Chapter 1

Introduction and Preliminaries

The notion of convexity plays an important role in different branches of mathematics.

Definition 1.1 *Let V be a real vector space.*

(a) A subset C of V is called convex, if for any two points $v_1, v_2 \in C$, the line segment between them also lies in C, that is $\lambda v_1 + (1-\lambda) v_2 \in C$ for all $\lambda \in [0,1]$.

(b) A function $f : C\,(\subset V) \to \mathbb{R}$ is called convex, if its domain C is a convex set, and for any two points $v_1, v_2 \in C$, and all $\lambda \in [0,1]$, we have that

$$f(\lambda v_1 + (1-\lambda) v_2) \leq \lambda f(v_1) + (1-\lambda) f(v_2).$$

The most important inequality concerning convex functions is the Jensen's inequality, named after the Danish mathematician Johan Jensen. It was proven by Jensen in [49]. We emphasize the following two variants of Jensen's inequality:

Theorem A. *(discrete Jensen's inequality, see [36]) Let C be a convex subset of a real vector space V, and let $f : C \to \mathbb{R}$ be a convex function. If $p_1, \ldots, p_n$ are nonnegative numbers with $\sum\limits_{i=1}^{n} p_i = 1$, and $v_1, \ldots, v_n \in C$, then*

$$f\left(\sum_{i=1}^{n} p_i v_i\right) \leq \sum_{i=1}^{n} p_i f(v_i) \tag{1.1}$$

holds. Particularly, we have

$$f\left(\frac{1}{n}\sum_{i=1}^{n} v_i\right) \leq \frac{1}{n}\sum_{i=1}^{n} f(v_i). \tag{1.2}$$

Theorem B. *(integral Jensen's inequality, see [36]) Let g be an integrable function on a probability space $(X, \mathscr{A}, \mu)$ taking values in an interval $I \subset \mathbb{R}$. Then $\int_X g d\mu$ lies in I. If f is a convex function on I such that $f \circ g$ is integrable, then*

$$f\left(\int_X g d\mu\right) \leq \int_X f \circ g d\mu.$$

Various attempts have been made by many authors to refine either the discrete or the integral Jensen's inequality (see the book [36] and the references therein). A multitude of applications underscores the importance of refinements of different Jensen's inequalities.

The following result which provide the starting point for our discussion is from Brnetić at al. [12].

Theorem 1.1 *Suppose I is a real interval. If $f : I \to \mathbb{R}$ is a convex function, then for all $t \in [0,1]$ we have*

$$f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) \leq \frac{1}{n}\sum_{i=1}^{n} f\left((1-t)x_i + tx_{i+1}\right) \leq \frac{\sum_{i=1}^{n} f(x_i)}{n},$$

where $x_i \in I$ $(1 \leq i \leq n)$ and $x_{n+1} = x_1$.

Recently, a lot of papers have been appeared dealing with generalizations of the previous theorem (see e.g. [13, 37]). The whole group of such results is now often known by the collective title "cyclic refinements". They find applications mainly in the theory of means and in information theory.

The title of this book indicates clearly the content of it. A synthesis of recent progress in the topic of cyclic refinements of different types of Jensen's inequalities is presented with the emphasis on their applications in information theory.

Let $2 \leq k \leq n$, and let $i \in \{1, \ldots, n\}$ and $j \in \{0, \ldots, k-1\}$. In further parts of this book $i + j$ always means $i + j - n$ in case of $i + j > n$.

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Chapter 2

Cyclic Improvement of Jensen's Inequality

In this chapter we give cyclic refinements of Jensen's inequality and their applications.

2.1 A refinement of Jensen's inequality

We start with the special following Jensen's inequality

$$f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) \le \frac{\sum_{i=1}^{n} f(x_i)}{n}.$$

Throughout this section we are going to use some of the following hypotheses:
($\mathscr{H}_1$) Let $I \subset \mathbb{R}$ be an interval, $\mathbf{x} := (x_1, \ldots, x_n) \in I^n$, and $\lambda := (\lambda_1, \ldots, \lambda_k)$ be a positive k-tuple such that $\sum_{i=1}^{k} \lambda_i = 1$ for some k, $2 \le k \le n$.
($\mathscr{H}_2$) Let $f : I \to \mathbb{R}$ be a convex function.
($\mathscr{H}_3$) Let h, $g : I \to \mathbb{R}$ be continuous and strictly monotone functions.

