MONOGRAPHS IN INEQUALITIES 9

Jensen Inequalities on Time Scales

Theory and Applications

Josipa Barić, Rabia Bibi, Martin Bohner, Ammara Nosheen and Josip Pečarić

Jensen Inequalities on Time Scales

Theory and Applications

Josipa Barić
Faculty of Electrical Engineering, Mechanical
Engineering and Naval Architecture
University of Split
Split, Croatia

Rabia Bibi
Center for Advanced Mathematics and Physics
National University of Sciences and Technology
Islamabad, Pakistan

Martin Bohner
Department of Mathematics and Statistics
Missouri University of Science and Technology
Rolla, USA

Ammara Nosheen
Department of Mathematics
University of Sargodha
Sargodha, Pakistan

Josip Pečarić
Faculty of Textile Technology
University of Zagreb
Zagreb, Croatia

ELEMENT

Zagreb, 2015

MONOGRAPHS IN INEQUALITIES 9

Jensen Inequalities on Time Scales

Theory and Applications

Josipa Barić, Rabia Bibi, Martin Bohner, Ammara Nosheen
and Josip Pečarić

This work has been supported by Croatian Science Foundation under the project 5435.

1st edition

Printed in Croatia.

A CIP catalogue record for this book is available from the National and University Library in Zagreb under 000908072.

ISBN 978-953-197-597-1

Preface

Jensen's inequality, named after the Danish mathematician Johan Ludwig William Valdemar Jensen (May 8, 1859 – March 5, 1925), relates the value of a convex function of an integral to the integral of that convex function. The discrete version of this inequality relates the value of a convex function of a sum to the sum of that convex function. Both continuous and discrete versions appear in numerous forms and have a rich history of applications in mathematics and many different sciences such as statistics, economics, physics, and information theory. As such, Jensen's inequality and Jensen-type inequalities together with their converses constitute an extremely important tool in mathematical analysis.

This book summarizes very recent research, published during the last three years by the authors and their collaborators, related to Jensen-type inequalities on time scales. Many of the presented results are proved via the theory of isotonic linear functionals. As such, this book combines three areas of classical and active current research:

1. Classical inequalities in analysis.

2. Dynamic equations on time scales.

3. Isotonic linear functionals.

Jensen-type inequalities have a long history of research, both in the continuous and the discrete case. Related inequalities involving functions and their integrals and derivatives are the following: Hardy's inequality, Hermite–Hadamard inequalities, converses of Jensen's inequality, Hölder's inequality, Minkowski's inequality, the Cauchy–Schwartz inequality, the Jensen–Steffensen inequality, Jessen's inequality, Jensen–Mercer inequalities, Beckenbach–Dresher inequalities, Bellman's inequality, Popovicu inequalities, Diaz–Metcalf inequalities, Slater's inequality, and Aczél's inequality, to name but a few. Many of these classical inequalities may be found in the monograph by Mitrinović, Pečarić, and Fink [103], both for the continuous and the discrete case. Time scales versions of all of the above and further inequalities are contained in the current book.

A time scale is an arbitrary nonempty closed subset of the real numbers. The concept of derivatives and integrals on time scales is designed in such a way that, if the time scale is equal to the set of all real numbers, then the derivative is the same as the usual derivative and the integral is the usual integral; if the time scale is equal to the set of all integers, then the derivative is the same as the usual forward difference and the integral is a sum; and if the time scale is equal to all integer powers of a fixed number bigger than one, then the derivative is the same as the usual Jackson derivative and the integral is the usual

Jackson integral. Of course there are many other examples of time scales, and as such, this theory allows for a unification of continuous and discrete calculus and for the study of cases “in between”. The theory of dynamic equations is rather young and goes back to the 1988 dissertation of Stefan Hilger. For an introduction with applications, we refer to the monograph by Bohner and Peterson [45].

The connection point between the previously mentioned two research areas is theory of isotonic linear functionals as presented in the monograph by Pečarić, Proschan, and Tong [119]. In there, many of the classical inequalities are proved for so-called isotonic linear functionals. It turns out now that the time scales integral is indeed an isotonic linear functionals, and thus the theory of isotonic linear functionals can be applied to it. This is true for many time scales integrals, such as the time scales Cauchy delta integral, the time scales Cauchy nabla integral, the α-diamond integral, the time scales Riemann and multiple Riemann integrals, and the time scales Lebesgue and multiple Lebesgue integrals.

The set up of this book is as follows: In Chapter 1, we give an introduction to convex functions, superquadratic functions, the theory of dynamic equations on time scales, and some basic elementary related inequalities. Chapter 2 discusses Jensen-type inequalities for convex and superquadratic functions. These results were published just recently in 2011 by Anwar, Bibi, Bohner, and Pečarić [20] and in 2013 by Barić, Bibi, Bohner, and Pečarić [27]. In Chapter 3, we introduce Jensen functionals, their properties, and applications, following closely the 2012 paper [21] by Anwar, Bibi, Bohner, and Pečarić. The case of several variables was published in 2014 by Anwar, Bibi, Bohner, and Pečarić [22] and is presented in Chapter 4. In Chapter 5, improvements of the Jensen–Steffensen inequality and its converses are given, summarizing results from the 2014 publication by Bibi, Pečaric, and Rodić Lipanović [39]. Chapters 6 and 7 contain Hermite–Hadamard inequalities in the single and several variables case, respectively, as derived in the soon-to-appear publications [38] and [37], respectively, by Bibi, Pečarić, and Perić. Cauchy-type means and exponential and logarithmic convexity for superquadratic functions on time scales, following the 2015 publication [35] by Bibi, Bohner, and Pečarić, are presented in Chapter 8. Chapter 9 features, among others, results from the 2013 paper [34] by Bibi, Bohner, Pečarić and Varošanec and discusses inequalities of Hölder and Minkowski type and related functionals. Finally, Chapter 10 presents the theory and applications of dynamic Hardy-type inequalities with general kernels. The results given in this chapter are based, amongh others, on the 2014 publication by Bohner, Nosheen, Pečarić, and Younus [44].

The authors would like to thank all of their collaborators and colleagues that helped in developing the results presented in this monograph. Moreover, we would like to thank the staff of the Element Publishing House as well as the referees who have looked at our manuscript in detail.

January 2015
Josipa Barić (Split, Croatia)
Rabia Bibi (Islamabad, Pakistan)
Martin Bohner (Rolla, USA)
Ammara Nosheen (Sargodha, Pakistan)
Josip Pečarić (Zagreb, Croatia)

Contents

Chapter 1

Introduction

1.1 Convex Functions

Definition 1.1 (SEE [119, DEFINITION 1.1]) (a) *Let I be an interval in $\mathbb{R}$. Then $\Phi : I \to \mathbb{R}$ is said to be convex if for all $x, y \in I$ and all $\alpha \in [0,1]$,*

$$\Phi(\alpha x + (1-\alpha)y) \leq \alpha\Phi(x) + (1-\alpha)\Phi(y) \tag{1.1}$$

holds. If (1.1) *is strict for all $x \neq y$ and $\alpha \in (0,1)$, then Φ is said to be strictly convex.*

(b) *If the inequality in* (1.1) *is reversed, then Φ is said to be concave. If it is strict for all $x \neq y$ and $\alpha \in (0,1)$, then Φ is said to be strictly concave.*

There are several equivalent ways to define convex functions, sometimes it is better to define convex function in one way than the other.

Remark 1.1 (SEE [119, REMARKS 1.2]) (a) For $x, y \in I$, $p, q \geq 0$, $p+q > 0$, (1.1) is equivalent to

$$\Phi\left(\frac{px+qy}{p+q}\right) \leq \frac{p\Phi(x)+q\Phi(y)}{p+q}.$$

(b) Let x_1, x_2, x_3 be three points in I such that $x_1 < x_2 < x_3$. Then (1.1) is equivalent to

$$\begin{vmatrix} x_1 & \Phi(x_1) & 1 \\ x_2 & \Phi(x_2) & 1 \\ x_3 & \Phi(x_3) & 1 \end{vmatrix} = (x_3-x_2)\Phi(x_1) + (x_1-x_3)\Phi(x_2) + (x_2-x_1)\Phi(x_3) \geq 0,$$

which is further equivalent to

$$\Phi(x_2) \leq \frac{x_2 - x_3}{x_1 - x_3}\Phi(x_1) + \frac{x_1 - x_2}{x_1 - x_3}\Phi(x_3). \tag{1.2}$$

More symmetrically and without the condition of monotonicity on x_1, x_2, x_3, we can write

$$\frac{\Phi(x_1)}{(x_1 - x_2)(x_1 - x_3)} + \frac{\Phi(x_2)}{(x_2 - x_3)(x_2 - x_1)} + \frac{\Phi(x_3)}{(x_3 - x_1)(x_3 - x_2)} \geq 0.$$

(c) Φ is both convex and concave if and only if

$$\Phi(x) = \lambda x + c$$

for some $\lambda, c \in \mathbb{R}$.

(d) Another way of writing (1.2) is instructive:

$$\frac{\Phi(x_1) - \Phi(x_2)}{x_1 - x_2} \leq \frac{\Phi(x_2) - \Phi(x_3)}{x_2 - x_3}, \tag{1.3}$$

where $x_1 < x_3$ and $x_1, x_3 \neq x_2$. Hence the following result is valid:

A function Φ is convex on I if and only if for every point $c \in I$, the function $(\Phi(x) - \Phi(c))/(x - c)$ is increasing on I $(x \neq c)$.

(e) By using (1.3), we can easily prove the following result:

If Φ is a convex function on I and if $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid:

$$\frac{\Phi(x_2) - \Phi(x_1)}{x_2 - x_1} \leq \frac{\Phi(y_2) - \Phi(y_1)}{y_2 - y_1}.$$

The following two theorems concern derivatives of convex functions.

Theorem 1.1 (SEE [119, THEOREM 1.3]) *Let I be an interval in $\mathbb{R}$ and $\Phi : I \to \mathbb{R}$ be convex. Then*

(i) *Φ is Lipschitz on any closed interval in I;*

(ii) *Φ'_+ and Φ'_- exist and are increasing in I, and $\Phi'_- \leq \Phi'_+$ (if Φ is strictly convex, then these derivatives are strictly increasing); and*

(iii) *Φ' exists, except possibly on a countable set, and on the complement of which it is continuous.*

Remark 1.2 (SEE [119, THEOREM 1.4]) In Theorem 1.1, if Φ'' exists on I, then Φ is convex if and only if $\Phi''(x) \geq 0$. If $\Phi''(x) > 0$, then Φ is strictly convex.

Theorem 1.2 (SEE [119, THEOREM 1.6]) *Let I be an open interval in $\mathbb{R}$.*

(a) *$\Phi : I \to \mathbb{R}$ is convex if and only if there is at least one line of support for Φ at each $x_0 \in I$, i.e.,*

$$\Phi(x) \geq \Phi(x_0) + \lambda(x - x_0) \quad \text{for all} \quad x \in (a,b),$$

where λ depends on x_0 and is given by $\lambda = \Phi'(x_0)$ when Φ' exists, and $\lambda \in [\Phi'_-(x_0), \Phi'_+(x_0)]$ when $\Phi'_-(x_0) \neq \Phi'_+(x_0)$.

(b) *$\Phi : I \to \mathbb{R}$ is convex if the function $\Phi(x) - \Phi(x_0) - \lambda(x - x_0)$ (the difference between the function and its support) is decreasing for $x < x_0$ and increasing for $x > x_0$.*

When dealing with functions with different degree of smoothness, divided differences are found to be very useful.

Definition 1.2 (SEE [119, PAGE 14]) *Let Φ be a real-valued function defined on $[a,b] \subset \mathbb{R}$. The kth-order divided difference of Φ at distinct points $x_0, \ldots, x_k$ in $[a,b]$ is defined recursively by*

$$[x_i; \Phi] = \Phi(x_i), \quad i \in \{0, 1, \ldots, k\}$$

and

$$[x_0, \ldots, x_k; \Phi] = \frac{[x_1, \ldots, x_k; \Phi] - [x_0, \ldots, x_{k-1}; \Phi]}{x_k - x_0}.$$

Remark 1.3 In Definition 1.2, the value $[x_0, \ldots, x_k; \Phi]$ is independent of the order of the points $x_0, \ldots, x_k$. This definition may be extended to include the case in which some or all of the points coincide by assuming that $x_0 \leq \ldots \leq x_k$ and letting

$$[\underbrace{x, \ldots, x}_{(j+1\, times)}; \Phi] = \Phi^{(j)}(x)/j!,$$

provided that $\Phi^{(j)}$ exists.

Definition 1.3 (SEE [119, PAGE 15]) *A real-valued function Φ defined on $[a,b] \subset \mathbb{R}$ is said to be n-convex, $n \geq 0$, on $[a,b]$ if and only if for all choices of $(n+1)$ distinct points in $[a,b]$,*

$$[x_0, \ldots, x_n; \Phi] \geq 0.$$

Remark 1.4 A function $\Phi : I \to \mathbb{R}$ is convex if and only if for every choice of three mutually different points $x_0, x_1, x_2 \in I$, $[x_0, x_1, x_2; \Phi] \geq 0$ holds.

The definition of a convex function has a very natural generalization to real-valued functions defined on $\mathbb{R}^n$. Here we merely require that the domain U of Φ be convex, i.e., $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in U$ whenever $\mathbf{x}, \mathbf{y} \in U$ and $\alpha \in [0,1]$.

Definition 1.4 *Let U be a convex set in $\mathbb{R}^n$. Then $\Phi : U \to \mathbb{R}$ is said to be convex if for all $\mathbf{x}, \mathbf{y} \in U$ and all $\alpha \in [0,1]$, we have*

$$\Phi(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \leq \alpha\Phi(\mathbf{x}) + (1-\alpha)\Phi(\mathbf{y}). \tag{1.4}$$

J-Convex Function

In the theory of convex functions, the most known case is that of J-convex functions, which deals with the arithmetic mean.

Definition 1.5 (SEE [119, DEFINITION 1.8]) *Let $I \subset \mathbb{R}$ be an interval. A function Φ : $I \to \mathbb{R}$ is called convex in the Jensen sense (or J-convex) on I if for all $x, y \in I$, the inequality*

$$\Phi\left(\frac{x+y}{2}\right) \leq \frac{\Phi(x)+\Phi(y)}{2} \tag{1.5}$$

holds. A J-convex function Φ is said to be strictly J-convex if for all pairs of points (x, y), $x \neq y$, strict inequality holds in (1.5)*.*

Remark 1.5 (SEE [119, THEOREM 1.10]) (i) It can be easily seen that a convex function is J-convex. For continuous functions, J-convex functions are equivalent to convex functions.

(ii) The inequality (1.5) can easily be extended to the convex combination of finitely many points and next to random variables associated to arbitrary probability spaces. These extensions are known as the discrete Jensen inequality and integral Jensen inequality, respectively.

Log-Convex Function

An important sub-class of convex functions is that of log-convex functions.

Definition 1.6 (SEE [119, DEFINITION 1.15]) *A function $\Phi: I \to \mathbb{R}$, I an interval in $\mathbb{R}$, is said to be log-convex, or multiplicative convex if $\log \Phi$ is convex, or equivalently if for all $x, y \in I$ and all $\alpha \in [0,1]$,*

$$\Phi(\alpha x + (1-\alpha)y) \leq \Phi(x)^{\alpha}\Phi(y)^{1-\alpha}. \tag{1.6}$$

It is said to be log-concave if the inequality in (1.6) *is reversed.*

Remark 1.6 (a) If we take $\alpha = 1/2$, then (1.6) becomes

$$\Phi\left(\frac{x+y}{2}\right)^2 \leq \Phi(x)\Phi(y),$$

and the function Φ is said to be log-convex in the Jensen sense. If the function Φ is log-convex in the Jensen sense and is continuous, then Φ is also log-convex.

(b) If $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$, then (1.6) is equivalent to

$$[\Phi(x_2)]^{(x_3-x_1)} \leq [\Phi(x_1)]^{(x_3-x_2)}[\Phi(x_3)]^{(x_2-x_1)}.$$

Furthermore, if $x_1, x_2, y_1, y_2 \in I$ such that $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then

$$\left(\frac{\Phi(x_2)}{\Phi(x_1)}\right)^{\frac{1}{x_2-x_1}} \leq \left(\frac{\Phi(y_2)}{\Phi(y_1)}\right)^{\frac{1}{y_2-y_1}}.$$

(c) $\Phi : I \to \mathbb{R}$ is log-convex in the Jensen sense if and only if

$$\alpha^2\Phi(x) + 2\alpha\beta\Phi\left(\frac{x+y}{2}\right) + \beta^2\Phi(y) \geq 0$$

holds for all $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$.

1.2 Exponential and n-Exponential Convexity

Exponentially convex functions were introduced by S. N. Bernstein [31] over eighty years ago and later D. V. Widder [132]. The notion of n-exponential convexity was introduced by J. Pečarić and J. Perić in [115] (see also [89, 78, 88]). Now we quote some definitions and results about exponential and n-exponential convexity.

Definition 1.7 *A function $\Phi : I \to \mathbb{R}$ $(I \subseteq \mathbb{R})$ is n-exponentially convex in the Jensen sense on I, if*

$$\sum_{i,j=1}^{n} \xi_i \xi_j \Phi\left(\frac{x_i + x_j}{2}\right) \geq 0$$

holds for all choices $\xi_i \in \mathbb{R}$ and $x_i \in I$, $i \in \{1, \ldots, n\}$. A function $\Phi : I \to \mathbb{R}$ is n-exponentially convex if it is n-exponentially convex in the Jensen sense and continuous on I.

Remark 1.7 It is clear from the definition that 1-exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $k \leq n$.

By definition of positive semi-definite matrices and some basic linear algebra, we have the following proposition.

Proposition 1.1 *If Φ is an n-exponentially convex function in the Jensen sense, then the matrix $\left[\Phi\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{k}$ is positive semi-definite for all $k \in \mathbb{N}$, $k \leq n$. Particularly,* $\det\left[\Phi\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{k} \geq 0$ *for all $k \in \mathbb{N}$, $k \leq n$.*

Definition 1.8 *A function $\Phi : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I, if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$. A function $\Phi : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.*

Proposition 1.2 *[See [19, Proposition 1]] Let $\Phi : (a,b) \to \mathbb{R}$. The following are equivalent:*

(i) *Φ is exponentially convex.*

(ii) Φ *is continuous and*

$$\sum_{i,j=1}^{n} v_i v_j \Phi\left(\frac{x_i+x_j}{2}\right) \geq 0$$

for all $n \in \mathbb{N}$, $v_i \in \mathbb{R}$, *and* $x_i + x_j \in (a,b)$, $1 \leq i, j \leq n$.

(iii) Φ *is continuous and*

$$\det\left[\Phi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{m} \geq 0, \quad 1 \leq m \leq n$$

for all $n \in \mathbb{N}$ *and for every* $x_i \in (a,b)$, $i \in \{1,\ldots,n\}$.

Remark 1.8 Some examples of exponentially convex functions are:

(i) $\Phi : I \to \mathbb{R}$ defined by $\Phi(x) = ce^{kx}$, where $c \geq 0$ and $k \in \mathbb{R}$.

(ii) $\Phi : \mathbb{R}^+ \to \mathbb{R}$ defined by $\Phi(x) = x^{-k}$, where $k > 0$.

(iii) $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\Phi(x) = e^{-k\sqrt{x}}$, where $k > 0$.

Remark 1.9 From Remark 1.6 (c) it follows that a positive function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense. Also, using basic convexity theory, it follows that a positive function is log-convex if and only if it is 2-exponentially convex.

1.3 Superquadratic Functions

The concept of superquadratic functions in one variable, as a generalization of the class of convex functions, was recently introduced by S. Abramovich, G. Jameson and G. Sinnamon in [6] and [5]. More examples and properties of superquadratic functions can be found in [1, 25, 26, 24] and its references.

Definition 1.9 *A function* $\Psi : [0,\infty) \to \mathbb{R}$ *is called superquadratic if there exists a function* $C : [0,\infty) \to \mathbb{R}$ *such that*

$$\Psi(y) - \Psi(x) - \Psi(|y-x|) \geq C(x)(y-x) \quad \textit{for all} \quad x, y \geq 0. \tag{1.7}$$

We say that Ψ *is subquadratic if* $-\Psi$ *is superquadratic. If for all* $x, y > 0$ *with* $x \neq y$, *there is strict inequality in* (1.7), *then* Ψ *is called strictly superquadratic.*

For example, the function $\Psi(x) = x^p$ is superquadratic for $p \geq 2$ and subquadratic for $p \in (0,2]$.

The following lemma shows essentially that positive superquadratic functions are also convex functions.

Lemma 1.1 *Let Ψ be a superquadratic function with $C(x)$ as in Definition 1.9. Then*

(i) $\Psi(0) \leq 0$;

(ii) *if* $\Psi(0) = \Psi'(0) = 0$, *then* $C(x) = \Psi'(x)$ *whenever* Ψ *is differentiable at* $x > 0$;

(iii) *if* $\Psi \geq 0$, *then* Ψ *is convex and* $\Psi(0) = \Psi'(0) = 0$.

In the following theorem, some characterizations of superquadratic functions are given analogous to the well-known characterizations of convex functions.

Theorem 1.3 (SEE [26, THEOREM 9]) *For the function* $\Psi : [0,\infty) \to \mathbb{R}$, *the following conditions are equivalent:*

(i) *The function* Ψ *is a superquadratic function, i.e.,* (1.7) *holds.*

(ii) *For any two nonnegative n-tuples* $(x_1,\dots,x_n)$ *and* $(p_1,\dots,p_n)$ *such that* $P_n = \sum_{i=1}^{n} p_i > 0$, *the inequality*

$$\Psi(\overline{x}) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i\Psi(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i\Psi(|x_i - \overline{x}|)$$

holds, where $\overline{x} = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i$.

(iii) *The inequality*

$$\begin{aligned}\Psi(\lambda y_1 + (1-\lambda)y_2) &\leq \lambda\Psi(y_1) + (1-\lambda)\Psi(y_2)\\ &\quad - \lambda\Psi(1-\lambda)|y_1 - y_2| - (1-\lambda)\Psi(\lambda|y_1 - y_2|)\end{aligned}$$

holds for all $y_1, y_2 \geq 0$ *and* $\lambda \in [0,1]$.

(iv) *For all* $x, y_1, y_2 \geq 0$, *such that* $y_1 < x < y_2$, *we have*

$$\Psi(x) \leq \frac{y_2 - x}{y_2 - y_1}\left(\Psi(y_1) - \Psi(x - y_1)\right) + \frac{x - y_1}{y_2 - y_1}\left(\Psi(y_2) - \Psi(y_2 - x)\right),$$

i.e.,

$$\frac{\Psi(y_1) - \Psi(x) - \Psi(x - y_1)}{y_1 - x} \leq \frac{\Psi(y_2) - \Psi(x) - \Psi(y_2 - x)}{y_2 - x}.$$

In the following, for any function $\Psi \in \mathrm{C}^1([0,\infty),\mathbb{R})$, we define an associated function $\overline{\Psi} \in \mathrm{C}^1((0,\infty),\mathbb{R})$ by

$$\overline{\Psi}(x) = \frac{\Psi'(x)}{x} \quad \text{for all} \quad x > 0. \tag{1.8}$$

Lemma 1.2 (SEE [3, LEMMA 1]) *Let* $\Psi \in \mathrm{C}^1([0,\infty),\mathbb{R})$ *such that* $\Psi(0) \leq 0$. *If* $\overline{\Psi}$ *is increasing (strictly increasing) or* Ψ' *is superadditive (strictly superadditive), then* Ψ *is superquadratic (strictly superquadratic).*

Lemma 1.3 *[See [3, Lemma 3]] Let* $\Psi \in C^2([0,\infty),\mathbb{R})$ *be such that*

$$m_1 \leq \frac{x\Psi''(x) - \Psi'(x)}{x^2} \leq M_1 \quad \text{for all} \quad x > 0.$$

Let the functions ϑ_1, ϑ_2 *be defined by*

$$\vartheta_1(x) = \frac{M_1 x^3}{3} - \Psi(x), \quad \vartheta_2(x) = \Psi(x) - \frac{m_1 x^3}{3}. \tag{1.9}$$

Then $\overline{\vartheta_1}, \overline{\vartheta_2}$ *are increasing. If also* $\Psi(0) = 0$*, then* ϑ_1, ϑ_2 *are superquadratic.*

Lemma 1.4 *Let* $s > 0$ *and* $\Psi_s : [0,\infty) \to \mathbb{R}$ *be defined by*

$$\Psi_s(x) = \begin{cases} \dfrac{x^s}{s(s-2)}, & s \neq 2, \\[2ex] \dfrac{x^2}{2}\log x, & s = 2. \end{cases} \tag{1.10}$$

Then Ψ_s *is superquadratic, with the convention* $0\log 0 := 0$*.*

Lemma 1.5 *Let* $s \in \mathbb{R}$ *and* $\varphi_s : [0,\infty) \to \mathbb{R}$ *be defined by*

$$\varphi_s(x) = \begin{cases} \dfrac{sxe^{sx} - e^{sx} + 1}{s^3}, & s \neq 0, \\[2ex] \dfrac{x^3}{3}, & s = 0. \end{cases} \tag{1.11}$$

Then φ_s *is superquadratic.*

1.4 Time Scales Theory

The theory of time scales was introduced by Stefan Hilger in his PhD thesis [69] in 1988 as a unification of the theory of difference equations with that of differential equations, unifying integral and differential calculus with the calculus of finite differences, extending to cases “in between”, and offering a formalism for studying hybrid discrete-continuous dynamic systems. It has applications in any field that requires simultaneous modelling of discrete and continuous data. Now, we briefly introduce the time scales calculus and refer to [70, 71] and the monograph [45] for further details.

By a time scale $\mathbb{T}$ we mean any nonempty closed subset of $\mathbb{R}$. The two most popular examples of time scales are the real numbers $\mathbb{R}$ and the integers $\mathbb{Z}$. Since the time scale $\mathbb{T}$ may or may not be connected, we need the concept of jump operators.

For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : \ s > t\}$$

and the backward jump operator by

$$\rho(t) = \sup\{s \in \mathbb{T} : \ s < t\}.$$

In this definition, the convention is $\inf\emptyset = \sup\mathbb{T}$ and $\sup\emptyset = \inf\mathbb{T}$.

If $\sigma(t) > t$, then we say that t is right-scattered, and if $\rho(t) < t$, then we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. Also, if $\sigma(t) = t$, then t is said to be right-dense, and if $\rho(t) = t$, then t is said to be left-dense. Points that are simultaneously right-dense and left-dense are called dense.

If $\mathbb{T}$ has a left-scattered maximum M_1, then we define $\mathbb{T}^\kappa = \mathbb{T} \setminus \{M_1\}$; otherwise $\mathbb{T}^\kappa = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum M_2, then we define $\mathbb{T}_\kappa = \mathbb{T} \setminus \{M_2\}$; otherwise $\mathbb{T}_\kappa = \mathbb{T}$. Finally we define $\mathbb{T}^* = \mathbb{T}^\kappa \cap \mathbb{T}_\kappa$.

The mappings $\mu, \nu : \mathbb{T} \to [0,\infty)$ defined by

$$\mu(t) = \sigma(t) - t \quad \text{and} \quad \nu(t) = t - \rho(t)$$

are called the forward and backward graininess functions, respectively.

In the following considerations, $\mathbb{T}$ will denote a time scale, $I_\mathbb{T} = I \cap \mathbb{T}$ will denote a time scale interval (for any open or closed interval I in $\mathbb{R}$), and $[0,\infty)_\mathbb{T}$ will be used for the time scale interval $[0,\infty) \cap \mathbb{T}$.

Definition 1.10 *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood $U_\mathbb{T}$ of t such that*

$$\left|(f(\sigma(t)) - f(s)) - f^\Delta(t)\,[\sigma(t) - s]\right| \le \varepsilon\,|\sigma(t) - s| \quad \textit{for all} \quad s \in U_\mathbb{T}.$$

We call $f^\Delta(t)$ the delta derivative of f at t. We say that f is delta differentiable on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

Definition 1.11 *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}_\kappa$. Then we define $f^\nabla(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood $U_\mathbb{T}$ of t such that*

$$\left|(f(\rho(t)) - f(s)) - f^\nabla(t)\,[\rho(t) - s]\right| \le \varepsilon\,|\rho(t) - s| \quad \textit{for all} \quad s \in U_\mathbb{T}.$$

We call $f^\nabla(t)$ the nabla derivative of f at t. We say that f is nabla differentiable on $\mathbb{T}_\kappa$ provided $f^\nabla(t)$ exists for all $t \in \mathbb{T}_\kappa$.

Example 1.1 (i) If $\mathbb{T} = \mathbb{R}$, then

$$f^\Delta(t) = f^\nabla(t) = f'(t).$$

(ii) If $\mathbb{T} = \mathbb{Z}$, then

$$f^{\Delta}(t) = f(t+1) - f(t)$$

is the forward difference operator, while

$$f^{\nabla}(t) = f(t) - f(t-1)$$

is the backward difference operator.

(iii) Let $h > 0$. If $\mathbb{T} = h\mathbb{Z}$, then

$$f^{\Delta}(t) = \frac{f(t+h) - f(t)}{h} \quad \text{and} \quad f^{\nabla}(t) = \frac{f(t) - f(t-h)}{h}$$

are the h-derivatives.

(iv) Let $q > 1$. If $\mathbb{T} = q^{\mathbb{N}_0}$, where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, then

$$f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q-1)t} \quad \text{and} \quad f^{\nabla}(t) = \frac{q(f(t) - f(t/q))}{(q-1)t}$$

are the q-derivatives (or Jackson derivatives).

Definition 1.12 *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^{\kappa}_{\kappa}$. Then we define $f^{\diamond_\alpha}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood $U_{\mathbb{T}}$ of t such that*

$$\begin{aligned}&|\alpha\,(f\,(\sigma(t)) - f(s))\,[\rho(t) - s] + (1-\alpha)\,(f\,(\rho(t)) - f(s))\,[\sigma(t) - s]\\ &\quad - f^{\diamond_\alpha}(t)\,[\rho(t) - s]\,[\sigma(t) - s]| \le \varepsilon\,|[\rho(t) - s]\,[\sigma(t) - s]| \quad \textit{for all} \quad s \in U_{\mathbb{T}}.\end{aligned}$$

We call $f^{\diamond_\alpha}(t)$ the diamond-α derivative of f at t. We say that f is diamond-α differentiable on $\mathbb{T}^{\kappa}_{\kappa}$ provided $f^{\diamond_\alpha}(t)$ exists for all $t \in \mathbb{T}^{\kappa}_{\kappa}$.

Remark 1.10 If $f : \mathbb{T} \to \mathbb{R}$ is differentiable on $\mathbb{T}$ in the sense of Δ and ∇, then f is diamond-α differentiable at $t \in \mathbb{T}^{\kappa}_{\kappa}$, and the diamond-$\alpha$ derivative is given by

$$f^{\diamond_\alpha}(t) = \alpha f^{\Delta}(t) + (1-\alpha) f^{\nabla}(t), \quad 0 \le \alpha \le 1.$$

Remark 1.11 From Definition 1.12, it is clear that f is diamond-α differentiable for $0 \le \alpha \le 1$ if and only if f is Δ and ∇ differentiable. It is obvious that for $\alpha = 1$, the diamond-α derivative reduces to the standard Δ derivative, and for $\alpha = 0$, the diamond-α derivative reduces to the standard ∇ derivative.

For all $t \in \mathbb{T}^{\kappa}$, we have the following properties of delta derivative.

Theorem 1.4 (SEE [45, THEOREM 1.16]) (i) *If f is delta differentiable at t, then f is continuous at t.*

(ii) *If f is continuous at t and t is right-scattered, then f is delta differentiable at t with $f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$.*

(iii) *If t is right-dense, then f is delta differentiable at t iff $\lim_{s \to t} \frac{f(t)-f(s)}{t-s}$ exists as a finite number. In this case, $f^{\Delta}(t) = \lim_{s \to t} \frac{f(t)-f(s)}{t-s}$.*

(iv) *If f is delta differentiable at t, then $f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t)$.*

In the same manner, for all $t \in \mathbb{T}_{\kappa}$, we have the following properties of nabla derivative.

Theorem 1.5 (SEE [45, THEOREM 1.16]) (i) *If f is nabla differentiable at t, then f is continuous at t.*

(ii) *If f is continuous at t and t is left-scattered, then f is nabla differentiable at t with $f^{\nabla}(t) = \frac{f(t)-f(\rho(t))}{\nu(t)}$.*

(iii) *If t is left-dense, then f is nabla differentiable at t if and only if $\lim_{s \to t} \frac{f(t)-f(s)}{t-s}$ exists as a finite number. In this case, $f^{\nabla}(t) = \lim_{s \to t} \frac{f(t)-f(s)}{t-s}$.*

(iv) *If f is nabla differentiable at t, then $f(\rho(t)) = f(t) + \nu(t) f^{\nabla}(t)$.*

Definition 1.13 *A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits are finite at all left-dense points in $\mathbb{T}$. We denote by C_{rd} the set of all rd-continuous functions. We say that f is rd-continuously delta differentiable (and write $f \in C^1_{rd}$) if $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{\kappa}$ and $f^{\Delta} \in C_{rd}$. A function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous if it is continuous at all left-dense points in $\mathbb{T}$ and its right-sided limits are finite at all right-dense points in $\mathbb{T}$. We denote by C_{ld} the set of all ld-continuous functions. We say that f is ld-continuously nabla differentiable (and write $f \in C^1_{ld}$) if $f^{\nabla}(t)$ exists for all $t \in \mathbb{T}_{\kappa}$ and $f^{\nabla} \in C_{ld}$.*

The set of all continuous functions on $\mathbb{T}$ contains both C_{rd} and C_{ld}.

Definition 1.14 *A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^{\kappa}$. Then we define the delta integral by*

$$\int_a^t f(s)\Delta s = F(t) - F(a).$$

A function $G : \mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $G^{\nabla}(t) = f(t)$ for all $t \in \mathbb{T}_{\kappa}$. Then we define the nabla integral by

$$\int_a^t f(s)\nabla s = G(t) - G(a).$$

The importance of rd-continuous and ld-continuous functions is revealed by the following result.

Theorem 1.6 (SEE [45, THEOREM 1.74, THEOREM 8.45]) *Every rd-continuous function has a delta antiderivative and every ld-continuous function has a nabla antiderivative.*

Definition 1.15 *Let $f : \mathbb{T} \to \mathbb{R}$ and $a, b \in \mathbb{T}$. Then the diamond-α integral of f from a to b is defined by*

$$\int_a^b f(t)\diamond_\alpha t = \alpha \int_a^b f(t)\Delta t + (1-\alpha)\int_a^b f(t)\nabla t, \quad 0 \le \alpha \le 1.$$

Remark 1.12 Since the diamond-α integral is a combined Δ and ∇ integral, in general

$$\left(\int_a^s f(t)\diamond_\alpha t\right)^{\diamond_\alpha} \neq f(t), \quad t \in \mathbb{R}.$$

Example 1.2 Let $\alpha = \frac{1}{2}$ and $\mathbb{T} = \{0,1,2,3\}$. Then the diamond-α derivative for a function on $\mathbb{T}$ is defined on the set $\mathbb{T}^\kappa_\kappa$ which is $\{1,2\}$. Define the function $f(t) \equiv 0$. Next define functions F and G as follows:

$$\begin{aligned} F(0) &= 0, \quad G(0) = 1;\\ F(1) &= 5, \quad G(1) = -3;\\ F(2) &= 0, \quad G(2) = 1;\\ F(3) &= 5, \quad G(3) = -3. \end{aligned}$$

Then

$$\begin{aligned} F^{\diamond_\alpha}(1) &= \frac{1}{2}\frac{F(2)-F(1)}{2-1} + \frac{1}{2}\frac{F(1)-F(0)}{1-0}\\ &= \frac{1}{2}(0-5) + \frac{1}{2}(5-0) = 0 = f(1) \end{aligned}$$

and

$$\begin{aligned} F^{\diamond_\alpha}(2) &= \frac{1}{2}\frac{F(3)-F(2)}{3-2} + \frac{1}{2}\frac{F(2)-F(1)}{2-1}\\ &= \frac{1}{2}(5-0) + \frac{1}{2}(0-5) = 0 = f(2). \end{aligned}$$

Also

$$\begin{aligned} G^{\diamond_\alpha}(1) &= \frac{1}{2}\frac{G(2)-G(1)}{2-1} + \frac{1}{2}\frac{G(1)-G(0)}{1-0}\\ &= \frac{1}{2}(1-(-3)) + \frac{1}{2}(-3-1) = 0 = f(1) \end{aligned}$$

and

$$\begin{aligned} G^{\diamond_\alpha}(2) &= \frac{1}{2}\frac{G(3)-G(2)}{3-2} + \frac{1}{2}\frac{G(2)-G(1)}{2-1}\\ &= \frac{1}{2}(-3-1) + \frac{1}{2}(1-(-3)) = 0 = f(2). \end{aligned}$$

Thus

$$F^{\diamond_{\alpha}(t)} = G^{\diamond_{\alpha}}(t) = f(t)$$

on $\mathbb{T}_{\kappa}^{\kappa}$. We see that both F and G are diamond-α antiderivatives of f on $\mathbb{T}_{\kappa}^{\kappa}$. However,

$$F(2) - F(1) = -5 \neq 4 = G(2) - G(1).$$

Example 1.2 can be generalized for any fixed α strictly between 0 and 1, and for any purely discrete time scale, such as $\mathbb{T} = \mathbb{Z}$.

Next, we present an example where no diamond-α antiderivative exists.

Example 1.3 Let $\alpha = \dfrac{1}{2}$. Let $\mathbb{T} = (-\infty, 1] \cup [2, \infty)$. Set

$$f(t) = \begin{cases} -1, & x \leq 1, \\ 5, & x \geq 2. \end{cases}$$

Assume a diamond-α antiderivative F of f exists on $\mathbb{T}_{\kappa}^{\kappa}$. On $(-\infty, 1]$, F must be of the form $-t + C_1$, where C_1 is a constant. On $[2, \infty)$, F must be of the form $5t + C_2$. It follows therefore

$$F^{\diamond_{\alpha}}(1) = \frac{1}{2}F^{\Delta}(1) + \frac{1}{2}F^{\nabla}(1) = f(1).$$

Thus

$$\frac{1}{2}[(5(2) + C_2) - (-1(1) + C_1)] + \frac{1}{2}(-1) = -1. \tag{1.12}$$

Also,

$$F^{\diamond_{\alpha}}(2) = \frac{1}{2}F^{\Delta}(2) + \frac{1}{2}F^{\nabla}(2) = f(2).$$

Thus

$$\frac{1}{2}(5) + \frac{1}{2}[(5(2) + C_2) - (-1(1) + C_1)] = 5. \tag{1.13}$$

From (1.12) and (1.13), we obtain the system of equations

$$C_1 - C_2 = 12,$$
$$C_1 - C_2 = 6,$$

which has no solution. Thus for the function f, which is continuous on $\mathbb{T}$, no diamond-α antiderivative exists on $\mathbb{T}_{\kappa}^{\kappa}$.

Now we give some properties of the delta integral.

Theorem 1.7 (See [45, Theorem 1.77]) *If $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and $f, g \in C_{rd}$, then*

(i) $\int_a^b (f(t) + g(t))\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$,

(ii) $\int_a^b \alpha f(t)\Delta t = \alpha \int_a^b f(t)\Delta t$,

(iii) $\int_b^a f(t)\Delta t = -\int_a^b f(t)\Delta t$,

(iv) $\int_a^a f(t)\Delta t = 0$,

(v) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$,

(vi) *if* $f(t) \geq 0$ *for all* t*, then* $\int_a^b f(t)\Delta t \geq 0$.

A similar theorem holds for the nabla integral, for $f, g \in \mathrm{C_{ld}}$, and for the diamond-α integral, for $f, g \in \mathrm{C}$.

Regarding integral calculus on time scales, the literature includes, among others, the Cauchy delta integral [45, 70], the Cauchy nabla integral [23, 45], the Riemann delta integral [46, 63, 64], the Riemann nabla integral [63]; the Cauchy diamond-α integral [16, 123], the Riemann diamond-α integral [97], the Lebesgue delta and nabla integrals [46, 62], the multiple Riemann and multiple Lebesgue delta, nabla and diamond-α integrals [42, 43].

Let $n \in \mathbb{N}$ be fixed. For each $i \in \{1, \ldots, n\}$, let $\mathbb{T}_i$ denote a time scale and

$$\Lambda^n = \mathbb{T}_1 \times \ldots \times \mathbb{T}_n = \{t = (t_1, \ldots, t_n) : t_i \in \mathbb{T}_i,\ 1 \leq i \leq n\} \tag{1.14}$$

an n-dimensional time scale. Let μ_Δ be the σ-additive Lebesgue Δ-measure on Λ^n and $\mathscr{F}$ be the family of Δ-measurable subsets of Λ^n. Let $\mathscr{E} \subset \Lambda^n$ and $(\mathscr{E}, \mathscr{F}, \mu_\Delta)$ be a time scale measure space. Then for a Δ-measurable function $f : \mathscr{E} \to \mathbb{R}$, the corresponding Δ-integral of f over $\mathscr{E}$ is denoted according to [43, (3.18)] by

$$\int_{\mathscr{E}} f(t_1, \ldots, t_n)\Delta_1 t_1 \ldots \Delta_n t_n, \quad \int_{\mathscr{E}} f(t)\Delta t, \quad \int_{\mathscr{E}} f \mathrm{d}\mu_\Delta, \quad \text{or} \quad \int_{\mathscr{E}} f(t)\mathrm{d}\mu_\Delta(t).$$

By [43, Section 3], all theorems of the general Lebesgue integration theory, including the Lebesgue dominated convergence theorem, hold also for Lebesgue Δ-integrals on Λ^n. Here we state Fubini's theorem for multiple Lebesgue Δ-integrals on time scales. It is used in Chapter 9.

Theorem 1.8 *Let* $(X, \mathscr{K}, \mu_\Delta)$ *and* $(Y, \mathscr{L}, \nu_\Delta)$ *be two finite-dimensional time scale measure spaces. If* $f : X \times Y \to \mathbb{R}$ *is a* Δ*-integrable function and if we define the functions*

$$\varphi(y) = \int_X f(x,y)\mathrm{d}\mu_\Delta(x) \quad \textit{for a.e.} \quad y \in Y$$

and

$$\psi(x) = \int_Y f(x,y)\mathrm{d}\nu_\Delta(y) \quad \textit{for a.e.} \quad x \in X,$$

then φ *is* Δ*-integrable on* Y *and* ψ *is* Δ*-integrable on* X *and*

$$\int_X \mathrm{d}\mu_\Delta(x) \int_Y f(x,y)\mathrm{d}\nu_\Delta(y) = \int_Y \mathrm{d}\nu_\Delta(y) \int_X f(x,y)\mathrm{d}\mu_\Delta(x).$$

1.5 Jensen's and Related Inequalities

Inequalities are used everywhere in mathematics. In 1934, Hardy, Littlewood, and Pólya [68] published a book on inequalities, since then, the theory of inequalities has become an important branch of mathematics. Among the inequalities, Jensen's inequality is one of the most important and extensively used inequality in various fields of modern mathematics, especially in mathematical analysis and statistics. It is a powerful tool of producing a large class of classical inequalities, e.g., the arithmetic mean-geometric mean-harmonic mean inequality, Young's inequality, Hölder's inequality, Minkowski's inequality, Beckenbach–Dresher inequality, the positivity of relative entropy in information theory, Shannon's inequality, Ky Fan's inequality, Levinson's inequality, etc. The improvements and generalizations of Jensen's inequality imply the improvements and generalizations of a whole series of other classical inequalities. A simple search in MathSciNet database of the American Mathematical Society with the key words "Jensen" and "inequalities" in the title reveals that there are more than 300 items intimately devoted to this word. However, the number of papers, where this inequality is used, is a lot larger and far more difficult to find.

In the following, we give a brief introduction to the Jensen and some of its related classical inequalities.

Jensen's Inequality

Let I be an interval in $\mathbb{R}$ and $\Phi : I \to \mathbb{R}$ a convex function on I. If

$$\mathbf{x} = (x_1, \ldots, x_n) \in I^n, \quad \mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{R}_+^n, \quad \text{and} \quad P_n = \sum_{i=1}^n p_i,$$

then the well-known Jensen inequality

$$\Phi\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) \le \frac{1}{P_n}\sum_{i=1}^n p_i \Phi(x_i) \tag{1.15}$$

holds. If Φ is strictly convex, then (1.15) is strict unless $x_i = c$ (constant) for all $i \in \{j : p_j > 0\}$.

In the following, we quote some history about the Jensen inequality from [104]. The Jensen inequality was proved under the assumption that Φ is a J-convex function by J. L. W. V. Jensen (see [82, 83] or for example [119, page 43]). He applied the famous inductive method used by Cauchy (1821) in the proof of the arithmetic mean-geometric mean inequality. However, inequality (1.15) appears, under different assumptions, much earlier. Jensen himself mentioned in the appendix to his paper that O. Hölder proved inequality (1.15) in 1889, supposing that Φ is a twice differentiable function on $[a,b]$ such that $\Phi''(x) \ge 0$ on that interval. This supposition is in the case of twice differentiable functions equivalent with the supposition that Φ is convex. The above inequality was proved, after Hölder, using the same assumptions, by R. Henderson in 1895. However, as far back as 1875, a particular case of the above inequality, the case when $p_1 = \ldots = p_n$, was proved by J. Grolous by an application of the centroid method. This is, as far as we could find, the first inequality for convex functions to appear in the mathematical literature. J. Grolous introduced the assumption that $\Phi''(x) \geq 0$, but it can be seen from the text itself that it is enough to assume that Φ is a convex function, in the geometric sense (see, for instance, D. S. Mitrinović [101, page 15]).

The original Jensen inequality for integrals can be stated as follows.

Theorem 1.9 (SEE [82, FORMULA (5′)]) *Let $a, b \in \mathbb{R}$ with $a < b$ and suppose $I \subset \mathbb{R}$ is an interval. If $\Phi \in \mathrm{C}(I, \mathbb{R})$ is convex and $f \in \mathrm{C}([a,b], I)$, then*

$$\Phi\left(\frac{\int_a^b f(t)\mathrm{d}t}{b-a}\right) \leq \frac{\int_a^b \Phi(f(t))\mathrm{d}t}{b-a}. \tag{1.16}$$

Note that in Jensen's inequality, we have nonnegative weights. It is reasonable to ask whether the condition "$\mathbf{p}$ is a nonnegative n-tuple" can be relaxed at the expense of restricting $\mathbf{x}$ more severely. An answer to this question was given by Steffensen in [127] (see also [119, page 57]).

Theorem 1.10 *Let I be an interval in $\mathbb{R}$ and $\Phi : I \to \mathbb{R}$ be a convex function. If $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ is a monotonic n-tuple and $\mathbf{p} = (p_1, \ldots, p_n)$ a real n-tuple such that*

$$0 \leq P_k \leq P_n, \quad k = 1, \ldots, n-1, \quad P_n > 0$$

is satisfied, where $P_k = \sum_{i=1}^{k} p_i, k \in \{1, \ldots, n\}$, then (1.15) *holds. If Φ is strictly convex, then inequality* (1.15) *is strict unless $x_1 = \ldots = x_n$.*

Inequality (1.15) under conditions from Theorem 1.10 is called the Jensen–Steffensen inequality. The integral version of the Jensen–Steffensen inequality is given by Boas [40] (see also [119, page 59]). Furthermore, for different refinements and generalizations of the Jensen–Steffensen inequality, see [51, 59, 119].

B. Jessen in 1931 (see [84] or see for example [119, page 47]) gave the generalization of Jensen's inequality for convex functions which involves positive normalized linear functionals. In 1937, E. J. McShane gave the generalization of Jessen's inequality for multivariables (see [99] or see for example [119, page 48–51]). S. Banić and S. Varošanec [26] refined Jessen's inequality for superquadratic functions.

In 2003, A. McD. Mercer in [100] gave a variant of Jensen's inequality, called the Jensen–Mercer inequality. Later, W. S. Cheung et al. generalized the Jensen–Mercer inequality for isotonic linear functionals, called Jessen–Mercer inequality (see [49]). Further in [2], S. Abramovich et al. gave the refinement of the Jessen–Mercer inequality for superquadratic functions.

There are also various generalizations of Jensen's inequality to the time scales theory, see Section 2.2.

Hermite–Hadamard Inequality

The Hermite–Hadamard inequality is strongly related to the Jensen inequality. It is also known as the first fundamental inequality for convex functions. It gives us an estimate for the integral arithmetic mean:

$$(b-a)\Phi\left(\frac{a+b}{2}\right) \leq \int_a^b \Phi(t)\mathrm{d}t \leq (b-a)\frac{\Phi(a)+\Phi(b)}{2}, \tag{1.17}$$

where $a,b \in \mathbb{R}$ with $a < b$ and $\Phi : [a,b] \to \mathbb{R}$ is a convex function. It was first established by Hermite in 1881. Also, Beckenbach, a leading expert on the history and theory of complex functions, wrote that the first inequality in (1.17) was proved in 1893 by Hadamard who apparently was not aware of Hermite's result (see [119]). In general, (1.17) is now known as the Hermite–Hadamard inequality.

Note that the first inequality in (1.17) is a Jensen inequality (1.16) when $f(t) = t$, and the second one gives a converse of Jensen's inequality. Various generalizations and refinements of the Hermite-Hadamard inequality and converses of Jensen's inequality are given in the literature for convex functions, superquadratic functions, as well as in time scales theory, see e.g., [3, 90, 89, 26, 50, 51, 103, 108, 119].

The first inequality in (1.17) is stronger than the second one: if Φ is convex on $[a,b]$, then

$$\frac{1}{(b-a)}\int_a^b \Phi(t)\mathrm{d}t - \Phi\left(\frac{a+b}{2}\right) \leq \frac{\Phi(a)+\Phi(b)}{2} - \frac{1}{(b-a)}\int_a^b \Phi(t)\mathrm{d}t. \tag{1.18}$$

A geometric proof of (1.18) is given in [65] and an analytic one in [47] (see also [119, page 140]). The inequality (1.18) is known as the Hammer–Bullen inequality.

Cauchy, Hölder, and Minkowski Inequalities

The three inequalities are well known to all studies of power means in mathematics (see for example [48]).

Augustin-Louis Cauchy published his famous inequality in 1821. Then in 1859, Viktor Yakovlevich Bunyakovsky derived a corresponding inequality for integrals, and in 1885, Hermann Schwarz proved a corresponding version for inner-product spaces. Therefore the Cauchy inequality sometimes also shows up under the name Schwarz inequality, or Cauchy-Schwarz inequality, or Cauchy-Bunyakovsky-Schwarz inequality. Hölder's generalization appeared in 1889. The Minkowski inequality was established in 1896 by Hermann Minkowski in his book Geometrie der Zahlen (Geometry of Numbers).

There are various versions of these inequalities given in the literature. For isotonic linear functionals, some generalizations and converses of the Hölder and Minkowski inequalities can be found in [119]. In any case, the discrete versions of Hölder's and Minkowski's inequalities are stated in the following two theorems, respectively.

Theorem 1.11 (HÖLDER'S INEQUALITY) *For $p \neq 1$, define q by $q = p/(p-1)$. If $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ are two positive n-tuples and $p > 1$, then*

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} y_i^q \right)^{\frac{1}{q}}. \tag{1.19}$$

If $p < 1$, $p \neq 0$, then (1.19) *holds in reverse order.*

In Theorem 1.11, if $p = q = 2$, then (1.19) becomes the Cauchy Schwarz inequality.

Theorem 1.12 (CAUCHY SCHWARZ INEQUALITY) *If $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ are two positive n-tuples, then*

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} y_i^2 \right)^{\frac{1}{2}}. \tag{1.20}$$

Theorem 1.13 (MINKOWSKI'S INEQUALITY) *If $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ are two positive n-tuples and $p > 1$, then*

$$\left(\sum_{i=1}^{n} (x_i + y_i)^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} y_i^p \right)^{\frac{1}{p}}. \tag{1.21}$$

If $p < 1$, $p \neq 0$, then (1.21) *holds in reverse order.*

Beckenbach–Dresher Inequality

In 1950, E. F. Beckenbach published an inequality which has aroused interest until nowadays. He proved that for positive real numbers $x_i, y_i > 0$, $i \in \{1, \dots, n\}$ and for $1 \leq p \leq 2$, the inequality

$$\frac{\sum_{i=1}^{n} (x_i + y_i)^p}{\sum_{i=1}^{n} (x_i + y_i)^{p-1}} \leq \frac{\sum_{i=1}^{n} x_i^p}{\sum_{i=1}^{n} x_i^{p-1}} + \frac{\sum_{i=1}^{n} y_i^p}{\sum_{i=1}^{n} y_i^{p-1}} \tag{1.22}$$

is valid. If $0 \leq p \leq 1$, then (1.22) is reversed.

Few years later, M. Dresher investigated moment spaces and stated that an integral analogue of (1.22) holds. In recent literature, (1.22) is called the Beckenbach–Dresher inequality. Some history and recent results about the Beckenbach–Dresher inequality can be found in [61, 129].

1.6 Hardy's Inequality

In 1920, G. H. Hardy [66] announced and proved in [67] (see also [68] and [95]) the following result: Let $p > 1$ and $f \in L^p(0,\infty)$ be a nonnegative function, then

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)\mathrm{d}t \right)^p \mathrm{d}x \le \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(x)\mathrm{d}x \tag{1.23}$$

holds. This interesting result is today referred to as the classical Hardy integral inequality. Inequality (1.23) has an interesting prehistory and history (see e.g., [68, 94, 95, 96] and the references given there).

Other important inequalities are the following: if $p > 1$ and f is a nonnegative function such that $f \in L^p(0,\infty)$, then

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} \mathrm{d}x \right)^p \mathrm{d}y \le \left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \right)^p \int_0^\infty f^p(y)\,\mathrm{d}y, \tag{1.24}$$

and if in addition $g \in L^q(0,\infty)$, where $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} \mathrm{d}x\,\mathrm{d}y \le \frac{\pi}{\sin\frac{\pi}{p}} \left(\int_0^\infty f^p(x)\,\mathrm{d}x \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)\,\mathrm{d}y \right)^{\frac{1}{q}}. \tag{1.25}$$

Moreover, (1.25) is sometimes called Hilbert's or Hardy–Hilbert's inequality even if Hilbert himself only considered the case $p = 2$ (L^p-spaces were defined much later).

We also note that (1.23) shows that the Hardy operator H, defined by setting

$$(Hf)(x) := \frac{1}{x} \int_0^x f(t)\,\mathrm{d}t,$$

maps L^p into itself with operator norm $\frac{p}{p-1}$. Similarly, (1.24) shows that the operator A, defined by setting

$$(Af)(y) := \int_0^\infty f(x)(x+y)^{-1}\,\mathrm{d}x,$$

maps L^p into itself with operator norm $\frac{\pi}{\sin\frac{\pi}{p}}$.

It is now natural to generalize the operators above to the following ones:

$$H_k f(x) := \frac{1}{K(x)} \int_0^x f(t)k(x,t)\,\mathrm{d}t, \tag{1.26}$$

where

$$K(x) := \int_0^x k(x,t)\,\mathrm{d}t \, < \infty$$

and (more generally)

$$A_k f(x) := \frac{1}{K(x)} \int_0^\infty f(t) k(x,t)\, \mathrm{d}t, \tag{1.27}$$

where now

$$K(x) := \int_0^\infty k(x,t)\, \mathrm{d}t \; < \infty.$$

Here, $k(x,y)$ is a general measurable and nonnegative function, a so-called kernel.

Now, let $(\Omega_1, \Sigma_1, \mu_1)$, $(\Omega_2, \Sigma_2, \mu_2)$ be σ-finite measure spaces and let A_k from (1.27) be generalized as follows:

$$A_k f(x) := \frac{1}{K(x)} \int_{\Omega_2} k(x,y) f(y)\, \mathrm{d}\mu_2(y), \tag{1.28}$$

where $f : \Omega_2 \to \mathbb{R}$ is a measurable function, $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is a measurable and nonnegative kernel and

$$K(x) := \int_{\Omega_2} k(x,y)\, \mathrm{d}\mu_2(y) < \infty,\, x \in \Omega_1. \tag{1.29}$$

Chapter 2

Jensen Type Inequalities for Convex and Superquadratic Functions

In this chapter, we apply the theory of isotonic linear functionals to derive a series of known inequalities, extensions of known inequalities, and new inequalities in the theory of dynamic equations on time scales. The original results presented in this chapter appeared in [20, 27].

2.1 Positive Linear Functionals and Time Scales Integrals

We recall the following definition from [119, page 47].

Definition 2.1 *Let E be a nonempty set and L be a linear class of real-valued functions $f : E \to \mathbb{R}$ having the following properties:*

(L_1) *If $f, g \in L$ and $a, b \in \mathbb{R}$, then $(af + bg) \in L$.*

(L_2) *If $f(t) = 1$ for all $t \in E$, then $f \in L$.*

An isotonic linear functional is a functional $A : L \to \mathbb{R}$ having the following properties:

(A_1) *If $f,g \in L$ and $a,b \in \mathbb{R}$, then $A(af+bg) = aA(f)+bA(g)$.*

(A_2) *If $f \in L$ and $f(t) \geq 0$ for all $t \in E$, then $A(f) \geq 0$.*

When we use the approach of isotonic linear functionals as given in Definition 2.1, it is not necessary to know many details from the calculus of dynamic equations on time scales. We only need to know that the time scales integral is such an isotonic linear functional.

Theorem 2.1 *Let $\mathbb{T}$ be a time scale. For $a,b \in \mathbb{T}$ with $a < b$, let*

$$E = [a,b)_{\mathbb{T}} \quad \text{and} \quad L = \mathrm{C_{rd}}([a,b)_{\mathbb{T}}, \mathbb{R}).$$

Then (L_1) and (L_2) are satisfied. Moreover, let

$$A(f) = \int_a^b f(t)\Delta t,$$

where the integral is the Cauchy delta time scales integral. Then (A_1) and (A_2) are satisfied.

Proof. This follows from Theorem 1.7. □

Now we give a few examples of the Cauchy delta time scales integral.

Example 2.1 (i) If $\mathbb{T} = \mathbb{R}$ in Theorem 2.1, then $L = \mathrm{C}([a,b),\mathbb{R})$ and

$$A(f) = \int_a^b f(t)\mathrm{d}t.$$

(ii) If $\mathbb{T} = \mathbb{Z}$ in Theorem 2.1, then L consists of all real-valued functions defined on $[a,b-1] \cap \mathbb{Z}$ and

$$A(f) = \sum_{t=a}^{b-1} f(t).$$

(iii) Let $h > 0$. If $\mathbb{T} = h\mathbb{Z}$ in Theorem 2.1, then L consists of all real-valued functions defined on $[a,b-h] \cap h\mathbb{Z}$ and

$$A(f) = h \sum_{k=a/h}^{b/h-1} f(kh).$$

(iv) Let $q > 1$. If $\mathbb{T} = q^{\mathbb{N}_0}$ in Theorem 2.1, then L consists of all real-valued functions defined on $[a,b/q] \cap q^{\mathbb{N}_0}$ and

$$A(f) = (q-1) \sum_{k=\log_q(a)}^{\log_q(b)-1} q^k f(q^k).$$

Note that Theorem 2.1 also has corresponding versions for the nabla and diamond-α integral, which are given next for completeness.

Theorem 2.2 *Let $\mathbb{T}$ be a time scale. For $a,b \in \mathbb{T}$ with $a < b$, let*

$$E = (a,b]_{\mathbb{T}} \quad \textit{and} \quad L = \mathrm{C}_{\mathrm{ld}}((a,b]_{\mathbb{T}}, \mathbb{R}).$$

Then (L_1) and (L_2) are satisfied. Moreover, let

$$A(f) = \int_a^b f(t)\nabla t,$$

where the integral is the Cauchy nabla time scales integral. Then (A_1) and (A_2) are satisfied.

Proof. This follows from [45, Definition 8.43 and Theorem 8.47]. □

Theorem 2.3 *Let $\mathbb{T}$ be a time scale. For $a,b \in \mathbb{T}$ with $a < b$, let*

$$E = [a,b]_{\mathbb{T}} \quad \textit{and} \quad L = \mathrm{C}([a,b]_{\mathbb{T}}, \mathbb{R}).$$

Then (L_1) and (L_2) are satisfied. Moreover, let

$$A(f) = \int_a^b f(t)\diamond_\alpha t,$$

where the integral is the Cauchy diamond-α time scales integral. Then (A_1) and (A_2) are satisfied.

Proof. This follows from [123, Definition 3.2 and Theorem 3.7]. □

Multiple Riemann integration on time scales was introduced in [42]. The Riemann integral introduced there is also an isotonic linear functional.

Theorem 2.4 *Let $\mathbb{T}_1, \ldots, \mathbb{T}_n$ be time scales. For $a_i, b_i \in \mathbb{T}_i$ with $a_i < b_i$, $1 \le i \le n$, let*

$$\mathscr{E} \subset [a_1, b_1)_{\mathbb{T}_1} \times \ldots \times [a_n, b_n)_{\mathbb{T}_n}$$

be Jordan Δ-measurable and let L be the set of all bounded Δ-integrable functions from $\mathscr{E}$ to $\mathbb{R}$. Then (L_1) and (L_2) are satisfied. Moreover, let

$$A(f) = \int_{\mathscr{E}} f(t)\Delta t,$$

where the integral is the multiple Riemann delta time scales integral. Then (A_1) and (A_2) are satisfied.

Proof. This follows from [42, Definition 4.13 and Theorem 3.4]. □

From [42, Remark 2.18], it is also clear that a theorem similar to Theorem 2.4 is also true for the nabla and diamond-α integrals in the multiple variable case.

Multiple Lebesgue integration on time scales was introduced in [43]. The Lebesgue integral introduced there is also an isotonic linear functional.

Theorem 2.5 *Let $\mathscr{E}$ be a Δ-measurable subset of Λ^n, defined as in* (1.14) *and let L be the set of all Δ-measurable functions from $\mathscr{E}$ to $\mathbb{R}$. Then* (L_1) *and* (L_2) *are satisfied. Moreover, let*

$$A(f) = \int_{\mathscr{E}} f(t)\mathrm{d}\mu_\Delta(t),$$

where the integral is the multiple Lebesgue delta time scales integral. Then (A_1) *and* (A_2) *are satisfied.*

Proof. This follows from [43, Section 3]. □

Theorem 2.6 *Under the assumptions of Theorem 2.5, let $A(f)$ be replaced by*

$$A(f) = \frac{\int_{\mathscr{E}} h(t)f(t)\mathrm{d}\mu_\Delta(t)}{\int_{\mathscr{E}} h(t)\mathrm{d}\mu_\Delta(t)},$$

where $h : \mathscr{E} \to \mathbb{R}$ is nonnegative Δ-integrable such that $\int_{\mathscr{E}} h(t)\mathrm{d}\mu_\Delta(t) > 0$. Then A is an isotonic linear functional satisfying $A(1) = 1$.

The monograph [119] contains numerous classical inequalities that are proved for isotonic linear functionals. Since the time scales integral is in fact an isotonic linear functional, the results from [119] can be applied to this setting. Our work shows that it is not necessary to prove such kinds of inequalities "from scratch" in the time scales setting as they can all be obtained easily from well-known inequalities for isotonic linear functionals.

For simplicity, in what follows, we use the following notations: $\mathscr{E}$ as Δ-measurable subset of Λ^n,

$$L_\Delta(f) = \int_{\mathscr{E}} f(t)\mathrm{d}\mu_\Delta(t), \quad \text{and} \quad \overline{L}_\Delta(f,h) = \frac{\int_{\mathscr{E}} f(t)h(t)\mathrm{d}\mu_\Delta(t)}{\int_{\mathscr{E}} h(t)\mathrm{d}\mu_\Delta(t)},$$

where $f : \mathscr{E} \to \mathbb{R}$ is Δ-integrable and $h : \mathscr{E} \to \mathbb{R}$ is nonnegative Δ-integrable such that $\int_{\mathscr{E}} h(t)\mathrm{d}\mu_\Delta(t) > 0$. Also we assume throughout that I and $[m,M]$ are nonempty intervals in $\mathbb{R}$ such that $-\infty < m < M < \infty$.

2.2 Jensen's Inequality

B. Jessen in [84] gave the following generalization of Jensen's inequality for isotonic linear functionals.

Theorem 2.7 (SEE [119, THEOREM 2.4]) *Let L satisfy properties* (L_1) *and* (L_2)*. Assume $\Phi \in \mathrm{C}(I,\mathbb{R})$ is convex. If A satisfies* (A_1) *and* (A_2) *such that $A(1) = 1$, then for all $f \in L$ such that $\Phi(f) \in L$, we have $A(f) \in I$ and*

$$\Phi(A(f)) \le A(\Phi(f)). \tag{2.1}$$

Now our first result is the following generalization of Jensen's inequality.

Theorem 2.8 *Assume $\Phi \in \mathrm{C}(I,\mathbb{R})$ is convex, $f : \mathscr{E} \to I$ is Δ-integrable, and $h : \mathscr{E} \to \mathbb{R}$ is nonnegative Δ-integrable such that $L_\Delta(h) > 0$. Then*

$$\Phi\left(\overline{L}_\Delta(f,h)\right) \le \overline{L}_\Delta(\Phi(f),h).$$

Proof. Just apply Theorem 2.7 and Theorem 2.6. □

Corollary 2.1 *Assume $\Phi \in \mathrm{C}''(I,\mathbb{R})$ such that $\Phi'' \ge 0$. Suppose $f : \mathscr{E} \to I$ is Δ-integrable and $h : \mathscr{E} \to \mathbb{R}$ is nonnegative Δ-integrable such that $L_\Delta(h) > 0$. Then*

$$\Phi\left(\overline{L}_\Delta(f,h)\right) \le \overline{L}_\Delta(\Phi(f),h).$$

Proof. This follows immediately from Theorem 2.8, by using the fact that a function Φ with $\Phi'' \ge 0$ is convex. □

Corollary 2.2 *Assume $\phi, \psi \in \mathrm{C}(I,\mathbb{R})$ such that ϕ^{-1} exists, ψ is strictly increasing, and $\psi \circ \phi^{-1}$ is convex. Suppose $f : \mathscr{E} \to I$ is Δ-integrable and $h : \mathscr{E} \to \mathbb{R}$ is nonnegative Δ-integrable such that $h\phi(f), h\psi(f)$ Δ-integrable and $L_\Delta(h) > 0$. Then*

$$\phi^{-1}\left(\overline{L}_\Delta(\phi(f),h)\right) \le \psi^{-1}\overline{L}_\Delta(\psi(f),h).$$

Proof. This follows from Theorem 2.8, by replacing Φ with $\psi \circ \phi^{-1}$ and f with $\phi \circ f$. □

Corollary 2.3 *Let $\alpha < 0$ or $\alpha > 1$. Suppose $f : \mathscr{E} \to I$ is positive Δ-integrable and $h : \mathscr{E} \to \mathbb{R}$ is nonnegative Δ-integrable such that $L_\Delta(h) > 0$. Then*

$$\left(\overline{L}_\Delta(f,h)\right)^\alpha \le \overline{L}_\Delta(f^\alpha,h).$$

Proof. This follows from Corollary 2.1, by choosing $\Phi(x) = x^\alpha,\ x > 0$. □

Corollary 2.4 *Suppose $f : \mathscr{E} \to I$ is Δ-integrable and $h : \mathscr{E} \to \mathbb{R}$ is nonnegative Δ-integrable such that $L_\Delta(h) > 0$. Then*

$$e^{\overline{L}_\Delta(f,h)} \le \overline{L}_\Delta(e^f,h).$$

Proof. This follows from Corollary 2.1, by choosing $\Phi(x) = e^x,\ x \in \mathbb{R}$. □

Corollary 2.5 *Suppose $f : \mathscr{E} \to I$ is positive Δ-integrable and $h : \mathscr{E} \to \mathbb{R}$ is nonnegative Δ-integrable such that $L_\Delta(h) > 0$. Then*

$$\ln \overline{L}_\Delta(f,h) \ge \overline{L}_\Delta(\ln f,h).$$

Proof. This follows from Corollary 2.1, by choosing $\Phi(x) = \ln x,\ x > 0$. □

Remark 2.1 Known results from time scales theory, which were proved by using time scales calculus, follow from Theorem 2.7 in the same way as Theorem 2.8 does. Note also that a similar theorem for the multiple Riemann integral can be stated and proved using Theorem 2.4. This will be the case for all inequalities stated in this section and the following sections; however, we only explicitly state each time the case for the multiple Lebesgue integral.

The Jensen inequality for Cauchy delta integrals has been obtained by Agarwal, Bohner, and Peterson [9].

Theorem 2.9 *Let $a,b \in \mathbb{T}$ with $a < b$. If $\Phi \in \mathrm{C}(I,\mathbb{R})$ is convex and $f \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}},I)$, then*

$$\Phi\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right) \le \frac{\int_a^b \Phi(f(t))\Delta t}{b-a}.$$

Remark 2.2 When $\mathbb{T} = \mathbb{R}$ in Theorem 2.9, then we obtain Theorem 1.9. When $\mathbb{T} = \mathbb{Z}$ in Theorem 2.9, then we get the discrete Jensen inequality (1.15).

The following result is given by Wong, Yeh, and Lian in [134]. When $h(t) \equiv 1$ in Theorem 2.10 below, then we obtain Theorem 2.9.

Theorem 2.10 *Let $a,b \in \mathbb{T}$ with $a < b$. Assume $h \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}},\mathbb{R})$ satisfies $\int_a^b |h(t)|\Delta t > 0$. If $\Phi \in \mathrm{C}(I,\mathbb{R})$ is convex and $f \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}},I)$, then*

$$\Phi\left(\frac{\int_a^b |h(t)|f(t)\Delta t}{\int_a^b |h(t)|\Delta t}\right) \le \frac{\int_a^b |h(t)|\Phi(f(t))\Delta t}{\int_a^b |h(t)|\Delta t}.$$

Proof. Just apply Theorem 2.7 and Theorem 2.1. □

In [112], Özkan, Sarikaya, and Yildirim proved that Theorem 2.10 is also true if we use the nabla integral (see [45, Section 8.4]) instead of the delta integral. The following result concerning the diamond-α integral is given by Ammi, Ferreira, and Torres in [16] (see also [112]).

Theorem 2.11 *Let $\alpha \in [0,1]$. Let $a,b \in \mathbb{T}$ with $a < b$. Assume $h \in \mathrm{C}([a,b]_{\mathbb{T}},\mathbb{R})$ satisfies $\int_a^b |h(t)|\diamond_\alpha t > 0$. If $\Phi \in \mathrm{C}(I,\mathbb{R})$ is convex and $f \in \mathrm{C}([a,b]_{\mathbb{T}},I)$, then*

$$\Phi\left(\frac{\int_a^b |h(t)|f(t)\diamond_\alpha t}{\int_a^b |h(t)|\diamond_\alpha t}\right) \le \frac{\int_a^b |h(t)|\Phi(f(t))\diamond_\alpha t}{\int_a^b |h(t)|\diamond_\alpha t}.$$

Proof. Just apply Theorem 2.7 and Theorem 2.3. □

2.3 Hermite–Hadamard Inequality

P. Beesack and J. Pečarić in [30] gave the following generalization of the converse of Jensen's inequality for isotonic linear functionals.

Theorem 2.12 (SEE [119, THEOREM 3.37]) *Let L satisfy properties (L_1) and (L_2). Assume $\Phi : I \to \mathbb{R}$ is convex, where $I = [m,M]$. If A satisfies (A_1) and (A_2) such that $A(1) = 1$, then for all $f \in L$ such that $\Phi(f) \in L$, we have*

$$A(\Phi(f)) \leq \frac{M - A(f)}{M - m}\Phi(m) + \frac{A(f) - m}{M - m}\Phi(M). \tag{2.2}$$

In the following theorem, we give a generalization of the converse of Jensen's inequality on time scales.

Theorem 2.13 *Assume $\Phi : I \to \mathbb{R}$ is convex, $f : \mathscr{E} \to [m,M]$ is Δ-integrable, where $[m,M] \subseteq I$, and $h : \mathscr{E} \to \mathbb{R}$ is nonnegative Δ-integrable such that $L_\Delta(h) > 0$. Then*

$$\overline{L}_\Delta(\Phi(f),h) \leq \frac{M - \overline{L}_\Delta(f,h)}{M - m}\Phi(m) + \frac{\overline{L}_\Delta(f,h) - m}{M - m}\Phi(M). \tag{2.3}$$

Proof. Just apply Theorem 2.12 and Theorem 2.6. □

Remark 2.3 If Φ is continuous in Theorem 2.13, then by combining Theorem 2.13 with Theorem 2.8, we obtain a generalization of the Hermite–Hadamard inequality (1.17):

$$\Phi\left(\overline{L}_\Delta(f,h)\right) \leq \overline{L}_\Delta(\Phi(f),h) \leq \frac{M - \overline{L}_\Delta(f,h)}{M - m}\Phi(m) + \frac{\overline{L}_\Delta(f,h) - m}{M - m}\Phi(M). \tag{2.4}$$

Note that the known result [50, Theorem 3.14] (see also [18, 51]) follows from Theorem 2.12 in the same way as Theorem 2.13 does, this time applying Theorem 2.3.

A combination of Theorem 2.7 and Theorem 2.12 in a slightly different form is given by Pečarić and Beesack as follows.

Theorem 2.14 (SEE [119, THEOREM 5.13]) *Let L satisfy properties (L_1) and (L_2). Assume $\Phi \in \mathrm{C}(I,\mathbb{R})$ is convex, where $[m,M] \subseteq I$. Suppose A satisfies (A_1) and (A_2) such that $A(1) = 1$. Let $f \in L$ be such that $f(E) \subseteq [m,M]$ and $\Phi(f) \in L$, and define $p,q \geq 0$ such that $p + q > 0$ and*

$$A(f) = \frac{pm + qM}{p + q}$$

holds. Then

$$\Phi\left(\frac{pm + qM}{p + q}\right) \leq A(\Phi(f)) \leq \frac{p\Phi(m) + q\Phi(M)}{p + q}.$$

Theorem 2.15 *Assume $\Phi \in \mathrm{C}(I,\mathbb{R})$ is convex, $f : \mathscr{E} \to [m,M]$ is Δ-integrable, where $[m,M] \subseteq I$, and $h : \mathscr{E} \to \mathbb{R}$ is nonnegative Δ-integrable such that $L_\Delta(h) > 0$. Let $p,q \geq 0$ be such that $p+q > 0$ and*

$$\overline{L}_\Delta(f,h) = \frac{pm+qM}{p+q}$$

holds. Then

$$\Phi\left(\frac{pm+qM}{p+q}\right) \leq \overline{L}_\Delta(\Phi(f),h) \leq \frac{p\Phi(m)+q\Phi(M)}{p+q}.$$

Proof. Just apply Theorem 2.14 and Theorem 2.6. □

R. Jakšić and J. Pečarić presented, in [80], new converses of Jensen's inequality for positive linear functionals. Their main result is given in the following theorem.

Theorem 2.16 *Let ϕ be a continuous convex function on an interval of real numbers I and $m,M \in \mathbb{R}$, $m < M$, with $[m,M] \subset \mathrm{Int}(I)$, where $\mathrm{Int}(I)$ is the interior of I. Let L satisfy conditions (L_1), (L_2) and let A be any isotonic linear functional on L with $A(1) = 1$. If $f \in L$ satisfies the bounds*

$$-\infty < m \leq f(t) \leq M < \infty \qquad \text{for every} \qquad t \in E$$

and $\phi \circ f \in L$, then

$$\begin{aligned}
0 &\leq A(\phi(f)) - \phi(A(f)) \\
&\leq (M - A(f))(A(f) - m)\frac{\phi'_-(M) - \phi'_+(m)}{M-m} \\
&\leq \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)),
\end{aligned}$$

where $\phi'_-(M) = \lim\limits_{x\to M^-} \frac{\phi(x)-\phi(M)}{x-M}$ is a left-hand derivative of ϕ at M, and $\phi'_+(M) = \lim\limits_{x\to M^+} \frac{\phi(x)-\phi(M)}{x-M}$ is a right-hand derivative of ϕ at M, $x \in I$. If ϕ is concave on I, then the above inequalities are reversed.

Theorem 2.16 can be generalized in terms of time scale calculus as follows.

Theorem 2.17 *Let $\phi \in C(I,\mathbb{R})$ be convex, where $I = [m,M] \subset \mathbb{R}$, with $m < M$. Assume $\mathscr{E}$ is as in Theorem 2.4 and suppose f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = I$. Moreover, let $h : \mathscr{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t > 0$. Then*

$$\begin{aligned}
0 &\leq \overline{L}_\Delta(\phi(f),h) - \phi\left(\overline{L}_\Delta(f,h)\right) \qquad (2.5)\\
&\leq \left(M - \overline{L}_\Delta(f,h)\right)\left(\overline{L}_\Delta(f,h) - m\right) \cdot \frac{\phi'_-(M) - \phi'_+(m)}{M-m} \\
&\leq \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)).
\end{aligned}$$

If ϕ is concave on I, then all inequalities in (2.5) are reversed.

Proof. Let ϕ be a convex function. The first inequality in (2.5) follows directly from Jessen's inequality on time scales given in Theorem 2.8. Now, let us take the inequality (2.3) from Theorem 2.13. Adding the term $-\phi\left(\overline{L}_\Delta(f,h)\right)$ on both sides of (2.3), we obtain

$$\begin{aligned}\overline{L}_\Delta(\phi(f),h) &- \phi\left(\overline{L}_\Delta(f,h)\right) \\ &\le \frac{M-\overline{L}_\Delta(f,h)}{M-m}\phi(m) + \frac{\overline{L}_\Delta(f,h)-m}{M-m}\phi(M) - \phi\left(\overline{L}_\Delta(f,h)\right) =: B. \quad (2.6)\end{aligned}$$

By the convexity of ϕ, it follows

$$\phi(x) - \phi(M) \ge \phi'_-(M)(x-M), \qquad x \in [m,M]. \quad (2.7)$$

Multiplying inequality (2.7) with $(x-m) \ge 0$, we get

$$(x-m)\phi(x) - (x-m)\phi(M) \ge \phi'_-(M)(x-M)(x-m), \qquad x \in [m,M]. \quad (2.8)$$

Similarly, multiplying the inequality $\phi(x) - \phi(m) \ge \phi'_+(m)(x-m)$ with $(M-x) \ge 0$, we obtain

$$(M-x)\phi(x) - (M-x)\phi(m) \ge \phi'_+(m)(x-m)(M-x), \qquad x \in [m,M]. \quad (2.9)$$

Adding (2.8) to (2.9) and dividing by $(M-m)$, for any $x \in [m,M]$, we have

$$\begin{aligned}\frac{(M-x)\phi(m)+(x-m)\phi(M)}{M-m} &- \phi(x) \\ &\le \frac{(M-x)(x-m)}{M-m}\left(\phi'_-(M) - \phi'_+(m)\right). \quad (2.10)\end{aligned}$$

Replacing x in (2.10) with $\overline{L}_\Delta(f,h)$, leads to

$$B \le \left(M-\overline{L}_\Delta(f,h)\right)\left(\overline{L}_\Delta(f,h)-m\right)\frac{\phi'_-(M)-\phi'_+(m)}{M-m}. \quad (2.11)$$

Combining (2.6) and (2.11) brings us to the second inequality in (2.5). The third inequality in (2.5) follows from the elementary estimate $\frac{(M-x)(x-m)}{M-m} \le \frac{1}{4}(M-m)$ for every $x \in \mathbb{R}$. If the function ϕ is concave, then $-\phi$ is convex, so applying (2.5) to $-\phi$ gives us the reversed inequalities in (2.5). □

Remark 2.4 The proof of Theorem 2.17 can be obtained directly from Theorem 2.16 since the multiple Riemann delta time scale integral is an isotonic linear functional, according to Theorem 2.4.

Theorem 2.18 *Suppose that all assumptions from Theorem 2.17 hold. Then*

$$0 \le \frac{M-\overline{L}_\Delta(f,h)}{M-m}\phi(m) + \frac{\overline{L}_\Delta(f,h)-m}{M-m}\phi(M) - \overline{L}_\Delta(\phi(f),h) \quad (2.12)$$

$$\leq \frac{\phi'_-(M)-\phi'_+(m)}{M-m} \cdot \frac{\int_{\mathscr{E}} h(t)\,(M-f(t))\,(f(t)-m)\,\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}$$

$$\leq \frac{\phi'_-(M)-\phi'_+(m)}{M-m}\left(M-\overline{L}_\Delta(f,h)\right)\left(\overline{L}_\Delta(f,h)-m\right)$$

$$\leq \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)).$$

If ϕ is concave on I, then all inequalities in (2.12) *are reversed.*

Proof. Assume that ϕ is convex. The first inequality in (2.12) follows directly from inequality (2.3) in Theorem 2.13. We now replace x in (2.10) by $f(t)$, $t\in E$ (notice that $m\leq f(t)\leq M$ since $f(E)=I$ by the assumptions) so that

$$\frac{M-f(t)}{M-m}\phi(m)+\frac{f(t)-m}{M-m}\phi(M)-\phi(f(t)) \leq \frac{(M-f(t))(f(t)-m)}{M-m}\left(\phi'_-(M)-\phi'_+(m)\right). \quad (2.13)$$

Since the multiple Riemann delta time scale integral is an isotonic linear functional, multiplying inequality (2.13) by $\frac{h(t)}{\int_E h(t)\Delta t}$ and integrating the resulting inequality, we get

$$\frac{M-\overline{L}_\Delta(f,h)}{M-m}\phi(m)+\frac{\overline{L}_\Delta(f,h)-m}{M-m}\phi(M)-\overline{L}_\Delta(\phi(f),h) \leq \frac{\int_{\mathscr{E}} h(t)\,(M-f(t))\,(f(t)-m)\,\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}\cdot\frac{\phi'_-(M)-\phi'_+(m)}{M-m}, \quad (2.14)$$

which is the second inequality in (2.12). Using the fact that the function $g:\mathbb{R}\to\mathbb{R}$ defined as $g(x)=(M-x)(x-m)$, is concave, and applying Theorem 2.8 to the function g instead of the function ϕ, we deduce

$$\frac{\int_{\mathscr{E}} h(t)\,(M-f(t))\,(f(t)-m)\,\Delta t}{\int_{\mathscr{E}} h(t)\Delta t} \leq \left(M-\overline{L}_\Delta(f,h)\right)\left(\overline{L}_\Delta(f,h)-m\right),$$

which implies the third inequality in (2.12). The last inequality in (2.12) is the same one as the last inequality in Theorem 2.17. If the function ϕ is concave, then $-\phi$ is convex, so applying (2.12) to $-\phi$ gives us the reversed inequalities in (2.12). □

In [79], the authors proved the following refinement of Theorem 2.16.

Theorem 2.19 *Let ϕ be a continuous convex function on an interval of real numbers I and $m,M\in\mathbb{R}$, $m<M$, with $[m,M]\subset Int(I)$, where $Int(I)$ is the interior of I. Let L satisfy*

conditions (L_1), (L_2) *on* $\mathscr{E}$ *and let* A *be any isotonic linear functional on* L *with* $A(1) = 1$. *If* $f \in L$ *satisfies the bounds*

$$-\infty < m \le f(t) \le M < \infty \qquad \text{for every} \qquad t \in \mathscr{E}$$

and $\phi \circ f \in L$, *then*

$$\begin{aligned} 0 &\le A(\phi(f)) - \phi(A(f)) \\ &\le (M - A(f))(A(f) - m) \sup_{t\in(m,M)} \Psi_\phi(t;m,M) \\ &\le (M - A(f))(A(f) - m)\frac{\phi'_-(M) - \phi'_+(m)}{M-m} \\ &\le \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)), \end{aligned} \tag{2.15}$$

where $\phi'_-(M) = \lim\limits_{x\to M^-} \frac{\phi(x)-\phi(M)}{x-M}$ *is a left-hand derivative of* ϕ *at* M, *and* $\phi'_+(M) = \lim\limits_{x\to M^+} \frac{\phi(x)-\phi(M)}{x-M}$ *is a right-hand derivative of* ϕ *at* M, $x \in I$. *We also have the inequalities*

$$\begin{aligned} 0 \le A(\phi(f)) - \phi(A(f)) &\le \frac{1}{4}(M-m)^2\Psi_\phi(A(f);m,M) \\ &\le \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)), \end{aligned} \tag{2.16}$$

where $\Psi_\Phi(\cdot;m,M)\colon (m,M) \to \mathbb{R}$ *is defined by*

$$\Psi_\Phi(t;m,M) = \frac{1}{M-m}\left(\frac{\phi(M)-\phi(t)}{M-t} - \frac{\phi(t)-\phi(m)}{t-m}\right).$$

If ϕ *is concave on* I, *then all inequalities in* (2.15) *and* (2.16) *are reversed.*

Using the result from Theorem 2.19, we obtain the following refinement of Theorem 2.17.

Theorem 2.20 *Let* $\phi \in C(I,\mathbb{R})$ *be convex, where* $I = [m,M] \subset \mathbb{R}$, *with* $m < M$. *Assume* f *is* Δ-*integrable on* $\mathscr{E}$ *such that* $f(\mathscr{E}) = I$. *Moreover, let* $h : \mathscr{E} \to \mathbb{R}$ *be nonnegative* Δ-*integrable such that* $\int_{\mathscr{E}} h(t)\Delta t > 0$. *Then*

$$\begin{aligned} 0 &\le \overline{L}_\Delta(\phi(f),h) - \phi\left(\overline{L}_\Delta(f,h)\right) \\ &\le \left(M - \overline{L}_\Delta(f,h)\right)\left(\overline{L}_\Delta(f,h) - m\right) \sup_{t\in(m,M)} \Psi_\phi(t;m,M) \\ &\le \left(M - \overline{L}_\Delta(f,h)\right)\left(\overline{L}_\Delta(f,h) - m\right)\cdot\frac{\phi'_-(M) - \phi'_+(m)}{M-m} \\ &\le \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)), \end{aligned} \tag{2.17}$$

where $\Psi_\phi(\cdot;m,M)\colon (m,M) \to \mathbb{R}$ *is defined by*

$$\Psi_\Phi(t;m,M) = \frac{1}{M-m}\left(\frac{\phi(M)-\phi(t)}{M-t} - \frac{\phi(t)-\phi(m)}{t-m}\right).$$

If ϕ is concave on I, then all inequalities in (2.17) *are reversed.*

Proof. Since ϕ is a convex function, the first inequality in (2.17) follows from Theorem 2.8. From Theorem 2.13, we have

$$\begin{aligned}
&\overline{L}_\Delta(\phi(f),h)-\phi\left(\overline{L}_\Delta(f,h)\right) \qquad (2.18)\\
&\le \frac{M-\overline{L}_\Delta(f,h)}{M-m}\phi(m)+\frac{\overline{L}_\Delta(f,h)-m}{M-m}\phi(M)-\phi\left(\overline{L}_\Delta(f,h)\right)\\
&= \frac{1}{M-m}\left(M-\overline{L}_\Delta(f,h)\right)\left(\overline{L}_\Delta(f,h)-m\right)\\
&\quad\cdot\left(\frac{\phi(M)-\phi\left(\overline{L}_\Delta(f,h)\right)}{M-\overline{L}_\Delta(f,h)}-\frac{\phi\left(\overline{L}_\Delta(f,h)\right)-\phi(m)}{\overline{L}_\Delta(f,h)-m}\right)\\
&= \left(M-\overline{L}_\Delta(f,h)\right)\left(\overline{L}_\Delta(f,h)-m\right)\Psi_\phi\left(\overline{L}_\Delta(f,h);m,M\right)\\
&\le \left(M-\overline{L}_\Delta(f,h)\right)\left(\overline{L}_\Delta(f,h)-m\right)\sup_{t\in(m,M)}\Psi_\phi(t;m,M),
\end{aligned}$$

which is the second inequality in (2.17), provided that $\overline{L}_\Delta(f,h)\neq m,M$. When $\overline{L}_\Delta(f,h)$ is equal to m or M, then inequality (2.17) is obvious. Since

$$\begin{aligned}
&\sup_{t\in(m,M)}\Psi_\phi(t;m,M)=\frac{1}{M-m}\sup_{t\in(m,M)}\left\{\frac{\phi(M)-\phi(t)}{M-t}-\frac{\phi(t)-\phi(m)}{t-m}\right\}\\
&\le\frac{1}{M-m}\left(\sup_{t\in(m,M)}\frac{\phi(M)-\phi(t)}{M-t}+\sup_{t\in(m,M)}\frac{-(\phi(t)-\phi(m))}{t-m}\right)\\
&=\frac{1}{M-m}\left(\sup_{t\in(m,M)}\frac{\phi(M)-\phi(t)}{M-t}-\inf_{t\in(m,M)}\frac{\phi(t)-\phi(m)}{t-m}\right)=\frac{\phi'_-(M)-\phi'_+(m)}{M-m},
\end{aligned}$$

the third inequality in (2.17) is true. The last inequality in (2.17) follows from the elementary estimate $\frac{(M-x)(x-m)}{M-m}\le\frac{1}{4}(M-m)$, for every $x\in\mathbb{R}$. If the function ϕ is concave, then $-\phi$ is convex, so applying (2.17) to $-\phi$ gives the reversed inequalities in (2.17). This completes the proof. □

Remark 2.5 According to (2.18), with the same assumptions as in Theorem 2.20, we also have

$$\begin{aligned}
0&\le\overline{L}_\Delta(\phi(f),h)-\phi\left(\overline{L}_\Delta(f,h)\right)\\
&\le\frac{1}{4}(M-m)^2\Psi_\phi\left(\overline{L}_\Delta(f,h);m,M\right)\\
&\le\frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)).
\end{aligned}$$

2.3.1 Applications

Now, we use the results from Theorem 2.17 and Theorem 2.18 to get new converse inequalities for generalized means and power means in the time scale setting.

Generalized Means

Applying classical results to the monotonicity properties of generalized means with respect to the functional A, found in [68, p. 75, Theorem 92] and [119, p. 108, Theorem 4.3], R. Jakšić and J. Pečarić proved, in [80], the following converse.

Theorem 2.21 *Let L satisfy properties (L_1), (L_2) and A satisfy (A_1), (A_2) with $A(1) = 1$. Suppose $\psi, \chi : I \to \mathbb{R}$ are continuous and strictly monotone and $\phi = \chi \circ \psi^{-1}$ is convex. Then, for every $f \in L$ such that $m \le f(t) \le M$, $t \in [m,M] \subset I$, $-\infty < m < M < \infty$ and $\psi(f), \chi(f) \in L$, we have*

$$\begin{aligned} 0 &\le \chi\left(M_\chi(f,A)\right) - \chi\left(M_\psi(f,A)\right) \\ &\le \left(M_\psi - A(\psi(f))\right)\left(A(\psi(f)) - m_\psi\right)\frac{\phi'_-(M_\psi) - \phi'_+(m_\psi)}{M_\psi - m_\psi} \\ &\le \frac{1}{4}\left(M_\psi - m_\psi\right)\left(\phi'_-(M_\psi) - \phi'_+(m_\psi)\right), \end{aligned} \tag{2.19}$$

where $M_\psi(f,A) = \psi^{-1}\left(A\left(\psi(f)\right)\right)$ is a generalized mean with respect to the operator A and function ψ and $[m_\psi, M_\psi] = \psi([m,M])$. If ϕ is concave, then all inequalities in (2.19) *are reversed.*

To get new converse inequalities, let us first define the generalized mean in terms of the multiple Riemann delta time scale integral using the definition of weighted generalized mean on time scales [21].

Definition 2.2 *Suppose $\Psi : I \to \mathbb{R}$ is continuous and strictly monotone and f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = I$, where $\mathscr{E} \subset \mathbb{R}^n$ is as in Theorem 2.4. Let $h : \mathscr{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t > 0$. The generalized mean with respect to the multiple Riemann delta time scale integral is defined by*

$$M_\Psi\left(f, \overline{L}_\Delta(f,h)\right) = \Psi^{-1}\left(\overline{L}_\Delta(\Psi(f),h)\right). \tag{2.20}$$

Since the multiple Riemann delta time scale integral is an isotonic linear functional, from Theorem 2.21 we deduce the following result.

Theorem 2.22 *Suppose $\psi, \chi : I \to \mathbb{R}$ are continuous and strictly monotone and $\phi = \chi \circ \psi^{-1}$ is convex, $I = [m,M]$, $-\infty < m < M < \infty$. Assume f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = I$, where $\mathscr{E} \subset \mathbb{R}^n$ is as in Theorem 2.4. Let $h : \mathscr{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t > 0$. Then,*

$$\begin{aligned} 0 &\le \chi\left(M_\chi\left(f, \overline{L}_\Delta(f,h)\right)\right) - \chi\left(M_\psi\left(f, \overline{L}_\Delta(f,h)\right)\right) \\ &\le \left(M_\psi - \overline{L}_\Delta(\Psi(f),h)\right)\left(\overline{L}_\Delta(\Psi(f),h) - m_\psi\right) \\ &\quad \cdot \frac{\phi'_-(M_\psi) - (\phi'_+(m_\psi)}{M_\psi - m_\psi} \\ &\le \frac{1}{4}\left(M_\psi - m_\psi\right)\left(\phi'_-(M_\psi) - \phi'_+(m_\psi)\right), \end{aligned} \tag{2.21}$$

where $[m_\psi, M_\psi] = \psi([m,M])$. If ϕ is concave, then all inequalities in (2.21) *are reversed.*

Proof. The claim follows from Theorem 2.4 and Theorem 2.21. □

Theorem 2.23 *Let all assumptions from Theorem 2.22 be valid. If the function $\phi = \chi\circ\psi^{-1}$ is convex, then*

$$\begin{aligned}
0 &\le \frac{M_\psi - \overline{L}_\Delta(\Psi(f),h)}{M_\psi - m_\psi}\phi(m) + \frac{\overline{L}_\Delta(\Psi(f),h) - m_\psi}{M_\psi - m_\psi}\phi(M) \\
&\quad -\chi\left(M_\chi\left(f,\overline{L}_\Delta(\Psi(f),h)\right)\right) \\
&\le \frac{\phi'_-(M_\psi)-\phi'_+(m_\psi)}{M_\psi - m_\psi}\cdot\frac{\int_{\mathscr{E}} h(t)\left(M_\psi - \psi(f(t))\right)\left(\psi(f(t)) - m_\psi\right)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t} \\
&\le \frac{\phi'_-(M_\psi)-\phi'_+(m_\psi)}{M_\psi - m_\psi}\cdot\left(M_\psi - \overline{L}_\Delta(\Psi(f),h)\right)\left(\overline{L}_\Delta(\Psi(f),h) - m_\psi\right) \\
&\le \frac{1}{4}\left(M_\psi - m_\psi\right)\left(\phi'_-(M_\psi)-\phi'_+(m_\psi)\right). \qquad (2.22)
\end{aligned}$$

where $[m_\psi,M_\psi] = \psi([m,M])$. If ϕ is concave on I, then all inequalities in (2.22) *are reversed.*

Proof. The inequalities in (2.22) follow directly from Theorem 2.18 by replacing m by m_ψ, M by M_ψ, ϕ by $\chi\circ\psi^{-1}$, and f by $\psi\circ f$. All conditions of Theorem 2.18 are satisfied because $\chi\circ\psi^{-1}$ is obviously continuous and convex by assumption. Also, we have $m_\psi \le \psi(f(t)) \le M_\psi$ for every $t\in[m,M]$ since $m_\psi = \psi(m)$ and $M_\psi = \psi(M)$ if ψ is increasing and $m_\psi = \psi(M)$ and $M_\psi = \psi(m)$ if ψ is decreasing. If the function $\phi = \chi\circ\psi^{-1}$ is concave, then the function $-\phi = -\chi\circ\psi^{-1}$ is convex so, replacing ϕ by $-\phi$ in (2.22), we obtain the reversed inequalities. □

Theorem 2.24 *Suppose $I=[m,M]$, $-\infty < m < M < \infty$, $\psi,\chi : I\to\mathbb{R}$ are continuous and strictly monotone and $\phi = \chi\circ\psi^{-1}$ is convex. Assume $f,h:\mathscr{E}\to\mathbb{R}$ are Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = I$ and $\int_{\mathscr{E}}|h(t)|\Delta t > 0$. Then,*

$$\begin{aligned}
0 &\le \chi\left(M_\chi\left(f,\overline{L}_\Delta\right)\right) - \chi\left(M_\psi\left(f,\overline{L}_\Delta\right)\right) \\
&\le \left(M_\psi - \overline{L}_\Delta(\psi(f),h)\right)\left(\overline{L}_\Delta(\psi(f),h) - m_\psi\right)\sup_{t\in(m,M)}\Psi_{\chi\circ\psi^{-1}}(\psi(t);m_\psi,M_\psi) \\
&\le \left(M_\psi - \overline{L}_\Delta(\psi(f),h)\right)\left(\overline{L}_\Delta(\psi(f),h) - m_\psi\right) \\
&\quad \cdot\frac{(\chi\circ\psi^{-1})'_-(M_\psi) - (\chi\circ\psi^{-1})'_+(m_\psi)}{M_\psi - m_\psi} \qquad (2.23) \\
&\le \frac{1}{4}\left(M_\psi - m_\psi\right)\left((\chi\circ\psi^{-1})'_-(M_\psi) - (\chi\circ\psi^{-1})'_+(m_\psi)\right),
\end{aligned}$$

where $[m_\psi,M_\psi] = \psi([m,M])$. If ϕ is concave, then all inequalities in (2.23) *are reversed.*

Proof. The claim follows from Theorem 2.20. □

The following result on power means with respect to an isotonic linear functional is proved in [80].

Theorem 2.25 *Let L satisfy properties (L_1), (L_2) and A satisfy (A_1), (A_2) with $A(1) = 1$. Let $m, M \in \mathbb{R}$, $f \in L$, be such that $-\infty < m < M < \infty$, $0 < m \le f(t) \le M < \infty$, for $t \in \mathscr{E}$, and $f^r, f^s, (\log f) \in L$, for $r, s \in \mathbb{R}$.*

(i) *If $0 < r < s$ or $r < 0 < s$, then*

$$\begin{aligned} 0 &\le \left(M^{[s]}(f,A)\right)^s - \left(M^{[r]}(f,A)\right)^s \\ &\le \frac{s}{r}\left(M^r - A(f^r)\right)\left(A(f^r) - m^r\right)\frac{M^{s-r} - m^{s-r}}{M^r - m^r} \\ &\le \frac{s}{4r}\left(M^r - m^r\right)\left(M^{s-r} - m^{s-r}\right). \end{aligned} \tag{2.24}$$

(ii) *If $r < s < 0$, then*

$$\begin{aligned} 0 &\ge \left(M^{[s]}(f,A)\right)^s - \left(M^{[r]}(f,A)\right)^s \\ &\ge \frac{s}{r}\left(M^r - A(f^r)\right)\left(A(f^r) - m^r\right)\frac{M^{s-r} - m^{s-r}}{M^r - m^r} \\ &\ge \frac{s}{4r}\left(M^r - m^r\right)\left(M^{s-r} - m^{s-r}\right). \end{aligned} \tag{2.25}$$

(iii) *If $s = 0$ and $r < 0$, then*

$$\begin{aligned} 0 &\le \log\left(M^{[0]}(f,A)\right) - \log\left(M^{[r]}(f,A)\right) \\ &\le -\frac{1}{r}\frac{\left(M^r - A(f^r)\right)\left(A(f^r) - m^r\right)}{M^r m^r} \\ &\le -\frac{1}{4r}\frac{\left(M^r - m^r\right)^2}{M^r m^r}. \end{aligned} \tag{2.26}$$

(iv) *If $r = 0$ or $s > 0$, then*

$$0 \le \left(M^{[s]}(f,A)\right)^s - \left(M^{[0]}(f,A)\right)^s \tag{2.27}$$

$$\begin{aligned} &\le \left(\log M - A(\log f)\right)\left(A(\log f) - \log m\right)\frac{s(e^{sM} - e^{sm})}{\log M - \log m} \\ &\le \frac{s}{4}(e^{sM} - e^{sm})\log\frac{M}{m}, \end{aligned}$$

where

$$M^{[r]}(f,A) = \begin{cases} \left(A(f^r)\right)^{\frac{1}{r}}, & \textit{if} \quad r \ne 0 \\ \exp\left(A(\log f)\right), & \textit{if} \quad r = 0 \end{cases}$$

is the power mean.

Now, we define the power mean in terms of the multiple Riemann delta time scale integral.

Definition 2.3 *Assume $\mathscr{E} \subset \mathbb{R}^n$ is as in Theorem 2.4 and f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = I$ and $f(t) > 0$, $t \in \mathscr{E}$. Let $h : \mathscr{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t > 0$. For $r \in \mathbb{R}$, suppose f^r and $(\log f)$ are Δ-integrable on $\mathscr{E}$. The power mean with respect to the multiple Riemann delta time scale integral is defined by*

$$M^{[r]}\left(f, \overline{L}_\Delta(f,h)\right) = \begin{cases} \left(\overline{L}_\Delta(f^r,h)\right)^{\frac{1}{r}}, & \text{if } \ r \neq 0 \\ \exp\left(\overline{L}_\Delta(\log f,h)\right), & \text{if } \ r = 0. \end{cases} \tag{2.28}$$

Using the fact that the multiple Riemann delta time scale integral is an isotonic linear functional, from Theorem 2.25, we derive the following result.

Theorem 2.26 *Let $\mathscr{E} \subset \mathbb{R}^n$ be as in Theorem 2.4, let f be Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = I$ and $0 < m \leq f(t) \leq M < \infty$, for $t \in \mathscr{E}$, $m, M \in \mathbb{R}$. Let $h : \mathscr{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t > 0$. For $r, s \in \mathbb{R}$ suppose f^r, f^s, $(\log f)$ are Δ-integrable on $\mathscr{E}$.*

(i) *If $0 < r < s$ or $r < 0 < s$, then*

$$\begin{aligned} 0 &\leq \left(M^{[s]}\left(f, \overline{L}_\Delta(f,h)\right)\right)^s - \left(M^{[r]}\left(f, \overline{L}_\Delta(f,h)\right)\right)^s \\ &\leq \frac{s}{r}\left(M^r - \overline{L}_\Delta(f^r,h)\right)\left(\overline{L}_\Delta(f^r,h) - m^r\right)\frac{M^{s-r} - m^{s-r}}{M^r - m^r} \\ &\leq \frac{s}{4r}\left(M^r - m^r\right)\left(M^{s-r} - m^{s-r}\right). \end{aligned} \tag{2.29}$$

(ii) *If $r < s < 0$, then*

$$\begin{aligned} 0 &\geq \left(M^{[s]}\left(f, \overline{L}_\Delta(f,h)\right)\right)^s - \left(M^{[r]}\left(f, \overline{L}_\Delta(f,h)\right)\right)^s \\ &\geq \frac{s}{r}\left(M^r - \overline{L}_\Delta(f^r,h)\right)\left(\overline{L}_\Delta(f^r,h) - m^r\right)\frac{M^{s-r} - m^{s-r}}{M^r - m^r} \\ &\geq \frac{s}{4r}\left(M^r - m^r\right)\left(M^{s-r} - m^{s-r}\right). \end{aligned} \tag{2.30}$$

(iii) *If $s = 0$ and $r < 0$, then*

$$\begin{aligned} 0 &\leq \log\left(M^{[0]}\left(f, \overline{L}_\Delta(f,h)\right)\right) - \log\left(M^{[r]}\left(f, \overline{L}_\Delta(f,h)\right)\right) \\ &\leq -\frac{1}{r}\frac{\left(M^r - \overline{L}_\Delta(f^r,h)\right)\left(\overline{L}_\Delta(f^r,h) - m^r\right)}{M^r m^r} \\ &\leq -\frac{1}{4r}\frac{(M^r - m^r)^2}{M^r m^r}. \end{aligned} \tag{2.31}$$

(iv) *If $r = 0$ and $s > 0$, then*

$$0 \leq \left(M^{[s]}\left(f, \overline{L}_\Delta(f,h)\right)\right)^s - \left(M^{[0]}\left(f, \overline{L}_\Delta(f,h)\right)\right)^s \tag{2.32}$$

$$\leq \left(\log M - \overline{L}_\Delta(\log f, h)\right)\left(\overline{L}_\Delta(\log f, h) - \log m\right) \cdot \frac{s(e^{sM} - e^{sm})}{\log M - \log m}$$
$$\leq \frac{s}{4}(e^{sM} - e^{sm}) \log \frac{M}{m}.$$

Proof. The claim follows from Theorem 2.4 and Theorem 2.25. □

Theorem 2.27 *Suppose the same hypotheses as in Theorem 2.26 are valid.*

(i) *If* $0 < r < s$ *or* $r < 0 < s$, *then*

$$\begin{aligned}
0 &\leq \frac{M^r - \overline{L}_\Delta(f^r, h)}{M^r - m^r} m^s + \frac{\overline{L}_\Delta(f^r, h) - m^r}{M^r - m^r} M^s \\
&\quad - \left(M^{[s]}\left(f, \overline{L}_\Delta(f, h)\right)\right)^s \\
&\leq \frac{s}{r} \cdot \frac{M^{s-r} - m^{s-r}}{M^r - m^r} \cdot \frac{\int_{\mathscr{E}} h(t)\left((M^r - f^r(t))(f^r(t) - m^r)\right)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t} \\
&\leq \frac{s}{r} \cdot \frac{M^{s-r} - m^{s-r}}{M^r - m^r}\left(M^r - \overline{L}_\Delta(f^r, h)\right)\left(\overline{L}_\Delta(f^r, h) - m^r\right) \\
&\leq \frac{s}{4r}(M^r - m^r)(M^{s-r} - m^{s-r}).
\end{aligned} \tag{2.33}$$

(ii) *If* $r < s < 0$, *then*

$$\begin{aligned}
0 &\geq \frac{M^r - \overline{L}_\Delta(f^r, h)}{M^r - m^r} m^s + \frac{\overline{L}_\Delta(f^r, h) - m^r}{M^r - m^r} M^s \\
&\quad - \left(M^{[s]}\left(f, \overline{L}_\Delta(f, h)\right)\right)^s \\
&\geq \frac{s}{r} \cdot \frac{M^{s-r} - m^{s-r}}{M^r - m^r} \cdot \frac{\int_{\mathscr{E}} h(t)\left((M^r - f^r(t))(f^r(t) - m^r)\right)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t} \\
&\geq \frac{s}{r} \cdot \frac{M^{s-r} - m^{s-r}}{M^r - m^r}\left(M^r - \overline{L}_\Delta(f^r, h)\right)\left(\overline{L}_\Delta(f^r, h) - m^r\right) \\
&\geq \frac{s}{4r}(M^r - m^r)(M^{s-r} - m^{s-r}).
\end{aligned} \tag{2.34}$$

(iii) *If* $s=0$ *and* $r<0$, *then*

$$\begin{aligned} 0 \le\ & \frac{M^r-\overline{L}_\Delta(f^r,h)}{M^r-m^r}\log m+\frac{s}{r}\cdot\frac{\overline{L}_\Delta(f^r,h)-m^r}{M^r-m^r}\log M \\ & -\log\left(M^{[0]}\left(f,\overline{L}_\Delta(f,h)\right)\right) \\ \le\ & -\frac{1}{r}\cdot\frac{\frac{\int_{\mathscr{E}} h(t)((M^r-f^r(t))(f^r(t)-m^r))\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}}{M^r m^r} \\ \le\ & -\frac{1}{r}\cdot\frac{\left(M^r-\overline{L}_\Delta(f^r,h)\right)\left(\overline{L}_\Delta(f^r,h)-m^r\right)}{M^r m^r} \\ \le\ & \frac{1}{4r}(M^r-m^r)\left(\frac{1}{M^r}-\frac{1}{m^r}\right). \end{aligned} \tag{2.35}$$

(iv) *If* $r=0$ *and* $s>0$, *then*

$$\begin{aligned} 0 \le\ & \frac{\log M-\overline{L}_\Delta(\log f,h)}{\log M-\log m}m^s+\frac{\overline{L}_\Delta(\log f,h)-\log m}{\log M-\log m}M^s \\ & -\left(M^{[s]}\left(f,\overline{L}_\Delta(f,h)\right)\right)^s \\ \le\ & \frac{s(e^{sM}-e^{sm})}{\log M-\log m}\cdot\frac{\int_{\mathscr{E}} h(t)\left((\log M-\log(f(t)))\left(\log(f(t))-\log m\right)\right)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t} \\ \le\ & \frac{s(e^{sM}-e^{sm})}{\log M-\log m}\left(\log M-\overline{L}_\Delta(\log f,h)\right)\left(\overline{L}_\Delta(\log f,h)-\log m\right) \\ \le\ & \frac{s}{4}(e^{sM}-e^{sm})\log\frac{M}{m}. \end{aligned} \tag{2.36}$$

Proof. The above inequalities follow from Theorem 2.18. Namely,

(i) if $0<r<s$ or $r<0<s$, then we can take the function ϕ to be of the form $\phi(t)=t^{\frac{s}{r}}$ because ϕ is now continuous and convex and all the conditions of Theorem 2.18 are satisfied. Now, inequality (2.33) follows from (2.12) with replacing m by m^r, M by M^r, and f by f^r if $r>0$ (because the function f^r is then strictly increasing) and with replacing M by m^r, m by M^r, and f by f^r if $r<0$ (because the function f^r is then strictly decreasing);

(ii) if $r<s<0$, then the function $\phi(t)=t^{\frac{s}{r}}$ is concave so we obtain inequality (2.34) from the inequalities reversed to (2.12) making following replacements: M by m^r, m by M^r, and f by f^r (f^r is now strictly decreasing);

(iii) if $s=0$ and $r<0$, then we take $\phi(t)=\frac{1}{r}\log t$ which is continuous and convex, and we deduce inequality (2.35) from (2.12) interchanging M by m^r, m by M^r, and f by f^r (f^r is now strictly decreasing);

(iv) if $r = 0$ and $s > 0$, then we take $\phi(t) = e^{st}$ which is continuous and convex, and inequality (2.36) follows from (2.12) interchanging m by $\log m$, M by $\log M$, and f by $\log f$ ($\log f$ is strictly increasing).

