Chapter 1

Difference type converses for linear functionals

This chapter begins with an overview of some important results related to the Jensen and Edmundson-Lah-Ribarič inequality for positive linear functionals which are known from earlier. Further, some difference type converses of the mentioned results will be shown, as well as their refinements and improvements. Finally, those improvements will be applied to generalized means and some famous classical inequalities (the ones of Hölder, Hermite-Hadamard, Giaccardi and Petrović). In that way we will get converses of listed inequalities that provide us with an upper bound for the difference of their right and left sides.

1.1 Introduction

Let E be a non-empty set and L a vector space of real functions $f\colon E \to \mathbb{R}$ with the following properties:

(L1): $f, g \in L \Rightarrow (af + bg) \in L$ for all $a, b \in \mathbb{R}$;

(L2): $\mathbf{1} \in L$, that is, if $f(t) = 1$ for every $t \in E$, then $f \in L$.

(L3): if $f, g \in L$, then $\min\{f, g\} \in L$ or $\max\{f, g\} \in L$.

Obviously, $(\mathbb{R}^E, \leq)$ (with standard ordering) is a lattice. It can also be easily verified that a subspace $(X \subseteq \mathbb{R}^E)$ is a lattice if and only if $x \in X$ implies $|x| \in X$. This is a simple consequence of the fact that for every $x \in X$ the functions $|x|$, x^- and x^+ can be defined by

$$|x|(t) = |x(t)|, \;\; x^+(t) = \max\{0, x(t)\}, \;\; x^-(t) = -\min\{0, x(t)\}, \;\; t \in E$$

and

$$x^+ + x^- = |x|, \;\; x^+ - x^- = x,$$

$$\min\{x,y\} = \frac{1}{2}(x+y-|x-y|), \;\; \max\{x,y\} = \frac{1}{2}(x+y+|x-y|).$$

We also study positive linear functionals $A\colon L \to \mathbb{R}$, that is, we assume:

(A1): $A(af+bg) = aA(f) + bA(g)$ for $f, g \in L$ and $a, b \in \mathbb{R}$;

(A2): $f \in L$, $f(t) \geq 0$ for every $t \in E \Rightarrow A(f) \geq 0$.

We say that a functional A is normalized if $A(\mathbf{1}) = 1$.

Throughout this chapter, if a function is defined on an interval $[m,M]$ without any further emphasis we assume that the bounds of that interval are finite.

Jessen [76] gave the following generalization of Jensen's inequality for convex functions (see also [124, p. 47]):

Theorem 1.1 ([76]) *Let L be a vector space of real functions defined on a non-empty set E that has properties $(L1)$ and $(L2)$, and let us assume that ϕ is a continuous convex function on an interval $I \subset \mathbb{R}$. If A is a normalized positive linear functional, then for every $f \in L$ such that $\phi(f) \in L$ we have $A(f) \in I$ and*

$$\phi(A(f)) \leq A(\phi(f)). \tag{1.1}$$

Next result is a generalization of the Edmundson-Lah-Ribarič inequality for linear functionals and it was proved by Beesack and Pečarić in [14] (see also [124, p. 98]):

Theorem 1.2 ([14]) *Let ϕ be a convex function on $I = [m,M]$, let L be a vector space of real functions defined on a non-empty set E that has properties $(L1)$ and $(L2)$, and let A be a normalized positive linear functional. Then for every $f \in L$ such that $\phi(f) \in L$ (so $m \leq f(t) \leq M$ for all $t \in E$), we have*

$$A(\phi(f)) \leq \frac{M - A(f)}{M-m}\phi(m) + \frac{A(f) - m}{M-m}\phi(M). \tag{1.2}$$

Dragomir in [37] studied a measure space $(\Omega, \mathscr{A}, \mu)$ which consists of a set Ω, σ-algebra $\mathscr{A}$ of subsets of Ω and a countably additive and positive measure μ on $\mathscr{A}$ with values in $\mathbb{R} \cup \{\infty\}$. For a μ-measurable function $w\colon \Omega \to \mathbb{R}$ such that $w(x) \geq 0$ for μ-a.e. (almost every) $x \in \Omega$, he considered a Lebesgue space

$$L_w(\Omega, \mu) := \{f\colon \Omega \to \mathbb{R}, f \text{ is } \mu-\text{measurable and } \textstyle\int_\Omega w(x)|f(x)|\mathrm{d}\mu(x) < \infty\},$$

and proved the following converse of Jensen's inequality.

