MONOGRAPHS IN INEQUALITIES 21

Edmundson-Lah-Ribarič Type Inequalities

Reverses of the Edmundson-Lah-Ribarič Inequality with Applications to Classical Inequalities

Mario Krnić, Rozarija Mikić and Josip Pečarić

ELEMENT

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Preface

Convex functions are one of the most important terms of the Theory of inequalities. We say that a function $f\colon I \to \mathbb{R}$ is convex on the interval $I \subseteq \mathbb{R}$ if for all $x, y \in I$ and every $\lambda \in [0,1]$ it holds

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y). \tag{a}$$

If for all $x \neq y$ and every $\lambda \in \langle 0,1 \rangle$ the inequality (a) is strict, then we say that the function f is strictly convex. If the inequality sign in (a) is reversed, then we say that f is a concave function. Geometrically speaking, function f is convex on the interval I if for any two points $x, y \in I$ part of the graph between points x and y is below the chord of the function f at these points.

Jensen's inequality for convex functions is one of the most important inequalities in contemporary mathematics since it results a whole series of other classical inequalities. It was named after a famous Danish mathematician Johan Ludwig Jensen (1859-1925) who proved it in 1906.

Theorem A *(Jensen's inequality,* [124]*) Let $f\colon I \to \mathbb{R}$ be a convex function on the interval $I \subseteq \mathbb{R}$. For $n \geq 2$, let $x_1, \ldots, x_n \in I$, and $p_1, .., p_n \in \mathbb{R}$ be such that $p_i > 0$ for every i. Then we have*

$$f\Big(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\Big) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i), \tag{b}$$

where $P_n = \sum_{i=1}^{n} p_i$. If the function f is strictly convex and if $x_1, \ldots, x_n$ are not all equal to each other, then the inequality in (b) is strict.

However, simpler variants of the inequality (b) appeared much earlier, but under different assumptions. In 1889 Hölder proved that for a function $f\colon [a,b] \to \mathbb{R}$ and $x, y \in [a,b]$ it holds

$$f\Big(\frac{x+y}{2}\Big) \leq \frac{f(x)+f(y)}{2} \tag{c}$$

assuming that the function f is twice differentiable on $[a,b]$, and $f''(x) \geq 0$ on that interval. If the function f is twice differentiable, then the condition $f''(x) \geq 0$ for $x \in [a,b]$ is equivalent to f being convex on $[a,b]$, but the concept of convexity was introduced later by Jensen in the paper [75] in which he also proved the inequality (b). In 1896 Henderson proved the inequality (b), but under Hölder's assumptions. Special case of the inequality

(b) for $p_1 = ... = p_n = 1$ was proved by Grolous yet in 1875 using centroid method in the paper [53]. This is also the first known inequality for convex functions in mathematical literature.

Jensen's inequality has a significant application in the various branches of mathematics, especially in mathematical analysis and statistics, where it is most commonly used to determine lower bounds for the expectation of convex functions. Over the centuries, Jensen's inequality has been extensively explored by many renowned mathematicians, and has been generalized in many ways. Integral version of the same inequlity was obtained by Beesack and Pečarić in 1984. Analogue inequality for positive linear functionals was proved by Jessen in 1931, and Davis in 1957 showed that Jensen's inequality is also true between operator algebras. Numerous well-known classical inequalities have arisen as a result of the Jensen inequality. One of the more known is the so-called Edmundson-Lah-Ribarič inequality:

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \leq \frac{M-\bar{x}}{M-m} f(m) + \frac{\bar{x}-m}{M-m} f(M), \tag{d}$$

where $f\colon [m,M] \to \mathbb{R}$ is a convex function and $\bar{x} = \sum_{i=1}^{n} p_i x_i$.

It was proved in 1973 by Lah and Ribarič. They also obtained a converse of the Jensen inequality in their paper [89] from 1971, which we state in its original form.

Theorem B ([89]) *Let μ be a positive measure on $[0,1]$ and let ϕ be a convex function on the interval $[m,M]$, where $-\infty < m < M < +\infty$. Then for every μ-measurable function f on the interval $[0,1]$ such that $m \leq f(x) \leq M$ holds for any $x \in [0,1]$ we have the following inequality*

$$\frac{\int_0^1 \Phi(f)d\mu}{\int_0^1 d\mu} \leq \frac{M-\bar{f}}{M-m}\Phi(m) + \frac{\bar{f}-m}{M-m}\Phi(M), \tag{e}$$

where $\bar{f} = \int_0^1 f d\mu / \int_0^1 d\mu$.

Since then, many papers have been written on the subject of generalizations and converses of the inequality (e). A whole series of monographs in inequalities ([3], [44], [48], [49], [84] and [85]) has been dedicated to classical inequalities, including the Lah-Ribarič inequality (e).

Beesack and Pečarić [14] (see also [124, p.98]) proved the following generalization of the Lah-Ribarič inequality (e) for positive linear functionals.

Theorem C ([14]) *Let ϕ be a convex function on the interval $I = [m,M]$, where $-\infty < m < M < \infty$. Let L be a vector space of all real functions defined on a non-empty set E such that $af + bg \in L$ holds for all $f,g \in L$, $a,b \in \mathbb{R}$ and $\mathbf{1} \in L$, and let A be any normalized positive linear functional on L. Then for every function $f \in L$ such that $\phi(f) \in L$ we have:*

$$A(\phi(f)) \leq \frac{M-A(f)}{M-m}\phi(m) + \frac{A(f)-m}{M-m}\phi(M). \tag{f}$$

Amongst most recent results, we should mention converses in the difference form of the integral Jensen's inequality obtained by Dragomir. We should also mention the improvement the of Edmundson-Lah-Ribarič inequalities for positive linear functionals obtained by Klaričić Bakula, Pečarić and Perić.

Probability version of the inequality (e) for the mathematical expectation of a random variable is given in the theorem which follows.

Theorem D ([42]) *Let $-\infty < a < b < +\infty$, and let $X\colon \Omega \to [a,b]$ be a random variable with finite expectation on a probability space (Ω, p). Let $f\colon [a,b] \to \mathbb{R}$ be a convex function such that $\mathbb{E}(f(X)) < \infty$. Then*

$$\mathbb{E}(f(X)) \leq \frac{b-\mathbb{E}(X)}{b-a} f(a) + \frac{\mathbb{E}(X)-a}{b-a} f(b). \tag{g}$$

Inequality (g) is often reffered to as the Edmundson-Madansky inequality, because it was proved in 1956 by Edmundson ([42]), and Madansky ([93]) in 1959 was the first one to start using it in context of stohastic programming for finding the best possible upper bound for the expectation of convex functions. A comprehensive list of some recent results concerning the Edmundson-Madansky inequality can be found in [18] and [87].

One can easily see that Theorem C is also a generalization of Theorem D, that is, the inequalities (e) and (g) are actually special cases of the same inequality, but in different settings. Therefore, from now on those inequalities will be united under the common name of the Edmundson-Lah-Ribarič inequality.

Jensen's inequality for mathematical expectation

$$f(\mathbb{E}(X)) \leq \mathbb{E}(f(X)) \tag{h}$$

is used in the same context for determining best possible lower bound for the expectation of a convex function, so it is clear that inequalities (g) and (h) are closely related.

Levinson in 1964 in the paper "Generalisation of an inequality of Ky Fan" obtained an important inequality of the Jensen type concerning two distinct series of numbers. This result today is known as Levinson's inequality. Many mathematicians have worked on weakening the conditions under which Levinson's inequality is valid. Thus, Bullen and Pečarić weakened the condition of symmetry, and Mercer completely replaced it with the condition of equality of variances for functions that have a nonnegative third derivation. Witkowski showed that in Mercer's assumptions it is enough for the function to be 3-convex. Further, Baloch, Pečarić and Praljak found the widest class of functions for which Levinson's inequality is valid under Mercer's assumptions. The mentioned class of functions is an extension of the class of 3-convex functions and can be viewed as a class of functions that are 3-convex at the point.

This book is based on several recent research papers on the subject of Jensen's inequality, its converses and their variants, with special emphasis on the Edmundson-Lah Ribarič inequality in different settings and under various conditions. In the first chapter we will show some difference type converses of the mentioned inequalities for positive linear functionals, together with their refinements, improvements and applications to many famous classical inequalities. In the second chapter different classes of inequalities of the Jensen and Edmundson-Lah-Ribarič type for functions with bounded second order divided differences, Lipschitzian functions and 3-convex functions are derived. Also, several representations of the left side in the Edmundson-Lah-Ribarič inequality via Hermite's interpolating polynomial in terms of divided differences are given and used for obtaining inequalities for the class of n-convex functions. Third chapter is dedicated to estimates for the Csiszár f-divergence functional and generalization of the f-divergence functional for different classes of functions via results from the previous two chapters. Application to Zipf and Zipf-Mandelbrot law is also given. In the fourth chapter we give difference type converses of the Jensen and Edmundson-Lah-Ribarič operator inequality for a unital field of positive linear mappings between C^*-algebras of operators in compact Hausdorff space and their further refinements and improvements. Several mutual bounds for the operator version of the Jensen and Edmundson-Lah-Ribarič inequality which hold for the classes of bounded real-valued functions, Lipschitzian functions and n-convex functions are also given. In the fifth chapter we show some converses of Ando's and Davis-Choi's inequality of different types, as well as the Edmundson-Lah-Ribarič inequality and its difference type converse for positive linear mappings. Some results are extended to the class of n-convex functions. Difference type converse for solidarities and connections are also given. In the sixth chapter, some converses of the Jensen and Edmundson-Lah-Ribarič inequality in terms of time scale calculus are proved together with new refinements of those converse relations with respect to the multiple Lebesgue delta integral for convex functions. In the last chapter we give a short historical comment on the connection between the Edmundson-Madansky and the Lah-Ribarič inequality, and an overview of some already known results. Also we give a Levinson's type generalization of the Edmundson-Lah-Ribarič inequality for a class of functions which contains the class of 3-convex functions, and analogous inequalities for the operator inequality in the Hilbert space and the scalar product of Hilbert space operators.

Authors

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Chapter 1

Difference type converses for linear functionals

This chapter begins with an overview of some important results related to the Jensen and Edmundson-Lah-Ribarič inequality for positive linear functionals which are known from earlier. Further, some difference type converses of the mentioned results will be shown, as well as their refinements and improvements. Finally, those improvements will be applied to generalized means and some famous classical inequalities (the ones of Hölder, Hermite-Hadamard, Giaccardi and Petrović). In that way we will get converses of listed inequalities that provide us with an upper bound for the difference of their right and left sides.

1.1 Introduction

Let E be a non-empty set and L a vector space of real functions $f\colon E \to \mathbb{R}$ with the following properties:

(L1): $f,g \in L \Rightarrow (af+bg) \in L$ for all $a,b \in \mathbb{R}$;

(L2): $\mathbf{1} \in L$, that is, if $f(t) = 1$ for every $t \in E$, then $f \in L$.

(L3): if $f,g \in L$, then $\min\{f,g\} \in L$ or $\max\{f,g\} \in L$.

Obviously, $(\mathbb{R}^E, \leq)$ (with standard ordering) is a lattice. It can also be easily verified that a subspace $(X \subseteq \mathbb{R}^E)$ is a lattice if and only if $x \in X$ implies $|x| \in X$. This is a simple consequence of the fact that for every $x \in X$ the functions $|x|$, x^- and x^+ can be defined by

$$|x|(t) = |x(t)|, \;\; x^+(t) = \max\{0, x(t)\}, \;\; x^-(t) = -\min\{0, x(t)\}, \;\; t \in E$$

and

$$x^+ + x^- = |x|, \;\; x^+ - x^- = x,$$

$$\min\{x,y\} = \frac{1}{2}(x+y-|x-y|), \;\; \max\{x,y\} = \frac{1}{2}(x+y+|x-y|).$$

We also study positive linear functionals $A\colon L \to \mathbb{R}$, that is, we assume:

(A1): $A(af+bg) = aA(f) + bA(g)$ for $f, g \in L$ and $a, b \in \mathbb{R}$;

(A2): $f \in L$, $f(t) \geq 0$ for every $t \in E \Rightarrow A(f) \geq 0$.

We say that a functional A is normalized if $A(\mathbf{1}) = 1$.

Throughout this chapter, if a function is defined on an interval $[m, M]$ without any further emphasis we assume that the bounds of that interval are finite.

Jessen [76] gave the following generalization of Jensen's inequality for convex functions (see also [124, p. 47]):

Theorem 1.1 ([76]) *Let L be a vector space of real functions defined on a non-empty set E that has properties $(L1)$ and $(L2)$, and let us assume that ϕ is a continuous convex function on an interval $I \subset \mathbb{R}$. If A is a normalized positive linear functional, then for every $f \in L$ such that $\phi(f) \in L$ we have $A(f) \in I$ and*

$$\phi(A(f)) \leq A(\phi(f)). \tag{1.1}$$

Next result is a generalization of the Edmundson-Lah-Ribarič inequality for linear functionals and it was proved by Beesack and Pečarić in [14] (see also [124, p. 98]):

Theorem 1.2 ([14]) *Let ϕ be a convex function on $I = [m, M]$, let L be a vector space of real functions defined on a non-empty set E that has properties $(L1)$ and $(L2)$, and let A be a normalized positive linear functional. Then for every $f \in L$ such that $\phi(f) \in L$ (so $m \leq f(t) \leq M$ for all $t \in E$), we have*

$$A(\phi(f)) \leq \frac{M - A(f)}{M - m}\phi(m) + \frac{A(f) - m}{M - m}\phi(M). \tag{1.2}$$

Dragomir in [37] studied a measure space $(\Omega, \mathscr{A}, \mu)$ which consists of a set Ω, σ-algebra $\mathscr{A}$ of subsets of Ω and a countably additive and positive measure μ on $\mathscr{A}$ with values in $\mathbb{R} \cup \{\infty\}$. For a μ-measurable function $w\colon \Omega \to \mathbb{R}$ such that $w(x) \geq 0$ for μ-a.e. (almost every) $x \in \Omega$, he considered a Lebesgue space

$$L_w(\Omega, \mu) := \{f\colon \Omega \to \mathbb{R}, f \text{ is } \mu-\text{measurable and } \textstyle\int_\Omega w(x)|f(x)|\mathrm{d}\mu(x) < \infty\},$$

and proved the following converse of Jensen's inequality.

Theorem 1.3 ([37]) *Let $\phi\colon I \to \mathbb{R}$ be a continuous convex function on an interval of real numbers I and let $m, M \in \mathbb{R}$, $m < M$ be such that the interval $[m,M]$ belongs to the interior of I. Let $w > 0$ be such that $\int w\mathrm{d}\mu = 1$. If $f\colon \Omega \to \mathbb{R}$ is μ-measurable, satisfies the bounds*

$$-\infty < m \le f(t) \le M < \infty \text{ for } \mu\text{-a.e. } t \in \Omega$$

and such that $f, \phi\circ f \in L_w(\Omega,\mu)$, then

$$\begin{aligned}
0 &\le \int_\Omega w(t)\phi(f(t))\mathrm{d}\mu(t) - \phi(\bar{f}_{\Omega,w}) \\
&\le (M-\bar{f}_{\Omega,w})(\bar{f}_{\Omega,w}-m)\frac{\phi'_-(M)-\phi'_+(m)}{M-m} \\
&\le \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)),
\end{aligned} \tag{1.3}$$

where $\bar{f}_{\Omega,w} := \int_\Omega w(t)f(t)\mathrm{d}\mu(t) \in [m,M]$.

In [38] Dragomir obtained a refinement of the previous result that we state in the following theorem.

Theorem 1.4 ([38]) *Let $\phi\colon I \to \mathbb{R}$ be a continuous convex function on an interval of real numbers I and let $m, M \in \mathbb{R}$, $m < M$ be such that the interval $[m,M]$ belongs to the interior of I. Let $w > 0$ be such that $\int w\mathrm{d}\mu = 1$. If $f\colon \Omega \to \mathbb{R}$ is μ-measurable, satisfies the bounds*

$$-\infty < m \le f(t) \le M < \infty \text{ for } \mu\text{-a.e. } t \in \Omega$$

and such that $f, \phi\circ f \in L_w(\Omega,\mu)$, then

$$\begin{aligned}
0 &\le \int_\Omega w(t)\phi(f(t))\mathrm{d}\mu(t) - \phi(\bar{f}_{\Omega,w}) \\
&\le \frac{(M-\bar{f}_{\Omega,w})(\bar{f}_{\Omega,w}-m)}{M-m}\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M) \\
&\le (M-\bar{f}_{\Omega,w})(\bar{f}_{\Omega,w}-m)\frac{\phi'_-(M)-\phi'_+(m)}{M-m} \\
&\le \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)),
\end{aligned} \tag{1.4}$$

where $\bar{f}_{\Omega,w} := \int_\Omega w(t)f(t)\mathrm{d}\mu(t) \in [m,M]$ and $\Psi_\phi(\cdot;m,M)\colon \langle m,M\rangle \to \mathbb{R}$ is defined by

$$\Psi_\phi(t;m,M) = \frac{\phi(M)-\phi(t)}{M-t} - \frac{\phi(t)-\phi(m)}{t-m}.$$

We also have inequalities

$$\begin{aligned}
0 &\le \int_\Omega w(t)\phi(f(t))\mathrm{d}\mu(t) - \phi(\bar{f}_{\Omega,w}) \le \frac{1}{4}(M-m)\Psi_\phi(t;m,M) \\
&\le \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)),
\end{aligned} \tag{1.5}$$

where $\bar{f}_{\Omega,w} \in \langle m,M\rangle$.

1.2 Converses of the Jensen and Edmundson-Lah--Ribarič inequality for linear functionals

Results that follow are obtained in [68] and they give an upper bound for the difference between the right and left side of the Jensen and Edmundson-Lah-Ribarič inequality respectively. First theorem is also a generalization of Dragomir's result (1.3) for linear functionals.

Theorem 1.5 *Let ϕ be a continuous convex function on the interval I whose interior contains interval $[m,M]$, let L be a vector space of real functions defined on a non-empty set E such that it has properties $(L1)$ and $(L2)$. Let A be any normalized positive linear functional on L. Then for every function $f \in L$ such that $\phi(f) \in L$ and which satisfies the bounds $m \leq f(t) \leq M$ for every $t \in E$ we have*

$$\begin{aligned}
0 &\leq A(\phi(f)) - \phi(A(f)) \\
&\leq (M - A(f))(A(f) - m)\frac{\phi'_-(M) - \phi'_+(m)}{M - m} \\
&\leq \frac{1}{4}(M - m)(\phi'_-(M) - \phi'_+(m)). \qquad (1.6)
\end{aligned}$$

If the function ϕ is concave on I, then the inequality signs in (1.6) are reversed.

Proof. Let ϕ be a convex function. The first inequality follows directly from Theorem 1.1. According to Theorem 1.2 we have

$$A(\phi(f)) - \phi(A(f)) \leq \frac{M - A(f)}{M - m}\phi(m) + \frac{A(f) - m}{M - m}\phi(M) - \phi(A(f)) =: z.$$

Because of the convexity of the function ϕ, the gradient inequality

$$\phi(t) - \phi(M) \geq \phi'_-(M)(t - M)$$

holds for every $t \in [m,M]$. If we multiply this inequality by $(t - m) \geq 0$, we get

$$(t - m)\phi(t) - (t - m)\phi(M) \geq \phi'_-(M)(t - M)(t - m), \quad t \in [m,M] \qquad (1.7)$$

In a similar manner we obtain:

$$(M - t)\phi(t) - (M - t)\phi(m) \geq \phi'_+(m)(t - m)(M - t), \quad t \in [m,M] \qquad (1.8)$$

When we add up (1.7) and (1.8) and then divide by $(m - M)$, we get that for every $t \in [m,M]$ it holds:

$$\frac{(t - m)\phi(M) + (M - t)\phi(m)}{M - m} - \phi(t) \leq \frac{(M - t)(t - m)}{M - m}(\phi'_-(M) - \phi'_+(m)). \qquad (1.9)$$

Since $A(f) \in [m,M]$, in the previous relation we can replace t with $A(f)$ and obtain the following

$$z \leq \frac{(M-A(f))(A(f)-m)}{M-m}(\phi'_-(M)-\phi'_+(m)),$$

what is exactly the second inequality in (1.6).

To prove the third inequlity in (1.6), we need to notice that inequality

$$\frac{1}{M-m}(M-t)(t-m) \leq \frac{1}{4}(M-m),$$

holds for every $t \in [m,M]$, and this proves the claim of the theorem.

If ϕ is a concave function, then the function $-\phi$ is convex, and we can apply inequalities (1.6) to the function $-\phi$, and reversed inequalities follow after multiplying by -1.

□

Remark 1.1 Observe that in the statement of Theorem 1.5 interval $[m,M]$ needs to belong to the interior of the interval I. This condition assures finiteness of the one-sided derivatives in (1.6). Without this assumption these derivatives might be infinite.

Theorem 1.6 *Let ϕ be a continuous convex function on the interval I whose interior contains interval $[m,M]$, let L be a vector space of real functions defined on a non-empty set E such that it has properties $(L1)$ and $(L2)$. Let A be any normalized positive linear functional on L. Then for every function $f \in L$ such that $\phi(f) \in L$ and which satisfies the bounds $m \leq f(t) \leq M$ for every $t \in E$ we have*

$$\begin{aligned} 0 &\leq \frac{M-A(f)}{M-m}\phi(m) + \frac{A(f)-m}{M-m}\phi(M) - A(\phi(f)) \\ &\leq \frac{\phi'_-(M)-\phi'_+(m)}{M-m}A([M-f][f-m]) \\ &\leq \frac{\phi'_-(M)-\phi'_+(m)}{M-m}(M-A(f))(A(f)-m) \\ &\leq \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)). \end{aligned} \tag{1.10}$$

If ϕ is concave, then the inequality signs in (1.10) are reversed.

Proof. Let ϕ be a convex function. The first inequality from (1.10) is obtained from (1.2) by subtracting $\phi(A(f))$ from both sides of the inequality. Since $f(t) \in [m,M]$, we can replace t by $f(t)$ i the relation (1.9), which gives us

$$\frac{M-f(t)}{M-m}\phi(m) + \frac{f(t)-m}{M-m}\phi(M) - \phi(f(t)) \leq \frac{(M-f(t))(f(t)-m)}{M-m}(\phi'_-(M)-\phi'_+(m)).$$

Function $h(t) = (M-t)(t-m)$ is concave on $[m,M]$, so when we apply the functional A to the previous inequality, because of its linearity and Jensen's inequality (1.1) we get the second inequality from (1.10):

$$
\begin{aligned}
&\frac{M-A(f)}{M-m}\phi(m)+\frac{A(f)-m}{M-m}\phi(M)-A(\phi(f))\\
&\quad\le \frac{(\phi'_-(m)-\phi'_+(m))}{M-m}A([M-f][f-m])\\
&\quad\le \frac{\phi'_-(M)-\phi'_+(m)}{M-m}(M-A(f))(A(f)-m).
\end{aligned}
$$

To prove the last inequality from (1.10), we need to notice that for every $t\in[m,M]$ we have $h(t)\le\frac{1}{4}(M-m)^2$. Since $A(f)\in[m,M]$, we also have

$$h(A(f))\le\frac{1}{4}(M-m)^2,$$

which completes the proof. □

Remark 1.2 Under the assumptions from the previous two theorems, let l be a linear function through points $(m,f(m))$ and $(M,f(M))$. Since ϕ is a convex function on $[m,M]$, the following relation

$$\phi(A(f))\le A(\phi(f))\le l(A(f))$$

holds for every $f\in L$ such that $\phi(f)\in L$.
From Theorem 1.5 and Theorem 1.6 we see that both differences

$$A(\phi(f))-\phi(A(f)) \text{ and } l(A(f))-A(\phi(f))$$

have the same estimation, so one can see that, in a weak sense, $A(\phi(f))$ is almost the mid point point between $\phi(A(f))$ and $l(A(f))$.

The following results are proved in [69], and they give refinements of sequences of inequalities obtained in Theorem 1.5 and Theorem 1.6. The first theorem that follows is also a generalization of Dragomir's results (1.4) and (1.5).

Theorem 1.7 *Let ϕ be a continuous convex function on the interval I whose interior contains interval $[m,M]$, let L be a vector space of real functions defined on a non-empty set E such that it has properties $(L1)$ and $(L2)$. Let A be any normalized positive linear functional on L. Then for every function $f\in L$ such that $\phi(f)\in L$ and which satisfies the bounds $m\le f(t)\le M$ for every $t\in E$ we have*

$$
\begin{aligned}
0&\le A(\phi(f))-\phi(A(f))\\
&\le (M-A(f))(A(f)-m)\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M)\\
&\le (M-A(f))(A(f)-m)\frac{\phi'_-(M)-\phi'_+(m)}{M-m}\\
&\le \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)).
\end{aligned}
\tag{1.11}
$$

We also have inequalities

$$
\begin{aligned}
0 \le A(\phi(f)) - \phi(A(f)) &\le \frac{1}{4}(M-m)^2 \Psi_\phi(A(f);m,M) \\
&\le \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)),
\end{aligned} \tag{1.12}
$$

where $\Psi_\phi(\cdot;m,M)\colon \langle m,M\rangle \to \mathbb{R}$ *is defined by*

$$
\Psi_\phi(t;m,M) = \frac{1}{M-m}\Big(\frac{\phi(M)-\phi(t)}{M-t} - \frac{\phi(t)-\phi(m)}{t-m}\Big). \tag{1.13}
$$

If ϕ *is concave on* I*, then the inequality signs are reversed.*

Proof. Let ϕ be a convex function. If $A(f) = m$ or $A(f) = M$, inequalities are trivial. Let us assume that $A(f) \in \langle m,M\rangle$.

The first inequality from (1.11) i (1.12) follows directly from Theorem 1.1. According to Theorem 1.2 we have

$$
\begin{aligned}
A(\phi(f)) - \phi(A(f)) &\le \frac{M-A(f)}{M-m}\phi(m) + \frac{A(f)-m}{M-m}\phi(M) - \phi(A(f)) \\
&= \frac{(M-A(f))(A(f)-m)}{M-m}\Big\{\frac{\phi(M)-\phi(A(f))}{M-A(f)} - \frac{\phi(A(f))-\phi(m)}{A(f)-m}\Big\} \\
&= (M-A(f))(A(f)-m)\Psi_\phi(A(f);m,M) \\
&\le (M-A(f))(A(f)-m)\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M),
\end{aligned}
$$

and we see that the second inequality from (1.11) holds. Further,

$$
\begin{aligned}
\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M) &= \frac{1}{M-m}\sup_{t\in\langle m,M\rangle}\Big\{\frac{\phi(M)-\phi(t)}{M-t} - \frac{\phi(t)-\phi(m)}{t-m}\Big\} \\
&\le \frac{1}{M-m}\Big(\sup_{t\in\langle m,M\rangle}\frac{\phi(M)-\phi(t)}{M-t} + \sup_{t\in\langle m,M\rangle}\frac{-(\phi(t)-\phi(m))}{t-m}\Big) \\
&= \frac{1}{M-m}\Big(\sup_{t\in\langle m,M\rangle}\frac{\phi(M)-\phi(t)}{M-t} - \inf_{t\in\langle m,M\rangle}\frac{\phi(t)-\phi(m)}{t-m}\Big) \\
&= \frac{\phi'_-(M)-\phi'_+(m)}{M-m},
\end{aligned}
$$

which proves the third inequality from (1.11). The last inequality in (1.11) follows from the fact that for every $t \in [m,M]$ we have $\frac{(M-t)(t-m)}{M-m} \le \frac{1}{4}(M-m)$. Since $A(f) \in [m,M]$, we can replace t with $A(f)$ in the previous inequality.

The proof for inequalities (1.12) is obvious from the proof for (1.11). □

Remark 1.3 Observe that $\Psi_\phi(\cdot;m,M)$, defined in (1.13), is actually second order divided difference $[m,t,M]\phi$ of the function ϕ in points m, t and M for every $t \in \langle m,M\rangle$.

In order to prove a converse of the Edmundson-Lah-Ribarič inequality, first we need the following result from [69].

Lemma 1.1 *Let ϕ be a convex function on an interval of real numbers I, and let $m,M \in \mathbb{R}$, $m < M$ be such that the interval $[m,M]$ belongs to the interior of I. Then for every $t \in [m,M]$ the following inequalities hold:*

$$\begin{aligned}\Delta_\phi(t;m,M) &= \frac{t-m}{M-m}\phi(M) + \frac{M-t}{M-m}\phi(m) - \phi(t)\\ &\leq (M-t)(t-m) \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M)\\ &\leq \frac{(M-t)(t-m)}{M-m}(\phi'_-(M) - \phi'_+(m))\\ &\leq \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)).\end{aligned} \tag{1.14}$$

We also have

$$\Delta_\phi(t;m,M) \leq \frac{1}{4}(M-m)^2\Psi_\phi(t;m,M) \leq \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)),$$

where $\Psi_\phi(\cdot;m,M)\colon \langle m,M\rangle \to \mathbb{R}$ is defined by (1.13) If the function ϕ is concave, then the inequality signs are reversed.

Proof. Let ϕ be a convex function. If $t = m$ or $t = M$, inequalities are trivial. For any $t \in \langle m,M\rangle$ it holds

$$\begin{aligned}\Delta_\phi(t;m,M) &= \frac{t-m}{M-m}\phi(M) + \frac{M-t}{M-m}\phi(m) - \phi(t)\\ &= \frac{(M-t)(t-m)}{M-m}\left[\frac{\phi(M)-\phi(t)}{M-t} - \frac{\phi(t)-\phi(m)}{t-m}\right]\\ &= (M-t)(t-m)\Psi_\phi(t;m,M)\\ &\leq (M-t)(t-m) \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M),\end{aligned}$$

which is exactly the first inequality from (1.14). The second inequality follows directly from:

$$\begin{aligned}\sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M) &= \frac{1}{M-m} \sup_{t\in\langle m,M\rangle} \left\{\frac{\phi(M)-\phi(t)}{M-t} - \frac{\phi(t)-\phi(m)}{t-m}\right\}\\ &\leq \frac{1}{M-m}\left(\sup_{t\in\langle m,M\rangle} \frac{\phi(M)-\phi(t)}{M-t} + \sup_{t\in\langle m,M\rangle} \frac{-(\phi(t)-\phi(m))}{t-m}\right)\\ &= \frac{1}{M-m}\left(\sup_{t\in\langle m,M\rangle} \frac{\phi(M)-\phi(t)}{M-t} - \inf_{t\in\langle m,M\rangle} \frac{\phi(t)-\phi(m)}{t-m}\right)\\ &= \frac{\phi'_-(M) - \phi'_+(m)}{M-m}.\end{aligned}$$

The last inequality from (1.14) follows directly from

$$\frac{(M-t)(t-m)}{M-m} \leq \frac{1}{4}(M-m) \text{ for every } t \in [m,M].$$

The proof of the inequalities (1.1) is clear from the proof of (1.14). □

Theorem 1.8 *Let ϕ be a continuous convex function on the interval I whose interior contains interval $[m,M]$, let L be a vector space of real functions defined on a non-empty set E such that it has properties $(L1)$ and $(L2)$. Let A be any normalized positive linear functional on L. Then for every function $f \in L$ such that $\phi(f) \in L$ and which satisfies the bounds $m \leq f(t) \leq M$ for every $t \in E$ we have*

(i)

$$\begin{aligned} 0 &\leq \frac{A(f)-m}{M-m}\phi(M) + \frac{M-A(f)}{M-m}\phi(m) - A(\phi(f)) \\ &\leq A[(M-f)(f-m)] \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M) \\ &\leq \frac{A[(M-f)(f-m)]}{M-m}(\phi'_-(M) - \phi'_+(m)) \\ &\leq \frac{(M-A(f))(A(f)-m)}{M-m}(\phi'_-(M) - \phi'_+(m)) \\ &\leq \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)) \end{aligned} \tag{1.15}$$

(ii)

$$\begin{aligned} 0 &\leq \frac{A(f)-m}{M-m}\phi(M) + \frac{M-A(f)}{M-m}\phi(m) - A(\phi(f)) \\ &\leq A[(M-f)(f-m)] \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M) \\ &\leq (M-A(f))(A(f)-m) \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M) \\ &\leq \frac{(M-A(f))(A(f)-m)}{M-m}(\phi'_-(M) - \phi'_+(m)) \\ &\leq \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)) \end{aligned} \tag{1.16}$$

(iii)

$$\begin{aligned} 0 &\leq \frac{A(f)-m}{M-m}\phi(M) + \frac{M-A(f)}{M-m}\phi(m) - A(\phi(f)) \\ &\leq \frac{1}{4}(M-m)^2 A(\Psi_\phi(t;m,M)) \\ &\leq \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)) \end{aligned} \tag{1.17}$$

where $\Psi_\phi(\cdot;m,M)\colon \langle m,M\rangle \to \mathbb{R}$ *is defined in (1.13). If the function* ϕ *is concave, then the inequality signs are reversed.*

Proof. Let ϕ be a convex function. The first inequalities from (1.15), (1.16) and (1.17) follow directly from Theorem 1.2.

Since f satisfies the bounds $m \le f(t) \le M$ for every $t \in [m,M]$, we can replace t with $f(t)$ in (1.14) and (1.1) from Lemma 1.1 and obtain

$$\begin{aligned}
&\frac{f(t)-m}{M-m}\phi(M) + \frac{M-f(t)}{M-m}\phi(m) - \phi(f(t))\\
&\quad \le (M-f(t))(f(t)-m) \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M)\\
&\quad \le \frac{(M-f(t))(f(t)-m)}{M-m}(\phi'_-(M)-\phi'_+(m))\\
&\quad \le \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m))
\end{aligned}$$

and

$$\begin{aligned}
&\frac{f(t)-m}{M-m}\phi(M) + \frac{M-f(t)}{M-m}\phi(m) - \phi(f(t))\\
&\quad \le \frac{1}{4}(M-m)^2\Psi_\phi(f;m,M)\\
&\quad \le \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)).
\end{aligned}$$

Next, we apply linear functional A, which is normalized, to the previous sequences of inequalities, and that gives us (1.17) and first three inequalities from (1.15) respectively. Since for every $t \in [m,M]$ we have $\dfrac{(M-t)(t-m)}{M-m} \le \dfrac{1}{4}(M-m)$, the same inequality holds for $A(f) \in [m,M]$. In that way we get the last inequality from (1.15).

The first inequality from (1.16) is the same as the first inequality from (1.15). Function $g(t) = (M-t)(t-m)$ is concave, so according to Jessen's inequality (1.1) we have

$$A([M-f][f-m]) \le (M-A(f))(A(f)-m),$$

which provides the second inequality from (1.16). In the proof of Lemma 1.1 we showed that

$$\sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M) \le \frac{\phi'_-(M)-\phi'_+(m)}{M-m},$$

so the third inequality from (1.16) easily follows. As before, the last inequality in (1.16) follows from $\dfrac{(M-A(f))(A(f)-m)}{M-m} \le \dfrac{1}{4}(M-m)$. □

Remark 1.4 The function ϕ is defined on the interval I whose interior contains the interval $[m,M]$. This condition ensures finiteness of the one-sided derivatives in points m and M. Then

$$\lim_{t\to m^+} \Psi_\phi(t;m,M) = \frac{1}{M-m}\left[\frac{\phi(M)-\phi(m)}{M-m} - \phi'_+(m)\right]$$

and

$$\lim_{t \to M^-} \Psi_\phi(t;m,M) = \frac{1}{M-m}\left[\phi'_-(M) - \frac{\phi(t)-\phi(m)}{t-m}\right],$$

so $\Psi_\phi(\cdot;m,M)$ can be observed as a continuous function (in parameter t) on the interval $[m,M]$. Therefore, if the function f satisfies bounds $m \le f(t) \le M$ for every $t \in E$, then the expression $\Psi_\phi(f(t);m,M)$ is meaningful.

In order to state an improvement of the Edmundson-Lah-Ribarič inequality (1.2) obtained by Klaričić Bakula, Pečarić and Perić in [80], the vector space of real functions L defined on a non-empty set E aditionally needs to satisfy the condition $(L3)$ stated in Introduction.

Theorem 1.9 ([80]) *Let L be a vector space of real functions defined on a non-empty set E that has properties $(L1)$, $(L2)$ and $(L3)$ and let A be a normalized positive linear functional on L. If ϕ is a convex function on $[m,M]$, then for every $f \in L$ such that $\phi(f) \in L$ we have $A(f) \in [m,M]$ and*

$$A(\phi(f)) \le \frac{M-A(f)}{M-m}\phi(m) + \frac{A(f)-m}{M-m}\phi(M) - A(\tilde{f})\delta_\phi, \tag{1.18}$$

where

$$\tilde{f} = \frac{1}{2}\mathbf{1} - \frac{1}{M-m}\left|f - \frac{m+M}{2}\mathbf{1}\right|, \quad \delta_\phi = \phi(m)+\phi(M)-2\phi\left(\frac{m+M}{2}\right). \tag{1.19}$$

Remark 1.5 When applied to an apropriate vector space of real functions L, inequality (1.18) from Theorem 1.9 is clearly an improvement of the Edmundson-Lah-Ribarič inequality (1.2), since under the required assumptions we have

$$A(\tilde{f})\delta_\phi = A\left(\frac{1}{2}\mathbf{1} - \frac{|f - \frac{m+M}{2}\mathbf{1}|}{M-m}\right)\left(\phi(m)+\phi(M)-2\phi\left(\frac{m+M}{2}\right)\right) \ge 0.$$

Next two results give improvements of Theorem 1.7 and Theorem 1.8 respectively. They are proved in an analogous way as the previous two theorems, only instead of Edmundson-Lah-Ribarič inequality, its improvement (1.18) was used.

Theorem 1.10 *Let ϕ be a continuous convex function on the interval I whose interior contains interval $[m,M]$, let L be a vector space of real functions defined on a non-empty set E such that it has properties $(L1)$, $(L2)$ and $(L3)$. Let A be any normalized positive linear functional on L. Then for every function $f \in L$ such that $\phi(f) \in L$ and which satisfies the bounds $m \le f(t) \le M$ for every $t \in E$ we have*

$$\begin{aligned}
0 &\le A(\phi(f)) - \phi(A(f)) \\
&\le (M-A(f))(A(f)-m)\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M) - A(\tilde{f})\delta_\phi \\
&\le (M-A(f))(A(f)-m)\frac{\phi'_-(M)-\phi'_+(m)}{M-m} - A(\tilde{f})\delta_\phi \\
&\le \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)) - A(\tilde{f})\delta_\phi.
\end{aligned} \tag{1.20}$$

We also have

$$0 \le A(\phi(f)) - \phi(A(f)) \le \frac{1}{4}(M-m)^2 \Psi_\phi(A(f);m,M) - A(\tilde{f})\delta_\phi$$
$$\le \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)) - A(\tilde{f})\delta_\phi, \tag{1.21}$$

where $\tilde{f}$ *and* δ_ϕ *are defined in (1.19), and we assume that* $\Psi_\phi(f;m,M) \in L$, *where* $\Psi_\phi(\cdot;m,M)\colon \langle m,M\rangle \to \mathbb{R}$ *is defined in (1.13). If* ϕ *is concave on* I, *then the inequality signs are reversed.*

Proof. First we need to note that according to the property ($L3$) it holds

$$\tilde{f} = \min\left\{\frac{M-f(x)}{M-m}, \frac{f(x)-m}{M-m}\right\} = \frac{1}{2}\mathbf{1} - \frac{|f - \frac{m+M}{2}\mathbf{1}|}{M-m} \in L.$$

If $A(f) = m$ or $A(f) = M$, inequalities are trivial. Let us assume that $A(f) \in \langle m,M\rangle$ and let ϕ be a convex function.

The first inequality in (1.20) is a direct consequence of Jessen's inequality (1.1). According to Theorem 1.9 we have

$$\begin{aligned}
A(\phi(f)) - \phi(A(f)) &\le \frac{M-A(f)}{M-m}\phi(m) + \frac{A(f)-m}{M-m}\phi(M) - \phi(A(f)) - A(\tilde{f})\delta_\phi \\
&= \frac{(M-A(f))(A(f)-m)}{M-m}\left\{\frac{\phi(M)-\phi(A(f))}{M-A(f)} - \frac{\phi(A(f))-\phi(m)}{A(f)-m}\right\} - A(\tilde{f})\delta_\phi \\
&= (M-A(f))(A(f)-m)\Psi_\phi(A(f);m,M) - A(\tilde{f})\delta_\phi \\
&\le (M-A(f))(A(f)-m)\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M) - A(\tilde{f})\delta_\phi,
\end{aligned}$$

which proves the second inequality in (1.20). Next,

$$\begin{aligned}
\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M) &= \frac{1}{M-m}\sup_{t\in\langle m,M\rangle}\left\{\frac{\phi(M)-\phi(t)}{M-t} - \frac{\phi(t)-\phi(m)}{t-m}\right\} \\
&\le \frac{1}{M-m}\left(\sup_{t\in\langle m,M\rangle}\frac{\phi(M)-\phi(t)}{M-t} + \sup_{t\in\langle m,M\rangle}\frac{-(\phi(t)-\phi(m))}{t-m}\right) \\
&= \frac{1}{M-m}\left(\sup_{t\in\langle m,M\rangle}\frac{\phi(M)-\phi(t)}{M-t} - \inf_{t\in\langle m,M\rangle}\frac{\phi(t)-\phi(m)}{t-m}\right) = \frac{\phi'_-(M)-\phi'_+(m)}{M-m},
\end{aligned}$$

and the third inequality from (1.20) is proved. We have already seen that

$$\frac{(M-A(f))(A(f)-m)}{M-m} \le \frac{1}{4}(M-m)$$

holds, so the last inequality from (1.20) follows directly.

Proof of the inequalities (1.21) is clear from the proof of inequalities (1.20). □

Theorem 1.11 *Let ϕ be a continuous convex function on the interval I whose interior contains interval $[m,M]$, let L be a vector space of real functions defined on a non-empty set E such that it has properties $(L1)$, $(L2)$ and $(L3)$. Let A be any normalized positive linear functional on L. Then for every function $f \in L$ such that $\phi(f) \in L$ and which satisfies the bounds $m \le f(t) \le M$ for every $t \in E$ we have the following sequences of inequalities*

(i)

$$\begin{aligned}
0 &\le \frac{A(f)-m}{M-m}\phi(M) + \frac{M-A(f)}{M-m}\phi(m) - A(\phi(f)) - A(\tilde{f})\delta_\phi \\
&\le A[(M-f)(f-m)] \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M) - A(\tilde{f})\delta_\phi \\
&\le \frac{A[(M-f)(f-m)]}{M-m}(\phi'_-(M)-\phi'_+(m)) - A(\tilde{f})\delta_\phi \\
&\le \frac{(M-A(f))(A(f)-m)}{M-m}(\phi'_-(M)-\phi'_+(m)) - A(\tilde{f})\delta_\phi \\
&\le \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)) - A(\tilde{f})\delta_\phi
\end{aligned} \tag{1.22}$$

(ii)

$$\begin{aligned}
0 &\le \frac{A(f)-m}{M-m}\phi(M) + \frac{M-A(f)}{M-m}\phi(m) - A(\phi(f)) - A(\tilde{f})\delta_\phi \\
&\le A[(M-f)(f-m)] \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M) - A(\tilde{f})\delta_\phi \\
&\le (M-A(f))(A(f)-m) \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M) - A(\tilde{f})\delta_\phi \\
&\le \frac{(M-A(f))(A(f)-m)}{M-m}(\phi'_-(M)-\phi'_+(m)) - A(\tilde{f})\delta_\phi \\
&\le \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)) - A(\tilde{f})\delta_\phi
\end{aligned} \tag{1.23}$$

(iii)

$$\begin{aligned}
0 &\le \frac{A(f)-m}{M-m}\phi(M) + \frac{M-A(f)}{M-m}\phi(m) - A(\phi(f)) - A(\tilde{f})\delta_\phi \\
&\le \frac{1}{4}(M-m)^2 A(\Psi_\phi(f;m,M)) - A(\tilde{f})\delta_\phi \\
&\le \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)) - A(\tilde{f})\delta_\phi
\end{aligned} \tag{1.24}$$

where $\tilde{f}$ and δ_ϕ are defined in (1.19), and $\Psi_\phi(\cdot;m,M)$ is defined in (1.13). If the function ϕ is concave, then the inequality signs are reversed.

Proof. In the proof of Theorem 1.10 it has been shown that $\tilde{f} \in L$. First inequalities in (1.22), (1.23) and (1.24) are obtained from (1.18) by subtracting $A(\phi(f))$ from both sides of the inequality. Because f satisfies the bounds $m \leq f(t) \leq M$ for every $t \in [m,M]$, we can replace t with $f(t)$ in inequalities (1.14) and (1.1) from Lemma 1.1, which gives

$$\begin{aligned}\frac{f(t)-m}{M-m}\phi(M)+\frac{M-f(t)}{M-m}\phi(m)-\phi(f(t)) &\leq (M-f(t))(f(t)-m)\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M)\\ &\leq \frac{(M-f(t))(f(t)-m)}{M-m}(\phi'_-(M)-\phi'_+(m))\\ &\leq \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m))\end{aligned}$$

and

$$\begin{aligned}\frac{f(t)-m}{M-m}\phi(M)+\frac{M-f(t)}{M-m}\phi(m)-\phi(f(t)) &\leq \frac{1}{4}(M-m)^2\Psi_\phi(f;m,M)\\ &\leq \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)).\end{aligned}$$

To prove (1.24) and first three inequalities from (1.22), we apply positive linear functional A to the previous sequences of inequalities, and then subtract $A(\tilde{f})\delta_\phi$ from each side of those inequalities. The fourth inequality in (1.22) follows from the fact that the function $g(t)=(M-t)(t-m)$ is concave, so Jessen's inequality gives us

$$A(g(f))-A(\tilde{f})\delta_\phi \leq g(A(f))-A(\tilde{f})\delta_\phi.$$

Since for every $t\in[m,M]$ we have

$$\frac{(M-t)(t-m)}{M-m}-A(\tilde{f})\delta_\phi \leq \frac{1}{4}(M-m)-A(\tilde{f})\delta_\phi,$$

and $A(f)\in[m,M]$, the last inequality from (1.22) follows.

First inequalities in (1.23) and (1.22) are the same. Again, we use the concavity of the function $g(t)=(M-t)(t-m)$. When we subtract $A(\tilde{f})\delta_\phi$ from both sides in Jessen's inequality, we get

$$A([M-f][f-m])-A(\tilde{f})\delta_\phi \leq (M-A(f))(A(f)-m)-A(\tilde{f})\delta_\phi,$$

which proves the second inequality in (1.23). In the proof of Lemma 1.1 we have shown that

$$\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M)\leq \frac{\phi'_-(M)-\phi'_+(m)}{M-m},$$

and the third inequality in (1.23) follows by subtracting $A(\tilde{f})\delta_\phi$ from both sides of the mentioned inequality. Since for every $t\in[m,M]$ it holds $\frac{(M-t)(t-m)}{M-m}\leq\frac{1}{4}(M-m)$, and since $A(f)\in[m,M]$, we immediately see that

$$\frac{(M-A(f))(A(f)-m)}{M-m}-A(\tilde{f})\delta_\phi \leq \frac{1}{4}(M-m)-A(\tilde{f})\delta_\phi,$$

and the last inequality from (1.23) is proved. □

Recently, Pečarić and Perić in [122], established even more accurate version of the Edmundson-Lah-Ribarič inequality. The corresponding result is derived by virtue of the refinement of the Jensen inequality via linear interpolation obtained by Choi et.al. [31].

Let the functions $r_n(v)$ be defined recursively:

$$r_0 = \min\{v, 1-v\}$$
$$r_n = \min\{2r_{n-1}(v), 1-2r_{n-1}(v)\}$$

for $0 \leq v \leq 1$. The functions r_n, $n \in \mathbb{N}$, are non-negative and it has been shown in [31] that they can be rewritten in an explicit form

$$r_n(t) = \begin{cases} 2^n t - k + 1, & \frac{k-1}{2^n} \leq t \leq \frac{2k-1}{2^{n+1}}, \\ k - 2^n t, & \frac{2k-1}{2^{n+1}} < t \leq \frac{k}{2^n}, \end{cases} \tag{1.25}$$

for $k = 1, 2, \ldots, 2^n$.

It has been shown in [31] that if N is a nonnegative integer and f is convex on $[0,1]$, then

$$(1-v)f(0) + vf(1) \geq f(v) + \sum_{n=0}^{N-1} r_n(v) \sum_{k=1}^{2^n} \Delta_f(n,k) \chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)}(v),$$

where

$$\Delta_f(n,k) = f\left(\frac{k-1}{2^n}\right) + f\left(\frac{k}{2^n}\right) - 2f\left(\frac{2k-1}{2^{n+1}}\right)$$

and χ represents the characteristic function of the corresponding interval. If $N = 0$ then sum is zero, that is we have convexity.

In the paper [31] previous relation is extended to hold for an arbitrary interval.

Lemma 1.2 ([31]) *Let N be a nonnegative integer and let f be convex on $[a,b]$. Then*

$$(1-v)f(a) + vf(b) \geq f((1-v)a + vb) + \sum_{n=0}^{N-1} r_n(v) \sum_{k=1}^{2^n} \Delta_f(a,b,n,k) \chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)}(v), \tag{1.26}$$

where

$$\Delta_f(a,b,n,k) = f\Big(\frac{(2^n-k+1)a + (k-1)b}{2^n}\Big) + f\Big(\frac{(2^n-k)a + kb}{2^n}\Big)$$
$$- 2f\Big(\frac{(2^{n+1}-2k+1)a + (2k-1)b}{2^{n+1}}\Big) \tag{1.27}$$

and χ represents the characteristic function of the corresponding interval.

Theorem 1.12 ([122]) *Let $\phi : [m,M] \to \mathbb{R}$ be a convex function and $f \in L$ be such that $\phi \circ f \in L$. Then, $A(f) \in [m,M]$ and*

$$A(\phi(f)) \leq \frac{M - A(f)}{M-m}\phi(m) + \frac{A(f) - m}{M-m}\phi(M) - R_{\phi,A}(m,M;f), \tag{1.28}$$

where

$$R_{\phi,A}(m,M;f) = \sum_{n=0}^{N-1}\sum_{k=1}^{2^n} \Delta_\phi(m,M,n,k) A\Big(r_n \chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}\Big(\frac{f-m\mathbf{1}}{M-m}\Big)\Big), \tag{1.29}$$

$\Delta_\phi(m,M,n,k)$ is defined in (1.27), and where χ stands for the characteristic function of the corresponding interval.

Proof. First observe that $\phi(f)\in L$ also means that the composition $\phi(f)$ is well defined, hence $f(E)\in[m,M]$. Now we have $m\mathbf{1}\le f\le M\mathbf{1}$ and

$$m = A(m\mathbf{1}) \le A(f) \le A(M\mathbf{1}) = M.$$

If we put $a=m$, $b=M$, $x=(1-v)a+vb$ in (1.26) from Lemma 1.2 using

$$v = \frac{x-m}{M-m}, \quad 1-v = \frac{M-x}{M-m}$$

we get

$$\begin{aligned}&\frac{M-x}{M-m}\phi(m) + \frac{x-m}{M-m}\phi(M)\\ &\quad \ge \phi(x) + \sum_{n=0}^{N-1} r_n\left(\frac{x-m}{M-m}\right)\sum_{k=1}^{2^n}\Delta_\phi(m,M,n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}\left(\frac{x-m}{M-m}\right).\end{aligned}$$

Let $f\in L$ be such that $\phi(f)\in L$. Applying the functional A to the above inequality with $x\longleftrightarrow f(x)$ we obtain

$$\begin{aligned}&\frac{M-A(f)}{M-m}\phi(m) + \frac{A(f)-m}{M-m}\phi(M)\\ &\quad \ge A(\phi(f)) + \sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\Delta_\phi(m,M,n,k)A\left(r_n\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}\left(\frac{f-m\mathbf{1}}{M-m}\right)\right)\end{aligned}$$

which is inequality (1.28). □

Remark 1.6 Any summation having $\sum_{n=0}^{N-1}$ is assumed to be zero for $N=0$, therefore inequality (1.28) may be regarded as a generalization of inequality (1.18). In addition, if $N\ge 1$, then $R_{\phi,A}(m,M;f)$ can be rewritten in the following way:

$$\begin{aligned}R_{\phi,A}(m,M;f) =&\Delta_\phi(m,M,0,1)A\Big(r_0\chi_{(0,1)}\Big(\frac{f-m}{M-m}\Big)\Big)\\ &+\sum_{n=1}^{N-1}\sum_{k=1}^{2^n}\Delta_\phi(m,M,n,k)A\Big(r_n\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}\Big(\frac{f-m}{M-m}\Big)\Big).\end{aligned}$$

Now, since $\chi_{(0,1)}\Big(\frac{f-m}{M-m}\Big)=1$, $\Delta_\phi(m,M,0,1)=\phi(m)+\phi(M)-2\phi\Big(\frac{m+M}{2}\Big)$, and

$$r_0\Big(\frac{f-m}{M-m}\Big) = \min\Big\{\frac{f-m}{M-m}, 1-\frac{f-m}{M-m}\Big\} = \frac{1}{2} - \frac{|f-\frac{m+M}{2}|}{M-m},$$

it follows that the inequality (1.28) provides sharper estimate for the Edmundson-Lah-Ribarič inequality than inequality (1.18).

Corollary 1.1 ([122]) *Let* $\boldsymbol{p}$ *be a nonnegative l-tuple with* $P_l = \sum_{i=1}^{l} p_i \neq 0$ *and* $\boldsymbol{x} \in [m,M]^l$. *If* $f\colon [m,M] \to \mathbb{R}$ *is a convex function then*

$$\begin{aligned}\frac{1}{P_l}\ \sum_{i=1}^{l}\ p_i f(x_i) &\le \frac{M-\overline{x}}{M-m} f(m) + \frac{\overline{x}-m}{M-m} f(M) \\ &\quad -\frac{1}{P_l}\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\sum_{i=1}^{l}\Delta_f(m,M,n,k)p_i\left[\left(r_n\cdot\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\right)\left(\frac{x_i-m}{M-m}\right)\right] \\ &= \frac{M-\overline{x}}{M-m} f(m) + \frac{\overline{x}-m}{M-m} f(M) \\ &\quad -\frac{1}{P_l}\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\sum_{i=1}^{l}\Delta_f(m,M,n,k)p_i\left[\left(2^n\frac{x_i-m}{M-m}-k+1\right)\cdot\chi_{\left(\frac{k-1}{2^n},\frac{2k-1}{2^{n+1}}\right)}\left(\frac{x_i-m}{M-m}\right)\right. \\ &\quad \left.+\left(k-2^n\frac{x_i-m}{M-m}\right)\chi_{\left(\frac{2k-1}{2^{n+1}},\frac{k}{2^n}\right)}\left(\frac{x_i-m}{M-m}\right)\right]\end{aligned} \tag{1.30}$$

where $\overline{x} = \frac{1}{P_l}\sum_{i=1}^{l} p_i x_i$.

Proof. If we consider $E = [m,M]$, $L = \mathbb{R}^{[m,M]}$, $g = id_E$, $A(f) = \frac{1}{P_l}\sum_{i=1}^{l} p_i f(x_i)$ in Theorem 1.12, then inequality (1.28) becomes (1.30). □

According to Remark 1.6 we can give strengthened Theorems 1.10 and 1.11. More precisely, following the lines of the proofs of Theorems 1.10 and 1.11 with a term $R_{\phi,A}(m,M;f)$ instead of $A(\tilde{f})\delta_\phi$, and taking into account relation (1.28), we give now sharper forms for converses of the Jensen and Edmundson-Lah-Ribarič inequalities than those established in Theorems 1.10 and 1.11.

Theorem 1.13 ([81]) *Let* $\phi : I \to \mathbb{R}$ *be a continuous convex function and* $[m,M] \subseteq \operatorname{Int} I$. *If* $f \in L$ *is such that* $f(E) \subseteq [m,M]$ *and* $\phi \circ f \in L$, *then*

$$\begin{aligned}0 &\le A(\phi(f)) - \phi(A(f)) \\ &\le (M-A(f))(A(f)-m)\sup_{t\in(m,M)}\Psi_\phi(t;m,M) - R_{\phi,A}(m,M;f) \\ &\le (M-A(f))(A(f)-m)\frac{\phi'_-(M)-\phi'_+(m)}{M-m} - R_{\phi,A}(m,M;f) \\ &\le \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)) - R_{\phi,A}(m,M;f)\end{aligned} \tag{1.31}$$

and

$$\begin{aligned}0 \le A(\phi(f)) - \phi(A(f)) &\le \frac{1}{4}(M-m)^2\Psi_\phi(A(f);m,M) - R_{\phi,A}(m,M;f) \\ &\le \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)) - R_{\phi,A}(m,M;f),\end{aligned} \tag{1.32}$$

where Ψ_ϕ *and* $R_{\phi,A}$ *are defined by* (1.13) *and* (1.29). *If* ϕ *is concave on* I, *then the inequality signs in* (1.31) *and* (1.32) *are reversed.*

Theorem 1.14 ([81]) *Let $\phi : I \to \mathbb{R}$ be a continuous convex function and $[m,M] \subseteq \operatorname{Int} I$. If $f \in L$ is such that $f(E) \subseteq [m,M]$ and $\phi \circ f \in L$, then*

$$\begin{aligned}
0 &\le \frac{A(f)-m}{M-m}\phi(M)+\frac{M-A(f)}{M-m}\phi(m)-A(\phi(f))-R_{\phi,A}(m,M;f)\\
&\le A[(M-f)(f-m)]\sup_{t\in(m,M)}\Psi_\phi(t;m,M)-R_{\phi,A}(m,M;f)\\
&\le \frac{A[(M-f)(f-m)]}{M-m}(\phi'_-(M)-\phi'_+(m))-R_{\phi,A}(m,M;f) \qquad (1.33)\\
&\le \frac{(M-A(f))(A(f)-m)}{M-m}(\phi'_-(M)-\phi'_+(m))-R_{\phi,A}(m,M;f)\\
&\le \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m))-R_{\phi,A}(m,M;f)
\end{aligned}$$

$$\begin{aligned}
0 &\le \frac{A(f)-m}{M-m}\phi(M)+\frac{M-A(f)}{M-m}\phi(m)-A(\phi(f))-R_{\phi,A}(m,M;f)\\
&\le A[(M-f)(f-m)]\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M)-R_{\phi,A}(m,M;f)\\
&\le (M-A(f))(A(f)-m)\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M)-R_{\phi,A}(m,M;f)\\
&\le \frac{(M-A(f))(A(f)-m)}{M-m}(\phi'_-(M)-\phi'_+(m))-R_{\phi,A}(m,M;f)\\
&\le \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m))-R_{\phi,A}(m,M;f) \qquad (1.34)
\end{aligned}$$

and

$$\begin{aligned}
0 &\le \frac{A(f)-m}{M-m}\phi(M)+\frac{M-A(f)}{M-m}\phi(m)-A(\phi(f))-R_{\phi,A}(m,M;f)\\
&\le \frac{1}{4}(M-m)^2A(\Psi_\phi(f;m,M))-R_{\phi,A}(m,M;f) \qquad (1.35)\\
&\le \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m))-R_{\phi,A}(m,M;f)
\end{aligned}$$

where Ψ_ϕ and $R_{\phi,A}$ are defined by (1.13) *and* (1.29). *If ϕ is concave on I, then the inequality signs in* (1.33), (1.34) *and* (1.35) *are reversed.*

Remark 1.7 Results presented in this section cover the classical discrete and integral case. Namely, common examples of positive linear functionals are $A(f)=\int_E f d\mu$ or $A(f)=\sum_{k\in E} p_k f_k$, where μ is positive measure on E in the first case, and in the other, $E=\mathbb{N}$ is a countable set with the discrete measure $\mu(k)=p_k \ge 0$, $0<\sum_{k\in E} p_k<\infty$, $f(k)=f_k$, defined on it.

Moreover, let $X : \Omega \to [m,M]$ be a random variable on a probability space (Ω, p) with finite expectation $\mathbb{E}[X]$. Then, setting $A=\mathbb{E}$ and $f=X$, all the theorems yield probabilistic versions of converses for the Jensen and Edmundson-Lah-Ribarič inequalities, provided that $\mathbb{E}[\phi(X)]<\infty$.

1.3 Applications

In this section we will apply Theorem 1.13 and Theorem 1.14 to some of the classical inequalities and in that way obtain upper bounds for the difference of right and left sides of those inequalities. Analogous applications can be obtained either from Theorem 1.5 and Theorem 1.6, or Theorem 1.7 and Theorem 1.8, or Theorem 1.10 and Theorem 1.11. Mentioned results can be found in [68], [69] and [81] respectively.

1.3.1 Generalized means

Definition 1.1 *Let $I = \langle a,b \rangle$, where $-\infty \leq a < b \leq \infty$, and let $\psi \colon I \to \mathbb{R}$ be a continuous and strictly monotone function. Let us assume that vector space of real functions L on a non-empty set E has properties $(L1)$, $(L2)$ and $(L3)$. Let A be a normalized positive linear functional on L, and let $\psi(f) \in L$ for a function $f \in L$. Generalized mean of the function $f \in L$ with respect to the functional A and function ψ is*

$$M_\psi(f,A) = \psi^{-1}(A(\psi(f))).$$

Note that if $\alpha \leq \psi(f(t)) \leq \beta$ for every $t \in E$, then because of the positivity of the functional A we have $\alpha \leq A(\psi(f)) \leq \beta$, so $M_\psi(f,A)$ is well defined. Also, note that because of the above assumptions we have $f(t) \in I$ for $t \in E$. From now on we assume that $f \in L$ satisfies the above assumptions, so the obtained result are valid only for such functions $f \in L$.

First we will state some already known results involving generalized means. Proofs of those results can be found in [124].

Theorem 1.15 ([124]) *Let $I = \langle a,b \rangle$, where $-\infty \leq a < b \leq \infty$, and let $\psi, \chi \colon I \to \mathbb{R}$ be continuous and strictly monotone functions. Assume that vector space of real functions L defined on a non-empty set E has properties $(L1)$ and $(L2)$. Let A be a normalized positive linear functional on L, and let $f \in L$ be such that $\psi(f), \chi(f) \in L$. Then we have*

$$M_\psi(f,A) \leq M_\chi(f,A),$$

under the assumption that either χ is increasing and $\phi = \chi \circ \psi^{-1}$ is convex, or χ is decreasing and $\phi = \chi \circ \psi^{-1}$ is concave.

Theorem 1.16 ([124]) *Let $I = [m,M]$, and let L, A, ψ and χ satisfy assumptions from Theorem 1.15. Then for every function $f \in L$ such that $m \leq f(t) \leq M$ for every $t \in E$ we have*

$$(\psi(M) - \psi(m))A(\chi(f)) - (\chi(M) - \chi(m))A(\psi(f)) \leq \psi(M)\chi(m) - \chi(M)\psi(m),$$

under the assumption that $\phi = \chi \circ \psi^{-1}$ is convex. Inequality is reversed if the function ϕ is concave.

The next results give us the upper bound for the difference between right and left side in the inequalities from Theorem 1.15 and Theorem 1.16 respectively.

Theorem 1.17 *Let L, A, ψ and χ satisfy assumptions from Theorem 1.16, and in addition, let L have property (L3). Let I be an interval of real numbers whose interior contains the interval $[m,M]$, and assume that function $\phi = \chi \circ \psi^{-1}$ is convex on I. Then for every function $f \in L$ such that $\psi(f), \chi(f) \in L$ and such that for every $t \in [m,M]$ it satisfies bounds $m \le f(t) \le M$ we have*

$$\begin{aligned}
0 &\le \chi(M_\chi(f,A)) - \chi(M_\psi(f,A)) \\
&\le (M_\psi - A(\psi(f)))(A(\psi(f)) - m_\psi) \sup_{t\in\langle m,M\rangle} \Psi_\phi(\psi(t); m_\psi, M_\psi) - R_{\phi,A}(m_\psi, M_\psi; \psi(f)) \\
&\le (M_\psi - A(\psi(f)))(A(\psi(f)) - m_\psi) \frac{\phi'_-(M_\psi) - \phi'_+(m_\psi)}{M_\psi - m_\psi} - R_{\phi,A}(m_\psi, M_\psi; \psi(f)) \\
&\le \frac{1}{4}(M_\psi - m_\psi)(\phi'_-(M_\psi) - \phi'_+(m_\psi)) - R_{\phi,A}(m_\psi, M_\psi; \psi(f)).
\end{aligned} \tag{1.36}$$

We also have

$$\begin{aligned}
0 &\le \chi(M_\chi(f,A)) - \chi(M_\psi(f,A)) \\
&\le \frac{1}{4}(M_\psi - m_\psi)^2 \Psi_\phi(A(\psi(f)); m_\psi, M_\psi) - R_{\phi,A}(m_\psi, M_\psi; \psi(f)) \\
&\le \frac{1}{4}(M_\psi - m_\psi)(\phi'_-(M_\psi) - \phi'_+(m_\psi)) - R_{\phi,A}(m_\psi, M_\psi; \psi(f)),
\end{aligned} \tag{1.37}$$

where $[m_\psi, M_\psi] = \psi([m,M])$ and Ψ_ϕ and $R_{\phi,A}$ are defined by (1.13) and (1.29). If the function ϕ is concave, inequality signs are reversed.

Proof. Since $f(E) \subseteq [m,M]$, it follows that $m_\psi \le \psi(f(t)) \le M_\psi$ for every $t \in E$ (if ψ is increasing, then $m_\psi = \psi(m)$ and $M_\psi = \psi(M)$; if ψ is decreasing, then $m_\psi = \psi(M)$ and $M_\psi = \psi(m)$). Therefore, the conditions as in Theorem 1.13 are fulfilled, so (1.36) and (1.37) are obtained by putting $m = m_\psi$, $M = M_\psi$ and replacing f with $\psi \circ f$ in inequalities (1.31) and (1.32) respectively. □

From Theorem 1.14, by utilizing the same substitutions as in the previous theorem, we get the following result.

Theorem 1.18 *Let the assumptions of Theorem 1.17 hold. If the function $\phi = \chi \circ \psi^{-1}$ is convex, then we have following sequences of inequalities:*

(i)

$$\begin{aligned}
R_{\phi,A}(m_\psi, M_\psi; \psi(f)) &\le \frac{A(\psi(f)) - \psi(m)}{\psi(M) - \psi(m)}\chi(M) + \frac{\psi(M) - A(\psi(f))}{\psi(M) - \psi(m)}\chi(m) - \chi(M_\chi(f,A)) \\
&\le A[(M_\psi - \psi(f))(\psi(f) - m_\psi)] \sup_{t\in\langle m,M\rangle} \Psi_\phi(\psi(t); m_\psi, M_\psi) \\
&\le \frac{A[(M_\psi - \psi(f))(\psi(f) - m_\psi)]}{M - m}(\phi'_-(M_\psi) - \phi'_+(m_\psi))
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(M_\psi - A(\psi(f)))(A(\psi(f)) - m_\psi)}{M_\psi - m_\psi}(\phi'_-(M_\psi) - \phi'_+(m_\psi)) \\
&\leq \frac{1}{4}(M_\psi - m_\psi)(\phi'_-(M_\psi) - \phi'_+(m_\psi)) \qquad (1.38)
\end{aligned}
$$

(ii)

$$
\begin{aligned}
R_{\phi,A}(m_\psi, M_\psi; \psi(f)) &\leq \frac{A(\psi(f)) - \psi(m)}{\psi(M) - \psi(m)}\chi(M) + \frac{\psi(M) - A(\psi(f))}{\psi(M) - \psi(m)}\chi(m) - \chi(M_\chi(f,A)) \\
&\leq A[(M_\psi - \psi(f))(\psi(f) - m_\psi)] \sup_{t\in\langle m,M\rangle} \Psi_\phi(\psi(t); m_\psi, M_\psi) \\
&\leq (M_\psi - A(\psi(f)))(A(\psi(f)) - m_\psi) \sup_{t\in\langle m,M\rangle} \Psi_\phi(\psi(t); m_\psi, M_\psi) \\
&\leq \frac{(M_\psi - A(\psi(f)))(A(\psi(f)) - m_\psi)}{M_\psi - m_\psi}(\phi'_-(M_\psi) - \phi'_+(m_\psi)) \\
&\leq \frac{1}{4}(M_\psi - m_\psi)(\phi'_-(M_\psi) - \phi'_+(m_\psi)) \qquad (1.39)
\end{aligned}
$$

(iii)

$$
\begin{aligned}
R_{\phi,A}(m_\psi, M_\psi; \psi(f)) &\leq \frac{A(\psi(f)) - \psi(m)}{\psi(M) - \psi(m)}\chi(M) + \frac{\psi(M) - A(\psi(f))}{\psi(M) - \psi(m)}\chi(m) - \chi(M_\chi(f,A)) \\
&\leq \frac{1}{4}(M_\psi - m_\psi)^2 A(\Psi_\phi(\psi(f); m_\psi, M_\psi)) \\
&\leq \frac{1}{4}(M_\psi - m_\psi)(\phi'_-(M_\psi) - \phi'_+(m_\psi)) \qquad (1.40)
\end{aligned}
$$

If the function ϕ is concave, inequality signs are reversed.

1.3.2 Power means

Definition 1.2 *Let E be a non-empty set, let L be vector space of real functions on E that has properties $(L1)$ and $(L2)$, and let A be a normalized positive linear functional that has properties $(A1)$ and $(A2)$ from Introduction. Power mean of the function $f \in L$ with respect to the normalized positive linear functional A is defined as*

$$
M^{[r]}(f,A) = \begin{cases} (A(f^r))^{1/r} & : r \neq 0 \\ \exp(A(\log f)) & : r = 0 \end{cases}
$$

where $r \in \mathbb{R}$, $f(t) > 0$ for $t \in E$, $f^r \in L$ and $\log f \in L$.

Since power means $M^{[r]}(f,A)$ are a special case of generalized means $M_\psi(f,A)$ for $\psi(t) = t^r$, from Theorem 1.15 ([57, p. 75, Theorem 92]), as a special case it follows:

Theorem 1.19 *Let $-\infty < r \leq s < \infty$ and let us assume that assumptions from Definition 1.2 hold. Then*

$$
M^{[r]}(f,A) \leq M^{[s]}(f,A).
$$

In the same manner, Goldman's inequality for positive linear functionals (see [27, p. 203]) can be obtained as a special case of Theorem 1.16:

$$(M^r - m^r)(M^{[s]}(f,A))^s - (M^s - m^s)(M^{[r]}(f,A))^r \le M^r m^r - M^s m^s$$

for $0 < r < s$ or $r < 0 < s$, and inequality is reversed for $r < s < 0$.
Similarly, for $r = 0$ and $s \in \mathbb{R}$ we have

$$\log \frac{M}{m}(M^{[s]}(f,A))^s - (M^s - m^s)\log(M^{[0]}(f,A)) \le m^s \log M - M^s \log m.$$

The results that follow are obtained by applying Theorem 1.17 and Theorem 1.18 on specially chosen functions ψ and χ, but they can also be proved by utilizing Theorem 1.13 and Theorem 1.14.

Corollary 1.2 *Let E be a non-empty set, let L be vector space of real functions on E that has properties $(L1)$ and $(L2)$, and let A be a normalized positive linear functional with properties $(A1)$ and $(A2)$. Let $f \in L$ and assume that $0 < m \le f(t) \le M < \infty$ for $t \in E$, $f^r, f^s \in L$ for $r,s \in \mathbb{R}$, $r < s$ and $\log f \in L$. Let us define function*

$$\phi(t) = \begin{cases} t^{s/r} & : r \ne 0, s \ne 0, \\ \frac{1}{r}\log t & : r \ne 0, s = 0, \\ e^{st} & : r = 0, s \ne 0. \end{cases}$$

If $0 < r < s$ then:

$$\begin{aligned} 0 &\le (M^{[s]}(f,A))^s - (M^{[r]}(f,A))^s \\ &\le (M^r - A(f^r))(A(f^r) - m^r) \sup_{t \in \langle m,M \rangle} \Psi_\phi(t^r; m^r, M^r) - R_{\phi,A}(m^r, M^r; f^r) \\ &\le \frac{s}{r}(M^r - A(f^r))(A(f^r) - m^r)\frac{M^{s-r} - m^{s-r}}{M^r - m^r} - R_{\phi,A}(m^r, M^r; f^r) \\ &\le \frac{s}{4r}(M^r - m^r)(M^{s-r} - m^{s-r}) - R_{\phi,A}(m^r, M^r; f^r) \end{aligned} \tag{1.41}$$

and we have

$$\begin{aligned} 0 &\le (M^{[s]}(f,A))^s - (M^{[r]}(f,A))^s \le \frac{1}{4}(M^r - m^r)^2 \Psi_\phi(A(f^r); m^r, M^r) - R_{\phi,A}(m^r, M^r; f^r) \\ &\le \frac{s}{4r}(M^r - m^r)(M^{s-r} - m^{s-r}) - R_{\phi,A}(m^r, M^r; f^r). \end{aligned} \tag{1.42}$$

If $r < 0 < s$, inequalities (1.41) hold with $R_{\phi,A}(M^r, m^r; f^r)$ instead of $R_{\phi,A}(m^r, M^r; f^r)$, and for $r < s < 0$ the inequality signs are reversed.
If $s = 0$ and $r < 0$, then:

$$\begin{aligned} 0 &\le \log(M^{[0]}(f,A)) - \log(M^{[r]}(f,A)) \\ &\le (M^r - A(f^r))(A(f^r) - m^r) \sup_{t \in \langle m,M \rangle} \Psi_\phi(t^r; M^r, m^r) - R_{\phi,A}(M^r, m^r; f^r) \end{aligned}$$

$$
\begin{aligned}
&\leq -\frac{1}{r}\frac{(M^r - A(f^r))(A(f^r)-m^r)}{M^r m^r} - R_{\phi,A}(M^r,m^r;f^r) \\
&\leq \frac{1}{4r}(m^r - M^r)\left(\frac{1}{m^r}-\frac{1}{M^r}\right) - R_{\phi,A}(M^r,m^r;f^r) \qquad (1.43)
\end{aligned}
$$

and we have

$$
\begin{aligned}
0 &\leq \log(M^{[0]}(f,A)) - \log(M^{[r]}(f,A)) \\
&\leq \frac{1}{4}(m^r - M^r)^2\Psi_\phi(A(f^r);M^r,m^r - R_{\phi,A}(M^r,m^r;f^r) \\
&\leq \frac{1}{4r}(m^r - M^r)\left(\frac{1}{m^r}-\frac{1}{M^r}\right) - R_{\phi,A}(M^r,m^r;f^r). \qquad (1.44)
\end{aligned}
$$

If $r = 0$ and $s > 0$, then:

$$
\begin{aligned}
0 &\leq (M^{[s]}(f,A))^s - (M^{[0]}(f,A))^s \\
&\leq (\log M - A(\log f))(A(\log f) - \log m) \sup_{t\in\langle m,M\rangle} \Psi_\phi(\log t;\log m,\log M) \\
&\qquad - R_{\phi,A}(\log m,\log M;\log f) \\
&\leq s(\log M - A(\log f))(A(\log f)-\log m)\frac{M^s - m^s}{\log M - \log m} - R_{\phi,A}(\log m,\log M;\log f) \\
&\leq s(M^s - m^s)\log\frac{M}{m} - R_{\phi,A}(\log m,\log M;\log f) \qquad (1.45)
\end{aligned}
$$

and we have

$$
\begin{aligned}
0 &\leq (M^{[s]}(f,A))^s - (M^{[0]}(f,A))^s \\
&\leq \frac{1}{4}(\log M - \log m)^2\Psi_\phi(A(\log f);\log m,\log M) - R_{\phi,A}(\log m,\log M;\log f) \\
&\leq \frac{s}{4}(M^s - m^s)\log\frac{M}{m} - R_{\phi,A}(\log m,\log M;\log f). \qquad (1.46)
\end{aligned}
$$

Proof. When we take $\chi(t) = t^s$ and $\psi(t) = t^r$, then the function $\phi(t) = \chi(\psi^{-1}(t)) = t^{s/r}$ is continuous, and convex for $0 < r < s$ and $r < 0 < s$. Function ψ is strictly increasing for $r > 0$, so the assumptions from Theorem 1.17 hold. Sequences of inequalities (1.41) and (1.42) are obtained by putting $m_\psi = \psi(m) = m^r$, $M_\psi = \psi(M) = M^r$, $\phi(t) = \chi\circ\psi^{-1}(t) = t^{s/r}$, $\psi(t) = t^r$ and $\psi(f) = f^r$ in (1.36) and (1.37) respectively. Function ψ is strictly decreasing for $r < 0$, so sequences of inequalities (1.41) and (1.42) are obtained by putting $M_\psi = \psi(m) = m^r$, $m_\psi = \psi(M) = M^r$, $\phi(t) = \chi\circ\psi^{-1}(t) = t^{s/r}$ $\psi(t) = t^r$ and $\psi(f) = f^r$ in (1.36) and (1.37) consecutively.

In case when $r < s < 0$, the function $\psi(t) = t^r$ is strictly decreasing and $\phi(t) = \chi(\psi^{-1}(t)) = t^{s/r}$ is concave, so we obtain inequalities (1.41) and (1.42) with reversed signs of inequality by putting $M_\psi = \psi(m) = m^r$, $m_\psi = \psi(M) = M^r$, $\phi(t) = -\chi \circ \psi^{-1}(t) = -t^{s/r}$, $\psi(t) = t^r$ and $\psi(f) = f^r$ in (1.36) and (1.37).

In case when $r < 0$ and $s = 0$ we take $\chi(t) = \log t$ and $\psi(t) = t^r$. Then the function $\phi(t) = \chi(\psi^{-1}(t)) = \dfrac{1}{r}\log t$ is continuous and convex, and ψ is strictly decreasing for $r < 0$, so by putting $M_\psi = \psi(m) = m^r$, $m_\psi = \psi(M) = M^r$, $\phi(t) = \chi \circ \psi^{-1}(t) = \dfrac{1}{r}\log t$, $\psi(t) = t^r$ and $\psi(f) = f^r$ in (1.36) and (1.37), we get inequalities (1.43) and (1.44) respectively.

When $r = 0$ and $s > 0$, we take $\chi(t) = t^s$ and $\psi(t) = \log t$. Then $\phi(t) = \chi(\psi^{-1}(t)) = e^{st}$ is continuous and convex function, and ψ is strictly increasing. Inequalities (1.45) and (1.46) follow by putting $m_\psi = \psi(m) = \log m$, $M_\psi = \psi(M) = \log M$, $\phi(t) = \chi \circ \psi^{-1}(t) = e^{st}$, $\psi(t) = \log t$ and $\psi(f) = \log f$ in (1.36) and (1.37) consecutively. □

By taking the same substitutions as in the proof of the previous corollary, from Theorem 1.18 we directly get our next result.

Corollary 1.3 *Let E be a non-empty set, let L be vector space of real functions on E that has properties $(L1)$ and $(L2)$, and let A be a normalized positive linear functional with properties $(A1)$ and $(A2)$. Let $f \in L$ and assume that $0 < m \le f(t) \le M < \infty$ for $t \in E$, $f^r, f^s \in L$ for $r, s \in \mathbb{R}$, $r < s$ and $\log f \in L$. Let ϕ be the function defined in the previous corollary.*
If $0 < r < s$, then:

(i)

$$\begin{aligned}
R_{\phi,A}(m^r, M^r; f^r) &\le \frac{A(f^r) - m^r}{M^r - m^r} M^s + \frac{M^r - A(f^r)}{M^r - m^r} m^s - (M^{[s]}(f, A))^s \\
&\le A[(M^r - f^r)(f^r - m^r)] \sup_{t \in \langle m, M \rangle} \Psi_\phi(t^r; m^r, M^r) \\
&\le \frac{s}{r} \frac{A[(M^r - f^r)(f^r - m^r)]}{M^r - m^r} (M^{s-r} - m^{s-r}) \\
&\le \frac{s}{r} \frac{(M^r - A(f^r))(A(f^r) - m^r)}{M^r - m^r} (M^{s-r} - m^{s-r}) \\
&\le \frac{s}{4r} (M^r - m^r)(M^{s-r} - m^{s-r})
\end{aligned}$$

(ii)

$$\begin{aligned}
A(\widetilde{f^r})\delta_\phi &\le \frac{A(f^r) - m^r}{M^r - m^r} M^s + \frac{M^r - A(f^r)}{M^r - m^r} m^s - (M^{[s]}(f, A))^s \\
&\le A[(M^r - f^r)(f^r - m^r)] \sup_{t \in \langle m, M \rangle} \Psi_\phi(t^r; m^r, M^r) \\
&\le (M^r - A(f^r))(A(f^r) - m^r) \sup_{t \in \langle m, M \rangle} \Psi_\phi(t^r; m^r, M^r) \\
&\le \frac{s}{r} \frac{(M^r - A(f^r))(A(f^r) - m^r)}{M^r - m^r} (M^{s-r} - m^{s-r}) \\
&\le \frac{s}{4r} (M^r - m^r)(M^{s-r} - m^{s-r})
\end{aligned}$$

(iii)

$$\begin{aligned}
A(\widetilde{f^r})\delta_\phi &\leq \frac{A(f^r)-m^r}{M^r-m^r}M^s + \frac{M^r-A(f^r)}{M^r-m^r}m^s - (M^{[s]}(f,A))^s \\
&\leq \frac{1}{4}(M^r-m^r)^2 A(\Psi_\phi(f^r;m^r,M^r)) \\
&\leq \frac{s}{4r}(M^r-m^r)(M^{s-r}-m^{s-r}).
\end{aligned}$$

If $r<0<s$, inequalities (1.41) hold with $R_{\phi,A}(M^r,m^r;f^r)$ instead of $R_{\phi,A}(m^r,M^r;f^r)$, and if $r<s<0$, then the inequality signs are reversed.
If $s=0$ and $r<0$, then:

(i)

$$\begin{aligned}
R_{\phi,A}(M^r,m^r;f^r) &\leq \frac{A(f^r)-m^r}{M^r-m^r}\log M + \frac{M^r-A(f^r)}{M^r-m^r}\log m - \log(M^{[0]}(f,A)) \\
&\leq A[(M^r-f^r)(f^r-m^r)] \sup_{t\in\langle m,M\rangle} \Psi_\phi(t^r;M^r,m^r) \\
&\leq \frac{1}{r}\frac{A[(M^r-f^r)(f^r-m^r)]}{m^r-M^r}\left(\frac{1}{m^r}-\frac{1}{M^r}\right) \\
&\leq \frac{1}{r}\frac{(M^r-A(f^r))(A(f^r)-m^r)}{m^r-M^r}\left(\frac{1}{m^r}-\frac{1}{M^r}\right) \\
&\leq \frac{1}{4r}(m^r-M^r)\left(\frac{1}{m^r}-\frac{1}{M^r}\right)
\end{aligned}$$

(ii)

$$\begin{aligned}
R_{\phi,A}(M^r,m^r;f^r) &\leq \frac{A(f^r)-m^r}{M^r-m^r}\log M + \frac{M^r-A(f^r)}{M^r-m^r}\log m - \log(M^{[0]}(f,A)) \\
&\leq A[(M^r-f^r)(f^r-m^r)] \sup_{t\in\langle m,M\rangle} \Psi_\phi(t^r;M^r,m^r) \\
&\leq (M^r-A(f^r))(A(f^r)-m^r) \sup_{t\in\langle m,M\rangle} \Psi_\phi(t^r;M^r,m^r) \\
&\leq \frac{1}{r}\frac{(M^r-A(f^r))(A(f^r)-m^r)}{m^r-M^r}\left(\frac{1}{m^r}-\frac{1}{M^r}\right) \\
&\leq \frac{1}{4r}(m^r-M^r)\left(\frac{1}{m^r}-\frac{1}{M^r}\right)
\end{aligned}$$

(iii)

$$\begin{aligned}
R_{\phi,A}(M^r,m^r;f^r) &\leq \frac{A(f^r)-m^r}{M^r-m^r}\log M + \frac{M^r-A(f^r)}{M^r-m^r}\log m - \log(M^{[0]}(f,A)) \\
&\leq \frac{1}{4}(m^r-M^r)^2 A(\Psi_\phi(f^r;M^r,m^r)) \\
&\leq \frac{1}{4r}(m^r-M^r)\left(\frac{1}{m^r}-\frac{1}{M^r}\right)
\end{aligned}$$

If $r = 0$ and $s > 0$, then:

(i)

$$
\begin{aligned}
R_{\phi,A}(\log m, \log M; \log f) &\leq \frac{A(\log f) - \log m}{\log M - \log m} M^s + \frac{\log M - A(\log f)}{\log M - \log m} m^s - (M^{[s]}(f,A))^s \\
&\leq A[(\log M - \log f)(\log f - \log m)] \sup_{t \in \langle m,M \rangle} \Psi_\phi(\log t; \log m, \log M) \\
&\leq s \frac{A[(\log M - \log f)(\log f - \log m)]}{\log M - \log m} (M^s - m^s) \\
&\leq s \frac{(\log M - A(\log f))(A(\log f) - \log m)}{\log M - \log m} (M^s - m^s) \\
&\leq \frac{s}{4}(M^s - m^s) \log \frac{M}{m}
\end{aligned}
$$

(ii)

$$
\begin{aligned}
R_{\phi,A}(\log m, \log M; \log f) &\leq \frac{A(\log f) - \log m}{\log M - \log m} M^s + \frac{\log M - A(\log f)}{\log M - \log m} m^s - (M^{[s]}(f,A))^s \\
&\leq A[(\log M - \log f)(\log f - \log m)] \sup_{t \in \langle m,M \rangle} \Psi_\phi(\log t; \log m, \log M) \\
&\leq (\log M - A(\log f))(A(\log f) - \log m) \sup_{t \in \langle m,M \rangle} \Psi_\phi(\log t; \log m, \log M) \\
&\leq s \frac{(\log M - A(\log f))(A(\log f) - \log m)}{\log M - \log m} (M^s - m^s) \\
&\leq \frac{s}{4}(M^s - m^s) \log \frac{M}{m}
\end{aligned}
$$

(iii)

$$
\begin{aligned}
R_{\phi,A}(\log m, \log M; \log f) &\leq \frac{A(\log f) - \log m}{\log M - \log m} M^s + \frac{\log M - A(\log f)}{\log M - \log m} m^s - (M^{[s]}(f,A))^s \\
&\leq \frac{1}{4}(\log M - \log m)^2 A(\Psi_\phi(\log f; \log m, \log M)) \\
&\leq \frac{s}{4r}(\log M - \log m)(M^s - m^s).
\end{aligned}
$$

Remark 1.8 It is easy to check that $M^{[r]}(f,A) = (M^{[-r]}(f^{-1},A))^{-1}$ holds for every function $f \in L$ and every $r \in \mathbb{R}$. Utilizing this relation, we can get sequences of inequalities analogous to those from Corollary 1.2 and Corollary 1.3 by making substitutions $f \longleftrightarrow f^{-1}$, $-r \longleftrightarrow s$ and $-s \longleftrightarrow r$.

1.3.3 Hölder's inequality

Theorem 1.20 [124] (Hölder's inequality for positive linear functionals) *Let E be a non-empty set, let L be vector space of real functions on E that has properties $(L1)$ and $(L2)$, and let A be a positive linear functional with properties $(A1)$ and $(A2)$. Let $p > 1$ and $q = p/(p-1)$. If $w, f, g \geq 0$ on E, and $wf^p, wg^q, wfg \in L$, then we have*

$$A(wfg) \leq A^{1/p}(wf^p)A^{1/q}(wg^q)$$

In case when $0 < p < 1$ and $A(wg^q) > 0$ (or $p < 0$ and $A(wf^p) > 0$) inequality is reversed.

Theorem 1.21 [124] *Let E be a non-empty set, let L be vector space of real functions on E that has properties $(L1)$ and $(L2)$, and let A be a positive linear functional on L. Let $p > 1$ and $q = p/(p-1)$. If the functions $w, f, g \geq 0$ on E are such that $wf^p, wg^q, wfg \in L$ and $0 < m \leq f(t)g^{-q/p}(t) \leq M$ for $t \in E$, then we have*

$$(M-m)A(wf^p) + (mM^p - Mm^p)A(wg^q) \leq (M^p - m^p)A(wfg).$$

If $p < 0$, then the upper inequality is also valid with the assumption that $A(wf^p) > 0$ or $A(wg^q) > 0$. If $0 < p < 1$, then reversed inequality holds under the assumption that $A(wf^p) > 0$ or $A(wg^q) > 0$.

Results that follow are converses of inequalities from Theorem 1.20 and Theorem 1.21 respectively, that is, they give an estimate of the difference between the right and left sides of the mentioned inequalities.

Theorem 1.22 *Let E be a non-empty set, let L be a vector space of real functions on E that has properties $(L1)$, $(L2)$ and $(L3)$, and let A be a positive linear functional on L. Let $p > 1$ and $q = p/(p-1)$. If the functions $w, f, g \geq 0$ on E are such that $wf^p, wg^q, wfg \in L$ and $0 < m \leq f(t)g^{-q/p}(t) \leq M$ for $t \in E$, then we have*

$$\begin{aligned}
0 &\leq A(wf^p)A^{p/q}(wg^q) - A^p(wfg) \\
&\leq (MA(wg^q) - A(wfg))(A(wfg) - mA(wg^q)) \sup_{t \in \langle m,M \rangle} \Psi_\phi(t;m,M)A^{p-2}(wg^q) \\
&\qquad - \tilde{R}_{\phi,A}(m,M; fg^{-\frac{p}{q}})A^{p-1}(wg^q) \\
&\leq (MA(wg^q) - A(wfg))(A(wfg) - mA(wg^q))p\frac{M^{p-1} - m^{p-1}}{M-m}A^{p-2}(wg^q) \\
&\qquad - \tilde{R}_{\phi,A}(m,M; fg^{-\frac{p}{q}})A^{p-1}(wg^q) \\
&\leq \frac{p}{4}(M-m)(M^{p-1} - m^{p-1})A^p(wg^q) - \tilde{R}_{\phi,A}(m,M; fg^{-\frac{p}{q}})A^{p-1}(wg^q). \qquad (1.47)
\end{aligned}$$

We also have

$$\begin{aligned}
0 &\le A(wf^p)A^{p/q}(wg^q) - A^p(wfg) \\
&\le \frac{1}{4}(M-m)^2\Psi_\phi\Big(\frac{A(wfg)}{A(wg^q)};m,M\Big)A^p(wg^q) - \tilde{R}_{\phi,A}(m,M;fg^{-\frac{p}{q}})A^{p-1}(wg^q) \\
&\le \frac{p}{4}(M-m)(M^{p-1}-m^{p-1})A^p(wg^q) - \tilde{R}_{\phi,A}(m,M;fg^{-\frac{p}{q}})A^{p-1}(wg^q), \qquad (1.48)
\end{aligned}$$

where $\phi(t) = t^p$, Ψ_ϕ *is defined by* (1.13), *and*

$$\tilde{R}_{\phi,A}(m,M;fg^{-\frac{p}{q}}) = \sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\Delta_\phi(m,M,n,k)A\Big(wg^q r_n\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}\Big(\frac{fg^{-\frac{q}{p}}-m}{M-m}\Big)\Big),$$

where $\Delta_\phi(m,M,n,k)$ *is defined in Theorem 1.12. If* $A(wfg) > 0$, *then the inequalities also hold for* $p < 0$, *while for* $0 < p < 1$ *the inequalities are reversed.*

Proof. Function $\phi(t) = t^p$ is continuous, convex for $p > 1$ and $p < 0$, concave for $0 < p < 1$. Let us define a linear functional $B(f) = \dfrac{A(wf)}{A(w)}$ for a function $w \in L$ such that $w \ge 0$ and $A(w) > 0$. Then it holds $B(\mathbf{1}) = \dfrac{A(w)}{A(w)} = 1$, so we see that B satisfies conditions from Theorem 1.13. Now inequalities (1.47) and (1.48) follow from (1.31) and (1.32) respectively after replacing functional A with functional B, and after putting wg^q instead of w, and $fg^{-q/p}$ instead of f. □

Using the same substitutions, from the Theorem 1.14 we get the following result.

Theorem 1.23 *Let E be a non-empty set, let L be a vector space of real functions on E that has properties* $(L1)$, $(L2)$ *and* $(L3)$, *and let A be a positive linear functional on L. Let* $p > 1$ *and* $q = p/(p-1)$. *If the functions* $w, f, g \ge 0$ *on E are such that* $wf^p, wg^q, wfg \in L$ *and* $0 < m \le f(t)g^{-q/p}(t) \le M$ *for* $t \in E$, *then we have*

(i)

$$\begin{aligned}
&\tilde{R}_{\phi,A}(m,M;fg^{-\frac{p}{q}})A^{p-1}(wg^q) \\
&\quad\le \frac{A(wfg)-mA(wg^q)}{M-m}M^p + \frac{MA(wg^q)-A(wfg)}{M-m}m^p - A(wf^p) \\
&\quad\le A(wg^q[(M-fg^{-q/p})(fg^{-q/p}-m)]) \sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M) \\
&\quad\le p\frac{A(wg^q[(M-fg^{-q/p})(fg^{-q/p}-m)])}{M-m}(M^{p-1}-m^{p-1}) \\
&\quad\le p\frac{(MA(wg^q)-A(wfg))(A(wfg)-mA(wg^q))}{(M-m)A(wg^q)}(M^{p-1}-m^{p-1}) \\
&\quad\le \frac{p}{4}(M-m)(M^{p-1}-m^{p-1})A(wg^q)
\end{aligned}$$

(ii)

$$
\begin{aligned}
&\tilde{R}_{\phi,A}(m,M;fg^{-\frac{p}{q}})A^{p-1}(wg^q)\\
&\quad\le \frac{A(wfg)-mA(wg^q)}{M-m}M^p+\frac{MA(wg^q)-A(wfg)}{M-m}m^p-A(wf^p)\\
&\quad\le A(wg^q[(M-fg^{-q/p})(fg^{-q/p}-m)])\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M)\\
&\quad\le \frac{(MA(wg^q)-A(wfg))(A(wfg)-mA(wg^q))}{A(wg^q)}\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M)\\
&\quad\le p\frac{(MA(wg^q)-A(wfg))(A(wfg)-mA(wg^q))}{(M-m)A(wg^q)}(M^{p-1}-m^{p-1})\\
&\quad\le \frac{p}{4}(M-m)(M^{p-1}-m^{p-1})A(wg^q)
\end{aligned}
$$

(iii)

$$
\begin{aligned}
&\tilde{R}_{\phi,A}(m,M;fg^{-\frac{p}{q}})A^{p-1}(wg^q)\\
&\quad\le \frac{A(wfg)-mA(wg^q)}{M-m}M^p+\frac{MA(wg^q)-A(wfg)}{M-m}m^p-A(wf^p)\\
&\quad\le \frac{1}{4}(M-m)^2A(wg^q\Psi_\phi(fg^{-q/p};m,M))\\
&\quad\le \frac{p}{4}(M-m)(M^{p-1}-m^{p-1})A(wg^q)
\end{aligned}
$$

where $\phi(t)=t^p$, Ψ_ϕ *is defined by* (1.13) *and* $\tilde{R}_{\phi,A}$ *is defined in the previous theorem. If* $A(wfg)>0$, *then the upper inequalities also hold for* $p<0$. *In case when* $0<p<1$, *inequality signs are reversed.*

1.3.4 Hermite-Hadamard's inequality

Hermite-Hadamard's inequality gives us an estimate of the (integral) mean value of a continuous convex function. It was first proved by Hermite [58] in 1883, and was rediscovered by Hadamard [54] ten years later. However, the note [58] has not been recorded anywhere, and it was only recently discovered that Hermite was the first one who proved it (for more historical details see [113]).

Theorem 1.24 ([54]) (Hermite-Hadamard's inequality) *Let* $-\infty<a<b<\infty$ *and* $f\colon[a,b]\to\mathbb{R}$. *If the function* f *is convex, then*

$$
f\Big(\frac{a+b}{2}\Big)\le\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\le\frac{f(a)+f(b)}{2}. \tag{1.49}
$$

If the function f *is concave, inequality signs in (1.49) are reversed.*

The first inequality in the sequence (1.49) is sharper than the second one, that is, so-called Bullen's inequality holds for a convex function (for proof see [26] or [124]):

$$\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - f\left(\frac{a+b}{2}\right) \le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t. \tag{1.50}$$

By applying Theorem 1.13 and Theorem 1.14 to Hermite-Hadamard's inequality (1.49) we get an estimate of the difference between left and right side and the value of $\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t$ respectively.

Theorem 1.25 *Let $a<b$ and let f be a convex function on the interval of real numbers I whose interior contains interval $[a,b]$. Then*

$$\begin{aligned}
0 &\le \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - f\left(\frac{a+b}{2}\right) \\
&\le \frac{1}{4}(b-a)^2 \sup_{t\in\langle a,b\rangle} \Psi_f(t;a,b) - R_f(a,b) \\
&\le \frac{1}{4}(b-a)(f'_-(b)-f'_+(a)) - R_f(a,b).
\end{aligned} \tag{1.51}$$

We also have

$$\begin{aligned}
0 &\le \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - f\left(\frac{a+b}{2}\right) \\
&\le \frac{1}{4}(b-a)^2 \Psi_f\left(\frac{a+b}{2};a,b\right) - R_f(a,b) \\
&\le \frac{1}{4}(b-a)(f'_-(b)-f'_+(a)) - R_f(a,b),
\end{aligned} \tag{1.52}$$

where

$$R_f(a,b) = \sum_{n=0}^{N-1} 2^{-n-2} \sum_{k=1}^{2^n} \Delta_f(a,b,n,k) \tag{1.53}$$

and $\Delta_f(a,b,n,k)$ is defined in Theorem 1.12. *If the function f is concave, inequality signs are reversed.*

Proof. Inequalities (1.51) and (1.52) are obtained from (1.31) and (1.32) respectively after putting $A(f) = \dfrac{1}{b-a}\int_a^b f(t)\mathrm{d}t$, $f(t)=t$ and then replacing ϕ with f. □

In the same manner, from Theorem 1.14 it follows:

Theorem 1.26 *Let $a<b$ and let f be a convex function on the interval of real numbers I whose interior contains interval $[a,b]$. Then*

$$\begin{aligned}
R_f(a,b) &\le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \le \frac{1}{6}(b-a)^2 \sup_{t\in\langle a,b\rangle} \Psi_f(t;a,b) \\
&\le \frac{1}{6}(b-a)(f'_-(b)-f'_+(a)).
\end{aligned} \tag{1.54}$$

where $R_f(a,b)$ is defined in (1.53). If the function f is concave, inequality signs are reversed.

Remark 1.9 For $N = 1$, when we take into account that

$$R_f(a,b) = 2^{-2}\Delta_f(a,b,0,1) = \frac{1}{4}\left(f(a)+f(b)-2f\left(\frac{a+b}{2}\right)\right),$$

we see that the first inequality from (1.54) is exactly Bullen's inequality (1.50).

Remark 1.10 Let $a < b$ and let f be a convex function on an interval of real numbers I whose interior contains the interval $[a,b]$. By combining the upper results we get

$$\begin{aligned}\frac{f(a)+f(b)}{2} - \frac{1}{6}(b-a)(f'_-(b)-f'_+(a)) &\leq \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \\ &\leq f\left(\frac{a+b}{2}\right) + \frac{1}{4}(b-a)(f'_-(b)-f'_+(a)) - R_f(a,b).\end{aligned} \tag{1.55}$$

where $R_f(a,b)$ is defined in (1.53). If the function f is concave, then inequality signs in (1.55) are reversed.

Remark 1.11 Similarly as in the proof of Bullen's inequality, if we apply the first inequality from (1.55) to the function f over the intervals $[a,(a+b)/2]$ and $[(a+b)/2,b]$ respectively, we get

$$\frac{1}{2}\left[f(a)+f\left(\frac{a+b}{2}\right)\right] - \frac{1}{12}(b-a)\left[f'_-\left(\frac{a+b}{2}\right) - f'_+(a)\right] \leq \frac{2}{b-a}\int_a^{(a+b)/2} f(t)\mathrm{d}t$$

and

$$\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+f(b)\right] - \frac{1}{12}(b-a)\left[f'_-(b) - f'_+\left(\frac{a+b}{2}\right)\right] \leq \frac{2}{b-a}\int_{(a+b)/2}^b f(t)\mathrm{d}t.$$

By summing these inequalities, we get a converse of Bullen's inequality:

$$\begin{aligned}&\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \\ &\quad\leq \frac{1}{b-a}\int_a^b f(t) - f\left(\frac{a+b}{2}\right) + \frac{1}{12}(b-a)\left[f'_-(b) - f'_+\left(\frac{a+b}{2}\right) + f'_-\left(\frac{a+b}{2}\right) - f'_+(a)\right],\end{aligned}$$

and if the function f is additionally differentiable in the mid-point $(a+b)/2$ of the interval $[a,b]$, then the upper relation becomes:

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \leq \frac{1}{b-a}\int_a^b f(t) - f\left(\frac{a+b}{2}\right) + \frac{1}{12}(b-a)(f'_-(b) - f'_+(a)).$$

Following generalization of the Hermite-Hadamard inequality for positive linear functionals is given in [121] (see also [124]).

Theorem 1.27 ([121]) *Let ϕ be a continuous convex function on an interval $I \supset [m,M]$, where $-\infty < m < M < \infty$. Suppose that $f\colon E \to \mathbb{R}$ satisfies $m \leq f(t) \leq M$ for every $t \in E$,*

$f \in L$ and $\phi(f) \in L$. Let $A\colon L \to \mathbb{R}$ be a normalized positive linear functional, and let $p = p_f$, $q = q_f$ be nonnegative real numbers (with $p+q > 0$) for which

$$A(f) = \frac{pm+qM}{p+q}. \tag{1.56}$$

Then

$$\phi\left(\frac{pm+qM}{p+q}\right) \leq A(\phi(f)) \leq \frac{p\phi(m)+q\phi(M)}{p+q}. \tag{1.57}$$

Applying Theorem 1.13 to the previous theorem, we obtain a converse of the first inequality from (1.57).

