Chapter 1

Definitions and basic results

1.1 Convex functions

Convex functions are very important in the theory of inequalities. The third chapter of the classical book of Hardy, Littlewood and Pólya [51] is devoted to the theory of convex functions (see also [82]). In this section we give some of the results concerning convex functions.

Definition 1.1 *Let I be an interval in $\mathbb{R}$. A function $\Phi\colon I \to \mathbb{R}$ is called* convex *if*

$$\Phi(\lambda x + (1-\lambda)y) \leq \lambda\Phi(x) + (1-\lambda)\Phi(y) \tag{1.1}$$

for all points $x, y \in I$ and all $\lambda \in [0,1]$. It is called strictly convex *if the inequality* (1.1) *holds strictly whenever x and y are distinct points and $\lambda \in (0,1)$.*

If $-\Phi$ is convex (respectively, strictly convex) then we say that Φ is concave *(respectively,* strictly concave*). If Φ is both convex and concave, Φ is said to be* affine.

Remark 1.1 (a) For $x, y \in I$, $p, q \geq 0$, $p+q > 0$, (1.1) is equivalent to

$$\Phi\left(\frac{px+qy}{p+q}\right) \leq \frac{p}{p+q}\Phi(x) + \frac{q}{p+q}\Phi(y).$$

(b) The simple geometrical interpretation of (1.1) is that the graph of Φ lies below its chords.

(c) If x_1, x_2, x_3 are three points in I such that $x_1 < x_2 < x_3$, then (1.1) is equivalent to

$$\begin{vmatrix} x_1 & \Phi(x_1) & 1 \\ x_2 & \Phi(x_2) & 1 \\ x_3 & \Phi(x_3) & 1 \end{vmatrix} = (x_3 - x_2)\Phi(x_1) + (x_1 - x_3)\Phi(x_2) + (x_2 - x_1)\Phi(x_3) \geq 0$$

which is equivalent to

$$\Phi(x_2) \leq \frac{x_2 - x_3}{x_1 - x_3}\Phi(x_1) + \frac{x_1 - x_2}{x_1 - x_3}\Phi(x_3),$$

or, more symmetrically and without the condition of monotonicity on x_1, x_2, x_3

$$\frac{\Phi(x_1)}{(x_1 - x_2)(x_1 - x_3)} + \frac{\Phi(x_2)}{(x_2 - x_3)(x_2 - x_1)} + \frac{\Phi(x_3)}{(x_3 - x_1)(x_3 - x_2)} \geq 0$$

Definition 1.2 *Let I be an interval in $\mathbb{R}$. A function $\Phi : I \to \mathbb{R}$ is called* convex in the Jensen sense*, or* J-convex on I (midconvex, midpoint convex) *if for all points $x, y \in I$ the inequality*

$$\Phi\left(\frac{x+y}{2}\right) \leq \frac{\Phi(x) + \Phi(y)}{2} \tag{1.2}$$

holds. A J-convex function is said to be strictly J-convex *if for all pairs of points $(x, y), x \neq y$, strict inequality holds in* (1.2).

In the context of continuity the following criteria of equivalence of (1.1) and (1.2) is valid.

Theorem 1.1 *Let $\Phi : I \to \mathbb{R}$ be a continuous function. Then Φ is a convex function if and only if Φ is a J-convex function.*

Inequality (1.1) can be extended to the convex combinations of finitely many points in I by mathematical induction. These extensions are known as discrete and integral Jensen's inequality.

Theorem 1.2 (THE DISCRETE CASE OF JENSEN'S INEQUALITY) *A function $\Phi : I \to \mathbb{R}$ is convex if and only if for all $x_1, ..., x_n \in I$ and all scalars $p_1, ..., p_n \in [0, 1]$ with $P_n = \sum_1^n p_i$ we have*

$$\Phi\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i \Phi(x_i). \tag{1.3}$$

Inequality (1.3) *is strict if Φ is a strictly convex function, all points x_i, $i = 1, ..., n$, $n \in \mathbb{N}$ are disjoint and all scalars p_i are positive.*

Now, we introduce some necessary notation and recall some basic facts about convex functions, log-convex functions (see e.g. [65], [82], [92]) as well as exponentially convex functions (see e.g [15], [79], [81]).

In 1929, S. N. Bernstein introduced the notion of exponentially convex function in [15]. Later on D.V. Widder in [100] introduced these functions as a sub-class of convex function in a given interval (a, b) (for details see [100], [101]).