This completes the proof. □

Theorem 2.28 *Suppose $\mathscr{E} \subset \mathbb{R}^n$ is as in Theorem 2.4 and f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = I$ and $0 < m \le f(t) \le M < \infty$, for $t \in \mathscr{E}$, $m, M \in \mathbb{R}$. Let $h : \mathscr{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t > 0$. For $r, s \in \mathbb{R}$ suppose f^r, f^s, $(\log f)$ are Δ-integrable on $\mathscr{E}$.*

(i) *If $r < 0 < s$ or $r < s < 0$, then*

$$\begin{aligned} 0 &\le \left(M^{[r]}\left(f, \overline{L}_\Delta(f,h)\right)\right)^r - \left(M^{[s]}\left(f, \overline{L}_\Delta(f,h)\right)\right)^r \\ &\le \frac{r}{s}\left(M^s - \overline{L}_\Delta(f^s,h)\right)\left(\overline{L}_\Delta(f^s,h) - m^s\right)\frac{M^{r-s} - m^{r-s}}{M^s - m^s} \\ &\le \frac{r}{4s}\left(M^s - m^s\right)\left(M^{r-s} - m^{r-s}\right). \end{aligned} \tag{2.37}$$

(ii) *If $0 < r < s$, then*

$$\begin{aligned} 0 &\ge \left(M^{[r]}\left(f, \overline{L}_\Delta(f,h)\right)\right)^r - \left(M^{[s]}\left(f, \overline{L}_\Delta(f,h)\right)\right)^r \\ &\ge \frac{r}{s}\left(M^s - \overline{L}_\Delta(f^s,h)\right)\left(\overline{L}_\Delta(f^s,h) - m^s\right)\frac{M^{r-s} - m^{r-s}}{M^s - m^s} \\ &\ge \frac{r}{4s}\left(M^s - m^s\right)\left(M^{r-s} - m^{r-s}\right). \end{aligned} \tag{2.38}$$

(iii) *If $s = 0$ and $r < 0$, then*

$$\begin{aligned} 0 &\le \left(M^{[r]}\left(f, \overline{L}_\Delta(f,h)\right)\right)^r - \left(M^{[0]}\left(f, \overline{L}_\Delta(f,h)\right)\right)^r \\ &\le \left(\log M - \overline{L}_\Delta(\log f,h)\right)\left(\overline{L}_\Delta(\log f,h) - \log m\right)\cdot\frac{r(M^r - m^r)}{\log M - \log m} \\ &\le \frac{r}{4}(M^r - m^r)\log\frac{M}{m}. \end{aligned} \tag{2.39}$$

(iv) *If $r = 0$ and $s > 0$, then*

$$\begin{aligned} 0 &\ge \log\left(M^{[0]}\left(f, \overline{L}_\Delta(f,h)\right)\right) - \log\left(M^{[s]}\left(f, \overline{L}_\Delta(f,h)\right)\right) \\ &\ge -\frac{1}{s}\left(M^s - \overline{L}_\Delta(f^s,h)\right)\left(\overline{L}_\Delta(f^s,h) - m^s\right)\frac{1}{M^r m^r} \\ &\ge \frac{1}{4s}(M^s - m^s)\left(\frac{1}{m^s} - \frac{1}{m^s}\right). \end{aligned} \tag{2.40}$$

Proof. The above inequalities follow directly from Theorem 2.17. Namely,

(i) if $r<0<s$ or $r<s<0$, then we can take the function ϕ to be of the form $\phi(t)=t^{\frac{r}{s}}$ because ϕ is now continuous and convex and all the conditions of Theorem 2.17 are satisfied. Now, inequality (2.37) follows from (2.5) with replacing m by m^s, M by M^s and f by f^s if $s>0$ (because the function f^s is then strictly increasing) and replacing M by m^s, m by M^s and f by f^s if $s<0$ (because the function f^s is then strictly decreasing);

(ii) if $0<r<s$, then the function $\phi(t)=t^{\frac{r}{s}}$ is concave so we obtain inequality (2.38) from the inequalities reversed to (2.5) making following replacements m by m^s, M by M^s and f by f^s (f^s is now strictly decreasing);

(iii) if $s=0$ and $r<0$, then we take $\phi(t)=e^{rt}$ which is continuous and convex, and we deduce inequality (2.39) from (2.5) replacing m by $\log m$, M by $\log M$ and f by $\log f$ ($\log f$ is strictly increasing);

(iv) if $r=0$ and $s>0$, then we take $\phi(t)=\frac{1}{s}\log t$ which is continuous and concave, and inequality (2.40) follows from (2.5) replacing m by m^s, M by M^s, and f by f^s, (f^s is now strictly increasing).

This completes the proof. □

Theorem 2.29 *Suppose the hypotheses of Theorem 2.28 hold.*

(i) *If $r<s<0$ or $r<0<s$, then*

$$\begin{aligned}0 &\leq \frac{M^s-\overline{L}_\Delta(f^s,h)}{M^s-m^s}m^r+\frac{\overline{L}_\Delta(f^s,h)-m^s}{M^s-m^s}M^r\\&\quad-\left(M^{[r]}\left(f,\overline{L}_\Delta(f,h)\right)\right)^r\\&\leq \frac{r}{s}\cdot\frac{M^{r-s}-m^{r-s}}{M^s-m^s}\cdot\frac{\int_{\mathscr{E}} h(t)\left(\left(M^s-f^s(t)\right)\left(f^s(t)-m^s\right)\right)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}\\&\leq \frac{r}{s}\cdot\frac{M^{r-s}-m^{r-s}}{M^s-m^s}\left(M^s-\overline{L}_\Delta(f^s,h)\right)\left(\overline{L}_\Delta(f^s,h)-m^s\right)\\&\leq \frac{r}{4s}(M^s-m^s)(M^{r-s}-m^{r-s}).\end{aligned}\tag{2.41}$$

(ii) *If $0<r<s$, then*

$$\begin{aligned}0 &\geq \frac{M^s-\overline{L}_\Delta(f^s,h)}{M^s-m^s}m^r+\frac{\overline{L}_\Delta(f^s,h)-m^s}{M^s-m^s}M^r\\&\quad-\left(M^{[r]}\left(f,\overline{L}_\Delta(f,h)\right)\right)^r\\&\geq \frac{r}{s}\cdot\frac{M^{r-s}-m^{r-s}}{M^s-m^s}\cdot\frac{\int_{\mathscr{E}} h(t)\left(\left(M^s-f^s(t)\right)\left(f^s(t)-m^s\right)\right)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}\end{aligned}\tag{2.42}$$

$$\geq \frac{r}{s}\cdot\frac{M^{r-s}-m^{r-s}}{M^s-m^s}\left(M^s-\overline{L}_\Delta(f^s,h)\right)\left(\overline{L}_\Delta(f^s,h)-m^s\right)$$
$$\geq \frac{r}{4s}(M^s-m^s)(M^{r-s}-m^{r-s}).$$

(iii) *If $s=0$ and $r<0$, then*

$$\begin{aligned}0 &\leq \frac{\log M-\overline{L}_\Delta(\log f,h)}{\log M-\log m}m^r+\frac{\overline{L}_\Delta(\log f,h)-\log m}{\log M-\log m}M^r \\ &\quad -\left(M^{[r]}\left(f,\overline{L}_\Delta(f,h)\right)\right)^r \\ &\leq \frac{r(M^r-m^r)}{\log M-\log m}\cdot\frac{\int_{\mathscr{E}} h(t)\left(\left(\log M-\log(f(t))\right)\left(\log(f(t))-\log m\right)\right)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t} \\ &\leq \frac{r(M^r-m^r)}{\log M-\log m}\left(\log M-\overline{L}_\Delta(\log f,h)\right)\left(\overline{L}_\Delta(\log f,h)-\log m\right) \\ &\leq \frac{r}{4}(M^r-m^r)\log\frac{M}{m}.\end{aligned} \tag{2.43}$$

(iv) *If $r=0$ and $s>0$, then*

$$\begin{aligned}0 &\geq \frac{M^s-\overline{L}_\Delta(f^s,h)}{M^s-m^s}\log m+\frac{\overline{L}_\Delta(f^s,h)-m^s}{M^s-m^s}\log M \\ &\quad -\log\left(M^{[0]}\left(f,\overline{L}_\Delta(f,h)\right)\right) \\ &\geq -\frac{1}{s}\cdot\frac{\frac{\int_{\mathscr{E}} h(t)((M^s-f^s(t))(f^s(t)-m^s))\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}}{M^s m^s} \\ &\geq -\frac{1}{s}\cdot\frac{\left(M^s-\overline{L}_\Delta(f^s,h)\right)\left(\overline{L}_\Delta(f^s,h)-m^s\right)}{M^s m^s} \\ &\geq \frac{1}{s}(M^s-m^s)\left(\frac{1}{M^s}-\frac{1}{m^s}\right).\end{aligned} \tag{2.44}$$

Proof. All the inequalities can be obtained directly from Theorem 2.18 using inequality (2.12) and the same technique and substitutions as in the proof of Theorem 2.28. □

From Theorem 2.20 and Theorem 2.25, the following refinement is obtained.

Theorem 2.30 *Suppose $\mathscr{E}\subset\mathbb{R}^n$ is as in Theorem 2.4, f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E})=I$ and $0<m\leq f(t)\leq M<\infty$, for $t\in\mathscr{E}$, $m,M\in\mathbb{R}$. Let $h:\mathscr{E}\to\mathbb{R}$ be Δ-integrable such that $\int_{\mathscr{E}}|h(t)|\Delta t>0$. For $r,s\in\mathbb{R}$ suppose f^r, f^s, $(\log f)$ are Δ-integrable on $\mathscr{E}$.*

(i) *If* $0 < r < s$ *or* $r < 0 < s$*, then*

$$\begin{aligned} 0 &\le \left(M^{[s]}\left(f,\overline{L}_\Delta\right)\right)^s - \left(M^{[r]}\left(f,\overline{L}_\Delta\right)\right)^s \\ &\le \left(M^r - \overline{L}_\Delta(f^r,h)\right)\left(\overline{L}_\Delta(f^r,h) - m^r\right) \sup_{t\in(m,M)} \Psi_\phi(t^r;m^r,M^r) \\ &\le \frac{s}{r}\left(M^r - \overline{L}_\Delta(f^r,h)\right)\left(\overline{L}_\Delta(f^r,h) - m^r\right)\frac{M^{s-r}-m^{s-r}}{M^r-m^r} \\ &\le \frac{s}{4r}\left(M^r-m^r\right)\left(M^{s-r}-m^{s-r}\right). \end{aligned} \tag{2.45}$$

(ii) *If* $r < s < 0$*, then*

$$\begin{aligned} 0 &\ge \left(M^{[s]}\left(f,\overline{L}_\Delta\right)\right)^s - \left(M^{[r]}\left(f,\overline{L}_\Delta\right)\right)^s \\ &\ge \left(M^r - \overline{L}_\Delta(f^r,h)\right)\left(\overline{L}_\Delta(f^r,h) - m^r\right) \sup_{t\in(m,M)} \Psi_\phi(t^r;m^r,M^r) \\ &\ge \frac{s}{r}\left(M^r - \overline{L}_\Delta(f^r,h)\right)\left(\overline{L}_\Delta(f^r,h) - m^r\right)\frac{M^{s-r}-m^{s-r}}{M^r-m^r} \\ &\ge \frac{s}{4r}\left(M^r-m^r\right)\left(M^{s-r}-m^{s-r}\right). \end{aligned} \tag{2.46}$$

(iii) *If* $s = 0$ *and* $r < 0$*, then*

$$\begin{aligned} 0 &\le \log\left(M^{[0]}\left(f,\overline{L}_\Delta\right)\right) - \log\left(M^{[r]}\left(f,\overline{L}_\Delta\right)\right) \\ &\le \left(M^r - \overline{L}_\Delta(f^r,h)\right)\left(\overline{L}_\Delta(f^r,h) - m^r\right) \sup_{t\in(m,M)} \Psi_\phi(t^r;M^r,m^r) \\ &\le -\frac{1}{r}\cdot\frac{\left(M^r - \overline{L}_\Delta(f^r,h)\right)\left(\overline{L}_\Delta(f^r,h) - m^r\right)}{M^r m^r} \\ &\le -\frac{1}{4r}\cdot\frac{(M^r-m^r)^2}{M^r m^r}. \end{aligned} \tag{2.47}$$

(iv) *If* $r = 0$ *and* $s > 0$*, then*

$$\begin{aligned} 0 &\le \left(M^{[s]}\left(f,\overline{L}_\Delta\right)\right)^s - \left(M^{[0]}\left(f,\overline{L}_\Delta\right)\right)^s \\ &\le \left(\log M - \overline{L}_\Delta(\log f,h)\right)\left(\overline{L}_\Delta(\log f,h) - \log m\right) \\ &\quad \cdot \sup_{t\in(m,M)} \Psi_\phi(\log t;\log m,\log M) \\ &\le \left(\log M - \overline{L}_\Delta(\log f,h)\right)\left(\overline{L}_\Delta(\log f,h) - \log m\right)\cdot\frac{s(M^s-m^s)}{\log M - \log m} \\ &\le s\left(M^s-m^s\right)\log\frac{M}{m}. \end{aligned} \tag{2.48}$$

Proof. The claim follows from Theorem 2.20 and Theorem 2.25. □

2.4 Inequalities Related to Jensen's Inequality

Hölder's Inequality

We first recall Hölder's inequality for isotonic linear functionals.

Theorem 2.31 (SEE [119, THEOREM 4.12]) *Let E, L, and A be such that (L_1), (L_2), (A_1), (A_2) are satisfied. For $p \neq 1$, define $q = p/(p-1)$. Assume w, f, g are nonnegative functions on E and $wf^p, wg^q, wfg \in L$. If $p > 1$, then*

$$A(wfg) \leq A^{1/p}(wf^p)A^{1/q}(wg^q).$$

This inequality is reversed if $0 < p < 1$ and $A(wg^q) > 0$, and it is also reversed if $p < 0$ and $A(wf^p) > 0$.

In the following theorem, we give the generalization of Hölder's inequality on time scales.

Theorem 2.32 *For $p \neq 1$, define $q = p/(p-1)$. Assume w, f, g are nonnegative functions on $\mathscr{E}$ and wf^p, wg^q, wfg are Δ-integrable on $\mathscr{E}$. If $p > 1$, then*

$$L_\Delta(wfg) \leq L_\Delta^{\frac{1}{p}}(wf^p)L_\Delta^{\frac{1}{q}}(wg^q).$$

This inequality is reversed if $0 < p < 1$ and $L_\Delta(wg^q) > 0$, and it is also reversed if $p < 0$ and $L_\Delta(wf^p) > 0$.

Proof. Just apply Theorem 2.31 and Theorem 2.5. □

Remark 2.6 Note that the known results from the time scales literature follow from Theorem 2.31 in the same way as Theorem 2.32 does: [45, Theorem 6.13] follows as in Theorem 2.1 and [16, Theorem 4.1] (see also [57, 17]) follows as in Theorem 2.3.

From Hölder's inequality follows the Cauchy–Schwarz inequality given in the next theorem.

Theorem 2.33 *If w, f, g are nonnegative functions on $\mathscr{E}$ and wf^2, wg^2, wfg are Δ-integrable on $\mathscr{E}$, then*

$$L_\Delta(wfg) \leq \sqrt{L_\Delta(wf^2)L_\Delta(wg^2)}.$$

Proof. Just let $p = 2$ in Theorem 2.32. □

Now, we can prove new converses of Hölder's inequality in terms of time scale calculus and Δ-integral using the results given in [80].

Theorem 2.34 *Let $p>1$ and define q by $\frac{1}{p}+\frac{1}{q}=1$. Assume $\mathscr{E}\subset\mathbb{R}^n$ is as in Theorem 2.4 and w,f,g are nonnegative real functions on $\mathscr{E}$ such that wf^p, wg^q, wfg are Δ-integrable on $\mathscr{E}$ and $\int_{\mathscr{E}} w(t)g^q(t)\Delta t>0$. Let $m,M\in\mathbb{R}$ such that $-\infty<m<M<\infty$ and $m\le f(t)g^q(t)\le M$, $t\in\mathscr{E}$. Then,*

$$\begin{aligned}0 &\le L_\Delta(wf^p)\cdot L_\Delta^{\frac{p}{q}}(wg^q)-L_\Delta^p(wfg)\\&\le (ML_\Delta(wg^q)-L_\Delta(wfg))\,(L_\Delta(wfg)-mL_\Delta(wg^q))\\&\quad\cdot\frac{p(M^{p-1}-m^{p-1})}{M-m}\cdot L_\Delta^{p-2}(wg^q)\\&\le \frac{p}{4}(M-m)(M^{p-1}-m^{p-1})L_\Delta^p(wg^q).\end{aligned}\tag{2.49}$$

For $p<0$, inequalities (2.49) *hold if $L_\Delta(wfg)>0$, $t\in\mathscr{E}$. In case $0<p<1$, all inequalities in* (2.49) *are reversed.*

Proof. Inequalities (2.49) follow directly from Theorem 2.17 by taking the function ϕ to be of the form $\phi(t)=t^p$ and replacing h by wg^q and f by $fg^{-\frac{q}{p}}$. For $p<0$ and $p>1$, the function t^p is convex, and inequalities (2.49) follow from inequalities (2.5). For $0<p<1$, the function t^p is concave, and, according to Theorem 2.17, all inequalities in (2.49) will be reversed. □

Theorem 2.35 *Let all assumptions of Theorem 2.34 hold. For $p<0$ or $p>1$, we have*

$$\begin{aligned}0 &\le \frac{ML_\Delta(wg^q)-L_\Delta(wfg)}{M-m}m^p+\frac{L_\Delta(wfg)-L_\Delta(wg^q)}{M-m}M^p-L_\Delta(wf^p)\\&\le \frac{p(M^{p-1}-m^{p-1})}{M-m}\\&\quad\cdot\int_{\mathscr{E}}(Mw(t)g^q(t)-w(t)f(t)g(t))\,(w(t)f(t)g(t)-mw(t)g^q(t))\,\Delta t\\&\le p\frac{M^{p-1}-m^{p-1}}{M-m}\cdot\frac{1}{L_\Delta(wg^q)}(ML_\Delta(wg^q)-L_\Delta(wfg))\\&\quad\cdot(L_\Delta(wfg)-mL_\Delta(wg^q))\\&\le \frac{p}{4}(M-m)(M^{p-1}-m^{p-1})L_\Delta(w,g^q).\end{aligned}\tag{2.50}$$

If $0<p<1$, then all inequalities in (2.50) *are reversed.*

Proof. Inequalities (2.50) follow directly from Theorem 2.18 by taking the function ϕ to be of the form $\phi(t)=t^p$ and replacing h by wg^q and f by $fg^{-\frac{q}{p}}$. If $p<0$ and $p>1$, then the function t^p is convex, and inequalities (2.50) follow from inequalities (2.12). For $0<p<1$, the function t^p is concave, and, according to Theorem 2.18, all inequalities in (2.50) will be reversed. □

Theorem 2.36 *Let $0 < p < 1$ and define q by $\frac{1}{p} + \frac{1}{q} = 1$. Assume $\mathscr{E} \subset \mathbb{R}^n$ is as in Theorem 2.4 and f, g are nonnegative real functions such that f^p, g^q, fg are Δ-integrable on $\mathscr{E}$ and $\int_{\mathscr{E}} g^q(t)\Delta t > 0$. Let $m, M \in \mathbb{R}$ such that $-\infty < m < M < \infty$ and $m \le f(t)g^{-q}(t) \le M$, $t \in \mathscr{E}$. Then,*

$$\begin{aligned}
0 &\le L_\Delta(fg) - L_\Delta^{\frac{1}{p}}(f^p) L_\Delta^{\frac{1}{q}}(g^q) \qquad (2.51)\\
&\le \frac{1}{p} \cdot \frac{M^{-\frac{1}{q}} - m^{-\frac{1}{q}}}{M-m} \cdot \frac{1}{L_\Delta(g^q)} \left(ML_\Delta(g^q) - L_\Delta(f^p)\right)\left(L_\Delta(f^p) - mL_\Delta(g^q)\right)\\
&\le \frac{1}{4p}(M-m)\left(M^{-\frac{1}{q}} - m^{-\frac{1}{q}}\right) L_\Delta(g^q).
\end{aligned}$$

For $p < 0$, inequalities (2.51) *hold if $L_\Delta(f^p) > 0$, $t \in \mathscr{E}$. In case $p > 1$, all inequalities in* (2.51) *are reversed.*

Proof. Inequalities (2.51) follow directly from Theorem 2.17 by taking the function ϕ to be of the form $\phi(t) = t^{\frac{1}{p}}$ and replacing h by g^q and f by $\frac{f^p}{g^q}$. Namely, when $p < 1$, the function $t^{\frac{1}{p}}$ is convex, and inequalities (2.51) follow from inequalities (2.5). For $p > 1$, the function t^p is concave, and, according to Theorem 2.17, all inequalities in (2.51) will be reversed. □

Theorem 2.37 *Let $p < 1$ and let the assumptions from Theorem 2.36 hold. Then,*

$$\begin{aligned}
0 &\le \frac{ML_\Delta(g^q) - L_\Delta(f^p)}{M-m} m^{\frac{1}{p}} + \frac{L_\Delta(f^p) - mL_\Delta(g^q)}{M-m} M^{\frac{1}{p}} - L_\Delta(fg)\\
&\le \frac{1}{p} \cdot \frac{M^{-\frac{1}{q}} - m^{-\frac{1}{q}}}{M-m} \int_{\mathscr{E}} \frac{(Mg^q(t) - f^p(t))(f^p(t) - mg^q(t))}{g^q(t)} \Delta t \qquad (2.52)\\
&\le \frac{1}{p} \cdot \frac{M^{-\frac{1}{q}} - m^{-\frac{1}{q}}}{M-m} \cdot \frac{1}{L_\Delta(g^q)} \left(ML_\Delta(g^q) - L_\Delta(f^p)\right)\\
&\quad \cdot \left(L_\Delta(f^p) - mL_\Delta(g^q)\right)\\
&\le \frac{1}{4p}(M-m)\left(M^{-\frac{1}{q}} - m^{-\frac{1}{q}}\right) L_\Delta(g^q).
\end{aligned}$$

If $p > 1$, then all inequalities in (2.52) *are reversed.*

Proof. Inequalities (2.52) follow directly from Theorem 2.18 by taking the function ϕ to be of the form $\phi(t) = t^{\frac{1}{p}}$ and replacing h by g^q and f by $\frac{f^p}{g^q}$. Namely, when $p < 1$, the function $t^{\frac{1}{p}}$ is convex, and inequalities (2.52) follow from inequalities (2.12). For $p > 1$, the function t^p is concave, and, according to Theorem 2.18, all inequalities in (2.52) will be reversed. □

Theorem 2.38 *For $p < 0$ or $p > 1$, define q by $\frac{1}{p} + \frac{1}{q} = 1$. Assume $\mathscr{E} \subset \mathbb{R}^n$ is as in Theorem 2.4 and f, g are nonnegative real functions such that g^q, fg are Δ-integrable on $\mathscr{E}$ and $L_\Delta(g^q) > 0$. Let $m, M \in \mathbb{R}$ such that $-\infty < m < M < \infty$ and $m \le f(t)g^{1-q}(t) \le M$, $t \in \mathscr{E}$. Then,*

$$\begin{aligned} 0 &\le L_\Delta(f^p) \cdot L_\Delta^{\frac{p}{q}}(g^q) - L_\Delta^p(f,g) \\ &\le (ML_\Delta(g^q) - L_\Delta(fg))(L_\Delta(fg) - mL_\Delta(g^q)) \\ &\quad \cdot \frac{p(M^{p-1} - m^{p-1})}{M-m} \cdot L_\Delta^{p-2}(g^q) \\ &\le \frac{p}{4}(M-m)(M^{p-1} - m^{p-1}) L_\Delta^p(g^q). \end{aligned} \tag{2.53}$$

In case $0 < p < 1$, all inequalities in (2.53) are reversed.

Proof. Inequalities (2.53) follow directly from Theorem 2.17 by taking the function ϕ to be of the form $\phi(t) = t^p$ and replacing h by g^q and f by fg^{1-q}. Namely, for $p < 0$ and $p > 1$, the function t^p is convex, and inequalities (2.53) follow from inequalities (2.5). For $0 < p < 1$, the function t^p is concave, and, according to Theorem 2.17, all inequalities in (2.53) will be reversed. □

Theorem 2.39 *Suppose that the assumptions from Theorem 2.34 hold. For $p < 0$ or $p > 1$, we have*

$$\begin{aligned} 0 &\le \frac{ML_\Delta(g^q) - L_\Delta(fg)}{M-m} m^p + \frac{L_\Delta(fg) - L_\Delta(g^q)}{M-m} M^p - L_\Delta(f^p) \qquad (2.54) \\ &\le p\frac{M^{p-1} - m^{p-1}}{M-m} \int_{\mathscr{E}} (Mg^q(t) - f(t)g(t))(f(t)g(t) - mg^q(t))\,\Delta t \\ &\le p\frac{M^{p-1} - m^{p-1}}{M-m} \cdot \frac{1}{L_\Delta(g^q)} (ML_\Delta(g^q) - L_\Delta(fg)) \\ &\quad \cdot (L_\Delta(fg) - mL_\Delta(g^q)) \\ &\le \frac{p}{4}(M-m)(M^{p-1} - m^{p-1}) L_\Delta(g^q). \end{aligned}$$

If $0 < p < 1$, all inequalities in (2.54) are reversed.

Proof. Inequalities (2.54) follow directly from Theorem 2.18 by taking the function ϕ to be of the form $\phi(t) = t^p$ and replacing h by g^q and f by fg^{1-q}. Namely, for $p < 0$ and $p > 1$, the function t^p is convex, and inequalities (2.54) follow from inequalities (2.12). For $0 < p < 1$, the function t^p is concave, and, according to Theorem 2.18, all inequalities in (2.54) will be reversed. □

Using the result of Theorem 2.20, we obtain the following refinements of previous converse Hölder's inequalities on time scales.

Theorem 2.40 *Assume* w,f,g *are real functions on* $\mathscr{E}$ *such that* $w,f,g\geq 0$. *For* $m,M\in\mathbb{R}$ *such that* $-\infty<m<M<\infty$, *let* $m\leq f(t)g^q(t)\leq M$, $t\in\mathscr{E}$. *If* wf^p, wg^q, wfg *are* Δ-*integrable on* $\mathscr{E}$ *and* $L_\Delta(wg^q)>0$, *where* $p>1$ *and* $q=p/(p-1)$, *then*

$$\begin{aligned}0 &\leq L_\Delta(wf^p)\cdot(L_\Delta(wg^q))^{\frac{p}{q}}-(L_\Delta(wfg))^p\\ &\leq (ML_\Delta(wg^q)-L_\Delta(wfg))\cdot(L_\Delta(wfg)-mL_\Delta(wg^q))\\ &\quad\cdot\sup_{t\in(m,M)}\Psi_\phi(t;m,M)\cdot(L_\Delta(wg^q))^{p-2}\\ &\leq (ML_\Delta(wg^q)-L_\Delta(wfg))\cdot(L_\Delta(wfg)-mL_\Delta(wg^q))\\ &\quad\cdot\frac{p(M^{p-1}-m^{p-1})}{M-m}\cdot(L_\Delta(wg^q))^{p-2}\\ &\leq \frac{p}{4}(M-m)(M^{p-1}-m^{p-1})\,(L_\Delta(wg^q))^p\,.\end{aligned}\tag{2.55}$$

For $p<0$, *inequalities* (2.55) *hold if* $L_\Delta(wfg)>0$, $t\in\mathscr{E}$. *In case* $0<p<1$, *all inequalities in* (2.55) *are reversed.*

Proof. Inequalities (2.55) follow directly from Theorem 2.20 by taking the function ϕ to be of the form $\phi(t)=t^p$ and replacing h by wg^q and f by $fg^{-\frac{q}{p}}$. For $p<0$ and $p>1$, the function t^p is convex, and inequalities (2.55) follow from inequalities (2.17). For $0<p<1$, the function t^p is concave, and, according to Theorem 2.20, all inequalities in (2.55) will be reversed. □

Theorem 2.41 *Assume* $f,g\geq 0$ *such that* f^p, g^q, fg *are* Δ-*integrable on* $\mathscr{E}$ *and* $\int_{\mathscr{E}} g^q(t)\Delta t>0$, *where* $0<p<1$ *and* $q=p/(p-1)$. *For* $m,M\in\mathbb{R}$ *such that* $-\infty<m<M<\infty$, *let* $m\leq f(t)g^{-q}(t)\leq M$, $t\in\mathscr{E}$. *Then,*

$$\begin{aligned}0 &\leq L_\Delta(fg)-(L_\Delta(f^p))^{\frac{1}{p}}\,(L_\Delta(g^q))^{\frac{1}{q}}\\ &\leq \frac{1}{L_\Delta(g^q)}(ML_\Delta(g^q)-L_\Delta(f^p))\cdot(L_\Delta(f^p)-mL_\Delta(g^q))\\ &\quad\cdot\sup_{t\in(m,M)}\Psi_\phi(t;m,M)\\ &\leq \frac{1}{p}\cdot\frac{M^{-\frac{1}{q}}-m^{-\frac{1}{q}}}{M-m}\cdot\frac{1}{L_\Delta(g^q)}(ML_\Delta(g^q)-L_\Delta(f^p))\\ &\quad\cdot(L_\Delta(f^p)-mL_\Delta(g^q))\\ &\leq \frac{1}{4p}(M-m)\left(M^{-\frac{1}{q}}-m^{-\frac{1}{q}}\right)L_\Delta(g^q).\end{aligned}\tag{2.56}$$

For $p<0$, *inequalities* (2.56) *hold if* $L_\Delta(f^p)>0$, $t\in\mathscr{E}$. *In case* $p>1$, *all inequalities in* (2.56) *are reversed.*

Proof. Inequalities (2.56) follow directly from Theorem 2.20 by taking the function ϕ to be of the form $\phi(t)=t^{\frac{1}{p}}$ and replacing h by g^q and f by $\frac{f^p}{g^q}$. Namely, when $p<1$, the function $t^{\frac{1}{p}}$ is convex, and inequalities (2.56) follow from inequalities (2.17). For $p > 1$, the function t^p is concave, and, according to Theorem 2.20, all inequalities in (2.56) will be reversed. □

Theorem 2.42 *Assume $f,g \geq 0$ such that g^q, fg are Δ-integrable on $\mathscr{E}$ and $\int_{\mathscr{E}} g^q(t)\Delta t > 0$, where $p < 0$ or $p > 1$ and $q = p/(p-1)$. Let $m,M \in \mathbb{R}$ such that $-\infty < m < M < \infty$ and $m \leq f(t)g^{1-q}(t) \leq M$, $t \in \mathscr{E}$. Then,*

$$\begin{aligned}
0 &\leq L_\Delta(f^p)\cdot(L_\Delta(g^q))^{\frac{p}{q}} - (L_\Delta(fg))^p \\
&\leq (ML_\Delta(g^q) - L_\Delta(fg))\left(L_\Delta(fg) - m\int_{\mathscr{E}} g^q(t)\Delta t\right) \\
&\quad \cdot (L_\Delta(g^q))^{p-2} \sup_{t\in(m,M)} \Psi_\phi(t;m,M) \qquad (2.57)\\
&\leq (ML_\Delta(g^q) - L_\Delta(fg))(L_\Delta(fg) - mL_\Delta(g^q)) \\
&\quad \cdot \frac{p(M^{p-1} - m^{p-1})}{M-m}\cdot(L_\Delta(g^q))^{p-2} \\
&\leq \frac{p}{4}(M-m)(M^{p-1} - m^{p-1})(L_\Delta(g^q))^p .
\end{aligned}$$

In case $0 < p < 1$, all inequalities in (2.57) are reversed.

Proof. Inequalities (2.57) follow from Theorem 2.20 by taking the function ϕ to be of the form $\phi(t) = t^p$ and replacing h by g^q and f by fg^{1-q}. Namely, for $p < 0$ or $p > 1$, the function t^p is convex, and inequalities (2.57) follow from inequalities (2.17). For $0 < p < 1$, the function t^p is concave, and, according to Theorem 2.20, all inequalities in (2.57) will be reversed. □

Additional Improvements

R. Jakšić and J. E. Pečarić proved in [81] a new refinement of the converse Jensen inequality for isotonic linear functionals, given in Theorem 2.19. Using that result, we derive the following theorem which refines inequality (2.17) from Theorem 2.20.

Theorem 2.43 *Let $\phi \in C(I,\mathbb{R})$ be convex, where $I = [m,M] \subset \mathbb{R}$, with $m < M$. Assume $\mathscr{E} \subset \mathbb{R}^n$ and L satisfies conditions (L_1), (L_2) with additional property that $\min\{f,g\} \in L$ and $\max\{f,g\} \in L$, for every $f,g \in L$. Let f be Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = I$. Moreover, let $h : \mathscr{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t > 0$. Then,*

$$\begin{aligned}
0 &\leq \overline{L}_\Delta(\phi(f),h) - \phi\left(\overline{L}_\Delta(f,h)\right) \\
&\leq \left(M - \overline{L}_\Delta(f,h)\right)\left(\overline{L}_\Delta(f,h) - m\right) \sup_{t\in(m,M)} \Psi_\phi(t;m,M) - \overline{L}_\Delta(\tilde{f},h)\delta_\phi \\
&\leq \left(M - \overline{L}_\Delta(f,h)\right)\left(\overline{L}_\Delta(f,h) - m\right)\cdot\frac{\phi'_-(M) - \phi'_+(m)}{M-m} - \overline{L}_\Delta(\tilde{f},h)\delta_\phi
\end{aligned}$$

$$\leq \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)) - \overline{L}_\Delta(\tilde{f},h)\delta_\phi, \tag{2.58}$$

where

$$\tilde{f} = \frac{1}{2} - \frac{|f - \frac{m+M}{2}|}{M-m}, \quad \delta_\phi = \phi(m)+\phi(M)-2\phi\left(\frac{m+M}{2}\right), \tag{2.59}$$

and $\Psi_\phi(\cdot;m,M)\colon (m,M) \to \mathbb{R}$ *is defined by*

$$\Psi_\Phi(t;m,M) = \frac{1}{M-m}\left(\frac{\phi(M)-\phi(t)}{M-t} - \frac{\phi(t)-\phi(m)}{t-m}\right). \tag{2.60}$$

If ϕ *is concave on I, then all inequalities in* (2.58) *are reversed.*

Proof. Inequality (2.58) follows from the main result of [81] and the fact that the multiple Lebesgue delta integral is an isotonic linear functional. □

Remark 2.7 Using Theorem 2.43, the refinements of the inequalities proved in Theorem 2.24 and Theorem 2.30 can be obtained.

Minkowski's Inequality

We first recall Minkowski's inequality for isotonic linear functionals.

Theorem 2.44 (SEE [119, THEOREM 4.13]) *Let* $E, L,$ *and* A *be such that* (L_1), (L_2), (A_1), (A_2) *are satisfied. For* $p \in \mathbb{R}$, *assume* w, f, g *are nonnegative functions on* E *and* $wf^p, wg^p, w(f+g)^p \in L$. *If* $p > 1$, *then*

$$A^{1/p}(w(f+g)^p) \leq A^{1/p}(wf^p) + A^{1/p}(wg^p).$$

This inequality is reversed if $0 < p < 1$ *or* $p < 0$ *provided* $A(wf^p) > 0$ *and* $A(wg^p) > 0$ *hold.*

In the following theorem, we give a generalization of the Minkowski inequality on time scales.

Theorem 2.45 *For* $p \in \mathbb{R}$, *assume* w, f, g *are nonnegative functions on* $\mathcal{E}$ *and* wf^p, wg^p, $w(f+g)^p$ *are* Δ*-integrable on* $\mathcal{E}$. *If* $p > 1$, *then*

$$L_\Delta^{\frac{1}{p}}(w(f+g)^p) \leq L_\Delta^{\frac{1}{p}}(wf^p) + L_\Delta^{\frac{1}{p}}(wg^p).$$

This inequality is reversed for $0 < p < 1$ *or* $p < 0$ *provided each of the two terms on the right-hand side is positive.*

Proof. Just apply Theorem 2.44 and Theorem 2.5. □

Remark 2.8 Note that the known results from the time scales literature follow from Theorem 2.44 in the same way as Theorem 2.45 does: [45, Theorem 6.16] follows as in Theorem 2.1 and [16, Theorem 4.4] (see also [57, 17]) follows as in Theorem 2.3.

Dresher's Inequality

If $n = 2$ in the following result, then we have the Dresher inequality (see [55, Section 7]). We first present the generalization of this inequality for isotonic linear functionals.

Theorem 2.46 (SEE [119, THEOREM 4.21]) *Let E and L be such that (L_1), (L_2) are satisfied and suppose that both A and B satisfy (A_1), (A_2). If f_i, u_i are nonnegative functions on E and wf_i^p, $w\left(\sum_{i=1}^n f_i\right)^p$, wg_i^r, $w\left(\sum_{i=1}^n g_i\right)^r \in L$, where $p \geq 1 > r > 0$ and $A(wg_i^r) > 0$ for $1 \leq i \leq n$, then*

$$\left(\frac{A\left(w\left(\sum_{i=1}^n f_i\right)^p\right)}{B\left(w\left(\sum_{i=1}^n g_i\right)^r\right)}\right)^{\frac{1}{p-r}} \leq \sum_{i=1}^n \left(\frac{A(wf_i^p)}{B(wg_i^r)}\right)^{\frac{1}{p-r}}.$$

In the following theorem, we give the Dresher inequality on time scales.

Theorem 2.47 *If f_i, u_i are nonnegative functions on $\mathscr{E}$ and wf_i^p, $w\left(\sum_{i=1}^n f_i\right)^p$, wg_i^r, $w\left(\sum_{i=1}^n g_i\right)^r$ are Δ-integrable on $\mathscr{E}$, where $p \geq 1 > r > 0$ and $L_\Delta(wg_i^r) > 0$ for $1 \leq i \leq n$, then*

$$\left(\frac{L_\Delta\left(w\left(\sum_{i=1}^n f_i\right)^p\right)}{L_\Delta\left(w\left(\sum_{i=1}^n g_i\right)^r\right)}\right)^{\frac{1}{p-r}} \leq \sum_{i=1}^n \left(\frac{L_\Delta(wf_i^p)}{L_\Delta(wg_i^r)}\right)^{\frac{1}{p-r}}.$$

Proof. Just apply Theorem 2.46 and Theorem 2.5. □

Remark 2.9 Dresher's inequality on time scales is new even for the cases of a single-variable Cauchy delta and nabla integral and also for the diamond-α integral.

Popoviciu's Inequality

We first recall Popoviciu's inequality for isotonic linear functionals.

Theorem 2.48 (SEE [119, THEOREM 4.27]) *Let E, L, and A be such that (L_1), (L_2), (A_1), (A_2) are satisfied. For $p \neq 1$, define $q = p/(p-1)$. Assume f, g are nonnegative functions on E and $f^p, g^q, fg \in L$. Suppose $f_0, g_0 > 0$ are such that*

$$f_0^p - A(f^p) > 0 \quad \textit{and} \quad g_0^q - A(g^q) > 0.$$

If $p > 1$, then

$$\left(f_0^p - A(f^p)\right)^{1/p}\left(g_0^q - A(g^q)\right)^{1/q} \leq f_0 g_0 - A(fg).$$

This inequality is reversed if $0 < p < 1$ and $A(g^q) > 0$, or if $p < 0$ and $A(f^p) > 0$.

In the following theorem, we give the Popoviciu inequality on time scales.

Theorem 2.49 *For $p \neq 1$, define $q = p/(p-1)$. Assume f, g are nonnegative functions on $\mathscr{E}$ and f^p, g^q, fg are Δ-integrable on $\mathscr{E}$. Suppose $f_0, g_0 > 0$ are such that*

$$f_0^p - L_\Delta(f^p) > 0 \quad and \quad g_0^q - L_\Delta(g^q) > 0.$$

If $p > 1$, then

$$f_0 g_0 - L_\Delta(fg) \geq \left(f_0^p - L_\Delta(f^p)\right)^{\frac{1}{p}} \left(g_0^q - L_\Delta(g^q)\right)^{\frac{1}{q}}.$$

This inequality is reversed if $0 < p < 1$ and $L_\Delta(g^q) > 0$, or if $p < 0$ and $L_\Delta(f^p) > 0$.

Proof. Just apply Theorem 2.48 and Theorem 2.5. □

From Popoviciu's inequality follows the Aczél inequality given in the next theorem.

Theorem 2.50 *Assume f, g are nonnegative functions on $\mathscr{E}$ and f^2, g^2, fg are Δ-integrable on $\mathscr{E}$. If $f_0, g_0 > 0$ are such that*

$$f_0^2 - L_\Delta(f^2) > 0 \quad and \quad g_0^2 - L_\Delta(g^2) > 0,$$

then

$$f_0 g_0 - L_\Delta(fg) \geq \sqrt{\left(f_0^2 - L_\Delta(f^2)\right)\left(g_0^2 - L_\Delta(g^2)\right)}.$$

Proof. Just let $p = 2$ in Theorem 2.49. □

Remark 2.10 The Aczél and Popoviciu inequalities on time scales are new even for the cases of a single-variable Cauchy delta and nabla integral and also for the diamond-α integral. The original Aczél inequality can be found in [8]. For a version of Aczél's inequality for isotonic linear functionals, we refer to [119, Theorem 4.26].

Bellman's Inequality

We first recall Bellman's inequality for isotonic linear functionals.

Theorem 2.51 (SEE [119, THEOREM 4.29]) *Let E, L, and A be such that (L_1), (L_2), (A_1), (A_2) are satisfied. For $p \in \mathbb{R}$, assume f, g are nonnegative functions on E and $f^p, g^p, (f+g)^p \in L$. Suppose $f_0, g_0 > 0$ are such that*

$$f_0^p - A(f^p) > 0 \quad and \quad g_0^p - A(g^p) > 0.$$

If $p > 1$, then

$$\left(\left(f_0^p - A(f^p)\right)^{1/p} + \left(g_0^p - A(g^p)\right)^{1/p}\right)^p \leq (f_0 + g_0)^p - A((f+g)^p).$$

This inequality is reversed if $0 < p < 1$ or $p < 0$ and $A(f^p) > 0$.

In the following theorem, we give the Bellman inequality on time scales.

Theorem 2.52 *For $p \in \mathbb{R}$, assume f, g are nonnegative functions on $\mathscr{E}$ and f^p, g^p, $(f+g)^p$ are Δ-integrable on $\mathscr{E}$. Suppose $f_0, g_0 > 0$ are such that*

$$f_0^p - L_\Delta(f^p) > 0 \quad \text{and} \quad g_0^p - L_\Delta(g^p) > 0.$$

If $p > 1$, then

$$\left(\left(f_0^p - L_\Delta(f^p)\right)^{\frac{1}{p}} + \left(g_0^p - L_\Delta(g^p)\right)^{\frac{1}{p}}\right)^p \leq (f_0 + g_0)^p - L_\Delta((f+g)^p).$$

This inequality is reversed if $0 < p < 1$ or $p < 0$ and $L_\Delta(f^p) > 0$.

Proof. Just apply Theorem 2.51 and Theorem 2.5. □

Diaz–Metcalf Inequality

If $p = q = 2$ and $w = 1$ in the following result, then we have the Diaz–Metcalf inequality. We first present the generalization of this inequality for isotonic linear functionals.

Theorem 2.53 (See [119, Theorem 4.14]) *Let E, L, and A be such that (L_1), (L_2), (A_1), (A_2) are satisfied. For $p \neq 1$, let $q = p/(p-1)$. Assume w, f, g are nonnegative functions on E such that $wf^p, wg^q, wfg \in L$ and, if $p \neq 0$,*

$$0 < m \leq f(t)g^{-q/p}(t) \leq M \quad \text{for all} \quad t \in E.$$

If $p > 1$, or if $p < 0$ and $A(wf^p) + A(wg^q) > 0$, then

$$(M-m)A(wf^p) + (mM^p - Mm^p)A(wg^q) \leq (M^p - m^p)A(wfg).$$

This inequality is reversed if $0 < p < 1$ and $A(wf^p) + A(wg^q) > 0$.

In the following theorem, we give the Diaz–Metcalf inequality on time scales.

Theorem 2.54 *For $p \neq 1$, let $q = p/(p-1)$. Assume w, f, g are nonnegative functions on $\mathscr{E}$ such that wf^p, wg^q, wfg are Δ-integrable on $\mathscr{E}$ and, if $p \neq 0$,*

$$0 < m \leq f(t)g^{-q/p}(t) \leq M \quad \text{for all} \quad t \in \mathscr{E}.$$

If $p > 1$, or if $p < 0$ and at least one of the two integrals on the left-hand side of the following inequality is positive, then

$$(M-m)L_\Delta(wf^p) + (mM^p - Mm^p)L_\Delta(wg^q) \leq (M^p - m^p)L_\Delta(wfg).$$

This inequality is reversed if $0 < p < 1$ and at least one of the two integrals on the left-hand side is positive.

Proof. Just apply Theorem 2.53 and Theorem 2.5. □

The following two inequalities follow from [119, Theorem 4.16 and Theorem 4.18] in the same way as Theorem 2.54 follows from Theorem 2.53.

Theorem 2.55 *Let $\mathscr{E}, p, q, w, f, g, m, M$ be as in Theorem 2.54. If $p > 1$, then*

$$L_\Delta(wfg) \geq K(p,m,M)\,(L_\Delta(wf^p))^{\frac{1}{p}}\,(L_\Delta(wg^q))^{\frac{1}{q}}, \tag{2.61}$$

where

$$K(p,m,M) = |p|^{1/p}|q|^{1/q}\frac{(M-m)^{1/p}|mM^p - Mm^p|^{1/q}}{|M^p - m^p|}. \tag{2.62}$$

The inequality (2.61) *is reversed if $p < 0$ or $0 < p < 1$, provided at least one of the two integrals on the right-hand side is positive.*

Proof. Just apply [119, Theorem 4.16] and Theorem 2.5. □

Theorem 2.56 *Let $\mathscr{E}, p, q, w, f, g, m, M$ be as in Theorem 2.54 and assume*

$$0 < m < F(t) \leq M \quad \text{and} \quad 0 \leq G(t) \leq M \quad \text{for all} \quad t \in \mathscr{E},$$

where $F = f(f+g)^{-q/p}$ and $G = g(f+g)^{-q/p}$. Let $K(p,m,M)$ be defined as in (2.62)*. If $p > 1$, then*

$$L_\Delta^{\frac{1}{p}}(w(f+g)^p) \geq |p|^{1/p}|q|^{1/q}\frac{(M-m)^{1/p}(mM^p - Mm^p)^{1/q}}{|M^p - m^p|}\times$$
$$\times\left\{(L_\Delta(wf^p))^{\frac{1}{p}} + L_\Delta^{\frac{1}{p}}(wg^p)\right\}.$$

This inequality is reversed if $0 < p < 1$, or if $p < 0$ and the integral on the left-hand side is positive.

Proof. Just apply [119, Theorem 4.18] and Theorem 2.5. □

2.5 Further Converses of the Jensen Inequality

Some converses of Jensen's inequality are obtained in the previous sections. This section is concerned with some further converses of Jensen's inequality. The five theorems presented follow from the specified results in [119] in the same way as Theorem 2.13 follows from Theorem 2.12.

Theorem 2.57 (a) *Assume $\Phi \in \mathrm{C}(I,\mathbb{R})$ is convex, where $I = [m,M]$, such that $\Phi''(x) \geq 0$ with equality for at most isolated points of I. Assume further that either*

(i) $\Phi(x) > 0$ *for all $x \in I$, or*

(i′) $\Phi(x) > 0$ *for all $m < x < M$ with either $\Phi(m) = 0$, $\Phi'(m) \neq 0$, or $\Phi(M) = 0$, $\Phi'(M) \neq 0$, or*

(ii) $\Phi(x) < 0$ *for all* $x \in I$, *or*

(ii$'$) $\Phi(x) < 0$ *for all* $m < x < M$ *with precisely one of* $\Phi(m) = 0$, $\Phi(M) = 0$.

Suppose f *is* Δ*-integrable on* $\mathscr{E}$ *such that* $f(\mathscr{E}) = I$ *and* $h : \mathscr{E} \to \mathbb{R}$ *is nonnegative* Δ*-integrable such that* $L_\Delta(h) > 0$. *Then*

$$\overline{L}_\Delta(\Phi(f),h) \le \lambda\Phi\left(\overline{L}_\Delta(f,h)\right)$$

holds for some $\lambda > 1$ *in cases* (i), (i$'$), *or* $\lambda \in (0,1)$ *in cases* (ii), (ii$'$). *More precisely, a value of* λ, *depending only on* m, M, Φ, *may be determined as follows: Define* $\nu = (\Phi(M) - \Phi(m))/(M - m)$. *If* $\nu = 0$, *let* $\tilde{x} \in (m,M)$ *be the unique solution of the equation* $\Phi'(x) = 0$; *then* $\lambda = \Phi(m)/\Phi(\tilde{x})$. *If* $\nu \neq 0$, *let* $\tilde{x} \in [m,M]$ *be the unique solution of the equation* $\nu\Phi(x) - \Phi'(x)(\Phi(m) + \nu(x-m)) = 0$; *then* $\lambda = \nu/\Phi'(\tilde{x})$. *Moreover, we have* $\tilde{x} \in (m,M)$ *in the cases* (i), (ii).

(b) *Let all the hypotheses of* (a) *hold except that* Φ *is concave on* I *with* $\Phi''(x) \le 0$ *with equality for at most isolated points of* I. *Then*

$$\overline{L}_\Delta(\Phi(f),h) \ge \lambda\Phi\left(\overline{L}_\Delta(f,h)\right),$$

where λ *is determined as in* (a). *Furthermore,* $\lambda > 1$ *holds if* $\Phi(x) < 0$ *for all* $x \in (m,M)$, *and* $0 < \lambda < 1$ *holds if* $\Phi(x) > 0$ *for all* $x \in (m,M)$.

Proof. Just apply [119, Theorem 3.39] and Theorem 2.6. □

Theorem 2.58 (a) *Let* f, I, m, M, h, ν *be as in Theorem 2.57 and* $\Phi \in \mathrm{C}(I,\mathbb{R})$ *be differentiable such that* Φ' *is strictly increasing on* I. *Then*

$$\overline{L}_\Delta(\Phi(f),h) \le \lambda + \Phi\left(\overline{L}_\Delta(f,h)\right)$$

for $\lambda = \Phi(m) - \Phi(\tilde{x}) + \nu(\tilde{x} - m) \in (0, (M-m)(\nu - \Phi'(m)))$, *where* $\tilde{x} \in (m,M)$ *is the unique solution of the equation* $\Phi'(x) = \nu$.

(b) *Let all the hypotheses of* (a) *hold except that* Φ' *is strictly decreasing on* I. *Then*

$$\Phi\left(\overline{L}_\Delta(f,h)\right) \le \lambda + \overline{L}_\Delta(\Phi(f),h)$$

for $\lambda = \Phi(\tilde{x}) - \Phi(m) - \nu(\tilde{x} - m) \in (0, (M-m)(\Phi'(m) - \nu))$ *with* $\tilde{x}$ *given in* (a).

Proof. Just apply [119, Theorem 3.41] and Theorem 2.6. □

Theorem 2.59 *In addition to the assumptions of Theorem 2.13, let* $J \subset \mathbb{R}$ *be an interval such that* $J \supset \Phi(I)$ *and assume that* $F : J \times J \to \mathbb{R}$ *is increasing in the first variable. Then*

$$\begin{aligned}
&F(\overline{L}_\Delta(\Phi(f),h), \Phi\left(\overline{L}_\Delta(f,h)\right)) \\
&\le \max_{x\in[m,M]} F\left(\frac{M-x}{M-m}\Phi(m) + \frac{x-m}{M-m}\Phi(M), \Phi(x)\right) \\
&= \max_{\sigma\in[0,1]} F(\sigma\Phi(m) + (1-\sigma)\Phi(M), \Phi(\sigma m + (1-\sigma)M)),
\end{aligned}$$

and the right-hand side of the inequality is an increasing function of M *and a decreasing function of* m.

Proof. Just apply [119, Theorem 3.42] and Theorem 2.6. □

Remark 2.11 The discrete version of Theorem 2.59 can be found in [103, Theorem 8, page 9–10].

Remark 2.12 If we choose $F(x,y) = x - y$, as a simple consequence of Theorem 2.59, it follows

$$\overline{L}_\Delta(\Phi(f),h) - \Phi\left(\overline{L}_\Delta(f,h)\right) \leq \max_{\sigma\in[0,1]} \left(\sigma\Phi(m) + (1-\sigma)\Phi(M) - \Phi(\sigma m + (1-\sigma)M)\right). \quad (2.63)$$

On the other hand, if we choose $F(x,y) = \dfrac{x}{y}$, then we get

$$\frac{\overline{L}_\Delta(\Phi(f),h)}{\Phi\left(\overline{L}_\Delta(f,h)\right)} \leq \max_{\sigma\in[0,1]} \left(\frac{\sigma\Phi(m) + (1-\sigma)\Phi(M)}{\Phi(\sigma m + (1-\sigma)M)}\right). \quad (2.64)$$

The inequalities (2.63) and (2.64) are generalizations of the results given in [124, 125, 126].

Theorem 2.60 *Under the same hypotheses as in Theorem 2.59 except that F is decreasing in its first variable, we have*

$$\begin{aligned} &F\left(\overline{L}_\Delta(\Phi(f),h), \Phi\left(\overline{L}_\Delta(f,h)\right)\right) \\ &\geq \min_{x\in[m,M]} F\left(\frac{M-x}{M-m}\Phi(m) + \frac{x-m}{M-m}\Phi(M), \Phi(x)\right) \\ &= \min_{\sigma\in[0,1]} F\left(\sigma\Phi(m) + (1-\sigma)\Phi(M), \Phi(\sigma m + (1-\sigma)M)\right). \end{aligned}$$

Moreover, the right-hand side of the above inequality is a decreasing function of M and an increasing function of m.

Proof. Just apply [119, Theorem 3.42′] and Theorem 2.6. □

Theorem 2.61 *Assume $\Phi : I \to \mathbb{R}$ is convex and $f : \mathscr{E} \to I$ is Δ-integrable. Let $h : \mathscr{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $0 < L_\Delta(h) < \alpha$ for some $\alpha \in \mathbb{R}$. If hf and $h(\Phi\circ f)$ are Δ-integrable on $\mathscr{E}$ and $a \in I$ is such that*

$$\frac{\alpha a - L_\Delta(hf)}{\alpha - L_\Delta(h)} \in I,$$

then

$$\Phi\left(\frac{\alpha a - L_\Delta(hf)}{\alpha - L_\Delta(h)}\right) \geq \frac{\alpha\Phi(a) - L_\Delta(h\Phi(f))}{\alpha - L_\Delta(h)}.$$

Proof. Just apply [119, Lemma 4.25] and Theorem 2.5. □

2.6 Jensen Type Inequalities for Superquadratic Functions

In this section, all the inequalities obtained are given for Cauchy delta time scales integrals, but they also hold for many other time scales integrals, such as Cauchy, Riemann, Lebesgue, multiple Riemann, and multiple Lebesgue delta, nabla, and diamond-α time scales integrals as we know that these integrals are isotonic linear functionals.

Jensen's Inequality

First we quote the following result of S. Banić and S. Varošanec.

Theorem 2.62 (SEE [26, THEOREM 10]) *Let E, L, and A be such that (L_1), (L_2), (A_1), (A_2) are satisfied. Suppose that $h \in L$ with $h \geq 0$ and $A(h) > 0$ and that $\Psi : [0,\infty) \to \mathbb{R}$ is a continuous superquadratic function. Then for all nonnegative $f \in L$ such that $hf, h\Psi(f), h\Psi\left(\left|f - \frac{A(hf)}{A(h)} \cdot 1\right|\right) \in L$, we have*

$$\Psi\left(\frac{A(hf)}{A(h)}\right) \leq \frac{A(h\Psi(f)) - A\left(h\Psi\left(\left|f - \frac{A(hf)}{A(h)} \cdot 1\right|\right)\right)}{A(h)}.$$

If Ψ is a subquadratic function, then the reversed inequality holds.

Now we will demonstrate how Jensen's inequality on time scales for superquadratic functions can be proved by two completely different approaches: The first approach uses the methods and techniques of time scales calculus and the second one follows from Theorem 2.62. According to the conclusion that comes out from the second way of proving Jensen's inequality, in the rest of this section, some new inequalities with delta integrals will be obtained.

In the next theorem, we present the Jensen inequality on time scales for superquadratic functions.

Theorem 2.63 *Let $a, b \in \mathbb{T}$. Assume $f \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}}, [0,\infty))$ and $\Psi \in \mathrm{C}([0,\infty), \mathbb{R})$ is superquadratic. Then*

$$\Psi\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right) \leq \frac{1}{b-a}\int_a^b \left[\Psi(f(s)) - \Psi\left(\left|f(s) - \frac{\int_a^b f(t)\Delta t}{b-a}\right|\right)\right]\Delta s. \tag{2.65}$$

Moreover, if Ψ is subquadratic, then (2.65) *holds in reverse order.*

Proof.[First Proof of Theorem 2.63] Let $\Psi : [0,\infty) \to \mathbb{R}$ be a superquadratic function and let $x_0 \in [0,\infty)$. According to (1.7), there is a constant $C(x_0)$ such that

$$\Psi(y) \geq \Psi(x_0) + C(x_0)(y - x_0) + \Psi(|y - x_0|). \tag{2.66}$$

Since f is rd-continuous,

$$x_0 = \frac{\int_a^b f(t)\Delta t}{b-a} \tag{2.67}$$

is well defined. The function $\Psi \circ f$ is also rd-continuous, so we may apply (2.66) with $y = f(s)$ and (2.67) to obtain

$$\Psi(f(s)) \geq \Psi\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right) + C(x_0)\left(f(s) - \frac{\int_a^b f(t)\Delta t}{b-a}\right) + \Psi\left(\left|f(s) - \frac{\int_a^b f(t)\Delta t}{b-a}\right|\right). \tag{2.68}$$

Integrating (2.68) from a to b, we get

$$\begin{aligned}
&\int_a^b \left[\Psi(f(s)) - \Psi\left(\left|f(s) - \frac{\int_a^b f(t)\Delta t}{b-a}\right|\right)\right]\Delta s - (b-a)\Psi\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right) \\
&= \int_a^b \Psi(f(s))\Delta s - \int_a^b \Psi\left(\left|f(s) - \frac{\int_a^b f(t)\Delta t}{b-a}\right|\right)\Delta s \\
&\quad - \int_a^b \Psi\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right)\Delta s \\
&\geq C(x_0)\int_a^b \left[f(s) - \frac{\int_a^b f(t)\Delta t}{b-a}\right]\Delta s \\
&= C(x_0)\left[\int_a^b f(s)\Delta s - (b-a)\cdot x_0\right] \\
&= 0,
\end{aligned}$$

from which (2.65) follows. If Ψ is subquadratic, then the reverse inequality in (2.65) can be obtained in a similar way. □

Proof. [Second Proof of Theorem 2.63] Substituting A from Theorem 2.1 into Theorem 2.62 and using $k(t) = 1$ for all $t \in [a,b)_{\mathbb{T}}$, we get inequality (2.65). □

Remark 2.13 Note that if Ψ is strictly superquadratic in Theorem 2.63, then strict inequality in (2.65) holds.

Remark 2.14 In the case when Ψ is a nonnegative superquadratic function and therefore (by Lemma 1.1) a convex one too, the result of Theorem 2.63 refines the result given in Theorem 2.9.

Hölder's Inequality

Let us recall the following refinement of the functional Hölder inequality.

Theorem 2.64 (SEE [26, THEOREM 13]) *For $p \neq 1$, define $q = p/(p-1)$. Let E, L, and A be such that (L_1), (L_2), (A_1), (A_2) are satisfied. If $p \geq 2$, then for all nonnegative functions $f, g \in L$ such that $fg, f^p, g^q, \left|f - g^{q-1}\frac{A(fg)}{A(g^q)}\right|^p \in L$, and $A(g^q) > 0$, the inequality*

$$A(fg) \leq \left[A(f^p) - A\left(\left|f - g^{q-1}\frac{A(fg)}{A(g^q)}\right|^p\right)\right]^{\frac{1}{p}} A^{\frac{1}{q}}(g^q) \tag{2.69}$$

holds. In the case $0 < p < 1$ or $1 < p < 2$, the inequality in (2.69) is reversed.

Now Hölder's inequality on time scales (see [9] and [45, Theorem 6.13]) can be refined as follows.

Theorem 2.65 *For $p \neq 1$, define $q = p/(p-1)$. Let $a, b \in \mathbb{T}$. If $p \geq 2$, then for $f, g \in C_{\mathrm{rd}}([a,b)_{\mathbb{T}}, [0,\infty))$, the inequality*

$$\begin{aligned}&\int_a^b (fg)(t)\Delta t \\ &\leq \left[\int_a^b f^p(t)\Delta t - \int_a^b \left(\left|f(s) - g^{q-1}(s)\frac{\int_a^b (fg)(t)\Delta t}{\int_a^b g^q(t)\Delta t}\right|^p\right)\Delta s\right]^{\frac{1}{p}} \left(\int_a^b g^q(t)\Delta t\right)^{\frac{1}{q}}\end{aligned} \tag{2.70}$$

holds. If $0 < p < 1$ or $1 < p < 2$, the inequality (2.70) holds in reverse order.

Proof. The inequality (2.70) follows from Theorem 2.69 and Theorem 2.1. □

Remark 2.15 Since the delta integral is an isotonic linear functional, we have

$$\int_a^b g^q(t)\Delta t \geq 0 \quad \text{and} \quad \int_a^b \left(\left|f(s) - g^{q-1}(s)\frac{\int_a^b (fg)(t)\Delta t}{\int_a^b g^q(t)\Delta t}\right|^p\right)\Delta s \geq 0,$$

so the inequality (2.70) represents a refinement of the classical Hölder inequality on time scales for nonnegative functions f and g.

Taking $p = q = 2$ in Theorem 2.65 gives the following special case of the above Hölder inequality that we can name the refinement of the Cauchy–Schwarz inequality on time scales.

Theorem 2.66 *Let $a, b \in \mathbb{T}$. For $f, g \in C_{\mathrm{rd}}([a,b)_{\mathbb{T}}, [0,\infty))$ with $\int_a^b g^2(t)\Delta t > 0$, the inequality*

$$\begin{aligned}&\int_a^b (fg)(t)\Delta t \\ &\leq \left[\int_a^b f^2(t)\Delta t - \int_a^b \left(\left|f(s) - g(s)\frac{\int_a^b (fg)(t)\Delta t}{\int_a^b g^2(t)\Delta t}\right|^2\right)\Delta s\right]^{\frac{1}{2}} \left(\int_a^b g^2(t)\Delta t\right)^{\frac{1}{2}}\end{aligned} \tag{2.71}$$

holds.

Minkowski's Inequality

First, we quote the functional Minkowski inequality for superquadratic functions.

Theorem 2.67 (SEE [26, THEOREM 14]) *Let E, L, and A be such that (L_1), (L_2), (A_1), (A_2) are satisfied. If $p \geq 2$, then for all nonnegative functions f, g on E such that $(f+g)^p, f^p, g^p \in L$ and $A(f+g)^p > 0$, the inequality*

$$A^{\frac{1}{p}}\left((f+g)^p\right) \leq \left(A\left(f^p\right) - A\left(\left|f-(f+g)\frac{A(f(f+g)^{p-1})}{A(f+g)^p}\right|^p\right)\right)^{\frac{1}{p}} + \left(A(g^p) - A\left(\left|g-(f+g)\frac{A(g(f+g)^{p-1})}{A(f+g)^p}\right|^p\right)\right)^{\frac{1}{p}}$$

holds.

Now, Minkowski's inequality on time scales (see [9] and [45, Theorem 6.16]) can be refined as follows.

Theorem 2.68 *Let $a, b \in \mathbb{T}$ and $p \geq 2$. For $f, g \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}}, [0,\infty))$ with $\int_a^b (f(s)+g(s))^p \Delta s > 0$, the inequality*

$$\begin{aligned} &\left(\int_a^b (f(t)+g(t))^p \,\Delta t\right)^{\frac{1}{p}} \\ &\leq \left(\int_a^b f^p(t)\Delta t \right. \\ &\left. -\int_a^b \left| f(t)-(f(t)+g(t))\frac{\int_a^b f(s)(f(s)+g(s))^{p-1}\Delta s}{\int_a^b (f(s)+g(s))^p \Delta s}\right|^p \Delta t\right)^{\frac{1}{p}} \\ &+\left(\int_a^b g^p(t)\Delta t \right. \\ &\left. -\int_a^b \left| g(t)-(f(t)+g(t))\frac{\int_a^b g(s)(f(s)+g(s))^{p-1}\Delta s}{\int_a^b (f(s)+g(s))^p \Delta s}\right|^p \Delta t\right)^{\frac{1}{p}} \end{aligned} \tag{2.72}$$

is valid.

Proof. The inequality (2.72) follows directly from Theorem 2.67 and Theorem 2.1. □

Remark 2.16 If the functions f and g in Theorem 2.68 are nonnegative, then inequality (2.72) represents a refinement of Minkowski's inequality on time scales as established in [9, Theorem 3.3].

Jensen–Mercer Inequality

A variant of Jensen's inequality of Mercer's type for superquadratic functions and isotonic linear functionals is given in the following theorem.

Theorem 2.69 (SEE [2, THEOREM 2.3]) *Let E, L, and A be such that (L_1), (L_2), (A_1), (A_2) are satisfied. Assume $\Psi : [0,\infty) \to \mathbb{R}$ is a continuous superquadratic function and let $0 \le m < M < \infty$. If $f \in L$ is such that $m \le f(t) \le M$ for all $t \in E$ and such that*

$$\Psi(f), \Psi(m+M-f), (M-f)\Psi(f-m), (f-m)\Psi(M-f) \in L,$$

then we have

$$\Psi(m+M-A(f)) \le \Psi(m)+\Psi(M)-A(\Psi(f)) \\ -\frac{2}{M-m}A((f-m)\Psi(M-f)+(M-f)\Psi(f-m))-A(\Psi(|f-A(f)|)).$$

If the function Ψ is subquadratic, then the above inequality is reversed.

Next, we state the time scales version of Jensen's inequality of Mercer's type for superquadratic functions and isotonic linear functionals which we will call the Jensen–Mercer inequality for superquadratic functions on time scales.

Theorem 2.70 *Let $a, b \in \mathbb{T}$. Assume $f \in C_{\mathrm{rd}}([a,b)_{\mathbb{T}}, [m,M])$, where $0 \le m < M < \infty$ and $\Psi \in C([0,\infty), \mathbb{R})$ is superquadratic. Then*

$$(b-a)\Psi\left(m+M-\frac{1}{b-a}\int_a^b f(t)\Delta t\right) \\ \le (b-a)(\Psi(m)+\Psi(M))-\int_a^b \Psi(f(t))\Delta t - K, \quad (2.73)$$

where

$$K = \frac{2}{M-m}\int_a^b [(f(t)-m)\Psi(M-f(t))+(M-f(t))\Psi(f(t)-m)]\Delta t \\ +\int_a^b \Psi\left(\left|f(u)-\frac{1}{b-a}\int_a^b f(t)\Delta t\right|\right)\Delta u. \quad (2.74)$$

Moreover, if Ψ is subquadratic then (2.73) *holds in reverse order.*

Proof. The result follows from Theorem 2.1 and Theorem 2.69. □

Remark 2.17 Note that if Ψ is strictly superquadratic in Theorem 2.70, then strict inequality in (2.73) holds.

Converses of Jensen's Inequality

In the following theorem, a functional version of the converse of Jensen's inequality for superquadratic functions is recalled.

Theorem 2.71 (SEE [26, THEOREM 15]) *Let E, L, and A be such that (L_1), (L_2), (A_1), (A_2) are satisfied. Let $h \in L$ be a nonnegative function. Suppose that $\Psi : [0,\infty) \to \mathbb{R}$ is a superquadratic function. Then for every $f \in L$, $f : E \to [m,M] \subseteq [0,\infty)$ such that $hf, h(\Psi \circ f) \in L$, we have*

$$A(h\Psi(f)) + \Delta_c \leq \frac{MA(h) - A(hf)}{M-m}\Psi(m) + \frac{A(hf) - mA(h)}{M-m}\Psi(M),$$

where

$$\Delta_c = \frac{1}{M-m}A((Mk - hf)\Psi(f - m\cdot 1) + (hf - mk)\Psi(M\cdot 1 - f)).$$

Now, we give a converse of Jensen's inequality for superquadratic functions on time scales.

Theorem 2.72 *Let $a, b \in \mathbb{T}$. Assume $f \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}}, [m,M])$, where $0 \leq m < M < \infty$ and $\Psi \in \mathrm{C}([0,\infty), \mathbb{R})$ is superquadratic. Then*

$$\int_a^b \Psi(f(t))\,\Delta t + R \leq \frac{M(b-a) - \int_a^b f(t)\Delta t}{M-m}\Psi(m) + \frac{\int_a^b f(t)\Delta t - m(b-a)}{M-m}\Psi(M), \quad (2.75)$$

where

$$R = \frac{1}{M-m}\int_a^b [(f(t)-m)\Psi(M-f(t)) + (M-f(t))\Psi(f(t)-m)]\,\Delta t. \quad (2.76)$$

Proof. Inequality (2.75) follows directly from Theorem 2.1 and Theorem 2.71 with $h(t) = 1$ for all $t \in [a,b)_{\mathbb{T}}$. □

Remark 2.18 Note that if Ψ is strictly superquadratic in Theorem 2.72, then strict inequality in (2.75) holds.

Slater's Inequality

A functional inequality of Slater type for superquadratic functions, which gives another estimate of the expression $A(\Psi(f))$, is given next.

Theorem 2.73 (SEE [26, THEOREM 17]) *Let E, L, and A be such that (L_1), (L_2), (A_1), (A_2) are satisfied. Suppose that $\Psi : [0,\infty) \to \mathbb{R}$ is a superquadratic function, C is as in*

Definition 1.9, and $h, f \in L$ are nonnegative functions such that $h\Psi(f)$, $hC(f)$, $hfC(f)$, $h\Psi(|f - S \cdot 1|) \in L$. If

$$S = \frac{A(hfC(f))}{A(hC(f))} \geq 0,$$

then

$$A(h\Psi(f)) \leq \Psi(S)A(h) - A(h\Psi(|f - S \cdot 1|)).$$

Now, we can state the inequality of Slater type for superquadratic functions on time scales.

Theorem 2.74 *Let $a, b \in \mathbb{T}$. Assume $\Psi : [0, \infty) \to \mathbb{R}$ is a superquadratic function, C is as in Definition 1.9 and $f : [a, b)_{\mathbb{T}} \to [0, \infty)$ such that $f, \Psi, C \in \mathrm{C_{rd}}$. If C is a nonnegative function, then*

$$\int_a^b \Psi(f(t))\Delta t \leq \Psi(S)(b - a) - \int_a^b \Psi(|f(t) - S|)\Delta t, \tag{2.77}$$

where

$$S = \frac{\int_a^b f(t)C(f(t))\Delta t}{\int_a^b C(f(t))\Delta t}.$$

Proof. Inequality (2.77) follows directly from Theorem 2.1 and Theorem 2.73 with $h(t) = 1$ for all $t \in [a, b)_{\mathbb{T}}$. □

Remark 2.19 Weighted version of all theorems, given in this section, also hold, i.e., we can take the weighted mean $\frac{\int_a^b h(t)f(t)\Delta t}{\int_a^b h(t)\Delta t}$ instead of $\frac{\int_a^b f(t)\Delta t}{b - a}$, where

$$h \in \mathrm{C_{rd}}([a, b)_{\mathbb{T}}, [0, \infty)) \quad \text{is such that} \quad \int_a^b h(t)\Delta t > 0.$$

Chapter 3

Jensen's Functionals, their Properties and Applications

In this chapter, we consider Jensen's functionals on time scales and discuss its properties and applications. Further, we define weighted generalized and power means on time scales. By applying the properties of Jensen's functionals on these means, we obtain several refinements and converses of Hölder's inequality on time scales. (See [21]).

We give all the results for Lebesgue Δ-integrals but they also hold for many other time scales integrals, such as Cauchy, Riemann, Lebesgue, multiple Riemann, and multiple Lebesgue delta, nabla, and diamond-α time scales integrals in a similar way. We use the same notations as in [46, Chapter 5].

3.1 Properties of Jensen's Functionals

First we recall Jensen's inequality on time scales for Lebesgue Δ-integrals.

Theorem 3.1 (SEE THEOREM 2.8) *Assume $\Phi \in C(I,\mathbb{R})$ is convex, $f : [a,b)_{\mathbb{T}} \to I$ is Δ-integrable and $p : [a,b)_{\mathbb{T}} \to \mathbb{R}$ is nonnegative and Δ-integrable such that $\int_{[a,b)} p\mathrm{d}\mu_\Delta > 0$. Then*

$$\Phi\left(\frac{\int_{[a,b)} pf\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}\right) \leq \frac{\int_{[a,b)} p(\Phi\circ f)\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}. \tag{3.1}$$

Definition 3.1 (JENSEN'S FUNCTIONAL) *Under the assumptions of Theorem 3.1, we define Jensen's functional on time scales by*

$$\mathscr{J}_\Delta(\Phi, f, p) = \int_{[a,b)} p(\Phi \circ f) \mathrm{d}\mu_\Delta - \int_{[a,b)} p \mathrm{d}\mu_\Delta \Phi \left(\frac{\int_{[a,b)} pf \mathrm{d}\mu_\Delta}{\int_{[a,b)} p \mathrm{d}\mu_\Delta} \right). \tag{3.2}$$

Remark 3.1 By Theorem 3.1, the following statements are obvious. If Φ is convex, then

$$\mathscr{J}_\Delta(\Phi, f, p) \geq 0,$$

while if Φ is concave, then

$$\mathscr{J}_\Delta(\Phi, f, p) \leq 0.$$

Theorem 3.2 *Assume* $\Phi \in \mathrm{C}(I, \mathbb{R})$ *and* $f : [a,b)_\mathbb{T} \to I$ *is* Δ*-integrable. Let* $p, q : [a,b)_\mathbb{T} \to \mathbb{R}$ *be nonnegative and* Δ*-integrable such that* $\int_{[a,b)} p \mathrm{d}\mu_\Delta > 0$ *and* $\int_{[a,b)} q \mathrm{d}\mu_\Delta > 0$*. If* Φ *is convex, then* $\mathscr{J}_\Delta(\Phi, f, \cdot)$ *is superadditive, i.e.,*

$$\mathscr{J}_\Delta(\Phi, f, p+q) \geq \mathscr{J}_\Delta(\Phi, f, p) + \mathscr{J}_\Delta(\Phi, f, q), \tag{3.3}$$

and $\mathscr{J}_\Delta(\Phi, f, \cdot)$ *is increasing, i.e.,* $p \geq q$ *with* $\int_{[a,b)} p \mathrm{d}\mu_\Delta > \int_{[a,b)} q \mathrm{d}\mu_\Delta$ *implies*

$$\mathscr{J}_\Delta(\Phi, f, p) \geq \mathscr{J}_\Delta(\Phi, f, q). \tag{3.4}$$

Moreover, if Φ *is concave, then* $\mathscr{J}_\Delta(\Phi, f, \cdot)$ *is subadditive and decreasing, i.e.,* (3.3) *and* (3.4) *hold in reverse order.*

Proof. Let Φ be convex. Because the time scales integral is linear, it follows from Definition 3.1 that

$$\begin{aligned}
&\mathscr{J}_\Delta(\Phi, f, p+q) \\
&= \int_{[a,b)} (p+q)(\Phi \circ f) \mathrm{d}\mu_\Delta - \int_{[a,b)} (p+q) \mathrm{d}\mu_\Delta \Phi \left(\frac{\int_{[a,b)} (p+q) f \mathrm{d}\mu_\Delta}{\int_{[a,b)} (p+q) \mathrm{d}\mu_\Delta} \right) \\
&= \int_{[a,b)} (p+q)(\Phi \circ f) \mathrm{d}\mu_\Delta - \int_{[a,b)} (p+q) \mathrm{d}\mu_\Delta \times \\
&\quad \times \Phi \left(\frac{\int_{[a,b)} p \mathrm{d}\mu_\Delta}{\int_{[a,b)} (p+q) \mathrm{d}\mu_\Delta} \frac{\int_{[a,b)} pf \mathrm{d}\mu_\Delta}{\int_{[a,b)} p \mathrm{d}\mu_\Delta} + \frac{\int_{[a,b)} q \mathrm{d}\mu_\Delta}{\int_{[a,b)} (p+q) \mathrm{d}\mu_\Delta} \frac{\int_{[a,b)} qf \mathrm{d}\mu_\Delta}{\int_{[a,b)} q \mathrm{d}\mu_\Delta} \right) \\
&\geq \int_{[a,b)} p(\Phi \circ f) \mathrm{d}\mu_\Delta + \int_{[a,b)} q(\Phi \circ f) \mathrm{d}\mu_\Delta - \int_{[a,b)} p \mathrm{d}\mu_\Delta \Phi \left(\frac{\int_{[a,b)} pf \mathrm{d}\mu_\Delta}{\int_{[a,b)} p \mathrm{d}\mu_\Delta} \right) \\
&\quad - \int_{[a,b)} q \mathrm{d}\mu_\Delta \Phi \left(\frac{\int_{[a,b)} qf \mathrm{d}\mu_\Delta}{\int_{[a,b)} q \mathrm{d}\mu_\Delta} \right) \\
&= \mathscr{J}_\Delta(\Phi, f, p) + \mathscr{J}_\Delta(\Phi, f, q).
\end{aligned}$$

If $p \geq q$, we have $p-q \geq 0$. Now, because Jensen's functional is superadditive (see above) and nonnegative, we have

$$\begin{aligned}
\mathscr{J}_\Delta(\Phi, f, p) &= \mathscr{J}_\Delta(\Phi, f, q+p-q) \\
&\geq \mathscr{J}_\Delta(\Phi, f, q) + \mathscr{J}_\Delta(\Phi, f, p-q) \\
&\geq \mathscr{J}_\Delta(\Phi, f, q).
\end{aligned}$$

On the other hand, if Φ is concave, then the reversed inequalities of (3.3) and (3.4) can be obtained in a similar way. □

Superadditivity (subadditivity) and monotonicity of Jensen's functional are very important properties, considering the numerous applications of the associated inequality. Regarding the monotonicity property, in the following corollaries we give some consequences of Theorem 3.2.

Corollary 3.1 *Let Φ, f, p, q satisfy the hypotheses of Theorem 3.2. Further, suppose there exist nonnegative constants m and M such that*

$$Mq(t) \geq p(t) \geq mq(t) \quad \textit{for all} \quad t \in [a,b)_{\mathbb{T}}$$

and

$$M \int_{[a,b)} q \mathrm{d}\mu_\Delta > \int_{[a,b)} p \mathrm{d}\mu_\Delta > m \int_{[a,b)} q \mathrm{d}\mu_\Delta.$$

If Φ is convex, then

$$M \mathscr{J}_\Delta(\Phi, f, q) \geq \mathscr{J}_\Delta(\Phi, f, p) \geq m \mathscr{J}_\Delta(\Phi, f, q), \tag{3.5}$$

while if Φ is concave, then the inequalities in (3.5) *hold in reverse order.*

Proof. By using Definition 3.1, we have

$$\mathscr{J}_\Delta(\Phi, f, mq) = m \mathscr{J}_\Delta(\Phi, f, q)$$

and

$$\mathscr{J}_\Delta(\Phi, f, Mq) = M \mathscr{J}_\Delta(\Phi, f, q).$$

Now the result follows from the second property of Theorem 3.2. □

Corollary 3.2 *Let Φ, f, p satisfy the hypotheses of Theorem 3.2. Further, assume that p attains its minimum value and its maximum value on its domain. If Φ is convex, then*

$$\left[\max_{t \in [a,b)_{\mathbb{T}}} p(t)\right] \mathfrak{J}_\Delta(\Phi, f) \geq \mathscr{J}_\Delta(\Phi, f, p) \geq \left[\min_{t \in [a,b)_{\mathbb{T}}} p(t)\right] \mathfrak{J}_\Delta(\Phi, f), \tag{3.6}$$

where

$$\mathfrak{J}_\Delta(\Phi, f) = \int_{[a,b)} (\Phi \circ f) \mathrm{d}\mu_\Delta - (b-a)\Phi\left(\frac{\int_{[a,b)} f \mathrm{d}\mu_\Delta}{b-a}\right).$$

Moreover, if Φ is concave, then the inequalities in (3.6) *hold in reverse order.*

Proof. Let p attain its minimum value $\underline{p}$ and its maximum value $\overline{p}$ on its domain $[a,b)_{\mathbb{T}}$. Then

$$\overline{p} = \max_{t\in[a,b)_{\mathbb{T}}} p(t) \geq p(x) \geq \min_{t\in[a,b)_{\mathbb{T}}} p(t) = \underline{p}.$$

By Definition 3.1, we have

$$\mathscr{J}_\Delta(\Phi, f, \overline{p}) = \overline{p}\mathfrak{J}_\Delta(\Phi, f)$$

and

$$\mathscr{J}_\Delta(\Phi, f, \underline{p}) = \underline{p}\mathfrak{J}_\Delta(\Phi, f).$$

Now the result follows from the second property of Theorem 3.2. □

Remark 3.2 The first inequality in (3.6) gives a converse of Jensen's inequality on time scales, and the second one gives a refinement of the observed inequality.

Example 3.1 (SEE [92, REMARK 4]) Let us take the discrete form of Jensen's functional (3.2). For this, let $\mathbb{T} = \mathbb{Z}$, $n \in \mathbb{N}$, $a = 1$, $b = n+1$ and $f(i) = x_i$, $p(i) = p_i$ for $i \in [a,b)_{\mathbb{T}} = \{1,2,\ldots,n\}$. Then (3.2) becomes

$$\mathscr{J}_n(\Phi, \mathbf{x}, \mathbf{p}) = \sum_{i=1}^{n} p_i\Phi(x_i) - P_n\Phi\left(\frac{\sum_{i=1}^{n} p_i x_i}{P_n}\right), \tag{3.7}$$

where

$$\mathbf{x} = (x_1, \ldots, x_n) \in I^n, \quad \mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{R}_+^n, \quad \text{and} \quad P_n = \sum_{i=1}^{n} p_i. \tag{3.8}$$

With these notations, (3.6) takes the form

$$\max_{1\leq i\leq n}\{p_i\}\mathfrak{J}_n(\Phi, \mathbf{x}) \geq \mathscr{J}_n(\Phi, \mathbf{x}, \mathbf{p}) \geq \min_{1\leq i\leq n}\{p_i\}\mathfrak{J}_n(\Phi, \mathbf{x}), \tag{3.9}$$

where

$$\mathfrak{J}_n(\Phi, \mathbf{x}) = \sum_{i=1}^{n} \Phi(x_i) - n\Phi\left(\frac{\sum_{i=1}^{n} x_i}{n}\right).$$

In addition to the above notations, let $q(i) = q_i > 0$ for $i \in [a,b)_{\mathbb{T}} = \{1,2,\ldots,n\}$ and put $\mathbf{q} = (q_1, q_2, \ldots, q_n)$. Using

$$m = \min_{1\leq i\leq n}\left\{\frac{p_i}{q_i}\right\} \quad \text{and} \quad M = \max_{1\leq i\leq n}\left\{\frac{p_i}{q_i}\right\}$$

in Corollary 3.1, (3.5) becomes

$$\max_{1\leq i\leq n}\left\{\frac{p_i}{q_i}\right\}\mathscr{J}_\Delta(\Phi, \mathbf{x}, \mathbf{q}) \geq \mathscr{J}_\Delta(\Phi, \mathbf{x}, \mathbf{p}) \geq \min_{1\leq i\leq n}\left\{\frac{p_i}{q_i}\right\}\mathscr{J}_\Delta(\Phi, \mathbf{x}, \mathbf{q}) \geq 0. \tag{3.10}$$

Dragomir et al., [54], investigated the properties of discrete Jensen's functionals (3.7) concerning superadditivity and monotonicity property of discrete Jensen's functional (see also [103, page 717]). In [53], Dragomir investigated boundedness of normalized Jensen's functionals, that is, functional (3.7) satisfying $\sum_{i=1}^{n} p_i = 1$. He obtained the lower and upper bound for the normalized functional given in (3.10).

Example 3.2 Suppose $\mathbb{T} = \mathbb{R}$ and $a, b \in \mathbb{R}$. Then Jensen's functional (3.2) becomes

$$\int_{[a,b)} h(t)\Phi(f(t))\mathrm{d}\mu(t) - \int_{[a,b)} h(t)\mathrm{d}\mu(t)\Phi\left(\frac{\int_{[a,b)} h(t)f(t)\mathrm{d}\mu(t)}{\int_{[a,b)} h(t)\mathrm{d}\mu(t)}\right).$$

3.2 Applications to Weighted Generalized Means

Definition 3.2 (WEIGHTED GENERALIZED MEAN) *Assume $\chi \in \mathrm{C}(I,\mathbb{R})$ is strictly monotone and $f : [a,b)_{\mathbb{T}} \to I$ is Δ-integrable. Let $p : [a,b)_{\mathbb{T}} \to \mathbb{R}$ be nonnegative and Δ-integrable such that $\int_{[a,b)} p\mathrm{d}\mu_\Delta > 0$. Then we define the weighted generalized mean on time scales by*

$$\mathscr{M}_\Delta(\chi, f, p) = \chi^{-1}\left(\frac{\int_{[a,b)} p(\chi \circ f)\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}\right). \tag{3.11}$$

Theorem 3.3 *Assume $\chi, \psi \in \mathrm{C}(I,\mathbb{R})$ are strictly monotone and $f : [a,b)_{\mathbb{T}} \to I$ is Δ-integrable. Let $p, q : [a,b)_{\mathbb{T}} \to \mathbb{R}$ be nonnegative and Δ-integrable such that the functional*

$$\int_{[a,b)} p\mathrm{d}\mu_\Delta \left[\chi\left(\mathscr{M}_\Delta(\chi, f, p)\right) - \chi\left(\mathscr{M}_\Delta(\psi, f, p)\right)\right] \tag{3.12}$$

is well defined. If $\chi \circ \psi^{-1}$ is convex, then (3.12) *is superadditive, i.e.,*

$$\begin{aligned} &\int_{[a,b)} (p+q)\mathrm{d}\mu_\Delta \left[\chi\left(\mathscr{M}_\Delta(\chi, f, p+q)\right) - \chi\left(\mathscr{M}_\Delta(\psi, f, p+q)\right)\right] \\ &\qquad \geq \int_{[a,b)} p\mathrm{d}\mu_\Delta \left[\chi\left(\mathscr{M}_\Delta(\chi, f, p)\right) - \chi\left(\mathscr{M}_\Delta(\psi, f, p)\right)\right] \\ &\qquad\qquad + \int_{[a,b)} q\mathrm{d}\mu_\Delta \left[\chi\left(\mathscr{M}_\Delta(\chi, f, q)\right) - \chi\left(\mathscr{M}_\Delta(\psi, f, q)\right)\right], \end{aligned} \tag{3.13}$$

and (3.12) *is increasing, i.e., $p \geq q$ with $\int_{[a,b)} p\mathrm{d}\mu_\Delta > \int_{[a,b)} q\mathrm{d}\mu_\Delta$ implies*

$$\begin{aligned} &\int_{[a,b)} p\mathrm{d}\mu_\Delta \left[\chi\left(\mathscr{M}_\Delta(\chi, f, p)\right) - \chi\left(\mathscr{M}_\Delta(\psi, f, p)\right)\right] \\ &\qquad\qquad \geq \int_{[a,b)} q\mathrm{d}\mu_\Delta \left[\chi\left(\mathscr{M}_\Delta(\chi, f, q)\right) - \chi\left(\mathscr{M}_\Delta(\psi, f, q)\right)\right]. \end{aligned} \tag{3.14}$$

Moreover, if $\chi\circ\psi^{-1}$ *is concave, then* (3.12) *is subadditive and decreasing, i.e.,* (3.13) *and* (3.14) *hold in reverse order.*

Proof. The functional defined in (3.12) is obtained by replacing Φ with $\chi\circ\psi^{-1}$ and f with $\psi\circ f$ in Jensen's functional (3.2), i.e.,

$$
\begin{aligned}
&\mathscr{J}_\Delta(\chi\circ\psi^{-1},\psi\circ f,p)\\
&=\int_{[a,b)} p(\chi\circ\psi^{-1}\circ\psi\circ f)\mathrm{d}\mu_\Delta-\int_{[a,b)} p\mathrm{d}\mu_\Delta\left(\chi\circ\psi^{-1}\left(\frac{\int_{[a,b)}p(\psi\circ f)\mathrm{d}\mu_\Delta}{\int_{[a,b)}p\mathrm{d}\mu_\Delta}\right)\right)\\
&=\int_{[a,b)} p(\chi\circ f)\mathrm{d}\mu_\Delta-\int_{[a,b)} p\mathrm{d}\mu_\Delta\chi\left(\mathscr{M}_\Delta(\psi,f,p)\right)\\
&=\int_{[a,b)} p\mathrm{d}\mu_\Delta\chi\left(\mathscr{M}_\Delta(\chi,f,p)\right)-\int_{[a,b)} p\mathrm{d}\mu_\Delta\chi\left(\mathscr{M}_\Delta(\psi,f,p)\right)\\
&=\int_{[a,b)} p\mathrm{d}\mu_\Delta\left[\chi\left(\mathscr{M}_\Delta(\chi,f,p)\right)-\chi\left(\mathscr{M}_\Delta(\psi,f,p)\right)\right].
\end{aligned}
$$

Now, all claims follow immediately from Theorem 3.2. □

Corollary 3.3 *Let* f,p,χ,ψ *satisfy the hypotheses of Theorem 3.3. Further, assume that* p *attains its minimum value and its maximum value on its domain. If* $\chi\circ\psi^{-1}$ *is convex, then*

$$
\begin{aligned}
&\left[\max_{t\in[a,b)_\mathbb{T}} p(t)\right](b-a)\left[\chi(\mathfrak{M}_\Delta(\chi,f))-\chi(\mathfrak{M}_\Delta(\psi,f))\right]\\
&\qquad\geq\int_{[a,b)} p\mathrm{d}\mu_\Delta\left[\chi\left(\mathscr{M}_\Delta(\chi,f,p)\right)-\chi\left(\mathscr{M}_\Delta(\psi,f,p)\right)\right]\\
&\qquad\qquad\geq\left[\min_{t\in[a,b)_\mathbb{T}} p(t)\right](b-a)\left[\chi(\mathfrak{M}_\Delta(\chi,f))-\chi(\mathfrak{M}_\Delta(\psi,f))\right],
\end{aligned}
\tag{3.15}
$$

where

$$
\mathfrak{M}_\Delta(\eta,f)=\eta^{-1}\left(\frac{\int_{[a,b)}(\eta\circ f)\mathrm{d}\mu_\Delta}{b-a}\right),\quad \eta\in\{\chi,\psi\}.
$$

Moreover, if $\chi\circ\psi^{-1}$ *is concave, then the inequalities in* (3.15) *hold in reverse order.*

Proof. The proof is omitted as it is similar to the proof of Corollary 3.2. □

3.3 Applications to Weighted Generalized Power Means

Definition 3.3 (WEIGHTED GENERALIZED POWER MEAN) *Let $r \in \mathbb{R}$. Assume $f : [a,b)_{\mathbb{T}} \to I$ is positive and Δ-integrable. Let $p : [a,b)_{\mathbb{T}} \to \mathbb{R}$ be nonnegative and Δ-integrable such that $\int_{[a,b)} p\mathrm{d}\mu_\Delta > 0$. Then we define the weighted generalized power mean on time scales by*

$$\mathscr{M}_\Delta^{[r]}(f,p) = \begin{cases} \left(\frac{\int_{[a,b)} pf^r \mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}\right)^{\frac{1}{r}} & \text{if} \quad r \neq 0, \\ \exp\left(\frac{\int_{[a,b)} p\ln(f)\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}\right) & \text{if} \quad r = 0. \end{cases} \tag{3.16}$$

Remark 3.3 The weighted generalized power mean defined in (3.16) follows from the weighted generalized mean defined in (3.11) by taking $\chi(x) = x^r$ $(x > 0)$ in the weighted generalized mean.

Theorem 3.4 *Let $r, s \in \mathbb{R}$ with $r \neq 0$. Assume $f : [a,b)_{\mathbb{T}} \to I$ is positive and Δ-integrable. Let $p, q : [a,b)_{\mathbb{T}} \to \mathbb{R}$ be nonnegative and Δ-integrable such that the functional*

$$\int_{[a,b)} p\mathrm{d}\mu_\Delta \left\{ \left[\mathscr{M}_\Delta^{[s]}(f,p)\right]^s - \left[\mathscr{M}_\Delta^{[r]}(f,p)\right]^s \right\} \tag{3.17}$$

is well defined. If $\min\{0,r\} > s > \max\{0,r\}$*, then* (3.17) *is superadditive (also if $r = 0$), i.e.,*

$$\begin{aligned} \int_{[a,b)} (p+q)\mathrm{d}\mu_\Delta & \left\{ \left[\mathscr{M}_\Delta^{[s]}(f,p+q)\right]^s - \left[\mathscr{M}_\Delta^{[r]}(f,p+q)\right]^s \right\} \\ & \geq \int_{[a,b)} p\mathrm{d}\mu_\Delta \left\{ \left[\mathscr{M}_\Delta^{[s]}(f,p)\right]^s - \left[\mathscr{M}_\Delta^{[r]}(f,p)\right]^s \right\} \\ & \quad + \int_{[a,b)} q\mathrm{d}\mu_\Delta \left\{ \left[\mathscr{M}_\Delta^{[s]}(f,q)\right]^s - \left[\mathscr{M}_\Delta^{[r]}(f,q)\right]^s \right\}, \end{aligned} \tag{3.18}$$

and (3.17) *is increasing, i.e., $p \geq q$ with $\int_{[a,b)} p\mathrm{d}\mu_\Delta > \int_{[a,b)} q\mathrm{d}\mu_\Delta$ implies*

$$\begin{aligned} \int_{[a,b)} p\mathrm{d}\mu_\Delta & \left\{ \left[\mathscr{M}_\Delta^{[s]}(f,p)\right]^s - \left[\mathscr{M}_\Delta^{[r]}(f,p)\right]^s \right\} \\ & \geq \int_{[a,b)} q\mathrm{d}\mu_\Delta \left\{ \left[\mathscr{M}_\Delta^{[s]}(f,q)\right]^s - \left[\mathscr{M}_\Delta^{[r]}(f,q)\right]^s \right\}. \end{aligned} \tag{3.19}$$

Moreover, if $r > s > 0$ or $0 > s > r$, then (3.17) *is subadditive and decreasing, i.e.,* (3.18) *and* (3.19) *hold in reverse order.*

Proof. If $r \neq 0$, then let $\chi(x) = x^s$ and $\psi(x) = x^r$ $(x > 0)$ in Theorem 3.3. Then $(\chi \circ \psi^{-1})(x) = x^{\frac{s}{r}}$ and therefore

$$(\chi \circ \psi^{-1})''(x) = \frac{s(s-r)}{r^2} x^{\frac{s}{r}-2}.$$

Thus $\chi \circ \psi^{-1}$ is convex if $\min\{0,r\} > s > \max\{0,r\}$ and concave if $r > s > 0$ or $0 > s > r$. If, however, $r = 0$, then let $\chi(x) = x^s$ and $\psi(x) = \ln(x)$ $(x > 0)$ in Theorem 3.3. Then $(\chi \circ \psi^{-1})(x) = e^{sx}$. Thus $\chi \circ \psi^{-1}$ is convex for $s \neq 0$. In either case the result follows now immediately from Theorem 3.3. □

Corollary 3.4 *Let r, s, f, p satisfy the hypotheses of Theorem 3.4. Further, assume that p attains its minimum value and its maximum value on its domain. If* $\min\{0,r\} > s > \max\{0,r\}$*, then*

$$\left[\max_{t\in[a,b)_{\mathbb{T}}} p(t)\right](b-a)\left\{\left[\mathfrak{M}_\Delta^{[s]}(f)\right]^s - \left[\mathfrak{M}_\Delta^{[r]}(f)\right]^s\right\}$$
$$\geq \int_{[a,b)} p\,\mathrm{d}\mu_\Delta \left\{\left[\mathscr{M}_\Delta^{[s]}(f,p)\right]^s - \left[\mathscr{M}_\Delta^{[r]}(f,p)\right]^s\right\}$$
$$\geq \left[\min_{t\in[a,b)_{\mathbb{T}}} p(t)\right](b-a)\left\{\left[\mathfrak{M}_\Delta^{[s]}(f)\right]^s - \left[\mathfrak{M}_\Delta^{[r]}(f)\right]^s\right\}, \quad (3.20)$$

where

$$\mathfrak{M}_\Delta^{[u]}(f) = \begin{cases} \left(\frac{\int_{[a,b)} f^u \mathrm{d}\mu_\Delta}{b-a}\right)^{\frac{1}{u}} & \text{if} \quad u \in \mathbb{R}\setminus\{0\}, \\ \exp\left(\frac{\int_{[a,b)} \ln(f)\mathrm{d}\mu_\Delta}{b-a}\right) & \text{if} \quad u = 0. \end{cases} \quad (3.21)$$

Moreover, if $r > s > 0$ or $0 > s > r$, then the inequalities in (3.20) *hold in reverse order.*

Proof. The proof is omitted as it is similar to the proof of Corollary 3.2 followed by Theorem 3.4. □

Example 3.3 From the discrete form of Corollary 3.4, i.e., by using $\mathbb{T} = \mathbb{Z}$, we get a refinement and a converse of the arithmetic-geometric mean inequality. Using the notation as introduced in Example 3.1, let $x_i > 0$ for all $i \in [a,b)_{\mathbb{T}}$ and $s = 1$, $r = 0$. Then (3.20) becomes

$$n \max_{1\leq i\leq n}\{p_i\}\left[A_n(\mathbf{x}) - G_n(\mathbf{x})\right] \geq P_n[\mathscr{M}_n^{[1]}(\mathbf{x},\mathbf{p}) - \mathscr{M}_n^{[0]}(\mathbf{x},\mathbf{p})]$$
$$\geq n \min_{1\leq i\leq n}\{p_i\}\left[A_n(\mathbf{x}) - G_n(\mathbf{x})\right] \geq 0, \quad (3.22)$$

where

$$\mathscr{M}_n^{[r]}(\mathbf{x},\mathbf{p}) = \begin{cases} \left(\dfrac{\sum\limits_{i=1}^n p_i x_i^r}{P_n}\right)^{\frac{1}{r}} & \text{if} \quad r \in \mathbb{R}\setminus\{0\}, \\ \left(\prod\limits_{i=1}^n x_i^{p_i}\right)^{\frac{1}{P_n}} & \text{if} \quad r = 0, \end{cases}$$

$$A_n(\mathbf{x}) = \frac{\sum_{i=1}^{n} x_i}{n}, \quad \text{and} \quad G_n(\mathbf{x}) = \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}.$$

The first inequality in (3.22) gives a converse and the second one gives a refinement of the arithmetic-geometric mean inequality of $\mathscr{M}_n^{[1]}(\mathbf{x},\mathbf{p})$ and $\mathscr{M}_n^{[0]}(\mathbf{x},\mathbf{p})$. Some variants of inequalities in (3.22) were recently studied by Aldaz in [14] (see also [10, 11, 12, 15, 13]).

Theorem 3.5 *Let r, f, p, q satisfy the hypotheses of Theorem 3.4. Suppose that the functional*

$$\int_{[a,b)} p\mathrm{d}\mu_\Delta \left\{ \frac{\int_{[a,b)} p\ln(f)\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta} - \ln\left(\mathscr{M}_\Delta^{[r]}(f,p)\right) \right\} \tag{3.23}$$

is well defined. If $r < 0$, then (3.23) *is superadditive, i.e.,*

$$\begin{aligned} \int_{[a,b)} (p+q)\mathrm{d}\mu_\Delta &\left\{ \frac{\int_{[a,b)} (p+q)\ln(f)\mathrm{d}\mu_\Delta}{\int_{[a,b)} (p+q)\mathrm{d}\mu_\Delta} - \ln\left(\mathscr{M}_\Delta^{[r]}(f,p+q)\right) \right\} \\ &\geq \int_{[a,b)} p\mathrm{d}\mu_\Delta \left\{ \frac{\int_{[a,b)} p\ln(f)\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta} - \ln\left(\mathscr{M}_\Delta^{[r]}(f,p)\right) \right\} \\ &\quad + \int_{[a,b)} q\mathrm{d}\mu_\Delta \left\{ \frac{\int_{[a,b)} q\ln(f)\mathrm{d}\mu_\Delta}{\int_{[a,b)} q\mathrm{d}\mu_\Delta} - \ln\left(\mathscr{M}_\Delta^{[r]}(f,q)\right) \right\}, \end{aligned} \tag{3.24}$$

and (3.23) *is increasing, i.e., $p \geq q$ with $\int_{[a,b)} p\mathrm{d}\mu_\Delta > \int_{[a,b)} q\mathrm{d}\mu_\Delta$ implies*

$$\begin{aligned} \int_{[a,b)} p\mathrm{d}\mu_\Delta &\left\{ \frac{\int_{[a,b)} p\ln(f)\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta} - \ln\left(\mathscr{M}_\Delta^{[r]}(f,p)\right) \right\} \\ &\geq \int_{[a,b)} q\mathrm{d}\mu_\Delta \left\{ \frac{\int_{[a,b)} q\ln(f)\mathrm{d}\mu_\Delta}{\int_{[a,b)} q\mathrm{d}\mu_\Delta} - \ln\left(\mathscr{M}_\Delta^{[r]}(f,q)\right) \right\}. \end{aligned} \tag{3.25}$$

Moreover, if $r > 0$, then (3.23) *is subadditive and decreasing, i.e.,* (3.24) *and* (3.25) *hold in reverse order.*

Proof. Let $\chi(x) = \ln(x)$ and $\psi(x) = x^r$ in Theorem 3.3. Then $\left(\chi \circ \psi^{-1}\right)(x) = \frac{1}{r}\ln(x)$. Thus $\chi \circ \psi^{-1}$ is convex if $r < 0$ and concave if $r > 0$. Now the rest of the proof follows immediately from Theorem 3.3. □

Corollary 3.5 *Let r, f, p satisfy the hypotheses of Theorem 3.4. Further, assume that p attains its minimum value and its maximum value on its domain. If $r < 0$, then*

$$\left[\max_{t\in[a,b)_{\mathbb{T}}} p(t)\right](b-a)\left\{\frac{\int_{[a,b)}\ln(f)\mathrm{d}\mu_\Delta}{b-a} - \ln\left(\mathfrak{M}_\Delta^{[r]}(f)\right)\right\} \tag{3.26}$$
$$\geq \int_{[a,b)} p\mathrm{d}\mu_\Delta \left\{\frac{\int_{[a,b)} p\ln(f)\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta} - \ln\left(\mathscr{M}_\Delta^{[r]}(f,p)\right)\right\}$$
$$\geq \left[\min_{t\in[a,b)_{\mathbb{T}}} p(t)\right](b-a)\left\{\frac{\int_{[a,b)}\ln(f)\mathrm{d}\mu_\Delta}{b-a} - \ln\left(\mathfrak{M}_\Delta^{[r]}(f)\right)\right\},$$

where $\mathfrak{M}_\Delta^{[r]}(f)$ *is defined in* (3.21)*. Moreover, if* $r>0$*, then the inequalities in* (3.26) *hold in reverse order.*

Proof. The proof is omitted as it is similar to the proof of Corollary 3.2 followed by Theorem 3.5. □

Example 3.4 (SEE [92, REMARK 8]) Again we consider $\mathbb{T}=\mathbb{Z}$. Using the notation as introduced in Example 3.1, the term $\frac{\int_{[a,b)} p\ln(f)\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}$ takes the form

$$\frac{\sum\limits_{i=1}^{n} p_i\ln(x_i)}{\sum\limits_{i=1}^{n} p_i} = \ln\left(\prod_{i=1}^{n} x_i^{p_i}\right)^{\frac{1}{P_n}} = \ln\left(\mathscr{M}_n^{[0]}(\mathbf{x},\mathbf{p})\right),$$

and (3.26) becomes

$$\left[\frac{G_n(\mathbf{x})}{A_n(\mathbf{x})}\right]^{n\max\limits_{1\le i\le n}\{p_i\}} \geq \left[\frac{\mathscr{M}_n^{[0]}(\mathbf{x},\mathbf{p})}{\mathscr{M}_n^{[1]}(\mathbf{x},\mathbf{p})}\right]^{P_n} \geq \left[\frac{G_n(\mathbf{x})}{A_n(\mathbf{x})}\right]^{n\min\limits_{1\le i\le n}\{p_i\}}. \tag{3.27}$$

The inequalities in (3.27) provide a refinement and a converse of the arithmetic-geometric mean inequality in quotient form.

Example 3.5 (SEE [92, REMARK 9]) The relations (3.22) and (3.27) also yield refinements and converses of Young's inequality. To see this, consider again $\mathbb{T}=\mathbb{Z}$. Using the notation as introduced in Example 3.1, define

$$\mathbf{x}^{\mathbf{p}} = (x_1^{p_1}, x_2^{p_2}, \ldots, x_n^{p_n}) \quad \text{and} \quad \mathbf{p}^{-1} = \left(\frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_n}\right),$$

where $\mathbf{x}$ and $\mathbf{p}$ are positive n-tuples such that $\sum\limits_{i=1}^{n}\frac{1}{p_i}=1$. Then (3.22) and (3.27) become

$$n\max_{1\le i\le n}\left\{\frac{1}{p_i}\right\}[A_n(\mathbf{x}^{\mathbf{p}})-G_n(\mathbf{x}^{\mathbf{p}})] \geq \mathscr{M}_n^{[1]}(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1}) - \mathscr{M}_n^{[0]}(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1}) \tag{3.28}$$
$$\geq n\min_{1\le i\le n}\left\{\frac{1}{p_i}\right\}[A_n(\mathbf{x}^{\mathbf{p}})-G_n(\mathbf{x}^{\mathbf{p}})],$$

and

$$\left[\frac{G_n(\mathbf{x}^{\mathbf{p}})}{A_n(\mathbf{x}^{\mathbf{p}})}\right]^{n\max\limits_{1\le i\le n}\left\{\frac{1}{p_i}\right\}} \geq \frac{\mathscr{M}_n^{[0]}(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1})}{\mathscr{M}_n^{[1]}(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1})} \geq \left[\frac{G_n(\mathbf{x}^{\mathbf{p}})}{A_n(\mathbf{x}^{\mathbf{p}})}\right]^{n\min\limits_{1\le i\le n}\left\{\frac{1}{p_i}\right\}}. \tag{3.29}$$

The inequalities in (3.28) and (3.29) provide refinements and converses of Young's inequality in difference and quotient form.

3.4 Improvements of Hölder's Inequality

Let us recall Hölder's inequality for Lebesgue Δ-integrals.

Theorem 3.6 (SEE THEOREM 2.32) *For $p \neq 1$, define $q = \frac{p}{p-1}$. Let w, f, g be nonnegative functions such that wf^p, wg^q, wfg are Δ-integrable on $[a,b)_{\mathbb{T}}$. If $p > 1$, then*

$$\int_{[a,b)} wfg\mathrm{d}\mu_\Delta \leq \left(\int_{[a,b)} wf^p\mathrm{d}\mu_\Delta\right)^{\frac{1}{p}} \left(\int_{[a,b)} wg^q\mathrm{d}\mu_\Delta\right)^{\frac{1}{q}}. \tag{3.30}$$

If $0 < p < 1$ and $\int_{[a,b)} wg^q\mathrm{d}\mu_\Delta > 0$, or if $p < 0$ and $\int_{[a,b)} wf^p\mathrm{d}\mu_\Delta > 0$, then (3.30) *is reversed.*

Remark 3.4 Let $n \in \mathbb{N}$ and let $f_i : [a,b)_{\mathbb{T}} \to \mathbb{R}$ be Δ-integrable for all $i \in \{1,2,\ldots,n\}$. Assume $p_i > 1$ for all $i \in \{1,2,\ldots,n\}$ are conjugate exponents, i.e., $\sum\limits_{i=1}^{n} \frac{1}{p_i} = 1$, and $\prod\limits_{i=1}^{n} f_i^{\frac{1}{p_i}}$ is Δ-integrable on $[a,b)_{\mathbb{T}}$. Hölder's inequality on time scales (Theorem 3.6) asserts that

$$\int_{[a,b)} \prod_{i=1}^{n} f_i^{\frac{1}{p_i}}\mathrm{d}\mu_\Delta \leq \prod_{i=1}^{n}\left(\int_{[a,b)} f_i\mathrm{d}\mu_\Delta\right)^{\frac{1}{p_i}}.$$

It is well known from the literature (see [103, 119]) that Hölder's inequality can easily be obtained from Young's inequality. Therefore, it is natural to expect that relations (3.28) and (3.29) also provide refinements and conversions of Hölder's inequality.

The first in a series of results refers to relation (3.28), that is, refinement and conversion of Hölder's inequality in difference form.

