Chapter 1

Basic results on convexity

1.1 Different types of convexity

In this section we give definitions and some properties of various types of convexity that are used in this book. Most of these material can be found in [53].

Definition 1.1 *Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is called convex if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y), \tag{1.1}$$

for every $x, y \in I$ and every $\lambda \in [0,1]$. If the inequality (1.1) *is reversed, then f i said to be concave.*

Definition 1.2 *Let f a real function defined on $[a,b]$. The $n-$th divided difference of f at mutually different knots $x_0, x_1, x_2, \ldots, x_n \in [a,b]$ is defined recursively by*

$$[x_i]f = f(x_i) \; i = 0, 1, \ldots, n,$$

and

$$[x_0, x_1, \ldots, x_k]f = \frac{[x_1, x_2, \ldots, x_n]f - [x_0, x_1, \ldots, x_{n-1}]f}{x_n - x_0}.$$

Definition 1.3 *Let $n \in \mathbb{N}_0$. A function $f : [a,b] \to \mathbb{R}$ is said to be $n-$convex on $[a,b]$ if and only if for every choice of $n+1$ distinct knots $x_0, x_1, x_2, \ldots, x_n \in [a,b]$*

$$[x_0, x_1, \ldots, x_k]f \geq 0. \tag{1.2}$$

If the inequality in (1.2) *is reversed, the function f is said to be $n-$concave on $[a,b]$.*

Remark 1.1 *Particulary, $0-$convex functions are nonnegative functions, $1-$convex functions are nondecreasing functions, $2-$convex functions are convex functions.*

Theorem 1.1 *If $f^{(n)}$ exists, then f is $n-$convex if and only if $f^{(n)} \geq 0$.*

Theorem 1.2 *If $f^{(n)}$ is $n-$convex on $[a,b]$, for $n \geq 2$, then $f^{(k)}$ exists and is $(n-k)-$convex for $1 \leq k \leq n-2$.*

Definition 1.4 *Let $I_1 = [a,b]$, $I_2 = [c,d]$. The $(n,m)-$divided difference of a function $f : I_1 \times I_2 \to \mathbb{R}$ at mutually different knots $x_0, x_1, \ldots, x_n \in I$ and $y_0, y_1, \ldots, y_m \in J$ is defined by*

$$\begin{aligned} \begin{bmatrix} x_0, x_1, \ldots, x_n \\ y_0, y_1, \ldots, y_m \end{bmatrix} f &= [x_0, x_1, \ldots, x_n]([y_0, y_1, \ldots, y_m] f) \\ &= [y_0, y_1, \ldots, y_m]([x_0, x_1, \ldots, x_n] f) \\ &= \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{f(x_i, x_j)}{\omega'(x_i)\omega'(y_j)}. \end{aligned}$$

where,

$$\omega(x) = \prod_{i=0}^{n}(x - x_i); \ \omega(y) = \prod_{j=0}^{m}(y - y_j).$$

Definition 1.5 *A function $f : I_1 \times I_2 \to \mathbb{R}$ is said to be $(n,m)-$convex or convex of order (n,m) if at mutually different knots $x_0, x_1, \ldots, x_n \in I$ and $y_0, y_1, \ldots, y_m \in J$*

$$\begin{bmatrix} x_0, x_1, \ldots, x_n \\ y_0, y_1, \ldots, y_m \end{bmatrix} f \geq 0.$$

Theorem 1.3 *If the partial derivative $f^{(n+m)}_{x^n y^m}$ of f exists, then f is (n,m)-convex if and only if $f^{(n+m)}_{x^n y^m} \geq 0$.*

Definition 1.6 *Let I be an interval in $\mathbb{R}$. The n-dimensional vector $F : I \to \mathbb{R}^n$*

$$F(x) = \big(f_1(x), f_2(x), \ldots, f_n(x)\big) \tag{1.3}$$

is called convex if

$$f_i(\lambda x + (1-\lambda)y) \leq \lambda f_i(x) + (1-\lambda) f(y)$$

for all $i = 1, 2, \ldots, n$, $\lambda \in [0,1]$ and all $x, y \in I$.

