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New Developments for Jensen and Lah-Ribarič Inequalities

Current Trends in Convex Analysis

Maja Andrić, Vera Čuljak, Đilda Pečarić, Josip Pečarić and Jurica Perić

ELEMENT

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Preface

Two known inequalities provide the bounds of the integral $\int_a^b p(x)f(w(x))dx$ – while the Jensen inequality gives the lower, the Lah-Ribarič inequality gives its upper bound:

The Jensen inequality. Let $w : [a,b] \to \mathbb{R}$ be an integrable function and let $p : [a,b] \to \mathbb{R}$ be a nonnegative function. If f is a convex function given on an interval I such that $w([a,b]) \subseteq I$, then

$$f\left(\frac{1}{\int_a^b p(x)dx}\int_a^b p(x)w(x)\,dx\right) \le \frac{1}{\int_a^b p(x)dx}\int_a^b p(x)f(w(x))\,dx.$$

The Lah-Ribarič inequality. Let $w : [a,b] \to \mathbb{R}$ be an integrable function such that $m \le w(x) \le M$ for $x \in [a,b]$, $m < M$, and let $p : [a,b] \to \mathbb{R}$ be a nonnegative function. If f is a convex function given on an interval I such that $[m,M] \subseteq I$, then

$$\frac{1}{\int_a^b p(x)dx}\int_a^b p(x)f(w(x))\,dx \le \frac{M-\bar{w}}{M-m}f(m) + \frac{\bar{w}-m}{M-m}f(M),$$

where

$$\bar{w} = \frac{\int_a^b p(x)w(x)dx}{\int_a^b p(x)dx}.$$

These two important inequalities are the motivation for this book, in which we want to present recent progress and current trends involving them.

This book is divided in three chapters. The first chapter gives refinements of the Jensen and the Lah-Ribarič inequality, for their discrete and integral forms. The technique used in proving these refinements can be found in the proof of the Jensen-Boas inequality.

Using this results, a refinement of the integral Hölder and discrete Hölder inequality, and refinements of some inequalities for power means and quasi arithmetic means are obtained. Also from the refinement of the integral forms we get a refinement of the famous Hermite-Hadamard inequality.

As applications of these refinements, in the last part of the chapter we give some interesting estimates for the Csiszár divergence (discrete and integral case), and for the discrete case we also consider the Zipf-Mandelbrot law.

In Chapter 2, we consider a generalization of the Jensen-McShane inequality for normalized positive isotonic linear functional and convex (concave) functions defined on a rectangle. We present the sequences of inequalities involving McShane generalization of Jensen's inequality. As applications of these inequalities, for various choices of the functional F, we present extensions of known inequalities: the Diaz-Metcalf type inequalities for bounded random variables, the Feyér and the Lupaş type inequalities for a function of two variables and inequalities of the Petrović type for two non-negative real n-tuples.

A conversion of the Jensen-McShane inequality is obtained by the two-variables function. Under special conditions, the Gheorghiu-type inequality is proved.

The last chapter is dedicated to the recently introduced class of $(h,g;m)$-convex functions, which unifies a certain range of convexity, thus allowing the generalizations of known results. For this class, we present several types of inequalities such as Hermite-Hadamard, Fejér, Lah-Ribarič and Jensen, which generalize and extend corresponding inequalities. From Lah-Ribarič type inequalities for $(h,g;m)$-convex functions we obtain inequalities of Giaccardi, Popoviciu and Petrović. We also point out some special refined results.

At the end, we use fractional calculus to obtain fractional version of the Hermite-Hadamard inequality, involving Riemann-Liouville fractional integral operators, which contain extended generalized Mittag-Leffler functions as their kernel. As an application, the upper bounds of fractional integral operators for $(h,g;m)$-convex functions are given.

Authors

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Chapter 1

Refinements of Jensen's and the Lah-Ribarič inequalities and applications to the Csiszár divergence

Research of the classical inequalities, such as the Jensen, the Hölder and similar, has experienced great expansion. These inequalities first appeared in discrete and integral forms, and then many generalizations and improvements have been proved. Lately, they are proven to be very useful in information theory.

Since all of these inequalities are related to the class of convex functions, we start with the definition of convex functions.

Definition 1.1 *Let I be an interval in $\mathbb{R}$. Function $f: I \to \mathbb{R}$ is said to be a convex function on I if for all $x, y \in I$ and all $\lambda \in [0,1]$*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

holds. If inequality is strict for all $x, y \in I$, $x \neq y$ and for all $\lambda \in (0,1)$, then f is said to be strictly convex. If the inequality is reversed, then f is said to be concave.

Jensen's inequality is one of the most famous inequalities in convex analysis, which special cases are other well-known inequalities (such as Hölder's inequality, A-G-H inequality, etc.). Beside mathematics, it has many applications in statistics, information theory and engineering.

Theorem 1.1 (JENSEN'S INEQUALITY) *Let I be an interval in $\mathbb{R}$ and $f\colon I \to \mathbb{R}$ a convex function. If $\boldsymbol{x} = (x_1,\ldots,x_n)$ is any n-tuple in I^n and $\boldsymbol{p} = (p_1,\ldots,p_n)$ a nonnegative n-tuple such that $P_n = \sum_{i=1}^n p_i > 0$, then the following inequality holds:*

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i). \tag{1.1}$$

If f is strictly convex then (1.1) is strict unless $x_i = c$ for all $i \in \{j\colon p_j > 0\}$. If f is concave, then (1.1) is reversed.

Strongly related to Jensen's inequality is the converse Jensen inequality. One of the most famous variants of the converse inequality is the Lah-Ribarič inequality (see [11]).

Theorem 1.2 (LAH-RIBARIČ INEQUALITY) *Let $f\colon I \to \mathbb{R}$ be a convex function on I, $[m,M] \subset I$, $-\infty < m < M < +\infty$. Let $\boldsymbol{p}$ be as in Theorem 1.1, $\boldsymbol{x} = (x_1,\ldots,x_n)$ is any n-tuple in $[m,M]^n$ and $\bar{x} = \frac{1}{P_n}\sum_{i=1}^n p_i x_i$. Then the following inequality holds:*

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \leq \frac{M-\bar{x}}{M-m} f(m) + \frac{\bar{x}-m}{M-m} f(M). \tag{1.2}$$

If f is strictly convex then (1.2) is strict unless $x_i \in \{m,M\}$ for all $i \in \{j\colon p_j > 0\}$.

The Lah-Ribarič inequality has been largely investigated and the interested reader can find many related results in the recent literature as well as in monographs such as [13] and [16]. It is interesting to find further refinements of the above inequality.

Integral form of the Jensen inequality is given in the following theorem (see [2], [7], or for example [8]).

Theorem 1.3 (INTEGRAL FORM OF JENSEN'S INEQUALITY) *Let $g\colon [a,b] \to \mathbb{R}$ be an integrable function and let $p\colon [a,b] \to \mathbb{R}$ be a nonnegative function. If f is a convex function given on an interval I that includes the image of g, then the following inequality holds*

$$f\left(\frac{1}{P(b)}\int_a^b p(t)g(t)dt\right) \leq \frac{1}{P(b)}\int_a^b p(t)f(g(t))dt, \tag{1.3}$$

where $P(t)$ is defined as

$$P(t) = \int_a^t p(x)dx.$$

Integral form of the Lah-Ribarič inequality is given in the following theorem.

Theorem 1.4 (INTEGRAL FORM OF THE LAH-RIBARIČ INEQUALITY) *Let $g\colon [a,b] \to \mathbb{R}$ be an integrable function such that $m \le g(t) \le M$, for all $t \in [a,b]$, $m < M$, and let $p\colon [a,b] \to \mathbb{R}$ be a nonnegative function. If f is a convex function given on an interval I such that $[m,M] \subseteq I$, then the following inequality holds*

$$\frac{1}{P(b)} \int_a^b p(t) f(g(t)) dt \le \frac{M-\bar{g}}{M-m} f(m) + \frac{\bar{g}-m}{M-m} f(M), \tag{1.4}$$

where P is defined as

$$P(t) = \int_a^t p(x) dx,$$

and $\bar{g}$ is defined as

$$\bar{g} = \frac{\int_a^b p(t) g(t) dt}{P(b)}.$$

We give a new refinement of the Lah-Ribarič inequality (1.2), and using the same technique we will give a refinement of the Jensen inequality (1.1) (see [17]).

We also give refinements of the integral form of Jensen's inequality (1.3) and the Lah-Ribarič inequality (1.4).

The idea for proving can be also found in a well known result (see [16, pages 55 - 60]). Refinement of the inequality on the interval is obtained by applying the same inequality on subintervals.

Using obtained results we give a refinement of the famous Hölder inequality and some new refinements for the weighted power means and quasi arithmetic means.

Also, we give a historical remark about the Jensen-Boas inequality.

In the last section, we deal with the notion of f-divergences, the Csiszár f-divergences in the first place, where by varying the generating functions we distinguish e.g. Jeffrey's distance, the KullbackLeibler divergence, the Hellinger distance, the Bhattacharyya distance. We deduce the relations for the mentioned f-divergences. In the discrete case, these results are further examined for the Zipf-Mandelbrot law.

1.1 New refinements

The starting point for this consideration is the following lemma.

Lemma 1.1 *Let f be a convex function on an interval I. If $a,b,c,d \in I$ such that $a \le b < c \le d$, then the inequality*

$$\frac{c-u}{c-b} f(b) + \frac{u-b}{c-b} f(c) \le \frac{d-u}{d-a} f(a) + \frac{u-a}{d-a} f(d)$$

holds for any $u \in [b,c]$.

Proof. We can write

$$b = \frac{d-b}{d-a}a + \frac{b-a}{d-a}d$$
$$c = \frac{d-c}{d-a}a + \frac{c-a}{d-a}d$$

and since f is convex, it follows that

$$f(b) \leq \frac{d-b}{d-a}f(a) + \frac{b-a}{d-a}f(d)$$
$$f(c) \leq \frac{d-c}{d-a}f(a) + \frac{c-a}{d-a}f(d).$$

Now we have

$$\begin{aligned}
&\frac{c-u}{c-b}f(b) + \frac{u-b}{c-b}f(c) \\
&\quad \leq \frac{c-u}{c-b}\left[\frac{d-b}{d-a}f(a) + \frac{b-a}{d-a}f(d)\right] + \frac{u-b}{c-b}\left[\frac{d-c}{d-a}f(a) + \frac{c-a}{d-a}f(d)\right] \\
&\quad = \frac{d-u}{d-a}f(a) + \frac{u-a}{d-a}f(d).
\end{aligned}$$

□

First main result is a refinement of the Lah-Ribarič inequality (1.2). As we will see, its proof is based on the idea from the proof of the Jensen-Boas inequality.

Theorem 1.5 *Let $f\colon I \to \mathbb{R}$ be a convex function on I, $[m,M] \subset I$, $-\infty < m < M < +\infty$, $\boldsymbol{p}$ is as in Theorem 1.1, $\boldsymbol{x} = (x_1,\ldots,x_n)$ be any n-tuple in $[m,M]^n$ and $\bar{x} = \frac{1}{P_n}\sum_{i=1}^n p_i x_i$. Let $N_i \subseteq \{1,2,\ldots,n\}$, $i = 1,\ldots,m$ where $N_i \cap N_j = \emptyset$ for $i \neq j$, $\cup_{i=1}^m N_i = \{1,2,\ldots,n\}$, $\sum_{j\in N_i} p_j > 0$, for $i = 1,\ldots,m$ and $m_i = \min\{x_j\colon j \in N_i\}$, $M_i = \max\{x_j\colon j \in N_i\}$, for $i = 1,\ldots,m$. Then*

$$\begin{aligned}
\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i) &\leq \frac{1}{P_n}\sum_{i=1}^m \left(\sum_{j\in N_i} p_j\right)\left[\frac{M_i-\bar{x}_i}{M_i-m_i}f(m_i) + \frac{\bar{x}_i-m_i}{M_i-m_i}f(M_i)\right] \\
&\leq \frac{M-\bar{x}}{M-m}f(m) + \frac{\bar{x}-m}{M-m}f(M)
\end{aligned} \tag{1.5}$$

holds, where

$$\bar{x}_i = \frac{1}{\sum_{j\in N_i} p_j}\sum_{j\in N_i} p_j x_j.$$

If f is concave on I, then the inequalities in (1.5) *are reversed.*

Proof. We have

$$\begin{aligned}
\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i) &= \frac{1}{P_n}\left[\sum_{i=1}^m \sum_{j\in N_i} p_j f(x_j)\right] \\
&= \frac{1}{P_n}\sum_{i=1}^m \left(\sum_{j\in N_i} p_j\right)\left[\frac{1}{\sum_{j\in N_i} p_j}\sum_{j\in N_i} p_j f(x_j)\right].
\end{aligned}$$

Using the Lah-Ribarič inequality (1.2) for each of the subsets N_i, we get

$$\begin{aligned}
&\frac{1}{P_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i} p_j\right)\left[\frac{1}{\sum_{j\in N_i} p_j}\sum_{j\in N_i} p_j f(x_j)\right] \\
&\leq \frac{1}{P_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i} p_j\right)\left[\frac{M_i-\frac{1}{\sum_{j\in N_i} p_j}\sum_{j\in N_i} p_j x_j}{M_i-m_i}f(m_i)+\frac{\frac{1}{\sum_{j\in N_i} p_j}\sum_{j\in N_i} p_j x_j-m_i}{M_i-m_i}f(M_i)\right] \\
&= \frac{1}{P_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i} p_j\right)\left[\frac{M_i-\overline{x}_i}{M_i-m_i}f(m_i)+\frac{\overline{x}_i-m_i}{M_i-m_i}f(M_i)\right].
\end{aligned}$$

Using $m \leq m_i \leq \overline{x}_i \leq M_i \leq M$, $m < M, m_i < M_i$ and Lemma 1.1, we get

$$\begin{aligned}
&\frac{1}{P_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i} p_j\right)\left[\frac{M_i-\overline{x}_i}{M_i-m_i}f(m_i)+\frac{\overline{x}_i-m_i}{M_i-m_i}f(M_i)\right] \\
&\leq \frac{1}{P_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i} p_j\right)\left[\frac{M-\overline{x}_i}{M-m}f(m)+\frac{\overline{x}_i-m}{M-m}f(M)\right] \\
&= \frac{M-\overline{x}}{M-m}f(m)+\frac{\overline{x}-m}{M-m}f(M).
\end{aligned}$$

□

Remark 1.1 If $N_i = \{x_j\}$ ($|N_i| = 1$), the related term in the sum on the right-hand side of the first inequality in the proof of Theorem 1.5 remains unaltered (i.e. is equal to $f(x_j)$).

Using the same technique, we obtain the following refinement of the Jensen inequality (1.1).

Theorem 1.6 *Let I be an interval in $\mathbb{R}$ and $f\colon I \to \mathbb{R}$ a convex function. Let $\boldsymbol{x} = (x_1,\ldots,x_n)$ be any n-tuple in I^n and $\boldsymbol{p} = (p_1,\ldots,p_n)$ a nonnegative n-tuple such that $P_n = \sum_{i=1}^{n} p_i > 0$. Let $N_i \subseteq \{1,2,\ldots,n\}$, $i = 1,\ldots,m$ where $N_i \cap N_j = \emptyset$ for $i \neq j$, $\cup_{i=1}^{m} N_i = \{1,2,\ldots,n\}$ and $\sum_{j\in N_i} p_j > 0$, $i = 1,\ldots,m$. Then*

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i} p_j\right) f\left(\frac{\sum_{j\in N_i} p_j x_j}{\sum_{j\in N_i} p_j}\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \tag{1.6}$$

holds.

If f is concave on I, then the inequalities in (1.6) *are reversed.*

Proof. We have

$$\begin{aligned}
f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) &= f\left(\frac{1}{P_n}\sum_{i=1}^{m}\left[\sum_{j\in N_i} p_j x_j\right]\right) \\
&= f\left(\frac{1}{P_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i} p_j\right)\frac{\sum_{j\in N_i} p_j x_j}{\sum_{j\in N_i} p_j}\right).
\end{aligned}$$

Using Jensen's inequality (1.1), we get

$$\begin{aligned} f\left(\frac{1}{P_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i}p_j\right)\frac{\sum_{j\in N_i}p_jx_j}{\sum_{j\in N_i}p_j}\right) &\leq \frac{1}{P_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i}p_j\right)f\left(\frac{\sum_{j\in N_i}p_jx_j}{\sum_{j\in N_i}p_j}\right)\\ &\leq \frac{1}{P_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i}p_j\right)\left[\frac{1}{\sum_{j\in N_i}p_j}\sum_{j\in N_i}p_jf(x_j)\right]\\ &= \frac{1}{P_n}\sum_{i=1}^{m}\sum_{j\in N_i}p_jf(x_j), \end{aligned}$$

which is (1.6). □

We can find this idea for proving the refinement of our main results (and the refinement of the Jensen inequality) in one other well-known result (see [16, pages 55–60]).

In Jensen's inequality there is a condition "$\boldsymbol{p} = (p_1,\ldots,p_n)$ a nonnegative n-tuple such that $P_n = \sum_{i=1}^{n} p_i > 0$". In 1919. Steffensen gave the same inequality (1.1) with a slightly relaxed conditions.

Theorem 1.7 (JENSEN-STEFFENSEN) *If $f\colon I \to \mathbb{R}$ is a convex function, $\mathbf{x}$ is a real monotonic n-tuple such that $x_i \in I$, $i = 1,\ldots,n$, and $\mathbf{p}$ is a real n-tuple such that*

$$0 \leq P_k \leq P_n, \quad k = 1,\ldots,n, \quad P_n > 0.$$

Then (1.1) *holds. If f is strictly convex, then inequality* (1.1) *is strict unless $x_1 = x_2 = \cdots = x_n$.*

One of many generalizations of the Jensen inequality is the Riemann-Stieltjes integral form of the Jensen inequality.

Theorem 1.8 (THE RIEMANN-STIELTJES FORM OF JENSEN'S INEQUALITY)
Let $\phi\colon I \to \mathbb{R}$ be a continuous convex function where I is a range of a continuous function $f\colon [a,b] \to \mathbb{R}$. The inequality

$$\phi\left(\frac{\int_a^b f(x)d\lambda(x)}{\int_a^b d\lambda(x)}\right) \leq \frac{\int_a^b \phi(f(x))\,d\lambda(x)}{\int_a^b d\lambda(x)} \tag{1.7}$$

holds, providing that $\lambda\colon [a,b] \to \mathbb{R}$ is increasing, bounded and $\lambda(a) \neq \lambda(b)$.

Analogously, integral form of the Jensen-Steffensen inequality is given.

Theorem 1.9 (THE JENSEN-STEFFENSEN) *If f is continuous and monotonic (either increasing or decreasing) and λ is either continuous or of bounded variation satisfying*

$$\lambda(a) \leq \lambda(x) \leq \lambda(b) \;\; \textit{for all} \;\; x \in [a,b], \quad \lambda(a) < \lambda(b),$$

then (1.7) *holds.*

In 1970. Boas gave the integral analogue of Jensen-Steffensen's inequality with slightly different conditions.

Theorem 1.10 (THE JENSEN-BOAS INEQUALITY) *If λ is continuous or of bounded variation satisfying*

$$\lambda(a) \leq \lambda(x_1) \leq \lambda(y_1) \leq \lambda(x_2) \leq \cdots \leq \lambda(y_{n-1}) \leq \lambda(x_n) \leq \lambda(b)$$

for all $x_k \in (y_{k-1}, y_k)$, and $\lambda(b) > \lambda(a)$, and if $f\colon [a,b] \to \mathbb{R}$ is continuous and monotonic (either increasing or decreasing) in each of the $n-1$ intervals (y_{k-1}, y_k), then inequality (1.7) *holds for a continuous convex function $\phi : I \to \mathbb{R}$, where I is the range of the function f.*

In 1982. J. Pečarić gave the following proof of the Jensen-Boas inequality.

Proof. If $\lambda(a) < \lambda(x_1) < \lambda(y_1) < \lambda(x_2) < \cdots < \lambda(y_{n-1}) < \lambda(x_n) < \lambda(b)$ with the notation

$$p_k = \int_{y_{k-1}}^{y_k} d\lambda(x), \ \ t_k = \frac{\int_{y_{k-1}}^{y_k} f(x) d\lambda(x)}{\int_{y_{k-1}}^{y_k} d\lambda(x)}, \ \ k = 1, \ldots, n,$$

we have

$$\phi\left(\frac{\int_a^b f(x) d\lambda(x)}{\int_a^b d\lambda(x)}\right) = \phi\left(\frac{\sum_{k=1}^n \int_{y_{k-1}}^{y_k} f(x) d\lambda(x)}{\sum_{k=1}^n \int_{y_{k-1}}^{y_k} d\lambda(x)}\right) = \phi\left(\frac{\sum_{k=1}^n p_k t_k}{\sum_{k=1}^n p_k}\right).$$

Using Jensen's inequality (1.1), we get

$$\begin{aligned} \phi\left(\frac{\sum_{k=1}^n p_k t_k}{\sum_{k=1}^n p_k}\right) &\leq \frac{1}{\sum_{k=1}^n p_k} \sum_{k=1}^n p_k \phi(t_k) \\ &= \frac{1}{\sum_{k=1}^n p_k} \left[\sum_{k=1}^n p_k \phi\left(\frac{\int_{y_{k-1}}^{y_k} f(x) d\lambda(x)}{\int_{y_{k-1}}^{y_k} d\lambda(x)}\right)\right]. \end{aligned}$$

Using the Jensen-Steffensen's inequality (1.7) on each subinterval $[y_{k-1}, y_k]$, $k = 1, \ldots, n$, we get

$$\begin{aligned} &\frac{1}{\sum_{k=1}^n p_k} \left[\sum_{k=1}^n p_k \phi\left(\frac{\int_{y_{k-1}}^{y_k} f(x) d\lambda(x)}{\int_{y_{k-1}}^{y_k} d\lambda(x)}\right)\right] \\ &\leq \frac{1}{\sum_{k=1}^n p_k} \left[\sum_{k=1}^n p_k \frac{1}{\int_{y_{k-1}}^{y_k} d\lambda(x)} \int_{y_{k-1}}^{y_k} \phi(f(x)) d\lambda(x)\right] \\ &= \frac{1}{\sum_{k=1}^n \int_{y_{k-1}}^{y_k} d\lambda(x)} \sum_{k=1}^n \int_{y_{k-1}}^{y_k} \phi(f(x)) d\lambda(x) \\ &= \frac{\int_a^b \phi(f(x)) d\lambda(x)}{\int_a^b d\lambda(x)}. \end{aligned}$$

If $\lambda(y_{j-1}) = \lambda(y_j)$, for some j, then $d\lambda(x) = 0$ on $[y_{j-1}, y_j]$ and we can easily prove that the Jensen-Boas inequality is valid. □

If we look at the previous proof, we see that the technique is the same as for our main results and the refinement of the Jensen inequality.

Our next main result will be a refinement of the integral form of the Jensen inequality (1.3).

Theorem 1.11 *Let g be an integrable function defined on an interval $[a,b]$, let $a_0, a_1, \ldots, a_{n-1}, a_n$ be arbitrary such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$. If f is a convex function given on an interval I that includes the image of g, then*

$$f\left(\frac{1}{P(b)}\int_a^b p(t)g(t)dt\right) \leq \frac{1}{P(b)}\sum_{i=1}^{n}\left(\int_{a_{i-1}}^{a_i} p(t)dt\right) f\left(\frac{\int_{a_{i-1}}^{a_i} p(t)g(t)dt}{\int_{a_{i-1}}^{a_i} p(t)dt}\right) \leq \frac{1}{P(b)}\int_a^b p(t)f(g(t))dt \tag{1.8}$$

is valid, where $p\colon [a,b] \to \mathbb{R}$ is nonnegative function and P is defined as

$$P(t) = \int_a^t p(x)dx.$$

Proof. Let $a_0, a_1, \ldots, a_{n-1}, a_n$ be arbitrary such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$. Applying Jensen's inequality, we have

$$\begin{aligned} f\left(\frac{1}{P(b)}\int_a^b p(t)g(t)dt\right) &= f\left(\frac{1}{P(b)}\sum_{i=1}^{n}\int_{a_{i-1}}^{a_i} p(t)g(t)dt\right) \\ &= f\left(\frac{1}{P(b)}\sum_{i=1}^{n}\left(\int_{a_{i-1}}^{a_i} p(t)dt\right)\frac{\int_{a_{i-1}}^{a_i} p(t)g(t)dt}{\int_{a_{i-1}}^{a_i} p(t)dt}\right) \\ &\leq \frac{1}{P(b)}\sum_{i=1}^{n}\left(\int_{a_{i-1}}^{a_i} p(t)dt\right) f\left(\frac{\int_{a_{i-1}}^{a_i} p(t)g(t)dt}{\int_{a_{i-1}}^{a_i} p(t)dt}\right), \end{aligned}$$

which is the left-hand side of (1.8).

Now we will use the inequality (1.3) on each of the subintervals $[a_{i-1}, a_i]$.

$$\begin{aligned} &\frac{1}{P(b)}\sum_{i=1}^{n}\left(\int_{a_{i-1}}^{a_i} p(t)dt\right) f\left(\frac{1}{\int_{a_{i-1}}^{a_i} p(t)dt}\int_{a_{i-1}}^{a_i} p(t)g(t)dt\right) \\ &\leq \frac{1}{P(b)}\sum_{i=1}^{n}\left(\int_{a_{i-1}}^{a_i} p(t)dt\right)\frac{1}{\int_{a_{i-1}}^{a_i} p(t)dt}\int_{a_{i-1}}^{a_i} p(t)f(g(t)dt, \end{aligned}$$

which is the right-hand side of (1.8). □

Last main result is a refinement of the integral form of the Lah-Ribarič inequality (1.4).

Theorem 1.12 *Let g be an integrable function defined on an interval $[a,b]$, let $a_0, a_1, \ldots, a_{n-1}, a_n$ be arbitrary such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$ and $m_i \le g(t) \le M_i$, for all $t \in [a_{i-1}, a_i]$, $m_i < M_i$, $i = 1, \ldots, n$, $m = \min_{1\le i\le n} m_i, M = \max_{1\le i\le n} M_i$. If f is a convex function given on an interval I that includes the image of g, then*

$$\begin{aligned}
&\frac{1}{P(b)}\int_a^b p(t)f(g(t))dt\\
&\le \frac{1}{P(b)}\sum_{i=1}^{n} p_i\left[\frac{M_i-\overline{g_i}}{M_i-m_i}f(m_i)+\frac{\overline{g_i}-m_i}{M_i-m_i}f(M_i)\right]\\
&\le \frac{M-\overline{g}}{M-m}f(m)+\frac{\overline{g}-m}{M-m}f(M)
\end{aligned} \tag{1.9}$$

is valid, where $p\colon [a,b] \to \mathbb{R}$ is nonnegative function, P is defined as

$$P(t) = \int_a^t p(x)dx$$

and $\overline{g}, \overline{g_i}, p_i$ are defined as

$$\overline{g} = \frac{\int_a^b p(t)g(t)dt}{P(b)}, \quad \overline{g_i} = \frac{\int_{a_{i-1}}^{a_i} p(t)g(t)dt}{\int_{a_{i-1}}^{a_i} p(t)dt}, \quad p_i = \int_{a_{i-1}}^{a_i} p(t)dt.$$

Proof. We will use (1.4) on each of the subintervals $[a_{i-1}, a_i]$.

$$\begin{aligned}
&\frac{1}{P(b)}\int_a^b p(t)f(g(t))dt\\
&= \frac{1}{P(b)}\sum_{i=1}^{n}\int_{a_{i-1}}^{a_i} p(t)f(g(t))dt\\
&\le \frac{1}{P(b)}\sum_{i=1}^{n}\left(\int_{a_{i-1}}^{a_i} p(t)dt\right)\left[\frac{M_i-\frac{\int_{a_{i-1}}^{a_i} p(t)g(t)dt}{\int_{a_{i-1}}^{a_i} p(t)dt}}{M_i-m_i}f(m_i)+\frac{\frac{\int_{a_{i-1}}^{a_i} p(t)g(t)dt}{\int_{a_{i-1}}^{a_i} p(t)dt}-m_i}{M_i-m_i}f(M_i)\right],
\end{aligned}$$

which is the left-hand side of inequality (1.9).

Since $m \le m_i \le \overline{g_i} \le M_i \le M, m < M, m_i < M_i$, then by Lemma 1.1 we get

$$\begin{aligned}
&\frac{1}{P(b)}\sum_{i=1}^{n} p_i\left[\frac{M_i-\overline{g_i}}{M_i-m_i}f(m_i)+\frac{\overline{g_i}-m_i}{M_i-m_i}f(M_i)\right]\\
&\le \frac{1}{P(b)}\sum_{i=1}^{n}\left[\frac{p_iM-\int_{a_{i-1}}^{a_i} p(t)g(t)dt}{M-m}f(m)+\frac{\int_{a_{i-1}}^{a_i} p(t)g(t)dt-p_im}{M-m}f(M)\right]\\
&= \frac{1}{P(b)}\left[\frac{\sum_{i=1}^{n} p_iM-\sum_{i=1}^{n}\int_{a_{i-1}}^{a_i} p(t)g(t)dt}{M-m}f(m)+\frac{\sum_{i=1}^{n}\int_{a_{i-1}}^{a_i} p(t)g(t)dt-\sum_{i=1}^{n} p_im}{M-m}f(M)\right]\\
&= \frac{M-\frac{\int_a^b p(t)g(t)dt}{P(b)}}{M-m}f(m)+\frac{\frac{\int_a^b p(t)g(t)dt}{P(b)}-m}{M-m}f(M),
\end{aligned}$$

which is the right-hand side of (1.9). □

1.1.1 The Hermite-Hadamard inequality

Another famous inequality established for the class of convex functions is the Hermite-Hadamard inequality.

Theorem 1.13 (HERMITE-HADAMARD) *Let f be a convex function on $[a,b] \subset \mathbb{R}$, where $a < b$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1.10}$$

This double inequality was first discovered by Hermite in 1881. This result was later incorrectly attributed to Hadamard who apparently was not aware of Hermite's discovery and today, when relating to (1.10), we use both names.

This result can be improved by applying (1.10) on each of the subintervals $[a, \frac{a+b}{2}]$, $[\frac{a+b}{2}, b]$ and the following result is obtained (see [14, p. 37]):

$$f\left(\frac{a+b}{2}\right) \leq l \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq L \leq \frac{f(a)+f(b)}{2}, \tag{1.11}$$

where $l = \frac{1}{2}\left(f\left(\frac{3b+a}{4}\right) + f\left(\frac{b+3a}{2}\right)\right)$ and $L = \frac{1}{2}\left(f\left(\frac{b+a}{2}\right) + \frac{f(a)+f(b)}{2}\right)$.

The following improvement of (1.11) is given in [3].

Theorem 1.14 *Assume that $f\colon I \to \mathbb{R}$ is a convex function on I. Then for all $\lambda \in [0,1]$ and $a,b \in I$, we have*

$$f\left(\frac{a+b}{2}\right) \leq l(\lambda) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq L(\lambda) \leq \frac{f(a)+f(b)}{2}, \tag{1.12}$$

where

$$l(\lambda) = \lambda f\left(\frac{\lambda b + (2-\lambda)a}{2}\right) + (1-\lambda)f\left(\frac{(1+\lambda)b + (1-\lambda)a}{2}\right)$$

and

$$L(\lambda) = \frac{1}{2}\left(f(\lambda b + (1-\lambda)a) + \lambda f(a) + (1-\lambda)f(b)\right).$$

The inequality (1.12) for $\lambda = \frac{1}{2}$ gives inequality (1.11). Further improvement was given in [4].

Theorem 1.15 *Let $I \subseteq \mathbb{R}$ be an interval and $f\colon I \to \mathbb{R}$ be a convex function. Let $\Phi\colon [a,b] \to I$ be such that $f \circ \Phi$ is also convex, where $a < b$. Then for $n \in \mathbb{N}$, $\lambda_0 = 0, \lambda_{n+1} = 1$ and arbitrary $0 \leq \lambda_1 \leq \cdots \leq \lambda_n \leq 1$, we have*

$$f\left(\frac{1}{b-a}\int_a^b \Phi(x)dx\right) \leq l(\lambda_1,\ldots,\lambda_n) \leq \frac{1}{b-a}\int_a^b f\circ\Phi(x)dx \tag{1.13}$$

$$\leq L(\lambda_1,\ldots,\lambda_n) \leq \frac{f\circ\Phi(a)+f\circ\Phi(b)}{2}, \tag{1.14}$$

where

$$l(\lambda_1,\ldots,\lambda_n)=\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)f\left(\frac{1}{(\lambda_{k+1}-\lambda_k)(b-a)}\int_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b}\Phi(x)dx\right)$$

and

$$L(\lambda_1,\ldots,\lambda_n)=\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\frac{f\circ\Phi((1-\lambda_k)a+\lambda_k b)+f\circ\Phi((1-\lambda_{k+1})a+\lambda_{k+1}b)}{2}.$$

Applying the previous theorem for $\Phi(x)=x$ and $n=1$, we get the inequality (1.12).

Using refinements of Jensen's and the Lah-Ribarič inequality we obtain a refinement of the Hermite-Hadamard inequality.

Remark 1.2 If we set $p(t)=1$ in Theorem 1.11 we get (1.13) in the form

$$\begin{aligned} f\left(\frac{1}{b-a}\int_a^b g(t)dt\right) &\leq \frac{1}{b-a}\sum_{i=1}^{n}(a_i-a_{i-1})f\left(\frac{\int_{a_{i-1}}^{a_i}g(t)dt}{a_i-a_{i-1}}\right)\\ &\leq \frac{1}{b-a}\int_a^b f(g(t))dt. \end{aligned}$$

This gives for $g(t)=t$

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\sum_{i=1}^{n}(a_i-a_{i-1})f\left(\frac{a_{i-1}+a_i}{2}\right)\leq\frac{1}{b-a}\int_a^b f(t)dt,$$

which is a refinement of the left-hand side of (1.12).

Analogously from Theorem 1.12, we have (for $p(t)=1$)

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(g(t))dt\\ &\leq\frac{1}{b-a}\sum_{i=1}^{n}(a_i-a_{i-1})\left[\frac{M_i-\frac{\int_{a_{i-1}}^{a_i}g(t)dt}{a_i-a_{i-1}}}{M_i-m_i}f(m_i)+\frac{\frac{\int_{a_{i-1}}^{a_i}g(t)dt}{a_i-a_{i-1}}-m_i}{M_i-m_i}f(M_i)\right]\\ &\leq\frac{M-\frac{\int_a^b g(t)dt}{b-a}}{M-m}f(m)+\frac{\frac{\int_a^b g(t)dt}{b-a}-m}{M-m}f(M), \end{aligned}$$

and for $g(t)=t$, $m_i=a_{i-1}$, $M_i=a_i$, we get

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(t)dt\\ &\leq\frac{1}{b-a}\sum_{i=1}^{n}(a_i-a_{i-1})\left[\frac{a_i-\frac{a_{i-1}+a_i}{2}}{a_i-a_{i-1}}f(a_{i-1})+\frac{\frac{a_{i-1}+a_i}{2}-a_{i-1}}{a_i-a_{i-1}}f(a_i)\right]\\ &=\frac{1}{b-a}\sum_{i=1}^{n}(a_i-a_{i-1})\frac{f(a_{i-1})+f(a_i)}{2}\\ &\leq\frac{f(a)+f(b)}{2}, \end{aligned}$$

which is a refinement of the right-hand side of (1.12).