Theorem 2.1 *Let ($\mathscr{H}_1$), ($\mathscr{H}_2$) be fulfilled. Then*

$$f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) \le \frac{1}{n} \sum_{i=1}^{n} f\left(\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j}\right) \le \frac{\sum_{i=1}^{n} f(x_i)}{n}. \tag{2.1}$$

Proof. First, since f is convex, by Jensen's inequality we have

$$\sum_{i=1}^{n} f\left(\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j}\right) \leq \sum_{i=1}^{n} \sum_{j=0}^{k-1} \lambda_{j+1} f(x_{i+j})$$

$$= \sum_{i=1}^{n} f(x_i) \sum_{j=1}^{k} \lambda_j = \sum_{i=1}^{n} f(x_i).$$

On the other hand, since f is convex, by Jensen's inequality, we have

$$\frac{1}{n} \sum_{i=1}^{n} f\left(\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j}\right) \geq f\left(\frac{\sum_{i=1}^{n} \sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j}}{n}\right)$$

$$= f\left(\frac{\sum_{i=1}^{n} x_i \sum_{j=1}^{k} \lambda_j}{n}\right) = f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right).$$

□

Theorem 2.1 is a generalization of Theorem 4 in [12].

2.1.1 Cyclic mixed symmetric means

Assume $(\mathscr{H}_1)$ for the positive n-tuple $\mathbf{x}$. We define the power means of order $r \in \mathbb{R}$ as follows:

$$M_r(x_i, \ldots, x_{i+k-1}; \lambda_1, \ldots, \lambda_k) = \begin{cases} \left(\sum\limits_{j=0}^{k-1} \lambda_{j+1} x_{i+j}^r\right)^{\frac{1}{r}}; & r \neq 0, \\ \prod\limits_{j=0}^{k-1} x_{i+j}^{\lambda_{j+1}}; & r = 0, \end{cases}$$

and cyclic mixed symmetric means corresponding to (2.1) are

$$M_{r,s}(\mathbf{x}, \lambda) := \begin{cases} \left(\frac{1}{n} \sum\limits_{i=1}^{n} M_r^s(x_i, \ldots, x_{i+k-1}; \lambda_1, \ldots, \lambda_k)\right)^{\frac{1}{s}}; & s \neq 0, \\ \left(\prod\limits_{i=1}^{n} M_r(x_i, \ldots, x_{i+k-1}; \lambda_1, \ldots, \lambda_k)\right)^{\frac{1}{n}}; & s = 0. \end{cases} \tag{2.2}$$

The standard power means of order $r \in \mathbb{R}$ for the positive n-tuple $\mathbf{x}$, are

$$M_r(x_1, \ldots, x_n) = M_r(\mathbf{x}) := \begin{cases} \left(\frac{1}{n} \sum\limits_{i=1}^{n} x_i^r\right)^{\frac{1}{r}}; & r \neq 0, \\ \left(\prod\limits_{i=1}^{n} x_i\right)^{\frac{1}{n}}; & r = 0. \end{cases}$$

The bounds for cyclic mixed symmetric means are power means, as given in the following result.

Corollary 2.1 *Assume ($\mathscr{H}_1$) for the positive n-tuple* $\mathbf{x}$. *Let $r,s \in \mathbb{R}$ such that $r \le s$. Then*

$$M_r(\mathbf{x}) \le M_{s,r}(\mathbf{x},\lambda) \le M_s(\mathbf{x}). \tag{2.3}$$

Proof. Assume r, $s \neq 0$. To obtain (2.3), we apply Theorem 2.1, either for the function $f(x) = x^{\frac{s}{r}}$ $(x > 0)$ and the n-tuples $(x_1^r,\dots,x_n^r)$ in (2.1) and then raising the power $\frac{1}{s}$, or $f(x) = x^{\frac{r}{s}}$ $(x > 0)$ and $(x_1^s,\dots,x_n^s)$ and raising the power $\frac{1}{r}$.
When $r = 0$ or $s = 0$, we get the required results by taking limit. □

