MONOGRAPHS IN INEQUALITIES 17

Weighted Steffensen's Inequality

Recent Advances in Generalizations of Steffensen's Inequality

Julije Jakšetić, Josip Pečarić, Anamarija Perušić Pribanić
and Ksenija Smoljak Kalamir

ELEMENT

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Preface

Since its invention in 1918, Steffensen's inequality is generalized in numerous directions under various settings. This book collects its most recent advances in generalizations. Under the vast diversities, in this book, Steffensen's inequality is connected with the following notions: convex functions, higher order convexity, exponential convexity, $h-$convex functions, interpolating polynomials, measure theoretic aspects, weighted Bellman-Steffensen type inequalities, Gauss type inequalities , Hölder type inequalities...
The book is organized as follows:

In the first chapter we consider the original version of Steffensen's inequality, its full characterization under relaxed assumptions, and then we overview tools and already known results that will serve for further generalizations in the later chapters.

In the second chapter, the original version of Steffensen's and reversed Steffensen's inequality is characterized in a measure theory settings (finite positive measures). In these settings, Bellman L^p generalization of the inequality is proved and further improvement of an extension of Bellman-Steffensen type of inequalities. Hölder inequality is also generalized in these settings, using modified Steffensen inequality.

In the third chapter results of Mercer, Pečarić and Wu-Srivastava are generalized, in terms of measure theory, as fixed bounds for function g are relaxed to functions. In this chaper we also cover Cerone's and Pachpatte's results and their generalizations.

In the fourth chapter we consider Steffensen inequality for convex and 3-convex functions. This chapter also covers Gauss-type and Gauss-Steffenesen type inequalities.

In the fifth chapter we consider weighted Steffensen inequality for n-convex functions using Taylor's formula, Euler-type identities, Montgomery's identity, Fink's identity, Lidstone polynomial and two-point Abel-Gontscharoff polynomials.

In every chapter of the book, after proving inequalities, appropriate linear functionals are constructed and corresponding Cauchy type means and exponentially convex functions are constructed.

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Chapter 1

Basic results and definitions

1.1 Steffensen's inequality

Since its appearance in 1918 Steffensen's inequality has been a subject of investigation by many mathematicians. The book [82] is devoted to generalizations and refinements of Steffensen's inequality and its connection to other inequalities, such as Gauss', Jensen-Steffensen's, Hölder's and Iyengar's inequality.

In this section we recall some important generalizations and refinements of Steffensen's inequality.

The original version from [85] has the following form.

Theorem 1.1 *Suppose that f and g are integrable functions defined on (a,b), f is non-increasing and for each $t \in (a,b)$ $0 \leq g \leq 1$. Then*

$$\int_{b-\lambda}^{b} f(t)dt \leq \int_{a}^{b} f(t)g(t)dt \leq \int_{a}^{a+\lambda} f(t)dt, \tag{1.1}$$

where

$$\lambda = \int_{a}^{b} g(t)dt. \tag{1.2}$$

Proof. The proof of the second inequality in (1.1) goes as follows.

$$\begin{aligned}\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt &= \int_a^{a+\lambda} [1-g(t)]f(t)dt - \int_{a+\lambda}^b f(t)g(t)dt \\ &\geq f(a+\lambda)\int_a^{a+\lambda} [1-g(t)]dt - \int_{a+\lambda}^b f(t)g(t)dt \\ &= f(a+\lambda)\left[\lambda - \int_a^{a+\lambda} g(t)dt\right] - \int_{a+\lambda}^b f(t)g(t)dt \\ &= f(a+\lambda)\int_{a+\lambda}^b g(t)dt - \int_{a+\lambda}^b f(t)g(t)dt \\ &= \int_{a+\lambda}^b g(t)[f(a+\lambda)-f(t)]dt \geq 0.\end{aligned}$$

The first inequality in (1.1) is proved in a similar way, but it also follows from the second one. One merely sets $G(t) = 1-g(t)$ and $\Lambda = \int_a^b G(t)dt$. Since $0 \leq g(t) \leq 1$ on (a,b) implies $0 \leq G(t) \leq 1$ on (a,b) and $b-a = \lambda + \Lambda$. Using the second inequality in (1.1) we obtain

$$\int_a^b f(t)G(t)dt \leq \int_a^{a+\Lambda} f(t)dt,$$

$$\int_a^b f(t)[1-g(t)]dt \leq \int_a^{b-\lambda} f(t)dt,$$

$$\int_a^b f(t)dt - \int_a^{b-\lambda} f(t)dt \leq \int_a^b f(t)g(t)dt.$$

Hence,

$$\int_{b-\lambda}^b f(t)dt \leq \int_a^b f(t)g(t)dt,$$

which is the first inequality in (1.1). □

Mitrinović stated in [48] (see also [82, p. 15]) that inequalities in (1.1) follow from the identities

$$\begin{aligned}&\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt \\ &= \int_a^{a+\lambda} [f(t)-f(a+\lambda)][1-g(t)]dt + \int_{a+\lambda}^b [f(a+\lambda)-f(t)]g(t)dt \qquad (1.3)\end{aligned}$$

and

$$\begin{aligned}&\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt \\ &= \int_a^{b-\lambda} [f(t)-f(b-\lambda)]g(t)dt + \int_{b-\lambda}^b [f(b-\lambda)-f(t)][1-g(t)]dt. \qquad (1.4)\end{aligned}$$

Applying Steffensen's inequality to appropriate functions, in [45] Masjed-Jamei, Qi and Srivastava obtained the following Steffensen type inequalities:

Theorem 1.2 *If f and g are integrable functions such that f is nonincreasing and*

$$-\frac{\sigma}{b-a}\left(1-\frac{1}{q}\right) \leq g(x) \leq 1-\frac{\sigma}{b-a}\left(1-\frac{1}{q}\right) \tag{1.5}$$

on (a,b), where $q \neq 0$ and

$$\sigma = q\int_a^b g(x)dx,$$

then

$$\begin{aligned}\int_{b-\sigma}^b f(x)dx - \frac{\sigma}{b-a}\left(1-\frac{1}{q}\right)\int_a^b f(x)dx &\leq \int_a^b f(x)g(x)dx \\ &\leq \int_a^{a+\sigma} f(x)dx - \frac{\sigma}{b-a}\left(1-\frac{1}{q}\right)\int_a^b f(x)dx.\end{aligned} \tag{1.6}$$

The inequalities (1.6) *are reversed for f nondecreasing.*

Identities (1.3) and (1.4) are starting points for researching the conditions of Steffensen's inequality and eventually changing them. Milovanović and Pečarić in their paper [47], using integration by parts in identities (1.3) and (1.4), obtained weaker conditions on the function g. Vasić and Pečarić in paper [87] showed that this weaker conditions are necessary and sufficient. Hence, we have the following theorem.

Theorem 1.3 *Let f and g be integrable functions on $[a,b]$ and let $\lambda = \int_a^b g(t)dt$.*

(a) The second inequality in (1.1) *holds for every nonincreasing function f if and only if*

$$\int_a^x g(t)dt \leq x-a \ \text{ and } \ \int_x^b g(t)dt \geq 0, \quad \text{for every } x \in [a,b]. \tag{1.7}$$

(b) The first inequality in (1.1) *holds for every nonincreasing function f if and only if*

$$\int_x^b g(t)dt \leq b-x \text{ and } \int_a^x g(t)dt \geq 0, \quad \text{for every } x \in [a,b]. \tag{1.8}$$

Using identities (1.3) and (1.4) and integration by parts, Pečarić in [55] also proved conditions for inverse inequalities in (1.1).

Theorem 1.4 *Let $f: I \to \mathbb{R}$, $g: [a,b] \to \mathbb{R}$ ($[a,b] \subseteq I$ where I is an interval in $\mathbb{R}$) be integrable functions, and $a+\lambda \in I$ where λ is given by* (1.2). *Then*

$$\int_a^{a+\lambda} f(t)dt \leq \int_a^b f(t)g(t)dt$$

holds for every nonincreasing function f if and only if

$$\int_a^x g(t)dt \geq x-a, \text{ for } x \in [a,a+\lambda] \quad \text{and} \quad \int_x^b g(t)dt \leq 0, \text{ for } x \in (a+\lambda,b],$$

and $0 \leq \lambda \leq b-a$;
or

$$\int_a^x g(t)dt \geq x-a, \quad \text{for } x \in [a,b],$$

and $\lambda > b-a$;
or

$$\int_x^b g(t)dt \leq 0, \quad \text{for } x \in [a,b]$$

and $\lambda < 0$.

Theorem 1.5 *Let $f : I \to \mathbb{R}$, $g : [a,b] \to \mathbb{R}$ ($[a,b] \subseteq I$ where I is an interval in $\mathbb{R}$) be integrable functions, and $b-\lambda \in I$ where λ is given by* (1.2). *Then*

$$\int_{b-\lambda}^b f(t)dt \geq \int_a^b f(t)g(t)dt$$

holds for every nonincreasing function f if and only if

$$\int_a^x g(t)dt \leq 0, \text{ for } x \in [a,b-\lambda] \quad \text{and} \quad \int_x^b g(t)dt \geq b-x, \quad \text{for } x \in (b-\lambda,b],$$

and $0 \leq \lambda \leq b-a$;
or

$$\int_x^b g(t)dt \geq b-x, \quad \text{for } x \in [a,b],$$

and $\lambda > b-a$;
or

$$\int_a^x g(t)dt \leq 0, \quad \text{for } x \in [a,b]$$

and $\lambda < 0$.

In 1982 Pečarić proved the following generalization of Steffensen's inequality (see [56]).

Theorem 1.6 *Let h be a positive integrable function on $[a,b]$ and f be an integrable function such that f/h is nondecreasing on $[a,b]$. If g is a real-valued integrable function such that $0 \leq g \leq 1$, then*

$$\int_a^b f(t)g(t)dt \geq \int_a^{a+\lambda} f(t)dt \tag{1.9}$$

holds, where λ is the solution of the equation

$$\int_a^{a+\lambda} h(t)dt = \int_a^b h(t)g(t)dt. \tag{1.10}$$

If f/h is a nonincreasing function, then the reverse inequality in (1.9) *holds.*

Theorem 1.7 *Let the conditions of Theorem 1.6 be fulfilled. Then*

$$\int_a^b f(t)g(t)dt \leq \int_{b-\lambda}^b f(t)dt,$$

where λ *is the solution of the equation*

$$\int_{b-\lambda}^b h(t)dt = \int_a^b h(t)g(t)dt. \tag{1.11}$$

For $h(x) = 1$ we have Steffensen's inequality.

In [46] Mercer proved the following generalization of Steffensen's inequality.

Theorem 1.8 *Let* f, g *and* h *be integrable functions on* (a,b) *with* f *nonincreasing and* $0 \leq g \leq h$*. Then*

$$\int_{b-\lambda}^b f(t)h(t)dt \leq \int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)h(t)dt, \tag{1.12}$$

where λ *is given by*

$$\int_a^{a+\lambda} h(t)dt = \int_a^b g(t)dt. \tag{1.13}$$

Wu and Srivastava in [93] and Liu in [44] noted that the generalization due to Mercer is incorrect as stated. They have proved that it is true if we add the condition:

$$\int_{b-\lambda}^b h(t)dt = \int_a^b g(t)dt. \tag{1.14}$$

As proven by Pečarić, Perušić and Smoljak in [61], a corrected version of Mercer's result follows from Theorems 1.6 and 1.7, and is stated as following.

Theorem 1.9 *Let* h *be a positive integrable function on* $[a,b]$ *and* f, g *be integrable functions on* $[a,b]$ *such that* f *is nonincreasing on* $[a,b]$ *and* $0 \leq g \leq h$*.*

a) Then

$$\int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)h(t)dt,$$

where λ *is given by* (1.13)*.*

b) Then

$$\int_{b-\lambda}^b f(t)h(t)dt \leq \int_a^b f(t)g(t)dt,$$

where λ *is given by* (1.14)*.*

In [46] Mercer also gave the following theorem.

Theorem 1.10 *Let f, g, h and k be integrable functions on (a,b) with $k>0$, f/k nonincreasing and $0 \le g \le h$. Then*

$$\int_a^b f(t)g(t)dt \le \int_a^{a+\lambda} f(t)h(t)dt,$$

where λ is given by

$$\int_a^{a+\lambda} h(t)k(t)dt = \int_a^b g(t)k(t)dt. \tag{1.15}$$

As showed in [82, p. 57] Theorem 1.10 is equivalent to Theorem 1.6.

Next, we recall a corrected and refined version of Mercer's result given by Wu and Srivastava in [93].

Theorem 1.11 *Let f, g and h be integrable functions on $[a,b]$ with f nonincreasing and let $0 \le g \le h$. Then the following integral inequalities hold true*

$$\begin{aligned}
\int_{b-\lambda}^b f(t)h(t)dt &\le \int_{b-\lambda}^b (f(t)h(t) - [f(t)-f(b-\lambda)][h(t)-g(t)])dt \\
&\le \int_a^b f(t)g(t)dt \le \int_a^{a+\lambda} (f(t)h(t) - [f(t)-f(a+\lambda)][h(t)-g(t)])dt \\
&\le \int_a^{a+\lambda} f(t)h(t)dt,
\end{aligned}$$

where λ satisfies

$$\int_a^{a+\lambda} h(t)dt = \int_a^b g(t)dt = \int_{b-\lambda}^b h(t)dt. \tag{1.16}$$

Motivated by refinement of Steffensen's inequality given in [93], Pečarić, Perušić and Smoljak [61] obtained the following refined version of results given in Theorems 1.6 and 1.7.

Corollary 1.1 *Let h be a positive integrable function on $[a,b]$ and f, g be integrable functions on $[a,b]$ such that f/h is nonincreasing and $0 \le g \le 1$. Then*

$$\begin{aligned}
\int_a^b f(t)g(t)dt &\le \int_a^{a+\lambda} \left(f(t) - \left[\frac{f(t)}{h(t)} - \frac{f(a+\lambda)}{h(a+\lambda)}\right] h(t)[1-g(t)]\right) dt \\
&\le \int_a^{a+\lambda} f(t)dt,
\end{aligned} \tag{1.17}$$

where λ is given by (1.10).
If f/h is a nondecreasing function, then the reverse inequality in (1.17) *holds.*

Corollary 1.2 *Let h be a positive integrable function on $[a,b]$ and f,g be integrable functions on $[a,b]$ such that f/h is nonincreasing and $0 \leq g \leq 1$. Then*

$$\begin{aligned}\int_{b-\lambda}^{b} f(t)dt &\leq \int_{b-\lambda}^{b} \left(f(t) - \left[\frac{f(t)}{h(t)} - \frac{f(b-\lambda)}{h(b-\lambda)} \right] h(t)[1-g(t)] \right) dt \\ &\leq \int_a^b f(t)g(t)dt\end{aligned} \tag{1.18}$$

where λ is given by (1.11).
If f/h is a nondecreasing function, then the reverse inequality in (1.18) *holds.*

Furthermore, in [93] Wu and Srivastava proved a new sharpened and generalized version of inequality (1.12). In [61] authors separated this result into two theorems to obtain weaker conditions on λ.

Theorem 1.12 *Let f,g,h and ψ be integrable functions on $[a,b]$ with f nonincreasing and let $0 \leq \psi(t) \leq g(t) \leq h(t) - \psi(t)$, $t \in [a,b]$. Then*

$$\int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)h(t)dt - \int_a^b |f(t) - f(a+\lambda)|\,\psi(t)dt$$

where λ is given by (1.13).

Theorem 1.13 *Let f,g,h and ψ be integrable functions on $[a,b]$ with f nonincreasing and let $0 \leq \psi(t) \leq g(t) \leq h(t) - \psi(t)$, $t \in [a,b]$. Then*

$$\int_{b-\lambda}^{b} f(t)h(t)dt + \int_a^b |f(t) - f(b-\lambda)|\,\psi(t)dt \leq \int_a^b f(t)g(t)dt$$

where λ is given by (1.14).

The following theorem is Cerone's generalization of Steffensen's inequality given in [15]. This generalization allows bounds that involve any two subintervals instead of restricting them to include the end points.

Theorem 1.14 *Let $f,g : [a,b] \to \mathbb{R}$ be integrable functions on $[a,b]$ and let f be nonincreasing. Further, let $0 \leq g \leq 1$ and*

$$\lambda = \int_a^b g(t)dt = d_i - c_i,$$

where $[c_i,d_i] \subseteq [a,b]$ for $i = 1,2$ and $d_1 \leq d_2$. Then

$$\int_{c_2}^{d_2} f(t)dt - r(c_2,d_2) \leq \int_a^b f(t)g(t)dt \leq \int_{c_1}^{d_1} f(t)dt + R(c_1,d_1)$$

holds, where

$$r(c_2,d_2) = \int_{d_2}^{b} (f(c_2) - f(t))g(t)dt \geq 0$$

and

$$R(c_1,d_1) = \int_a^{c_1} (f(t) - f(d_1))g(t)dt \geq 0.$$

In 1959 Bellman gave an L^p generalization of Steffensen's inequality (see [11]). As noted by many mathematicians Bellman's result is incorrect as stated. A comprehensive survey of corrected versions and generalizations of Bellman's result can be found in [82]. In the following theorem we recall generalization of Bellman's result obtained by Pachpatte in [53].

Theorem 1.15 *Let f,g,h be real-valued integrable functions defined on $[0,1]$ such that $f(t) \geq 0$, $h(t) \geq 0$, $t \in [0,1]$, f/h is nonincreasing on $[0,1]$ and $0 \leq g \leq A$, where A is a real positive constant. If $p \geq 1$, then*

$$\left(\int_0^1 g(t)f(t)dt\right)^p \leq A^p \int_0^\lambda f^p(t)dt, \tag{1.19}$$

where λ is the solution of the equation

$$\int_0^\lambda h^p(t)dt = \frac{1}{A^p}\left(\int_0^1 h^p(t)g(t)dt\right)\left(\int_0^1 g(t)dt\right)^{p-1}.$$

In [24] Gauss mentioned the following inequality:

Theorem 1.16 *If f is a nonnegative nonincreasing function and $k > 0$, then*

$$\int_k^\infty f(x)dx \leq \frac{4}{9k^2}\int_0^\infty x^2 f(x)dx. \tag{1.20}$$

In [59] Pečarić proved the following result.

Theorem 1.17 *Let $G : [a,b] \to \mathbb{R}$ be an increasing function and let $f : I \to \mathbb{R}$ be a non-increasing function (I is an interval from $\mathbb{R}$ such that $a,b,G(a),G(b) \in I$). If $G(x) \geq x$ then*

$$\int_{G(a)}^{G(b)} f(x)dx \leq \int_a^b f(x)G'(x)dx. \tag{1.21}$$

If $G(x) \leq x$, the reverse inequality in (1.21) *is valid.*
If f is a nondecreasing function and $G(x) \geq x$ then the inequality (1.21) *is reversed.*

Inequality (1.21) is usually called Gauss-Steffensen's inequality. As pointed out in [82] Gauss-Steffensen's inequality includes as special cases three famous inequalities: Volkov's, Steffensen's and Ostrowski's inequality.

In [9] Alzer gave a lower bound for Gauss' inequality (1.20). In fact, he proved the following theorem.

Theorem 1.18 *Let $g : [a,b] \to \mathbb{R}$ be increasing, convex and differentiable, and let $f : I \to \mathbb{R}$ be nonincreasing function. Then*

$$\int_a^b f(s(x))g'(x)dx \leq \int_{g(a)}^{g(b)} f(x)dx \leq \int_a^b f(t(x))g'(x)dx, \tag{1.22}$$

where

$$s(x) = \frac{g(b)-g(a)}{b-a}(x-a)+g(a), \tag{1.23}$$

and

$$t(x) = g'(x_0)(x - x_0) + g(x_0), \quad x_0 \in [a,b]. \tag{1.24}$$

(I is an interval containing $a, b, g(a), g(b), t(a)$ and $t(b)$.)

If either g is concave or f is nondecreasing, then the reversed inequalities hold.

Remark 1.1 If we consider only the left-hand side inequality in (1.22), interval I should only contain $a, b, g(a)$ and $g(b)$. When considering the right-hand side inequality in (1.22), interval I should also contain $t(a)$ and $t(b)$.

1.2 Convex functions

In this section we give definitions and some properties of convex functions. Convex functions are very important in the theory of inequalities. The third chapter of the classical book of Hardy, Littlewood and Pólya [27] is devoted to the theory of convex functions (see also [52]).

Definition 1.1 *Let I be an interval in $\mathbb{R}$. A function $f\colon I \to \mathbb{R}$ is called* convex *if*

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y) \tag{1.25}$$

for all points $x, y \in I$ and all $\lambda \in [0,1]$. It is called strictly convex *if the inequality* (1.25) *holds strictly whenever x and y are distinct points and $\lambda \in (0,1)$.*

If the inequality in (1.25) *is reversed, then f is said to be* concave. *It is called* strictly concave *if the reversed inequality* (1.25) *holds strictly whenever x and y are distinct points and $\lambda \in (0,1)$.*

If f is both convex and concave, f is said to be affine.

Remark 1.2 (a) For $x, y \in I$, $p, q \ge 0$, $p + q > 0$, (1.25) is equivalent to

$$f\left(\frac{px+qy}{p+q}\right) \le \frac{pf(x)+qf(y)}{p+q}.$$

(b) The simple geometrical interpretation of (1.25) is that the graph of f lies below its chords.

(c) If x_1, x_2, x_3 are three points in I such that $x_1 < x_2 < x_3$, then (1.25) is equivalent to

$$\begin{vmatrix} x_1 & f(x_1) & 1 \\ x_2 & f(x_2) & 1 \\ x_3 & f(x_3) & 1 \end{vmatrix} = (x_3 - x_2) f(x_1) + (x_1 - x_3) f(x_2) + (x_2 - x_1) f(x_3) \ge 0$$

which is equivalent to

$$f(x_2) \le \frac{x_2 - x_3}{x_1 - x_3} f(x_1) + \frac{x_1 - x_2}{x_1 - x_3} f(x_3),$$

or, more symmetrically and without the condition of monotonicity on x_1, x_2, x_3

$$\frac{f(x_1)}{(x_1-x_2)(x_1-x_3)}+\frac{f(x_2)}{(x_2-x_3)(x_2-x_1)}+\frac{f(x_3)}{(x_3-x_1)(x_3-x_2)}\geq 0.$$

Proposition 1.1 *If f is a convex function on I and if $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid*

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}\leq\frac{f(y_2)-f(y_1)}{y_2-y_1}.$$

If the function f is concave, the inequality is reversed.

Definition 1.2 *Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is called* convex in the Jensen sense*, or* J-convex on I (midconvex, midpoint convex) *if for all points $x, y \in I$ the inequality*

$$f\left(\frac{x+y}{2}\right)\leq\frac{f(x)+f(y)}{2} \tag{1.26}$$

holds. A J-convex function is said to be strictly J-convex *if for all pairs of points $(x,y), x \neq y$, strict inequality holds in* (1.26)*.*

In the context of continuity the following criteria of equivalence of (1.25) and (1.26) is valid.

Theorem 1.19 *Let $f : I \to \mathbb{R}$ be a continuous function. Then f is a convex function if and only if f is a J-convex function.*

Definition 1.3 *Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is called* Wright convex *function if for each $x \leq y$, $z \geq 0$, $x, y+z \in I$, the inequality*

$$f(x+z)-f(x)\leq f(y+z)-f(y)$$

holds.

Next, we want do define convex functions of higher order, but first we need to define divided differences.

Definition 1.4 *Let f be a function defined on $[a,b]$. The n*-th order divided difference *of f at distinct points $x_0, x_1, \ldots, x_n$ in $[a,b]$ is defined recursively by*

$$[x_j; f] = f(x_j), \quad j = 0, \ldots, n$$

and

$$[x_0, x_1, \ldots, x_n; f] = \frac{[x_1, \ldots, x_n; f]-[x_0, \ldots, x_{n-1}; f]}{x_n - x_0}. \tag{1.27}$$

Remark 1.3 The value $[x_0,x_1,\ldots,x_n;f]$ is independent of the order of the points $x_0,\ldots,x_n$. Previous definition can be extended to include the case in which some or all of the points coincide by assuming that $x_0 \leq \cdots \leq x_k$ and letting

$$\underbrace{[x,\ldots,x}_{j+1 \text{ times}};f] = \frac{f^{(j)}(x)}{j!},$$

provided that $f^{(j)}(x)$ exists. Note that (1.27) is equivalent to

$$[x_0,\ldots,x_n;f] = \sum_{k=0}^{n} \frac{f(x_k)}{\omega'(x_k)}, \text{ where } \omega'(x_k) = \prod_{\substack{j=0 \\ j\neq k}}^{n} (x_k - x_j).$$

Definition 1.5 *Let $n \in \mathbb{N}$. Function $f : [a,b] \to R$ is said to be n-*convex *on $[a,b]$ if and only if for every choice of $n+1$ distinct points $x_0,x_1,\ldots,x_n$ in $[a,b]$*

$$[x_0,x_1,\ldots,x_n;f] \geq 0. \tag{1.28}$$

If the inequality in (1.28) *is reversed, function f is said to be n-*concave *on $[a,b]$. If the inequality is strict, f is said to be* strictly $n-$convex ($n-$concave) *function.*

Remark 1.4 Specially, $0-$convex function is nonnegative function, $1-$convex function is nondecreasing function, 2-convex function is convex function.

Theorem 1.20 *If $f^{(n)}$ exists, then f is $n-$convex if and only if $f^{(n)} \geq 0$.*

Definition 1.6 *A positive function f is said to be* logarithmically convex *on an interval $I \subseteq \mathbb{R}$ if $\log f$ is a convex function on I, or equivalently if for all $x,y \in I$ and all $\alpha \in [0,1]$*

$$f(\alpha x + (1-\alpha)y) \leq f^{\alpha}(x) f^{1-\alpha}(y). \tag{1.29}$$

For such function f, we shortly say f is log-convex.
It is said to be log-concave *if the inequality in* (1.29) *is reversed.*

Definition 1.7 *A positive function f is said to be* log-convex in the Jensen sense *if for each $x,y \in I$*

$$f^2\left(\frac{x+y}{2}\right) \leq f(x)f(y)$$

holds, i.e. if $\log f$ is convex in the Jensen sense.

As a consequence of results from Remark 1.2 (c) and Proposition 1.1 we get the following inequality for log-convex function:

$$[f(b)]^{c-a} \leq [f(a)]^{c-b} [f(c)]^{b-a}. \tag{1.30}$$

Corollary 1.3 *For a log-convex function f on interval I and $p,q,r,s \in I$ such that $p \le r,\ q \le s,\ p \ne q,\ r \ne s$, it holds*

$$\left(\frac{f(p)}{f(q)}\right)^{\frac{1}{p-q}} \le \left(\frac{f(r)}{f(s)}\right)^{\frac{1}{r-s}}. \tag{1.31}$$

Inequality (1.31) is known as Galvani's theorem for log-convex functions $f \colon I \to \mathbb{R}$.

At the end of this introductory section we overview one subclass of convex functions, so-called $s-$convex functions (see [14]).

Definition 1.8 *Let s be a real number, $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is said to be $s-$convex if*

$$f(\alpha x + (1-\alpha)y) \le \alpha^s f(x) + (1-\alpha)^s f(y). \tag{1.32}$$

for all $x,y \in [0,\infty)$ and $\alpha \in [0,1]$

This class of function is recently even further refined (for details see [88]).

Definition 1.9 *Let J be an open interval and $h : J \to \mathbb{R}$ non-negative function, $h \not\equiv 0$. We say that $f : I \to \mathbb{R}$ is an $h-$convex function if f is non-negative and for all $x,y \in I$ and $\alpha \in (0,1)$ we have*

$$f(\alpha x + (1-\alpha)y) \le h(\alpha) f(x) + h(1-\alpha) f(y). \tag{1.33}$$

1.3 Exponentially convex functions

In this section we introduce definition of exponential convexity as given by Bernstein in [12] (see also [7], [50], [51]). In this section I is an open interval in $\mathbb{R}$.

Definition 1.10 *A function $h : I \to \mathbb{R}$ is said to be* exponentially convex *on I if it is continuous and*

$$\sum_{i,j=1}^{n} \xi_i \xi_j h(x_i + x_j) \ge 0$$

holds for every $n \in \mathbb{N}$ and all sequences $(\xi_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ of real numbers, such that $x_i + x_j \in I$, $1 \le i,j \le n$.

The following Proposition follows directly from the previous Definition.

Proposition 1.2 *For function $h : I \to \mathbb{R}$ the following statements are equivalent:*

(i) *h is exponentially convex*

(ii) *h is continuous and*

$$\sum_{i,j=1}^{n} \xi_i \xi_j h\left(\frac{x_i + x_j}{2}\right) \geq 0, \tag{1.34}$$

for all $n \in \mathbb{N}$, all sequences $(\xi_n)_{n\in\mathbb{N}}$ of real numbers, and all sequences $(x_n)_{n\in\mathbb{N}}$ in I.

Note that for $n = 1$, it follows from (1.34) that exponentially convex function is non-negative.

Directly from a definition of positive semi-definite matrix and inequality (1.34) we get the following result.

Corollary 1.4 *If h is exponentially convex on I, then the matrix*

$$\left[h\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{n}$$

is a positive semi-definite matrix. Specially,

$$\det\left[h\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{n} \geq 0, \tag{1.35}$$

for every $n \in \mathbb{N}$ and every choice $x_i \in I$, $i = 1, \ldots, n$.

Remark 1.5 Note that for $n = 2$ from (1.35) we obtain

$$h(x_1)h(x_2) - h^2\left(\frac{x_1 + x_2}{2}\right) \geq 0.$$

Hence, exponentially convex function is log-convex in the Jensen sense, and, being continuous, it is also log-convex function.

We continue with the definition of n-exponentially convex function.

Definition 1.11 *A function $h : I \to \mathbb{R}$ is* n-exponentially convex in the Jensen sense *on I if*

$$\sum_{i,j=1}^{n} \xi_i \xi_j h\left(\frac{x_i + x_j}{2}\right) \geq 0$$

holds for all choices of $\xi_i \in \mathbb{R}$ and $x_i \in I$, $i = 1, \ldots, n$.

A function $h : I \to \mathbb{R}$ is *n-exponentially convex* on I if it is n-exponentially convex in the Jensen sense and continuous on I.

Remark 1.6 It is clear from the definition that 1-exponentially convex functions in the Jensen sense are nonnegative functions.

Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \le n$, $k \in \mathbb{N}$.

A function $h : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is $n-$exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

One of the most important properties of exponentially convex functions is their integral representation.

Theorem 1.21 *The function $\psi : I \to \mathbb{R}$ is exponentially convex on I if and only if*

$$\psi(x) = \int_{-\infty}^{\infty} e^{tx} d\sigma(t), \ \ x \in I$$

for some non-decreasing function $\sigma : \mathbb{R} \to \mathbb{R}$.

Proof. See [7, p. 211]. □

Remark 1.7 A function $\psi : I \to \mathbb{R}$ is log-convex in the Jensen sense, i.e.

$$\psi\left(\frac{x+y}{2}\right)^2 \le \psi(x)\psi(y), \qquad \text{for all } x, y \in I, \tag{1.36}$$

if and only if

$$\alpha^2\psi(x) + 2\alpha\beta\psi\left(\frac{x+y}{2}\right) + \beta^2\psi(y) \ge 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$, i.e., if and only if ψ is 2-exponentially convex in the Jensen sense. By induction from (1.36) we have

$$\psi\left(\frac{1}{2^k}x + \left(1 - \frac{1}{2^k}\right)y\right) \le \psi(x)^{\frac{1}{2^k}}\psi(y)^{1-\frac{1}{2^k}}.$$

Therefore, if ψ is continuous and $\psi(x) = 0$ for some $x \in I$, then from the last inequality and nonnegativity of ψ (see Remark 1.6) we get

$$\psi(y) = \lim_{k\to\infty} \psi\left(\frac{1}{2^k}x + \left(1 - \frac{1}{2^k}\right)y\right) = 0 \qquad \text{for all } y \in I.$$

Hence, 2-exponentially convex function is either identically equal to zero or it is strictly positive and log-convex.

1.4 Functions convex at point c

In this section we introduce definition of a class of functions that extends the class of convex functions as given by Pečarić and Smoljak in [75]

Definition 1.12 *Let $f : [a,b] \to \mathbb{R}$ be a function and $c \in (a,b)$. We say that f belongs to class $\mathcal{M}_1^c[a,b]$ (f belongs to class $\mathcal{M}_2^c[a,b]$) if there exists a constant A such that the function $F(x) = f(x) - Ax$ is nonincreasing (nondecreasing) on $[a,c]$ and nondecreasing (nonincreasing) on $[c,b]$.*

If $f \in \mathcal{M}_1^c[a,b]$ or $f \in \mathcal{M}_2^c[a,b]$ and $f'(c)$ exists, then $f'(c) = A$.
Let us show this for $f \in \mathcal{M}_1^c[a,b]$. Since F is nonincreasing on $[a,c]$ and nondecreasing on $[c,b]$ for any distinct points $x_1, x_2 \in [a,c]$ and $y_1, y_2 \in [c,b]$ we have

$$[x_1,x_2;F] = [x_1,x_2;f] - A \le 0 \le [y_1,y_2;f] - A = [y_1,y_2;F].$$

Therefore, since $f'_-(c)$ and $f'_+(c)$ exist, letting $x_1 = y_1 = c$, $x_2 \nearrow c$ and $y_2 \searrow c$ we get

$$f'_-(c) \le A \le f'_+(c). \tag{1.37}$$

Remark 1.8 We mention here that Florea and Păltănea recently introduced (see [21]) the following more general definition of the convexity of a function $f : [a,b] \to \mathbb{R}$ at a point $c \in (a,b)$:

$$f(c) + f(x+y-c) \le f(x) + f(y),$$

for all $x, y \in [a,b]$ such that $x \le c \le y$. This property is denoted by $f \in \mathrm{Conv}_c([a,b])$. We can easily state that $\mathcal{M}_1^c[a,b] \subset \mathrm{Conv}_c([a,b])$, but the two classes of punctual convex functions are not equal. For example, consider the function

$$f(x) = \begin{cases} |x|, & x \in [-1,1]; \\ 2 - |x|, & x \in [-2,2] \setminus [-1,1]. \end{cases}$$

We have $f \in \mathrm{Conv}_0([-2,2])$ (see Example 2 in [21]). On the other hand, clearly $f \notin \mathcal{M}_1^0[-2,2]$.

In the following lemma and theorem we give a connection between the class of functions $\mathcal{M}_1^c[a,b]$ and the class of convex functions which was obtained in [75].

Lemma 1.1 *If $f : [a,b] \to \mathbb{R}$ is convex (concave), then $f \in \mathcal{M}_1^c[a,b]$ ($f \in \mathcal{M}_2^c[a,b]$) for every $c \in (a,b)$.*

Proof. If f is convex, then f'_- and f'_+ exist (see [71]). Hence, for every $x_1, x_2 \in [a,c]$ and $y_1, y_2 \in [c,b]$ it holds

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \le f'_-(c) \le f'_+(c) \le \frac{f(y_2) - f(y_1)}{y_2 - y_1}.$$

Therefore, for every $A \in [f'_-(c), f'_+(c)]$ the function $F(x) = f(x) - Ax$ satisfies

$$\frac{F(x_2) - F(x_1)}{x_2 - x_1} \le 0 \le \frac{F(y_2) - F(y_1)}{y_2 - y_1},$$

so F is nonincreasing on $[a,c]$ and nondecreasing on $[c,b]$. □

Theorem 1.22 *If $f \in \mathcal{M}_1^c[a,b]$ ($f \in \mathcal{M}_2^c[a,b]$) for every $c \in (a,b)$, then f is convex (concave).*

Proof. We will give the proof for $f \in \mathcal{M}_1^c[a,b]$. First, let us recall the characterization of convexity given in [71]: the function g is convex if and only if the function

$$(x,y) \mapsto [x,y;g] = \frac{g(x) - g(y)}{x - y}$$

is nondecreasing in both variables.

For every $c \in (a,b)$ there exists constant A_c such that the function $F_c(x) = f(x) - A_c x$ is nonincreasing on $[a,c]$ and nondecreasing on $[c,b]$. So for every $x_1 \neq x_2 \le c \le y_1 \neq y_2$ we have

$$\frac{F_c(x_2) - F_c(x_1)}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} - A_c \le 0 \le \frac{f(y_2) - f(y_1)}{y_2 - y_1} - A_c = \frac{F_c(y_2) - F_c(y_1)}{y_2 - y_1}.$$

Particularly, for $u < v < w$ we have

$$\frac{f(v) - f(u)}{v - u} \le A_v \le \frac{f(w) - f(v)}{w - v}. \tag{1.38}$$

Now, let $x_1, x_2, y \in [a,b]$ be arbitrary. If $y < x_1 < x_2$, applying (1.38) we get

$$\frac{f(x_1) - f(y)}{x_1 - y} \le A_{x_1} \le \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_2) - f(y)}{x_2 - x_1} - \frac{f(x_1) - f(y)}{x_2 - x_1}.$$

By multiplying the above inequality with $\frac{x_2 - x_1}{x_2 - y} > 0$ and simplifying we get

$$\frac{f(x_1) - f(y)}{x_1 - y} \le \frac{f(x_2) - f(y)}{x_2 - y}.$$

Similarly for the cases $x_1 < y < x_2$ and $x_1 < x_2 < y$. So we can conclude that the function $(x,y) \mapsto [x,y;f]$ is nondecreasing in variable x. By symmetry, the same thing holds for variable y, so the proof is completed. □

Taking into account Lemma 1.1 and Theorem 1.22, we can describe the property from the Definition 1.12 as "convexity at point c". Therefore, function f is convex on $[a,b]$ if and only if it is convex at every $c \in (a,b)$.

1.5 Čebyšev functional bounds

In this section we give definition of the Čebyšev functional and some results which we will use in the book.

The Čebyšev functional is defined by

$$T(f,h) = \frac{1}{b-a}\int_a^b f(t)h(t)dt - \frac{1}{b-a}\int_a^b f(t)dt \cdot \frac{1}{b-a}\int_a^b h(t)dt$$

where $f,h:[a,b]\to\mathbb{R}$ are two Lebesgue integrable functions.

In 1882, Čebyšev proved that

$$|T(f,h)| \leq \frac{1}{12}\|f'\|_\infty \|h'\|_\infty (b-a)^2,$$

provided that f',h' exists and are continuous on $[a,b]$ and $\|f'\|_\infty = \sup_{t\in[a,b]}|f'(t)|$. It also holds if $f,h:[a,b]\to\mathbb{R}$ are absolutely continuous and $f',g' \in L_\infty[a,b]$ while $\|f'\|_\infty = \sup_{t\in[a,b]}|f'(t)|$.

In 1935, Grüss in his paper [26] proved the following inequality

$$\left|\frac{1}{b-a}\int_a^b f(t)h(t)dt - \frac{1}{b-a}\int_a^b f(t)dt \cdot \frac{1}{b-a}\int_a^b h(t)dt\right| \leq \frac{1}{4}(M-m)(N-n),$$

having that f and h are two integrable functions on $[a,b]$ satisfying the condition

$$m \leq f(t) \leq M, \quad n \leq h(t) \leq N \quad \text{for all } t\in[a,b]$$

The constant $1/4$ is the best possible.

When considering the above Grüss inequality, we observe that on the left hand side of the estimate is the Čebyšev functional, while the right side is of Grüss-type.
There exist lot of estimations for the Čebyšev functional T. Recently, Cerone and Dragomir in [16] proved the following theorems:

Theorem 1.23 *Let $f,h:[a,b]\to\mathbb{R}$ be two absolutely continuous functions on $[a,b]$ with $(\cdot-a)(b-\cdot)[f']^2$, $(\cdot-a)(b-\cdot)[h']^2 \in L_1[a,b]$. Then we have the inequality*

$$|T(f,h)| \leq \frac{1}{\sqrt{2}}[T(f,f)]^{\frac{1}{2}}\frac{1}{\sqrt{b-a}}\left(\int_a^b (x-a)(b-x)[h'(x)]^2dx\right)^{\frac{1}{2}}. \tag{1.39}$$

The constant $\frac{1}{\sqrt{2}}$ in (1.39) *is the best possible.*

Theorem 1.24 *Assume that $h : [a,b] \to \mathbb{R}$ is monotonic nondecreasing on $[a,b]$ and $f : [a,b] \to \mathbb{R}$ is absolutely continuous with $f' \in L_\infty[a,b]$. Then we have the inequality*

$$|T(f,h)| \le \frac{1}{2(b-a)} \|f'\|_\infty \int_a^b (x-a)(b-x)dh(x). \tag{1.40}$$

The constant $\frac{1}{2}$ in (1.40) is the best possible.

1.6 Interpolating polynomials

In this section definitions and some properties of Lidstone, Hermite and two-point Abel-Gontscharoff interpolating polynomials are given, following the terminology from [2].

1.6.1 Lidstone interpolating polynomials

The Lidstone polynomial was introduced independently by Lidstone [42] in 1929 and Poritsky [83] in 1932 as a generalization of the Taylor series that approximates a given function in the neighborhood of two points instead of one.

In [92] Widder proved the following fundamental lemma:

Lemma 1.2 *If $f \in C^{(2n)}[0,1]$, then*

$$f(t) = \sum_{k=0}^{n-1} \left[f^{(2k)}(0)\Lambda_k(1-t) + f^{(2k)}(1)\Lambda_k(t) \right] + \int_0^1 G_n(t,s) f^{(2n)}(s)ds, \tag{1.41}$$

where G_n is Green's function defined by

$$G_1(t,s) = G(t,s) = \begin{cases} (t-1)s, & \text{if} \quad s < t, \\ (s-1)t, & \text{if} \quad t \le s, \end{cases} \tag{1.42}$$

$$G_n(t,s) = \int_0^1 G_1(t,p) G_{n-1}(p,s)dp, \qquad n \ge 2 \tag{1.43}$$

and Λ_n is the unique polynomial (Lidstone polynomial) of degree $2n+1$, $n \in \mathbb{N}$, defined on an interval $[0,1]$ by

$$\Lambda_0(t) = t,$$
$$\Lambda_n''(t) = \Lambda_{n-1}(t),$$
$$\Lambda_n(0) = \Lambda_n(1), \quad n \ge 1.$$

The Lidstone polynomial $\Lambda_n(t)$ can be expressed in the terms of $G_n(t,s)$ as

$$\Lambda_n(t) = \int_0^1 G_n(t,s)sds \quad n \ge 1.$$

The following lemma is given in [2].

Lemma 1.3 *The following holds*

$$G_n(t,s) = \begin{cases} -\sum\limits_{k=0}^{n-1} \Lambda_k(t)\frac{(1-s)^{2n-2k-1}}{(2n-2k-1)!}, & t < s, \\ -\sum\limits_{k=0}^{n-1} \Lambda_k(1-t)\frac{s^{2n-2k-1}}{(2n-2k-1)!}, & s \le t. \end{cases} \tag{1.44}$$

1.6.2 Hermite interpolating polynomials

Let $-\infty < a \le a_1 < a_2 < \ldots < a_r \le b < \infty$, $(r \ge 2)$ be given. For $f \in C^n[a,b]$ there exists a unique polynomial P_H of degree $n-1$, called the Hermite interpolating polynomial of the function f, satisfying the following **Hermite conditions**:

$$P_H^{(i)}(a_j) = f^{(i)}(a_j), \quad 0 \le i \le k_j,\ 1 \le j \le r,\ \sum_{j=1}^{r} k_j + r = n.$$

The Hermite conditions include the following particular cases:

Simple Hermite or Osculatory conditions $(n = 2m,\ r = m,\ k_j = 1$ for all $j)$

$$P_O(a_j) = f(a_j),\ P_O'(a_j) = f'(a_j),\ 1 \le j \le m,$$

Lagrange conditions $(r = n,\ k_j = 0$ for all $j)$

$$P_L(a_j) = f(a_j),\ 1 \le j \le n,$$

Type $(m, n-m)$ **conditions** $(r = 2, a_1 = a, a_2 = b, 1 \le m \le n-1,\ \ k_1 = m-1,\ k_2 = n-m-1)$

$$P_{mn}^{(i)}(a) = f^{(i)}(a),\ 0 \le i \le m-1,$$
$$P_{mn}^{(i)}(b) = f^{(i)}(b),\ 0 \le i \le n-m-1,$$

One-point Taylor conditions $(r = 1, k_1 = n-1)$

$$P_T^{(i)}(a) = f^{(i)}(a),\ 0 \le i \le n-1,$$

Two-point Taylor conditions $(n = 2m,\ r = 2,\ a_1 = a,\ a_2 = b,\ k_1 = k_2 = m-1)$

$$P_{2T}^{(i)}(a) = f^{(i)}(a),\ P_{2T}^{(i)}(b) = f^{(i)}(b),\ 0 \le i \le m-1.$$

The associated error $|e_H(t)|$ can be represented in the terms of the Green function $G_{H,n}(t,s)$ for the multipoint boundary value problem

$$z^{(n)}(t) = 0, z^{(i)}(a_j) = 0, 0 \le i \le k_j,\ 1 \le j \le r,$$

that is, the following result given in [2] holds:

Theorem 1.25 *Let $f \in C^n[a,b]$, and let P_H be its Hermite interpolating polynomial. Then*

$$\begin{aligned} f(t) &= P_H(t) + e_H(t) \\ &= \sum_{j=1}^{r}\sum_{i=0}^{k_j} H_{ij}(t) f^{(i)}(a_j) + \int_a^b G_{H,n}(t,s) f^{(n)}(s)ds, \end{aligned} \tag{1.45}$$

where H_{ij} are fundamental polynomials of the Hermite basis defined by

$$H_{ij}(t) = \frac{1}{i!}\frac{\omega(t)}{(t-a_j)^{k_j+1-i}} \sum_{k=0}^{k_j-i} \frac{1}{k!}\frac{d^k}{dt^k}\left(\frac{(t-a_j)^{k_j+1}}{\omega(t)}\right)\Big|_{t=a_j}(t-a_j)^k, \tag{1.46}$$

where

$$\omega(t) = \prod_{j=1}^{r}(t-a_j)^{k_j+1}, \tag{1.47}$$

and $G_{H,n}$ is Green's function for the Hemite interpolation given by

$$G_{H,n}(t,s) = \begin{cases} \sum\limits_{j=1}^{\ell}\sum\limits_{i=0}^{k_j} \frac{(a_j-s)^{n-i-1}}{(n-i-1)!} H_{ij}(t), \; s \le t, \\ -\sum\limits_{j=\ell+1}^{r}\sum\limits_{i=0}^{k_j} \frac{(a_j-s)^{n-i-1}}{(n-i-1)!} H_{ij}(t), \; s \ge t \end{cases} \tag{1.48}$$

for all $a_\ell \le s \le a_{\ell+1}, \ell = 0,1,\ldots,r$ ($a_0 = a,\ a_{r+1} = b$).

Remark 1.9 *In the particular case, for type $(m, n-m)$ conditions: $r = 2, a_1 = a$, $a_2 = b$, $1 \le m \le n-1$, $k_1 = m-1$, $k_2 = n-m-1$ we have*

$$f(x) = \sum_{i=0}^{m-1} \tau_i(x) f^{(i)}(a) + \sum_{i=0}^{n-m-1} \eta_i(x) f^{(i)}(b) + \int_a^b G_{m,n}(x,s) f^{(n)}(s)ds,$$

where

$$\tau_i(x) = \frac{1}{i!}(x-a)^i \left(\frac{x-b}{a-b}\right)^{n-m} \sum_{k=0}^{m-1-i} \binom{n-m+k-1}{k} \left(\frac{x-a}{b-a}\right)^k, \tag{1.49}$$

$$\eta_i(x) = \frac{1}{i!}(x-b)^i \left(\frac{x-a}{b-a}\right)^{m} \sum_{k=0}^{n-m-1-i} \binom{m+k-1}{k} \left(\frac{x-b}{a-b}\right)^k, \tag{1.50}$$

and Green's function $G_{m,n}$ is of the form

$$G_{m,n}(x,s) = \begin{cases} \sum_{j=0}^{m-1}\left[\sum_{p=0}^{m-1-j}\binom{n-m+p-1}{p}\left(\frac{x-a}{b-a}\right)^p\right]\frac{(x-a)^j(a-s)^{n-j-1}}{j!(n-j-1)!}\left(\frac{b-x}{b-a}\right)^{n-m}, & s \le x, \\ -\sum_{i=0}^{n-m-1}\left[\sum_{q=0}^{n-m-1-i}\binom{m+q-1}{q}\left(\frac{b-x}{b-a}\right)^q\right]\frac{(x-b)^i(b-s)^{n-i-1}}{i!(n-i-1)!}\left(\frac{x-a}{b-a}\right)^{m}, & s \ge x. \end{cases} \tag{1.51}$$

In the following lemma the positivity of Green's function given by (1.48) is described (see Beesack [10] and Levin [41] for more information).

Lemma 1.4 *The Green's function $G_{H,n}(t,s)$ has the following properties:*

(i) $\dfrac{G_{H,n}(t,s)}{\omega(t)} > 0, \quad a_1 \le t \le a_r,\ a_1 < s < a_r;$

(ii) $G_{H,n}(t,s) \le \dfrac{1}{(n-1)!(b-a)}|\omega(t)|;$

(iii) $\displaystyle\int_a^b G_{H,n}(t,s)ds = \frac{\omega(t)}{n!}.$

1.6.3 The two-point Abel-Gontscharoff interpolating polynomials

The two-point Abel-Gontscharoff interpolation problem is a particular case of Abel-Gontscharoff interpolation problem introduced in 1935 by Whittaker [90] and subsequently by Gontscharoff [25] and Davis [17]. In [2] this interpolation problem is also reffered to as the two-point right focal interpolation problem. Let $f \in C^n[a,b]$ $(n \ge 2)$ and let P_{AG2} be its two-point Abel-Gontscharoff interpolating polynomial then

$$f(t) = P_{AG2}(t) + e_{AG2}(t) \tag{1.52}$$

where P_{AG2} is the polynomial of the degree $n-1$ defined by

$$\begin{aligned} P_{AG2}(t) = &\sum_{i=0}^{\alpha} \frac{(t-a_1)^i}{i!} f^{(i)}(a_1) \\ &+ \sum_{j=0}^{n-\alpha-2} \left[\sum_{i=0}^{j} \frac{(t-a_1)^{\alpha+1+i}(a_1-a_2)^{j-i}}{(\alpha+1+i)!(j-i)!} \right] f^{(\alpha+1+j)}(a_2). \end{aligned}$$

The associated error can be expressed by

$$e_{AG2}(t) = \int_a^b g_{AG2}(t,s) f^{(n)}(s)ds. \tag{1.53}$$

The corresponding Green function $g_{AG2}(t,s)$ from (1.53) is defined by

$$g_{AG2}(t,s) = \frac{1}{(n-1)!} \begin{cases} \sum\limits_{i=0}^{\alpha} \binom{n-1}{i} (t-a_1)^i (a_1-s)^{n-i-1}, & a \le s \le t, \\ -\sum\limits_{i=\alpha+1}^{n-1} \binom{n-1}{i} (t-a_1)^i (a_1-s)^{n-i-1}, & t \le s \le b. \end{cases}$$

The polynomial P_{AG2} satisfies the following conditions called the two-point right focal conditions (see [2, p. 172]):

$$P_{AG2}^{(i)}(a_1) = f^{(i)}(a_1),\ 0 \le i \le \alpha,$$
$$P_{AG2}^{(i)}(a_2) = f^{(i)}(a_2),\ \alpha+1 \le i \le n-1,\ a \le a_1 < a_2 \le b.$$

These conditions are a particular case of the general Abel-Gontscharoff interpolation conditions

$$P_{AG}^{(i)}(a_{i+1}) = f^{(i)}(a_{i+1}), \quad 0 \le i \le n-1,\ a \le a_1 \le a_2 \le \cdots \le a_n \le b.$$

Chapter 2

Weighted Steffensen's inequality

2.1 Steffensen's inequality for positive measures

This section is devoted to Steffensen's inequality for positive finite measures on Borel $\sigma-$algebra defined on segment $[a,b]$ denoted by $\mathscr{B}([a,b])$. Results given in this section were obtained by Jakšetić and Pečarić in [31]. Generalizations of Steffensen's inequality in a measure theory settings were also considered in the papers [19], [22], [23] and book [82].

In the following theorems we give Steffensen's inequality for positive measures.

Theorem 2.1 *Let μ be a finite, positive measure on $\mathscr{B}([a,b])$ and let f and g be measurable functions such that f is nonincreasing and $0 \leq g \leq 1$. If there exists $\lambda \in \mathbb{R}_+$ such that*

$$\mu((b-\lambda,b]) = \int_{[a,b]} g(t)d\mu(t), \tag{2.1}$$

then

$$\int_{(b-\lambda,b]} f(t)d\mu(t) \leq \int_{[a,b]} f(t)g(t)d\mu(t). \tag{2.2}$$

Proof.

$$
\begin{aligned}
&\int_{[a,b]} f(t)g(t)d\mu(t) - \int_{(b-\lambda,b]} f(t)d\mu(t) \\
&= \int_{[a,b-\lambda]} f(t)g(t)d\mu(t) - \int_{(b-\lambda,b]} f(t)(1-g(t))d\mu(t) \\
&\geq \int_{[a,b-\lambda]} f(t)g(t)d\mu(t) - f(b-\lambda)\int_{(b-\lambda,b]} (1-g(t))d\mu(t) \qquad (2.3) \\
&= \int_{[a,b-\lambda]} f(t)g(t)d\mu(t) - f(b-\lambda)\int_{[a,b-\lambda]} g(t)d\mu(t) \qquad (2.4) \\
&= \int_{[a,b-\lambda]} (f(t)-f(b-\lambda))g(t)d\mu(t) \geq 0.
\end{aligned}
$$

□

Theorem 2.2 *Let μ be a finite, positive measure on $\mathscr{B}([a,b])$ and let f and g be measurable functions such that f is nonincreasing and nonnegative and $0 \leq g \leq 1$. If there exists $\lambda \in \mathbb{R}_+$ that satisfies*

$$
\mu((b,b-\lambda]) \leq \int_{[a,b]} g(t)d\mu(t), \tag{2.5}
$$

then (2.2) *holds.*

Proof. We re-adjust the proof of Theorem 2.1 in the following way:
the condition (2.5) together with $f(b-\lambda) > 0$ ensures us the transition from the line (2.3) to (2.4). □

Theorem 2.3 *Let μ be a finite, positive measure on $\mathscr{B}([a,b])$ and let f and g be measurable functions such that f is nonincreasing and $0 \leq g \leq 1$. If there exists $\lambda \in \mathbb{R}_+$ such that*

$$
\mu([a,a+\lambda]) = \int_{[a,b]} g(t)d\mu(t), \tag{2.6}
$$

then

$$
\int_{[a,a+\lambda]} f(t)d\mu(t) \geq \int_{[a,b]} f(t)g(t)d\mu(t). \tag{2.7}
$$

Proof.

$$
\begin{aligned}
&\int_{[a,a+\lambda]} f(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) \\
&= \int_{[a,a+\lambda]} f(t)(1-g(t))d\mu(t) - \int_{(a+\lambda,b]} f(t)g(t)d\mu(t) \\
&\geq f(a+\lambda)\int_{[a,a+\lambda]} (1-g(t))d\mu(t) - \int_{(a+\lambda,b]} f(t)g(t)d\mu(t) \\
&= f(a+\lambda)\int_{(a+\lambda,b]} g(t)d\mu(t) - \int_{(a+\lambda,b]} f(t)g(t)d\mu(t) \\
&= \int_{(a+\lambda,b]} (f(a+\lambda)-f(t))g(t)d\mu(t) \geq 0.
\end{aligned}
$$

□

Theorem 2.4 *Let μ be a finite, positive measure on $\mathscr{B}([a,b])$ and let f and g be measurable functions such that f is nonincreasing and nonnegative and $0 \le g \le 1$. If there exists $\lambda \in \mathbb{R}_+$ such that*

$$\mu([a,a+\lambda]) \ge \int_{[a,b]} g(t)d\mu(t), \tag{2.8}$$

then (2.7) *holds.*

Proof. Similar to the proof of Theorem 2.2. □

Remark 2.1 For the sake of applications we observe here that if f is an increasing nonnegative function and if (2.8) is valid, then (2.7) is reversed. We have the same conclusion if $f(a) = 0$ and f is an increasing function.

If we consider the Lebesgue measure in the conditions (2.1) and (2.6), we obtain the standard constant λ in Steffensen's inequality given by (1.2).

If $\mu \ll \nu$ and $h = \frac{d\mu}{d\nu}$, the condition (2.5) becomes

$$\int_{(b-\lambda,b]} h(t)d\nu(t) \le \int_{[a,b]} h(t)g(t)d\nu(t),$$

while the condition (2.8) becomes

$$\int_{[a,a+\lambda]} h(t)d\nu(t) \ge \int_{[a,b]} h(t)g(t)d\nu(t).$$

Clearly, both conditions have the same form as Steffensen's inequality altough the monotonicity request on the function h is dropped.

The following families of functions will be useful in constructing exponentially convex functions.

Lemma 2.1 *For $p \in \mathbb{R}$ let $\phi_p : (0,\infty) \to \mathbb{R}$ be defined by*

$$\phi_p(x) = \begin{cases} \frac{x^p}{p}, & p \neq 0; \\ \log x, & p = 0. \end{cases}$$

Then $x \mapsto \phi_p(x)$ is increasing on $\mathbb{R}$ for each $p \in \mathbb{R}$ and $p \mapsto \phi_p(x)$ is exponentially convex on $(0,\infty)$, for each $x \in (0,\infty)$.

Proof. First part: follows from $\frac{d}{dx}(\phi_p(x)) = x^{p-1} > 0$ on $(0,\infty)$, for each $p \in \mathbb{R}$.
Second part: $p \mapsto \frac{x^p}{p} = e^{p\log x} \cdot \frac{1}{p}$. Since $p \mapsto e^{p\log x}$ and $p \mapsto \frac{1}{p}$ are exponentially convex functions (see [30]), according to the above comment, the conclusion follows. □

Lemma 2.2 *For $p \in \mathbb{R}$ let $\varphi_p : \mathbb{R} \to [0,\infty)$ be defined by*

$$\varphi_p(x) = \begin{cases} \frac{e^{px}}{p}, & p \neq 0; \\ x, & p = 0. \end{cases}$$

Then $x \mapsto \varphi_p(x)$ is increasing on $\mathbb{R}$ for each $p \in \mathbb{R}$, and $p \mapsto \varphi_p(x)$ is exponentially convex on $(0,\infty)$, for each $x \in \mathbb{R}$.

Proof. First part: follows from $\frac{d}{dx}(\varphi_p(x)) = e^{px} > 0$ on $\mathbb{R}$, for each $p \in \mathbb{R}$.
Second part: follows from the fact $\frac{e^{px}}{p} = e^{px} \cdot \frac{1}{p}$. □

Using the characterization of a convexity by the monotonicity of the first order divided differences it follows (see [71, p. 4]):

Theorem 2.5 *Let $I \subseteq \mathbb{R}$ be an open interval. Let $f : I \to (0,\infty)$ be log-convex, differentiable function on I and $M : I \times I \to (0,\infty)$ be defined by*

$$M(x,y) = \begin{cases} \left(\frac{f(x)}{f(y)}\right)^{\frac{1}{x-y}}, & x \neq y; \\ \exp\left(\frac{f'(x)}{f(x)}\right), & x = y. \end{cases}$$

If $x_1, x_2, y_1, y_2 \in I$ such that $x_1 \leq x_2,\ y_1 \leq y_2$ then

$$M(x_1,y_1) \leq M(x_2,y_2).$$

To obtain some applications of the results concerning Steffensen's inequality for positive measures let us observe the following linear functionals from Theorems 2.1 and 2.3:

$$\mathfrak{L}_1(f) = \int_{(b-\lambda,b]} f(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) \tag{2.9}$$

and

$$\mathfrak{L}_2(f) = \int_{[a,b]} f(t)g(t)d\mu(t) - \int_{[a,a+\lambda]} f(t)d\mu(t). \tag{2.10}$$

Theorem 2.6 *Let $f \mapsto \mathfrak{L}_i(f),\ i = 1,2$, be linear functionals defined by* (2.9) *and* (2.10) *and let $F_i : (0,\infty) \to \mathbb{R},\ i = 1,2$, be defined by*

$$F_i(p) = \mathfrak{L}_i(\phi_p)$$

where ϕ_p is defined in Lemma 2.1. Then the following statements hold for every $i = 1,2$.

(i) The function F_i is continuous on $\mathbb{R}$.

(ii) If $n \in \mathbb{N}$ and $p_1, \ldots, p_n \in \mathbb{R}$ are arbitrary, then the matrix

$$\left[F_i\left(\frac{p_j + p_k}{2}\right)\right]_{j,k=1}^n$$

is positive semidefinite. Particularly,

$$det\left[F_i\left(\frac{p_j + p_k}{2}\right)\right]_{j,k=1}^n \geq 0.$$

(iii) The function F_i is exponentially convex on $\mathbb{R}$.

(iv) The function F_i is log-convex on $\mathbb{R}$.

(v) If $p,q,r \in \mathbb{R}$ are such that $p < q < r$, then

$$F_i(q)^{r-p} \leq F_i(p)^{r-q} F_i(r)^{q-p}. \tag{2.11}$$

Proof. (i) Continuity of the function $p \mapsto F_i(p)$ is obvious for $p \in \mathbb{R} \setminus \{0\}$. For $p = 0$ it is directly checked using the Heine characterization.
(ii) Let $n \in \mathbb{N}$, $p_i \in \mathbb{R}$ $(i = 1,\dots,n)$ be arbitrary and define an auxiliary function $\psi : (0,\infty) \to \mathbb{R}$ by

$$\psi(x) = \sum_{j,k=1}^{n} \xi_j \xi_k \phi_{\frac{p_j+p_k}{2}}(x).$$

Now

$$\psi'(x) = \left(\sum_{j=1}^{n} \xi_j x^{\frac{p_j-1}{2}} \right)^2 \geq 0$$

implies that ψ is a nondecreasing function on $(0,\infty)$ and then

$$\mathfrak{L}_i(\psi) \geq 0, \; i = 1,2.$$

This means that the matrix

$$\left[F_i\left(\frac{p_i + p_j}{2} \right) \right]_{j,k=1}^{n}$$

is positive semi-definite.
(iii), (iv), (v) are simple consequences of (i) and (ii). □

By using similar arguments we obtain the following theorem.

Theorem 2.7 *Theorem 2.6 is still valid for φ_p given in Lemma 2.2.*

We now use the mean value theorems to produce Cauchy means.

Theorem 2.8 *Let $f \mapsto \mathfrak{L}_i(f)$, $i = 1,2$ be linear functionals defined by* (2.9) *and* (2.10) *and $\psi \in C^1[a,b]$. Then there exist $\xi_i \in [a,b]$, $i = 1,2$ such that*

$$\mathfrak{L}_i(\psi) = \psi'(\xi_i)\mathfrak{L}_i(id),$$

where $id(x) = x$.

Proof. Since $\psi \in C^1[a,b]$ there exist $m = \min_{x \in [a,b]} \psi'(x)$ and $M = \max_{x \in [a,b]} \psi'(x)$. Denote $h_1(x) = Mx - \psi(x)$ and $h_2(x) = \psi(x) - mx$. Then

$$h_1'(x) = M - \psi'(x) \geq 0$$
$$h_2'(x) = \psi'(x) - m \geq 0$$

which means that $\mathfrak{L}_i(h_1)$, $\mathfrak{L}_i(h_2) \geq 0$, $i = 1,2$ i.e.

$$m\mathfrak{L}_i(id) \leq \mathfrak{L}_i(\psi) \leq M\mathfrak{L}_i(id).$$

If $\mathfrak{L}_i(id) = 0$, the proof is complete. If $\mathfrak{L}_i(id) > 0$, then

$$m \leq \frac{\mathfrak{L}_i(\psi)}{\mathfrak{L}_i(id)} \leq M$$

and the existence of $\xi_i \in [a,b]$ follows. □

Using the standard Cauchy type mean value theorem we obtain the following corollary.

Corollary 2.1 *Let $f \mapsto \mathfrak{L}_i(f)$, $i = 1,2$ be linear functionals defined by* (2.9) *and* (2.10) *and ψ_1, $\psi_2 \in C^1[a,b]$ such that $\psi_2'(x)$ does not vanish for any value of $x \in [a,b]$, then there exist $\xi_i \in [a,b]$, $i = 1,2$ such that*

$$\frac{\psi_1'(\xi_i)}{\psi_2'(\xi_i)} = \frac{\mathfrak{L}_i(\psi_1)}{\mathfrak{L}_i(\psi_2)}, \tag{2.12}$$

provided that the denominator on right side is non-zero.

If the inverse of ψ_1'/ψ_2' exists then various kinds of means can be defined by (2.12). That is

$$\xi_i = \left(\frac{\psi_1'}{\psi_2'}\right)^{-1}\left(\frac{\mathfrak{L}_i(\psi_1)}{\mathfrak{L}_i(\psi_2)}\right), \; i = 1,2. \tag{2.13}$$

Particularly, if we substitute $\psi_1(x) = \phi_p(x)$, $\psi_2(x) = \phi_q(x)$ in (2.13) and use the continuous extension, the following expressions are obtained ($i = 1,2$):

$$M_i(p,q) = \begin{cases} \left(\frac{\mathfrak{L}_i(\phi_p)}{\mathfrak{L}_i(\phi_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(-\frac{1}{p} + \frac{\mathfrak{L}_i(\phi_0\phi_p)}{\mathfrak{L}_i(\phi_p)}\right), & p = q \neq 0; \\ \exp\left(\frac{\mathfrak{L}_i(\phi_0^2)}{2\mathfrak{L}_i(\phi_0)}\right), & p = q = 0. \end{cases}$$

By Theorem 2.5, if $p,q,u,v \in \mathbb{R}$ such that $p \leq u$, $q \leq v$ then,

$$M_i(p,q) \leq M_i(u,v).$$

Similarly, if we substitute $\psi_1(x) = \varphi_p(x)$, $\psi_2(x) = \varphi_q(x)$ in (2.13) and use the continuous extension, the following expressions are obtained ($i = 1,2$):

$$\overline{M}_i(p,q) = \begin{cases} \left(\frac{\mathfrak{L}_i(\varphi_p)}{\mathfrak{L}_i(\varphi_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(-\frac{1}{p} + \frac{\mathfrak{L}_i(\varphi_0\varphi_p)}{\mathfrak{L}_i(\varphi_p)}\right), & p = q \neq 0; \\ \exp\left(\frac{\mathfrak{L}_i(\varphi_0^2)}{2\mathfrak{L}_i(\varphi_0)}\right), & p = q = 0. \end{cases}$$

Again, using Theorem 2.5, if $p,q,u,v \in \mathbb{R}$ such that $p \leq u$, $q \leq v$ then,

$$\overline{M}_i(p,q) \leq \overline{M}_i(u,v).$$

In [31] Jakšetić and Pečarić further refined obtained results by dropping some of the analytical properties of the families of functions from Lemmas 2.1 and 2.2. Therefore they defined

$$\mathscr{C} = \{\psi_p : \ \psi_p : [a,b] \to \mathbb{R}, \ p \in J\},$$

a family of functions from $C([a,b])$ such that $p \mapsto [x_0, x_1; \psi_p]$ is log-convex in the Jensen sense on J for every choice of two distinct points $x_0, x_1 \in [a,b]$.

Theorem 2.9 *Let $f \mapsto \mathfrak{L}_i(f), \ i = 1,2$ be linear functionals defined by* (2.9) *and* (2.10) *and let $G_i : J \to \mathbb{R}$, be defined by*

$$G_i(p) = \mathfrak{L}_i(\psi_p)$$

where $\psi_p \in \mathscr{C}$. Then the following statements hold, for every $i = 1,2$.

(i) G_i is log-convex in the Jensen sense on J.

(ii) If G_i is continuous on J, then it is log-convex on J and for $p,q,r \in J$ such that $p < q < r$, we have

$$G_i(q)^{r-p} \le G_i(p)^{r-q} G_i(r)^{q-p}.$$

(iii) If G_i is positive and differentiable on J, then for every $p,q,u,v \in J$ such that $p \le u, \ q \le v$, we have

$$\widetilde{M}_i(p,q) \le \widetilde{M}_i(u,v) \tag{2.14}$$

where $\widetilde{M}_i(p,q)$ is defined by

$$\widetilde{M}_i(p,q) = \begin{cases} \left(\dfrac{G_i(p)}{G_i(q)}\right)^{\frac{1}{p-q}}, & p \ne q; \\ \exp\left(\dfrac{\frac{d}{dp}(G_i(p))}{G_i(p)}\right), & p = q. \end{cases} \tag{2.15}$$

Proof. (i) We prove our claim for the case $i = 1$, the second case is treated similarly. Choose any two distinct points $x_0, x_1 \in [a,b]$, any $\xi_1, \ \xi_2 \in \mathbb{R}$ and any $p, \ q \in J$. Define an auxiliary function $\psi : [a,b] \to \mathbb{R}$ by

$$\psi(x) = \xi_1^2 \psi_p(x) + 2\xi_1\xi_2 \psi_{\frac{p+q}{2}}(x) + \xi_2^2 \psi_q(x), \tag{2.16}$$

where $\psi_p, \psi_{\frac{p+q}{2}}$ and ψ_q are from the class C_1. Then

$$\begin{aligned} [x_0, x_1; \psi] =& \xi_1^2 [x_0, x_1; \psi_p] + 2\xi_1\xi_2 [x_0, x_1; \psi_{\frac{p+q}{2}}] \\ &+ \xi_2^2 [x_0, x_1, \ x_2; \psi_q] \ge 0 \end{aligned}$$

by the definition of $\mathscr{C}$ and the characterization of log-convexity. This implies that ψ is a nondecreasing function on $[a,b]$. Hence $\mathfrak{L}_1(\psi) \ge 0$ which is equivalent to

$$\xi_1^2 G_1(p) + 2\xi_1\xi_2 G_1\left(\frac{p+q}{2}\right) + \xi_2^2 G_1(q) \ge 0.$$

This proves that G_1 is log-convex in the Jensen sense on J.
(ii) Since G_i is continuous on J, then it is log-convex.
(iii) This is a simple consequence of Theorem 2.5. $\square$

Let us continue by introducing the following family of functions. Let

$$\mathscr{D} = \{\psi_p : \ \psi_p : [a,b] \to \mathbb{R}, \ p \in J\},$$

be a family of functions from $C([a,b])$ such that $p \mapsto [x_0, x_1; \psi_p]$ is exponentially convex on J for every choice of two distinct points $x_0, x_1 \in [a,b]$.

Theorem 2.10 *Let $f \mapsto \mathfrak{L}_i(f), \ i = 1,2$ be linear functionals defined by* (2.9) *and* (2.10) *and let $H_i : J \to \mathbb{R}$, be defined by*

$$H_i(p) = \mathfrak{L}_i(\psi_p) \tag{2.17}$$

where $\psi_p \in \mathscr{D}$. Then the following statements hold for every $i = 1,2$.

(i) If $n \in \mathbb{N}$ and $p_1, \ldots, p_n \in \mathbb{R}$ are arbitrary, then the matrix

$$\left[H_i\left(\frac{p_k + p_m}{2}\right)\right]_{k,m=1}^{n}$$

is positive semidefinite. Particularly,

$$det\left[H_i\left(\frac{p_k + p_m}{2}\right)\right]_{k,m=1}^{n} \geq 0.$$

(ii) If the function H_i is continuous on J, then H_i is exponentially convex on J.

(iii) If H_i is positive and differentiable on J, then for every $p,q,u,v \in J$ such that $p \leq u, \ q \leq v$, we have

$$\widehat{M}_i(p,q) \leq \widehat{M}_i(u,v)$$

where $\widehat{M}_i(p,q)$ is defined by

$$\widehat{M}_i(p,q) = \begin{cases} \left(\frac{H_i(p)}{H_i(q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{\frac{d}{dp}(H_i(p))}{H_i(p)}\right), & p = q. \end{cases}$$

Proof. (i) We prove our claim for the case $i = 1$, the second case is treated similarly. Let $n \in \mathbb{N}, \ p_1, \ldots p_n \in \mathbb{R}$ be arbitrary and define an auxiliary function $\psi : [a,b] \to \mathbb{R}$ by

$$\psi(x) = \sum_{k,m=1}^{n} \xi_k \xi_m \psi_{\frac{p_k + p_m}{2}}(x).$$

Then

$$[x_0,x_1;\psi]=\sum_{k,m=1}^{n}\xi_k\xi_m[x_0,x_1;\psi_{\frac{p_k+p_m}{2}}]\geq 0$$

by the definition of $\mathscr{D}$ and exponential convexity. This implies that ψ is a nondecreasing function on $[a,b]$ and then $\mathfrak{L}_1(\psi)\geq 0$ which is equivalent to

$$\sum_{k,m=1}^{n}\xi_i\xi_j H_1\left(\frac{p_k+p_m}{2}\right)\geq 0.$$

(ii) Follows from (i).
(iii) This is a simple consequence of Theorem 2.5. □

Families of exponentially convex functions similar to families given in Lemmas 2.1 and 2.2 can be easily constructed because of an application of Theorem 1.21.

Example 2.1 Consider a family of functions $h_p:(0,\infty)\to(0,\infty), p>0$, defined by

$$h_p(x)=\begin{cases}-\frac{p^{-x}}{\log p}, & p\neq 1;\\ x, & p=1.\end{cases}$$

Since $p\mapsto\frac{d}{dx}(h_p(x))=p^{-x}$ is the Laplace transform of a nonnegative function (see [84] p. 210), it is exponentially convex according to Theorem 1.21.

Obviously $x\mapsto h_p(x)$ are nondecreasing functions for every $p>0$. It is easy to prove that the function $p\mapsto[x_0,x_1;h_p]$ is also exponentially convex for arbitrary positive x_0,x_1 (see also [30]). Using Theorem 2.10 it follows that for linear functionals $f\mapsto\mathfrak{L}_i(f)$, $i=1,2$ defined by (2.9) and (2.10) we have that $p\mapsto\mathfrak{L}_i(h_p)$ are exponentially convex (it is easy to verify that they are continuous), for $i=1,2$.
Further using Theorem 2.10 we conclude that

$$R_i(p,q)=\begin{cases}\left(\frac{\mathfrak{L}_i(h_p)}{\mathfrak{L}_i(h_q)}\right)^{\frac{1}{p-q}}, & p\neq q;\\ \exp\left(-\frac{\mathfrak{L}_i(h_1\cdot h_p)}{p\mathfrak{L}_i(h_p)}-\frac{1}{p}\log p\right), & p=q\neq 1;\\ \exp\left(-\frac{\mathfrak{L}_i(h_1^2)}{2\mathfrak{L}_i(h_1)}\right), & p=q\neq 1;\end{cases}$$

satisfies

$$R_i(p,q)\leq R_i(u,v).$$

for $p,q,u,v\in\mathbb{R}$ such that $p\leq u,\ q\leq v$.

From Example 2.1 and Theorem 2.10 it is clear that in [31] the authors have presented a new way how to generate exponentially convex functions, aside from Laplace transform and Theorem 1.21.

2.2 Some measure theoretic aspects of Steffensen's and reversed Steffensen's inequality

We begin this section with necessary and sufficient conditions for Steffensen's and reversed Steffensen's inequality obtained by Jakšetić, Pečarić and Smoljak Kalamir in [34].

Theorem 2.11 *Let μ be a finite, positive measure on $\mathscr{B}([a,b])$, let $g : [a,b] \to \mathbb{R}$ be a $\mu-$integrable function.*

(a) Let λ be a positive constant such that $\mu([a,a+\lambda]) = \int_{[a,b]} g(t)d\mu(t)$. The inequality

$$\int_{[a,b]} f(t)g(t)d\mu(t) \leq \int_{[a,a+\lambda]} f(t)d\mu(t) \tag{2.18}$$

holds for every nonincreasing, right-continuous function $f : [a,b] \to \mathbb{R}$ if and only if

$$\int_{[a,x)} g(t)d\mu(t) \leq \mu([a,x)) \quad \text{and} \quad \int_{[x,b]} g(t)d\mu(t) \geq 0, \quad \text{for every } x \in [a,b]. \tag{2.19}$$

(b) Let λ be a positive constant such that $\mu((b-\lambda,b]) = \int_{[a,b]} g(t)d\mu(t)$. The inequality

$$\int_{(b-\lambda,b]} f(t)d\mu(t) \leq \int_{[a,b]} f(t)g(t)d\mu(t)$$

holds for every nonincreasing, right-continuous function $f : [a,b] \to \mathbb{R}$ if and only if

$$\int_{[x,b]} g(t)d\mu(t) \leq \mu([x,b]) \quad \text{and} \quad \int_{[a,x)} g(t)d\mu(t) \geq 0, \quad \text{for every } x \in [a,b].$$

Proof.
(a) For the sufficiency part we use the identity

$$\begin{aligned}&\int_{[a,a+\lambda]} f(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) \\ &= \int_{[a,a+\lambda]} [f(t)-f(a+\lambda)][1-g(t)]d\mu(t) + \int_{(a+\lambda,b]} [f(a+\lambda)-f(t)]g(t)d\mu(t)\end{aligned} \tag{2.20}$$

similar to (1.3). We define a new measure ν on $\sigma-$algebra $\mathscr{B}((a,b])$ such that, on an algebra of finite disjoint unions of half open intervals, we set $\nu((c,d]) = f(c) - f(d)$, for $a < c < d \leq b$, and then we pass to $\mathscr{B}((a,b])$ in a unique way (for details see, for example, [13, p. 21]).

