x_i).$$

Proof. (a) We take M to be the following multiset (actually a set): $M := \{1,\ldots,n\}$. In this case $\nu_j(M) = 1$ $(1 \le j \le n)$, and

$$\sum_{I\subset M,|I|=k} f\left(\frac{1}{k}x_I\right) = \binom{n}{k} f_{k,n}, \quad k = 1,\ldots,n-1.$$

By (1.14), this implies that

$$\binom{n}{k+1} f_{k+1,n} \le \frac{1}{k+1} \sum_{I\subset M,|I|=k} c_I f\left(\frac{1}{k}x_I\right)$$

$$= \frac{1}{k+1}(n-k)\binom{n}{k} f_{k,n} = \binom{n}{k+1} f_{k,n}, \quad k = 1,\ldots,n-1,$$

finishes the proof of the first claim.

(b) Let the integers $k \ge 1$ and $l \ge k+1$ be fixed, and let M be the following multiset: $M := \{1^l,\ldots,n^l\}$ (the multiplicity of j is l for $1 \le j \le n$). Then

$$\sum_{I\subset M,|I|=k} f\left(\frac{1}{k}x_I\right) = \binom{n+k-1}{k} \overline{f}_{k,n}, \quad k = 1,\ldots,l.$$

This yields by (1.14)

$$\binom{n+k}{k+1} \overline{f}_{k+1,n} \le \frac{1}{k+1} \sum_{I\subset M,|I|=k} c_I f\left(\frac{1}{k}x_I\right)$$

$$= \frac{1}{k+1}(k+n)\binom{n+k-1}{k} \overline{f}_{k,n} = \binom{n+k}{k+1} \overline{f}_{k,n},$$

and therefore

$$\overline{f}_{k+1,n} \le \overline{f}_{k,n}, \quad k \ge 1.$$

□

The following result is given in [20]:

Theorem 1.11 *Let C be a convex subset of a real vector space V, and let $f : C \to \mathbb{R}$ be a convex function. Suppose $\mathbf{x} = (x_1,\ldots,x_n) \in C^n$, and $\mathbf{p} = (p_1,\ldots,p_n)$ is a nonnegative n-tuple such that $\sum_{i=1}^n p_i = 1$. If*

$$f^3_{k,n} = f^3_{k,n}(\mathbf{x},\mathbf{p}) := \sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\ldots p_{i_k} f\left(\frac{1}{k}\sum_{j=1}^{k} x_{i_j}\right), \quad k \ge 1, \tag{1.15}$$

then

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \le \ldots \le f^3_{k+1,n} \le f^3_{k,n} \le \ldots \le f^3_{1,n} = \sum_{i=1}^{n} p_i f(x_i), \quad k \ge 1. \tag{1.16}$$

The next result comes from [19] and [74] (see also Theorem 3.36 in [69, p.97]).

Theorem 1.12 *Let $I \subset \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ be a convex function, σ be an increasing function on $[0,1]$ such that $\int_0^1 d\sigma(x) = 1$, and $u : [0,1] \to I$ be σ-integrable. If $f \circ u$ is also σ-integrable, then*

$$\begin{aligned} f\left(\int_0^1 u(x)d\sigma(x)\right) &\leq \int_{[0,1]^{k+1}} f\left(\frac{1}{k+1}\sum_{i=1}^{k+1} u(x_i)\right)\prod_{i=1}^{k+1} d\sigma(x_i) \\ &\leq \int_{[0,1]^k} f\left(\frac{1}{k}\sum_{i=1}^{k} u(x_i)\right)\prod_{i=1}^{k} d\sigma(x_i) \leq \ldots \\ &\leq \int_{[0,1]^2} f\left(\frac{1}{2}\sum_{i=1}^{2} u(x_i)\right)\prod_{i=1}^{2} d\sigma(x_i) \\ &\leq \int_0^1 f\left(u(x)\right) d\sigma(x), \end{aligned} \tag{1.17}$$

for all positive integers k.