Theorem 3.7 *Let $p_i > 1$, $i \in \{1,2,\ldots,n\}$, be conjugate exponents. Let f_i, $i \in \{1,2,\ldots,n\}$, be nonnegative Δ-integrable functions such that $\prod\limits_{i=1}^{n} f_i^{\frac{1}{p_i}}$ and $\prod\limits_{i=1}^{n} f_i^{\frac{1}{n}}$ are nonnegative and Δ-integrable. Then the following inequalities hold:*

$$n\max_{1\le i\le n}\left\{\frac{1}{p_i}\right\}\left[\prod_{i=1}^{n}\left(\int_{[a,b)} f_i\mathrm{d}\mu_\Delta\right)^{\frac{1}{p_i}} - \prod_{i=1}^{n}\left(\int_{[a,b)} f_i\mathrm{d}\mu_\Delta\right)^{\frac{1}{p_i}-\frac{1}{n}}\left(\int_{[a,b)} \prod_{i=1}^{n} f_i^{\frac{1}{n}}\mathrm{d}\mu_\Delta\right)\right]$$

$$\geq \prod_{i=1}^{n}\left(\int_{[a,b)} f_i \mathrm{d}\mu_\Delta\right)^{\frac{1}{p_i}} - \left(\int_{[a,b)} \prod_{i=1}^{n} f_i^{\frac{1}{p_i}} \mathrm{d}\mu_\Delta\right)$$

$$\geq n \min_{1\leq i\leq n}\left\{\frac{1}{p_i}\right\}\left[\prod_{i=1}^{n}\left(\int_{[a,b)} f_i \mathrm{d}\mu_\Delta\right)^{\frac{1}{p_i}}\right.$$

$$\left. - \prod_{i=1}^{n}\left(\int_{[a,b)} f_i \mathrm{d}\mu_\Delta\right)^{\frac{1}{p_i}-\frac{1}{n}}\left(\int_{[a,b)} \prod_{i=1}^{n} f_i^{\frac{1}{n}} \mathrm{d}\mu_\Delta\right)\right].$$

Proof. Let $x_i = \left[\frac{f_i}{\int_{[a,b)} f_i \mathrm{d}\mu_\Delta}\right]^{\frac{1}{p_i}}$, $i \in \{1,2,\ldots,n\}$, in Example 3.5. Then the expressions in (3.28) become

$$\mathscr{M}_n^{[1]}(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1}) - \mathscr{M}_n^{[0]}(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1}) = \sum_{i=1}^{n}\frac{f_i}{p_i \int_{[a,b)} f_i \mathrm{d}\mu_\Delta} - \prod_{i=1}^{n}\frac{f_i^{\frac{1}{p_i}}}{\left(\int_{[a,b)} f_i \mathrm{d}\mu_\Delta\right)^{\frac{1}{p_i}}}$$

and

$$A_n(\mathbf{x}^{\mathbf{p}}) - G_n(\mathbf{x}^{\mathbf{p}}) = \frac{1}{n}\sum_{i=1}^{n}\frac{f_i}{\int_{[a,b)} f_i \mathrm{d}\mu_\Delta} - \prod_{i=1}^{n}\frac{f_i^{\frac{1}{n}}}{\left(\int_{[a,b)} f_i \mathrm{d}\mu_\Delta\right)^{\frac{1}{n}}}.$$

Now, by applying the Δ-integral to the last two equations, we get

$$\int_{[a,b)}\left[\mathscr{M}_n^{[1]}(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1}) - \mathscr{M}_n^{[0]}(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1})\right]\mathrm{d}\mu_\Delta$$

$$= \sum_{i=1}^{n}\frac{\int_{[a,b)} f_i \mathrm{d}\mu_\Delta}{p_i \int_{[a,b)} f_i \mathrm{d}\mu_\Delta} - \frac{\int_{[a,b)}\left(\prod_{i=1}^{n} f_i^{\frac{1}{p_i}}\right)\mathrm{d}\mu_\Delta}{\prod_{i=1}^{n}\left(\int_{[a,b)} f_i \mathrm{d}\mu_\Delta\right)^{\frac{1}{p_i}}}$$

$$= 1 - \frac{\int_{[a,b)}\left(\prod_{i=1}^{n} f_i^{\frac{1}{p_i}}\right)\mathrm{d}\mu_\Delta}{\prod_{i=1}^{n}\left(\int_{[a,b)} f_i \mathrm{d}\mu_\Delta\right)^{\frac{1}{p_i}}}$$

and

$$\int_{[a,b)}\left[A_n(\mathbf{x}^{\mathbf{p}}) - G_n(\mathbf{x}^{\mathbf{p}})\right]\mathrm{d}\mu_\Delta = \frac{1}{n}\sum_{i=1}^{n}\frac{\int_{[a,b)} f_i \mathrm{d}\mu_\Delta}{\int_{[a,b)} f_i \mathrm{d}\mu_\Delta} - \frac{\int_{[a,b)}\left(\prod_{i=1}^{n} f_i^{\frac{1}{n}}\right)\mathrm{d}\mu_\Delta}{\prod_{i=1}^{n}\left(\int_{[a,b)} f_i \mathrm{d}\mu_\Delta\right)^{\frac{1}{n}}}$$

$$= 1 - \frac{\int_{[a,b)}\left(\prod_{i=1}^{n} f_i^{\frac{1}{n}}\right)\mathrm{d}\mu_\Delta}{\prod_{i=1}^{n}\left(\int_{[a,b)} f_i \mathrm{d}\mu_\Delta\right)^{\frac{1}{n}}}.$$

By applying the Δ-integral to the series of inequalities in (3.28), we obtain the required inequalities. $\square$

Remark 3.5 The first inequality in Theorem 3.7 gives a converse and the second one gives a refinement of Hölder's inequality on time scales.

Now we give refinement and conversion of Hölder's inequality in quotient form, deduced from relation (3.29).

Theorem 3.8 *Under the same assumptions as in Theorem 3.7, the following inequalities hold:*

$$\begin{aligned}
&\left[\frac{n^n}{\prod_{i=1}^{n}\int_{[a,b)} f_i \mathrm{d}\mu_\Delta}\right]^{\min\limits_{1\le i\le n}\left\{\frac{1}{p_i}\right\}} \times \\
&\times \int_{[a,b)} \left[\sum_{i=1}^{n} \frac{f_i}{p_i \int_{[a,b)} f_i \mathrm{d}\mu_\Delta}\right] \left[\frac{\prod_{i=1}^{n} f_i^{\frac{1}{n}}}{\sum_{i=1}^{n} \frac{f_i}{\int_{[a,b)} f_i \mathrm{d}\mu_\Delta}}\right]^{n \min\limits_{1\le i\le n}\left\{\frac{1}{p_i}\right\}} \mathrm{d}\mu_\Delta \\
&\ge \frac{\int_{[a,b)} \prod_{i=1}^{n} f_i^{\frac{1}{p_i}} \mathrm{d}\mu_\Delta}{\prod_{i=1}^{n}\left(\int_{[a,b)} f_i \mathrm{d}\mu_\Delta\right)^{\frac{1}{p_i}}} \\
&\ge \left[\frac{n^n}{\prod_{i=1}^{n}\int_{[a,b)} f_i \mathrm{d}\mu_\Delta}\right]^{\max\limits_{1\le i\le n}\left\{\frac{1}{p_i}\right\}} \times \\
&\times \int_{[a,b)} \left[\sum_{i=1}^{n} \frac{f_i}{p_i \int_{[a,b)} f_i \mathrm{d}\mu_\Delta}\right] \left[\frac{\prod_{i=1}^{n} f_i^{\frac{1}{n}}}{\sum_{i=1}^{n} \frac{f_i}{\int_{[a,b)} f_i \mathrm{d}\mu_\Delta}}\right]^{n \max\limits_{1\le i\le n}\left\{\frac{1}{p_i}\right\}} \mathrm{d}\mu_\Delta
\end{aligned}$$

provided that all expressions are well defined.

Proof. We consider relation (3.29) in the same settings as in Theorem 3.7. By inverting, (3.29) can be rewritten in the form

$$\begin{aligned}
\mathscr{M}_n^{[1]}(\mathbf{x}^{\mathbf{p}}, \mathbf{p}^{-1}) \left[\frac{G_n(\mathbf{x}^{\mathbf{p}})}{A_n(\mathbf{x}^{\mathbf{p}})}\right]^{n \min\limits_{1\le i\le n}\left\{\frac{1}{p_i}\right\}} &\ge \mathscr{M}_n^{[0]}(\mathbf{x}^{\mathbf{p}}, \mathbf{p}^{-1}) \\
&\ge \mathscr{M}_n^{[1]}(\mathbf{x}^{\mathbf{p}}, \mathbf{p}^{-1}) \left[\frac{G_n(\mathbf{x}^{\mathbf{p}})}{A_n(\mathbf{x}^{\mathbf{p}})}\right]^{n \max\limits_{1\le i\le n}\left\{\frac{1}{p_i}\right\}}.
\end{aligned} \tag{3.31}$$

Now, if we consider the n-tuple $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, where

$$x_i = \left[\frac{f_i}{\int_{[a,b)} f_i \mathrm{d}\mu_\Delta}\right]^{\frac{1}{p_i}} \quad \text{for all} \quad i \in \{1,2,\ldots,n\},$$

then the expressions that represent the means in (3.31) become

$$\mathscr{M}_n^{[1]}(\mathbf{x}^{\mathbf{p}}, \mathbf{p}^{-1}) = \sum_{i=1}^{n} \frac{f_i}{p_i \int_{[a,b)} f_i \mathrm{d}\mu_\Delta}, \quad \mathscr{M}_n^{[0]}(\mathbf{x}^{\mathbf{p}}, \mathbf{p}^{-1}) = \prod_{i=1}^{n} \frac{f_i^{\frac{1}{p_i}}}{\left(\int_{[a,b)} f_i \mathrm{d}\mu_\Delta\right)^{\frac{1}{p_i}}}$$

and

$$A_n(\mathbf{x}^{\mathbf{p}}) = \frac{1}{n} \sum_{i=1}^{n} \frac{f_i}{\int_{[a,b)} f_i \mathrm{d}\mu_\Delta}, \quad G_n(\mathbf{x}^{\mathbf{p}}) = \prod_{i=1}^{n} \frac{f_i^{\frac{1}{n}}}{\left(\int_{[a,b)} f_i \mathrm{d}\mu_\Delta\right)^{\frac{1}{n}}}.$$

Now, by taking the Δ-integral on (3.31) in the described setting, we obtain the required inequalities. □

Remark 3.6 The first inequality in Theorem 3.8 gives a refinement and the second one gives a converse of Hölder's inequality on time scales.

Corollary 3.6 *Let $r, s \in \mathbb{R}$ such that $\frac{1}{r} + \frac{1}{s} = 1$. Further, assume that f, g are positive and Δ-integrable such that f attains its minimum value and its maximum value on its domain. If $r > 1$, then*

$$\begin{aligned}
&\left[\max_{t \in [a,b)_\mathbb{T}} f(t)\right] \left[(b-a)^{\frac{1}{r}} \left(\int_{[a,b)} \frac{g}{f} \mathrm{d}\mu_\Delta\right)^{\frac{1}{s}} - \int_{[a,b)} \left(\frac{g}{f}\right)^{\frac{1}{s}} \mathrm{d}\mu_\Delta\right] \\
&\geq \left(\int_{[a,b)} f \mathrm{d}\mu_\Delta\right)^{\frac{1}{r}} \left(\int_{[a,b)} g \mathrm{d}\mu_\Delta\right)^{\frac{1}{s}} - \int_{[a,b)} f^{\frac{1}{r}} g^{\frac{1}{s}} \mathrm{d}\mu_\Delta \\
&\geq \left[\min_{t \in [a,b)_\mathbb{T}} f(t)\right] \left[(b-a)^{\frac{1}{r}} \left(\int_{[a,b)} \frac{g}{f} \mathrm{d}\mu_\Delta\right)^{\frac{1}{s}} - \int_{[a,b)} \left(\frac{g}{f}\right)^{\frac{1}{s}} \mathrm{d}\mu_\Delta\right].
\end{aligned} \tag{3.32}$$

Moreover, if $0 < r < 1$, then the inequalities in (3.32) hold in reverse order.

Proof. The result follows from Corollary 3.2 by replacing f with $\frac{g}{f}$, p with f and letting $\Phi(x) = -rsx^{\frac{1}{s}}$. Then Φ is convex on $(0, \infty)$ and we have

$$\begin{aligned}
\mathscr{J}_\Delta\left(\Phi, \frac{g}{f}, f\right) &= \int_{[a,b)} f \Phi\left(\frac{g}{f}\right) \mathrm{d}\mu_\Delta - \int_{[a,b)} f \mathrm{d}\mu_\Delta \Phi\left(\frac{\int_{[a,b)} g \mathrm{d}\mu_\Delta}{\int_{[a,b)} f \mathrm{d}\mu_\Delta}\right) \\
&= rs\left[\left(\int_{[a,b)} f \mathrm{d}\mu_\Delta\right)^{1-\frac{1}{s}} \left(\int_{[a,b)} g \mathrm{d}\mu_\Delta\right)^{\frac{1}{s}} - \int_{[a,b)} f^{1-\frac{1}{s}} g^{\frac{1}{s}} \mathrm{d}\mu_\Delta\right]
\end{aligned}$$

$$= rs\left[\left(\int_{[a,b)} f \mathrm{d}\mu_\Delta\right)^{\frac{1}{r}} \left(\int_{[a,b)} g \mathrm{d}\mu_\Delta\right)^{\frac{1}{s}} - \int_{[a,b)} f^{\frac{1}{r}} g^{\frac{1}{s}} \mathrm{d}\mu_\Delta\right]$$

and

$$\begin{aligned}
\mathfrak{J}_\Delta\left(\Phi, \frac{g}{f}\right) &= \int_{[a,b)} \Phi\left(\frac{g}{f}\right) \mathrm{d}\mu_\Delta - (b-a)\Phi\left(\frac{\int_{[a,b)} \frac{g}{f} \mathrm{d}\mu_\Delta}{b-a}\right) \\
&= rs\left[(b-a)^{1-\frac{1}{s}} \left(\int_{[a,b)} \frac{g}{f} \mathrm{d}\mu_\Delta\right)^{\frac{1}{s}} - \int_{[a,b)} \left(\frac{g}{f}\right)^{\frac{1}{s}} \mathrm{d}\mu_\Delta\right] \\
&= rs\left[(b-a)^{\frac{1}{r}} \left(\int_{[a,b)} \frac{g}{f} \mathrm{d}\mu_\Delta\right)^{\frac{1}{s}} - \int_{[a,b)} \left(\frac{g}{f}\right)^{\frac{1}{s}} \mathrm{d}\mu_\Delta\right].
\end{aligned}$$

If $r > 1$, then by substituting $\mathscr{J}_\Delta\left(\Phi, \frac{g}{f}, f\right)$ and $\mathfrak{J}_\Delta(\Phi, \frac{g}{f})$ in (3.6), we get (3.32). If $0 < r < 1$, then $rs < 0$, and since the expressions $\mathscr{J}_\Delta\left(\Phi, \frac{g}{f}, f\right)$ and $\mathfrak{J}_\Delta(\Phi, \frac{g}{f})$ contain the factor rs, we conclude that the inequalities in (3.32) hold in reverse order in that case. □

Remark 3.7 The first inequality in (3.32) gives a converse and the second one gives a refinement of Hölder's inequality on time scales.

Since Hölder's inequality can directly be deduced from Jensen's inequality in the case of two functions (see [103]), Corollary 3.2 also provides another class of refinements and conversions of Hölder's inequality.

Corollary 3.7 *Let $r, s \in \mathbb{R}$ such that $r > 0$ and $\frac{1}{r} + \frac{1}{s} = 1$. Further, assume that f, g are positive and Δ-integrable such that f attains its minimum value and its maximum value on its domain. Then*

$$\begin{aligned}
&\left[\max_{t\in[a,b)_\mathbb{T}} f(t)\right] \times \\
&\times \left[\left(\int_{[a,b)} f \mathrm{d}\mu_\Delta\right)^{s-1} \int_{[a,b)} \frac{g}{f} \mathrm{d}\mu_\Delta - \left(\frac{\int_{[a,b)} f \mathrm{d}\mu_\Delta}{b-a}\right)^{s-1} \left(\int_{[a,b)} \left(\frac{g}{f}\right)^{\frac{1}{s}} \mathrm{d}\mu_\Delta\right)^{s}\right] \\
&\geq \left[\left(\int_{[a,b)} f \mathrm{d}\mu_\Delta\right)^{\frac{1}{r}} \left(\int_{[a,b)} g \mathrm{d}\mu_\Delta\right)^{\frac{1}{s}}\right]^{s} - \left[\int_{[a,b)} f^{\frac{1}{r}} g^{\frac{1}{s}} \mathrm{d}\mu_\Delta\right]^{s} \\
&\geq \left[\min_{t\in[a,b)_\mathbb{T}} f(t)\right] \times \\
&\times \left[\left(\int_{[a,b)} f \mathrm{d}\mu_\Delta\right)^{s-1} \int_{[a,b)} \frac{g}{f} \mathrm{d}\mu_\Delta - \left(\frac{\int_{[a,b)} f \mathrm{d}\mu_\Delta}{b-a}\right)^{s-1} \left(\int_{[a,b)} \left(\frac{g}{f}\right)^{\frac{1}{s}} \mathrm{d}\mu_\Delta\right)^{s}\right].
\end{aligned}$$

Proof. In Corollary 3.2, replace f with $\left(\frac{g}{f}\right)^{\frac{1}{s}}$, p with f and let $\Phi(x)=\frac{x^s}{s(s-1)}$. Then Φ is convex on $(0,\infty)$. We get

$$\begin{aligned}
&\mathscr{J}_\Delta\left(\Phi,\left(\frac{g}{f}\right)^{\frac{1}{s}},f\right)\\
&=\int_{[a,b)} f\Phi\left(\left(\frac{g}{f}\right)^{\frac{1}{s}}\right)\mathrm{d}\mu_\Delta-\int_{[a,b)} f\mathrm{d}\mu_\Delta\Phi\left(\frac{\int_{[a,b)} f^{\frac{1}{r}}g^{\frac{1}{s}}\mathrm{d}\mu_\Delta}{\int_{[a,b)} f\mathrm{d}\mu_\Delta}\right)\\
&=\frac{1}{s(s-1)}\left[\int_{[a,b)} g\mathrm{d}\mu_\Delta-\left(\int_{[a,b)} f\mathrm{d}\mu_\Delta\right)^{1-s}\left(\int_{[a,b)} f^{\frac{1}{r}}g^{\frac{1}{s}}\mathrm{d}\mu_\Delta\right)^{s}\right]
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{J}_\Delta\left(\Phi,\left(\frac{g}{f}\right)^{\frac{1}{s}}\right)&=\int_{[a,b)}\Phi\left(\left(\frac{g}{f}\right)^{\frac{1}{s}}\right)\mathrm{d}\mu_\Delta-(b-a)\Phi\left(\frac{\int_{[a,b)}\left(\frac{g}{f}\right)^{\frac{1}{s}}\mathrm{d}\mu_\Delta}{b-a}\right)\\
&=\frac{1}{s(s-1)}\left[\int_{[a,b)}\frac{g}{f}\mathrm{d}\mu_\Delta-(b-a)^{1-s}\left(\int_{[a,b)}\left(\frac{g}{f}\right)^{\frac{1}{s}}\mathrm{d}\mu_\Delta\right)^{s}\right].
\end{aligned}$$

Now, the result follows immediately from (3.6). □

Remark 3.8 Similarly as in Chapter 2, we can apply the theory of isotonic linear functionals. The related results for isotonic linear functionals are given in [91].

Chapter 4

Jensen's Functionals for Several Variables, their Properties and Applications

In this chapter, we define the Jensen functional and related generalized means for several variables on time scales. We derive properties of Jensen functionals and apply them to generalized means. In this setting, we obtain generalizations, refinements, and conversions of many remarkable inequalities. The results presented in this chapter are taken from [22].

In the single-variable case, the obtained results coincide with the results given in Chapter 3. Moreover, we give all results for Lebesgue Δ-integrals, but they also hold for many other time scales integrals such as Cauchy, Riemann, Lebesgue, multiple Riemann, and multiple Lebesgue delta, nabla, and diamond-α time scales integrals in a similar way. We use the same notations as in Chapter 3.

4.1 Jensen's Inequality and Jensen's Functionals

Let $\mathbf{f}(t) = (f_1(t), \ldots, f_n(t))$ be an n-tuple of functions such that $f_1, \ldots, f_n$ are Δ-integrable on $[a,b)_{\mathbb{T}}$. Then $\int_{[a,b)} \mathbf{f} \mathrm{d}\mu_\Delta$ denotes the n-tuple

$$\left(\int_{[a,b)} f_1 \mathrm{d}\mu_\Delta, \ldots, \int_{[a,b)} f_n \mathrm{d}\mu_\Delta \right),$$

i.e., Δ-integral acts on each component of $\mathbf{f}$.

Theorem 4.1 *Assume $\Phi \in C(U,\mathbb{R})$ is convex, where $U \subseteq \mathbb{R}^n$ is a closed convex set. Suppose f_i, $i \in \{1,\dots,n\}$, are Δ-integrable on $[a,b)_{\mathbb{T}}$ such that $\mathbf{f}(t) = (f_1(t),\dots,f_n(t)) \in U$ for all $t \in [a,b)_{\mathbb{T}}$. Moreover, let $p : [a,b)_{\mathbb{T}} \to \mathbb{R}$ be nonnegative and Δ-integrable such that $\int_{[a,b)} p\mathrm{d}\mu_\Delta > 0$. Then*

$$\Phi\left(\frac{\int_{[a,b)} p\mathbf{f}\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}\right) \le \frac{\int_{[a,b)} p\Phi(\mathbf{f})\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}. \tag{4.1}$$

Proof. Since Φ is convex on $U \subset \mathbb{R}^n$, for every point $\mathbf{x}_0 \in U$ there exists a point $\lambda \in \mathbb{R}^n$ (see [119, Theorem 1.31]) such that

$$\Phi(\mathbf{x}) - \Phi(\mathbf{x}_0) \ge \langle \lambda, \mathbf{x} - \mathbf{x}_0 \rangle. \tag{4.2}$$

Let $\lambda = (\lambda_1,\dots,\lambda_n)$. By (4.2), we get

$$\begin{aligned}
\frac{\int_{[a,b)} p\Phi(\mathbf{f})\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta} - \Phi\left(\frac{\int_{[a,b)} p\mathbf{f}\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}\right) &= \frac{\int_{[a,b)} p\left\{\Phi(\mathbf{f}) - \Phi\left(\frac{\int_{[a,b)} p\mathbf{f}\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}\right)\right\}\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta} \\
&\ge \frac{\int_{[a,b)} p\left\langle \lambda, \mathbf{f} - \frac{\int_{[a,b)} p\mathbf{f}\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}\right\rangle \mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta} \\
&= \frac{\int_{[a,b)} p\sum_{i=1}^{n} \lambda_i\left(f_i - \frac{\int_{[a,b)} pf_i\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}\right)\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta} \\
&= 0,
\end{aligned}$$

and hence the proof is completed. □

Remark 4.1 By using the fact that the time scale integral is an isotonic linear functional, Theorem 4.1 can also be obtained from [119, Theorem 2.6].

Definition 4.1 (JENSEN'S FUNCTIONAL) *Assume $\Phi \in C(U,\mathbb{R})$, where $U \subseteq \mathbb{R}^n$ is a closed convex set. Suppose f_i, $i \in \{1,\dots,n\}$, are Δ-integrable functions on $[a,b)_{\mathbb{T}}$ such that $\mathbf{f}(t) = (f_1(t),\dots,f_n(t)) \in U$ for all $t \in [a,b)_{\mathbb{T}}$. Moreover, let $p : [a,b)_{\mathbb{T}} \to \mathbb{R}$ be nonnegative and Δ-integrable such that $\int_{[a,b)} p\mathrm{d}\mu_\Delta > 0$. Then we define the Jensen functional on time scales for several variables by*

$$\mathbf{J}_\Delta(\Phi,\mathbf{f},p) = \int_{[a,b)} p\Phi(\mathbf{f})\mathrm{d}\mu_\Delta - \int_{[a,b)} p\mathrm{d}\mu_\Delta \Phi\left(\frac{\int_{[a,b)} p\mathbf{f}\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}\right). \tag{4.3}$$

Remark 4.2 By Theorem 4.1, the following statements are obvious. If Φ is continuous and convex, then

$$\mathbf{J}_\Delta(\Phi,\mathbf{f},p) \ge 0,$$

while if Φ is continuous and concave, then

$$\mathbf{J}_\Delta(\Phi,\mathbf{f},p) \le 0.$$

Example 4.1 Let $[a,b)_{\mathbb{T}} = \{1,2,\ldots,n\}$, $f_1(i) = x_{1_i},\ldots,f_n(i) = x_{n_i}$, $p(i) = p_i$, $i \in \{1,\ldots,n\}$, in (4.3). Then Jensen's functional (4.3) becomes

$$\mathbf{J}_n(\Phi,\mathbf{X},\mathbf{p}) = \sum_{i=1}^{n} p_i\Phi(\mathbf{x}_i) - P_n\Phi\left(\frac{\sum_{i=1}^{n} p_i\mathbf{x}_i}{P_n}\right),$$

where $\mathbf{X} = (\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)$ with $\mathbf{x}_i = (x_{i_1},x_{i_2},\ldots,x_{i_n})$, $\mathbf{p} = (p_1,\ldots,p_n)$, and $P_n = \sum_{i=1}^{n} p_i > 0$.

Example 4.2 If $[a,b)_{\mathbb{T}} = [a,b)$, then Jensen's functional (4.3) becomes

$$\int_{[a,b)} p(t)\Phi(f_1(t),f_2(t),\ldots,f_n(t))\mathrm{d}\mu(t) - \int_{[a,b)} p(t)\mathrm{d}\mu(t)$$
$$\times\Phi\left(\frac{\int_{[a,b)} p(t)f_1(t)\mathrm{d}\mu(t)}{\int_{[a,b)} p(t)\mathrm{d}\mu(t)},\frac{\int_{[a,b)} p(t)f_2(t)\mathrm{d}\mu(t)}{\int_{[a,b)} p(t)\mathrm{d}\mu(t)},\ldots,\frac{\int_{[a,b)} p(t)f_n(t)\mathrm{d}\mu(t)}{\int_{[a,b)} p(t)\mathrm{d}\mu(t)}\right).$$

4.2 Properties of Jensen's Functionals

In the following theorem, we give our main result concerning the properties of the Jensen functional (4.3).

Theorem 4.2 *Assume $\Phi \in \mathrm{C}(U,\mathbb{R})$, where $U \subseteq \mathbb{R}^n$ is a closed convex set. Suppose f_i, $i \in \{1,\ldots,n\}$, are Δ-integrable on $[a,b)_{\mathbb{T}}$ such that $\mathbf{f}(t) = (f_1(t),\ldots,f_n(t)) \in U$ for all $t \in [a,b)_{\mathbb{T}}$. Let $p,q : [a,b)_{\mathbb{T}} \to \mathbb{R}$ be nonnegative and Δ-integrable such that $\int_{[a,b)} p\mathrm{d}\mu_\Delta > 0$ and $\int_{[a,b)} q\mathrm{d}\mu_\Delta > 0$. If Φ is convex, then $\mathbf{J}_\Delta(\Phi,\mathbf{f},\cdot)$ is superadditive, i.e.,*

$$\mathbf{J}_\Delta(\Phi,\mathbf{f},p+q) \geq \mathbf{J}_\Delta(\Phi,\mathbf{f},p) + \mathbf{J}_\Delta(\Phi,\mathbf{f},q), \tag{4.4}$$

and $\mathbf{J}_\Delta(\Phi,\mathbf{f},\cdot)$ is increasing, i.e., $p \geq q$ with $\int_{[a,b)} p\mathrm{d}\mu_\Delta > \int_{[a,b)} q\mathrm{d}\mu_\Delta$ implies

$$\mathbf{J}_\Delta(\Phi,\mathbf{f},p) \geq \mathbf{J}_\Delta(\Phi,\mathbf{f},q). \tag{4.5}$$

Moreover, if Φ is concave, then $\mathbf{J}_\Delta(\Phi,\mathbf{f},\cdot)$ is subadditive and decreasing, i.e., (4.4) *and* (4.5) *hold in reverse order.*

Proof. We omit the proof because it is similar to the proof of Theorem 3.2. □

Corollary 4.1 *Let $\Phi,\mathbf{f},p,q$ satisfy the hypotheses of Theorem 4.2. Further, suppose there exist nonnegative constants m and M such that*

$$Mq(t) \geq p(t) \geq mq(t) \quad \textit{for all} \quad t \in [a,b)_{\mathbb{T}}$$

and

$$M\int_{[a,b)} q\mathrm{d}\mu_\Delta > \int_{[a,b)} p\mathrm{d}\mu_\Delta > m\int_{[a,b)} q\mathrm{d}\mu_\Delta.$$

If Φ is convex, then

$$M\mathbf{J}_\Delta(\Phi,\mathbf{f},q) \geq \mathbf{J}_\Delta(\Phi,\mathbf{f},p) \geq m\mathbf{J}_\Delta(\Phi,\mathbf{f},q), \tag{4.6}$$

while if Φ is concave, then the inequalities in (4.6) *hold in reverse order.*

Proof. The proof is similar to the proof of Corollary 3.1. □

Corollary 4.2 *Let $\Phi,\mathbf{f},p$ satisfy the hypotheses of Theorem 4.2. Further, assume that p attains its minimum value and its maximum value on its domain. If Φ is convex, then*

$$\left[\max_{t\in[a,b)_{\mathbb{T}}} p(t)\right]\mathbf{j}_\Delta(\Phi,\mathbf{f}) \geq \mathbf{J}_\Delta(\Phi,\mathbf{f},p) \geq \left[\min_{t\in[a,b)_{\mathbb{T}}} p(t)\right]\mathbf{j}_\Delta(\Phi,\mathbf{f}), \tag{4.7}$$

where

$$\mathbf{j}_\Delta(\Phi,\mathbf{f}) = \int_{[a,b)} \Phi(\mathbf{f})\mathrm{d}\mu_\Delta - (b-a)\Phi\left(\frac{\int_{[a,b)}\mathbf{f}\mathrm{d}\mu_\Delta}{b-a}\right).$$

Moreover, if Φ is concave, then the inequalities in (4.7) *hold in reverse order.*

Proof. The proof is similar to the proof of Corollary 3.2. □

Example 4.3 Let the functional $\mathbf{J}_n(\Phi,\mathbf{X},\mathbf{p})$ be defined as in Example 4.1. Let $\mathbf{q} = (q_1,\ldots,q_n)$ with $q_i \geq 0$ and $\sum_{i=1}^{n} q_i = Q_n > 0$. If Φ is convex, then Theorem 4.2 implies that $\mathbf{J}_n(\Phi,\mathbf{X},.)$ is superadditive, i.e.,

$$\mathbf{J}_n(\Phi,\mathbf{X},\mathbf{p}+\mathbf{q}) \geq \mathbf{J}_n(\Phi,\mathbf{X},\mathbf{p}) + \mathbf{J}_n(\Phi,\mathbf{X},\mathbf{q}), \tag{4.8}$$

and that $\mathbf{J}_n(\Phi,\mathbf{X},.)$ is increasing, i.e., if $\mathbf{p} \geq \mathbf{q}$ such that $P_n > Q_n$, then

$$\mathbf{J}_n(\Phi,\mathbf{X},\mathbf{p}) \geq \mathbf{J}_n(\Phi,\mathbf{X},\mathbf{q}). \tag{4.9}$$

Moreover, if Φ is concave, then the inequalities in (4.8) and (4.9) hold in reverse order. If $\mathbf{p}$ attains minimum and maximum value on its domain, then Corollary 4.2 yields

$$\max_{1\leq i\leq n}\{p_i\}\mathbf{j}_n(\Phi,\mathbf{X}) \geq \mathbf{J}_n(\Phi,\mathbf{X},\mathbf{p}) \geq \min_{1\leq i\leq n}\{p_i\}\mathbf{j}_n(\Phi,\mathbf{X}), \tag{4.10}$$

where

$$\mathbf{j}_n(\Phi,\mathbf{X}) = \sum_{i=1}^{n}\Phi(\mathbf{x}_i) - n\Phi\left(\frac{\sum_{i=1}^{n}\mathbf{x}_i}{n}\right),$$

if Φ is convex. Further, the inequalities in (4.10) hold in reverse order if Φ is concave.

4.3 Applications to Weighted Generalized Means

We start this section by applying the obtained results on the properties of Jensen's functionals to weighted generalized means. In the sequel, $U \subseteq \mathbb{R}^n$ is closed and convex.

Definition 4.2 *Assume $\chi \in \mathrm{C}(I,\mathbb{R})$ is strictly monotone and $\varphi : U \to I$ is a function of n variables. Suppose f_i, $i \in \{1,\dots,n\}$, are Δ-integrable on $[a,b)_{\mathbb{T}}$ such that $\mathbf{f}(t) = (f_1(t),\dots,f_n(t)) \in U$ for all $t \in [a,b)_{\mathbb{T}}$. Let $p : [a,b)_{\mathbb{T}} \to \mathbb{R}$ be a nonnegative Δ-integrable function such that $p\chi(\varphi(\mathbf{f}))$ is Δ-integrable and $\int_{[a,b)} p\mathrm{d}\mu_\Delta > 0$. Then we define the weighted generalized mean on time scales by*

$$\mathbf{M}_\Delta(\chi,\varphi(\mathbf{f}),p) = \chi^{-1}\left(\frac{\int_{[a,b)} p\chi(\varphi(\mathbf{f}))\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}\right). \tag{4.11}$$

Theorem 4.3 *Assume $\chi, \psi_i \in \mathrm{C}(I,\mathbb{R})$, $i \in \{1,\dots,n\}$, are strictly monotone and $\varphi : U \to I \subseteq \mathbb{R}$ is a function of n variables. Suppose $f_i : [a,b)_{\mathbb{T}} \to I$, $i \in \{1,\dots,n\}$, are Δ-integrable such that $\mathbf{f}(t) = (f_1(t),\dots,f_n(t)) \in U$ for all $t \in [a,b)_{\mathbb{T}}$. Let $p,q : [a,b)_{\mathbb{T}} \to \mathbb{R}$ be nonnegative and Δ-integrable such that $p\chi(\varphi(\mathbf{f})), q\chi(\varphi(\mathbf{f})), p\psi_i(f_i)), q\psi_i(f_i))$, $i \in \{1,\dots,n\}$, are Δ-integrable and $\int_{[a,b)} p\mathrm{d}\mu_\Delta > 0$, $\int_{[a,b)} q\mathrm{d}\mu_\Delta > 0$. If H defined by*

$$H(s_1,\dots,s_n) = (\chi\circ\varphi)\left(\psi_1^{-1}(s_1),\dots,\psi_n^{-1}(s_n)\right)$$

is convex, then the functional

$$\int_{[a,b)} p\mathrm{d}\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi,\varphi(\mathbf{f}),p)\right) - (\chi\circ\varphi)\left(\mathbf{M}_\Delta(\psi_1,f_1,p),\dots,\mathbf{M}_\Delta(\psi_n,f_n,p)\right)\right] \tag{4.12}$$

satisfies

$$\begin{aligned}\int_{[a,b)} &(p+q)\mathrm{d}\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi,\varphi(\mathbf{f}),p+q)\right)\right.\\ &\qquad\qquad \left.-(\chi\circ\varphi)\left(\mathbf{M}_\Delta(\psi_1,f_1,p+q),\dots,\mathbf{M}_\Delta(\psi_n,f_n,p+q)\right)\right]\\ &\geq \int_{[a,b)} p\mathrm{d}\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi,\varphi(\mathbf{f}),p)\right) - (\chi\circ\varphi)\left(\mathbf{M}_\Delta(\psi_1,f_1,p),\dots,\mathbf{M}_\Delta(\psi_n,f_n,p)\right)\right]\\ &+ \int_{[a,b)} q\mathrm{d}\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi,\varphi(\mathbf{f}),q)\right) - (\chi\circ\varphi)\left(\mathbf{M}_\Delta(\psi_1,f_1,q),\dots,\mathbf{M}_\Delta(\psi_n,f_n,q)\right)\right].\end{aligned} \tag{4.13}$$

If $p \geq q$ with $\int_{[a,b)} p\mathrm{d}\mu_\Delta > \int_{[a,b)} q\mathrm{d}\mu_\Delta$, then

$$\begin{aligned}&\int_{[a,b)} p\mathrm{d}\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi,\varphi(\mathbf{f}),p)\right) - (\chi\circ\varphi)\left(\mathbf{M}_\Delta(\psi_1,f_1,p),\dots,\mathbf{M}_\Delta(\psi_n,f_n,p)\right)\right]\\ &\geq \int_{[a,b)} q\mathrm{d}\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi,\varphi(\mathbf{f}),q)\right) - (\chi\circ\varphi)\left(\mathbf{M}_\Delta(\psi_1,f_1,q),\dots,\mathbf{M}_\Delta(\psi_n,f_n,q)\right)\right].\end{aligned} \tag{4.14}$$

Moreover, if H is concave, then (4.13) *and* (4.14) *hold in reverse order, i.e., the functional*

$$\int_{[a,b)} \cdot \mathrm{d}\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi,\varphi(\mathbf{f}),\cdot)\right) - (\chi\circ\varphi)\left(\mathbf{M}_\Delta(\psi_1,f_1,\cdot),\ldots,\mathbf{M}_\Delta(\psi_n,f_n,\cdot)\right)\right]$$

is subadditive and decreasing.

Proof. The functional defined in (4.12) is obtained by replacing Φ with H and f_i with $\psi_i(f_i)$, $i\in\{1,\ldots,n\}$, in the Jensen functional (4.3), and letting $\Psi(\mathbf{f}) = (\psi_1(f_1),\ldots,\psi_n(f_n))$, i.e.,

$$\begin{aligned}
&\mathbf{J}_\Delta(H,\Psi(\mathbf{f}),p)\\
&= \int_{[a,b)} p(\chi\circ\varphi)(f_1,\ldots,f_n)\mathrm{d}\mu_\Delta\\
&\quad - \int_{[a,b)} p\mathrm{d}\mu_\Delta H\left(\frac{\int_{[a,b)} p\psi_1(f_1)\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta},\ldots,\frac{\int_{[a,b)} p\psi_n(f_n)\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}\right)\\
&= \int_{[a,b)} p\mathrm{d}\mu_\Delta \chi(\mathbf{M}_\Delta(\chi,\varphi(\mathbf{f}),p))\\
&\quad - \int_{[a,b)} p\mathrm{d}\mu_\Delta (\chi\circ\varphi)\left(\mathbf{M}_\Delta(\psi_1,f_1,p),\ldots,\mathbf{M}_\Delta(\psi_n,f_n,p)\right)\\
&= \int_{[a,b)} p\mathrm{d}\mu_\Delta \left[\chi(\mathbf{M}_\Delta(\chi,\varphi(\mathbf{f}),p)) - (\chi\circ\varphi)\left(\mathbf{M}_\Delta(\psi_1,f_1,p),\ldots,\mathbf{M}_\Delta(\psi_n,f_n,p)\right)\right].
\end{aligned}$$

Now, all claims follow immediately from Theorem 4.2. □

Corollary 4.3 *Let $H,\varphi,\mathbf{f},p,\chi,f_i,\psi_i$, $i\in\{1,\ldots,n\}$, satisfy the hypotheses of Theorem 4.3. Further, assume that p attains its minimum value and its maximum value on its domain. If H is convex, then*

$$\begin{aligned}
&\left[\max_{t\in[a,b)_\mathbb{T}} p(t)\right](b-a) \qquad (4.15)\\
&\quad \times \left[\chi(\mathbf{m}_\Delta(\chi,\varphi(\mathbf{f}))) - (\chi\circ\varphi)\left(\mathbf{m}_\Delta(\psi_1,f_1),\ldots,\mathbf{m}_\Delta(\psi_n,f_n)\right)\right]\\
&\geq \int_{[a,b)} p\mathrm{d}\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi,\varphi(\mathbf{f}),p)\right) - (\chi\circ\varphi)\left(\mathbf{M}_\Delta(\psi_1,f_1,p),\ldots,\mathbf{M}_\Delta(\psi_n,f_n,p)\right)\right]\\
&\geq \times \left[\min_{t\in[a,b)_\mathbb{T}} p(t)\right](b-a)\\
&\quad \left[\chi(\mathbf{m}_\Delta(\chi,\varphi(\mathbf{f}))) - (\chi\circ\varphi)\left(\mathbf{m}_\Delta(\psi_1,f_1),\ldots,\mathbf{m}_\Delta(\psi_n,f_n)\right)\right].
\end{aligned}$$

where

$$\mathbf{m}_\Delta(\chi,\varphi(\mathbf{f})) = \chi^{-1}\left(\frac{\int_{[a,b)} \chi(\varphi(\mathbf{f}))\mathrm{d}\mu_\Delta}{b-a}\right). \qquad (4.16)$$

Moreover, if H is concave, then the inequalities in (4.15) *hold in reverse order.*

Proof. The proof is similar to the proof of Corollary 3.2. □

Remark 4.3 If we take the discrete form of weighted generalized mean (4.11) with $\int_{[a,b)} p \mathrm{d}\mu_\Delta = 1$, then we obtain the quasi-arithmetic mean. Namely, let $\psi : I \subseteq \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be a continuous and strictly monotone function, $\mathbf{a} = (a_1, \ldots, a_n)$ with $a_k \in I$, $k \in \{1, \ldots, n\}$, and $\mathbf{w} = (w_1, \ldots, w_n)$ with $w_k \geq 0$ and $\sum_{k=1}^{n} w_k = 1$. Then the quasi-arithmetic mean of $\mathbf{a}$ with weight $\mathbf{w}$ is defined by

$$\mathbf{M}_n = \psi^{-1}\left(\sum_{k=1}^{n} w_k \psi(a_k)\right). \tag{4.17}$$

Now the following examples connects the quasi-arithmetic mean (4.17) and the properties of Jensen functionals.

Example 4.4 (SEE [92, COROLLARY 3]) Let $\mathbf{w}$ and ψ be defined as in Remark 4.3 such that ψ is a strictly increasing, strictly convex function with continuous derivatives of second order and $\frac{\psi'}{\psi''}$ is a concave function. Further, let $\mathbf{X}, \mathbf{p}, \mathbf{x_i}$, $i \in \{1, \ldots, n\}$, be defined as in Example 4.1, and $\mathbf{q} = (q_1, \ldots, q_n)$ with $q_i \geq 0$, $i \in \{1, \ldots, n\}$, and $\sum_{i=1}^{n} q_i = Q_n > 0$. Then, $\Phi_{M_n}(\mathbf{x_i}) = \psi^{-1}\left(\sum_{k=1}^{n} w_k \psi(x_{i_k})\right)$ is a convex function (see [103, Theorem 1]). Hence by Theorem 4.2, the functional

$$\mathbf{J}_n(\Phi_{\mathbf{M}_n}, \mathbf{X}, \mathbf{p}) = \sum_{i=1}^{n} p_i \Phi_{\mathbf{M}_n}(\mathbf{x_i}) - P_n \Phi_{\mathbf{M}_n}\left(\frac{\sum_{i=1}^{n} p_i \mathbf{x_i}}{P_n}\right)$$

is superadditive, i.e.,

$$\mathbf{J}_n(\Phi_{\mathbf{M}_n}, \mathbf{X}, \mathbf{p} + \mathbf{q}) \geq \mathbf{J}_n(\Phi_{\mathbf{M}_n}, \mathbf{X}, \mathbf{p}) + \mathbf{J}_n(\Phi_{\mathbf{M}_n}, \mathbf{X}, \mathbf{q}),$$

and increasing, i.e., if $\mathbf{p} \geq \mathbf{q}$ such that $P_n > Q_n$, then

$$\mathbf{J}_n(\Phi_{M_n}, \mathbf{X}, \mathbf{p}) \geq \mathbf{J}_n(\Phi_{\mathbf{M}_n}, \mathbf{X}, \mathbf{q}).$$

Also, by Corollary 4.2, we have

$$\max_{1 \leq i \leq n}\{p_i\} \mathbf{j}_n(\Phi_{\mathbf{M}_n}, \mathbf{X}) \geq \mathbf{J}_n(\Phi_{M_n}, \mathbf{X}, \mathbf{p}) \geq \min_{1 \leq i \leq n}\{p_i\} \mathbf{j}_n(\Phi_{\mathbf{M}_n}, \mathbf{X}),$$

where

$$\mathbf{j}_n(\Phi_{\mathbf{M}_n}, \mathbf{X}) = \sum_{i=1}^{n} \Phi_{\mathbf{M}_n}(\mathbf{x_i}) - n \Phi_{\mathbf{M}_n}\left(\frac{\sum_{i=1}^{n} \mathbf{x_i}}{n}\right).$$

Example 4.5 (SEE [92, COROLLARY 4]) Consider (4.17), but with different conditions on ψ and $\mathbf{w}$. Namely, if

(i) $w_i \geq 1$, for $i \in \{1,\ldots,n\}$;

(ii) $\psi : \mathbb{R}^+ \to \mathbb{R}^+$;

(iii) $\lim_{x\to 0} \psi(x) = +\infty$ or $\lim_{x\to\infty} \psi(x) = +\infty$,

then we have the following definition:

$$\widetilde{\mathbf{M}}_n = \psi^{-1}\left(\sum_{k=1}^{n} w_k \psi(a_k)\right).$$

Let $\mathbf{X}, \mathbf{p}, \mathbf{x_i},\ i \in \{1,\ldots,n\}$, be defined as in Example 4.1 and $\mathbf{q} = (q_1,\ldots,q_n)$ with $q_i \geq 0$ and $\sum_{i=1}^{n} q_i = Q_n > 0$. Let ψ be strictly increasing and strictly convex with continuous derivatives of second order, such that $\frac{\psi}{\psi'}$ is convex. Then $\Phi_{\widetilde{\mathbf{M}}_n}(\mathbf{x_i}) = \psi^{-1}\left(\sum_{k=1}^{n} w_k \psi(x_{i_k})\right)$ is a convex function (see [103, Theorem 2]). Hence by Theorem 4.2, the functional

$$\mathbf{J}_n(\Phi_{\widetilde{\mathbf{M}}_n}, \mathbf{X}, \mathbf{p}) = \sum_{i=1}^{n} p_i \Phi_{\widetilde{\mathbf{M}}_n}(x_{i_k}) - P_n \Phi_{\widetilde{\mathbf{M}}_n}\left(\frac{\sum_{i=1}^{n} p_i x_{i_k}}{P_n}\right)$$

is superadditive, i.e.,

$$\mathbf{J}_n(\Phi_{\widetilde{\mathbf{M}}_n}, \mathbf{X}, \mathbf{p}+\mathbf{q}) \geq \mathbf{J}_n(\Phi_{\widetilde{M}_n}, \mathbf{X}, \mathbf{p}) + \mathbf{J}_n(\Phi_{\widetilde{\mathbf{M}}_n}, \mathbf{X}, \mathbf{q}),$$

and increasing, i.e., if $p \geq q$, then

$$\mathbf{J}_n(\Phi_{\widetilde{\mathbf{M}}_n}, \mathbf{X}, \mathbf{p}) \geq \mathbf{J}_n(\Phi_{\widetilde{\mathbf{M}}_n}, \mathbf{X}, \mathbf{q}).$$

Also, by Corollary 4.2, we have

$$\max_{1\leq i\leq n}\{p_i\}\mathbf{j}_n(\Phi_{\widetilde{\mathbf{M}}_n}, \mathbf{X}) \geq \mathbf{J}_n(\Phi_{\mathbf{M}_n}, \mathbf{X}, \mathbf{p}) \geq \min_{1\leq i\leq n}\{p_i\}\mathbf{j}_n(\Phi_{\widetilde{\mathbf{M}}_n}, \mathbf{X}),$$

where

$$\mathbf{j}_n(\Phi_{\widetilde{\mathbf{M}}_n}, \mathbf{X}) = \sum_{i=1}^{n} \Phi_{\widetilde{\mathbf{M}}_n}(x_{i_k}) - n\Phi_{\widetilde{\mathbf{M}}_n}\left(\frac{\sum_{i=1}^{n} x_{i_k}}{n}\right).$$

Example 4.6 (SEE [92, COROLLARY 5]) Let $\mathbf{X}, \mathbf{p}, \mathbf{x_i},\ i \in \{1,\ldots,n\}$, be defined as in Example 4.1 and $\mathbf{q} = (q_1,\ldots,q_n)$, with $q_i \geq 0$ and $\sum_{i=1}^{n} q_i = Q_n > 0$. Let $\varphi : I \to \mathbb{R}$ be a

$(n+1)$-convex function, where I is a closed and bounded interval in $\mathbb{R}$. Then by Theorem 4.2, for $\Phi_G(\mathbf{x_i}) = [x_{i_1},\ldots,x_{i_n};\varphi]$, the functional

$$\mathbf{J}_n(\Phi_G,\mathbf{X},\mathbf{p}) = \sum_{i=1}^{n} p_i\Phi_G(\mathbf{x_i}) - P_n\Phi_G\left(\frac{\sum_{i=1}^{n} p_i\mathbf{x_i}}{P_n}\right)$$

is superadditive, i.e.,

$$\mathbf{J}_n(\Phi_G,\mathbf{X},\mathbf{p}+\mathbf{q}) \geq \mathbf{J}_n(\Phi_G,\mathbf{X},\mathbf{p}) + \mathbf{J}_n(\Phi_G,\mathbf{X},\mathbf{q}),$$

and increasing, i.e., if $\mathbf{p} \geq \mathbf{q}$ such that $P_n > Q_n$, then

$$\mathbf{J}_n(\Phi_G,\mathbf{X},\mathbf{p}) \geq \mathbf{J}_n(\Phi_G,\mathbf{X},\mathbf{q}).$$

Also, by Corollary 4.2, we have

$$\max_{1\leq i\leq n}\{p_i\}\mathbf{j}_n(\Phi_G,\mathbf{X}) \geq \mathbf{J}_n(\Phi_G,\mathbf{X},\mathbf{p}) \geq \min_{1\leq i\leq n}\{p_i\}\mathbf{j}_n(\Phi_G,\mathbf{X}),$$

where

$$\mathbf{j}_n(\Phi_G,\mathbf{X}) = \sum_{i=1}^{n}\Phi_G(\mathbf{x}_i) - n\Phi_G\left(\frac{\sum_{i=1}^{n}\mathbf{x}_i}{n}\right).$$

4.4 Applications to Additive and Multiplicative Type Inequalities

In this section we give some applications of Theorem 4.2 to additive and multiplicative type mean inequalities.

Corollary 4.4 *Assume $\chi,\psi_1,\psi_2 \in \mathrm{C}^2(I,\mathbb{R})$ are strictly monotone. Suppose $f_1,f_2 : [a,b)_{\mathbb{T}} \to I$ are Δ-integrable such that $f_1(t)+f_2(t) \in I$ for all $t \in [a,b)_{\mathbb{T}}$. Let $p,q : [a,b)_{\mathbb{T}} \to \mathbb{R}$ be non-negative and Δ-integrable such that $p\chi(f_1+f_2), q\chi(f_1+f_2), p\psi_i(f_i)), q\psi_i(f_i)),\ i=1,2,$ are Δ-integrable and $\int_{[a,b)} p\mathrm{d}\mu_\Delta > 0$, $\int_{[a,b)} q\mathrm{d}\mu_\Delta > 0$. Furthermore, let*

$$E = \frac{\psi_1'}{\psi_1''}, \quad F = \frac{\psi_2'}{\psi_2''}, \quad G = \frac{\chi'}{\chi''}.$$

If ψ_1',ψ_2',χ' are positive and ψ_1'',ψ_2'',χ'' are negative, then the functional

$$\int_{[a,b)} p\mathrm{d}\mu_\Delta\left[\chi\left(\mathbf{M}_\Delta(\chi,f_1+f_2,p)\right) - \chi\left(\mathbf{M}_\Delta(\psi_1,f_1,p)+\mathbf{M}_\Delta(\psi_2,f_2,p)\right)\right] \tag{4.18}$$

is superadditive, i.e.,

$$\begin{aligned}
&\int_{[a,b)}(p+q)\mathrm{d}\mu_\Delta\left[\chi\left(\mathbf{M}_\Delta(\chi,f_1+f_2,p+q)\right)\right. \\
&\qquad\left.-\chi\left(\mathbf{M}_\Delta(\psi_1,f_1,p+q)+\mathbf{M}_\Delta(\psi_2,f_2,p+q)\right)\right] \\
&\geq\int_{[a,b)}p\mathrm{d}\mu_\Delta\left[\chi\left(\mathbf{M}_\Delta(\chi,f_1+f_2,p)\right)-\chi\left(\mathbf{M}_\Delta(\psi_1,f_1,p)+\mathbf{M}_\Delta(\psi_2,f_2,p)\right)\right] \\
&\quad+\int_{[a,b)}q\mathrm{d}\mu_\Delta\left[\chi\left(\mathbf{M}_\Delta(\chi,f_1+f_2,q)\right)-\chi\left(\mathbf{M}_\Delta(\psi_1,f_1,q)+\mathbf{M}_\Delta(\psi_2,f_2,q)\right)\right].
\end{aligned}\tag{4.19}$$

If $p\geq q$ such that $\int_{[a,b)}p\mathrm{d}\mu_\Delta>\int_{[a,b)}q\mathrm{d}\mu_\Delta$, then

$$\begin{aligned}
&\int_{[a,b)}p\mathrm{d}\mu_\Delta\left[\chi\left(\mathbf{M}_\Delta(\chi,f_1+f_2,p)\right)-\chi\left(\mathbf{M}_\Delta(\psi_1,f_1,p)+\mathbf{M}_\Delta(\psi_2,f_2,p)\right)\right] \\
&\quad\geq\int_{[a,b)}q\mathrm{d}\mu_\Delta\left[\chi\left(\mathbf{M}_\Delta(\chi,f_1+f_2,q)\right)-\chi\left(\mathbf{M}_\Delta(\psi_1,f_1,q)+\mathbf{M}_\Delta(\psi_2,f_2,q)\right)\right]
\end{aligned}\tag{4.20}$$

if and only if $G(x+y)\leq E(x)+F(y)$. If p attains its minimum and maximum value on its domain $[a,b)_{\mathbb{T}}$, then (4.20) *yields*

$$\begin{aligned}
&\left[\max_{t\in[a,b)_{\mathbb{T}}}p(t)\right](b-a)\left[\chi\left(\mathbf{m}_\Delta(\chi,f_1+f_2)\right)-\chi\left(\mathbf{m}_\Delta(\psi_1,f_1)+\mathbf{m}_\Delta(\psi_2,f_2)\right)\right] \\
&\geq\int_{[a,b)}p\mathrm{d}\mu_\Delta\left[\chi\left(\mathbf{M}_\Delta(\chi,f_1+f_2,p)\right)-\chi\left(\mathbf{M}_\Delta(\psi_1,f_1,p)+\mathbf{M}_\Delta(\psi_2,f_2,p)\right)\right] \\
&\geq\left[\min_{t\in[a,b)_{\mathbb{T}}}p(t)\right](b-a)\left[\chi\left(\mathbf{m}_\Delta(\chi,f_1+f_2)\right)-\chi\left(\mathbf{m}_\Delta(\psi_1,f_1)+\mathbf{m}_\Delta(\psi_2,f_2)\right)\right].
\end{aligned}\tag{4.21}$$

Moreover, if $\psi_1',\psi_2',\chi',\psi_1'',\psi_2'',\chi''$ are all positive, then the inequalities in (4.19), (4.20), *and* (4.21) *are reversed if and only if $G(x+y)\geq E(x)+F(y)$.*

Proof. Let $n=2$ in Theorem 4.3. By setting $\varphi(x,y)=x+y$, we have

$$H(s_1,s_2)=\chi(\psi_1^{-1}(s_1)+\psi_2^{-1}(s_2)).$$

If ψ_1',ψ_2',χ' are positive and ψ_1'',ψ_2'',χ'' are negative, then H is convex if and only if $G(x+y)\leq E(x)+F(y)$ (see [29]). If $\psi_1',\psi_2',\chi',\psi_1'',\psi_2'',\chi''$ are all positive, then H is concave if and only if $G(x+y)\geq E(x)+F(y)$ (see [29]). Now, all claims follow immediately from Theorem 4.3. □

Corollary 4.5 *Assume $\chi,\psi_1,\psi_2\in C^2(I,\mathbb{R})$ are strictly monotone. Suppose $f_1,f_2:[a,b)_{\mathbb{T}}\to I$ are Δ-integrable such that $f_1(t)f_2(t)\in I$ for all $t\in[a,b)_{\mathbb{T}}$. Let $p,q:[a,b)_{\mathbb{T}}\to\mathbb{R}$ be nonnegative and Δ-integrable such that $p\chi(f_1\cdot f_2),q\chi(f_1\cdot f_2),p\psi_i(f_i)),q\psi_i(f_i))$, $i=1,2$, are Δ-integrable and $\int_{[a,b)}p\mathrm{d}\mu_\Delta>0$, $\int_{[a,b)}q\mathrm{d}\mu_\Delta>0$. Furthermore, let*

$$A(t)=\frac{\psi_1'(t)}{\psi_1'(t)+t\psi_1''(t)},\quad B(t)=\frac{\psi_2'(t)}{\psi_2'(t)+t\psi_2''(t)},\quad C(t)=\frac{\chi'(t)}{\chi'(t)+t\chi''(t)}.$$

If ψ_1', ψ_2', χ' are positive and A, B, C are negative, then the functional

$$\int_{[a,b)} p\mathrm{d}\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi, f_1 \cdot f_2, p)\right) - \chi\left(\mathbf{M}_\Delta(\psi_1, f_1, p) \cdot \mathbf{M}_\Delta(\psi_2, f_2, p)\right)\right] \tag{4.22}$$

is superadditive, i.e.,

$$\begin{aligned}
&\int_{[a,b)} (p+q)\mathrm{d}\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi, f_1 f_2, p+q)\right)\right. \\
&\quad \left. -\chi\left(\mathbf{M}_\Delta(\psi_1, f_1, p+q)\mathbf{M}_\Delta(\psi_2, f_2, p+q)\right)\right] \\
&\geq \int_{[a,b)} p\mathrm{d}\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi, f_1 f_2, p)\right) - \chi\left(\mathbf{M}_\Delta(\psi_1, f_1, p)\mathbf{M}_\Delta(\psi_2, f_2, p)\right)\right] \\
&\quad + \int_{[a,b)} q\mathrm{d}\mu_\Delta \left[\chi\left(M(\chi, f_1 f_2, q)\right) - \chi\left(\mathbf{M}_\Delta(\psi_1, f_1, q)\mathbf{M}_\Delta(\psi_2, f_2, q)\right)\right].
\end{aligned} \tag{4.23}$$

If $p \geq q$ such that $\int_{[a,b)} p\mathrm{d}\mu_\Delta > \int_{[a,b)} q\mathrm{d}\mu_\Delta$, then

$$\begin{aligned}
&\int_{[a,b)} p\mathrm{d}\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi, f_1 f_2, p)\right) - \chi\left(\mathbf{M}_\Delta(\psi_1, f_1, p)\mathbf{M}_\Delta(\psi_2, f_2, p)\right)\right] \\
&\qquad \geq \int_{[a,b)} q\mathrm{d}\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi, f_1 f_2, q)\right) - \chi\left(\mathbf{M}_\Delta(\psi_1, f_1, q)\mathbf{M}_\Delta(\psi_2, f_2, q)\right)\right]
\end{aligned} \tag{4.24}$$

if and only if $C(xy) \leq A(x) + B(y)$. If p attains its minimum and maximum value on its domain $[a,b)_{\mathbb{T}}$, then (4.24) *yields*

$$\begin{aligned}
&\left[\max_{t \in [a,b)_{\mathbb{T}}} p(t)\right] (b-a) \left[\chi\left(\mathbf{m}_\Delta(\chi, f_1 f_2)\right) - \chi\left(\mathbf{m}_\Delta(\psi_1, f_1)\mathbf{m}_\Delta(\psi_2, f_2)\right)\right] \\
&\geq \int_{[a,b)} p\mathrm{d}\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi, f_1 f_2, p)\right) - \chi\left(\mathbf{M}_\Delta(\psi_1, f_1, p)\mathbf{M}_\Delta(\psi_2, f_2, p)\right)\right] \\
&\geq \left[\min_{t \in [a,b)_{\mathbb{T}}} p(t)\right] (b-a) \left[\chi\left(\mathbf{m}_\Delta(\chi, f_1 f_2)\right) - \chi\left(\mathbf{m}_\Delta(\psi_1, f_1)\mathbf{m}_\Delta(\psi_2, f_2)\right)\right].
\end{aligned} \tag{4.25}$$

If $\psi_1', \psi_2', \chi', A, B, C$ are all positive, then the inequalities in (4.23)*,* (4.24)*, and* (4.25) *are reversed if and only if $C(xy) \geq A(x) + B(y)$.*

Proof. Let $n = 2$ in Theorem 4.3. By setting $\varphi(x,y) = xy$, we have

$$H(s_1, s_2) = \chi(\psi_1^{-1}(s_1)\psi_2^{-1}(s_2)).$$

If ψ_1', ψ_2', χ' are positive and A, B, C are negative, then H is convex if and only if $C(xy) \leq A(x) + B(y)$. If $\psi_1', \psi_2', \chi', A, B, C$ are all positive, then H is concave if and only if $C(xy) \geq A(x) + B(y)$ (see [29]). Now, all claims follow immediately from Theorem 4.3. $\square$

Corollary 4.6 *Let $\lambda, \omega, \nu \in \mathbb{R}$ be such that*

(a) $\lambda < 0 < \omega, \nu$, *or* $\omega, \nu < 0 < \lambda$;

(b) $\lambda < \omega, \nu < 0$, *or* $\nu < 0 < \omega < \lambda$, *or* $\omega < 0 < \nu < \lambda$, *for* $\frac{1}{\lambda} \leq \frac{1}{\omega} + \frac{1}{\nu}$;

(c) $\lambda < \omega < 0 < \nu$, *or* $\lambda < \nu < 0 < \omega$, *for* $\frac{1}{\lambda} \geq \frac{1}{\omega} + \frac{1}{\nu}$.

Suppose $f_1, f_2 : [a,b)_{\mathbb{T}} \to \mathbb{R}$ are Δ-integrable. Let $p, q : [a,b)_{\mathbb{T}} \to \mathbb{R}$ be nonnegative and Δ-integrable such that $pf_1^\lambda f_2^\lambda$, $qf_1^\lambda f_2^\lambda$, $pf_1^\omega, qf_1^\omega, pf_2^\nu, qf_2^\nu$ are Δ-integrable and $\int_{[a,b)} p\mathrm{d}\mu_\Delta > 0$, $\int_{[a,b)} q\mathrm{d}\mu_\Delta > 0$. Then the functional

$$\int_{[a,b)} pf_1^\lambda f_2^\lambda \mathrm{d}\mu_\Delta - \int_{[a,b)} p\mathrm{d}\mu_\Delta \left[\left(\frac{\int_{[a,b)} pf_1^\omega \mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta} \right)^{\frac{1}{\omega}} \left(\frac{\int_{[a,b)} pf_2^\nu \mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta} \right)^{\frac{1}{\nu}} \right]^\lambda \tag{4.26}$$

is superadditive, i.e.,

$$\begin{aligned}
&\int_{[a,b)} (p+q) f_1^\lambda f_2^\lambda \mathrm{d}\mu_\Delta \\
&\quad - \int_{[a,b)} (p+q)\mathrm{d}\mu_\Delta \left[\left(\frac{\int_{[a,b)} (p+q) f_1^\omega \mathrm{d}\mu_\Delta}{\int_{[a,b)} (p+q)\mathrm{d}\mu_\Delta} \right)^{\frac{1}{\omega}} \left(\frac{\int_{[a,b)} (p+q) f_2^\nu \mathrm{d}\mu_\Delta}{\int_{[a,b)} (p+q)\mathrm{d}\mu_\Delta} \right)^{\frac{1}{\nu}} \right]^\lambda \\
&\geq \int_{[a,b)} pf_1^\lambda f_2^\lambda \mathrm{d}\mu_\Delta - \int_{[a,b)} p\mathrm{d}\mu_\Delta \left[\left(\frac{\int_{[a,b)} pf_1^\omega \mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta} \right)^{\frac{1}{\omega}} \left(\frac{\int_{[a,b)} pf_2^\nu \mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta} \right)^{\frac{1}{\nu}} \right]^\lambda \\
&+ \int_{[a,b)} qf_1^\lambda f_2^\lambda \mathrm{d}\mu_\Delta - \int_{[a,b)} q\mathrm{d}\mu_\Delta \left[\left(\frac{\int_{[a,b)} qf_1^\omega \mathrm{d}\mu_\Delta}{\int_{[a,b)} q\mathrm{d}\mu_\Delta} \right)^{\frac{1}{\omega}} \left(\frac{\int_{[a,b)} qf_2^\nu \mathrm{d}\mu_\Delta}{\int_{[a,b)} q\mathrm{d}\mu_\Delta} \right)^{\frac{1}{\nu}} \right]^\lambda .
\end{aligned} \tag{4.27}$$

If $p \geq q$ such that $\int_{[a,b)} p\mathrm{d}\mu_\Delta > \int_{[a,b)} q\mathrm{d}\mu_\Delta$, then

$$\begin{aligned}
&\int_{[a,b)} pf_1^\lambda f_2^\lambda \mathrm{d}\mu_\Delta - \int_{[a,b)} p\mathrm{d}\mu_\Delta \left[\left(\frac{\int_{[a,b)} pf_1^\omega \mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta} \right)^{\frac{1}{\omega}} \left(\frac{\int_{[a,b)} pf_2^\nu \mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta} \right)^{\frac{1}{\nu}} \right]^\lambda \\
&\geq \int_{[a,b)} qf_1^\lambda f_2^\lambda \mathrm{d}\mu_\Delta - \int_{[a,b)} q\mathrm{d}\mu_\Delta \left[\left(\frac{\int_{[a,b)} qf_1^\omega \mathrm{d}\mu_\Delta}{\int_{[a,b)} q\mathrm{d}\mu_\Delta} \right)^{\frac{1}{\omega}} \left(\frac{\int_{[a,b)} qf_2^\nu \mathrm{d}\mu_\Delta}{\int_{[a,b)} q\mathrm{d}\mu_\Delta} \right)^{\frac{1}{\nu}} \right]^\lambda .
\end{aligned} \tag{4.28}$$

If p attains its minimum and maximum value on its domain, then

$$
\begin{aligned}
&\max_{t\in[a,b)_{\mathbb{T}}} p(t) \\
&\left[\int_{[a,b)} f_1^{\lambda} f_2^{\lambda}\,\mathrm{d}\mu_\Delta - (b-a)\left[\left(\frac{\int_{[a,b)} f_1^{\omega}\,\mathrm{d}\mu_\Delta}{b-a}\right)^{\frac{1}{\omega}}\left(\frac{\int_{[a,b)} f_2^{\nu}\,\mathrm{d}\mu_\Delta}{b-a}\right)^{\frac{1}{\nu}}\right]^{\lambda}\right] \\
&\geq \int_{[a,b)} p f_1^{\lambda} f_2^{\lambda}\,\mathrm{d}\mu_\Delta - \int_{[a,b)} p\,\mathrm{d}\mu_\Delta\left[\left(\frac{\int_{[a,b)} p f_1^{\omega}\,\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\,\mathrm{d}\mu_\Delta}\right)^{\frac{1}{\omega}}\left(\frac{\int_{[a,b)} p f_2^{\nu}\,\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\,\mathrm{d}\mu_\Delta}\right)^{\frac{1}{\nu}}\right]^{\lambda} \\
&\geq \min_{t\in[a,b)_{\mathbb{T}}} p(t) \\
&\left[\int_{[a,b)} f_1^{\lambda} f_2^{\lambda}\,\mathrm{d}\mu_\Delta - (b-a)\left[\left(\frac{\int_{[a,b)} f_1^{\omega}\,\mathrm{d}\mu_\Delta}{b-a}\right)^{\frac{1}{\omega}}\left(\frac{\int_{[a,b)} f_2^{\nu}\,\mathrm{d}\mu_\Delta}{b-a}\right)^{\frac{1}{\nu}}\right]^{\lambda}\right].
\end{aligned}
\tag{4.29}
$$

Moreover, the inequalities in (4.27)*,* (4.28)*, and* (4.29) *are reversed provided*

(a′) $\omega,\nu>\lambda>0$*, for* $\frac{1}{\lambda}\geq\frac{1}{\omega}+\frac{1}{\nu}$*;*

(b′) $\omega,\nu<\lambda<0$*, for* $\frac{1}{\lambda}\leq\frac{1}{\omega}+\frac{1}{\nu}$*.*

Proof. Let $n=2$ in Theorem 4.3. By setting $\varphi(x,y)=xy$, $\chi(t)=t^{\lambda}$, $\psi_1(t)=t^{\omega}$, and $\psi_2(t)=t^{\nu}$, we have

$$H(s_1,s_2)=\chi(\psi_1^{-1}(s_1)\psi_2^{-1}(s_2))=\left(s_1^{\frac{1}{\omega}}s_2^{\frac{1}{\nu}}\right)^{\lambda}.$$

Now, H is convex if and only if $\mathrm{d}^2H\geq 0$, which implies

$$\frac{\lambda}{\omega}\left(\frac{\lambda}{\omega}-1\right)\geq 0,\quad \frac{\lambda}{\nu}\left(\frac{\lambda}{\nu}-1\right)\geq 0,\quad\text{and}\quad \frac{\lambda^3}{\omega\nu}\left(\frac{1}{\lambda}-\frac{1}{\omega}-\frac{1}{\nu}\right)\geq 0,$$

and these are satisfied if λ,ω, and ν satisfy conditions (a),(b), and (c). H is concave if and only if $\mathrm{d}^2H\leq 0$, and this implies

$$\frac{\lambda}{\omega}\left(\frac{\lambda}{\omega}-1\right)\leq 0,\quad \frac{\lambda}{\nu}\left(\frac{\lambda}{\nu}-1\right)\leq 0,\quad\text{and}\quad \frac{\lambda^3}{\omega\nu}\left(\frac{1}{\lambda}-\frac{1}{\omega}-\frac{1}{\nu}\right)\geq 0.$$

These are satisfied if λ,ω, and ν satisfy conditions (a′) and (b′). Now, all claims follow immediately from Theorem 4.3. □

Corollary 4.7 *Let $\lambda,\omega,\nu\in\mathbb{R}$ be such that $\lambda,\omega,\nu>0$, $\lambda,\omega,\nu\neq 1$ and*

(a) $\lambda<1<\omega,\nu$, *or* $\omega,\nu<1<\lambda$;

(b) $\lambda<\omega,\nu<1$, *or* $\nu<1<\omega<\lambda$, *or* $\omega<1<\nu<\lambda$, *for* $\frac{1}{\log\lambda}\leq\frac{1}{\log\omega}+\frac{1}{\log\nu}$;

(c) $\lambda<\omega<1<\nu$, *or* $\lambda<\nu<1<\omega$, *for* $\frac{1}{\log\lambda}\geq\frac{1}{\log\omega}+\frac{1}{\log\nu}$.

Suppose $f_1,f_2:[a,b)_{\mathbb{T}}\to\mathbb{R}$ are Δ-integrable. Let $p,q:[a,b)_{\mathbb{T}}\to\mathbb{R}$ be nonnegative and Δ-integrable such that $p\lambda^{f_1+f_2},q\lambda^{f_1+f_2},p\omega^{f_1},q\omega^{f_1},p\nu^{f_2},q\nu^{f_2}$ are Δ-integrable and $\int_{[a,b)}p\mathrm{d}\mu_\Delta>0$, $\int_{[a,b)}q\mathrm{d}\mu_\Delta>0$. Then the functional

$$\int_{[a,b)}p\lambda^{f_1+f_2}\mathrm{d}\mu_\Delta-\int_{[a,b)}p\mathrm{d}\mu_\Delta\lambda^{\log_\omega\frac{\int_{[a,b)}p\omega^{f_1}\mathrm{d}\mu_\Delta}{\int_{[a,b)}p\mathrm{d}\mu_\Delta}+\log_\nu\frac{\int_{[a,b)}p\nu^{f_2}\mathrm{d}\mu_\Delta}{\int_{[a,b)}p\mathrm{d}\mu_\Delta}} \tag{4.30}$$

is superadditive, i.e.,

$$\begin{aligned}&\int_{[a,b)}(p+q)\lambda^{f_1+f_2}\mathrm{d}\mu_\Delta\\&\quad-\int_{[a,b)}(p+q)\mathrm{d}\mu_\Delta\lambda^{\log_\omega\frac{\int_{[a,b)}(p+q)\omega^{f_1}\mathrm{d}\mu_\Delta}{\int_{[a,b)}(p+q)\mathrm{d}\mu_\Delta}+\log_\nu\frac{\int_{[a,b)}(p+q)\nu^{f_2}\mathrm{d}\mu_\Delta}{\int_{[a,b)}(p+q)\mathrm{d}\mu_\Delta}}\\&\geq\int_{[a,b)}p\lambda^{f_1+f_2}\mathrm{d}\mu_\Delta-\int_{[a,b)}p\mathrm{d}\mu_\Delta\lambda^{\log_\omega\frac{\int_{[a,b)}p\omega^{f_1}\mathrm{d}\mu_\Delta}{\int_{[a,b)}p\mathrm{d}\mu_\Delta}+\log_\nu\frac{\int_{[a,b)}p\nu^{f_2}\mathrm{d}\mu_\Delta}{\int_{[a,b)}p\mathrm{d}\mu_\Delta}}\\&\quad+\int_{[a,b)}q\lambda^{f_1+f_2}\mathrm{d}\mu_\Delta-\int_{[a,b)}q\mathrm{d}\mu_\Delta\lambda^{\log_\omega\frac{\int_{[a,b)}q\omega^{f_1}\mathrm{d}\mu_\Delta}{\int_{[a,b)}q\mathrm{d}\mu_\Delta}+\log_\nu\frac{\int_{[a,b)}q\nu^{f_2}\mathrm{d}\mu_\Delta}{\int_{[a,b)}q\mathrm{d}\mu_\Delta}}.\end{aligned} \tag{4.31}$$

If $p\geq q$ such that $\int_{[a,b)}p\mathrm{d}\mu_\Delta>\int_{[a,b)}q\mathrm{d}\mu_\Delta$, then

$$\begin{aligned}&\int_{[a,b)}p\lambda^{f_1+f_2}\mathrm{d}\mu_\Delta-\int_{[a,b)}p\mathrm{d}\mu_\Delta\lambda^{\log_\omega\frac{\int_{[a,b)}p\omega^{f_1}\mathrm{d}\mu_\Delta}{\int_{[a,b)}p\mathrm{d}\mu_\Delta}+\log_\nu\frac{\int_{[a,b)}p\nu^{f_2}\mathrm{d}\mu_\Delta}{\int_{[a,b)}p\mathrm{d}\mu_\Delta}}\\&\geq\int_{[a,b)}q\lambda^{f_1+f_2}\mathrm{d}\mu_\Delta-\int_{[a,b)}q\mathrm{d}\mu_\Delta\lambda^{\log_\omega\frac{\int_{[a,b)}q\omega^{f_1}\mathrm{d}\mu_\Delta}{\int_{[a,b)}q\mathrm{d}\mu_\Delta}+\log_\nu\frac{\int_{[a,b)}q\nu^{f_2}\mathrm{d}\mu_\Delta}{\int_{[a,b)}q\mathrm{d}\mu_\Delta}}.\end{aligned} \tag{4.32}$$

If p attains its minimum and maximum value on its domain, then

$$\begin{aligned}&\max_{t\in[a,b)_{\mathbb{T}}}p(t)\left[\int_{[a,b)}\lambda^{f_1+f_2}\mathrm{d}\mu_\Delta-(b-a)\lambda^{\log_\omega\frac{\int_{[a,b)}\omega^{f_1}\mathrm{d}\mu_\Delta}{b-a}+\log_\nu\frac{\int_{[a,b)}\nu^{f_2}\mathrm{d}\mu_\Delta}{b-a}}\right]\\&\geq\int_{[a,b)}p\lambda^{f_1+f_2}\mathrm{d}\mu_\Delta-\int_{[a,b)}p\mathrm{d}\mu_\Delta\lambda^{\log_\omega\frac{\int_{[a,b)}p\omega^{f_1}\mathrm{d}\mu_\Delta}{\int_{[a,b)}p\mathrm{d}\mu_\Delta}+\log_\nu\frac{\int_{[a,b)}p\nu^{f_2}\mathrm{d}\mu_\Delta}{\int_{[a,b)}p\mathrm{d}\mu_\Delta}}\\&\geq\min_{t\in[a,b)_{\mathbb{T}}}p(t)\left[\int_{[a,b)}\lambda^{f_1+f_2}\mathrm{d}\mu_\Delta-(b-a)\lambda^{\log_\omega\frac{\int_{[a,b)}\omega^{f_1}\mathrm{d}\mu_\Delta}{b-a}+\log_\nu\frac{\int_{[a,b)}\nu^{f_2}\mathrm{d}\mu_\Delta}{b-a}}\right].\end{aligned} \tag{4.33}$$

Moreover, the inequalities in (4.31), (4.32), *and* (4.33) *are reversed provided*

(a′) $\omega, \nu > \lambda > 1$, *for* $\frac{1}{\log\lambda} \geq \frac{1}{\log\omega} + \frac{1}{\log\nu}$;

(b′) $\omega, \nu < \lambda < 0$, *for* $\frac{1}{\log\lambda} \leq \frac{1}{\log\omega} + \frac{1}{\log\nu}$.

Proof. Let $n = 2$ in Theorem 4.3. By setting $\varphi(x,y) = x+y$, $\chi(t) = \lambda^t$, $\psi_1(t) = \omega^t$, and $\psi_2(t) = \nu^t$, we have

$$H(s_1,s_2) = \left(s_1^{\frac{1}{\log\omega}} s_2^{\frac{1}{\log\nu}}\right)^{\log\lambda}.$$

Now, the proof is similar to the proof of Corollary 4.6. □

Corollary 4.8 *Let* $\lambda, \omega, \nu \in \mathbb{R}$ *be such that*

(a) $0 < \omega, \nu \leq \lambda < 1$, *for all* $f_1, f_2 > 0$;

(b) $0 < \nu \leq \lambda \leq \omega < 1$, *for* $f_2 \geq \frac{(\omega-\lambda)(1-\nu)}{(\lambda-\nu)(1-\omega)} f_1 \geq 0$;

(c) $0 < \omega \leq \lambda \leq \nu < 1$, *for* $\frac{(\lambda-\omega)(1-\nu)}{(\nu-\lambda)(1-\omega)} f_1 \geq f_2 \geq 0$.

Suppose $f_1, f_2 : [a,b)_{\mathbb{T}} \to \mathbb{R}$ *are* Δ*-integrable. Let* $p,q : [a,b)_{\mathbb{T}} \to \mathbb{R}$ *be nonnegative and* Δ*-integrable such that* $p(f_1+f_2)^\lambda, q(f_1+f_2)^\lambda, pf_1^\omega, qf_1^\omega, pf_2^\nu, qf_2^\nu$ *are* Δ*-integrable and* $\int_{[a,b)} p\mathrm{d}\mu_\Delta > 0$, $\int_{[a,b)} q\mathrm{d}\mu_\Delta > 0$. *Then the functional*

$$\int_{[a,b)} p(f_1+f_2)^\lambda \mathrm{d}\mu_\Delta - \int_{[a,b)} p\mathrm{d}\mu_\Delta \left[\left(\frac{\int_{[a,b)} pf_1^\omega \mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}\right)^{\frac{1}{\omega}} + \left(\frac{\int_{[a,b)} pf_2^\nu \mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}\right)^{\frac{1}{\nu}}\right]^\lambda \tag{4.34}$$

is superadditive, i.e.,

$$\begin{aligned}
&\int_{[a,b)} (p+q)(f_1+f_2)^\lambda \mathrm{d}\mu_\Delta \qquad (4.35)\\
&\quad - \int_{[a,b)} (p+q)\mathrm{d}\mu_\Delta \left[\left(\frac{\int_{[a,b)} (p+q)f_1^\omega \mathrm{d}\mu_\Delta}{\int_{[a,b)} (p+q)\mathrm{d}\mu_\Delta}\right)^{\frac{1}{\omega}} + \left(\frac{\int_{[a,b)} (p+q)f_2^\nu \mathrm{d}\mu_\Delta}{\int_{[a,b)} (p+q)\mathrm{d}\mu_\Delta}\right)^{\frac{1}{\nu}}\right]^\lambda \\
&\geq \int_{[a,b)} p(f_1+f_2)^\lambda \mathrm{d}\mu_\Delta \\
&\quad - \int_{[a,b)} p\mathrm{d}\mu_\Delta \left[\left(\frac{\int_{[a,b)} pf_1^\omega \mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}\right)^{\frac{1}{\omega}} + \left(\frac{\int_{[a,b)} pf_2^\nu \mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}\right)^{\frac{1}{\nu}}\right]^\lambda \\
&\quad + \int_{[a,b)} q(f_1+f_2)^\lambda \mathrm{d}\mu_\Delta \\
&\quad - \int_{[a,b)} q\mathrm{d}\mu_\Delta \left[\left(\frac{\int_{[a,b)} qf_1^\omega \mathrm{d}\mu_\Delta}{\int_{[a,b)} q\mathrm{d}\mu_\Delta}\right)^{\frac{1}{\omega}} + \left(\frac{\int_{[a,b)} qf_2^\nu \mathrm{d}\mu_\Delta}{\int_{[a,b)} q\mathrm{d}\mu_\Delta}\right)^{\frac{1}{\nu}}\right]^\lambda.
\end{aligned}$$

If $p \geq q$ such that $\int_{[a,b)} p\mathrm{d}\mu_\Delta > \int_{[a,b)} q\mathrm{d}\mu_\Delta$, then

$$\begin{aligned}
&\int_{[a,b)} p(f_1+f_2)^\lambda \mathrm{d}\mu_\Delta \\
&\quad - \int_{[a,b)} p\mathrm{d}\mu_\Delta \left[\left(\frac{\int_{[a,b)} p f_1^\omega \mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta} \right)^{\frac{1}{\omega}} + \left(\frac{\int_{[a,b)} p f_2^\nu \mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta} \right)^{\frac{1}{\nu}} \right]^\lambda \\
&\geq \int_{[a,b)} q(f_1+f_2)^\lambda \mathrm{d}\mu_\Delta \\
&\quad - \int_{[a,b)} q\mathrm{d}\mu_\Delta \left[\left(\frac{\int_{[a,b)} q f_1^\omega \mathrm{d}\mu_\Delta}{\int_{[a,b)} q\mathrm{d}\mu_\Delta} \right)^{\frac{1}{\omega}} + \left(\frac{\int_{[a,b)} q f_2^\nu \mathrm{d}\mu_\Delta}{\int_{[a,b)} q\mathrm{d}\mu_\Delta} \right)^{\frac{1}{\nu}} \right]^\lambda .
\end{aligned} \tag{4.36}$$

If p attains its minimum and maximum value on its domain, then

$$\begin{aligned}
&\max_{t\in[a,b)_{\mathbb{T}}} p(t) \left[\int_{[a,b)} (f_1+f_2)^\lambda \mathrm{d}\mu_\Delta \right. \\
&\quad \left. -(b-a) \left[\left(\frac{\int_{[a,b)} p f_1^\omega \mathrm{d}\mu_\Delta}{b-a} \right)^{\frac{1}{\omega}} + \left(\frac{\int_{[a,b)} p f_2^\nu \mathrm{d}\mu_\Delta}{b-a} \right)^{\frac{1}{\nu}} \right]^\lambda \right] \\
&\geq \int_{[a,b)} p(f_1+f_2)^\lambda \mathrm{d}\mu_\Delta \\
&\quad - \int_{[a,b)} p\mathrm{d}\mu_\Delta \left[\left(\frac{\int_{[a,b)} p f_1^\omega \mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta} \right)^{\frac{1}{\omega}} + \left(\frac{\int_{[a,b)} p f_2^\nu \mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta} \right)^{\frac{1}{\nu}} \right]^\lambda \\
&\geq \min_{t\in[a,b)_{\mathbb{T}}} p(t) \left[\int_{[a,b)} (f_1+f_2)^\lambda \mathrm{d}\mu_\Delta \right. \\
&\quad \left. -(b-a) \left[\left(\frac{\int_{[a,b)} p f_1^\omega \mathrm{d}\mu_\Delta}{b-a} \right)^{\frac{1}{\omega}} + \left(\frac{\int_{[a,b)} p f_2^\nu \mathrm{d}\mu_\Delta}{b-a} \right)^{\frac{1}{\nu}} \right]^\lambda \right].
\end{aligned} \tag{4.37}$$

Moreover, the inequalities in (4.35)*,* (4.36)*, and* (4.37) *are reversed provided*

(a′) $1 < \lambda \leq \omega, \nu$, *for all* $f_1, f_2 > 0$*;*

(b′) $1 < \nu \leq \lambda \leq \omega$, *for* $0 \leq f_2 \leq \frac{(\omega-\lambda)(\nu-1)}{(\lambda-\nu)(\omega-1)} f_1$*;*

(b′) $1 < \omega \leq \lambda \leq \nu$, *for* $f_2 \geq \frac{(\lambda-\omega)(\nu-1)}{(\nu-\lambda)(\omega-1)} f_1 \geq 0$.

Proof. Let $n = 2$ in Theorem 4.3. By setting $\varphi(x,y) = x+y$, $\chi(t) = t^\lambda$, $\psi_1(t) = t^\omega$, and $\psi_2(t) = t^\nu$, we have

$$H(s_1,s_2) = \left(s_1^{\frac{1}{\omega}} + s_2^{\frac{1}{\nu}} \right)^\lambda .$$

Now, the proof is similar to the proof of Corollary 4.4, with some extra considerations of the definitions of the functions E,F, and G. □

Corollary 4.9 *Suppose $f_1,f_2:[a,b)_{\mathbb{T}}\to[0,\frac{\pi}{4}]$ are Δ-integrable. Let $p,q:[a,b)_{\mathbb{T}}\to\mathbb{R}$ be nonnegative and Δ-integrable such that $p\cos(f_1+f_2), q\cos(f_1+f_2), p\cos(f_i), q\cos(f_i)$, $i=1,2$, are Δ-integrable and $\int_{[a,b)}p\mathrm{d}\mu_\Delta>0$, $\int_{[a,b)}q\mathrm{d}\mu_\Delta>0$. Then the functional*

$$\int_{[a,b)}p\mathrm{d}\mu_\Delta\cos\left[\arccos\left(\frac{\int_{[a,b)}p\cos(f_1)\mathrm{d}\mu_\Delta}{\int_{[a,b)}p\mathrm{d}\mu_\Delta}\right)+\arccos\left(\frac{\int_{[a,b)}p\cos(f_2)\mathrm{d}\mu_\Delta}{\int_{[a,b)}p\mathrm{d}\mu_\Delta}\right)\right]-\int_{[a,b)}p\cos(f_1+f_2)\mathrm{d}\mu_\Delta \tag{4.38}$$

is subadditive, i.e.,

$$\begin{aligned}&\int_{[a,b)}(p+q)\mathrm{d}\mu_\Delta\cos\left[\arccos\left(\frac{\int_{[a,b)}(p+q)\cos(f_1)\mathrm{d}\mu_\Delta}{\int_{[a,b)}(p+q)\mathrm{d}\mu_\Delta}\right)\right.\\&\quad\left.+\arccos\left(\frac{\int_{[a,b)}(p+q)\cos(f_2)\mathrm{d}\mu_\Delta}{\int_{[a,b)}(p+q)\mathrm{d}\mu_\Delta}\right)\right]-\int_{[a,b)}(p+q)\cos(f_1+f_2)\mathrm{d}\mu_\Delta\\&\le\int_{[a,b)}p\mathrm{d}\mu_\Delta\cos\left[\arccos\left(\frac{\int_{[a,b)}p\cos(f_1)\mathrm{d}\mu_\Delta}{\int_{[a,b)}p\mathrm{d}\mu_\Delta}\right)\right.\\&\quad\left.+\arccos\left(\frac{\int_{[a,b)}p\cos(f_2)\mathrm{d}\mu_\Delta}{\int_{[a,b)}p\mathrm{d}\mu_\Delta}\right)\right]-\int_{[a,b)}p\cos(f_1+f_2)\mathrm{d}\mu_\Delta\\&+\int_{[a,b)}q\mathrm{d}\mu_\Delta\cos\left[\arccos\left(\frac{\int_{[a,b)}q\cos(f_1)\mathrm{d}\mu_\Delta}{\int_{[a,b)}q\mathrm{d}\mu_\Delta}\right)\right.\\&\quad\left.+\arccos\left(\frac{\int_{[a,b)}q\cos(f_2)\mathrm{d}\mu_\Delta}{\int_{[a,b)}q\mathrm{d}\mu_\Delta}\right)\right]-\int_{[a,b)}q\cos(f_1+f_2)\mathrm{d}\mu_\Delta.\end{aligned}\tag{4.39}$$

If $p\ge q$ such that $\int_{[a,b)}p\mathrm{d}\mu_\Delta>\int_{[a,b)}q\mathrm{d}\mu_\Delta$, then

$$\begin{aligned}&\int_{[a,b)}p\mathrm{d}\mu_\Delta\cos\left[\arccos\left(\frac{\int_{[a,b)}p\cos(f_1)\mathrm{d}\mu_\Delta}{\int_{[a,b)}p\mathrm{d}\mu_\Delta}\right)\right.\\&\quad\left.+\arccos\left(\frac{\int_{[a,b)}p\cos(f_2)\mathrm{d}\mu_\Delta}{\int_{[a,b)}p\mathrm{d}\mu_\Delta}\right)\right]-\int_{[a,b)}p\cos(f_1+f_2)\mathrm{d}\mu_\Delta\\&\le\int_{[a,b)}q\mathrm{d}\mu_\Delta\cos\left[\arccos\left(\frac{\int_{[a,b)}q\cos(f_1)\mathrm{d}\mu_\Delta}{\int_{[a,b)}q\mathrm{d}\mu_\Delta}\right)\right.\\&\quad\left.+\arccos\left(\frac{\int_{[a,b)}q\cos(f_2)\mathrm{d}\mu_\Delta}{\int_{[a,b)}q\mathrm{d}\mu_\Delta}\right)\right]-\int_{[a,b)}q\cos(f_1+f_2)\mathrm{d}\mu_\Delta.\end{aligned}\tag{4.40}$$

If p attains its minimum and maximum value on its domain, then

$$\begin{aligned}
&\left[\max_{t\in[a,b)_{\mathbb{T}}} p(t)\right](b-a)\cos\left[\arccos\left(\frac{\int_{[a,b)}\cos(f_1)\mathrm{d}\mu_\Delta}{b-a}\right)\right. \\
&\quad\left.+\arccos\left(\frac{\int_{[a,b)}\cos(f_2)\mathrm{d}\mu_\Delta}{b-a}\right)\right]-\int_{[a,b)}\cos(f_1+f_2)\mathrm{d}\mu_\Delta \\
&\le \int_{[a,b)} p\mathrm{d}\mu_\Delta\cos\left[\arccos\left(\frac{\int_{[a,b)} p\cos(f_1)\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}\right)\right. \\
&\quad\left.+\arccos\left(\frac{\int_{[a,b)} p\cos(f_2)\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}\right)\right]-\int_{[a,b)} p\cos(f_1+f_2)\mathrm{d}\mu_\Delta \\
&\le \left[\min_{t\in[a,b)_{\mathbb{T}}} p(t)\right](b-a)\cos\left[\arccos\left(\frac{\int_{[a,b)}\cos(f_1)\mathrm{d}\mu_\Delta}{b-a}\right)\right. \\
&\quad\left.+\arccos\left(\frac{\int_{[a,b)}\cos(f_2)\mathrm{d}\mu_\Delta}{b-a}\right)\right]-\int_{[a,b)}\cos(f_1+f_2)\mathrm{d}\mu_\Delta.
\end{aligned} \tag{4.41}$$

Proof. Let $n=2$ in Theorem 4.3. By setting $\varphi(x,y)=x+y$ and $\chi(t)=\psi_1(t)=\psi_2(t)=-\cos(t)$, we have

$$H(s_1,s_2)=-\cos(\arccos(-s_1)+\arccos(-s_2)).$$

Now, the proof is similar to the proof of Corollary 4.4. □

4.5 Applications to Hölder's Inequality

Remark 4.4 Suppose f_i, $i\in\{1,\dots,n\}$, are nonnegative and Δ-integrable functions on $[a,b)_{\mathbb{T}}$ such that $\prod_{i=1}^{n} f_i^{\alpha_i}$ is Δ-integrable, where $\alpha_i\ge 0$, $i\in\{1,\dots,n\}$, are such that $\sum_{i=1}^{n}\alpha_i=1$. Then, by using Theorem 3.6 (Hölder's inequality for Lebesgue Δ-integrals), we have

$$\int_{[a,b)}\prod_{i=1}^{n} f_i^{\alpha_i}\mathrm{d}\mu_\Delta \le \prod_{i=1}^{n}\left(\int_{[a,b)} f_i\mathrm{d}\mu_\Delta\right)^{\alpha_i}. \tag{4.42}$$

If $\sum_{i=1}^{n}\alpha_i=\mathscr{A}_n>0$, then (4.42) implies

$$\int_{[a,b)}\prod_{i=1}^{n} f_i^{\frac{\alpha_i}{\mathscr{A}_n}}\mathrm{d}\mu_\Delta \le \prod_{i=1}^{n}\left(\int_{[a,b)} f_i\mathrm{d}\mu_\Delta\right)^{\frac{\alpha_i}{\mathscr{A}_n}} \tag{4.43}$$

or

$$\left(\int_{[a,b)}\prod_{i=1}^{n}f_i^{\frac{\alpha_i}{\mathscr{A}_n}}\mathrm{d}\mu_\Delta\right)^{\mathscr{A}_n}\leq\prod_{i=1}^{n}\left(\int_{[a,b)}f_i\mathrm{d}\mu_\Delta\right)^{\alpha_i}. \tag{4.44}$$

In this section, we discuss properties of the functional, deduced from the Hölder inequality (4.43), defined in the following way.

Definition 4.3 *Suppose* $\mathbf{f}=(f_1,\ldots,f_n)$ *is such that* f_i, $i\in\{1,\ldots,n\}$, *are nonnegative and* Δ*-integrable functions on* $[a,b)_{\mathbb{T}}$. *Let* $\alpha=(\alpha_1,\ldots,\alpha_n)$ *be such that* $\alpha_i\geq 0$ *and* $\sum_{i=1}^{n}\alpha_i=\mathscr{A}_n>0$. *Then we define the functional* $\mathbf{H}_\Delta$ *by*

$$\mathbf{H}_\Delta(\mathbf{f},\alpha)=\frac{\prod_{i=1}^{n}\left(\int_{[a,b)}f_i\mathrm{d}\mu_\Delta\right)^{\alpha_i}}{\left(\int_{[a,b)}\prod_{i=1}^{n}f_i^{\frac{\alpha_i}{\mathscr{A}_n}}\mathrm{d}\mu_\Delta\right)^{\mathscr{A}_n}}. \tag{4.45}$$

Theorem 4.4 *Let* $\alpha=(\alpha_1,\ldots,\alpha_n)$ *and* $\beta=(\beta_1,\ldots,\beta_n)$ *be real n-tuples with* $\alpha_i\geq 0$, $\beta_i\geq 0$ *and* $\sum_{i=1}^{n}\alpha_i=\mathscr{A}_n>0$, $\sum_{i=1}^{n}\beta_i=\mathscr{B}_n>0$. *Suppose* f_i, $i\in\{1,\ldots,n\}$, *are nonnegative and* Δ*-integrable functions on* $[a,b)_{\mathbb{T}}$ *such that* $\prod_{i=1}^{n}f_i^{\frac{\alpha_i}{\mathscr{A}_n}}$ *and* $\prod_{i=1}^{n}f_i^{\frac{\beta_i}{\mathscr{B}_n}}$ *are* Δ*-integrable. Then*

$$\mathbf{H}_\Delta(\mathbf{f},\alpha+\beta)\geq\mathbf{H}_\Delta(\mathbf{f},\alpha)\mathbf{H}_\Delta(\mathbf{f},\beta), \tag{4.46}$$

and $\mathbf{H}_\Delta(\mathbf{f},\cdot)$ *is increasing, i.e., if* $\alpha\geq\beta$ *such that* $\mathscr{A}_n>\mathscr{B}_n$, *then*

$$\mathbf{H}_\Delta(\mathbf{f},\alpha)\geq\mathbf{H}_\Delta(\mathbf{f},\beta). \tag{4.47}$$

Proof. By Definition 4.3, we have

$$\mathbf{H}_\Delta(\mathbf{f},\alpha+\beta)=\frac{\prod_{i=1}^{n}\left(\int_{[a,b)}f_i\mathrm{d}\mu_\Delta\right)^{\alpha_i+\beta_i}}{\left(\int_{[a,b)}\prod_{i=1}^{n}f_i^{\frac{\alpha_i+\beta_i}{\mathscr{A}_n+\mathscr{B}_n}}\mathrm{d}\mu_\Delta\right)^{\mathscr{A}_n+\mathscr{B}_n}} \tag{4.48}$$

where

$$\begin{aligned}&\left(\int_{[a,b)}\prod_{i=1}^{n}f_i^{\frac{\alpha_i+\beta_i}{\mathscr{A}_n+\mathscr{B}_n}}\mathrm{d}\mu_\Delta\right)^{\mathscr{A}_n+\mathscr{B}_n} \qquad (4.49)\\&=\left[\int_{[a,b)}\left(\prod_{i=1}^{n}f_i^{\frac{\alpha_i}{\mathscr{A}_n}}\right)^{\frac{\mathscr{A}_n}{\mathscr{A}_n+\mathscr{B}_n}}\left(\prod_{i=1}^{n}f_i^{\frac{\beta_i}{\mathscr{B}_n}}\right)^{\frac{\mathscr{B}_n}{\mathscr{A}_n+\mathscr{B}_n}}\mathrm{d}\mu_\Delta\right]^{\mathscr{A}_n+\mathscr{B}_n}\\&\leq\left(\int_{[a,b)}\prod_{i=1}^{n}f_i^{\frac{\alpha_i}{\mathscr{A}_n}}\mathrm{d}\mu_\Delta\right)^{\mathscr{A}_n}\left(\int_{[a,b)}\prod_{i=1}^{n}f_i^{\frac{\beta_i}{\mathscr{B}_n}}\mathrm{d}\mu_\Delta\right)^{\mathscr{B}_n}.\end{aligned}$$

Now, by combining (4.48) and (4.49), we have

$$\begin{aligned}\mathbf{H}_\Delta(\mathbf{f},\alpha+\beta) &\geq \frac{\prod_{i=1}^{n}\left(\int_{[a,b)} f_i \mathrm{d}\mu_\Delta\right)^{\alpha_i}\prod_{i=1}^{n}\left(\int_{[a,b)} f_i \mathrm{d}\mu_\Delta\right)^{\beta_i}}{\left(\int_{[a,b)}\prod_{i=1}^{n} f_i^{\frac{\alpha_i}{\mathscr{A}_n}}\mathrm{d}\mu_\Delta\right)^{\mathscr{A}_n}\left(\int_{[a,b)}\prod_{i=1}^{n} f_i^{\frac{\beta_i}{\mathscr{B}_n}}\mathrm{d}\mu_\Delta\right)^{\mathscr{B}_n}}\\ &= \mathbf{H}_\Delta(\mathbf{f},\alpha)\mathbf{H}_\Delta(\mathbf{f},\beta).\end{aligned}$$

If $\alpha \geq \beta$, then $\alpha-\beta \geq 0$ and therefore

$$\begin{aligned}\mathbf{H}_\Delta(\mathbf{f},\alpha) &= \mathbf{H}_\Delta(\mathbf{f},(\alpha-\beta)+\beta)\\ &\geq \mathbf{H}_\Delta(\mathbf{f},\alpha-\beta)\mathbf{H}_\Delta(\mathbf{f},\beta)\\ &\geq \mathbf{H}_\Delta(\mathbf{f},\beta).\end{aligned}$$

The proof is complete. □

Corollary 4.10 *Let* $\mathbf{f}$ *and* α *satisfy the hypotheses of Theorem 4.4. Then*

$$\left[\frac{\prod_{i=1}^{n}\int_{[a,b)} f_i \mathrm{d}\mu_\Delta}{\left(\int_{[a,b)}\prod_{i=1}^{n} f_i^{\frac{1}{n}}\mathrm{d}\mu_\Delta\right)^{n}}\right]^{\max\limits_{1\leq i\leq n}\{\alpha_i\}} \geq \mathbf{H}_\Delta(\mathbf{f},\alpha) \geq \left[\frac{\prod_{i=1}^{n}\int_{[a,b)} f_i \mathrm{d}\mu_\Delta}{\left(\int_{[a,b)}\prod_{i=1}^{n} f_i^{\frac{1}{n}}\mathrm{d}\mu_\Delta\right)^{n}}\right]^{\min\limits_{1\leq i\leq n}\{\alpha_i\}}. \tag{4.50}$$

Proof. Let

$$\alpha_{\max} = \left(\max_{1\leq i\leq n}\{\alpha_i\},\ldots,\max_{1\leq i\leq n}\{\alpha_i\}\right) \quad\text{and}\quad \alpha_{\min} = \left(\min_{1\leq i\leq n}\{\alpha_i\},\ldots,\min_{1\leq i\leq n}\{\alpha_i\}\right).$$

Now by Definition 4.3, we have

$$\mathbf{H}_\Delta(\mathbf{f},\alpha_{\max}) = \left[\frac{\prod_{i=1}^{n}\int_{[a,b)} f_i \mathrm{d}\mu_\Delta}{\left(\int_{[a,b)}\prod_{i=1}^{n} f_i^{\frac{1}{n}}\mathrm{d}\mu_\Delta\right)^{n}}\right]^{\max\limits_{1\leq i\leq n}\{\alpha_i\}}$$

and

$$\mathbf{H}_\Delta(\mathbf{f},\alpha_{\min}) = \left[\frac{\prod_{i=1}^{n}\int_{[a,b)} f_i \mathrm{d}\mu_\Delta}{\left(\int_{[a,b)}\prod_{i=1}^{n} f_i^{\frac{1}{n}}\mathrm{d}\mu_\Delta\right)^{n}}\right]^{\min\limits_{1\leq i\leq n}\{\alpha_i\}}.$$

Since $\alpha_{\max} \geq \alpha \geq \alpha_{\min}$, the result follows from the second property of Theorem 4.4. □

Corollary 4.11 *Let* $\mathbf{f}, \alpha$, *and* β *satisfy the hypotheses of Theorem 4.4 with* $\mathscr{A}_n = \mathscr{B}_n = 1$. *If there exist constants* $M > 1 > m$ *such that* $M\beta \geq \alpha \geq m\beta$, *then*

$$\mathbf{H}_\Delta(\mathbf{f},M\beta) \geq \mathbf{H}_\Delta(\mathbf{f},\alpha) \geq \mathbf{H}_\Delta(\mathbf{f},m\beta). \tag{4.51}$$

Proof. By Definition 4.3, we have

$$\mathbf{H}_\Delta(\mathbf{f}, M\beta) = M\mathbf{H}_\Delta(\mathbf{f}, \beta) \quad \text{and} \quad \mathbf{H}_\Delta(\mathbf{f}, m\beta) = m\mathbf{H}_\Delta(\mathbf{f}, \beta).$$

Now the result follows from the second property of Theorem 4.4. □

Remark 4.5 Similarly as in Chapter 2, we can apply the theory of isotonic linear functionals. The related results for isotonic linear functionals are given in [92].

Chapter 5

Improvements of the Jensen–Steffensen Inequality and its Converse

In this chapter, we give a generalization of the Jensen–Steffensen inequality and its converse on time scales. These results also generalize the Jensen–Steffensen inequality and its converse given for the discrete and continuous cases. Further, we investigate the exponential and logarithmic convexity of the functionals defined as differences of the left-hand and the right-hand sides of these inequalities. Finally, we present several families of functions for which these results can be applied. The results presented in this chapter are taken from [39].

5.1 α-SP and α-HH Weights

In order to give a better version of the Jensen inequality (Theorem 2.11) on time scales, C. Dinu in [51] gives the definition of an α-Steffensen–Popoviciu (α-SP) weight.

Definition 5.1 (α-SP WEIGHT) *Let $g \in \mathrm{C}(\mathbb{T},\mathbb{R})$. Then $w \in \mathrm{C}(\mathbb{T},\mathbb{R})$ is an α-Steffensen–Popoviciu (α-SP) weight for g on $[a,b]_{\mathbb{T}}$ if*

$$\int_a^b w(t)\diamond_\alpha t > 0 \quad \text{and} \quad \int_a^b \Phi^+(g(t))w(t)\diamond_\alpha t \geq 0 \tag{5.1}$$

for every convex function $\Phi \in C([m,M],\mathbb{R})$*, where*

$$m = \inf_{t\in[a,b]_{\mathbb{T}}} g(t) \quad \textit{and} \quad M = \sup_{t\in[a,b]_{\mathbb{T}}} g(t).$$

In the following lemma, he gives a characterization for α-SP weight for a nondecreasing function g on time scales.

Lemma 5.1 *Let* $w \in C(\mathbb{T},\mathbb{R})$ *be such that* $\int_a^b w(t)\diamond_\alpha t > 0$*. Then* w *is an* α*-SP weight for a nondecreasing function* $g \in C([a,b]_{\mathbb{T}},\mathbb{R})$ *if and only if it satisfies*

$$\int_a^s (g(s)-g(t))w(t)\diamond_\alpha t \geq 0 \quad \textit{and} \quad \int_s^b (g(t)-g(s))w(t)\diamond_\alpha t \geq 0, \tag{5.2}$$

for every $s \in [a,b]_{\mathbb{T}}$*. If the stronger (but more suitable) condition*

$$0 \leq \int_a^s w(t)\diamond_\alpha t \leq \int_a^b w(t)\diamond_\alpha t \quad \textit{for every} \quad s \in [a,b]_{\mathbb{T}} \tag{5.3}$$

holds, then w *is also an* α*-SP weight for the nondecreasing continuous function* g*.*

As given in [51], all positive weights are α-SP weights, for any continuous function g and every $\alpha \in [0,1]$. But there are some α-SP weights that are allowed to take negative values. The Jensen inequality on time scales (where it is allowed that the weight function takes some negative values) is given in the following theorem from [51].

Theorem 5.1 *Let* $g \in C([a,b]_{\mathbb{T}},[m,M])$ *and let* $w \in C([a,b]_{\mathbb{T}},\mathbb{R})$ *be such that* $\int_a^b w(t)\diamond_\alpha t > 0$*. Then the following two statements are equivalent:*

(i) *w is an α-SP weight for g on $[a,b]_{\mathbb{T}}$;*

(ii) *for every convex function $\Phi \in C([m,M],\mathbb{R})$, we have*

$$\Phi\left(\frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\right) \leq \frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}. \tag{5.4}$$

Remark 5.1 Let g be a nondecreasing function. If $\mathbb{T} = \mathbb{N}$, then Theorem 5.1 is equivalent to Theorem 1.10 (Jensen–Steffensen inequality). On the other hand, if we take $\mathbb{T} = \mathbb{R}$ in Theorem 5.1, we obtain the integral version of Jensen–Steffensen inequality given by Boas [40].

Considering the converse of the Jensen inequality, C. Dinu gives the following definition of α-Hermite–Hadamard (α-HH) weight. He gives the characterization for a nondecreasing function g on time scales and the improvement of the converse of the Jensen inequality for some negative weights.

Definition 5.2 (α-HH WEIGHT) *Let $g \in \mathrm{C}(\mathbb{T},\mathbb{R})$. Then $w \in \mathrm{C}(\mathbb{T},\mathbb{R})$ is an α-Hermite–Hadamard (α-HH) weight for g on $[a,b]_{\mathbb{T}}$ if*

$$\int_a^b w(t)\diamond_\alpha t > 0 \quad \text{and} \quad \frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} \le \frac{M-\overline{g}_\alpha}{M-m}\Phi(m) + \frac{\overline{g}_\alpha - m}{M-m}\Phi(M),$$

for every convex function $\Phi \in \mathrm{C}([m,M],\mathbb{R})$, where

$$m = \inf_{t\in[a,b]_{\mathbb{T}}} g(t),\; M = \sup_{t\in[a,b]_{\mathbb{T}}} g(t), \quad \text{and} \quad \overline{g}_\alpha = \frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}.$$

Lemma 5.2 *Let $w \in \mathrm{C}([a,b]_{\mathbb{T}},\mathbb{R})$ be such that $\int_a^b w(t)\diamond_\alpha t > 0$. Then w is an α-HH weight for a nondecreasing function $g \in \mathrm{C}([a,b]_{\mathbb{T}},\mathbb{R})$ if and only if it satisfies*

$$\frac{g(b)-g(s)}{g(b)-g(a)}\int_a^s (g(t)-g(a))w(t)\diamond_\alpha t + \frac{g(s)-g(a)}{g(b)-g(a)}\int_s^b (g(b)-g(t))w(t)\diamond_\alpha t \ge 0 \quad (5.5)$$

for every $s \in [a,b]_{\mathbb{T}}$.

In the next result, C. Dinu gives the connection between these two classes of weights on a time scale.

Theorem 5.2 *Let $g \in \mathrm{C}(\mathbb{T},\mathbb{R})$. Then every α-SP weight for g on $[a,b]_{\mathbb{T}}$ is an α-HH weight for g on $[a,b]_{\mathbb{T}}$, for all $\alpha \in [0,1]$.*

Corollary 5.1 *Let $g \in \mathrm{C}([a,b]_{\mathbb{T}},[m,M])$. Let $\Phi \in \mathrm{C}([m,M],\mathbb{R})$ be a convex function and $w \in \mathrm{C}([a,b]_{\mathbb{T}},\mathbb{R})$ an α-SP weight for g on $[a,b]_{\mathbb{T}}$. Then*

$$\Phi(\overline{g}_\alpha) \le \frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} \le \frac{M-\overline{g}_\alpha}{M-m}\Phi(m) + \frac{\overline{g}_\alpha - m}{M-m}\Phi(M), \quad (5.6)$$

where $\overline{g}_\alpha = \dfrac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}$.

5.2 Jensen–Steffensen Inequality

Let $m, M \in \mathbb{R}$, where $m \ne M$. Consider the Green function $G : [m,M]\times[m,M] \to \mathbb{R}$ defined by

$$G(x,y) = \begin{cases} \dfrac{(x-M)(y-m)}{M-m} & \text{for} \quad m \le y \le x, \\[2ex] \dfrac{(y-M)(x-m)}{M-m} & \text{for} \quad x \le y \le M. \end{cases} \quad (5.7)$$

The function G is convex and continuous with respect to both x and y.

Remark 5.2 Note that the condition (5.5) is equivalent to

$$\int_a^b G(g(t),g(s))w(t)\diamond_\alpha t \le 0,$$

where the function G is defined in (5.7).

It is well known that (see for example [88, 108, 120, 133]) any function $\Phi \in \mathrm{C}^2([m,M],\mathbb{R})$ can be represented by

$$\Phi(x) = \frac{M-x}{M-m}\Phi(m) + \frac{x-m}{M-m}\Phi(M) + \int_m^M G(x,y)\Phi''(y)\mathrm{d}y, \tag{5.8}$$

where the function G is defined in (5.7). Using (5.8), we now derive several interesting results concerning inequalities of Jensen type.