Special cases of Corollary 2.1 can be found in [11] (see Theorem 4 with Corollaries 4.1–4.4). Namely, the result of this theorem is an inequality (2.3) for $r = 0$, $s = 1$, $n = 3$ and $k = 3$.
Assume ($\mathscr{H}_1$) and ($\mathscr{H}_3$). Then we define the generalized means with respect to (2.1) as follows:

$$M_{g,h}(\mathbf{x},\lambda) := g^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}(g\circ h^{-1})(\sum_{j=0}^{k-1}\lambda_{j+1}h(x_{i+j}))\right).$$

Let $q : I \to \mathbb{R}$ be a continuous and strictly monotone function then the cyclic quasi-arithmetic means are given by

$$M_q(\mathbf{x}) := q^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}q(x_i)\right).$$

The relation among the generalized means and cyclic quasi-arithmetic means is given in the next corollary.

Corollary 2.2 *Assume ($\mathscr{H}_1$) and ($\mathscr{H}_3$). Then*

$$M_h(\mathbf{x}) \le M_{g,h}(\mathbf{x},\lambda) \le M_g(\mathbf{x}) \tag{2.4}$$

if either $g\circ h^{-1}$ is convex and g is strictly increasing or $g\circ h^{-1}$ is concave and g is strictly decreasing.

Proof. First, we can apply Theorem 2.1 to the function $g\circ h^{-1}$ and the n-tuples $(h(x_1),\dots,h(x_n))$, then we can apply g^{-1} to the inequality coming from (2.1). This gives (2.4). □

For instance, if we put $g(x) = x$ and $h(x) = \ln x$ in Corollary 2.2 we obtain

$$M_0(x_1,...,x_n) \le \frac{1}{n}\sum_{i=1}^{n}M_0(x_i,...,x_{i+k-1};\lambda_1,...,\lambda_k) \le M_1(x_1,...,x_n).$$

which is a special case of Corollary 2.1 as well.

Remark 2.1 Under the conditions ($\mathscr{H}_1$), we define

$$\Upsilon_1(f) = \Upsilon_1(\mathbf{x},\lambda,f) := \frac{1}{n}\sum_{i=1}^{n}f(x_i) - \frac{1}{n}\sum_{i=1}^{n}f(\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j}),$$

$$\Upsilon_2(f) = \Upsilon_2(\mathbf{x}, \lambda, f) := \frac{1}{n}\sum_{i=1}^{n} f(\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j}) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right),$$

where $f : I \to \mathbb{R}$ is a function and $2 \le k \le n$. The functionals $f \to \Upsilon_i(f)$ are linear, $i = 1,2$, and Theorem 2.1 implies that

$$\Upsilon_i(f) \ge 0, \quad i = 1,2$$

if $f : I \to \mathbb{R}$ is a convex function.

2.1.2 *m*-Exponential convexity

For log-convexity, exponential convexity and m-exponential convexity of the functionals obtained from the interpolations of the discrete Jensen's inequality, we refer [36] and references therein.

We apply the method given in [83], to prove the m-exponential convexity and exponential convexity of the functionals $f \to \Upsilon_i(f)$ for $i = 1,2$, together with the Lagrange type and Cauchy type mean value theorems.

Definition 2.1 *(see [83]) A function $g : I \to \mathbb{R}$ is called* m-exponentially convex *in the Jensen sense if*

$$\sum_{i,j=1}^{m} a_i a_j g\left(\frac{x_i + x_j}{2}\right) \ge 0$$

holds for every $a_i \in \mathbb{R}$ and every $x_i \in I$, $i = 1,2,...,m$.

A function $g : I \to \mathbb{R}$ is m-exponentially convex *if it is* m-exponentially convex *in the Jensen sense and continuous on I.*

Note that 1-exponentially convex functions in the Jensen sense are in fact the nonnegative functions. Also, m-exponentially convex functions in the Jensen sense are n-exponentially convex in the Jensen sense for every $n \in \mathbb{N}$, $n \le m$.

Proposition 2.1 *If $g : I \to \mathbb{R}$ is an m-exponentially convex function, then for every $x_i \in I$, $i = 1,2,...,m$ and for all $n \in \mathbb{N}$, $n \le m$ the matrix $\left[g\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{n}$ is a positive semi-definite matrix. Particularly,*

$$\det\left[g\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{n} \ge 0$$

for all $n \in \mathbb{N}$, $n = 1,2,...,m$.