Now, using Fubini, we have

$$\begin{aligned}
&\int_{[a,a+\lambda]} [f(t)-f(a+\lambda)][1-g(t)]d\mu(t) \\
&= \int_{[a,a+\lambda]} \left[\int_{(t,a+\lambda]} d\nu(x)\right][1-g(t)]d\mu(t) = \int_{(a,a+\lambda]} \left[\int_{[a,x)} (1-g(t))d\mu(t)\right] d\nu(x).
\end{aligned}$$

Similarly,

$$\int_{(a+\lambda,b]} [f(a+\lambda)-f(t)]g(t)\mu(t) = \int_{(a+\lambda,b]} \left[\int_{[x,b]} g(t)d\mu(t)\right] d\nu(x).$$

This means that (2.20) is in fact

$$\begin{aligned}
&\int_{[a,a+\lambda]} f(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) \\
&= \int_{(a,a+\lambda]} \left[\int_{[a,x)} (1-g(t))d\mu(t)\right] d\nu(x) + \int_{(a+\lambda,b]} \left[\int_{[x,b]} g(t)d\mu(t)\right] d\nu(x), \qquad (2.21)
\end{aligned}$$

concluding (2.18) under the assumptions (2.19).

The previous conditions are also necessary. In fact, if x is any element of $[a,b]$, then let f be the function defined by

$$f(t) = \begin{cases} 1, & t < x; \\ 0, & t \geq x. \end{cases}$$

Using the inequality (2.7) from Theorem 2.3 we obtain

$$\begin{aligned}
\int_{[a,x)} g(t)d\mu(t) = \int_{[a,b]} f(t)g(t)d\mu(t) &\leq \int_{[a,a+\lambda]} f(t)d\mu(t) \\
&= \begin{cases} \mu([a,x)), & x \in [a,a+\lambda]; \\ \mu([a,a+\lambda]), & x \in (a+\lambda,b]. \end{cases} \qquad (2.22)
\end{aligned}$$

If $x \in (a+\lambda,b]$ then $\mu([a,x)) \geq \mu([a,a+\lambda])$, from (2.22), we have

$$\int_{[a,x)} g(t)d\mu(t) \leq \mu([a,x)), \quad \text{for every } x \in [a,b].$$

Also, if $x \in (a+\lambda,b]$, from (2.22) we have
$\int_{[a,x)} g(t)d\mu(t) \leq \mu([a,a+\lambda]) = \int_{[a,b]} g(t)d\mu(t)$, concluding

$$\int_{[x,b]} g(t)d\mu(t) \geq 0, \quad \text{for every } x \in (a+\lambda,b].$$

Finally, if $x \in [a,a+\lambda]$, then

$$\begin{aligned}
\int_{[x,b]} g(t)d\mu(t) &= \int_{[a,b]} g(t)d\mu(t) - \int_{[a,x)} g(t)d\mu(t) \\
&\geq \mu([a,a+\lambda]) - \mu([a,x)) = \mu([x,a+\lambda]) \geq 0,
\end{aligned}$$

concluding

$$\int_{[x,b]} g(t)d\mu(t) \geq 0, \quad \text{for every } x \in [a,b].$$

(b) The proof of this part is similar to the proof of the (a)-part so we omit the details. □

In the following theorems we give necessary and sufficient conditions for reversed Steffensen's inequality.

Theorem 2.12 *Let μ be a finite, positive measure on $\mathscr{B}(I)$, $g:[a,b]\to\mathbb{R}$ ($[a,b]\subseteq I$, I is an interval in $\mathbb{R}$) be a $\mu-$integrable function, and $a+\lambda\in I$ where*

$$\mu([a,a+\lambda]) = \int_{[a,b]} g(t)d\mu(t), \tag{2.23}$$

for $\lambda \geq 0$, and

$$-\mu([a+\lambda,a]) = \int_{[a,b]} g(t)d\mu(t),$$

for $\lambda < 0$.
Then, for $\lambda \geq 0$,

$$\int_{[a,a+\lambda]} f(t)d\mu(t) \leq \int_{[a,b]} f(t)g(t)d\mu(t); \tag{2.24}$$

and for $\lambda < 0$,

$$-\int_{[a+\lambda,a]} f(t)d\mu(t) \leq \int_{[a,b]} f(t)g(t)d\mu(t); \tag{2.25}$$

for every nonincreasing, right continuous function $f:I\to\mathbb{R}$ if and only if either

$$\int_{[a,x)} g(t)d\mu(t) \geq \mu([a,x)), \text{ for } x\in[a,a+\lambda] \text{ and } \int_{[x,b]} g(t)d\mu(t) \leq 0, \text{ for } x\in(a+\lambda,b], \tag{2.26}$$

where $0 \leq \lambda \leq b-a$;
or

$$\int_{[a,x)} g(t)d\mu(t) \geq \mu([a,x)), \quad \text{for } x\in[a,b], \tag{2.27}$$

where $\lambda > b-a$;
or

$$\int_{[x,b]} g(t)d\mu(t) \leq 0, \quad \text{for } x\in[a,b], \tag{2.28}$$

where $\lambda < 0$.

Proof. Necessity part. Putting

$$f(t) = \begin{cases} 1, & t < x; \\ 0, & t \geq x \end{cases}$$

in (2.24) we get (2.26), (2.27) and (2.28) for choices $\lambda \in [0,b-a]$, $\lambda \in (b-a,+\infty)$, and $\lambda \in (-\infty,0)$, respectively.
Sufficiency part. Let $\lambda \in [0,b-a]$. Then from (2.21) we have

$$\begin{aligned}
&\int_{[a,a+\lambda]} f(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) \\
&= \int_{(a,a+\lambda]} \left[\int_{[a,x)} (1-g(t))d\mu(t)\right] d\nu(x) + \int_{(a+\lambda,b]} \left[\int_{[x,b]} g(t)d\mu(t)\right] d\nu(x) \leq 0,
\end{aligned}$$

where ν is a measure defined on $\mathscr{B}((a,b])$, with $\nu((c,d]) = f(c) - f(d)$, for $c<d$, $c,d \in I$.
If $\lambda \in (b-a,+\infty)$ then

$$\begin{aligned}
&\int_{[a,a+\lambda]} f(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) \\
&= \int_{[a,b]} f(t)(1-g(t))d\mu(t) + \int_{(b,a+\lambda]} f(t)d\mu(t) \\
&= \int_{[a,b]} (f(t)-f(b))(1-g(t))d\mu(t) + \int_{(b,a+\lambda]} (f(t)-f(b))d\mu(t) \\
&= \int_{[a,b]} \int_{[a,x)} (1-g(t))d\mu(t)d\nu(x) - \int_{(b,a+\lambda]} \int_{[x,a+\lambda]} d\mu(t)d\nu(x) \\
&= \int_{[a,b]} \left(\mu([a,x)) - \int_{[a,x)} g(t)d\mu(t)\right) d\nu(x) - \int_{(b,a+\lambda]} \mu([x,a+\lambda])d\nu(x) \leq 0.
\end{aligned}$$

If $\lambda \in (-\infty,0)$ then

$$\begin{aligned}
&-\int_{[a+\lambda,a]} f(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) \\
&= \int_{[a,b]} (f(a)-f(t))g(t)d\mu(t) + \int_{[a+\lambda,a]} (f(a)-f(t))d\mu(t) \\
&= \int_{[a,b]} \int_{[x,b]} g(t)d\mu(t)d\nu(x) - \int_{[a+\lambda,a]} \mu([a+\lambda,x])d\nu(x) \leq 0.
\end{aligned}$$

□

Theorem 2.13 *Let μ be a finite, positive measure on $\mathscr{B}(I)$, $g:[a,b] \to \mathbb{R}$ ($[a,b] \subseteq I$, I is an interval in $\mathbb{R}$) be a $\mu-$integrable function, and $b-\lambda \in I$ where*

$$\mu((b-\lambda,b]) = \int_{[a,b]} g(t)d\mu(t),$$

for $\lambda \geq 0$, and

$$-\mu([b,b-\lambda)) = \int_{[a,b]} g(t)d\mu(t),$$

for $\lambda < 0$.
Then, for $\lambda \geq 0$

$$\int_{(b-\lambda,b]} f(t)d\mu(t) \geq \int_{[a,b]} f(t)g(t)d\mu(t); \tag{2.29}$$

and for $\lambda < 0$,

$$-\int_{[b,b-\lambda)} f(t)d\mu(t) \geq \int_{[a,b]} f(t)g(t)d\mu(t); \tag{2.30}$$

for every nonincreasing, right continuous function $f : I \to \mathbb{R}$ *if and only if either*

$$\int_{[a,x)} g(t)d\mu(t) \leq 0, \text{ for } x \in [a,b-\lambda] \text{ and } \int_{[x,b]} g(t)d\mu(t) \geq \mu([x,b]), \text{ for } x \in (b-\lambda,b],$$

where $0 \leq \lambda \leq b-a$;
or

$$\int_{[x,b]} g(t)d\mu(t) \geq \mu([x,b]), \quad \text{for } x \in [a,b]$$

where $\lambda > b-a$;
or

$$\int_{[a,x)} g(t)d\mu(t) \leq 0, \quad \text{for } x \in [a,b]$$

where $\lambda < 0$.

Proof. Similar to the proof of Theorem 2.12. □

Theorem 2.14 *Let* μ *be a finite, positive measure on* $\mathscr{B}(I)$, $g : [a,b] \to \mathbb{R}$ *be a* $\mu-$*integrable function for which there exists* $c \in [a,b]$ *such that* $g(x) \geq 1$ *for* $x \in [a,c]$ *and* $g(x) \leq 0$ *for* $x \in (c,b]$. *Then* (2.24) *(resp.* (2.25)*) for* $\lambda \geq 0$ *(resp.* $\lambda < 0$*) is valid for every nonincreasing function* $f : I \to \mathbb{R}$ *provided that* $[a,b] \subseteq I$ *and* $a+\lambda \in I$.

Proof. Let $\lambda \in [0,b-a]$. Suppose that $c \leq a+\lambda$. Then it is obvious that

$$\int_{[a,x)} g(t)d\mu(t) \geq \mu([a,x)), \quad \text{for } x \in [a,c]$$

and

$$\int_{[x,b]} g(t)d\mu(t) \leq 0, \quad \text{for } x \in (a+\lambda,b].$$

Suppose that for some $x_0 \in (c,a+\lambda]$ we have $\int_{[a,x_0)} g(t)d\mu(t) < \mu([a,x_0))$. Since $\int_{[x_0,b]} g(t)d\mu(t) \leq 0$, it follows $\mu([a,a+\lambda]) = \int_{[a,b]} g(t)d\mu(t) < \mu([a,x_0))$, hence $a+\lambda < x_0$, what is, evidently, a contradiction. Analogously, in the case $c > a+\lambda$ we can also prove that (2.26) holds.
Let $\lambda \in (b-a,\infty)$. Then $\int_{[a,x)} g(t)d\mu(t) \geq \mu([a,x))$ for $x \in [a,c]$. For $x \in (c,b]$ we have

$$\begin{aligned}\int_{[a,x)} g(t)d\mu(t) &= \int_{[a,b]} g(t)d\mu(t) - \int_{[x,b]} g(t)d\mu(t)\\ &\geq \int_{[a,b]} g(t)d\mu(t) = \mu([a,a+\lambda]) \geq \mu([a,x)),\end{aligned}$$

and the condition (2.27) is fulfilled.

If $\lambda \in (-\infty, 0)$ then if $x \in [a,c]$

$$\int\limits_{[x,b]} g(t)d\mu(t) = \int\limits_{[a,b]} g(t)d\mu(t) - \int\limits_{[a,x)} g(t)d\mu(t) = -\mu([a+\lambda,a]) - \mu([a,x)) \leq 0;$$

if $x \in (c,b]$ then $\int_{[x,b]} g(t)d\mu(t) \leq 0$ so (2.28) is again valid. $\square$

Similar to Theorem 2.14 we can prove the next theorem.

Theorem 2.15 *Let μ be a finite, positive measure on $\mathscr{B}(I)$, $g : [a,b] \to \mathbb{R}$ be a $\mu-$integrable function for which there exists $c \in [a,b]$ such that $g(x) \leq 0$ for $x \in [a,c]$ and $g(x) \geq 1$ for $x \in (c,b]$. Then* (2.29) *(resp.* (2.30)*) for $\lambda \geq 0$ (resp. $\lambda < 0$) is valid for every nonincreasing function $f : I \to \mathbb{R}$ provided that $[a,b] \subseteq I$ and $b - \lambda \in I$.*

It is obvious that the choice of a measure μ in the previous results covers some known results: the Lebesgue measure gives us the classic Steffensen inequality, the counting measure gives us Jensen-Steffensen's inequality (even with relaxed conditions, see [29]), and the Lebesgue-Stieltjes measure gives us results for Steffensen's inequality from [43].

2.3 Exponential convexity induced by Steffensen's inequality and positive measures

In this section we give generalizations of results from [49] obtained by Jakšetić, Pečarić and Smoljak Kalamir in [37].

Theorem 2.16 *Let μ be a finite, positive measure on $\mathscr{B}([a,b])$, $f : [a,b] \to \mathbb{R}$ a nonincreasing, right-continuous function . Then*

$$\frac{\int_{[a,b]} f(t)G(t)d\mu(t)}{\int_{[a,b]} G(t)d\mu(t)} \leq \frac{\int_{[a,a+\lambda]} f(t)d\mu(t)}{\mu([a,a+\lambda])} \tag{2.31}$$

if and only if $G : [a,b] \to \mathbb{R}$ is a $\mu-$integrable function and λ is a positive constant such that

$$\frac{\int_{[a,x)} G(t)d\mu(t)}{\int_{[a,b]} G(t)d\mu(t)} \leq \frac{\mu([a,x))}{\mu([a,a+\lambda])} \quad \text{and} \quad \int\limits_{[x,b]} G(t)d\mu(t) \geq 0, \tag{2.32}$$

for every $x \in [a,b]$, assuming $\int_{[a,b]} G(t)d\mu(t) > 0$.
For an increasing, right-continuous function $f : [a,b] \to \mathbb{R}$ the inequality (2.31) *is reversed.*

Proof. Sufficiency. Let us define the function

$$g(t) = \frac{G(t)\mu([a,a+\lambda])}{\int_{[a,b]} G(t)d\mu(t)}.$$

Since the conditions $\mu([a,a+\lambda]) = \int_{[a,b]} g(t)d\mu(t)$ and (2.19) are fulfilled we can apply (2.18) and (2.31) is valid.
Necessity. If we put the function

$$f(t) = \begin{cases} 1, & t < x; \\ 0, & t \geq x, \end{cases}$$

for $a \leq x \leq a+\lambda$ in (2.31) we get (2.32). □

In the following theorem we use the following property of sub-linearity of a class of convex functions.

Lemma 2.3 *If $\phi : [0,\infty) \to \mathbb{R}$ is a convex function such that $\phi(0) = 0$ then for any $0 \leq a \leq 1$*

$$\phi(ax) \leq a\phi(x), \text{ for any } x \in [0,\infty).$$

Proof. $\phi(ax) = \phi(ax + (1-a)\cdot 0) \leq a\phi(x) + (1-a)\phi(0) = a\phi(x)$. □

The following theorem gives us a connection between Jensen's and Steffensen's inequality and it generalizes Theorem 2.16.

Theorem 2.17 *Let μ be a finite, positive measure on $\mathscr{B}([a,b])$. Let f be a nonnegative nonincreasing function on $[a,b]$, and let ϕ be an increasing convex function on $[0,\infty)$ with $\phi(0) = 0$. If G is a nonnegative nondecreasing function on $[a,b]$ such that there exists a nonnegative function g_1, defined by the equation*

$$\int_{[a,b]} g_1(t)\phi\left(\frac{G(t)}{g_1(t)}\right) d\mu(t) \leq \mu([a,b]) \tag{2.33}$$

and $\int_{[a,b]} g_1(t)d\mu(t) \leq 1$, then the following inequality is valid:

$$\phi\left(\frac{\int_{[a,b]} f(t)G(t)d\mu(t)}{\int_{[a,b]} G(t)d\mu(t)}\right) \leq \frac{\int_{[a,a+\lambda]} \phi(f(t))d\mu(t)}{\mu([a,a+\lambda])}, \tag{2.34}$$

where $\mu([a,a+\lambda]) = \phi\left(\int_{[a,b]} G(t)d\mu(t)\right)$.

Proof. From Jensen's inequality

$$\phi\left(\frac{\int_{[a,b]} f(t)G(t)d\mu(t)}{\int_{[a,b]} G(t)d\mu(t)}\right) \leq \frac{\int_{[a,b]} \phi(f(t))G(t)d\mu(t)}{\int_{[a,b]} G(t)d\mu(t)}, \tag{2.35}$$

and since $\phi \circ f$ is nonincreasing, we only have to check the conditions in (2.32). Since G is nonnegative, it is obvious that $\int_{[x,b]} G(t)d\mu(t) \geq 0$, so we only have to show

$$\phi\left(\int_{[a,b]} G(t)d\mu(t)\right) \int_{[a,x)} G(t)d\mu(t) \leq \mu([a,x)) \int_{[a,b]} G(t)d\mu(t). \tag{2.36}$$

We calculate, using sub-linearity and Jensen's inequality

$$
\begin{aligned}
\phi\left(\int_{[a,b]} G(t)d\mu(t)\right) &= \phi\left(\int_{[a,b]} g_1(t)d\mu(t)\frac{\int_{[a,b]} G(t)d\mu(t)}{\int_{[a,b]} g_1(t)d\mu(t)}\right)\\
&\leq \int_{[a,b]} g_1(t)d\mu(t)\phi\left(\frac{\int_{[a,b]} g_1(t)\frac{G(t)}{g_1(t)}d\mu(t)}{\int_{[a,b]} g_1(t)d\mu(t)}\right)\\
&\leq \int_{[a,b]} g_1(t)\phi\left(\frac{G(t)}{g_1(t)}\right)d\mu(t) \leq \mu([a,b]). \qquad (2.37)
\end{aligned}
$$

Since G is a nondecreasing function,

$$
\frac{\int_{[a,x)} G(t)d\mu(t)}{\mu([a,x))} \leq \frac{\int_{[a,b]} G(t)d\mu(t)}{\mu([a,b])}
$$

i.e.

$$
\mu([a,b])\int_{[a,x)} G(t)d\mu(t) \leq \mu([a,x))\int_{[a,b]} G(t)d\mu(t),
$$

so along with (2.37) we proved (2.36) and the theorem is proved. □

The condition (2.33) is a version of a more general condition that is given in [43].

Now let us give some applications of the previous results. Using (2.31), under the assumptions of Theorem 2.16, we can produce the linear functional

$$
\mathfrak{M}(f) = \frac{\int_{[a,b]} f(t)G(t)d\mu(t)}{\int_{[a,b]} G(t)d\mu(t)} - \frac{\int_{[a,a+\lambda]} f(t)d\mu(t)}{\mu([a,a+\lambda])} \qquad (2.38)
$$

which is nonnegative on the class of increasing, right-continuous functions $f : [a,b] \to \mathbb{R}$.

Theorem 2.18 *Let $f \mapsto \mathfrak{M}(f)$ be the linear functional defined by (2.38) and let $\Phi : \mathbb{R} \to \mathbb{R}$ be defined by*

$$
\Phi(p) = \mathfrak{M}(\varphi_p)
$$

where φ_p is defined in Lemma 2.2. Then the following statements hold.

(i) The function Φ is continuous on $\mathbb{R}$.

(ii) If $n \in \mathbb{N}$ and $p_1, \ldots, p_n \in \mathbb{R}$ are arbitrary, then the matrix

$$
\left[\Phi\left(\frac{p_j+p_k}{2}\right)\right]_{j,k=1}^{n}
$$

is positive semidefinite. Particularly,

$$
\det\left[\Phi\left(\frac{p_j+p_k}{2}\right)\right]_{j,k=1}^{n} \geq 0.
$$

(iii) The function Φ *is exponentially convex on* $\mathbb{R}$.

(iv) The function Φ *is log-convex on* $\mathbb{R}$.

(v) If $p,q,r \in \mathbb{R}$ *are such that* $p<q<r$, *then*

$$\Phi(q)^{r-p} \le \Phi(p)^{r-q}\Phi(r)^{q-p}.$$

Proof. Similar to the proof of Theorem 2.6. □

The following theorem is just a stepping stone for the future theorems and it is a slight variant of the preceding theorem.
Firstly consider the family $\{\theta_p : \ p \in (0,\infty)\}$ of functions defined on $[0,\infty)$ by

$$\theta_p(x) = \frac{x^p}{p}. \tag{2.39}$$

Similar as in Lemmas 2.1 and 2.2 we conclude that $x \mapsto \theta_p(x)$ is increasing on $[0,\infty)$ for each $p \in \mathbb{R}$ and $p \mapsto \theta_p(x)$ is exponentially convex on $(0,\infty)$ for each $x \in [0,\infty)$.
Secondly, from Remark 2.1 we form the linear functional

$$\mathfrak{N}(f) = \int_{[a,b]} f(t)g(t)d\mu(t) - \int_{[a,a+\lambda]} f(t)d\mu(t), \tag{2.40}$$

which is nonnegative, acting on an increasing functions $f : [a,b] \to \mathbb{R}$ with property $f(a) = 0$.

Theorem 2.19 *Let*
$f \mapsto \mathfrak{N}(f)$ *be the linear functional defined by* (2.40) *and let* $F : (0,\infty) \to \mathbb{R}$ *be defined by*

$$F(p) = \mathfrak{N}(\theta_p)$$

where θ_p *is defined by* (2.39)*. Then the following statements hold.*

(i) The function F *is continuous on* $(0,\infty)$.

(ii) If $n \in \mathbb{N}$ *and* $p_1,\dots,p_n \in (0,\infty)$ *are arbitrary, then the matrix*

$$\left[F\left(\frac{p_j+p_k}{2}\right)\right]_{j,k=1}^{n}$$

is positive semidefinite. Particularly,

$$det\left[F\left(\frac{p_j+p_k}{2}\right)\right]_{j,k=1}^{n} \ge 0.$$

(iii) The function F *is exponentially convex on* $(0,\infty)$.

(iv) The function F *is log-convex on* $(0,\infty)$.

(v) If $p,q,r \in (0,\infty)$ are such that $p<q<r$, then

$$F(q)^{r-p} \leq F(p)^{r-q} F(r)^{q-p}.$$

Proof. (i) The continuity of the function $p \mapsto F(p)$ is obvious.
(ii) Let $n \in \mathbb{N}$, $p_i \in (0,\infty)$ $(i=1,\ldots,n)$ be arbitrary and define an auxiliary function $\Psi : [0,\infty) \to \mathbb{R}$ by

$$\Psi(x) = \sum_{j,k=1}^{n} \xi_j \xi_k \theta_{\frac{p_j+p_k}{2}}(x).$$

Now

$$\Psi'(x) = \left(\sum_{j=1}^{n} \xi_j x^{\frac{p_j-1}{2}} \right)^2 \geq 0$$

implies that Ψ is increasing on $[0,\infty)$ and nonnegative since $\Psi(0)=0$. Then $\mathfrak{N}(\Psi) \geq 0$ and we conclude that

$$\left[F\left(\frac{p_j+p_k}{2} \right) \right]_{j,k=1}^{n}$$

is a positive semi-definite matrix.
(iii), (iv), (v) are simple consequences of (i) and (ii). $\square$

Observe here that the above proof is futile for the family of functions given in Lemma 2.2 since $\varphi_p(0)$ is not defined for $p \leq 0$.

We now use the mean value theorems to produce Cauchy means.

Theorem 2.20 *Let $f \mapsto \mathfrak{N}(f)$ be the linear functional defined by* (2.40) *and $\psi \in C^1[a,b]$ such that $\psi(a)=0$. Then there exists $\xi \in [a,b]$ such that*

$$\mathfrak{N}(\psi) = \psi'(\xi)\mathfrak{N}(id),$$

where $id(x)=x$.

Proof. Similar to the proof of Theorem 2.8. $\square$

Using the standard Cauchy type mean value theorem we obtain the following corollary.

Corollary 2.2 *Let $f \mapsto \mathfrak{N}(f)$ be the linear functional defined by* (2.40)*, $\psi_1,\ \psi_2 \in C^1[a,b]$ such that $\psi_1(a)=\psi_2(a)=0$, then there exists $\xi \in [a,b]$, such that*

$$\frac{\psi_1'(\xi)}{\psi_2'(\xi)} = \frac{\mathfrak{N}(\psi_1)}{\mathfrak{N}(\psi_2)}, \tag{2.41}$$

provided that the denominator on right side is non-zero.

If the inverse of ψ_1'/ψ_2' exists then various kinds of means can be defined by (2.41). That is

$$\xi = \left(\frac{\psi_1'}{\psi_2'} \right)^{-1} \left(\frac{\mathfrak{N}(\psi_1)}{\mathfrak{N}(\psi_2)} \right). \tag{2.42}$$

Particularly, if we substitute $\psi_1(x) = \theta_p(x),\ \psi_2(x) = \theta_q(x)$ in (2.42) and use the continuous extension, the following expressions are obtained:

$$M(p,q) = \begin{cases} \left(\frac{\mathfrak{N}(\theta_p)}{\mathfrak{N}(\theta_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(-\frac{1}{p} + \frac{\mathfrak{N}(\theta_0\theta_p)}{\mathfrak{N}(\theta_p)}\right), & p = q, \end{cases}$$

where $\theta_0(x) = \log x$ and $p,q \in (0,\infty)$. By Theorem 2.5, if $p,q,u,v \in (0,\infty)$ such that $p \le u,\ q \le v$ then,

$$M(p,q) \le M(u,v).$$

Now we make a step further using (2.34), under the assumptions of Theorem 2.17, we can produce the linear functional

$$\mathfrak{L}(\phi) = \frac{\int_{[a,a+\lambda]} \phi(f(t))d\mu(t)}{\mu([a,a+\lambda])} - \phi\left(\frac{\int_{[a,b]} f(t)G(t)d\mu(t)}{\int_{[a,b]} G(t)d\mu(t)}\right) \tag{2.43}$$

which is nonnegative on the class of increasing convex functions on $[0,\infty)$ with property $\phi(0) = 0$.

Theorem 2.21 *Let $f \mapsto \mathfrak{L}(f)$ be the linear functional defined by* (2.43) *and let $H : (1,\infty) \to \mathbb{R}$ be defined by*

$$H(p) = \mathfrak{L}(\phi_p)$$

where ϕ_p is defined in Lemma 2.1. Then the following statements hold.

(i) The function H is continuous on $(1,\infty)$.

(ii) If $n \in \mathbb{N}$ and $p_1,\ldots,p_n \in (1,\infty)$ are arbitrary, then the matrix

$$\left[H\left(\frac{p_j+p_k}{2}\right)\right]_{j,k=1}^{n}$$

is positive semidefinite. Particularly,

$$det\left[H\left(\frac{p_j+p_k}{2}\right)\right]_{j,k=1}^{n} \ge 0.$$

(iii) The function H is exponentially convex on $(1,\infty)$.

(iv) The function H is log-convex on $(1,\infty)$.

(v) If $p,q,r \in (1,\infty)$ are such that $p < q < r$, then

$$H(q)^{r-p} \le H(p)^{r-q} H(r)^{q-p}.$$

Proof. (i) The continuity of the function $p \mapsto H(p)$ is obvious.
(ii) Let $n \in \mathbb{N}$, $p_i \in (1,\infty)$ $(i = 1,\dots,n)$ be arbitrary and define an auxiliary function $\psi : [0,\infty) \to \mathbb{R}$ by

$$\psi(x) = \sum_{j,k=1}^{n} \xi_j \xi_k \phi_{\frac{p_j+p_k}{2}}(x). \tag{2.44}$$

Now

$$\psi'(0) = \sum_{j,k=1}^{n} \xi_j \xi_k \phi'_{\frac{p_j+p_k}{2}}(0) = 0. \tag{2.45}$$

Further

$$\psi''(x) = \left(\sum_{j=1}^{n} \xi_j x^{\frac{p_j-2}{2}} \right)^2 \geq 0. \tag{2.46}$$

(2.45) and (2.46) together with $\psi(0) = 0$ imply that ψ is a convex increasing function and then

$$\mathfrak{L}(\psi) \geq 0.$$

This means that the matrix

$$\left[H\left(\frac{p_j+p_k}{2} \right) \right]_{j,k=1}^{n}$$

is positive semi-definite.
(iii), (iv), (v) are simple consequences of (i) and (ii). □

Proof. [Alternative proof.] The monotonicity of the function ψ defined by (2.44) can be proved directly:

$$\psi'(x) = \sum_{j,k=1}^{n} \xi_j \xi_k \phi'_{\frac{p_j+p_k}{2}}(x)$$

and since $p \mapsto \phi'_p(x)$, according to Lemma 2.1, is exponentially convex on $(1,\infty)$, we conclude $\psi'(x) \geq 0$. The rest is as in the previous proof. □

Theorem 2.22 *Let $f \mapsto \mathfrak{L}(f)$ be the linear functional defined by* (2.43) *and $\psi \in C^2[0,a]$ such that $\psi(0) = \psi'(0) = 0$. Then there exists $\xi \in [0,a]$ such that*

$$\mathfrak{L}(\psi) = \psi''(\xi)\mathfrak{L}(e_2),$$

where $e_2(x) = \frac{x^2}{2}$.

Proof. Since $\psi \in C^2[0,a]$ there exist $m = \min_{x\in[0,a]} \psi''(x)$ and $M = \max_{x\in[0,a]} \psi''(x)$. Denote $h_1(x) = M\frac{x^2}{2} - \psi(x)$ and $h_2(x) = \psi(x) - m\frac{x^2}{2}$. Then $h_1(0) = h_1'(0) = 0$, $h_2(0) = h_2'(0) = 0$. This, together with

$$h_1''(x) = M - \psi''(x) \geq 0$$
$$h_2''(x) = \psi''(x) - m \geq 0$$

implies that h_1 and h_2 are an increasing convex functions.

Then $\mathfrak{L}(h_1),\ \mathfrak{L}(h_2) \geq 0$ i.e.

$$m\mathfrak{L}(e_2) \leq \mathfrak{L}(\psi) \leq M\mathfrak{L}(e_2).$$

If $\mathfrak{L}(e_2) = 0$, the proof is complete. If $\mathfrak{L}(e_2) > 0$, then

$$m \leq \frac{\mathfrak{L}(\psi)}{\mathfrak{L}(e_2)} \leq M$$

and the existence of $\xi \in [0,a]$ now follows. □

Similar to Corollary 2.2 we also obtain the following corollary.

Corollary 2.3 *Let $f \mapsto \mathfrak{L}(f)$ be the linear functional defined by* (2.43) *and $\psi_1,\ \psi_2 \in C^2[0,a]$ such that $\psi_1(0) = \psi_2(0) = \psi_1'(0) = \psi_2'(0) = 0$ and such that $\psi_2''(x)$ does not vanish for any value of $x \in [0,a]$, then there exists $\xi \in [0,a]$ such that*

$$\frac{\psi_1''(\xi)}{\psi_2''(\xi)} = \frac{\mathfrak{L}(\psi_1)}{\mathfrak{L}(\psi_2)}, \tag{2.47}$$

provided that the denominator on right side is non-zero.

If the inverse of ψ_1''/ψ_2'' exists then various kinds of means can be defined by (2.12). That is

$$\xi = \left(\frac{\psi_1''}{\psi_2''}\right)^{-1} \left(\frac{\mathfrak{L}(\psi_1)}{\mathfrak{L}(\psi_2)}\right). \tag{2.48}$$

Particularly, if we substitute $\psi_1(x) = \phi_p(x),\ \psi_2(x) = \phi_q(x)$ in (2.48) and use the continuous extension, the following expressions are obtained:

$$N(p,q) = \begin{cases} \left(\frac{\mathfrak{L}(\phi_p)}{\mathfrak{L}(\phi_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{3-2p}{(p-1)(p-2)} + \frac{\mathfrak{L}(\phi_0\phi_p)}{\mathfrak{L}(\phi_p)}\right), & p = q \neq 2; \\ \exp(-\frac{3}{2} - \frac{\mathfrak{L}_i(\phi_0\phi_2)}{\mathfrak{L}_i(\phi_2)}), & p = q = 2, \end{cases}$$

where $\phi_0(x) = \log x$ and $p,q \in (1,\infty)$. By Theorem 2.5, if $p,q,u,v \in (1,\infty)$ such that $p \leq u,\ q \leq v$ then,

$$N(p,q) \leq N(u,v).$$

We can generalize the above construction. For $n \geq 2$, fixed, let us define

$$\mathscr{C}_n = \{\psi_p : \ \ p \in J\},$$

a family of functions from $C([0,a])$ such that $\psi_p(0) = \psi_p'(0) = 0$, and $p \mapsto \psi_p''(x)$ is n-exponentially convex in the Jensen sense on J for every $x \in [0,a]$.

Theorem 2.23 *Let $f \mapsto \mathfrak{L}(f)$ be the linear functional defined by* (2.43) *and let $S : J \to \mathbb{R}$, be defined by*

$$S(p) = \mathfrak{L}(\psi_p)$$

where $\psi_p \in \mathscr{C}_n$. Then the following statements hold.

(i) S is n-exponentially convex in the Jensen sense on J.

(ii) If S is continuous on J, then it is $n-$exponentially convex on J and for $p,q,r \in J$ such that $p < q < r$, we have

$$S(q)^{r-p} \leq S(p)^{r-q} S(r)^{q-p}.$$

(iii) If S is positive and differentiable on J, then for every $p,q,u,v \in J$ such that $p \leq u$, $q \leq v$, we have

$$\widetilde{M}(p,q) \leq \widetilde{M}(u,v) \tag{2.49}$$

where $\widetilde{M}(p,q)$ is defined by

$$\widetilde{M}(p,q) = \begin{cases} \left(\frac{S(p)}{S(q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{\frac{d}{dp}(S(p))}{S(p)}\right), & p = q. \end{cases} \tag{2.50}$$

Proof. (i) Choose any n points $\xi_1, \ldots, \xi_n \in \mathbb{R}$, any $p_1, \ldots, p_n \in J$. Define an auxiliary function $\Psi : [0,a] \to \mathbb{R}$ by

$$\Psi(x) = \sum_{k,m=1}^{n} \xi_k \xi_m \psi_{\frac{p_k+p_m}{2}}(x). \tag{2.51}$$

Then $\Psi(0) = \Psi'(0) = 0$ and

$$\Psi''(x) = \sum_{k,m=1}^{n} \xi_k \xi_m \psi''_{\frac{p_k+p_m}{2}}(x) \geq 0$$

by the definition of $\mathscr{C}_n$. Hence, Ψ is an increasing convex function and then $\mathfrak{L}(\Psi) \geq 0$ which is equivalent to

$$\sum_{k,m=1}^{n} \xi_k \xi_m S\left(\frac{p_k+p_m}{2}\right) \geq 0.$$

(ii) Since S is continuous on J, then it is $n-$exponentially convex.
(iii) This is a simple consequence of Theorem 2.5. □

We can further refine the obtained results by dropping some of the analytical properties of the families of functions and using just divided differences.

Let us introduce the following family of functions:

$$\mathscr{D} = \{\chi_p : \ p \in J\},$$

a family of functions from $C([0,a])$ such that $\chi_p(0) = 0$, $p \mapsto [x,y;\chi_p]$ is exponentially convex on J for every choice of two distinct points $x,y \in [0,a]$, and $p \mapsto [x_0,x_1,x_2;\chi_p]$ is exponentially convex on J for every choice of three distinct points $x_0,x_1,x_2 \in [0,a]$.

Theorem 2.24 *Let $f \mapsto \mathfrak{L}(f)$ be the linear functional defined by* (2.43) *and let $H : J \to \mathbb{R}$ be defined by*

$$H(p) = \mathfrak{L}(\chi_p) \tag{2.52}$$

where $\chi_p \in \mathscr{D}$. Then the following statements hold.

(i) If $n \in \mathbb{N}$ and $p_1, \dots, p_n \in \mathbb{R}$ are arbitrary, then the matrix

$$\left[H\left(\frac{p_k + p_m}{2}\right)\right]_{k,m=1}^{n}$$

is positive semidefinite. Particularly,

$$det\left[H\left(\frac{p_k + p_m}{2}\right)\right]_{k,m=1}^{n} \geq 0.$$

(ii) If the function H is continuous on J, then H is exponentially convex on J.

(iii) If H is positive and differentiable on J, then for every $p,q,u,v \in J$ such that $p \leq u,\ q \leq v$, we have

$$\widehat{M}(p,q) \leq \widehat{M}(u,v)$$

where $\widehat{M}(p,q)$ is defined by

$$\widehat{M}(p,q) = \begin{cases} \left(\frac{H(p)}{H(q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{\frac{d}{dp}(H(p))}{H(p)}\right), & p = q. \end{cases}$$

Proof. (i) Let $n \in \mathbb{N},\ p_1, \dots p_n \in \mathbb{R}$ be arbitrary and define an auxiliary function $\Psi : [0,a] \to \mathbb{R}$ by

$$\Psi(x) = \sum_{k,m=1}^{n} \xi_k \xi_m \chi_{\frac{p_k+p_m}{2}}(x).$$

Then

$$[x,y;\Psi] = \sum_{k,m=1}^{n} \xi_k \xi_m [x,y;\chi_{\frac{p_k+p_m}{2}}] \geq 0$$

by the definition of $\mathscr{D}$ and exponential convexity. This implies that Ψ is a nondecreasing function on $[0,a]$. Similarly, $[x_0,x_1,x_2;\Psi] \geq 0$, for every choice of three distinct points $x_0,x_1,x_2 \in [0,a]$. This implies that Ψ is a nondecreasing, convex function on $[0,a]$ such that $\Psi(0) = 0$. Hence $\mathfrak{L}(\Psi) \geq 0$, which is equivalent to

$$\sum_{k,m=1}^{n} \xi_k \xi_m H\left(\frac{p_k + p_m}{2}\right) \geq 0.$$

(ii) Follows from (i).
(iii) This is a simple consequence of Theorem 2.5. □

2.4 Bellman-Steffensen type inequalities

In [11] Bellman introduced an L^p generalization of Steffensen's inequality. Since it was incorrect as stated it has been a subject of investigation by many mathematicians giving corrected versions and generalizations of Bellman's result. In [57] Pečarić showed that with very simple modifications of conditions Bellman's generalization is true. Using some substitutions in his corrected version Pečarić also proved some modification of Steffensen's inequality (see [58]) for which Mitrinović and Pečarić gave necessary and sufficient conditions (see [49]). In this section we give some generalizations of Bellman-Steffensen type inequalities for positive measures obtained by Jakšetić, Pečarić and Smoljak Kalamir in [38].

Theorem 2.25 *Let μ be a finite, positive measure on $\mathscr{B}([a,b])$, f, h be $\mu-$integrable functions on $[a,b]$ such that h is positive and f nonincreasing and right-continuous. Then*

$$\frac{\int_{[a,b]} f(t)G(t)d\mu(t)}{\int_{[a,b]} G(t)d\mu(t)} \leq \frac{\int_{[a,a+\lambda]} f(t)h(t)d\mu(t)}{\int_{[a,a+\lambda]} h(t)d\mu(t)} \tag{2.53}$$

if and only if $G : [a,b] \to \mathbb{R}$ is $\mu-$integrable and λ is a positive constant such that

$$\frac{\int_{[a,x)} G(t)d\mu(t)}{\int_{[a,b]} G(t)d\mu(t)} \leq \frac{\int_{[a,x)} h(t)d\mu(t)}{\int_{[a,a+\lambda]} h(t)d\mu(t)} \quad \text{and} \quad \int_{[x,b]} G(t)d\mu(t) \geq 0, \tag{2.54}$$

for every $x \in [a,b]$, assuming $\int_{[a,b]} G(t)d\mu(t) > 0$.
For a nondecreasing, right-continuous function $f : [a,b] \to \mathbb{R}$ the inequality (2.53) *is reversed.*

Proof. *Sufficiency*. Let us define the function

$$g(t) = \frac{G(t)\int_{[a,a+\lambda]} h(t)d\mu(t)}{\int_{[a,b]} G(t)d\mu(t)}.$$

Since $\int_{[a,b]} g(t)d\mu(t) = \int_{[a,a+\lambda]} h(t)d\mu(t)$ and (3.84) are fulfilled we can apply (3.83) and (2.53) is valid.
Necessity. If we put the function

$$f(t) = \begin{cases} 1, & t < x; \\ 0, & t \geq x, \end{cases}$$

for $a \leq x \leq a+\lambda$ in the inequality (2.53) we obtain the conditions in (2.54). $\square$

Theorem 2.26 *Let μ be a finite, positive measure on $\mathscr{B}([a,b])$. Let f and h be nonnegative nonincreasing functions on $[a,b]$, and let ϕ be an increasing convex function on $[0,\infty)$ with $\phi(0)=0$. If G is a nonnegative nondecreasing function on $[a,b]$ such that there exists a nonnegative function g_1, defined by the equation*

$$\int_{[a,b]} g_1(t)\phi\left(\frac{G(t)}{g_1(t)}\right)d\mu(t) \le \int_{[a,b]} h(t)d\mu(t)$$

and $\int_{[a,b]} g_1(t)d\mu(t) \le 1$, then the following inequality is valid:

$$\phi\left(\frac{\int_{[a,b]} f(t)G(t)d\mu(t)}{\int_{[a,b]} G(t)d\mu(t)}\right) \le \frac{\int_{[a,a+\lambda]} \phi(f(t))h(t)d\mu(t)}{\int_{[a,a+\lambda]} h(t)d\mu(t)},$$

where $\int_{[a,a+\lambda]} h(t)d\mu(t) = \phi\left(\int_{[a,b]} G(t)d\mu(t)\right)$.

Proof. Using Jensen's inequality we have

$$\phi\left(\frac{\int_{[a,b]} f(t)G(t)d\mu(t)}{\int_{[a,b]} G(t)d\mu(t)}\right) \le \frac{\int_{[a,b]} \phi(f(t))G(t)d\mu(t)}{\int_{[a,b]} G(t)d\mu(t)},$$

and since $\phi \circ f$ is nonincreasing, we only have to check conditions in (2.54). Since G is nonnegative, obviously $\int_{[x,b]} G(t)d\mu(t) \ge 0$. So we only have to show

$$\phi\left(\int_{[a,b]} G(t)d\mu(t)\right)\int_{[a,x)} G(t)d\mu(t) \le \int_{[a,x)} h(t)d\mu(t)\int_{[a,b]} G(t)d\mu(t). \tag{2.55}$$

Using sub-linearity from Lemma 2.3 and Jensen's inequality we have

$$\begin{aligned}
\phi\left(\int_{[a,b]} G(t)d\mu(t)\right) &= \phi\left(\int_{[a,b]} g_1(t)d\mu(t)\frac{\int_{[a,b]} G(t)d\mu(t)}{\int_{[a,b]} g_1(t)d\mu(t)}\right)\\
&\le \int_{[a,b]} g_1(t)d\mu(t)\phi\left(\frac{\int_{[a,b]} g_1(t)\frac{G(t)}{g_1(t)}d\mu(t)}{\int_{[a,b]} g_1(t)d\mu(t)}\right)\\
&\le \int_{[a,b]} g_1(t)\phi\left(\frac{G(t)}{g_1(t)}\right)d\mu(t) \le \int_{[a,b]} h(t)d\mu(t).
\end{aligned} \tag{2.56}$$

Since G is a nonnegative nondecreasing function and h is a nonnegative nonincreasing function, we see that for each $x \in [a,b]$

$$\frac{\int_{[a,x)} G(t)d\mu(t)}{\int_{[a,x)} h(t)d\mu(t)} \le \frac{\int_{[a,b]} G(t)d\mu(t)}{\int_{[a,b]} h(t)d\mu(t)}$$

i.e.

$$\int_{[a,b]} h(t)d\mu(t)\int_{[a,x)} G(t)d\mu(t) \le \int_{[a,x)} h(t)d\mu(t)\int_{[a,b]} G(t)d\mu(t),$$

so along with (2.56) we proved (2.55). Hence, the proof is completed. □

In Theorems 2.25 and 2.26 we proved similar results to the ones obtained by Liu in [43] but we only need μ to be finite and positive instead of finite continuous and strictly increasing as in [43].

Taking $h \equiv 1$ in Theorems 2.25 and 2.26 we obtain results given in [37].

2.5 Further improvement of an extension of Hölder-type inequality

In [54] Pearce and Pečarić proved an extension of Hölder's inequality using the following generalization of Steffensen's inequality.

Theorem 2.27 *(i) Suppose that f and g are integrable functions on $[a,b]$, f is nonincreasing and $\lambda > 0$. If a positive function g satisfies the condition*

$$\lambda \int_a^x g(t)dt \leq (x-a) \int_a^b g(t)dt \tag{2.57}$$

for every $x \in [a,b]$, then

$$\frac{\int_a^b f(t)g(t)dt}{\int_a^b g(t)dt} \leq \lambda^{-1} \int_a^{a+\lambda} f(t)dt, \tag{2.58}$$

while if a positive function g satisfies

$$\lambda \int_x^b g(t)dt \leq (b-x) \int_a^b g(t)dt \tag{2.59}$$

for every $x \in [a,b]$, then

$$\lambda^{-1} \int_{b-\lambda}^b f(t)dt \leq \frac{\int_a^b f(t)g(t)dt}{\int_a^b g(t)dt}. \tag{2.60}$$

In either case equality holds if f is constant.

(ii) If f is nondecreasing, the reverse inequalities hold in (2.58) *and* (2.60).

In the following theorem we recall the aforementioned extension of Hölder's inequality from [54].

Theorem 2.28 *Let f and g be two integrable and positive functions defined on $[a,b]$ and let M, K be real numbers satisfying $a \leq K < M \leq b$.*

(i) Suppose that for every $x \in [K,b]$ we have

$$\frac{1}{x-K} \int_K^x g(t)dt \leq \frac{1}{M-K} \int_K^b g(t)dt, \tag{2.61}$$

that $p > 1$, $p^{-1}+q^{-1} = 1$ *and that* f *is nonincreasing. Then*

$$\int_a^b f(t)g(t)dt \leq \left(\int_a^M f^p(t)dt\right)^{1/p} \left(\int_a^M \hat{g}^q(t)dt\right)^{1/q}, \tag{2.62}$$

where

$$\hat{g}(t) = \begin{cases} g(t), & a \leq t < K \\ \frac{1}{M-K}\int_K^b g(t)dt, & K \leq t \leq M. \end{cases} \tag{2.63}$$

The inequality in (2.62) *is reversed if* $p < 1$ *and* f *is a nondecreasing function. In both cases, equality holds in* (2.62) *if*

$$f^p(t) = c\hat{g}^q(t), \quad a \leq t \leq M$$

(where c *is constant) and*

$$f(t) = f(K), \quad t \in [K,b].$$

(ii) Suppose that for every $x \in [a,M]$ *we have*

$$\frac{1}{M-x}\int_x^M g(t)dt \leq \frac{1}{M-K}\int_a^M g(t)dt, \tag{2.64}$$

that $p > 1$, $p^{-1}+q^{-1} = 1$ *and that* f *is nondecreasing. Then*

$$\int_a^b f(t)g(t)dt \leq \left(\int_K^b f^p(t)dt\right)^{1/p} \left(\int_K^b \hat{g}^q(t)dt\right)^{1/q}, \tag{2.65}$$

where

$$\hat{g}(t) = \begin{cases} \frac{1}{M-K}\int_a^M g(t)dt, & K \leq t \leq M \\ g(t), & M < t \leq b. \end{cases} \tag{2.66}$$

The inequality in (2.65) *is reversed if* $p < 1$ *and* f *is a nonincreasing function. In both cases, equality holds in* (2.65) *if*

$$f^p(t) = c\hat{g}^q(t), \quad K \leq t \leq b$$

(where c *is constant) and*

$$f(t) = f(M), \quad t \in [a,M].$$

In [73] Pečarić and Smoljak improved the above extension of Hölder's inequality using log-convexity.

By simple modification of Theorem 2.16 we have the following generalization of Steffensen's inequality for positive measures given in [39].

Theorem 2.29 *(i) Let μ be a finite, positive measure on $\mathscr{B}([a,b])$. Suppose that g is a $\mu-$integrable function on $[a,b]$, f is nonincreasing, right-continuous function on $[a,b]$ and λ is a positive constant. If a positive function g satisfies the condition*

$$\mu([a,a+\lambda])\int_{[a,x)} g(t)d\mu(t) \leq \mu([a,x))\int_{[a,b]} g(t)d\mu(t) \tag{2.67}$$

for every $x \in [a,b]$, then

$$\frac{\int_{[a,b]} f(t)g(t)d\mu(t)}{\int_{[a,b]} g(t)d\mu(t)} \leq \frac{\int_{[a,a+\lambda]} f(t)d\mu(t)}{\mu([a,a+\lambda])}, \tag{2.68}$$

while if a positive function g satisfies the condition

$$\mu((b-\lambda,b])\int_{[x,b]} g(t)d\mu(t) \leq \mu([x,b])\int_{[a,b]} g(t)d\mu(t) \tag{2.69}$$

for every $x \in [a,b]$, then

$$\frac{\int_{(b-\lambda,b]} f(t)d\mu(t)}{\mu((b-\lambda,b])} \leq \frac{\int_{[a,b]} f(t)g(t)d\mu(t)}{\int_{[a,b]} g(t)d\mu(t)}. \tag{2.70}$$

In either case the equality holds if f is constant.

(ii) If f is a nondecreasing, right-continuous function, the reverse inequalities hold in (2.68) *and* (2.70)*.*

Using the above generalization of Steffensen's inequality the following extension of Hölder's inequality for positive measures was obtained in [39].

Theorem 2.30 *Let μ be a finite, positive measure on $\mathscr{B}([a,b])$. Let f and g be two $\mu-$integrable and positive functions defined on $[a,b]$ and let M, K be real numbers satisfying $a \leq K < M \leq b$.*

(i) Suppose that for every $x \in [K,b]$ we have

$$\frac{1}{\mu([K,x))}\int_{[K,x)} g(t)d\mu(t) \leq \frac{1}{\mu([K,M])}\int_{[K,b]} g(t)d\mu(t), \tag{2.71}$$

that $p > 1$, $p^{-1}+q^{-1} = 1$ and that f is a nonincreasing, right-continuous function. Then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \leq \left(\int_{[a,M]} f^p(t)d\mu(t)\right)^{1/p}\left(\int_{[a,M]} \hat{g}^q(t)d\mu(t)\right)^{1/q}, \tag{2.72}$$

where

$$\hat{g}(t) = \begin{cases} g(t), & a \leq t < K \\ \frac{1}{\mu([K,M])}\int_{[K,b]} g(t)d\mu(t), & K \leq t \leq M. \end{cases} \tag{2.73}$$

The inequality in (2.72) *is reversed if* $p < 1$ *and* f *is a nondecreasing, right-continuous function. In both cases, the equality holds in* (2.72) *if*

$$f^p(t) = c\hat{g}^q(t), \quad a \le t \le M$$

where c *is a constant and* $f(t) = f(K)$, $t \in [K,b]$.

(ii) Suppose that for every $x \in [a,M]$ *we have*

$$\frac{1}{\mu([x,M])}\int_{[x,M]} g(t)d\mu(t) \le \frac{1}{\mu((K,M])}\int_{[a,M]} g(t)d\mu(t), \tag{2.74}$$

that $p > 1$, $p^{-1}+q^{-1} = 1$ *and* f *is a nondecreasing, right-continuous function. Then*

$$\int_{[a,b]} f(t)g(t)d\mu(t) \le \left(\int_{(K,b]} f^p(t)d\mu(t)\right)^{1/p}\left(\int_{(K,b]} \hat{g}^q(t)d\mu(t)\right)^{1/q}, \tag{2.75}$$

where

$$\hat{g}(t) = \begin{cases} \frac{1}{\mu((K,M])}\int_{[a,M]} g(t)d\mu(t), & K \le t \le M \\ g(t), & M < t \le b. \end{cases} \tag{2.76}$$

The inequality in (2.75) *is reversed if* $p < 1$ *and* f *is a nonincreasing, right-continuous function. In both cases, the equality holds in* (2.75) *if*

$$f^p(t) = c\hat{g}^q(t), \quad K \le t \le b$$

where c *is a constant and* $f(t) = f(M)$, $t \in [a,M]$.

Proof.

(i) Let $\lambda = M - K$ and replace a by K in Theorem 2.29 (i). Now, by (2.71) we have that the condition (2.67) is satisfied. Hence, (2.68) holds, that is

$$\begin{aligned}\int_{[K,b]} f(t)g(t)d\mu(t) &\le \frac{1}{\mu([K,M])}\int_{[K,b]} g(t)d\mu(t)\int_{[K,M]} f(t)d\mu(t)\\ &= \int_{[K,M]} f(t)\hat{g}(t)d\mu(t).\end{aligned}$$

So,

$$\begin{aligned}\int_{[a,b]} f(t)g(t)d\mu(t) &= \int_{[a,K)} f(t)g(t)d\mu(t) + \int_{[K,b]} f(t)g(t)d\mu(t)\\ &\le \int_{[a,K)} f(t)\hat{g}(t)d\mu(t) + \int_{[K,M]} f(t)\hat{g}(t)d\mu(t)\\ &= \int_{[a,M]} f(t)\hat{g}(t)d\mu(t).\end{aligned}$$

Now using Hölder's inequality, the inequality (2.72) follows.

(ii) Let $\lambda = M - K$ and replace b by M in Theorem 2.29 (ii). Now, by (2.74) we have that the condition (2.69) is satisfied. Since f is nondecreasing the reversed inequality in (2.70) holds, that is

$$\begin{aligned}\int_{[a,M]} f(t)g(t)d\mu(t) &\leq \frac{1}{\mu((K,M])}\int_{[a,M]} g(t)d\mu(t)\int_{(K,M]} f(t)d\mu(t)\\ &= \int_{(K,M]} f(t)\hat{g}(t)d\mu(t).\end{aligned}$$

So

$$\begin{aligned}\int_{[a,b]} f(t)g(t)d\mu(t) &= \int_{[a,M]} f(t)g(t)d\mu(t) + \int_{(M,b]} f(t)g(t)d\mu(t)\\ &\leq \int_{(K,M]} f(t)\hat{g}(t)d\mu(t) + \int_{(M,b]} f(t)\hat{g}(t)d\mu(t)\\ &= \int_{(K,b]} f(t)\hat{g}(t)d\mu(t).\end{aligned}$$

Now using Hölder's inequality, the inequality (2.75) follows.

The other cases follow similarly, while the statement of equality follows from the condition for the equality in Steffensen's and Hölder's inequalities. □

Corollary 2.4 *(i) Suppose the assumptions of Theorem 2.30(i) are satisfied and further g is nonincreasing. Then Theorem 2.30(i) is also valid if the condition* (2.71) *is replaced by*

$$g(K) \leq \frac{1}{\mu([K,M])}\int_{[K,b]} g(t)d\mu(t).$$

(ii) Suppose the assumptions of Theorem 2.30(ii) are satisfied and further g is nondecreasing. Then Theorem 2.30(ii) is also valid if the condition (2.74) *is replaced by*

$$g(M) \leq \frac{1}{\mu((K,M])}\int_{[a,M]} g(t)d\mu(t).$$

Proof. If g is nonincreasing, then

$$\frac{1}{\mu([K,x])}\int_{[K,x)} g(t)d\mu(t) \leq g(K) \leq \frac{1}{\mu([K,M])}\int_{[K,b]} g(t)d\mu(t),$$

that is, (2.71) holds. Similarly, if g is nondecreasing, then

$$\frac{1}{\mu([x,M])}\int_{[x,M]} g(t)d\mu(t) \leq g(M) \leq \frac{1}{\mu((K,M])}\int_{[a,M]} g(t)dt,$$

that is, (2.74) holds. □

We continue with some applications of previous results given in [39].

Let φ_p be defined as in Lemma 2.2. Under the assumptions of Theorem 2.30 (i) and (ii), respectively, let us define the following linear functionals

$$\mathfrak{N}_1(\varphi_p \circ f) = \int_{[a,b]} \varphi_p(f(t))g(t)d\mu(t) - \int_{[a,M]} \varphi_p(f(t))\hat{g}(t)d\mu(t) \tag{2.77}$$

and

$$\mathfrak{N}_2(\varphi_p \circ f) = \int_{(K,b]} \varphi_p(f(t))\hat{g}(t)d\mu(t) - \int_{[a,b]} \varphi_p(f(t))g(t)d\mu(t), \tag{2.78}$$

which are positive on a class of nondecreasing, right-continuous functions f.

Also, we have that $-\mathfrak{N}_1(\varphi_p \circ f)$ and $-\mathfrak{N}_2(\varphi_p \circ f)$ are positive on a class of nonincreasing, right-continuous functions f.

Theorem 2.31 *Let $\Phi_i : \mathbb{R} \to \mathbb{R}$, $i = 1,2$, be defined by*

$$\Phi_i(p) = \mathfrak{N}_i(\varphi_p \circ f),$$

where $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are linear functionals defined by (2.77) and (2.78), φ_p is defined in Lemma 2.2 and f is a nondecreasing, right-continuous function. Then the following statements hold for every $i = 1,2$.

(i) The function Φ_i is continuous on $\mathbb{R}$.

(ii) If $n \in \mathbb{N}$ and $p_1,\ldots,p_n \in \mathbb{R}$ are arbitrary, then the matrix

$$\left[\Phi_i\left(\frac{p_j+p_k}{2}\right)\right]_{j,k=1}^n$$

is positive semidefinite. Particularly,

$$det\left[\Phi_i\left(\frac{p_j+p_k}{2}\right)\right]_{j,k=1}^n \geq 0.$$

(iii) The function Φ_i is exponentially convex on $\mathbb{R}$.

(iv) The function Φ_i is log-convex on $\mathbb{R}$.

(v) If $r,s,t \in \mathbb{R}$ are such that $r < s < t$, then

$$\Phi_i(s)^{t-r} \leq \Phi_i(r)^{t-s}\Phi_i(t)^{s-r}.$$

Proof. (i) Continuity of the function $p \mapsto \Phi_i(p)$, $i = 1,2$ is obvious for $p \in \mathbb{R}\setminus\{0\}$. For $p = 0$ it is directly checked using the Heine characterization.
(ii) First, let us prove this for $i = 1$. Let $n \in \mathbb{N}$, $\xi_j, p_j \in \mathbb{R}, (j = 1,\ldots,n)$ be arbitrary and define an auxiliary function $\Psi : (0,\infty) \to \mathbb{R}$ by

$$\Psi(x) = \sum_{j,k=1}^n \xi_j\xi_k\varphi_{\frac{p_j+p_k}{2}}(x).$$

Since

$$\Psi'(x) = \left(\sum_{j=1}^{n} \xi_j x^{\frac{p_j-1}{2}} \right)^2 \geq 0$$

we have that Ψ is increasing on $(0,\infty)$.

By (2.71), the condition (2.67) is satisfied with $\lambda = M - K$ and a replaced by K. Hence, by Theorem 2.29, the reverse inequality in (2.68) holds, so for a nondecreasing function $\Psi \circ f$ we obtain

$$\int_{[K,b]} \Psi(f(t))g(t)d\mu(t) \geq \int_{[K,M]} \Psi(f(t))\hat{g}(t)d\mu(t)t.$$

By definition

$$\int_{[a,K)} \Psi(f(t))g(t)d\mu(t) = \int_{[a,K)} \Psi(f(t))\hat{g}(t)d\mu(t),$$

so we obtain

$$\int_{[a,b]} \Psi(f(t))g(t)d\mu(t) \geq \int_{[a,M]} \Psi(f(t))\hat{g}(t)d\mu(t),$$

that is, $\mathfrak{N}_1(\Psi \circ f) \geq 0$. This is means that

$$\left[\Phi_1 \left(\frac{p_j + p_k}{2} \right) \right]_{j,k=1}^{n}$$

is a positive semi-definite matrix.
Similarly, we can prove this for $i = 2$.
(iii), (iv), (v) are simple consequences of (i) and (ii). □

Similarly as in Theorem 2.31 we obtain that for a nonincreasing, right-continuous function f statements of Theorem 2.31 hold for $-\Phi_i(p)$, $i = 1,2$.

Hence, the following inequality holds true

$$|\Phi_i(s)|^{t-r} \leq |\Phi_i(r)|^{t-s} |\Phi_i(t)|^{s-r}, \quad i = 1,2, \tag{2.79}$$

for every choice $r,s,t \in \mathbb{R}$ such that $r < s < t$.

In the following theorems we obtain an improvement of Hölder-type inequality in measure theory settings.

Theorem 2.32 *Let μ be a finite, positive measure on $\mathscr{B}([a,b])$. Let f and g be two $\mu-$integrable and positive functions defined on $[a,b]$, let $\hat{g}$ be defined by* (2.73) *and let M, K be real numbers satisfying $a \leq K < M \leq b$. Suppose that for every $x \in [K,b]$ we have* (2.71).

(i) Suppose that $p > 1$, $p^{-1} + q^{-1} = 1$, $1 < s < t$ and that f is a nonincreasing, right-continuous function. Then

$$\left(\int_{[a,M]} f^p(t)d\mu(t) \right)^{1/p} \left(\int_{[a,M]} \hat{g}^q(t)d\mu(t) \right)^{1/q} - \int_{[a,b]} f(t)g(t)d\mu(t) \geq \left[-\Phi_1(s) \right]^{\frac{t-1}{t-s}} \left[-\Phi_1(t) \right]^{\frac{1-s}{t-s}}. \tag{2.80}$$

If $p < 1$ and f is a nondecreasing, right-continuous function, then

$$\int_{[a,b]} f(t)g(t)d\mu(t) - \left(\int_{[a,M]} f^p(t)d\mu(t)\right)^{1/p} \left(\int_{[a,M]} \hat{g}^q(t)d\mu(t)\right)^{1/q} \geq [\Phi_1(s)]^{\frac{t-1}{t-s}} [\Phi_1(t)]^{\frac{1-s}{t-s}}. \tag{2.81}$$

(ii) *Suppose that $p > 1$, $p^{-1} + q^{-1} = 1$, $r < s < 1$ and that f is a nonincreasing, right-continuous function. Then*

$$\left(\int_{[a,M]} f^p(t)d\mu(t)\right)^{1/p} \left(\int_{[a,M]} \hat{g}^q(t)d\mu(t)\right)^{1/q} - \int_{[a,b]} f(t)g(t)d\mu(t) \geq [-\Phi_1(s)]^{\frac{1-r}{s-r}} [-\Phi_1(r)]^{\frac{s-1}{s-r}}.$$

If $p < 1$ and f is a nondecreasing, right-continuous function, then

$$\int_{[a,b]} f(t)g(t)d\mu(t) - \left(\int_{[a,M]} f^p(t)d\mu(t)\right)^{1/p} \left(\int_{[a,M]} \hat{g}^q(t)d\mu(t)\right)^{1/q} \geq [\Phi_1(s)]^{\frac{1-r}{s-r}} [\Phi_1(r)]^{\frac{s-1}{s-r}}.$$

Proof.

(i) Taking substitution $r \to 1$ in (2.79) (for $i = 1$) and then raising both sides of the inequality (2.79) to the power $\frac{1}{t-s}$ we obtain

$$|\Phi_1(1)| \geq |\Phi_1(s)|^{\frac{t-1}{t-s}} |\Phi_1(t)|^{\frac{1-s}{t-s}}.$$

For a nonincreasing function f, we have

$$|\Phi_1(1)| = -\Phi_1(1) = \int_{[a,M]} f(t)\hat{g}(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) \geq 0.$$

Now by Hölder's inequality we have

$$\begin{aligned}\left(\int_{[a,M]} f^p(t)d\mu(t)\right)^{1/p} \left(\int_{[a,M]} \hat{g}^q(t)d\mu(t)\right)^{1/q} &- \int_{[a,b]} f(t)g(t)d\mu(t) \\ &\geq \int_{[a,M]} f(t)\hat{g}(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) \\ &= -\Phi_1(1) \geq [-\Phi_1(s)]^{\frac{t-1}{t-s}} [-\Phi_1(t)]^{\frac{1-s}{t-s}}.\end{aligned}$$

Hence, we obtain (2.80).

For a nondecreasing function f, we have

$$|\Phi_1(1)| = \Phi_1(1) = \int_{[a,b]} f(t)g(t)d\mu(t) - \int_{[a,M]} f(t)\hat{g}(t)d\mu(t) \geq 0.$$

Now by Hölder's inequality for $p < 1$ we have

$$\begin{aligned}\int_{[a,b]} f(t)g(t)d\mu(t) - \left(\int_{[a,M]} f^p(t)d\mu(t)\right)^{1/p} \left(\int_{[a,M]} \hat{g}^q(t)d\mu(t)\right)^{1/q}\\ \geq \int_{[a,b]} f(t)g(t)d\mu(t) - \int_{[a,M]} f(t)\hat{g}(t)d\mu(t)\\ = \Phi_1(1) \geq [\Phi_1(s)]^{\frac{t-1}{t-s}} [\Phi_1(t)]^{\frac{1-s}{t-s}}.\end{aligned}$$

Hence, we obtain (2.81).

(ii) Similar to the proof of (i), taking substitution $t \to 1$.

□

Theorem 2.33 *Let μ be a finite, positive measure on $\mathscr{B}([a,b])$. Let f and g be two $\mu-$integrable and positive functions defined on $[a,b]$, let $\hat{g}$ be defined by* (2.76) *and let M, K be real numbers satisfying $a \leq K < M \leq b$. Suppose that for every $x \in [a,M]$ we have* (2.74)*.*

(i) Suppose that $p > 1$, $p^{-1} + q^{-1} = 1$, $1 < s < t$ and that f is a nondecreasing, right-continuous function. Then

$$\begin{aligned}\left(\int_{(K,b]} f^p(t)d\mu(t)\right)^{1/p} \left(\int_{(K,b]} \hat{g}^q(t)d\mu(t)\right)^{1/q} - \int_{[a,b]} f(t)g(t)d\mu(t)\\ \geq [\Phi_2(s)]^{\frac{t-1}{t-s}} [\Phi_2(t)]^{\frac{1-s}{t-s}}.\end{aligned}$$

If $p < 1$ and f is a nonincreasing, right-continuous function, then

$$\begin{aligned}\int_{[a,b]} f(t)g(t)d\mu(t) - \left(\int_{(K,b]} f^p(t)d\mu(t)\right)^{1/p} \left(\int_{(K,b]} \hat{g}^q(t)d\mu(t)\right)^{1/q}\\ \geq [-\Phi_2(s)]^{\frac{t-1}{t-s}} [-\Phi_2(t)]^{\frac{1-s}{t-s}}.\end{aligned}$$

(ii) Suppose that $p > 1$, $p^{-1} + q^{-1} = 1$, $r < s < 1$ and that f is a nondecreasing, right-continuous function. Then

$$\begin{aligned}\left(\int_{(K,b]} f^p(t)d\mu(t)\right)^{1/p} \left(\int_{(K,b]} \hat{g}^q(t)d\mu(t)\right)^{1/q} - \int_{[a,b]} f(t)g(t)d\mu(t)\\ \geq [\Phi_2(s)]^{\frac{1-r}{s-r}} [\Phi_2(r)]^{\frac{s-1}{s-r}}.\end{aligned}$$

If $p < 1$ and f is a nonincreasing, right-continuous function, then

$$\begin{aligned}\int_{[a,b]} f(t)g(t)d\mu(t) - \left(\int_{(K,b]} f^p(t)d\mu(t)\right)^{1/p} \left(\int_{(K,b]} \hat{g}^q(t)d\mu(t)\right)^{1/q}\\ \geq [-\Phi_2(s)]^{\frac{1-r}{s-r}} [-\Phi_2(t)]^{\frac{s-1}{s-r}}.\end{aligned}$$

Proof. Similar to the proof of Theorem 2.32. □

We continue with the Lagrange-type mean value theorem.

Theorem 2.34 *Let f be a nondecreasing, right-continuous function and let $\psi \in C^1[f(a), f(b)]$. Let $\mathfrak{N}_i$, $i = 1,2$, be linear functionals defined by* (2.77) *and* (2.78). *Then there exist $\xi_i \in [f(a), f(b)]$, $i = 1,2$, such that*

$$\mathfrak{N}_i(\psi \circ f) = \psi'(\xi_i)\mathfrak{N}_i(\mathrm{id} \circ f),$$

where $\mathrm{id}(x) = x$.

Proof. Since $\psi \in C^1[f(a), f(b)]$ there exist

$$m = \min_{x \in [f(a), f(b)]} \psi'(x) \quad \text{and} \quad M = \max_{x \in [f(a), f(b)]} \psi'(x).$$

Denote $h_1(x) = Mx - \psi(x)$ and $h_2(x) = \psi(x) - mx$. Then

$$\begin{aligned} h_1'(x) &= M - \psi'(x) \geq 0 \\ h_2'(x) &= \psi'(x) - m \geq 0 \end{aligned}$$

so h_1 and h_2 are nondecreasing on $[f(a), f(b)]$, which means that $\mathfrak{N}_i(h_1 \circ f) \geq 0$ and $\mathfrak{N}_i(h_2 \circ f) \geq 0$ i.e.

$$m\mathfrak{N}_i(\mathrm{id} \circ f) \leq \mathfrak{N}_i(\psi \circ f) \leq M\mathfrak{N}_i(\mathrm{id} \circ f).$$

If $\mathfrak{N}_i(\mathrm{id} \circ f) = 0$, the proof is complete. If $\mathfrak{N}_i(\mathrm{id} \circ f) > 0$, then

$$m \leq \frac{\mathfrak{N}_i(\psi \circ f)}{\mathfrak{N}_i(\mathrm{id} \circ f)} \leq M$$

and the existence of $\xi_i \in [f(a), f(b)]$ follows. □

Corollary 2.5 *Let f be a nondecreasing, right-continuous function and let $\psi_1, \psi_2 \in C^1[f(a), f(b)]$. Then there exist $\xi_i \in [f(a), f(b)]$, such that*

$$\frac{\psi_1'(\xi)}{\psi_2'(\xi)} = \frac{\mathfrak{N}_i(\psi_1 \circ f)}{\mathfrak{N}_i(\psi_2 \circ f)} \tag{2.82}$$

provided that the denominator on right sides is non-zero, where $\mathfrak{N}_i$, $i = 1,2$, are linear functionals defined by (2.77) *and* (2.78).

If the inverse of ψ_1'/ψ_2' exists then various kinds of means can be defined by (2.82). That is

$$\xi_i = \left(\frac{\psi_1'}{\psi_2'}\right)^{-1} \left(\frac{\mathfrak{N}_i(\psi_1 \circ f)}{\mathfrak{N}_i(\psi_2 \circ f)}\right), \quad i = 1,2. \tag{2.83}$$

Particularly, if we substitute $\psi_1(x) = \varphi_p(x)$, $\psi_2(x) = \varphi_q(x)$, where φ_p is defined in Lemma 2.2, in (2.83) and use the continuous extension, the following expressions are obtained ($i = 1,2$):

$$M_i(p,q) = \begin{cases} \left(\frac{\mathfrak{N}_i(\varphi_p \circ f)}{\mathfrak{N}_i(\varphi_q \circ f)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{\mathfrak{N}_i(\varphi_0 \cdot (\varphi_p \circ f))}{\mathfrak{N}_i(\varphi_p \circ f)} - \frac{1}{p}\right), & p = q. \end{cases}$$

By Theorem 2.5, if $p,q,u,v \in (0,\infty)$ such that $p \leq u,\ q \leq v$ then,

$$M_i(p,q) \leq M_i(u,v), \quad i = 1,2.$$

Similar as in Section 2.1 we see that we can further refine obtained results by dropping some of the analytical properties of a family of functions from Lemma 2.2. Proofs are similar to the ones in Section 2.1 so we omit the details.

By

$$\mathscr{C} = \{\psi_p : \ \psi_p : [a,b] \to \mathbb{R},\ p \in J\}$$

let us define a family of functions from $C([a,b])$ such that $p \mapsto [x_0,x_1;\psi_p]$ is log-convex in the Jensen sense on J for every choice of two distinct points $x_0,x_1 \in [a,b]$.

Theorem 2.35 *Let $G_i : J \to \mathbb{R}$, be defined by*

$$G_i(p) = \mathfrak{N}_i(\psi_p \circ f), \quad i = 1,2,$$

where functionals $\mathfrak{N}_i$, $i = 1,2$ are defined by (2.77) *and* (2.78)*, $\psi_p \in \mathscr{C}$ and f is a nondecreasing right-continuous function. Then the following statements hold, for every $i = 1,2$.*

(i) G_i is log-convex in the Jensen sense on J.

(ii) If G_i is continuous on J, then it is log-convex on J and for $p,q,r \in J$ such that $p < q < r$, we have

$$G_i(q)^{r-p} \leq G_i(p)^{r-q} G_i(r)^{q-p}.$$

(iii) If G_i is positive and differentiable on J, then for every $p,q,r \in J$ such that $p \leq u$, $q \leq v$, we have

$$\widetilde{M}_i(p,q) \leq \widetilde{M}_i(u,v) \tag{2.84}$$

where $\widetilde{M}_i(p,q)$ is defined by

$$\widetilde{M}_i(p,q) = \begin{cases} \left(\frac{G_i(p)}{G_i(q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{\frac{d}{dp}(G_i(p))}{G_i(p)}\right), & p = q. \end{cases} \tag{2.85}$$

By

$$\mathscr{D} = \{\psi_p : \ \psi_p : [a,b] \to \mathbb{R},\ p \in J\},$$

let us define a family of functions from $C([a,b])$ such that $p \mapsto [x_0,x_1;\psi_p]$ is exponentially convex on J for every choice of two distinct points $x_0,x_1 \in [a,b]$.

Theorem 2.36 *Let $H_i : J \to \mathbb{R}$, be defined by*

$$H_i(p) = \mathfrak{N}_i(\psi_p \circ f), \quad i = 1,2,$$

where functionals $\mathfrak{N}_i$, $i = 1,2$ are defined by (2.77) *and* (2.78)*, $\psi_p \in \mathscr{D}$ and f is a nondecreasing right-continuous function. Then the following statements hold for every $i = 1,2$.*

(i) If $n \in \mathbb{N}$ and $p_1, \ldots, p_n \in \mathbb{R}$ are arbitrary, then the matrix

$$\left[H_i\left(\frac{p_k+p_m}{2}\right)\right]_{k,m=1}^{n}$$

is positive semidefinite. Particularly,

$$det\left[H_i\left(\frac{p_k+p_m}{2}\right)\right]_{k,m=1}^{n} \geq 0.$$

(ii) If the function H_i is continuous on J, then H_i is exponentially convex on J.

(iii) If H_i is positive and differentiable on J, then for every $p,q,r \in J$ such that $p \leq u$, $q \leq v$, we have

$$\widehat{M}_i(p,q) \leq \widehat{M}_i(u,v)$$

where $\widehat{M}_i(p,q)$ is defined by

$$\widehat{M}_i(p,q) = \begin{cases} \left(\frac{H_i(p)}{H_i(q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{\frac{d}{dp}(H_i(p))}{H_i(p)}\right), & p = q. \end{cases}$$

Chapter 3

Weighted Pečarić, Mercer and Wu-Srivastava results

3.1 Measure theoretic generalization of Pečarić, Mercer and Wu-Srivastava results

In 2000 Mercer gave a generalization of Steffensen's inequality and he noted that his generalization contains various already known generalizations, one of which is a generalization given by Pečarić in 1982. It was noted by Wu and Srivastava in 2007 that Mercer's result is incorrect as stated and they have not only corrected it but also gave a refinement of Steffensen's inequality and a sharpened version of Mercer's result. Furthermore, in 1979 Milovanović and Pečarić gave weaker conditions on the function g in Steffensen's inequality and Mercer obtained weaker conditions for his generalizations. In [61] Pečarić, Perušić and Smoljak related generalizations of Steffensen's inequality given by Pečarić, Mercer and Wu-Srivastava. Moreover, using Wu-Srivastava refinements of Steffensen's inequality they obtained refined versions of Pečarić and Mercer's results and gave this results with weaker conditions. The aim of this section is to give a measure theoretic generalizations of previously mentioned generalizations of Steffensen's inequality. Results presented in this section were obtained by Jakšetić, Pečarić and Smoljak Kalamir in [35].

Firstly we give measure theoretic version of Theorems 1.6 and 1.7.