1.1.2 Hölder's inequality

One of the most important special cases of the Jensen inequality is the Hölder inequality.

Theorem 1.16 (DISCRETE HÖLDER'S INEQUALITY) *Let $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$. Let $\boldsymbol{a}=(a_1,a_2,\ldots,a_n)$, $\boldsymbol{b}=(b_1,b_2,\ldots,b_n)$ such that $a_i,b_i>0$, $i=1,\ldots,n$. Then:*

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}.$$

By using Theorem 1.6 and Theorem 1.11, we obtain the following refinements of the discrete Hölder and the integral Hölder inequality (for more about the Hölder inequality see [16]).

First we give the refinement of the discrete Hölder inequality.

Corollary 1.1 *Let $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$. Let $\boldsymbol{a}=(a_1,a_2,\ldots,a_n)$, $\boldsymbol{b}=(b_1,b_2,\ldots,b_n)$ such that $a_i,b_i>0$, $i=1,\ldots,n$. Then:*

$$\begin{aligned}\sum_{i=1}^{n} a_i b_i &\leq \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}} \left[\sum_{i=1}^{m}\left(\sum_{j\in N_i} b_j^q\right)^{1-p}\left(\sum_{j\in N_i} a_j b_j\right)^{p}\right]^{\frac{1}{p}} \\ &\leq \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}. \end{aligned} \tag{1.15}$$

Proof. We use Theorem 1.6 with $p_i=b_i^q>0$, $x_i=a_i b_i^{-\frac{q}{p}}>0$. Then $p_i x_i = b_i^q a_i b_i^{-\frac{q}{p}} = a_i b_i^{q-\frac{q}{p}} = a_i b_i^{q(1-\frac{1}{p})} = a_i b_i^{q\frac{1}{q}} = a_i b_i$ and from (1.6), we get

$$\begin{aligned} f\left(\frac{1}{\sum_{i=1}^n b_i^q}\sum_{i=1}^{n} a_i b_i\right) &\leq \frac{1}{\sum_{i=1}^n b_i^q}\sum_{i=1}^{m}\left(\sum_{j\in N_i} b_j^q\right) f\left(\frac{\sum_{j\in N_i} a_j b_j}{\sum_{j\in N_i} b_j^q}\right) \\ &\leq \frac{1}{\sum_{i=1}^n b_i^q}\sum_{i=1}^{n} b_i^q f\left(a_i b_i^{-\frac{q}{p}}\right). \end{aligned} \tag{1.16}$$

For the function $f(t)=t^p$ from (1.16), we get

$$\begin{aligned} \left(\frac{1}{\sum_{i=1}^n b_i^q}\sum_{i=1}^{n} a_i b_i\right)^p &\leq \frac{1}{\sum_{i=1}^n b_i^q}\sum_{i=1}^{m}\left(\sum_{j\in N_i} b_j^q\right)\left(\frac{\sum_{j\in N_i} a_j b_j}{\sum_{j\in N_i} b_j^q}\right)^p \\ &\leq \frac{1}{\sum_{i=1}^n b_i^q}\sum_{i=1}^{n} b_i^q \left(a_i b_i^{-\frac{q}{p}}\right)^p \\ &= \frac{1}{\sum_{i=1}^n b_i^q}\sum_{i=1}^{n} a_i^p. \end{aligned}$$

Multiplying with $\left(\sum_{i=1}^{n} b_i^q\right)^p$, and raising to the power of $\frac{1}{p}$, we get

$$\begin{aligned}\sum_{i=1}^{n} a_i b_i &\leq \left(\sum_{i=1}^{n} b_i^q\right)^{1-\frac{1}{p}} \left[\sum_{i=1}^{m}\left(\sum_{j\in N_i} b_j^q\right)^{1-p}\left(\sum_{j\in N_i} a_j b_j\right)^{p}\right]^{\frac{1}{p}} \\ &\leq \left(\sum_{i=1}^{n} b_i^q\right)^{1-\frac{1}{p}} \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}},\end{aligned}$$

which is (1.15). □

Corollary 1.2 *Using the same conditions as in previous corollary for $p \in \mathbb{R}$, $p < 1$, $p \neq 0$, we get*

$$\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}} \leq \sum_{i=1}^{m}\left(\sum_{j\in N_i} b_j^q\right)^{\frac{1}{q}}\left(\sum_{j\in N_i} a_j^p\right)^{\frac{1}{p}} \leq \sum_{i=1}^{n} a_i b_i. \tag{1.17}$$

Proof. Let us consider the case $0 < p < 1$. We use Theorem 1.6 with $p_i = b_i^q > 0$, $x_i = a_i^p b_i^{-q} > 0$. Then $p_i x_i = b_i^q a_i^p b_i^{-q} = a_i^p$ and from (1.6), we get

$$\begin{aligned} f\left(\frac{1}{\sum_{i=1}^{n} b_i^q}\sum_{i=1}^{n} a_i^p\right) &\leq \frac{1}{\sum_{i=1}^{n} b_i^q}\sum_{i=1}^{m}\left(\sum_{j\in N_i} b_j^q\right) f\left(\frac{\sum_{j\in N_i} a_j^p}{\sum_{j\in N_i} b_j^q}\right) \\ &\leq \frac{1}{\sum_{i=1}^{n} b_i^q}\sum_{i=1}^{n} b_i^q f\left(a_i^p b_i^{-q}\right).\end{aligned}$$

For the function $f(t) = t^{\frac{1}{p}}$, we get

$$\begin{aligned} \left(\frac{1}{\sum_{i=1}^{n} b_i^q}\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} &\leq \frac{1}{\sum_{i=1}^{n} b_i^q}\sum_{i=1}^{m}\left(\sum_{j\in N_i} b_j^q\right)\left(\frac{\sum_{j\in N_i} a_j^p}{\sum_{j\in N_i} b_j^q}\right)^{\frac{1}{p}} \\ &\leq \frac{1}{\sum_{i=1}^{n} b_i^q}\sum_{i=1}^{n} b_i^q \left(a_i^p b_i^{-q}\right)^{\frac{1}{p}}.\end{aligned}$$

Multiplying with $\left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{p}}$, and then with $\left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}$, we get

$$\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}} \leq \sum_{i=1}^{m}\left(\sum_{j\in N_i} b_j^q\right)^{\frac{1}{q}}\left(\sum_{j\in N_i} a_j^p\right)^{\frac{1}{p}} \leq \sum_{i=1}^{n} a_i b_i,$$

which is (1.17).

If $p < 0$, then $0 < q < 1$, and the same result follows from symmetry. □

Now using Theorem 1.11 we give a refinement of the integral Hölder inequality.

Corollary 1.3 *Let $p,q \in \mathbb{R}$ such that $\frac{1}{p}+\frac{1}{q}=1$. Let w, g_1 and g_2 be nonnegative functions defined on $[a,b]$ such that $wg_1^p, wg_2^q, wg_1g_2 \in L^1([a,b])$.*

(i) If $p>1$, then

$$\begin{aligned}
&\int_a^b w(t)g_1(t)g_2(t)dt \\
&\leq \left(\int_a^b w(t)g_2^q(t)dt\right)^{\frac{1}{q}} \left(\sum_{i=1}^n \left(\int_{a_{i-1}}^{a_i} w(t)g_2^q(t)dt\right)^{1-p} \left(\int_{a_{i-1}}^{a_i} w(t)g_1(t)g_2(t)dt\right)^p\right)^{\frac{1}{p}} \\
&\leq \left(\int_a^b w(t)g_1^p(t)dt\right)^{\frac{1}{p}} \left(\int w(t)g_2^q(t)dt\right)^{\frac{1}{q}}.
\end{aligned}$$

(ii) If $p<1$, $p \neq 0$, then

$$\begin{aligned}
&\left(\int_a^b w(t)g_1^p(t)dt\right)^{\frac{1}{p}} \left(\int w(t)g_2^q(t)dt\right)^{\frac{1}{q}} \\
&\leq \sum_{i=1}^n \left(\int_{a_{i-1}}^{a_i} w(t)g_1^p(t)dt\right)^{\frac{1}{p}} \left(\int_{a_{i-1}}^{a_i} w(t)g_2^q(t)dt\right)^{\frac{1}{q}} \\
&\leq \int_a^b w(t)g_1(t)g_2(t)dt.
\end{aligned}$$

Proof. For the case $p>1$ we use Theorem 1.11 with $p(t)=w(t)g_2^q(t)$, $g(t)=g_1(t)g_2^{-\frac{q}{p}}$ and with the function $f(x)=x^p$ which is convex for $x>0$, $p>1$. From (1.8) we get

$$\begin{aligned}
&\left(\frac{1}{\int_a^b w(t)g_2^q(t)dt}\int_a^b w(t)g_2^q(t)g_1(t)g_2^{-\frac{q}{p}}dt\right)^p \\
&\leq \frac{1}{\int_a^b w(t)g_2^q(t)dt}\sum_{i=1}^n \left(\int_{a_{i-1}}^{a_i} w(t)g_2^q(t)dt\right)\left(\frac{\int_{a_{i-1}}^{a_i} w(t)g_2^q(t)g_1(t)g_2^{-\frac{q}{p}}(t)dt}{\int_{a_{i-1}}^{a_i} w(t)g_2^q(t)dt}\right)^p \\
&\leq \frac{1}{\int_a^b w(t)g_2^q(t)dt}\int_a^b w(t)g_2^q(t)\left(g_1(t)g_2^{-\frac{q}{p}}(t)\right)^p dt.
\end{aligned}$$

Using $q-\frac{q}{p}=1$, multiplying with $\int_a^b w(t)g_2^q(t)dt$ and raising to the power of $\frac{1}{p}$, we have

$$\begin{aligned}
&\left(\int_a^b w(t)g_2^q(t)dt\right)^{\frac{1}{p}-1}\left(\int_a^b w(t)g_1(t)g_2(t)dt\right) \\
&\leq \left(\sum_{i=1}^n \left(\int_{a_{i-1}}^{a_i} w(t)g_2^q(t)dt\right)^{1-p}\left(\int_{a_{i-1}}^{a_i} w(t)g_1(t)g_2(t)dt\right)^p\right)^{\frac{1}{p}} \\
&\leq \left(\int_a^b w(t)g_1^p(t)dt\right)^{\frac{1}{p}}.
\end{aligned}$$

Now multiplying with $\left(\int_a^b w(t)g_2^q(t)dt\right)^{\frac{1}{q}}$, we get

$$\begin{aligned}
&\int_a^b w(t)g_1(t)g_2(t)dt \\
&\quad \leq \left(\int_a^b w(t)g_2^q(t)dt\right)^{\frac{1}{q}} \left(\sum_{i=1}^n \left(\int_{a_{i-1}}^{a_i} w(t)g_2^q(t)dt\right)^{1-p} \left(\int_{a_{i-1}}^{a_i} w(t)g_1(t)g_2(t)dt\right)^p\right)^{\frac{1}{p}} \\
&\quad \leq \left(\int_a^b w(t)g_2^q(t)dt\right)^{\frac{1}{q}} \left(\int_a^b w(t)g_1^p(t)dt\right)^{\frac{1}{p}}.
\end{aligned}$$

For $0 < p < 1$ we use Theorem 1.11 with $p(t) = w(t)g_2^q(t)$, $g(t) = g_1^p(t)g_2^{-q}(t)$ and with the function $f(x) = x^{\frac{1}{p}}$ which is convex for $x > 0$, $0 < p < 1$. From (1.8) we get

$$\begin{aligned}
&\left(\frac{1}{\int_a^b w(t)g_2^q(t)dt} \int_a^b w(t)g_2^q(t)g_1^p(t)g_2^{-q}dt\right)^{\frac{1}{p}} \\
&\quad \leq \frac{1}{\int_a^b w(t)g_2^q(t)dt} \sum_{i=1}^n \left(\int_{a_{i-1}}^{a_i} w(t)g_2^q(t)dt\right) \left(\frac{\int_{a_{i-1}}^{a_i} w(t)g_2^q(t)g_1^p(t)g_2^{-q}dt}{\int_{a_{i-1}}^{a_i} w(t)g_2^q(t)dt}\right)^{\frac{1}{p}} \\
&\quad \leq \frac{1}{\int_a^b w(t)g_2^q(t)dt} \int_a^b w(t)g_2^q(t) \left(g_1^p(t)g_2^{-q}\right)^{\frac{1}{p}} dt.
\end{aligned}$$

Now using $q - \frac{q}{p} = 1$ and multiplying with $\int_a^b w(t)g_2^q(t)dt$ we have

$$\begin{aligned}
&\left(\int_a^b w(t)g_1^p(t)dt\right)^{\frac{1}{p}} \left(\int w(t)g_2^q(t)dt\right)^{\frac{1}{q}} \\
&\quad \leq \sum_{i=1}^n \left(\int_{a_{i-1}}^{a_i} w(t)g_2^q(t)dt\right)^{\frac{1}{q}} \left(\int_{a_{i-1}}^{a_i} w(t)g_1^p(t)dt\right)^{\frac{1}{p}} \\
&\quad \leq \int_a^b w(t)g_1(t)g_2(t)dt.
\end{aligned}$$

If $p < 0$ then we have $0 < q < 1$ and we have the same result from symmetry. □

1.1.3 Power and quasi arithmetic means

It is interesting to show how the previously obtained results impact on the study of the weighted power means and the weighted quasi arithmetic means.

First we look at the discrete cases.

Let $n \in \mathbb{N}$, $n \geq 2$, $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\boldsymbol{p} = (p_1, \ldots, p_n)$, $x_i, p_i \mathbb{R}^+ = (0, \infty]$. The weighted discrete power means of order $r \in \mathbb{R}$ are defined as:

$$M_r(\boldsymbol{x}, \boldsymbol{p}) = \begin{cases} \left(\frac{1}{P_n} \sum_{i=1}^n p_i x_i^r\right)^{\frac{1}{r}} & , \ r \neq 0, \\ \left(\prod_{i=1}^n x_i^{p_i}\right)^{\frac{1}{P_n}} & , \ r = 0. \end{cases}$$

Using Theorem 1.6, we obtain the following inequalities for the weighted discrete power means. Let's notice that the left hand side and right hand sides of both inequalities are the same, only mixed means in the middle, which are a refinement, change.

Corollary 1.4 *Let $n \in \mathbb{N}$, $n \geq 2$, $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\boldsymbol{p} = (p_1, \ldots, p_n)$, $x_i, p_i \in \mathbb{R}^+$. Let $s,t \in \mathbb{R}$ such that $s \leq t$. Then*

$$M_s(\boldsymbol{x},\boldsymbol{p}) \leq \left[\frac{1}{P_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i} p_j\right) M_s^t(\boldsymbol{x}_{N_i},\boldsymbol{p}_{N_i})\right]^{\frac{1}{t}} \leq M_t(\boldsymbol{x},\boldsymbol{p}), \tag{1.18}$$

$$M_s(\boldsymbol{x},\boldsymbol{p}) \leq \left[\frac{1}{P_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i} p_j\right) M_t^s(\boldsymbol{x}_{N_i},\boldsymbol{p}_{N_i})\right]^{\frac{1}{s}} \leq M_t(\boldsymbol{x},\boldsymbol{p}), \tag{1.19}$$

where $\boldsymbol{x}_{N_i} = (x^i_{j_1}, \ldots, x^i_{j_{k_i}})$, $\boldsymbol{p}_{N_i} = (p^i_{j_1}, \ldots, p^i_{j_{k_i}})$, $k_i = |N_i|$, $N_i = \{j^i_1, \ldots, j^i_{k_i}\}$, *for* $i = 1, \ldots, m$.

Proof. We use Theorem 1.6 with $f(x) = x^{\frac{t}{s}}$ for $x > 0$, $s,t \in \mathbb{R}$, $t > 0$, $s \neq 0$, $s \leq t$. From (1.6), we get

$$\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right)^{\frac{t}{s}} \leq \frac{1}{P_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i} p_j\right)\left(\frac{\sum_{j\in N_i} p_j x_j}{\sum_{j\in N_i} p_j}\right)^{\frac{t}{s}} \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i^{\frac{t}{s}}.$$

Substituing x_i with x_i^s, and then raising to the power of $\frac{1}{t}$ we get

$$\left[\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i^s\right)^{\frac{1}{s}}\right]^t \leq \frac{1}{P_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i} p_j\right)\left[\left(\frac{\sum_{j\in N_i} p_j x_j^s}{\sum_{j\in N_i} p_j}\right)^{\frac{1}{s}}\right]^t \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i (x_i^s)^{\frac{t}{s}},$$

which is (1.18).

Similarly, we use Theorem 1.6 with $f(x) = x^{\frac{s}{t}}$ for $x > 0$, $s,t \in \mathbb{R}$, $s,t > 0$, $s \leq t$. We get

$$\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right)^{\frac{s}{t}} \geq \frac{1}{P_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i} p_j\right)\left(\frac{\sum_{j\in N_i} p_j x_j}{\sum_{j\in N_i} p_j}\right)^{\frac{s}{t}} \geq \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i^{\frac{s}{t}}.$$

Substituing x_i with x_i^t, and then raising to the power of $\frac{1}{s}$, inequality (1.19) easily follows. Other cases follow similarly. □

Let I be an interval in $\mathbb{R}$. Let $n \in \mathbb{N}$, $n \geq 2$, $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\boldsymbol{p} = (p_1, \ldots, p_n)$, $x_i \in I$, $p_i \in \mathbb{R}^+$, and $P_n = \sum_{i=1}^n p_i$. Then for a strictly monotone continuous function $h\colon I \to \mathbb{R}$ the discrete weighted quasi arithmetic mean is defined as:

$$M_h(x,p) = h^{-1}\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i h(x_i)\right).$$

Using Theorem 1.6, we obtain the following inequalities for quasi arithmetic means.

Corollary 1.5 *Let I be an interval in $\mathbb{R}$. Let $n \in \mathbb{N}$, $n \geq 2$, $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\boldsymbol{p} = (p_1, \ldots, p_n)$, $x_i \in I$, $p_i \in \mathbb{R}^+$. Let $h\colon I \to \mathbb{R}$ be a strictly monotone continuous function such that $f \circ h^{-1}$ convex. Let $N_i \subseteq \{1, 2, \ldots, n\}$, $i = 1, \ldots, m$ where $N_i \cap N_j = \emptyset$ for $i \neq j$, $\cup_{i=1}^m N_i = \{1, 2, \ldots, n\}$ and $\sum_{j \in N_i} p_j > 0$, $i = 1, \ldots, m$. Then*

$$f\left(M_h(\boldsymbol{x}, \boldsymbol{p})\right) \leq \frac{1}{P_n} \sum_{i=1}^m \left(\sum_{j \in N_i} p_j\right) f\left(M_h(\boldsymbol{x}_{N_i}, \boldsymbol{p}_{N_i})\right) \leq \frac{1}{P_n} \sum_{i=1}^n p_i f(x_i),$$

where $\boldsymbol{x}_{N_i} = (x^i_{j_1}, \ldots, x^i_{j_{k_i}})$, $\boldsymbol{p}_{N_i} = (p^i_{j_1}, \ldots, p^i_{j_{k_i}})$, $k_i = |N_i|$, $N_i = \{j^i_1, \ldots, j^i_{k_i}\}$, for $i = 1, \ldots, m$.

Proof. Theorem 1.6 with f substituting with $f \circ h^{-1}$ and x_i with $h(x_i)$ gives:

$$\begin{aligned} f\left(h^{-1}\left(\frac{1}{P_n}\sum_{i=1}^n p_i h(x_i)\right)\right) &\leq \frac{1}{P_n}\sum_{i=1}^m \left(\sum_{j \in N_i} p_j\right) f\left(h^{-1}\left(\frac{\sum_{j \in N_i} p_j h(x_j)}{\sum_{j \in N_i} p_j}\right)\right) \\ &\leq \frac{1}{P_n}\sum_{i=1}^n p_i f(x_i). \end{aligned}$$

□

Now we give results for the integral variants.

Let p and g be positive integrable functions defined on $[a,b]$. Then the integral power means of order $r \in \mathbb{R}$ are defined as follows:

$$M_r(g;p;a,b) = \begin{cases} \left(\frac{1}{\int_a^b p(x)dx} \int_a^b p(x) g^r(x) dx\right)^{\frac{1}{r}} &, r \neq 0, \\ \exp\left(\frac{\int_a^b p(x)\log g(x)dx}{\int_a^b p(x)dx}\right) &, r = 0. \end{cases}$$

Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$ be positive n-tuples. The weighted power mean (of the n-tuple $\mathbf{x}$ with the weight $\mathbf{w}$) of order $r \in \mathbb{R}$ is defined as

$$M_r(\mathbf{x}; \mathbf{w}) = \begin{cases} \left(\frac{1}{\sum_{i=1}^n w_i} \sum_{i=1}^n w_i x_i^r\right)^{\frac{1}{r}} &, r \neq 0, \\ e^{\frac{1}{\sum_{i=1}^n w_i} \sum_{i=1}^n w_i \log x_i} = \left(\prod_{i=1}^n x_i^{w_i}\right)^{\frac{1}{\sum_{i=1}^n w_i}} &, r = 0. \end{cases}$$

We'll use more suitable notation $M_r\left(x_i; w_i; \overline{1,n}\right)$ for $M_r(\mathbf{x}; \mathbf{w})$.

Using Theorem 1.11 we obtain following inequalities for integral power means.

Corollary 1.6 *Let p and g be positive integrable functions defined on $[a,b]$ and let $a_0, a_1, \ldots, a_{n-1}, a_n$ be arbitrary such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$. Let $s, t \in \mathbb{R}$ be such that $s \leq t$. Then*

$$\begin{aligned} M_s(g;p;a,b) &\leq M_t\left(M_s(g;p;a_{i-1},a_i); \int_{a_{i-1}}^{a_i} p(x)dx; \overline{1,n}\right) \\ &\leq M_t(g;p;a,b), \end{aligned} \tag{1.20}$$

$$\begin{aligned} M_t(g;p;a,b) &\geq M_s\left(M_t(g;p;a_{i-1},a_i); \int_{a_{i-1}}^{a_i} p(x)dx; \overline{1,n}\right) \\ &\geq M_s(g;p;a,b). \end{aligned} \tag{1.21}$$

Proof. We use Theorem 1.11 with $f(x) = x^{\frac{t}{s}}$ for $x > 0$, $s,t \in \mathbb{R}$, $s,t \neq 0$, $s \leq t$ (convex on $\langle 0, +\infty \rangle$). From (1.8) we get

$$\left(\frac{1}{P(b)}\int_a^b p(x)g(x)dx\right)^{\frac{t}{s}} \leq \frac{1}{P(b)}\sum_{i=1}^n \left(\int_{a_{i-1}}^{a_i} p(x)dx\right)\left(\frac{1}{\int_{a_{i-1}}^{a_i} p(x)dx}\int_{a_{i-1}}^{a_i} p(x)g(x)dx\right)^{\frac{t}{s}}$$
$$\leq \frac{1}{P(b)}\int_a^b p(x)g^{\frac{t}{s}}(x)dx.$$

Substituting g with g^s and raising to the power of $\frac{1}{t}$, we get the result.

Similarly, we use Theorem 1.11 with $f(x) = x^{\frac{s}{t}}$ for $x > 0$, $s,t \in \mathbb{R}$, $s,t \neq 0$, $s \leq t$ (concave on $\langle 0, +\infty \rangle$). From (1.8) we get

$$\left(\frac{1}{P(b)}\int_a^b p(x)g(x)dx\right)^{\frac{s}{t}} \geq \frac{1}{P(b)}\sum_{i=1}^n \left(\int_{a_{i-1}}^{a_i} p(x)dx\right)\left(\frac{1}{\int_{a_{i-1}}^{a_i} p(x)dx}\int_{a_{i-1}}^{a_i} p(x)g(x)dx\right)^{\frac{s}{t}}$$
$$\geq \frac{1}{P(b)}\int_a^b p(x)g^{\frac{s}{t}}(x)dx.$$

Substituting g with g^t and raising to the power of $\frac{1}{s}$, we get the result.

Cases $t = 0$ or $s = 0$ follows from the inequalities (1.20) and (1.21) by simple limiting process. □

Means of the type

$$M_t\left(M_s\left(g;p;a_{i-1},a_i\right); \int_{a_{i-1}}^{a_i} p(x)dx; \overline{1,n}\right)$$

can be regarded as mixed means.

Let p be positive integrable function defined on $[a,b]$ and g be any integrable function defined on $[a,b]$. Then for a strictly monotone continuous function h whose domain belongs to the image of g, the quasi arithmetic mean is defined as follows:

$$M_h(g;p;a,b) = h^{-1}\left(\frac{1}{\int_a^b p(x)dx}\int_a^b p(x)h(g(x))dx\right).$$

Using Theorem 1.11 we obtain the following inequalities for quasi arithmetic means.

Corollary 1.7 *Let p be a positive integrable function defined on $[a,b]$, g any integrable function defined on $[a,b]$ and let $a_0, a_1, \ldots, a_{n-1}, a_n$ be arbitrary such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$. Also assume that h is a strictly monotone continuous function whose domain belongs to the image of g. If $f \circ h^{-1}$ is convex function then*

$$f\left(M_h(g;p;a,b)\right) \leq \frac{1}{P(b)}\sum_{i=1}^n \left(\int_{a_{i-1}}^{a_i} p(x)dx\right) f\left(M_h(g;p;a_{i-1},a_i)\right)$$
$$\leq \frac{1}{P(b)}\int_a^b p(x)f(g(x))dx.$$

Proof. We use Theorem 1.11 with f substituting with $f \circ h^{-1}$ and g with $h \circ g$. □

1.2 Applications in information theory

In this section we give some interesting estimates concerning the discrete and the integral Csiszár f-divergence, and also for its important special cases (see for example [1], [5], [6], [10], [12], [15]).

Also, in the discrete case bounds for the Zipf-Mandelbrot law divergence are obtained.

First we consider the discrete case.

Let us denote the set of all probability densities by $\mathbb{P}$, i.e. $\boldsymbol{p} = (p_1, \ldots, p_n) \in \mathbb{P}$ if $p_i \in [0,1]$ for $i = 1, \ldots, n$ and $\sum_{i=1}^n p_i = 1$.

In [1] Csiszár introduced the $f-$divergence functional as

$$D_f(\boldsymbol{p}, \boldsymbol{q}) = \sum_{i=1}^n q_i f\left(\frac{p_i}{q_i}\right), \tag{1.22}$$

where $f\colon [0, +\infty\rangle$ is a convex function, and it represents a "distance function" on the set of probability distributions $\mathbb{P}$.

In order to use nonnegative probability distributions in the f-divergence functional, we assume as usual

$$f(0) := \lim_{t \to 0+} f(t), \quad 0 \cdot f\left(\frac{0}{0}\right) := 0, \quad 0 \cdot f\left(\frac{a}{0}\right) := \lim_{t \to 0+} t f\left(\frac{a}{t}\right),$$

and the following definition of a generalized f-divergence functional $\hat{D}$ is given.

Definition 1.2 (THE CSISZÁR $f-$DIVERGENCE FUNCTIONAL) *Let $J \subset \mathbb{R}$ be an interval, and let $f\colon J \to \mathbb{R}$ be a function. Let $\boldsymbol{p} = (p_1, \ldots, p_n)$ be an n-tuple of real numbers and $\boldsymbol{q} = (q_1, \ldots, q_n)$ be an n-tuple of nonnegative real numbers such that $p_i/q_i \in J$ for every $i = 1, \ldots, n$. The Csiszár f-divergence functional is defined as*

$$\hat{D}_f(\boldsymbol{p}, \boldsymbol{q}) := \sum_{i=1}^n q_i f\left(\frac{p_i}{q_i}\right). \tag{1.23}$$

Theorem 1.17 *Let I be an interval in $\mathbb{R}$ and $f\colon I \to \mathbb{R}$ a convex function. Let $\boldsymbol{p} = (p_1, \ldots, p_n)$ be an n-tuple of real numbers and $\boldsymbol{q} = (q_1, \ldots, q_n)$ be an n-tuple of nonnegative real numbers such that $p_i/q_i \in I$ for every $i = 1, \ldots, n$, and $P_n = \sum_{i=1}^n p_i, Q_n = \sum_{i=1}^n q_i$. Let $N_i \subseteq \{1, 2, \ldots, n\}$, $i = 1, \ldots, m$ where $N_i \cap N_j = \emptyset$ for $i \neq j$, $\cup_{i=1}^m N_i = \{1, 2, \ldots, n\}$, $\sum_{j \in N_i} q_j > 0$, $i = 1, \ldots, m$ and $\frac{\sum_{j \in N_i} p_j}{\sum_{j \in N_i} q_j} \in I$, $i = 1, \ldots, m$. Then*

$$f\left(\frac{P_n}{Q_n}\right) \leq \frac{1}{P_n} \sum_{i=1}^m \left(\sum_{j \in N_i} q_j\right) f\left(\frac{\sum_{j \in N_i} p_j}{\sum_{j \in N_i} q_j}\right) \leq \frac{1}{Q_n} \hat{D}_f(\boldsymbol{p}, \boldsymbol{q}) \tag{1.24}$$

holds.

Proof. Using Theorem 1.6 with p_i substituting with q_i and x_i with $\frac{p_i}{q_i}$, we get

$$f\left(\frac{1}{Q_n}\sum_{i=1}^{n} q_i \frac{p_i}{q_i}\right) \leq \frac{1}{P_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i} q_j\right) f\left(\frac{\sum_{j\in N_i} q_j \frac{p_j}{q_j}}{\sum_{j\in N_i} q_j}\right) \leq \frac{1}{Q_n}\sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right),$$

which is (1.24). □

Corollary 1.8 *If in the previous theorem we take* $\boldsymbol{p}$ *and* $\boldsymbol{q}$ *to be probability distributions, we directly get the following result:*

$$f(1) \leq \sum_{i=1}^{m}\left(\sum_{j\in N_i} q_j\right) f\left(\frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i} q_j}\right) \leq D_f(\boldsymbol{p},\boldsymbol{q}). \tag{1.25}$$

Theorem 1.18 *Let* $f: I \to \mathbb{R}$ *be a convex function on* I, $[m,M] \subset I$, $-\infty < m < M < +\infty$. *Let* $\boldsymbol{p} = (p_1,\ldots,p_n)$ *be an n-tuple of real numbers and* $\boldsymbol{q} = (q_1,\ldots,q_n)$ *be an n-tuple of nonnegative real numbers such that* $m \leq \frac{p_i}{q_i} \leq M$, $i = 1,\ldots,n$, *and* $P_n = \sum_{i=1}^{n} p_i, Q_n = \sum_{i=1}^{n} q_i$. *Let* $N_i \subseteq \{1,2,\ldots,n\}$, $i = 1,\ldots,m$ *where* $N_i \cap N_j = \emptyset$ *for* $i \neq j$, $\cup_{i=1}^{m} N_i = \{1,2,\ldots,n\}$, $\sum_{j\in N_i} q_j > 0$, *for* $i = 1,\ldots,m$ *and* $m_i = \min\{p_j/q_j : j \in N_i\}$, $M_i = \max\{p_j/q_j : j \in N_i\}$, *for* $i = 1,\ldots,m$. *Then*

$$\begin{aligned}\hat{D}_f(\boldsymbol{p},\boldsymbol{q}) &\leq \sum_{i=1}^{m}\left(\sum_{j\in N_i} q_j\right)\left[\frac{M_i - \frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i} q_j}}{M_i - m_i} f(m_i) + \frac{\frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i} q_j} - m_i}{M_i - m_i} f(M_i)\right] \\ &\leq \frac{M - \frac{P_n}{Q_n}}{M-m} f(m) + \frac{\frac{P_n}{Q_n} - m}{M-m} f(M)\end{aligned} \tag{1.26}$$

holds.

Proof. Using Theorem 1.5 with $p_i \to q_i$ and $x_i \to \frac{p_i}{q_i}$, we get

$$\begin{aligned}&\frac{1}{Q_n}\sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right) \\ &\leq \frac{1}{Q_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i} q_j\right)\left[\frac{M_i - \frac{1}{\sum_{j\in N_i} q_j}\sum_{j\in N_i} q_j \frac{p_j}{q_j}}{M_i - m_i} f(m_i) + \frac{\frac{1}{\sum_{j\in N_i} q_j}\sum_{j\in N_i} q_j \frac{p_j}{q_j} - m_i}{M_i - m_i} f(M_i)\right] \\ &\leq \frac{M - \frac{1}{\sum_{i=1}^{n} q_i}\sum_{i=1}^{n} q_i \frac{p_i}{q_i}}{M-m} f(m) + \frac{\frac{1}{\sum_{i=1}^{n} q_i}\sum_{i=1}^{n} q_i \frac{p_i}{q_i} - m}{M-m} f(M),\end{aligned}$$

which is (1.26) □

Corollary 1.9 *If in the previous theorem we take* $\boldsymbol{p}$ *and* $\boldsymbol{q}$ *to be probability distributions, we directly get the following result:*

$$\begin{aligned} D_f(\boldsymbol{p},\boldsymbol{q}) &\leq \sum_{i=1}^{m}\left(\sum_{j\in N_i} q_j\right)\left[\frac{M_i-\frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i} q_j}}{M_i-m_i}f(m_i)+\frac{\frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i} q_j}-m_i}{M_i-m_i}f(M_i)\right] \\ &\leq \frac{M-1}{M-m}f(m)+\frac{1-m}{M-m}f(M). \end{aligned} \tag{1.27}$$

If $\boldsymbol{p}$ and $\boldsymbol{q}$ are probability distributions, the Kullback-Leibler divergence, also called relative entropy is defined as

$$D_{KL}(\boldsymbol{p},\boldsymbol{q}) := \sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right).$$

Next corollary provides the bound for the Kullback-Leibler divergence of two probability distributions.

Corollary 1.10 *Let* $N_i \subseteq \{1,2,\ldots,n\}$, $i=1,\ldots,m$ *where* $N_i \cap N_j = \emptyset$ *for* $i \neq j$, $\cup_{i=1}^{m} N_i = \{1,2,\ldots,n\}$ *and* $\sum_{j\in N_i} q_j > 0$, $i=1,\ldots,m$.

- *Let* $\boldsymbol{p}=(p_1,\ldots,p_n)$, $\boldsymbol{q}=(q_1,\ldots,q_n)$ *be n-tuples of nonnegative real numbers,* $P_n = \sum_{i=1}^{n} p_i, Q_n = \sum_{i=1}^{n} q_i.$. *Then*

$$\frac{P_n}{Q_n}\log\frac{P_n}{Q_n} \leq \frac{1}{P_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i} p_j\right)\log\frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i} q_j} \leq \frac{1}{Q_n}\sum_{i=1}^{n} p_i \log\frac{p_i}{q_i}.$$

- *Let* $\boldsymbol{p}=(p_1,\ldots,p_n)$, $\boldsymbol{q}=(q_1,\ldots,q_n) \in \mathbb{P}$ *be probability distributions. Then*

$$0 \leq \sum_{i=1}^{m}\left(\sum_{j\in N_i} p_j\right)\log\frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i} q_j} \leq D_{KL}(\boldsymbol{p},\boldsymbol{q}).$$

Proof. Let $\boldsymbol{p}=(p_1,\ldots,p_n)$ and $\boldsymbol{q}=(q_1,\ldots,q_n)$ be n-tuples of nonnegative real numbers. Since the function $t \mapsto t\log t$ is convex, the first inequality follows from Theorem 1.17 by setting $f(t) = t\log t$.