Theorem 1.3 ([37]) *Let $\phi\colon I\to\mathbb{R}$ be a continuous convex function on an interval of real numbers I and let $m,M\in\mathbb{R}$, $m<M$ be such that the interval $[m,M]$ belongs to the interior of I. Let $w>0$ be such that $\int w\,\mathrm{d}\mu=1$. If $f\colon\Omega\to\mathbb{R}$ is μ-measurable, satisfies the bounds*

$$-\infty<m\le f(t)\le M<\infty \text{ for } \mu\text{-a.e. } t\in\Omega$$

and such that $f,\phi\circ f\in L_w(\Omega,\mu)$, then

$$\begin{aligned}
0&\le\int_\Omega w(t)\phi(f(t))\mathrm{d}\mu(t)-\phi(\bar f_{\Omega,w})\\
&\le (M-\bar f_{\Omega,w})(\bar f_{\Omega,w}-m)\frac{\phi'_-(M)-\phi'_+(m)}{M-m}\\
&\le\frac14(M-m)(\phi'_-(M)-\phi'_+(m)),
\end{aligned}\tag{1.3}$$

where $\bar f_{\Omega,w}:=\int_\Omega w(t)f(t)\mathrm{d}\mu(t)\in[m,M]$.

In [38] Dragomir obtained a refinement of the previous result that we state in the following theorem.

Theorem 1.4 ([38]) *Let $\phi\colon I\to\mathbb{R}$ be a continuous convex function on an interval of real numbers I and let $m,M\in\mathbb{R}$, $m<M$ be such that the interval $[m,M]$ belongs to the interior of I. Let $w>0$ be such that $\int w\,\mathrm{d}\mu=1$. If $f\colon\Omega\to\mathbb{R}$ is μ-measurable, satisfies the bounds*

$$-\infty<m\le f(t)\le M<\infty \text{ for } \mu\text{-a.e. } t\in\Omega$$

and such that $f,\phi\circ f\in L_w(\Omega,\mu)$, then

$$\begin{aligned}
0&\le\int_\Omega w(t)\phi(f(t))\mathrm{d}\mu(t)-\phi(\bar f_{\Omega,w})\\
&\le\frac{(M-\bar f_{\Omega,w})(\bar f_{\Omega,w}-m)}{M-m}\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M)\\
&\le (M-\bar f_{\Omega,w})(\bar f_{\Omega,w}-m)\frac{\phi'_-(M)-\phi'_+(m)}{M-m}\\
&\le\frac14(M-m)(\phi'_-(M)-\phi'_+(m)),
\end{aligned}\tag{1.4}$$

where $\bar f_{\Omega,w}:=\int_\Omega w(t)f(t)\mathrm{d}\mu(t)\in[m,M]$ and $\Psi_\phi(\cdot;m,M)\colon\langle m,M\rangle\to\mathbb{R}$ is defined by

$$\Psi_\phi(t;m,M)=\frac{\phi(M)-\phi(t)}{M-t}-\frac{\phi(t)-\phi(m)}{t-m}.$$

We also have inequalities

$$\begin{aligned}
0&\le\int_\Omega w(t)\phi(f(t))\mathrm{d}\mu(t)-\phi(\bar f_{\Omega,w})\le\frac14(M-m)\Psi_\phi(t;m,M)\\
&\le\frac14(M-m)(\phi'_-(M)-\phi'_+(m)),
\end{aligned}\tag{1.5}$$

where $\bar f_{\Omega,w}\in\langle m,M\rangle$.

1.2 Converses of the Jensen and Edmundson-Lah-Ribarič inequality for linear functionals

Results that follow are obtained in [68] and they give an upper bound for the difference between the right and left side of the Jensen and Edmundson-Lah-Ribarič inequality respectively. First theorem is also a generalization of Dragomir's result (1.3) for linear functionals.

Theorem 1.5 *Let ϕ be a continuous convex function on the interval I whose interior contains interval $[m,M]$, let L be a vector space of real functions defined on a non-empty set E such that it has properties $(L1)$ and $(L2)$. Let A be any normalized positive linear functional on L. Then for every function $f \in L$ such that $\phi(f) \in L$ and which satisfies the bounds $m \leq f(t) \leq M$ for every $t \in E$ we have*

$$\begin{aligned} 0 &\leq A(\phi(f)) - \phi(A(f)) \\ &\leq (M - A(f))(A(f) - m)\frac{\phi'_-(M) - \phi'_+(m)}{M - m} \\ &\leq \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)). \end{aligned} \tag{1.6}$$

If the function ϕ is concave on I, then the inequality signs in (1.6) are reversed.