Theorem 3.1 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let f,g and h be measurable functions on $[a,b]$ such that h is positive, f/h is nonincreasing and $0 \le g \le 1$.*

(a) If there exists $\lambda \in \mathbb{R}_+$ such that

$$\int_{[a,a+\lambda]} h(t)d\mu(t) = \int_{[a,b]} h(t)g(t)d\mu(t), \tag{3.1}$$

then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \le \int_{[a,a+\lambda]} f(t)d\mu(t). \tag{3.2}$$

(b) If there exists $\lambda \in \mathbb{R}_+$ such that

$$\int_{(b-\lambda,b]} h(t)d\mu(t) = \int_{[a,b]} h(t)g(t)d\mu(t), \tag{3.3}$$

then

$$\int_{(b-\lambda,b]} f(t)d\mu(t) \le \int_{[a,b]} f(t)g(t)d\mu(t). \tag{3.4}$$

Proof. Let us prove the (a)-part. Transformation of the difference between the right-hand side and the left-hand side of inequality (3.2) gives

$$\begin{aligned}
&\int_{[a,a+\lambda]} f(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) \\
&= \int_{[a,a+\lambda]} (1-g(t))f(t)d\mu(t) - \int_{(a+\lambda,b]} f(t)g(t)d\mu(t) \\
&\ge \frac{f(a+\lambda)}{h(a+\lambda)} \int_{[a,a+\lambda]} h(t)(1-g(t))d\mu(t) - \int_{(a+\lambda,b]} f(t)g(t)d\mu(t) \\
&= \frac{f(a+\lambda)}{h(a+\lambda)} \left(\int_{[a,b]} h(t)g(t)d\mu(t) - \int_{[a,a+\lambda]} h(t)g(t)d\mu(t) \right) - \int_{(a+\lambda,b]} f(t)g(t)d\mu(t) \\
&= \int_{(a+\lambda,b]} g(t)h(t) \left(\frac{f(a+\lambda)}{h(a+\lambda)} - \frac{f(t)}{h(t)} \right) d\mu(t) \ge 0,
\end{aligned}$$

where we use (3.1).
Proof of the (b)-part is similar so we omit the details. □

Theorem 3.2 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let f,g and h be measurable functions on $[a,b]$ such that h is positive, f is nonnegative, $0 \le g \le 1$ and f/h is nonincreasing.*

(a) If there exists $\lambda \in \mathbb{R}_+$ such that

$$\int_{[a,a+\lambda]} h(t)d\mu(t) \ge \int_{[a,b]} h(t)g(t)d\mu(t), \tag{3.5}$$

then (3.2) holds.

(b) If there exists $\lambda \in \mathbb{R}_+$ *such that*

$$\int_{(b-\lambda,b]} h(t)d\mu(t) \leq \int_{[a,b]} h(t)g(t)d\mu(t),$$

then (3.4) *holds.*

Proof. From the conditions of theorem we have that f/h is nonnegative. Hence, condition (3.5) together with $f(a+\lambda)/h(a+\lambda) \geq 0$ enables us to re-adjust the proof of Theorem 3.1 (a) to prove the (a)-part. Similarly we obtain the (b)-part. □

Taking $h \equiv 1$ in Theorems 3.1 and 3.2 we have Steffensen's inequality for positive measures given in Section 2.1.

Using approach from [61] in the following theorems we obtain corrected version of Mercer's generalization in measure theory settings.

Theorem 3.3 *Let* μ *be a positive finite measure on* $\mathscr{B}([a,b])$*, let* f,g *and* h *be measurable functions on* $[a,b]$ *such that* h *is positive,* f *is nonincreasing and* $0 \leq g \leq h$*.*

(a) If there exists $\lambda \in \mathbb{R}_+$ *such that*

$$\int_{[a,a+\lambda]} h(t)d\mu(t) = \int_{[a,b]} g(t)d\mu(t), \tag{3.6}$$

then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \leq \int_{[a,a+\lambda]} f(t)h(t)d\mu(t). \tag{3.7}$$

(b) If there exists $\lambda \in \mathbb{R}_+$ *such that*

$$\int_{(b-\lambda,b]} h(t)d\mu(t) = \int_{[a,b]} g(t)d\mu(t), \tag{3.8}$$

then

$$\int_{(b-\lambda,b]} f(t)h(t)d\mu(t) \leq \int_{[a,b]} f(t)g(t)d\mu(t). \tag{3.9}$$

Proof. Putting substitutions $g \mapsto g/h$ and $f \mapsto fh$ in Theorem 3.1 we obtain statements of this theorem. □

Theorem 3.4 *Let* μ *be a positive finite measure on* $\mathscr{B}([a,b])$*, let* f,g *and* h *be measurable functions on* $[a,b]$ *such that* h *is positive,* $0 \leq g \leq h$*, and* f *is nonnegative and nonincreasing.*

(a) If there exists $\lambda \in \mathbb{R}_+$ *such that*

$$\int_{[a,a+\lambda]} h(t)d\mu(t) \geq \int_{[a,b]} g(t)d\mu(t), \tag{3.10}$$

then (3.7) *holds.*

(b) If there exists $\lambda \in \mathbb{R}_+$ such that

$$\int_{(b-\lambda,b]} h(t)d\mu(t) \leq \int_{[a,b]} g(t)d\mu(t), \tag{3.11}$$

then (3.9) *holds.*

Proof. Putting substitutions $g \mapsto g/h$ and $f \mapsto fh$ in Theorem 3.2 we obtain statements of this theorem. □

In the following theorem we obtain generalization of Theorem 1.10 in measure theory settings.

Theorem 3.5 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let f,g,h and k be measurable functions on $[a,b]$ such that k is positive, $0 \leq g \leq h$ and f/k is nonincreasing.*

(a) If there exists $\lambda \in \mathbb{R}_+$ such that

$$\int_{[a,a+\lambda]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t), \tag{3.12}$$

then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \leq \int_{[a,a+\lambda]} f(t)h(t)d\mu(t). \tag{3.13}$$

(b) If there exists $\lambda \in \mathbb{R}_+$ such that

$$\int_{(b-\lambda,b]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t), \tag{3.14}$$

then

$$\int_{(b-\lambda,b]} f(t)h(t)d\mu(t) \leq \int_{[a,b]} f(t)g(t)d\mu(t). \tag{3.15}$$

Proof. Take $h \mapsto kh$, $g \mapsto g/h$ and $f \mapsto fh$ in Theorem 3.1. □

Theorem 3.6 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let f,g,h and k be measurable functions on $[a,b]$ such that k is positive, f is nonnegative, $0 \leq g \leq h$ and f/k is nonincreasing.*

(a) If there exists $\lambda \in \mathbb{R}_+$ such that

$$\int_{[a,a+\lambda]} h(t)k(t)d\mu(t) \geq \int_{[a,b]} g(t)k(t)d\mu(t), \tag{3.16}$$

then (3.13) *holds.*

(b) If there exists $\lambda \in \mathbb{R}_+$ such that

$$\int_{(b-\lambda,b]} h(t)k(t)d\mu(t) \leq \int_{[a,b]} g(t)k(t)d\mu(t), \tag{3.17}$$

then (3.15) *holds.*

Proof. Take $h \mapsto kh$, $g \mapsto g/h$ and $f \mapsto fh$ in Theorem 3.2. □

Taking $k \equiv 1$ in Theorems 3.5 and 3.6 we obtain results given in Theorems 3.3 and 3.4.

Motivated by corrected and refined version of Mercer's results given in Theorem 1.11 we obtain the following results for positive measure.

Theorem 3.7 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let f, g and h be measurable functions on $[a,b]$ such that $0 \leq g \leq h$ and f is nonincreasing.*

(a) If there exists $\lambda \in \mathbb{R}_+$ such that

$$\int_{[a,a+\lambda]} h(t)d\mu(t) = \int_{[a,b]} g(t)d\mu(t),$$

then

$$\begin{aligned} \int_{[a,b]} f(t)g(t)d\mu(t) &\leq \int_{[a,a+\lambda]} (f(t)h(t) - [f(t) - f(a+\lambda)][h(t) - g(t)])d\mu(t) \\ &\leq \int_{[a,a+\lambda]} f(t)h(t)d\mu(t). \end{aligned} \tag{3.18}$$

(b) If there exists $\lambda \in \mathbb{R}_+$ such that

$$\int_{(b-\lambda,b]} h(t)d\mu(t) = \int_{[a,b]} g(t)d\mu(t),$$

then

$$\begin{aligned} \int_{(b-\lambda,b]} f(t)h(t)d\mu(t) &\leq \int_{(b-\lambda,b]} (f(t)h(t) - [f(t) - f(b-\lambda)][h(t) - g(t)])d\mu(t) \\ &\leq \int_{[a,b]} f(t)g(t)d\mu(t). \end{aligned} \tag{3.19}$$

Proof. The proof is based on the following identities:

$$\begin{aligned} \int_{[a,b]} f(t)g(t)d\mu(t) = &\int_{[a,a+\lambda]} (f(t)h(t) - [f(t) - f(a+\lambda)][h(t) - g(t)])d\mu(t) \\ &+ \int_{(a+\lambda,b]} [f(t) - f(a+\lambda)]g(t)d\mu(t) \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} \int_{[a,b]} f(t)g(t)d\mu(t) = &\int_{(b-\lambda,b]} (f(t)h(t) - [f(t) - f(b-\lambda)][h(t) - g(t)])d\mu(t) \\ &+ \int_{[a,b-\lambda]} [f(t) - f(b-\lambda)]g(t)d\mu(t). \end{aligned} \tag{3.21}$$

Let us prove the first one. Transformation of the right-hand side of identity (3.20) gives the following

$$
\begin{aligned}
&\int_{[a,a+\lambda]} (f(t)h(t) - [f(t) - f(a+\lambda)][h(t) - g(t)])\,d\mu(t) \\
&+ \int_{(a+\lambda,b]} [f(t) - f(a+\lambda)]g(t)\,d\mu(t) = \int_{(a+\lambda,b]} f(t)g(t)\,d\mu(t) \\
&+ \int_{[a,a+\lambda]} [f(t)g(t) + f(a+\lambda)(h(t) - g(t))]\,d\mu(t) - f(a+\lambda)\int_{(a+\lambda,b]} g(t)\,d\mu(t) \\
&= \int_{[a,b]} f(t)g(t)\,d\mu(t) + f(a+\lambda)\left[\int_{[a,a+\lambda]} (h(t) - g(t))\,d\mu(t) - \int_{(a+\lambda,b]} g(t)\,d\mu(t)\right] \\
&= \int_{[a,b]} f(t)g(t)\,d\mu(t) + f(a+\lambda)\left[\int_{[a,a+\lambda]} h(t)\,d\mu(t) - \int_{[a,b]} g(t)\,d\mu(t)\right] \\
&= \int_{[a,b]} f(t)g(t)\,d\mu(t)
\end{aligned}
$$

where in the last equality we use a definition of λ i.e.

$$\int_{[a,a+\lambda]} h(t)\,d\mu(t) = \int_{[a,b]} g(t)\,d\mu(t).$$

The second identity can be proved in a similar manner.

Since f is nonincreasing on $[a,b]$ we have $f(t) \geq f(a+\lambda)$ for all $t \in [a,a+\lambda]$ and $f(t) \leq f(a+\lambda)$ for all $t \in [a+\lambda,b]$. Then

$$\int_{(a+\lambda,b]} [f(t) - f(a+\lambda)]g(t)\,d\mu(t) \leq 0$$

and

$$\int_{[a,a+\lambda]} [f(t) - f(a+\lambda)][h(t) - g(t)]\,d\mu(t) \geq 0.$$

Using (3.20) and above inequalities we obtain

$$
\begin{aligned}
\int_{[a,b]} f(t)g(t)\,d\mu(t) &\leq \int_{[a,a+\lambda]} (f(t)h(t) - [f(t) - f(a+\lambda)][h(t) - g(t)])\,d\mu(t) \\
&\leq \int_{[a,a+\lambda]} f(t)h(t)\,d\mu(t).
\end{aligned}
$$

Similarly, we obtain (3.19) using identity (3.21). □

Theorem 3.8 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let f, g and h be measurable functions on $[a,b]$ such that $0 \leq g \leq h$ and f is nonnegative and nonincreasing.*

(a) If there exists $\lambda \in \mathbb{R}_+$ such that

$$\int_{[a,a+\lambda]} h(t)\,d\mu(t) \geq \int_{[a,b]} g(t)\,d\mu(t),$$

then (3.18) *holds.*

(b) *If there exists $\lambda \in \mathbb{R}_+$ such that*

$$\int_{(b-\lambda,b]} h(t)d\mu(t) \leq \int_{[a,b]} g(t)d\mu(t),$$

then (3.19) *holds.*

Proof. Re-adjusting proof of Theorem 3.7 we have that the proof is based on the following inequalities:

$$\begin{aligned}\int_{[a,b]} f(t)g(t)d\mu(t) &\leq \int_{[a,a+\lambda]} (f(t)h(t) - [f(t)-f(a+\lambda)][h(t)-g(t)])d\mu(t) \\ &+ \int_{(a+\lambda,b]} [f(t)-f(a+\lambda)]g(t)d\mu(t)\end{aligned}$$

and

$$\begin{aligned}\int_{[a,b]} f(t)g(t)d\mu(t) &\geq \int_{(b-\lambda,b]} (f(t)h(t) - [f(t)-f(b-\lambda)][h(t)-g(t)])d\mu(t) \\ &+ \int_{[a,b-\lambda]} [f(t)-f(b-\lambda)]g(t)d\mu(t).\end{aligned}$$

□

In the following theorems we obtain a refined version of results given in Theorems 3.5 and 3.6.

Theorem 3.9 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let f,g,h and k be measurable functions on $[a,b]$ such that $0 \leq g \leq h$ and f/k is nonincreasing.*

(a) *If there exists $\lambda \in \mathbb{R}_+$ such that* (3.12) *holds, then*

$$\begin{aligned}&\int_{[a,b]} f(t)g(t)d\mu(t) \\ &\leq \int_{[a,a+\lambda]} \left(f(t)h(t) - \left[\frac{f(t)}{k(t)} - \frac{f(a+\lambda)}{k(a+\lambda)} \right] k(t)[h(t)-g(t)] \right) d\mu(t) \\ &\leq \int_{[a,a+\lambda]} f(t)h(t)d\mu(t).\end{aligned} \tag{3.22}$$

(b) *If there exists $\lambda \in \mathbb{R}_+$ such that* (3.14) *holds, then*

$$\begin{aligned}&\int_{(b-\lambda,b]} f(t)h(t)d\mu(t) \\ &\leq \int_{(b-\lambda,b]} \left(f(t)h(t) - \left[\frac{f(t)}{k(t)} - \frac{f(b-\lambda)}{k(b-\lambda)} \right] k(t)[h(t)-g(t)] \right) d\mu(t) \\ &\leq \int_{[a,b]} f(t)g(t)d\mu(t).\end{aligned} \tag{3.23}$$

Proof. Take $h \mapsto kh$, $g \mapsto kg$ and $f \mapsto f/k$ in Theorem 3.7. □

Theorem 3.10 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let f,g,h and k be measurable functions on $[a,b]$ such that f is nonnegative, $0 \le g \le h$ and f/k is nonincreasing.*

(a) If there exists $\lambda \in \mathbb{R}_+$ such that (3.16) *holds, then* (3.22) *holds.*

(b) If there exists $\lambda \in \mathbb{R}_+$ such that (3.17) *holds, then* (3.23) *holds.*

In the following theorem we extend Theorems 1.12 and 1.13 to Borel σ-algebra.

Theorem 3.11 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let f,g,h and ψ be measurable functions on $[a,b]$ such that $0 \le \psi \le g \le h-\psi$ and f is nonincreasing.*

(a) If there exists $\lambda \in \mathbb{R}_+$ such that (3.6) *holds, then*

$$\int_{[a,b]} f(t)g(t)d\mu(t) \le \int_{[a,a+\lambda]} f(t)h(t)d\mu(t) - \int_{[a,b]} |f(t)-f(a+\lambda)|\,\psi(t)d\mu(t). \tag{3.24}$$

(b) If there exists $\lambda \in \mathbb{R}_+$ such that (3.8) *holds, then*

$$\int_{(b-\lambda,b]} f(t)h(t)d\mu(t) + \int_{[a,b]} |f(t)-f(b-\lambda)|\,\psi(t)d\mu(t) \le \int_{[a,b]} f(t)g(t)d\mu(t). \tag{3.25}$$

Proof. Since f is nonincreasing on $[a,b]$ we have $f(t) \ge f(a+\lambda)$ for all $t \in [a,a+\lambda]$ and $f(t) \le f(a+\lambda)$ for all $t \in [a+\lambda,b]$. Now using identity (3.20) we get

$$\begin{aligned}
&\int_{[a,a+\lambda]} f(t)h(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t)\\
&= \int_{[a,a+\lambda]} [f(t)-f(a+\lambda)][h(t)-g(t)]d\mu(t) - \int_{(a+\lambda,b]} [f(t)-f(a+\lambda)]g(t)d\mu(t)\\
&\ge \int_{[a,a+\lambda]} |f(t)-f(a+\lambda)|\psi(t)d\mu(t) + \int_{(a+\lambda,b]} |f(a+\lambda)-f(t)|\psi(t)d\mu(t)\\
&= \int_{[a,b]} |f(t)-f(a+\lambda)|\psi(t)d\mu(t)
\end{aligned}$$

and the proof of the first statement is established.

The second statement can be proved in a similar manner using identity (3.21). □

Theorem 3.12 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let f,g,h and ψ be measurable functions on $[a,b]$ such that $0 \le \psi \le g \le h-\psi$ and f is nonnegative and nonincreasing.*

(a) If there exists $\lambda \in \mathbb{R}_+$ such that (3.10) *holds, then* (3.24) *holds.*

(b) If there exists $\lambda \in \mathbb{R}_+$ such that (3.11) *holds, then* (3.25) *holds.*

In the following theorems we obtain sharpening of results given in Theorems 3.5 and 3.6.

Theorem 3.13 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let f,g,h,k and ψ be measurable functions on $[a,b]$ such that k is positive, $0 \leq \psi \leq g \leq h-\psi$ and f/k is nonincreasing.*

(a) If there exists $\lambda \in \mathbb{R}_+$ such that (3.12) *holds, then*

$$\begin{aligned}\int_{[a,b]} f(t)g(t)d\mu(t) \leq & \int_{[a,a+\lambda]} f(t)h(t)d\mu(t) \\ & - \int_{[a,b]} \left| \left(\frac{f(t)}{k(t)} - \frac{f(a+\lambda)}{k(a+\lambda)} \right) \right| k(t)\psi(t)d\mu(t).\end{aligned} \tag{3.26}$$

(b) If there exists $\lambda \in \mathbb{R}_+$ such that (3.14) *holds, then*

$$\begin{aligned}\int_{(b-\lambda,b]} f(t)h(t)d\mu(t) + \int_{[a,b]} \left| \left(\frac{f(t)}{k(t)} - \frac{f(b-\lambda)}{k(b-\lambda)} \right) \right| k(t)\psi(t)d\mu(t) \\ \leq \int_{[a,b]} f(t)g(t)d\mu(t).\end{aligned} \tag{3.27}$$

Proof. Take $g \mapsto kg$, $f \mapsto f/k$, $h \mapsto kh$ and $\psi \mapsto k\psi$ in Theorem 3.11. □

Theorem 3.14 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let f,g,h,k and ψ be measurable functions on $[a,b]$ such that k is positive, $0 \leq \psi \leq g \leq h-\psi$, f is nonnegative and f/k is nonincreasing.*

(a) If there exists $\lambda \in \mathbb{R}_+$ such that (3.16) *holds, then* (3.26) *holds.*

(b) If there exists $\lambda \in \mathbb{R}_+$ such that (3.17) *holds, then* (3.27) *holds.*

Proof. Take $g \mapsto kg$, $f \mapsto f/k$, $h \mapsto kh$ and $\psi \mapsto k\psi$ in Theorem 3.12. □

3.1.1 Weaker conditions

Motivated by weaker conditions given in Theorem 1.3 in the following theorems we obtain weaker conditions for some generalizations and refinements given in the previous results.

The following theorem gives weaker conditions for Theorem 3.5 and more general version of analog theorem from [43].

Theorem 3.15 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let g,h and k be $\mu-$integrable functions on $[a,b]$ such that k is positive and h is nonnegative.*

(a) Let λ be a positive constant such that $\int_{[a,a+\lambda]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t)$. The inequality

$$\int_{[a,b]} f(t)g(t)d\mu(t) \leq \int_{[a,a+\lambda]} f(t)h(t)d\mu(t) \tag{3.28}$$

holds for every nonincreasing, right-continuous function $f/k : [a,b] \to \mathbb{R}$ if and only if

$$\int_{[a,x)} k(t)g(t)d\mu(t) \leq \int_{[a,x)} k(t)h(t)d\mu(t) \quad \text{and} \quad \int_{[x,b]} k(t)g(t)d\mu(t) \geq 0, \tag{3.29}$$

for every $x \in [a,b]$.

(b) Let λ be a positive constant such that $\int_{(b-\lambda,b]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t)$. The inequality

$$\int_{(b-\lambda,b]} f(t)h(t)d\mu(t) \leq \int_{[a,b]} f(t)g(t)d\mu(t) \tag{3.30}$$

holds for every nonincreasing, right-continuous function $f/k : [a,b] \to \mathbb{R}$ if and only if

$$\int_{[x,b]} k(t)g(t)d\mu(t) \leq \int_{[x,b]} k(t)h(t)d\mu(t) \quad \text{and} \quad \int_{[a,x)} k(t)g(t)d\mu(t) \geq 0,$$

for every $x \in [a,b]$.

Proof.
(a) For the sufficiency part we use the identity

$$\begin{aligned}
&\int_{[a,a+\lambda]} f(t)h(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) \\
&= \int_{[a,a+\lambda]} \left[\frac{f(t)}{k(t)} - \frac{f(a+\lambda)}{k(a+\lambda)}\right] k(t)[h(t)-g(t)]d\mu(t) \\
&+ \int_{(a+\lambda,b]} \left[\frac{f(a+\lambda)}{k(a+\lambda)} - \frac{f(t)}{k(t)}\right] k(t)g(t)d\mu(t).
\end{aligned} \tag{3.31}$$

We define a new measure ν on $\sigma-$algebra $\mathscr{B}((a,b])$ such that, on an algebra of finite disjoint unions of half open intervals, we set

$$\nu((c,d]) = \frac{f(c)}{k(c)} - \frac{f(d)}{k(d)}, \quad \text{for } a < c < d \leq b,$$

and then we pass to $\mathscr{B}((a,b])$ in a unique way (for details see, for example, [13, p. 21]).

Now, using Fubini, we have

$$\begin{aligned}
&\int_{[a,a+\lambda]} \left[\frac{f(t)}{k(t)} - \frac{f(a+\lambda)}{k(a+\lambda)}\right] k(t)[h(t)-g(t)]d\mu(t) \\
&= \int_{[a,a+\lambda]} \left[\int_{(t,a+\lambda]} d\nu(x)\right] k(t)[h(t)-g(t)]d\mu(t) \\
&= \int_{(a,a+\lambda]} \left[\int_{[a,x)} k(t)[h(t)-g(t)]d\mu(t)\right] d\nu(x).
\end{aligned} \tag{3.32}$$

Similarly,

$$\int_{(a+\lambda,b]}\left[\frac{f(a+\lambda)}{k(a+\lambda)}-\frac{f(t)}{k(t)}\right]k(t)g(t)\mu(t)=\int_{(a+\lambda,b]}\left[\int_{[x,b]}k(t)g(t)d\mu(t)\right]d\nu(x). \quad (3.33)$$

Now using (3.32) and (3.33) we have that (3.31) is in fact

$$\begin{aligned}&\int_{[a,a+\lambda]}f(t)h(t)d\mu(t)-\int_{[a,b]}f(t)g(t)d\mu(t)\\&=\int_{(a,a+\lambda]}\left[\int_{[a,x)}k(t)(h(t)-g(t))d\mu(t)\right]d\nu(x)+\int_{(a+\lambda,b]}\left[\int_{[x,b]}k(t)g(t)d\mu(t)\right]d\nu(x),\end{aligned}$$

concluding (3.28) under assumptions (3.29).

The previous conditions are also necessary. In fact, if x is any element of $[a,b]$, then let f be the function defined by

$$f(t)=\begin{cases}k(t), & t<x;\\ 0, & t\geq x.\end{cases}$$

We have that f/k is a nonincreasing function. Using inequality (3.28) we obtain

$$\begin{aligned}\int_{[a,x)}k(t)g(t)d\mu(t)&=\int_{[a,b]}f(t)g(t)d\mu(t)\leq\int_{[a,a+\lambda]}f(t)h(t)d\mu(t)\\&=\begin{cases}\int_{[a,x)}k(t)h(t)d\mu(t), & x\in[a,a+\lambda];\\ \int_{[a,a+\lambda]}k(t)h(t)d\mu(t), & x\in(a+\lambda,b].\end{cases}\end{aligned} \quad (3.34)$$

If $x\in(a+\lambda,b]$ then $\int_{[a,x)}k(t)h(t)d\mu(t)\geq\int_{[a,a+\lambda]}k(t)h(t)d\mu(t)$, so from (3.34), we have

$$\int_{[a,x)}k(t)g(t)d\mu(t)\leq\int_{[a,x)}k(t)h(t)d\mu(t),\quad \text{for every } x\in[a,b].$$

Also, if $x\in(a+\lambda,b]$, from (3.34) and definition of λ we have $\int_{[a,x)}k(t)g(t)d\mu(t)\leq\int_{[a,a+\lambda]}k(t)h(t)d\mu(t)=\int_{[a,b]}k(t)g(t)d\mu(t)$, concluding

$$\int_{[x,b]}k(t)g(t)d\mu(t)\geq 0,\quad \text{for every } x\in(a+\lambda,b].$$

Finally, if $x\in[a,a+\lambda]$, then

$$\begin{aligned}&\int_{[x,b]}k(t)g(t)d\mu(t)=\int_{[a,b]}k(t)g(t)d\mu(t)-\int_{[a,x)}k(t)g(t)d\mu(t)\\&\geq\int_{[a,a+\lambda]}k(t)h(t)d\mu(t)-\int_{[a,x)}k(t)h(t)d\mu(t)=\int_{[x,a+\lambda]}k(t)h(t)d\mu(t)\geq 0,\end{aligned}$$

concluding

$$\int_{[x,b]}k(t)g(t)d\mu(t)\geq 0,\quad \text{for every } x\in[a,b].$$

(b) The proof of this part is similar to the proof of (a)-part so we omit the details. □

Taking $h\equiv 1$ in Theorem 3.15 we obtain weaker conditions for Theorem 3.1.

In the following theorem we give weaker conditions for Theorem 3.9.

Theorem 3.16 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let g,h and k be $\mu-$integrable functions on $[a,b]$ such that k is positive and h is nonnegative.*

(a) Let λ be a positive constant such that $\int_{[a,a+\lambda]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t)$. If conditions (3.29) hold for every $x \in [a,b]$, then (3.22) holds for every nonincreasing, right-continuous function $f/k : [a,b] \to \mathbb{R}$.

(b) Let λ be a positive constant such that $\int_{(b-\lambda,b]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t)$. If conditions (3.30) hold for every $x \in [a,b]$, then (3.23) holds for every nonincreasing, right-continuous function $f/k : [a,b] \to \mathbb{R}$.

Proof. Let us prove (a)-part. Using identity (3.31) and a measure ν on $\sigma-$algebra $\mathscr{B}((a,b])$ as in the proof of Theorem 3.15 we have

$$\begin{aligned}
&\int_{[a,a+\lambda]} f(t)h(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) \\
&- \int_{[a,a+\lambda]} \left[\frac{f(t)}{k(t)} - \frac{f(a+\lambda)}{k(a+\lambda)}\right] k(t)[h(t)-g(t)]d\mu(t) \\
&= \int_{(a+\lambda,b]} \left[\frac{f(a+\lambda)}{k(a+\lambda)} - \frac{f(t)}{k(t)}\right] k(t)g(t)d\mu(t) = \int_{(a+\lambda,b]} \left[\int_{[x,b]} k(t)g(t)d\mu(t)\right] d\nu(x).
\end{aligned}$$

From here we conclude that the left-hand side inequality in (3.22) holds when conditions (3.29) hold.

Further, we have

$$\begin{aligned}
&\int_{[a,a+\lambda]} \left[\frac{f(t)}{k(t)} - \frac{f(a+\lambda)}{k(a+\lambda)}\right] k(t)[h(t)-g(t)]d\mu(t) \\
&\qquad = \int_{(a,a+\lambda]} \left[\int_{[a,x)} k(t)[h(t)-g(t)]d\mu(t)\right] d\nu(x) \geq 0
\end{aligned}$$

if the first condition in (3.29) is satisfied. Hence, the right-hand side inequality in (3.22) holds.

Proof of (b)-part is similar so we omit the details. □

Theorem 3.17 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let g,h and k be $\mu-$integrable functions on $[a,b]$ such that k is positive, h is nonnegative and f/k is nonincreasing, right-continuous function.*

(a) Let λ be a positive constant such that $\int_{[a,a+\lambda]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t)$. If

$$\int_{[x,b]} k(t)g(t)d\mu(t) \geq 0, \quad \text{for } x \in (a+\lambda,b],$$

then

$$\begin{aligned}
\int_{[a,b]} f(t)g(t)d\mu(t) \leq &\int_{[a,a+\lambda]} f(t)h(t)d\mu(t) \\
&- \int_{[a,a+\lambda]} \left[\frac{f(t)}{k(t)} - \frac{f(a+\lambda)}{k(a+\lambda)}\right] k(t)[h(t)-g(t)]d\mu(t).
\end{aligned}$$

If we additionally have

$$\int_{[a,x)} k(t)g(t)d\mu(t) \leq \int_{[a,x)} k(t)h(t)d\mu(t), \quad \textit{for } x \in [a, a+\lambda],$$

then (3.22) *holds.*

(b) Let λ be a positive constant such that $\int_{(b-\lambda,b]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t)$. If

$$\int_{[a,x)} k(t)g(t)d\mu(t) \geq 0, \quad \textit{for } x \in [a, b-\lambda],$$

then

$$\int_{(b-\lambda,b]} f(t)h(t)d\mu(t) - \int_{(b-\lambda,b]} \left[\frac{f(t)}{k(t)} - \frac{f(b-\lambda)}{k(b-\lambda)}\right] k(t)[h(t)-g(t)]d\mu(t) \leq \int_{[a,b]} f(t)g(t)d\mu(t).$$

If we additionally have

$$\int_{[x,b]} k(t)g(t)d\mu(t) \leq \int_{[x,b]} k(t)h(t)d\mu(t), \quad \textit{for } x \in (b-\lambda, b],$$

then (3.23) *holds.*

Proof. Similar to the proof of Theorem 3.16. □

3.2 On some bounds for the parameter λ in Steffensen's inequality

Results given in this section were obtained by Pečarić and Smoljak Kalamir in [79].

In the following theorems we obtain weaker condition for the parameter λ in generalizations of Steffensen's inequality given in Theorems 1.2 and 1.3.

Theorem 3.18 *Let h be a positive integrable function on $[a,b]$ and f be a nonnegative integrable function such that f/h is nonincreasing on $[a,b]$. Let g be an integrable function on $[a,b]$ with $0 \leq g \leq 1$. Then*

$$\int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)dt \tag{3.35}$$

holds, where λ is given by

$$\int_a^{a+\lambda} h(t)dt \geq \int_a^b h(t)g(t)dt. \tag{3.36}$$

If f/h is nondecreasing, then the reverse inequality in (3.35) *holds, where λ is given by* (3.36) *with the reverse inequality.*

Proof. Since f/h is nonincreasing transformation of the difference between the right-hand side and the left-hand side of inequality (3.35) gives

$$\begin{aligned}
&\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt = \int_a^{a+\lambda}(1-g(t))f(t)dt - \int_{a+\lambda}^b f(t)g(t)dt \\
&\geq \frac{f(a+\lambda)}{h(a+\lambda)}\int_a^{a+\lambda} h(t)(1-g(t))dt - \int_{a+\lambda}^b f(t)g(t)dt \\
&\geq \frac{f(a+\lambda)}{h(a+\lambda)}\left(\int_a^b h(t)g(t)dt - \int_a^{a+\lambda} h(t)g(t)dt\right) - \int_{a+\lambda}^b f(t)g(t)dt \\
&= \int_{a+\lambda}^b g(t)h(t)\left(\frac{f(a+\lambda)}{h(a+\lambda)} - \frac{f(t)}{h(t)}\right)dt \geq 0,
\end{aligned}$$

where we use (3.36) and nonnegativity of function f. □

Theorem 3.19 *Let h be a positive integrable function on $[a,b]$ and f be a nonnegative integrable function such that f/h is nonincreasing on $[a,b]$. Let g be an integrable function on $[a,b]$ with $0 \leq g \leq 1$. Then*

$$\int_a^b f(t)g(t)dt \geq \int_{b-\lambda}^b f(t)dt \tag{3.37}$$

holds, where λ is given by

$$\int_{b-\lambda}^b h(t)dt \leq \int_a^b h(t)g(t)dt. \tag{3.38}$$

If f/h is nondecreasing, then the reverse inequality in (3.37) *holds, where λ is given by* (3.38) *with the reverse inequality.*

Proof. Similar to the proof of Theorem 2.1. □

Taking $h \equiv 1$ in Theorems 3.18 and 3.19 we obtain the following weaker conditions for the parameter λ in Steffensen's inequality.

Corollary 3.1 *Let f be a nonnegative nonincreasing function on $[a,b]$ and g be an integrable function on $[a,b]$ with $0 \leq g \leq 1$. Then*

$$\int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)dt \tag{3.39}$$

holds, where

$$\lambda \geq \int_a^b g(t)dt. \tag{3.40}$$

If f is nondecreasing, then the reverse inequality in (3.39) *holds, where λ is given by* (3.40) *with the reverse inequality.*

Corollary 3.2 *Let f be a nonnegative nonincreasing function on $[a,b]$ and g be an integrable function on $[a,b]$ with $0 \leq g \leq 1$. Then*

$$\int_a^b f(t)g(t)dt \geq \int_{b-\lambda}^b f(t)dt \tag{3.41}$$

holds, where

$$\lambda \le \int_a^b g(t)dt. \tag{3.42}$$

If f is nondecreasing, then the reverse inequality in (3.41) *holds, where λ is given by* (3.42) *with the reverse inequality.*

In order to obtain Bellman-type inequality we need the following generalization of result given in Theorem 3.18.

Theorem 3.20 *Let h be a positive integrable function on $[a,b]$ and f be a nonnegative integrable function such that f/h is nonincreasing on $[a,b]$. Let g be an integrable function on $[a,b]$ with $0 \le g \le 1$. If $p \ge 1$ then*

$$\int_a^b f^p(t)g(t)dt \le \int_a^{a+\lambda} f^p(t)dt \tag{3.43}$$

holds, where λ is given by

$$\int_a^{a+\lambda} h^p(t)dt \ge \int_a^b h^p(t)g(t)dt. \tag{3.44}$$

If f/h is nondecreasing, then the reverse inequality in (3.43) *holds, where λ is given by* (3.44) *with the reverse inequality.*

Proof. Since f/h is nonincreasing, we have that f^p/h^p is nonincreasing. Hence, we can apply Theorem 3.18 to the function f^p/h^p. □

Similarly, we have the following generalization of result given in Theorem 3.19.

Theorem 3.21 *Let h be a positive integrable function on $[a,b]$ and f be a nonnegative integrable function such that f/h is nonincreasing on $[a,b]$. Let g be an integrable function on $[a,b]$ with $0 \le g \le 1$. If $p \ge 1$ then*

$$\int_a^b f^p(t)g(t)dt \ge \int_{b-\lambda}^b f^p(t)dt \tag{3.45}$$

holds, where λ is given by

$$\int_{b-\lambda}^b h^p(t)dt \ge \int_a^b h^p(t)g(t)dt. \tag{3.46}$$

If f/h is nondecreasing, then the reverse inequality in (3.45) *holds, where λ is given by* (3.46) *with the reverse inequality.*

Proof. Applying Theorem 3.19 to the function f^p/h^p. □

We continue with the following Bellman-type inequality which allows us to obtain better estimation for the parameter λ in Pachpatte's result given in Theorem 1.15.

Theorem 3.22 *Let h be a positive integrable function on $[a,b]$ and f be a nonnegative integrable function such that f/h is nonincreasing on $[a,b]$. Let g be an integrable function on $[a,b]$ with $0 \le g \le 1$. If $p \ge 1$ then*

$$\frac{1}{(b-a)^{p-1}}\left(\int_a^b f(t)g(t)dt\right)^p \le \int_a^{a+\lambda} f^p(t)dt \tag{3.47}$$

holds, where λ is given by (3.44).

Proof. Using the Jensen inequality for convex function $\Phi(x) = x^p$ $(p \ge 1)$, we have

$$\left(\int_a^b f(t)g(t)dt\right)^p \le \left(\int_a^b g(t)dt\right)^{p-1}\int_a^b f^p(t)g(t)dt. \tag{3.48}$$

Since $0 \le g \le 1$ we have

$$\left(\int_a^b g(t)dt\right)^{p-1}\int_a^b f^p(t)g(t)dt \le (b-a)^{p-1}\int_a^b f^p(t)g(t)dt. \tag{3.49}$$

Combining (3.48) and (3.49), and using (3.43) we obtain

$$\frac{1}{(b-a)^{p-1}}\left(\int_a^b f(t)g(t)dt\right)^p \le \int_a^b f^p(t)g(t)dt \le \int_a^{a+\lambda} f^p(t)dt.$$

□

Taking $[a,b] = [0,1]$ in Theorem 3.22 we obtain the following corollary.

Corollary 3.3 *Let h be a positive integrable function on $[0,1]$ and f be a nonnegative integrable function such that f/h is nonincreasing on $[0,1]$. Let g be an integrable function on $[0,1]$ with $0 \le g \le 1$. If $p \ge 1$ then*

$$\left(\int_0^1 f(t)g(t)dt\right)^p \le \int_0^\lambda f^p(t)dt \tag{3.50}$$

holds, where λ is given by

$$\int_0^\lambda h^p(t)dt \ge \int_0^1 h^p(t)g(t)dt. \tag{3.51}$$

Taking $A = 1$ in Theorem 1.15 we obtain the following corollary.

Corollary 3.4 *Let f,g,h be real-valued integrable functions defined on $[0,1]$ such that $f(t) \ge 0$, $h(t) \ge 0$, $t \in [0,1]$, f/h is nonincreasing on $[0,1]$ and $0 \le g(t) \le 1$, $t \in [0,1]$. If $p \ge 1$, then* (3.50) *holds, where λ is the solution of the equation*

$$\int_0^\lambda h^p(t)dt = \left(\int_0^1 h^p(t)g(t)dt\right)\left(\int_0^1 g(t)dt\right)^{p-1}.$$

Since $0 \le g \le 1$ from (3.51) we have the following

$$\int_0^\lambda h^p(t)dt \ge \int_0^1 h^p(t)g(t)dt \ge \left(\int_0^1 g(t)dt\right)^{p-1}\int_0^1 h^p(t)g(t)dt. \tag{3.52}$$

Hence, the estimation for λ in Corollary 3.3 is better then the one in Pachpatte's result for the case $A = 1$ given in Corollary 3.4.

3.2.1 Weaker conditions

In the following theorem we obtain weaker conditions for the parameter λ in the corrected version of Mercer's results which follows from Theorems 3.18 and 3.19.

Theorem 3.23 *Let h be a positive integrable function on $[a,b]$ and f be a nonnegative nonincreasing function on $[a,b]$. Let g be an integrable function on $[a,b]$ with $0 \leq g \leq h$.*

a) Then

$$\int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)h(t)dt$$

holds, where λ is given by

$$\int_a^{a+\lambda} h(t)dt \geq \int_a^b g(t)dt.$$

b) Then

$$\int_{b-\lambda}^b f(t)h(t)dt \leq \int_a^b f(t)g(t)dt,$$

where λ is given by

$$\int_{b-\lambda}^b h(t)dt \leq \int_a^b g(t)dt.$$

Proof. Using substitutions $g \mapsto g/h$ and $f \mapsto fh$ in Theorems 3.18 and 3.19 we obtain the statements of this theorem. □

In the following theorem we relax condition (1.15) for the Mercer's result (Theorem 1.10) and the corresponding condition for result equivalent to Theorem 1.7.

Theorem 3.24 *Let k be a positive integrable function on $[a,b]$ and f be a nonnegative integrable function such that f/k is nonincreasing on $[a,b]$. Let g,h be integrable functions on $[a,b]$ with $0 \leq g \leq h$.*

a) Then

$$\int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)h(t)dt$$

holds, where λ is given by

$$\int_a^{a+\lambda} h(t)k(t)dt \geq \int_a^b g(t)k(t)dt.$$

b) Then

$$\int_{b-\lambda}^b f(t)h(t)dt \leq \int_a^b f(t)g(t)dt,$$

where λ is given by

$$\int_{b-\lambda}^b h(t)k(t)dt \leq \int_a^b g(t)k(t)dt.$$

Proof. Using substitutions $h \mapsto kh$, $g \mapsto g/h$ and $f \mapsto fh$ in Theorems 3.18 and 3.19 we obtain the statements of this theorem. □

In the following theorem we relax condition (1.16) by separating the above result into two parts and assuming nonnegativity of the function f.

Theorem 3.25 *Let h be a positive integrable function on $[a,b]$ and f be a nonnegative nonincreasing function on $[a,b]$. Let g be an integrable function on $[a,b]$ with $0 \leq g \leq h$.*

a) Then

$$\int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} (f(t)h(t) - [f(t) - f(a+\lambda)][h(t) - g(t)])dt$$
$$\leq \int_a^{a+\lambda} f(t)h(t)dt$$

holds, where λ is given by

$$\int_a^{a+\lambda} h(t)dt \geq \int_a^b g(t)dt.$$

b) Then

$$\int_{b-\lambda}^b f(t)h(t)dt \leq \int_{b-\lambda}^b (f(t)h(t) - [f(t) - f(b-\lambda)][h(t) - g(t)])dt$$
$$\leq \int_a^b f(t)g(t)dt$$

holds, where λ is given by

$$\int_{b-\lambda}^b h(t)dt \leq \int_a^b g(t)dt.$$

Proof. The proof is based on inequalities (3.20) and (3.21). Let us prove the first one. Transformation of the right-hand side of (3.20) gives the following

$$\int_a^{a+\lambda} (f(t)h(t) - [f(t) - f(a+\lambda)][h(t) - g(t)])dt + \int_{a+\lambda}^b [f(t) - f(a+\lambda)]g(t)dt$$
$$= \int_a^{a+\lambda} f(t)g(t)dt + f(a+\lambda)\int_a^{a+\lambda} (h(t) - g(t))dt + \int_{a+\lambda}^b f(t)g(t)dt - f(a+\lambda)\int_{a+\lambda}^b g(t)dt$$
$$= \int_a^b f(t)g(t)dt + f(a+\lambda)\left[\int_a^{a+\lambda} (h(t) - g(t))dt - \int_{a+\lambda}^b g(t)dt\right]$$
$$= \int_a^b f(t)g(t)dt + f(a+\lambda)\left[\int_a^{a+\lambda} h(t)dt - \int_a^b g(t)dt\right]$$
$$\geq \int_a^b f(t)g(t)dt + f(a+\lambda)\left[\int_a^b g(t)dt - \int_a^b g(t)dt\right]$$
$$= \int_a^b f(t)g(t)dt$$

where in the inequality we use nonnegativity of f and a definition of λ, i.e. $\int_a^{a+\lambda} h(t)dt \geq \int_a^b g(t)dt$.

Inequality (3.21) can be proved in a similar manner.

Since f is nonincreasing on $[a,b]$ we get $f(t) \geq f(a+\lambda)$ for all $t \in [a,a+\lambda]$ and $f(t) \leq f(a+\lambda)$ for all $t \in [a+\lambda,b]$. Then

$$\int_{a+\lambda}^{b} [f(t) - f(a+\lambda)]g(t)dt \leq 0$$

and

$$\int_{a}^{a+\lambda} [f(t) - f(a+\lambda)][h(t) - g(t)]dt \geq 0.$$

Using (3.20) and above inequalities we obtain

$$\begin{aligned}\int_a^b f(t)g(t)dt &\leq \int_a^{a+\lambda} (f(t)h(t) - [f(t) - f(a+\lambda)][h(t) - g(t)])dt \\ &\leq \int_a^{a+\lambda} f(t)h(t)dt.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\int_a^b f(t)g(t)dt &\geq \int_{b-\lambda}^{b} (f(t)h(t) - [f(t) - f(b-\lambda)][h(t) - g(t)])dt \\ &\geq \int_{b-\lambda}^{b} f(t)h(t)dt.\end{aligned}$$

□

If we additionally assume that f is nonnegative in Theorems 1.12 and 1.13 we obtain the following weaker conditions for the parameter λ.

Theorem 3.26 *Let f, g, h and ψ be integrable functions on $[a,b]$ with f nonnegative nonincreasing and let $0 \leq \psi \leq g \leq h - \psi$.*

a) Then

$$\int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)h(t)dt - \int_a^b |f(t) - f(a+\lambda)|\,\psi(t)dt$$

holds, where λ is given by

$$\int_a^{a+\lambda} h(t)dt \geq \int_a^b g(t)dt.$$

b) Then

$$\int_{b-\lambda}^{b} f(t)h(t)dt + \int_a^b |f(t) - f(b-\lambda)|\psi(t)dt \leq \int_a^b f(t)g(t)dt,$$

where λ is given by

$$\int_{b-\lambda}^{b} h(t)dt \leq \int_a^b g(t)dt.$$

Proof. Since f is nonincreasing on $[a,b]$ we get $f(t) \geq f(a+\lambda)$ for all $t \in [a,a+\lambda]$ and $f(t) \leq f(a+\lambda)$ for all $t \in [a+\lambda,b]$. Hence, using inequality (3.20) and the fact that $0 \leq \psi \leq g \leq h-\psi$ we get

$$\begin{aligned}
&\int_a^{a+\lambda} f(t)h(t)dt - \int_a^b f(t)g(t)dt \\
&\quad \geq \int_a^{a+\lambda} [f(t)-f(a+\lambda)][h(t)-g(t)]dt - \int_{a+\lambda}^b [f(t)-f(a+\lambda)]g(t)dt \\
&\quad = \int_a^{a+\lambda} |f(t)-f(a+\lambda)|[h(t)-g(t)]dt + \int_{a+\lambda}^b |f(t)-f(a+\lambda)|g(t)dt \\
&\quad \geq \int_a^{a+\lambda} |f(t)-f(a+\lambda)|\psi(t)dt + \int_{a+\lambda}^b |f(a+\lambda)-f(t)|\psi(t)dt \\
&\quad = \int_a^b |f(t)-f(a+\lambda)|\psi(t)dt
\end{aligned}$$

and the proof is established. □

3.3 Extension of Cerone's generalizations of Steffensen's inequality

Results given in this section were obtained by Jakšetić, Pečarić and Smoljak Kalamir in [36].

We begin with an extension of Cerone's result given in Theorem 1.14 to positive finite measures.

Theorem 3.27 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let f be nonincreasing and g be measurable function on $[a,b]$ such that $0 \leq g \leq 1$. Let $[c,d] \subseteq [a,b]$ and*

$$\mu([c,d]) = \int_{[a,b]} g(t)d\mu(t). \tag{3.53}$$

Then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \leq \int_{[c,d]} f(t)d\mu(t) + R_\mu(c,d) \tag{3.54}$$

holds, where

$$R_\mu(c,d) = \int_{[a,c)} (f(t)-f(d))g(t)d\mu(t) \geq 0.$$

Proof. Let us consider a corresponding difference:

$$
\begin{aligned}
&\int_{[c,d]} f(t)d\mu(t) + R_\mu(c,d) - \int_{[a,b]} f(t)g(t)d\mu(t) \\
&= \int_{[c,d]} f(t)d\mu(t) + \int_{[a,c)} (f(t)-f(d))g(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) \\
&= \int_{[c,d]} f(t)d\mu(t) - \int_{[c,b]} f(t)g(t)d\mu(t) - f(d)\int_{[a,b]} g(t)d\mu(t) + f(d)\int_{[c,b]} g(t)d\mu(t) \\
&= \int_{[c,d]} f(t)d\mu(t) - \int_{[c,b]} f(t)g(t)d\mu(t) - f(d)\int_{[c,d]} d\mu(t) + f(d)\int_{[c,b]} g(t)d\mu(t) \\
&= \int_{[c,d]} (f(t)-f(d))(1-g(t))d\mu(t) + \int_{(d,b]} (f(d)-f(t))g(t)d\mu(t)
\end{aligned}
\tag{3.55}
$$

where we used (3.53).

Since $0 \le g \le 1$, f is nonincreasing and μ is positive, terms under the integral sign are nonnegative, hence the first sum in this chain is nonnegative, i.e.

$$\int_{[c,d]} f(t)d\mu(t) + R_\mu(c,d) \ge \int_{[a,b]} f(t)g(t)d\mu(t).$$

□

Theorem 3.28 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let f be nonincreasing and g be measurable function on $[a,b]$ such that $0 \le g \le 1$. Let $[c,d] \subseteq [a,b]$ and*

$$\mu((c,d]) = \int_{[a,b]} g(t)d\mu(t). \tag{3.56}$$

Then

$$\int_{(c,d]} f(t)d\mu(t) - r_\mu(c,d) \le \int_{[a,b]} f(t)g(t)d\mu(t) \tag{3.57}$$

holds, where

$$r_\mu(c,d) = \int_{(d,b]} (f(c)-f(t))g(t)d\mu(t) \ge 0.$$

Proof. Similar to the proof of Theorem 3.27 we obtain the corresponding difference

$$
\begin{aligned}
&\int_{[a,b]} f(t)g(t)d\mu(t) - \int_{(c,d]} f(t)d\mu(t) + r_\mu(c,d) \\
&\quad = \int_{(c,d]} (f(c)-f(t))(1-g(t))d\mu(t) + \int_{[a,c]} (f(t)-f(c))g(t)d\mu(t) \ge 0.
\end{aligned}
$$

□

Motivated by Pachpatte's result from [53] under the additional assumption on the function f we can replace conditions (3.53) and (3.56) by weaker conditions given in the following theorems.

Theorem 3.29 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let f be nonincreasing and nonnegative, and g be measurable function on $[a,b]$ such that $0 \leq g \leq 1$. Let $[c,d] \subseteq [a,b]$ and*

$$\mu([c,d]) \geq \int_{[a,b]} g(t)d\mu(t). \tag{3.58}$$

Then (3.54) *holds.*

Proof. Since $f(d) \geq 0$ using (3.58) in (3.55) we obtain the claim of this theorem. □

Theorem 3.30 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let f be nonincreasing and nonnegative, and g be measurable function on $[a,b]$ such that $0 \leq g \leq 1$. Let $[c,d] \subseteq [a,b]$ and*

$$\mu((c,d]) \leq \int_{[a,b]} g(t)d\mu(t).$$

Then (3.57) *holds.*

Proof. Similar to the proof of Theorem 3.29. □

Taking $c = a$ and $d = a + \lambda$ in Theorems 3.27 and 3.29 or taking $c = b - \lambda$ and $d = b$ in Theorems 3.28 and 3.30 we obtain results given in Section 2.1. Further, if we take $c = a$, $d = a + \lambda$ and consider the Lebesgue measure in Theorem 3.27 or take $c = b - \lambda$, $d = b$ and consider the Lebesgue measure in Theorem 3.28 we obtain Cerone's result given in Theorem 1.14.

In the sequel we need the following lemmas to generalize the above results for the function f/k.

Lemma 3.1 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let f,g,h and k be measurable functions on $[a,b]$ such that k is positive. Further, let $[c,d] \subseteq [a,b]$ with $\int_{[c,d]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t)$. Then the following identity holds:*

$$\begin{aligned}\int_{[c,d]} f(t)h(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) &= \int_{[a,c)} \left(\frac{f(d)}{k(d)} - \frac{f(t)}{k(t)}\right) g(t)k(t)d\mu(t) \\ + \int_{[c,d]} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) k(t)[h(t) - g(t)]d\mu(t) &+ \int_{(d,b]} \left(\frac{f(d)}{k(d)} - \frac{f(t)}{k(t)}\right) g(t)k(t)d\mu(t).\end{aligned} \tag{3.59}$$

Proof. We have

$$\begin{aligned}&\int_{[c,d]} f(t)h(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) = \int_{[c,d]} \frac{f(t)}{k(t)} k(t)[h(t) - g(t)]d\mu(t) \\ &- \int_{[a,c)} \frac{f(t)}{k(t)} g(t)k(t)d\mu(t) - \int_{(d,b]} \frac{f(t)}{k(t)} g(t)k(t)d\mu(t) \\ &= \int_{[a,c)} \left(\frac{f(d)}{k(d)} - \frac{f(t)}{k(t)}\right) g(t)k(t)d\mu(t) + \int_{(d,b]} \left(\frac{f(d)}{k(d)} - \frac{f(t)}{k(t)}\right) g(t)k(t)d\mu(t) \\ &+ \int_{[c,d]} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) k(t)[h(t) - g(t)]d\mu(t) + \frac{f(d)}{k(d)} \left[\int_{[c,d]} k(t)h(t)d\mu(t)\right. \\ &\left. - \int_{[a,c)} g(t)k(t)dt - \int_{[c,d]} k(t)g(t)d\mu(t) - \int_{(d,b]} g(t)k(t)d\mu(t)\right].\end{aligned} \tag{3.60}$$

Since

$$\int_{[c,d]} k(t)h(t)d\mu(t) = \int_{[a,b]} k(t)g(t)d\mu(t),$$

we have

$$\int_{[c,d]} k(t)h(t)d\mu(t) - \int_{[a,c)} g(t)k(t)d\mu(t) - \int_{[c,d]} k(t)g(t)d\mu(t) - \int_{(d,b]} g(t)k(t)d\mu(t) = 0.$$

Hence, (3.59) follows from (3.60). □

Lemma 3.2 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let f,g,h and k be measurable functions on $[a,b]$ such that k is positive. Further, let $[c,d] \subseteq [a,b]$ with $\int_{(c,d]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t)$. Then the following identity holds:*

$$\begin{aligned}\int_{[a,b]} f(t)g(t)d\mu(t) - \int_{(c,d]} f(t)h(t)d\mu(t) &= \int_{[a,c]} \left(\frac{f(t)}{k(t)} - \frac{f(c)}{k(c)}\right) g(t)k(t)d\mu(t) \\ + \int_{(c,d]} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right) k(t)[h(t)-g(t)]d\mu(t) &+ \int_{(d,b]} \left(\frac{f(t)}{k(t)} - \frac{f(c)}{k(c)}\right) g(t)k(t)d\mu(t).\end{aligned} \tag{3.61}$$

Proof. Similar to the proof of Lemma 3.1. □

Theorem 3.31 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let f,g,h and k be measurable functions on $[a,b]$ such that k is positive, $0 \le g \le h$ and f/k is nonincreasing. Further, let $[c,d] \subseteq [a,b]$ with*

$$\int_{[c,d]} h(t)k(t)dt = \int_{[a,b]} g(t)k(t)dt. \tag{3.62}$$

Then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \le \int_{[c,d]} f(t)h(t)d\mu(t) + \mathfrak{R}_\mu(c,d) \tag{3.63}$$

holds, where

$$\mathfrak{R}_\mu(c,d) = \int_{[a,c)} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) g(t)k(t)d\mu(t) \ge 0. \tag{3.64}$$

Proof. Since f/k is nonincreasing, k and μ are positive and $0 \le g \le h$ we have

$$\int_{[c,d]} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) k(t)[h(t)-g(t)]d\mu(t) \ge 0, \tag{3.65}$$

$$\int_{(d,b]} \left(\frac{f(d)}{k(d)} - \frac{f(t)}{k(t)}\right) g(t)k(t)d\mu(t) \ge 0 \tag{3.66}$$

and $\mathfrak{R}_\mu(c,d) \geq 0$. Now, from (3.59), (3.65) and (3.66) we have

$$\int_{[c,d]} f(t)h(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) + \int_{[a,c)} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) g(t)k(t)d\mu(t)$$
$$= \int_{[c,d]} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) k(t)[h(t)-g(t)]d\mu(t) + \int_{(d,b]} \left(\frac{f(d)}{k(d)} - \frac{f(t)}{k(t)}\right) g(t)k(t)d\mu(t) \geq 0. \tag{3.67}$$

Hence, (3.63) holds. □

Theorem 3.32 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let f,g,h and k be measurable functions on $[a,b]$ such that k is positive, $0 \leq g \leq h$ and f/k is nonincreasing. Further, let $[c,d] \subseteq [a,b]$ with*

$$\int_{(c,d]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t). \tag{3.68}$$

Then

$$\int_{(c,d]} f(t)h(t)d\mu(t) - \mathfrak{r}_\mu(c,d) \leq \int_{[a,b]} f(t)g(t)d\mu(t) \tag{3.69}$$

holds, where

$$\mathfrak{r}_\mu(c,d) = \int_{(d,b]} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right) g(t)k(t)d\mu(t) \geq 0. \tag{3.70}$$

Proof. Since f/k is nonincreasing, k and μ are positive and $0 \leq g \leq h$ we have

$$\int_{[a,c]} \left(\frac{f(t)}{k(t)} - \frac{f(c)}{k(c)}\right) k(t)g(t)d\mu(t) \geq 0, \tag{3.71}$$

$$\int_{(c,d]} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right) k(t)[h(t)-g(t)]d\mu(t) \geq 0 \tag{3.72}$$

and $\mathfrak{r}_\mu(c,d) \geq 0$. Now, from (3.61), (3.71) and (3.72) we have

$$\int_{[a,b]} f(t)g(t)d\mu(t) - \int_{(c,d]} f(t)h(t)d\mu(t) + \int_{(d,b]} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right) g(t)k(t)d\mu(t)$$
$$= \int_{[a,c]} \left(\frac{f(t)}{k(t)} - \frac{f(c)}{k(c)}\right) g(t)k(t)d\mu(t) + \int_{(c,d]} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right) k(t)[h(t)-g(t)]d\mu(t) \geq 0. \tag{3.73}$$

Hence, (3.69) holds. □

If we additionally assume that the function f is nonnegative, conditions (3.62) and (3.68) can be replaced by weaker conditions

$$\int_{[c,d]} h(t)k(t)dt \geq \int_{[a,b]} g(t)k(t)dt \text{ and } \int_{(c,d]} h(t)k(t)d\mu(t) \leq \int_{[a,b]} g(t)k(t)d\mu(t).$$

If we take $c = a$, $d = a + \lambda$ and consider the Lebesgue measure in Theorem 3.31 we obtain Mercer's generalization of the right-hand Steffensen's inequality (see Theorem 1.10). If we take $c = b - \lambda$, $d = b$ and consider the Lebesgue measure in Theorem 3.32 we obtain a similar generalization of the left-hand Steffensen's inequality which is obtained in [61] from a generalization given by Pečarić in [56].

For $h \equiv 1$ and $k \equiv 1$ Theorems 3.31 and 3.32 reduce to Theorems 3.27 and 3.28.

Inequalities (3.63) and (3.69) can be refined using similar reasoning as in [61] and [93]. These refinements are given in the following theorems.

Theorem 3.33 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let f, g, h and k be measurable functions on $[a,b]$ such that k is positive, $0 \leq g \leq h$ and f/k is nonincreasing. Further, let $[c,d] \subseteq [a,b]$ with $\int_{[c,d]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t)$. Then*

$$\begin{aligned}
&\int_{[a,b]} f(t)g(t)d\mu(t) \\
&\leq \int_{[c,d]} f(t)h(t)d\mu(t) - \int_{[c,d]} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) k(t)[h(t) - g(t)]d\mu(t) + \mathfrak{R}_\mu(c,d) \\
&\leq \int_{[c,d]} f(t)h(t)d\mu(t) + \mathfrak{R}_\mu(c,d)
\end{aligned}$$

holds, where $\mathfrak{R}_\mu(c,d)$ is defined by (3.64).

Proof. Similar to the proof of Theorem 3.31. □

Theorem 3.34 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let f, g, h and k be measurable functions on $[a,b]$ such that k is positive, $0 \leq g \leq h$ and f/k is nonincreasing. Further, let $[c,d] \subseteq [a,b]$ with $\int_{(c,d]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t)$. Then*

$$\begin{aligned}
&\int_{(c,d]} f(t)h(t)d\mu(t) - \mathfrak{r}_\mu(c,d) \\
&\leq \int_{(c,d]} f(t)h(t)d\mu(t) + \int_{(c,d]} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right) k(t)[h(t) - g(t)]d\mu(t) - \mathfrak{r}_\mu(c,d) \\
&\leq \int_{[a,b]} f(t)g(t)d\mu(t)
\end{aligned}$$

holds, where $\mathfrak{r}_\mu(c,d)$ is defined by (3.70).

Proof. Similar to the proof of Theorem 3.32. □

If we take $c = a$, $d = a + \lambda$ and consider the Lebesgue measure in Theorem 3.33, or $c = b - \lambda$, $d = b$ and consider the Lebesgue measure in Theorem 3.34, we obtain generalizations of Wu and Srivastava refinement of Steffensen's inequality given in [61]. Additionally taking $k \equiv 1$ we obtain Wu and Srivastava refinement given in Theorem 1.11.

Furthermore, from Theorems 3.33 and 3.34 (taking $h \equiv 1$ and $k \equiv 1$) we can obtain a refinement of Theorems 3.27 and 3.28.

3.3.1 Weaker conditions

Here we give results obtained by replacing conditions on the function g in previous results with weaker conditions as in Theorem 1.3.

Theorem 3.35 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let f,g,h and k be μ-integrable functions on $[a,b]$ such that k is positive, h is nonnegative and f/k is nonincreasing and right-continuous. Further, let $[c,d] \subseteq [a,b]$ with $\int_{[c,d]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t)$. If*

$$\int_{[c,x)} k(t)g(t)d\mu(t) \leq \int_{[c,x)} k(t)h(t)d\mu(t), \quad c \leq x \leq d \tag{3.74}$$

and

$$\int_{[x,b]} k(t)g(t)d\mu(t) \geq 0, \quad d < x \leq b, \tag{3.75}$$

then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \leq \int_{[c,d]} f(t)h(t)d\mu(t) + \int_{[a,c)} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) g(t)k(t)d\mu(t). \tag{3.76}$$

Proof. We use identity (3.67) and define a new measure ν on σ-algebra $\mathscr{B}((a,b])$ such that, on an algebra of finite disjoint unions of half open intervals, we set

$$\nu((x,y]) = \frac{f(x)}{k(x)} - \frac{f(y)}{k(y)}, \qquad \text{for } a < x < y \leq b,$$

and then we pass to $\mathscr{B}((a,b])$ in a unique way (for details see [13, p. 21]). Hence, using Fubini, we have

$$\begin{aligned}
&\int_{[c,d]} f(t)h(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) + \int_{[a,c)} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) g(t)k(t)d\mu(t) \\
&= \int_{[c,d]} \left(\int_{(t,d]} d\nu(x)\right) k(t)[h(t)-g(t)]d\mu(t) + \int_{(d,b]} \left(\int_{(d,t]} d\nu(x)\right) g(t)k(t)d\mu(t) \\
&= \int_{[c,d]} \left(\int_{[c,x)} k(t)[h(t)-g(t)]d\mu(t)\right) d\nu(x) + \int_{(d,b]} \left(\int_{[x,b]} g(t)k(t)d\mu(t)\right) d\nu(x).
\end{aligned}$$

So we have (3.76) when (3.74) and (3.75) hold. □

Theorem 3.36 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let f,g,h and k be μ-integrable functions on $[a,b]$ such that k is positive, h is nonnegative and f/k is nonincreasing and right-continuous. Further, let $[c,d] \subseteq [a,b]$ with $\int_{(c,d]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t)$. If*

$$\int_{[x,d]} k(t)g(t)d\mu(t) \leq \int_{[x,d]} k(t)h(t)d\mu(t), \quad c < x \leq d \tag{3.77}$$

and

$$\int_{[a,x)} k(t)g(t)d\mu(t) \geq 0, \quad a \leq x \leq c, \tag{3.78}$$

then

$$\int_{(c,d]} f(t)h(t)d\mu(t) - \int_{(d,b]} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right) g(t)k(t)d\mu(t) \leq \int_{[a,b]} f(t)g(t)d\mu(t). \tag{3.79}$$

Proof. Defining a new measure ν as in the proof of Theorem 3.35 and using identity (3.73) and Fubini we obtain

$$\begin{aligned}
&\int_{[a,b]} f(t)g(t)d\mu(t) - \int_{(c,d]} f(t)h(t)d\mu(t) + \int_{(d,b]} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right) g(t)k(t)d\mu(t) \\
&= \int_{[a,c]} \left(\int_{[a,x)} g(t)k(t)d\mu(t)\right) d\nu(x) + \int_{(c,d]} \left(\int_{[x,d]} k(t)[h(t)-g(t)]d\mu(t)\right) d\nu(x).
\end{aligned}$$

So we have (3.79) when (3.77) and (3.78) hold. □

Taking $k \equiv 1$ and $h \equiv 1$ in Theorems 3.35 and 3.36 we obtain weaker conditions for the function g in an extension of Cerone's result obtained in Theorems 3.27 and 3.28.

Theorem 3.37 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let f and g be μ-integrable functions on $[a,b]$ such that f is nonincreasing and right-continuous. Further, let $[c,d] \subseteq [a,b]$ with $\mu([c,d]) = \int_{[a,b]} g(t)d\mu(t)$. If*

$$\int_{[c,x)} g(t)d\mu(t) \leq \mu([c,x)), \quad c \leq x \leq d \quad \textit{and} \quad \int_{[x,b]} g(t)d\mu(t) \geq 0, \quad d < x \leq b, \tag{3.80}$$

then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \leq \int_{[c,d]} f(t)d\mu(t) + \int_{[a,c)} (f(t)-f(d))\, g(t)d\mu(t).$$

Theorem 3.38 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let f and g be μ-integrable functions on $[a,b]$ such that f is nonincreasing and right-continuous. Further, let $[c,d] \subseteq [a,b]$ with $\mu((c,d]) = \int_{[a,b]} g(t)d\mu(t)$. If*

$$\int_{[x,d]} g(t)d\mu(t) \leq \mu([x,d]), \quad c < x \leq d \quad \textit{and} \quad \int_{[a,x)} g(t)d\mu(t) \geq 0, \quad a \leq x \leq c, \tag{3.81}$$

then

$$\int_{(c,d]} f(t)d\mu(t) - \int_{(d,b]} (f(c)-f(t))\, g(t)d\mu(t) \leq \int_{[a,b]} f(t)g(t)d\mu(t).$$

If we take $c = a$ and $d = a+\lambda$ conditions (3.80) become

$$\int_{[a,x)} g(t)d\mu(t) \leq \mu([a,x)), a \leq x \leq a+\lambda \text{ and } \int_{[x,b]} g(t)d\mu(t) \geq 0, a+\lambda < x \leq b.$$

For $a+\lambda < x \leq b$ we have

$$\begin{aligned}\int_{[a,x)} g(t)d\mu(t) &= \int_{[a,b]} g(t)d\mu(t) - \int_{[x,b]} g(t)d\mu(t) = \mu([a,a+\lambda]) - \int_{[x,b]} g(t)d\mu(t)\\ &\leq \mu([a,a+\lambda]) \leq \mu([a,x)).\end{aligned}$$

Also, for $a \leq x \leq a+\lambda$ we have

$$\begin{aligned}\int_{[x,b]} g(t)d\mu(t) &= \int_{[a,b]} g(t)d\mu(t) - \int_{[a,x)} g(t)d\mu(t) = \mu([a,a+\lambda]) - \int_{[a,x)} g(t)d\mu(t)\\ &\geq \mu([a,a+\lambda]) - \mu([a,x)) = \mu([x,a+\lambda]) \geq 0.\end{aligned}$$

Hence, for $c = a$ and $d = a+\lambda$ conditions (3.80) are equivalent to

$$\int_{[a,x)} g(t)d\mu(t) \leq \mu([a,x)) \quad \text{and} \quad \int_{[x,b]} g(t)d\mu(t) \geq 0, \quad \text{for every } x \in [a,b].$$

Similarly, if we take $c = b-\lambda$ and $d = b$ conditions (3.81) are equivalent to

$$\int_{[x,b]} g(t)d\mu(t) \leq \mu([x,b]) \quad \text{and} \quad \int_{[a,x)} g(t)d\mu(t) \geq 0, \quad \forall x \in [a,b].$$

Therefore, if we take $c = a$ and $d = a+\lambda$ in Theorem 3.37, or $c = b-\lambda$ and $d = b$ in Theorem 3.38 we obtain sufficient conditions from Section 2.2.

In the following theorems we obtain weaker conditions for refinements given in Theorems 3.33 and 3.34.

Theorem 3.39 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let f,g,h and k be μ-integrable functions on $[a,b]$ such that k is positive, h is nonnegative and f/k is nonincreasing and right-continuous. Further, let $[c,d] \subseteq [a,b]$ with $\int_{[c,d]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t)$. If* (3.74) *and* (3.75) *hold, then*

$$\begin{aligned}\int_{[a,b]} f(t)g(t)d\mu(t) &\leq \int_{[c,d]} f(t)h(t)d\mu(t) + \int_{[a,c)} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) g(t)k(t)d\mu(t)\\ &\quad - \int_{[c,d]} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) k(t)[h(t)-g(t)]d\mu(t) \qquad (3.82)\\ &\leq \int_{[c,d]} f(t)h(t)d\mu(t) + \int_{[a,c)} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) g(t)k(t)d\mu(t).\end{aligned}$$

Proof. Using identity (3.59), defining a new measure ν as in the proof of Theorem 3.35 and using Fubini we have

$$\begin{aligned}
&\int_{[c,d]} f(t)h(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) + \int_{[a,c)} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) g(t)k(t)d\mu(t) \\
&- \int_{[c,d]} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) k(t)[h(t)-g(t)]d\mu(t) = \int_{(d,b]} \left(\frac{f(d)}{k(d)} - \frac{f(t)}{k(t)}\right) g(t)k(t)d\mu(t) \\
&= \int_{(d,b]} \left(\int_{(d,t]} d\nu(x)\right) g(t)k(t)d\mu(t) = \int_{(d,b]} \left(\int_{[x,b]} g(t)k(t)d\mu(t)\right) d\nu(x) \geq 0
\end{aligned}$$

when

$$\int_{[x,b]} g(t)k(t)d\mu(t) \geq 0, \quad d < x \leq b.$$

Furthermore,

$$\begin{aligned}
&\int_{[c,d]} \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) k(t)[h(t)-g(t)]d\mu(t) \\
&\qquad = \int_{[c,d]} \left(\int_{[c,x)} k(t)[h(t)-g(t)]d\mu(t)\right) d\nu(x) \geq 0
\end{aligned}$$

when

$$\int_{[c,x)} k(t)g(t)d\mu(t) \leq \int_{[c,x)} k(t)h(t)d\mu(t), \quad c \leq x \leq d.$$

Hence (3.82) holds when (3.74) and (3.75) hold. □

Theorem 3.40 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let f,g,h and k be μ-integrable functions on $[a,b]$ such that k is positive, h is nonnegative and f/k is nonincreasing and right-continuous. Further, let $[c,d] \subseteq [a,b]$ with $\int_{(c,d]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t)$. If* (3.77) *and* (3.78) *hold, then*

$$\begin{aligned}
&\int_{(c,d]} f(t)h(t)d\mu(t) - \int_{(d,b]} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right) g(t)k(t)d\mu(t) \leq \int_{(c,d]} f(t)h(t)d\mu(t) \\
&- \int_{(d,b]} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right) g(t)k(t)d\mu(t) + \int_{(c,d]} \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right) k(t)[h(t)-g(t)]d\mu(t) \\
&\leq \int_{[a,b]} f(t)g(t)d\mu(t).
\end{aligned}$$

Proof. Similar to the proof of Theorem 3.39 using identity (3.61). □

3.4 Weighted Bellman-Steffensen type inequalities

In this section we give more general results than the results given in the Section 2.4, which are related to the function f/k. Results given in this section were obtained by Jakšetić, Pečarić and Smoljak Kalamir in [38].

Firstly, let us give a corollary obtained from Theorem 3.15 by taking $k \equiv 1$.

Corollary 3.5 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$, let g and h be $\mu-$integrable functions on $[a,b]$ such that h is nonnegative. Let λ be a positive constant such that $\int_{[a,a+\lambda]} h(t)d\mu(t) = \int_{[a,b]} g(t)d\mu(t)$. The inequality*

$$\int_{[a,b]} f(t)g(t)d\mu(t) \leq \int_{[a,a+\lambda]} f(t)h(t)d\mu(t) \tag{3.83}$$

holds for every nonincreasing, right-continuous function $f : [a,b] \to \mathbb{R}$ if and only if

$$\int_{[a,x)} g(t)d\mu(t) \leq \int_{[a,x)} h(t)d\mu(t) \quad \text{and} \quad \int_{[x,b]} g(t)d\mu(t) \geq 0, \tag{3.84}$$

for every $x \in [a,b]$.

Theorem 3.41 *Let μ be a finite, positive measure on $\mathscr{B}([a,b])$, f,h and k be $\mu-$integrable functions on $[a,b]$ such that h is nonnegative, k is positive and f/k is nonincreasing, right-continuous. Then*

$$\frac{\int_{[a,b]} f(t)G(t)d\mu(t)}{\int_{[a,b]} k(t)G(t)d\mu(t)} \leq \frac{\int_{[a,a+\lambda]} f(t)h(t)d\mu(t)}{\int_{[a,a+\lambda]} k(t)h(t)d\mu(t)} \tag{3.85}$$

if and only if $G : [a,b] \to \mathbb{R}$ is a $\mu-$integrable function and λ is a positive constant such that

$$\frac{\int_{[a,x)} k(t)G(t)d\mu(t)}{\int_{[a,b]} k(t)G(t)d\mu(t)} \leq \frac{\int_{[a,x)} k(t)h(t)d\mu(t)}{\int_{[a,a+\lambda]} k(t)h(t)d\mu(t)} \quad \text{and} \quad \int_{[x,b]} k(t)G(t)d\mu(t) \geq 0, \tag{3.86}$$

for every $x \in [a,b]$, assuming $\int_{[a,b]} k(t)G(t)d\mu(t) > 0$.
For a nondecreasing, right-continuous function f/k the inequality (3.85) *is reversed.*

Proof. *Sufficiency*. Let us define the function

$$g(t) = \frac{G(t)\int_{[a,a+\lambda]} k(t)h(t)d\mu(t)}{\int_{[a,b]} k(t)G(t)d\mu(t)}.$$

Since $\int_{[a,b]} k(t)g(t)d\mu(t) = \int_{[a,a+\lambda]} k(t)h(t)d\mu(t)$ and the conditions (3.29) are fulfilled we can apply (3.28), and (3.85) is valid.
Necessity. If we put the function

$$f(t) = \begin{cases} k(t), & t < x; \\ 0, & t \geq x, \end{cases}$$

for $a \leq x \leq a+\lambda$ in the inequality (3.85) we get the conditions (3.86). □

Theorem 3.42 *Let μ be a finite, positive measure on $\mathscr{B}([a,b])$. Let h and f/k be nonnegative nonincreasing functions on $[a,b]$ such that k is positive, and let ϕ be an increasing convex function on $[0,\infty)$ with $\phi(0)=0$. If G is a nonnegative nondecreasing function on $[a,b]$ such that there exists a nonnegative function g_1 defined by the equation*

$$\int_{[a,b]} g_1(t)\phi\left(\frac{k(t)G(t)}{g_1(t)}\right)d\mu(t) \leq \int_{[a,b]} k(t)h(t)d\mu(t)$$

and $\int_{[a,b]} g_1(t)d\mu(t) \leq 1$, then the following inequality is valid:

$$\phi\left(\frac{\int_{[a,b]} f(t)G(t)d\mu(t)}{\int_{[a,b]} k(t)G(t)d\mu(t)}\right) \leq \frac{\int_{[a,a+\lambda]} \phi\left(\frac{f(t)}{k(t)}\right) k(t)h(t)d\mu(t)}{\int_{[a,a+\lambda]} k(t)h(t)d\mu(t)}, \tag{3.87}$$

where $\int_{[a,a+\lambda]} k(t)h(t)d\mu(t) = \phi\left(\int_{[a,b]} k(t)G(t)d\mu(t)\right)$.

Proof. Using Jensen's inequality we have

$$\begin{aligned}\phi\left(\frac{\int_{[a,b]} f(t)G(t)d\mu(t)}{\int_{[a,b]} k(t)G(t)d\mu(t)}\right) &= \phi\left(\frac{\int_{[a,b]} \frac{f(t)}{k(t)}k(t)G(t)d\mu(t)}{\int_{[a,b]} k(t)G(t)d\mu(t)}\right)\\ &\leq \frac{\int_{[a,b]} \phi\left(\frac{f(t)}{k(t)}\right) k(t)G(t)d\mu(t)}{\int_{[a,b]} k(t)G(t)d\mu(t)}.\end{aligned}$$

From (3.85) for $f \mapsto (\phi\circ(f/k))\cdot k$, since $\phi\circ(f/k)$ is nonincreasing, we have

$$\frac{\int_{[a,b]} \phi\left(\frac{f(t)}{k(t)}\right) k(t)G(t)d\mu(t)}{\int_{[a,b]} k(t)G(t)d\mu(t)} \leq \frac{\int_{[a,a+\lambda]} \phi\left(\frac{f(t)}{k(t)}\right) k(t)h(t)d\mu(t)}{\int_{[a,a+\lambda]} k(t)h(t)d\mu(t)}$$

if conditions in (3.86) are satisfied. Obviously, $\int_{[x,b]} k(t)G(t)d\mu(t) \geq 0$ since k and μ are positive and G is nonnegative. Hence, we have to show

$$\begin{aligned}\phi\left(\int_{[a,b]} k(t)G(t)d\mu(t)\right)&\int_{[a,x)} k(t)G(t)d\mu(t)\\ &\leq \int_{[a,b]} k(t)G(t)d\mu(t)\int_{[a,x)} k(t)h(t)d\mu(t).\end{aligned} \tag{3.88}$$

Using sub-linearity from Lemma 2.3 and Jensen's inequality we have

$$\begin{aligned}\phi\left(\int_{[a,b]} k(t)G(t)d\mu(t)\right) &= \phi\left(\int_{[a,b]} g_1(t)d\mu(t)\frac{\int_{[a,b]} k(t)G(t)d\mu(t)}{\int_{[a,b]} g_1(t)d\mu(t)}\right)\\ &\leq \int_{[a,b]} g_1(t)d\mu(t)\phi\left(\frac{\int_{[a,b]} g_1(t)\frac{k(t)G(t)}{g_1(t)}d\mu(t)}{\int_{[a,b]} g_1(t)d\mu(t)}\right)\\ &\leq \int_{[a,b]} g_1(t)\phi\left(\frac{k(t)G(t)}{g_1(t)}\right)d\mu(t) \leq \int_{[a,b]} k(t)h(t)d\mu(t).\end{aligned} \tag{3.89}$$

Since G is nonnegative nondecreasing, h is nonnegative nonincreasing and k is positive, we have

$$\frac{\int_{[a,x)} k(t)G(t)d\mu(t)}{\int_{[a,x)} k(t)h(t)d\mu(t)} \le \frac{\int_{[a,b]} k(t)G(t)d\mu(t)}{\int_{[a,b]} k(t)h(t)d\mu(t)}$$

i.e.

$$\int_{[a,b]} k(t)h(t)d\mu(t) \int_{[a,x)} k(t)G(t)d\mu(t) \le \int_{[a,x)} k(t)h(t)d\mu(t) \int_{[a,b]} k(t)G(t)d\mu(t).$$

So, along with (3.89) we have proved (3.88). Hence the theorem is proved. □

We continue with some applications given in [38].