Definition 1.3 *A positive function Φ is said to be* logarithmically convex *on an interval $I \subseteq \mathbb{R}$ if $\log \Phi$ is a convex function on I, or equivalently if for all $x, y \in I$ and all $\alpha \in [0,1]$*

$$\Phi(\alpha x + (1-\alpha)y) \leq \Phi^{\alpha}(x)\Phi^{1-\alpha}\Phi(y).$$

For such function Φ, we shortly say Φ is log-convex. A positive function Φ is log-convex in the Jensen sense *if for each $x, y \in I$*

$$\Phi^2\left(\frac{x+y}{2}\right) \leq \Phi(x)\Phi(y)$$

holds, i.e., if $\log \Phi$ is convex in the Jensen sense.

Remark 1.2 A function Φ is log-convex on an interval I, if and only if for all $x_1, x_2, x_3 \in I$, $x_1 < x_2 < x_3$, it holds

$$[\Phi(x_2)]^{x_3-x_1} \leq [\Phi(x_1)]^{x_3-x_2}[\Phi(x_3)]^{x_2-x_1}. \tag{1.4}$$

Furthermore, if $x_1, x_2, y_1, y_2 \in I$ are such that $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then

$$\left(\frac{\Phi(x_2)}{\Phi(x_1)}\right)^{\frac{1}{x_2-x_1}} \leq \left(\frac{\Phi(y_2)}{\Phi(y_1)}\right)^{\frac{1}{y_2-y_1}}. \tag{1.5}$$

Inequality (1.5) is known as Galvani's theorem for log-convex functions $\Phi\colon I \to \mathbb{R}$.

We continue with the definition of exponentially convex function as originally given in [15] by Berstein (see also [9], [79], [81]).

Definition 1.4 *A function $\Phi\colon (a,b) \to \mathbb{R}$ is* exponentially convex *if it is continuous and*

$$\sum_{i,j=1}^{n} t_i t_j \Phi(x_i + x_j) \geq 0, \tag{1.6}$$

holds for every $n \in \mathbb{N}$ and all sequences $(t_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ of real numbers, such that $x_i + x_j \in (a,b)$, $1 \leq i, j \leq n$.

Moreover, the condition (1.6) can be replaced with a more suitable condition

$$\sum_{i,j=1}^{n} t_i t_j \Phi\left(\frac{x_i + x_j}{2}\right) \geq 0, \tag{1.7}$$

which has to hold for all $n \in \mathbb{N}$, all sequences $(t_n)_{n\in\mathbb{N}}$ of real numbers, and all sequences $(x_n)_{n\in\mathbb{N}}$ in (a,b). More precisely, a function $\Phi\colon (a,b) \to \mathbb{R}$ is exponentially convex if and only if it is continuous and fulfils (1.7). Condition (1.7) means that the matrix $\left[\Phi(\frac{x_i+x_j}{2})\right]_{i,j=1}^{n}$ is positive semi-definite matrix. Hence, its determinant must be non-negative. For $n = 2$ this means that it holds

$$\Phi(x_1)\Phi(x_2) - \Phi^2\left(\frac{x_1+x_2}{2}\right) \geq 0,$$

hence, exponentially convex function is log-convex in the Jensen sense, and, being continuous, it is also log-convex function.

We continue with the definition of n-exponentially convex function.

Definition 1.5 *A function $\Phi : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if*

$$\sum_{i,j=1}^{n} t_i t_j \Phi\left(\frac{x_i + x_j}{2}\right) \geq 0$$

holds for all choices of $t_i \in \mathbb{R}$, $x_i \in I$, $i = 1, \dots, n$.
A function $\Phi : I \to \mathbb{R}$ is n-exponentially convex on I if it is n-exponentially convex in the Jensen sense and continuous on I.

Remark 1.3 *It is clear from the definition that 1-exponentially convex functions in the Jensen sense are in fact non-negative functions. Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $k \leq n$.*

Proposition 1.1 *Let I be an open interval in $\mathbb{R}$. If Φ is n-exponentially convex in the Jensen sense on J then the matrix $\left[\Phi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k}$ is positive semi-definite matrix for all $k \in \mathbb{N}$, $k \leq n$. Particularly*

$$\det\left[\Phi\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{k} \geq 0, \; for\, all\; k \in \mathbb{N},\; k \leq n.$$

Definition 1.6 *Let I be an open interval in $\mathbb{R}$. A function $\Phi : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense on I for all $n \in \mathbb{N}$.*

Remark 1.4 *It follows that a function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense.*
Also, using basic convexity theory it follows that a function is log-convex if and only if it is 2-exponentially convex.