Proof. Let ϕ be a convex function. The first inequality follows directly from Theorem 1.1. According to Theorem 1.2 we have

$$A(\phi(f)) - \phi(A(f)) \leq \frac{M - A(f)}{M - m}\phi(m) + \frac{A(f) - m}{M - m}\phi(M) - \phi(A(f)) =: z.$$

Because of the convexity of the function ϕ, the gradient inequality

$$\phi(t) - \phi(M) \geq \phi'_-(M)(t - M)$$

holds for every $t \in [m,M]$. If we multiply this inequality by $(t-m) \geq 0$, we get

$$(t-m)\phi(t) - (t-m)\phi(M) \geq \phi'_-(M)(t-M)(t-m), \quad t \in [m,M] \tag{1.7}$$

In a similar manner we obtain:

$$(M-t)\phi(t) - (M-t)\phi(m) \geq \phi'_+(m)(t-m)(M-t), \quad t \in [m,M] \tag{1.8}$$

When we add up (1.7) and (1.8) and then divide by $(m-M)$, we get that for every $t \in [m,M]$ it holds:

$$\frac{(t-m)\phi(M) + (M-t)\phi(m)}{M-m} - \phi(t) \leq \frac{(M-t)(t-m)}{M-m}(\phi'_-(M) - \phi'_+(m)). \tag{1.9}$$

Since $A(f) \in [m,M]$, in the previous relation we can replace t with $A(f)$ and obtain the following

$$z \leq \frac{(M-A(f))(A(f)-m)}{M-m}(\phi'_-(M)-\phi'_+(m)),$$

what is exactly the second inequality in (1.6).

To prove the third inequlity in (1.6), we need to notice that inequality

$$\frac{1}{M-m}(M-t)(t-m) \leq \frac{1}{4}(M-m),$$

holds for every $t \in [m,M]$, and this proves the claim of the theorem.

If ϕ is a concave function, then the function $-\phi$ is convex, and we can apply inequalities (1.6) to the function $-\phi$, and reversed inequalities follow after multiplying by -1.

□

Remark 1.1 Observe that in the statement of Theorem 1.5 interval $[m,M]$ needs to belong to the interior of the interval I. This condition assures finiteness of the one-sided derivatives in (1.6). Without this assumption these derivatives might be infinite.

Theorem 1.6 *Let ϕ be a continuous convex function on the interval I whose interior contains interval $[m,M]$, let L be a vector space of real functions defined on a non-empty set E such that it has properties $(L1)$ and $(L2)$. Let A be any normalized positive linear functional on L. Then for every function $f \in L$ such that $\phi(f) \in L$ and which satisfies the bounds $m \leq f(t) \leq M$ for every $t \in E$ we have*

$$\begin{aligned} 0 &\leq \frac{M-A(f)}{M-m}\phi(m)+\frac{A(f)-m}{M-m}\phi(M)-A(\phi(f)) \\ &\leq \frac{\phi'_-(M)-\phi'_+(m)}{M-m}A([M-f][f-m]) \\ &\leq \frac{\phi'_-(M)-\phi'_+(m)}{M-m}(M-A(f))(A(f)-m) \\ &\leq \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)). \end{aligned} \tag{1.10}$$

If ϕ is concave, then the inequality signs in (1.10) are reversed.

Proof. Let ϕ be a convex function. The first inequality from (1.10) is obtained from (1.2) by subtracting $\phi(A(f))$ from both sides of the inequality. Since $f(t) \in [m,M]$, we can replace t by $f(t)$ i the relation (1.9), which gives us

$$\frac{M-f(t)}{M-m}\phi(m)+\frac{f(t)-m}{M-m}\phi(M)-\phi(f(t)) \leq \frac{(M-f(t))(f(t)-m)}{M-m}(\phi'_-(M)-\phi'_+(m)).$$

Function $h(t)=(M-t)(t-m)$ is concave on $[m,M]$, so when we apply the functional A to the previous inequality, because of its linearity and Jensen's inequality (1.1) we get the second inequality from (1.10):

$$\begin{aligned}
&\frac{M-A(f)}{M-m}\phi(m)+\frac{A(f)-m}{M-m}\phi(M)-A(\phi(f)) \\
&\quad\leq \frac{(\phi'_-(m)-\phi'_+(m))}{M-m}A([M-f][f-m]) \\
&\quad\leq \frac{\phi'_-(M)-\phi'_+(m)}{M-m}(M-A(f))(A(f)-m).
\end{aligned}$$