The second inequality is a special case of the first inequality for probability distributions $\boldsymbol{p}$ and $\boldsymbol{q}$. □

Corollary 1.11 *Let* $N_i \subseteq \{1,2,\ldots,n\}$, $i=1,\ldots,m$ *where* $N_i \cap N_j = \emptyset$ *for* $i \neq j$, $\cup_{i=1}^{m} N_i = \{1,2,\ldots,n\}$ *and* $\sum_{j\in N_i} q_j > 0$, *for* $i=1,\ldots,m$.

- *Let* $\boldsymbol{p}=(p_1,\ldots,p_n)$ *and* $\boldsymbol{q}=(q_1,\ldots,q_n)$ *be n-tuples of nonnegative real numbers. Let* $m=\min\{p_i/q_i\colon i=1,\ldots,n\}$, $M=\max\{p_i/q_i\colon i=1,\ldots,n\}$, $m_i=\min\{p_j/q_j\colon j\in N_i\}$ *and* $M_i=\max\{p_j/q_j\colon j\in N_i\}$, *for* $i=1,\ldots,m$. *Then*

$$\begin{aligned} \sum_{i=1}^{n} p_i \log\frac{p_i}{q_i} &\leq \sum_{i=1}^{m}\left(\sum_{j\in N_i} q_j\right)\frac{1}{M_i-m_i}\log\left(m_i^{m_i\left(M_i-\frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i} q_j}\right)} M_i^{M_i\left(\frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i} q_j}-m_i\right)}\right) \\ &\leq \frac{1}{M-m}\log\left(m^{m\left(M-\frac{P_n}{Q_n}\right)} M^{M\left(\frac{P_n}{Q_n}-m\right)}\right). \end{aligned}$$

- *Let $\boldsymbol{p} = (p_1, \ldots, p_n)$ and $\boldsymbol{q} = (q_1, \ldots, q_n) \in \mathbb{P}$ be probability distributions. Let $m = \min\{p_i/q_i \colon i = 1, \ldots, n\}$, $M = \max\{p_i/q_i \colon i = 1, \ldots, n\}$, $m_i = \min\{p_j/q_j \colon j \in N_i\}$ and $M_i = \max\{p_j/q_j \colon j \in N_i\}$, for $i = 1, \ldots, m$. Then*

$$\begin{aligned} D_{KL}(\boldsymbol{p},\boldsymbol{q}) &\leq \sum_{i=1}^{m} \left(\sum_{j\in N_i} q_j \right) \frac{1}{M_i - m_i} \log \left(m_i^{m_i\left(M_i - \frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i} q_j}\right)} M_i^{M_i\left(\frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i} q_j} - m_i\right)} \right) \\ &\leq \frac{1}{M-m} \log \left(m^{m(M-1)} M^{M(1-m)} \right). \end{aligned}$$

Proof. Let $\boldsymbol{p} = (p_1, \ldots, p_n)$ and $\boldsymbol{q} = (q_1, \ldots, q_n)$ be n-tuples of nonnegative real numbers. Since the function $t \mapsto t \log t$ is convex, the first inequality follows from Theorem 1.18 by setting $f(t) = t \log t$.

The second inequality is a special case of the first inequality for probability distributions $\boldsymbol{p}$ and $\boldsymbol{q}$. □

Now we deduce the relations for some more special cases of the Csiszár f-divergence.

Definition 1.3 (JEFFREY'S DISTANCE) *For $\boldsymbol{p},\boldsymbol{q} \in \mathbb{P}$ the discrete Jeffrey distance is defined as*

$$J_d(\boldsymbol{p},\boldsymbol{q}) = \sum_{i=1}^{n} (p_i - q_i) \log \frac{p_i}{q_i}.$$

Corollary 1.12 *Let $\boldsymbol{p},\boldsymbol{q} \in \mathbb{P}$. Let $N_i \subseteq \{1,2,\ldots,n\}$, $i = 1,\ldots,m$ where $N_i \cap N_j = \emptyset$ for $i \neq j$, $\cup_{i=1}^{m} N_i = \{1,2,\ldots,n\}$ and $\sum_{j\in N_i} q_j > 0$, $i = 1,\ldots,m$. Then*

$$0 \leq \sum_{i=1}^{m} \left(\sum_{j\in N_i} p_j - \sum_{j\in N_i} q_j \right) \log \frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i} q_j} \leq J_d(\boldsymbol{p},\boldsymbol{q}). \tag{1.28}$$

Proof. Using Corollary 1.8 with $f(t) = (t-1)\log t$, $t \in \mathbb{R}^+$, we get

$$\begin{aligned} (1-1)\log 1 &\leq \sum_{i=1}^{m} \left(\sum_{j\in N_i} q_j \right) \left(\frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i} q_j} - 1 \right) \log \frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i} q_j} \\ &\leq \sum_{i=1}^{n} q_i \left(\frac{p_i}{q_i} - 1 \right) \log \frac{p_i}{q_i}, \end{aligned}$$

and (1.28) easily follows. □

Corollary 1.13 *Let m, M such that $0 < m < M < +\infty$, $\boldsymbol{p},\boldsymbol{q} \in \mathbb{P}$ such that $m \leq \frac{p_i}{q_i} \leq M, i = 1,\ldots,n$. Let $N_i \subseteq \{1,2,\ldots,n\}$, $i = 1,\ldots,m$ where $N_i \cap N_j = \emptyset$ for $i \neq j$, $\cup_{i=1}^{m} N_i = \{1,2,\ldots,n\}$, $\sum_{j\in N_i} q_j > 0$, for $i = 1,\ldots,m$ and $m_i = \min\{p_j/q_j \colon j \in N_i\}$, $M_i = \max\{p_j/q_j \colon$*

$j \in N_i\}$, *for* $i = 1, \ldots, m$. *Then*

$$J_d(\boldsymbol{p},\boldsymbol{q}) \le \sum_{i=1}^{m}\left(\sum_{j\in N_i} q_j\right)\left[\frac{M_i - \frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i} q_j}}{M_i - m_i}(m_i - 1)\log m_i + \frac{\frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i} q_j} - m_i}{M_i - m_i}(M_i - 1)\log M_i\right]$$
$$\le \log\left(\frac{M}{m}\right)^{\frac{(1-m)(M-1)}{M-m}} \tag{1.29}$$

holds.

Proof. Using Corollary 1.9 with $f(t) = (t-1)\log t$, $t \in \mathbb{R}^+$, we get

$$\sum_{i=1}^{n} q_i\left(\frac{p_i}{q_i} - 1\right)\log\frac{p_i}{q_i}$$
$$\le \sum_{i=1}^{m}\left(\sum_{j\in N_i} q_j\right)\left[\frac{M_i - \frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i} q_j}}{M_i - m_i}(m_i - 1)\log m_i + \frac{\frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i} q_j} - m_i}{M_i - m_i}(M_i - 1)\log M_i\right]$$
$$\le \frac{M-1}{M-m}(m-1)\log m + \frac{1-m}{M-m}(M-1)\log M,$$

and (1.29) easily follows. □

Definition 1.4 (THE HELLINGER DISTANCE) *For* $\boldsymbol{p},\boldsymbol{q} \in \mathbb{P}$ *the discrete Hellinger distance is defined as*

$$H_d(\boldsymbol{p},\boldsymbol{q}) = \sum_{i=1}^{n}(\sqrt{p_i} - \sqrt{q_i})^2.$$

Corollary 1.14 *Let* $\boldsymbol{p},\boldsymbol{q} \in \mathbb{P}$. *Let* $N_i \subseteq \{1,2,\ldots,n\}$, $i = 1,\ldots,m$ *where* $N_i \cap N_j = \emptyset$ *for* $i \neq j$, $\cup_{i=1}^{m} N_i = \{1,2,\ldots,n\}$ *and* $\sum_{j\in N_i} q_j > 0$, $i = 1,\ldots,m$. *Then*

$$0 \le \sum_{i=1}^{m}\left(\sqrt{\sum_{j\in N_i} p_j} - \sqrt{\sum_{j\in N_i} q_j}\right)^2 \le H_d(\boldsymbol{p},\boldsymbol{q}). \tag{1.30}$$

Proof. Using Corollary 1.8 with $f(t) = (\sqrt{t} - 1)^2$, $t \in \mathbb{R}^+$ (1.30) follows. □

Corollary 1.15 *Let* m, M *such that* $0 < m < M < +\infty$, $\boldsymbol{p},\boldsymbol{q} \in \mathbb{P}$ *such that* $m \le \frac{p_i}{q_i} \le M$, $i = 1,\ldots,n$. *Let* $N_i \subseteq \{1,2,\ldots,n\}$, $i = 1,\ldots,m$ *where* $N_i \cap N_j = \emptyset$ *for* $i \neq j$, $\cup_{i=1}^{m} N_i = \{1,2,\ldots,n\}$, $\sum_{j\in N_i} q_j > 0$, *for* $i = 1,\ldots,m$ *and* $m_i = \min\{p_j/q_j : j \in N_i\}$, $M_i = \max\{p_j/q_j : j \in N_i\}$, *for* $i = 1,\ldots,m$. *Then*

$$H_d(\boldsymbol{p},\boldsymbol{q}) \le \sum_{i=1}^{m}\left(\sum_{j\in N_i} q_j\right)\left[\frac{M_i - \frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i} q_j}}{M_i - m_i}(\sqrt{m_i} - 1)^2 + \frac{\frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i} q_j} - m_i}{M_i - m_i}(\sqrt{M_i} - 1)^2\right]$$
$$\le \frac{M-1}{M-m}(\sqrt{m} - 1)^2 + \frac{1-m}{M-m}(\sqrt{M} - 1)^2 \tag{1.31}$$

holds.

Proof. Using Corollary 1.9 with $f(t) = (\sqrt{t}-1)^2$, $t \in \mathbb{R}^+$ (1.31) follows. □

Definition 1.5 (BHATTACHARYYA DISTANCE) *For $\boldsymbol{p},\boldsymbol{q} \in \mathbb{P}$ the discrete Bhattacharyya distance is defined as*

$$B_d(\boldsymbol{p},\boldsymbol{q}) = \sum_{i=1}^{n} \sqrt{p_i q_i}.$$

Corollary 1.16 *Let $\boldsymbol{p},\boldsymbol{q} \in \mathbb{P}$. Let $N_i \subseteq \{1,2,\ldots,n\}$, $i = 1,\ldots,m$ where $N_i \cap N_j = \emptyset$ for $i \neq j$, $\cup_{i=1}^{m} N_i = \{1,2,\ldots,n\}$ and $\sum_{j\in N_i} q_j > 0$, $i = 1,\ldots,m$. Then*

$$-1 \leq -\sum_{i=1}^{m} \sqrt{\sum_{j\in N_i} p_j \sum_{j\in N_i} q_j} \leq -B_d(\boldsymbol{p},\boldsymbol{q}). \tag{1.32}$$

Proof. Using Corollary 1.8 with $f(t) = -\sqrt{t}$, $t \in \mathbb{R}^+$ (1.32) follows. □

Corollary 1.17 *Let m,M such that $0 < m < M < +\infty$, $\boldsymbol{p},\boldsymbol{q} \in \mathbb{P}$ such that $m \leq \frac{p_i}{q_i} \leq M, i = 1,\ldots,n$. Let $N_i \subseteq \{1,2,\ldots,n\}$, $i = 1,\ldots,m$ where $N_i \cap N_j = \emptyset$ for $i \neq j$, $\cup_{i=1}^{m} N_i = \{1,2,\ldots,n\}$, $\sum_{j\in N_i} q_j > 0$, for $i = 1,\ldots,m$ and $m_i = \min\{p_j/q_j : j \in N_i\}$, $M_i = \max\{p_j/q_j : j \in N_i\}$, for $i = 1,\ldots,m$. Then*

$$\begin{aligned} -B_d(\boldsymbol{p},\boldsymbol{q}) &\leq \sum_{i=1}^{m} \left(\sum_{j\in N_i} q_j\right) \frac{\left(\sqrt{m_i M_i} + \frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i} q_j}\right)\left(\sqrt{m_i} - \sqrt{M_i}\right)}{M_i - m_i} \\ &\leq \frac{(\sqrt{mM}+1)(\sqrt{m}-\sqrt{M})}{M-m} \end{aligned} \tag{1.33}$$

holds.

Proof. Using Corollary 1.9 with $f(t) = -\sqrt{t}$, $t \in \mathbb{R}^+$, (1.33) follows. □

Now we are going to derive the results from Theorems (1.17) and (1.18) for the Zipf-Mandelbrot law.

The Zipf-Mandelbrot law is a discrete probability distribution and is defined by the following probability mass function

$$f(i;M,s,t) = \frac{1}{(i+t)^s H_{M,s,t}}, \quad i = 1,\ldots,M,$$

where

$$H_{M,s,t} = \sum_{i=1}^{M} \frac{1}{(i+t)^s}$$

is a generalization of the hormonic number and $M \in \mathbb{N}$, $s > 0$ and $t \in [0,\infty\rangle$ are parameters.

If we define **q** as a Zipf-Mandelbrot law M-tuple, we have

$$q_i = \frac{1}{(i+t_2)^{s_2} H_{M,s_2,t_2}}, \quad i = 1,\dots,M,$$

where

$$H_{M,s_2,t_2} = \sum_{i=1}^{M} \frac{1}{(i+t_2)^{s_2}},$$

and the Csiszar functional becomes

$$\hat{D}_f(\mathbf{p},i,M,s_2,t_2) = \sum_{i=1}^{M} \frac{1}{(i+t_2)^{s_2} H_{M,s_2,t_2}} f\left(p_i (i+t_2)^{s_2} H_{M,s_2,t_2}\right),$$

where $f\colon I \to \mathbb{R}$, $I \subseteq \mathbb{R}$, and the parameters $M \in \mathbb{N}$, $s_2 > 0$, $t_2 \geq 0$ are such that $p_i(i+t_2)^{s_2} H_{M,s_2,t_2} \in I$, $i = 1,\dots,M$.

If **p** and **q** are both defined as Zipf-Mandelbrot law M-tuples, then the Csiszar functional becomes

$$\hat{D}_f(i,M,s_1,s_2,t_1,t_2) = \sum_{i=1}^{M} \frac{1}{(i+t_2)^{s_2} H_{M,s_2,t_2}} f\left(\frac{(i+t_2)^{s_2} H_{M,s_2,t_2}}{(i+t_1)^{s_1} H_{M,s_1,t_1}}\right),$$

where $f\colon I \to \mathbb{R}, I \subseteq \mathbb{R}$, and the parameters $M \in \mathbb{N}$, $s_1,s_2 > 0$, $t_1,t_2 \geq 0$ are such that $\frac{(i+t_2)^{s_2} H_{M,s_2,t_2}}{(i+t_1)^{s_1} H_{M,s_1,t_1}} \in I$, $i = 1,\dots,M$.

Now from Theorem 1.17 we have the following result.

Corollary 1.18 *Let I be an interval in $\mathbb{R}$ and $f\colon I \to \mathbb{R}$ a convex function. Let $\mathbf{p} = (p_1,\dots,p_n)$ be an n-tuple of real numbers, $P_n = \sum_{i=1}^{n} p_i$, and $\mathbf{q} = (q_1,\dots,q_n)$ be an n-tuple of nonnegative real numbers such that $p_i/q_i \in I$ for every $i = 1,\dots,n$. Let $N_i \subseteq \{1,2,\dots,n\}$, $i = 1,\dots,m$ where $N_i \cap N_j = \emptyset$ for $i \neq j$, $\cup_{i=1}^{m} N_i = \{1,2,\dots,n\}$. Suppose $s_2 > 0$, $t_2 \geq 0$ are such that $p_i(i+t_2)^{s_2} H_{n,s_2,t_2} \in I$, $i = 1,\dots,n$, $\sum_{j \in N_i} p_j (j+t_2)^{s_2} H_{n,s_2,t_2} \in I, i = 1,\dots,m$. Then*

$$\begin{aligned} f(P_n) &\leq \frac{1}{P_n} \sum_{i=1}^{m} \left(\sum_{j \in N_i} \frac{1}{(j+t_2)^{s_2} H_{n,s_2,t_2}} \right) f\left(\frac{\sum_{j \in N_i} p_j}{\sum_{j \in N_i} \frac{1}{(j+t_2)^{s_2} H_{n,s_2,t_2}}} \right) \\ &\leq \hat{D}_f(\mathbf{p},i,n,s_2,t_2) \end{aligned} \tag{1.34}$$

holds.

Proof. If we define **q** as a Zipf-Mandelbrot law n-tuple with parameters s_2,t_2, then from Theorem 1.17 it follows

$$\begin{aligned} f(P_n) &\leq \frac{1}{P_n} \sum_{i=1}^{m} \left(\sum_{j \in N_i} \frac{1}{(j+t_2)^{s_2} H_{n,s_2,t_2}} \right) f\left(\frac{\sum_{j \in N_i} p_j}{\sum_{j \in N_i} \frac{1}{(j+t_2)^{s_2} H_{n,s_2,t_2}}} \right) \\ &\leq \sum_{i=1}^{n} \frac{1}{(j+t_2)^{s_2} H_{n,s_2,t_2}} f\left(p_i (i+t_2)^{s_2} H_{n,s_2,t_2}\right), \end{aligned}$$

which is (1.34). □

From Theorem 1.18 we have the following result.

Corollary 1.19 *Let $f: I \to \mathbb{R}$ be a convex function on I, $[m,M] \subset I$, $-\infty < m < M < +\infty$. Let $\mathbf{p} = (p_1, \ldots, p_n)$ be an n-tuple of real numbers, $P_n = \sum_{i=1}^{n} p_i$.. Suppose $s_2 > 0$, $t_2 \geq 0$ are such that $m \leq p_i(i+t_2)^{s_2} H_{n,s_2,t_2} \leq M$, $i = 1, \ldots, n$. Let $N_i \subseteq \{1,2,\ldots,n\}$, $i = 1,\ldots,m$ where $N_i \cap N_j = \emptyset$ for $i \neq j$, $\cup_{i=1}^{m} N_i = \{1,2,\ldots,n\}$, $p_i(i+t_2)^{s_2} H_{n,s_2,t_2} \in I$, $i = 1,\ldots,n$, $\frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i} \frac{1}{(j+t_2)^{s_2} H_{n,s_2,t_2}}} \in I$, $i = 1,\ldots,m$ and $m_i = \min\{p_j/(j+t_2)^{s_2} H_{n,s_2,t_2} : j \in N_i\}$, $M_i = \max\{p_j/(j + t_2)^{s_2} H_{n,s_2,t_2} : j \in N_i\}$, for $i = 1,\ldots,m$. Then*

$$\begin{aligned} \hat{D}_f(\mathbf{p}, i, n, s_2, t_2) &\leq \sum_{i=1}^{m} \left(\sum_{j\in N_i} \frac{1}{(j+t_2)^{s_2} H_{n,s_2,t_2}} \right) \\ &\quad \times \left[\frac{M_i - \frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i} \frac{1}{(j+t_2)^{s_2} H_{n,s_2,t_2}}}}{M_i - m_i} f(m_i) + \frac{\frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i} \frac{1}{(j+t_2)^{s_2} H_{n,s_2,t_2}}} - m_i}{M_i - m_i} f(M_i) \right] \\ &\leq \frac{M - P_n}{M - m} f(m) + \frac{P_n - m}{M - m} f(M) \end{aligned} \tag{1.35}$$

holds.

Proof. If we define $\mathbf{q}$ as a Zipf-Mandelbrot law n-tuple with parameters s_2, t_2, then from Theorem 1.18 it follows

$$\begin{aligned} &\sum_{i=1}^{n} \frac{1}{(j+t_2)^{s_2} H_{n,s_2,t_2}} f\left(p_i(i+t_2)^{s_2} H_{n,s_2,t_2}\right) \\ &\leq \sum_{i=1}^{m} \left(\sum_{j\in N_i} \frac{1}{(j+t_2)^{s_2} H_{n,s_2,t_2}} \right) \\ &\quad \times \left[\frac{M_i - \frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i} \frac{1}{(j+t_2)^{s_2} H_{n,s_2,t_2}}}}{M_i - m_i} f(m_i) + \frac{\frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i} \frac{1}{(j+t_2)^{s_2} H_{n,s_2,t_2}}} - m_i}{M_i - m_i} f(M_i) \right] \\ &\leq \frac{M - \frac{P_n}{1}}{M - m} f(m) + \frac{\frac{P_n}{1} - m}{M - m} f(M), \end{aligned}$$

which is (1.35) □

Now from Theorem 1.17 we also have the following result.

Corollary 1.20 *Let I be an interval in $\mathbb{R}$ and $f\colon I\to\mathbb{R}$ a convex function. Let $N_i\subseteq\{1,2,\ldots,n\}$, $i=1,\ldots,m$ where $N_i\cap N_j=\emptyset$ for $i\neq j$, $\cup_{i=1}^m N_i=\{1,2,\ldots,n\}$. Suppose $s_1,s_2>0,t_1,t_2\geq 0$ are such that $\frac{(i+t_2)^{s_2}H_{n,s_2,t_2}}{(i+t_1)^{s_1}H_{n,s_1,t_1}}\in I$, $i=1,\ldots,n$, $\sum_{j\in N_i}\frac{(j+t_2)^{s_2}H_{n,s_2,t_2}}{(j+t_1)^{s_1}H_{n,s_1,t_1}}\in I$, $\frac{\sum_{j\in N_i}\frac{1}{(j+t_1)^{s_1}H_{n,s_1,t_1}}}{\sum_{j\in N_i}\frac{1}{(j+t_2)^{s_2}H_{n,s_2,t_2}}}\in I$, $i=1,\ldots,m$. Then*

$$f(1)\leq\frac{1}{P_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i}\frac{1}{(j+t_2)^{s_2}H_{n,s_2,t_2}}\right)f\left(\frac{\sum_{j\in N_i}\frac{1}{(j+t_1)^{s_1}H_{n,s_1,t_1}}}{\sum_{j\in N_i}\frac{1}{(j+t_2)^{s_2}H_{n,s_2,t_2}}}\right)$$
$$\leq\hat{D}_f(i,n,s_1,s_2,t_1,t_2)\tag{1.36}$$

holds.

Proof. If we define $\mathbf{p},\mathbf{q}$ as a Zipf-Mandelbrot law n-tuples with parameters s_1,t_1,s_2,t_2, then from Theorem 1.17, we get (1.36). □

From Theorem 1.18 we have the following result.

Corollary 1.21 *Let $f\colon I\to\mathbb{R}$ be a convex function on I, $[m,M]\subset I$, $-\infty<m<M<+\infty$. Suppose $s_1,s_2>0,t_1,t_2\geq 0$ are such that $m\leq\frac{(i+t_2)^{s_2}H_{n,s_2,t_2}}{(i+t_1)^{s_1}H_{n,s_1,t_1}}\leq M$, $i=1,\ldots,n$. Let $N_i\subseteq\{1,2,\ldots,n\}$, $i=1,\ldots,m$ where $N_i\cap N_j=\emptyset$ for $i\neq j$, $\cup_{i=1}^m N_i=\{1,2,\ldots,n\}$, $\frac{(i+t_2)^{s_2}H_{n,s_2,t_2}}{(i+t_1)^{s_1}H_{n,s_1,t_1}}\in I$, $i=1,\ldots,n$, $\frac{\sum_{j\in N_i}\frac{1}{(j+t_1)^{s_1}H_{n,s_1,t_1}}}{\sum_{j\in N_i}\frac{1}{(j+t_2)^{s_2}H_{n,s_2,t_2}}}\in I$, $i=1,\ldots,m$ and $m_i=\min\left\{\frac{(j+t_2)^{s_2}H_{n,s_2,t_2}}{(j+t_1)^{s_1}H_{n,s_1,t_1}}:\ j\in N_i\right\}$, $M_i=\max\left\{\frac{(j+t_2)^{s_2}H_{n,s_2,t_2}}{(j+t_1)^{s_1}H_{n,s_1,t_1}}:\ j\in N_i\right\}$, for $i=1,\ldots,m$. Then*

$$\hat{D}_f(i,n,s_1,s_2,t_1,t_2)$$
$$\leq\sum_{i=1}^{m}\left(\sum_{j\in N_i}\frac{1}{(j+t_2)^{s_2}H_{n,s_2,t_2}}\right)$$
$$\times\left[\frac{M_i-\frac{\sum_{j\in N_i}\frac{1}{(j+t_1)^{s_1}H_{n,s_1,t_1}}}{\sum_{j\in N_i}\frac{1}{(j+t_2)^{s_2}H_{n,s_2,t_2}}}}{M_i-m_i}f(m_i)+\frac{\frac{\sum_{j\in N_i}\frac{1}{(j+t_1)^{s_1}H_{n,s_1,t_1}}}{\sum_{j\in N_i}\frac{1}{(j+t_2)^{s_2}H_{n,s_2,t_2}}}-m_i}{M_i-m_i}f(M_i)\right]$$
$$\leq\frac{M-1}{M-m}f(m)+\frac{1-m}{M-m}f(M)\tag{1.37}$$

holds.

Proof. If we define $\mathbf{p},\mathbf{q}$ as a Zipf-Mandelbrot law n-tuples with parameters s_1,t_1,s_2,t_2, then from Theorem 1.18, we get (1.37). □

Since $\min_{1\leq i\leq n}\{q_i\}=\frac{1}{(n+t_2)^{s_2}H_{l,s_2,t_2}}$ and $\max_{1\leq i\leq n}\{q_i\}=\frac{1}{(1+t_2)^{s_2}H_{l,s_2,t_2}}$, from the right-hand side of (1.34) and the left-hand side of (1.35), we get the following result.

Corollary 1.22 *Let $f\colon I \to \mathbb{R}^+$ be a convex function on I, $[m,M] \subset I$, $-\infty < m < M < +\infty$. Let $\boldsymbol{p} = (p_1,\ldots,p_n)$ be an n-tuple of real numbers, $P_n = \sum_{i=1}^n p_i$. Suppose $s_2 > 0$, $t_2 \geq 0$ are such that $m \leq p_i(i+t_2)^{s_2}H_{n,s_2,t_2} \leq M$, $i = 1,\ldots,n$. Let $N_i \subseteq \{1,2,\ldots,n\}$, $i = 1,\ldots,m$ where $N_i \cap N_j = \emptyset$ for $i \neq j$, $\cup_{i=1}^m N_i = \{1,2,\ldots,n\}$, $p_i(i+t_2)^{s_2}H_{n,s_2,t_2} \in I$, $i = 1,\ldots,n$, $\frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i}\frac{1}{(j+t_2)^{s_2}H_{n,s_2,t_2}}} \in I$, $i = 1,\ldots,m$ and $m_i = \min\{p_j/(j+t_2)^{s_2}H_{n,s_2,t_2}\colon j \in N_i\}$, $M_i = \max\{p_j/(j+t_2)^{s_2}H_{n,s_2,t_2}\colon j \in N_i\}$, for $i = 1,\ldots,m$. Then*

$$\begin{aligned}
&\frac{1}{P_n(n+t_2)^{s_2}H_{n,s_2,t_2}}\sum_{i=1}^m |N_i|\, f\left(\frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i}\frac{1}{(j+t_2)^{s_2}H_{n,s_2,t_2}}}\right)\\
&\quad\leq \hat{D}_f(\mathbf{p},i,n,s_2,t_2)\\
&\quad\leq \frac{1}{(1+t_2)^{s_2}H_{n,s_2,t_2}}\sum_{i=1}^m\left[\frac{M_i\,|N_i| - (1+t_2)^{s_2}H_{n,s_2,t_2}\sum_{j\in N_i} p_j}{M_i - m_i} f(m_i)\right.\\
&\qquad\left.+\frac{(n+t_2)^{s_2}H_{n,s_2,t_2}\sum_{j\in N_i} p_j - m_i\,|N_i|}{M_i - m_i} f(M_i)\right] \qquad (1.38)
\end{aligned}$$

holds.

Proof. Using $\min_{1\leq i\leq n}\{q_i\} = \frac{1}{(n+t_2)^{s_2}H_{l,s_2,t_2}}$ and $\max_{1\leq i\leq n}\{q_i\} = \frac{1}{(1+t_2)^{s_2}H_{l,s_2,t_2}}$ from the right-hand side of (1.34) and the left-hand side of (1.35), we get

$$\begin{aligned}
&\frac{1}{P_n}\sum_{i=1}^m\left(\sum_{j\in N_i}\frac{1}{(n+t_2)^{s_2}H_{n,s_2,t_2}}\right) f\left(\frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i}\frac{1}{(j+t_2)^{s_2}H_{n,s_2,t_2}}}\right)\\
&\leq \hat{D}_f(\mathbf{p},i,n,s_2,t_2)\\
&\leq \sum_{i=1}^m\left(\sum_{j\in N_i}\frac{1}{(1+t_2)^{s_2}H_{n,s_2,t_2}}\right)\\
&\quad\times\left[\frac{M_i - \frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i}\frac{1}{(1+t_2)^{s_2}H_{n,s_2,t_2}}}}{M_i - m_i} f(m_i) + \frac{\frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i}\frac{1}{(n+t_2)^{s_2}H_{n,s_2,t_2}}} - m_i}{M_i - m_i} f(M_i)\right],
\end{aligned}$$

and (1.38) follows. □

Now we consider the integral case.

Definition 1.6 (CSISZÁR DIVERGENCE) *Let $f\colon I \to \mathbb{R}$ be a function defined on some positive interval I and let $p,q\colon [a,b] \to \mathbb{R}^+$ be two probability density functions such that $\frac{p(t)}{q(t)} \in I$, for all $t \in [a,b]$. The Csiszár divergence is defined as*

$$C_d(p,q) = \int_a^b q(t) f\left(\frac{p(t)}{q(t)}\right) dt.$$

Theorem 1.19 *Let $f: I \to \mathbb{R}$ be a convex function defined on some positive interval I, let $p, q\colon [a,b] \to \mathbb{R}^+$ be probability density functions such that $\frac{p(t)}{q(t)} \in I$, for all $t \in [a,b]$, and let $a_0, a_1, \ldots, a_{n-1}, a_n$ be arbitrary such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$. Then*

$$f(1) \leq \sum_{i=1}^{n} \left(\int_{a_{i-1}}^{a_i} q(t)dt \right) f\left(\frac{\int_{a_{i-1}}^{a_i} p(t)dt}{\int_{a_{i-1}}^{a_i} q(t)dt} \right) \leq C_d(p,q).$$

Proof. Using Theorem 1.11 with p substituting with q and g with $\frac{p}{q}$ we obtain the result.

The condition $\frac{p(t)}{q(t)} \in I$, for all $t \in [a,b]$ obviously implies $1 \in I$ and $\frac{\int_{a_{i-1}}^{a_i} p(t)dt}{\int_{a_{i-1}}^{a_i} q(t)dt} \in I$ for every $i = 1, \ldots, n$. □

Theorem 1.20 *Let $f: I \to \mathbb{R}$ be a convex function defined on some positive interval I, let $p, q\colon [a,b] \to \mathbb{R}^+$ be probability density functions such that $\frac{p(t)}{q(t)} \in I$, for all $t \in [a,b]$, and let $a_0, a_1, \ldots, a_{n-1}, a_n$ be arbitrary such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$. Let $m_i \leq \frac{p(t)}{q(t)} \leq M_i$, for all $t \in [a_{i-1}, a_i]$, $m_i < M_i$, $i = 1, \ldots, n$, $m = \min_{1 \leq i \leq n} m_i, M = \max_{1 \leq i \leq n} M_i$. Then*

$$\begin{aligned} C_d(p,q) &\leq \sum_{i=1}^{n} \left(\int_{a_{i-1}}^{a_i} q(t)dt \right) \left[\frac{M_i - \frac{\int_{a_{i-1}}^{a_i} p(t)dt}{\int_{a_{i-1}}^{a_i} q(t)dt}}{M_i - m_i} f(m_i) + \frac{\frac{\int_{a_{i-1}}^{a_i} p(t)dt}{\int_{a_{i-1}}^{a_i} q(t)dt} - m_i}{M_i - m_i} f(M_i) \right] \\ &\leq \frac{M-1}{M-m} f(m) + \frac{1-m}{M-m} f(M). \end{aligned}$$

Proof. Using Theorem 1.12 with p substituting q and g with $\frac{p}{q}$ we obtain the result. □

Definition 1.7 (KULLBACK-LEIBLER DIVERGENCE) *Let $p, q\colon [a,b] \to \mathbb{R}^+$ be two probability density functions. The Kullback-Leibler divergence is defined as*

$$KL_d(p,q) = \int_a^b p(t) \log\left(\frac{p(t)}{q(t)} \right) dt.$$

Corollary 1.23 *Let $p, q\colon [a,b] \to \mathbb{R}^+$ be probability density functions and let $a_0, a_1, \ldots, a_{n-1}, a_n$ be arbitrary such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$. Then*

$$0 \leq \sum_{i=1}^{n} \left(\int_{a_{i-1}}^{a_i} p(t)dt \right) \log\left(\frac{\int_{a_{i-1}}^{a_i} p(t)dt}{\int_{a_{i-1}}^{a_i} q(t)dt} \right) \leq KL_d(p,q).$$

Proof. Using Theorem 1.19 with $f(t) = t \log t$, $t \in \mathbb{R}^+$, we obtain the result. □

Corollary 1.24 *Let $p, q\colon [a,b] \to \mathbb{R}^+$ be probability density functions, let $a_0, a_1, \ldots, a_{n-1}, a_n$ be arbitrary such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$ and let $m_i \leq \frac{1}{q(t)} \leq M_i$,* *for all* $t \in [a_{i-1}, a_i]$, $m_i < M_i$, $i = 1, \ldots, n$, $m = \min_{1 \le i \le n} m_i, M = \max_{1 \le i \le n} M_i$. *Then*

$$KL_d(p,q) \le \sum_{i=1}^{n} \left(\int_{a_{i-1}}^{a_i} q(t)dt \right) \left[\frac{M_i - \frac{\int_{a_{i-1}}^{a_i} p(t)dt}{\int_{a_{i-1}}^{a_i} q(t)dt}}{M_i - m_i} m_i \log m_i + \frac{\frac{\int_{a_{i-1}}^{a_i} p(t)dt}{\int_{a_{i-1}}^{a_i} q(t)dt} - m_i}{M_i - m_i} M_i \log M_i \right]$$
$$\le \frac{M-1}{M-m} m \log m + \frac{1-m}{M-m} M \log M.$$

Proof. Using Theorem 1.20 with $f(t) = t \log t$, $t \in \mathbb{R}^+$ we obtain the result. □

Definition 1.8 (VARIATIONAL DISTANCE) *Let* $p,q\colon [a,b] \to \mathbb{R}^+$ *be two probability density functions. The variational distance is defined by*

$$V_d(p,q) = \int_a^b |p(t) - q(t)| dt.$$

The following corollary can be also proved elementary by using the triangle inequality for integrals.