Theorem 1.28 *Let ϕ be a continuous convex function on an open interval of real numbers $I \supset [m,M]$, where $-\infty < m < M < \infty$. Suppose that $f\colon E \to \mathbb{R}$ satisfies $m \leq f(t) \leq M$ for every $t \in E$, $f \in L$ and $\phi(f) \in L$. Let $A\colon L \to \mathbb{R}$ be a normalized positive linear functional, and let $p = p_f$, $q = q_f$ be nonnegative real numbers (with $p+q > 0$) for which (1.56) holds. Then*

$$\begin{aligned}
0 &\leq A(\phi(f)) - \phi\left(\frac{pm+qM}{p+q}\right) \\
&\leq \frac{pq}{(p+q)^2}(M-m) \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M) - R_{\phi,A}(m,M;f) \\
&\leq \frac{pq}{(p+q)^2}(M-m)(\phi'_-(M)-\phi'_+(m)) - R_{\phi,A}(m,M;f) \\
&\leq \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)) - R_{\phi,A}(m,M;f)
\end{aligned} \tag{1.58}$$

and

$$\begin{aligned}
0 &\leq A(\phi(f)) - \phi\left(\frac{pm+qM}{p+q}\right) \\
&\leq \frac{1}{4}(M-m)^2\Psi_\phi\left(\frac{pm+qM}{p+q};m,M\right) - R_{\phi,A}(m,M;f) \\
&\leq \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)) - R_{\phi,A}(m,M;f),
\end{aligned} \tag{1.59}$$

where Ψ_ϕ and $R_{\phi,A}$ are defined by (1.13) *and* (1.29) *respectively. If ϕ is concave, the inequalities in (1.58) and (1.59) are reversed.*

Proof. First observe that $\phi(f) \in$L implies $A(f) \in [m,M]$, hence there exists a unique nonnegative real number $\lambda \in [0,1]$ such that $A(f) = \lambda m + (1-\lambda)M$. If p,q are nonnegative real numbers satisfying (1.56), then obviously

$$\frac{p}{p+q} = \lambda, \quad \frac{q}{p+q} = 1-\lambda.$$

Inequalities (1.58) and (1.59) are now obtained from (1.31) and (1.32) by replacing $A(f)$ with $\dfrac{pm+qM}{p+q}$. □

Further, applying Theorem 1.14 to Theorem 1.27, we obtain converses of the second inequality from (1.57).

Theorem 1.29 *Let ϕ be a continuous convex function on an open interval of real numbers $I \supset [m,M]$, where $-\infty < m < M < \infty$. Suppose that $f\colon E \to \mathbb{R}$ satisfies $m \le f(t) \le M$ for every $t \in E$, $f \in L$ and $\phi(f) \in L$. Let $A\colon L \to \mathbb{R}$ be a positive linear functional with $A(1) = 1$, and let $p = p_f$, $q = q_f$ be nonnegative real numbers (with $p+q > 0$) for which (1.56) holds. Then*

$$\begin{aligned}
0 &\le \frac{p\phi(m)+q\phi(M)}{p+q} - A(\phi(f)) - R_{\phi,A}(m,M;f) \\
&\le A[(M-f)(f-m)] \sup_{t\in(m,M)} \Psi_\phi(t;m,M) - R_{\phi,A}(m,M;f) \\
&\le \frac{\phi'_-(M)-\phi'_+(m)}{M-m} A([M-f][f-m]) - R_{\phi,A}(m,M;f) \\
&\le \frac{pq}{(p+q)^2}(M-m)(\phi'_-(M)-\phi'_+(m)) - R_{\phi,A}(m,M;f) \\
&\le \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)) - R_{\phi,A}(m,M;f).
\end{aligned} \tag{1.60}$$

$$\begin{aligned}
0 &\le \frac{p\phi(m)+q\phi(M)}{p+q} - A(\phi(f)) - R_{\phi,A}(m,M;f) \\
&\le A[(M-f)(f-m)] \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M) - R_{\phi,A}(m,M;f) \\
&\le pq\frac{(M-m)^2}{(p+q)^2} \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M) - R_{\phi,A}(m,M;f) \\
&\le \frac{pq}{(p+q)^2}(M-m)(\phi'_-(M)-\phi'_+(m)) - R_{\phi,A}(m,M;f) \\
&\le \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)) - R_{\phi,A}(m,M;f)
\end{aligned} \tag{1.61}$$

and

$$\begin{aligned}
0 &\le \frac{p\phi(m)+q\phi(M)}{p+q} - A(\phi(f)) - R_{\phi,A}(m,M;f) \\
&\le \frac{1}{4}(M-m)^2 A(\Psi_\phi(f;m,M)) - R_{\phi,A}(m,M;f) \\
&\le \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)) - R_{\phi,A}(m,M;f)
\end{aligned} \tag{1.62}$$

where Ψ_ϕ and $R_{\phi,A}$ are defined by (1.13) *and* (1.29) *respectively. If ϕ is concave, the inequalities are reversed.*

Proof. Like in the proof of the previous theorem, there exist unique nonnegative real numbers p,q satisfying (1.56). Since

$$\frac{M-A(f)}{M-m}\phi(m) + \frac{A(f)-m}{M-m}\phi(M) = \frac{p}{p+q}\phi(m) + \frac{q}{p+q}\phi(M)$$

we obtain inequalities (1.60), (1.61) and (1.62) from (1.33), (1.34) and (1.35) respectively by replacing $A(f)$ with $\dfrac{pm+qM}{p+q}$. □

The following result arises from combining previous two theorems. It was also proved in [122] by applying Theorem 1.12 to generalized Hermite-Hamadard's inequality.

Remark 1.12 Under the same assumptions as in last two theorems, we have

$$\phi\Big(\frac{pm+qM}{p+q}\Big) \leq A(\phi(f)) \leq \frac{p\phi(m)+q\phi(M)}{p+q} - R_{\phi,A}(m,M;f).$$

1.3.5 Inequalities of Giaccardi and Petrović

We start this subsection with the inequality of Giaccardi.

Theorem 1.30 ([130]) *Let $\mathbf{p}$ be an $n-$tuple of nonnegative real numbers and $\mathbf{x}$ be an $n-$tuple of real numbers such that*

$$(x_i - x_0)\Big(\sum_{j=1}^{n} p_j x_j - x_i\Big) \geq 0,\ i = 1,\ldots,n;\ \sum_{k=1}^{n} p_k x_k \neq x_0;\ x_0, \sum_{i=1}^{n} p_i x_i \in [a,b]. \tag{1.63}$$

If $f\colon [a,b] \to \mathbb{R}$ is a convex function, then

$$\sum_{i=1}^{n} p_i f(x_i) \leq A f\Big(\sum_{i=1}^{n} p_i x_i\Big) + B\Big(\sum_{i=1}^{n} p_i - 1\Big) f(x_0)$$

where

$$A = \frac{\sum_{i=1}^{n} p_i(x_i - x_0)}{\sum_{i=1}^{n} p_i x_i - x_0},\ B = \frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i x_i - x_0}. \tag{1.64}$$

Next result is a converse of the Giaccardi inequality, and it follows from Theorem 1.14:

Theorem 1.31 *Let $\mathbf{p}$ be an $n-$tuple of nonnegative real numbers and let $\mathbf{x}$ be an $n-$tuple of real numbers such that (1.63) holds. Let I be an interval of real numbers such that its interior contains $[a,b]$. If $f\colon I \to \mathbb{R}$ is a convex function, then we have*

(i)

$$\begin{aligned}
R_f(m,M;\mathbf{x}) &\leq A f\Big(\sum_{i=1}^{n} p_i x_i\Big) + B\Big(\sum_{i=1}^{n} p_i - 1\Big) f(x_0) - \sum_{i=1}^{n} p_i f(x_i) \\
&\leq \sum_{j=1}^{n} p_j \Big(\sum_{i=1}^{n} p_i x_i - x_j\Big)(x_j - x_0) \sup_{t\in\langle m,M\rangle} \Psi_f\Big(t; x_0, \sum_{i=1}^{n} p_i x_i\Big) \\
&\leq \frac{\sum_{j=1}^{n} p_j(\sum_{i=1}^{n} p_i x_i - x_j)(x_j - x_0)}{M-m}(f'_-(M) - f'_+(m)) \\
&\leq \Big(M - \frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i}\Big)\Big(\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i} - m\Big)\frac{f'_-(M) - f'_+(m)}{M-m}\sum_{i=1}^{n} p_i \\
&\leq \frac{1}{4}(M-m)(f'_-(M) - f'_+(m))\sum_{i=1}^{n} p_i
\end{aligned}$$

(ii)

$$\begin{aligned}
R_f(m,M;\mathbf{x}) &\le Af\Big(\sum_{i=1}^n p_ix_i\Big)+B\Big(\sum_{i=1}^n p_i-1\Big)f(x_0)-\sum_{i=1}^n p_if(x_i)\\
&\le \sum_{j=1}^n p_j\Big(\sum_{i=1}^n p_ix_i-x_j\Big)(x_j-x_0)\sup_{t\in\langle m,M\rangle}\Psi_f\Big(t;x_0,\sum_{i=1}^n p_ix_i\Big)\\
&\le \Big(M-\frac{\sum_{i=1}^n p_ix_i}{\sum_{i=1}^n p_i}\Big)\Big(\frac{\sum_{i=1}^n p_ix_i}{\sum_{i=1}^n p_i}-m\Big)\sup_{t\in\langle m,M\rangle}\Psi_f\Big(t;x_0,\sum_{i=1}^n p_ix_i\Big)\sum_{i=1}^n p_i\\
&\le \Big(M-\frac{\sum_{i=1}^n p_ix_i}{\sum_{i=1}^n p_i}\Big)\Big(\frac{\sum_{i=1}^n p_ix_i}{\sum_{i=1}^n p_i}-m\Big)\frac{f'_-(M)-f'_+(m)}{M-m}\sum_{i=1}^n p_i\\
&\le \frac{1}{4}(M-m)(f'_-(M)-f'_+(m))\sum_{i=1}^n p_i
\end{aligned}$$

(iii)

$$\begin{aligned}
R_f(m,M;\mathbf{x}) &\le Af\Big(\sum_{i=1}^n p_ix_i\Big)+B\Big(\sum_{i=1}^n p_i-1\Big)f(x_0)-\sum_{i=1}^n p_if(x_i)\\
&\le \frac{1}{4}(M-m)^2\sum_{i=1}^n p_i\Psi_f\Big(x_i;x_0,\sum_{i=1}^n p_ix_i\Big)\\
&\le \frac{1}{4}(M-m)(f'_-(M)-f'_+(m))\sum_{i=1}^n p_i
\end{aligned}$$

where $m=\min\{x_0,\sum_{i=1}^n p_ix_i\}$, $M=\max\{x_0,\sum_{i=1}^n p_ix_i\}$,

$$R_f(m,M;\mathbf{x})=\sum_{i=1}^r\sum_{n=0}^{N-1}\sum_{k=1}^{2^n} p_i\Delta_f(m,M,n,k)r_n\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}\Big(\frac{x_i-m}{M-m}\Big),$$

$\Delta_f(m,M,n,k)$ is defined in Theorem 1.12 and A, B are defined in (1.64). If the function f is concave, inequality signs are reversed.

Proof. Let f be a convex function. The inequalities from above are obtained directly from Theorem 1.14 for $A(\mathbf{x})=\dfrac{\sum_{i=1}^n p_ix_i}{\sum_{i=1}^n p_i}$ and $\phi=f$. □

The well-known Petrović inequality [125] for a convex function $f\colon [0,a]\to\mathbb{R}$ is

$$\sum_{i=1}^n f(x_i)\le f\Big(\sum_{i=1}^n x_i\Big)+(n-1)f(0)$$

where $x_i, i=1,\dots,n$ are nonnegative real numbers such that $x_1,\dots,x_n,\sum_{i=1}^n x_i\in[0,a]$.

The next result is a special case of Theorem 1.31 for $p_1=\dots=p_n=1$ and $x_0=0$. Likewise, it can be obtained by applying Theorem 1.14 to the Petrović inequality.

Theorem 1.32 *Let f be a convex function on an interval od real numbers I whose interior contains $[0,a]$. If $x_1,\dots,x_n \in [0,a]$ are real numbers such that $\sum_{i=1}^n x_i \in \langle 0,a]$, then we have*

(i)

$$\begin{aligned}
R_f(\mathbf{x}) &\le f\Big(\sum_{i=1}^n x_i\Big) + (n-1)f(0) - \sum_{i=1}^n f(x_i) \\
&\le \sum_{j=1}^n x_j\Big(\sum_{i=1}^n x_i - x_j\Big) \sup_{t\in\langle 0,\sum_{i=1}^n x_i\rangle} \Psi_f\Big(t;0,\sum_{i=1}^n x_i\Big) \\
&\le \frac{\sum_{j=1}^n x_j(\sum_{i=1}^n x_i - x_j)}{\sum_{i=1}^n x_i}\Big(f'_-\Big(\sum_{i=1}^n x_i\Big) - f'_+(0)\Big) \\
&\le \frac{n-1}{n}\Big(\sum_{i=1}^n x_i\Big)\Big(f'_-\Big(\sum_{i=1}^n x_i\Big) - f'_+(0)\Big) \\
&\le \frac{n}{4}\Big(\sum_{i=1}^n x_i\Big)\Big(f'_-\Big(\sum_{i=1}^n x_i\Big) - f'_+(0)\Big)
\end{aligned}$$

(ii)

$$\begin{aligned}
R_f(\mathbf{x}) &\le f\Big(\sum_{i=1}^n x_i\Big) + (n-1)f(0) - \sum_{i=1}^n f(x_i) \\
&\le \sum_{j=1}^n x_j\Big(\sum_{i=1}^n x_i - x_j\Big) \sup_{t\in\langle 0,\sum_{i=1}^n x_i\rangle} \Psi_f\Big(t;0,\sum_{i=1}^n x_i\Big) \\
&\le \frac{n-1}{n}\Big(\sum_{i=1}^n x_i\Big)^2 \sup_{t\in\langle 0,\sum_{i=1}^n x_i\rangle} \Psi_f\Big(t;0,\sum_{i=1}^n x_i\Big) \\
&\le \frac{n-1}{n}\Big(\sum_{i=1}^n x_i\Big)\Big(f'_-\Big(\sum_{i=1}^n x_i\Big) - f'_+(0)\Big) \\
&\le \frac{n}{4}\Big(\sum_{i=1}^n x_i\Big)\Big(f'_-\Big(\sum_{i=1}^n x_i\Big) - f'_+(0)\Big)
\end{aligned}$$

(iii)

$$\begin{aligned}
R_f(\mathbf{x}) &\le f\Big(\sum_{i=1}^n x_i\Big) + (n-1)f(0) - \sum_{i=1}^n f(x_i) \\
&\le \frac{1}{4}\Big(\sum_{i=1}^n x_i\Big)^2 \sum_{i=1}^n \Psi_f\Big(x_i;0,\sum_{i=1}^n x_i\Big) \\
&\le \frac{n}{4}\Big(\sum_{i=1}^n x_i\Big)\Big(f'_-\Big(\sum_{i=1}^n x_i\Big) - f'_+(0)\Big)
\end{aligned}$$

where

$$R_f(\mathbf{x}) = \sum_{i=1}^r \sum_{n=0}^{N-1} \sum_{k=1}^{2^n} \Delta_f(0,\sum_{i=1}^r x_i,n,k) r_n \chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}\Big(\frac{x_i}{\sum_{i=1}^r x_i}\Big),$$

and $\Delta_f(0,\sum_{i=1}^r x_i,n,k)$ is defined in Theorem 1.12. If the function f is concave, inequality signs are reversed.

Chapter 2

Inequalities of the Jensen and Edmundson-Lah-Ribarič type without convexity in the classical sense

In this chapter first we will derive classes of inequalities of the Jensen and Edmundson-Lah-Ribarič type which are valid for functions with bounded second order divided differences and for Lipschitzian functions. This is a significant improvement compared to the results from the previous chapter because the results from this chapter hold for a much wider class of functions than the class of convex functions. Next, we will derive different classes of inequalities of the Jensen and Edmundson-Lah-Ribarič type that hold for 3-convex functions. Finally, we will derive several representations of the left side in the Edmundson-Lah-Ribarič inequality by using Hermite's interpolating polynomial written in terms of divided differences. Those representations are then utilized for obtaining different Edmundson-Lah-Ribarič type inequalities for positive linear functionals and n-convex functions. General results are applied to generalized means. Also, examples with power means are given.

2.1 Introduction

Let E be a nonempty set and let L be a vector space of real-valued functions $f\colon E \to \mathbb{R}$ having the properties:

(L1) $f,g \in L \Rightarrow (af+bg) \in L$ for all $a,b \in \mathbb{R}$;

(L2) $\mathbf{1} \in L$, i.e., if $f(t) = 1$ for every $t \in E$, then $f \in L$.

We also consider positive linear functionals $A\colon L \to \mathbb{R}$. That is, we assume that:

(A1) $A(af+bg) = aA(f)+bA(g)$ for $f,g \in L$ and $a,b \in \mathbb{R}$;

(A2) $f \in L$, $f(t) \geq 0$ for every $t \in E \Rightarrow A(f) \geq 0$ (A is positive).

We say that a functional A is normalized if $A(\mathbf{1}) = 1$.

Throughout this chapter, if a function is defined on an interval $[m,M]$, we assume that the bounds of that interval are finite.

Unlike the results from the previous chapter, which require convexity of the involved functions, the main objective of this chapter is to derive a class of inequalities of the Jensen and Edmundson-Lah-Ribarič type that hold for n-convex functions.

Definition of the n-convex function is characterized by nth-order divided difference. The nth-order divided difference of a function $f\colon [a,b] \to \mathbb{R}$ at mutually distinct points $t_0,t_1,\dots,t_n \in [a,b]$ is defined recursively by

$$[t_i]f = f(t_i), \quad i = 0,\dots,n,$$
$$[t_0,\dots,t_n]f = \frac{[t_1,\dots,t_n]f - [t_0,\dots,t_{n-1}]f}{t_n - t_0}.$$

The value $[t_0,\dots,t_n]f$ is independent of the order of the points $t_0,\dots,t_n$. Definition of divided differences can be extended to include the cases in which some or all the points coincide (see e.g. [2], [124]):

$$f[\underbrace{a,\dots,a}_{n \text{ times}}] = \frac{1}{(n-1)!} f^{(n-1)}(a), \quad n \in \mathbb{N}.$$

Regarding third order divided differences, in the case in which some or all the points coincide they are defined in the following way.

- If the function f is differentiable on $[a,b]$ and $t,t_0,t_1 \in [a,b]$ are mutually different points, then

$$\begin{aligned}[t,t,t_0,t_1]f = {} & \frac{f'(t)}{(t-t_0)(t-t_1)} + \frac{f(t)(t_0+t_1-2t)}{(t-t_0)^2(t-t_1)^2} \\ & + \frac{f(t_0)}{(t_0-t)^2(t_0-t_1)} + \frac{f(t_1)}{(t_1-t)^2(t_1-t_0)}. \end{aligned} \tag{2.1}$$

- If the function f is differentiable on $[a,b]$ and $t,t_0 \in [a,b]$ are mutually different points, then

$$[t,t,t_0,t_0]f = \frac{1}{(t_0-t)^3}\left[(t_0-t)(f'(t_0)+f'(t)) + 2(f(t)-f(t_0))\right]. \tag{2.2}$$

- If the function f is twice differentiable on $[a,b]$ and $t,t_0 \in [a,b]$ are mutually different points, then

$$[t,t,t,t_0]f = \frac{1}{(t_0-t)^3}\left[f(t_0) - \sum_{k=0}^{2}\frac{f^{(k)}(t)}{k!}(t_0-t)^k\right]. \tag{2.3}$$

- If the function f is three times differentiable on $[a,b]$ and $t \in [a,b]$, then

$$[t,t,t,t]f = \frac{f'''(t)}{3!}. \tag{2.4}$$

A function $f\colon [a,b] \to \mathbb{R}$ is said to be n-convex ($n \geq 0$) if and only if for all choices of $(n+1)$ distinct points $t_0,t_1,\ldots,t_n \in [a,b]$, we have $[t_0,\ldots,t_n]f \geq 0$.

We can extend the definition of 3-convex functions by including the cases in which some or all of the points coincide. This is given in the following theorem which can be easily proven by using the mean value theorem for divided differences (see e.g. [64]).

Theorem 2.1 *Let a function f be defined on an interval $I \subseteq \mathbb{R}$. The following equivalences hold.*

(i) If $f \in \mathscr{C}(I)$, then f is 3-convex if and only if $[t,t,t_0,t_1]f \geq 0$ for all mutually different points $t,t_0,t_1 \in I$.

(ii) If $f \in \mathscr{C}(I)$, then f is 3-convex if and only if $[t,t,t_0,t_0]f \geq 0$ for all mutually different points $t,t_0 \in I$.

(iii) If $f \in \mathscr{C}^2(I)$, then f is 3-convex if and only if $[t,t,t,t_0]f \geq 0$ for all mutually different points $t,t_0 \in I$.

(iv) If $f \in \mathscr{C}^3(I)$, then f is 3-convex if and only if $[t,t,t,t]f \geq 0$ for every $t \in I$.

2.2 Inequalities for functions with bounded second order divided differences

In this section we derive a class of inequalities of the Jensen and Edmundson-Lah-Ribarič type which are valid for functions with bounded second order divided differences. This is a significant improvement compared to the results from the previous chapter, because these hold for a much wider class of functions than the class of convex functions.

Throughout this section, whenever mentioning the interval $[m,M]$, we assume that $-\infty < m < M < \infty$ holds.

Theorem 2.2 ([105]) *Let ϕ be a function on an interval of real numbers $[m,M]$ such that there exist $\gamma,\Gamma \in \mathbb{R}$ such that $\gamma \le [m,t,M]\phi \le \Gamma$ holds for every $t \in [m,M]$, that is, such that its second order divided difference in m,t and M is bounded for every $t \in [m,M]$. Let L satisfy conditions (L1) and (L2) on E and let A be any positive linear functional on L with $A(\mathbf{1}) = 1$. Then*

$$\begin{aligned}\gamma A\left[(M\mathbf{1}-f)(f-m\mathbf{1})\right] &\le \frac{M-A(f)}{M-m}\phi(m)+\frac{A(f)-m}{M-m}\phi(M)-A\left(\phi(f)\right)\\ &\le \Gamma A\left[(M\mathbf{1}-f)(f-m\mathbf{1})\right] \qquad (2.5)\end{aligned}$$

holds for any $f \in L$ such that $\phi \circ f \in L$.

Proof. We start with a scalar identity for $t \in [m,M]$:

$$\begin{aligned}\frac{M-t}{M-m}\phi(m)+\frac{t-m}{M-m}\phi(M)-\phi(t) =& \frac{M-t}{M-m}\left(\phi(m)-\phi(t)\right)+\frac{t-m}{M-m}\left(\phi(M)-\phi(t)\right)\\ =& \frac{(M-t)(t-m)}{M-m}\left(\frac{\phi(M)-\phi(t)}{M-t}-\frac{\phi(t)-\phi(m)}{t-m}\right)\\ =& (M-t)(t-m)[m,t,M]\phi.\end{aligned}$$

It follows that

$$\frac{M-t}{M-m}\phi(m)+\frac{t-m}{M-m}\phi(M)-\phi(t) = (M-t)(t-m)[m,t,M]\phi$$

holds for every $t \in [m,M]$. Since the second order divided difference of the function ϕ in m,t and M is bounded, from the previous relation we have

$$\begin{aligned}(M-t)(t-m)\gamma &\le \frac{M-t}{M-m}\phi(m)+\frac{t-m}{M-m}\phi(M)-\phi(t)\\ &\le (M-t)(t-m)\Gamma \qquad (2.6)\end{aligned}$$

for any $t \in [m,M]$. The function $\phi \circ f$ belongs to L, which means that the function f satisfies the bounds

$$m \le f(t) \le M,$$

so we can replace t with $f(t)$ in (2.6) and obtain:

$$\begin{aligned}(M-f(t))(f(t)-m)\gamma &\leq \frac{M-f(t)}{M-m}\phi(m)+\frac{f(t)-m}{M-m}\phi(M)-\phi(f(t))\\ &\leq (M-f(t))(f(t)-m)\Gamma\end{aligned}$$

Functional A is linear and positive, and such that $A(\mathbf{1})=1$, so when we apply it to the previous inequalities we get the following:

$$\begin{aligned}\gamma A\left[(M\mathbf{1}-f)(f-m\mathbf{1})\right] &\leq \frac{M-A(f)}{M-m}\phi(m)+\frac{A(f)-m}{M-m}\phi(M)-A\left(\phi(f)\right)\\ &\leq \Gamma A\left[(M\mathbf{1}-f)(f-m\mathbf{1})\right],\end{aligned}$$

and the proof is complete. □

Corollary 2.1 ([105]) *Let us suppose that the assumptions from Theorem 2.2 hold. If in addition we have $-\infty<\gamma<0<\Gamma<\infty$, then the following inequalities*

$$\begin{aligned}\frac{\gamma}{4}(M-m)^2 \leq \gamma(M-A(f))(A(f)-m) &\leq \gamma A\left[(M\mathbf{1}-f)(f-m\mathbf{1})\right]\\ &\leq \frac{M-A(f)}{M-m}\phi(m)+\frac{A(f)-m}{M-m}\phi(M)-A\left(\phi(f)\right)\\ &\leq \Gamma A\left[(M\mathbf{1}-f)(f-m\mathbf{1})\right]\\ &\leq \Gamma(M-A(f))(A(f)-m)\\ &\leq \frac{\Gamma}{4}(M-m)^2\end{aligned} \tag{2.7}$$

hold for any $f\in L$ such that $\phi\circ f\in L$.

Proof. The function $t\mapsto -t^2+(m+M)t-mM$ is concave, so from Jensen's inequality (1.1) we have

$$A\left[(M\mathbf{1}-f)(f-m\mathbf{1})\right]\leq (M-A(f))(A(f)-m). \tag{2.8}$$

Since Γ is positive, when we multiply (2.8) by $\Gamma/(M-m)$ we get

$$\Gamma A\left[(M\mathbf{1}-f)(f-m\mathbf{1})\right]\leq \Gamma(M-A(f))(A(f)-m), \tag{2.9}$$

and since γ is negative, when we multiply (2.8) by $\gamma/(M-m)$ we get

$$\gamma A\left[(M\mathbf{1}-f)(f-m\mathbf{1})\right]\geq \gamma(M-A(f))(A(f)-m). \tag{2.10}$$

Inequalities (2.7) now follow from Theorem 2.2, relations (2.9) and (2.10) and relation

$$(M-t)(t-m)\leq \frac{1}{4}(M-m)^2 \quad \text{for any } t\in[m,M].$$

□

Corollary 2.2 ([105]) *Let us suppose that the assumptions from Theorem 2.2 hold. If in addition we have $\gamma = -\Gamma$, then the following inequalities*

$$\begin{aligned} &\left| \frac{M - A(f)}{M - m} \phi(m) + \frac{A(f) - m}{M - m} \phi(M) - A(\phi(f)) \right| \\ &\quad \leq \Gamma A[(M\mathbf{1} - f)(f - m\mathbf{1})] \\ &\quad \leq \Gamma(M - A(f))(A(f) - m) \leq \frac{\Gamma}{4}(M - m)^2 \end{aligned} \tag{2.11}$$

hold for any $f \in L$ such that $\phi \circ f \in L$.

Proof. Inequalities (2.11) follow directly from the definition of the absolute value and Corollary 2.1. □

Remark 2.1 There are two more cases that need to be considered.

- If $0 \leq \gamma < \Gamma < \infty$, then the function ϕ is convex, so from the Edmundson-Lah-Ribarič inequality (1.2) and (2.9) it follows that

$$\begin{aligned} 0 &\leq \gamma A[(M\mathbf{1} - f)(f - m\mathbf{1})] \\ &\leq \frac{M - A(f)}{M - m} \phi(m) + \frac{A(f) - m}{M - m} \phi(M) - A(\phi(f)) \\ &\leq \Gamma A[(M\mathbf{1} - f)(f - m\mathbf{1})] \\ &\leq \Gamma(M - A(f))(A(f) - m) \leq \frac{\Gamma}{4}(M - m)^2 \end{aligned} \tag{2.12}$$

  holds for any $f \in L$ such that $\phi \circ f \in L$.

- If $-\infty < \gamma < \Gamma \leq 0$, then the function ϕ is concave, so from the Edmundson-Lah-Ribarič inequality (1.2) and (2.10) it follows that

$$\begin{aligned} \frac{\gamma}{4}(M - m)^2 &\leq \gamma(M - A(f))(A(f) - m) \\ &\leq \gamma A[(M\mathbf{1} - f)(f - m\mathbf{1})] \\ &\leq \frac{M - A(f)}{M - m} \phi(m) + \frac{A(f) - m}{M - m} \phi(M) - A(\phi(f)) \\ &\leq \Gamma A[(M\mathbf{1} - f)(f - m\mathbf{1})] \leq 0 \end{aligned} \tag{2.13}$$

  holds for any $f \in L$ such that $\phi \circ f \in L$.

Theorem 2.2 can be utilized for obtaining Jensen-type inequalities for functions with bounded second order divided differences.

Theorem 2.3 ([105]) *Let ϕ be a function on an interval of real numbers $[m, M]$ such that there exist $\gamma, \Gamma \in \mathbb{R}$ such that $\gamma \leq [m, t, M]\phi \leq \Gamma$ holds for every $t \in [m, M]$, that is, such that its second order divided difference in m, t and M is bounded for every $t \in [m, M]$. Let L*

satisfy conditions (L1) and (L2) on E and let A be any positive linear functional on L with $A(\mathbf{1}) = 1$. *Then*

$$\begin{aligned}\gamma(M-A(f))(A(f)-m) &- \Gamma A\left[(M\mathbf{1}-f)(f-m\mathbf{1})\right] \\ &\leq A(\phi(f)) - \phi(A(f)) \leq \Gamma(M-A(f))(A(f)-m) - \gamma A\left[(M\mathbf{1}-f)(f-m\mathbf{1})\right]\end{aligned} \tag{2.14}$$

holds for any $f \in L$ *such that* $\phi \circ f \in L$.

Proof. Function $\phi \circ f$ belongs to L, which means that the function f satisfies the bounds $m \leq f(t) \leq M$. It follows that $m \leq A(f) \leq M$, so we can replace t with $A(f)$ in the relation (2.6) and obtain

$$\begin{aligned}\gamma(M-A(f))(A(f)-m) &\leq \frac{M-A(f)}{M-m}\phi(m) + \frac{A(f)-m}{M-m}\phi(M) - \phi(A(f)) \\ &\leq \Gamma(M-A(f))(A(f)-m).\end{aligned} \tag{2.15}$$

When we multiply the relation (2.5) from Theorem 2.2 by -1 we get

$$\begin{aligned}-\Gamma A\left[(M\mathbf{1}-f)(f-m\mathbf{1})\right] &\leq -\frac{M-A(f)}{M-m}\phi(m) - \frac{A(f)-m}{M-m}\phi(M) + A(\phi(f)) \\ &\leq -\gamma A\left[(M\mathbf{1}-f)(f-m\mathbf{1})\right].\end{aligned} \tag{2.16}$$

Inequalities (2.14) follow by adding (2.15) to (2.16). □

In an analogous way as in Corollary 2.1 and Corollary 2.2, depending on the positivity and negativity of the bounds γ and Γ, inequalities (2.14) can be extended in the following way.

Corollary 2.3 ([105]) *Let us suppose that the assumptions from Theorem 2.3 hold. If in addition we have* $-\infty < \gamma < 0 < \Gamma < \infty$, *then the following inequalities*

$$\begin{aligned}\frac{\gamma-\Gamma}{4}(M-m)^2 &\leq (\gamma-\Gamma)(M-A(f))(A(f)-m) \\ &\leq \gamma(M-A(f))(A(f)-m) - \Gamma A\left[(M\mathbf{1}-f)(f-m\mathbf{1})\right] \\ &\leq A(\phi(f)) - \phi(A(f)) \\ &\leq \Gamma(M-A(f))(A(f)-m) - \gamma A\left[(M\mathbf{1}-f)(f-m\mathbf{1})\right] \\ &\leq (\Gamma-\gamma)(M-A(f))(A(f)-m) \leq \frac{\Gamma-\gamma}{4}(M-m)^2\end{aligned} \tag{2.17}$$

hold for any $f \in L$ *such that* $\phi \circ f \in L$.

Corollary 2.4 ([105]) *Let us suppose that the assumptions from Theorem 2.3 hold. If in addition we have* $\gamma = -\Gamma$, *then the following inequalities*

$$\begin{aligned}|A(\phi(f)) - \phi(A(f))| &\leq \Gamma\left(A(f^2) - (A(f))^2\right) \\ &\leq 2\Gamma(M-A(f))(A(f)-m) \leq \frac{\Gamma}{2}(M-m)^2\end{aligned} \tag{2.18}$$

hold for any $f \in L$ *such that* $\phi \circ f \in L$.

In our next result, which is based on the method from the paper [29], we give inequalities of the Jensen and Edmundson-Lah-Ribarič type for Lipschitzian mappings.

Theorem 2.4 ([105]) *Let $\phi\colon [m,M] \to \mathbb{R}$ be a Lipschitzian function with the Lipschitz constant L. Let L satisfy conditions (L1) and (L2) on E and let A be any positive linear functional on L with $A(\mathbf{1}) = 1$. Then the inequalities*

$$|A(\phi(f)) - \phi(A(f))| \leq LA(|f - A(f)\mathbf{1}|) \tag{2.19}$$

and

$$\begin{aligned}
&\left|\frac{M-A(f)}{M-m}\phi(m) + \frac{A(f)-m}{M-m}\phi(M) - A(\phi(f))\right| \\
&\quad \leq \frac{2L}{M-m}A([M\mathbf{1}-f][f-m\mathbf{1}]) \\
&\quad \leq \frac{2L}{M-m}(M-A(f))(A(f)-m) \leq \frac{L}{2}(M-m)
\end{aligned} \tag{2.20}$$

hold for any $f \in L$ such that $\phi \circ f \in L$.

Proof. By using the properties of linear functionals and absolute value, and the fact that because function ϕ is a Lipschitzian with the Lipschitz constant L we have

$$|\phi(x) - \phi(y)| \leq L|x-y| \quad \text{for every } x, y \in [m,M],$$

we can calculate

$$\begin{aligned}
|A(\phi(f)) - \phi(A(f))| &= |A[\phi(f) - \phi(A(f))\mathbf{1}]| \\
&\leq A(|\phi(f) - \phi(A(f))\mathbf{1}|) \leq LA(|f - A(f)\mathbf{1}|),
\end{aligned}$$

which is (2.19). In a similar way we have

$$\begin{aligned}
&\left|\frac{M-A(f)}{M-m}\phi(m) + \frac{A(f)-m}{M-m}\phi(M) - A(\phi(f))\right| \\
&\quad = \left|A\left(\frac{M\mathbf{1}-f}{M-m}\phi(m) + \frac{f-m\mathbf{1}}{M-m}\phi(M) - \phi(f)\right)\right| \\
&\quad \leq A\left(\left|\frac{M\mathbf{1}-f}{M-m}\phi(m) + \frac{f-m\mathbf{1}}{M-m}\phi(M) - \phi(f)\right|\right) \\
&\quad = A\left(\left|\frac{M\mathbf{1}-f}{M-m}(\phi(m) - \phi(f)) + \frac{f-m\mathbf{1}}{M-m}(\phi(M) - \phi(f))\right|\right) \\
&\quad \leq A\left(\frac{M\mathbf{1}-f}{M-m}|\phi(m) - \phi(f)| + \frac{f-m\mathbf{1}}{M-m}|\phi(M) - \phi(f)|\right) \\
&\quad \leq \frac{2L}{M-m}A([M\mathbf{1}-f][f-m\mathbf{1}]),
\end{aligned}$$

which gives us first inequality in (2.20). Last two inequalities in (2.20) follow by applying Jensen's inequality to the concave function $t \mapsto -t^2 + (m+M)t - mM$, and the fact that $(M-t)(t-m) \leq (M-m)^2/4$ holds for every $t \in [m,M]$. □

2.3 Inequalities for 3-convex functions

Theorem 2.5 ([103]) *Let L satisfy conditions (L1) and (L2) on a non-empty set E and let A be any positive linear functional on L with $A(\mathbf{1}) = 1$. Let ϕ be a 3-convex function on an interval of real numbers I whose interior contains the interval $[m,M]$. Then*

$$
\begin{aligned}
&\frac{A[(M\mathbf{1}-f)(f-m\mathbf{1})]}{M-m}\left(\frac{\phi(M)-\phi(m)}{M-m}-\phi'_{+}(m)\right)\\
&\quad\le \frac{M-A(f)}{M-m}\phi(m)+\frac{A(f)-m}{M-m}\phi(M)-A(\phi(f)) \qquad (2.21)\\
&\quad\le \frac{A[(M\mathbf{1}-f)(f-m\mathbf{1})]}{M-m}\left(\phi'_{-}(M)-\frac{\phi(M)-\phi(m)}{M-m}\right)
\end{aligned}
$$

holds for any $f \in L$ such that $\phi \circ f \in L$ and $m \le f(t) \le M$ for $t \in E$. If the function $-\phi$ is 3-convex, then the inequalities are reversed.

Proof. We start with a scalar identity for $t \in [m,M]$:

$$
\begin{aligned}
\frac{M-t}{M-m}\phi(m)+\frac{t-m}{M-m}\phi(M)-\phi(t) &= \frac{M-t}{M-m}(\phi(m)-\phi(t))+\frac{t-m}{M-m}(\phi(M)-\phi(t))\\
&= \frac{(M-t)(t-m)}{M-m}\left(\frac{\phi(M)-\phi(t)}{M-t}-\frac{\phi(t)-\phi(m)}{t-m}\right)\\
&= (M-t)(t-m)[m,t,M]\phi.
\end{aligned}
$$

It follows that

$$
\frac{M-t}{M-m}\phi(m)+\frac{t-m}{M-m}\phi(M)-\phi(t) = (M-t)(t-m)[m,t,M]\phi \qquad (2.22)
$$

holds for every $t \in [m,M]$.

Since the function ϕ is 3-convex, we have $[t_0,t_1,t_2,t_3]\phi \ge 0$ for every choice of the points $t_0,t_1,t_2,t_3 \in [m,M]$. Let $t_0 = m$, $t_3 = M$ and $t_1 < t_2$. From the definition and main properties of the divided differences we get the following relation:

$$
\begin{aligned}
0 \le [m,t_1,t_2,M]\phi = [t_1,m,M,t_2]\phi &= \frac{[m,M,t_2]\phi-[t_1,m,M]\phi}{t_2-t_1}\\
&= \frac{[m,t_2,M]\phi-[m,t_1,M]\phi}{t_2-t_1},
\end{aligned}
$$

so we have obtained that

$$
[m,t_2,M]\phi-[m,t_1,M]\phi \ge 0
$$

holds for any $t_1 < t_2$, that is, the function $[m,t,M]\phi$ is non-decreasing on $[m,M]$. It follows that the function $[m,t,M]\phi$ attains its minimal and maximal value in the points m and M respectively. We can calculate those bounds:

$$
[m,m,M]\phi = \frac{1}{M-m}\left(\frac{\phi(M)-\phi(m)}{M-m}-\phi'_{+}(m)\right)
$$

$$[m,M,M]\phi = \frac{1}{M-m}\left(\phi'_-(M) - \frac{\phi(M)-\phi(m)}{M-m}\right) \tag{2.23}$$

Now, from (2.22) and (2.23) we have

$$\begin{aligned}\frac{(M-t)(t-m)}{M-m}\left(\frac{\phi(M)-\phi(m)}{M-m} - \phi'_+(m)\right) &\le \frac{M-t}{M-m}\phi(m) + \frac{t-m}{M-m}\phi(M) - \phi(t) \\ &\le \frac{(M-t)(t-m)}{M-m}\left(\phi'_-(M) - \frac{\phi(M)-\phi(m)}{M-m}\right) \end{aligned} \tag{2.24}$$

for any $t \in [m,M]$. The function f satisfies the bounds

$$m \le f(t) \le M,$$

so we can replace t with $f(t)$ in (2.24) and obtain:

$$\begin{aligned}&\frac{(M-f(t))(f(t)-m)}{M-m}\left(\frac{\phi(M)-\phi(m)}{M-m} - \phi'_+(m)\right) \\ &\le \frac{M-f(t)}{M-m}\phi(m) + \frac{f(t)-m}{M-m}\phi(M) - \phi(f(t)) \\ &\le \frac{(M-f(t))(f(t)-m)}{M-m}\left(\phi'_-(M) - \frac{\phi(M)-\phi(m)}{M-m}\right).\end{aligned}$$

Functional A is linear and positive, and such that $A(\mathbf{1}) = 1$, so when we apply it to the previous inequalities we get the following:

$$\begin{aligned}&\frac{A[(M\mathbf{1}-f)(f-m\mathbf{1})]}{M-m}\left(\frac{\phi(M)-\phi(m)}{M-m} - \phi'_+(m)\right) \\ &\le \frac{M-A(f)}{M-m}\phi(m) + \frac{A(f)-m}{M-m}\phi(M) - A(\phi(f)) \\ &\le \frac{A[(M\mathbf{1}-f)(f-m\mathbf{1})]}{M-m}\left(\phi'_-(M) - \frac{\phi(M)-\phi(m)}{M-m}\right),\end{aligned}$$

which concludes the proof. □

Theorem 2.5 can be utilized for obtaining Jensen-type inequalities for 3-convex functions.

Theorem 2.6 ([103]) *Let L satisfy conditions (L1) and (L2) on a non-empty set E and let A be any positive linear functional on L with $A(\mathbf{1}) = 1$. Let ϕ be a 3-convex function on an interval of real numbers I whose interior contains the interval $[m,M]$. Then*

$$\begin{aligned}&\frac{(M-A(f))(A(f)-m)}{M-m}\left(\frac{\phi(M)-\phi(m)}{M-m} - \phi'_+(m)\right) \\ &\quad - \frac{A[(M\mathbf{1}-f)(f-m\mathbf{1})]}{M-m}\left(\phi'_-(M) - \frac{\phi(M)-\phi(m)}{M-m}\right)\end{aligned} \tag{2.25}$$

$$\leq A(\phi(f)) - \phi(A(f)) \leq \frac{(M-A(f))(A(f)-m)}{M-m}\left(\phi'_-(M) - \frac{\phi(M)-\phi(m)}{M-m}\right)$$
$$-\frac{A\left[(M\mathbf{1}-f)(f-m\mathbf{1})\right]}{M-m}\left(\frac{\phi(M)-\phi(m)}{M-m} - \phi'_+(m)\right)$$

holds for any $f \in L$ such that $\phi \circ f \in L$ and $m \leq f(t) \leq M$ for $t \in E$. If the function $-\phi$ is 3-convex, then the inequalities are reversed.

Proof. Function $\phi \circ f$ belongs to L, which means that the function f satisfies the bounds $m \leq f(t) \leq M$. It follows that $m \leq A(f) \leq M$, so we can replace t with $A(f)$ in the relation (2.24) and obtain

$$\begin{aligned}
&\frac{(M-A(f))(A(f)-m)}{M-m}\left(\frac{\phi(M)-\phi(m)}{M-m} - \phi'_+(m)\right)\\
&\quad\leq \frac{M-A(f)}{M-m}\phi(m) + \frac{A(f)-m}{M-m}\phi(M) - \phi(A(f)) \qquad (2.26)\\
&\quad\leq \frac{(M-A(f))(A(f)-m)}{M-m}\left(\phi'_-(M) - \frac{\phi(M)-\phi(m)}{M-m}\right).
\end{aligned}$$

When we multiply the relation (2.21) from Theorem 2.5 by -1 we get

$$\begin{aligned}
&-\frac{A\left[(M\mathbf{1}-f)(f-m\mathbf{1})\right]}{M-m}\left(\phi'_-(M) - \frac{\phi(M)-\phi(m)}{M-m}\right)\\
&\quad\leq -\frac{M-A(f)}{M-m}\phi(m) - \frac{A(f)-m}{M-m}\phi(M) + A(\phi(f)) \qquad (2.27)\\
&\quad\leq -\frac{A\left[(M\mathbf{1}-f)(f-m\mathbf{1})\right]}{M-m}\left(\frac{\phi(M)-\phi(m)}{M-m} - \phi'_+(m)\right).
\end{aligned}$$

Inequalities (2.25) follow by adding (2.26) to (2.27). □

Following results are obtained by virtue of Theorem 2.1, and they represent different Edmundson-Lah-Ribarič type inequalities for 3-convex functions.

Theorem 2.7 ([106]) *Let L satisfy conditions (L1) and (L2) on a non-empty set E and let A be any positive linear functional on L with $A(\mathbf{1}) = 1$. Let ϕ be a 3-convex function defined on an interval of real numbers I whose interior contains the interval $[m,M]$ and differentiable on $\langle m,M\rangle$. Then*

$$\begin{aligned}
&(A(f)-m)\left[\frac{\phi(M)-\phi(m)}{M-m} - \frac{\phi'_+(m)}{2}\right] - \frac{1}{2}A[(f-m\mathbf{1})\phi'(f)]\\
&\quad\leq \frac{M-A(f)}{M-m}\phi(m) + \frac{A(f)-m}{M-m}\phi(M) - A(\phi(f)) \qquad (2.28)\\
&\quad\leq \frac{1}{2}A[(M\mathbf{1}-f)\phi'(f)] - (M-A(f))\left[\frac{\phi(M)-\phi(m)}{M-m} - \frac{\phi'_-(M)}{2}\right]
\end{aligned}$$

holds for any $f \in L$ such that $\phi \circ f \in L$ and $m \leq f(t) \leq M$ for $t \in E$. If the function $-\phi$ is 3-convex, then the inequalities are reversed.

Proof. Let ϕ be a 3-convex function. From Theorem 2.1 (ii) we have that $[t,t,t_0,t_0]\phi \geq 0$ for all mutually different points $t,t_0 \in I$. When we take $t = m$ and $t_0 = x$ in (2.2), we obtain that

$$0 \leq \frac{1}{(x-m)^3}\left[(x-m)(\phi'(x)+\phi'_+(m))+2(\phi(m)-\phi(x))\right]$$

holds for every $x \in \langle m,M\rangle$. After multiplying by $(x-m)^3$ and rearranging, the relation from above becomes

$$\begin{aligned}(x-m)&\left[\frac{\phi(M)-\phi(m)}{M-m}-\frac{1}{2}\left(\phi'(x)+\phi'_+(m)\right)\right]\\ &\leq \frac{M-x}{M-m}\phi(m)+\frac{x-m}{M-m}\phi(M)-\phi(x).\end{aligned} \tag{2.29}$$

Similarly, when we put $t = M$ and $t_0 = x$ in (2.2) and rearrange the obtained relation, we get that

$$\begin{aligned}&\frac{M-x}{M-m}\phi(m)+\frac{x-m}{M-m}\phi(M)-\phi(x)\\ &\quad\leq (M-x)\left[\frac{1}{2}\left(\phi'(x)+\phi'_-(M)\right)-\frac{\phi(M)-\phi(m)}{M-m}\right]\end{aligned} \tag{2.30}$$

holds for every $x \in \langle m,M\rangle$. Now, we see that (2.29) and (2.30) together give the following sequence of inequalities:

$$\begin{aligned}(x-m)&\left[\frac{\phi(M)-\phi(m)}{M-m}-\frac{1}{2}\left(\phi'(x)+\phi'_+(m)\right)\right]\\ &\leq \frac{M-x}{M-m}\phi(m)+\frac{x-m}{M-m}\phi(M)-\phi(x)\\ &\leq (M-x)\left[\frac{1}{2}\left(\phi'(x)+\phi'_-(M)\right)-\frac{\phi(M)-\phi(m)}{M-m}\right].\end{aligned} \tag{2.31}$$

Since the function $f \in L$ satisfies the bounds $m \leq f(t) \leq M$, we can replace x with $f(t)$ in (2.31), and get

$$\begin{aligned}(f(t)-m)&\left[\frac{\phi(M)-\phi(m)}{M-m}-\frac{1}{2}\left(\phi'(f(t))+\phi'_+(m)\right)\right]\\ &\leq \frac{M-f(t)}{M-m}\phi(m)+\frac{f(t)-m}{M-m}\phi(M)-\phi(f(t))\\ &\leq (M-f(t))\left[\frac{1}{2}\left(\phi'(f(t))+\phi'_-(M)\right)-\frac{\phi(M)-\phi(m)}{M-m}\right].\end{aligned}$$

The inequalities (2.28) follow after applying linear functional A to the previous relation taking into account linearity of the functional A and condition $A(\mathbf{1}) = 1$. □

Remark 2.2 If it exists, the first derivative ϕ' of a 3-convex function ϕ is a convex function. It is known that convex functions are continuous on every open interval, and their one-sided derivatives exist and are finite.

Theorem 2.8 ([106]) *Let L satisfy conditions (L1) and (L2) on a non-empty set E and let A be any positive linear functional on L with $A(\mathbf{1}) = 1$. Let ϕ be a 3-convex function defined on an interval of real numbers I whose interior contains the interval $[m,M]$ and differentiable on $\langle m,M\rangle$. Then*

$$\begin{aligned}
&(M - A(f))\left[\phi'_-(M) - \frac{\phi(M)-\phi(m)}{M-m}\right] - \frac{\phi''_-(M)}{2}A[(M\mathbf{1}-f)^2] \\
&\leq \frac{M-A(f)}{M-m}\phi(m) + \frac{A(f)-m}{M-m}\phi(M) - A(\phi(f)) \qquad (2.32)\\
&\leq (A(f)-m)\left[\frac{\phi(M)-\phi(m)}{M-m} - \phi'_+(m)\right] - \frac{\phi''_+(m)}{2}A[(f-m\mathbf{1})^2]
\end{aligned}$$

holds for any $f \in L$ such that $\phi \circ f \in L$ and $m \leq f(t) \leq M$ for $t \in E$. If the function $-\phi$ is 3-convex, then the inequalities are reversed.

Proof. The function ϕ is 3-convex on $[m,M]$ and twice differentiable, so from Theorem 2.1 (iii) we have that $[t,t,t_0,t_1]\phi \geq 0$ for all mutually different points $t,t,t_0 \in [m,M]$. When we take $t = m$ and $t_0 = x$ in (2.3), we obtain that

$$0 \leq \frac{1}{(x-m)^3}\left[\phi(x) - \sum_{k=0}^{2}\frac{\phi_+^{(k)}(m)}{k!}(x-m)^k\right]$$

holds for every $x \in \langle m,M\rangle$. After multiplying by $(x-m)^3$ and rearranging, the upper relation becomes

$$\begin{aligned}
&\frac{M-x}{M-m}\phi(m) + \frac{x-m}{M-m}\phi(M) - \phi(x) \\
&\quad\leq (x-m)\left[\frac{\phi(M)-\phi(m)}{M-m} - \phi'_+(m) - \frac{\phi''_+(m)}{2}(x-m)\right]. \qquad (2.33)
\end{aligned}$$

In a similar manner, when we put $t = M$ and $t_0 = x$ in (2.3), after rearranging the relation thus obtained, we get that

$$\begin{aligned}
&(M-x)\left[\phi'_-(M) - \frac{\phi(M)-\phi(m)}{M-m} - \frac{\phi''_-(M)}{2}(M-x)\right] \\
&\quad\leq \frac{M-x}{M-m}\phi(m) + \frac{x-m}{M-m}\phi(M) - \phi(x) \qquad (2.34)
\end{aligned}$$

holds for every $x \in \langle m,M\rangle$. Now, we see that (2.33) and (2.34) give the following sequence of inequalities:

$$\begin{aligned}
&(M-x)\left[\phi'_-(M) - \frac{\phi(M)-\phi(m)}{M-m} - \frac{\phi''_-(M)}{2}(M-x)\right] \\
&\quad\leq \frac{M-x}{M-m}\phi(m) + \frac{x-m}{M-m}\phi(M) - \phi(x) \\
&\quad\leq (x-m)\left[\frac{\phi(M)-\phi(m)}{M-m} - \phi'_+(m) - \frac{\phi''_+(m)}{2}(x-m)\right]. \qquad (2.35)
\end{aligned}$$

Since the function $f \in L$ satisfies the bounds $m \le f(t) \le M$, we can replace x with $f(t)$ in (2.35), and get

$$
\begin{aligned}
&(M-f(t))\left[\phi_-'(M)-\frac{\phi(M)-\phi(m)}{M-m}-\frac{\phi_-''(M)}{2}(M-f(t))\right]\\
&\le \frac{M-f(t)}{M-m}\phi(m)+\frac{f(t)-m}{M-m}\phi(M)-\phi(f(t))\\
&\le (f(t)-m)\left[\frac{\phi(M)-\phi(m)}{M-m}-\phi_+'(m)-\frac{\phi_+''(m)}{2}(f(t)-m)\right].
\end{aligned}
$$

The inequalities (2.32) follow after applying linear functional A to the previous relation taking into account linearity of the functional A and condition $A(\mathbf{1})=1$. □

Remark 2.3 For the sake of completeness, we give an alternative proof of Theorem 2.5, which is shorter and more elegant ([106]):

Proof. The function ϕ is 3-convex, so from Theorem 2.1 (i) we have that $[t,t,t_0,t_1]\phi \ge 0$ for all mutually different points $t,t_0,t_1 \in I$. When we take $t=m$, $t_0=x$ and $t_1=M$ in (2.1), we obtain that

$$
0 \le \frac{\phi_+'(m)}{(m-x)(m-M)}+\frac{\phi(m)(x+M-2m)}{(m-x)^2(m-M)^2}+\frac{\phi(x)}{(x-m)^2(x-M)}+\frac{\phi(M)}{(M-m)^2(M-x)}
$$

holds for every $x \in \langle m,M\rangle$. After multiplying by $(x-m)^2(x-M)$ and rearranging, the upper relation becomes

$$
\begin{aligned}
&\frac{(M-x)(x-m)}{M-m}\left(\frac{\phi(M)-\phi(m)}{M-m}-\phi_+'(m)\right)\\
&\le \frac{M-x}{M-m}\phi(m)+\frac{x-m}{M-m}\phi(M)-\phi(x).
\end{aligned} \tag{2.36}
$$

In a similar manner, when we put $t=M$, $t_0=x$ and $t_1=m$ in (2.1), after arranging the relation thus obtained, we get that

$$
\begin{aligned}
&\frac{M-x}{M-m}\phi(m)+\frac{x-m}{M-m}\phi(M)-\phi(x)\\
&\le \frac{(M-x)(x-m)}{M-m}\left(\phi_-'(M)-\frac{\phi(M)-\phi(m)}{M-m}\right)
\end{aligned} \tag{2.37}
$$

holds for every $x \in \langle m,M\rangle$. Now, we see that (2.36) and (2.37) give the following sequence of inequalities:

$$
\begin{aligned}
&\frac{(M-x)(x-m)}{M-m}\left(\frac{\phi(M)-\phi(m)}{M-m}-\phi_+'(m)\right)\\
&\le \frac{M-x}{M-m}\phi(m)+\frac{x-m}{M-m}\phi(M)-\phi(x)\\
&\le \frac{(M-x)(x-m)}{M-m}\left(\phi_-'(M)-\frac{\phi(M)-\phi(m)}{M-m}\right).
\end{aligned} \tag{2.38}
$$

Since the function $f \in L$ satisfies the bounds $m \leq f(t) \leq M$, we can replace x with $f(t)$ in (2.38), and get

$$\begin{aligned}
&\frac{(M-f(t))(f(t)-m)}{M-m}\left(\frac{\phi(M)-\phi(m)}{M-m}-\phi'_+(m)\right) \\
&\leq \frac{M-f(t)}{M-m}\phi(m)+\frac{f(t)-m}{M-m}\phi(M)-\phi(f(t)) \\
&\leq \frac{(M-f(t))(f(t)-m)}{M-m}\left(\phi'_-(M)-\frac{\phi(M)-\phi(m)}{M-m}\right).
\end{aligned}$$

The inequalities (2.21) follow after applying linear functional A to the previous relation taking into account linearity of the functional A and condition $A(\mathbf{1})=1$. □

Remark 2.4 Theorems 2.7 and 2.8 can be utilized for obtaining following Jensen-type inequalities for 3-convex functions.

(i) When we put $x=A(f)$ in scalar inequalities (2.31) and then subtract the inequalities from Theorem 2.7, we get

$$\begin{aligned}
&(A(f)-m)\left[\frac{\phi(M)-\phi(m)}{M-m}-\frac{1}{2}\left(\phi'(A(f))+\phi'_+(m)\right)\right] \\
&\quad -\frac{1}{2}A[(M\mathbf{1}-f)\phi'(f)]-(M-A(f))\left[\frac{\phi(M)-\phi(m)}{M-m}+\frac{\phi'_-(M)}{2}\right] \\
&\leq A(\phi(f))-\phi(A(f)) \\
&\leq (M-A(f))\left[\frac{1}{2}\left(\phi'(A(f))+\phi'_-(M)\right)-\frac{\phi(M)-\phi(m)}{M-m}\right] \\
&\quad -(A(f)-m)\left[\frac{\phi(M)-\phi(m)}{M-m}-\frac{\phi'_+(m)}{2}\right]+\frac{1}{2}A[(f-m\mathbf{1})\phi'(f)].
\end{aligned}$$

(ii) When we put $x=A(f)$ in scalar inequalities (2.35) and then subtract the inequalities from Theorem 2.8, we get

$$\begin{aligned}
&(M-A(f))\left[\phi'_-(M)-\frac{\phi(M)-\phi(m)}{M-m}-\frac{\phi''_-(M)}{2}(M-A(f))\right] \\
&\quad -(A(f)-m)\left[\frac{\phi(M)-\phi(m)}{M-m}-\phi'_+(m)\right]+\frac{\phi''_+(m)}{2}A[(f-m\mathbf{1})^2] \\
&\leq A(\phi(f))-\phi(A(f)) \\
&\leq (A(f)-m)\left[\frac{\phi(M)-\phi(m)}{M-m}-\phi'_+(m)-\frac{\phi''_+(m)}{2}(A(f)-m)\right] \\
&\quad -(M-A(f))\left[\phi'_-(M)-\frac{\phi(M)-\phi(m)}{M-m}\right]+\frac{\phi''_-(M)}{2}A[(M\mathbf{1}-f)^2].
\end{aligned}$$

2.4 Inequalities for n-convex functions

The results in this section are obtained by utilizing Hermite's interpolating polynomial, so first we need to give a definition and some properties (see [2]).

Let $-\infty < a < b < \infty$ and let $a \leq a_1 < a_2 < \ldots < a_r \leq b$, where $r \geq 2$, be given points. For $f \in \mathscr{C}^n([a,b])$ there exists a unique polynomial $P_H(t)$, called Hermite's interpolating polynomial, of degree $(n-1)$ fulfilling Hermite's conditions:

$$P_H^{(i)}(a_j) = f^{(i)}(a_j) : \ 0 \leq i \leq k_j, \ 1 \leq j \leq r, \ \sum_{j=1}^{r} k_j + r = n.$$

Among other special cases, these conditions include type $(m, n-m)$ conditions, which will be of special interest to us:

$$(r = 2, \ 1 \leq m \leq n-1, \ k_1 = m-1, \ k_2 = n-m-1)$$

$$\begin{aligned} P_{mn}^{(i)}(a) &= f^{(i)}(a), \ 0 \leq i \leq m-1 \\ P_{mn}^{(i)}(b) &= f^{(i)}(b), \ 0 \leq i \leq n-m-1. \end{aligned}$$

To give a development of the interpolating polynomial in terms of divided differences, first let us assume that the function f is also defined at a point $t \neq a_j, \ 1 \leq j \leq n$. In [2] it is shown that

$$f(t) = P(t) + R(t), \tag{2.39}$$

where

$$\begin{aligned} P(t) = f(a_1) &+ (t-a_1) f[a_1,a_2] + (t-a_1)(t-a_2) f[a_1,a_2,a_3] \\ &+ \ldots + (t-a_1) \cdots (t-a_{n-1}) f[a_1, \ldots, a_n] \end{aligned} \tag{2.40}$$

and

$$R(t) = (t-a_1) \cdots (t-a_n) f[t, a_1, \ldots, a_n]. \tag{2.41}$$

In case of $(m, n-m)$ conditions, (2.40) and (2.41) become

$$\begin{aligned} P_{mn}(t) = f(a) &+ (t-a) f[a,a] + \ldots + (t-a)^{m-1} f[\underbrace{a, \ldots, a}_{m \text{ times}}] \\ &+ (t-a)^m f[\underbrace{a, \ldots, a}_{m \text{ times}}; b] + (t-a)^m (t-b) f[\underbrace{a, \ldots, a}_{m \text{ times}}; b, b] \\ &+ \ldots + (t-a)^m (t-b)^{n-m-1} f[\underbrace{a, \ldots, a}_{m \text{ times}}; \underbrace{b, b, \ldots, b}_{(n-m) \text{ times}}] \end{aligned} \tag{2.42}$$

and

$$R_m(t) = (t-a)^m(t-b)^{n-m} f[t; \underbrace{a,\dots,a}_{m \text{ times}};\ \underbrace{b,b,\dots,b}_{(n-m) \text{ times}}]. \tag{2.43}$$

Throughout this section, whenever mentioning the interval $[a,b]$, we assume that $-\infty < a < b < \infty$ holds.

Let L satisfy conditions $(L1)$ and $(L2)$ on a non-empty set E, let A be any positive linear functional on L with $A(\mathbf{1}) = 1$, and let $g \in L$ be any function such that $g(E) \subseteq [a,b]$. For a given function $f\colon [a,b] \to \mathbb{R}$ denote:

$$LR(f,g,a,b,A) = A(f(g)) - \frac{b-A(g)}{b-a} f(a) - \frac{A(g)-a}{b-a} f(b). \tag{2.44}$$

Following representations of the left side in the Edmundson-Lah-Ribarič inequality is obtained by using Hermite's interpolating polynomials in terms of divided differences (2.42).

Lemma 2.1 ([107]) *Let L satisfy conditions (L1) and (L2) on a non-empty set E and let A be any positive linear functional on L with $A(\mathbf{1}) = 1$. Let $f \in \mathscr{C}^n([a,b])$, and let $g \in L$ be any function such that $f \circ g \in L$. Then the following identities hold:*

- $$LR(f,g,a,b,A) = \sum_{k=2}^{n-1} f[a; \underbrace{b,\dots,b}_{k \text{ times}}] A\left[(g-a\mathbf{1})(g-b\mathbf{1})^{k-1}\right] + A(R_1(g)) \tag{2.45}$$

- $$\begin{aligned} LR(f,g,a,b,A) &= f[a,a;b] A[(g-a\mathbf{1})(g-b\mathbf{1})] \\ &+ \sum_{k=2}^{n-2} f[a,a; \underbrace{b,\dots,b}_{k \text{ times}}] A\left[(g-a\mathbf{1})^2(g-b\mathbf{1})^{k-1}\right] + A(R_2(g)) \end{aligned} \tag{2.46}$$

- $$\begin{aligned} LR(f,g,a,b,A) &= (A(g)-a)\,(f[a,a]-f[a,b]) + \sum_{k=2}^{m-1} \frac{f^{(k)}(a)}{k!} A\left[(g-a\mathbf{1})^k\right] \\ &+ \sum_{k=1}^{n-m} f[\underbrace{a,\dots,a}_{m \text{ times}}; \underbrace{b,\dots,b}_{k \text{ times}}] A\left[(g-a\mathbf{1})^m(g-b\mathbf{1})^{k-1}\right] + A(R_m(g)), \end{aligned} \tag{2.47}$$

where $m \geq 3$ and $R_m(\cdot)$ is defined in (2.43).

Proof. From representation (2.39) of every function $f \in \mathscr{C}^n([a,b])$ and its Hermite interpolating polynomial of type $(m, n-m)$ conditions in terms of divided differences (2.42) we have

$$\begin{aligned} f(t) &= f(a) + (t-a) f[a,a] + \dots + (t-a)^{m-1} f[\underbrace{a,\dots,a}_{m \text{ times}}] \\ &+ (t-a)^m f[\underbrace{a,\dots,a}_{m \text{ times}}; b] + (t-a)^m (t-b) f[\underbrace{a,\dots,a}_{m \text{ times}}; b,b] \\ &+ \dots + (t-a)^m (t-b)^{n-m-1} f[\underbrace{a,\dots,a}_{m \text{ times}};\ \underbrace{b,b,\dots,b}_{(n-m) \text{ times}}] + R_m(t), \end{aligned} \tag{2.48}$$

where $R_m(\cdot)$ is defined in (2.43). After some straightforward calculations, for different choices of $1 \le m \le n-1$, from (2.48) we get the following:

- for $m = 1$ it holds

$$\begin{aligned} LR(f,\mathbf{1},a,b,\mathrm{id}) &= (t-a)(t-b)f[a;b,b] + (t-a)(t-b)^2 f[a;b,b,b] \\ &\quad + \dots + (t-a)(t-b)^{n-2} f[a;\underbrace{b,b,\dots,b}_{(n-1)\text{ times}}] + R_1(t) \end{aligned} \tag{2.49}$$

- for $m = 2$ it holds

$$\begin{aligned} LR(f,\mathbf{1},a,b,\mathrm{id}) &= (t-a)(t-b)f[a,a;b] + (t-a)^2(t-b)f[a,a;b,b] \\ &\quad + \dots + (t-a)^2(t-b)^{n-3} f[a,a;\underbrace{b,b,\dots,b}_{(n-2)\text{ times}}] + R_2(t) \end{aligned} \tag{2.50}$$

- for $3 \le m \le n-1$ it holds

$$\begin{aligned} LR(f,\mathbf{1},a,b,\mathrm{id}) &= (t-a)\left(f[a,a]-f[a,b]\right) + \dots + (t-a)^{m-1} f[\underbrace{a,\dots,a}_{m\text{ times}}] \\ &\quad + (t-a)^m f[\underbrace{a,\dots,a}_{m\text{ times}};b] + (t-a)^m(t-b) f[\underbrace{a,\dots,a}_{m\text{ times}};b,b] \\ &\quad + \dots + (t-a)^m(t-b)^{n-m-1} f[\underbrace{a,\dots,a}_{m\text{ times}};\underbrace{b,b,\dots,b}_{(n-m)\text{ times}}] + R_m(t). \end{aligned} \tag{2.51}$$

Since $f \circ g \in L$ it holds $g(E) \subseteq [a,b]$, so we can replace t with $g(t)$ in (2.49), (2.50) and (2.51), and thus obtain:

$$LR(f,g,a,b,\mathrm{id}) = \sum_{k=2}^{n-1} (g(t)-a)(g(t)-b)^{k-1} f[a;\underbrace{b,\dots,b}_{k\text{ times}}] + R_1(g(t)),$$

$$\begin{aligned} LR(f,g,a,b,\mathrm{id}) &= (g(t)-a)(g(t)-b)f[a,a;b] \\ &\quad + \sum_{k=2}^{n-2} (g(t)-a)^2(g(t)-b)^{k-1} f[a,a;\underbrace{b,\dots,b}_{k\text{ times}}] + R_2(g(t)) \end{aligned}$$

and

$$\begin{aligned} LR(f,g,a,b,\mathrm{id}) &= (g(t)-a)\left(f[a,a]-f[a,b]\right) + \sum_{k=3}^{m} (g(t)-a)^{k-1} f[\underbrace{a,\dots,a}_{k\text{ times}}] \\ &\quad + \sum_{k=1}^{n-m} (g(t)-a)^m (g(t)-b)^{k-1} f[\underbrace{a,\dots,a}_{m\text{ times}};\underbrace{b,\dots,b}_{k\text{ times}}] + R_m(g(t)). \end{aligned}$$

Identities (2.45), (2.46) and (2.47) follow by applying positive normalized linear functional A to the previous equalities respectively. $\square$

Lemma 2.2 ([107]) *Let L satisfy conditions (L1) and (L2) on a non-empty set E and let A be any positive linear functional on L with $A(\mathbf{1}) = 1$. Let $f \in \mathscr{C}^n([a,b])$, and let $g \in L$ be any function such that $f \circ g \in L$. Then the following identities hold:*

- $$LR(f,g,a,b,A) = \sum_{k=2}^{n-1} f[b;\underbrace{a,\dots,a}_{k \text{ times}}]A[(g-b\mathbf{1})(g-a\mathbf{1})^{k-1}] + A(R_1^*(g)) \tag{2.52}$$
- $$\begin{aligned} LR(f,g,a,b,A) =& f[b,b;a]A[(g-b\mathbf{1})(g-a\mathbf{1})] \\ &+ \sum_{k=2}^{n-2} f[b,b;\underbrace{a,\dots,a}_{k \text{ times}}]A[(g-b\mathbf{1})^2(g-a\mathbf{1})^{k-1}] + A(R_2^*(g)) \end{aligned} \tag{2.53}$$
- $$\begin{aligned} LR(f,g,a,b,A) =& (b-A(g))\,(f[a,b]-f[b,b]) + \sum_{k=2}^{m-1} \frac{f^{(k)}(b)}{k!} A[(g-b\mathbf{1})^k] \\ &+ \sum_{k=1}^{n-m} f[\underbrace{b,\dots,b}_{m \text{ times}};\underbrace{a,\dots,a}_{k \text{ times}}]A[(g-b\mathbf{1})^m(g-a\mathbf{1})^{k-1}] + A(R_m^*(g)) \end{aligned} \tag{2.54}$$

where $m \geq 3$ and

$$A(R_m^*(g)) = A[f[g;\underbrace{b\mathbf{1},\dots,b\mathbf{1}}_{m \text{ times}};\underbrace{a\mathbf{1},\dots,a\mathbf{1}}_{(n-m) \text{ times}}](g-b\mathbf{1})^m(g-a\mathbf{1})^{n-m}]. \tag{2.55}$$

Proof. Let us define an auxiliary function $F\colon [a,b] \to \mathbb{R}$ with

$$F(t) = f(a+b-t).$$

Since $f \in \mathscr{C}^n([a,b])$ we immediately have $F \in \mathscr{C}^n([a,b])$, so we can apply (2.49), (2.50) and (2.51) to F and obtain respectively

$$LR(F,\mathbf{1},a,b,\mathrm{id}) = \sum_{k=2}^{n-1} F[a;\underbrace{b,\dots,b}_{k \text{ times}}](t-a)(t-b)^{k-1} + R_1(t) \tag{2.56}$$

$$\begin{aligned} LR(F,\mathbf{1},a,b,\mathrm{id}) =& F[a,a;b](t-a)(t-b) \\ &+ \sum_{k=2}^{n-2} F[a,a;\underbrace{b,\dots,b}_{k \text{ times}}](t-a)^2(t-b)^{k-1} + R_2(t) \end{aligned} \tag{2.57}$$

$$\begin{aligned} LR(F,\mathbf{1},a,b,\mathrm{id}) =& (t-a)\,(F[a,a]-F[a,b]) + \sum_{k=2}^{m-1} \frac{F^{(k)}(a)}{k!}(t-a)^k \\ &+ \sum_{k=1}^{n-m} F[\underbrace{a,\dots,a}_{m \text{ times}};\underbrace{b,\dots,b}_{k \text{ times}}](t-a)^m(t-b)^{k-1} + R_m(t). \end{aligned} \tag{2.58}$$

We can calculate divided differences of the function F in terms of divided differences of the function f:

$$F[\underbrace{a,\dots,a}_{k \text{ times}};\underbrace{b,\dots,b}_{i \text{ times}}] = (-1)^{k+i-1} f[\underbrace{b,\dots,b}_{k \text{ times}};\underbrace{a,\dots,a}_{i \text{ times}}].$$

Now (2.56), (2.57) and (2.58) become

$$LR(F,\mathbf{1},a,b,\mathrm{id}) = \sum_{k=2}^{n-1} (-1)^k f[b;\underbrace{a,\dots,a}_{k \text{ times}}](t-a)(t-b)^{k-1} + \bar{R}_1(t) \tag{2.59}$$

$$\begin{aligned} LR(F,\mathbf{1},a,b,\mathrm{id}) =& (-1)^2 f[b,b;a](t-a)(t-b) \\ &+ \sum_{k=2}^{n-2} (-1)^{k+1} f[b,b;\underbrace{a,\dots,a}_{k \text{ times}}](t-a)^2(t-b)^{k-1} + \bar{R}_2(t) \end{aligned} \tag{2.60}$$

$$\begin{aligned} LR(F,\mathbf{1},a,b,\mathrm{id}) =& (t-a)\left(-f[b,b]+f[a,b]\right) + \sum_{k=2}^{m-1} \frac{(-1)^k f^{(k)}(b)}{k!}(t-a)^k \\ &+ \sum_{k=1}^{n-m} (-1)^{m+k-1} f[\underbrace{b,\dots,b}_{m \text{ times}};\underbrace{a,\dots,a}_{k \text{ times}}](t-a)^m(t-b)^{k-1} + \bar{R}_m(t), \end{aligned} \tag{2.61}$$

where

$$\bar{R}_m(t) = (t-a)^m(t-b)^{n-m}(-1)^n f[a+b-t;\underbrace{b,\dots,b}_{m \text{ times}};\ \underbrace{a,a,\dots,a}_{(n-m) \text{ times}}].$$

Let $g \in L$ be any function such that $f \circ g \in L$, that is, $a \le g(t) \le b$ for every $t \in E$. Let us define a function $\bar{g}(t) = a+b-g(t)$. Trivially, we have $a \le \bar{g}(t) \le b$ and $\bar{g} \in L$. Since

$$\begin{aligned} LR(F,\bar{g},a,b,\mathrm{id}) =& f(a+b-(a+b-g(t))) - \frac{b-(a+b-g(t))}{b-a} f(a+b-a) \\ &- \frac{a+b-g(t)-a}{b-a} f(a+b-b) = LR(f,g,a,b,\mathrm{id}), \end{aligned}$$

after putting $\bar{g}(t)$ in (2.59), (2.60) and (2.61) instead of t, we get

$$LR(f,g,a,b,\mathrm{id}) = \sum_{k=2}^{n-1} (-1)^k f[b;\underbrace{a,\dots,a}_{k \text{ times}}](b-g(t))(a-g(t))^{k-1} + \bar{R}_1(a+b-g(t))$$

$$\begin{aligned} LR(f,g,a,b,\mathrm{id}) =& (-1)^2 f[b,b;a](b-g(t))(a-g(t)) \\ &+ \sum_{k=2}^{n-2} (-1)^{k+1} f[b,b;\underbrace{a,\dots,a}_{k \text{ times}}](b-g(t))^2(a-g(t))^{k-1} + \bar{R}_2(a+b-g(t)) \end{aligned}$$

$$\begin{aligned} LR(f,g,a,b,\mathrm{id}) =& (b-g(t))\left(-f[b,b]+f[a,b]\right) + \sum_{k=2}^{m-1} \frac{(-1)^k f^{(k)}(b)}{k!}(b-g(t))^k \\ &+ \sum_{k=1}^{n-m} (-1)^{m+k-1} f[\underbrace{b,\dots,b}_{m \text{ times}};\underbrace{a,\dots,a}_{k \text{ times}}](b-g(t))^m(a-g(t))^{k-1} + \bar{R}_m(a+b-g(t)). \end{aligned}$$

Identities (2.52), (2.53) and (2.54) follow after applying a normalized positive linear functional A to previous equalities respectively. □

Following representations of the left side in the scalar Edmundson-Lah-Ribarič inequality are special cases of Lemma 2.1 and Lemma 2.2 for $A = \mathrm{id}$ and $g = \mathbf{1}$.

Lemma 2.3 *Let a,b be real numbers such that $a < b$. For a function $f \in \mathscr{C}^n([a,b])$, $n \geq 3$, the following identities hold:*

- $$f(t) - \frac{b-t}{b-a} f(a) - \frac{t-a}{b-a} f(b) = \sum_{k=2}^{n-1} f[a;\underbrace{b,\ldots,b}_{k\ \text{times}}](t-a)(t-b)^{k-1} + R_1(t) \tag{2.62}$$
- $$f(t) - \frac{b-t}{b-a} f(a) - \frac{t-a}{b-a} f(b) = f[a,a;b](t-a)(t-b) + \sum_{k=2}^{n-2} f[a,a;\underbrace{b,\ldots,b}_{k\ \text{times}}](t-a)^2(t-b)^{k-1} + R_2(t) \tag{2.63}$$

where

$$R_m(t) = (t-a)^m (t-b)^{n-m} f[t;\underbrace{a,\ldots,a}_{m\ \text{times}};\underbrace{b,b,\ldots,b}_{(n-m)\ \text{times}}]. \tag{2.64}$$

Additionally, if $n > m \geq 3$, then we have

- $$f(t) - \frac{b-t}{b-a} f(a) - \frac{t-a}{b-a} f(b) = (t-a)\left(f[a,a] - f[a,b]\right) + \sum_{k=2}^{m-1} \frac{f^{(k)}(a)}{k!}(t-a)^k + \sum_{k=1}^{n-m} f[\underbrace{a,\ldots,a}_{m\ \text{times}};\underbrace{b,\ldots,b}_{k\ \text{times}}](t-a)^m(t-b)^{k-1} + R_m(t). \tag{2.65}$$

Lemma 2.4 *Let a,b be real numbers such that $a < b$. For a function $f \in \mathscr{C}^n([a,b])$, $n \geq 3$, the following identities hold:*

- $$f(t) - \frac{b-t}{b-a} f(a) - \frac{t-a}{b-a} f(b) = \sum_{k=2}^{n-1} f[b;\underbrace{a,\ldots,a}_{k\ \text{times}}](t-b)(t-a)^{k-1} + R_1^*(t) \tag{2.66}$$
- $$f(t) - \frac{b-t}{b-a} f(a) - \frac{t-a}{b-a} f(b) = f[b,b;a](t-b)(t-a) + \sum_{k=2}^{n-2} f[b,b;\underbrace{a,\ldots,a}_{k\ \text{times}}](t-b)^2(t-a)^{k-1} + R_2^*(t) \tag{2.67}$$

where

$$R_m^*(t) = f[t;\underbrace{b,\ldots,b}_{m\ \text{times}};\underbrace{a,a,\ldots,a}_{(n-m)\ \text{times}}](t-b)^m(t-a)^{n-m}. \tag{2.68}$$

Also, if $n > m \geq 3$, then

- $$f(t) - \frac{b-t}{b-a} f(a) - \frac{t-a}{b-a} f(b) = (b-t)\left(f[a,b] - f[b,b]\right) + \sum_{k=2}^{m-1} \frac{f^{(k)}(b)}{k!}(t-b)^k + \sum_{k=1}^{n-m} f[\underbrace{b,\ldots,b}_{m\ \text{times}};\underbrace{a,\ldots,a}_{k\ \text{times}}](t-b)^m(t-a)^{k-1} + R_m^*(t). \tag{2.69}$$

A generalization of the Edmundson-Lah-Ribarič inequality by Hermite's interpolating polynomials in terms of divided differences, obtained from Lemma 2.1, is given in the following theorem.

Theorem 2.9 ([107]) *Let L satisfy conditions (L1) and (L2) on a non-empty set E and let A be any positive linear functional on L with $A(\mathbf{1}) = 1$. Let $f \in \mathscr{C}^n([a,b])$, and let $g \in L$ be any function such that $f \circ g \in L$. If the function f is n-convex and if n and $m \geq 3$ are of different parity, then*

$$\begin{aligned} LR(f,g,a,b,A) \leq (A(g)-a)\left(f[a,a]-f[a,b]\right) + \sum_{k=2}^{m-1} \frac{f^{(k)}(a)}{k!} A\left[(g-a\mathbf{1})^k\right] \\ + \sum_{k=1}^{n-m} f[\underbrace{a,\ldots,a}_{m\ times};\underbrace{b,\ldots,b}_{k\ times}] A\left[(g-a\mathbf{1})^m (g-b\mathbf{1})^{k-1}\right]. \end{aligned} \tag{2.70}$$

Inequality (2.70) also holds when the function f is n-concave and n and m are of equal parity. In case when the function f is n-convex and n and m are of equal parity, or when the function f is n-concave and n and m are of different parity, the inequality sign in (2.70) is reversed.

Proof. We start with the representation of the left side in the Edmundson-Lah-Ribarič inequality (2.47) from Lemma 2.1 with a special focus on the last term:

$$A(R(g)) = A\left((g-a\mathbf{1})^m (g-b\mathbf{1})^{n-m} f[g;\underbrace{a\mathbf{1},\ldots,a\mathbf{1}}_{m \text{ times}};\underbrace{b\mathbf{1},\ldots,b\mathbf{1}}_{(n-m) \text{ times}}]\right).$$

Since A is positive, it preserves the sign, so we need to study the sign of the expression:

$$(g(t)-a)^m (g(t)-b)^{n-m} f[g(t);\underbrace{a,\ldots,a}_{m \text{ times}};\underbrace{b,b,\ldots,b}_{(n-m) \text{ times}}].$$

Since $a \leq g(t) \leq b$ for every $t \in E$, we have $(g(t)-a)^m \geq 0$ for every $t \in E$ and any choice of m. For the same reason we have $(g(t)-b) \leq 0$. Trivially it follows that $(g(t)-b)^{n-m} \leq 0$ when n and m are of different parity, and $(g(t)-b)^{n-m} \geq 0$ when n and m are of equal parity.

If the function f is n-convex, then $f[g(t);\underbrace{a,\ldots,a}_{m \text{ times}};\underbrace{b,b,\ldots,b}_{(n-m) \text{ times}}] \geq 0$, and if the function f is n-concave, then $f[g(t);\underbrace{a,\ldots,a}_{m \text{ times}};\underbrace{b,b,\ldots,b}_{(n-m) \text{ times}}] \leq 0$.

Now (2.70) easily follows from (2.1). □

Following generalization of the Edmundson-Lah-Ribarič inequality by Hermite's interpolating polynomials in terms of divided differences is obtained from Lemma 2.2.

Theorem 2.10 ([107]) *Let L satisfy conditions (L1) and (L2) on a non-empty set E and let A be any positive linear functional on L with $A(\mathbf{1}) = 1$. Let $f \in \mathscr{C}^n([a,b])$, and let $g \in L$ be any function such that $f \circ g \in L$. If the function f is n-convex and if $m \geq 3$ is odd, then*

$$LR(f,g,a,b,A) \leq (b - A(g))\,(f[a,b] - f[b,b]) + \sum_{k=2}^{m-1} \frac{f^{(k)}(b)}{k!} A[(g - b\mathbf{1})^k]$$
$$+ \sum_{k=1}^{n-m} f[\underbrace{b,\dots,b}_{m\ times};\underbrace{a,\dots,a}_{k\ times}]A[(g - b\mathbf{1})^m (g - a\mathbf{1})^{k-1}] \tag{2.71}$$

Inequality (2.71) also holds when the function f is n-concave and m is even. In case when the function f is n-convex and m is even, or when the function f is n-concave and m is odd, the inequality sign in (2.71) is reversed.

Proof. Similarly as in the proof of the previous theorem, we start with the representation of the left side in the Edmundson-Lah-Ribarič inequality (2.54) from Lemma 2.2 with a special focus on the last term:

$$A(R_m^*(g)) = A\left(f[g;\underbrace{b\mathbf{1},\dots,b\mathbf{1}}_{m\text{ times}};\underbrace{a\mathbf{1},\dots,a\mathbf{1}}_{(n-m)\text{ times}}](g - b\mathbf{1})^m (g - a\mathbf{1})^{n-m} \right)$$

As before, because of the positivity of the linear functional A, we only need to study the sign of the expression:

$$(g(t) - b)^m (g(t) - a)^{n-m} f[g(t);\underbrace{b,\dots,b}_{m\text{ times}};\underbrace{a,a,\dots,a}_{(n-m)\text{ times}}].$$

Since $a \leq g(t) \leq b$ for every $t \in E$, we have $(g(t) - a)^{n-m} \geq 0$ for every $t \in E$ and any choice of m. For the same reason we have $(g(t) - b) \leq 0$. Trivially it follows that $(g(t) - b)^m \leq 0$ when m is odd, and $(g(t) - b)^m \geq 0$ when m is even.

If the function f is n-convex, then its n-th order divided differences are greater of equal to zero, and if the function f is n-concave, then its n-th order divided differences are less or equal to zero.

Now (2.71) easily follows from Lemma (2.2). □

Corollary 2.5 ([107]) *Let L satisfy conditions (L1) and (L2) on a non-empty set E and let A be any positive linear functional on L with $A(\mathbf{1}) = 1$. Let n be an odd number, let $f \in \mathscr{C}^n([a,b])$, and let $g \in L$ be any function such that $f \circ g \in L$. If the function f is n-convex and if $m \geq 3$ is odd, then*

$$(A(g) - a)\,(f[a,a] - f[a,b]) + \sum_{k=2}^{m-1} \frac{f^{(k)}(a)}{k!} A\left[(g - a\mathbf{1})^k\right]$$
$$+ \sum_{k=1}^{n-m} f[\underbrace{a,\dots,a}_{m\ times};\underbrace{b,\dots,b}_{k\ times}]A\left[(g - a\mathbf{1})^m (g - b\mathbf{1})^{k-1}\right]$$

$$\leq LR(f,g,a,b,A) \leq (b-A(g))\,(f[a,b]-f[b,b]) + \sum_{k=2}^{m-1} \frac{f^{(k)}(b)}{k!} A[(g-b\mathbf{1})^k]$$
$$+\sum_{k=1}^{n-m} f[\underbrace{b,\ldots,b}_{m\ times};\underbrace{a,\ldots,a}_{k\ times}]A[(g-b\mathbf{1})^m(g-a\mathbf{1})^{k-1}]. \tag{2.72}$$

Inequality (2.72) also holds when the function f is n-concave and m is even. In case when the function f is n-convex and m is even, or when the function f is n-concave and m is odd, the inequality signs in (2.72) are reversed.

Remark 2.5 In Theorem 2.8 it is shown that for a 3-convex functions we have

$$(A(g)-a)\left[f'(a)-\frac{f(b)-f(a)}{b-a}\right]+\frac{f''(a)}{2}A[(g-a\mathbf{1})^2],$$
$$\leq LR(f,g,a,b,A) \leq (b-A(g))\left[\frac{f(b)-f(a)}{b-a}-f'(b)\right]+\frac{f''(b)}{2}A[(b\mathbf{1}-g)^2]$$

and if the function f is 3-concave, then the inequality signs are reversed. It is obvious that inequalities (2.72) from Corollary 2.5 provide us with a generalization of the result stated above.

The following result is another generalization of the Edmundson-Lah-Ribarič inequality by Hermite's interpolating polynomials in terms of divided differences obtained from Lemma 2.1.

Theorem 2.11 ([107]) *Let L satisfy conditions (L1) and (L2) on a non-empty set E and let A be any positive linear functional on L with $A(\mathbf{1})=1$. Let $f \in \mathscr{C}^n([a,b])$, and let $g \in L$ be any function such that $f \circ g \in L$. If the function f is n-convex and if n is odd, then*

$$\sum_{k=2}^{n-1} f[a;\underbrace{b,\ldots,b}_{k\ times}]A\left[(g-a\mathbf{1})(g-b\mathbf{1})^{k-1}\right] \leq LR(f,g,a,b,A) \tag{2.73}$$
$$\leq f[a,a;b]A[(g-a\mathbf{1})(g-b\mathbf{1})]+\sum_{k=2}^{n-2} f[a,a;\underbrace{b,\ldots,b}_{k\ times}]A\left[(g-a\mathbf{1})^2(g-b\mathbf{1})^{k-1}\right].$$

Inequalities (2.73) also hold when the function f is n-concave and n is even. In case when the function f is n-convex and n is even, or when the function f is n-concave and n is odd, the inequality signs in (2.73) are reversed.

Proof. From the discussion about positivity and negativity of the term $A(R_m(g))$ in the proof of Theorem 2.9, for $m=1$ it follows that

* $A(R_1(g)) \geq 0$ when the function f is n-convex and n is odd, or when f is n-concave and n even;
* $A(R_1(g)) \leq 0$ when the function f is n-concave and n is odd, or when f is n-convex and n even.

Now the identity (2.45) gives us

$$\begin{aligned}LR(f,g,a,b,A) &\geq f[a;b,b]A[(g-a\mathbf{1})(g-b\mathbf{1})]+f[a;b,b,b]A[(g-a\mathbf{1})(g-b\mathbf{1})^2]\\ &+\dots+f[a;\underbrace{b,b,\dots,b}_{(n-1)\text{ times}}]A\left[(g-a\mathbf{1})(g-b\mathbf{1})^{n-2}\right]\end{aligned}$$

for $A(R_1(g))\geq 0$, and in case $A(R_1(g))\leq 0$ the inequality sign is reversed.

In the same manner, for $m=2$ it follows that

* $A(R_2(g))\leq 0$ when the function f is n-convex and n is odd, or when f is n-concave and n even;
* $A(R_2(g))\geq 0$ when the function f is n-concave and n is odd, or when f is n-convex and n even.

In this case the identity (2.46) for $A(R_2(g))\leq 0$ gives us

$$\begin{aligned}LR(f,g,a,b,A) &\leq f[a,a;b]A[(g-a\mathbf{1})(g-b\mathbf{1})]+f[a,a;b,b]A[(g-a\mathbf{1})^2(g-b\mathbf{1})]\\ &+\dots+f[a,a;\underbrace{b,b,\dots,b}_{(n-2)\text{ times}}]A\left[(g-a\mathbf{1})^2(g-b\mathbf{1})^{n-3}\right]\end{aligned}$$

and in case $A(R_2(g))\geq 0$ the inequality sign is reversed.

When we combine the two results from above, we get exactly (2.73). □

By utilizing Lemma 2.2 we can get a similar generalization of the Edmundson-Lah-Ribarič inequality that holds for all $n\in\mathbb{N}$, not only the odd ones.

Theorem 2.12 ([107]) *Let L satisfy conditions (L1) and (L2) on a non-empty set E and let A be any positive linear functional on L with $A(\mathbf{1})=1$. Let $f\in\mathscr{C}^n([a,b])$, and let $g\in L$ be any function such that $f\circ g\in L$. If the function f is n-convex, then*

$$\begin{aligned}&f[b,b;a]A[(g-b\mathbf{1})(g-a\mathbf{1})]+\sum_{k=2}^{n-2}f[b,b;\underbrace{a,\dots,a}_{k\text{ times}}]A[(g-b\mathbf{1})^2(g-a\mathbf{1})^{k-1}]\\ &\leq LR(f,g,a,b,A)\leq\sum_{k=1}^{n-1}f[b;\underbrace{a,\dots,a}_{k\text{ times}}]A[(g-b\mathbf{1})(g-a\mathbf{1})^{k-1}]. \qquad (2.74)\end{aligned}$$

If the function f is n-concave, the inequality signs in (2.74) are reversed.

Proof. We return to the discussion about positivity and negativity of the term $A(R_m^*(g))$ in the proof of Theorem 2.10. For $m=1$ we have

$$(g(t)-b)^1(g(t)-a)^{n-1}\leq 0 \quad\text{for every } t\in E,$$

so $A(R_1^*(g))\geq 0$ when the function f is n-concave, and $A(R_1^*(g))\leq 0$ when the function f is n-convex. Now the identity (2.52) for a n-convex function f gives us

$$
\begin{aligned}
LR(f,g,a,b,A) \geq\ & f[b,b;a]A[(g-b\mathbf{1})(g-a\mathbf{1})] + f[b,b;a,a]A[(g-b\mathbf{1})^2(g-a\mathbf{1})] \\
& + \ldots + f[b,b;\underbrace{a,a,\ldots,a}_{(n-2)\text{ times}}]A[(g-b\mathbf{1})^2(g-a\mathbf{1})^{n-3}]
\end{aligned}
$$

and if the function f is n-concave, the inequality sign is reversed.

Similarly, for $m=2$ we have

$$
(g(t)-b)^2(g(t)-a)^{n-2} \geq 0 \quad \text{for every } t \in E,
$$

so $A(R_2^*(g)) \geq 0$ when the function f is n-convex, and $A(R_2^*(g)) \leq 0$ when the function f is n-concave. In this case the identity (2.53) for a n-convex function f gives us

$$
\begin{aligned}
LR(f,g,a,b,A) \leq\ & f[b;a,a]A[(g-b\mathbf{1})(g-a\mathbf{1})] + f[b;a,a,a]A[(g-b\mathbf{1})(g-a\mathbf{1})^2] \\
& + \ldots + f[b;\underbrace{a,a,\ldots,a}_{(n-1)\text{ times}}]A[(g-b\mathbf{1})(g-a\mathbf{1})^{n-2}]
\end{aligned}
$$

and if the function f is n-concave, the inequality sign is reversed.

When we combine the two results from above, we get exactly (2.74). □

Remark 2.6 Since

$$
\begin{aligned}
f[a;b,b] &= \frac{1}{b-a}\left(f'(b) - \frac{f(b)-f(a)}{b-a}\right) \\
f[a,a;b] &= \frac{1}{b-a}\left(f'(b) - \frac{f(b)-f(a)}{b-a}\right),
\end{aligned}
$$

when we take $n=3$ in (2.73) or (2.74), we get that

$$
\begin{aligned}
& \frac{A[(g-a\mathbf{1})(g-b\mathbf{1})]}{b-a}\left(f'(b) - \frac{f(b)-f(a)}{b-a}\right) \qquad (2.75) \\
& \quad \leq LR(f,g,a,b,A) \leq \frac{A[(g-a\mathbf{1})(g-b\mathbf{1})]}{b-a}\left(f'(b) - \frac{f(b)-f(a)}{b-a}\right)
\end{aligned}
$$

holds for a 3-convex function, and for a 3-concave function the inequality signs are reversed. Inequalities (2.75) are proved in Theorem 2.5, so it follows that Theorem 2.11 and Theorem 2.12 give a generalization of that result.

2.5 Applications to generalized means

Let $I = \langle a,b\rangle$, $-\infty \leq a < b \leq \infty$, and let $\psi \colon I \to \mathbb{R}$ be continuous and strictly monotonic. Suppose that L and A satisfy the conditions $L1, L2$ and $A1, A2$ with $A(\mathbf{1}) = 1$ on a nonempty set E, and that $\psi(f) \in L$ for some $f \in L$. Generalized mean for $f \in L$ with respect to the operator A and the function ψ is defined by

$$
M_\psi(f,A) = \psi^{-1}\left(A(\psi(f))\right). \tag{2.76}
$$

The following results give us inequalities of the Edmundson-Lah-Ribarič and Jensen type respectively for the generalized means.

Theorem 2.13 ([103]) *Let $I \subset \mathbb{R}$ be such that its interior contains the interval $[m,M]$, and let $\psi,\chi : I \to \mathbb{R}$ be continuous and strictly monotonic. Suppose that L and A satisfy the conditions $L1,L2$ and $A1,A2$ with $A(\mathbf{1}) = 1$ on a non-empty set E, and let $f \in L$ be such that $\psi(f),\chi(f) \in L$. Let us assume that the function $\phi = \chi \circ \psi^{-1}$ is 3-convex. Then*

$$\begin{aligned}
&\frac{A\left([M_\psi \mathbf{1} - \psi(f)][\psi(f) - m_\psi \mathbf{1}]\right)}{M_\psi - m_\psi}\left(\frac{\chi(M)-\chi(m)}{\psi(M)-\psi(m)} - [\chi\circ\psi^{-1}]'_+(m_\psi)\right)\\
&\quad\le \frac{\psi(M)-A(\psi(f))}{\psi(M)-\psi(m)}\chi(m) + \frac{A(\psi(f))-\psi(m)}{\psi(M)-\psi(m)}\chi(M) - \chi(M_\chi(f,A)) \qquad (2.77)\\
&\quad\le \frac{A\left([M_\psi \mathbf{1} - \psi(f)][\psi(f) - m_\psi \mathbf{1}]\right)}{M_\psi - m_\psi}\left([\chi\circ\psi^{-1}]'_-(M_\psi) - \frac{\chi(M)-\chi(m)}{\psi(M)-\psi(m)}\right)
\end{aligned}$$

for every $f \in L$ such that $m \le f(t) \le M$ for $t \in [m,M]$, where $[m_\psi, M_\psi] = \psi([m,M])$. If $-\phi$ is 3-convex, then the inequalities in (2.77) are reversed.

Proof. Function ψ is strictly monotonic. If ψ is increasing, then $m_\psi = \psi(m)$ and $M_\psi = \psi(M)$, and if ψ is decreasing, then $m_\psi = \psi(M)$ and $M_\psi = \psi(m)$. Since $m \le f(t) \le M$ for $t \in [m,M]$, we have $m_\psi \le \psi(f(t)) \le M_\psi$ for every $t \in [m,M]$. We see that the conditions of Theorem 2.5 are satisfied, so we can obtain (2.77) by making substitutions

$$m = m_\psi,\ M = M_\psi,\ \phi = \chi\circ\psi^{-1} \ \text{ and } \ f = \psi\circ f$$

in (2.21). □

Theorem 2.14 ([103]) *Let $I \subset \mathbb{R}$ be such that its interior contains the interval $[m,M]$, and let $\psi,\chi : I \to \mathbb{R}$ be continuous and strictly monotonic. Suppose that L and A satisfy the conditions $L1,L2$ and $A1,A2$ with $A(\mathbf{1}) = 1$ on a non-empty set E, and let $f \in L$ be such that $\psi(f),\chi(f) \in L$. Let us assume that the function $\phi = \chi \circ \psi^{-1}$ is 3-convex. Then*

$$\begin{aligned}
&\frac{(M_\psi - A(\psi(f)))(A(\psi(f)) - m_\psi)}{M_\psi - m_\psi}\left(\frac{\chi(M)-\chi(m)}{\psi(M)-\psi(m)} - [\chi\circ\psi^{-1}]'_+(m_\psi)\right)\\
&\quad - \frac{A\left[(M_\psi \mathbf{1} - \psi(f))(\psi(f) - m_\psi \mathbf{1})\right]}{M_\psi - m_\psi}\left([\chi\circ\psi^{-1}]'_-(M_\psi) - \frac{\chi(M)-\chi(m)}{\psi(M)-\psi(m)}\right)\\
&\le \chi(M_\chi(f,A)) - \chi(M_\psi(f,A)) \qquad (2.78)\\
&\le \frac{(M_\psi - A(\psi(f)))(A(\psi(f)) - m_\psi)}{M_\psi - m_\psi}\left([\chi\circ\psi^{-1}]'_-(M_\psi) - \frac{\chi(M)-\chi(m)}{\psi(M)-\psi(m)}\right)\\
&\quad - \frac{A\left([M_\psi \mathbf{1} - \psi(f)][\psi(f) - m_\psi \mathbf{1}]\right)}{M_\psi - m_\psi}\left(\frac{\chi(M)-\chi(m)}{\psi(M)-\psi(m)} - [\chi\circ\psi^{-1}]'_+(m_\psi)\right)
\end{aligned}$$

for every $f \in L$ such that $m \le f(t) \le M$ for $t \in [m,M]$, where $[m_\psi, M_\psi] = \psi([m,M])$. If $-\phi$ is 3-convex, then the inequalities in (2.78) are reversed.

Proof. The inequalities (2.78) are obtained by making the same substitutions in the relation (2.25) from Theorem 2.6 as in the proof of the previous theorem. □

Remark 2.7 With notations as in Theorems 2.13 and 2.14, suppose that the function $\chi \circ \psi^{-1}$ is differentiable in points ψ_m and ψ_M. In this case, expressions ψ_m and ψ_M can respectively be replaced by $\psi(m)$ and $\psi(M)$, due to the symmetry. In addition, utilizing the chain rule, the expressions

$$[\chi \circ \psi^{-1}]'_{-}(\psi(M)) \text{ and } [\chi \circ \psi^{-1}]'_{+}(\psi(m))$$

can be rewritten in a more suitable form, that is,

$$[\chi \circ \psi^{-1}]'_{-}(\psi(M)) = \frac{\chi'(M)}{\psi'(M)} \text{ and } (\chi \circ \psi^{-1})'_{+}(\psi(m)) = \frac{\chi'(m)}{\psi'(m)}.$$

2.5.1 Examples with power means

Suppose that L and A satisfy the conditions $L1, L2$ and $A1, A2$ with $A(\mathbf{1}) = 1$, on a non-empty set E. The power mean of a function $f \in L$ with respect to the operator A is a special case of the generalized mean, and it is defined for $r \in \mathbb{R}$ with:

$$M^{[r]}(f,A) = \begin{cases} (A(f^r))^{1/r} & : r \neq 0 \\ \exp(A(\log f)) & : r = 0 \end{cases} \tag{2.79}$$

where $f(t) > 0$ for $t \in E$, $f^r \in L$ and $\log f \in L$.

Following two results are simple consequences of Theorem 2.13 and Theorem 2.14, that is, the series of inequalities in (2.77) and (2.78) with particular choices of functions χ and ψ respectively. The first result is a Edmundson-Lah-Ribarič type inequality for power means.

Corollary 2.6 ([103]) *Let $I \subset \mathbb{R}$ be such that its interior contains the interval $[m,M]$. Suppose that L and A satisfy the conditions $L1, L2$ and $A1, A2$ with $A(\mathbf{1}) = 1$ on a non-empty set E, and let $f \in L$ be such that $0 < m \leq f(t) \leq M$ for $t \in E$, $f^r, f^s \in L$ for $r,s \in \mathbb{R}$ and $\log f \in L$.*

- *If any of the relations $0 \leq s \leq r$ or $0 \leq 2r \leq s$ or $r < 0 < s$ or $2r < s < r < 0$ hold, then*

$$\begin{aligned}
&\frac{A\left([M^r\mathbf{1} - f^r][f^r - m^r\mathbf{1}]\right)}{M^r - m^r}\left(\frac{M^s - m^s}{M^r - m^r} - m^{s-r}\right) \\
&\leq \frac{M^r - M^{[r]}(f,A)^r}{M^r - m^r} m^s + \frac{M^{[r]}(f,A)^r - m^r}{M^r - m^r} M^s - M^{[s]}(f,A)^s \\
&\leq \frac{A\left([M_\psi \mathbf{1} - \psi(f)][\psi(f) - m_\psi \mathbf{1}]\right)}{M^r - m^r}\left(M^{s-r} - \frac{M^s - m^s}{M^r - m^r}\right).
\end{aligned} \tag{2.80}$$

If $r \leq s \leq 0$ or $s \leq 2r \leq 0$ or $s < 0 < r$ or $0 < r < s < 2r$, then the inequalities in (2.80) are reversed.