It is easily seen that a convex function is continuous on the interior of its domain, but it may not be continuous at the boundary points of the domain.

Theorem 1.3 *If $\Phi : I \to \mathbb{R}$ is a convex function, then Φ satisfies the Lipschitz condition on any closed interval $[a,b]$ contained in the interior of I, that is, there exists a constant K so that for any two points $x, y \in [a,b]$,*

$$|\Phi(x) - \Phi(y)| \leq K|x - y|.$$

Now, we continue with derivative of a convex function. The derivative of a convex function is best studied in terms of the left and right derivatives defined by

$$\Phi'_{-}(x) = \lim_{y \nearrow x} \frac{\Phi(y) - \Phi(x)}{y - x}, \; \Phi'_{+}(x) = \lim_{y \searrow x} \frac{\Phi(y) - \Phi(x)}{y - x}.$$

The following result concerning the left and the right derivative of a convex function can be seen e.g. in [92].

Theorem 1.4 *Let I be an interval in $\mathbb{R}$ and $\Phi\colon I \to \mathbb{R}$ be convex. Then*

(i) *$\Phi_-^{'}$ and $\Phi_+^{'}$ exist and are increasing on I, and $\Phi_-^{'} \leq \Phi_+^{'}$ (if Φ is strictly convex, then these derivatives are strictly increasing);*

(ii) *$\Phi^{'}$ exists, except possibly on a countable set, and on the complement of which it is continuous.*

Theorem 1.5 (a) *$\Phi\colon [a,b] \to \mathbb{R}$ is (strictly) convex if there exists an (strictly) increasing function $f\colon [a,b] \to \mathbb{R}$ and a real number c $(a < c < b)$ such that for all x and $a < x < b$,*

$$\Phi(x) = \Phi(c) + \int_c^x f(t)dt.$$

(b) *If Φ is differentiable, then Φ is (strictly) convex if $\Phi^{'}$ is (strictly) increasing.*

(c) *If $\Phi^{''}$ exists on (a,b), then Φ is convex if $\Phi^{''}(x) \geq 0$. If $\Phi^{''}(x) > 0$, then Φ is strictly convex.*

Example 1.1 (a) The exponential function $\Phi : \mathbb{R} \to \mathbb{R}$, $\Phi(x) = e^x$ is a strictly convex function.

(b) Let $\Phi : \mathbb{R}_+ \to \mathbb{R}$ be defined by $\Phi(x) = x^p$, $p \in \mathbb{R} \setminus \{0\}$. Obviously, $\Phi'(x) = px^{p-1}$ and the function Φ is convex for $p \in \mathbb{R} \setminus [0,1)$, concave for $p \in (0,1]$, and affine for $p = 1$.

Remark 1.5 Let I be an open interval and let $h \in C^2(I)$ be such that h'' is bounded, that is, $m \leq h'' \leq M$. Then the functions Φ_1, Φ_2 defined by

$$\Phi_1(t) = \frac{M}{2}t^2 - h(t), \qquad \Phi_2(t) = h(t) - \frac{m}{2}t^2$$

are convex.

The geometric characterization depends upon the idea of a support line. The following result can be seen e.g. in [92].

Theorem 1.6 (a) *$\Phi\colon (a,b) \to \mathbb{R}$ is convex if there is at least one line of support for Φ at each $x_0 \in (a,b)$, i.e.,*

$$\Phi(x) \geq \Phi(x_0) + \lambda(x - x_0), \forall x \in (a,b),$$

where λ depends on x_0 and is given by $\lambda = \Phi^{'}(x_0)$ when $\Phi^{'}$ exists, and $\lambda \in [\Phi_-^{'}(x_0), \Phi_+^{'}(x_0)]$ when $\Phi_-^{'}(x_0) \neq \Phi_+^{'}(x_0)$.