Corollary 1.25 *Let* $p,q\colon [a,b] \to \mathbb{R}^+$ *be probability density functions and let* $a_0, a_1, \ldots, a_{n-1}, a_n$ *be arbitrary such that* $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$. *Then*

$$0 \le \sum_{i=1}^{n} \left| \int_{a_{i-1}}^{a_i} p(t)dt - \int_{a_{i-1}}^{a_i} q(t)dt \right| \le V_d(p,q).$$

Proof. Using Theorem 1.19 with $f(t) = |t-1|$, $t \in \mathbb{R}^+$ we obtain the result. □

Corollary 1.26 *Let* $p,q\colon [a,b] \to \mathbb{R}^+$ *be probability density functions, let* $a_0, a_1, \ldots, a_{n-1}, a_n$ *be arbitrary such that* $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$ *and let* $m_i \le \frac{p(t)}{q(t)} \le M_i$, *for all* $t \in [a_{i-1}, a_i]$, $m_i < M_i$, $i = 1, \ldots, n$, $m = \min_{1 \le i \le n} m_i, M = \max_{1 \le i \le n} M_i$. *Then*

$$\int_a^b |p(t) - q(t)| dt$$
$$\le \sum_{i=1}^{n} \left[\frac{M_i \int_{a_{i-1}}^{a_i} q(t)dt - \int_{a_{i-1}}^{a_i} p(t)dt}{M_i - m_i} |m_i - 1| + \frac{\int_{a_{i-1}}^{a_i} p(t)dt - m_i \int_{a_{i-1}}^{a_i} q(t)dt}{M_i - m_i} |M_i - 1| \right]$$
$$\le \frac{2(M-1)(1-m)}{M-m}.$$

Proof. Using Theorem 1.20 with $f(t) = |t-1|$, $t \in \mathbb{R}^+$ and $m \le 1 \le M$ we obtain the result. □

Definition 1.9 (JEFFREY'S DISTANCE) *Let* $p,q\colon [a,b] \to \mathbb{R}^+$ *be two probability density functions. The Jeffrey distance is defined by*

$$J_d(p,q) = \int_a^b (p(t) - q(t)) \log \left(\frac{p(t)}{q(t)} \right) dt.$$

Corollary 1.27 *Let $p,q\colon [a,b] \to \mathbb{R}^+$ be probability density functions and let $a_0, a_1, \ldots, a_{n-1}, a_n$ be arbitrary such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$. Then*

$$0 \leq \sum_{i=1}^{n} \left(\int_{a_{i-1}}^{a_i} p(t)dt - \int_{a_{i-1}}^{a_i} q(t)dt \right) \log \left(\frac{\int_{a_{i-1}}^{a_i} p(t)dt}{\int_{a_{i-1}}^{a_i} q(t)dt} \right) \leq J_d(p,q).$$

Proof. Using Theorem 1.19 with $f(t) = (t-1)\log t$, $t \in \mathbb{R}^+$ we obtain the result. □

Corollary 1.28 *Let $p,q\colon [a,b] \to \mathbb{R}^+$ be probability density functions, let $a_0, a_1, \ldots, a_{n-1}, a_n$ be arbitrary such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$ and let $m_i \leq \frac{p(t)}{q(t)} \leq M_i$, for all $t \in [a_{i-1}, a_i]$, $m_i < M_i$, $i = 1, \ldots, n$, $m = \min_{1 \leq i \leq n} m_i, M = \max_{1 \leq i \leq n} M_i$.*

$$\begin{aligned} J_d(p,q) \leq & \sum_{i=1}^{n} \left[\frac{M_i \int_{a_{i-1}}^{a_i} q(t)dt - \int_{a_{i-1}}^{a_i} p(t)dt}{M_i - m_i} (m_i - 1)\log m_i \right. \\ & \left. + \frac{\int_{a_{i-1}}^{a_i} p(t)dt - m_i \int_{a_{i-1}}^{a_i} q(t)dt}{M_i - m_i} (M_i - 1)\log M_i \right] \\ \leq & \frac{(M-1)(1-m)}{M-m} \log \frac{M}{m}. \end{aligned}$$

Proof. Using Theorem 1.20 with $f(t) = (t-1)\log t$, $t \in \mathbb{R}^+$ we obtain the result. □

Definition 1.10 (BHATTACHARYYA DISTANCE) *Let $p,q\colon [a,b] \to \mathbb{R}^+$ be two probability density functions. The Bhattacharyya distance is defined by*

$$B_d(p,q) = \int_a^b \sqrt{p(t)q(t)}dt.$$

Corollary 1.29 *Let $p,q\colon [a,b] \to \mathbb{R}^+$ be probability density functions and let $a_0, a_1, \ldots, a_{n-1}, a_n$ be arbitrary such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$. Then*

$$1 \geq \sum_{i=1}^{n} \sqrt{\int_{a_{i-1}}^{a_i} p(t)dt \int_{a_{i-1}}^{a_i} q(t)dt} \geq B_d(p,q).$$

Proof. Using Theorem 1.19 with $f(t) = -\sqrt{t}$, $t \in \mathbb{R}^+$ we obtain the result. □

Corollary 1.30 *Let $p,q\colon [a,b] \to \mathbb{R}^+$ be probability density functions, let $a_0, a_1, \ldots, a_{n-1}, a_n$ be arbitrary such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$ and let $m_i \leq \frac{p(t)}{q(t)} \leq M_i$, for all $t \in [a_{i-1}, a_i]$, $m_i < M_i$, $i = 1, \ldots, n$, $m = \min_{1 \leq i \leq n} m_i, M = \max_{1 \leq i \leq n} M_i$. Then*

$$\begin{aligned} B_d(p,q) \geq & \sum_{i=1}^{n} \left[\frac{M_i \int_{a_{i-1}}^{a_i} q(t)dt - \int_{a_{i-1}}^{a_i} p(t)dt}{M_i - m_i} \sqrt{m_i} + \frac{\int_{a_{i-1}}^{a_i} p(t)dt - m_i \int_{a_{i-1}}^{a_i} q(t)dt}{M_i - m_i} \sqrt{M_i} \right] \\ \geq & \frac{1 + \sqrt{mM}}{\sqrt{m} + \sqrt{M}}. \end{aligned}$$

Proof. Using Theorem 1.20 with $f(t) = -\sqrt{t}$, $t \in \mathbb{R}^+$ we obtain the result. □

Definition 1.11 (HELLINGER DISTANCE) *Let $p,q\colon [a,b] \to \mathbb{R}^+$ be two probability density functions. The Hellinger distance is defined by*

$$H_d(p,q) = \int_a^b (\sqrt{p(t)} - \sqrt{q(t)})^2 dt.$$

Corollary 1.31 *Let $p,q\colon [a,b] \to \mathbb{R}^+$ be probability density functions and let $a_0, a_1, \ldots, a_{n-1}, a_n$ be arbitrary such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$. Then*

$$0 \leq \sum_{i=1}^{n} \left(\sqrt{\int_{a_{i-1}}^{a_i} p(t)dt} - \sqrt{\int_{a_{i-1}}^{a_i} q(t)dt} \right)^2 \leq H_d(p,q)$$

Proof. Using Theorem 1.19 with $f(t) = (\sqrt{t} - 1)^2$, $t \in \mathbb{R}^+$ we obtain the result. □

Corollary 1.32 *Let $p,q\colon [a,b] \to \mathbb{R}^+$ be probability density functions, let $a_0, a_1, \ldots, a_{n-1}, a_n$ be arbitrary such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$ and let $m_i \leq \frac{p(t)}{q(t)} \leq M_i$, for all $t \in [a_{i-1}, a_i]$, $m_i < M_i$, $i = 1, \ldots, n$, $m = \min_{1\leq i\leq n} m_i, M = \max_{1\leq i\leq n} M_i$. Then*

$$\begin{aligned} H_d(p,q) \leq \sum_{i=1}^{n} \Bigg[& \frac{M_i \int_{a_{i-1}}^{a_i} q(t)dt - \int_{a_{i-1}}^{a_i} p(t)dt}{M_i - m_i} (\sqrt{m_i} - 1)^2 \\ & + \frac{\int_{a_{i-1}}^{a_i} p(t)dt - m_i \int_{a_{i-1}}^{a_i} q(t)dt}{M_i - m_i} (\sqrt{M_i} - 1)^2 \Bigg] \\ \leq\ & 2\frac{(\sqrt{M} - 1)(1 - \sqrt{m})}{\sqrt{m} + \sqrt{M}}. \end{aligned}$$

Proof. Using Theorem 1.20 with $f(t) = (\sqrt{t} - 1)^2$, $t \in \mathbb{R}^+$ we obtain the result. □

Definition 1.12 (TRIANGULAR DISCRIMINATION) *Let $p,q\colon [a,b] \to \mathbb{R}^+$ be two probability density functions. The triangular discrimination between p and q is defined by*

$$T_d(p,q) = \int_a^b \frac{(p(t) - q(t))^2}{p(t) + q(t)} dt.$$

Corollary 1.33 *Let $p,q\colon [a,b] \to \mathbb{R}^+$ be probability density functions and let $a_0, a_1, \ldots, a_{n-1}, a_n$ be arbitrary such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$. Then*

$$0 \leq \sum_{i=1}^{n} \frac{\left(\int_{a_{i-1}}^{a_i} p(t)dt - \int_{a_{i-1}}^{a_i} q(t)dt \right)^2}{\int_{a_{i-1}}^{a_i} p(t)dt + \int_{a_{i-1}}^{a_i} q(t)dt} \leq T_d(p,q).$$

Proof. Using Theorem 1.19 with $f(t) = \frac{(t-1)^2}{t+1}$, $t \in \mathbb{R}^+$ we obtain the result. □

Corollary 1.34 *Let* $p,q\colon [a,b] \to \mathbb{R}^+$ *be probability density functions, let* $a_0, a_1, \dots, a_{n-1}, a_n$ *be arbitrary such that* $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$ *and let* $m_i \le \frac{p(t)}{q(t)} \le M_i$, *for all* $t \in [a_{i-1}, a_i]$, $m_i < M_i$, $i = 1, \dots, n$, $m = \min_{1\le i\le n} m_i, M = \max_{1\le i\le n} M_i$. *Then*

$$\begin{aligned} T_d(p,q) \;&\le\; \sum_{i=1}^{n}\left[\frac{M_i\int_{a_{i-1}}^{a_i} q(t)dt - \int_{a_{i-1}}^{a_i} p(t)dt}{M_i - m_i}\frac{(m_i-1)^2}{m_i+1}\right.\\ &\qquad \left. + \frac{\int_{a_{i-1}}^{a_i} p(t)dt - m_i\int_{a_{i-1}}^{a_i} q(t)dt}{M_i - m_i}\frac{(M_i-1)^2}{M_i+1}\right]\\ &\le \frac{2(M-1)(1-m)}{(M+1)(m+1)}. \end{aligned}$$

Proof. Using Theorem 1.20 with $f(t) = \frac{(t-1)^2}{t+1}$, $t \in \mathbb{R}^+$ we obtain the result. □

A note on the Shannon entropy

After the concept of information theory, Shannon entropy is defined as

$$SE(\boldsymbol{p}) = -\sum_{i=1}^{n} p_i \log p_i,$$

where $\boldsymbol{p} \in \mathbb{P}$ in its discrete case.

Integral form of the Shannon entropy assumes the following form

$$SE(p) = -\int_a^b p(t)\log p(t)dt,$$

where $p\colon [a,b] \to \mathbb{R}^+$ be a probability density function.

The corresponding bounds for the Shannon entropy in its discrete form are given as follows.

Corollary 1.35 *Let* $\boldsymbol{q} \in \mathbb{P}$. *Let* $N_i \subseteq \{1,2,\dots,n\}, i = 1,\dots,m$ *where* $N_i \cap N_j = \emptyset$ *for* $i \ne j$, $\cup_{i=1}^{m} N_i = \{1,2,\dots,n\}$ *and* $\sum_{j\in N_i} q_j > 0, i = 1,\dots,m$. *Then*

$$-\log n \le \sum_{i=1}^{m}\left(\sum_{j\in N_i} q_j\right)\left(\log\sum_{j\in N_i} q_j - \log|N_i|\right) \le -SE(\boldsymbol{q}).$$

Proof. Using Theorem 1.17 with $f(t) = -\log t, t \in \mathbb{R}^+$ and $\boldsymbol{q} \in \mathbb{P}$, we get

$$-\log(P_n) \le \sum_{i=1}^{m}\left(\sum_{j\in N_i} q_j\right)\left(-\log\left(\frac{\sum_{j\in N_i} p_j}{\sum_{j\in N_i} q_j}\right)\right) \le \sum_{i=1}^{n} q_i\left(-\log\frac{p_i}{q_i}\right).$$

For $p_i = 1, i = 1,\dots,n$ inequality (1.39) follows.

Corollary 1.36 *Let m, M such that $0 < m < M < +\infty$, $\boldsymbol{q} \in \mathbb{P}$ such that $m \leq \frac{1}{q_i} \leq M, i = 1,\dots,n$. Let $N_i \subseteq \{1,2,\dots,n\}, i = 1,\dots,m$ where $N_i \cap N_j = \emptyset$ for $i \neq j$, $\cup_{i=1}^{m} N_i = \{1,2,\dots,n\}$, $\sum_{j\in N_i} q_j > 0$, for $i = 1,\dots,m$ and $m_i = \min\{1/q_j\colon j \in N_i\}$, $M_i = \max\{1/q_j\colon j \in N_i\}$, for $i = 1,\dots,m$. Then*

$$
\begin{aligned}
-SE(\boldsymbol{q}) &\leq \sum_{i=1}^{m}\left(\sum_{j\in N_i} q_j\right)\left[\frac{\frac{|N_i|}{\sum_{j\in N_i} q_j} - M_i}{M_i - m_i}\log m_i + \frac{\frac{m_i - |N_i|}{\sum_{j\in N_i} q_j}}{M_i - m_i}\log M_i\right] \\
&\leq \frac{n-M}{M-m}\log m + \frac{m-n}{M-m}\log M
\end{aligned}
\tag{1.39}
$$

holds.

Proof. Using Theorem 1.18 with $f(t) = -\log t, t \in \mathbb{R}^+$, $\boldsymbol{q} \in \mathbb{P}$ and $p_i = 1,\dots,n$, we get

$$
\begin{aligned}
&\sum_{i=1}^{n} q_i\left(-\log\frac{1}{q_i}\right) \\
&\leq \sum_{i=1}^{m}\left(\sum_{j\in N_i} q_j\right)\left[\frac{M_i - \frac{|N_i|}{\sum_{j\in N_i} q_j}}{M_i - m_i}(-\log m_i) + \frac{\frac{|N_i|}{\sum_{j\in N_i} q_j} - m_i}{M_i - m_i}(-\log M_i)\right],
\end{aligned}
$$

and (1.39) easily follows.

Now we consider the integral case.

In its integral form, analogous results are as follows.

Corollary 1.37 *Let $q\colon [a,b] \to \mathbb{R}^+$ be a probability density function and let $a_0, a_1, \dots, a_{n-1}, a_n$ be arbitrary such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$. Then*

$$
-\log(b-a) \leq \sum_{i=1}^{n}\left(\int_{a_{i-1}}^{a_i} q(t)dt\right)\log\left(\frac{\int_{a_{i-1}}^{a_i} q(t)dt}{a_i - a_{i-1}}\right) \leq -SE(q).
$$

Proof. Using Theorem 1.19 with $f(t) = -\log t, t \in \mathbb{R}^+$ and $p(t) = \frac{1}{b-a}, t \in [a,b]$ we obtain the result.

Corollary 1.38 *Let $q\colon [a,b] \to \mathbb{R}^+$ be a probability density function, let $a_0, a_1, \dots, a_{n-1}, a_n$ be arbitrary such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$, and let $m_i \leq \frac{1}{q(t)} \leq M_i$ for all $t \in [a_{i-1}, a_i]$, $m_i < M_i, i = 1,\dots,n$, $m = \min_{1\leq i\leq n} m_i, M = \max_{1\leq i\leq n} M_i$. Then*

$$
\begin{aligned}
&-SE(q) + \log(b-a) \\
&\leq \sum_{i=1}^{n}\left(\int_{a_{i-1}}^{a_i} q(t)dt\right)\left[\frac{\frac{a_i - a_{i-1}}{(b-a)\int_{a_{i-1}}^{a_i} q(t)dt} - M_i}{M_i - m_i}\log m_i + \frac{m_i - \frac{a_i - a_{i-1}}{(b-a)\int_{a_{i-1}}^{a_i} q(t)dt}}{M_i - m_i}\log M_i\right] \\
&\leq \frac{1-M}{M-m}\log m + \frac{m-1}{M-m}\log M.
\end{aligned}
$$

Proof. Using Theorem 1.20 with $f(t) = -\log t, t \in \mathbb{R}^+$ and $p(t) = \frac{1}{b-a}, t \in [a,b]$ we obtain the result.

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Chapter 2

Jensen-McShane type inequalities on a rectangle and applications

Let Ω be a nonempty set and L be a linear class of real-valued functions

$$f : \Omega \to \mathbb{R},$$

having the properties:

L1: $f,g \in L \Rightarrow (\alpha f + \beta g) \in L$, for all $\alpha, \beta \in \mathbb{R}$;

L2: $1 \in L$, i.e., if $f(t) = 1$ for all $t \in \Omega$, then $f \in L$.

Throughout this chapter we consider normalized isotonic positive linear functional

$$F : L \to \mathbb{R},$$

that is, we assume

A1: $F(\alpha f + \beta g) = \alpha F(f) + \beta F(g)$ for $f,g \in L, \alpha, \beta \in \mathbb{R}$ (linearity);

A2: $f \in L$, $f \geq 0$ on $\Omega \Rightarrow F(f) \geq 0$ (positive isotonicity);

A3: $F(1) = 1$.

For instance, some normalized linear positive functionals are:

- $F(f) = \frac{1}{\nu(\Omega)} \int_\Omega f d\nu$, for positive measure ν on Ω;
- $F(f) = \dfrac{1}{\sum_{k\in\Omega} p_k} \sum\limits_{k\in\Omega} f_k p_k$, for discrete measure on $\Omega = \{1,2,\ldots\}$, where $0 < \sum\limits_{k\in\Omega} p_k < \infty$ and $p_k \geq 0$.

The Jensen inequality for concave (convex) functions is one of the most important inequalities in mathematics and statistics. There are many forms of this famous inequality (discrete form, integral form, etc.). We will consider the McShane generalizations of the Jensen inequality (see [13], [14, p. 48–49]).

Theorem A 1 (THE MCSHANE INEQUALITY) *Let φ be a continuous concave function on a closed convex set K in $\mathbb{R}^n$ and F be a normalized isotonic positive linear functional on L. Let g_i be functions in $L, i = 1,\ldots,n$, such that $(g_1(t),\ldots,g_n(t))$ is in K for all $t \in \Omega$ and the components of $\varphi(g_i)$ are in the class L. Then $(F(g_1),\ldots,F(g_n))$ is in K, $\varphi(F(g_1),\ldots,F(g_n))$ is defined and this inequality holds*

$$F(\varphi(g_1),\ldots,\varphi(g_n)) \leq \varphi(F(g_1),\ldots,F(g_n)). \tag{2.1}$$

If φ is a continuous convex function then the reverse inequality holds.

Note that Raşa in [16] pointed out that φ has to be continuous.

In this chapter we provide an extension of the McShane inequality for φ being a concave (convex) function defined on a rectangle $D = [a,A] \times [b,B]$ and functions $g_1, g_2 \in L$ such that $g_1(t) \in [a,A]$, $g_2(t) \in [b,B]$ for all $t \in \Omega$. The lower bound for $F(\varphi(g_1),\varphi(g_2))$ is obtained by geometrical consideration of the secant planes of the surface $z = \varphi(x,y)$.

This chapter is based on the results from the papers [3], [8] and [9].

Notation will be our first issue for clarifications purposes. We are observing rectangle $D = [a,A] \times [b,B]$ separated into triangles in the two different ways:

(i) $D = \Delta_1 \cup \Delta_2$, where Δ_1 is a triangle with vertices $(a,b),(A,b)$ and (a,B), and $\Delta_2 = \Delta((A,B),(a,B),(A,b))$. Note that the following is valid:

$$\Delta_1 \cap \Delta_2 = \{(x,y) : (A-a)y + (B-b)x - AB + ab = 0\},$$

$$(x,y) \in \Delta_1 \Leftrightarrow (A-a)y + (B-b)x - AB + ab \leq 0, \tag{2.2}$$

$$(x,y) \in \Delta_2 \Leftrightarrow (A-a)y + (B-b)x - AB + ab \geq 0. \tag{2.3}$$

(ii) $D = \Delta_3 \cup \Delta_4$, where Δ_3 is a triangle determined with vertices $(a,b),(A,B)$ and (a,B), while the Δ_4 is determined with $(a,b),(A,B)$ and (A,b). Note that the following is valid

$$\Delta_3 \cap \Delta_4 = \{(x,y) : (A-a)y - (B-b)x - Ab + aB = 0\},$$

$$(x,y) \in \Delta_3 \Leftrightarrow (A-a)y - (B-b)x - Ab + aB \geq 0, \tag{2.4}$$

$$(x,y) \in \Delta_4 \Leftrightarrow (A-a)y - (B-b)x - Ab + aB \leq 0. \tag{2.5}$$

For the continuous concave (convex) function φ we denote the vertices

$$T_1(a,b,\varphi(a,b)),\, T_2(A,B,\varphi(A,B)),\, T_3(a,B,\varphi(a,b)) \quad \text{and} \quad T_4(A,b,\varphi(A,B))$$

and the planes Π_k determined by the vertices as follows:

$$\Pi_1(T_1,T_3,T_4),\, \Pi_2(T_2,T_3,T_4),\, \Pi_3(T_3,T_1,T_2) \quad \text{and} \quad \Pi_4(T_4,T_1,T_2).$$

These planes are the graphs of affine functions $\pi_k : D \to \mathbb{R}$:

$$\pi_k(x,y) = \lambda_k x + \mu_k y + \nu_k, \quad k \in \{1,2,3,4\} \tag{2.6}$$

with the coefficients:

$$\begin{aligned} \lambda_1 = \lambda_4 &= \frac{\varphi(A,b)-\varphi(a,b)}{A-a}, \quad \mu_1 = \mu_3 = \frac{\varphi(a,B)-\varphi(a,b)}{B-b}, \\ \lambda_2 = \lambda_3 &= \frac{\varphi(A,B)-\varphi(a,B)}{A-a}, \quad \mu_2 = \mu_4 = \frac{\varphi(A,B)-\varphi(A,b)}{B-b}, \\ \nu_1 &= \varphi(a,b) - \lambda_1 a - \mu_1 b, \quad \nu_2 = \varphi(A,B) - \lambda_2 A - \mu_2 B, \\ \nu_3 &= \varphi(a,B) - \lambda_3 a - \mu_3 B, \quad \nu_4 = \varphi(A,b) - \lambda_4 A - \mu_4 b. \end{aligned} \tag{2.7}$$

Let us denote

$$\Delta\varphi = \varphi(a,b) - \varphi(a,B) - \varphi(A,b) + \varphi(A,B). \tag{2.8}$$

In this geometrical setting, a condition $\Delta\varphi > 0$ means that the edge T_3T_4 lies below the edge T_1T_2.

Let $M_{ij}, m_{ij} : D \to \mathbb{R}$, $(i,j) \in \{(1,2),(3,4)\}$ denote functions defined by

$$\begin{aligned} M_{ij}(x,y) &= \max\{\pi_i(x,y), \pi_j(x,y), \\ m_{ij}(x,y) &= \min\{\pi_i(x,y), \pi_j(x,y)\}. \end{aligned} \tag{2.9}$$

The compositions of functions $M_{ij}(g_1,g_2) : \Omega \to \mathbb{R}$ and $m_{ij}(g_1,g_2) : \Omega \to \mathbb{R}$ are well defined for $g_1, g_2 \in L$ such that $g_1(t) \in [a,A], g_2(t) \in [b,B]$ for all $t \in \Omega$ by

$$M_{ij}(g_1,g_2)(t) = M_{ij}(g_1(t),g_2(t)) = \max\{\pi_i(g_1(t),g_2(t)), \pi_j(g_2(t),g_2(t))\},$$

$$m_{ij}(g_1,g_2)(t) = M_{ij}(g_1(t),g_2(t)) = \min\{\pi_i(g_1(t),g_2(t)), \pi_j(g_2(t),g_2(t))\}.$$

These functions $M_{ij}, m_{ij} : D \to \mathbb{R}$ can be defined also as follows:

$$M_{ij}(x,y) = \frac{(\lambda_i+\lambda_j)x + (\mu_i+\mu_j)y + \nu_i + \nu_j}{2} + \frac{|(\lambda_i-\lambda_j)x + (\mu_i-\mu_j)y + \nu_i - \nu_j|}{2}$$

and

$$m_{ij}(x,y) = \frac{(\lambda_i+\lambda_j)x + (\mu_i+\mu_j)y + \nu_i + \nu_j}{2} - \frac{|(\lambda_i-\lambda_j)x + (\mu_i-\mu_j)y + \nu_i - \nu_j|}{2}.$$

We introduce the functions:

$$\pi_{12}(x,y) = \begin{cases} \pi_1(x,y), & (x,y) \in \Delta_1 \\ \pi_2(x,y), & (x,y) \in \Delta_2 \end{cases} \quad \text{and} \quad \pi_{34}(x,y) = \begin{cases} \pi_3(x,y), & (x,y) \in \Delta_3 \\ \pi_4(x,y), & (x,y) \in \Delta_4. \end{cases} \tag{2.10}$$

The compsite functions $\pi_{12}(g_1,g_2) : \Omega \to \mathbb{R}$ and $\pi_{34}(g_1,g_2) : \Omega \to \mathbb{R}$, are well defined for $g_1, g_2 \in L$ such that $g_1(t) \in [a,A]$, $g_2(t) \in [b,B]$ for all $t \in \Omega$.

The following lemma integrates the previously presented relations.

Lemma 2.1 *Let* M_{ij}, m_{ij} , π_{12} *and* π_{34} *be functions defined in* (2.9) *and* (2.10)*. For a function* $\varphi : D \to \mathbb{R}$ *and* $\Delta\varphi$ *defined by* (2.8) *we have*

(i) *if* $\Delta\varphi \geq 0$*, then for all* $(x,y) \in D$

$$\pi_{12}(x,y) \leq \pi_{34}(x,y), \tag{2.11}$$

and

$$\pi_{12}(x,y) = M_{12}(x,y) \quad \text{and} \quad \pi_{34}(x,y) = m_{34}(x,y); \tag{2.12}$$

(ii) *if* $\Delta\varphi \leq 0$*, then for all* $(x,y) \in D$

$$\pi_{12}(x,y) \geq \pi_{34}(x,y), \tag{2.13}$$

and

$$\pi_{12}(x,y) = m_{12}(x,y) \quad \text{and} \quad \pi_{34}(x,y) = M_{34}(x,y). \tag{2.14}$$

Proof. Using elementary algebra, we can obtain some convenient formulas. Namely, in the term of $\Delta\varphi$ there exist relations:

$$\pi_2(x,y) - \pi_1(x,y) = \Delta\varphi \cdot \frac{(A-a)y + (B-b)x - AB + ab}{(B-b)(A-a)}; \tag{2.15}$$

$$\pi_4(x,y) - \pi_1(x,y) = \Delta\varphi \cdot \frac{y-b}{B-b}; \tag{2.16}$$

$$\pi_3(x,y) - \pi_1(x,y) = \Delta\varphi \cdot \frac{x-a}{A-a}; \tag{2.17}$$

$$\pi_3(x,y) - \pi_2(x,y) = \Delta\varphi \cdot \frac{B-y}{B-b}; \tag{2.18}$$

$$\pi_4(x,y) - \pi_2(x,y) = \Delta\varphi \cdot \frac{A-x}{A-a}; \tag{2.19}$$

$$\pi_4(x,y) - \pi_3(x,y) = \Delta\varphi \cdot \frac{(A-a)y - (B-b)x - Ab + aB}{(B-b)(A-a)}. \tag{2.20}$$

According to (2.16), (2.17), (2.18) and (2.19) for all (x,y) in D we have

$$\pi_j(x,y) - \pi_i(x,y) \geq 0 \quad \text{for} \quad j \in \{3,4\},\ i \in \{1,2\},$$

and (2.11) holds by (2.10).

To prove the claims in expression (2.12), we check that for $(x,y) \in \Delta_1$, (2.2) and (2.15) entail that $\pi_1 \geq \pi_2$ and consequently $M_{12}(x,y) = \pi_1(x,y) = \pi_{12}(x,y)$.

If $(x,y) \in \Delta_2$, then (2.3) and (2.15) give us that $\pi_1 \leq \pi_2$, and therefore $M_{12}(x,y) = \pi_2(x,y) = \pi_{12}(x,y)$.

Furthermore, we note that for $(x,y) \in \Delta_3$, (2.4) and (2.20) entail that $\pi_4 \geq \pi_3$ and $m_{34}(x,y) = \pi_3(x,y) = \pi_{34}(x,y)$.

Finally, for $(x,y) \in \Delta_4$, (2.5) and (2.20) ensure that $\pi_4 \leq \pi_3$, so $m_{34}(x,y) = \pi_4(x,y) = \pi_{34}(x,y)$ according to definitions (2.10), as previously mentioned. □

2.1 Jensen-McShane type inequality on a rectangle

2.1.1 Main result

Here we state the basic result of this chapter, a refinement of the Jensen-McShane type inequality on a rectangle proved in [3].

Theorem 2.1 *Let $F : L \to \mathbb{R}$ be a normalized isotonic positive linear functional, where L is a linear space of real-valued functions defined on a nonempty set Ω. Moreover, let $g_1, g_2 \in L$ be functions such that $g_1(t) \in [a,A]$, $g_2(t) \in [b,B]$ for all $t \in \Omega$ and π_{12}, π_{34} be functions defined by* (2.10).

If $\varphi : D = [a,A] \times [b,B] \to \mathbb{R}$ is a continuous concave function then

$$\begin{aligned} &\max\{F(\pi_{12}(g_1,g_2)), F(\pi_{34}(g_1,g_2))\} \\ &\quad \leq F(\max\{\pi_{12}(g_1,g_2), \pi_{34}(g_1,g_2)\}) \\ &\quad \leq F(\varphi(g_1,g_2)) \leq \varphi(F(g_1),F(g_2)), \end{aligned} \tag{2.21}$$

and if $\varphi : D \to \mathbb{R}$ is a continuous convex function then

$$\begin{aligned} &\min\{F(\pi_{12}(g_1,g_2)), F(\pi_{34}(g_1,g_2))\} \\ &\quad \geq F(\min\{\pi_{12}(g_1,g_2), \pi_{34}(g_1,g_2)\}) \\ &\quad \geq F(\varphi(g_1,g_2)) \geq \varphi(F(g_1),F(g_2)). \end{aligned} \tag{2.22}$$

Proof. Note that from the property A1 we can obtain

A1': $F(l(g_1,g_2,\ldots,g_n)) = l(F(g_1),F(g_2),\ldots,F(g_n))$

for every function l that is linear on $\mathbb{R}^n$.

So, for linear functions π_i defined by (2.6) we conclude that

$$F(\pi_i(g_1,g_2)) = \pi_i(F(g_1,g_2)), \quad i = 1,\ldots,4,$$

and $F(\max\{\pi_{12}(g_1,g_2), \pi_{34}(g_1,g_2)\})$ is well defined. The statement of the McShane theorem ensures that $(F(g_1),F(g_2)) \in D$.

The first inequality (2.21) follows by applying the monotonicity property of the functional F:

$$\begin{aligned} \pi_{12}(g_1,g_2) &\leq \max\{\pi_{12}(g_1,g_2), \pi_{34}(g_1,g_2)\}, \\ \pi_{34}(g_1,g_2) &\leq \max\{\pi_{12}(g_1,g_2), \pi_{34}(g_1,g_2)\}, \\ F(\pi_{12}(g_1,g_2)) &\leq F(\max\{\pi_{12}(g_1,g_2), \pi_{34}(g_1,g_2)\}), \\ F(\pi_{34}(g_1,g_2)) &\leq F(\max\{\pi_{12}(g_1,g_2), \pi_{34}(g_1,g_2)\}). \end{aligned}$$

The concavity of $\varphi : D \to \mathbb{R}$ provides that for all $t \in \Omega$ and $(g_1(t), g_2(t)) \in \Delta_i$, $i = 1,\ldots,4$ it holds:

$$\pi_i(g_1(t), g_2(t)) \leq \varphi(g_1(t), g_2(t)). \tag{2.23}$$

In the case of φ being a convex function, the inequalities are reversed.

Using (2.10) and inequalities (2.23) we show that the following inequalities hold for concave functions φ, for all $t \in \Omega$:

$$\pi_{12}(g_1,g_2) \le \varphi(g_1,g_2) \quad \text{and} \quad \pi_{34}(g_1,g_2) \le \varphi(g_1,g_2).$$

Hence we have

$$\max\{\pi_{12}(g_1,g_2),\pi_{34}(g_1,g_2)\} \le \varphi(g_1,g_2).$$

Applying the normalized positive linear functional F we obtain the second inequality in (2.21).

The third inequality in (2.21) is the well-known Jensen inequality which was modified by Jessen and generalized by McShane (2.1).

To prove (2.22), note that if φ is convex, then $-\varphi$ is a concave function. □

By setting conditions $\Delta\varphi \ge 0$ or $\Delta\varphi \le 0$ then Theorem 2.1 can be generalized as follows.

Theorem 2.2 *Let $F : L \to \mathbb{R}$ be a normalized isotonic positive linear functional, where L is a linear space of real-valued functions defined on a nonempty set Ω. Moreover, let $g_1,g_2 \in L$ be functions such that $g_1(t) \in [a,A]$, $g_2(t) \in [b,B]$ for all $t \in \Omega$, π_{12},π_{34} be functions defined by* (2.10) *and M_{ij},m_{ij} be functions defined by (2.9).*

(i) Suppose that $\varphi : D = [a,A] \times [b,B] \to \mathbb{R}$ is a continuous and concave function and $\Delta\varphi$ is defined by (2.8).

(i_1) *If $\Delta\varphi \ge 0$, then*

$$\begin{aligned} M_{12}(F(g_1),F(g_2)) &\le F(M_{12}(g_1,g_2)) \\ &\le \max\{F(\pi_{12}(g_1,g_2)),F(\pi_{34}(g_1,g_2))\} \\ &\le F(\max\{\pi_{12}(g_1,g_2),\pi_{34}(g_1,g_2)\}) \\ &= F(m_{34}(g_1,g_2)) \le F(\varphi(g_1,g_2)) \\ &\le \varphi(F(g_1),F(g_2)). \end{aligned} \tag{2.24}$$

(i_2) *If $\Delta\varphi \le 0$, then*

$$\begin{aligned} M_{34}(F(g_1),F(g_2)) &\le F(M_{34}(g_1,g_2)) \\ &\le \max\{F(\pi_{12}(g_1,g_2)),F(\pi_{34}(g_1,g_2))\} \\ &\le F(\max\{\pi_{12}(g_1,g_2),\pi_{34}(g_1,g_2)\}) \\ &= F(m_{12}(g_1,g_2)) \le F(\varphi(g_1,g_2)) \\ &\le \varphi(F(g_1),F(g_2)). \end{aligned} \tag{2.25}$$

(ii) Suppose that $\varphi : D = [a,A] \times [b,B] \to \mathbb{R}$ is a continuous and convex function and $\Delta\varphi$ is defined by (2.8).