- *If $r \neq 0$, then*

$$\begin{aligned}
&\frac{A\left([M^r\mathbf{1}-f^r][f^r-m^r\mathbf{1}]\right)}{M^r-m^r}\left(\frac{\log M-\log m}{M^r-m^r}-\frac{1}{rm^r}\right)\\
&\leq \frac{M^r-M^{[r]}(f,A)^r}{M^r-m^r}\log m+\frac{M^{[r]}(f,A)^r-m^r}{M^r-m^r}\log M-\log[M^{[0]}(f,A)] \qquad (2.81)\\
&\leq \frac{A\left([M^r\mathbf{1}-f^r][f^r-m^r\mathbf{1}]\right)}{M^r-m^r}\left(\frac{1}{rM^r}-\frac{\log M-\log m}{M^r-m^r}\right).
\end{aligned}$$

- *If $s > 0$, then*

$$\begin{aligned}
&\frac{A\left([\log M\mathbf{1}-\log f][\log f-\log m\mathbf{1}]\right)}{\log M-\log m}\left(\frac{M^s-m^s}{\log M-\log m}-sm^s\right)\\
&\leq \frac{\log M-\log[M^{[0]}(f,A)]}{\log M-\log m}m^s+\frac{\log[M^{[0]}(f,A)]-\log m}{\log M-\log m}M^s-M^{[s]}(f,A)^s \qquad (2.82)\\
&\leq \frac{A\left([\log M\mathbf{1}-\log f][\log f-\log m\mathbf{1}]\right)}{\log M-\log m}\left(sM^s-\frac{M^s-m^s}{\log M-\log m}\right),
\end{aligned}$$

and if $s < 0$, the inequality signs in (2.82) are reversed.

Proof. Let us set $\chi(t)=t^s$ and $\psi(t)=t^r$, where s and r are mutually different real parameters not equal to zero. Then the function $\left(\chi\circ\psi^{-1}\right)(t)=t^{\frac{s}{r}}$ is 3-convex on $\mathbb{R}_+$ if $0\leq\frac{s}{r}\leq 1$ or $\frac{s}{r}\geq 2$. It is possible in each of the following four cases: $0\leq s\leq r$ or $r\leq s\leq 0$ or $0\leq 2r\leq s$ or $s\leq 2r\leq 0$. We calculate $\left(\chi\circ\psi^{-1}\right)'(t)=\frac{s}{r}t^{\frac{s-r}{r}}$. Since the function $\psi(t)=t^r$ is increasing for $r>0$ we have $m_\psi=\psi(m)$ and $M_\psi=\psi(M)$. Now, considering (2.77) with the above functions χ and ψ on the interval $[m,M]$, we obtain (2.80). For $r<0$ the function $\psi(t)=t^r$ is decreasing, which means that $m_\psi=\psi(M)$ and $M_\psi=\psi(m)$, so those inequalities are reversed.

On the other hand, the function $-\left(\chi\circ\psi^{-1}\right)(t)=-t^{\frac{s}{r}}$ is 3-convex on $\mathbb{R}_+$ if $0\leq\frac{s}{r}<0$ or $1<\frac{s}{r}<2$, which is possible in any of the following cases: $r<0<s$ or $s<0<r$ or $0<r<s<2r$ or $2r<s<r<0$. Again, if $r>0$ the function $\psi(t)=t^r$ is increasing, so we get the inequalities (2.80) with the reversed sign of inequality by setting $\chi(t)=t^s$ and $\psi(t)=t^r$ in the reversed inequalities (2.77), and if $r<0$, we get exactly inequalities (2.80).

It remains to consider the cases when one of the parameters r and s is equal to zero. If $s=0$, then setting $\chi(t)=\log t$ and $\psi(t)=t^r$, it follows that $\left(\chi\circ\psi^{-1}\right)(t)=\frac{1}{r}\log t$. Clearly, this function is 3-convex for $r>0$, while $-\chi\circ\psi^{-1}$ is 3-convex for $r<0$. Moreover, since $\left(\chi\circ\psi^{-1}\right)'(t)=\frac{1}{rt}$, after a straightforward computation and taking into account that the function $\psi(t)=t^r$ is increasing for $r>0$ and decreasing for $r<0$, we obtain (2.81).

Finally, if $r=0$, then setting $\chi(t)=t^s$ and $\psi(t)=\log t$, it follows that the function $\left(\chi\circ\psi^{-1}\right)(t)=\exp(st)$ is 3-convex for $s>0$. The function $\psi(t)=\log t$ is increasing, so after calculating $\left(\chi\circ\psi^{-1}\right)'(t)=s\exp(st)$, from (2.77) we get (2.82). □

Next result is a Jensen type inequality for power means, and it is obtained from Theorem 2.14 in an analogous way as described in the proof of the previous corollary.

Corollary 2.7 ([103]) *Let $I \subset \mathbb{R}$ be such that its interior contains the interval $[m,M]$. Suppose that L and A satisfy the conditions $L1, L2$ and $A1, A2$ with $A(\mathbf{1}) = 1$ on a non-empty set E, and let $f \in L$ be such that $0 < m \leq f(t) \leq M$ for $t \in E$, $f^r, f^s \in L$ for $r,s \in \mathbb{R}$ and $\log f \in L$.*

- *If any of the relations $0 \leq s \leq r$ or $0 \leq 2r \leq s$ or $r < 0 < s$ or $2r < s < r < 0$ hold, then*

$$\begin{aligned}
&\frac{(M^r - M^{[r]}(f,A)^r)(M^{[r]}(f,A)^r - m^r)}{M^r - m^r}\left(\frac{M^s - m^s}{M^r - m^r} - \frac{s}{r}m^{s-r}\right) \\
&\quad - \frac{A\left[(M^r\mathbf{1} - f^r)(f^r - m^r\mathbf{1})\right]}{M^r - m^r}\left(\frac{s}{r}M^{s-r} - \frac{M^s - m^s}{M^r - m^r}\right) \\
&\leq M^{[s]}(f,A)^s - M^{[r]}(f,A)^s \qquad (2.83)\\
&\leq \frac{(M^r - M^{[r]}(f,A)^r)(M^{[r]}(f,A)^r - m^r)}{M^r - m^r}\left(\frac{s}{r}M^{s-r} - \frac{M^s - m^s}{M^r - m^r}\right) \\
&\quad - \frac{A\left[(M^r\mathbf{1} - f^r)(f^r - m^r\mathbf{1})\right]}{M^r - m^r}\left(\frac{M^s - m^s}{M^r - m^r} - \frac{s}{r}m^{s-r}\right)
\end{aligned}$$

  If $r \leq s \leq 0$ or $s \leq 2r \leq 0$ or $s < 0 < r$ or $0 < r < s < 2r$, then the inequalities in (2.83) are reversed.

- *If $r \neq 0$, then*

$$\begin{aligned}
&\frac{(M^r - M^{[r]}(f,A)^r)(M^{[r]}(f,A)^r - m^r)}{M^r - m^r}\left(\frac{\log M - \log m}{M^r - m^r} - \frac{1}{rm^r}\right) \\
&\quad - \frac{A\left([M^r\mathbf{1} - f^r][f^r - m^r\mathbf{1}]\right)}{M^r - m^r}\left(\frac{1}{rM^r} - \frac{\log M - \log m}{M^r - m^r}\right) \\
&\leq \log[M^{[0]}(f,A)] - \log[M^{[r]}(f,A)] \qquad (2.84)\\
&\leq \frac{(M^r - M^{[r]}(f,A)^r)(M^{[r]}(f,A)^r - m^r)}{M^r - m^r}\left(\frac{1}{rM^r} - \frac{\log M - \log m}{M^r - m^r}\right) \\
&\quad - \frac{A\left([M^r\mathbf{1} - f^r][f^r - m^r\mathbf{1}]\right)}{M^r - m^r}\left(\frac{\log M - \log m}{M^r - m^r} - \frac{1}{rm^r}\right).
\end{aligned}$$

- *If $s > 0$, then*

$$\begin{aligned}
&\frac{(\log M - \log[M^{[0]}(f,A))(\log[M^{[0]}(f,A) - \log m)}{\log M - \log m}\left(\frac{M^s - m^s}{\log M - \log m} - sm^s\right) \\
&\quad - \frac{A\left([\log M\mathbf{1} - \log f][\log f - \log m\mathbf{1}]\right)}{\log M - \log m}\left(sM^s - \frac{M^s - m^s}{\log M - \log m}\right) \\
&\leq M^{[s]}(f,A)^s - M^{[0]}(f,A)^s \qquad (2.85)\\
&\leq \frac{(\log M - \log[M^{[0]}(f,A))(\log[M^{[0]}(f,A) - \log m)}{\log M - \log m}\left(sM^s - \frac{M^s - m^s}{\log M - \log m}\right) \\
&\quad - \frac{A\left([\log M\mathbf{1} - \log f][\log f - \log m\mathbf{1}]\right)}{\log M - \log m}\left(\frac{M^s - m^s}{\log M - \log m} - sm^s\right)
\end{aligned}$$

  and if $s < 0$, the inequality signs in (2.85) are reversed.

Chapter 3

Jensen and Edmundson-Lah-Ribarič type inequalities for f-divergence

Numerous theoretic divergence measures between two probability distributions have been introduced and comprehensively studied. Their applications can be found in the analysis of contingency tables, in approximation of probability distributions, in signal processing, and in pattern recognition. Csiszár introduced the $f-$divergence functional which represent a "distance function" on the set of probability distributions. A great number of theoretic divergences are special cases of Csiszár f-divergence for different choices of the function f.

In this chapter first we will obtain some estimates for the f-divergence functional via converses of the Jensen and Edmundson-Lah-Ribarič inequalities for convex functions. Then we will study a generalization of the f-divergence functional for different classes of functions (functions with bounded second order divided differences, Lipschitzian functions, 3-convex functions and n-convex functions). We also utilize our results regarding Csiszár divergence in order to obtain different inequalities for the Zipf and Zipf-Mandelbrot law.

The Zipf law has and continues to attract considerable attention in a wide variety of scientific disciplines - from astronomy to demographics to software structure to economics to zoology, and even to warfare [45]. It is one of the basic laws in information science and bibliometrics, but it is also often used in linguistics. Same law in mathematical sense is also used in other scientific disciplines, but name of the law can be different, since regularities in different scientific fields are discovered independently from each other. Typically one is dealing with integer-valued observables (numbers of objects, people, cities, words, animals, corpses) and the frequency of their occurrence. Benoit Mandelbrot in 1966 gave an improvement of the Zipf law for the count of the low-rank words. Various scientific fields use this law for different purposes, for example information sciences use it for indexing, ecological field studies in predictability of ecosystem, in music it is used to determine aesthetically pleasing music.

3.1 Introduction

Let us denote the set of all probability densities by $\mathbb{P}$, i.e. $\boldsymbol{p} = (p_1, \ldots, p_n) \in \mathbb{P}$ if $p_i \in [0,1]$ for $i = 1, \ldots, n$ and $\sum_{i=1}^{n} p_i = 1$. One of the numerous applications of Probability Theory is finding an appropriate measure of distance (difference or divergence) between two probability distributions.

Consequently, many different divergence measures have been introduced and extensively studied, for example Kullback-Leibler divergence, Hellinger divergence, Renyi divergence, Bhattacharyya divergence, harmonic divergence, Jeffrey divergence, triangular divergence etc. All of the mentioned divergences are special cases of the Csiszár f-divergence.

These measures of distance between two probability distributions have an important application in a great number of fields such as: anthropology, genetics, economics and political science, biology, approximation of probability distributions ([32], [92]), signal processing ([77]) and pattern recognition ([15], [28]), analysis of contingency tables ([52]), ecological studies, music etc.

A large number of papers has been written on the subject of inequalities for different types of divergences. Since the functions that are used to define most of the divergences are convex, Jensen's inequality and its converses play an important role in the mentioned inequalities.

Csiszár [33]–[34] introduced the $f-$divergence functional as

$$D_f(\boldsymbol{p}, \boldsymbol{q}) = \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right), \tag{3.1}$$

where $f : [0, +\infty\rangle$ is a convex function, and it represents a "distance function" on the set of probability distributions $\mathbb{P}$.

Dragomir [39] gave the following upper bound for the Csiszár divergence functional

$$D_f(\boldsymbol{p}, \boldsymbol{q}) \leq \frac{M-1}{M-m} f(m) + \frac{1-m}{M-m} f(M), \tag{3.2}$$

where f is a convex function on the interval $[m, M]$, $\boldsymbol{p} = (p_1, \ldots, p_n), \boldsymbol{q} = (q_1, \ldots, q_n) \in \mathbb{P}$ and $m \leq p_i/q_i \leq M$ for every $i = 1, \ldots, n$ (then it easily follows that $1 \in [m, M]$).

The Kullback-Leibler divergence, also called relative entropy or KL divergence

$$D_{KL}(\boldsymbol{p},\boldsymbol{q}) := \sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right)$$

is a measure of the non-symmetric difference between two probability distributions $\boldsymbol{p}$ and $\boldsymbol{q}$, but it is not a true metric because it does not obey the triangle inequality and in general $D_{KL}(\boldsymbol{p},\boldsymbol{q}) \neq D_{KL}(\boldsymbol{q},\boldsymbol{p})$. The Kullback-Leibler divergence was introduced by Kullback and Leibler in [88], and it is a special case of the Csiszár divergence for $f(t) = t\log t$.

In order to use nonnegative probability distributions in the f-divergence functional, Horvath et. al. in [62] defined

$$f(0) := \lim_{t\to 0+} f(t),\ 0\cdot f\left(\frac{0}{0}\right) := 0,\ 0\cdot f\left(\frac{a}{0}\right) := \lim_{t\to 0+} tf\left(\frac{a}{t}\right)$$

and gave the following definition of a generalized f-divergence functional.

Definition 3.1 *Let $J \subset \mathbb{R}$ be an interval, and let $f\colon J \to \mathbb{R}$ be a function. Let $P = (p_1,\dots,p_n)$ be an n-tuple of real numbers and $Q = (q_1,\dots,q_n)$ be an n-tuple of nonnegative real numbers such that $p_i/q_i \in J$ for every $i = 1,\dots,n$. Then*

$$\hat{D}_f(\boldsymbol{p},\boldsymbol{q}) := \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right). \tag{3.3}$$

Let $\boldsymbol{p} = (p_1,\dots,p_n)$ and $\boldsymbol{p} = (q_1,\dots,q_n)$ be probability distributions. Examples of some well-known divergences and their generating functions are as follows.

- **Kullback-Leibler divergence** of the probability distributions $\boldsymbol{p}$ and $\boldsymbol{q}$ is defined as

$$D_{KL}(\boldsymbol{p},\boldsymbol{q}) = \sum_{i=1}^{n} q_i \log\frac{q_i}{p_i},$$

  and the corresponding generating function is $f(t) = t\log t$, $t > 0$.

- **Hellinger divergence** of the probability distributions $\boldsymbol{p}$ and $\boldsymbol{q}$ is defined as

$$D_H(\boldsymbol{p},\boldsymbol{q}) = \frac{1}{2}\sum_{i=1}^{n}(\sqrt{q_i}-\sqrt{p_i})^2,$$

  and the corresponding generating function is $f(t) = \frac{1}{2}(1-\sqrt{t})^2$, $t > 0$.

- **Renyi divergence** of the probability distributions $\boldsymbol{p}$ and $\boldsymbol{q}$ is defined as

$$D_\alpha(\boldsymbol{p},\boldsymbol{q}) = \sum_{i=1}^{n} q_i^{\alpha-1} p_i^{\alpha},\ \alpha \in \mathbb{R},$$

  and the corresponding generating function is $f(t) = t^{\alpha}$, $t > 0$.

▷ **Harmonic divergence** of the probability distributions $\boldsymbol{p}$ and $\boldsymbol{q}$ is defined as

$$D_{Ha}(\boldsymbol{p},\boldsymbol{q}) = \sum_{i=1}^{n} \frac{2p_iq_i}{p_i+q_i},$$

and the corresponding generating function is $f(t) = \frac{2t}{1+t}$.

▷ **Jeffrey divergence** of the probability distributions $\boldsymbol{p}$ and $\boldsymbol{q}$ is defined as

$$D_J(\boldsymbol{p},\boldsymbol{q}) = \frac{1}{2}\sum_{i=1}^{n}(q_i-p_i)\log\frac{q_i}{p_i},$$

and the corresponding generating function is $f(t) = (1-t)\log\frac{1}{t}$, $t>0$.

3.2 Inequalities for generalized f-divergence

Our first result in this section is an improved version of Dragomir's result (3.2) for the generalized f-divergence functional, and it provides an upper bound for the mentioned functional.

Theorem 3.1 ([104]) *Let $[m,M] \subset \mathbb{R}$ be an interval, let $f\colon [m,M] \to \mathbb{R}$ be a function and let δ_f be defined in (1.19). Let $\boldsymbol{p} = (p_1,\dots,p_n)$ be an n-tuple of real numbers and $\boldsymbol{q} = (q_1,\dots,q_n)$ be an n-tuple of nonnegative real numbers such that $p_i/q_i \in [m,M]$ for every $i = 1,\dots,n$. If the function f is convex, we have*

$$\begin{aligned}\hat{D}_f(\boldsymbol{p},\boldsymbol{q}) \le{}& \frac{MQ_n-P_n}{M-m}f(m) + \frac{P_n-mQ_n}{M-m}f(M)\\ &-\left(\frac{Q_n}{2} - \frac{1}{M-m}\sum_{i=1}^{n}\left|p_i - \frac{m+M}{2}q_i\right|\right)\delta_f,\end{aligned} \tag{3.4}$$

where $P_n = \sum_{i=1}^n p_i$ and $Q_n = \sum_{i=1}^n q_i$. If the function f is concave, then the inequality sign is reversed.

Proof. Let $f\colon [m,M] \to \mathbb{R}$ be a convex function. For an n-tuple of real numbers $\boldsymbol{x} = (x_1,\dots,x_n)$, an n-tuple of positive numbers $\boldsymbol{p} = (p_1,\dots,p_n)$ and a normalized positive linear functional $A(\boldsymbol{x}) = \frac{1}{P_n}\sum_{i=1}^n p_ix_i$, from Theorem 1.9 we have

$$\begin{aligned}\frac{1}{P_n}\sum_{i=1}^{n}p_if(x_i) \le{}& \frac{M-\bar{x}}{M-m}f(m) + \frac{\bar{x}-m}{M-m}f(M)\\ &-\frac{1}{P_n}\sum_{i=1}^{n}p_i\left(\frac{1}{2} - \frac{1}{M-m}\left|x_i - \frac{m+M}{2}\right|\right)\delta_f,\end{aligned} \tag{3.5}$$

where $\bar{x} = \frac{1}{P_n}\sum_{i=1}^n p_i x_i$. Since $\boldsymbol{q} = (q_1, \ldots, q_n)$ are nonnegative real numbers, we can put

$$p_i = q_i \text{ and } x_i = \frac{p_i}{q_i}$$

in (3.5) and get

$$\begin{aligned}\frac{1}{Q_n}\sum_{i=1}^n q_i f\left(\frac{p_i}{q_i}\right) \le{}& \frac{M - \frac{1}{Q_n}\sum_{i=1}^n q_i \frac{p_i}{q_i}}{M-m} f(m) + \frac{\frac{1}{Q_n}\sum_{i=1}^n q_i \frac{p_i}{q_i} - m}{M-m} f(M) \\ & - \frac{1}{Q_n}\left(\frac{Q_n}{2} - \frac{1}{M-m}\sum_{i=1}^n q_i \left|\frac{p_i}{q_i} - \frac{m+M}{2}\right|\right)\delta_f,\end{aligned}$$

and after multiplying by Q_n we get (3.4). □

Remark 3.1 From $m \le p_i/q_i \le M$ it easily follows that (see [80])

$$-\frac{M-m}{2}q_i \le p_i - \frac{m+M}{2}q_i \le \frac{M-m}{2}q_i, \text{ i.e. } \left|p_i - \frac{m+M}{2}q_i\right| \le \frac{M-m}{2}q_i$$

which together with $\delta_f \ge 0$ for a convex function f gives us

$$\left(\frac{Q_n}{2} - \frac{1}{M-m}\sum_{i=1}^n \left|p_i - \frac{m+M}{2}q_i\right|\right)\delta_f \ge 0.$$

Remark 3.2 If in the previous theorem we take $\boldsymbol{p}$ and $\boldsymbol{q}$ to be probability distributions, we directly get an improvement of Dragomir's result for the Csiszár f-divergence functional:

$$\begin{aligned}D_f(\boldsymbol{p},\boldsymbol{q}) \le{}& \frac{M-1}{M-m}f(m) + \frac{1-m}{M-m}f(M) \\ & - \left(\frac{1}{2} - \frac{1}{M-m}\sum_{i=1}^n \left|p_i - \frac{m+M}{2}q_i\right|\right)\delta_f.\end{aligned}$$

Next result is a special case of Theorem 3.1, and provides with bounds for the Kullback-Leibler divergence of two probability distributions.

Corollary 3.1 ([104]) *Let $[m,M] \subset \mathbb{R}$ be an interval and let us assume that the base of the logarithm is greater than* 1.

- *Let $\boldsymbol{p} = (p_1, \ldots, p_n)$ and $\boldsymbol{q} = (q_1, \ldots, q_n)$ be n-tuples of nonnegative real numbers such that $p_i/q_i \in [m,M]$ for every $i = 1, \ldots, n$. Then*

$$\begin{aligned}\sum_{i=1}^n p_i \log\left(\frac{p_i}{q_i}\right) \le{}& Q_n \frac{Mm}{M-m}\log\left(\frac{m}{M}\right) + \frac{P_n}{M-m}\log\left(\frac{M^M}{m^m}\right) \\ & - \left(\frac{Q_n}{2} - \frac{1}{M-m}\sum_{i=1}^n \left|p_i - \frac{m+M}{2}q_i\right|\right)\left(m\log\frac{2m}{m+M} + M\log\frac{2M}{m+M}\right).\end{aligned} \tag{3.6}$$

- *Let $\boldsymbol{p} = (p_1, \ldots, p_n)$ and $\boldsymbol{q} = (q_1, \ldots, q_n) \in \mathbb{P}$ be probability distributions such that $m \le p_i/q_i \le M$ holds for every $i = 1, \ldots, n$. Then*

$$D_{KL}(\boldsymbol{p},\boldsymbol{q}) \le \frac{Mm}{M-m}\log\left(\frac{m}{M}\right) + \frac{1}{M-m}\log\left(\frac{M^M}{m^m}\right) - \left(\frac{1}{2} - \frac{1}{M-m}\sum_{i=1}^{n}\left|p_i - \frac{m+M}{2}q_i\right|\right)\left(m\log\frac{2m}{m+M} + M\log\frac{2M}{m+M}\right). \tag{3.7}$$

If the base of the logarithm is less than 1, the inequality sign in the inequalities above is reversed.

Proof. Let $\boldsymbol{p} = (p_1, \ldots, p_n)$ and $\boldsymbol{q} = (q_1, \ldots, q_n)$ be n-tuples of nonnegative real numbers. Since the function $t \mapsto t\log t$ is convex when the base of the logarithm is greater than 1, the inequality (3.6) follows from Theorem 3.1, inequality (3.1), by setting $f(t) = t\log t$.

Inequality (3.7) is a special case of the inequality (3.6) for probability distributions $\boldsymbol{p}$ and $\boldsymbol{q}$. $\square$

Next result is obtained by utilizing Theorem 1.10, and it also gives us bounds for the generalized f-divergence functional. Concurrently, it represents an improvement of bounds for f-divergence functional obtained by Dragomir in the paper [39].

Theorem 3.2 ([104]) *Let $I \subset \mathbb{R}$ be an interval such that its interior contains the interval $[m,M]$, let $f\colon I \to \mathbb{R}$ be a continuous function and let δ_f be defined in (1.19). Let $\boldsymbol{p} = (p_1, \ldots, p_n)$ be an n-tuple of real numbers and $\boldsymbol{q} = (q_1, \ldots, q_n)$ be an n-tuple of nonnegative real numbers such that $p_i/q_i \in [m,M]$ for every $i = 1, \ldots, n$. Let Ψ_f be defined in (1.13). If the function f is convex, then*

$$\begin{aligned}
0 &\le \hat{D}_f(\boldsymbol{p},\boldsymbol{q}) - Q_n f\left(\frac{P_n}{Q_n}\right) \\
&\le Q_n\left(M - \frac{P_n}{Q_n}\right)\left(\frac{P_n}{Q_n} - m\right)\sup_{t\in\langle m,M\rangle}\Psi_f(t;m,M) \\
&\quad - \left(\frac{Q_n}{2} - \frac{1}{M-m}\sum_{i=1}^{n}\left|p_i - \frac{m+M}{2}q_i\right|\right)\delta_f \\
&\le \frac{Q_n}{M-m}\left(M - \frac{P_n}{Q_n}\right)\left(\frac{P_n}{Q_n} - m\right)\left(f'_-(M) - f'_+(m)\right) \\
&\quad - \left(\frac{Q_n}{2} - \frac{1}{M-m}\sum_{i=1}^{n}\left|p_i - \frac{m+M}{2}q_i\right|\right)\delta_f \\
&\le \frac{Q_n}{4}(M-m)(f'_-(M) - f'_+(m)) - \left(\frac{Q_n}{2} - \frac{1}{M-m}\sum_{i=1}^{n}\left|p_i - \frac{m+M}{2}q_i\right|\right)\delta_f.
\end{aligned} \tag{3.8}$$

If the function f is concave, the inequality signs are reversed.

Proof. Let $f\colon [m,M] \to \mathbb{R}$ be a convex function. Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be an n-tuple of real numbers and let $\boldsymbol{p} = (p_1, \ldots, p_n)$ be an n-tuple of positive numbers. Then $A(\boldsymbol{x}) =$

$\frac{1}{P_n}\sum_{i=1}^n p_i x_i$ is a normalized positive linear functional, so from Theorem 1.10, inequality (1.20), we have

$$\begin{aligned}
0 &\le \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right) \\
&\le (M-\overline{x})(\overline{x}-m) \sup_{t\in\langle m,M\rangle} \Psi_f(t;m,M) - \left(\frac{1}{2} - \frac{1}{M-m}\sum_{i=1}^n p_i \left|x_i - \frac{m+M}{2}\right|\right)\delta_f \\
&\le \frac{(M-\overline{x})(\overline{x}-m)}{M-m}(f'_-(M) - f'_+(m)) - \left(\frac{1}{2} - \frac{1}{M-m}\sum_{i=1}^n p_i \left|x_i - \frac{m+M}{2}\right|\right)\delta_f \\
&\le \frac{1}{4}(M-m)(f'_-(M) - f'_+(m)) - \left(\frac{1}{2} - \frac{1}{M-m}\sum_{i=1}^n p_i \left|x_i - \frac{m+M}{2}\right|\right)\delta_f, \qquad (3.9)
\end{aligned}$$

Since $\boldsymbol{q} = (q_1,\dots,q_n)$ are nonnegative real numbers, we can put

$$p_i = \frac{q_i}{\sum_{i=1}^n q_i} = \frac{q_i}{Q_n} \quad \text{and} \quad x_i = \frac{p_i}{q_i}$$

in (3.9) and get

$$\begin{aligned}
0 &\le \sum_{i=1}^n \frac{q_i}{\sum_{i=1}^n q_i} f\left(\frac{p_i}{q_i}\right) - f\left(\sum_{i=1}^n \frac{q_i}{\sum_{i=1}^n q_i}\frac{p_i}{q_i}\right) \\
&\le \left(M - \sum_{i=1}^n \frac{q_i}{\sum_{i=1}^n q_i}\frac{p_i}{q_i}\right)\left(\sum_{i=1}^n \frac{q_i}{\sum_{i=1}^n q_i}\frac{p_i}{q_i} - m\right) \sup_{t\in\langle m,M\rangle} \Psi_f(t;m,M) \\
&\quad - \left(\frac{1}{2} - \frac{1}{M-m}\sum_{i=1}^n \frac{q_i}{\sum_{i=1}^n q_i}\left|\frac{p_i}{q_i} - \frac{m+M}{2}\right|\right)\delta_f \\
&\le \frac{f'_-(M) - f'_+(m)}{M-m}\left(M - \sum_{i=1}^n \frac{q_i}{\sum_{i=1}^n q_i}\frac{p_i}{q_i}\right)\left(\sum_{i=1}^n \frac{q_i}{\sum_{i=1}^n q_i}\frac{p_i}{q_i} - m\right) \\
&\quad - \left(\frac{1}{2} - \frac{1}{M-m}\sum_{i=1}^n \frac{q_i}{\sum_{i=1}^n q_i}\left|\frac{p_i}{q_i} - \frac{m+M}{2}\right|\right)\delta_f \\
&\le \frac{1}{4}(M-m)(f'_-(M) - f'_+(m)) - \left(\frac{1}{2} - \frac{1}{M-m}\sum_{i=1}^n \frac{q_i}{\sum_{i=1}^n q_i}\left|\frac{p_i}{q_i} - \frac{m+M}{2}\right|\right)\delta_f,
\end{aligned}$$

and after multiplying by Q_n we get (3.8). □

The result that follows is a special case of Theorem 3.2. It gives us different bounds of those that we have already obtained for the Kullback-Leibler divergence of two probability distributions.

Corollary 3.2 ([104]) *Let $[m,M] \subset \mathbb{R}$ be an interval and let us assume that the base of the logarithm is greater than* 1.

- *Let $\boldsymbol{p}=(p_1,\dots,p_n)$ and $\boldsymbol{q}=(q_1,\dots,q_n)$ be n-tuples of nonnegative real numbers such that $p_i/q_i \in [m,M]$ for every $i=1,\dots,n$. Then*

$$\begin{aligned}
0 &\le \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i} - P_n \log\left(\frac{P_n}{Q_n}\right) \\
&\le Q_n\left(M-\frac{P_n}{Q_n}\right)\left(\frac{P_n}{Q_n}-m\right)\sup_{t\in\langle m,M\rangle}\Psi_{id\cdot\log}(t;m,M) \\
&\quad -\left(\frac{Q_n}{2}-\frac{1}{M-m}\sum_{i=1}^{n}\left|p_i-\frac{m+M}{2}q_i\right|\right)\left(m\log\frac{2m}{m+M}+M\log\frac{2M}{m+M}\right) \\
&\le \frac{Q_n}{M-m}\left(M-\frac{P_n}{Q_n}\right)\left(\frac{P_n}{Q_n}-m\right)\log\frac{M}{m} \qquad (3.10)\\
&\quad -\left(\frac{Q_n}{2}-\frac{1}{M-m}\sum_{i=1}^{n}\left|p_i-\frac{m+M}{2}q_i\right|\right)\left(m\log\frac{2m}{m+M}+M\log\frac{2M}{m+M}\right) \\
&\le \frac{Q_n}{4}(M-m)\log\frac{M}{m}-\left(\frac{Q_n}{2}-\frac{1}{M-m}\sum_{i=1}^{n}\left|p_i-\frac{m+M}{2}q_i\right|\right)\left(m\log\frac{2m}{m+M}+M\log\frac{2M}{m+M}\right).
\end{aligned}$$

- *Let $\boldsymbol{p}=(p_1,\dots,p_n)$ and $\boldsymbol{q}=(q_1,\dots,q_n)\in\mathbb{P}$ be probability distributions such that $m\le p_i/q_i\le M$ holds for every $i=1,\dots,n$. Then*

$$\begin{aligned}
0 &\le D_{KL}(\boldsymbol{p},\boldsymbol{q}) \\
&\le (M-1)(1-m)\sup_{t\in\langle m,M\rangle}\Psi_{id\cdot\log}(t;m,M) \\
&\quad -\left(\frac{1}{2}-\frac{1}{M-m}\sum_{i=1}^{n}\left|p_i-\frac{m+M}{2}q_i\right|\right)\left(m\log\frac{2m}{m+M}+M\log\frac{2M}{m+M}\right) \\
&\le \frac{1}{M-m}(M-1)(1-m)\log\frac{M}{m} \qquad (3.11)\\
&\quad -\left(\frac{1}{2}-\frac{1}{M-m}\sum_{i=1}^{n}\left|p_i-\frac{m+M}{2}q_i\right|\right)\left(m\log\frac{2m}{m+M}+M\log\frac{2M}{m+M}\right) \\
&\le \frac{1}{4}(M-m)\log\frac{M}{m}-\left(\frac{1}{2}-\frac{1}{M-m}\sum_{i=1}^{n}\left|p_i-\frac{m+M}{2}q_i\right|\right)\left(m\log\frac{2m}{m+M}+M\log\frac{2M}{m+M}\right).
\end{aligned}$$

If the base of the logarithm is less than 1*, the inequality sign in the inequalities above is reversed.*

Proof. Let $\boldsymbol{p}=(p_1,\dots,p_n)$ and $\boldsymbol{q}=(q_1,\dots,q_n)$ be n-tuples of nonnegative real numbers. Function $t\mapsto t\log t$ is convex, so inequality (3.10) follows from Theorem 3.2, inequality (3.8), by setting $f(t)=t\log t$.

Inequality (3.11) is a special case of the inequality (3.10) for probability distributions $\boldsymbol{p}$ and $\boldsymbol{q}$. □

Remark 3.3 If in Theorem 3.2, inequality (3.8), we set $f(t) = -\log t$ with the base greater than 1, we get the following:

- for n-tuples of nonnegative real numbers $\boldsymbol{p} = (p_1, \dots, p_n)$ and $\boldsymbol{q} = (q_1, \dots, q_n)$ such that $p_i/q_i \in [m, M]$ for every $i = 1, \dots, n$ we have

$$\begin{aligned}
0 &\leq \sum_{i=1}^{n} q_i \log\left(\frac{q_i}{p_i}\right) + Q_n \log\left(\frac{P_n}{Q_n}\right) \\
&\leq Q_n\left(M - \frac{P_n}{Q_n}\right)\left(\frac{P_n}{Q_n} - m\right) \sup_{t\in\langle m,M\rangle} \Psi_{-\log}(t; m, M) \\
&\quad - \left(\frac{Q_n}{2} - \frac{1}{M-m}\sum_{i=1}^{n}\left|p_i - \frac{m+M}{2}q_i\right|\right)\log\frac{(m+M)^2}{4mM} \qquad (3.12)\\
&\leq \frac{Q_n}{Mm}\left(M - \frac{P_n}{Q_n}\right)\left(\frac{P_n}{Q_n} - m\right) \\
&\quad - \left(\frac{Q_n}{2} - \frac{1}{M-m}\sum_{i=1}^{n}\left|p_i - \frac{m+M}{2}q_i\right|\right)\log\frac{(m+M)^2}{4mM} \\
&\leq \frac{Q_n(M-m)^2}{4Mm} - \left(\frac{Q_n}{2} - \frac{1}{M-m}\sum_{i=1}^{n}\left|p_i - \frac{m+M}{2}q_i\right|\right)\log\frac{(m+M)^2}{4mM}.
\end{aligned}$$

- for probability distributions $\boldsymbol{p} = (p_1, \dots, p_n)$ and $\boldsymbol{q} = (q_1, \dots, q_n) \in \mathbb{P}$ such that $m \leq p_i/q_i \leq M$ holds for every $i = 1, \dots, n$ we have

$$\begin{aligned}
0 &\leq D_{KL}(\boldsymbol{q}, \boldsymbol{p}) \\
&\leq (M-1)(1-m) \sup_{t\in\langle m,M\rangle} \Psi_{-\log}(t; m, M) \\
&\quad - \left(\frac{1}{2} - \frac{1}{M-m}\sum_{i=1}^{n}\left|p_i - \frac{m+M}{2}q_i\right|\right)\log\frac{(m+M)^2}{4mM} \qquad (3.13)\\
&\leq \frac{1}{Mm}(M-1)(1-m) \\
&\quad - \left(\frac{1}{2} - \frac{1}{M-m}\sum_{i=1}^{n}\left|p_i - \frac{m+M}{2}q_i\right|\right)\log\frac{(m+M)^2}{4mM} \\
&\leq \frac{(M-m)^2}{4Mm} - \left(\frac{1}{2} - \frac{1}{M-m}\sum_{i=1}^{n}\left|p_i - \frac{m+M}{2}q_i\right|\right)\log\frac{(m+M)^2}{4mM}.
\end{aligned}$$

If the base of the logarithm is less than 1, the inequality signs in the inequalities above are reversed.

By following the same steps as in the proof of Theorem 3.2, but starting from Theorem 1.11, we get lower and upper bounds for the difference in the results from Theorem 3.1, and consequently in Dragomir's result (3.2).

Theorem 3.3 ([104]) *Let $I \subset \mathbb{R}$ be an interval such that its interior contains the interval $[m,M]$, let $f\colon I \to \mathbb{R}$ be a continuous function and let δ_f be defined in (1.19). Let $\boldsymbol{p} = (p_1,\dots,p_n)$ be an n-tuple of real numbers and $\boldsymbol{q} = (q_1,\dots,q_n)$ be an n-tuple of nonnegative real numbers such that $p_i/q_i \in [m,M]$ for every $i = 1,\dots,n$. Let Ψ_f be defined in (1.13). If the function f is convex, then we have*

$$\begin{aligned}
&\left(\frac{Q_n}{2} - \frac{1}{M-m}\sum_{i=1}^{n}\left|p_i - \frac{m+M}{2}q_i\right|\right)\delta_f \\
&\quad \le \frac{MQ_n - P_n}{M-m} f(m) + \frac{P_n - mQ_n}{M-m} f(M) - \hat{D}_f(P,Q) \\
&\quad \le \sup_{t\in\langle m,M\rangle} \Psi_f(t;m,M) \sum_{i=1}^{n}\left(M - \frac{p_i}{q_i}\right)\left(\frac{p_i}{q_i} - m\right) \\
&\quad \le Q_n\left(M - \frac{P_n}{Q_n}\right)\left(\frac{P_n}{Q_n} - m\right) \sup_{t\in\langle m,M\rangle} \Psi_f(t;m,M) \\
&\quad \le Q_n\left(M - \frac{P_n}{Q_n}\right)\left(\frac{P_n}{Q_n} - m\right) \frac{f'_-(M) - f'_+(m)}{M-m} \\
&\quad \le \frac{Q_n}{4}(M-m)(f'_-(M) - f'_+(m)).
\end{aligned} \tag{3.14}$$

If the function f is concave, the inequality signs are reversed.

We can utilize Theorem 3.3 to obtain lower and upper bounds for the difference in the results from Corollary 3.1, as well as for the reversed Kullback-Leibler divergence.

Corollary 3.3 ([104]) *Let $[m,M] \subset \mathbb{R}$ be an interval and let us assume that the base of the logarithm is greater than* 1.

- *Let $\boldsymbol{p} = (p_1,\dots,p_n)$ and $\boldsymbol{q} = (q_1,\dots,q_n)$ be n-tuples of nonnegative real numbers such that $p_i/q_i \in [m,M]$ for every $i = 1,\dots,n$. Then*

$$\begin{aligned}
&\left(\frac{Q_n}{2} - \frac{1}{M-m}\sum_{i=1}^{n}\left|p_i - \frac{m+M}{2}q_i\right|\right)\left(m\log\frac{2m}{m+M} + M\log\frac{2M}{m+M}\right) \\
&\quad \le Q_n\frac{Mm}{M-m}\log\left(\frac{m}{M}\right) + \frac{P_n}{M-m}\log\left(\frac{M^M}{m^m}\right) - \sum_{i=1}^{n} p_i\log\left(\frac{p_i}{q_i}\right) \\
&\quad \le \sup_{t\in\langle m,M\rangle} \Psi_{id\cdot\log}(t;m,M) \sum_{i=1}^{n}\left(M - \frac{p_i}{q_i}\right)\left(\frac{p_i}{q_i} - m\right) \\
&\quad \le Q_n\left(M - \frac{P_n}{Q_n}\right)\left(\frac{P_n}{Q_n} - m\right) \sup_{t\in\langle m,M\rangle} \Psi_{id\cdot\log}(t;m,M) \qquad (3.15)\\
&\quad \le \frac{Q_n}{M-m}\left(M - \frac{P_n}{Q_n}\right)\left(\frac{P_n}{Q_n} - m\right)\log\left(\frac{M}{m}\right) \le \frac{Q_n}{4}(M-m)\log\left(\frac{M}{m}\right).
\end{aligned}$$

- *Let $\boldsymbol{p} = (p_1, \ldots, p_n)$ and $\boldsymbol{q} = (q_1, \ldots, q_n) \in \mathbb{P}$ be probability distributions such that $m \le p_i/q_i \le M$ holds for every $i = 1, \ldots, n$. Then*

$$\begin{aligned}
&\left(\frac{1}{2} - \frac{1}{M-m}\sum_{i=1}^{n}\left|p_i - \frac{m+M}{2}q_i\right|\right)\left(m\log\frac{2m}{m+M} + M\log\frac{2M}{m+M}\right) \\
&\quad\le \frac{Mm}{M-m}\log\left(\frac{m}{M}\right) + \frac{1}{M-m}\log\left(\frac{M^M}{m^m}\right) - D_{KL}(\boldsymbol{p},\boldsymbol{q}) \\
&\quad\le \sup_{t\in\langle m,M\rangle}\Psi_{id\cdot\log}(t;m,M)\sum_{i=1}^{n}\left(M - \frac{p_i}{q_i}\right)\left(\frac{p_i}{q_i} - m\right) \\
&\quad\le (M-1)(1-m)\sup_{t\in\langle m,M\rangle}\Psi_{id\cdot\log}(t;m,M) \qquad (3.16) \\
&\quad\le \frac{1}{M-m}(M-1)(1-m)\log\left(\frac{M}{m}\right) \le \frac{1}{4}(M-m)\log\left(\frac{M}{m}\right).
\end{aligned}$$

If the base of the logarithm is less than 1*, the inequality signs in the inequalities above are reversed.*

Remark 3.4 As in Remark 3.3, we can set $f(t) = -\log t$ with the base greater than 1 in Theorem 3.3, inequality (3.14), and obtain the following inequalities for the reversed Kullback-Leibler divergence:

- for n-tuples of nonnegative real numbers $\boldsymbol{p} = (p_1, \ldots, p_n)$ and $\boldsymbol{q} = (q_1, \ldots, q_n)$ such that $p_i/q_i \in [m, M]$ for every $i = 1, \ldots, n$ we have

$$\begin{aligned}
&\left(\frac{Q_n}{2} - \frac{1}{M-m}\sum_{i=1}^{n}\left|p_i - \frac{m+M}{2}q_i\right|\right)\log\frac{(m+M)^2}{4mM} \\
&\quad\le \frac{Q_n}{M-m}\log\left(\frac{M^m}{m^M}\right) + \frac{P_n}{M-m}\log\left(\frac{m}{M}\right) - \sum_{i=1}^{n}q_i\log\left(\frac{q_i}{p_i}\right) \\
&\quad\le \sup_{t\in\langle m,M\rangle}\Psi_{\log}(t;m,M)\sum_{i=1}^{n}\left(M - \frac{p_i}{q_i}\right)\left(\frac{p_i}{q_i} - m\right) \\
&\quad\le Q_n\left(M - \frac{P_n}{Q_n}\right)\left(\frac{P_n}{Q_n} - m\right)\sup_{t\in\langle m,M\rangle}\Psi_{\log}(t;m,M) \\
&\quad\le -\frac{Q_n}{Mm}\left(M - \frac{P_n}{Q_n}\right)\left(\frac{P_n}{Q_n} - m\right) \le -\frac{Q_n}{4Mm}(M-m)^2. \qquad (3.17)
\end{aligned}$$

- for probability distributions $\boldsymbol{p} = (p_1, \ldots, p_n)$ and $\boldsymbol{q} = (q_1, \ldots, q_n) \in \mathbb{P}$ such that $m \le p_i/q_i \le M$ holds for every $i = 1, \ldots, n$ we have

$$\begin{aligned}
&\left(\frac{1}{2} - \frac{1}{M-m}\sum_{i=1}^{n}\left|p_i - \frac{m+M}{2}q_i\right|\right)\log\frac{(m+M)^2}{4mM} \\
&\quad\le \frac{1}{M-m}\log\left(\frac{M^{m-1}}{m^{M-1}}\right) - D_{KL}(\boldsymbol{q},\boldsymbol{p})
\end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{t\in\langle m,M\rangle} \Psi_{\log}(t;m,M)\sum_{i=1}^{n}\left(M-\frac{p_i}{q_i}\right)\left(\frac{p_i}{q_i}-m\right)\\
&\leq (M-1)(1-m)\sup_{t\in\langle m,M\rangle} \Psi_{\log}(t;m,M)\\
&\leq -\frac{1}{Mm}(M-1)(1-m)\leq -\frac{1}{4Mm}(M-m)^2. \qquad (3.18)
\end{aligned}
$$

If the base of the logarithm is less than 1, the inequality signs in the inequalities above are reversed.

By using Corollary 1.1 we get an improvement of Theorem 3.1.

Theorem 3.4 [122] *Let $[m,M]\subset\mathbb{R}$ be an interval and let $f\colon [m,M]\to\mathbb{R}$ be a function. Let $\boldsymbol{p}=(p_1,\ldots,p_l)$ be an l-tuple of real numbers and $\boldsymbol{q}=(q_1,\ldots,q_l)$ be an l-tuple of nonnegative real numbers such that $p_i/q_i\in[m,M]$ for every $i=1,\ldots,l$. If the function f is convex, we have*

$$
\begin{aligned}
\hat{D}_f(\boldsymbol{p},\boldsymbol{q}) \leq\ & \frac{MQ_l-P_l}{M-m}f(m)+\frac{P_l-mQ_l}{M-m}f(M)\\
&-\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\sum_{i=1}^{l}\Delta_f(m,M,n,k)q_i\left[\left(r_n\cdot\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\right)\left(\frac{\frac{p_i}{q_i}-m}{M-m}\right)\right]\\
=\ & \frac{MQ_l-P_l}{M-m}f(m)+\frac{P_l-mQ_l}{M-m}f(M)\\
&-\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\sum_{i=1}^{l}\Delta_f(m,M,n,k)\frac{1}{M-m}\cdot\\
&\cdot\left[(2^n(p_i-mq_i)-q_i(M-m)(k-1))\cdot\chi_{\left(\frac{k-1}{2^n},\frac{2k-1}{2^{n+1}}\right)}\left(\frac{\frac{p_i}{q_i}-m}{M-m}\right)\right.\\
&\left.+\,(q_i(M-m)k-2^n(p_i-mq_i))\,\chi_{\left(\frac{2k-1}{2^{n+1}},\frac{k}{2^n}\right)}\left(\frac{\frac{p_i}{q_i}-m}{M-m}\right)\right]
\end{aligned}
$$

where $P_l=\sum_{i=1}^{l}p_i$ and $Q_l=\sum_{i=1}^{l}q_i$. If the function f is concave, then the inequality sign is reversed.

Proof. Let $f\colon [m,M]\to\mathbb{R}$ be a convex function. For an l-tuple of real numbers $\boldsymbol{x}=(x_1,\ldots,x_l)$ and an l-tuple of nonnegative numbers $\boldsymbol{p}=(p_1,\ldots,p_l)$ from Corollary 1.1 we have

$$
\begin{aligned}
\frac{1}{P_l}\sum_{i=1}^{l}p_if(x_i) \leq\ & \frac{M-\bar{x}}{M-m}f(m)+\frac{\bar{x}-m}{M-m}f(M)\\
&-\frac{1}{P_l}\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\sum_{i=1}^{l}\Delta_f(m,M,n,k)p_i\left[\left(r_n\cdot\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\right)\left(\frac{x_i-m}{M-m}\right)\right]\\
=\ & \frac{M-\bar{x}}{M-m}f(m)+\frac{\bar{x}-m}{M-m}f(M)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{P_l}\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\sum_{i=1}^{l}\Delta_f(m,M,n,k)p_i\left[\left(2^n\frac{x_i-m}{M-m}-k+1\right)\cdot\chi_{\left(\frac{k-1}{2^n},\frac{2k-1}{2^{n+1}}\right)}\left(\frac{x_i-m}{M-m}\right)\right.\\
&\left.+\left(k-2^n\frac{x_i-m}{M-m}\right)\chi_{\left(\frac{2k-1}{2^{n+1}},\frac{k}{2^n}\right)}\left(\frac{x_i-m}{M-m}\right)\right]
\end{aligned}
$$

where $\bar{x}=\frac{1}{P_l}\sum_{i=1}^{l}p_ix_i$. Since $\boldsymbol{q}=(q_1,\ldots,q_l)$ are nonnegative real numbers, we can put

$$
p_i=q_i \ \text{ and } \ x_i=\frac{p_i}{q_i}
$$

in previous inequality and get

$$
\begin{aligned}
\frac{1}{Q_l}\sum_{i=1}^{l}q_if\left(\frac{p_i}{q_i}\right) &\leq \frac{M-\frac{1}{Q_l}\sum_{i=1}^{l}q_i\frac{p_i}{q_i}}{M-m}f(m)+\frac{\frac{1}{Q_l}\sum_{i=1}^{l}q_i\frac{p_i}{q_i}-m}{M-m}f(M)\\
&\quad-\frac{1}{Q_l}\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\sum_{i=1}^{l}\Delta_f(m,M,n,k)q_i\left[r_n\left(\frac{\frac{p_i}{q_i}-m}{M-m}\right)\cdot\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{\frac{p_i}{q_i}-m}{M-m}\right)\right]\\
&=\frac{M-\frac{1}{Q_l}\sum_{i=1}^{l}q_i\frac{p_i}{q_i}}{M-m}f(m)+\frac{\frac{1}{Q_l}\sum_{i=1}^{l}q_i\frac{p_i}{q_i}-m}{M-m}f(M)\\
&\quad-\frac{1}{Q_l}\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\sum_{i=1}^{l}\Delta_f(m,M,n,k)\frac{1}{M-m}\cdot\\
&\quad\cdot\left[\left(2^n(p_i-mq_i)-q_i(M-m)(k-1)\right)\cdot\chi_{\left(\frac{k-1}{2^n},\frac{2k-1}{2^{n+1}}\right)}\left(\frac{\frac{p_i}{q_i}-m}{M-m}\right)\right.\\
&\quad\left.+\left(q_i(M-m)k-2^n(p_i-mq_i)\right)\chi_{\left(\frac{2k-1}{2^{n+1}},\frac{k}{2^n}\right)}\left(\frac{\frac{p_i}{q_i}-m}{M-m}\right)\right]
\end{aligned}
$$

and after multiplying by Q_l we get the result. □

Remark 3.5 If in the previous theorem we take $\boldsymbol{p}$ and $\boldsymbol{q}$ to be probability distributions, we directly get following result for the Csiszár f-divergence functional.

$$
\begin{aligned}
D_f(\boldsymbol{p},\boldsymbol{q}) &\leq \frac{M-1}{M-m}f(m)+\frac{1-m}{M-m}f(M)\\
&\quad-\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\sum_{i=1}^{l}\Delta_f(m,M,n,k)q_i\left[r_n\left(\frac{\frac{p_i}{q_i}-m}{M-m}\right)\cdot\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{\frac{p_i}{q_i}-m}{M-m}\right)\right]\\
&=\frac{M-1}{M-m}f(m)+\frac{1-m}{M-m}f(M)-\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\sum_{i=1}^{l}\Delta_f(m,M,n,k)\frac{1}{M-m}\cdot\\
&\quad\cdot\left[\left(2^n(p_i-mq_i)-q_i(M-m)(k-1)\right)\cdot\chi_{\left(\frac{k-1}{2^n},\frac{2k-1}{2^{n+1}}\right)}\left(\frac{\frac{p_i}{q_i}-m}{M-m}\right)\right.\\
&\quad\left.+\left(q_i(M-m)k-2^n(p_i-mq_i)\right)\chi_{\left(\frac{2k-1}{2^{n+1}},\frac{k}{2^n}\right)}\left(\frac{\frac{p_i}{q_i}-m}{M-m}\right)\right]
\end{aligned}
$$

Next result provides with an improvement of the bounds for the Kullback-Leibler divergence of two probability distributions, that is result from Corollary 3.1.

Corollary 3.4 [122] *Let $[m,M] \subset \mathbb{R}$ be an interval and let us assume that the base of the logarithm is greater than* 1.

- *Let $\boldsymbol{p} = (p_1,...,p_n)$ and $\boldsymbol{q} = (q_1,...,q_l)$ be l-tuples of nonnegative real numbers such that $\frac{p_i}{q_i} \in [m,M]$ for every $i = 1,\dots,l$. Then*

$$\begin{aligned}
\sum_{i=1}^{l} p_i \log\left(\frac{p_i}{q_i}\right) &\leq Q_l \frac{Mm}{M-m}\log\left(\frac{m}{M}\right) + \frac{P_l}{M-m}\log\left(\frac{M^M}{m^m}\right) \\
&\quad - \sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\sum_{i=1}^{l} \Delta_{log}(m,M,n,k) q_i \left[\left(r_n \cdot \chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\right)\left(\frac{\frac{p_i}{q_i}-m}{M-m}\right)\right] \\
&= Q_l \frac{Mm}{M-m}\log\left(\frac{m}{M}\right) + \frac{P_l}{M-m}\log\left(\frac{M^M}{m^m}\right) \\
&\quad - \sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\sum_{i=1}^{l} \Delta_{log}(m,M,n,k) \frac{1}{M-m} \cdot \\
&\quad \cdot \left[(2^n(p_i - mq_i) - q_i(M-m)(k-1)) \cdot \chi_{\left(\frac{k-1}{2^n},\frac{2k-1}{2^{n+1}}\right)}\left(\frac{\frac{p_i}{q_i}-m}{M-m}\right)\right. \\
&\quad \left. + (q_i(M-m)k - 2^n(p_i - mq_i))\, \chi_{\left(\frac{2k-1}{2^{n+1}},\frac{k}{2^n}\right)}\left(\frac{\frac{p_i}{q_i}-m}{M-m}\right)\right]
\end{aligned}$$

- *Let $\boldsymbol{p} = (p_1,...,p_n)$ and $\boldsymbol{q} = (q_1,...,q_n) \in \mathbb{P}$ be probability distributions such that $m \leq p_i/q_i \leq M$ holds for every $i = 1,...,n$. Then*

$$\begin{aligned}
D_{KL}(\boldsymbol{p},\boldsymbol{q}) &\leq \frac{Mm}{M-m}\log\left(\frac{m}{M}\right) + \frac{1}{M-m}\log\left(\frac{M^M}{m^m}\right) \\
&\quad - \sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\sum_{i=1}^{l} \Delta_{log}(m,M,n,k) q_i \left[r_n\left(\frac{\frac{p_i}{q_i}-m}{M-m}\right) \cdot \chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{\frac{p_i}{q_i}-m}{M-m}\right)\right] \\
&= \frac{Mm}{M-m}\log\left(\frac{m}{M}\right) + \frac{1}{M-m}\log\left(\frac{M^M}{m^m}\right) \\
&\quad - \sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\sum_{i=1}^{l} \Delta_{log}(m,M,n,k) \frac{1}{M-m} \cdot \\
&\quad \cdot \left[(2^n(p_i - mq_i) - q_i(M-m)(k-1)) \cdot \chi_{\left(\frac{k-1}{2^n},\frac{2k-1}{2^{n+1}}\right)}\left(\frac{\frac{p_i}{q_i}-m}{M-m}\right)\right. \\
&\quad \left. + (q_i(M-m)k - 2^n(p_i - mq_i))\, \chi_{\left(\frac{2k-1}{2^{n+1}},\frac{k}{2^n}\right)}\left(\frac{\frac{p_i}{q_i}-m}{M-m}\right)\right]
\end{aligned}$$

If the base of the logarithm is less than 1*, the inequality sign in the inequalities above is reversed.*

Proof. Let $\boldsymbol{p} = (p_1, \ldots, p_n)$ and $\boldsymbol{q} = (q_1, \ldots, q_n)$ be n-tuples of nonnegative real numbers. Since the function $t \mapsto t\log t$ is convex when the base of the logarithm is greater than 1, first inequality follows from Theorem 3.4 by setting $f(t) = t\log t$.

Second inequality is a special case of the first inequality for probability distributions $\boldsymbol{p}$ and $\boldsymbol{q}$. □

Unlike previous results, the following results do not require convexity in the classical sense of the function f. We start with an Edmundson-Lah-Ribarič type inequality for the generalized f-divergence functional $\tilde{D}_f(\boldsymbol{p}, \boldsymbol{q})$, where the function f has bounded second order divided differences. This is a significant progress in relation to the previous results, since the class of functions with bounded second order divided differences is much greater then the class of convex functions.

Theorem 3.5 ([105]) *Let $[m,M] \subset \mathbb{R}$ be an interval such that $m \le 1 \le M$ and let $f\colon [m,M] \to \mathbb{R}$ be a function with $\gamma \le [m,t,M]f \le \Gamma$. Let $\boldsymbol{p} = (p_1, \ldots, p_n)$ and $\boldsymbol{p} = (q_1, \ldots, q_n)$ be probability distributions such that $p_i/q_i \in [m,M]$ for every $i = 1, \ldots, n$. Then we have*

$$\begin{aligned}\gamma\sum_{i=1}^{n} q_i\left(M - \frac{p_i}{q_i}\right)\left(\frac{p_i}{q_i} - m\right) &\le \frac{M-1}{M-m}f(m) + \frac{1-m}{M-m}f(M) - \tilde{D}_f(\boldsymbol{p}, \boldsymbol{q}) \\ &\le \Gamma\sum_{i=1}^{n} q_i\left(M - \frac{p_i}{q_i}\right)\left(\frac{p_i}{q_i} - m\right). \qquad (3.19)\end{aligned}$$

Proof. The function f has bounded second order divided difference with bounds γ and Γ, so when we set linear functional A from (2.5) to be a discrete sum, we get

$$\begin{aligned}\gamma\sum_{i=1}^{n} p_i(M - x_i)(x_i - m) &\le \frac{M - \bar{x}}{M-m}\phi(m) + \frac{\bar{x} - m}{M-m}\phi(M) - \sum_{i=1}^{n} p_i\phi(x_i) \\ &\le \Gamma\sum_{i=1}^{n} p_i(M - x_i)(x_i - m), \qquad (3.20)\end{aligned}$$

where $\boldsymbol{x} = (x_1, \ldots, x_n)$ is an n-tuple of real numbers from $[m,M]$, $\boldsymbol{p} = (p_1, \ldots, p_n)$ is an n-tuple of nonnegative real numbers such that $\sum_{i=1}^{n} p_i = 1$, and $\bar{x} = \sum_{i=1}^{n} p_i x_i$. Now, in the relation (3.20) we can put

$$\phi = f, \quad p_i = q_i \text{ and } x_i = \frac{p_i}{q_i},$$

and after calculating

$$\bar{x} = \sum_{i=1}^{n} q_i\frac{p_i}{q_i} = \sum_{i=1}^{n} p_i = 1$$

we get (3.19). □

By following the same idea as in the proof of the previous theorem, but starting with the relation (2.14) from Theorem 2.3, we get the following result, which is a Jensen type inequality for the generalized f-divergence functional $\tilde{D}_f(\boldsymbol{p}, \boldsymbol{q})$.

Theorem 3.6 *Let $[m,M] \subset \mathbb{R}$ be an interval such that $m \le 1 \le M$ and let $f : [m,M] \to \mathbb{R}$ be a function with $\gamma \le [m,t,M]f \le \Gamma$. Let $\boldsymbol{p} = (p_1,\dots,p_n)$ and $\boldsymbol{p} = (q_1,\dots,q_n)$ be probability distributions such that $p_i/q_i \in [m,M]$ for every $i = 1,\dots,n$. Then we have*

$$\begin{aligned} &\gamma(M-1)(1-m) - \Gamma\sum_{i=1}^{n} q_i\left(M-\frac{p_i}{q_i}\right)\left(\frac{p_i}{q_i}-m\right) \\ &\quad \le \tilde{D}_f(\boldsymbol{p},\boldsymbol{q}) - f(1) \le \Gamma(M-1)(1-m) - \gamma\sum_{i=1}^{n} q_i\left(M-\frac{p_i}{q_i}\right)\left(\frac{p_i}{q_i}-m\right). \end{aligned} \tag{3.21}$$

Remark 3.6 If the function $f : [m,M] \to \mathbb{R}$ is additionally convex, then from (2.12), by following the same idea as in the proof of Theorem 3.5, we get Edmundson-Lah-Ribarič type inequality for the Csiszár f-divergence functional:

$$\begin{aligned} 0 &\le \gamma\sum_{i=1}^{n} q_i\left(M-\frac{p_i}{q_i}\right)\left(\frac{p_i}{q_i}-m\right) \\ &\le \frac{M-1}{M-m}f(m) + \frac{1-m}{M-m}f(M) - D_f(\boldsymbol{p},\boldsymbol{q}) \\ &\le \Gamma\sum_{i=1}^{n} q_i\left(M-\frac{p_i}{q_i}\right)\left(\frac{p_i}{q_i}-m\right) \\ &\le \Gamma(M-1)(1-m) \le \frac{\Gamma}{4}(M-m)^2. \end{aligned} \tag{3.22}$$

Jensen type inequality for the Csiszár divergence functional is a special case of Theorem 3.6 for a convex function.

The generating function of the Kullback-Leibler divergence $f(t) = t\log t$ is convex, and its second order divided difference $[m,t,M]f$ is a continuous and decreasing function, which means that it attains its maximal and minimal value in the points m and M respectively.

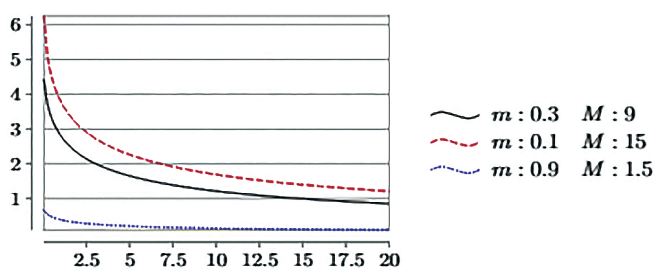

Figure 3.1: Graphs of the Function $-[m,t,M]\mathrm{id}\circ\log$ for different Choices of the Points m and M.

We calculate the bounds for the second order divided difference of the function $f(t) = t\log t$:

$$\begin{aligned}
\Gamma &= [m,m,M]\mathrm{id}\cdot\log = \frac{1}{M-m}\left(\frac{M\log M - m\log m}{M-m} - [(\mathrm{id}\cdot\log)(m)]'_+\right) \\
&= \frac{1}{M-m}\left(\frac{1}{M-m}\log\frac{M^M}{m^m} - \log m - 1\right) \\
\gamma &= [m,M,M]\mathrm{id}\cdot\log\frac{1}{M-m}\left([(\mathrm{id}\cdot\log)(m)]'_- - \frac{M\log M - m\log m}{M-m}\right) \\
&= \frac{1}{M-m}\left(\log M + 1 - \frac{1}{M-m}\log\frac{M^M}{m^m}\right).
\end{aligned}$$

Now, as a special case of Theorem 3.5 and Theorem 3.6 for $f(t) = t\log t$, taking into account convexity of the function f, we have obtained Jensen and Edmundson-Lah-Ribarič type inequalities for Kullback-Leibler divergence $D_{KL}(\boldsymbol{p},\boldsymbol{q})$.

Corollary 3.5 ([105]) *Let $[m,M] \subset \mathbb{R}$ be an interval such that $m \le 1 \le M$. Let $\boldsymbol{p} = (p_1,\ldots,p_n)$ and $\boldsymbol{p} = (q_1,\ldots,q_n)$ be probability distributions such that $p_i/q_i \in [m,M]$ for every $i = 1,\ldots,n$. Then we have*

$$\begin{aligned}
0 &\le \frac{1}{M-m}\left(\log M + 1 - \frac{1}{M-m}\log\frac{M^M}{m^m}\right)\sum_{i=1}^{n} q_i\left(M - \frac{p_i}{q_i}\right)\left(\frac{p_i}{q_i} - m\right) \\
&\le \frac{mM}{M-m}\log\frac{m}{M} + \frac{1}{M-m}\log\frac{M^M}{m^m} - D_{KL}(\boldsymbol{p},\boldsymbol{q}) \qquad (3.23)\\
&\le \frac{1}{M-m}\left(\frac{1}{M-m}\log\frac{M^M}{m^m} - \log m - 1\right)\sum_{i=1}^{n} q_i\left(M - \frac{p_i}{q_i}\right)\left(\frac{p_i}{q_i} - m\right) \\
&\le \frac{1}{M-m}\left(\frac{1}{M-m}\log\frac{M^M}{m^m} - \log m - 1\right)(M-1)(1-m) \\
&\le \frac{1}{4}\left(\log\frac{M^M}{m^m} - (\log m + 1)(M-m)\right).
\end{aligned}$$

Corollary 3.6 ([105]) *Let $[m,M] \subset \mathbb{R}$ be an interval such that $m \le 1 \le M$. Let $\boldsymbol{p} = (p_1,\ldots,p_n)$ and $\boldsymbol{p} = (q_1,\ldots,q_n)$ be probability distributions such that $p_i/q_i \in [m,M]$ for every $i = 1,\ldots,n$. Then we have*

$$\begin{aligned}
0 \le D_{KL}(\boldsymbol{p},\boldsymbol{q}) &\le \frac{1}{M-m}\left[\left(\frac{1}{M-m}\log\frac{M^M}{m^m} - \log m - 1\right)(M-1)(1-m)\right. \\
&\left. - \left(\log M + 1 - \frac{1}{M-m}\log\frac{M^M}{m^m}\right)\sum_{i=1}^{n} q_i\left(M - \frac{p_i}{q_i}\right)\left(\frac{p_i}{q_i} - m\right)\right]. \qquad (3.24)
\end{aligned}$$

The function $f(t) = -\log t$ is also convex, and its second order divided difference $[m,t,M]f$ is a continuous and decreasing function, which means that it attains its maximal and minimal value in the points m and M respectively.

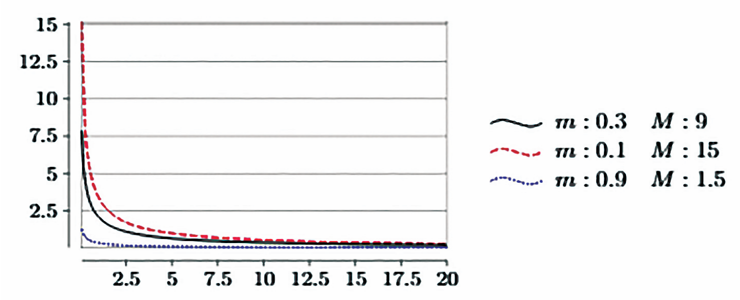

Figure 3.2: Graphs of the function $-[m,t,M]\log$ for different choices of the points m and M.

We calculate the bounds for the second order divided difference of the function $f(t) = -\log t$:

$$\Gamma = -[m,m,M]\log = \frac{1}{M-m}\left(\frac{-\log M + \log m}{M-m} - (-\log)'_{+}(m)\right)$$
$$= \frac{1}{M-m}\left(\frac{1}{M-m}\log\frac{m}{M} + \frac{1}{m}\right)$$
$$\gamma = -[m,M,M]\log = \frac{1}{M-m}\left((-\log)'_{-}(M) - \frac{-\log M + \log m}{M-m}\right)$$
$$= -\frac{1}{M-m}\left(\frac{1}{M} + \frac{1}{M-m}\log\frac{m}{M}\right).$$

As a special case of Theorem 3.5 and Theorem 3.6 for $f(t) = -\log t$, taking into account convexity of the function f, we get Jensen and Edmundson-Lah-Ribarič type inequalities for the reversed Kullback-Leibler divergence $D_{KL}(\boldsymbol{q},\boldsymbol{p})$.

Corollary 3.7 ([105]) *Let $[m,M] \subset \mathbb{R}$ be an interval such that $m \le 1 \le M$. Let $\boldsymbol{p} = (p_1,\ldots,p_n)$ and $\boldsymbol{p} = (q_1,\ldots,q_n)$ be probability distributions such that $p_i/q_i \in [m,M]$ for every $i = 1,\ldots,n$. Then we have*

$$\begin{aligned}
0 &\le -\frac{1}{M-m}\left(\frac{1}{M} + \frac{1}{M-m}\log\frac{m}{M}\right)\sum_{i=1}^{n} q_i\left(M - \frac{p_i}{q_i}\right)\left(\frac{p_i}{q_i} - m\right) \\
&\le \frac{1}{M-m}\log\frac{M^m}{m^M} + \frac{1}{M-m}\log\frac{m}{M} - D_{KL}(\boldsymbol{q},\boldsymbol{p}) \\
&\le \frac{1}{M-m}\left(\frac{1}{M-m}\log\frac{m}{M} + \frac{1}{m}\right)\sum_{i=1}^{n} q_i\left(M - \frac{p_i}{q_i}\right)\left(\frac{p_i}{q_i} - m\right) \\
&\le \frac{1}{M-m}\left(\frac{1}{M-m}\log\frac{m}{M} + \frac{1}{m}\right)(M-1)(1-m) \\
&\le \frac{1}{4}\left(\log\frac{m}{M} + \frac{M}{m} - 1\right).
\end{aligned} \tag{3.25}$$

Corollary 3.8 ([105]) *Let $[m,M] \subset \mathbb{R}$ be an interval such that $m \leq 1 \leq M$. Let $\boldsymbol{p} = (p_1,\dots,p_n)$ and $\boldsymbol{p} = (q_1,\dots,q_n)$ be probability distributions such that $p_i/q_i \in [m,M]$ for every $i = 1,\dots,n$. Then we have*

$$
\begin{aligned}
0 \leq D_{KL}(\boldsymbol{q},\boldsymbol{p}) \leq \frac{1}{M-m}\Bigg[&\left(\frac{1}{M-m}\log\frac{m}{M}+\frac{1}{m}\right)(M-1)(1-m) \\
&+\left(\frac{1}{M}+\frac{1}{M-m}\log\frac{m}{M}\right)\sum_{i=1}^{n} q_i\left(M-\frac{p_i}{q_i}\right)\left(\frac{p_i}{q_i}-m\right)\Bigg]. \qquad (3.26)
\end{aligned}
$$

Following results are applications of Theorem 2.5 and Theorem 2.6, and they provide us with an Edmundson-Lah-Ribarič type and Jensen type inequality respectively for the generalized f-divergence functional for 3-convex function.

Theorem 3.7 ([103]) *Let $[m,M] \subset \mathbb{R}$ be an interval such that $m \leq 1 \leq M$ and let $f\colon [m,M] \to \mathbb{R}$ be a 3-convex function. Let $\boldsymbol{p} = (p_1,\dots,p_n)$ and $\boldsymbol{p} = (q_1,\dots,q_n)$ be probability distributions such that $p_i/q_i \in [m,M]$ for every $i = 1,\dots,n$. Then we have*

$$
\begin{aligned}
&\frac{1}{M-m}\sum_{i=1}^{n} q_i\left(M-\frac{p_i}{q_i}\right)\left(\frac{p_i}{q_i}-m\right)\left(\frac{f(M)-f(m)}{M-m}-f'_+(m)\right) \\
&\leq \frac{M-1}{M-m}f(m)+\frac{1-m}{M-m}f(M)-\tilde{D}_f(\boldsymbol{p},\boldsymbol{q}) \qquad (3.27) \\
&\leq \frac{1}{M-m}\sum_{i=1}^{n} q_i\left(M-\frac{p_i}{q_i}\right)\left(\frac{p_i}{q_i}-m\right)\left(f'_-(M)-\frac{f(M)-f(m)}{M-m}\right)
\end{aligned}
$$

Proof. Let $\boldsymbol{x} = (x_1,\dots,x_n)$ such that $x_i \in [m,M]$ for $i = 1,\dots,n$. For a 3-convex function ϕ, in the relation (2.21) we can replace

$$f \longleftrightarrow \boldsymbol{x}, \quad \text{and} \quad A(\boldsymbol{x}) = \sum_{i=1}^{n} p_i x_i.$$

In that way we get

$$
\begin{aligned}
&\frac{\sum_{i=1}^{n} p_i(M-x_i)(x_i-m)}{M-m}\left(\frac{\phi(M)-\phi(m)}{M-m}-\phi'_+(m)\right) \\
&\leq \frac{M-\bar{x}}{M-m}\phi(m)+\frac{\bar{x}-m}{M-m}\phi(M)-\sum_{i=1}^{n} p_i\phi(x_i) \\
&\leq \frac{\sum_{i=1}^{n} p_i(M-x_i)(x_i-m)}{M-m}\left(\phi'_-(M)-\frac{\phi(M)-\phi(m)}{M-m}\right),
\end{aligned}
$$

where $\bar{x} = \sum_{i=1}^{n} p_i x_i$. Since the function f is 3-convex, in the previous relation we can set

$$\phi = f, \quad p_i = q_i \quad \text{and} \quad x_i = \frac{p_i}{q_i},$$

and after calculating

$$\bar{x} = \sum_{i=1}^{n} q_i\frac{p_i}{q_i} = \sum_{i=1}^{n} p_i = 1$$

we get (3.27). □

Theorem 3.8 ([103]) *Let $[m,M] \subset \mathbb{R}$ be an interval such that $m \leq 1 \leq M$ and let $f\colon [m,M] \to \mathbb{R}$ be a 3-convex function. Let $\boldsymbol{p} = (p_1,\ldots,p_n)$ and $\boldsymbol{p} = (q_1,\ldots,q_n)$ be probability distributions such that $p_i/q_i \in [m,M]$ for every $i = 1,\ldots,n$. Then we have*

$$\begin{aligned}
&\frac{(M-1)(1-m)}{M-m}\left(\frac{f(M)-f(m)}{M-m} - f'_+(m)\right) \\
&\quad - \frac{1}{M-m}\sum_{i=1}^{n} q_i\left(M-\frac{p_i}{q_i}\right)\left(\frac{p_i}{q_i}-m\right)\left(f'_-(M)-\frac{f(M)-f(m)}{M-m}\right) \\
&\leq \tilde{D}_f(\boldsymbol{p},\boldsymbol{q}) - f(1) \leq \frac{(M-1)(1-m)}{M-m}\left(f'_-(M)-\frac{f(M)-f(m)}{M-m}\right) \\
&\quad - \frac{1}{M-m}\sum_{i=1}^{n} q_i\left(M-\frac{p_i}{q_i}\right)\left(\frac{p_i}{q_i}-m\right)\left(\frac{f(M)-f(m)}{M-m} - f'_+(m)\right).
\end{aligned} \tag{3.28}$$

Proof. As in the proof of the previous theorem, let $\boldsymbol{x} = (x_1,\ldots,x_n)$ such that $x_i \in [m,M]$ for $i = 1,\ldots,n$. For a 3-convex function ϕ, in the relation (2.25) we can replace

$$f \longleftrightarrow \boldsymbol{x}, \quad \text{and} \quad A(\boldsymbol{x}) = \sum_{i=1}^{n} p_i x_i$$

and obtain the following discrete sequence of inequalities:

$$\begin{aligned}
&\frac{(M-\bar{x})(\bar{x}-m)}{M-m}\left(\frac{\phi(M)-\phi(m)}{M-m} - \phi'_+(m)\right) \\
&\quad - \frac{\sum_{i=1}^{n} p_i(M-x_i)(x_i-m)}{M-m}\left(\phi'_-(M)-\frac{\phi(M)-\phi(m)}{M-m}\right) \\
&\leq \sum_{i=1}^{n} p_i\phi(x_i) - \phi(\bar{x}) \leq \frac{(M-\bar{x})(\bar{x}-m)}{M-m}\left(\phi'_-(M)-\frac{\phi(M)-\phi(m)}{M-m}\right) \\
&\quad - \frac{\sum_{i=1}^{n} p_i(M-x_i)(x_i-m)}{M-m}\left(\frac{\phi(M)-\phi(m)}{M-m} - \phi'_+(m)\right).
\end{aligned}$$

The function f is 3-convex, so in the previous relation we can set

$$\phi = f, \quad p_i = q_i \quad \text{and} \quad x_i = \frac{p_i}{q_i},$$

and after calculating

$$\bar{x} = \sum_{i=1}^{n} q_i \frac{p_i}{q_i} = \sum_{i=1}^{n} p_i = 1$$

we get (3.28). □

Next two results are obtained as an application of Theorem 2.7 and Theorem 2.8 respectively, and they give us different Edmundson-Lah-Ribarič type inequalities for the generalized f-divergence functional.

Theorem 3.9 ([106]) *Let $[m,M] \subset \mathbb{R}$ be an interval such that $m \leq 1 \leq M$. Let f be a 3-convex function on the interval I whose interior contains $[m,M]$ and differentiable on $\langle m,M \rangle$. Let $\boldsymbol{p} = (p_1,...,p_n)$ and $\boldsymbol{p} = (q_1,...,q_n)$ be probability distributions such that $p_i/q_i \in [m,M]$ for every $i = 1,...,n$. Then we have*

$$
\begin{aligned}
&(1-m)\left[\frac{f(M)-f(m)}{M-m} - \frac{f'_+(m)}{2}\right] - \frac{1}{2}\sum_{i=1}^{n}(p_i - mq_i)f'\left(\frac{p_i}{q_i}\right) \\
&\leq \frac{M-1}{M-m}f(m) + \frac{1-m}{M-m}f(M) - \tilde{D}_f(\boldsymbol{p},\boldsymbol{q}) \qquad (3.29)\\
&\leq \frac{1}{2}\sum_{i=1}^{n}(Mq_i - p_i)f'\left(\frac{p_i}{q_i}\right) - (M-1)\left[\frac{f(M)-f(m)}{M-m} - \frac{f'_-(M)}{2}\right].
\end{aligned}
$$

Proof. Let $\boldsymbol{x} = (x_1,...,x_n)$ such that $x_i \in [m,M]$ for $i = 1,...,n$. Let ϕ be a 3-convex function on the interval I whose interior contains $[m,M]$ and differentiable on $\langle m,M \rangle$. In the relation (2.28) we can replace

$$f \longleftrightarrow \boldsymbol{x}, \text{ and } A(\boldsymbol{x}) = \sum_{i=1}^{n} p_i x_i.$$

In that way we get

$$
\begin{aligned}
&(\overline{x}-m)\left[\frac{\phi(M)-\phi(m)}{M-m} - \frac{\phi'_+(m)}{2}\right] - \frac{1}{2}\sum_{i=1}^{n} p_i(x_i - m)\phi'(x_i) \\
&\leq \frac{M-\overline{x}}{M-m}\phi(m) + \frac{\overline{x}-m}{M-m}\phi(M) - \sum_{i=1}^{n} p_i\phi(x_i) \\
&\leq \frac{1}{2}\sum_{i=1}^{n} p_i(M - x_i)\phi'(x_i) - (M-\overline{x})\left[\frac{\phi(M)-\phi(m)}{M-m} - \frac{\phi'_-(M)}{2}\right]
\end{aligned}
$$

where $\overline{x} = \sum_{i=1}^{n} p_i x_i$. Since the function f satisfies the same assumtions as ϕ, in the previous relation we can set

$$\phi = f, \ \ p_i = q_i \ \text{ and } \ x_i = \frac{p_i}{q_i},$$

and after calculating

$$\overline{x} = \sum_{i=1}^{n} q_i \frac{p_i}{q_i} = \sum_{i=1}^{n} p_i = 1$$

we get (3.29). □

By utilizing Theorem 2.8 in the analogous way as above, we get a different Edmundson-Lah-Ribarič type inequality for the generalized f-divergence functional, and it is given in the following theorem.

Theorem 3.10 ([106]) *Let $[m,M] \subset \mathbb{R}$ be an interval such that $m \leq 1 \leq M$. Let f be a 3-convex function on the interval I whose interior contains $[m,M]$ and differentiable on $\langle m,M \rangle$. Let $\boldsymbol{p} = (p_1,...,p_n)$ and $\boldsymbol{p} = (q_1,...,q_n)$ be probability distributions such that $p_i/q_i \in [m,M]$ for every $i = 1,...,n$. Then we have*

$$
\begin{aligned}
&(M-1)\left[f'_{-}(M)-\frac{f(M)-f(m)}{M-m}\right]-\frac{f''_{-}(M)}{2}\sum_{i=1}^{n}\frac{(Mq_i-p_i)^2}{q_i}\\
&\le \frac{M-1}{M-m}f(m)+\frac{1-m}{M-m}f(M)-\tilde{D}_f(\boldsymbol{p},\boldsymbol{q}) \qquad (3.30)\\
&\le (1-m)\left[\frac{f(M)-f(m)}{M-m}-f'_{+}(m)\right]-\frac{f''_{+}(m)}{2}\sum_{i=1}^{n}\frac{(p_i-mq_i)^2}{q_i}.
\end{aligned}
$$

Remark 3.7 Let $\boldsymbol{p}=(p_1,...,p_n)$ and $\boldsymbol{p}=(q_1,...,q_n)$ be probability distributions and let $[m,M]\subset\mathbb{R}$ be an interval such that $m\le 1\le M$ and $p_i/q_i\in[m,M]$ for every $i=1,...,n$.

▷ **Kullback-Leibler divergence** of the probability distributions $\boldsymbol{p}$ and $\boldsymbol{q}$ is defined by means of the generating function $f(t)=t\log t$, $t>0$. We can calculate $f'''(t)=-\frac{1}{t^2}<0$, so the function $-f(t)=-t\log t$ is 3-convex. It is obvious that for the Kullback-Leibler divergence the inequalities (3.27), (3.28), (3.29) and (3.30) hold with reversed signs of inequality, with

$$f'_{+}(m)=\log m+1,\ \ f'_{-}(M)=\log M+1$$

and

$$f''_{+}(m)=\frac{1}{m},\ \ f''_{-}(M)=\frac{1}{M}.$$

▷ **Hellinger divergence** of the probability distributions $\boldsymbol{p}$ and $\boldsymbol{q}$ is defined by means of the generating function $f(t)=\frac{1}{2}(1-\sqrt{t})^2$, $t>0$. We see that $f'''(t)=-\frac{3}{8}t^{-\frac{5}{2}}<0$, so the function $-f(t)=-\frac{1}{2}(1-\sqrt{t})^2$ is 3-convex. For the Hellinger divergence the inequalities (3.27), (3.28), (3.29) and (3.30) hold with reversed signs of inequality, with

$$f'_{+}(m)=-\frac{1}{2\sqrt{m}}+\frac{1}{2},\ \ f'_{-}(M)=-\frac{1}{2\sqrt{M}}+\frac{1}{2}$$

and

$$f''_{+}(m)=\frac{1}{4\sqrt{m^3}},\ \ f''_{-}(M)=\frac{1}{4\sqrt{M^3}}.$$

▷ **Renyi divergence** of the probability distributions $\boldsymbol{p}$ and $\boldsymbol{q}$ is defined via the generating function is $f(t)=t^{\alpha}$, $t>0$. We calculate that $f'''(t)=\alpha(\alpha-1)(\alpha-2)t^{\alpha-3}$, which is 3-convex for $0\le\alpha\le 1$ and $\alpha\ge 2$, and $-f(t)=-t^{\alpha}$ is 3-convex for $\alpha\le 0$ and $1<\alpha<2$. We have

$$f'_{+}(m)=\alpha m^{\alpha-1},\ \ f'_{-}(M)=\alpha M^{\alpha-1},$$

$$f''_{+}(m)=\alpha(\alpha-1)m^{\alpha-2}\ \text{ and }\ f''_{-}(M)=\alpha(\alpha-1)M^{\alpha-2}.$$

As regards the Renyi divergence, the inequalities (3.27), (3.28), (3.29) and (3.30) hold for $0\le\alpha\le 1$ and $\alpha\ge 2$, and if $\alpha\le 0$ or $1<\alpha<2$ the signs of inequality are reversed.

▷ **Harmonic divergence** of the probability distributions $\boldsymbol{p}$ and $\boldsymbol{q}$ is defined using the generating function $f(t) = \frac{2t}{1+t}$. We can calculate $f'''(t) = \frac{12}{(1+t)^4} > 0$, so the function f is 3-convex. It is obvious that for the harmonic divergence the inequalities (3.27), (3.28), (3.29) and (3.30) hold with

$$f'_+(m) = \frac{2}{(1+m)^2}, \quad f'_-(M) = \frac{2}{(1+M)^2}$$

and

$$f''_+(m) = -\frac{4}{(1+m)^3}, \quad f''_-(M) = -\frac{4}{(1+M)^3}.$$

▷ **Jeffrey divergence** of the probability distributions $\boldsymbol{p}$ and $\boldsymbol{q}$ is defined using the generating function $f(t) = (1-t)\log\frac{1}{t}$, $t > 0$. We see that $f'''(t) = -\frac{1}{t^2} - \frac{2}{t^3} < 0$, so the function $-f(t) = (1-t)\log t$ is 3-convex, and we instantly get that for the Jeffrey divergence the inequalities (3.27), (3.28), (3.29) and (3.30) hold with reversed signs of inequality, with

$$f'_+(m) = \log m - \frac{1}{m} + 1, \quad f'_-(M) = \log M - \frac{1}{M} + 1$$

and

$$f''_+(m) = \frac{1}{m} + \frac{1}{m^2}, \quad f''_-(M) = \frac{1}{M} + \frac{1}{M^2}.$$

The results that follow are a generalization of the previous results which hold for the class of 3-convex functions. Until the end of this section, when mentioning the interval $[a,b]$, we assume that $[a,b] \subseteq \mathbb{R}_+$.

We can utilize Theorem 2.9 to get an Edmundson-Lah-Ribarič type inequality for the above defined generalized f-divergence functional.

Theorem 3.11 *Let $[a,b] \subset \mathbb{R}$ be an interval such that $a \leq 1 \leq b$. Let $f \in \mathscr{C}^n([a,b])$ and let $\boldsymbol{p} = (p_1,...,p_r)$ and $\boldsymbol{p} = (q_1,...,q_r)$ be probability distributions such that $p_i/q_i \in [a,b]$ for every $i = 1,...,r$. If the function f is n-convex and if n and $3 \leq m \leq n-1$ are of different parity, then*

$$\begin{aligned} &\frac{b-1}{b-a} f(a) + \frac{1-a}{b-a} f(b) - \tilde{D}_f(\boldsymbol{p},\boldsymbol{q}) \\ &\quad \leq (1-a)\left(f[a,a] - f[a,b]\right) + \sum_{k=2}^{m-1} \frac{f^{(k)}(a)}{k!} \sum_{i=1}^{r} \frac{(p_i - aq_i)^k}{q_i^{k-1}} \\ &\qquad + \sum_{k=1}^{n-m} f[\underbrace{a,...,a}_{m\ times}; \underbrace{b,...,b}_{k\ times}] \sum_{i=1}^{r} \frac{(p_i - aq_i)^m (p_i - aq_i)^{k-1}}{q_i^{m+k-2}}. \end{aligned} \tag{3.31}$$

Inequality (3.31) also holds when the function f is n-concave and n and m are of equal parity. In case when the function f is n-convex and n and m are of equal parity, or when the function f is n-concave and n and m are of different parity, the inequality sign in (3.31) is reversed.

Proof. Let $\boldsymbol{x} = (x_1, \ldots, x_r)$ be such that $x_i \in [a,b]$ for $i = 1, \ldots, r$. In the relation (2.70) we can replace

$$g \longleftrightarrow \boldsymbol{x}, \ \text{ and } \ A(\boldsymbol{x}) = \sum_{i=1}^{r} p_i x_i.$$

In that way we get

$$\begin{aligned}
&\frac{b-\bar{x}}{b-a} f(a) + \frac{\bar{x}-a}{b-a} f(b) - \sum_{i=1}^{r} p_i f(x_i) \\
&\leq (\bar{x}-a)\left(f[a,a] - f[a,b]\right) + \sum_{k=2}^{m-1} \frac{f^{(k)}(a)}{k!} \sum_{i=1}^{r} p_i (x_i - a)^k \\
&\quad + \sum_{k=1}^{n-m} f[\underbrace{a,\ldots,a}_{m \text{ times}}; \underbrace{b,\ldots,b}_{k \text{ times}}] \sum_{i=1}^{r} p_i (x_i - a)^m (x_i - b)^{k-1},
\end{aligned}$$

where $\bar{x} = \sum_{i=1}^{n} p_i x_i$. In the previous relation we can set

$$p_i = q_i \ \text{ and } \ x_i = \frac{p_i}{q_i},$$

and after calculating

$$\bar{x} = \sum_{i=1}^{n} q_i \frac{p_i}{q_i} = \sum_{i=1}^{n} p_i = 1$$

we get (3.31). □

By utilizing Theorem 2.10 in the analogous way as above, we get an Edmundson-Lah-Ribarič type inequality for the generalized f-divergence functional (3.3) which does not depend on parity of n, and it is given in the following theorem.

Theorem 3.12 *Let $[a,b] \subset \mathbb{R}$ be an interval such that $a \leq 1 \leq b$. Let $f \in \mathscr{C}^n([a,b])$ and let $\boldsymbol{p} = (p_1, \ldots, p_r)$ and $\boldsymbol{p} = (q_1, \ldots, q_r)$ be probability distributions such that $p_i/q_i \in [a,b]$ for every $i = 1, \ldots, r$. If the function f is n-convex and if $3 \leq m \leq n-1$ is odd, then*

$$\begin{aligned}
&\frac{b-1}{b-a} f(a) + \frac{1-a}{b-a} f(b) - \tilde{D}_f(\boldsymbol{p}, \boldsymbol{q}) \\
&\leq (b-1)\left(f[a,b] - f[b,b]\right) + \sum_{k=2}^{m-1} \frac{f^{(k)}(b)}{k!} \sum_{i=1}^{r} \frac{(p_i - bq_i)^k}{q_i^{k-1}} \\
&\quad + \sum_{k=1}^{n-m} f[\underbrace{b,\ldots,b}_{m \text{ times}}; \underbrace{a,\ldots,a}_{k \text{ times}}] \sum_{i=1}^{r} \frac{(p_i - bq_i)^m (p_i - aq_i)^{k-1}}{q_i^{m+k-2}}
\end{aligned} \tag{3.32}$$

Inequality (3.32) also holds when the function f is n-concave and m is even. In case when the function f is n-convex and m is even, or when the function f is n-concave and m is odd, the inequality sign in (3.32) is reversed.

Another generalization of the Edmundson-Lah-Ribarič inequality, which provides with a lower and an upper bound for the generalized f-divergence functional, is given in the following theorem.

Theorem 3.13 *Let $[a,b] \subset \mathbb{R}$ be an interval such that $a \leq 1 \leq b$. Let $f \in \mathscr{C}^n([a,b])$ and let $\boldsymbol{p} = (p_1,...,p_r)$ and $\boldsymbol{p} = (q_1,...,q_r)$ be probability distributions such that $p_i/q_i \in [a,b]$ for every $i = 1,...,r$. If the function f is n-convex and if n is odd, then we have*

$$\sum_{k=2}^{n-1} f[a;\underbrace{b,b,...,b}_{k\ times}] \sum_{i=1}^{r} \frac{(p_i - aq_i)(p_i - bq_i)^{k-1}}{q_i^{k-1}} \leq \frac{b-1}{b-a} f(a) + \frac{1-a}{b-a} f(b) - \tilde{D}_f(\boldsymbol{p},\boldsymbol{q})$$
$$\leq f[a,a;b] \sum_{i=1}^{r} \frac{(p_i - aq_i)(p_i - bq_i)}{q_i} + \sum_{k=2}^{n-2} f[a,a;\underbrace{b,...,b}_{k\ times}] \sum_{i=1}^{r} \frac{(p_i - aq_i)^2 (p_i - bq_i)^{k-1}}{q_i^k}. \tag{3.33}$$

Inequalities (3.33) also hold when the function f is n-concave and n is even. In case when the function f is n-convex and n is even, or when the function f is n-concave and n is odd, the inequality signs in (3.33) are reversed.

Proof. We start with inequalities (2.73) from Theorem 2.11, and follow the steps from the proof of Theorem 3.11. □

By utilizing Theorem 2.12 in an analogous way, we can get similar bounds for the generalized f-divergence functional that hold for all $n \in \mathbb{N}$, not only the odd ones.

Theorem 3.14 *Let $[a,b] \subset \mathbb{R}$ be an interval such that $a \leq 1 \leq b$. Let $f \in \mathscr{C}^n([a,b])$ and let $\boldsymbol{p} = (p_1,...,p_r)$ and $\boldsymbol{p} = (q_1,...,q_r)$ be probability distributions such that $p_i/q_i \in [a,b]$ for every $i = 1,...,r$. If the function f is n-convex, then we have*

$$f[b,b;a] \sum_{i=1}^{r} \frac{(p_i - aq_i)(p_i - bq_i)}{q_i} + \sum_{k=2}^{n-2} f[b,b;\underbrace{a,a,...,a}_{k\ times}] \sum_{i=1}^{r} \frac{(p_i - aq_i)^{k-1} (p_i - bq_i)^2}{q_i^k}$$
$$\leq \frac{b-1}{b-a} f(a) + \frac{1-a}{b-a} f(b) - \tilde{D}_f(\boldsymbol{p},\boldsymbol{q}) \leq \sum_{k=2}^{n-1} f[b;\underbrace{a,...,a}_{k\ times}] \sum_{i=1}^{r} \frac{(p_i - aq_i)^{k-1}(p_i - bq_i)}{q_i^{k-1}}. \tag{3.34}$$

If the function f is n-concave, the inequality signs in (3.34) are reversed.

3.3 Applications to Zipf-Mandelbrot law

The Zipf-Mandelbrot law is a discrete probability distribution with parameters $N \in \mathbb{N}$, $q,s \in \mathbb{R}$ such that $q \geq 0$ and $s > 0$, possible values $\{1,2,...,N\}$ and probability mass function

$$f(i;N,q,s) = \frac{1/(i+q)^s}{H_{N,q,s}}, \quad \text{where} \quad H_{N,q,s} = \sum_{i=1}^{N} \frac{1}{(i+q)^s}.$$

It is used in various scientific fields: linguistics [115], information sciences [43, 129], ecological field studies [116] and music [94]. Benoit Mandelbrot in 1966 gave improvement of the Zipf law for the count of the low-rank words. Various scientific fields use this law for different purposes, for example information sciences use it for indexing [43, 129], ecological field studies in predictability of ecosystem [116], in music is used to determine aesthetically pleasing music [94].

Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be Zipf-Mandelbrot laws with parameters $N \in \mathbb{N}$, $q_1, q_2 \geq 0$ and $s_1, s_2 > 0$ respectively. We can use Corollary 3.2 and Corollary 3.3 in a similar way as described above in order to obtain inequalities for the Kullback-Leibler divergence. Let us denote

$$
\begin{aligned}
&H_{N,q_1,s_1} = H_1,\ H_{N,q_2,s_2} = H_2 \\
&m_{\boldsymbol{p},\boldsymbol{q}} := \min\left\{\frac{p_i}{q_i}\right\} = \frac{H_2}{H_1}\min\left\{\frac{(i+q_2)^{s_2}}{(i+q_1)^{s_1}}\right\} \\
&M_{\boldsymbol{p},\boldsymbol{q}} := \max\left\{\frac{p_i}{q_i}\right\} = \frac{H_2}{H_1}\max\left\{\frac{(i+q_2)^{s_2}}{(i+q_1)^{s_1}}\right\}
\end{aligned} \tag{3.35}
$$

Corollary 3.9 ([104]) *Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be Zipf-Mandelbrot laws with parameters $N \in \mathbb{N}$, $q_1, q_2 \geq 0$ and $s_1, s_2 > 0$ respectively. If the base of the logarithm is greater than one, we have*

$$
\begin{aligned}
0 &\leq D_{KL}(\boldsymbol{p},\boldsymbol{q}) \\
&\leq (M_{\boldsymbol{p},\boldsymbol{q}}-1)(1-m_{\boldsymbol{p},\boldsymbol{q}}) \sup_{t\in\langle m_{\boldsymbol{p},\boldsymbol{q}},M_{\boldsymbol{p},\boldsymbol{q}}\rangle} \Psi_{id\cdot\log}(t;m_{\boldsymbol{p},\boldsymbol{q}},M_{P,Q}) - \Delta_{\boldsymbol{p},\boldsymbol{q}} \\
&\leq \frac{1}{M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}}}(M_{\boldsymbol{p},\boldsymbol{q}}-1)(1-m_{\boldsymbol{p},\boldsymbol{q}})\log\frac{M_{\boldsymbol{p},\boldsymbol{q}}}{m_{\boldsymbol{p},\boldsymbol{q}}} - \Delta_{\boldsymbol{p},\boldsymbol{q}} \\
&\leq \frac{1}{4}(M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}})\log\frac{M_{\boldsymbol{p},\boldsymbol{q}}}{m_{\boldsymbol{p},\boldsymbol{q}}} - \Delta_{\boldsymbol{p},\boldsymbol{q}}
\end{aligned} \tag{3.36}
$$

and

$$
\begin{aligned}
\Delta_{\boldsymbol{p},\boldsymbol{q}} &\leq \frac{M_{\boldsymbol{p},\boldsymbol{q}}m_{\boldsymbol{p},\boldsymbol{q}}}{M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}}}\log\left(\frac{m_{\boldsymbol{p},\boldsymbol{q}}}{M_{\boldsymbol{p},\boldsymbol{q}}}\right) + \frac{1}{M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}}}\log\left(\frac{M_{\boldsymbol{p},\boldsymbol{q}}^{M_{\boldsymbol{p},\boldsymbol{q}}}}{m_{\boldsymbol{p},\boldsymbol{q}}^{m_{\boldsymbol{p},\boldsymbol{q}}}}\right) - D_{KL}(\boldsymbol{p},\boldsymbol{q}) \\
&\leq (M_{\boldsymbol{p},\boldsymbol{q}}-1)(1-m_{\boldsymbol{p},\boldsymbol{q}}) \sup_{t\in\langle m_{\boldsymbol{p},\boldsymbol{q}},M_{\boldsymbol{p},\boldsymbol{q}}\rangle} \Psi_{id\cdot\log}(t;m_{\boldsymbol{p},\boldsymbol{q}},M_{\boldsymbol{p},\boldsymbol{q}}) \\
&\leq \frac{1}{M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}}}(M_{\boldsymbol{p},\boldsymbol{q}}-1)(1-m_{\boldsymbol{p},\boldsymbol{q}})\log\left(\frac{M_{\boldsymbol{p},\boldsymbol{q}}}{m_{\boldsymbol{p},\boldsymbol{q}}}\right) \leq \frac{1}{4}(M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}})\log\left(\frac{M_{\boldsymbol{p},\boldsymbol{q}}}{m_{\boldsymbol{p},\boldsymbol{q}}}\right),
\end{aligned} \tag{3.37}
$$

where $D_{KL}(\boldsymbol{p},\boldsymbol{q})$ is the Kullback-Leibler divergence of distributions $\boldsymbol{p}$ and $\boldsymbol{q}$, $m_{\boldsymbol{p},\boldsymbol{q}}$ and $M_{\boldsymbol{p},\boldsymbol{q}}$ are defined in (3.35), and

$$
\begin{aligned}
\Delta_{\boldsymbol{p},\boldsymbol{q}} = &\left(\frac{1}{2} - \frac{1}{M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}}}\sum_{i=1}^{N}\left|\frac{1}{H_1(i+q_1)^{s_1}} - \frac{m_{\boldsymbol{p},\boldsymbol{q}}+M_{\boldsymbol{p},\boldsymbol{q}}}{2}\cdot\frac{1}{H_2(i+q_2)^{s_2}}\right|\right) \\
&\times\left(m_{\boldsymbol{p},\boldsymbol{q}}\log\frac{2m_{\boldsymbol{p},\boldsymbol{q}}}{m_{\boldsymbol{p},\boldsymbol{q}}+M_{\boldsymbol{p},\boldsymbol{q}}} + M_{\boldsymbol{p},\boldsymbol{q}}\log\frac{2M_{\boldsymbol{p},\boldsymbol{q}}}{m_{\boldsymbol{p},\boldsymbol{q}}+M_{\boldsymbol{p},\boldsymbol{q}}}\right)
\end{aligned}
$$

Remark 3.8 If we utilize Remark 3.3 and Remark 3.4 in the same way as described above, we can obtain companion inequalities for the reversed Kullback-Leibler divergence $D_{KL}(\boldsymbol{q},\boldsymbol{p})$ of these distributions.

For finite N and $q=0$ the Zipf-Mandelbrot law becomes Zipf's law. It is one of the basic laws in information science and bibliometrics, but it is also often used in linguistics. George Zipf's in 1932 found that we can count how many times each word appears in the text. So if we ranked (r) word according to the frequency of word occurrence (f), the product of these two numbers is a constant $C = r*f$. Same law in mathematical sense is also used in other scientific disciplines, but name of the law can be different, since regularities in different scientific fields are discovered independently from each other. In economics same law or regularity are called Pareto's law which analyze and predicts the distribution of the wealthiest members of the community [36]. The same type of distribution that we have in Zipf's and Pareto's law, also known as the Power law, can be found in wide variety of scientific disciplines, such as: physics, biology, earth and planetary sciences, computer science, demography and the social sciences [117] and many others. At this point of time we will not explain usage and their importance of this law in each scientific field, but we will retain on frequency of the word usage. Since, words are one of basic properties in human communication system. That frequency of used word and human communication system can be explained with plain mathematical formula is extremely interesting and useful in analysis of language and their usage. Since this law is applicable in indexing and text mining, it is quite useful in today's world in which we use Internet to retrive most of the information that we need.

Probability mass function of Zipf's law is:

$$f(k;N,s) = \frac{1/k^s}{H_{N,s}}, \quad \text{where } H_{N,s} = \sum_{i=1}^{N} \frac{1}{i^s}.$$

Since Zipf's law is a special case of the Zipf-Mandelbrot law, all of the results from above hold for $q=0$.

Gelbukh and Sidorov in [50] observed the difference between the coefficients s_1 and s_2 in Zipf's law for the Russian and English language. They processed 39 literature texts for each language, chosen randomly from different genres, with the requirement that the size is greater than 10,000 running words each. They calculated coefficients for each of the mentioned texts and as the result they obtained the average of $s_1 = 0,892869$ for the Russian language, and $s_2 = 0,973863$ for the English language.