To prove the last inequality from (1.10), we need to notice that for every $t\in[m,M]$ we have $h(t)\leq\frac{1}{4}(M-m)^2$. Since $A(f)\in[m,M]$, we also have

$$h(A(f))\leq\frac{1}{4}(M-m)^2,$$

which completes the proof. □

Remark 1.2 Under the assumptions from the previous two theorems, let l be a linear function through points $(m,f(m))$ and $(M,f(M))$. Since ϕ is a convex function on $[m,M]$, the following relation

$$\phi(A(f))\leq A(\phi(f))\leq l(A(f))$$

holds for every $f\in L$ such that $\phi(f)\in L$.
From Theorem 1.5 and Theorem 1.6 we see that both differences

$$A(\phi(f))-\phi(A(f)) \text{ and } l(A(f))-A(\phi(f))$$

have the same estimation, so one can see that, in a weak sense, $A(\phi(f))$ is almost the mid point point between $\phi(A(f))$ and $l(A(f))$.

The following results are proved in [69], and they give refinements of sequences of inequalities obtained in Theorem 1.5 and Theorem 1.6. The first theorem that follows is also a generalization of Dragomir's results (1.4) and (1.5).

Theorem 1.7 *Let ϕ be a continuous convex function on the interval I whose interior contains interval $[m,M]$, let L be a vector space of real functions defined on a non-empty set E such that it has properties $(L1)$ and $(L2)$. Let A be any normalized positive linear functional on L. Then for every function $f\in L$ such that $\phi(f)\in L$ and which satisfies the bounds $m\leq f(t)\leq M$ for every $t\in E$ we have*

$$\begin{aligned}
0&\leq A(\phi(f))-\phi(A(f)) \\
&\leq (M-A(f))(A(f)-m)\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M) \\
&\leq (M-A(f))(A(f)-m)\frac{\phi'_-(M)-\phi'_+(m)}{M-m} \\
&\leq \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)).
\end{aligned} \tag{1.11}$$

We also have inequalities

$$
\begin{aligned}
0 \le A(\phi(f)) - \phi(A(f)) &\le \frac{1}{4}(M-m)^2 \Psi_\phi(A(f);m,M) \\
&\le \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)),
\end{aligned} \tag{1.12}
$$

where $\Psi_\phi(\cdot;m,M)\colon \langle m,M\rangle \to \mathbb{R}$ *is defined by*

$$
\Psi_\phi(t;m,M) = \frac{1}{M-m}\Big(\frac{\phi(M)-\phi(t)}{M-t} - \frac{\phi(t)-\phi(m)}{t-m}\Big). \tag{1.13}
$$

If ϕ *is concave on I, then the inequality signs are reversed.*

Proof. Let ϕ be a convex function. If $A(f) = m$ or $A(f) = M$, inequalities are trivial. Let us assume that $A(f) \in \langle m,M\rangle$.

The first inequality from (1.11) i (1.12) follows directly from Theorem 1.1. According to Theorem 1.2 we have

$$
\begin{aligned}
A(\phi(f)) - \phi(A(f)) &\le \frac{M-A(f)}{M-m}\phi(m) + \frac{A(f)-m}{M-m}\phi(M) - \phi(A(f)) \\
&= \frac{(M-A(f))(A(f)-m)}{M-m}\Big\{\frac{\phi(M)-\phi(A(f))}{M-A(f)} - \frac{\phi(A(f))-\phi(m)}{A(f)-m}\Big\} \\
&= (M-A(f))(A(f)-m)\Psi_\phi(A(f);m,M) \\
&\le (M-A(f))(A(f)-m) \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M),
\end{aligned}
$$