(b) *$\Phi\colon (a,b) \to \mathbb{R}$ is convex if the function $\Phi(x) - \Phi(x_0) - \lambda(x - x_0)$ (the difference between the function and its support) is decreasing for $x < x_0$ and increasing for $x > x_0$.*

Definition 1.7 *Let $\Phi : I \longrightarrow \mathbb{R}$ be a convex function, then the sub-differential of Φ at x, denoted by $\partial\Phi(x)$, is defined as*

$$\partial\Phi(x) = \{\alpha \in \mathbb{R} : \Phi(y) - \Phi(x) - \alpha(y-x) \geq 0, y \in I\}.$$

There is a connection between a convex function and its sub-differential. It is well-known that $\partial\Phi(x) \neq \emptyset$ for all $x \in \mathrm{Int}I$. More precisely, at each point $x \in \mathrm{Int}I$ we have $-\infty < \Phi'_-(x) \leq \Phi'_+(x) < \infty$ and

$$\partial\Phi(x) = [\Phi'_-(x), \Phi'_+(x)],$$

while the set on which Φ is not differentiable is at most countable. Moreover, each function $\varphi : I \longrightarrow \mathbb{R}$ such that $\varphi(x) \in \partial\Phi(x)$, whenever $x \in \mathrm{Int}I$, is increasing on $\mathrm{Int}I$. For any such function φ and arbitrary $x \in \mathrm{Int}I$, $y \in I$ we have

$$\Phi(y) - \Phi(x) - \varphi(x)(y-x) \geq 0$$

and further

$$\begin{aligned} \Phi(y) - \Phi(x) - \varphi(x)(y-x) &= |\Phi(y) - \Phi(x) - \varphi(x)(y-x)| \\ &\geq ||\Phi(y) - \Phi(x)| - |\varphi(x)| \cdot |y-x||\,. \end{aligned} \tag{1.8}$$

On the other hand, if $\Phi : I \to \mathbb{R}$ is a concave function, that is, $-\Phi$ is convex, then $\partial\Phi(x) = \{\alpha \in \mathbb{R} : \Phi(x) - \Phi(y) - \alpha(x-y) \geq 0, y \in I\}$ denotes the superdifferential of Φ at the point $x \in I$. For all $x \in \mathrm{Int}I$, in this setting we have $-\infty < \Phi'_+(x) \leq \Phi'_-(x) < \infty$ and $\partial\Phi(x) = [\Phi'_+(x), \Phi'_-(x)] \neq \emptyset$. Hence, the inequality

$$\Phi(x) - \Phi(y) - \varphi(x)(x-y) \geq 0$$

holds for all $x \in \mathrm{Int}I$, $y \in I$, and all real functions φ on I, such that $\varphi(z) \in \partial\Phi(z)$, $z \in \mathrm{Int}I$. Finally, we get

$$\begin{aligned} \Phi(x) - \Phi(y) - \varphi(x)(x-y) &= |\Phi(x) - \Phi(y) - \varphi(x)(x-y)| \\ &\geq ||\Phi(y) - \Phi(x)| - |\varphi(x)| \cdot |y-x||\,. \end{aligned} \tag{1.9}$$

Note that, although the symbol $\partial\Phi(x)$ has two different notions, it will be clear from the context whether it applies to a convex or to a concave function Φ. Many further information on convex and concave functions can be found e.g. in the monographs [82] and [92] and in references cited therein.

1.2 Superquadratic and subquadratic functions

The concept of superquadratic and subquadratic functions is introduced by Abramovich, Jameson and Sinnamon in [4] (see also [3]).

Definition 1.8 *(See* [4, Definition 2.1].*) A function* $\varphi : [0,\infty) \to \mathbb{R}$ *is superquadratic provided that for all* $x \geq 0$ *there exists a constant* $C_x \in \mathbb{R}$ *such that*

$$\varphi(y) - \varphi(x) - \varphi(|y-x|) \geq C_x\,(y-x)\,, \tag{1.10}$$

for all $y \geq 0$. *We say that* φ *is subquadratic if* $-\varphi$ *is superquadratic. We say that* φ *is a strictly superquadratic function if for* $x \neq y, x, y \neq 0$ *there is strict inequality in* (1.10). *We say that* φ *is a strictly subquadratic function if* $-\varphi$ *is a strictly superquadratic function.*

Lemma 1.1 *(See* [4, Theorem 2.3].) Let (Ω,μ) be a probability measure space. The inequality

$$\varphi\left(\int_\Omega f(s)d\mu(s)\right) \leq \int_\Omega \varphi(f(s))d\mu(s) - \int_\Omega \varphi\left(\left|f(s) - \int_\Omega f(s)d\mu(s)\right|\right) d\mu(s) \tag{1.11}$$

holds for all probability measures μ and all non-negative $\mu-$integrable functions f if and only if φ is superquadratic. Moreover, (1.11) holds in the reversed direction if and only if φ is subquadratic.