If we take $q_1 = q_2 = 0$, we can use the results from the above regarding the Kullback-Leibler divergence of two Zipf-Mandelbrot distributions in order to give estimates for the Kullback-Leibler divergence of the distributions associated to the Russian and English language. For those experimental values of s_1 and s_2 we have

$$m_N = \min\left\{\frac{p_i}{q_i}\right\} = \frac{H_{N,s_2}}{H_{N,s_1}} \min\left\{\frac{i^{s_2}}{i^{s_1}}\right\} = \frac{H_{N,s_2}}{H_{N,s_1}} \min\left\{i^{s_2-s_1}\right\} = \frac{H_{N,s_2}}{H_{N,s_1}}$$

and

$$M_N = \max\left\{\frac{p_i}{q_i}\right\} = \frac{H_{N,s_2}}{H_{N,s_1}} \max\left\{\frac{i^{s_2}}{i^{s_1}}\right\} = \frac{H_{N,s_2}}{H_{N,s_1}} \max\left\{i^{s_2-s_1}\right\} = \frac{H_{N,s_2}}{H_{N,s_1}} N^{0,080994}.$$

Hence the following bounds for the mentioned divergence, arising from Corollary 3.9 and depending only on the parameter N, hold.

$$\begin{aligned}
0 &\le D_{KL}(\boldsymbol{p},\boldsymbol{q}) \\
&\le (M_N-1)(1-m_N)\sup_{t\in\langle m_N,M_N\rangle}\Psi_{id\cdot\log}(t;m_N,M_N)-\Delta_N \\
&\le \frac{0,080994}{M_N-m_N}(M_N-1)(1-m_N)\log N-\Delta_N \\
&\le 0,020249(M_N-m_N)\log N-\Delta_N
\end{aligned}$$

We also have

$$\begin{aligned}
\Delta_N &\le \frac{0,080994N^{0,080994}}{N^{0,080994}-1}\left(1-\frac{H_{N;0,973863}}{H_{N;0,892869}}\right)\log N+\log\left(\frac{H_{N;0,973863}}{H_{N;0,892869}}\right)-D_{KL}(\boldsymbol{p},\boldsymbol{q}) \\
&\le (M_N-1)(1-m_N)\sup_{t\in\langle m_N,M_N\rangle}\Psi_{id\cdot\log}(t;m_N,M_N) \\
&\le \frac{0,080994}{M_N-m_N}(M_N-1)(1-m_N)\log N \le 0,020249(M_N-m_N)\log N,
\end{aligned}$$

where

$$\begin{aligned}
\Delta_N = &\left(\frac{1}{2}-\frac{1}{H_{N;0,973863}(N^{0,080994}-1)}\sum_{i=1}^{N}\left|\frac{1}{i^{0,892869}}-\frac{N^{0,080994}+1}{2i^{0,973863}}\right|\right) \\
&\times\left(\log\frac{2}{N^{0,080994}+1}+N^{0,080994}\log\frac{2N^{0,080994}}{N^{0,080994}+1}\right)\frac{H_{N;0,973863}}{H_{N;0,892869}}
\end{aligned}$$

By calculating the above results for the Kullback-Leibler divergence of the distributions associated to the Russian ($\boldsymbol{p}$) and English ($\boldsymbol{q}$) language for different values of the parameter N, we obtained the following bounds:

- from the first series of inequalities:

N	5000	10000	50000	100000
$D_{KL}(\boldsymbol{p},\boldsymbol{q}) \le$	0,0862934	0,100855	0,138862	0,157016

- from the second series of inequalities:

N	5000	10000	50000	100000
$D_{KL}(\boldsymbol{p},\boldsymbol{q}) \le$	0,00106	0,001274	0,0018269	0,002091

The base of the logarithm used in our calculations is 2.

Again, $\boldsymbol{p}$ and $\boldsymbol{q}$ are Zipf-Mandelbrot laws with parameters $N\in\mathbb{N}$, $q_1,q_2\ge 0$ and $s_1,s_2>0$ respectively, and let $m_{\boldsymbol{p},\boldsymbol{q}}$ and $M_{\boldsymbol{p},\boldsymbol{q}}$ be defined in (3.35).

In the following three results we will denote parameters in Zipf-Mandelbrot law as l, t_1, s_1. If we define $\mathbf{q}$ as a Zipf-Mandelbrot law l-tuple, we have

$$q_i = \frac{1}{(i+t_2)^{s_2} H_{l,s_2,t_2}}, \quad i = 1, \ldots, l$$

where

$$H_{l,s_2,t_2} = \sum_{i=1}^{l} \frac{1}{(k+t_2)^{s_2}}$$

and Csiszar functional becomes

$$\hat{D}_f(\mathbf{p}, i, l, s_2, t_2) = \sum_{i=1}^{l} \frac{1}{(i+t_2)^{s_2} H_{l,s_2,t_2}} f\left(p_i (i+t_2)^{s_2} H_{l,s_2,t_2}\right),$$

where $f\colon I \to \mathbb{R}, I \subseteq \mathbb{R}$, and the parameters $l \in \mathbb{N}$, $s_2 > 0$, $t_2 \geq 0$ are such that $p_i(i+t_2)^{s_2} H_{l,s_2,t_2} \in I$, $i = 1, \ldots, l$.

If $\mathbf{p}$ and $\mathbf{q}$ are both defined as Zipf-Mandelbrot law l-tuples, then Csiszar functional becomes

$$\hat{D}_f(i, l, s_1, s_2, t_1, t_2) = \sum_{i=1}^{l} \frac{1}{(i+t_2)^{s_2} H_{l,s_2,t_2}} f\left(\frac{(i+t_2)^{s_2} H_{l,s_2,t_2}}{(i+t_1)^{s_1} H_{l,s_1,t_1}}\right),$$

where $f\colon I \to \mathbb{R}, I \subseteq \mathbb{R}$, and the parameters $l \in \mathbb{N}$, $s_1, s_2 > 0$, $t_1, t_2 \geq 0$ are such that $\frac{(i+t_2)^{s_2} H_{l,s_2,t_2}}{(i+t_1)^{s_1} H_{l,s_1,t_1}} \in I$, $i = 1, \ldots, l$.

Since the minimal value for q_i is $\min\{q_i\} = \frac{1}{(l+t_2)^{s_2} H_{l,s_2,t_2}}$, then from Theorem 3.4 we have the following result.

Corollary 3.10 [122] *Let $\mathbf{p} = (\mathbf{p_1}, \ldots, \mathbf{p_l})$ be an l-tuple of real numbers with $P_l = \sum_{i=1}^{l} p_i$. Suppose $I \subseteq \mathbb{R}$ is an interval, $l \in \mathbb{N}$ and $s_2 > 0$, $t_2 \geq 0$ are such that $p_i(i+t_2)^{s_2} H_{l,s_2,t_2} \in I$, $i = 1, \ldots, l$. If $f\colon I \to \mathbb{R}$ is a convex function, then*

$$\begin{aligned}
\hat{D}_f(\boldsymbol{p}, i, l, s_2, t_2) &\leq \frac{M-P_l}{M-m} f(m) + \frac{P_l - m}{M-m} f(M) \\
&\quad - \sum_{n=0}^{N-1} \sum_{k=1}^{2^n} \sum_{i=1}^{l} \Delta_f(m, M, n, k) \frac{1}{(l+t_2)^{s_2} H_{l,s_2,t_2}} \left[\left(r_n \cdot \chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)}\right)\left(\frac{p_i (i+t_2)^{s_2} H_{M,s_2,t_2} - m}{M-m}\right)\right] \\
&\leq \frac{M-P_l}{M-m} f(m) + \frac{P_l - m}{M-m} f(M) - \sum_{n=0}^{N-1} \sum_{k=1}^{2^n} \sum_{i=1}^{l} \Delta_f(m, M, n, k) \frac{1}{M-m} \\
&\quad \left[(2^n (p_i - m \min\{q_i\}) - \min\{q_i\}(M-m)(k-1)) \cdot \chi_{\left(\frac{k-1}{2^n}, \frac{2k-1}{2^{n+1}}\right)} \left(\frac{p_i (i+t_2)^{s_2} H_{M,s_2,t_2} - m}{M-m}\right)\right. \\
&\quad \left. + \, (\min\{q_i\}(M-m)k - 2^n (p_i - m \min\{q_i\}))\, \chi_{\left(\frac{2k-1}{2^{n+1}}, \frac{k}{2^n}\right)} \left(\frac{p_i (i+t_2)^{s_2} H_{M,s_2,t_2} - m}{M-m}\right)\right]
\end{aligned} \tag{3.38}$$

Proof. Follows easily from Theorem 3.4. □

Now let's denote

$$H_{l,s_1,t_1} = H_1, H_{l,s_2,t_2} = H_2,$$
$$m_{\mathbf{p},\mathbf{q}} := \min\left\{\frac{p_i}{q_i}\right\} = \frac{H_2}{H_1}\min\left\{\frac{(i+q_2)^{s_2}}{(i+q_1)^{s_1}}\right\}.$$

Corollary 3.11 [122] *Let $I \subseteq \mathbb{R}$ be an interval and suppose $N \in \mathbb{N}$, $s_1, s_2 > 0$, $q_1, q_2 \geq 0$ are such that $\frac{(i+t_2)^{s_2} H_{l,s_2,t_2}}{(i+t_1)^{s_1} H_{l,s_1,t_1}} \in I$, $i = 1, \ldots, l$.*
If $f : I \to \mathbb{R}$ is a convex function, then

$$\hat{D}_f(i,l,s_1,s_2,t_1,t_2) \leq \frac{M-1}{M-m} f(m) + \frac{1-m}{M-m} f(M)$$
$$- \sum_{n=0}^{N-1} \sum_{k=1}^{2^n} \sum_{i=1}^{l} \Delta_f(m,M,n,k) \frac{1}{(l+t_2)^{s_2} H_{l,s_2,t_2}} \left[\left(r_n \cdot \chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)} \right) \left(\frac{\frac{(i+t_2)^{s_2} H_{l,s_2,t_2}}{(i+t_1)^{s_1} H_{l,s_1,t_1}} - m}{M-m} \right) \right]$$
$$\leq \frac{M-1}{M-m} f(m) + \frac{1-m}{M-m} f(M) - \sum_{n=0}^{N-1} \sum_{k=1}^{2^n} \sum_{i=1}^{l} \Delta_f(m,M,n,k) \frac{1}{(l+t_2)^{s_2} H_{l,s_2,t_2}} \cdot$$
$$\cdot \left[\left(2^n \left(\frac{m_{\mathbf{p},\mathbf{q}} - m}{M-m} \right) - k + 1 \right) \cdot \chi_{\left(\frac{k-1}{2^n}, \frac{2k-1}{2^{n+1}}\right)} \left(\frac{\frac{(i+t_2)^{s_2} H_{l,s_2,t_2}}{(i+t_1)^{s_1} H_{l,s_1,t_1}} - m}{M-m} \right) \right.$$
$$\left. + \left(k - 2^n \left(\frac{m_{\mathbf{p},\mathbf{q}} - m}{M-m} \right) \right) \chi_{\left(\frac{2k-1}{2^{n+1}}, \frac{k}{2^n}\right)} \left(\frac{\frac{(i+t_2)^{s_2} H_{l,s_2,t_2}}{(i+t_1)^{s_1} H_{l,s_1,t_1}} - m}{M-m} \right) \right]. \tag{3.39}$$

Proof. Follows easily from Theorem 3.4. □

We denote Kullback-Leibler divergence for **p** and **q** both defined as Zipf-Mandelbrot law l-tuples as $D_{KL}(i,l,s_1,s_2,t_1,t_2)$.

Corollary 3.12 [122] *Let $l \in \mathbb{N}$ and $s_1, s_2 > 0$, $t_1, t_2 \geq 0$.*
If the logarithm base is greater than 1, then

$$D_{KL}(\boldsymbol{p},\boldsymbol{q}) \leq \frac{Mm}{M-m} \log\left(\frac{m}{M}\right) + \frac{1}{M-m} \log\left(\frac{M^M}{m^m}\right) \tag{3.40}$$
$$- \sum_{n=0}^{N-1} \sum_{k=1}^{2^n} \sum_{i=1}^{l} \Delta_{log}(m,M,n,k) \frac{1}{(l+t_2)^{s_2} H_{l,s_2,t_2}} \left[\left(r_n \cdot \chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)} \right) \left(\frac{\frac{(i+t_2)^{s_2} H_{l,s_2,t_2}}{(i+t_1)^{s_1} H_{l,s_1,t_1}} - m}{M-m} \right) \right]$$
$$\leq \frac{Mm}{M-m} \log\left(\frac{m}{M}\right) + \frac{1}{M-m} \log\left(\frac{M^M}{m^m}\right)$$
$$- \sum_{n=0}^{N-1} \sum_{k=1}^{2^n} \sum_{i=1}^{l} \Delta_{log}(m,M,n,k) \frac{1}{(l+t_2)^{s_2} H_{l,s_2,t_2}} \cdot$$

$$\cdot \left[\left(2^n \left(\frac{m_{\mathbf{p},\mathbf{q}} - m}{M - m} \right) - k + 1 \right) \cdot \chi_{\left(\frac{k-1}{2^n}, \frac{2k-1}{2^{n+1}} \right)} \left(\frac{\frac{(i+t_2)^{s_2} H_{l,s_2,t_2}}{(i+t_1)^{s_1} H_{l,s_1,t_1}} - m}{M - m} \right) \right.$$

$$\left. + \left(k - 2^n \left(\frac{m_{\mathbf{p},\mathbf{q}} - m}{M - m} \right) \right) \chi_{\left(\frac{2k-1}{2^{n+1}}, \frac{k}{2^n} \right)} \left(\frac{\frac{(i+t_2)^{s_2} H_{l,s_2,t_2}}{(i+t_1)^{s_1} H_{l,s_1,t_1}} - m}{M - m} \right) \right]. \tag{3.41}$$

If the base of the logarithm is less than 1*, the inequality signs in the inequalities above are reversed.*

Proof. Follows easily from Corollary 3.4. □

Next result is a special case of Corollary 3.5 and Corollary 3.6, and it gives us Edmundson-Lah-Ribarič and Jensen type inequalities for the Kullback-Leibler divergence of two Zipf-Mandelbrot laws. In contrast to previous results, function f is not necessarily convex in the classical sense.

Corollary 3.13 ([105]) *Let* $\boldsymbol{p}$ *and* $\boldsymbol{q}$ *be Zipf-Mandelbrot laws with parameters* $N \in \mathbb{N}$, $q_1, q_2 \geq 0$ *and* $s_1, s_2 > 0$ *respectively. Then we have*

$$\begin{aligned}
0 &\leq \frac{1}{M_{\boldsymbol{p},\boldsymbol{q}} - m_{\boldsymbol{p},\boldsymbol{q}}} \left(\log M_{\boldsymbol{p},\boldsymbol{q}} + 1 - \frac{1}{M_{\boldsymbol{p},\boldsymbol{q}} - m_{\boldsymbol{p},\boldsymbol{q}}} \log \frac{M_{\boldsymbol{p},\boldsymbol{q}}^{M_{\boldsymbol{p},\boldsymbol{q}}}}{m_{\boldsymbol{p},\boldsymbol{q}}^{m_{\boldsymbol{p},\boldsymbol{q}}}} \right) \times \\
&\quad \sum_{i=1}^{N} \frac{1}{(i+q_2)^{s_2} H_2} \left(M_{\boldsymbol{p},\boldsymbol{q}} - \frac{H_2}{H_1} \frac{(i+q_2)^{s_2}}{(i+q_1)^{s_1}} \right) \left(\frac{H_2}{H_1} \frac{(i+q_2)^{s_2}}{(i+q_1)^{s_1}} - m_{\boldsymbol{p},\boldsymbol{q}} \right) \\
&\leq \frac{m_{\boldsymbol{p},\boldsymbol{q}} M_{\boldsymbol{p},\boldsymbol{q}}}{M_{\boldsymbol{p},\boldsymbol{q}} - m_{\boldsymbol{p},\boldsymbol{q}}} \log \frac{m_{\boldsymbol{p},\boldsymbol{q}}}{M_{\boldsymbol{p},\boldsymbol{q}}} + \frac{1}{M_{\boldsymbol{p},\boldsymbol{q}} - m_{\boldsymbol{p},\boldsymbol{q}}} \log \frac{M_{\boldsymbol{p},\boldsymbol{q}}^{M_{\boldsymbol{p},\boldsymbol{q}}}}{m_{\boldsymbol{p},\boldsymbol{q}}^{m_{\boldsymbol{p},\boldsymbol{q}}}} - D_{KL}(\boldsymbol{p},\boldsymbol{q}) \\
&\leq \frac{1}{M_{\boldsymbol{p},\boldsymbol{q}} - m_{\boldsymbol{p},\boldsymbol{q}}} \left(\frac{1}{M_{\boldsymbol{p},\boldsymbol{q}} - m_{\boldsymbol{p},\boldsymbol{q}}} \log \frac{M_{\boldsymbol{p},\boldsymbol{q}}^{M_{\boldsymbol{p},\boldsymbol{q}}}}{m_{\boldsymbol{p},\boldsymbol{q}}^{m_{\boldsymbol{p},\boldsymbol{q}}}} - \log m_{\boldsymbol{p},\boldsymbol{q}} - 1 \right) \times \\
&\quad \sum_{i=1}^{N} \frac{1}{(i+q_2)^{s_2} H_2} \left(M_{\boldsymbol{p},\boldsymbol{q}} - \frac{H_2}{H_1} \frac{(i+q_2)^{s_2}}{(i+q_1)^{s_1}} \right) \left(\frac{H_2}{H_1} \frac{(i+q_2)^{s_2}}{(i+q_1)^{s_1}} - m_{\boldsymbol{p},\boldsymbol{q}} \right) \\
&\leq \frac{1}{M_{\boldsymbol{p},\boldsymbol{q}} - m_{\boldsymbol{p},\boldsymbol{q}}} \left(\frac{1}{M_{\boldsymbol{p},\boldsymbol{q}} - m_{\boldsymbol{p},\boldsymbol{q}}} \log \frac{M_{\boldsymbol{p},\boldsymbol{q}}^{M_{\boldsymbol{p},\boldsymbol{q}}}}{m_{\boldsymbol{p},\boldsymbol{q}}^{m_{\boldsymbol{p},\boldsymbol{q}}}} - \log m_{\boldsymbol{p},\boldsymbol{q}} - 1 \right) (M_{\boldsymbol{p},\boldsymbol{q}} - 1)(1 - m_{\boldsymbol{p},\boldsymbol{q}}) \\
&\leq \frac{1}{4} \left(\log \frac{M_{\boldsymbol{p},\boldsymbol{q}}^{M_{\boldsymbol{p},\boldsymbol{q}}}}{m_{\boldsymbol{p},\boldsymbol{q}}^{m_{\boldsymbol{p},\boldsymbol{q}}}} - (\log m_{\boldsymbol{p},\boldsymbol{q}} + 1)(M_{\boldsymbol{p},\boldsymbol{q}} - m_{\boldsymbol{p},\boldsymbol{q}}) \right)
\end{aligned}$$

and

$$\begin{aligned}
0 &\leq D_{KL}(\boldsymbol{p},\boldsymbol{q}) \\
&\leq \frac{1}{M_{\boldsymbol{p},\boldsymbol{q}} - m_{\boldsymbol{p},\boldsymbol{q}}} \left[\left(\frac{1}{M_{\boldsymbol{p},\boldsymbol{q}} - m_{\boldsymbol{p},\boldsymbol{q}}} \log \frac{M_{\boldsymbol{p},\boldsymbol{q}}^{M_{\boldsymbol{p},\boldsymbol{q}}}}{m_{\boldsymbol{p},\boldsymbol{q}}^{m_{\boldsymbol{p},\boldsymbol{q}}}} - \log m - 1 \right) (M_{\boldsymbol{p},\boldsymbol{q}} - 1)(1 - m_{\boldsymbol{p},\boldsymbol{q}}) \right.
\end{aligned}$$

$$- \left(\log M_{\boldsymbol{p},\boldsymbol{q}} + 1 - \frac{1}{M_{\boldsymbol{p},\boldsymbol{q}} - m_{\boldsymbol{p},\boldsymbol{q}}} \log \frac{M_{\boldsymbol{p},\boldsymbol{q}}^{M_{\boldsymbol{p},\boldsymbol{q}}}}{m_{\boldsymbol{p},\boldsymbol{q}}^{m_{\boldsymbol{p},\boldsymbol{q}}}} \right) \times$$
$$\sum_{i=1}^{N} \frac{1}{(i+q_2)^{s_2} H_2} \left(M_{\boldsymbol{p},\boldsymbol{q}} - \frac{H_2}{H_1} \frac{(i+q_2)^{s_2}}{(i+q_1)^{s_1}} \right) \left(\frac{H_2}{H_1} \frac{(i+q_2)^{s_2}}{(i+q_1)^{s_1}} - m_{\boldsymbol{p},\boldsymbol{q}} \right) \Bigg].$$

Remark 3.9 From Corollary 3.7 and Corollary 3.8 we can obtain the same type of inequalities, but for the reversed Kullback-Leibler divergence $D_{KL}(\boldsymbol{q},\boldsymbol{p})$ of the Zipf-Mandelbrot distributions $\boldsymbol{p}$ and $\boldsymbol{q}$.

Since Zipf's law is a special case of the Zipf-Mandelbrot law, two previous results hold for Zipf's law with $q = 0$.

Again, if we take $q_1 = q_2 = 0$, we can use the results from the above regarding the Kullback-Leibler divergence of two Zipf-Mandelbrot distributions in order to give estimates for the Kullback-Leibler divergence of the distributions associated to the Russian and English language. As said before, for those experimental values of s_1 and s_2 we have

$$m_N = \min\left\{\frac{p_i}{q_i}\right\} = \frac{H_{N,s_2}}{H_{N,s_1}} \min\left\{\frac{i^{s_2}}{i^{s_1}}\right\} = \frac{H_{N,s_2}}{H_{N,s_1}} \min\left\{i^{s_2-s_1}\right\} = \frac{H_{N,s_2}}{H_{N,s_1}}$$

and

$$M_N = \max\left\{\frac{p_i}{q_i}\right\} = \frac{H_{N,s_2}}{H_{N,s_1}} \max\left\{\frac{i^{s_2}}{i^{s_1}}\right\} = \frac{H_{N,s_2}}{H_{N,s_1}} \max\left\{i^{s_2-s_1}\right\} = \frac{H_{N,s_2}}{H_{N,s_1}} N^{0,080994}.$$

Hence the following bounds for the mentioned divergence, depending only on the parameter N, hold.

$$\begin{aligned}
0 &\le \frac{1}{M_N - m_N} \left(\log M_N + 1 - \frac{1}{M_N - m_N} \log \frac{M_N^{M_N}}{m_N^{m_N}} \right) \times \\
&\quad \sum_{i=1}^{N} \frac{H_{N,0,973863}}{i^{0,973863} H_{N,0,892869}^2} \left(N^{0,080994} - i^{0,080994}\right) \left(i^{0,080994} - 1\right) \\
&\le \frac{m_N M_N}{M_N - m_N} \log \frac{m_N}{M_N} + \frac{1}{M_N - m_N} \log \frac{M_N^{M_N}}{m_N^{m_N}} - D_{KL}(\boldsymbol{p},\boldsymbol{q}) \\
&\le \frac{1}{M_N - m_N} \left(\frac{1}{M_N - m_N} \log \frac{M_N^{M_N}}{m_N^{m_N}} - \log m_N - 1 \right) \times \\
&\quad \sum_{i=1}^{N} \frac{H_{N,0,973863}}{i^{0,973863} H_{N,0,892869}^2} \left(N^{0,080994} - i^{0,080994}\right) \left(i^{0,080994} - 1\right) \\
&\le \frac{1}{M_N - m_N} \left(\frac{1}{M_N - m_N} \log \frac{M_N^{M_N}}{m_N^{m_N}} - \log m_N - 1 \right) (M_N - 1)(1 - m_N) \\
&\le \frac{1}{4} \left(\log \frac{M_N^{M_N}}{m_N^{m_N}} - (\log m_N + 1)(M_N - m_N) \right)
\end{aligned}$$

$$0 \le D_{KL}(\boldsymbol{p},\boldsymbol{q}) \le \frac{1}{M_N - m_N}\left[\left(\frac{1}{M_N - m_N}\log\frac{M_N^{M_N}}{m_N^{m_N}} - \log m_N - 1\right)(M_N-1)(1-m_N)\right.$$
$$-\left(\log M + 1 - \frac{1}{M-m}\log\frac{M^M}{m^m}\right)\times$$
$$\left.\sum_{i=1}^{N}\frac{H_{N,0,973863}}{i^{0,973863}H_{N,0,892869}^2}\left(N^{0,080994} - i^{0,080994}\right)\left(i^{0,080994}-1\right)\right].$$

By calculating the above results for the Kullback-Leibler divergence of the distributions associated to the Russian ($\boldsymbol{p}$) and English ($\boldsymbol{q}$) language for different values of the parameter N, we obtained the following bounds:

- from the first series of inequalities:

N	5000	10000	50000	100000
$D_{KL}(\boldsymbol{p},\boldsymbol{q}) \le$	1.19101	1.16826	1.12176	1.10408

- from the second series of inequalities:

N	5000	10000	50000	100000
$D_{KL}(\boldsymbol{p},\boldsymbol{q}) \le$	0.170194	0.189118	0.236439	0.258335

The base of the logarithm used in our calculations is 2.

The result that follows is a special case of Theorem 3.7, and it gives us Edmundson-Lah-Ribarič type inequality for the generalized f-divergence of the Zipf-Mandelbrot law.

Corollary 3.14 ([103]) *Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be Zipf-Mandelbrot laws with parameters $N \in \mathbb{N}$, $q_1, q_2 \ge 0$ and $s_1, s_2 > 0$ respectively, and let $m_{\boldsymbol{p},\boldsymbol{q}}$ and $M_{\boldsymbol{p},\boldsymbol{q}}$ be defined in (3.35). Let $f\colon [m_{\boldsymbol{p},\boldsymbol{q}}, M_{\boldsymbol{p},\boldsymbol{q}}] \to \mathbb{R}$ be a 3-convex function. Then we have*

$$\begin{aligned}
&\frac{1}{M_{\boldsymbol{p},\boldsymbol{q}} - m_{\boldsymbol{p},\boldsymbol{q}}}\left(\frac{f(M_{\boldsymbol{p},\boldsymbol{q}}) - f(m_{\boldsymbol{p},\boldsymbol{q}})}{M_{\boldsymbol{p},\boldsymbol{q}} - m_{\boldsymbol{p},\boldsymbol{q}}} - f'_+(m_{\boldsymbol{p},\boldsymbol{q}})\right)\times\\
&\quad\sum_{i=1}^{n}\frac{1}{(i+q_2)^{s_2}H_2}\left(M_{\boldsymbol{p},\boldsymbol{q}} - \frac{(i+q_2)^{s_2}H_2}{(i+q_1)^{s_1}H_1}\right)\left(\frac{(i+q_2)^{s_2}H_2}{(i+q_1)^{s_1}H_1} - m_{\boldsymbol{p},\boldsymbol{q}}\right)\\
&\le \frac{M_{\boldsymbol{p},\boldsymbol{q}} - 1}{M_{\boldsymbol{p},\boldsymbol{q}} - m_{\boldsymbol{p},\boldsymbol{q}}} f(m_{\boldsymbol{p},\boldsymbol{q}}) + \frac{1 - m_{\boldsymbol{p},\boldsymbol{q}}}{M_{\boldsymbol{p},\boldsymbol{q}} - m_{\boldsymbol{p},\boldsymbol{q}}} f(M_{\boldsymbol{p},\boldsymbol{q}}) - \tilde{D}_f(\boldsymbol{p},\boldsymbol{q}) \qquad (3.42)\\
&\le \frac{1}{M_{\boldsymbol{p},\boldsymbol{q}} - m_{\boldsymbol{p},\boldsymbol{q}}}\left(f'_-(M_{\boldsymbol{p},\boldsymbol{q}}) - \frac{f(M_{\boldsymbol{p},\boldsymbol{q}}) - f(m_{\boldsymbol{p},\boldsymbol{q}})}{M_{\boldsymbol{p},\boldsymbol{q}} - m_{\boldsymbol{p},\boldsymbol{q}}}\right)\times\\
&\quad\sum_{i=1}^{n}\frac{1}{(i+q_2)^{s_2}H_2}\left(M_{\boldsymbol{p},\boldsymbol{q}} - \frac{(i+q_2)^{s_2}H_2}{(i+q_1)^{s_1}H_1}\right)\left(\frac{(i+q_2)^{s_2}H_2}{(i+q_1)^{s_1}H_1} - m_{\boldsymbol{p},\boldsymbol{q}}\right).
\end{aligned}$$

Next result follows directly from Theorem 3.8, and it represents a Jensen type inequality for the generalized f-divergence of the Zipf-Mandelbrot law without the assumption on the convexity of the function f in the classical sense.

Corollary 3.15 ([103]) *Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be Zipf-Mandelbrot laws with parameters $N \in \mathbb{N}$, $q_1, q_2 \geq 0$ and $s_1, s_2 > 0$ respectively, and let $m_{\boldsymbol{p},\boldsymbol{q}}$ and $M_{\boldsymbol{p},\boldsymbol{q}}$ be defined in (3.35). Let $f\colon [m_{\boldsymbol{p},\boldsymbol{q}}, M_{\boldsymbol{p},\boldsymbol{q}}] \to \mathbb{R}$ be a 3-convex function. Then we have*

$$\begin{aligned}
&\frac{(M_{\boldsymbol{p},\boldsymbol{q}}-1)(1-m_{\boldsymbol{p},\boldsymbol{q}})}{M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}}}\left(\frac{f(M_{\boldsymbol{p},\boldsymbol{q}})-f(m_{\boldsymbol{p},\boldsymbol{q}})}{M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}}}-f'_+(m_{\boldsymbol{p},\boldsymbol{q}})\right)\\
&\quad-\frac{1}{M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}}}\left(f'_-(M_{\boldsymbol{p},\boldsymbol{q}})-\frac{f(M_{\boldsymbol{p},\boldsymbol{q}})-f(m_{\boldsymbol{p},\boldsymbol{q}})}{M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}}}\right)\times\\
&\quad\sum_{i=1}^{n}\frac{1}{(i+q_2)^{s_2}H_2}\left(M_{\boldsymbol{p},\boldsymbol{q}}-\frac{(i+q_2)^{s_2}H_2}{(i+q_1)^{s_1}H_1}\right)\left(\frac{(i+q_2)^{s_2}H_2}{(i+q_1)^{s_1}H_1}-m_{\boldsymbol{p},\boldsymbol{q}}\right)\\
&\quad\leq \tilde{D}_f(\boldsymbol{p},\boldsymbol{q})-f(1) \qquad\qquad (3.43)\\
&\quad\leq \frac{(M_{\boldsymbol{p},\boldsymbol{q}}-1)(1-m_{\boldsymbol{p},\boldsymbol{q}})}{M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}}}\left(f'_-(M_{\boldsymbol{p},\boldsymbol{q}})-\frac{f(M_{\boldsymbol{p},\boldsymbol{q}})-f(m_{\boldsymbol{p},\boldsymbol{q}})}{M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}}}\right)\\
&\quad-\frac{1}{M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}}}\left(\frac{f(M_{\boldsymbol{p},\boldsymbol{q}})-f(m_{\boldsymbol{p},\boldsymbol{q}})}{M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}}}-f'_+(m_{\boldsymbol{p},\boldsymbol{q}})\right)\times\\
&\quad\sum_{i=1}^{n}\frac{1}{(i+q_2)^{s_2}H_2}\left(M_{\boldsymbol{p},\boldsymbol{q}}-\frac{(i+q_2)^{s_2}H_2}{(i+q_1)^{s_1}H_1}\right)\left(\frac{(i+q_2)^{s_2}H_2}{(i+q_1)^{s_1}H_1}-m_{\boldsymbol{p},\boldsymbol{q}}\right).
\end{aligned}$$

Remark 3.10 Corollary 3.14 and Corollary 3.15 can easily be applied to Kullback-Leibler divergence, Hellinger divergence, Renyi divergence, harmonic divergence or Jeffrey divergence considering Remark 3.7.

The following result is a special case of Theorem 3.9, and it gives us Edmundson-Lah-Ribarič type inequality for the generalized f-divergence of the Zipf-Mandelbrot law.

Corollary 3.16 ([106]) *Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be Zipf-Mandelbrot laws with parameters $N \in \mathbb{N}$, $q_1, q_2 \geq 0$ and $s_1, s_2 > 0$ respectively, and let $m_{\boldsymbol{p},\boldsymbol{q}}$ and $M_{\boldsymbol{p},\boldsymbol{q}}$ be defined in (3.35). Let $f\colon [m_{\boldsymbol{p},\boldsymbol{q}}, M_{\boldsymbol{p},\boldsymbol{q}}] \to \mathbb{R}$ be a 3-convex function. Then we have*

$$\begin{aligned}
&(1-m_{\boldsymbol{p},\boldsymbol{q}})\left[\frac{f(M_{\boldsymbol{p},\boldsymbol{q}})-f(m_{\boldsymbol{p},\boldsymbol{q}})}{M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}}}-\frac{f'_+(m_{\boldsymbol{p},\boldsymbol{q}})}{2}\right]\\
&\quad-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{1}{(i+q_1)^{s_1}H_1}-\frac{m_{\boldsymbol{p},\boldsymbol{q}}}{(i+q_2)^{s_2}H_2}\right)f'\left(\frac{H_2}{H_1}\frac{(i+q_2)^{s_2}}{(i+q_1)^{s_1}}\right)\\
&\quad\leq \frac{M_{\boldsymbol{p},\boldsymbol{q}}-1}{M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}}}f(m_{\boldsymbol{p},\boldsymbol{q}})+\frac{1-m_{\boldsymbol{p},\boldsymbol{q}}}{M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}}}f(M_{\boldsymbol{p},\boldsymbol{q}})-\tilde{D}_f(\boldsymbol{p},\boldsymbol{q}) \qquad\qquad (3.44)\\
&\quad\leq \frac{1}{2}\sum_{i=1}^{n}\left(\frac{M_{\boldsymbol{p},\boldsymbol{q}}}{(i+q_2)^{s_2}H_2}-\frac{1}{(i+q_1)^{s_1}H_1}\right)f'\left(\frac{H_2}{H_1}\frac{(i+q_2)^{s_2}}{(i+q_1)^{s_1}}\right)\\
&\quad-(M_{\boldsymbol{p},\boldsymbol{q}}-1)\left[\frac{f(M_{\boldsymbol{p},\boldsymbol{q}})-f(m_{\boldsymbol{p},\boldsymbol{q}})}{M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}}}-\frac{f'_-(M_{\boldsymbol{p},\boldsymbol{q}})}{2}\right].
\end{aligned}$$

Next result follows directly from Theorem 2.8, and it gives us another Edmundson-Lah-Ribarič type inequality for the generalized f-divergence of the Zipf-Mandelbrot law.

Corollary 3.17 ([106]) *Let* $\boldsymbol{p}$ *and* $\boldsymbol{q}$ *be Zipf-Mandelbrot laws with parameters* $N \in \mathbb{N}$, $q_1, q_2 \geq 0$ *and* $s_1, s_2 > 0$ *respectively, and let* $m_{\boldsymbol{p},\boldsymbol{q}}$ *and* $M_{\boldsymbol{p},\boldsymbol{q}}$ *be defined in (3.35). Let* $f\colon [m_{\boldsymbol{p},\boldsymbol{q}}, M_{\boldsymbol{p},\boldsymbol{q}}] \to \mathbb{R}$ *be a 3-convex function. Then we have*

$$\begin{aligned}
&(M_{\boldsymbol{p},\boldsymbol{q}}-1)\left[f'_{-}(M_{\boldsymbol{p},\boldsymbol{q}}) - \frac{f(M_{\boldsymbol{p},\boldsymbol{q}})-f(m_{\boldsymbol{p},\boldsymbol{q}})}{M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}}}\right] \\
&\quad - \frac{f''_{-}(M_{\boldsymbol{p},\boldsymbol{q}})}{2}\sum_{i=1}^{n}(i+q_2)^{s_2}H_2\left(\frac{M_{\boldsymbol{p},\boldsymbol{q}}}{(i+q_2)^{s_2}H_2} - \frac{1}{(i+q_1)^{s_1}H_1}\right)^2 \\
&\leq \frac{M_{\boldsymbol{p},\boldsymbol{q}}-1}{M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}}}f(m_{\boldsymbol{p},\boldsymbol{q}}) + \frac{1-m_{\boldsymbol{p},\boldsymbol{q}}}{M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}}}f(M_{\boldsymbol{p},\boldsymbol{q}}) - \tilde{D}_f(\boldsymbol{p},\boldsymbol{q}) \qquad (3.45)\\
&\leq (1-m_{\boldsymbol{p},\boldsymbol{q}})\left[\frac{f(M_{\boldsymbol{p},\boldsymbol{q}})-f(m_{\boldsymbol{p},\boldsymbol{q}})}{M_{\boldsymbol{p},\boldsymbol{q}}-m_{\boldsymbol{p},\boldsymbol{q}}} - f'_{+}(m_{\boldsymbol{p},\boldsymbol{q}})\right] \\
&\quad - \frac{f''_{+}(m_{\boldsymbol{p},\boldsymbol{q}})}{2}\sum_{i=1}^{n}(i+q_2)^{s_2}H_{N,q_2,s_2}\left(\frac{1}{(i+q_1)^{s_1}H_1} - \frac{m_{\boldsymbol{p},\boldsymbol{q}}}{(i+q_2)^{s_2}H_2}\right)^2.
\end{aligned}$$

Remark 3.11 Again, by taking into consideration Remark 3.7 one can see that Corollary 3.16 and Corollary 3.17 can easily be applied to any of the following divergences: Kullback-Leibler divergence, Hellinger divergence, Renyi divergence, harmonic divergence or Jeffrey divergence.

The following results are special cases of Theorems 3.11, 3.12, 3.13 and 3.14 respectively, and they gives us Edmundson-Lah-Ribarič type inequality for the generalized f-divergence of the Zipf-Mandelbrot law.

Corollary 3.18 ([107]) *Let* $\boldsymbol{p}$ *and* $\boldsymbol{q}$ *be Zipf-Mandelbrot laws with parameters* $N \in \mathbb{N}$, $q_1, q_2 \geq 0$ *and* $s_1, s_2 > 0$ *respectively, and let* H_1, H_2, $a_{\boldsymbol{p},\boldsymbol{q}}$ *and* $b_{\boldsymbol{p},\boldsymbol{q}}$ *be defined in (3.35). Let* $f \in \mathscr{C}^n([a_{\boldsymbol{p},\boldsymbol{q}}, b_{\boldsymbol{p},\boldsymbol{q}}])$ *be a n-convex function. If n and* $3 \leq m \leq n-1$ *are of different parity, then*

$$\begin{aligned}
&\frac{b_{\boldsymbol{p},\boldsymbol{q}}-1}{b_{\boldsymbol{p},\boldsymbol{q}}-a_{\boldsymbol{p},\boldsymbol{q}}}f(a_{\boldsymbol{p},\boldsymbol{q}}) + \frac{1-a_{\boldsymbol{p},\boldsymbol{q}}}{b_{\boldsymbol{p},\boldsymbol{q}}-a_{\boldsymbol{p},\boldsymbol{q}}}f(b_{\boldsymbol{p},\boldsymbol{q}}) - \tilde{D}_f(\boldsymbol{p},\boldsymbol{q}) \\
&\quad \leq (1-a_{\boldsymbol{p},\boldsymbol{q}})\left(f'(a_{\boldsymbol{p},\boldsymbol{q}}) - f[a_{\boldsymbol{p},\boldsymbol{q}}, b_{\boldsymbol{p},\boldsymbol{q}}]\right) + \sum_{k=2}^{m-1}\frac{f^{(k)}(a_{\boldsymbol{p},\boldsymbol{q}})}{H_2 k!}\sum_{i=1}^{r}\frac{\left(\frac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}} - a_{\boldsymbol{p},\boldsymbol{q}}\right)^k}{(i+q_2)^{s_2}} \\
&\qquad + \sum_{k=1}^{n-m} f[\underbrace{a_{\boldsymbol{p},\boldsymbol{q}}, \ldots, a_{\boldsymbol{p},\boldsymbol{q}}}_{m \text{ times}}; \underbrace{b_{\boldsymbol{p},\boldsymbol{q}}, \ldots, b_{\boldsymbol{p},\boldsymbol{q}}}_{k \text{ times}}]\sum_{i=1}^{r}\frac{\left(\frac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}} - a_{\boldsymbol{p},\boldsymbol{q}}\right)^m \left(\frac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}} - b_{\boldsymbol{p},\boldsymbol{q}}\right)^{k-1}}{H_2(i+q_2)^{s_2}}.
\end{aligned}$$

This inequality also holds when the function f is n-concave and n and m are of equal parity. In case when the function f is n-convex and n and m are of equal parity, or when the function f is n-concave and n and m are of different parity, the inequality sign is reversed.

Corollary 3.19 ([107]) *Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be Zipf-Mandelbrot laws with parameters $N \in \mathbb{N}$, $q_1, q_2 \geq 0$ and $s_1, s_2 > 0$ respectively, and let H_1, H_2, $a_{\boldsymbol{p},\boldsymbol{q}}$ and $b_{\boldsymbol{p},\boldsymbol{q}}$ be defined in (3.35). Let Let $f \in \mathscr{C}^n([a_{\boldsymbol{p},\boldsymbol{q}}, b_{\boldsymbol{p},\boldsymbol{q}}])$ be a n-convex function. If the function f is n-convex and if $3 \leq m \leq n-1$ are of different parity, then*

$$\begin{aligned}
&\frac{b_{\boldsymbol{p},\boldsymbol{q}}-1}{b_{\boldsymbol{p},\boldsymbol{q}}-a_{\boldsymbol{p},\boldsymbol{q}}} f(a_{\boldsymbol{p},\boldsymbol{q}}) + \frac{1-a_{\boldsymbol{p},\boldsymbol{q}}}{b_{\boldsymbol{p},\boldsymbol{q}}-a_{\boldsymbol{p},\boldsymbol{q}}} f(b_{\boldsymbol{p},\boldsymbol{q}}) - \tilde{D}_f(\boldsymbol{p},\boldsymbol{q}) \\
&\leq (b_{\boldsymbol{p},\boldsymbol{q}}-1)\left(f[a_{\boldsymbol{p},\boldsymbol{q}}, b_{\boldsymbol{p},\boldsymbol{q}}] - f'(b_{\boldsymbol{p},\boldsymbol{q}})\right) + \sum_{k=2}^{m-1} \frac{f^{(k)}(b_{\boldsymbol{p},\boldsymbol{q}})}{H_2 k!} \sum_{i=1}^{r} \frac{\left(\frac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}} - b_{\boldsymbol{p},\boldsymbol{q}}\right)^k}{(i+q_2)^{s_2}} \\
&\quad + \sum_{k=1}^{n-m} f[\underbrace{b_{\boldsymbol{p},\boldsymbol{q}}, \ldots, b_{\boldsymbol{p},\boldsymbol{q}}}_{m \text{ times}}; \underbrace{a_{\boldsymbol{p},\boldsymbol{q}}, \ldots, a_{\boldsymbol{p},\boldsymbol{q}}}_{k \text{ times}}] \sum_{i=1}^{r} \frac{\left(\frac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}} - b_{\boldsymbol{p},\boldsymbol{q}}\right)^m \left(\frac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}} - a_{\boldsymbol{p},\boldsymbol{q}}\right)^{k-1}}{H_2(i+q_2)^{s_2}}.
\end{aligned}$$

The inequality above also holds when the function f is n-concave and m is even. In case when the function f is n-convex and m is even, or when the function f is n-concave and m is odd, the inequality sign is reversed.

Corollary 3.20 ([107]) *Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be Zipf-Mandelbrot laws with parameters $N \in \mathbb{N}$, $q_1, q_2 \geq 0$ and $s_1, s_2 > 0$ respectively, and let H_1, H_2, $a_{\boldsymbol{p},\boldsymbol{q}}$ and $b_{\boldsymbol{p},\boldsymbol{q}}$ be defined in (3.35). Let Let $f \in \mathscr{C}^n([a_{\boldsymbol{p},\boldsymbol{q}}, b_{\boldsymbol{p},\boldsymbol{q}}])$ be a n-convex function. If the function f is n-convex and if n is odd, then we have*

$$\begin{aligned}
&\sum_{k=2}^{n-1} f[a_{\boldsymbol{p},\boldsymbol{q}}; \underbrace{b_{\boldsymbol{p},\boldsymbol{q}}, \ldots, b_{\boldsymbol{p},\boldsymbol{q}}}_{k \text{ times}}] \sum_{i=1}^{r} \frac{\left(\frac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}} - a_{\boldsymbol{p},\boldsymbol{q}}\right)\left(\frac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}} - b_{\boldsymbol{p},\boldsymbol{q}}\right)^{k-1}}{H_2(i+q_2)^{s_2}} \\
&\leq \frac{b_{\boldsymbol{p},\boldsymbol{q}}-1}{b_{\boldsymbol{p},\boldsymbol{q}}-a_{\boldsymbol{p},\boldsymbol{q}}} f(a_{\boldsymbol{p},\boldsymbol{q}}) + \frac{1-a_{\boldsymbol{p},\boldsymbol{q}}}{b_{\boldsymbol{p},\boldsymbol{q}}-a_{\boldsymbol{p},\boldsymbol{q}}} f(b_{\boldsymbol{p},\boldsymbol{q}}) - \tilde{D}_f(\boldsymbol{p},\boldsymbol{q}) \\
&\leq f[a_{\boldsymbol{p},\boldsymbol{q}}, a_{\boldsymbol{p},\boldsymbol{q}}; b_{\boldsymbol{p},\boldsymbol{q}}] \sum_{i=1}^{r} \frac{\left(\frac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}} - a_{\boldsymbol{p},\boldsymbol{q}}\right)\left(\frac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}} - b_{\boldsymbol{p},\boldsymbol{q}}\right)}{H_2(i+q_2)^{s_2}} \\
&\quad + \sum_{k=2}^{n-2} f[a_{\boldsymbol{p},\boldsymbol{q}}, a_{\boldsymbol{p},\boldsymbol{q}}; \underbrace{b_{\boldsymbol{p},\boldsymbol{q}}, \ldots, b_{\boldsymbol{p},\boldsymbol{q}}}_{k \text{ times}}] \sum_{i=1}^{r} \frac{\left(\frac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}} - a_{\boldsymbol{p},\boldsymbol{q}}\right)^2 \left(\frac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}} - b_{\boldsymbol{p},\boldsymbol{q}}\right)^{k-1}}{H_2(i+q_2)^{s_2}}.
\end{aligned}$$

These inequalities also hold when the function f is n-concave and n is even. In case when the function f is n-convex and n is even, or when the function f is n-concave and n is odd, the inequality signs are reversed.

Corollary 3.21 ([107]) *Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be Zipf-Mandelbrot laws with parameters $N \in \mathbb{N}$, $q_1, q_2 \geq 0$ and $s_1, s_2 > 0$ respectively, and let H_1, H_2, $a_{\boldsymbol{p},\boldsymbol{q}}$ and $b_{\boldsymbol{p},\boldsymbol{q}}$ be defined in (3.35). Let Let $f \in \mathscr{C}^n([a_{\boldsymbol{p},\boldsymbol{q}}, b_{\boldsymbol{p},\boldsymbol{q}}])$ be a n-convex function. If the function f is n-convex, then we have*

$$f[b_{\boldsymbol{p},\boldsymbol{q}}, b_{\boldsymbol{p},\boldsymbol{q}}; a_{\boldsymbol{p},\boldsymbol{q}}] \sum_{i=1}^{r} \frac{\left(\frac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}} - a_{\boldsymbol{p},\boldsymbol{q}}\right)\left(\frac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}} - b_{\boldsymbol{p},\boldsymbol{q}}\right)}{H_2(i+q_2)^{s_2}}$$

$$+\sum_{k=2}^{n-2} f[b_{\boldsymbol{p},\boldsymbol{q}}, b_{\boldsymbol{p},\boldsymbol{q}}; \underbrace{a_{\boldsymbol{p},\boldsymbol{q}}, \ldots, a_{\boldsymbol{p},\boldsymbol{q}}}_{k \text{ times}}] \sum_{i=1}^{r} \frac{\left(\frac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}} - a_{\boldsymbol{p},\boldsymbol{q}}\right)^{k-1} \left(\frac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}} - b_{\boldsymbol{p},\boldsymbol{q}}\right)^2}{H_2(i+q_2)^{s_2}}$$

$$\leq \frac{b-1}{b-a} f(a) + \frac{1-a}{b-a} f(b) - \tilde{D}_f(\boldsymbol{p},\boldsymbol{q})$$

$$\leq \sum_{k=2}^{n-1} f[b_{\boldsymbol{p},\boldsymbol{q}}; \underbrace{a_{\boldsymbol{p},\boldsymbol{q}}, \ldots, a_{\boldsymbol{p},\boldsymbol{q}}}_{k \text{ times}}] \sum_{i=1}^{r} \frac{\left(\frac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}} - a_{\boldsymbol{p},\boldsymbol{q}}\right)^{k-1} \left(\frac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}} - b_{\boldsymbol{p},\boldsymbol{q}}\right)}{H_2(i+q_2)^{s_2}}.$$

If the function f is n-concave, the inequality signs are reversed.

Chapter 4

Converse inequalities in compact Hausdorff space

In this chapter we will prove difference type converses of the Jensen and Edmundson-Lah-Ribarič operator inequality for a unital field of positive linear mappings between C^*-algebras of operators in compact Hausdorff space, as well as further refinements and improvements thereto. Obtained general result will be applied to quasi-arithmetic operator means and to potential operator means with the aim of obtaining a better estimate of the difference between these means. Likewise, the mutual bounds for the Jensen operator inequality and the Lah-Ribarič operator inequality for the classes of bounded real-valued functions and Lipschitzian functions will be studied. The connection with the classical convexity will also be discussed. In the last section we give several mutual bounds for the operator version of the Edmundson-Lah-Ribarič inequality which hold for the class of n-convex functions. By virtue of the established estimates, we then derive several mutual bounds for the Jensen operator inequality which are also related to n-convex functions. As an application, we obtain mutual bounds for the differences of quasi-arithmetic and power operator means based on n-convexity.

4.1 Introduction

Let T be a locally compact Hausdorff space and let $\mathscr{A}$ be a C^*-algebra. We say that a field $(x_t)_{t\in T}$ of elements in $\mathscr{A}$ is continuous if the function $t \to x_t$ is norm continuous on T. Additionally, if T is equipped with a Radon measure μ and the function $t \to ||x_t||$ is integrable, then, the so-called Bochner integral $\int_T x_t d\mu(t)$ can be formed. More precisely, the Bochner integral is the unique element in $\mathscr{A}$ such that the relation

$$\varphi\left(\int_T x_t d\mu(t)\right) = \int_T \varphi(x_t)\, d\mu(t)$$

holds for every linear functional φ in the norm dual $\mathscr{A}^*$ (see [55]).

Assume furthermore that there is a field $(\phi_t)_{t\in T}$ of positive linear mappings $\phi_t : \mathscr{A} \to \mathscr{B}$ from $\mathscr{A}$ to another C^*-algebra $\mathscr{B}$. Such field is said to be continuous if the function $t \to \phi_t(x)$ is continuous for every $x \in \mathscr{A}$. If the C^*-algebras are unital and the field $t \to \phi_t(\mathbf{1})$ is integrable with integral $\mathbf{1}$, we say that $(\phi_t)_{t\in T}$ is unital. We assume that such field is continuous.

Let x and y be operators (acting) on an infinite dimensional Hilbert space $\mathscr{H}$. The ordering is defined by setting $x \le y$ if $y - x$ is a positive semi-definite operator.

A continuous function $f : I \to \mathbb{R}$ is operator convex if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$$

holds for each $\lambda \in [0,1]$ and every pair of self-adjoint operators x and y (acting) on an infinite dimensional Hilbert space $\mathscr{H}$ with spectra in I. When the inequality sign is reversed, function f is operator concave.

If $f : I \to \mathbb{R}$ is operator convex function, where I is a real interval of any type, and $(\phi_t)_{t\in T}$ is a unital field, then the Jensen operator inequality (see Hansen *et.al.*, [56]) asserts that

$$f\left(\int_T \phi_t(x_t) d\mu(t)\right) \le \int_T \phi_t(f(x_t)) d\mu(t) \tag{4.1}$$

holds for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathscr{A}$ with spectra contained in I. If $f : I \to \mathbb{R}$ is operator concave function, then the sign of inequality in (4.1) is reversed.

In the same paper, Hansen *et.al.* obtained the following inequality which holds for an usual convex function $f : [m,M] \to \mathbb{R}$ (see [56], proof of Theorem 2):

$$\int_T \phi_t(f(x_t)) d\mu(t) \le \alpha_f \int_T \phi_t(x_t) d\mu(t) + \beta_f \mathbf{1}. \tag{4.2}$$

In this matter, the usual notation is used:

$$\alpha_f = \frac{f(M) - f(m)}{M - m} \quad \text{and} \quad \beta_f = \frac{Mf(m) - mf(M)}{M - m}.$$

Inequality (4.2) is referred to as the Edmundson-Lah-Ribarič operator inequality. Observe that the operator inequality (4.2) is established by applying the functional calculus to the well-known inequality

$$f(t) \leq \alpha_f t + \beta_t, \tag{4.3}$$

which holds for every convex function on the interval $[m,M]$. Recall that $l(t) = \alpha_f t + \beta_t$ is the linear function limiting convex function $f(t)$ on interval $[m,M]$ from the above.

On the other hand, Mićić, Pečarić and Perić in [99] obtained the following improvement of the Edmundson-Lah-Ribarič operator inequality

$$\int_T \phi_t(f(\mathbf{x}_t))d\mu(t) \leq \alpha_f \int_T \phi_t(\mathbf{x}_t)d\mu(t) + \beta_f \mathbf{1} - \delta_f \underline{\mathbf{x}}, \tag{4.4}$$

where

$$\underline{\mathbf{x}} = \frac{1}{2}\mathbf{1} - \frac{1}{M-m}\int_T \phi_t\left(\left|\mathbf{x}_t - \frac{m+M}{2}\mathbf{1}\right|\right) d\mu(t) \tag{4.5}$$

and

$$\delta_f = f(m) + f(M) - 2f\left(\frac{m+M}{2}\right). \tag{4.6}$$

Since $f : [m,M] \to \mathbb{R}$ is convex function, it follows that $\underline{\mathbf{x}} \geq \mathbf{0}$ and $\delta_f \geq 0$.

The techniques that will be used in the proofs are mainly based on the classical real and functional calculus, especially on the well-known monotonicity principle for self-adjoint elements of a C^*-algebra $\mathscr{A}$: If $\mathbf{x} \in \mathscr{A}$ with a spectra $\mathrm{Sp}(\mathbf{x})$, then

$$f(t) \geq g(t),\ t \in \mathrm{Sp}(x) \quad \Longrightarrow \quad f(x) \geq g(x), \tag{4.7}$$

where f and g are real continuous functions (for more details see [49]). Moreover, all the results that follow include the Bochner integral, defined in this Introduction. If nothing else is explicitly stated, $(\mathbf{x}_t)_{t\in T}$ is a bounded continuous field of self-adjoint elements in unital C^*-algebra whose spectra belongs to a domain of the corresponding function and $(\phi_t)_{t\in T}$ is a unital field of positive linear mappings between the corresponding unital C^*-algebras.

The following results refer to functions that are convex in the classical sense. Although regarding different inequalities, it appears that these two series of converses are closely connected.

4.2 Converses of the Jensen and Edmundson-Lah--Ribarič operator inequality

First we give a series of converses for the Jensen operator inequality obtained in [65]. It should be noticed here that the following theorem in the classical real case was proved by Dragomir in the recent paper [37], and generalization of the same inequality for linear functionals was proved by Jakšić and Pečarić in [68]. In fact, such series of scalar inequalities will be exploited in establishing the corresponding operator form.

Theorem 4.1 *Let $f : I \to \mathbb{R}$ be a continuous convex function, and let $m, M \in \mathbb{R}$, $m < M$, be such that interval $[m,M]$ belongs to the interior of interval I. Further, suppose $\mathscr{A}$ and $\mathscr{B}$ are unital C^*-algebras, and $(\phi_t)_{t\in T}$ is a unital field of positive linear mappings $\phi_t : \mathscr{A} \to \mathscr{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ. Then the series of inequalities*

$$\begin{aligned}
&\int_T \phi_t(f(x_t))d\mu(t) - f\left(\int_T \phi_t(x_t)d\mu(t)\right) \\
&\leq \frac{f'_-(M) - f'_+(m)}{M-m}\left(M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)\right)\left(\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1}\right) \\
&\leq \frac{1}{4}(M-m)(f'_-(M) - f'_+(m))\mathbf{1} \qquad (4.8)
\end{aligned}$$

holds for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathscr{A}$ with spectra contained in $[m,M]$. If f is concave on I, then the signs of inequalities in (4.8) *are reversed.*

Proof. Taking into account the operator version of the Lah-Ribarič inequality (4.2), it follows that

$$\begin{aligned}
&\int_T \phi_t(f(x_t))d\mu(t) - f\left(\int_T \phi_t(x_t)d\mu(t)\right) \\
&\leq \alpha_f \int_T \phi_t(x_t)d\mu(t) + \beta_f\mathbf{1} - f\left(\int_T \phi_t(x_t)d\mu(t)\right). \qquad (4.9)
\end{aligned}$$

On the other hand, regarding convexity of f, we have the so-called gradient inequality,

$$f(t) - f(M) \geq f'_-(M)(t-M),$$

which holds for every $t \in [m,M]$, that is,

$$(t-m)f(t) - (t-m)f(M) \geq f'_-(M)(t-M)(t-m), \quad t \in [m,M],$$

after multiplying with $t-m$. In the same way, it follows that

$$(M-t)f(t) - (M-t)f(m) \geq f'_+(m)(M-t)(t-m), \quad t \in [m,M].$$

Now, adding the above two inequalities, and then, dividing by $m-M$, we have

$$\alpha_f t + \beta_f - f(t) \leq \frac{f'_-(M) - f'_+(m)}{M-m}(M-t)(t-m). \qquad (4.10)$$

Moreover, taking into account the arithmetic-geometric mean inequality, the following series of inequalities holds for all $t \in [m,M]$ (see also [37]):

$$\begin{aligned}
\alpha_f t + \beta_f - f(t) &\leq \frac{f'_-(M) - f'_+(m)}{M-m}(M-t)(t-m) \\
&\leq \frac{1}{4}(M-m)(f'_-(M) - f'_+(m)). \qquad (4.11)
\end{aligned}$$

Now, since $m\mathbf{1} \leq x_t \leq M\mathbf{1}$ for every $t \in T$, it follows that $m\phi_t(\mathbf{1}) \leq \phi_t(x_t) \leq M\phi_t(\mathbf{1})$, that is, $m\mathbf{1} \leq \int_T \phi_t(x_t)d\mu(t) \leq M\mathbf{1}$. Hence, applying the functional calculus to the above series of inequalities, that is, setting $\int_T \phi_t(x_t)d\mu(t)$ instead of t, we have

$$\begin{aligned}
&\alpha_f \int_T \phi_t(x_t)d\mu(t) + \beta_f\mathbf{1} - f\left(\int_T \phi_t(x_t)d\mu(t)\right) \\
&\leq \frac{f'_-(M) - f'_+(m)}{M-m}\left(M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)\right)\left(\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1}\right) \\
&\leq \frac{1}{4}(M-m)(f'_-(M) - f'_+(m))\mathbf{1}. \qquad (4.12)
\end{aligned}$$

Finally, comparing (4.9) and (4.12), we obtain (4.8), as claimed. □

Remark 4.1 Observe that in the statement of Theorem 4.1 the interval $[m,M]$ belongs to the interior of the interval I. This condition assures finiteness of the one-sided derivatives in (4.8). Without this assumption these derivatives might be infinite.

Remark 4.2 It should be noticed here that the first expression in the series of inequalities (4.8), that is, the element $\int_T \phi_t(f(x_t))d\mu(t) - f\left(\int_T \phi_t(x_t)d\mu(t)\right)$ is not positive in general. This element is positive if f is in addition operator convex function, due to the Jensen operator inequality (4.1).

The following result was also proved in [65] and it represents converses of the Edmundson-Lah-Ribarič operator inequality (4.2):

Theorem 4.2 *Suppose $f : I \to \mathbb{R}$ is a continuous convex function, and $m, M \in \mathbb{R}$, $m < M$, are such that interval $[m,M]$ belongs to the interior of interval I. Further, let $(\phi_t)_{t\in T}$ be a unital field of positive linear mappings $\phi_t : \mathscr{A} \to \mathscr{B}$, where $\mathscr{A}$ and $\mathscr{B}$ are unital C^*-algebras, defined on a locally compact Hausdorff space T with a bounded Radon measure μ. Then the series of inequalities*

$$\begin{aligned}
0 &\leq \alpha_f \int_T \phi_t(x_t)d\mu(t) + \beta_f\mathbf{1} - \int_T \phi_t(f(x_t))d\mu(t) \\
&\leq \frac{f'_-(M) - f'_+(m)}{M-m}\int_T \phi_t\left([M\mathbf{1} - x_t][x_t - m\mathbf{1}]\right)d\mu(t) \\
&\leq \frac{f'_-(M) - f'_+(m)}{M-m}\left(M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)\right)\left(\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1}\right) \\
&\leq \frac{1}{4}(M-m)(f'_-(M) - f'_+(m))\mathbf{1} \qquad (4.13)
\end{aligned}$$

holds for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathscr{A}$ with spectra contained in $[m,M]$. If f is concave on I, then the signs of inequalities in (4.13) are reversed.

Proof. The first inequality in (4.13) holds by virtue of the Edmundson-Lah-Ribarič inequality (4.2). Further, starting from the scalar inequality (4.10), it follows that relation

$$\alpha_f x_t + \beta_f\mathbf{1} - f(x_t) \leq \frac{f'_-(M) - f'_+(m)}{M-m}(M\mathbf{1} - x_t)(x_t - m\mathbf{1})$$

holds for every $t \in T$. Now, applying the positive linear mappings ϕ_t to the above relation, we obtain

$$\alpha_f \phi_t(x_t) + \beta_f \phi_t(\mathbf{1}) - \phi_t(f(x_t)) \leq \frac{f'_-(M) - f'_+(m)}{M-m} \phi_t([M\mathbf{1} - x_t][x_t - m\mathbf{1}]),$$

while integrating yields

$$\begin{aligned} &\alpha_f \int_T \phi_t(x_t) d\mu(t) + \beta_f \mathbf{1} - \int_T \phi_t(f(x_t)) d\mu(t) \\ &\leq \frac{f'_-(M) - f'_+(m)}{M-m} \int_T \phi_t\left([M\mathbf{1} - x_t][x_t - m\mathbf{1}]\right) d\mu(t), \end{aligned}$$

so the second inequality in (4.13) holds.

Taking into account Theorem 4.1, it is enough to justify the third inequality sign in (4.13). To prove our assertion, we note that the function

$$h(t) = (M-t)(t-m) = -t^2 + (M+m)t - Mm, \quad t \in [m,M]$$

is operator concave (see e.g. [49]). Finally, applying the Jensen operator inequality (4.1) to the above function h, it follows that

$$\begin{aligned} &\int_T \phi_t\left([M\mathbf{1} - x_t][x_t - m\mathbf{1}]\right) d\mu(t) \\ &\leq \left(M\mathbf{1} - \int_T \phi_t(x_t) d\mu(t)\right)\left(\int_T \phi_t(x_t) d\mu(t) - m\mathbf{1}\right), \end{aligned}$$

and the proof is completed. □

Following results are obtained in [66] and they represent more precise converses of the Jensen and Edmundson-Lah-Ribarič operator inequality, and they represent also refinements of the inequalities (4.8) and (4.13) respectively. Such improved relations are also accompanied with a convexity in the classical real sense.

In order to present our basic results, we define

$$\Delta_f(t;m,M) = \frac{1}{M-m}\left[\frac{f(M)-f(t)}{M-t} - \frac{f(t)-f(m)}{t-m}\right], \tag{4.14}$$

where $m < M$ and $f : I \to \mathbb{R}$ is a continuous convex function such that the interval $[m,M]$ belongs to the interior of interval I. Observe that expression (4.14) is actually the second order divided difference of the function f at points m, t, and M, for every $t \in (m,M)$.

Remark 4.3 Observe that the function f is defined on the interval I whose interior contains interval $[m,M]$. This condition ensures finiteness of one-sided derivatives at points m and M. Then,

$$\lim_{t \to m^+} \Delta_f(t;m,M) = \frac{1}{M-m}\left[\frac{f(M)-f(m)}{M-m} - f'_+(m)\right]$$

and

$$\lim_{t\to M^-} \Delta_f(t;m,M) = \frac{1}{M-m}\left[f'_-(M) - \frac{f(M)-f(m)}{M-m}\right],$$

so $\Delta_f(\cdot;m,M)$ may be regarded as a continuous function (in parameter t) on the interval $[m,M]$. Therefore, if x is a self-adjoint element in C^*-algebra with spectra contained in $[m,M]$, then the expression $\Delta_f(x;m,M)$ is also meaningful. Clearly, this assertion holds due to functional calculus.

Now we give two series of converses for the Jensen operator inequality. One of them refines series (4.8). The classical real version of the following theorem was proved by Dragomir in recent paper [38]. In fact, such scalar series of inequalities will be exploited in establishing the corresponding operator forms.

Theorem 4.3 *Let $f: I \to \mathbb{R}$ be a continuous convex function, and let $m,M \in \mathbb{R}$, $m < M$, be such that interval $[m,M]$ belongs to the interior of interval I. Further, suppose $\mathscr{A}$ and $\mathscr{B}$ are unital C^*-algebras, and $(\phi_t)_{t\in T}$ is a unital field of positive linear mappings $\phi_t : \mathscr{A} \to \mathscr{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ. Then the series of inequalities*

$$\begin{aligned}
&\int_T \phi_t(f(x_t))d\mu(t) - f\left(\int_T \phi_t(x_t)d\mu(t)\right)\\
&\le \sup_{m<t<M} \Delta_f(t;m,M)\left(M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)\right)\left(\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1}\right)\\
&\le \frac{f'_-(M)-f'_+(m)}{M-m}\left(M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)\right)\left(\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1}\right)\\
&\le \frac{1}{4}(M-m)(f'_-(M)-f'_+(m))\mathbf{1}
\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}
&\int_T \phi_t(f(x_t))d\mu(t) - f\left(\int_T \phi_t(x_t)d\mu(t)\right)\\
&\le \frac{1}{4}(M-m)^2\Delta_f\left(\int_T \phi_t(x_t)d\mu(t);m,M\right)\\
&\le \frac{1}{4}(M-m)(f'_-(M)-f'_+(m))\mathbf{1}
\end{aligned} \tag{4.16}$$

hold for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathscr{A}$ with spectra contained in $[m,M]$. If f is concave on I, then the signs of inequalities in (4.15) *and* (4.16) *are reversed.*

Proof. Taking into account the operator version of the Edmundson-Lah-Ribarič inequality (4.2), it follows that

$$\begin{aligned}
&\int_T \phi_t(f(x_t))d\mu(t) - f\left(\int_T \phi_t(x_t)d\mu(t)\right)\\
&\le \alpha_f \int_T \phi_t(x_t)d\mu(t) + \beta_f\mathbf{1} - f\left(\int_T \phi_t(x_t)d\mu(t)\right).
\end{aligned} \tag{4.17}$$

On the other hand, the scalar inequality

$$\begin{aligned}
\alpha_f t+\beta_f-f(t) &= \frac{M-t}{M-m}f(m)+\frac{t-m}{M-m}f(M)-f(t)\\
&= \frac{(M-t)(t-m)}{M-m}\left[\frac{f(M)-f(t)}{M-t}-\frac{f(t)-f(m)}{t-m}\right]\\
&= (M-t)(t-m)\Delta_f(t;m,M)\\
&\le (M-t)(t-m)\sup_{m<t<M}\Delta_f(t;m,M) \qquad (4.18)
\end{aligned}$$

holds for all $t\in[m,M]$. In addition, since

$$\begin{aligned}
\sup_{m<t<M}\Delta_f(t;m,M) &= \frac{1}{M-m}\sup_{m<t<M}\left[\frac{f(M)-f(t)}{M-t}-\frac{f(t)-f(m)}{t-m}\right]\\
&\le \frac{1}{M-m}\left[\sup_{m<t<M}\frac{f(M)-f(t)}{M-t}+\sup_{m<t<M}\left(-\frac{f(t)-f(m)}{t-m}\right)\right]\\
&= \frac{1}{M-m}\left[\sup_{m<t<M}\frac{f(M)-f(t)}{M-t}-\inf_{m<t<M}\frac{f(t)-f(m)}{t-m}\right]\\
&= \frac{f'_-(M)-f'_+(m)}{M-m}, \qquad (4.19)
\end{aligned}$$

we have the following series of inequalities:

$$\begin{aligned}
\alpha_f t+\beta_f-f(t) &\le (M-t)(t-m)\sup_{m<t<M}\Delta_f(t;m,M)\\
&\le \frac{f'_-(M)-f'_+(m)}{M-m}(M-t)(t-m)\\
&\le \frac{1}{4}(M-m)(f'_-(M)-f'_+(m)). \qquad (4.20)
\end{aligned}$$

Clearly, the last inequality sign in (4.20) holds due to the arithmetic-geometric mean inequality, that is, $(M-t)(t-m)\le\frac{1}{4}(M-m)^2$.

Now, since $m\mathbf{1}\le x_t\le M\mathbf{1}$ for every $t\in T$, it follows that $m\phi_t(\mathbf{1})\le\phi_t(x_t)\le M\phi_t(\mathbf{1})$, that is, $m\mathbf{1}\le\int_T\phi_t(x_t)d\mu(t)\le M\mathbf{1}$. Hence, applying the functional calculus to the above series of inequalities, that is, putting $\int_T\phi_t(x_t)d\mu(t)$ instead of t, we have

$$\begin{aligned}
&\alpha_f\int_T\phi_t(x_t)d\mu(t)+\beta_f\mathbf{1}-f\left(\int_T\phi_t(x_t)d\mu(t)\right)\\
&\le \sup_{m<t<M}\Delta_f(t;m,M)\left(M\mathbf{1}-\int_T\phi_t(x_t)d\mu(t)\right)\left(\int_T\phi_t(x_t)d\mu(t)-m\mathbf{1}\right)\\
&\le \frac{f'_-(M)-f'_+(m)}{M-m}\left(M\mathbf{1}-\int_T\phi_t(x_t)d\mu(t)\right)\left(\int_T\phi_t(x_t)d\mu(t)-m\mathbf{1}\right)\\
&\le \frac{1}{4}(M-m)(f'_-(M)-f'_+(m))\mathbf{1}. \qquad (4.21)
\end{aligned}$$

Finally, comparing (4.17) and (4.21), we obtain (4.15), as claimed.

To prove (4.16), we start with the scalar series of inequalities

$$\begin{aligned}\alpha_f t+\beta_f-f(t) &\leq \frac{1}{4}(M-m)^2\Delta_f(t;m,M)\\ &\leq \frac{1}{4}(M-m)(f'_-(M)-f'_+(m)),\ t\in[m,M],\end{aligned}\tag{4.22}$$

which obviously follows from (4.18), (4.19), and the arithmetic-geometric mean inequality. Finally, setting $\int_T \phi_t(x_t)d\mu(t)$ in (4.22) and utilizing (4.17), we obtain (4.16) and the proof is completed. □

Remark 4.4 Observe that the series of inequalities in (4.15) refines the series (4.8), since $\sup_{m<t<M}\Delta_f(t;m,M)\leq\frac{f'_-(M)-f'_+(m)}{M-m}$. For example, if $f(t)=t^2$ and $m<M$, then

$$1=\sup_{m<t<M}\Delta_f(t;m,M)<\frac{f'_-(M)-f'_+(m)}{M-m}=2,$$

while for $f(t)=t^3$ we have

$$m+2M=\sup_{m<t<M}\Delta_f(t;m,M)<\frac{f'_-(M)-f'_+(m)}{M-m}=3(m+M),$$

provided that $0<m<M$. However, a convex function needs not to be differentiable. To see the corresponding example, let $m<0<M$ and let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(t)=\begin{cases}t^2, & t\geq 0\\ -t, & t<0\end{cases}.$$

Then,

$$\Delta_f(t;m,M)=\begin{cases}1-\frac{m(m+1)}{(M-m)(t-m)}, & t\geq 0\\ \frac{M(M+1)}{(M-m)(M-t)}, & t<0\end{cases},$$

and consequently,

$$\sup_{m<t<M}\Delta_f(t;m,M)=\begin{cases}\frac{M^2-2Mm-m}{(M-m)^2}, & \text{if } m<-1\\ \frac{M+1}{M-m}, & \text{if } -1\leq m<0\end{cases}.$$

On the other hand,

$$\frac{f'_-(M)-f'_+(m)}{M-m}=\frac{2M+1}{M-m},$$

which implies that $\sup_{m<t<M}\Delta_f(t;m,M)<\frac{f'_-(M)-f'_+(m)}{M-m}$, since $M>0$.

Remark 4.5 It should be noticed here that the first line in the series of inequalities (4.15) and (4.16), that is, the element $\int_T\phi_t(f(x_t))d\mu(t)-f\left(\int_T\phi_t(x_t)d\mu(t)\right)$ is not positive in general. This element is positive if f is in addition operator convex function, due to the Jensen operator inequality (4.1).

The following result provides several converse series of inequalities for the Edmundson-Lah-Ribarič operator inequality (4.2). As we shall see below, one of them improves the series (4.13).

Theorem 4.4 *Suppose $f : I \to \mathbb{R}$ is a continuous convex function, and $m, M \in \mathbb{R}$, $m < M$, are such that interval $[m,M]$ belongs to the interior of interval I. Further, let $(\phi_t)_{t\in T}$ be a unital field of positive linear mappings $\phi_t : \mathscr{A} \to \mathscr{B}$, where $\mathscr{A}$ and $\mathscr{B}$ are unital C^*-algebras, defined on a locally compact Hausdorff space T with a bounded Radon measure μ. Then the series of inequalities*

$$\begin{aligned}
0 \leq &\alpha_f \int_T \phi_t(x_t)d\mu(t) + \beta_f \mathbf{1} - \int_T \phi_t(f(x_t))d\mu(t) \\
\leq &\sup_{m<t<M} \Delta_f(t;m,M) \int_T \phi_t\left([M\mathbf{1} - x_t][x_t - m\mathbf{1}]\right) d\mu(t) \\
\leq &\frac{f'_-(M) - f'_+(m)}{M-m} \int_T \phi_t\left([M\mathbf{1} - x_t][x_t - m\mathbf{1}]\right) d\mu(t) \\
\leq &\frac{f'_-(M) - f'_+(m)}{M-m} \left(M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)\right)\left(\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1}\right) \\
\leq &\frac{1}{4}(M-m)(f'_-(M) - f'_+(m))\mathbf{1},
\end{aligned} \tag{4.23}$$

$$\begin{aligned}
0 \leq &\alpha_f \int_T \phi_t(x_t)d\mu(t) + \beta_f \mathbf{1} - \int_T \phi_t(f(x_t))d\mu(t) \\
\leq &\sup_{m<t<M} \Delta_f(t;m,M) \int_T \phi_t\left([M\mathbf{1} - x_t][x_t - m\mathbf{1}]\right) d\mu(t) \\
\leq &\sup_{m<t<M} \Delta_f(t;m,M) \left(M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)\right)\left(\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1}\right) \\
\leq &\frac{f'_-(M) - f'_+(m)}{M-m} \left(M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)\right)\left(\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1}\right) \\
\leq &\frac{1}{4}(M-m)(f'_-(M) - f'_+(m))\mathbf{1},
\end{aligned} \tag{4.24}$$

and

$$\begin{aligned}
0 \leq &\alpha_f \int_T \phi_t(x_t)d\mu(t) + \beta_f \mathbf{1} - \int_T \phi_t(f(x_t))d\mu(t) \\
\leq &\frac{1}{4}(M-m)^2 \int_T \phi_t\left(\Delta_f(x_t;m,M)\right) d\mu(t) \\
\leq &\frac{1}{4}(M-m)(f'_-(M) - f'_+(m))\mathbf{1}
\end{aligned} \tag{4.25}$$

hold for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathscr{A}$ with spectra contained in $[m,M]$. Moreover, if f is concave on I, then the signs of inequalities in (4.23)*,* (4.24)*, and* (4.25) *are reversed.*

Proof. The first inequality in (4.23) holds by virtue of the Edmundson-Lah-Ribarič inequality (4.2). Further, starting from the scalar inequalities (4.18) and (4.19), it follows that the relation

$$\begin{aligned}\alpha_f x_t + \beta_f \mathbf{1} - f(x_t) &\le \sup_{m<t<M} \Delta_f(t;m,M)(M\mathbf{1} - x_t)(x_t - m\mathbf{1}) \\ &\le \frac{f'_-(M) - f'_+(m)}{M-m}(M\mathbf{1} - x_t)(x_t - m\mathbf{1})\end{aligned}$$

holds for every $t \in T$. Now, applying the positive linear mappings ϕ_t to the above relation, we obtain

$$\begin{aligned}\alpha_f \phi_t(x_t) + \beta_f \phi_t(\mathbf{1}) - \phi_t(f(x_t)) &\le \sup_{m<t<M} \Delta_f(t;m,M)\phi_t([M\mathbf{1} - x_t][x_t - m\mathbf{1}]) \\ &\le \frac{f'_-(M) - f'_+(m)}{M-m}\phi_t([M\mathbf{1} - x_t][x_t - m\mathbf{1}]),\end{aligned}$$

while integrating yields

$$\begin{aligned}&\alpha_f \int_T \phi_t(x_t)d\mu(t) + \beta_f \mathbf{1} - \int_T \phi_t(f(x_t))d\mu(t) \\ &\le \sup_{m<t<M} \Delta_f(t;m,M) \int_T \phi_t\left([M\mathbf{1} - x_t][x_t - m\mathbf{1}]\right) d\mu(t) \\ &\le \frac{f'_-(M) - f'_+(m)}{M-m} \int_T \phi_t\left([M\mathbf{1} - x_t][x_t - m\mathbf{1}]\right) d\mu(t).\end{aligned}$$

Therefore, the second and the third inequality sign in (4.23) hold.

Taking into account Theorem 4.3, that is, the series of inequalities in (4.15), it suffices to motivate the fourth inequality sign in (4.23). To prove our assertion, we note that the function

$$h(t) = (M-t)(t-m) = -t^2 + (M+m)t - Mm, \quad t \in [m,M]$$

is operator concave (see e.g. [49]). Finally, applying the Jensen operator inequality (4.1) to the above function h, it follows that

$$\begin{aligned}&\int_T \phi_t\left([M\mathbf{1} - x_t][x_t - m\mathbf{1}]\right) d\mu(t) \\ &\le \left(M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)\right)\left(\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1}\right),\end{aligned}$$

and the proof of (4.23) is completed.

Further, the series of inequalities in (4.24) is established in the same way as the proof of (4.23), except that we apply the above functional calculus to inequality (4.18) and utilize scalar inequality (4.19).

Finally, the series of inequalities in (4.25) follows by applying the functional calculus to the scalar series of inequalities in (4.22). □

Remark 4.6 Taking into account the Remark 4.4, the series of inequalities in (4.23) refines converse series (4.13) from Theorem 4.2.

Improved version of the Edmundson-Lah-Ribarič operator inequality (4.4) can be utilized for obtaining more accurate refinements of the Jensen and Edmundson-Lah-Ribarič inequality than those from Theorem 4.3 and Theorem 4.4. Results that follow are proved in [67].

Theorem 4.5 *Let $f : I \to \mathbb{R}$ be a continuous convex function, and let $m, M \in \mathbb{R}$, $m < M$, be such that interval $[m, M]$ belongs to the interior of interval I. Further, suppose $\mathscr{A}$ and $\mathscr{B}$ are unital C^*-algebras, and $(\phi_t)_{t\in T}$ is a unital field of positive linear mappings $\phi_t : \mathscr{A} \to \mathscr{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ. Let $\underline{\mathbf{x}}$ and δ_f be defined in (4.5) and (4.6) respectively. Then the series of inequalities*

$$\begin{aligned}
&\int_T \phi_t(f(\mathbf{x}_t))d\mu(t) - f\left(\int_T \phi_t(\mathbf{x}_t)d\mu(t)\right) \\
&\le \sup_{m<t<M} \Delta_f(t;m,M)\left(M\mathbf{1} - \int_T \phi_t(\mathbf{x}_t)d\mu(t)\right)\left(\int_T \phi_t(\mathbf{x}_t)d\mu(t) - m\mathbf{1}\right) - \delta_f\underline{\mathbf{x}} \\
&\le \frac{f'_-(M) - f'_+(m)}{M-m}\left(M\mathbf{1} - \int_T \phi_t(\mathbf{x}_t)d\mu(t)\right)\left(\int_T \phi_t(\mathbf{x}_t)d\mu(t) - m\mathbf{1}\right) - \delta_f\underline{\mathbf{x}} \\
&\le \frac{1}{4}(M-m)(f'_-(M) - f'_+(m))\mathbf{1} - \delta_f\underline{\mathbf{x}}
\end{aligned} \tag{4.26}$$

and

$$\begin{aligned}
&\int_T \phi_t(f(\mathbf{x}_t))d\mu(t) - f\left(\int_T \phi_t(\mathbf{x}_t)d\mu(t)\right) \\
&\le \frac{1}{4}(M-m)^2\Delta_f\left(\int_T \phi_t(\mathbf{x}_t)d\mu(t); m, M\right) - \delta_f\underline{\mathbf{x}} \\
&\le \frac{1}{4}(M-m)(f'_-(M) - f'_+(m))\mathbf{1} - \delta_f\underline{\mathbf{x}},
\end{aligned} \tag{4.27}$$

hold for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathscr{A}$ with spectra contained in $[m, M]$. Moreover, if f is concave on I, then the signs of inequalities in (4.26) *and* (4.27) *are reversed.*

Proof. Utilizing the improved version (4.4) of the Lah-Ribarič inequality, it follows that

$$\begin{aligned}
&\int_T \phi_t(f(\mathbf{x}_t))d\mu(t) - f\left(\int_T \phi_t(\mathbf{x}_t)d\mu(t)\right) \\
&\le \alpha_f \int_T \phi_t(\mathbf{x}_t)d\mu(t) + \beta_f\mathbf{1} - f\left(\int_T \phi_t(\mathbf{x}_t)d\mu(t)\right) - \delta_f\underline{\mathbf{x}}.
\end{aligned} \tag{4.28}$$

On the other hand, it is easy to see that the scalar expression $\alpha_f t + \beta_f - f(t)$ can be rewritten as $(M-t)(t-m)\Delta_f(t;m,M)$, so, it follows that the inequality

$$\alpha_f t + \beta_f - f(t) \le (M-t)(t-m)\sup_{m<t<M} \Delta_f(t;m,M) \tag{4.29}$$

holds for all $t \in [m,M]$. Moreover, since

$$
\begin{aligned}
\sup_{m<t<M} \Delta_f(t;m,M) &= \frac{1}{M-m} \sup_{m<t<M} \left[\frac{f(M)-f(t)}{M-t} - \frac{f(t)-f(m)}{t-m} \right] \\
&\leq \frac{1}{M-m} \left[\sup_{m<t<M} \frac{f(M)-f(t)}{M-t} + \sup_{m<t<M} \left(-\frac{f(t)-f(m)}{t-m} \right) \right] \\
&= \frac{1}{M-m} \left[\sup_{m<t<M} \frac{f(M)-f(t)}{M-t} - \inf_{m<t<M} \frac{f(t)-f(m)}{t-m} \right] \\
&= \frac{f'_-(M)-f'_+(m)}{M-m}, \qquad (4.30)
\end{aligned}
$$

i $(M-t)(t-m) \leq \frac{1}{4}(M-m)^2$, and $(M-t)(t-m) \leq \frac{1}{4}(M-m)^2$, due to the arithmetic-geometric mean inequality, we have the following set of inequalities:

$$
\begin{aligned}
\alpha_f t + \beta_f - f(t) &\leq (M-t)(t-m) \sup_{m<t<M} \Delta_f(t;m,M) \\
&\leq \frac{f'_-(M)-f'_+(m)}{M-m}(M-t)(t-m) \\
&\leq \frac{1}{4}(M-m)(f'_-(M)-f'_+(m)). \qquad (4.31)
\end{aligned}
$$

Now, since $m\mathbf{1} \leq \mathbf{x}_t \leq M\mathbf{1}$ for every $t \in T$, it follows that $m\phi_t(\mathbf{1}) \leq \phi_t(\mathbf{x}_t) \leq M\phi_t(\mathbf{1})$, that is, $m\mathbf{1} \leq \int_T \phi_t(\mathbf{x}_t)d\mu(t) \leq M\mathbf{1}$. Consequently, applying the functional calculus to above set of scalar inequalities and subtracting $\delta_f \underline{\mathbf{x}}$, it follows that

$$
\begin{aligned}
&\alpha_f \int_T \phi_t(\mathbf{x}_t)d\mu(t) + \beta_f \mathbf{1} - f\left(\int_T \phi_t(\mathbf{x}_t)d\mu(t) \right) - \delta_f \underline{\mathbf{x}} \\
&\leq \sup_{m<t<M} \Delta_f(t;m,M) \left(M\mathbf{1} - \int_T \phi_t(\mathbf{x}_t)d\mu(t) \right) \left(\int_T \phi_t(\mathbf{x}_t)d\mu(t) - m\mathbf{1} \right) - \delta_f \underline{\mathbf{x}} \\
&\leq \frac{f'_-(M)-f'_+(m)}{M-m} \left(M\mathbf{1} - \int_T \phi_t(\mathbf{x}_t)d\mu(t) \right) \left(\int_T \phi_t(\mathbf{x}_t)d\mu(t) - m\mathbf{1} \right) - \delta_f \underline{\mathbf{x}} \\
&\leq \frac{1}{4}(M-m)(f'_-(M)-f'_+(m))\mathbf{1} - \delta_f \underline{\mathbf{x}}. \qquad (4.32)
\end{aligned}
$$

Finally, comparing (4.28) and (4.32), we obtain (4.26), as claimed.

To prove (4.27), we start with the scalar series of inequalities

$$
\begin{aligned}
\alpha_f t + \beta_f - f(t) &\leq \frac{1}{4}(M-m)^2 \Delta_f(t;m,M) \\
&\leq \frac{1}{4}(M-m)(f'_-(M)-f'_+(m)), \; t \in [m,M], \qquad (4.33)
\end{aligned}
$$

following from (4.30) and the arithmetic-geometric mean inequality.

Now, inserting $\int_T \phi_t(\mathbf{x}_t)d\mu(t)$ in (4.33) and utilizing (4.28), we obtain (4.27). □

Remark 4.7 Observe that the series of inequalities in (4.26) refines the converse set of inequalities from Theorem 4.3, since $\sup_{m<t<M}\Delta_f(t;m,M) \le \frac{f'_-(M)-f'_+(m)}{M-m}$, $\delta_f \ge 0$, and $\underline{\mathbf{x}} \ge \mathbf{0}$. For example, if $f(t)=t^2$ and $m<M$, then

$$1 = \sup_{m<t<M} \Delta_f(t;m,M) < \frac{f'_-(M)-f'_+(m)}{M-m} = 2,$$

and

$$\delta_f = \frac{(M-m)^2}{2} > 0.$$

Besides the improved Edmundson-Lah-Ribarič operator inequality, the crucial step in proving Theorem 4.5 was in estimating the scalar expression $\alpha_f t+\beta_f - f(t)$ from above. Our next goal is to derive a different kind of upper bound for this scalar expression, which will result in another converse of the Jensen operator inequality (4.1).

It should be noticed here that the improved version (4.4) of the Edmundson-Lah-Ribarič inequality was derived by applying functional calculus to the left inequality in

$$\min\{p_1,p_2\}\delta_f \le p_1 f(m)+p_2 f(M) - f(p_1 m + p_2 M) \le \max\{p_1,p_2\}\delta_f, \tag{4.34}$$

where $f:[m,M]\to\mathbb{R}$ is an arbitrary convex function and $p_1,p_2\in[0,1]$ are real parameters such that $p_1+p_2=1$. Obviously, the left inequality in (4.34) represents a refinement of the classical Jensen inequality, while the second inequality sign means the converse of the Jensen inequality (for more details, see [114, Theorem 1, p. 717]). Now, we show that this converse relation can also be utilized to obtain another type of converses for the inequality (4.1).

Theorem 4.6 *Let the assumptions from the previous theorem hold. If $f:[m,M]\to\mathbb{R}$ is a continuous convex function, then*

$$\begin{aligned}
&\int_T \phi_t(f(\mathbf{x}_t))d\mu(t) - f\left(\int_T \phi_t(\mathbf{x}_t)d\mu(t)\right)\\
&\le \frac{\delta_f}{M-m}\left(\left|\int_T \phi_t(\mathbf{x}_t)d\mu(t) - \frac{m+M}{2}\mathbf{1}\right| + \int_T \phi_t\left(\left|\mathbf{x}_t - \frac{m+M}{2}\mathbf{1}\right|\right)d\mu(t)\right)\\
&\le \delta_f \mathbf{1}.
\end{aligned} \tag{4.35}$$

Moreover, if f is concave, then the signs of inequalities in (4.35) *are reversed.*

Proof. The starting point is relation (4.28) from the proof of Theorem 4.5, this time accompanied with another method for estimating the expression $\alpha_f \int_T \phi_t(\mathbf{x}_t)d\mu(t)+\beta_f\mathbf{1} - f\left(\int_T \phi_t(\mathbf{x}_t)d\mu(t)\right)$. More precisely, we utilize the right inequality in (4.34), i.e. the converse of the classical Jensen inequality, which reduces to

$$\alpha_f t+\beta_f - f(t) \le \max\left\{\frac{M-t}{M-m},\frac{t-m}{M-m}\right\}\delta_f,$$

after setting $p_1 = \frac{M-t}{M-m}$ and $p_2 = \frac{t-m}{M-m}$, $t \in [m,M]$. Now, since

$$\max\left\{\frac{M-t}{M-m}, \frac{t-m}{M-m}\right\} = \frac{1}{2} + \frac{1}{M-m}\left|t - \frac{m+M}{2}\right|,$$

we have

$$\alpha_f t + \beta_f - f(t) \leq \left(\frac{1}{2} + \frac{1}{M-m}\left|t - \frac{m+M}{2}\right|\right)\delta_f, \tag{4.36}$$

and consequently,

$$\begin{aligned} &\alpha_f \int_T \phi_t(\mathbf{x}_t)d\mu(t) + \beta_f \mathbf{1} - f\left(\int_T \phi_t(\mathbf{x}_t)d\mu(t)\right) \\ &\leq \left(\frac{1}{2}\mathbf{1} + \frac{1}{M-m}\left|\int_T \phi_t(\mathbf{x}_t)d\mu(t) - \frac{m+M}{2}\mathbf{1}\right|\right)\delta_f, \end{aligned} \tag{4.37}$$

after applying the functional calculus. Now, the first inequality sign in (4.35) holds due to (4.28), (4.37), and the definition of element $\underline{\mathbf{x}}$. Finally, the second inequality sign in (4.35) holds due to obvious relations $\left|\int_T \phi_t(\mathbf{x}_t)d\mu(t) - \frac{m+M}{2}\mathbf{1}\right| \leq \frac{M-m}{2}\mathbf{1}$ and $\int_T \phi_t\left(\left|\mathbf{x}_t - \frac{m+M}{2}\mathbf{1}\right|\right) d\mu(t) \leq \frac{M-m}{2}\mathbf{1}$. □

Remark 4.8 Note that the inequality (4.36) represents the converse of the classical Lah-Ribarič inequality (4.3).

Remark 4.9 For the sake of completeness, let us mention that the results presented here also cover the discrete case. For example, if $T = \{1,2,\dots,n\}$ and μ is a counting measure, then the relation (4.26) reduces to

$$\begin{aligned} &\sum_{i=1}^n \phi_i(f(\mathbf{x}_i)) - f\left(\sum_{i=1}^n \phi_i(\mathbf{x}_i)\right) \\ &\leq \sup_{m<t<M} \Delta_f(t;m,M)\left(M\mathbf{1} - \sum_{i=1}^n \phi_i(\mathbf{x}_i)\right)\left(\sum_{i=1}^n \phi_i(\mathbf{x}_i) - m\mathbf{1}\right) - \delta_f \underline{\mathbf{x}} \\ &\leq \frac{f'_-(M) - f'_+(m)}{M-m}\left(M\mathbf{1} - \sum_{i=1}^n \phi_i(\mathbf{x}_i)\right)\left(\sum_{i=1}^n \phi_i(\mathbf{x}_i) - m\mathbf{1}\right) - \delta_f \underline{\mathbf{x}} \\ &\leq \frac{1}{4}(M-m)(f'_-(M) - f'_+(m))\mathbf{1} - \delta_f \underline{\mathbf{x}}, \end{aligned}$$

where $\underline{\mathbf{x}} = \frac{1}{2}\mathbf{1} - \frac{1}{M-m}\sum_{i=1}^n \phi_i\left(\left|\mathbf{x}_i - \frac{m+M}{2}\mathbf{1}\right|\right)$, f is a continuous convex function, $\mathbf{x}_i$, $i = 1,2,\dots,n$, are self-adjoint elements in C^*-algebra with spectra contained in $[m,M]$, and ϕ_i, $i = 1,2,\dots,n$, are positive linear mappings provided that $\sum_{i=1}^n \phi_i(\mathbf{1}) = \mathbf{1}$.

Applying the right inequality in (4.34), we can also establish a relation that is in some way complementary to (4.2). This complementary relation represents the converse of the

Edmundson-Lah-Ribarič inequality (4.2). In order to state the corresponding result, we define $\overline{\mathbf{x}}$ as

$$\overline{\mathbf{x}} = \frac{1}{2}\mathbf{1} + \frac{1}{M-m}\int_T \phi_t\left(\left|\mathbf{x}_t - \frac{m+M}{2}\mathbf{1}\right|\right)d\mu(t), \tag{4.38}$$

where $(\mathbf{x}_t)_{t\in T}$ is a bounded continuous field of self-adjoint elements in unital C^*-algebra whose spectra belongs to the interval $[m,M]$. Observe that $\frac{1}{2}\mathbf{1} \leq \overline{\mathbf{x}} \leq \mathbf{1}$ and $\underline{\mathbf{x}} + \overline{\mathbf{x}} = \mathbf{1}$.

Theorem 4.7 *Let $f: I \to \mathbb{R}$ be a continuous convex function, and let $m,M \in \mathbb{R}$, $m<M$, be such that interval $[m,M]$ belongs to the interior of interval I. Further, suppose $\mathscr{A}$ and $\mathscr{B}$ are unital C^*-algebras, and $(\phi_t)_{t\in T}$ is a unital field of positive linear mappings $\phi_t: \mathscr{A} \to \mathscr{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ. Let $\underline{\mathbf{x}}$ and δ_f be defined in (4.38) and (4.6) respectively.*

$$\alpha_f \int_T \phi_t(\mathbf{x}_t)d\mu(t) + \beta_f\mathbf{1} - \int_T \phi_t(f(\mathbf{x}_t))d\mu(t) \leq \delta_f\overline{\mathbf{x}} \tag{4.39}$$

holds for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathscr{A}$ with spectra contained in $[m,M]$. Moreover, if f is concave on I, then the sign of inequality in (4.39) *is reversed.*

Proof. The starting point in this proof is the scalar inequality (4.36), derived in the proof of Theorem 4.6. Now, since $m\mathbf{1} \leq \mathbf{x}_t \leq M\mathbf{1}$ for every $t \in T$, applying the functional calculus to (4.36), it follows that

$$\alpha_f\mathbf{x}_t + \beta_f\mathbf{1} - f(\mathbf{x}_t) \leq \left(\frac{1}{2}\mathbf{1} + \frac{1}{M-m}\left|\mathbf{x}_t - \frac{m+M}{2}\right|\right)\delta_f,$$

i.e.

$$\alpha_f\phi_t(\mathbf{x}_t) + \beta_f\phi_t(\mathbf{1}) - \phi_t(f(\mathbf{x}_t)) \leq \left[\frac{1}{2}\phi_t(\mathbf{1}) + \frac{1}{M-m}\phi_t\left(\left|\mathbf{x}_t - \frac{m+M}{2}\right|\right)\right]\delta_f,$$

after applying a linear mapping ϕ_t. Finally, integrating the previous relation and using that $\int_T \phi_t(\mathbf{1})d\mu(t) = \mathbf{1}$, we get (4.39) and the proof is completed. □

Remark 4.10 At the first glance, it seems that the relation (4.39) may be utilized to obtain refinements of the Jensen operator inequality (4.1). Namely, the converse relation (4.39) yields the inequality

$$\begin{aligned} &\int_T \phi_t(f(\mathbf{x}_t))d\mu(t) - f\left(\int_T \phi_t(\mathbf{x}_t)d\mu(t)\right) \\ &\geq \alpha_f \int_T \phi_t(\mathbf{x}_t)d\mu(t) + \beta_f\mathbf{1} - f\left(\int_T \phi_t(\mathbf{x}_t)d\mu(t)\right) - \delta_f\overline{\mathbf{x}}. \end{aligned} \tag{4.40}$$

Unfortunately, it turns out that the methods used in Theorems 4.5 and 4.6 are not really applicable in obtaining the corresponding refinements.

More precisely, since $\inf_{m<t<M} \Delta_f(t;m,M) = 0$, the method used in Theorem 4.5 accompanied with relation (4.40) yields the inequality

$$\int_T \phi_t(f(\mathbf{x}_t))d\mu(t) - f\left(\int_T \phi_t(\mathbf{x}_t)d\mu(t)\right) \geq -\delta_f \overline{\mathbf{x}},$$

which does not represent refinement of (4.1) since $-\delta_f \overline{\mathbf{x}} \leq \mathbf{0}$.

On the other hand, following the lines as in the proof of Theorem 4.6, with the left inequality in (4.34) instead of the right one, it follows that

$$\begin{aligned}
&\alpha_f \int_T \phi_t(\mathbf{x}_t)d\mu(t) + \beta_f \mathbf{1} - f\left(\int_T \phi_t(\mathbf{x}_t)d\mu(t)\right) \\
&\geq \left(\frac{1}{2}\mathbf{1} - \frac{1}{M-m}\left|\int_T \phi_t(\mathbf{x}_t)d\mu(t) - \frac{m+M}{2}\mathbf{1}\right|\right)\delta_f,
\end{aligned}$$

which together with (4.40) yields

$$\begin{aligned}
&\int_T \phi_t(f(\mathbf{x}_t))d\mu(t) - f\left(\int_T \phi_t(\mathbf{x}_t)d\mu(t)\right) \\
&\geq -\frac{\delta_f}{M-m}\left(\left|\int_T \phi_t(\mathbf{x}_t)d\mu(t) - \frac{m+M}{2}\mathbf{1}\right| + \int_T \phi_t\left(\left|\mathbf{x}_t - \frac{m+M}{2}\mathbf{1}\right|\right)d\mu(t)\right) \\
&\geq -\delta_f \mathbf{1}.
\end{aligned}$$

Obviously, this set of inequalities does not improve the Jensen operator inequality due to negative elements that are the right side of the inequality signs.

If we follow the proof of Theorem 4.4, but instead the Edmundson-Lah-Ribarič inequality (4.2) we take its improvement (4.4), we obtain the following result that provides an improvement of Theorem 4.4.

Corollary 4.1 *Let $f : I \to \mathbb{R}$ be a continuous convex function, and let $m, M \in \mathbb{R}$, $m < M$, be such that interval $[m,M]$ belongs to the interior of interval I. Further, suppose $\mathscr{A}$ and $\mathscr{B}$ are unital C^*-algebras, and $(\phi_t)_{t\in T}$ is a unital field of positive linear mappings $\phi_t : \mathscr{A} \to \mathscr{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ. Let $\underline{\mathbf{x}}$ and δ_f be defined in (4.5) and (4.6) respectively.*

$$\begin{aligned}
\delta_f \underline{\mathbf{x}} \leq &\alpha_f \int_T \phi_t(x_t)d\mu(t) + \beta_f \mathbf{1} - \int_T \phi_t(f(x_t))d\mu(t) \\
\leq &\sup_{m<t<M} \Delta_f(t;m,M) \int_T \phi_t\left([M\mathbf{1} - x_t][x_t - m\mathbf{1}]\right)d\mu(t) \\
\leq &\frac{f'_-(M) - f'_+(m)}{M-m} \int_T \phi_t\left([M\mathbf{1} - x_t][x_t - m\mathbf{1}]\right)d\mu(t) \\
\leq &\frac{f'_-(M) - f'_+(m)}{M-m}\left(M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)\right)\left(\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1}\right) \\
\leq &\frac{1}{4}(M-m)(f'_-(M) - f'_+(m))\mathbf{1}, \qquad (4.41)
\end{aligned}$$

$$\begin{aligned}
\delta_f \underline{\mathbf{x}} \leq & \alpha_f \int_T \phi_t(x_t) d\mu(t) + \beta_f \mathbf{1} - \int_T \phi_t(f(x_t)) d\mu(t) \\
\leq & \sup_{m<t<M} \Delta_f(t;m,M) \int_T \phi_t\left([M\mathbf{1} - x_t][x_t - m\mathbf{1}]\right) d\mu(t) \\
\leq & \sup_{m<t<M} \Delta_f(t;m,M) \left(M\mathbf{1} - \int_T \phi_t(x_t) d\mu(t)\right)\left(\int_T \phi_t(x_t) d\mu(t) - m\mathbf{1}\right) \\
\leq & \frac{f'_-(M) - f'_+(m)}{M-m}\left(M\mathbf{1} - \int_T \phi_t(x_t) d\mu(t)\right)\left(\int_T \phi_t(x_t) d\mu(t) - m\mathbf{1}\right) \\
\leq & \frac{1}{4}(M-m)(f'_-(M) - f'_+(m))\mathbf{1},
\end{aligned} \tag{4.42}$$

and

$$\begin{aligned}
\delta_f \underline{\mathbf{x}} \leq & \alpha_f \int_T \phi_t(x_t) d\mu(t) + \beta_f \mathbf{1} - \int_T \phi_t(f(x_t)) d\mu(t) \\
\leq & \frac{1}{4}(M-m)^2 \int_T \phi_t\left(\Delta_f(x_t;m,M)\right) d\mu(t) \\
\leq & \frac{1}{4}(M-m)(f'_-(M) - f'_+(m))\mathbf{1}
\end{aligned} \tag{4.43}$$

hold for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathscr{A}$ with spectra contained in $[m,M]$. Moreover, if f is concave on I, then the signs of inequalities in (4.41), (4.42) *and* (4.43) *are reversed.*

4.3 Applications to quasi-arithmetic operator means

The main objective of this section is application of general converses from the previous section to the so-called quasi-arithmetic operator means. A generalized quasi-arithmetic operator mean with regard to the Bochner integral (see Introduction), is defined by

$$M_\psi(\mathbf{x},\phi) = \psi^{-1}\left(\int_T \phi_t(\psi(\mathbf{x}_t)) d\mu(t)\right), \tag{4.44}$$

where $(\phi_t)_{t\in T}$ is a unital field of positive linear mappings, $(\mathbf{x}_t)_{t\in T}$ is a bounded continuous field of self-adjoint elements in the corresponding unital C^*-algebra with spectra in $[m,M] \subseteq \mathbb{R}$, and $\psi : [m,M] \to \mathbb{R}$ is a continuous strictly monotone function.