Firstly, let us give an example and lemma which will be useful in our applications.

Example 3.1

(i) $f(x) = e^{\alpha x}$ *is exponentially convex on* $\mathbb{R}$, *for any* $\alpha \in \mathbb{R}$.

(ii) $g(x) = x^{-\alpha}$ *is exponentially convex on* $(0,\infty)$, *for any* $\alpha > 0$.

Lemma 3.3 *Let k be a positive function and $p \in \mathbb{R}$. Let $\varphi_p : (0,\infty) \to \mathbb{R}$ be defined by*

$$\varphi_p(x) = \begin{cases} \frac{x^p}{p} k(x), & p \neq 0; \\ k(x)\log x, & p = 0. \end{cases} \tag{3.90}$$

Then $x \mapsto (\varphi_p/k)(x)$ is increasing on $(0,\infty)$ for each $p \in \mathbb{R}$ and $p \mapsto (\varphi_p/k)(x)$ is exponentially convex on $(0,\infty)$ for each $x \in (0,\infty)$.

Proof. The first part follows from $\frac{d}{dx}\left(\frac{\varphi_p(x)}{k(x)}\right) = x^{p-1} > 0$ on $(0,\infty)$ for each $p \in \mathbb{R}$. The second part follows from $p \mapsto \frac{x^p}{p} = e^{p\log x} \cdot \frac{1}{p}$. Since $p \mapsto e^{p\log x}$ and $p \mapsto \frac{1}{p}$ are exponentially convex according to Example 3.1 and according to the above comment, conclusion follows. □

Lemma 3.4 *For $p \in \mathbb{R}$ let $\phi_p : [0,\infty) \to \mathbb{R}$ be defined by*

$$\phi_p(x) = \frac{x^p}{p(p-1)}, \quad p > 1. \tag{3.91}$$

Then $x \mapsto \phi_p(x)$ is convex on $[0,\infty)$ for each $p > 1$ and $p \mapsto \phi_p(x)$ and $p \mapsto \phi_p'(x)$ are exponentially convex on $(1,\infty)$ for each $x \in [0,\infty)$.

Using (3.85), under the assumptions of Theorem 3.41, we can define a linear functional $\mathfrak{L}$ by

$$\mathfrak{L}(f) = \frac{\int_{[a,b]} f(t)G(t)d\mu(t)}{\int_{[a,b]} k(t)G(t)d\mu(t)} - \frac{\int_{[a,a+\lambda]} f(t)h(t)d\mu(t)}{\int_{[a,a+\lambda]} k(t)h(t)d\mu(t)}. \tag{3.92}$$

We have that the functional $\mathfrak{L}$ is nonnegative on the class of nondecreasing, right-continuous functions $f/k : [a,b] \to \mathbb{R}$.

Theorem 3.43 *Let $f \mapsto \mathfrak{L}(f)$ be the linear functional defined by* (3.92) *and let $\Phi : \mathbb{R} \to \mathbb{R}$ be defined by*

$$\Phi(p) = \mathfrak{L}(\varphi_p)$$

where φ_p is defined by (3.90)*. Then the following statements hold:*

(i) The function Φ is continuous on $\mathbb{R}$.

(ii) If $n \in \mathbb{N}$ and $p_1, \ldots, p_n \in \mathbb{R}$ are arbitrary, then the matrix

$$\left[\Phi\left(\frac{p_j + p_k}{2}\right)\right]_{j,k=1}^{n}$$

is positive semidefinite. Particularly,

$$det\left[\Phi\left(\frac{p_j + p_k}{2}\right)\right]_{j,k=1}^{n} \geq 0.$$

(iii) The function Φ is exponentially convex on $\mathbb{R}$.

(iv) The function Φ is log-convex on $\mathbb{R}$.

(v) If $p, q, r \in \mathbb{R}$ are such that $p < q < r$, then

$$\Phi(q)^{r-p} \leq \Phi(p)^{r-q} \Phi(r)^{q-p}.$$

Proof. (i) Continuity of the function $p \mapsto \Phi(p)$ is obvious for $p \in \mathbb{R} \setminus \{0\}$. For $p = 0$ it is directly checked using Heine characterization.
(ii) Let $n \in \mathbb{N}$, $p_i \in \mathbb{R}$, $i = 1, \ldots, n$ be arbitrary. Let us define an auxiliary function $\Psi : (0,\infty) \to \mathbb{R}$ by

$$\Psi(x) = \sum_{j,k=1}^{n} \xi_j \xi_k \varphi_{\frac{p_j + p_k}{2}}(x).$$

Now

$$\left(\frac{\Psi(x)}{k(x)}\right)' = \sum_{j,k=1}^{n} \xi_j \xi_k x^{\frac{p_j+p_k}{2}-1} = \left(\sum_{j=1}^{n} \xi_j x^{\frac{p_j-1}{2}}\right)^2 \geq 0$$

implies that Ψ/k is nondecreasing on $(0,\infty)$, so $\mathfrak{L}(\Psi) \geq 0$. This means that

$$\left[\Phi\left(\frac{p_j + p_k}{2}\right)\right]_{j,k=1}^{n}$$

is a positive semi-definite matrix.
(iii), (iv), (v) are simple consequences of (i) and (ii). □

Let k be a positive function and let $\{\theta_p/k : \ p \in (0,\infty)\}$ be the family of functions defined on $[0,\infty)$ with

$$\theta_p(x) = \frac{x^p}{p} k(x). \tag{3.93}$$

Similar as in Lemma 2.2 we conclude that $x \mapsto (\theta_p/k)(x)$ is increasing on $[0,\infty)$ for each $p \in \mathbb{R}$ and $p \mapsto (\theta_p/k)(x)$ is exponentially convex on $(0,\infty)$ for each $x \in [0,\infty)$.

Let us define a linear functional $\mathfrak{M}$ by

$$\mathfrak{M}(f) = \int_{[a,b]} f(t)g(t)d\mu(t) - \int_{[a,a+\lambda]} f(t)h(t)d\mu(t). \tag{3.94}$$

Under the assumptions of Remark 1.6 we have that the linear functional $\mathfrak{M}$ is nonnegative acting on nondecreasing functions $f/k : [a,b] \to \mathbb{R}$ with the property $(f/k)(a) = 0$.

Theorem 3.44 *Let $f \mapsto \mathfrak{M}(f)$ be the linear functional defined by* (3.94) *and let $F : (0,\infty) \to \mathbb{R}$ be defined by*

$$F(p) = \mathfrak{M}(\theta_p)$$

where θ_p is defined by (3.93)*. Then the following statements hold:*

(i) The function F is continuous on $(0,\infty)$.

(ii) If $n \in \mathbb{N}$ and $p_1,\dots,p_n \in (0,\infty)$ are arbitrary, then the matrix

$$\left[F\left(\frac{p_j+p_k}{2}\right)\right]_{j,k=1}^n$$

is positive semidefinite. Particularly,

$$det\left[F\left(\frac{p_j+p_k}{2}\right)\right]_{j,k=1}^n \geq 0.$$

(iii) The function F is exponentially convex on $(0,\infty)$.

(iv) The function F is log-convex on $(0,\infty)$.

(v) If $p,q,r \in (0,\infty)$ are such that $p<q<r$, then

$$F(q)^{r-p} \leq F(p)^{r-q} F(r)^{q-p}.$$

Proof. (i) Continuity of the function $p \mapsto F(p)$ is obvious.

(ii) Let $n \in \mathbb{N}$, $p_i \in (0,\infty)$, $i = 1,\dots,n$ be arbitrary. Let us define an auxiliary function $\Psi : [0,\infty) \to \mathbb{R}$ by

$$\Psi(x) = \sum_{j,k=1}^n \xi_j \xi_k \theta_{\frac{p_j+p_k}{2}}(x).$$

Now

$$\left(\frac{\Psi(x)}{k(x)}\right)' = \left(\sum_{j=1}^n \xi_j x^{\frac{p_j-1}{2}}\right)^2 \geq 0$$

implies that Ψ/k is nondecreasing on $[0,\infty)$ and nonnegative since $(\Psi/k)(0) = 0$. Hence, $\mathfrak{M}(\Psi) \geq 0$ and we conclude that

$$\left[F\left(\frac{p_j+p_k}{2}\right)\right]_{j,k=1}^n$$

is a positive semi-definite matrix.

(iii), (iv), (v) are simple consequences of (i) and (ii). □

Theorem 3.45 *Let $f \mapsto \mathfrak{M}(f)$ be the linear functional defined by* (3.94)*, let k be a positive function on $[a,b]$ and $\psi/k \in C^1[a,b]$ such that $(\psi/k)(a) = 0$. Then there exists $\xi \in [a,b]$ such that*

$$\mathfrak{M}(\psi) = \left(\frac{\psi(\xi)}{k(\xi)}\right)' \mathfrak{M}(e_1),$$

where $e_1(x) = (x-a)k(x)$.

Proof. Since $\psi/k \in C^1[a,b]$ there exist

$$m = \min_{x\in[a,b]} \frac{\psi'(x)k(x) - \psi(x)k'(x)}{k^2(x)} \text{ and } M = \max_{x\in[a,b]} \frac{\psi'(x)k(x) - \psi(x)k'(x)}{k^2(x)}.$$

Denote

$$h_1(x) = M(x-a)k(x) - \psi(x) \text{ and } h_2(x) = \psi(x) - m(x-a)k(x).$$

Then $(h_1/k)(a) = (h_2/k)(a) = 0$ and

$$\left(\frac{h_1(x)}{k(x)}\right)' = M - \frac{\psi'(x)k(x) - \psi(x)k'(x)}{k^2(x)} \geq 0$$
$$\left(\frac{h_2(x)}{k(x)}\right)' = \frac{\psi'(x)k(x) - \psi(x)k'(x)}{k^2(x)} - m \geq 0$$

so h_1/k and h_2/k are nondecreasing, nonnegative functions on $[a,b]$, which means that $\mathfrak{M}(h_1)$, $\mathfrak{M}(h_2) \geq 0$ i.e.

$$m\mathfrak{M}(e_1) \leq \mathfrak{M}(\psi) \leq M\mathfrak{M}(e_1).$$

If $\mathfrak{M}(e_1) = 0$, the proof is complete. If $\mathfrak{M}(e_1) > 0$, then

$$m \leq \frac{\mathfrak{M}(\psi)}{\mathfrak{M}(e_1)} \leq M$$

and the existence of $\xi \in [a,b]$ follows. □

Using the standard Cauchy type mean value theorem we obtain the following corollary.

Corollary 3.6 *Let $f \mapsto \mathfrak{M}(f)$ be linear functional defined by* (3.94)*, let k be a positive function on $[a,b]$ and ψ_1/k, $\psi_2/k \in C^1[a,b]$ such that $(\psi_1/k)(a) = (\psi_2/k)(a) = 0$, then there exists $\xi \in [a,b]$, such that*

$$\frac{\left(\frac{\psi_1(\xi)}{k(\xi)}\right)'}{\left(\frac{\psi_2(\xi)}{k(\xi)}\right)'} = \frac{\mathfrak{M}(\psi_1)}{\mathfrak{M}(\psi_2)}, \tag{3.95}$$

provided that the denominator on the right side is non-zero.

If the inverse of $(\psi_1/k)'/(\psi_2/k)'$ exists then various kinds of means can be defined by (3.95). That is

$$\xi = \left(\frac{\left(\frac{\psi_1}{k}\right)'}{\left(\frac{\psi_2}{k}\right)'} \right)^{-1} \left(\frac{\mathfrak{M}(\psi_1)}{\mathfrak{M}(\psi_2)} \right). \tag{3.96}$$

Particularly, if we substitute $\psi_1(x) = \theta_p(x),\ \psi_2(x) = \theta_q(x)$ in (3.96) and use the continuous extension, the following expressions are obtained.

$$M(p,q) = \begin{cases} \left(\frac{\mathfrak{M}(\theta_p)}{\mathfrak{M}(\theta_q)} \right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{\mathfrak{M}(\theta_0\theta_p)}{\mathfrak{M}(\theta_p)} - \frac{1}{p} \right), & p = q, \end{cases}$$

where $\theta_0(x) = \log x$ and $p,q \in (0,\infty)$. By Theorem 2.5, if $p,q,u,v \in (0,\infty)$ such that $p \le u,\ q \le v$ then,

$$M(p,q) \le M(u,v).$$

Using (3.87), under the assumptions of Theorem 3.42, we can define a linear functional $\mathfrak{N}$ by

$$\mathfrak{N}(\phi) = \frac{\int_{[a,a+\lambda]} \phi\left(\frac{f(t)}{k(t)}\right) k(t)h(t)d\mu(t)}{\int_{[a,a+\lambda]} k(t)h(t)d\mu(t)} - \phi\left(\frac{\int_{[a,b]} f(t)G(t)d\mu(t)}{\int_{[a,b]} k(t)G(t)d\mu(t)} \right). \tag{3.97}$$

We have that the linear functional $\mathfrak{N}$ is nonnegative on the class of increasing convex functions ϕ on $[0,\infty)$ with the property $\phi(0) = 0$.

Theorem 3.46 *Let $f \mapsto \mathfrak{N}(f)$ be the linear functional defined by* (3.97) *and let $H : (1,\infty) \to \mathbb{R}$ be defined by*

$$H(p) = \mathfrak{N}(\phi_p)$$

where ϕ_p is defined by (3.91)*. Then the following statements hold:*

(i) The function H is continuous on $(1,\infty)$.

(ii) If $n \in \mathbb{N}$ and $p_1,\ldots,p_n \in (1,\infty)$ are arbitrary, then the matrix

$$\left[H\left(\frac{p_j + p_k}{2} \right) \right]_{j,k=1}^{n}$$

is positive semidefinite. Particularly,

$$det \left[H\left(\frac{p_j + p_k}{2} \right) \right]_{j,k=1}^{n} \ge 0.$$

(iii) The function H is exponentially convex on $(1,\infty)$.

(iv) The function H is log-convex on $(1,\infty)$.

(v) If $p,q,r \in (1,\infty)$ are such that $p<q<r$, then

$$H(q)^{r-p} \le H(p)^{r-q} H(r)^{q-p}.$$

Proof. Similar to the proof of Theorem 2.18. □

Similar to Corollary 3.6 we also have the following corollary.

Corollary 3.7 *Let $f \mapsto \mathfrak{N}(f)$ be the linear functional defined by* (3.97) *and $\psi_1,\ \psi_2 \in C^2[0,a]$ such that $\psi_1(0)=\psi_2(0)=\psi_1'(0)=\psi_2'(0)=0$ and such that $\psi_2''(x)$ does not vanish for any value of $x \in [0,a]$, then there exists $\xi \in [0,a]$ such that*

$$\frac{\psi_1''(\xi)}{\psi_2''(\xi)} = \frac{\mathfrak{N}(\psi_1)}{\mathfrak{N}(\psi_2)}, \tag{3.98}$$

provided that the denominator on the right side is non-zero.

If the inverse of ψ_1''/ψ_2'' exists then various kinds of means can be defined by (3.98). That is

$$\xi = \left(\frac{\psi_1''}{\psi_2''}\right)^{-1} \left(\frac{\mathfrak{N}(\psi_1)}{\mathfrak{N}(\psi_2)}\right). \tag{3.99}$$

Particularly, if we substitute $\psi_1(x) = \phi_p(x),\ \psi_2(x) = \phi_q(x)$ in (3.99) and use continuous extension, the following expressions are obtained:

$$N(p,q) = \begin{cases} \left(\frac{\mathfrak{N}(\phi_p)}{\mathfrak{N}(\phi_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{\mathfrak{N}(\phi_0\phi_p)}{\mathfrak{N}(\phi_p)} + \frac{3-2p}{(p-1)(p-2)}\right), & p = q, \end{cases}$$

where $\phi_0(x) = \log x$ and $p,q \in (1,\infty)$. By Theorem 2.5, if $p,q,u,v \in (1,\infty)$ such that $p \le u,\ q \le v$ then,

$$N(p,q) \le N(u,v).$$

We can generalize the above construction. For a fixed $n \ge 2$, let us define

$$\mathscr{C}_n = \{\psi_p :\ p \in J\},$$

a family of functions from $C([0,a])$ such that $\psi_p(0) = \psi_p'(0) = 0$, and $p \mapsto \psi_p''(x)$ is n-exponentially convex in the Jensen sense on J for every $x \in [0,a]$.

Theorem 3.47 *Let $f \mapsto \mathfrak{N}(f)$ be the linear functional defined by* (3.97) *and let $S : J \to \mathbb{R}$, be defined by*

$$S(p) = \mathfrak{N}(\psi_p)$$

where $\psi_p \in \mathscr{C}_n$. Then the following statements hold:

(i) S is n-exponentially convex in the Jensen sense on J.

(ii) If S is continuous on J, then it is $n-$exponentially convex on J and for $p,q,r \in J$ such that $p < q < r$, we have

$$S(q)^{r-p} \leq S(p)^{r-q} S(r)^{q-p}.$$

(iii) If S is positive and differentiable on J, then for every $p,q,u,v \in J$ such that $p \leq u$, $q \leq v$, we have

$$\widetilde{M}(p,q) \leq \widetilde{M}(u,v)$$

where $\widetilde{M}(p,q)$ is defined by

$$\widetilde{M}(p,q) = \begin{cases} \left(\frac{S(p)}{S(q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{\frac{d}{dp}(S(p))}{S(p)}\right), & p = q. \end{cases}$$

Proof. Similar to the proof of Theorem 2.23. □

Using divided differences we can further refine obtained results. Let

$$\mathscr{D} = \{\chi_p : \ p \in J\},$$

be a family of functions from $C([0,a])$ such that $\chi_p(0) = 0$, $p \mapsto [x,y;\chi_p]$ is exponentially convex on J for every choice of two distinct points $x,y \in [0,a]$, and $p \mapsto [x_0,x_1,x_2;\chi_p]$ is exponentially convex on J for every choice of three distinct points $x_0,x_1,x_2 \in [0,a]$.

Theorem 3.48 *Let $f \mapsto \mathfrak{N}(f)$ be the linear functional defined by* (3.97) *and let $H : J \to \mathbb{R}$ be defined by*

$$H(p) = \mathfrak{N}(\chi_p)$$

where $\chi_p \in \mathscr{D}$. Then the following statements hold:

(i) If $n \in \mathbb{N}$ and $p_1,\ldots,p_n \in \mathbb{R}$ are arbitrary, then the matrix

$$\left[H\left(\frac{p_k+p_m}{2}\right)\right]_{k,m=1}^{n}$$

is positive semidefinite. Particularly,

$$det\left[H\left(\frac{p_k+p_m}{2}\right)\right]_{k,m=1}^{n} \geq 0.$$

(ii) If the function H is continuous on J, then H is exponentially convex on J.

(iii) If H is positive and differentiable on J, then for every $p,q,u,v \in J$ such that $p \leq u$, $q \leq v$, we have

$$\widehat{M}(p,q) \leq \widehat{M}(u,v)$$

where $\widehat{M}(p,q)$ is defined by

$$\widehat{M}(p,q) = \begin{cases} \left(\frac{H(p)}{H(q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{\frac{d}{dp}(H(p))}{H(p)}\right), & p = q. \end{cases}$$

Proof. Similar to the proof of Theorem 2.24. □

Chapter 4

Steffensen type inequalities involving convex and 3-convex functions

4.1 Weighted Steffensen type inequalities

Results given in this section were obtained by Pečarić and Smoljak in [81].

Let us begin by giving definition of class $\mathscr{M}_1^c[a,b]$ for function f/h which is similar to Definition 1.12.

Definition 4.1 *Let $h : [a,b] \to \mathbb{R}$ be a positive function, $f : [a,b] \to \mathbb{R}$ be a function and $c \in (a,b)$. We say that f/h belongs to the class $\mathscr{M}_1^c[a,b]$ (f/h belongs to the class $\mathscr{M}_2^c[a,b]$) if there exists a constant A such that the function $\frac{F(x)}{h(x)} = \frac{f(x)}{h(x)} - Ax$ is nonincreasing (nondecreasing) on $[a,c]$ and nondecreasing (nonincreasing) on $[c,b]$.*

As noted in Section 1.4 we can describe the property from Definition 4.1 as "convexity at point c". In the following theorem we give connection between the class of functions $\mathscr{M}_1^c[a,b]$ and the class of convex functions.

Theorem 4.1 *The function f/h is convex (concave) on $[a,b]$ if and only if $f/h \in \mathscr{M}_1^c[a,b]$ ($f/h \in \mathscr{M}_2^c[a,b]$) for every $c \in (a,b)$.*

Applying measure theoretic generalizations of Steffensen's inequality given in Theorem 3.1 to a class of functions that are convex at point c we obtain the following results.

Theorem 4.2 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$ and let $c \in (a,b)$. Let f,g and h be measurable functions on $[a,b]$ such that h is positive and $0 \leq g \leq 1$. Let $\lambda_1, \lambda_2 \in \mathbb{R}_+$ be such that*

$$\int_{[a,a+\lambda_1]} h(t)d\mu(t) = \int_{[a,c]} h(t)g(t)d\mu(t) \tag{4.1}$$

and

$$\int_{(b-\lambda_2,b]} h(t)d\mu(t) = \int_{[c,b]} h(t)g(t)d\mu(t). \tag{4.2}$$

If $f/h \in \mathscr{M}_1^c[a,b]$ and

$$\int_{[a,b]} th(t)g(t)d\mu(t) = \int_{[a,a+\lambda_1]} th(t)d\mu(t) + \int_{(b-\lambda_2,b]} th(t)d\mu(t), \tag{4.3}$$

then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \leq \int_{[a,a+\lambda_1]} f(t)d\mu(t) + \int_{(b-\lambda_2,b]} f(t)d\mu(t). \tag{4.4}$$

If $f/h \in \mathscr{M}_2^c[a,b]$ and (4.3) *holds, the inequality in* (4.4) *is reversed.*

Proof. Let us prove this for $f/h \in \mathscr{M}_1^c[a,b]$. Let $F(x) = f(x) - Axh(x)$, where A is the constant from Definition 4.1. Since $F/h : [a,c] \to \mathbb{R}$ is nonincreasing, inequality (3.2) implies

$$\begin{aligned}
0 &\leq \int_{[a,a+\lambda_1]} F(t)d\mu(t) - \int_{[a,c]} F(t)g(t)d\mu(t) \\
&= \int_{[a,a+\lambda_1]} f(t)d\mu(t) - \int_{[a,c]} f(t)g(t)d\mu(t) \\
&\quad - A\left(\int_{[a,a+\lambda_1]} th(t)d\mu(t) - \int_{[a,c]} th(t)g(t)d\mu(t)\right).
\end{aligned} \tag{4.5}$$

Similarly, $F/h : [c,b] \to \mathbb{R}$ is nondecreasing, so inequality (3.4) implies

$$\begin{aligned}
0 &\leq \int_{(b-\lambda_2,b]} F(t)d\mu(t) - \int_{[c,b]} F(t)g(t)d\mu(t) \\
&= \int_{(b-\lambda_2,b]} f(t)d\mu(t) - \int_{[c,b]} f(t)g(t)d\mu(t) \\
&\quad - A\left(\int_{(b-\lambda_2,b]} th(t)d\mu(t) - \int_{[c,b]} th(t)g(t)d\mu(t)\right).
\end{aligned} \tag{4.6}$$

Adding up (4.5) and (4.6) we obtain

$$\begin{aligned}
&\int_{[a,a+\lambda_1]} f(t)d\mu(t) + \int_{(b-\lambda_2,b]} f(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) \\
&\geq A\left(\int_{[a,a+\lambda_1]} th(t)d\mu(t) + \int_{(b-\lambda_2,b]} th(t)d\mu(t) - \int_{[a,b]} th(t)g(t)d\mu(t)\right) = 0
\end{aligned}$$

which completes the proof.
Proof for $f/h \in \mathscr{M}_2^c[a,b]$ is similar so we omit the details. $\square$

Remark 4.1 It is obvious from the proof that for $f/h \in \mathscr{M}_1^c[a,b]$ inequality (4.4) holds if equality (4.3) is replaced by the weaker condition

$$A\left(\int_{[a,a+\lambda_1]} th(t)d\mu(t) + \int_{(b-\lambda_2,b]} th(t)d\mu(t) - \int_{[a,b]} th(t)g(t)d\mu(t)\right) \geq 0, \tag{4.7}$$

where A is the constant from Definition 4.1.

Moreover, condition (4.7) can be further weakened if the function f/h is monotonic.

First, let us show that for $f/h \in \mathscr{M}_1^c[a,b]$ we have

$$(f/h)'_-(c) \leq A \leq (f/h)'_+(c). \tag{4.8}$$

Since F/h is nonincreasing on $[a,c]$ and nondecreasing on $[c,b]$ for every distinct points $x_1,x_2 \in [a,c]$ and $y_1,y_2 \in [c,b]$ we have

$$[x_1,x_2;F/h] = [x_1,x_2;f/h] - A \leq 0 \leq [y_1,y_2;f/h] - A = [y_1,y_2;F/h].$$

Therefore, if $(f/h)'_-(c)$ and $(f/h)'_+(c)$ exist, letting $x_i \nearrow c$ and $y_i \searrow c$, $i = 1,2$ we get (4.8). Similarly, for $f/h \in \mathscr{M}_2^c[a,b]$ we have (4.8) with the reverse inequality.

Hence if we additionally assume that $f/h \in \mathscr{M}_1^c[a,b]$ is nondecreasing, condition (4.7) can be further weakened to

$$\int_{[a,b]} th(t)g(t)d\mu(t) \leq \int_{[a,a+\lambda_1]} th(t)d\mu(t) + \int_{(b-\lambda_2,b]} th(t)d\mu(t). \tag{4.9}$$

Further, if $f/h \in \mathscr{M}_1^c[a,b]$ is nonincreasing, condition (4.7) can be further weakened to (4.9) with the reverse inequality.

Theorem 4.3 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$ and let $c \in (a,b)$. Let f,g and h be measurable functions on $[a,b]$ such that h is positive and $0 \leq g \leq 1$. Let $\lambda_1,\lambda_2 \in \mathbb{R}_+$ be such that*

$$\int_{(c-\lambda_1,c]} h(t)d\mu(t) = \int_{[a,c]} h(t)g(t)d\mu(t) \tag{4.10}$$

and

$$\int_{[c,c+\lambda_2]} h(t)d\mu(t) = \int_{[c,b]} h(t)g(t)d\mu(t). \tag{4.11}$$

If $f/h \in \mathscr{M}_1^c[a,b]$ and

$$\int_{[a,b]} th(t)g(t)d\mu(t) = \int_{(c-\lambda_1,c+\lambda_2]} th(t)d\mu(t), \tag{4.12}$$

then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \geq \int_{(c-\lambda_1,c+\lambda_2]} f(t)d\mu(t). \tag{4.13}$$

If $f/h \in \mathscr{M}_2^c[a,b]$ and (4.12) *holds, the inequality in* (4.13) *is reversed.*

Proof. Let $f/h \in \mathscr{M}_1^c[a,b]$ and let $F(x) = f(x) - Axh(x)$, where A is the constant from Definition 4.1. Since $F/h : [a,c] \to \mathbb{R}$ is nonincreasing, inequality (3.4) implies

$$\begin{aligned} 0 \leq & \int_{[a,c]} f(t)g(t)d\mu(t) - \int_{(c-\lambda_1,c]} f(t)d\mu(t) \\ & - A\left(\int_{[a,c]} th(t)g(t)d\mu(t) - \int_{(c-\lambda_1,c]} th(t)d\mu(t)\right). \end{aligned} \tag{4.14}$$

Similarly, $F/h : [c,b] \to \mathbb{R}$ is nondecreasing, so inequality (3.2) implies

$$\begin{aligned} 0 \leq & \int_{[c,b]} f(t)g(t)d\mu(t) - \int_{[c,c+\lambda_2]} f(t)d\mu(t) \\ & - A\left(\int_{[c,b]} th(t)g(t)d\mu(t) - \int_{[c,c+\lambda_2]} th(t)d\mu(t)\right). \end{aligned} \tag{4.15}$$

Adding up (4.14) and (4.15) we obtain

$$\begin{aligned} & \int_{[a,b]} f(t)g(t)d\mu(t) - \int_{(c-\lambda_1,c+\lambda_2]} f(t)d\mu(t) \\ & \geq A\left(\int_{[a,b]} th(t)g(t)d\mu(t) - \int_{(c-\lambda_1,c+\lambda_2]} th(t)d\mu(t)\right) = 0 \end{aligned}$$

which completes the proof for $f/h \in \mathscr{M}_1^c[a,b]$. Similarly for $f/h \in \mathscr{M}_2^c[a,b]$. □

Remark 4.2 Similarly as in Remark 4.1, it is obvious from the proof that for $f/h \in \mathscr{M}_1^c[a,b]$ inequality (4.13) holds if equality (4.12) is replaced by the weaker condition

$$A\left(\int_{[a,b]} th(t)g(t)d\mu(t) - \int_{(c-\lambda_1,c+\lambda_2]} th(t)d\mu(t)\right) \geq 0, \tag{4.16}$$

where A is the constant from Definition 4.1.

If we additionally assume that $f/h \in \mathscr{M}_1^c[a,b]$ is nondecreasing, condition (4.16) can be further weakened to

$$\int_{[a,b]} th(t)g(t)d\mu(t) \geq \int_{(c-\lambda_1,c+\lambda_2]} th(t)d\mu(t). \tag{4.17}$$

Further, if $f/h \in \mathscr{M}_1^c[a,b]$ is nonincreasing, condition (4.16) can be further weakened to (4.17) with the reverse inequality.

As a consequence of Theorems 4.2 and 4.3 we obtain the following weighted Steffensen type inequalities that involve convex functions.

Corollary 4.1 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$ and let $c \in (a,b)$. Let f,g and h be measurable functions on $[a,b]$ such that h is positive and $0 \leq g \leq 1$. Let $\lambda_1, \lambda_2 \in \mathbb{R}_+$ be such that* (4.1) *and* (4.2) *hold. If $f/h : [a,b] \to \mathbb{R}$ is convex and* (4.3) *holds, then the inequality* (4.4) *holds.*
If $f/h : [a,b] \to \mathbb{R}$ is concave, the inequality in (4.4) *is reversed.*

Proof. Since f/h is convex, we have that $f/h \in \mathscr{M}_1^c[a,b]$ for every $c \in (a,b)$. Hence, we can apply Theorem 4.2. □

Corollary 4.2 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$ and let $c \in (a,b)$. Let f,g and h be measurable functions on $[a,b]$ such that h is positive and $0 \le g \le 1$. Let $\lambda_1, \lambda_2 \in \mathbb{R}_+$ be such that* (4.10) *and* (4.11) *hold. If $f/h : [a,b] \to \mathbb{R}$ is convex and* (4.12) *holds, then the inequality* (4.13) *holds.*
If $f/h : [a,b] \to \mathbb{R}$ is concave, the inequality in (4.13) *is reversed.*

Proof. Similar to the proof of Corollary 4.1 applying Theorem 4.3. □

Motivated by Theorem 3.15 we obtain the following weaker conditions for weighted Steffensen type inequalities for convex functions at a point.

Theorem 4.4 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$ and let $c \in (a,b)$. Let g and h be $\mu-$integrable functions on $[a,b]$ such that h is positive and*

$$\int_{[a,x)} h(t)g(t)d\mu(t) \le \int_{[a,x)} h(t)d\mu(t) \quad \text{and} \quad \int_{[x,c]} h(t)g(t)d\mu(t) \ge 0,$$

for every $x \in [a,c]$ and

$$\int_{[x,b]} h(t)g(t)d\mu(t) \le \int_{[x,b]} h(t)d\mu(t) \quad \text{and} \quad \int_{[c,x)} h(t)g(t)d\mu(t) \ge 0,$$

for every $x \in [c,b]$.

Let f/h be a right-continuous function on $[a,b]$ and let $\lambda_1, \lambda_2 \in \mathbb{R}_+$ be such that (4.1) *and* (4.2) *hold. If $f/h \in \mathscr{M}_1^c[a,b]$ and* (4.3) *holds, the inequality* (4.4) *holds. If $f/h \in \mathscr{M}_2^c[a,b]$ and* (4.3) *holds, the inequality in* (4.4) *is reversed.*

Proof. Similar to the proof of Theorem 4.2 using weaker conditions from Theorem 3.15. □

Theorem 4.5 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$ and let $c \in (a,b)$. Let g and h be $\mu-$integrable functions on $[a,b]$ such that h is positive and*

$$\int_{[x,c]} h(t)g(t)d\mu(t) \le \int_{[x,c]} h(t)d\mu(t) \quad \text{and} \quad \int_{[a,x)} h(t)g(t)d\mu(t) \ge 0,$$

for every $x \in [a,c]$ and

$$\int_{[c,x)} h(t)g(t)d\mu(t) \le \int_{[c,x)} h(t)d\mu(t) \quad \text{and} \quad \int_{[x,b]} h(t)g(t)d\mu(t) \ge 0,$$

for every $x \in [c,b]$.

Let f/h be a right-continuous function on $[a,b]$ and let $\lambda_1, \lambda_2 \in \mathbb{R}_+$ be such that (4.10) *and* (4.11) *hold. If $f/h \in \mathscr{M}_1^c[a,b]$ and* (4.12) *holds, the inequality* (4.13) *holds. If $f/h \in \mathscr{M}_2^c[a,b]$ and* (4.12) *holds, the inequality in* (4.13) *is reversed.*

Proof. Similar to the proof of Theorem 4.3 using weaker conditions from Theorem 3.15. □

Condition (4.3) (resp. (4.12)) in Corollary 4.3 and Theorem 4.4 (resp. Corollary 4.4 and Theorem 4.5) can be replaced by weaker conditions given in Remark 4.1 (resp. Remark 4.2).

Similar as in Corollaries 4.3 and 4.4 we have that Theorems 4.4 and 4.5 still hold if f/h is a convex function.

4.1.1 Further generalizations of weighted Steffensen type inequalities

Making substitutions $g \mapsto g/h$ and $f \mapsto fh$ in Theorems 4.2 and 4.3 we obtain the following weighted Steffensen type inequalities for convex functions at a point related to corrected version of Mercer's result given in Theorem 1.9.

Theorem 4.6 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$ and let $c \in (a,b)$. Let f, g and h be measurable functions on $[a,b]$ such that h is positive and $0 \leq g \leq h$. Let $\lambda_1, \lambda_2 \in \mathbb{R}_+$ be such that $\int_{[a,a+\lambda_1]} h(t)d\mu(t) = \int_{[a,c]} g(t)d\mu(t)$ and $\int_{(b-\lambda_2,b]} h(t)d\mu(t) = \int_{[c,b]} g(t)d\mu(t)$. If $f \in \mathscr{M}_1^c[a,b]$ and*

$$\int_{[a,b]} tg(t)d\mu(t) = \int_{[a,a+\lambda_1]} th(t)d\mu(t) + \int_{(b-\lambda_2,b]} th(t)d\mu(t), \tag{4.18}$$

then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \leq \int_{[a,a+\lambda_1]} f(t)h(t)d\mu(t) + \int_{(b-\lambda_2,b]} f(t)h(t)d\mu(t). \tag{4.19}$$

If $f \in \mathscr{M}_2^c[a,b]$ and (4.18) holds, the inequality in (4.19) is reversed.

Theorem 4.7 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$ and let $c \in (a,b)$. Let f, g and h be measurable functions on $[a,b]$ such that h is positive and $0 \leq g \leq h$. Let $\lambda_1, \lambda_2 \in \mathbb{R}_+$ be such that $\int_{(c-\lambda_1,c]} h(t)d\mu(t) = \int_{[a,c]} g(t)d\mu(t)$ and $\int_{[c,c+\lambda_2]} h(t)d\mu(t) = \int_{[c,b]} g(t)d\mu(t)$. If $f \in \mathscr{M}_1^c[a,b]$ and*

$$\int_{[a,b]} tg(t)d\mu(t) = \int_{(c-\lambda_1,c+\lambda_2]} th(t)d\mu(t), \tag{4.20}$$

then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \geq \int_{(c-\lambda_1,c+\lambda_2]} f(t)h(t)d\mu(t). \tag{4.21}$$

If $f \in \mathscr{M}_2^c[a,b]$ and (4.20) holds, the inequality in (4.21) is reversed.

Using substitutions $h \mapsto kh$, $g \mapsto g/k$ and $f \mapsto fk$ in Theorems 4.2 and 4.3 we obtain related Steffensen type inequalities given in the following theorems.

Theorem 4.8 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$ and let $c \in (a,b)$. Let f,g,h and k be measurable functions on $[a,b]$ such that h is positive and $0 \le g \le k$. Let $\lambda_1, \lambda_2 \in \mathbb{R}_+$ be such that $\int_{[a,a+\lambda_1]} k(t)h(t)d\mu(t) = \int_{[a,c]} h(t)g(t)d\mu(t)$ and $\int_{(b-\lambda_2,b]} k(t)h(t)d\mu(t) = \int_{[c,b]} h(t)g(t)d\mu(t)$. If $f/h \in \mathscr{M}_1^c[a,b]$ and*

$$\int_{[a,b]} th(t)g(t)d\mu(t) = \int_{[a,a+\lambda_1]} tk(t)h(t)d\mu(t) + \int_{(b-\lambda_2,b]} tk(t)h(t)d\mu(t), \tag{4.22}$$

then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \le \int_{[a,a+\lambda_1]} f(t)k(t)d\mu(t) + \int_{(b-\lambda_2,b]} f(t)k(t)d\mu(t). \tag{4.23}$$

If $f/h \in \mathscr{M}_2^c[a,b]$ and (4.22) holds, the inequality in (4.23) is reversed.

Theorem 4.9 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$ and let $c \in (a,b)$. Let f,g,h and k be measurable functions on $[a,b]$ such that h is positive and $0 \le g \le k$. Let $\lambda_1, \lambda_2 \in \mathbb{R}_+$ be such that $\int_{(c-\lambda_1,c]} k(t)h(t)d\mu(t) = \int_{[a,c]} h(t)g(t)d\mu(t)$ and $\int_{[c,c+\lambda_2]} k(t)h(t)d\mu(t) = \int_{[c,b]} h(t)g(t)d\mu(t)$. If $f/h \in \mathscr{M}_1^c[a,b]$ and*

$$\int_{[a,b]} th(t)g(t)d\mu(t) = \int_{(c-\lambda_1,c+\lambda_2]} tk(t)h(t)d\mu(t), \tag{4.24}$$

then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \ge \int_{(c-\lambda_1,c+\lambda_2]} f(t)k(t)d\mu(t). \tag{4.25}$$

If $f/h \in \mathscr{M}_2^c[a,b]$ and (4.24) holds, the inequality in (4.25) is reversed.

The following theorems give refined versions of results given in Theorems 4.2 and 4.3.

Theorem 4.10 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$ and let $c \in (a,b)$. Let f,g and h be measurable functions on $[a,b]$ such that h is positive and $0 \le g \le 1$. Let $\lambda_1, \lambda_2 \in \mathbb{R}_+$ be such that (4.1) and (4.2) hold. If $f/h \in \mathscr{M}_1^c[a,b]$ and*

$$\begin{aligned}\int_{[a,b]} th(t)g(t)d\mu(t) = &\int_{[a,a+\lambda_1]} (th(t) - [t-a-\lambda_1]h(t)[1-g(t)])d\mu(t) \\ &+ \int_{(b-\lambda_2,b]} (th(t) - [t-b+\lambda_2]h(t)[1-g(t)])d\mu(t),\end{aligned} \tag{4.26}$$

then

$$\begin{aligned}\int_{[a,b]} f(t)g(t)d\mu(t) \le &\int_{[a,a+\lambda_1]} \left(f(t) - \left[\frac{f(t)}{h(t)} - \frac{f(a+\lambda_1)}{h(a+\lambda_1)} \right] h(t)[1-g(t)] \right) d\mu(t) \\ &+ \int_{(b-\lambda_2,b]} \left(f(t) - \left[\frac{f(t)}{h(t)} - \frac{f(b-\lambda_2)}{h(b-\lambda_2)} \right] h(t)[1-g(t)] \right) d\mu(t).\end{aligned} \tag{4.27}$$

If $f/h \in \mathscr{M}_2^c[a,b]$ and (4.26) holds, the inequality in (4.27) is reversed.

Proof. Similar to the proof of Theorem 4.2 applying Theorem 3.11(a) for $F/h : [a,c] \to \mathbb{R}$ nonincreasing and Theorem 3.11(b) for $F/h : [c,b] \to \mathbb{R}$ nondecreasing. $\square$

Theorem 4.11 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$ and let $c \in (a,b)$. Let f,g and h be measurable functions on $[a,b]$ such that h is positive and $0 \leq g \leq 1$. Let $\lambda_1, \lambda_2 \in \mathbb{R}_+$ be such that* (4.10) *and* (4.11) *hold. If $f/h \in \mathscr{M}_1^c[a,b]$ and*

$$\begin{aligned}\int_{[a,b]} th(t)g(t)d\mu(t) &= \int_{(c-\lambda_1,c]} (th(t) - [t-c+\lambda_1]h(t)[1-g(t)])\,d\mu(t) \\ &\quad + \int_{[c,c+\lambda_2]} (th(t) - [t-c-\lambda_2]h(t)[1-g(t)])\,d\mu(t),\end{aligned} \tag{4.28}$$

then

$$\begin{aligned}\int_{[a,b]} f(t)g(t)d\mu(t) &\geq \int_{(c-\lambda_1,c]} \left(f(t) - \left[\frac{f(t)}{h(t)} - \frac{f(c-\lambda_1)}{h(c-\lambda_1)}\right] h(t)[1-g(t)]\right) d\mu(t) \\ &\quad + \int_{[c,c+\lambda_2]} \left(f(t) - \left[\frac{f(t)}{h(t)} - \frac{f(c+\lambda_2)}{h(c+\lambda_2)}\right] h(t)[1-g(t)]\right) d\mu(t).\end{aligned} \tag{4.29}$$

If $f/h \in \mathscr{M}_2^c[a,b]$ and (4.28) *holds, the inequality in* (4.29) *is reversed.*

Proof. Similar to the proof of Theorem 4.3 applying Theorem 3.11(b) for $F/h : [a,c] \to \mathbb{R}$ nonincreasing and Theorem 3.11(a) for $F/h : [c,b] \to \mathbb{R}$ nondecreasing. □

Motivated by sharpened and generalized version of Theorem 3.1 obtained by Jakšetić, Pečarić and Smoljak in [35] we obtain the following results.

Theorem 4.12 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$ and let $c \in (a,b)$. Let f,g,h and ψ be measurable functions on $[a,b]$ such that h is positive and $0 \leq \psi \leq g \leq 1-\psi$. Let $\lambda_1, \lambda_2 \in \mathbb{R}_+$ be such that* (4.1) *and* (4.2) *hold. If $f/h \in \mathscr{M}_1^c[a,b]$ and*

$$\begin{aligned}\int_{[a,b]} th(t)g(t)d\mu(t) &= \int_{[a,a+\lambda_1]} th(t)d\mu(t) - \int_{[a,c]} |t-a-\lambda_1| h(t)\psi(t)d\mu(t) \\ &\quad + \int_{(b-\lambda_2,b]} th(t)d\mu(t) + \int_{[c,b]} |t-b+\lambda_2| h(t)\psi(t)d\mu(t),\end{aligned} \tag{4.30}$$

then

$$\begin{aligned}\int_{[a,b]} f(t)g(t)d\mu(t) &\leq \int_{[a,a+\lambda_1]} f(t)d\mu(t) - \int_{[a,c]} \left|\frac{f(t)}{h(t)} - \frac{f(a+\lambda_1)}{h(a+\lambda_1)}\right| h(t)\psi(t)d\mu(t) \\ &\quad + \int_{(b-\lambda_2,b]} f(t)d\mu(t) + \int_{[c,b]} \left|\frac{f(t)}{h(t)} - \frac{f(b-\lambda_2)}{h(b-\lambda_2)}\right| h(t)\psi(t)d\mu(t).\end{aligned} \tag{4.31}$$

If $f/h \in \mathscr{M}_2^c[a,b]$ and (4.30) *holds, the inequality in* (4.31) *is reversed.*

Proof. We use the following inequalities, which hold for f/h nonincreasing, proved in [35] (see also Theorem 3.13):

$$\int_{[a,b]} f(t)g(t)d\mu(t) \leq \int_{[a,a+\lambda]} f(t)d\mu(t) - \int_{[a,b]} \left|\frac{f(t)}{h(t)} - \frac{f(a+\lambda)}{h(a+\lambda)}\right| h(t)\psi(t)d\mu(t) \tag{4.32}$$

and

$$\int_{(b-\lambda,b]} f(t)d\mu(t) + \int_{[a,b]} \left|\frac{f(t)}{h(t)} - \frac{f(b-\lambda)}{h(b-\lambda)}\right| h(t)\psi(t)d\mu(t) \leq \int_{[a,b]} f(t)g(t)d\mu(t). \tag{4.33}$$

For f/h nondecreasing inequalities in (4.32) and (4.33) are reversed.
The proof is similar to that of Theorem 4.2 applying (4.32) for $F/h : [a,c] \to \mathbb{R}$ nonincreasing and (4.33) for $F/h : [c,b] \to \mathbb{R}$ nondecreasing, where $F(x) = f(x) - Axh(x)$. □

Theorem 4.13 *Let μ be a positive finite measure on $\mathscr{B}([a,b])$ and let $c \in (a,b)$. Let f, g, h and ψ be measurable functions on $[a,b]$ such that h is positive and $0 \leq \psi \leq g \leq 1 - \psi$. Let $\lambda_1, \lambda_2 \in \mathbb{R}_+$ be such that* (4.10) *and* (4.11) *hold. If $f/h \in \mathscr{M}_1^c[a,b]$ and*

$$\begin{aligned}\int_{[a,b]} th(t)g(t)d\mu(t) = \int_{(c-\lambda_1,c+\lambda_2]} th(t)d\mu(t) - \int_{[a,c]} |t - c + \lambda_1| h(t)\psi(t)d\mu(t) \\ + \int_{[c,b]} |t - c - \lambda_2| h(t)\psi(t)d\mu(t),\end{aligned} \tag{4.34}$$

then

$$\begin{aligned}\int_{[a,b]} f(t)g(t)d\mu(t) \geq \int_{(c-\lambda_1,c+\lambda_2]} f(t)d\mu(t) + \int_{[a,c]} \left| \frac{f(t)}{h(t)} - \frac{f(c-\lambda_1)}{h(c-\lambda_1)} \right| h(t)\psi(t)d\mu(t) \\ - \int_{[c,b]} \left| \frac{f(t)}{h(t)} - \frac{f(c+\lambda_2)}{h(c+\lambda_2)} \right| h(t)\psi(t)d\mu(t).\end{aligned} \tag{4.35}$$

If $f/h \in \mathscr{M}_2^c[a,b]$ and (4.34) *holds, the inequality in* (4.35) *is reversed.*

Proof. Similar to the proof of Theorem 4.12. □

Steffensen type inequalities obtained in this subsection also hold if the function f/h is convex (resp. concave).

4.2 Generalized Steffensen type inequalities

In this section we give results obtained in [76]. Applying generalizations of Steffensen's inequality given in Section 1.1 to functions that are convex at point c we obtain the following results.

Theorem 4.14 *Let $h : [a,b] \to \mathbb{R}$ be a positive integrable function, $f : [a,b] \to \mathbb{R}$ be an integrable function and let $c \in (a,b)$. Let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq 1$. Let λ_1 be the solution of the equation*

$$\int_a^{a+\lambda_1} h(t)dt = \int_a^c h(t)g(t)dt \tag{4.36}$$

and λ_2 be the solution of the equation

$$\int_{b-\lambda_2}^b h(t)dt = \int_c^b h(t)g(t)dt. \tag{4.37}$$

If $f/h \in \mathscr{M}_1^c[a,b]$ and

$$\int_a^b th(t)g(t)dt = \int_a^{a+\lambda_1} th(t)dt + \int_{b-\lambda_2}^b th(t)dt, \tag{4.38}$$

then

$$\int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda_1} f(t)dt + \int_{b-\lambda_2}^b f(t)dt. \tag{4.39}$$

If $f/h \in \mathscr{M}_2^c[a,b]$ and (4.38) *holds, the inequality in* (4.39) *is reversed.*

Proof. Similar to the proof of Theorem 4.2 taking $d\mu(t) = dt$. □

Remark 4.3 From the proof we deduce that condition (4.38) can be weakened. So, for $f/h \in \mathscr{M}_1^c[a,b]$ inequality (4.39) still holds if (4.38) is replaced by the weaker condition

$$A\left(\int_a^{a+\lambda_1} th(t)dt + \int_{b-\lambda_2}^b th(t)dt - \int_a^b th(t)g(t)dt\right) \geq 0, \tag{4.40}$$

where A is the constant from Definition 4.1. For $f/h \in \mathscr{M}_2^c[a,b]$ the reverse inequality in (4.39) holds if (4.38) is replaced by (4.40) with the reverse inequality.

Moreover, condition (4.40) can be further weakened if the function f/h is monotonic. Hence, if the function $f/h \in \mathscr{M}_1^c[a,b]$ is nondecreasing or $f/h \in \mathscr{M}_2^c[a,b]$ is nonincreasing, from (4.40) we obtain that (4.38) can be weakened to

$$\int_a^b th(t)g(t)dt \leq \int_a^{a+\lambda_1} th(t)dt + \int_{b-\lambda_2}^b th(t)dt. \tag{4.41}$$

Further, if $f/h \in \mathscr{M}_1^c[a,b]$ is nonincreasing or $f/h \in \mathscr{M}_2^c[a,b]$ is nondecreasing, (4.38) can be weakened to (4.41) with the reverse inequality.

Theorem 4.15 *Let $h : [a,b] \to \mathbb{R}$ be a positive integrable function, $f : [a,b] \to \mathbb{R}$ be an integrable function and let $c \in (a,b)$. Let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq 1$. Let λ_1 be the solution of the equation*

$$\int_{c-\lambda_1}^c h(t)dt = \int_a^c h(t)g(t)dt \tag{4.42}$$

and λ_2 be the solution of the equation

$$\int_c^{c+\lambda_2} h(t)dt = \int_c^b h(t)g(t)dt. \tag{4.43}$$

If $f/h \in \mathscr{M}_1^c[a,b]$ and

$$\int_a^b th(t)g(t)dt = \int_{c-\lambda_1}^{c+\lambda_2} th(t)dt, \tag{4.44}$$

then

$$\int_a^b f(t)g(t)dt \geq \int_{c-\lambda_1}^{c+\lambda_2} f(t)dt. \tag{4.45}$$

If $f/h \in \mathscr{M}_2^c[a,b]$ and (4.44) *holds, the inequality in* (4.45) *is reversed.*

Proof. Similar to the proof of Theorem 4.3 taking $d\mu(t) = dt$. □

Remark 4.4 For $f/h \in \mathscr{M}_1^c[a,b]$ inequality (4.45) still holds if condition (4.44) is replaced by the weaker condition

$$A\left(\int_a^b th(t)g(t)dt - \int_{c-\lambda_1}^{c+\lambda_2} th(t)dt\right) \geq 0, \tag{4.46}$$

where A is the constant from Definition 4.1. Also, for $f/h \in \mathscr{M}_2^c[a,b]$ the reverse inequality in (4.45) holds if (4.44) is replaced by (4.46) with the reverse inequality.

Additionally, condition (4.46) can be further weakened if the function f/h is monotonic. Similar as in Remark 4.3, if the function $f/h \in \mathscr{M}_1^c[a,b]$ is nondecreasing or $f/h \in \mathscr{M}_2^c[a,b]$ is nonincreasing, from (4.46) we obtain that (4.44) can be weakened to

$$\int_a^b th(t)g(t)dt \geq \int_{c-\lambda_1}^{c+\lambda_2} th(t)dt. \tag{4.47}$$

Further, if $f/h \in \mathscr{M}_1^c[a,b]$ is nonincreasing or $f/h \in \mathscr{M}_2^c[a,b]$ is nondecreasing, (4.44) can be weakened to (4.47) with the reverse inequality.

As a consequence of Theorems 4.14 and 4.15 we obtain generalized Steffensen type inequalities that involve convex functions.

Corollary 4.3 *Let $h : [a,b] \to \mathbb{R}$ be a positive integrable function, $f : [a,b] \to \mathbb{R}$ be an integrable function and let $c \in (a,b)$. Let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq 1$. Let λ_1 be the solution of the equation* (4.36) *and λ_2 be the solution of the equation* (4.37)*. If $f/h : [a,b] \to \mathbb{R}$ is convex and* (4.38) *holds, then the inequality* (4.39) *holds.*
If $f/h : [a,b] \to \mathbb{R}$ is concave, the inequality in (4.39) *is reversed.*

Proof. Since f/h is convex, from Theorem 4.1, we have that $f/h \in \mathscr{M}_1^c[a,b]$ for every $c \in (a,b)$. Hence, we can apply Theorem 4.14. □

Corollary 4.4 *Let $h : [a,b] \to \mathbb{R}$ be a positive integrable function, $f : [a,b] \to \mathbb{R}$ be an integrable function and let $c \in (a,b)$. Let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq 1$. Let λ_1 be the solution of the equation* (4.42) *and λ_2 be the solution of the equation* (4.43)*. If $f/h : [a,b] \to \mathbb{R}$ is convex and* (4.44) *holds, then the inequality* (4.45) *holds.*
If $f/h : [a,b] \to \mathbb{R}$ is concave, the inequality in (4.45) *is reversed.*

Proof. Similar to the proof of Corollary 4.3 applying Theorem 4.15. □

Similar as in Remarks 4.3 and 4.4 we obtain that conditions (4.38) and (4.44) in Corollaries 4.3 and 4.4 can be weakened if, additionally, the function f/h is monotonic.

For $h \equiv 1$ in Theorems 4.14 and 4.15 and Corollaries 4.3 and 4.4 we obtain the results given in [75].

In Theorem 1.9 Pečarić, Perušić and Smoljak gave a corrected version of Mercer's result which follows from Theorems 1.6 and 1.7. In the following theorems we obtain generalizations of these results for functions from the class $\mathscr{M}_1^c[a,b]$.

Theorem 4.16 *Let $h : [a,b] \to \mathbb{R}$ be a positive integrable function, $f : [a,b] \to \mathbb{R}$ be an integrable function and let $c \in (a,b)$. Let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \le g \le h$. Let λ_1 be the solution of the equation $\int_a^{a+\lambda_1} h(t)dt = \int_a^c g(t)dt$ and λ_2 be the solution of the equation $\int_{b-\lambda_2}^b h(t)dt = \int_c^b g(t)dt$. If $f \in \mathscr{M}_1^c[a,b]$ and*

$$\int_a^b tg(t)dt = \int_a^{a+\lambda_1} th(t)dt + \int_{b-\lambda_2}^b th(t)dt, \tag{4.48}$$

then

$$\int_a^b f(t)g(t)dt \le \int_a^{a+\lambda_1} f(t)h(t)dt + \int_{b-\lambda_2}^b f(t)h(t)dt. \tag{4.49}$$

If $f \in \mathscr{M}_2^c[a,b]$ and (4.48) *holds, the inequality in* (4.49) *is reversed.*

Proof. Take $g \mapsto g/h$ and $f \mapsto fh$ in Theorem 4.14. □

Theorem 4.17 *Let $h : [a,b] \to \mathbb{R}$ be a positive integrable function, $f : [a,b] \to \mathbb{R}$ be an integrable function and let $c \in (a,b)$. Let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \le g \le h$. Let λ_1 be the solution of the equation $\int_{c-\lambda_1}^c h(t)dt = \int_a^c g(t)dt$ and λ_2 be the solution of the equation $\int_c^{c+\lambda_2} h(t)dt = \int_c^b g(t)dt$. If $f \in \mathscr{M}_1^c[a,b]$ and*

$$\int_a^b tg(t)dt = \int_{c-\lambda_1}^{c+\lambda_2} th(t)dt, \tag{4.50}$$

then

$$\int_a^b f(t)g(t)dt \ge \int_{c-\lambda_1}^{c+\lambda_2} f(t)h(t)dt. \tag{4.51}$$

If $f \in \mathscr{M}_2^c[a,b]$ and (4.50) *holds, the inequality in* (4.51) *is reversed.*

Proof. Take $g \mapsto g/h$ and $f \mapsto fh$ in Theorem 4.15. □

In [61, Theorem 2.6] Pečarić, Perušić and Smoljak showed that Mercer's generalization [46, Theorem 3] is equivalent to Theorem 1.6. Further, in [61, Theorem 2.7] they obtained analogue theorem equivalent to Theorem 1.7. Motivated by mentioned generalizations in the following theorems we obtain generalizations for functions from class $\mathscr{M}_1^c[a,b]$.

Theorem 4.18 *Let $h : [a,b] \to \mathbb{R}$ be a positive integrable function, $f : [a,b] \to \mathbb{R}$ be an integrable function and let $c \in (a,b)$. Let $g,k : [a,b] \to \mathbb{R}$ be integrable functions such that $0 \le g \le k$. Let λ_1 be the solution of the equation*

$$\int_a^{a+\lambda_1} k(t)h(t)dt = \int_a^c h(t)g(t)dt$$

and λ_2 be the solution of the equation

$$\int_{b-\lambda_2}^b k(t)h(t)dt = \int_c^b h(t)g(t)dt.$$

If $f/h \in \mathscr{M}_1^c[a,b]$ *and*

$$\int_a^b th(t)g(t)dt = \int_a^{a+\lambda_1} tk(t)h(t)dt + \int_{b-\lambda_2}^b tk(t)h(t)dt, \tag{4.52}$$

then

$$\int_a^b f(t)g(t)dt \le \int_a^{a+\lambda_1} f(t)k(t)dt + \int_{b-\lambda_2}^b f(t)k(t)dt. \tag{4.53}$$

If $f/h \in \mathscr{M}_2^c[a,b]$ *and* (4.52) *holds, the inequality in* (4.53) *is reversed.*

Proof. Take $h \mapsto kh$, $g \mapsto g/k$ and $f \mapsto fk$ in Theorem 4.14. □

Theorem 4.19 *Let* $h : [a,b] \to \mathbb{R}$ *be a positive integrable function,* $f : [a,b] \to \mathbb{R}$ *be an integrable function and let* $c \in (a,b)$*. Let* $g,k : [a,b] \to \mathbb{R}$ *be integrable functions such that* $0 \le g \le k$*. Let* λ_1 *be the solution of the equation*

$$\int_{c-\lambda_1}^c k(t)h(t)dt = \int_a^c h(t)g(t)dt$$

and λ_2 *be the solution of the equation*

$$\int_c^{c+\lambda_2} k(t)h(t)dt = \int_c^b h(t)g(t)dt.$$

If $f/h \in \mathscr{M}_1^c[a,b]$ *and*

$$\int_a^b th(t)g(t)dt = \int_{c-\lambda_1}^{c+\lambda_2} tk(t)h(t)dt, \tag{4.54}$$

then

$$\int_a^b f(t)g(t)dt \ge \int_{c-\lambda_1}^{c+\lambda_2} f(t)k(t)dt. \tag{4.55}$$

If $f/h \in \mathscr{M}_2^c[a,b]$ *and* (4.54) *holds, the inequality in* (4.55) *is reversed.*

Proof. Take $h \mapsto kh$, $g \mapsto g/k$ and $f \mapsto fk$ in Theorem 4.15. □

Remark 4.5 Taking $k \equiv 1$ in Theorems 4.18 and 4.19 we obtain Theorems 4.14 and 4.15, respectively.

In the following theorems we obtain refined version of results given in Theorems 4.14 and 4.15.

Theorem 4.20 *Let* $h : [a,b] \to \mathbb{R}$ *be a positive integrable function,* $f : [a,b] \to \mathbb{R}$ *be an integrable function and let* $c \in (a,b)$*. Let* $g : [a,b] \to \mathbb{R}$ *be an integrable function such that* $0 \le g \le 1$*. Let* λ_1 *be the solution of the equation* (4.36) *and* λ_2 *be the solution of the equation* (4.37)*. If* $f/h \in \mathscr{M}_1^c[a,b]$ *and*

$$\begin{aligned}\int_a^b th(t)g(t)dt = &\int_a^{a+\lambda_1} (th(t) - [t-a-\lambda_1]h(t)[1-g(t)])\,dt \\ &+ \int_{b-\lambda_2}^b (th(t) - [t-b+\lambda_2]h(t)[1-g(t)])\,dt,\end{aligned} \tag{4.56}$$

then

$$\begin{aligned}\int_a^b f(t)g(t)dt \leq &\int_a^{a+\lambda_1}\left(f(t)-\left[\frac{f(t)}{h(t)}-\frac{f(a+\lambda_1)}{h(a+\lambda_1)}\right]h(t)[1-g(t)]\right)dt\\&+\int_{b-\lambda_2}^b\left(f(t)-\left[\frac{f(t)}{h(t)}-\frac{f(b-\lambda_2)}{h(b-\lambda_2)}\right]h(t)[1-g(t)]\right)dt.\end{aligned}\tag{4.57}$$

If $f/h \in \mathscr{M}_2^c[a,b]$ and (4.56) *holds, the inequality in* (4.57) *is reversed.*

Proof. Similar to the proof of Theorem 4.14 applying Corollary 1.1 for $F/h : [a,c] \to \mathbb{R}$ nonincreasing and Corollary 1.2 for $F/h : [c,b] \to \mathbb{R}$ nondecreasing. □

Theorem 4.21 *Let $h : [a,b] \to \mathbb{R}$ be a positive integrable function, $f : [a,b] \to \mathbb{R}$ be an integrable function and let $c \in (a,b)$. Let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq 1$. Let λ_1 be the solution of the equation* (4.42) *and λ_2 be the solution of the equation* (4.43)*. If $f/h \in \mathscr{M}_1^c[a,b]$ and*

$$\begin{aligned}\int_a^b th(t)g(t)dt = &\int_{c-\lambda_1}^c (th(t)-[t-c+\lambda_1]h(t)[1-g(t)])dt\\&+\int_c^{c+\lambda_2}(th(t)-[t-c-\lambda_2]h(t)[1-g(t)])dt,\end{aligned}\tag{4.58}$$

then

$$\begin{aligned}\int_a^b f(t)g(t)dt \geq &\int_{c-\lambda_1}^c\left(f(t)-\left[\frac{f(t)}{h(t)}-\frac{f(c-\lambda_1)}{h(c-\lambda_1)}\right]h(t)[1-g(t)]\right)dt\\&+\int_c^{c+\lambda_2}\left(f(t)-\left[\frac{f(t)}{h(t)}-\frac{f(c+\lambda_2)}{h(c+\lambda_2)}\right]h(t)[1-g(t)]\right)dt.\end{aligned}\tag{4.59}$$

If $f/h \in \mathscr{M}_2^c[a,b]$ and (4.58) *holds, the inequality in* (4.59) *is reversed.*

Proof. Similar to the proof of Theorem 4.15 applying Corollary 1.2 for $F/h : [a,c] \to \mathbb{R}$ nonincreasing and Corollary 1.1 for $F/h : [c,b] \to \mathbb{R}$ nondecreasing. □

Motivated by sharpened and generalized versions of Theorems 1.6 and 1.7 obtained by Pečarić, Perušić and Smoljak in [61, Corollaries 2.4 and 2.5] we obtain the following results.

Theorem 4.22 *Let $h : [a,b] \to \mathbb{R}$ be a positive integrable function, $f : [a,b] \to \mathbb{R}$ be an integrable function and let $c \in (a,b)$. Let $g,\psi : [a,b] \to \mathbb{R}$ be integrable functions such that $0 \leq \psi \leq g \leq 1-\psi$. Let λ_1 be the solution of the equation* (4.36) *and λ_2 be the solution of the equation* (4.37)*. If $f/h \in \mathscr{M}_1^c[a,b]$ and*

$$\begin{aligned}\int_a^b th(t)g(t)dt = &\int_a^{a+\lambda_1} th(t)dt - \int_a^c |t-a-\lambda_1|h(t)\psi(t)dt\\&+\int_{b-\lambda_2}^b th(t)dt + \int_c^b |t-b+\lambda_2|h(t)\psi(t)dt,\end{aligned}\tag{4.60}$$

then

$$\begin{aligned}\int_a^b f(t)g(t)dt \le \int_a^{a+\lambda_1} f(t)dt - \int_a^c \left|\frac{f(t)}{h(t)} - \frac{f(a+\lambda_1)}{h(a+\lambda_1)}\right| h(t)\psi(t)dt \\ + \int_{b-\lambda_2}^b f(t)dt + \int_c^b \left|\frac{f(t)}{h(t)} - \frac{f(b-\lambda_2)}{h(b-\lambda_2)}\right| h(t)\psi(t)dt.\end{aligned} \tag{4.61}$$

If $f/h \in \mathscr{M}_2^c[a,b]$ and (4.60) *holds, the inequality in* (4.61) *is reversed.*

Proof. Similar to the proof of Theorem 4.14 applying [61, Corollary 2.3] for $F/h : [a,c] \to \mathbb{R}$ nonincreasing and [61, Corollary 2.4] for $F/h : [c,b] \to \mathbb{R}$ nondecreasing. □

Theorem 4.23 *Let $h : [a,b] \to \mathbb{R}$ be a positive integrable function, $f : [a,b] \to \mathbb{R}$ be an integrable function and let $c \in (a,b)$. Let $g, \psi : [a,b] \to \mathbb{R}$ be integrable functions such that $0 \le \psi \le g \le 1 - \psi$. Let λ_1 be the solution of the equation* (4.42) *and λ_2 be the solution of the equation* (4.43). *If $f/h \in \mathscr{M}_1^c[a,b]$ and*

$$\begin{aligned}\int_a^b th(t)g(t)dt = \int_{c-\lambda_1}^{c+\lambda_2} th(t)dt - \int_a^c |t-c+\lambda_1| h(t)\psi(t)dt \\ + \int_c^b |t-c-\lambda_2| h(t)\psi(t)dt,\end{aligned} \tag{4.62}$$

then

$$\begin{aligned}\int_a^b f(t)g(t)dt \ge \int_{c-\lambda_1}^{c+\lambda_2} f(t)dt + \int_a^c \left|\frac{f(t)}{h(t)} - \frac{f(c-\lambda_1)}{h(c-\lambda_1)}\right| h(t)\psi(t)dt \\ - \int_c^b \left|\frac{f(t)}{h(t)} - \frac{f(c+\lambda_2)}{h(c+\lambda_2)}\right| h(t)\psi(t)dt.\end{aligned} \tag{4.63}$$

If $f/h \in \mathscr{M}_2^c[a,b]$ and (4.62) *holds, the inequality in* (4.63) *is reversed.*

Proof. Similar to the proof of Theorem 4.15 applying [61, Corollary 2.4] for $F/h : [a,c] \to \mathbb{R}$ nonincreasing and [61, Corollary 2.3] for $F/h : [c,b] \to \mathbb{R}$ nondecreasing. □

Remark 4.6 Generalized Steffensen type inequalities obtained in Theorems 4.18-4.23 also hold if the function f/h is convex (concave). This follows from Theorem 4.1, i.e. if f/h is a convex function then $f/h \in \mathscr{M}_1^c[a,b]$ for every $c \in (a,b)$.

Similar as in Remarks 4.3 and 4.4 we obtain that conditions (4.48), (4.50), (4.52), (4.54), (4.56), (4.58), (4.60), (4.62) can be weakened, but here we omit the details.

4.3 New Steffensen type inequalities

Motivated by weaker conditions given in Theorem 1.3 in the following theorem we obtain weaker conditions on the function g for Steffensen type inequalities obtained by Masjed-Jamei, Qi and Srivastava given in Theorem 1.2. Results presented in this section were obtained by Pečarić and Smoljak Kalamir in [77].

Theorem 4.24 *Let g be an integrable function on $[a,b]$ with $\sigma = q\int_a^b g(t)dt \in [0, b-a]$, where $q \neq 0$.*

a) The second inequality in (1.6) holds for every nonincreasing function f on $[a,b]$ if and only if

$$\int_a^x g(t)dt \leq (x-a)\left[1 - \frac{\sigma}{b-a}\left(1-\frac{1}{q}\right)\right] \text{ and } \int_x^b g(t)dt \geq -\frac{\sigma(b-x)}{b-a}\left(1-\frac{1}{q}\right),$$

for every $x \in [a,b]$.

b) The first inequality in (1.6) holds for every nonincreasing function f on $[a,b]$ if and only if

$$\int_x^b g(t)dt \leq (b-x)\left[1 - \frac{\sigma}{b-a}\left(1-\frac{1}{q}\right)\right] \text{ and } \int_a^x g(t)dt \geq -\frac{\sigma(x-a)}{b-a}\left(1-\frac{1}{q}\right),$$

for every $x \in [a,b]$.

Proof. We apply Theorem 1.3 to the function

$$G(x) = g(x) + \frac{q-1}{b-a}\int_a^b g(t)dt, \quad x \in [a,b]$$

with $\int_a^b G(x)dx = \sigma$. □

Note that for $q = 1$ Theorem 4.24 reduces to Theorem 1.3.

In the following theorems we obtain new Steffensen type inequalities for class of functions that are convex at point c.

Theorem 4.25 *Let $g : [a,b] \to \mathbb{R}$ be an integrable function. For given $c \in (a,b)$ and $q \neq 0$, denote $\sigma_1 = q\int_a^c g(t)dt$ and $\sigma_2 = q\int_c^b g(t)dt$. Assume*

$$-\frac{\sigma_1}{c-a}\left(1-\frac{1}{q}\right) \leq g(x) \leq 1 - \frac{\sigma_1}{c-a}\left(1-\frac{1}{q}\right), \text{ for } x \in (a,c), \tag{4.64}$$

$$-\frac{\sigma_2}{b-c}\left(1-\frac{1}{q}\right) \leq g(x) \leq 1 - \frac{\sigma_2}{b-c}\left(1-\frac{1}{q}\right), \text{ for } x \in (c,b), \tag{4.65}$$

and

$$\int_a^b tg(t)dt = a\sigma_1 + b\sigma_2 + \frac{\sigma_1^2 - \sigma_2^2}{2} - \left(1-\frac{1}{q}\right)\frac{\sigma_1(a+c)+\sigma_2(c+b)}{2}. \tag{4.66}$$

If $f \in \mathscr{M}_1^c[a,b]$ then

$$\begin{aligned}\int_a^b f(t)g(t)dt \leq \int_a^{a+\sigma_1} f(t)dt - \frac{\sigma_1}{c-a}\left(1-\frac{1}{q}\right)\int_a^c f(t)dt \\ + \int_{b-\sigma_2}^b f(t)dt - \frac{\sigma_2}{b-c}\left(1-\frac{1}{q}\right)\int_c^b f(t)dt.\end{aligned} \tag{4.67}$$

If $f \in \mathscr{M}_2^c[a,b]$ then the inequality in (4.67) *is reversed.*

Proof. We give the proof for $f \in \mathscr{M}_1^c[a,b]$, the proof for $f \in \mathscr{M}_2^c[a,b]$ is similar so we omit the details. Let A be the constant from Definition 1.12 and let us consider the function $F:[a,b]\to\mathbb{R}$, $F(x) = f(x) - Ax$. Since F is nonincreasing on $[a,c]$ we can apply the second inequality in (1.6) to functions F and g. So we have

$$\int_a^c F(t)g(t)dt \leq \int_a^{a+\sigma_1} F(t)dt - \frac{\sigma_1}{c-a}\left(1-\frac{1}{q}\right)\int_a^c F(t)dt.$$

Hence, we obtain

$$\begin{aligned}0 &\leq \int_a^{a+\sigma_1} F(t)dt - \frac{\sigma_1}{c-a}\left(1-\frac{1}{q}\right)\int_a^c F(t)dt - \int_a^c F(t)g(t)dt \\ &= \int_a^{a+\sigma_1} f(t)dt - \frac{\sigma_1}{c-a}\left(1-\frac{1}{q}\right)\int_a^c f(t)dt - \int_a^c f(t)g(t)dt \\ &-A\left(a\sigma_1 + \frac{\sigma_1^2}{2} - \frac{\sigma_1(a+c)}{2}\left(1-\frac{1}{q}\right) - \int_a^c tg(t)dt\right).\end{aligned} \tag{4.68}$$

Further, since F is nondecreasing on $[c,b]$ the first inequality in (1.6) applied to the functions F and g is reversed. So we have

$$\int_c^b F(t)g(t)dt \leq \int_{b-\sigma_2}^b F(t)dt - \frac{\sigma_2}{b-c}\left(1-\frac{1}{q}\right)\int_c^b F(t)dt.$$

Hence, we obtain

$$\begin{aligned}0 &\leq \int_{b-\sigma_2}^b f(t)dt - \frac{\sigma_2}{b-c}\left(1-\frac{1}{q}\right)\int_c^b f(t)dt - \int_c^b f(t)g(t)dt \\ &-A\left(b\sigma_2 - \frac{\sigma_2^2}{2} - \frac{\sigma_2(c+b)}{2}\left(1-\frac{1}{q}\right) - \int_c^b tg(t)dt\right).\end{aligned} \tag{4.69}$$

Now combining (4.68) and (4.69) we obtain

$$\begin{aligned}&\int_a^{a+\sigma_1} f(t)dt - \frac{\sigma_1}{c-a}\left(1-\frac{1}{q}\right)\int_a^c f(t)dt + \int_{b-\sigma_2}^b f(t)dt \\ &- \frac{\sigma_2}{b-c}\left(1-\frac{1}{q}\right)\int_c^b f(t)dt - \int_a^b f(t)g(t)dt \\ &\geq A\left(a\sigma_1 + b\sigma_2 + \frac{\sigma_1^2-\sigma_2^2}{2} - \left(1-\frac{1}{q}\right)\frac{\sigma_1(a+c)+\sigma_2(c+b)}{2} - \int_a^b tg(t)dt\right).\end{aligned}$$

Now, from (4.66), we conclude that (4.67) holds. □

Remark 4.7 From the proof we see that the condition (4.66) in Theorem 4.25 can be replaced by the weaker condition

$$A\left(a\sigma_1+b\sigma_2+\frac{\sigma_1^2-\sigma_2^2}{2}-\left(1-\frac{1}{q}\right)\frac{\sigma_1(a+c)+\sigma_2(c+b)}{2}-\int_a^b tg(t)dt\right)\geq 0 \quad (4.70)$$

for $f\in\mathscr{M}_1^c[a,b]$, where A is the constant from Definition 1.12. Further, for $f\in\mathscr{M}_2^c[a,b]$ the condition (4.66) can be replaced by (4.70) with the reverse inequality.

Additionaly, the condition (4.66) can be further weakened if the function f is monotonic. For example, if $f\in\mathscr{M}_1^c[a,b]$ is nondecreasing, then clearly $A\geq 0$. So, from from (4.70) we obtain that (4.66) can be weakened to

$$\int_a^b tg(t)dt\leq a\sigma_1+b\sigma_2+\frac{\sigma_1^2-\sigma_2^2}{2}-\left(1-\frac{1}{q}\right)\frac{\sigma_1(a+c)+\sigma_2(c+b)}{2}. \quad (4.71)$$

Further, if $f\in\mathscr{M}_1^c[a,b]$ is nonincreasing or $f\in\mathscr{M}_2^c[a,b]$ is nondecreasing, (4.66) can be weakened to (4.71) with the reverse inequality.