Definition 2.2 *A function $g : I \to \mathbb{R}$ is exponentially convex in the Jensen sense, if it is m-exponentially convex in the Jensen sense for all $m \in \mathbb{N}$.*

A function $g : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Remark 2.2 It is easy to see that a positive function $g: I \to \mathbb{R}$ is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense, that is

$$a_1^2 g(x) + 2a_1 a_2 g\left(\frac{x+y}{2}\right) + a_2^2 g(y) \geq 0$$

holds for every $a_1, a_2 \in \mathbb{R}$ and $x, y \in I$.

Similarly, if g is 2-exponentially convex, then g is log-convex. On the other hand, if g is log-convex and continuous, then g is 2-exponentially convex.

In sequel, we need the well known notion of "Divided difference".

Definition 2.3 *The second order divided difference of a function $g: I \to \mathbb{R}$ at mutually different points $y_0, y_1, y_2 \in I$ is defined recursively by*

$$[y_i; g] = g(y_i), \quad i = 0, 1, 2$$

$$[y_i, y_{i+1}; g] = \frac{g(y_{i+1}) - g(y_i)}{y_{i+1} - y_i}, \quad i = 0, 1$$

$$[y_0, y_1, y_2; g] = \frac{[y_1, y_2; g] - [y_0, y_1; g]}{y_2 - y_0}. \tag{2.5}$$

Remark 2.3 The value $[y_0, y_1, y_2; g]$ is independent of the order of the points y_0, y_1, and y_2. By taking limits this definition may be extended to include the cases in which any two or all three points coincide as follows: for all y_0, y_1, $y_2 \in I$ such that $y_2 \neq y_0$

$$\lim_{y_1 \to y_0} [y_0, y_1, y_2; g] = [y_0, y_0, y_2; g] = \frac{g(y_2) - g(y_0) - g'(y_0)(y_2 - y_0)}{(y_2 - y_0)^2}$$

provided that g' exists, and furthermore, taking the limits $y_i \to y_0$, $i = 1, 2$ in (2.5), we get

$$[y_0, y_0, y_0; g] = \lim_{y_i \to y_0} [y_0, y_1, y_2; g] = \frac{g''(y_0)}{2} \text{ for } i = 1, 2$$

provided that g'' exist on I.

Now, we give the m-exponential convexity for the linear functionals $\Upsilon_i(f)$ $(i = 1, 2)$.

Theorem 2.2 *Assume $I \subset \mathbb{R}$ is an interval, and assume $\Lambda = \{\phi_t \mid t \in J\}$ is a family of functions defined on an interval $I \subset \mathbb{R}$, such that the function $t \to [y_0, y_1, y_2; \phi_t]$ $(t \in J)$ is m-exponentially convex in the Jensen sense on I for every three mutually different points $y_0, y_1, y_2 \in I$. Let $\Upsilon_i(f)$ $(i = 1, 2)$ be the linear functionals constructed in Remark 2.1. Then $t \to \Upsilon_i(\phi_t)$ $(t \in J)$ is an m-exponentially convex function in the Jensen sense on I for each $i = 1, 2$. If the function $t \to \Upsilon_i(\phi_t)$ $(t \in J)$ is continuous for $i = 1, 2$, then it is m-exponentially convex on I for $i = 1, 2$.*

Proof. Fix $i = 1,2$.

Let $t_k, t_l \in J$, $t_{kl} := \frac{t_k + t_l}{2}$ and $b_k, b_l \in \mathbb{R}$ for $k,l = 1,2,...,n$, and define the function ω on I by

$$\omega := \sum_{k,l=1}^{n} b_k b_l \phi_{t_{kl}}.$$

Since the function $t \to [y_0, y_1, y_2; \phi_t]$ $(t \in J)$ is m-exponentially convex in the Jensen sense, we have

$$[y_0, y_1, y_2; \omega] = \sum_{k,l=1}^{n} b_k b_l [y_0, y_1, y_2; \phi_{t_{kl}}] \geq 0.$$

Hence ω is a convex function on I. Therefore we have $\Upsilon_i(\omega) \geq 0$, which yields by the linearity of Υ_i, that

$$\sum_{k,l=1}^{n} b_k b_l \Upsilon_i(\phi_{t_{kl}}) \geq 0.$$

We conclude that the function $t \to \Upsilon_i(\phi_t)$ $(t \in J)$ is an m-exponentially convex function in the Jensen sense on I.