Roughly speaking, an arbitrary C^*-algebra is isomorphic to a C^*-algebra of bounded operators on a Hilbert space $\mathscr{H}$, denoted by $\mathfrak{B}(\mathscr{H})$. It is a consequence of the well-known Gelfand-Naimark theorem (see [51]). Hence, for the reader convenience, from now on, C^*-algebras will be regarded as algebras of bounded operators on a Hilbert space.

Now, for the Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, let $P[\mathfrak{B}(\mathscr{H}),\mathfrak{B}(\mathscr{K})]$ denote the set of all fields $(\phi_t)_{t\in T}$ of positive linear mappings $\phi_t : \mathfrak{B}(\mathscr{H}) \to \mathfrak{B}(\mathscr{K})$, defined on a locally compact Hausdorff space T with a bounded Radon measure μ, which are unital.

Recently, Mićić *et.al.* in [102], [101] and [96] investigated an order among the above quasi-arithmetic means $M_{\psi}(\mathbf{x},\phi)$ and $M_{\chi}(\mathbf{x},\phi)$. Such order was established by virtue of the operator convexity and monotonicity of the corresponding functions appearing in these means.

As before, let $(\mathbf{x}_t)_{t\in T}$ be a bounded continuous field of self-adjoint operators, let $(\phi_t)_{t\in T}$ be a unital field of positive linear mappings, and let $\chi,\psi : I\supset[m,M]\to\mathbb{R}$ be continuous, strictly monotone functions. Then the inequality

$$M_{\psi}(x,\phi)\le M_{\chi}(x,\phi) \tag{4.45}$$

holds if one of the following two conditions is fulfilled:
(i) $\chi\circ\psi^{-1}$ is operator convex and χ^{-1} is operator monotone,
(ii) $\chi\circ\psi^{-1}$ is operator concave and $-\chi^{-1}$ is operator monotone.
On the other hand, if
(i') $\chi\circ\psi^{-1}$ is operator concave and χ^{-1} is operator monotone,
(ii') $\chi\circ\psi^{-1}$ is operator convex and $-\chi^{-1}$ is operator monotone,
then the sign of inequality in (4.45) is reversed.

Moreover, if ψ^{-1} is operator convex and χ^{-1} is operator concave, then

$$M_{\psi}(x,\phi)\le M_{1}(x,\phi)\le M_{\chi}(x,\phi), \tag{4.46}$$

while for operator concave function ψ^{-1} and operator convex function χ^{-1} the signs of inequalities in series (4.46) are reversed.

In contrast to above reference related to the order among quasi-arithmetic means, the corresponding converses are derived by virtue of convexity and monotonicity in the classical real sense.

In order to state the corresponding results, we first present some notation arising from this particular setting. Throughout this section we denote

$$\psi_m=\min\{\psi(m),\psi(M)\},\quad \psi_M=\max\{\psi(m),\psi(M)\},$$

where $\psi:[m,M]\to\mathbb{R}$ is a continuous strictly monotone function. Moreover, with this regard, it will be more convenient to use a slightly altered notation for the divided difference of second order (4.14). More precisely, if $\chi,\psi : I\to\mathbb{R}$ are continuous strictly monotone functions such that $\chi\circ\psi^{-1}$ is well-defined and convex on $\psi(I)$, and the interval $[m,M]$ belongs to the interior of interval I, we define

$$\Delta_{\psi}^{\chi}(t;m,M)=\frac{1}{\psi(M)-\psi(m)}\left[\frac{\chi(M)-\chi(t)}{\psi(M)-\psi(t)}-\frac{\chi(t)-\chi(m)}{\psi(t)-\psi(m)}\right].$$

Remark 4.11 Taking into account the discussion from the previous section which concerned the relation (4.14), the expression $\Delta_{\psi}^{\chi}(\cdot;m,M)$ may be regarded as a continuous function (in parameter t) on the interval $[m,M]$. Consequently, the operator expression $\Delta_{\psi}^{\chi}(\mathbf{x};m,M)$ is meaningful whenever $m\mathbf{1}\le\mathbf{x}\le M\mathbf{1}$.

Finally, with the abbreviations

$$\underline{\mathbf{x}}_{\psi}=\frac{1}{2}\mathbf{1}-\frac{1}{\psi_M-\psi_m}\int_T\phi_t\left(\left|\psi(\mathbf{x}_t)-\frac{\psi_m+\psi_M}{2}\mathbf{1}\right|\right)d\mu(t)$$

and

$$\delta_{\psi}^{\chi} = \chi(m) + \chi(M) - 2\chi \circ \psi^{-1}\left(\frac{\psi_m + \psi_M}{2}\right),$$

we have the following consequence of Theorem 4.5, providing two series of converses for quasi-arithmetic operator means.

Theorem 4.8 *Let $\chi, \psi : I \to \mathbb{R}$ be continuous strictly monotone functions and let the interval $[m,M]$ belongs to the interior of interval I. If the function $\chi \circ \psi^{-1}$ is well-defined and convex on $\psi(I)$, then the series of inequalities*

$$\begin{aligned}
&\chi\left(M_{\chi}(\mathbf{x},\phi)\right) - \chi\left(M_{\psi}(\mathbf{x},\phi)\right) \\
&\le \sup_{t\in(m,M)} \Delta_{\psi}^{\chi}(t;m,M)\left[\psi_M\mathbf{1} - \psi\left(M_{\psi}(\mathbf{x},\phi)\right)\right]\left[\psi\left(M_{\psi}(\mathbf{x},\phi)\right) - \psi_m\mathbf{1}\right] - \delta_{\psi}^{\chi}\underline{\mathbf{x}}_{\psi} \\
&\le \frac{(\chi\circ\psi^{-1})'_-(\psi_M) - (\chi\circ\psi^{-1})'_+(\psi_m)}{\psi_M - \psi_m}\left[\psi_M\mathbf{1} - \psi\left(M_{\psi}(\mathbf{x},\phi)\right)\right] \\
&\qquad \times\left[\psi\left(M_{\psi}(\mathbf{x},\phi)\right) - \psi_m\mathbf{1}\right] - \delta_{\psi}^{\chi}\underline{\mathbf{x}}_{\psi} \\
&\le \frac{1}{4}(\psi_M - \psi_m)\left[(\chi\circ\psi^{-1})'_-(\psi_M) - (\chi\circ\psi^{-1})'_+(\psi_m)\right]\mathbf{1} - \delta_{\psi}^{\chi}\underline{\mathbf{x}}_{\psi}
\end{aligned} \tag{4.47}$$

and

$$\begin{aligned}
&\chi\left(M_{\chi}(\mathbf{x},\phi)\right) - \chi\left(M_{\psi}(\mathbf{x},\phi)\right) \\
&\le \frac{1}{4}(\psi_M - \psi_m)^2\Delta_{\psi}^{\chi}\left(M_{\psi}(\mathbf{x},\phi);m,M\right) - \delta_{\psi}^{\chi}\underline{\mathbf{x}}_{\psi} \\
&\le \frac{1}{4}(\psi_M - \psi_m)\left[(\chi\circ\psi^{-1})'_-(\psi_M) - (\chi\circ\psi^{-1})'_+(\psi_m)\right]\mathbf{1} - \delta_{\psi}^{\chi}\underline{\mathbf{x}}_{\psi}
\end{aligned} \tag{4.48}$$

holds for every bounded continuous field $(\mathbf{x}_t)_{t\in T}$ of self-adjoint elements in C^*-algebra, with spectra contained in $[m,M]$. Moreover, if $\chi \circ \psi^{-1}$ is concave on $\psi(I)$, then the signs of inequalities in (4.47) *and* (4.48) *are reversed.*

Proof. Since $\psi_m \le \psi(t) \le \psi_M$, for all $t \in [m,M]$, it follows that $\psi_m\mathbf{1} \le \psi(\mathbf{x}_t) \le \psi_M\mathbf{1}$, for every $t \in T$. This means that the spectra of the field $(\mathbf{y}_t)_{t\in T} = (\psi(\mathbf{x}_t))_{t\in T}$ is contained in the interval $[\psi_m, \psi_M]$. In addition, since the function $\chi \circ \psi^{-1}$ is continuous on $\psi(I)$, the interval $[\psi_m, \psi_M]$ belongs to the interior of $\psi(I)$.

Finally, utilizing the series of inequalities in (4.26) and (4.27) with ψ_m, ψ_M, $\chi \circ \psi^{-1}$, $(\mathbf{y}_t)_{t\in T}$ respectively instead of m, M, f, $(\mathbf{x}_t)_{t\in T}$, noting that

$$\Delta_{\chi\circ\psi^{-1}}(\psi(t);\psi_m,\psi_M) = \Delta_{\psi}^{\chi}(t;m,M),$$

and with definition (4.44) of quasi-arithmetic operator mean, we obtain (4.47) and (4.48). □

Remark 4.12 Clearly, with assumptions from the Theorem 4.8, operator $\chi\left(M_{\chi}(x,\phi)\right) - \chi\left(M_{\psi}(x,\phi)\right)$ generally is not necessarily positive. It is positive if the function $\chi \circ \psi^{-1}$ is additionally operator convex on the appropriate interval.

With the same setting as in the proof of Theorem 4.8, we also obtain another series of converses of quasi-arithmetic means, which follows immediately from Theorem 4.6.

Corollary 4.2 *Suppose $\chi,\psi:[m,M]\to\mathbb{R}$, are continuous strictly monotone functions. If the function $\chi\circ\psi^{-1}$ is well-defined and convex on $[\psi_m,\psi_M]$, then*

$$\begin{aligned}&\chi\left(M_\chi(\mathbf{x},\phi)\right)-\chi\left(M_\psi(\mathbf{x},\phi)\right)\\&\le\frac{\delta_\psi^\chi}{\psi_M-\psi_m}\left[\left|\psi\left(M_\psi(\mathbf{x},\phi)\right)-\frac{\psi_m+\psi_M}{2}\mathbf{1}\right|+\int_T\phi_t\left(\left|\psi(\mathbf{x}_t)-\frac{\psi_m+\psi_M}{2}\mathbf{1}\right|\right)d\mu(t)\right]\\&\le\delta_\psi^\chi\mathbf{1}.\end{aligned}\tag{4.49}$$

Moreover, if $\chi\circ\psi^{-1}$ is concave on $[\psi_m,\psi_M]$, then the signs of inequalities in (4.49) *are reversed.*

In the same manner, Corollary 4.1 can be also utilized for obtaining converses of the Edmundson-Lah-Ribarič inequality related to quasi-arithmetic operator means.

Corollary 4.3 *Let $\chi,\psi:[m,M]\to\mathbb{R}$ be continuous strictly monotone functions. If the function $\chi\circ\psi^{-1}$ is well-defined and convex on $[\psi_m,\psi_M]$, then we have*

$$\begin{aligned}\delta_\psi^\chi\underline{\mathbf{x}}_\psi&\le\frac{\chi(M)-\chi(m)}{\psi(M)-\psi(m)}\psi\left(M_\psi(x,\phi)\right)+\frac{\psi(M)\chi(m)-\psi(m)\chi(M)}{\psi(M)-\psi(m)}\mathbf{1}-\chi\left(M_\chi(x,\phi)\right)\\&\le\sup_{t\in(m,M)}\delta_\psi^\chi(t;m,M)\int_T\phi_t\left([\psi_M\mathbf{1}-\psi(x_t)][\psi(x_t)-\psi_m\mathbf{1}]\right)d\mu(t)\\&\le\frac{(\chi\circ\psi^{-1})'_-(\psi_M)-(\chi\circ\psi^{-1})'_+(\psi_m)}{\psi_M-\psi_m}\\&\quad\times\int_T\phi_t\left([\psi_M\mathbf{1}-\psi(x_t)][\psi(x_t)-\psi_m\mathbf{1}]\right)d\mu(t)\\&\le\frac{(\chi\circ\psi^{-1})'_-(\psi_M)-(\chi\circ\psi^{-1})'_+(\psi_m)}{\psi_M-\psi_m}\left[\psi_M\mathbf{1}-\psi\left(M_\psi(x,\phi)\right)\right]\left[\psi\left(M_\psi(x,\phi)\right)-\psi_m\mathbf{1}\right]\\&\le\frac{1}{4}(\psi_M-\psi_m)\left[(\chi\circ\psi^{-1})'_-(\psi_M)-(\chi\circ\psi^{-1})'_+(\psi_m)\right]\mathbf{1},\end{aligned}\tag{4.50}$$

$$\begin{aligned}\delta_\psi^\chi\underline{\mathbf{x}}_\psi&\le\frac{\chi(M)-\chi(m)}{\psi(M)-\psi(m)}\psi\left(M_\psi(x,\phi)\right)+\frac{\psi(M)\chi(m)-\psi(m)\chi(M)}{\psi(M)-\psi(m)}\mathbf{1}-\chi\left(M_\chi(x,\phi)\right)\\&\le\sup_{t\in(m,M)}\delta_\psi^\chi(t;m,M)\int_T\phi_t\left([\psi_M\mathbf{1}-\psi(x_t)][\psi(x_t)-\psi_m\mathbf{1}]\right)d\mu(t)\\&\le\sup_{t\in(m,M)}\delta_\psi^\chi(t;m,M)\left[\psi_M\mathbf{1}-\psi\left(M_\psi(x,\phi)\right)\right]\left[\psi\left(M_\psi(x,\phi)\right)-\psi_m\mathbf{1}\right]\\&\le\frac{(\chi\circ\psi^{-1})'_-(\psi_M)-(\chi\circ\psi^{-1})'_+(\psi_m)}{\psi_M-\psi_m}\left[\psi_M\mathbf{1}-\psi\left(M_\psi(x,\phi)\right)\right]\left[\psi\left(M_\psi(x,\phi)\right)-\psi_m\mathbf{1}\right]\\&\le\frac{1}{4}(\psi_M-\psi_m)\left[(\chi\circ\psi^{-1})'_-(\psi_M)-(\chi\circ\psi^{-1})'_+(\psi_m)\right]\mathbf{1},\end{aligned}\tag{4.51}$$

and

$$\begin{aligned}
\delta_{\psi}^{\chi}\underline{\mathbf{x}}_{\psi} \leq & \frac{\chi(M)-\chi(m)}{\psi(M)-\psi(m)}\psi\left(M_{\psi}(x,\phi)\right)+\frac{\psi(M)\chi(m)-\psi(m)\chi(M)}{\psi(M)-\psi(m)}\mathbf{1}-\chi\left(M_{\chi}(x,\phi)\right) \\
\leq & \frac{1}{4}(\psi_M-\psi_m)^2\int_T \phi_t\left(\delta_{\psi}^{\chi}(x_t;m,M)\right)d\mu(t) \\
\leq & \frac{1}{4}(\psi_M-\psi_m)\left[(\chi\circ\psi^{-1})'_-(\psi_M)-(\chi\circ\psi^{-1})'_+(\psi_m).\right]\mathbf{1}
\end{aligned} \tag{4.52}$$

Moreover, if $\chi\circ\psi^{-1}$ is concave on $[\psi_m,\psi_M]$, then the signs of inequalities in (4.50)*,* (4.51) *and* (4.52) *are reversed.*

Remark 4.13 The first inequality from the relation (4.50) can be rewritten as

$$\begin{aligned}
&(\psi(M)-\psi(m))\chi\left(M_{\chi}(x,\phi)\right)-(\chi(M)-\chi(m))\psi\left(M_{\psi}(x,\phi)\right) \\
&\leq(\psi(M)\chi(m)-\psi(m)\chi(M))\mathbf{1}-(\psi(M)-\psi(m))\delta_{\psi}^{\chi}\underline{\mathbf{x}}_{\psi},
\end{aligned}$$

which results in the improvement of the operator analogous of corresponding inequality for the linear functionals (see [124], Theorem 4.3, p. 108).

Remark 4.14 Suppose that the function χ is differentiable at the points m and M, and ψ^{-1} is differentiable at ψ_m and ψ_M, so that $\chi\circ\psi^{-1}$ is differentiable at the points ψ_m and ψ_M. In this case, the points ψ_m and ψ_M in (4.47), (4.50) and (4.51) may respectively be replaced by $\psi(m)$ and $\psi(M)$, due to the symmetry. In addition, utilizing a chain rule, the expression

$$(\chi\circ\psi^{-1})'_-(\psi(M))-(\chi\circ\psi^{-1})'_+(\psi(m))$$

may be rewritten in a more suitable form, that is,

$$(\chi\circ\psi^{-1})'_-(\psi(M))-(\chi\circ\psi^{-1})'_+(\psi(m))=\frac{\chi'(M)}{\psi'(M)}-\frac{\chi'(m)}{\psi'(m)}.$$

This formula will frequently be used in the following subsection.

4.3.1 Examples with power operator means

The most common example of a quasi-arithmetic mean (4.44) is a power operator mean (see e.g. [101]):

$$M_r(\mathbf{x},\phi)=\begin{cases}\left(\int_T \phi_t(\mathbf{x}_t^r)d\mu(t)\right)^{\frac{1}{r}}, & r\neq 0\\ \exp\left(\int_T \phi_t(\log\mathbf{x}_t)d\mu(t)\right), & r=0.\end{cases} \tag{4.53}$$

In this subsection, our intention is to derive converses for power operator means by utilizing results from the previous section. Having regard to this particular setting, we define:

$$\Delta_r^s(t;m,M)=\frac{1}{M^r-m^r}\left[\frac{M^s-t^s}{M^r-t^r}-\frac{t^s-m^s}{t^r-m^r}\right],\ s\in\mathbb{R},r\neq 0,$$

$$\Delta_r^*(t;m,M) = \frac{1}{M^r - m^r}\left[\frac{\log M - \log t}{M^r - t^r} - \frac{\log t - \log m}{t^r - m^r}\right],\ r \neq 0,$$
$$\Delta_*^s(t;m,M) = \frac{1}{\log M - \log m}\left[\frac{M^s - t^s}{\log M - \log t} - \frac{t^s - m^s}{\log t - \log m}\right],\ s \in \mathbb{R}.$$

Due to Remark 4.11, the operator expressions $\Delta_r^s(\mathbf{x};m,M)$, $\Delta_r^*(\mathbf{x};m,M)$, and $\Delta_*^s(\mathbf{x};m,M)$ are well-defined whenever $m\mathbf{1} \leq \mathbf{x} \leq M\mathbf{1}$.

Now, with the abbreviations

$$\underline{\mathbf{x}}_r = \begin{cases} \frac{1}{2}\mathbf{1} - \frac{1}{|M^r - m^r|}\int_T \phi_t\left(\left|\mathbf{x}_t^r - \frac{m^r+M^r}{2}\mathbf{1}\right|\right)d\mu(t), & r \neq 0 \\ \frac{1}{2}\mathbf{1} - \frac{1}{\log M - \log m}\int_T \phi_t\left(\left|\log \mathbf{x}_t - \log\sqrt{mM}\mathbf{1}\right|\right)d\mu(t), & r = 0, \end{cases}$$

and

$$\delta_r^s = m^s + M^s - 2\left(\frac{m^r + M^r}{2}\right)^{\frac{s}{r}},\ s \in \mathbb{R}, r \neq 0,$$
$$\delta_r^* = \frac{2}{r}\log\frac{2\sqrt{m^r M^r}}{m^r + M^r},\ r \neq 0,$$
$$\delta_*^s = \left(\sqrt{M^s} - \sqrt{m^s}\right)^2,\ s \in \mathbb{R},$$

we obtain a whole range of converses of power operator means, arising from Theorem 4.8.

Corollary 4.4 *Let $(\mathbf{x}_t)_{t\in T}$ be a bounded continuous field of self-adjoint elements in C^*-algebra with spectra in $[m,M]$, $0 < m < M$.*

(i) If either $s \leq 0 < r$ or $r < 0 \leq s$ or $0 < r \leq s$ or $s \leq r < 0$, then the series of inequalities

$$\begin{aligned} &[M_s(\mathbf{x},\phi)]^s - [M_r(\mathbf{x},\phi)]^s \\ &\leq \sup_{t\in(m,M)} \Delta_r^s(t;m,M)\left[M^r\mathbf{1} - [M_r(\mathbf{x},\phi)]^r\right]\left[[M_r(\mathbf{x},\phi)]^r - m^r\mathbf{1}\right] - \delta_r^s\underline{\mathbf{x}}_r \\ &\leq \frac{s}{r}\cdot\frac{M^{s-r} - m^{s-r}}{M^r - m^r}\left[M^r\mathbf{1} - [M_r(\mathbf{x},\phi)]^r\right]\left[[M_r(\mathbf{x},\phi)]^r - m^r\mathbf{1}\right] - \delta_r^s\underline{\mathbf{x}}_r \\ &\leq \frac{s}{4r}\left(M^r - m^r\right)\left(M^{s-r} - m^{s-r}\right)\mathbf{1} - \delta_r^s\underline{\mathbf{x}}_r \end{aligned} \tag{4.54}$$

and

$$\begin{aligned} &[M_s(\mathbf{x},\phi)]^s - [M_r(\mathbf{x},\phi)]^s \leq \frac{1}{4}\left(M^r - m^r\right)^2\Delta_r^s\left(M_r(\mathbf{x},\phi);m,M\right) - \delta_r^s\underline{\mathbf{x}}_r \\ &\leq \frac{s}{4r}\left(M^r - m^r\right)\left(M^{s-r} - m^{s-r}\right)\mathbf{1} - \delta_r^s\underline{\mathbf{x}}_r \end{aligned} \tag{4.55}$$

hold. Further, if $0 \leq s \leq r \neq 0$ or $0 \neq r \leq s \leq 0$, then the signs of inequalities in (4.54) *and* (4.55) *are reversed.*

(ii) If $r<0$, then

$$\begin{aligned}
0 &\le \log\left[M_0(\mathbf{x},\phi)\right]-\log\left[M_r(\mathbf{x},\phi)\right]\\
&\le \sup_{t\in(m,M)} \Delta_r^*(t;m,M)\left[M^r\mathbf{1}-\left[M_r(\mathbf{x},\phi)\right]^r\right]\left[\left[M_r(\mathbf{x},\phi)\right]^r-m^r\mathbf{1}\right]-\delta_r^*\underline{\mathbf{x}}_r\\
&\le -\frac{1}{rM^rm^r}\left[M^r\mathbf{1}-\left[M_r(\mathbf{x},\phi)\right]^r\right]\left[\left[M_r(\mathbf{x},\phi)\right]^r-m^r\mathbf{1}\right]-\delta_r^*\underline{\mathbf{x}}_r\\
&\le -\frac{(M^r-m^r)^2}{4rM^rm^r}\mathbf{1}-\delta_r^*\underline{\mathbf{x}}_r
\end{aligned}\tag{4.56}$$

and

$$\begin{aligned}
0 &\le \log\left[M_0(\mathbf{x},\phi)\right]-\log\left[M_r(\mathbf{x},\phi)\right]\\
&\le \frac{1}{4}(M^r-m^r)^2\,\Delta_r^*\left(M_r(\mathbf{x},\phi);m,M\right)-\delta_r^*\underline{\mathbf{x}}_r\\
&\le -\frac{(M^r-m^r)^2}{4rM^rm^r}\mathbf{1}-\delta_r^*\underline{\mathbf{x}}_r,
\end{aligned}\tag{4.57}$$

while for $r>0$ the signs of inequalities in (4.56) *and* (4.57) *are reversed.*

(iii) The series of inequalities

$$\begin{aligned}
&\left[M_s(\mathbf{x},\phi)\right]^s-\left[M_0(\mathbf{x},\phi)\right]^s\\
&\le \sup_{t\in(m,M)} \Delta_*^s(t;m,M)\left[\log M\mathbf{1}-\log\left[M_0(\mathbf{x},\phi)\right]\right]\left[\log\left[M_0(\mathbf{x},\phi)\right]-\log m\mathbf{1}\right]-\delta_*^s\underline{\mathbf{x}}_0\\
&\le \frac{s\,(M^s-m^s)}{\log M-\log m}\left[\log M\mathbf{1}-\log\left[M_0(\mathbf{x},\phi)\right]\right]\left[\log\left[M_0(\mathbf{x},\phi)\right]-\log m\mathbf{1}\right]-\delta_*^s\underline{\mathbf{x}}_0\\
&\le \frac{s}{4}(\log M-\log m)\,(M^s-m^s)\,\mathbf{1}-\delta_*^s\underline{\mathbf{x}}_0
\end{aligned}\tag{4.58}$$

and

$$\begin{aligned}
\left[M_s(\mathbf{x},\phi)\right]^s-\left[M_0(\mathbf{x},\phi)\right]^s &\le \frac{1}{4}(\log M-\log m)^2\Delta_*^s\left(M_0(\mathbf{x},\phi);m,M\right)-\delta_*^s\underline{\mathbf{x}}_0\\
&\le \frac{s}{4}(\log M-\log m)\,(M^s-m^s)\,\mathbf{1}-\delta_*^s\underline{\mathbf{x}}_0
\end{aligned}\tag{4.59}$$

hold for all $s\in\mathbb{R}$.

Proof. The proof is a consequence of Theorem 4.8, accompanied with particular choices of functions χ and ψ.

First, set $\chi(t)=t^s$ and $\psi(t)=t^r$, where s and r are real parameters such that $r\neq 0$. The function $\left(\chi\circ\psi^{-1}\right)(t)=t^{\frac{s}{r}}$ is convex on $\mathbb{R}_+$ if $\frac{s}{r}\le 0$ or $\frac{s}{r}\ge 1$, which is possible in each of the following four cases: $s\le 0<r$ or $r<0\le s$ or $0<r\le s$ or $s\le r<0$. Finally, since $\left(\chi\circ\psi^{-1}\right)'(t)=\frac{s}{r}t^{\frac{s-r}{r}}$, considering (4.47) and (4.48) with above functions χ and ψ on the interval $[m,M]$, we obtain (4.54) and (4.55).

Conversely, the function $\left(\chi\circ\psi^{-1}\right)(t)=t^{\frac{s}{r}}$ is concave on $\mathbb{R}_+$ provided that $0\le\frac{s}{r}\le 1$, therefore, if $0\le s\le r\neq 0$ or $0\neq r\le s\le 0$, we obtain relations (4.54) and (4.55) with reversed signs of inequalities.

It remains to consider non-trivial cases when one of parameters r and s is equal to zero. If $s=0$, then, putting $\chi(t)=\log t$ and $\psi(t)=t^r$, it follows that $\left(\chi\circ\psi^{-1}\right)(t)=\frac{1}{r}\log t$. Obviously, this function is convex (concave) for $r<0$ ($r>0$). In addition, since $\left(\chi\circ\psi^{-1}\right)'(t)=\frac{1}{rt}$, we obtain (4.56) and (4.57) without the first inequality sign. The first inequality sign in (4.56) and (4.57), as well as in the corresponding reversed inequalities, holds due to the operator convexity (concavity) of the function $\frac{1}{r}\log t$ when $r<0$ ($r>0$).

Finally, if $r=0$, then, setting $\chi(t)=t^s$ and $\psi(t)=\log t$, it follows that the function $\left(\chi\circ\psi^{-1}\right)(t)=\exp(st)$ is convex for every $s\in\mathbb{R}$. Moreover, since $\left(\chi\circ\psi^{-1}\right)'(t)=s\exp(st)$, we obtain (4.58) and (4.59), and the proof is completed. □

Remark 4.15 Generally speaking, the element $[M_s(\mathbf{x},\phi)]^s-[M_r(\mathbf{x},\phi)]^s$, appearing in (4.54) and (4.55), is not positive semi-definite. Certainly, positivity of this element depends on the operator convexity of a power function. It is well-known that the function $f(t)=t^r$ is operator convex on $\mathbb{R}_+$ if either $1\le r\le 2$ or $-1\le r\le 0$, and is operator concave on $\mathbb{R}_+$ when $0\le r\le 1$ (for more details, see e.g. [49]). Therefore, discussing the operator convexity of the function $\left(\chi\circ\psi^{-1}\right)(t)=t^{\frac{s}{r}}$, as in the proof of Corollary 4.4, we obtain conditions for parameters r and s under which the operator $[M_s(\mathbf{x},\phi)]^s-[M_r(\mathbf{x},\phi)]^s$ is positive semi-definite. More precisely,

$$0\le [M_s(\mathbf{x},\phi)]^s-[M_r(\mathbf{x},\phi)]^s \tag{4.60}$$

holds if either

$$0<r\le s\le 2r \text{ or } 2r\le s\le r<0 \text{ or } 0\le s+r\le r\ne 0 \text{ or } 0\ne r\le r+s\le 0. \tag{4.61}$$

On the other hand, if

$$0\ne r\le s\le 0 \text{ or } 0\le s\le r\ne 0, \tag{4.62}$$

then the sign of inequality in (4.60) is reversed. Moreover, since the operator convexity (concavity) of a power function implies its usual convexity (concavity), it follows that relations (4.54), (4.55), and (4.60) simultaneously hold under conditions as in (4.61). The reverse relations simultaneously hold provided that conditions as in (4.62) are fulfilled.

Remark 4.16 Above discussion with regard to operator convexity can not be applied to relations (4.58) and (4.59), since the exponential function $f(t)=\exp t$ is not operator convex (see e.g. [16]).

For the same choices of functions χ and ψ as above, but in Corollary 4.2 and Corollary 4.3, we get the following results respectively.

Corollary 4.5 *Let $(\mathbf{x}_t)_{t\in T}$ be a bounded continuous field of self-adjoint elements in C^*-algebra with spectra in $[m,M]$, $0<m<M$.*

(i) If either $s\le 0<r$ or $r<0\le s$ or $0<r\le s$ or $s\le r<0$, then

$$\begin{aligned}&[M_s(\mathbf{x},\phi)]^s-[M_r(\mathbf{x},\phi)]^s\\ &\le \frac{\delta_r^s}{|M^r-m^r|}\left[\left|[M_r(\mathbf{x},\phi)]^r-\frac{m^r+M^r}{2}\mathbf{1}\right|+\int_T\phi_t\left(\left|\mathbf{x}_t^r-\frac{m^r+M^r}{2}\mathbf{1}\right|\right)d\mu(t)\right]\\ &\le \delta_r^s\mathbf{1}.\end{aligned}$$

Moreover, if $0 \le s \le r \neq 0$ *or* $0 \neq r \le s \le 0$*, then the signs of inequalities are reversed.*

(ii) If $r < 0$*, then*

$$\begin{aligned} 0 &\le \log\left[M_0(\mathbf{x},\phi)\right] - \log\left[M_r(\mathbf{x},\phi)\right] \\ &\le \frac{\delta_r^*}{|M^r - m^r|}\left[\left|[M_r(\mathbf{x},\phi)]^r - \frac{m^r+M^r}{2}\mathbf{1}\right| + \int_T \phi_t\left(\left|\mathbf{x}_t^r - \frac{m^r+M^r}{2}\mathbf{1}\right|\right)d\mu(t)\right] \\ &\le \delta_r^*\mathbf{1}, \end{aligned}$$

while for $r > 0$ *the signs of inequalities are reversed.*

(iii) The series of inequalities

$$\begin{aligned} &[M_s(\mathbf{x},\phi)]^s - [M_0(\mathbf{x},\phi)]^s \\ &\le \frac{\delta_*^s}{\log M - \log m}\left[\left|\log\left[M_0(\mathbf{x},\phi)\right] - \log\sqrt{mM}\mathbf{1}\right| + \int_T \phi_t\left(\left|\log\mathbf{x}_t - \log\sqrt{mM}\mathbf{1}\right|\right)d\mu(t)\right] \\ &\le \delta_*^s\mathbf{1} \end{aligned}$$

hold for all $s \in \mathbb{R}$*.*

Corollary 4.6 *Let* $(\mathbf{x}_t)_{t\in T}$ *be a bounded continuous field of self-adjoint elements in* C^**-algebra with spectra in* $[m,M]$*,* $0 < m < M$*.*

(i) If any of the relations $s \le 0 < r$ *or* $r < 0 \le s$ *or* $0 < r \le s$ *or* $s \le r < 0$ *holds, then we have*

$$\begin{aligned} \delta_r^s \underline{\mathbf{x}}_r &\le \frac{M^s - m^s}{M^r - m^r}\left[M_r(x,\phi)\right]^r + \frac{M^r m^s - m^r M^s}{M^r - m^r}\mathbf{1} - [M_s(x,\phi)]^s \\ &\le \sup_{t\in(m,M)} \delta_r^s(t;m,M)\int_T \phi_t\left([M^r\mathbf{1} - x_t^r][x_t^r - m^r\mathbf{1}]\right)d\mu(t) \\ &\le \frac{s}{r}\cdot\frac{M^{s-r} - m^{s-r}}{M^r - m^r}\int_T \phi_t\left([M^r\mathbf{1} - x_t^r][x_t^r - m^r\mathbf{1}]\right)d\mu(t) \\ &\le \frac{s}{r}\cdot\frac{M^{s-r} - m^{s-r}}{M^r - m^r}\left[M^r\mathbf{1} - [M_r(x,\phi)]^r\right]\left[[M_r(x,\phi)]^r - m^r\mathbf{1}\right] \\ &\le \frac{s}{4r}\left(M^r - m^r\right)\left(M^{s-r} - m^{s-r}\right)\mathbf{1}, \end{aligned}$$

$$\begin{aligned} \delta_r^s \underline{\mathbf{x}}_r &\le \frac{M^s - m^s}{M^r - m^r}\left[M_r(x,\phi)\right]^r + \frac{M^r m^s - m^r M^s}{M^r - m^r}\mathbf{1} - [M_s(x,\phi)]^s \\ &\le \sup_{t\in(m,M)} \delta_r^s(t;m,M)\int_T \phi_t\left([M^r\mathbf{1} - x_t^r][x_t^r - m^r\mathbf{1}]\right)d\mu(t) \\ &\le \sup_{t\in(m,M)} \delta_r^s(t;m,M)\left[M^r\mathbf{1} - [M_r(x,\phi)]^r\right]\left[[M_r(x,\phi)]^r - m^r\mathbf{1}\right] \\ &\le \frac{s}{r}\cdot\frac{M^{s-r} - m^{s-r}}{M^r - m^r}\left[M^r\mathbf{1} - [M_r(x,\phi)]^r\right]\left[[M_r(x,\phi)]^r - m^r\mathbf{1}\right] \\ &\le \frac{s}{4r}\left(M^r - m^r\right)\left(M^{s-r} - m^{s-r}\right)\mathbf{1}, \end{aligned}$$

and

$$
\begin{aligned}
\delta_r^s \underline{\mathbf{x}}_r \leq & \frac{M^s - m^s}{M^r - m^r} \left[M_r\left(x,\phi\right)\right]^r + \frac{M^r m^s - m^r M^s}{M^r - m^r} \mathbf{1} - \left[M_s(x,\phi)\right]^s \\
\leq & \frac{1}{4}\left(M^r - m^r\right)^2 \int_T \phi_t\left(\delta_r^s\left(x_t; m, M\right)\right) d\mu(t) \\
\leq & \frac{s}{4r}\left(M^r - m^r\right)\left(M^{s-r} - m^{s-r}\right) \mathbf{1}.
\end{aligned}
$$

Additionally, if either $0 \leq s \leq r \neq 0$ *or* $0 \neq r \leq s \leq 0$ *holds, then the signs of inequalities are reversed.*

(ii) If $r < 0$*, then*

$$
\begin{aligned}
\delta_r^* \underline{\mathbf{x}}_r \leq & \frac{\log M - \log m}{M^r - m^r} \left[M_r\left(x,\phi\right)\right]^r + \frac{M^r \log m - m^r \log M}{M^r - m^r} \mathbf{1} - \log\left[M_0(x,\phi)\right] \\
\leq & \sup_{t \in (m,M)} \delta_r^*(t; m, M) \int_T \phi_t\left([M^r \mathbf{1} - x_t^r][x_t^r - m^r \mathbf{1}]\right) d\mu(t) \\
\leq & -\frac{1}{rM^r m^r} \int_T \phi_t\left([M^r \mathbf{1} - x_t^r][x_t^r - m^r \mathbf{1}]\right) d\mu(t) \\
\leq & -\frac{1}{rM^r m^r} \left[M^r \mathbf{1} - \left[M_r\left(x,\phi\right)\right]^r\right]\left[\left[M_r\left(x,\phi\right)\right]^r - m^r \mathbf{1}\right] \\
\leq & -\frac{\left(M^r - m^r\right)^2}{4rM^r m^r} \mathbf{1},
\end{aligned}
$$

$$
\begin{aligned}
\delta_r^* \underline{\mathbf{x}}_r \leq & \frac{\log M - \log m}{M^r - m^r} \left[M_r\left(x,\phi\right)\right]^r + \frac{M^r \log m - m^r \log M}{M^r - m^r} \mathbf{1} - \log\left[M_0(x,\phi)\right] \\
\leq & \sup_{t \in (m,M)} \delta_r^*(t; m, M) \int_T \phi_t\left([M^r \mathbf{1} - x_t^r][x_t^r - m^r \mathbf{1}]\right) d\mu(t) \\
\leq & \sup_{t \in (m,M)} \delta_r^*(t; m, M) \left[M^r \mathbf{1} - \left[M_r\left(x,\phi\right)\right]^r\right]\left[\left[M_r\left(x,\phi\right)\right]^r - m^r \mathbf{1}\right] \\
\leq & -\frac{1}{rM^r m^r} \left[M^r \mathbf{1} - \left[M_r\left(x,\phi\right)\right]^r\right]\left[\left[M_r\left(x,\phi\right)\right]^r - m^r \mathbf{1}\right] \\
\leq & -\frac{\left(M^r - m^r\right)^2}{4rM^r m^r} \mathbf{1},
\end{aligned}
$$

and

$$
\begin{aligned}
\delta_r^* \underline{\mathbf{x}}_r \leq & \frac{\log M - \log m}{M^r - m^r} \left[M_r\left(x,\phi\right)\right]^r + \frac{M^r \log m - m^r \log M}{M^r - m^r} \mathbf{1} - \log\left[M_0(x,\phi)\right] \\
\leq & \frac{1}{4}\left(M^r - m^r\right)^2 \int_T \phi_t\left(\delta_r^*\left(x_t; m, M\right)\right) d\mu(t) \\
\leq & -\frac{\left(M^r - m^r\right)^2}{4rM^r m^r} \mathbf{1},
\end{aligned}
$$

while for $r > 0$ *the signs of inequalities are reversed.*

(iii) Series of inequalities

$$\begin{aligned}
\delta_*^s \underline{\mathbf{x}}_0 &\leq \frac{M^s - m^s}{\log M - \log m} \log\left[M_0\left(x,\phi\right)\right] + \frac{m^s \log M - M^s \log m}{\log M - \log m} \mathbf{1} - \left[M_s(x,\phi)\right]^s \\
&\leq \sup_{t \in (m,M)} \delta_*^s(t;m,M) \int_T \phi_t\left(\left[\log M\mathbf{1} - \log x_t\right]\left[\log x_t - \log m\mathbf{1}\right]\right) d\mu(t) \\
&\leq \frac{s\left(M^s - m^s\right)}{\log M - \log m} \int_T \phi_t\left(\left[\log M\mathbf{1} - \log x_t\right]\left[\log x_t - \log m\mathbf{1}\right]\right) d\mu(t) \\
&\leq \frac{s\left(M^s - m^s\right)}{\log M - \log m} \left[\log M\mathbf{1} - \log\left[M_0\left(x,\phi\right)\right]\right]\left[\log\left[M_0\left(x,\phi\right)\right] - \log m\mathbf{1}\right] \\
&\leq \frac{s}{4}\left(\log M - \log m\right)\left(M^s - m^s\right)\mathbf{1},
\end{aligned}$$

$$\begin{aligned}
\delta_*^s \underline{\mathbf{x}}_0 &\leq \frac{M^s - m^s}{\log M - \log m} \log\left[M_0\left(x,\phi\right)\right] + \frac{m^s \log M - M^s \log m}{\log M - \log m} \mathbf{1} - \left[M_s(x,\phi)\right]^s \\
&\leq \sup_{t \in (m,M)} \delta_*^s(t;m,M) \int_T \phi_t\left(\left[\log M\mathbf{1} - \log x_t\right]\left[\log x_t - \log m\mathbf{1}\right]\right) d\mu(t) \\
&\leq \sup_{t \in (m,M)} \delta_*^s(t;m,M) \left[\log M\mathbf{1} - \log\left[M_0\left(x,\phi\right)\right]\right]\left[\log\left[M_0\left(x,\phi\right)\right] - \log m\mathbf{1}\right] \\
&\leq \frac{s\left(M^s - m^s\right)}{\log M - \log m} \left[\log M\mathbf{1} - \log\left[M_0\left(x,\phi\right)\right]\right]\left[\log\left[M_0\left(x,\phi\right)\right] - \log m\mathbf{1}\right] \\
&\leq \frac{s}{4}\left(\log M - \log m\right)\left(M^s - m^s\right)\mathbf{1},
\end{aligned}$$

and

$$\begin{aligned}
\delta_*^s \underline{\mathbf{x}}_0 &\leq \frac{M^s - m^s}{\log M - \log m} \log\left[M_0\left(x,\phi\right)\right] + \frac{m^s \log M - M^s \log m}{\log M - \log m} \mathbf{1} - \left[M_s(x,\phi)\right]^s \\
&\leq \frac{1}{4}\left(\log M - \log m\right)^2 \int_T \phi_t\left(\delta_*^s(x_t;m,M)\right) \\
&\leq \frac{s}{4}\left(\log M - \log m\right)\left(M^s - m^s\right)\mathbf{1}
\end{aligned}$$

hold for every $s \in \mathbb{R}$.

4.4 Jensen-type inequalities for bounded and Lipschitzian functions

It is interesting that the estimates, mentioned and proved in the previous sections, can be established even for some more general classes of functions. The main objective of this section is to establish mutual bounds for the Jensen operator inequality (4.1) and the Lah-Ribarič operator inequality (4.2) in the case of bounded real-valued functions and Lipschitzian functions. Further, obtained results are then applied to quasi-arithmetic operator means, with a particular emphasis to power operator means. In such a way, we obtain some new reverse relations for quasi-arithmetic and power operator means.

In order to do so, we first need to state the corresponding estimate for the scalar Lah-Ribarič inequality obtained by Dragomir [40]: If $f:[m,M]\to[\gamma,\Gamma]$ is a boundexd real-valued function, then the inequality

$$\left|\alpha_f t+\beta_f-f(t)\right|\le\Gamma-\gamma \tag{4.63}$$

holds for every $t\in[m,M]$. In addition, the constant $\Gamma-\gamma$ is the best possible in the sense that it cannot be replaced by a smaller quantity.

In the same paper Dragomir also established a similar relation that correspond to L-Lipschitzian functions. Recall that a function $f: I\to\mathbb{R}$ is said to be L-Lipschitzian if there exists a constant L such that

$$|f(x)-f(y)|\le L|x-y|$$

holds for all $x,y\in I$. Now, if $f\colon [m,M]\to\mathbb{R}$ is a L-Lipschitzian function, Dragomir showed that the relation

$$\left|\alpha_f t+\beta_f-f(t)\right|\le 2L\frac{(M-t)(t-m)}{M-m}\le\frac{L}{2}(M-m) \tag{4.64}$$

holds for every $t\in[m,M]$.

Scalar inequalities (4.63) and (4.64) will be crucial in establishing our main results.

The techniques that will be used in the proofs are mainly based on the classical real and functional analysis, and on a bounded Borel functional calculus, especially on the monotonicity principle for self-adjoint operators on a Hilbert space: If X has a spectra $\mathrm{Sp}(X)$, then

$$f(t)\ge g(t),\ t\in\mathrm{Sp}(X)\quad\Longrightarrow\quad f(X)\ge g(X),$$

where f and g are bounded Borel functions. For more details about the bounded Borel functional calculus, the reader is referred to [8] or [132]. It should be noticed here that, throughout this section, all the functions are assumed to be Borel measurable.

Our first result refers to an operator extension of the scalar inequality (4.63) with respect to the Bochner integral defined in the Introduction.

Theorem 4.9 ([82]) *Let $f:[m,M]\to[\gamma,\Gamma]$ be a real-valued function. Further, suppose $\mathscr{A}$ and $\mathscr{B}$ are unital C^*-algebras and $(\phi_t)_{t\in T}$ is a unital field of positive linear mappings ϕ_t :*

$\mathscr{A} \to \mathscr{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ. Then, inequalities

$$-(\Gamma-\gamma)\mathbf{1} \leq \alpha_f \int_T \phi_t(X_t)d\mu(t) + \beta_f \mathbf{1} - \int_T \phi_t(f(X_t))d\mu(t) \leq (\Gamma-\gamma)\mathbf{1} \tag{4.65}$$

and

$$\left| \alpha_f \int_T \phi_t(X_t)d\mu(t) + \beta_f \mathbf{1} - \int_T \phi_t(f(X_t))d\mu(t) \right| \leq (\Gamma-\gamma)\mathbf{1} \tag{4.66}$$

hold for every bounded continuous field $(X_t)_{t\in T}$ of self-adjoint elements in $\mathscr{A}$ with spectra contained in $[m,M]$. In addition, the constant $\Gamma-\gamma$ cannot be replaced by a smaller quantity.

Proof. Since $(X_t)_{t\in T}$ is a bounded continuous field of self-adjoint operators with spectra contained in $[m,M]$, it follows that $m\mathbf{1} \leq X_t \leq M\mathbf{1}$, for every $t \in T$. Hence, applying the functional calculus to scalar inequality (4.63), that is, by setting X_t instead of t, we get

$$-(\Gamma-\gamma)\mathbf{1} \leq \alpha_f X_t + \beta_f \mathbf{1} - f(X_t) \leq (\Gamma-\gamma)\mathbf{1}.$$

Now, (4.65) follows after applying ϕ_t to the previous relation and then integrating.

As regards the inequality (4.66), by squaring the inequality (4.63) it follows that

$$\left(\alpha_f t + \beta_f - f(t)\right)^2 \leq (\Gamma-\gamma)^2$$

holds for every $t \in [m,M]$. Further, applaying the functional calculus to the above squared inequality, we have

$$\left(\alpha_f X_t + \beta_f \mathbf{1} - f(X_t)\right)^2 \leq (\Gamma-\gamma)^2\mathbf{1}.$$

Now, since the linear mapping preserves the order, applying ϕ_t to the above relation yields

$$\phi_t\left(\left[\alpha_f X_t + \beta_f \mathbf{1} - f(X_t)\right]^2\right) \leq (\Gamma-\gamma)^2\phi_t(\mathbf{1}),$$

while integrating yields

$$\int_T \phi_t\left(\left[\alpha_f X_t + \beta_f \mathbf{1} - f(X_t)\right]^2\right) d\mu(t) \leq (\Gamma-\gamma)^2\mathbf{1},$$

due to $\int_T \phi_t(\mathbf{1})\, d\mu(t) = \mathbf{1}$.

It should be noticed here that the function $g(t) = t^2$ is operator convex (see e.g. [16, 49]), so utilizing the Jensen operator inequality (4.1) it follows that

$$\left(\alpha_f \int_T \phi_t(X_t)d\mu(t) + \beta_f \phi_t(\mathbf{1}) - \int_T \phi_t(f(X_t))d\mu(t)\right)^2$$
$$\leq \int_T \phi_t\left(\left[\alpha_f X_t + \beta_f \mathbf{1} - f(X_t)\right]^2\right) d\mu(t),$$

and consequently

$$\left(\alpha_f \int_T \phi_t(X_t)d\mu(t) + \beta_f \phi_t(\mathbf{1}) - \int_T \phi_t(f(X_t))d\mu(t)\right)^2 \leq (\Gamma - \gamma)^2 \mathbf{1}.$$

Finally, relation (4.66) follows from definition of the absolute value of a self-adjoint operator and due to the operator monotonicity of the function $h(t) = \sqrt{t}$ (see e.g. [16, 49]).

Now, in order to show that the constant $\Gamma - \gamma$ is the best possible in (4.66), assume that inequality (4.66) holds with a multiplicative constant C as follows:

$$\left|\alpha_f \int_T \phi_t(X_t)d\mu(t) + \beta_f \mathbf{1} - \int_T \phi_t(f(X_t))d\mu(t)\right| \leq C(\Gamma - \gamma)\mathbf{1}. \tag{4.67}$$

Let us consider the function

$$f(t) = \begin{cases} \frac{M-m}{2}, & t = m, \\ 0, & t \in (m, M), \\ \frac{M-m}{2}, & t = M. \end{cases}$$

Clearly, the function f is bounded, and we can set $\Gamma = \frac{M-m}{2}$ and $\gamma = 0$. Now, if we substitute $X_t = \frac{m+M}{2}\mathbf{1}$ and the above function f in (4.67), it follows that

$$\left|\frac{M - \frac{m+M}{2}}{M-m} \cdot \frac{M-m}{2}\mathbf{1} + \frac{\frac{m+M}{2} - m}{M-m} \cdot \frac{M-m}{2}\mathbf{1} - 0\right| \leq C\frac{M-m}{2}\mathbf{1},$$

which reduces to

$$\left|\frac{M-m}{2}\mathbf{1}\right| \leq C\frac{M-m}{2}\mathbf{1}.$$

The above inequality implies that $C \geq 1$ and shows that $C = 1$ is the best possible constant in (4.67). □

Remark 4.17 By applying positive linear mappings ϕ_t, $t \in T$, to $m\mathbf{1} \leq X_t \leq M\mathbf{1}$ and then, integrating, we get $m\mathbf{1} \leq \int_T \phi_t(X_t)d\mu(t) \leq M\mathbf{1}$. Since the function f is bounded, in the same way it follows that $\int_T \phi_t(f(X_t))d\mu(t)$ is bounded as well. This means that the operator $X := \alpha_f \int_T \phi_t(X_t)d\mu(t) + \beta_f \phi_t(\mathbf{1}) - \int_T \phi_t(f(X_t))d\mu(t)$ is bounded. Therefore, the absolute value of X is $|X| = \sqrt{X^*X} = \sqrt{X^2}$, since $X^* = X$.

Remark 4.18 In Theorem 4.9 we have established inequalities of the form $-Y \leq X \leq Y$ and $|X| \leq Y$, where X, Y are bounded self-adjoint operators. Generally speaking, if $-Y \leq X \leq Y$, then it need not be true that $|X| \leq Y$. For more details, the reader is referred to [79].

By virtue of Theorem 4.9, we can establish mutual bounds for the Jensen operator inequality (4.1), for the class of bounded real-valued functions.

Theorem 4.10 ([82]) *Let $f:[m,M]\to[\gamma,\Gamma]$ be a real-valued function. Further, suppose $\mathscr{A}$ and $\mathscr{B}$ are unital C^*-algebras, and $(\phi_t)_{t\in T}$ is a unital field of positive linear mappings $\phi_t:\mathscr{A}\to\mathscr{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ. Then*

$$-2(\Gamma-\gamma)\mathbf{1}\leq\int_T\phi_t(f(X_t))d\mu(t)-f\left(\int_T\phi_t(X_t)d\mu(t)\right)\leq 2(\Gamma-\gamma)\mathbf{1} \tag{4.68}$$

holds for every bounded continuous field $(X_t)_{t\in T}$ of self-adjoint elements in $\mathscr{A}$ with spectra contained in $[m,M]$.

Proof. Since $(X_t)_{t\in T}$ is a bounded continuous field of self-adjoint operators with spectra contained in $[m,M]$, we have $m\mathbf{1}\leq X_t\leq M\mathbf{1}$ for every $t\in T$, so applying the positive linear mapping ϕ_t and then integrating, we have

$$m\mathbf{1}\leq\int_T\phi_t(X_t)d\mu(t)\leq M\mathbf{1}.$$

Therefore, applying the functional calculus to (4.63) by setting $\int_T\phi_t(X_t)d\mu(t)$ instead of t, it follows that

$$-(\Gamma-\gamma)\mathbf{1}\leq\alpha_f\int_T\phi_t(X_t)d\mu(t)+\beta_f\mathbf{1}-f\left(\int_T\phi_t(X_t)d\mu(t)\right)\leq(\Gamma-\gamma)\mathbf{1}. \tag{4.69}$$

After adding up (4.69) and (4.65) we get (4.68), and the proof is complete. □

Our next result refers to an operator extension of the scalar inequality (4.64) for a class of Lipschitzian functions.

Theorem 4.11 ([82]) *Let $f:[m,M]\to\mathbb{R}$ be a L-Lipschitzian function. Further, suppose $\mathscr{A}$ and $\mathscr{B}$ are unital C^*-algebras, and $(\phi_t)_{t\in T}$ is a unital field of positive linear mappings $\phi_t:\mathscr{A}\to\mathscr{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ. Then the set of inequalities*

$$\begin{aligned}
&-\frac{L}{2}(M-m)\mathbf{1}\\
&\leq\frac{-2L}{M-m}\left(M\mathbf{1}-\int_T\phi_t(X_t)d\mu(t)\right)\left(\int_T\phi_t(X_t)d\mu(t)-m\mathbf{1}\right)\\
&\leq\frac{-2L}{M-m}\int_T\phi_t\left[(M\mathbf{1}-X_t)(X_t-m\mathbf{1})\right]d\mu(t)\\
&\leq\alpha_f\int_T\phi_t(X_t)d\mu(t)+\beta_f\mathbf{1}-\int_T\phi_t(f(X_t))d\mu(t)\\
&\leq\frac{2L}{M-m}\int_T\phi_t\left[(M\mathbf{1}-X_t)(X_t-m\mathbf{1})\right]d\mu(t)\\
&\leq\frac{2L}{M-m}\left(M\mathbf{1}-\int_T\phi_t(X_t)d\mu(t)\right)\left(\int_T\phi_t(X_t)d\mu(t)-m\mathbf{1}\right)\\
&\leq\frac{L}{2}(M-m)\mathbf{1}
\end{aligned} \tag{4.70}$$

holds for every bounded continuous field $(X_t)_{t\in T}$ *of self-adjoint elements in* $\mathscr{A}$ *with spectra contained in* $[m,M]$.

Proof. Since $m\mathbf{1} \le X_t \le M\mathbf{1}$ for every $t \in T$, applying the functional calculus to scalar inequality (4.64), it follows that

$$-\frac{2L}{M-m}(M\mathbf{1}-X_t)(X_t-m\mathbf{1}) \le \alpha_f X_t + \beta_f - f(X_t) \le \frac{2L}{M-m}(M\mathbf{1}-X_t)(X_t-m\mathbf{1}).$$

Furthermore, applying linear mappings ϕ_t to the above set of inequalities and then, integrating, we have

$$\begin{aligned}
&-\frac{2L}{M-m}\int_T \phi_t\left[(M\mathbf{1}-X_t)(X_t-m\mathbf{1})\right]d\mu(t)\\
&\le \alpha_f \int_T \phi_t(X_t)d\mu(t) + \beta_f\mathbf{1} - \int_T \phi_t(f(X_t))d\mu(t)\\
&\le \frac{2L}{M-m}\int_T \phi_t\left[(M\mathbf{1}-X_t)(X_t-m\mathbf{1})\right]d\mu(t).
\end{aligned}$$

Now, taking into account that the function $g(t) = -t^2+(m+M)t-mM$ is operator concave (see e.g. [49]), application of the Jensen operator inequality (4.1) yields

$$\begin{aligned}
&\frac{2L}{M-m}\int_T \phi_t\left[(M\mathbf{1}-X_t)(X_t-m\mathbf{1})\right]d\mu(t)\\
&\le \frac{2L}{M-m}\left(M\mathbf{1}-\int_T \phi_t(X_t)d\mu(t)\right)\left(\int_T \phi_t(X_t)d\mu(t)-m\mathbf{1}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{-2L}{M-m}\int_T \phi_t\left[(M\mathbf{1}-X_t)(X_t-m\mathbf{1})\right]d\mu(t)\\
&\ge \frac{-2L}{M-m}\left(M\mathbf{1}-\int_T \phi_t(X_t)d\mu(t)\right)\left(\int_T \phi_t(X_t)d\mu(t)-m\mathbf{1}\right).
\end{aligned}$$

Finally, the first and the last inequality in (4.70) are direct consequences of the arithmetic-geometric mean inequality

$$(M-t)(t-m) \le \frac{1}{4}(M-m)^2,$$

where $t \in [m,M]$. □

In a similar way as before, Theorem 4.11 can also be utilized for establishing double precision of the Jensen operator inequality (4.1), this time for a class of L-Lipschitzian functions.

Theorem 4.12 ([82]) *Let* $f : [m,M] \to \mathbb{R}$ *be a L-Lipschitzian function. Further, suppose* $\mathscr{A}$ *and* $\mathscr{B}$ *are unital* C^**-algebras, and* $(\phi_t)_{t\in T}$ *is a unital field of positive linear mappings* $\phi_t : \mathscr{A} \to \mathscr{B}$ *defined on a locally compact Hausdorff space* T *with a bounded Radon measure* μ*. Then the set of inequalities*

$$
\begin{aligned}
& -L(M-m)\mathbf{1} \\
& \leq \frac{-4L}{M-m}\left(M\mathbf{1}-\int_T \phi_t(X_t)d\mu(t)\right)\left(\int_T \phi_t(X_t)d\mu(t)-m\mathbf{1}\right) \\
& \leq \frac{-2L}{M-m}\left[\int_T \phi_t\left[(M\mathbf{1}-X_t)(X_t-m\mathbf{1})\right]d\mu(t)\right. \\
& \qquad \left. +\left(M\mathbf{1}-\int_T \phi_t(X_t)d\mu(t)\right)\left(\int_T \phi_t(X_t)d\mu(t)-m\mathbf{1}\right)\right] \\
& \leq \int_T \phi_t(f(X_t))d\mu(t)-f\left(\int_T \phi_t(X_t)d\mu(t)\right) \qquad (4.71) \\
& \leq \frac{2L}{M-m}\left[\int_T \phi_t\left[(M\mathbf{1}-X_t)(X_t-m\mathbf{1})\right]d\mu(t)\right. \\
& \qquad \left. +\left(M\mathbf{1}-\int_T \phi_t(X_t)d\mu(t)\right)\left(\int_T \phi_t(X_t)d\mu(t)-m\mathbf{1}\right)\right] \\
& \leq \frac{4L}{M-m}\left(M\mathbf{1}-\int_T \phi_t(X_t)d\mu(t)\right)\left(\int_T \phi_t(X_t)d\mu(t)-m\mathbf{1}\right) \\
& \leq L(M-m)\mathbf{1}
\end{aligned}
$$

holds for every bounded continuous field $(X_t)_{t\in T}$ of self-adjoint elements in $\mathscr{A}$ with spectra contained in $[m,M]$.

Proof. In the same way as in the proof of the previous theorem, it follows that

$$
m\mathbf{1} \leq \int_T \phi_t(X_t)d\mu(t) \leq M\mathbf{1},
$$

and so, by putting $\int_T \phi_t(X_t)d\mu(t)$ in (4.64) we obtain

$$
\begin{aligned}
& -\frac{2L}{M-m}\left(M-\int_T \phi_t(X_t)d\mu(t)\right)\left(\int_T \phi_t(X_t)d\mu(t)-m\right) \\
& \leq \alpha_f \int_T \phi_t(X_t)d\mu(t)+\beta_f\mathbf{1}-f\left(\int_T \phi_t(X_t)d\mu(t)\right) \qquad (4.72) \\
& \leq \frac{2L}{M-m}\left(M-\int_T \phi_t(X_t)d\mu(t)\right)\left(\int_T \phi_t(X_t)d\mu(t)-m\right).
\end{aligned}
$$

Now, taking into account the set of inequalities (4.70) multiplied by -1, it follows that

$$
\begin{aligned}
& -\frac{2L}{M-m}\int_T \phi_t\left[(M\mathbf{1}-X_t)(X_t-m\mathbf{1})\right]d\mu(t) \\
& \leq -\alpha_f \int_T \phi_t(X_t)d\mu(t)-\beta_f\mathbf{1}+\int_T \phi_t(f(X_t))d\mu(t) \qquad (4.73) \\
& \leq \frac{2L}{M-m}\int_T \phi_t\left[(M\mathbf{1}-X_t)(X_t-m\mathbf{1})\right]d\mu(t).
\end{aligned}
$$

When we add up inequalities (4.72) and (4.73), we get the first inequality sign in (4.71). As in the proof of the previous theorem, the remaining inequality signs in (4.71) follow by applying the Jensen operator inequality with respect to the operator concave function $t \mapsto -t^2 + (m+M)t - mM$ and due to the arithmetic-geometric mean inequality

$$(M-t)(t-m) \leq \frac{1}{4}(M-m)^2$$

which holds for every $t \in [m,M]$. □

It is well known that every convex function is bounded and Lipschitz continuous on any compact subset of the interior of its domain (see e.g. [126]), so all of the results from this section can be applied to the class of convex functions. The following consequence is a special case of Theorems 4.9 and 4.10.

Corollary 4.7 ([82]) *Let $f : I \to \mathbb{R}$ be a convex function, and let $m, M \in \mathbb{R}$, $m < M$, be such that interval $[m,M]$ belongs to the interior of interval I. Suppose $\mathscr{A}$ and $\mathscr{B}$ are unital C^*-algebras, and $(\phi_t)_{t\in T}$ is a unital field of positive linear mappings $\phi_t : \mathscr{A} \to \mathscr{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ. Then there exist $\gamma, \Gamma \in \mathbb{R}$ such that relations*

$$\mathbf{0} \leq \alpha_f \int_T \phi_t(X_t)d\mu(t) + \beta_f \mathbf{1} - \int_T \phi_t(f(X_t))d\mu(t) \leq (\Gamma - \gamma)\mathbf{1} \tag{4.74}$$

and

$$-(\Gamma-\gamma)\mathbf{1} \leq \int_T \phi_t(f(X_t))d\mu(t) - f\left(\int_T \phi_t(X_t)d\mu(t)\right) \leq (\Gamma-\gamma)\mathbf{1} \tag{4.75}$$

hold for every bounded continuous field $(X_t)_{t\in T}$ of self-adjoint elements in $\mathscr{A}$ with spectra contained in $[m,M]$.

Proof. Since the interval $[m,M]$ is compact, there exist γ, Γ such that $\gamma \leq f(t) \leq \Gamma$ for every $t \in [m,M]$. Function f is convex, so by virtue of the scalar Lah-Ribarič inequality, the relation (4.63) becomes

$$0 \leq \alpha_f t + \beta_f - f(t) \leq \Gamma - \gamma.$$

Now, inequalities (4.74) and (4.75) are obtained by following the lines as in the proofs of Theorems 4.9 and 4.10, except that we utilize the above scalar relation instead of (4.63). The last part follows from the Jensen operator inequality. □

Remark 4.19 If f is in addition an operator convex function, then the left term $-(\Gamma-\gamma)\mathbf{1}$ in (4.75) can be replaced by $\mathbf{0}$.

Our next result is a special case of Theorem 4.11, and it follows directly from the Lah-Ribarič operator inequality (4.2).

Corollary 4.8 ([82]) *Let $f: I \to \mathbb{R}$ be a convex function, and let $m, M \in \mathbb{R}$, $m < M$, be such that interval $[m, M]$ belongs to the interior of interval I. Let L be the Lipschitz constant of the function f on $[m, M]$. Suppose $\mathscr{A}$ and $\mathscr{B}$ are unital C^*-algebras, and $(\phi_t)_{t\in T}$ is a unital field of positive linear mappings $\phi_t : \mathscr{A} \to \mathscr{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ. Then the set of inequalities*

$$\begin{aligned} \mathbf{0} \leq & \alpha_f \int_T \phi_t(X_t) d\mu(t) + \beta_f \mathbf{1} - \int_T \phi_t(f(X_t)) d\mu(t) \\ \leq & \frac{2L}{M-m} \int_T \phi_t\left[(M\mathbf{1} - X_t)(X_t - m\mathbf{1})\right] d\mu(t) \\ \leq & \frac{2L}{M-m} \left(M\mathbf{1} - \int_T \phi_t(X_t) d\mu(t)\right)\left(\int_T \phi_t(X_t) d\mu(t) - m\mathbf{1}\right) \\ \leq & \frac{L}{2}(M-m)\mathbf{1} \end{aligned} \tag{4.76}$$

holds for every bounded continuous field $(X_t)_{t\in T}$ of self-adjoint elements in $\mathscr{A}$ with spectra contained in $[m, M]$.

In order to conclude this section, we give the following result which is a special case of Theorem 4.12.

Corollary 4.9 ([82]) *Let $f: I \to \mathbb{R}$ be a convex function, and let $m, M \in \mathbb{R}$, $m < M$, be such that interval $[m, M]$ belongs to the interior of interval I. Let L be the Lipschitz constant of the function f on $[m, M]$. Suppose $\mathscr{A}$ and $\mathscr{B}$ are unital C^*-algebras, and $(\phi_t)_{t\in T}$ is a unital field of positive linear mappings $\phi_t : \mathscr{A} \to \mathscr{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ. Then the set of inequalities*

$$\begin{aligned} & -\frac{2L}{M-m} \int_T \phi_t\left[(M\mathbf{1} - X_t)(X_t - m\mathbf{1})\right] d\mu(t) \\ & \leq \int_T \phi_t(f(X_t)) d\mu(t) - f\left(\int_T \phi_t(X_t) d\mu(t)\right) \\ & \leq \frac{2L}{M-m} \left(M\mathbf{1} - \int_T \phi_t(X_t) d\mu(t)\right)\left(\int_T \phi_t(X_t) d\mu(t) - m\mathbf{1}\right) \\ & \leq \frac{L}{2}(M-m)\mathbf{1} \end{aligned} \tag{4.77}$$

holds for every bounded continuous field $(X_t)_{t\in T}$ of self-adjoint elements in $\mathscr{A}$ with spectra contained in $[m, M]$.

Proof. Since f is a convex function, it follows that

$$0 \leq \alpha_f t + \beta_f - f(t) \leq 2L \frac{(M-t)(t-m)}{M-m}.$$

Now the proof follows the lines of the proof of Theorem 4.12 except that we use the above scalar inequality instead of (4.64). □

Remark 4.20 If the function f is additionally operator convex, then the first line in (4.77) can be replaced by $\mathbf{0}$.

4.4.1 Applications to quasi-arithmetic operator means

It is well-known that an arbitrary C^*-algebra is isomorphic to a C^*-algebra of bounded operators on a Hilbert space $\mathscr{H}$ (see, e.g. [51]). Hence, in order to simplify our further discussion, from now on, C^*-algebras will be regarded as algebras of bounded operators on a Hilbert space, denoted here by $\mathfrak{B}(\mathscr{H})$.

The main goal in this subsection is an application of obtained general Jensen-type inequalities to the so-called quasi-arithmetic operator means. As in the previous section, generalized quasi-arithmetic operator mean is defined by

$$M_\psi(X,\phi) = \psi^{-1}\left(\int_T \phi_t(\psi(X_t))d\mu(t)\right),$$

where $(X_t)_{t\in T}$ is a continuous field of positive operators in $\mathfrak{B}(\mathscr{H})$ with spectra in $[m,M]$ for some scalars $0<m<M$, $(\phi_t)_{t\in T} \in P[\mathfrak{B}(\mathscr{H}),\mathfrak{B}(\mathscr{K})]$, and $\psi : [m,M] \to \mathbb{R}$ is a strictly monotone bounded function.

In Section 4.3 we have established reverse relations for quasi-arithmetic operator means which rely on convexity and monotonicity in the classical real sense. Now, our intention is to derive mutual bounds for quasi-arithmetic means in described setting. In such a way, we will obtain some new reverse relations for quasi-arithmetic means that correspond to bounded and Lipschitzian functions. Before we state such results, we have to introduce some notations arising from this particular setting. Throughout this section we denote

$$\psi_m = \min\{\psi(m),\psi(M)\}, \quad \psi_M = \max\{\psi(m),\psi(M)\}$$

and

$$\gamma_\chi = \min\{\chi(t),\ t\in m,M]\}, \quad \Gamma_\chi = \max\{\chi(t),\ t\in[m,M]\},$$

where $\psi,\chi : [m,M] \to \mathbb{R}$ are strictly monotone bounded functions.

The first result in this subsection is carried out by virtue of our Theorem 4.9.

Theorem 4.13 ([82]) *Let $\chi,\psi : [m,M] \to \mathbb{R}$ be strictly monotone bounded functions, where $0<m<M$. Further, suppose that $\chi\circ\psi^{-1}$ is well-defined on $[\psi_m,\psi_M]$. If $(\phi_t)_{t\in T} \in P[\mathfrak{B}(\mathscr{H}),\mathfrak{B}(\mathscr{K})]$, where $\mathscr{H}$, $\mathscr{K}$ are Hilbert spaces and T is a locally compact Hausdorff space with a bounded Radon measure μ, then the series of inequalities*

$$\begin{aligned} -(\Gamma_\chi-\gamma_\chi)\mathbf{1} \le \frac{\chi(M)-\chi(m)}{\psi(M)-\psi(m)}\psi\left(M_\psi(X,\phi)\right) + \frac{\psi(M)\chi(m)-\psi(m)\chi(M)}{\psi(M)-\psi(m)}\mathbf{1} \\ -\chi\left(M_\chi(X,\phi)\right) \le (\Gamma_\chi-\gamma_\chi)\mathbf{1} \end{aligned} \tag{4.78}$$

holds for every continuous field $(X_t)_{t\in T}$ of positive operators in $\mathfrak{B}(\mathscr{H})$ with spectra in $[m,M]$.

Proof. Since $\psi : [m,M] \to \mathbb{R}$ is a bounded strictly monotone function, it follows that $\psi_m \le \psi(t) \le \psi_M$, for all $t\in[m,M]$. Moreover, by virtue of the functional calculus, it follows that $\psi_m\mathbf{1} \le \psi(X_t) \le \psi_M\mathbf{1}$ for every $t\in T$. This means that the spectra of the field $(Y_t)_{t\in T} = (\psi(X_t))_{t\in T}$ is contained in the interval $[\psi_m,\psi_M]$.

Now, regarding Theorem 4.9, that is, utilizing the series of inequalities in (4.65) with ψ_m, ψ_M, $\chi \circ \psi^{-1}$, $(Y_t)_{t\in T}$ respectively instead of m, M, f, $(X_t)_{t\in T}$, and with definition (4.44) of the quasi-arithmetic mean, we obtain (4.78), as claimed. □

Following the lines as in the proof of the previous result, Theorem 4.10 can also be exploited in establishing some new reverses of the Jensen operator inequality related to quasi-arithmetic means. First, we establish the corresponding mutual bounds quasi-arithmetic means.

Theorem 4.14 ([82]) *Let $\chi, \psi : [m,M] \to \mathbb{R}$ be strictly monotone bounded functions, where $0 < m < M$. Further, suppose that $\chi \circ \psi^{-1}$ is well-defined on $[\psi_m, \psi_M]$. If $(\phi_t)_{t\in T} \in P[\mathfrak{B}(\mathscr{H}), \mathfrak{B}(\mathscr{K})]$, where $\mathscr{H}$, $\mathscr{K}$ are Hilbert spaces and T is a locally compact Hausdorff space with a bounded Radon measure μ, then the series of inequalities*

$$-2(\Gamma_\chi - \gamma_\chi)\mathbf{1} \le \chi\left(M_\chi(X,\phi)\right) - \chi\left(M_\psi(X,\phi)\right) \le 2(\Gamma_\chi - \gamma_\chi)\mathbf{1} \tag{4.79}$$

holds for every continuous field $(X_t)_{t\in T}$ of positive operators in $\mathfrak{B}(\mathscr{H})$ with spectra in $[m,M]$.

Remark 4.21 Let $\chi, \psi : I \to \mathbb{R}$ be strictly monotone functions and let the interval $[m,M]$ belongs to the interior of interval I. If the function $\chi \circ \psi^{-1} : I \to \mathbb{R}$ is additionally convex on $[\psi_m, \psi_M]$, inequalities in (4.78) become

$$\begin{aligned}\mathbf{0} \le & \frac{\chi(M) - \chi(m)}{\psi(M) - \psi(m)}\psi\left(M_\psi(X,\phi)\right) + \frac{\psi(M)\chi(m) - \psi(m)\chi(M)}{\psi(M) - \psi(m)}\mathbf{1} \\ & - \chi\left(M_\chi(X,\phi)\right) \le (\Gamma_\chi - \gamma_\chi)\mathbf{1},\end{aligned} \tag{4.80}$$

while inequalities in (4.79) become

$$-(\Gamma_\chi - \gamma_\chi)\mathbf{1} \le \chi\left(M_\chi(X,\phi)\right) - \chi\left(M_\psi(X,\phi)\right) \le (\Gamma_\chi - \gamma_\chi)\mathbf{1}. \tag{4.81}$$

If the function $\chi \circ \psi^{-1}$ is additionally operator convex, then the left term $-(\Gamma_\chi - \gamma_\chi)\mathbf{1}$ in (4.81) can be replaced by $\mathbf{0}$. This is a consequence of Corollary 4.7.

Our next result arises from Theorem 4.11 and it provides Lah-Ribarič-type estimates for quasi-arithmetic operator means, for a class of Lipschitzian functions.

Theorem 4.15 ([82]) *Let $\psi : [m,M] \to \mathbb{R}$ be a strictly monotone bounded function and let $\chi : [m,M] \to \mathbb{R}$ be an L_χ-Lipschitzian function, where $0 < m < M$. Further, suppose that $\chi \circ \psi^{-1}$ is well-defined on $[\psi_m, \psi_M]$. If $(\phi_t)_{t\in T} \in P[\mathfrak{B}(\mathscr{H}), \mathfrak{B}(\mathscr{K})]$, where $\mathscr{H}$, $\mathscr{K}$ are Hilbert spaces and T is a locally compact Hausdorff space with a bounded Radon measure μ, then the series of inequalities*

$$
\begin{aligned}
& -\frac{L_\chi}{2}(\psi_M-\psi_m)\mathbf{1} \\
& \leq \frac{-2L_\chi}{\psi_M-\psi_m}\left(\psi_M\mathbf{1}-\psi\left(M_\psi(X,\phi)\right)\right)\left(\psi\left(M_\psi(X,\phi)\right)-\psi_m\mathbf{1}\right) \\
& \leq \frac{-2L}{M-m}\int_T \phi_t\left([\psi_M\mathbf{1}-\psi(X_t)][\psi(X_t)-\psi_m\mathbf{1}]\right)d\mu(t) \\
& \leq \frac{\chi(M)-\chi(m)}{\psi(M)-\psi(m)}\psi\left(M_\psi(X,\phi)\right)+\frac{\psi(M)\chi(m)-\psi(m)\chi(M)}{\psi(M)-\psi(m)}\mathbf{1}-\chi\left(M_\chi(X,\phi)\right) \qquad (4.82) \\
& \leq \frac{2L_\chi}{\psi_M-\psi_m}\int_T \phi_t\left([\psi_M\mathbf{1}-\psi(X_t)][\psi(X_t)-\psi_m\mathbf{1}]\right)d\mu(t) \\
& \leq \frac{2L_\chi}{\psi_M-\psi_m}\left(\psi_M\mathbf{1}-\psi\left(M_\psi(X,\phi)\right)\right)\left(\psi\left(M_\psi(X,\phi)\right)-\psi_m\mathbf{1}\right) \\
& \leq \frac{L_\chi}{2}(\psi_M-\psi_m)
\end{aligned}
$$

holds for every continuous field $(X_t)_{t\in T}$ of positive operators in $\mathfrak{B}(\mathscr{H})$ with spectra in $[m,M]$.

Proof. As in the proof of Theorem 4.13, since $\psi:[m,M]\to\mathbb{R}$ is a bounded strictly monotone function, it follows that $\psi_m\leq\psi(t)\leq\psi_M$, for all $t\in[m,M]$, and by virtue of the functional calculus we have $\psi_m\mathbf{1}\leq\psi(X_t)\leq\psi_M\mathbf{1}$, for every $t\in T$. This means that the spectra of the field $(Y_t)_{t\in T}=(\psi(X_t))_{t\in T}$ belongs to the interval $[\psi_m,\psi_M]$.

On the other hand, the function $\chi\circ\psi^{-1}$ is obviously L_χ-Lipschitzian on $\psi([m,M])$, so we can utilize the series of inequalities in (4.71) with ψ_m, ψ_M, $\chi\circ\psi^{-1}$, $(Y_t)_{t\in T}$ respectively instead of m, M, f, $(X_t)_{t\in T}$. Finally, taking into account (4.44) we obtain (4.82). □

In the same manner as described above, Theorem 4.12 enables us to establish mutual bounds of the Jensen operator inequality for Lipschitzian functions, related to quasi-arithmetic means.

Theorem 4.16 ([82]) *Let $\psi:[m,M]\to\mathbb{R}$ be a strictly monotone bounded function and let $\chi:[m,M]\to\mathbb{R}$ be an L_χ-Lipschitzian function, where $0<m<M$. Further, suppose that $\chi\circ\psi^{-1}$ is well-defined on $[\psi_m,\psi_M]$. If $(\phi_t)_{t\in T}\in P[\mathfrak{B}(\mathscr{H}),\mathfrak{B}(\mathscr{K})]$, where $\mathscr{H}$, $\mathscr{K}$ are Hilbert spaces and T is a locally compact Hausdorff space with a bounded Radon measure μ, then the series of inequalities*

$$
\begin{aligned}
& -L_\chi(\psi_M-\psi_m)\mathbf{1} \\
& \leq \frac{-4L_\chi}{\psi_M-\psi_m}\left(\psi_M\mathbf{1}-\psi\left(M_\psi(X,\phi)\right)\right)\left(\psi\left(M_\psi(X,\phi)\right)-\psi_m\mathbf{1}\right) \\
& \leq \frac{-2L_\chi}{\psi_M-\psi_m}\Bigg[\int_T \phi_t\left([\psi_M\mathbf{1}-\psi(X_t)][\psi(X_t)-\psi_m\mathbf{1}]\right)d\mu(t) \\
& \qquad +\left(\psi_M\mathbf{1}-\psi\left(M_\psi(X,\phi)\right)\right)\left(\psi\left(M_\psi(X,\phi)\right)-\psi_m\mathbf{1}\right)\Bigg]
\end{aligned}
$$

$$\begin{aligned}
&\le \chi\left(M_\chi(X,\phi)\right) - \chi\left(M_\psi(X,\phi)\right) \qquad (4.83)\\
&\le \frac{2L_\chi}{\psi_M-\psi_m}\left[\int_T \phi_t\left([\psi_M\mathbf{1}-\psi(X_t)][\psi(X_t)-\psi_m\mathbf{1}]\right)d\mu(t)\right.\\
&\qquad\left. + \left(\psi_M\mathbf{1}-\psi\left(M_\psi(X,\phi)\right)\right)\left(\psi\left(M_\psi(X,\phi)\right)-\psi_m\mathbf{1}\right)\right]\\
&\le \frac{4L_\chi}{\psi_M-\psi_m}\left(\psi_M\mathbf{1}-\psi\left(M_\psi(X,\phi)\right)\right)\left(\psi\left(M_\psi(X,\phi)\right)-\psi_m\mathbf{1}\right)\\
&\le L_\chi(\psi_M-\psi_m)\mathbf{1}
\end{aligned}$$

holds for every continuous field $(X_t)_{t\in T}$ *of positive operators in* $\mathfrak{B}(\mathscr{H})$ *with spectra in* $[m,M]$.

Remark 4.22 If the function $\chi\circ\psi^{-1}$ is convex, then, due to the fact that every convex function on a compact set that belongs to the interior of its domain is Lipschitzian, the inequalities in (4.82) become

$$\begin{aligned}
\mathbf{0} &\le \frac{\chi(M)-\chi(m)}{\psi(M)-\psi(m)}\psi\left(M_\psi(X,\phi)\right) + \frac{\psi(M)\chi(m)-\psi(m)\chi(M)}{\psi(M)-\psi(m)}\mathbf{1} - \chi\left(M_\chi(X,\phi)\right)\\
&\le \frac{2L_\chi}{\psi_M-\psi_m}\int_T \phi_t\left([\psi_M\mathbf{1}-\psi(X_t)][\psi(X_t)-\psi_m\mathbf{1}]\right)d\mu(t) \qquad (4.84)\\
&\le \frac{2L_\chi}{\psi_M-\psi_m}\left(\psi_M\mathbf{1}-\psi\left(M_\psi(X,\phi)\right)\right)\left(\psi\left(M_\psi(X,\phi)\right)-\psi_m\mathbf{1}\right)\\
&\le \frac{L_\chi}{2}(\psi_M-\psi_m),
\end{aligned}$$

while inequalities in (4.83) read

$$\begin{aligned}
&-\frac{2L_\chi}{\psi_M-\psi_m}\int_T \phi_t\left([\psi_M\mathbf{1}-\psi(X_t)][\psi(X_t)-\psi_m\mathbf{1}]\right)d\mu(t)\\
&\le \chi\left(M_\chi(X,\phi)\right) - \chi\left(M_\psi(X,\phi)\right) \qquad (4.85)\\
&\le \frac{2L_\chi}{\psi_M-\psi_m}\left(\psi_M\mathbf{1}-\psi\left(M_\psi(X,\phi)\right)\right)\left(\psi\left(M_\psi(X,\phi)\right)-\psi_m\mathbf{1}\right)\\
&\le \frac{L_\chi}{2}(\psi_M-\psi_m)\mathbf{1}
\end{aligned}$$

In addition, if $\chi\circ\psi^{-1}$ is operator convex, then the first line in (4.85) can be replaced by $\mathbf{0}$. These inequalities follow from Corollaries 4.8 and 4.9.

4.4.2 Examples with power operator means

Let us recall, a common example of a quasi-arithmetic mean (4.44) is a power operator mean

$$M_r(X,\phi) = \left\{ \begin{array}{ll} \left(\int_T \phi_t(X_t^r)d\mu(t)\right)^{\frac{1}{r}}, & r\neq 0\\ \exp\left(\int_T \phi_t(\log X_t)d\mu(t)\right), & r=0 \end{array}\right\}$$

already defined in the previous section by (4.53).

Clearly, the method developed in this section can be applied to the above power means. More precisely, as a consequence of results from the previous section we obtain a whole series of estimates for power operator means. In particular, we obtain some new reverse relations for power operator means.

Corollary 4.10 ([82]) *Let $(\phi_t)_{t\in T} \in P[\mathfrak{B}(\mathscr{H}),\mathfrak{B}(\mathscr{K})]$, where $\mathscr{H}$, $\mathscr{K}$ are Hilbert spaces and T is a locally compact Hausdorff space with a bounded Radon measure μ, and let $(X_t)_{t\in T}$ be a continuous field of positive operators in $\mathfrak{B}(\mathscr{H})$ with spectra in $[m,M] \subseteq \mathbb{R}_+$.*

(i) If either $s<0<r$ or $r<0<s$ or $0<r<s$ or $s<r<0$, then

$$\mathbf{0} \leq \frac{M^s-m^s}{M^r-m^r}\left(M_r(X,\phi)\right)^r + \frac{M^r m^s - m^r M^s}{M^r-m^r}\mathbf{1} - \left(M_s(X,\phi)\right)^s \leq |M^s-m^s|\mathbf{1}, \quad (4.86)$$

and if $0 \leq s < r$ or $r < s \leq 0$, then

$$-|M^s-m^s|\mathbf{1} \leq \frac{M^s-m^s}{M^r-m^r}\left(M_r(X,\phi)\right)^r + \frac{M^r m^s - m^r M^s}{M^r-m^r}\mathbf{1} - \left(M_s(X,\phi)\right)^s \leq \mathbf{0}. \quad (4.87)$$

(ii) For $r \in \mathbb{R}$, $r<0$ we have

$$\mathbf{0} \leq \frac{\log M - \log m}{M^r-m^r}\left(M_r(X,\phi)\right)^r + \frac{M^r \log m - m^r \log M}{M^r-m^r}\mathbf{1} - \log\left(M_0(X,\phi)\right) \leq \log\frac{M}{m}\mathbf{1}, \quad (4.88)$$

and for $r>0$ we have

$$\log\frac{m}{M}\mathbf{1} \leq \frac{\log M - \log m}{M^r-m^r}\left(M_r(X,\phi)\right)^r + \frac{M^r \log m - m^r \log M}{M^r-m^r}\mathbf{1} - \log\left(M_0(X,\phi)\right) \leq \mathbf{0}, \quad (4.89)$$

Proof. The proof is a consequence of Theorem 4.13 and Remark 4.21. More precisely, we utilize series of inequalities in (4.80) with particular choices of functions χ and ψ.

First, we set $\chi(t)=t^s$ and $\psi(t)=t^r$, where s and r are real parameters such that $sr \neq 0$. Further, the function $\left(\chi\circ\psi^{-1}\right)(t)=t^{\frac{s}{r}}$ is convex on $\mathbb{R}_+$ if $\frac{s}{r}\leq 0$ or $\frac{s}{r}\geq 1$, which is possible in each of the following four cases: $s\leq 0<r$, $r<0\leq s$, $0<r\leq s$, $s\leq r<0$. Now, utilizing inequalities in (4.80) with the above functions χ and ψ on the interval $[m,M]$, we obtain (4.86).

On the other hand, the function $\left(\chi\circ\psi^{-1}\right)(t)=t^{\frac{s}{r}}$ is concave on $\mathbb{R}_+$ provided that $0\leq\frac{s}{r}\leq 1$, hence, if $0\leq s\leq r\neq 0$ or $0\neq r\leq s\leq 0$, we obtain (4.87).

It remains to consider non-trivial cases when one of parameters r or s is equal to zero. Without loss of generality, let $s=0$. Here we set $\chi(t)=\log t$ and $\psi(t)=t^r$, and it follows that $\left(\chi\circ\psi^{-1}\right)(t)=\frac{1}{r}\log t$. Clearly, this function is convex for $r<0$, while it is concave for $r>0$. Now, relations (4.88) and (4.89) follow from inequalities in (4.80) and the proof is completed. □

By following the same procedure as in the proof of the previous corollary and taking into account inequalities in (4.81), we can also obtain a consequence of Theorem 4.14 that corresponds to Jensen-type inequalities for power operator means.

Corollary 4.11 ([82]) *Let* $(\phi_t)_{t\in T} \in P[\mathfrak{B}(\mathscr{H}),\mathfrak{B}(\mathscr{K})]$, *where* $\mathscr{H}$, $\mathscr{K}$ *are Hilbert spaces and* T *is a locally compact Hausdorff space with a bounded Radon measure* μ, *and let* $(X_t)_{t\in T}$ *be a continuous field of positive operators in* $\mathfrak{B}(\mathscr{H})$ *with spectra in* $[m,M] \subseteq \mathbb{R}_+$.

(i) For $s,r \in \mathbb{R}$ *such that* $sr \neq 0$ *we have*

$$-|M^s - m^s|\mathbf{1} \leq (M_s(X,\phi))^s - (M_r(X,\phi))^s \leq |M^s - m^s|\mathbf{1}, \tag{4.90}$$

(ii) For $r \in \mathbb{R},\ r < 0$ *we have*

$$\mathbf{0} \leq \log(M_0(X,\phi)) - \log(M_r(X,\phi)) \leq \log\frac{M}{m}\mathbf{1}, \tag{4.91}$$

and for $r > 0$ *we have*

$$\log\frac{m}{M}\mathbf{1} \leq \log(M_0(X,\phi)) - \log(M_r(X,\phi)) \leq \mathbf{0}. \tag{4.92}$$

Remark 4.23 It is well-known that the function $f(t) = t^r$ is operator convex on $\mathbb{R}_+$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$, and is operator concave on $\mathbb{R}_+$ when $0 \leq r \leq 1$. Hence, discussing the operator convexity of the function $\left(\chi \circ \psi^{-1}\right)(t) = t^{\frac{s}{r}}$, as in the proof of Corollary 4.10, we obtain conditions on parameters r and s under which one of the outer terms in (4.90) is equal to $\mathbf{0}$. If either

$$0 < r \leq s \leq 2r \text{ or } 2r \leq s \leq r < 0 \text{ or } 0 \leq s+r \leq r \neq 0 \text{ or } 0 \neq r \leq r+s \leq 0, \tag{4.93}$$

then (4.90) reads

$$\mathbf{0} \leq (M_s(X,\phi))^s - (M_r(X,\phi))^s \leq |M^s - m^s|\mathbf{1},$$

and if

$$0 \neq r \leq s \leq 0 \text{ or } 0 \leq s \leq r \neq 0, \tag{4.94}$$

then (4.90) becomes

$$-|M^s - m^s|\mathbf{1} \leq (M_s(X,\phi))^s - (M_r(X,\phi))^s \leq \mathbf{0}.$$

The following results rely on Theorem 4.15, Theorem 4.16 and Remark 4.22, and they provide a different class of Lah-Ribarič and Jensen-type inequalities for power operator means then those obtained above.

Corollary 4.12 ([82]) *Let* $(\phi_t)_{t\in T} \in P[\mathfrak{B}(\mathscr{H}),\mathfrak{B}(\mathscr{K})]$, *where* $\mathscr{H}$, $\mathscr{K}$ *are Hilbert spaces and* T *is a locally compact Hausdorff space with a bounded Radon measure* μ, *and let* $(X_t)_{t\in T}$ *be a continuous field of positive operators in* $\mathfrak{B}(\mathscr{H})$ *with spectra in* $[m,M] \subseteq \mathbb{R}_+$.

(i) If either $0 < r < s$ *or* $s < r < 0$*, then*

$$\begin{aligned}
\mathbf{0} &\leq \frac{M^s - m^s}{M^r - m^r}(M_r(X,\phi))^r + \frac{M^r m^s - m^r M^s}{M^r - m^r}\mathbf{1} - (M_s(X,\phi))^s \\
&\leq \frac{2sM^{\frac{s-r}{r}}}{r|M^r - m^r|}\int_T \phi_t\left([M^r\mathbf{1} - X_t^r][X_t^r - m^r\mathbf{1}]\right)d\mu(t) \\
&\leq \frac{2sM^{\frac{s-r}{r}}}{r|M^r - m^r|}\left(M^r\mathbf{1} - (M_r(X,\phi))^r\right)\left((M_r(X,\phi))^r - m^r\mathbf{1}\right) \\
&\leq \frac{sM^{\frac{s-r}{r}}}{2r}|M^r - m^r|\mathbf{1},
\end{aligned} \tag{4.95}$$

if $0 \leq s < r$ *or* $r < s \leq 0$*, then*

$$\begin{aligned}
&-\frac{sm^{\frac{s-r}{r}}}{2r}|M^r - m^r|\mathbf{1} \\
&\leq -\frac{2sm^{\frac{s-r}{r}}}{r|M^r - m^r|}\left(M^r\mathbf{1} - (M_r(X,\phi))^r\right)\left((M_r(X,\phi))^r - m^r\mathbf{1}\right) \\
&\leq -\frac{2sm^{\frac{s-r}{r}}}{r|M^r - m^r|}\int_T \phi_t\left([M^r\mathbf{1} - X_t^r][X_t^r - m^r\mathbf{1}]\right)d\mu(t) \\
&\leq \frac{M^s - m^s}{M^r - m^r}(M_r(X,\phi))^r + \frac{M^r m^s - m^r M^s}{M^r - m^r}\mathbf{1} - (M_s(X,\phi))^s \leq \mathbf{0},
\end{aligned} \tag{4.96}$$

and if $s < 0 < r$ *or* $r < 0 < s$ *then the inequality signs in* (4.96) are reversed.

(ii) For $r \in \mathbb{R},\ r < 0$ we have

$$\begin{aligned}
\mathbf{0} &\leq \frac{\log M - \log m}{M^r - m^r}(M_r(X,\phi))^r + \frac{M^r\log m - m^r\log M}{M^r - m^r}\mathbf{1} - \log(M_0(X,\phi)) \\
&\leq \frac{2}{rm(M^r - m^r)}\int_T \phi_t\left([M^r\mathbf{1} - X_t^r][X_t^r - m^r\mathbf{1}]\right)d\mu(t) \\
&\leq \frac{2L}{rm(M^r - m^r)}\left(M^r\mathbf{1} - (M_r(X,\phi))^r\right)\left((M_r(X,\phi))^r - m^r\mathbf{1}\right) \leq \frac{M^r - m^r}{2rm}\mathbf{1},
\end{aligned} \tag{4.97}$$

and if $r > 0$, then the inequality signs in (4.97) are reversed.

(iii) For $s \in \mathbb{R},\ s > 0$ we have

$$\begin{aligned}
\mathbf{0} &\leq \frac{M^s - m^s}{\log M - \log m}\log(M_0(X,\phi)) + \frac{m^s\log M - M^s\log m}{\log M - \log m}\mathbf{1} - (M_s(X,\phi))^s \\
&\leq \frac{2se^{sM}}{\log M - \log m}\int_T \phi_t\left([\log M\mathbf{1} - \log X_t][\log X_t - \log m\mathbf{1}]\right)d\mu(t) \\
&\leq \frac{2se^{sM}}{\log M - \log m}\left(\log M\mathbf{1} - \log(M_0(X,\phi))\right)\left(\log(M_0(X,\phi)) - \log m\mathbf{1}\right) \\
&\leq \frac{se^{sM}}{2}\log\frac{M}{m}\mathbf{1},
\end{aligned} \tag{4.98}$$

and for $s < 0$ we have

$$
\begin{aligned}
\mathbf{0} \leq & \frac{M^s - m^s}{\log M - \log m} \log\left(M_0(X,\phi)\right) + \frac{m^s \log M - M^s \log m}{\log M - \log m}\mathbf{1} - \left(M_s(X,\phi)\right)^s \\
\leq & \frac{2se^{sm}}{\log m - \log M} \int_T \phi_t\left([\log M\mathbf{1} - \log X_t][\log X_t - \log m\mathbf{1}]\right) d\mu(t) \qquad (4.99) \\
\leq & \frac{2se^{sm}}{\log m - \log M}\left(\log M\mathbf{1} - \log\left(M_0(X,\phi)\right)\right)\left(\log\left(M_0(X,\phi)\right) - \log m\mathbf{1}\right) \\
\leq & \frac{se^{sm}}{2} \log \frac{m}{M}\mathbf{1}.
\end{aligned}
$$

Proof. The proof is similar to the proof of Corollary 4.10 except that we use relations in (4.84) instead of inequalities in (4.80).

More precisely, let $\chi(t) = t^s$ and $\psi(t) = t^r$, where s and r are mutually different real parameters not equal to zero. As noticed before, the function $\left(\chi \circ \psi^{-1}\right)(t) = t^{\frac{s}{r}}$ is convex on $\mathbb{R}_+$ if either $s < 0 < r$ or $r < 0 < s$ or $0 < r < s$ or $s < r < 0$, and it is concave if $0 < s < r$ or $r < s < 0$.

Since the function $\left(\chi \circ \psi^{-1}\right)(t) = t^{\frac{s}{r}}$ is differentiable, we can use the mean value theorem

$$
\frac{\left(\chi \circ \psi^{-1}\right)(x) - \left(\chi \circ \psi^{-1}\right)(y)}{x - y} = \left(\chi \circ \psi^{-1}\right)'(t), \quad x < t < y,
$$

and deduce that for finding its Lipschitz constant it is sufficient to find a bound L such that

$$
\left|\left(\chi \circ \psi^{-1}\right)'(t)\right| \leq L \quad \text{for all } t \in [m,M].
$$

We have

$$
\left(\chi \circ \psi^{-1}\right)'(t) = \frac{s}{r} t^{\frac{s-r}{r}},
$$

and since the function $f(t) = t^a$ is increasing for $a \geq 0$ and decreasing for $a < 0$, it follows that

$$
\begin{aligned}
L_\chi &= \frac{s}{r} M^{\frac{s-r}{r}} \quad \text{for } 0 < r \leq s \text{ or } s \leq r < 0; \\
L_\chi &= -\frac{s}{r} m^{\frac{s-r}{r}} \quad \text{for } s < 0 < r \text{ or } r < 0 < s; \\
L_\chi &= \frac{s}{r} m^{\frac{s-r}{r}} \quad \text{for } 0 < s < r \text{ or } r < s < 0.
\end{aligned}
$$

In first two cases the function $\chi \circ \psi^{-1}$ is convex, so relations (4.95) and (4.96) with reversed signs of inequality follow directly from (4.84). In the third case the function $\chi \circ \psi^{-1}$ is concave, so inequalities in (4.96) follow from (4.84) and due to convexity of $-\chi \circ \psi^{-1}$.

We still need to consider the cases when one of the parameters r and s is equal to zero. If $s = 0$, then setting $\chi(t) = \log t$ and $\psi(t) = t^r$, it follows that $\left(\chi \circ \psi^{-1}\right)(t) = \frac{1}{r}\log t$. Clearly, this function is convex for $r < 0$, while it is concave for $r > 0$. Moreover, since it is differentiable we can calculate $\left(\chi \circ \psi^{-1}\right)'(t) = \frac{1}{rt}$, and from the mean value theorem we get

$$
L_\chi = -\frac{1}{rm} \quad \text{for } r < 0 \quad \text{and} \quad L_\chi = \frac{1}{rm} \quad \text{for } r > 0.
$$

Since in the first case the function $\chi \circ \psi^{-1}$ is convex and $\psi_M = m^r,\ \psi_m = M^r$, inequalities in (4.97) follow directly from (4.84), and in the second case the reversed inequalities in (4.97) follow from (4.84) due to the convexity of the function $-\frac{1}{r}\log t$ when $r > 0$.

Finally, if $r = 0$, then setting $\chi(t) = t^s$ and $\psi(t) = \log t$, it follows that the function $\left(\chi \circ \psi^{-1}\right)(t) = e^{st}$ is convex for every $s \neq 0$. In addition, $\left(\chi \circ \psi^{-1}\right)'(t) = se^{st}$, so the mean value theorem yields

$$L_\chi = se^{sM} \ \text{ for } s > 0 \quad \text{and} \quad L_\chi = -se^{sm} \ \text{ for } s < 0.$$

Now relations (4.98) and (4.99) follow from (4.84). □

In the same manner as described above and relying on inequalities in (4.85), we obtain the following result with which we conclude the paper.

Corollary 4.13 ([82]) *Let $(\phi_t)_{t\in T} \in P[\mathfrak{B}(\mathscr{H}), \mathfrak{B}(\mathscr{K})]$, where $\mathscr{H}$, $\mathscr{K}$ are Hilbert spaces and T is a locally compact Hausdorff space with a bounded Radon measure μ, and let $(X_t)_{t\in T}$ be a continuous field of positive operators in $\mathfrak{B}(\mathscr{H})$ with spectra in $[m,M] \subseteq \mathbb{R}_+$.*

(i) If either $0 < r < s$ or $s < r < 0$, then

$$\begin{aligned}
&-\frac{2sM^{\frac{s-r}{r}}}{r|M^r - m^r|}\int_T \phi_t\left([M^r\mathbf{1} - X_t^r][X_t^r - m^r\mathbf{1}]\right)d\mu(t)\\
&\le (M_s(X,\phi))^s - (M_r(X,\phi))^s \qquad\qquad (4.100)\\
&\le \frac{2sM^{\frac{s-r}{r}}}{r|M^r - m^r|}\left(M^r\mathbf{1} - (M_r(X,\phi))^r\right)\left((M_r(X,\phi))^r - m^r\mathbf{1}\right)\\
&\le \frac{sM^{\frac{s-r}{r}}}{2r}|M^r - m^r|\mathbf{1},
\end{aligned}$$

if $0 \le s < r$ or $r < s < 0$, then

$$\begin{aligned}
&-\frac{sm^{\frac{s-r}{r}}}{2r}|M^r - m^r|\mathbf{1}\\
&\le -\frac{2sm^{\frac{s-r}{r}}}{r|M^r - m^r|}\left(M^r\mathbf{1} - (M_r(X,\phi))^r\right)\left((M_r(X,\phi))^r - m^r\mathbf{1}\right)\\
&\le (M_s(X,\phi))^s - (M_r(X,\phi))^s \qquad\qquad (4.101)\\
&\le \frac{2sm^{\frac{s-r}{r}}}{r|M^r - m^r|}\int_T \phi_t\left([M^r\mathbf{1} - X_t^r][X_t^r - m^r\mathbf{1}]\right)d\mu(t),
\end{aligned}$$

and if $s < 0 < r$ or $r < 0 < s$ then the inequality signs in (4.101) are reversed.

(ii) For $r \in \mathbb{R},\ r < 0$ we have

$$\begin{aligned}
\mathbf{0} &\le \log(M_0(X,\phi)) - \log(M_r(X,\phi))\\
&\le \frac{2}{rm(M^r - m^r)}\left(M^r\mathbf{1} - (M_r(X,\phi))^r\right)\left((M_r(X,\phi))^r - m^r\mathbf{1}\right) \qquad (4.102)\\
&\le \frac{M^r - m^r}{2rm}\mathbf{1}
\end{aligned}$$

and if $r > 0$, then the inequality signs in (4.102) are reversed.

(iii) For $s \in \mathbb{R},\ s > 0$ we have

$$\begin{aligned} & -\frac{2se^{sM}}{\log M - \log m}\int_T \phi_t\left([\log M\mathbf{1} - \log X_t][\log X_t - \log m\mathbf{1}]\right) d\mu(t) \\ & \leq (M_s(X,\phi))^s - (M_0(X,\phi))^s \qquad (4.103) \\ & \leq \frac{2se^{sm}}{\log m - \log M}\left(\log M\mathbf{1} - \log\left(M_0\left(X,\phi\right)\right)\right)\left(\log M\mathbf{1} - \log\left(M_0\left(X,\phi\right)\right)\right) \\ & \leq \frac{se^{sM}}{2}\log\frac{M}{m}\mathbf{1}, \end{aligned}$$

and for $s < 0$ we have

$$\begin{aligned} & \frac{2se^{sm}}{\log M - \log m}\int_T \phi_t\left([\log M\mathbf{1} - \log X_t][\log X_t - \log m\mathbf{1}]\right) d\mu(t) \\ & \leq (M_s(X,\phi))^s - (M_0(X,\phi))^s \qquad (4.104) \\ & \leq \frac{2se^{sm}}{\log m - \log M}\left(\log M\mathbf{1} - \log\left(M_0\left(X,\phi\right)\right)\right)\left(\log\left(M_0\left(X,\phi\right)\right) - \log m\mathbf{1}\right) \\ & \leq \frac{se^{sm}}{2}\log\frac{m}{M}\mathbf{1}. \end{aligned}$$

Remark 4.24 Inequalities (4.100) and (4.101) can be further altered in accordance with the positivity and negativity of the term $(M_s(X,\phi))^s - (M_r(X,\phi))^s$ which has already been discussed in Remark 4.23.

4.5 Mutual bounds for Jensen-type operator inequalities related to higher order convexity

The main objective of this section is to establish lower and upper bounds for the difference between the left-hand side and the right-hand side of the Lah-Ribarič operator inequality (4.2), which hold for a class of n-convex functions. The results that follow will be established in a general setting, as described in the Introduction. Therefore, throughout, we assume that $\mathscr{A}$ and $\mathscr{B}$ are unital C^*-algebras, and $(\phi_t)_{t\in T}$ is a unital field of positive linear mappings $\phi_t : \mathscr{A} \to \mathscr{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ.