Theorem 4.26 *Let $g:[a,b]\to\mathbb{R}$ be an integrable function. For given $c\in(a,b)$ and $q\neq 0$, denote $\sigma_1=q\int_a^c g(t)dt$ and $\sigma_2=q\int_c^b g(t)dt$. Assume* (4.64) *and* (4.65) *hold and*

$$\int_a^b tg(t)dt=c(\sigma_1+\sigma_2)+\frac{\sigma_2^2-\sigma_1^2}{2}-\frac{\sigma_2(c+b)+\sigma_1(a+c)}{2}\left(1-\frac{1}{q}\right). \quad (4.72)$$

If $f\in\mathscr{M}_1^c[a,b]$ then

$$\int_a^b f(t)g(t)dt\geq\int_{c-\sigma_1}^{c+\sigma_2} f(t)dt-\left(\frac{\sigma_1}{c-a}\int_a^c f(t)dt+\frac{\sigma_2}{b-c}\int_c^b f(t)dt\right)\left(1-\frac{1}{q}\right). \quad (4.73)$$

If $f\in\mathscr{M}_2^c[a,b]$ then the inequality in (4.73) *is reversed.*

Proof. We give the proof for $f\in\mathscr{M}_1^c[a,b]$. Let A be the constant from Definition 1.12 and let us consider the function $F:[a,b]\to\mathbb{R}$, $F(x)=f(x)-Ax$. Since F is nonincreasing on $[a,c]$ we can apply the first inequality in (1.6) to the functions F and g. So we have

$$\begin{aligned}0\leq&\int_a^c f(t)g(t)dt-\int_{c-\sigma_1}^c f(t)dt+\frac{\sigma_1}{c-a}\left(1-\frac{1}{q}\right)\int_a^c f(t)dt\\&-A\left(\int_a^c tg(t)dt-c\sigma_1+\frac{\sigma_1^2}{2}+\frac{\sigma_1(a+c)}{2}\left(1-\frac{1}{q}\right)\right).\end{aligned} \quad (4.74)$$

Further, since F is nondecreasing on $[c,b]$ the second inequality in (1.6) applied to the functions F and g is reversed. So we have

$$\begin{aligned}0\leq&\int_c^b f(t)g(t)dt-\int_c^{c+\sigma_2} f(t)dt+\frac{\sigma_2}{b-c}\left(1-\frac{1}{q}\right)\int_c^b f(t)dt\\&-A\left(\int_c^b tg(t)dt-c\sigma_2-\frac{\sigma_2^2}{2}+\frac{\sigma_2(c+b)}{2}\left(1-\frac{1}{q}\right)\right).\end{aligned} \quad (4.75)$$

Now from (4.74) and (4.75) we obtain

$$\int_a^b f(t)g(t)dt - \int_{c-\sigma_1}^{c+\sigma_2} f(t)dt + \left(\frac{\sigma_1}{c-a}\int_a^c f(t)dt + \frac{\sigma_2}{b-c}\int_c^b f(t)dt\right)\left(1-\frac{1}{q}\right)$$
$$\geq A\left(\int_a^b tg(t)dt - c(\sigma_1+\sigma_2) + \frac{\sigma_1^2-\sigma_2^2}{2} + \frac{\sigma_1(a+c)+\sigma_2(c+b)}{2}\left(1-\frac{1}{q}\right)\right).$$

Now, from (4.72), we conclude that (4.73) holds.
Proof for $f \in \mathscr{M}_2^c[a,b]$ is similar so we omit the details. □

Remark 4.8 Similarly as in Remark 4.7, the condition (4.72) in Theorem 4.26 can be replaced by the weaker condition

$$A\left(\int_a^b tg(t)dt - c(\sigma_1+\sigma_2) + \frac{\sigma_1^2-\sigma_2^2}{2} + \frac{\sigma_1(a+c)+\sigma_2(c+b)}{2}\left(1-\frac{1}{q}\right)\right) \geq 0 \quad (4.76)$$

for $f \in \mathscr{M}_1^c[a,b]$ and by (4.76) with the reverse inequality for $f \in \mathscr{M}_2^c[a,b]$.

Additionaly, it can be further weakened if the function f is monotonic. For a nondecreasing function $f \in \mathscr{M}_1^c[a,b]$ or nonincreasing function $f \in \mathscr{M}_2^c[a,b]$, from (4.76) we obtain that (4.72) can be weakened to

$$\int_a^b tg(t)dt \geq c(\sigma_1+\sigma_2) + \frac{\sigma_2^2-\sigma_1^2}{2} - \frac{\sigma_2(c+b)+\sigma_1(a+c)}{2}\left(1-\frac{1}{q}\right). \quad (4.77)$$

Further, if $f \in \mathscr{M}_1^c[a,b]$ is nonincreasing or $f \in \mathscr{M}_2^c[a,b]$ is nondecreasing, (4.72) can be weakened to (4.77) with the reverse inequality.

As a consequence of Theorems 4.25 and 4.26 we obtain the following new Steffensen type inequalities that involve convex functions.

Corollary 4.5 *Let $g:[a,b]\to\mathbb{R}$ be an integrable function. For given $c\in(a,b)$ and $q\neq 0$ denote $\sigma_1 = q\int_a^c g(t)dt$ and $\sigma_2 = q\int_c^b g(t)dt$. Assume* (4.64), (4.65) *and* (4.66) *hold. If $f:[a,b]\to\mathbb{R}$ is convex function, then* (4.67) *holds.*
If $f:[a,b]\to\mathbb{R}$ is concave function, the inequality in (4.67) *is reversed.*

Proof. Since f is convex from Lemma 1.1 we have that $f\in\mathscr{M}_1^c[a,b]$, for every $c\in(a,b)$. Hence we can apply Theorem 4.25. □

Corollary 4.6 *Let $g:[a,b]\to\mathbb{R}$ be an integrable function. For given $c\in(a,b)$ and $q\neq 0$ denote $\sigma_1 = q\int_a^c g(t)dt$ and $\sigma_2 = q\int_c^b g(t)dt$. Assume* (4.64), (4.65) *and* (4.72) *hold. If $f:[a,b]\to\mathbb{R}$ is convex function, then* (4.73) *holds.*
If $f:[a,b]\to\mathbb{R}$ is concave function, the inequality in (4.73) *is reversed.*

Proof. Similar to the proof of Corollary 4.5. □

Motivated by weaker conditions obtained in Theorem 4.24 in the following theorems we give weaker conditions for previously obtained Steffensen type inequalities involving the class $\mathscr{M}_1^c[a,b]$.

Theorem 4.27 *Let $g:[a,b]\to\mathbb{R}$ be an integrable function. Consider $c\in(a,b)$ and $q\neq 0$. Assume $\sigma_1 = q\int_a^c g(t)dt \in [0,c-a]$ and $\sigma_2 = q\int_c^b g(t)dt\in[0,b-c]$, such that* (4.66) *holds. Then* (4.67) *holds for all $f\in\mathscr{M}_1^c[a,b]$ if and only if*

$$\int_a^x g(t)dt \leq (x-a)\left[1-\frac{\sigma_1}{c-a}\left(1-\frac{1}{q}\right)\right] \text{ and } \int_x^c g(t)dt \geq -\frac{\sigma_1(c-x)}{c-a}\left(1-\frac{1}{q}\right), \tag{4.78}$$

for every $x\in[a,c]$ and

$$\int_x^b g(t)dt \leq (b-x)\left[1-\frac{\sigma_2}{b-c}\left(1-\frac{1}{q}\right)\right] \text{ and } \int_c^x g(t)dt \geq -\frac{\sigma_2(x-c)}{b-c}\left(1-\frac{1}{q}\right), \tag{4.79}$$

for every $x\in[c,b]$.
Further, the reverse inequality in (4.67) *holds for all $f\in\mathscr{M}_2^c[a,b]$ if and only if* (4.78) *and* (4.79) *hold.*

Proof. Similar to the proof of Theorem 4.25 using weaker conditions obtained in Theorem 4.24. □

Theorem 4.28 *Let $g:[a,b]\to\mathbb{R}$ be an integrable function. Consider $c\in(a,b)$ and $q\neq 0$. Assume $\sigma_1 = q\int_a^c g(t)dt \in [0,c-a]$ and $\sigma_2 = q\int_c^b g(t)dt\in[0,b-c]$, such that* (4.72) *holds. Then* (4.73) *holds for all $f\in\mathscr{M}_1^c[a,b]$ if and only if*

$$\int_x^c g(t)dt \leq (c-x)\left[1-\frac{\sigma_1}{c-a}\left(1-\frac{1}{q}\right)\right] \text{ and } \int_a^x g(t)dt \geq -\frac{\sigma_1(x-a)}{c-a}\left(1-\frac{1}{q}\right), \tag{4.80}$$

for every $x\in[a,c]$ and

$$\int_c^x g(t)dt \leq (x-c)\left[1-\frac{\sigma_2}{b-c}\left(1-\frac{1}{q}\right)\right] \text{ and } \int_x^b g(t)dt \geq -\frac{\sigma_2(b-x)}{b-c}\left(1-\frac{1}{q}\right), \tag{4.81}$$

for every $x\in[c,b]$.
Further, the reverse inequality in (4.73) *holds for all $f\in\mathscr{M}_2^c[a,b]$ if and only if* (4.80) *and* (4.81) *hold.*

Proof. Similar to the proof of Theorem 4.27. □

Further, as a consequence of Theorems 4.27 and 4.28 we obtain the following Steffensen type inequalities that involve convex functions.

Corollary 4.7 *Let $g:[a,b]\to\mathbb{R}$ be an integrable function. Consider $c\in(a,b)$ and $q\neq 0$. Assume $\sigma_1 = q\int_a^c g(t)dt$ and $\sigma_2 = q\int_c^b g(t)dt$, such that* (4.66) *holds. Then* (4.67) *holds for all convex functions $f:[a,b]\to\mathbb{R}$ if and only if* (4.78) *and* (4.79) *hold. Further, the reverse inequality in* (4.67) *holds for all concave functions $f:[a,b]\to\mathbb{R}$ if and only if* (4.78) *and* (4.79) *hold.*

Proof. Similar to the proof of Corollary 4.5. □

Corollary 4.8 *Let $g : [a,b] \to \mathbb{R}$ be an integrable function. Consider $c \in (a,b)$ and $q \neq 0$. Assume $\sigma_1 = q\int_a^c g(t)dt$ and $\sigma_2 = q\int_c^b g(t)dt$, such that* (4.72) *holds. Then* (4.73) *holds for all convex functions $f : [a,b] \to \mathbb{R}$ if and only if* (4.80) *and* (4.81) *hold. Further, the reverse inequality in* (4.73) *holds for all concave functions $f : [a,b] \to \mathbb{R}$ if and only if* (4.80) *and* (4.81) *hold.*

Proof. Similar to the proof of Corollary 4.5. □

4.4 Gauss-Steffensen type inequalities

In this section we present some new Gauss-Steffensen type inequalities which involve convex functions which were motivated by Gauss-Steffensen's inequality given in Theorem 1.17. These results were obtained in [78].

In the following theorem we obtain Gauss-Steffensen type inequality for class of functions that are "convex at point c".

Theorem 4.29 *Let $G : [a,b] \to \mathbb{R}$ be an increasing function such that $G(x) \geq x$ and let $c \in (a,b)$. If $f \in \mathcal{M}_1^c(I)$ and*

$$\int_a^c G(t)dt - \int_c^b G(t)dt = 2cG(c) - aG(a) - bG(b) + \frac{G^2(b) + G^2(a) - 2G^2(c)}{2}, \quad (4.82)$$

then

$$\int_a^c f(t)G'(t)dt - \int_c^b f(t)G'(t)dt \geq \int_{G(a)}^{G(c)} f(t)dt - \int_{G(c)}^{G(b)} f(t)dt \quad (4.83)$$

holds.
If $f \in \mathcal{M}_2^c(I)$ and (4.82) *holds, the inequality in* (4.83) *is reversed.*

Proof. Let A be the constant from Definition 1.12 and let $f \in \mathcal{M}_1^c(I)$. We have $c \in (a,b) \subseteq I^\circ$. Let us consider the function $F : I \to \mathbb{R}$, $F(x) = f(x) - Ax$. Since F is nonincreasing on $I \cap (-\infty, c]$ we can apply inequality (1.21) to the function F, so

$$\int_{G(a)}^{G(c)} F(t)dt \leq \int_a^c F(t)G'(t)dt.$$

Hence, we obtain

$$\begin{aligned} 0 &\leq \int_a^c F(t)G'(t)dt - \int_{G(a)}^{G(c)} F(t)dt \\ &= \int_a^c f(t)G'(t)dt - \int_{G(a)}^{G(c)} f(t)dt - A\left(cG(c) - aG(a) - \int_a^c G(t)dt - \frac{G^2(c) - G^2(a)}{2}\right). \end{aligned} \quad (4.84)$$

Further, the function F is nondecreasing on $I \cap [c, \infty)$ so we can apply the reverse inequality (1.21), so we have

$$\int_c^b F(t)G'(t)dt \leq \int_{G(c)}^{G(b)} F(t)dt.$$

Hence, we obtain

$$\begin{aligned}0 &\leq \int_{G(c)}^{G(b)} F(t)dt - \int_c^b F(t)G'(t)dt \\ &= \int_{G(c)}^{G(b)} f(t)dt - \int_c^b f(t)G'(t)dt - A\left(\frac{G^2(b)-G^2(c)}{2} - bG(b) + cG(c) + \int_c^b G(t)dt\right).\end{aligned} \tag{4.85}$$

Now combining (4.84) and (4.85) we obtain

$$\begin{aligned}&\int_a^c f(t)G'(t)dt - \int_{G(a)}^{G(c)} f(t)dt - \int_c^b f(t)G'(t)dt + \int_{G(c)}^{G(b)} f(t)dt \\ &\geq A\left(2cG(c) - aG(a) - bG(b) - \int_a^c G(t)dt + \int_c^b G(t)dt + \frac{G^2(a)+G^2(b)-2G^2(c)}{2}\right).\end{aligned}$$

Now, from (4.82), we conclude that (4.83) holds.
Proof for $f \in \mathscr{M}_2^c(I)$ is similar so we omit the details. □

As a consequence of previous theorem we obtain the following Gauss-Steffensen type inequality for class of convex functions.

Corollary 4.9 *Let $G : [a,b] \to \mathbb{R}$ be an increasing function such that $G(x) \geq x$ and let $c \in (a,b)$. If $f : I \to \mathbb{R}$ is convex and* (4.82) *holds then* (4.83) *holds. If $f : I \to \mathbb{R}$ is concave and* (4.82) *holds, the inequality in* (4.83) *is reversed.*

Proof. Since the function f is convex, from Lemma 1.1, we have that $f \in \mathscr{M}_1^c(I)$ for every $c \in (a,b) \subseteq I^\circ$. Hence, we can apply Theorem 4.29. □

If the function G in Theorem 4.29 and Corollary 4.9 is such that $G(x) \leq x$, then the reverse inequality in (4.83) holds.

Remark 4.9 Condition (4.82) can be weakened. From the proof of Theorem 4.29 we have that for $f \in \mathscr{M}_1^c(I)$ condition (4.82) can be replaced by the weaker condition

$$A\left(2cG(c) - aG(a) - bG(b) - \int_a^c G(t)dt + \int_c^b G(t)dt + \frac{G^2(a)+G^2(b)-2G^2(c)}{2}\right) \geq 0, \tag{4.86}$$

where A is the constant from Definition 1.12. Also, for $f \in \mathscr{M}_2^c(I)$ condition (4.82) can be replaced by condition (4.86) with the reverse inequality.

Furthermore, condition (4.82) can be further weakened if the function f is monotonic. Since (1.37) holds, if $f \in \mathscr{M}_1^c(I)$ is nondecreasing or $f \in \mathscr{M}_2^c(I)$ is nonincreasing, from (4.86) we obtain that (4.82) can be weakened to

$$\int_a^c G(t)dt - \int_c^b G(t)dt \leq 2cG(c) - aG(a) - bG(b) + \frac{G^2(a)+G^2(b)-2G^2(c)}{2}. \tag{4.87}$$

Also, if $f \in \mathscr{M}_1^c(I)$ is nonincreasing or $f \in \mathscr{M}_2^c(I)$ is nondecreasing, (4.82) can be weakened to (4.87) with the reverse inequality.

4.5 Gauss-type inequalities

Motivated by Alzer's lower bound for Gauss' inequality given in Theorem 1.18 this section is devoted to Gauss-type inequalities for convex functions obtained by Pečarić and Smoljak Kalamir in [80].

In the following theorems we obtain Gauss type inequalities for the class of functions that are convex at point c.

Theorem 4.30 *Let $c \in (a,b)$ and let $g : [a,b] \to \mathbb{R}$ be increasing, convex and differentiable such that $g(c) = c$. Assume*

$$s_1(x) = \frac{g(b) - g(c)}{b - c}(x - c) + g(c), \tag{4.88}$$

$$t_1(x) = g'(x_0)(x - x_0) + g(x_0), \;\; x_0 \in [a,c] \tag{4.89}$$

and

$$\int_a^c t_1(x)g'(x)dx + \int_c^b s_1(x)g'(x)dx = \frac{g^2(b) - g^2(a)}{2}. \tag{4.90}$$

If $f \in \mathscr{M}_1^c(I)$, then

$$\int_a^c f(t_1(x))g'(x)dx + \int_c^b f(s_1(x))g'(x)dx \geq \int_{g(a)}^{g(b)} f(x)dx. \tag{4.91}$$

If $f \in \mathscr{M}_2^c(I)$, then the inequality in (4.91) is reversed.
(I is an interval containing $a, b, g(a), g(b), t_1(a)$ and $t_1(c)$.)

Proof. From $g(c) = c$ and other conditions of theorem it follows that $g(a), t_1(a), t_1(c) \leq c$ and $g(b) \geq c$, where $g(a) \leq c$, $t_1(a) \leq c$ and $g(b) \geq c$ follow from the fact that the function g is increasing, and $t_1(c) \leq c$ follows from the convexity of the function g. Since interval I contains $a, b, g(a), g(b), t_1(a)$ and $t_1(c)$, these conditions imply $g(a), g(c), t_1(a), t_1(c) \in I \cap (-\infty, c]$ and $g(c), g(b) \in I \cap [c, \infty)$.

Let $f \in \mathscr{M}_1^c(I)$. Let A be the constant from Definition 1.12 and let us consider the function $F : I \to \mathbb{R}$, $F(x) = f(x) - Ax$. Since F is nonincreasing on $I \cap (-\infty, c]$ and $g(a), g(c), t_1(a), t_1(c) \in I \cap (-\infty, c]$, we can apply the right-hand side of inequality (1.22) to the function F, so

$$\int_{g(a)}^{g(c)} F(x)dx \leq \int_a^c F(t_1(x))g'(x)dx.$$

Hence, we obtain

$$\begin{aligned} 0 &\leq \int_a^c F(t_1(x))g'(x)dx - \int_{g(a)}^{g(c)} F(x)dx = \\ &= \int_a^c f(t_1(x))g'(x)dx - \int_{g(a)}^{g(c)} f(x)dx - A\left(\int_a^c t_1(x)g'(x)dx - \frac{g^2(c) - g^2(a)}{2}\right). \end{aligned} \tag{4.92}$$

Further, since F is nondecreasing on $I \cap [c,\infty)$ and $g(c), g(b) \in I \cap [c,\infty)$, the left-hand side of inequality (1.22) applied to the function F is reversed. So we have

$$\int_c^b F(s_1(x))g'(x)dx \geq \int_{g(c)}^{g(b)} F(x)dx.$$

Hence, we obtain

$$\begin{aligned} 0 &\leq \int_c^b F(s_1(x))g'(x)dx - \int_{g(c)}^{g(b)} F(x)dx = \\ &= \int_c^b f(s_1(x))g'(x)dx - \int_{g(c)}^{g(b)} f(x)dx - A\left(\int_c^b s_1(x)g'(x)dx - \frac{g^2(b)-g^2(c)}{2}\right). \end{aligned} \tag{4.93}$$

Now combining (4.92) and (4.93) we obtain

$$\begin{aligned} &\int_a^c f(t_1(x))g'(x)dx + \int_c^b f(s_1(x))g'(x)dx - \int_{g(a)}^{g(b)} f(x)dx \\ &\geq A\left(\int_a^c t_1(x)g'(x)dx + \int_c^b s_1(x)g'(x)dx - \frac{g^2(b)-g^2(a)}{2}\right). \end{aligned}$$

Hence, from (4.90) we conclude that (4.91) holds.
Proof for $f \in \mathcal{M}_2^c(I)$ is similar, so we omit the details. □

An example of a function $g : [a,b] \to \mathbb{R}$ satisfying conditions of Theorem 4.30 is the function

$$g(x) = \frac{x^2}{c},$$

where $0 \leq a < c < b$.

Theorem 4.31 *Let $c \in (a,b)$ and let $g : [a,b] \to \mathbb{R}$ be increasing, convex and differentiable such that $g(c) = c$. Assume*

$$s_2(x) = \frac{g(c)-g(a)}{c-a}(x-a) + g(a), \tag{4.94}$$

$$t_2(x) = g'(x_0)(x-x_0) + g(x_0), \quad x_0 \in [c,b], \tag{4.95}$$

and

$$\int_a^c s_2(x)g'(x)dx + \int_c^b t_2(x)g'(x)dx = \frac{g^2(b)-g^2(a)}{2}. \tag{4.96}$$

If $f \in \mathcal{M}_1^c(I)$, then

$$\int_a^c f(s_2(x))g'(x)dx + \int_c^b f(t_2(x))g'(x)dx \leq \int_{g(a)}^{g(b)} f(x)dx. \tag{4.97}$$

If $f \in \mathcal{M}_2^c(I)$, then the inequality in (4.97) is reversed.
(I is an interval containing $a, b, g(a), g(b), t_2(c)$ and $t_2(b)$.)

Proof. Similar to the proof of Theorem 4.30, it follows that $g(a) \le c$ and $g(b), t_2(c), t_2(b) \ge c$. Since interval I contains $a, b, g(a), g(b), t_2(c)$ and $t_2(b)$, these conditions imply $g(a), g(c) \in I \cap (-\infty, c]$ and $g(c), g(b), t_2(c), t_2(b) \in I \cap [c, \infty)$.

Let $f \in \mathscr{M}_1^c(I)$. Let A be the constant from Definition 1.12 and let us consider the function $F : I \to \mathbb{R}$, $F(x) = f(x) - Ax$. Since F is nonincreasing on $I \cap (-\infty, c]$ and $g(a), g(c) \in I \cap (-\infty, c]$, we can apply the left-hand side of inequality (1.22) to the function F. So we obtain

$$0 \le \int_{g(a)}^{g(c)} f(x)dx - \int_a^c f(s_2(x))g'(x)dx - A\left(\frac{g^2(c) - g^2(a)}{2} - \int_a^c s_2(x)g'(x)dx\right). \quad (4.98)$$

Further, since F is nondecreasing on $I \cap [c, \infty)$ and $g(c), g(b), t_2(c), t_2(b) \in I \cap [c, \infty)$, the right-hand side of inequality (1.22) applied to the function F is reversed. So we have

$$0 \le \int_{g(c)}^{g(b)} f(x)dx - \int_c^b f(t_2(x))g'(x)dx - A\left(\frac{g^2(b) - g^2(c)}{2} - \int_c^b t_2(x)g'(x)dx\right). \quad (4.99)$$

Now combining (4.98) and (4.99) we obtain

$$\begin{aligned} &\int_{g(a)}^{g(b)} f(x)dx - \int_a^c f(s_2(x))g'(x)dx - \int_c^b f(t_2(x))g'(x)dx \ge \\ &\ge A\left(\frac{g^2(b) - g^2(a)}{2} - \int_a^c s_2(x)g'(x)dx - \int_c^b t_2(x)g'(x)dx\right). \end{aligned}$$

Hence, from (4.96) we conclude that (4.97) holds.
Proof for $f \in \mathscr{M}_2^c(I)$ is similar, so we omit the details. □

As a consequence of Theorems 4.30 and 4.31, we obtain Gauss type inequalities that involve convex functions.

Corollary 4.10 *Let $c \in (a,b)$ and let $g : [a,b] \to \mathbb{R}$ be increasing, convex and differentiable such that $g(c) = c$. Assume* (4.88), (4.89) *and* (4.90) *hold and I is an interval as in Theorem 4.30. If $f : I \to \mathbb{R}$ is convex, then* (4.91) *holds. If $f : I \to \mathbb{R}$ is concave, then the inequality in* (4.91) *is reversed.*

Proof. Since the function f is convex, from Lemma 1.1 we have $f \in \mathscr{M}_1^c(I)$ for every $c \in (a,b) \subseteq I^\circ$. Hence, we can apply Theorem 4.30. □

Corollary 4.11 *Let $c \in (a,b)$ and let $g : [a,b] \to \mathbb{R}$ be increasing, convex and differentiable such that $g(c) = c$. Assume* (4.94), (4.95) *and* (4.96) *hold and I is an interval as in Theorem 4.31. If $f : I \to \mathbb{R}$ is convex, then* (4.97) *holds. If $f : I \to \mathbb{R}$ is concave, then the inequality in* (4.97) *is reversed.*

Proof. Similar to the proof of Corollary 4.10. □

Remark 4.10 Conditions (4.90) and (4.96) can be relaxed. For $f \in \mathscr{M}_1^c(I)$ condition (4.90) can be replaced by the weaker condition

$$A\left(\int_a^c t_1(x)g'(x)dx+\int_c^b s_1(x)g'(x)dx-\frac{g^2(b)-g^2(a)}{2}\right)\geq 0, \tag{4.100}$$

and condition (4.96) can be replaced by the weaker condition

$$A\left(\frac{g^2(b)-g^2(a)}{2}-\int_a^c s_2(x)g'(x)dx-\int_c^b t_2(x)g'(x)dx\right)\geq 0, \tag{4.101}$$

where A is the constant from Definition 1.12. Also, for $f \in \mathscr{M}_2^c(I)$ condition (4.90) (resp. (4.96)) can be replaced by condition (4.100) (resp. (4.101)) with the reverse inequality.

Additionaly, conditions (4.90) and (4.96) can be further weakened if the function f is monotonic. Since (1.37) holds, if $f \in \mathscr{M}_1^c(I)$ is nondecreasing or $f \in \mathscr{M}_2^c(I)$ is nonincreasing, from (4.100) we obtain that (4.90) can be weakened to

$$\int_a^c t_1(x)g'(x)dx+\int_c^b s_1(x)g'(x)dx\geq\frac{g^2(b)-g^2(a)}{2}, \tag{4.102}$$

and that (4.96) can be weakened to

$$\frac{g^2(b)-g^2(a)}{2}\geq\int_a^c s_2(x)g'(x)dx+\int_c^b t_2(x)g'(x)dx. \tag{4.103}$$

Also, if $f \in \mathscr{M}_1^c(I)$ is nonincreasing or $f \in \mathscr{M}_2^c(I)$ is nondecreasing, (4.90) (resp. (4.96)) can be weakened to (4.102) (resp. (4.103)) with the reverse inequality.

4.6 Steffensen's inequality for 3-convex functions

According Definition 1.4, a function $f: I \to \mathbb{R}$ is 3-convex if for pairwise distinct points $x_0,\ x_1,\ x_2,\ x_3 \in I$:

$$[x_0,\ x_1,\ x_2,\ x_3; f]\geq 0.$$

A 3rd order divided difference of f at points $x_0,\ x_1,\ x_2,\ x_3 \in I$ can be expressed in the following forms

(1) If $x_0,\ x_1,\ x_2,\ x_3 \in I$ such that $x_i \neq x_j,\ i\neq j,\ i,j=0,1,2,3$ then

$$[x_0,\ x_1,\ x_2,\ x_3; f]=\sum_{i=0}^{3}\frac{f(x_i)}{q'(x_i)};\quad q(x)=\prod_{i=0}^{3}(x-x_i)$$

(2) If f is differentiable on I and $x,\ x_0,\ x_1 \in I$ such that $x\neq x_0\neq x_1\neq x$ then

$$[x,\ x,\ x_0,\ x_1; f]=\frac{f'(x)}{(x-x_0)(x-x_1)}+\frac{f(x)(x_0+x_1-2x)}{(x-x_0)^2(x-x_1)^2}+\frac{f(x_0)}{(x-x_0)^2(x_0-x_1)}+\frac{f(x_1)}{(x-x_1)^2(x_1-x_0)}$$

(3) If f is differentiable on I and $x,\ x_0 \in I$ such that $x \neq x_0$ then

$$[x,\ x,\ x_0,\ x_0; f] = \frac{(x_0-x)(f'(x_0)+f'(x))+2(f(x)-f(x_0))}{(x-x_0)^3}$$

(4) If f is twice differentiable on I and $x,\ x_0 \in I$ such that $x \neq x_0$ then

$$[x,\ x,\ x,\ x_0; f] = \frac{1}{(x_0-x)^3}\left[f(x_0) - \sum_{i=0}^{2} \frac{f^{(i)}(x)}{i!}(x_0-x)^i\right].$$

(5) If f is three times differentiable on I and $x \in I$ then

$$[x,\ x,\ x,\ x; f] = \frac{f'''(x)}{3!}.$$

We can extend the definition of 3-convex function by including the cases in which some or all of the points coincide. This is given in the following theorem which can be easily proven by using the mean value theorem for divided differences (see for example [28]).

Theorem 4.32 *Let f be defined on interval I in $\mathbb{R}$. The following equivalences hold.*

(i) If $f \in C(I)$ then f is 3-convex if and only if $[x,\ x,\ x_0,\ x_1; f] \geq 0$ for every $x \neq x_0 \neq x_1 \neq x$ in I.

(ii) If $f \in C^1(I)$ then f is 3-convex if and only if $[x,\ x,\ x_0,\ x_0; f] \geq 0$ for every $x \neq x_0$ in I.

(iii) If $f \in C^2(I)$ then f is 3-convex if and only if $[x,\ x,\ x,\ x_0; f] \geq 0$ for every $x \neq x_0$ in I.

(iv) If $f \in C^3(I)$ then f is 3-convex if and only if $[x,\ x,\ x,\ x; f] \geq 0$ for every $x \in I$.

The following families of functions will be useful in constructing exponentially convex functions.

Lemma 4.1 *For $p \in \mathbb{R}$ let $\phi_p : (0,\infty) \to \mathbb{R}$ be defined with*

$$\phi_p(x) = \begin{cases} \frac{x^p}{p(p-1)(p-2)}, & p \neq 0,1,2; \\ \frac{1}{2}\log x, & p = 0; \\ -x\log x, & p = 1; \\ \frac{1}{2}x^2\log x, & p = 2. \end{cases} \tag{4.104}$$

Then ϕ_p is 3-convex on $\mathbb{R}$ for each $p \in \mathbb{R}$.

Proof. Follows from $\frac{d^3}{dx^3}(\phi_p(x)) = x^{p-3} > 0$ on $(0,\infty)$, for each $p \in \mathbb{R}$. □

Lemma 4.2 *For $p \in \mathbb{R}$ let $\varphi_p : \mathbb{R} \to [0,\infty)$ be defined with*

$$\varphi_p(x) = \begin{cases} \frac{e^{px}}{p^3}, & p \neq 0; \\ \frac{x^3}{6}, & p = 0. \end{cases} \tag{4.105}$$

Then φ_p is 3-convex on $\mathbb{R}$ for each $p \in R$.

Proof. Follows from $\frac{d^3}{dx^3}(\varphi_p(x)) = e^{px}$ on $\mathbb{R}$, for each $p \in \mathbb{R}$. □

Remark 4.11 *It is convenient here to note here that, for fixed x, $p \mapsto \phi_p(x)$ and $p \mapsto \varphi_p(x)$ are exponentially convex on $(2,\infty)$ and $(0,\infty)$ respectively. For details see [30].*

Theorem 4.33 *Let $f : [a,b] \to \mathbb{R}$ be a 3-convex function and let $g : [a,b] \to \mathbb{R}$ be an integrable function, such that $\lambda = \int_a^b g(t)dt$, and such that (1.7) is valid.*
Then

$$\begin{aligned}
&\left(\int_a^b xg(x)dx - a\lambda - \frac{\lambda^2}{2}\right) f'\left(\frac{\int_a^b x^2 g(x)dx - a^2\lambda - \lambda^2 a - \frac{\lambda^3}{3}}{2\int_a^b xg(x)dx - 2a\lambda - \lambda^2}\right)\\
&\leq \int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt \leq \\
&\leq \frac{f'(b)-f'(a)}{2(b-a)} \int_a^b x^2 g(x)dx + \frac{bf'(a)-af'(b)}{b-a}\int_a^b xg(x)dx\\
&+ \frac{f'(b)-f'(a)}{b-a}\left(\frac{a^2\lambda}{2} - \frac{\lambda^3}{6}\right) + \frac{f'(a)}{b-a}\left(ab + (b-a)\frac{\lambda}{2}\right)
\end{aligned} \tag{4.106}$$

Proof. We use the following identity (proof of identity can be found in [71] p. 183.), as a basic tool in our proof.
Assume $F : [a,b] \to \mathbb{R}$ is integrable function. Then

$$\int_a^b F(t)g(t)dt - \int_a^{a+\lambda} F(t)dt = \int_a^b G_1(x)F'(x)dx, \tag{4.107}$$

where

$$G_1(x) = \begin{cases} \int_a^x (1-g(t))dt, & x \in [a, a+\lambda], \\ \int_x^b g(t)dt, & x \in [a+\lambda, b]; \end{cases} \tag{4.108}$$

We observe here that assumption (1.7) ensures us that $G_1(x) \geq 0$, for $x \in [a,b]$, in fact these two conditions are equivalent (see for instance [71], p. 184).
Now, since f is 3-convex, f' is convex, using integral Jensen's inequality we have

$$\int_a^b G_1(x)f'(x)dx \geq \int_a^b G_1(x)dx f'\left(\frac{\int_a^b xG_1(x)dx}{\int_a^b G_1(x)dx}\right). \tag{4.109}$$

Applying identity (4.107) for functions $F(x) = x$ and $F(x) = \frac{x^2}{2}$ we get

$$\int_a^b G_1(x)dx = \int_a^b xg(x)dx - \int_a^{a+\lambda} xdx$$

and

$$\int_a^b xG_1(x)dx = \int_a^b \frac{x^2}{2}g(x)dx - \int_a^{a+\lambda} \frac{x^2}{2}dx$$

respectively. Applying (4.107) once again, for choice $F(x) = f(x)$, we get left hand side of (4.106).

We now prove second inequality in (4.106) using discrete version of Jensen's inequality. Namely,

$$\int_a^b G_1(x)f'(x)dx = \int_a^b G_1(x)f'\left(\frac{b-x}{b-a}\cdot a+\frac{x-a}{b-a}\cdot b\right)dx$$
$$\leq \frac{f'(a)}{b-a}\int_a^b G_1(x)(b-x)dx+\frac{f'(b)}{b-a}\int_a^b G_1(x)(x-a)dx.$$

For two summands in the last inequality we apply identity (4.107) for functions $F(x) = -\frac{(b-x)^2}{2}$ and $F(x) = \frac{(x-a)^2}{2}$, respectively, concluding

$$\int_a^b G_1(x)f'(x)dx \leq \frac{f'(a)}{b-a}\left(\int_a^b -\frac{(b-x)^2}{2}g(x)dx+\int_a^{a+\lambda}\frac{(b-x)^2}{2}dx\right)$$
$$+\frac{f'(b)}{b-a}\left(\int_a^b \frac{(x-a)^2}{2}g(x)dx-\int_a^{a+\lambda}\frac{(x-a)^2}{2}dx\right).$$

After rearrangement and grouping expressions we get right hand side of (4.106). □

Remark 4.12 *From the proof of Theorem 4.33 we see that assumptions on the function g can be relaxed for the first inequality in* (4.106)*. It is enough to have*

$$0 \leq \int_a^x G_1(t)dt \leq \int_a^b G_1(t)dt, \qquad x\in[a,b] \tag{4.110}$$

and then use integral version of Jensen-Steffensen inequality to conclude (4.109)*. Here*

$$\int_a^x G_1(t)dt = \begin{cases} \frac{(x-a)^2}{2}+\int_a^x(t-x)g(t)dt, & x\in[a,a+\lambda],\\ \lambda(x-a)-\frac{\lambda^2}{2}+\int_a^x(t-x)g(t)dt, & x\in[a+\lambda,b]. \end{cases} \tag{4.111}$$

Theorem 4.34 *Let $f:[a,b]\to\mathbb{R}$ be a 3-convex function and let $g:[a,b]\to\mathbb{R}$ be an integrable function, such that $\lambda = \int_a^b g(t)dt$, and such that* (1.8) *is valid. Then*

$$\left(\int_a^b xg(x)dx+b\lambda-\frac{\lambda^2}{2}\right)f'\left(\frac{\int_a^b x^2g(x)dx-b^2\lambda+\lambda^2b-\frac{\lambda^3}{3}}{2\int_a^b xg(x)dx+2b\lambda-\lambda^2}\right)$$
$$\leq \int_{b-\lambda}^b f(t)dt-\int_a^b f(t)g(t)dt \leq$$
$$\frac{f'(b)-f'(a)}{2(b-a)}\int_a^b x^2g(x)dx+\frac{bf'(a)-af'(b)}{b-a}\int_a^b xg(x)dx+$$
$$\frac{f'(b)-f'(a)}{b-a}\left(\frac{a^2\lambda}{2}-\frac{\lambda^3}{6}\right)+\frac{f'(a)}{b-a}\left(ab+(b-a)\frac{\lambda}{2}\right). \tag{4.112}$$

Proof. Similar to proof of Theorem 4.33: instead of identity (4.107) we use

$$\int_{b-\lambda}^{b} F(t)dt - \int_{a}^{b} F(t)g(t)dt = \int_{a}^{b} G_2(x)F'(x)dx, \tag{4.113}$$

where

$$G_2(x) = \begin{cases} \int_a^x g(t)dt, & x \in [a, b-\lambda], \\ \int_x^b (1-g(t))dt, & x \in [b-\lambda, b]; \end{cases} \tag{4.114}$$

Now assumption (1.8) ensures us that $G_2(x) \geq 0$, for $x \in [a,b]$, in fact these two conditions are equivalent (see for instance [71], p. 184). The rest of the proof is similar to the proof of Theorem 4.33, so is omitted. □

Remark 4.13 *Similar to Remark 4.12 we see that assumptions on the function g can be relaxed for the first inequality in* (4.112)*. It is enough to have*

$$0 \leq \int_a^x G_2(t)dt \leq \int_a^b G_2(t)dt, \qquad x \in [a,b] \tag{4.115}$$

where

$$\int_a^x G_2(t)dt = \begin{cases} \int_a^x (x-t)g(t)dt, & x \in [a, b-\lambda], \\ \int_a^x (x-t)g(t)dt - \frac{(x-b+\lambda)^2}{2}, & x \in [b-\lambda, b]. \end{cases} \tag{4.116}$$

In the next two theorems we get similar results to Alomari (see [8]).

Theorem 4.35 *Let $f, g : [a,b] \subset \mathbb{R}_+ \to \mathbb{R}$ be integrable such that* (1.7) *is valid and that $|f'|$ is $s-$convex on $[a,b]$. Then*

$$\begin{aligned} \left| \int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt \right| \leq \frac{|f'(a)|}{(b-a)^{s+1}} \left(\int_a^b -\frac{(b-x)^{s+1}}{s+1} g(x)dx + \frac{(b-a-\lambda)^{s+2}}{(s+1)(s+2)} - \frac{(b-a)^{s+2}}{(s+1)(s+2)} \right) \\ + \frac{|f'(b)|}{(b-a)^{s+1}} \left(\int_a^b \frac{(x-a)^{s+1}}{s+1} g(x)dx - \frac{\lambda^{s+2}}{(s+1)(s+2)} \right). \end{aligned} \tag{4.117}$$

Proof. Using triangle inequality and the same logic as in proof of the second part of Theorem 4.33. □

Theorem 4.36 *Let $f, g : [a,b] \subset \mathbb{R}_+ \to \mathbb{R}$ be integrable such that* (1.7) *is valid and that $|f'|$ is $s-$convex on $[a,b]$. Then*

$$\begin{aligned} \left| \int_a^b f(t)g(t)dt - \int_{b-\lambda}^{b} f(t)dt \right| \leq \frac{|f'(a)|}{(b-a)^{s+1}} \left(\int_a^b \frac{(b-x)^{s+1}}{s+1} g(x)dx - \frac{\lambda^{s+2}}{(s+1)(s+2)} \right) \\ + \frac{|f'(b)|}{(b-a)^{s+1}} \left(\frac{(b-a)^{s+2}}{(s+1)(s+2)} - \frac{(b-a-\lambda)^{s+2}}{(s+1)(s+2)} - \int_a^b \frac{(x-a)^{s+1}}{s+1} g(x)dx \right). \end{aligned} \tag{4.118}$$

Proof. Using triangle inequality and the proof of the second part of Theorem 4.34. □

Identities (4.107) and (4.113) enables us to extend previous two theorems to $h-$convex functions introduced in Definition 1.9.

Theorem 4.37 *Let $f,g:[a,b]\to\mathbb{R}$ be integrable such that* (1.7) *is valid and that $|f'|$ is $h-$convex on $[a,b]$, where $h:J\to\mathbb{R}$ for some open interval $J\supseteq(0,1)$. Then*

$$\left|\int_a^b f(t)g(t)dt-\int_a^{a+\lambda}f(t)dt\right|\le|f'(a)|\left(\int_a^b H\left(\frac{b-x}{b-a}\right)g(x)dx-\int_a^{a+\lambda}H\left(\frac{b-x}{b-a}\right)dx\right)$$
$$+|f'(b)|\left(\int_a^b H\left(\frac{x-a}{b-a}\right)g(x)dx-\int_a^{a+\lambda}H\left(\frac{b-x}{b-a}\right)dx\right), \tag{4.119}$$

where the function H is antiderivative of h i.e. $H'=h$.

Theorem 4.38 *Let $f,g:[a,b]\to\mathbb{R}$ be integrable such that* (1.7) *is valid and that $|f'|$ is $h-$convex on $[a,b]$, where $h:J\to\mathbb{R}$ for some open interval $J\supseteq(0,1)$. Then*

$$\left|\int_a^b f(t)g(t)dt-\int_{b-\lambda}^{b}f(t)dt\right|\le|f'(a)|\left(\int_{b-\lambda}^b H\left(\frac{b-x}{b-a}\right)dx-\int_a^b H\left(\frac{b-x}{b-a}\right)g(x)dx\right)$$
$$+|f'(b)|\left(\int_{b-\lambda}^b H\left(\frac{x-a}{b-a}\right)dx-\int_a^b H\left(\frac{x-a}{b-a}\right)g(x)dx\right), \tag{4.120}$$

where the function H is antiderivative of h i.e. $H'=h$.

Now we define linear functionals that will generate exponential convexity. First, using Theorem 4.33, we get two functionals

$$\mathfrak{L}_1(f)=\int_a^b f(t)g(t)dt-\int_a^{a+\lambda}f(t)dt \tag{4.121}$$
$$-\left(\int_a^b xg(x)dx-a\lambda-\frac{\lambda^2}{2}\right)f'\left(\frac{\int_a^b x^2g(x)dx-a^2\lambda-\lambda^2a-\frac{\lambda^3}{3}}{2\int_a^b xg(x)dx-2a\lambda-\lambda^2}\right)$$

and

$$\mathfrak{L}_2(f)=\frac{f'(b)-f'(a)}{2(b-a)}\int_a^b x^2g(x)dx+\frac{bf'(a)-af'(b)}{b-a}\int_a^b xg(x)dx$$
$$+\frac{f'(b)-f'(a)}{b-a}\left(\frac{a^2\lambda}{2}-\frac{\lambda^3}{6}\right)+\frac{f'(a)}{b-a}\left(ab+(b-a)\frac{\lambda}{2}\right) \tag{4.122}$$
$$+\int_a^{a+\lambda}f(t)dt-\int_a^b f(t)g(t)dt.$$

Also, from Theorem 4.34 we get, additionally, two more linear functionals:

$$\mathfrak{L}_3(f) = \int_{b-\lambda}^{b} f(t)dt - \int_a^b f(t)g(t)dt \tag{4.123}$$
$$- \left(\int_a^b xg(x)dx + b\lambda - \frac{\lambda}{2}^2 \right) f' \left(\frac{\int_a^b x^2 g(x)dx - b^2\lambda + \lambda^2 b - \frac{\lambda}{3}^3}{2\int_a^b xg(x)dx + 2b\lambda - \lambda^2} \right)$$

and

$$\mathfrak{L}_4(f) = \frac{f'(b)-f'(a)}{2(b-a)} \int_a^b x^2 g(x)dx + \frac{bf'(a)-af'(b)}{b-a} \int_a^b xg(x)dx$$
$$+ \frac{f'(b)-f'(a)}{b-a} \left(\frac{a^2\lambda}{2} - \frac{\lambda}{6}^3 \right) + \frac{f'(a)}{b-a} \left(ab + (b-a)\frac{\lambda}{2} \right) \tag{4.124}$$
$$- \int_{b-\lambda}^{b} f(t)dt + \int_a^b f(t)g(t)dt.$$

Functionals (4.121)-(4.124) are defined on the real vector space of integrable functions on $[a,b]$. From Theorems 4.33 and 4.34 we see that these functionals are nonnegative on the convex cone of 3-convex functions.

Theorem 4.39 *Let $f \mapsto \mathfrak{L}_i(f),\ i = 1,2,3,4$ be linear functionals defined with* (4.121)-(4.124) *and let $F_i : \mathbb{R} \to \mathbb{R},\ i = 1,2,3,4$ be defined with*

$$F_i(p) = \mathfrak{L}_i(\phi_p) \tag{4.125}$$

where ϕ_p is defined in Lemma 2.1. Then the following hold for every $i = 1,2,3,4$.

(i) The function F_i is continuous on $\mathbb{R}$.

(ii) If $n \in \mathbb{N}$ and $p_1,\ldots,p_n \in \mathbb{R}$ are arbitrary, then the matrix

$$\left[F_i\left(\frac{p_j + p_k}{2} \right) \right]_{j,k=1}^{n}$$

is positive semidefinite. Particularly,

$$det \left[F_i\left(\frac{p_j + p_k}{2} \right) \right]_{j,k=1}^{n} \geq 0.$$

(iii) The function F_i is exponentially convex on $\mathbb{R}$.

(iv) The function F_i is log-convex on $\mathbb{R}$.

(v) If $p,q,r \in \mathbb{R}$ such that $p < q < r$, then

$$F_i(q)^{r-p} \leq F_i(p)^{r-q} F_i(r)^{q-p}. \tag{4.126}$$

Proof. (i) Continuity of the function $p \mapsto F_i(p)$ is obvious for $p \in \mathbb{R} \setminus \{0,1,2\}$. For $p = 0,1,2$ it is directly checked using Heine characterization.
(ii) Let $n \in \mathbb{N}$, $p_i \in \mathbb{R}$ $(i = 1,\dots,n)$ be arbitrary and define auxiliary function $\psi : (0,\infty) \to \mathbb{R}$ by

$$\psi(x) = \sum_{j,k=1}^{n} \xi_j \xi_k \phi_{\frac{p_j+p_k}{2}}(x). \tag{4.127}$$

Now

$$\psi'''(x) = \left(\sum_{j=1}^{n} \xi_j x^{\frac{p_j-3}{2}} \right)^2 \geq 0$$

implies that ψ is 3-convex function on $(0,\infty)$ and then

$$\mathfrak{L}_i(\psi) \geq 0, \ i = 1,2,3,4. \tag{4.128}$$

This means that the matrix

$$\left[F_i\left(\frac{p_i + p_j}{2} \right) \right]_{j,k=1}^{n}$$

is positive semi-definite.
(iii), (iv), (v) are simple consequences of (i), (ii). □

With similar arguments we deduce the next theorem.

Theorem 4.40 *Theorem 4.39 is still valid for φ_p given in Lemma 4.2.*

We now use mean value theorems to produce Cauchy means.

Theorem 4.41 *Let $f \mapsto \mathfrak{L}_i(f)$, $i = 1,2,3,4$ be linear functionals defined with* (4.121)-(4.124) *and $\psi \in C^3[a,b]$. Then there exists $\xi_i \in [a,b]$, $i = 1,2,3,4$, such that*

$$\mathfrak{L}_i(\psi) = \frac{\psi'''(\xi_i)}{6} \mathfrak{L}_i(\psi_0), \tag{4.129}$$

where $\psi_0(x) = x^3$.

Proof. Since $psi \in C^3[a,b]$ there exist $m = \min_{x\in[a,b]} \psi'''(x)$ and $M = \max_{x\in[a,b]} \psi'''(x)$. Denote $h_1(x) = \frac{Mx^3}{6} - \psi(x)$ and $h_2(x) = \psi(x) - \frac{mx^3}{6}$. Then

$$h_1'''(x) = M - \psi'''(x) \geq 0$$
$$h_1'''(x) = \psi'''(x) - m \geq 0$$

which means that $\mathfrak{L}_i(h_1)$, $\mathfrak{L}_i(h_2) \geq 0$, $i = 1,2,3,4$, i.e.

$$\frac{m}{6} \mathfrak{L}_i(\psi_0) \leq \mathfrak{L}_i(\psi) \leq \frac{M}{6} \mathfrak{L}_i(\psi_0). \tag{4.130}$$

If $\mathfrak{L}_i(\psi_0) = 0$, the proof is complete. If $\mathfrak{L}_i(\psi_0) > 0$, then

$$m \leq \frac{6\mathfrak{L}_i(\psi)}{\mathfrak{L}_i(\psi_0)} \leq M$$

and the existence of $\xi_i \in [a,b]$ follows. □

Using, standard, Cauchy type mean value theorem we get the next corollary.

Corollary 4.12 *Let $f \mapsto \mathfrak{L}_i(f),\ i = 1,2,3,4$ be linear functionals defined with* (4.121)-(4.124) *and $\psi_1,\ \psi_2 \in C^3[a,b]$ such that $\psi_2'''(x)$ does not vanish for any value of $x \in [a,b]$, then there exists $\xi \in [a,b]$ such that*

$$\frac{\psi_1'''(\xi_i)}{\psi_2'''(\xi_i)} = \frac{\mathfrak{L}_i(\psi_1)}{\mathfrak{L}_i(\psi_2)}, \tag{4.131}$$

provided that denominator on right side is non-zero.

Remark 4.14 *If the inverse of $\frac{\psi_1'''}{\psi_2'''}$ exists then various kinds of means can be defined by* (4.131)*. That is*

$$\xi_i = \left(\frac{\psi_1'''}{\psi_2'''}\right)^{-1}\left(\frac{\mathfrak{L}_i(\psi_1)}{\mathfrak{L}_i(\psi_2)}\right),\ i = 1,2,3,4. \tag{4.132}$$

Particularly, if we substitute $\psi_1(x) = \phi_p(x),\ \psi_2(x) = \phi_q(x)$ in (4.132) *and use continuous extension, the following expressions are obtained ($i = 1,2,3,4$).*

$$M_i(p,q) = \begin{cases} \left(\frac{\mathfrak{L}_i(\phi_p)}{\mathfrak{L}_i(\phi_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(-\frac{3p^2-6p+2}{p(p-1)(p-2)} + \frac{2\mathfrak{L}_i(\phi_p\phi_0)}{\mathfrak{L}_i(\phi_p)}\right), & p = q \neq 0,1,2; \\ \exp(\frac{3}{2} + \frac{\mathfrak{L}_i(\phi_0^2)}{\mathfrak{L}_i(\phi_0)}), & p = q = 0; \\ \exp(\frac{\mathfrak{L}_i(\phi_0\phi_1)}{\mathfrak{L}_i(\phi_1)}), & p = q = 1; \\ \exp(-\frac{3}{2} - \frac{\mathfrak{L}_i(\phi_0\phi_2)}{\mathfrak{L}_i(\phi_2)}), & p = q = 2. \end{cases} \tag{4.133}$$

By Theorem 2.5, if $p,q,u,v \in \mathbb{R}$ such that $p \leq u,\ q \leq v$ then,

$$M_i(p,q) \leq M_i(u,v). \tag{4.134}$$

Remark 4.15 *Similarly, if we substitute $\psi_1(x) = \varphi_p(x),\ \psi_2(x) = \varphi_q(x)$ in* (4.132) *and use continuous extension, the following expressions are obtained ($i = 1,2,3,4$).*

$$\overline{M}_i(p,q) = \begin{cases} \left(\frac{\mathfrak{L}_i(\phi_p)}{\mathfrak{L}_i(\phi_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(-\frac{3}{p} + \frac{\mathfrak{L}_i(id\cdot\varphi_p)}{\mathfrak{L}_i(\varphi_p)}\right), & p = q \neq 0; \\ \exp(\frac{\mathfrak{L}_i(id\cdot\varphi_0)}{4\mathfrak{L}_i(\varphi_0)}), & p = q = 0. \end{cases} \tag{4.135}$$

Again, using Theorem 2.5, if $p,q,u,v \in \mathbb{R}$ such that $p \leq u,\ q \leq v$ then,

$$\overline{M}_i(p,q) \leq \overline{M}_i(u,v). \tag{4.136}$$

Obtained results we can now further refine by dropping some of analytical properties of families of functions from Lemmas 4.1 and 4.105. Using Theorem 4.32 we therefore define the following classes

- $C_1 = \{\psi_p : \ \psi_p : [a,b] \to \mathbb{R}, \ p \in J\}$, a family of functions from $C([a,b])$ such that $p \mapsto [x_0, \ x_1, \ x_2, \ x_3; \psi_p]$ is log-convex in the Jensen sense on J for every choice of four distinct points $x_0, x_1, x_2, x_3 \in [a,b]$.
- $C_2 = \{\psi_p : \ \psi_p : [a,b] \to \mathbb{R}, \ p \in J\}$, a family of functions from $C^1([a,b])$ such that $p \mapsto [x_0, \ x_0, \ x_1, \ x_2; \psi_p]$ is log-convex in the Jensen sense on J for every choice of three distinct points $x_0, x_1, x_2 \in [a,b]$.
- $C_3 = \{\psi_p : \ \psi_p : [a,b] \to \mathbb{R}, \ p \in J\}$, a family of functions from $C^1([a,b])$ such that $p \mapsto [x_0, \ x_0, \ x_1, \ x_1; \psi_p]$ is log-convex in the Jensen sense on J for every choice of two distinct points $x_0, x_1 \in [a,b]$.
- $C_4 = \{\psi_p : \ \psi_p : [a,b] \to \mathbb{R}, \ p \in J\}$, a family of functions from $C^2([a,b])$ such that $p \mapsto [x_0, \ x_0, \ x_0, \ x_1; \psi_p]$ is log-convex in the Jensen sense on J for every choice of two distinct points $x_0, x_1 \in [a,b]$.
- $C_5 = \{\psi_p : \ \psi_p : [a,b] \to \mathbb{R}, \ p \in J\}$, a family of functions from $C^3([a,b])$ such that $p \mapsto [x_0, \ x_0, \ x_0, \ x_0; \psi_p]$ is log-convex in the Jensen sense on J for every choice $x_0 \in [a,b]$.

Theorem 4.42 *Let $f \mapsto \mathfrak{L}_i(f), \ i = 1,2,3,4$ be linear functionals defined with* (4.121)-(4.124) *and let $F_{i,j} : J \to \mathbb{R}$, be defined with*

$$F_{i,j}(p) = \mathfrak{L}_i(\psi_p) \tag{4.137}$$

where $\psi_p \in C_j, \ j = 1,2,3,4,5$. Then the following hold, for every for every $i = 1,2,3,4$, $j = 1,2,3,4,5$.

(i) $F_{i,j}$ is log-convex in the Jensen sense on J.

(ii) If $F_{i,j}$ is continuous on J, then it is log-convex on J and for $p,q,r \in J$ such that $p < q < r$, we have

$$F_{i,j}(q)^{r-p} \le F_{i,j}(p)^{r-q} F_{i,j}(r)^{q-p}. \tag{4.138}$$

(iii) If $F_{i,j}$ is positive and differentiable on J, then for every $p,q,r \in J$ such that $p \le u, \ q \le v$, we have

$$M_{i,j}(p,q) \le M_i(u,v) \tag{4.139}$$

where $M_{i,j}(p,q)$ is defined with

$$M_{i,j}(p,q) = \begin{cases} \left(\frac{F_{i,j}(p)}{F_{i,j}(q)} \right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{\frac{d}{dp}\left(F_{i,j}(p)\right)}{F_{i,j}(p)} \right), & p = q. \end{cases} \tag{4.140}$$

Proof. (i) We prove our claim for the case $i=1,\ j=1$, other cases are treated similarly. Choose any four distinct points $x_0,x_1,x_2,x_3 \in [a,b]$, any $\xi_1,\ \xi_2 \in \mathbb{R}$ and any $p,\ q \in J$. Define auxiliary function $\psi : [a,b] \to \mathbb{R}$ by

$$\psi(x) = \xi_1^2 \psi_p(x) + 2\xi_1\xi_2 \psi_{\frac{p+q}{2}}(x) + \xi_2^2 \psi_q(x), \tag{4.141}$$

where $\psi_p, \psi_{\frac{p+q}{2}}$ and ψ_q are from class C_1. Then

$$\begin{aligned}[x_0,\ x_1,\ x_2,\ x_3; \psi] =& \xi_1^2[x_0,\ x_1,\ x_2,\ x_3; \psi_p] + 2\xi_1\xi_2[x_0,\ x_1,\ x_2,\ x_3; \psi_{\frac{p+q}{2}}] \\ &+ \xi_2^2[x_0,\ x_1,\ x_2,\ x_3; \psi_q] \geq 0\end{aligned}$$

by definition of C_1 and characterization of log-convexity. This implies that h is 3-convex function on $[a,b]$. Hence $\mathfrak{L}_1(\psi) \geq 0$ which is equivalent to

$$\xi_1^2 F_{1,1}(p) + 2\xi_1\xi_2 F_{1,1}\left(\frac{p+q}{2}\right) + \xi_2^2 F_{1,1}(q) \geq 0.$$

This proves that $F_{1,1}$ is log-convex in the Jensen sense on J.
(ii) Since $F_{1,1}$ is continuous on J, then it is log-convex.
(iii) This is a simple consequence of Theorem 2.5. □

Let us introduce the following families of functions which will be used in the next theorem.

- $D_1 = \{\psi_p :\ \psi_p : [a,b] \to \mathbb{R},\ p \in J\}$, a family of functions from $C([a,b])$ such that $p \mapsto [x_0,\ x_1,\ x_2,\ x_3; \psi_p]$ is exponentially convex on J for every choice of four distinct points $x_0,x_1,x_2,x_3 \in [a,b]$.
- $D_2 = \{\psi_p :\ \psi_p : [a,b] \to \mathbb{R},\ p \in J\}$, a family of functions from $C^1([a,b])$ such that $p \mapsto [x_0,\ x_0,\ x_1,\ x_2; \psi_p]$ is exponentially convex on J for every choice of three distinct points $x_0,x_1,x_2 \in [a,b]$.
- $D_3 = \{\psi_p :\ \psi_p : [a,b] \to \mathbb{R},\ p \in J\}$, a family of functions from $C^1([a,b])$ such that $p \mapsto [x_0,\ x_0,\ x_1,\ x_1; \psi_p]$ is exponentially convex on J for every choice of two distinct points $x_0,x_1 \in [a,b]$.
- $D_4 = \{\psi_p :\ \psi_p : [a,b] \to \mathbb{R},\ p \in J\}$, a family of functions from $C^2([a,b])$ such that $p \mapsto [x_0,\ x_0,\ x_0,\ x_1; \psi_p]$ is exponentially convex on J for every choice of two distinct points $x_0,x_1 \in [a,b]$.
- $D_5 = \{\psi_p :\ \psi_p : [a,b] \to \mathbb{R},\ p \in J\}$, a family of functions from $C^3([a,b])$ such that $p \mapsto [x_0,\ x_0,\ x_0,\ x_0; \psi_p]$ is exponentially convex on J for every choice $x_0 \in [a,b]$.

Theorem 4.43 *Let $f \mapsto \mathfrak{L}_i(f),\ i=1,2,3,4$ be linear functionals defined with* (4.121)–(4.124) *and let $\overline{F}_{i,j} : J \to \mathbb{R}$, be defined with*

$$\overline{F}_{i,j}(p) = \mathfrak{L}_i(\psi_p) \tag{4.142}$$

where $\psi_p \in D_j,\ j=1,2,3,4,5$. Then the following hold for every for every $i=1,2,3,4$, $j=1,2,3,4,5$.

(i) If $n \in \mathbb{N}$ and $p_1, \ldots, p_n \in J$ are arbitrary, then the matrix

$$\left[\overline{F}_{i,j}\left(\frac{p_k+p_m}{2}\right)\right]_{k,m=1}^{n}$$

is positive semidefinite. Particularly,

$$det\left[\overline{F}_{i,j}\left(\frac{p_k+p_m}{2}\right)\right]_{k,m=1}^{n} \geq 0.$$

(ii) If the function $\overline{F}_{i,j}$ is continuous on J, then $\overline{F}_{i,j}$ is exponentially convex on J.

(iii) If $\overline{F}_{i,j}$ is positive and differentiable on J, then for every $p,q,r \in J$ such that $p \leq u,\ q \leq v$, we have

$$\overline{M}_{i,j}(p,q) \leq \overline{M}_i(u,v) \tag{4.143}$$

where $\overline{M}_{i,j}(p,q)$ is defined with

$$\overline{M}_{i,j}(p,q) = \begin{cases} \left(\frac{\overline{F}_{i,j}(p)}{\overline{F}_{i,j}(q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{\frac{d}{dp}\left(\overline{F}_{i,j}(p)\right)}{\overline{F}_{i,j}(p)}\right), & p = q. \end{cases} \tag{4.144}$$

Proof. (i) We prove our claim for the case $i = 1,\ j = 1$, other cases are treated similarly. Let $n \in \mathbb{N},\ p_1, \ldots p_n \in J$ be arbitrary and define auxiliary function $\psi : [a,b] \to \mathbb{R}$ by

$$\psi(x) = \sum_{k,m=1}^{n} \xi_k \xi_m \psi_{\frac{p_k+p_m}{2}}(x). \tag{4.145}$$

Then

$$[x_0,\ x_1,\ x_2,\ x_3; \psi] = \sum_{k,m=1}^{n} \xi_k \xi_m [x_0,\ x_1,\ x_2,\ x_3; \psi_{\frac{p_k+p_m}{2}}] \geq 0$$

by definition of D_1 and exponential convexity. This implies that h is 3-convex function on $[a,b]$ and then $\mathfrak{L}_1(\psi) \geq 0$ which is equivalent to

$$\sum_{k,m=1}^{n} \xi_i \xi_j \overline{F}_{1,1}\left(\frac{p_k+p_m}{2}\right) \geq 0.$$

(ii) Follows from (i).
(iii) This is a simple consequence of Theorem 2.5. □

Families of exponentially convex functions similar to families Lemmas 2.1 and 2.1 can be easily constructed because of application of Theorem 1.21 :

Example 4.1 *Consider a family of functions $h_p : (0,\infty) \to (0,\infty), p > 0$, defined with*

$$h_p(x) = \begin{cases} -\frac{p^{-x}}{\log^3 p}, & p \neq 1; \\ \frac{x^3}{6}, & p = 1. \end{cases} \tag{4.146}$$

Since $p \mapsto \frac{d^3}{dx^3}(h_p(x)) = p^{-x} > 0$is the Laplace transform of a non-negative function (see [84] p. 210), it is exponentially convex according Theorem 1.21. Obviously h_p are 3-convex functions for every $p > 0$. It is easy to prove that the function $p \mapsto [x_0,x_1,x_2,x_3;hp]$ is also

exponentially convex for arbitrary positive x_0, x_1, x_2, x_3 *(see also [30]). Using Theorem 4.43 it follows that for linear functionals* $f \mapsto \mathfrak{L}_i(f)$, $i = 1,2,3,4$ *defined with* (4.121)-(4.124) *we have that* $p \mapsto \mathfrak{L}_i(h_p)$ *is exponentially convex (it is easy to verify that it is continuous), for* $i = 1,2,3,4$.
Using further Theorem 4.43 we conclude that

$$R_i(p,q) = \begin{cases} \left(\frac{\mathfrak{L}_i(h_p)}{\mathfrak{L}_i(h_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(-\frac{3}{p\log p} - \frac{\mathfrak{L}_i(id \cdot h_p)}{p\mathfrak{L}_i(h_p)}\right), & p = q \neq 1; \\ \exp(-\frac{\mathfrak{L}_i(id \cdot h_1)}{4\mathfrak{L}_i(h_1)}), & p = q = 1; \end{cases} \tag{4.147}$$

satisfies

$$R_i(p,q) \leq R_i(u,v). \tag{4.148}$$

for $p,q,u,v \in \mathbb{R}$ *such that* $p \leq u$, $q \leq v$.

Remark 4.16 *From Example 4.1 and Theorem 4.40 it clear that we presented a new way how to generate exponentially convex functions, aside from Laplace transform and Theorem 1.21.*

Remark 4.17 *Notion of exponential convexity can be even further refined. For details see [60].*

4.7 Applications to Stolarsky type means

In this section we generate $n-$exponentially and exponentially convex functions from functionals associated with Steffensen, Gauss and Gauss-Steffensen type inequalities involving convex functions given in Sections 4.2 - 4.5. We use generated exponentially convex functions to construct new Stolarsky type means. This results were obtained in [76], [77], [78] and [80].

Results related to convex function f/h

By using generalizations of Steffensen type inequalities given by (4.39), (4.45), (4.53), (4.55), (4.57) and (4.59) we can define the following linear functionals:

$$L_1(f) = \int_a^{a+\lambda_1} f(t)dt + \int_{b-\lambda_2}^b f(t)dt - \int_a^b f(t)g(t)dt, \tag{4.149}$$

$$L_2(f) = \int_a^b f(t)g(t)dt - \int_{c-\lambda_1}^{c+\lambda_2} f(t)dt, \tag{4.150}$$

$$L_3(f) = \int_a^{a+\lambda_1} f(t)k(t)dt + \int_{b-\lambda_2}^b f(t)k(t)dt - \int_a^b f(t)g(t)dt, \tag{4.151}$$

$$L_4(f) = \int_a^b f(t)g(t)dt - \int_{c-\lambda_1}^{c+\lambda_2} f(t)k(t)dt, \tag{4.152}$$

$$\begin{aligned} L_5(f) = &\int_a^{a+\lambda_1} \left(f(t) - \left[\frac{f(t)}{h(t)} - \frac{f(a+\lambda_1)}{h(a+\lambda_1)} \right] h(t)[1-g(t)] \right) dt \\ &+ \int_{b-\lambda_2}^{b} \left(f(t) - \left[\frac{f(t)}{h(t)} - \frac{f(b-\lambda_2)}{h(b-\lambda_2)} \right] h(t)[1-g(t)] \right) dt - \int_a^b f(t)g(t)dt, \end{aligned} \tag{4.153}$$

and

$$\begin{aligned} L_6(f) = &\int_a^b f(t)g(t)dt - \int_{c-\lambda_1}^{c} \left(f(t) - \left[\frac{f(t)}{h(t)} - \frac{f(c-\lambda_1)}{h(c-\lambda_1)} \right] h(t)[1-g(t)] \right) dt \\ &- \int_c^{c+\lambda_2} \left(f(t) - \left[\frac{f(t)}{h(t)} - \frac{f(c+\lambda_2)}{h(c+\lambda_2)} \right] h(t)[1-g(t)] \right) dt. \end{aligned} \tag{4.154}$$

Under the assumptions of Theorems 4.14, 4.15, 4.18, 4.19, 4.20 and 4.21 we have that $L_i(f) \geq 0$, $i = 1,\ldots,6$ for $f/h \in \mathcal{M}_1^c[a,b]$. Further, Corollaries 4.3, 4.4 and Remark 4.6 assure that $L_i(f) \geq 0$, $i = 1,\ldots,6$ for any convex function f/h.

Let us begin by giving a Lagrange type mean value theorem for the functional L_1.

Theorem 4.44 *Let $h : [a,b] \to \mathbb{R}$ be a positive integrable function and let $c \in (a,b)$. Let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq 1$. Let λ_1 be the solution of* (4.36) *and λ_2 be the solution of* (4.37)*. Let $f : [a,b] \to \mathbb{R}$ be such that $f/h \in C^2[a,b]$. If* (4.38) *holds, then there exists $\xi \in [a,b]$ such that*

$$L_1(f) = \frac{1}{2} \left(\frac{f(\xi)}{h(\xi)} \right)'' \left[\int_a^b t^2 h(t)g(t)dt - \int_a^{a+\lambda_1} t^2 h(t)dt - \int_{b-\lambda_2}^{b} t^2 h(t)dt \right]. \tag{4.155}$$

where L_1 is defined by (4.149)*.*

Proof. Since $f/h \in C^2[a,b]$ there exist

$$m = \min_{x\in[a,b]} \left(\frac{f(x)}{h(x)} \right)'' \text{ and } M = \max_{x\in[a,b]} \left(\frac{f(x)}{h(x)} \right)''.$$

Let

$$\Phi_1(x) = f(x) - \frac{m}{2}x^2 h(x) \quad \text{and} \quad \Phi_2(x) = \frac{M}{2}x^2 h(x) - f(x).$$

Functions Φ_1/h and Φ_2/h are convex since $\Phi_i/h \geq 0$, $i = 1,2$. Hence, $L_1(\Phi_i) \geq 0$, $i = 1,2$ and we obtain

$$\frac{m}{2} L_1(\tilde{f}) \leq L_1(f) \leq \frac{M}{2} L_1(\tilde{f}) \tag{4.156}$$

where $\tilde{f}(x) = x^2 h(x)$. Since $\tilde{f}/h$ is convex we have $L_1(\tilde{f}) \geq 0$.

If $L_1(\tilde{f}) = 0$, then (4.156) implies $L_1(f) = 0$ and (4.155) holds for every $\xi \in [a,b]$. Otherwise, multiplying (4.156) by $2/L_1(\tilde{f})$ we obtain

$$m \leq \frac{2L_1(f)}{L_1(\tilde{f})} \leq M,$$

so continuinity of $(f/h)''$ ensures the existence of $\xi \in [a,b]$ satisfying (4.155). □

We continue with a Cauchy type mean value theorem for the functional L_1.

Theorem 4.45 *Let $h:[a,b]\to\mathbb{R}$ be a positive integrable function and let $c\in(a,b)$. Let $g:[a,b]\to\mathbb{R}$ be an integrable function such that $0\le g\le 1$. Let λ_1 be the solution of* (4.36) *and λ_2 be the solution of* (4.37). *Let the functions f and F be such that $f/h, F/h\in C^2[a,b]$. If* (4.38) *holds and $L_1(F)\neq 0$, then there exists $\xi\in[a,b]$ such that*

$$\frac{L_1(f)}{L_1(F)}=\frac{(f/h)''(\xi)}{(F/h)''(\xi)}$$

where L_1 is defined by (4.149).

Proof. Define $\Psi(x)=L_1(F)f(x)-L_1(f)F(x)$. Due to linearity of L_1 we have $L_1(\Psi)=0$. Now by Theorem 4.44 there exist $\xi,\xi_1\in[a,b]$ such that

$$\begin{aligned}0\ &=L_1(\Psi)=\tfrac{1}{2}\left(\tfrac{\Psi(\xi)}{h(\xi)}\right)''L_1(\tilde{f})\\ 0\ &\neq L_1(F)=\tfrac{1}{2}\left(\tfrac{F(\xi_1)}{h(\xi_1)}\right)''L_1(\tilde{f})\end{aligned}$$

where $\tilde{f}(x)=x^2h(x)$. Therefore, $L_1(\tilde{f})\neq 0$ and

$$0=\left(\frac{\Psi(\xi)}{h(\xi)}\right)''=L_1(F)\left(\frac{f(\xi)}{h(\xi)}\right)''-L_1(f)\left(\frac{F(\xi)}{h(\xi)}\right)'',$$

which gives the claim of the theorem. □

As in Theorems 4.44 and 4.45 we can obtain the Lagrange and the Cauchy type mean value theorems for the functionals L_i, $i=2,\ldots,6$. Hence, we obtain that there exist $\xi_i\in[a,b]$, $i=1,\ldots,6$ such that

$$\frac{L_i(f)}{L_i(F)}=\frac{(f/h)''(\xi_i)}{(F/h)''(\xi_i)},\quad i=1,\ldots 6.$$

We continue with results related to exponential convexity.

Theorem 4.46 *Let $\Omega=\{f_p/h:I\to\mathbb{R}\,|\,p\in J\}$ be a family of functions such that for every mutually different points $x_0,x_1,x_2\in I$ the mapping $p\mapsto[x_0,x_1,x_2;f_p/h]$ is $n-$exponentially convex in the Jensen sense on J. Let L_i, $i=1,\ldots,6$ be linear functionals defined by* (4.149)-(4.154). *Then the mapping $p\mapsto L_i(f_p)$ is $n-$exponentially convex in the Jensen sense on J.*
If the mapping $p\mapsto L_i(f_p)$ is continuous on J, then it is $n-$exponentially convex on J.

Proof. For $\xi_j\in\mathbb{R}$ and $p_j\in J$, $j=1,\ldots,n$, we define the function

$$\Phi(x)=\sum_{j,k=1}^{n}\xi_j\xi_k f_{\frac{p_j+p_k}{2}}(x).$$

Since the mapping $p\mapsto[x_0,x_1,x_2;f_p/h]$ is n-exponentially convex in the Jensen sense we have

$$\left[x_0,x_1,x_2;\frac{\Phi}{h}\right]=\sum_{j,k=1}^{n}\xi_j\xi_k\left[x_0,x_1,x_2;\frac{f_{\frac{p_j+p_k}{2}}}{h}\right]\geq 0.$$

So Φ/h is a convex function and

$$0 \le L_i(\Phi) = \sum_{j,k=1}^{n} \xi_j \xi_k L_i\left(f_{\frac{p_j+p_k}{2}}\right), \quad i = 1, \ldots, 6.$$

Therefore, the mapping $p \mapsto L_i(f_p)$ is n-exponentially convex on J in the Jensen sense.

If the mapping $p \mapsto L_i(f_p)$ is also continuous on J, then $p \mapsto L_i(f_p)$ is n-exponentially convex by definition. □

If the assumptions of Theorem 4.46 hold for all $n \in \mathbb{N}$, then we have the following corollary.

Corollary 4.13 *Let $\Omega = \{f_p/h : I \to \mathbb{R} \mid p \in J\}$ be a family of functions such that for every mutually different points $x_0, x_1, x_2 \in I$ the mapping $p \mapsto [x_0, x_1, x_2; f_p/h]$ is exponentially convex in the Jensen sense on J. Let L_i, $i = 1, \ldots, 6$ be linear functionals defined by* (4.149)-(4.154)*. Then the mapping $p \mapsto L_i(f_p)$ is exponentially convex in the Jensen sense on J. If the mapping $p \mapsto L_i(f_p)$ is continuous on J, then it is exponentially convex on J.*

The following corollary enables us to obtain applications to Stolarsky type means.

Corollary 4.14 *Let $\Omega = \{f_p/h : I \to \mathbb{R} \mid p \in J\}$ be a family of functions such that for every mutually different points $x_0, x_1, x_2 \in I$ the mapping $p \mapsto [x_0, x_1, x_2; f_p/h]$ is* 2*-exponentially convex in the Jensen sense on J. Let L_i, $i = 1, \ldots, 6$ be linear functionals defined by* (4.149)-(4.154)*. Then the following statements hold:*

(i) If the mapping $p \mapsto L_i(f_p)$ is continuous on J, then for $r, s, t \in J$, such that $r < s < t$, we have

$$[L_i(f_s)]^{t-r} \le [L_i(f_r)]^{t-s} [L_i(f_t)]^{s-r}, \quad i = 1, \ldots, 6. \tag{4.157}$$

(ii) If the mapping $p \mapsto L_i(f_p)$ is positive and differentiable on J, then for every $p, q, u, v \in J$ such that $p \le u$ and $q \le v$ we have

$$\mu_{p,q}(L_i, \Omega) \le \mu_{u,v}(L_i, \Omega), \tag{4.158}$$

where

$$\mu_{p,q}(L_i, \Omega) = \begin{cases} \left(\frac{L_i(f_p)}{L_i(f_q)}\right)^{\frac{1}{p-q}}, & p \ne q, \\ \exp\left(\frac{\frac{d}{dp} L_i(f_p)}{L_i(f_p)}\right), & p = q. \end{cases} \tag{4.159}$$

Proof.

(i) By Theorem 4.46 the mapping $p \mapsto L_i(f_p)$ is 2-exponentially convex. Hence, by Remark 1.7, this mapping is either identically equal to zero, in which case inequality (4.157) holds trivially with zeros on both sides, or it is strictly positive and log-convex. Therefore, for $r, s, t \in J$ such that $r < s < t$ Remark 1.2(c) gives

$$(t-s)\log L_i(f_r) + (r-t)\log L_i(f_s) + (s-r)\log L_i(f_t) \ge 0,$$

which is equivalent to inequality (4.157).

(ii) By (i) we have that the mapping $p \mapsto L_i(f_p)$ is log-convex on J, that is, the function $p \mapsto \log L_i(f_p)$ is convex on J. Applying Proposition 1.1 with $p \leq u$, $q \leq v$, $p \neq q$, $u \neq v$, we obtain

$$\frac{\log L_i(f_p) - \log L_i(f_q)}{p-q} \leq \frac{\log L_i(f_u) - \log L_i(f_v)}{u-v},$$

that is

$$\mu_{p,q}(L_i, \Omega) \leq \mu_{u,v}(L_i, \Omega).$$

The limit cases $p = q$ and $u = v$ are obtained by taking the limits $p \to q$ and $u \to v$.

□

Results stated in Theorem 4.46 and Corollaries 4.13 and 4.14 still hold when some or all of the points $x_0, x_1, x_2 \in I$ coincide. The proofs are obtained by recalling Remark 1.3 and a suitable characterization of convexity.