Proof. See [4] and [3] for the details. □

Definition 1.9 *A function* $f : [0,\infty) \to \mathbb{R}$ *is superadditive provided* $f(x+y) \geq f(x) + f(y)$ *for all* $x,\, y \geq 0$. *If the reverse inequality holds, then* f *is said to be subadditive.*

Lemma 1.2 *(See* [4, Lemma 3.1].) Suppose $\varphi : [0,\infty) \to \mathbb{R}$ is continuously differentiable and $\varphi(0) \leq 0$. If φ' is superadditive or $\frac{\varphi'(x)}{x}$ is nondecreasing, then φ is superquadratic.

Proof. See [4] for details. □

Remark 1.6 By Lemma 1.2, the function $\varphi(x) = x^p$ is superquadratic for $p \geq 2$ and subquadratic for $1 < p \leq 2$. Therefore, by Lemma 1.1, for $p \geq 2$ the inequality

$$\left(\int_\Omega f(s)d\mu(s)\right)^p \leq \int_\Omega f^p(s)d\mu(s) - \int_\Omega \left|f(s) - \int_\Omega f(s)d\mu(s)\right|^p d\mu(s)$$

holds and the reversed inequality holds when $1 < p \leq 2$ (see also [2, Example 1, p. 1448]).

1.3 Operator convex functions

We shall first recall the definition of an operator convex function.

Definition 1.10 *Let I be a real interval of any type. A continuous function $f : I \to \mathbb{R}$ is said to be operator convex if*

$$f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y)$$

holds for each $\lambda \in [0,1]$ and every pair of self-adjoint x and y (acting) on an infinite dimensional Hilbert space H with spectra in I (the ordering is defined by setting $x \leq y$ if $y-x$ is positive semi-definite).

Let f be an operator convex function defined on an interval I. Ch. Davis [34] proved a Schwartz type inequality

$$f(\Phi(x)) \leq \Phi(f(x)),$$

where $\Phi : A \to B(H)$ is a unital completely positive linear map from a C^*-algebra A to linear operators on a Hilbert space H and x is a self-adjoint element in A with spectrum in I.

Let us recall the definition of a unital field. Assume that there is a field $(\Phi_t)_{t\in T}$ of positive linear mappings $\Phi_t : A \to B$ from A to another C^*-algebra B. We say that such a field is continuous if the function $t \to \Phi_t(x)$ is continuous for every $x \in A$. If the C^*-algebras are unital and the field $t \to \Phi_t(\mathbf{1})$ is integrable with integral $\mathbf{1}$, that is $\int_T \Phi_t(\mathbf{1})d\mu(t) = \mathbf{1}$, we say that $(\Phi_t)_{t\in T}$ is unital.

In particular, F. Hansen et al. [46] proved the following result:

Theorem 1.7 *Let $f : I \to \mathbb{R}$ be an operator convex function defined on an interval I, and let A and B be unital C^*-algebras. If $(\Phi_t)_{t\in T}$ is a unital field of positive linear mappings $\Phi_t : A \to B$ defined on a locally compact space T with a bounded positive Radon measure μ, then the Jensen type inequality*

$$f\left(\int_T \Phi_t(x_t))d\mu(t)\right) \leq \int_T \Phi_t(f(x_t))d\mu(t)$$

holds for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in A with spectra contained in I.

1.4 Fractional integrals and fractional derivatives

First, let us recall some facts about fractional derivatives needed in the sequel, for more details see e.g. [9], [43].

Let $0 < a < b \leq \infty$. By $C^m([a,b])$ we denote the space of all functions on $[a,b]$ which have continuous derivatives up to order m, and $AC([a,b])$ is the space of all absolutely continuous functions on $[a,b]$. By $AC^m([a,b])$ we denote the space of all functions $g \in C^{m-1}([a,b])$ with $g^{(m-1)} \in AC([a,b])$. For any $\alpha \in \mathbb{R}$ we denote by $[\alpha]$ the integral part of α (the integer k satisfying $k \leq \alpha < k+1$) and $\lceil \alpha \rceil$ is the ceiling of α ($\min\{n \in \mathbb{N}, n \geq \alpha\}$). By $L_1(a,b)$ we denote the space of all functions integrable on the interval (a,b), and by $L_\infty(a,b)$ the set of all functions measurable and essentially bounded on (a,b). Clearly, $L_\infty(a,b) \subset L_1(a,b)$.