(ii_1) *If* $\Delta\varphi \le 0$, *then*

$$\begin{aligned} m_{12}(F(g_1),F(g_2)) &\ge F(m_{12}(g_1,g_2)) \\ &\ge \min\{F(\pi_{12}(g_1,g_2)),F(\pi_{34}(g_1,g_2))\} \\ &\ge F(\min\{\pi_{12}(g_1,g_2),\pi_{34}(g_1,g_2)\}) \\ &= F(M_{34}(g_1,g_2)) \ge F(\varphi(g_1,g_2)) \\ &\ge \varphi(F(g_1),F(g_2)). \end{aligned}$$

(ii_2) *If* $\Delta\varphi \ge 0$, *then*

$$\begin{aligned} m_{34}(F(g_1),F(g_2)) &\ge F(m_{34}(g_1,g_2)) \\ &\ge \min\{F(\pi_{12}(g_1,g_2)),F(\pi_{34}(g_1,g_2))\} \\ &\ge F(\min\{\pi_{12}(g_1,g_2),\pi_{34}(g_1,g_2)\}) \\ &= F(M_{12}(g_1,g_2)) \ge F(\varphi(g_1,g_2)) \\ &\ge \varphi(F(g_1),F(g_2)). \end{aligned}$$

Proof. (i) First, we consider a concave function $\varphi : D \to \mathbb{R}$.

(i_1) Since $\pi_1(g_1,g_2) \le M_{12}(g_1,g_2)$ and $\pi_2(g_1,g_2) \le M_{12}(g_1,g_2)$, properties of functional F ensure that

$$F(\pi_1(g_1,g_2)) = \pi_1(F(g_1),F(g_2)) \le F(M_{12}(g_1,g_2))$$

and

$$F(\pi_2(g_1,g_2)) = \pi_2(F(g_1),F(g_2)) \le F(M_{12}(g_1,g_2)),$$

so the first inequality in (2.24) states.

Since φ is a concave function with $\Delta\varphi \ge 0$, the second, fourth and fifth inequalities in (2.24) are consequence of (2.11) and (2.12) in Lemma 2.1.

The third and the last inequality are rewritten from (2.21).

(i_2) If we assume that φ is a concave function with $\Delta\varphi \le 0$, the first inequality in (2.25) is consequence of isotonicity.

The second, fourth and fifth inequalities in (2.25) are consequence of (2.13) and (2.14) in Lemma 2.1.

The third and the last inequality are rewritten from (2.21).

(ii) Similarly we can prove (ii_1) and (ii_2). □

Remark 2.1 The figure 2.1 visualises the Jensen-McShane type inequality on the rectangle D for linear isotonic positive functional E, and for a concave function φ such that $\Delta\varphi \ge 0$ (Theorem 2.2 case (i_1)).

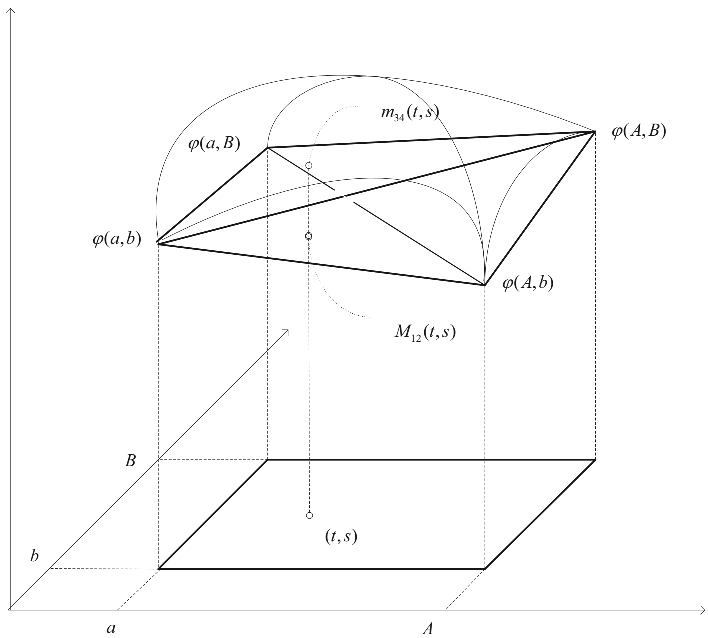

Figure 2.1: *Jensen-McShane type inequality on a rectangle*

Remark 2.2 Let $\varphi : D \to \mathbb{R}$ be a continuous and concave function and $\Delta\varphi = 0$. Then holds

$$\lambda F(g_1) + \mu F(g_2) + \nu \leq F(\varphi(g_1, g_2)) \leq \varphi(F(g_1), F(g_2)),$$

with $\lambda = \lambda_k$, $\mu = \mu_k$ and $\nu = \nu_k$, $k = 1, \ldots, 4$.

Let $\varphi : D \to \mathbb{R}$ be a continuous and convex function and $\Delta\varphi = 0$. Then holds

$$\lambda F(g_1) + \mu F(g_2) + \nu \geq F(\varphi(g_1, g_2)) \leq \varphi(F(g_1), F(g_2)),$$

with $\lambda = \lambda_k$, $\mu = \mu_k$ and $\nu = \nu_k$, $k = 1, \ldots, 4$.

2.2 Applied results

In this section we present applied results proved in [3], the sequence of inequalities which include the McShane generalization of the Jensen inequality on a rectangle for different choices of the functional F.

2.2.1 Diaz-Metcalf type inequalities

The results of this section were inspired by the research of V. Csiszár and T. F. Móri based on the Diaz-Metcalf inequality in the probability settings (see [2]. [15]). They got a result related to the random variables, for the expectation E and for the concave function defined on a rectangle.

For the random variables ξ, η defined on a probability space (Ω, Σ, P) with $P(m_1 \leq \xi \leq M_1) = 1$, $P(m_2 \leq \eta \leq M_2) = 1, M_1, M_2, m_1, m_2 > 0$, Diaz-Metcalf inequality holds [5] :

$$m_2 M_2 E[\xi^2] + m_1 M_1 E[\eta^2] \leq (m_1 m_2 + M_1 M_2) E[\xi \eta].$$

Csiszár and Móri in [2] obtained the lower bound for $E[\xi \eta]$ for known $E[\xi^2]$ and $E[\eta^2]$ as follows:

$$\lambda E[\xi^2] + \mu E[\eta^2] + \nu \leq E[\xi \eta]. \tag{2.26}$$

This result can be interpreted according to our investigation:

Theorem A 2 (THE DIAZ-METCALF TYPE INEQUALITY) *Suppose that $\varphi : D = [a,A] \times [b,B] \to \mathbb{R}$ is a concave function. Let (X,Y) be a random vector, $P[(X,Y) \in D] = 1$ and $E[X], E[Y]$ be the expectations of random variables X and Y with respect to probability P.*

If $\Delta\varphi \geq 0$, then

$$M_{12}(E[X], E[Y]) \leq E[\varphi(X,Y)] \leq \varphi(E[X], E[Y])$$

holds and if $\Delta\varphi \leq 0$, then

$$M_{34}(E[X], E[Y]) \leq E[\varphi(X,Y)] \leq \varphi(E[X], E[Y])$$

holds, where M_{12} and M_{34} are defined by (2.9) *and $\Delta\varphi$ is defined by (2.8).*

As an application of Theorem 2.2 for mathematical expectations and bounded random variables $X, Y : \Omega \to \mathbb{R}$ we obtain the following refinement of the Theorem A2.

Theorem 2.3 *Suppose that $\varphi : D = [a,A] \times [b,B] \to \mathbb{R}$ is a concave function. Let (X,Y) be a random vector with $P[(X,Y) \in D] = 1$ and $E[X], E[Y]$ be the expectations of random variables X and Y with respect to probability P.*

If $\Delta\varphi \geq 0$, then

$$\begin{aligned} M_{12}(E[X], E[Y]) &\leq E(M_{12}(X,Y)) \\ &\leq \max\{E[\pi_{12}(X,Y)], E[\pi_{34}(X,Y)]\} \\ &\leq E[\max\{\pi_{12}(X,Y), \pi_{34}(X,Y)\}] \\ &= E[m_{34}(X,Y)] \leq E[\varphi(X,Y)] \\ &\leq \varphi(E[X], E[Y]), \end{aligned}$$

and if $\Delta\varphi \leq 0$, then

$$\begin{aligned} M_{34}(E[X],E[Y]) &\le E[M_{34}(X,Y)] \\ &\le \max\{E[\pi_{12}(X,Y)],E[\pi_{34}(f,g)]\} \\ &\le E[\max\{\pi_{12}(X,Y),\pi_{34}(X,Y)\}] \\ &= E[m_{12}(X,Y)] \le E[\varphi(X,Y)] \\ &\le \varphi(E[X],E[Y]). \end{aligned}$$

Remark 2.3 By substituting $\varphi(x,y) = (xy)^{\frac{1}{2}}$, $a = m_1^2$, $A = M_1^2$, $b = m_2^2$ and $B = M_2^2$ in Theorem A2, we get the Csiszár and Móri's coefficients:

(i) If

$$(M_2^2 - m_2^2)E[\xi^2] + (M_1^2 - m_1^2)E[\eta^2] \le M_1^2M_2^2 - m_1^2m_2^2$$

holds, then Csiszár and Móri's coefficients are:

$$\lambda = \lambda_1 = \frac{m_2}{m_1+M_1}, \quad \mu = \mu_1 = \frac{m_1}{m_2+M_2} \quad \text{and} \quad \nu = \nu_1 = (M_1M_2 - m_1m_2)\lambda_1\mu_1.$$

(ii) If

$$(M_2^2 - m_2^2)E[\xi^2] + (M_1^2 - m_1^2)E[\eta^2] \le M_1^2M_2^2 - m_1^2m_2^2$$

holds, then Csiszár and Móri's coefficients are:

$$\lambda = \lambda_2 = \frac{M_2}{m_1+M_1}, \quad \mu = \mu_2 = \frac{M_1}{m_2+M_2} \quad \text{and} \quad \nu = \nu_2 = (m_1m_2 - M_1M_2)\lambda_2\mu_2.$$

Remark 2.4 Theorem 2.3 improves Csiszár and Móri's result:

(i) If

$$(M_2^2 - m_2^2)E[\xi^2] - (M_1^2 - m_1^2)E[\eta^2] \le m_1^2M_2^2 - M_1^2m_2^2,$$

holds, then Csiszár and Móri's coefficients are:

$$\lambda = \lambda_3 = \frac{M_2}{m_1+M_1}, \quad \mu = q_3 = \frac{m_1}{m_2+M_2} \quad \text{and} \quad \nu = r_3 = (M_1m_2 - m_1M_2)\lambda_3\mu_3.$$

(ii) If

$$(M_2^2 - m_2^2)E[\xi^2[-(M_1^2 - m_1^2)E[\eta^2] \ge m_1^2M_2^2 - M_1^2m_2^2$$

holds, then Csiszár and Móri's coefficients are:

$$\lambda = p_4 = \frac{m_2}{m_1+M_1}, \quad \mu = q_4 = \frac{M_1}{m_2+M_2} \quad \text{and} \quad \nu = r_4 = (m_1M_2 - M_1m_2)\lambda_4\mu_4.$$

According to Lemma 2.1 for functional E we have that new lower bound for $E[\xi\eta]$ is greater than bound in Remark 2.3:

$$\lambda_iE[\xi^2] + \mu_iE[\eta^2] + \nu_i \le \lambda_jE[\xi^2] + \mu_jE[\eta^2] + \nu_j, \quad (i,j) \in \{(1,2),(3,4)\}.$$

2.2.2 Hadamard and Fejér type inequalities

In this section we obtain a refinement of Feyér type inequalities calculated by $|\Delta\varphi|$ as an application for a functional defined as weighted integral over the rectangle D.

In [10] authors obtained the following result considering the extension of the weighted version of the Hadamard inequality known as by Fejér's inequality for functions of two-variables defined on a rectangle (see [7], [14, p. 138]).

Theorem A 3 (THE FEJÉR TYPE INEQUALITY) *Let $w : D = [a,A] \times [b,B] \to \mathbb{R}$ be a nonnegative integrable function such that $w(s,t) = u(s)v(t)$, where $u : [a,A] \to \mathbb{R}$ is an integrable function such that $\int_a^A u(s)ds = 1$, $u(s) = u(a+A-s)$, for all $s \in [a,A]$, and $v : [b,B] \to \mathbb{R}$ is an integrable function such that $\int_b^B v(t)dt = 1$, $v(t) = v(b+B-t)$,for all $s \in [a,A]$. If $\varphi : D \to \mathbb{R}$ is a concave function, then*

$$\frac{\varphi(A,b)+\varphi(a,B)}{2} \le \int_D w(\mathbf{x})\varphi(\mathbf{x})\,d\mathbf{x} \le \varphi\left(\frac{a+A}{2},\frac{b+B}{2}\right).$$

As an application of Theorem 2.1 for a functional defined as a weighted integral over the rectangle D, we obtain a refinement of Feyér's inequalities calculated by $|\Delta\varphi|$.

Theorem 2.4 *Let $w : D = [a,A] \times [b,B] \to \mathbb{R}$ be a nonnegative integrable function such that $w(s,t) = u(s)v(t)$, where $u : [a,A] \to \mathbb{R}$ is an integrable function, $\int_a^A u(s)ds = 1$, $u(s) = u(a+A-s)$, for all $s \in [a,A]$, and $v : [b,B] \to \mathbb{R}$ is an integrable function, $\int_b^B v(t)dt = 1$, $v(t) = v(b+B-t)$, for all $t \in [b,B]$. If $\varphi : D \to \mathbb{R}$ is a continuous concave function, then*

$$\begin{aligned}\max&\left\{\frac{\varphi(a,b)+\varphi(A,B)}{2},\frac{\varphi(A,b)+\varphi(a,B)}{2}\right\} - O(|\Delta\varphi|)\\ &\le \int_D w(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x} \le \varphi\left(\frac{a+A}{2},\frac{b+B}{2}\right).\end{aligned} \tag{2.27}$$

If $\varphi : D \to \mathbb{R}$ is a continuous convex function, then

$$\begin{aligned}\varphi\left(\frac{a+A}{2},\frac{b+B}{2}\right) &\le \int_D w(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x}\\ &\le \min\left\{\frac{\varphi(a,b)+\varphi(A,B)}{2},\frac{\varphi(A,b)+\varphi(a,B)}{2}\right\} + O(|\Delta\varphi|),\end{aligned} \tag{2.28}$$

where

$$\begin{aligned}O(|\Delta\varphi|) = \frac{|\Delta\varphi|}{2}&\left\{\frac{1}{A-a}\int_a^A su(s)\left(\int_{b+\frac{B-b}{A-a}(s-a)}^{B-\frac{B-b}{A-a}(s-a)} v(t)dt\right)ds\right.\\ &\left.+\frac{1}{B-b}\int_b^B tv(t)\left(\int_{a+\frac{A-a}{B-b}(t-b)}^{A-\frac{A-a}{B-b}(t-b)} u(s)ds\right)dt\right\}.\end{aligned}$$

Proof. Suppose that $\varphi : D \to \mathbb{R}$ is a concave function. We apply Theorem 2.1 and Lemma 2.1 for a functional F defined on L, the class of integrable real functions on $\Omega = D$ by $F(f) = \int_D w(\mathbf{x})f(\mathbf{x})d\mathbf{x}$, for $f : \Omega = D \to \mathbb{R}$.

Let $g_1, g_2 \in L$ be such functions that $g_1(s,t) = s$, $g_2(s,t) = t$, for all (s,t) in $\Omega = D$. Using the properties of functions w, u and v, we may check that $F(g_1) = \frac{a+A}{2}$ as follows:

$$\begin{aligned}
F(g_1) &= \int_D w(\mathbf{x}) g_1(\mathbf{x}) d\mathbf{x} = \int_a^A \int_b^B w(s,t) s ds dt \\
&= \int_a^A s u(s) \Big(\int_b^B v(t) dt \Big) ds \\
&= \int_a^{\frac{a+A}{2}} s u(s) ds + \int_{\frac{a+A}{2}}^A s u(a+A-s) ds \\
&= \text{use the substitution } a+A-s = x \\
&= \int_a^{\frac{a+A}{2}} s u(s) ds + \int_a^{\frac{a+A}{2}} (a+A-x) u(x) dx \\
&= (a+A) \int_a^{\frac{a+A}{2}} u(x) dx = \frac{a+A}{2}.
\end{aligned}$$

Similarly, we can show that $F(g_2) = \frac{b+B}{2}$.

If $\Delta\varphi \geq 0$, then we calculate $F(m_{34}(g_1, g_2))$ using a fact that $\min\{a,b\} = \dfrac{a+b-|a-b|}{2}$:

$$\begin{aligned}
F(m_{34}(g_1,g_2)) &= \frac{1}{2} \int_D [(\lambda_3 + \lambda_4)s + (\mu_3 + \mu_4)t + \nu_3 + \nu_4] u(s) v(t) ds dt \\
&\quad - \frac{1}{2} \int_D |(\lambda_3 - \lambda_4)s + (\mu_3 - \mu_4)t + \nu_3 - \nu_4| u(s) v(t) ds dt \\
&= \frac{\varphi(a,b) + \varphi(A,B)}{2} - O(\Delta\varphi) \\
&= \frac{\varphi(a,b) + \varphi(A,B)}{2} - O(|\Delta\varphi|).
\end{aligned}$$

We obtain the final expression for $O(\Delta\varphi)$ by elementary calculus as follows.

$$O(\Delta\varphi) = \frac{1}{2} \int_D |(\lambda_3 - \lambda_4)s + (\mu_3 - \mu_4)t + \nu_3 - \nu_4| u(s) v(t) ds dt.$$

For $\Delta\varphi \geq 0$, $(s,t) \in \Delta_3$ implies

$$(\lambda_3 - \lambda_4)s + (\mu_3 - \mu_4)t + \nu_3 - \nu_4 \geq 0,$$

and $(s,t) \in \Delta_4$ implies that

$$(\lambda_3 - \lambda_4)s + (\mu_3 - \mu_4)t + \nu_3 - \nu_4 \leq 0.$$

$$\begin{aligned}
O(\Delta\varphi) &= \frac{1}{2} \int_{\Delta_3} [(\lambda_3 - \lambda_4)s + (\mu_3 - \mu_4)t + \nu_3 - \nu_4] u(s) v(t) ds dt \\
&\quad - \frac{1}{2} \int_{\Delta_4} [(\lambda_3 - \lambda_4)s + (\mu_3 - \mu_4)t + \nu_3 - \nu_4] u(s) v(t) ds dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\Delta\varphi}{2}\Big\{\frac{1}{(A-a)}\Big[\int_{\Delta_3} su(s)v(t)dsdt - \int_{\Delta_4} su(s)v(t)dsdt\Big] \\
&\quad -\frac{1}{(B-b)}\Big[\int_{\Delta_3} tu(s)v(t)dsdt - \int_{\Delta_4} tu(s)v(t)dsdt\Big] \\
&\quad -\frac{aB-Ab}{(A-a)(B-b)}\Big[\int_{\Delta_3} u(s)v(t)dsdt - \int_{\Delta_4} u(s)v(t)dsdt)\Big]\Big\} \\
&= \frac{\Delta\varphi}{2}\Big\{\frac{1}{(A-a)}\Big[\int_a^A su(s)\Big(\int_{\frac{B-b}{A-a}(s-a)+b}^{B} v(t)dt\Big)ds \\
&\quad -\int_a^A su(s)\Big(\int_b^{\frac{B-b}{A-a}(s-a)+b} v(t)dt\Big)ds\Big] \\
&\quad -\frac{1}{(B-b)}\Big[\int_b^B tv(t)\Big(\int_a^{\frac{A-a}{B-b}(t-b)+a} u(s)ds\Big)dt \\
&\quad -\int_b^B tv(t)\Big(\int_{\frac{A-a}{B-b}(t-b)+a}^{A} u(s)ds\Big)dt\Big] \\
&\quad -\frac{aB-Ab}{(A-a)(B-b)}\Big[\int_a^A u(s)\Big(\int_{\frac{B-b}{A-a}(s-a)+b}^{B} v(t)dt\Big)ds \\
&\quad -\int_a^A u(s)\Big(\int_b^{\frac{B-b}{A-a}(s-a)+b} v(t)dt\Big)ds\Big]\Big\} \\
&= \frac{\Delta\varphi}{2}\Big\{\frac{1}{(A-a)}\Big[\int_a^A su(s)\Big(\int_{\frac{B-b}{A-a}(s-a)+b}^{B} v(t)dt - \int_b^{\frac{B-b}{A-a}(s-a)+b} v(t)dt\Big)ds\Big] \\
&\quad -\frac{1}{(B-b)}\Big[\int_b^B tv(t)\Big(\int_a^{\frac{A-a}{B-b}(t-b)+a} u(s)ds - \int_{\frac{A-a}{B-b}(t-b)+a}^{A} u(s)ds\Big)dt\Big] \\
&\quad -\frac{aB-Ab}{(A-a)(B-b)}\Big[\int_a^A u(s)\Big(\int_{\frac{B-b}{A-a}(s-a)+b}^{B} v(t)dt \\
&\quad -\int_b^{\frac{B-b}{A-a}(s-a)+b} v(t)dt\Big)ds\Big]\Big\}.
\end{aligned}
$$

Since $u(s) = u(a+A-s)$ for all $s \in [a,A]$ and $v(t) = v(b+B-t)$ for all $t \in [b,B]$, we have

$$
\begin{aligned}
O(\Delta\varphi) &= \frac{\Delta\varphi}{2}\Big\{\frac{1}{(A-a)}\Big[\int_a^A su(s)\Big(\int_{\frac{B-b}{A-a}(s-a)+b}^{B} v(t)dt - \int_b^{\frac{B-b}{A-a}(s-a)+b} v(b+B-t)dt\Big)ds\Big] \\
&\quad -\frac{1}{(B-b)}\Big[\int_b^B tv(t)\Big(\int_a^{\frac{A-a}{B-b}(t-b)+a} u(s)ds + \int_{\frac{A-a}{B-b}(t-b)+a}^{A} u(a+A-s)ds\Big)dt\Big] \\
&\quad -\frac{aB-Ab}{(A-a)(B-b)}\Big[\int_a^A u(s)\Big(\int_{\frac{B-b}{A-a}(s-a)+b}^{B} v(t)dt - \int_b^{\frac{B-b}{A-a}(s-a)+b} v(b+B-t)dt\Big)ds\Big]\Big\}.
\end{aligned}
$$

We use substitutions $a+A-s = x$, $b+B-t = y$ and calculate:

$$
\begin{aligned}
O(\Delta\varphi) \;=\; & \frac{\Delta\varphi}{2}\Bigg\{\frac{1}{(A-a)}\Big[\int_a^A su(s)\Big(\int_{\frac{B-b}{A-a}(s-a)+b}^{B} v(t)dt+\int_B^{B-\frac{B-b}{A-a}(s-a)} v(y)dy\Big)ds\Big] \\
& -\frac{1}{(B-b)}\Big[\int_b^B tv(t)\Big(\int_a^{\frac{A-a}{B-b}(t-b)+a} u(s)ds+\int_{A-\frac{A-a}{B-b}(t-b)}^{a} u(x)dx\Big)dt\Big] \\
& -\frac{aB-Ab}{(A-a)(B-b)}\Big[\int_a^A u(s)\Big(\int_{\frac{B-b}{A-a}(s-a)+b}^{B} v(t)dt+\int_B^{B-\frac{B-b}{A-a}(s-a)} v(y)dy\Big)ds\Big]\Bigg\} \\
=\; & \frac{\Delta\varphi}{2}\Bigg\{\frac{1}{(A-a)}\Big[\int_a^A su(s)\Big(\int_{\frac{B-b}{A-a}(s-a)+b}^{B-\frac{B-b}{A-a}(s-a)} v(t)dt\Big)ds\Big] \\
& +\frac{1}{(B-b)}\Big[\int_b^B tv(t)\Big(\int_{\frac{A-a}{B-b}(t-b)+a}^{A-\frac{A-a}{B-b}(t-b)} u(s)ds\Big)dt\Big] \\
& -\frac{aB-Ab}{(A-a)(B-b)}\Big[\int_a^A u(s)\Big(\int_{\frac{B-b}{A-a}(s-a)+b}^{B-\frac{B-b}{A-a}(s-a)} v(t)dt\Big)ds\Big]\Bigg\}.
\end{aligned}
$$

It is easy to see that third integral equals to zero.
For $\Delta\varphi \leq 0$ we have to calculate $F(m_{12}(g_1,g_2))$:

$$
\begin{aligned}
F(m_{12}(g_1,g_2)) \;=\; & \frac{1}{2}\int_D [(\lambda_1+\lambda_2)s+(\mu_1+\mu_2)t+\nu_1+\nu_2]u(s)v(t)dsdt \\
& -\frac{1}{2}\int_D |(\lambda_1-\lambda_2)s+(\mu_1-\mu_2)t+\nu_1-\nu_2|u(s)v(t)dsdt \\
=\; & \frac{\varphi(a,B)+\varphi(A,b)}{2}+O(\Delta\varphi)=\frac{\varphi(a,B)+\varphi(A,b)}{2}-O(|\Delta\varphi|).
\end{aligned}
$$

Now, the inequalities in (2.21)

$$\max\{F(\pi_{12}(g_1,g_2)),F(\pi_{34}(g_1,g_2))\} \;\leq\; F(\varphi(g_1,g_2)) \leq \varphi(F(g_1),F(g_2)),$$

Lemma 2.1, relations (2.12) and (2.14) imply (2.27).

In the case of φ being a convex one, note that $-\varphi$ is concave and we use the previous proof. The term $O(|\Delta\varphi|)$ is a consequence of (2.22), (2.14) and the fact that $\max\{a,b\} = \dfrac{a+b+|a-b|}{2}$. □

Special choice of u,v in Theorem 2.4 gives a refinement of the Hadamard inequality for a concave and convex function of two variables obtained in [10].

Corollary 2.1 *Suppose that $\varphi : D=[a,A]\times[b,B]\to\mathbb{R}$ is a continuous concave function.*
(i) If $\Delta\varphi \geq 0$, then it holds

$$
\begin{aligned}
\frac{2\varphi(a,b)+2\varphi(A,B)+\varphi(a,B)+\varphi(A,b)}{6} \;\leq\; & \frac{\int_a^A\int_b^B \varphi(t,s)dtds}{(A-a)(B-b)} \\
\leq\; & \varphi\left(\frac{a+A}{2},\frac{b+B}{2}\right). \qquad (2.29)
\end{aligned}
$$

(*ii*) *If* $\Delta\varphi \leq 0$, *then it holds*

$$\frac{2\varphi(a,B)+2\varphi(A,b)+\varphi(a,b)+\varphi(A,B)}{6} \leq \frac{\int_a^A\int_b^B \varphi(t,s)dtds}{(A-a)(B-b)} \leq \varphi\left(\frac{a+A}{2},\frac{b+B}{2}\right). \tag{2.30}$$

Proof. Substituting $u(s) = \frac{1}{A-a}$ and $v(t) = \frac{1}{B-b}$ in (2.28) and (2.27) one can get $O(|\Delta\varphi|) = \frac{|\Delta\varphi|}{6}$. □

Corollary 2.2 *Suppose that* $\varphi : D = [a,A]\times[b,B] \to \mathbb{R}$ *is a continuous convex function.*
(*i*) *If* $\Delta\varphi \geq 0$, *then*

$$\varphi\left(\frac{a+A}{2},\frac{b+B}{2}\right) \leq \frac{\int_a^A\int_b^B \varphi(t,s)dtds}{(A-a)(B-b)} \leq \frac{2\varphi(a,B)+2\varphi(A,b)+\varphi(a,b)+\varphi(A,B)}{6}. \tag{2.31}$$

(*ii*) *If* $\Delta\varphi \leq 0$, *then*

$$\varphi\left(\frac{a+A}{2},\frac{b+B}{2}\right) \leq \frac{\int_a^A\int_b^B \varphi(t,s)dtds}{(A-a)(B-b)} \leq \frac{2\varphi(a,b)+2\varphi(A,B)+\varphi(a,B)+\varphi(A,b)}{6}. \tag{2.32}$$

Remark 2.5 Allasia in [1, Theorem 1] gave the Hermite-Hadamard inequality for a triangle which implies our result in Corollary 2.2. The right side of the inequality in Theorem 1, for a convex function φ and for the special choice of triangles Δ_1,Δ_2 and Δ_3,Δ_4 gives

$$\begin{aligned}
&\frac{\int_a^A\int_b^B \varphi(s,t)dsdt}{(A-a)(B-b)} \\
&\leq \min\left\{\frac{\varphi(a,b)+2\varphi(A,b)+2\varphi(a,B)+\varphi(A,B)}{6},\frac{2\varphi(a,b)+\varphi(A,b)+\varphi(a,B)+2\varphi(A,B)}{6}\right. \\
&= \begin{cases} \frac{1}{6}[\varphi(a,b)+2\varphi(A,b)+2\varphi(a,B)+\varphi(A,B)], \ \Delta\varphi \geq 0 \\ \frac{1}{6}[2\varphi(a,b)+\varphi(A,b)+\varphi(a,B)+2\varphi(A,B)], \ \Delta\varphi \leq 0. \end{cases}
\end{aligned}$$

2.2.3 Lupaş type inequalities

In this section we apply normalized positive linear functional defined by

$$F(f) = \frac{\displaystyle\int_{x_1-h}^{x_1+h}\int_{y_i-k}^{y_i+k} f(s,t)\cdot w(s,t)dsdt}{\displaystyle\int_{x_1-h}^{x_1+h}\int_{y_2-k}^{y_2+k} w(s,t)dsdt}, \quad i=1,2, \quad \text{for } f:\Omega = D \to \mathbb{R}. \tag{2.33}$$

A local property of concave functions inspired by the Feyér inequality has been given by Vasić, Lacković (1974, 1976) and Lupaş (1976), (see [6, p. 5]).

Theorem A 4 (The Lupaş inequality) *Let p,q be given positive real numbers and $a_1 \le a \le b \le b_1$. Moreover, let $w : [a_1,b_1] \to \mathbb{R}$ be a positive symmetric function with respect to $x_0 = \dfrac{pa+qb}{p+q}$, i.e. $w(x_0+s) = w(x_0-s)$, for $0 \le s \le h$.*

Then the inequalities

$$\frac{p\varphi(a)+q\varphi(b)}{p+q} \le \frac{\int_{x_0-h}^{x_0+h} w(x)\varphi(x)dx}{\int_{x_0-h}^{x_0+h} w(x)dx} \le \varphi(x_0) \tag{2.34}$$

hold for all continuous concave functions $\varphi : [a_1,b_1] \to \mathbb{R}$ if and only if $h \le \frac{b-a}{p+q} \cdot \min\{p,q\}$.

Theorem A4 inspired us to give the following result related to Theorem 2.1.

Theorem 2.5 *Let L be a linear space of real-valued functions defined on a nonempty set Ω and $g_1, g_2 \in L$ be functions such that $g_1(t) \in [a,A]$, $g_2(t) \in [b,B]$ for all $t \in \Omega$. Moreover, let $F : L \to \mathbb{R}$ be a normalized isotonic positive linear functional such that*

$$F(g_1) = \frac{pa+qA}{p+q} \quad \text{and} \quad F(g_2) = \frac{qb+pB}{p+q} \quad \text{or} \quad F(g_2) = \frac{pb+qB}{p+q} \tag{2.35}$$

where $p,q \ge 0$ and $p^2+q^2 > 0$.

(i) Suppose $\varphi : D = [a,A] \times [b,B] \to \mathbb{R}$ is a concave continuous function. Then

$$\begin{aligned}
&\max\left\{\frac{p\varphi(a,B)+q\varphi(A,b)}{p+q}, \frac{p\varphi(a,b)+q\varphi(A,B)}{p+q} - \frac{|\Delta\varphi|}{2}\right\} \\
&\le F(\varphi(g_1,g_2)) \\
&\le \min\left\{\varphi\left(\frac{pa+qA}{p+q}, \frac{pb+qB}{p+q}\right), \varphi\left(\frac{pa+qA}{p+q}, \frac{qb+pB}{p+q}\right)\right\}.
\end{aligned} \tag{2.36}$$

(ii) Suppose $\varphi : D = [a,A] \times [b,B] \to \mathbb{R}$ is a continuous convex function. Then

$$\begin{aligned}
&\max\left\{\varphi\left(\frac{pa+qA}{p+q}, \frac{pb+qB}{p+q}\right), \varphi\left(\frac{pa+qA}{p+q}, \frac{qb+pB}{p+q}\right)\right\} \\
&\le F(\varphi(g_1,g_2)) \\
&\le \min\left\{\frac{p\varphi(a,B)+q\varphi(A,b)}{p+q}, \frac{p\varphi(a,b)+q\varphi(A,B)}{p+q} - \frac{|\Delta\varphi|}{2}\right\}.
\end{aligned} \tag{2.37}$$

Proof. *(i)* First we suppose that $\varphi : D \to \mathbb{R}$ is a concave continuous function, $\Delta\varphi \ge 0$ and $F(g_2) = \dfrac{qb+pB}{p+q}$. According Theorem 2.1, Theorem 2.2 and Lemma 2.1, we have to calculate

$$F(m_{34}(g_1,g_2)) = F\left(\frac{\pi_3(g_1,g_2)+\pi_4(g_1,g_2)-|\pi_3(g_1,g_2)-\pi_4(g_1,g_2)|}{2}\right).$$

The properties of F and (2.7) enable us to continue with

$$\begin{aligned} F(m_{34}(g_1,g_2)) &= \frac{1}{2}\left(\lambda_3\frac{pa+qA}{p+q}+\mu_3\frac{qb+pB}{p+q}+\varphi(a,B)-\lambda_3 a-\mu_3 B\right) \\ &+\frac{1}{2}\left(\lambda_4\frac{pa+qA}{p+q}+\mu_4\frac{qb+pB}{p+q}+\varphi(A,b)-\lambda_3 A-\mu_3 b\right) \\ &-\frac{1}{2}F(|\pi_3(g_1,g_2)-\pi_4(g_1,g_2)|). \end{aligned}$$

Using some algebra operations with (2.20) we obtain

$$\begin{aligned} F(m_{34}(g_1,g_2)) &= \frac{(p+q)(\varphi(a,b)+\varphi(A,B))+(p-q)(\varphi(a,B)-\varphi(A,b))}{2(p+q)} \\ &-\frac{1}{2}\frac{|\Delta\varphi|}{(B-b)(A-a)}F(|(A-a)g_2-(B-b)g_1-Ab+aB|). \end{aligned}$$

Note that the maximum value of $|(A-a)g_2-(B-b)g_1-Ab+aB|$ is $(B-b)(A-a)$ so we can claim that

$$F(m_{34}(g_1,g_2)) \geq \frac{(p+q)(\varphi(a,b)+\varphi(A,B))+(p-q)(\varphi(a,B)-\varphi(A,b))}{2(p+q)}-\frac{\Delta\varphi}{2}.$$

Using Theorem 2.2 and (2.35) we have

$$\frac{p\varphi(a,B)+q\varphi(A,b)}{p+q} \leq F(\varphi(g_1,g_2)) \leq \varphi\left(\frac{pa+qA}{p+q},\frac{qb+pB}{p+q}\right). \tag{2.38}$$

The same analysis can be used with the assumption that $F(g_2)=\dfrac{pb+qB}{p+q}$, to prove:

$$\frac{p\varphi(a,b)+q\varphi(A,B)}{p+q}-\frac{|\Delta\varphi|}{2} \leq F(\varphi(g_1,g_2)) \leq \varphi\left(\frac{pa+qA}{p+q},\frac{pb+qB}{p+q}\right). \tag{2.39}$$

Taking the maximum of (2.38) and (2.39), we obtain the desired inequality (2.36).

Very similar procedures for $\Delta\varphi \leq 0$ give the same result (2.36).

For the convex case we use Theorem 2.1, Theorem 2.2, Lemma 2.1, and calculate

$$F(M_{12}(g_1,g_2)) = F\left(\frac{\pi_1(g_1,g_2)+\pi_2(g_1,g_2)+|\pi_1(g_1,g_2)-\pi_2(g_1,g_2)|}{2}\right).$$

□

Generalization of Theorem A4 for concave functions of two variables is obtained in the following corollary.