In [86] Stolarsky obtained a two variable homogenous mean. For positive numbers a, b and real numbers r, s Stolarsky means are defined by

$$E_{r,s}(a,b) = \begin{cases} \left(\frac{s(b^r - a^r)}{r(b^s - a^s)}\right)^{\frac{1}{r-s}}, & rs(r-s) \neq 0; \\ \exp\left(-\frac{1}{r} + \frac{a^r \log a - b^r \log b}{a^r - b^r}\right)^{\frac{1}{a^r - b^r}}, & s = r \neq 0; \\ \left(\frac{b^r - a^r}{r(\log b - \log a)}\right)^{\frac{1}{r}}, & s = 0, r \neq 0; \\ \sqrt{ab}, & r = s = 0 \end{cases}$$

for $a \neq b$ and $E_{r,s}(a,a) = a$. Stolarsky means have been the subject of intensive research by many mathematicians. Since they play an important role in the application of inequalities in various branches of mathematics we apply our results to obtain new Stolarsky type means.

Here we give a family of functions for which we use Corollaries 4.15 and 4.16 to construct exponentially convex functions and new Stolarsky type means.

Let h be a positive integrable function and

$$\Upsilon = \{f_p/h \colon \mathbb{R} \to (0,\infty) \,|\, p \in \mathbb{R}\}$$

be a family of functions where f_p is defined by

$$f_p(x) = \begin{cases} \frac{1}{p^2} e^{px} h(x), & p \neq 0, \\ \frac{1}{2} x^2 h(x), & p = 0. \end{cases}$$

We have $\frac{d^2}{dx^2} \frac{f_p(x)}{h(x)} = e^{px} > 0$, so f_p/h is convex on $\mathbb{R}$ for every $p \in \mathbb{R}$ and $p \mapsto \frac{d^2}{dx^2} \frac{f_p(x)}{h(x)}$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 4.46 we have that $p \mapsto [t_0, t_1, t_2; f_p/h]$ is exponentially convex (and so exponentially convex in the Jensen sense). We see that the family Υ satisfies the assumptions of Corollary 4.13, so mappings $p \mapsto L_i(f_p)$ are exponentially convex in the Jensen sense. It is easy to verify that this mappings are continuous, so they are exponentially convex.

For this family of functions $\mu_{p,q}(L_i,\Upsilon)$, $i = 1,\dots,6$ from (4.159) becomes

$$\mu_{p,q}(L_i,\Upsilon) = \begin{cases} \left(\frac{L_i(f_p)}{L_i(f_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{L_i(\mathrm{id}\cdot f_p)}{L_i(f_p)} - \frac{2}{p}\right), & p = q \neq 0, \\ \exp\left(\frac{L_i(\mathrm{id}\cdot f_0)}{3L_i(f_0)}\right), & p = q = 0. \end{cases}$$

Explicitly for $\mu_{p,q}(L_1,\Upsilon)$ we have:

∗ for $p \neq q$

$$\mu_{p,q}(L_1,\Upsilon) = \left(\frac{q^2}{p^2}\frac{\int_a^b e^{pt}h(t)g(t)dt - \int_a^{a+\lambda_1} e^{pt}h(t)dt - \int_{b-\lambda_2}^b e^{pt}h(t)dt}{\int_a^b e^{qt}h(t)g(t)dt - \int_a^{a+\lambda_1} e^{qt}h(t)dt - \int_{b-\lambda_2}^b e^{qt}h(t)dt}\right)^{\frac{1}{p-q}}$$

∗ for $p = q \neq 0$

$$\mu_{p,p}(L_1,\Upsilon) = \exp\left(\frac{\int_a^b te^{pt}h(t)g(t)dt - \int_a^{a+\lambda_1} te^{pt}h(t)dt - \int_{b-\lambda_2}^b te^{pt}h(t)dt}{\int_a^b e^{pt}h(t)g(t)dt - \int_a^{a+\lambda_1} e^{pt}h(t)dt - \int_{b-\lambda_2}^b e^{pt}h(t)dt} - \frac{2}{p}\right)$$

∗ for $p = q = 0$

$$\mu_{0,0}(L_1,\Upsilon) = \exp\left(\frac{1}{3}\frac{\int_a^b t^3h(t)g(t)dt - \int_a^{a+\lambda_1} t^3h(t)dt - \int_{b-\lambda_2}^b t^3h(t)dt}{\int_a^b t^2h(t)g(t)dt - \int_a^{a+\lambda_1} t^2h(t)dt - \int_{a-\lambda_2}^b t^2h(t)dt}\right).$$

Theorem 4.45 applied on functions $f_p/h, f_q/h \in \Upsilon$ implies that

$$M_{p,q}(L_1,\Upsilon) = \log \mu_{p,q}(L_1,\Upsilon)$$

satisfies $a \leq M_{p,q}(L_1,\Upsilon) \leq b$. Hence $M_{p,q}$ is a monotonic mean by (4.158).

Results related to convex function f

Let us define the following linear functionals

$$\begin{aligned} L_1(f) = & \int_a^{a+\sigma_1} f(t)dt - \frac{\sigma_1}{c-a}\left(1-\frac{1}{q}\right)\int_a^c f(t)dt + \int_{b-\sigma_2}^b f(t)dt \\ & - \frac{\sigma_2}{b-c}\left(1-\frac{1}{q}\right)\int_c^b f(t)dt - \int_a^b f(t)g(t)dt \end{aligned} \tag{4.160}$$

and

$$\begin{aligned} L_2(f) = & \int_a^b f(t)g(t)dt - \int_{c-\sigma_1}^{c+\sigma_2} f(t)dt \\ & + \left(\frac{\sigma_1}{c-a}\int_a^c f(t)dt + \frac{\sigma_2}{b-c}\int_c^b f(t)dt\right)\left(1-\frac{1}{q}\right). \end{aligned} \tag{4.161}$$

Under the assumptions of Theorems 4.25 - 4.28 we have that $L_i(f) \geq 0$, $i = 1,2$ for $f \in \mathscr{M}_1^c[a,b]$. Further, under the assumptions of Corollaries 4.5 - 4.8 we have that $L_i(f) \geq 0$, $i = 1,2$ for any convex function f. In the following theorems we give Lagrange type mean value theorems.

Theorem 4.47 *Let $g : [a,b] \to \mathbb{R}$ be an integrable function. For given $c \in (a,b)$ and $q \neq 0$, denote $\sigma_1 = q\int_a^c g(t)dt$ and $\sigma_2 = q\int_c^b g(t)dt$. Assume* (4.64), (4.65) *and* (4.66) *hold. Then for any $f \in C^2[a,b]$ there exists $\xi \in [a,b]$ such that*

$$\begin{aligned} L_1(f) = \frac{f''(\xi)}{2} \Bigg[& a^2\sigma_1 + a\sigma_1^2 + b^2\sigma_2 - b\sigma_2^2 + \frac{\sigma_1^3 + \sigma_2^3}{3} \\ & - \left(\sigma_1 \frac{a^2 + ac + c^2}{3} + \sigma_2 \frac{b^2 + cb + c^2}{3} \right) \left(1 - \frac{1}{q} \right) - \int_a^b t^2 g(t)dt \Bigg], \end{aligned} \tag{4.162}$$

where L_1 is defined by (4.160)*.*

Proof. Since $f \in C^2[a,b]$ there exist

$$m = \min_{x \in [a,b]} f''(x) \quad \text{and} \quad M = \max_{x \in [a,b]} f''(x).$$

The functions

$$\Psi_1(x) = f(x) - \frac{m}{2}x^2 \quad \text{and} \quad \Psi_2(x) = \frac{M}{2}x^2 - f(x)$$

are convex since $\Psi_i''(x) \geq 0$, $i = 1,2$. Using Corollary 4.5 we have that $L_1(\Psi_i) \geq 0$, $i = 1,2$, so

$$\frac{m}{2}L_1(x^2) \leq L_1(f) \leq \frac{M}{2}L_1(x^2), \tag{4.163}$$

where

$$\begin{aligned} L_1(x^2) = {} & a^2\sigma_1 + a\sigma_1^2 + b^2\sigma_2 - b\sigma_2^2 + \frac{\sigma_1^3 + \sigma_2^3}{3} \\ & - \left(\sigma_1 \frac{a^2 + ac + c^2}{3} + \sigma_2 \frac{b^2 + cb + c^2}{3} \right) \left(1 - \frac{1}{q} \right) - \int_a^b t^2 g(t)dt. \end{aligned}$$

Since x^2 is convex, by Corollary 4.5 we have that $L_1(x^2) \geq 0$.

If $L_1(x^2) = 0$, then (4.163) implies $L_1(f) = 0$ and (4.162) holds for every $\xi \in [a,b]$. Otherwise, dividing (4.163) by $L_1(x^2)/2 > 0$ we get

$$m \leq \frac{2L_1(f)}{L_1(x^2)} \leq M,$$

so continuinity of f'' ensures existence of $\xi \in [a,b]$ satisfying (4.162). □

Theorem 4.48 *Let $g : [a,b] \to \mathbb{R}$ be an integrable function. For given $c \in (a,b)$ and $q \neq 0$, denote $\sigma_1 = q\int_a^c g(t)dt$ and $\sigma_2 = q\int_c^b g(t)dt$. Assume* (4.64), (4.65) *and* (4.72) *hold. Then for any $f \in C^2[a,b]$ there exists $\xi \in [a,b]$ such that*

$$L_2(f) = \frac{f''(\xi)}{2}\left[\int_a^b t^2 g(t)dt - c^2(\sigma_1+\sigma_2) - c(\sigma_2^2-\sigma_1^2) - \frac{\sigma_1^3+\sigma_2^3}{3}\right.$$
$$\left. + \left(\sigma_1\frac{c^2+ac+a^2}{3} + \sigma_2\frac{b^2+bc+c^2}{3}\right)\left(1-\frac{1}{q}\right)\right],$$

where L_2 is defined by (4.161).

Proof. Similar to the proof of Theorem 4.47. □

We continue with the Cauchy type mean value theorems related to functionals L_i, $i = 1,2$.

Theorem 4.49 *Let $g : [a,b] \to \mathbb{R}$ be an integrable function. For given $c \in (a,b)$ and $q \neq 0$, denote $\sigma_1 = q\int_a^c g(t)dt$ and $\sigma_2 = q\int_c^b g(t)dt$. Assume* (4.64), (4.65) *and* (4.66) *hold. Then for any $f,h \in C^2[a,b]$ such that $h''(x) \neq 0$, for every $x \in [a,b]$, there exists $\xi \in [a,b]$ such that*

$$\frac{L_1(f)}{L_1(h)} = \frac{f''(\xi)}{h''(\xi)}$$

holds, where L_1 is defined by (4.160).

Proof. We define $\Psi \in C^2[a,b]$ by $\Psi(x) = L_1(h)f(x) - L_1(f)h(x)$. Due to linearity of L_1 we have $L_1(\Psi) = 0$. By Theorem 4.47 there exist $\xi,\eta \in [a,b]$ such that

$$0 = L_1(\Psi) = \tfrac{\Psi''(\xi)}{2}L_1(x^2)$$
$$0 \neq L_1(h) = \tfrac{h''(\eta)}{2}L_1(x^2).$$

Therefore, $L_1(x^2) \neq 0$ and $0 = \Psi''(\xi) = L_1(h)f''(\xi) - L_1(f)h''(\xi)$, which proves the theorem. □

Theorem 4.50 *Let $g : [a,b] \to \mathbb{R}$ be an integrable function. For given $c \in (a,b)$ and $q \neq 0$, denote $\sigma_1 = q\int_a^c g(t)dt$ and $\sigma_2 = q\int_c^b g(t)dt$. Assume* (4.64), (4.65) *and* (4.72) *hold. Then for any $f,h \in C^2[a,b]$ such that $h''(x) \neq 0$, for every $x \in [a,b]$, there exists $\xi \in [a,b]$ such that*

$$\frac{L_2(f)}{L_2(h)} = \frac{f''(\xi)}{h''(\xi)}$$

holds, where L_2 is defined by (4.161).

Proof. Similar to the proof of Theorem 4.49. □

Conditions (4.64), (4.65), (4.66) and (4.72) in Theorems 4.47 – 4.50 can be replaced by weaker conditions obtained in Section 4.3.

Similar as in Theorem 4.46 we have the following result.

Theorem 4.51 *Let $\Omega = \{f_p : I \to \mathbb{R} \mid p \in J\}$ be a family of functions such that for every mutually different points $x_0, x_1, x_2 \in I$ the mapping $p \mapsto [x_0, x_1, x_2; f_p]$ is $n-$exponentially convex in the Jensen sense on J. Let L_i, $i = 1,2$ be linear functionals defined by* (4.160) *and* (4.161). *Then the mapping $p \mapsto L_i(f_p)$ is $n-$exponentially convex in the Jensen sense on J.*
If the mapping $p \mapsto L_i(f_p)$ is continuous on J, then it is $n-$exponentially convex on J.

If the assumptions of Theorem 4.51 hold for all $n \in \mathbb{N}$, the following corollary is valid.

Corollary 4.15 *Let $\Omega = \{f_p : I \to \mathbb{R} \mid p \in J\}$ be a family of functions such that for every mutually different points $x_0, x_1, x_2 \in I$ the mapping $p \mapsto [x_0, x_1, x_2; f_p]$ is exponentially convex in the Jensen sense on J. Let L_i, $i = 1,2$ be linear functionals defined by* (4.160) *and* (4.161). *Then the mapping $p \mapsto L_i(f_p)$ is exponentially convex in the Jensen sense on J.*
If the mapping $p \mapsto L_i(f_p)$ is continuous on J, then it is exponentially convex on J.

We continue with the result needed for the application to Stolarsky type means.

Corollary 4.16 *Let $\Omega = \{f_p : I \to \mathbb{R} \mid p \in J\}$ be a family of functions such that for every mutually different points $x_0, x_1, x_2 \in I$ the mapping $p \mapsto [x_0, x_1, x_2; f_p]$ is 2-exponentially convex in the Jensen sense on J. Let L_i, $i = 1,2$ be linear functionals defined by* (4.160) *and* (4.161). *Then the following statements hold:*

(i) If the mapping $p \mapsto L_i(f_p)$ is continuous on J, then for $r,s,t \in J$, such that $r < s < t$, we have

$$[L_i(f_s)]^{t-r} \le [L_i(f_r)]^{t-s} [L_i(f_t)]^{s-r}, \quad i = 1,2.$$

(ii) If the mapping $p \mapsto L_i(f_p)$ is strictly positive and differentiable on J, then for every $p,s,u,v \in J$ such that $p \le u$ and $s \le v$ we have

$$\mu_{p,s}(L_i, \Omega) \le \mu_{u,v}(L_i, \Omega), \tag{4.164}$$

where

$$\mu_{p,s}(L_i, \Omega) = \begin{cases} \left(\frac{L_i(f_p)}{L_i(f_s)}\right)^{\frac{1}{p-s}}, & p \neq s; \\ \exp\left(\frac{\frac{d}{dp} L_i(f_p)}{L_i(f_p)}\right), & p = s. \end{cases} \tag{4.165}$$

Proof. Similar to the proof of Corollary 4.14. □

In the following examples we give some families of functions for which we use Corollaries 4.15 and 4.16 to construct new Stolarsky type means.

Example 4.2 Let

$$\Upsilon_1 = \{f_p : I \subset (0,\infty) \to \mathbb{R} \mid p \in \mathbb{R}\}$$

be a family of functions defined by

$$f_p(x) = \begin{cases} \frac{x^p}{p(p-1)}, & p \neq 0,1; \\ -\log x, & p = 0; \\ x \log x, & p = 1. \end{cases} \tag{4.166}$$

We have that f_p is a convex function on $\mathbb{R}^+$ since $\frac{d^2}{dx^2} f_p(x) = x^{p-2} > 0$ for $x > 0$. Furthermore, $p \mapsto \frac{d^2}{dx^2} f_p(x)$ is exponentially convex by definition. Similar as in proof of Theorem 4.46 we conclude that $p \mapsto [x_0, x_1, x_2; f_p]$ is exponentially convex (and so exponentially convex in the Jensen sense). Using Corollary 4.15 we obtain that $p \mapsto L_i(f_p)$, $i = 1,2$ is exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous, so it is exponentially convex. Hence, for this family of functions, from Corollary 4.16 we have that $\mu_{p,s}(L_i, \Upsilon_1)$, $i = 1,2$ is given by

$$\mu_{p,s}(L_i, \Upsilon_1) = \begin{cases} \left(\frac{L_i(f_p)}{L_i(f_s)}\right)^{\frac{1}{p-s}}, & p \neq s; \\ \exp\left(\frac{-L_i(f_p f_0)}{L_i(f_p)} - \frac{2p-1}{p(p-1)}\right), & p = s \neq 0,1; \\ \exp\left(\frac{-L_i(f_0^2)}{2L_i(f_0)} + 1\right), & p = s = 0; \\ \exp\left(\frac{-L_i(f_0 f_1)}{2L_i(f_1)} - 1\right), & p = s = 1. \end{cases}$$

Explicitly, for $p \neq s$ we have

$$\mu_{p,s}(L_1, \Upsilon_1) =$$

$$\left(\frac{s(s-1)}{p(p-1)} \cdot \frac{\frac{(a+\sigma_1)^{p+1} - a^{p+1}}{p+1} - \frac{\sigma_1}{c-a}\left(1 - \frac{1}{q}\right)\frac{c^{p+1} - a^{p+1}}{p+1} + \frac{b^{p+1} - (b-\sigma_2)^{p+1}}{p+1} - \int_a^b t^p g(t)\,dt}{\frac{(a+\sigma_1)^{s+1} - a^{s+1}}{s+1} - \frac{\sigma_1}{c-a}\left(1 - \frac{1}{q}\right)\frac{c^{s+1} - a^{s+1}}{s+1} + \frac{b^{s+1} - (b-\sigma_2)^{s+1}}{s+1} - \int_a^b t^s g(t)\,dt}\right)^{\frac{1}{p-s}}.$$

The limiting cases are:

* for $p = s \neq 0,1$

$$\mu_{p,p}(L_1, \Upsilon_1) = \exp\left(\frac{\alpha_1 - \int_a^b t^p \log t\, g(t)\,dt}{\alpha_2 - \int_a^b t^s g(t)\,dt} - \frac{2p-1}{p(p-1)}\right)$$

where

$$\begin{aligned} \alpha_1 = & \frac{(a+\sigma_1)^{p+1}\log(a+\sigma_1) - a^{p+1}\log a}{p+1} - \frac{(a+\sigma_1)^{p+1} - a^{p+1}}{(p+1)^2} \\ & - \frac{\sigma_1}{c-a}\left(1 - \frac{1}{q}\right)\left(\frac{c^{p+1}\log c - a^{p+1}\log a}{p+1} - \frac{c^{p+1} - a^{p+1}}{(p+1)^2}\right) \\ & + \frac{b^{p+1}\log b - (b-\sigma_2)^{p+1}\log(b-\sigma_2)}{p+1} - \frac{b^{p+1} - (b-\sigma_2)^{p+1}}{(p+1)^2} \\ & - \frac{\sigma_2}{b-c}\left(1 - \frac{1}{q}\right)\left(\frac{b^{p+1}\log b - c^{p+1}\log c}{p+1} - \frac{b^{p+1} - c^{p+1}}{(p+1)^2}\right), \end{aligned}$$

$$\alpha_2 = \frac{(a+\sigma_1)^{s+1} - a^{s+1}}{s+1} - \frac{\sigma_1}{c-a}\left(1 - \frac{1}{q}\right)\frac{c^{s+1} - a^{s+1}}{s+1} + \frac{b^{s+1} - (b-\sigma_2)^{s+1}}{s+1}$$

* for $p = s = 0$

$$\mu_{0,0}(L_1, \Upsilon_1) = \exp\left(\frac{1}{2} \cdot \frac{\beta_1 - \int_a^b \log^2 t g(t)dt}{\beta_2 - \int_a^b \log t g(t)dt} + 1\right)$$

where

$$\begin{aligned}\beta_1 &= (a+\sigma_1)\log^2(a+\sigma_1) - a\log^2 a - 2(a+\sigma_1)\log(a+\sigma_1) + 2a\log a + 2\sigma_1 \\ &\quad - \frac{\sigma_1}{c-a}\left(1-\frac{1}{q}\right)\left(c\log^2 c - a\log^2 a - 2c\log c + 2a\log a + 2c - 2a\right) \\ &\quad + b\log^2 b - (b-\sigma_2)\log^2(b-\sigma_2) - 2b\log b + 2(b-\sigma_2)\log(b-\sigma_2) + 2\sigma_2 \\ &\quad - \frac{\sigma_2}{b-c}\left(1-\frac{1}{q}\right)\left(b\log^2 b - c\log^2 c - 2b\log b + 2c\log c + 2b - 2c\right)\end{aligned}$$

and

$$\begin{aligned}\beta_2 &= (a+\sigma_1)\log(a+\sigma_1) - a\log a - \sigma_1 - \frac{\sigma_1}{c-a}\left(1-\frac{1}{q}\right)(c\log c - a\log a - c + a) \\ &\quad + b\log b - (b-\sigma_2)\log(b-\sigma_2) - \sigma_2 - \frac{\sigma_2}{b-c}\left(1-\frac{1}{q}\right)(b\log b - c\log c - b + c)\end{aligned}$$

* for $p = s = 1$

$$\mu_{1,1}(L_1, \Upsilon_1) = \exp\left(\frac{1}{2} \cdot \frac{\gamma_1 - \int_a^b t\log^2 t g(t)dt}{\gamma_2 - \int_a^b t\log t g(t)dt} - 1\right)$$

where

$$\begin{aligned}\gamma_1 &= \frac{(a+\sigma_1)^2}{2}(\log^2(a+\sigma_1) - \log(a+\sigma_1)) - \frac{a^2}{2}(\log^2 a - \log a) + \frac{b^2}{2}(\log^2 b - \log b) \\ &\quad - \frac{(b-\sigma_2)^2}{2}(\log^2(b-\sigma_2) - \log(b-\sigma_2)) + \frac{a\sigma_1 + b\sigma_2}{2} + \frac{\sigma_1^2 - \sigma_2^2}{4} \\ &\quad - \frac{\sigma_1}{c-a}\left(1-\frac{1}{q}\right)\left(\frac{c^2}{2}(\log^2 c - \log c) - \frac{a^2}{2}(\log^2 a - \log a) + \frac{c^2 - a^2}{4}\right) \\ &\quad - \frac{\sigma_2}{b-c}\left(1-\frac{1}{q}\right)\left(\frac{b^2}{2}(\log^2 b - \log b) - \frac{c^2}{2}(\log^2 c - \log c) + \frac{b^2 - c^2}{4}\right)\end{aligned}$$

and

$$\begin{aligned}\gamma_2 &= \frac{(a+\sigma_1)^2}{2}\log(a+\sigma_1) - \frac{a^2}{2}\log a + \frac{b^2}{2}\log b - \frac{(b-\sigma_2)^2}{2}\log(b-\sigma_2) \\ &\quad - \frac{a\sigma_1 + b\sigma_2}{2} - \frac{\sigma_1 - \sigma_2^2}{4} - \frac{\sigma_1}{c-a}\left(1-\frac{1}{q}\right)\left(\frac{c^2}{2}\log c - \frac{a^2}{2}\log a - \frac{c^2 - a^2}{4}\right) \\ &\quad - \frac{\sigma_2}{b-c}\left(1-\frac{1}{q}\right)\left(\frac{b^2}{2}\log b - \frac{c^2}{2}\log c - \frac{b^2 - c^2}{4}\right).\end{aligned}$$

Applying Theorems 4.49 and 4.50 on functions $f_p, f_s \in \Upsilon_1$ and functionals L_i, $i = 1,2$ we conclude that there exist $\xi_i \in [a,b]$ such that

$$\xi_i^{p-s} = \frac{L_i(f_p)}{L_i(f_s)}, \quad i = 1,2.$$

Since the function $\xi \mapsto \xi^{p-s}$ is invertible for $p \neq s$ we have

$$a \leq \left(\frac{L_i(f_p)}{L_i(f_s)}\right)^{\frac{1}{p-s}} \leq b$$

which together with the fact that $\mu_{p,s}(L_i, \Upsilon_1)$ is continuous, symetric and monotonic shows that $\mu_{p,s}(L_i, \Upsilon_1)$, $i = 1,2$ are means.

Example 4.3 Let

$$\Upsilon_2 = \{k_p \colon \mathbb{R} \to (0,\infty) \,|\, p \in \mathbb{R}\}$$

be a family of functions defined by

$$k_p(x) = \begin{cases} \frac{e^{px}}{p^2}, & p \neq 0; \\ \frac{x^2}{2}, & p = 0. \end{cases}$$

Similar as in Example 4.2 we conclude that $p \mapsto L_i(k_p)$, $i = 1,2$ are exponentially convex. For this family of functions, from Corollary 4.16 we have

$$\mu_{p,s}(L_i, \Upsilon_2) = \begin{cases} \left(\frac{L_i(k_p)}{L_i(k_s)}\right)^{\frac{1}{p-s}}, & p \neq s; \\ \exp\left(\frac{L_i(\mathrm{id} \cdot k_p)}{L_i(k_p)} - \frac{2}{p}\right), & p = s \neq 0; \\ \exp\left(\frac{1}{3}\frac{L_i(\mathrm{id} \cdot k_0)}{L_i(k_0)}\right), & p = s = 0. \end{cases}$$

Applying Theorems 4.49 and 4.50 on functions $k_p, k_s \in \Upsilon_2$ and functionals L_i, $i = 1,2$ it follows that $M_{p,s}(L_i, \Upsilon_2) = \log \mu_{p,s}(L_i, \Upsilon_2)$ satisfy $a \leq M_{p,s}(L_i, \Upsilon_2) \leq b$. So $M_{p,s}(L_i, \Omega_2)$, $i = 1,2$ are monotonic means by (4.164).

Example 4.4 Let

$$\Upsilon_3 = \{\phi_p \colon (0,\infty) \to (0,\infty) \,|\, p \in (0,\infty)\}$$

be a family of functions defined by

$$\phi_p(x) = \begin{cases} \frac{p^{-x}}{\log^2 p}, & p \neq 1; \\ \frac{x^2}{2}, & p = 1. \end{cases}$$

Similar as in Example 4.2 we conclude that $p \mapsto L_i(\phi_p)$, $i = 1,2$ are exponentially convex. For this family of functions, from Corollary 4.16 we have

$$\mu_{p,s}(L_i, \Upsilon_3) = \begin{cases} \left(\frac{L_i(\phi_p)}{L_i(\phi_s)}\right)^{\frac{1}{p-s}}, & p \neq s; \\ \exp\left(-\frac{L_i(\mathrm{id} \cdot \phi_p)}{pL_i(\phi_p)} - \frac{2}{p \log p}\right), & p = s \neq 1; \\ \exp\left(-\frac{L_i(\mathrm{id} \cdot \phi_1)}{3L_i(\phi_1)}\right), & p = s = 1. \end{cases}$$

Applying Theorems 4.49 and 4.50 on functions $\phi_p, \phi_s \in \Upsilon_3$ and functionals L_i, $i = 1,2$ it follows that $M_{p,s}(L_i, \Upsilon_3) = -L(p,s)\log \mu_{p,s}(L_i, \Upsilon_3)$ satisfy $a \le M_{p,s}(L_i, \Upsilon_3) \le b$, where $L(p,s)$ is logarithmic mean defined by $L(p,s) = \frac{p-s}{\log p - \log s}$. So $M_{p,s}(L_i, \Upsilon_3)$, $i = 1,2$ are means and by (4.164) they are monotonic.

Example 4.5 Let

$$\Upsilon_4 = \{\psi_p \colon (0,\infty) \to (0,\infty) \,|\, p \in (0,\infty)\}$$

be a family of functions defined by

$$\psi_p(x) = \frac{e^{-x\sqrt{p}}}{p}.$$

Similar as in Example 4.2 we conclude that $p \mapsto L_i(\psi_p)$, $i = 1,2$ are exponentially convex. For this family of functions, from Corollary 4.16 we have

$$\mu_{p,s}(L_i, \Upsilon_4) = \begin{cases} \left(\frac{L_i(\psi_p)}{L_i(\psi_s)}\right)^{\frac{1}{p-s}}, & p \neq s; \\ \exp\left(\frac{-1}{2\sqrt{p}}\frac{L_i(\mathrm{id}\cdot\psi_p)}{L_i(\psi_p)} - \frac{1}{p}\right), & p = s. \end{cases}$$

Applying Theorems 4.49 and 4.50 on functions $\psi_p, \psi_s \in \Upsilon_4$ and functionals L_1, L_2 it follows that $M_{p,s}(L_i, \Upsilon_4) = -(\sqrt{p} + \sqrt{s})\log \mu_{p,s}(L_i, \Upsilon_4)$ satisfy $a \le M_{p,s}(L_i, \Upsilon_4) \le b$. So $M_{p,s}(L_i, \Upsilon_4)$, $i = 1,2$ are monotonic means by (4.164).

Using Gauss type and Gauss-Steffensen type inequalities obtained in Sections 4.4 and 4.5 we can define the following linear functionals:

$$L_1(f) = \int_a^c f(t_1(x))g'(x)dx + \int_c^b f(s_1(x))g'(x)dx - \int_{g(a)}^{g(b)} f(x)dx, \tag{4.167}$$

$$L_2(f) = \int_{g(a)}^{g(b)} f(x)dx - \int_a^c f(s_2(x))g'(x)dx - \int_c^b f(t_2(x))g'(x)dx, \tag{4.168}$$

and

$$L_3(f) = \int_a^c f(t)G'(t)dt - \int_c^b f(t)G'(t)dt - \int_{G(a)}^{G(c)} f(t)dt + \int_{G(c)}^{G(b)} f(t)dt. \tag{4.169}$$

Remark 4.18 Under assumptions of Theorems 4.30, 4.31 and 4.29 we have that $L_i(f) \ge 0$, $i = 1,2,3$ for $f \in \mathscr{M}_1^c(I)$. Further, under assumptions of Corollaries 4.10, 4.11 and 4.9 we have that $L_i(f) \ge 0$, $i = 1,2,3$ for any convex function f.

We continue with the Lagrange type mean value theorems. Proofs are similar to the proof of Theorem 4.47 so we omit the details.

Theorem 4.52 *Let $c \in (a,b)$ and let $g : [a,b] \to \mathbb{R}$ be increasing, convex and differentiable such that $g(c) = c$. Assume* (4.88), (4.89) *and* (4.90) *hold and I is an interval as in Theorem 4.30. Then for any $f \in C^2(I)$ there exists $\xi \in I$ such that*

$$L_1(f) = \frac{f''(\xi)}{2}\left[\int_a^c t_1^2(x)g'(x)dx + \int_c^b s_1^2(x)g'(x)dx - \frac{g^3(b)-g^3(a)}{3}\right], \qquad (4.170)$$

where L_1 is defined by (4.167).

Theorem 4.53 *Let $c \in (a,b)$ and let $g : [a,b] \to \mathbb{R}$ be increasing, convex and differentiable such that $g(c) = c$. Assume* (4.94), (4.95) *and* (4.96) *hold and I is an interval as in Theorem 4.31, Then for any $f \in C^2(I)$ there exists $\xi \in I$ such that*

$$L_2(f) = \frac{f''(\xi)}{2}\left[\frac{g^3(b)-g^3(a)}{3} - \int_a^c s^2(x)g'(x)dx - \int_c^b t^2(x)g'(x)dx\right], \qquad (4.171)$$

where L_2 is defined by (4.168).

Theorem 4.54 *Let $G : [a,b] \to \mathbb{R}$ be an increasing function such that $G(x) \geq x$ and let $c \in (a,b)$. Assume that* (4.82) *holds. Then for any $f \in C^2(I)$ there exists $\xi \in I$ such that*

$$L_3(f) = \frac{f''(\xi)}{2}\left[\int_a^c x^2 G'(x)dx - \int_c^b x^2 G'(x)dx + \frac{G^3(b)+G^3(a)-2G^3(c)}{3}\right], \qquad (4.172)$$

where L_3 is defined by (4.169).

We continue with the Cauchy type mean value theorems related to functionals L_i, $i = 1,2,3$. Proofs are similar to the proof of Theorem 4.49.

Theorem 4.55 *Let $c \in (a,b)$ and let $g : [a,b] \to \mathbb{R}$ be increasing, convex and differentiable such that $g(c) = c$. Assume* (4.88), (4.89) *and* (4.90) *hold and I is an interval as in Theorem 4.30. Then for any $f,h \in C^2(I)$ such that $h''(x) \neq 0$ for every $x \in I$, there exists $\xi \in I$ such that*

$$\frac{L_1(f)}{L_1(h)} = \frac{f''(\xi)}{h''(\xi)} \qquad (4.173)$$

holds, where L_1 is defined by (4.167).

Theorem 4.56 *Let $c \in (a,b)$ and let $g : [a,b] \to \mathbb{R}$ be increasing, convex and differentiable such that $g(c) = c$. Assume* (4.94), (4.95) *and* (4.96) *hold and I is an interval as in Theorem 4.31. Then for any $f,h \in C^2(I)$ such that $h''(x) \neq 0$ for every $x \in I$, there exists $\xi \in I$ such that*

$$\frac{L_2(f)}{L_2(h)} = \frac{f''(\xi)}{h''(\xi)}, \qquad (4.174)$$

holds, where L_2 is defined by (4.150).

Theorem 4.57 *Let $G : [a,b] \to \mathbb{R}$ be an increasing function such that $G(x) \geq x$ and let $c \in (a,b)$. Assume that* (4.82) *holds. Then for any $f,h \in C^2(I)$ such that $h''(x) \neq 0$ for every $x \in I$, there exists $\xi \in I$ such that*

$$\frac{L_3(f)}{L_3(h)} = \frac{f''(\xi)}{h''(\xi)}$$

holds, where L_3 is defined by (4.169).

Conditions (4.90) and (4.96) in Theorems 4.52, 4.53, 4.55 and 4.56 can be replaced by weaker conditions given in Remark 4.10.

In the following theorem we show $n-$exponential convexity of functionals L_1 and L_2. Proof is similar to the proof of Theorem 4.46. In the sequel, J, K denote intervals in $\mathbb{R}$.

Theorem 4.58 *Let $\Omega = \{f_p : J \to \mathbb{R} \,|\, p \in K\}$ be a family of functions such that for every mutually different points $x_0, x_1, x_2 \in J$ the mapping $p \mapsto [x_0, x_1, x_2; f_p]$ is $n-$exponentially convex in the Jensen sense on K. Let L_i, $i = 1,2,3$ be linear functionals defined by* (4.167), (4.168) *and* (4.169). *Then the mapping $p \mapsto L_i(f_p)$ is $n-$exponentially convex in the Jensen sense on K.*
If the mapping $p \mapsto L_i(f_p)$ is continuous on K, then it is $n-$exponentially convex on K.

If the assumptions of Theorem 4.58 hold for all $n \in \mathbb{N}$, then we have the following corollary.

Corollary 4.17 *Let $\Omega = \{f_p : J \to \mathbb{R} \,|\, p \in K\}$ be a family of functions such that for every mutually different points $x_0, x_1, x_2 \in J$ the mapping $p \mapsto [x_0, x_1, x_2; f_p]$ is exponentially convex in the Jensen sense on K. Let L_i, $i = 1,2,3$ be linear functionals defined by* (4.167), (4.168) *and* (4.169). *Then the mapping $p \mapsto L_i(f_p)$ is exponentially convex in the Jensen sense on K.*
If the mapping $p \mapsto L_i(f_p)$ is continuous on K, then it is exponentially convex on K.

We continue with the result which is useful for the application to Stolarsky type means. Proof is similar to the proof of Corollary 4.14.

Corollary 4.18 *Let $\Omega = \{f_p : J \to \mathbb{R} \,|\, p \in K\}$ be a family of functions such that for every mutually different points $x_0, x_1, x_2 \in J$ the mapping $p \mapsto [x_0, x_1, x_2; f_p]$ is 2-exponentially convex in the Jensen sense on K. Let L_i, $i = 1,2,3$ be linear functionals defined by* (4.167), (4.168) *and* (4.169). *Then the following statements hold:*

(i) If the mapping $p \mapsto L_i(f_p)$ is continuous on K, then for $r,s,t \in K$, such that $r < s < t$, we have

$$[L_i(f_s)]^{t-r} \leq [L_i(f_r)]^{t-s} [L_i(f_t)]^{s-r}, \quad i = 1,2,3. \tag{4.175}$$

(ii) If the mapping $p \mapsto L_i(f_p)$ is strictly positive and differentiable on K, then for every $p,q,u,v \in K$ such that $p \leq u$ and $q \leq v$ we have

$$\mu_{p,q}(L_i, \Omega) \leq \mu_{u,v}(L_i, \Omega), \tag{4.176}$$

where

$$\mu_{p,q}(L_i,\Omega)=\begin{cases}\left(\frac{L_i(f_p)}{L_i(f_q)}\right)^{\frac{1}{p-q}}, & p\neq q,\\ \exp\left(\frac{\frac{d}{dp}L_i(f_p)}{L_i(f_p)}\right), & p=q.\end{cases}\tag{4.177}$$

Results from Theorem 4.58, Corollaries 4.17 and 4.18 still hold when two of the points $x_0,x_1,x_2\in J$ coincide, say $x_1=x_0$, for a family of differentiable functions f_p such that the function $p\to[x_0,x_1,x_2;f_p]$ is $n-$exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, they still hold when all three points coincide for a family of twice differentiable functions with the same property. The proofs are obtained by recalling Remark 1.3 and suitable characterization of convexity.

We continue with some families of functions $\Upsilon=\{f_p:J\to\mathbb{R}\,|\,p\in\mathbb{R}\}$ for which we use Corollaries 4.17 and 4.18 to construct exponentially convex functions and Stolarsky type means related to Gauss type and Gauss-Steffensen tpye inequalities.

Example 4.6 Let

$$\Upsilon_1=\{f_p\colon\mathbb{R}\to[0,\infty)\,|\,p\in\mathbb{R}\},$$

be a family of functions defined by

$$f_p(x)=\begin{cases}\frac{e^{px}}{p^2}, & p\neq 0;\\ \frac{x^2}{2}, & p=0.\end{cases}$$

Similar as in Example 4.3 for this family of functions, from Corollary 4.18 we have that $\mu_{p,q}(L_i,\Upsilon_1)$, $i=1,2,3$ are given by

$$\mu_{p,q}(L_i,\Upsilon_1)=\begin{cases}\left(\frac{L_i(f_p)}{L_i(f_q)}\right)^{\frac{1}{p-q}}, & p\neq q;\\ \exp\left(\frac{L_i(\mathrm{id}\cdot f_p)}{L_i(f_p)}-\frac{2}{p}\right), & p=q\neq 0;\\ \exp\left(\frac{1}{3}\frac{L_i(\mathrm{id}\cdot f_0)}{L_i(f_0)}\right), & p=q=0.\end{cases}$$

Explicitly, for $\mu_{p,q}(L_1,\Upsilon_1)$ we have the following:

* for $p\neq q$, $p,q\neq 0$:

$$\mu_{p,q}(L_1,\Upsilon_1)=\left(\frac{q^2}{p^2}\frac{\int_a^c e^{pt_1(x)}g'(x)dx+\int_c^b e^{ps_1(x)}g'(x)dx-\frac{e^{pg(b)}-e^{pg(a)}}{p}}{\int_a^c e^{qt_1(x)}g'(x)dx+\int_c^b e^{qs_1(x)}g'(x)dx-\frac{e^{qg(b)}-e^{qg(a)}}{q}}\right)^{\frac{1}{p-q}}$$

* for $p\neq q$, $q=0$ (or $p=0$):

$$\begin{aligned}\mu_{p,0}(L_1,\Upsilon_1)&=\left(\frac{2}{p^2}\frac{\int_a^c e^{pt_1(x)}g'(x)dx+\int_c^b e^{ps_1(x)}g'(x)dx-\frac{e^{pg(b)}-e^{pg(a)}}{p}}{\int_a^c t_1^2(x)g'(x)dx+\int_c^b s_1^2(x)g'(x)dx-\frac{g^3(b)-g^3(a)}{3}}\right)^{\frac{1}{p}}\\&=\mu_{0,p}(L_1,\Upsilon_1)\end{aligned}$$

∗ for $p = q \neq 0$:

$$\mu_{p,p}(L_1, \Upsilon_1) = \exp\left(\frac{A-B}{C} - \frac{2}{p}\right),$$

where

$$A = \int_a^c e^{pt_1(x)} t_1(x) g'(x)dx + \int_c^b e^{ps_1(x)} s_1(x) g'(x)dx,$$

$$B = \frac{1}{p}\left(g(b)e^{pg(b)} - g(a)e^{pg(a)} - \frac{e^{pg(b)} - e^{pg(a)}}{p}\right),$$

$$C = \int_a^c e^{pt_1(x)} g'(x)dx + \int_c^b e^{ps_1(x)} g'(x)dx - \frac{e^{pg(b)} - e^{pg(a)}}{p}.$$

∗ for $p = q = 0$:

$$\mu_{0,0}(L_1, \Upsilon_1) = \exp\left(\frac{1}{3}\frac{\int_a^c t_1^3(x)g'(x)dx + \int_c^b s_1^3(x)g'(x)dx - \frac{g^4(b)-g^4(a)}{4}}{\int_a^c t_1^2(x)g'(x)dx + \int_c^b s_1^2(x)g'(x)dx - \frac{g^3(b)-g^3(a)}{3}}\right).$$

For $\mu_{p,q}(L_3, \Upsilon_1)$ we have the following:

∗ for $p \neq q$, $p, q \neq 0$:

$$\mu_{p,q}(L_3, \Upsilon_1) = \left(\frac{q^2}{p^2}\frac{\int_a^c e^{px}G'(x)dx - \int_c^b e^{px}G'(x)dx + \frac{e^{pG(b)}+e^{pG(a)}-2e^{pG(c)}}{p}}{\int_a^c e^{qx}G'(x)dx - \int_c^b e^{qx}G'(x)dx + \frac{e^{qG(b)}+e^{qG(a)}-2e^{qG(c)}}{q}}\right)^{\frac{1}{p-q}}$$

∗ for $p \neq q$, $q = 0$ (or $p = 0$):

$$\begin{aligned}\mu_{p,0}(L_3, \Upsilon_1) &= \left(\frac{2}{p^2}\frac{\int_a^c e^{px}G'(x)dx - \int_c^b e^{px}G'(x)dx + \frac{e^{pG(b)}+e^{pG(a)}-2e^{pG(c)}}{p}}{\int_a^c x^2G'(x)dx - \int_c^b x^2G'(x)dx + \frac{G^3(a)+G^3(b)-2G^3(c)}{3}}\right)^{\frac{1}{p}} \\ &= \mu_{0,p}(L, \Upsilon_1)\end{aligned}$$

∗ for $p = q \neq 0$:

$$\mu_{p,p}(L_3, \Upsilon_1) = \exp\left(\frac{\int_a^c xe^{px}G'(x)dx - \int_c^b xe^{px}G'(x)dx + \alpha}{\int_a^c e^{px}G'(x)dx - \int_c^b e^{px}G'(x)dx + \beta} - \frac{2}{p}\right)$$

where

$$\alpha = \frac{e^{pG(b)}(pG(b)-1) + e^{pG(a)}(pG(a)-1) - 2e^{pG(c)}(pG(c)-1)}{p^2},$$

$$\beta = \frac{e^{pG(b)} + e^{pG(a)} - 2e^{pG(c)}}{p}$$

* for $p = q = 0$:

$$\mu_{0,0}(L_3,\Upsilon_1) = \exp\left(\frac{1}{3}\,\frac{\int_a^c x^3G'(x)dx - \int_c^b x^3G'(x)dx + \frac{G^4(b)+G^4(a)-2G^4(c)}{4}}{\int_a^c x^2G'(x)dx - \int_c^b x^2G'(x)dx + \frac{G^3(b)+G^3(a)-2G^3(c)}{3}}\right).$$

Applying Theorems 4.55, 4.56 and 4.57 on functions $f_p, f_q \in \Upsilon_1$ and functionals L_i, $i = 1,2,3$, we obtain that for $i = 1,2,3$

$$M_{p,q}(L_i,\Upsilon_1) = \log \mu_{p,q}(L_i,\Upsilon_1),$$

satisfy $\min I \le M_{p,q}(L_i,\Upsilon_1) \le \max I$. So $M_{p,q}(L_i,\Upsilon_1)$, $i = 1,2$ are monotonic means by (4.176).

Example 4.7 Let

$$\Upsilon_2 = \{h_p\colon (0,\infty) \to \mathbb{R} \,|\, p \in \mathbb{R}\},$$

be a family of functions defined by

$$h_p(x) = \begin{cases} \frac{x^p}{p(p-1)}, & p \ne 0,1; \\ -\log x, & p = 0; \\ x\log x, & p = 1. \end{cases} \tag{4.178}$$

Similar as in Example 4.2, for this family of functions, from Corollary 4.18 we have that $\mu_{p,q}(L_i,\Upsilon_2)$, $i = 1,2,3$ are given by

$$\mu_{p,q}(L_i,\Upsilon_2) = \begin{cases} \left(\frac{L_i(h_p)}{L_i(h_q)}\right)^{\frac{1}{p-q}}, & p \ne q; \\ \exp\left(\frac{-L_i(h_ph_0)}{L_i(h_p)} - \frac{2p-1}{p(p-1)}\right), & p = q \ne 0,1; \\ \exp\left(\frac{-L_i(h_0^2)}{2L_i(h_0)} + 1\right), & p = q = 0; \\ \exp\left(\frac{-L_i(h_0h_1)}{2L_i(h_1)} - 1\right), & p = q = 1. \end{cases}$$

Applying Theorems 4.55, 4.56 and 4.57 on functions $h_p, h_q \in \Upsilon_2$ and functionals L_i, $i = 1,2,3$, we conclude that there exist $\xi_i \in I$ such that

$$\xi_i^{p-q} = \frac{L_i(h_p)}{L_i(h_q)}, \quad i = 1,2,3.$$

Since the function $\xi \mapsto \xi^{p-q}$ is invertible, for $p \ne q$ we have

$$\min I \le \left(\frac{L_i(h_p)}{L_i(h_q)}\right)^{\frac{1}{p-q}} \le \max I,$$

which together with the fact that $\mu_{p,q}(L_i,\Upsilon_2)$, $i = 1,2,3$ are continuous, symmetric and monotonic (by (4.176)) shows that $\mu_{p,q}(L_i,\Upsilon_2)$, $i = 1,2,3$ are means.

Example 4.8 Let

$$\Upsilon_3 = \{\phi_p \colon (0,\infty) \to (0,\infty) \,|\, p \in (0,\infty)\},$$

be a family of functions defined by

$$\phi_p(x) = \begin{cases} \frac{p^{-x}}{\log^2 p}, & p \neq 1; \\ \frac{x^2}{2}, & p = 1. \end{cases}$$

Similar as in Example 4.4, for this family of functions, from Corollary 4.18 we have

$$\mu_{p,q}(L_i, \Upsilon_3) = \begin{cases} \left(\frac{L_i(\phi_p)}{L_i(\phi_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(-\frac{L_i(\mathrm{id}\cdot\phi_p)}{pL_i(\phi_p)} - \frac{2}{p\log p}\right), & p = q \neq 1; \\ \exp\left(-\frac{L_i(\mathrm{id}\cdot\phi_1)}{3L_i(\phi_1)}\right), & p = q = 1. \end{cases}$$

Applying Theorems 4.55, 4.56 and 4.57 on functions $\phi_p, \phi_q \in \Upsilon_3$ and functionals L_i, $i = 1,2,3$, we obtain that

$$M_{p,q}(L_i, \Upsilon_3) = -L(p,q) \log \mu_{p,q}(L_i, \Upsilon_3),$$

satisfy $\min I \leq M_{p,q}(L_i, \Upsilon_3) \leq \max I$, where $L(p,q)$ is logarithmic mean defined by $L(p,q) = \frac{p-q}{\log p - \log q}$. So $M_{p,q}(L_i, \Upsilon_3)$, $i = 1,2,3$ are means and by (4.158) they are monotonic.

Example 4.9 Let

$$\Upsilon_4 = \{\psi_p \colon (0,\infty) \to (0,\infty) \,|\, p \in (0,\infty)\},$$

be a family of functions defined by

$$\psi_p(x) = \frac{e^{-x\sqrt{p}}}{p}.$$

Similar as in Example 4.5, for this family of functions, from Corollary 4.18 we have

$$\mu_{p,q}(L_i, \Upsilon_4) = \begin{cases} \left(\frac{L_i(\psi_p)}{L_i(\psi_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{-1}{2\sqrt{p}} \frac{L_i(\mathrm{id}\cdot\psi_p)}{L_i(\psi_p)} - \frac{1}{p}\right), & p = q. \end{cases}$$

Applying Theorems 4.55, 4.56 and 4.57 on functions $\psi_p, \psi_q \in \Upsilon_4$ and functionals L_i, $i = 1,2,3$, we obtain

$$M_{p,q}(L_i, \Upsilon_4) = -(\sqrt{p} + \sqrt{q}) \log \mu_{p,q}(L_i, \Upsilon_4),$$

satisfy $\min I \leq M_{p,q}(L_i, \Upsilon_4) \leq \max I$. So $M_{p,q}(L_i, \Upsilon_4)$, $i = 1,2,3$ are monotonic means by (4.176).

Chapter 5

Weighted Steffensen inequality for *n*-convex functions

5.1 Generalizations via Taylor's formula

In [32], the authors show that Taylor's formula is natural choice to connect higher order convexity with Steffensen's type inequality. In this section, using different approach we give some new generalizations of Steffensen's inequality via Taylor's formula. First, we start with the proof of some identities related to generalizations of Steffensen's inequality. Results given in this section were obtained by Pečarić, Perušić Pribanić and Smoljak Kalamir in [64].

Theorem 5.1 *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n\geq 2$ and let $g:[a,b]\to\mathbb{R}$ be an integrable function such that $0\leq g\leq 1$. Let $\lambda=\int_a^b g(t)dt$ and let the function G_1 be defined by*

$$G_1(x)=\begin{cases}\int_a^x(1-g(t))dt, & x\in[a,a+\lambda],\\ \int_x^b g(t)dt, & x\in[a+\lambda,b].\end{cases} \tag{5.1}$$

Then

$$\begin{aligned}\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!} \int_a^b G_1(x)(x-a)^i dx \\ = -\frac{1}{(n-2)!} \int_a^b \left(\int_t^b G_1(x)(x-t)^{n-2} dx \right) f^{(n)}(t)dt\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!} \int_a^b G_1(x)(x-b)^i dx \\ = \frac{1}{(n-2)!} \int_a^b \left(\int_a^t G_1(x)(x-t)^{n-2} dx \right) f^{(n)}(t)dt.\end{aligned} \tag{5.3}$$

Proof. Applying Taylor's formula

$$f(x) = \sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{i!}(x-a)^i + \frac{1}{(n-1)!} \int_a^x f^{(n)}(t)(x-t)^{n-1} dt.$$

to the function f' and replacing n with $n-1$ $(n \geq 2)$ we have

$$f'(x) = \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!}(x-a)^i + \int_a^x f^{(n)}(t) \frac{(x-t)^{n-2}}{(n-2)!} dt. \tag{5.4}$$

Applying integration by parts to the identity (4.107) and then using the definition of the function G_1, we obtain

$$\begin{aligned}&\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt \\ &\quad = -\int_a^{a+\lambda} \left(\int_a^x (1-g(t)dt \right) df(x) - \int_{a+\lambda}^b \left(\int_x^b g(t)dt \right) df(x) \\ &\quad = -\int_a^b G_1(x) f'(x) dx.\end{aligned}$$

Hence, using (5.4) we obtain

$$\begin{aligned}\int_a^b G_1(x) f'(x) dx = \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!} \int_a^b G_1(x)(x-a)^i dx \\ + \frac{1}{(n-2)!} \int_a^b G_1(x) \left(\int_a^x (x-t)^{n-2} f^{(n)}(t) dt \right) dx.\end{aligned} \tag{5.5}$$

After applying Fubini's theorem on the last term in (5.5) we obtain (5.2).
Similarly, applying Taylor's formula at the point b to the function f' we obtain (5.3). □

Theorem 5.2 *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n\geq 2$ and let $g:[a,b]\to\mathbb{R}$ be an integrable function such that $0\leq g\leq 1$. Let $\lambda=\int_a^b g(t)dt$ and let the function G_2 be defined by*

$$G_2(x)=\begin{cases}\int_a^x g(t)dt, & x\in[a,b-\lambda],\\ \int_x^b(1-g(t))dt, & x\in[b-\lambda,b].\end{cases} \tag{5.6}$$

Then

$$\begin{aligned}\int_a^b f(t)g(t)dt-\int_{b-\lambda}^b f(t)dt+\sum_{i=0}^{n-2}\frac{f^{(i+1)}(a)}{i!}\int_a^b G_2(x)(x-a)^i dx\\ =-\frac{1}{(n-2)!}\int_a^b\left(\int_t^b G_2(x)(x-t)^{n-2}dx\right)f^{(n)}(t)dt\end{aligned} \tag{5.7}$$

and

$$\begin{aligned}\int_a^b f(t)g(t)dt-\int_{b-\lambda}^b f(t)dt+\sum_{i=0}^{n-2}\frac{f^{(i+1)}(b)}{i!}\int_a^b G_2(x)(x-b)^i dx\\ =\frac{1}{(n-2)!}\int_a^b\left(\int_a^t G_2(x)(x-t)^{n-2}dx\right)f^{(n)}(t)dt.\end{aligned} \tag{5.8}$$

Proof. Similar to the proof of Theorem 5.1 applying integration by parts to the identity (1.4) and then using the identity (5.4). □

In the following theorems, generalizations of Steffensen's inequality for n-convex functions are given.

Theorem 5.3 *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n\geq 2$ and let $g:[a,b]\to\mathbb{R}$ be an integrable function such that $0\leq g\leq 1$. Let $\lambda=\int_a^b g(t)dt$ and let the function G_1 be defined by* (5.1).

(i) If f is n-convex, then

$$\int_a^b f(t)g(t)dt\geq\int_a^{a+\lambda}f(t)dt+\sum_{i=0}^{n-2}\frac{f^{(i+1)}(a)}{i!}\int_a^b G_1(x)(x-a)^i dx. \tag{5.9}$$

(ii) If f is n-convex and

$$\int_a^t G_1(x)(x-t)^{n-2}dx\leq 0,\quad t\in[a,b], \tag{5.10}$$

then

$$\int_a^b f(t)g(t)dt\geq\int_a^{a+\lambda}f(t)dt+\sum_{i=0}^{n-2}\frac{f^{(i+1)}(b)}{i!}\int_a^b G_1(x)(x-b)^i dx. \tag{5.11}$$

Proof. If the function f is n-convex, without loss of generality we can assume that f is n-times differentiable and $f^{(n)} \geq 0$ see [71, p. 16 and p. 293]. Since $0 \leq g \leq 1$, the function G_1 is nonnegative and for every $n \geq 2$ we have

$$\int_t^b G_1(x)(x-t)^{n-2}dx \geq 0, \quad t \in [a,b].$$

Hence, we can apply Theorem 5.1 to obtain (5.9) and (5.11) respectively. □

Theorem 5.4 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$ and let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq 1$. Let $\lambda = \int_a^b g(t)dt$ and let the function G_2 be defined by* (5.6)*.*

(i) If f is n-convex, then

$$\int_a^b f(t)g(t)dt \leq \int_{b-\lambda}^b f(t)dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!} \int_a^b G_2(x)(x-a)^i dx. \tag{5.12}$$

(ii) If f is n-convex and

$$\int_a^t G_2(x)(x-t)^{n-2}dx \leq 0, \quad t \in [a,b], \tag{5.13}$$

then

$$\int_a^b f(t)g(t)dt \leq \int_{b-\lambda}^b f(t)dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!} \int_a^b G_2(x)(x-b)^i dx. \tag{5.14}$$

Proof. Similar as in the proof of Theorem 5.3, we can apply Theorem 5.2 to obtain (5.12) and (5.14). Again, since $0 \leq g \leq 1$, the function G_2 is nonnegative and for every $n \geq 2$ we have

$$\int_t^b G_2(x)(x-t)^{n-2}dx \geq 0, \quad t \in [a,b].$$

□

Taking $n = 2$ in Theorems 5.3 and 5.4, in the next corollaries the special cases for convex functions are given.

Corollary 5.1 *Let $f : [a,b] \to \mathbb{R}$ be such that f' is absolutely continuous, let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq 1$ and let $\lambda = \int_a^b g(t)dt$.*

(i) If f is convex, then

$$\int_a^b f(t)g(t)dt \geq \int_a^{a+\lambda} f(t)dt + f'(a)\left(\int_a^b tg(t)dt - \lambda a - \frac{\lambda^2}{2}\right).$$

(ii) If f is convex and

$$\int_a^t (t-x)g(x)dx \geq \frac{(t-a)^2}{2}, \quad t\in[a,a+\lambda],$$

$$\int_a^t (t-x)g(x)dx \geq \frac{\lambda^2}{2} + \lambda(t-a-\lambda), \quad t\in[a+\lambda,b],$$

then

$$\int_a^b f(t)g(t)dt \geq \int_a^{a+\lambda} f(t)dt + f'(b)\left(\int_a^b tg(t)dt - \lambda a - \frac{\lambda^2}{2}\right).$$

Corollary 5.2 *Let $f:[a,b]\to\mathbb{R}$ be such that f' is absolutely continuous, let $g:[a,b]\to\mathbb{R}$ be an integrable function such that $0\leq g\leq 1$ and let $\lambda=\int_a^b g(t)dt$.*

(i) If f is convex, then

$$\int_a^b f(t)g(t)dt \leq \int_{b-\lambda}^b f(t)dt - f'(a)\left(b\lambda - \frac{\lambda^2}{2} - \int_a^b tg(t)dt\right).$$

(ii) If f is convex and

$$\int_a^t (t-x)g(x)dx \leq 0, \quad t\in[a,b-\lambda],$$

$$\int_a^b (b-x)g(x)dx \leq \frac{(b-t)^2-\lambda^2}{2} + \lambda(t-b+\lambda), \quad t\in[b-\lambda,b],$$

then

$$\int_a^b f(t)g(t)dt \leq \int_{b-\lambda}^b f(t)dt - f'(b)\left(b\lambda - \frac{\lambda^2}{2} - \int_a^b tg(t)dt\right).$$

In the sequel we use Theorems 1.23 and 1.24 to obtain some new bounds for integrals on the left hand side in the perturbed version of the previously obtained identities.

Firstly, let us denote

$$\Phi_i(t) = \int_t^b G_i(x)(x-t)^{n-2}dx, \quad i=1,2 \tag{5.15}$$

and

$$\Omega_i(t) = \int_a^t G_i(x)(x-t)^{n-2}dx, \quad i=1,2. \tag{5.16}$$

We have that Čebyšev functionals $T(\Phi_i,\Phi_i)$ and $T(\Omega_i,\Omega_i), i=1,2$ are given by:

$$\begin{aligned} T(\Phi_1,\Phi_1) = {} & \frac{1}{(n-1)^2(b-a)}\left[\int_a^b \Psi^2(t)dt - \frac{2}{n}\int_a^{a+\lambda}(a+\lambda-t)^n\Psi(t)dt + \frac{\lambda^{2n+1}}{(2n+1)n^2}\right] \\ & - \frac{1}{(b-a)^2(n-1)^2n^2}\left(\int_a^b g(x)(x-a)^n dx - \frac{\lambda^{n+1}}{n+1}\right)^2, \end{aligned}$$

$$T(\Phi_2,\Phi_2) = \frac{1}{(n-1)^2(b-a)}\left[\frac{(b-a)^{2n+1}-(b-\lambda-a)^{2n+1}}{(2n+1)n^2} + \int_a^b \Psi^2(t)dt\right.$$
$$\left. -\frac{2}{n}\left(\frac{1}{n}\int_a^{b-\lambda}(b-t)^n(b-\lambda-t)^n dt + \int_a^b (b-t)^n\Psi(t)dt - \int_a^{b-\lambda}(b-\lambda-t)^n\Psi(t)dt\right)\right]$$
$$-\frac{1}{(b-a)^2(n-1)^2n^2}\left(\frac{(b-\lambda-a)^{n+1}-(b-a)^{n+1}}{n+1} + \int_a^b g(x)(x-a)^n dx\right)^2,$$

where $\Psi(t) = \int_t^b g(x)(x-t)^{n-1}dx$,

$$T(\Omega_1,\Omega_1) = \frac{1}{(n-1)^2(b-a)}\left[\frac{(a+\lambda-b)^{2n+1}-(a-b)^{2n+1}}{(2n+1)n^2} + \int_a^b \Upsilon^2(t)dt\right.$$
$$\left. -\frac{2}{n}\left(\frac{1}{n}\int_{a+\lambda}^{b}(a-t)^n(a+\lambda-t)^n dt + \int_a^b (a-t)^n\Upsilon(t)dt - \int_{a+\lambda}^{b}(a+\lambda-t)^n\Upsilon(t)dt\right)\right]$$
$$-\frac{1}{(b-a)^2(n-1)^2n^2}\left(\frac{(a+\lambda-b)^{n+1}-(a-b)^{n+1}}{n+1} + \int_a^b g(x)(x-b)^n dx\right)^2$$

and

$$T(\Omega_2,\Omega_2) = \frac{1}{(n-1)^2(b-a)}\left[\int_a^b \Upsilon^2(t)dt - \frac{2}{n}\int_{b-\lambda}^{b}(b-\lambda-t)^n\Upsilon(t)dt + \frac{\lambda^{2n+1}}{(2n+1)n^2}\right]$$
$$-\frac{1}{(b-a)^2(n-1)^2n^2}\left(\int_a^b g(x)(x-b)^n dx + \frac{(-\lambda)^{n+1}}{n+1}\right)^2,$$

where $\Upsilon(t) = \int_a^t g(x)(x-t)^{n-1}dx$.

Theorem 5.5 *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous function for some $n\geq 2$ with $(\cdot-a)(b-\cdot)[f^{(n+1)}]^2 \in L[a,b]$ and let g be an integrable function on $[a,b]$ such that $0\leq g\leq 1$. Let $\lambda = \int_a^b g(t)dt$ and let the functions G_1, Φ_1 and Ω_1 be defined by* (5.1)*,* (5.15) *and* (5.16)*.*

(i) Then

$$\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt + \sum_{i=0}^{n-2}\frac{f^{(i+1)}(a)}{i!}\int_a^b G_1(x)(x-a)^i dx$$
$$+\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)(n-2)!}\int_a^b \Phi_1(t)dt = H_n^1(f;a,b) \tag{5.17}$$

where the remainder $H_n^1(f;a,b)$ satisfies the estimation

$$\left|H_n^1(f;a,b)\right| \leq \frac{\sqrt{b-a}}{\sqrt{2}(n-2)!}\left[T(\Phi_1,\Phi_1)\right]^{\frac{1}{2}}\left|\int_a^b (t-a)(b-t)[f^{(n+1)}(t)]^2 dt\right|^{\frac{1}{2}}. \tag{5.18}$$

(ii) Then

$$\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!} \int_a^b G_1(x)(x-b)^i dx \\ - \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{(b-a)(n-2)!} \int_a^b \Omega_1(t)dt = H_n^2(f;a,b) \tag{5.19}$$

where the remainder $H_n^2(f;a,b)$ satisfies the estimation

$$\left|H_n^2(f;a,b)\right| \leq \frac{\sqrt{b-a}}{\sqrt{2}(n-2)!} \left[T(\Omega_1,\Omega_1)\right]^{\frac{1}{2}} \left| \int_a^b (t-a)(b-t)[f^{(n+1)}(t)]^2 dt \right|^{\frac{1}{2}}.$$

Proof.

(i) If we apply Theorem 1.23 for $f \to \Phi_1$ and $h \to f^{(n)}$ we obtain

$$\left| \frac{1}{b-a} \int_a^b \Phi_1(t) f^{(n)}(t)dt - \frac{1}{b-a} \int_a^b \Phi_1(t)dt \cdot \frac{1}{b-a} \int_a^b f^{(n)}(t)dt \right| \\ \leq \frac{1}{\sqrt{2}} \left[T(\Phi_1,\Phi_1)\right]^{\frac{1}{2}} \frac{1}{\sqrt{b-a}} \left| \int_a^b (t-a)(b-t)[f^{(n+1)}(t)]^2 dt \right|^{\frac{1}{2}}. \tag{5.20}$$

Therefore we have

$$\frac{1}{(b-a)(n-2)!} \int_a^b \Phi_1(t)dt \cdot \frac{1}{b-a} \int_a^b f^{(n)}(t)dt \\ = \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{(b-a)(n-2)!} \int_a^b \Phi_1(t)dt.$$

Now if we add that to the both sides of the identity (5.2) and use inequality (5.20) we obtain representation (5.17) and bound (5.18).

(ii) Similar to the first part.

□

Theorem 5.6 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous function for some $n \geq 2$ with $(\cdot - a)(b - \cdot)[f^{(n+1)}]^2 \in L[a,b]$ and let g be an integrable function on $[a,b]$ such that $0 \leq g \leq 1$. Let $\lambda = \int_a^b g(t)dt$ and let the functions G_2, Φ_2 and Ω_2 be defined by* (5.6), (5.15) *and* (5.16)*.*

(i) Then

$$\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!} \int_a^b G_2(x)(x-a)^i dx \\ + \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{(b-a)(n-2)!} \int_a^b \Phi_2(t)dt = H_n^3(f;a,b) \tag{5.21}$$

where the remainder $H_n^3(f;a,b)$ satisfies the estimation

$$\left|H_n^3(f;a,b)\right| \leq \frac{\sqrt{b-a}}{\sqrt{2}(n-2)!}\left[T(\Phi_2,\Phi_2)\right]^{\frac{1}{2}}\left|\int_a^b (t-a)(b-t)[f^{(n+1)}(t)]^2dt\right|^{\frac{1}{2}}.$$

(ii) Then

$$\begin{aligned}\int_a^b f(t)g(t)dt-\int_{b-\lambda}^b f(t)dt+\sum_{i=0}^{n-2}\frac{f^{(i+1)}(b)}{i!}\int_a^b G_2(x)(x-b)^i dx\\ -\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)(n-2)!}\int_a^b \Omega_2(t)dt=H_n^4(f;a,b)\end{aligned} \tag{5.22}$$

where the remainder $H_n^4(f;a,b)$ satisfies the estimation

$$\left|H_n^4(f;a,b)\right| \leq \frac{\sqrt{b-a}}{\sqrt{2}(n-2)!}\left[T(\Omega_2,\Omega_2)\right]^{\frac{1}{2}}\left|\int_a^b (t-a)(b-t)[f^{(n+1)}(t)]^2dt\right|^{\frac{1}{2}}.$$

Proof. Similar to the proof of Theorem 5.5 □

Taking $n=2$ in Theorems 5.5 and 5.6 we obtain the following corollaries.

Corollary 5.3 *Let $f:[a,b]\to\mathbb{R}$ be such that f'' is absolutely continuous function with $(\cdot-a)(b-\cdot)[f''']^2\in L[a,b]$, let g be an integrable function on $[a,b]$ such that $0\leq g\leq 1$ and let $\lambda=\int_a^b g(t)dt$.*

(i) Then

$$\begin{aligned}\int_a^{a+\lambda} f(t)dt-\int_a^b f(t)g(t)dt+f'(a)\left(\int_a^b tg(t)dt-\lambda a-\frac{\lambda^2}{2}\right)\\ +\frac{f'(b)-f'(a)}{2(b-a)}\left(\int_a^b g(t)(t-a)^2dt-\frac{\lambda^3}{3}\right)=H_2^1(f;a,b),\end{aligned}$$

where the remainder $H_2^1(f;a,b)$ satisfies the estimation

$$\left|H_2^1(f;a,b)\right| \leq \frac{\sqrt{b-a}}{\sqrt{2}}\left[T(\phi_1,\phi_1)\right]^{\frac{1}{2}}\left|\int_a^b (t-a)(b-t)[f'''(t)]^2dt\right|^{\frac{1}{2}}$$

and

$$\begin{aligned}T(\phi_1,\phi_1)=\frac{1}{b-a}&\left[\int_a^b\left(\int_t^b g(x)(x-t)dx\right)^2dt+\frac{\lambda^5}{20}\right.\\ &\left.-\int_a^{a+\lambda}(a+\lambda-t)^2\left(\int_t^b g(x)(x-t)dx\right)dt\right]\\ &-\frac{1}{4(b-a)^2}\left(\int_a^b g(x)(x-a)^2dx-\frac{\lambda^3}{3}\right)^2.\end{aligned}$$

(ii) Then

$$\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt + f'(b)\left(\int_a^b tg(t)dt - \lambda a - \frac{\lambda^2}{2}\right)$$
$$-\frac{f'(b)-f'(a)}{2(b-a)}\left(\frac{(b-a)^3-(b-a-\lambda)^3}{3} - \int_a^b g(t)(b-t)^2dt\right) = H_2^2(f;a,b),$$

where the remainder $H_2^2(f;a,b)$ satisfies the estimation

$$\left|H_2^2(f;a,b)\right| \le \frac{\sqrt{b-a}}{\sqrt{2}}\left[T(\omega_1,\omega_1)\right]^{\frac{1}{2}}\left|\int_a^b (t-a)(b-t)[f'''(t)]^2dt\right|^{\frac{1}{2}}$$

and

$$T(\omega_1,\omega_1) = \frac{1}{b-a}\left[\frac{(a+\lambda-b)^5-(a-b)^5}{20} - \left(\frac{1}{2}\int_{a+\lambda}^b (a-t)^2(a+\lambda-t)^2dt\right.\right.$$
$$+\int_a^b (a-t)^2\left(\int_a^t g(x)(x-t)dx\right)dt$$
$$\left.-\int_{a+\lambda}^b (a+\lambda-t)^2\left(\int_a^t g(x)(x-t)dx\right)dt\right)$$
$$\left.+\int_a^b\left(\int_a^t g(x)(x-t)dx\right)^2 dt\right]$$
$$-\frac{1}{4(b-a)^2}\left(\frac{(a+\lambda-b)^3-(a-b)^3}{3} + \int_a^b g(x)(x-b)^2dx\right)^2.$$

Corollary 5.4 *Let $f:[a,b]\to\mathbb{R}$ be such that f'' is absolutely continuous function with $(\cdot-a)(b-\cdot)[f''']^2 \in L[a,b]$, let g be an integrable function on $[a,b]$ such that $0\le g\le 1$ and let $\lambda = \int_a^b g(t)dt$.*

(i) Then

$$\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt + f'(a)\left(b\lambda - \frac{\lambda^2}{2} - \int_a^b tg(t)dt\right)$$
$$+\frac{f'(b)-f'(a)}{2(b-a)}\left(\frac{(b-a)^3}{3} - \frac{(b-a-\lambda)^3}{3} - \frac{1}{2}\int_a^b g(x)(x-a)^2\right) = H_2^3(f;a,b),$$

where the remainder $H_2^3(f;a,b)$ satisfies the estimation

$$\left|H_2^3(f;a,b)\right| \le \frac{\sqrt{b-a}}{\sqrt{2}}\left[T(\phi_2,\phi_2)\right]^{\frac{1}{2}}\left|\int_a^b (t-a)(b-t)[f'''(t)]^2dt\right|^{\frac{1}{2}}.$$

and

$$
\begin{aligned}
T(\phi_2,\phi_2) = \frac{1}{b-a}\Bigg[&\frac{(b-a)^5-(b-\lambda-a)^5}{20} - \left(\frac{1}{2}\int_a^{b-\lambda}(b-t)^2(b-\lambda-t)^2 dt\right.\\
&+\int_a^b (b-t)^2\left(\int_t^b g(x)(x-t)dx\right)dt\\
&\left.-\int_a^{b-\lambda}(b-\lambda-t)^2\left(\int_t^b g(x)(x-t)dx\right)dt\right)\\
&+\int_a^b\left(\int_t^b g(x)(x-t)dx\right)^2 dt\Bigg]\\
&-\frac{1}{4(b-a)^2}\left(\frac{(b-\lambda-a)^3-(b-a)^3}{3}+\int_a^b g(x)(x-a)^2dx\right)^2.
\end{aligned}
$$

(ii) Then

$$
\begin{aligned}
\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt + f'(b)\left(b\lambda - \frac{\lambda^2}{2} - \int_a^b tg(t)dt\right)&\\
-\frac{f'(b)-f'(a)}{2(b-a)}\left(\int_a^b g(t)(b-t)^2dt - \frac{\lambda^3}{3}\right) &= H_2^4(f;a,b),
\end{aligned}
$$

where the remainder $H_2^4(f;a,b)$ satisfies the estimation

$$
\left|H_2^4(f;a,b)\right| \le \frac{\sqrt{b-a}}{\sqrt{2}}\left[T(\omega_2,\omega_2)\right]^{\frac{1}{2}}\left|\int_a^b (t-a)(b-t)[f'''(t)]^2dt\right|^{\frac{1}{2}}.
$$

and

$$
\begin{aligned}
T(\omega_2,\omega_2) = \frac{1}{b-a}\Bigg[&\int_a^b\left(\int_a^t g(x)(x-t)dx\right)^2 dt + \frac{\lambda^5}{20}\\
&-\int_{b-\lambda}^b (b-\lambda-t)^2\left(\int_a^t g(x)(x-t)dx\right)dt\Bigg]\\
&-\frac{1}{4(b-a)^2}\left(\int_a^b g(x)(x-b)^2dx - \frac{\lambda^3}{3}\right)^2.
\end{aligned}
$$

Using Theorem 1.24 we obtain the following Grüss type inequalities for the Čebyšev functional.

Theorem 5.7 *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ $(n\ge 2)$ is absolutely continuous function and $f^{(n+1)}\ge 0$ on $[a,b]$. Let the functions Φ_i and Ω_i, $i=1,2$ be defined by* (5.15) *and* (5.16) .

(i) Then we have the representation (5.17) *and the remainder $H_n^1(f;a,b)$ satisfies the bound*

$$
\begin{aligned}
&\left|H_n^1(f;a,b)\right|\\
&\le \frac{1}{(n-2)!}\|\Phi_1'\|_\infty\left\{\frac{f^{(n-1)}(b)+f^{(n-1)}(a)}{2} - \frac{f^{(n-2)}(b)-f^{(n-2)}(a)}{b-a}\right\}.
\end{aligned}
\tag{5.23}
$$

(ii) Then we have the representation (5.19) *and the remainder* $H_n^2(f;a,b)$ *satisfies the bound*

$$\left|H_n^2(f;a,b)\right| \le \frac{1}{(n-2)!}\|\Omega_1'\|_\infty \left\{\frac{f^{(n-1)}(b)+f^{(n-1)}(a)}{2} - \frac{f^{(n-2)}(b)-f^{(n-2)}(a)}{b-a}\right\}.$$

(iii) Then we have the representation (5.21) *and the remainder* $H_n^3(f;a,b)$ *satisfies the bound*

$$\left|H_n^3(f;a,b)\right| \le \frac{1}{(n-2)!}\|\Phi_2'\|_\infty \left\{\frac{f^{(n-1)}(b)+f^{(n-1)}(a)}{2} - \frac{f^{(n-2)}(b)-f^{(n-2)}(a)}{b-a}\right\}.$$

(iv) Then we have the representation (5.22) *and the remainder* $H_n^4(f;a,b)$ *satisfies the bound*

$$\left|H_n^4(f;a,b)\right| \le \frac{1}{(n-2)!}\|\Omega_2'\|_\infty \left\{\frac{f^{(n-1)}(b)+f^{(n-1)}(a)}{2} - \frac{f^{(n-2)}(b)-f^{(n-2)}(a)}{b-a}\right\}.$$

Proof.

(i) Applying Theorem 1.24 for $f \to \Phi_1$ and $h \to f^{(n)}$ we obtain

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b \Phi_1(t)f^{(n)}(t)dt - \frac{1}{b-a}\int_a^b \Phi_1(t)dt \cdot \frac{1}{b-a}\int_a^b f^{(n)}(t)dt\right| \\ &\le \frac{1}{2(b-a)}\|\Phi_1'\|_\infty \int_a^b (t-a)(b-t)f^{(n+1)}(t)dt.\end{aligned} \tag{5.24}$$

Since

$$\begin{aligned}\int_a^b (t-a)(b-t)f^{(n+1)}(t)dt &= \int_a^b [2t-(a+b)]f^{(n)}(t)dt \\ &= (b-a)\left[f^{(n-1)}(b)+f^{(n-1)}(a)\right] - 2\left(f^{(n-2)}(b)-f^{(n-2)}(a)\right),\end{aligned}$$

using the representation (5.2) and the inequality (5.24) we deduce (5.23). Similarly,

we can prove the other parts. □

Taking $n=2$ in the previous theorem we obtain the corollary:

Corollary 5.5 *Let* $f:[a,b]\to\mathbb{R}$ *be such that* f'' *is absolutely continuous function and* $f'''\ge 0$ *on* $[a,b]$. *Let* g *be an integrable function such that* $0\le g\le 1$, $\lambda=\int_a^b g(t)dt$ *and let the functions* $G_i,\ i=1,2$ *be defined by* (5.1) *and* (5.6).