and we see that the second inequality from (1.11) holds. Further,

$$
\begin{aligned}
\sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M) &= \frac{1}{M-m} \sup_{t\in\langle m,M\rangle} \Big\{\frac{\phi(M)-\phi(t)}{M-t} - \frac{\phi(t)-\phi(m)}{t-m}\Big\} \\
&\le \frac{1}{M-m}\Big(\sup_{t\in\langle m,M\rangle} \frac{\phi(M)-\phi(t)}{M-t} + \sup_{t\in\langle m,M\rangle} \frac{-(\phi(t)-\phi(m))}{t-m}\Big) \\
&= \frac{1}{M-m}\Big(\sup_{t\in\langle m,M\rangle} \frac{\phi(M)-\phi(t)}{M-t} - \inf_{t\in\langle m,M\rangle} \frac{\phi(t)-\phi(m)}{t-m}\Big) \\
&= \frac{\phi'_-(M) - \phi'_+(m)}{M-m},
\end{aligned}
$$

which proves the third inequality from (1.11). The last inequality in (1.11) follows from the fact that for every $t \in [m,M]$ we have $\dfrac{(M-t)(t-m)}{M-m} \le \dfrac{1}{4}(M-m)$. Since $A(f) \in [m,M]$, we can replace t with $A(f)$ in the previous inequality.

The proof for inequalities (1.12) is obvious from the proof for (1.11). □

Remark 1.3 Observe that $\Psi_\phi(\cdot;m,M)$, defined in (1.13), is actually second order divided difference $[m,t,M]\phi$ of the function ϕ in points m, t and M for every $t \in \langle m,M\rangle$.

In order to prove a converse of the Edmundson-Lah-Ribarič inequality, first we need the following result from [69].

Lemma 1.1 *Let ϕ be a convex function on an interval of real numbers I, and let $m, M \in \mathbb{R}$, $m < M$ be such that the interval $[m,M]$ belongs to the interior of I. Then for every $t \in [m,M]$ the following inequalities hold:*

$$\begin{aligned}\Delta_\phi(t;m,M) &= \frac{t-m}{M-m}\phi(M) + \frac{M-t}{M-m}\phi(m) - \phi(t) \\ &\le (M-t)(t-m) \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M) \\ &\le \frac{(M-t)(t-m)}{M-m}(\phi'_-(M) - \phi'_+(m)) \\ &\le \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)). \qquad (1.14)\end{aligned}$$

We also have

$$\Delta_\phi(t;m,M) \le \frac{1}{4}(M-m)^2\Psi_\phi(t;m,M) \le \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)),$$

where $\Psi_\phi(\cdot;m,M)\colon \langle m,M\rangle \to \mathbb{R}$ is defined by (1.13) If the function ϕ is concave, then the inequality signs are reversed.

Proof. Let ϕ be a convex function. If $t = m$ or $t = M$, inequalities are trivial. For any $t \in \langle m,M\rangle$ it holds

$$\begin{aligned}\Delta_\phi(t;m,M) &= \frac{t-m}{M-m}\phi(M) + \frac{M-t}{M-m}\phi(m) - \phi(t) \\ &= \frac{(M-t)(t-m)}{M-m}\left[\frac{\phi(M)-\phi(t)}{M-t} - \frac{\phi(t)-\phi(m)}{t-m}\right] \\ &= (M-t)(t-m)\Psi_\phi(t;m,M) \\ &\le (M-t)(t-m) \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M),\end{aligned}$$

which is exactly the first inequality from (1.14). The second inequality follows directly from:

$$\begin{aligned}\sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M) &= \frac{1}{M-m} \sup_{t\in\langle m,M\rangle} \left\{\frac{\phi(M)-\phi(t)}{M-t} - \frac{\phi(t)-\phi(m)}{t-m}\right\} \\ &\le \frac{1}{M-m}\left(\sup_{t\in\langle m,M\rangle} \frac{\phi(M)-\phi(t)}{M-t} + \sup_{t\in\langle m,M\rangle} \frac{-(\phi(t)-\phi(m))}{t-m}\right) \\ &= \frac{1}{M-m}\left(\sup_{t\in\langle m,M\rangle} \frac{\phi(M)-\phi(t)}{M-t} - \inf_{t\in\langle m,M\rangle} \frac{\phi(t)-\phi(m)}{t-m}\right) \\ &= \frac{\phi'_-(M)-\phi'_+(m)}{M-m}.\end{aligned}$$

The last inequality from (1.14) follows directly from

$$\frac{(M-t)(t-m)}{M-m} \leq \frac{1}{4}(M-m) \text{ for every } t \in [m,M].$$

The proof of the inequalities (1.1) is clear from the proof of (1.14). □

Theorem 1.8 *Let ϕ be a continuous convex function on the interval I whose interior contains interval $[m,M]$, let L be a vector space of real functions defined on a non-empty set E such that it has properties $(L1)$ and $(L2)$. Let A be any normalized positive linear functional on L. Then for every function $f \in L$ such that $\phi(f) \in L$ and which satisfies the bounds $m \leq f(t) \leq M$ for every $t \in E$ we have*