Firstly, we give a generalization of the Jensen inequality on time scales, where negative weights are also allowed.

Theorem 5.3 *Assume $g \in \mathrm{C}([a,b]_\mathbb{T},\mathbb{R})$ is such that $g([a,b]_\mathbb{T}) \subseteq [m,M]$. Let $w \in \mathrm{C}([a,b]_\mathbb{T},\mathbb{R})$ be such that $\int_a^b w(t)\diamond_\alpha t \neq 0$ and $\frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} \in [m,M]$. Then the following two statements are equivalent:*

(i) *For every convex $\Phi \in \mathrm{C}([m,M],\mathbb{R})$, we have*

$$\Phi\left(\frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\right) \le \frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}. \tag{5.9}$$

(ii) *For all $y \in [m,M]$, we have*

$$G\left(\frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}, y\right) \le \frac{\int_a^b G(g(t),y)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}, \tag{5.10}$$

where $G : [m,M] \times [m,M] \to \mathbb{R}$ is defined in (5.7).

Furthermore, the statements (i) *and* (ii) *are also equivalent if we reverse the inequality in both* (5.9) *and* (5.10).

Proof. (i) $\Rightarrow$ (ii): Let (i) hold. As the function $G(\cdot,y)$, where $y \in [m,M]$, is also continuous and convex, it follows that (5.10) holds.

(ii) $\Rightarrow$ (i): Let (ii) hold. Let $\Phi \in \mathrm{C}^2([m,M],\mathbb{R})$. By using (5.8), we get

$$\begin{aligned}&\frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} - \Phi\left(\frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\right)\\ &\quad= \int_m^M \left[\frac{\int_a^b G(g(t),y)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} - G\left(\frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}, y\right)\right]\Phi''(y)\mathrm{d}y. \end{aligned}\tag{5.11}$$

If the function Φ is also convex, then $\Phi''(y) \geq 0$ for all $y \in [m,M]$, and hence it follows that for every convex function $\Phi \in C^2([m,M],\mathbb{R})$, inequality (5.9) holds. Moreover, it is not necessary to demand the existence of the second derivative of the function Φ (see [119, page 172]). The differentiability condition can be directly eliminated by using the fact that it is possible to approximate uniformly a continuous convex function by convex polynomials.

The last part can be proved analogously. □

Remark 5.3 Let the conditions of Theorem 5.3 hold. Then the following two statements are equivalent:

(i′) For every concave $\Phi \in C([m,M],\mathbb{R})$, the reverse inequality in (5.9) holds.

(ii′) For all $y \in [m,M]$, inequality (5.10) holds.

Moreover, the statements (i′) and (ii′) are also equivalent if we reverse the inequality in both statements (i′) and (ii′).

Remark 5.4 Consider (5.11). Suppose that g is nondecreasing and that it has a first derivative. Let $m = g(a)$, $M = g(b)$ and make the substitution $y = g(s)$. Then we get

$$\frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} - \Phi\left(\frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\right)$$
$$= \int_a^b \left[\frac{\int_a^b G(g(t),g(s))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} - G\left(\frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}, g(s)\right)\right]\Phi''(g(s))g'(s)\mathrm{d}s. \tag{5.12}$$

Since g is nondecreasing, so is $g'(s) \geq 0$. If $\Phi \in C^2([m,M],\mathbb{R})$ is convex, then $\Phi''(g(s)) \geq 0$ for all $s \in [a,b]_\mathbb{T}$. Hence every continuous and convex Φ satisfies (5.9) if and only if

$$G\left(\frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}, g(s)\right) \leq \frac{\int_a^b G(g(t),g(s))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}$$

holds for all $s \in [a,b]_\mathbb{T}$.

Combining the result from Theorem 5.3 with Theorem 5.1 and Lemma 5.1, we get the following two corollaries.

Corollary 5.2 *Let $g \in C([a,b]_\mathbb{T},[m,M])$ and $w \in C([a,b]_\mathbb{T},\mathbb{R})$ such that $\int_a^b w(t)\diamond_\alpha t > 0$. Then w is an α-SP weight for g on $[a,b]_\mathbb{T}$ if and only if* (5.10) *holds for all $y \in [m,M]$.*

Corollary 5.3 *Let $g \in C([a,b]_\mathbb{T},[m,M])$ is nondecreasing and $w \in C([a,b]_\mathbb{T},\mathbb{R})$ be such that $\int_a^b w(t)\diamond_\alpha t > 0$. Then* (5.2) *holds for all $s \in [a,b]_\mathbb{T}$ if and only if* (5.10) *holds for all $y \in [m,M]$.*

If $\mathbb{T} = \mathbb{R}$, then from Theorem 5.3 we obtain the following result.

Corollary 5.4 *Assume* $g \in \mathrm{C}([a,b],\mathbb{R})$ *is such that* $g([a,b]) \subseteq [m,M]$. *Let* $w \in \mathrm{C}([a,b],\mathbb{R})$ *be such that* $\int_a^b w(t)\mathrm{d}t \neq 0$ *and* $\frac{\int_a^b g(t)w(t)\mathrm{d}t}{\int_a^b w(t)\mathrm{d}t} \in [m,M]$. *Then the following two statements are equivalent:*

(i) *For every convex* $\Phi \in \mathrm{C}([m,M],\mathbb{R})$, *we have*

$$\Phi\left(\frac{\int_a^b g(t)w(t)\mathrm{d}t}{\int_a^b w(t)\mathrm{d}t}\right) \leq \frac{\int_a^b \Phi(g(t))w(t)\mathrm{d}t}{\int_a^b w(t)\mathrm{d}t}. \tag{5.13}$$

(ii) *For all* $y \in [m,M]$,

$$G\left(\frac{\int_a^b g(t)w(t)\mathrm{d}t}{\int_a^b w(t)\mathrm{d}t}, y\right) \leq \frac{\int_a^b G(g(t),y)w(t)\mathrm{d}t}{\int_a^b w(t)\mathrm{d}t} \tag{5.14}$$

holds, where $G : [m,M] \times [m,M] \to \mathbb{R}$ *is defined in* (5.7).

Furthermore, the statements (i) *and* (ii) *are also equivalent if we reverse the inequality in both* (5.13) *and* (5.14).

When $\mathbb{T} = \mathbb{Z}$, Theorem 5.3 yields the following result.

Corollary 5.5 *Let* $p_i \in \mathbb{R}$, $x_i \in [m,M]$, $i \in \{1,\ldots,n+1\}$, *such that*

$$\alpha p_1 + (1-\alpha)p_{n+1} + \sum_{i=2}^{n} p_i > 0$$

and

$$\frac{\alpha p_1 x_1 + (1-\alpha)p_{n+1}x_{n+1} + \sum_{i=2}^{n} p_i x_i}{\alpha p_1 + (1-\alpha)p_{n+1} + \sum_{i=2}^{n} p_i} \in [m,M],$$

where $\alpha \in [0,1]$. *Then the following two statements are equivalent:*

(i) *For every convex* $\Phi \in \mathrm{C}([m,M],\mathbb{R})$, *we have*

$$\Phi\left(\frac{\alpha p_1 x_1 + (1-\alpha)p_{n+1}x_{n+1} + \sum_{i=2}^{n} p_i x_i}{\alpha p_1 + (1-\alpha)p_{n+1} + \sum_{i=2}^{n} p_i}\right)$$
$$\leq \frac{\alpha p_1 \Phi(x_1) + (1-\alpha)p_{n+1}\Phi(x_{n+1}) + \sum_{i=2}^{n} p_i \Phi(x_i)}{\alpha p_1 + (1-\alpha)p_{n+1} + \sum_{i=2}^{n} p_i}. \tag{5.15}$$

(ii) *For all* $y \in [m,M]$,

$$G\left(\frac{\alpha p_1 x_1 + (1-\alpha) p_{n+1} x_{n+1} + \sum_{i=2}^{n} p_i x_i}{\alpha p_1 + (1-\alpha) p_{n+1} + \sum_{i=2}^{n} p_i}, y\right) \leq \frac{\alpha p_1 G(x_1,y) + (1-\alpha) p_{n+1} G(x_{n+1},y) + \sum_{i=2}^{n} p_i G(x_i,y)}{\alpha p_1 + (1-\alpha) p_{n+1} + \sum_{i=2}^{n} p_i} \tag{5.16}$$

holds, where $G : [m,M] \times [m,M] \to \mathbb{R}$ *is defined in* (5.7).

Furthermore, the statements (i) *and* (ii) *are also equivalent if we reverse the inequality in both* (5.15) *and* (5.16).

Example 5.1 (i) If $\alpha = 1$ and $p_i = 1$, $i \in \{1,\dots,n+1\}$, then Corollary 5.5 is equivalent to Jensen's inequality.

(ii) Let $[m,M] \subset (0,\infty)$ and $\Phi(x) = x^{\beta}$, where $\beta < 0$ or $\beta > 1$. Then (5.15) becomes

$$\left(\frac{\alpha p_1 x_1 + (1-\alpha) p_{n+1} x_{n+1} + \sum_{i=2}^{n} p_i x_i}{\alpha p_1 + (1-\alpha) p_{n+1} + \sum_{i=2}^{n} p_i}\right)^{\beta} \leq \frac{\alpha p_1 x_1^{\beta} + (1-\alpha) p_{n+1} x_{n+1}^{\beta} + \sum_{i=2}^{n} p_i x_i^{\beta}}{\alpha p_1 + (1-\alpha) p_{n+1} + \sum_{i=2}^{n} p_i}.$$

(iii) Let $\Phi(x) = \exp(x)$. Then (5.15) becomes

$$\exp\left(\frac{\alpha p_1 x_1 + (1-\alpha) p_{n+1} x_{n+1} + \sum_{i=2}^{n} p_i x_i}{\alpha p_1 + (1-\alpha) p_{n+1} + \sum_{i=2}^{n} p_i}\right) \leq \frac{\alpha p_1 \exp(x_1) + (1-\alpha) p_{n+1} \exp(x_{n+1}) + \sum_{i=2}^{n} p_i \exp(x_i)}{\alpha p_1 + (1-\alpha) p_{n+1} + \sum_{i=2}^{n} p_i}.$$

(iv) Let $[m,M] \subset (0,\infty)$ and $\Phi(x) = \ln x$. Then (5.15) becomes

$$\frac{\alpha p_1 x_1 + (1-\alpha)p_{n+1}x_{n+1} + \sum_{i=2}^{n} p_i x_i}{\alpha p_1 + (1-\alpha)p_{n+1} + \sum_{i=2}^{n} p_i}$$
$$\geq x_1^{\frac{\alpha p_1}{\alpha p_1 + (1-\alpha)p_{n+1} + \sum_{i=2}^{n} p_i}} x_{n+1}^{\frac{(1-\alpha)p_{n+1}}{\alpha p_1 + (1-\alpha)p_{n+1} + \sum_{i=2}^{n} p_i}} \prod_{i=2}^{n} x_i^{\frac{p_i}{\alpha p_1 + (1-\alpha)p_{n+1} + \sum_{i=2}^{n} p_i}}.$$

To shorten the notation, in the sequel we will use the notation

$$\overline{g}_\alpha = \frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}.$$

Under the assumptions of Theorem 5.3, we define the functional $\mathscr{J}_{\alpha 1}(g,\Phi)$ by

$$\mathscr{J}_{\alpha 1}(g,\Phi) = \begin{cases} \dfrac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} - \Phi\left(\dfrac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\right) \\ \text{if (5.10) holds for all } y \in [m,M], \\ \Phi\left(\dfrac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\right) - \dfrac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} \\ \text{if the reverse of (5.10) holds for all } y \in [m,M], \end{cases} \tag{5.17}$$

where the function Φ is defined on $[m,M]$. Clearly, if Φ is continuous and convex, then $\mathscr{J}_{\alpha 1}(g,\Phi)$ is nonnegative.

Theorem 5.4 *Let g, w, and $\overline{g}_\alpha$ satisfy the assumptions of Theorem 5.3. Let $\Phi \in C^2([m,M],\mathbb{R})$ and let $\mathscr{J}_{\alpha 1}$ be the functional defined in* (5.17)*. Then there exists $\xi \in [m,M]$ such that*

$$\mathscr{J}_{\alpha 1}(g,\Phi) = \Phi''(\xi)\mathscr{J}_{\alpha 1}(g,\Phi_0) \tag{5.18}$$

holds, where $\Phi_0(x) = \frac{x^2}{2}$.

Proof. The function Φ'' is continuous and

$$\frac{\int_a^b G(g(t),y)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} - G\left(\frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}, y\right)$$

does not change positivity on $[m,M]$. For Φ, (5.11) holds, and now applying the integral mean value theorem, we get that there exists $\xi \in [m,M]$ such that

$$\frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} - \Phi\left(\frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\right)$$
$$= \Phi''(\xi)\int_m^M \left[\frac{\int_a^b G(g(t),y)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} - G\left(\frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}, y\right)\right] dy. \tag{5.19}$$

As in [120], it can be easily checked that

$$\int_m^M G(x,y)\mathrm{d}y = \int_m^x \frac{(x-M)(y-m)}{M-m}\mathrm{d}y + \int_x^M \frac{(y-M)(x-m)}{M-m}\mathrm{d}y$$
$$= \frac{1}{2}(x-m)(x-M)$$

holds. Calculating the integral on the right-hand side in (5.19), we get

$$\frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} - \Phi\left(\frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\right)$$
$$= \Phi''(\xi)\left[\frac{1}{\int_a^b w(t)\diamond_\alpha t}\int_a^b\left(\int_m^M G(g(t),y)\mathrm{d}y\right)w(t)\diamond_\alpha t - \int_m^M G(\overline{g}_\alpha,y)\mathrm{d}y\right]$$
$$= \Phi''(\xi)$$
$$\left[\frac{1}{\int_a^b w(t)\diamond_\alpha t}\int_a^b \frac{1}{2}(g(t)-m)(g(t)-M)w(t)\diamond_\alpha t - \frac{1}{2}(\overline{g}_\alpha - m)(\overline{g}_\alpha - M)\right]$$
$$= \frac{1}{2}\Phi''(\xi)\left[\frac{\int_a^b (g(t))^2 w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} - \overline{g}_\alpha^2\right],$$

and the proof is completed. □

Remark 5.5 Theorem 5.4 can also be proved by using the two convex functions

$$\phi_1(x) = \frac{\Phi^*}{2}x^2 - \Phi(x) \quad \text{and} \quad \phi_2(x) = \Phi(x) - \frac{\Phi_*}{2}x^2,$$

where

$$\Phi_* = \min_{x\in[m,M]} \Phi''(x) \quad \text{and} \quad \Phi^* = \max_{x\in[m,M]} \Phi''(x).$$

Since ϕ_1 and ϕ_2 are continuous and convex, we have

$$\mathscr{J}_{\alpha 1}(g,\phi_1) \geq 0 \quad \text{and} \quad \mathscr{J}_{\alpha 1}(g,\phi_2) \geq 0.$$

This implies that

$$\Phi_* \mathscr{J}_{\alpha 1}(g,\Phi_0) \leq \mathscr{J}_{\alpha 1}(g,\Phi) \leq \Phi^* \mathscr{J}_{\alpha 1}(g,\Phi_0).$$

Hence, as Φ'' is continuous, there exists $\xi \in [m,M]$ such that (5.18) holds.

Theorem 5.5 *Let g, w, and $\overline{g}_\alpha$ satisfy the assumptions of Theorem 5.3. Let $\Phi,\Psi \in C^2([m,M],\mathbb{R})$ and $\mathscr{J}_{\alpha 1}$ be the functional defined in* (5.17)*. Then there exists $\xi \in [m,M]$ such that*

$$\frac{\mathscr{J}_{\alpha 1}(g,\Phi)}{\mathscr{J}_{\alpha 1}(g,\Psi)} = \frac{\Phi''(\xi)}{\Psi''(\xi)} \tag{5.20}$$

holds, provided that the denominator in the left-hand side of (5.20) *is nonzero.*

Proof. Consider the following function χ, defined as the linear combination of functions Φ and Ψ by

$$\chi(x) = \mathscr{J}_{\alpha 1}(g,\Psi)\Phi(x) - \mathscr{J}_{\alpha 1}(g,\Phi)\Psi(x).$$

Clearly, $\chi \in \mathrm{C}^2([m,M],\mathbb{R})$. By applying Theorem 5.4 to χ, it follows that there exists $\xi \in [m,M]$ such that

$$\mathscr{J}_{\alpha 1}(g,\chi) = \chi''(\xi)\mathscr{J}_{\alpha 1}(g,\Phi_0). \tag{5.21}$$

Thus, $\mathscr{J}_{\alpha 1}(g,\chi) = 0$, and by hypotheses $\mathscr{J}_{\alpha 1}(g,\Phi_0) \neq 0$ (otherwise we have a contradiction with $\mathscr{J}_{\alpha 1}(g,\Psi) \neq 0$). It follows that

$$\chi''(\xi) = 0,$$

which is equivalent to (5.20). □

Remark 5.6 In Theorem 5.5, if the inverse of the function $\frac{\Phi''}{\Psi''}$ exists, then (5.20) gives

$$\xi = \left(\frac{\Phi''}{\Psi''}\right)^{-1}\left(\frac{\mathscr{J}_{\alpha 1}(g,\Phi)}{\mathscr{J}_{\alpha 1}(g,\Psi)}\right).$$

Remark 5.7 Note that setting the function Ψ as $\Psi(x) = \frac{x^2}{2}$ in Theorem 5.5, we get the statement of Theorem 5.4.

As a consequence of Theorem 5.4 and Theorem 5.5, the following corollaries easily follow.

Corollary 5.6 *Let $g \in \mathrm{C}([a,b]_{\mathbb{T}},[m,M])$, $\Phi,\Psi : [m,M] \to \mathbb{R}$ and let $w \in \mathrm{C}([a,b]_{\mathbb{T}},\mathbb{R})$ be an α-SP weight for g. Let $\mathscr{J}_{\alpha 1}$ be the functional defined in* (5.17)*. Then the following two statements hold:*

(i) *If $\Phi \in \mathrm{C}^2([m,M],\mathbb{R})$, then there exists $\xi \in [m,M]$ such that* (5.18) *holds.*

(ii) *If $\Phi,\Psi \in \mathrm{C}^2([m,M],\mathbb{R})$, then there exists $\xi \in [m,M]$ such that* (5.20) *holds.*

Proof. The statement (i) (statement (ii), respectively) directly follows from Theorem 5.4 (Theorem 5.5, respectively) and Corollary 5.2. □

Corollary 5.7 *Let $g \in \mathrm{C}([a,b]_{\mathbb{T}},[m,M])$ be monotone, $\Phi,\Psi : [m,M] \to \mathbb{R}$, and $w \in \mathrm{C}([a,b]_{\mathbb{T}},\mathbb{R})$ such that $\int_a^b w(t)\diamond_\alpha t > 0$. Suppose* (5.2) *holds for all $s \in [a,b]_{\mathbb{T}}$. Let $\mathscr{J}_{\alpha 1}$ be the functional defined in* (5.17)*. Then the following two statements hold:*

(i) *If $\Phi \in \mathrm{C}^2([m,M],\mathbb{R})$, then there exists $\xi \in [m,M]$ such that* (5.18) *holds.*

(ii) *If $\Phi,\Psi \in \mathrm{C}^2([m,M],\mathbb{R})$, then there exists $\xi \in [m,M]$ such that* (5.20) *holds.*

Proof. The statement (i) (statement (ii), respectively) directly follows from Theorem 5.4 (Theorem 5.5, respectively) and Corollary 5.3. □

5.3 Converse of Jensen–Steffensen Inequality

Using a similar method as in Section 5.2, in the following theorem, we obtain a generalization of the converse of the Jensen–Steffensen inequality on time scales, where negative weights are also allowed.

Theorem 5.6 *Let $g \in \mathrm{C}([a,b]_{\mathbb{T}},\mathbb{R})$ be such that $g([a,b]_{\mathbb{T}}) \subseteq [m,M]$ and let $c,d \in [m,M]$, where $c \neq d$, be such that $c \leq g(t) \leq d$ for all $t \in [a,b]_{\mathbb{T}}$. Let $w \in \mathrm{C}([a,b]_{\mathbb{T}},\mathbb{R})$ be such that $\int_a^b w(t)\diamond_\alpha t \neq 0$. Then the following two statements are equivalent:*

(i) *For every convex $\Phi \in \mathrm{C}([m,M],\mathbb{R})$, we have*

$$\frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} \leq \frac{d-\overline{g}_\alpha}{d-c}\Phi(c) + \frac{\overline{g}_\alpha - c}{d-c}\Phi(d). \tag{5.22}$$

(ii) *For all $y \in [m,M]$,*

$$\frac{\int_a^b G(g(t),y)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} \leq \frac{d-\overline{g}_\alpha}{d-c}G(c,y) + \frac{\overline{g}_\alpha - c}{d-c}G(d,y) \tag{5.23}$$

holds, where $G : [m,M] \times [m,M] \to \mathbb{R}$ is defined in (5.7).

Furthermore, the statements (i) *and* (ii) *are also equivalent if we reverse the inequality in both* (5.22) *and* (5.23).

Proof. The idea of the proof is very similar to the proof of Theorem 5.3. Suppose that (i) holds. As $G(\cdot,y)$, where $y \in [m,M]$, is also continuous and convex, it follows that (5.23) holds. Suppose now that (ii) holds. Let $\Phi \in \mathrm{C}^2([m,M],\mathbb{R})$. Then, by using (5.8), we get

$$\begin{aligned} &\frac{d-\overline{g}_\alpha}{d-c}\Phi(c) + \frac{\overline{g}_\alpha - c}{d-c}\Phi(d) - \frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} \\ &\quad = \int_m^M \left[\frac{d-\overline{g}_\alpha}{d-c}G(c,y) + \frac{\overline{g}_\alpha - c}{d-c}G(d,y) - \frac{\int_a^b G(g(t),y)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\right]\Phi''(y)\mathrm{d}y. \end{aligned} \tag{5.24}$$

If Φ is also convex, then $\Phi''(y) \geq 0$ for all $y \in [m,M]$, and hence it follows that for every convex $\Phi \in \mathrm{C}^2([m,M],\mathbb{R})$, (5.22) holds. Moreover, it is not necessary to demand the existence of the second derivative of Φ (see [119, page 172]). The differentiability condition can be directly eliminated by using the fact that it is possible to approximate a continuous convex function uniformly by convex polynomials. The last part of our theorem can be proved analogously. □

Remark 5.8 Let the conditions of Theorem 5.6 hold. Then the following two statements are equivalent:

(i′) For every concave $\Phi \in \mathrm{C}([m,M],\mathbb{R})$, the reverse inequality in (5.22) holds.

(ii′) For all $y \in [m,M]$, (5.23) holds.

Moreover, the statements (i′) and (ii′) are also equivalent if we reverse the inequality in both statements (i′) and (ii′).

Remark 5.9 Note that in all the results in this section, we allow that the mean value $\overline{g}_\alpha$ leaves the interval $[m,M]$, while in the results from Section 5.2, we demanded that $\overline{g}_\alpha \in [m,M]$.

Setting $c = m$ and $d = M$ in Theorem 5.6, we get the following result.

Corollary 5.8 *Let $g \in \mathrm{C}([a,b]_{\mathbb{T}},\mathbb{R})$ be such that $g([a,b]_{\mathbb{T}}) \subseteq [m,M]$. Let $w \in \mathrm{C}([a,b]_{\mathbb{T}},\mathbb{R})$ be such that $\int_a^b w(t)\diamond_\alpha t \neq 0$. Then the following two statements are equivalent:*

(i) *For every convex $\Phi \in \mathrm{C}([m,M],\mathbb{R})$, we have*

$$\frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} \leq \frac{M-\overline{g}_\alpha}{M-m}\Phi(m) + \frac{\overline{g}_\alpha - m}{M-m}\Phi(M). \tag{5.25}$$

(ii) *For all $y \in [m,M]$,*

$$\frac{\int_a^b G(g(t),y)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} \leq 0 \tag{5.26}$$

holds, where $G : [m,M] \times [m,M] \to \mathbb{R}$ is defined in (5.7).

Furthermore, the statements (i) *and* (ii) *are also equivalent if we reverse the inequality in both* (5.25) *and* (5.26).

Remark 5.10 As a consequence of Corollary 5.8, we obtain Lemma 5.2. Let $c = m$ and $d = M$. Then (5.24) transforms into

$$\frac{M-\overline{g}_\alpha}{M-m}\Phi(m) + \frac{\overline{g}_\alpha - m}{M-m}\Phi(M) - \frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} = -\int_m^M \frac{\int_a^b G(g(t),y)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\Phi''(y)\mathrm{d}y. \tag{5.27}$$

Let $\int_a^b w(t)\diamond_\alpha t > 0$ and suppose that g is nondecreasing and that it has the first derivative. Now, similarly as in [51], we obtain Lemma 5.2. Let $m = g(a)$, $M = g(b)$ and make the substitution $y = g(s)$. Then we get

$$\frac{M-\overline{g}_{\alpha}}{M-m}\Phi(m)+\frac{\overline{g}_{\alpha}-m}{M-m}\Phi(M)-\frac{\int_a^b \Phi(g(t))w(t)\diamond_{\alpha}t}{\int_a^b w(t)\diamond_{\alpha}t}$$
$$=-\frac{1}{\int_a^b w(t)\diamond_{\alpha}t}\int_a^b\left(\int_a^b G(g(t),g(s))w(t)\diamond_{\alpha}t\right)\Phi''(g(s))g'(s)\mathrm{d}s. \quad (5.28)$$

Since g is nondecreasing, $g'(s)\geq 0$. If $\Phi\in \mathrm{C}^2([m,M],\mathbb{R})$ is convex, then $\Phi''(g(s))\geq 0$, for all $s\in[a,b]_{\mathbb{T}}$. Hence every continuous and convex Φ satisfies (5.25) if and only if

$$\int_a^b G(g(t),g(s))w(t)\diamond_{\alpha}t\leq 0$$

holds for all $s\in[a,b]_{\mathbb{T}}$.

Corollary 5.9 *Let $g\in \mathrm{C}([a,b]_{\mathbb{T}},[m,M])$ and $w\in \mathrm{C}([a,b]_{\mathbb{T}},\mathbb{R})$ be an α-SP weight for g on $[a,b]_{\mathbb{T}}$. Then* (5.26) *holds for all $y\in[m,M]$.*

Proof. The proof follows directly from Theorem 5.2 and Corollary 5.8. □

Under the assumptions of Theorem 5.6, we define the following functional $\mathscr{J}_{\alpha 2}(g,\Phi)$:

$$\mathscr{J}_{\alpha 2}(g,\Phi)=\begin{cases}\dfrac{d-\overline{g}_{\alpha}}{d-c}\Phi(c)+\dfrac{\overline{g}_{\alpha}-c}{d-c}\Phi(d)-\dfrac{\int_a^b \Phi(g(t))w(t)\diamond_{\alpha}t}{\int_a^b w(t)\diamond_{\alpha}t} \\ \qquad \text{if (5.23) holds for all } y\in[m,M], \\ \\ \dfrac{\int_a^b \Phi(g(t))w(t)\diamond_{\alpha}t}{\int_a^b w(t)\diamond_{\alpha}t}-\dfrac{d-\overline{g}_{\alpha}}{d-c}\Phi(c)-\dfrac{\overline{g}_{\alpha}-c}{d-c}\Phi(d) \\ \qquad \text{if the reverse of (5.23) holds for all } y\in[m,M],\end{cases} \quad (5.29)$$

where the function Φ is defined on $[m,M]$. Clearly, if Φ is continuous and convex, then $\mathscr{J}_{\alpha 2}(g,\Phi)$ is nonnegative.

Theorem 5.7 *Let $c,d,g,$ and w satisfy the assumptions of Theorem 5.6. Let $\Phi\in \mathrm{C}^2([m,M],\mathbb{R})$ and $\mathscr{J}_{\alpha 2}$ be the functional defined in* (5.29)*. Then there exists $\xi\in[m,M]$ such that*

$$\mathscr{J}_{\alpha 2}(g,\Phi)=\Phi''(\xi)\mathscr{J}_{\alpha 2}(g,\Phi_0) \quad (5.30)$$

holds, where $\Phi_0(x)=\frac{x^2}{2}$.

Proof. The idea of the proof is very similar to the proof of Theorem 5.4. Following the assumptions of our theorem, we have that Φ'' is continuous and that

$$\frac{d-\overline{g}_{\alpha}}{d-c}G(c,y)+\frac{\overline{g}_{\alpha}-c}{d-c}G(d,y)-\frac{\int_a^b G(g(t),y)w(t)\diamond_{\alpha}t}{\int_a^b w(t)\diamond_{\alpha}t}$$

does not change positivity on $[m,M]$. For Φ, (5.24) holds, and now applying the integral mean value theorem, we get that there exists $\xi\in[m,M]$ such that

$$
\begin{aligned}
&\frac{d-\overline{g}_\alpha}{d-c}\Phi(c)+\frac{\overline{g}_\alpha-c}{d-c}\Phi(d)-\frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} \\
&\quad =\Phi''(\xi)\int_m^M\left[\frac{d-\overline{g}_\alpha}{d-c}G(c,y)+\frac{\overline{g}_\alpha-c}{d-c}G(d,y)-\frac{\int_a^b G(g(t),y)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\right]\mathrm{d}y. \qquad (5.31)
\end{aligned}
$$

Calculating the integral on the right-hand side in (5.31), we get

$$
\begin{aligned}
&\frac{d-\overline{g}_\alpha}{d-c}\Phi(c)+\frac{\overline{g}_\alpha-c}{d-c}\Phi(d)-\frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} \\
&\quad =\frac{1}{2}\Phi''(\xi)\left[\frac{d-\overline{g}_\alpha}{d-c}\cdot c^2+\frac{\overline{g}_\alpha-c}{d-c}\cdot d^2-\frac{\int_a^b (g(t))^2 w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\right], \qquad (5.32)
\end{aligned}
$$

and we obtain (5.30). □

Remark 5.11 Note that (5.32) can also be expressed as

$$
\begin{aligned}
&\frac{d-\overline{g}_\alpha}{d-c}\Phi(c)+\frac{\overline{g}_\alpha-c}{d-c}\Phi(d)-\frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} \\
&\quad =\frac{1}{2}\Phi''(\xi)\left[\overline{g}_\alpha(c+d)-cd-\frac{\int_a^b (g(t))^2 w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\right].
\end{aligned}
$$

Theorem 5.8 *Let $c,d,g,$ and w satisfy the assumptions of Theorem 5.6. Let $\Phi,\Psi\in C^2([m,M],\mathbb{R})$ and $\mathscr{J}_{\alpha 2}$ be the functional defined in* (5.29)*. Then there exists $\xi\in[m,M]$ such that*

$$
\frac{\mathscr{J}_{\alpha 2}(g,\Phi)}{\mathscr{J}_{\alpha 2}(g,\Psi)}=\frac{\Phi''(\xi)}{\Psi''(\xi)}, \qquad (5.33)
$$

provided that the denominator in the left-hand side of (5.33) *is nonzero.*

Proof. The proof is very similar to the proof of Theorem 5.5. □

5.4 Exponential Convexity and Logarithmic Convexity

We use an idea from [78] to give a method of producing n-exponentially convex and exponentially convex functions, applying the functionals $\mathscr{J}_{\alpha 1}$ and $\mathscr{J}_{\alpha 2}$ to a given family of functions with the same property.

Theorem 5.9 *Let $\mathscr{J}_{\alpha i}$, $i = 1,2$, be the linear functionals defined in* (5.17) *and* (5.29)*, respectively. Let $\Omega = \{\Phi_\rho : \rho \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions $\Phi_\rho \in \mathrm{C}([m,M],\mathbb{R})$ such that the function $\rho \mapsto [x_0,x_1,x_2;\Phi_\rho]$ is n-exponentially convex in the Jensen sense on J for every choice of three mutually different points $x_0,x_1,x_2 \in [m,M]$. Then $\rho \mapsto \mathscr{J}_{\alpha i}(g,\Phi_\rho)$ is an n-exponentially convex function in the Jensen sense on J. If the function $\rho \mapsto \mathscr{J}_{\alpha i}(g,\Phi_\rho)$ is also continuous on J, then it is n-exponentially convex on J.*

Proof. Define the function $\nu : I \to \mathbb{R}$ by

$$\nu(x) = \sum_{j,k=1}^{n} \xi_j \xi_k \Phi_{r_{jk}}(x),$$

where $\xi_j \in \mathbb{R}$, $r_j, r_k \in J$, $1 \le j,k \le n$, $r_{jk} = \dfrac{r_j + r_k}{2}$, and $\Phi_{r_{jk}} \in \Omega$. Using the assumption that for every choice of three mutually different points $x_0,x_1,x_2 \in [m,M]$, $\rho \mapsto [x_0,x_1,x_2;\Phi_\rho]$ is n-exponentially convex in the Jensen sense on J, we obtain that

$$[x_0,x_1,x_2;\nu] = \sum_{j,k=1}^{n} \xi_j \xi_k [x_0,x_1,x_2;\Phi_{r_{jk}}] \ge 0.$$

Therefore ν is convex (and continuous) on I. Hence $\mathscr{J}_{\alpha i}(g,\nu) \ge 0$, $i = 1,2$, which implies that

$$\sum_{j,k=1}^{n} \xi_j \xi_k \mathscr{J}_{\alpha i}(g,\Phi_{r_{jk}}) \ge 0.$$

We conclude that the function $\rho \mapsto \mathscr{J}_{\alpha i}(g,\Phi_\rho)$ is n-exponentially convex on J in the Jensen sense. If $\rho \mapsto \mathscr{J}_{\alpha i}(g,\Phi_\rho)$ is continuous on J, then $\rho \mapsto \mathscr{J}_{\alpha i}(g,\Phi_\rho)$ is n-exponentially convex by definition. □

The following corollary is an immediate consequence of Theorem 5.9.

Corollary 5.10 *Let $\mathscr{J}_{\alpha i}$, $i = 1,2$, be the linear functionals defined in* (5.17) *and* (5.29)*, respectively. Let $\Omega = \{\Phi_\rho : \rho \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions $\Phi_\rho \in \mathrm{C}([m,M],\mathbb{R})$ such that the function $\rho \mapsto [x_0,x_1,x_2;\Phi_\rho]$ is exponentially convex in the Jensen sense on J for every choice of three mutually different points $x_0,x_1,x_2 \in [m,M]$. Then $\rho \mapsto \mathscr{J}_{\alpha i}(g,\Phi_\rho)$ is an exponentially convex function in the Jensen sense on J. If the function $\rho \mapsto \mathscr{J}_{\alpha i}(g,\Phi_\rho)$ is also continuous on J, then it is exponentially convex on J.*

Corollary 5.11 *Let $\mathscr{J}_{\alpha i}$, $i=1,2$, be the linear functionals defined in* (5.17) *and* (5.29)*, respectively. Let $\Omega=\{\Phi_\rho:\ \rho\in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions $\Phi_\rho\in \mathrm{C}([m,M],\mathbb{R})$ such that the function $\rho\mapsto[x_0,x_1,x_2;\Phi_\rho]$ is* 2*-exponentially convex in the Jensen sense on J for every choice of three mutually different points $x_0,x_1,x_2\in[m,M]$. Then the following statements hold:*

(i) *$\rho\mapsto\mathscr{J}_{\alpha i}(g,\Phi_\rho)$ is a* 2*-exponentially convex function in the Jensen sense on J.*

(ii) *If $\rho\mapsto\mathscr{J}_{\alpha i}(g,\Phi_\rho)$ is continuous on J, then it is also* 2*-exponentially convex on J. If $\rho\mapsto\mathscr{J}_{\alpha i}(g,\Phi_\rho)$ is additionally strictly positive, then it is also log-convex on J.*

(iii) *If $\rho\mapsto\mathscr{J}_{\alpha i}(g,\Phi_\rho)$ is strictly positive and differentiable function on J, then for every $p,q,u,v\in J$ such that $p\le u$, $q\le v$, we have*

$$\mathscr{M}_{p,q}(g,\mathscr{J}_{\alpha i},\Omega)\le\mathscr{M}_{u,v}(g,\mathscr{J}_{\alpha i},\Omega),\quad i=1,2, \tag{5.34}$$

where

$$\mathscr{M}_{p,q}(g,\mathscr{J}_{\alpha i},\Omega)=\begin{cases}\left(\dfrac{\mathscr{J}_{\alpha i}(g,\Phi_p)}{\mathscr{J}_{\alpha i}(g,\Phi_q)}\right)^{\frac{1}{p-q}}, & p\neq q;\\[2ex] \exp\left(\dfrac{\frac{\mathrm{d}}{\mathrm{d}p}\mathscr{J}_{\alpha i}(g,\Phi_p)}{\mathscr{J}_{\alpha i}(g,\Phi_p)}\right), & p=q,\end{cases} \tag{5.35}$$

for $\Phi_p,\Phi_q\in\Omega$.

Proof. (i) and (ii) are immediate consequences of Theorem 5.9. To prove (iii), let $\rho\mapsto\mathscr{J}_{\alpha i}(g,\Phi_\rho)$ be strictly positive and differentiable and therefore continuous, too. By (ii), the function $\rho\mapsto\mathscr{J}_{\alpha i}(g,\Phi_\rho)$ is log-convex on J, and by Remark 1.1 (e), we obtain

$$\frac{\log\mathscr{J}_{\alpha i}(g,\Phi_p)-\log\mathscr{J}_{\alpha i}(g,\Phi_q)}{p-q}\le\frac{\log\mathscr{J}_{\alpha i}(g,\Phi_u)-\log\mathscr{J}_{\alpha i}(g,\Phi_v)}{u-v} \tag{5.36}$$

for $p\le u$, $q\le v$, $p\neq q$, $u\neq v$, concluding

$$\mathscr{M}_{p,q}(g,\mathscr{J}_{\alpha i},\Omega)\le\mathscr{M}_{u,v}(g,\mathscr{J}_{\alpha i},\Omega).$$

The cases $p=q$ and $u=v$ follow from (5.36) as limit cases. □

Remark 5.12 Note that the results from Theorem 5.9, Corollary 5.10, and Corollary 5.11 still hold when two of the points $x_0,x_1,x_2\in[m,M]$ coincide, for a family of differentiable functions Φ_ρ such that the function $\rho\mapsto[x_0,x_1,x_2;\Phi_\rho]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense) and furthermore, they still hold when all three points coincide for a family of twice differentiable functions with the same property. The proofs are obtained by recalling Remark 1.3 and a suitable characterization of convexity.

5.5 Examples

In this section, we will vary the choice of a family $\Omega = \{\Phi_\rho : \rho \in J\}$ in order to construct different examples of exponentially convex functions and construct some means.

Example 5.2 Consider the family of functions

$$\Omega_1 = \{\kappa_\rho : \mathbb{R} \to [0,\infty);\ \rho \in \mathbb{R}\}$$

defined by

$$\kappa_\rho(x) = \begin{cases} \frac{1}{\rho^2} e^{\rho x}, & \rho \neq 0; \\ \frac{1}{2} x^2, & \rho = 0. \end{cases}$$

We have $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\kappa_\rho(x) = e^{\rho x} > 0$, which shows that κ_ρ is convex on $\mathbb{R}$ for every $\rho \in \mathbb{R}$. From Remark 1.8, it follows that $\rho \mapsto \frac{\mathrm{d}^2}{\mathrm{d}x^2}\kappa_\rho(x)$ is exponentially convex. Therefore, $\rho \mapsto [x_0,x_1,x_2;\kappa_\rho]$ is exponentially convex (see [78]) (and so exponentially convex in the Jensen sense). Now using Corollary 5.10, we conclude that $\rho \mapsto \mathscr{J}_{\alpha i}(g,\kappa_\rho),\ i = 1,2$, are exponentially convex in the Jensen sense. It is easy to verify that these mappings are continuous, so they are exponentially convex. For this family of functions, $\mathscr{M}_{p,q}(g,\mathscr{J}_{\alpha i},\Omega)$ from (5.35) becomes

$$\mathscr{M}_{p,q}(g,\mathscr{J}_{\alpha i},\Omega_1) = \begin{cases} \left(\frac{\mathscr{J}_{\alpha i}(g,\kappa_p)}{\mathscr{J}_{\alpha i}(g,\kappa_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{\mathscr{J}_{\alpha i}(g,\mathrm{id}\cdot\kappa_p)}{\mathscr{J}_{\alpha i}(g,\kappa_p)} - \frac{2}{p}\right), & p = q \neq 0; \\ \exp\left(\frac{\mathscr{J}_{\alpha i}(g,\mathrm{id}\cdot\kappa_0)}{3\,\mathscr{J}_{\alpha i}(g,\kappa_0)}\right), & p = q = 0, \end{cases}$$

and using (5.34), we have that it is monotone in p and q. If $\mathscr{J}_{\alpha i},\ i = 1,2$, are positive, using Theorem 5.5 and Theorem 5.8 applied for $\Phi = \kappa_p \in \Omega_1$ and $\Psi = \kappa_q \in \Omega_1$, it follows that

$$\aleph_{p,q}(g,\mathscr{J}_{\alpha i},\Omega_1) = \log \mathscr{M}_{p,q}(g,\mathscr{J}_{\alpha i},\Omega_1) \qquad i = 1,2$$

satisfy $\aleph_{p,q}(g,\mathscr{J}_{\alpha i},\Omega_1) \in [m,M]$. If we set $g([a,b]_{\mathbb{T}}) = [m,M]$, then we have that $\aleph_{p,q}(g,\mathscr{J}_{\alpha i},\Omega_1)$ are means (of the function g). Note that by (5.34) they are monotone means.

Example 5.3 Consider the family of functions

$$\Omega_2 = \{\beta_\rho : (0,\infty) \to \mathbb{R};\ \rho \in \mathbb{R}\}$$

defined by

$$\beta_\rho(x) = \begin{cases} \dfrac{x^\rho}{\rho(\rho-1)}, & \rho \neq 0,1; \\ -\log x, & \rho = 0; \\ x\log x, & \rho = 1. \end{cases}$$

We have $\frac{d^2}{dx^2}\beta_\rho(x) = x^{\rho-2} = e^{(\rho-2)\log x} > 0$, which shows that β_ρ is convex for $x > 0$. Also, from Remark 1.8, it follows that $\rho \mapsto \frac{d^2}{dx^2}\beta_\rho(x)$ is exponentially convex. Therefore, $\rho \mapsto [x_0, x_1, x_2; \beta_\rho]$ is exponentially convex (and so exponentially convex in the Jensen sense). Here we assume that $[m,M] \subset (0,\infty)$, so Ω_2 fulfills the conditions of Corollary 5.10. In this case, $\mathscr{M}_{p,q}(g, \mathscr{J}_{\alpha i}, \Omega)$ from (5.35) becomes

$$\mathscr{M}_{p,q}(g, \mathscr{J}_{\alpha i}, \Omega_2) = \begin{cases} \left(\dfrac{\mathscr{J}_{\alpha i}(g,\beta_p)}{\mathscr{J}_{\alpha i}(g,\beta_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\dfrac{1-2p}{p(p-1)} - \dfrac{\mathscr{J}_{\alpha i}(g,\beta_0\beta_p)}{\mathscr{J}_{\alpha i}(g,\beta_p)}\right), & p = q \neq 0,1; \\ \exp\left(1 - \dfrac{\mathscr{J}_{\alpha i}(g,\beta_0^2)}{2\mathscr{J}_{\alpha i}(g,\beta_0)}\right), & p = q = 0; \\ \exp\left(-1 - \dfrac{\mathscr{J}_{\alpha i}(g,\beta_0\beta_1)}{2\mathscr{J}_{\alpha i}(g,\beta_1)}\right), & p = q = 1. \end{cases}$$

If $\mathscr{J}_{\alpha i}$, $i = 1,2$, are positive, by applying Theorem 5.5 and Theorem 5.8 for $\Phi = \beta_p \in \Omega_2$ and $\Psi = \beta_q \in \Omega_2$, it follows that for $i = 1,2$, there exist $\xi_i \in [m,M]$ such that

$$\xi_i^{p-q} = \frac{\mathscr{J}_{\alpha i}(g,\beta_p)}{\mathscr{J}_{\alpha i}(g,\beta_q)}.$$

Since the function $\xi_i \mapsto \xi_i^{p-q}$ is invertible for $p \neq q$, we have

$$m \leq \left(\frac{\mathscr{J}_{\alpha i}(g,\beta_p)}{\mathscr{J}_{\alpha i}(g,\beta_q)}\right)^{\frac{1}{p-q}} \leq M. \tag{5.37}$$

Also, $\mathscr{M}_{p,q}(g, \mathscr{J}_{\alpha i}, \Omega_2)$ is continuous, symmetric and monotone (by (5.34)). If we set $g([a,b]_{\mathbb{T}}) = [m,M]$, then we have that

$$m = \min_{t\in[a,b]_{\mathbb{T}}}\{g(t)\} \leq \left(\frac{\mathscr{J}_{\alpha i}(g,\beta_p)}{\mathscr{J}_{\alpha i}(g,\beta_q)}\right)^{\frac{1}{p-q}} \leq \max_{t\in[a,b]_{\mathbb{T}}}\{g(t)\} = M, \quad \text{for } i = 1,2, \tag{5.38}$$

which shows that $\mathscr{M}_{p,q}(g, \mathscr{J}_{\alpha i}, \Omega_2)$ are means (of the function g). Now we impose one additional parameter r in case $g([a,b]_{\mathbb{T}}) = [m,M]$. For $r \neq 0$, by substituting $g \mapsto g^r$,

$p \mapsto p/r$ and $q \mapsto q/r$ in (5.3), we get

$$m = \min_{t\in[a,b]_{\mathbb{T}}} \{g^r(t)\} \leq \left(\frac{\mathscr{J}_{\alpha i}(g^r,\beta_p)}{\mathscr{J}_{\alpha i}(g^r,\beta_q)}\right)^{\frac{r}{p-q}} \leq \max_{t\in[a,b]_{\mathbb{T}}} \{g^r(t)\} = M, \quad \text{for } i = 1,2. \quad (5.39)$$

We define new generalized means by

$$\mathscr{M}_{p,q;r}(g, \mathscr{J}_{\alpha i}, \Omega_2) = \begin{cases} \left(\mathscr{M}_{\frac{p}{r},\frac{q}{r}}(g^r, \mathscr{J}_{\alpha i}, \Omega_2)\right)^{\frac{1}{r}}, & r \neq 0; \\ \left(\mathscr{M}_{\frac{p}{r},\frac{q}{r}}(\log g, \mathscr{J}_{\alpha i}, \Omega_1)\right), & r = 0. \end{cases}$$

These new generalized means are also monotone.

Example 5.4 Consider the family of functions

$$\Omega_3 = \{\gamma_\rho : (0,\infty) \to (0,\infty) : \rho \in (0,\infty)\}$$

defined by

$$\gamma_\rho(x) = \begin{cases} \dfrac{\rho^{-x}}{(\log\rho)^2}, & \rho \neq 1; \\ \dfrac{x^2}{2}, & \rho = 1. \end{cases}$$

We have $\frac{d^2}{dx^2}\gamma_\rho(x) = \rho^{-x} > 0$, which shows that γ_ρ is convex for $\rho > 0$. Also, from Remark 1.8, it follows that $\rho \mapsto \frac{d^2}{dx^2}\gamma_\rho(x)$ is exponentially convex. Therefore, $\rho \mapsto [x_0,x_1,x_2;\gamma_\rho]$ is exponentially convex (and so exponentially convex in the Jensen sense). Here we assume that $[m,M] \subset (0,\infty)$, so Ω_3 fulfills the conditions of Corollary 5.10. In this case, $\mathscr{M}_{p,q}(g, \mathscr{J}_{\alpha i}, \Omega)$ from (5.35) becomes

$$\mathscr{M}_{p,q}(g, \mathscr{J}_{\alpha i}, \Omega_3) = \begin{cases} \left(\dfrac{\mathscr{J}_{\alpha i}(g,\gamma_p)}{\mathscr{J}_{\alpha i}(g,\gamma_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(-\dfrac{\mathscr{J}_{\alpha i}(g,\mathrm{id}\cdot\gamma_p)}{p\,\mathscr{J}_{\alpha i}(g,\gamma_p)} - \dfrac{2}{p\log p}\right), & p = q \neq 1; \\ \exp\left(\dfrac{-\mathscr{J}_{\alpha i}(g,\mathrm{id}\cdot\gamma_1)}{3\,\mathscr{J}_{\alpha i}(g,\gamma_1)}\right), & p = q = 1, \end{cases}$$

and by (5.34) it is monotone in p and q. Using Theorem 5.5 and Theorem 5.8, it follows that for $i = 1,2$,

$$\aleph_{p,q}(g, \mathscr{J}_{\alpha i}, \Omega_3) = -L(p,q)\log \mathscr{M}_{p,q}(g, \mathscr{J}_{\alpha i}, \Omega_3)$$

satisfy $\aleph_{p,q}(g, \mathscr{J}_{\alpha i}, \Omega_3) \in [m,M]$. Here, $L(p,q)$ is the logarithmic mean defined by

$$L(p,q) = \frac{p-q}{\log p - \log q}, \quad p \neq q, \quad L(p,p) = p.$$

Example 5.5 Consider the family of functions

$$\Omega_4 = \{\delta_\rho : (0,\infty) \to (0,\infty) : \rho \in (0,\infty)\}$$

defined by

$$\delta_\rho(x) = \frac{e^{-x\sqrt{\rho}}}{\rho}.$$

We have $\frac{d^2}{dx^2}\delta_\rho(x) = e^{-x\sqrt{\rho}} > 0$, which shows that δ_ρ is convex for $\rho > 0$. Also, from Remark 1.8, it follows that $\rho \mapsto \frac{d^2}{dx^2}\delta_\rho(x)$ is exponentially convex. Therefore, $\rho \mapsto [x_0, x_1, x_2; \delta_\rho]$ is exponentially convex (and so exponentially convex in the Jensen sense). Here we assume that $[m,M] \subset (0,\infty)$, so Ω_4 fulfills the conditions of Corollary 5.10. In this case, $\mathscr{M}_{p,q}(g, \mathscr{J}_{\alpha i}, \Omega)$ from (5.35) becomes

$$\mathscr{M}_{p,q}(g, \mathscr{J}_{\alpha i}, \Omega_4) = \begin{cases} \left(\dfrac{\mathscr{J}_{\alpha i}(g, \delta_p)}{\mathscr{J}_{\alpha i}(g, \delta_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(-\dfrac{\mathscr{J}_{\alpha i}(g, \mathrm{id}\cdot\delta_p)}{2\sqrt{p}\,\mathscr{J}_{\alpha i}(g, \delta_p)} - \dfrac{1}{p}\right), & p = q, \end{cases}$$

and it is monotone in p and q by (5.34). Using Theorem 5.5 and Theorem 5.8, it follows that for $i = 1, 2$,

$$\aleph_{p,q}(g, \mathscr{J}_{\alpha i}, \Omega_4) = -(\sqrt{s} + \sqrt{q}) \log \mathscr{M}_{p,q}(g, \mathscr{J}_{\alpha i}, \Omega_4)$$

satisfies $\aleph_{p,q}(g, \mathscr{J}_{\alpha i}, \Omega_4) \in [m,M]$.

Chapter 6

Improvements of the Hermite–Hadamard Inequality

In this chapter, we give several refinements of the converses of Jensen's inequality as well as of the Hermite–Hadamard inequality on time scales. We give mean value theorems and investigate logarithmic and exponential convexity of linear functionals related to the obtained refinements. We also give several examples which illustrate possible applications for our results. Our presentation closely follows [38].

6.1 Converses of Jensen's Inequality

To prove our main results, we need the following lemma, which is a simple consequence of [103, page 717, Theorem 1].

Lemma 6.1 *Let Φ be a convex function on I, $x,y \in I$, and $p,q \in [0,1]$ such that $p+q=1$. Then*

$$\begin{aligned} &\min\{p,q\}\left[\Phi(x)+\Phi(y)-2\Phi\left(\frac{x+y}{2}\right)\right] \\ &\leq p\Phi(x)+q\Phi(y)-\Phi(px+qy) \\ &\leq \max\{p,q\}\left[\Phi(x)+\Phi(y)-2\Phi\left(\frac{x+y}{2}\right)\right]. \end{aligned} \tag{6.1}$$

Throughout this chapter, we use the same notation as in Chapter 2.

Theorem 6.1 *Assume* $f:\mathscr{E}\to[m,M]$ *is* Δ*-integrable, where* $[m,M]\subseteq I$*, and* $h:\mathscr{E}\to\mathbb{R}$ *is nonnegative and* Δ*-integrable such that* $L_\Delta(h)>0$*. If* Φ *is convex on I, then*

$$\overline{L}_\Delta(\Phi(f),h)\le\frac{M-\overline{L}_\Delta(f,h)}{M-m}\Phi(m)+\frac{\overline{L}_\Delta(f,h)-m}{M-m}\Phi(M)-\overline{L}_\Delta\left(\tilde{f},h\right)\delta_\Phi,\tag{6.2}$$

where

$$\tilde{f}=\frac{1}{2}-\frac{|f-(m+M)/2|}{M-m},\quad \delta_\Phi=\Phi(m)+\Phi(M)-2\Phi\left(\frac{m+M}{2}\right).\tag{6.3}$$

Moreover, if Φ *is concave, then the inequality in* (6.2) *holds in reverse order.*

Proof. Let $p,q:[m,M]\to\mathbb{R}$ be defined by

$$p(x)=\frac{M-x}{M-m},\quad q(x)=\frac{x-m}{M-m}.\tag{6.4}$$

For any $x\in[m,M]$, we can write

$$\Phi(x)=\Phi\left(\frac{M-x}{M-m}m+\frac{x-m}{M-m}M\right)=\Phi(p(x)m+q(x)M).$$

Suppose Φ is convex. By Lemma 6.1, we have

$$\Phi(x)\le p(x)\Phi(m)+q(x)\Phi(M)\\ -\min\{p(x),q(x)\}\left(\Phi(m)+\Phi(M)-2\Phi\left(\frac{m+M}{2}\right)\right).$$

Since

$$\min\{x,y\}=\frac{1}{2}(x+y-|x-y|),$$

by replacing x with $f(s)$, for $s\in\mathscr{E}$, we obtain

$$\Phi(f(s))\le p(f(s))\Phi(m)+q(f(s))\Phi(M)-\tilde{f}(s)\delta_\Phi\tag{6.5}$$

where the function $\tilde{f}$ is defined on $\mathscr{E}$ by

$$\tilde{f}(s)=\frac{1}{2}-\frac{|f(s)-(m+M)/2|}{M-m}.$$

Since h is nonnegative and Δ-integrable and $L_\Delta(h)>0$, multiplying (6.5) by h, integrating, and then dividing by $L_\Delta(h)$, we obtain

$$\overline{L}_\Delta(\Phi(f),h)\le\overline{L}_\Delta(p(f),h)\Phi(m)+\overline{L}_\Delta(q(f),h)\Phi(M)-\overline{L}_\Delta\left(\tilde{f},h\right)\delta_\Phi,$$

from which (6.2) follows. If Φ is concave, the reverse inequality in (6.2) follows immediately by using the fact that if Φ is concave, then $-\Phi$ is convex. □

Remark 6.1 Theorem 6.1 gives a refinement of Theorem 2.13 as under the required assumptions, we have

$$\overline{L}_\Delta\left(\tilde{f},h\right)\delta_\Phi \geq 0.$$

Theorem 6.2 *Assume $\Phi \in C(I,\mathbb{R})$ is convex and $f : \mathscr{E} \to [m,M]$ is Δ-integrable with $[m,M] \subseteq I$. Also, let $h : \mathscr{E} \to \mathbb{R}$ be nonnegative and Δ-integrable such that $L_\Delta(h) > 0$. Then*

$$\begin{aligned}
&\overline{L}_\Delta(\Phi(f),h) - \Phi\left(\overline{L}_\Delta(f,h)\right) \qquad (6.6)\\
&\leq \max_{x\in[m,M]}\left\{\frac{M-x}{M-m}\Phi(m) + \frac{x-m}{M-m}\Phi(M) - \Phi(x)\right\} - \overline{L}_\Delta\left(\tilde{f},h\right)\delta_\Phi\\
&= \max_{\sigma\in[0,1]}\left\{\sigma\Phi(m) + (1-\sigma)\Phi(M) - \Phi(\sigma m + (1-\sigma)M)\right\} - \overline{L}_\Delta\left(\tilde{f},h\right)\delta_\Phi,
\end{aligned}$$

where $\tilde{f}$ and δ_Φ are defined in (6.3)*.*

Proof. This is an immediate consequence of Theorem 6.1. The identity follows from the change of variables $\sigma = (M-x)/(M-m)$, so that for $x \in [m,M]$, we have $\sigma \in [0,1]$ and $x = \sigma m + (1-\sigma)M$. □

Remark 6.2 Arguing as in Remark 6.1, (6.6) is a refinement of (2.63).

Theorem 6.3 *Assume $\Phi \in C(I,\mathbb{R})$, $f : \mathscr{E} \to [m,M]$ is Δ-integrable, where $[m,M] \subseteq I$, and $h : \mathscr{E} \to \mathbb{R}$ is nonnegative and Δ-integrable such that $L_\Delta(h) > 0$. If Φ is convex, then*

$$\begin{aligned}
&\overline{L}_\Delta(\Phi(f),h) - \Phi\left(\overline{L}_\Delta(f,h)\right)\\
&\qquad \leq \frac{1}{M-m}\left\{\left|\frac{m+M}{2} - \overline{L}_\Delta(f,h)\right| + \overline{L}_\Delta\left(\left|\frac{m+M}{2} - f\right|,h\right)\right\}\delta_\Phi, \qquad (6.7)
\end{aligned}$$

where δ_Φ is defined in (6.3)*. Moreover, if Φ is concave, then* (6.7) *holds in reverse order.*

Proof. Let $p,q : [m,M] \to \mathbb{R}$ be defined as in (6.4). Then for any $x \in [m,M]$, we can write

$$\Phi(x) = \Phi(p(x)m + q(x)M).$$

Since $\overline{L}_\Delta(f,h) \in [m,M]$, the above equation implies that

$$\Phi\left(\overline{L}_\Delta(f,h)\right) = \Phi\left(p\left(\overline{L}_\Delta(f,h)\right)m + q\left(\overline{L}_\Delta(f,h)\right)M\right).$$

Suppose Φ is convex. By Lemma 6.1, we get

$$\begin{aligned}
\Phi\left(\overline{L}_\Delta(f,h)\right) &\geq p\left(\overline{L}_\Delta(f,h)\right)\Phi(m) + q\left(\overline{L}_\Delta(f,h)\right)\Phi(M) \qquad (6.8)\\
&\quad - \max\left\{p\left(\overline{L}_\Delta(f,h)\right), q\left(\overline{L}_\Delta(f,h)\right)\right\}\delta_\Phi\\
&= p\left(\overline{L}_\Delta(f,h)\right)\Phi(m) + q\left(\overline{L}_\Delta(f,h)\right)\Phi(M)\\
&\quad - \left\{\frac{1}{2} + \frac{\left|(m+M)/2 - \overline{L}_\Delta(f,h)\right|}{M-m}\right\}\delta_\Phi.
\end{aligned}$$

Again by Lemma 6.1, we get

$$\Phi(f) \le p(f)\Phi(m) + q(f)\Phi(M) - \min\{p(f), q(f)\}\delta_\Phi,$$

which implies that

$$\begin{aligned}\overline{L}_\Delta(\Phi(f),h) &\le \overline{L}_\Delta(p(f),h)\Phi(m) + \overline{L}_\Delta(q(f),h)\Phi(M) \\ &\quad - \overline{L}_\Delta(\min\{p(f),q(f)\},h)\delta_\Phi \\ &= p\left(\overline{L}_\Delta(f,h)\right)\Phi(m) + q\left(\overline{L}_\Delta(f,h)\right)\Phi(M) \\ &\quad - \left\{\frac{1}{2} - \frac{\overline{L}_\Delta(|f-(m+M)/2|,h)}{M-m}\right\}\delta_\Phi.\end{aligned} \tag{6.9}$$

Now, from inequalities (6.8) and (6.9), we get the desired inequality (6.7). If Φ is concave, the reverse inequality in (6.2) follows immediately by using the fact that if Φ is concave, then $-\Phi$ is convex. □

Corollary 6.1 *Let all the assumptions of Theorem 6.3 be satisfied. If Φ is convex, then*

$$\begin{aligned}&\overline{L}_\Delta(\Phi(f),h) - \Phi\left(\overline{L}_\Delta(f,h)\right) \\ &\qquad \le \left\{\frac{1}{2} + \frac{1}{M-m}\left|\frac{m+M}{2} - \overline{L}_\Delta(f,h)\right|\right\}\delta_\Phi.\end{aligned} \tag{6.10}$$

Moreover, if Φ is concave, then (6.10) *holds in reverse order.*

Proof. Since

$$\frac{1}{M-m}\left|\frac{m+M}{2} - f\right| \le \frac{1}{2},$$

we have

$$\frac{1}{M-m}\overline{L}_\Delta\left(\left|\frac{m+M}{2} - f\right|, h\right) \le \frac{1}{2}.$$

Now inequality (6.10) directly follows from Theorem 6.3. □

Theorem 6.4 *Assume $\Phi : [m,M] \to \mathbb{R}$ is differentiable such that Φ' is strictly increasing on $[m,M]$. Suppose $f : \mathscr{E} \to [m,M]$ is Δ-integrable and $h : \mathscr{E} \to \mathbb{R}$ is nonnegative and Δ-integrable such that $L_\Delta(h) > 0$. If $\tilde{f}$ and δ_Φ are defined as in* (6.3)*, then*

$$\overline{L}_\Delta(\Phi(f),h) \le \lambda + \Phi\left(\overline{L}_\Delta(f,h)\right) - \overline{L}_\Delta\left(\tilde{f},h\right)\delta_\Phi \tag{6.11}$$

holds for some λ satisfying $0 < \lambda < (M-m)(\nu - \Phi'(m))$, where $\nu = (\Phi(M) - \Phi(m))/(M-m)$. More precisely, λ may be determined as follows: Let $\tilde{x}$ be the unique solution of the equation $\Phi'(x) = \nu$. Then

$$\lambda = \Phi(m) - \Phi(\tilde{x}) + \nu(\tilde{x} - m)$$

satisfies (6.11).

Proof. By Theorem 6.2, we have

$$\overline{L}_\Delta(\Phi(f),h) - \Phi\left(\overline{L}_\Delta(f,h)\right) \leq \max_{x\in[m,M]} g(x) - \overline{L}_\Delta\left(\tilde{f},h\right)\delta_\Phi,$$

where

$$g(x) = \frac{M-x}{M-m}\Phi(m) + \frac{x-m}{M-m}\Phi(M) - \Phi(x).$$

Then

$$g'(x) = \nu - \Phi'(x),$$

which is strictly decreasing on I with $g'(\tilde{x}) = 0$ for a unique $\tilde{x} \in I$. Consequently g achieves its maximum value at $\tilde{x}$. Hence the result follows. □

Remark 6.3 Theorem 6.4 gives a refinement of Theorem 2.58.

Corollary 6.2 *Suppose $f : \mathscr{E} \to [m,M]$ is Δ-integrable such that $[m,M] \subset (0,\infty)$ and $h : \mathscr{E} \to \mathbb{R}$ is nonnegative and Δ-integrable such that $L_\Delta(h) > 0$. Then*

$$\overline{L}_\Delta(f,h) \leq \exp\left(\overline{L}_\Delta(\log f,h)\right) \frac{\exp\left(S\left(M/m\right)\right)}{\left[(m+M)^2/4mM\right]^{\overline{L}_\Delta(\tilde{f},h)}}, \tag{6.12}$$

where $S(\cdot)$ is the Specht ratio defined by

$$S(a) = \frac{a^{1/(a-1)}}{e\log a^{1/(a-1)}}, \quad a \in (0,\infty)\setminus\{1\},$$

and $\tilde{f}$ is defined in Theorem 6.1.

Proof. This is a special case of Theorem 6.4 for $\Phi = -\log$. In this case, (6.11) becomes

$$-\overline{L}_\Delta(\log f,h) \leq \lambda - \log(\overline{L}_\Delta(f,h)) - \overline{L}_\Delta\left(\tilde{f},h\right)\delta_{-\log},$$

that is,

$$\begin{aligned}\overline{L}_\Delta(f,h) &\leq \exp\left(\overline{L}_\Delta(\log f,h) + \lambda - \overline{L}_\Delta\left(\tilde{f},h\right)\delta_{-\log}\right)\\ &= \exp\left(\overline{L}_\Delta(\log f,h)\right)\frac{\exp\lambda}{\exp\left(\overline{L}_\Delta\left(\tilde{f},h\right)\delta_{-\log}\right)},\end{aligned}$$

where

$$\delta_{-\log} = -\log m - \log M + 2\log\frac{m+M}{2} = \log\frac{(m+M)^2}{4mM},$$

$$\nu = \frac{\log m - \log M}{M-m}, \quad \tilde{x} = -\frac{1}{\nu} = \frac{M-m}{\log M - \log m},$$

and

$$\lambda = -\log m + \nu(\tilde{x}-m) + \log\tilde{x} = \log\frac{(M/m)^{m/(M-m)}}{e\log(M/m)^{m/(M-m)}} = S\left(\frac{M}{m}\right).$$

Considering all this, we obtain (6.12). □

Corollary 6.3 *Suppose $f : \mathscr{E} \to [m,M]$ is Δ-integrable such that $[m,M] \subset (0,\infty)$ and $h : \mathscr{E} \to \mathbb{R}$ is nonnegative and Δ-integrable such that $L_\Delta(h) > 0$. Then*

$$\begin{aligned}\overline{L}_\Delta(f,h) \le &\exp\left(\overline{L}_\Delta(\log f,h)\right) + \frac{M-m}{\log(M/m)} S\left(\frac{M}{m}\right) \\ &- \overline{L}_\Delta(\tilde{f}_2,h)(m+M-2\sqrt{mM}),\end{aligned} \tag{6.13}$$

where $S(\cdot)$ is the Specht ratio and $\tilde{f}_2$ is defined by

$$\tilde{f}_2 = \frac{1}{2} - \frac{\left|\log f - \log\sqrt{mM}\right|}{\log M - \log m}. \tag{6.14}$$

Proof. This is a special case of Theorem 6.4 for $\Phi = \exp$ and $f = \log f$. In this case, (6.11) becomes

$$\overline{L}_\Delta(\exp\log f,h) \le \lambda + \exp\left(\overline{L}_\Delta(\log f,h)\right) - \overline{L}_\Delta(\tilde{f}_2,h)\delta_{\exp},$$

where

$$\begin{aligned}\delta_{\exp} &= \exp\log m + \exp\log M - 2\exp\frac{\log m + \log M}{2} = m + M - 2\sqrt{mM}, \\ \nu &= \frac{M-m}{\log M - \log m}, \quad \tilde{x} = \log\nu = \log\frac{M-m}{\log M - \log m},\end{aligned}$$

and

$$\begin{aligned}\lambda &= \exp\log m + \nu(\tilde{x} - \log m) - \exp\tilde{x} \\ &= m + \frac{M-m}{\log M - \log m}\left(\log\frac{M-m}{\log M - \log m} - \log m - 1\right) \\ &= \frac{M-m}{\log(M/m)} S\left(\frac{M}{m}\right).\end{aligned}$$

Considering all this, we obtain (6.13). □

6.2 Improvements of the Hermite–Hadamard Inequality

If Φ is continuous in Theorem 6.1, then by combining this theorem with Theorem 2.8, we obtain the refinement of the generalized Hermite–Hadamard inequality (2.4). In the following two theorems, we give improvements of Theorem 2.15.

Theorem 6.5 *Assume $f : \mathscr{E} \to [m,M]$ is Δ-integrable, where $[m,M] \subseteq I$, and $h : \mathscr{E} \to \mathbb{R}$ is nonnegative and Δ-integrable such that $L_\Delta(h) > 0$. Moreover, let p,q be positive numbers such that*

$$\overline{L}_\Delta(f,h) = \frac{pm+qM}{p+q}$$

holds. If Φ is convex on I, then

$$\Phi\left(\frac{pm+qM}{p+q}\right) \leq \overline{L}_\Delta(\Phi(f),h) \leq \frac{p\Phi(m)+q\Phi(M)}{p+q} - \overline{L}_\Delta\left(\tilde{f},h\right)\delta_\Phi, \tag{6.15}$$

where $\tilde{f}$ and δ_Φ are defined as in (6.3). *Moreover, if Φ is concave, then the inequalities in* (6.15) *hold in reverse order.*

Proof. The first inequality in (6.15) follows from Theorem 2.8, and the second one follows from Theorem 6.1. □

Theorem 6.6 *Assume $f : \mathscr{E} \to [m,M]$ is Δ-integrable, where $[m,M] \subseteq I$, and $h : \mathscr{E} \to \mathbb{R}$ is nonnegative and Δ-integrable such that $L_\Delta(h) > 0$. Moreover, let p,q be positive numbers such that*

$$\overline{L}_\Delta(f,h) = \frac{pm+qM}{p+q}, \quad 0 < y \leq \frac{M-m}{p+q}\min\{p,q\} \tag{6.16}$$

holds. If Φ is convex on I, then

$$\begin{aligned}\Phi\left(\frac{pm+qM}{p+q}\right) &\leq \overline{L}_\Delta(\Phi(f),h)\\ &\leq \frac{p\Phi(m)+q\Phi(M)}{p+q} - 2\overline{L}_\Delta\left(\tilde{f}_1,h\right)\left(\frac{p\Phi(m)+q\Phi(M)}{p+q} - \Phi\left(\frac{pm+qM}{p+q}\right)\right),\end{aligned} \tag{6.17}$$

where

$$\tilde{f}_1 = \frac{1}{2} - \frac{|f-(pm+qM)/(p+q)|}{2y}. \tag{6.18}$$

Moreover, if Φ is concave, then (6.17) *holds in reverse order.*

Proof. The first inequality in (6.17) follows from Theorem 2.8. By using (6.16), we have

$$m \leq \overline{L}_\Delta(f,h) - y < \overline{L}_\Delta(f,h) + y \leq M.$$

Suppose $m_1 = \overline{L}_\Delta(f,h) - y$ and $M_1 = \overline{L}_\Delta(f,h) + y$. Then

$$\overline{L}_\Delta(f,h) = \frac{\overline{L}_\Delta(f,h) - y + \overline{L}_\Delta(f,h) + y}{2} = \frac{m_1+M_1}{2}.$$

Now by Theorem 6.5 with $p = q = 1$, we obtain

$$\overline{L}_\Delta(\Phi(f),h) \leq \frac{\Phi(\overline{L}_\Delta(f,h) - y) + \Phi(\overline{L}_\Delta(f,h) + y)}{2}$$

$$-\overline{L}_\Delta\left(\tilde{f}_1,h\right)\left(\Phi\left(\overline{L}_\Delta(f,h)-y\right)+\Phi\left(\overline{L}_\Delta(f,h)+y\right)-2\Phi\left(\overline{L}_\Delta(f,h)\right)\right)$$
$$=\left(1-2\overline{L}_\Delta\left(\tilde{f}_1,h\right)\right)\frac{\Phi\left(\overline{L}_\Delta(f,h)-y\right)+\Phi\left(\overline{L}_\Delta(f,h)+y\right)}{2}$$
$$+2\overline{L}_\Delta\left(\tilde{f}_1,h\right)\Phi\left(\overline{L}_\Delta(f,h)\right).$$

Suppose Φ is convex. By Theorem 2.13, we get

$$\Phi\left(\overline{L}_\Delta(f,h)-y\right)\leq\frac{M-\left(\overline{L}_\Delta(f,h)-y\right)}{M-m}\Phi(m)+\frac{\overline{L}_\Delta(f,h)-y-m}{M-m}\Phi(M),$$
$$\Phi\left(\overline{L}_\Delta(f,h)+y\right)\leq\frac{M-\left(\overline{L}_\Delta(f,h)+y\right)}{M-m}\Phi(m)+\frac{\overline{L}_\Delta(f,h)+y-m}{M-m}\Phi(M).$$

Hence

$$\frac{\Phi\left(\overline{L}_\Delta(f,h)-y\right)+\Phi\left(\overline{L}_\Delta(f,h)+y\right)}{2}\leq\frac{M-\overline{L}_\Delta(f,h)}{M-m}\Phi(m)+\frac{\overline{L}_\Delta(f,h)-m}{M-m}\Phi(M).$$

If p and q are any nonnegative numbers such that (6.16) holds (observe that they are different from those we started with), we obtain

$$\frac{\Phi\left(\overline{L}_\Delta(f,h)-y\right)+\Phi\left(\overline{L}_\Delta(f,h)+y\right)}{2}\leq\frac{p\Phi(m)+q\Phi(M)}{p+q}.$$

Considering all this and the fact that $1-2\overline{L}_\Delta\left(\tilde{f}_1,h\right)\geq 0$, we deduce

$$\overline{L}_\Delta(\Phi(f),h)\leq\left(1-2\overline{L}_\Delta\left(\tilde{f}_1,h\right)\right)\frac{p\Phi(m)+q\Phi(M)}{p+q}+2\overline{L}_\Delta\left(\tilde{f}_1,h\right)\Phi\left(\overline{L}_\Delta(f,h)\right)$$
$$=\frac{p\Phi(m)+q\Phi(M)}{p+q}$$
$$-2\overline{L}_\Delta\left(\tilde{f}_1,h\right)\left[\frac{p\Phi(m)+q\Phi(M)}{p+q}-\Phi\left(\frac{pm+qM}{p+q}\right)\right].$$

If Φ is concave, then the reverse inequality in (6.17) holds immediately by using the fact that if Φ is concave, then $-\Phi$ is convex. □

From (6.17), an inequality of Hammer–Bullen type for multiple Lebesgue Δ-integrals easily follows.

Corollary 6.4 *Let all the assumptions of Theorem 6.6 be satisfied. If Φ is convex, then*

$$\left(1-2\overline{L}_\Delta\left(\tilde{f}_1,h\right)\right)\left[\frac{p\Phi(m)+q\Phi(M)}{p+q}-\overline{L}_\Delta(\Phi(f),h)\right]$$
$$\geq 2\overline{L}_\Delta\left(\tilde{f}_1,h\right)\left[\overline{L}_\Delta(\Phi(f),h)-\Phi\left(\frac{pm+qM}{p+q}\right)\right].\quad(6.19)$$

Moreover, if Φ is concave, then (6.19) *holds in reverse order.*

Proof. It follows directly from Theorem 6.6. □

Remark 6.4 We can also prove all the results of this section by using the fact that the time scales integral is an isotonic linear functional. Using Theorem 2.6, Theorem 6.1 follows from [90, Theorem 12]; Theorem 6.2 follows from [90, Theorem 13]; Theorem 6.3 follows from [117, Theorem 8]; Corollary 6.1 follows from [117, Theorem 6]; Theorem 6.4 follows from [90, Theorem 14]; Corollary 6.2 follows from [90, Corollary 2]; Corollary 6.3 follows from [90, Corollary 3]; Theorem 6.5 follows from [89, Theorem 5]; Theorem 6.6 follows from [89, Theorem 6] and Corollary 6.4 follows from [89, Corollary 1].

6.3 Mean Value Theorems

We assume throughout this section and the next section that $f:\mathscr{E}\to[m,M]$ is Δ-integrable and $h:\mathscr{E}\to\mathbb{R}$ is nonnegative and Δ-integrable such that $L_\Delta(h)>0$. If $\Phi:I\to\mathbb{R}$ is such that $[m,M]\subseteq I$ and $\Phi(f)$ is Δ-integrable, then motivated by Theorems 6.1, 6.3, and Corollary 6.1, we define the linear functionals $\mathscr{J}_{\Delta i}$, $i\in\{1,2,3\}$, by

$$\begin{aligned}\mathscr{J}_{\Delta 1}(\Phi)=\frac{M-\overline{L}_\Delta(f,h)}{M-m}\Phi(m)+\frac{\overline{L}_\Delta(f,h)-m}{M-m}\Phi(M)-\overline{L}_\Delta(\Phi(f),h)\\ -\overline{L}_\Delta\left(\tilde{f},h\right)\delta_\Phi,\end{aligned}\tag{6.20}$$

$$\begin{aligned}\mathscr{J}_{\Delta 2}(\Phi)=\Phi\left(\overline{L}_\Delta(f,h)\right)-\overline{L}_\Delta(\Phi(f),h)\\ +\frac{1}{M-m}\left\{\left|\frac{m+M}{2}-\overline{L}_\Delta(f,h)\right|+\overline{L}_\Delta(|(m+M)/2-f|,h)\right\}\delta_\Phi,\end{aligned}\tag{6.21}$$

$$\begin{aligned}\mathscr{J}_{\Delta 3}(\Phi)=\Phi\left(\overline{L}_\Delta(f,h)\right)-\overline{L}_\Delta(\Phi(f),h)\\ +\left\{\frac{1}{2}+\frac{1}{M-m}\left|\frac{m+M}{2}-\overline{L}_\Delta(f,h)\right|\right\}\delta_\Phi,\end{aligned}\tag{6.22}$$

where $\tilde{f}$ and δ_Φ are defined in (6.3). If p,q, and $\tilde{f}_1$ are as in Theorems 6.5 and 6.6, we define linear functionals $\mathscr{J}_{\Delta 4}$ and $\mathscr{J}_{\Delta 5}$ by

$$\mathscr{J}_{\Delta 4}(\Phi)=\frac{p\Phi(m)+q\Phi(M)}{p+q}-\overline{L}_\Delta(\Phi(f),h)-\overline{L}_\Delta\left(\tilde{f},h\right)\delta_\Phi,\tag{6.23}$$

$$\mathscr{J}_{\Delta 5}(\Phi) = \frac{p\Phi(m)+q\Phi(M)}{p+q} - \overline{L}_\Delta(\Phi(f),h)$$
$$- 2\overline{L}_\Delta\left(\tilde{f}_1,h\right)\left(\frac{p\Phi(m)+q\Phi(M)}{p+q} - \Phi\left(\frac{pm+qM}{p+q}\right)\right). \quad (6.24)$$

If Φ is additionally continuous and convex on I, then by using Theorem 6.1, Theorem 6.3, Corollary 6.1, Theorem 6.5, and Theorem 6.6, respectively, we have

$$\mathscr{J}_{\Delta i}(\Phi) \geq 0, \quad i \in \{1,2,3,4,5\}.$$

Theorem 6.7 *Assume* $\Phi \in C^2(I,\mathbb{R})$, *where* $[m,M] \subseteq I$. *Then there exist* $\xi_i \in [m,M]$, $i \in \{1,2,3,4,5\}$, *such that*

$$\mathscr{J}_{\Delta i}(\Phi) = \frac{\Phi''(\xi_i)}{2}\mathscr{J}_{\Delta i}(\Phi_0), \quad i \in \{1,2,3,4,5\} \quad (6.25)$$

where $\Phi_0(x) = x^2$.

Proof. We give a proof for the functional $\mathscr{J}_{\Delta 1}$. Since $\Phi \in C^2(I)$, there exist $\Phi_*, \Phi^* \in \mathbb{R}$ such that

$$\Phi_* = \min_{x\in[m,M]} \Phi''(x) \quad \text{and} \quad \Phi^* = \max_{x\in[m,M]} \Phi''(x).$$

Let

$$\phi_1(x) = \frac{\Phi^*}{2}x^2 - \Phi(x) \quad \text{and} \quad \phi_2(x) = \Phi(x) - \frac{\Phi_*}{2}x^2$$

Then ϕ_1 and ϕ_2 are continuous and convex on $[m,M]$, and we have

$$\mathscr{J}_{\Delta 1}(\phi_1) \geq 0, \quad \mathscr{J}_{\Delta 1}(\phi_2) \geq 0,$$

which implies

$$\frac{\Phi_*}{2}\mathscr{J}_{\Delta 1}(\Phi_0) \leq \mathscr{J}_{\Delta 1}(\Phi) \leq \frac{\Phi^*}{2}\mathscr{J}_{\Delta 1}(\Phi_0).$$

If $\mathscr{J}_{\Delta 1}(\Phi_0) = 0$, then there is nothing to prove. Suppose $\mathscr{J}_{\Delta 1}(\Phi_0) > 0$. Then we have

$$\Phi_* \leq \frac{2\mathscr{J}_{\Delta 1}(\Phi)}{\mathscr{J}_{\Delta 1}(\Phi_0)} \leq \Phi^*.$$

Hence, there exists $\xi_1 \in [m,M]$ such that

$$\frac{2\mathscr{J}_{\Delta 1}(\Phi)}{\mathscr{J}_{\Delta 1}(\Phi_0)} = \Phi''(\xi_1),$$

and the result follows. □

Theorem 6.8 *Assume* $\Phi, \Psi \in C^2(I,\mathbb{R})$, *where* $[m,M] \subseteq I$. *Then there exist* $\xi_i \in [m,M]$, $i \in \{1,2,3,4,5\}$, *such that*

$$\frac{\mathscr{J}_{\Delta i}(\Phi)}{\mathscr{J}_{\Delta i}(\Psi)} = \frac{\Phi''(\xi_i)}{\Psi''(\xi_i)}, \quad i \in \{1,2,3,4,5\}, \quad (6.26)$$

provided that the denominators in (6.26) *are nonzero.*

Proof. The proof is similar to the proof of Theorem 5.5. □

Remark 6.5 If the inverse of the function $\frac{\Phi''}{\Psi''}$ exists, then (6.26) gives

$$\xi_i = \left(\frac{\Phi''}{\Psi''}\right)^{-1}\left(\frac{\mathscr{J}_{\Delta i}(\Phi)}{\mathscr{J}_{\Delta i}(\Psi)}\right), \quad i \in \{1,2,3,4,5\}.$$

6.4 Exponential Convexity and Logarithmic Convexity

Now we study log-convexity, n-exponential convexity, and exponential-convexity of the functionals $\mathscr{J}_{\Delta i}$, $i \in \{1,2,3,4,5\}$, similarly as in Section 5.4.

Theorem 6.9 *Let J be an interval in $\mathbb{R}$ and I be an open interval in $\mathbb{R}$. Assume $\Omega = \{\Phi_\rho : \rho \in J\}$ is a family of functions $\Phi_\rho : I \to \mathbb{R}$ such that $\Phi_\rho(f)$ is Δ-integrable and that the function $\rho \mapsto [x_0,x_1,x_2;\Phi_\rho]$ is n-exponentially convex in the Jensen sense on J for every choice of mutually different numbers $x_0,x_1,x_2 \in I$. Then $\rho \mapsto \mathscr{J}_{\Delta i}(\Phi_\rho)$, $i \in \{1,2,3,4,5\}$, is an n-exponentially convex function in the Jensen sense on J. If the function $\rho \mapsto \mathscr{J}_{\Delta i}(\Phi_\rho)$, $i \in \{1,2,3,4,5\}$, is also continuous on J, then it is n-exponentially convex on J.*

Proof. The proof is similar to the proof of Theorem 5.9. □

The following corollary is an immediate consequence of Theorem 6.9.

Corollary 6.5 *Let J be an interval in $\mathbb{R}$ and I be an open interval in $\mathbb{R}$. Assume $\Omega = \{\Phi_\rho : \rho \in J\}$ is a family of functions $\Phi_\rho : I \to \mathbb{R}$ such that $\Phi_\rho(f)$ is Δ-integrable and that the function $\rho \mapsto [x_0,x_1,x_2;\Phi_\rho]$ is exponentially convex in the Jensen sense on J for every choice of mutually different numbers $x_0,x_1,x_2 \in I$. Then $\rho \mapsto \mathscr{J}_{\Delta i}(\Phi_\rho)$, $i \in \{1,2,3,4,5\}$, is an exponentially convex function in the Jensen sense on J. If the function $\rho \mapsto \mathscr{J}_{\Delta i}(\Phi_\rho)$, $i \in \{1,2,3,4,5\}$, is also continuous on J, then it is exponentially convex on J.*

Corollary 6.6 *Let J be an interval in $\mathbb{R}$ and I be an open interval in $\mathbb{R}$. Assume $\Omega = \{\Phi_\rho : \rho \in J\}$ is a family of functions $\Phi_\rho : I \to \mathbb{R}$ such that $\Phi_\rho(f)$ is Δ-integrable and that the function $\rho \mapsto [x_0,x_1,x_2;\Phi_\rho]$ is 2-exponentially convex in the Jensen sense on J for every choice of mutually different numbers $x_0,x_1,x_2 \in I$. Then the following statements hold:*

(i) *The function $\rho \mapsto \mathscr{J}_{\Delta i}(\Phi_\rho)$, $i \in \{1,2,3,4,5\}$, is 2-exponentially convex in the Jensen sense on J.*

(ii) *If* $\rho \mapsto \mathscr{J}_{\Delta i}(\Phi_\rho)$, $i \in \{1,2,3,4,5\}$, *is continuous on J, then it is also* 2*-exponentially convex on J. If* $\rho \mapsto \mathscr{J}_{\Delta i}(\Phi_\rho)$, $i \in \{1,2,3,4,5\}$, *is additionally strictly positive, then it is also log-convex on J.*

(iii) *If* $\rho \mapsto \mathscr{J}_{\Delta i}(\Phi_\rho)$, $i \in \{1,2,3,4,5\}$, *is strictly positive and differentiable on J, then for* $p \le u$, $q \le v$, $p,q,u,v \in J$, *we have*

$$\mathscr{M}_{p,q}(\mathscr{J}_{\Delta i},\Omega) \le \mathscr{M}_{u,v}(\mathscr{J}_{\Delta i},\Omega), \quad i \in \{1,2,3,4,5\}, \tag{6.27}$$

where

$$\mathscr{M}_{p,q}(\mathscr{J}_{\Delta i},\Omega) = \begin{cases} \left(\dfrac{\mathscr{J}_{\Delta i}(\Phi_p)}{\mathscr{J}_{\Delta i}(\Phi_q)}\right)^{\frac{1}{p-q}}, & p \ne q; \\[2ex] \exp\left(\dfrac{\frac{\mathrm{d}}{\mathrm{d}p}\mathscr{J}_{\Delta i}(\Phi_p)}{\mathscr{J}_{\Delta i}(\Phi_p)}\right), & p = q. \end{cases} \tag{6.28}$$

Proof. The proof is similar to the proof of Corollary 5.11. □

Remark 6.6 Note that the results from Theorem 6.9, Corollary 6.5, and Corollary 6.6 still hold when two of the points $x_0, x_1, x_2 \in I$ coincide, for a family of differentiable functions Φ_ρ such that the function $\rho \mapsto [x_0,x_1,x_2;\Phi_\rho]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, they still hold when all three points coincide for a family of twice differentiable functions with the same property. The proofs are obtained by recalling Remark 1.3 and a suitable characterization of convexity.

Example 6.1 Consider the family of functions

$$\Omega_1 = \{\kappa_\rho : \mathbb{R} \to [0,\infty);\ \rho \in \mathbb{R}\}$$

defined in Example 5.2. Then by using Corollary 6.5, we conclude that $\rho \mapsto \mathscr{J}_{\Delta i}(\kappa_\rho)$, $i \in \{1,2,3,4,5\}$, are exponentially convex in the Jensen sense. It is easy to verify that these mappings are continuous, so they are exponentially convex. For this family of functions, $\mathscr{M}_{p,q}(\mathscr{J}_{\Delta i},\Omega)$, $i \in \{1,2,3,4,5\}$, from (6.28) becomes

$$\mathscr{M}_{p,q}(\mathscr{J}_{\Delta i},\Omega_1) = \begin{cases} \left(\dfrac{\mathscr{J}_{\Delta i}(\kappa_p)}{\mathscr{J}_{\Delta i}(\kappa_q)}\right)^{\frac{1}{p-q}}, & p \ne q; \\[2ex] \exp\left(\dfrac{\mathscr{J}_{\Delta i}(\mathrm{id}\cdot\kappa_p)}{\mathscr{J}_{\Delta i}(\kappa_p)} - \dfrac{2}{p}\right), & p = q \ne 0; \\[2ex] \exp\left(\dfrac{\mathscr{J}_{\Delta i}(\mathrm{id}\cdot\kappa_0)}{3\,\mathscr{J}_{\Delta i}(\kappa_0)}\right), & p = q = 0, \end{cases}$$

and by (6.27), it is monotone in p and q. Using Theorem 6.8, it follows that for $i \in \{1,2,3,4,5\}$,

$$\aleph_{p,q}(\mathscr{I}_{\Delta i},\Omega_1) = \log \mathscr{M}_{p,q}(\mathscr{I}_{\Delta i},\Omega_1)$$

satisfy $\aleph_{p,q}(\mathscr{I}_{\Delta i},\Omega_1) \in [m,M]$, which shows that $\aleph_{p,q}(\mathscr{I}_{\Delta i},\Omega_1)$ are means (of the function f). Note that by (6.27), they are monotone means.

Example 6.2 Consider the family of functions

$$\Omega_2 = \{\beta_\rho : (0,\infty) \to \mathbb{R};\ \rho \in \mathbb{R}\}$$

defined in Example 5.3. Arguing as in Example 6.1, we have $\rho \mapsto \mathscr{I}_{\Delta i}(\beta_\rho)$, $i \in \{1,2,3,4,5\}$, are exponentially convex. In this case, $\mathscr{M}_{p,q}(\mathscr{I}_{\Delta i},\Omega)$, $i \in \{1,2,3,4, 5\}$, from (6.28) becomes

$$\mathscr{M}_{p,q}(\mathscr{I}_{\Delta i},\Omega_2) = \begin{cases} \left(\dfrac{\mathscr{I}_{\Delta i}(\beta_p)}{\mathscr{I}_{\Delta i}(\beta_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\[2ex] \exp\left(\dfrac{1-2p}{p(p-1)} - \dfrac{\mathscr{I}_{\Delta i}(\beta_p\beta_0)}{\mathscr{I}_{\Delta i}(\beta_p)}\right), & p = q \neq 0,1; \\[2ex] \exp\left(1 - \dfrac{\mathscr{I}_{\Delta i}(\beta_0^2)}{2\mathscr{I}_{\Delta i}(\beta_0)}\right), & p = q = 0; \\[2ex] \exp\left(-1 - \dfrac{\mathscr{I}_{\Delta i}(\beta_0\beta_1)}{2\mathscr{I}_{\Delta i}(\beta_1)}\right), & p = q = 1. \end{cases}$$

As $\mathscr{I}_{\Delta i}$, $i \in \{1,2,3,4,5\}$, is positive, by Theorem 6.8 for $\Phi = \beta_p \in \Omega_2$ and $\Psi = \beta_q \in \Omega_2$, there exist $\xi_i \in [m,M]$, $i \in \{1,2,3,4,5\}$, such that

$$(\xi_i)^{p-q} = \frac{\mathscr{I}_{\Delta i}(\beta_p)}{\mathscr{I}_{\Delta i}(\beta_q)}, \quad i \in \{1,2,3,4,5\}.$$

Since the function $\xi_i \mapsto (\xi_i)^{p-q}$, $i \in \{1,2,3,4,5\}$, is invertible for $p \neq q$, we have

$$m \leq \left(\frac{\mathscr{I}_{\Delta i}(\beta_p)}{\mathscr{I}_{\Delta i}(\beta_q)}\right)^{\frac{1}{p-q}} \leq M, \quad i \in \{1,2,3,4,5\}.$$

Also $\mathscr{M}_{p,q}(\mathscr{I}_{\Delta i},\Omega_2)$, $i \in \{1,2,3,4,5\}$, is continuous, symmetric, and monotone (by (6.27)), shows that $\mathscr{M}_{p,q}(\mathscr{I}_{\Delta i},\Omega_2)$, $i \in \{1,2,3,4,5\}$, is a mean (of the function f).

Example 6.3 Consider the family of functions

$$\Omega_3 = \{\gamma_\rho : (0,\infty) \to (0,\infty) : \rho \in (0,\infty)\}$$

defined in Example 5.4. For this family of functions, $\mathscr{M}_{p,q}(\mathscr{J}_{\Delta i},\Omega)$, $i \in \{1,2,3,4,5\}$, from (6.28) becomes

$$\mathscr{M}_{p,q}(\mathscr{J}_{\Delta i},\Omega_3) = \begin{cases} \left(\dfrac{\mathscr{J}_{\Delta i}(\gamma_p)}{\mathscr{J}_{\Delta i}(\gamma_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(-\dfrac{\mathscr{J}_{\Delta i}(\mathrm{id}\cdot\gamma_p)}{p\,\mathscr{J}_{\Delta i}(\gamma_p)} - \dfrac{2}{p\ln p}\right), & p = q \neq 0,1; \\ \exp\left(\dfrac{-2\,\mathscr{J}_{\Delta i}(\mathrm{id}\cdot\gamma_1)}{3\,\mathscr{J}_{\Delta i}(\gamma_1)}\right), & p = q = 1, \end{cases}$$

and by (6.27), it is monotone in s and q. Using Theorem 6.8, it follows that for $i \in \{1,2,3,4,5\}$,

$$\aleph_{p,q}(\mathscr{J}_{\Delta i},\Omega_3) = -L(p,q)\log \mathscr{M}_{p,q}(\mathscr{J}_{\Delta i},\Omega_3)$$

satisfies $\aleph_{p,q}(\mathscr{J}_{\Delta i},\Omega_3) \in [m,M]$, which shows that $\aleph_{p,q}(\mathscr{J}_{\Delta i},\Omega_3)$ is a mean (of the function f). Here $L(p,q)$ is the logarithmic mean defined by

$$L(p,q) = \frac{p-q}{\log p - \log q}, \quad p \neq q \quad ,L(p,p) = p.$$

Example 6.4 Consider the family of functions

$$\Omega_4 = \{\delta_\rho : (0,\infty) \to (0,\infty) : \rho \in (0,\infty)\}$$

defined in Example 5.5. For this family of functions, $\mathscr{M}_{p,q}(\mathscr{J}_{\Delta i},\Omega)$, $i \in \{1,2,3,4,5\}$, from (6.28) becomes

$$\mathscr{M}_{p,q}(\mathscr{J}_{\Delta i},\Omega_4) = \begin{cases} \left(\dfrac{\mathscr{J}_{\Delta i}(\delta_p)}{\mathscr{J}_{\Delta i}(\delta_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(-\dfrac{\mathscr{J}_{\Delta i}(\mathrm{id}\cdot\delta_p)}{2\sqrt{p}\,\mathscr{J}_{\Delta i}(\delta_p)} - \dfrac{1}{p}\right), & p = q, \end{cases}$$

and it is monotone in p and q by (6.27). Using Theorem 6.8, it follows that for $i \in \{1,2,3,4,5\}$,

$$\aleph_{p,q}(\mathscr{J}_{\Delta i},\Omega_4) = -(\sqrt{s}+\sqrt{q})\log \mathscr{M}_{p,q}(\mathscr{J}_{\Delta i},\Omega_4)$$

satisfies $\aleph_{p,q}(\mathscr{J}_{\Delta i},\Omega_4) \in [m,M]$, which shows that $\aleph_{p,q}(\mathscr{J}_{\Delta i},\Omega_4)$ is a mean (of the function f).

Chapter 7

Hermite–Hadamard and Jensen–Mercer Inequalities on Time Scales for Several Variables

In this chapter, we obtain many improvements and generalizations of Jensen–Mercer and Hermite–Hadamard inequalities on time scales. We also generalize these inequalities for convex hulls in $\mathbb{R}^k$. Moreover, we investigate logarithmic and exponential convexity of the linear functionals obtained by the new results concerning Jensen–Mercer inequality. The results presented in this chapter are taken from [37].

7.1 Preliminaries

The convex hull of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^k$ is the set

$$\left\{ \sum_{i=1}^{n} \alpha_i \mathbf{x}_i \mid \alpha_i \in \mathbb{R},\ \alpha_i \geq 0,\ \sum_{i=1}^{n} \alpha_i = 1 \right\}$$

and it is represented by $K = \mathrm{co}(\{\mathbf{x}_1, \ldots, \mathbf{x}_n\})$.

Barycentric coordinates over K are continuous real functions $\lambda_1, \dots, \lambda_n$ on K with the following properties:

$$\lambda_i(\mathbf{x}) \geq 0,\ i \in \{1, \dots, n\}, \quad \sum_{i=1}^{n} \lambda_i(\mathbf{x}) = 1, \quad \text{and} \quad \mathbf{x} = \sum_{i=1}^{n} \lambda_i(\mathbf{x})\mathbf{x}_i. \tag{7.1}$$

If $\mathbf{x}_2 - \mathbf{x}_1, \dots, \mathbf{x}_n - \mathbf{x}_1$ are linearly independent vectors, then each $\mathbf{x} \in K$ can be written in a unique way as a convex combination of $\mathbf{x}_1, \dots, \mathbf{x}_n$ in the form (7.1). We also consider a k-simplex $S = \mathrm{co}(\{\mathbf{w}_1, \dots, \mathbf{w}_{k+1}\})$ in $\mathbb{R}^k$ which is a convex hull of its vertices $\mathbf{w}_1, \dots, \mathbf{w}_{k+1} \in \mathbb{R}^k$, where vertices $\mathbf{w}_2 - \mathbf{w}_1, \dots, \mathbf{w}_{k+1} - \mathbf{w}_1 \in \mathbb{R}^k$ are linearly independent. In this case, we denote a k-simplex by $S = [\mathbf{w}_1, \dots, \mathbf{w}_{k+1}]$. Barycentric coordinates $\lambda_1, \dots, \lambda_{k+1}$ over S are nonnegative linear polynomials on S and have a special form (see [32]). Moreover, in what follows, we denote

$$\Omega_n = \left\{ (v_1, \dots, v_n) : \ v_i \geq 0,\ i \in \{1, \dots, n\},\ \sum_{i=1}^{n} v_i = 1 \right\}$$

for $n \in \mathbb{N}$. Also, if Φ is a function defined on a convex subset $U \subseteq \mathbb{R}^k$ and $\mathbf{x}_1, \dots, \mathbf{x}_n \in U$, we denote

$$S_{\Phi}^{n}(\mathbf{x}_1, \dots, \mathbf{x}_n) = \sum_{i=1}^{n} \Phi(\mathbf{x}_i) - n\Phi\left(\frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \right).$$

Obviously, if Φ is convex, then $S_{\Phi}^{n}(\mathbf{x}_1, \dots, \mathbf{x}_n) \geq 0$.

To prove our main results we need the following lemma, which is a simple consequence of [103, page 717, Theorem 1].

Lemma 7.1 *Let U be a convex set in $\mathbb{R}^k$ and $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in U^n$. Suppose $p = (p_1, \dots, p_n)$ is a nonnegative n-tuple such that $\sum_{i=1}^{n} p_i = 1$. If Φ is a convex function on U, then*

$$\begin{aligned} \min_{1 \leq i \leq n} \{p_i\} S_{\Phi}^{n}(\mathbf{x}_1, \dots, \mathbf{x}_n) &\leq \sum_{i=1}^{n} p_i \Phi(\mathbf{x}_i) - \Phi\left(\sum_{i=1}^{n} p_i \mathbf{x}_i \right) \\ &\leq \max_{1 \leq i \leq n} \{p_i\} S_{\Phi}^{n}(\mathbf{x}_1, \dots, \mathbf{x}_n). \end{aligned}$$

From the discrete Jensen inequality, a reversed Jensen inequality easily follows (see [119]).

Theorem 7.1 *Let $\mathbf{p}$ be a real n-tuple such that*

$$p_1 > 0, \quad p_i \leq 0,\ i \in \{2, \dots, n\}, \quad \textit{and} \quad P_n > 0,$$

where $P_n = \sum_{i=1}^{n} p_i$. Let U be a convex set in $\mathbb{R}^K$, $\mathbf{x}_i \in U$, $i \in \{1, \dots, n\}$, and $\frac{1}{P_n} \sum_{i=1}^{n} p_i \mathbf{x}_i \in U$. If $f : U \to \mathbb{R}$ is a convex function, then

$$f\left(\frac{1}{P_n} \sum_{i=1}^{n} p_i \mathbf{x}_i \right) \geq \frac{1}{P_n} \sum_{i=1}^{n} p_i f(\mathbf{x}_i).$$

7.2 Generalizations of the Hermite–Hadamard Inequality

In this section, we obtain generalizations of the converses of Jensen's inequality, Theorems 2.13, 2.59, and 6.2. As a consequence, we obtain generalizations of the Hermite–Hadamard's inequality (1.17). Moreover, since time scales integrals are isotonic linear functionals, we can also use the approach of isotonic linear functionals. Results for isotonic linear functionals analogous to the results of this section are given in [77, 118].

First we present generalizations of Theorem 2.13 and Theorem 6.1.

Lemma 7.2 *Let $\mathbf{x}_1,\dots,\mathbf{x}_n \in \mathbb{R}^k$ and $K = \operatorname{co}(\{\mathbf{x}_1,\dots,\mathbf{x}_n\})$. Let $\lambda_1,\dots,\lambda_n$ be barycentric coordinates over K. Suppose $\mathbf{f}$ is a Δ-integrable function on $\mathscr{E}$ such that $\mathbf{f}(\mathscr{E}) \subseteq K$ and $h\colon \mathscr{E} \to \mathbb{R}$ is nonnegative and Δ-integrable. If Φ is a convex function on K, then*

$$L_\Delta(h\Phi(\mathbf{f})) \le \sum_{i=1}^{n} L_\Delta(h\lambda_i(\mathbf{f}))\Phi(\mathbf{x}_i) - L_\Delta\left(h \min_{1\le i\le n}\{\lambda_i(\mathbf{f})\}\right) S_\Phi^n(\mathbf{x}_1,\dots,\mathbf{x}_n). \tag{7.2}$$

Moreover, if Φ is concave, then the inequality in (7.2) *holds in reverse order.*

Proof. By using the properties of barycentric coordinates, we have

$$\lambda_i(\mathbf{f}(t)) \ge 0,\ i \in \{1,\dots,n\}, \quad \sum_{i=1}^{n} \lambda_i(\mathbf{f}(t)) = 1, \quad \text{and} \quad \mathbf{f}(t) = \sum_{i=1}^{n} \lambda_i(\mathbf{f}(t))\mathbf{x}_i.$$

Suppose Φ is convex on K. Then by using Lemma 7.1, we have

$$\begin{aligned} \Phi(\mathbf{f}(t)) &= \Phi\left(\sum_{i=1}^{n} \lambda_i(\mathbf{f}(t))\mathbf{x}_i\right) \\ &\le \sum_{i=1}^{n} \lambda_i(\mathbf{f}(t))\Phi(\mathbf{x}_i) - \min_{1\le i\le n}\{\lambda_i(\mathbf{f}(t))\} S_\Phi^n(\mathbf{x}_1,\dots,\mathbf{x}_n). \end{aligned} \tag{7.3}$$

Since h is nonnegative and Δ-integrable, multiplying (7.3) by h and integrating, we obtain the inequality (7.2). If Φ is concave, the reverse inequality in (7.2) holds immediately by using the fact that if Φ is concave, then $-\Phi$ is convex. □

Theorem 7.2 *Let $\mathbf{x}_1,\dots,\mathbf{x}_n \in \mathbb{R}^k$ and $K = \operatorname{co}(\{\mathbf{x}_1,\dots,\mathbf{x}_n\})$. Let $\lambda_1,\dots,\lambda_n$ be barycentric coordinates over K. Suppose $\mathbf{f}$ is a Δ-integrable function on $\mathscr{E}$ such that $\mathbf{f}(\mathscr{E}) \subseteq K$ and $h\colon \mathscr{E} \to \mathbb{R}$ is nonnegative and Δ-integrable such that $L_\Delta(h) > 0$. If Φ is a convex function on K, then*

$$\overline{L}_\Delta(\Phi(\mathbf{f}),h) \le \sum_{i=1}^{n} \overline{L}_\Delta(\lambda_i(\mathbf{f}),h)\Phi(\mathbf{x}_i) - \overline{L}_\Delta\left(\min_{1\le i\le n}\{\lambda_i(\mathbf{f})\},h\right) S_\Phi^n(\mathbf{x}_1,\dots,\mathbf{x}_n). \tag{7.4}$$

Moreover, if Φ is concave, then the inequality in (7.4) *holds in reverse order.*

Proof. Since all assumptions of Lemma 7.2 are satisfied and additionally we have $L_\Delta(h) > 0$, after dividing (7.2) by $L_\Delta(h)$, we obtain (7.4). □

Remark 7.1 Since the second term in (7.4) is nonnegative, we have

$$\overline{L}_\Delta(\Phi(\mathbf{f}),h) \leq \sum_{i=1}^{n} \overline{L}_\Delta(\lambda_i(\mathbf{f}),h)\Phi(\mathbf{x}_i). \tag{7.5}$$

Now (7.5) generalizes (2.3) and (7.4) generalizes (6.2).

Remark 7.2 If all assumptions of Theorem 7.2 are satisfied and additionally Φ is continuous, then by combining (7.4) with (4.1), we have

$$\begin{aligned}\Phi(\overline{L}_\Delta(\mathbf{f},h)) &\leq \overline{L}_\Delta(\Phi(\mathbf{f}),h) \\ &\leq \sum_{i=1}^{n} \overline{L}_\Delta(\lambda_i(\mathbf{f}),h)\Phi(\mathbf{x}_i) - \overline{L}_\Delta\left(\min_{1\leq i\leq n}\{\lambda_i(\mathbf{f})\},h\right) S_\Phi^n(\mathbf{x}_1,\ldots,\mathbf{x}_n).\end{aligned}$$

Remark 7.3 Let all assumptions of Theorem 7.2 be satisfied and additionally $\Phi > 0$. Dividing (7.4) by

$$\Phi(\overline{L}_\Delta(\mathbf{f},h)) = \Phi\left(\sum_{i=1}^{n} \overline{L}_\Delta(\lambda_i(\mathbf{f})\mathbf{x}_i,h)\right),$$

we obtain

$$\begin{aligned}\frac{\overline{L}_\Delta(\Phi(\mathbf{f}),h)}{\Phi(\overline{L}_\Delta(\mathbf{f},h))} &\leq \frac{\sum_{i=1}^{n} \overline{L}_\Delta(\lambda_i(\mathbf{f}),h)\Phi(\mathbf{x}_i)}{\Phi\left(\sum_{i=1}^{n} \overline{L}_\Delta(\lambda_i(\mathbf{f})\mathbf{x}_i,h)\right)} - \frac{\overline{L}_\Delta\left(\min_{1\leq i\leq n}\{\lambda_i(\mathbf{f})\},h\right)}{\Phi(\overline{L}_\Delta(\mathbf{f},h))} S_\Phi^n(\mathbf{x}_1,\ldots,\mathbf{x}_n) \\ &\leq \max_{\Omega_n} \frac{\sum_{i=1}^{n} \nu_i\Phi(\mathbf{x}_i)}{\Phi\left(\sum_{i=1}^{n} \nu_i\mathbf{x}_i\right)} - \frac{\overline{L}_\Delta\left(\min_{1\leq i\leq n}\{\lambda_i(\mathbf{f})\},h\right)}{\Phi(\overline{L}_\Delta(\mathbf{f},h))} S_\Phi^n(\mathbf{x}_1,\ldots,\mathbf{x}_n),\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\overline{L}_\Delta(\Phi(\mathbf{f}),h) \\ &\quad\leq \max_{\Omega_n} \frac{\sum_{i=1}^{n} \nu_i\Phi(\mathbf{x}_i)}{\Phi\left(\sum_{i=1}^{n} \nu_i\mathbf{x}_i\right)} \Phi(\overline{L}_\Delta(\mathbf{f},h)) - \overline{L}_\Delta\left(\min_{1\leq i\leq n}\{\lambda_i(\mathbf{f})\},h\right) S_\Phi^n(\mathbf{x}_1,\ldots,\mathbf{x}_n).\end{aligned} \tag{7.6}$$

Now inequality (7.6) is a refinement and generalization of (2.64).

In the following theorem, we give a generalization and refinement of Theorem 2.59.