Our first extension of the Lah-Ribarič inequality (4.2) that holds for n-convex functions follows by virtue of Lemma 2.3, and is given in the following theorem. Throughout this paper, whenever mentioning the interval $[a,b]$, we assume that a,b are real numbers such that $a < b$.

Theorem 4.17 ([83]) *Let $f \in \mathscr{C}^n([a,b])$ be n-convex function. If $n > m \geq 3$ are of different parity, then the inequality*

$$
\begin{aligned}
&\int_T \phi_t(f(x_t))d\mu(t) - \alpha_f \int_T \phi_t(x_t)d\mu(t) - \beta_f \mathbf{1} \\
&\leq (f[a,a] - f[a,b]) \left(\int_T \phi_t(x_t)d\mu(t) - a\mathbf{1} \right) \\
&\quad + \sum_{k=2}^{m-1} \frac{f^{(k)}(a)}{k!} \int_T \phi_t\left((x_t - a\mathbf{1})^k \right) d\mu(t) \\
&\quad + \sum_{k=1}^{n-m} f[\underbrace{a,\dots,a}_{m \text{ times}};\underbrace{b,\dots,b}_{k \text{ times}}] \int_T \phi_t \left((x_t - a\mathbf{1})^m (x_t - b\mathbf{1})^{k-1} \right) d\mu(t)
\end{aligned}
\tag{4.105}
$$

holds for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathscr{A}$ with spectra contained in $[a,b]$. Inequality (4.105) holds also when f is n-concave and n and m are of equal parity. Moreover, when f is n-convex and n and m are of equal parity, or when f is n-concave and n and m are of different parity, then the inequality sign in (4.105) is reversed.

Proof. Since $f \in \mathscr{C}^n([a,b])$, it follows that its n-th order divided difference $f_n(t) = f[t;\underbrace{a,\dots,a}_{m \text{ times}};\ \underbrace{b,b,\dots,b}_{(n-m) \text{ times}}]$ is continuous, so consequently the function $R_m(t)$ defined by (2.64) is also continuous. Therefore, applying the functional calculus to scalar relation (2.65), i.e. setting x_t instead of scalar t, we have

$$
\begin{aligned}
f(x_t) - \alpha_f x_t - \beta_f \mathbf{1} &= (x_t - a\mathbf{1})(f[a,a] - f[a,b]) + \sum_{k=2}^{m-1} \frac{f^{(k)}(a)}{k!}(x_t - a\mathbf{1})^k \\
&\quad + \sum_{k=1}^{n-m} f[\underbrace{a,\dots,a}_{m \text{ times}};\underbrace{b,\dots,b}_{k \text{ times}}](x_t - a\mathbf{1})^m (x_t - b\mathbf{1})^{k-1} + R_m(x_t).
\end{aligned}
$$

Now, applying the positive linear mapping ϕ_t to the above relation and then by integrating, it follows that

$$
\begin{aligned}
&\int_T \phi_t(f(x_t))d\mu(t) - \alpha_f \int_T \phi_t(x_t)d\mu(t) - \beta_f \mathbf{1} \\
&= (f[a,a] - f[a,b]) \left(\int_T \phi_t(x_t)d\mu(t) - a\mathbf{1} \right) \\
&\quad + \sum_{k=2}^{m-1} \frac{f^{(k)}(a)}{k!} \int_T \phi_t\left((x_t - a\mathbf{1})^k \right) d\mu(t) \\
&\quad + \sum_{k=1}^{n-m} f[\underbrace{a,\dots,a}_{m \text{ times}};\underbrace{b,\dots,b}_{k \text{ times}}] \int_T \phi_t \left((x_t - a\mathbf{1})^m (x_t - b\mathbf{1})^{k-1} \right) d\mu(t) \\
&\quad + \int_T \phi_t(R_m(x_t))d\mu(t).
\end{aligned}
\tag{4.106}
$$

In order to complete our proof, we discuss positivity of the term $\int_T \phi_t\left(R_m(x_t)\right) d\mu(t)$. Due to the monotonicity property, it suffices to study the sign of the function

$$R_m(t) = (t-a)^m (t-b)^{n-m} f[t;\underbrace{a,\dots,a}_{m\text{ times}};\underbrace{b,b,\dots,b}_{(n-m)\text{ times}}].$$

Since $a \le t \le b$, it follows that $(t-a)^m \ge 0$ for any choice of m. Similarly, $t-b \le 0$ implies that $(t-b)^{n-m} \le 0$ when n and m are of different parity, and $(t-b)^{n-m} \ge 0$ when n and m are of the same parity. Finally, according to the definition, $f[t;\underbrace{a,\dots,a}_{m\text{ times}};\underbrace{b,b,\dots,b}_{(n-m)\text{ times}}] \ge 0 (\le 0)$ for n-convex (n-concave) function, so the inequality (4.105) easily follows from relation (4.106) and the proof is completed. □

Our next result provides another extension of the Lah-Ribarič operator inequality (4.2) for n-convex functions in terms of divided differences, and it follows by virtue of Lemma 2.4.

Theorem 4.18 ([83]) *Let $f \in \mathscr{C}^n([a,b])$ be n-convex function. If $m \ge 3$ is odd and $m < n$, then the inequality*

$$\begin{aligned}
&\int_T \phi_t(f(x_t))d\mu(t) - \alpha_f \int_T \phi_t(x_t)d\mu(t) - \beta_f \mathbf{1} \\
&\le (f[a,b]-f[b,b])\left(b\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)\right) \\
&\quad + \sum_{k=2}^{m-1} \frac{f^{(k)}(b)}{k!} \int_T \phi_t\left((x_t - b\mathbf{1})^k\right) d\mu(t) \\
&\quad + \sum_{k=1}^{n-m} f[\underbrace{b,\dots,b}_{m\ times};\underbrace{a,\dots,a}_{k\ times}] \int_T \phi_t\left((x_t-b\mathbf{1})^m (x_t - a\mathbf{1})^{k-1}\right) d\mu(t)
\end{aligned} \tag{4.107}$$

holds for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathscr{A}$ with spectra contained in $[a,b]$. Inequality (4.107) also holds when f is n-concave and m is even. Moreover, when f is n-convex and m is even, or when f is n-concave and m is odd, then the inequality sign in (4.107) is reversed.

Proof. In a similar manner as in the proof of the previous theorem, since every involved function is continuous, we can replace t with operator x_t in (2.69), and then apply positive linear mapping ϕ_t to the obtained relation and integrate it. In such a way we have

$$\begin{aligned}
&\int_T \phi_t(f(x_t))d\mu(t) - \alpha_f \int_T \phi_t(x_t)d\mu(t) - \beta_f \mathbf{1} \\
&= (f[a,b]-f[b,b])\left(b\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)\right) \\
&\quad + \sum_{k=2}^{m-1} \frac{f^{(k)}(b)}{k!} \int_T \phi_t\left((x_t - b\mathbf{1})^k\right) d\mu(t)
\end{aligned} \tag{4.108}$$

$$+\sum_{k=1}^{n-m} f[\underbrace{b,\dots,b}_{m \text{ times}};\underbrace{a,\dots,a}_{k \text{ times}}] \int_T \phi_t\left((x_t-b\mathbf{1})^m(x_t-a\mathbf{1})^{k-1}\right)d\mu(t)$$
$$+\int_T \phi_t\left(R_m^*(x_t)\right)d\mu(t).$$

Following the lines of the previous theorem, it remains to discuss positivity of the term $\int_T \phi_t\left(R_m^*(x_t)\right)d\mu(t)$. Again, due to the monotonicity property, it is enough to study the sign of the function

$$R_m^*(t) = (t-b)^m(t-a)^{n-m} f[t;\underbrace{b,\dots,b}_{m \text{ times}};\ \underbrace{a,a,\dots,a}_{(n-m) \text{ times}}].$$

Since $t \in [a,b]$, we have $(t-a)^{n-m} \geq 0$ for every t and for any choice of m. Similarly, $(t-b)^m \leq 0$ when m is odd, and $(t-b)^m \geq 0$ when m is even. Finally, taking into account the definition of n-convex (n-concave) function and relation (4.108), we obtain (4.107), as claimed. □

Remark 4.25 In the proofs of previous two theorems, when discussing positivity of terms $\int_T \phi_t\left(R_m(x_t)\right)d\mu(t)$ and $\int_T \phi_t\left(R_m^*(x_t)\right)d\mu(t)$, it was enough to discuss the sign of functions $R_m(t)$ and $R_m^*(t)$ since for a continuous and positive function f and a self-adjoint operator x_t, the operator $f(x_t)$ is positive definite. Moreover, since a positive linear mapping ϕ_t preserves positivity, it follows that $\int_T \phi_t\left(f(x_t)\right)d\mu(t) \geq 0$.

By combining results from Theorem 4.17 and Theorem 4.18, we get the following bounds for the difference between the left-hand side and the right-hand side of the Lah-Ribarič operator inequality (4.2), which are valid for the class of n-convex functions.

Corollary 4.14 ([83]) *Let $f \in \mathscr{C}^n([a,b])$ be n-convex function, where n is an odd number. If $m \geq 3$ is odd and $m < n$, then the series of inequalities*

$$
\begin{aligned}
&(f[a,a]-f[a,b])\left(\int_T \phi_t(x_t)d\mu(t)-a\mathbf{1}\right)\\
&\quad+\sum_{k=2}^{m-1}\frac{f^{(k)}(a)}{k!}\int_T \phi_t\left((x_t-a\mathbf{1})^k\right)d\mu(t)\\
&\quad+\sum_{k=1}^{n-m} f[\underbrace{a,\dots,a}_{m \text{ times}};\underbrace{b,\dots,b}_{k \text{ times}}]\int_T \phi_t\left((x_t-a\mathbf{1})^m(x_t-b\mathbf{1})^{k-1}\right)d\mu(t)\\
&\leq \int_T \phi_t(f(x_t))d\mu(t)-\alpha_f\int_T \phi_t(x_t)d\mu(t)-\beta_f\mathbf{1} \qquad (4.109)\\
&\leq (f[a,b]-f[b,b])\left(b\mathbf{1}-\int_T \phi_t(x_t)d\mu(t)\right)\\
&\quad+\sum_{k=2}^{m-1}\frac{f^{(k)}(b)}{k!}\int_T \phi_t\left((x_t-b\mathbf{1})^k\right)d\mu(t)\\
&\quad+\sum_{k=1}^{n-m} f[\underbrace{b,\dots,b}_{m \text{ times}};\underbrace{a,\dots,a}_{k \text{ times}}]\int_T \phi_t\left((x_t-b\mathbf{1})^m(x_t-a\mathbf{1})^{k-1}\right)d\mu(t)
\end{aligned}
$$

holds for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathscr{A}$ with spectra contained in $[a,b]$. Inequality (4.109) also holds when f is n-concave and m is even. Moreover, when f is n-convex and m is even, or when f is n-concave and m is odd, then the inequality signs in (4.109) are reversed.

The following result also provides mutual bounds for the difference between the left-hand side and the right-hand side of the Lah-Ribarič operator inequality, and it relies on scalar relations (2.62) and (2.63).

Theorem 4.19 ([83]) *Let $f \in \mathscr{C}^n([a,b])$ be n-convex function, where $n \geq 3$ is an odd number. Then the inequalities*

$$
\begin{aligned}
&\sum_{k=2}^{n-1} f[a;\underbrace{b,\dots,b}_{k\ times}]\int_T \phi_t\left((x_t-a\mathbf{1})(x_t-b\mathbf{1})^{k-1}\right)d\mu(t)\\
&\leq \int_T \phi_t(f(x_t))d\mu(t)-\alpha_f\int_T \phi_t(x_t)d\mu(t)-\beta_f\mathbf{1}\\
&\leq f[a,a;b]\int_T \phi_t\left((x_t-a\mathbf{1})(x_t-b\mathbf{1})\right)d\mu(t)\\
&\qquad+\sum_{k=2}^{n-2} f[a,a;\underbrace{b,\dots,b}_{k\ times}]\int_T \phi_t\left((x_t-a\mathbf{1})^2(x_t-b\mathbf{1})^{k-1}\right)d\mu(t)
\end{aligned}
\tag{4.110}
$$

hold for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathscr{A}$ with spectra contained in $[a,b]$. Inequalities in (4.110) also hold when f is n-concave and n is even. In the case when f is n-convex and n is even, or when f is n-concave and n is odd, then the inequality signs in (4.110) are reversed.

Proof. Following the lines as in the proofs of the previous two theorems, we can replace t by x_t in (2.62) and (2.63) respectively, and then, apply a positive linear mapping ϕ_t to the established relations and integrate them. By doing so, we get

$$
\begin{aligned}
&\int_T \phi_t(f(x_t))d\mu(t)-\alpha_f\int_T \phi_t(x_t)d\mu(t)-\beta_f\mathbf{1}\\
&=\sum_{k=2}^{n-1} f[a;\underbrace{b,\dots,b}_{k\text{ times}}]\int_T \phi_t\left((x_t-a\mathbf{1})(x_t-b\mathbf{1})^{k-1}\right)d\mu(t)+\int_T \phi_t(R_1(x_t))d\mu(t)
\end{aligned}
\tag{4.111}
$$

and

$$
\begin{aligned}
&\int_T \phi_t(f(x_t))d\mu(t)-\alpha_f\int_T \phi_t(x_t)d\mu(t)-\beta_f\mathbf{1}\\
&=f[a,a;b]\int_T \phi_t\left((x_t-a\mathbf{1})(x_t-b\mathbf{1})\right)d\mu(t)\\
&\qquad+\sum_{k=2}^{n-2} f[a,a;\underbrace{b,\dots,b}_{k\text{ times}}]\int_T \phi_t\left((x_t-a\mathbf{1})^2(x_t-b\mathbf{1})^{k-1}\right)d\mu(t)\\
&\qquad+\int_T \phi_t\left(R_2(x_t)\right)d\mu(t).
\end{aligned}
\tag{4.112}
$$

Now, taking into account the discussion about positivity of term $\int_T \phi_t\left(R_m(x_t)\right) d\mu(t)$ as in the proof of Theorem 4.17, it follows that if $m=1$, then

- $\int_T \phi_t\left(R_1(x_t)\right) d\mu(t) \geq 0$ when f is n-convex and n is odd, or when f is n-concave and n even;

- $\int_T \phi_t\left(R_1(x_t)\right) d\mu(t) \leq 0$ when f is n-concave and n is odd, or when f is n-convex and n even.

Hence, if $\int_T \phi_t\left(R_1(x_t)\right) d\mu(t) \geq 0$, the relation (4.111) yields the first inequality sign in (4.110). Moreover, if $\int_T \phi_t\left(R_1(x_t)\right) d\mu(t) \leq 0$ the corresponding inequality sign is reversed.

In the same manner, if $m=2$, then

- $\int_T \phi_t\left(R_2(x_t)\right) d\mu(t) \leq 0$ when f is n-convex and n is odd, or when f is n-concave and n even;

- $\int_T \phi_t\left(R_2(x_t)\right) d\mu(t) \geq 0$ when f is n-concave and n is odd, or when f is n-convex and n even.

Consequently, if $\int_T \phi_t\left(R_2(x_t)\right) d\mu(t) \leq 0$, the relation (4.112) provides the second inequality sign in (4.110), while for $\int_T \phi_t\left(R_2(x_t)\right) d\mu(t) \geq 0$ the corresponding sign is reversed. This completes the proof. □

In order to conclude this topic, we give yet another pair of mutual bounds for the difference between the left-hand side and the right-hand side of the Lah-Ribarič operator inequality. The corresponding result relies on Lemma 2.4 and it is interesting since it holds for all $n \in \mathbb{N}$, not only for the odd ones.

Theorem 4.20 ([83]) *If $f \in \mathscr{C}^n([a,b])$ is n-convex function, where $n \geq 3$, then the inequalities*

$$\begin{aligned}
& f[b,b;a] \int_T \phi_t\left((x_t - b\mathbf{1})(x_t - a\mathbf{1})\right) d\mu(t) \\
& \qquad + \sum_{k=2}^{n-2} f[b,b;\underbrace{a,\dots,a}_{k \text{ times}}] \int_T \phi_t\left((x_t - b\mathbf{1})^2 (x_t - a\mathbf{1})^{k-1}\right) d\mu(t) \\
& \leq \int_T \phi_t(f(x_t)) d\mu(t) - \alpha_f \int_T \phi_t(x_t) d\mu(t) - \beta_f \mathbf{1} \\
& \leq \sum_{k=2}^{n-1} f[b;\underbrace{a,\dots,a}_{k \text{ times}}] \int_T \phi_t\left((x_t - b\mathbf{1})(x_t - a\mathbf{1})^{k-1}\right) d\mu(t)
\end{aligned} \tag{4.113}$$

hold for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathscr{A}$ with spectra contained in $[a,b]$. If the function f is n-concave, the inequality signs in (4.113) are reversed.

Proof. We follow the same procedure as in the proof of Theorem 4.19 except that we utilize relations (2.66) and (2.67) instead of (2.62) and (2.63). In such a way we obtain relations

$$\begin{aligned}
&\int_T \phi_t(f(x_t))d\mu(t) - \alpha_f \int_T \phi_t(x_t)d\mu(t) - \beta_f \mathbf{1} \\
&= \sum_{k=2}^{n-1} f[b;\underbrace{a,\ldots,a}_{k \text{ times}}] \int_T \phi_t\left((x_t - b\mathbf{1})(x_t - a\mathbf{1})^{k-1}\right) d\mu(t) + \int_T \phi_t\left(R_1^*(x_t)\right) d\mu(t)
\end{aligned} \tag{4.114}$$

and

$$\begin{aligned}
&\int_T \phi_t(f(x_t))d\mu(t) - \alpha_f \int_T \phi_t(x_t)d\mu(t) - \beta_f \mathbf{1} \\
&= f[b,b;a] \int_T \phi_t\left((x_t - b\mathbf{1})(x_t - a\mathbf{1})\right) d\mu(t) \\
&\quad + \sum_{k=2}^{n-2} f[b,b;\underbrace{a,\ldots,a}_{k \text{ times}}] \int_T \phi_t\left((x_t - b\mathbf{1})^2(x_t - a\mathbf{1})^{k-1}\right) d\mu(t) \\
&\quad + \int_T \phi_t\left(R_2^*(x_t)\right) d\mu(t).
\end{aligned} \tag{4.115}$$

Now, we refer to discussion about positivity of the term $\int_T \phi_t\left(R_m^*(x_t)\right) d\mu(t)$, as in the proof of Theorem 4.18. If $m = 1$, then $(t-b)^1(t-a)^{n-1} \leq 0$ for every $t \in [a,b]$, so $\int_T \phi_t\left(R_1^*(x_t)\right) d\mu(t) \geq 0$ when the function f is n-concave, and $\int_T \phi_t\left(R_1^*(x_t)\right) d\mu(t) \leq 0$ when f is n-convex. Therefore the relation (4.114) yields the second inequality sign in (4.113) for an n-convex function f, while for n-concave function f the corresponding sign is reversed.

Similarly, for $m = 2$ we have $(t-b)^2(t-a)^{n-2} \geq 0$ for every $t \in [a,b]$, so $\int_T \phi_t\left(R_2^*(x_t)\right) d\mu(t) \geq 0$ when f is n-convex, and $\int_T \phi_t\left(R_2^*(x_t)\right) d\mu(t) \leq 0$ when f is n-concave. In this case the identity (4.115) yields the first inequality sign in (4.113), which completes the proof. □

Remark 4.26 It should be noticed here that if f is 3-convex function, then the series of inequalities in (4.110) and (4.113) coincide, providing the relation

$$\begin{aligned}
&f[b,b;a] \int_T \phi_t\left((x_t - b\mathbf{1})(x_t - a\mathbf{1})\right) d\mu(t) \\
&\leq \int_T \phi_t(f(x_t))d\mu(t) - \alpha_f \int_T \phi_t(x_t)d\mu(t) - \beta_f \mathbf{1} \\
&\leq f[a,a;b] \int_T \phi_t\left((x_t - b\mathbf{1})(x_t - a\mathbf{1})\right) d\mu(t).
\end{aligned} \tag{4.116}$$

In the remainder of this section we will utilize the results from above, as well Lemma 2.3 and Lemma 2.4, in order to obtain several Jensen-type operator inequalities that correspond to n-convex functions. More precisely, we will establish several mutual bounds for the difference between the right-hand side and the left hand side of the Jensen operator inequality (4.1). The results that follows will be derived in the same setting as in the previous section.

Our first estimate for the difference in the Jensen inequality (4.1) relies on Corollary 4.14.

Theorem 4.21 ([83]) *Let $f \in \mathscr{C}^n([a,b])$ be n-convex function, where n is an odd number. If $m \geq 3$ is odd and $m < n$, then the series of inequalities*

$$H_f(a,b) \leq \int_T \phi_t(f(x_t))\,d\mu(t) - f\left(\int_T \phi_t(x_t)d\mu(t)\right) \leq H_f(b,a), \tag{4.117}$$

where

$$\begin{aligned}
H_f(a,b) &= \big(f(a)-f(b)+bf'(b)-af'(a)\big)\mathbf{1} + (f'(a)-f'(b))\int_T \phi_t(x_t)d\mu(t) \\
&+\sum_{k=2}^{m-1}\left[\frac{f^{(k)}(a)}{k!}\int_T \phi_t\left((x_t-a\mathbf{1})^k\right)d\mu(t) - \frac{f^{(k)}(b)}{k!}\left(\int_T \phi_t(x_t)d\mu(t)-b\mathbf{1}\right)^k\right] \\
&+\sum_{k=1}^{n-m} f[\underbrace{a,\ldots,a}_{m\ times};\underbrace{b,\ldots,b}_{k\ times}]\int_T \phi_t\left((x_t-a\mathbf{1})^m(x_t-b\mathbf{1})^{k-1}\right)d\mu(t) \\
&-\sum_{k=1}^{n-m} f[\underbrace{b,\ldots,b}_{m\ times};\underbrace{a,\ldots,a}_{k\ times}]\left(\int_T \phi_t(x_t)d\mu(t)-b\mathbf{1}\right)^m\left(\int_T \phi_t(x_t)d\mu(t)-a\mathbf{1}\right)^{k-1},
\end{aligned}$$

hold for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathscr{A}$ with spectra contained in $[a,b]$. Inequalities in (4.117) also hold when f is n-concave and m is even. In the cases when f is n-convex and m is even, or when f is n-concave and m is odd, the inequality signs in (4.117) are reversed.

Proof. Since $a\mathbf{1} \leq x_t \leq b\mathbf{1}$ for every $t \in T$, it follows that $a\phi_t(\mathbf{1}) \leq \phi_t(x_t) \leq b\phi_t(\mathbf{1})$, i.e. $a\mathbf{1} \leq \int_T \phi_t(x_t)d\mu(t) \leq b\mathbf{1}$. Hence, applying the functional calculus to relation (2.65), i.e. setting $\int_T \phi_t(x_t)d\mu(t)$ instead of t, it follows that

$$\begin{aligned}
&f\left(\int_T \phi_t(x_t)d\mu(t)\right) - \alpha_f\int_T \phi_t(x_t)d\mu(t) - \beta_f\mathbf{1} \\
&= \left(\int_T \phi_t(x_t)d\mu(t)-a\mathbf{1}\right)(f[a,a]-f[a,b]) + \sum_{k=2}^{m-1}\frac{f^{(k)}(a)}{k!}\left(\int_T \phi_t(x_t)d\mu(t)-a\mathbf{1}\right)^k \\
&\quad+\sum_{k=1}^{n-m} f[\underbrace{a,\ldots,a}_{m\text{ times}};\underbrace{b,\ldots,b}_{k\text{ times}}]\left(\int_T \phi_t(x_t)d\mu(t)-a\mathbf{1}\right)^m\left(\int_T \phi_t(x_t)d\mu(t)-b\mathbf{1}\right)^{k-1} \\
&\quad+R_m\left(\int_T \phi_t(x_t)d\mu(t)\right).
\end{aligned}$$

Now, we study positivity of the term $R_m\left(\int_T \phi_t(x_t)d\mu(t)\right)$. Namely, since $a\mathbf{1} \leq \int_T \phi_t(x_t)d\mu(t) \leq b\mathbf{1}$, due to the monotonicity property, it suffices to study positivity of the scalar function

$R_m(t)$ for $t \in [a,b]$, which we have already done in the proof of Theorem 4.17. Hence, if f is an n-convex function and n and $m \geq 3$ are of different parity, or if f is n-concave and n and $m \geq 3$ are of the same parity, the above relation yields the inequality

$$\begin{aligned}
& f\left(\int_T \phi_t(x_t)d\mu(t)\right) - \alpha_f \int_T \phi_t(x_t)d\mu(t) - \beta_f \mathbf{1} \\
& \leq \left(\int_T \phi_t(x_t)d\mu(t) - a\mathbf{1}\right)(f[a,a]-f[a,b]) + \sum_{k=2}^{m-1} \frac{f^{(k)}(a)}{k!}\left(\int_T \phi_t(x_t)d\mu(t) - a\mathbf{1}\right)^k \\
& \quad + \sum_{k=1}^{n-m} f[\underbrace{a,\ldots,a}_{m \text{ times}};\underbrace{b,\ldots,b}_{k \text{ times}}]\left(\int_T \phi_t(x_t)d\mu(t) - a\mathbf{1}\right)^m \left(\int_T \phi_t(x_t)d\mu(t) - b\mathbf{1}\right)^{k-1}.
\end{aligned}$$

Clearly, if f is n-convex and n and $m \geq 3$ are of the same parity, or if f is n-concave and n and $m \geq 3$ are of different parity, the inequality sign is reversed.

Now, in the same way as above, applying the functional calculus to (2.69) and taking into account discussion about the sign of the scalar function $R_m^*(t)$ for $t \in [a,b]$ (see the proof of the Theorem 4.18), we obtain the inequality

$$\begin{aligned}
& f\left(\int_T \phi_t(x_t)d\mu(t)\right) - \alpha_f \int_T \phi_t(x_t)d\mu(t) - \beta_f \mathbf{1} \\
& \leq \left(b\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)\right)(f[a,b]-f[b,b]) + \sum_{k=2}^{m-1} \frac{f^{(k)}(b)}{k!}\left(\int_T \phi_t(x_t)d\mu(t) - b\mathbf{1}\right)^k \\
& \quad + \sum_{k=1}^{n-m} f[\underbrace{b,\ldots,b}_{m \text{ times}};\underbrace{a,\ldots,a}_{k \text{ times}}]\left(\int_T \phi_t(x_t)d\mu(t) - b\mathbf{1}\right)^m \left(\int_T \phi_t(x_t)d\mu(t) - a\mathbf{1}\right)^{k-1},
\end{aligned}$$

which holds for n-convex function f and an odd number $m \geq 3$ or for n-concave function f and an even number $m \geq 3$. If f is n-convex and m is even, or if f is n-concave and m is odd, the inequality sign is reversed.

By combining the previous two inequalities, it follows that the series of inequalities

$$\begin{aligned}
& \left(\int_T \phi_t(x_t)d\mu(t) - a\mathbf{1}\right)(f[a,a]-f[a,b]) + \sum_{k=2}^{m-1} \frac{f^{(k)}(a)}{k!}\left(\int_T \phi_t(x_t)d\mu(t) - a\mathbf{1}\right)^k \\
& \quad + \sum_{k=1}^{n-m} f[\underbrace{a,\ldots,a}_{m \text{ times}};\underbrace{b,\ldots,b}_{k \text{ times}}]\left(\int_T \phi_t(x_t)d\mu(t) - a\mathbf{1}\right)^m \left(\int_T \phi_t(x_t)d\mu(t) - b\mathbf{1}\right)^{k-1} \\
& \leq f\left(\int_T \phi_t(x_t)d\mu(t)\right) - \alpha_f \int_T \phi_t(x_t)d\mu(t) - \beta_f \mathbf{1} \\
& \leq \left(b\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)\right)(f[a,b]-f[b,b]) + \sum_{k=2}^{m-1} \frac{f^{(k)}(b)}{k!}\left(\int_T \phi_t(x_t)d\mu(t) - b\mathbf{1}\right)^k \\
& \quad + \sum_{k=1}^{n-m} f[\underbrace{b,\ldots,b}_{m \text{ times}};\underbrace{a,\ldots,a}_{k \text{ times}}]\left(\int_T \phi_t(x_t)d\mu(t) - b\mathbf{1}\right)^m \left(\int_T \phi_t(x_t)d\mu(t) - a\mathbf{1}\right)^{k-1}
\end{aligned}$$

holds if f is n-convex and m is odd, or if f is n-concave and m is even. If f is n-convex and m is even, or f is n-concave and m is odd, then the inequality signs are reversed.

Finally, multiplying the above series of inequalities by -1 and then, adding it to (4.109), we obtain exactly the relation (4.117), as claimed. □

Our next result provides yet another lower and upper bound for the difference in the Jensen operator inequality for n-convex functions, this time obtained by virtue of Lemma 2.3 and Theorem 4.19.

Theorem 4.22 ([83]) *Let $f \in \mathscr{C}^n([a,b])$ be n-convex function, where $n \geq 3$ is an odd number. Then the series of inequalities*

$$\begin{aligned}
& f[a,a;b]\left(b\mathbf{1}-\int_T \phi_t(x_t)d\mu(t)\right)\left(\int_T \phi_t(x_t)d\mu(t)-a\mathbf{1}\right) \\
& \quad +\sum_{k=2}^{n-1} f[a;\underbrace{b,\ldots,b}_{k\ times}]\int_T \phi_t\left((x_t-a\mathbf{1})(x_t-b\mathbf{1})^{k-1}\right)d\mu(t) \\
& \quad -\left(\int_T \phi_t(x_t)d\mu(t)-a\mathbf{1}\right)^2\sum_{k=2}^{n-2} f[a,a;\underbrace{b,\ldots,b}_{k\ times}]\left(\int_T \phi_t(x_t)d\mu(t)-b\mathbf{1}\right)^{k-1} \\
& \leq \int_T \phi_t\left(f(x_t)\right)d\mu(t)-f\left(\int_T \phi_t(x_t)d\mu(t)\right) \qquad (4.118)\\
& \leq f[a,a;b]\int_T \phi_t\left((x_t-a\mathbf{1})(x_t-b\mathbf{1})\right)d\mu(t) \\
& \quad -\left(\int_T \phi_t(x_t)d\mu(t)-a\mathbf{1}\right)\sum_{k=2}^{n-1} f[a;\underbrace{b,\ldots,b}_{k\ times}]\left(\int_T \phi_t(x_t)d\mu(t)-b\mathbf{1}\right)^{k-1} \\
& \quad +\sum_{k=2}^{n-2} f[a,a;\underbrace{b,\ldots,b}_{k\ times}]\int_T \phi_t\left((x_t-a\mathbf{1})^2(x_t-b\mathbf{1})^{k-1}\right)d\mu(t)
\end{aligned}$$

holds for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathscr{A}$ with spectra contained in $[a,b]$. Inequalities in (4.118) also hold when f is n-concave and n is even. In the cases when f is n-convex and n is even, or when f is n-concave and n is odd, the inequality signs in (4.118) are reversed.

Proof. By following the same procedure as in the proof of the previous theorem, we start by replacing t with $\int_T \phi_t(x_t)d\mu(t)$ in relations (2.62) and (2.63). Therefore we get

$$\begin{aligned}
& f\left(\int_T \phi_t(x_t)d\mu(t)\right)-\alpha_f\int_T \phi_t(x_t)d\mu(t)-\beta_f\mathbf{1} \\
& =\sum_{k=2}^{n-1} f[a;\underbrace{b,\ldots,b}_{k\text{ times}}]\left(\int_T \phi_t(x_t)d\mu(t)-a\mathbf{1}\right)\left(\int_T \phi_t(x_t)d\mu(t)-b\mathbf{1}\right)^{k-1}+R_1\left(\int_T \phi_t(x_t)d\mu(t)\right)
\end{aligned}$$

and

$$
\begin{aligned}
& f\left(\int_T \phi_t(x_t)d\mu(t)\right) - \alpha_f \int_T \phi_t(x_t)d\mu(t) - \beta_f \mathbf{1} \\
&= f[a,a;b]\left(\int_T \phi_t(x_t)d\mu(t) - a\mathbf{1}\right)\left(\int_T \phi_t(x_t)d\mu(t) - b\mathbf{1}\right) \\
&\quad + \sum_{k=2}^{n-2} f[a,a;\underbrace{b,\ldots,b}_{k \text{ times}}]\left(\int_T \phi_t(x_t)d\mu(t) - a\mathbf{1}\right)^2\left(\int_T \phi_t(x_t)d\mu(t) - b\mathbf{1}\right)^{k-1} \\
&\quad + R_2\left(\int_T \phi_t(x_t)d\mu(t)\right),
\end{aligned}
$$

respectively. Now, taking into account discussion about the sign of scalar terms $R_1(t)$ and $R_2(t)$, $t \in [a,b]$, from Theorem 4.19, the above relations imply that the series of inequalities

$$
\begin{aligned}
& \sum_{k=2}^{n-1} f[a;\underbrace{b,\ldots,b}_{k \text{ times}}]\left(\int_T \phi_t(x_t)d\mu(t) - a\mathbf{1}\right)\left(\int_T \phi_t(x_t)d\mu(t) - b\mathbf{1}\right)^{k-1} \\
&\leq f\left(\int_T \phi_t(x_t)d\mu(t)\right) - \alpha_f \int_T \phi_t(x_t)d\mu(t) - \beta_f \mathbf{1} \\
&\leq f[a,a;b]\left(\int_T \phi_t(x_t)d\mu(t) - a\mathbf{1}\right)\left(\int_T \phi_t(x_t)d\mu(t) - b\mathbf{1}\right) \\
&\quad + \sum_{k=2}^{n-2} f[a,a;\underbrace{b,\ldots,b}_{k \text{ times}}]\left(\int_T \phi_t(x_t)d\mu(t) - a\mathbf{1}\right)^2\left(\int_T \phi_t(x_t)d\mu(t) - b\mathbf{1}\right)^{k-1}
\end{aligned}
$$

holds when n is odd and f is n-convex, or when n is even and f is n-concave. If n is odd and f is n-concave, or if n is even and f is n-convex, then the corresponding inequality signs are reversed.

Finally, inequalities in (4.118) follow after multiplying the above series by -1 and adding it to (4.110). □

In an analogous way as described in the proof of the previous theorem, this time by virtue of Lemma 2.4 and Theorem 4.20, we can get a similar lower and upper bound for the difference between the right-hand side and the left-hand side of (4.1) that holds for all $n \geq 3$, not only for the odd ones.

Theorem 4.23 ([83]) *If $f \in \mathscr{C}^n([a,b])$ is n-convex function, $n \geq 3$, then the inequalities*

$$
\begin{aligned}
& f[b,b;a]\int_T \phi_t\left((x_t - b\mathbf{1})(x_t - a\mathbf{1})\right)d\mu(t) \\
&\quad - \left(\int_T \phi_t(x_t)d\mu(t) - b\mathbf{1}\right)\sum_{k=2}^{n-1} f[b;\underbrace{a,\ldots,a}_{k \text{ times}}]\left(\int_T \phi_t(x_t)d\mu(t) - a\mathbf{1}\right)^{k-1} \\
&\quad + \sum_{k=2}^{n-2} f[b,b;\underbrace{a,\ldots,a}_{k \text{ times}}]\int_T \phi_t\left((x_t - b\mathbf{1})^2(x_t - a\mathbf{1})^{k-1}\right)d\mu(t)
\end{aligned}
$$

$$\begin{aligned}
&\leq \int_T \phi_t(f(x_t))\,d\mu(t) - f\left(\int_T \phi_t(x_t)\,d\mu(t)\right) \qquad (4.119)\\
&\leq f[b,b;a]\left(b\mathbf{1} - \int_T \phi_t(x_t)\,d\mu(t)\right)\left(\int_T \phi_t(x_t)\,d\mu(t) - a\mathbf{1}\right)\\
&\quad + \sum_{k=2}^{n-1} f[b;\underbrace{a,\ldots,a}_{k\ times}] \int_T \phi_t\left((x_t - b\mathbf{1})(x_t - a\mathbf{1})^{k-1}\right) d\mu(t)\\
&\quad - \left(\int_T \phi_t(x_t)\,d\mu(t) - b\mathbf{1}\right)^2 \sum_{k=2}^{n-2} f[b,b;\underbrace{a,\ldots,a}_{k\ times}]\left(\int_T \phi_t(x_t)\,d\mu(t) - a\mathbf{1}\right)^{k-1}
\end{aligned}$$

hold for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathscr{A}$ with spectra contained in $[a,b]$. In addition, if f is n-concave, then the inequality signs in (4.119) are reversed.

Remark 4.27 It should be noticed here that if f is 3-convex function, then the relations (4.118) and (4.119) coincide, giving the series of inequalities

$$\begin{aligned}
&f[a,a;b]\left(b\mathbf{1} - \int_T \phi_t(x_t)\,d\mu(t)\right)\left(\int_T \phi_t(x_t)\,d\mu(t) - a\mathbf{1}\right)\\
&\quad + f[b,b;a]\int_T \phi_t((x_t - a\mathbf{1})(x_t - b\mathbf{1}))\,d\mu(t)\\
&\leq \int_T \phi_t(f(x_t))\,d\mu(t) - f\left(\int_T \phi_t(x_t)\,d\mu(t)\right) \qquad (4.120)\\
&\leq f[b,b;a]\left(b\mathbf{1} - \int_T \phi_t(x_t)\,d\mu(t)\right)\left(\int_T \phi_t(x_t)\,d\mu(t) - a\mathbf{1}\right)\\
&\quad + f[a,a;b]\int_T \phi_t((x_t - a\mathbf{1})(x_t - b\mathbf{1}))\,d\mu(t).
\end{aligned}$$

4.5.1 Applications to quasi-arithmetic operator means

Our goal in this subsection is an application of general Jensen-type inequalities established above to the so-called quasi-arithmetic operator means. In such a way, we shall obtain mutual bounds for the differences of quasi-arithmetic means.

A few years ago, Mićić et al. [101], investigated an order among quasi-arithmetic means $M_\chi(x,\phi)$ and $M_\psi(x,\phi)$. Such order was derived by virtue of operator convexity and operator monotonicity of the corresponding functions appearing in these means. A similar conclusion can be drawn for quasi-arithmetic means in a view of higher convexity. More precisely, utilizing the Jensen-type inequalities from above, we will establish several mutual bounds for the difference $\chi\left(M_\chi(x,\phi)\right) - \chi\left(M_\psi(x,\phi)\right)$ of quasi-arithmetic means.

Before we state our results, we will first introduce some notation arising from this particular setting. Throughout this section we denote $F = \chi\circ\psi^{-1}$ and $\psi_a = \min\{\psi(a),\psi(b)\}$, $\psi_b = \max\{\psi(a),\psi(b)\}$, where χ and ψ are strictly monotone functions. It is obvious that if ψ is increasing, then $\psi_a = \psi(a)$, $\psi_b = \psi(b)$, and if ψ is decreasing, then $\psi_a = \psi(b)$, $\psi_b = \psi(a)$. Furthermore, since $\psi : [a,b] \to \mathbb{R}$ is a continuous strictly monotone function, it follows that $\psi_a\mathbf{1} \leq \psi(x_t) \leq \psi_b\mathbf{1}$, for every $t \in T$, which means that the spectra of the field $(y_t)_{t\in T} = (\psi(x_t))_{t\in T}$ is contained in the interval $[\psi_a, \psi_b]$.

Now, rewriting Theorem 4.21 with $F = \chi \circ \psi^{-1}$, $(\psi(x_t))_{t\in T}$ and $[\psi_a, \psi_b]$ instead of f, $(x_t)_{t\in T}$, and $[a,b]$, respectively, we obtain the following result:

Corollary 4.15 ([83]) *Let $\chi, \psi : [a,b] \to \mathbb{R}$ be continuous strictly monotone functions such that $F = \chi \circ \psi^{-1} \in \mathscr{C}^n([a,b])$. If the function F is n-convex, $m \geq 3$ is odd and $m < n$, then the series of inequalities*

$$H_F(\psi_a, \psi_b) \leq \chi\left(M_\chi(x,\phi)\right) - \chi\left(M_\psi(x,\phi)\right) \leq H_F(\psi_b, \psi_a), \tag{4.121}$$

where $H_f(a,b)$ is defined in the statement of Theorem 4.21, hold for every continuous field $(x_t)_{t\in T}$ of self-adjoint operators in $\mathfrak{B}(\mathscr{H})$ with spectra in $[a,b]$. Inequalities in (4.121) also hold when F is n-concave and m is even. In the cases when F is n-convex and m is even, or when the F is n-concave and m is odd, the inequality signs in (4.121) are reversed.

Similarly to the previous corollary, Theorems 4.22 and 4.23 also provide mutual bounds for the difference $\chi\left(M_\chi(x,\phi)\right) - \chi\left(M_\psi(x,\phi)\right)$. In order to simplify our further discussion, we give the corresponding result for the case when Theorems 4.22 and 4.23 coincide, i.e. for the case of 3-convex function. Namely, rewriting relation (4.120) with $F = \chi \circ \psi^{-1}$, $(\psi(x_t))_{t\in T}$ and $[\psi_a, \psi_b]$ instead of f, $(x_t)_{t\in T}$, and $[a,b]$, respectively, we obtain the following result:

Corollary 4.16 ([83]) Let $\chi, \psi : [a,b] \to \mathbb{R}$ be continuous strictly monotone functions such that $F = \chi \circ \psi^{-1} \in \mathscr{C}^3([a,b])$. If F is 3-convex function, then the series of inequalities

$$\begin{aligned}
&F[\psi_a, \psi_a; \psi_b]\left(\psi_b\mathbf{1} - \int_T \phi_t(\psi(x_t))\,d\mu(t)\right)\left(\int_T \phi_t(\psi(x_t))\,d\mu(t) - \psi_a\mathbf{1}\right) \\
&\qquad + F[\psi_b, \psi_b; \psi_a]\int_T \phi_t\left((\psi(x_t) - \psi_a\mathbf{1})(\psi(x_t) - \psi_b\mathbf{1})\right)d\mu(t) \\
&\leq \chi\left(M_\chi(x,\phi)\right) - \chi\left(M_\psi(x,\phi)\right) \\
&\leq F[\psi_b, \psi_b; \psi_a]\left(\psi_b\mathbf{1} - \int_T \phi_t(\psi(x_t))\,d\mu(t)\right)\left(\int_T \phi_t(\psi(x_t))\,d\mu(t) - \psi_a\mathbf{1}\right) \\
&\qquad + F[\psi_a, \psi_a; \psi_b]\int_T \phi_t\left((\psi(x_t) - \psi_a\mathbf{1})(\psi(x_t) - \psi_b\mathbf{1})\right)d\mu(t)
\end{aligned}$$

holds for every continuous field $(x_t)_{t\in T}$ of positive operators in $\mathfrak{B}(\mathscr{H})$ with spectra in $[a,b]$. If F is 3-concave, the corresponding inequality signs are reversed.

The simplest example of a quasi-arithmetic mean (4.44) is a power operator mean defined by (see e.g. [101]):

$$M_r(x,\phi) = \begin{cases} \left(\int_T \phi_t(x_t^r)\,d\mu(t)\right)^{\frac{1}{r}}, & r \neq 0 \\ \exp\left(\int_T \phi_t(\log x_t)\,d\mu(t)\right), & r = 0. \end{cases}$$

Since power operator means are special cases of quasi-arithmetic operator means for particular choices of functions χ and ψ, let us first set $\chi(t) = t^s$ and $\psi(t) = t^r$, $t > 0$, where s and r are real parameters such that $r, s \neq 0$.

Now, the function $F(t) = (\chi \circ \psi^{-1})(t) = t^{s/r}$ belongs to the class $\mathscr{C}^n(\mathbb{R})$ for any $n \in \mathbb{N}$, and we have

$$F^{(n)}(t) = \frac{s}{r}\left(\frac{s}{r}-1\right)\left(\frac{s}{r}-2\right)\cdots\left(\frac{s}{r}-n+1\right)t^{\frac{s}{r}-n}.$$

It is straightforward to check that:

- if $r < 0 < s$ or $s < 0 < r$, then the function F is n-convex for any even $n \in \mathbb{N}$, and n-concave for any odd number n;
- if $0 < s < r$ or $r < s < 0$, then the function F is n-convex for any odd $n \in \mathbb{N}$, and n-concave for any even number n;
- if $0 < r < s$ or $s < r < 0$, then the function F is n-convex when $\lfloor \frac{s}{r} \rfloor$ is even and n is odd, or when $\lfloor \frac{s}{r} \rfloor$ is odd and n is even, and F is n-concave when $\lfloor \frac{s}{r} \rfloor$ and n are both either even or odd.

It remains to consider the cases when one of the parameters r and s is equal to zero. If $s = 0$, then setting $\chi(t) = \log t$ and $\psi(t) = t^r$, it follows that $F(t) = (\chi \circ \psi^{-1})(t) = \dfrac{1}{r}\log t$ belongs to the class $\mathscr{C}^n(\mathbb{R})$ for any $n \in \mathbb{N}$, and we have

$$F^{(n)}(t) = \frac{1}{r}(-1)^{n-1}(n-1)!\,t^{-n}.$$

It is easy to see that:

- the function F is n-convex if $r > 0$ and $n \in \mathbb{N}$ is odd, or if $r < 0$ and $n \in \mathbb{N}$ is even;
- the function F is n-concave if $r > 0$ and $n \in \mathbb{N}$ is even, or if $r < 0$ and $n \in \mathbb{N}$ is odd.

In cases when $r < 0$ the function $\psi(t) = t^r$ is strictly decreasing, so we have $\psi_a = b^r$ and $\psi_b = a^r$, and in cases when $0 < r$ the function ψ is strictly increasing, so we have $\psi_a = a^r$ and $\psi_b = b^r$.

Finally, if $r = 0$, then setting $\chi(t) = t^s$ and $\psi(t) = \log t$, it follows that the function $F(t) = (\chi \circ \psi^{-1})(t) = e^{st}$ belongs to the class $\mathscr{C}^n(\mathbb{R})$ for any $n \in \mathbb{N}$, and we have

$$F^{(n)}(t) = s^n\, e^{st}.$$

Trivially,

- if $s > 0$, then the function F is n-convex for any $n \in \mathbb{N}$;
- if $s < 0$, then F is n-convex for any even number n, and n-concave for any odd number n.

The function $\psi(t) = \log t$ is strictly increasing, so in this case we have $\psi_a = \log a$ and $\psi_b = \log b$.

We see that all of our results regarding quasi-arithmetic means can be applied to power operator means, considering our discussion from above (see also [108]).

• If $0 < s \le r$ or $0 < 2r \le s$ holds, then

$$\begin{aligned}
&\frac{1}{b^r - a^r}\Bigg[\left(\frac{s}{r}b^{s-r} - \frac{b^s - a^s}{b^r - a^r}\right)\int_T \phi_t\left((x_t^r - b^r\mathbf{1})(x_t^r - a^r\mathbf{1})\right)d\mu(t) \\
&\qquad + \left(\frac{b^s - a^s}{b^r - a^r} - \frac{s}{r}a^{s-r}\right)\left(b^r\mathbf{1} - \int_T \phi_t(x_t^r)\,d\mu(t)\right)\left(\int_T \phi_t(x_t^r)\,d\mu(t) - a^r\mathbf{1}\right)\Bigg] \\
&\le (M_s(x,\phi))^s - (M_r(x,\phi))^s \\
&\le \frac{1}{b^r - a^r}\Bigg[\left(\frac{s}{r}b^{s-r} - \frac{b^s - a^s}{b^r - a^r}\right)\left(b^r\mathbf{1} - \int_T \phi_t(x_t^r)\,d\mu(t)\right)\left(\int_T \phi_t(x_t^r)\,d\mu(t) - a^r\mathbf{1}\right) \\
&\qquad + \left(\frac{b^s - a^s}{b^r - a^r} - \frac{s}{r}a^{s-r}\right)\int_T \phi_t\left((x_t^r - b^r\mathbf{1})(x_t^r - a^r\mathbf{1})\right)d\mu(t)\Bigg]
\end{aligned}$$

If $s < 0 < r$ or $0 < r \le s \le 2r$, then the above inequalities are reversed.

• If $r \le s < 0$ or $s \le 2r < 0$ holds, then

$$\begin{aligned}
&\frac{1}{b^r - a^r}\Bigg[\left(\frac{b^s - a^s}{b^r - a^r} - \frac{s}{r}a^{s-r}\right)\int_T \phi_t\left((x_t^r - b^r\mathbf{1})(x_t^r - a^r\mathbf{1})\right)d\mu(t) \\
&\qquad + \left(\frac{s}{r}b^{s-r} - \frac{b^s - a^s}{b^r - a^r}\right)\left(b^r\mathbf{1} - \int_T \phi_t(x_t^r)\,d\mu(t)\right)\left(\int_T \phi_t(x_t^r)\,d\mu(t) - a^r\mathbf{1}\right)\Bigg] \\
&\le (M_s(x,\phi))^s - (M_r(x,\phi))^s \\
&\le \frac{1}{b^r - a^r}\Bigg[\left(\frac{b^s - a^s}{b^r - a^r} - \frac{s}{r}a^{s-r}\right)\left(b^r\mathbf{1} - \int_T \phi_t(x_t^r)\,d\mu(t)\right)\left(\int_T \phi_t(x_t^r)\,d\mu(t) - a^r\mathbf{1}\right) \\
&\qquad + \left(\frac{s}{r}b^{s-r} - \frac{b^s - a^s}{b^r - a^r}\right)\int_T \phi_t\left((x_t^r - b^r\mathbf{1})(x_t^r - a^r\mathbf{1})\right)d\mu(t)\Bigg]
\end{aligned}$$

If $r < 0 < s$ or $2r \le s \le r < 0$, then the inequalities are reversed.

• If $r \ne 0$, then

$$\begin{aligned}
&\frac{1}{b^r - a^r}\Bigg[\left(\frac{1}{rb^r} - \frac{\log b - \log a}{b^r - a^r}\right)\int_T \phi_t\left((x_t^r - b^r\mathbf{1})(x_t^r - a^r\mathbf{1})\right)d\mu(t) \\
&\qquad + \left(\frac{\log b - \log a}{b^r - a^r} - \frac{1}{ra^r}\right)\left(b^r\mathbf{1} - \int_T \phi_t(x_t^r)\,d\mu(t)\right)\left(\int_T \phi_t(x_t^r)\,d\mu(t) - a^r\mathbf{1}\right) \\
&\le \log(M_0(x,\phi)) - \log(M_r(x,\phi)) \\
&\le \frac{1}{b^r - a^r}\Bigg[\left(\frac{1}{rb^r} - \frac{\log b - \log a}{b^r - a^r}\right)\left(b^r\mathbf{1} - \int_T \phi_t(x_t^r)\,d\mu(t)\right)\left(\int_T \phi_t(x_t^r)\,d\mu(t) - a^r\mathbf{1}\right) \\
&\qquad + \left(\frac{\log b - \log a}{b^r - a^r} - \frac{1}{ra^r}\right)\int_T \phi_t\left((x_t^r - b^r\mathbf{1})(x_t^r - a^r\mathbf{1})\right)d\mu(t)\Bigg]
\end{aligned}$$

• If $s > 0$, then

$$\frac{1}{\log\frac{b}{a}}\left[\left(sb^s-\frac{b^s-a^s}{\log\frac{b}{a}}\right)\int_T \phi_t\left((\psi(x_t)-\psi_b\mathbf{1})(\log x_t-\log a\mathbf{1})\right)d\mu(t)\right.$$
$$\left.+\left(\frac{b^s-a^s}{\log\frac{b}{a}}-sa^s\right)\left(\log b\mathbf{1}-\int_T \phi_t\left(\log x_t\right)d\mu(t)\right)\left(\int_T \phi_t\left(\log x_t\right)d\mu(t)-\log a\mathbf{1}\right)\right]$$
$$\leq \left(M_s\left(x,\phi\right)\right)^s-\left(M_0\left(x,\phi\right)\right)^s$$
$$\leq \frac{1}{\log\frac{b}{a}}\left[\left(sb^s-\frac{b^s-a^s}{\log\frac{b}{a}}\right)\left(\log b\mathbf{1}-\int_T \phi_t\left(\log x_t\right)\right)d\mu(t)\right)\left(\int_T \phi_t\left(\log x_t\right)d\mu(t)-\log a\mathbf{1}\right)$$
$$\left.+\left(\frac{b^s-a^s}{\log\frac{b}{a}}-sa^s\right)\int_T \phi_t\left((\log x_t-\log b\mathbf{1})(\log x_t-\log a\mathbf{1})\right)d\mu(t)\right]$$

and if $s < 0$, the inequality signs are reversed.

Chapter 5

Converses of Ando's and Davis-Choi's inequality

In this chapter, several converses of Ando's inequalitiy and Davis-Choi's inequality of different types have been proved, as well as the Edmundson-Lah-Ribarič inequality and its difference type converse for positive linear mappings. Also, a difference type converse for solidarities, one of which is of a special type that includes connections, and a quotient reverse inequality (or a reverse of the operator Hölder inequality) for connections and for the special type of solidarities that includes connections are given. In the case of converses in the form of a difference, the estimations are expressed using a kind of variation of the involved family of operators. As an application of the obtained results, operator reverses of inequalities for the general weighted power mean are given in a difference and a quotient form.

Also, by exploiting different scalar equalities obtained via Hermite's interpolating polynomial, we will obtain lower and upper bounds for the difference in Ando's inequality and in the Edmundson-Lah-Ribarič inequality for solidarities that hold for the class of n-convex functions. As an example, main results are applied to some operator means and relative operator entropy.

5.1 Introduction

Let H be Hilbert space and let A be a linear operator on H. Operator norm of A is defined as

$$||A|| := \sup\{||Ax|| : ||x|| \leq 1, x \in H\}.$$

Adjoint operator A^* of A is defined as a unique operator on H such that $\langle Ax, y\rangle = \langle x, A^*y\rangle$ holds for every $x, y \in H$. It follows that $||A|| = ||A^*|| = ||AA^*||^{1/2}$.

Operator A is bounded if $||A|| < \infty$. With $\mathscr{B}(H)$ we denote a C^*-algebra of all bounded (that is continuous) linear operators on H.

Spectrum of an operator A is defined as a set

$$\mathrm{Sp(A)} = \{\lambda \in \mathbb{C} : A - \lambda \mathbf{1}_H \text{ not invertible in } \mathscr{B}(H)\}.$$

We say that a bounded linear operator $A \in \mathscr{B}(H)$ is self-adjoint if $A = A^*$. Operator A is self-adjoint if and only if $\langle Ax, x\rangle \in \mathbb{R}$ holds for every $x \in H$. We say that a self-adjoint operator A is positive semi-definite (or simply positive) and write $A \geq 0$ if $\langle Ax, x\rangle \geq 0$ holds for every vector $x \in H$.

The theory for connections and means of pairs of positive operators has been developed by Kubo and Ando in [86]. Connection σ, as a binary operation on the set of positive definite operators, is characterized by the relation

$$A\sigma B = A^{1/2} f\left(A^{-1/2} B A^{-1/2}\right) A^{1/2}, \tag{5.1}$$

where f is a positive operator monotone function on $(0,\infty)$ called the representing function for σ. The axiomatic properties of connections are as follows:

(1) $A \leq C$, $B \leq D$ implies $A\sigma B \leq C\sigma D$,

(2) $C(A\sigma B)C \leq (CAC)\sigma(CBC)$,

(3) from $A_k \downarrow A$ and $B_k \downarrow B$ it follows that $A_k\sigma B_k \downarrow A\sigma B$.

A mean is a connection with normalization condition:

(4) $I\sigma I = I$.

A binary operation σ on the set of positive definite operators is called solidarity if the representing function f in (5.1) is just operator monotone on $(0,\infty)$. The theory of solidarities has been developed in [46]. The relative operator entropy

$$S(A|B) = A^{1/2} \log\left(A^{-1/2} B A^{-1/2}\right) A^{1/2}$$

is an example of solidarity.

The following properties are proved in [46].

Theorem 5.1 ([46]) *If σ is a solidarity, then*

- $(A+B)\sigma(C+D) \geq A\sigma C + B\sigma D$ *(subaditivity)*
- $(\lambda A_1 + (1-\lambda)A_2)\,\sigma\,(\lambda B_1 + (1-\lambda)B_2) \geq \lambda\,(A_1\sigma B_1) + (1-\lambda)\,(A_2\sigma B_2)$, $0 \leq \lambda \leq 1$ *(joint concavity).*

A simple consequence of the stated properties is the following Jensen type inequality.

Corollary 5.1 ([46]) *Let $p_i \geq 0$, $A_i, B_i > 0$, $i = 1,\ldots,n$. Then*

$$\sum_{i=1}^{n} p_i A_i \sigma B_i \leq \left(\sum_{i=1}^{n} p_i A_i\right) \sigma \left(\sum_{i=1}^{n} p_i B_i\right) \tag{5.2}$$

for any solidarity σ.

The basic examples of connections and their representing functions are:

- The weighted arithmetic mean
$$A\nabla_\alpha B = (1-\alpha)A + \alpha B, 0 \leq \alpha \leq 1,$$
with representing function $t \mapsto (1-\alpha) + \alpha t$.
- The weighted harmonic mean
$$A\,!_\alpha B = \left[(1-\alpha)A^{-1} + \alpha B^{-1}\right]^{-1}, 0 \leq \alpha \leq 1,$$
with representing function $t \mapsto \frac{t}{(1-\alpha)t+\alpha}$.
- The weighted geometric mean
$$A\#_\alpha B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\alpha} A^{1/2}, 0 \leq \alpha \leq 1,$$
with representing function $t \mapsto t^{\alpha}$.
- The weighted power mean
$$A\#_{p,\alpha}B = A^{1/2}\left[(1-\alpha)I + \alpha\left(A^{-1/2}BA^{-1/2}\right)^{p}\right]^{1/p} A^{1/2}, 0 \leq \alpha \leq 1, -1 \leq p \leq 1,$$
with representing function $t \mapsto [(1-\alpha) + \alpha t^{p}]^{1/p}$.

In this way the Hölder inequality for positive definite operators A_i, B_i and $p_i \geq 0$, $i = 1,\ldots,n$

$$\sum_{i=1}^{n} p_i A_i \#_{p,\alpha} B_i \leq \left(\sum_{i=1}^{n} p_i A_i\right) \#_{p,\alpha} \left(\sum_{i=1}^{n} p_i B_i\right) \tag{5.3}$$

holds, where $0 \leq \alpha \leq 1$, $-1 \leq p \leq 1$.

Although it is common to call inequality (5.3) the Hölder inequality, it is worthwhile to mention that in the real case inequality (5.3) reduces to

$$\sum_{i=1}^{n} p_i \left[(1-\alpha)a_i^p + \alpha b_i^p\right]^{\frac{1}{p}} \leq \left[(1-\alpha)\left(\sum_{i=1}^{n} p_i a_i\right)^p + \alpha\left(\sum_{i=1}^{n} p_i b_i\right)^p\right]^{\frac{1}{p}},$$

which holds for $p < 1$ and the reversed inequality holds for $p > 1$. This is discrete Minkowski's inequality. A more general form of which is

$$\sum_{i=1}^{n} p_i \left(\sum_{j=1}^{m} q_j a_{i,j}^p\right)^{\frac{1}{p}} \leq \left[\sum_{j=1}^{m} q_j \left(\sum_{i=1}^{n} p_i a_{i,j}\right)^p\right]^{\frac{1}{p}}, \tag{5.4}$$

where $a_{i,j} > 0$, $p_i, q_j \geq 0$, $i = 1,\dots,n$, $j = 1,\dots,m$, $p < 1$. For $p > 1$ the reversed inequality holds in (5.4).

Note that inequality (5.4) is, due to homogeneous property, equivalent to inequality

$$\sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_{i,j}^p\right)^{\frac{1}{p}} \leq \left[\sum_{j=1}^{m} \left(\sum_{i=1}^{n} a_{i,j}\right)^p\right]^{\frac{1}{p}}.$$

In the operator case the only known result of this type is proven in [4] for the harmonic mean:

$$\sum_{i=1}^{n} \left(\sum_{j=1}^{m} A_{i,j}^{-1}\right)^{-1} \leq \left[\sum_{j=1}^{m} \left(\sum_{i=1}^{n} A_{i,j}\right)^{-1}\right]^{-1},$$

where $A_{i,j}$, $i = 1,\dots,n$, $j = 1,\dots,m$ are positive invertible operators.

Main goal of the first section is to give reverse inequalities to (5.2) of a difference and quotient type for the special type of solidarities that includes connections. In the case of the weighted power mean the explicit estimations of reverse inequalities of (5.3) in a difference and quotient form are obtained. The difference case for the relative operator entropy is also given. The methods used in this paper and related results in this area can be found in the monographs [48] and [49]. The next result has been proved [24].

Theorem 5.2 ([24]) *Let $A_i, B_i > 0$, $i = 1,\dots n$ be such that $mA_i \leq B_i \leq MA_i$ for some scalars $0 < m < M$. Then, if $0 < \alpha < 1$*

$$\left(\sum_{i=1}^{n} A_i\right) \#_\alpha \left(\sum_{i=1}^{n} B_i\right) \leq \frac{1}{K(m,M,\alpha)} \sum_{i=1}^{n} A_i \#_\alpha B_i, \tag{5.5}$$

where

$$K(m,M,\alpha) = \frac{Mm^\alpha - mM^\alpha}{(1-\alpha)(M-m)} \left(\frac{1-\alpha}{\alpha} \frac{M^\alpha - m^\alpha}{Mm^\alpha - mM^\alpha}\right)^\alpha$$

is the Kantorovich constant.

A reverse type result is given in [47].

Theorem 5.3 ([47]) *Let A_i, B_i be positive definite matrices such that $mA_i \le B_i \le MA_i$ for some $0 < m \le M$ and $i = 1, \dots, n$. Then for every $\alpha \in [0,1]$*

$$\left(\sum_{i=1}^{n} A_i\right) \#_\alpha \left(\sum_{i=1}^{n} B_i\right) - \sum_{i=1}^{n} A_i \#_\alpha B_i \le C(m,M,\alpha) \sum_{i=1}^{n} A_i, \tag{5.6}$$

where

$$C(m,M,\alpha) = (1-\alpha)\left(\frac{M^\alpha - m^\alpha}{\alpha(M-m)}\right)^{\frac{\alpha}{1-\alpha}} - \frac{Mm^\alpha - mM^\alpha}{M-m}.$$

In the same paper, using Theorem 5.3 and $S(A|B) = \lim_{\alpha\to 0} \frac{A\#_\alpha B - A}{A}$, the following corollary is proven.

Corollary 5.2 ([47]) *Let A_i, B_i be positive definite matrices such that $mA_i \le B_i \le MA_i$ for some $0 < m \le M$ and $i = 1, \dots, n$. Then*

$$S\left(\sum_{i=1}^{n} A_i | \sum_{i=1}^{n}\right) - \sum_{i=1}^{n} S(A_i|B_i) \le \log S(h) \sum_{i=1}^{n} A_i, \tag{5.7}$$

where

$$S(h) = \frac{(h-1)h^{\frac{1}{h-1}}}{e \log h}$$

is the Specht ratio and $h = \frac{M}{m}$.

A mapping $\Phi\colon \mathscr{B}(H) \to \mathscr{B}(K)$ is said to be linear if:

* $\Phi(X+Y) = \Phi(X) + \Phi(Y)$ for all $X, Y \in \mathscr{B}(H)$ (additivity);
* $\Phi(\lambda X) = \lambda\Phi(X)$ for all $X \in \mathscr{B}(H)$ i $\lambda \in \mathbb{C}$ (homogeneity).

A linear mapping $\Phi\colon \mathscr{B}(H) \to \mathscr{B}(K)$ is positive if it preserves the operator order, that is if $A \ge 0$ implies $\Phi(A) \ge 0$. A linear mapping $\Phi\colon \mathscr{B}(H) \to \mathscr{B}(K)$ is unital if it preserves the identity operator, that is, if $\Phi(\mathbf{1}_H) = \mathbf{1}_K$.

It is clear from the definition that a positive linear mapping preserves the adjoint operation, that is $\Phi(A^*) = \Phi(A)^*$, and if Φ is additionally unital, then from $\alpha\mathbf{1}_H \le A \le \beta\mathbf{1}_H$ it follows $\alpha\mathbf{1}_K \le \Phi(A) \le \beta\mathbf{1}_K$, where $\alpha, \beta \in \mathbb{C}$.

Main goal of the second section is to obtain a converse of the well-known Davis-Choi inequality, which states that for an operator convex function $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, and for a positive unital linear mapping Φ we have

$$f(\Phi(A)) \le \Phi(f(A)), \tag{5.8}$$

where A is a self-adjoint operator such that $\mathrm{Sp}(A) \subseteq I$ (see [30], [35]).

Next result is another Jensen type inequality, but for connections (see [10]).

Theorem 5.4 (Ando's inequality, [10]) *If Φ is a positive linear mapping, then for any connection σ and for any positive definite operators A and B we have*

$$\Phi(A\sigma B) \le \Phi(A)\sigma\Phi(B). \tag{5.9}$$

From Theorem 5.4 it is easy to draw the conclusion that also holds a generalization of Ando's inequality

$$\sum_{j=1}^{n} p_j\Phi_j(A_j\sigma B_j) \leq \Big(\sum_{j=1}^{n} p_j\Phi_j(A_j)\Big)\sigma\Big(\sum_{j=1}^{n} p_j\Phi_j(B_j)\Big), \tag{5.10}$$

where σ is a connection, A_j, B_j are positive definite operators, Φ_j are positive linear mappings and we have $p_j \geq 0$, $j = 1,\dots,n$.

Another goal of the second section is to obtain an estimation for the upper bounds for the differences generated by Ando's inequality and by the generalization of the Edmundson-Lah-Ribarič inequality using a kind of variation of the involved family of operators. Method is based on the method from [38].

5.2 Converse inequalities of the quotient type for connections and solidarities

First result is a generalization of the Edmundson-Lah-Ribarič inequality for the special type of solidarities that includes connections.

Theorem 5.5 *Let A_i, B_i be positive definite operators such that $mA_i \leq B_i \leq MA_i$ for some $0 < m \leq M$, $p_i \geq 0$, and let Φ_i be positive linear maps, $i = 1,\dots,n$. Suppose that σ is a solidarity generated by an operator monotone and operator concave function f. Then*

$$\begin{aligned}&\sum_{i=1}^{n} p_i\Phi_i(A_i\sigma B_i) \\ &\geq \frac{M\sum_{i=1}^{n} p_i\Phi_i(A_i) - \sum_{i=1}^{n} p_i\Phi_i(B_i)}{M-m}f(m) + \frac{\sum_{i=1}^{n} p_i\Phi_i(B_i) - m\sum_{i=1}^{n} p_i\Phi_i(A_i)}{M-m}f(M).\end{aligned} \tag{5.11}$$

Proof. Since f is concave on $(0,\infty)$, by the Edmundson-Lah-Ribarič inequality (see [124]) we have

$$f(t) \geq \frac{M-t}{M-m}f(m) + \frac{t-m}{M-m}f(M),\ t \in [m,M]. \tag{5.12}$$

From $mA_i \leq B_i \leq MA_i$ easily follows $mI \leq A_i^{-\frac{1}{2}}B_iA_i^{-\frac{1}{2}} \leq MI$, $i = 1,\dots,n$. Using the functional calculus and (5.12), we obtain

$$f\left(A_i^{-\frac{1}{2}}B_iA_i^{-\frac{1}{2}}\right) \geq \frac{M\mathbf{1}_H - A_i^{-\frac{1}{2}}B_iA_i^{-\frac{1}{2}}}{M-m}f(m) + \frac{A_i^{-\frac{1}{2}}B_iA_i^{-\frac{1}{2}} - m\mathbf{1}_H}{M-m}f(M),\ i = 1,\dots,n. \tag{5.13}$$

Multiplying (5.13) twice by $A_i^{\frac{1}{2}}$, acting by Φ_i, then multiplying by p_i and summing, it follows

$$\sum_{i=1}^{n} p_i\Phi_i\left(A_i^{\frac{1}{2}} f\left(A_i^{-\frac{1}{2}} B_i A_i^{-\frac{1}{2}}\right) A_i^{\frac{1}{2}}\right)$$
$$\geq \frac{M\sum_{i=1}^{n} p_i\Phi_i(A_i) - \sum_{i=1}^{n} p_i\Phi_i(B_i)}{M-m} f(m) + \frac{\sum_{i=1}^{n} p_i\Phi_i(B_i) - m\sum_{i=1}^{n} p_i\Phi_i(A_i)}{M-m} f(M).$$

□

A difference counterpart of the operator Hölder inequality (5.1) is given in the following theorem.

Theorem 5.6 *Let A_i, B_i be positive definite operators such that $mA_i \leq B_i \leq MA_i$ for some $0 < m \leq M$, $p_i \geq 0$, $\lambda \in \mathbb{R}$, and let Φ_i be positive linear maps, $i = 1, \ldots, n$. Suppose that σ_1 is a solidarity generated by f_1 and σ_2 is a solidarity generated by an operator monotone and operator concave function f_2. Then*

$$\lambda\left(\sum_{i=1}^{n} p_i\Phi_i(A_i)\right)\sigma_1\left(\sum_{i=1}^{n} p_i\Phi_i(B_i)\right) - \sum_{i=1}^{n} p_i\Phi_i(A_i\sigma_2 B_i)$$
$$\leq \max_{m\leq t\leq M}\left[\lambda f_1(t) - \left(\frac{M-t}{M-m} f_2(m) + \frac{t-m}{M-m} f_2(M)\right)\right]\sum_{i=1}^{n} p_i\Phi_i(A_i). \tag{5.14}$$

Proof. Using (5.11) we have:

$$\lambda\left(\sum_{i=1}^{n} p_i\Phi_i(A_i)\right)\sigma_1\left(\sum_{i=1}^{n} p_i\Phi_i(B_i)\right) - \sum_{i=1}^{n} p_i\Phi_i(A_i\sigma_2 B_i)$$
$$\leq \lambda\left(\sum_{i=1}^{n} p_i\Phi_i(A_i)\right)^{\frac{1}{2}}$$
$$f_1\left(\left(\sum_{i=1}^{n} p_i\Phi_i(A_i)\right)^{-\frac{1}{2}}\left(\sum_{i=1}^{n} p_i\Phi_i(B_i)\right)\left(\sum_{i=1}^{n} p_i\Phi_i(A_i)\right)^{-\frac{1}{2}}\right)\left(\sum_{i=1}^{n} p_i\Phi_i(A_i)\right)^{\frac{1}{2}}$$
$$-\left[\frac{M\sum_{i=1}^{n} p_i\Phi_i(A_i) - \sum_{i=1}^{n} p_i\Phi_i(B_i)}{M-m} f_2(m) + \frac{\sum_{i=1}^{n} p_i\Phi(B_i) - m\sum_{i=1}^{n} p_i\Phi_i(A_i)}{M-m} f_2(M)\right]$$
$$= \lambda\left(\sum_{i=1}^{n} p_i\Phi_i(A_i)\right)^{\frac{1}{2}}\left[f_1\left(\left(\sum_{i=1}^{n} p_i\Phi(A_i)\right)^{-\frac{1}{2}}\left(\sum_{i=1}^{n} p_i\Phi_i(B_i)\right)\left(\sum_{i=1}^{n} p_i\Phi_i(A_i)\right)^{-\frac{1}{2}}\right)\right.$$
$$-\left[\frac{M - \left(\sum_{i=1}^{n} p_i\Phi_i(A_i)\right)^{-\frac{1}{2}}\left(\sum_{i=1}^{n} p_i\Phi_i(B_i)\right)\left(\sum_{i=1}^{n} p_i\Phi(A_i)\right)^{-\frac{1}{2}}}{M-m} f_2(m)\right.$$
$$\left.\left.+\frac{\left(\sum_{i=1}^{n} p_i\Phi_i(A_i)\right)^{-\frac{1}{2}}\left(\sum_{i=1}^{n} p_i\Phi(B_i)\right)\left(\sum_{i=1}^{n} p_i\Phi_i(A_i)\right)^{-\frac{1}{2}} - m}{M-m} f_2(M)\right]\right]\left(\sum_{i=1}^{n} p_i\Phi_i(A_i)\right)^{\frac{1}{2}}$$
$$\leq \max_{m\leq t\leq M}\left[\lambda f_1(t) - \left(\frac{M-t}{M-m} f_2(m) + \frac{t-m}{M-m} f_2(M)\right)\right]\sum_{i=1}^{n} p_i\Phi_i(A_i).$$

□

As a corollary we give a reverse of the operator Hölder inequality.

Corollary 5.3 *Let A_i, B_i be positive definite operators such that $mA_i \leq B_i \leq MA_i$ for some $0 < m \leq M$, $p_i \geq 0$, and let Φ_i be positive linear maps, $i = 1, \ldots, n$. Suppose that σ_1 is a connection generated by f_1 and σ_2 is a solidarity generated by an operator monotone and operator concave function f_2. Then*

$$\sum_{i=1}^{n} p_i \Phi(A_i \sigma_2 B_i)$$
$$\geq \min_{m \leq t \leq M} \frac{\frac{M-t}{M-m} f_2(m) + \frac{t-m}{M-m} f_2(M)}{f_1(t)} \left(\sum_{i=1}^{n} p_i \Phi_i(A_i) \right) \sigma_1 \left(\sum_{i=1}^{n} p_i \Phi_i(B_i) \right). \tag{5.15}$$

Proof. Notice that $f_1 > 0$. Set in (5.14)

$$\lambda = \min_{m \leq t \leq M} \frac{\frac{M-t}{M-m} f_2(m) + \frac{t-m}{M-m} f_2(M)}{f_1(t)}.$$

Notice that for λ chosen in this way, it follows

$$\lambda f_1(t) - \frac{M-t}{M-m} f_2(m) - \frac{t-m}{M-m} f_2(M) \leq 0, \; t \in [m, M].$$

Since the function f_1 is continuous on $[m, M]$, there exists $t_0 \in [m, M]$ such that

$$\lambda = \frac{\frac{M-t_0}{M-m} f_2(m) + \frac{t_0-m}{M-m} f_2(M)}{f_1(t_0)},$$

which implies that for this λ

$$\max_{m \leq t \leq M} \left[\lambda f_1(t) - \left(\frac{M-t}{M-m} f_2(m) + \frac{t-m}{M-m} f_2(M) \right) \right] = 0,$$

which obviously gives (5.15). $\square$

An alternative approach can be given using Mond-Pečarić method described in [49]. The following results deal with reverses of the Davis-Choi and Ando's inequality (5.9). The proofs of matrix reverses of Hölder's inequality given in [24], [90], [78] are based on Gelfand-Naimark-Segal construction.

Lemma 5.1 *Let A be a self-adjoint operator with $\mathrm{Sp}(A) \subseteq [m, M]$ for some $m < M$. Suppose that $f, g \in C([m, M])$, where f is a concave function and Φ is a normalized positive linear map. Then*

$$\Phi(f(A)) \geq \alpha g(\Phi(A)) + \beta \mathbf{1}_K,$$

where $\beta = \min_{m \leq t \leq M} \left[\mu_f t + \nu_f - \alpha g(t) \right]$, $\mu_f = \frac{f(M)-f(m)}{M-m}$, $\nu_f = \frac{Mf(m)-mf(M)}{M-m}$.

Proof. Since f is concave, it follows $f(t) \geq \mu_f t + \nu_f$. This implies $f(A) \geq \mu_f A + \nu_f I$. Applying the normalized positive linear map Φ it follows

$$\Phi(f(A)) \geq \mu_f \Phi(A) + \nu_f I,$$

which gives

$$\Phi(f(A)) - \alpha g(\Phi(A)) \geq \mu_f \Phi(A) + \nu_f I - \alpha g(\Phi(A)) \geq \beta I.$$ $\square$

Theorem 5.7 *Let A, B be positive definite operators such that $mA \leq B \leq MA$ for some $0 < m \leq M$. Let σ be a connection generated by f and τ a connection generated by g. Suppose that Φ is a normalized positive linear map and $\alpha \in \mathbb{R}$. Then*

$$\Phi(A\sigma B) \geq \alpha\Phi(A)\tau\Phi(B) + \beta\Phi(A),$$

where $\beta = \min_{m \leq t \leq M}\left[\mu_f t + \nu_f - \alpha g(t)\right]$.

Proof. Define a normalized positive linear map Ψ by

$$\Psi(X) = \Phi(A)^{-\frac{1}{2}}\Phi\left(A^{\frac{1}{2}}XA^{\frac{1}{2}}\right)\Phi(A)^{-\frac{1}{2}}.$$

Using Lemma 5.1 we have:

$$\begin{aligned}
\Phi(A\sigma B)) &= \Phi\left(A^{\frac{1}{2}}f\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}\right)\\
&= \Phi(A)^{\frac{1}{2}}\Psi\left(f\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)\Phi(A)^{\frac{1}{2}}\\
&\geq \Phi(A)^{\frac{1}{2}}\left[\alpha g\left(\Psi\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right) + \beta I\right]\Phi(A)^{\frac{1}{2}}\\
&= \alpha\Phi(A)^{\frac{1}{2}}g\left(\Psi\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)\Phi(A)^{\frac{1}{2}} + \beta\Phi(A)\\
&= \alpha\Phi(A)\tau\Phi(B) + \beta\Phi(A).
\end{aligned}$$

□

Corollary 5.4 *Let A, B be positive definite operators such that $mA \leq B \leq MA$ for some $0 < m \leq M$. Let σ be a connection generated by f and τ a connection generated by g. Suppose that Φ is a normalized positive linear map. Then*

$$\Phi(A\sigma B) \geq \min_{m \leq t \leq M}\frac{\mu_f t + \nu_f}{g(t)}\Phi(A)\tau\Phi(B).$$

Proof. Set in Theorem 5.7 $\alpha = \min_{m \leq t \leq M}\frac{\mu_f t + \nu_f}{g(t)}$. Since g is continuous, there exists $t_1 \in [m, M]$ such that $\min_{m \leq t \leq M}\frac{\mu_f t + \nu_f}{g(t)} = \frac{\mu_f t_1 + \nu_f}{g(t_1)}$. Notice that for this α it holds $0 \leq \mu_f t - \nu_f - \alpha g(t)$, but $\mu_f t_1 + \nu_f - \alpha g(t_1) = 0$, so $\beta = 0$ and (5.4) is proven. □

As an application we give the following theorem which is a generalization of Theorem 5.2.

Theorem 5.8 *Let A_i, B_i be positive definite operators such that $mA_i \leq B_i \leq MA_i$ for some $0 < m \leq M$, Φ_i positive linear maps, $p_i \geq 0$, $i = 1, \ldots, n$, and let $-1 \leq p \leq 1$, $0 \leq \alpha \leq 1$. Then*

$$\left(\sum_{i=1}^{n} p_i\Phi_i(A_i)\right)\#_{p,\alpha}\left(\sum_{i=1}^{n} p_i\Phi_i(B_i)\right) \leq K(M, m, p, \alpha)\sum_{i=1}^{n} p_i\Phi_i(A_i\#_{p,\alpha}B_i),$$

where

$$
\begin{aligned}
&K(m,M,p,\alpha) \\
&= \frac{M-m}{M\left(1-\alpha+\alpha m^p\right)^{\frac{1}{p}} - m\left(1-\alpha+\alpha M^p\right)^{\frac{1}{p}}} \\
&\quad \cdot \left[(1-\alpha)^{\frac{1}{1-p}} + \alpha^{\frac{1}{1-p}} \left(\frac{M\left(1-\alpha+\alpha m^p\right)^{\frac{1}{p}} - m\left(1-\alpha+\alpha M^p\right)^{\frac{1}{p}}}{\left(1-\alpha+\alpha M^p\right)^{\frac{1}{p}} - \left(1-\alpha+\alpha m^p\right)^{\frac{1}{p}}} \right)^{\frac{p}{1-p}} \right]^{\frac{1-p}{p}} \\
&= \frac{1}{\nu_f} \left[(1-\alpha)^{\frac{1}{1-p}} + \alpha^{\frac{1}{1-p}} \left(\frac{\nu_f}{\mu_f} \right)^{\frac{p}{1-p}} \right]^{\frac{1-p}{p}},
\end{aligned}
$$

and $f(t) = [1-\alpha+\alpha t^p]^{1/p}$.

Proof. We give a sketch of long but routine calculations. Set in (5.15) $f_1(t) = f_2(t) = (1-\alpha+\alpha t^p)^{1/p}$. Define

$$
F(t) = \frac{(M-m)\left(1-\alpha+\alpha t^p\right)^{1/p}}{(M-t)\left(1-\alpha+\alpha m^p\right)^{1/p} + (t-m)\left(1-\alpha+\alpha M^p\right)^{1/p}}.
$$

By straightforward calculations $F'(t) = 0$ is equivalent to equation

$$
\begin{aligned}
&\alpha t^{p-1} \left[(M-t)\left(1-\alpha+\alpha m^p\right)^{1/p} + (t-m)\left(1-\alpha+\alpha M^p\right)^{1/p} \right] \\
&= \left(1-\alpha+\alpha t^p\right) \left[\left(1-\alpha+\alpha M^p\right)^{1/p} - \left(1-\alpha+\alpha m^p\right)^{1/p} \right],
\end{aligned}
$$

which by obvious reduction is equivalent to equation

$$
\begin{aligned}
&\alpha \left(M\left(1-\alpha+\alpha m^p\right)^{1/p} - m\left(1-\alpha+\alpha M^p\right)^{1/p} \right) \\
&= (1-\alpha) t^{1-p} \left(\left(1-\alpha+\alpha M^p\right)^{1/p} - \left(1-\alpha+\alpha m^p\right)^{1/p} \right),
\end{aligned}
$$

which finally gives

$$
t = \left(\frac{\alpha}{1-\alpha} \right)^{\frac{1}{1-p}} \left(\frac{M\left(1-\alpha+\alpha m^p\right)^{1/p} - m\left(1-\alpha+\alpha M^p\right)^{1/p}}{\left(1-\alpha+\alpha M^p\right)^{1/p} - \left(1-\alpha+\alpha m^p\right)^{1/p}} \right)^{\frac{1}{1-p}}.
$$

Plugging this value in F and rearranging, the constant $K(m,M,p,\alpha)$ can be easily obtained. □

The following corollary is proven by setting $\alpha = -1$ in the previous theorem.

Corollary 5.5 *Let A_i, B_i be positive definite operators such that $mA_i \le B_i \le MA_i$ for some $0 < m \le M$, Φ_i positive linear maps, $0 \le \alpha \le 1$ and $p_i \ge 0$, $i = 1, \dots, n$. Then*

$$
\left(\sum_{i=1}^{n} p_i \Phi_i (A_i) \right) !_\alpha \left(\sum_{i=1}^{n} p_i \Phi_i (B_i) \right) \le K(M,m,-1,\alpha) \sum_{i=1}^{n} p_i \Phi_i (A_i !_\alpha B_i),
$$

where

$$K(m,M,-1,\alpha) = \frac{((1-\alpha)m+\alpha)((1-\alpha)M+\alpha)}{\left((1-\alpha)\sqrt{mM}+\alpha\right)^2}.$$

Generalization of Theorem 5.3 is given in the following theorem.

Theorem 5.9 *Let A_i, B_i be positive definite operators such that $mA_i \le B_i \le MA_i$ for some $0 < m \le M$, Φ_i positive linear maps, $p_i \ge 0$, $i = 1,\dots,n$ and let $-1 \le p \le 1$, $0 \le \alpha \le 1$. Then*

$$\left(\sum_{i=1}^n A_i\right) \#_{p,\alpha} \left(\sum_{i=1}^n B_i\right) - \sum_{i=1}^n A_i \#_{p,\alpha} B_i \le C(m,M,p,\alpha) \sum_{i=1}^n A_i,$$

where

$$C(m,M,p,\alpha) = \left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{p}} \frac{\mu_f}{\left(\alpha^{\frac{1}{p-1}} \mu_f^{\frac{p}{1-p}} - 1\right)^{\frac{1-p}{p}}} - \nu_f,$$

and $f(t) = [1-\alpha+\alpha t^p]^{1/p}$.

Proof. We use Theorem 5.14 for $\lambda = 1$, $f_1(t) = f_2(t) = f(t) = [1-\alpha+\alpha t^p]^{1/p}$. Set

$$F(t) = [1-\alpha+\alpha t^p]^{1/p} - \mu_f t - \nu_f.$$

It is easy to see that equation $F'(t) = 0$ gives

$$t = \left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{p}} \frac{1}{\left[\alpha^{\frac{1}{p-1}} \mu_f^{\frac{p}{1-p}} - 1\right]^{\frac{1}{p}}}.$$

Plugging this value in F and rearranging, the constant $C(m,M,p,\alpha)$ easily follows. □

Now we can give a direct proof of Corollary 5.2 using Theorem 5.6.

Corollary 5.6 *Let A_i, B_i be positive definite operators such that $mA_i \le B_i \le MA_i$ for some $0 < m \le M$, Φ_i positive linear maps and $p_i \ge 0$, $i = 1,\dots,n$. Then*

$$S\left(\sum_{i=1}^n p_i\Phi_i(A_i) | \sum_{i=1}^n p_i\Phi_i(B_i)\right) - \sum_{i=1}^n p_i\Phi_i(S(A_i|B_i)) \le \log S(h) \sum_{i=1}^n p_i\Phi_i(A_i),$$

where $S(h)$ is defined in Corollary 5.2.

Proof. Set in Theorem 5.6 $f_1(t) = f_2(t) = \log t$ and $\lambda = 1$. Define

$$F(t) = \log t - \frac{M-t}{M-m}\log m - \frac{t-m}{M-m}\log M.$$

Trivially $F'(t) = 0$ is equivalent to

$$t = \frac{M-m}{\log M - \log m}.$$

It is straightforward to check that

$$F\left(\frac{M-m}{\log M-\log m}\right)=\log S(h),$$

where $h=M/m$. □

5.3 Converses of Ando's and Davis-Choi's inequality in a difference form

The following theorem is given in [66] in integral form. For the sake of completeness we will prove the discrete version. It is about a difference type convese of the generalized Davis-Choi's inequality (5.8).

Theorem 5.10 *Let A_j be self-adjoint operators such that $Sp(A_j)\subseteq[m,M]$ for some scalars $m<M$ and let Φ_j be normalized positive linear maps, $j=1,\ldots,n$. If f is a concave function on an interval I whose interior contains $[m,M]$ and $p_1,\ldots,p_n$ are positive real numbers such that $\sum_{j=1}^n p_j=1$, then*

$$\begin{aligned}
&f\Big(\sum_{j=1}^n p_j\Phi_j(A_j)\Big)-\sum_{j=1}^n p_j\Phi_j(f(A_j))\\
&\le -\inf_{t\in\langle m,M\rangle}\Psi_f(t;m,M)\Big(M\mathbf{1}_K-\sum_{j=1}^n p_j\Phi_j(A_j)\Big)\Big(\sum_{j=1}^n p_j\Phi_j(A_j)-m\mathbf{1}_K\Big)\\
&\le \frac{f'_+(m)-f'_-(M)}{M-m}\Big(M\mathbf{1}_K-\sum_{j=1}^n p_j\Phi_j(A_j)\Big)\Big(\sum_{j=1}^n p_j\Phi_j(A_j)-m\mathbf{1}_K\Big)\\
&\le \frac{1}{4}(M-m)(f'_+(m)-f'_-(M))\mathbf{1}_K,
\end{aligned}\tag{5.16}$$

where $\Psi_f(\cdot;m,M)\colon\langle m,M\rangle\to\mathbb{R}$ is defined by

$$\Psi_f(t;m,M)=\frac{1}{M-m}\left(\frac{f(M)-f(t)}{M-t}-\frac{f(t)-f(m)}{t-m}\right)=[m,t,M;f],\tag{5.17}$$

and $[m,t,M;f]$ denotes second order divided difference.

Proof. Since f is concave, from the Edmundson-Lah-Ribarič inequality we have

$$f(t)\ge\frac{M-t}{M-m}f(m)+\frac{t-m}{M-m}f(M)\tag{5.18}$$

for every $t\in[m,M]$, so we can replace t with A_j in (5.18) and then apply Φ_j:

$$\Phi_j(f(A_j))\ge\frac{M\mathbf{1}_K-\Phi_j(A_j)}{M-m}f(m)+\frac{\Phi_j(A_j)-m\mathbf{1}_K}{M-m}f(M).$$

Multiplying the previous inequality with p_j, and then summing it, we get

$$\sum_{j=1}^{n} p_j \Phi_j(f(A_j)) \geq \frac{M\mathbf{1}_K - \sum_{j=1}^{n} p_j \Phi_j(A_j)}{M-m} f(m) + \frac{\sum_{j=1}^{n} p_j \Phi_j(A_j) - m\mathbf{1}_K}{M-m} f(M). \quad (5.19)$$

Using functional calculus and (5.19) we obtain

$$\begin{aligned}
& f\Big(\sum_{j=1}^{n} p_j \Phi_j(A_j)\Big) - \sum_{j=1}^{n} p_j \Phi_j(f(A_j)) \\
& \leq f\Big(\sum_{j=1}^{n} p_j \Phi_j(A_j)\Big) - \frac{M\mathbf{1}_K - \sum_{j=1}^{n} p_j \Phi_j(A_j)}{M-m} f(m) - \frac{\sum_{j=1}^{n} p_j \Phi_j(A_j) - m\mathbf{1}_K}{M-m} f(M) \\
& = -\frac{1}{M-m}\Big(M\mathbf{1}_K - \sum_{j=1}^{n} p_j \Phi_j(A_j)\Big)\Big(\sum_{j=1}^{n} p_j \Phi_j(A_j) - m\mathbf{1}_K\Big) \\
& \quad \times \Bigg[\bigg(f(M)\mathbf{1}_K - f\Big(\sum_{j=1}^{n} p_j \Phi_j(A_j)\Big)\bigg)\bigg(M\mathbf{1}_K - \sum_{j=1}^{n} p_j \Phi_j(A_j)\bigg)^{-1} - \\
& \qquad \bigg(f\Big(\sum_{j=1}^{n} p_j \Phi_j(A_j)\Big) - f(m)\mathbf{1}_K\bigg)\bigg(\sum_{j=1}^{n} p_j \Phi_j(A_j) - m\mathbf{1}_K\bigg)^{-1}\Bigg] \\
& = \Big(M\mathbf{1}_K - \sum_{j=1}^{n} p_j \Phi_j(A_j)\Big)\Big(\sum_{j=1}^{n} p_j \Phi_j(A_j) - m\mathbf{1}_K\Big)\Big(-\Psi_f\Big(\sum_{j=1}^{n} p_j \Phi_j(A_j); m, M\Big)\Big) \\
& \leq -\inf_{t \in \langle m,M \rangle}(\Psi_f(t;m,M))\Big(M\mathbf{1}_K - \sum_{j=1}^{n} p_j \Phi_j(A_j)\Big)\Big(\sum_{j=1}^{n} p_j \Phi_j(A_j) - m\mathbf{1}_K\Big),
\end{aligned}$$

since $\sum_{j=1}^{n} p_j \Phi_j(A_j) \in (m,M)$. The last two inequalities in (5.16) follow directly from:

$$-\inf_{t \in (m,M)} \Psi_f(t;m,M) \leq \frac{f'_+(m) - f'_-(M)}{M-m}$$

and $(M-t)(t-m) \leq \frac{1}{4}(M-m)^2$. □

Remark 5.1 We need to observe that if in Theorem 5.10 we take $p_1 = 1$, we obtain a difference type reverse of the Davis-Choi inequality:

$$\begin{aligned}
& f(\Phi(A)) - \Phi(f(A)) \\
& \leq -\inf_{t \in \langle m,M \rangle} \Psi_f(t;m,M)(M\mathbf{1}_K - \Phi(A))(\Phi(A) - m\mathbf{1}_K) \\
& \leq (M\mathbf{1}_K - \Phi(A))(\Phi(A) - m\mathbf{1}_K)\frac{f'_+(m) - f'_-(M)}{M-m} \\
& \leq \frac{1}{4}(M-m)(f'_+(m) - f'_-(M))\mathbf{1}_K. \qquad (5.20)
\end{aligned}$$

Theorem 5.11 *Let A_j, B_j, $j = 1,\dots,n$, be positive definite operators such that $mA_j \le B_j \le MA_j$ for some $0 < m < M < \infty$ and $p_j \ge 0$ such that $\sum_{j=1}^n p_j = 1$. Let σ be a solidarity generated by an operator monotone and operator concave function f. If Φ_j are normalized positive linear maps for $j = 1,\dots,n$, then we have*

$$
\begin{aligned}
&\left(\sum_{j=1}^n p_j\Phi_j(A_j)\right)\sigma\left(\sum_{j=1}^n p_j\Phi_j(B_j)\right) - \frac{M\sum_{j=1}^n p_j\Phi_j(A_j) - \sum_{j=1}^n p_j\Phi_j(B_j)}{M-m} f(m) \\
&\quad - \frac{\sum_{j=1}^n p_j\Phi_j(B_j) - m\sum_{j=1}^n p_j\Phi_j(A_j)}{M-m} f(M) \\
&\le -\inf_{t\in\langle m,M\rangle} \Psi_f(t;m,M)\Big(M\sum_{j=1}^n p_j\Phi_j(A_j) - \sum_{j=1}^n p_j\Phi_j(B_j)\Big)\Big(\sum_{j=1}^n p_j\Phi_j(A_j)\Big)^{-1} \\
&\quad \times \Big(\sum_{j=1}^n p_j\Phi_j(B_j) - m\sum_{j=1}^n p_j\Phi_j(A_j)\Big) \\
&\le \frac{f'_+(m) - f'_-(M)}{M-m}\Big(M\sum_{j=1}^n p_j\Phi_j(A_j) - \sum_{j=1}^n p_j\Phi_j(B_j)\Big)\Big(\sum_{j=1}^n p_j\Phi_j(A_j)\Big)^{-1} \\
&\quad \times \Big(\sum_{j=1}^n p_j\Phi_j(B_j) - m\sum_{j=1}^n p_j\Phi_j(A_j)\Big) \\
&\le \frac{1}{4}(M-m)(f'_+(m) - f'_-(M))\Big(\sum_{j=1}^n p_j\Phi_j(A_j)\Big),
\end{aligned}
\tag{5.21}
$$

where $\Psi_f(\cdot;m,M)$ is defined in (5.17).

Proof. Let I be an interval of real numbers such that $[m,M]$ belongs to the interior of I and let us suppose that $\phi : I \to \mathbb{R}$ is a concave function. Then from the Edmundson-Lah-Ribarič inequality we easily get

$$
\begin{aligned}
&\phi(t) - \frac{M-t}{M-m}\phi(m) - \frac{t-m}{M-m}\phi(M) \\
&= -(M-t)(t-m)\frac{1}{M-m}\Big(\frac{\phi(M)-\phi(t)}{M-t} - \frac{\phi(t)-\phi(m)}{t-m}\Big) \\
&= -(M-t)(t-m)\Psi_\Phi(t;m,M) \le (M-t)(t-m)\sup_{t\in\langle m,M\rangle}(-\Psi_\Phi(t;m,M)) \\
&\le (M-t)(t-m)\frac{\phi'_+(m) - \phi'_-(M)}{M-m} \le \frac{1}{4}(M-m)(\phi'_+(m) - \phi'_-(M))
\end{aligned}
\tag{5.22}
$$

for every $t \in \langle m,M\rangle$.

From $mA_j \le B_j \le MA_j$ easily follows that

$$
m \le \Big(\sum_{j=1}^n p_j\Phi_j(A_j)\Big)^{-\frac{1}{2}}\Big(\sum_{j=1}^n p_j\Phi_j(B_j)\Big)\Big(\sum_{j=1}^n p_j\Phi_j(A_j)\Big)^{-\frac{1}{2}} \le M,
$$

so using functional calculus and (5.22) we get the following inequalities:

$$
\begin{aligned}
&f\Big(\Big(\sum_{j=1}^{n} p_j\Phi_j(A_j)\Big)^{-\frac{1}{2}}\Big(\sum_{j=1}^{n} p_j\Phi_j(B_j)\Big)\Big(\sum_{j=1}^{n} p_j\Phi_j(A_j)\Big)^{-\frac{1}{2}}\Big)\\
&\quad -\frac{M\mathbf{1}_K-\Big(\sum_{j=1}^{n} p_j\Phi_j(A_j)\Big)^{-\frac{1}{2}}\Big(\sum_{j=1}^{n} p_j\Phi_j(B_j)\Big)\Big(\sum_{j=1}^{n} p_j\Phi_j(A_j)\Big)^{-\frac{1}{2}}}{M-m}f(m)\\
&\quad -\frac{\Big(\sum_{j=1}^{n} p_j\Phi_j(A_j)\Big)^{-\frac{1}{2}}\Big(\sum_{j=1}^{n} p_j\Phi_j(B_j)\Big)\Big(\sum_{j=1}^{n} p_j\Phi_j(A_j)\Big)^{-\frac{1}{2}}-m\mathbf{1}_K}{M-m}f(M)\\
&\le -\inf_{t\in\langle m,M\rangle}\Psi_f(t;m,M)\Big(M\mathbf{1}_K-\Big(\sum_{j=1}^{n} p_j\Phi_j(A_j)\Big)^{-\frac{1}{2}}\Big(\sum_{j=1}^{n} p_j\Phi_j(B_j)\Big)\Big(\sum_{j=1}^{n} p_j\Phi_j(A_j)\Big)^{-\frac{1}{2}}\Big)\\
&\qquad\cdot\Big(\Big(\sum_{j=1}^{n} p_j\Phi_j(A_j)\Big)^{-\frac{1}{2}}\Big(\sum_{j=1}^{n} p_j\Phi_j(B_j)\Big)\Big(\sum_{j=1}^{n} p_j\Phi_j(A_j)\Big)^{-\frac{1}{2}}-m\mathbf{1}_K\Big)\\
&\le \frac{f'_+(m)-f'_-(M)}{M-m}\Big(M\mathbf{1}_K-\Big(\sum_{j=1}^{n} p_j\Phi_j(A_j)\Big)^{-\frac{1}{2}}\Big(\sum_{j=1}^{n} p_j\Phi_j(B_j)\Big)\Big(\sum_{j=1}^{n} p_j\Phi_j(A_j)\Big)^{-\frac{1}{2}}\Big)\\
&\qquad\cdot\Big(\Big(\sum_{j=1}^{n} p_j\Phi_j(A_j)\Big)^{-\frac{1}{2}}\Big(\sum_{j=1}^{n} p_j\Phi_j(B_j)\Big)\Big(\sum_{j=1}^{n} p_j\Phi_j(A_j)\Big)^{-\frac{1}{2}}-m\mathbf{1}_K\Big)\\
&\le \frac{1}{4}(M-m)(f'_+(m)-f'_-(M))\mathbf{1}_K.
\end{aligned}
$$

Now, if we multiply the inequalities above twice by $\Big(\sum_{k=1}^{n} p_k\Phi_k(A_k)\Big)^{\frac{1}{2}}$, inequalities (5.21) follow. □

As an immediate consequence of Theorems 5.5 and 5.11 we have the following corollary which is a difference type converse of Ando's inequality (5.10).

Corollary 5.7 *Let A_j, B_j, $j=1,\ldots,n$, be positive definite operators such that $mA_j \le B_j \le MA_j$ for some $0<m<M<\infty$ and $p_j \ge 0$ such that $\sum_{j=1}^{n} = 1$. Let σ be a solidarity generated by an operator monotone and operator concave function f. If Φ_j are normalized positive linear maps for $j=1,\ldots,n$, then we have*

$$
\begin{aligned}
&\Big(\sum_{j=1}^{n} p_j\Phi_j(A_j)\Big)\sigma\Big(\sum_{j=1}^{n} p_j\Phi_j(B_j)\Big)-\sum_{j=1}^{n} p_j\Phi_j(A_j\sigma B_j)\\
&\le -\inf_{t\in\langle m,M\rangle}\Psi_f(t;m,M)\Big(M\sum_{j=1}^{n} p_j\Phi_j(A_j)-\sum_{j=1}^{n} p_j\Phi_j(B_j)\Big)\Big(\sum_{j=1}^{n} p_j\Phi_j(A_j)\Big)^{-1}\\
&\qquad\times\Big(\sum_{j=1}^{n} p_j\Phi_j(B_j)-m\sum_{j=1}^{n} p_j\Phi_j(A_j)\Big)\\
&\le \frac{f'_+(m)-f'_-(M)}{M-m}\Big(M\sum_{j=1}^{n} p_j\Phi_j(A_j)-\sum_{j=1}^{n} p_j\Phi_j(B_j)\Big)\Big(\sum_{j=1}^{n} p_j\Phi_j(A_j)\Big)^{-1}\\
&\qquad\times\Big(\sum_{j=1}^{n} p_j\Phi_j(B_j)-m\sum_{j=1}^{n} p_j\Phi_j(A_j)\Big)
\end{aligned}
$$

$$\leq \frac{1}{4}(M-m)(f'_+(m)-f'_-(M))\Big(\sum_{j=1}^{n} p_j\Phi_j(A_j)\Big),$$

where $\Psi_f(\cdot;m,M)$ *is defined in (5.17).*

Result that follows is a special case of the previous corollary for $n=1$ and $p_1=1$, but we will give an alternative proof using the converse of Davis-Choi's inequality (5.20).