1.3 Quotients for samples without repetitions

Let $I \subset \mathbb{R}$ be an interval, and $f : I \to \mathbb{R}$. Consider the following notations: for $x_i \in I$ $(1 \leq i \leq n)$

$$\begin{aligned} &\mathbf{x} := (x_1, \ldots, x_n);\ f(\mathbf{x}) := (f(x_1), \ldots, f(x_n)); \\ &\text{arithmetic mean: } A(\mathbf{x}) := \tfrac{1}{n}(x_1 + \cdots + x_n); \\ &\text{geometric mean: } G(\mathbf{x}) := \sqrt[n]{x_1 \cdots x_n} \quad (I \subset [0,\infty)). \end{aligned}$$

Then the discrete Jensen's inequality for equal weights is

$$f(A(\mathbf{x})) \leq A(f(\mathbf{x})), \tag{1.18}$$

where $f : I \to \mathbb{R}$ is a convex function, and $\mathbf{x} \in I^n$. The inequality is clearly reversed if $f : I \to \mathbb{R}$ is concave function.

In this context, (1.7) can be written as

$$f(A(\mathbf{x})) = f_{n,n} \leq \ldots \leq f_{k+1,n} \leq f_{k,n} \cdots \leq f_{1,n} = A(f(\mathbf{x})), \quad 1 \leq k \leq n-1. \tag{1.19}$$

In 2003, Tang and Wen [76] obtained the following inequalities which contain (1.19): For all $1 \leq r \leq j \leq s \leq i \leq n$, the following refinement holds:

$$f_{r,s,n} \geq \cdots \geq f_{r,s,i} \geq \cdots \geq f_{r,s,s} \geq \cdots \geq f_{r,j,j} \geq \cdots \geq f_{r,r,r} = 0, \tag{1.20}$$

where

$$f_{r,s,n} := \binom{n}{r}\binom{n}{s}(f_{r,n} - f_{s,n}).$$

Equality conditions are also considered.

In 2008, Gao and Wen [22] obtained the following results in this direction:

Theorem 1.13 *Let $I \subset \mathbb{R}$ be an interval. If $f : I \to \mathbb{R}$, $\mathbf{a} = (a_1,\dots,a_n)$, $\mathbf{b} = (b_1,\dots,b_n) \in I^n$ $(n \geq 2)$ and*

(i) $a_1 \leq \cdots \leq a_n \leq b_n \leq \ldots \leq b_1, a_1 + b_1 \leq \cdots \leq a_n + b_n$,

(ii) $f(t) > 0, f'(t) > 0,\ f''(t) > 0,\ f'''(t) < 0$ for every $t \in I$,

then

$$\frac{f(A(\mathbf{a}))}{f(A(\mathbf{b}))} = \frac{f_{n,n}(\mathbf{a})}{f_{n,n}(\mathbf{b})} \leq \cdots \leq \frac{f_{k+1,n}(\mathbf{a})}{f_{k+1,n}(\mathbf{b})} \leq \frac{f_{k,n}(\mathbf{a})}{f_{k,n}(\mathbf{b})} \leq \cdots \leq \frac{f_{1,n}(\mathbf{a})}{f_{1,n}(\mathbf{b})} = \frac{A(f(\mathbf{a}))}{A(f(\mathbf{b}))}, \quad 1 \leq k \leq n-1. \tag{1.21}$$

The inequalities are reversed for $f''(t) < 0,\ f'''(t) > 0$ $(t \in I)$. Equality signs hold if and only if $a_1 = \cdots = a_n$ and $b_1 = \cdots = b_n$.

Moreover, Wen and Wang [82] considered some inequalities for linear combinations involving $f_{k,n}$.