Corollary 2.3 *Let $w : D=[a,A]\times[b,B]\to\mathbb{R}$ be a nonnegative integrable function such that $w(s,t)=u(s)v(t)$, where $u:[a,A]\to\mathbb{R}$ and $v:[b,B]\to\mathbb{R}$ are integrable functions with properties: $u(x_1+s)=u(x_1-s)$, for all $s\in[0,h]$, $v(y_i+t)=v(y_i-t)$, $i=1,2$, for* *all* $t \in [0,k]$ *where* $x_1 = \dfrac{pa+qA}{p+q}$, $y_1 \dfrac{pb+qB}{p+q}$ *and* $y_2 = \dfrac{qb+pB}{p+q}$ *are fixed and determined with given numbers* $p,q \geq 0$, $p^2+q^2 > 0$. *For all* $h,k > 0$ *such that*

$$0 \leq h \leq \frac{A-a}{p+q}\min\{p,q\}, \quad 0 \leq k \leq \frac{B-b}{p+q}\min\{p,q\}, \tag{2.40}$$

it holds

(i) *if* $\varphi : D \to \mathbb{R}$ *is a continuous concave function, then*

$$\frac{p\varphi(a,B)+q\varphi(A,b)}{p+q} + \frac{\Delta\varphi - |\Delta\varphi|}{4} \leq \frac{\displaystyle\int_{x_1-h}^{x_1+h}\int_{y_2-k}^{y_2+k} w(s,t)\varphi(s,t)dsdt}{\displaystyle\int_{x_1-h}^{x_1+h}\int_{y_2-k}^{y_2+k} w(s,t)dsdt} \leq \varphi(x_1,y_2) \tag{2.41}$$

and

$$\frac{p\varphi(a,b)+q\varphi(A,B)}{p+q} - \frac{\Delta\varphi + |\Delta\varphi|}{4} \leq \frac{\displaystyle\int_{x_1-h}^{x_1+h}\int_{y_1-k}^{y_1+k} w(s,t)\varphi(s,t)dsdt}{\displaystyle\int_{x_1-h}^{x_1+h}\int_{y_1-k}^{y_1+k} w(s,t)dtds} \leq \varphi(x_1,y_1); \tag{2.42}$$

(ii) *if* $\varphi : D \to \mathbb{R}$ *is a continuous convex function then*

$$\frac{p\varphi(a,b)+q\varphi(A,B)}{p+q} - \frac{\Delta\varphi - |\Delta\varphi|}{4} \leq \frac{\displaystyle\int_{x_1-h}^{x_1+h}\int_{y_1-k}^{y_1+k} w(s,t)\varphi(s,t)dsdt}{\displaystyle\int_{x_1-h}^{x_1+h}\int_{y_1-k}^{y_1+k} w(s,t)dsdt} \leq \varphi(x_1,y_1)$$

and

$$\frac{p\varphi(a,B)+q\varphi(A,b)}{p+q} + \frac{\Delta\varphi + |\Delta\varphi|}{4} \leq \frac{\displaystyle\int_{x_1-h}^{x_1+h}\int_{y_2-k}^{y_2+k} w(s,t)\varphi(s,t)dsdt}{\displaystyle\int_{x_1-h}^{x_1+h}\int_{y_2-k}^{y_2+k} w(s,t)dsdt} \leq \varphi(x_1,y_2).$$

Proof. (i) To prove (2.41) we check that conditions $[x_1-h, x_2+h] \subset [a,A]$ and $[y_i - k, y_i+k] \subset [b,B]$ are satisfied by (2.40). We use Lemma 2.1 and Theorem 2.1 for $\Omega = D$, functions $g_1(s,t) = s$ and $g_2(s,t) = t$. The functional is defined by

$$F(f) = \frac{\displaystyle\int_{x_1-h}^{x_1+h}\int_{y_i-k}^{y_i+k} f(s,t)\cdot w(s,t)dsdt}{\displaystyle\int_{x_1-h}^{x_1+h}\int_{y_2-k}^{y_2+k} w(s,t)dsdt}, \quad i=1,2, \text{ for } f : \Omega = D \to \mathbb{R}. \tag{2.43}$$

The functional (2.43) is positive, linear and $F(1) = 1$, so the proof is similar to the proof of Theorem 2.5. □

Remark 2.6 In the cases $p = q$, $h = \dfrac{A-a}{2}$ and $k = \dfrac{B-b}{2}$, Theorem 2.3 is expressed as the Corollary 3.1 in [10].

Integral version of Hadamard's inequalities for concave function of two variables is a consequence of Theorem 2.3.

Remark 2.7 Using Corollary 2.3 for $w(s,t) = u(s) = v(t) = 1$ the functional acquires the shape

$$F(g_1) = \frac{1}{4hk}\int_{x_1-h}^{x_1+h}\int_{y_2-k}^{y_2+k} sdsdt = x_1.$$

The inequalities obtained from Corollary 2.3 with the mentioned shape of the functional, show how the local feature from the Hadamard inequality is enlarged on the functions of two variables.

Remark 2.8 For $p = q$, $h = \dfrac{A-a}{2}$, $k = \dfrac{B-b}{2}$, u, v such that $\int_a^A u(s)ds = \int_b^B v(t)dt = 1$ one can get Corollary 3.2 in [10].

2.2.4 Petrović type inequalities

In this section we present a refinement of one discrete generalization of the Petrović inequality as an application of Theorem 2.2 in a discrete case including the special choice of Ω and F.

In [10] authors achieved the following generalization of the famous Petrović inequality.

Theorem A 5 (THE PETROVIĆ TYPE INEQUALITY) *Let* $\mathbf{p} = (p_1, \ldots, p_n)$ *and* $\mathbf{q} = (q_1, \ldots, q_n)$ *be n-tuples of nonnegative real numbers with* $P_n := \sum_{i=1}^n p_i\ (> 0)$ *and* $Q_n := \sum_{j=1}^n q_j\ (> 0)$. *Suppose that* $\mathbf{x} = (x_1, \ldots, x_n)$ *and* $\mathbf{y} = (y_1, \ldots, y_n)$ *are n-tuples of nonnegative real numbers such that* $0 \le x_k \le \sum_{i=1}^n p_i x_i \le c$ *and* $0 \le y_k \le \sum_{j=1}^n q_j y_j \le d$, *for* $k = 1, 2, \ldots, n$.

Let $\varphi : [0,c] \times [0,d] \to R$ *be a concave function.*

(i) *Suppose that*

$$\varphi(0,0) + \varphi\left(\sum_{i=1}^n p_i x_i, \sum_{j=1}^n q_j y_j\right) \ge \varphi\left(\sum_{i=1}^n p_i x_i, 0\right) + \varphi\left(0, \sum_{j=1}^n q_j y_j\right).$$

If $\dfrac{1}{P_n} + \dfrac{1}{Q_n} \le 1$, *then*

$$\frac{1}{P_n}\varphi\left(\sum_{i=1}^n p_i x_i, 0\right) + \frac{1}{Q_n}\varphi\left(0, \sum_{j=1}^n q_j y_j\right) + \left(1 - \frac{1}{P_n} - \frac{1}{Q_n}\right)\varphi(0,0)$$
$$\le \frac{1}{P_n Q_n}\sum_{i=1}^n\sum_{j=1}^n p_i q_j \varphi(x_i, y_j).$$

If $\frac{1}{P_n}+\frac{1}{Q_n}\geq 1$, *then*

$$\left(\frac{1}{P_n}+\frac{1}{Q_n}-1\right)\varphi\left(\sum_{i=1}^{n}p_ix_i,\sum_{j=1}^{n}q_jy_j\right)+\left(1-\frac{1}{Q_n}\right)\varphi\left(\sum_{i=1}^{n}p_ix_i,0\right)$$
$$+\left(1-\frac{1}{P_n}\right)\varphi\left(0,\sum_{j=1}^{n}q_jy_j\right)$$
$$\leq\frac{1}{P_nQ_n}\sum_{i=1}^{n}\sum_{j=1}^{n}p_iq_j\varphi(x_i,y_j).$$

(ii) *Suppose that*

$$\varphi(0,0)+\varphi\left(\sum_{i=1}^{n}p_ix_i,\sum_{j=1}^{n}q_jy_j\right)\leq\varphi\left(\sum_{i=1}^{n}p_ix_i,0\right)+\varphi\left(0,\sum_{j=1}^{n}q_jy_j\right).$$

If $P_n\geq Q_n$, *then*

$$\frac{1}{P_n}\varphi\left(\sum_{i=1}^{n}p_ix_i,\sum_{j=1}^{n}q_jy_j\right)+\left(\frac{1}{Q_n}-\frac{1}{P_n}\right)\varphi\left(0,\sum_{j=1}^{n}q_jy_j\right)+\left(1-\frac{1}{Q_n}\right)\varphi(0,0)$$
$$\leq\frac{1}{P_nQ_n}\sum_{i=1}^{n}\sum_{j=1}^{n}p_iq_j\varphi(x_i,y_j).$$

If $Q_n\geq P_n$, *then*

$$\frac{1}{Q_n}\varphi\left(\sum_{i=1}^{n}p_ix_i,\sum_{j=1}^{n}q_jy_j\right)-\left(\frac{1}{Q_n}-\frac{1}{P_n}\right)\varphi\left(\sum_{i=1}^{n}p_ix_i,0\right)+\left(1-\frac{1}{P_n}\right)\varphi(0,0)$$
$$\leq\frac{1}{P_nQ_n}\sum_{i=1}^{n}\sum_{j=1}^{n}p_iq_j\varphi(x_i,y_j).$$

We obtain a refinement of Theorem A5 as an application of Theorem 2.2 in a discrete case including the special choice of Ω and F.

Theorem 2.6 *Let* $\mathbf{p}=(p_1,\ldots,p_n)$ *and* $\mathbf{q}=(q_1,\ldots,q_n)$ *be n-tuples of nonnegative real numbers with* $P_n:=\sum_{i=1}^{n}p_i\ (>0)$ *and* $Q_n:=\sum_{j=1}^{n}q_j\ (>0)$. *Suppose that* $\mathbf{x}=(x_1,\ldots,x_n)$ *and* $\mathbf{y}=(y_1,\ldots,y_n)$ *are n-tuples of nonnegative real numbers such that* $0\leq x_k\leq\sum_{i=1}^{n}p_ix_i=A\leq c$ *and* $0\leq y_k\leq\sum_{j=1}^{n}q_jy_j=B\leq d$, *for* $k=1,2,\ldots,n$.

Let $\varphi:[0,c)\times[0,d)\to\mathbb{R}$ *be a concave function.*

(i) *If*

$$\varphi(0,0)+\varphi\left(\sum_{i=1}^{n}p_ix_i,\sum_{j=1}^{n}q_jy_j\right)\geq\varphi\left(\sum_{i=1}^{n}p_ix_i,0\right)+\varphi\left(0,\sum_{j=1}^{n}q_jy_j\right),$$

then

$$\varphi(0,0)+\frac{\varphi(A,B)-\varphi(0,0)}{2}\left(\frac{1}{P_n}+\frac{1}{Q_n}\right)+\frac{\varphi(A,0)-\varphi(0,B)}{2}\left(\frac{1}{P_n}-\frac{1}{Q_n}\right)$$
$$-\frac{\Delta\varphi}{2P_nQ_n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\frac{x_i}{A}-\frac{y_j}{B}\right|$$
$$\leq\frac{1}{P_nQ_n}\sum_{i=1}^{n}\sum_{j=1}^{n}p_iq_j\varphi(x_i,y_j). \tag{2.44}$$

(ii) *If*

$$\varphi(0,0)+\varphi\left(\sum_{i=1}^{n}p_ix_i,\sum_{j=1}^{n}q_jy_j\right)\geq\varphi\left(\sum_{i=1}^{n}p_ix_i,0\right)+\varphi\left(0,\sum_{j=1}^{n}q_jy_j\right),$$

then

$$\varphi(0,0)+\frac{\varphi(A,B)-\varphi(0,0)}{2}\left(\frac{1}{P_n}+\frac{1}{Q_n}\right)+\frac{\varphi(A,0)-\varphi(0,B)}{2}\left(\frac{1}{P_n}-\frac{1}{Q_n}\right)$$
$$-\frac{\Delta\varphi}{2}\left[1-\frac{1}{P_nQ_n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\frac{x_i}{A}+\frac{y_j}{B}-1\right|\right]$$
$$\leq\frac{1}{P_nQ_n}\sum_{i=1}^{n}\sum_{j=1}^{n}p_iq_j\varphi(x_i,y_j). \tag{2.45}$$

Proof. We use Theorem 2.2. Let L be a linear class of real-valued functions defined on $\Omega^2=\{1,2,\dots,n\}\times\{1,2,\dots,n\}$ having the conditions L1 and L2. We consider a functional F on L defined by

$$F(h)=\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}p_iq_jh(i,j)}{\sum_{i=1}^{n}p_i\sum_{j=1}^{n}q_j},\quad\text{for}\quad h:\Omega^2\to\mathbb{R} \tag{2.46}$$

where $\mathbf{p}=(p_1,p_2,\dots,p_n)$ and $\mathbf{q}=(q_1,q_2,\dots,q_n)$ are given nonnegative n-tuples.

Let's put $a=b=0$, $A=\sum_{i=1}^{n}p_ix_i$, $B=\sum_{j=1}^{n}q_jy_j$ and define $g_1,g_2:\Omega^2\to\mathbb{R}$ by $g_1(i,j)=x_i$ and $g_2(i,j)=y_j$.

Using definition (2.46) we have $F(g_1)=\frac{\sum_{i=1}^{n}p_ix_i}{P_n}=\frac{A}{P_n}$ and $F(g_2)=\frac{B}{Q_n}$.

If $\Delta\varphi\geq 0$ then we obtain the result in (2.44) using the inequality (2.24) $F(m_{34}(g_1,g_2))\leq F(\varphi(g_1,g_2))$ in Theorem 2.2.

Similarly, if $\Delta\varphi\leq 0$, the inequality in (2.45) we obtain using the inequality (2.25) for $F(m_{12}(g_1,g_2))$. □

Remark 2.9 The left side of (2.44) can be rewritten as:

$$\varphi(0,0)\left(1-\frac{1}{P_n}-\frac{1}{Q_n}\right)+\frac{1}{P_n}\varphi(A,0)+\frac{1}{Q_n}\varphi(0,B)$$
$$+\frac{\Delta\varphi}{2}\left[\frac{1}{P_n}+\frac{1}{Q_n}-\frac{1}{P_nQ_n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\frac{x_i}{A}-\frac{y_j}{B}\right|\right].$$

Note that for $\Delta\varphi \geq 0$ and $\dfrac{1}{P_n} + \dfrac{1}{Q_n} \leq 1$ in Theorem A5 there is the similar right side. According to Theorem 2.2 we obtain:

$$\frac{1}{P_n} + \frac{1}{Q_n} - \frac{1}{P_n Q_n} \sum_{i=1}^{n} \sum_{j=1}^{n} \left| \frac{x_i}{A} - \frac{y_i}{B} \right| \geq 0.$$

Similarly, for all cases in Theorem A5 we can obtain the following results as the consequences of Theorem 2.2

$$1 - \left| \frac{1}{P_n} + \frac{1}{Q_n} - 1 \right| - \frac{1}{P_n Q_n} \sum_{i=1}^{n} \sum_{j=1}^{n} \left| \frac{x_i}{A} - \frac{y_j}{B} \right| \geq 0;$$

$$1 - \left| \frac{1}{Q_n} - \frac{1}{P_n} \right| - \frac{1}{P_n Q_n} \sum_{i=1}^{n} \sum_{j=1}^{n} \left| \frac{x_i}{A} + \frac{y_j}{B} - 1 \right| \geq 0.$$

We obtain the following estimations as a consequence of the refinement made in Remark 2.9.

Corollary 2.4 *Let* $\mathbf{p} = (p_1, \ldots, p_n)$ *and* $\mathbf{q} = (q_1, \ldots, q_n)$ *be n-tuples of nonnegative real numbers. Suppose that* $\mathbf{x} = (x_1, \ldots, x_n)$ *and* $\mathbf{y} = (y_1, \ldots, y_n)$ *are n-tuples of nonnegative real numbers such that* $0 \leq x_k \leq \sum_{i=1}^{n} p_i x_i$ *and* $0 \leq y_k \leq \sum_{j=1}^{n} q_j y_j$, *for* $k = 1, 2, \ldots, n$. *Then*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \left| \frac{x_i}{\sum_{k=1}^{n} x_k p_k} - \frac{y_j}{\sum_{k=1}^{n} y_k q_k} \right| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} p_i q_j - \left| \sum_{i=1}^{n} \sum_{j=1}^{n} p_i q_j - \sum_{i=1}^{n} p_i - \sum_{i=1}^{n} q_i \right|;$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \left| \frac{x_i}{\sum_{k=1}^{n} x_k p_k} + \frac{y_j}{\sum_{k=1}^{n} y_k q_k} - 1 \right| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} p_i q_j - \left| \sum_{i=1}^{n} p_i - \sum_{j=1}^{n} q_j \right|.$$

2.3 Advanced conversion

In this section, the conversion of the Jensen-McShane inequality by a two-variable function is given. Under the special conditions the Gheorghiu-type inequality is proven. The following results are given in [8] and [9]. More general conversion and refinement in Theorem 2.2 is proven in [4].

2.3.1 Main result

In this subsection, the conversion of the Jensen-McShane inequality, which was considered in Theorem 2.2, is given by functions of two variables $\mathscr{F}$.

Theorem 2.7 *Let $\varphi, \psi : D = [a,A] \times [b,B] \to \mathbb{R}$ be continuous and φ be concave with the notation $\Delta\varphi = \varphi(a,b) + \varphi(A,B) - \varphi(A,b) - \varphi(a,B)$ and M_{ij}, m_{ij} are defined by (2.9). For $g_1, g_2 \in L$ assume that $(g_1(t), g_2(t)) \in D$ for all $t \in \Omega$. Let F be a normalized isotonic positive linear functional on L. Suppose that $\varphi(D) \subseteq U$ and $\psi(D) \subseteq V$ and suppose that $\mathscr{F} : U \times V \subseteq \mathbb{R}^2 \to \mathbb{R}$ is increasing in the first variable.*

(i) If $\Delta\varphi \geq 0$ then

$$\min_{(t,s)\in D} \mathscr{F}\left(M_{12}(t,s), \psi(t,s)\right) \leq \mathscr{F}\left(F(\varphi(g_1,g_2)), \psi(F(g_1),F(g_2))\right).$$

(ii) If $\Delta\varphi \leq 0$, then

$$\min_{(t,s)\in D} \mathscr{F}\left(M_{34}(t,s), \psi(t,s)\right) \leq \min_{(t,s)\in D} \mathscr{F}\left(F(\varphi(g_1,g_2)), \psi(F(g_1),F(g_2))\right).$$

Proof. Since $(F(g_1), F(g_2)) \in D$, then

$$\min_{(t,s)\in D} \mathscr{F}\left(M_{12}(t,s), \psi(t,s)\right) \leq \mathscr{F}\left((M_{12}(F(g_1),F(g_2)), \psi(F(g_1),F(g_2))\right).$$

If $\Delta\varphi \geq 0$, then by Theorem 2.2 we have that

$$M_{12}(F(g_1),F(g_2)) \leq F(\varphi(g_1,g_2)).$$

Since $\mathscr{F}$ is increasing in the first variable, we get

$$\min_{(t,s)\in D} \mathscr{F}\left(M_{12}(t,s), \psi(t,s)\right) \leq \mathscr{F}\left(F(\varphi(g_1,g_2)), \psi(F(g_1),F(g_2))\right)$$

and obtain the desired inequality. □

A multiplicative conversion is made by taking $\mathscr{F}(x,y) = \dfrac{x}{y}$.

Corollary 2.5 *Suppose that the assumptions of Theorem 2.7 hold with $\varphi(D) > 0$ additionally.*

If $\Delta\varphi \geq 0$, then

$$\min_{(t,s)\in D} \frac{M_{12}(t,s)}{\varphi(t,s)} \cdot \varphi(F(g_1),F(g_2)) \leq F(\varphi(g_1,g_2)) \leq \varphi(F(g_1),F(g_2)).$$

In opposite, if $\varphi(a,b) + \varphi(A,B) - \varphi(A,b) - \varphi(a,B) \leq 0$, then

$$\min_{(t,s)\in D} \frac{M_{34}(t,s)}{\varphi(t,s)} \cdot \varphi(F(g_1),F(g_2)) \leq F(\varphi(g_1,g_2)) \leq \varphi(F(g_1),F(g_2)).$$

Proof. According to Theorem 2.7 with assumption that $\varphi(D) \subset V$ we have that holds

$$\min_{(t,s)\in D} \mathscr{F}\left(M_{12}(t,s), \varphi(t,s)\right) \leq \mathscr{F}\left(F(\varphi(g_1,g_2)), \varphi(F(g_1),F(g_2))\right).$$

By taking $\mathscr{F}(x,y) = \dfrac{x}{y}$ we obtain conversion (i). □

A conversion by medium value is given by the next lemma.

Lemma 2.2 *Assume that φ, g_1, g_2 and F are as in Theorem 2.7 and λ_i, μ_i andν_i, $i \in \{1,\dots,4\}$ are defined by (2.7). Let $\alpha, \beta \geq 0$ be such that $\alpha+\beta = 1$.*

If $\Delta\varphi \geq 0$ then

$$(\alpha\lambda_1+\beta\lambda_2)F(g_1)+(\alpha\mu_1+\beta\mu_2)F(g_2)+\alpha\nu_1+\beta\nu_2 \leq \mathbf{F}(\varphi(g_1,g_2)).$$

If $\Delta\varphi \leq 0$, then

$$(\alpha\lambda_3+\beta\lambda_4)F(g_1)+(\alpha\mu_3+\beta\mu_4)F(g_2)+\alpha\nu_3+\beta\nu_4 \leq F(\varphi(g_1,g_2)).$$

Proof. Considering that

$$M_{12}(F(g_1),F(g_2) = \max\{\lambda_1 F(g_1)+\mu_1 F(g_1)+\nu_1, \lambda_2 F(g_1)+\mu_2 F(g_2)+\nu_2\},$$

we obtain the first inequality if $\Delta\varphi \geq 0$. □

A conversion with a very special condition is given bellow.

Proposition 2.1 *Suppose that the assumptions of Lemma 2.2 hold.*

(i) If $\Delta\varphi \geq 0$ and $\nu_1 \cdot \nu_2 < 0$, then

$$U_{12}F(g_1)+V_{12}F(g_2) \leq F(\varphi(g_1,g_2)) \leq \varphi(F(g_1),F(g_2)),$$

where:

$$U_{12} = \frac{\nu_2\lambda_1-\nu_1\lambda_2}{\nu_2-\nu_1}, \quad V_{12} = \frac{\nu_2\mu_1-\nu_1\mu_2}{\nu_2-\nu_1}.$$

(ii) If $\Delta\varphi \leq 0$ and $\nu_3 \cdot \nu_4 < 0$, then

$$U_{34} = \frac{\nu_4\lambda_3-\nu_3\lambda_4}{\nu_4-\nu_3}, \quad V_{34} = \frac{\nu_4\mu_3-\nu_3\mu_4}{\nu_4-\nu_3}.$$

Proof. Solving the system $\begin{cases} \alpha+\beta = 1 \\ \alpha\nu_1+\beta\nu_2 = 0 \end{cases}$ by α, β we obtain that $\alpha\lambda_1+\beta\lambda_2 = U_{12}$ and $\alpha\mu_1+\beta\mu_2 = V_{12}$. □

Considering Corollary 2.5, the next proposition is given.

Proposition 2.2 *Let $\varphi : D \to \mathbb{R}$ be a continuous concave positive function, $g_1, g_2 \in L$ and F normalised positive linear functional on L.*

(i) If $\Delta\varphi \geq 0$ and $\nu_1 \cdot \nu_2 < 0$, then:

$$\min_{(t,s)\in D} \frac{U_{12}t+V_{12}s}{\varphi(t,s)} \cdot \varphi(F(g_1),F(g_2)) \leq F(\varphi(g_1,g_2)) \leq \varphi(F(g_1),F(g_2)). \tag{2.47}$$

(ii) If $\Delta\varphi \leq 0$ and $\nu_3 \cdot \nu_4 < 0$, then:

$$\min_{(t,s)\in D} \frac{U_{34}t+V_{34}s}{\varphi(t,s)} \cdot \varphi(F(g_1),F(g_2)) \leq F(\varphi(g_1,g_2)) \leq \varphi(F(g_1),F(g_2)),$$

where values U_{12} V_{12}, U_{34} and V_{34} are given in Proposition 2.1.

Remark 2.10 The figures 2.2 and 2.3 visualise the conversion of the Jensen-McShane type inequalities on a rectangle D for linear isotonic positive functional E.

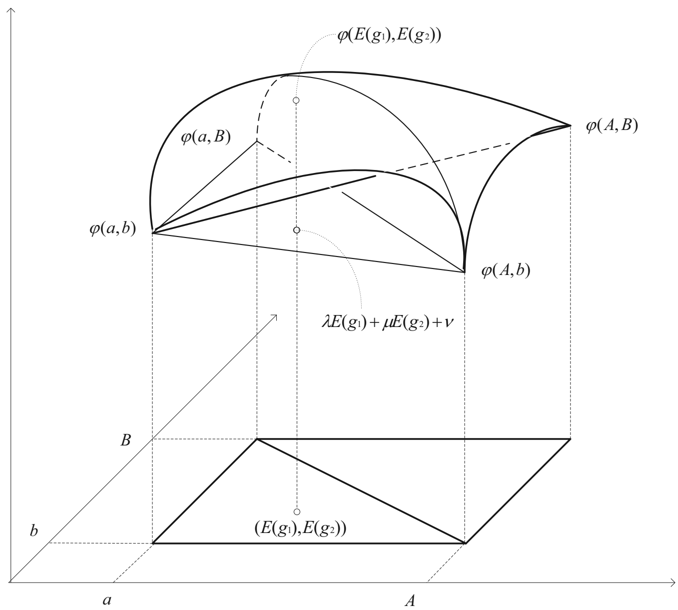

Figure 2.2: *Conversion by Csiszár and Móri*

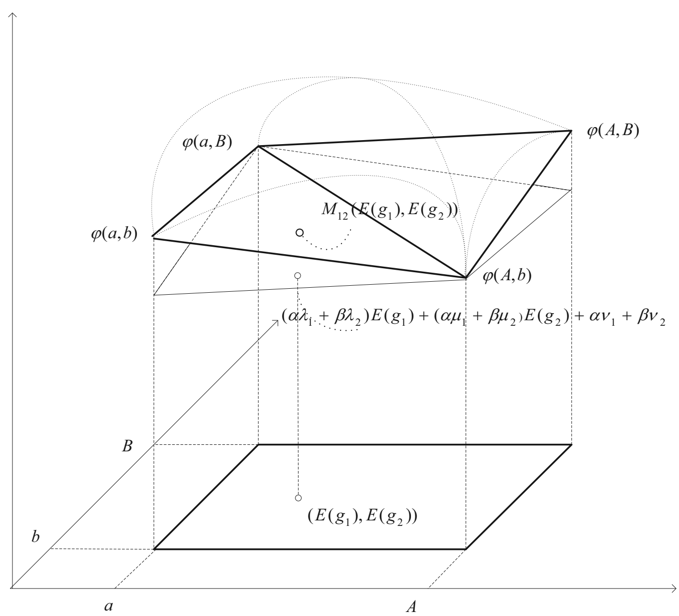

Figure 2.3: *Conversion by a medium value*

2.3.2 Gheorghiu type inequality

The Gheorghiu type inequality is a converse of Hölder's type inequality. In this section a proof for a refinement is presented.

The original Gheorghiu inequality from [17] can be expressed as follows:

Suppose that $a_1, a_2, \dots a_n$, $b_1, b_2, \dots b_n$ are given positive real numbers. Let the pair (a, A) represents the minimal and maximal number among $a_1, a_2, \dots, a_n$ and in the same manner let (b, B) be the pair of those among the $b_1, b_2, \dots, b_n$. Assume that p is a real number greater than 1. Then we have

$$1 \le \frac{\left(\sum_{k=1}^n a_k^p\right)\left(\sum_{k=1}^n b_k^{\frac{p}{p-1}}\right)^{p-1}}{\left(\sum_{k=1}^n a_k b_k\right)^p} \le \mu, \tag{2.48}$$

where

$$\mu = \frac{(p-1)^{p-1}}{p^p} \cdot \frac{A^{p-1}}{a^{p-1}} \cdot \frac{B}{b} \cdot \frac{\left(1 - \frac{a^p b^{\frac{p}{p-1}}}{A^p B^{\frac{p}{p-1}}}\right)^p}{\left(1 - \frac{ab^{\frac{1}{p-1}}}{AB^{\frac{1}{p-1}}}\right)\left(1 - \frac{a^{p-1}b}{A^{p-1}B}\right)^{p-1}}. \tag{2.49}$$

The left inequality has been demonstrated by Hölder and Jensen. This inequality could be presented in the terms that are given in the introduction of this chapter.

Theorem A 6 (THE GHEORGHIU INEQUALITY) *Suppose that* $\Omega = \{1, 2, 3, \dots, n\}$ *and* $g_1, g_2 : \Omega \to \mathbb{R}$ *are given real functions. Let* $a = \min\{g_1(k), k \in \Omega\}$, $A = \max\{g_2(k), k \in \Omega\}$, $b = \min\{g_2(k), k \in \Omega\}$ *and* $B = \max\{g_2(k), k \in \Omega\}$. *Let* $F(g) = \frac{1}{n}\sum_{k=1}^n g(k)$.

If $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$1 \le \frac{F(g_1^p)^{\frac{1}{p}} \cdot F(g_2^q)^{\frac{1}{q}}}{F(g_1 \cdot g_2)} \le \mu^{\frac{1}{p}}, \tag{2.50}$$

where μ *is given by* (2.49).

Inequality (2.50) can be expressed as the chain of inequalities:

$$\mu^{-\frac{1}{p}} \cdot F(g_1^p)^{\frac{1}{p}} \cdot F(g_2^q)^{\frac{1}{q}} \le F(g_1 \cdot g_2) \le F(g_1^p)^{\frac{1}{p}} \cdot F(g_2^q)^{\frac{1}{q}}. \tag{2.51}$$

The next theorem is a refinement of Theorem A6.

Theorem 2.8 *Let* F *be a normalized isotonic positive linear functional on* L *and for* $g_1, g_2 \in L$ *let us assume that* $g_1(\Omega) \subset [a, A]$ *and* $g_2(\Omega) \subset [b, B]$ *for positive real numbers* a, b. *Let* p, q *be positive real numbers such that* $\frac{1}{p} + \frac{1}{q} = 1$ *holds. Then*

$$\begin{aligned} &\frac{p^{\frac{1}{p}} q^{\frac{1}{q}} (abAB)^{\frac{1}{pq}} \left((AB)^{\frac{1}{p}} - (ab)^{\frac{1}{p}}\right)^{\frac{1}{p}} \left((AB)^{\frac{1}{q}} - (ab)^{\frac{1}{q}}\right)^{\frac{1}{q}}}{AB - ab} \cdot (F(g_1))^{\frac{1}{p}} (F(g_2))^{\frac{1}{q}} \\ &\le F\left(g_1^{\frac{1}{p}} \cdot g_2^{\frac{1}{q}}\right) \le (F(g_1))^{\frac{1}{p}} (F(g_2))^{\frac{1}{q}}. \end{aligned} \tag{2.52}$$

Proof. The function $\varphi(x,y) = x^{\frac{1}{p}} y^{\frac{1}{q}}$ is continuous, concave and positive for all $(x,y) \in [a,A] \times [b,B]$. Because $\left(A^{\frac{1}{p}} - a^{\frac{1}{p}}\right)\left(B^{\frac{1}{q}} - b^{\frac{1}{q}}\right) > 0$, for application of Proposition 2.2 it is enough to prove that

$$v_1 = a^{\frac{1}{p}} b^{\frac{1}{q}} - ab^{\frac{1}{q}} \frac{A^{\frac{1}{p}} - a^{\frac{1}{p}}}{A-a} - a^{\frac{1}{p}} b \frac{B^{\frac{1}{q}} - b^{\frac{1}{q}}}{B-b} \geq 0$$

and

$$v_2 = A^{\frac{1}{p}} B^{\frac{1}{q}} - AB^{\frac{1}{q}} \frac{A^{\frac{1}{p}} - a^{\frac{1}{p}}}{A-a} - A^{\frac{1}{p}} B \frac{B^{\frac{1}{q}} - b^{\frac{1}{q}}}{B-b} \leq 0.$$

Since the function $f(x) = x^{\frac{1}{p}}$ is concave for $p > 1$, $f'(x)$ is continuous and decreasing. So there exists $c \in [a,A]$ such that $f'(c) = \dfrac{A^{\frac{1}{p}} - a^{\frac{1}{p}}}{A-a}$ and $f'(a) \geq f'(c) \geq f'(A)$ which gives

$$\frac{1}{p} a^{\frac{1}{p}-1} \geq \frac{A^{\frac{1}{p}} - a^{\frac{1}{p}}}{A-a} \geq \frac{1}{p} A^{\frac{1}{p}-1}.$$

Multiplying with $ab^{\frac{1}{q}}$ and $AB^{\frac{1}{q}}$ we get

$$\frac{a^{\frac{1}{p}} b^{\frac{1}{q}}}{p} \geq ab^{\frac{1}{q}} \frac{A^{\frac{1}{p}} - a^{\frac{1}{p}}}{A-a} \quad \text{and} \quad AB^{\frac{1}{q}} \frac{A^{\frac{1}{p}} - a^{\frac{1}{p}}}{A-a} \geq \frac{A^{\frac{1}{p}} B^{\frac{1}{q}}}{p}.$$

Similar consideration on $f(x) = x^{\frac{1}{q}}$ gives

$$\frac{1}{q} b^{\frac{1}{q}-1} \geq \frac{B^{\frac{1}{q}} - b^{\frac{1}{q}}}{B-b} \geq \frac{1}{p} A^{\frac{1}{p}-1}.$$

Multiplying with $a^{\frac{1}{p}} b$ and $A^{\frac{1}{p}} B$ we get

$$\frac{a^{\frac{1}{p}} b^{\frac{1}{q}}}{q} \geq a^{\frac{1}{p}} b \frac{B^{\frac{1}{q}} - b^{\frac{1}{q}}}{B-b} \quad \text{and} \quad A^{\frac{1}{p}} B \frac{B^{\frac{1}{q}} - b^{\frac{1}{q}}}{B-b} \geq \frac{A^{\frac{1}{p}} B^{\frac{1}{q}}}{q}.$$

Now we have

$$v_1 = a^{\frac{1}{p}} b^{\frac{1}{q}} - ab^{\frac{1}{q}} \frac{A^{\frac{1}{p}} - a^{\frac{1}{p}}}{A-a} - a^{\frac{1}{p}} b \frac{B^{\frac{1}{q}} - b^{\frac{1}{q}}}{B-b}$$

$$v_1 \geq a^{\frac{1}{p}} b^{\frac{1}{q}} - \frac{a^{\frac{1}{p}} b^{\frac{1}{q}}}{p} - \frac{a^{\frac{1}{p}} b^{\frac{1}{q}}}{q} = a^{\frac{1}{p}} b^{\frac{1}{q}} \left(1 - \frac{1}{p} - \frac{1}{q}\right) = 0.$$

$$v_2 = A^{\frac{1}{p}} B^{\frac{1}{q}} - AB^{\frac{1}{q}} \frac{A^{\frac{1}{p}} - a^{\frac{1}{p}}}{A-a} - A^{\frac{1}{p}} B \frac{B^{\frac{1}{q}} - b^{\frac{1}{q}}}{B-b}$$

$$v_2 \leq A^{\frac{1}{p}} B^{\frac{1}{q}} - \frac{A^{\frac{1}{p}} B^{\frac{1}{q}}}{p} - \frac{A^{\frac{1}{p}} B^{\frac{1}{q}}}{q} = A^{\frac{1}{p}} B^{\frac{1}{q}} \left(1 - \frac{1}{p} - \frac{1}{q}\right) = 0.$$

It is necessary to minimize $\dfrac{U_{12}t+V_{12}s}{t^{\frac{1}{p}}s^{\frac{1}{q}}} = \min\limits_{(t,s)\in D}\left(U_{12}\cdot\left(\dfrac{t}{s}\right)^{\frac{1}{q}}+V_{12}\cdot\left(\dfrac{s}{t}\right)^{\frac{1}{p}}\right)$.