Corollary 5.8 *Let* A,B *be positive definite operators such that* $mA \leq B \leq MA$ *for some* $0<m<M<\infty$. *Let* σ *be a solidarity generated by an operator monotone and operator concave function* f. *If* Φ *is a normalized positive linear map, then we have*

$$\begin{aligned}
&\Phi(A)\sigma\Phi(B)-\Phi(A\sigma B)\\
&\leq -\inf_{t\in\langle m,M\rangle}\Psi_f(t;m,M)(M\Phi(A)-\Phi(B))\Phi(A)^{-1}(\Phi(B)-m\Phi(A))\\
&\leq (M\Phi(A)-\Phi(B))\Phi(A)^{-1}(\Phi(B)-m\Phi(A))\frac{f'_+(m)-f'_-(M)}{M-m}\\
&\leq \frac{1}{4}(M-m)(f'_+(m)-f'_-(M))\Phi(A),
\end{aligned}$$

where $\Psi_f(\cdot;m,M)$ *is defined in (5.17).*

Proof. Let us define an auxiliary linear map

$$\Psi(X)=\Phi(A)^{-\frac{1}{2}}\Phi(A^{\frac{1}{2}}XA^{\frac{1}{2}})\Phi(A)^{-\frac{1}{2}}.$$

It is straightforward to check that Ψ is unital and positive. Now from the inequality (5.20) it follows that

$$\begin{aligned}
&f(\Psi(X))-\Psi(f(X))\\
&\leq -\inf_{t\in\langle m,M\rangle}\Psi_f(t;m,M)(M\mathbf{1}_K-\Psi(X))(\Psi(X)-m\mathbf{1}_K)\\
&\leq \frac{f'_+(m)-f'_-(M)}{M-m}(M\mathbf{1}_K-\Psi(X))(\Psi(X)-m\mathbf{1}_K)\\
&\leq \frac{1}{4}(M-m)(f'_+(m)-f'_-(M))\mathbf{1}_K, \qquad (5.23)
\end{aligned}$$

holds for a self-adjoint operator X such that $\mathrm{Sp}(X)\in[m,M]$. From $mA\leq B\leq MA$ we easily get $m\leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\leq M$, so we can put $X=A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in (5.23) and obtain:

$$\begin{aligned}
&f(\Phi(A)^{-\frac{1}{2}}\Phi(B)\Phi(A)^{-\frac{1}{2}})-\Phi(A)^{-\frac{1}{2}}\Phi(A^{\frac{1}{2}}f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}})\Phi(A)^{-\frac{1}{2}}\\
&\leq -\inf_{t\in\langle m,M\rangle}\Psi_f(t;m,M)(M\mathbf{1}_K-\Phi(A)^{-\frac{1}{2}}\Phi(B)\Phi(A)^{-\frac{1}{2}})(\Phi(A)^{-\frac{1}{2}}\Phi(B)\Phi(A)^{-\frac{1}{2}}-m\mathbf{1}_K)\\
&\leq \frac{f'_+(m)-f'_-(M)}{M-m}\left(M\mathbf{1}_K-\Phi(A)^{-\frac{1}{2}}\Phi(B)\Phi(A)^{-\frac{1}{2}}\right)\left(\Phi(A)^{-\frac{1}{2}}\Phi(B)\Phi(A)^{-\frac{1}{2}}-m\mathbf{1}_K\right)\\
&\leq \frac{1}{4}(M-m)(f'_+(m)-f'_-(M))\mathbf{1}_K.
\end{aligned}$$

Sought inequalities are obtained by multiplying the inequalities from above twice with $\Phi(A)^{\frac{1}{2}}$. □

As applications of Corollary 5.7 we give reverses of this type for basic examples of operator means and relative operator entropy.

Corollary 5.9 *Let A_i, B_i, $i = 1, \ldots, n$, be positive definite operators such that $mA_i \le B_i \le MA_i$ for some $0 < m < M < \infty$ and let $p_i \ge 0$ be real numbers such that $\sum_{i=1}^n p_i = 1$.*

- *If $\alpha \in [0,1]$ and $p \in [-1,1]$, then*

$$\begin{aligned}
&\left(\sum_{i=1}^n p_i A_i\right) \sharp_{p,\alpha} \left(\sum_{i=1}^n p_i B_i\right) - \sum_{i=1}^n p_i A_i \sharp_{p,\alpha} B_i \\
&\le - \inf_{t \in \langle m,M \rangle} \Psi_f(t;m,M) \\
&\qquad \times \left(M \sum_{i=1}^n p_i A_i - \sum_{i=1}^n p_i B_i\right) \left(\sum_{i=1}^n p_i A_i\right)^{-1} \left(\sum_{i=1}^n p_i B_i - m \sum_{i=1}^n p_i A_i\right) \\
&\le \alpha \frac{(\alpha + (1-\alpha)m^{-p})^{\frac{1-p}{p}} - (\alpha + (1-\alpha)M^{-p})^{\frac{1-p}{p}}}{M-m} \\
&\qquad \times \left(M \sum_{i=1}^n p_i A_i - \sum_{i=1}^n p_i B_i\right) \left(\sum_{i=1}^n p_i A_i\right)^{-1} \left(\sum_{i=1}^n p_i B_i - m \sum_{i=1}^n p_i A_i\right).
\end{aligned}$$

- *If $\alpha \in [0,1]$, then*

$$\begin{aligned}
&\left(\sum_{i=1}^n p_i A_i\right) \sharp_{\alpha} \left(\sum_{i=1}^n p_i B_i\right) - \sum_{i=1}^n p_i A_i \sharp_{\alpha} B_i \\
&\le - \inf_{t \in \langle m,M \rangle} \Psi_\alpha(t;m,M) \\
&\qquad \times \left(M \sum_{i=1}^n p_i A_i - \sum_{i=1}^n p_i B_i\right) \left(\sum_{i=1}^n p_i A_i\right)^{-1} \left(\sum_{i=1}^n p_i B_i - m \sum_{i=1}^n p_i A_i\right) \\
&\le \frac{\alpha(m^{\alpha-1} - M^{\alpha-1})}{M-m} \left(M \sum_{i=1}^n p_i A_i - \sum_{i=1}^n p_i B_i\right) \left(\sum_{i=1}^n p_i A_i\right)^{-1} \left(\sum_{i=1}^n p_i B_i - m \sum_{i=1}^n p_i A_i\right).
\end{aligned}$$

- *If $\alpha \in [0,1]$, then*

$$\begin{aligned}
&\left(\sum_{i=1}^n p_i A_i\right) !_{\alpha} \left(\sum_{i=1}^n p_i B_i\right) - \sum_{i=1}^n p_i A_i !_{\alpha} B_i \\
&\le - \inf_{t \in \langle m,M \rangle} \Psi_f(t;m,M) \\
&\qquad \times \left(M \sum_{i=1}^n p_i A_i - \sum_{i=1}^n p_i B_i\right) \left(\sum_{i=1}^n p_i A_i\right)^{-1} \left(\sum_{i=1}^n p_i B_i - m \sum_{i=1}^n p_i A_i\right)
\end{aligned}$$

$$\leq \alpha \frac{(\alpha+(1-\alpha)m)^{-2}-(\alpha+(1-\alpha)M)^{-2}}{M-m}$$
$$\times \Big(M\sum_{i=1}^{n} p_iA_i - \sum_{i=1}^{n} p_iB_i\Big)\Big(\sum_{i=1}^{n} p_iA_i\Big)^{-1}\Big(\sum_{i=1}^{n} p_iB_i - m\sum_{i=1}^{n} p_iA_i\Big).$$

- *We also have*

$$S\Big(\sum_{i=1}^{n} p_iA_i \Big| \sum_{i=1}^{n} p_iB_i\Big) - \sum_{i=1}^{n} p_iS(A_i|B_i)$$
$$\leq -\inf_{t\in\langle m,M\rangle} \Psi_f(t;m,M)$$
$$\times \Big(M\sum_{i=1}^{n} p_iA_i - \sum_{i=1}^{n} p_iB_i\Big)\Big(\sum_{i=1}^{n} p_iA_i\Big)^{-1}\Big(\sum_{i=1}^{n} p_iB_i - m\sum_{i=1}^{n} p_iA_i\Big)$$
$$\leq \frac{1}{Mm}\Big(M\sum_{i=1}^{n} p_iA_i - \sum_{i=1}^{n} p_iB_i\Big)\Big(\sum_{i=1}^{n} p_iA_i\Big)^{-1}\Big(\sum_{i=1}^{n} p_iB_i - m\sum_{i=1}^{n} p_iA_i\Big).$$

In each case $\Psi_f(\cdot;m,M)\colon (m,M) \to \mathbb{R}$ is defined in (5.17), where f is the appropriate generating function.

5.4 Inequalities of Ando's type for n-convex functions

In this section the difference generated by the Edmundson-Lah-Ribarič inequality is estimated from below and from above by Hermite's interpolating polynomials in terms of divided differences.

For the rest of the chapter, let Φ_i be normalized positive linear maps and let A_i, B_i, $i = 1,\ldots,r$, be positive definite operators such that $aA_i \leq B_i \leq bA_i$ for some $0 < a < b < \infty$ and $p_i \geq 0$ such that $\sum_{i=1}^{r} p_i = 1$. Let σ be a solidarity generated by an operator monotone function $f \in \mathscr{C}^n([a,b])$. In order to simplify the obtained relations, we introduce the following notations:

$$\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) = \sum_{i=1}^{r} p_i\Phi_i(A_i); \quad \Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};g(t)) = \sum_{i=1}^{r} p_i\Phi_j\left(A_i^{\frac{1}{2}} g\left(A_i^{-\frac{1}{2}}B_iA_i^{-\frac{1}{2}}\right)A_i^{\frac{1}{2}}\right).$$

Our first result is a generalization of the Edmundson-Lah-Ribarič inequality for solidarities that holds for the class of n-convex functions.

Theorem 5.12 ([109]) *If the function f is n-convex and if $n > m \geq 3$ are of different parity, then we have*

$$\begin{aligned}
&\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}\sigma\boldsymbol{B}) - \alpha_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - \beta_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \qquad (5.24)\\
&\leq (f[a,a]-f[a,b])\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - a\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right) + \sum_{k=2}^{m-1} \frac{f^{(k)}(a)}{k!} \Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)^k\right)\\
&\quad + \sum_{k=1}^{n-m} f[\underbrace{a,\dots,a}_{m \text{ times}};\underbrace{b,\dots,b}_{k \text{ times}}] \Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)^m(t-b)^{k-1}\right).
\end{aligned}$$

Inequality (5.24) also holds when the function f is n-concave and n and m are of equal parity. In case when the function f is n-convex and n and m are of equal parity, or when the function f is n-concave and n and m are of different parity, the inequality sign in (5.24) is reversed.

Proof. Since $f \in \mathscr{C}^n([a,b])$, it is continuous and its n-th order divided difference $f_n(t) = f[t;\underbrace{a,\dots,a}_{m \text{ times}};\underbrace{b,b,\dots,b}_{(n-m) \text{ times}}]$ is also continuous, so consequently the function $R_m(t)$ defined in (2.64) is also continuous.

From $aA_i \leq B_i \leq bA_i$ easily follows $a\mathbf{1} \leq A_i^{-\frac{1}{2}} B_i A_i^{-\frac{1}{2}} \leq b\mathbf{1}$, $i = 1,\dots,n$. Using the functional calculus and (2.65), we get

$$\begin{aligned}
&f\left(A_i^{-\frac{1}{2}} B_i A_i^{-\frac{1}{2}}\right) - \alpha_f A_i^{-\frac{1}{2}} B_i A_i^{-\frac{1}{2}} - \beta_f \mathbf{1}\\
&= (f[a,a]-f[a,b])\left(A_i^{-\frac{1}{2}} B_i A_i^{-\frac{1}{2}} - a\mathbf{1}\right) + \sum_{k=2}^{m-1} \frac{f^{(k)}(a)}{k!}\left(A_i^{-\frac{1}{2}} B_i A_i^{-\frac{1}{2}} - a\mathbf{1}\right)^k\\
&\quad + \sum_{k=1}^{n-m} f[\underbrace{a,\dots,a}_{m \text{ times}};\underbrace{b,\dots,b}_{k \text{ times}}]\left(A_i^{-\frac{1}{2}} B_i A_i^{-\frac{1}{2}} - a\mathbf{1}\right)^m \left(A_i^{-\frac{1}{2}} B_i A_i^{-\frac{1}{2}} - b\mathbf{1}\right)^{k-1}\\
&\quad + R_m\left(A_i^{-\frac{1}{2}} B_i A_i^{-\frac{1}{2}}\right).
\end{aligned}$$

After multiplying the obtained relation twice by $A_i^{\frac{1}{2}}$, acting by Φ_i, then finally multiplying by p_i and summing, it follows:

$$\begin{aligned}
&\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}\sigma\boldsymbol{B}) - \alpha_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - \beta_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \qquad (5.25)\\
&= (f[a,a]-f[a,b])\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - a\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right) + \sum_{k=2}^{m-1} \frac{f^{(k)}(a)}{k!} \Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)^k\right)\\
&\quad + \sum_{k=1}^{n-m} f[\underbrace{a,\dots,a}_{m \text{ times}};\underbrace{b,\dots,b}_{k \text{ times}}] \Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)^m(t-b)^{k-1}\right) + \Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_m(t)).
\end{aligned}$$

We want to remove the term $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_m(t))$ from the equality above, so we need to check its positivity (negativity). Due to the monotonicity property, it is actually enough to study positivity and negativity of the function:

$$R_m(t) = (t-a)^m (t-b)^{n-m} f[t; \underbrace{a,\dots,a}_{m \text{ times}}; \underbrace{b,b,\dots,b}_{(n-m) \text{ times}}].$$

Since $a \le t \le b$, we have $(t-a)^m \ge 0$ for any choice of m. For the same reason we have $(t-b) \le 0$. Trivially it follows that $(t-b)^{n-m} \le 0$ when n and m are of different parity, and $(t-b)^{n-m} \ge 0$ when n and m are of equal parity. When the function f is n-convex, then its nth order divided differences are nonnegative, and when it is n-concave, then those divided differences are less or equal to zero.

Now we see that $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_m(t)) \le 0$ when the function f is n-convex and n and m are of different parity or when f is n-concave and n and m are of equal parity, and $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_m(t)) \ge 0$ in the remaining cases, so inequality (5.24) easily follows from (5.25).
□

Remark 5.2 Sum of positive definite operators is again positive, and for positive definite operators A and B, operator $A^{\frac{1}{2}}BA^{\frac{1}{2}}$ is positive definite, so in the proof of Theorem 5.12, in the discussion about the positivity and negativity of the term $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_m(t))$ it was enough to discuss the positivity and negativity of function $R_m(t)$ because for a continuous and positive function R_m and a selfadjoint operator A, the operator $R_m(A)$ is positive definite.

Following result provides with a similar generalization of the Edmundson-Lah-Ribarič inequality, and it is obtained from Lemma 2.4.

Theorem 5.13 ([109]) *If the function f is n-convex and if $n > m$, where $m \ge 3$ is an odd number, then*

$$\begin{aligned}
&\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}\sigma\boldsymbol{B}) - \alpha_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - \beta_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \qquad (5.26)\\
&\le (f[a,b]-f[b,b])\left(b\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) - \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right) + \sum_{k=2}^{m-1} \frac{f^{(k)}(b)}{k!} \Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-b)^k\right)\\
&\quad + \sum_{k=1}^{n-m} f[\underbrace{b,\dots,b}_{m \text{ times}};\underbrace{a,\dots,a}_{k \text{ times}}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-b)^m(t-a)^{k-1}\right).
\end{aligned}$$

Inequality (5.26) also holds when the function f is n-concave and m is even. In case when the function f is n-convex and m is even, or when the function f is n-concave and m is odd, the inequality sign in (5.26) is reversed.

Proof. As in the proof of Theorem 5.12, since all the involved functions are continuous, we can replace t with operator $A_i^{-\frac{1}{2}}B_iA_i^{-\frac{1}{2}}$ in (2.69), multiply the obtained relation twice by $A_i^{\frac{1}{2}}$, act by Φ_i, then finally multiply it by p_i and sum it and get

$$\begin{aligned}
&\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}\sigma\boldsymbol{B}) - \alpha_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - \beta_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \qquad (5.27)\\
&= (f[a,b] - f[b,b])\left(b\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) - \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right) + \sum_{k=2}^{m-1} \frac{f^{(k)}(b)}{k!} \Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-b)^k\right)\\
&\quad + \sum_{k=1}^{n-m} f[\underbrace{b,\ldots,b}_{m \text{ times}};\underbrace{a,\ldots,a}_{k \text{ times}}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-b)^m(t-a)^{k-1}\right) + \Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_m^*(t)).
\end{aligned}$$

As before, in order to remove the term $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_m^*(t))$, we need to know when it is positive, and when it is negative. Due to the monotonicity property, it is enough to check positivity and negativity of the function:

$$R_m^*(t) = (t-b)^m(t-a)^{n-m} f[t;\underbrace{b,\ldots,b}_{m \text{ times}};\ \underbrace{a,a,\ldots,a}_{(n-m) \text{ times}}].$$

Since $t \in [a,b]$, we have $(t-a)^{n-m} \geq 0$ for every t and any choice of m. For the same reason we have $(t-b) \leq 0$. Trivially it follows that $(t-b)^m \leq 0$ when m is odd, and $(t-b)^m \geq 0$ when m is even. If the function f is n-convex, then its n-th order divided differences are greater of equal to zero, and if the function f is n-concave, then its n-th order divided differences are less or equal to zero.

Now it follows that $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_m^*(t)) \leq 0$ when the function f is n-convex and m is odd or f is n-concave and m is even. In the remaining cases the inequality sign is reversed, (5.26) easily follows from (5.27). □

As a direct consequence of Theorem 5.12 and Theorem 5.13, we get lower and upper bounds for the difference in the Edmundson-Lah-Ribarič inequality that hold for the class of n-convex functions.

Corollary 5.10 ([109]) *If the function f is n-convex, where n is an odd number, and if $m \geq 3$ is odd, then*

$$\begin{aligned}
&(f[a,a] - f[a,b])\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - a\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right) + \sum_{k=2}^{m-1} \frac{f^{(k)}(a)}{k!} \Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)^k\right)\\
&\quad + \sum_{k=1}^{n-m} f[\underbrace{a,\ldots,a}_{m \text{ times}};\underbrace{b,\ldots,b}_{k \text{ times}}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)^m(t-b)^{k-1}\right)\\
&\leq \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}\sigma\boldsymbol{B}) - \alpha_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - \beta_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \qquad (5.28)\\
&\leq (f[a,b] - f[b,b])\left(b\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) - \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right) + \sum_{k=2}^{m-1} \frac{f^{(k)}(b)}{k!} \Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-b)^k\right)\\
&\quad + \sum_{k=1}^{n-m} f[\underbrace{b,\ldots,b}_{m \text{ times}};\underbrace{a,\ldots,a}_{k \text{ times}}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-b)^m(t-a)^{k-1}\right).
\end{aligned}$$

Inequality (5.28) also holds when the function f is n-concave and m is even. In case when the function f is n-convex and m is even, or when the function f is n-concave and m is odd, the inequality signs in (5.28) are reversed.

The following result also provides with a lower and upper bound for the difference in the Edmundson-Lah-Ribarič inequality, and it is obtained from Lemma 2.3.

Theorem 5.14 ([109]) *If the function f is n-convex and if $n \geq 3$ is odd, then*

$$\begin{aligned}
&\sum_{k=2}^{n-1} f[a;\underbrace{b,\ldots,b}_{k \text{ times}}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)(t-b)^{k-1}\right) \\
&\leq \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}\sigma\boldsymbol{B}) - \alpha_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - \beta_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \\
&\leq f[a,a;b]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};(t-a)(t-b)) \\
&\qquad + \sum_{k=2}^{n-2} f[a,a;\underbrace{b,\ldots,b}_{k \text{ times}}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)^2(t-b)^{k-1}\right),
\end{aligned} \tag{5.29}$$

where $x \in H$ is a unit vector. Inequalities (5.29) also hold when the function f is n-concave and n is even. In case when the function f is n-convex and n is even, or when the function f is n-concave and n is odd, the inequality signs in (5.29) are reversed.

Proof. Again, using the functional calculus, we can replace t with $A_i^{-\frac{1}{2}} B_i A_i^{-\frac{1}{2}}$ in (2.62) and (2.63), multiply obtained relations twice by $A_i^{\frac{1}{2}}$, act by Φ_i, then finally multiply them by p_i and sum them. In that way we get

$$\begin{aligned}
&\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}\sigma\boldsymbol{B}) - \alpha_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - \beta_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \\
&= \sum_{k=2}^{n-1} f[a;\underbrace{b,\ldots,b}_{k \text{ times}}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)(t-b)^{k-1}\right) + \Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_1(t))
\end{aligned} \tag{5.30}$$

and

$$\begin{aligned}
&\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}\sigma\boldsymbol{B}) - \alpha_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - \beta_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \\
&= f[a,a;b]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};(t-a)(t-b)) \\
&\qquad + \sum_{k=2}^{n-2} f[a,a;\underbrace{b,\ldots,b}_{k \text{ times}}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)^2(t-b)^{k-1}\right) + \Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_2(t)).
\end{aligned} \tag{5.31}$$

We have already discussed positivity and negativity of the term $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_m(t))$ in the proof of Theorem 5.12. For $m=1$ it follows that $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_1(t)) \geq 0$ when the function f is n-convex and n is odd, or when f is n-concave and n even, and when the function f is n-concave and n is odd, or when f is n-convex and n even the inequality sign is reversed, so the relation (5.30) gives us

$$\begin{aligned}
&\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}\sigma\boldsymbol{B}) - \alpha_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - \beta_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \\
&\geq \sum_{k=2}^{n-1} f[a;\underbrace{b,\ldots,b}_{k \text{ times}}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)(t-b)^{k-1}\right)
\end{aligned}$$

for $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_1(t)) \geq 0$, and in case $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_1(t)) \leq 0$ the inequality sign is reversed.

In the same way, for $m = 2$ it follows that $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_2(t)) \leq 0$ when the function f is n-convex and n is odd, or when f is n-concave and n even, and $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_2(t)) \geq 0$ in the remaining cases.

The relation (5.31) for $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_2(t)) \leq 0$ gives us

$$\begin{aligned}
&\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}\sigma\boldsymbol{B}) - \alpha_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - \beta_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \\
&\leq f[a,a;b]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};(t-a)(t-b)) \\
&\qquad + \sum_{k=2}^{n-2} f[a,a;\underbrace{b,\ldots,b}_{k \text{ times}}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)^2(t-b)^{k-1}\right),
\end{aligned}$$

and when $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_2(t)) \geq 0$ the inequality sign is reversed.

When we combine the two inequalities obtained above, we get exactly (5.29). □

By utilizing Lemma 2.4 we can get similar lower and upper bounds for the difference in the Edmundson-Lah-Ribarič operator inequality that hold for all $n \in \mathbb{N}$, not only the odd ones.

Theorem 5.15 ([109]) *If the function f is n-convex, $n \geq 3$, then*

$$\begin{aligned}
&f[b,b;a]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};(t-b)(t-a)) \\
&\qquad + \sum_{k=2}^{n-2} f[b,b;\underbrace{a,\ldots,a}_{k \text{ times}}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-b)^2(t-a)^{k-1}\right) \\
&\leq \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}\sigma\boldsymbol{B}) - \alpha_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - \beta_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \\
&\leq \sum_{k=2}^{n-1} f[b;\underbrace{a,\ldots,a}_{k \text{ times}}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-b)(t-a)^{k-1}\right).
\end{aligned} \tag{5.32}$$

If the function f is n-concave, the inequality signs in (5.32) are reversed.

Proof. This proof follows the lines of the proof of Theorem 5.14. We start with replacing t with operator $A_i^{-\frac{1}{2}}B_iA_i^{-\frac{1}{2}}$ in (2.66) and (2.67) respectively, and after multiplying obtained relations twice by $A_i^{\frac{1}{2}}$, acting by Φ_i, then finally multiplying them by p_i and summing them we get

$$\begin{aligned}
&\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}\sigma\boldsymbol{B}) - \alpha_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - \beta_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \\
&= \sum_{k=2}^{n-1} f[b;\underbrace{a,\ldots,a}_{k \text{ times}}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-b)(t-a)^{k-1}\right) + \Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_1^*(t))
\end{aligned} \tag{5.33}$$

and

$$
\begin{aligned}
&\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}\sigma\boldsymbol{B}) - \alpha_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - \beta_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \\
&= f[b,b;a]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};(t-b)(t-a)) \\
&\quad + \sum_{k=2}^{n-2} f[b,b;\underbrace{a,\dots,a}_{k \text{ times}}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-b)^2(t-a)^{k-1}\right) + \Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_2^*(t)).
\end{aligned} \tag{5.34}
$$

Now we return to the discussion about positivity and negativity of the term $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_m^*(t))$ from the proof of Theorem 5.13. For $m=1$ we have $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_1^*(t)) \geq 0$ when the function f is n-concave, and $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_1^*(t)) \leq 0$ when the function f is n-convex, so the relation (5.33) for a n-convex function f gives us

$$
\begin{aligned}
&\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}\sigma\boldsymbol{B}) - \alpha_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - \beta_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \\
&\leq \sum_{k=2}^{n-1} f[b;\underbrace{a,\dots,a}_{k \text{ times}}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-b)(t-a)^{k-1}\right),
\end{aligned}
$$

and if the function f is n-concave, the inequality sign is reversed.

Similarly, for $m=2$ we have $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_2^*(t)) \geq 0$ when the function f is n-convex, and $\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};R_2^*(t)) \leq 0$ when the function f is n-concave. In this case the identity (5.34) for a n-convex function f gives us

$$
\begin{aligned}
&\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}\sigma\boldsymbol{B}) - \alpha_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) - \beta_f \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \\
&\geq f[b,b;a]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};(t-b)(t-a)) \\
&\quad + \sum_{k=2}^{n-2} f[b,b;\underbrace{a,\dots,a}_{k \text{ times}}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-b)^2(t-a)^{k-1}\right),
\end{aligned}
$$

and if the function f is n-concave, the inequality sign is reversed.

When we combine the two results from above, we get exactly (5.32). □

In the rest of this section we will utilize the results from above, as well as Lemma 2.3 and Lemma 2.4, in order to obtain some Jensen-type inequalities that hold for the class of n-convex functions. in that way we will obtain lower and upper bounds for the difference generated by the Jensen inequality for solidarities.

Again, let Φ_i be normalized positive linear maps and let A_i, B_i, $i = 1,\dots,r$, be positive definite operators such that $aA_i \leq B_i \leq bA_i$ for some $0 < a < b < \infty$ and $p_i \geq 0$ such that $\sum_{i=1}^r p_i = 1$. Let σ be a solidarity generated by an operator monotone function $f \in \mathscr{C}^n([a,b])$.

Our first result is a consequence of Corollary 5.10 and Lemma 2.3 and 2.4.

Theorem 5.16 ([109]) *Let n be an odd number. If the function f is n-convex, $n > m$, and if $m \geq 3$ is odd, then*

$$
\begin{aligned}
&(f[a,a]-f[b,b])\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})+(b(f[b,b]-f[a,b])-a(f[a,a]-f[a,b]))\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\\
&\quad+\sum_{k=2}^{m-1}\Bigg[\frac{f^{(k)}(a)}{k!}\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)^k\right)-\frac{f^{(k)}(b)}{k!}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\\
&\qquad\times\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)^k\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\Bigg]\\
&\quad+\sum_{k=1}^{n-m}f[\underbrace{a,\dots,a}_{m\ times};\underbrace{b,\dots,b}_{k\ times}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)^m(t-b)^{k-1}\right)\\
&\quad-\sum_{k=1}^{n-m}f[\underbrace{b,\dots,b}_{m\ times};\underbrace{a,\dots,a}_{k\ times}]\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)^m\\
&\qquad\times\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)^{k-1}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\\
&\le\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}\sigma\boldsymbol{B})-\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\,\sigma\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) \qquad (5.35)\\
&\le(f[b,b]-f[a,a])\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})+(b(f[a,b]-f[b,b])-a(f[a,b]-f[a,a]))\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\\
&\quad+\sum_{k=2}^{m-1}\Bigg[\frac{f^{(k)}(b)}{k!}\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-b)^k\right)-\frac{f^{(k)}(a)}{k!}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\\
&\qquad\times\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)^k\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\Bigg]\\
&\quad+\sum_{k=1}^{n-m}f[\underbrace{b,\dots,b}_{m\ times};\underbrace{a,\dots,a}_{k\ times}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-b)^m(t-a)^{k-1}\right).\\
&\quad-\sum_{k=1}^{n-m}f[\underbrace{a,\dots,a}_{m\ times};\underbrace{b,\dots,b}_{k\ times}]\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)^m\\
&\qquad\times\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)^{k-1}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}
\end{aligned}
$$

Inequalities (5.35) also hold when the function f is n-concave and m is even. In case when the function f is n-convex and m is even, or when the function f is n-concave and m is odd, the inequality signs in (5.35) are reversed.

Proof. From $aA_j \le B_j \le bA_j$ it follows that

$$
a\mathbf{1}\le\left(\sum_{j=1}^{r}p_j\Phi_j(A_j)\right)^{-\frac{1}{2}}\left(\sum_{j=1}^{r}p_j\Phi_j(B_j)\right)\left(\sum_{j=1}^{r}p_j\Phi_j(A_j)\right)^{-\frac{1}{2}}\le b\mathbf{1},
$$

so using functional calculus and (2.65), and then multiplying twice by $\left(\sum_{i=1}^{r}p_i\Phi_i(A_1)\right)^{\frac{1}{2}}$ we get the following relation:

$$
\begin{aligned}
&\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\,\sigma\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})-\alpha_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})-\beta_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\\
&=(f[a,a]-f[a,b])\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})-a\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)\\
&\quad+\sum_{k=2}^{m-1}\frac{f^{(k)}(a)}{k!}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac12}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac12}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac12}-a\mathbf{1}\right)^{k}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac12}\\
&\quad+\sum_{k=1}^{n-m}f[\underbrace{a,\dots,a}_{m\text{ times}};\underbrace{b,\dots,b}_{k\text{ times}}]\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac12}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac12}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac12}-a\mathbf{1}\right)^{m}\\
&\qquad\times\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac12}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac12}-b\mathbf{1}\right)^{k-1}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac12}\\
&\quad+\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac12}R_m\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac12}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac12}\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac12}.
\end{aligned}
$$

We want to remove the last term from the equality above, so we need to check its positivity (negativity). Again, due to the monotonicity property and Remark 5.2, it is actually enough to study positivity and negativity of the function $R_m(t)$, and we already have that discussion in the proof of Theorem 5.12, so for n-convex function f and n and $m\geq 3$ of different parity, or n-concave function f and n and $m\geq 3$ of the same parity, from the previous equality it follows

$$
\begin{aligned}
&\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\,\sigma\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})-\alpha_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})-\beta_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\\
&\leq(f[a,a]-f[a,b])\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})-a\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)\\
&\quad+\sum_{k=2}^{m-1}\frac{f^{(k)}(a)}{k!}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac12}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac12}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac12}-a\mathbf{1}\right)^{k}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac12}\\
&\quad+\sum_{k=1}^{n-m}f[\underbrace{a,\dots,a}_{m\text{ times}};\underbrace{b,\dots,b}_{k\text{ times}}]\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac12}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac12}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac12}-a\mathbf{1}\right)^{m}\\
&\qquad\times\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac12}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac12}-b\mathbf{1}\right)^{k-1}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac12},
\end{aligned}
\tag{5.36}
$$

and for n-convex function f and n and $m\geq 3$ of the same parity, or n-concave function f and n and $m\geq 3$ of different parity, the inequality sign is reversed.

In the same way we can replace t with $\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac12}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac12}$ in (2.69) and then multiplying twice by $(\sum_{i=1}^{r}p_i\Phi_i(A_1))^{\frac12}$ we get

$$
\begin{aligned}
&\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\,\sigma\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})-\alpha_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})-\beta_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\\
&=(f[a,b]-f[b,b])\left(b\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})-\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\\
&\quad+\sum_{k=2}^{m-1}\frac{f^{(k)}(b)}{k!}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac12}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac12}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac12}-b\mathbf{1}\right)^{k}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac12}
\end{aligned}
$$

$$+\sum_{k=1}^{n-m} f[\underbrace{b,\dots,b}_{m\text{ times}};\underbrace{a,\dots,a}_{k\text{ times}}]\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)^{m}$$
$$\times\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)^{k-1}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}$$
$$+\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}R_m^*\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}.$$

To remove the last term from the previous equality, we need to study its positivity and negativity. For the same reasons as before, it is enough to check positivity and negativity of the function R_m^*, and we have that discussion in the proof of Theorem 5.13. The equality above now turns into

$$\begin{aligned}
&\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\,\sigma\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})-\alpha_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})-\beta_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \qquad (5.37)\\
&\le (f[a,b]-f[b,b])\left(b\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})-\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\\
&\quad+\sum_{k=2}^{m-1}\frac{f^{(k)}(b)}{k!}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)^{k}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\\
&\quad+\sum_{k=1}^{n-m} f[\underbrace{b,\dots,b}_{m\text{ times}};\underbrace{a,\dots,a}_{k\text{ times}}]\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)^{m}\\
&\qquad\times\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)^{k-1}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}
\end{aligned}$$

for n-convex function f and an odd number $m\ge 3$ or n-concave function f and an even number $m\ge 3$. If f is n-convex and m is even, or if f is n-concave and m is odd, the inequality is reversed.

By combining inequalities (5.36) and (5.37) we get that

$$\begin{aligned}
&(f[a,a]-f[a,b])\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})-a\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)\\
&\quad+\sum_{k=2}^{m-1}\frac{f^{(k)}(a)}{k!}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)^{k}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\\
&\quad+\sum_{k=1}^{n-m} f[\underbrace{a,\dots,a}_{m\text{ times}};\underbrace{b,\dots,b}_{k\text{ times}}]\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)^{m}\\
&\qquad\times\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)^{k-1}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\\
&\le\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\,\sigma\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})-\alpha_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})-\beta_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}) \qquad (5.38)\\
&\quad\le (f[a,b]-f[b,b])\left(b\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})-\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\\
&\quad+\sum_{k=2}^{m-1}\frac{f^{(k)}(b)}{k!}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)^{k}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}
\end{aligned}$$

$$+\sum_{k=1}^{n-m} f[\underbrace{b,\ldots,b}_{m\text{ times}};\underbrace{a,\ldots,a}_{k\text{ times}}]\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)^{m}$$
$$\times\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)^{k-1}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}$$

holds if n is odd and f is n-convex and m is odd, or f is n-concave and m is even. If f is n-convex and m is even, or f is n-concave and m is odd, then the inequality signs are reversed.

When we multiply series of inequalities (5.38) by -1 and add to (5.28), we get exactly (5.35), and the proof is complete. □

Next result also provides with an estimate from below and from above for the difference generated by Ando's inequality, and it is obtained from Theorem 5.14 and Lemma 2.3.

Theorem 5.17 ([109]) *If the function f is n-convex and if $n \geq 3$ is odd, then*

$$\begin{aligned}
&\sum_{k=2}^{n-1} f[a;\underbrace{b,\ldots,b}_{k\text{ times}}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)(t-b)^{k-1}\right)\\
&\quad - f[a,a;b]\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)\\
&\qquad\times\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\\
&\quad - \sum_{k=2}^{n-2} f[a,a;\underbrace{b,\ldots,b}_{k\text{ times}}]\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)^{2}\\
&\qquad\times\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)^{k-1}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\\
&\leq \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}\sigma\boldsymbol{B})-\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\,\sigma\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) \qquad (5.39)\\
&\leq f[a,a;b]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};(t-a)(t-b))\\
&\quad+\sum_{k=2}^{n-2} f[a,a;\underbrace{b,\ldots,b}_{k\text{ times}}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-a)^{2}(t-b)^{k-1}\right)\\
&\quad-\sum_{k=2}^{n-1} f[a;\underbrace{b,\ldots,b}_{k\text{ times}}]\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)\\
&\qquad\times\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)^{k-1}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}
\end{aligned}$$
$$(5.40)$$

Inequalities (5.39) also hold when the function f is n-concave and n is even. In case when the function f is n-convex and n is even, or when the function f is n-concave and n is odd, the inequality signs in (5.39) are reversed.

Proof. By following a similar procedure as in the proof of the previous theorem, we start by replacing t with $\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}$ in with relations (2.62) and (2.63) from Lemma 2.3, and then multiplying them twice with $\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}$. We get

$$
\begin{aligned}
&\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\,\sigma\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})-\alpha_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})-\beta_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\\
&=\sum_{k=2}^{n-1} f[a;\underbrace{b,\dots,b}_{k\text{ times}}]\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)\\
&\qquad\times\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)^{k-1}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\\
&\quad+\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}R_1\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\,\sigma\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})-\alpha_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})-\beta_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\\
&=f[a,a;b]\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)\\
&\qquad\times\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\\
&\quad+\sum_{k=2}^{n-2} f[a,a;\underbrace{b,\dots,b}_{k\text{ times}}]\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)^2\\
&\qquad\times\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)^{k-1}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\\
&\quad+\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}R_2\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}
\end{aligned}
$$

respectively. After discussing the positivity an negativity of the last terms in the equalities from above and removing them in the same way as in the proof Theorem 5.16, we get a series of inequalities

$$
\begin{aligned}
&\sum_{k=2}^{n-1} f[a;\underbrace{b,\dots,b}_{k\text{ times}}]\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)\\
&\qquad\times\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)^{k-1}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\\
&\le\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\,\sigma\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})-\alpha_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})-\beta_f\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})
\end{aligned}
\tag{5.41}
$$

$$
\begin{aligned}
&\leq f[a,a;b]\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)\\
&\qquad\times\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\\
&\quad+\sum_{k=2}^{n-2}f[a,a;\underbrace{b,\dots,b}_{k\text{ times}}]\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)^{2}\\
&\qquad\times\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)^{k-1}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}
\end{aligned}
$$

that holds when n is odd and f is n-convex, or when n is even and f is n-concave. If n is odd and f is n-concave, or if n is even and f is n-convex, then the inequality signs in (5.41) are reversed.

Inequalities (5.39) are obtained after multiplying (5.41) by -1 and adding it to (5.29). □

In the analogous way as described in the proof of the previous theorem, but this time utilizing Lemma 2.4 and Theorem 5.15, we can get similar lower and upper bounds for the difference generated by Ando's inequality that hold for all $n \in \mathbb{N}$, not only the odd ones.

Theorem 5.18 ([109]) *Let $A \in \mathscr{B}_h(H)$ be a selfadjoint operator with $Sp(A) \subseteq [a,b]$ and let $f \in \mathscr{C}^n([a,b])$. If the function f is n-convex, $n \geq 3$, then*

$$
\begin{aligned}
&f[b,b;a]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A},\boldsymbol{B};(t-b)(t-a))\\
&\quad-\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)\\
&\qquad\times\sum_{k=2}^{n-1}f[b;\underbrace{a,\dots,a}_{k\ times}]\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)^{k-1}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\\
&\quad+\sum_{k=2}^{n-2}f[b,b;\underbrace{a,\dots,a}_{k\ times}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-b)^2(t-a)^{k-1}\right)\\
&\leq \Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A}\sigma\boldsymbol{B})-\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\,\sigma\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B}) \qquad (5.42)\\
&\leq f[b,b;a]\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(b\mathbf{1}-\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\right)(\langle Ax,x\rangle-a\mathbf{1})\\
&\quad+\sum_{k=2}^{n-1}f[b;\underbrace{a,\dots,a}_{k\ times}]\Lambda_{\boldsymbol{p}}^{\boldsymbol{\Phi}}\left(\boldsymbol{A},\boldsymbol{B};(t-b)(t-a)^{k-1}\right)\\
&\quad-\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-b\mathbf{1}\right)^{2}\\
&\qquad\times\sum_{k=2}^{n-2}f[b,b;\underbrace{a,\dots,a}_{k\ times}]\left(\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{B})\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{-\frac{1}{2}}-a\mathbf{1}\right)^{k-1}\left(\Delta_{\boldsymbol{p}}^{\boldsymbol{\Phi}}(\boldsymbol{A})\right)^{\frac{1}{2}}.
\end{aligned}
$$

If the function f is n-concave, the inequality signs in (5.42) are reversed.

5.4.1 Applications

As applications of the results from this section, we give reverses of this type for basic examples of some operator means and relative operator entropy.

Let A_i, B_i, $i = 1,\ldots,r$, be positive definite operators such that $aA_i \le B_i \le bA_i$ for some $0 < a < b < \infty$ and $p_i \ge 0$ such that $\sum_{i=1}^r p_i = 1$.

Some examples of connections and solidarities and their representing functions to which our results are applicable are as follows.

- The weighted harmonic mean

$$A\,!_\alpha B = \left[(1-\alpha)A^{-1} + \alpha B^{-1}\right]^{-1}, 0 \le \alpha \le 1,$$

has representing function $f(t) = \frac{t}{(1-\alpha)t+\alpha}$. We can calculate that for $n \in \mathbb{N}$

$$f^{(n)}(t) = \alpha(-1)^{n-1} n!(1-\alpha)^{n-1} \left((1-\alpha)t+\alpha\right)^{-n-1}.$$

Since $\alpha \in [0,1]$, it is easy to see that this generating function is n-convex when n is odd, and it is n-concave when n is an even number.

- The weighted geometric mean

$$A\#_\alpha B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\alpha A^{1/2}, 0 \le \alpha \le 1,$$

has representing function $f(t) = t^\alpha$. After an easy calculation we get that

$$f^{(n)}(t) = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)t^{\alpha-n}.$$

Because $\alpha \in [0,1]$, we see that this generating function is n-convex when n is odd, and it is n-concave when n is an even number.

- The relative operator entropy

$$S(A|B) = A^{1/2}\log\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}$$

has representing function $f(t) = \log t$. After an easy calculation we get that

$$f^{(n)}(t) = (-1)^{n-1}(n-1)!t^{-n}.$$

We immediately see that this generating function is n-convex when n is odd, and it is n-concave when n is an even number.

Chapter 6

Inequalities on time scales

In this chapter, some converses of the Jensen and Edmundson-Lah-Ribarič inequality in terms of time scale calculus are proved. We will also obtain new refinements of those converse relations with respect to the multiple Lebesgue delta integral for convex functions. The applicability of these results is illustrated in refinements of converse inequalities regarding monotonicity properties of generalized means, power means and some refinements of converse Hölder's inequality, which are all proved in the time scale setting.

Additionally, by utilizing some scalar inequalities obtained via Hermite's interpolating polynomial, we will obtain lower and upper bounds for the difference in Jensen's inequality and in the Edmundson-Lah-Ribarič inequality in time scale calculus that hold for the class of n-convex functions. Those results are later applied to generalized means, with a particular emphasis to power means, and in that way some new reverse relations for generalized and power means that correspond to n-convex functions are obtained.

6.1 Introduction

The theory of time scales was introduced by Stefan Hilger in his PhD thesis [59] in 1988 as a unification of the theory of difference equations with that of differential equations, unifying integral and differential calculus with the calculus of finite differences, extending to cases "in between" and offering a formalism for studying hybrid discrete-continuous dynamic systems. It has applications in any field that requires simultaneous modelling of discrete and continuous time. Now, we briefly introduce the time scales calculus and refer to [1, 60, 61] and the books [22, 23] for further details.

By a time scale $\mathbb{T}$ we mean any closed subset of $\mathbb{R}$. The two most popular examples of time scales are the real numbers $\mathbb{R}$ and the integers $\mathbb{Z}$. Since the time scale $\mathbb{T}$ may or may not be connected, we need the concept of jump operators.

For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

and the *backward jump operator* by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

In this definition, the convention is $\inf \emptyset = \sup \mathbb{T}$ (that is, $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t). If $\sigma(t) > t$, then we say that t is right-scattered, and if $\rho(t) < t$, then we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. Also, if $\sigma(t) = t$, then t is said to be right-dense, and if $\rho(t) = t$, then t is said to be left-dense. Points that are simultaneously right-dense and left-dense are called dense. The mapping $\mu : \mathbb{T} \to [0,\infty)$ defined by

$$\mu(t) = \sigma(t) - t$$

is called the graininess function. If $\mathbb{T}$ has a left-scattered maximum M, then we define $\mathbb{T}^\kappa = \mathbb{T} \setminus \{M\}$; otherwise $\mathbb{T}^\kappa = \mathbb{T}$. If $f : \mathbb{T} \to \mathbb{R}$ is a function, then we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by

$$f^\sigma(t) = f(\sigma(t)) \quad \text{for all} \quad t \in \mathbb{T}.$$

In the following considerations, $\mathbb{T}$ will denote a time scale, $I_\mathbb{T} = I \cap \mathbb{T}$ will denote a time scale interval (for any open or closed interval I in $\mathbb{R}$), and $[0,\infty)_\mathbb{T}$ will be used for the time scale interval $[0,\infty) \cap \mathbb{T}$.

Definition 6.1 *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that*

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s) \right| \leq \varepsilon |\sigma(t) - s| \quad \text{for all} \quad s \in U_\mathbb{T}.$$

We call $f^\Delta(t)$ the delta derivative of f at t. We say that f is delta differentiable on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

For all $t \in \mathbb{T}^\kappa$, we have the following properties:

(i) If f is delta differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is delta differentiable at t with $f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$.

(iii) If t is right-dense, then f is delta differentiable at t iff the limit $\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$ exists as a finite number. In this case, $f^{\Delta}(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$.

(iv) If f is delta differentiable at t, then $f(\sigma(t))=f(t)+\mu(t)f^{\Delta}(t)$.

Definition 6.2 *A function $f:\mathbb{T}\to\mathbb{R}$ is called rd-continuous if it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits are finite at all left-dense points in $\mathbb{T}$. We denote by* C_{rd} *the set of all rd-continuous functions. We say that f is rd-continuously delta differentiable (and write $f\in\mathrm{C}^1_{\mathrm{rd}}$) if $f^{\Delta}(t)$ exists for all $t\in\mathbb{T}^{\kappa}$ and $f^{\Delta}\in\mathrm{C}_{\mathrm{rd}}$.*

Definition 6.3 *A function $F:\mathbb{T}\to\mathbb{R}$ is called a delta antiderivative of $f:\mathbb{T}\to\mathbb{R}$ if $F^{\Delta}(t)=f(t)$ for all $t\in\mathbb{T}^{\kappa}$. Then we define the* delta integral *by*

$$\int_a^t f(s)\Delta s=F(t)-F(a).$$

The importance of rd-continuous function is revealed by the following result.

Theorem 6.1 *Every rd-continuous function has a delta antiderivative.*

Now we give some properties of the delta integral.

Theorem 6.2 *If $a,b,c\in\mathbb{T}$, $\beta\in\mathbb{R}$ and $f,g\in\mathrm{C}_{\mathrm{rd}}$, then*

(i) $\int_a^b (f(t)+g(t))\Delta t=\int_a^b f(t)\Delta t+\int_a^b g(t)\Delta t$;

(ii) $\int_a^b \alpha f(t)\Delta t=\alpha\int_a^b f(t)\Delta t$;

(iii) $\int_a^b f(t)\Delta t=-\int_b^a f(t)\Delta t$;

(iv) $\int_a^b f(t)\Delta t=\int_a^c f(t)\Delta t+\int_c^b f(t)\Delta t$;

(v) $\int_a^a f(t)\Delta t=0$;

(vi) if $f(t)\geq 0$ for all t, then $\int_a^b f(t)\Delta t\geq 0$.

In order to show the connection between positive linear functionals and time scale integrals, we first need to define the appropriate settings.

Let E be a nonempty set and L be a linear class of real-valued functions $f:E\to\mathbb{R}$ having the following properties.

(L_1) If $f,g\in L$ and $\alpha,\beta\in\mathbb{R}$, then $(\alpha f+\beta g)\in L$.

(L_2) If $f(t) = 1$ for all $t \in E$, then $f \in L$.

A positive linear functional is a functional $A : L \to \mathbb{R}$ having the following properties.

(A_1) If $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$, then $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$.

(A_2) If $f \in L$ and $f(t) \geq 0$ for all $t \in E$, then $A(f) \geq 0$.

In [6, 17, 9, 21], the authors presented a series of inequalities for the time scale integral and showed that it is not necessary to prove such kind of inequalities "from scratch" in the time scale setting as they can be obtained easily from well-known inequalities for positive linear functionals since the time scale integral is in fact a positive linear functional. The results on classical inequalities that are proved for the positive linear functionals, given in the monograph [124], are used to get new inequalities for the time scale integral.

Now we quote three theorems from [6] that we need in our research.

Theorem 6.3 *Let $\mathbb{T}$ be a time scale. For $a, b \in \mathbb{T}$ with $a < b$, let*

$$E = [a,b) \cap \mathbb{T} \quad \text{and} \quad L = \mathrm{C}_{\mathrm{rd}}(E, \mathbb{R}).$$

Then (L_1) and (L_2) are satisfied. Moreover, the delta integral $\int_a^b f(t)\Delta t$ is a positive linear functional which satisfies conditions (A_1) and (A_2).

Corresponding versions of Theorem 6.3 for nabla and α-diamond integrals are also given in [6].

Multiple Riemann integration and multiple Lebesgue integration on time scale was introduced in [19] and [20], respectively, and both integrals are also positive linear functionals.

Theorem 6.4 *Let $\mathbb{T}_1, \ldots, \mathbb{T}_n$ be time scales. For $a_i, b_i \in \mathbb{T}_i$ with $a_i < b_i$, $1 \leq i \leq n$, let*

$$\mathscr{E} \subset ([a_1, b_1) \cap \mathbb{T}_1) \times \cdots \times ([a_n, b_n) \cap \mathbb{T}_n)$$

be Lebesgue Δ-measurable and let L be the set of all Δ-measurable functions from $\mathscr{E}$ to $\mathbb{R}$. Then (L_1) and (L_2) are satisfied. Moreover, the multiple Lebesgue delta integral on time scales $\int_{\mathscr{E}} f(t)\Delta t$ is a positive linear functional and satisfies conditions (A_1) and (A_2).

Theorem 6.5 *Under the assumptions of Theorem 6.4, the delta integral $\frac{\int_{\mathscr{E}} h(t)f(t)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}$, where $h : \mathscr{E} \to \mathbb{R}$ is nonnegative, Δ-integrable and $\int_{\mathscr{E}} h(t)\Delta t > 0$, is also a positive linear functional satisfying (A_1), (A_2) and $A(1) = 1$.*

Using the known Jessen inequality for positive linear functionals ([124, Theorem 2.4]) and Theorem 6.5, M. Anwar, R. Bibi, M. Bohner and J. Pečarić proved in [6] the following generalization of Jessen's inequality on time scales.

Theorem 6.6 *Assume $\phi \in C(I,\mathbb{R})$ is convex, where $I \subset \mathbb{R}$ is an interval. Let $\mathscr{E} \subset \mathbb{R}^n$ be as in Theorem 6.4 and suppose f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = I$. Moreover, let $h : \mathscr{E} \to \mathbb{R}$ be nonnegative, Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t > 0$. Then*

$$\phi\left(\frac{\int_{\mathscr{E}} h(t)f(t)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}\right) \leq \frac{\int_{\mathscr{E}} h(t)\phi(f(t))\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}. \tag{6.1}$$

Lah and Ribarič proved in [89] the converse of Jensen's inequality for convex functions (see also [120]). Beesack and Pečarić gave in [14] the generalization of Lah–Ribarič's inequality for positive linear functionals. Applying the fact that the multiple Lebesgue delta time scale integral is a positive linear functional (Theorem 6.5) to Beesack–Pečarić's result from [14], the following theorem is proved in [6].

Theorem 6.7 *Assume $\phi \in C(I,\mathbb{R})$ is convex, where $I = [m,M] \subset \mathbb{R}$, with $m < M$. Let $\mathscr{E} \subset \mathbb{R}^n$ be as in Theorem 6.4 and suppose f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = I$. Moreover, let $h : \mathscr{E} \to \mathbb{R}$ be nonnegative, Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t > 0$. Then*

$$\frac{\int_{\mathscr{E}} h(t)\phi(f(t))\Delta t}{\int_{\mathscr{E}} h(t)\Delta t} \leq \frac{M - \frac{\int_{\mathscr{E}} h(t)f(t)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}}{M-m}\phi(m) + \frac{\frac{\int_{\mathscr{E}} h(t)f(t)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t} - m}{M-m}\phi(M). \tag{6.2}$$

6.2 Converses of the Jensen and Edmundson-Lah--Ribarič inequalities

In this section, we prove new converses of Jensen's inequality on time scales. For simplicity, we introduce the notations

$$L_\Delta(f) = \int_{\mathscr{E}} f(t)\Delta t \quad \text{and} \quad \overline{L}_\Delta(f,h) = \frac{\int_{\mathscr{E}} f(t)h(t)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t},$$

where $f : \mathscr{E} \to \mathbb{R}$ is Δ-integrable and $h : \mathscr{E} \to \mathbb{R}$ is nonnegative Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t > 0$.

Theorem 6.8 ([12]) *Let $\phi \in C(I,\mathbb{R})$ be convex, where $I = [m,M] \subset \mathbb{R}$, with $m < M$. Assume $\mathscr{E}$ is as in Theorem 6.4 and suppose f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = I$. Moreover, let $h : \mathscr{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t > 0$. Then*

$$0 \leq \overline{L}_\Delta(\phi(f),h) - \phi\left(\overline{L}_\Delta(f,h)\right) \tag{6.3}$$

$$\leq \left(M - \overline{L}_\Delta(f,h)\right)\left(\overline{L}_\Delta(f,h) - m\right) \cdot \frac{\phi'_-(M) - \phi'_+(m)}{M - m}$$
$$\leq \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)).$$

If ϕ is concave on I, then all inequalities in (6.3) *are reversed.*

Proof. Let ϕ be a convex function. The first inequality in (6.3) follows directly from Jensen's inequality on time scales given in Theorem 6.6. Now, let us take the inequality (6.2) from Theorem 6.7. Adding the term $-\phi\left(\overline{L}_\Delta(f,h)\right)$ on both sides of (6.2), we obtain

$$\begin{aligned} &\overline{L}_\Delta(\phi(f),h) - \phi\left(\overline{L}_\Delta(f,h)\right) \\ &\quad \leq \frac{M - \overline{L}_\Delta(f,h)}{M-m}\phi(m) + \frac{\overline{L}_\Delta(f,h) - m}{M-m}\phi(M) - \phi\left(\overline{L}_\Delta(f,h)\right) =: B. \end{aligned} \tag{6.4}$$

By the convexity of ϕ, it follows

$$\phi(x) - \phi(M) \geq \phi'_-(M)(x-M), \qquad x \in [m,M]. \tag{6.5}$$

Multiplying inequality (6.5) with $(x-m) \geq 0$, we get

$$(x-m)\phi(x) - (x-m)\phi(M) \geq \phi'_-(M)(x-M)(x-m), \qquad x \in [m,M]. \tag{6.6}$$

Similarly, multiplying the inequality $\phi(x) - \phi(m) \geq \phi'_+(m)(x-m)$ with $(M-x) \geq 0$, we obtain

$$(M-x)\phi(x) - (M-x)\phi(m) \geq \phi'_+(m)(x-m)(M-x), \qquad x \in [m,M]. \tag{6.7}$$

Adding (6.6) to (6.7) and dividing by $(M-m)$, for any $x \in [m,M]$, we have

$$\frac{(M-x)\phi(m) + (x-m)\phi(M)}{M-m} - \phi(x) \leq \frac{(M-x)(x-m)}{M-m}\left(\phi'_-(M) - \phi'_+(m)\right). \tag{6.8}$$

Replacing x in (6.8) with $\overline{L}_\Delta(f,h)$, leads to

$$B \leq \left(M - \overline{L}_\Delta(f,h)\right)\left(\overline{L}_\Delta(f,h) - m\right)\frac{\phi'_-(M) - \phi'_+(m)}{M-m}. \tag{6.9}$$

Combining (6.4) and (6.9) brings us to the second inequality in (6.3). The third inequality in (6.3) follows from the elementary estimate $\frac{(M-x)(x-m)}{M-m} \leq \frac{1}{4}(M-m)$ for every $x \in \mathbb{R}$. If the function ϕ is concave, then $-\phi$ is convex, so applying (6.3) to $-\phi$ gives us the reversed inequalities in (6.3). □

Remark 6.1 The proof of Theorem 6.8 can be obtained directly from Theorem 1.5 since the multiple Lebesgue delta time scale integral is a positive linear functional, according to Theorem 6.5.

Theorem 6.9 ([12]) *Suppose that all assumptions from Theorem 6.8 hold. Then*

$$\begin{aligned}
0 &\leq \frac{M-\overline{L}_\Delta(f,h)}{M-m}\phi(m)+\frac{\overline{L}_\Delta(f,h)-m}{M-m}\phi(M)-\overline{L}_\Delta(\phi(f),h) \\
&\leq \frac{\phi_-'(M)-\phi_+'(m)}{M-m}\cdot\frac{\int_{\mathscr{E}} h(t)\,(M-f(t))\,(f(t)-m)\,\Delta t}{\int_{\mathscr{E}} h(t)\Delta t} \\
&\leq \frac{\phi_-'(M)-\phi_+'(m)}{M-m}\left(M-\overline{L}_\Delta(f,h)\right)\left(\overline{L}_\Delta(f,h)-m\right) \\
&\leq \frac{1}{4}(M-m)(\phi_-'(M)-\phi_+'(m)).
\end{aligned} \tag{6.10}$$

If ϕ is concave on I, then all inequalities in (6.10) *are reversed.*

Proof. Assume that ϕ is convex. The first inequality in (6.10) follows directly from inequality (6.2) in Theorem 6.7. We now replace x in (6.8) by $f(t)$, $t\in\mathscr{E}$ (notice that $m\leq f(t)\leq M$ since $f(\mathscr{E})=I$ by the assumptions) so that

$$\begin{aligned}
&\frac{M-f(t)}{M-m}\phi(m)+\frac{f(t)-m}{M-m}\phi(M)-\phi(f(t)) \\
&\leq \frac{(M-f(t))(f(t)-m)}{M-m}\left(\phi_-'(M)-\phi_+'(m)\right).
\end{aligned} \tag{6.11}$$

Since the multiple Lebesgue delta time scale integral is a positive linear functional, multiplying inequality (6.11) by $\frac{h(t)}{\int_{\mathscr{E}} h(t)\Delta t}$ and integrating the resulting inequality, we get

$$\begin{aligned}
&\frac{M-\overline{L}_\Delta(f,h)}{M-m}\phi(m)+\frac{\overline{L}_\Delta(f,h)-m}{M-m}\phi(M)-\overline{L}_\Delta(\phi(f),h) \\
&\leq \frac{\int_{\mathscr{E}} h(t)\,(M-f(t))\,(f(t)-m)\,\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}\cdot\frac{\phi_-'(M)-\phi_+'(m)}{M-m}
\end{aligned}$$

which is the second inequality in (6.10). Using the fact that the function $g:\mathbb{R}\to\mathbb{R}$, defined as $g(x)=(M-x)(x-m)$, is concave and applying Theorem 6.6 to the function g instead of the function ϕ, we deduce

$$\frac{\int_{\mathscr{E}} h(t)\,(M-f(t))\,(f(t)-m)\,\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}\leq\left(M-\overline{L}_\Delta(f,h)\right)\left(\overline{L}_\Delta(f,h)-m\right),$$

which implies the third inequality in (6.10). The last inequality in (6.10) is the same one as the last inequality in Theorem 6.8. If the function ϕ is concave, then $-\phi$ is convex, so applying (6.10) to $-\phi$ gives us the reversed inequalities in (6.10). □

The following results are converses of Jensen's inequality on time scales which refine the results from the above for multiple Lebesgue delta integral. For simplicity, we introduce the following notations

$$L_\Delta(f) = \int_{\mathscr{E}} f(t)\Delta t \quad \text{and} \quad \overline{L}_\Delta(f,h) = \frac{\int_{\mathscr{E}} f(t)|h(t)|\Delta t}{\int_{\mathscr{E}} |h(t)|\Delta t},$$

where $f,h : \mathscr{E} \to \mathbb{R}$ are Δ-integrable and $\int_{\mathscr{E}} |h(t)|\Delta t > 0$.

Theorem 6.10 ([13]) *Let $\phi \in C(I,\mathbb{R})$ be convex, where $I = [m,M] \subset \mathbb{R}$, with $m < M$. Assume $\mathscr{E} \subset \mathbb{R}^n$ is as in Theorem 6.4 and suppose f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = I$. Moreover, let $h : \mathscr{E} \to \mathbb{R}$ be Δ-integrable such that $\int_{\mathscr{E}} |h(t)|\Delta t > 0$. Then*

$$\begin{aligned} 0 &\le \overline{L}_\Delta(\phi(f),h) - \phi\left(\overline{L}_\Delta(f,h)\right) \\ &\le \left(M - \overline{L}_\Delta(f,h)\right)\left(\overline{L}_\Delta(f,h) - m\right) \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M) \\ &\le \left(M - \overline{L}_\Delta(f,h)\right)\left(\overline{L}_\Delta(f,h) - m\right) \cdot \frac{\phi'_-(M) - \phi'_+(m)}{M-m} \\ &\le \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)), \end{aligned} \tag{6.12}$$

where $\Psi_\phi(\cdot;m,M)\colon \langle m,M\rangle \to \mathbb{R}$ is defined by

$$\Psi_\Phi(t;m,M) = \frac{1}{M-m}\Big(\frac{\phi(M)-\phi(t)}{M-t} - \frac{\phi(t)-\phi(m)}{t-m}\Big).$$

If ϕ is concave on I, then all inequalities in (6.3) *are reversed.*

Proof. Since ϕ is a convex function, first inequality in (6.12) follows from Theorem 6.6. From Theorem 6.7, we have

$$\begin{aligned} &\overline{L}_\Delta(\phi(f),h) - \phi\left(\overline{L}_\Delta(f,h)\right) \\ &\le \frac{M - \overline{L}_\Delta(f,h)}{M-m}\phi(m) + \frac{\overline{L}_\Delta(f,h) - m}{M-m}\phi(M) - \phi\left(\overline{L}_\Delta(f,h)\right) \\ &= \frac{1}{M-m}\left(M - \overline{L}_\Delta(f,h)\right)\left(\overline{L}_\Delta(f,h) - m\right) \\ &\quad \cdot \left(\frac{\phi(M) - \phi\left(\overline{L}_\Delta(f,h)\right)}{M - \overline{L}_\Delta(f,h)} - \frac{\phi\left(\overline{L}_\Delta(f,h)\right) - \phi(m)}{\overline{L}_\Delta(f,h) - m}\right) \\ &= \left(M - \overline{L}_\Delta(f,h)\right)\left(\overline{L}_\Delta(f,h) - m\right)\Psi_\phi\left(\overline{L}_\Delta(f,h);m,M\right) \\ &\le \left(M - \overline{L}_\Delta(f,h)\right)\left(\overline{L}_\Delta(f,h) - m\right) \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M), \end{aligned} \tag{6.13}$$

which is the second inequality in (6.12), provided that $\overline{L}_\Delta(f,h) \ne m,M$. When $\overline{L}_\Delta(f,h)$ is equal to m or M then inequality (6.12) is obvious.

Since,

$$\sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M) = \frac{1}{M-m} \sup_{t\in\langle m,M\rangle} \left\{\frac{\phi(M)-\phi(t)}{M-t} - \frac{\phi(t)-\phi(m)}{t-m}\right\}$$

$$
\begin{aligned}
&\leq \frac{1}{M-m}\left(\sup_{t\in\langle m,M\rangle}\frac{\phi(M)-\phi(t)}{M-t}+\sup_{t\in\langle m,M\rangle}\frac{-(\phi(t)-\phi(m))}{t-m}\right)\\
&=\frac{1}{M-m}\left(\sup_{t\in\langle m,M\rangle}\frac{\phi(M)-\phi(t)}{M-t}-\inf_{t\in\langle m,M\rangle}\frac{\phi(t)-\phi(m)}{t-m}\right)=\frac{\phi'_-(M)-\phi'_+(m)}{M-m},
\end{aligned}
$$

the third inequality in (6.12) is true. The last inequality in (6.12) follows from the elementary estimate $\frac{(M-x)(x-m)}{M-m}\leq \frac{1}{4}(M-m)$, for every $x\in\mathbb{R}$. If the function ϕ is concave, then $-\phi$ is convex, so applying (6.12) to $-\phi$ gives the reversed inequalities in (6.12). This completes the proof. □

Remark 6.2 According to (6.13), with the same assumptions as in Theorem 6.10, following inequalities are also true

$$
\begin{aligned}
0 &\leq \overline{L}_\Delta(\phi(f),h)-\phi\left(\overline{L}_\Delta(f,h)\right)\\
&\leq \frac{1}{4}(M-m)^2\Psi_\phi\left(\overline{L}_\Delta(f,h);m,M\right)\\
&\leq \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)).
\end{aligned}
$$

Using the refinement of the converse Jensen inequality for normalized positive linear functionals, given in Theorem 1.10, we derive the following theorem which refines inequality (6.12) from Theorem 6.10.

Theorem 6.11 ([13]) *Let $\phi\in C(I,\mathbb{R})$ be convex, where $I=[m,M]\subset\mathbb{R}$, with $m<M$. Assume $\mathscr{E}\subset\mathbb{R}^n$ and L are as in Theorem 6.4 with additional property that for every $f,g\in L$ we have that $\min\{f,g\}\in L$ and $\max\{f,g\}\in L$. Let f be Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E})=I$. Moreover, let $h:\mathscr{E}\to\mathbb{R}$ be Δ-integrable such that $\int_{\mathscr{E}}|h(t)|\Delta t>0$. Then,*

$$
\begin{aligned}
0 &\leq \overline{L}_\Delta(\phi(f),h)-\phi\left(\overline{L}_\Delta(f,h)\right)\\
&\leq \left(M-\overline{L}_\Delta(f,h)\right)\left(\overline{L}_\Delta(f,h)-m\right)\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M)-\overline{L}_\Delta(\tilde{f},h)\delta_\phi\\
&\leq \left(M-\overline{L}_\Delta(f,h)\right)\left(\overline{L}_\Delta(f,h)-m\right)\cdot\frac{\phi'_-(M)-\phi'_+(m)}{M-m}-\overline{L}_\Delta(\tilde{f},h)\delta_\phi \qquad (6.14)\\
&\leq \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m))-\overline{L}_\Delta(\tilde{f},h)\delta_\phi,
\end{aligned}
$$

where

$$
\tilde{f}=\tfrac{1}{2}-\frac{|f-\frac{m+M}{2}|}{M-m},\quad \delta_\phi=\phi(m)+\phi(M)-2\phi\left(\tfrac{m+M}{2}\right)
$$

and $\Psi_\phi(\cdot;m,M)\colon\langle m,M\rangle\to\mathbb{R}$ *is defined by*

$$
\Psi_\Phi(t;m,M)=\tfrac{1}{M-m}\left(\tfrac{\phi(M)-\phi(t)}{M-t}-\tfrac{\phi(t)-\phi(m)}{t-m}\right).
$$

If ϕ is concave on I, then the above inequalities are reversed.

Proof. Inequality (6.14) follows directly from the result from Theorem 1.10 and the fact that multiple Lebesgue delta integral is a positive linear functional. □

6.3 Inequalities of the Jensen and Edmundson-Lah-Ribarič type on time scales for n-convex functions

For simplicity, we introduce the notations

$$L_\Delta(f) = \int_{\mathscr{E}} f(t)\Delta t \quad \text{and} \quad \overline{L}_\Delta(f,h) = \frac{\int_{\mathscr{E}} f(t)h(t)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t},$$

where $f : \mathscr{E} \to \mathbb{R}$ is Δ-integrable and $h : \mathscr{E} \to \mathbb{R}$ is nonnegative Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t > 0$.

Throughout this section, whenever mentioning the interval $[a,b]$, we assume that a,b are finite real numbers such that $a < b$. We can write the Edmundson-Lah-Ribarič inequality (C) in the form

$$\alpha_\phi \overline{L}_\Delta(f,h) + \beta_\phi - \overline{L}_\Delta(\phi(f),h) \geq 0 \tag{6.15}$$

with standard notation

$$\alpha_\phi = \frac{\phi(b)-\phi(a)}{b-a} \quad \text{and} \quad \beta_\phi = \frac{b\phi(a)-a\phi(b)}{b-a}.$$

A generalization of the Edmundson-Lah-Ribarič inequality (C) obtained from Lemma 2.3 is given in the following theorem.

Theorem 6.12 ([108]) *Let $\phi \in \mathscr{C}^n([a,b])$ be an n-convex function. Assume $\mathscr{E}$ is as in Theorem 6.4 and suppose f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = [a,b]$. Moreover, let $h : \mathscr{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t > 0$. If $n > m \geq 3$ are of different parity, then*

$$\begin{aligned}
&\overline{L}_\Delta(\phi(f),h) - \alpha_\phi \overline{L}_\Delta(f,h) - \beta_\phi \\
&\leq (\overline{L}_\Delta(f,h) - a)\left(\phi'(a) - \phi[a,b]\right) + \sum_{k=2}^{m-1} \frac{\phi^{(k)}(a)}{k!} \overline{L}_\Delta\left((f-a\mathbf{1})^k, h\right) \\
&\quad + \sum_{k=1}^{n-m} \phi[\underbrace{a,\ldots,a}_{m \text{ times}}; \underbrace{b,\ldots,b}_{k \text{ times}}] \overline{L}_\Delta\left((f-a\mathbf{1})^m (f-b\mathbf{1})^{k-1}, h\right).
\end{aligned} \tag{6.16}$$

Inequality (6.16) also holds when the function ϕ is n-concave and n and m are of equal parity. In case when the function ϕ is n-convex and n and m are of equal parity, or when the function ϕ is n-concave and n and m are of different parity, the inequality sign in (6.16) is reversed.

Proof. Since $f(\mathscr{E}) = [a,b]$ by the assumptions, we have $a \le f(t) \le b$, so we can replace t with $f(t)$ in (2.65) and obtain:

$$\phi(f(t)) - \alpha_\phi f(t) - \beta_\phi = (f(t)-a)\left(\phi'(a) - \phi[a,b]\right) + \sum_{k=2}^{m-1} \frac{\phi^{(k)}(a)}{k!}(f(t)-a)^k$$
$$+ \sum_{k=1}^{n-m} \phi[\underbrace{a,\dots,a}_{m \text{ times}};\underbrace{b,\dots,b}_{k \text{ times}}](f(t)-a)^m (f(t)-b)^{k-1} + R_m(f(t)).$$

Since the multiple Lebesgue delta time scale integral is a positive linear functional, multiplying the previous inequality by $\frac{h(t)}{\int_{\mathscr{E}} h(t)\Delta t}$ and then integrating the resulting inequality yields

$$\overline{L}_\Delta(\phi(f),h) - \alpha_\phi \overline{L}_\Delta(f,h) - \beta_\phi \tag{6.17}$$
$$= (\overline{L}_\Delta(f,h) - a)\left(\phi'(a) - \phi[a,b]\right) + \sum_{k=2}^{m-1} \frac{\phi^{(k)}(a)}{k!} \cdot \frac{\int_{\mathscr{E}} (f(t)-a)^k h(t)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}$$
$$+ \sum_{k=1}^{n-m} \phi[\underbrace{a,\dots,a}_{m \text{ times}};\underbrace{b,\dots,b}_{k \text{ times}}] \frac{\int_{\mathscr{E}} (f(t)-a)^m (f(t)-b)^{k-1} h(t)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t} + \frac{\int_{\mathscr{E}} R_m(f(t))h(t)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}.$$

Now we set our focus on positivity and negativity of the term

$$\frac{\int_{\mathscr{E}} R_m(f(t))h(t)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}.$$

Because multiple Lebesgue delta time scale integral takes nonnegative values for positive functions, it is enough to study positivity and negativity of:

$$R_m(f(t)) = (f(t)-a)^m (f(t)-b)^{n-m} \phi[f(t);\underbrace{a,\dots,a}_{m \text{ times}};\ \underbrace{b,b,\dots,b}_{(n-m) \text{ times}}].$$

Since by assumptions we have $a \le f(t) \le b$, we have $(f(t)-a)^m \ge 0$ for any choice of m. For the same reason we have $(f(t)-b) \le 0$. Trivially it follows that $(f(t)-b)^{n-m} \le 0$ when n and m are of different parity, and $(f(t)-b)^{n-m} \ge 0$ when n and m are of equal parity.

If the function ϕ is n-convex, then $\phi[f(t);\underbrace{a,\dots,a}_{m \text{ times}};\ \underbrace{b,b,\dots,b}_{(n-m) \text{ times}}] \ge 0$, and if the function ϕ is n-concave, then $\phi[f(t);\underbrace{a,\dots,a}_{m \text{ times}};\ \underbrace{b,b,\dots,b}_{(n-m) \text{ times}}] \le 0$ for any $t \in [a,b]$. Inequality (6.16) now easily follows from (6.17). □

Next result is another generalization of the Edmundson-Lah-Ribarič inequality in terms of divided differences, obtained from Lemma 2.4 that also holds for the class of n-convex functions.

Theorem 6.13 ([108]) *Let $\phi \in \mathscr{C}^n([a,b])$ be an n-convex function. Assume $\mathscr{E}$ is as in Theorem 6.4 and suppose f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = [a,b]$. Moreover, let $h : \mathscr{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t > 0$. For an odd number $m \geq 3$ such that $m < n$ we have*

$$\begin{aligned}
&\overline{L}_\Delta(\phi(f),h) - \alpha_\phi \overline{L}_\Delta(f,h) - \beta_\phi \qquad (6.18)\\
&\leq (b - \overline{L}_\Delta(f,h))\left(\phi[a,b] - \phi'(b)\right) + \sum_{k=2}^{m-1} \frac{\phi^{(k)}(b)}{k!} \overline{L}_\Delta\left((f - b\mathbf{1})^k, h\right)\\
&\quad + \sum_{k=1}^{n-m} \phi[\underbrace{b,\ldots,b}_{m \text{ times}};\underbrace{a,\ldots,a}_{k \text{ times}}] \overline{L}_\Delta\left((f - b\mathbf{1})^m (f - a\mathbf{1})^{k-1}, h\right).
\end{aligned}$$

Inequality (6.18) also holds when the function ϕ is n-concave and m is even. In case when the function ϕ is n-convex and m is even, or when the function ϕ is n-concave and m is odd, the inequality sign in (6.18) is reversed.

Proof. In a similar manner as in the proof of the previous theorem, we can replace t with $f(t)$ in (2.69), multiply the obtained inequality by $\frac{h(t)}{\int_{\mathscr{E}} h(t)\Delta t}$ and then integrate the resulting inequality. In that way we get

$$\begin{aligned}
&\overline{L}_\Delta(\phi(f),h) - \alpha_\phi \overline{L}_\Delta(f,h) - \beta_\phi \qquad (6.19)\\
&= (b - \overline{L}_\Delta(f,h))\left(\phi[a,b] - \phi'(b)\right) + \sum_{k=2}^{m-1} \frac{\phi^{(k)}(b)}{k!} \cdot \frac{\int_{\mathscr{E}} (f(t) - b)^k h(t)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}\\
&\quad + \sum_{k=1}^{n-m} \phi[\underbrace{b,\ldots,b}_{m \text{ times}};\underbrace{a,\ldots,a}_{k \text{ times}}] \frac{\int_{\mathscr{E}} (f(t) - b)^m (f(t) - a)^{k-1} h(t)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t} + \frac{\int_{\mathscr{E}} R_m^*(f(t))h(t)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}.
\end{aligned}$$

Next, we study positivity and negativity of the term

$$\frac{\int_{\mathscr{E}} R_m^*(f(t))h(t)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}.$$

Again, it is enough to study positivity and negativity of the function:

$$R_m^*(f(t)) = (f(t) - b)^m (f(t) - a)^{n-m} \phi[f(t);\underbrace{b,\ldots,b}_{m \text{ times}};\ \underbrace{a,a,\ldots,a}_{(n-m) \text{ times}}].$$

Since $f(t) \in [a,b]$, we have $(f(t) - a)^{n-m} \geq 0$ for every t and any choice of m. For the same reason we have $(f(t) - b) \leq 0$. Trivially it follows that $(f(t) - b)^m \leq 0$ when m is odd, and $(f(t) - b)^m \geq 0$ when m is even. If the function ϕ is n-convex, then its n-th order divided differences are greater of equal to zero, and if the function ϕ is n-concave, then its n-th order divided differences are less or equal to zero. Now (6.18) easily follows from (6.19). □

The following corollary is a direct consequence of the previous two theorems, and it provides with a lower and an upper bound for the difference in the Edmundson-Lah-Ribarič inequality for time scales.

Corollary 6.1 *Let $\phi \in \mathscr{C}^n([a,b])$ be an n-convex function. Assume $\mathscr{E}$ is as in Theorem 6.4 and suppose f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = [a,b]$. Moreover, let $h : \mathscr{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t > 0$. If $m \geq 3$ is odd and $m < n$, then*

$$
\begin{aligned}
&(\overline{L}_\Delta(f,h) - a)\left(\phi'(a) - \phi[a,b]\right) + \sum_{k=2}^{m-1} \frac{\phi^{(k)}(a)}{k!} \overline{L}_\Delta((f - a\mathbf{1})^k, h) \\
&\qquad + \sum_{k=1}^{n-m} \phi[\underbrace{a,\dots,a}_{m \text{ times}}; \underbrace{b,\dots,b}_{k \text{ times}}] \overline{L}_\Delta((f - a\mathbf{1})^m (f - b\mathbf{1})^{k-1}, h) \\
&\leq \overline{L}_\Delta(\phi(f),h) - \alpha_\phi \overline{L}_\Delta(f,h) - \beta_\phi \qquad (6.20)\\
&\leq (b - \overline{L}_\Delta(f,h))\left(\phi[a,b] - \phi'(b)\right) + \sum_{k=2}^{m-1} \frac{\phi^{(k)}(b)}{k!} \overline{L}_\Delta((f - b\mathbf{1})^k, h) \\
&\qquad + \sum_{k=1}^{n-m} \phi[\underbrace{b,\dots,b}_{m \text{ times}}; \underbrace{a,\dots,a}_{k \text{ times}}] \overline{L}_\Delta((f - b\mathbf{1})^m (f - a\mathbf{1})^{k-1}, h).
\end{aligned}
$$

Inequality (6.20) also holds when the function ϕ is n-concave and m is even. In case when the function ϕ is n-convex and m is even, or when the function ϕ is n-concave and m is odd, the inequality signs in (6.20) are reversed.

In our next result we establish another set of bounds for the difference in the Edmundson-Lah-Ribarič inequality. It is obtained from Lemma 2.3.

Theorem 6.14 ([108]) *Let $\phi \in \mathscr{C}^n([a,b])$ be an n-convex function. Assume $\mathscr{E}$ is as in Theorem 6.4 and suppose f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = [a,b]$. Moreover, let $h : \mathscr{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t > 0$. If $n \geq 3$ is odd, then*

$$
\begin{aligned}
&\sum_{k=2}^{n-1} \phi[a; \underbrace{b,\dots,b}_{k \text{ times}}] \overline{L}_\Delta((f - a\mathbf{1})(f - b\mathbf{1})^{k-1}, h) \\
&\leq \overline{L}_\Delta(\phi(f),h) - \alpha_\phi \overline{L}_\Delta(f,h) - \beta_\phi \leq \phi[a,a;b] \overline{L}_\Delta((f - a\mathbf{1})(f - b\mathbf{1}), h) \qquad (6.21)\\
&\qquad + \sum_{k=2}^{n-2} \phi[a,a; \underbrace{b,\dots,b}_{k \text{ times}}] \overline{L}_\Delta((f - a\mathbf{1})^2 (f - b\mathbf{1})^{k-1}, h).
\end{aligned}
$$

Inequalities (6.21) also hold when the function ϕ is n-concave and n is even. In case when the function ϕ is n-convex and n is even, or when the function ϕ is n-concave and n is odd, the inequality signs in (6.21) are reversed.

Proof. Again, we can replace t with $f(t)$ in (2.62) and (2.63), multiply the obtained inequality by $\frac{h(t)}{\int_{\mathscr{E}} h(t)\Delta t}$ and then integrate the resulting inequality. In that way we get

$$\overline{L}_\Delta(\phi(f),h) - \alpha_\phi \overline{L}_\Delta(f,h) - \beta_\phi \tag{6.22}$$
$$= \sum_{k=2}^{n-1} \phi[a;\underbrace{b,\dots,b}_{k \text{ times}}] \frac{\int_{\mathscr{E}} h(t)(f(t)-a)(f(t)-b)^{k-1}\Delta t}{\int_{\mathscr{E}} h(t)\Delta t} + \frac{\int_{\mathscr{E}} h(t)R_1(f(t))\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}$$

and

$$\overline{L}_\Delta(\phi(f),h) - \alpha_\phi \overline{L}_\Delta(f,h) - \beta_\phi = \phi[a,a;b] \frac{\int_{\mathscr{E}} h(t)(f(t)-a)(f(t)-b)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}$$
$$+ \sum_{k=2}^{n-2} \phi[a,a;\underbrace{b,\dots,b}_{k \text{ times}}] \frac{\int_{\mathscr{E}} h(t)(f(t)-a)^2(f(t)-b)^{k-1}\Delta t}{\int_{\mathscr{E}} h(t)\Delta t} + \frac{\int_{\mathscr{E}} h(t)R_2(f(t))\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}. \tag{6.23}$$

From the discussion about positivity and negativity of the term

$$\overline{L}_\Delta(R_m(f),h) = \frac{\int_{\mathscr{E}} h(t)R_m(f(t))\Delta t}{\int_{\mathscr{E}} h(t)\Delta t},$$

that is, about positivity and negativity of the function $R_m(f(t))$ in the proof of Theorem 6.12, for $m = 1$ it follows that

* $\overline{L}_\Delta(R_1(f),h) \geq 0$ when the function ϕ is n-convex and n is odd, or when ϕ is n-concave and n even;
* $\overline{L}_\Delta(R_1(f),h) \leq 0$ when the function ϕ is n-concave and n is odd, or when ϕ is n-convex and n even.

Now the relation (6.22) becomes inequality

$$\overline{L}_\Delta(\phi(f),h) - \alpha_\phi \overline{L}_\Delta(f,h) - \beta_\phi \geq \sum_{k=2}^{n-1} \phi[a;\underbrace{b,\dots,b}_{k \text{ times}}] \overline{L}_\Delta((f-a\mathbf{1})(f-b\mathbf{1})^{k-1},h)$$

that holds for $\overline{L}_\Delta(R_1(f),h) \geq 0$, and in case $\overline{L}_\Delta(R_1(f),h) \leq 0$ the inequality sign is reversed.

In the same manner, for $m = 2$ it follows that

* $\overline{L}_\Delta(R_2(f),h) \leq 0$ when the function ϕ is n-convex and n is odd, or when ϕ is n-concave and n even;
* $\overline{L}_\Delta(R_2(f),h) \geq 0$ when the function ϕ is n-concave and n is odd, or when ϕ is n-convex and n even.

In this case the relation (6.23) for $\overline{L}_\Delta(R_2(f),h) \leq 0$ gives us

$$\overline{L}_\Delta(\phi(f),h) - \alpha_\phi \overline{L}_\Delta(f,h) - \beta_\phi \leq \phi[a,a;b] \overline{L}_\Delta((f-a\mathbf{1})(f-b\mathbf{1}),h)$$
$$+ \sum_{k=2}^{n-2} \phi[a,a;\underbrace{b,\dots,b}_{k \text{ times}}] \overline{L}_\Delta((f-a\mathbf{1})^2(f-b\mathbf{1})^{k-1},h),$$

and when $\overline{L}_\Delta(R_2(f),h) \geq 0$ the inequality sign is reversed.

When we combine the two inequalities obtained above, we get exactly (6.21). □

By utilizing Lemma 2.4 we can get a similar lower and upper bound for the difference in the Edmundson-Lah-Ribarič inequality that holds for all $n \in \mathbb{N}$, not only the odd ones.

Theorem 6.15 ([108]) *Let $\phi \in \mathscr{C}^n([a,b])$ be an n-convex function, $n \geq 3$. Assume $\mathscr{E}$ is as in Theorem 6.4 and suppose f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = [a,b]$. Moreover, let $h : \mathscr{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t > 0$. Then we have*

$$\begin{aligned}
&\phi[b,b;a]\overline{L}_\Delta((f-b\mathbf{1})(f-a\mathbf{1}),h) + \sum_{k=2}^{n-2} \phi[b,b;\underbrace{a,\dots,a}_{k\ times}]\overline{L}_\Delta((f-b\mathbf{1})^2(f-a\mathbf{1})^{k-1},h) \\
&\leq \overline{L}_\Delta(\phi(f),h) - \alpha_\phi \overline{L}_\Delta(f,h) - \beta_\phi \leq \sum_{k=2}^{n-1} \phi[b;\underbrace{a,\dots,a}_{k\ times}]\overline{L}_\Delta((f-b\mathbf{1})(f-a\mathbf{1})^{k-1},h) \quad (6.24)
\end{aligned}$$

If the function ϕ is n-concave, the inequality signs in (6.24) are reversed.

Proof. We follow the lines from the proof of Theorem 6.14, with the difference that we start with equalities (2.66) and (2.67) from Lemma 2.4, and then we return to the discussion about positivity and negativity of the term $\overline{L}_\Delta(R_m^*(f),h)$ from the proof of Theorem 6.13 for $m = 1$ and $m = 2$. □

In the rest of this section we will utilize the results from above, as well Lemma 2.3 and Lemma 2.4, in order to obtain some Jensen-type inequalities that hold for n-convex functions.

Our first result is a consequence of Corollary 6.1, and it provides with a lower and an upper bound for the difference in the Jensen inequality for time scales (6.1).

Theorem 6.16 ([108]) *Let $\phi \in \mathscr{C}^n([a,b])$ be an n-convex function. Assume $\mathscr{E}$ is as in Theorem 6.4 and suppose f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = [a,b]$. Moreover, let $h : \mathscr{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t > 0$. If $m \geq 3$ is odd and $m < n$, then*

$$\begin{aligned}
&\phi(a) - \phi(b) + b\phi'(b) - a\phi'(a) + (\phi'(a) - \phi'(b))\overline{L}_\Delta(f,h) \\
&\quad + \sum_{k=2}^{m-1} \left(\frac{\phi^{(k)}(a)}{k!}\overline{L}_\Delta((f-a\mathbf{1})^k,h) - \frac{\phi^{(k)}(b)}{k!}(\overline{L}_\Delta(f,h)-b)^k \right) \\
&\quad + \sum_{k=1}^{n-m} \phi[\underbrace{a,\dots,a}_{m\ times};\underbrace{b,\dots,b}_{k\ times}]\overline{L}_\Delta((f-a\mathbf{1})^m(f-b\mathbf{1})^{k-1},h) \\
&\quad - \sum_{k=1}^{n-m} \phi[\underbrace{b,\dots,b}_{m\ times};\underbrace{a,\dots,a}_{k\ times}](\overline{L}_\Delta(f,h)-b)^m(\overline{L}_\Delta(f,h)-a)^{k-1} \\
&\leq \overline{L}_\Delta(\phi(f),h) - \phi(\overline{L}_\Delta(f,h)) \quad (6.25)
\end{aligned}$$

$$
\begin{aligned}
&\leq \phi(b) - \phi(a) + a\phi'(a) - b\phi'(b) + (\phi'(b) - \phi'(a))\overline{L}_\Delta(f,h) \\
&\quad + \sum_{k=2}^{m-1}\left(\frac{\phi^{(k)}(b)}{k!}\overline{L}_\Delta((f-b\mathbf{1})^k,h) - \frac{\phi^{(k)}(a)}{k!}(\overline{L}_\Delta(f,h)-a)^k\right) \\
&\quad + \sum_{k=1}^{n-m}\phi[\underbrace{b,\dots,b}_{m\ times};\underbrace{a,\dots,a}_{k\ times}]\overline{L}_\Delta((f-b\mathbf{1})^m(f-a\mathbf{1})^{k-1},h) \\
&\quad - \sum_{k=1}^{n-m}\phi[\underbrace{a,\dots,a}_{m\ times};\underbrace{b,\dots,b}_{k\ times}](\overline{L}_\Delta(f,h)-a)^m(\overline{L}_\Delta(f,h)-b)^{k-1}
\end{aligned}
$$

Inequalities (6.25) also hold when the function ϕ is n-concave and m is even. In case when the function ϕ is n-convex and m is even, or when the function ϕ is n-concave and m is odd, the inequality signs in (6.25) are reversed.

Proof. Because $f(\mathscr{E}) = [a,b]$, we have $ah(t) \leq f(t)h(t) \leq bh(t)$, and consequently $\overline{L}_\Delta(f,h) \in [a,b]$, so we can substitute t with $\overline{L}_\Delta(f,h)$ in (2.65) and obtain

$$
\begin{aligned}
&\phi(\overline{L}_\Delta(f,h)) - \alpha_\phi\overline{L}_\Delta(f,h) - \beta_\phi \qquad (6.26)\\
&= (\overline{L}_\Delta(f,h)-a)\left(\phi'(a) - \phi[a,b]\right) + \sum_{k=2}^{m-1}\frac{\phi^{(k)}(a)}{k!}(\overline{L}_\Delta(f,h)-a)^k \\
&\quad + \sum_{k=1}^{n-m}\phi[\underbrace{a,\dots,a}_{m\text{ times}};\underbrace{b,\dots,b}_{k\text{ times}}](\overline{L}_\Delta(f,h)-a)^m(\overline{L}_\Delta(f,h)-b)^{k-1} + R_m(\overline{L}_\Delta(f,h)).
\end{aligned}
$$

We need to study positivity and negativity of the term:

$$
R_m(\overline{L}_\Delta(f,h)) = \left(\overline{L}_\Delta(f,h)-a\right)^m\left(\overline{L}_\Delta(f,h)-b\right)^{n-m}\phi[\overline{L}_\Delta(f,h);\underbrace{a,\dots,a}_{m\text{ times}};\ \underbrace{b,b,\dots,b}_{(n-m)\text{ times}}].
$$

Since $\overline{L}_\Delta(f,h) \in [a,b]$, we have $\left(\overline{L}_\Delta(f,h)-a\right)^m \geq 0$ for any choice of m, and $(\overline{L}_\Delta(f,h)-b)^{n-m} \leq 0$ when n and m are of different parity, and $(\overline{L}_\Delta(f,h)-b)^{n-m} \geq 0$ when n and m are of equal parity.

If the function ϕ is n-convex, then $\phi[\overline{L}_\Delta(f,h);\underbrace{a,\dots,a}_{m\text{ times}};\ \underbrace{b,b,\dots,b}_{(n-m)\text{ times}}] \geq 0$, and if the function ϕ is n-concave, then the inequality sign is reversed.

Now the relation (6.26) for n-convex function ϕ and n and $m \geq 3$ of different parity, or n-concave function ϕ and n and $m \geq 3$ of the same parity, becomes

$$
\begin{aligned}
&\phi(\overline{L}_\Delta(f,h)) - \alpha_\phi\overline{L}_\Delta(f,h) - \beta_\phi \qquad (6.27)\\
&\leq (\overline{L}_\Delta(f,h)-a)\left(\phi'(a) - \phi[a,b]\right) + \sum_{k=2}^{m-1}\frac{\phi^{(k)}(a)}{k!}(\overline{L}_\Delta(f,h)-a)^k \\
&\quad + \sum_{k=1}^{n-m}\phi[\underbrace{a,\dots,a}_{m\text{ times}};\underbrace{b,\dots,b}_{k\text{ times}}](\overline{L}_\Delta(f,h)-a)^m(\overline{L}_\Delta(f,h)-b)^{k-1},
\end{aligned}
$$

and for n-convex function ϕ and n and $m \geq 3$ of the same parity, or n-concave function ϕ and n and $m \geq 3$ of different parity, the inequality sign is reversed.

In the same way we can replace t with $\overline{L}_\Delta(f,h)$ in (2.69) and get

$$\begin{aligned}
&\phi(\overline{L}_\Delta(f,h)) - \alpha_\phi \overline{L}_\Delta(f,h) - \beta_\phi \qquad (6.28)\\
&= (b - \overline{L}_\Delta(f,h))\left(\phi[a,b] - \phi'(b)\right) + \sum_{k=2}^{m-1} \frac{\phi^{(k)}(b)}{k!}(\overline{L}_\Delta(f,h) - b)^k\\
&\quad + \sum_{k=1}^{n-m} \phi[\underbrace{b,\dots,b}_{m \text{ times}};\underbrace{a,\dots,a}_{k \text{ times}}](\overline{L}_\Delta(f,h) - b)^m(\overline{L}_\Delta(f,h) - a)^{k-1} + R_m^*(\overline{L}_\Delta(f,h)).
\end{aligned}$$

As before, we study positivity and negativity of the term $R_m^*(\overline{L}_\Delta(f,h))$:

$$R_m^*(\overline{L}_\Delta(f,h)) = (\overline{L}_\Delta(f,h) - b)^m(\overline{L}_\Delta(f,h) - a)^{n-m}\phi[\overline{L}_\Delta(f,h);\underbrace{b,\dots,b}_{m \text{ times}};\underbrace{a,a,\dots,a}_{(n-m) \text{ times}}].$$

Again, since $\overline{L}_\Delta(f,h) \in [a,b]$, we have $\left(\overline{L}_\Delta(f,h) - a\right)^{n-m} \geq 0$ for any choice of m, and $(\overline{L}_\Delta(f,h) - b)^m \leq 0$ when m is odd, and $(\overline{L}_\Delta(f,h) - b)^m \geq 0$ when m is even. If the function ϕ is n-convex, then its n-th order divided differences are greater of equal to zero, and if the function ϕ is n-concave, then its n-th order divided differences are less or equal to zero.

Equality (6.28) now turns into

$$\begin{aligned}
&\phi(\overline{L}_\Delta(f,h)) - \alpha_\phi \overline{L}_\Delta(f,h) - \beta_\phi \qquad (6.29)\\
&\leq (b - \overline{L}_\Delta(f,h))\left(\phi[a,b] - \phi'(b)\right) + \sum_{k=2}^{m-1} \frac{\phi^{(k)}(b)}{k!}(\overline{L}_\Delta(f,h) - b)^k\\
&\quad + \sum_{k=1}^{n-m} \phi[\underbrace{b,\dots,b}_{m \text{ times}};\underbrace{a,\dots,a}_{k \text{ times}}](\overline{L}_\Delta(f,h) - b)^m(\overline{L}_\Delta(f,h) - a)^{k-1}
\end{aligned}$$

for n-convex function ϕ and an odd number $m \geq 3$ or n-concave function ϕ and an even number $m \geq 3$. If ϕ is n-convex and m is even, or if ϕ is n-concave and m is odd, the inequality is reversed.

By combining inequalities (6.27) and (6.29) we get that

$$\begin{aligned}
&(\overline{L}_\Delta(f,h) - a)\left(\phi'(a) - \phi[a,b]\right) + \sum_{k=2}^{m-1} \frac{\phi^{(k)}(a)}{k!}(\overline{L}_\Delta(f,h) - a)^k\\
&\quad + \sum_{k=1}^{n-m} \phi[\underbrace{a,\dots,a}_{m \text{ times}};\underbrace{b,\dots,b}_{k \text{ times}}](\overline{L}_\Delta(f,h) - a)^m(\overline{L}_\Delta(f,h) - b)^{k-1}\\
&\leq \phi(\overline{L}_\Delta(f,h)) - \alpha_\phi \overline{L}_\Delta(f,h) - \beta_\phi \qquad (6.30)\\
&\leq (b - \overline{L}_\Delta(f,h))\left(\phi[a,b] - \phi'(b)\right) + \sum_{k=2}^{m-1} \frac{\phi^{(k)}(b)}{k!}(\overline{L}_\Delta(f,h) - b)^k\\
&\quad + \sum_{k=1}^{n-m} \phi[\underbrace{b,\dots,b}_{m \text{ times}};\underbrace{a,\dots,a}_{k \text{ times}}](\overline{L}_\Delta(f,h) - b)^m(\overline{L}_\Delta(f,h) - a)^{k-1}
\end{aligned}$$

holds if n is odd and ϕ is n-convex and m is odd, or ϕ is n-concave and m is even. If ϕ is n-convex and m is even, or ϕ is n-concave and m is odd, then the inequality signs are reversed.

When we multiply series of inequalities (6.30) by -1 and add to (6.20), we get exactly (6.25), and the proof is complete. □

Next result also provides with a lower and upper bound for the difference in the Jensen inequality for time scales, and it is obtained from Theorem 6.14 and Lemma 2.3.

Theorem 6.17 ([108]) *Let $\phi \in \mathscr{C}^n([a,b])$ be an n-convex function. Assume $\mathscr{E}$ is as in Theorem 6.4 and suppose f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = [a,b]$. Moreover, let $h : \mathscr{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t > 0$. If $n \geq 3$ is odd, then*

$$
\begin{aligned}
&\phi[a,a;b](b-\overline{L}_\Delta(f,h))(\overline{L}_\Delta(f,h)-a)+\sum_{k=2}^{n-1}\phi[a;\underbrace{b,\dots,b}_{k\ times}]\overline{L}_\Delta((f-a\mathbf{1})(f-b\mathbf{1})^{k-1},h)\\
&\qquad-(\overline{L}_\Delta(f,h)-a)^2\sum_{k=2}^{n-2}\phi[a,a;\underbrace{b,\dots,b}_{k\ times}](\overline{L}_\Delta(f,h)-b)^{k-1}\\
&\leq \overline{L}_\Delta(\phi(f),h)-\phi(\overline{L}_\Delta(f,h)) \qquad (6.31)\\
&\leq \phi[a,a;b]\overline{L}_\Delta((f-a\mathbf{1})(f-b\mathbf{1}),h)-(\overline{L}_\Delta(f,h)-a)\sum_{k=2}^{n-1}\phi[a;\underbrace{b,\dots,b}_{k\ times}](\overline{L}_\Delta(f,h)-b)^{k-1}\\
&\qquad+\sum_{k=2}^{n-2}\phi[a,a;\underbrace{b,\dots,b}_{k\ times}]\overline{L}_\Delta((f-a\mathbf{1})^2(f-b\mathbf{1})^{k-1},h).
\end{aligned}
$$

Inequalities (6.31) also hold when the function ϕ is n-concave and n is even. In case when the function ϕ is n-convex and n is even, or when the function ϕ is n-concave and n is odd, the inequality signs in (6.31) are reversed.

Proof. By following a similar procedure as in the proof of the previous theorem, we start by replacing t with $\overline{L}_\Delta(f,h)$ in with relations (2.62) and (2.63) from Lemma 2.3. We get

$$
\begin{aligned}
&\phi(\overline{L}_\Delta(f,h))-\alpha_\phi\overline{L}_\Delta(f,h)-\beta_\phi\\
&\quad=\sum_{k=2}^{n-1}\phi[a;\underbrace{b,\dots,b}_{k\text{ times}}](\overline{L}_\Delta(f,h)-a)(\overline{L}_\Delta(f,h)-b)^{k-1}+R_1(\overline{L}_\Delta(f,h)) \qquad (6.32)
\end{aligned}
$$

and

$$\begin{aligned}
&\phi(\overline{L}_\Delta(f,h)) - \alpha_\phi \overline{L}_\Delta(f,h) - \beta_\phi \qquad (6.33)\\
&= \phi[a,a;b](\overline{L}_\Delta(f,h)-a)(\overline{L}_\Delta(f,h)-b)\\
&\quad + \sum_{k=2}^{n-2} \phi[a,a;\underbrace{b,\dots,b}_{k \text{ times}}](\overline{L}_\Delta(f,h)-a)^2(\overline{L}_\Delta(f,h)-b)^{k-1} + R_2(\overline{L}_\Delta(f,h))
\end{aligned}$$

respectively. After discussing the positivity an negativity of terms $R_1(\overline{L}_\Delta(f,h))$ and $R_2(\overline{L}_\Delta(f,h))$ in the same way as in the proof Theorem 6.16, from relations (6.32) and (6.33) we get a series of inequalities

$$(\overline{L}_\Delta(f,h)-a)\sum_{k=2}^{n-1}\phi[a;\underbrace{b,\dots,b}_{k \text{ times}}](\overline{L}_\Delta(f,h)-b)^{k-1} \le \phi(\overline{L}_\Delta(f,h)) - \alpha_\phi \overline{L}_\Delta(f,h) - \beta_\phi \quad (6.34)$$

$$\le \phi[a,a;b](\overline{L}_\Delta(f,h)-a)(\overline{L}_\Delta(f,h)-b) + (\overline{L}_\Delta(f,h)-a)^2\sum_{k=2}^{n-2}\phi[a,a;\underbrace{b,\dots,b}_{k \text{ times}}](\overline{L}_\Delta(f,h)-b)^{k-1}$$

that holds when n is odd and ϕ is n-convex, or when n is even and ϕ is n-concave. If n is odd and ϕ is n-concave, or if n is even and ϕ is n-convex, then the inequality signs in (6.34) are reversed.

Inequalities (6.31) are obtained after multiplying (6.34) by -1 and adding it to (6.21). □

In the analogous way as described in the proof of the previous theorem, but with utilizing Lemma 2.4 and Theorem 6.15, we can get a similar lower and upper bound for the difference in the Jensen inequality (6.1) that holds for all $n \in \mathbb{N}$, not only the odd ones.

Theorem 6.18 ([108]) *Let $\phi \in \mathscr{C}^n([a,b])$ be an n-convex function, $n \ge 3$. Assume $\mathscr{E}$ is as in Theorem 6.4 and suppose f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = [a,b]$. Moreover, let $h : \mathscr{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t > 0$. We have*

$$\begin{aligned}
&\phi[b,b;a]\overline{L}_\Delta((f-b\mathbf{1})(f-a\mathbf{1}),h) - (\overline{L}_\Delta(f,h)-b)\sum_{k=2}^{n-1}\phi[b;\underbrace{a,\dots,a}_{k\ times}](\overline{L}_\Delta(f,h)-a)^{k-1}\\
&\quad + \sum_{k=2}^{n-2}\phi[b,b;\underbrace{a,\dots,a}_{k\ times}]\overline{L}_\Delta((f-b\mathbf{1})^2(f-a\mathbf{1})^{k-1},h)\\
&\le \overline{L}_\Delta(\phi(f),h) - \phi(\overline{L}_\Delta(f,h)) \qquad (6.35)\\
&\le f[b,b;a](b-\overline{L}_\Delta(f,h))(\overline{L}_\Delta(f,h)-a) + \sum_{k=2}^{n-1}\phi[b;\underbrace{a,\dots,a}_{k\ times}]\overline{L}_\Delta((f-b\mathbf{1})(f-a\mathbf{1})^{k-1},h)\\
&\quad - (\overline{L}_\Delta(f,h)-b)^2\sum_{k=2}^{n-2}\phi[b,b;\underbrace{a,\dots,a}_{k\ times}](\overline{L}_\Delta(f,h)-a)^{k-1}.
\end{aligned}$$

If the function ϕ is n-concave, the inequality signs in (6.35) are reversed.

6.4 Applications

In this section, we use the results obtained in the previous sections from this chapter in order to get new converse inequalities for generalized means, power means and the Hölder inequality in the time scale setting.

6.4.1 Generalized means

Let us define the generalized mean in terms of the multiple Lebesgue delta time scale integral using the definition of weighted generalized mean on time scales proved in [7].