(i)

$$\begin{aligned} 0 &\leq \frac{A(f)-m}{M-m}\phi(M) + \frac{M-A(f)}{M-m}\phi(m) - A(\phi(f)) \\ &\leq A[(M-f)(f-m)] \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M) \\ &\leq \frac{A[(M-f)(f-m)]}{M-m}(\phi'_-(M) - \phi'_+(m)) \\ &\leq \frac{(M-A(f))(A(f)-m)}{M-m}(\phi'_-(M) - \phi'_+(m)) \\ &\leq \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)) \end{aligned} \tag{1.15}$$

(ii)

$$\begin{aligned} 0 &\leq \frac{A(f)-m}{M-m}\phi(M) + \frac{M-A(f)}{M-m}\phi(m) - A(\phi(f)) \\ &\leq A[(M-f)(f-m)] \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M) \\ &\leq (M-A(f))(A(f)-m) \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M) \\ &\leq \frac{(M-A(f))(A(f)-m)}{M-m}(\phi'_-(M) - \phi'_+(m)) \\ &\leq \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)) \end{aligned} \tag{1.16}$$

(iii)

$$\begin{aligned} 0 &\leq \frac{A(f)-m}{M-m}\phi(M) + \frac{M-A(f)}{M-m}\phi(m) - A(\phi(f)) \\ &\leq \frac{1}{4}(M-m)^2 A(\Psi_\phi(t;m,M)) \\ &\leq \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)) \end{aligned} \tag{1.17}$$

where $\Psi_\phi(\cdot;m,M)\colon \langle m,M\rangle \to \mathbb{R}$ is defined in (1.13). If the function ϕ is concave, then the inequality signs are reversed.

Proof. Let ϕ be a convex function. The first inequalities from (1.15), (1.16) and (1.17) follow directly from Theorem 1.2.

Since f satisfies the bounds $m \le f(t) \le M$ for every $t \in [m,M]$, we can replace t with $f(t)$ in (1.14) and (1.1) from Lemma 1.1 and obtain

$$\begin{aligned}
&\frac{f(t)-m}{M-m}\phi(M)+\frac{M-f(t)}{M-m}\phi(m)-\phi(f(t))\\
&\quad\le (M-f(t))(f(t)-m)\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M)\\
&\quad\le \frac{(M-f(t))(f(t)-m)}{M-m}(\phi'_-(M)-\phi'_+(m))\\
&\quad\le \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m))
\end{aligned}$$

and

$$\begin{aligned}
&\frac{f(t)-m}{M-m}\phi(M)+\frac{M-f(t)}{M-m}\phi(m)-\phi(f(t))\\
&\quad\le \frac{1}{4}(M-m)^2\Psi_\phi(f;m,M)\\
&\quad\le \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)).
\end{aligned}$$

Next, we apply linear functional A, which is normalized, to the previous sequences of inequalities, and that gives us (1.17) and first three inequalities from (1.15) respectively. Since for every $t \in [m,M]$ we have $\dfrac{(M-t)(t-m)}{M-m} \le \dfrac{1}{4}(M-m)$, the same inequality holds for $A(f) \in [m,M]$. In that way we get the last inequality from (1.15).

The first inequality from (1.16) is the same as the first inequality from (1.15). Function $g(t) = (M-t)(t-m)$ is concave, so according to Jessen's inequality (1.1) we have

$$A([M-f][f-m]) \le (M-A(f))(A(f)-m),$$

which provides the second inequality from (1.16). In the proof of Lemma 1.1 we showed that

$$\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M) \le \frac{\phi'_-(M)-\phi'_+(m)}{M-m},$$

so the third inequality from (1.16) easily follows. As before, the last inequality in (1.16) follows from $\dfrac{(M-A(f))(A(f)-m)}{M-m} \le \dfrac{1}{4}(M-m)$. □

Remark 1.4 The function ϕ is defined on the interval I whose interior contains the interval $[m,M]$. This condition ensures finiteness of the one-sided derivatives in points m and M. Then

$$\lim_{t\to m^+}\Psi_\phi(t;m,M) = \frac{1}{M-m}\left[\frac{\phi(M)-\phi(m)}{M-m}-\phi'_+(m)\right]$$