Definition 1.7 *The n-dimensional vector $F : I \to \mathbb{R}^n$ is called smooth convex if*

$$\frac{d^2}{dx^2} f_i(x) \geq 0, \ for \ all \ i = 1, 2, \ldots, n.$$

The vector addition and scalar multiplication is defined in the usual way:
if

$$F(x) = \big(f_1(x), f_2(x), \ldots, f_n(x)\big)$$

and

$$G(x) = \big(g_1(x), g_2(x), \ldots, g_n(x)\big),$$

then the vector addition is defined as

$$F(x) + G(x) = \Big(f_1(x) + g_1(x), f_2(x) + g_2(x), \ldots, f_n(x) + g_n(x)\Big)$$

and scalar multiplication as

$$\alpha F(x) = \Big(\alpha f_1(x), \alpha f_2(x), \ldots, \alpha f_n(x)\Big).$$

The vector composition is defined as follows

$$F \circ G(x) = F(G(x)) = \Big(f_1(g_1(x)), f_2(g_2(x)), \ldots, f_n(g_n(x))\Big).$$

Definition 1.8 *The vector F is said to be increasing vector if f_i are increasing functions for all $i = 1, 2, \ldots, n$.*
The vector F is said to be decreasing vector if f_i are decreasing functions for all $i = 1, 2, \ldots, n$.

Let $\chi_{[1,j]}^{[j+1,n]}[a,b]$ be the class of vectors having convex function on its first j components and remaining $n-j$ components are concave on the interval $[a,b]$ and let $\chi_{[j+1,n]}^{[1,j]}[a,b]$ be the class of vectors having concave functions on its first j components and remaining are convex on the interval $[a,b]$. It is obvious that if $F \in \chi_{[1,j]}^{[j+1,n]}[a,b]$ then $-F(x) \in \chi_{[j+1,n]}^{[1,j]}[a,b]$.

The proofs of two following propositions can be found in [51] and [53].

Proposition 1.1 *For convex vectors, we have*

(i) Adding two convex vectors, we obtain also a convex vector.

(ii) Multiplying a convex vector by a positive scalar is also a convex vector.

(iii) If $F : I \to \mathbb{R}$ is a convex vector and $G : \mathbb{R} \to \mathbb{R}$ is increasing vector then GoF is also convex vector.

Proposition 1.2 *Let F , $G \in \chi_{[1,j]}^{[j+1,n]}[a,b]$ then*

(i) $F + G \in \chi_{[1,j]}^{[j+1,n]}[a,b]$.

(ii) For any positive scalar α

$$\alpha F \in \chi_{[1,j]}^{[j+1,n]}[a,b].$$

(iii) Let $F \in \chi_{[1,j]}^{[j+1,n]}[a,b]$, and G is the vector such that f_i are increasing function, $i = 1,\dots,j$, and f_i are decreasing functions for all $i = j+1,\dots,n$. Then

$$G \circ F \in \chi_{[1,j]}^{[j+1,n]}[a,b].$$

1.2 Convexity of a mollification

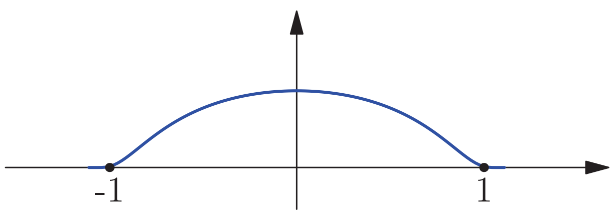
In this book we rely heavily on mollification technique. This is just tool that will allow us to build smooth approximations to given functions.

Definition 1.9 *The function $\eta \in C^\infty(\mathbb{R})$,*

$$\eta(x) = \begin{cases} C\exp\left(\frac{1}{x^2-1}\right), & x \le 1, \\ 0, & x > 1, \end{cases}$$

where C is a constant such that $\int\limits_{\mathbb{R}} \eta(x)dx = 1$, is called standard mollifier.

The graph of this function is shown below.



For each $\varepsilon > 0$, let

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon}\eta\left(\frac{x}{\varepsilon}\right),$$

and

$$I_\varepsilon = \{x \in I| \ \operatorname{dist}(x,\partial I) > \varepsilon\}.$$

Definition 1.10 *Let I be an open interval in $\mathbb{R}$. For a locally integrable function $f : I \to \mathbb{R}$ its mollification is*

$$f_\varepsilon(x) = (\eta_\varepsilon * f)(x), \ \ x \in I_\varepsilon,$$

i.e.

$$f_\varepsilon(x) = \int_{-\varepsilon}^{\varepsilon} f(x-y)\eta_\varepsilon(y)dy = \int_I f(y)\eta_\varepsilon(x-y)dy, \ \ x \in I_\varepsilon.$$

Proof of the next theorem can be found in [14].

Theorem 1.4

(i) $f_\varepsilon \in C^\infty(I_\varepsilon)$.

(ii) $f_\varepsilon \to f$ *a.e. as* $\varepsilon \to 0$.

(iii) *If* $f \in C(I)$, *then* $f_\varepsilon \to f$ *uniformly on compact subsets of* I.

(iv) *If* $1 \le p < \infty$ *and* $f \in L^p_{loc}(I)$, *then* $f_\varepsilon \to f$ *in* $L^p_{loc}(I)$.

