Preface

Despite of the title, the main motivation for writing this book (and the papers from which this book has grown) was presenting some aspects of generalizations, refinements, variants of three famous inequalities (actually four). The first inequality is the Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2},$$

which holds for a convex function f on $[a,b] \subset \mathbb{R}$. The second inequality is the Ostrowski inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{1}{(b-a)^2}\left(x - \frac{a+b}{2}\right)^2\right](b-a)L,$$

which holds for a $L-$Lipschitzian function f on $[a,b] \subset \mathbb{R}$, and the third one is the Iyengar inequality

$$\left| \frac{1}{b-a}\int_a^b f(x)dx - \frac{f(a)+f(b)}{2} \right| \leq \left[1 - \left(\frac{f(b)-f(a)}{L(b-a)}\right)^2\right]\frac{b-a}{4}L,$$

which also holds for a $L-$Lipschitzian function f on $[a,b] \subset \mathbb{R}$.

Generalizations of the Ostrowski inequality are mainly given in Chapter 1, but related results are scattered throughout the book (especially for the case $x = (a+b)/2$). Variants of the Hermite-Hadamard inequality are given for some pairs of quadrature formulae (called dual formulae) and refinements are given in the sense of the Bullen inequalities for higher convex functions. The basic Bullen inequality

$$0 \leq \frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx$$

holds for a convex function f on $[a,b] \subset \mathbb{R}$.

The book contains generalizations of many classical quadrature formulae such as Simpson, dual Simpson, Maclaurin, Gauss, Lobatto, Radau. Standard methods in deducing these formulae are very different, spanning from Lagrange, Newton interpolation polynomials to orthogonal polynomials such as Legendre, Chebyshev, Jacobi. The specific

feature of this book (regarded nevertheless as a book in numerical integration) is that the unique method is used. This method is based on the, so called, *Euler integral identities* expressing expansion of a function in Bernoulli polynomials proved by V. I. Krylov in [79] as a generalization of the first and the second Euler-Maclaurin sum formula (for details see Chapter 1). The Iyengar inequality is the exception. Although related to the Hermite-Hadamard inequality in the same way as the Ostrowski inequality, Iyengar type inequalities are, it seems, beyond the reach of methods based on the Euler integral identities. This is the reason why generalizations of the Iyengar inequality are given in the Addendum.