Now, we give well known definitions of the Riemann-Liouville fractional integrals, see [67]. Let $[a,b]$ be a finite interval on real axis $\mathbb{R}$. The Riemann-Liouville fractional integrals $I^\alpha_{a+} f$ and $I^\alpha_{b-} f$ of order $\alpha > 0$ are defined by

$$I^\alpha_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(y)(x-y)^{\alpha-1} dy, \ \ (x > a)$$

and

$$I^\alpha_{b-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b f(y)(y-x)^{\alpha-1} dy, \ \ (x < b)$$

respectively. Here $\Gamma(\alpha)$ is the Gamma function. These integrals are called the right-sided and left-sided fractional integrals. Some recent results involving Riemann-Liouville fractional integrals can be seen in e.g [10], [11], [61] and [63]. We denote some properties of the operators $I^\alpha_{a_+} f$ and $I^\alpha_{b_-} f$ of order $\alpha > 0$, see also [96]. The first result yields that the fractional integral operators $I^\alpha_{a_+} f$ and $I^\alpha_{b_-} f$ are bounded in $L_p(a,b)$, $1 \leq p \leq \infty$, that is

$$\|I^\alpha_{a_+} f\|_p \leq K\|f\|_p, \quad \|I^\alpha_{b_-} f\|_p \leq K\|f\|_p, \tag{1.12}$$

where

$$K = \frac{(b-a)^\alpha}{\alpha\Gamma(\alpha)}.$$

Inequality (1.12), that is the result involving the left-sided fractional integral, was proved by G. H. Hardy in one of his first papers, see [49]. He did not write down the constant, but the calculation of the constant was hidden inside his proof. Inequality (1.12) is referred to as inequality of G. H. Hardy.

Next we give result with respect to the generalized Riemann-Liouville fractional derivative. Let us recall the definition.

Let $\alpha > 0$ and $n = [\alpha] + 1$ where $[\cdot]$ is the integral part. We define the generalized Riemann-Liouville fractional derivative of f of order α by

$$(D_a^\alpha f)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-y)^{n-\alpha-1} f(y)\, dy .$$

In addition, we stipulate

$$D_a^0 f := f =: I_a^0 f, \quad I_a^{-\alpha} f := D_a^\alpha f \text{ if } \alpha > 0.$$

If $\alpha \in \mathbb{N}$ then $D_a^\alpha f = \frac{d^\alpha f}{dx^\alpha}$, the ordinary α-order derivative.

The space $I_a^\alpha(L(a,b))$ is defined as the set of all functions f on $[a,b]$ of the form $f = I_a^\alpha \varphi$ for some $\varphi \in L(a,b)$, [96, Chapter 1, Definition 2.3]. According to Theorem 2.3 in [96, p. 43], the latter characterization is equivalent to the condition

$$I_a^{n-\alpha} f \in AC^n[a,b], \tag{1.13}$$

$$\frac{d^j}{dx^j} I_a^{n-\alpha} f(a) = 0, \quad j = 0, 1, \dots, n-1 .$$

A function $f \in L(a,b)$ satisfying (1.13) is said to have an integrable fractional derivative $D_a^\alpha f$, [96, Chapter1, Definition 2.4].

The following lemma summarizes conditions in identity for generalized Riemann-Liouville fractional derivative.

Lemma 1.3 *Let $\beta > \alpha \geq 0$, $n = [\beta] + 1$, $m = [\alpha] + 1$. Identity*

$$D_a^\alpha f(x) = \frac{1}{\Gamma(\beta-\alpha)} \int_a^x (x-y)^{\beta-\alpha-1} D_a^\beta f(y)\, dy, \quad x \in [a,b] .$$

is valid if one of the following conditions holds:

(i) $f \in I_a^\beta(L(a,b))$.

(ii) $I_a^{n-\beta} f \in AC^n[a,b]$ *and* $D_a^{\beta-k} f(a) = 0$ *for* $k = 1, \dots n$.

(iii) $D_a^{\beta-k} f \in C[a,b]$ *for* $k = 1, \dots, n$, $D_a^{\beta-1} f \in AC[a,b]$ *and* $D_a^{\beta-k} f(a) = 0$ *for* $k = 1, \dots n$.