Another type of generalization is due to Wen [80]: Let $I \subset \mathbb{R}$ be an interval, and let $f : I \to \mathbb{R}$ be twice continuously differentiable such that f'' is convex. Then

$$f''(D_3(\mathbf{x})) \leq \frac{2J[f(\mathbf{x})]}{J[\mathbf{x}^2]} \leq \frac{1}{3}\left[\max_{1\leq i\leq n}\{f''(x_i)\} + A\left(f''(\mathbf{x})\right) + f''(A(\mathbf{x}))\right], \tag{1.22}$$

where

$$D_3(\mathbf{x}) := \frac{1}{3}\frac{A(\mathbf{x}^3) - A^3(\mathbf{x})}{A(\mathbf{x}^2) - A^2(\mathbf{x})},$$
$$J[f(\mathbf{x})] := A(f(\mathbf{x})) - f(A(\mathbf{x})), \quad J\left[\mathbf{x}^2\right] := A(\mathbf{x}^2) - A^2(\mathbf{x}).$$

In [81] an other kind of interesting inequalities, centering about the topic of refinements involving quotients of two functions, are given.

Theorem 1.14 *Let the functions*

$$f : [a,b] \to (0,\infty), g : [a,b] \to (0,\infty)$$

satisfying

$$\sup_{t\in[a,b]}\left\{\left|\frac{g''(t)}{f''(t)}\right|\right\} < \inf_{t\in[a,b]}\left\{\frac{g(t)}{f(t)}\right\}.$$

If $f''(t) > 0$ for each $t \in [a,b]$, then for any $\mathbf{x} \in [a,b]^n$, we have the following inequalities of Jensen-Pečarić-Svrtan-Fan (Abbreviated as J-P-S-F) type:

$$\frac{f(A(\mathbf{x}))}{g(A(\mathbf{x}))} = \frac{f_{n,n}(A(\mathbf{x}))}{g_{n,n}(A(\mathbf{x}))} \leq \cdots \leq \frac{f_{k+1,n}(A(\mathbf{x}))}{g_{k+1,n}(A(\mathbf{x}))} \tag{1.23}$$

$$\leq \frac{f_{k,n}(A(\mathbf{x}))}{g_{k,n}(A(\mathbf{x}))} \leq \cdots \leq \frac{f_{1,n}(A(\mathbf{x}))}{g_{1,n}(A(\mathbf{x}))} = \frac{A(f(\mathbf{x}))}{A(g(\mathbf{x}))}, \quad 1 \leq k \leq n-1.$$

If $f''(t) < 0$ for each $t \in [a,b]$, then the above inequalities are reversed. In each case, the sign of the equality holding throughout if and only if $x_1 = \cdots = x_n$.

Proof of Theorem 1.14: To prove Theorem 1.14, we set

$$\alpha := (\alpha_1, \ldots, \alpha_n); \quad \Omega_n := \{\alpha \in [0,1]^n | \alpha_1 + \cdots + \alpha_n = 1\},$$

$$S_f(\alpha, \mathbf{x}) := \frac{1}{n!} \sum_{i_1 \cdots i_n} f(\alpha_1 x_{i_1} + \cdots + \alpha_n x_{i_n}); \quad F(\alpha) := \log \frac{S_f(\alpha, \mathbf{x})}{S_g(\alpha, \mathbf{x})};$$

$$u_{\mathbf{i}}(\mathbf{x}) := \alpha_1 x_{i_1} + \alpha_2 x_{i_2} + \sum_{j=3}^{n} \alpha_j x_{i_j}; \quad v_{\mathbf{i}}(\mathbf{x}) := \alpha_1 x_{i_2} + \alpha_2 x_{i_1} + \sum_{j=3}^{n} \alpha_j x_{i_j}. \tag{1.24}$$

Here and in the sequel $\mathbf{x} \in [a,b]^n$, $\alpha \in \Omega_n$, $\mathbf{i} = (i_1, \ldots, i_n)$, and let $i_1 \cdots i_n$ and $i_3 \cdots i_n$ denote the possible permutations of $N_n = \{1, \ldots, n\}$ and the possible permutations of $N_n \setminus \{i_1, i_2\}$, respectively.