Definition 6.4 *Suppose $\Psi : I \to \mathbb{R}$ is continuous and strictly monotone and f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = I$, where $\mathscr{E} \subset \mathbb{R}^n$ is as in Theorem 6.4. Let $h : \mathscr{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $\int\limits_{\mathscr{E}} h(t)\Delta t > 0$. The generalized mean with respect to the multiple Lebesgue delta time scale integral is defined by*

$$M_\Psi\left(f, \overline{L}_\Delta(f,h)\right) = \Psi^{-1}\left(\overline{L}_\Delta(\Psi(f),h)\right). \tag{6.36}$$

Theorem 6.19 ([13]) *Suppose $I = [m,M]$, $-\infty < m < M < \infty$, $\psi, \chi : I \to \mathbb{R}$ are continuous and strictly monotone and $\phi = \chi \circ \psi^{-1}$ is convex. Assume $\mathscr{E} \subset \mathbb{R}^n$ is as in Theorem 6.4 and $f,h : \mathscr{E} \to \mathbb{R}$ are Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = I$ and $\int_{\mathscr{E}} |h(t)|\Delta t > 0$. Then,*

$$\begin{aligned}
0 &\le \chi\left(M_\chi\left(f,\overline{L}_\Delta\right)\right) - \chi\left(M_\psi\left(f,\overline{L}_\Delta\right)\right) \\
&\le \left(M_\psi - \overline{L}_\Delta(\psi(f),h)\right)\left(\overline{L}_\Delta(\psi(f),h) - m_\psi\right) \sup_{t\in\langle m,M\rangle} \Psi_{\chi\circ\psi^{-1}}(\psi(t); m_\psi, M_\psi) \\
&\le \left(M_\psi - \overline{L}_\Delta(\psi(f),h)\right)\left(\overline{L}_\Delta(\psi(f),h) - m_\psi\right) \\
&\quad \cdot \frac{(\chi\circ\psi^{-1})'_-(M_\psi) - (\chi\circ\psi^{-1})'_+(m_\psi)}{M_\psi - m_\psi} \\
&\le \frac{1}{4}\left(M_\psi - m_\psi\right)\left((\chi\circ\psi^{-1})'_-(M_\psi) - (\chi\circ\psi^{-1})'_+(m_\psi)\right),
\end{aligned} \tag{6.37}$$

where $[m_\psi, M_\psi] = \psi([m,M])$. If ϕ is concave, then all inequalities in (6.37) *are reversed.*

Proof. The claim follows from Theorem 6.4, Theorem 6.10 and Theorem 1.17. □

Theorem 6.20 ([12]) *Let all assumptions from Theorem 6.19 be valid. If the function $\phi = \chi \circ \psi^{-1}$ is convex, then*

$$\begin{aligned}
0 \le\ & \frac{M_\Psi - \overline{L}_\Delta(\Psi(f),h)}{M_\psi - m_\psi}\phi(m) + \frac{\overline{L}_\Delta(\Psi(f),h) - m_\Psi}{M_\psi - m_\psi}\phi(M) \\
& -\chi\left(M_\chi\left(f,\overline{L}_\Delta(\Psi(f),h)\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\phi'_-(M_\psi)-\phi'_+(m_\psi)}{M_\psi - m_\psi} \cdot \frac{\int_{\mathscr{E}} h(t)\left(M_\psi - \psi(f(t))\right)\left(\psi(f(t)) - m_\psi\right)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t} \\
&\leq \frac{\phi'_-(M_\psi)-\phi'_+(m_\psi)}{M_\psi - m_\psi} \cdot \left(M_\psi - \overline{L}_\Delta(\Psi(f),h)\right)\left(\overline{L}_\Delta(\Psi(f),h) - m_\psi\right) \qquad (6.38) \\
&\leq \frac{1}{4}\left(M_\psi - m_\psi\right)\left(\phi'_-(M_\psi)-\phi'_+(m_\psi)\right).
\end{aligned}
$$

where $[m_\psi, M_\psi] = \psi([m,M])$. *If* ϕ *is concave on* I, *then all inequalities in* (6.38) *are reversed.*

Proof. The inequalities in (6.38) follow directly from Theorem 6.9 by replacing m by m_ψ, M by M_ψ, ϕ by $\chi \circ \psi^{-1}$, and f by $\psi \circ f$. All conditions of Theorem 6.9 are satisfied because $\chi \circ \psi^{-1}$ is obviously continuous and convex by assumption. Also, we have $m_\psi \leq \psi(f(t)) \leq M_\psi$ for every $t \in [m,M]$ since $m_\psi = \psi(m)$ and $M_\psi = \psi(M)$ if ψ is increasing and $m_\psi = \psi(M)$ and $M_\psi = \psi(m)$ if ψ is decreasing. If the function $\phi = \chi \circ \psi^{-1}$ is concave, then the function $-\phi = -\chi \circ \psi^{-1}$ is convex so, replacing ϕ by $-\phi$ in (6.38), we obtain the reversed inequalities. □

Next, our intention is to obtain some new reverse relations for generalized means that correspond to n-convex functions and in that way get some mutual bounds for generalized means.

Before we state such results, we have to introduce some notations arising from this particular setting. Throughout this section we denote

$$
\Phi = \chi \circ \psi^{-1}, \quad \alpha_\Phi = \frac{\chi(b)-\chi(a)}{\psi(b)-\psi(a)}, \quad \beta_\Phi = \frac{\psi(b)\chi(a)-\psi(a)\chi(b)}{\psi(b)-\psi(a)}
$$

and

$$
\psi_a = \min\{\psi(a), \psi(b)\}, \quad \psi_b = \max\{\psi(a), \psi(b)\},
$$

where χ and ψ are strictly monotone functions. It is obvious that if the function ψ is increasing, then $\psi_a = \psi(a)$, $\psi_b = \psi(b)$, and if ψ is decreasing, then $\psi_a = \psi(b)$, $\psi_b = \psi(a)$.

Since for a Δ-integrable function f on $\mathscr{E}$ such that $f(\mathscr{E}) = [a,b]$ we have $\psi(f(\mathscr{E})) = [\psi_a, \psi_b]$, all of the results from previous sections can be exploited in establishing some new reverses of Jensen's inequality and the Edmundson-Lah-Ribarič inequality for selfadjoint operators related to quasi-arithmetic means by substituting ϕ with $\Phi = \chi \circ \psi^{-1}$ and f with $\psi(f)$.

We start with some Edmundson-Lah-Ribarič type inequalities for generalized means which arise from the previous section. The first result of this type is carried out by virtue of our Theorem 6.12.

Corollary 6.2 ([108]) *Suppose* $\psi, \chi : [a,b] \to \mathbb{R}$ *are continuous and strictly monotone and* $\Phi = \chi \circ \psi^{-1} \in \mathscr{C}^n([a,b])$ *is n-convex. Assume* f *is* Δ*-integrable on* $\mathscr{E}$ *such that* $f(\mathscr{E}) =$

$[a,b]$, where $\mathscr{E} \subset \mathbb{R}^n$ is as in Theorem 6.4. Let $h : \mathscr{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t > 0$. If $n > m \geq 3$ are of different parity, then

$$\begin{aligned} &\chi\left(M_\chi(f,\overline{L}_\Delta(f,h))\right) - \alpha_\Phi \psi\left(M_\psi(f,\overline{L}_\Delta(f,h))\right) - \beta_\Phi \qquad (6.39)\\ &\leq (\overline{L}_\Delta(\psi(f),h) - \psi_a)\left(\Phi'(\psi_a) - \Phi[\psi_a,\psi_b]\right) + \sum_{k=2}^{m-1} \frac{\Phi^{(k)}(\psi_a)}{k!}\overline{L}_\Delta\left((\psi(f)-\psi_a\mathbf{1})^k,h\right)\\ &\quad + \sum_{k=1}^{n-m} \Phi[\underbrace{\psi_a,\dots,\psi_a}_{m\ times};\underbrace{\psi_b,\dots,\psi_b}_{k\ times}]\overline{L}_\Delta\left((\psi(f)-\psi_a\mathbf{1})^m(\psi(f)-\psi_b\mathbf{1})^{k-1},h\right). \end{aligned}$$

Inequality (6.39) also holds when the function Φ is n-concave and n and m are of equal parity. In case when the function Φ is n-convex and n and m are of equal parity, or when the function Φ is n-concave and n and m are of different parity, the inequality sign in (6.39) is reversed.

The following result is a direct consequence of Theorem 6.13.

Corollary 6.3 ([108]) *Suppose $\psi,\chi : [a,b] \to \mathbb{R}$ are continuous and strictly monotone and $\Phi = \chi \circ \psi^{-1} \in \mathscr{C}^n([a,b])$ is n-convex. Assume f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = [a,b]$, where $\mathscr{E} \subset \mathbb{R}^n$ is as in Theorem 6.4. Let $h : \mathscr{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t > 0$. For an odd number $m \geq 3$ such that $m < n$, we have*

$$\begin{aligned} &\chi\left(M_\chi(f,\overline{L}_\Delta(f,h))\right) - \alpha_\Phi \psi\left(M_\psi(f,\overline{L}_\Delta(f,h))\right) - \beta_\Phi \qquad (6.40)\\ &\leq (\psi_b - \overline{L}_\Delta(\psi(f),h))\left(\Phi[\psi_a,\psi_b] - \Phi'(\psi_b)\right) + \sum_{k=2}^{m-1} \frac{\Phi^{(k)}(\psi_b)}{k!}\overline{L}_\Delta((\psi(f)-\psi_b\mathbf{1})^k,h)\\ &\quad + \sum_{k=1}^{n-m} \Phi[\underbrace{\psi_b,\dots,\psi_b}_{m\ times};\underbrace{\psi_a,\dots,\psi_a}_{k\ times}]\overline{L}_\Delta(f,h)(\psi(f)-\psi_b\mathbf{1})^m(\psi(f)-\psi_a\mathbf{1})^{k-1}. \end{aligned}$$

Inequality (6.40) also holds when the function Φ is n-concave and m is even. In case when the function Φ is n-convex and m is even, or when the function Φ is n-concave and m is odd, the inequality sign in (6.40) is reversed.

Our next result arises from Theorem 6.14.

Corollary 6.4 ([108]) *Suppose $\psi,\chi : [a,b] \to \mathbb{R}$ are continuous and strictly monotone and $\Phi = \chi \circ \psi^{-1} \in \mathscr{C}^n([a,b])$ is n-convex. Assume f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = [a,b]$, where $\mathscr{E} \subset \mathbb{R}^n$ is as in Theorem 6.4. Let $h : \mathscr{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t > 0$. If $n \geq 3$ is odd, then*

$$\begin{aligned} &\sum_{k=2}^{n-1} \Phi[\psi_a;\underbrace{\psi_b,\dots,\psi_b}_{k\ times}]\overline{L}_\Delta((\psi(f)-\psi_a\mathbf{1})(\psi(f)-\psi_b\mathbf{1})^{k-1},h)\\ &\leq \chi\left(M_\chi(f,\overline{L}_\Delta(f,h))\right) - \alpha_\Phi \psi\left(M_\psi(f,\overline{L}_\Delta(f,h))\right) - \beta_\Phi \qquad (6.41) \end{aligned}$$

$$\leq \Phi[\psi_a,\psi_a;\psi_b]\overline{L}_\Delta((\psi(f)-\psi_a\mathbf{1})(\psi(f)-\psi_b\mathbf{1}),h)$$
$$+\sum_{k=2}^{n-2}\Phi[\psi_a,\psi_a;\underbrace{\psi_b,\ldots,\psi_b}_{k\ times}]\overline{L}_\Delta((\psi(f)-\psi_a\mathbf{1})^2(\psi(f)-\psi_b\mathbf{1})^{k-1},h).$$

Inequalities (6.41) also hold when the function Φ is n-concave and n is even. In case when the function Φ is n-convex and n is even, or when the function Φ is n-concave and n is odd, the inequality signs in (6.41) are reversed.

As a consequence of Theorem 6.15, we have the following result.

Corollary 6.5 ([108]) *Suppose $\psi,\chi:[a,b]\to\mathbb{R}$ are continuous and strictly monotone and $\Phi=\chi\circ\psi^{-1}\in\mathscr{C}^n([a,b])$ is n-convex, $n\geq 3$. Assume f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E})=[a,b]$, where $\mathscr{E}\subset\mathbb{R}^n$ is as in Theorem 6.4. Let $h:\mathscr{E}\to\mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t>0$. Then we have*

$$\Phi[\psi_b,\psi_b;\psi_a]\overline{L}_\Delta((\psi(f)-\psi_b\mathbf{1})(\psi(f)-\psi_a\mathbf{1}),h)$$
$$+\sum_{k=2}^{n-2}\Phi[\psi_b,\psi_b;\underbrace{\psi_a,\ldots,\psi_a}_{k\ times}]\overline{L}_\Delta((\psi(f)-\psi_b\mathbf{1})^2(\psi(f)-\psi_a\mathbf{1})^{k-1},h)$$
$$\leq \chi\left(M_\chi(f,\overline{L}_\Delta(f,h))\right)-\alpha_\Phi\psi\left(M_\psi(f,\overline{L}_\Delta(f,h))\right)-\beta_\Phi \tag{6.42}$$
$$\leq \sum_{k=2}^{n-1}\Phi[\psi_b;\underbrace{\psi_a,\ldots,\psi_a}_{k\ times}]\overline{L}_\Delta((\psi(f)-\psi_b\mathbf{1})(\psi(f)-\psi_a\mathbf{1})^{k-1},h).$$

If the function Φ is n-concave, the inequality signs in (6.42) are reversed.

The corollaries below arise from the Jensen-type inequalities on time scales for n-convex functions and give us Jensen type inequalities for quasi-arithmetic means. They are obtained from Theorem 6.16, 6.17 and 6.18 respectively.

Corollary 6.6 ([108]) *Suppose $\psi,\chi:[a,b]\to\mathbb{R}$ are continuous and strictly monotone and $\Phi=\chi\circ\psi^{-1}\in\mathscr{C}^n([a,b])$ is n-convex. Assume f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E})=[a,b]$, where $\mathscr{E}\subset\mathbb{R}^n$ is as in Theorem 6.4. Let $h:\mathscr{E}\to\mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t>0$. If $m\geq 3$ is odd and $m<n$, then*

$$\Phi(\psi_a)-\Phi(\psi_b)+\psi_b\Phi'(\psi_b)-\psi_a\Phi'(\psi_a)+(\Phi'(\psi_a)-\Phi'(\psi_b))\overline{L}_\Delta(\psi(f),h)$$
$$+\sum_{k=2}^{m-1}\left(\frac{\Phi^{(k)}(\psi_a)}{k!}\overline{L}_\Delta((\psi(f)-\psi_a\mathbf{1})^k,h)-\frac{\Phi^{(k)}(g_b)}{k!}(\overline{L}_\Delta(\psi(f),h)-\psi_b)^k\right)$$
$$+\sum_{k=1}^{n-m}\Phi[\underbrace{\psi_a,\ldots,\psi_a}_{m\ times};\underbrace{\psi_b,\ldots,\psi_b}_{k\ times}]\overline{L}_\Delta((\psi(f)-\psi_a\mathbf{1})^m(\psi(f)-\psi_b\mathbf{1})^{k-1},h)$$
$$-\sum_{k=1}^{n-m}\Phi[\underbrace{\psi_b,\ldots,\psi_b}_{m\ times};\underbrace{\psi_a,\ldots,\psi_a}_{k\ times}](\overline{L}_\Delta(\psi(f),h)-\psi_b)^m(\overline{L}_\Delta(\psi(f),h)-\psi_a)^{k-1}$$

$$
\begin{aligned}
&\leq \chi\left(M_\chi(f,\overline{L}_\Delta(f,h))\right) - \chi\left(M_\psi(f,\overline{L}_\Delta(f,h))\right) && (6.43)\\
&\leq \Phi(\psi_b) - \Phi(\psi_a) + \psi_a\Phi'(\psi_a) - \psi_b\Phi'(\psi_b) + (\Phi'(\psi_b) - \Phi'(\psi_a))\overline{L}_\Delta(\psi(f),h)\\
&\quad + \sum_{k=2}^{m-1}\left(\frac{\Phi^{(k)}(\psi_b)}{k!}\overline{L}_\Delta((\psi(f)-\psi_b\mathbf{1})^k,h) - \frac{\Phi^{(k)}(\psi_a)}{k!}(\overline{L}_\Delta(\psi(f),h)-\psi_a)^k\right)\\
&\quad + \sum_{k=1}^{n-m}\Phi[\underbrace{\psi_b,\dots,\psi_b}_{m \text{ times}};\underbrace{\psi_a,\dots,\psi_a}_{k \text{ times}}]\overline{L}_\Delta((\psi(f)-\psi_b\mathbf{1})^m(\psi(f)-\psi_a\mathbf{1})^{k-1},h)\\
&\quad - \sum_{k=1}^{n-m}\Phi[\underbrace{\psi_a,\dots,\psi_a}_{m \text{ times}};\underbrace{\psi_b,\dots,\psi_b}_{k \text{ times}}](\overline{L}_\Delta(\psi(f),h)-\psi_a)^m(\overline{L}_\Delta(\psi(f),h)-\psi_b)^{k-1}
\end{aligned}
$$

Inequalities (6.43) also hold when the function Φ is n-concave and m is even. In case when the function Φ is n-convex and m is even, or when the function Φ is n-concave and m is odd, the inequality signs in (6.43) are reversed.

Corollary 6.7 ([108]) *Suppose $\psi,\chi : [a,b] \to \mathbb{R}$ are continuous and strictly monotone and $\Phi = \chi\circ\psi^{-1} \in \mathscr{C}^n([a,b])$ is n-convex. Assume f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = [a,b]$, where $\mathscr{E} \subset \mathbb{R}^n$ is as in Theorem 6.4. Let $h : \mathscr{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t > 0$. If $n \geq 3$ is odd, then*

$$
\begin{aligned}
&\Phi[\psi_a,\psi_a;\psi_b](\psi_b - \overline{L}_\Delta(\psi(f),h))(\overline{L}_\Delta(\psi(f),h)-\psi_a)\\
&\quad + \sum_{k=2}^{n-1}\Phi[\psi_a;\underbrace{\psi_b,\dots,\psi_b}_{k \text{ times}}]\overline{L}_\Delta((\psi(f)-\psi_a\mathbf{1})(\psi(f)-\psi_b\mathbf{1})^{k-1},h)\\
&\quad - (\overline{L}_\Delta(\psi(f),h)-\psi_a)^2\sum_{k=2}^{n-2}\Phi[\psi_a,\psi_a;\underbrace{\psi_b,\dots,\psi_b}_{k \text{ times}}](\overline{L}_\Delta(\psi(f),h)-\psi_b)^{k-1}\\
&\leq \chi\left(M_\chi(f,\overline{L}_\Delta(f,h))\right) - \chi\left(M_\psi(f,\overline{L}_\Delta(f,h))\right) && (6.44)\\
&\leq \Phi[\psi_a,\psi_a;\psi_b]\overline{L}_\Delta((\psi(f)-\psi_a\mathbf{1})(\psi(f)-\psi_b\mathbf{1}),h)\\
&\quad - (\overline{L}_\Delta(\psi(f),h)-\psi_a)\sum_{k=2}^{n-1}\Phi[\psi_a;\underbrace{\psi_b,\dots,\psi_b}_{k \text{ times}}](\overline{L}_\Delta(\psi(f),h)-\psi_b)^{k-1}\\
&\quad + \sum_{k=2}^{n-2}\Phi[\psi_a,\psi_a;\underbrace{\psi_b,\dots,\psi_b}_{k \text{ times}}](\overline{L}_\Delta((\psi(f)-\psi_a\mathbf{1})^2(\psi(f)-\psi_b\mathbf{1})^{k-1},h).
\end{aligned}
$$

Inequalities (6.44) also hold when the function Φ is n-concave and n is even. In case when the function Φ is n-convex and n is even, or when the function Φ is n-concave and n is odd, the inequality signs in (6.44) are reversed.

Corollary 6.8 ([108]) *Suppose $\psi,\chi : [a,b] \to \mathbb{R}$ are continuous and strictly monotone and $\Phi = \chi\circ\psi^{-1} \in \mathscr{C}^n([a,b])$ is n-convex, $n \geq 3$. Assume f is Δ-integrable on $\mathscr{E}$ such* *that $f(\mathscr{E}) = [a,b]$, where $\mathscr{E} \subset \mathbb{R}^n$ is as in Theorem 6.4. Let $h : \mathscr{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t > 0$. Then we have*

$$
\begin{aligned}
&\Phi[\psi_b,\psi_b;\psi_a]\overline{L}_\Delta((\psi(f)-\psi_b\mathbf{1})(\psi(f)-\psi_a\mathbf{1}),h)\\
&\quad -(\overline{L}_\Delta(\psi(f),h)-\psi_b)\sum_{k=2}^{n-1}\Phi[\psi_b;\underbrace{\psi_a,\dots,\psi_a}_{k\ times}](\overline{L}_\Delta(\psi(f),h)-\psi_a)^{k-1}\\
&\quad +\sum_{k=2}^{n-2}\Phi[\psi_b,\psi_b;\underbrace{\psi_a,\dots,\psi_a}_{k\ times}]\overline{L}_\Delta((\psi(f)-\psi_b\mathbf{1})^2(\psi(f)-\psi_a\mathbf{1})^{k-1},h)\\
&\leq \chi\left(M_\chi(f,\overline{L}_\Delta(f,h))\right)-\chi\left(M_\psi(f,\overline{L}_\Delta(f,h))\right) \qquad (6.45)\\
&\leq \Phi[\psi_b,\psi_b;\psi_a](\psi_b-\overline{L}_\Delta(\psi(f),h))(\overline{L}_\Delta(\psi(f),h)-\psi_a)\\
&\quad +\sum_{k=2}^{n-1}\Phi[\psi_b;\underbrace{\psi_a,\dots,\psi_a}_{k\ times}]\overline{L}_\Delta((\psi(f)-\psi_b\mathbf{1})(\psi(f)-\psi_a\mathbf{1})^{k-1},h)\\
&\quad -(\overline{L}_\Delta(\psi(f),h)-\psi_b)^2\sum_{k=2}^{n-2}\Phi[\psi_b,\psi_b;\underbrace{\psi_a,\dots,\psi_a}_{k\ times}](\overline{L}_\Delta(\psi(f),h)-\psi_a)^{k-1}.
\end{aligned}
$$

If the function Φ is n-concave, the inequality signs in (6.45) are reversed.

6.4.2 Power means

First we need to define the power mean in terms of the multiple Lebesgue delta time scale integral.

Definition 6.5 *Assume $\mathscr{E} \subset \mathbb{R}^n$ is as in Theorem 6.4 and f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = I$ and $f(t) > 0$, $t \in \mathscr{E}$. Let $h : \mathscr{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t > 0$. For $r \in \mathbb{R}$, suppose f^r and $(\log f)$ are Δ-integrable on $\mathscr{E}$. The power mean with respect to the multiple Riemann delta time scale integral is defined by*

$$
M^{[r]}\left(f,\overline{L}_\Delta(f,h)\right)=\begin{cases}\left(\overline{L}_\Delta(f^r,h)\right)^{\frac{1}{r}}, & if \quad r\neq 0\\ \exp\left(\overline{L}_\Delta(\log f,h)\right), & if \quad r=0.\end{cases} \qquad (6.46)
$$

According to definition of power mean on time scales with respect of the multiple Lebesgue delta integral ([12]), we derive the following result.

Theorem 6.21 ([13]) *Suppose $\mathscr{E} \subset \mathbb{R}^n$ is as in Theorem 6.4, f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = I$ and $0 < m \leq f(t) \leq M < \infty$, for $t \in \mathscr{E}$, $m,M \in \mathbb{R}$. Let $h : \mathscr{E} \to \mathbb{R}$ be Δ-integrable such that $\int_{\mathscr{E}} |h(t)|\Delta t > 0$. For $r,s \in \mathbb{R}$ suppose f^r, f^s, $(\log f)$ are Δ-integrable on $\mathscr{E}$.*

(i) If $0 < r < s$ or $r < 0 < s$, then

$$
0 \leq \left(M^{[s]}\left(f,\overline{L}_\Delta\right)\right)^s-\left(M^{[r]}\left(f,\overline{L}_\Delta\right)\right)^s
$$

$$
\begin{aligned}
&\leq \left(M^r - \overline{L}_\Delta(f^r,h)\right)\left(\overline{L}_\Delta(f^r,h) - m^r\right) \sup_{t\in\langle m,M\rangle} \Psi_\phi(t^r;m^r,M^r) \\
&\leq \frac{s}{r}\left(M^r - \overline{L}_\Delta(f^r,h)\right)\left(\overline{L}_\Delta(f^r,h) - m^r\right) \frac{M^{s-r}-m^{s-r}}{M^r-m^r} \qquad (6.47)\\
&\leq \frac{s}{4r}\left(M^r - m^r\right)\left(M^{s-r}-m^{s-r}\right).
\end{aligned}
$$

(ii) If $r < s < 0$, then

$$
\begin{aligned}
0 &\geq \left(M^{[s]}\left(f,\overline{L}_\Delta\right)\right)^s - \left(M^{[r]}\left(f,\overline{L}_\Delta\right)\right)^s \\
&\geq \left(M^r - \overline{L}_\Delta(f^r,h)\right)\left(\overline{L}_\Delta(f^r,h) - m^r\right) \sup_{t\in\langle m,M\rangle} \Psi_\phi(t^r;m^r,M^r) \\
&\geq \frac{s}{r}\left(M^r - \overline{L}_\Delta(f^r,h)\right)\left(\overline{L}_\Delta(f^r,h) - m^r\right) \frac{M^{s-r}-m^{s-r}}{M^r-m^r} \qquad (6.48)\\
&\geq \frac{s}{4r}\left(M^r - m^r\right)\left(M^{s-r}-m^{s-r}\right).
\end{aligned}
$$

(iii) If $s = 0$ and $r < 0$, then

$$
\begin{aligned}
0 &\leq \log\left(M^{[0]}\left(f,\overline{L}_\Delta\right)\right) - \log\left(M^{[r]}\left(f,\overline{L}_\Delta\right)\right) \\
&\leq \left(M^r - \overline{L}_\Delta(f^r,h)\right)\left(\overline{L}_\Delta(f^r,h) - m^r\right) \sup_{t\in\langle m,M\rangle} \Psi_\phi(t^r;M^r,m^r) \\
&\leq -\frac{1}{r}\cdot\frac{\left(M^r - \overline{L}_\Delta(f^r,h)\right)\left(\overline{L}_\Delta(f^r,h) - m^r\right)}{M^r m^r} \qquad (6.49)\\
&\leq -\frac{1}{4r}\cdot\frac{(M^r-m^r)^2}{M^r m^r}.
\end{aligned}
$$

(iv) If $r = 0$ and $s > 0$, then

$$
\begin{aligned}
0 &\leq \left(M^{[s]}\left(f,\overline{L}_\Delta\right)\right)^s - \left(M^{[0]}\left(f,\overline{L}_\Delta\right)\right)^s \\
&\leq \left(\log M - \overline{L}_\Delta(\log f,h)\right)\left(\overline{L}_\Delta(\log f,h) - \log m\right) \sup_{t\in\langle m,M\rangle} \Psi_\phi(\log t;\log m,\log M) \\
&\leq \left(\log M - \overline{L}_\Delta(\log f,h)\right)\left(\overline{L}_\Delta(\log f,h) - \log m\right)\cdot\frac{s(M^s-m^s)}{\log M-\log m} \qquad (6.50)\\
&\leq s\left(M^s-m^s\right)\log\frac{M}{m}.
\end{aligned}
$$

Proof. The claim follows from Theorem 6.4, Theorem 6.10 and Theorem 6.19. □

Theorem 6.22 ([12]) *Suppose that the same hypotheses as in Theorem 6.21 are valid.*

(i) *If $0 < r < s$ or $r < 0 < s$, then*

$$
0 \leq \frac{M^r - \overline{L}_\Delta(f^r,h)}{M^r-m^r} m^s + \frac{\overline{L}_\Delta(f^r,h) - m^r}{M^r-m^r} M^s - \left(M^{[s]}\left(f,\overline{L}_\Delta(f,h)\right)\right)^s \qquad (6.51)
$$

$$
\begin{aligned}
&\le \frac{s}{r}\cdot\frac{M^{s-r}-m^{s-r}}{M^r-m^r}\cdot\frac{\int_{\mathscr{E}} h(t)\left((M^r-f^r(t))\,(f^r(t)-m^r)\right)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}\\
&\le \frac{s}{r}\cdot\frac{M^{s-r}-m^{s-r}}{M^r-m^r}\left(M^r-\overline{L}_\Delta(f^r,h)\right)\left(\overline{L}_\Delta(f^r,h)-m^r\right)\\
&\le \frac{s}{4r}(M^r-m^r)(M^{s-r}-m^{s-r}).
\end{aligned}
$$

(ii) *If $r<s<0$, then*

$$
\begin{aligned}
0 &\ge \frac{M^r-\overline{L}_\Delta(f^r,h)}{M^r-m^r}m^s+\frac{\overline{L}_\Delta(f^r,h)-m^r}{M^r-m^r}M^s-\left(M^{[s]}\left(f,\overline{L}_\Delta(f,h)\right)\right)^s \qquad (6.52)\\
&\ge \frac{s}{r}\cdot\frac{M^{s-r}-m^{s-r}}{M^r-m^r}\cdot\frac{\int_{\mathscr{E}} h(t)\left((M^r-f^r(t))\,(f^r(t)-m^r)\right)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}\\
&\ge \frac{s}{r}\cdot\frac{M^{s-r}-m^{s-r}}{M^r-m^r}\left(M^r-\overline{L}_\Delta(f^r,h)\right)\left(\overline{L}_\Delta(f^r,h)-m^r\right)\\
&\ge \frac{s}{4r}(M^r-m^r)(M^{s-r}-m^{s-r}).
\end{aligned}
$$

(iii) *If $s=0$ and $r<0$, then*

$$
\begin{aligned}
0 &\le \frac{M^r-\overline{L}_\Delta(f^r,h)}{M^r-m^r}\log m+\frac{s}{r}\cdot\frac{\overline{L}_\Delta(f^r,h)-m^r}{M^r-m^r}\log M-\log\left(M^{[0]}\left(f,\overline{L}_\Delta(f,h)\right)\right) \qquad (6.53)\\
&\le -\frac{1}{r}\cdot\frac{\frac{\int_{\mathscr{E}} h(t)((M^r-f^r(t))(f^r(t)-m^r))\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}}{M^r m^r}\\
&\le -\frac{1}{r}\cdot\frac{\left(M^r-\overline{L}_\Delta(f^r,h)\right)\left(\overline{L}_\Delta(f^r,h)-m^r\right)}{M^r m^r}\\
&\le \frac{1}{4r}(M^r-m^r)\left(\frac{1}{M^r}-\frac{1}{m^r}\right).
\end{aligned}
$$

(iv) *If $r=0$ and $s>0$, then*

$$
\begin{aligned}
0 &\le \frac{\log M-\overline{L}_\Delta(\log f,h)}{\log M-\log m}m^s+\frac{\overline{L}_\Delta(\log f,h)-\log m}{\log M-\log m}M^s-\left(M^{[s]}\left(f,\overline{L}_\Delta(f,h)\right)\right)^s \qquad (6.54)\\
&\le \frac{s(e^{sM}-e^{sm})}{\log M-\log m}\cdot\frac{\int_{\mathscr{E}} h(t)\left((\log M-\log(f(t)))\,(\log(f(t))-\log m)\right)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}\\
&\le \frac{s(e^{sM}-e^{sm})}{\log M-\log m}\left(\log M-\overline{L}_\Delta(\log f,h)\right)\left(\overline{L}_\Delta(\log f,h)-\log m\right)\\
&\le \frac{s}{4}(e^{sM}-e^{sm})\log\frac{M}{m}.
\end{aligned}
$$

Proof. The above inequalities follow from Theorem 6.9. Namely,

(i) if $0 < r < s$ or $r < 0 < s$, then we can take the function ϕ to be of the form $\phi(t) = t^{\frac{s}{r}}$ because ϕ is now continuous and convex and all the conditions of Theorem 6.9 are satisfied. Now, inequality (6.51) follows from (6.10) with replacing m by m^r, M by M^r, and f by f^r if $r > 0$ (because the function f^r is then strictly increasing) and with replacing M by m^r, m by M^r, and f by f^r if $r < 0$ (because the function f^r is then strictly decreasing);

(ii) if $r < s < 0$, then the function $\phi(t) = t^{\frac{s}{r}}$ is concave so we obtain inequality (6.52) from the inequalities reversed to (6.10) making following replacements: M by m^r, m by M^r, and f by f^r (f^r is now strictly decreasing);

(iii) if $s = 0$ and $r < 0$, then we take $\phi(t) = \frac{1}{r}\log t$ which is continuous and convex and we deduce inequality (6.53) from (6.10) interchanging M by m^r, m by M^r, and f by f^r (f^r is now strictly decreasing);

(iv) if $r = 0$ and $s > 0$, then we take $\phi(t) = e^{st}$ which is continuous and convex and inequality (6.54) follows from (6.10) interchanging m by $\log m$, M by $\log M$, and f by $\log f$ ($\log f$ is strictly increasing).

This completes the proof. $\square$

Theorem 6.23 *Suppose $\mathscr{E} \subset \mathbb{R}^n$ is as in Theorem 6.4 and f is Δ-integrable on $\mathscr{E}$ such that $f(\mathscr{E}) = I$ and $0 < m \le f(t) \le M < \infty$, for $t \in \mathscr{E}$, $m, M \in \mathbb{R}$. Let $h : \mathscr{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $\int_{\mathscr{E}} h(t)\Delta t > 0$. For $r, s \in \mathbb{R}$ suppose f^r, f^s, $(\log f)$ are Δ-integrable on $\mathscr{E}$.*

(i) *If $r < 0 < s$ or $r < s < 0$, then*

$$
\begin{aligned}
0 &\le \left(M^{[r]}\left(f, \overline{L}_\Delta(f,h)\right)\right)^r - \left(M^{[s]}\left(f, \overline{L}_\Delta(f,h)\right)\right)^r \\
&\le \frac{r}{s}\left(M^s - \overline{L}_\Delta(f^s,h)\right)\left(\overline{L}_\Delta(f^s,h) - m^s\right)\frac{M^{r-s} - m^{r-s}}{M^s - m^s} \\
&\le \frac{r}{4s}\left(M^s - m^s\right)\left(M^{r-s} - m^{r-s}\right).
\end{aligned} \tag{6.55}
$$

(ii) *If $0 < r < s$, then*

$$
\begin{aligned}
0 &\ge \left(M^{[r]}\left(f, \overline{L}_\Delta(f,h)\right)\right)^r - \left(M^{[s]}\left(f, \overline{L}_\Delta(f,h)\right)\right)^r \\
&\ge \frac{r}{s}\left(M^s - \overline{L}_\Delta(f^s,h)\right)\left(\overline{L}_\Delta(f^s,h) - m^s\right)\frac{M^{r-s} - m^{r-s}}{M^s - m^s} \\
&\ge \frac{r}{4s}\left(M^s - m^s\right)\left(M^{r-s} - m^{r-s}\right).
\end{aligned} \tag{6.56}
$$

(iii) *If* $s = 0$ *and* $r < 0$, *then*

$$
\begin{aligned}
0 &\leq \left(M^{[r]}\left(f, \overline{L}_{\Delta}(f,h)\right)\right)^{r} - \left(M^{[0]}\left(f, \overline{L}_{\Delta}(f,h)\right)\right)^{r} \\
&\leq \left(\log M - \overline{L}_{\Delta}(\log f, h)\right)\left(\overline{L}_{\Delta}(\log f, h) - \log m\right) \cdot \frac{r(M^r - m^r)}{\log M - \log m} \\
&\leq \frac{r}{4}(M^r - m^r)\log\frac{M}{m}. \qquad (6.57)
\end{aligned}
$$

(iv) *If* $r = 0$ *and* $s > 0$, *then*

$$
\begin{aligned}
0 &\geq \log\left(M^{[0]}\left(f, \overline{L}_{\Delta}(f,h)\right)\right) - \log\left(M^{[s]}\left(f, \overline{L}_{\Delta}(f,h)\right)\right) \\
&\geq -\frac{1}{s}\left(M^s - \overline{L}_{\Delta}(f^s,h)\right)\left(\overline{L}_{\Delta}(f^s,h) - m^s\right)\frac{1}{M^r m^r} \\
&\geq \frac{1}{4s}(M^s - m^s)\left(\frac{1}{m^s} - \frac{1}{m^s}\right). \qquad (6.58)
\end{aligned}
$$

Proof. The above inequalities follow directly from Theorem 6.8. Namely,

(i) if $r < 0 < s$ or $r < s < 0$, then we can take the function ϕ to be of the form $\phi(t) = t^{\frac{r}{s}}$ because ϕ is now continuous and convex and all the conditions of Theorem 6.8 are satisfied. Now, inequality (6.55) follows from (6.3) with replacing m by m^s, M by M^s and f by f^s if $s > 0$ (because the function f^s is then strictly increasing) and replacing M by m^s, m by M^s and f by f^s if $s < 0$ (because the function f^s is then strictly decreasing);

(ii) if $0 < r < s$, then the function $\phi(t) = t^{\frac{r}{s}}$ is concave so we obtain inequality (6.56) from the inequalities reversed to (6.3) making following replacements m by m^s, M by M^s and f by f^s (f^s is now strictly decreasing);

(iii) if $s = 0$ and $r < 0$, then we take $\phi(t) = e^{rt}$ which is continuous and convex and we deduce inequality (6.57) from (6.3) replacing m by $\log m$, M by $\log M$ and f by $\log f$ ($\log f$ is strictly increasing);

(iv) if $r = 0$ and $s > 0$, then we take $\phi(t) = \frac{1}{s}\log t$ which is continuous and concave and inequality (6.58) follows from (6.3) replacing m by m^s, M by M^s, and f by f^s (f^s is now strictly increasing).

This completes the proof. □

Theorem 6.24 *Suppose the hypotheses of Theorem 6.23 hold.*

(i) *If $r < s < 0$ or $r < 0 < s$, then*

$$\begin{aligned}
0 &\le \frac{M^s - \overline{L}_\Delta(f^s,h)}{M^s - m^s} m^r + \frac{\overline{L}_\Delta(f^s,h) - m^s}{M^s - m^s} M^r - \left(M^{[r]}\left(f, \overline{L}_\Delta(f,h)\right)\right)^r \qquad (6.59)\\
&\le \frac{r}{s} \cdot \frac{M^{r-s} - m^{r-s}}{M^s - m^s} \cdot \frac{\int_{\mathscr{E}} h(t)\left(\left(M^s - f^s(t)\right)\left(f^s(t) - m^s\right)\right)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}\\
&\le \frac{r}{s} \cdot \frac{M^{r-s} - m^{r-s}}{M^s - m^s} \left(M^s - \overline{L}_\Delta(f^s,h)\right)\left(\overline{L}_\Delta(f^s,h) - m^s\right)\\
&\le \frac{r}{4s}(M^s - m^s)(M^{r-s} - m^{r-s}).
\end{aligned}$$

(ii) *If $0 < r < s$, then*

$$\begin{aligned}
0 &\ge \frac{M^s - \overline{L}_\Delta(f^s,h)}{M^s - m^s} m^r + \frac{\overline{L}_\Delta(f^s,h) - m^s}{M^s - m^s} M^r - \left(M^{[r]}\left(f, \overline{L}_\Delta(f,h)\right)\right)^r \qquad (6.60)\\
&\ge \frac{r}{s} \cdot \frac{M^{r-s} - m^{r-s}}{M^s - m^s} \cdot \frac{\int_{\mathscr{E}} h(t)\left(\left(M^s - f^s(t)\right)\left(f^s(t) - m^s\right)\right)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}\\
&\ge \frac{r}{s} \cdot \frac{M^{r-s} - m^{r-s}}{M^s - m^s} \left(M^s - \overline{L}_\Delta(f^s,h)\right)\left(\overline{L}_\Delta(f^s,h) - m^s\right)\\
&\ge \frac{r}{4s}(M^s - m^s)(M^{r-s} - m^{r-s}).
\end{aligned}$$

(iii) *If $s = 0$ and $r < 0$, then*

$$\begin{aligned}
0 &\le \frac{\log M - \overline{L}_\Delta(\log f,h)}{\log M - \log m} m^r + \frac{\overline{L}_\Delta(\log f,h) - \log m}{\log M - \log m} M^r - \left(M^{[r]}\left(f, \overline{L}_\Delta(f,h)\right)\right)^r \qquad (6.61)\\
&\le \frac{r(M^r - m^r)}{\log M - \log m} \cdot \frac{\int_{\mathscr{E}} h(t)\left(\left(\log M - \log(f(t))\right)\left(\log(f(t)) - \log m\right)\right)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}\\
&\le \frac{r(M^r - m^r)}{\log M - \log m}\left(\log M - \overline{L}_\Delta(\log f,h)\right)\left(\overline{L}_\Delta(\log f,h) - \log m\right)\\
&\le \frac{r}{4}(M^r - m^r)\log\frac{M}{m}.
\end{aligned}$$

(iv) *If $r = 0$ and $s > 0$, then*

$$\begin{aligned}
0 &\ge \frac{M^s - \overline{L}_\Delta(f^s,h)}{M^s - m^s}\log m + \frac{\overline{L}_\Delta(f^s,h) - m^s}{M^s - m^s}\log M - \log\left(M^{[0]}\left(f, \overline{L}_\Delta(f,h)\right)\right)\\
&\ge -\frac{1}{s} \cdot \frac{\dfrac{\int_{\mathscr{E}} h(t)\left((M^s - f^s(t))(f^s(t) - m^s)\right)\Delta t}{\int_{\mathscr{E}} h(t)\Delta t}}{M^s m^s}
\end{aligned}$$

$$\geq -\frac{1}{s}\cdot\frac{\left(M^s-\overline{L}_\Delta(f^s,h)\right)\left(\overline{L}_\Delta(f^s,h)-m^s\right)}{M^s m^s} \tag{6.62}$$
$$\geq \frac{1}{s}(M^s-m^s)\left(\frac{1}{M^s}-\frac{1}{m^s}\right).$$

Proof. All the inequalities can be obtained directly from Theorem 6.9 using inequality (6.10) and the same technique and substitutions as in the proof of Theorem 6.23. □

Since power means are a special case of generalized means for particular choices of functions χ and ψ, in order to utilize our results on generalized means for obtaining similar results on power means, first let us set $\chi(t)=t^s$ and $\psi(t)=t^r$, where s and r are real parameters such that $r\neq 0$ and $t>0$.

Now, the function $\Phi(t)=(\chi\circ\psi^{-1})(t)=t^{s/r}$ belongs to the class $\mathscr{C}^n(\mathbb{R})$ for any $n\in\mathbb{N}$, and we have

$$\Phi^{(n)}(t)=\frac{s}{r}\left(\frac{s}{r}-1\right)\left(\frac{s}{r}-2\right)\cdots\left(\frac{s}{r}-n+1\right)t^{\frac{s}{r}-n}.$$

It is straightforward to check that:

- if $r<0<s$ or $s<0<r$, then the function Φ is n-convex for any even $n\in\mathbb{N}$, and n-concave for any odd number n;
- if $0<s<r$ or $r<s<0$, then the function Φ is n-convex for any odd $n\in\mathbb{N}$, and n-concave for any even number n;
- if $0<r<s$ or $s<r<0$, then the function Φ is n-convex when $\lfloor\frac{s}{r}\rfloor$ is even and n is odd, or when $\lfloor\frac{s}{r}\rfloor$ is odd and n is even, and Φ is n-concave when $\lfloor\frac{s}{r}\rfloor$ and n are both either even or odd.

It remains to consider the cases when one of the parameters r and s is equal to zero. If $s=0$, then setting $\chi(t)=\log t$ and $\psi(t)=t^r$, it follows that $\Phi(t)=(\chi\circ\psi^{-1})(t)=\dfrac{1}{r}\log t$ belongs to the class $\mathscr{C}^n(\mathbb{R})$ for any $n\in\mathbb{N}$, and we have

$$\Phi^{(n)}(t)=\frac{1}{r}(-1)^{n-1}(n-1)!\,t^{-n}.$$

It is easy to see that:

- the function Φ is n-convex if $r>0$ and $n\in\mathbb{N}$ is odd, or if $r<0$ and $n\in\mathbb{N}$ is even;
- the function Φ is n-concave if $r>0$ and $n\in\mathbb{N}$ is even, or if $r<0$ and $n\in\mathbb{N}$ is odd.

In cases when $r<0$ the function $\psi(t)=t^r$ is strictly decreasing, so we have $\psi_a=b^r$ and $\psi_b=a^r$, and in cases when $0<r$ the function ψ is strictly increasing, so we have $\psi_a=a^r$ and $\psi_b=b^r$.

Finally, if $r=0$, then setting $\chi(t)=t^s$ and $\psi(t)=\log t$, it follows that the function $\Phi(t)=(\phi\circ\psi^{-1})(t)=e^{st}$ belongs to the class $\mathscr{C}^n(\mathbb{R})$ for any $n\in\mathbb{N}$, and we have

$$\Phi^{(n)}(t)=s^n\,e^{st}.$$

Trivially,

- if $s > 0$, then the function Φ is n-convex for any $n \in \mathbb{N}$;
- if $s < 0$, then Φ is n-convex for any even number n, and n-concave for any odd number n.

The function $\psi(t) = \log t$ is strictly increasing, so in this case we have $\psi_a = \log a$ and $\psi_b = \log b$.

We see that all of the results regarding generalized means obtained in the previous subsection can be applied to power means, considering our discussion from above.

6.4.3 Hölder's inequality

In this subsection, we deduce some converses of Hölder's inequality on time scales using the results from the previous section and the following Hölder inequality for delta time scale integrals proved in [6].

Theorem 6.25 ([6]) *For $p > 1$, define $q = p/(p-1)$. Let $\mathscr{E} \subset \mathbb{R}^n$ be as in Theorem 6.4. Assume $|w||f|^p$, $|w||g|^q$, $|wfg|$ are Δ-integrable on $\mathscr{E}$. Then,*

$$\int_{\mathscr{E}} |w(t)f(t)g(t)|\Delta t \leq \left(\int_{\mathscr{E}} |w(t)||f(t)|^p \Delta t\right)^{\frac{1}{p}} \left(\int_{\mathscr{E}} |w(t)||g(t)|^q \Delta t\right)^{\frac{1}{q}}.$$

This inequality is reversed if $0 < p < 1$ and $\int_{\mathscr{E}} |w(t)||g(t)|^q \Delta t > 0$, and it is also reversed if $p < 0$ and $\int_{\mathscr{E}} |w(t)||f(t)|^p \Delta t > 0$.

Now, we can prove new converses of Hölder's inequality in terms of time scale calculus and Δ-integral.

Theorem 6.26 ([13]) *Assume $\mathscr{E} \subset \mathbb{R}^n$ is as in Theorem 6.4 and w, f, g are real functions on $\mathscr{E}$ such that $w, f, g \geq 0$. For $m, M \in \mathbb{R}$ such that $-\infty < m < M < \infty$ let $m \leq f(t)g^q(t) \leq M$, $t \in \mathscr{E}$. If wf^p, wg^q, wfg are Δ-integrable on $\mathscr{E}$ and $\int_{\mathscr{E}} w(t)g^q(t)\Delta t > 0$, where $p > 1$ and $q = p/(p-1)$, then*

$$\begin{aligned}
0 &\leq L_\Delta(wf^p)\cdot L_\Delta^{\frac{p}{q}}(wg^q) - L_\Delta^p(wfg) \\
&\leq (ML_\Delta(wg^q) - L_\Delta(wfg))\,(L_\Delta(wfg) - mL_\Delta(wg^q)) \qquad (6.63)\\
&\quad \cdot \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M)\cdot L_\Delta^{p-2}(wg^q) \\
&\leq (ML_\Delta(wg^q) - L_\Delta(wfg))\,(L_\Delta(wfg) - mL_\Delta(wg^q)) \qquad (6.64)\\
&\quad \cdot \frac{p(M^{p-1} - m^{p-1})}{M-m}\cdot L_\Delta^{p-2}(wg^q) \\
&\leq \frac{p}{4}(M-m)(M^{p-1} - m^{p-1})L_\Delta^{p-2}(wg^q).
\end{aligned}$$

For $p < 0$, inequalities (6.64) hold if $\int_{\mathscr{E}} w(t)f(t)g(t)\Delta t > 0$, $t \in \mathscr{E}$. In case $0 < p < 1$, all inequalities in (6.64) are reversed.

Proof. Inequalities (6.64) follow directly from Theroem 6.10 by taking the function ϕ to be of the form $\phi(t)=t^p$ and replacing h by wg^q and f by $fg^{-\frac{q}{p}}$. For $p<0$ and $p>1$, the function t^p is convex and inequalities (6.64) follow from inequalities (6.12). For $0<p<1$, the function t^p is concave and, according to Theorem 6.10, all inequalities in (6.64) will be reversed. □

Theorem 6.27 ([12]) *Let all assumptions of Theorem 6.26 hold. For $p<0$ or $p>1$, we have*

$$\begin{aligned}
0 &\leq \frac{ML_\Delta(wg^q)-L_\Delta(wfg)}{M-m}m^p+\frac{L_\Delta(wfg)-L_\Delta(wg^q)}{M-m}M^p-L_\Delta(wf^p) \\
&\leq \frac{p(M^{p-1}-m^{p-1})}{M-m} \\
&\quad \cdot \int_{\mathscr{E}} (Mw(t)g^q(t)-w(t)f(t)g(t))\,(w(t)f(t)g(t)-mw(t)g^q(t))\,\Delta t \\
&\leq p\frac{M^{p-1}-m^{p-1}}{M-m}\cdot\frac{1}{L_\Delta(wg^q)}\,(ML_\Delta(wg^q)-L_\Delta(wfg)) \\
&\quad \cdot (L_\Delta(wfg)-mL_\Delta(wg^q)) \\
&\leq \frac{p}{4}(M-m)(M^{p-1}-m^{p-1})L_\Delta(w,g^q).
\end{aligned} \tag{6.65}$$

If $0<p<1$, then all inequalities in (6.65) *are reversed.*

Proof. Inequalities (6.65) follow directly from Theorem 6.9 by taking the function ϕ to be of the form $\phi(t)=t^p$ and replacing h by wg^q and f by $fg^{-\frac{q}{p}}$. If $p<0$ and $p>1$, then the function t^p is convex and inequalities (6.65) follow from inequalities (6.10). For $0<p<1$, the function t^p is concave and, according to Theorem 6.9, all inequalities in (6.65) will be reversed. □

Theorem 6.28 ([13]) *Assume $\mathscr{E}\subset\mathbb{R}^n$ is as in Theorem 6.4 and $f,g\geq 0$ such that f^p, g^q, fg are Δ-integrable on $\mathscr{E}$ and $\int_{\mathscr{E}} g^q(t)\Delta t>0$, where $0<p<1$ and $q=p/(p-1)$. For $m,M\in\mathbb{R}$ such that $-\infty<m<M<\infty$, let $m\leq f(t)g^{-q}(t)\leq M$, $t\in\mathscr{E}$. Then,*

$$\begin{aligned}
0 &\leq L_\Delta(fg)-L_\Delta^{\frac{1}{p}}(f^p)L_\Delta^{\frac{1}{q}}(g^q) \\
&\leq \frac{1}{L_\Delta(g^q)}\,(ML_\Delta(g^q)-L_\Delta(f^p))\,(L_\Delta(f^p)-mL_\Delta(g^q))\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M) \\
&\leq \frac{1}{p}\cdot\frac{M^{-\frac{1}{q}}-m^{-\frac{1}{q}}}{M-m}\cdot\frac{1}{L_\Delta(g^q)}\,(ML_\Delta(g^q)-L_\Delta(f^p))\,(L_\Delta(f^p)-mL_\Delta(g^q)) \\
&\leq \frac{1}{4p}(M-m)\left(M^{-\frac{1}{q}}-m^{-\frac{1}{q}}\right)L_\Delta(g^q).
\end{aligned} \tag{6.66}$$

For $p<0$, inequalities (6.66) *hold if $\int_{\mathscr{E}} f^p(t)\Delta t>0$, $t\in\mathscr{E}$. In case $p>1$, all inequalities in* (6.66) *are reversed.*

Proof. Inequalities (6.66) follow directly from Theroem 6.10 by taking the function ϕ to be of the form $\phi(t) = t^{\frac{1}{p}}$ and replacing h by g^q and f by $\frac{f^p}{g^q}$. Namely, when $p < 1$, the function $t^{\frac{1}{p}}$ is convex and inequalities (6.66) follow from inequalities (6.12). For $p > 1$, the function t^p is concave and, according to Theorem 6.10, all inequalities in (6.66) will be reversed. □

Theorem 6.29 ([12]) *Let $p < 1$ and let the assumptions from Theorem 6.28 hold. Then,*

$$\begin{aligned}
0 &\le \frac{ML_\Delta(g^q) - L_\Delta(f^p)}{M-m} m^{\frac{1}{p}} + \frac{L_\Delta(f^p) - mL_\Delta(g^q)}{M-m} M^{\frac{1}{p}} - L_\Delta(fg) \\
&\le \frac{1}{p} \cdot \frac{M^{-\frac{1}{q}} - m^{-\frac{1}{q}}}{M-m} \int_{\mathscr{E}} \frac{(Mg^q(t) - f^p(t))(f^p(t) - mg^q(t))}{g^q(t)} \Delta t \\
&\le \frac{1}{p} \cdot \frac{M^{-\frac{1}{q}} - m^{-\frac{1}{q}}}{M-m} \cdot \frac{1}{L_\Delta(g^q)} (ML_\Delta(g^q) - L_\Delta(f^p)) \\
&\quad \cdot (L_\Delta(f^p) - mL_\Delta(g^q)) \\
&\le \frac{1}{4p}(M-m)\left(M^{-\frac{1}{q}} - m^{-\frac{1}{q}}\right) L_\Delta(g^q).
\end{aligned} \tag{6.67}$$

If $p > 1$, then all inequalities in (6.67) *are reversed.*

Proof. Inequalities (6.67) follow directly from Theroem 6.9 by taking the function ϕ to be of the form $\phi(t) = t^{\frac{1}{p}}$ and replacing h by g^q and f by $\frac{f^p}{g^q}$. Namely, when $p < 1$, the function $t^{\frac{1}{p}}$ is convex and inequalities (6.67) follow from inequalities (6.10). For $p > 1$, the function t^p is concave and, according to Theorem 6.9, all inequalities in (6.67) will be reversed. □

Theorem 6.30 ([13]) *Assume $\mathscr{E} \subset \mathbb{R}^n$ is as in Theorem 6.4 and $f, g \ge 0$ such that g^q, fg are Δ-integrable on $\mathscr{E}$ and $\int_{\mathscr{E}} g^q(t)\Delta t > 0$, where $p < 0$ or $p > 1$ and $q = p/(p-1)$. Let $m, M \in \mathbb{R}$ such that $-\infty < m < M < \infty$ and $m \le f(t)g^{1-q}(t) \le M$, $t \in \mathscr{E}$. Then,*

$$\begin{aligned}
0 &\le L_\Delta(f^p) \cdot L_\Delta^{\frac{p}{q}}(g^q) - L_\Delta^p(f,g) \qquad (6.68) \\
&\le \sup_{t \in \langle m,M \rangle} \Psi_\phi(t; m, M) \, (ML_\Delta(g^q) - L_\Delta(fg)) \, (L_\Delta(fg) - mL_\Delta(g^q)) \, L_\Delta^{p-2}(g^q) \\
&\le \frac{p(M^{p-1} - m^{p-1})}{M-m} (ML_\Delta(g^q) - L_\Delta(fg)) \, (L_\Delta(fg) - mL_\Delta(g^q)) \, L_\Delta^{p-2}(g^q) \\
&\le \frac{p}{4}(M-m)(M^{p-1} - m^{p-1}) L_\Delta^{p-2}(g^q).
\end{aligned}$$

In case $0 < p < 1$, all inequalities in (6.68) *are reversed.*

Proof. Inequalities (6.68) follow directly from Theroem 6.10 by taking the function ϕ to be of the form $\phi(t) = t^p$ and replacing h by g^q and f by fg^{1-q}. Namely, for $p < 0$ or $p > 1$, the function t^p is convex and inequalities (6.68) follow from inequalities (6.12). For $0 < p < 1$, the function t^p is concave and, according to Theorem 6.10, all inequalities in (6.68) will be reversed. □

Theorem 6.31 ([12]) *Suppose that the assumptions from Theorem 6.30 hold. For $p < 0$ or $p > 1$, we have*

$$\begin{aligned}
0 &\le \frac{ML_\Delta(g^q) - L_\Delta(fg)}{M-m} m^p + \frac{L_\Delta(fg) - L_\Delta(g^q)}{M-m} M^p - L_\Delta(f^p) \\
&\le p\frac{M^{p-1} - m^{p-1}}{M-m} \int\limits_{\mathscr{E}} (Mg^q(t) - f(t)g(t))(f(t)g(t) - mg^q(t))\Delta t \\
&\le p\frac{M^{p-1} - m^{p-1}}{M-m} \cdot \frac{1}{L_\Delta(g^q)} (ML_\Delta(g^q) - L_\Delta(fg)) \\
&\quad \cdot (L_\Delta(fg) - mL_\Delta(g^q)) \qquad (6.69) \\
&\le \frac{p}{4}(M-m)(M^{p-1} - m^{p-1}) L_\Delta(g^q).
\end{aligned}$$

If $0 < p < 1$, all inequalities in (6.69) *are reversed.*

Proof. Inequalities (6.69) follow directly from Theroem 6.9 by taking the function ϕ to be of the form $\phi(t) = t^p$ and replacing h by g^q and f by fg^{1-q}. Namely, for $p < 0$ and $p > 1$, the function t^p is convex and inequalities (6.69) follow from inequalities (6.10). For $0 < p < 1$, the function t^p is concave and, according to Theorem 6.9, all inequalities in (6.69) will be reversed. □

Chapter 7

Inequalities of the Levinson type

This chapter begins with a short historical comment on the connection between the Edmundson-Madansky and the Lah-Ribarič inequality, and an overview of some already known results. Both inequalities are special cases of the same inequality, so they have been united under the name of Edmundson-Lah-Ribarič inequality. Further, a Levinson's type generalization of the Edmundson-Lah-Ribarič inequality for a class of functions which contains the class of 3-convex functions will be proved, and it will be examined under what conditions the mentioned inequality is valid. Also it will be shown that analogous inequalities hold for the operator Edmundson-Lah-Ribarič inequality in Hilbert space, as well as for the scalar product of Hilbert space operators.

7.1 Introduction

In 1960, Levinson [91], obtained a very important inequality concerning two different sequences of real numbers. That result, today known as the Levinson inequality, is stated in the next theorem.

Theorem 7.1 ([91]) *Let the function $f\colon \langle 0,2c\rangle \to \mathbb{R}$ satisfy $f''' \geq 0$ and let p_i, x_i, y_i for $i = 1,\ldots,n$ be such that $p_i > 0$, $\sum_{i=1}^{n} p_i = 1$, $0 \leq x_i \leq c$ and*

$$x_1 + y_1 = x_2 + y_2 = \ldots = x_n + y_n = 2c. \tag{7.1}$$

Then we have the following inequality:

$$\sum_{i=1}^{n} p_i f(x_i) - f(\bar{x}) \le \sum_{i=1}^{n} p_i f(y_i) - f(\bar{y}), \tag{7.2}$$

where $\bar{x} = \sum_{i=1}^{n} p_i x_i$ *and* $\bar{y} = \sum_{i=1}^{n} p_i y_i$ *denote weighted arithemic means.*

In order to weaken the assumption on the differentiability of the function f, divided differences were observed. Divided difference of the k-th order of a function $f\colon I \to \mathbb{R}$ defined on the interval I in mutually different points $x_0, x_1, \ldots, x_k \in I$ is defined recursively by relation

$$[x_i]f = f(x_i), \quad \text{for } i = 0, \ldots, k$$

$$[x_0, \ldots, x_k]f = \frac{[x_1, \ldots, x_k]f - [x_0, \ldots, x_{k-1}]f}{x_k - x_0}.$$

We say that a function $f\colon I \to \mathbb{R}$ is k-convex if $[x_0, \ldots, x_k]f \ge 0$ holds for every choice of $k+1$ mutually different points $x_0, x_1, \ldots, x_k \in I$. If k-th derivation of a convex function exists, then $f^{(k)} \ge 0$, but $f^{(k)}$ doesn't need to exist (for properties of the divided differences and k-convex functions see [124]).

Numerous mathematicians have dealt with the weakening of the conditions under which Levinson's inequality (7.2) holds. Bullen in his paper [25] generalized Levinson's inequality to an interval $[a,b]$ and showed that if the function f is 3-convex and if p_i, x_i, y_i $(i = 1, \ldots, n)$ are such that $p_i > 0$, $\sum_{i=1}^{n} p_i = 1$, $a \le x_i, y_i \le b$ holds, and if we have (7.1) and

$$\max\{x_1, \ldots, x_n\} \le \max\{y_1, \ldots, y_n\}, \tag{7.3}$$

then the inequality (7.2) holds. He also showed that the reverse is true as well, that is, he showed that the function f is 3-convex if for p_i, x_i, y_i $(i = 1, \ldots, n)$ which satisfy the conditions from above the inequality (7.2) holds.

Pečarić in paper [119] additionally weakened Bullen's conditions, that is, he proved that the inequality (7.2) is valid if the condition (7.3) is replaced with a weaker one:

$$x_i + x_{n-i+1} \le 2c \;\text{ and }\; \frac{p_i x_i + p_{n-i+1} x_{n-i+1}}{p_i + p_{n-i+1}} \le c, \text{ for } i = 1, 2, \ldots, n$$

where $x_1 + y_1 = x_2 + y_2 = \ldots = x_n + y_n = 2c$.

Mercer in [95] obtained a significant improvement by completely replacing the symmetry condition (7.1) by equality of variances, that is he proved that the inequality (7.2) holds under the following conditions:

$$f''' \ge 0,\; p_i > 0,\; \sum_{i=1}^{n} p_i = 1,\; a \le x_i, y_i \le b,\; \max\{x_1, \ldots, x_n\} \le \max\{y_1, \ldots, y_n\}$$

$$\sum_{i=1}^{n} p_i (x_i - \bar{x})^2 = \sum_{i=1}^{n} p_i (y_i - \bar{y})^2. \tag{7.4}$$

Witkowski in [131] showed that in Mercer's assumptions, instead of the condition $f''' \ge 0$, it is enough to assume that the function f is 3-convex. Further, Witkowski even additionally weakened the condition (7.4) and showed that the sign of equality can be replaced by a sign of inequality in a certain direction.

In paper [11] Baloch, Pečarić and Praljak introduced a new class of functions $\mathscr{K}_c^1(a,b)$ which extends the class of 3-convex functions and can be interpreted as functions that are "3-convex in the point $c \in \langle a,b\rangle$". They showed that $\mathscr{K}_c^1(a,b)$ is the largest class of functions for which Levinson's inequality (7.2) holds under Mercer's assumptions, that is, they showed that $f \in \mathscr{K}_c^1(a,b)$ if and only if the inequality (7.2) holds for arbitrary weights $p_i > 0$, $\sum_{i=1}^n p_i = 1$ and sequences x_i and y_i that satisfy $x_i \leq c \leq y_i$ for $i = 1,2,\ldots,n$.

Here, a definition of the class $\mathscr{K}_c^1(a,b)$ extended to an arbitrary interval $I \subset \mathbb{R}$ is given.

Definition 7.1 *Let $f: I \to \mathbb{R}$ and let c be an arbitrary point from the interior of the interval I. We say that $f \in \mathscr{K}_c^1(I)$ ($f \in \mathscr{K}_c^2(I)$) if there exists a constant D such that function $F(x) = f(x) - \frac{D}{2}x^2$ is concave (convex) on $\langle -\infty, c] \cap I$ and convex (concave) on $[c, +\infty\rangle \cap I$.*

7.2 Levinson's type generalization of the Edmundson-Lah-Ribarič inequality

Throughout this section, $\mathbb{E}(Z)$ and $\mathrm{Var}(Z)$ denote expectation and variance of a random variable Z respectively, and without further emphasis, we assume these values are finite. Pečarić, Praljak and Witkowski in [123] proved the following probabilistic version of Levinson's inequality.

Theorem 7.2 ([123]) *Let $X: \Omega_1 \to I$ and $Y: \Omega_2 \to I$ be two random variables defined on probability spaces (Ω_1, p) and (Ω_2, q) respecively, and let us assume that there exists c from the interior of the interval I such that*

$$\operatorname*{ess\ sup}_{\omega\in\Omega_1} X(\omega) \leq c \leq \operatorname*{ess\ inf}_{\omega\in\Omega_2} Y(\omega) \tag{7.5}$$

and

$$\mathrm{Var}(X) = \mathrm{Var}(Y) < \infty.$$

Then for every function $f \in \mathscr{K}_c^1(I)$ such that $\mathbb{E}(f(X))$ and $\mathbb{E}(f(Y))$ are finite we have:

$$\mathbb{E}(f(X)) - f(\mathbb{E}(X)) \leq \mathbb{E}(f(Y)) - f(\mathbb{E}(Y)).$$

As a simple consequence of the previous theorem, they obtained the following generalization of the results from the paper [11]. This resulted in a significant improvement of Levinson's inequality, because not only they additionally weakened the conditions, but the sequences of the involved real numbers are of mutually different lenght and with different weights.

Corollary 7.1 ([123]) *If $x_i \in I \cap \langle -\infty, c]$, $y_j \in I \cap [c, +\infty\rangle$, $p_i > 0$, $q_j > 0$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ are such that $\sum_{i=1}^n p_i = \sum_{j=1}^m q_j = 1$ and $\sum_{i=1}^n p_i (x_i - \overline{x})^2 = \sum_{j=1}^m q_j (y_j - \overline{y})^2$, then*

$$\sum_{i=1}^n p_i f(x_i) - f(\overline{x}) \leq \sum_{j=1}^m q_j f(y_j) - f(\overline{y}) \tag{7.6}$$

holds for every $f \in \mathscr{K}_c^1(I)$.

Since the Edmundson-Lah-Ribarič inequality resulted from the Jensen iequality, they are closely related. Jensen's inequality gives us a lower, while the Edmundson-Lah-Ribarič inequality gives us an upper bound for the expectation of convex functions. Therefore it is natural to expect that a generalization of the Edmundson-Lah-Ribarič inequality analogous to the one from Theorem 7.2 will hold. Aim of this section is to find such generalization, and to see under what different condition that inequality holds.

The main result in this section is a Levinson's type generalization of the Edmundson-Lah-Ribarič inequality for the mathematical expectation obtained in paper [70].

Theorem 7.3 *Let $-\infty < a \leq A \leq b \leq B < +\infty$. Let $X \colon \Omega_1 \to [a, A]$ and $Y \colon \Omega_2 \to [b, B]$ be two random variables on probability spaces (Ω_1, p) and (Ω_2, q) respectively such that (7.5) holds and*

$$\frac{A - \mathbb{E}(X)}{A - a} a^2 + \frac{\mathbb{E}(X) - a}{A - a} A^2 - \mathbb{E}(X^2) = \frac{B - \mathbb{E}(Y)}{B - b} b^2 + \frac{\mathbb{E}(Y) - b}{B - b} B^2 - \mathbb{E}(Y^2). \tag{7.7}$$

Then for every function $f \in \mathscr{K}_c^1(a, B)$ such that $\mathbb{E}(f(X))$ and $\mathbb{E}(f(Y))$ are finite we have

$$\begin{aligned} &\frac{A - \mathbb{E}(X)}{A - a} f(a) + \frac{\mathbb{E}(X) - a}{A - a} f(A) - \mathbb{E}(f(X)) \\ &\leq \frac{B - \mathbb{E}(Y)}{B - b} f(b) + \frac{\mathbb{E}(Y) - b}{B - b} f(B) - \mathbb{E}(f(Y)). \end{aligned} \tag{7.8}$$

Proof. Let $F(x) = f(x) - \frac{D}{2} x^2$, where D is the constant from Definition 7.1. Since $F \colon [a, A] \to \mathbb{R}$ is concave, from Edmundson-Madansky inequality (g) immediately follows

$$\begin{aligned} 0 &\geq \frac{A - \mathbb{E}(X)}{A - a} F(a) + \frac{\mathbb{E}(X) - a}{A - a} F(A) - \mathbb{E}(F(X)) \\ &= \frac{A - \mathbb{E}(X)}{A - a} f(a) + \frac{\mathbb{E}(X) - a}{A - a} f(A) - \mathbb{E}(f(X)) \\ &\quad - \frac{D}{2} \Big(\frac{A - \mathbb{E}(X)}{A - a} a^2 + \frac{\mathbb{E}(X) - a}{A - a} A^2 - \mathbb{E}(X^2) \Big). \end{aligned}$$

After resetting the previous inequality we get

$$\begin{aligned} &- \frac{D}{2} \Big(\frac{A - \mathbb{E}(X)}{A - a} a^2 + \frac{\mathbb{E}(X) - a}{A - a} A^2 - \mathbb{E}(X^2) \Big) \\ &\leq - \frac{A - \mathbb{E}(X)}{A - a} f(a) - \frac{\mathbb{E}(X) - a}{A - a} f(A) + \mathbb{E}(f(X)). \end{aligned} \tag{7.9}$$

Similarly, $F: [b,B] \to \mathbb{R}$ is convex, so in the same manner we obtain

$$\begin{aligned}
0 &\leq \frac{B-\mathbb{E}(Y)}{B-b}F(b) + \frac{\mathbb{E}(Y)-b}{B-b}F(B) - \mathbb{E}(F(Y)) \\
&= \frac{B-\mathbb{E}(Y)}{B-b}f(b) + \frac{\mathbb{E}(Y)-b}{B-b}f(B) - \mathbb{E}(f(Y)) \\
&\quad - \frac{D}{2}\Big(\frac{B-\mathbb{E}(Y)}{B-b}b^2 + \frac{\mathbb{E}(Y)-b}{B-b}B^2 - \mathbb{E}(Y^2)\Big),
\end{aligned}$$

and after resetting we have

$$\begin{aligned}
&\frac{D}{2}\Big(\frac{B-\mathbb{E}(Y)}{B-b}b^2 + \frac{\mathbb{E}(Y)-b}{B-b}B^2 - \mathbb{E}(Y^2)\Big) \\
&\leq \frac{B-\mathbb{E}(Y)}{B-b}f(b) + \frac{\mathbb{E}(Y)-b}{B-b}f(B) - \mathbb{E}(f(Y)).
\end{aligned} \tag{7.10}$$

After summing up (7.9) and (7.10), we get

$$\begin{aligned}
0 &= \frac{D}{2}\Big(\frac{B-\mathbb{E}(Y)}{B-b}b^2 + \frac{\mathbb{E}(Y)-b}{B-b}B^2 - \mathbb{E}(Y^2) \\
&\quad - \frac{A-\mathbb{E}(X)}{A-a}a^2 - \frac{\mathbb{E}(X)-a}{A-a}A^2 + \mathbb{E}(X^2)\Big) \\
&\leq \frac{B-\mathbb{E}(Y)}{B-b}f(b) + \frac{\mathbb{E}(Y)-b}{B-b}f(B) - \mathbb{E}(f(Y)) \\
&\quad - \frac{A-\mathbb{E}(X)}{A-a}f(a) - \frac{\mathbb{E}(X)-a}{A-a}f(A) + \mathbb{E}(f(X)).
\end{aligned}$$

Thus the theorem assertion is proved. □

Remark 7.1 One can see from the proof of the previous theorem that the inequality (7.8) holds even if we replace the equality condition (7.7) with a weaker one

$$\begin{aligned}
&D\Big(\frac{B-\mathbb{E}(Y)}{B-b}b^2 + \frac{\mathbb{E}(Y)-b}{B-b}B^2 - \mathbb{E}(Y^2) \\
&\quad - \frac{A-\mathbb{E}(X)}{A-a}a^2 - \frac{\mathbb{E}(X)-a}{A-a}A^2 + \mathbb{E}(X^2)\Big) \geq 0.
\end{aligned}$$

Since $f''_-(c) \leq D \leq f''_+(c)$ (for details see [11]), if additionally the function f is convex (respectively concave), this condition can be further weakened to

$$\begin{aligned}
&\frac{B-\mathbb{E}(Y)}{B-b}b^2 + \frac{\mathbb{E}(Y)-b}{B-b}B^2 - \mathbb{E}(Y^2) \\
&\quad - \frac{A-\mathbb{E}(X)}{A-a}a^2 - \frac{\mathbb{E}(X)-a}{A-a}A^2 + \mathbb{E}(X^2) \geq 0 \text{ (respectively } \leq 0\text{)}.
\end{aligned}$$

From (7.9) and (7.10) one can see that the inequality (7.8) can be also written as

$$\begin{aligned}\frac{A-\mathbb{E}(X)}{A-a}f(a)+\frac{\mathbb{E}(X)-a}{A-a}f(A)-\mathbb{E}(f(X)) &\leq 0\\ \leq \frac{B-\mathbb{E}(Y)}{B-b}f(b)+\frac{\mathbb{E}(Y)-b}{B-b}f(B)-\mathbb{E}(f(Y))&\end{aligned}$$

or

$$\begin{aligned}\frac{A-\mathbb{E}(X)}{A-a}f(a)+\frac{\mathbb{E}(X)-a}{A-a}f(A)-\mathbb{E}(f(X)) &\leq \frac{D}{2}C\\ \leq \frac{B-\mathbb{E}(Y)}{B-b}f(b)+\frac{\mathbb{E}(Y)-b}{B-b}f(B)-\mathbb{E}(f(Y)),&\end{aligned}$$

where C is equal to any of the sides in the equality (7.7).

Next result is a discrete version of the Levinson's type generalization of the Edmundson-Lah-Ribarič inequality, and it is easily obtained as a simple consequence of the previous theorem.

Corollary 7.2 *Let $-\infty < a \leq A \leq c \leq b \leq B < +\infty$. If $x_i \in [a,A]$, $y_j \in [b,B]$, $p_i > 0$, $q_j > 0$ for $i = 1,\ldots,n$ i $j = 1,\ldots,m$ are real numbers such that $\sum_{i=1}^{n} p_i = \sum_{j=1}^{m} q_j = 1$ and*

$$\frac{A-\overline{x}}{A-a}a^2+\frac{\overline{x}-a}{A-a}A^2-\sum_{i=1}^{n}p_i x_i^2=\frac{B-\overline{y}}{B-b}b^2+\frac{\overline{y}-b}{B-b}B^2-\sum_{j=1}^{m}q_j y_j^2, \tag{7.11}$$

where $\overline{x} = \sum_{i=1}^{n} p_i x_i$ and $\overline{y} = \sum_{j=1}^{m} q_j y_j$, then for every function $f \in \mathscr{K}_c^1(a,B)$ we have

$$\frac{A-\overline{x}}{A-a}f(a)+\frac{\overline{x}-a}{A-a}f(A)-\sum_{i=1}^{n}p_i f(x_i)\leq\frac{B-\overline{y}}{B-b}f(b)+\frac{\overline{y}-b}{B-b}f(B)-\sum_{j=1}^{m}q_j f(y_j). \tag{7.12}$$

Proof. Let X be a discrete random variable taking the value x_i with probability p_i for every $i = 1,2,\ldots,n$ and let Y be a discrete random variable taking the value y_j with probability q_j for every $j = 1,2,\ldots,m$. It can be seen immediately that randon variables X and Y satisfy the conditions from Theorem 7.3, so inequality (7.12) follows directly from (7.8). □

7.3 Levinson's type generalization of the operator Edmundson-Lah-Ribarič inequality

In this section we consider a general form of the Edmundson-Lah-Ribarič inequality for self-adjoint operators in Hilbert space.

Recall that if A is a self-adjoint operator and f is a continuous real function defined on the spectrum $Sp(A)$ of the operator A, then from $f(t) \geq 0$ for every $t \in Sp(A)$ it follows that $f(A) \geq 0$, that is, $f(A)$ is a positive operator on H. Equivalently, if f and g are continuous real functions on $Sp(A)$, then the following property is valid:

$$\text{from } f(t) \geq g(t) \text{ for every } t \in Sp(A) \text{ it follows that } f(A) \geq g(A) \tag{7.13}$$

in the operator order of $\mathscr{B}(H)$.

Lower and upper bound of a self-adjoint operator $X \in \mathscr{B}(H)$ is defined respectively as:

$$m_X = \inf_{||\xi||=1} \langle X\xi, \xi \rangle \quad \text{and} \quad M_X = \sup_{||\xi||=1} \langle X\xi, \xi \rangle.$$

A mapping $\Phi\colon \mathscr{B}(H) \to \mathscr{B}(K)$ is linear if it is additive and homogeneous, that is, if we have $\Phi(\alpha X + \beta Y) = \alpha\Phi(X) + \beta\Phi(Y)$ for every $\alpha, \beta \in \mathbb{C}$ and $X, Y \in \mathscr{B}(H)$. Linear mapping $\Phi\colon \mathscr{B}(H) \to \mathscr{B}(K)$ is positive if it preserves the operator order $\geq$, that is if for a positive operator $A \in \mathscr{B}(H)$ the operator $\Phi(A)$ is also positive. Linear mapping $\Phi\colon \mathscr{B}(H) \to \mathscr{B}(K)$ is unital if it preserves the identity operator, that is if the relation $\Phi(\mathbf{1}_H) = \mathbf{1}_K$ holds.

A continuous function $f : I \to \mathbb{R}$ is operator convex if

$$f(\lambda X + (1-\lambda)Y) \leq \lambda f(X) + (1-\lambda) f(Y)$$

holds for each $\lambda \in [0,1]$ and every pair of self-adjoint operators X and Y on Hilbert space H with spectra in I. When the inequality sign is reversed, function f is operator concave.

If a function $f\colon I \to \mathbb{R}$ is operator convex, then the Jensen operator inequality

$$f(\Phi(X)) \leq \Phi(f(X)) \tag{7.14}$$

holds for every positive unital linear mapping Φ on $\mathscr{B}(H)$ and for every operator $X \in \mathscr{B}_h(H)$ with spectrum contained in the interval I.

Mičić Hot, Pečarić and Praljak in [100] proved a generalization of Levinson's inequality for self-adjoint operators in Hilbert space. Since their result is based on the operator convexity and concavity, before showing the mentioned result, we need to state definition of the class $\mathring{\mathscr{K}}_c^1(I)$ from [100].

Definition 7.2 *Let $f\colon I \to \mathbb{R}$, and let the point c belong to the interior of the interval I. We say that $f \in \mathring{\mathscr{K}}_c^1(I)$ (that is $f \in \mathring{\mathscr{K}}_c^2(I)$) if there exists a constant D such that the function $F(x) = f(x) - \frac{D}{2}x^2$ is operator concave (that is operator convex) on $\langle -\infty, c] \cap I$, and operator convex (that is operator concave) on $[c, +\infty\rangle \cap I$.*

Theorem 7.4 ([100]) *Let $X_i, Y_j \in \mathscr{B}_h(H)$, $i = 1,\dots,n$, $j = 1,\dots,k$ be self-adjoint operators with spectra contained in intervals $[m_X, M_X]$ and $[m_Y, M_Y]$ respectively such that $m_X < M_X \le c \le m_Y < M_Y$. Let $\Phi_i, \Psi_j \colon \mathscr{B}(H) \to \mathscr{B}(K)$, $i = 1,\dots,n$, $j = 1,\dots,k$ be positive linear mappings such that $\sum_{i=1}^{n} \Phi_i(\mathbf{1}_H) = \mathbf{1}_K$ and $\sum_{j=1}^{k} \Psi_j(\mathbf{1}_H) = \mathbf{1}_K$. Let $f \in \mathring{\mathscr{K}}_c^1(m_X, M_Y)$. If*

$$C_1 := \frac{D}{2}\Big[\sum_{i=1}^{n} \Phi_i(X_i^2) - \Big(\sum_{i=1}^{n} \Phi_i(X_i)\Big)^2\Big] \le C_2 := \frac{D}{2}\Big[\sum_{j=1}^{k} \Psi_j(Y_j^2) - \Big(\sum_{j=1}^{k} \Psi_j(Y_j)\Big)^2\Big]$$

holds, then we have

$$\sum_{i=1}^{n} \Phi_i(f(X_i)) - f\Big(\sum_{i=1}^{n} \Phi_i(X_i)\Big) \le C_1 \le C_2 \le \sum_{j=1}^{k} \Psi_j(f(Y_j)) - f\Big(\sum_{j=1}^{k} \Psi_j(Y_j)\Big). \tag{7.15}$$

The following generalization of the Edmundson-Lah-Ribarič inequality for self-adjoint operators in Hilbert space is proved in [49].

Theorem 7.5 ([49]) *Let $A_j \in \mathscr{B}_h(H)$ be self-adjoint operators with spectra contained in the interval $[m, M]$ for some scalars $m < M$, and let $\Phi_j \colon \mathscr{B}(H) \to \mathscr{B}(K)$ be positive linear mappings for $j = 1,\dots,n$ such that $\sum_{j=1}^{n} \Phi_j(\mathbf{1}_H) = \mathbf{1}_K$. If $f \colon [m, M] \to \mathbb{R}$ is a continuous convex function, then we have*

$$\sum_{j=1}^{n} \Phi_j(f(A_j)) \le \frac{M\mathbf{1}_K - \sum_{j=1}^{n} \Phi_j(A_j)}{M - m} f(m) + \frac{\sum_{j=1}^{n} \Phi_j(A_j) - m\mathbf{1}_K}{M - m} f(M). \tag{7.16}$$

First and main result in this section is a Levinson's type generalization of the Edmundson-Lah-Ribarič inequality for operators in Hilbert space, and it was proved in [71] by using a similar method as the one from the previous section.

Theorem 7.6 *Let $X_i, Y_j \in \mathscr{B}_h(H)$ for $i = 1,\dots,n$ and $j = 1,\dots,k$ be self-adjoint operators with spectra contained in the intervals $[m_X, M_X]$ and $[m_Y, M_Y]$ respectively, where $m_X < M_X \le c \le m_Y < M_Y$ are some scalars. Let $\Phi_i, \Psi_j \colon \mathscr{B}(H) \to \mathscr{B}(K)$, $i = 1,\dots,n$, $j = 1,\dots,k$ be positive linear mappings such that $\sum_{i=1}^{n} \Phi_i(\mathbf{1}_H) = \mathbf{1}_K$ and $\sum_{j=1}^{k} \Psi_i(\mathbf{1}_H) = \mathbf{1}_K$ hold. Let $f \in \mathscr{K}_c^1(m_X, M_Y)$. If*

$$\begin{aligned} &\frac{D}{2}\Big[\frac{M_X\mathbf{1}_K - \sum_{i=1}^{n} \Phi_i(X_i)}{M_X - m_X} m_X^2 + \frac{\sum_{i=1}^{n} \Phi_i(X_i) - m_X\mathbf{1}_K}{M_X - m_X} M_X^2 - \sum_{i=1}^{n} \Phi_i(X_i^2)\Big] \\ &= C_1 \le C_2 = \frac{D}{2}\Big[\frac{M_Y\mathbf{1}_K - \sum_{j=1}^{k} \Psi_j(Y_j)}{M_Y - m_Y} m_Y^2 + \frac{\sum_{j=1}^{k} \Psi_j(Y_j) - m_Y\mathbf{1}_K}{M_Y - m_Y} M_Y^2 - \sum_{j=1}^{k} \Psi_j(Y_j^2)\Big] \end{aligned} \tag{7.17}$$

holds, then we have

$$\frac{M_X\mathbf{1}_K - \sum_{i=1}^{n} \Phi_i(X_i)}{M_X - m_X} f(m_X) + \frac{\sum_{i=1}^{n} \Phi_i(X_i) - m_X\mathbf{1}_K}{M_X - m_X} f(M_X) - \sum_{i=1}^{n} \Phi_i(f(X_i))$$

$$\leq C_1 \leq C_2 \leq \frac{M_Y \mathbf{1}_K - \sum_{j=1}^k \Psi_j(Y_j)}{M_Y - m_Y} f(m_Y) + \frac{\sum_{j=1}^k \Psi_j(Y_j) - m_Y \mathbf{1}_K}{M_Y - m_Y} f(M_Y) - \sum_{j=1}^k \Psi_j(f(Y_j)). \tag{7.18}$$

If $f \in \mathscr{K}_c^2(m_X, M_Y)$ and $C_1 \geq C_2$, then the inequality signs in (7.18) are reversed.

Proof. We will only prove the case when $f \in \mathscr{K}_c^1(m_X, M_Y)$. Let $F(x) = f(x) - \frac{D}{2}x^2$, where D is the constant from Definition 7.1. Since the function $F\colon [m_X, c] \to \mathbb{R}$ is concave, the Edmundson-Lah-Ribarič inequality for operators in Hilbert space (7.16) immediately gives us:

$$\begin{aligned}
0 &\geq \frac{M_X \mathbf{1}_K - \sum_{i=1}^n \Phi_i(X_i)}{M_X - m_X} F(m_X) + \frac{\sum_{i=1}^n \Phi_i(X_i) - m_X \mathbf{1}_K}{M_X - m_X} F(M_X) - \sum_{i=1}^n \Phi_i(F(X_i)) \\
&= \frac{M_X \mathbf{1}_K - \sum_{i=1}^n \Phi_i(X_i)}{M_X - m_X} f(m_X) + \frac{\sum_{i=1}^n \Phi_i(X_i) - m_X \mathbf{1}_K}{M_X - m_X} f(M_X) - \sum_{i=1}^n \Phi_i(f(X_i)) \\
&\quad - \frac{D}{2}\left[\frac{M_X \mathbf{1}_K - \sum_{i=1}^n \Phi_i(X_i)}{M_X - m_X} m_X^2 + \frac{\sum_{i=1}^n \Phi_i(X_i) - m_X \mathbf{1}_K}{M_X - m_X} M_X^2 - \sum_{i=1}^n \Phi_i(X_i^2)\right],
\end{aligned}$$

that is, we obtained

$$\frac{M_X \mathbf{1}_K - \sum_{i=1}^n \Phi_i(X_i)}{M_X - m_X} f(m_X) + \frac{\sum_{i=1}^n \Phi_i(X_i) - m_X \mathbf{1}_K}{M_X - m_X} f(M_X) - \sum_{i=1}^n \Phi_i(f(X_i)) \leq C_1. \tag{7.19}$$

In the same way, because the function $F\colon [c, M_Y] \to \mathbb{R}$ is convex, we get

$$\begin{aligned}
0 &\leq \frac{M_Y \mathbf{1}_K - \sum_{j=1}^k \Psi_j(Y_j)}{M_Y - m_Y} F(m_Y) + \frac{\sum_{j=1}^k \Psi_j(Y_j) - m_Y \mathbf{1}_K}{M_Y - m_Y} F(M_Y) - \sum_{j=1}^k \Psi_j(F(Y_j)) \\
&= \frac{M_Y \mathbf{1}_K - \sum_{j=1}^k \Psi_j(Y_j)}{M_Y - m_Y} f(m_Y) + \frac{\sum_{j=1}^k \Psi_j(Y_j) - m_Y \mathbf{1}_K}{M_Y - m_Y} f(M_Y) - \sum_{j=1}^k \Psi_j(f(Y_j)) \\
&\quad - \frac{D}{2}\left[\frac{M_Y \mathbf{1}_K - \sum_{j=1}^k \Psi_j(Y_j)}{M_Y - m_Y} m_Y^2 + \frac{\sum_{j=1}^k \Psi_j(Y_j) - m_Y \mathbf{1}_K}{M_Y - m_Y} M_Y^2 - \sum_{j=1}^k \Psi_j(Y_j^2)\right],
\end{aligned}$$

and after resetting the obtained relation we have

$$C_2 \leq \frac{M_Y \mathbf{1}_K - \sum_{j=1}^k \Psi_j(Y_j)}{M_Y - m_Y} f(m_Y) + \frac{\sum_{j=1}^k \Psi_j(Y_j) - m_Y \mathbf{1}_K}{M_Y - m_Y} f(M_Y) - \sum_{j=1}^k \Psi_j(f(Y_j)). \tag{7.20}$$

Finally, by combining the inequalities (7.19) and (7.20), and taking into account (7.17), we get exactly (7.18), and the claim of the theorem is proved. $\square$

Remark 7.2 Condition (7.17) can be replaced by a stronger one:

$$(D \geq 0 \text{ i } \Delta_X \leq \delta_Y) \ \text{ or } \ (D \leq 0 \text{ and } \Delta_Y \leq \delta_X),$$

where $\delta_X \leq \Delta_X$ (respectively $\delta_Y \leq \Delta_Y$) lower and upper bound of the positive operator X (respectively Y) defined as

$$X = \frac{M_X \mathbf{1}_K - \sum_{i=1}^n \Phi_i(X_i)}{M_X - m_X} m_X^2 + \frac{\sum_{i=1}^n \Phi_i(X_i) - m_X \mathbf{1}_K}{M_X - m_X} M_X^2 - \sum_{i=1}^n \Phi_i(X_i^2)$$

(respectively

$$Y = \frac{M_Y \mathbf{1}_K - \sum_{j=1}^k \Psi_j(Y_j)}{M_Y - m_Y} m_Y^2 + \frac{\sum_{j=1}^k \Psi_j(Y_j) - m_Y \mathbf{1}_K}{M_Y - m_Y} M_Y^2 - \sum_{j=1}^k \Psi_j(Y_j^2)\Big).$$

Remark 7.3 If in the addition the function f is convex (respectively concave), then we have $f''_-(c) \leq D \leq f''_+(c)$ (respectively $f''_+(c) \leq D \leq f''_-(c)$), so the condition (7.17) can be weakened to (see [11])

$$X \leq Y, \quad (\text{respectively } \; Y \leq X).$$

The next result is a simpler version of inequality (7.18).

Corollary 7.3 *Let $X_i, Y_j \in \mathscr{B}_h(H)$, $i = 1,\ldots,n$, $j = 1,\ldots,k$ be self-adjoint operators with spectra contained in $[m_X, M_X]$ and $[m_Y, M_Y]$ respectively, where $m_X < M_X \leq c \leq m_Y < M_Y$ are some scalars. Let $\Phi_i, \Psi_j \colon \mathscr{B}(H) \to \mathscr{B}(K)$, $i = 1,\ldots,n$, $j = 1,\ldots,k$ be positive linear mappings such that $\sum_{i=1}^n \Phi_i(\mathbf{1}_H) = \mathbf{1}_K$ and $\sum_{j=1}^k \Psi_j(\mathbf{1}_H) = \mathbf{1}_K$. Let $f \in \mathscr{K}_c^1(m_X, M_Y)$. If*

$$\begin{aligned} C :&= \frac{M_X \mathbf{1}_K - \sum_{i=1}^n \Phi_i(X_i)}{M_X - m_X} m_X^2 + \frac{\sum_{i=1}^n \Phi_i(X_i) - m_X \mathbf{1}_K}{M_X - m_X} M_X^2 - \sum_{i=1}^n \Phi_i(X_i^2) \\ &= \frac{M_Y \mathbf{1}_K - \sum_{j=1}^k \Psi_j(Y_j)}{M_Y - m_Y} m_Y^2 + \frac{\sum_{j=1}^k \Psi_j(Y_j) - m_Y \mathbf{1}_K}{M_Y - m_Y} M_Y^2 - \sum_{j=1}^k \Psi_j(Y_j^2) \end{aligned} \tag{7.21}$$

holds, then

$$\begin{aligned} &\frac{M_X \mathbf{1}_K - \sum_{i=1}^n \Phi_i(X_i)}{M_X - m_X} f(m_X) + \frac{\sum_{i=1}^n \Phi_i(X_i) - m_X \mathbf{1}_K}{M_X - m_X} f(M_X) - \sum_{i=1}^n \Phi_i(f(X_i)) \\ &\leq C \leq \frac{M_Y \mathbf{1}_K - \sum_{j=1}^k \Psi_j(Y_j)}{M_Y - m_Y} f(m_Y) + \frac{\sum_{j=1}^k \Psi_j(Y_j) - m_Y \mathbf{1}_K}{M_Y - m_Y} f(M_Y) \\ &\qquad - \sum_{j=1}^k \Psi_j(f(Y_j)). \end{aligned} \tag{7.22}$$

If $f \in \mathscr{K}_c^2(m_X, M_Y)$, then the inequality signs in (7.22) are reversed.

7.3.1 Results on scalar product

Mičić Hot, Pečarić and Praljak in [100] also proved a generalization of Levinson's inequality for scalar product of self-adjoint operators in Hilbert space. Unlike their previously stated result, where operator convexity or concavity was required, this result only requires convexity or concavity in the classical sense.

Theorem 7.7 ([100]) *Let $X_i, Y_j \in \mathscr{B}_h(H)$, $i = 1,\dots,n$, $j = 1,\dots,k$ be self-adjoint operators with spectra contained in the intervals $[m_X, M_X]$ and $[m_Y, M_Y]$ respectively such that $m_X < M_X \le c \le m_Y < M_Y$. Let $z_i, w_j \in H$, $i = 1,\dots,n$, $j = 1,\dots,k$ be vectors such that $\sum_{i=1}^{n} ||z_i||^2 = 1$ and $\sum_{j=1}^{k} ||w_j||^2 = 1$. Let $f \in \mathscr{K}_c^1(m_X, M_Y)$. If*

$$C_1 := \frac{D}{2}\sum_{i=1}^{n}\langle (X_i - \bar{X}\mathbf{1}_H)^2 z_i, z_i\rangle \le C_2 := \frac{D}{2}\sum_{j=1}^{k}\langle (Y_j - \bar{Y}\mathbf{1}_H)^2 w_j, w_j\rangle$$

holds, then

$$\sum_{i=1}^{n}\langle f(X_i)z_i, z_i\rangle - f(\bar{X}) \le C_1 \le C_2 \le \sum_{j=1}^{k}\langle f(Y_j)w_j, w_j\rangle - f(\bar{Y}), \tag{7.23}$$

where $\bar{X} = \sum_{i=1}^{n}\langle X_i z_i, z_i\rangle$ and $\bar{Y} = \sum_{j=1}^{k}\langle Y_j w_j, w_j\rangle$.

In this subsection we will derive a Levinson's type generalization of the scalar Edmundson-Lah-Ribarič inequality for operators in Hilbert space. In order to state our assertions, first we need to state a generalized version of the Edmundson-Lah-Ribarič inequality for scalar product which we will need in the proof of the mentioned assertion.

Theorem 7.8 ([49]) *Let $A_1,\dots,A_n$ be self-adjoint operators on Hilbert space H with spectra contained in the interval $[m, M]$ for some scalars $m < M$. If f is a convex function on $[m, M]$, then*

$$\sum_{i=1}^{n}\langle f(A_i)x_i, x_i\rangle \le \frac{M - \sum_{i=1}^{n}\langle A_i x_i, x_i\rangle}{M - m} f(m) + \frac{\sum_{i=1}^{n}\langle A_i x_i, x_i\rangle - m}{M - m} f(M) \tag{7.24}$$

holds for every n-tuple of vectors $x_1,\dots,x_n \in H$ such that $\sum_{i=1}^{n}\|x_i\|^2 = 1$.

The technique used in the proof of the following result is analogous to the one used in the proof of Theorem 7.6, but for the sake of completeness we give it in full.

Theorem 7.9 *Let $X_i, Y_j \in \mathscr{B}_h(H)$, $i = 1,\dots,n$, $j = 1,\dots,k$ be self-adjoint operators with spectra contained in the intervals $[m_X, M_X]$ and $[m_Y, M_Y]$ respectively such that $m_X < M_X \le c \le m_Y < M_Y$. Let $z_i, w_j \in H$, $i = 1,\dots,n$, $j = 1,\dots,k$ be vectors such that $\sum_{i=1}^{n}\|z_i\|^2 = 1$ and $\sum_{j=1}^{k}\|w_j\|^2 = 1$. Let $f \in \mathscr{K}_c^1(m_X, M_Y)$. If*

$$\begin{aligned}&\frac{D}{2}\Big[\frac{M_X - \sum_{i=1}^{n}\langle X_i z_i, z_i\rangle}{M_X - m_X} m_X^2 + \frac{\sum_{i=1}^{n}\langle X_i z_i, z_i\rangle - m_X}{M_X - m_X} M_X^2 - \sum_{i=1}^{n}\langle X_i^2 z_i, z_i\rangle\Big]\\ &= C_1 \le C_2 = \frac{D}{2}\Big[\frac{M_Y - \sum_{j=1}^{k}\langle Y_j w_j, w_j\rangle}{M_Y - m_Y} m_Y^2 + \frac{\sum_{j=1}^{k}\langle Y_j w_j, w_j\rangle - m_Y}{M_Y - m_Y} M_Y^2 - \sum_{j=1}^{k}\langle Y_j^2 w_j, w_j\rangle\Big]\end{aligned} \tag{7.25}$$

holds, then we have

$$\frac{M_X - \sum_{i=1}^{n}\langle X_i z_i, z_i\rangle}{M_X - m_X} f(m_X) + \frac{\sum_{i=1}^{n}\langle X_i z_i, z_i\rangle - m_X}{M_X - m_X} f(M_X) - \sum_{i=1}^{n}\langle f(X_i)z_i, z_i\rangle$$

$$\leq C_1 \leq C_2 \leq \frac{M_Y - \sum_{j=1}^k \langle Y_j w_j, w_j\rangle}{M_Y - m_Y} f(m_Y) + \frac{\sum_{j=1}^k \langle Y_j w_j, w_j\rangle - m_Y}{M_Y - m_Y} f(M_Y) - \sum_{j=1}^k \langle f(Y_j) w_j, w_j\rangle. \tag{7.26}$$

If $f \in \mathscr{K}_c^2(m_X, M_Y)$ and $C_1 \geq C_2$, then th inequality signs in (7.26) reversed.

Proof. As before, we will only prove the case when $f \in \mathscr{K}_c^1(m_X, M_Y)$. Let $F(x) = f(x) - \frac{D}{2}x^2$, where D is the constant from the Definition 7.1. Since $F\colon [m_X, c] \to \mathbb{R}$ is a concave function, the generalized Edmundson-Lah-Ribarič inequality for scalar product (7.24) implies

$$\begin{aligned}
0 &\geq \frac{M_X - \sum_{i=1}^n \langle X_i z_i, z_i\rangle}{M_X - m_X} F(m_X) + \frac{\sum_{i=1}^n \langle X_i z_i, z_i\rangle - m_X}{M_X - m_X} F(M_X) - \sum_{i=1}^n \langle F(X_i) z_i, z_i\rangle \\
&= \frac{M_X - \sum_{i=1}^n \langle X_i z_i, z_i\rangle}{M_X - m_X} f(m_X) + \frac{\sum_{i=1}^n \langle X_i z_i, z_i\rangle - m_X}{M_X - m_X} f(M_X) - \sum_{i=1}^n \langle f(X_i) z_i, z_i\rangle \\
&\quad - \frac{D}{2}\left[\frac{M_X - \sum_{i=1}^n \langle X_i z_i, z_i\rangle}{M_X - m_X} m_X^2 + \frac{\sum_{i=1}^n \langle X_i z_i, z_i\rangle - m_X}{M_X - m_X} M_X^2 - \sum_{i=1}^n \langle X_i^2 z_i, z_i\rangle\right],
\end{aligned}$$

whereby we get

$$\frac{M_X - \sum_{i=1}^n \langle X_i z_i, z_i\rangle}{M_X - m_X} f(m_X) + \frac{\sum_{i=1}^n \langle X_i z_i, z_i\rangle - m_X}{M_X - m_X} f(M_X) - \sum_{i=1}^n \langle f(X_i) z_i, z_i\rangle \leq C_1. \tag{7.27}$$

Due to convexity of $F : [c, M_Y] \to \mathbb{R}$, in a similar way we get

$$\begin{aligned}
0 &\leq \frac{M_Y - \sum_{j=1}^k \langle Y_j w_j, w_j\rangle}{M_Y - m_Y} F(m_Y) + \frac{\sum_{j=1}^k \langle Y_j w_j, w_j\rangle - m_Y}{M_Y - m_Y} F(M_Y) - \sum_{j=1}^k \langle F(Y_j) w_j, w_j\rangle \\
&= \frac{M_Y - \sum_{j=1}^k \langle Y_j w_j, w_j\rangle}{M_Y - m_Y} f(m_Y) + \frac{\sum_{j=1}^k \langle Y_j w_j, w_j\rangle - m_Y}{M_Y - m_Y} f(M_Y) - \sum_{j=1}^k \langle f(Y_j) w_j, w_j\rangle \\
&\quad - \frac{D}{2}\left[\frac{M_Y - \sum_{j=1}^k \langle Y_j w_j, w_j\rangle}{M_Y - m_Y} m_Y^2 + \frac{\sum_{j=1}^k \langle Y_j w_j, w_j\rangle - m_Y}{M_Y - m_Y} M_Y^2 - \sum_{j=1}^k \langle Y_j^2 w_j, w_j\rangle\right],
\end{aligned}$$

and after resetting, the upper relation becomes

$$C_2 \leq \frac{M_Y - \sum_{j=1}^k \langle Y_j w_j, w_j\rangle}{M_Y - m_Y} f(m_Y) + \frac{\sum_{j=1}^k \langle Y_j w_j, w_j\rangle - m_Y}{M_Y - m_Y} f(M_Y) - \sum_{j=1}^k \langle f(Y_j) w_j, w_j\rangle. \tag{7.28}$$

Inequality (7.26) directly follows as a combination of the inequalities (7.27) and (7.28) and taking into account the condition (7.25). □

Remark 7.4 Mappings $\Phi_i, \Psi_J\colon \mathscr{B}(H) \to \mathscr{B}(K)$, $i = 1, \dots, n$, $j = 1, \dots, k$ from the previous section can be chosen in the following way. Let $z_1, \dots, z_n$ i $w_1, \dots, w_k$ be vectors from H such that $\sum_{i=1}^{n}\|z_i\|^2=1$ and $\sum_{j=1}^{k}\|w_j\|^2=1$. For any $A\in\mathscr{B}(H)$ we define a mapping Φ_i with $\Phi_i(A)=\langle Az_i,z_i\rangle$, and a mapping Ψ_j with $\Psi_j(A)=\langle Aw_j,w_j\rangle$. Then $\Phi_i,\Psi_j\colon\mathscr{B}(H)\to\mathbb{R}$ are positive linear functionals such that $\sum_{i=1}^{n}\Phi_i(\mathbf{1}_H)=\sum_{j=1}^{k}\Psi_j(\mathbf{1}_H)=1$. Now we see that $\sum_{i=1}^{n}\Phi_i(X_i)=\sum_{i=1}^{n}\langle X_iz_i,z_i\rangle$ i $\sum_{j=1}^{k}\Psi_j(Y_j)=\sum_{j=1}^{k}\langle Y_jw_j,w_j\rangle$ holds. That way Theorem 7.9 directly follows from Theorem 7.6 (see [49]).

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