Theorem 1.5 *If function f is convex, then its mollification f_ε is also convex.*

Proof. For $x_1,\ x_2 \in I_\varepsilon,\ \lambda \in [0,1]$, we have

$$\begin{aligned}
f_\varepsilon\left(\lambda x_1 + (1-\lambda)x_2\right) &= \int_{-\varepsilon}^{\varepsilon} f\left(\lambda x_1 + (1-\lambda)x_2 - y\right)\eta_\varepsilon(y)dy \\
&= \int_{-\varepsilon}^{\varepsilon} f\left(\lambda(x_1-y) + (1-\lambda)(x_2-y)\right)\eta_\varepsilon(y)dy \\
&\le \int_{-\varepsilon}^{\varepsilon} \left[\lambda f(x_1-y) + (1-\lambda)f(x_2-y)\right]\eta_\varepsilon(y)dy \\
&= \int_{-\varepsilon}^{\varepsilon} \lambda f(x_1-y)\eta_\varepsilon(y)dy + \int_{-\varepsilon}^{\varepsilon} (1-\lambda)f(x_2-y)\eta_\varepsilon(y)dy \\
&= \lambda f_\varepsilon(x_1) + (1-\lambda)f_\varepsilon(x_2).
\end{aligned}$$

□

Chapter 2

The weighted energy inequalities for convex functions

2.1 The weighted square integral inequalities for the first derivative of the function of a real variable

We consider the pair of twice continuously differential functions f and g defined on the closed bounded interval $[a,b]$. We assume that the function g is convex and the following requirement is satisfied:

$$|f''(x)| \leq g''(x), \quad a \leq x \leq b. \tag{2.1}$$

Let us introduce a family of nonnegative twice continuously differentiable weight functions $H : [a,b] \to \mathbb{R}$ which satisfy the following conditions

$$H(a) = H(b) = 0, \quad H'(a) = H'(b) = 0. \tag{2.2}$$

Theorem 2.1 *Let $f,\ g : [a,b] \to \mathbb{R}$ be two twice continuously differentiable functions which satisfy the requirement* (2.1) *and let $H : [a,b] \to \mathbb{R}$ be arbitrary nonnegative weight* *function such that condition* (2.2) *is fulfilled. Then the following inequality is valid*

$$\int_a^b \big(f'(x)\big)^2 H(x)dx \le \int_a^b \left[\left(\frac{f(x)}{2}\right)^2 + \big(\sup_{a\le t\le b}|f(t)|\big)g(x)\right]|H''(x)|dx. \tag{2.3}$$

Proof. Using the integration by parts

$$\begin{aligned}\int_a^b \big(f'(x)\big)^2 H(x)dx &= f(x)f'(x)H(x)\big|_a^b - \int_a^b (f'H)'(x)f(x)dx\\ &= -\int_a^b f(x)f'(x)H'(x)dx - \int_a^b f(x)f''(x)H(x)dx\\ &= -\frac{1}{2}\int_a^b (f^2)'(x)H'(x)dx - \int_a^b f(x)f''(x)H(x)dx. \end{aligned}\tag{2.4}$$

Similarly, using $H'(a) = H'(b) = 0$,

$$\int_a^b (f^2)'(x)H'(x)dx = -\int_a^b f^2(x)H''(x)dx.$$

Now (2.4) becomes

$$\begin{aligned}\int_a^b \big(f'(x)\big)^2 H(x)dx &= \frac{1}{2}\int_a^b f^2(x)H''(x)dx - \int_a^b f(x)f''(x)H(x)dx\\ &\le \frac{1}{2}\int_a^b f^2(x)H''(x)dx + \int_a^b |f(x)||f''(x)|H(x)dx\\ &\le \frac{1}{2}\int_a^b f^2(x)H''(x)dx + \sup_{a\le t\le b}|f(t)|\int_a^b |f''(x)|H(x)dx\\ &\le \frac{1}{2}\int_a^b f^2(x)H''(x)dx + \sup_{a\le t\le b}|f(t)|\int_a^b g''(x)H(x)dx\\ \text{(repeated int. by parts)} &= \frac{1}{2}\int_a^b f^2(x)H''(x)dx + \sup_{a\le t\le b}|f(t)|\int_a^b g(x)H''(x)dx.\end{aligned}$$

□

Corollary 2.1 *Under the same conditions as in the Theorem* 2.1, *the following bound is valid*

$$\int_a^b \left(f'(x)\right)^2 H(x)dx \le \|f\|_\infty \left(\frac{1}{2}\|f\|_p + \|g\|_p\right)\|H''\|_q \tag{2.5}$$

where $1 \le p \le \infty$, *and* p *and* q *are conjugate exponents.*

Proof. We apply Hölder inequality to the right-hand side of estimate (2.3).