We start with two lemmas.

Lemma 1.3 *Under the hypotheses of Theorem 1.14, there exist $\xi_{\mathbf{i}}$ and $\xi_{\mathbf{i}}^*$ between $u_{\mathbf{i}}(\mathbf{x})$ and $v_{\mathbf{i}}(\mathbf{x})$ such that*

$$\begin{aligned} (\alpha_1 - \alpha_2)\left(\frac{\partial F(\alpha)}{\partial \alpha_1} - \frac{\partial F(\alpha)}{\partial \alpha_2}\right) &= \frac{1}{n!} \sum_{i_3 \cdots i_n} \sum_{1 \leq i_1 < i_2 \leq n} \frac{f''(\xi_{\mathbf{i}})(u_{\mathbf{i}}(\mathbf{x}) - v_{\mathbf{i}}(\mathbf{x}))^2}{S_f(\alpha, \mathbf{x})} \\ &\quad \times \left(1 - \frac{g''(\xi_{\mathbf{i}}{}^*)}{f''(\xi_{\mathbf{i}}{}^*)} \frac{S_f(\alpha, \mathbf{x})}{S_g(\alpha, \mathbf{x})}\right). \end{aligned} \tag{1.25}$$

Proof. Note the following identities:

$$\begin{aligned} S_f(\alpha, \mathbf{x}) &:= \frac{1}{n!} \sum_{i_3 \cdots i_n} \sum_{1 \leq i_1 \neq i_2 \leq n} f(\alpha_1 x_{i_1} + \cdots + \alpha_n x_{i_n}) \\ &= \frac{1}{n!} \sum_{i_3 \cdots i_n} \sum_{1 \leq i_1 < i_2 \leq n} [f(u_{\mathbf{i}}(\mathbf{x})) - f(v_{\mathbf{i}}(\mathbf{x}))]; \end{aligned}$$

similarly,

$$S_g(\alpha, \mathbf{x}) = \frac{1}{n!} \sum_{i_3 \cdots i_n} \sum_{1 \leq i_1 < i_2 \leq n} [g(u_{\mathbf{i}}(\mathbf{x})) - g(v_{\mathbf{i}}(\mathbf{x}))];$$

$$\begin{aligned} &\frac{\partial}{\partial \alpha_1}[f(u_{\mathbf{i}}(\mathbf{x})) + f(v_{\mathbf{i}}(\mathbf{x}))] - \frac{\partial}{\partial \alpha_2}[f(u_{\mathbf{i}}(\mathbf{x})) + f(v_{\mathbf{i}}(\mathbf{x}))] \\ &= [x_{i_1} f'(u_{\mathbf{i}}(\mathbf{x})) + x_{i_2} f'(v_{\mathbf{i}}(\mathbf{x}))] - [x_{i_2} f'(u_{\mathbf{i}}(\mathbf{x})) + x_{i_1} f'(v_{\mathbf{i}}(\mathbf{x}))] \\ &= [f'(u_{\mathbf{i}}(\mathbf{x})) - f'(v_{\mathbf{i}}(\mathbf{x}))](x_{i_1} - x_{i_2}); \end{aligned}$$

similarly,

$$\begin{aligned} &\frac{\partial}{\partial \alpha_1}[g(u_{\mathbf{i}}(\mathbf{x})) + f(v_{\mathbf{i}}(\mathbf{x}))] - \frac{\partial}{\partial \alpha_2}[g(u_{\mathbf{i}}(\mathbf{x})) + g(v_{\mathbf{i}}(\mathbf{x}))] \\ &= [g'(u_{\mathbf{i}}(\mathbf{x})) - g'(v_{\mathbf{i}}(\mathbf{x}))](x_{i_1} - x_{i_2}). \end{aligned}$$