Theorem 7.3 *Let $\mathbf{x}_1,\ldots,\mathbf{x}_n \in \mathbb{R}^k$ and $K = \mathrm{co}(\{\mathbf{x}_1,\ldots,\mathbf{x}_n\})$. Let $\lambda_1,\ldots,\lambda_n$ be barycentric coordinates over K. Suppose $\mathbf{f}$ is a Δ-integrable function on $\mathscr{E}$ such that $\mathbf{f}(\mathscr{E}) \subseteq K$ and $h\colon \mathscr{E} \to \mathbb{R}$ is nonnegative and Δ-integrable such that $L_\Delta(h) > 0$. Let J be an interval in $\mathbb{R}$. Suppose Φ is a convex function on K such that $\Phi(K) \subseteq J$ and $F\colon J \times J \to \mathbb{R}$ is increasing in the first variable. Then*

$$\begin{aligned}
&F\left(\overline{L}_\Delta(\Phi(\mathbf{f}),h), \Phi\left(\overline{L}_\Delta(\mathbf{f},h)\right)\right) \qquad (7.7)\\
&\le F\left(\sum_{i=1}^n \overline{L}_\Delta(\lambda_i(\mathbf{f}),h)\Phi(\mathbf{x}_i)\right.\\
&\qquad \left. - \overline{L}_\Delta\left(\min_{1\le i\le n}\{\lambda_i(\mathbf{f})\}, h\right) S_\Phi^n(\mathbf{x}_1,\ldots,\mathbf{x}_n), \Phi\left(\overline{L}_\Delta(\mathbf{f},h)\right)\right)\\
&\le \max_{\Omega_n} F\left(\sum_{i=1}^n \nu_i \Phi(\mathbf{x}_i)\right.\\
&\qquad \left. - \overline{L}_\Delta\left(\min_{1\le i\le n}\{\lambda_i(\mathbf{f})\}, h\right) S_\Phi^n(\mathbf{x}_1,\ldots,\mathbf{x}_n), \Phi\left(\sum_{i=1}^n \nu_i \mathbf{x}_i\right)\right).
\end{aligned}$$

Proof. By using the properties of barycentric coordinates, we have

$$\lambda_i(\mathbf{f}(t)) \ge 0,\ i \in \{1,\ldots,n\}, \quad \sum_{i=1}^n \lambda_i(\mathbf{f}(t)) = 1,$$

and

$$\mathbf{f}(t) = \sum_{i=1}^n \lambda_i(\mathbf{f}(t))\mathbf{x}_i. \qquad (7.8)$$

Since h is nonnegative, Δ-integrable, and $L_\Delta(h) > 0$, multiplying (7.8) by h, integrating, and then dividing by $L_\Delta(h)$, we obtain

$$\overline{L}_\Delta(\mathbf{f},h) = \sum_{i=1}^n \overline{L}_\Delta(\lambda_i(\mathbf{f}),h)\mathbf{x}_i,$$

where

$$\overline{L}_\Delta(\lambda_i(\mathbf{f}),h) \ge 0, \quad i \in \{1,\ldots,n\}$$

and

$$\sum_{i=1}^n \overline{L}_\Delta(\lambda_i(\mathbf{f}),h) = \overline{L}_\Delta\left(\sum_{i=1}^n \lambda_i(\mathbf{f}),h\right) = 1.$$

Therefore, $\overline{L}_\Delta(\mathbf{f},h) \in K$. Since $F\colon J \times J \to \mathbb{R}$ is increasing in the first variable, using (7.4), we have

$$\begin{aligned}
F\left(\overline{L}_\Delta(\Phi(\mathbf{f}),h), \Phi\left(\overline{L}_\Delta(\mathbf{f},h)\right)\right) &\le F\left(\sum_{i=1}^n \overline{L}_\Delta(\lambda_i(\mathbf{f}),h)\Phi(\mathbf{x}_i)\right.\\
&\left. - \overline{L}_\Delta\left(\min_{1\le i\le n}\{\lambda_i(\mathbf{f})\}, h\right) S_\Phi^n(\mathbf{x}_1,\ldots,\mathbf{x}_n), \Phi\left(\overline{L}_\Delta(\mathbf{f},h)\right)\right). \qquad (7.9)
\end{aligned}$$

By the substitutions $\overline{L}_\Delta(\lambda_i(\mathbf{f}),h) = v_i,\ i \in \{1,\ldots,n\}$, we get

$$\overline{L}_\Delta(\mathbf{f},h) = \sum_{i=1}^{n} v_i \mathbf{x}_i.$$

Now we have

$$F\left(\sum_{i=1}^{n} \overline{L}_\Delta(\lambda_i(\mathbf{f}),h)\Phi(\mathbf{x}_i) - \overline{L}_\Delta\left(\min_{1\le i\le n}\{\lambda_i(\mathbf{f})\},h\right) S^n_\Phi(\mathbf{x}_1,\ldots,\mathbf{x}_n), \Phi\left(\overline{L}_\Delta(\mathbf{f},h)\right)\right) \tag{7.10}$$
$$= F\left(\sum_{i=1}^{n} v_i\Phi(\mathbf{x}_i) - \overline{L}_\Delta\left(\min_{1\le i\le n}\{\lambda_i(\mathbf{f})\},h\right) S^n_\Phi(\mathbf{x}_1,\ldots,\mathbf{x}_n), \Phi\left(\sum_{i=1}^{n} v_i\mathbf{x}_i\right)\right)$$
$$\le \max_{\Omega_n} F\left(\sum_{i=1}^{n} v_i\Phi(\mathbf{x}_i) - \overline{L}_\Delta\left(\min_{1\le i\le n}\{\lambda_i(\mathbf{f})\},h\right) S^n_\Phi(\mathbf{x}_1,\ldots,\mathbf{x}_n), \Phi\left(\sum_{i=1}^{n} v_i\mathbf{x}_i\right)\right).$$

By combining (7.9) and (7.10), we get (7.7). □

Remark 7.4 In Theorem 7.3, let $F(x,y) = x - y$. Then we have

$$\overline{L}_\Delta(\Phi(\mathbf{f}),h) - \Phi(\overline{L}_\Delta(\mathbf{f},h))$$
$$\le \max_{\Omega_n}\left(\sum_{i=1}^{n} v_i\Phi(\mathbf{x}_i) - \Phi\left(\sum_{i=1}^{n} v_i\mathbf{x}_i\right) - \overline{L}_\Delta\left(\min_{1\le i\le n}\{\lambda_i(\mathbf{f})\},h\right) S^n_\Phi(\mathbf{x}_1,\ldots,\mathbf{x}_n)\right).$$

If $F(x,y) = \dfrac{x}{y}$, for $\Phi > 0$, then

$$\frac{\overline{L}_\Delta(\Phi(\mathbf{f}),h)}{\Phi(\overline{L}_\Delta(\mathbf{f},h))} \le \max_{\Omega_n}\left(\frac{\sum_{i=1}^{n} v_i\Phi(\mathbf{x}_i) - \overline{L}_\Delta\left(\min_{1\le i\le n}\{\lambda_i(\mathbf{f})\},h\right) S^n_\Phi(\mathbf{x}_1,\ldots,\mathbf{x}_n)}{\Phi\left(\sum_{i=1}^{n} v_i\mathbf{x}_i\right)}\right).$$

These inequalities are refinements and generalizations of the inequalities given in Remark 2.12.

By replacing F with $-F$ in Theorem 7.3, we get the next result.

Theorem 7.4 *Let $\mathbf{x}_1,\ldots,\mathbf{x}_n \in \mathbb{R}^k$ and $K = \mathrm{co}(\{\mathbf{x}_1,\ldots,\mathbf{x}_n\})$. Let $\lambda_1,\ldots,\lambda_n$ be barycentric coordinates over K. Suppose $\mathbf{f}$ is a Δ-integrable function on $\mathscr{E}$ such that $\mathbf{f}(\mathscr{E}) \subseteq K$ and $h\colon \mathscr{E} \to \mathbb{R}$ is nonnegative and Δ-integrable such that $L_\Delta(h) > 0$. Let J be an interval in $\mathbb{R}$. Suppose Φ is a convex function on K such that $\Phi(K) \subseteq J$ and $F\colon J \times J \to \mathbb{R}$ is decreasing in the first variable. Then*

$$F\left(\overline{L}_\Delta(\Phi(\mathbf{f}),h), \Phi\left(\overline{L}_\Delta(\mathbf{f},h)\right)\right) \tag{7.11}$$
$$\ge F\left(\sum_{i=1}^{n} \overline{L}_\Delta(\lambda_i(\mathbf{f}),h)\Phi(\mathbf{x}_i) - \overline{L}_\Delta\left(\min_{1\le i\le n}\{\lambda_i(\mathbf{f})\},h\right) S^n_\Phi(\mathbf{x}_1,\ldots,\mathbf{x}_n), \Phi\left(\overline{L}_\Delta(\mathbf{f},h)\right)\right)$$

$$\geq \min_{\Omega_n} F\left(\sum_{i=1}^{n} \nu_i \Phi(\mathbf{x}_i) - \overline{L}_\Delta\left(\min_{1\leq i\leq n}\{\lambda_i(\mathbf{f})\}, h\right) S_\Phi^n(\mathbf{x}_1,\ldots,\mathbf{x}_n), \Phi\left(\sum_{i=1}^{n}\nu_i \mathbf{x}_i\right)\right).$$

Let $S = [\mathbf{w}_1,\ldots,\mathbf{w}_{k+1}]$ be a k-simplex in $\mathbb{R}^k$ with vertices $\mathbf{w}_1,\ldots,\mathbf{w}_{k+1} \in \mathbb{R}^k$. The barycentric coordinates $\lambda_1,\ldots,\lambda_{k+1}$ over S are nonnegative linear polynomials that satisfy Lagrange's property

$$\lambda_i(\mathbf{w}_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases} \tag{7.12}$$

Therefore, it is known that for each $\mathbf{x} \in S$, the barycentric coordinates $\lambda_1(\mathbf{x}),\ldots,\lambda_{k+1}(\mathbf{x})$ have the form

$$\begin{aligned}
\lambda_1(\mathbf{x}) &= \frac{\mathrm{Vol}_k([\mathbf{x},\mathbf{w}_2,\ldots,\mathbf{w}_{k+1}])}{\mathrm{Vol}_k(S)}\\
\lambda_2(\mathbf{x}) &= \frac{\mathrm{Vol}_k([\mathbf{w}_1,\mathbf{x},\ldots,\mathbf{w}_{k+1}])}{\mathrm{Vol}_k(S)}\\
&\vdots\\
\lambda_{k+1}(\mathbf{x}) &= \frac{\mathrm{Vol}_k([\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_k,\mathbf{x}])}{\mathrm{Vol}_k(S)},
\end{aligned}$$

where Vol_k denotes k-dimensional Lebesgue measure on S. Here, for example, $[\mathbf{w}_1,\mathbf{x},\ldots,\mathbf{w}_{k+1}]$ denotes the subsimplex obtained by replacing $\mathbf{w}_2$ by $\mathbf{x}$.

In other words, we see that the barycentric coordinates $\lambda_1,\ldots,\lambda_{k+1}$ for each $\mathbf{x} \in S$ can be presented as the ratios of the volume of a subsimplex with one vertex in $\mathbf{x}$ and the volume of S.

The signed volume $\mathrm{Vol}_k(S)$ is given by the $(k+1)\times(k+1)$ determinant

$$\mathrm{Vol}_k(S) = \frac{1}{k!}\begin{vmatrix} 1 & 1 & \ldots & 1\\ w_{11} & w_{21} & & w_{k+1,1}\\ w_{12} & w_{22} & & w_{k+1,2}\\ \vdots & \vdots & & \vdots\\ w_{1k} & w_{2k} & \ldots & w_{k+1,k}\end{vmatrix},$$

where $\mathbf{w}_1 = (w_{11},w_{12},\ldots,w_{1k}),\ldots,\mathbf{w}_{k+1} = (w_{k+1,1},w_{k+1,2},\ldots,w_{k+1,k})$. Since the vectors $\mathbf{w}_2-\mathbf{w}_1,\ldots,\mathbf{w}_{k+1}-\mathbf{w}_1$ are linearly independent, each $\mathbf{x} \in S$ can be written in a unique way as a convex combination of $\mathbf{w}_1,\ldots,\mathbf{w}_{k+1}$ in the form

$$\begin{aligned}
\mathbf{x} = \frac{\mathrm{Vol}_k([\mathbf{x},\mathbf{w}_2,\ldots,\mathbf{w}_{k+1}])}{\mathrm{Vol}_k(S)}\mathbf{w}_1 + \frac{\mathrm{Vol}_k([\mathbf{w}_1,\mathbf{x},\ldots,\mathbf{w}_{k+1}])}{\mathrm{Vol}_k(S)}\mathbf{w}_2\\
+\ldots+\frac{\mathrm{Vol}_k([\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_k,\mathbf{x}])}{\mathrm{Vol}_k(S)}\mathbf{w}_{k+1}.
\end{aligned} \tag{7.13}$$

Now we present an analogue of Theorem 7.2 for convex functions defined on k-simplices in $\mathbb{R}^k$.

Theorem 7.5 *Let $S=[\mathbf{w}_1,\ldots,\mathbf{w}_{k+1}]$ be a k-simplex in $\mathbb{R}^k$ and let $\lambda_1,\ldots,\lambda_{k+1}$ be barycentric coordinates over S. Suppose $\mathbf{f}$ is a Δ-integrable function on $\mathscr{E}$ such that $\mathbf{f}(\mathscr{E})\subseteq S$ and $h\colon \mathscr{E}\to\mathbb{R}$ is nonnegative and Δ-integrable such that $L_\Delta(h)>0$. If Φ is a convex function on S, then*

$$\begin{aligned}
&\overline{L}_\Delta(\Phi(\mathbf{f}),h) \qquad (7.14)\\
&\le \sum_{i=1}^{k+1}\overline{L}_\Delta(\lambda_i(\mathbf{f}),h)\Phi(\mathbf{w}_i)-\overline{L}_\Delta\left(\min_{1\le i\le k+1}\{\lambda_i(\mathbf{f})\},h\right)S_\Phi^{k+1}(\mathbf{w}_1,\ldots,\mathbf{w}_{k+1})\\
&=\frac{\mathrm{Vol}_k([\overline{L}_\Delta(\mathbf{f},h),\mathbf{w}_2,\ldots,\mathbf{w}_{k+1}])}{\mathrm{Vol}_k(S)}\Phi(\mathbf{w}_1)+\ldots\\
&\quad+\frac{\mathrm{Vol}_k([\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_k,\overline{L}_\Delta(\mathbf{f},h)])}{\mathrm{Vol}_k(S)}\Phi(\mathbf{w}_{k+1})\\
&\quad-\overline{L}_\Delta\left(\min_{1\le i\le k+1}\{\lambda_i(\mathbf{f})\},h\right)S_\Phi^{k+1}(\mathbf{w}_1,\ldots,\mathbf{w}_{k+1}).
\end{aligned}$$

Moreover, if Φ is concave, then the inequality in (7.14) *holds in reverse order.*

Proof. The proof is analogous to the proof of Theorem 7.2 with

$$\lambda_1(\mathbf{f}(t))=\frac{\mathrm{Vol}_k([\mathbf{f}(t),\mathbf{w}_2,\ldots,\mathbf{w}_{k+1}])}{\mathrm{Vol}_k(S)}=\frac{\frac{1}{k!}\begin{vmatrix}1&1&\ldots&1\\ f_1(t)&w_{21}&&w_{k+1,1}\\ \vdots&\vdots&&\vdots\\ f_k(t)&w_{2k}&\ldots&w_{k+1,k}\end{vmatrix}}{\frac{1}{k!}\begin{vmatrix}1&1&\ldots&1\\ w_{11}&w_{21}&&w_{k+1,1}\\ w_{12}&w_{22}&&w_{k+1,2}\\ \vdots&\vdots&&\vdots\\ w_{1k}&w_{2k}&\ldots&w_{k+1,k}\end{vmatrix}}$$

$$\vdots$$

$$\lambda_{k+1}(\mathbf{f}(t))=\frac{\mathrm{Vol}_k([\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_k,\mathbf{f}(t)])}{\mathrm{Vol}_k(S)}=\frac{\frac{1}{k!}\begin{vmatrix}1&1&\ldots&1\\ w_{11}&w_{21}&&f_1(t)\\ \vdots&\vdots&&\vdots\\ w_{1k}&w_{2k}&\ldots&f_k(t)\end{vmatrix}}{\frac{1}{k!}\begin{vmatrix}1&1&\ldots&1\\ w_{11}&w_{21}&&w_{k+1,1}\\ w_{12}&w_{22}&&w_{k+1,2}\\ \vdots&\vdots&&\vdots\\ w_{1k}&w_{2k}&\ldots&w_{k+1,k}\end{vmatrix}}$$

and

$$\overline{L}_\Delta(\lambda_1(\mathbf{f}),h) = \frac{\frac{1}{k!}\begin{vmatrix} 1 & 1 & \dots & 1 \\ \overline{L}_\Delta(f_1,h) & w_{21} & & w_{k+1,1} \\ \vdots & \vdots & & \vdots \\ \overline{L}_\Delta(f_k,h) & w_{2k} & \dots & w_{k+1,k} \end{vmatrix}}{\frac{1}{k!}\begin{vmatrix} 1 & 1 & \dots & 1 \\ w_{11} & w_{21} & & w_{k+1,1} \\ w_{12} & w_{22} & & w_{k+1,2} \\ \vdots & \vdots & & \vdots \\ w_{1k} & w_{2k} & \dots & w_{k+1,k} \end{vmatrix}}$$

$$= \frac{\mathrm{Vol}_k\left(\left[\overline{L}_\Delta(\mathbf{f},h),\mathbf{w}_2,\dots,\mathbf{w}_{k+1}\right]\right)}{\mathrm{Vol}_k(S)}$$

$$\vdots$$

$$\overline{L}_\Delta(\lambda_{k+1}(\mathbf{f}),h) = \frac{\frac{1}{k!}\begin{vmatrix} 1 & 1 & \dots & 1 \\ w_{11} & w_{21} & & \overline{L}_\Delta(f_1,h) \\ \vdots & \vdots & & \vdots \\ w_{1k} & w_{2k} & \dots & \overline{L}_\Delta(f_k,h) \end{vmatrix}}{\frac{1}{k!}\begin{vmatrix} 1 & 1 & \dots & 1 \\ w_{11} & w_{21} & & w_{k+1,1} \\ w_{12} & w_{22} & & w_{k+1,2} \\ \vdots & \vdots & & \vdots \\ w_{1k} & w_{2k} & \dots & w_{k+1,k} \end{vmatrix}}$$

$$= \frac{\mathrm{Vol}_k\left(\left[\mathbf{w}_1,\mathbf{w}_2,\dots,\mathbf{w}_k,\overline{L}_\Delta(\mathbf{f},h)\right]\right)}{\mathrm{Vol}_k(S)}.$$

□

Using Theorem 7.5, in the following theorem, we present an analogue of Theorem 7.3.

Theorem 7.6 *Let $S = [\mathbf{w}_1,\dots,\mathbf{w}_{k+1}]$ be a k-simplex in $\mathbb{R}^k$ and let $\lambda_1,\dots,\lambda_{k+1}$ be barycentric coordinates over S. Suppose $\mathbf{f}$ is a Δ-integrable function on $\mathscr{E}$ such that $\mathbf{f}(\mathscr{E}) \subseteq S$ and $h\colon \mathscr{E} \to \mathbb{R}$ is nonnegative and Δ-integrable such that $L_\Delta(h) > 0$. Let J be an interval in $\mathbb{R}$. Suppose Φ is a convex function on K such that $\Phi(S) \subseteq J$ and $F : J \times J \to \mathbb{R}$ is increasing in the first variable. Then*

$$\begin{aligned} &F\left(\overline{L}_\Delta(\Phi(\mathbf{f}),h),\Phi\left(\overline{L}_\Delta(\mathbf{f},h)\right)\right) \\ &\leq \max_{\mathbf{x}\in S} F\left(\frac{\mathrm{Vol}_k([\mathbf{x},\mathbf{w}_2,\dots,\mathbf{w}_{k+1}])}{\mathrm{Vol}_k(S)}\Phi(\mathbf{w}_1)+\dots\right. \\ &\quad + \frac{\mathrm{Vol}_k([\mathbf{w}_1,\mathbf{w}_2,\dots,\mathbf{w}_k,\mathbf{x}])}{\mathrm{Vol}_k(S)}\Phi(\mathbf{w}_{k+1}) \end{aligned} \tag{7.15}$$

$$
\begin{aligned}
&-\overline{L}_{\Delta}\left(\min_{1\le i\le k+1}\{\lambda_i(\mathbf{f})\},h\right)S_{\Phi}^{k+1}(\mathbf{w}_1,\dots,\mathbf{w}_{k+1}),\Phi(\mathbf{x})\Big)\\
&=\max_{\Omega_{k+1}}F\left(\sum_{i=1}^{k+1}\nu_i\Phi(\mathbf{w}_i)\right.\\
&\left.-\overline{L}_{\Delta}\left(\min_{1\le i\le k+1}\{\lambda_i(\mathbf{f})\},h\right)S_{\Phi}^{k+1}(\mathbf{w}_1,\dots,\mathbf{w}_{k+1}),\Phi\left(\sum_{i=1}^{k+1}\nu_i\mathbf{w}_i\right)\right).
\end{aligned}
$$

Proof. By putting

$$
\nu_1=\frac{\mathrm{Vol}_k([\mathbf{x},\mathbf{w}_2,\dots,\mathbf{w}_{k+1}])}{\mathrm{Vol}_k(S)},\dots,\nu_{k+1}=\frac{\mathrm{Vol}_k([\mathbf{w}_1,\mathbf{w}_2,\dots,\mathbf{w}_k,\mathbf{x}])}{\mathrm{Vol}_k(S)},
$$

and

$$
\mathbf{x}=\sum_{i=1}^{k+1}\nu_i\mathbf{w}_i,
$$

the proof is analogous to the proof of Theorem 7.3. □

Remark 7.5 By replacing F with $-F$ in Theorem 7.6, we get an analogue of Theorem 7.4 for convex functions defined on k-simplices in $\mathbb{R}^k$.

Remark 7.6 If all assumptions of Theorem 7.5 are satisfied and if Φ is continuous and convex, then

$$
\begin{aligned}
&\Phi(\overline{L}_{\Delta}(\mathbf{f},h))\le\overline{L}_{\Delta}(\Phi(\mathbf{f}),h) \qquad (7.16)\\
&\le\sum_{i=1}^{k+1}\overline{L}_{\Delta}(\lambda_i(\mathbf{f}),h)\Phi(\mathbf{w}_i)-\overline{L}_{\Delta}\left(\min_{1\le i\le k+1}\{\lambda_i(\mathbf{f})\},h\right)S_{\Phi}^{k+1}(\mathbf{w}_1,\dots,\mathbf{w}_{k+1})\\
&=\frac{\mathrm{Vol}_k([\overline{L}_{\Delta}(\mathbf{f},h),\mathbf{w}_2,\dots,\mathbf{w}_{k+1}])}{\mathrm{Vol}_k(S)}\Phi(\mathbf{w}_1)+\dots\\
&\quad+\frac{\mathrm{Vol}_k([\mathbf{w}_1,\mathbf{w}_2,\dots,\mathbf{w}_k,\overline{L}_{\Delta}(\mathbf{f},h)])}{\mathrm{Vol}_k(S)}\Phi(\mathbf{w}_{k+1})\\
&\quad-\overline{L}_{\Delta}\left(\min_{1\le i\le k+1}\{\lambda_i(\mathbf{f})\},h\right)S_{\Phi}^{k+1}(\mathbf{w}_1,\dots,\mathbf{w}_{k+1}).
\end{aligned}
$$

The first inequality is (4.1) and the second one is (7.14). If we take the discrete form of inequality (7.16), then its related results are given in [32, 131]. On the other hand, if we take the real case, related results are obtained as a consequence of Choquet's theory (see [58, 105, 106, 107, 128]).

7.3 Jensen–Mercer Inequality

The discrete form of Jensen–Mercer's inequality and some of its applications are given in [100]. In this section and the next section, we give refinements and generalizations on time scales. Moreover, related results for isotonic linear functionals are given in [98].

Throughout this section and the following sections, we take $[m,M]$ an interval in $\mathbb{R}$ such that $-\infty < m < M < \infty$.

Theorem 7.7 *Suppose f is a Δ-integrable function on $\mathscr{E}$ such that $f(\mathscr{E}) \subseteq [m,M]$ and $h\colon \mathscr{E} \to \mathbb{R}$ is nonnegative and Δ-integrable such that $L_\Delta(h) > 0$. Let $\Phi \in \mathrm{C}([m,M],\mathbb{R})$. If Φ is convex, then*

$$\begin{aligned}\Phi(m+M-\overline{L}_\Delta(f,h)) &\le \overline{L}_\Delta(\Phi(m+M-f),h) \\ &\le \frac{M-\overline{L}_\Delta(f,h)}{M-m}\Phi(M) + \frac{\overline{L}_\Delta(f,h)-m}{M-m}\Phi(m) \\ &\le \Phi(m)+\Phi(M)-\overline{L}_\Delta(\Phi(f),h).\end{aligned} \tag{7.17}$$

Moreover, if Φ is concave, then the inequalities in (7.17) *hold in reverse order.*

Proof. Suppose Φ is convex. By applying the Jensen inequality on time scales to the function $g = m+M-f$, we obtain

$$\Phi\left(\overline{L}_\Delta(g,h)\right) \le \overline{L}_\Delta(\Phi(g),h),$$

i.e.,

$$\Phi\left(m+M-\overline{L}_\Delta(f,h)\right) \le \overline{L}_\Delta(\Phi(m+M-f),h).$$

Now by applying Theorem 2.13 to g, and then to f, we obtain

$$\begin{aligned}&\overline{L}_\Delta(\Phi(m+M-f,h) \\ &\le \frac{M-\overline{L}_\Delta(f,h)}{M-m}\Phi(M) + \frac{\overline{L}_\Delta(f,h)-m}{M-m}\Phi(m) \\ &= \Phi(m)+\Phi(M) - \left[\frac{M-\overline{L}_\Delta(f,h)}{M-m}\Phi(m) + \frac{\overline{L}_\Delta(f,h)-m}{M-m}\Phi(M)\right] \\ &\le \Phi(m)+\Phi(M)-\overline{L}_\Delta(\Phi(f),h).\end{aligned}$$

If Φ is concave, the reverse inequalities in (7.17) hold immediately by using the fact that if Φ is concave, then $-\Phi$ is convex. □

Theorem 7.8 *Suppose f is a Δ-integrable function on $\mathscr{E}$ such that $f(\mathscr{E}) \subseteq [m,M]$ and $h\colon \mathscr{E} \to \mathbb{R}$ is nonnegative and Δ-integrable such that $L_\Delta(h) > 0$. Let $\Phi \in \mathrm{C}([m,M],\mathbb{R})$. If Φ is convex, then*

$$\Phi(m+M-\overline{L}_\Delta(f,h)) \tag{7.18}$$

$$\begin{aligned}
&\leq \overline{L}_\Delta(\Phi(m+M-f),h)\\
&\leq \frac{M-\overline{L}_\Delta(f,h)}{M-m}\Phi(M)+\frac{\overline{L}_\Delta(f,h)-m}{M-m}\Phi(m)-\overline{L}_\Delta(\tilde{f},h)\delta_\Phi\\
&\leq \Phi(m)+\Phi(M)-\overline{L}_\Delta(\Phi(f),h)\\
&\quad-\left[1-\frac{2}{M-m}\overline{L}_\Delta\left(\left|f-\frac{m+M}{2}\right|,h\right)\right]\delta_\Phi\\
&\leq \Phi(m)+\Phi(M)-\overline{L}_\Delta(\Phi(f),h),
\end{aligned}$$

where $\tilde{f}$ and δ_Φ are defined as in (6.3)*. Moreover, if Φ is concave, then the inequalities in* (7.18) *hold in reverse order.*

Proof. Suppose Φ is convex. Using the first inequality from the series (7.17) and applying inequality (6.2) first to the function $g = m+M-f$, and then to the function f, we obtain

$$\begin{aligned}
&\Phi(m+M-\overline{L}_\Delta(f,h))\\
&\leq \overline{L}_\Delta(\Phi(m+M-f),h)\\
&\leq \frac{M-\overline{L}_\Delta(f,h)}{M-m}\Phi(M)+\frac{\overline{L}_\Delta(f,h)-m}{M-m}\Phi(m)-\overline{L}_\Delta(\tilde{f},h)\delta_\Phi\\
&=\Phi(m)+\Phi(M)-\left[\frac{M-\overline{L}_\Delta(f,h)}{M-m}\Phi(m)+\frac{\overline{L}_\Delta(f,h)-m}{M-m}\Phi(M)\right]-\overline{L}_\Delta(\tilde{f},h)\delta_\Phi\\
&\leq \Phi(m)+\Phi(M)-\overline{L}_\Delta(\Phi(f),h)-2\overline{L}_\Delta(\tilde{f},h)\delta_\Phi\\
&=\Phi(m)+\Phi(M)-\overline{L}_\Delta(\Phi(f),h)-\left[1-\frac{2}{M-m}\overline{L}_\Delta\left(\left|f-\frac{m+M}{2}\right|,h\right)\right]\delta_\Phi\\
&\leq \Phi(m)+\Phi(M)-\overline{L}_\Delta(\Phi(f),h).
\end{aligned}$$

The last inequality follows from the facts that $\delta_\Phi \geq 0$ and

$$1-\frac{2}{M-m}\overline{L}_\Delta\left(\left|f-\frac{m+M}{2}\right|,h\right)\geq 0.$$

If Φ is concave, then the reverse inequalities in (7.17) hold immediately by using the fact that if Φ is concave, then $-\Phi$ is convex. □

Theorem 7.9 *Suppose f is a Δ-integrable function on $\mathscr{E}$ such that $f(\mathscr{E}) \subseteq [m,M]$ and $h\colon \mathscr{E} \to \mathbb{R}$ is nonnegative and Δ-integrable such that $L_\Delta(h) > 0$. Let $\Phi \in \mathrm{C}([m,M],\mathbb{R})$. If Φ is convex, then*

$$\begin{aligned}
&\Phi(m+M-\overline{L}_\Delta(f,h)) \qquad (7.19)\\
&\leq \frac{M-\overline{L}_\Delta(f,h)}{M-m}\Phi(M)+\frac{\overline{L}_\Delta(f,h)-m}{M-m}\Phi(m)\\
&\quad-\left(\frac{1}{2}-\frac{1}{M-m}\left|\overline{L}_\Delta(f,h)-\frac{m+M}{2}\right|\right)\delta_\Phi
\end{aligned}$$

$$
\begin{aligned}
&\leq \Phi(m)+\Phi(M)-\overline{L}_\Delta(\Phi(f),h) \\
&\quad -\left[1-\frac{1}{M-m}\left(\overline{L}_\Delta\left(\left|f-\frac{m+M}{2}\right|,h\right)+\left|\overline{L}_\Delta(f,h)-\frac{m+M}{2}\right|\right)\right]\delta_\Phi \\
&\leq \Phi(m)+\Phi(M)-\overline{L}_\Delta(\Phi(f),h)-\left[1-\frac{2}{M-m}\overline{L}_\Delta\left(\left|f-\frac{m+M}{2}\right|,h\right)\right]\delta_\Phi \\
&\leq \Phi(m)+\Phi(M)-\overline{L}_\Delta(\Phi(f),h).
\end{aligned}
$$

where δ_Φ *is defined as in* (6.3)*. Moreover, if* Φ *is concave, then the inequalities in* (7.19) *hold in reverse order.*

Proof. Let the functions $p,q\colon [m,M]\to\mathbb{R}$ be defined by

$$
p(x)=\frac{M-x}{M-m} \quad \text{and} \quad q(x)=\frac{x-m}{M-m}.
$$

For any $x\in[m,M]$, we can write

$$
\Phi(x)=\Phi\left(\frac{M-x}{M-m}m+\frac{x-m}{M-m}M\right)=\Phi(p(x)m+q(x)M).
$$

By Lemma 7.1 for $n=2$, we get

$$
\Phi(x)\leq p(x)\Phi(m)+q(x)\Phi(M)-\min\{p(x),q(x)\}\delta_\Phi.
$$

Since

$$
\min\{x,y\}=\frac{1}{2}(x+y-|x-y|),
$$

we have

$$
\Phi(x)\leq\frac{M-x}{M-m}\Phi(m)+\frac{x-m}{M-m}\Phi(M)-\left(\frac{1}{2}-\frac{1}{M-m}\left|x-\frac{m+M}{2}\right|\right)\delta_\Phi.
$$

Substituting x by $\overline{L}_\Delta(g,h)$ such that g is Δ-integrable on $\mathscr{E}$, we get

$$
\begin{aligned}
\Phi\left(\overline{L}_\Delta(g,h)\right)\leq{}&\frac{M-\overline{L}_\Delta(g,h)}{M-m}\Phi(m)+\frac{\overline{L}_\Delta(g,h)-m}{M-m}\Phi(M) \\
&-\left(\frac{1}{2}-\frac{1}{M-m}\left|\overline{L}_\Delta(g,h)-\frac{m+M}{2}\right|\right)\delta_\Phi. \qquad (7.20)
\end{aligned}
$$

Now, applying inequality (7.20) to $g=m+M-f$, and then using inequality (6.2), we have

$$
\begin{aligned}
&\Phi(m+M-\overline{L}_\Delta(f,h)) \\
&\leq\frac{M-\overline{L}_\Delta(f,h)}{M-m}\Phi(M)+\frac{\overline{L}_\Delta(f,h)-m}{M-m}\Phi(m) \\
&\quad-\left(\frac{1}{2}-\frac{1}{M-m}\left|\overline{L}_\Delta(f,h)-\frac{m+M}{2}\right|\right)\delta_\Phi
\end{aligned}
$$

$$
\begin{aligned}
&= \Phi(m)+\Phi(M)-\left[\frac{M-\overline{L}_{\Delta}(f,h)}{M-m}\Phi(m)+\frac{\overline{L}_{\Delta}(f,h)-m}{M-m}\Phi(M)\right] \\
&\quad -\left(\frac{1}{2}-\frac{1}{M-m}\left|\overline{L}_{\Delta}(f,h)-\frac{m+M}{2}\right|\right)\delta_{\Phi} \\
&\leq \Phi(m)+\Phi(M)-\overline{L}_{\Delta}(\Phi(f),h)-\overline{L}_{\Delta}(\tilde{f},h)\delta_{\Phi} \\
&\quad -\left(\frac{1}{2}-\frac{1}{M-m}\left|\overline{L}_{\Delta}(f,h)-\frac{m+M}{2}\right|\right)\delta_{\Phi} \\
&= \Phi(m)+\Phi(M)-\overline{L}_{\Delta}(\Phi(f),h) \\
&\quad -\left[1-\frac{1}{M-m}\left(\overline{L}_{\Delta}\left(\left|f-\frac{m+M}{2}\right|,h\right)+\left|\overline{L}_{\Delta}(f,h)-\frac{m+M}{2}\right|\right)\right]\delta_{\Phi} \\
&\leq \Phi(m)+\Phi(M)-\overline{L}_{\Delta}(\Phi(f),h)-\left[1-\frac{2}{M-m}\overline{L}_{\Delta}\left(\left|f-\frac{m+M}{2}\right|,h\right)\right]\delta_{\Phi}.
\end{aligned}
$$

The last inequality is obtained by applying Jensen's inequality to the continuous and convex function $|x|$, so that

$$
\begin{aligned}
\left|\overline{L}_{\Delta}(f,h)-\frac{m+M}{2}\right| &= \left|\overline{L}_{\Delta}\left(\left(f-\frac{m+M}{2}\right),h\right)\right| \\
&\leq \overline{L}_{\Delta}\left(\left|\left(f-\frac{m+M}{2}\right)\right|,h\right).
\end{aligned}
$$

If Φ is concave, then the reverse inequalities in (7.19) hold immediately by using the fact that if Φ is concave, then $-\Phi$ is convex. □

Remark 7.7 Using Theorem 7.9, we get an upper bound for the difference $\overline{L}_{\Delta}(\Phi(f),h)-\Phi\left(\overline{L}_{\Delta}(f,h)\right)$ obtained in [38, Theorem 2.6]. From (7.19), we have

$$
\begin{aligned}
&\overline{L}_{\Delta}(\Phi(f),h) \\
&\leq \Phi(m)+\Phi(M)-\Phi(m+M-\overline{L}_{\Delta}(f,h)) \\
&\quad -\left[1-\frac{1}{M-m}\left(\overline{L}_{\Delta}\left(\left|f-\frac{m+M}{2}\right|,h\right)+\left|\overline{L}_{\Delta}(f,h)-\frac{m+M}{2}\right|\right)\right]\delta_{\Phi}.
\end{aligned} \tag{7.21}
$$

Since Φ is convex, we get

$$
\Phi(m+M-\overline{L}_{\Delta}(f,h))+\Phi\left(\overline{L}_{\Delta}(f,h)\right)\geq 2\Phi\left(\frac{m+M}{2}\right). \tag{7.22}
$$

Combining inequalities (7.21) and (7.22), we obtain

$$
\begin{aligned}
&\overline{L}_{\Delta}(\Phi(f),h)-\Phi\left(\overline{L}_{\Delta}(f,h)\right) \\
&\leq \Phi(m)+\Phi(M)-\left[\Phi(m+M-\overline{L}_{\Delta}(f,h))+\Phi\left(\overline{L}_{\Delta}(f,h)\right)\right] \\
&\quad -\left[1-\frac{1}{M-m}\left(\overline{L}_{\Delta}\left(\left|f-\frac{m+M}{2}\right|,h\right)+\left|\overline{L}_{\Delta}(f,h)-\frac{m+M}{2}\right|\right)\right]\delta_{\Phi}
\end{aligned}
$$

$$\leq \Phi(m)+\Phi(M)-2\Phi\left(\frac{m+M}{2}\right)$$
$$-\left[1-\frac{1}{M-m}\left(\overline{L}_\Delta\left(\left|f-\frac{m+M}{2}\right|,h\right)+\left|\overline{L}_\Delta(f,h)-\frac{m+M}{2}\right|\right)\right]\delta_\Phi$$
$$=\frac{1}{M-m}\left(\overline{L}_\Delta\left(\left|f-\frac{m+M}{2}\right|,h\right)+\left|\overline{L}_\Delta(f,h)-\frac{m+M}{2}\right|\right)\delta_\Phi.$$

7.4 Generalizations of Jensen–Mercer Inequality

Theorem 7.10 *Let $\mathbf{x}_1,\ldots,\mathbf{x}_n \in \mathbb{R}^k$ and $K=\mathrm{co}(\{\mathbf{x}_1,\ldots,\mathbf{x}_n\})$. Let $\lambda_1,\ldots,\lambda_n$ be barycentric coordinates over K. Suppose $\mathbf{f}$ is a Δ-integrable function on $\mathscr{E}$ such that $\mathbf{f}(\mathscr{E})\subseteq K$ and $h\colon \mathscr{E}\to\mathbb{R}$ is nonnegative and Δ-integrable such that $L_\Delta(h)>0$. Let $p_1,\ldots,p_n$ be positive real numbers such that*

$$P_n=\sum_{i=1}^n p_i > L_\Delta(h) \quad \textit{and} \quad p_i \geq L_\Delta(h),\ i\in\{1,\ldots,n\}.$$

If Φ is a convex function on K, then

$$\Phi\left(\frac{\sum\limits_{i=1}^n p_i\mathbf{x}_i - L_\Delta(h\mathbf{f})}{P_n-L_\Delta(h)}\right) \tag{7.23}$$
$$\leq \frac{1}{P_n-L_\Delta(h)}\left(\sum_{i=1}^n p_i\Phi(\mathbf{x}_i)-\sum_{i=1}^n L_\Delta(h\lambda_i(\mathbf{f}))\Phi(\mathbf{x}_i)\right.$$
$$\left.-\min_{1\leq i\leq n}\{p_i-L_\Delta(h\lambda_i(\mathbf{f}))\}\,S_\Phi^n(\mathbf{x}_1,\ldots,\mathbf{x}_n)\right)$$
$$\leq \frac{1}{P_n-L_\Delta(h)}\left(\sum_{i=1}^n p_i\Phi(\mathbf{x}_i)-L_\Delta(h\Phi(\mathbf{f}))\right.$$
$$\left.-S_\Phi^n(\mathbf{x}_1,\ldots,\mathbf{x}_n)\left[\min_{1\leq i\leq n}\{p_i-L_\Delta(h\lambda_i(\mathbf{f}))\}+L_\Delta\left(h\min_{1\leq i\leq n}\{\lambda_i(\mathbf{f})\}\right)\right]\right).$$

Moreover, if Φ is concave, then the inequalities in (7.23) *hold in reverse order.*

Proof. By using the properties of barycentric coordinates, we have

$$\lambda_i(\mathbf{f}(t))\geq 0,\ i\in\{1,\ldots,n\},\quad \sum_{i=1}^n\lambda_i(\mathbf{f}(t))=1,$$

and

$$\mathbf{f}(t)=\sum_{i=1}^n\lambda_i(\mathbf{f}(t))\mathbf{x}_i. \tag{7.24}$$

Since h is nonnegative and Δ-integrable, multiplying (7.24) by h and integrating, we obtain

$$L_\Delta(h\mathbf{f}) = \sum_{i=1}^{n} L_\Delta(h\lambda_i(\mathbf{f}))\mathbf{x}_i.$$

Now we can write

$$\frac{\sum_{i=1}^{n} p_i\mathbf{x}_i - L_\Delta(h\mathbf{f})}{P_n - L_\Delta(h)} = \frac{\sum_{i=1}^{n} p_i\mathbf{x}_i - \sum_{i=1}^{n} L_\Delta(h\lambda_i(\mathbf{f}))\mathbf{x}_i}{P_n - L_\Delta(h)} = \sum_{i=1}^{n} \frac{p_i - L_\Delta(h\lambda_i(\mathbf{f}))}{P_n - L_\Delta(h)}\mathbf{x}_i,$$

where

$$\sum_{i=1}^{n} \frac{p_i - L_\Delta(h\lambda_i(\mathbf{f}))}{P_n - L_\Delta(h)} = 1 \quad \text{and} \quad \frac{p_i - L_\Delta(h\lambda_i(\mathbf{f}))}{P_n - L_\Delta(h)} \geq 0,\ i \in \{1,\dots,n\},$$

since

$$p_i \geq L_\Delta(h) \geq L_\Delta(h\lambda_i(\mathbf{f})) \quad \text{for all} \quad i \in \{1,\dots,n\}.$$

Therefore, the expression

$$\frac{\sum_{i=1}^{n} p_i\mathbf{x}_i - L_\Delta(h\mathbf{f})}{P_n - L_\Delta(h)}$$

is a convex combination of the vectors $\mathbf{x}_1,\dots,\mathbf{x}_n$ and belongs to K. Suppose Φ is convex on K. By using Lemma 7.1 and inequality (7.2), we get

$$\begin{aligned}
&\Phi\left(\frac{\sum_{i=1}^{n} p_i\mathbf{x}_i - L_\Delta(h\mathbf{f})}{P_n - L_\Delta(h)}\right) = \Phi\left(\sum_{i=1}^{n} \frac{p_i - L_\Delta(h\lambda_i(\mathbf{f}))}{P_n - L_\Delta(h)}\mathbf{x}_i\right)\\
&\leq \sum_{i=1}^{n} \frac{p_i - L_\Delta(h\lambda_i(\mathbf{f}))}{P_n - L_\Delta(h)}\Phi(\mathbf{x}_i) - \min_{1\leq i\leq n}\left\{\frac{p_i - L_\Delta(h\lambda_i(\mathbf{f}))}{P_n - L_\Delta(h)}\right\} S_\Phi^n(\mathbf{x}_1,\dots,\mathbf{x}_n)\\
&= \frac{\sum_{i=1}^{n} p_i\Phi(\mathbf{x}_i) - \sum_{i=1}^{n} L_\Delta(h\lambda_i(\mathbf{f}))\Phi(\mathbf{x}_i) - \min_{1\leq i\leq n}\{p_i - L_\Delta(h\lambda_i(\mathbf{f}))\} S_\Phi^n(\mathbf{x}_1,\dots,\mathbf{x}_n)}{P_n - L_\Delta(h)}\\
&\leq \frac{1}{P_n - L_\Delta(h)}\left(\sum_{i=1}^{n} p_i\Phi(\mathbf{x}_i) - L_\Delta(h\Phi(\mathbf{f}))\right.\\
&\quad \left. - S_\Phi^n(\mathbf{x}_1,\dots,\mathbf{x}_n)\left[\min_{1\leq i\leq n}\{p_i - L_\Delta(h\lambda_i(\mathbf{f}))\} + L_\Delta\left(h \min_{1\leq i\leq n}\{\lambda_i(\mathbf{f})\}\right)\right]\right).
\end{aligned}$$

If Φ is concave, then the reverse inequality in (7.23) holds immediately by using the fact that if Φ is concave, then $-\Phi$ is convex. □

Theorem 7.11 *Let $\mathbf{x}_1,\dots,\mathbf{x}_n \in \mathbb{R}^k$ and $K = \text{co}(\{\mathbf{x}_1,\dots,\mathbf{x}_n\})$. Let $\lambda_1,\dots,\lambda_n$ be barycentric coordinates over K. Suppose $\mathbf{f}$ is a Δ-integrable function on $\mathscr{E}$ such that $\mathbf{f}(\mathscr{E}) \subseteq K$ and*

$h\colon \mathscr{E} \to \mathbb{R}$ is nonnegative and Δ-integrable such that $L_\Delta(h) > 0$. Let $p_1, \dots, p_n$ be positive real numbers such that

$$P_n = \sum_{i=1}^{n} p_i > 1 \quad \text{and} \quad p_i \geq 1,\ i \in \{1, \dots, n\}. \tag{7.25}$$

If Φ is a convex function on K, then

$$\begin{aligned}
&\Phi\left(\frac{1}{P_n - 1}\left(\sum_{i=1}^{n} p_i \mathbf{x}_i - \overline{L}_\Delta(\mathbf{f}, h)\right)\right) \\
&\leq \frac{1}{P_n - 1}\left(\sum_{i=1}^{n} p_i \Phi(\mathbf{x}_i) - \sum_{i=1}^{n} \overline{L}_\Delta(\lambda_i(\mathbf{f}), h)\Phi(\mathbf{x}_i)\right. \\
&\quad \left. - \min_{1\leq i\leq n} \left\{p_i - \overline{L}_\Delta(\lambda_i(\mathbf{f}), h)\right\} S^n_\Phi(\mathbf{x}_1, \dots, \mathbf{x}_n)\right) \\
&\leq \frac{1}{P_n - 1}\left(\sum_{i=1}^{n} p_i \Phi(\mathbf{x}_i) - \overline{L}_\Delta(\mathbf{f}, h)\right. \\
&\quad \left. - S^n_\Phi(\mathbf{x}_1, \dots, \mathbf{x}_n)\left[\min_{1\leq i\leq n} \left\{p_i - \overline{L}_\Delta(\lambda_i(\mathbf{f}), h)\right\} + \overline{L}_\Delta\left(\min_{1\leq i\leq n} \{\lambda_i(\mathbf{f})\}, h\right)\right]\right).
\end{aligned} \tag{7.26}$$

Moreover, if Φ is concave, then the inequalities in (7.26) *hold in reverse order.*

Proof. The proof is similar to the proof of Theorem 7.10. Here we have

$$\sum_{i=1}^{n} \overline{L}_\Delta(\lambda_i(\mathbf{f}), h) = 1, \quad 0 \leq \overline{L}_\Delta(\lambda_i(\mathbf{f}), h) \leq 1,\ i \in \{1, \dots, n\},$$

and

$$\overline{L}_\Delta(\mathbf{f}, h) = \sum_{i=1}^{n} \overline{L}_\Delta(\lambda_i(\mathbf{f}), h)\mathbf{x}_i.$$

Now we can write

$$\frac{\sum_{i=1}^{n} p_i \mathbf{x}_i - \overline{L}_\Delta(\mathbf{f}, h)}{P_n - 1} = \frac{\sum_{i=1}^{n} p_i \mathbf{x}_i - \sum_{i=1}^{n} \overline{L}_\Delta(\lambda_i(\mathbf{f}), h)\mathbf{x}_i}{P_n - 1} = \sum_{i=1}^{n} \frac{p_i - \overline{L}_\Delta(\lambda_i(\mathbf{f}), h)}{P_n - 1}\mathbf{x}_i,$$

where

$$\sum_{i=1}^{n} \frac{p_i - \overline{L}_\Delta(\lambda_i(\mathbf{f}), h)}{P_n - 1} = 1 \quad \text{and} \quad \frac{p_i - \overline{L}_\Delta(\lambda_i(\mathbf{f}), h)}{P_n - 1} \geq 0,\ i \in \{1, \dots, n\},$$

since

$$p_i \geq 1 \geq \overline{L}_\Delta(\lambda_i(\mathbf{f}), h),\ i \in \{1, \dots, n\}.$$

Therefore, the expression

$$\frac{\sum_{i=1}^{n} p_i \mathbf{x}_i - \overline{L}_\Delta(\mathbf{f}, h)}{P_n - 1}$$

is a convex combination of the vectors $\mathbf{x}_1,\ldots,\mathbf{x}_n$ and belongs to K. Suppose Φ is convex on K. By using Lemma 7.1 and Theorem 7.2, we have

$$\begin{aligned}
&\Phi\left(\frac{\sum_{i=1}^{n} p_i\mathbf{x}_i - \overline{L}_{\Delta}(\mathbf{f},h)}{P_n-1}\right) = \Phi\left(\sum_{i=1}^{n}\frac{p_i-\overline{L}_{\Delta}(\lambda_i(\mathbf{f}),h)}{P_n-1}\mathbf{x}_i\right)\\
&\le \sum_{i=1}^{n}\frac{p_i-\overline{L}_{\Delta}(\lambda_i(\mathbf{f}),h)}{P_n-1}\Phi(\mathbf{x}_i) - \min_{1\le i\le n}\left\{\frac{p_i-\overline{L}_{\Delta}(\lambda_i(\mathbf{f}),h)}{P_n-1}\right\}S^n_{\Phi}(\mathbf{x}_1,\ldots,\mathbf{x}_n)\\
&= \frac{\sum_{i=1}^{n} p_i\Phi(\mathbf{x}_i) - \sum_{i=1}^{n}\overline{L}_{\Delta}(\lambda_i(\mathbf{f}),h)\Phi(\mathbf{x}_i) - \min_{1\le i\le n}\left\{p_i-\overline{L}_{\Delta}(\lambda_i(\mathbf{f}),h)\right\}S^n_{\Phi}(\mathbf{x}_1,\ldots,\mathbf{x}_n)}{P_n-1}\\
&\le \frac{1}{P_n-1}\left(\sum_{i=1}^{n} p_i\Phi(\mathbf{x}_i) - \overline{L}_{\Delta}(\mathbf{f},h)\right.\\
&\quad \left. -S^n_{\Phi}(\mathbf{x}_1,\ldots,\mathbf{x}_n)\left[\min_{1\le i\le n}\left\{p_i-\overline{L}_{\Delta}(\lambda_i(\mathbf{f}),h)\right\} + \overline{L}_{\Delta}\left(\min_{1\le i\le n}\{\lambda_i(\mathbf{f})\},h\right)\right]\right).
\end{aligned}$$

If Φ is concave, then the reverse inequality in (7.26) holds immediately by using the fact that if Φ is concave, then $-\Phi$ is convex. □

Remark 7.8 Theorems 7.10 and 7.11 can also be obtained by using Theorem 2.5 and [98, Theorem 4].

Remark 7.9 Theorem 7.11 is a generalization of Theorem 7.9 for convex hulls. Since the interval $I=[m,M]$ is a 1-simplex with vertices m and M, the barycentric coordinates have the special form

$$\lambda_1(f(t)) = \frac{M-f(t)}{M-m} \quad \text{and} \quad \lambda_2(f(t)) = \frac{f(t)-m}{M-m}. \tag{7.27}$$

Therefore we have

$$\overline{L}_{\Delta}(\lambda_1(f),h) = \frac{M-\overline{L}_{\Delta}(f,h)}{M-m} \quad \text{and} \quad \overline{L}_{\Delta}(\lambda_2(f),h) = \frac{\overline{L}_{\Delta}(f,h)-m}{M-m}.$$

Choosing $n=2$, $p_1=p_2=1$, $x_1=m$, $x_2=M$, from (7.26), we get

$$\begin{aligned}
&\Phi(m+M-\overline{L}_{\Delta}(f,h))\\
&\le \Phi(m)+\Phi(M) - \left[\frac{M-\overline{L}_{\Delta}(f,h)}{M-m}\Phi(m) + \frac{\overline{L}_{\Delta}(f,h)-m}{M-m}\Phi(M)\right]\\
&\quad -\left(\frac{1}{2}-\frac{1}{M-m}\left|\overline{L}_{\Delta}(f,h)-\frac{m+M}{2}\right|\right)S^2_{\Phi}(m,M)\\
&= \frac{M-\overline{L}_{\Delta}(f,h)}{M-m}\Phi(M) + \frac{\overline{L}_{\Delta}(f,h)-m}{M-m}\Phi(m)
\end{aligned}$$

$$
\begin{aligned}
&-\left(\frac{1}{2}-\frac{1}{M-m}\left|\overline{L}_\Delta(f,h)-\frac{m+M}{2}\right|\right)S_\Phi^2(m,M)\\
&\le \Phi(m)+\Phi(M)-\overline{L}_\Delta(\Phi(f),h)-\left[\frac{1}{2}-\frac{1}{M-m}\left|\overline{L}_\Delta(f,h)-\frac{m+M}{2}\right|\right.\\
&\quad\left.+\overline{L}_\Delta\left(\left(\frac{1}{2}-\frac{1}{M-m}\left|f-\frac{m+M}{2}\right|\right),h\right)\right]S_\Phi^2(m,M)\\
&=\Phi(m)+\Phi(M)-\overline{L}_\Delta(\Phi(f),h)\\
&\quad-\left[1-\frac{1}{M-m}\left(\left|\overline{L}_\Delta(f,h)-\frac{m+M}{2}\right|+\overline{L}_\Delta\left(\left|f-\frac{m+M}{2}\right|,h\right)\right)\right]\\
&\quad S_\Phi^2(m,M).
\end{aligned}
$$

Theorem 7.12 *Let $\mathbf{x}_1,\ldots,\mathbf{x}_n\in\mathbb{R}^k$ and $K=\mathrm{co}(\{\mathbf{x}_1,\ldots,\mathbf{x}_n\})$. Let $\lambda_1,\ldots,\lambda_n$ be barycentric coordinates over K. Suppose $\mathbf{f}$ is a Δ-integrable function on $\mathscr{E}$ such that $\mathbf{f}(\mathscr{E})\subseteq K$ and $h\colon\mathscr{E}\to\mathbb{R}$ is nonnegative and Δ-integrable such that $L_\Delta(h)>0$. Let $p_1,\ldots,p_n$ be positive real numbers such that*

$$
P_n>L_\Delta(h)\quad\text{and}\quad\frac{\sum_{i=1}^n p_i\mathbf{x}_i-L_\Delta(h\mathbf{f})}{P_n-L_\Delta(h)}\in K.
$$

If Φ is a convex function on K, then

$$
\begin{aligned}
&\Phi\left(\frac{\sum_{i=1}^n p_i\mathbf{x}_i-L_\Delta(h\mathbf{f})}{P_n-L_\Delta(h)}\right) \qquad (7.28)\\
&\ge\frac{1}{P_n-L_\Delta(h)}\left(P_n\Phi\left(\frac{\sum_{i=1}^n p_i\mathbf{x}_i}{P_n}\right)-L_\Delta(h)\Phi\left(\overline{L}_\Delta(\mathbf{f},h)\right)\right)\\
&\ge\frac{1}{P_n-L_\Delta(h)}\left(P_n\Phi\left(\frac{\sum_{i=1}^n p_i\mathbf{x}_i}{P_n}\right)-\sum_{i=1}^n L_\Delta(h\lambda_i(\mathbf{f}))\Phi(\mathbf{x}_i)\right.\\
&\quad\left.+\min_{1\le i\le n}\{L_\Delta(h\lambda_i(\mathbf{f}))\}S_\Phi^n(\mathbf{x}_1,\ldots,\mathbf{x}_n)\right).
\end{aligned}
$$

Moreover, if Φ is concave, then the inequalities in (7.28) *hold in reverse order.*

Proof. The proof is similar to the proof of Theorem 7.10. Here we use Theorem 7.1 instead of Lemma 7.1. If Φ is convex, then we have

$$
\Phi\left(\frac{1}{P_n-L_\Delta(h)}\left(P_n\left(\frac{\sum_{i=1}^n p_i\mathbf{x}_i}{P_n}\right)-L_\Delta(h)\left(\overline{L}_\Delta(\mathbf{f},h)\right)\right)\right)
$$

$$\geq \frac{1}{P_n - L_\Delta(h)} \left(P_n \Phi\left(\frac{\sum_{i=1}^{n} p_i \mathbf{x}_i}{P_n} \right) - L_\Delta(h) \Phi\left(\overline{L}_\Delta(\mathbf{f}, h) \right) \right)$$

$$\geq \frac{1}{P_n - L_\Delta(h)} \left(P_n \Phi\left(\frac{\sum_{i=1}^{n} p_i \mathbf{x}_i}{P_n} \right) - \sum_{i=1}^{n} L_\Delta(h\lambda_i(\mathbf{f})) \Phi(\mathbf{x}_i) \right.$$

$$\left. + \min_{1 \leq i \leq n} \{L_\Delta(h\lambda_i(\mathbf{f}))\} S_\Phi^n(\mathbf{x}_1, \ldots, \mathbf{x}_n) \right).$$

If Φ is concave, then the reverse inequality in (7.28) holds immediately by using the fact that if Φ is concave, then $-\Phi$ is convex. □

Theorem 7.13 *Let $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^k$ and $K = \mathrm{co}(\{\mathbf{x}_1, \ldots, \mathbf{x}_n\})$. Let $\lambda_1, \ldots, \lambda_n$ be barycentric coordinates over K. Suppose $\mathbf{f}$ is a Δ-integrable function on $\mathscr{E}$ such that $\mathbf{f}(\mathscr{E}) \subseteq K$ and $h \colon \mathscr{E} \to \mathbb{R}$ is nonnegative and Δ-integrable such that $L_\Delta(h) > 0$. Let $p_1, \ldots, p_n$ be positive real numbers such that*

$$P_n = \sum_{i=1}^{n} p_i > 1 \quad \textit{and} \quad \frac{\left(\sum_{i=1}^{n} p_i \mathbf{x}_i - \overline{L}_\Delta(\mathbf{f}, h) \right)}{P_n - 1} \in K. \tag{7.29}$$

If Φ is a convex function on K, then

$$\Phi\left(\frac{\sum_{i=1}^{n} p_i \mathbf{x}_i - \overline{L}_\Delta(\mathbf{f}, h)}{P_n - 1} \right) \tag{7.30}$$

$$\geq \frac{1}{P_n - 1} \left(P_n \Phi\left(\frac{\sum_{i=1}^{n} p_i \mathbf{x}_i}{P_n} \right) - \Phi\left(\overline{L}_\Delta(\mathbf{f}, h) \right) \right)$$

$$\geq \frac{1}{P_n - 1} \left(P_n \Phi\left(\frac{\sum_{i=1}^{n} p_i \mathbf{x}_i}{P_n} \right) - \sum_{i=1}^{n} \overline{L}_\Delta(\lambda_i(\mathbf{f}), h) \Phi(\mathbf{x}_i) \right.$$

$$\left. + \min_{1 \leq i \leq n} \left\{ \overline{L}_\Delta(\lambda_i(\mathbf{f}), h) \right\} S_\Phi^n(\mathbf{x}_1, \ldots, \mathbf{x}_n) \right).$$

Moreover, if Φ is concave, then the inequalities in (7.30) *hold in reverse order.*

Proof. The proof is similar to the proof of Theorem 7.11; only here we use Theorem 7.1 instead of Lemma 7.1. □

Remark 7.10 If positive real numbers $p_1,\dots,p_n$ satisfy condition (7.25), then condition (7.29) is also satisfied since K is a convex set. Hence (7.26) can be extended as

$$\begin{aligned}
&\frac{1}{P_n-1}\left(P_n\Phi\left(\frac{\sum_{i=1}^n p_i\mathbf{x}_i}{P_n}\right)-\sum_{i=1}^n \overline{L}_\Delta(\lambda_i(\mathbf{f}),h)\Phi(\mathbf{x}_i)\right.\\
&\left.+\min_{1\le i\le n}\left\{\overline{L}_\Delta(\lambda_i(\mathbf{f}),h)\right\}S_\Phi^n(\mathbf{x}_1,\dots,\mathbf{x}_n)\right)\\
&\le \frac{1}{P_n-1}\left(P_n\Phi\left(\frac{\sum_{i=1}^n p_i\mathbf{x}_i}{P_n}\right)-\Phi\left(\overline{L}_\Delta(\mathbf{f},h)\right)\right)\\
&\le \Phi\left(\frac{1}{P_n-1}\left(\sum_{i=1}^n p_i\mathbf{x}_i-\overline{L}_\Delta(\mathbf{f},h)\right)\right)\\
&\le \frac{1}{P_n-1}\left(\sum_{i=1}^n p_i\Phi(\mathbf{x}_i)-\sum_{i=1}^n \overline{L}_\Delta(\lambda_i(\mathbf{f}),h)\Phi(\mathbf{x}_i)\right.\\
&\left.-\min_{1\le i\le n}\left\{p_i-\overline{L}_\Delta(\lambda_i(\mathbf{f}),h)\right\}S_\Phi^n(\mathbf{x}_1,\dots,\mathbf{x}_n)\right)\\
&\le \frac{1}{P_n-1}\left(\sum_{i=1}^n p_i\Phi(\mathbf{x}_i)-\overline{L}_\Delta(\Phi(\mathbf{f}),h)\right.\\
&\left.-S_\Phi^n(\mathbf{x}_1,\dots,\mathbf{x}_n)\left[\min_{1\le i\le n}\left\{p_i-\overline{L}_\Delta(\lambda_i(\mathbf{f}),h)\right\}+\overline{L}_\Delta\left(\min_{1\le i\le n}\{\lambda_i(\mathbf{f})\},h\right)\right]\right).
\end{aligned}$$

Corollary 7.1 *Suppose* $\mathbf{f}$ *is a* Δ*-integrable function on* $\mathscr{E}$ *such that* $\mathbf{f}(\mathscr{E})\subseteq[m,M]$ *and* $h\colon\mathscr{E}\to\mathbb{R}$ *is nonnegative and* Δ*-integrable such that* $L_\Delta(h)>0$*. If* Φ *is a convex function on* $[m,M]$*, then*

$$\begin{aligned}
&\Phi(m+M-\overline{L}_\Delta(f,h)) \qquad (7.31)\\
&\ge 2\Phi\left(\frac{m+M}{2}\right)-\Phi\left(\overline{L}_\Delta(f,h)\right)\\
&\ge 2\Phi\left(\frac{m+M}{2}\right)-\left[\frac{M-\overline{L}_\Delta(f,h)}{M-m}\Phi(m)+\frac{\overline{L}_\Delta(f,h)-m}{M-m}\Phi(M)\right]\\
&\quad+\left(\frac{1}{2}-\frac{1}{M-m}\left|\overline{L}_\Delta(f,h)-\frac{m+M}{2}\right|\right)S_\Phi^2(m,M).
\end{aligned}$$

Moreover, if Φ *is concave, then the inequalities in* (7.31) *hold in reverse order.*

Proof. Choosing $n=2,\ x_1=m,\ x_2=M,\ p_1=p_2=1$, and using (7.27), the inequalities in (7.31) easily follow from (7.30). □

Corollary 7.2 *Let $S=[\mathbf{w}_1,\ldots,\mathbf{w}_{k+1}]$ be a k-simplex in $\mathbb{R}^k$ and let $\lambda_1,\ldots,\lambda_{k+1}$ be barycentric coordinates over S. Suppose $\mathbf{f}$ is a Δ-integrable function on $\mathscr{E}$ such that $\mathbf{f}(\mathscr{E})\subseteq S$ and $h\colon \mathscr{E}\to\mathbb{R}$ is nonnegative and Δ-integrable such that $L_\Delta(h)>0$. If Φ is a convex function on S, then*

$$\begin{aligned}
&\frac{1}{k}\left((k+1)\Phi\left(\frac{\sum_{i=1}^{k+1}\mathbf{w}_i}{k+1}\right)-\sum_{i=1}^{k+1}\lambda_i\left(\overline{L}_\Delta(\mathbf{f},h)\right)\Phi(\mathbf{w}_i)\right.\\
&\left.+\min_{1\le i\le n}\left\{\lambda_i\left(\overline{L}_\Delta(\mathbf{f},h)\right)\right\}S_\Phi^{k+1}(\mathbf{w}_1,\ldots,\mathbf{w}_{k+1})\right)\\
&\le\frac{1}{k}\left((k+1)\Phi\left(\frac{\sum_{i=1}^{k+1}\mathbf{w}_i}{k+1}\right)-\Phi\left(\overline{L}_\Delta(\mathbf{f},h)\right)\right)\\
&\le\Phi\left(\frac{1}{k}\left(\sum_{i=1}^{k+1}\mathbf{w}_i-\overline{L}_\Delta(\mathbf{f},h)\right)\right)\\
&\le\frac{1}{k}\left(\sum_{i=1}^{k+1}\Phi(\mathbf{w}_i)-\sum_{i=1}^{k+1}\lambda_i\left(\overline{L}_\Delta(\mathbf{f},h)\right)\Phi(\mathbf{w}_i)\right.\\
&\left.-\min_{1\le i\le n}\left\{1-\lambda_i\left(\overline{L}_\Delta(\mathbf{f},h)\right)\right\}S_\Phi^{k+1}(\mathbf{w}_1,\ldots,\mathbf{w}_{k+1})\right)\\
&\le\frac{1}{k}\left(\sum_{i=1}^{k+1}\Phi(\mathbf{w}_i)-\overline{L}_\Delta(\mathbf{f},h)-S_\Phi^{k+1}(\mathbf{w}_1,\ldots,\mathbf{w}_{k+1})\right.\\
&\left.\left[\min_{1\le i\le n}\left\{1-\lambda_i\left(\overline{L}_\Delta(\mathbf{f},h)\right)\right\}+\overline{L}_\Delta\left(\min_{1\le i\le n}\lambda_i(\mathbf{f}),h\right)\right]\right).
\end{aligned}\tag{7.32}$$

Moreover, if Φ is concave, then the inequalities in (7.32) *hold in reverse order.*

Proof. Since barycentric coordinates $\lambda_1,\ldots,\lambda_{k+1}$ over a k-simplex S in $\mathbb{R}^k$ are nonnegative linear polynomials, we have

$$\overline{L}_\Delta(\lambda_i(\mathbf{f}),h)=\lambda_i\left(\overline{L}_\Delta(\mathbf{f},h)\right)\quad\text{for all}\quad i=1,\ldots,k+1.$$

Choosing $\mathbf{x}_i=\mathbf{w}_i$ for all $i=1,\ldots,k+1$ and $p_1=p_2=\ldots=p_{k+1}=1$, the inequalities in (7.32) easily follow from (7.26) and (7.30). □

7.5 Exponential Convexity

Suppose $\Phi\colon I \to \mathbb{R}$ is such that $\Phi(f)$ is Δ-integrable, where $[m,M] \subseteq I$. Then motivated by Theorems 7.8 and 7.9, we define the functionals $\mathscr{J}_{\Delta i},\ i \in \{6,\dots,10\}$, by

$$\mathscr{J}_{\Delta 6}(\Phi) = \overline{L}_\Delta(\Phi(m+M-f),h) - \Phi(m+M-\overline{L}_\Delta(f,h)), \tag{7.33}$$

$$\begin{aligned}\mathscr{J}_{\Delta 7}(\Phi) = \frac{M-\overline{L}_\Delta(f,h)}{M-m}\Phi(M) + \frac{\overline{L}_\Delta(f,h)-m}{M-m}\Phi(m) - \overline{L}_\Delta(\tilde{f},h)\delta_\Phi \\ - \overline{L}_\Delta(\Phi(m+M-f),h),\end{aligned} \tag{7.34}$$

$$\begin{aligned}\mathscr{J}_{\Delta 8}(\Phi) = \frac{M-\overline{L}_\Delta(f,h)}{M-m}\Phi(M) + \frac{\overline{L}_\Delta(f,h)-m}{M-m}\Phi(m) \\ - \left(\frac{1}{2} - \frac{1}{M-m}\left|\overline{L}_\Delta(f,h) - \frac{m+M}{2}\right|\right)\delta_\Phi - \Phi(m+M-\overline{L}_\Delta(f,h)),\end{aligned} \tag{7.35}$$

$$\begin{aligned}\mathscr{J}_{\Delta 9}(\Phi) = \Phi(m)+\Phi(M) - \Phi(m+M-\overline{L}_\Delta(f,h)) - \overline{L}_\Delta(\Phi(f),h) \\ - \left[1 - \frac{2}{M-m}\overline{L}_\Delta\left(\left|f - \frac{m+M}{2}\right|,h\right)\right]\delta_\Phi,\end{aligned} \tag{7.36}$$

and

$$\begin{aligned}\mathscr{J}_{\Delta 10}(\Phi) = \Phi(m)+\Phi(M) - \Phi(m+M-\overline{L}_\Delta(f,h)) - \overline{L}_\Delta(\Phi(f),h) \\ - \left[1 - \frac{1}{M-m}\left(\overline{L}_\Delta\left(\left|f - \frac{m+M}{2}\right|,h\right) + \left|\overline{L}_\Delta(f,h) - \frac{m+M}{2}\right|\right)\right]\delta_\Phi.\end{aligned} \tag{7.37}$$

where $\tilde{f}$ and δ_Φ are defined as in (6.3). Obviously, $\mathscr{J}_{\Delta i},\ i \in \{6,\dots,10\}$, are linear. If Φ is additionally continuous and convex, then Theorems 7.8 and 7.9 imply $\mathscr{J}_{\Delta i}(\Phi) \geq 0,\ i \in \{6,\dots,10\}$.

In the following, we denote by Φ_0 the function defined by $\Phi_0(x) = x^2$ on a suitable domain.

Now, we give Cauchy mean value type theorems for the functionals $\mathscr{J}_{\Delta i}(\Phi),\ i \in \{6,\dots,10\}$.

Theorem 7.14 *Let $\varphi \in C^2(I,\mathbb{R})$, where $[m,M] \subseteq I$. Suppose $\mathscr{J}_{\Delta i},\ i \in \{6,\dots,10\}$, are defined as in* (7.33),…,(7.37). *Then there exist $\xi_i \in [m,M],\ i \in \{6,\dots,10\}$, such that*

$$\mathscr{J}_{\Delta i}(\varphi) = \frac{\varphi''(\xi_i)}{2}\mathscr{J}_{\Delta i}(\Phi_0), \quad i \in \{6,\dots,10\}. \tag{7.38}$$

Proof. Since $\varphi \in C^2(I)$, there exist $\eta, \zeta \in \mathbb{R}$ such that

$$\eta = \min_{x\in[m,M]} \varphi''(x) \quad \text{and} \quad \zeta = \max_{x\in[m,M]} \varphi''(x).$$

Let

$$\phi_1(x) = \frac{\zeta}{2}x^2 - \varphi(x) \quad \text{and} \quad \phi_2(x) = \varphi(x) - \frac{\eta}{2}x^2.$$

Then ϕ_1 and ϕ_2 are continuous and convex. Therefore, we have

$$\mathscr{J}_{\Delta i}(\phi_1) \geq 0 \quad \text{and} \quad \mathscr{J}_{\Delta i}(\phi_2) \geq 0, \quad i \in \{6,\ldots,10\},$$

which implies

$$\frac{\eta}{2}\mathscr{J}_{\Delta i}(\Phi_0) \leq \mathscr{J}_{\Delta i}(\varphi) \leq \frac{\zeta}{2}\mathscr{J}_{\Delta i}(\Phi_0), \quad i \in \{6,\ldots,10\}.$$

If $\mathscr{J}_{\Delta i}(\Phi_0) = 0$, then there is nothing to prove. If $\mathscr{J}_{\Delta i}(\Phi_0) > 0$, then we have

$$\eta \leq \frac{2\mathscr{J}_{\Delta i}(\varphi)}{\mathscr{J}_{\Delta i}(\Phi_0)} \leq \zeta, \quad i \in \{6,\ldots,10\}.$$

Hence, there exist $\xi_i \in [m,M]$, $i \in \{6,\ldots,10\}$, such that

$$\frac{2\mathscr{J}_{\Delta i}(\varphi)}{\mathscr{J}_{\Delta i}(\Phi_0)} = \varphi''(\xi_i), \quad i \in \{6,\ldots,10\},$$

and the result follows. □

Theorem 7.15 *Let $\varphi, \psi \in C^2(I,\mathbb{R})$, where $[m,M] \subseteq I$. Suppose $\mathscr{J}_{\Delta i}$, $i \in \{6,\ldots,10\}$, are defined as in* (7.33),...,(7.37). *Then there exist $\xi_i \in [m,M]$, $i \in \{6,\ldots,10\}$, such that*

$$\frac{\mathscr{J}_{\Delta i}(\varphi)}{\mathscr{J}_{\Delta i}(\psi)} = \frac{\varphi''(\xi_i)}{\psi''(\xi_i)}, \quad i \in \{6,\ldots,10\}, \tag{7.39}$$

provided that the denominators in (7.39) *are nonzero.*

Proof. Consider the function χ defined by

$$\chi(t) = \mathscr{J}_{\Delta i}(\psi)\varphi(t) - \mathscr{J}_{\Delta i}(\varphi)\psi(t).$$

As the function χ is a linear combination of φ and ψ, we get $\chi \in C^2(I)$. Now by applying Theorem 7.14 to χ, there exists $\xi_i \in [m,M]$ such that

$$\mathscr{J}_{\Delta i}(\chi) = \frac{\chi''(\xi_i)}{2}\mathscr{J}_{\Delta i}(\Phi_0).$$

But $\mathscr{J}_{\Delta i}(\chi) = 0$ and $\mathscr{J}_{\Delta i}(\Phi_0) \neq 0$ (otherwise we have a contradiction with $\mathscr{J}_{\Delta i}(\psi) \neq 0$, by Theorem 7.14). Therefore

$$\chi''(\xi_i) = 0.$$

From here the result follows. □

Remark 7.11 If the inverse of the function $\frac{\varphi''}{\psi''}$ exists, then (7.39) gives

$$\xi_i = \left(\frac{\varphi''}{\psi''}\right)^{-1} \left(\frac{\mathscr{J}_{\Delta i}(\varphi)}{\mathscr{J}_{\Delta i}(\psi)}\right).$$

Now we study the log-convexity, n-exponential convexity, and exponential-convexity of the functionals $\mathscr{J}_{\Delta i}$, $i \in \{6,\ldots,10\}$, similarly as in Section 5.4.

Theorem 7.16 *Let $\mathscr{J}_{\Delta i}$, $i \in \{6,\ldots,10\}$, be linear functionals defined as in* (7.33), ..., (7.37). *Suppose J is an interval in $\mathbb{R}$ and $\Omega = \{\Phi_\rho : \rho \in J\}$ is a family of functions defined on an open interval I such that $[m,M] \subset I$. If the function $\rho \mapsto [x_0,x_1,x_2;\Phi_\rho]$ is n-exponentially convex in the Jensen sense on J for every choice of mutually different numbers $x_0,x_1,x_2 \in I$, then $\rho \mapsto \mathscr{J}_{\Delta i}(\Phi_\rho)$, $i \in \{6,\ldots,10\}$, is an n-exponentially convex function in the Jensen sense on J. If the function $\rho \mapsto \mathscr{J}_{\Delta i}(\Phi_\rho)$, $i \in \{6,\ldots,10\}$, is also continuous on J, then it is n-exponentially convex on J.*

Proof. The proof is similar to the proof of Theorem 5.9. □

The following corollary is an immediate consequence of Theorem 7.16.

Corollary 7.3 *Let $\mathscr{J}_{\Delta i}$, $i \in \{6,\ldots,10\}$, be linear functionals defined as in* (7.33), ..., (7.37). *Suppose J is an interval in $\mathbb{R}$ and $\Omega = \{\Phi_\rho : \rho \in J\}$ is a family of functions defined on an open interval I such that $[m,M] \subset I$. If the function $\rho \mapsto [x_0,x_1,x_2;\Phi_\rho]$ is exponentially convex in the Jensen sense on J for every choice of mutually different numbers $x_0,x_1,x_2 \in I$, then $\rho \mapsto \mathscr{J}_{\Delta i}(\Phi_\rho)$, $i \in \{6,\ldots,10\}$, is an exponentially convex function in the Jensen sense on J. If the function $\rho \mapsto \mathscr{J}_{\Delta i}(\Phi_\rho)$, $i \in \{6,\ldots,10\}$, is also continuous on J, then it is exponentially convex on J.*

Corollary 7.4 *Let $\mathscr{J}_{\Delta i}$, $i \in \{6,\ldots,10\}$, be linear functionals defined as in* (7.33), ..., (7.37). *Suppose J is an interval in $\mathbb{R}$ and $\Omega = \{\Phi_\rho : \rho \in J\}$ is a family of functions defined on an open interval I such that $[m,M] \subset I$. If the function $\rho \mapsto [x_0,x_1,x_2;\Phi_\rho]$ is 2-exponentially convex in the Jensen sense on J for every choice of mutually different numbers $x_0,x_1,x_2 \in I$, then the following statements hold:*

(i) *The function $\rho \mapsto \mathscr{J}_{\Delta i}(\Phi_\rho)$, $i \in \{6,\ldots,10\}$, is 2-exponentially convex in the Jensen sense on J.*

(ii) *If $\rho \mapsto \mathscr{J}_{\Delta i}(\Phi_\rho)$, $i \in \{6,\ldots,10\}$, is continuous on J, then it is also 2-exponentially convex on J. If $\rho \mapsto \mathscr{J}_{\Delta i}(\Phi_\rho)$, $i \in \{6,\ldots,10\}$, is additionally strictly positive, then it is also log-convex on J.*

(iii) *If $\rho \mapsto \mathscr{J}_{\Delta i}(\Phi_\rho)$, $i \in \{6,\ldots,10\}$, is a strictly positive differentiable function on J, then for any $p \le u$, $q \le v$, $p,q,u,v \in J$, we have*

$$\mathscr{M}_{p,q}(\mathscr{J}_{\Delta i},\Omega) \le \mathscr{M}_{u,v}(\mathscr{J}_{\Delta i},\Omega), \quad i \in \{6,\ldots,10\}, \tag{7.40}$$

where

$$\mathscr{M}_{p,q}(\mathscr{J}_{\Delta i},\Omega)=\begin{cases}\left(\dfrac{\mathscr{J}_{\Delta i}(\Phi_p)}{\mathscr{J}_{\Delta i}(\Phi_q)}\right)^{\frac{1}{p-q}}, & p\neq q;\\ \exp\left(\dfrac{\frac{d}{dp}\mathscr{J}_{\Delta i}(\Phi_p)}{\mathscr{J}_{\Delta i}(\Phi_p)}\right), & p=q.\end{cases} \tag{7.41}$$

Proof. (i) and (ii) are immediate consequences of Theorem 7.16. To prove (iii), note that $\rho\mapsto\mathscr{J}_{\Delta i}(\Phi_\rho)$, $i\in\{6,\ldots,10\}$, is positive and differentiable and therefore continuous too. By (ii), the function $\rho\mapsto\mathscr{J}_{\Delta i}(\Phi_\rho)$ is log-convex, and by Remark 1.6 (b), we obtain

$$\frac{\log\mathscr{J}_{\Delta i}(\Phi_p)-\log\mathscr{J}_{\Delta i}(\Phi_q)}{p-q}\leq\frac{\log\mathscr{J}_{\Delta i}(\Phi_u)-\log\mathscr{J}_{\Delta i}(\Phi_v)}{u-v}$$

for $p\leq u$, $q\leq v$, $p\neq q$, $u\neq v$, concluding

$$\mathscr{M}_{p,q}(\mathscr{J}_{\Delta i},\Omega)\leq\mathscr{M}_{u,v}(\mathscr{J}_{\Delta i},\Omega). \tag{7.42}$$

The cases $p=q$ and $u=v$ follow from (7.42) as limit cases. □

Remark 7.12 Note that the results from Theorem 7.16, Corollary 7.3, and Corollary 7.4 still hold when two of the points $x_0,x_1,x_2\in I$ coincide, for a family of differentiable functions Φ_ρ such that the function $\rho\mapsto[x_0,x_1,x_2;\Phi_\rho]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, they still hold when all three points coincide for a family of twice differentiable functions with the same property. The proofs are obtained by recalling Remark 1.4 and suitably characterizing convexity.

Now, we present several families of functions which fulfil the conditions of Theorem 7.16, Corollary 7.3, and Corollary 7.4 (and Remark 7.12). This enables us to construct large families of functions which are exponentially convex. For a discussion related to this problem, see [56]. In the following, we denote by id the identity function.

Example 7.1 Consider the family of functions

$$\Omega_1=\{\kappa_\rho:\ \mathbb{R}\to[0,\infty);\ \rho\in\mathbb{R}\}$$

defined in Example 5.2. Then by using Corollary 7.3, we conclude that $\rho\mapsto\mathscr{J}_{\Delta i}(\kappa_\rho)$, $i\in\{6,\ldots,10\}$, are exponentially convex in the Jensen sense. It is easy to verify that these mappings are continuous, so they are exponentially convex. For this family of functions,

$\mathscr{M}_{p,q}(\mathscr{J}_{\Delta i},\Omega)$, $i \in \{6,\ldots,10\}$, from (6.28) becomes

$$\mathscr{M}_{p,q}(\mathscr{J}_{\Delta i},\Omega_1) = \begin{cases} \left(\dfrac{\mathscr{J}_{\Delta i}(\kappa_p)}{\mathscr{J}_{\Delta i}(\kappa_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\dfrac{\mathscr{J}_{\Delta i}(\mathrm{id}\cdot\kappa_p)}{\mathscr{J}_{\Delta i}(\kappa_p)} - \dfrac{2}{p}\right), & p = q \neq 0; \\ \exp\left(\dfrac{\mathscr{J}_{\Delta i}(\mathrm{id}\cdot\kappa_0)}{3\,\mathscr{J}_{\Delta i}(\kappa_0)}\right), & p = q = 0, \end{cases}$$

and using (7.40), it is monotone in p and q. Using Theorem 7.15, it follows that for $i \in \{6,\ldots,10\}$,

$$\aleph_{p,q}(\mathscr{J}_{\Delta i},\Omega_1) = \log \mathscr{M}_{p,q}(\mathscr{J}_{\Delta i},\Omega_1)$$

satisfy $\aleph_{p,q}(\mathscr{J}_{\Delta i},\Omega_1) \in [m,M]$, which shows that $\aleph_{p,q}(\mathscr{J}_{\Delta i},\Omega_1)$ are means (of the function f). Note that by (7.40) they are monotone means.

Example 7.2 Consider the family of functions

$$\Omega_2 = \{\beta_\rho\colon\ (0,\infty) \to \mathbb{R};\ \rho \in \mathbb{R}\}$$

defined in Example 5.3. Arguing as in Example 7.1, we have $\rho \mapsto \mathscr{J}_{\Delta i}(\beta_\rho)$, $i \in \{6,\ldots,10\}$, are exponentially convex. In this case $\mathscr{M}_{p,q}(\mathscr{J}_{\Delta i},\Omega)$, $i \in \{6,\ldots,10\}$, from (7.41) becomes

$$\mathscr{M}_{p,q}(\mathscr{J}_{\Delta i},\Omega_2) = \begin{cases} \left(\dfrac{\mathscr{J}_{\Delta i}(\beta_p)}{\mathscr{J}_{\Delta i}(\beta_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\dfrac{1-2p}{p(p-1)} - \dfrac{\mathscr{J}_{\Delta i}(\beta_p\beta_0)}{\mathscr{J}_{\Delta i}(\beta_p)}\right), & p = q \neq 0,1; \\ \exp\left(1 - \dfrac{\mathscr{J}_{\Delta i}(\beta_0^2)}{2\,\mathscr{J}_{\Delta i}(\beta_0)}\right), & p = q = 0; \\ \exp\left(-1 - \dfrac{\mathscr{J}_{\Delta i}(\beta_0\beta_1)}{2\,\mathscr{J}_{\Delta i}(\beta_1)}\right), & p = q = 1. \end{cases}$$

As $\mathscr{J}_{\Delta i}$ is positive, by applying Theorem 7.15 for $\varphi = \beta_p \in \Omega_2$ and $\psi = \beta_q \in \Omega_2$, there exist $\xi_i \in [m,M]$ such that

$$\xi_i^{p-q} = \frac{\mathscr{J}_{\Delta i}(\beta_p)}{\mathscr{J}_{\Delta i}(\beta_q)}, \quad i \in \{6,\ldots,10\}.$$

Since the function $\xi_i \mapsto (\xi_i)^{p-q}$ is invertible for $p \neq q$, we have

$$m \leq \left(\frac{\mathscr{J}_{\Delta i}(\beta_p)}{\mathscr{J}_{\Delta i}(\beta_q)}\right)^{\frac{1}{p-q}} \leq M, \quad i \in \{6,\ldots,10\}.$$

Also $\mathscr{M}_{p,q}(\mathscr{J}_{\Delta i},\Omega_2)$ is continuous, symmetric, and monotone (by (7.40)) shows that $\mathscr{M}_{p,q}(\mathscr{J}_{\Delta i},\Omega_2)$ is a mean (of the function f).

Example 7.3 Consider the family of functions

$$\Omega_3 = \{\gamma_\rho : \ (0,\infty) \to (0,\infty) : \ \rho \in (0,\infty)\}$$

defined in Example 5.4. For this family of functions, $\mathscr{M}_{p,q}(\mathscr{J}_{\Delta i},\Omega),\ i \in \{6,\dots,10\}$, from (7.41) become

$$\mathscr{M}_{p,q}(\mathscr{J}_{\Delta i},\Omega_3) = \begin{cases} \left(\dfrac{\mathscr{J}_{\Delta i}(\gamma_p)}{\mathscr{J}_{\Delta i}(\gamma_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\[2ex] \exp\left(-\dfrac{\mathscr{J}_{\Delta i}(\mathrm{id}\cdot\gamma_p)}{p\,\mathscr{J}_{\Delta i}(\gamma_p)} - \dfrac{2}{p\ln p}\right), & p = q \neq 1; \\[2ex] \exp\left(\dfrac{-2\,\mathscr{J}_{\Delta i}(\mathrm{id}\cdot\gamma_1)}{3\,\mathscr{J}_{\Delta i}(\gamma_1)}\right), & p = q = 1, \end{cases}$$

and by (7.40), it is monotone in p and q. Using Theorem 7.15, it follows that for $i \in \{6,\dots,10\}$,

$$\aleph_{p,q}(\mathscr{J}_{\Delta i},\Omega_3) = -L(p,q)\log \mathscr{M}_{p,q}(\mathscr{J}_{\Delta i},\Omega_3)$$

satisfies $\aleph_{p,q}(\mathscr{J}_{\Delta i},\Omega_3) \in [m,M]$, which shows that $\aleph_{p,q}(\mathscr{J}_{\Delta i},\Omega_3)$ is a mean (of the function f). Here $L(p,q)$ is the logarithmic mean defined by

$$L(p,q) = \frac{p-q}{\log p - \log q}, \quad p \neq q, \quad L(p,p) = p.$$

Example 7.4 Consider the family of functions

$$\Omega_4 = \{\delta_\rho : \ (0,\infty) \to (0,\infty) : \ \rho \in (0,\infty)\}$$

For this family of functions, $\mathscr{M}_{p,q}(\mathscr{J}_{\Delta i},\Omega),\ i \in \{6,\dots,10\}$, from (7.41) become

$$\mathscr{M}_{p,q}(\mathscr{J}_{\Delta i},\Omega_4) = \begin{cases} \left(\dfrac{\mathscr{J}_{\Delta i}(\delta_p)}{\mathscr{J}_{\Delta i}(\delta_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\[2ex] \exp\left(-\dfrac{\mathscr{J}_{\Delta i}(\mathrm{id}\cdot\delta_p)}{2\sqrt{p}\,\mathscr{J}_{\Delta i}(\delta_p)} - \dfrac{1}{p}\right), & p = q, \end{cases}$$

and it is monotone in p and q by (7.40). Using Theorem 7.15, it follows that for $i \in \{6,\dots,10\}$,

$$\aleph_{p,q}(\mathscr{J}_{\Delta i},\Omega_4) = -(\sqrt{s}+\sqrt{q})\log \mathscr{M}_{p,q}(\mathscr{J}_{\Delta i},\Omega_4)$$

satisfies $\aleph_{p,q}(\mathscr{J}_{\Delta i},\Omega_4) \in [m,M]$, which shows that $\aleph_{p,q}(\mathscr{J}_{\Delta i},\Omega_4)$ is a mean (of the function f).

Chapter 8

Cauchy Type Means and Exponential and Logarithmic Convexity for Superquadratic Functions

In this chapter, we define positive functionals by using Jensen's inequality, the converse of Jensen's inequality, and Jensen–Mercer's inequality on time scales for superquadratic functions. We give mean-value theorems and introduce related Cauchy type means by using the functionals mentioned above and show the monotonicity of these means. We also show that these functionals are exponentially convex and give some applications of them by using log-convexity and exponential convexity. The presentation of the results in this chapter closely follows [35].

8.1 Mean Value Theorems

Under the assumptions of Theorems 2.63, 2.70, and 2.72, we define functionals $\mathscr{J}_\Psi$, $\widetilde{\mathscr{J}_\Psi}$, and $\widehat{\mathscr{J}_\Psi}$ by

$$\mathscr{J}_\Psi = \int_a^b \left[\Psi(f(u)) - \Psi\left(\left|f(u) - \frac{\int_a^b f(t)\Delta t}{b-a}\right|\right)\right]\Delta u - (b-a)\Psi\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right), \quad (8.1)$$

$$\widetilde{\mathscr{J}_\Psi} = (b-a)(\Psi(m)+\Psi(M)) - \int_a^b \Psi(f(t))\Delta t - K \tag{8.2}$$
$$- (b-a)\Psi\left(m+M-\frac{1}{b-a}\int_a^b f(t)\Delta t\right),$$

$$\widehat{\mathscr{J}_\Psi} = \frac{M(b-a)-\int_a^b f(t)\Delta t}{M-m}\Psi(m) + \frac{\int_a^b f(t)\Delta t - m(b-a)}{M-m}\Psi(M) \tag{8.3}$$
$$-\int_a^b \Psi(f(t))\,\Delta t - R.$$

From the inequalities (2.65), (2.73), and (2.75), it is clear that, subject to the relevant assumptions, $\mathscr{J}_\Psi$, $\widetilde{\mathscr{J}_\Psi}$, and $\widehat{\mathscr{J}_\Psi}$ are nonnegative.

In the sequel, we consider $\overline{\Psi} \in \mathrm{C}^1((0,\infty),\mathbb{R})$ defined as in (1.8).

Theorem 8.1 *Let $a,b \in \mathbb{T}$. Suppose $f \in \mathrm{C_{rd}}([a,b)_\mathbb{T},[0,\infty))$ and $\Psi \in \mathrm{C}^1([0,\infty),\mathbb{R})$ is such that $\Psi(0)=0$ and $\overline{\Psi} \in \mathrm{C}^1((0,\infty),\mathbb{R})$. Then*

$$\mathscr{J}_\Psi = \frac{\rho\Psi''(\rho)-\Psi'(\rho)}{\rho^2}\mathscr{J}_{\Psi_3} \tag{8.4}$$

holds for some $\rho > 0$, provided that $\mathscr{J}_{\Psi_3} \neq 0$, where Ψ_3 is defined in (1.10)*.*

Proof. Define

$$\psi_* := \inf_{x\in(0,\infty)} \overline{\Psi}'(x) \quad \text{and} \quad \psi^* := \sup_{x\in(0,\infty)} \overline{\Psi}'(x).$$

Case 1: Suppose

$$\psi_* = \min_{x\in(0,\infty)} \overline{\Psi}'(x) \quad \text{and} \quad \psi^* = \max_{x\in(0,\infty)} \overline{\Psi}'(x).$$

Then

$$\psi_* \le \frac{x\Psi''(x)-\Psi'(x)}{x^2} \le \psi^* \quad \text{for all} \quad x>0. \tag{8.5}$$

Hence by Lemma 1.3, ϑ_1 and ϑ_2 defined in (1.9) are superquadratic. By Theorem 2.63, we have $\mathscr{J}_{\vartheta_1}, \mathscr{J}_{\vartheta_2} \ge 0$. Thus, since $\mathscr{J}_{\vartheta_1} = \psi^*\mathscr{J}_{\Psi_3} - \mathscr{J}_\Psi$ and $\mathscr{J}_{\vartheta_2} = \mathscr{J}_\Psi - \psi_*\mathscr{J}_{\Psi_3}$, we obtain

$$\psi_*\mathscr{J}_{\Psi_3} \le \mathscr{J}_\Psi \le \psi^*\mathscr{J}_{\Psi_3}. \tag{8.6}$$

Now, (8.5) and (8.6) imply that there exists $\rho > 0$ such that (8.4) holds.

Case 2: Suppose

$$\psi_* = \min_{x\in(0,\infty)} \overline{\Psi}'(x) \quad \text{and} \quad \psi^* \neq \max_{x\in(0,\infty)} \overline{\Psi}'(x).$$

In this case, ϑ_1 is strictly superquadratic. Therefore $\mathscr{J}_{\vartheta_1} > 0$ and $\mathscr{J}_{\vartheta_2} \ge 0$. Hence

$$\psi_* \le \frac{x\Psi''(x)-\Psi'(x)}{x^2} < \psi^* \tag{8.7}$$

and thus

$$\psi_* \mathscr{J}_{\Psi_3} \leq \mathscr{J}_\Psi < \psi^* \mathscr{J}_{\Psi_3}. \tag{8.8}$$

Now, (8.7) and (8.8) imply that (8.4) holds for some $\rho > 0$.

Case 3: Suppose

$$\psi_* \neq \min_{x\in(0,\infty)} \overline{\Psi}'(x) \quad \text{and} \quad \psi^* = \max_{x\in(0,\infty)} \overline{\Psi}'(x).$$

In this case, ϑ_2 is strictly superquadratic. The rest of the proof is analogous to the proof in Case 2.

Case 4: Suppose

$$\psi_* \neq \min_{x\in(0,\infty)} \overline{\Psi}'(x) \quad \text{and} \quad \psi^* \neq \max_{x\in(0,\infty)} \overline{\Psi}'(x).$$

In this case, ϑ_1 and ϑ_2 both are strictly superquadratic. The rest of the proof is analogous to the proof in Case 2.

In the case where $\psi^* = \infty$ (i.e., $\overline{\Psi}'$ is not bounded above) and ψ_* exists, using just ϑ_2, we obtain

$$\psi_* \leq \frac{x\Psi''(x) - \Psi'(x)}{x^2}$$

in the case of minimum, and strong inequality in the case where ψ_* is infimum. The rest of the proof is as above. The remaining cases can be treated analogously. □

Theorem 8.2 *Let* $a,b \in \mathbb{T}$ *and* $f \in \mathrm{C_{rd}}([a,b)_\mathbb{T},[0,\infty))$ *such that* $\mathscr{J}_{\Psi_3} \neq 0$. *Suppose* $\Psi,\Phi \in \mathrm{C}^1([0,\infty),\mathbb{R})$ *are such that* $\Psi(0) = \Phi(0) = 0$ *and* $\overline{\Psi},\overline{\Phi} \in \mathrm{C}^1((0,\infty),\mathbb{R})$. *Then there exists* $\rho > 0$ *such that*

$$\frac{\mathscr{J}_\Psi}{\mathscr{J}_\Phi} = \frac{\rho\Psi''(\rho) - \Psi'(\rho)}{\rho\Phi''(\rho) - \Phi'(\rho)} \tag{8.9}$$

holds, provided that the denominators in (8.9) *are nonzero.*

Proof. Define $\chi \in \mathrm{C}^1([0,\infty),\mathbb{R})$ by

$$\chi(x) = \mathscr{J}_\Phi \Psi(x) - \mathscr{J}_\Psi \Phi(x) \quad \text{for} \quad x \geq 0.$$

Then $\overline{\chi} \in \mathrm{C}^1((0,\infty),\mathbb{R})$, $\chi(0) = 0$, and $\mathscr{J}_\chi = 0$. Therefore, by using χ instead of Ψ in Theorem 8.1, we obtain that there exists $\rho > 0$ such that

$$0 = \rho\chi''(\rho) - \chi'(\rho) = \mathscr{J}_\Phi(\rho\Psi''(\rho) - \Psi'(\rho)) - \mathscr{J}_\Psi(\rho\Phi''(\rho) - \Phi'(\rho)),$$

from which (8.9) follows. □

Remark 8.1 In Theorem 8.2, let

$$\mathscr{G}(\rho) = \frac{\rho\Psi''(\rho) - \Psi'(\rho)}{\rho\Phi''(\rho) - \Phi'(\rho)}$$

and suppose $\mathscr{G}$ is invertible. Then we obtain another mean defined by

$$\rho = \mathscr{G}^{-1}\left(\frac{\mathscr{J}_\Psi}{\mathscr{J}_\Phi}\right).$$

Theorem 8.3 *Let $a,b \in \mathbb{T}$. Suppose $f \in C_{rd}([a,b)_{\mathbb{T}},[m,M])$, where $0 \le m < M < \infty$, and $\Psi \in C^1([0,\infty),\mathbb{R})$ is such that $\Psi(0)=0$ and $\overline{\Psi} \in C^1((0,\infty),\mathbb{R})$. Then*

$$\widetilde{\mathscr{J}_{\Psi}} = \frac{\rho\Psi''(\rho)-\Psi'(\rho)}{\rho^2}\widetilde{\mathscr{J}_{\Psi_3}} \tag{8.10}$$

holds for some $\rho > 0$, provided that $\widetilde{\mathscr{J}_{\Psi_3}} \neq 0$.

Proof. The proof is analogous to the proof of Theorem 8.1, where, instead of using Theorem 2.63, we apply Theorem 2.70 to ϑ_1 and ϑ_2. □

Theorem 8.4 *Let $a,b \in \mathbb{T}$ and $f \in C_{rd}([a,b)_{\mathbb{T}},[m,M])$, where $0 \le m < M < \infty$, such that $\widetilde{\mathscr{J}_{\Psi_3}} \neq 0$. Suppose $\Psi,\Phi \in C^1([0,\infty),\mathbb{R})$ are such that $\Psi(0)=\Phi(0)=0$ and $\overline{\Psi},\overline{\Phi} \in C^1((0,\infty),\mathbb{R})$. Then there exists $\rho > 0$ such that*

$$\frac{\widetilde{\mathscr{J}_{\Psi}}}{\widetilde{\mathscr{J}_{\Phi}}} = \frac{\rho\Psi''(\rho)-\Psi'(\rho)}{\rho\Phi''(\rho)-\Phi'(\rho)} \tag{8.11}$$

holds, provided that the denominators in (8.11) *are nonzero.*

Proof. The proof is analogous to the proof of Theorem 8.2, where, instead of using Theorem 8.1, we apply Theorem 8.3 to χ. □

Remark 8.2 In Theorem 8.4, let

$$\widetilde{\mathscr{G}}(\rho) = \frac{\rho\Psi''(\rho)-\Psi'(\rho)}{\rho\Phi''(\rho)-\Phi'(\rho)}$$

and suppose $\widetilde{\mathscr{G}}$ is invertible. Then we obtain another mean defined by

$$\rho = \widetilde{\mathscr{G}}^{-1}\left(\frac{\widetilde{\mathscr{J}_{\Psi}}}{\widetilde{\mathscr{J}_{\Phi}}}\right).$$

Theorem 8.5 *Let $a,b \in \mathbb{T}$. Suppose $f \in C_{rd}([a,b)_{\mathbb{T}},[m,M])$, where $0 \le m < M < \infty$, and $\Psi \in C^1([0,\infty),\mathbb{R})$ is such that $\Psi(0)=0$ and $\overline{\Psi} \in C^1((0,\infty),\mathbb{R})$. Then*

$$\widehat{\mathscr{J}_{\Psi}} = \frac{\rho\Psi''(\rho)-\Psi'(\rho)}{\rho^2}\widehat{\mathscr{J}_{\Psi_3}} \tag{8.12}$$

holds for some $\rho > 0$, provided that $\widehat{\mathscr{J}_{\Psi_3}} \neq 0$.

Proof. The proof is analogous to the proof of Theorem 8.1, where, instead of using Theorem 2.63, we apply Theorem 2.72 to ϑ_1 and ϑ_2. □

Theorem 8.6 *Let* $a,b \in \mathbb{T}$ *and* $f \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}},[m,M])$, *where* $0 \leq m < M < \infty$, *such that* $\widehat{\mathscr{J}_{\Psi_3}} \neq 0$. *Suppose* $\Psi, \Phi \in \mathrm{C}^1([0,\infty),\mathbb{R})$ *are such that* $\Psi(0) = \Phi(0) = 0$ *and* $\overline{\Psi}, \overline{\Phi} \in \mathrm{C}^1((0,\infty),\mathbb{R})$. *Then there exists* $\rho > 0$ *such that*

$$\frac{\widehat{\mathscr{J}_\Psi}}{\widehat{\mathscr{J}_\Phi}} = \frac{\rho\Psi''(\rho) - \Psi'(\rho)}{\rho\Phi''(\rho) - \Phi'(\rho)} \tag{8.13}$$

holds, provided that the denominators in (8.13) *are nonzero.*

Proof. The proof is analogous to the proof of Theorem 8.2, where, instead of using Theorem 8.1, we apply Theorem 8.5 to χ. □

Remark 8.3 In Theorem 8.6, let

$$\widehat{\mathscr{G}}(\rho) = \frac{\rho\Psi''(\rho) - \Psi'(\rho)}{\rho\Phi''(\rho) - \Phi'(\rho)}$$

and suppose $\widehat{\mathscr{G}}$ is invertible. Then we obtain another mean defined by

$$\rho = \widehat{\mathscr{G}}^{-1}\left(\frac{\widehat{\mathscr{J}_\Psi}}{\widehat{\mathscr{J}_\Phi}}\right).$$

8.2 Generalized Means

First we recall the definition of generalized means for Cauchy Δ-integrals (See Definition 3.2).

Definition 8.1 *Let* $a,b \in \mathbb{T}$. *Let* $\alpha \in \mathrm{C}(I,\mathbb{R})$ *be strictly monotone, where* $I \subset \mathbb{R}$ *is an interval. If* $f \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}},I)$, *then the* generalized mean *of* f *is defined by*

$$\mathfrak{M}_\alpha(f) = \alpha^{-1}\left(\frac{\int_a^b (\alpha \circ f)(t)\Delta t}{b-a}\right), \tag{8.14}$$

provided that (8.14) *is well defined.*

Theorem 8.7 *Let* $a,b \in \mathbb{T}$ *and* $f \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}},[0,\infty))$. *Suppose that* $\alpha,\beta,\gamma \in \mathrm{C}^2([0,\infty),\mathbb{R})$ *are strictly monotone such that*

$$\overline{\alpha \circ \gamma^{-1}}, \overline{\beta \circ \gamma^{-1}} \in \mathrm{C}^1((0,\infty),\mathbb{R}) \quad \text{and} \quad (\alpha \circ \gamma^{-1})(0) = (\beta \circ \gamma^{-1})(0) = 0.$$

If

$$\int_a^b \left((\gamma\circ f)^3(u) - \left| (\gamma\circ f)(u) - \frac{\int_a^b (\gamma\circ f)(t)\Delta t}{b-a} \right|^3 \right) \Delta u - \frac{\left(\int_a^b (\gamma\circ f)(t)\Delta t\right)^3}{(b-a)^2} \neq 0,$$

then

$$\frac{\alpha(\mathfrak{M}_\alpha(f)) - \alpha(\mathfrak{M}_\alpha(\gamma^{-1}(|(\gamma\circ f) - \gamma(\mathfrak{M}_\gamma(f))|))) - \alpha(\mathfrak{M}_\gamma(f))}{\beta(\mathfrak{M}_\beta(f)) - \beta(\mathfrak{M}_\beta(\gamma^{-1}(|(\gamma\circ f) - \gamma(\mathfrak{M}_\gamma(f))|))) - \beta(\mathfrak{M}_\gamma(f))} = \frac{\gamma(\zeta)(\alpha''(\zeta)\gamma'(\zeta) - \alpha'(\zeta)\gamma''(\zeta)) - \alpha'(\zeta)(\gamma'(\zeta))^2}{\gamma(\zeta)(\beta''(\zeta)\gamma'(\zeta) - \beta'(\zeta)\gamma''(\zeta)) - \beta'(\zeta)(\gamma'(\zeta))^2} \quad (8.15)$$

holds for some $\zeta \in f([a,b)_{\mathbb{T}})$*, provided that the denominators in* (8.15) *are nonzero.*

Proof. Replace the functions f, Ψ, and Φ in Theorem 8.2 by $\gamma\circ f$, $\alpha\circ\gamma^{-1}$, and $\beta\circ\gamma^{-1}$, respectively. So there exists $\rho > 0$ such that

$$\frac{\alpha(\mathfrak{M}_\alpha(f)) - \alpha(\mathfrak{M}_\alpha(\gamma^{-1}(|(\gamma\circ f) - \gamma(\mathfrak{M}_\gamma(f))|))) - \alpha(\mathfrak{M}_\gamma(f))}{\beta(\mathfrak{M}_\beta(f)) - \beta(\mathfrak{M}_\beta(\gamma^{-1}(|(\gamma\circ f) - \gamma(\mathfrak{M}_\gamma(f))|))) - \beta(\mathfrak{M}_\gamma(f))}$$
$$= \frac{\rho(\alpha''(\gamma^{-1}(\rho))\gamma'(\gamma^{-1}(\rho)) - \alpha'(\gamma^{-1}(\rho))\gamma''(\gamma^{-1}(\rho))) - \alpha'(\gamma^{-1}(\rho))(\gamma'(\gamma^{-1}(\rho)))^2}{\rho(\beta''(\gamma^{-1}(\rho))\gamma'(\gamma^{-1}(\rho)) - \beta'(\gamma^{-1}(\rho))\gamma''(\gamma^{-1}(\rho))) - \beta'(\gamma^{-1}(\rho))(\gamma'(\gamma^{-1}(\rho)))^2}.$$

By putting $\gamma^{-1}(\rho) = \zeta$, there exists $\zeta \in f([a,b)_{\mathbb{T}})$ such that (8.15) holds. □

Remark 8.4 In Theorem 8.7, let

$$\mathscr{F}(\zeta) = \frac{\gamma(\zeta)(\alpha''(\zeta)\gamma'(\zeta) - \alpha'(\zeta)\gamma''(\zeta)) - \alpha'(\zeta)(\gamma'(\zeta))^2}{\gamma(\zeta)(\beta''(\zeta)\gamma'(\zeta) - \beta'(\zeta)\gamma''(\zeta)) - \beta'(\zeta)(\gamma'(\zeta))^2}$$

and suppose $\mathscr{F}$ is invertible. Then, since ζ is in the image of f, we obtain a new mean defined by

$$\mathscr{F}^{-1}\left(\frac{\alpha(\mathfrak{M}_\alpha(f)) - \alpha(\mathfrak{M}_\alpha(\gamma^{-1}(|(\gamma\circ f) - \gamma(\mathfrak{M}_\gamma(f))|))) - \alpha(\mathfrak{M}_\gamma(f))}{\beta(\mathfrak{M}_\beta(f)) - \beta(\mathfrak{M}_\beta(\gamma^{-1}(|(\gamma\circ f) - \gamma(\mathfrak{M}_\gamma(f))|))) - \beta(\mathfrak{M}_\gamma(f))} \right).$$

Now we recall the definition of generalized power means for Cauchy Δ-integrals (See Definition 3.3).

Definition 8.2 *Let* $a,b \in \mathbb{T}$ *and* $f \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}}, I)$*, where* $I \subset \mathbb{R}$ *is an interval. If* $r \in \mathbb{R}$*, then the generalized power mean of* f *is defined by*

$$\mathfrak{M}_r(f) = \begin{cases} \left(\dfrac{\int_a^b f^r(t)\Delta t}{b-a}\right)^{\frac{1}{r}}, & r \neq 0, \\ \exp\left(\dfrac{\int_a^b \log f(t)\Delta t}{b-a}\right), & r = 0, \end{cases} \quad (8.16)$$

provided that (8.16) *is well defined.*

Corollary 8.1 *Let $a,b \in \mathbb{T}$ and $f \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}}, I)$ be positive. Suppose $r,l,s > 0$ are such that $r \neq l$, $r \neq 2s$, $l \neq 2s$, and*

$$\int_a^b \left(f^{3s}(u) - \left| f^s(u) - \frac{\int_a^b f^s(t)\Delta t}{b-a} \right|^3 \right) \Delta u - \frac{\left(\int_a^b f^s(t)\Delta t\right)^3}{(b-a)^2} \neq 0.$$

Then

$$\frac{\mathfrak{M}_r^r(f) - \mathfrak{M}_r^r(|f^s - \mathfrak{M}_s^s(f)|^{\frac{1}{s}}) - \mathfrak{M}_s^r(f)}{\mathfrak{M}_l^l(f) - \mathfrak{M}_l^l(|f^s - \mathfrak{M}_s^s(f)|^{\frac{1}{s}}) - \mathfrak{M}_s^l(f)} = \frac{r(r-2s)}{l(l-2s)} \zeta^{r-l} \tag{8.17}$$

holds for some $\zeta \in f([a,b)_{\mathbb{T}})$, provided that the denominators in (8.17) are nonzero.