(iv) $f \in AC^n[a,b]$, $D_a^\beta f \in L(a,b)$, $D_a^\alpha f \in L(a,b)$, $\beta - \alpha \notin \mathbb{N}$, $D_a^{\beta-k} f(a) = 0$ *for* $k = 1, \dots, n$ *and* $D_a^{\alpha-k} f(a) = 0$ *for* $k = 1, \dots, m$.

(v) $f \in AC^n[a,b]$, $D_a^\beta f \in L(a,b)$, $D_a^\alpha f \in L(a,b)$, $\beta - \alpha = l \in \mathbb{N}$, $D_a^{\beta-k} f(a) = 0$ *for* $k = 1, \dots, l$.

(vi) $f \in AC^n[a,b]$, $D_a^\beta f \in L(a,b)$, $D_a^\alpha f \in L(a,b)$ *and* $f(a) = f'(a) = \dots = f^{(n-2)}(a) = 0$.

(vii) $f \in AC^n[a,b]$, $D_a^\beta f \in L(a,b)$, $D_a^\alpha f \in L(a,b)$, $\beta \notin \mathbb{N}$ *and* $D_a^{\beta-1} f$ *is bounded in a neighborhood of* $t = a$.

The definition of Canavati-type fractional derivative is given in [9] but we will use the Canavati-type fractional derivative given in [13] with some new conditions. Now we define Canavati-type fractional derivative ($\nu-$fractional derivative of f). We consider

$$C^{\nu}([0,1]) = \{f \in C^{n}([0,1]) : I_{1-\overline{\nu}} f^{(n)} \in C^{1}([0,1])\},$$

$\nu > 0$, $n = [\nu]$, $[.]$ is the integral part, and $\overline{\nu} = \nu - n, 0 \leq \overline{\nu} < 1$.
For $f \in C^{\nu}([0,1])$, the Canavati-ν fractional derivative of f is defined by

$$D^{\nu} f = D I_{1-\overline{\nu}} f^{(n)},$$

where $D = d/dx$.

Lemma 1.4 *Let $\nu > \gamma \geq 0$, $n = [\nu]$, $m = [\gamma]$. Let $f \in C^{\nu}([0,1])$, be such that $f^{(i)}(0) = 0$, $i = m, m+1, \ldots, n-1$. Then*

(*i*) $f \in C^{\gamma}([0,1])$

(*ii*) $(D^{\gamma} f)(x) = \frac{1}{\Gamma(\nu-\gamma)} \int\limits_{0}^{x} (x-t)^{\nu-\gamma-1} (D^{\nu} f)(t) dt,$

for every $x \in [a,b]$.

Next, we define Caputo fractional derivative, for details see [9, p. 449]. Let $\nu \geq 0$, $n = \lceil \nu \rceil$, $g \in AC^{n}([a,b])$. The Caputo fractional derivative is given by

$$D_{*a}^{\nu} g(t) = \frac{1}{\Gamma(n-\nu)} \int\limits_{a}^{x} \frac{g^{(n)}(y)}{(x-y)^{\nu-n+1}} dy,$$

for all $x \in [a,b]$. The above function exists almost everywhere for $x \in [a,b]$.

We continue with the following lemma that is given in [12].

Lemma 1.5 *Let $\nu > \gamma \geq 0$, $n = [\nu] + 1$, $m = [\gamma] + 1$ and $f \in AC^{n}([a,b])$. Suppose that one of the following conditions hold:*

(*a*) $\nu, \gamma \notin \mathbb{N}_0$ *and* $f^{(i)}(a) = 0$ *for* $i = m, \ldots, n-1$

(*b*) $\nu \in \mathbb{N}_0, \gamma \notin \mathbb{N}_0$ *and* $f^{(i)}(a) = 0$ *for* $i = m, \ldots, n-2$

(*c*) $\nu \notin \mathbb{N}_0, \gamma \in \mathbb{N}_0$ *and* $f^{(i)}(a) = 0$ *for* $i = m-1, \ldots, n-1$

(*d*) $\nu \in \mathbb{N}_0, \gamma \in \mathbb{N}_0$ *and* $f^{(i)}(a) = 0$ *for* $i = m-1, \ldots, n-2$.