Supstitution $z=\dfrac{t}{s}$ and easy calculus provide

$$\min_{(t,s)\in D}\frac{U_{12}t+V_{12}s}{\varphi(t,s)} = U_{12}^{\frac{1}{p}}V_{12}^{\frac{1}{q}}p^{\frac{1}{p}}q^{\frac{1}{q}}.$$

Note that (2.52) is equal to (2.47) by coefficients

$$U_{12}=\frac{B^{\frac{1}{q}}b^{\frac{1}{q}}\left((AB)^{\frac{1}{p}}-(ab)^{\frac{1}{p}}\right)}{AB-ab} \text{ and } V_{12}=\frac{A^{\frac{1}{p}}a^{\frac{1}{p}}\left((AB)^{\frac{1}{q}}-(ab)^{\frac{1}{q}}\right)}{AB-ab}.$$

Applying the inequality (2.47) we get (2.52) and the proof is done. □

Remark 2.11 Using substitutions $g_1 \rightarrowtail g_1^p$ and $g_2 \rightarrowtail g_2^q$ in the previous Theorem 2.8 we get the following Gheorghiu type inequality:

$$\begin{aligned}\frac{p^{\frac{1}{p}}q^{\frac{1}{q}}\left(AbB^q-ab^qB\right)^{\frac{1}{p}}\left(A^paB-a^pbA\right)^{\frac{1}{q}}}{A^pB^q-a^pb^q}&\cdot\left(F(g_1^p)\right)^{\frac{1}{p}}\left(F(g_2^q)\right)^{\frac{1}{q}}\\ \le F\left(g_1\cdot g_2\right)\le \left(F(g_1^p)\right)^{\frac{1}{p}}\left(F(g_2^q)\right)^{\frac{1}{q}}.&\end{aligned} \tag{2.53}$$

Remark 2.12 Theorem 2.8 is refinement of Theorem A6. The next lemma shows that left inequality in (2.53) is better than the left inequality in (2.51).

Lemma 2.3 *Under the assumptions of Theorem 2.8 and using Remark 2.11, the next is valid:*

$$\frac{p^{\frac{1}{p}}q^{\frac{1}{q}}\left(AbB^q-ab^qB\right)^{\frac{1}{p}}\left(A^paB-a^pbA\right)^{\frac{1}{q}}}{A^pB^q-a^pb^q}=p\cdot\mu^{-\frac{1}{p}}. \tag{2.54}$$

Proof. Using elementary algebra for (2.49) we get

$$\mu^{-\frac{1}{p}}=\frac{q^{\frac{1}{q}}a^{\frac{1}{q}}b^{\frac{1}{p}}\left(AB^{\frac{q}{p}}-ab^{\frac{q}{p}}\right)^{\frac{1}{p}}\left(A^{\frac{p}{q}}B-a^{\frac{p}{q}}b\right)^{\frac{1}{q}}}{p^{\frac{1}{q}}A^{\frac{p}{q^2}+1-p}B^{\frac{q}{p^2}+1-q}\left(A^pB^q-a^pb^q\right)}.$$

Separately, using relation $p-1=\dfrac{p}{q}$, we have $\dfrac{p}{q^2}+1-p=-\dfrac{1}{q}$ and $\dfrac{q}{p^2}+1-q=-\dfrac{1}{p}$. The proof is prolonging with

$$\mu^{-\frac{1}{p}}=\frac{q^{\frac{1}{q}}A^{\frac{1}{q}}a^{\frac{1}{q}}B^{\frac{1}{p}}b^{\frac{1}{p}}\left(AB^{\frac{q}{p}}-ab^{\frac{q}{p}}\right)^{\frac{1}{p}}\left(A^{\frac{p}{q}}B-a^{\frac{p}{q}}b\right)^{\frac{1}{q}}}{p^{\frac{1}{q}}\left(A^pB^q-a^pb^q\right)}.$$

By selective multiplying factors and brackets with the same exponent we have

$$\mu^{-\frac{1}{p}}=\frac{q^{\frac{1}{q}}\left(AbB^{\frac{q}{p}+1}-ab^{\frac{q}{p}+1}B\right)^{\frac{1}{p}}\left(aA^{\frac{p}{q}+1}B-a^{\frac{p}{q}+1}bA\right)^{\frac{1}{q}}}{p^{\frac{1}{q}}\left(A^pB^q-a^pb^q\right)}.$$

Considering that $\frac{q}{p}+1=q$ and $\frac{p}{q}+1=q$ we finally obtain that

$$\mu^{-\frac{1}{p}}=\frac{p^{\frac{1}{p}}q^{\frac{1}{q}}\left(AbB^q-ab^qB\right)^{\frac{1}{p}}\left(aA^pB-a^pbA\right)^{\frac{1}{q}}}{p\left(A^pB^q-a^pb^q\right)}.$$

The last equation is the same as (2.54) and the proof is finished. □

Remark 2.13 As an application of Theorem 2.8 we get a refinement of normalized Gheorghiu inequality in [11]. Let (Ω,Σ,P) be a probability space, functions $g_1=X$ and $g_2=Y$ be random variables and functional $F(g_1)=E[X]$ be the mathematical expectation of random variable X.

Corollary 2.6 *Suppose that random variables X and Y capture their values $0<\alpha\leq X\leq 1$ and $0<\beta\leq Y\leq 1$. Equality $\frac{1}{p}+\frac{1}{q}=1$ implies*

$$\frac{p^{\frac{1}{p}}q^{\frac{1}{q}}(\beta-\alpha\beta^q)^{\frac{1}{p}}(\alpha-\alpha^p\beta)^{\frac{1}{q}}}{1-\alpha^p\beta^q}(E[X^p])^{\frac{1}{p}}(E[Y^q])^{\frac{1}{q}}\leq E[XY]\leq(E[X^p])^{\frac{1}{p}}(E[Y^q])^{\frac{1}{q}}$$

in the case of positive p and q.

In [12] author proved the converse of the Hölder inequality for a measure space.

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Chapter 3

Class of $(h, g; m)$-convex functions and certain types of inequalities

A convex function is one whose epigraph is a convex set, or, as in the basic definition:

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex function if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \tag{3.1}$$

holds for all points x and y in I and all $\lambda \in [0,1]$.

It is called strictly convex if the inequality (3.1) *holds strictly whenever x and y are distinct points and $\lambda \in (0,1)$. If $-f$ is convex (respectively, strictly convex) then we say that f is concave (respectively, strictly concave). If f is both convex and concave, then f is said to be affine.*

Motivated by a large number of different classes of convexity, we present a new convexity that unifies a certain range of them. Starting from the above convex function up to a recent convexity [27]:

A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called exponentially (s,m)-convex in the second sense if the following inequality holds

$$f(\lambda x + m(1-\lambda)y) \leq \frac{\lambda^s}{e^{\alpha x}} f(x) + \frac{(1-\lambda)^s}{e^{\alpha y}} m f(y) \tag{3.2}$$

for all $x, y \in I$ and all $\lambda \in [0,1]$, where $\alpha \in \mathbb{R}$, $s, m \in (0,1]$.

we noticed that the whole range in-between could be covered if we use on the right-hand side functions h and g in a form

$$f(\lambda x+m(1-\lambda)y) \leq h(\lambda)f(x)g(x)+mh(1-\lambda)f(y)g(y).$$

We named this convexity an $(h,g;m)$-convexity.

Here are several more varieties of convexity that will be generalized with this:

- A non-negative function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called P-function if the inequality holds
$$f(\lambda x+(1-\lambda)y) \leq f(x)+f(y)$$
for all $x,y \in I$ and all $\lambda \in [0,1]$.
- A function $f : [0,\infty) \to [0,\infty)$ is called s-convex in the second sense if the inequality holds
$$f(\lambda x+(1-\lambda)y) \leq \lambda^s f(x)+(1-\lambda)^s f(y)$$
for all $x,y \in [0,\infty)$ and all $\lambda \in [0,1]$, where $s \in (0,1]$.
- A non-negative function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called Godunova-Levin function if the inequality holds
$$f(\lambda x+(1-\lambda)y) \leq \frac{f(x)}{\lambda}+\frac{f(y)}{1-\lambda}$$
for all $x,y \in I$ and all $\lambda \in (0,1)$.
- A non-negative function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called h-convex if the inequality holds
$$f(\lambda x+(1-\lambda)y) \leq h(\lambda)f(x)+h(1-\lambda)f(y)$$
for all $x,y \in I$ and all $\lambda \in (0,1)$, where $h : J \to \mathbb{R}$ is a non-negative function, $h \not\equiv 0$, $(0,1) \subseteq J$.
- A function $f : [0,b] \to \mathbb{R}$ is called m-convex if the inequality holds
$$f(\lambda x+m(1-\lambda)y) \leq \lambda f(x)+m(1-\lambda)f(y)$$
for all $x,y \in [0,b]$ and all $\lambda \in [0,1]$, where $m \in [0,1]$.
- A non-negative function $f : [0,b] \to \mathbb{R}$ is called $(h-m)$-convex if the inequality holds
$$f(\lambda x+m(1-\lambda)y) \leq h(\lambda)f(x)+mh(1-\lambda)f(y)$$
for all $x,y \in [0,b]$ and all $\lambda \in (0,1)$, where $h : J \to \mathbb{R}$ is a non-negative function, $h \not\equiv 0$, $(0,1) \subseteq J$ and $m \in [0,1]$.
- A non-negative function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called (s,m)-Godunova-Levin function of the second kind if the inequality holds
$$f(\lambda x+m(1-\lambda)y) \leq \frac{f(x)}{\lambda^s}+\frac{mf(y)}{(1-\lambda)^s}$$
for all $x,y \in I$ and all $\lambda \in (0,1)$, where $m \in (0,1]$, $s \in [0,1]$.

- A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called exponential convex if the inequality holds

$$f(\lambda x + (1-\lambda)y) \leq \frac{\lambda}{e^{\alpha x}} f(x) + \frac{1-\lambda}{e^{\alpha y}} f(y)$$

  for all $x, y \in I$ and all $\lambda \in [0,1]$, where $\alpha \in \mathbb{R}$.

- A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called exponentially s-convex in the second sense if the inequality holds

$$f(\lambda x + (1-\lambda)y) \leq \frac{\lambda^s}{e^{\alpha x}} f(x) + \frac{(1-\lambda)^s}{e^{\alpha y}} f(y)$$

  for all $x, y \in I$ and all $\lambda \in [0,1]$, where $\alpha \in \mathbb{R}$, $s \in (0,1]$.

More detailed information may be found in [8, 10, 12, 15, 20, 23, 24, 27, 35, 36].

Furthermore, recall that a real valued function f on the interval I is said to be starshaped if

$$f(\lambda x) \leq \lambda f(x)$$

whenever $x \in I, \lambda x \in I$ and $\lambda \in [0,1]$.

This chapter is based on our results from [1], [2], [3], [6] and [7].

3.1 A class of $(h,g;m)$-convex functions

Definition 3.1 *Let h be a nonnegative function on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$ and let g be a positive function on $I \subseteq \mathbb{R}$. Furthermore, let $m \in (0,1]$. A function $f : I \to \mathbb{R}$ is said to be an $(h,g;m)$-convex function if it is nonnegative and if*

$$f(\lambda x + m(1-\lambda)y) \leq h(\lambda) f(x) g(x) + m h(1-\lambda) f(y) g(y) \tag{3.3}$$

holds for all $x, y \in I$ and all $\lambda \in (0,1)$.

If (3.3) holds in the reversed sense, then f is said to be an $(h,g;m)$-concave function.

Remark 3.1 For different choices of functions h, g and parameter m in (3.3), we can obtain corresponding convexity, e.g., if we set $h(\lambda) = \lambda^s$, $s \in (0,1]$, $g(x) = e^{-\alpha x}$, $\alpha \in \mathbb{R}$, then $(h,g;m)$-convexity reduces to exponentially (s,m)-convexity in the second sense (3.2).

Lemma 3.1 *If $f : I \to [0,\infty)$ is an $(h,g;m)$-convex function such that $f(0) = 0$, $g(x) \leq 1$ and $h(\lambda) \leq \lambda$, then f is starshaped.*

Proof. Let f be an $(h,g;m)$-convex function. Then we have

$$\begin{aligned} f(\lambda x) &= f(\lambda x + m(1-\lambda)0) \\ &\leq h(\lambda) f(x) g(x) + m h(1-\lambda) f(0) g(0) \\ &\leq \lambda f(x). \end{aligned}$$

Therefore, f is a starshaped. □

Remark 3.2 Let g be a positive function such that $g(x) \geq 1$. If f is a nonnegative $(h-m)$-convex function on $[0,\infty)$, then we have

$$\begin{aligned} f(\lambda x + m(1-\lambda)y) &\leq h(\lambda)f(x) + mh(1-\lambda)f(y) \\ &\leq h(\lambda)f(x)g(x) + mh(1-\lambda)f(y)g(y). \end{aligned}$$

Hence, f is an $(h,g;m)$-convex function.

If additionally $h(\lambda) \geq \lambda$, then for nonnegative m-convex function f on $[0,\infty)$ we have

$$\begin{aligned} f(\lambda x + m(1-\lambda)y) &\leq \lambda f(x) + m(1-\lambda)f(y) \\ &\leq h(\lambda)f(x) + mh(1-\lambda)f(y) \\ &\leq h(\lambda)f(x)g(x) + mh(1-\lambda)f(y)g(y), \end{aligned}$$

i.e., f is an $(h,g;m)$-convex function. An example of a function that satisfies $h(\lambda) \geq \lambda$ is $h(\lambda) = \lambda^k$, where $k \leq 1$ and $\lambda \in (0,1)$.

Similarly, if $g(x) \leq 1$, then all nonnegative $(h-m)$-concave functions are $(h,g;m)$-concave functions on $[0,\infty)$. Furthermore, if $g(x) \leq 1$ and $h(\lambda) \leq \lambda$, then all nonnegative m-concave functions are $(h,g;m)$-concave functions on $[0,\infty)$.

Proposition 3.1 *Let h_1, h_2 be nonnegative functions on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h_1, h_2 \not\equiv 0$, such that*

$$h_2(\lambda) \leq h_1(\lambda), \quad \lambda \in (0,1).$$

Let g be a positive function on $I \subseteq \mathbb{R}$ and $m \in (0,1]$. If $f : I \to [0,\infty)$ is an $(h_2,g;m)$-convex function, then f is $(h_1,g;m)$-convex.

If $f : I \to [0,\infty)$ is an $(h_1,g;m)$-concave function, then f is $(h_2,g;m)$-concave.

Proof. Let f be an $(h_2,g;m)$-convex function. Then we have

$$\begin{aligned} f(\lambda x + m(1-\lambda)y) &\leq h_2(\lambda)f(x)g(x) + mh_2(1-\lambda)f(y)g(y) \\ &\leq h_1(\lambda)f(x)g(x) + mh_1(1-\lambda)f(y)g(y). \end{aligned}$$

Hence, f is an $(h_1,g;m)$-convex function.

If f is an $(h_1,g;m)$-concave function, then analogously follows that f is $(h_2,g;m)$-concave. □

Proposition 3.2 *Let h be a nonnegative function on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$ and g be a positive function on $I \subseteq \mathbb{R}$. Furthermore, let $m \in (0,1]$ and $\alpha > 0$. If $f_1, f_2 : I \to [0,\infty)$ are $(h,g;m)$-convex functions, then $f_1 + f_2$ and αf_1 are $(h,g;m)$-convex.*

If $f_1, f_2 : I \to [0,\infty)$ are $(h,g;m)$-concave functions, then $f_1 + f_2$ and αf_1 are $(h,g;m)$-concave.

Proof. Let f_1, f_2 be $(h,g;m)$-convex functions and $\alpha > 0$. Then we have

$$f_1(\lambda x + m(1-\lambda)y) \leq h(\lambda)f_1(x)g(x) + mh(1-\lambda)f_1(y)g(y)$$

and

$$f_2(\lambda x+m(1-\lambda)y) \le h(\lambda)f_2(x)g(x)+mh(1-\lambda)f_2(y)g(y).$$

Adding the above we obtain

$$[f_1+f_2](\lambda x+m(1-\lambda)y) \le h(\lambda)[f_1+f_2](x)g(x)+mh(1-\lambda)[f_1+f_2](y)g(y).$$

Furthermore,

$$\begin{aligned}[\alpha f_1](\lambda x+m(1-\lambda)y) &\le \alpha h(\lambda)f_1(x)g(x)+\alpha mh(1-\lambda)f_1(y)g(y)\\ &= h(\lambda)[\alpha f_1](x)g(x)+mh(1-\lambda)[\alpha f_1](y)g(y).\end{aligned}$$

We conclude that f_1+f_2 and αf_1 are $(h_1,g;m)$-convex.

If $f_1,f_2 : I \to [0,\infty)$ are $(h,g;m)$-concave functions, then analogously follows that f_1+f_2 and αf_1 are $(h,g;m)$-concave. □

Proposition 3.3 *Let h be a nonnegative function on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$ and g be a positive increasing function on $I \subseteq \mathbb{R}$. Furthermore, let $0 < n < m \le 1$. If $f : I \to [0,\infty)$ is an $(h,g;m)$-convex function such that $f(0)=0$, $g(x) \le 1$ and $h(\lambda) \le \lambda$, then f is $(h,g;n)$-convex.*

Proof. Let f be an $(h,g;m)$-convex function. From $f(0)=0$, $g(x)\le 1$ and $h(\lambda)\le\lambda$ by Lemma 3.1 follows $f(\lambda x) \le \lambda f(x)$. Considering also that g is an increasing function, we obtain

$$\begin{aligned}f(\lambda x+n(1-\lambda)y) &= f\left(\lambda x+m(1-\lambda)\left(\frac{n}{m}y\right)\right)\\ &\le h(\lambda)f(x)g(x)+mh(1-\lambda)f\left(\frac{n}{m}y\right)g\left(\frac{n}{m}y\right)\\ &\le h(\lambda)f(x)g(x)+mh(1-\lambda)\frac{n}{m}f(y)g(y),\end{aligned}$$

which proves that f is $(h,g;n)$-convex. □

Proposition 3.4 *Let h_1,h_2 be nonnegative functions on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h_1,h_2 \not\equiv 0$ and let*

$$h(t)=\max\{h_1(t),h_2(t)\}, \quad t \in J.$$

Let g_1,g_2 be positive functions on $I \subseteq \mathbb{R}$ and let $m_1,m_2 \in (0,1]$. For $i=1,2$, let $f_i : I \to [0,\infty)$ be $(h_i,g_i;m_i)$-convex functions. If the functions f_1g_1 and f_2g_2 are monotonic in the same sense, i.e.

$$[f_1(x)g_1(x)-f_1(y)g_1(y)][f_2(x)g_2(x)-f_2(y)g_2(y)] \ge 0, \quad x,y \in I,$$

and if $c>0$ such that

$$h(\lambda)+mh(1-\lambda) \le c, \quad \lambda \in (0,1),$$

where $m=\max\{m_1,m_2\}$, then f_1f_2 is a $(ch,g_1g_2;m)$-convex function.

Proof. Let $f_i : I \to [0,\infty)$ be $(h_i, g_i; m_i)$-convex functions, $i = 1,2$. From hypotheses on functions, for $x, y \in I$ we have

$$\begin{aligned} & f_1(x)g_1(x)f_2(x)g_2(x) + f_1(y)g_1(y)f_2(y)g_2(y) \\ & \quad \geq f_1(x)g_1(x)f_2(y)g_2(y) + f_1(y)g_1(y)f_2(x)g_2(x). \end{aligned}$$

Let α and $\beta > 0$ be positive numbers such that $\alpha + \beta = 1$. Then we have

$$\begin{aligned} & f_1 f_2(\alpha x + \beta y) \\ & \quad \leq [h_1(\alpha)f_1(x)g_1(x) + m_1 h_1(\beta)f_1(y)g_1(y)] \\ & \qquad \times [h_2(\alpha)f_2(x)g_2(x) + m_2 h_2(\beta)f_2(y)g_2(y)] \\ & \quad \leq [h(\alpha)f_1(x)g_1(x) + mh(\beta)f_1(y)g_1(y)] \\ & \qquad \times [h(\alpha)f_2(x)g_2(x) + mh(\beta)f_2(y)g_2(y)] \\ & \quad = h^2(\alpha)f_1(x)g_1(x)f_2(x)g_2(x) + mh(\alpha)h(\beta)f_1(x)g_1(x)f_2(y)g_2(y) \\ & \qquad + mh(\alpha)h(\beta)f_1(y)g_1(y)f_2(x)g_2(x) + m^2h^2(\beta)f_1(y)g_1(y)f_2(y)g_2(y), \end{aligned}$$

hence

$$\begin{aligned} & f_1 f_2(\alpha x + \beta y) \\ & \quad \leq h^2(\alpha)f_1(x)g_1(x)f_2(x)g_2(x) + mh(\alpha)h(\beta)f_1(x)g_1(x)f_2(x)g_2(x) \\ & \qquad + mh(\alpha)h(\beta)f_1(y)g_1(y)f_2(y)g_2(y) + m^2h^2(\beta)f_1(y)g_1(y)f_2(y)g_2(y) \\ & \quad = [h(\alpha) + mh(\beta)] \\ & \qquad \times [h(\alpha)f_1(x)f_2(x)g_1(x)g_2(x) + mh(\beta)f_1(y)f_2(y)g_1(y)g_2(y)] \\ & \quad \leq ch(\alpha)f_1(x)f_2(x)g_1(x)g_2(x) + mch(\beta)f_1(y)f_2(y)g_1(y)g_2(y). \end{aligned}$$

This proves that $f_1 f_2$ is $(ch, g_1g_2; m)$-convex. □

Analogously follows the following proposition.

Proposition 3.5 *Let h_1, h_2 be nonnegative functions on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h_1, h_2 \not\equiv 0$ and let*

$$h(t) = \min\{h_1(t), h_2(t)\}, \quad t \in J.$$

Let g_1, g_2 be positive functions on $I \subseteq \mathbb{R}$ and let $m_1, m_2 \in (0,1]$. For $i = 1,2$, let $f_i : I \to [0,\infty)$ be $(h_i, g_i; m_i)$-concave functions. If the functions $f_1 g_1$ and $f_2 g_2$ are monotonic in the opposite sense, i.e.

$$[f_1(x)g_1(x) - f_1(y)g_1(y)]\,[f_2(x)g_2(x) - f_2(y)g_2(y)] \leq 0, \quad x, y \in I,$$

and if $c > 0$ such that

$$h(\lambda) + mh(1-\lambda) \geq c, \quad \lambda \in (0,1),$$

where $m = \min\{m_1, m_2\}$, then $f_1 f_2$ is a $(ch, g_1g_2; m)$-concave function.

3.2 Hermite-Hadamard type inequalities for $(h,g;m)$-convex functions

The famous Hermite-Hadamard inequality gives us an estimate of the (integral) mean value of a continuous convex function.

Theorem 3.1 (THE HERMITE-HADAMARD INEQUALITY) *Let $f : [a,b] \to \mathbb{R}$ be a continuous convex function. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

Of course, equality holds in either side only for affine functions. In this section we prove the Hermite-Hadamard inequality for $(h,g;m)$-convex functions and we point out some special results. Furthermore, several known inequalities are improved.

Recall, by $L_p[a,b]$, $1 \leq p < \infty$, the space of all Lebesgue measurable functions f for which $|f^p|$ is Lebesgue integrable on $[a,b]$ is denoted.

Theorem 3.2 *Let f be a nonnegative $(h,g;m)$-convex function on $[0,\infty)$ where h is a nonnegative function on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$, g is a positive function on $[0,\infty)$ and $m \in (0,1]$. If $f,g,h \in L_1[a,b]$, where $0 \leq a < b < \infty$, then the following inequalities hold*

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq \frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b \left[f(x)g(x)+mf\left(\frac{x}{m}\right)g\left(\frac{x}{m}\right)\right]dx \\
&\leq \frac{h\left(\frac{1}{2}\right)f(a)g(a)}{b-a}\int_a^b h\left(\frac{b-x}{b-a}\right)g(x)\,dx \\
&\quad+\frac{mh\left(\frac{1}{2}\right)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)}{b-a}\int_a^b h\left(\frac{x-a}{b-a}\right)g(x)\,dx \\
&\quad+\frac{mh\left(\frac{1}{2}\right)f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)}{b-a}\int_a^b h\left(\frac{b-x}{b-a}\right)g\left(\frac{x}{m}\right)dx \\
&\quad+\frac{m^2h\left(\frac{1}{2}\right)f\left(\frac{b}{m^2}\right)g\left(\frac{b}{m^2}\right)}{b-a}\int_a^b h\left(\frac{x-a}{b-a}\right)g\left(\frac{x}{m}\right)dx. \qquad (3.4)
\end{aligned}$$

Proof. Let f be an $(h,g;m)$-convex function. Then for $\lambda = \frac{1}{2}$ we have

$$f\left(\frac{x+my}{2}\right) \leq h\left(\frac{1}{2}\right)\left[f(x)g(x)+mf(y)g(y)\right].$$

Choosing $y \equiv \frac{y}{m}$ we obtain

$$f\left(\frac{x+y}{2}\right) \leq h\left(\frac{1}{2}\right)\left[f(x)g(x)+mf\left(\frac{y}{m}\right)g\left(\frac{y}{m}\right)\right]. \qquad (3.5)$$

Let $x=\lambda a+(1-\lambda)b$ and $y=(1-\lambda)a+\lambda b$. Then

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq h\left(\frac{1}{2}\right)\Big[f(\lambda a+(1-\lambda)b)g(\lambda a+(1-\lambda)b) \\ &\quad +mf\left((1-\lambda)\frac{a}{m}+\lambda\frac{b}{m}\right)g\left((1-\lambda)\frac{a}{m}+\lambda\frac{b}{m}\right)\Big]. \end{aligned}$$

In the following step we will need to integrate the above over $\lambda\in[0,1]$. From

$$\int_0^1 f(\lambda a+(1-\lambda)b)g(\lambda a+(1-\lambda)b)\,d\lambda=\frac{1}{b-a}\int_a^b f(u)g(u)\,du$$

and

$$\int_0^1 f\left((1-\lambda)\frac{a}{m}+\lambda\frac{b}{m}\right)g\left((1-\lambda)\frac{a}{m}+\lambda\frac{b}{m}\right)d\lambda=\frac{1}{b-a}\int_a^b f\left(\frac{u}{m}\right)g\left(\frac{u}{m}\right)du$$

we obtain

$$f\left(\frac{a+b}{2}\right)\leq\frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b\left[f(u)g(u)+mf\left(\frac{u}{m}\right)g\left(\frac{u}{m}\right)\right]du. \tag{3.6}$$

By $(h,g;m)$-convexity of f we have

$$f(\lambda a+(1-\lambda)b)\leq h(\lambda)f(a)g(a)+mh(1-\lambda)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right).$$

Multiplying the above inequality by $g(\lambda a+(1-\lambda)b)$ and integrating over $\lambda\in[0,1]$ we obtain

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(u)g(u)du &\leq f(a)g(a)\int_0^1 h(\lambda)g(\lambda a+(1-\lambda)b)\,d\lambda \\ &\quad +mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\int_0^1 h(1-\lambda)g(\lambda a+(1-\lambda)b)\,d\lambda \\ &= \frac{f(a)g(a)}{b-a}\int_a^b h\left(\frac{b-u}{b-a}\right)g(u)\,du \\ &\quad +\frac{mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)}{b-a}\int_a^b h\left(\frac{u-a}{b-a}\right)g(u)\,du. \end{aligned} \tag{3.7}$$

Again, by $(h,g;m)$-convexity of f we have

$$f\left((1-\lambda)\frac{a}{m}+\lambda\frac{b}{m}\right)\leq h(1-\lambda)f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)+mh(\lambda)f\left(\frac{b}{m^2}\right)g\left(\frac{b}{m^2}\right)$$

and if we multiply above inequality by $g\left((1-\lambda)\frac{a}{m}+\lambda\frac{b}{m}\right)$ and integrate over $\lambda \in [0,1]$ we obtain

$$\begin{aligned}\frac{1}{b-a}\int_a^b f\left(\frac{u}{m}\right)g\left(\frac{u}{m}\right)du &\leq f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\int_0^1 h(1-\lambda)g\left((1-\lambda)\frac{a}{m}+\lambda\frac{b}{m}\right)d\lambda\\ &\quad +mf\left(\frac{b}{m^2}\right)g\left(\frac{b}{m^2}\right)\int_0^1 h(\lambda)g\left((1-\lambda)\frac{a}{m}+\lambda\frac{b}{m}\right)d\lambda\\ &= \frac{f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)}{b-a}\int_a^b h\left(\frac{b-u}{b-a}\right)g\left(\frac{u}{m}\right)du\\ &\quad +\frac{mf\left(\frac{b}{m^2}\right)g\left(\frac{b}{m^2}\right)}{b-a}\int_a^b h\left(\frac{u-a}{b-a}\right)g\left(\frac{u}{m}\right)du. \end{aligned} \tag{3.8}$$

Now from (3.6), (3.7) and (3.8) we obtain (3.4). □

In the sequel we state several corollaries, using special functions for h and/or g, and choosing the parameter m. We start with the first special case: if $g \equiv 1$, then we have the Hermite-Hadamard inequality for $(h-m)$-convex functions.

Corollary 3.1 *Let f be a nonnegative $(h-m)$-convex function on $[0,\infty)$ where h is a nonnegative function on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$ and $m \in (0,1]$. If $f,h \in L_1[a,b]$, where $0 \leq a < b < \infty$, then the following inequalities hold*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b \left[f(x)+mf\left(\frac{x}{m}\right)\right]dx\\ &\leq h\left(\frac{1}{2}\right)\int_0^1 h(x)\,dx\left[f(a)+mf\left(\frac{b}{m}\right)+mf\left(\frac{a}{m}\right)+m^2 f\left(\frac{b}{m^2}\right)\right]. \end{aligned} \tag{3.9}$$

Proof. We use

$$\int_a^b h\left(\frac{b-x}{b-a}\right)dx = \int_a^b h\left(\frac{x-a}{b-a}\right)dx = (b-a)\int_0^1 h(u)\,du.$$

□

Remark 3.3 In [24, Theorem 9] authors gave the following Hermite-Hadamard type inequality for $(h-m)$-convex functions:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b \left[f(x)+mf\left(\frac{x}{m}\right)\right]dx\\ &\leq h\left(\frac{1}{2}\right)\left[f(a)+mf\left(\frac{b}{m}\right)+mf\left(\frac{a}{m}\right)+m^2 f\left(\frac{b}{m^2}\right)\right]. \end{aligned} \tag{3.10}$$

For all functions h such that $\int_0^1 h(x)\,dx \leq 1$, our result (3.9) will improve (3.10).

If $g \equiv 1$ and $m = 1$, then we have the Hermite-Hadamard inequality for h-convex functions ([36]):

Corollary 3.2 *Let f be a nonnegative h-convex function on $[0,\infty)$ where h is a nonnegative function on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$. If $f,h \in L_1[a,b]$, where $0 \le a < b < \infty$, then the following inequalities hold*

$$\begin{aligned} \frac{1}{2} f\left(\frac{a+b}{2}\right) &\le \frac{h\left(\frac{1}{2}\right)}{b-a} \int_a^b f(x)\,dx \\ &\le h\left(\frac{1}{2}\right) [f(a)+f(b)] \int_0^1 h(x)\,dx. \end{aligned} \tag{3.11}$$

For h being identity and $g \equiv 1$, the Hermite-Hadamard type inequality for m-convex functions holds ([11]):

Corollary 3.3 *Let f be a nonnegative m-convex function on $[0,\infty)$ with $m \in (0,1]$. If $f \in L_1[a,b]$, where $0 \le a < b < \infty$, then the following inequalities hold*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{1}{2(b-a)} \int_a^b \left[f(x) + mf\left(\frac{x}{m}\right)\right] dx \\ &\le \frac{1}{4}\left[f(a) + mf\left(\frac{b}{m}\right) + mf\left(\frac{a}{m}\right) + m^2 f\left(\frac{b}{m^2}\right)\right]. \end{aligned}$$

Of course, if $h(x) = x$, $g \equiv 1$ and $m = 1$, then we have the Hermite-Hadamard inequality given in Theorem 3.1.

An interesting Hermite-Hadamard type inequality follows if h is an identity.

Corollary 3.4 *Suppose that assumptions of Theorem 3.2 hold and let $h(x) = x$. Then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{1}{2(b-a)} \int_a^b \left[f(x)g(x) + mf\left(\frac{x}{m}\right) g\left(\frac{x}{m}\right)\right] dx \\ &\le \frac{f(a)g(a)}{2(b-a)^2} \int_a^b (b-x)g(x)\,dx \\ &\quad + \frac{mf\left(\frac{b}{m}\right) g\left(\frac{b}{m}\right)}{2(b-a)^2} \int_a^b (x-a)g(x)\,dx \\ &\quad + \frac{mf\left(\frac{a}{m}\right) g\left(\frac{a}{m}\right)}{2(b-a)^2} \int_a^b (b-x) g\left(\frac{x}{m}\right) dx \\ &\quad + \frac{m^2 f\left(\frac{b}{m^2}\right) g\left(\frac{b}{m^2}\right)}{2(b-a)^2} \int_a^b (x-a) g\left(\frac{x}{m}\right) dx. \end{aligned} \tag{3.12}$$

Next we use $h(\lambda) = \lambda^s$, $s \in (0,1]$ and a special case of a positive function $g(x) = e^{-\alpha x}$, $\alpha \in \mathbb{R}$, to obtain a following new Hermite-Hadamard inequality for exponentially (s,m)-convex functions in the second sense.