Proof. Equation (8.17) directly follows from Theorem 8.7 by taking $\alpha(x) = x^r$, $\beta(x) = x^l$, and $\gamma(x) = x^s$ in Theorem 8.7. □

Remark 8.5 From Corollary 8.1, since $\zeta \in f([a,b)_{\mathbb{T}})$, we obtain a new mean defined by

$$\mathfrak{M}_{r,l}^{[s]}(f) = \left(\frac{l(l-2s)}{r(r-2s)} \frac{\mathfrak{M}_r^r(f) - \mathfrak{M}_r^r(|f^s - \mathfrak{M}_s^s(f)|^{\frac{1}{s}}) - \mathfrak{M}_s^r(f)}{\mathfrak{M}_l^l(f) - \mathfrak{M}_l^l(|f^s - \mathfrak{M}_s^s(f)|^{\frac{1}{s}}) - \mathfrak{M}_s^l(f)} \right)^{\frac{1}{r-l}},$$

where $r,l,s > 0$, $r \neq 2s$, $l \neq 2s$. We can extend these means to the limiting cases. To do so, let $r,l,s > 0$. We define

$$\mathfrak{M}_{l,l}^{[s]}(f) = \exp\left(\frac{P}{Q} - \frac{2(l-s)}{l(l-2s)} \right), \quad l \neq 2s,$$

$$\mathfrak{M}_{l,2s}^{[s]}(f) = \mathfrak{M}_{2s,l}^{[s]}(f) = \exp\left(\frac{2sQ}{l(l-2s)P_1} \right)^{\frac{1}{l-2s}}, \quad l \neq 2s,$$

$$\mathfrak{M}_{2s,2s}^{[s]}(f) = \exp\left(\frac{Q_1}{2P_1} - \frac{1}{2s} \right),$$

where P, Q, P_1, and Q_1 are given by

$$\begin{aligned}
P =& \frac{1}{b-a} \int_a^b f^l(t) \log f(t) \Delta t - \mathfrak{M}_s^l(f) \log \mathfrak{M}_s(f) \\
& - \frac{1}{s(b-a)} \int_a^b |f^s(t) - \mathfrak{M}_s^s(f)|^{\frac{l}{s}} \log |f^s(t) - \mathfrak{M}_s^s(f)| \Delta t, \\
Q =& \mathfrak{M}_l^l(f) - \mathfrak{M}_l^l(|f^s - \mathfrak{M}_s^s(f)|^{\frac{1}{s}}) - \mathfrak{M}_s^l(f), \\
P_1 =& \frac{1}{b-a} \int_a^b f^{2s}(t) \log f(t) \Delta t - \mathfrak{M}_s^{2s}(f) \log \mathfrak{M}_s(f) \\
& - \frac{1}{s(b-a)} \int_a^b |f^s(t) - \mathfrak{M}_s^s(f)|^2 \log |f^s(t) - \mathfrak{M}_s^s(f)| \Delta t,
\end{aligned}$$

$$
\begin{aligned}
Q_1 =&\frac{1}{b-a}\int_a^b f^{2s}(t)(\log f(t))^2\Delta t-\mathfrak{M}_s^{2s}(f)(\log\mathfrak{M}_s(f))^2\\
&-\frac{1}{s^2(b-a)}\int_a^b |f^s(t)-\mathfrak{M}_s^s(f)|^2\left(\log|f^s(t)-\mathfrak{M}_s^s(f)|\right)^2\Delta t.
\end{aligned}
$$

Theorem 8.8 *Let $a,b\in\mathbb{T}$ and $f\in \mathrm{C}_{\mathrm{rd}}([a,b)_{\mathbb{T}},[m,M])$, where $0\le m<M<\infty$. Suppose $\alpha,\beta,\gamma\in\mathrm{C}^2([0,\infty),\mathbb{R})$ are strictly monotone such that*

$$\overline{\alpha\circ\gamma^{-1}},\overline{\beta\circ\gamma^{-1}}\in\mathrm{C}^1((0,\infty),\mathbb{R})\quad\text{and}\quad(\alpha\circ\gamma^{-1})(0)=(\beta\circ\gamma^{-1})(0)=0.$$

If

$$
\begin{aligned}
&(b-a)((\gamma(m))^3+(\gamma(M))^3)-\int_a^b(\gamma\circ f)^3(t)\Delta t\\
&-(b-a)\left(\gamma(m)+\gamma(M)-\frac{1}{b-a}\int_a^b(\gamma\circ f)(t)\Delta t\right)^3\\
&-\frac{2}{\gamma(M)-\gamma(m)}\int_a^b\left[((\gamma\circ f)(t)-\gamma(m))(\gamma(M)-(\gamma\circ f)(t))^3\right.\\
&\left.+(\gamma(M)-(\gamma\circ f)(t))((\gamma\circ f)(t)-\gamma(m))^3\right]\Delta t\\
&-\int_a^b\left|(\gamma\circ f)(u)-\frac{1}{b-a}\int_a^b(\gamma\circ f)(t)\Delta t\right|^3\Delta u\neq 0,
\end{aligned}
$$

then

$$
\begin{aligned}
&\frac{W_\alpha-X_\alpha-\frac{Y}{b-a}\int_a^b[\mathfrak{g}(t)(\alpha\circ\gamma^{-1})(\mathfrak{h}(t))+\mathfrak{h}(t)(\alpha\circ\gamma^{-1})(\mathfrak{g}(t))]\Delta t-Z_\alpha}{W_\beta-X_\beta-\frac{Y}{b-a}\int_a^b[\mathfrak{g}(t)(\beta\circ\gamma^{-1})(\mathfrak{h}(t))+\mathfrak{h}(t)(\beta\circ\gamma^{-1})(\mathfrak{g}(t))]\Delta t-Z_\beta}\\
&\qquad=\frac{\gamma(\zeta)(\alpha''(\zeta)\gamma'(\zeta)-\alpha'(\zeta)\gamma''(\zeta))-\alpha'(\zeta)(\gamma'(\zeta))^2}{\gamma(\zeta)(\beta''(\zeta)\gamma'(\zeta)-\beta'(\zeta)\gamma''(\zeta))-\beta'(\zeta)(\gamma'(\zeta))^2}
\end{aligned}
\tag{8.18}
$$

holds for some $\zeta\in f([a,b)_{\mathbb{T}})$, provided that the denominators in (8.18) *are nonzero, where*

$$
\begin{aligned}
W_\alpha&=\alpha(m)+\alpha(M)-\alpha(\mathfrak{M}_\alpha(f)),\\
X_\alpha&=(\alpha\circ\gamma^{-1})(\gamma(m)+\gamma(M)-\gamma(\mathfrak{M}_\gamma(f))),\\
Z_\alpha&=\alpha(\mathfrak{M}_\alpha(\gamma^{-1}(|(\gamma\circ f)-\gamma(\mathfrak{M}_\gamma(f))|))),\quad Y=\frac{2}{\gamma(M)-\gamma(m)},\\
\mathfrak{g}&=(\gamma\circ f)-\gamma(m),\quad \mathfrak{h}=\gamma(M)-(\gamma\circ f).
\end{aligned}
$$

Proof. Replace the functions f, Ψ, and Φ in Theorem 8.4 by $\gamma\circ f$, $\alpha\circ\gamma^{-1}$, and $\beta\circ\gamma^{-1}$, respectively. The rest of the proof is analogous to the proof of Theorem 8.7. □

Remark 8.6 In Theorem 8.8, let

$$\widetilde{\mathscr{F}}(\zeta)=\frac{\gamma(\zeta)(\alpha''(\zeta)\gamma'(\zeta)-\alpha'(\zeta)\gamma''(\zeta))-\alpha'(\zeta)(\gamma'(\zeta))^2}{\gamma(\zeta)(\beta''(\zeta)\gamma'(\zeta)-\beta'(\zeta)\gamma''(\zeta))-\beta'(\zeta)(\gamma'(\zeta))^2}$$

and suppose $\widetilde{\mathscr{F}}$ is invertible. Then, since ζ is in the image of f, we obtain a new mean defined by

$$\widetilde{\mathscr{F}}^{-1}\left(\frac{W_\alpha - X_\alpha - \frac{Y}{b-a}\int_a^b[\mathfrak{g}(t)(\alpha\circ\gamma^{-1})(\mathfrak{h}(t))+\mathfrak{h}(t)(\alpha\circ\gamma^{-1})(\mathfrak{g}(t))]\Delta t - Z_\alpha}{W_\beta - X_\beta - \frac{Y}{b-a}\int_a^b[\mathfrak{g}(t)(\beta\circ\gamma^{-1})(\mathfrak{h}(t))+\mathfrak{h}(t)(\beta\circ\gamma^{-1})(\mathfrak{g}(t))]\Delta t - Z_\beta}\right).$$

Corollary 8.2 *Let $a,b\in\mathbb{T}$ and $f\in \mathrm{C_{rd}}([a,b)_\mathbb{T},[m,M])$, where $0\le m<M<\infty$. Suppose $r,l,s>0$ are such that $r\ne l$, $r\ne 2s$, $l\ne 2s$, and*

$$\begin{aligned}&(b-a)(m^{3s}+M^{3s})-\int_a^b f^{3s}(t)\Delta t-(b-a)\left(m^s+M^s-\frac{1}{b-a}\int_a^b f^s(t)\Delta t\right)^3\\&-\frac{2}{M^s-m^s}\int_a^b\left[(f^s(t)-m^s)(M^s-f^s(t))^3\right.\\&\left.+(M^s-f^s(t))(f^s(t)-m^s)^3\right]\Delta t-\int_a^b\left|f^s(u)-\frac{1}{b-a}\int_a^b f^s(t)\Delta t\right|^3\Delta u\ne 0.\end{aligned}$$

Then

$$\frac{W_r-X_r-Y_s(\mathfrak{M}_r^r(\mathfrak{g}_s^{\frac1r}\mathfrak{h}_s^{\frac1s})+\mathfrak{M}_r^r(\mathfrak{h}_s^{\frac1r}\mathfrak{g}_s^{\frac1s}))-Z_r}{W_l-X_l-Y_s(\mathfrak{M}_l^l(\mathfrak{g}_s^{\frac1l}\mathfrak{h}_s^{\frac1s})+\mathfrak{M}_l^l(\mathfrak{h}_s^{\frac1l}\mathfrak{g}_s^{\frac1s}))-Z_l}=\frac{r(r-2s)}{l(l-2s)}\zeta^{r-l} \tag{8.19}$$

holds for some $\zeta\in f([a,b)_\mathbb{T})$, provided that the denominators in (8.19) *are nonzero, where*

$$\begin{gathered}W_r=m^r+M^r-\mathfrak{M}_r^r(f),\quad X_r=(m^s+M^s-\mathfrak{M}_s^s(f))^{\frac rs},\\ Z_r=\mathfrak{M}_r^r(|f^s-\mathfrak{M}_s^s(f)|^{\frac1s}),\quad Y_s=\frac{2}{M^s-m^s},\\ \mathfrak{g}_s=f^s-m^s,\quad \mathfrak{h}_s=M^s-f^s.\end{gathered}$$

Remark 8.7 From Corollary 8.2, since $\zeta\in f([a,b)_\mathbb{T})$, we obtain a new mean defined by

$$\widetilde{\mathfrak{M}}_{r,l}^{[s]}(f)=\left(\frac{l(l-2s)}{r(r-2s)}\frac{W_r-X_r-Y_s(\mathfrak{M}_r^r(\mathfrak{g}_s^{\frac1r}\mathfrak{h}_s^{\frac1s})+\mathfrak{M}_r^r(\mathfrak{h}_s^{\frac1r}\mathfrak{g}_s^{\frac1s}))-Z_r}{W_l-X_l-Y_s(\mathfrak{M}_l^l(\mathfrak{g}_s^{\frac1l}\mathfrak{h}_s^{\frac1s})+\mathfrak{M}_l^l(\mathfrak{h}_s^{\frac1l}\mathfrak{g}_s^{\frac1s}))-Z_l}\right)^{\frac{1}{r-l}},$$

where $r,l,s>0$, $r\ne 2s$, $l\ne 2s$. We can extend these means to the limiting cases. To do so, let $r,l,s>0$. We define

$$\begin{aligned}\widetilde{\mathfrak{M}}_{l,l}^{[s]}(f)&=\exp\left(\frac{\widetilde{P}}{\widetilde{Q}}-\frac{2(l-s)}{l(l-2s)}\right),\quad l\ne 2s,\\ \widetilde{\mathfrak{M}}_{l,2s}^{[s]}(f)=\widetilde{\mathfrak{M}}_{2s,l}^{[s]}(f)&=\exp\left(\frac{2s\widetilde{Q}}{l(l-2s)\widetilde{P}_1}\right)^{\frac{1}{l-2s}},\quad l\ne 2s,\\ \widetilde{\mathfrak{M}}_{2s,2s}^{[s]}(f)&=\exp\left(\frac{\widetilde{Q}_1}{2\widetilde{P}_1}-\frac{1}{2s}\right),\end{aligned}$$

where $\widetilde{P}$, $\widetilde{Q}$, $\widetilde{P}_1$, and $\widetilde{Q}_1$ are defined by

$$\begin{aligned}
\widetilde{P} =& m^l \log m + M^l \log M - \frac{1}{b-a}\int_a^b f^l(t)\log f(t)\Delta t \\
&- \frac{1}{s}X_l \log(m^s + M^s - \mathfrak{M}_s^s(f)) \\
&- \frac{Y_s}{s(b-a)}\int_a^b [\mathfrak{g}_s(t)\mathfrak{h}_s^{\frac{l}{s}}(t)\log(\mathfrak{h}_s(t)) + \mathfrak{h}_s(t)\mathfrak{g}_s^{\frac{l}{s}}(t)\log(\mathfrak{g}_s(t))]\Delta t \\
&- \frac{1}{s(b-a)}\int_a^b |f^s(t) - \mathfrak{M}_s^s(f)|^{\frac{l}{s}} \log|f^s(t) - \mathfrak{M}_s^s(f)|\Delta t, \\
\widetilde{Q} =& W_l - X_l - Y_s(\mathfrak{M}_l^l(\mathfrak{g}_s^{\frac{1}{l}}\mathfrak{h}_s^{\frac{1}{s}}) + \mathfrak{M}_l^l(\mathfrak{h}_s^{\frac{1}{l}}\mathfrak{g}_s^{\frac{1}{s}})) - Z_l, \\
\widetilde{P}_1 =& m^{2s}\log m + M^{2s}\log M - \frac{1}{b-a}\int_a^b f^{2s}(t)\log f(t)\Delta t \\
&- \frac{1}{s}X_{2s}\log(m^s + M^s - \mathfrak{M}_s^s(f)) \\
&- \frac{Y_s}{s(b-a)}\int_a^b [\mathfrak{g}_s(t)\mathfrak{h}_s^2(t)\log(\mathfrak{h}_s(t)) + \mathfrak{h}_s(t)\mathfrak{g}_s^2(t)\log(\mathfrak{g}_s(t))]\Delta t \\
&- \frac{1}{s(b-a)}\int_a^b |f^s(t) - \mathfrak{M}_s^s(f)|^2 \log|f^s(t) - \mathfrak{M}_s^s(f)|\Delta t, \\
\widetilde{Q}_1 =& m^{2s}(\log m)^2 + M^{2s}(\log M)^2 - \frac{1}{b-a}\int_a^b f^{2s}(t)(\log f(t))^2\Delta t \\
&- \frac{1}{s^2}X_{2s}(\log(m^s + M^s - \mathfrak{M}_s^s(f)))^2 \\
&- \frac{Y_s}{s^2(b-a)}\int_a^b [\mathfrak{g}_s(t)\mathfrak{h}_s^2(t)(\log(\mathfrak{h}_s(t)))^2 + \mathfrak{h}_s(t)\mathfrak{g}_s^2(t)(\log(\mathfrak{g}_s(t)))^2]\Delta t \\
&- \frac{1}{s^2(b-a)}\int_a^b |f^s(t) - \mathfrak{M}_s^s(f)|^2(\log|f^s(t) - \mathfrak{M}_s^s(f)|)^2\Delta t.
\end{aligned}$$

Theorem 8.9 *Let $a, b \in \mathbb{T}$ and $f \in \mathrm{C}_{\mathrm{rd}}([a,b)_{\mathbb{T}}, [m,M])$, where $0 \le m < M < \infty$. Suppose $\alpha, \beta, \gamma \in \mathrm{C}^2([0,\infty), \mathbb{R})$ are strictly monotone such that*

$$\overline{\alpha \circ \gamma^{-1}}, \overline{\beta \circ \gamma^{-1}} \in \mathrm{C}^1((0,\infty), \mathbb{R}) \quad \textit{and} \quad (\alpha \circ \gamma^{-1})(0) = (\beta \circ \gamma^{-1})(0) = 0.$$

If

$$\int_a^b (\gamma\circ f)(t)\Delta t\left((\gamma(M))^2+(\gamma(m))^2+\gamma(M)\gamma(m)\right)$$
$$-(b-a)\gamma(M)\gamma(m)(\gamma(M)+\gamma(m))-\int_a^b(\gamma\circ f)^3(t)\Delta t$$
$$-\frac{1}{\gamma(M)-\gamma(m)}\int_a^b\left[((\gamma\circ f)(t)-\gamma(m))(\gamma(M)-(\gamma\circ f)(t))^3\right.$$
$$\left.+(\gamma(M)-(\gamma\circ f)(t))((\gamma\circ f)(t)-\gamma(m))^3\right]\Delta t\neq 0,$$

then

$$\frac{(b-a)E_\alpha-\int_a^b[\mathfrak{g}(t)(\alpha\circ\gamma^{-1})(\mathfrak{h}(t))+\mathfrak{h}(t)(\alpha\circ\gamma^{-1})(\mathfrak{g}(t))]\Delta t-(b-a)F_\alpha}{(b-a)E_\beta-\int_a^b[\mathfrak{g}(t)(\beta\circ\gamma^{-1})(\mathfrak{h}(t))+\mathfrak{h}(t)(\beta\circ\gamma^{-1})(\mathfrak{g}(t))]\Delta t-(b-a)F_\beta}$$
$$=\frac{\gamma(\zeta)(\alpha''(\zeta)\gamma'(\zeta)-\alpha'(\zeta)\gamma''(\zeta))-\alpha'(\zeta)(\gamma'(\zeta))^2}{\gamma(\zeta)(\beta''(\zeta)\gamma'(\zeta)-\beta'(\zeta)\gamma''(\zeta))-\beta'(\zeta)(\gamma'(\zeta))^2} \tag{8.20}$$

holds for some $\zeta\in f([a,b)_{\mathbb{T}})$, provided that the denominators in (8.20) *are nonzero, where $\mathfrak{g}$ and $\mathfrak{h}$ are defined as in Theorem 8.8 and*

$$E_\alpha=(\gamma(M)-\gamma(\mathfrak{M}_\gamma(f)))\alpha(m)+(\gamma(\mathfrak{M}_\gamma(f))-\gamma(m))\alpha(M),$$
$$F_\alpha=(\gamma(M)-\gamma(m))\alpha(\mathfrak{M}_\alpha(f)).$$

Proof. Replace the functions f, Ψ, and Φ in Theorem 8.4 by $\gamma\circ f$, $\alpha\circ\gamma^{-1}$, and $\beta\circ\gamma^{-1}$, respectively. The rest of the proof is analogous to the proof of Theorem 8.7. □

Remark 8.8 In Theorem 8.9, let

$$\widehat{\mathscr{F}}(\zeta)=\frac{\gamma(\zeta)(\alpha''(\zeta)\gamma'(\zeta)-\alpha'(\zeta)\gamma''(\zeta))-\alpha'(\zeta)(\gamma'(\zeta))^2}{\gamma(\zeta)(\beta''(\zeta)\gamma'(\zeta)-\beta'(\zeta)\gamma''(\zeta))-\beta'(\zeta)(\gamma'(\zeta))^2}$$

and suppose $\widehat{\mathscr{F}}$ is invertible. Then, since ζ is in the image of f, we obtain a new mean defined by

$$\widehat{\mathscr{F}}^{-1}\left(\frac{(b-a)E_\alpha-\int_a^b[\mathfrak{g}(t)(\alpha\circ\gamma^{-1})(\mathfrak{h}(t))+\mathfrak{h}(t)(\alpha\circ\gamma^{-1})(\mathfrak{g}(t))]\Delta t-(b-a)F_\alpha}{(b-a)E_\beta-\int_a^b[\mathfrak{g}(t)(\beta\circ\gamma^{-1})(\mathfrak{h}(t))+\mathfrak{h}(t)(\beta\circ\gamma^{-1})(\mathfrak{g}(t))]\Delta t-(b-a)F_\beta}\right).$$

Corollary 8.3 *Let $a,b\in\mathbb{T}$ and $f\in \mathrm{C}_{\mathrm{rd}}([a,b)_{\mathbb{T}},[m,M])$, where $0\leq m<M<\infty$. Suppose $r,l,s>0$ are such that $r\neq l$, $r\neq 2s$, $l\neq 2s$, and*

$$\int_a^b f^s(t)\Delta t\left(M^{2s}+m^{2s}+(Mm)^s\right)-(b-a)(Mm)^s(M^s+m^s)-\int_a^b f^{3s}(t)\Delta t$$
$$-\frac{\int_a^b\left[(f^s(t)-m^s)(M^s-f^s(t))^3+(M^s-f^s(t))(f^s(t)-m^s)^3\right]\Delta t}{M^s-m^s}\neq 0.$$

Then

$$\frac{E_r - \mathfrak{M}_r^r(\mathfrak{g}_s^{\frac{1}{r}}\mathfrak{h}_s^{\frac{1}{s}}) - \mathfrak{M}_r^r(\mathfrak{h}_s^{\frac{1}{r}}\mathfrak{g}_s^{\frac{1}{s}}) - F_r}{E_l - \mathfrak{M}_l^l(\mathfrak{g}_s^{\frac{1}{l}}\mathfrak{h}_s^{\frac{1}{s}}) - \mathfrak{M}_l^l(\mathfrak{h}_s^{\frac{1}{l}}\mathfrak{g}_s^{\frac{1}{s}}) - F_l} = \frac{r(r-2s)}{l(l-2s)}\zeta^{r-l} \tag{8.21}$$

holds for some $\zeta \in f([a,b)_{\mathbb{T}})$*, provided that the denominators in* (8.21) *are nonzero, where* $\mathfrak{g}_s$ *and* $\mathfrak{h}_s$ *are defined as in Corollary 8.2 and*

$$E_r = (M^s - \mathfrak{M}_s^s(f))m^r + (\mathfrak{M}_s^s(f) - m^s)M^r, \quad F_r = (M^s - m^s)\mathfrak{M}_r^r(f).$$

Remark 8.9 From Corollary 8.3, since $\zeta \in f([a,b)_{\mathbb{T}})$, we obtain a new mean defined by

$$\widehat{\mathfrak{M}}_{r,l}^{[s]}(f) = \left(\frac{l(l-2s)}{r(r-2s)}\frac{E_r - \mathfrak{M}_r^r(\mathfrak{g}_s^{\frac{1}{r}}\mathfrak{h}_s^{\frac{1}{s}}) - \mathfrak{M}_r^r(\mathfrak{h}_s^{\frac{1}{r}}\mathfrak{g}_s^{\frac{1}{s}}) - F_r}{E_l - \mathfrak{M}_l^l(\mathfrak{g}_s^{\frac{1}{l}}\mathfrak{h}_s^{\frac{1}{s}}) - \mathfrak{M}_l^l(\mathfrak{h}_s^{\frac{1}{l}}\mathfrak{g}_s^{\frac{1}{s}}) - F_l}\right)^{\frac{1}{r-l}},$$

where $r,l,s > 0$, $r \neq 2s$, $l \neq 2s$. We can extend these means to the limiting cases. To do so, let $r,l,s > 0$. We define

$$\widehat{\mathfrak{M}}_{l,l}^{[s]}(f) = \exp\left(\frac{\widehat{P}}{\widehat{Q}} - \frac{2(l-s)}{l(l-2s)}\right), \quad l \neq 2s,$$

$$\widehat{\mathfrak{M}}_{l,2s}^{[s]}(f) = \widehat{\mathfrak{M}}_{2s,l}^{[s]}(f) = \exp\left(\frac{2s\widehat{Q}}{l(l-2s)\widehat{P}_1}\right)^{\frac{1}{l-2s}}, \quad l \neq 2s,$$

$$\widehat{\mathfrak{M}}_{2s,2s}^{[s]}(f) = \exp\left(\frac{\widehat{Q}_1}{2\widehat{P}_1} - \frac{1}{2s}\right),$$

where $\widehat{P}$, $\widehat{Q}$, $\widehat{P}_1$, and $\widehat{Q}_1$ are defined by

$$\begin{aligned}
\widehat{P} =& (M^s - \mathfrak{M}_s^s(f))m^l \log m + (\mathfrak{M}_s^s(f) - m^s)M^l \log M \\
&- \frac{1}{s(b-a)}\int_a^b [\mathfrak{g}_s(t)\mathfrak{h}_s^{\frac{l}{s}}(t)\log(\mathfrak{h}_s(t)) + \mathfrak{h}_s(t)\mathfrak{g}_s^{\frac{l}{s}}(t)\log(\mathfrak{g}_s(t))]\Delta t \\
&- \frac{M^s - m^s}{b-a}\int_a^b f^l(t)\log f(t)\Delta t, \\
\widehat{Q} =& E_l - \mathfrak{M}_l^l(\mathfrak{g}_s^{\frac{1}{l}}\mathfrak{h}_s^{\frac{1}{s}}) - \mathfrak{M}_l^l(\mathfrak{h}_s^{\frac{1}{l}}\mathfrak{g}_s^{\frac{1}{s}}) - F_l, \\
\widehat{P}_1 =& (M^s - \mathfrak{M}_s^s(f))m^{2s} \log m + (\mathfrak{M}_s^s(f) - m^s)M^{2s} \log M \\
&- \frac{1}{s(b-a)}\int_a^b [\mathfrak{g}_s(t)\mathfrak{h}_s^2(t)\log(\mathfrak{h}_s(t)) + \mathfrak{h}_s(t)\mathfrak{g}_s^2(t)\log(\mathfrak{g}_s(t))]\Delta t \\
&- \frac{M^s - m^s}{b-a}\int_a^b f^{2s}(t)\log f(t)\Delta t,
\end{aligned}$$

$$
\begin{aligned}
\widehat{Q}_1 =&(M^s - \mathfrak{M}_s^s(f))m^{2s}(\log m)^2 + (\mathfrak{M}_s^s(f) - m^s)M^{2s}(\log M)^2 \\
&- \frac{1}{s^2(b-a)} \int_a^b [\mathfrak{g}_s(t)\mathfrak{h}_s^2(t)(\log(\mathfrak{h}_s(t)))^2 + \mathfrak{h}_s(t)\mathfrak{g}_s^2(t)(\log(\mathfrak{g}_s(t)))^2]\Delta t \\
&- \frac{M^s - m^s}{b-a} \int_a^b f^{2s}(t)(\log f(t))^2 \Delta t.
\end{aligned}
$$

8.3 Exponential Convexity and Logarithmic Convexity

Applying the functional $\mathscr{J}_\Psi$ to the function Ψ_s defined in Lemma 1.4, we obtain

$$
\begin{aligned}
\mathscr{J}_{\Psi_s} =&\frac{1}{s(s-2)} \left\{ \int_a^b \left[f^s(u) - \left| f(u) - \frac{\int_a^b f(t)\Delta t}{b-a} \right|^s \right] \Delta u \right. \\
&\left. -(b-a)\left(\frac{\int_a^b f(t)\Delta t}{b-a} \right)^s \right\}, \quad s \neq 2
\end{aligned} \tag{8.22}
$$

and

$$
\begin{aligned}
\mathscr{J}_{\Psi_2} =&\frac{1}{2} \left\{ \int_a^b \left[f^2(u)\log f(u) \right.\right. \\
&\left. - \left| f(u) - \frac{\int_a^b f(t)\Delta t}{b-a} \right|^2 \log \left| f(u) - \frac{\int_a^b f(t)\Delta t}{b-a} \right| \right] \Delta u \\
&\left. - \frac{1}{b-a} \left(\int_a^b f(t)\Delta t \right)^2 \log \left(\frac{\int_a^b f(t)\Delta t}{b-a} \right) \right\}.
\end{aligned} \tag{8.23}
$$

Theorem 8.10 *Let $\mathscr{J}_{\Psi_s}$ be defined as in* (8.22)–(8.23). *Then*

(i) *for all $n \in \mathbb{N}$ and for all $p_i > 0$, $p_{ij} = \dfrac{p_i + p_j}{2}$, $1 \leq i, j \leq n$, the matrix $\left[\mathscr{J}_{\Psi_{p_{ij}}} \right]_{i,j=1}^n$ is positive semidefinite;*

(ii) *the function $s \mapsto \mathscr{J}_{\Psi_s}$ is exponentially convex;*

(iii) *if $\mathscr{J}_{\Psi_s} > 0$, then the function $s \mapsto \mathscr{J}_{\Psi_s}$ is* log-*convex, i.e., for $0 < r < s < w$, we have*

$$
(\mathscr{J}_{\Psi_s})^{w-r} \leq (\mathscr{J}_{\Psi_r})^{w-s} (\mathscr{J}_{\Psi_w})^{s-r}.
$$

Proof. To show (i), let

$$\Lambda(x) = \sum_{i,j=1}^{n} v_i v_j \Psi_{p_{ij}}(x).$$

Then

$$\overline{\Lambda}'(x) = \sum_{i,j=1}^{n} v_i v_j x^{\frac{p_{ij}}{2}-3} = \left(\sum_{i=1}^{n} v_i x^{\frac{p_i-3}{2}}\right)^2 \geq 0$$

and $\Lambda(0) = 0$. Thus Λ is superquadratic. Now using Λ instead of Ψ in (8.1), we obtain

$$\mathscr{J}_\Lambda = \sum_{i,j=1}^{n} v_i v_j \mathscr{J}_{\Psi_{p_{ij}}} \geq 0. \tag{8.24}$$

Hence the matrix $\left[\mathscr{J}_{\Psi_{p_{ij}}}\right]_{i,j=1}^{n}$ is positive semidefinite.

Now we show (ii). Because $\lim_{s\to 2} \mathscr{J}_{\Psi_s} = \mathscr{J}_{\Psi_2}$, the function $s \mapsto \mathscr{J}_{\Psi_s}$ is continuous on $\mathbb{R}_+$. Hence by (8.24) and Proposition 1.2, the function $s \mapsto \mathscr{J}_{\Psi_s}$ is exponentially convex.

Finally, we show (iii). Because the function $s \mapsto \mathscr{J}_{\Psi_s}$ is exponentially convex, if $\mathscr{J}_{\Psi_s} > 0$, then by Remark 1.9, the function $s \mapsto \mathscr{J}_{\Psi_s}$ is log-convex. □

Corollary 8.4 *Let $a,b \in \mathbb{T}$ and $f \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}}, I)$ be positive and define*

$$\mathscr{D}_s = \begin{cases} \int_a^b \left[f^s(u) - \left| f(u) - \frac{\int_a^b f(t)\Delta t}{b-a} \right|^s \right] \Delta u \\ -(b-a)\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right)^s, & s \neq 2 \\ \\ \int_a^b \left[f^2(u)\log f(u) - \left| f(u) - \frac{\int_a^b f(t)\Delta t}{b-a} \right|^2 \right. \\ \left. \log\left| f(u) - \frac{\int_a^b f(t)\Delta t}{b-a} \right| \right] \Delta u \\ -\frac{1}{b-a}\left(\int_a^b f(t)\Delta t\right)^2 \log\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right), & s = 2. \end{cases}$$

Then

(i) *for* $s > 4$,

$$\frac{\int_a^b f^s(t)\Delta t}{b-a} \geq \left(\frac{\int_a^b f(t)\Delta t}{b-a}\right)^s + \frac{1}{b-a}\int_a^b \left| f(u) - \frac{\int_a^b f(t)\Delta t}{b-a} \right|^s \Delta u$$
$$+ \frac{s(s-2)}{3(b-a)}\left(\frac{3\mathscr{D}_4}{8\mathscr{D}_3}\right)^{s-3} \mathscr{D}_3;$$

(ii) *for* $1 < s < 2$,

$$\frac{\int_a^b f^s(t)\Delta t}{b-a} \leq \left(\frac{\int_a^b f(t)\Delta t}{b-a}\right)^s + \frac{1}{b-a}\int_a^b \left|f(u) - \frac{\int_a^b f(t)\Delta t}{b-a}\right|^s \Delta u$$

$$+\frac{s(s-2)}{b-a}\left(\frac{\mathscr{D}_2}{2\int_a^b \left|f(u)-\frac{\int_a^b f(t)\Delta t}{b-a}\right|\Delta u}\right)^{s-1} \int_a^b \left|f(u) - \frac{\int_a^b f(t)\Delta t}{b-a}\right| \Delta u;$$

(iii) *for* $2 < s < 3$,

$$\frac{\int_a^b f^s(t)\Delta t}{b-a} \leq \left(\frac{\int_a^b f(t)\Delta t}{b-a}\right)^s + \frac{1}{b-a}\int_a^b \left|f(u) - \frac{\int_a^b f(t)\Delta t}{b-a}\right|^s \Delta u$$

$$+\frac{s(s-2)}{2(b-a)}\left(\frac{2\mathscr{D}_3}{3\mathscr{D}_2}\right)^{s-2}\mathscr{D}_2;$$

(iv) *for* $3 < s < 4$,

$$\frac{\int_a^b f^s(t)\Delta t}{b-a} \leq \left(\frac{\int_a^b f(t)\Delta t}{b-a}\right)^s + \frac{1}{b-a}\int_a^b \left|f(u) - \frac{\int_a^b f(t)\Delta t}{b-a}\right|^s \Delta u$$

$$+\frac{s(s-2)}{3(b-a)}\left(\frac{3\mathscr{D}_4}{8\mathscr{D}_3}\right)^{s-3}\mathscr{D}_3.$$

Proof. The results follow from Theorem 8.10 (iii). □

Example 8.1 Let us consider the discrete form of $\mathscr{D}_s$. For this, let $[a,b) = \{1,2\}$, $f(1) = x$, $f(2) = y$ such that $y \geq x \geq 0$. Then $\mathscr{D}_s$ becomes

$$\mathscr{D}_s = d_s = x^s + y^s - 2\left(\frac{x+y}{2}\right)^s - 2\left(\frac{y-x}{2}\right)^s.$$

For $s > 4$, we obtain the inequality

$$d_s \geq \frac{s(s-2)}{3}\left(\frac{3d_4}{8d_3}\right)^{s-3} d_3 = \frac{s(s-2)}{3}\left(\frac{3^2(y+x)^2}{4^2(y+2x)}\right)^{s-3}\frac{(y-x)^2(y+2x)}{2}.$$

If $3 < s < 4$, we have

$$d_s \leq \frac{s(s-2)}{3}\left(\frac{3^2(y+x)^2}{4^2(y+2x)}\right)^{s-3}\frac{(y-x)^2(y+2x)}{2}.$$

Therefore for $s = 1$, the inequality becomes

$$-(y-x) \leq -\frac{1}{3\cdot 2}\left(\frac{4^2}{3^2}\right)^2 \frac{(y+2x)^3(y-x)^2}{(y+x)^4}.$$

Theorem 8.11 *Suppose $p,q \in \mathbb{R}$ are such that $1 < p < 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $a,b \in \mathbb{T}$ and $f,g \in C_{rd}([a,b)_{\mathbb{T}},[0,\infty))$ be such that $\int_a^b g^q(t)\Delta t > 0$. Then*

$$\frac{1}{p(p-2)}\left(\left(\left(\int_a^b f^p(t)\Delta t - \int_a^b g(u)h^p(u)\Delta u\right)^{\frac{1}{p}}\left(\int_a^b g^q(t)\Delta t\right)^{\frac{1}{q}}\right)^p \right. \tag{8.25}$$
$$\left. - \left(\int_a^b f(t)g(t)\Delta t\right)^p\right)$$
$$\leq \frac{1}{2^{p-1}}\left(\int_a^b g(u)h(u)\Delta u\right)^{2-p}\left(\int_a^b g^q(t)\Delta t\left(\int_a^b f^2(t)g^{2-q}(t)\log(f(t)g^{1-q}(t))\Delta t\right.\right.$$
$$\left. - \int_a^b g^{2-q}(u)h^2(u)\log(g^{1-q}(u)h(u))\Delta u\right)$$
$$\left. - \left(\int_a^b f(t)g(t)\Delta t\right)^2 \log\left(\frac{\int_a^b f(t)g(t)\Delta t}{\int_a^b g^q(t)\Delta t}\right)\right)^{p-1}$$

holds, where

$$h(u) = \left| f(u) - g^{q-1}(u)\frac{\int_a^b f(t)g(t)\Delta t}{\int_a^b g^q(t)\Delta t}\right|.$$

Proof. In Theorem 8.10 (iii), let $r = 1$, $s = p$, $w = 2$, so that $1 < p < 2$. Then we have

$$\left(\mathscr{I}_{\Psi_p}\right)^1 \leq \left(\mathscr{I}_{\Psi_1}\right)^{2-p}\left(\mathscr{I}_{\Psi_2}\right)^{p-1}.$$

By replacing $\frac{\int_a^b f(t)\Delta t}{b-a}$ with $\frac{\int_a^b k(t)f(t)\Delta t}{\int_a^b k(t)\Delta t}$, where $k \in C_{rd}([a,b)_{\mathbb{T}},[0,\infty))$ is such that $\int_a^b k(t)\Delta t > 0$, we get

$$\frac{1}{p(p-2)}\left(\int_a^b k(t)f^p(t)\Delta t - \int_a^b k(u)\left|f(u) - \frac{\int_a^b k(t)f(t)\Delta t}{\int_a^b k(t)\Delta t}\right|^p \Delta u\right.$$
$$\left. - \int_a^b k(t)\Delta t\left(\frac{\int_a^b k(t)f(t)\Delta t}{\int_a^b k(t)\Delta t}\right)^p\right)$$
$$\leq \frac{1}{2^{p-1}}\left\{\left(\int_a^b k(t)\left|f(u) - \frac{\int_a^b k(t)f(t)\Delta t}{\int_a^b k(t)\Delta t}\right|\Delta u\right)^{2-p}\left(\int_a^b k(t)f^2(t)\log f(t)\Delta t\right.\right.$$
$$- \int_a^b k(u)\left|f(u) - \frac{\int_a^b k(t)f(t)\Delta t}{\int_a^b k(t)\Delta t}\right|^2 \log\left|f(u) - \frac{\int_a^b k(t)f(t)\Delta t}{\int_a^b k(t)\Delta t}\right|\Delta u$$
$$\left.\left. - \frac{1}{\int_a^b k(t)\Delta t}\left(\int_a^b k(t)f(t)\Delta t\right)^2 \log\left(\frac{\int_a^b k(t)f(t)\Delta t}{\int_a^b k(t)\Delta t}\right)\right)^{p-1}\right\}.$$

Now replacing k by g^q and f by fg^{1-q}, we get the required result. □

Remark 8.10 Theorem 8.11 refines the time scales Hölder inequality for superquadratic functions (Theorem 2.65).

Theorem 8.12 *Let $\mathscr{J}_{\Psi_s}$ and $\mathscr{J}_{\Psi_2}$ be positive. Then for $r,l,v,w>0$ such that $r\le v$, $l\le w$, we have*

$$\mathfrak{M}^{[s]}_{r,l}(f)\le \mathfrak{M}^{[s]}_{v,w}(f). \tag{8.26}$$

Proof. Since $\mathscr{J}_{\Psi_s}$ is positive, by Theorem 8.10, $\mathscr{J}_{\Psi_s}$ is log-convex. Now by using Remark 1.6 (b), for $r,l,v,w>0$ such that $r\le v$, $l\le w$, $r\ne l$, $v\ne w$, we have

$$\left(\frac{\mathscr{J}_{\Psi_r}}{\mathscr{J}_{\Psi_l}}\right)^{\frac{1}{r-l}}\le\left(\frac{\mathscr{J}_{\Psi_v}}{\mathscr{J}_{\Psi_w}}\right)^{\frac{1}{v-w}}.$$

By substituting $\frac{r}{s}$ for r, $\frac{l}{s}$ for l, $\frac{u}{s}$ for u, $\frac{v}{s}$ for v, f^s for f, and from the continuity of $\mathscr{J}_{\Psi_s}$, we obtain the required result. □

Theorem 8.13 *Theorem 8.10 is still valid if we replace Ψ_s by φ_s as defined in Lemma 1.5.*

Proof. As in the proof of Theorem 8.10, consider

$$\Omega(x)=\sum_{i,j=1}^{n}v_iv_j\varphi_{p_{ij}}(x).$$

Then

$$\overline{\Omega}'(x)=\left(\sum_{i=1}^{n}v_ie^{\frac{p_i}{2}x}\right)^2\ge 0$$

and $\Omega(0)=0$. Thus Ω is superquadratic. Now using Ω instead of Ψ in (8.1), we obtain the required result. □

Corollary 8.5 *Let $a,b\in\mathbb{T}$ and $f\in \mathrm{C_{rd}}([a,b)_{\mathbb{T}},I)$ be positive. Let $r,s\in\mathbb{R}$, $r\ne s$. Then we have*

$$\mathfrak{M}_{r,s}(f)$$
$$=\left(\frac{s^3\left(r\int_a^b f^r(t)\log f(t)\Delta t-A_r-r\int_a^b\mathscr{B}(t)e^{r\mathscr{B}(t)}\Delta t+\int_a^b e^{r\mathscr{B}(t)}\Delta t-1\right)}{r^3\left(s\int_a^b f^s(t)\log f(t)\Delta t-A_s-s\int_a^b\mathscr{B}(t)e^{s\mathscr{B}(t)}\Delta t+\int_a^b e^{s\mathscr{B}(t)}\Delta t-1\right)}\right)^{\frac{1}{r-s}},$$

provided that the occurring denominators are nonzero, where

$$A_r=(b-a)(\mathfrak{M}_r^r(f)+\mathfrak{M}_0^r(f)\log(\mathfrak{M}_0^r(f))-\mathfrak{M}_0^r(f)),\quad \mathscr{B}(t)=\left|\log\left(\frac{f(t)}{\mathfrak{M}_0(f)}\right)\right|.$$

Proof. The proof follows from Theorem 8.2 by replacing Ψ, Φ, and f with φ_r, φ_s, and $\log f$, respectively. □

Remark 8.11 For the limiting cases of Cauchy type means defined in Corollary 8.5, we have

$$\mathfrak{M}_{s,s}(f) = \exp\left(\frac{B}{C} - \frac{3}{s}\right), \quad s \neq 0 \quad \text{and} \quad \mathfrak{M}_{0,0}(f) = \exp\left(\frac{3B_1}{8C_1}\right),$$

where

$$\begin{aligned} B =& s\left(\int_a^b f^s(t)(\log f(t))^2\Delta t - (b-a)\mathfrak{M}_0^s(f)\,(\log(\mathfrak{M}_0(f)))^2 \right.\\ &\left. - \int_a^b \mathscr{B}^2(t)e^{s\mathscr{B}(t)}\Delta t\right), \\ C =& s\int_a^b f^s(t)\log f(t)\Delta t - (b-a)A_s - s\int_a^b \mathscr{B}(t)e^{s\mathscr{B}(t)}\Delta t + \int_a^b e^{s\mathscr{B}(t)}\Delta t - 1, \\ B_1 =& \int_a^b (\log f(t))^4\Delta t - (b-a)(\log(\mathfrak{M}_0(f)))^4 - \int_a^b \mathscr{B}^4(t)\Delta t, \\ C_1 =& \int_a^b (\log f(t))^3\Delta t - (b-a)(\log(\mathfrak{M}_0(f)))^3 - \int_a^b \mathscr{B}^3(t)\Delta t. \end{aligned}$$

Theorem 8.14 *Let $\mathscr{J}_{\Psi_s}$ be positive. Then for $r,l,v,w>0$ such that $r \le v$, $l \le w$, we have*

$$\mathfrak{M}_{r,l}(f) \le \mathfrak{M}_{v,w}(f). \tag{8.27}$$

Proof. See the proof of Theorem 8.12. □

We can obtain corresponding results for $\widetilde{\mathscr{J}}_{\Psi_s}$ and $\widehat{\mathscr{J}}_{\Psi_s}$ analogously as in the case of $\mathscr{J}_{\Psi_s}$.

Theorem 8.15 (i) *For all $n \in \mathbb{N}$ and for all $p_i > 0$, $p_{ij} = \dfrac{p_i + p_j}{2}$, $1 \le i,j \le n$, the matrix $\left[\widetilde{\mathscr{J}}_{\Psi_{p_{ij}}}\right]_{i,j=1}^n$ is positive semidefinite;*

(ii) *the function $s \mapsto \widetilde{\mathscr{J}}_{\Psi_s}$ is exponentially convex;*

(iii) *if $\widetilde{\mathscr{J}}_{\Psi_s} > 0$, then the function $s \mapsto \widetilde{\mathscr{J}}_{\Psi_s}$ is* log-convex, *i.e., for $0 < r < s < w$, we have*

$$\widetilde{\mathscr{J}}_{\Psi_s}^{\,w-r} \le \widetilde{\mathscr{J}}_{\Psi_r}^{\,w-s}\, \widetilde{\mathscr{J}}_{\Psi_w}^{\,s-r}.$$

Corollary 8.6 *Let* $a,b\in\mathbb{T}$ *and* $f\in C_{\mathrm{rd}}([a,b)_{\mathbb{T}},[m,M])$, *where* $0\leq m<M<\infty$. *Suppose*

$$\widetilde{\mathscr{D}}_s=\begin{cases}(b-a)(m^s+M^s)-\int_a^b f^s(t)\Delta t\\ -(b-a)\left(m+M-\frac{1}{b-a}\int_a^b f(t)\Delta t\right)^s-K_s, & s\neq 2\\ \\ (b-a)(m^2\log m+M^2\log M)-\int_a^b f^2(t)\log f(t)\Delta t\\ -(b-a)\left(m+M-\frac{1}{b-a}\int_a^b f(t)\Delta t\right)^2\\ \log\left(m+M-\frac{1}{b-a}\int_a^b f(t)\Delta t\right)-K_2, & s=2,\end{cases}$$

where

$$K_s=\frac{2}{M-m}\int_a^b\left[(f(t)-m)(M-f(t))^s+(M-f(t))(f(t)-m)^s\right]\Delta t\\ +\int_a^b\left|f(u)-\frac{1}{b-a}\int_a^b f(t)\Delta t\right|^s\Delta u$$

and

$$K_2=\frac{2}{M-m}\int_a^b\Big[(f(t)-m)(M-f(t))^2\log(M-f(t))\\ +(M-f(t))(f(t)-m)^2\log(f(t)-m)\Big]\Delta t\\ +\int_a^b\left|f(u)-\frac{1}{b-a}\int_a^b f(t)\Delta t\right|^2\log\left|f(u)-\frac{1}{b-a}\int_a^b f(t)\Delta t\right|\Delta u.$$

Then

(i) *for* $s>4$,

$$\frac{\widetilde{\mathscr{D}}_s}{s(s-2)}\geq\frac{\widetilde{\mathscr{D}}_3}{3}\left(\frac{3\widetilde{\mathscr{D}}_4}{8\widetilde{\mathscr{D}}_3}\right)^{s-3};$$

(ii) *for* $1<s<2$,

$$\frac{\widetilde{\mathscr{D}}_s}{s(s-2)}\leq-\widetilde{\mathscr{D}}_1\left(-\frac{\widetilde{\mathscr{D}}_2}{2\widetilde{\mathscr{D}}_1}\right)^{s-1};$$

(iii) *for* $2<s<3$,

$$\frac{\widetilde{\mathscr{D}}_s}{s(s-2)}\leq\frac{\widetilde{\mathscr{D}}_2}{2}\left(\frac{2\widetilde{\mathscr{D}}_3}{3\widetilde{\mathscr{D}}_2}\right)^{s-2};$$

(iv) *for* $3<s<4$,

$$\frac{\widetilde{\mathscr{D}}_s}{s(s-2)}\leq\frac{\widetilde{\mathscr{D}}_3}{3}\left(\frac{3\widetilde{\mathscr{D}}_4}{8\widetilde{\mathscr{D}}_3}\right)^{s-3}.$$

Theorem 8.16 *Suppose $p,q \in \mathbb{R}$ are such that $1 < p < 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $a,b \in \mathbb{T}$ and $f,g \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}},[m,M])$, where $0 \le m < M < \infty$, be such that $\int_a^b g^q(t)\Delta t > 0$. Then*

$$\begin{aligned}
&\frac{1}{p(p-2)}\Bigg(\left(\left(\int_a^b g^q(t)\Delta t\right)^p (m^p + M^p)\right. \qquad (8.28)\\
&-\left(\int_a^b g^q(t)\Delta t\right)^{p-1}\int_a^b f^p(t)\Delta t - U_1^p - \frac{2}{M-m}\left(\int_a^b g^q(t)\Delta t\right)^{p-1} U_2\\
&-\int_a^b g(u)h^p(u)\Delta u\left(\int_a^b g^q(t)\Delta t\right)^{p-1}\Bigg)\\
&\le \frac{1}{2^{p-1}}V_1^{2-p}\Bigg(\left(m^2\log m + M^2\log M\right)\left(\int_a^b g^q(t)\Delta t\right)^2\\
&-\int_a^b g^q(t)\Delta t\int_a^b f^2(t)g^{2-q}(t)\log(f(t)g^{1-q}(t))\Delta t\\
&-\left((m+M)\int_a^b g^q(t)\Delta t - \int_a^b f(t)g(t)\Delta t\right)^2\\
&\log\left(m+M+\frac{\int_a^b f(t)g(t)\Delta t}{\int_a^b g^q(t)\Delta t}\right) - \frac{2}{M-m}\int_a^b g^q(t)\Delta t V_2\\
&-\int_a^b g^q(t)\Delta t\int_a^b g^{2-q}(u)h^2(u)\log(g^{1-q}(u)h(u))\Delta u\Bigg)^{p-1}
\end{aligned}$$

holds, where

$$\begin{aligned}
U_1 =&(m+M)\int_a^b g^q(t)\Delta t - \int_a^b f(t)g(t)\Delta t,\\
U_2 =&\int_a^b g^q(t)\left(f(t)g^{1-q}(t)-m\right)\left(M-f(t)g^{1-q}(t)\right)^p\Delta t\\
&+\int_a^b g^q(t)\left(M-f(t)g^{1-q}(t)\right)\left(f(t)g^{1-q}(t)-m\right)^p\Delta t,\\
V_1 =&\int_a^b g(u)h(u)\Delta u + \frac{4}{M-m}\int_a^b g^q(t)\left(M-f(t)g^{1-q}(t)\right)\left(f(t)g^{1-q}(t)-m\right)\Delta t,\\
V_2 =&\int_a^b g^q(t)\Big[\left(f(t)g^{1-q}(t)-m\right)\left(M-f(t)g^{1-q}(t)\right)^2\log\left(M-f(t)g^{1-q}(t)\right)\\
&+\left(M-f(t)g^{1-q}(t)\right)\left(f(t)g^{1-q}(t)-m\right)^2\log\left(f(t)g^{1-q}(t)-m\right)\Big]\Delta t.
\end{aligned}$$

Theorem 8.17 *Let $\widetilde{\mathscr{J}}_{\Psi_s}$ be positive. Then for $r,l,v,w > 0$ such that $r \le v$, $l \le w$, we have*

$$\widetilde{\mathfrak{M}}_{r,l}^{[s]}(f) \le \widetilde{\mathfrak{M}}_{v,w}^{[s]}(f). \tag{8.29}$$

Theorem 8.18 (i) *For all* $n \in \mathbb{N}$ *and for all* $p_i > 0$, $p_{ij} = \frac{p_i + p_j}{2}$, $1 \leq i, j \leq n$, *the matrix* $\left[\widehat{\mathscr{J}}_{\Psi_{p_{ij}}}\right]_{i,j=1}^{n}$ *is positive semidefinite;*

(ii) *the function* $s \mapsto \widehat{\mathscr{J}}_{\Psi_s}$ *is exponentially convex;*

(iii) *if* $\widehat{\mathscr{J}}_{\Psi_s} > 0$, *then the function* $s \mapsto \widehat{\mathscr{J}}_{\Psi_s}$ *is* log-*convex, i.e., for* $0 < r < s < w$, *we have*

$$\widehat{\mathscr{J}}_{\Psi_s}^{w-r} \leq \widehat{\mathscr{J}}_{\Psi_r}^{w-s}\, \widehat{\mathscr{J}}_{\Psi_w}^{s-r}.$$

Corollary 8.7 *Let* $a, b \in \mathbb{T}$ *and* $f \in \mathrm{C}_{\mathrm{rd}}([a,b)_{\mathbb{T}}, [m,M])$, *where* $0 \leq m < M < \infty$. *Suppose*

$$\widehat{\mathscr{D}}_s = \begin{cases} \frac{M(b-a)-\int_a^b f(t)\Delta t}{M-m} m^s + \frac{\int_a^b f(t)\Delta t - m(b-a)}{M-m} M^s \\ -R_s - \int_a^b f^s(t)\Delta t, & s \neq 2 \\ \\ \frac{M(b-a)-\int_a^b f(t)\Delta t}{M-m} m^2 \log m + \frac{\int_a^b f(t)\Delta t - m(b-a)}{M-m} M^2 \log M \\ -R_2 - \int_a^b f^2(t) \log f(t)\Delta t, & s = 2, \end{cases}$$

where

$$R_s = \frac{1}{M-m} \int_a^b \left[(f(t)-m)(M-f(t))^s + (M-f(t))(f(t)-m)^s\right]\Delta t$$

and

$$R_2 = \frac{1}{M-m} \int_a^b \big[(f(t)-m)(M-f(t))^2 \log(M-f(t)) \\ +(M-f(t))(f(t)-m)^2 \log(f(t)-m)\big]\,\Delta t.$$

Then

(i) *for* $s > 4$,

$$\frac{\widehat{\mathscr{D}}_s}{s(s-2)} \geq \frac{\widehat{\mathscr{D}}_3}{3} \left(\frac{3\widehat{\mathscr{D}}_4}{8\widehat{\mathscr{D}}_3}\right)^{s-3};$$

(ii) *for* $1 < s < 2$,

$$\frac{\widehat{\mathscr{D}}_s}{s(s-2)} \leq -\widehat{\mathscr{D}}_1 \left(-\frac{\widehat{\mathscr{D}}_2}{2\widehat{\mathscr{D}}_1}\right)^{s-1};$$

(iii) *for* $2 < s < 3$,

$$\frac{\widehat{\mathscr{D}}_s}{s(s-2)} \leq \frac{\widehat{\mathscr{D}}_2}{2} \left(\frac{2\widehat{\mathscr{D}}_3}{3\widehat{\mathscr{D}}_2}\right)^{s-2};$$

(iv) *for* $3 < s < 4$,

$$\frac{\widehat{\mathscr{D}}_s}{s(s-2)} \leq \frac{\widehat{\mathscr{D}}_3}{3} \left(\frac{3\widehat{\mathscr{D}}_4}{8\widehat{\mathscr{D}}_3}\right)^{s-3}.$$

Theorem 8.19 *Suppose $p,q \in \mathbb{R}$ are such that $1 < p < 2$ and $\frac{1}{p}+\frac{1}{q}=1$. Let $a,b \in \mathbb{T}$ and $f,g \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}},[m,M])$, where $0 \le m < M < \infty$, be such that $\int_a^b g^q(t)\Delta t > 0$. Then*

$$\begin{aligned}
&\frac{1}{p(p-2)}\Big(W_1 m^p + W_2 M^p \\
&-\int_a^b g^q(t)\left[\mathfrak{a}(t)\mathfrak{b}^p(t)+\mathfrak{b}(t)\mathfrak{a}^p(t)\right]\Delta t - (M-m)\int_a^b f^p(t)\Delta t\Big) \\
&\le \frac{1}{2^{2p-3}}\left(\int_a^b g^q(t)\mathfrak{a}(t)\mathfrak{b}(t)\Delta t\right)^{2-p}\Big(W_1 m^2\log m + W_2 M^2 \log M \\
&-\int_a^b g^q(t)\left[\mathfrak{a}(t)\mathfrak{b}^2(t)\log\mathfrak{b}(t)+\mathfrak{b}(t)\mathfrak{a}^2(t)\log\mathfrak{a}(t)\right]\Delta t \\
&-(M-m)\int_a^b f^2(t)g^{2-q}(t)\Delta t\Big)^{p-1}
\end{aligned} \tag{8.30}$$

holds, where

$$\mathfrak{a} = fg^{1-q}-m, \quad \mathfrak{b} = M - fg^{1-q},$$
$$W_1 = M\int_a^b g^q(t)\Delta t - \int_a^b f(t)g(t)\Delta t, \quad W_2 = \int_a^b f(t)g(t)\Delta t - m\int_a^b g^q(t)\Delta t.$$

Theorem 8.20 *Let $\widehat{\mathscr{J}}_{\Psi_s}$ be positive. Then for $r,l,v,w > 0$ such that $r \le v$, $l \le w$, we have*

$$\widehat{\mathfrak{M}}_{r,l}^{[s]}(f) \le \widehat{\mathfrak{M}}_{v,w}^{[s]}(f). \tag{8.31}$$

Remark 8.12 Similarly as in Chapter 2, we can apply the theory of isotonic linear functionals. The related results for isotonic linear functionals are given in [3, 4].

Chapter 9

Hölder and Minkowski Type Inequalities and Functionals

In this chapter, we give integral forms of Minkowski's inequality, a converse Minkowski inequality, and Beckenbach–Dresher's inequality on time scales and investigate the properties concerning superadditivity and monotonicity of several functions arising from the these inequalities. We give refinements of the generalized Popoviciu, Bellman and Diaz–Metcalf inequalities. Further, we give some new integral inequalities by using Popoviciu's inequality and Diaz–Metcalf's inequality. The presentation in this chapter is based on [33, 34, 36].

9.1 Integral Minkowski Inequality and Functionals

Theorem 2.45 also holds if we have a finite number of functions. The next theorem gives an inequality of Minkowski type for infinitely many functions. We assume throughout that all occurring integrals are finite.

Theorem 9.1 (INTEGRAL MINKOWSKI INEQUALITY) *Let $(X,\mathscr{K},\mu_\Delta)$ and $(Y,\mathscr{L},\nu_\Delta)$ be time scale measure spaces and let u, v, and f be nonnegative functions on X, Y, and $X\times Y$, respectively. If $p\geq 1$, then*

$$\left[\int_X\left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{1}{p}} \leq \int_Y\left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y) \quad (9.1)$$

holds provided all integrals in (9.1) *exists. If* $0 < p < 1$ *and*

$$\int_X \left(\int_Y f v \mathrm{d}\nu_\Delta\right)^p u \mathrm{d}\mu_\Delta > 0, \quad \int_Y f v \mathrm{d}\nu_\Delta > 0 \tag{9.2}$$

holds, then (9.1) *is reversed. If* $p < 0$ *and* (9.2) *and*

$$\int_X f^p u \mathrm{d}\mu_\Delta > 0, \tag{9.3}$$

hold, then (9.1) *is reversed as well.*

Proof. Let $p \geq 1$. Put

$$H(x) = \int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y).$$

Now, by using Fubini's theorem (Theorem 1.8) and Hölder's inequality (Theorem 2.32) on time scales, we have

$$\begin{aligned}
&\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x) = \int_X H(x)H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x) \\
&= \int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right) H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x) \\
&= \int_Y \left(\int_X f(x,y)H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x)\right) v(y)\mathrm{d}\nu_\Delta(y) \\
&\leq \int_Y \left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} \left(\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{p-1}{p}} v(y)\mathrm{d}\nu_\Delta(y) \\
&= \int_Y \left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y) \left(\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{p-1}{p}}
\end{aligned}$$

and hence

$$\left(\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} \leq \int_Y \left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y).$$

For $p < 0$ and $0 < p < 1$, the corresponding results can be obtained similarly. □

Remark 9.1 Theorem 9.1 is a generalization of Theorem 2.44 (Minkowski inequality on time scales). Moreover, if $X, Y \subseteq \mathbb{R}^n$, then (9.1) becomes

$$\begin{aligned}
&\left[\int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu(y)\right)^p u(x)\mathrm{d}\mu(x)\right]^{\frac{1}{p}} \\
&\qquad \leq \int_Y \left(\int_X f^p(x,y)u(x)\mathrm{d}\mu(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu(y).
\end{aligned} \tag{9.4}$$

Now we consider some functionals which arise from the Minkowski inequality. Similar results (but not for time scales measure spaces) can be found in [76].

Let f and v be fixed functions satisfying the assumptions of Theorem 9.1. Let us consider the functional M_1 defined by

$$\mathsf{M}_1(u) = \left[\int_Y \left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y)\right]^p - \int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x),$$

where u is a nonnegative function on X such that all occurring integrals exist. Also, if we fix the functions f and u, then we can consider the functional

$$\mathsf{M}_2(v) = \int_Y \left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y) - \left[\int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{1}{p}},$$

where v is a nonnegative function on Y such that all occurring integrals exist.

Remark 9.2 (i) It is obvious that M_1 and M_2 are positive homogeneous, i.e., $\mathsf{M}_1(au) = a\mathsf{M}_1(u)$, and $\mathsf{M}_2(av) = a\mathsf{M}_2(v)$, for any $a > 0$.

(ii) If $p \geq 1$ or $p < 0$, then $\mathsf{M}_1(u) \geq 0$, and if $0 < p < 1$, then $\mathsf{M}_1(u) \leq 0$.

(iii) If $p \geq 1$, then $\mathsf{M}_2(v) \geq 0$, and if $p < 1$ and $p \neq 0$, then $\mathsf{M}_2(v) \leq 0$.

Theorem 9.2 (i) *If $p \geq 1$ or $p < 0$, then M_1 is superadditive. If $0 < p < 1$, then M_1 is subadditive.*

(ii) *If $p \geq 1$, then M_2 is superadditive. If $p < 1$ and $p \neq 0$, then M_2 is subadditive.*

(iii) *Suppose u_1 and u_2 are nonnegative functions such that $u_2 \geq u_1$. If $p \geq 1$ or $p < 0$, then*

$$0 \leq \mathsf{M}_1(u_1) \leq \mathsf{M}_1(u_2), \tag{9.5}$$

and if $0 < p < 1$, then (9.5) *is reversed.*

(iv) *Suppose v_1 and v_2 are nonnegative functions such that $v_2 \geq v_1$. If $p \geq 1$, then*

$$0 \leq \mathsf{M}_2(v_1) \leq \mathsf{M}_2(v_2), \tag{9.6}$$

and if $p < 1$ and $p \neq 0$, then (9.6) *is reversed.*

Proof. First we show (i). We have

$$\begin{aligned}
&\mathsf{M}_1(u_1+u_2)-\mathsf{M}_1(u_1)-\mathsf{M}_1(u_2)\\
&=\left[\int_Y\left(\int_X f^p(x,y)(u_1+u_2)(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}}v(y)\mathrm{d}\nu_\Delta(y)\right]^p\\
&\quad-\int_X\left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p(u_1+u_2)(x)\mathrm{d}\mu_\Delta(x)\\
&\quad-\left[\int_Y\left(\int_X f^p(x,y)u_1(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}}v(y)\mathrm{d}\nu_\Delta(y)\right]^p\\
&\quad+\int_X\left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u_1(x)\mathrm{d}\mu_\Delta(x)\\
&\quad-\left[\int_Y\left(\int_X f^p(x,y)u_2(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}}v(y)\mathrm{d}\nu_\Delta(y)\right]^p\\
&\quad+\int_X\left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u_2(x)\mathrm{d}\mu_\Delta(x)\\
&=\left[\int_Y\left(\int_X f^p(x,y)(u_1+u_2)(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}}v(y)\mathrm{d}\nu_\Delta(y)\right]^p\\
&\quad-\left[\int_Y\left(\int_X f^p(x,y)u_1(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}}v(y)\mathrm{d}\nu_\Delta(y)\right]^p\\
&\quad-\left[\int_Y\left(\int_X f^p(x,y)u_2(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}}v(y)\mathrm{d}\nu_\Delta(y)\right]^p.
\end{aligned}$$

Using the Minkowski inequality (2.68) for integrals (Theorem 2.45) with p replaced by $1/p$, we have

$$\mathsf{M}_1(u_1+u_2)-\mathsf{M}_1(u_1)-\mathsf{M}_1(u_2)\begin{cases}\geq 0 & \text{if} \quad p\geq 1 \text{ or } p<0,\\ \leq 0 & \text{if} \quad 0<p\leq 1.\end{cases} \tag{9.7}$$

So, M_1 is superadditive for $p\geq 1$ or $p<0$, and it is subadditive for $0<p\leq 1$. The proof of (ii) is similar: We have

$$\begin{aligned}
&\mathsf{M}_2(v_1+v_2)-\mathsf{M}_2(v_1)-\mathsf{M}_2(v_2)\\
&=\left[\int_X\left(\int_Y f(x,y)v_1(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{1}{p}}\\
&\quad+\left[\int_X\left(\int_Y f(x,y)v_2(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{1}{p}}\\
&\quad-\left[\int_X\left(\int_Y f(x,y)(v_1+v_2)(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{1}{p}}.
\end{aligned}$$

Using the Minkowski inequality for integrals (Theorem 9.1), we have that this is nonnegative for $p \geq 1$ and nonpositive for $p < 1$ and $p \neq 0$. Now we show (iii). If $p \geq 1$ or $p < 0$, then using superadditivity and positivity of M_1, $u_2 \geq u_1$ implies

$$\mathsf{M}_1(u_2) = \mathsf{M}_1(u_1 + (u_2 - u_1)) \geq \mathsf{M}_1(u_1) + \mathsf{M}_1(u_2 - u_1) \geq \mathsf{M}_1(u_1),$$

and the proof of (9.5) is established. If $0 < p < 1$, then using subadditivity and negativity of M_1, $u_2 \geq u_1$ implies

$$\mathsf{M}_1(u_2) \leq \mathsf{M}_1(u_1) + \mathsf{M}_1(u_2 - u_1) \leq \mathsf{M}_1(u_1).$$

The proof of (iv) is similar. □

Remark 9.3 From Theorem 9.2, we obtain a refinement of the discrete Minkowski inequality given in [76]. Namely, put $X, Y \subseteq \mathbb{N}$ and let u be Δ-measurable on X and v_1 and v_2 be Δ-measurable on Y such that $u(i) = u_i \geq 0$, $i \in X$, $v_1(j) = n_j \geq 0$, $v_2(j) = p_j \geq 0$, $j \in Y$. Then, for fixed f and u, the function M_2 has the form

$$\mathsf{M}_2(v_1) = \sum_{j\in Y} n_j \left(\sum_{i\in X} u_i a_{ij}^p\right)^{1/p} - \left(\sum_{i\in X} u_i \left(\sum_{j\in Y} n_j a_{ij}\right)^p\right)^{1/p},$$

where $f(i,j) = a_{ij} \geq 0$. If $p \geq 1$, then the mapping M_2 is superadditive, and $p_j \geq n_j$ for all $j \in Y$ implies

$$\begin{aligned} 0 &\leq \sum_{j\in Y} n_j \left(\sum_{i\in X} u_i a_{ij}^p\right)^{1/p} - \left(\sum_{i\in X} u_i \left(\sum_{j\in Y} n_j a_{ij}\right)^p\right)^{1/p} \\ &\leq \sum_{j\in Y} p_j \left(\sum_{i\in X} u_i a_{ij}^p\right)^{1/p} - \left(\sum_{i\in X} u_i \left(\sum_{j\in Y} p_j a_{ij}\right)^p\right)^{1/p} \end{aligned}$$

provided all occurring sums are finite.

Corollary 9.1 (i) *Suppose u_1 and u_2 are nonnegative functions such that $Cu_2 \geq u_1 \geq cu_2$, where $c, C \geq 0$. If $p \geq 1$ or $p < 0$, then*

$$c\mathsf{M}_1(u_2) \leq \mathsf{M}_1(u_1) \leq C\mathsf{M}_1(u_2),$$

and if $0 < p < 1$, then the above inequality is reversed.

(ii) *Suppose v_1 and v_2 are nonnegative functions such that $Cv_2 \geq v_1 \geq cv_2$, where $c, C \geq 0$. If $p \geq 1$, then*

$$c\mathsf{M}_2(v_2) \leq \mathsf{M}_2(v_1) \leq C\mathsf{M}_2(v_2),$$

and if $p < 1$ and $p \neq 0$, then the above inequality is reversed.

Let the functions f,u,v be defined as in Theorem 9.5. Now we define the rth power mean $\overline{\mathscr{M}}_{\Delta}^{[r]}(f,u)$ of the function f with weight function u and measure μ_Δ by

$$\overline{\mathscr{M}}_{\Delta}^{[r]}(f,u)=\begin{cases}\left(\frac{\int_X f^r(x,y)u(x)\mathrm{d}\mu_\Delta(x)}{\int_X u(x)\mathrm{d}\mu_\Delta(x)}\right)^{\frac{1}{r}} & \text{if} \quad r\neq 0,\\ \exp\left(\frac{\int_X \log f(x,y)u(x)\mathrm{d}\mu_\Delta(x)}{\int_X u(x)\mathrm{d}\mu_\Delta(x)}\right) & \text{if} \quad r=0,\end{cases} \tag{9.8}$$

where $\int_X u\mathrm{d}\mu_\Delta>0$.

Corollary 9.2 *If v_1 and v_2 are nonnegative functions such that $v_2\geq v_1$, then*

$$\begin{aligned}\overline{\mathscr{M}}_{\Delta}^{[0]}\left(\int_Y f(x,y)v_1(y)\mathrm{d}\nu_\Delta(y),u\right)-\int_Y \overline{\mathscr{M}}_{\Delta}^{[0]}(f,u)v_1(y)\mathrm{d}\nu_\Delta(y)\\ \leq \overline{\mathscr{M}}_{\Delta}^{[0]}\left(\int_Y f(x,y)v_2(y)\mathrm{d}\nu_\Delta(y),u\right)-\int_Y \overline{\mathscr{M}}_{\Delta}^{[0]}(f,u)v_2(y)\mathrm{d}\nu_\Delta(y),\end{aligned} \tag{9.9}$$

where $\overline{\mathscr{M}}_{\Delta}^{[0]}(f,u)$ is defined in (9.8)*.*

Remark 9.4 If the measures are discrete, then from Corollary 9.2, we get the following result: Let $u_j,v_i,w_i,a_{ij}>0$ for all $i\in\{1,\ldots,n\}$ and all $j\in\{1,\ldots,k\}$. Put $U=\sum_{j=1}^{k}u_j$. If $v_i\leq w_i$ for all $i\in\{1,\ldots,n\}$, then

$$\begin{aligned}\prod_{j=1}^{k}\left(\sum_{i=1}^{n}v_ia_{ij}\right)^{\frac{u_j}{U}}-\sum_{i=1}^{n}v_i\left(\prod_{j=1}^{k}a_{ij}^{\frac{u_j}{U}}\right)\\ \leq \prod_{j=1}^{k}\left(\sum_{i=1}^{n}w_ia_{ij}\right)^{\frac{u_j}{U}}-\sum_{i=1}^{n}w_i\left(\prod_{j=1}^{k}a_{ij}^{\frac{u_j}{U}}\right).\end{aligned}$$

This inequality is a refinement of the discrete Hölder inequality

$$\prod_{j=1}^{k}\left(\sum_{i=1}^{n}w_ia_{ij}\right)^{\frac{u_j}{U}}\geq\sum_{i=1}^{n}w_i\left(\prod_{j=1}^{k}a_{ij}^{\frac{u_j}{U}}\right).$$

The next result gives another property of M_1, but a similar result can also be stated for M_2.

Theorem 9.3 *Let $\varphi:[0,\infty)\to[0,\infty)$ be a concave function. Suppose u_1 and u_2 are nonnegative functions such that*

$$\varphi\circ u_1, \quad \varphi\circ u_2, \quad \varphi\circ(\alpha u_1+(1-\alpha)u_2)$$

are Δ-integrable for $\alpha\in[0,1]$. If $p\geq 1$, then

$$\mathsf{M}_1(\varphi\circ(\alpha u_1+(1-\alpha)u_2))\geq\alpha\mathsf{M}_1(\varphi\circ u_1)+(1-\alpha)\mathsf{M}_1(\varphi\circ u_2),$$

and if $0<p<1$, then the above inequality is reversed.

Proof. We show this only for $p \geq 1$ as the other case follows similarly. Since φ is concave, we have

$$\varphi(\alpha u_1 + (1-\alpha)u_2)) \geq \alpha\varphi(u_1) + (1-\alpha)\varphi(u_2).$$

Now, from (9.5) and (9.7), we have

$$\begin{aligned} \mathsf{M}_1(\varphi\circ(\alpha u_1 + (1-\alpha)u_2)) &\geq \mathsf{M}_1(\alpha(\varphi\circ u_1) + (1-\alpha)(\varphi\circ u_2)) \\ &\geq \mathsf{M}_1(\alpha(\varphi\circ u_1)) + \mathsf{M}_1((1-\alpha)(\varphi\circ u_2)) \\ &\geq \alpha\mathsf{M}_1(\varphi\circ u_1) + (1-\alpha)\mathsf{M}_1(\varphi\circ u_2), \end{aligned}$$

and the proof is established. □

Let f, u, and v be fixed functions satisfying the assumptions of Theorem 9.1. Let us define functionals M_3 and M_4 by

$$\begin{aligned} \mathsf{M}_3(A) = {}& \left[\int_Y \left(\int_A f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y)\right]^p \\ &- \int_A \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x) \end{aligned}$$

and

$$\begin{aligned} \mathsf{M}_4(B) = {}& \int_B \left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y) \\ &- \left[\int_X \left(\int_B f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{1}{p}}, \end{aligned}$$

where $A \subseteq X$ and $B \subseteq Y$.

The following theorem establishes superadditivity and monotonicity of the mappings M_3 and M_4.

Theorem 9.4 (i) *Suppose $A_1, A_2 \subseteq X$ and $A_1 \cap A_2 = \emptyset$. If $p \geq 1$ or $p < 0$, then*

$$\mathsf{M}_3(A_1 \cup A_2) \geq \mathsf{M}_3(A_1) + \mathsf{M}_3(A_2),$$

and if $0 < p < 1$, then the above inequality is reversed.

(ii) *Suppose $A_1, A_2 \subseteq X$ and $A_1 \subseteq A_2$. If $p \geq 1$ or $p < 0$, then*

$$\mathsf{M}_3(A_1) \leq \mathsf{M}_3(A_2),$$

and if $0 < p < 1$, then the above inequality is reversed.

(iii) *Suppose $B_1, B_2 \subseteq Y$ and $B_1 \cap B_2 = \emptyset$. If $p \geq 1$, then*

$$\mathsf{M}_4(B_1 \cup B_2) \geq \mathsf{M}_4(B_1) + \mathsf{M}_4(B_2),$$

and if $p < 1$ and $p \neq 0$, then the above inequality is reversed.

(iv) *Suppose* $B_1, B_2 \subseteq Y$ *and* $B_1 \subseteq B_2$*. If* $p \geq 1$*, then*

$$\mathsf{M}_4(B_1) \leq \mathsf{M}_4(B_2),$$

and if $p < 1$ *and* $p \neq 0$*, then the above inequality is reversed.*

The proof of Theorem 9.4 is omitted as it is similar to the proof of Theorem 9.2.

Remark 9.5 For $p \geq 1$, if S_m is a subset of Y with m elements and if $S_m \supseteq S_{m-1} \supseteq \ldots \supseteq S_2$, then we have

$$\mathsf{M}_4(S_m) \geq \mathsf{M}_4(S_{m-1}) \geq \ldots \geq \mathsf{M}_4(S_2) \geq 0$$

and $\mathsf{M}_4(S_m) \geq \max\{\mathsf{M}_4(S_2) : S_2$ is any subset of S_m with 2 elements$\}$.

9.2 Converse Integral Minkowski Inequality and Functionals

In the following theorem, we give a converse of Theorem 9.1 (integral Minkowski inequality).

Theorem 9.5 (Converse of integral Minkowski inequality) *Let* $(X, \mathscr{K}, \mu_\Delta)$ *and* $(Y, \mathscr{L}, \nu_\Delta)$ *be time scale measure spaces and let* u, v*, and* f *be nonnegative functions on* X, Y*, and* $X \times Y$*, respectively. Suppose*

$$0 < m \leq \frac{f(x,y)}{\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)} \leq M \quad \textit{for all} \quad x \in X,\ y \in Y.$$

If $p \geq 1$*, then*

$$\left[\int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{1}{p}}$$
$$\geq K(p,m,M)\int_Y \left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y) \quad (9.10)$$

provided all integrals in (9.10) *exist, where* $K(p,m,M)$ *is defined by* (2.62)*. If* $0 < p < 1$ *and* (9.2) *holds, then* (9.10) *is reversed. If* $p < 0$ *and* (9.2) *and* (9.3) *hold, then* (9.10) *is reversed as well.*

Proof. Let $p \geq 1$. Put

$$H(x) = \int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y).$$

Then by using Fubini's theorem (Theorem 1.8) and the converse Hölder inequality (Theorem 2.55) on time scales, we get

$$\begin{aligned}\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x) &= \int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right) H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x)\\ &= \int_Y \left(\int_X f(x,y)H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x)\right) v(y)\mathrm{d}\nu_\Delta(y)\\ &\geq K(p,m,M)\int_Y \left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{1/p}\\ &\quad\times\left(\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{p-1}{p}} v(y)\mathrm{d}\nu_\Delta(y).\end{aligned}$$

Dividing both sides by $(\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x))^{\frac{p-1}{p}}$, we obtain (9.10). For $0<p<1$ and $p<0$, the corresponding results can be obtained similarly. □

Corollary 9.3 *Let* $0<s\leq r$. *Then*

$$\overline{\mathscr{M}}_\Delta^{[r]}(\overline{\mathscr{M}}_\Delta^{[s]}(f,v),u) \geq K\left(\frac{r}{s},m,M\right)\overline{\mathscr{M}}_\Delta^{[s]}(\overline{\mathscr{M}}_\Delta^{[r]}(f,u),v).$$

Proof. By putting $p=r/s$ and replacing f by f^s in (9.10), raising to the power of $\frac{1}{s}$ and dividing by

$$\left(\int_X u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{r}}\left(\int_Y v(y)\mathrm{d}\nu_\Delta(y)\right)^{\frac{1}{s}},$$

we get the result. □

Let f and v be fixed functions satisfying the assumptions of Theorem 9.5. Let us consider the functional CM_1 defined by

$$\begin{aligned}\mathsf{CM}_1(u) &= \int_X\left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\\ &\quad - K^p(p,m,M)\left[\int_Y\left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y)\right]^p,\end{aligned}$$

where u is a nonnegative function on X such that all occurring integrals exist. Also, if we fix the functions f and u, then we can consider the functional

$$\begin{aligned}\mathsf{CM}_2(v) &= \left[\int_X\left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{1}{p}}\\ &\quad - K(p,m,M)\int_Y\left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y),\end{aligned}$$

where v is a nonnegative function on Y such that all occurring integrals exist.

Remark 9.6 (i) It is obvious that CM_1 and CM_2 are positive homogeneous, i.e., $\mathsf{CM}_1(au) = a\mathsf{CM}_1(u)$, and $\mathsf{CM}_2(av)=a\mathsf{CM}_2(v)$, for any $a>0$.

(ii) If $p \geq 1$ or $p < 0$, then $\mathsf{CM}_1(u) \geq 0$, and if $0 < p < 1$, then $\mathsf{CM}_1(u) \leq 0$.

(iii) If $p \geq 1$, then $\mathsf{CM}_2(v) \geq 0$, and if $p < 1$ and $p \neq 0$, then $\mathsf{CM}_2(v) \leq 0$.

Theorem 9.6 (i) *If $p \geq 1$ or $p < 0$, then CM_1 is subadditive. If $0 < p < 1$, then CM_1 is superadditive.*

(ii) *If $p \geq 1$, then CM_2 is subadditive. If $p < 1$ and $p \neq 0$, then CM_2 is superadditive.*

Proof. First we show (i). We have

$$\begin{aligned}
&\mathsf{CM}_1(u_1+u_2) - \mathsf{CM}_1 u_1 - \mathsf{CM}_1(u_2) \\
&= \int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p (u_1+u_2)(x)\mathrm{d}\mu_\Delta(x) \\
&\quad -K^p(p,m,M)\left[\int_Y \left(\int_X f^p(x,y)(u_1+u_2)(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y)\right]^p \\
&\quad -\int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u_1(x)\mathrm{d}\mu_\Delta(x) \\
&\quad +K^p(p,m,M)\left[\int_Y \left(\int_X f^p(x,y)u_1(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y)\right]^p \\
&\quad -\int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u_2(x)\mathrm{d}\mu_\Delta(x) \\
&\quad +K^p(p,m,M)\left[\int_Y \left(\int_X f^p(x,y)u_2(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y)\right]^p \\
&= -K^p(p,m,M)\left(\left[\int_Y \left(\int_X f^p(x,y)(u_1+u_2)(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y)\right]^p\right. \\
&\quad -\left[\int_Y \left(\int_X f^p(x,y)u_1(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y)\right]^p \\
&\quad \left.-\left[\int_Y \left(\int_X f^p(x,y)u_2(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y)\right]^p\right).
\end{aligned}$$

Using the Minkowski inequality for integrals (Theorem 9.1) with p replaced by $1/p$, we have

$$\mathsf{CM}_1(u_1+u_2) - \mathsf{CM}_1(u_1) - \mathsf{CM}_1(u_2) \begin{cases} \leq 0 & \text{if} \quad p \geq 1 \text{ or } p < 0, \\ \geq 0 & \text{if} \quad 0 < p \leq 1. \end{cases}$$

So, CM_1 is subadditive for $p \geq 1$ or $p < 0$, and it is superadditive for $0 < p \leq 1$. The proof of (ii) is similar: We have

$$\mathsf{CM}_2(v_1+v_2) - \mathsf{CM}_2(v_1) - \mathsf{CM}_2(v_2)$$

$$= \left[\int_X \left(\int_Y f(x,y) v_1(y) \mathrm{d}\nu_\Delta(y) \right)^p u(x) \mathrm{d}\mu_\Delta(x) \right]^{\frac{1}{p}}$$
$$+ \left[\int_X \left(\int_Y f(x,y) v_2(y) \mathrm{d}\nu_\Delta(y) \right)^p u(x) \mathrm{d}\mu_\Delta(x) \right]^{\frac{1}{p}}$$
$$- \left[\int_X \left(\int_Y f(x,y) (v_1+v_2)(y) \mathrm{d}\nu_\Delta(y) \right)^p u(x) \mathrm{d}\mu_\Delta(x) \right]^{\frac{1}{p}}.$$

Using the Minkowski inequality (9.1) for integrals (Theorem 9.1), we have that this is nonpositive for $p \geq 1$ and nonnegative for $p < 1$ and $p \neq 0$. □

Let f, u, and v be fixed functions satisfying the assumptions of Theorem 9.5. Let us define functionals CM_3 and CM_4 by

$$\mathsf{CM}_3(A) = \int_A \left(\int_Y f(x,y) v(y) \mathrm{d}\nu_\Delta(y) \right)^p u(x) \mathrm{d}\mu_\Delta(x) - K^p(p,m,M) \left[\int_Y \left(\int_A f^p(x,y) u(x) \mathrm{d}\mu_\Delta(x) \right)^{\frac{1}{p}} v(y) \mathrm{d}\nu_\Delta(y) \right]^p,$$

and

$$\mathsf{CM}_4(B) = \left[\int_X \left(\int_B f(x,y) v(y) \mathrm{d}\nu_\Delta(y) \right)^p u(x) \mathrm{d}\mu_\Delta(x) \right]^{\frac{1}{p}} - K(p,m,M) \int_B \left(\int_X f^p(x,y) u(x) \mathrm{d}\mu_\Delta(x) \right)^{\frac{1}{p}} v(y) \mathrm{d}\nu_\Delta(y).$$

where $A \subseteq X$ and $B \subseteq Y$.

The following theorem establishes superadditivity of the mappings CM_3 and CM_4.

Theorem 9.7 (i) *Suppose $A_1, A_2 \subseteq X$ and $A_1 \cap A_2 = \emptyset$. If $p \geq 1$ or $p < 0$, then*

$$\mathsf{CM}_3(A_1 \cup A_2) \leq \mathsf{CM}_3(A_1) + \mathsf{CM}_3(A_2),$$

and if $0 < p < 1$, then the above inequality is reversed.

(ii) *Suppose $B_1, B_2 \subseteq Y$ and $B_1 \cap B_2 = \emptyset$. If $p \geq 1$, then*

$$\mathsf{CM}_4(B_1 \cup B_2) \leq \mathsf{CM}_4(B_1) + \mathsf{CM}_4(B_2),$$

and if $p < 1,\ p \neq 0$, then the above inequality is reversed.

The proof of Theorem 9.4 is omitted as it is similar to the proof of Theorem 9.2.

9.3 Beckenbach–Dresher Inequality and Functionals

Theorem 9.8 *Let $(X,\mathscr{K},\mu_\Delta)$, $(X,\mathscr{K},\lambda_\Delta)$, and $(Y,\mathscr{L},\nu_\Delta)$ be time scale measure spaces. Suppose u and w are nonnegative functions on X, v is a nonnegative function on Y, f is a nonnegative function on $X\times Y$ with respect to the measure $(\mu_\Delta\times\nu_\Delta)$, and g is a nonnegative function on $X\times Y$ with respect to the measure $(\lambda_\Delta\times\nu_\Delta)$. If*

$$s\geq 1,\quad q\leq 1\leq p,\quad \text{and}\quad q\neq 0 \tag{9.11}$$

or

$$s<0,\quad p\leq 1\leq q,\quad \text{and}\quad p\neq 0, \tag{9.12}$$

then

$$\frac{\left[\int_X\left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{s}{p}}}{\left[\int_X\left(\int_Y g(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^q w(x)\mathrm{d}\lambda_\Delta(x)\right]^{\frac{s-1}{q}}} \leq \int_Y \frac{\left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{s}{p}}}{\left(\int_X g^q(x,y)w(x)\mathrm{d}\lambda_\Delta(x)\right)^{\frac{s-1}{q}}} v(y)\mathrm{d}\nu_\Delta(y) \tag{9.13}$$

provided all occurring integrals in (9.13) *exist. If*

$$0<s\leq 1,\quad p\leq 1,\quad q\leq 1,\quad \text{and}\quad p,q\neq 0, \tag{9.14}$$

then (9.13) *is reversed.*

Proof. Assume (9.11) or (9.12). By using the integral Minkowski inequality (9.1) and Hölder's inequality (2.65), we have

$$\begin{aligned}
&\frac{\left[\int_X\left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{s}{p}}}{\left[\int_X\left(\int_Y g(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^q w(x)\mathrm{d}\lambda_\Delta(x)\right]^{\frac{s-1}{q}}} \\
&\leq \frac{\left[\int_Y\left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y)\right]^s}{\left[\int_Y\left(\int_X g^q(x,y)w(x)\mathrm{d}\lambda_\Delta(x)\right)^{\frac{1}{q}} v(y)\mathrm{d}\nu_\Delta(y)\right]^{s-1}} \\
&= \left[\int_Y\left(\left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{s}{p}}\right)^{\frac{1}{s}} v(y)\mathrm{d}\nu_\Delta(y)\right]^s \\
&\quad\times\left[\int_Y\left(\left(\int_X g^q(x,y)w(x)\mathrm{d}\lambda_\Delta(x)\right)^{\frac{1-s}{q}}\right)^{\frac{1}{1-s}} v(y)\mathrm{d}\nu_\Delta(y)\right]^{1-s}
\end{aligned}$$

$$\leq \int_Y \left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{s}{p}} \left(\int_X g^q(x,y)w(x)\mathrm{d}\lambda_\Delta(x)\right)^{\frac{1-s}{q}} v(y)\mathrm{d}\nu_\Delta(y).$$

If (9.14) holds, then the reversed inequality in (9.13) can be proved in a similar way. □

Remark 9.7 Theorem 9.8 is a generalization of Theorem 2.47 (Beckenbach–Dresher inequality on time scales).

Let f, g, u, w be fixed functions satisfying the assumptions of Theorem 9.8. We define the Beckenbach–Dresher functional $\mathsf{BD}(v)$ by

$$\mathsf{BD}(v) = \int_Y \frac{\left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{s}{p}}}{\left(\int_X g^q(x,y)w(x)\mathrm{d}\lambda_\Delta(x)\right)^{\frac{s-1}{q}}} v(y)\mathrm{d}\nu_\Delta(y) - \frac{\left[\int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{s}{p}}}{\left[\int_X \left(\int_Y g(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^q w(x)\mathrm{d}\lambda_\Delta(x)\right]^{\frac{s-1}{q}}},$$

where we suppose that all occurring integrals exist.

Theorem 9.9 *If* (9.11) *or* (9.12) *holds, then*

$$\mathsf{BD}(v_1+v_2) \geq \mathsf{BD}(v_1)+\mathsf{BD}(v_2). \tag{9.15}$$

If $v_2 \geq v_1$, then

$$\mathsf{BD}(v_1) \leq \mathsf{BD}(v_2). \tag{9.16}$$

If $C, c \geq 0$ and $Cv_2 \geq v_1 \geq cv_2$, then

$$C\mathsf{BD}(v_2) \geq \mathsf{BD}(v_1) \geq c\mathsf{BD}(v_1). \tag{9.17}$$

If (9.14) *holds, then* (9.15), (9.16), *and* (9.17) *are reversed.*

Proof. Assume (9.11) or (9.12). Then we have

$$\begin{aligned}
&\mathsf{BD}(v_1+v_2) - \mathsf{BD}(v_1) - \mathsf{BD}(v_2) \\
&= \frac{\left[\int_X \left(\int_Y f(x,y)v_1(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{s}{p}}}{\left[\int_X \left(\int_Y g(x,y)v_1(y)\mathrm{d}\nu_\Delta(y)\right)^q w(x)\mathrm{d}\lambda_\Delta(x)\right]^{\frac{s-1}{q}}} \\
&\quad + \frac{\left[\int_X \left(\int_Y f(x,y)v_2(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{s}{p}}}{\left[\int_X \left(\int_Y g(x,y)v_2(y)\mathrm{d}\nu_\Delta(y)\right)^q w(x)\mathrm{d}\lambda_\Delta(x)\right]^{\frac{s-1}{q}}} \\
&\quad - \frac{\left[\int_X \left(\int_Y f(x,y)v_1(y)\mathrm{d}\nu_\Delta(y) + \int_Y f(x,y)v_2(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{s}{p}}}{\left[\int_X \left(\int_Y g(x,y)v_1(y)\mathrm{d}\nu_\Delta(y) + \int_Y g(x,y)v_2(y)\mathrm{d}\nu_\Delta(y)\right)^q w(x)\mathrm{d}\lambda_\Delta(x)\right]^{\frac{s-1}{q}}} \\
&\geq 0,
\end{aligned}$$

where in the last inequality we used (9.13) from Theorem 9.8. Using Theorem 9.8 again, $v_2 \geq v_1$ implies

$$\mathsf{BD}(v_2) = \mathsf{BD}(v_1 + (v_2 - v_1)) \geq \mathsf{BD}(v_1) + \mathsf{BD}(v_2 - v_1) \geq \mathsf{BD}(v_1).$$

The proof of (9.17) is similar. If (9.14) holds, then the reversed inequalities of (9.15), (9.16), and (9.17) can be proved in a similar way. □

Let f, g, u, v, w be fixed functions. We define a functional BD_1 by

$$\mathsf{BD}_1(A) = \int_A \frac{\left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{s}{p}}}{\left(\int_X g^q(x,y)w(x)\mathrm{d}\lambda_\Delta(x)\right)^{\frac{s-1}{q}}} v(y)\mathrm{d}\nu_\Delta(y) - \frac{\left[\int_X \left(\int_A f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{s}{p}}}{\left[\int_X \left(\int_A g(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^q w(x)\mathrm{d}\lambda_\Delta(x)\right]^{\frac{s-1}{q}}},$$

where $A \subseteq Y$.

For BD_1, the following result holds.

Theorem 9.10 (i) *Suppose* $A_1, A_2 \subseteq Y$ *and* $A_1 \cap A_2 = \emptyset$. *If* (9.11) *or* (9.12) *holds, then*

$$\mathsf{BD}_1(A_1 \cup A_2) \geq \mathsf{BD}_1(A_1) + \mathsf{BD}_1(A_2), \tag{9.18}$$

and if (9.14) *holds, then the inequality in* (9.18) *is reversed.*

(ii) *Suppose* $A_1, A_2 \subseteq Y$ *and* $A_1 \subseteq A_2$. *If* (9.11) *or* (9.12) *holds, then*

$$\mathsf{BD}_1(A_1) \leq \mathsf{BD}_1(A_2), \tag{9.19}$$

and if (9.14) *holds, then the inequality in* (9.19) *is reversed.*

The proof of Theorem 9.10 is omitted as it is similar to the proof of Theorem 9.9.

Remark 9.8 If $S_k \subseteq Y$ has k elements and if $S_m \supseteq S_{m-1} \supseteq \ldots \supseteq S_2$, then (9.11) or (9.12) implies

$$\mathsf{BD}_1(S_m) \geq \mathsf{BD}_1(S_{m-1}) \geq \ldots \geq \mathsf{BD}_1(S_2) \geq 0$$

and $\mathsf{BD}_1(S_m) \geq \max\{\mathsf{BD}_1(S_2) : S_2$ is any subset of S_m with 2 elements$\}$, while (9.14) implies the reversed inequalities with max replaced by min.

9.4 Refinement of the Integral Minkowski Inequality

In the following theorems, we recall Theorem 2.64 and Theorem 2.68 (Hölder and Minkowski inequalities) for multiple Lebesgue Δ-integrals.

Theorem 9.11 *For $p \neq 1$, define q by $\frac{1}{p}+\frac{1}{q}=1$. Let $(X, \mathscr{M}, \mu_\Delta)$ be a time scale measure space. Assume w, g, h are nonnegative functions such that wg^p, wh^q, wgh are Δ-integrable on X and $\int_X wh^q \mathrm{d}\mu_\Delta > 0$. If $p \geq 2$, then*

$$\begin{aligned}
&\int_X (wgh)(t)\mathrm{d}\mu_\Delta(t) \qquad (9.20)\\
&\leq \left[\int_X w(t)g^p(t)\mathrm{d}\mu_\Delta(t)\right.\\
&\left.\quad - \int_X w(s)\left(\left|g(s)-h^{q-1}(s)\frac{\int_X (wgh)(t)\mathrm{d}\mu_\Delta(t)}{\int_X w(t)h^q(t)\mathrm{d}\mu_\Delta(t)}\right|^p\right)\mathrm{d}\mu_\Delta(s)\right]^{\frac{1}{p}}\\
&\quad \left(\int_X w(t)h^q(t)\mathrm{d}\mu_\Delta(t)\right)^{\frac{1}{q}}
\end{aligned}$$

holds. In the case $0 < p < 2$ and $p \neq 1$, the inequality in (9.20) holds in reverse order.

Proof. The inequality (9.20) follows from Theorem 2.69 and Theorem 2.5. □

Remark 9.9 Note that for $p \geq 2$, Theorem 9.11 is a refinement of Theorem 2.32, and for $1 < p < 2$, we have

$$\begin{aligned}
&\left[\int_X w(t)g^p(t)\mathrm{d}\mu_\Delta(t)\right.\\
&\left.\quad - \int_X w(s)\left(\left|g(s)-h^{q-1}(s)\frac{\int_X (wgh)(t)\mathrm{d}\mu_\Delta(t)}{\int_X w(t)h^q(t)\mathrm{d}\mu_\Delta(t)}\right|^p\right)\mathrm{d}\mu_\Delta(s)\right]^{\frac{1}{p}}\\
&\left(\int_X w(t)h^q(t)\mathrm{d}\mu_\Delta(t)\right)^{\frac{1}{q}} \leq \int_X (wgh)(t)\mathrm{d}\mu_\Delta(t)\\
&\leq \left(\int_X w(t)g^p(t)\mathrm{d}\mu_\Delta(t)\right)^{1/p}\left(\int_X w(t)h^q(t)\mathrm{d}\mu_\Delta(t)\right)^{1/q}.
\end{aligned}$$

Theorem 9.12 *Let $(X, \mathscr{M}, \mu_\Delta)$ be a time scale measure space. For $p \in \mathbb{R}$, assume w, g, h are nonnegative functions such that $wg^p, wh^p, w(g+h)^p$ are Δ-integrable on X and $\int_X w(g+h)^p \mathrm{d}\mu_\Delta > 0$. If $p \geq 2$, then*

$$\begin{aligned}
&\left(\int_X w(t)\,(g(t)+h(t))^p\,\mathrm{d}\mu_\Delta(t)\right)^{\frac{1}{p}} \qquad (9.21)\\
&\leq \left(\int_X w(t)g^p(t)\mathrm{d}\mu_\Delta(t) - \int_X w(t)\,\Bigg|g(t)\right.\\
&\qquad\left. -\,(g(t)+h(t))\,\frac{\int_X w(s)g(s)\,(g(s)+h(s))^{p-1}\,\mathrm{d}\mu_\Delta(s)}{\int_X w(s)(g(s)+h(s))^p\mathrm{d}\mu_\Delta(s)}\Bigg|^p \mathrm{d}\mu_\Delta(t)\right)^{\frac{1}{p}}\\
&+\left(\int_X w(t)h^p(t)\mathrm{d}\mu_\Delta(t) - \int_X w(t)\,\Bigg|h(t)\right.\\
&\qquad\left. -(g(t)+h(t))\frac{\int_X w(s)h(s)(g(s)+h(s))^{p-1}\mathrm{d}\mu_\Delta(s)}{\int_X w(s)(g(s)+h(s))^p\mathrm{d}\mu_\Delta(s)}\Bigg|^p \mathrm{d}\mu_\Delta(t)\right)^{\frac{1}{p}}
\end{aligned}$$

is valid.

Proof. The inequality (9.21) follows from Theorem 2.67 and Theorem 2.5. □

Remark 9.10 In Theorem 9.12, if we take $0 < p < 2$ and $p \neq 1$, then the inequality in (9.21) holds in reverse order.

Theorem 9.12 also holds if we take a finite number of functions, as given in the following corollary.

Corollary 9.4 *Let $(X, \mathscr{M}, \mu_\Delta)$ be a time scale measure space. For $p \in \mathbb{R}$, assume w, g_i, $i \in \{1,\ldots,n\}$, are nonnegative functions such that $wg_i^p, w\left(\sum_{i=1}^n g_i\right)^p$ are Δ-integrable on X and $\int_X w\left(\sum_{i=1}^n g_i\right)^p \mathrm{d}\mu_\Delta > 0$. If $p \geq 2$, then*

$$\begin{aligned}
&\left(\int_X w(t)\left(\sum_{i=1}^n g_i(t)\right)^p \mathrm{d}\mu_\Delta(t)\right)^{\frac{1}{p}} \qquad (9.22)\\
&\leq \sum_{i=1}^n \left(\int_X w(t)g_i^p(t)\mathrm{d}\mu_\Delta(t) - \int_X w(t)\,\Bigg|g_i(t)\right.\\
&\qquad\left. -\left(\sum_{i=1}^n g_i(t)\right)\frac{\int_X w(s)g_i(s)\left(\sum_{i=1}^n g_i(s)\right)^{p-1}\mathrm{d}\mu_\Delta(s)}{\int_X w(s)\left(\sum_{i=1}^n g_i(s)\right)^p \mathrm{d}\mu_\Delta(s)}\Bigg|^p \mathrm{d}\mu_\Delta(t)\right)^{\frac{1}{p}}
\end{aligned}$$

is valid. If $0 < p < 2$ and $p \neq 1$, then the inequality in (9.22) holds in reverse order.

In the following theorem, we give a refinement of Theorem 9.1 and a generalization of Corollary 9.4.

Theorem 9.13 *Let $(X,\mathscr{M},\mu_\Delta)$ and $(Y,\mathscr{L},\nu_\Delta)$ be time scale measure spaces and let u,v, and f be nonnegative functions on X,Y, and $X\times Y$, respectively. If $p\geq 2$, then*

$$\left(\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} \tag{9.23}$$
$$\leq \int_Y \left[\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x) - \int_X \left| f(x,y) \right.\right.$$
$$\left.\left. -H(x)\frac{\int_X f(x,y)H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x)}{\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x)}\right|^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y)$$

holds provided that all integrals in (9.23) *exist, where*

$$H(x) = \int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y).$$

If $0<p<2$ and $p\neq 1$, then the inequality in (9.23) *holds in reverse order.*

Proof. Let $p\geq 2$. By using Fubini's theorem (Theorem 1.8) and Theorem 9.11, we obtain

$$\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x) = \int_X H(x)H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x)$$
$$= \int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right) H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x)$$
$$= \int_Y \left(\int_X f(x,y)H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x)\right) v(y)\mathrm{d}\nu_\Delta(y)$$
$$\leq \int_Y \left[\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right.$$
$$\left. - \int_X \left| f(x,y) - H(x)\frac{\int_X f(x,y)H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x)}{\int_X H^p(x)u(x)}\right|^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{1}{p}}$$
$$\left(\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{p-1}{p}} v(y)\mathrm{d}\nu_\Delta(y)$$
$$= \int_Y \left[\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right.$$
$$\left. - \int_X \left| f(x,y) - H(x)\frac{\int_X f(x,y)H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x)}{\int_X H^p(x)u(x)}\right|^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y)$$
$$\left(\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{p-1}{p}},$$

and hence

$$\begin{aligned}
&\left(\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} \\
&\leq \int_Y \left[\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right. \\
&\quad \left. - \int_X \left| f(x,y) - H(x)\frac{\int_X f(x,y)H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x)}{\int_X H^p(x)u(x)}\right|^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{1}{p}} \\
&\quad v(y)\mathrm{d}\nu_\Delta(y).
\end{aligned}$$

For $0 < p < 2$ and $p \neq 1$, the reverse inequality in (9.23) can be obtained similarly. □

9.5 Refinements of the Converse Hölder and Minkowski Inequalities

In [121], the following refinements of the converse Hölder and and Minkowski inequalities are given for isotonic linear functionals.

Theorem 9.14 *Let A be a linear positive functional on a linear class L. Let $p \in \mathbb{R}$, $q = p/(p-1)$ and $w, f, g \geq 0$ on E with $wg^p, wh^q, wgh \in L$. Let m, M be such that $0 < m \leq g(t)h^{-q/p}(t) \leq M$ for all $t \in E$. If $p > 1$, then*

$$\begin{aligned}
A(wgh) &\geq K(p,m,M)A^{\frac{1}{p}}(wg^p)A^{\frac{1}{q}}(wh^q) + \Omega(g^q, fg)N(p,m,M) \qquad (9.24) \\
&\geq K(p,m,M)A^{\frac{1}{p}}(wg^p)A^{\frac{1}{q}}(wh^q),
\end{aligned}$$

where K is defined as in (2.62),

$$N(p,m,M) = \frac{m^p + M^p - 2\left((m+M)/2\right)^p}{M^p - m^p} \qquad (9.25)$$

and

$$\Omega(h^q, gh) = A\left(w\left(\frac{M-m}{2}h^q(t) - \left|g(t)h(t) - \frac{m+M}{2}h^q(t)\right|\right)\right). \qquad (9.26)$$

If $0 < p < 1$ and $A(wh^q) > 0$, or $p < 0$ and $A(wg^p) > 0$, then the inequalities in (9.24) hold in reverse order.

Theorem 9.15 *Let $A, p, q, K, N, w, g, h, \Omega$ be as in Theorem 9.14 with additional property $w(g+h)^p \in L$. Let m and M be such that $0 < m < g(t)(g(t)+h(t))^{-1} \leq M$ and $0 < m < h(t)(g(t)+h(t))^{-1} \leq M$ for $t \in E$. Then for $p > 1$,*

$$A^{\frac{1}{p}}(w(g+h)^p) \geq K(p,m,M)\left(A^{\frac{1}{p}}(wg^p)+A^{\frac{1}{p}}(wh^p)\right)+N(p,m,M)$$
$$\frac{\Omega((f+g)^p, f(f+g)^{p-1})+\Omega((f+g)^p, g(f+g)^{p-1})}{A^{1-\frac{1}{p}}(w(g+h)^p)}, \quad (9.27)$$

and for $p<1$ $(p \neq 0)$, *the reversed inequality holds.*

From the above theorems follow the refinements of the converse Hölder and Minkowski inequalities on time scales.

Theorem 9.16 *For* $p \neq 1$, *define* $q = p/(p-1)$. *Let* $(X, \mathscr{M}, \mu_\Delta)$ *be a time scale measure space. Assume* w, g, h *are nonnegative functions such that* wg^p, wh^q, wgh *are* Δ*-integrable on* X. *Let*

$$0 < m \leq g(t)h^{-q/p}(t) \leq M \quad \text{for all} \quad t \in X.$$

If $p>1$, *then*

$$\begin{aligned} &\int_X w(t)g(t)h(t)\mathrm{d}\mu_\Delta(t) \qquad (9.28)\\ &\geq K(p,m,M)\left(\int_X w(t)g^p(t)\mathrm{d}\mu_\Delta(t)\right)^{1/p}\left(\int_X w(t)h^q(t)\mathrm{d}\mu_\Delta(t)\right)^{1/q}\\ &\quad +\Lambda(h^q, gh)N(p,m,M)\\ &\geq K(p,m,M)\left(\int_X w(t)g^p(t)\mathrm{d}\mu_\Delta(t)\right)^{1/p}\left(\int_X w(t)h^q(t)\mathrm{d}\mu_\Delta(t)\right)^{1/q}, \end{aligned}$$

where K *is defined as in* (2.62), N *is defined as in* (9.25), *and*

$$\Lambda(h^q, gh) = \int_X w(t)\left(\frac{M-m}{2}h^q(t) - \left|g(t)h(t) - \frac{m+M}{2}h^q(t)\right|\right)\mathrm{d}\mu_\Delta(t). \qquad (9.29)$$

If $0<p<1$ *and* $\int_X wh^q \mathrm{d}\mu_\Delta > 0$, *or* $p<0$ *and* $\int_X wg^p\mathrm{d}\mu_\Delta > 0$, *then the inequalities in* (9.28) *hold in reverse order.*

Proof. Just apply Theorems 2.5 and 9.14. □

Theorem 9.17 *Let* $p, w, g, h, K, N, \Lambda$ *be defined as in Theorem 9.16 with additional property* $w(g+h)^p$ *be* Δ*-integrable. Let* m *and* M *be such that* $0<m<g(t)(g(t)+h(t))^{-1} \leq M$ *and* $0<m<h(t)(g(t)+h(t))^{-1} \leq M$ *for* $t \in X$. *Then for* $p>1$,

$$\begin{aligned} &\left[\int_X w(t)(g(t)+h(t))^p \mathrm{d}\mu_\Delta(t)\right]^{\frac{1}{p}}\\ &\geq K(p,m,M)\left(\left(\int_X w(t)g^p(t)\mathrm{d}\mu_\Delta(t)\right)^{1/p}+\left(\int_X w(t)h^p(t)\mathrm{d}\mu_\Delta(t)\right)^{1/p}\right)\\ &\quad +N(p,m,M)\frac{\Lambda\left((f+g)^p, f(f+g)^{p-1}\right)+\Lambda\left((f+g)^p, g(f+g)^{p-1}\right)}{\left(\int_X w(t)(f(t)+g(t))^p\mathrm{d}\mu_\Delta(t)\right)^{1-\frac{1}{p}}}, \end{aligned}$$

and for $p<1$ $(p \neq 0)$, *the reversed inequality holds.*

Proof. Just apply Theorems 2.5 and 9.15. □

In next theorem, we generalize Theorem 9.17 to infinitely many functions.

Theorem 9.18 *Let $(X,\mathscr{M},\mu_\Delta)$ and $(Y,\mathscr{L},\nu_\Delta)$ be time scale measure spaces and let u,v, and f be nonnegative functions on X,Y, and $X\times Y$, respectively. Suppose*

$$0<m\leq \frac{f(x,y)}{\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)}\leq M \quad \textit{for all} \quad x\in X,\ y\in Y.$$

If $p\geq 1$, then

$$\begin{aligned}
&\left[\int_X\left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{1}{p}} \qquad (9.30)\\
&\geq K(p,m,M)\int_Y\left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y)\\
&\quad +N(p,m,M)\left[\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{1-p}{p}}\Lambda_1\\
&\geq K(p,m,M)\int_Y\left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y),
\end{aligned}$$

where K is defined in (2.62), N is defined in (9.25), $H(x)=\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)$, and

$$\Lambda_1=\int_Y\int_X\left(\frac{m-M}{2}H^p(x)-\left|f(x,y)H^{p-1}(x)-\frac{m+M}{2}H^p(x)\right|\right) u(x)\mathrm{d}\mu_\Delta(x)v(y)\mathrm{d}\nu_\Delta(y).$$

If $0<p<1$ with (9.2) or $p<0$ with (9.2) and (9.3), then the reversed inequality holds.

Proof. Using Fubini's theorem on time scales and (9.28), we get

$$\begin{aligned}
&\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x)=\int_X H(x)H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x)\\
&=\int_X\left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x)\\
&=\int_Y\left(\int_X f(x,y)H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x)\right)v(y)\mathrm{d}\nu_\Delta(y)\\
&\geq K(p,m,M)\int_Y\left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}}\\
&\left(\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{p-1}{p}} v(y)\mathrm{d}\nu_\Delta(y)+N(p,m,M)\Lambda_1
\end{aligned}$$

$$\geq K(p,m,M)\int_Y\left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}}$$
$$\left(\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{p-1}{p}} v(y)\mathrm{d}\nu_\Delta(y).$$

Dividing by $\left(\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{p-1}{p}}$, we get (9.30). □

Remark 9.11 If $\Lambda^n = \mathbb{R}^n$ in (1.14) and $X,Y \subset \mathbb{R}^n$, then Theorem 9.18 is equivalent to [121, Theorem 8].

9.6 Refinements of Bellman's inequality

In the following theorem, we give a refinement of the functional Bellman inequality (Theorem 2.51).

Theorem 9.19 *Let L satisfy conditions (L_1), (L_2) and A satisfy (A_1), (A_2). For $p \geq 2$, assume f,g are nonnegative functions on E such that $(f+g)^p, f^p, g^p \in L$ and $A((f+g)^p) > 0$. Suppose $f_0, g_0 > 0$ are such that*

$$f_0^p - A(f^p) > 0 \quad \textit{and} \quad g_0^p - A(g^p) > 0. \tag{9.31}$$

Then we have

$$\left(\left(f_0^p - A(f^p)\right)^{1/p} + \left(g_0^p - A(g^p)\right)^{1/p}\right)^p \tag{9.32}$$
$$\leq \left[\left(f_0^p - A\left(\left|f-(f+g)\frac{A\left(f(f+g)^{p-1}\right)}{A\left((f+g)^p\right)}\right|^p\right)\right)^{\frac{1}{p}}\right.$$
$$\left.+\left(g_0^p - A\left(\left|g-(f+g)\frac{A\left(g(f+g)^{p-1}\right)}{A\left((f+g)^p\right)}\right|^p\right)\right)^{\frac{1}{p}}\right]^p - A((f+g)^p).$$

Proof. Let x_1,x_2,y_1,y_2 be nonnegative real numbers. Now from the discrete Minkowski inequality, we have

$$((x_1+y_1)^p + (x_2+y_2)^p)^{\frac{1}{p}} \leq \left(x_1^p + x_2^p\right)^{\frac{1}{p}} + \left(y_1^p + y_2^p\right)^{\frac{1}{p}}. \tag{9.33}$$

By applying the substitution

$$x_1^p \to f_0^p - A(f^p), \quad y_1^p \to g_0^p - A(g^p),$$

$$x_2^p \to A(f^p) - A\left(\left|f-(f+g)\frac{A\left(f(f+g)^{p-1}\right)}{A\left((f+g)^p\right)}\right|^p\right),$$
$$y_2^p \to A(g^p) - A\left(\left|g-(f+g)\frac{A\left(g(f+g)^{p-1}\right)}{A\left((f+g)^p\right)}\right|^p\right)$$

in (9.33), and by using Theorem 2.67 and (9.31), we have

$$\begin{aligned}
&\left(\left(f_0^p - A(f^p)\right)^{1/p} + \left(g_0^p - A(g^p)\right)^{1/p}\right)^p \\
&\leq \left[\left(f_0^p - A\left(\left|f-(f+g)\frac{A\left(f(f+g)^{p-1}\right)}{A\left((f+g)^p\right)}\right|^p\right)\right)^{\frac{1}{p}}\right. \\
&\quad\left. + \left(g_0^p - A\left(\left|g-(f+g)\frac{A\left(g(f+g)^{p-1}\right)}{A\left((f+g)^p\right)}\right|^p\right)\right)^{\frac{1}{p}}\right]^p \\
&\quad - \left[\left(A(f^p) - A\left(\left|f-(f+g)\frac{A\left(f(f+g)^{p-1}\right)}{A\left((f+g)^p\right)}\right|^p\right)\right)^{1/p}\right. \\
&\quad\left. + \left(A(g^p) - A\left(\left|g-(f+g)\frac{A\left(g(f+g)^{p-1}\right)}{A\left((f+g)^p\right)}\right|^p\right)\right)^{1/p}\right]^p \\
&\leq \left[\left(f_0^p - A\left(\left|f-(f+g)\frac{A\left(f(f+g)^{p-1}\right)}{A\left((f+g)^p\right)}\right|^p\right)\right)^{\frac{1}{p}}\right. \\
&\quad\left. + \left(g_0^p - A\left(\left|g-(f+g)\frac{A\left(g(f+g)^{p-1}\right)}{A\left((f+g)^p\right)}\right|^p\right)\right)^{\frac{1}{p}}\right]^p - A((f+g)^p),
\end{aligned}$$

i.e., (9.32) holds. □

Remark 9.12 Since

$$f_0^p > A(f^p) \geq A\left(\left|f-(f+g)\frac{A\left(f(f+g)^{p-1}\right)}{A\left((f+g)^p\right)}\right|^p\right) \geq 0,$$

we have

$$\left(f_0^p - A\left(\left|f-(f+g)\frac{A\left(f(f+g)^{p-1}\right)}{A\left((f+g)^p\right)}\right|^p\right)\right)^{\frac{1}{p}} \leq f_0.$$

Similarly,

$$\left(g_0^p - A\left(\left|g-(f+g)\frac{A\left(g(f+g)^{p-1}\right)}{A\left((f+g)^p\right)}\right|^p\right)\right)^{\frac{1}{p}} \leq g_0.$$

It follows that (9.32) is a refinement of the Bellman inequality.