(i) Then we have

$$\begin{aligned}&\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt + f'(a)\left(\int_a^b tg(t)dt - \lambda a - \frac{\lambda^2}{2}\right) \\ &\qquad + \frac{f'(b)-f'(a)}{b-a}\int_a^b (x-a)G_1(x)dx = H_2^1(f;a,b)\end{aligned}$$

and the remainder $H_2^1(f;a,b)$ satisfies the bound

$$\left|H_2^1(f;a,b)\right| \leq \|\Phi_1'\|_\infty \left\{\frac{f'(b)+f'(a)}{2} - \frac{f(b)-f(a)}{b-a}\right\}$$

where

$$\Phi_1'(t) = -\int_a^t (1-g(x))dx.$$

(ii) Then we have

$$\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt + f'(b)\left(\int_a^b tg(t)dt - \lambda a - \frac{\lambda^2}{2}\right)$$
$$- \frac{f'(b)-f'(a)}{b-a}\int_a^b (b-x)G_1(x)dx = H_2^2(f;a,b)$$

and the remainder $H_2^2(f;a,b)$ satisfies the bound

$$\left|H_2^2(f;a,b)\right| \leq \|\Omega_1'\|_\infty \left\{\frac{f'(b)+f'(a)}{2} - \frac{f(b)-f(a)}{b-a}\right\}$$

where

$$\Omega_1'(t) = \int_t^b g(x)dx.$$

(iii) Then we have

$$\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt + f'(a)\left(b\lambda - \frac{\lambda^2}{2} - \int_a^b tg(t)dt\right)$$
$$+ \frac{f'(b)-f'(a)}{b-a}\int_a^b (x-a)G_2(x)dxdt = H_2^3(f;a,b)$$

and the remainder $H_2^3(f;a,b)$ satisfies the bound

$$\left|H_2^3(f;a,b)\right| \leq \|\Phi_2'\|_\infty \left\{\frac{f'(b)+f'(a)}{2} - \frac{f(b)-f(a)}{b-a}\right\}$$

where

$$\Phi_2'(t) = -\int_a^t g(x)dx.$$

(iv) Then we have

$$\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt + f'(b)\left(b\lambda - \frac{\lambda^2}{2} - \int_a^b tg(t)dt\right)$$
$$- \frac{f'(b)-f'(a)}{b-a}\int_a^b (b-x)G_2(x)dx = H_2^4(f;a,b)$$

and the remainder $H_2^4(f;a,b)$ *satisfies the bound*

$$\left|H_2^4(f;a,b)\right| \leq \|\Omega_2'\|_\infty \left\{ \frac{f'(b)+f'(a)}{2} - \frac{f(b)-f(a)}{b-a} \right\}$$

where

$$\Omega_2'(t) = \int_t^b (1-g(x))dx.$$

Using identities from Theorems 5.1 and 5.2, the Ostrowski type inequalities are obtained.

Theorem 5.8 *Suppose that all assumptions of Theorem 5.1 hold. Assume* (p,q) *is a pair of conjugate exponents, that is* $1 \leq p,q \leq \infty$, $1/p+1/q=1$. *Let* $f^{(n)} \in L_p[a,b]$ *for some* $n \geq 2$. *Then we have:*

(i)

$$\begin{aligned} &\left| \int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!} \int_a^b G_1(x)(x-a)^i dx \right| \\ &\leq \frac{1}{(n-2)!} \left\| f^{(n)} \right\|_p \left\| \int_t^b G_1(x)(x-t)^{n-2} dx \right\|_q \end{aligned} \tag{5.25}$$

The constant on the right-hand side of (5.25) *is sharp for* $1 < p \leq \infty$ *and the best possible for* $p=1$.

(ii)

$$\begin{aligned} &\left| \int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!} \int_a^b G_1(x)(x-b)^i dx \right| \\ &\leq \frac{1}{(n-2)!} \left\| f^{(n)} \right\|_p \left\| \int_a^t G_1(x)(x-t)^{n-2} dx \right\|_q . \end{aligned} \tag{5.26}$$

The constant on the right-hand side of (5.26) *is sharp for* $1 < p \leq \infty$ *and the best possible for* $p=1$.

Proof.

(i) Let us denote

$$C(t) = \frac{1}{(n-2)!} \int_t^b G_1(x)(x-t)^{n-2} dx.$$

Since $0 \leq g \leq 1$, the function G_1 is nonnegative and for every $n \geq 2$ we have $C(t) \geq 0, \forall t \in [a,b]$. Using the identity (5.2) and applying Hölder's inequality we obtain

$$\begin{aligned} &\left| \int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)(a)}}{i!} \int_a^b G_1(x)(x-a)^i dx \right| \\ &= \left| - \int_a^b C(t) f^{(n)}(t) dt \right| \leq \left\| f^{(n)} \right\|_p \|C(t)\|_q . \end{aligned}$$

For the proof of the sharpness we will find a function f for which the equality in (5.25) is obtained.
For $1 < p < \infty$ take f to be such that

$$f^{(n)}(t) = \operatorname{sgn} C(t)\, |C(t)|^{\frac{1}{p-1}}.$$

For $p = \infty$ take $f^{(n)}(t) = \operatorname{sgn} C(t)$.
For $p = 1$ we prove that

$$\left| \int_a^b C(t) f^{(n)}(t)dt \right| \leq \max_{t\in[a,b]} |C(t)| \left(\int_a^b \left| f^{(n)}(t) \right| dt \right) \tag{5.27}$$

is the best possible inequality. Suppose that $|C(t)|$ attains its maximum at $t_0 \in [a,b]$ and we have $C(t_0) > 0$. For ε small enough we define $f_\varepsilon(t)$ by

$$f_\varepsilon(t) = \begin{cases} 0, & a \leq t \leq t_0, \\ \frac{1}{\varepsilon n!}(t-t_0)^n, & t_0 \leq t \leq t_0+\varepsilon, \\ \frac{1}{n!}(t-t_0)^{n-1}, & t_0+\varepsilon \leq t \leq b. \end{cases}$$

Then for ε small enough

$$\left| \int_a^b C(t) f^{(n)}(t)dt \right| = \left| \int_{t_0}^{t_0+\varepsilon} C(t)\frac{1}{\varepsilon}dt \right| = \frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} C(t)dt.$$

Now from the inequality (5.27) we have

$$\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} C(t)dt \leq C(t_0)\int_{t_0}^{t_0+\varepsilon} \frac{1}{\varepsilon}dt = C(t_0).$$

Since,

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} C(t)dt = C(t_0)$$

the statement follows.

(ii) Here, we denote $C(t) = \frac{1}{(n-2)!}\int_a^t G_1(x)(x-t)^{n-2}dx$. Thus we have one more case when $|C(t)|$ attains its maximum at $t_0 \in [a,b]$ and $C(t_0) < 0$. In the case $C(t_0) < 0$, we define $f_\varepsilon(t)$ by

$$f_\varepsilon(t) = \begin{cases} \frac{1}{n!}(t-t_0-\varepsilon)^{n-1},, & a \leq t \leq t_0, \\ -\frac{1}{\varepsilon n!}(t-t_0-\varepsilon)^n, & t_0 \leq t \leq t_0+\varepsilon, \\ 0, & t_0+\varepsilon \leq t \leq b. \end{cases}$$

The rest of the proof is the same as above.

$\square$

Theorem 5.9 *Suppose that all assumptions of Theorem 5.2 hold. Assume (p,q) is a pair of conjugate exponents, that is $1 \leq p,q \leq \infty$, $1/p+1/q=1$. Let $f^{(n)} \in L_p[a,b]$ for some $n \geq 2$. Then we have:*

(i)

$$\begin{aligned} &\left| \int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!} \int_a^b G_2(x)(x-a)^i dx \right| \\ &\leq \frac{1}{(n-2)!} \left\| f^{(n)} \right\|_p \left\| \int_t^b G_2(x)(x-t)^{n-2} dx \right\|_q . \end{aligned} \tag{5.28}$$

The constant on the right-hand side of (5.28) *is sharp for $1 < p \leq \infty$ and the best possible for $p=1$.*

(ii)

$$\begin{aligned} &\left| \int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!} \int_a^b G_2(x)(x-b)^i dx \right| \\ &\leq \frac{1}{(n-2)!} \left\| f^{(n)} \right\|_p \left\| \int_a^t G_2(x)(x-t)^{n-2} dx \right\|_q . \end{aligned} \tag{5.29}$$

The constant on the right-hand side of (5.29) *is sharp for $1 < p \leq \infty$ and the best possible for $p=1$.*

Proof. Similar to the proof of Theorem 5.8. □

Taking $n=2$ in the previous theorems we obtain the following corollaries.

Corollary 5.6 *Let $f : [a,b] \to \mathbb{R}$ be such that f' is absolutely continuous, let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq 1$, and let $\lambda = \int_a^b g(t)dt$. Assume (p,q) is a pair of conjugate exponents, that is $1 \leq p,q \leq \infty$, $1/p+1/q=1$. Let $f'' \in L_p[a,b]$. Then we have:*

(i)

$$\begin{aligned} &\left| \int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt + f'(a)\left(\int_a^b tg(t)dt - \lambda a - \frac{\lambda^2}{2} \right) \right| \\ &\leq \left\| f'' \right\|_p \left(\int_a^{a+\lambda} \left| t\int_a^t g(x)dx + \int_t^b xg(x)dx - \lambda a - \frac{\lambda^2}{2} - \frac{(t-a)^2}{2} \right|^q dt \right. \\ &\left. + \int_{a+\lambda}^b \left| \int_t^b xg(x)dx - t\int_t^b g(x)dx \right|^q dt \right)^{\frac{1}{q}} . \end{aligned} \tag{5.30}$$

The constant on the right-hand side of (5.30) *is sharp for $1 < p \leq \infty$ and the best possible for $p=1$.*

(ii)

$$\begin{aligned}
&\left|\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt + f'(b)\left(\int_a^b tg(t)dt - \lambda a - \frac{\lambda^2}{2}\right)\right| \\
&\leq \|f''\|_p \left(\int_a^{a+\lambda} \left|\frac{(t-a)^2}{2} - \int_a^t (t-x)g(x)dx\right|^q dt\right. \qquad (5.31)\\
&\left. + \int_{a+\lambda}^b \left|\frac{\lambda^2}{2} + \lambda(t-a-\lambda) - \int_a^t (t-x)g(x)dx\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

The constant on the right-hand side of (5.31) *is sharp for* $1 < p \leq \infty$ *and the best possible for* $p = 1$.

Corollary 5.7 *Let* $f : [a,b] \to \mathbb{R}$ *be such that* f' *is absolutely continuous, let* $g : [a,b] \to \mathbb{R}$ *be an integrable function such that* $0 \leq g \leq 1$, *and let* $\lambda = \int_a^b g(t)dt$. *Assume* (p,q) *is a pair of conjugate exponents, that is* $1 \leq p,q \leq \infty$, $1/p + 1/q = 1$. *Let* $f'' \in L_p[a,b]$. *Then we have:*

(i)

$$\begin{aligned}
&\left|\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt + f'(a)\left(b\lambda - \frac{\lambda^2}{2} - \int_a^b tg(t)dt\right)\right| \\
&\leq \|f''\|_p \left(\int_a^{b-\lambda} \left|b\lambda - \frac{\lambda^2}{2} - t\int_a^t g(x)dx - \int_t^b xg(x)dx\right|^q dt\right. \qquad (5.32)\\
&\left. + \int_{b-\lambda}^b \left|\frac{(b-t)^2}{2} - \int_t^b (x-t)g(x)dx\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

The constant on the right-hand side of (5.32) *is sharp for* $1 < p \leq \infty$ *and the best possible for* $p = 1$.

(ii)

$$\begin{aligned}
&\left|\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt + f'(b)\left(b\lambda - \frac{\lambda^2}{2} - \int_a^b tg(t)dt\right)\right| \\
&\leq \|f''\|_p \left(\int_a^{b-\lambda} \left|\int_a^t (t-x)g(x)dx\right|^q dt\right. \qquad (5.33)\\
&\left. + \int_{b-\lambda}^b \left|\int_a^b (b-x)g(x)dx - \frac{(b-t)^2 - \lambda^2}{2} - \lambda(t-b+\lambda)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

The constant on the right-hand side of (5.33) *is sharp for* $1 < p \leq \infty$ *and the best possible for* $p = 1$.

5.2 Generalizations via Montgomery's identitiy

In [3], the authors obtain the following extension of Montgomery identity using Taylor's formula:

Theorem 5.10 *Let $f : I \to \mathbb{R}$ be suct that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$. Then the following identity holds*

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt - \sum_{i=0}^{n-2} f^{(i+1)}(x)\,\frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!\,(b-a)} + \frac{1}{(n-1)!}\int_a^b T_n(x,s)\,f^{(n)}(s)\,ds \tag{5.34}$$

where

$$T_n(x,s) = \begin{cases} \frac{-1}{n(b-a)}(a-s)^n, & a \leq s \leq x; \\ \frac{-1}{n(b-a)}(b-s)^n, & x < s \leq b. \end{cases}$$

Remark 5.1 *The last identity holds also for $n = 1$. In this special case, we assume that $\sum_{i=0}^{n-2} \cdots$ is an empty sum. Thus* (5.34) *reduces to well known **Montgomery identity** (e.g. [4])*

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + \int_a^b T_1(x,s)f'(s)ds$$

where the Peano kernel is

$$T_1(x,s) = \begin{cases} \frac{s-a}{b-a}, & a \leq s \leq x; \\ \frac{s-b}{b-a}, & x < s \leq b. \end{cases}$$

We begin this section with some new identities related to Steffensen's inequality. Using these new identities, Steffensen's inequality for n-convex functions is generalized and further, some new Ostrowski type inequalities are obtained. Some generalizations of Steffensen's inequality via weighted Montgomery's identitiy are given in [5]. Results from this section are published in [6].

Theorem 5.11 *Let $f : I \to \mathbb{R}$ be suct that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$. Let $g, p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$ and let the function $\mathcal{G}_1$ be defined by*

$$\mathcal{G}_1(x) = \begin{cases} \int_a^x (1-g(t))p(t)dt, & x \in [a, a+\lambda], \\ \int_x^b g(t)p(t)dt, & x \in [a+\lambda, b]. \end{cases} \tag{5.35}$$

Then

$$\begin{aligned}
&\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt \\
&+\int_a^b \mathscr{G}_1(x)\left(\frac{f(b)-f(a)}{b-a} - \sum_{i=0}^{n-3} f^{(i+2)}(x)\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!\,(b-a)}\right)dx \qquad (5.36)\\
&= -\frac{1}{(n-2)!}\int_a^b \left(\int_a^b \mathscr{G}_1(x)T_{n-1}(x,s)dx\right) f^{(n)}(s)ds.
\end{aligned}$$

Proof. Using identity

$$\begin{aligned}
&\int_{[a,a+\lambda]} f(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) \\
&\qquad = \int_{[a,a+\lambda]} f(t)(1-g(t))d\mu(t) - \int_{(a+\lambda,b]} f(t)g(t)d\mu(t)
\end{aligned} \tag{5.37}$$

for $d\mu(t) = p(t)dt$ and integration by parts we have

$$\begin{aligned}
&\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt \\
&= \int_a^{a+\lambda} [f(t)-f(a+\lambda)][1-g(t)]p(t)dt + \int_{a+\lambda}^b [f(a+\lambda)-f(t)]g(t)p(t)dt \\
&= -\int_a^{a+\lambda}\left[\int_a^x (1-g(t))p(t)dt\right]df(x) - \int_{a+\lambda}^b \left[\int_x^b g(t)p(t)dt\right]df(x) \\
&= -\int_a^b \mathscr{G}_1(x)df(x) = -\int_a^b \mathscr{G}_1(x)f'(x)dx.
\end{aligned}$$

Applying identity (5.34) to f' and replacing n with $n-1$ we have

$$\begin{aligned}
f'(x) = \frac{f(b)-f(a)}{b-a} &- \sum_{i=0}^{n-3} f^{(i+1)}(x)\,\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!\,(b-a)} \\
&+ \frac{1}{(n-2)!}\int_a^b T_{n-1}(x,s)f^{(n)}(s)ds.
\end{aligned} \tag{5.38}$$

Now we obtain

$$\begin{aligned}
\int_a^b \mathscr{G}_1(x)f'(x)dx = &\int_a^b \mathscr{G}_1(x)\left(\frac{f(b)-f(a)}{b-a}\right. \\
&\left.- \sum_{i=0}^{n-3} f^{(i+2)}(x)\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!\,(b-a)}\right)dx \\
&+ \frac{1}{(n-2)!}\int_a^b \mathscr{G}_1(x)\left(\int_a^b T_{n-1}(x,s)f^{(n)}(s)ds\right)dx.
\end{aligned} \tag{5.39}$$

After applying Fubini's theorem on the last term in (5.39) we obtain (5.36. □

Theorem 5.12 *Let $f : I \to \mathbb{R}$ be suct that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$. Let $g, p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let $\int_{b-\lambda}^{b} p(t)dt = \int_a^b g(t)p(t)dt$ and let the function $\mathscr{G}_2$ be defined by*

$$\mathscr{G}_2(x) = \begin{cases} \int_a^x g(t)p(t)dt, & x \in [a, b-\lambda], \\ \int_x^b (1-g(t))p(t)dt, & x \in [b-\lambda, b]. \end{cases} \tag{5.40}$$

Then

$$\begin{aligned} &\int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^{b} f(t)p(t)dt \\ &+ \int_a^b \mathscr{G}_2(x)\left(\frac{f(b)-f(a)}{b-a} - \sum_{i=0}^{n-3} f^{(i+2)}(x)\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!\,(b-a)}\right)dx \\ &= -\frac{1}{(n-2)!}\int_a^b \left(\int_a^b \mathscr{G}_2(x)T_{n-1}(x,s)dx\right) f^{(n)}(s)ds. \end{aligned} \tag{5.41}$$

Proof. Similarly as in the proof of Theorem 5.11, we use the identity

$$\begin{aligned} &\int_{[a,b]} f(t)g(t)d\mu(t) - \int_{(b-\lambda,b]} f(t)d\mu(t) \\ &\qquad = \int_{[a,b-\lambda]} f(t)g(t)d\mu(t) - \int_{(b-\lambda,b]} f(t)(1-g(t))d\mu(t). \end{aligned} \tag{5.42}$$

for $d\mu(t) = p(t)dt$. □

Further, using the above obtained identites we give generalization of Steffensen's inequality for n-convex functions.

Theorem 5.13 *Let $f : I \to \mathbb{R}$ be suct that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$. Let $g, p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$ and let the function $\mathscr{G}_1$ be defined by* (5.35)*. If f is n-convex and*

$$\int_a^b \mathscr{G}_1(x)T_{n-1}(x,s)dx \geq 0, \quad s \in [a,b], \tag{5.43}$$

then

$$\begin{aligned} &\int_a^b f(t)g(t)p(t)dt \geq \int_a^{a+\lambda} f(t)p(t)dt \\ &+ \int_a^b \mathscr{G}_1(x)\left(\frac{f(b)-f(a)}{b-a} - \sum_{i=0}^{n-3} f^{(i+2)}(x)\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!\,(b-a)}\right)dx. \end{aligned} \tag{5.44}$$

Proof. If the function f is n-convex, without loss of generality we can assume that f is n-times differentiable and $f^{(n)} \geq 0$ see [71, p. 16 and p. 293]. Hence we can apply Theorem 5.11 to obtain (5.44. □

Theorem 5.14 *Let $f: I \to \mathbb{R}$ be suct that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$, $I \subset \mathbb{R}$ an open interval, $a,b \in I$, $a < b$. Let $g,p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let $\int_{b-\lambda}^{b} p(t)dt = \int_a^b g(t)p(t)dt$ and let the function $\mathcal{G}_2$ be defined by* (5.40)*. If f is n-convex and*

$$\int_a^b \mathcal{G}_2(x) T_{n-1}(x,s)dx \geq 0, \quad s \in [a,b], \tag{5.45}$$

then

$$\begin{aligned}\int_a^b f(t)g(t)p(t)dt \leq & \int_{b-\lambda}^{b} f(t)p(t)dt \\ & - \int_a^b \mathcal{G}_2(x) \left(\frac{f(b)-f(a)}{b-a} - \sum_{i=0}^{n-3} f^{(i+2)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!\,(b-a)} \right) dx.\end{aligned} \tag{5.46}$$

Proof. Similar to the proof of Theorem 5.13. □

Remark 5.2 *If the integrals in* (5.43) *and* (5.45) *are nonpositive, then the reverse inequalities in* (5.44) *and* (5.46) *hold. Note that, in this case for some odd $n \geq 3$, functions $\mathcal{G}_i$, $i = 1,2$ are nonnegative so integrals in* (5.43) *and* (5.45) *are nonpositive. Hence, inequalities* (5.44) *and* (5.46) *are reversed.*

In sequel we give Ostrowski type inequalities for previous results. The proofs are analogous to the proof of Theorem 5.8.

Theorem 5.15 *Suppose that all assumptions of Theorem* 5.11 *hold. Assume also that (p,q) is a pair of conjugate exponents, that is $1 \leq p,q \leq \infty$, $1/p + 1/q = 1$ and $f^{(n)} \in L_p[a,b]$ for some $n \geq 2$. Then we have*

$$\begin{aligned}&\left| \int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt \right. \\ &\left. + \int_a^b \mathcal{G}_1(x) \left(\frac{f(b)-f(a)}{b-a} - \sum_{i=0}^{n-3} f^{(i+2)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!\,(b-a)} \right) dx \right| \\ &\leq \frac{1}{(n-2)!} \left\| f^{(n)} \right\|_p \left\| \int_a^b \mathcal{G}_1(x) T_{n-1}(x,\cdot)dx \right\|_q .\end{aligned} \tag{5.47}$$

The constant on the right-hand side of (5.47) *is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.*

Theorem 5.16 *Suppose that all assumptions of Theorem* 5.12 *hold. Assume also* (p,q) *is a pair of conjugate exponents, that is* $1 \leq p,q \leq \infty$, $1/p+1/q=1$. *Let* $f^{(n)} \in L_p[a,b]$ *for some* $n \geq 2$. *Then we have*

$$\begin{aligned}
&\left| \int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^b f(t)p(t)dt \right. \\
&\left. + \int_a^b \mathcal{G}_2(x) \left(\frac{f(b)-f(a)}{b-a} - \sum_{i=0}^{n-3} f^{(i+2)}(x) \frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!\,(b-a)} \right) dx \right| \qquad (5.48) \\
&\leq \frac{1}{(n-2)!} \left\| f^{(n)} \right\|_p \left\| \int_a^b \mathcal{G}_2(x) T_{n-1}(x,\cdot) dx \right\|_q .
\end{aligned}$$

The constant on the right-hand side of (5.48) *is sharp for* $1 < p \leq \infty$ *and the best possible for* $p=1$.

We conclude this section with some new bounds for the identities, using the Čebyšev and Grüss type inequalities.

By $\Omega_i(s)$ we will denote

$$\Omega_i(s) = \int_a^b \mathcal{G}_i(x) T_{n-1}(x,s) dx, \quad i=1,2. \qquad (5.49)$$

Using Theorems 1.23 and 1.24 we obtain the following results.

Theorem 5.17 *Let* $f: I \to \mathbb{R}$ *be suct that* $f^{(n)}$ *is absolutely continuous for some* $n \geq 2$, $I \subset \mathbb{R}$ *an open interval,* $a,b \in I$, $a<b$ *and* $(\cdot - a)(b - \cdot)[f^{(n+1)}]^2 \in L_1[a,b]$. *Let* $g,p : [a,b] \to \mathbb{R}$ *be integrable functions such that* p *is positive and* $0 \leq g \leq 1$. *Let* $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$ *and let the functions* $\mathcal{G}_1$ *and* Ω_1 *be defined by* (5.35) *and* (5.49). *Then*

$$\begin{aligned}
&\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt \\
&+ \int_a^b \mathcal{G}_1(x) \left(\frac{f(b)-f(a)}{b-a} - \sum_{i=0}^{n-3} f^{(i+2)}(x) \frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!\,(b-a)} \right) dx \qquad (5.50) \\
&+ \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)(n-2)!} \int_a^b \Omega_1(s) ds = S_n^1(f;a,b),
\end{aligned}$$

where the remainder $S_n^1(f;a,b)$ *satisfies the estimation*

$$\left| S_n^1(f;a,b) \right| \leq \frac{\sqrt{b-a}}{\sqrt{2}(n-2)!} \left[T(\Omega_1,\Omega_1) \right]^{\frac{1}{2}} \left(\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2 ds \right)^{\frac{1}{2}}. \qquad (5.51)$$

Proof. Similar to the proof of Theorem 5.5. □

Theorem 5.18 *Let $f : I \to \mathbb{R}$ be suct that $f^{(n)}$ is absolutely continuous for some $n \geq 2$, $I \subset \mathbb{R}$ an open interval, $a,b \in I$, $a < b$ and $(\cdot - a)(b - \cdot)[f^{(n+1)}]^2 \in L_1[a,b]$. Let $g, p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let $\int_{b-\lambda}^{b} p(t)dt = \int_a^b g(t)p(t)dt$ and let the functions $\mathscr{G}_2$ and Ω_2 be defined by* (5.40) *and* (5.49)*. Then*

$$\begin{aligned}
&\int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^{b} f(t)p(t)dt \\
&+ \int_a^b \mathscr{G}_2(x) \left(\frac{f(b)-f(a)}{b-a} - \sum_{i=0}^{n-3} f^{(i+2)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!\,(b-a)} \right) dx \qquad (5.52)\\
&+ \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{(b-a)(n-2)!} \int_a^b \Omega_2(s)ds = S_n^2(f;a,b),
\end{aligned}$$

where the remainder $S_n^2(f;a,b)$ satisfies the estimation

$$\left| S_n^2(f;a,b) \right| \leq \frac{\sqrt{b-a}}{\sqrt{2}(n-2)!} \left[T(\Omega_2, \Omega_2) \right]^{\frac{1}{2}} \left(\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2 ds \right)^{\frac{1}{2}}.$$

Proof. Similar to the proof of Theorem 5.5. □

Theorem 5.19 *Let $f : I \to \mathbb{R}$ be suct that $f^{(n)}$ is absolutely continuous for some $n \geq 2$ and $f^{(n+1)} \geq 0$ on $[a,b]$. Let functions Ω_i, $i = 1,2$ be defined by* (5.49)*.*

(a) Let $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$. Then we have representation (5.50) *and the remainder $S_n^1(f;a,b)$ satisfies the bound*

$$\begin{aligned}
&\left| S_n^1(f;a,b) \right| \\
&\leq \frac{b-a}{(n-2)!} \|\Omega_1'\|_\infty \left\{ \frac{f^{(n-1)}(b) + f^{(n-1)}(a)}{2} - \frac{f^{(n-2)}(b) - f^{(n-2)}(a)}{b-a} \right\}.
\end{aligned}$$

(b) Let $\int_{b-\lambda}^{b} p(t)dt = \int_a^b g(t)p(t)dt$. Then we have representation (5.52) *and the remainder $S_n^2(f;a,b)$ satisfies the bound*

$$\begin{aligned}
&\left| S_n^2(f;a,b) \right| \\
&\leq \frac{b-a}{(n-2)!} \|\Omega_2'\|_\infty \left\{ \frac{f^{(n-1)}(b) + f^{(n-1)}(a)}{2} - \frac{f^{(n-2)}(b) - f^{(n-2)}(a)}{b-a} \right\}.
\end{aligned}$$

Proof. Similar to the proof of Theorem 5.7. □

5.3 Generalizations via some Euler-type identities

Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is continuous function of bounded variation on $[a,b]$ for some $n \geq 1$ and for every $x \in [a,b]$. In the paper [18] the following two identities have been proved:

$$f(x) = \frac{1}{b-a} \int_a^b f(t)dt + T_n(x) + R_n^1(x) \tag{5.53}$$

and

$$f(x) = \frac{1}{b-a} \int_a^b f(t)dt + T_{n-1}(x) + R_n^2(x), \tag{5.54}$$

where $T_0(x) = 0$, and for $1 \leq m \leq n$

$$T_m(x) = \sum_{k=1}^{m} \frac{(b-a)^{k-1}}{k!} B_k\left(\frac{x-a}{b-a}\right)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right],$$

$$R_n^1(x) = -\frac{(b-a)^{n-1}}{n!} \int_{[a,b]} B_n^*\left(\frac{x-t}{b-a}\right) df^{(n-1)}(t),$$

$$R_n^2(x) = -\frac{(b-a)^{n-1}}{n!} \int_{[a,b]} \left[B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right)\right] df^{(n-1)}(t).$$

Here, $B_k(x)$, $k \geq 0$ are the Bernoulli polynomials, B_k, $k \geq 0$ are the Bernoulli numbers and $B_k^*(x)$, $k \geq 0$ are periodic functions of period one, related to the Bernoulli polynomials as

$$B_k^*(x) = B_k(x), \quad 0 \leq x < 1$$

and

$$B_k^*(x+1) = B_k^*(x), \quad x \in \mathbb{R}.$$

Let us recall some properties of the Bernoulli polynomials. The first three Bernoulli polynomials are

$$B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6},$$

and

$$B_n'(x) = nB_{n-1}(x), n \in \mathbb{N}.$$

$B_0^*(x)$ is a constant equal to 1, while $B_1^*(x)$ is a discontinuous function with a jump of -1 at each integer. For $k \geq 2$, $B_k^*(x)$ is a continuous function.

For more details on Bernoulli polynomials and Bernoulli numbers we refer the reader to [1, 40]. The expresions (5.53) and (5.54) are extensions of the Euler integral formula (see [40]).

After brief introduction we give, generalizations of Steffensen's inequality for n-convex functions using the identities (5.53) and (5.54), which are the main results of this section. The results in this section are given in [63].

Theorem 5.20 *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is continuous function of bounded variation on $[a,b]$ for some $n\geq 2$ and let $g:[a,b]\to\mathbb{R}$ be an integrable function such that $0\leq g\leq 1$. Let $\lambda=\int_a^b g(t)dt$ and let the function G_1 be defined by* (5.1).

(a) *Then*

$$\begin{aligned}&\int_a^{a+\lambda} f(t)dt-\int_a^b f(t)g(t)dt\\&+\sum_{k=1}^{n}\frac{(b-a)^{k-2}}{(k-1)!}\left(\int_a^b G_1(x)B_{k-1}\left(\frac{x-a}{b-a}\right)dx\right)[f^{(k-1)}(b)-f^{(k-1)}(a)]\\&=\frac{(b-a)^{n-2}}{(n-1)!}\int_a^b\left(\int_a^b G_1(x)B^*_{n-1}\left(\frac{x-t}{b-a}\right)dx\right)f^{(n)}(t)dt.\end{aligned}\tag{5.55}$$

(b) *Then*

$$\begin{aligned}&\int_a^{a+\lambda} f(t)dt-\int_a^b f(t)g(t)dt\\&+\sum_{k=1}^{n-1}\frac{(b-a)^{k-2}}{(k-1)!}\left(\int_a^b G_1(x)B_{k-1}\left(\frac{x-a}{b-a}\right)dx\right)[f^{(k-1)}(b)-f^{(k-1)}(a)]\\&=\frac{(b-a)^{n-2}}{(n-1)!}\int_a^b\left(\int_a^b G_1(x)\left[B^*_{n-1}\left(\frac{x-t}{b-a}\right)-B_{n-1}\left(\frac{x-a}{b-a}\right)\right]dx\right)f^{(n)}(t)dt.\end{aligned}\tag{5.56}$$

Proof. (a) Similar to the proof of Theorem 5.1 using the identity (5.53) on the function f'.
(b) Similar to the proof of Theorem 5.1 using the identity (5.54) on the function f'. □

We continue with the results related to the identity given by (1.4).

Theorem 5.21 *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is continuous function of bounded variation on $[a,b]$ for some $n\geq 2$ and let $g:[a,b]\to\mathbb{R}$ be an integrable function such that $0\leq g\leq 1$. Let $\lambda=\int_a^b g(t)dt$ and let the function G_2 be defined by* (5.6).

(a) *Then*

$$\begin{aligned}&\int_a^b f(t)g(t)dt-\int_{b-\lambda}^b f(t)dt\\&+\sum_{k=1}^{n}\frac{(b-a)^{k-2}}{(k-1)!}\left(\int_a^b G_2(x)B_{k-1}\left(\frac{x-a}{b-a}\right)dx\right)[f^{(k-1)}(b)-f^{(k-1)}(a)]\\&=\frac{(b-a)^{n-2}}{(n-1)!}\int_a^b\left(\int_a^b G_2(x)B^*_{n-1}\left(\frac{x-t}{b-a}\right)dx\right)f^{(n)}(t)dt.\end{aligned}\tag{5.57}$$

(b) *Then*

$$\begin{aligned}
&\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt \\
&+ \sum_{k=1}^{n-1} \frac{(b-a)^{k-2}}{(k-1)!} \left(\int_a^b G_2(x) B_{k-1}\left(\frac{x-a}{b-a}\right) dx \right) [f^{(k-1)}(b) - f^{(k-1)}(a)] \\
&= \frac{(b-a)^{n-2}}{(n-1)!} \int_a^b \left(\int_a^b G_2(x) \left[B^*_{n-1}\left(\frac{x-t}{b-a}\right) - B_{n-1}\left(\frac{x-a}{b-a}\right)\right] dx \right) f^{(n)}(t)dt.
\end{aligned} \tag{5.58}$$

Proof. (a) Similar to the proof of Theorem 5.1 applying integration by parts on the identity (1.4) and then using the identity (5.53) on the function f'. The proof of part (b) is similar to the first part, we use identity (5.54) on the function f'. □

Now we give the generalizations of Steffensen's inequality for n-convex functions.

Theorem 5.22 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is continuous function of bounded variation on $[a,b]$ for some $n \geq 2$ and let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq 1$. Let $\lambda = \int_a^b g(t)dt$ and let the function G_1 be defined by* (5.1)*.*

(i) If f is n-convex and

$$\int_a^b G_1(x) B^*_{n-1}\left(\frac{x-t}{b-a}\right) dx \geq 0, \quad t \in [a,b], \tag{5.59}$$

then

$$\begin{aligned}
&\int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)dt \\
&+ \sum_{k=1}^{n} \frac{(b-a)^{k-2}}{(k-1)!} \left(\int_a^b G_1(x) B_{k-1}\left(\frac{x-a}{b-a}\right) dx \right) [f^{(k-1)}(b) - f^{(k-1)}(a)].
\end{aligned} \tag{5.60}$$

(ii) If f is n-convex and

$$\int_a^b G_1(x) \left[B^*_{n-1}\left(\frac{x-t}{b-a}\right) - B_{n-1}\left(\frac{x-a}{b-a}\right)\right] dx \geq 0, \quad t \in [a,b], \tag{5.61}$$

then

$$\begin{aligned}
&\int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)dt \\
&+ \sum_{k=1}^{n-1} \frac{(b-a)^{k-2}}{(k-1)!} \left(\int_a^b G_1(x) B_{k-1}\left(\frac{x-a}{b-a}\right) dx \right) [f^{(k-1)}(b) - f^{(k-1)}(a)].
\end{aligned} \tag{5.62}$$

Proof. If the function f is n-convex, without loss of generality we can assume that f is n-times differentiable and $f^{(n)} \geq 0$ see [71, p. 16 and p. 293]. Hence. we can apply Theorem 5.20 to obtain (5.60) and (5.62). □

Theorem 5.23 *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is continuous function of bounded variation on $[a,b]$ for some $n\geq 2$ and let $g:[a,b]\to\mathbb{R}$ be an integrable function such that $0\leq g\leq 1$. Let $\lambda=\int_a^b g(t)dt$ and let the function G_2 be defined by* (5.6).

(i) If f is n-convex and

$$\int_a^b G_2(x)B^*_{n-1}\left(\frac{x-t}{b-a}\right)dx\geq 0,\quad t\in[a,b], \tag{5.63}$$

then

$$\begin{aligned}\int_a^b f(t)g(t)dt &\geq \int_{b-\lambda}^b f(t)dt\\ &-\sum_{k=1}^{n}\frac{(b-a)^{k-2}}{(k-1)!}\left(\int_a^b G_2(x)B_{k-1}\left(\frac{x-a}{b-a}\right)dx\right)[f^{(k-1)}(b)-f^{(k-1)}(a)].\end{aligned} \tag{5.64}$$

(ii) If f is n-convex and

$$\int_a^b G_2(x)\left[B^*_{n-1}\left(\frac{x-t}{b-a}\right)-B_{n-1}\left(\frac{x-a}{b-a}\right)\right]dx\geq 0,\quad t\in[a,b], \tag{5.65}$$

then

$$\begin{aligned}\int_a^b f(t)g(t)dt &\geq \int_{b-\lambda}^b f(t)dt\\ &-\sum_{k=1}^{n-1}\frac{(b-a)^{k-2}}{(k-1)!}\left(\int_a^b G_2(x)B_{k-1}\left(\frac{x-a}{b-a}\right)dx\right)[f^{(k-1)}(b)-f^{(k-1)}(a)].\end{aligned} \tag{5.66}$$

Proof. Similar to the proof of Theorem 5.22, applying Theorem 5.21. □

We continue with Ostrowski type inequalities related to the results given in Theorems 5.20 and 5.21.

Theorem 5.24 *Suppose that all assumptions of Theorem* 5.20 *hold. Assume (p,q) is a pair of conjugate exponents, that is $1\leq p,q\leq\infty$, $1/p+1/q=1$. Let $\left|f^{(n)}\right|^p:[a,b]\to\mathbb{R}$ be an R-integrable function for some $n\geq 2$.*

(a) *Then we have*

$$\begin{aligned}&\left|\int_a^{a+\lambda} f(t)dt-\int_a^b f(t)g(t)dt\right.\\ &\left.+\sum_{k=1}^{n}\frac{(b-a)^{k-2}}{(k-1)!}\left(\int_a^b G_1(x)B_{k-1}\left(\frac{x-a}{b-a}\right)dx\right)[f^{(k-1)}(b)-f^{(k-1)}(a)]\right|\\ &\leq\frac{(b-a)^{n-2}}{(n-1)!}\left\|f^{(n)}\right\|_p\left(\int_a^b\left|\int_a^b G_1(x)B^*_{n-1}\left(\frac{x-t}{b-a}\right)dx\right|^q dt\right)^{\frac{1}{q}}.\end{aligned} \tag{5.67}$$

The constant on the right-hand side of (5.67) *is sharp for $1<p\leq\infty$ and the best possible for $p=1$.*

(b) *Then we have*

$$
\begin{aligned}
&\left|\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt\right.\\
&\left.+\sum_{k=1}^{n-1}\frac{(b-a)^{k-2}}{(k-1)!}\left(\int_a^b G_1(x)B_{k-1}\left(\frac{x-a}{b-a}\right)dx\right)[f^{(k-1)}(b)-f^{(k-1)}(a)]\right|\\
&\le \frac{(b-a)^{n-2}}{(n-1)!}\left\|f^{(n)}\right\|_p\left(\int_a^b\left|\int_a^b G_1(x)\left[B^*_{n-1}\left(\frac{x-t}{b-a}\right)-B_{n-1}\left(\frac{x-a}{b-a}\right)\right]dx\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}
\tag{5.68}
$$

The constant on the right-hand side of (5.68) *is sharp for* $1<p\le\infty$ *and the best possible for* $p=1$.

Proof. Similar to the proof of the Theorem 5.8 using identities obtained in Theorem 5.20. □

Theorem 5.25 *Suppose that all assumptions of Theorem* 5.21 *hold. Assume* (p,q) *is a pair of conjugate exponents, that is* $1\le p,q\le\infty$, $1/p+1/q=1$. *Let* $\left|f^{(n)}\right|^p : [a,b]\to\mathbb{R}$ *be an R-integrable function for some* $n\ge 2$.

(a) *Then we have*

$$
\begin{aligned}
&\left|\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt\right.\\
&\left.+\sum_{k=1}^{n}\frac{(b-a)^{k-2}}{(k-1)!}\left(\int_a^b G_2(x)B_{k-1}\left(\frac{x-a}{b-a}\right)dx\right)[f^{(k-1)}(b)-f^{(k-1)}(a)]\right|\\
&\le \frac{(b-a)^{n-2}}{(n-1)!}\left\|f^{(n)}\right\|_p\left(\int_a^b\left|\int_a^b G_2(x)B^*_{n-1}\left(\frac{x-t}{b-a}\right)dx\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}
\tag{5.69}
$$

The constant on the right-hand side of (5.69) *is sharp for* $1<p\le\infty$ *and the best possible for* $p=1$.

(b)

$$
\begin{aligned}
&\left|\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt\right.\\
&\left.+\sum_{k=1}^{n-1}\frac{(b-a)^{k-2}}{(k-1)!}\left(\int_a^b G_2(x)B_{k-1}\left(\frac{x-a}{b-a}\right)dx\right)[f^{(k-1)}(b)-f^{(k-1)}(a)]\right|\\
&\le \frac{(b-a)^{n-2}}{(n-1)!}\left\|f^{(n)}\right\|_p\left(\int_a^b\left|\int_a^b G_2(x)\left[B^*_{n-1}\left(\frac{x-t}{b-a}\right)-B_{n-1}\left(\frac{x-a}{b-a}\right)\right]dx\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}
\tag{5.70}
$$

The constant on the right-hand side of (5.70) *is sharp for* $1<p\le\infty$ *and the best possible for* $p=1$.

At the end of this section we give some new bounds for integrals on the left hand side in perturbed versions of identities obtained in Theorems 5.20 and 5.21.

Let us denote

$$H_1(t) = \int_a^b G_1(x) B_{n-1}^*\left(\frac{x-t}{b-a}\right) dx. \tag{5.71}$$

and

$$\Phi_1(t) = \int_a^b G_1(x)\left[B_{n-1}^*\left(\frac{x-t}{b-a}\right) - B_{n-1}\left(\frac{x-a}{b-a}\right)\right] dx. \tag{5.72}$$

Using Theorems 1.23 and 1.24 the following bounds are obtained.

Theorem 5.26 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous function for some $n \geq 2$ with $(\cdot - a)(b - \cdot)[f^{(n+1)}]^2 \in L[a,b]$ and let g be an integrable function on $[a,b]$. Let $\lambda = \int_a^b g(t)dt$ and let the functions G_1, H_1 and Φ_1 be defined by* (5.1), (5.71) *and* (5.72), *respectively.*

(a) *Then*

$$\begin{aligned} &\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt \\ &+ \sum_{k=1}^{n} \frac{(b-a)^{k-2}}{(k-1)!}\left(\int_a^b G_1(x)B_{k-1}\left(\frac{x-a}{b-a}\right)dx\right)[f^{(k-1)}(b) - f^{(k-1)}(a)] \\ &- \frac{(b-a)^{n-3}[f^{(n-1)}(b) - f^{(n-1)}(a)]}{(n-1)!}\int_a^b H_1(t)dt = S_n^1(f;a,b) \end{aligned} \tag{5.73}$$

where the remainder $S_n^1(f;a,b)$ satisfies the estimation

$$\left|S_n^1(f;a,b)\right| \leq \frac{(b-a)^{n-\frac{3}{2}}}{\sqrt{2}(n-1)!}\left[T(H_1,H_1)\right]^{\frac{1}{2}}\left|\int_a^b (t-a)(b-t)[f^{(n+1)}(t)]^2 dt\right|^{\frac{1}{2}}. \tag{5.74}$$

(b) *Then*

$$\begin{aligned} &\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt \\ &+ \sum_{k=1}^{n-1} \frac{(b-a)^{k-2}}{(k-1)!}\left(\int_a^b G_1(x)B_{k-1}\left(\frac{x-a}{b-a}\right)dx\right)[f^{(k-1)}(b) - f^{(k-1)}(a)] \\ &- \frac{(b-a)^{n-3}[f^{(n-1)}(b) - f^{(n-1)}(a)]}{(n-1)!}\int_a^b \Phi_1(t)dt = S_n^2(f;a,b) \end{aligned} \tag{5.75}$$

where the remainder $S_n^2(f;a,b)$ satisfies the estimation

$$\left|S_n^2(f;a,b)\right| \leq \frac{(b-a)^{n-\frac{3}{2}}}{\sqrt{2}(n-1)!}\left[T(\Phi_1,\Phi_1)\right]^{\frac{1}{2}}\left|\int_a^b (t-a)(b-t)[f^{(n+1)}(t)]^2 dt\right|^{\frac{1}{2}}.$$

Proof. Similar to the proof od Theorem 5.5.
Applying Theorem 1.23 for $f \to H_1$ and $h \to f^{(n)}$ we obtain

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b H_1(t)f^{(n)}(t)dt - \frac{1}{b-a}\int_a^b H_1(t)dt \cdot \frac{1}{b-a}\int_a^b f^{(n)}(t)dt\right| \\
&\leq \frac{1}{\sqrt{2}}[T(H_1,H_1)]^{\frac{1}{2}}\frac{1}{\sqrt{b-a}}\left|\int_a^b (t-a)(b-t)[f^{(n+1)}(t)]^2 dt\right|^{\frac{1}{2}}.
\end{aligned} \tag{5.76}$$

Hence, if we subtract

$$\begin{aligned}
&\frac{(b-a)^{n-1}}{(n-1)!}\cdot\frac{1}{b-a}\int_a^b H_1(t)dt \cdot \frac{1}{b-a}\int_a^b f^{(n)}(t)dt \\
&\qquad = \frac{(b-a)^{n-3}}{(n-1)!}[f^{(n-1)}(b)-f^{(n-1)}(a)]\int_a^b H_1(t)dt
\end{aligned}$$

from both side of the identity (5.55) and use the inequality (5.76) we obtain the representation (5.73). The second statement can be proved in a similar manner using identity (5.56). □

We continue with the results related to the identities (5.57) and (5.58). Let us denote

$$H_2(t) = \int_a^b G_2(x)B_{n-1}^*\left(\frac{x-t}{b-a}\right)dx \tag{5.77}$$

and

$$\Phi_2(t) = \int_a^b G_2(x)\left[B_{n-1}^*\left(\frac{x-t}{b-a}\right) - B_{n-1}\left(\frac{x-a}{b-a}\right)\right]dx. \tag{5.78}$$

Theorem 5.27 *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous function for some $n\geq 2$ with $(\cdot-a)(b-\cdot)[f^{(n+1)}]^2 \in L[a,b]$ and let g be an integrable function on $[a,b]$. Let $\lambda = \int_a^b g(t)dt$ and let the functions G_2, H_2 and Φ_2 be defined by* (5.6), (5.77) *and* (5.78) *respectively. Then*

(i)

$$\begin{aligned}
&\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt \\
&+\sum_{k=1}^n \frac{(b-a)^{k-2}}{(k-1)!}\left(\int_a^b G_2(x)B_{k-1}\left(\frac{x-a}{b-a}\right)dx\right)[f^{(k-1)}(b)-f^{(k-1)}(a)] \\
&-\frac{(b-a)^{n-3}[f^{(n-1)}(b)-f^{(n-1)}(a)]}{(n-1)!}\int_a^b H_2(t)dt = S_n^3(f;a,b)
\end{aligned} \tag{5.79}$$

where the remainder $S_n^3(f;a,b)$ satisfies the estimation

$$\left|S_n^3(f;a,b)\right| \leq \frac{(b-a)^{n-\frac{3}{2}}}{\sqrt{2}(n-1)!}[T(H_2,H_2)]^{\frac{1}{2}}\left|\int_a^b (t-a)(b-t)[f^{(n+1)}(t)]^2 dt\right|^{\frac{1}{2}}.$$

(ii)

$$\begin{aligned}&\int_a^b f(t)g(t)dt-\int_{b-\lambda}^b f(t)dt\\&+\sum_{k=1}^{n-1}\frac{(b-a)^{k-2}}{(k-1)!}\left(\int_a^b G_2(x)B_{k-1}\left(\frac{x-a}{b-a}\right)dx\right)[f^{(k-1)}(b)-f^{(k-1)}(a)]\\&-\frac{(b-a)^{n-3}[f^{(n-1)}(b)-f^{(n-1)}(a)]}{(n-1)!}\int_a^b \Phi_2(t)dt=S_n^4(f;a,b)\end{aligned}\tag{5.80}$$

where the remainder $S_n^4(f;a,b)$ satisfies the estimation

$$\left|S_n^4(f;a,b)\right|\leq\frac{(b-a)^{n-\frac{3}{2}}}{\sqrt{2}(n-1)!}\left[T(\Phi_2,\Phi_2)\right]^{\frac{1}{2}}\left|\int_a^b(t-a)(b-t)[f^{(n+1)}(t)]^2dt\right|^{\frac{1}{2}}.$$

Proof. Similar to the proof of Theorem 5.26. □

The following Grüss type inequalities also hold.

Theorem 5.28 *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ $(n\geq 2)$ is absolutely continuous function and $f^{(n+1)}\geq 0$ on $[a,b]$. Let H_1, H_2, Φ_1 and Φ_2 be defined by* (5.71), (5.77), (5.72) *and* (5.78), *respectively. Then we have the representations* (5.73) (5.75), (5.79) *and* (5.80) *where the remainders $S_n^i(f;a,b), i=1,2,3,4$ satisfy the bounds*

$$\left|S_n^1(f;a,b)\right|\leq\frac{(b-a)^{n-1}}{(n-1)!}\|H_1'\|_\infty\left\{\frac{f^{(n-1)}(b)+f^{(n-1)}(a)}{2}-\left[a,b;f^{(n-2)}\right]\right\}.$$

$$\left|S_n^2(f;a,b)\right|\leq\frac{(b-a)^{n-1}}{(n-1)!}\|\Phi_1'\|_\infty\left\{\frac{f^{(n-1)}(b)+f^{(n-1)}(a)}{2}-\left[a,b;f^{(n-2)}\right]\right\},$$

$$\left|S_n^3(f;a,b)\right|\leq\frac{(b-a)^{n-1}}{(n-1)!}\|H_2'\|_\infty\left\{\frac{f^{(n-1)}(b)+f^{(n-1)}(a)}{2}-\left[a,b;f^{(n-2)}\right]\right\}$$

and

$$\left|S_n^4(f;a,b)\right|\leq\frac{(b-a)^{n-1}}{(n-1)!}\|\Phi_2'\|_\infty\left\{\frac{f^{(n-1)}(b)+f^{(n-1)}(a)}{2}-\left[a,b;f^{(n-2)}\right]\right\}.$$

Proof. Similar to the of Theorem 5.7. □

5.4 Generalizations via Fink's identitiy

In the paper [20] A. M. Fink obtained the following identity

$$\begin{aligned}\frac{1}{n}\left(f(x)+\sum_{k=1}^{n-1}F_k(x)\right) &- \frac{1}{b-a}\int_a^b f(t)dt \\ &= \frac{1}{n!(b-a)}\int_a^b (x-t)^{n-1}k(t,x)f^{(n)}(t)dt,\end{aligned} \tag{5.81}$$

where

$$F_k(x)=\frac{n-k}{k!}\frac{f^{(k-1)}(a)(x-a)^k-f^{(k-1)}(b)(x-b)^k}{b-a},$$

$$k(t,x)=\begin{cases} t-a, & a\le t\le x\le b,\\ t-b, & a\le x<t\le b.\end{cases}$$

In the authors [68] give some generalizations of Steffensen's inequality using an extension of weighted Montgomery identity via Fink's identity. In this section we use the identity given by (5.81) to obtain generalization of Steffensen's inequality for n-convex functions using different reasoning from the one used in [68]. These results are given in [69].

First we obtain some new identities related to Steffensen's inequality.

Here, by $T_k(x)$ we will denote

$$T_k(x)=\frac{n-1-k}{k!}\frac{f^{(k)}(a)(x-a)^k-f^{(k)}(b)(x-b)^k}{b-a}.$$

Theorem 5.29 *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n\ge 2$ and let g,p be integrable functions on $[a,b]$ such that p is positive and $0\le g\le 1$ on $[a,b]$. Let $\int_a^{a+\lambda}p(t)dt=\int_a^b g(t)p(t)dt$ and let the function $\mathscr{G}_1$ be defined by* (5.35) *Then*

$$\begin{aligned}&\int_a^{a+\lambda}f(t)p(t)dt-\int_a^b f(t)g(t)p(t)dt-\sum_{k=0}^{n-2}\int_a^b T_k(x)\mathscr{G}_1(x)dx\\ &=-\frac{1}{(b-a)(n-2)!}\int_a^b\left(\int_a^b \mathscr{G}_1(x)(x-t)^{n-2}k(t,x)dx\right)f^{(n)}(t)dt.\end{aligned} \tag{5.82}$$

Proof. Similar to the proof of Theorem 5.11 using identity (5.37) for $d\mu(t)=p(t)dt$. □

Theorem 5.30 *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n\ge 2$ and let g,p be integrable functions on $[a,b]$ such that p is positive and $0\le g\le 1$ on $[a,b]$. Let $\int_{b-\lambda}^b p(t)dt=\int_a^b g(t)p(t)dt$ and let the function $\mathscr{G}_2$ be defined by* (5.40) *Then*

$$\begin{aligned}&\int_a^b f(t)g(t)p(t)dt-\int_{b-\lambda}^b f(t)p(t)dt-\sum_{k=0}^{n-2}\int_a^b T_k(x)\mathscr{G}_2(x)dx\\ &=-\frac{1}{(b-a)(n-2)!}\int_a^b\left(\int_a^b \mathscr{G}_2(x)(x-t)^{n-2}k(t,x)dx\right)f^{(n)}(t)dt.\end{aligned} \tag{5.83}$$

Proof. Similar to the proof of Theorem 5.11 using identity (5.42) for $d\mu(t) = p(t)dt$. □

Using above theorems we obtain the following generalizations of Steffensen's inequality for n-convex functions.

Theorem 5.31 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$ and let g, p be integrable functions on $[a,b]$ such that p is positive and $0 \leq g \leq 1$ on $[a,b]$. Let $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$ and let the function $\mathcal{G}_1$ be defined by* (5.35). *If f is n-convex and*

$$\int_a^b \mathcal{G}_1(x)(x-t)^{n-2}k(t,x)dx \leq 0, \quad t \in [a,b], \tag{5.84}$$

then

$$\int_a^b f(t)g(t)p(t)dt \leq \int_a^{a+\lambda} f(t)p(t)dt - \sum_{k=0}^{n-2} \int_a^b T_k(x)\mathcal{G}_1(x)dx. \tag{5.85}$$

Proof. If the function f is n-convex, without loss of generality we can assume that f is n-times differentiable and $f^{(n)} \geq 0$ see [71, p. 16 and p. 293]. Now we can apply Theorem 5.29 to obtain (5.85). □

Theorem 5.32 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$ and let g, p be integrable functions on $[a,b]$ such that u is positive and $0 \leq g \leq 1$ on $[a,b]$. Let $\int_{b-\lambda}^b p(t)dt = \int_a^b g(t)p(t)dt$ and let the function $\mathcal{G}_2$ be defined by* (5.40).

If f is n-convex and

$$\int_a^b \mathcal{G}_2(x)(x-t)^{n-2}k(t,x)dx \leq 0, \quad t \in [a,b], \tag{5.86}$$

then

$$\int_a^b f(t)g(t)p(t)dt \geq \int_{b-\lambda}^b f(t)p(t)dt + \sum_{k=0}^{n-2} \int_a^b T_k(x)\mathcal{G}_2(x)dx. \tag{5.87}$$

Proof. Similar to the proof of Theorem 5.31. □

Taking $p \equiv 1$ and $n = 2$ in Theorems 5.31 and 5.32 we obtain following corollary.

Corollary 5.8 *Let $f : [a,b] \to \mathbb{R}$ be such that f' is absolutely continuous. Let g be an integrable function on $[a,b]$ with $0 \leq g \leq 1$ and let $\lambda = \int_a^b g(t)dt$.*

(i) If f is convex and

$$\begin{aligned} t(b-a)\int_a^t g(x)dx + (t-b)\int_a^t xg(x)dx + (t-a)\int_t^b xg(x)dx \\ \leq (t-a)\left(\frac{\lambda^2}{2} + \lambda a\right) + \frac{(b-a)(t-a)^2}{2}, \quad t \in [a, a+\lambda], \end{aligned}$$

$$\begin{aligned} -t(b-a)\int_t^b g(x)dx + (t-b)\int_a^t xg(x)dx + (t-a)\int_t^b xg(x)dx \\ \leq (t-b)\left(\frac{\lambda^2}{2} + \lambda a\right), \quad t \in [a+\lambda, b], \end{aligned}$$

then

$$\int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)dt - (n-1)\frac{f(a)-f(b)}{b-a}\left(\int_a^b tg(t)dt - \lambda a - \frac{\lambda^2}{2}\right).$$

(ii) If f is convex and

$$-t(b-a)\int_a^t g(x)dx+(b-t)\int_a^t xg(x)dx+(a-t)\int_t^b xg(x)dx$$
$$\le (t-a)\left(\frac{\lambda^2}{2}-\lambda b\right),\quad t\in[a,b-\lambda],$$

$$t(b-a)\int_t^b g(x)dx+(b-t)\int_a^t xg(x)dx+(a-t)\int_t^b xg(x)dx$$
$$\le (t-b)\left(\frac{\lambda^2}{2}-\lambda b\right)-\frac{(b-a)(t-b)^2}{2},\quad t\in[b-\lambda,b],$$

then

$$\int_a^b f(t)g(t)dt \ge \int_{b-\lambda}^b f(t)dt+(n-1)\frac{f(a)-f(b)}{b-a}\left(b\lambda-\frac{\lambda^2}{2}-\int_a^b tg(t)dt\right).$$

Now we give the Ostrowski type inequalities for previous results.

Theorem 5.33 *Suppose that all assumptions of Theorem* 5.29 *hold. Assume* (p,q) *is a pair of conjugate exponents, that is* $1\le p,q\le\infty$, $1/p+1/q=1$. *Let* $\left|f^{(n)}\right|^p : [a,b]\to\mathbb{R}$ *be an R-integrable function for some* $n\ge 2$. *Then we have*

$$\begin{aligned}&\left|\int_a^{a+\lambda} f(t)p(t)dt-\int_a^b f(t)g(t)p(t)dt-\sum_{k=0}^{n-2}\int_a^b T_k(x)\mathscr{G}_1(x)dx\right|\\&\le\frac{1}{(b-a)(n-2)!}\left\|f^{(n)}\right\|_p\left(\int_a^b\left|\int_a^b\mathscr{G}_1(x)(x-t)^{n-2}k(t,x)dx\right|^q dt\right)^{\frac{1}{q}}.\end{aligned}\tag{5.88}$$

The constant on the right-hand side of (5.88) *is sharp for* $1<p\le\infty$ *and the best possible for* $p=1$.

Proof. Let's denote

$$C(t)=\frac{-1}{(b-a)(n-2)!}\int_a^b\mathscr{G}_1(x)(x-t)^{n-2}k(t,x)dx.$$

By taking the modulus on (5.82) and applying Hölder's inequality we obtain

$$\left|\int_a^{a+\lambda} f(t)p(t)dt-\int_a^b f(t)g(t)p(t)dt-\sum_{k=0}^{n-2}\int_a^b T_k(x)\mathscr{G}_1(x)dx\right|$$
$$=\left|\int_a^b C(t)f^{(n)}(t)dt\right|\le\left\|f^{(n)}\right\|_p\left(\int_a^b|C(t)|^q dt\right)^{\frac{1}{q}}.$$

The proof of the sharpness of the constant $\left(\int_a^b|C(t)|^q dt\right)^{\frac{1}{q}}$ is similar to the proof of the sharpness in Theorem 5.8. □

Theorem 5.34 *Suppose that all assumptions of Theorem* 5.30 *hold. Assume* (p,q) *is a pair of conjugate exponents, that is* $1 \leq p,q \leq \infty$, $1/p+1/q=1$. *Let* $\left|f^{(n)}\right|^p : [a,b] \to \mathbb{R}$ *be an R-integrable function for some* $n \geq 2$. *Then we have*

$$\begin{aligned}
&\left|\int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^b f(t)p(t)dt - \sum_{k=0}^{n-2}\int_a^b T_k(x)\mathscr{G}_2(x)dx\right| \\
&\leq \frac{1}{(b-a)(n-2)!}\left\|f^{(n)}\right\|_p \left(\int_a^b \left|\int_a^b \mathscr{G}_2(x)(x-t)^{n-2}k(t,x)dx\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned} \tag{5.89}$$

The constant on the right-hand side of (5.89) *is sharp for* $1 < p \leq \infty$ *and the best possible for* $p=1$.

Proof. Similar to the proof of Theorem 5.33. □

In the sequel we provide some new bounds for the integrals on the left hand side in the perturbed version of identities (5.82) and (5.83).

We use the following notation:

$$\Phi_i(t) = \int_a^b \mathscr{G}_i(x)(x-t)^{n-2}k(t,x)dx, \quad i=1,2. \tag{5.90}$$

Theorem 5.35 *Let* $f:[a,b]\to\mathbb{R}$ *be such that* $f^{(n)}$ *is absolutely continuous function for some* $n\geq 2$ *with* $(\cdot-a)(b-\cdot)[f^{(n+1)}]^2 \in L[a,b]$ *and let* g,p *be integrable functions on* $[a,b]$ *such that* p *is positive and* $0\leq g\leq 1$ *on* $[a,b]$. *Let* $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$ *and let the functions* $\mathscr{G}_1$ *and* Φ_1 *be defined by* (5.35) *and* (5.90). *Then*

$$\begin{aligned}
&\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt - \sum_{k=0}^{n-2}\int_a^b T_k(x)\mathscr{G}_1(x)dx \\
&+\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)^2(n-2)!}\int_a^b \Phi_1(t)dt = S^1_{p,n}(f;a,b),
\end{aligned} \tag{5.91}$$

where the remainder $S^1_{p,n}(f;a,b)$ *satisfies the estimation*

$$\left|S^1_{p,n}(f;a,b)\right| \leq \frac{1}{\sqrt{2}(n-2)!}\left[T(\Phi_1,\Phi_1)\right]^{\frac{1}{2}}\frac{1}{\sqrt{b-a}}\left|\int_a^b (t-a)(b-t)[f^{(n+1)}(t)]^2dt\right|^{\frac{1}{2}}.$$

Proof. Similar to the proof of Theorem 5.5. □

Theorem 5.36 *Let* $f:[a,b]\to\mathbb{R}$ *be such that* $f^{(n)}$ *is absolutely continuous function for some* $n\geq 2$ *with* $(\cdot-a)(b-\cdot)[f^{(n+1)}]^2 \in L[a,b]$ *and let* g,p *be integrable functions on* $[a,b]$ *such that* p *is positive and* $0\leq g\leq 1$ *on* $[a,b]$. *Let* $\int_{b-\lambda}^{b} p(t)dt = \int_a^b g(t)p(t)dt$ *and let the functions* $\mathscr{G}_2$ *and* Φ_2 *be defined by* (5.40) *and* (5.90). *Then*

$$\begin{aligned}
&\int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^b f(t)p(t)dt - \sum_{k=0}^{n-2}\int_a^b T_k(x)\mathscr{G}_2(x)dx \\
&+\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)^2(n-2)!}\int_a^b \Phi_2(t)dt = S^2_{p,n}(f;a,b),
\end{aligned} \tag{5.92}$$

where the remainder $S^2_{p,n}(f;a,b)$ satisfies the estimation

$$\left|S^2_{p,n}(f;a,b)\right| \le \frac{1}{\sqrt{2}(n-2)!}\left[T(\Phi_2,\Phi_2)\right]^{\frac{1}{2}} \frac{1}{\sqrt{b-a}}\left|\int_a^b (t-a)(b-t)[f^{(n+1)}(t)]^2 dt\right|^{\frac{1}{2}}.$$

Proof. Similar to the proof of Theorem 5.5. □

The following Grüss type inequalities hold.

Theorem 5.37 *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ $(n\ge 2)$ is absolutely continuous function and $f^{(n+1)}\ge 0$ on $[a,b]$. Let the functions Φ_i, $i=1,2$ be defined by* (5.90)*.*

(a) Let $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$.
Then we have the representation (5.91) *and the remainder $S^1_{p,n}(f;a,b)$ satisfies the bound*

$$\left|S^1_{p,n}(f;a,b)\right| \le \frac{1}{(n-2)!}\|\Phi_1'\|_\infty \left\{\frac{f^{(n-1)}(b)+f^{(n-1)}(a)}{2} - \frac{f^{(n-2)}(b)-f^{(n-2)}(a)}{b-a}\right\}.$$

(b) Let $\int_{b-\lambda}^b p(t)dt = \int_a^b g(t)p(t)dt$.
Then we have the representation (5.92) *and the remainder $S^2_{p,n}(f;a,b)$ satisfies the bound*

$$\left|S^2_{p,n}(f;a,b)\right| \le \frac{1}{(n-2)!}\|\Phi_2'\|_\infty \left\{\frac{f^{(n-1)}(b)+f^{(n-1)}(a)}{2} - \frac{f^{(n-2)}(b)-f^{(n-2)}(a)}{b-a}\right\}.$$

Proof. Similar to the proof of Theorem 5.7. □

Corollary 5.9 *Let $f:[a,b]\to\mathbb{R}$ be such that f'' is absolutely continuous function and $f'''\ge 0$ on $[a,b]$. Let $\lambda=\int_a^b g(t)dt$.*

(i) Then we have

$$\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt - (n-1)\frac{f(a)-f(b)}{b-a}\left(\int_a^b xg(x)dx - \frac{\lambda^2}{2} - \lambda a\right) + \frac{f'(b)-f'(a)}{(b-a)^2}\int_a^b \Phi_1(t)dt = S^1_{1,2}(f;a,b)$$

and the remainder $S^1_{1,2}(f;a,b)$ satisfies the bound

$$\left|S^1_{1,2}(f;a,b)\right| \le \|\Phi_1'\|_\infty \left\{\frac{f'(b)+f'(a)}{2} - \frac{f(b)-f(a)}{b-a}\right\}$$

where

$$\Phi_1'(t) = \begin{cases} \int_a^b xg(x)dx + (b-a)\int_a^t g(x)dx \\ -(t-a)(b-a) - \frac{\lambda^2}{2} - \lambda a, & t\in[a,a+\lambda]; \\ \\ \int_a^b xg(x)dx - (b-a)\int_t^b g(x)dx - \frac{\lambda^2}{2} - \lambda a, & t\in[a+\lambda,b]. \end{cases}$$

(ii) Then we have

$$\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt - (n-1)\frac{f(a)-f(b)}{b-a}\left(b\lambda - \frac{\lambda^2}{2} - \int_a^b xg(x)dx\right)$$
$$+\frac{f'(b)-f'(a)}{(b-a)^2}\int_a^b \Phi_2(t)dt = S_{1,2}^2(f;a,b)$$

and the remainder $S_{1,2}^2(f;a,b)$ satisfies the bound

$$\left|S_{1,2}^2(f;a,b)\right| \le \|\Phi_2'\|_\infty \left\{\frac{f'(b)+f'(a)}{2} - \frac{f(b)-f(a)}{b-a}\right\}$$

where

$$\Phi_2'(t) = \begin{cases} b\lambda - \frac{\lambda^2}{2} - \int_a^b xg(x)dx - (b-a)\int_a^t g(x)dx & t \in [a,b-\lambda]; \\ b\lambda - \frac{\lambda^2}{2} - (b-a)(b-t) - \int_a^b xg(x)dx & \\ +(b-a)\int_t^b g(x)dx, & t \in [b-\lambda,b]. \end{cases}$$

Proof. *Apply Theorem 5.37 with $p \equiv 1$ and $n = 2$.* □

5.5 Generalizations via Lidstone polynomial

In this section we give new generalizations of Steffensen's inequality via Lidstone polynomials. Results given in this section were obtained by Pečarić, Perušić Pribanić and Vukelić in [70]. In [65], using different approach and ideas, the authors introduced and proved some generalizations of Steffensen's inequality via Lidstone polynomials.

First we give the proofs of identities related to generalization of Steffensen's ineguality using Lidstone's interpolating polynomial.

Theorem 5.38 *Let $f : I \to \mathbb{R}$ be such that $f^{(2n-1)}$ is absolutely continuous for some $n \ge 1$, $I \subset \mathbb{R}$ an open interval, $a,b \in I$, $a < b$. Let $g,p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive and $0 \le g \le 1$. Let $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$ and let the function $\mathscr{G}_1$ be defined by* (5.35)*. Then*

$$\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt$$
$$+\sum_{k=0}^{n-1}(b-a)^{2k-1}\int_a^b \mathscr{G}_1(x)\left[f^{(2k)}(b)\Lambda_k'\left(\frac{x-a}{b-a}\right) - f^{(2k)}(a)\Lambda_k'\left(\frac{b-x}{b-a}\right)\right]dx \tag{5.93}$$
$$= -(b-a)^{2n-1}\int_a^b\left(\int_a^b \mathscr{G}_1(x)\frac{dG_n}{dx}\left(\frac{x-a}{b-a},\frac{s-a}{b-a}\right)dx\right)f^{(2n)}(s)ds.$$

Proof. Using identity (5.37) for $d\mu(t) = p(t)dt$ and integration by parts we have

$$\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt = -\int_a^b \mathscr{G}_1(x)f'(x)dx.$$

By Lemma 1.2 every function $f \in C^{(2n)}([a,b])$ can be represented as

$$\begin{aligned} f(x) = \sum_{k=0}^{n-1}(b-a)^{2k}\left[f^{(2k)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right) + f^{(2k)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right] \\ + (b-a)^{2n-1}\int_a^b G_n\left(\frac{x-a}{b-a},\frac{s-a}{b-a}\right)f^{(2n)}(s)ds. \end{aligned} \tag{5.94}$$

Its derivative is

$$\begin{aligned} f'(x) = \sum_{k=0}^{n-1}(b-a)^{2k-1}\left[f^{(2k)}(b)\Lambda_k'\left(\frac{x-a}{b-a}\right) - f^{(2k)}(a)\Lambda_k'\left(\frac{b-x}{b-a}\right)\right] \\ + (b-a)^{2n-1}\int_a^b \frac{dG_n}{dx}\left(\frac{x-a}{b-a},\frac{s-a}{b-a}\right)f^{(2n)}(s)ds \end{aligned} \tag{5.95}$$

where

$$\frac{dG_n}{dx}\left(\frac{x-a}{b-a},\frac{s-a}{b-a}\right) = \begin{cases} -\sum_{k=0}^{n-1}\Lambda_k'\left(\frac{x-a}{b-a}\right)\frac{(b-s)^{2n-2k-1}}{(b-a)^{2n-2k}(2n-2k-1)!}, & x < s, \\ \sum_{k=0}^{n-1}\Lambda_k'\left(\frac{b-x}{b-a}\right)\frac{(s-a)^{2n-2k-1}}{(b-a)^{2n-2k}(2n-2k-1)!}, & s \le x. \end{cases}$$

Now we have

$$\begin{aligned} &\int_a^b \mathscr{G}_1(x)f'(x)dx \\ &= \sum_{k=0}^{n-1}(b-a)^{2k-1}\int_a^b \mathscr{G}_1(x)\left[f^{(2k)}(b)\Lambda_k'\left(\frac{x-a}{b-a}\right) - f^{(2k)}(a)\Lambda_k'\left(\frac{b-x}{b-a}\right)\right]dx \\ &+ (b-a)^{2n-1}\int_a^b \mathscr{G}_1(x)\left(\int_a^b \frac{dG_n}{dx}\left(\frac{x-a}{b-a},\frac{s-a}{b-a}\right)f^{(2n)}(s)ds\right)dx. \end{aligned} \tag{5.96}$$

After applying Fubini's theorem on the last term in (5.96) we obtain (5.93). □

Theorem 5.39 *Let $f : I \to \mathbb{R}$ be such that $f^{(2n-1)}$ is absolutely continuous for some $n \ge 1$, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$. Let $g, p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive and $0 \le g \le 1$. Let $\int_{b-\lambda}^b p(t)dt = \int_a^b g(t)p(t)dt$ and let the function $\mathscr{G}_2$ be defined by* (5.40)*. Then*

$$\begin{aligned} &\int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^b f(t)p(t)dt \\ &+ \sum_{k=0}^{n-1}(b-a)^{2k-1}\int_a^b \mathscr{G}_2(x)\left[f^{(2k)}(b)\Lambda_k'\left(\frac{x-a}{b-a}\right) - f^{(2k)}(a)\Lambda_k'\left(\frac{b-x}{b-a}\right)\right]dx \\ &= -(b-a)^{2n-1}\int_a^b\left(\int_a^b \mathscr{G}_2(x)\frac{dG_n}{dx}\left(\frac{x-a}{b-a},\frac{s-a}{b-a}\right)dx\right)f^{(2n)}(s)ds \end{aligned} \tag{5.97}$$

Proof. Similar to the proof of Theorem 5.38 using identity (5.42). $\square$

Theorem 5.40 *Let $f : I \to \mathbb{R}$ be such that $f^{(2n)}$ is absolutely continuous for some $n \geq 1$, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$. Let $g, p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$ and let the function $\mathscr{G}_1$ be defined by* (5.35)*. Then*

$$\begin{aligned}
&\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt \\
&+ \sum_{k=0}^{n-1} (b-a)^{2k} \int_a^b \mathscr{G}_1(x) \left[f^{(2k+1)}(a) \Lambda_k \left(\frac{b-x}{b-a} \right) + f^{(2k+1)}(b) \Lambda_k \left(\frac{x-a}{b-a} \right) \right] dx \qquad (5.98) \\
&= -(b-a)^{2n-1} \int_a^b \left(\int_a^b \mathscr{G}_1(x) G_n \left(\frac{x-a}{b-a}, \frac{s-a}{b-a} \right) dx \right) f^{(2n+1)}(s) ds.
\end{aligned}$$

Proof. Similar to the proof of Theorem 5.38 using the identity (5.94) on the function f'. $\square$

Theorem 5.41 *Let $f : I \to \mathbb{R}$ be such that $f^{(2n)}$ is absolutely continuous for some $n \geq 1$, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$. Let $g, p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let $\int_{b-\lambda}^b p(t)dt = \int_a^b g(t)p(t)dt$ and let the function $\mathscr{G}_2$ be defined by* (5.40)*. Then*

$$\begin{aligned}
&\int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^b f(t)p(t)dt \\
&+ \sum_{k=0}^{n-1} (b-a)^{2k} \int_a^b \mathscr{G}_2(x) \left[f^{(2k+1)}(a) \Lambda_k \left(\frac{b-x}{b-a} \right) + f^{(2k+1)}(b) \Lambda_k \left(\frac{x-a}{b-a} \right) \right] dx \qquad (5.99) \\
&= -(b-a)^{2n-1} \int_a^b \left(\int_a^b \mathscr{G}_2(x) G_n \left(\frac{x-a}{b-a}, \frac{s-a}{b-a} \right) dx \right) f^{(2n+1)}(s) ds.
\end{aligned}$$

Proof. Similar to the proof of Theorem 5.39 using (5.94) on the function f'. $\square$

Now we give generalizations of Steffensen's inequality for $(2n)$-convex and $(2n+1)$-convex functions.

Theorem 5.42 *Let $f : I \to \mathbb{R}$ be such that $f^{(2n-1)}$ is absolutely continuous for some $n \geq 1$, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$. Let $g, p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let the functions $\mathscr{G}_1, \mathscr{G}_2$ be defined by* (5.35) *and* (5.40) *respectively.*

(i) Let $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$. If f is $(2n)$-convex function and

$$\int_a^b \mathscr{G}_1(x) \frac{dG_n}{dx} \left(\frac{x-a}{b-a}, \frac{s-a}{b-a} \right) dx \geq 0 \qquad (5.100)$$

then

$$\int_a^b f(t)g(t)p(t)dt \geq \int_a^{a+\lambda} f(t)p(t)dt$$
$$+\sum_{k=0}^{n-1}(b-a)^{2k-1}\int_a^b \mathcal{G}_1(x)\left[f^{(2k)}(b)\Lambda_k'\left(\frac{x-a}{b-a}\right)-f^{(2k)}(a)\Lambda_k'\left(\frac{b-x}{b-a}\right)\right]dx. \tag{5.101}$$

(ii) Let $\int_{b-\lambda}^b p(t)dt = \int_a^b g(t)p(t)dt$. *If* f *is* $(2n)$*-convex function and*

$$\int_a^b \mathcal{G}_2(x)\frac{dG_n}{dx}\left(\frac{x-a}{b-a},\frac{s-a}{b-a}\right)dx \geq 0 \tag{5.102}$$

then

$$\int_a^b f(t)g(t)p(t)dt \leq \int_{b-\lambda}^{b} f(t)p(t)dt$$
$$-\sum_{k=0}^{n-1}(b-a)^{2k-1}\int_a^b \mathcal{G}_2(x)\left[f^{(2k)}(b)\Lambda_k'\left(\frac{x-a}{b-a}\right)-f^{(2k)}(a)\Lambda_k'\left(\frac{b-x}{b-a}\right)\right]dx. \tag{5.103}$$

The reversed inequalities in (5.100) *and* (5.102) *implies the reversed inequalities in* (5.101) *and* (5.103) *respectively.*

Proof. If the function f is $(2n)$-convex, without loss of generality we can assume that f is $2n-$times differentiable and $f^{(2n)} \geq 0$ see [71, p. 16 and p. 293]. Now we can apply Theorem 5.38 and Theorem 5.39 to obtain (5.101) and (5.103) respectively. □

Theorem 5.43 *Let* $f: I \to \mathbb{R}$ *be such that* $f^{(2n)}$ *is absolutely continuous for some* $n > 1$, $I \subset \mathbb{R}$ *an open interval,* $a,b \in I$, $a < b$. *Let* $g,p:[a,b]\to\mathbb{R}$ *be integrable functions such that* p *is positive and* $0 \leq g \leq 1$. *Let the functions* $\mathcal{G}_1, \mathcal{G}_2$ *be defined by* (5.35) *and* (5.40) *respectively.*

(i) Let $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$. *If* n *is even, then for* $(2n+1)$*-convex function* f *we have*

$$\int_a^b f(t)g(t)p(t)dt \geq \int_a^{a+\lambda} f(t)p(t)dt$$
$$+\sum_{k=0}^{n-1}(b-a)^{2k}\int_a^b \mathcal{G}_1(x)\left[f^{(2k+1)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right)+f^{(2k+1)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right]dx \tag{5.104}$$

(ii) Let $\int_{b-\lambda}^b p(t)dt = \int_a^b g(t)p(t)dt$. *If* n *is even, then for* $(2n+1)-$*convex function* f *we have*

$$\int_a^b f(t)g(t)p(t)dt \leq \int_{b-\lambda}^{b} f(t)p(t)dt$$
$$-\sum_{k=0}^{n-1}(b-a)^{2k}\int_a^b \mathcal{G}_2(x)\left[f^{(2k+1)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right)+f^{(2k+1)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right]dx. \tag{5.105}$$

If n *is odd, then the reversed inequalities in* (5.104) *and* (5.105) *hold.*

Proof. Since $0 \leq g \leq 1$ from definition of $\mathcal{G}_i$, $i = 1,2$ it follows that functions $\mathcal{G}_i$, $i = 1,2$ are nonnegative. Since n is even from (1.43), it follows that $G_n\left(\frac{x-a}{b-a}, \frac{x-a}{b-a}\right) \geq 0$. Also, if the function f is $(2n+1)$-convex, without loss of generality we can assume that f is $(2n+1)-$times differentiable and $f^{(2n+1)} \geq 0$ see [71, p. 16 and p. 293]. Now we can apply Theorem 5.40 and Theorem 5.41 to obtain (5.104) and (5.105) respectively. □

As in the previous sections, we give the Ostrowski type inequalities related to generalizations of Steffensen's inequality. Using the same approch as in Theorem 5.8 it is easy to see that the following theorem holds.