If the function $t \to \Upsilon_i(\phi_t)$ $(t \in J)$ is continuous on I, then it is m-exponentially convex on I by definition. □

As a consequence of the above theorem we can give the following corollaries.

Corollary 2.3 *Assume $I \subset \mathbb{R}$ is an interval, and assume $\Lambda = \{\phi_t \mid t \in J\}$ is a family of functions defined on an interval $I \subset \mathbb{R}$, such that the function $t \to [y_0, y_1, y_2; \phi_t]$ $(t \in J)$ is exponentially convex in the Jensen sense on I for every three mutually different points $y_0, y_1, y_2 \in I$. Let $\Upsilon_i(f)$ $(i = 1,2)$ be the linear functionals constructed in Remark 2.1. Then $t \to \Upsilon_i(\phi_t)$ $(t \in J)$ is an exponentially convex function in the Jensen sense on I for $i = 1,2$. If the function $t \to \Upsilon_i(\phi_t)$ $(t \in J)$ is continuous, then it is exponentially convex on I for $i = 1,2$.*

Corollary 2.4 *Assume $I \subset \mathbb{R}$ is an interval, and assume $\Lambda = \{\phi_t : t \in J\}$ is a family of functions defined on an interval $I \subset \mathbb{R}$, such that the function $t \to [y_0, y_1, y_2; \phi_t]$ $(t \in J)$ is 2-exponentially convex in the Jensen sense on I for every three mutually different points $y_0, y_1, y_2 \in I$. Let $\Upsilon_i(f)$ $(i = 1,2)$ be the linear functionals constructed in Remark 2.1. Then the following two statements hold for $i = 1,2$:*

(i) If the function $t \to \Upsilon_i(\phi_t)$ $(t \in J)$ is positive and continuous, then it is 2-exponentially convex on I, and thus log-*convex.*

(ii) If the function $t \to \Upsilon_i(\phi_t)$ $(t \in J)$ is positive and differentiable, then for every $s,t,u,v \in J$, such that $s \leq u$ and $t \leq v$, we have

$$\mathfrak{u}_{s,t}(\Upsilon_i, \Lambda) \leq \mathfrak{u}_{u,v}(\Upsilon_i, \Lambda) \tag{2.6}$$

where

$$\mathfrak{u}_{s,t}(\Upsilon_i, \Lambda) := \begin{cases} \left(\frac{\Upsilon_i(\phi_s)}{\Upsilon_i(\phi_t)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(\frac{\frac{d}{ds}\Upsilon_i(\phi_s)}{\Upsilon_i(\phi_s)}\right), & s = t \end{cases} \tag{2.7}$$

for $\phi_s, \phi_t \in \Lambda$.

Proof. Fix $i = 1,2$.

(i) The proof follows by Remark 2.2 and Theorem 2.2.

(ii) From the definition of a convex function ψ on I, we have the following inequality (see [82, page 2])

$$\frac{\psi(s) - \psi(t)}{s - t} \leq \frac{\psi(u) - \psi(v)}{u - v}, \tag{2.8}$$

$\forall s,t,u,v \in J$ such that $s \leq u$, $t \leq v$, $s \neq t$, $u \neq v$.
By (i), $s \to \Upsilon_i(\phi_s)$, $s \in J$ is log-convex, and hence (2.8) shows with $\psi(s) = \log\Upsilon_i(\phi_s)$, $s \in J$ that

$$\frac{\log\Upsilon_i(\phi_s) - \log\Upsilon_i(\phi_t)}{s-t} \leq \frac{\log\Upsilon_i(\phi_u) - \log\Upsilon_i(\phi_v)}{u-v} \tag{2.9}$$

for $s \leq u$, $t \leq v$, $s \neq t$, $u \neq v$, which is equivalent to (2.6). For $s = t$ or $u = v$ (2.6) follows from (2.9) by taking limit.