Remark 2.1 *Let us notice that dominance* (2.1) *is equivalent to the existence of decomposition of the function* f *as the difference of two twice continuously differentiable convex functions,* f_1 *and* f_2, *such that,* $f(x) = f_1(x) - f_2(x),\ a \le x \le b$ *and* $g(x) = f_1(x) + f_2(x)$. *Indeed,* $|f''(x)| \le g''(x)$ *is equivalent* $-g''(x) \le f''(x) \le g''(x)$, *that is,*

$$f''(x) + g''(x) \ge 0, \quad g''(x) - f''(x) \ge 0.$$

The latter means that the functions

$$f_1(x) = \frac{1}{2}(f(x) + g(x)), \qquad f_2(x) = \frac{1}{2}(g(x) - f(x))$$

are convex functions such that

$$f(x) = f_1(x) - f_2(x), \qquad g(x) = f_1(x) + f_2(x). \tag{2.6}$$

Conversely, if f_1 *and* f_2 *are two twice continuously differentiable convex such that* (2.6) *is valid, then it is obvious that we have dominance* (2.1).

This remark suggests to write inequality (2.5) in a different form:

$$\begin{aligned}\int_a^b (f_1'(x) - f_2'(x))^2 H(x)dx \ \le\ & \|f_1 - f_2\|_\infty \left[\frac{1}{2}\|f_1 - f_2\|_p\right.\\ & +\ \left.\|f_1 + f_2\|_p\right] \|H''\|_q,\end{aligned} \tag{2.7}$$

where $1 \le p \le \infty$.

Corollary 2.2 *Let* f_1 *and* f_2 *be twice continuously differentiable convex functions defined on a closed bounded interval* $[a,b]$ *and let the weight function* H *be equal to*

$$H(x) = (x-a)^2(b-x)^2, \quad a \le x \le b.$$

Then the following estimate holds

$$\begin{aligned}\int_a^b (f_1'(x) - f_2'(x))^2 H(x)dx \ \le\ & \|f_1 - f_2\|_\infty [\frac{4\sqrt{3}}{9}\|f_1 + f_2\|_\infty\\ & +\ \frac{2\sqrt{3}}{9}\|f_1 - f_2\|_\infty](b-a)^3.\end{aligned} \tag{2.8}$$

Proof. We have

$$H''(x) = 12x^2 - 12(a+b)x + 2(a^2 + 4ab + b^2),$$

and then,

$$\int_a^b |H''(x)| = 2(b-a)^3 \int_0^1 |6u^2 - 6u + 1| du = \frac{4\sqrt{3}}{9}(b-a)^3.$$

Finally, taking into account the latter expression in estimate (2.7), we come to the desired inequality (2.8). □

Remark 2.2 *Comparing the result stated in Corollary 2.2 with Theorem 2.1 from K. Shashiashvili and M. Shashiashvili [50], we come to the conclusion that the constant factor* $\frac{4\sqrt{3}}{9}$ *is twice less than the constant factor obtained in the latter paper.*

2.1.1 The weighted square integral estimates for the difference of derivatives of two convex functions

Now we consider two arbitrary bounded convex functions f and g on an infinite interval $[0,\infty)$. It is well known that they are continuous and have finite left and right hand derivatives $f'(x-)$, $f'(x+)$ and $g'(x-)$, $g'(x+)$ inside the open interval $(0,\infty)$. We will assume that there exists a positive number A such that if $x \geq A$, we have

$$|f'(x-)| \leq C, \quad |g'(x-)| \leq C \tag{2.9}$$

where C is a certain positive constant.
Let us assume also that the difference of the functions f and g is bounded on the infinite interval $[0,\infty)$:

$$\sup_{x\geq 0} |f(x) - g(x)| < \infty. \tag{2.10}$$

Introduce now the family of nonnegative twice continuously differentiable weight functions $H(x)$ defined on the open interval $(0,\infty)$, which satisfy the following conditions:

$$\lim_{x\to 0+} H(x) = 0, \quad \lim_{x\to\infty} H(x) = 0, \quad \lim_{x\to 0+} H'(x) = 0, \quad \lim_{x\to\infty} H'(x) = 0, \tag{2.11}$$

and

$$\int_0^\infty (|f(x)| + |g(x)|)|H''(x)| dx < \infty. \tag{2.12}$$

Theorem 2.2 *For arbitrary bounded convex functions f and g defined on $[0,\infty)$ satisfying conditions* (2.9) *and* (2.10) *and for any nonnegative twice continuously differentiable*