Theorem 5.44 *Suppose that all assumptions of Theorem* 5.38 *hold. Assume also that* (p,q) *is a pair of conjugate exponents, that is* $1 \leq p,q \leq \infty$, $1/p + 1/q = 1$ *and* $f^{(2n)} \in L_p[a,b]$. *Then we have*

$$\left| \int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt \right.$$
$$\left. + \sum_{k=0}^{n-1} (b-a)^{2k-1} \int_a^b \mathcal{G}_1(x) \left[f^{(2k)}(b)\Lambda_k'\left(\frac{x-a}{b-a}\right) - f^{(2k)}(a)\Lambda_k'\left(\frac{b-x}{b-a}\right)\right] dx \right| \tag{5.106}$$
$$\leq (b-a)^{2n-1} \left\| f^{(2n)} \right\|_p \left\| \int_a^b \mathcal{G}_1(x) \frac{dG_n}{dx}\left(\frac{x-a}{b-a}, \frac{s-a}{b-a}\right) dx \right\|_q .$$

The constant on the right-hand side of (5.106) *is sharp for* $1 < p \leq \infty$ *and the best possible for* $p = 1$.

Theorem 5.45 *Suppose that all assumptions of Theorem* 5.39 *hold. Assume also that* (p,q) *is a pair of conjugate exponents, that is* $1 \leq p,q \leq \infty$, $1/p + 1/q = 1$ *and* $f^{(2n)} \in L_p[a,b]$. *Then we have*

$$\left| \int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^b f(t)p(t)dt \right.$$
$$\left. + \sum_{k=0}^{n-1} (b-a)^{2k-1} \int_a^b \mathcal{G}_2(x) \left[f^{(2k)}(b)\Lambda_k'\left(\frac{x-a}{b-a}\right) - f^{(2k)}(a)\Lambda_k'\left(\frac{b-x}{b-a}\right)\right] dx \right| \tag{5.107}$$
$$\leq (b-a)^{2n-1} \left\| f^{(2n)} \right\|_p \left\| \int_a^b \mathcal{G}_2(x) \frac{dG_n}{dx}\left(\frac{x-a}{b-a}, \frac{s-a}{b-a}\right) dx \right\|_q .$$

The constant on the right-hand side of (5.107) *is sharp for* $1 < p \leq \infty$ *and the best possible for* $p = 1$.

Theorem 5.46 *Suppose that all assumptions of Theorem* 5.40 *hold. Assume also that* (p,q) *is a pair of conjugate exponents, that is* $1 \leq p,q \leq \infty$, $1/p + 1/q = 1$ *and* $f^{(2n+1)} \in L_p[a,b]$. *Then we have*

$$\left| \int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt \right.$$
$$+ \left. \sum_{k=0}^{n-1} (b-a)^{2k} \int_a^b \mathcal{G}_1(x) \left[f^{(2k+1)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right) + f^{(2k+1)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right] dx \right|$$
$$\leq (b-a)^{2n-1} \left\| f^{(2n+1)} \right\|_p \left\| \int_a^b \mathcal{G}_1(x) G_n\left(\frac{x-a}{b-a}, \frac{s-a}{b-a}\right) dx \right\|_q . \tag{5.108}$$

The constant on the right-hand side of (5.108) *is sharp for* $1 < p \leq \infty$ *and the best possible for* $p = 1$.

Theorem 5.47 *Suppose that all assumptions of Theorem* 5.41 *hold. Assume also that* (p,q) *is a pair of conjugate exponents, that is* $1 \leq p,q \leq \infty$, $1/p + 1/q = 1$ *and* $f^{(2n+1)} \in L_p[a,b]$. *Then we have*

$$\left| \int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^b f(t)p(t)dt \right.$$
$$+ \left. \sum_{k=0}^{n-1} (b-a)^{2k} \int_a^b \mathcal{G}_2(x) \left[f^{(2k+1)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right) + f^{(2k+1)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right] dx \right|$$
$$\leq (b-a)^{2n-1} \left\| f^{(2n+1)} \right\|_p \left\| \int_a^b \mathcal{G}_2(x) G_n\left(\frac{x-a}{b-a}, \frac{s-a}{b-a}\right) dx \right\|_q . \tag{5.109}$$

The constant on the right-hand side of (5.109) *is sharp for* $1 < p \leq \infty$ *and the best possible for* $p = 1$.

In this section by $\Omega_i(s)$ and $\Phi_i(s)$ we will denote

$$\Omega_i(s) = \int_a^b \mathcal{G}_i(x) \frac{dG_n}{dx}\left(\frac{x-a}{b-a}, \frac{s-a}{b-a}\right) dx, \quad i = 1,2 \tag{5.110}$$

and

$$\Phi_i(s) = \int_a^b \mathcal{G}_i(x) G_n\left(\frac{x-a}{b-a}, \frac{s-a}{b-a}\right) dx, \quad i = 1,2. \tag{5.111}$$

Similarly as in previous sections, using Theorems 1.23 and 1.24 it is easy to see that the following theorems hold.

Theorem 5.48 *Let* $f : [a,b] \to \mathbb{R}$ *be such that* $f \in C^{2n}([a,b])$ *and* $(\cdot - a)(b - \cdot)[f^{(2n+1)}]^2 \in L_1[a,b]$. *Let* $g, p : [a,b] \to \mathbb{R}$ *be integrable functions such that* p *is positive and* $0 \leq g \leq 1$. *Let the functions* $\mathcal{G}_1$, $\mathcal{G}_2$ *and* Ω_i *be defined by* (5.35),(5.40) *and* (5.110), *respectively.*

(a) *Let* $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$. *Then*

$$\begin{aligned}&\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt \\ &+\sum_{k=0}^{n-1}(b-a)^{2k-1}\int_a^b \mathscr{G}_1(x)\left[f^{(2k)}(b)\Lambda_k'\left(\frac{x-a}{b-a}\right)-f^{(2k)}(a)\Lambda_k'\left(\frac{b-x}{b-a}\right)\right]dx \\ &+(b-a)^{2n-2}\left(f^{(2n-1)}(b)-f^{(2n-1)}(a)\right)\int_a^b \Omega_1(s)ds = S_n^1(f;a,b),\end{aligned} \tag{5.112}$$

where the remainder $S_n^1(f;a,b)$ *satisfies the estimation*

$$\left|S_n^1(f;a,b)\right| \le \frac{(b-a)^{2n}}{\sqrt{2}}\left[T(\Omega_1,\Omega_1)\right]^{\frac{1}{2}} \times \frac{1}{\sqrt{b-a}}\left(\int_a^b (s-a)(b-s)[f^{(2n+1)}(s)]^2 ds\right)^{\frac{1}{2}}.$$

(b) *Let* $\int_{b-\lambda}^{b} p(t)dt = \int_a^b g(t)p(t)dt$. *Then*

$$\begin{aligned}&\int_a^{b} f(t)g(t)p(t)dt - \int_{b-\lambda}^b f(t)p(t)dt \\ &+\sum_{k=0}^{n-1}(b-a)^{2k-1}\int_a^b \mathscr{G}_2(x)\left[f^{(2k)}(b)\Lambda_k'\left(\frac{x-a}{b-a}\right)-f^{(2k)}(a)\Lambda_k'\left(\frac{b-x}{b-a}\right)\right]dx \\ &+(b-a)^{2n-2}\left(f^{(2n-1)}(b)-f^{(2n-1)}(a)\right)\int_a^b \Omega_2(s)ds = S_n^2(f;a,b),\end{aligned} \tag{5.113}$$

where the remainder $S_n^2(f;a,b)$ *satisfies the estimation*

$$\left|S_n^2(f;a,b)\right| \le \frac{(b-a)^{2n}}{\sqrt{2}}\left[T(\Omega_2,\Omega_2)\right]^{\frac{1}{2}} \times \frac{1}{\sqrt{b-a}}\left(\int_a^b (s-a)(b-s)[f^{(2n+1)}(s)]^2 ds\right)^{\frac{1}{2}}.$$

Theorem 5.49 *Let* $f:[a,b]\to\mathbb{R}$ *be such that* $f\in C^{2n+1}([a,b])$ *and* $(\cdot-a)(b-\cdot)[f^{(2n+2)}]^2 \in L_1[a,b]$. *Let* $g,p:[a,b]\to\mathbb{R}$ *be integrable functions such that* p *is positive and* $0\le g\le 1$. *Let the functions* $\mathscr{G}_1$, $\mathscr{G}_2$ *and* Φ_i *be defined by* (5.35), (5.40) *and* (5.111).

(a) *Let* $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$. *Then*

$$\begin{aligned}&\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt \\ &+\sum_{k=0}^{n-1}(b-a)^{2k}\int_a^b \mathscr{G}_1(x)\left[f^{(2k+1)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right)+f^{(2k+1)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right]dx \\ &+(b-a)^{2n-2}\left(f^{(2n)}(b)-f^{(2n)}(a)\right)\int_a^b \Phi_1(s)ds = S_n^3(f;a,b),\end{aligned} \tag{5.114}$$

where the remainder $S_n^3(f;a,b)$ *satisfies the estimation*

$$\left|S_n^3(f;a,b)\right| \le \frac{(b-a)^{2n}}{\sqrt{2}}\left[T(\Phi_1,\Phi_1)\right]^{\frac{1}{2}} \times \frac{1}{\sqrt{b-a}}\left(\int_a^b (s-a)(b-s)[f^{(2n+2)}(s)]^2 ds\right)^{\frac{1}{2}}.$$

(b) *Let* $\int_{b-\lambda}^{b} p(t)dt = \int_a^b g(t)p(t)dt$. *Then*

$$\begin{aligned}
&\int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^{b} f(t)p(t)dt \\
&+ \sum_{k=0}^{n-1} (b-a)^{2k} \int_a^b \mathscr{G}_2(x) \left[f^{(2k+1)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right) + f^{(2k+1)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right] dx \\
&+ (b-a)^{2n-2}\left(f^{(2n)}(b) - f^{(2n)}(a)\right)\int_a^b \Phi_2(s)ds = S_n^4(f;a,b),
\end{aligned} \tag{5.115}$$

where the remainder $S_n^4(f;a,b)$ *satisfies the estimation*

$$\left|S_n^4(f;a,b)\right| \leq \frac{(b-a)^{2n}}{\sqrt{2}}\left[T(\Phi_2,\Phi_2)\right]^{\frac{1}{2}} \times \frac{1}{\sqrt{b-a}}\left(\int_a^b (s-a)(b-s)[f^{(2n+2)}(s)]^2 ds\right)^{\frac{1}{2}}.$$

Theorem 5.50 *Let* $f:[a,b]\to\mathbb{R}$ *be such that* $f\in C^{2n}([a,b])$ *and* $f^{(2n+1)}\geq 0$ *on* $[a,b]$. *Let functions* Ω_i, $i=1,2$ *be defined by* (5.110).

(a) Let $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$. *Then we have representation* (5.112) *and the remainder* $S_n^1(f;a,b)$ *satisfies the bound*

$$\left|S_n^1(f;a,b)\right| \leq (b-a)^{2n}\|\Omega_1'\|_\infty \left\{\frac{f^{(2n-1)}(b)+f^{(2n-1)}(a)}{2} - \left[a,b;f^{(2n-2)}\right]\right\}.$$

(b) Let $\int_{b-\lambda}^{b} p(t)dt = \int_a^b g(t)p(t)dt$. *Then we have representation* (5.113) *and the remainder* $S_n^2(f;a,b)$ *satisfies the bound*

$$\left|S_n^2(f;a,b)\right| \leq (b-a)^{2n}\|\Omega_2'\|_\infty \left\{\frac{f^{(2n-1)}(b)+f^{(2n-1)}(a)}{2} - \left[a,b;f^{(2n-2)}\right]\right\}.$$

Theorem 5.51 *Let* $f:[a,b]\to\mathbb{R}$ *be such that* $f\in C^{2n+1}([a,b])$ *and* $f^{(2n+2)}\geq 0$ *on* $[a,b]$. *Let functions* Φ_i, $i=1,2$ *be defined by* (5.111).

(a) Let $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$. *Then we have representation* (5.114) *and the remainder* $S_n^3(f;a,b)$ *satisfies the bound*

$$\left|S_n^3(f;a,b)\right| \leq (b-a)^{2n}\|\Phi_1'\|_\infty \left\{\frac{f^{(2n)}(b)+f^{(2n)}(a)}{2} - \left[a,b;f^{(2n-1)}\right]\right\}.$$

(b) Let $\int_{b-\lambda}^{b} p(t)dt = \int_a^b g(t)p(t)dt$. *Then we have representation* (5.115) *and the remainder* $S_n^4(f;a,b)$ *satisfies the bound*

$$\left|S_n^4(f;a,b)\right| \leq (b-a)^{2n}\|\Phi_2'\|_\infty \left\{\frac{f^{(2n)}(b)+f^{(2n)}(a)}{2} - \left[a,b;f^{(2n-1)}\right]\right\}.$$

5.6 Generalizations via Hermite polynomial

We begin this section with representations of Steffensen's inequality that are obtained by using Hermite interpolating polynomials. Results given in this section were shown in [66]. For further reading on some different generalizations of Steffensen's inequality using Hermite expansions with integral remainder we refer the reader to [33].

Theorem 5.52 *Let $-\infty < a \leq a_1 < a_2 \ldots < a_r \leq b < \infty$, $(r \geq 2)$ be given points and $f \in C^n[a,b]$. Let $g, p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive, $0 \leq g \leq 1$ and $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$. Let the function $\mathscr{G}_1$ be defined by* (5.35). *Then*

$$\begin{aligned}&\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt + \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+1)}(a_j)\int_a^b \mathscr{G}_1(x)H_{ij}(x)dx \\ &= -\int_a^b \left(\int_a^b \mathscr{G}_1(x)G_{H,n-1}(x,s)dx\right) f^{(n)}(s)ds\end{aligned} \tag{5.116}$$

where H_{ij} are defined on $[a,b]$ by (1.46) *and $G_{H,n-1}$ is Green's function defined by* (1.48).

Proof. Similar to the proof of Theorem 5.11 using identity (1.45) on the function f'. □

Theorem 5.53 *Let $-\infty < a \leq a_1 < a_2 \ldots < a_r \leq b < \infty$, $(r \geq 2)$ be given points and $f \in C^n[a,b]$. Let $g, p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive, $0 \leq g \leq 1$ and $\int_{b-\lambda}^b p(t)dt = \int_a^b g(t)p(t)dt$. Let the function $\mathscr{G}_2$ be defined by* (5.40). *Then*

$$\begin{aligned}&\int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^b f(t)p(t)dt + \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+1)}(a_j)\int_a^b \mathscr{G}_2(x)H_{ij}(x)dx \\ &= -\int_a^b \left(\int_a^b \mathscr{G}_2(x)G_{H,n-1}(x,s)dx\right) f^{(n)}(s)ds\end{aligned} \tag{5.117}$$

where H_{ij} are defined on $[a,b]$ by (1.46) *and $G_{H,n-1}$ is Green's function defined by* (1.48).

Proof. Similar to the proof of the previous theorem using identity (5.42). □

In the following theorems we give generalizations of Steffensen's inequality for n-convex functions.

Theorem 5.54 *Let $-\infty < a \leq a_1 < a_2 \ldots < a_r \leq b < \infty$, $(r \geq 2)$ be given points and $f \in C^n[a,b]$. Let $g, p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive, $0 \leq g \leq 1$ and $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$. Let the function $\mathscr{G}_1$ be defined by* (5.35). *If f is n-convex and*

$$\int_a^b \mathscr{G}_1(x)G_{H,n-1}(x,s)dx \geq 0, \quad s \in [a,b], \tag{5.118}$$

then

$$\int_a^b f(t)g(t)p(t)dt \geq \int_a^{a+\lambda} f(t)p(t)dt + \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+1)}(a_j)\int_a^b \mathscr{G}_1(x)H_{ij}(x)dx, \quad (5.119)$$

where H_{ij} are defined on $[a,b]$ by (1.46) *and $G_{H,n-1}$ is Green's function defined by* (1.48)*. If the reverse inequality in* (5.118) *holds, then the reverse inequality in* (5.119) *holds.*

Proof. If the function f is n-convex, without loss of generality we can assume that f is n-times differentiable and $f^{(n)} \geq 0$ see [71, p. 16 and p. 293]. Hence we can apply Theorem 5.52 to obtain (5.119). □

Theorem 5.55 *Let $-\infty < a \leq a_1 < a_2 \ldots < a_r \leq b < \infty$, $(r \geq 2)$ be given points and $f \in C^n[a,b]$. Let $g,p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive, $0 \leq g \leq 1$ and $\int_{b-\lambda}^b p(t)dt = \int_a^b g(t)p(t)dt$. Let the function $\mathscr{G}_2$ be defined by* (5.40)*. If f is n-convex and*

$$\int_a^b \mathscr{G}_2(x)G_{H,n-1}(x,s)dx \geq 0, \quad s \in [a,b], \quad (5.120)$$

then

$$\int_a^b f(t)g(t)p(t)dt \leq \int_{b-\lambda}^b f(t)p(t)dt - \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+1)}(a_j)\int_a^b \mathscr{G}_2(x)H_{ij}(x)dx, \quad (5.121)$$

where H_{ij} are defined on $[a,b]$ by (1.46) *and $G_{H,n-1}$ is Green's function defined by* (1.48)*. If the reverse inequality in* (5.120) *holds, then the reverse inequality in* (5.121) *holds.*

Proof. Analogous to the proof of Theorem 5.54. □

Remark 5.3 *Note that functions $\mathscr{G}_i$, $i = 1,2$ defined by* (5.35) *and* (5.40) *are nonnegative. If all $k_1,\ldots,k_r$ are odd then $\omega(x) = \prod_{j=1}^r (x-a_j)^{k_j+1} \geq 0$ and according to (i)-part of Lemma 1.4 $G_{H,n}(x,s) \geq 0$. Therefore, in Theorems 5.54 and 5.55 it is enough to assume that the function f is n-convex. For the case when only one k_j is even and others are odd we have $\omega(x) = \prod_{j=1}^r (x-a_j)^{k_j+1} \leq 0$ and by Lemma 1.4, $G_{H,n}(x,s) \leq 0$. Hence, integrals in* (5.118) *and* (5.120) *are nonpositive and the reverse inequalities in* (5.119) *and* (5.121) *hold.*

By using $(m,n-m)$ type conditions we obtain following representations of Steffensen's inequality.

Corollary 5.10 *Let $-\infty < a < b < \infty$ be given points and $f \in C^n[a,b]$. Let $g,p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive, $0 \leq g \leq 1$ and $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$. Let the function $\mathscr{G}_1$ be defined by* (5.35) *and τ_i, η_i be defined by* (1.49) *and* (1.50)*, respectively. Then*

$$\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt + \sum_{i=0}^{m-1} f^{(i+1)}(a)\int_a^b \mathscr{G}_1(x)\tau_i(x)dx$$
$$+ \sum_{i=0}^{n-m-2} f^{(i+1)}(b)\int_a^b \mathscr{G}_1(x)\eta_i(x)dx = -\int_a^b\left(\int_a^b \mathscr{G}_1(x)G_{m,n-1}(x,s)dx\right) f^{(n)}(s)ds,$$

where $G_{m,n-1}$ is Green's function defined by (1.51).

Corollary 5.11 *Let $-\infty < a < b < \infty$ be given points and $f \in C^n[a,b]$. Let $g,p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive, $0 \le g \le 1$ and $\int_{b-\lambda}^{b} p(t)dt = \int_a^b g(t)p(t)dt$. Let the function $\mathscr{G}_2$ be defined by* (5.40) *and τ_i, η_i be defined by* (1.49) *and* (1.50), *respectively. Then*

$$\int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^{b} f(t)p(t)dt + \sum_{i=0}^{m-1} f^{(i+1)}(a)\int_a^b \mathscr{G}_2(x)\tau_i(x)dx$$
$$+ \sum_{i=0}^{n-m-2} f^{(i+1)}(b)\int_a^b \mathscr{G}_2(x)\eta_i(x)dx = -\int_a^b \left(\int_a^b \mathscr{G}_2(x)G_{m,n-1}(x,s)dx\right) f^{(n)}(s)ds,$$

where $G_{m,n-1}$ is Green's function defined by (1.51).

Further, using $(m,n-m)$ type conditions the following generalizations of Steffensen's inequality are obtained.

Corollary 5.12 *Let $-\infty < a < b < \infty$ be given points and $f \in C^n[a,b]$. Let $g,p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive, $0 \le g \le 1$ and $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$. Let the function $\mathscr{G}_1$ be defined by* (5.35) *and τ_i, η_i be defined by* (1.49) *and* (1.50), *respectively. If f is n-convex and*

$$\int_a^b \mathscr{G}_1(x)G_{m,n-1}(x,s)dx \ge 0, \quad s \in [a,b],$$

then

$$\int_a^b f(t)g(t)p(t)dt \ge \int_a^{a+\lambda} f(t)p(t)dt + \sum_{i=0}^{m-1} f^{(i+1)}(a)\int_a^b \mathscr{G}_1(x)\tau_i(x)dx$$
$$+ \sum_{i=0}^{n-m-2} f^{(i+1)}(b)\int_a^b \mathscr{G}_1(x)\eta_i(x)dx,$$

where $G_{m,n-1}$ is Green's function defined by (1.51).

Corollary 5.13 *Let $-\infty < a < b < \infty$ be given points and $f \in C^n[a,b]$. Let $g,p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive, $0 \le g \le 1$ and $\int_{b-\lambda}^{b} p(t)dt = \int_a^b g(t)p(t)dt$. Let the function $\mathscr{G}_2$ be defined by* (5.40) *and τ_i, η_i be defined by* (1.49) *and* (1.50), *respectively. If f is n-convex and*

$$\int_a^b \mathscr{G}_2(x)G_{m,n-1}(x,s)dx \ge 0, \quad s \in [a,b],$$

then

$$\int_a^b f(t)g(t)p(t)dt \le \int_{b-\lambda}^{b} f(t)p(t)dt - \sum_{i=0}^{m-1} f^{(i+1)}(a)\int_a^b \mathscr{G}_2(x)\tau_i(x)dx$$
$$- \sum_{i=0}^{n-m-2} f^{(i+1)}(b)\int_a^b \mathscr{G}_2(x)\eta_i(x)dx,$$

where $G_{m,n-1}$ is Green's function defined by (1.51).

The Ostrowski-type inequalities related to previously obtained generalizations are given.

Theorem 5.56 *Suppose that all assumptions of Theorem 5.52 hold. Assume also that (p,q) is a pair of conjugate exponents, that is $1 \le p,q \le \infty$, $1/p+1/q=1$ and $f^{(n)} \in L_p[a,b]$ for some $n \ge 2$. Then we have*

$$\begin{aligned}&\left|\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt + \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+1)}(a_j)\int_a^b \mathcal{G}_1(x)H_{ij}(x)dx\right| \\ &\le \left\|f^{(n)}\right\|_p \left\|\int_a^b \mathcal{G}_1(x)G_{H,n-1}(x,\cdot)dx\right\|_q .\end{aligned} \tag{5.122}$$

The constant on the right-hand side of (5.122) *is sharp for $1 < p \le \infty$ and the best possible for $p=1$.*

Proof. Similar to the proof of the Theorem 5.8. □

Theorem 5.57 *Suppose that all assumptions of Theorem 5.53 hold. Assume also that (p,q) is a pair of conjugate exponents, that is $1 \le p,q \le \infty$, $1/p+1/q=1$. Let $f^{(n)} \in L_p[a,b]$ for some $n \ge 2$. Then we have*

$$\begin{aligned}&\left|\int_a^{b} f(t)g(t)p(t)dt - \int_{b-\lambda}^b f(t)p(t)dt + \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+1)}(a_j)\int_a^b \mathcal{G}_2(x)H_{ij}(x)dx\right| \\ &\le \left\|f^{(n)}\right\|_p \left\|\int_a^b \mathcal{G}_2(x)G_{H,n-1}(x,\cdot)dx\right\|_q .\end{aligned} \tag{5.123}$$

The constant on the right-hand side of (5.123) *is sharp for $1 < p \le \infty$ and the best possible for $p=1$.*

Proof. Similar to the proof of Theorem 5.8. □

By using $(m,n-m)$ type conditions we obtain the following results.

Corollary 5.14 *Suppose that all assumptions of Corollary 5.10 hold. Assume also that (p,q) is a pair of conjugate exponents, that is $1 \le p,q \le \infty$, $1/p+1/q=1$ and $f^{(n)} \in L_p[a,b]$ for some $n \ge 2$. Then we have*

$$\begin{aligned}&\left|\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt + \sum_{i=0}^{m-1} f^{(i+1)}(a)\int_a^b \mathcal{G}_1(x)\tau_i(x)dx\right. \\ &\left.+\sum_{i=0}^{n-m-2} f^{(i+1)}(b)\int_a^b \mathcal{G}_1(x)\eta_i(x)dx\right| \le \left\|f^{(n)}\right\|_p \left\|\int_a^b \mathcal{G}_1(x)G_{m,n-1}(x,\cdot)dx\right\|_q .\end{aligned} \tag{5.124}$$

The constant on the right-hand side of (5.124) *is sharp for $1 < p \le \infty$ and the best possible for $p=1$.*

Corollary 5.15 *Suppose that all assumptions of Corollary* 5.11 *hold. Assume also that (p,q) is a pair of conjugate exponents, that is $1 \leq p,q \leq \infty$, $1/p+1/q=1$ and $f^{(n)} \in L_p[a,b]$ for some $n \geq 2$. Then we have*

$$\begin{aligned}&\left|\int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^b f(t)p(t)dt + \sum_{i=0}^{m-1} f^{(i+1)}(a)\int_a^b \mathscr{G}_2(x)\tau_i(x)dx\right.\\&\left.+\sum_{i=0}^{n-m-2} f^{(i+1)}(b)\int_a^b \mathscr{G}_2(x)\eta_i(x)dx\right| \leq \left\|f^{(n)}\right\|_p \left\|\int_a^b \mathscr{G}_2(x)G_{m,n-1}(x,\cdot)dx\right\|_q.\end{aligned} \tag{5.125}$$

The constant on the right-hand side of (5.125) *is sharp for $1 < p \leq \infty$ and the best possible for $p=1$.*

In the sequel, similarly as in previous sections, we give some new bounds for integrals on the left hand side in the perturbed versions of representations obtained in Theorems 5.52 and 5.53.

Let us denote

$$\Omega_i(s) = \int_a^b \mathscr{G}_i(x)G_{H,n-1}(x,s)dx, \quad i=1,2 \tag{5.126}$$

$$\Phi_i(s) = \int_a^b \mathscr{G}_i(x)G_{m,n-1}(x,s)dx, \quad i=1,2, \tag{5.127}$$

for $\mathscr{G}_i$ defined by (5.35) and (5.40) and $G_{H,n-1}$, $G_{m,n-1}$ defined by (1.48) and (1.51), respectively.

Using Theorem 1.23 we obtain the following results.

Proofs are similar to the proof of Theorem 5.5 so we omit them here.

Theorem 5.58 *Let $-\infty < a \leq a_1 < a_2 \ldots < a_r \leq b < \infty$, $(r \geq 2)$ be given points, $f \in C^{n+1}[a,b]$ and $(\cdot - a)(b - \cdot)[f^{(n+1)}]^2 \in L_1[a,b]$. Let $g,p:[a,b] \to \mathbb{R}$ be integrable functions such that p is positive, $0 \leq g \leq 1$ and $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$. Let $\mathscr{G}_1$ and Ω_1 be defined by* (5.35) *and* (5.126)*, respectively. Then*

$$\begin{aligned}&\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt + \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+1)}(a_j)\int_a^b \mathscr{G}_1(x)H_{ij}(x)dx\\&+\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\int_a^b \Omega_1(s)ds = S_n^1(f;a,b),\end{aligned} \tag{5.128}$$

where the remainder $S_n^1(f;a,b)$ satisfies the estimation

$$\left|S_n^1(f;a,b)\right| \leq \frac{\sqrt{b-a}}{\sqrt{2}}\left[T(\Omega_1,\Omega_1)\right]^{\frac{1}{2}}\left(\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2 ds\right)^{\frac{1}{2}}.$$

Theorem 5.59 *Let* $-\infty < a \leq a_1 < a_2 \ldots < a_r \leq b < \infty$, $(r \geq 2)$ *be given points,* $f \in C^{n+1}[a,b]$ *and* $(\cdot - a)(b - \cdot)[f^{(n+1)}]^2 \in L_1[a,b]$. *Let* $g,p : [a,b] \to \mathbb{R}$ *be integrable functions such that* p *is positive,* $0 \leq g \leq 1$ *and* $\int_{b-\lambda}^{b} p(t)dt = \int_a^b g(t)p(t)dt$. *Let* $\mathscr{G}_2$ *and* Ω_2 *be defined by* (5.40) *and* (5.126), *respectively.*

Then

$$
\begin{aligned}
&\int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^{b} f(t)p(t)dt + \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+1)}(a_j)\int_a^b \mathscr{G}_2(x)H_{ij}(x)dx \\
&+ \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}\int_a^b \Omega_2(s)ds = S_n^2(f;a,b),
\end{aligned}
\tag{5.129}
$$

where the remainder $S_n^2(f;a,b)$ *satisfies the estimation*

$$
\left|S_n^2(f;a,b)\right| \leq \frac{\sqrt{b-a}}{\sqrt{2}}\left[T(\Omega_2,\Omega_2)\right]^{\frac{1}{2}}\left(\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2 ds\right)^{\frac{1}{2}}.
$$

Using Theorem 1.24 we obtain the following Grüss type inequalities.

Theorem 5.60 *Let* $-\infty < a \leq a_1 < a_2 \ldots < a_r \leq b < \infty$, $(r \geq 2)$ *be given points,* $f \in C^{n+1}[a,b]$ *and* $f^{(n+1)} \geq 0$ *on* $[a,b]$. *Let functions* Ω_i, $i = 1,2$ *be defined by* (5.126).

(a) Let $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$. *Then the representation* (5.128) *holds and the remainder* $S_n^1(f;a,b)$ *satisfies the bound*

$$
\left|S_n^1(f;a,b)\right| \leq (b-a)\|\Omega_1'\|_\infty \left\{\frac{f^{(n-1)}(b) + f^{(n-1)}(a)}{2} - \frac{f^{(n-2)}(b) - f^{(n-2)}(a)}{b-a}\right\}.
$$

(b) Let $\int_{b-\lambda}^{b} p(t)dt = \int_a^b g(t)p(t)dt$. *Then the representation* (5.129) *holds and the remainder* $S_n^2(f;a,b)$ *satisfies the bound*

$$
\left|S_n^2(f;a,b)\right| \leq (b-a)\|\Omega_2'\|_\infty \left\{\frac{f^{(n-1)}(b) + f^{(n-1)}(a)}{2} - \frac{f^{(n-2)}(b) - f^{(n-2)}(a)}{b-a}\right\}.
$$

Proof. Similar to the proof of Theorem 5.7. □

Similarly, using the $(m, n-m)$ conditions we obtain the following results.

Corollary 5.16 *Let* $-\infty < a < b < \infty$ *be given points,* $f \in C^{n+1}[a,b]$ *and* $(\cdot - a)(b - \cdot)[f^{(n+1)}]^2 \in L_1[a,b]$. *Let* $g,p : [a,b] \to \mathbb{R}$ *be integrable functions such that* p *is positive,* $0 \leq g \leq 1$ *and* $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$. *Let* $\mathscr{G}_1$, Φ_1, τ_i *and* η_i *be defined by* (5.35), (5.127),(1.49) *and* (1.50) *respectively. Then*

$$
\begin{aligned}
&\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt + \sum_{i=0}^{m-1} f^{(i+1)}(a)\int_a^b \mathscr{G}_1(x)\tau_i(x)dx \\
&+ \sum_{i=0}^{n-m-2} f^{(i+1)}(b)\int_a^b \mathscr{G}_1(x)\eta_i(x)dx + \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}\int_a^b \Phi_1(s)ds \\
&= S_n^3(f;a,b),
\end{aligned}
\tag{5.130}
$$

where the remainder $S_n^3(f;a,b)$ satisfies the estimation

$$\left|S_n^3(f;a,b)\right| \leq \frac{\sqrt{b-a}}{\sqrt{2}}\left[T(\Phi_1,\Phi_1)\right]^{\frac{1}{2}}\left(\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2 ds\right)^{\frac{1}{2}}.$$

Corollary 5.17 *Let $-\infty < a < b < \infty$ be given points, $f \in C^{n+1}[a,b]$ and $(\cdot - a)(b-\cdot)[f^{(n+1)}]^2 \in L_1[a,b]$. Let $g,p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive, $0 \leq g \leq 1$ and $\int_{b-\lambda}^b p(t)dt = \int_a^b g(t)p(t)dt$. Let $\mathcal{G}_2$, Φ_2, τ_i and η_i be defined by* (5.40), (5.127),(1.49) *and* (1.50) *respectively. Then*

$$\begin{aligned}
&\int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^b f(t)p(t)dt + \sum_{i=0}^{m-1} f^{(i+1)}(a)\int_a^b \mathcal{G}_2(x)\tau_i(x)dx \\
&+ \sum_{i=0}^{n-m-2} f^{(i+1)}(b)\int_a^b \mathcal{G}_2(x)\eta_i(x)dx + \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\int_a^b \Phi_2(s)ds \qquad (5.131)\\
&= S_n^4(f;a,b),
\end{aligned}$$

where the remainder $S_n^4(f;a,b)$ satisfies the estimation

$$\left|S_n^4(f;a,b)\right| \leq \frac{\sqrt{b-a}}{\sqrt{2}}\left[T(\Phi_2,\Phi_2)\right]^{\frac{1}{2}}\left(\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2 ds\right)^{\frac{1}{2}}.$$

Corollary 5.18 *Let $-\infty < a < b < \infty$ be given points, $f \in C^{n+1}[a,b]$ and $f^{(n+1)} \geq 0$ on $[a,b]$. Let functions Φ_i, $i = 1,2$ be defined by* (5.127).

(a) Let $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$. Then the representation (5.130) *holds and the remainder $S_n^3(f;a,b)$ satisfies the bound*

$$\begin{aligned}
&\left|S_n^3(f;a,b)\right| \\
&\leq (b-a)\|\Phi_1'\|_\infty \left\{\frac{f^{(n-1)}(b)+f^{(n-1)}(a)}{2} - \frac{f^{(n-2)}(b)-f^{(n-2)}(a)}{b-a}\right\}.
\end{aligned}$$

(b) Let $\int_{b-\lambda}^b p(t)dt = \int_a^b g(t)p(t)dt$. Then the representation (5.131) *holds and the remainder $S_n^4(f;a,b)$ satisfies the bound*

$$\begin{aligned}
&\left|S_n^4(f;a,b)\right| \\
&\leq (b-a)\|\Phi_2'\|_\infty \left\{\frac{f^{(n-1)}(b)+f^{(n-1)}(a)}{2} - \frac{f^{(n-2)}(b)-f^{(n-2)}(a)}{b-a}\right\}.
\end{aligned}$$

5.7 Generalizations via two-point Abel-Gontscharoff polynomial

Some generalizations of Steffensen's inequality via Abel-Gontscharoff polynomial using the difference of integrals on two intervals were obtained in [62]. Using different approach from the one used in [62], the authors obtained following results, given in [67].

First, by using two-point Abel-Gontscharoff polynomial some identities for Steffensen's difference were obtained.

Theorem 5.61 *Let $f \in C^n[a,b]$ for $n \geq 3$ and let $g, p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$ and let the function $\mathscr{G}_1$ be defined by* (5.35)*. Then*

$$\begin{aligned}
&\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt + \sum_{i=0}^{\alpha} \frac{f^{(i+1)}(a)}{i!} \int_a^b \mathscr{G}_1(x)(x-a)^i dx \\
&+ \sum_{j=0}^{n-\alpha-3} f^{(\alpha+j+2)}(a+\lambda) \left[\sum_{i=0}^{j} \frac{(-\lambda)^{j-i}}{(\alpha+1+i)!(j-i)!} \left(\int_a^b (x-a)^{\alpha+1+i} \mathscr{G}_1(x)dx \right) \right] \qquad (5.132) \\
&= -\int_a^b \left(\int_a^b \mathscr{G}_1(x) g_{AG2}(x,s)dx \right) f^{(n)}(s)ds.
\end{aligned}$$

Proof. Similar to the proof of Theorem 5.11 using identity (1.52) on the function f'. □

Theorem 5.62 *Let $f \in C^n[a,b]$ for $n \geq 3$ and let $g, p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let $\int_{b-\lambda}^{b} p(t)dt = \int_a^b g(t)p(t)dt$ and let the function $\mathscr{G}_2$ be defined by* (5.40)*. Then*

$$\begin{aligned}
&\int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^b f(t)p(t)dt + \sum_{i=0}^{\alpha} \frac{f^{(i+1)}(b-\lambda)}{i!} \int_a^b \mathscr{G}_2(x)(x-b+\lambda)^i dx \\
&+ \sum_{j=0}^{n-\alpha-3} f^{(\alpha+j+2)}(b) \left(\sum_{i=0}^{j} \frac{(-\lambda)^{j-i}}{(\alpha+1+i)!(j-i)!} \left(\int_a^b (x-b+\lambda)^{\alpha+1+i} \mathscr{G}_2(x)dx \right) \right) \\
&= -\int_a^b \left(\int_a^b \mathscr{G}_2(x) g_{AG2}(x,s)dx \right) f^{(n)}(s)ds.
\end{aligned} \qquad (5.133)$$

Proof. Similar to the proof of Theorem 5.61 using identity (5.42) for $d\mu(t) = p(t)dt$. □

Now, from these identities the generalizations of Steffensen's inequality for n-convex functions are obtained.

Theorem 5.63 *Let $f \in C^n[a,b]$ for $n \geq 3$ and let $g,p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$ and let the function $\mathscr{G}_1$ be defined by* (5.35). *If f is n-convex and*

$$\int_a^b \mathscr{G}_1(x) g_{AG2}(x,s)dx \leq 0, \quad s \in [a,b], \tag{5.134}$$

then

$$\int_a^b f(t)g(t)p(t)dt \leq \int_a^{a+\lambda} f(t)p(t)dt + \sum_{i=0}^{\alpha} \frac{f^{(i+1)}(a)}{i!} \int_a^b \mathscr{G}_1(x)(x-a)^i dx$$
$$+ \sum_{j=0}^{n-\alpha-3} f^{(\alpha+j+2)}(a+\lambda) \left[\sum_{i=0}^{j} \frac{(-\lambda)^{j-i}}{(\alpha+1+i)!(j-i)!} \left(\int_a^b (x-a)^{\alpha+1+i} \mathscr{G}_1(x)dx \right) \right]. \tag{5.135}$$

Proof. If the function f is n-convex, without loss of generality we can assume that f is n-times differentiable and $f^{(n)} \geq 0$ see [71, p. 16 and p. 293]. Now we can apply Theorem 5.61 to obtain (5.135). □

Theorem 5.64 *Let $f \in C^n[a,b]$ for $n \geq 3$ and let $g,p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let $\int_{b-\lambda}^b p(t)dt = \int_a^b g(t)p(t)dt$ and let the function $\mathscr{G}_2$ be defined by* (5.40). *If f is n-convex and*

$$\int_a^b \mathscr{G}_2(x) g_{AG2}(x,s)dx \leq 0, \quad s \in [a,b], \tag{5.136}$$

then

$$\int_a^b f(t)g(t)p(t)dt \geq \int_{b-\lambda}^b f(t)p(t)dt - \sum_{i=0}^{\alpha} \frac{f^{(i+1)}(b-\lambda)}{i!} \int_a^b \mathscr{G}_2(x)(x-b+\lambda)^i dx$$
$$- \sum_{j=0}^{n-\alpha-3} f^{(\alpha+j+2)}(b) \left(\sum_{i=0}^{j} \frac{(-\lambda)^{j-i}}{(\alpha+1+i)!(j-i)!} \left(\int_a^b (x-b+\lambda)^{\alpha+1+i} \mathscr{G}_2(x)dx \right) \right) \tag{5.137}$$

Proof. Similar to the proof of Theorem 5.63. □

Remark 5.4 *If the integrals in* (5.134) *and* (5.136) *are nonnegative, then the reverse inequalities in* (5.135) *and* (5.137) *hold.*

Taking $p \equiv 1$ and $n = 3$ in previous theorems we obtain the following corollary.

Corollary 5.19 *Let $f \in C^3[a,b]$ and let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq 1$. Let $\lambda = \int_a^b g(t)dt$.*

(i) If f is 3-convex and

$$-\frac{1}{3}\int_a^s x^3 g(x)dx + s\int_a^s x^2 g(x)dx + s^2\int_s^b xg(x)dx + (a^2 - 2as)\int_a^b xg(x)dx + \frac{s^3}{3}\int_a^s g(x)dx$$
$$\leq \frac{(s-a)^4}{12} - \frac{(s-a)^2}{2}\left(\lambda a - \frac{\lambda^2}{2}\right), \quad s \in [a, a+\lambda],$$

$$-\frac{1}{3}\int_a^s x^3 g(x)dx + s\int_a^s x^2 g(x)dx + s^2\int_s^b xg(x)dx + (a^2 - 2as)\int_a^b xg(x)dx - \frac{s^3}{3}\int_s^b g(x)dx$$
$$\leq \left(\frac{2a^3}{3} - sa^2\right)\lambda + \frac{(s-a)\lambda^3}{3} - \frac{\lambda^4}{12}, \quad s \in [a+\lambda, b],$$

then

$$\int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)dt + f'(a)\left(\int_a^b xg(x)dx - a\lambda - \frac{\lambda^2}{2}\right)$$
$$+ \frac{f''(a+\lambda)}{2}\left(\int_a^b g(x)(x-a)^2 dx - \frac{\lambda^3}{3}\right).$$

(ii) If f is 3-convex and

$$\frac{1}{3}\int_a^s x^3 g(x)dx - s\int_a^s x^2 g(x)dx - ((b-\lambda)^2 - 2s(b-\lambda))\int_a^b xg(x)dx$$
$$-s^2\int_s^b xg(x)dx - \frac{s^3}{3}\int_a^s g(x)dx \leq \frac{(b-\lambda-s)^2}{2}\left(\frac{\lambda^2}{2} - \lambda b\right), \quad s \in [a, b-\lambda],$$

$$\frac{1}{3}\int_a^s x^3 g(x)dx - s\int_a^s x^2 g(x)dx - \left[(b-\lambda)^2 - 2s(b-\lambda)\right]\int_a^b xg(x)dx$$
$$-s^2\int_s^b xg(x)dx + \frac{s^3}{3}\int_s^b g(x)dx \leq -\frac{(s-b+\lambda)^4}{12}$$
$$-\frac{(s-b+\lambda)^3\lambda}{2}\left[\frac{\lambda}{3} + \frac{2}{3}(b-s)\right] - \lambda(b-\lambda)^2\left[\frac{2}{3}(b-\lambda) - s\right], \quad s \in [b-\lambda, b],$$

then

$$\int_a^b f(t)g(t)dt \geq \int_{b-\lambda}^b f(t)dt + f'(b-\lambda)\left(\frac{\lambda^2}{2} - b\lambda + \int_a^b xg(x)dx\right)$$
$$-\frac{f''(b)}{2}\left(\frac{\lambda^3}{3} - \int_a^b g(x)(x-b+\lambda)^2 dx\right).$$

Now we give Ostrowski type inequalities for the identities associated with generalized inequalities. Using identities (5.132), (5.133) respectively, we obtain the following results.

Theorem 5.65 *Suppose that all assumptions of Theorem* 5.61 *hold. Assume* (p,q) *is a pair of conjugate exponents, that is* $1 \le p,q \le \infty$, $1/p+1/q=1$. *Let* $\left|f^{(n)}\right|^p : [a,b] \to \mathbb{R}$ *be an R-integrable function for some* $n \ge 3$. *Then we have*

$$\begin{aligned}
&\left| \int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt + \sum_{i=0}^{\alpha} \frac{f^{(i+1)}(a)}{i!} \int_a^b \mathscr{G}_1(x)(x-a)^i dx \right. \\
&\left. + \sum_{j=0}^{n-\alpha-3} f^{(\alpha+j+2)}(a+\lambda) \left[\sum_{i=0}^{j} \frac{(-\lambda)^{j-i}}{(\alpha+1+i)!(j-i)!} \left(\int_a^b (x-a)^{\alpha+1+i} \mathscr{G}_1(x)dx \right) \right] \right| \\
&\le \left\| f^{(n)} \right\|_p \left(\int_a^b \left| \int_a^b \mathscr{G}_1(x) g_{AG2}(x,s)dx \right|^q ds \right)^{\frac{1}{q}}.
\end{aligned} \tag{5.138}$$

The constant on the right-hand side of (5.138) *is sharp for* $1 < p \le \infty$ *and the best possible for* $p = 1$.

Proof. Proof of theorem can be done using the same approach as given in Theorem 5.8. □

Theorem 5.66 *Suppose that all assumptions of Theorem* 5.12 *hold. Assume* (p,q) *is a pair of conjugate exponents, that is* $1 \le p,q \le \infty$, $1/p+1/q=1$. *Let* $\left|f^{(n)}\right|^p : [a,b] \to \mathbb{R}$ *be an R-integrable function for some* $n \ge 3$. *Then we have*

$$\begin{aligned}
&\left| \int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^b f(t)p(t)dt + \sum_{i=0}^{\alpha} \frac{f^{(i+1)}(b-\lambda)}{i!} \int_a^b \mathscr{G}_2(x)(x-b+\lambda)^i dx \right. \\
&\left. + \sum_{j=0}^{n-\alpha-3} f^{(\alpha+j+2)}(b) \left(\sum_{i=0}^{j} \frac{(-\lambda)^{j-i}}{(\alpha+1+i)!(j-i)!} \left(\int_a^b (x-b+\lambda)^{\alpha+1+i} \mathscr{G}_2(x)dx \right) \right) \right| \\
&\le \left\| f^{(n)} \right\|_p \left(\int_a^b \left| \int_a^b \mathscr{G}_2(x) g_{AG2}(x,s)dx \right|^q ds \right)^{\frac{1}{q}}.
\end{aligned} \tag{5.139}$$

The constant on the right-hand side of (5.139) *is sharp for* $1 < p \le \infty$ *and the best possible for* $p = 1$.

Proof. Similar to the proof of Theorem 5.8. □

Taking $p \equiv 1$ and $n = 3$ in Theorems 5.65 and 5.66 we obtain the following corollaries.

Corollary 5.20 *Let* $f : [a,b] \to \mathbb{R}$ *be such that* $f \in C^3[a,b]$, *let* $g : [a,b] \to \mathbb{R}$ *be an integrable function such that* $0 \le g \le 1$ *and let* $\lambda = \int_a^b g(t)dt$. *Assume* (p,q) *is a pair of conjugate exponents, that is* $1 \le p,q \le \infty$, $1/p+1/q=1$. *Let* $|f'''|^p : [a,b] \to \mathbb{R}$ *be an R-integrable function. Then for* $1 < p \le \infty$ *we have*

$$
\begin{aligned}
&\left|\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt + f'(a)\left(\int_a^b xg(x)dx - a\lambda - \frac{\lambda^2}{2}\right)\right. \\
&\left.+\frac{f''(a+\lambda)}{2}\left(\int_a^b g(x)(x-a)^2dx - \frac{\lambda^3}{3}\right)\right| \le \|f'''\|_p \left(\int_a^{a+\lambda}\left|-\frac{1}{3}\int_a^s x^3g(x)dx\right.\right. \\
&+s\int_a^s x^2g(x)dx + s^2\int_s^b xg(x)dx + (a^2-2as)\int_a^b xg(x)dx + \frac{s^3}{3}\int_a^s g(x)dx \\
&\left.-\frac{(s-a)^4}{12} + \frac{(s-a)^2}{2}\left(\lambda a - \frac{\lambda^2}{2}\right)\right|^q ds + \int_{a+\lambda}^b \left|-\frac{1}{3}\int_a^s x^3g(x)dx\right. \\
&+s\int_a^s x^2g(x)dx + s^2\int_s^b xg(x)dx + (a^2-2as)\int_a^b xg(x)dx \\
&\left.\left.-\frac{s^3}{3}\int_s^b g(x)dx - \left(\frac{2a^3}{3} - sa^2\right)\lambda - \frac{(s-a)\lambda^3}{3} + \frac{\lambda^4}{12}\right|^q ds\right)^{\frac{1}{q}}.
\end{aligned} \tag{5.140}
$$

and the constant on the right-hand side of (5.140) *is sharp, while for* $p = 1$ *we have*

$$
\begin{aligned}
&\left|\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt + f'(a)\left(\int_a^b xg(x)dx - a\lambda - \frac{\lambda^2}{2}\right)\right. \\
&\left.+\frac{f''(a+\lambda)}{2}\left(\int_a^b g(x)(x-a)^2dx - \frac{\lambda^3}{3}\right)\right| \le \|f'''\|_1 \max\{M_1, M_2\}
\end{aligned} \tag{5.141}
$$

where

$$
\begin{aligned}
M_1 = \max_{s\in[a,a+\lambda]} &\left\{-\frac{1}{3}\int_a^s x^3g(x)dx + s\int_a^s x^2g(x)dx + s^2\int_s^b xg(x)dx\right. \\
&\left.+(a^2-2as)\int_a^b xg(x)dx + \frac{s^3}{3}\int_a^s g(x)dx - \frac{(s-a)^4}{12} + \frac{(s-a)^2}{2}\left(\lambda a - \frac{\lambda^2}{2}\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
M_2 = \max_{s\in[a+\lambda,b]} &\left\{-\frac{1}{3}\int_a^s x^3g(x)dx + s\int_a^s x^2g(x)dx + s^2\int_s^b xg(x)dx\right. \\
&\left.+(a^2-2as)\int_a^b xg(x)dx - \frac{s^3}{3}\int_s^b g(x)dx - \left(\frac{2a^3}{3} - sa^2\right)\lambda - \frac{(s-a)\lambda^3}{3} + \frac{\lambda^4}{12}\right\}.
\end{aligned}
$$

and the constant on the right-hand side of (5.141) *is the best possible.*

Corollary 5.21 *Let* $f : [a,b] \to \mathbb{R}$ *be such that* $f \in C^3[a,b]$, *let* $g : [a,b] \to \mathbb{R}$ *be an integrable function such that* $0 \le g \le 1$ *and let* $\lambda = \int_a^b g(t)dt$. *Assume* (p,q) *is a pair of conjugate exponents, that is* $1 \le p,q \le \infty$, $1/p + 1/q = 1$. *Let* $|f'''|^p : [a,b] \to \mathbb{R}$ *be an R-integrable function. Then for* $1 < p \le \infty$ *we have*

$$
\begin{aligned}
&\left|\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt - f'(b-\lambda)\left(\frac{\lambda^2}{2} - b\lambda + \int_a^b xg(x)dx\right)\right. \\
&\left.+\frac{f''(b)}{2}\left(\frac{\lambda^3}{3} - \int_a^b g(x)(x-b+\lambda)^2 dx\right)\right| \leq \|f'''\|_p \left(\int_a^{b-\lambda}\left|\frac{1}{3}\int_a^s x^3 g(x)dx\right.\right. \\
&-s\int_a^s x^2 g(x)dx - \left[(b-\lambda)^2 - 2s(b-\lambda)\right]\int_a^b xg(x)dx - s^2\int_s^b xg(x)dx \\
&\left.-\frac{s^3}{3}\int_a^s g(x)dx - \frac{(b-\lambda-s)^2}{2}\left(\frac{\lambda^2}{2} - \lambda b\right)\right|^q ds + \int_{b-\lambda}^b \left|\frac{1}{3}\int_a^s x^3 g(x)dx\right. \qquad (5.142)\\
&-s\int_a^s x^2 g(x)dx - \left[(b-\lambda)^2 - 2s(b-\lambda)\right]\int_a^s xg(x)dx - s^2\int_s^b xg(x)dx \\
&+\frac{s^3}{3}\int_s^b g(x)dx + \frac{(s-b+\lambda)^4}{12} + \frac{(s-b+\lambda)^3\lambda}{2}\left[\frac{\lambda}{3} + \frac{2}{3}(b-s)\right] \\
&\left.\left.+\frac{2}{3}\lambda(b-\lambda)^3 - s\lambda(b-\lambda)^2\right|^q ds\right).
\end{aligned}
$$

and the constant on the right-hand side of (5.142) *is sharp, while for* $p = 1$ *we have*

$$
\begin{aligned}
&\left|\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt - f'(b-\lambda)\left(\frac{\lambda^2}{2} - b\lambda + \int_a^b xg(x)dx\right)\right. \\
&\left.+\frac{f''(b)}{2}\left(\frac{\lambda^3}{3} - \int_a^b g(x)(x-b+\lambda)^2 dx\right)\right| \leq \|f'''\|_1 \max\{M_1, M_2\} \qquad (5.143)
\end{aligned}
$$

where

$$
\begin{aligned}
M_1 = \max_{s\in[a,b-\lambda]} &\left\{\frac{1}{3}\int_a^s x^3 g(x)dx - s\int_a^s x^2 g(x)dx - \left[(b-\lambda)^2 - 2s(b-\lambda)\right]\int_a^b xg(x)dx\right. \\
&\left.-s^2\int_s^b xg(x)dx - \frac{s^3}{3}\int_a^s g(x)dx - \frac{(b-\lambda-s)^2}{2}\left(\frac{\lambda^2}{2} - \lambda b\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
M_2 = \max_{s\in[b-\lambda,b]} &\left\{\frac{1}{3}\int_a^s x^3 g(x)dx - s\int_a^s x^2 g(x)dx\right. \\
&-\left[(b-\lambda)^2 - 2s(b-\lambda)\right]\int_a^s xg(x)dx - s^2\int_s^b xg(x)dx + \frac{s^3}{3}\int_s^b g(x)dx \\
&+\frac{(s-b+\lambda)^4}{12} + \frac{(s-b+\lambda)^3\lambda}{2}\left[\frac{\lambda}{3} + \frac{2}{3}(b-s)\right] \\
&\left.+\frac{2}{3}\lambda(b-\lambda)^3 - s\lambda(b-\lambda)^2\right\}.
\end{aligned}
$$

and the constant on the right-hand side of (5.143) *is the best possible.*

In this section by $\Omega_i(s)$ we will denote

$$\Omega_i(s) = \int_a^b \mathscr{G}_i(x) g_{AG2}(x,s)dx, \quad i = 1,2. \tag{5.144}$$

We continue by giving the bounds for identities related to obtained generalizations of Steffensen's inequality using some Čebyšev and Grüss type inequalities.

Theorem 5.67 *Let $f \in C^{n+1}[a,b]$ for some $n \geq 3$ with $(\cdot - a)(b - \cdot)[f^{(n+1)}]^2 \in L_1[a,b]$. Let $g,p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$ and let functions $\mathscr{G}_1$ and Ω_1 be defined by* (5.35) *and* (5.144)*. Then*

$$\begin{aligned}
&\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt + \sum_{i=0}^{\alpha} \frac{f^{(i+1)}(a)}{i!} \int_a^b \mathscr{G}_1(x)(x-a)^i dx \\
&+ \sum_{j=0}^{n-\alpha-3} f^{(\alpha+j+2)}(a+\lambda) \left[\sum_{i=0}^{j} \frac{(-\lambda)^{j-i}}{(\alpha+1+i)!(j-i)!} \left(\int_a^b (x-a)^{\alpha+1+i} \mathscr{G}_1(x)dx \right) \right] \\
&+ \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \int_a^b \Omega_1(s)ds = S_n^1(f;a,b),
\end{aligned} \tag{5.145}$$

where the remainder $S_n^1(f;a,b)$ satisfies the estimation

$$\left| S_n^1(f;a,b) \right| \leq \frac{\sqrt{b-a}}{\sqrt{2}} [T(\Omega_1,\Omega_1)]^{\frac{1}{2}} \left| \int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2 ds \right|^{\frac{1}{2}}. \tag{5.146}$$

Proof. Analogous to the proof of Theorem 5.5. After applying Theorem 1.23 for $f \to \Omega_1$ and $h \to f^{(n)}$ we obtain

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b \Omega_1(s) f^{(n)}(s)ds - \frac{1}{b-a} \int_a^b \Omega_1(s)ds \cdot \frac{1}{b-a} \int_a^b f^{(n)}(s)ds \right| \\
&\leq \frac{1}{\sqrt{2}} [T(\Omega_1,\Omega_1)]^{\frac{1}{2}} \frac{1}{\sqrt{b-a}} \left| \int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2 dt \right|^{\frac{1}{2}}.
\end{aligned} \tag{5.147}$$

Now if we add

$$\frac{1}{b-a} \int_a^b \Omega_1(s)ds \int_a^b f^{(n)}(s)ds = \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \int_a^b \Omega_1(s)ds$$

to both sides of identity (5.132) and use inequality (5.147) we obtain representation (5.145) and bound (5.146). □

Similarly, using the identity (5.133) we obtain the following result:

Theorem 5.68 *Let $f \in C^{n+1}[a,b]$ for some $n \geq 3$ with $(\cdot - a)(b - \cdot)[f^{(n+1)}]^2 \in L_1[a,b]$. Let $g, p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let $\int_{b-\lambda}^{b} p(t)dt = \int_a^b g(t)p(t)dt$ and let functions $\mathscr{G}_2$ and Ω_2 be defined by* (5.40) *and* (5.144)*. Then*

$$\begin{aligned}
&\int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^{b} f(t)p(t)dt + \sum_{i=0}^{\alpha} \frac{f^{(i+1)}(b-\lambda)}{i!} \int_a^b \mathscr{G}_2(x)(x-b+\lambda)^i dx \\
&+ \sum_{j=0}^{n-\alpha-3} f^{(\alpha+j+2)}(b) \left(\sum_{i=0}^{j} \frac{(-\lambda)^{j-i}}{(\alpha+1+i)!(j-i)!} \left(\int_a^b (x-b+\lambda)^{\alpha+1+i} \mathscr{G}_2(x)dx \right) \right) \\
&+ \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \int_a^b \Omega_2(s)ds = S_n^2(f;a,b),
\end{aligned} \tag{5.148}$$

where the remainder $S_n^2(f;a,b)$ satisfies the estimation

$$\left| S_n^2(f;a,b) \right| \leq \frac{\sqrt{b-a}}{\sqrt{2}} \left[T(\Omega_2,\Omega_2) \right]^{\frac{1}{2}} \left| \int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2 ds \right|^{\frac{1}{2}}.$$

Proof. Analogous to the previous theorem. □

Using the representation (5.132) and Theorem 1.24 we obtain the following Grüss-type inequalities.

Theorem 5.69 *Let $f \in C^{n+1}[a,b]$ for some $n \geq 3$ and $f^{(n+1)} \geq 0$ on $[a,b]$. Let functions Ω_i, $i = 1,2$ be defined by* (5.144)*.*

(a) Let $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$. Then we have representation (5.145) *and the remainder $S_n^1(f;a,b)$ satisfies the bound*

$$\begin{aligned}
&\left| S_n^1(f;a,b) \right| \\
&\leq (b-a)\|\Omega_1'\|_\infty \left\{ \frac{f^{(n-1)}(b) + f^{(n-1)}(a)}{2} - \frac{f^{(n-2)}(b) - f^{(n-2)}(a)}{b-a} \right\}.
\end{aligned}$$

(b) Let $\int_{b-\lambda}^{b} p(t)dt = \int_a^b g(t)p(t)dt$. Then we have representation (5.148) *and the remainder $S_n^2(f;a,b)$ satisfies the bound*

$$\begin{aligned}
&\left| S_n^2(f;a,b) \right| \\
&\leq (b-a)\|\Omega_2'\|_\infty \left\{ \frac{f^{(n-1)}(b) + f^{(n-1)}(a)}{2} - \frac{f^{(n-2)}(b) - f^{(n-2)}(a)}{b-a} \right\}.
\end{aligned}$$

5.8 *k*-exponential convexity of generalizations of Steffensen's inequality

In this section we generate $k-$exponentially and exponentially convex functions from functionals associated with generalizations of Steffensen's inequality for n-convex functions. This generalizations are given in Sections 5.1, 5.3 and 5.4.

Motivated by inequalities (5.9), (5.11), (5.12) and (5.14), under the assumptions of Theorems 5.3 and 5.4 we define the following linear functionals:

$$L_1(f) = \int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!} \int_a^b G_1(x)(x-a)^i dx, \tag{5.149}$$

$$L_2(f) = \int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!} \int_a^b G_1(x)(x-b)^i dx, \tag{5.150}$$

$$L_3(f) = \int_{b-\lambda}^b f(t)dt - \int_a^b f(t)g(t)dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!} \int_a^b G_2(x)(x-a)^i dx, \tag{5.151}$$

$$L_4(f) = \int_{b-\lambda}^b f(t)dt - \int_a^b f(t)g(t)dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!} \int_a^b G_2(x)(x-b)^i dx. \tag{5.152}$$

Under the assumptions of Theorems 5.3 and 5.4 we have that $L_i(f) \geq 0$, $i = 1,\ldots,4$ for all n-convex functions f.

First we will state and prove mean value theorems related to defined functionals. These results were obtained by Pečarić, Perušić Pribanić and Smoljak Kalamir in [64].

Theorem 5.70 *Let $f : [a,b] \to \mathbb{R}$ be such that $f \in C^n[a,b]$. If the inequalities in* (5.10) $(i = 2)$ *and* (5.13) $(i = 4)$ *hold, then there exist $\xi_i \in [a,b]$ such that*

$$L_i(f) = f^{(n)}(\xi_i)L_i(\varphi), \quad i = 1,\ldots,4$$

where $\varphi(x) = \frac{x^n}{n!}$ and L_i, $i = 1,\ldots,4$ are defined by (5.149) – (5.152).

Proof. Since $f \in C^n[a,b]$ there exist $m = \min_{x\in[a,b]} f^{(n)}(x)$ and $M = \max_{x\in[a,b]} f^{(n)}(x)$. For a given function $f \in C^n[a,b]$ we define the functions $F_1, F_2 : [a,b] \to \mathbb{R}$ with $F_1(x) = M\varphi(x) - f(x)$ and $F_2(x) = f(x) - m\varphi(x)$. Then

$$F_1^{(n)}(x) = M - f^{(n)}(x) \geq 0$$

$$F_2^{(n)}(x) = f^{(n)}(x) - m \geq 0$$

which means that $L_i(F_1), L_i(F_2) \geq 0$, $i = 1,\ldots,4$ i.e.

$$m \cdot L_i(\varphi) \leq L_i(f) \leq M \cdot L_i(\varphi).$$

If $L_i(\varphi) = 0$, the proof is complete. If $L_i(\varphi) > 0$, then

$$m \leq \frac{L_i(f)}{L_i(\varphi)} \leq M, \quad i = 1,\ldots,4$$

and existence of $\xi_i \in [a,b]$, $i = 1,\ldots,4$ now follows. □

Theorem 5.71 *Let $f, \hat{f} : [a,b] \to \mathbb{R}$ be such that $f, \hat{f} \in C^n[a,b]$ and $\hat{f}^{(n)} \neq 0$. If the inequalities in* (5.10) $(i = 2)$ *and* (5.13) $(i = 4)$ *hold, then there exist $\xi_i \in [a,b]$ such that*

$$\frac{L_i(f)}{L_i(\hat{f})} = \frac{f^{(n)}(\xi_i)}{\hat{f}^{(n)}(\xi_i)}, \quad i = 1,\ldots,4 \tag{5.153}$$

where L_i, $i = 1,\ldots,4$ are defined by (5.149) – (5.152).

Proof. We define functions $\phi_i(x) = f(x)L_i(\hat{f}) - \hat{f}(x)L_i(f)$, $i = 1,\ldots,4$. According to Theorem 5.70 there exist $\xi_i \in [a,b]$ such that

$$L_i(\phi_i) = \phi_i^{(n)}(\xi_i)L_i(\varphi), \quad i = 1,\ldots,4.$$

Since $L_i(\phi_i) = 0$ it follows that $f^{(n)}(\xi_i)L_i(\hat{f}) - \hat{f}^{(n)}(\xi_i)L_i(f) = 0$. □

If the inverse of $f^{(n)}/\hat{f}^{(n)}$ exists then the various types of means can be defined by (5.153). That is

$$\xi_i = \left(\frac{f^{(n)}}{\hat{f}^{(n)}}\right)^{-1} \left(\frac{L_i(f)}{L_i(\hat{f})}\right), \quad i = 1,\ldots,4.$$

Now, by using the same idea as in [30] and [60] we generate $k-$exponentially and exponentially convex functions applying above defined functionals.

Theorem 5.72 *Let $\Lambda = \{f_p : p \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$ such that the function $p \mapsto [x_0,\ldots,x_n; f_p]$ is $k-$exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0,\ldots,x_n \in I$. Let L_i, $i = 1,\ldots,4$ be linear functionals defined by* (5.149) – (5.152). *Then $p \mapsto L_i(f_p)$ is $k-$exponentially convex function in the Jensen sense on J.*
If the function $p \mapsto L_i(f_p)$ is continuous on J, then it is $k-$exponentially convex on J.

Proof. For $\xi_j \in \mathbb{R}$ and $p_j \in J$, $j = 1,\ldots,k$, we define the function

$$h(x) = \sum_{j,l=1}^{k} \xi_j \xi_l f_{\frac{p_j+p_l}{2}}(x).$$

Using the assumption that the function $p \mapsto [x_0, \dots, x_n; f_p]$ is k-exponentially convex in the Jensen sense, we have

$$[x_0, \dots, x_n; h] = \sum_{j,l=1}^{k} \xi_j \xi_l [x_0, \dots, x_n; f_{\frac{p_j+p_l}{2}}] \geq 0,$$

which in turn implies that h is n-convex function on J, so $L_i(h) \geq 0$, $i = 1, \dots, 4$. Hence

$$\sum_{j,l=1}^{k} \xi_j \xi_l L_i \left(f_{\frac{p_j+p_l}{2}} \right) \geq 0.$$

We conclude that the function $p \mapsto L_i(f_p)$ is k-exponentially convex on J in the Jensen sense.

If the function $p \mapsto L_i(f_p)$ is also continuous on J, then $p \mapsto L_i(f_p)$ is k-exponentially convex by definition. □

The following corollaries follow directly from above theorem.

Corollary 5.22 *Let $\Lambda = \{f_p : p \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $p \mapsto [x_0, \dots, x_n; f_p]$ is exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0, \dots, x_n \in I$. Let L_i, $i = 1, \dots, 4$, be linear functionals defined by* (5.149) - (5.152)*. Then $p \mapsto L_i(f_p)$ is an exponentially convex function in the Jensen sense on J. If the function $p \mapsto L_i(f_p)$ is continuous on J, then it is exponentially convex on J.*

Corollary 5.23 *Let $\Lambda = \{f_p : p \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $p \mapsto [x_0, \dots, x_n; f_p]$ is 2-exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0, \dots, x_n \in I$. Let L_i, $i = 1, \dots, 4$ be linear functionals defined by* (5.149) - (5.152)*. Then the following statements hold:*

(i) If the function $p \mapsto L_i(f_p)$ is continuous on J, then it is 2-exponentially convex function on J. If $p \mapsto L_i(f_p)$ is additionally strictly positive, then it is also log-convex on J. Furthermore, the following inequality holds true:

$$[L_i(f_s)]^{t-r} \leq [L_i(f_r)]^{t-s} [L_i(f_t)]^{s-r}, \quad i = 1, \dots, 4$$

for every choice $r, s, t \in J$, such that $r < s < t$.

(ii) If the function $p \mapsto L_i(f_p)$ is strictly positive and differentiable on J, then for every $p, q, u, v \in J$, such that $p \leq u$ and $q \leq v$, we have

$$\mu_{p,q}(L_i, \Lambda) \leq \mu_{u,v}(L_i, \Lambda), \tag{5.154}$$

where

$$\mu_{p,q}(L_i, \Lambda) = \begin{cases} \left(\frac{L_i(f_p)}{L_i(f_q)} \right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{\frac{d}{dp} L_i(f_p)}{L_i(f_p)} \right), & p = q, \end{cases} \tag{5.155}$$

for $f_p, f_q \in \Lambda$.

Proof.

(i) This is an immediate consequence of Theorem 5.72 and Remark 1.7.

(ii) Since $p \mapsto L_i(f_p)$ is positive and continuous, by (i) we have that $p \mapsto L_i(f_p)$ is log-convex on J, that is, the function $p \mapsto \log L_i(f_p)$ is convex on J. Hence we get

$$\frac{\log L_i(f_p) - \log L_i(f_q)}{p-q} \leq \frac{\log L_i(f_u) - \log L_i(f_v)}{u-v}, \tag{5.156}$$

for $p \leq u, q \leq v, p \neq q, u \neq v$. So, we conclude that

$$\mu_{p,q}(L_i, \Lambda) \leq \mu_{u,v}(L_i, \Lambda).$$

Cases $p = q$ and $u = v$ follow from (5.156) as limit cases.

□

In the sequel we give some families of functions for which we use Corollaries 5.22 and 5.23 to construct exponentially convex functions and Stolarsky type means.

Example 5.1 Let us consider a family of functions

$$\Lambda_1 = \{f_p : \mathbb{R} \to \mathbb{R} : p \in \mathbb{R}\}$$

defined by

$$f_p(x) = \begin{cases} \frac{e^{px}}{p^n}, & p \neq 0, \\ \frac{x^n}{n!}, & p = 0. \end{cases}$$

Since $\frac{d^n f_p}{dx^n}(x) = e^{px} > 0$, the function f_p is n-convex on $\mathbb{R}$ for every $p \in \mathbb{R}$ and $p \mapsto \frac{d^n f_p}{dx^n}(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 5.72 we also have that $p \mapsto [x_0, \ldots, x_n; f_p]$ is exponentially convex (and so exponentially convex in the Jensen sense). Now, using Corollary 5.22 we conclude that $p \mapsto L_i(f_p), i = 1, \ldots, 4$, are exponentially convex in the Jensen sense. It is easy to verify that this mappings are continuous (although the mapping $p \mapsto f_p$ is not continuous for $p = 0$), so they are exponentially convex. For this family of functions, $\mu_{p,q}(L_i, \Lambda_1)$, $i = 1, \ldots, 4$, from (5.155), becomes

$$\mu_{p,q}(L_i, \Lambda_1) = \begin{cases} \left(\frac{L_i(f_p)}{L_i(f_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{L_i(id \cdot f_p)}{L_i(f_p)} - \frac{n}{p}\right), & p = q \neq 0, \\ \exp\left(\frac{1}{n+1}\frac{L_i(id \cdot f_0)}{L_i(f_0)}\right), & p = q = 0, \end{cases}$$

where id is the identity function. By Corollary 5.23 $\mu_{p,q}(L_i, \Lambda_1)$, $i = 1, \ldots, 4$ are monotonic functions in parameters p and q.

Since

$$\left(\frac{\frac{d^n f_p}{dx^n}}{\frac{d^n f_q}{dx^n}}\right)^{\frac{1}{p-q}} (\log x) = x,$$

using Theorem 5.71 it follows that:

$$M_{p,q}(L_i,\Lambda_1) = \log \mu_{p,q}(L_i,\Lambda_1), \quad i=1,\dots,4$$

satisfy

$$a \le M_{p,q}(L_i,\Lambda_1) \le b, \quad i=1,\dots,4.$$

So, $M_{p,q}(L_i,\Lambda_1)$, $i=1,\dots,4$ are monotonic means.

Example 5.2 Let us consider a family of functions

$$\Lambda_2 = \{g_p : (0,\infty) \to \mathbb{R} : p \in \mathbb{R}\}$$

defined by

$$g_p(x) = \begin{cases} \frac{x^p}{p(p-1)\cdots(p-n+1)}, & p \notin \{0,1,\dots,n-1\}, \\ \frac{x^j \log x}{(-1)^{n-1-j} j!(n-1-j)!}, & p = j \in \{0,1,\dots,n-1\}. \end{cases}$$

Since $\frac{d^n g_p}{dx^n}(x) = x^{p-n} > 0$, the function g_p is n-convex for $x > 0$ and $p \mapsto \frac{d^n g_p}{dx^n}(x)$ is exponentially convex by definition. Arguing as in Example 5.1 we get that the mappings $p \mapsto L_i(g_p), i=1,\dots,4$ are exponentially convex. Hence, for this family of functions $\mu_{p,q}(L_i,\Lambda_2)$, $i=1,\dots,4$, from (5.155), is equal to

$$\mu_{p,q}(L_i,\Lambda_2) = \begin{cases} \left(\frac{L_i(g_p)}{L_i(g_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left((-1)^{n-1}(n-1)!\frac{L_i(g_0 g_p)}{L_i(g_p)} + \sum\limits_{k=0}^{n-1}\frac{1}{k-p}\right), & p = q \notin \{0,1,\dots,n-1\}, \\ \exp\left((-1)^{n-1}(n-1)!\frac{L_i(g_0 g_p)}{2L_i(g_p)} + \sum\limits_{\substack{k=0 \\ k\neq p}}^{n-1}\frac{1}{k-p}\right), & p = q \in \{0,1,\dots,n-1\}. \end{cases}$$

Again, using Theorem 5.71 we conclude that

$$a \le \left(\frac{L_i(g_p)}{L_i(g_q)}\right)^{\frac{1}{p-q}} \le b, \quad i=1,\dots,4.$$

So, $\mu_{p,q}(L_i,\Lambda_2)$, $i=1,\dots,4$ are means and by (5.154) they are monotonic.

Similarly, we can generate $k-$exponentially and exponentially convex functions from functionals related to generalized Steffensen's inequality given in Sections 5.3 and 5.4. Motivated by inequalities (5.60), (5.62), (5.64) and (5.66) we can define the following functionals:

$$\begin{aligned} A_1(f) &= \int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt \\ &+ \sum_{k=1}^{n} \frac{(b-a)^{k-2}}{(k-1)!}\left(\int_a^b G_1(x)B_{k-1}\left(\frac{x-a}{b-a}\right)dx\right)[f^{(k-1)}(b) - f^{(k-1)}(a)] \end{aligned}$$

$$A_2(f) = \int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt$$
$$+ \sum_{k=1}^{n-1} \frac{(b-a)^{k-2}}{(k-1)!} \left(\int_a^b G_1(x) B_{k-1}\left(\frac{x-a}{b-a}\right) dx \right) [f^{(k-1)}(b) - f^{(k-1)}(a)]$$

$$A_3(f) = \int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt$$
$$+ \sum_{k=1}^{n} \frac{(b-a)^{k-2}}{(k-1)!} \left(\int_a^b G_2(x) B_{k-1}\left(\frac{x-a}{b-a}\right) dx \right) [f^{(k-1)}(b) - f^{(k-1)}(a)]$$

$$A_4(f) = \int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt$$
$$+ \sum_{k=1}^{n-1} \frac{(b-a)^{k-2}}{(k-1)!} \left(\int_a^b G_2(x) B_{k-1}\left(\frac{x-a}{b-a}\right) dx \right) [f^{(k-1)}(b) - f^{(k-1)}(a)].$$

Motivated by inequalities (5.85),(5.87) and under the assumptions of Theorems 5.31 and 5.32, respectively, we can define functionals:

$$L_1(f) = \int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt - \sum_{k=0}^{n-2} \int_a^b T_k(x)\mathscr{G}_1(x)dx,$$

$$L_2(f) = \int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^b f(t)p(t)dt - \sum_{k=0}^{n-2} \int_a^b T_k(x)\mathscr{G}_2(x)dx.$$

Furthermore, we can generate means from defined functionals. For more details we refer the reader to [63, 69].

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