The following corollary is an immediate consequence of Theorem 9.19 by using the fact that the Δ-integral is an isotonic linear functional.

Corollary 9.5 *Let $(X,\mathscr{M},\mu_\Delta)$ be a time scale measure space. For $p\geq 2$, assume f,g are nonnegative functions on X such that $(f+g)^p, f^p, g^p$ are Δ-integrable on X and $\int_X(f(t)+g(t))^p\Delta t>0$. Suppose $f_0,g_0>0$ are such that*

$$f_0^p-\int_X f^p(t)\Delta t>0 \quad \text{and} \quad g_0^p-\int_X g^p(t)\Delta t>0.$$

Then we have

$$\begin{aligned}&\left(\left(f_0^p-\int_X f^p(t)\Delta t\right)^{1/p}+\left(g_0^p-\int_X g^p(t)\Delta t\right)^{1/p}\right)^p \qquad (9.34)\\
&\leq\left[\left(f_0^p-\int_X\left|f(s)-(f(s)+g(s))\frac{\int_X f(t)(f(t)+g(t))^{p-1}\Delta t}{\int_X(f(t)+g(t))^p\Delta t}\right|^p\Delta s\right)^{\frac{1}{p}}\right.\\
&\quad\left.+\left(g_0^p-\int_X\left|g(s)-(f(s)+g(s))\frac{\int_X g(t)(f(t)+g(t))^{p-1}\Delta t}{\int_X(f(t)+g(t))^p\Delta t}\right|^p\Delta s\right)^{\frac{1}{p}}\right]^p\\
&\quad-\int_X(f(t)+g(t))^p\Delta t.\end{aligned}$$

Remark 9.13 Corollary 9.5 is a refinement of Theorem 2.52.

9.7 Integral Inequalities of Popoviciu Type

In the following theorem, we give the weighted version of Popoviciu's inequality (Theorem 2.49).

Theorem 9.20 *Let $(X,\mathscr{M},\mu_\Delta)$ be a time scale measure space. For $p\neq 1$, define $q=p/(p-1)$. Assume that w,f,g are nonnegative functions such that wf^p, wg^q, wfg are Δ-integrable on X. Suppose $f_0,g_0>0$ are such that*

$$f_0^p-\int_X wf^p\mathrm{d}\mu_\Delta>0 \quad \text{and} \quad g_0^q-\int_X wg^q\mathrm{d}\mu_\Delta>0.$$

If $p>1$, then we have

$$\begin{aligned}&f_0g_0-\int_X w(t)f(t)g(t)\mathrm{d}\mu_\Delta(t)\\
&\geq\left(f_0^p-\int_X w(t)f^p(t)\mathrm{d}\mu_\Delta(t)\right)^{\frac{1}{p}}\left(g_0^q-\int_X w(t)g^q(t)\mathrm{d}\mu_\Delta(t)\right)^{\frac{1}{q}}. \qquad (9.35)\end{aligned}$$

This inequality is reversed if $0<p<1$ and $\int_X wg^q\mathrm{d}\mu_\Delta>0$, or if $p<0$ and $\int_X wf^p\mathrm{d}\mu_\Delta>0$.

Proof. Just apply [119, Theorem 4.27] and Theorem 2.5. □

In the next theorem, we obtain a new inequality by using Popoviciu's inequality on time scales (Theorem 9.20).

Theorem 9.21 *Let $(X,\mathcal{M},\mu_\Delta)$ and $(Y,\mathcal{L},\nu_\Delta)$ be time scale measure spaces. For $p \neq 1$, define $q = p/(p-1)$. Assume that u,v,f are nonnegative functions on X,Y, and $X \times Y$, respectively. Let $f_0, g_0 > 0$. If $p > 1$, then*

$$\int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y) \right)^p u(x)\mathrm{d}\mu_\Delta(x) \tag{9.36}$$
$$\leq \int_Y \left[f_0 g_0 - \left(f_0^p - \int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x) \right)^{\frac{1}{p}} \left(g_0^q \right.\right.$$
$$\left.\left. - \int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y) \right)^p u(x)\mathrm{d}\mu_\Delta(x) \right)^{\frac{1}{q}} \right] v(y)\mathrm{d}\nu_\Delta(y)$$

holds, provided all integrals in (9.36) *exist and*

$$f_0^p - \int_X f^p u \mathrm{d}\mu_\Delta > 0, \quad g_0^q - \int_X \left(\int_Y f v \mathrm{d}\nu_\Delta \right)^p u \mathrm{d}\mu_\Delta > 0.$$

The inequality (9.36) *is reversed if $0 < p < 1$ and $\int_X H^p u \mathrm{d}\mu_\Delta > 0$, or if $p < 0$ and $\int_X f^p u \mathrm{d}\mu_\Delta >$ 0.*

Proof. Let $p > 1$. Put

$$H(x) = \int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y).$$

Now, by using Fubini's theorem (Theorem 1.8) and Popoviciu's inequality (Theorem 9.20), we have

$$\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x) = \int_X H(x)H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x)$$
$$= \int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y) \right) H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x)$$
$$= \int_Y \left(\int_X f(x,y)H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x) \right) v(y)\mathrm{d}\nu_\Delta(y)$$
$$\leq \int_Y \left[f_0 g_0 - \left(f_0^p - \int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x) \right)^{\frac{1}{p}} \right.$$
$$\left. \left(g_0^q - \int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x) \right)^{\frac{1}{q}} \right] v(y)\mathrm{d}\nu_\Delta(y).$$

If $0 < p < 1$ or $p < 0$, then the corresponding results can be obtained similarly. □

Now we recall the refinement of Popoviciu's inequality for isotonic linear functionals as given in [26].

Theorem 9.22 (SEE [26, THEOREM 21]) *Let L satisfy conditions (L_1), (L_2) and A satisfy (A_1), (A_2). For $p \geq 2$, let $q = p/(p-1)$. Assume f, g are nonnegative functions on E such that*

$$f^p,\ g^q,\ fg,\ \left|g^{\frac{q}{p}}A(fg) - fA(g^q)\right|^p \in L,$$

and $f_0, g_0 > 0$ are such that

$$f_0^p - A(f^p) > 0,\ g_0^q - A(g^q) > 0, \quad \text{and} \quad A(g^q) > 0.$$

Then

$$\begin{aligned} f_0 g_0 - A(fg) &\geq \left[\left(f_0^p - A(f^p)\right)\left(g_0^q - A(g^q)\right)^{\frac{p}{q}} + N_p\right]^{1/p} \\ &\geq \left(f_0^p - A(f^p)\right)^{1/p}\left(g_0^q - A(g^q)\right)^{1/q} \end{aligned}$$

holds, where

$$\begin{aligned} N_p = \frac{\left(g_0^q - A(g^q)\right)^{\frac{p}{q}}}{A^p(g^q)} A\left(\left|g^{\frac{q}{p}}A(fg) - fA(g^q)\right|^p\right) \\ + \left|f_0 g_0^{-\frac{q}{p}} A(g^q) - A(fg)\right|^p \left(1 + \frac{\left(g_0^q - A(g^q)\right)^{\frac{p}{q}}}{A^{\frac{p}{q}}(g^q)}\right). \end{aligned}$$

In the following theorem, we present a refinement of Popoviciu's inequality for Δ-integrals.

Theorem 9.23 *Let $(X, \mathscr{M}, \mu_\Delta)$ be a time scale measure space. For $p \geq 2$, let $q = p/(p-1)$. Assume f, g are nonnegative functions on X such that*

$$f^p,\ g^q,\ fg,\ \left|g^{\frac{q}{p}}\int_X f(t)g(t)\Delta t - f\int_X g^q(t)\Delta t\right|^p$$

are Δ-integrable on X. Suppose $f_0, g_0 > 0$ are such that

$$f_0^p - \int_X f^p(t)\Delta t > 0,\ g_0^q - \int_X g^q(t)\Delta t > 0, \quad \text{and} \quad \int_X g^q(t)\Delta t > 0.$$

Then

$$\begin{aligned} & f_0 g_0 - \int_X f(t)g(t)\Delta t && (9.37) \\ & \geq \left[\left(f_0^p - \int_X f^p(t)\Delta t\right)\left(g_0^q - \int_X g^q(t)\Delta t\right)^{\frac{p}{q}} + R_p\right]^{\frac{1}{p}} \\ & \geq \left(f_0^p - \int_X f^p(t)\Delta t\right)^{\frac{1}{p}}\left(g_0^q - \int_X g^q(t)\Delta t\right)^{\frac{1}{q}} \end{aligned}$$

holds, where

$$R_p = \frac{\left(g_0^q - \int_X g^q(t)\Delta t\right)^{\frac{p}{q}}}{\left(\int_X g^q(t)\Delta t\right)^p} \int_X \left| g^{\frac{q}{p}}(s) \int_X f(t)g(t)\Delta t - f(s) \int_X g^q(t)\Delta t \right|^p \Delta s$$
$$+ \left| f_0 g_0^{-\frac{q}{p}} \int_X g^q(t)\Delta t - \int_X f(t)g(t)\Delta t \right|^p \left(1 + \frac{\left(g_0^q - \int_X g^q(t)\Delta t\right)^{\frac{p}{q}}}{\left(\int_X g^q(t)\Delta t\right)^{\frac{p}{q}}} \right).$$

Proof. The inequality (9.37) follows from Theorem 9.22 by using the fact that the Δ-integral is an isotonic linear functional. □

Remark 9.14 Theorem 9.23 is a refinement of Theorem 2.49.

For $p = 2$, Theorem 9.23 gives the refinement of Aczél's inequality on time scales (Theorem 2.50).

Theorem 9.24 *Let $(X, \mathscr{M}, \mu_\Delta)$ be a time scale measure space. Assume f, g are nonnegative functions on X such that*

$$f^2,\ g^2,\ fg,\ \left| g \int_X f(t)g(t)\Delta t - f \int_X g^2(t)\Delta t \right|^2$$

are Δ-integrable on X. Suppose $f_0, g_0 > 0$ are such that

$$f_0^2 - \int_X f^2(t)\Delta t > 0,\ g_0^2 - \int_X g^2(t)\Delta t > 0, \quad \text{and} \quad \int_X g^2(t)\Delta t > 0.$$

Then

$$f_0 g_0 - \int_X f(t)g(t)\Delta t \tag{9.38}$$
$$\geq \sqrt{\left[\left(f_0^2 - \int_X f^2(t)\Delta t \right) \left(g_0^2 - \int_X g^2(t)\Delta t \right) + R_2 \right]}$$
$$\geq \sqrt{\left(f_0^2 - \int_X f^2(t)\Delta t \right) \left(g_0^2 - \int_X g^2(t)\Delta t \right)}$$

holds, where

$$R_2 = \frac{g_0^2 - \int_X g^2(t)\Delta t}{\left(\int_X g^2(t)\Delta t\right)^2} \int_X \left(g(s) \int_X f(t)g(t)\Delta t - f(s) \int_X g^2(t)\Delta t \right)^2 \Delta s$$
$$+ \left(f_0 g_0^{-1} \int_X g^2(t)\Delta t - \int_X f(t)g(t)\Delta t \right)^2 \left(1 + \frac{g_0^2 - \int_X g^2(t)\Delta t}{\int_X g^2(t)\Delta t} \right).$$

9.8 Integral Inequalities of Diaz–Metcalf Type

In the following theorem, we obtain a new inequality by using Diaz–Metcalf's inequality (Theorem 2.54) on time scales.

Theorem 9.25 *Let $(X, \mathscr{M}, \mu_\Delta)$ and $(Y, \mathscr{L}, \nu_\Delta)$ be time scale measure spaces. For $p \neq 1$, define $q = p/(p-1)$. Assume that u, v, f are nonnegative functions on X, Y, and $X \times Y$, respectively and $p \neq 0$,*

$$0 < m \leq \frac{f(x,y)}{\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)} \leq M \quad \text{for all} \quad x \in X,\ y \in Y.$$

If $p > 1$, or if $p < 0$, then

$$\begin{aligned} &\int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p (x)u(x)\mathrm{d}\mu_\Delta(x) \qquad (9.39)\\ &\geq \frac{1}{(M^p - m^p)} \int_Y \Bigg[(M-m)\int_X f^p(x)u(x)\mathrm{d}\mu_\Delta(x) \\ &+(mM^p - Mm^p)\int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p (x)u(x)\mathrm{d}\mu_\Delta(x)\Bigg] v(y)\mathrm{d}\nu_\Delta(y) \end{aligned}$$

holds, provided all integrals in (9.39) *exist and*

$$\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x) > 0 \quad \text{or}$$

$$\int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x) > 0. \qquad (9.40)$$

The inequality (9.39) *is reversed if* $0 < p < 1$ *and* (9.40) *holds.*

Proof. Let $p > 1$ or $p < 0$ and suppose (9.40) holds. Put

$$H(x) = \int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y).$$

Now, by using Fubini's theorem (Theorem 1.8) and the Diaz–Metcalf inequality (Theorem 2.54), we obtain

$$\begin{aligned} &\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x) = \int_X H(x)H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x)\\ &= \int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right) H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x)\\ &= \int_Y \left(\int_X f(x,y)H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x)\right) v(y)\mathrm{d}\nu_\Delta(y) \end{aligned}$$

$$\geq \frac{1}{(M^p - m^p)} \int_Y \left[(M-m) \int_X f^p(x)u(x)\mathrm{d}\mu_\Delta(x) \right.$$
$$\left. +(mM^p - Mm^p) \int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x) \right] v(y)\mathrm{d}\nu_\Delta(y).$$

If $0 < p < 1$, then the corresponding result can be obtained similarly. □

In the following theorem, we give a refinement of the functional Diaz–Metcalf inequality (Theorem 2.53).

Theorem 9.26 *Let L satisfy conditions (L_1), (L_2) and A satisfy (A_1), (A_2). For $p \geq 2$, let $q = p/(p-1)$. Assume w, f, g are nonnegative functions on E such that $wf^p, wg^q, wfg \in L$ and*

$$0 < m \leq f(t)g^{-q/p}(t) \leq M \quad \text{for all} \quad t \in E.$$

Then we have

$$\begin{aligned}
&(M-m)A(wf^p) + (mM^p - Mm^p)A(wg^q) \\
&\leq (M^p - m^p)A(wfg) \\
&\quad - A\left[wg^q \left(\left(M - fg^{-\frac{q}{p}}\right)\left(fg^{-\frac{q}{p}} - m\right)^p + \left(fg^{-\frac{q}{p}} - m\right)\left(M - fg^{-\frac{q}{p}}\right)^p \right) \right].
\end{aligned} \tag{9.41}$$

Proof. Applying the substitution

$$\Psi(x) \to x^p, \; h \to fg^{-q/p}, \quad \text{and} \quad k \to wg^q,$$

the inequality (9.41) follows from Theorem 2.71. □

The following corollary is an immediate consequence of Theorem 9.26 by using the fact that Δ-integral is an isotonic linear functional.

Corollary 9.6 *Let $(X, \mathscr{M}, \mu_\Delta)$ be a time scale measure space. For $p \geq 2$, let $q = p/(p-1)$. Assume w, f, g are nonnegative functions on X such that wf^p, wg^q, wfg are Δ-integrable on X and*

$$0 < m \leq f(t)g^{-q/p}(t) \leq M \quad \text{for all} \quad t \in X.$$

Then we have

$$\begin{aligned}
&(M-m)\int_X w(t)f^p(t)\Delta t + (mM^p - Mm^p)\int_X w(t)g^q(t)\Delta t \\
&\leq (M^p - m^p)\int_X w(t)f(t)g(t)\Delta t \\
&\quad - \int_X w(t)g^q(t)\left(\left(M - f(t)g^{-\frac{q}{p}}(t)\right)\left(f(t)g^{-\frac{q}{p}}(t) - m\right)^p \right. \\
&\quad \left. + \left(f(t)g^{-\frac{q}{p}}(t) - m\right)\left(M - f(t)g^{-\frac{q}{p}}(t)\right)^p \right)\Delta t.
\end{aligned} \tag{9.42}$$

Remark 9.15 Corollary 9.6 is a refinement of Theorem 2.54.

Chapter 10

Some Dynamic Hardy-Type Inequalities with General Kernels

The well-known Hardy inequality as presented in [68] (both in the continuous and discrete settings) has been extensively studied and used as a model for investigation of more general integral inequalities [60, 85, 86, 93, 102]. Recently, several papers have treated the unification and extension of Hardy's continuous and discrete integral inequalities by means of the theory of time scales [113, 114, 122]. Measure spaces and measurable functions for time scales are discussed in [42, 43, 62]. The aim of this chapter is to extend some inequalities of Hardy type with certain kernels to arbitrary time scales. Certain classical and some new integral and discrete inequalities are deduced in seek of applications.

10.1 Hardy Type Inequalities via Convexity in One Variable

In this section, inequalities of Hardy type using convex function of one variable with general kernels on arbitrary time scales are studied. The results of this section are contained in [44].

10.1.1 Inequalities with General Kernels

Let us consider the following hypotheses:

(H_1) $(X, \mathscr{K}, \mu_\Delta)$ and $(Y, \mathscr{L}, \nu_\Delta)$ are two time scale measure spaces.

(H_2) $k : X \times Y \to \mathbb{R}$ is a nonnegative kernel and

$$K(x) := \int_Y k(x,y)\Delta y < \infty, \quad x \in X.$$

(H_3) $\xi : X \to \mathbb{R}_+$ is μ_Δ-integrable and denote

$$w(y) = \int_X \frac{k(x,y)\xi(x)}{K(x)}\Delta x, \quad y \in Y.$$

Theorem 10.1 *Assume (H_1)–(H_3). If $\Phi \in \mathrm{C}(I,\mathbb{R})$ is convex, where $I \subset \mathbb{R}$ is an interval, then*

$$\int_X \xi(x)\Phi\left(\frac{1}{K(x)}\int_Y k(x,y)f(y)\Delta y\right)\Delta x \le \int_Y w(y)\Phi(f(y))\Delta y \tag{10.1}$$

holds for all ν_Δ-integrable $f : Y \to \mathbb{R}$ such that $f(Y) \subset I$.

Proof. By using Jensen's inequality given in Theorem 2.10 and the Fubini Theorem 1.8 on time scales, we find that

$$\begin{aligned}
&\int_X \xi(x)\Phi\left(\frac{1}{K(x)}\int_Y k(x,y)f(y)\Delta y\right)\Delta x\\
&= \int_X \xi(x)\Phi\left(\frac{\int_Y |k(x,y)|f(y)\Delta y}{\int_Y |k(x,y)|\Delta y}\right)\Delta x\\
&\le \int_X \xi(x)\frac{\int_Y |k(x,y)|\Phi(f(y))\Delta y}{\int_Y |k(x,y)|\Delta y}\Delta x\\
&= \int_X \frac{\xi(x)}{K(x)}\left(\int_Y k(x,y)\Phi(f(y))\Delta y\right)\Delta x\\
&= \int_Y \Phi(f(y))\left(\int_X \frac{k(x,y)\xi(x)}{K(x)}\Delta x\right)\Delta y\\
&= \int_Y w(y)\Phi(f(y))\Delta y,
\end{aligned}$$

and the proof is complete. □

Corollary 10.1 *Assume (H_1)–(H_3). If $p > 1$, then*

$$\int_X \xi(x)\left(\frac{1}{K(x)}\int_Y k(x,y)f(y)\Delta y\right)^p \Delta x \le \int_Y w(y)(f(y))^p\Delta y$$

holds for all ν_Δ-integrable $f : Y \to \mathbb{R}_+$.

Proof. Use $\Phi(r) = r^p$ and $I = \mathbb{R}_+$ in Theorem 10.1. □

Corollary 10.2 *Assume* (H_1)–(H_3). *If* $p > 1$, *then*

$$\int_X \xi(x) e^{\frac{p}{K(x)} \int_Y k(x,y)\ln(g(y))\Delta y} \Delta x \le \int_Y w(y)(g(y))^p \Delta y$$

holds for all ν_Δ*-integrable* $g : Y \to (0,\infty)$.

Proof. Use $\Phi(r) = e^r$ and $I = \mathbb{R}$ and let $f = \ln(g^p)$ in Theorem 10.1. □

Corollary 10.3 *Assume* (H_1)–(H_3). *Then*

$$\int_X \xi(x) e^{\frac{1}{K(x)} \int_Y k(x,y)\ln(g(y))\Delta y} \Delta x \le \int_Y w(y) g(y) \Delta y$$

holds for all ν_Δ*-integrable* $g : Y \to (0,\infty)$.

Proof. Use $p = 1$ in Corollary 10.2. □

Further, we assume the following hypotheses:

(H_1') Let $X = Y = [a_1,b_1)_\mathbb{T} \times [a_2,b_2)_\mathbb{T} \times \ldots \times [a_n,b_n)_\mathbb{T}$, $0 \le a_i < b_i \le \infty$ for all $i \in \{1,\ldots,n\}$, where $\mathbb{T}$ is an arbitrary time scale.

Theorem 10.2 *Assume* (H_1') *and* (H_2). *Suppose*

$u : X \to \mathbb{R}_+$ *is such that*

$$v(\mathbf{y}) := \int_X \frac{y_1 \ldots y_n k(\mathbf{x},\mathbf{y}) u(\mathbf{x})}{\sigma(x_1)\ldots\sigma(x_n) K(\mathbf{x})} \Delta \mathbf{x} < \infty, \quad \mathbf{y} \in Y. \tag{10.2}$$

If $\Phi \in C(I,\mathbb{R})$ *is convex, where* $I \subset \mathbb{R}$ *is an interval, then*

$$\begin{aligned}\int_{a_1}^{b_1} \ldots \int_{a_n}^{b_n} u(\mathbf{x}) \Phi((A_k f)(\mathbf{x})) \frac{\Delta x_1 \ldots \Delta x_n}{\sigma(x_1)\ldots\sigma(x_n)} \\ \le \int_{a_1}^{b_1} \ldots \int_{a_n}^{b_n} v(\mathbf{y}) \Phi(f(\mathbf{y})) \frac{\Delta y_1 \ldots \Delta y_n}{y_1 \ldots y_n}\end{aligned} \tag{10.3}$$

holds for all ν_Δ*-integrable* $f : Y \to \mathbb{R}$ *such that* $f(Y) \subset I$, *where*

$$(A_k f)(\mathbf{x}) := \frac{1}{K(\mathbf{x})} \int_{a_1}^{b_1} \ldots \int_{a_n}^{b_n} k(\mathbf{x},\mathbf{y}) f(\mathbf{y}) \Delta y_1 \ldots \Delta y_n.$$

Proof. We replace $\xi(\mathbf{x})$ by $\frac{u(\mathbf{x})}{\sigma(x_1)\ldots\sigma(x_n)}$ in Theorem 10.1 and notice that therefore

$$w(\mathbf{y}) = \frac{v(\mathbf{y})}{y_1 \ldots y_n}$$

holds. An application of Theorem 10.1 completes the proof. □

Corollary 10.4 *Assume* (H_1'), (H_2), *and* (10.2). *If* $p>1$, *then*

$$\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} u(\mathbf{x})((A_k f)(\mathbf{x}))^p \frac{\Delta x_1\ldots\Delta x_n}{\sigma(x_1)\ldots\sigma(x_n)} \le \int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} v(\mathbf{y})(f(\mathbf{y}))^p \frac{\Delta y_1\ldots\Delta y_n}{y_1\ldots y_n}$$

holds for all ν_Δ*-integrable* $f : Y \to \mathbb{R}_+$.

Proof. Use $\Phi(r)=r^p$ and $I=\mathbb{R}_+$ in Theorem 10.2. □

Corollary 10.5 *Assume* (H_1'), (H_2), *and* (10.2). *If* $p>1$, *then*

$$\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} u(\mathbf{x})e^{p(A_k\ln(g))(\mathbf{x})} \frac{\Delta x_1\ldots\Delta x_n}{\sigma(x_1)\ldots\sigma(x_n)} \le \int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} v(\mathbf{y})(g(\mathbf{y}))^p \frac{\Delta y_1\ldots\Delta y_n}{y_1\ldots y_n} \tag{10.4}$$

holds for all ν_Δ*-integrable* $g : Y \to (0,\infty)$.

Proof. Use $\Phi(r)=e^r$ and $I=\mathbb{R}$ and let $f=\ln(g^p)$ in Theorem 10.2. □

Example 10.1 If in Corollary 10.5 we take $\mathbb{T}=\mathbb{R}$ and $a_i=0$ for all $1\le i\le n$, then (10.4) takes the form

$$\int_{0}^{b_1}\cdots\int_{0}^{b_n} u(\mathbf{x})e^{p(A_k\ln(g))(\mathbf{x})} \frac{\mathrm{d}x_1\ldots\mathrm{d}x_n}{x_1\ldots x_n} \le \int_{0}^{b_1}\cdots\int_{0}^{b_n} v(\mathbf{y})(g(\mathbf{y}))^p \frac{\mathrm{d}y_1\ldots\mathrm{d}y_n}{y_1\ldots y_n}.$$

Corollary 10.6 *Assume* (H_1'), (H_2), *and* (10.2). *Then*

$$\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} u(\mathbf{x})e^{(A_k\ln(g))(\mathbf{x})} \frac{\Delta x_1\ldots\Delta x_n}{\sigma(x_1)\ldots\sigma(x_n)} \le \int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} v(\mathbf{y})g(\mathbf{y}) \frac{\Delta y_1\ldots\Delta y_n}{y_1\ldots y_n}$$

holds for all ν_Δ*-integrable* $g : Y \to (0,\infty)$.

Proof. Use $p=1$ in Corollary 10.5. □

10.1.2 Inequalities with Special Kernels

Corollary 10.7 *Assume* (H_1'), (H_2), *and* (10.2) *with the kernel k such that*

$$k(\mathbf{x},\mathbf{y}) = 0 \quad \text{if} \quad a_i \le y_i \le \sigma(x_i) \le b_i,\ 1 \le i \le n. \tag{10.5}$$

If $\Phi \in \mathrm{C}(I,\mathbb{R})$ *is convex, where* $I \subset \mathbb{R}$ *is an interval, then* (10.3) *holds for all* ν_Δ*-integrable* $f : Y \to \mathbb{R}$ *such that* $f(Y) \subset I$, *where*

$$K(\mathbf{x}) = \int_{\sigma(x_1)}^{b_1} \dots \int_{\sigma(x_n)}^{b_n} k(\mathbf{x},\mathbf{y})\Delta y_1 \dots \Delta y_n,$$

$$v(\mathbf{y}) = y_1 \dots y_n \int_{a_1}^{y_1} \dots \int_{a_n}^{y_n} \frac{k(\mathbf{x},\mathbf{y})u(\mathbf{x})}{\sigma(x_1)\dots\sigma(x_n)K(\mathbf{x})} \Delta x_1 \dots \Delta x_n,$$

and

$$(A_k f)(\mathbf{x}) = \frac{1}{K(\mathbf{x})} \int_{\sigma(x_1)}^{b_1} \dots \int_{\sigma(x_n)}^{b_n} k(\mathbf{x},\mathbf{y}) f(\mathbf{y})\Delta y_1 \dots \Delta y_n.$$

Proof. The statement follows from Theorem 10.2 by using (10.5). □

Example 10.2 If in Corollary 10.7 we take $\mathbb{T} = \mathbb{R}$ and $b_i = \infty$ for all $1 \le i \le n$, then (10.3) takes the form

$$\int_{a_1}^{\infty} \dots \int_{a_n}^{\infty} u(\mathbf{x})\Phi((A_k f)(\mathbf{x})) \frac{\mathrm{d}x_1 \dots \mathrm{d}x_n}{x_1 \dots x_n} \le \int_{a_1}^{\infty} \dots \int_{a_n}^{\infty} v(\mathbf{y})\Phi(f(\mathbf{y})) \frac{\mathrm{d}y_1 \dots \mathrm{d}y_n}{y_1 \dots y_n},$$

where

$$K(\mathbf{x}) = \int_{x_1}^{\infty} \dots \int_{x_n}^{\infty} k(\mathbf{x},\mathbf{y})\mathrm{d}y_1 \dots \mathrm{d}y_n,$$

$$v(\mathbf{y}) = y_1 \dots y_n \int_{a_1}^{y_1} \dots \int_{a_n}^{y_n} \frac{k(\mathbf{x},\mathbf{y})u(\mathbf{x})}{x_1 \dots x_n K(\mathbf{x})} \mathrm{d}x_1 \dots \mathrm{d}x_n,$$

and

$$(A_k f)(\mathbf{x}) = \frac{1}{K(\mathbf{x})} \int_{x_1}^{\infty} \dots \int_{x_n}^{\infty} k(\mathbf{x},\mathbf{y}) f(\mathbf{y})\mathrm{d}y_1 \dots \mathrm{d}y_n.$$

This result is the same as [110, inequality (2.2)].

Corollary 10.8 *Assume* (H_1'), (H_2), *and* (10.2) *with the kernel k such that*

$$k(\mathbf{x},\mathbf{y}) = 0 \quad \text{if} \quad a_i \le \sigma(x_i) \le y_i \le b,\ 1 \le i \le n. \tag{10.6}$$

If $\Phi \in \mathrm{C}(I,\mathbb{R})$ *is convex, where* $I \subset \mathbb{R}$ *is an interval, then* (10.3) *holds for all* ν_Δ*-integrable* $f : Y \to \mathbb{R}$ *such that* $f(Y) \subset I$*, where*

$$K(\mathbf{x}) = \int_{a_1}^{\sigma(x_1)} \dots \int_{a_n}^{\sigma(x_n)} k(\mathbf{x},\mathbf{y})\Delta y_1 \dots \Delta y_n,$$
$$v(\mathbf{y}) = y_1 \dots y_n \int_{y_1}^{b_1} \dots \int_{y_n}^{b_n} \frac{k(\mathbf{x},\mathbf{y})u(\mathbf{x})}{\sigma(x_1)\dots\sigma(x_n)K(\mathbf{x})}\Delta x_1 \dots \Delta x_n,$$

and

$$(A_k f)(\mathbf{x}) = \frac{1}{K(\mathbf{x})} \int_{a_1}^{\sigma(x_1)} \dots \int_{a_n}^{\sigma(x_n)} k(\mathbf{x},\mathbf{y}) f(\mathbf{y})\Delta y_1 \dots \Delta y_n.$$

Proof. The statement follows from Theorem 10.2 by using (10.6). □

Example 10.3 If in Corollary 10.8 we take $\mathbb{T} = \mathbb{R}$ and $a_i = 0$ for all $1 \le i \le n$, then (10.3) takes the form

$$\int_0^{b_1} \dots \int_0^{b_n} u(\mathbf{x})\Phi\left((A_k f)(\mathbf{x})\right) \frac{\mathrm{d}x_1 \dots \mathrm{d}x_n}{x_1 \dots x_n} \le \int_0^{b_1} \dots \int_0^{b_n} v(\mathbf{y})\Phi(f(\mathbf{y}))\frac{\mathrm{d}y_1 \dots \mathrm{d}y_n}{y_1 \dots y_n},$$

where

$$K(\mathbf{x}) = \int_0^{x_1} \dots \int_0^{x_n} k(\mathbf{x},\mathbf{y})\mathrm{d}y_1 \dots \mathrm{d}y_n,$$
$$v(\mathbf{y}) = y_1 \dots y_n \int_{y_1}^{b_1} \dots \int_{y_n}^{b_n} \frac{k(\mathbf{x},\mathbf{y})u(\mathbf{x})}{x_1 \dots x_n K(\mathbf{x})}\mathrm{d}x_1 \dots \mathrm{d}x_n,$$

and

$$(A_k f)(\mathbf{x}) = \frac{1}{K(\mathbf{x})} \int_0^{x_1} \dots \int_0^{x_n} k(\mathbf{x},\mathbf{y}) f(\mathbf{y})\mathrm{d}y_1 \dots \mathrm{d}y_n.$$

This result is the same as [110, inequality (2.5)]. Special cases are given (for $n = 1$) in [85, Theorem 4.1] and (for $k(\mathbf{x},\mathbf{y}) = 1$) in [109, inequality (2.2)].

Remark 10.1 Using (10.6) in Example 10.1, we obtain [85, inequality (4.2)] (for $n = 1$).

10.1.3 Examples and Special Cases

Theorem 10.3 *Assume* (H_1') *and*

$$\xi : X \to \mathbb{R}_+ \text{ is such that}$$
$$\widetilde{w}(\mathbf{y}) := \int_{y_1}^{b_1} \dots \int_{y_n}^{b_n} \frac{\xi(\mathbf{x})}{\prod_{i=1}^{n}(\sigma(x_i) - a_i)} \Delta x_1 \dots \Delta x_n < \infty, \ \mathbf{y} \in Y. \quad (10.7)$$

If $\Phi \in \mathrm{C}(I, \mathbb{R})$ *is convex, where* $I \subset \mathbb{R}$ *is an interval, then*

$$\int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} \xi(\mathbf{x}) \Phi\left((\widetilde{A}f)(\mathbf{x})\right) \Delta x_1 \dots \Delta x_n \le \int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} \widetilde{w}(\mathbf{y}) \Phi(f(\mathbf{y})) \Delta y_1 \dots \Delta y_n \quad (10.8)$$

holds for all ν_Δ*-integrable* $f : Y \to \mathbb{R}$ *such that* $f(Y) \subset I$*, where*

$$(\widetilde{A}f)(\mathbf{x}) := \frac{1}{\prod_{i=1}^{n}(\sigma(x_i) - a_i)} \int_{a_1}^{\sigma(\mathbf{x}_1)} \dots \int_{a_n}^{\sigma(\mathbf{x}_n)} f(\mathbf{y}) \Delta y_1 \dots \Delta y_n.$$

Proof. Let K and $A_k f$ be defined as in the statements of Theorem 10.1 and Theorem 10.2, respectively. The statement follows from Theorem 10.1 by using

$$k(\mathbf{x}, \mathbf{y}) = \begin{cases} 1 & \text{if} \quad a_i \le y_i < \sigma(x_i) \le b_i, \ 1 \le i \le n \\ 0 & \text{otherwise,} \end{cases} \quad (10.9)$$

since in this case we have

$$K(\mathbf{x}) = \int_{a_1}^{\sigma(x_1)} \dots \int_{a_n}^{\sigma(x_n)} \Delta y_1 \dots \Delta y_n = \prod_{i=1}^{n}(\sigma(x_i) - a_i)$$

and thus $A_k = \widetilde{A}$ and $w = \widetilde{w}$. □

Remark 10.2 By using k of the form (10.9), we may also give results corresponding to Corollary 10.1, Corollary 10.2, Corollary 10.3, Theorem 10.2, Corollary 10.4, Corollary 10.5, Corollary 10.6, and Corollary 10.6.

Corollary 10.9 *Assume* (H_1') *with* $a_i = 0$ *for all* $1 \le i \le n$*. If* $\Phi \in \mathrm{C}(I, \mathbb{R})$ *is convex, where* $I \subset \mathbb{R}$ *is an interval, then*

$$\int_{0}^{b_1} \dots \int_{0}^{b_n} \Phi\left((\widetilde{A}f)(\mathbf{x})\right) \frac{\Delta x_1 \dots \Delta x_n}{x_1 \dots x_n} \le \int_{0}^{b_1} \dots \int_{0}^{b_n} \left\{\prod_{i=1}^{n}\left(\frac{1}{y_i} - \frac{1}{b_i}\right)\right\} \Phi(f(\mathbf{y})) \Delta y_1 \dots \Delta y_n \quad (10.10)$$

holds for all ν_Δ-integrable $f : Y \to \mathbb{R}$ such that $f(Y) \subset I$, where

$$(\widetilde{A}f)(\mathbf{x}) := \frac{1}{\prod_{i=1}^{n} \sigma(x_i)} \int_0^{\sigma(x_1)} \dots \int_0^{\sigma(x_n)} f(\mathbf{y}) \Delta y_1 \dots \Delta y_n.$$

Proof. The statement follows from Theorem 10.3 by using

$$\xi(\mathbf{x}) = \frac{1}{x_1 \dots x_n},$$

since in this case we have

$$\widetilde{w}(\mathbf{y}) = \int_{y_1}^{b_1} \dots \int_{y_n}^{b_n} \frac{1}{\prod_{i=1}^{n} x_i \sigma(x_i)} \Delta x_1 \dots \Delta x_n = \prod_{i=1}^{n} \left(\frac{1}{y_i} - \frac{1}{b_i} \right)$$

as the function $h(x) = 1/x$ is known [45, Example 1.25] to have the time scales derivative $h^\Delta(x) = -1/(x\sigma(x))$. □

Example 10.4 If $b_i = \infty$ for all $1 \le i \le n$ in addition to the assumptions of Corollary 10.9, then (10.10) takes the form

$$\int_0^\infty \dots \int_0^\infty \Phi\left((\widetilde{A}f)(\mathbf{x}) \right) \frac{\Delta x_1 \dots \Delta x_n}{x_1 \dots x_n} \le \int_0^\infty \dots \int_0^\infty \Phi(f(\mathbf{y})) \frac{\Delta y_1 \dots \Delta y_n}{y_1 \dots y_n}.$$

For $\mathbb{T} = \mathbb{N}$ and $n = 1$, this result is given in [41, 86].

Now we give an inequality of Hardy–Hilbert type on time scales.

Theorem 10.4 *Assume (H_1') with $n = 1$, $a_1 = 0$, and $b_1 = \infty$. If we define*

$$K_1(x) = \int_0^\infty \frac{\left(\frac{y}{x}\right)^{-\frac{1}{p}}}{x+y} \Delta y \quad \text{and} \quad K_2(y) = \int_0^\infty \frac{\left(\frac{y}{x}\right)^{1-\frac{1}{p}}}{x+y} \Delta x,$$

then

$$\int_0^\infty (K_1(x))^{1-p} \left(\int_0^\infty \frac{g(y)}{x+y} \Delta y \right)^p \Delta x \le \int_0^\infty K_2(y)(g(y))^p \Delta y \tag{10.11}$$

holds for all ν_Δ-integrable $g : Y \to \mathbb{R}_+$.

Proof. We use

$$\xi(x) = \frac{K_1(x)}{x} \quad \text{and} \quad k(x,y) = \begin{cases} \frac{\left(\frac{y}{x}\right)^{-1/p}}{x+y} & \text{if} \quad x \neq 0,\ y \neq 0,\ x+y \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

in Corollary 10.1 to obtain

$$\int_0^\infty (K_1(x))^{1-p} \left(\int_0^\infty \frac{\left(\frac{y}{x}\right)^{-1/p} f(y)}{x+y} \Delta y \right)^p \frac{\Delta x}{x} \le \int_0^\infty w(y)(f(y))^p \Delta y, \tag{10.12}$$

where

$$\begin{aligned} w(y) &= \int_0^\infty \frac{k(x,y)\xi(x)}{K_1(x)}\Delta x = \int_0^\infty \frac{k(x,y)\Delta x}{x} \\ &= \frac{1}{y}\int_0^\infty \frac{\left(\frac{y}{x}\right)^{1-1/p}}{x+y}\Delta x = \frac{K_2(y)}{y}. \end{aligned}$$

Using this in (10.12) and letting $f(y) = g(y)y^{-\frac{1}{p}}$, we obtain (10.11). □

Example 10.5 If we take $\mathbb{T} = \mathbb{R}$ in Theorem 10.4 and use the known fact that

$$\int_0^\infty \frac{\left(\frac{y}{x}\right)^{-1/p}}{x+y}\mathrm{d}y = \int_0^\infty \frac{\left(\frac{y}{x}\right)^{1-1/p}}{x+y}\mathrm{d}x = \frac{\pi}{\sin(\pi/p)},$$

then (10.11) turns into the classical Hilbert inequality (see e.g., [68])

$$\int_0^\infty \left(\int_0^\infty \frac{g(y)}{x+y}\mathrm{d}y\right)^p \mathrm{d}x \le \left(\frac{\pi}{\sin(\pi/p)}\right)^p \int_0^\infty (g(y))^p \mathrm{d}y.$$

Now we consider some generalizations of the inequalities of Pólya–Knopp type.

Corollary 10.10 *Assume* (H_1') *with* $n = 1$, $a_1 = a \ge 0$, $b_1 = \infty$. *Suppose* (10.7). *If* $\Phi \in \mathrm{C}(I,\mathbb{R})$ *is convex, where* $I \subset \mathbb{R}$ *is an interval, then*

$$\int_a^\infty \xi(x)\Phi\left(\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(y)\Delta y\right)\Delta x \le \int_a^\infty \left(\int_y^\infty \frac{\xi(x)\Delta x}{\sigma(x)-a}\right)\Phi(f(y))\Delta y \quad (10.13)$$

holds for all ν_Δ*-integrable* $f : Y \to \mathbb{R}$ *such that* $f(Y) \subset I$.

Proof. The statement follows from Theorem 10.3 by using $n = 1$. □

Example 10.6 In addition to the assumptions of Corollary 10.10, if $\mathbb{T}$ consists of only isolated points, then (10.13) takes the form

$$\sum_{x\in[a,\infty)_\mathbb{T}} \xi(x)\Phi\left(\frac{1}{\sigma(x)-a}\sum_{y\in[a,x]_\mathbb{T}} f(y)(\sigma(y)-y)\right)(\sigma(x)-x) \le \sum_{y\in[a,\infty)_\mathbb{T}}\left(\sum_{x\in[y,\infty)_\mathbb{T}} \xi(x)\frac{\sigma(x)-x}{\sigma(x)-a}\right)\Phi(f(y))(\sigma(y)-y). \quad (10.14)$$

This result is the same as [130, Theorem 1.1], but here we use time scales notation instead of the notation given in [130].

Remark 10.3 As in Example 10.6, one can write the discrete version of (10.8).

In the following three examples, we consider Example 10.6 with $\Phi(r) = r^p$, where $p > 1$.

Example 10.7 For $\mathbb{T} = h\mathbb{N} = \{hn : n \in \mathbb{N}\}$ with $h > 0$, $a = h$, and

$$\xi(x) = \frac{1}{\sigma(x)},$$

(10.14) takes the form

$$\sum_{n=1}^{\infty} \frac{1}{n+1} \left(\frac{1}{n} \sum_{k=1}^{n} f(kh) \right)^p \leq \sum_{n=1}^{\infty} \frac{(f(nh))^p}{n}. \tag{10.15}$$

Example 10.8 For $\mathbb{T} = \mathbb{N}^2 = \{n^2 : n \in \mathbb{N}\}$ with $a = 1$ and

$$\xi(x) = \frac{2(\sigma(x)-1)}{(\sigma(x)-x)^2(2\sqrt{x}+3)},$$

(10.14) takes the form

$$\sum_{n=1}^{\infty} \frac{2\,(n(n+2))^{1-p}}{(2n+1)(2n+3)} \left(\sum_{k=1}^{n} (2k+1) f(k^2) \right)^p \leq \sum_{n=1}^{\infty} (f(n^2))^p.$$

If instead

$$\xi(x) = \frac{\sigma(x)-1}{x\sigma(x)},$$

then (10.14) takes the form

$$\sum_{n=1}^{\infty} \frac{(2n+1)(n+2)^{1-p}}{n^{p+1}(n+1)^2} \left(\sum_{k=1}^{n} (2k+1) f(k^2) \right)^p \leq \sum_{n=1}^{\infty} \frac{2n+1}{n^2} (f(n^2))^p.$$

Example 10.9 For $\mathbb{T} = q^{\mathbb{N}} = \{q^n : n \in \mathbb{N}\}$ with $q > 1$, $a = q$, and

$$\xi(x) = \frac{\sigma(x)-a}{\sigma(x)(\sigma(x)-x)},$$

(10.14) takes the form

$$\sum_{n=1}^{\infty} q^{-n}(q-1)^p(q^n-1)^{1-p} \left(\sum_{k=1}^{n} q^{k-1} f(q^k) \right)^p \leq \sum_{n=1}^{\infty} (f(q^n))^p. \tag{10.16}$$

In the following three examples, we consider Example 10.6 with $\Phi(r) = e^r$ and $f(y) = \ln(g(y))$ for $g(y) > 0$.

Example 10.10 For $\mathbb{T}$, a, and ξ as in Example 10.7, (10.14) takes the form

$$\sum_{n=1}^{\infty} \frac{1}{n+1} \left(\prod_{k=1}^{n} g(kh) \right)^{\frac{1}{n}} \leq \sum_{n=1}^{\infty} \frac{g(nh)}{n}. \tag{10.17}$$

If we let $\varphi(y) = g(y)/y$ in (10.17), then we get

$$\sum_{n=1}^{\infty} \frac{1}{n+1} \left(n! \prod_{k=1}^{n} \varphi(kh) \right)^{\frac{1}{n}} \leq \sum_{n=1}^{\infty} \varphi(nh). \tag{10.18}$$

Since $e^{-1} < (n!)^{\frac{1}{n}}/(n+1)$, from (10.18) we obtain

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} \varphi(kh) \right)^{\frac{1}{n}} \leq e \sum_{n=1}^{\infty} \varphi(nh),$$

which is the well-known Carleman inequality [85, p. 141].

Example 10.11 For $\mathbb{T}$, a, and the two choices of ξ as in Example 10.8, (10.14) takes the forms

$$\sum_{n=1}^{\infty} \frac{2n(n+2)}{(2n+1)(2n+3)} \left(\prod_{k=1}^{n} (g(k^2))^{2k+1} \right)^{\frac{1}{n(n+2)}} \leq \sum_{n=1}^{\infty} g(n^2)$$

and

$$\sum_{n=1}^{\infty} \frac{(2n+1)(n+2)}{n(n+1)^2} \left(\prod_{k=1}^{n} (g(k^2))^{2k+1} \right)^{\frac{1}{n(n+2)}} \leq \sum_{n=1}^{\infty} \frac{2n+1}{n^2} g(n^2).$$

Example 10.12 For $\mathbb{T}$, a, and ξ as in Example 10.9, (10.14) takes the form

$$\sum_{n=1}^{\infty} q^{-n}(q^n - 1) \left(\prod_{k=1}^{n} (g(q^k))^{q^{k-1}} \right)^{\frac{q-1}{q^n-1}} \leq \sum_{n=1}^{\infty} g(q^n). \tag{10.19}$$

Remark 10.4 For $h = 1$, inequalities (10.15) and (10.18) are given in [102, (12.6), (12.7), p. 153]. Also, (10.16) and (10.19) are the same as [102, (12.1), (12.2), p. 153].

10.2 Hardy-Type Inequalities via Convexity in Several Variables

Here we extend the results of Section 10.1 using convex functions of multivariable with general kernels to arbitrary time scales. The main result of this section is a direct consequence of Theorem 10.1, but with different interesting applications. Results of this section are contained in [52].

10.2.1 Inequalities with General Kernels

Theorem 10.5 *Assume* (H_1)–(H_3). *If* $U \subset \mathbb{R}^m$ *is a closed convex set such that* $\Phi \in C(U,\mathbb{R})$ *is convex, then*

$$\int_X \xi(x)\Phi\left(\frac{1}{K(x)}\int_Y k(x,y)\mathbf{f}(y)\Delta y\right)\Delta x \le \int_Y w(y)\Phi(\mathbf{f}(y))\Delta y \tag{10.20}$$

holds for all ν_Δ*-integrable functions* $\mathbf{f}: Y \to \mathbb{R}^m$ *such that* $\mathbf{f}(Y) \subset U$.

Proof. Using Jensen's inequality (4.1) for several variables and the Fubini theorem on time scales, we find that

$$\begin{aligned}
&\int_X \xi(x)\Phi\left(\frac{1}{K(x)}\int_Y k(x,y)\mathbf{f}(y)\Delta y\right)\Delta x\\
&= \int_X \xi(x)\Phi\left(\frac{1}{K(x)}\int_Y k(x,y)f_1(y)\Delta y,\ldots,\frac{1}{K(x)}\int_Y k(x,y)f_m(y)\Delta y\right)\Delta x\\
&\le \int_X \frac{\xi(x)}{K(x)}\left(\int_Y k(x,y)\Phi(\mathbf{f}(y))\Delta y\right)\Delta x\\
&= \int_Y \Phi(\mathbf{f}(y))\left(\int_X \frac{k(x,y)\xi(x)}{K(x)}\Delta x\right)\Delta y\\
&= \int_Y w(y)\Phi(\mathbf{f}(y))\Delta y.
\end{aligned}$$

The proof is therefore complete. □

Remark 10.5 If Φ is concave, then (10.20) holds in reverse direction.

Corollary 10.11 *Assume* (H_1)–(H_3). *Let* $\Psi : [l_1,l_1') \times \ldots \times [l_m,l_m') \to \mathbb{R}_+$ *be continuous and define*

$$\tilde{L}_j(f_j,Y) = L_j^{-1}\left(\frac{\int_Y k(x,y)L_j(f_j(y))\Delta y}{K(x)}\right)$$

for all $j \in \{1,\ldots,m\}$. *If* $\Phi(s_1,\ldots,s_m) = \Psi(L_1^{-1}(s_1),\ldots,L_m^{-1}(s_m))$ *is convex, then*

$$\int_X \xi(x)\Psi\left(\tilde{L}_1(f_1,Y),\ldots,\tilde{L}_m(f_m,Y)\right)\Delta x \le \int_Y w(y)\Psi(f_1(y),\ldots,f_m(y))\Delta y$$

holds for all $f_j(Y) \subset [l_j,l_j')$ *and continuous monotone functions* $L_j : [l_j,l_j') \to \mathbb{R}$ *such that* $L_j \circ f_j$ *are* ν_Δ*-integrable for all* $j \in \{1,\ldots,m\}$.

Proof. Replace in Theorem 10.5 $f_j(y)$ by $L_j(f_j(y))$ for all $j \in \{1,\ldots,m\}$ and $\Phi(s_1,\ldots,s_m)$ by $\Psi(L_1^{-1}(s_1),\ldots,L_m^{-1}(s_m))$. □

Remark 10.6 In case $\mathbb{T} = \mathbb{N}$ and $m = 2$, Corollary 10.11 is [130, Corollary 1.2].

Remark 10.7 In case $m=2$, we can use the results of E. Beck [29] (see also [103, p. 194]) as applications of Corollary 10.11, which correspond to the generalizations of Hölder's and Minkowski's inequalities. In the classical case, many authors have studied these types of generalizations, see, e.g., [74, 73, 92].

Further in this section, we use $p>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 10.12 *Assume* (H_1)–(H_3). *If* $\boldsymbol{f}(x)=(f_1(x),f_2(x))$, *then*

$$\int_X \xi(x)\left(\frac{1}{K(x)}\int_Y k(x,y)(f_1(y))^p\Delta y\right)^{\frac{1}{p}} \left(\frac{1}{K(x)}\int_Y k(x,y)(f_2(y))^q\Delta y\right)^{\frac{1}{q}}\Delta x \geq \int_Y w(y)f_1(y)f_2(y)\Delta y$$

holds for all ν_Δ*-integrable* $f_j: Y\to\mathbb{R}_+$, *where* $j\in\{1,2\}$.

Proof. Use $m=2, \Psi(s_1,s_2)=s_1s_2, L_1(t_1)=t_1^p, L_2(t_2)=t_2^q$ in Corollary 10.11. Then $\Phi(s_1,s_2)=s_1^{\frac{1}{p}}s_2^{\frac{1}{q}}$ is concave in Theorem 10.5. □

Corollary 10.13 *Assume* (H_1)–(H_3). *If* $\boldsymbol{f}(x)=(f_1(x),f_2(x))$, *then*

$$\int_X \xi(x)\left(\left(\frac{1}{K(x)}\int_Y k(x,y)(f_1(y))^p\Delta y\right)^{\frac{1}{p}} + \left(\frac{1}{K(x)}\int_Y k(x,y)(f_2(y))^p\Delta y\right)^{\frac{1}{p}}\right)^p\Delta x \geq \int_Y w(y)(f_1(y)+f_2(y))^p\Delta y$$

holds for all ν_Δ*-integrable* $f_j: Y\to\mathbb{R}_+$, *where* $j\in\{1,2\}$.

Proof. Use $m=2, \Psi(s_1,s_2)=(s_1+s_2)^p, L_1(t_1)=t_1^p, L_2(t_2)=t_2^p$ in Corollary 10.11. Then $\Phi(s_1,s_2)=(s_1^{\frac{1}{p}}+s_2^{\frac{1}{p}})^p$ is concave in Theorem 10.5. □

Remark 10.8 If $p<1$, then the reverse inequalities hold in Corollary 10.11, Corollary 10.12, and Corollary 10.13.

10.2.2 Inequalities with Special Kernels

Corollary 10.14 *Assume* (H_1') *and*

$$\xi : X \to \mathbb{R}_+ \quad \textit{is such that}$$

$$w(\mathbf{y}) = \int_{y_1}^{b_1} \dots \int_{y_n}^{b_n} \frac{k(\mathbf{x},\mathbf{y})\xi(\mathbf{x})}{K(\mathbf{x})} \Delta x_1 \dots \Delta x_n, \quad \mathbf{y} \in Y. \tag{10.21}$$

If $\Phi \in \mathrm{C}(U,\mathbb{R})$ *is convex, where* $U \subset \mathbb{R}^m$ *is a closed convex set, then*

$$\int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} \xi(\mathbf{x})\Phi((A_k\mathbf{f})(\mathbf{x}))\Delta x_1 \dots \Delta x_n \\ \le \int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} w(\mathbf{y})\Phi(\mathbf{f}(\mathbf{y}))\Delta y_1 \dots \Delta y_n \tag{10.22}$$

holds for all ν_Δ*-integrable* $\mathbf{f} : Y \to \mathbb{R}^m$ *such that* $\mathbf{f}(Y) \subset U$*, where*

$$(A_k\mathbf{f})(\mathbf{x}) = \frac{1}{K(\mathbf{x})} \int_{a_1}^{\sigma(x_1)} \dots \int_{a_n}^{\sigma(x_n)} k(\mathbf{x},\mathbf{y})\mathbf{f}(\mathbf{y})\Delta y_1 \dots \Delta y_n.$$

Proof. The statement follows from Theorem 10.5 by using

$$k(\mathbf{x},\mathbf{y}) = 0, \quad \text{if} \quad a_i \le y_i \le \sigma(x_i) \quad \text{for all} \quad i \in \{1,\dots,n\},$$

since in this case

$$K(\mathbf{x}) = \int_{a_1}^{\sigma(x_1)} \dots \int_{a_n}^{\sigma(x_n)} k(\mathbf{x},\mathbf{y})\Delta y_1 \dots \Delta y_n.$$

This completes the proof. □

Corollary 10.15 *Assume*

$$\xi : X \to \mathbb{R}_+ \quad \textit{is such that}$$

$$w(\mathbf{y}) = \int_{a_1}^{y_1} \dots \int_{a_n}^{y_n} \frac{k(\mathbf{x},\mathbf{y})\xi(\mathbf{x})}{K(\mathbf{x})} \Delta x_1 \dots \Delta x_n, \quad \mathbf{y} \in Y. \tag{10.23}$$

If $\Phi \in \mathrm{C}(U,\mathbb{R})$ *is convex, where* $U \subset \mathbb{R}^m$ *is a closed convex set, then* (10.22) *holds for all* ν_Δ*-integrable* $\mathbf{f} : Y \to \mathbb{R}^m$ *such that* $\mathbf{f}(Y) \subset U$*, and*

$$(A_k\mathbf{f})(\mathbf{x}) = \frac{1}{K(\mathbf{x})} \int_{\sigma(x_1)}^{b_1} \dots \int_{\sigma(x_n)}^{b_n} k(\mathbf{x},\mathbf{y})\mathbf{f}(\mathbf{y})\Delta y_1 \dots \Delta y_n.$$

Proof. The statement follows from Theorem 10.5 by using

$$k(\mathbf{x},\mathbf{y}) = 0, \quad \text{if } a_i \le \sigma(x_i) \le y_i \text{ for all } i \in \{1,\dots,n\},$$

since in this case

$$K(\mathbf{x}) = \int_{\sigma(x_1)}^{b_1} \dots \int_{\sigma(x_n)}^{b_n} k(\mathbf{x},\mathbf{y})\Delta y_1 \dots \Delta y_n.$$

The proof is complete. □

Theorem 10.6 *Assume* (H_1') *and*

$\xi : X \to \mathbb{R}_+$ *is such that*

$$\widetilde{w}(\mathbf{y}) = \int_{y_1}^{b_1} \dots \int_{y_n}^{b_n} \frac{\xi(\mathbf{x})}{\prod_{i=1}^{n}(\sigma(x_i)-a_i)} \Delta x_1 \dots \Delta x_n, \quad \mathbf{y} \in Y. \tag{10.24}$$

If $\Phi \in \mathrm{C}(U,\mathbb{R})$ *is convex, where* $U \subset \mathbb{R}^m$ *is a closed convex set, then*

$$\int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} \xi(\mathbf{x})\Phi\left((\widetilde{A}_k\mathbf{f})(\mathbf{x})\right)\Delta x_1 \dots \Delta x_n \le \int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} \widetilde{w}(\mathbf{y})\Phi(\mathbf{f}(\mathbf{y}))\Delta y_1 \dots \Delta y_n \tag{10.25}$$

holds for all ν_Δ*-integrable* $\mathbf{f} : Y \to \mathbb{R}^m$ *such that* $\mathbf{f}(Y) \subset U$*, where*

$$(\widetilde{A}_k\mathbf{f})(\mathbf{x}) = \frac{1}{\prod_{i=1}^{n}(\sigma(x_i)-a_i)} \int_{a_1}^{\sigma(x_1)} \dots \int_{a_n}^{\sigma(x_n)} \widetilde{\mathbf{f}}(\mathbf{y})\Delta y_1 \dots \Delta y_n.$$

Proof. The statement follows from Theorem 10.5 by using

$$k(\mathbf{x},\mathbf{y}) = \left\{ \begin{array}{ll} 1 & \text{if } a_i \le y_i < \sigma(x_i) \le b_i,\ i \in \{1,\dots,n\} \\ 0 & \text{otherwise} \end{array} \right\},$$

since in this case

$$K(\mathbf{x}) = \int_{a_1}^{\sigma(x_1)} \dots \int_{a_n}^{\sigma(x_n)} \Delta y_1 \dots \Delta y_n = \prod_{i=1}^{n}(\sigma(x_i)-a_i).$$

Thus $A_k = \widetilde{A}_k, w = \widetilde{w}$. □

Corollary 10.16 *Assume* (H_1') *with* $a_i = 0$ *for all* $i \in \{1,\dots,n\}$*. If* $\Phi \in \mathrm{C}(U,\mathbb{R})$ *is convex, where* $U \subset \mathbb{R}^m$ *is a closed convex set, then*

$$\int_0^{b_1} \dots \int_0^{b_n} \Phi((A_k\mathbf{f})(\mathbf{x})) \frac{\Delta x_1 \dots \Delta x_n}{x_1 \dots x_n}$$
$$\leq \int_0^{b_1} \dots \int_0^{b_n} \prod_{i=1}^n \left(\frac{1}{y_i} - \frac{1}{b_i}\right) \Phi(\mathbf{f}(\mathbf{y}))\Delta y_1 \dots \Delta y_n \quad (10.26)$$

holds for all ν_Δ*-integrable* $\mathbf{f}: Y \to \mathbb{R}^m$ *such that* $\mathbf{f}(Y) \subset U$*, where*

$$(A_k\mathbf{f})(\mathbf{x}) = \frac{1}{\prod_{i=1}^n \sigma(x_i)} \int_0^{\sigma(x_1)} \dots \int_0^{\sigma(x_n)} \mathbf{f}(\mathbf{y})\Delta y_1 \dots \Delta y_n.$$

Proof. The statement follows from Theorem 10.6 by using

$$\xi(\mathbf{x}) = \frac{1}{x_1 \dots x_n},$$

since in this case

$$w(\mathbf{y}) = \int_{y_1}^{b_1} \dots \int_{y_n}^{b_n} \frac{1}{\prod_{i=1}^n x_i \sigma(x_i)} \Delta x_1 \dots \Delta x_n = \prod_{i=1}^n \left(\frac{1}{y_i} - \frac{1}{b_i}\right) \quad (10.27)$$

holds. □

Example 10.13 If $b_i = \infty$ for all $i \in \{1, \dots, n\}$ in addition to the assumptions of Corollary 10.16, then (10.26) takes the form

$$\int_0^{\infty} \dots \int_0^{\infty} \Phi((A_k\mathbf{f})(\mathbf{x})) \frac{\Delta x_1 \dots \Delta x_n}{x_1 \dots x_n}$$
$$\leq \int_0^{\infty} \dots \int_0^{\infty} \Phi(\mathbf{f}(\mathbf{y})) \frac{\Delta y_1 \dots \Delta y_n}{y_1 \dots y_n}. \quad (10.28)$$

Remark 10.9 Clearly, if the left-hand side is ∞ in (10.26), then the right-hand side is also ∞.

Remark 10.10 For $\mathbb{T} \nleftrightarrow \mathbb{R}$ and $m = 1$, the inequality (10.28) is proved in [41, 86].

10.2.3 Some Particular Cases

Here, let us start with the inequality of Hilbert type on time scales.

Theorem 10.7 *Assume* (H_1') *with* $n = 1, a_1 = 0,$ *and* $b_1 = \infty$*. If we define*

$$K_1(x) = \int_0^{\infty} \frac{\left(\frac{y}{x}\right)^{-\frac{1}{q}}}{x+y} \Delta y, \; K_2(y) = \int_0^{\infty} \frac{\left(\frac{y}{x}\right)^{1-\frac{1}{q}}}{x+y} \Delta x \quad (10.29)$$

for $q > 1$*, then*

$$\int_0^\infty \left(\int_0^\infty \frac{(\frac{y}{x})^{1-1/q}(g_1(y))^p}{x+y} \Delta y \right)^{\frac{1}{p}} \left(\int_0^\infty \frac{(\frac{y}{x})^{1-1/q}(g_2(y))^{p'}}{x+y} \Delta y \right)^{\frac{1}{p'}} \Delta x$$
$$\geq \int_0^\infty K_2(y) g_1(y) g_2(y) \Delta y \quad (10.30)$$

holds for all ν_Δ*-integrable* $g_j : Y \to \mathbb{R}_+$, *where* $j \in \{1,2\}$.

Proof. We use

$$\xi(x) = \frac{K_1(x)}{x} \quad \text{and} \quad k(x,y) = \left\{ \begin{array}{ll} \frac{(\frac{y}{x})^{-1/q}}{x+y} & x \neq 0, y \neq 0, x+y \neq 0 \\ 0 & \text{otherwise} \end{array} \right\},$$

in Corollary 10.12 to obtain

$$\int_0^\infty \frac{K_1(x)}{x} \left(\frac{1}{K_1(x)} \int_0^\infty \frac{(\frac{y}{x})^{-1/q}(f_1(y))^p}{x+y} \Delta y \right)^{\frac{1}{p}} \left(\frac{1}{K_1(x)} \int_0^\infty \frac{(\frac{y}{x})^{-1/q}(f_2(y))^{p'}}{x+y} \Delta y \right)^{\frac{1}{p'}} \Delta x$$
$$\geq \int_0^\infty w(y) f_1(y) f_2(y) \Delta y, \quad (10.31)$$

where

$$w(y) = \int_0^\infty \frac{k(x,y)\xi(x)}{K_1(x)} \Delta x = \int_0^\infty \frac{k(x,y)}{x} \Delta x = \frac{1}{y} \int_0^\infty \frac{(\frac{y}{x})^{1-1/q}}{x+y} \Delta x = \frac{K_2(y)}{y}.$$

Using this value in (10.31), we obtain

$$\int_0^\infty \left(\int_0^\infty \frac{(\frac{y}{x})^{-1/q}(f_1(y))^p}{x+y} \Delta y \right)^{\frac{1}{p}} \left(\int_0^\infty \frac{(\frac{y}{x})^{-1/q}(f_2(y))^{p'}}{x+y} \Delta y \right)^{\frac{1}{p'}} \frac{\Delta x}{x}$$
$$\geq \int_0^\infty K_2(y) f_1(y) f_2(y) \frac{\Delta y}{y}. \quad (10.32)$$

Now, if we replace $f_1(y)$ by $g_1(y)y^{1/p}$ and $f_2(y)$ by $g_2(y)y^{1/p'}$, then we obtain (10.30). □

Proof.[Another Proof of (10.32)] Consider the left-hand side of (10.31) and apply Hölder inequality on time scale [20, Theorem 6.2] and Fubini theorem on time scale [34, Theorem 1.1].Then we have

$$\int_0^\infty \frac{K_1(x)}{x}\left(\frac{1}{K_1(x)}\int_0^\infty \frac{(\frac{y}{x})^{-1/q}(f_1(y))^p}{x+y}\Delta y\right)^{\frac{1}{p}}\left(\frac{1}{K_1(x)}\int_0^\infty \frac{(\frac{y}{x})^{-1/q}(f_2(y))^{p'}}{x+y}\Delta y\right)^{\frac{1}{p'}}\Delta x$$
$$\geq \int_0^\infty\left(\int_0^\infty \frac{(\frac{y}{x})^{-1/q}f_1(y)f_2(y)}{x+y}\Delta y\right)\frac{\Delta x}{x} = \int_0^\infty f_1(y)f_2(y)\left(\int_0^\infty \frac{(\frac{y}{x})^{-1/q}}{x+y}\frac{\Delta x}{x}\right)\Delta y$$
$$= \int_0^\infty \frac{f_1(y)f_2(y)}{y}\left(\int_0^\infty \frac{(\frac{y}{x})^{1-1/q}}{x+y}\Delta x\right)\Delta y = \int_0^\infty K_2(y)f_1(y)f_2(y)\frac{\Delta y}{y},$$

i.e., (10.32) holds. □

Example 10.14 It is known that

$$\int_0^\infty \frac{(\frac{y}{x})^{-1/q}}{x+y}\mathrm{d}y = \int_0^\infty \frac{(\frac{y}{x})^{1-1/q}}{x+y}\mathrm{d}x = \frac{\pi}{\sin(\pi/q)} \tag{10.33}$$

for all $x,y\in\mathbb{R}_+=(0,\infty)$ with $q>1$. If $\mathbb{T}=\mathbb{R}$, then from (10.30) we obtain

$$\int_0^\infty\left(\int_0^\infty \frac{(\frac{y}{x})^{1-1/q}(f_1(y))^p}{x+y}\mathrm{d}y\right)^{\frac{1}{p}}\left(\int_0^\infty \frac{(\frac{y}{x})^{1-1/q}(f_2(y))^{p'}}{x+y}\mathrm{d}y\right)^{\frac{1}{p'}}\mathrm{d}x$$
$$\geq \frac{\pi}{\sin(\pi/q)}\int_0^\infty f_1(y)f_2(y)\mathrm{d}y.$$

In the rest of this section, we take $n=1, a\geq 0, b=\infty$ in (H_1').

Theorem 10.8 *If* (10.29) *is satisfied, then*

$$\int_0^\infty\left(\left(\int_0^\infty \frac{(\frac{y}{x})^{1-1/q}(f_1(y))^p}{x+y}\Delta y\right)^{\frac{1}{p}}+\left(\int_0^\infty \frac{(\frac{y}{x})^{1-1/q}(f_2(y))^p}{x+y}\Delta y\right)^{\frac{1}{p}}\right)^p\Delta x$$
$$\geq \int_0^\infty K_2(y)(f_1(y)+f_2(y))^p\Delta y \tag{10.34}$$

holds for all ν_Δ-integrable $f_j: Y\to\mathbb{R}_+$, where $j\in\{1,2\}$.

Proof. We use

$$\xi(x)=\frac{K_1(x)}{x} \quad\text{and}\quad k(x,y)=\left\{\begin{array}{ll}\frac{(\frac{y}{x})^{-1/q}}{x+y} & x\neq 0, y\neq 0, x+y\neq 0\\ 0 & \text{otherwise}\end{array}\right\},$$

in Corollary 10.12 to obtain

$$\int_0^\infty \left(\left(\int_0^\infty \frac{(\frac{y}{x})^{-1/q}(f_1(y))^p}{x+y}\Delta y \right)^{\frac{1}{p}} + \left(\int_0^\infty \frac{(\frac{y}{x})^{-1/q}(f_2(y))^p}{x+y}\Delta y \right)^{\frac{1}{p}} \right)^p \frac{\Delta x}{x}$$

$$\geq \int_0^\infty K_2(y)(f_1(y)+f_2(y))^p \frac{\Delta y}{y}. \quad (10.35)$$

Now, if we replace $f_1(y)$ by $f_1(y)y^{1/p}$ and $f_2(y)$ by $f_2(y)y^{1/p}$, then we obtain (10.34). □

Remark 10.11 (a) We can give another proof of (10.35) using Minkowski's inequality on time scales [20, Theorem 7.2].

(b) If $p < 1$, then we have reverse inequalities.

Now we consider some generalization of the inequalities of Pólya–Knopp type.

Corollary 10.17 *Let* (10.24) *hold. If* $\Phi \in \mathrm{C}(U,\mathbb{R})$ *is convex, where* $U \subset \mathbb{R}^m$ *is a closed convex set, then*

$$\int_a^\infty \xi(x)\Phi\left(\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} \mathbf{f}(y)\Delta y\right)\Delta x \leq \int_a^\infty w(y)\Phi(\mathbf{f}(y))\Delta y$$

holds for all ν_Δ*-integrable* $\mathbf{f}: Y \to \mathbb{R}^m$ *such that* $\mathbf{f}(Y) \subset U$.

Proof. The statement follows from Theorem 10.6 by using $n = 1$. □

Corollary 10.18 *Assume* (10.24). *Then*

$$\int_a^\infty \xi(x)\left(\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} (f_1(y))^p\Delta y\right)^{\frac{1}{p}}$$

$$\left(\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} (f_1(y))^{p'}\Delta y\right)^{\frac{1}{p'}}\Delta x \geq \int_a^\infty w(y)f_1(y)f_2(y)\Delta y$$

holds for all ν_Δ*-integrable* $f_j : Y \to \mathbb{R}_+$, *where* $j \in \{1,2\}$.

Proof. The statement follows from Corollary 10.12 by using $m = 2$. □

Corollary 10.19 *Assume* (10.24). *Then*

$$\int_a^\infty \xi(x)\left(\left(\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)}(f_1(y))^p\Delta y\right)^{\frac{1}{p}}\right.$$

$$\left. +\left(\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} (f_2(y))^p\Delta y\right)^{\frac{1}{p}}\right)^p \Delta x \geq \int_a^\infty w(y)(f_1(y)+f_2(y))^p\Delta y$$

holds for all ν_Δ*-integrable* $f_j : Y \to \mathbb{R}_+$, *where* $j \in \{1,2\}$.

Proof. The statement follows from Corollary 10.13 by using $m = 2$. □

Example 10.15 If $\mathbb{T}$ consists of isolated points, then from Corollary 10.18 we have

$$\sum_{x\in[a,\infty)} \frac{\xi(x)(\sigma(x)-x)}{\sigma(x)-a}\left(\sum_{y\in[a,\sigma(x))}(f_1(y))^p(\sigma(y)-y)\right)^{\frac{1}{p}} \left(\sum_{y\in[a,\sigma(x))}(f_2(y))^{p'}(\sigma(y)-y)\right)^{\frac{1}{p'}} \geq \sum_{y\in[a,\infty)} w(y)f_1(y)f_2(y)(\sigma(y)-y), \quad (10.36)$$

where $w(y) = \sum_{x\in[y,\infty)} \frac{\xi(x)(\sigma(x)-x)}{(\sigma(x)-a)}$.

Example 10.16 If $\mathbb{T}$ consists of isolated points, then from Corollary 10.19 we have

$$\sum_{x\in[a,\infty)} \frac{\xi(x)(\sigma(x)-x)}{\sigma(x)-a}\left(\left(\sum_{y\in[a,\sigma(x))}(f_1(y))^p(\sigma(y)-y)\right)^{\frac{1}{p}} + \left(\sum_{y\in[a,\sigma(x))}(f_2(y))^p(\sigma(y)-y)\right)^{\frac{1}{p}}\right)^p \geq \sum_{y\in[a,\infty)} w(y)(f_1(y)+f_2(y))^p(\sigma(y)-y), \quad (10.37)$$

where $w(y)$ is the same as in Corollary 10.15.

Example 10.17 For $\mathbb{T} = h\mathbb{N} = \{hn : n \in \mathbb{N}\}$ with $h > 0$, $a = 1$, and $\xi(x) = \frac{1}{\sigma(x)}$, (10.36) takes the form

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}\left(\sum_{k=1}^{n}(f_1(kh))^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n}(f_2(kh))^{p'}\right)^{\frac{1}{p'}} \geq \sum_{n=1}^{\infty}\frac{1}{n}f_1(nh)f_2(nh), \quad (10.38)$$

and (10.37) takes the form

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}\left(\left(\sum_{k=1}^{n}(f_1(kh))^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n}(f_2(kh))^p\right)^{\frac{1}{p}}\right)^p \geq \sum_{n=1}^{\infty}\frac{1}{n}(f_1(nh)+f_2(nh))^p. \quad (10.39)$$

.

Example 10.18 For $\mathbb{T}\leftrightarrow\mathbb{N}^2=\{n^2:n\in\mathbb{N}\}$, $a=1$, and

$$\xi(x)=\frac{2(\sigma(x)-1)}{(\sigma(x)-x)^2(2\sqrt{x}+3)},$$

(10.36) takes the form

$$\sum_{n=1}^{\infty}\frac{2}{(2n+1)(2n+3)}\left(\sum_{k=1}^{n}(2k+1)(f_1(k^2))^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n}(2k+1)(f_2(k^2))^{p'}\right)^{\frac{1}{p'}}\geq\sum_{k=1}^{\infty}f_1(k^2)f_2(k^2),\quad(10.40)$$

and (10.37) takes the form

$$\sum_{n=1}^{\infty}\frac{2}{(2n+1)(2n+3)}\left(\left(\sum_{k=1}^{n}(2k+1)(f_1(k^2))^p\right)^{\frac{1}{p}}+\left(\sum_{k=1}^{n}(2k+1)(f_2(k^2))^p\right)^{\frac{1}{p}}\right)^p\geq\sum_{k=1}^{\infty}\left(f_1(k^2)+f_2(k^2)\right)^p.\quad(10.41)$$

If we take

$$\xi(x)=\frac{\sigma(x)-1}{x\sigma(x)},$$

then (10.36) takes the form

$$\sum_{n=1}^{\infty}\frac{(2n+1)}{n^2(n+1)^2}\left(\sum_{k=1}^{n}(2k+1)(f_1(k^2))^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n}(2k+1)(f_2(k^2))^{p'}\right)^{\frac{1}{p'}}\geq\sum_{k=1}^{\infty}\frac{(2k+1)}{k^2}f_1(k^2)f_2(k^2),\quad(10.42)$$

and (10.37) takes the form

$$\sum_{n=1}^{\infty}\frac{(2n+1)}{n^2(n+1)^2}\left(\left(\sum_{k=1}^{n}(2k+1)(f_1(k^2))^p\right)^{\frac{1}{p}}+\left(\sum_{k=1}^{n}(2k+1)(f_2(k^2))^p\right)^{\frac{1}{p}}\right)^p\geq\sum_{k=1}^{\infty}\frac{(2k+1)}{k^2}\left(f_1(k^2)+f_2(k^2)\right)^p.\quad(10.43)$$

By replacing $f_1(k^2)$ with $\left(\frac{k^2}{2k+1}\right)^{\frac{1}{p}}f_1(k^2)$ and $f_2(k^2)$ with $\left(\frac{k^2}{2k+1}\right)^{\frac{1}{p'}}f_2(k^2)$ in (10.42) and (10.43), respectively, we have

$$\sum_{n=1}^{\infty}\frac{(2n+1)}{n^2(n+1)^2}\left(\sum_{k=1}^{n}k^2(f_1(k^2))^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n}k^2(f_2(k^2))^{p'}\right)^{\frac{1}{p'}} \geq \sum_{k=1}^{\infty}f_1(k^2)f_2(k^2) \quad (10.44)$$

and

$$\sum_{n=1}^{\infty}\frac{(2n+1)}{n^2(n+1)^2}\left(\left(\sum_{k=1}^{n}k^2(f_1(k^2))^p\right)^{\frac{1}{p}}+\left(\sum_{k=1}^{n}k^2(f_2(k^2))^p\right)^{\frac{1}{p}}\right)^p$$
$$\geq \sum_{k=1}^{\infty}\left(f_1(k^2)+f_2(k^2)\right)^p. \quad (10.45)$$

Example 10.19 For $\mathbb{T}=q^{\mathbb{N}}=\{q^n : n\in\mathbb{N}\}$ with $q>1, a=q$, and

$$\xi(x)=\frac{\sigma(x)-a}{\sigma(x)(\sigma(x)-x)},$$

(10.36) takes the form

$$\sum_{n=1}^{\infty}\frac{(q-1)}{q^n}\left(\sum_{k=1}^{n}q^{k-1}(f_1(q^k))^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n}q^{k-1}(f_2(q^k))^{p'}\right)^{\frac{1}{p'}}$$
$$\geq \sum_{n=1}^{\infty}f_1(q^n)f_2(q^n), \quad (10.46)$$

and (10.37) takes the form

$$\sum_{n=1}^{\infty}\frac{(q-1)}{q^n}\left(\left(\sum_{k=1}^{n}q^{k-1}(f_1(q^k))^p\right)^{\frac{1}{p}}+\left(\sum_{k=1}^{n}q^{k-1}(f_2(q^k))^p\right)^{\frac{1}{p}}\right)^p$$
$$\geq \sum_{n=1}^{\infty}(f_1(q^n)+f_2(q^n))^p. \quad (10.47)$$

Remark 10.12 (a) In classical case, for $h=1$, the inequalities (10.38), (10.39) are the same as (1.7), (1.9). Also (10.46), (10.47) are the same as (1.6), (1.8) in [130, Corollary 1.3], respectively, while according to the authors knowledge (10.40), (10.41), (10.42), (10.43), (10.44), and (10.45) are not existing in the literature.

(b) For $p<1$, we get the reverse inequalities.

Remark 10.13 The results given in Section 10.2.3 can be proved analogously for $X=Y=[a_1,\infty)_{\mathbb{T}}\times\ldots\times[a_n,\infty)_{\mathbb{T}}$.

Remark 10.14 The results given in Corollary 10.12, Corollary 10.13, and in their given applications can also be obtained analogously for a finite value of $m>2$.

10.3 Hardy-Type Inequalities via Superquadratic Functions

In this section, we give extensions of the inequalities of Hardy type with general kernel for superquadratic functions to arbitrary time scales. Results of this section are contained in [28].

Before presenting the next results, which were recently proved by Oguntuase et al. [111], it is necessary to introduce some further notation: We use bold letters to denote n-tuples of real numbers, e.g., $\mathbf{x} = (x_1, \dots, x_n)$. Also, we set $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^n$ and $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$. Furthermore, the relations $<$, $\leq$, $>$, and $\geq$ are, as usual, defined componentwise, for example, for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we write $\mathbf{x} < \mathbf{y}$ if $x_i < y_i$ for all $i \in \{1, \dots, n\}$. Finally, we denote

$$(\mathbf{0}, \mathbf{b}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{0} < \mathbf{x} < \mathbf{b}\} \quad \text{and} \quad (\mathbf{b}, \infty) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{b} < \mathbf{x} < \infty\}.$$

Proposition 10.1 *Let $b \in (0, \infty)$, $u : (0, b) \to \mathbb{R}$ be a weight function which is locally integrable in $(0, b)$, and define v by*

$$v(t) = t_1 \dots t_n \int_{t_1}^{b_1} \dots \int_{t_n}^{b_n} \frac{u(x)}{x_1^2 \dots x_n^2} \mathrm{d}x, \quad t \in (\mathbf{0}, \mathbf{b}). \tag{10.48}$$

Suppose $I = (a, c)$, $0 \leq a < c \leq \infty$, $\varphi : I \to \mathbb{R}$, and $f : (0, b) \to \mathbb{R}$ is an integrable function such that $f(x) \in I$ for all $x \in (0, b)$.

(i) *If φ is superquadratic, then the inequality*

$$\begin{aligned} &\int_0^{b_1} \dots \int_0^{b_n} u(x) \varphi\left(\frac{1}{x_1 \dots x_n} \int_0^{x_1} \dots \int_0^{x_n} f(t) \mathrm{d}t\right) \frac{\mathrm{d}x}{x_1 \dots x_n} \\ &+ \int_0^{b_1} \dots \int_0^{b_n} \int_{t_1}^{b_1} \dots \int_{t_n}^{b_n} \varphi\left(\left|f(t) - \frac{1}{x_1 \dots x_n} \int_0^{x_1} \dots \int_0^{x_n} f(t) \mathrm{d}t\right|\right) \\ &\frac{u(x)}{x_1^2 \dots x_n^2} dx\mathrm{d}t \leq \int_0^{b_1} \dots \int_0^{b_n} v(x) \varphi(f(x)) \frac{\mathrm{d}x}{x_1 \dots x_n} \end{aligned} \tag{10.49}$$

holds.

(ii) *If φ is subquadratic, then* (10.49) *is reversed.*

Proposition 10.2 *Let $b \in [0, \infty)$, $u : (b, \infty) \to \mathbb{R}$ be a weight function which is locally integrable in $(0, b)$, and define v by*

$$v(t) = \frac{1}{t_1 \dots t_n} \int_{b_1}^{t_1} \dots \int_{b_n}^{t_n} u(x) \mathrm{d}x < \infty, t \in (\mathbf{b}, \infty). \tag{10.50}$$

Suppose $I = (a, c)$, $0 \leq a < c \leq \infty$, $\varphi : I \to \mathbb{R}$, and $f : (b, \infty) \to \mathbb{R}$ is integrable such that $f(x) \in I$ for all $x \in (b, \infty)$.

(i) *If φ is superquadratic, then the inequality*

$$\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty} u(x)\varphi\left(x_1\ldots x_n\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty} f(t)\frac{\mathrm{d}t}{t_1^2\ldots t_n^2}\right)\frac{\mathrm{d}x}{x_1\ldots x_n} \tag{10.51}$$
$$+\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty}\int_{b_1}^{t_1}\cdots\int_{b_n}^{t_n}\varphi\left(\left|f(t)-x_1\ldots x_n\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty} f(t)\frac{\mathrm{d}t}{t_1^2\ldots t_n^2}\right|\right)$$
$$u(x)\mathrm{d}x\frac{\mathrm{d}t}{t_1^2\ldots t_n^2}\le\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty} v(x)\varphi(f(x))\frac{\mathrm{d}x}{x_1\ldots x_n}$$

holds.

(ii) *If φ is subquadratic, then* (10.51) *is reversed.*

Recently, S. Abramovich, K. Krulić, J. Pečarić, and L. E. Persson proved in [7] that for superquadratic function φ and an integral operator $A_k f$ defined by (1.28) the following theorem holds.

Theorem 10.9 *Let (X_1,Σ_1,μ_1) and (X_2,Σ_2,μ_2) be measure spaces with positive σ-finite measures, u be a weight function on X_1, k a nonnegative measurable function on $X_1\times X_2$, and K be defined on X_1 by* (1.29). *Suppose that $K(x)>0$ for all $x\in X_1$ and that the function $x\mapsto u(x)\dfrac{k(x,y)}{K(x)}$ is integrable on X_1 for each fixed $y\in X_2$. Suppose $I=(0,c)$, $c\le\infty$, $\varphi: I\to\mathbb{R}$. If φ is a superquadratic function, then the inequality*

$$\int_{\Omega_1} u(x)\varphi((A_k f)(x))\,\mathrm{d}\mu_1(x)+\int_{\Omega_2}\int_{\Omega_1}\varphi\left(|f(y)-(A_k f)(x)|\right)$$
$$\frac{u(x)k(x,y)}{K(x)}\,\mathrm{d}\mu_1(x)\,\mathrm{d}\mu_2(y)\le\int_{\Omega_2} v(y)\varphi(f(y))\,\mathrm{d}\mu_2(y) \tag{10.52}$$

holds for all measurable functions $f: X_2\to\mathbb{R}$ such that $f(X_2)\subseteq I$, where A_k is defined by (1.28) *and*

$$v(y)=\int_{\Omega_1}\frac{k(x,y)}{K(x)}u(x)\mathrm{d}\mu_1(x)<\infty. \tag{10.53}$$

If φ is subquadratic, then (10.52) *is reversed.*

10.3.1 Inequalities with General Kernel

In the next theorem, we give an analogue of Theorem 10.9 for arbitrary time scale measure spaces.

Theorem 10.10 *Suppose that hypotheses (H_1)–(H_3) are valid. Let $I=[a,c)$, $0\le a<c\le\infty$, and $\varphi: I\to\mathbb{R}$. If $\varphi\in C(I,\mathbb{R})$ is superquadratic, then*

$$\int_X \varphi((A_k f)(x))\xi(x)\Delta x + \int_Y \int_X \varphi(|f(y)-(A_k f)(x)|)\frac{\xi(x)k(x,y)}{K(x)}\Delta x\Delta y \le \int_Y \varphi(f(y))w(y)\Delta y \quad (10.54)$$

holds for all $Y_\triangle$-integrable functions $f : Y \to \mathbb{R}$ such that $f(Y) \subset I$. If φ is subquadratic, then (10.54) *is reversed.*

Proof. Using Jensen's inequality (2.65) for superquadratic functions on time scales and the Fubini theorem on time scales, we have

$$\begin{aligned}
&\int_X \varphi((A_k f)(x))\xi(x)\Delta x \\
&= \int_X \xi(x)\varphi\left(\frac{1}{K(x)}\int_Y k(x,y)f(y)\Delta y\right)\Delta x \\
&\le \int_X \xi(x)\frac{1}{K(x)}\left(\int_Y k(x,y)\varphi(f(y))\Delta y\right)\Delta x \\
&\quad - \int_X \frac{\xi(x)}{K(x)}\int_Y k(x,y)\varphi(|f(y)-(A_k f)(x)|)\Delta y\Delta x \\
&= \int_Y \varphi(f(y))\left(\int_X \frac{k(x,y)}{K(x)}\xi(x)\Delta x\right)\Delta y \\
&\quad - \int_Y\int_X \varphi(|f(y)-(A_k f)(x)|)\frac{\xi(x)k(x,y)}{K(x)}\Delta x\Delta y,
\end{aligned}$$

which is equivalent to (10.54). If φ is subquadratic, then (2.65) is reversed, which implies, according to the conclusions made above, that (10.54) is reversed. □

Remark 10.15 If φ is nonnegative in Theorem 10.10, then according to Lemma 1.1 φ is convex and therefore Theorem 10.10 gives a refinement of Theorem 10.1.

Corollary 10.20 *Assume* (H_1)–(H_3). *If $p \ge 2$, then*

$$\int_X A_k^p f(x)\xi(x)\Delta x + \int_Y\int_X |f(y)-(A_k f)(x)|^p \frac{\xi(x)k(x,y)}{K(x)}\Delta x\Delta y \le \int_Y f^p(y)w(y)\Delta y \quad (10.55)$$

holds for all ν_Δ-integrable $f : Y \to \mathbb{R}_+$. If $0 < p \le 2$, then (10.55) *is reversed.*

Proof. We use $\varphi(x) = x^p$ in Theorem 10.10. □

Remark 10.16 In particular, if $p = 2$ in Corollary 10.20, then we get

$$\int_X A_k^2 f(x)\xi(x)\,\Delta x \;+\; \int_Y \int_X |f(y) - (A_k f)(x)|^2 \frac{\xi(x)k(x,y)}{K(x)}\,\Delta x\,\Delta y \;=\; \int_Y f^2(y)w(y)\Delta y.$$

Corollary 10.21 *Assume* (H_1)–(H_3). *Then*

$$\int_X \exp(A_k g(x))\xi(x)\,\Delta x + I \le \int_Y g(y)w(y)\Delta y \tag{10.56}$$

holds for all ν_Δ*-integrable* $g : Y \to (0,\infty)$ *with*

$$A_k g(x) = \frac{1}{K(x)} \int_Y k(x,y)\ln g(y)\,\Delta y$$

and

$$I = \int_Y \int_X \left(\exp\left(|\ln g(y) - A_k g(x)|\right) - |\ln g(y) - A_k g(x)|\right) \frac{\xi(x)k(x,y)}{K(x)}\,\Delta x\,\Delta y$$
$$+ \int_Y \ln g(y)w(y)\Delta y - \int_X (1 + A_k g(x))\Delta x.$$

Proof. Use $\varphi(x) = e^x - x - 1$ and $f(x) = \ln g(x)$ in Theorem 10.10. □

Corollary 10.22 *Assume* (H_1)–(H_3). *Let* $\int_X \Delta y = |X|$, $\int_Y \Delta y = |Y|$, *such that* $|X|, |Y| < \infty$. *If* $\varphi \in C(I,\mathbb{R})$ *is superquadratic, then*

$$\int_X \varphi\left(\frac{1}{|Y|}\int_Y f(y)\Delta y\right)\Delta x$$
$$+ \frac{1}{|Y|}\int_Y \int_X \varphi\left(\left|f(y) - \frac{1}{|Y|}\int_Y f(y)\Delta y\right|\right)\Delta x\Delta y \le \frac{|X|}{|Y|}\int_Y \varphi(f(y))\Delta y \tag{10.57}$$

holds for all ν_Δ*-integrable* $f : Y \to \mathbb{R}$ *such that* $f(Y) \subset I$. *If* φ *is subquadratic, then* (10.57) *is reversed.*

Proof. By taking $k(x,y) = 1$ and $\xi(x) = 1$ so that

$$K(x) = \int_Y \Delta y = |Y| \quad \text{and} \quad w(y) = \int_X \frac{1}{|Y|}\,\Delta x = \frac{|X|}{|Y|},$$

the statement follows directly from Theorem 10.10. □

10.3.2 Inequalities with Special Kernels

Throughout this section, we assume (H_1') holds.

Theorem 10.11 *Assume (H_1'). Let*

$$\widetilde{w}(\mathbf{y}) = \int_{y_1}^{b_1} \cdots \int_{y_n}^{b_n} \frac{\xi(\mathbf{x})}{\prod_{i=1}^n (\sigma(x_i) - a_i)} \Delta x_1 \ldots \Delta x_n \tag{10.58}$$

such that $\xi : X \to \mathbb{R}_+$ is a μ_Δ-integrable function. If $\varphi \in C(I,\mathbb{R})$, $I \subset \mathbb{R}$, is superquadratic, then

$$\begin{aligned}
&\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \xi(\mathbf{x})\varphi\left((\widetilde{A}_k f)(\mathbf{x})\right) \Delta x_1 \ldots \Delta x_n + \\
&\quad \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \int_{y_1}^{b_1} \cdots \int_{y_n}^{b_n} \frac{\xi(\mathbf{x})}{\prod_{i=1}^n (\sigma(x_i) - a_i)} \varphi\left(\left|f(\mathbf{y}) - (\widetilde{A}_k f)(\mathbf{x})\right|\right) \\
&\qquad \Delta x_1 \ldots \Delta x_n \Delta y_1 \ldots \Delta y_n \le \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \widetilde{w}(\mathbf{y})\varphi(f(\mathbf{y}))\Delta y_1 \ldots \Delta y_n
\end{aligned} \tag{10.59}$$

holds for all ν_Δ-integrable $f : Y \to \mathbb{R}$ such that $f(Y) \subset I$, where

$$(\widetilde{A}_k f)(\mathbf{x}) = \frac{1}{\prod_{i=1}^n (\sigma(x_i) - a_i)} \int_{a_1}^{\sigma(x_1)} \cdots \int_{a_n}^{\sigma(x_n)} f(\mathbf{y})\Delta y_1 \ldots \Delta y_n.$$

If φ is subquadratic, then (10.59) *is reversed.*

Proof. The statement follows from Theorem 10.10 by using

$$k(\mathbf{x},\mathbf{y}) = \left\{ \begin{array}{ll} 1 & \text{if } a_i \le y_i < \sigma(x_i) \le b_i,\ i \in \{1,\ldots,n\} \\ 0 & \text{otherwise,} \end{array} \right\},$$

since in this case

$$K(\mathbf{x}) = \int_{a_1}^{\sigma(x_1)} \cdots \int_{a_n}^{\sigma(x_n)} \Delta y_1 \ldots \Delta y_n = \prod_{i=1}^n (\sigma(x_i) - a_i),$$

and thus $A_k = \widetilde{A}_k$, $w = \widetilde{w}$. □

Corollary 10.23 *Assume (H_1') with $a_i = 0$ for all $i \in \{1,\ldots,n\}$. If $\varphi \in C(I,\mathbb{R})$ is superquadratic for $I \subset \mathbb{R}$, then*

$$\begin{aligned}
&\int_0^{b_1} \cdots \int_0^{b_n} \varphi((A_k f)(\mathbf{x})) \frac{\Delta x_1 \ldots \Delta x_n}{\prod_{i=1}^n x_i} + \\
&\quad \int_0^{b_1} \cdots \int_0^{b_n} \int_{y_1}^{b_1} \cdots \int_{y_n}^{b_n} \varphi(|f(\mathbf{y}) - (A_k f)(\mathbf{x})|) \frac{\Delta x_1 \ldots \Delta x_n}{\prod_{i=1}^n x_i \sigma(x_i)} \Delta y_1 \ldots \Delta y_n \\
&\qquad \le \int_0^{b_1} \cdots \int_0^{b_n} w(\mathbf{y})\varphi(f(\mathbf{y}))\Delta y_1 \ldots \Delta y_n
\end{aligned} \tag{10.60}$$

holds for all ν_Δ*-integrable* $f : Y \to \mathbb{R}$ *such that* $f(Y) \subset I$*, where*

$$(A_k f)(\mathbf{x}) = \frac{1}{\prod_{i=1}^n \sigma(x_i)} \int_0^{\sigma(x_1)} \dots \int_0^{\sigma(x_n)} f(\mathbf{y})\Delta y_1 \dots \Delta y_n.$$

Proof. The statement follows from Theorem 10.11 by using

$$\xi(\mathbf{x}) = \frac{1}{x_1 \dots x_n},$$

since in this case

$$w(\mathbf{y}) = \int_{y_1}^{b_1} \dots \int_{y_n}^{b_n} \frac{1}{\prod_{i=1}^n x_i \sigma(x_i)} \Delta x_1 \dots \Delta x_n = \prod_{i=1}^n \left(\frac{1}{y_i} - \frac{1}{b_i} \right).$$

This completes the proof □

Remark 10.17 If $b_i = \infty$ for all $i \in \{1, \dots, n\}$ in (H_1'), then (10.60) takes the form

$$\int_0^\infty \dots \int_0^\infty \varphi\left((A_k f)(\mathbf{x})\right) \frac{\Delta x_1 \dots \Delta x_n}{\prod_{i=1}^n x_i} +$$
$$\int_0^\infty \dots \int_0^\infty \int_{y_1}^\infty \dots \int_{y_n}^\infty \varphi\left(|f(\mathbf{y}) - (A_k f)(\mathbf{x})|\right) \frac{\Delta x_1 \dots \Delta x_n}{\prod_{i=1}^n x_i \sigma(x_i)} \Delta y_1 \dots \Delta y_n$$
$$\le \int_0^\infty \dots \int_0^\infty \varphi(f(\mathbf{y})) \frac{\Delta y_1 \dots \Delta y_n}{\prod_{i=1}^n y_i}. \quad (10.61)$$

If φ is subquadratic, then (10.61) is reversed.

Theorem 10.12 *Let* $a \ge 0, b = \infty$ *in* (H_1')*, and* $v : Y \to \mathbb{R}_+$ *be defined by*

$$v(\mathbf{y}) = \frac{1}{\prod_{i=1}^n \sigma(y_i)} \int_{a_1}^{\sigma(y_1)} \dots \int_{a_n}^{\sigma(y_n)} \Delta x_1 \dots \Delta x_n = \prod_{i=1}^n \left(1 - \frac{a_i}{\sigma(y_i)}\right).$$

If $\varphi \in C(I, \mathbb{R})$ *is superquadratic, then*

$$\int_{a_1}^\infty \dots \int_{a_n}^\infty \varphi\left(\prod_{i=1}^n x_i \int_{x_1}^\infty \dots \int_{x_n}^\infty \frac{f(\mathbf{y})}{\prod_{i=1}^n y_i \sigma(y_i)} \Delta y_1 \dots \Delta y_n\right) \frac{\Delta x_1 \dots \Delta x_n}{\prod_{i=1}^n x_i} \quad (10.62)$$
$$+ \int_{a_1}^\infty \dots \int_{a_n}^\infty \int_{a_1}^{\sigma(y_1)} \dots \int_{a_n}^{\sigma(y_n)} \varphi\left(\left| f(\mathbf{y}) \right.\right.$$
$$\left.\left. - \prod_{i=1}^n x_i \int_{x_1}^\infty \dots \int_{x_n}^\infty \frac{f(\mathbf{y})}{\prod_{i=1}^n y_i \sigma(y_i)} \Delta y_1 \dots \Delta y_n \right| \right) \Delta x_1 \dots \Delta x_n \frac{\Delta y_1 \dots \Delta y_n}{\prod_{i=1}^n y_i \sigma(y_i)}$$
$$\le \int_{a_1}^\infty \dots \int_{a_n}^\infty v(\mathbf{y}) \varphi(f(\mathbf{y})) \frac{\Delta y_1 \dots \Delta y_n}{\prod_{i=1}^n y_i}$$

holds for all ν_Δ*-integrable* $f : Y \to \mathbb{R}$ *such that* $f(Y) \subset I$*. If* φ *is subquadratic, then* (10.62) *is reversed.*

Proof. The statement follows from Theorem 10.11 by using

$$k(\mathbf{x},\mathbf{y}) = \left\{ \begin{array}{ll} \prod_{i=1}^{n} \frac{1}{y_i\sigma(y_i)}, & y_i \geq x_i \text{ for all } i \in \{1,\ldots,n\}, \\ 0 \; otherwise, & \end{array} \right\},$$

since in this case

$$K(\mathbf{x}) = \frac{1}{\prod_{i=1}^{n} x_i},$$

and we replace $\xi(\mathbf{x})$ with $1/\prod_{i=1}^{n} x_i$ to obtain

$$w(\mathbf{y}) = \frac{v(\mathbf{y})}{\prod_{i=1}^{n} y_i}.$$

This completes the proof. □

Remark 10.18 If $\varphi(u) = u^p$ in Theorem 10.11 and Theorem 10.12, then for $p \geq 2$, the corresponding inequalities are preserved. However, for $p \in (0,2]$, the corresponding inequalities are reversed.

10.3.3 Some Particular Cases

Corollary 10.24 *Let $\mathbb{T}$ be an isolated time scale. Suppose $X = Y = [a,\infty)_{\mathbb{T}}$, $a \geq 0$, and $\xi : X \to \mathbb{R}_+$ is a μ_Δ-integrable function. If $p \geq 2$, then*

$$\begin{aligned} &\sum_{x\in[a,\infty)_{\mathbb{T}}} \xi(x)\left((A_k f)(x)\right)^p \mu(x) \\ &+ \sum_{y\in[a,\infty)_{\mathbb{T}}} \sum_{x\in[y,\infty)_{\mathbb{T}}} \frac{\xi(x)}{\sigma(x)-a}\left(|f(y)-(A_k f)(x)|^p\right)\mu(x)\mu(y) \\ &\leq \sum_{y\in[a,\infty)_{\mathbb{T}}} w(y)(f(y))^p \mu(y) \end{aligned} \tag{10.63}$$

holds for all ν_Δ-integrable $f : Y \to \mathbb{R}_+$, where

$$(A_k f)(x) = \frac{1}{\sigma(x)-a} \sum_{y\in[a,\sigma(x))_{\mathbb{T}}} f(y)\mu(y)$$

and

$$w(y) = \sum_{x\in[y,\infty)_{\mathbb{T}}} \frac{\xi(x)}{\sigma(x)-a}\mu(x).$$

Proof. We use $\varphi(r) = r^p$ in Theorem 10.11 with $p \geq 2$ and $n = 1$. □

Corollary 10.25 *Let $\mathbb{T}$ be an isolated time scale. Suppose $X = Y = [a,\infty)_{\mathbb{T}}$, $a \geq 0$, and $\xi : X \to \mathbb{R}_+$ is a μ_Δ-integrable function. Then*

$$\sum_{x\in[a,\infty)_{\mathbb{T}}} \xi(x)\left((A_k f)(x)\right)\mu(x) + I \leq \sum_{y\in[a,\infty)_{\mathbb{T}}} w(y)f(y)\mu(y)$$

holds for all ν_Δ-integrable $f : Y \to (0,\infty)$, where

$$I = \sum_{y\in[a,\infty)_{\mathbb{T}}} \sum_{x\in[y,\infty)_{\mathbb{T}}} \frac{\mu(x)\mu(y)\xi(x)}{\sigma(x)-a}\left(\exp\left|\ln\left(\frac{g(y)}{\overline{g}(x)}\right)\right| - \left|\ln\left(\frac{g(y)}{\overline{g}(x)}\right)\right|\right)$$
$$+\ln\left(\frac{\prod_{y\in[a,(\sigma(x))_{\mathbb{T}}} (g(y))^{w(y)\mu(y)}}{\prod_{x\in[a,\infty)_{\mathbb{T}}} e^{\mu(x)}\overline{g}(x)^{\mu(x)}}\right), \quad (10.64)$$

$$\overline{g}(x) = \left(\prod_{y\in[a,\sigma(x))_{\mathbb{T}}} (g(y))^{\mu(y)}\right)^{\frac{1}{\sigma(x)-a}} \quad (10.65)$$

for a ν_Δ-integrable function $g : Y \to (0,\infty)$, and $w(y)$ is as in Corollary 10.24.

Proof. We use $\varphi(x) = e^x - x - 1$ and $f(x) = \ln g(x)$ in Theorem 10.11. □

Example 10.20 For $\mathbb{T} = h\mathbb{N} = \{hn : n \in \mathbb{N}\}$ with $h > 0$ and

$$\xi(x) = \frac{1}{\sigma(x)},$$

(10.63) takes the form

$$\sum_{n=1}^{\infty} \frac{1}{n+1}\left(\frac{1}{n}\sum_{k=1}^{n} f(kh)\right)^p$$
$$+\sum_{k=1}^{\infty}\sum_{n=k}^{\infty} \frac{1}{n(n+1)}\left|f(kh) - \frac{1}{n}\sum_{k=1}^{n} f(kh)\right|^p \leq \sum_{k=1}^{\infty} \frac{(f(kh))^p}{k}. \quad (10.66)$$

Example 10.21 For $\mathbb{T} \leftrightarrow \mathbb{N}^2 = \{n^2 : n \in \mathbb{N}\}, a = 1$, and

$$\xi(x) = \frac{2(\sigma(x)-1)}{(\sigma(x)-x)^2(2\sqrt{x}+3)},$$

(10.63) takes the form

$$\sum_{n=1}^{\infty} \frac{2\left(n(n+2)\right)^{1-p}}{(2n+1)(2n+3)}\left(\sum_{k=1}^{n}(2k+1)f(k^2)\right)^p$$
$$+\sum_{k=1}^{\infty}\sum_{n=k}^{\infty} \frac{2(2k+1)}{(2n+1)(2n+3)}\left|f(k^2) - \frac{1}{n(n+2)}\sum_{k=1}^{n}(2k+1)f(k^2)\right|^p \leq \sum_{k=1}^{\infty}\left(f(k^2)\right)^p. \quad (10.67)$$

Example 10.22 For $\mathbb{T} = q^{\mathbb{N}} = \{q^n : n \in \mathbb{N}\}$ with $q > 1$ and

$$\xi(x) = \frac{2(\sigma(x)-1)}{(\sigma(x)-x)^2(2\sqrt{x}+3)},$$

(10.63) takes the form

$$\sum_{n=1}^{\infty} \frac{(q-1)^p(q^n-1)^{1-p}}{q^n}\left(\sum_{k=1}^{n} q^{k-1} f(q^k)\right)^p + \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} \frac{q^{k-1}(q-1)}{q^n}\left|f(q^k) - \frac{q-1}{q^n-1}\sum_{k=1}^{n} q^{k-1} f(q^k)\right|^p \leq \sum_{k=1}^{\infty}\left(f(q^k)\right)^p. \quad (10.68)$$

Example 10.23 For $\mathbb{T}$ and ξ as in Example 10.20, (10.64) takes the form

$$\sum_{n=1}^{\infty} \frac{1}{n+1}\left(\prod_{k=1}^{n} g(kh)\right)^{\frac{1}{n}} \quad (10.69)$$
$$+ \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} \frac{1}{n(n+1)}\left(\exp\left|\ln\left(\frac{g(kh)}{\prod_{k=1}^{n}(g(kh))^{\frac{1}{n}}}\right)\right| - \left|\ln\left(\frac{g(kh)}{\prod_{k=1}^{n}(g(kh))^{\frac{1}{n}}}\right)\right|\right) + \ln\left(\frac{\prod_{k=1}^{\infty}(g(kh))^{\frac{1}{k}}}{e^h}\prod_{n=1}^{\infty}\left(\prod_{k=1}^{n} g(kh)\right)^{\frac{h}{n}}\right)$$
$$\leq \sum_{k=1}^{\infty} \frac{g(kh)}{k}.$$

Example 10.24 For $\mathbb{T}$ and ξ as in Example 10.21, (10.64) takes the form

$$\sum_{n=1}^{\infty} \frac{2n(n+2)}{(2n+1)(2n+3)}\left(\prod_{k=1}^{n}\left(g(k^2)\right)^{2k+1}\right)^{\frac{1}{n(n+2)}} \quad (10.70)$$
$$+ \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} \frac{2(2k+1)}{(2n+1)(2n+3)}\left(\exp\left|\ln\left(\frac{g(k^2)}{(\prod_{k=1}^{n} g(k^2)^{2k+1})^{\frac{1}{n(n+2)}}}\right)\right| - \left|\ln\left(\frac{g(k^2)}{(\prod_{k=1}^{n} g(k^2)^{2k+1})^{\frac{1}{n(n+2)}}}\right)\right|\right)$$
$$+ \ln\left(\frac{\prod_{k=1}^{\infty} g(k^2)}{\prod_{n=1}^{\infty} e^{2n+1}(\prod_{k=1}^{n} g(k^2)^{2k+1})^{\frac{2n+1}{n(n+2)}}}\right)$$
$$\leq \sum_{k=1}^{\infty} g(k^2).$$

Example 10.25 For $\mathbb{T}$ and $\xi(x)$ as in Example 10.22, (10.64) takes the form

$$\begin{aligned}
&\sum_{n=1}^{\infty} q^{-n}(q^n-1)\left(\prod_{k=1}^{n}\left(g(q^k)\right)^{q^{k-1}}\right)^{\frac{q-1}{q^n-1}} \\
&+\sum_{k=1}^{\infty}\sum_{n=k}^{\infty}\frac{q^{k-1}(q-1)}{q^n}\left(\exp\left|\ln\left(\frac{g(q^k)}{(\prod_{k=1}^{n} g(q^k)^{q^{k-1}})^{\frac{q-1}{q^n-1}}}\right)\right|\right. \\
&\left.-\left|\ln\left(\frac{g(q^k)}{(\prod_{k=1}^{n} g(q^k)^{q^{k-1}})^{\frac{q-1}{q^n-1}}}\right)\right|\right) \\
&+\ln\left(\frac{\prod_{k=1}^{\infty} g(q^k)}{\prod_{n=1}^{\infty} e^{q^n(q-1)}(\prod_{k=1}^{n} g(q^k)^{q^{k-1}})^{\frac{q^n(q-1)^2}{q^n-1}}}\right) \\
&\le \sum_{k=1}^{\infty} g(q^k).
\end{aligned} \tag{10.71}$$

10.4 n-Exponential Convexity of some Dynamic Hardy-Type Functionals

Here we use the isotonic linear functionals obtained from the results given in Section 10.1 to give nontrivial examples of n-exponentially convex functions. The results in this section are taken from [87].

Remark 10.19 Under the assumptions of Theorem 10.1, we have

$$\Upsilon_1(\Phi) = \int_Y w(y)\Phi(f(y))\Delta y - \int_X \xi(x)\Phi(A_k f)(x))\Delta x \ge 0. \tag{10.72}$$

From Theorem 10.2, we have

$$\begin{aligned}
\Upsilon_2(\Phi) = &\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} v(y)\Phi(f(y))\frac{\Delta y_1\ldots\Delta y_n}{y_1\ldots y_n} \\
&-\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} u(x)\Phi((\hat{A}_k f)(x))\frac{\Delta x_1\ldots\Delta x_n}{\sigma(x_1)\ldots\sigma(x_n)} \ge 0.
\end{aligned} \tag{10.73}$$

From Theorem 10.3, we have

$$\Upsilon_3(\Phi) = \int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} \widetilde{w}(\mathbf{y})\Phi(f(\mathbf{y}))\Delta y_1 \dots \Delta y_n \\ - \int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} \xi(\mathbf{x})\Phi\left((\widetilde{A}f)(\mathbf{x})\right)\Delta x_1 \dots \Delta x_n \geq 0. \quad (10.74)$$

For simplicity, we use $\Upsilon(\Phi)$ instead of $\Upsilon_i(\Phi)\ \forall\ i \in \{1,2,3\}$. Hence, for any convex $\Phi \in C(I,\mathbb{R}\supsetneq,$

$$\Upsilon(\Phi) \geq 0.$$

In [75], the authors describe the n-exponential convexity for the functionals obtained from the inequalities of Hardy and Boas types. In this section, we utilize the functional $\Upsilon(\Phi)$ given in Remark 10.19 to establish the n-exponential convexity via the theory of time scales. Therefore our work is a continuation of the results in [75].