Then

$$D_{*a}^{\gamma} f(x) = \frac{1}{\Gamma(\nu-\gamma)} \int\limits_{a}^{x} (x-y)^{\nu-\gamma-1} D_{*a}^{\nu} f(y) dy$$

for all $a \leq x \leq b$.

Now, we define Hadamard-type fractional integrals. Let (a,b) be finite or infinite interval of $\mathbb{R}_+$ and $\alpha > 0$. The left- and right-sided Hadamard-type fractional integrals of order $\alpha > 0$ are given by

$$(J^{\alpha}_{a+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log\frac{x}{y}\right)^{\alpha-1} \frac{f(y)dy}{y}, \ \ x > a$$

and

$$(J^{\alpha}_{b-} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\log\frac{y}{x}\right)^{\alpha-1} \frac{f(y)dy}{y}, \ \ x < b$$

respectively.

We continue with definitions and some properties of the fractional integrals of a function f with respect to a given function g. For details see e.g. [67, p. 99].

Let (a,b), $-\infty \le a < b \le \infty$ be a finitive or infinitive interval of the real line $\mathbb{R}$ and $\alpha > 0$. Also let g be an increasing function on $(a,b]$ such that g' is continuous on (a,b). The left- and right-sided fractional integrals of a function f with respect to another function g in (a,b) are given by

$$(I^{\alpha}_{a+;g} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t)f(t)dt}{[g(x)-g(t)]^{1-\alpha}}, x > a \tag{1.14}$$

and

$$(I^{\alpha}_{b-;g} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t)f(t)dt}{[g(t)-g(x)]^{1-\alpha}}, x < b, \tag{1.15}$$

respectively.

Remark 1.7 If $g(x) = x$, then $I^{\alpha}_{a+;x} f$ reduces to $I^{\alpha}_{a+} f$ and $I^{\alpha}_{b-;x} f$ reduces to $I^{\alpha}_{b-} f$, that is to Riemann-Liouville fractional integrals. Notice also that Hadamard fractional integrals of order α are special case of the left- and right-sided fractional integrals of a function f with respect to another function $g(x) = \log(x)$ in $[a,b]$ where $0 \le a < b \le \infty$.

We also recall the definition of the Erdelyi-Kóber type fractional integrals. For details see [96] (also see [35, p, 154]).

Let $(a,b), (0 \le a < b \le \infty)$ be finite or infinite interval of $\mathbb{R}_+$ Let $\alpha > 0, \sigma > 0$, and $\eta \in \mathbb{R}$. The left- and right-sided Erdelyi-Kóber type fractional integral of order $\alpha > 0$ are defined by

$$(I^{\alpha}_{a+;\sigma;\eta} f)(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^x \frac{t^{\sigma\eta+\sigma-1} f(t)dt}{(x^{\sigma}-t^{\sigma})^{1-\alpha}}, \ \ (x > a)$$

and

$$(I^{\alpha}_{b-;\sigma;\eta} f)(x) = \frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)} \int_x^b \frac{t^{\sigma(1-\eta-\alpha)-1} f(t)dt}{(t^{\sigma}-x^{\sigma})^{1-\alpha}}, \ \ (x < b)$$

respectively.

We conclude this section with multidimensional fractional integrals. Such type of fractional integrals are usually generalization of the corresponding one-dimensional fractional integral and fractional derivative.

For $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, ..., \alpha_n)$, we use the following notations:

$$\Gamma(\alpha) = (\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)),\ [\mathbf{a},\mathbf{b}] = [a_1,b_1]\times\cdots\times[a_n,b_n],$$

and by $\mathbf{x} > \mathbf{a}$ we mean $x_1 > a_1, ..., x_n > a_n$.

We define the mixed Riemann-Liouville fractional integrals of order $\alpha > 0$ as

$$(I^{\alpha}_{\mathbf{a}+} f)(\mathbf{x}) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^{x_1} \cdots \int_{a_n}^{x_n} f(\mathbf{t})(\mathbf{x}-\mathbf{t})^{\alpha-1} d\mathbf{t},\ (\mathbf{x} > \mathbf{a})$$

and

$$(I^{\alpha}_{\mathbf{b}-} f)(\mathbf{x}) = \frac{1}{\Gamma(\alpha)} \int_{x_1}^{b_1} \cdots \int_{x_n}^{b_n} f(\mathbf{t})(\mathbf{t}-\mathbf{x})^{\alpha-1} d\mathbf{t},\ (\mathbf{x} < \mathbf{b}).$$