□

Remark 2.4 Note that the results from Theorem 2.2, Corollary 2.3, Corollary 2.4 are valid when two of the points $y_0, y_1, y_2 \in I$ coincide, say $y_1 = y_0$, for a family of differentiable functions ϕ_t such that the function $t \to [y_0, y_1, y_2; \phi_t]$ is m-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and moreover, they are are also valid when all three points coincide for a family of twice differentiable functions with the same property. The proofs can be obtained by recalling Remark 2.3 and suitable characterization of convexity.

The following result given in [35] is related to the first condition of Theorem 2.2.

Theorem 2.3 *Assume $I \subset \mathbb{R}$ is an interval, and assume $\Lambda = \{\phi_t \mid t \in J\}$ is a family of twice differentiable functions defined on an interval $I \subset \mathbb{R}$ such that the function $t \mapsto \phi_t''(x)$ $(t \in J)$ is exponentially convex for every fixed $x \in I$. Then the function $t \mapsto [y_0, y_1, y_2; \phi_t]$ $(t \in J)$ is exponentially convex in the Jensen sense for any three points y_0, y_1, $y_2 \in I$.*

Remark 2.5 It comes from either the conditions of Theorem 2.3 or the proof of this theorem that the functions ϕ_t, $t \in J$ are convex.

2.1.3 Mean value theorems

Now we formulate mean value theorems of Lagrange and Cauchy type for the linear functionals $\Upsilon_i(f)$ $(i = 1,2)$ defined in Remark 2.1.

Theorem 2.4 *Let $\Upsilon_i(f)$ $(i = 1,2)$ be the linear functionals constructed in Remark 2.1 and $g \in C^2[a,b]$. Then there exists $\xi \in [a,b]$ such that*

$$\Upsilon_i(g) = \frac{1}{2} g''(\xi)\Upsilon_i\left(x^2\right); \quad i = 1,2.$$

Proof. Fix $i = 1,2$.

Since $g \in C^2[a,b]$, there exist the real numbers $m = \min_{x\in[a,b]} g''(x)$ and $M = \max_{x\in[a,b]} g''(x)$. It is easy to show that the functions ϕ_1 and ϕ_2 defined on $[a,b]$ by

$$\phi_1(x) = \frac{M}{2}x^2 - g(x),$$

and

$$\phi_2(x) = g(x) - \frac{m}{2}x^2,$$

are convex.

By applying the functional Υ_i to the functions ϕ_1 and ϕ_2, we have the properties of Υ_i that

$$\Upsilon_i\left(\frac{M}{2}x^2 - g(x)\right) \geq 0,$$

$$\Rightarrow \Upsilon_i(g) \leq \frac{M}{2}\Upsilon_i\left(x^2\right), \tag{2.10}$$

and

$$\Upsilon_i\left(g(x) - \frac{m}{2}x^2\right) \geqslant 0$$

$$\Rightarrow \frac{m}{2}\Upsilon_i\left(x^2\right) \leq \Upsilon_i(g). \tag{2.11}$$

From (2.10) and from (2.11), we get

$$\frac{m}{2}\Upsilon_i\left(x^2\right) \leq \Upsilon_i(g) \leq \frac{M}{2}\Upsilon_i\left(x^2\right).$$

If $\Upsilon_i\left(x^2\right) = 0$, then nothing to prove. If $\Upsilon_i\left(x^2\right) \neq 0$, then

$$m \leq \frac{2\Upsilon_i(g)}{\Upsilon_i(x^2)} \leq M.$$

Hence we have

$$\Upsilon_i(g) = \frac{1}{2}g''(\xi)\,\Upsilon_i\left(x^2\right).$$

□

Theorem 2.5 *Let $\Upsilon_i(f)$ $(i = 1,2)$ be the linear functionals constructed in Remark 2.1 and $g,h \in C^2[a,b]$. Then there exists $\xi \in [a,b]$ such that*

$$\frac{\Upsilon_i(g)}{\Upsilon_i(h)} = \frac{g''(\xi)}{h''(\xi)}; \quad i = 1,2,$$

provided that $\Upsilon_i(h) \neq 0$ $(i = 1,2)$.