Theorem 10.13 *Let J be an interval in $\mathbb{R}$ and I be an open interval in $\mathbb{R}$. Let $\Upsilon(\Phi)$ be as given in Remark 10.19. Assume $Y = \{\phi_t \mid t \in J\}$ is a family of continuous functions $\phi_t : I \to \mathbb{R}$ such that the function $t \to [y_0,y_1,y_2;\phi_t]$ is n-exponentially convex in the Jensen sense on J for every choice of mutually different points $y_0,y_1,y_2 \in I$. Then $t \to \Upsilon(\phi_t)$ is an n-exponentially convex function in the Jensen sense on J. Also the function $t \to \Upsilon(\phi_t)$ is continuous, therefore it is n-exponentially convex on J.*

Proof. The proof is similar to the proof of Theorem 5.9. □

The following corollary is an immediate consequence of Theorem 6.9.

Corollary 10.26 *Let J be an interval in $\mathbb{R}$ and I be an open interval in $\mathbb{R}$. Let $\Upsilon(\Phi)$ be as given in Remark 10.19. Assume $Y = \{\phi_t \mid t \in J\}$ is a family of continuous functions $\phi_t : I \to \mathbb{R}$ such that the function $t \to [y_0,y_1,y_2;\phi_t]$ is exponentially convex in the Jensen sense on J for every choice of mutually different points $y_0,y_1,y_2 \in I$. Then $t \to \Upsilon(\phi_t)$ is an exponentially convex function in the Jensen sense on J. As the function $t \to \Upsilon(\phi_t)$ is continuous, therefore it is exponentially convex on J.*

Corollary 10.27 *Let J be an interval in $\mathbb{R}$ and I be an open interval in $\mathbb{R}$. Let $\Upsilon(\Phi)$ be as given in Remark 10.19. Assume $Y = \{\phi_t \mid t \in J\}$ is a family of continuous functions $\phi_t : I \to \mathbb{R}$ such that the function $t \to [y_0,y_1,y_2;\phi_t]$ is 2-exponentially convex in the Jensen sense on J for every choice of mutually different points $y_0,y_1,y_2 \in I$. Then the following statements hold:*

(i) *The function $t \to \Upsilon(\phi_t)$ is 2-exponentially convex on J. If $t \to \Upsilon(\phi_t)$ $(t \in J)$ is additionally strictly positive, then it is also log-convex on J, i.e.,*

$$\Upsilon^{(r-p)}(\phi_q) \leq \Upsilon^{(r-q)}(\phi_p)\Upsilon^{(q-p)}(\phi_r) \quad (10.75)$$

for $p,q,r \in J$ such that $p < q < r$.

(ii) *If the function $t \to \Upsilon(\phi_t)$ is strictly positive and differentiable on J, then for $s \le u$, $t \le v$, $s,t,u,v \in J$, we have*

$$\mathfrak{u}_{s,t}(\Upsilon,Y) \le \mathfrak{u}_{u,v}(\Upsilon,Y), \tag{10.76}$$

where

$$\mathfrak{u}_{s,t}(\Upsilon,Y) = \begin{cases} \left(\dfrac{\Upsilon(\phi_s)}{\Upsilon(\phi_t)}\right)^{\frac{1}{s-t}}, & s \neq t; \\ \exp\left(\dfrac{\frac{d}{ds}\Upsilon(\phi_s)}{\Upsilon(\phi_s)}\right), & s = t \end{cases} \tag{10.77}$$

for $\phi_s, \phi_t \in Y$.

Proof. The proof is similar to the proof of Corollary 5.11. □

Remark 10.20 Note that the results from Theorem 10.13, Corollary 10.26, and Corollary 10.27 are valid when two of the points $y_0, y_1, y_2 \in I$ coincide, for a family of differentiable functions ϕ_t such that the function $t \to [y_0, y_1, y_2; \phi_t]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and moreover, they are also valid when all three points coincide for a family of twice differentiable functions with the same property. The proofs can be obtained by recalling Remark 1.3 and a suitable characterization of convexity.

The following result is given in [72].

Theorem 10.14 *Assume $J \subset \mathbb{R}$ is an interval, and assume $Y = \{\phi_t \mid t \in J\}$ is a family of twice differentiable functions defined on an interval $I \subset \mathbb{R}$ such that the function $t \mapsto \phi_t''(x)$, $(t \in J)$, is exponentially convex for every fixed $x \in I$. Then the function $t \mapsto [y_0, y_1, y_2; \phi_t]$, $(t \in J)$, is exponentially convex in the Jensen sense for any three points $y_0, y_1, y_2 \in I$.*

10.4.1 Applications to Cauchy Means

In this section, first we give mean value theorems corresponding to the Hardy-type functional $\Upsilon(\Phi)$ given in Remark 10.19.

Theorem 10.15 *Let the linear functional Υ be defined as in Remark 10.19. Assume $g \in (C^2[a,b], \mathbb{R})$, where $[a,b] \subset \mathbb{R}$. Then there exists $\xi \in [a,b]$ such that*

$$\Upsilon(g) = \frac{1}{2} g''(\xi) \Upsilon\left(x^2\right).$$

Proof. The proof is similar to the proof of Theorem 6.7. □

Theorem 10.16 *Let the linear functional Υ be defined as in Remark 10.19. Assume $g, h \in (C^2[a,b], \mathbb{R})$, where $[a,b] \subset \mathbb{R}$. Then there exists $\xi \in [a,b]$ such that*

$$\Upsilon(g) = \frac{1}{2} g''(\xi) \Upsilon\left(x^2\right),$$

$$\frac{\Upsilon(g)}{\Upsilon(h)} = \frac{g''(\xi)}{h''(\xi)}, \tag{10.78}$$

provided that $\Upsilon(h) \neq 0$.

Proof. The proof is similar to the proof of Theorem 5.5. □

Remark 10.21 If the inverse of the function $\frac{g''}{h''}$ exists, then (10.78) gives

$$\xi = \left(\frac{g''}{h''}\right)^{-1} \left(\frac{\Upsilon(g)}{\Upsilon(h)}\right). \tag{10.79}$$

Example 10.26 Consider the family of functions

$$\Omega_1 = \{\kappa_\rho : \mathbb{R} \to [0,\infty);\ \rho \in \mathbb{R}\}$$

defined in Example 5.2. Then by using Corollary 10.26, we conclude that $\rho \mapsto \Upsilon(\kappa_\rho)$ is exponentially convex in the Jensen sense. This mapping is also continuous, so they are exponentially convex. For this family of functions, the expression in (10.77) becomes

$$\mathfrak{u}_{s,t}(\Upsilon,\Omega_1) = \begin{cases} \left(\dfrac{\Upsilon(\kappa_s)}{\Upsilon(\kappa_t)}\right)^{\frac{1}{s-t}}, & s \neq t; \\ \exp\left(\dfrac{\Upsilon(\mathrm{id}\cdot\kappa_s)}{\Upsilon(\kappa_s)} - \dfrac{2}{s}\right), & s = t \neq 0; \\ \exp\left(\dfrac{\Upsilon(\mathrm{id}\cdot\kappa_0)}{3\Upsilon(\kappa_0)}\right), & s = t = 0, \end{cases}$$

and by (10.76), it is monotone in s and t. Using (10.79), it follows that

$$\mathfrak{M}_{s,t}(\Upsilon,\Omega_1) = \log \mathfrak{u}_{s,t}(\Upsilon,\Omega_1)$$

satisfy $\mathfrak{M}_{s,t}(\Upsilon,\Omega_1) \in [a,b]$, which shows that $\mathfrak{M}_{s,t}(\Upsilon,\Omega_1)$ are means (of the function f). Note that by (10.76), they are monotone means.

Example 10.27 Consider the family of functions

$$\Omega_2 = \{\beta_\rho : (0,\infty) \to \mathbb{R};\ \rho \in \mathbb{R}\}$$

defined in Example 5.3. Arguing as in Example 10.26, we have $\rho \mapsto \Upsilon(\beta_\rho))$ is exponentially convex. In this case, (10.77) becomes

$$\mathfrak{u}_{s,t}(\Upsilon,\Omega_2)=\begin{cases}\left(\dfrac{\Upsilon(\beta_s)}{\Upsilon(\beta_t)}\right)^{\frac{1}{s-t}}, & s\neq t;\\[2ex] \exp\left(\dfrac{1-2s}{s(s-1)}-\dfrac{\Upsilon(\beta_s\beta_0)}{\Upsilon(\beta_s)}\right), & s=t\neq 0,1;\\[2ex] \exp\left(1-\dfrac{\Upsilon(\beta_0^2)}{2\Upsilon(\beta_0)}\right), & s=t=0;\\[2ex] \exp\left(-1-\dfrac{\Upsilon(\beta_0\beta_1)}{2\Upsilon(\beta_1)}\right), & s=t=1.\end{cases}$$

Also $\mathfrak{u}_{s,t}(\Upsilon,\Omega_2)$ is continuous, symmetric, and monotone in both parameters (by (10.76)). For the class Ω_2, we have

$$\Upsilon_1(\beta_p)=\begin{cases}\frac{1}{p(p-1)}\left(\int_Y w(y)f^p(y)\Delta y-\int_X \xi(x)\left(R(x)\right)^p\Delta x\right), & p\neq 0,1;\\ -\int_Y w(y)\log f(y)\Delta y+\int_X \xi(x)\log\left(R(x)\right)\Delta x, & p=0;\\ \int_Y w(y)f(y)\log f(y)\Delta y-\int_X \xi(x)R(x)\log\left(R(x)\right)\Delta x, & p=1,\end{cases}\tag{10.80}$$

where

$$R(x)=\frac{1}{K(x)}\int_Y k(x,y)f(y)\Delta y \quad\text{and}\quad w(y)=\int_X\frac{k(x,y)\xi(x)}{K(x)}\Delta x.$$

For (10.80), (10.75) gives

$$\begin{aligned}&\frac{1}{q(q-1)}\left(\int_Y w(y)f^q(y)\Delta y-\int_X \xi(x)(R(x))^q\Delta x\right)\\ &\leq\left(-\int_Y w(y)\log f(y)\Delta y+\int_X \xi(x)\log\left(R(x)\right)\Delta x\right)^{1-q}\\ &\times\left(\int_Y w(y)f(y)\log f(y)\Delta y-\int_X \xi(x)R(x)\log\left(R(x)\right)\Delta x\right)^{q},\end{aligned}\tag{10.81}$$

where $p=0<q<1=r$. If $q<0<1$ or $0<1<q$, then (10.81) holds in reverse order. Observe that (10.81) is a refinement of the inequality in Corollary 10.1. Similar results can be written for $i\in\{2,3\}$. Particularly, for $i=3$, $n=1$, we have

$$\Upsilon_3(\beta_p) =$$

$$\begin{cases} \frac{1}{p(p-1)}\left(\int_a^b \widetilde{w}(y) f^p(y)\Delta y - \int_a^b \xi(x)\left(\widetilde{R}(x)\right)^p \Delta x\right), & p \neq 0,1; \\ -\int_a^b \widetilde{w}(y)\log f(y)\Delta y + \int_a^b \xi(x)\log\left(\widetilde{R}(x)\right)\Delta x, & p = 0; \\ \int_a^b \widetilde{w}(y) f(y)\log f(y)\Delta y - \int_a^b \xi(x)\widetilde{R}(x)\log\left(\widetilde{R}(x)\right)\Delta x, & p = 1, \end{cases} \tag{10.82}$$

where

$$\widetilde{R}(x) = \frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(y)\Delta y \quad \text{and} \quad \widetilde{w}(y) = \int_y^{\infty} \frac{\xi(x)\Delta x}{\sigma(x)-a}.$$

For $0 < q < 1$, using (10.82) in (10.75), we have

$$\begin{aligned} &\frac{1}{q(q-1)}\left(\int_a^b \widetilde{w}(y) f^q(y)\Delta y - \int_a^b \xi(x)\left(\widetilde{R}(x)\right)^q \Delta x\right) \\ &\leq \left(-\int_a^b \widetilde{w}(y)\log f(y)\Delta y + \int_a^b \xi(x)\log\left(\widetilde{R}(x)\right)\Delta x\right)^{1-q} \\ &\times \left(\int_a^b \widetilde{w}(y) f(y)\log f(y)\Delta y - \int_a^b \xi(x)\widetilde{R}(x)\log\left(\widetilde{R}(x))\right)\Delta x\right)^q. \end{aligned} \tag{10.83}$$

If $q < 0 < 1$ or $0 < 1 < q$, then (10.83) is reversed.

Example 10.28 Consider the family of functions

$$\Omega_3 = \{\gamma_\rho : (0,\infty) \to (0,\infty) : \rho \in (0,\infty)\}$$

defined in Example 5.4. For this family of functions, (10.77) has the form

$$\mathfrak{u}_{s,t}(\Upsilon,\Omega_3) = \begin{cases} \left(\frac{\Upsilon(\gamma_s)}{\Upsilon(\gamma_t)}\right)^{\frac{1}{s-t}}, & s \neq t; \\ \exp\left(-\frac{\Upsilon(id\cdot\gamma_s)}{s\Upsilon(\gamma_s)} - \frac{2}{s\ln s}\right), & s = t \neq 0,1; \\ \exp\left(\frac{-2\Upsilon(id\cdot\gamma_1)}{3\Upsilon(\gamma_1)}\right), & s = t = 1, \end{cases}$$

and by (10.76), it is monotone in s and t. Using Theorem 10.16, it follows that

$$\aleph_{s,t}(\Upsilon,\Omega_3) = -L(s,t)\log\mathfrak{u}_{s,t}(\Upsilon,\Omega_3)$$

satisfies $\aleph_{s,t}(\Upsilon,\Omega_3) \in [a,b]$, which shows that $\aleph_{s,t}(\Upsilon,\Omega_3)$ is a mean. Here $L(s,t)$ is the logarithmic mean defined by

$$L(s,t) = \frac{s-t}{\log s - \log t}, \quad s \neq t, \quad L(s,s) = s.$$

Example 10.29 Consider the family of functions

$$\Omega_4 = \{\delta_\rho : (0,\infty) \to (0,\infty) : \rho \in (0,\infty)\}$$

defined in Example 5.5. For this family of functions, (10.77) becomes

$$\mathfrak{u}_{s,t}(\Upsilon,\Omega_4) = \begin{cases} \left(\dfrac{\Upsilon(\delta_s)}{\Upsilon(\delta_t)}\right)^{\frac{1}{s-t}}, & s \neq t; \\ \exp\left(-\dfrac{\Upsilon(id\cdot\delta_s)}{2\sqrt{s}\Upsilon(\delta_s)} - \dfrac{1}{s}\right), & s = t, \end{cases}$$

and it is monotone function in s and t by (10.76). Using Theorem 10.16, it follows that

$$\aleph_{s,t}(\Upsilon,\Omega_4) = -(\sqrt{s}+\sqrt{t})\log \mathfrak{u}_{s,t}(\Upsilon,\Omega_4)$$

satisfies $\aleph_{s,t}(\Upsilon,\Omega_4) \in [a,b]$, which shows that $\aleph_{p,q}(\Upsilon,\Omega_4)$ is a mean.

10.4.2 Applications to Isolated Time Scales

Now, we consider some particular cases corresponding to the examples from Section 10.1.

Let us take $X = Y = [a,\infty)_{\mathbb{T}} = [0,\infty) \cap \mathbb{T}$, $a \geq 0$. Further assume that the time scale $\mathbb{T}$ is isolated. In this case, (10.82) takes the form

$$\Upsilon_3(\beta_p) = \begin{cases} \dfrac{1}{p(p-1)}\left(\sum\limits_{[a,b)_{\mathbb{T}}} \widetilde{w}(y)(f(y))^p\mu(y) - \sum\limits_{[a,b)_{\mathbb{T}}} \xi(x)(\widetilde{R}(x))^p\mu(x)\right), & p \neq 0,1; \\ -\log\left(\prod\limits_{[a,b)_{\mathbb{T}}} (f(y))^{\widetilde{w}(y)\mu(y)}\right) + \log\left(\prod\limits_{[a,b)_{\mathbb{T}}} (\widetilde{R}(x))^{\xi(x)\mu(x)}\right), & p = 0; \\ \log\left(\prod\limits_{[a,b)_{\mathbb{T}}} (f(y))^{\widetilde{w}(y)f(y)\mu(y)}\right) - \log\left(\prod\limits_{[a,b)_{\mathbb{T}}} (\widetilde{R}(x))^{\xi(x)\widetilde{R}(x)\mu(x)}\right), & p = 1, \end{cases} \tag{10.84}$$

where

$$\widetilde{R}(x) = \left(\frac{1}{\sigma(x)-a}\sum_{y\in[a,x]_{\mathbb{T}}} f(y)\mu(y)\right) \quad \text{and} \quad \widetilde{w}(y) = \left(\sum_{x\in[y,\infty)_{\mathbb{T}}} \xi(x)\frac{\mu(x)}{\sigma(x)-a}\right).$$

Also, for $0 < q < 1$, (10.83) takes the form

$$\frac{1}{q(q-1)}\left(\sum_{[a,b)_{\mathbb{T}}} \widetilde{w}(y)(f(y))^q\mu(y) - \sum_{[a,b)_{\mathbb{T}}} \xi(x)(\widetilde{R}(x))^q\mu(x)\right)$$
$$\leq \left(\log\left(\frac{\prod_{[a,b)_{\mathbb{T}}} (R(x))^{\xi(x)\mu(x)}}{\prod_{[a,b)_{\mathbb{T}}} (f(y))^{\widetilde{w}(y)\mu(y)}}\right)\right)^{1-q} \left(\log\left(\frac{\prod_{[a,b)_{\mathbb{T}}} (f(y))^{\widetilde{w}(y)f(y)\mu(y)}}{\prod_{[a,b)_{\mathbb{T}}} (\widetilde{R}(x))^{\xi(x)\widetilde{R}(x)\mu(x)}}\right)\right)^q. \quad (10.85)$$

For $\mathbb{T} = h\mathbb{N} = \{hn : n \in \mathbb{N}\}$, with $h > 0$, $a = h$, and $\xi(x) = \frac{1}{\sigma(x)}$, (10.84) takes the form

$$\Upsilon_3(\beta_p) = \begin{cases} \frac{1}{q(q-1)}\left(\sum_{n=1}^{\infty} \frac{(f(nh))^p}{n} - \sum_{n=1}^{\infty} \frac{1}{n+1}\left(\widetilde{R}(nh)\right)^p\right), & p \neq 0,1; \\ -\log\left(\prod_{n=1}^{\infty} (f(nh))^{\frac{1}{n}}\right) + \log\left(\prod_{n=1}^{\infty} (\widetilde{R}(nh))^{\frac{1}{n+1}}\right), & p = 0; \\ \log\left(\prod_{n=1}^{\infty} (f(nh))^{\frac{(f(nh))}{n}}\right) - \log\left(\prod_{n=1}^{\infty} (\widetilde{R}(nh))^{\frac{\widetilde{R}(nh)}{n+1}}\right), & p = 1. \end{cases}$$

For $0 < q < 1$, (10.85) takes the form

$$\frac{1}{q(q-1)}\left(\sum_{n=1}^{\infty} \frac{(f(nh))^q}{n} - \sum_{n=1}^{\infty} \frac{1}{n+1}\left(\widetilde{R}(nh)\right)^q\right)$$
$$\leq \left(\log\left(\frac{\prod_{n=1}^{\infty} (\widetilde{R}(nh))^{\frac{1}{n+1}}}{\prod_{n=1}^{\infty} (f(nh))^{\frac{1}{n}}}\right)\right)^{1-q} \left(\log\left(\frac{\prod_{n=1}^{\infty} (f(nh))^{\frac{(f(nh))}{n}}}{\prod_{n=1}^{\infty} (\widetilde{R}(nh))^{\frac{\widetilde{R}(nh)}{n+1}}}\right)\right)^q, \quad (10.86)$$

where

$$\widetilde{R}(nh) = \frac{1}{n}\sum_{k=1}^{n} f(kh).$$

For $\mathbb{T} = \mathbb{N}^2 = \{n^2 : n \in \mathbb{N}\}$ with $a = 1$ and

$$\xi(x) = \frac{2(\sigma(x)-1)}{(\sigma(x)-x)^2(2\sqrt{x}+3)},$$

(10.84) takes the form

$$\Upsilon_3(\beta_p) = \begin{cases} \dfrac{1}{p(p-1)}\left(\sum\limits_{n=1}^{\infty}(f(n^2))^p - \sum\limits_{n=1}^{\infty}\dfrac{2n(n+2)}{(}2n+1)(2n+3)\left(\widetilde{R}(n^2)\right)^p\right), & p \neq 0,1; \\ -\log\left(\prod\limits_{n=1}^{\infty} f(n^2)\right) + \log\left(\prod\limits_{n=1}^{\infty}(\widetilde{R}(n^2))^{\frac{2n(n+2)}{(2n+1)(2n+3)}}\right), & p = 0; \\ \log\left(\prod\limits_{n=1}^{\infty} f(n^2)^{f(n^2)}\right) - \log\left(\prod\limits_{n=1}^{\infty}(\widetilde{R}(n^2))^{\frac{2n(n+2)}{(2n+1)(2n+3)}\widetilde{R}(n^2)}\right), & p = 1. \end{cases} \tag{10.87}$$

For $0 < q < 1$, (10.85) takes the form

$$\frac{1}{q(q-1)}\left(\sum_{n=1}^{\infty}(f(n^2))^q - \sum_{n=1}^{\infty}\frac{2n(n+2)}{(2n+1)(2n+3)}\left(\widetilde{R}(n^2)\right)^q\right)$$
$$\leq \left(\log\left(\frac{\prod\limits_{n=1}^{\infty}(\widetilde{R}(n^2))^{\frac{2n(n+2)}{(2n+1)(2n+3)}}}{\prod\limits_{n=1}^{\infty} f(n^2)}\right)\right)^{1-q}\left(\log\left(\frac{\prod\limits_{n=1}^{\infty} f(n^2)^{f(n^2)}}{\prod\limits_{n=1}^{\infty}(\widetilde{R}(n^2))^{\frac{2n(n+2)}{(2n+1)(2n+3)}\widetilde{R}(n^2)}}\right)\right)^q, \tag{10.88}$$

where

$$\widetilde{R}(n^2) = \frac{\sum\limits_{k=1}^{n}(2k+1)f(k^2)}{n(n+2)}.$$

For $\mathbb{T} = Y^{\mathbb{N}} = \{Y^n : n \in \mathbb{N}\}$ with $Y > 1$, $a = Y$, and

$$\xi(x) = \frac{\sigma(x) - a}{\sigma(x)(\sigma(x) - x)},$$

(10.84) takes the form

$$\Upsilon_3(\beta_p) = \begin{cases} \dfrac{1}{p(p-1)}\left(\sum\limits_{n=1}^{\infty}(f(Y^n))^p - \sum\limits_{n=1}^{\infty} Y^{-n}(Y^n - 1)(\widetilde{R}(Y^n))^p\right), & p \neq 0,1; \\ -\log\left(\prod\limits_{n=1}^{\infty} f(Y^n)\right) + \log\left(\prod\limits_{n=1}^{\infty}(\widetilde{R}(Y^n))^{Y^{-n}(Y^n-1)}\right), & p = 0; \\ \log\left(\prod\limits_{n=1}^{\infty}(f(Y^n))^{f(Y^n)}\right) - \log\left(\prod\limits_{n=1}^{\infty}(\widetilde{R}(Y^n))^{Y^{-n}(Y^n-1)\widetilde{R}(Y^n)}\right), & p = 1. \end{cases} \tag{10.89}$$

For $0 < q < 1$, (10.85) takes the form

$$\frac{1}{q(q-1)}\left(\sum_{n=1}^{\infty}(f(Y^n))^q - \sum_{n=1}^{\infty}Y^{-n}(Y^n-1)(\widetilde{R}(Y^n))^q\right)$$
$$\leq \left(\log\left(\frac{\prod_{n=1}^{\infty}(\widetilde{R}(Y^n))^{Y^{-n}(Y^n-1)}}{\prod_{n=1}^{\infty}f(Y^n)}\right)\right)^{1-q}\left(\log\left(\frac{\prod_{n=1}^{\infty}(f(Y^n))^{f(Y^n)}}{\prod_{n=1}^{\infty}(\widetilde{R}(Y^n))^{Y^{-n}(Y^n-1)\widetilde{R}(Y^n)}}\right)\right)^q, \quad (10.90)$$

where

$$\widetilde{R}(Y^n) = \frac{(Y-1)\sum_{k=1}^{n}Y^{k-1}f(Y^k)}{Y^n-1}.$$

Remark 10.22 (a) If $q < 0 < 1$ or $0 < 1 < q$, then (10.86), (10.88), and (10.90) are reversed.

(b) The inequalities (10.86), (10.88), and (10.90) are refinements of (5.9), the first inequality given in Example 5.11, and (5.11) of [44], respectively.

10.5 Refinements of Hardy-Type Inequalities on Time Scales

In this section, we give many refinements and generalizations of the inequalities of Hardy type on time scales for convex functions, nonnegative convex functions, monotone convex functions, and nonnegative monotone convex functions. Further we give refinements for power and exponential functions. Finally we present several examples of these inequalities on different time scales. Results of this section are contained in [28].

10.5.1 Results using General Kernels

Let us consider the following additional hypotheses.

($\check{\mathrm{H}}_3$) $0 < p \leq q < \infty$ or $-\infty < q \leq p < 0$ and $\xi : X \to \mathbb{R}_+$ is such that

$$\mathscr{T}(y) = \left(\int_X \xi(x)\left(\frac{k(x,y)}{\mathscr{K}(x)}\right)^{\frac{q}{p}}\Delta x\right)^{\frac{p}{q}} < \infty, \quad y \in Y.$$

(H_4) $\Phi \in \mathrm{C}(I,\mathbb{R})$ is nonnegative and convex, where $I \subset \mathbb{R}$, and $\varphi : I \to \mathbb{R}$ is such that $\varphi(x) \in \partial\Phi(x) = [\Phi'_+(x), \Phi'_-(x)]$ for all $x \in \operatorname{Int} I$.

Theorem 10.17 *Assume* (H$_1$), (H$_2$), ($\check{\text{H}}_3$)*, and* (H$_4$)*. Then*

$$\left(\int_Y \Phi(f(y))\mathscr{T}(y)\Delta y\right)^{\frac{q}{p}} - \int_X \xi(x)\Phi^{\frac{q}{p}}(\mathscr{A}_k f(x))\Delta x \\ \geq \frac{q}{p}\int_X \frac{\xi(x)}{\mathscr{K}(x)}\Phi^{\frac{q}{p}-1}(\mathscr{A}_k f(x))\int_Y k(x,y)\mathscr{R}_k(x,y)\Delta y\Delta x \quad (10.91)$$

holds for ν_Δ*-integrable function* f *on* Y *such that* $f(Y) \subset I$*, where*

$$\mathscr{A}_k f(x) = \frac{1}{\mathscr{K}(x)}\int_Y k(x,y)f(y)\Delta y, \quad x \in X, \quad (10.92)$$

and

$$\mathscr{R}_k(x,y) = ||\Phi(f(y)) - \Phi(\mathscr{A}_k f(x))| - |\varphi(\mathscr{A}_k f(x))||f(y) - \mathscr{A}_k f(x)||. \quad (10.93)$$

If Φ *is nonnegative, monotone, and convex on* I *in* (H$_4$) *and* $f(y) > \mathscr{A}_k f(x)$ *for* $y \in Y' \subset Y$*, then*

$$\left(\int_Y \Phi(f(y))\mathscr{T}(y)\Delta y\right)^{\frac{q}{p}} - \int_X \xi(x)\Phi^{\frac{q}{p}}(\mathscr{A}_k f(x))\Delta x \\ \geq \frac{q}{p}\left|\int_X \frac{\xi(x)}{\mathscr{K}(x)}\Phi^{\frac{q}{p}-1}(\mathscr{A}_k f(x))\int_Y \operatorname{sgn}(f(y) - \mathscr{A}_k f(x))k(x,y)\mathscr{S}_k(x,y)\Delta y\Delta x\right| \quad (10.94)$$

holds, where

$$\mathscr{S}_k(x,y) = \Phi(f(y)) - \Phi(\mathscr{A}_k f(x)) - |\varphi(\mathscr{A}_k f(x))|(f(y) - \mathscr{A}_k f(x)). \quad (10.95)$$

Proof. Since Φ is convex on I and $\varphi(x) \in \partial\Phi(x)$ for all $x \in \text{Int}\, I$, we have

$$\Phi(s) - \Phi(r) - \varphi(r)(s-r) \geq 0$$

for all $r \in \text{Int}\, I$ and $s \in I$. Now

$$\Phi(s) - \Phi(r) - \varphi(r)(s-r) = |\Phi(s) - \Phi(r) - \varphi(r)(s-r)| \\ \geq ||\Phi(s) - \Phi(r)| - |\varphi(r)|\,|s-r||. \quad (10.96)$$

Since $\mathscr{A}_k f(x) \in I$ for all $x \in X$, let $\mathscr{A}_k f(x) \in \text{Int}\, I$. Then by substituting $r = \mathscr{A}_k f(x)$ and $s = f(y)$ in (10.96), we get

$$\Phi(f(y)) - \Phi(\mathscr{A}_k f(x)) - \varphi(\mathscr{A}_k f(x))(f(y) - \mathscr{A}_k f(x)) \\ \geq ||\Phi(f(y)) - \Phi(\mathscr{A}_k f(x))| - |\varphi(\mathscr{A}_k f(x))||f(y) - \mathscr{A}_k f(x)|| = \mathscr{R}_k(x,y). \quad (10.97)$$

If $\mathscr{A}_k f(x)$ is an end point of I, then (10.97) holds with value zero on both sides of the inequality for ν_Δ-a.e. $y \in Y$. Multiplying (10.97) by $k(x,y)/\mathscr{K}(x) \geq 0$ and integrating it over Y with respect to the measure ν_Δ, we obtain

$$\begin{aligned}\frac{1}{\mathcal{K}(x)}\int_Y k(x,y)\Phi(f(y))\Delta y - \frac{1}{\mathcal{K}(x)}\int_Y k(x,y)\Phi(\mathscr{A}_k f(x))\Delta y \\ - \frac{1}{\mathcal{K}(x)}\int_Y k(x,y)\varphi(\mathscr{A}_k f(x))(f(y)-\mathscr{A}_k f(x))\Delta y \\ \geq \frac{1}{\mathcal{K}(x)}\int_Y k(x,y)\mathscr{R}_k(x,y)\Delta y.\end{aligned} \tag{10.98}$$

The second integral on the left-hand side of (10.98) becomes

$$\frac{1}{\mathcal{K}(x)}\int_Y k(x,y)\Phi(\mathscr{A}_k f(x))\Delta y = \frac{\Phi(\mathscr{A}_k f(x))}{\mathcal{K}(x)}\int_Y k(x,y)\Delta y = \Phi(\mathscr{A}_k f(x)),$$

and for the third integral we have

$$\frac{1}{\mathcal{K}(x)}\int_Y k(x,y)\varphi(\mathscr{A}_k f(x))(f(y)-\mathscr{A}_k f(x))\Delta y = 0.$$

Hence (10.98) becomes

$$\Phi(\mathscr{A}_k f(x)) + \frac{1}{\mathcal{K}(x)}\int_Y k(x,y)\mathscr{R}_k(x,y)\Delta y \leq \frac{1}{\mathcal{K}(x)}\int_Y k(x,y)\Phi(f(y))\Delta y.$$

Since Φ is nonnegative, for $\frac{q}{p} \geq 1$, we have

$$\left(\Phi(\mathscr{A}_k f(x)) + \frac{1}{\mathcal{K}(x)}\int_Y k(x,y)\mathscr{R}_k(x,y)\Delta y\right)^{\frac{q}{p}} \leq \left(\frac{1}{\mathcal{K}(x)}\int_Y k(x,y)\Phi(f(y))\Delta y\right)^{\frac{q}{p}}.$$

By applying Bernoulli's inequality on the left-hand side of the above inequality, we get

$$\begin{aligned}\Phi^{\frac{q}{p}}(\mathscr{A}_k f(x)) + \frac{q}{p}\frac{\Phi^{\frac{q}{p}-1}(\mathscr{A}_k f(x))}{\mathcal{K}(x)}\int_Y k(x,y)\mathscr{R}_k(x,y)\Delta y \\ \leq \left(\Phi(\mathscr{A}_k f(x)) + \frac{1}{\mathcal{K}(x)}\int_Y k(x,y)\mathscr{R}_k(x,y)\Delta y\right)^{\frac{q}{p}} \\ \leq \left(\frac{1}{\mathcal{K}(x)}\int_Y k(x,y)\Phi(f(y))\Delta y\right)^{\frac{q}{p}}.\end{aligned} \tag{10.99}$$

Multiplying (10.99) by $\xi(x)$, integrating it over X with respect to the measure μ_Δ, and applying the integral Minkowski inequality on time scales, we have

$$\begin{aligned}\int_X \xi(x)\Phi^{\frac{q}{p}}(\mathscr{A}_k f(x))\Delta x + \frac{q}{p}\int_X \frac{\xi(x)}{\mathcal{K}(x)}\Phi^{\frac{q}{p}-1}(\mathscr{A}_k f(x))\int_Y k(x,y)\mathscr{R}_k(x,y)\Delta y\Delta x \\ \leq \int_X \xi(x)\left(\frac{1}{\mathcal{K}(x)}\int_Y k(x,y)\Phi(f(y))\Delta y\right)^{\frac{q}{p}}\Delta x \\ = \left(\left(\int_X \xi(x)\left(\frac{1}{\mathcal{K}(x)}\int_Y k(x,y)\Phi(f(y))\Delta y\right)^{\frac{q}{p}}\Delta x\right)^{\frac{p}{q}}\right)^{\frac{q}{p}}\end{aligned}$$

$$\leq \left(\int_Y \Phi(f(y)) \left(\int_X \xi(x) \left(\frac{k(x,y)}{\mathscr{K}(x)} \right)^{\frac{q}{p}} \Delta x \right)^{\frac{p}{q}} \Delta y \right)^{\frac{q}{p}}.$$

If Φ is nondecreasing on the interval I, for a fixed $x \in X$, then let $Y' = \{y \in Y : f(y) > \mathscr{A}_k f(x)\}$. Then

$$\begin{aligned} &\int_Y k(x,y)|\Phi(f(y)) - \Phi(\mathscr{A}_k f(x))|\Delta y \\ &= \int_{Y'} k(x,y)[\Phi(f(y)) - \Phi(\mathscr{A}_k f(x))]\Delta y \\ &\quad + \int_{Y\setminus Y'} k(x,y)[\Phi(\mathscr{A}_k f(x)) - \Phi(f(y))]\Delta y \\ &= \int_{Y'} k(x,y)\Phi(f(y))\Delta y - \int_{Y\setminus Y'} k(x,y)\Phi(f(y))\Delta y \\ &\quad - \Phi(\mathscr{A}_k f(x)) \int_{Y'} k(x,y)\Delta y + \Phi(\mathscr{A}_k f(x)) \int_{Y\setminus Y'} k(x,y)\Delta y \\ &= \int_Y \operatorname{sgn}(f(y) - \mathscr{A}_k f(x)) k(x,y)[\Phi(f(y)) - \Phi(\mathscr{A}_k f(x))]\Delta y. \end{aligned} \tag{10.100}$$

Similarly, we can write

$$\int_Y k(x,y)|f(y) - \mathscr{A}_k f(x)|\Delta y = \int_Y \operatorname{sgn}(f(y) - \mathscr{A}_k f(x)) k(x,y)(f(y) - \mathscr{A}_k f(x))\Delta y. \tag{10.101}$$

From (10.91), (10.100), and (10.101), we get (10.94). The case when Φ is nonincreasing can be discussed in a similar way. □

Remark 10.23 (i) Suppose Φ is concave (that is, $-\Phi$ is convex) in (H_4). Then for all $r \in \operatorname{Int} I$ and $s \in I$, we have

$$\Phi(r) - \Phi(s) - \varphi(r)(r-s) \geq 0,$$

and (10.96) reads

$$\begin{aligned} \Phi(r) - \Phi(s) - \varphi(r)(r-s) &= |\Phi(r) - \Phi(s) - \varphi(r)(r-s)| \\ &\geq ||\Phi(s) - \Phi(r)| - |\varphi(r)||. \end{aligned}$$

Hence, in this setting, (10.91) takes the form

$$\begin{aligned} &\int_X \xi(x)\Phi^{\frac{q}{p}}(\mathscr{A}_k f(x))\Delta x - \left(\int_Y \Phi(f(y))\mathscr{T}(y)\Delta y \right)^{\frac{q}{p}} \\ &\qquad \geq \frac{q}{p} \int_X \frac{\xi(x)}{\mathscr{K}(x)} \Phi^{\frac{q}{p}-1}(\mathscr{A}_k f(x)) \int_Y k(x,y)\mathscr{R}_k(x,y)\Delta y \Delta x. \end{aligned}$$

(ii) If Φ is nonnegative, monotone, and concave in (H_4), then the order of terms on the left-hand side of (10.94) is reversed.

Corollary 10.28 *Assume* (H_1), (H_2), $(\check{\mathrm{H}}_3)$, *and* (H_4), *where* $0 < p \leq q < \infty$ *and* f *is a* ν_Δ*-integrable function on* Y *such that* $f(Y) \subset I$. *Then*

$$\left(\int_Y \Phi^p(f(y))\mathscr{T}(y)\Delta y\right)^{\frac{q}{p}} - \int_X \xi(x)\Phi^q(\mathscr{A}_k f(x))\Delta x$$
$$\geq \frac{q}{p}\int_X \frac{\xi(x)}{\mathscr{K}(x)}\Phi^{q-p}(\mathscr{A}_k f(x))\int_Y k(x,y)$$
$$||\Phi^p(f(y)) - \Phi^p(\mathscr{A}_k f(x))| - |\varphi(\mathscr{A}_k f(x))||f(y) - \mathscr{A}_k f(x)||\Delta y\Delta x$$

holds. Moreover, if Φ *is nonnegative, monotone, and convex on I, and* $f(y) > \mathscr{A}_k f(x)$ *for* $y \in Y' \subset Y$, *then*

$$\left(\int_Y \Phi^p(f(y))\mathscr{T}(y)\Delta y\right)^{\frac{q}{p}} - \int_X \xi(x)\Phi^q(\mathscr{A}_k f(x))\Delta x$$
$$\geq \frac{q}{p}\left|\int_X \frac{\xi(x)}{\mathscr{K}(x)}\Phi^{q-p}(\mathscr{A}_k f(x))\int_Y \operatorname{sgn}(f(y) - \mathscr{A}_k f(x))k(x,y)\right.$$
$$\left.[\Phi^p(f(y)) - \Phi^p(\mathscr{A}_k f(x)) - |\varphi(\mathscr{A}_k f(x))|(f(y) - \mathscr{A}_k f(x))]\,\Delta y\Delta x\right|$$

holds.

Proof. The result follows from Theorem 10.17 by replacing Φ with Φ^p. □

Corollary 10.29 *Assume* (H_1), (H_2), *and* $(\check{\mathrm{H}}_3)$. *Suppose* f *is a nonnegative* ν_Δ*-integrable function (positive for* $p < 0$*) on* Y *and* $A_k f$ *is defined in* (10.92).

(i) *If* $1 < p \leq q < \infty$, *or* $-\infty < q \leq p < 0$, *then*

$$\left(\int_Y f^p(y)\mathscr{T}(y)\Delta y\right)^{\frac{q}{p}} - \int_X \xi(x)(\mathscr{A}_k f(x))^q\Delta x$$
$$\geq \frac{q}{p}\int_X \frac{\xi(x)}{\mathscr{K}(x)}(\mathscr{A}_k f(x))^{q-p}\int_Y k(x,y)\mathscr{R}_{p,k}(x,y)\Delta y\Delta x \quad (10.102)$$

holds, where

$$\mathscr{R}_{p,k}(x,y) = \left|\left|f^p(y) - \mathscr{A}_k^p f(x)\right| - |p|\,|\mathscr{A}_k f(x)|^{p-1}\,|f(y) - \mathscr{A}_k f(x)|\right|. \quad (10.103)$$

If $p \in (0,1)$ *and* $p \leq q < \infty$, *then the order of terms on the left-hand side of* (10.102) *is reversed.*

(ii) *Let $f(y) > \mathscr{A}_k f(x)$ for $y \in Y' \subset Y$. If $1 < p \le q < \infty$, or $-\infty < q \le p < 0$, then*

$$\left(\int_Y f^p(y)\mathscr{T}(y)\Delta y\right)^{\frac{q}{p}} - \int_X \xi(x)(\mathscr{A}_k f(x))^q \Delta x$$
$$\ge \frac{q}{p}\left|\int_X \frac{\xi(x)}{\mathscr{K}(x)}(\mathscr{A}_k f(x))^{q-p}\int_Y \operatorname{sgn}(f(y)-\mathscr{A}_k f(x))k(x,y)\mathscr{S}_{p,k}(x,y)\Delta y\Delta x\right| \tag{10.104}$$

holds, where

$$\mathscr{S}_{p,k}(x,y) = f^p(y) - \mathscr{A}_k^p f(x) - |p|(\mathscr{A}_k f(x))^{p-1}(f(y)-\mathscr{A}_k f(x)). \tag{10.105}$$

If $p \in (0,1)$ and $p \le q < \infty$, then the order of terms on the left-hand side of (10.104) *is reversed.*

Proof. We use $\Phi(x) = x^p$, $x \ge 0$, in Theorem 10.17, which is nonnegative, monotone, and convex for $p \in \mathbb{R}\setminus[0,1)$, concave for $p \in (0,1]$, and affine, that is, both convex and concave for $p = 1$. Obviously, in this case $\varphi(x) = \Phi'(x) = px^{p-1}$. □

Corollary 10.30 *Assume* (H_1), (H_2), *and* ($\check{H}_3$) *hold with $0 < p \le q < \infty$ and g is a positive ν_Δ-integrable function on Y. Then*

$$\left(\int_Y g^p(y)\mathscr{T}(y)\Delta y\right)^{\frac{q}{p}} - \int_X \xi(x)\mathscr{G}_k^q(x)\Delta x$$
$$\ge \frac{q}{p}\int_X \frac{\xi(x)}{\mathscr{K}(x)}\mathscr{G}_k^{q-p}(x)\int_Y k(x,y)\mathscr{Q}_{p,k}(x,y)\Delta y\Delta x$$

holds, where

$$\mathscr{G}_k(x) = \exp\left(\frac{1}{\mathscr{K}(x)}\int_Y k(x,y)\ln g(y)\Delta y\right) \tag{10.106}$$

and

$$\mathscr{Q}_{p,k}(x,y) = \left|\left|g^p(y) - \mathscr{G}_k^p(x)\right| - p|\mathscr{G}_k^p(x)|\left|\ln\frac{g(y)}{\mathscr{G}_k(x)}\right|\right|. \tag{10.107}$$

Moreover, if $g(y) > \mathscr{G}_k(x)$ for $y \in Y' \subset Y$, then

$$\left(\int_Y g^p(y)\mathscr{T}(y)\Delta y\right)^{\frac{q}{p}} - \int_X \xi(x)\mathscr{G}_k^q(x)\Delta x$$
$$\ge \frac{q}{p}\left|\int_X \frac{\xi(x)}{\mathscr{K}(x)}\mathscr{G}_k^{q-p}(x)\int_Y \operatorname{sgn}(g(y)-\mathscr{G}_k(x))k(x,y)\mathscr{U}_{p,k}(x,y)\Delta y\Delta x\right|$$

holds, where

$$\mathscr{U}_{p,k}(x,y) = g^p(y) - \mathscr{G}_k^p(x) - p|\mathscr{G}_k^p(x)|\ln\frac{g(y)}{\mathscr{G}_k(x)}. \tag{10.108}$$

Proof. We use $\Phi(x) = e^x$, $x > 0$, and $f(x) = p\ln g(x)$ in Theorem 10.17, to obtain the required result. Note that $\mathscr{G}_k(x) = \exp(\mathscr{A}_k(\ln g(x)))$. □

Theorem 10.18 *Assume* (H$_1$), (H$_2$), (H$_3$), *and* (H$_4$) *hold and f is a ν_Δ-integrable function on Y such that $f(Y) \subset I$.*

(i) *If Φ is convex (need not to be nonnegative) in* (H$_4$), *then*

$$\int_Y \Phi(f(y))w(y)\Delta y - \int_X \xi(x)\Phi(\mathscr{A}_k f(x))\Delta x \geq \int_X \frac{\xi(x)}{\mathscr{K}(x)} \int_Y k(x,y)\mathscr{R}_k(x,y)\Delta y\Delta x \quad (10.109)$$

holds, where $\mathscr{R}_k$ is defined in (10.93). *If Φ is concave, then the order of terms on the left-hand side of* (10.109) *is reversed.*

(ii) *If Φ is monotone and convex, and $f(y) > \mathscr{A}_k f(x)$ for $y \in Y' \subset Y$, then*

$$\int_Y \Phi(f(y))w(y)\Delta y - \int_X \xi(x)\Phi(\mathscr{A}_k f(x))\Delta x \geq \left| \int_X \frac{\xi(x)}{\mathscr{K}(x)} \int_Y \operatorname{sgn}(f(y) - \mathscr{A}_k f(x))k(x,y)\mathscr{S}_k(x,y)\Delta y\Delta x \right| \quad (10.110)$$

holds, where $\mathscr{S}_k$ is defined in (10.95). *If Φ is monotone and concave, then the order of terms on the left-hand side of* (10.110) *is reversed.*

Proof. The proof is similar to the proof of Theorem 10.17, just use $q = p$ in the proof of Theorem 10.17. □

Remark 10.24 In Theorem 10.18, since the right-hand side of (10.109) is nonnegative, we get the refinement of Theorem 10.1.

Corollary 10.31 *Assume* (H$_1$), (H$_2$), *and* (H$_3$) *hold and f is a positive ν_Δ-integrable function on Y.*

(i) *If $p \geq 1$ or $p < 0$, then*

$$\int_Y f^p(y)w(y)\Delta y - \int_X \xi(x)\mathscr{A}_k^p f(x)\Delta x \geq \int_X \frac{\xi(x)}{\mathscr{K}(x)} \int_Y k(x,y)\mathscr{R}_{p,k}(x,y)\Delta y\Delta x \quad (10.111)$$

holds, where $\mathscr{R}_{p,k}$ is defined in (10.103). *If $p \in (0,1)$, then the order of terms on the left-hand side of* (10.111) *is reversed.*

(ii) *Let $f(y) > \mathscr{A}_k f(x)$ for $y \in Y' \subset Y$. If $p \geq 1$ or $p < 0$, then*

$$\begin{aligned}\int_Y f^p(y)w(y)\Delta y - \int_X \xi(x)\mathscr{A}_k^p f(x)\Delta x \\ \geq \left| \int_X \frac{\xi(x)}{\mathscr{K}(x)} \int_Y \operatorname{sgn}(f(y) - \mathscr{A}_k f(x)) k(x,y) \mathscr{S}_{p,k}(x,y)\Delta y \Delta x \right| \end{aligned} \quad (10.112)$$

holds, where $\mathscr{S}_{p,k}$ is defined in (10.105)*. If $p \in (0,1)$, then the order of terms on the left-hand side of* (10.112) *is reversed.*

Proof. We use $\Phi(x) = x^p$, $x \geq 0$, in Theorem 10.18. □

Remark 10.25 (10.111) is a refinement of the inequality in Corollary 10.1.

Corollary 10.32 *Assume* (H_1), (H_2), *and* (H_3)*. If g is a positive ν_Δ-integrable function on Y, then*

$$\int_Y g^p(y)w(y)\Delta y - \int_X \xi(x)\mathscr{G}_k^p(x)\Delta x \geq \int_X \frac{\xi(x)}{\mathscr{K}(x)} \int_Y k(x,y)\mathscr{Q}_{p,k}(x,y)\Delta y \Delta x \quad (10.113)$$

holds, where $\mathscr{G}_k$ is defined in (10.106) *and $\mathscr{Q}_{p,k}$ is defined in* (10.107)*. Moreover, if $g(y) > \mathscr{G}_k(x)$ for $y \in Y' \subset Y$, then*

$$\begin{aligned}\int_Y g^p(y)w(y)\Delta y - \int_X \xi(x)\mathscr{G}_k^p(x)\Delta x \\ \geq \left| \int_X \frac{\xi(x)}{\mathscr{K}(x)} \int_Y \operatorname{sgn}(g(y) - \mathscr{G}_k(x)) k(x,y) \mathscr{U}_{p,k}(x,y)\Delta y \Delta x \right| \end{aligned} \quad (10.114)$$

holds, where $\mathscr{U}_{p,k}$ is defined in (10.108)*.*

Proof. We use $\Phi(x) = e^x$, $x > 0$, and $f(x) = p \ln g(x)$ in Theorem 10.18. □

Remark 10.26 (10.113) is a refinement of the inequality in Corollary 10.2.

10.5.2 Results using Special Kernels

Let us give two new hypotheses for our next result.

($\overline{H}_1$) $X = Y$ in (H_1).

($\overline{H}_2$) $m : Y \to \mathbb{R}_+$ is such that $\int_Y m(y)\Delta y < \infty$ for all $y \in Y$.

Theorem 10.19 *Assume* ($\overline{H}_1$), ($\overline{H}_2$), *and* (H_4)*. If $0 < p \leq q < \infty$ or $-\infty < q \leq p < 0$ and f is a ν_Δ-integrable function on Y such that $f(Y) \subset I$, then*

$$\left(\frac{\int_Y m(y)\Phi(f(y))\Delta y}{\int_Y m(y)\Delta y} \right)^{\frac{q}{p}} - \Phi^{\frac{q}{p}}(\mathscr{A}_m f(y)) \geq \frac{q}{p} \frac{1}{\int_Y m(y)\Delta y} \int_Y m(y)\mathscr{M}(y)\Delta y \quad (10.115)$$

holds, where

$$\mathscr{A}_m f(y) = \frac{1}{\int_Y m(y)\Delta y} \int_Y m(y) f(y) \Delta y \tag{10.116}$$

and

$$\mathscr{M}(y) = ||\Phi(f(y)) - \Phi(\mathscr{A}_m f(y))| - |\varphi(\mathscr{A}_m f(y))|\,|f(y) - \mathscr{A}_m f||\,. \tag{10.117}$$

If Φ *is nonnegative and concave, then the order of terms on the left-hand side of* (10.115) *is reversed. Moreover, if* Φ *is a nonnegative, monotone, and convex, and* $f(y) > \mathscr{A}_m f(y)$ *for* $y \in Y' \subset Y$, *then*

$$\begin{aligned} &\left(\frac{\int_Y m(y)\Phi(f(y))\Delta y}{\int_Y m(y)\Delta y}\right)^{\frac{q}{p}} - \Phi^{\frac{q}{p}}(\mathscr{A}_m f(y)) \\ &\qquad \geq \frac{q}{p}\left|\frac{1}{\int_Y m(y)\Delta y}\int_Y \operatorname{sgn}(f(y) - \mathscr{A}_m f(y)) m(y) \mathscr{N}(y)\Delta y\right| \end{aligned} \tag{10.118}$$

holds, where

$$\mathscr{N}(y) = \Phi(f(y)) - \Phi(\mathscr{A}_m f(y)) - |\varphi(\mathscr{A}_m f(y))|\,(f(y) - \mathscr{A}_m f). \tag{10.119}$$

If Φ *is nonnegative, monotone, and concave, then the order of terms on the left-hand side of* (10.118) *is reversed.*

Proof. The result follows from Theorem 10.17 by taking $k(x,y) = \xi(x)m(y)$ for some positive μ_Δ-integrable function ξ and positive ν_Δ-integrable function m. □

Theorem 10.20 *Assume* $(\overline{H}_1)$, $(\overline{H}_2)$, *and* (H_4). *If* f *is a* ν_Δ*-integrable function on* Y *such that* $f(Y) \subset I$ *and* Φ *is a convex function in* (H_4), *then*

$$\frac{\int_Y m(y)\Phi(f(y))\Delta y}{\int_Y m(y)\Delta y} - \Phi(\mathscr{A}_m f(y)) \geq \frac{1}{\int_Y m(y)\Delta y}\int_Y m(y)\mathscr{M}(y)\Delta y \tag{10.120}$$

holds, where $\mathscr{A}_m f$ *is defined in* (10.116) *and* $\mathscr{M}$ *is defined in* (10.117). *If* Φ *is concave, then the order of terms on the left-hand side of* (10.120) *is reversed. Moreover, if* Φ *is monotone and convex on I, and* $f(y) > \mathscr{A}_m f(y)$ *for* $y \in Y' \subset Y$, *then*

$$\begin{aligned} &\frac{\int_Y m(y)\Phi(f(y))\Delta y}{\int_Y m(y)\Delta y} - \Phi(\mathscr{A}_m f(y)) \\ &\qquad \geq \left|\frac{1}{\int_Y m(y)\Delta y}\int_Y \operatorname{sgn}(f(y) - \mathscr{A}_m f(y)) m(y) \mathscr{N}(y)\Delta y\right| \end{aligned} \tag{10.121}$$

holds, where $\mathscr{N}$ *is defined in* (10.119). *If* Φ *is monotone and concave, then the order of terms on the left-hand side of* (10.121) *is reversed.*

Proof. The result follows from Theorem 10.18 by taking $k(x,y) = \xi(x)m(y)$ for some positive μ_Δ-integrable function ξ and positive ν_Δ-integrable function m. □

Remark 10.27 Since the right-hand side of (10.120) is nonnegative, it gives a refinement of Jensen's inequality on time scales.

Further our new hypotheses are:

($\widetilde{H}_1$) $X = Y = [a,b)_{\mathbb{T}}$, where $\mathbb{T}$ is an arbitrary time scale.

($\widetilde{H}_2$) Let $0 < p \le q < \infty$ or $-\infty < q \le p < 0$, and

$\xi : X \to \mathbb{R}_+$ is such that

$$\widetilde{\mathscr{T}}(y) = \left(\int_y^b \xi(x) \left(\frac{1}{\sigma(x) - a} \right)^{\frac{q}{p}} \Delta x \right)^{\frac{p}{q}} < \infty, \quad y \in Y.$$

Theorem 10.21 *Assume* ($\widetilde{H}_1$), ($\widetilde{H}_2$), *and* (H$_4$). *If f is a ν_Δ-integrable function on Y such that $f(Y) \subset I$, then*

$$\left(\int_a^b \Phi(f(y)) \widetilde{\mathscr{T}}(y) \Delta y \right)^{\frac{q}{p}} - \int_a^b \xi(x) \Phi^{\frac{q}{p}}(\mathscr{A}_1 f(x)) \Delta x \\ \ge \frac{q}{p} \int_a^b \frac{\xi(x)}{\sigma(x) - a} \Phi^{\frac{q}{p}-1}(\mathscr{A}_1 f(x)) \int_a^{\sigma(x)} \mathscr{R}_1(x,y) \Delta y \Delta x \quad (10.122)$$

holds, where

$$\mathscr{A}_1 f(x) = \frac{1}{\sigma(x) - a} \int_a^{\sigma(x)} f(y) \Delta y, \quad x \in X \quad (10.123)$$

and

$$\mathscr{R}_1(x,y) = ||\Phi(f(y)) - \Phi(\mathscr{A}_1 f(x))| - |\varphi(\mathscr{A}_1 f(x))||f(y) - \mathscr{A}_1 f(x)||. \quad (10.124)$$

If Φ is nonnegative, monotone, and convex, and $f(y) > \mathscr{A}_1 f(x)$ for $y \in Y' \subset Y$, then

$$\left(\int_a^b \Phi(f(y)) \widetilde{\mathscr{T}}(y) \Delta y \right)^{\frac{q}{p}} - \int_a^b \xi(x) \Phi^{\frac{q}{p}}(\mathscr{A}_1 f(x)) \Delta x \\ \ge \frac{q}{p} \left| \int_a^b \frac{\xi(x)}{\sigma(x) - a} \Phi^{\frac{q}{p}-1}(\mathscr{A}_1 f(x)) \int_a^{\sigma(x)} \operatorname{sgn}(f(y) - \mathscr{A}_1 f(x)) \mathscr{S}_1(x,y) \Delta y \Delta x \right| \quad (10.125)$$

holds, where

$$\mathscr{S}_1(x,y) = \Phi(f(y)) - \Phi(\mathscr{A}_1 f(x)) - |\varphi(\mathscr{A}_1 f(x))|(f(y) - \mathscr{A}_1 f(x)). \quad (10.126)$$

Proof. The statement follows from Theorem 10.17, by using

$$k(x,y) = \begin{cases} 1 & \text{if} \quad a \le y < \sigma(x) \le b, \\ 0 & \text{otherwise,} \end{cases} \quad (10.127)$$

since in this case we have

$$\mathscr{K}(x) = \int_a^{\sigma(x)} \Delta y = (\sigma(x) - a).$$

This completes the proof. □

For $p = q$, $(\widetilde{H}_2)$ becomes

$(\hat{H}_2)$ $\xi : X \to \mathbb{R}_+$ is such that $\widetilde{w}(y) = \int_y^b \frac{\xi(x)}{\sigma(x)-a}\Delta x < \infty, \quad y \in Y.$

Theorem 10.22 *Assume* $(\widetilde{H}_1)$, $(\hat{H}_2)$, *and* (H_4) *hold, and* f *is a* ν_Δ*-integrable function on* Y *such that* $f(Y) \subset I$.

(i) *If* Φ *is convex in* (H_4), *then*

$$\int_a^b \Phi(f(y))\widetilde{w}(y)\Delta y - \int_a^b \xi(x)\Phi(\mathscr{A}_1 f(x))\Delta x \geq \int_a^b \frac{\xi(x)}{\sigma(x)-a}\int_a^{\sigma(x)} \mathscr{R}_1(x,y)\Delta y \Delta x \quad (10.128)$$

holds, where $\mathscr{A}_1 f$ *and* $\mathscr{R}_1$ *are defined in* (10.123) *and* (10.124), *respectively. If* Φ *is concave, then the order of terms on the left-hand side of* (10.128) *is reversed.*

(ii) *If* Φ *is monotone and convex, and* $f(y) > \mathscr{A}_1 f(x)$ *for* $y \in Y' \subset Y$, *then*

$$\int_a^b \Phi(f(y))\widetilde{w}(y)\Delta y - \int_a^b \xi(x)\Phi(\mathscr{A}_1 f(x))\Delta x \geq \left| \int_a^b \frac{\xi(x)}{\sigma(x)-a}\int_a^{\sigma(x)} \operatorname{sgn}(f(y) - \mathscr{A}_1 f(x))\mathscr{S}_1(x,y)\Delta y \Delta x \right| \quad (10.129)$$

holds, where $\mathscr{S}_1$ *is defined in* (10.126). *If* Φ *is monotone and concave, then the order of terms on the left-hand side of* (10.129) *is reversed.*

Proof. The statement follows from Theorem 10.18 with k defined as in (10.127). □

Corollary 10.33 *Assume* $(\widetilde{H}_1)$ *and* $(\hat{H}_2)$ *hold and* f *is a nonnegative* ν_Δ*-integrable function on* Y *such that* $f(Y) \subset I$.

(i) *If* $p \geq 1$ *or* $p < 0$, *then*

$$\int_a^b f^p(y)\widetilde{w}(y)\Delta y - \int_a^b \xi(x)\mathscr{A}_1^p f(x)\Delta x \geq \int_a^b \frac{\xi(x)}{\sigma(x)-a}\int_a^{\sigma(x)} \mathscr{R}_{p,1}(x,y)\Delta y \Delta x \quad (10.130)$$

holds, where $\mathscr{A}_1 f$ *is defined in* (10.123) *and*

$$\mathscr{R}_{p,1}(x,y) = \left| \left| f^p(y) - \mathscr{A}_1^p f(x) \right| - |p| \, |\mathscr{A}_1 f(x)|^{p-1} \, |f(y) - \mathscr{A}_1 f(x)| \right|.$$

If $p \in (0,1)$, *then the order of terms on the left-hand side of* (10.130) *is reversed.*

(ii) *Let $f(y) > \mathscr{A}_1 f(x)$ for $y \in Y' \subset Y$. If $p \geq 1$ or $p < 0$, then*

$$\int_a^b f^p(y)\widetilde{w}(y)\Delta y - \int_a^b \xi(x)\mathscr{A}_1^p f(x)\Delta x \geq \left| \int_a^b \frac{\xi(x)}{\sigma(x)-a} \int_a^{\sigma(x)} \operatorname{sgn}(f(y) - \mathscr{A}_1 f(x)) \mathscr{S}_{p,1}(x,y)\Delta y \Delta x \right| \quad (10.131)$$

holds, where

$$\mathscr{S}_{p,1}(x,y) = f^p(y) - \mathscr{A}_1^p f(x) - |p|(\mathscr{A}_1 f(x))^{p-1}(f(y) - \mathscr{A}_1 f(x)).$$

If $p \in (0,1)$,then the order of terms on the left-hand side of (10.131) *is reversed.*

Proof. We use $\Phi(x) = x^p$, $x > 0$, in Theorem 10.22. □

Corollary 10.34 *Assume $(\widetilde{H}_1)$ and $(\hat{H}_2)$. If g is a positive ν_Δ-integrable function on Y, then*

$$\int_a^b g(y)\widetilde{w}(y)\Delta y - \int_a^b \xi(x)\mathscr{G}_1(x)\Delta x \geq \int_a^b \frac{\xi(x)}{\sigma(x)-a} \int_a^{\sigma(x)} \mathscr{Q}_1(x,y)\Delta y \Delta x \quad (10.132)$$

holds, where

$$\mathscr{G}_1(x) = exp\left(\frac{1}{\sigma(x)-a} \int_a^{\sigma(x)} \ln g(y)\Delta y \right)$$

and

$$\mathscr{Q}_1(x,y) = \left| |g(y) - \mathscr{G}_1(x)| - |\mathscr{G}_1(x)| \left| \ln \frac{g(y)}{\mathscr{G}_1(x)} \right| \right|.$$

If $g(y) > \mathscr{G}_1(x)$ for $y \in Y' \subset Y$, then

$$\int_a^b g(y)\widetilde{w}(y)\Delta y - \int_a^b \xi(x)\mathscr{G}_1(x)\Delta x \geq \left| \int_a^b \frac{\xi(x)}{\sigma(x)-a} \int_a^{\sigma(x)} \operatorname{sgn}(g(y) - \mathscr{G}_1(x)) \mathscr{U}_1(x,y)\Delta y \Delta x \right| \quad (10.133)$$

holds, where

$$\mathscr{U}_1(x,y) = g(y) - \mathscr{G}_1(x) - |\mathscr{G}_1(x)| \ln \frac{g(y)}{\mathscr{G}_1(x)}.$$

Proof. We use $\Phi(x) = e^x$, $x > 0$, and $f(x) = \ln g(x)$ in Theorem 10.22. □

10.5.3 Examples

Example 10.30 In addition to the assumptions of Theorem 10.21, if $\mathbb{T}$ consists of only isolated points and $b=\infty$, then (10.122) takes the form

$$\left(\sum_{y\in[a,\infty)_{\mathbb{T}}}\Phi(f(y))\hat{\mathscr{T}}(y)(\sigma(y)-y)\right)^{\frac{q}{p}}-\sum_{x\in[a,\infty)_{\mathbb{T}}}\xi(x)\Phi^{\frac{q}{p}}(\hat{\mathscr{A}}_1f(x))(\sigma(x)-x)$$
$$\geq\frac{q}{p}\sum_{x\in[a,\infty)_{\mathbb{T}}}\frac{\xi(x)}{\sigma(x)-a}\Phi^{\frac{q}{p}-1}(\hat{\mathscr{A}}_1f(x))\sum_{y\in[a,\sigma(x))_{\mathbb{T}}}\hat{\mathscr{R}}_1(x,y)(\sigma(y)-y)(\sigma(x)-x),$$

where

$$\hat{\mathscr{T}}(y)=\left(\sum_{x\in[y,\infty)_{\mathbb{T}}}\xi(x)\left(\frac{1}{\sigma(x)-a}\right)^{\frac{q}{p}}(\sigma(x)-x)\right)^{\frac{p}{q}},\quad y\in Y\quad\text{and}\quad p,q\in\mathbb{R},$$

$$\hat{\mathscr{A}}_1f(x)=\frac{1}{\sigma(x)-a}\sum_{y\in[a,\sigma(x))_{\mathbb{T}}}f(y)(\sigma(y)-y),\quad x\in X \tag{10.134}$$

and

$$\hat{\mathscr{R}}_1(x,y)=||\Phi(f(y))-\Phi(\hat{\mathscr{A}}_1f(x))|-|\varphi(\hat{\mathscr{A}}_1f(x))||f(y)-\hat{\mathscr{A}}_1f(x)||,$$

and (10.125) takes the form

$$\left(\sum_{y\in[a,\infty)_{\mathbb{T}}}\Phi(f(y))\hat{\mathscr{T}}(y)(\sigma(y)-y)\right)^{\frac{q}{p}}-\sum_{x\in[a,\infty)_{\mathbb{T}}}\xi(x)\Phi^{\frac{q}{p}}(\hat{\mathscr{A}}_1f(x))(\sigma(x)-x)$$
$$\geq\frac{q}{p}\left|\sum_{x\in[a,\infty)_{\mathbb{T}}}\frac{\xi(x)}{\sigma(x)-a}\Phi^{\frac{q}{p}-1}(\hat{\mathscr{A}}_1f(x))\right.$$
$$\left.\sum_{y\in[a,\sigma(x))_{\mathbb{T}}}\operatorname{sgn}(f(y)-\hat{\mathscr{A}}_1f(x))\hat{\mathscr{S}}_1(x,y)(\sigma(y)-y)(\sigma(x)-x)\right|,$$

where

$$\hat{\mathscr{S}}_1(x,y)=\Phi(f(y))-\Phi(\hat{\mathscr{A}}_1f(x))-|\varphi(\hat{\mathscr{A}}_1f(x))|(f(y)-\hat{\mathscr{A}}_1f(x)).$$

Example 10.31 In addition to the hypotheses of Corollary 10.33, let $\mathbb{T}$ consist of only isolated points and $b=\infty$.

(i) If $p\geq 1$ or $p<0$, then

$$\sum_{y\in[a,\infty)_{\mathbb{T}}} f^p(y)\hat{w}(y)(\sigma(y)-y) - \sum_{x\in[a,\infty)_{\mathbb{T}}} \xi(x)\hat{\mathscr{A}}_1^p f(x)(\sigma(x)-x)$$
$$\geq \sum_{x\in[a,\infty)_{\mathbb{T}}} \frac{\xi(x)}{\sigma(x)-a} \sum_{y\in[a,\sigma(x))_{\mathbb{T}}} \hat{\mathscr{R}}_{p,1}(x,y)(\sigma(y)-y)(\sigma(x)-x) \quad (10.135)$$

holds, where $\hat{\mathscr{A}}_1 f$ is defined in (10.134) and

$$\hat{w}(y) = \sum_{x\in[y,\infty)_{\mathbb{T}}} \frac{\xi(x)}{\sigma(x)-a}(\sigma(x)-x), \quad y\in Y, \quad (10.136)$$

$$\hat{\mathscr{R}}_{p,1}(x,y) = \left| \left| f^p(y) - \hat{\mathscr{A}}_1^p f(x) \right| - |p| \left| \hat{\mathscr{A}}_1 f(x) \right|^{p-1} \left| f(y) - \hat{\mathscr{A}}_1 f(x) \right| \right|.$$

If $p \in (0,1)$, then the order of terms on the left-hand side of (10.135) is reversed.

(ii) Let $f(y) > \hat{\mathscr{A}}_1 f(x)$ for $y \in Y' \subset Y$. If $p \geq 1$ or $p < 0$, then

$$\sum_{y\in[a,\infty)_{\mathbb{T}}} f^p(y)\hat{w}(y)(\sigma(y)-y) - \sum_{x\in[a,\infty)_{\mathbb{T}}} \xi(x)\hat{\mathscr{A}}_1^p f(x)(\sigma(x)-x) \quad (10.137)$$
$$\geq \left| \sum_{x\in[a,\infty)_{\mathbb{T}}} \frac{\xi(x)}{\sigma(x)-a} \right.$$
$$\left. \sum_{y\in[a,\sigma(x))_{\mathbb{T}}} \operatorname{sgn}(f(y)-\hat{\mathscr{A}}_1 f(x))\hat{\mathscr{S}}_{p,1}(x,y)(\sigma(y)-y)(\sigma(x)-x) \right|$$

holds, where

$$\hat{\mathscr{S}}_{p,1}(x,y) = f^p(y) - \hat{\mathscr{A}}_1^p f(x) - |p|(\hat{\mathscr{A}}_1 f(x))^{p-1}(f(y)-\hat{\mathscr{A}}_1 f(x)).$$

If $p \in (0,1)$, then the order of terms on the left-hand side of (10.137) is reversed.

Example 10.32 In addition to the hypotheses of Corollary 10.34, if $\mathbb{T}$ consists of only isolated points, then (10.132) becomes

$$\sum_{y\in[a,\infty)_{\mathbb{T}}} g(y)\hat{w}(y)(\sigma(y)-y) - \sum_{x\in[a,\infty)_{\mathbb{T}}} \xi(x)\hat{\mathscr{G}}_1(x)(\sigma(x)-x)$$
$$\geq \sum_{x\in[a,\infty)_{\mathbb{T}}} \frac{\xi(x)}{\sigma(x)-a} \sum_{y\in[a,\sigma(x))_{\mathbb{T}}} \hat{\mathscr{Q}}_1(x,y)(\sigma(y)-y)(\sigma(x)-x), \quad (10.138)$$

where $\hat{w}$ is defined in (10.136) and

$$\hat{\mathscr{G}}_1(x) = \left(\prod_{y\in[a,\sigma(x))_{\mathbb{T}}} (g(y))^{(\sigma(y)-y)} \right)^{\frac{1}{\sigma(x)-a}},$$

$$\hat{\mathscr{Q}}_1(x,y) = \left| \left| g(y) - \hat{\mathscr{G}}_1(x) \right| - |\hat{\mathscr{G}}_1(x)| \left| \ln \frac{g(y)}{\hat{\mathscr{G}}_1(x)} \right| \right|.$$

If $g(y) > \hat{\mathscr{G}}_1(x)$ for $y \in Y' \subset Y$, then (10.133) becomes

$$\sum_{y\in[a,\infty)_{\mathbb{T}}} g(y)\hat{w}(y)(\sigma(y)-y) - \sum_{x\in[a,\infty)_{\mathbb{T}}} \xi(x)\hat{\mathscr{G}}_1(x)(\sigma(x)-x) \tag{10.139}$$
$$\geq \left| \sum_{x\in[a,\infty)_{\mathbb{T}}} \frac{\xi(x)}{\sigma(x)-a} \sum_{y\in[a,\sigma(x))_{\mathbb{T}}} \operatorname{sgn}(g(y)-\hat{\mathscr{G}}_1(x))\hat{\mathscr{U}}_1(x,y)(\sigma(y)-y)(\sigma(x)-x) \right|,$$

where

$$\hat{\mathscr{U}}_1(x,y) = g(y) - \hat{\mathscr{G}}_1(x) - |\hat{\mathscr{G}}_1(x)| \ln \frac{g(y)}{\hat{\mathscr{G}}_1(x)}.$$

Example 10.33 For $\mathbb{T} = h\mathbb{N} = \{hn : n \in \mathbb{N}\}$ with $h > 0$, $a = h$, and

$$\xi(x) = \frac{1}{\sigma(x)},$$

(10.135) takes the form

$$\sum_{m=1}^{\infty} \frac{f^p(mh)}{m} - \sum_{n=1}^{\infty} \frac{1}{n+1} \left(\frac{1}{n} \sum_{m=1}^{n} f(mh) \right)^p \geq \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \sum_{m=1}^{n} \hat{\mathscr{R}}_{p,1}(nh,mh),$$

where

$$\hat{\mathscr{R}}_{p,1}(nh,mh) = \left| \left| f^p(mh) - \left(\frac{1}{n} \sum_{m=1}^{n} f(mh) \right)^p \right| - |p| \left| \frac{1}{n} \sum_{m=1}^{n} f(mh) \right|^{p-1} \left| f(mh) - \frac{1}{n} \sum_{m=1}^{n} f(mh) \right| \right|,$$

and (10.137) takes the form

$$\sum_{m=1}^{\infty} \frac{f^p(mh)}{m} - \sum_{n=1}^{\infty} \frac{1}{n+1} \left(\frac{1}{n} \sum_{m=1}^{n} f(mh) \right)^p$$
$$\geq \left| \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \sum_{m=1}^{n} \operatorname{sgn}\left(f(mh) - \frac{1}{n} \sum_{m=1}^{n} f(mh) \right) \hat{\mathscr{S}}_{p,1}(nh,mh) \right|,$$

where

$$\hat{\mathscr{S}}_{p,1}(nh,mh) = f^p(mh) - \left(\frac{1}{n}\sum_{m=1}^{n} f(mh)\right)^p$$
$$- |p|\left(\frac{1}{n}\sum_{m=1}^{n} f(mh)\right)^{p-1}\left(f(mh) - \frac{1}{n}\sum_{m=1}^{n} f(mh)\right).$$

Example 10.34 For $\mathbb{T} = \mathbb{N}^2 = \{n^2 : n \in \mathbb{N}\}$ with $a = 1$ and

$$\xi(x) = \frac{2(\sigma(x)-1)}{(\sigma(x)-x)^2(2\sqrt{x}+3)},$$

(10.135) takes the form

$$\sum_{m=1}^{\infty} f^p(m^2) - \sum_{n=1}^{\infty} \frac{2\,(n(n+2))^{1-p}}{(2n+1)(2n+3)}\left(\sum_{m=1}^{n}(2m+1)f(m^2)\right)^p$$
$$\geq \sum_{n=1}^{\infty} \frac{2}{(2n+1)(2n+3)}\sum_{m=1}^{n} \hat{\mathscr{R}}_{p,1}(n^2,m^2)(2m+1),$$

where

$$\hat{\mathscr{R}}_{p,1}(n^2,m^2) = \left|\left|f^p(m^2) - \left(\frac{1}{n(n+2)}\sum_{m=1}^{n}(2m+1)f(m^2)\right)^p\right|\right.$$
$$- |p|\left|\frac{1}{n(n+2)}\sum_{m=1}^{n}(2m+1)f(m^2)\right|^{p-1}$$
$$\left.\left|f(mh) - \frac{1}{n(n+2)}\sum_{m=1}^{n}(2m+1)f(m^2)\right|\right|,$$

and (10.137) takes the form

$$\sum_{m=1}^{\infty} f^p(m^2) - \sum_{n=1}^{\infty} \frac{2\,(n(n+2))^{1-p}}{(2n+1)(2n+3)}\left(\sum_{m=1}^{n}(2m+1)f(m^2)\right)^p$$
$$\geq \left|\sum_{n=1}^{\infty} \frac{2}{(2n+1)(2n+3)}\sum_{m=1}^{n} \operatorname{sgn}\left(f(m^2) - \frac{1}{n(n+2)}\sum_{m=1}^{n} f(m^2)\right)\right.$$
$$\left.\hat{\mathscr{S}}_{p,1}(n^2,m^2)(2m+1)\right|,$$

where

$$\hat{\mathscr{S}}_{p,1}(n^2,m^2) = f^p(m^2) - \left(\frac{1}{n(n+2)}\sum_{m=1}^{n}(2m+1)f(m^2)\right)^p$$
$$- |p|\left(\frac{1}{n(n+2)}\sum_{m=1}^{n}(2m+1)f(m^2)\right)^{p-1}\left(f(m^2) - \frac{1}{n(n+2)}\sum_{m=1}^{n}(2m+1)f(m^2)\right).$$

Example 10.35 For $\mathbb{T} = q^{\mathbb{N}} = \{q^n : n \in \mathbb{N}\}$ with $q > 1$, $a = q$, and

$$\xi(x) = \frac{\sigma(x) - a}{\sigma(x)(\sigma(x) - x)},$$

(10.135) takes the form

$$\sum_{m=1}^{\infty} f^p(q^m) - \sum_{n=1}^{\infty} q^{-n}(q-1)^p(q^n-1)^{1-p}\left(\sum_{m=1}^{n} q^{m-1} f(q^m)\right)^p \geq \frac{q-1}{q}\sum_{n=1}^{\infty}\frac{1}{q^n}\sum_{m=1}^{n}\hat{\mathscr{R}}_{p,1}(q^n,q^m)q^m,$$

where

$$\hat{\mathscr{R}}_{p,1}(q^n,q^m) = \left|\left|f^p(q^m) - \left(\frac{q-1}{q^n-1}\sum_{m=1}^{n} q^{m-1} f(q^m)\right)^p\right| - |p|\left|\left(\frac{q-1}{q^n-1}\sum_{m=1}^{n} q^{m-1} f(q^m)\right)\right|^{p-1}\left|f(q^m) - \frac{q-1}{q^n-1}\sum_{m=1}^{n} q^{m-1} f(q^m)\right|\right|,$$

and (10.137) takes the form

$$\sum_{m=1}^{\infty} f^p(q^m) - \sum_{n=1}^{\infty} q^{-n}(q-1)^p(q^n-1)^{1-p}\left(\sum_{m=1}^{n} q^{m-1} f(q^m)\right)^p \geq \frac{q-1}{q}\left|\sum_{n=1}^{\infty}\frac{1}{q^n}\sum_{m=1}^{n}\operatorname{sgn}\left(f(q^m) - \frac{q-1}{q^n-1}\sum_{m=1}^{n} q^{m-1} f(q^m)\right)\hat{\mathscr{S}}_{p,1}(q^n,q^m)q^m\right|,$$

where

$$\hat{\mathscr{S}}_{p,1}(q^n,q^m) = f^p(q^m) - \left(\frac{q-1}{q^n-1}\sum_{m=1}^{n} q^{m-1} f(q^m)\right)^p - |p|\left(\frac{q-1}{q^n-1}\sum_{m=1}^{n} q^{m-1} f(q^m)\right)^{p-1}\left(f(q^m) - \frac{q-1}{q^n-1}\sum_{m=1}^{n} q^{m-1} f(q^m)\right).$$

Example 10.36 For $\mathbb{T}$, a, and ξ as in Example 10.33, (10.138) takes the form

$$\sum_{m=1}^{\infty}\frac{g(mh)}{m} - \sum_{n=1}^{\infty}\frac{1}{n+1}\left(\prod_{m=1}^{n} g(mh)\right)^{\frac{1}{n}} \geq \sum_{n=1}^{\infty}\frac{1}{n(n+1)}\sum_{m=1}^{n}\hat{\mathscr{Q}}_1(nh,mh),$$

where

$$\hat{\mathscr{Q}}_1(nh,mh) = \left\|\left|g(mh) - \left(\prod_{m=1}^{n} g(mh)\right)^{\frac{1}{n}}\right| - \left|\left(\prod_{m=1}^{n} g(mh)\right)^{\frac{1}{n}}\right|\left|\ln \frac{g(mh)}{(\prod_{m=1}^{n} g(mh))^{\frac{1}{n}}}\right\|\right|,$$

and (10.139) takes the form

$$\sum_{m=1}^{\infty} \frac{g(mh)}{m} - \sum_{n=1}^{\infty} \frac{1}{n+1}\left(\prod_{m=1}^{n} g(mh)\right)^{\frac{1}{n}}$$
$$\geq \left|\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \sum_{m=1}^{n} \frac{1}{m} \operatorname{sgn}\left(g(mh) - \left(\prod_{m=1}^{n} g(mh)\right)^{\frac{1}{n}}\right) \hat{\mathscr{U}}_1(nh,mh)\right|,$$

where

$$\hat{\mathscr{U}}_1(nh,mh) = g(mh) - \left(\prod_{m=1}^{n} g(mh)\right)^{\frac{1}{n}} - \left|\left(\prod_{m=1}^{n} g(mh)\right)^{\frac{1}{n}}\right| \ln \frac{f(mh)}{(\prod_{m=1}^{n} g(mh))^{\frac{1}{n}}}.$$

Example 10.37 For $\mathbb{T}$, a, and ξ as in Example 10.34, (10.138) takes the form

$$\sum_{m=1}^{\infty} g(m^2) - \sum_{n=1}^{\infty} \frac{2n(n+2)}{(2n+1)(2n+3)}\left(\prod_{m=1}^{n} (g(m^2))^{2m+1}\right)^{\frac{1}{n(n+2)}}$$
$$\geq \sum_{n=1}^{\infty} \frac{2}{(2n+1)(2n+3)} \sum_{m=1}^{n} \hat{\mathscr{Q}}_1(n^2,m^2)(2m+1),$$

where

$$\hat{\mathscr{Q}}_1(n^2,m^2) = \left\|\left|g(m^2) - \left(\prod_{m=1}^{n} (g(m^2))^{2m+1}\right)^{\frac{1}{n(n+2)}}\right|\right.$$
$$\left. - \left|\left(\prod_{m=1}^{n} (g(m^2))^{2m+1}\right)^{\frac{1}{n(n+2)}}\right|\left|\ln \frac{g(m^2)}{(\prod_{m=1}^{n} (g(m^2))^{2m+1})^{\frac{1}{n(n+2)}}}\right\|\right|,$$

and (10.139) takes the form

$$\sum_{m=1}^{\infty} g(m^2) - \sum_{n=1}^{\infty} \frac{2n(n+2)}{(2n+1)(2n+3)}\left(\prod_{m=1}^{n} (g(m^2))^{2m+1}\right)^{\frac{1}{n(n+2)}}$$
$$\geq \left|\sum_{n=1}^{\infty} \frac{2}{(2n+1)(2n+3)} \sum_{m=1}^{n} \operatorname{sgn}\left(g(m^2) - \left(\prod_{m=1}^{n} (g(m^2))^{2m+1}\right)^{\frac{1}{n(n+2)}}\right) \hat{\mathscr{U}}_1(n^2,m^2)(2m+1)\right|,$$

where

$$\hat{\mathscr{U}}_1(n^2,m^2) = g(m^2) - \left(\prod_{m=1}^{n}(g(m^2))^{2m+1}\right)^{\frac{1}{n(n+2)}} - \left|\left(\prod_{m=1}^{n}(g(m^2))^{2m+1}\right)^{\frac{1}{n(n+2)}}\right| \ln \frac{g(m^2)}{\left(\prod_{m=1}^{n}(g(m^2))^{2m+1}\right)^{\frac{1}{n(n+2)}}}.$$

Example 10.38 For $\mathbb{T}$, a, and ξ as in Example 10.35, (10.138) takes the form

$$\sum_{m=1}^{\infty} g(q^m) - \sum_{n=1}^{\infty} q^{-n}(q^n-1)\left(\prod_{m=1}^{n}(g(q^m))^{q^{m-1}}\right)^{\frac{q-1}{q^n-1}} \geq \frac{q-1}{q}\sum_{n=1}^{\infty}\frac{1}{q^n}\sum_{m=1}^{n}\hat{\mathscr{Q}}_1(q^n,q^m)q^m,$$

where

$$\hat{\mathscr{Q}}_1(q^n,q^m) = \left\|g(q^m) - \left(\prod_{m=1}^{n}(g(q^m))^{q^{m-1}}\right)^{\frac{q-1}{q^n-1}}\right| - \left|\left(\prod_{m=1}^{n}(g(q^m))^{q^{m-1}}\right)^{\frac{q-1}{q^n-1}}\right|\left|\ln \frac{g(q^m)}{\left(\prod_{m=1}^{n}(g(q^m))^{q^{m-1}}\right)^{\frac{q-1}{q^n-1}}}\right\|,$$

and (10.139) takes the form

$$\sum_{m=1}^{\infty} g(q^m) - \sum_{n=1}^{\infty} q^{-n}(q^n-1)\left(\prod_{m=1}^{n}(g(q^m))^{q^{m-1}}\right)^{\frac{q-1}{q^n-1}} \geq \frac{q-1}{q}\left|\sum_{n=1}^{\infty}\frac{1}{q^n}\sum_{m=1}^{n}\operatorname{sgn}\left(g(q^m) - \left(\prod_{m=1}^{n}(g(q^m))^{q^{m-1}}\right)^{\frac{q-1}{q^n-1}}\right)\hat{\mathscr{U}}_1(q^n,q^m)q^m\right|,$$

where

$$\hat{\mathscr{U}}_1(q^n,q^m) = g(q^m) - \left(\prod_{m=1}^{n}(g(q^m))^{q^{m-1}}\right)^{\frac{q-1}{q^n-1}} - \left|\left(\prod_{m=1}^{n}(g(q^m))^{q^{m-1}}\right)^{\frac{q-1}{q^n-1}}\right| \ln \frac{g(m^2)}{\left(\prod_{m=1}^{n}(g(q^m))^{q^{m-1}}\right)^{\frac{q-1}{q^n-1}}}.$$

Bibliography

[1] Shoshana Abramovich, Senka Banić, Marko Matić, and Josip E. Pečarić. Jensen–Steffensen's and related inequalities for superquadratic functions. *Math. Inequal. Appl.*, 11(1):23–41, 2008.

[2] Shoshana Abramovich, Josipa Barić, and Josip E. Pečarić. A variant of Jessen's inequality of Mercer's type for superquadratic functions. *JIPAM. J. Inequal. Pure Appl. Math.*, 9(3):Article 62, 13, 2008.

[3] Shoshana Abramovich, Ghulam Farid, Slavica Ivelić, and Josip E. Pečarić. More on Cauchy's means and generalization of Hadamard inequality via converses of Jensen's inequality and superquadracity. *Int. J. Pure Appl. Math.*, 69(1):97–116, 2011.

[4] Shoshana Abramovich, Ghulam Farid, and Josip E. Pečarić. More about Jensen's inequality and Cauchy's means for superquadratic functions. *J. Math. Inequal.*, 7(1):11–24, 2013.

[5] Shoshana Abramovich, Graham Jameson, and Gord Sinnamon. Inequalities for averages of convex and superquadratic functions. *JIPAM. J. Inequal. Pure Appl. Math.*, 5(4):Article 91, 14 pp. (electronic), 2004.

[6] Shoshana Abramovich, Graham Jameson, and Gord Sinnamon. Refining Jensen's inequality. *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)*, 47(95)(1-2):3–14, 2004.

[7] Shoshana Abramovich, Kristina Krulić, Josip E. Pečarić, and Lars-Erik Persson. Some new refined Hardy type inequalities with general kernels and measures. *Aequationes Math.*, 79(1-2):157–172, 2010.

[8] János D. Aczé. Some general methods in the theory of functional equations in one variable. New applications of functional equations. *Uspehi Mat. Nauk (N.S.)*, 11(3(69)):3–68, 1956.

[9] Ravi P. Agarwal, Martin Bohner, and Allan C. Peterson. Inequalities on time scales: a survey. *Math. Inequal. Appl.*, 4(4):535–557, 2001.

[10] Jesus M. Aldaz. A refinement of the inequality between arithmetic and geometric means. *J. Math. Inequal.*, 2(4):473–477, 2008.

[11] Jesus M. Aldaz. A stability version of Hölder's inequality. *J. Math. Anal. Appl.*, 343(2):842–852, 2008.

[12] Jesus M. Aldaz. Self-improvement of the inequality between arithmetic and geometric means. *J. Math. Inequal.*, 3(2):213–216, 2009.

[13] Jesus M. Aldaz. Concentration of the ratio between the geometric and arithmetic means. *J. Theoret. Probab.*, 23(2):498–508, 2010.

[14] Jesus M. Aldaz. Comparison of differences between arithmetic and geometric means. *Tamkang J. Math.*, 42(4):453–462, 2011.

[15] Jesus M. Aldaz. A measure-theoretic version of the Dragomir–Jensen inequality. *Proc. Amer. Math. Soc.*, 140(7):2391–2399, 2012.

[16] Moulay Rchid Sidi Ammi, Rui A. C. Ferreira, and Delfim F. M. Torres. Diamond-α Jensen's inequality on time scales. *J. Inequal. Appl.*, pages Art. ID 576876, 13 pages, 2008.

[17] Moulay Rchid Sidi Ammi and Delfim F. M. Torres. Hölder's and Hardy's two dimensional diamond-alpha inequalities on time scales. *An. Univ. Craiova Ser. Mat. Inform.*, 37(1):1–11, 2010.

[18] Douglas R. Anderson. Time-scale integral inequalities. *JIPAM. J. Inequal. Pure Appl. Math.*, 6(3):Article 66, 15 pp. (electronic), 2005.

[19] Matloob Anwar, Julije Jakšetić, and Josip E. Pečarić. Exponential convexity, positive semi-definite matrices and fundamental inequalities. *J. Math. Inequal.*, 4(2):171–189, 2010.

[20] Matloob Anwar, Rabia Bibi, Martin Bohner, and Josip E. Pečarić. Integral inequalities on time scales via the theory of isotonic linear functionals. *Abstr. Appl. Anal.*, pages Art. ID 483595, 16 pages, 2011.

[21] Matloob Anwar, Rabia Bibi, Martin Bohner, and Josip E. Pečarić. Jensen's functionals on time scales. *J. Funct. Spaces Appl.*, pages Art. ID 384045, 17 pages, 2012.

[22] Matloob Anwar, Rabia Bibi, Martin Bohner, and Josip E. Pečarić. Jensen functionals on time scales for several variables. *Int. J. Anal.*, pages Art. ID 126797, 14 pages, 2014.

[23] Ferhan M. Atici and Gusein Sh. Guseinov. On Green's functions and positive solutions for boundary value problems on time scales. *J. Comput. Appl. Math.*, 141(1-2):75–99, 2002. Dynamic equations on time scales.

[24] Senka Banić. *Superquadratic functions*. PhD thesis, University of Zagreb, Zagreb, Croatia, 2007. In Croatian.

[25] Senka Banić, Josip E. Pečarić, and Sanja Varošanec. Superquadratic functions and refinements of some classical inequalities. *J. Korean Math. Soc.*, 45(2):513–525, 2008.

[26] Senka Banić and Sanja Varošanec. Functional inequalities for superquadratic functions. *Int. J. Pure Appl. Math.*, 43(4):537–549, 2008.

[27] Josipa Barić, Rabia Bibi, Martin Bohner, and Josip E. Pečarić. Time scales integral inequalities for superquadratic functions. *J. Korean Math. Soc.*, 50(3):465–477, 2013.

[28] Josipa Barić, Ammara Nosheen, and Josip E. Pečarić. Time scale Hardy-type inequalities with general kernel for superquadratic functions. *Proceedings of A. Razmadze Mathematical Institute* (1512-0007) 165 (2014); 1–18.

[29] Eugen Beck. Über Ungleichungen von der Form $f(M_\phi(x;\alpha),\ M_\psi(y;\alpha)) \geqq M_\chi(f(x,y);\alpha)$. *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No.*, 320-328:1–14, 1970.

[30] Paul R. Beesack and Josip E. Pečarić. On Jessen's inequality for convex functions. *J. Math. Anal. Appl.*, 110(2):536–552, 1985.

[31] Serge Bernstein. Sur les fonctions absolument monotones. *Acta Math.*, 52(1):1–66, 1929.

[32] Mihály Bessenyei. The Hermite–Hadamard inequality on simplices. *Amer. Math. Monthly*, 115(4):339–345, 2008.

[33] Rabia Bibi. Some improvements of the Popoviciu, Bellman and Diaz–Metcalf inequalities via superquadratic fuctions. 2014. Submitted.

[34] Rabia Bibi, Martin Bohner, Josip E. Pečarić, and Sanja Varošanec. Minkowski and Beckenbach–Dresher inequalities and functionals on time scales. *J. Math. Inequal.*, 7(3):299–312, 2013.

[35] Rabia Bibi, Martin Bohner, and Josip E. Pečarić. Cauchy-type means and exponential and logarithmic convexity for superquadratic functions on time scales. *Ann. Funct. Anal.*, 6(1), 2015. To appear.

[36] Rabia Bibi, Josip E. Pečarić, and Jurica Perić. Hölder and Minkowski type inequalities on time scales. 2014. In progress.

[37] Rabia Bibi, Josip E. Pečarić, and Jurica Perić. Improvements and generalizations of Hermite–Hadamard and Jensen–Mercer inequality on time scales. 2014. In progress.

[38] Rabia Bibi, Josip E. Pečarić, and Jurica Perić. Improvements of Hermite–Hadamard's inequality on time scales. *J. Math. Inequal.*, 2014. To appear.

[39] Rabia Bibi, Josip E. Pečarić, and Mirna Rodić Lipanović. Improvements of Jensen-type inequalities for diamond-α integrals. 2014. Submitted.

[40] Ralph P. Boas, Jr. The Jensen–Steffensen inequality. *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, (302-319):1–8, 1970.

[41] Ralph P. Boas, Jr. and Christopher O. Imoru. Elementary convolution inequalities. *SIAM J. Math. Anal.*, 6:457–471, 1975.

[42] Martin Bohner and Gusein Sh. Guseinov. Multiple integration on time scales. *Dynam. Systems Appl.*, 14(3-4):579–606, 2005.

[43] Martin Bohner and Gusein Sh. Guseinov. Multiple Lebesgue integration on time scales. *Adv. Difference Equ.*, pages Art. ID 26391, 12 pages, 2006.

[44] Martin Bohner, Ammara Nosheen, Josip E. Pečarić, and Awais Younus. Some dynamic Hardy-type inequalities with general kernel. *J. Math. Inequal.*, 8(1):185–199, 2014.

[45] Martin Bohner and Allan C. Peterson. *Dynamic equations on time scales*. Birkhäuser Boston, Inc., Boston, MA, 2001. An introduction with applications.

[46] Martin Bohner and Allan C. Peterson. *Advances in dynamic equations on time scales*. Birkhäuser Boston, Inc., Boston, MA, 2003.

[47] Peter S. Bullen. Error estimates for some elementary quadrature rules. *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, (602-633):97–103 (1979), 1978.

[48] Peter S. Bullen. *Handbook of means and their inequalities*, volume 560 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 2003. Revised from the 1988 original [P. S. Bullen, D. S. Mitrinović and P. M. Vasić, *Means and their inequalities*, Reidel, Dordrecht; MR0947142].

[49] Wing-Sum Cheung, Anita Matković, and Josip E. Pečarić. A variant of Jessen's inequality and generalized means. *JIPAM. J. Inequal. Pure Appl. Math.*, 7(1):Article 10, 8 pages, 2006.

[50] Cristian Dinu. Hermite–Hadamard inequality on time scales. *J. Inequal. Appl.*, pages Art. ID 287947, 24 pages, 2008.

[51] Cristian Dinu. A weighted Hermite–Hadamard inequality for Steffensen–Popoviciu and Hermite–Hadamard weights on time scales. *An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat.*, 17(1):77–90, 2009.

[52] Tzanko Donchev, Ammara Nosheen, and Josip E. Pečarić. Hardy-type inequalities on time scale via convexity in several variables. *ISRN Math. Anal.*, pages Art. ID 903196, 9 pages, 2013.

[53] Sever S. Dragomir. Bounds for the normalised Jensen functional. *Bull. Austral. Math. Soc.*, 74(3):471–478, 2006.

[54] Sever S. Dragomir, Josip E. Pečarić, and Lars-Erik Persson. Properties of some functionals related to Jensen's inequality. *Acta Math. Hungar.*, 70(1-2):129–143, 1996.

[55] Melvin Dresher. Moment spaces and inequalities. *Duke Math. J.*, 20:261–271, 1953.

[56] Werner Ehm, Marc G. Genton, and Tilmann Gneiting. Stationary covariances associated with exponentially convex functions. *Bernoulli*, 9(4):607–615, 2003.

[57] Rui A. C. Ferreira, Moulay Rchid Sidi Ammi, and Delfim F. M. Torres. Diamond-alpha integral inequalities on time scales. *Int. J. Math. Stat.*, 5(A09):52–59, 2009.

[58] Aurelia Florea and Constantin P. Niculescu. A Hermite–Hadamard inequality for convex-concave symmetric functions. *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)*, 50(98)(2):149–156, 2007.

[59] Iva Franjić, Sadia Khalid, and Josip E. Pečarić. On the refinements of the Jensen–Steffensen inequality. *J. Inequal. Appl.*, pages 2011:12, 11 pages, 2011.

[60] E. K. Godunova. Inequalities based on convex functions. *Izv. Vysš. UČebn. Zaved. Matematika*, 1965(4 (47)):45–53, 1965.

[61] Boris Guljaš, Charles E. M. Pearce, and Josip E. Pečarić. Some generalizations of the Beckenbach–Dresher inequality. *Houston J. Math.*, 22(3):629–638, 1996.

[62] Gusein Sh. Guseinov. Integration on time scales. *J. Math. Anal. Appl.*, 285(1):107–127, 2003.

[63] Gusein Sh. Guseinov and Billûr Kaymakçalan. Basics of Riemann delta and nabla integration on time scales. *J. Difference Equ. Appl.*, 8(11):1001–1017, 2002. Special issue in honour of Professor Allan Peterson on the occasion of his 60th birthday, Part I.

[64] Gusein Sh. Guseinov and Billûr Kaymakçalan. On the Riemann integration on time scales. In *Proceedings of the Sixth International Conference on Difference Equations*, pages 289–298. CRC, Boca Raton, FL, 2004.

[65] Preston C. Hammer. The midpoint method of numerical integration. *Math. Mag.*, 31:193–195, 1957/1958.

[66] Godfrey H. Hardy. Note on a theorem of Hilbert. *Math. Z.*, 6(3-4):314–317, 1920.

[67] Godfrey H. Hardy. Notes on some points in the integral caluclus, LX. An inequality between integrals. *Messenger Math.*, 54:150–156, 1925.

[68] Godfrey H. Hardy, John E. Littlewood, and George Pólya. *Inequalities*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1988. Reprint of the 1952 edition.

[69] Stefan Hilger. *Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*. PhD thesis, Universität Würzburg, Würzburg, Germany, 1988.

[70] Stefan Hilger. Analysis on measure chains—a unified approach to continuous and discrete calculus. *Results Math.*, 18(1-2):18–56, 1990.

[71] Stefan Hilger. Differential and difference calculus—unified! In *Proceedings of the Second World Congress of Nonlinear Analysts, Part 5 (Athens, 1996)*, volume 30, pages 2683–2694, 1997.

[72] László Horváth, Ali Kh. Khan, and Josip E. Pečarić. Further refinement of results about mixed symmetric means and Cauchy means. *Adv. Inequl. Appl.*, 1(1):12–32, 2012.

[73] László Horváth, Ali Kh. Khan, and Josip E. Pečarić. Refinements of Hölder and Minkowski inequalities with weights. *Proc. A. Razmadze Math. Inst.*, 158:33–56, 2012.

[74] László Horváth, Ali Kh. Khan, and Josip E. Pečarić. New refinements of Hölder and Minkowski inequalities with weights. *Proc. A. Razmadze Math. Inst.*, 2014. To appear.

[75] Sajid Iqbal, Kristina Krulić Himmelreich, Josip E. Pečarić, and Dora Pokaz. n-exponential convexity of Hardy-type and Boas-type functionals. *J. Math. Inequal.*, 7(4):739–750, 2013.

[76] Božidar Ivanković, Josip E. Pečarić, and Sanja Varošanec. Properties of mappings related to the Minkowski inequality. *Mediterr. J. Math.*, 8(4):543–551, 2011.

[77] Slavica Ivelić and Josip E. Pečarić. Generalizations of converse Jensen's inequality and related results. *J. Math. Inequal.*, 5(1):43–60, 2011.

[78] Julije Jakšetić and Josip E. Pečarić. Exponential convexity method. *J. Convex Anal.*, 20(1):181–197, 2013.

[79] Rozarija Jakšić and Josip E. Pečarić. New converses of the Jessen and Lah–Ribarič inequalities II. *J. Math. Inequal.*, 7(4):617–645, 2013.

[80] Rozarija Jakšić and Josip E. Pečarić. New converses of the Jessen and Lah–Ribarič inequalities. *Math. Inequal. Appl.*, 17(1):197–216, 2014.

[81] Rozarija Jakšić and Josip E. Pečarić. New converses of the Jessen and Lah–Ribarič inequalities III. 2014. In preparation.

[82] J. L. W. V. Jensen. Om konvekse Funktioner og Uligheder mellem Middelværdier. *Nyt Tidsskrift for Matematik*, 16(B):49–69, 1905.

[83] J. L. W. V. Jensen. Sur les fonctions convexes et les inégalités entre les valeurs moyennes. *Acta Math.*, 30(1):175–193, 1906.

[84] Børge Jessen. Bemærkninger om konvekse Funktioner og Uligheder mellem Middelværdier. *Matematisk Tidsskrift*, 2(B):17–28, 1931.

[85] Sten Kaijser, Ludmila Nikolova, Lars-Erik Persson, and Anna Wedestig. Hardy-type inequalities via convexity. *Math. Inequal. Appl.*, 8(3):403–417, 2005.

[86] Sten Kaijser, Lars-Erik Persson, and Anders Öberg. On Carleman and Knopp's inequalities. *J. Approx. Theory*, 117(1):140–151, 2002.

[87] Ali Kh. Khan, Ammara Nosheen, and Josip E. Pečarić. n-exponential convexity of some dynamic Hardy-type functionals. *J. Math. Inequal.*, 8(2):331–347, 2014.

[88] Asif R. Khan, Josip E. Pečarić, and Mirna Rodić Lipanović. n-exponential convexity for Jensen-type inequalities. *J. Math. Inequal.*, 7(3):313–335, 2013.

[89] Milica Klaričić Bakula, Josip E. Pečarić, and Jurica Perić. Extensions of the Hermite–Hadamard inequality with applications. *Math. Inequal. Appl.*, 15(4):899–921, 2012.

[90] Milica Klaričić Bakula, Josip E. Pečarić, and Jurica Perić. On the converse Jensen inequality. *Appl. Math. Comput.*, 218(11):6566–6575, 2012.

[91] Mario Krnić, Neda Lovričević, and Josip E. Pečarić. Jessen's functional, its properties and applications. *An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat.*, 20(1):225–247, 2012.

[92] Mario Krnić, Neda Lovričević, and Josip E. Pečarić. On the properties of McShane's functional and their applications. *Period. Math. Hungar.*, 66(2):159–180, 2013.

[93] Kristina Krulić, Josip E. Pečarić, and Lars-Erik Persson. Some new Hardy type inequalities with general kernels. *Math. Inequal. Appl.*, 12(3):473–485, 2009.

[94] Alois Kufner, Lech Maligranda, and Lars-Erik Persson. The prehistory of the Hardy inequality. *Amer. Math. Monthly*, 113(8):715–732, 2006.

[95] Alois Kufner, Lech Maligranda, and Lars-Erik Persson. *The Hardy inequality*. Vydavatelský Servis, Plzeň, 2007. About its history and some related results.

[96] Alois Kufner and Lars-Erik Persson. *Weighted inequalities of Hardy type*. World Scientific Publishing Co., Inc., River Edge, NJ, 2003.

[97] Agnieszka B. Malinowska and Delfim F. M. Torres. On the diamond-alpha Riemann integral and mean value theorems on time scales. *Dynam. Systems Appl.*, 18(3-4):469–481, 2009.

[98] Anita Matković, Josip E. Pečarić, and Jurica Perić. A refinement of the Jessen–Mercer inequality and a generalization of convex hulls in $\mathbb{R}^k$. 2014. To appear.

[99] Edward J. McShane. Jensen's inequality. *Bull. Amer. Math. Soc.*, 43(8):521–527, 1937.

[100] A. McD. Mercer. A variant of Jensen's inequality. *JIPAM. J. Inequal. Pure Appl. Math.*, 4(4):Article 73, 2 pp. (electronic), 2003.

[101] Dragoslav S. Mitrinović. *Analytic inequalities*, volume 165 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, New York-Berlin, 1970. In cooperation with P. M. Vasić.

[102] Dragoslav S. Mitrinović, Josip E. Pečarić, and Arlington M. Fink. *Inequalities involving functions and their integrals and derivatives*, volume 53 of *Mathematics and its Applications (East European Series)*. Kluwer Academic Publishers Group, Dordrecht, 1991.

[103] Dragoslav S. Mitrinović, Josip E. Pečarić, and Arlington M. Fink. *Classical and new inequalities in analysis*, volume 61 of *Mathematics and its Applications (East European Series)*. Kluwer Academic Publishers Group, Dordrecht, 1993.

[104] Dragoslav S. Mitrinović and Petar M. Vasić. The centroid method in inequalities. *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, (498-541):3–16, 1975.

[105] Constantin P. Niculescu. The Hermite–Hadamard inequality for convex functions of a vector variable. *Math. Inequal. Appl.*, 5(4):619–623, 2002.

[106] Constantin P. Niculescu. The Hermite–Hadamard inequality for convex functions on a global NPC space. *J. Math. Anal. Appl.*, 356(1):295–301, 2009.

[107] Constantin P. Niculescu and Lars-Erik Persson. Old and new on the Hermite–Hadamard inequality. *Real Anal. Exchange*, 29(2):663–685, 2003/04.

[108] Constantin P. Niculescu and Lars-Erik Persson. *Convex functions and their applications*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 23. Springer, New York, 2006. A contemporary approach.

[109] James A. Oguntuase, Christopher A. Okpoti, Lars-Erik Persson, and Francis K. A. Allotey. Weighted multidimensional Hardy type inequalities via Jensen's inequality. *Proc. A. Razmadze Math. Inst.*, 144:91–105, 2007.

[110] James A. Oguntuase, Lars-Erik Persson, and Emmanuel K. Essel. Multidimensional Hardy-type inequalities with general kernels. *J. Math. Anal. Appl.*, 348(1):411–418, 2008.

[111] James A. Oguntuase, Lars-Erik Persson, Emmanuel K. Essel, and B. A. Popoola. Refined multidimensional Hardy-type inequalities via superquadracity. *Banach J. Math. Anal.*, 2(2):129–139, 2008.

[112] Umut Mutlu Özkan, Mehmet Zeki Sarikaya, and Hüseyin Yildirim. Extensions of certain integral inequalities on time scales. *Appl. Math. Lett.*, 21(10):993–1000, 2008.

[113] Umut Mutlu Özkan and Hüseyin Yildirim. Hardy–Knopp-type inequalities on time scales. *Dynam. Systems Appl.*, 17(3-4):477–486, 2008.

[114] Umut Mutlu Özkan and Hüseyin Yildirim. Time scale Hardy–Knopp type integral inequalities. *Commun. Math. Anal.*, 6(1):36–41, 2009.

[115] Josip Pečarić and Jurica Perić. Improvements of the Giaccardi and the Petrović inequality and related Stolarsky type means. *An. Univ. Craiova Ser. Mat. Inform.*, 39(1):65–75, 2012.

[116] Josip E. Pečarić and Paul R. Beesack. On Jessen's inequality for convex functions. II. *J. Math. Anal. Appl.*, 118(1):125–144, 1986.

[117] Josip E. Pečarić and Jurica Perić. Remarks on the paper "Jensen's inequality and new entropy bounds" of S. Simić. *J. Math. Inequal.*, 6(4):631–636, 2012.

[118] Josip E. Pečarić and Jurica Perić. Generalizations and improvements of converse Jensen's inequality for convex hulls in $\mathbb{R}^k$. *Math. Inequal. Appl.*, 17(3):1125–1137, 2014.

[119] Josip E. Pečarić, Frank Proschan, and Yung Liang Tong. *Convex functions, partial orderings, and statistical applications*, volume 187 of *Mathematics in Science and Engineering*. Academic Press, Inc., Boston, MA, 1992.

[120] Josip E. Pečarić, Jurica Perić, and Mirna Rodić Lipanović. Uniform treatment of Jensen type inequalities. *Math. Rep. (Bucur.)*, 16(66)(2):183–205, 2014.

[121] Josip E. Pečarić, Jurica Perić, and Sanja Varošanec. Refinements of the converse Hölder and Minkowski inequalities. 2014. In preparation.

[122] Pavel Řehák. Hardy inequality on time scales and its application to half-linear dynamic equations. *J. Inequal. Appl.*, (5):495–507, 2005.

[123] Qin Sheng, Mai Fadag, Johnny Henderson, and John M. Davis. An exploration of combined dynamic derivatives on time scales and their applications. *Nonlinear Anal. Real World Appl.*, 7(3):395–413, 2006.

[124] Slavko Simić. On a converse of Jensen's discrete inequality. *J. Inequal. Appl.*, pages Art. ID 153080, 6 pages, 2009.

[125] Slavko Simić. On a new converse of Jensen's inequality. *Publ. Inst. Math. (Beograd) (N.S.)*, 85(99):107–110, 2009.

[126] Slavko Simić. On an upper bound for Jensen's inequality. *JIPAM. J. Inequal. Pure Appl. Math.*, 10(2):Article 60, 5 pages, 2009.

[127] J. F. Steffensen. On certain inequalities and methods of approximation. *J. Inst. Actuar.*, 51:274–297, 1919.

[128] Tiberiu Trif. Characterizations of convex functions of a vector variable via Hermite–Hadamard's inequality. *J. Math. Inequal.*, 2(1):37–44, 2008.

[129] Sanja Varošanec. A generalized Beckenbach–Dresher inequality and related results. *Banach J. Math. Anal.*, 4(1):13–20, 2010.

[130] Petar M. Vasić and Josip E. Pečarić. Notes on some inequalities for convex functions. *Mat. Vesnik*, 6(19)(34)(2):185–193, 1982.

[131] Szymon Wąsowicz. Hermite–Hadamard-type inequalities in the approximate integration. *Math. Inequal. Appl.*, 11(4):693–700, 2008.

[132] David V. Widder. Necessary and sufficient conditions for the representation of a function by a doubly infinite Laplace integral. *Bull. Amer. Math. Soc.*, 40(4):321–326, 1934.

[133] David V. Widder. *The Laplace Transform*. Princeton Mathematical Series, v. 6. Princeton University Press, Princeton, N. J., 1941.

[134] Fu-Hsiang Wong, Cheh-Chih Yeh, and Wei-Cheng Lian. An extension of Jensen's inequality on time scales. *Adv. Dyn. Syst. Appl.*, 1(1):113–120, 2006.

Index

Zagreb, October 2015