Corollary 3.5 *Let f be a nonnegative exponentially (s,m)-convex function in the second sense on $[0,\infty)$ where $s,m \in (0,1]$. If $f \in L_1[a,b]$, where $0 \le a < b < \infty$, then the following inequalities hold*

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\le \frac{1}{2^s(b-a)}\int_a^b \left[f(x)e^{-\alpha x}+mf\left(\frac{x}{m}\right)e^{-\frac{\alpha x}{m}}\right]dx \\
&\le \frac{f(a)e^{-\alpha a}}{2^s(b-a)^{s+1}}\int_a^b (b-x)^s e^{-\alpha x}\,dx \\
&\quad +\frac{mf\left(\frac{b}{m}\right)e^{-\frac{\alpha b}{m}}}{2^s(b-a)^{s+1}}\int_a^b (x-a)^s e^{-\alpha x}\,dx \\
&\quad +\frac{mf\left(\frac{a}{m}\right)e^{-\frac{\alpha a}{m}}}{2^s(b-a)^{s+1}}\int_a^b (b-x)^s e^{-\frac{\alpha x}{m}}\,dx \\
&\quad +\frac{m^2f\left(\frac{b}{m^2}\right)e^{-\frac{\alpha b}{m^2}}}{2^s(b-a)^{s+1}}\int_a^b (x-a)^s e^{-\frac{\alpha x}{m}}\,dx. \qquad (3.13)
\end{aligned}
$$

The next result is for exponentially convex functions: a special case of Theorem 3.2 when $h(\lambda)=\lambda$, $g(x)=e^{-\alpha x}$ and $m=1$.

Corollary 3.6 *Let f be a nonnegative exponentially convex function, $\alpha \neq 0$. If $f \in L_1[a,b]$, where $0 \le a < b < \infty$, then the following inequalities hold*

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\le \frac{1}{b-a}\int_a^b f(x)e^{-\alpha x}\,dx \\
&\le \frac{e^{-2\alpha a}f(a)-e^{-2\alpha b}f(b)}{\alpha(b-a)} \\
&\quad +\left[e^{-\alpha b}-e^{-\alpha a}\right]\frac{e^{-\alpha a}f(a)-e^{-\alpha b}f(b)}{\alpha^2(b-a)^2}. \qquad (3.14)
\end{aligned}
$$

Proof. The second part of (3.14) follows from

$$
\begin{aligned}
&\frac{e^{-\alpha a}f(a)}{(b-a)^2}\int_a^b (b-x)e^{-\alpha x}\,dx+\frac{e^{-\alpha b}f(b)}{(b-a)^2}\int_a^b (x-a)e^{-\alpha x}\,dx \\
&= \frac{e^{-2\alpha a}f(a)-e^{-2\alpha b}f(b)}{\alpha(b-a)} \\
&\quad +\left[e^{-\alpha b}-e^{-\alpha a}\right]\frac{e^{-\alpha a}f(a)-e^{-\alpha b}f(b)}{\alpha^2(b-a)^2}.
\end{aligned}
$$

□

Remark 3.4 In [8, Theorem 1] authors gave the following Hermite-Hadamard type inequality for an exponentially convex function

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\le \frac{1}{b-a}\int_a^b f(x)e^{-\alpha x}\,dx \\
&\le \frac{e^{-\alpha a}f(a)+e^{-\alpha b}f(b)}{2}.
\end{aligned}
$$

Actually, this result holds only for $\alpha > 0$ (not for all $\lambda \in \mathbb{R}$, as the authors stated). In that case, our result (3.14) is an improvement of the above inequality since

$$\begin{aligned}
&\frac{e^{-2\alpha a}f(a) - e^{-2\alpha b}f(b)}{\alpha(b-a)} + \left[e^{-\alpha b} - e^{-\alpha a}\right]\frac{e^{-\alpha a}f(a) - e^{-\alpha b}f(b)}{\alpha^2(b-a)^2} \\
&\quad = \frac{e^{-\alpha a}f(a)}{(b-a)^2}\int_a^b (b-x)e^{-\alpha x}\,dx + \frac{e^{-\alpha b}f(b)}{(b-a)^2}\int_a^b (x-a)e^{-\alpha x}\,dx \\
&\quad \le \frac{e^{-\alpha a}f(a)}{(b-a)^2}\int_a^b (b-x)\,dx + \frac{e^{-\alpha b}f(b)}{(b-a)^2}\int_a^b (x-a)\,dx \\
&\quad = \frac{e^{-\alpha a}f(a) + e^{-\alpha b}f(b)}{2}.
\end{aligned}$$

We use the basic property of the exponential function: $e^{-\alpha x} \le 1$ for $x \in [0,\infty)$ and $\alpha > 0$.

Next we present few other inequalities of the Hermite-Hadamard type.

Theorem 3.3 *Let f be a nonnegative $(h,g;m)$-convex function on $[0,\infty)$ where h is a nonnegative function on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$, g is a positive function on $[0,\infty)$ and $m \in (0,1]$. If $f,g,h \in L_1[a,b]$, where $0 \le a < b < \infty$, then the following inequality holds*

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)dx \\
&\quad \le \min\left\{f(a)g(a) + mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right); f(b)g(b) + mf\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\right\} \\
&\quad \times \int_0^1 h(x)dx. \qquad (3.15)
\end{aligned}$$

Proof. Let f be an $(h,g;m)$-convex function on $[0,\infty)$, $m \in (0,1]$ and $\lambda \in (0,1)$. Then

$$\begin{aligned}
f(\lambda a + (1-\lambda)b) &= f\left(\lambda a + m(1-\lambda)\frac{b}{m}\right) \\
&\le h(\lambda)f(a)g(a) + mh(1-\lambda)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)
\end{aligned}$$

and

$$f(\lambda b + (1-\lambda)a) \le h(\lambda)f(b)g(b) + mh(1-\lambda)f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right).$$

Integrating on $[0,1]$ with respect to the variable λ we obtain

$$\begin{aligned}
&\int_0^1 f(\lambda a + (1-\lambda)b)d\lambda \\
&\quad \le f(a)g(a)\int_0^1 h(\lambda)d\lambda + mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\int_0^1 h(1-\lambda)d\lambda
\end{aligned}$$

and

$$\int_0^1 f(\lambda b+(1-\lambda)a)d\lambda \\ \leq f(b)g(b)\int_0^1 h(\lambda)d\lambda + mf\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\int_0^1 h(1-\lambda)d\lambda.$$

The inequality (3.15) now follows from $\int_0^1 h(\lambda)d\lambda = \int_0^1 h(1-\lambda)d\lambda$ and

$$\int_0^1 f(\lambda a+(1-\lambda)b)d\lambda = \int_0^1 f(\lambda b+(1-\lambda)a)d\lambda = \frac{1}{b-a}\int_a^b f(x)dx.$$

□

Theorem 3.4 *Suppose that the assumptions of Theorem 3.3 hold with $f,g,h \in L_1[ma,b]$. Then*

$$\frac{1}{mb-a}\int_a^{mb} f(x)dx + \frac{1}{b-ma}\int_{ma}^b f(x)dx \\ \leq (m+1)\left[f(a)g(a)+f(b)g(b)\right]\int_0^1 h(x)dx. \tag{3.16}$$

Proof. Let f be an $(h,g;m)$-convex function on $[0,\infty)$, $m \in (0,1]$ and $\lambda \in (0,1)$. Then

$$f(\lambda a+m(1-\lambda)b) \leq h(\lambda)f(a)g(a)+mh(1-\lambda)f(b)g(b), \\ f((1-\lambda)a+m\lambda b) \leq h(1-\lambda)f(a)g(a)+mh(\lambda)f(b)g(b)$$

and

$$f(\lambda b+m(1-\lambda)a) \leq h(\lambda)f(b)g(b)+mh(1-\lambda)f(a)g(a), \\ f((1-\lambda)b+m\lambda a) \leq h(1-\lambda)f(b)g(b)+mh(\lambda)f(a)g(a).$$

Next we add the above inequalities

$$f(\lambda a+m(1-\lambda)b)+f((1-\lambda)a+m\lambda b) \\ +f(\lambda b+m(1-\lambda)a)+f((1-\lambda)b+m\lambda a) \\ \leq (m+1)\left[f(a)g(a)+f(b)g(b)\right]\left[h(\lambda)+h(1-\lambda)\right]$$

and integrate to obtain

$$\int_0^1 f(\lambda a+m(1-\lambda)b)d\lambda + \int_0^1 f((1-\lambda)a+m\lambda b)d\lambda \\ \int_0^1 f(\lambda b+m(1-\lambda)a)d\lambda + \int_0^1 f((1-\lambda)b+m\lambda a)d\lambda \\ \leq (m+1)\left[f(a)g(a)+f(b)g(b)\right]\int_0^1 \left[h(\lambda)+h(1-\lambda)\right]d\lambda.$$

The inequality (3.16) now follows from

$$\int_0^1 f(\lambda a+m(1-\lambda)b)d\lambda=\int_0^1 f((1-\lambda)a+m\lambda b)d\lambda=\frac{1}{mb-a}\int_a^{mb} f(x)dx$$

and

$$\int_0^1 f(\lambda b+m(1-\lambda)a)d\lambda=\int_0^1 f((1-\lambda)b+m\lambda a)d\lambda=\frac{1}{b-ma}\int_{ma}^{b} f(x)dx.$$

□

Remark 3.5 For h being identity and $q\equiv 1$, $(h,g;m)$-convexity reduces to m-convexity. Therefore, Theorems 3.3 and 3.4 generalize [11, Theorem 2, Theorem 5], respectively.

Notice, if $m\in(0,1]$ then $ma<b$, but there is no guarantee that $a<mb$, so the resulting inequality from [11, Theorem 5]

$$\frac{1}{m+1}\left[\int_a^{mb} f(x)dx+\frac{mb-a}{b-ma}\int_{ma}^{b} f(x)dx\right]\leq(mb-a)\frac{f(a)+f(b)}{2}$$

should not be multiplied with $(mb-a)$.

We emphasize that all other corresponding Hermite-Hadamard inequalities for different types of convexity, which follow from this section, can be done analogously.

3.3 Fejér type inequalities for $(h,g;m)$-convex functions

The Fejér inequality is a weighted version of the Hermite-Hadamard inequality:

Theorem 3.5 (THE FEJÉR INEQUALITY) *Let $f:[a,b]\to\mathbb{R}$ be a convex function and $w:[a,b]\to\mathbb{R}$ nonnegative, integrable and symmetric about $\frac{a+b}{2}$. Then*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx\leq\int_a^b f(x)w(x)\,dx\leq\frac{f(a)+f(b)}{2}\int_a^b w(x)dx. \tag{3.17}$$

Here we prove the Fejér inequality for an $(h,g;m)$-convex function and give some similar results.

Theorem 3.6 (THE SECOND FEJÉR INEQUALITY FOR $(h,g;m)$-CONVEX FUNCTIONS) *Let f be a nonnegative $(h,g;m)$-convex function on $[0,\infty)$ where h is a nonnegative function on $J\subseteq\mathbb{R}$, $(0,1)\subseteq J$, $h\not\equiv 0$, g is a positive function on $[0,\infty)$ and $m\in(0,1]$. Let*

$0 \leq a < b < \infty$ and $f,g,h \in L_1[a,b]$. Furthermore, let $w : [a,b] \to \mathbb{R}$ be a nonnegative, integrable and symmetric about $\frac{a+b}{2}$. Then the following inequality holds

$$\begin{aligned}
&\int_a^b f(x)w(x)dx \\
&\quad \leq \frac{1}{2}\left[f(a)g(a)+f(b)g(b)+mf\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)+mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right] \\
&\quad \times \int_a^b h\left(\frac{x-a}{b-a}\right)w(x)dx. \qquad (3.18)
\end{aligned}$$

Proof. Let f be an $(h,g;m)$-convex function on $[0,\infty)$, $m \in (0,1]$ and $x \in [a,b]$. First we use

$$f(x) = f\left(\frac{b-x}{b-a}a + m\frac{x-a}{b-a}\frac{b}{m}\right),$$

$$f(a+b-x) = f\left(\frac{b-x}{b-a}b + m\frac{x-a}{b-a}\frac{a}{m}\right).$$

Next, since $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ and w is symmetric with respect to $\frac{a+b}{2}$, i.e. $w(x) = w(a+b-x)$, we have

$$\begin{aligned}
\int_a^b f(x)w(x)dx &= \frac{1}{2}\left[\int_a^b f(x)w(x)dx + \int_a^b f(a+b-x)w(a+b-x)dx\right] \\
&= \frac{1}{2}\int_a^b [f(x)+f(a+b-x)]w(x)dx \\
&= \frac{1}{2}\int_a^b \left[f\left(\frac{b-x}{b-a}a + m\frac{x-a}{b-a}\frac{b}{m}\right) + f\left(\frac{b-x}{b-a}b + m\frac{x-a}{b-a}\frac{a}{m}\right)\right]w(x)dx.
\end{aligned}$$

Applying $(h,g;m)$-convexity of f, we obtain

$$\begin{aligned}
\int_a^b f(x)w(x)dx &\leq \frac{1}{2}\int_a^b \left[h\left(\frac{b-x}{b-a}\right)f(a)g(a) + mh\left(\frac{x-a}{b-a}\right)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right. \\
&\quad \left. + h\left(\frac{b-x}{b-a}\right)f(b)g(b) + mh\left(\frac{x-a}{b-a}\right)f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\right]w(x)dx \\
&= \frac{1}{2}[f(a)g(a)+f(b)g(b)]\int_a^b h\left(\frac{b-x}{b-a}\right)w(x)dx \\
&\quad + \frac{m}{2}\left[f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right]\int_a^b h\left(\frac{x-a}{b-a}\right)w(x)dx.
\end{aligned}$$

From symmetric property of w and with suitable substitution we obtain

$$\begin{aligned}
\int_a^b h\left(\frac{b-x}{b-a}\right)w(x)dx &= \int_a^b h\left(\frac{b-x}{b-a}\right)w(a+b-x)dx \\
&= \int_a^b h\left(\frac{x-a}{b-a}\right)w(x)dx. \qquad (3.19)
\end{aligned}$$

Hence, the inequality (3.18) is proved. □

Remark 3.6 If $q \equiv 1$ and $m = 1$, then $(h,g;m)$-convexity reduces to h-convexity. Therefore, Theorem 3.6 generalizes [31, Theorem 5].

Next we have two similar results.

Theorem 3.7 *Suppose that the assumptions of Theorem 3.6 hold. Then*

$$\int_a^b f(x)w(x)dx \leq \left[f(a)g(a)+mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right]\int_a^b h\left(\frac{x-a}{b-a}\right)w(x)dx. \tag{3.20}$$

Proof. Let f be an $(h,g;m)$-convex function on $[0,\infty)$, $m \in (0,1]$ and $\lambda \in (0,1)$. Then, as before,

$$f(\lambda a+(1-\lambda)b) \leq h(\lambda)f(a)g(a)+mh(1-\lambda)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right).$$

If we multiply the above with $w(\lambda a+(1-\lambda)b)$ and integrate on $[0,1]$ we obtain

$$\begin{aligned}
&\int_0^1 f(\lambda a+(1-\lambda)b)w(\lambda a+(1-\lambda)b)d\lambda \\
&\quad\leq \int_0^1 \left[h(\lambda)f(a)g(a)+mh(1-\lambda)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right]w(\lambda a+(1-\lambda)b)d\lambda \\
&\quad= f(a)g(a)\int_0^1 h(\lambda)w(\lambda a+(1-\lambda)b)d\lambda \\
&\qquad+mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\int_0^1 h(1-\lambda)w(\lambda a+(1-\lambda)b)d\lambda.
\end{aligned}$$

Of course,

$$\int_0^1 f(\lambda a+(1-\lambda)b)w(\lambda a+(1-\lambda)b)d\lambda = \frac{1}{b-a}\int_a^b f(x)w(x)dx.$$

From the symmetry of the weight w it follows $w(\lambda a+(1-\lambda)b) = w((1-\lambda)a+\lambda b)$ which gives us

$$\begin{aligned}
\int_0^1 h(\lambda)w(\lambda a+(1-\lambda)b)d\lambda &= \int_0^1 h(1-\lambda)w((1-\lambda)a+\lambda b)d\lambda \\
&= \frac{1}{b-a}\int_a^b h\left(\frac{x-a}{b-a}\right)w(x)dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)w(x)dx &\leq \frac{1}{b-a}f(a)g(a)\int_a^b h\left(\frac{x-a}{b-a}\right)w(x)dx \\
&\quad+\frac{m}{b-a}f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\int_a^b h\left(\frac{x-a}{b-a}\right)w(x)dx,
\end{aligned}$$

from which we obtain (3.20). □

Theorem 3.8 *Suppose that the assumptions of Theorem 3.6 hold. Then*

$$\int_a^b f(x)w(x)dx \leq \left[f(b)g(b)+mf\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\right]\int_a^b h\left(\frac{b-x}{b-a}\right)w(x)dx. \quad (3.21)$$

Proof. We start with

$$f(\lambda a+(1-\lambda)b) = f\left(m\lambda\frac{a}{m}+(1-\lambda)b\right) \leq h(1-\lambda)f(b)g(b)+mh(\lambda)f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)$$

and following the same steps as in the proof of the previous theorem we obtain the inequality (3.21). □

Remark 3.7 If we use (3.19) and add inequalities (3.20) and (3.21), then we obtain (3.18).

Theorems 3.7 and 3.8 both generalize [36, Theorem 3] – the second Fejér inequality for an h-convex function.

Furthermore, if we apply the symmetry of w and suitable substitution, we can see that [31, Theorem 5] and [36, Theorem 3] are the same results.

We continue with the first part of the Fejér inequality.

Theorem 3.9 (THE FIRST FEJÉR INEQUALITY FOR $(h,g;m)$-CONVEX FUNCTIONS) *Suppose that the assumptions of Theorem 3.6 hold. Then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx \\ \leq h\left(\frac{1}{2}\right)\int_a^b f(x)g(x)w(x)dx+mh\left(\frac{1}{2}\right)\int_a^b f\left(\frac{x}{m}\right)g\left(\frac{x}{m}\right)w(x)dx. \quad (3.22)\end{aligned}$$

Proof. Let f be an $(h,g;m)$-convex function on $[0,\infty)$, $m\in(0,1]$ and $x\in[a,b]$. Applying $(h,g;m)$-convexity of f and

$$f\left(\frac{a+b}{2}\right)=f\left(\frac{1}{2}(a+b-x)+m\frac{1}{2}\frac{x}{m}\right)$$

we obtain

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx \\ = \int_a^b f\left(\frac{1}{2}(a+b-x)+m\frac{1}{2}\frac{x}{m}\right)w(x)dx \\ \leq \int_a^b \left[h\left(\frac{1}{2}\right)f(a+b-x)g(a+b-x)+mh\left(\frac{1}{2}\right)f\left(\frac{x}{m}\right)g\left(\frac{x}{m}\right)\right]w(x)dx.\end{aligned}$$

Now from the above and

$$\begin{aligned}\int_a^b f(a+b-x)g(a+b-x)w(x)dx &= \int_a^b f(a+b-x)g(a+b-x)w(a+b-x)dx\\ &= \int_a^b f(x)g(x)w(x)dx\end{aligned}$$

where we use the symmetry of w, follows the inequality (3.22). □

Next we give a lemma that will be used in the last given Fejér type inequality.

Lemma 3.2 *Let f be a nonnegative $(h,g;m)$-convex function on $[0,\infty)$ where h is a nonnegative function on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$, g is a positive function on $[0,\infty)$ and $m \in (0,1]$. Then for all $x \in (a,b) \subset [0,\infty)$ there exists $\lambda \in (0,1)$ such that*

$$f(a+b-x) \le [h(\lambda)+h(1-\lambda)]\left[f(a)g(a)+mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right]-f(x). \tag{3.23}$$

If f is an $(h,g;m)$-concave function, then the reversed inequality holds.

Proof. Let f be an $(h,g;m)$-convex function on $[0,\infty)$, $m \in (0,1]$ and $x \in (a,b)$. Then there exists $\lambda \in (0,1)$ such that $x = \lambda a + (1-\lambda)b$. Therefore

$$\begin{aligned}f(a+b-x) &= f((1-\lambda)a+\lambda b)\\ &\le h(1-\lambda)f(a)g(a)+mh(\lambda)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\\ &= h(1-\lambda)f(a)g(a)+mh(\lambda)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\\ &\quad \pm h(\lambda)f(a)g(a) \pm mh(1-\lambda)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\\ &= [h(\lambda)+h(1-\lambda)\left[f(a)g(a)+mh(\lambda)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right]\\ &\quad -\left[h(\lambda)f(a)g(a)+mh(1-\lambda)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right]\\ &\le [h(\lambda)+h(1-\lambda)]\left[f(a)g(a)+mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right]-f(x).\end{aligned}$$

□

Theorem 3.10 *Suppose that the assumptions of Theorem 3.6 hold. Then for every $\lambda \in (0,1)$ there is the representation*

$$\begin{aligned}&\int_a^b f(x)w(x)dx\\ &\quad \le \frac{h(\lambda)+h(1-\lambda)}{2}\left[f(a)g(a)+mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right]\int_a^b w(x)dx.\end{aligned} \tag{3.24}$$

Proof. Let f be an $(h,g;m)$-convex function on $[0,\infty)$, $m \in (0,1]$ and $\lambda \in (0,1)$. Set $x = \lambda a + (1-\lambda)b$. If we multiply the inequality (3.23) with $w(x)$ and integrate we obtain

$$\int_a^b f(a+b-x)w(x)dx \le [(h(\lambda)+h(1-\lambda)]\left[f(a)g(a)+mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right]\int_a^b w(x)dx - \int_a^b f(x)w(x)dx.$$

If we use the symmetry of w and

$$\int_a^b f(a+b-x)w(a+b-x)dx = \int_a^b f(x)w(x)dx$$

then we obtain the inequality (3.24). □

Remark 3.8 If we set $q \equiv 1$ and $m = 1$, then Theorems 3.9 and 3.10 generalize the inequalities for an h-convex function in [31, Theorem 6].

Using special functions for h and/or g, as well as choosing a fixed parameter for m, inequalities for other different types of convexity can be derived from results of this section.

3.4 Lah-Ribarič type inequalities for $(h,g;m)$-convex functions

Two known inequalities provide bounds of the integral $\int_a^b p(x)f(w(x))dx$. While the Jensen inequality gives the lower, the Lah-Ribarič inequality gives its upper bound:

Theorem 3.11 (THE JENSEN INEQUALITY) *Let $w : [a,b] \to \mathbb{R}$ be an integrable function and let $p : [a,b] \to \mathbb{R}$ be a nonnegative function. If f is a convex function given on an interval I such that $w([a,b]) \subseteq I$, then*

$$f\left(\frac{1}{\int_a^b p(x)dx}\int_a^b p(x)w(x)\,dx\right) \le \frac{1}{\int_a^b p(x)dx}\int_a^b p(x)f(w(x))\,dx.$$

Theorem 3.12 (THE LAH-RIBARIČ INEQUALITY, [19]) *Let $w : [a,b] \to \mathbb{R}$ be an integrable function such that $m \le w(x) \le M$ for $x \in [a,b]$, $m < M$, and let $p : [a,b] \to \mathbb{R}$ be a nonnegative function. If f is a convex function given on an interval I such that $[m,M] \subseteq I$, then*

$$\frac{1}{\int_a^b p(x)dx}\int_a^b p(x)f(w(x))\,dx \le \frac{M-\bar{w}}{M-m}f(m) + \frac{\bar{w}-m}{M-m}f(M), \tag{3.25}$$

where

$$\bar{w} = \frac{\int_a^b p(x)w(x)dx}{\int_a^b p(x)dx}.$$

The Lah-Ribarič inequality derives certain results of Hermite-Hadamard, Fejér, Giaccardi, Popoviciu and Petrović (see [21, 25]), which was the main motivation for this section. We want to prove these inequalities in a more general setting, using the recently introduced class of $(h,g;m)$-convex functions ([6]).

We have already investigated certain Hermite-Hadamard and Fejér type inequalities for $(h,g;m)$-convex functions, but here we will observe those that we can obtain from the Lah-Ribarič inequality.

First we prove the Lah-Ribarič inequality for an $(h,g;m)$-convex function.

Theorem 3.13 (THE LAH-RIBARIČ INEQUALITY FOR $(h,g;m)$-CONVEX FUNCTIONS) *Let $w:[a,b]\to\mathbb{R}$ be an integrable function such that $0\le a<b<\infty$, $\mu\le w(x)\le M$ for $x\in[a,b]$, $\mu<M$, and let $p:[a,b]\to\mathbb{R}$ be a nonnegative function. Let f be a nonnegative $(h,g;m)$-convex function on $[0,\infty)$ such that $[\mu,M]\subseteq[0,\infty)$, where h is a nonnegative function on $J\subseteq\mathbb{R}$, $(0,1)\subseteq J$, $h\not\equiv 0$, g is a positive function on $[0,\infty)$, $m\in(0,1]$ and $f,g,h\in L_1[a,b]$. Then the following inequality holds*

$$\begin{aligned}\int_a^b p(x)f(w(x))dx \le\; & f(\mu)g(\mu)\int_a^b p(x)h\left(\frac{M-w(x)}{M-\mu}\right)dx\\ &+mf\left(\frac{M}{m}\right)g\left(\frac{M}{m}\right)\int_a^b p(x)h\left(\frac{w(x)-\mu}{M-\mu}\right)dx. \qquad (3.26)\end{aligned}$$

Proof. For $x\in[a,b]$ we have

$$f(x)=f\left(\frac{M-x}{M-\mu}\mu+m\frac{x-\mu}{M-\mu}\frac{M}{m}\right).$$

Applying $(h,g;m)$-convexity of f, we obtain

$$f(x)\le h\left(\frac{M-x}{M-\mu}\right)f(\mu)g(\mu)+mh\left(\frac{x-\mu}{M-\mu}\right)f\left(\frac{M}{m}\right)g\left(\frac{M}{m}\right).$$

If we substitute x with $w(x)$, multiply the above with $p(x)$ and integrate on $[a,b]$, then we obtain (3.26). □

Analogous inequality can be obtain for an $(h,g;m)$-concave function.

Interesting results arise for $(h,g;m)$-convex (concave) functions if we set h to be a super(sub)multiplicative function. For this we need the following definition.

Definition 3.2 *A function $h:J\to\mathbb{R}$ is said to be a supermultiplicative function if*

$$h(xy)\ge h(x)h(y) \qquad (3.27)$$

for all $x,y\in J$.

If the inequality (3.27) is reversed, then h is said to be a submultiplicative function. If the equality holds in (3.27), then h is said to be a multiplicative function.

Theorem 3.14 *Suppose that assumptions of Theorem 3.13 hold and let h be a supermultiplicative function with $M-w(x), w(x)-\mu, M-\mu \in J$. Then*

$$\int_a^b p(x)f(w(x))dx \le \frac{f(\mu)g(\mu)}{h(M-\mu)}\int_a^b p(x)h(M-w(x))dx \\ +\frac{m}{h(M-\mu)}f\left(\frac{M}{m}\right)g\left(\frac{M}{m}\right)\int_a^b p(x)h(w(x)-\mu)dx. \tag{3.28}$$

If additionally

$$h(\lambda)+h(1-\lambda)\le 1,\ \lambda\in(0,1), \tag{3.29}$$

then

$$\int_a^b p(x)f(w(x))dx \le f(\mu)g(\mu)\left[\int_a^b p(x)dx - \frac{1}{h(M-\mu)}\int_a^b p(x)h(w(x)-\mu)dx\right] \\ +\frac{m}{h(M-\mu)}f\left(\frac{M}{m}\right)g\left(\frac{M}{m}\right)\int_a^b p(x)h(w(x)-\mu)dx. \tag{3.30}$$

Proof. For $x\in[a,b]$ we have

$$\frac{M-w(x)}{M-\mu}\in(0,1)\subseteq J,\quad \frac{w(x)-\mu}{M-\mu}\in(0,1)\subseteq J.$$

Since h is a supermultiplicative function, and $M-w(x)$, $w(x)-\mu$, $M-\mu$ are in J, we have

$$h(M-w(x)) = h\left(\frac{M-w(x)}{M-\mu}\cdot(M-\mu)\right)\ge h\left(\frac{M-w(x)}{M-\mu}\right)h(M-\mu)$$

and

$$h(w(x)-\mu)\ge h\left(\frac{w(x)-\mu}{M-\mu}\right)h(M-\mu).$$

Hence, (3.28) is proved.

If we apply the condition (3.29), then

$$h\left(\frac{M-w(x)}{M-\mu}\right) = h\left(1-\frac{w(x)-\mu}{M-\mu}\right)\le 1-h\left(\frac{w(x)-\mu}{M-\mu}\right)\le 1-\frac{h(w(x)-\mu)}{h(M-\mu)},$$

which proves the inequality (3.30). □

Next we present several corollaries of Theorems 3.13 and 3.14, using special functions for h and/or g, as well as choosing a fixed parameter for m. We start with the first special case: Lah-Ribarič type inequalities for $(h-m)$-convex functions, obtained by setting $g\equiv 1$.

Corollary 3.7 *Let $w:[a,b]\to\mathbb{R}$ be an integrable function such that $0\le a<b<\infty$, $\mu\le w(x)\le M$ for $x\in[a,b]$, $\mu<M$, and let $p:[a,b]\to\mathbb{R}$ be a nonnegative function. Let f be a nonnegative $(h-m)$-convex function on $[0,\infty)$ such that $[\mu,M]\subseteq[0,\infty)$, where h is*

a nonnegative function on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$, $m \in (0,1]$ and $f,h \in L_1[a,b]$. Then the following inequality holds

$$\int_a^b p(x)f(w(x))dx \leq f(\mu)\int_a^b p(x)h\left(\frac{M-w(x)}{M-\mu}\right)dx + mf\left(\frac{M}{m}\right)\int_a^b p(x)h\left(\frac{w(x)-\mu}{M-\mu}\right)dx.$$

Corollary 3.8 *Suppose that assumptions of Corollary 3.7 hold and let h be a supermultiplicative function with $M-w(x), w(x)-\mu, M-\mu \in J$. Then*

$$\int_a^b p(x)f(w(x))dx \leq \frac{f(\mu)}{h(M-\mu)}\int_a^b p(x)h(M-w(x))dx + \frac{m}{h(M-\mu)}f\left(\frac{M}{m}\right)\int_a^b p(x)h(w(x)-\mu)dx.$$

If additionally (3.29) *holds, then*

$$\int_a^b p(x)f(w(x))dx \leq f(\mu)\left[\int_a^b p(x)dx - \frac{1}{h(M-\mu)}\int_a^b p(x)h(w(x)-\mu)dx\right] + \frac{m}{h(M-\mu)}f\left(\frac{M}{m}\right)\int_a^b p(x)h(w(x)-\mu)dx.$$

If $g \equiv 1$ and $m = 1$, then we have Lah-Ribarič type inequalities for h-convex functions.

Corollary 3.9 *Let $w : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leq a < b < \infty$, $\mu \leq w(x) \leq M$ for $x \in [a,b]$, $\mu < M$, and let $p : [a,b] \to \mathbb{R}$ be a nonnegative function. Let f be a nonnegative h-convex function on $[0,\infty)$ such that $[\mu, M] \subseteq [0,\infty)$, where h is a nonnegative function on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$ and $f,h \in L_1[a,b]$. Then the following inequality holds*

$$\int_a^b p(x)f(w(x))dx \leq f(\mu)\int_a^b p(x)h\left(\frac{M-w(x)}{M-\mu}\right)dx + f(M)\int_a^b p(x)h\left(\frac{w(x)-\mu}{M-\mu}\right)dx.$$

Corollary 3.10 *Suppose that assumptions of Corollary 3.9 hold and let h be a supermultiplicative function with $M-w(x), w(x)-\mu, M-\mu \in J$. Then*

$$\int_a^b p(x)f(w(x))dx \leq \frac{f(\mu)}{h(M-\mu)}\int_a^b p(x)h(M-w(x))dx + \frac{f(M)}{h(M-\mu)}\int_a^b p(x)h(w(x)-\mu)dx.$$

If additionally (3.29) *holds, then*

$$\int_a^b p(x)f(w(x))dx \leq f(\mu)\left[\int_a^b p(x)dx - \frac{1}{h(M-\mu)}\int_a^b p(x)h(w(x)-\mu)dx\right] + \frac{f(M)}{h(M-\mu)}\int_a^b p(x)h(w(x)-\mu)dx.$$

Finally, Lah-Ribarič type inequality for m-convex functions holds for h being an identity and $g \equiv 1$.

Corollary 3.11 *Let $w : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \le a < b < \infty$, $\mu \le w(x) \le M$ for $x \in [a,b]$, $\mu < M$, and let $p : [a,b] \to \mathbb{R}$ be a nonnegative function. Let f be a nonnegative m-convex function on $[0,\infty)$ such that $[\mu,M] \subseteq [0,\infty)$ with $m \in (0,1]$ and $f \in L_1[a,b]$. Then the following inequality holds*

$$\frac{1}{\int_a^b p(x)dx}\int_a^b p(x)f(w(x))\,dx \le \frac{M-\bar{w}}{M-\mu}f(\mu)+m\frac{\bar{w}-\mu}{M-\mu}f\left(\frac{M}{m}\right),$$

where

$$\bar{w} = \frac{\int_a^b p(x)w(x)dx}{\int_a^b p(x)dx}.$$

Of course, if $h(x) = x$, $g(x) = 1$ and $m = 1$, then we have the Lah-Ribarič inequality for a convex function, as given in (3.25).

3.4.1 Obtaining Hermite-Hadamard and Fejér inequalities

From the Lah-Ribarič inequality given in Theorem 3.13, we can deduce the right Hermite-Hadamard inequality for an $(h,g;m)$-convex function as follows.

Theorem 3.15 *Let f be a nonnegative $(h,g;m)$-convex function on $[0,\infty)$, where h is a nonnegative function on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$, g is a positive function on $[0,\infty)$ and $m \in (0,1]$. If $f,g,h \in L_1[a,b]$, where $0 \le a < b < \infty$, then following inequality holds*

$$\frac{1}{b-a}\int_a^b f(x)dx \le \left[f(a)g(a)+mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right]\int_0^1 h(x)dx. \tag{3.31}$$

Proof. If we set $M = b$, $\mu = a$ with $p(x) = 1$ and $w(x) = x$ on $[a,b]$ in Theorem 3.13, then we obtain

$$\begin{aligned}\int_a^b f(x)dx \;\le\; & f(a)g(a)\int_a^b h\left(\frac{b-x}{b-a}\right)dx \\ & +mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\int_a^b h\left(\frac{x-a}{b-a}\right)dx.\end{aligned}$$

Now (3.31) follows from

$$\int_a^b h\left(\frac{b-x}{b-a}\right)dx = \int_a^b h\left(\frac{x-a}{b-a}\right)dx = (b-a)\int_0^1 h(t)\,dt.$$

□

Remark 3.9 In addition to the inequalities already obtained in Theorems 3.2 and 3.3, the above theorem gives us yet another inequality of Hermite-Hadamard type.

A weighted version of the Hermite-Hadamard inequality is the Fejér inequality, also known in literature as the Hermite-Hadamard-Fejér inequality. If we add the symmetry of the weight p, then we can obtain one of the Fejér type inequality for $(h,g;m)$-convex functions.

Theorem 3.16 *Let f be a nonnegative $(h,g;m)$-convex function on $[0,\infty)$, where h is a nonnegative function on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$, g is a positive function on $[0,\infty)$ and $m \in (0,1]$. Let $0 \le a < b < \infty$ and $f,g,h \in L_1[a,b]$. Furthermore, let $p:[a,b] \to \mathbb{R}$ be a nonnegative, integrable and symmetric about $\frac{a+b}{2}$. Then the following inequality holds*

$$\int_a^b p(x)f(x)dx \le \left[f(a)g(a) + mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right]\int_a^b p(x)h\left(\frac{x-a}{b-a}\right)dx. \tag{3.32}$$

Proof. If we set $M = b$, $\mu = a$ with $w(x) = x$ on $[a,b]$, then from (3.26) we obtain

$$\begin{aligned}\int_a^b p(x)f(x)dx \le\ & f(a)g(a)\int_a^b p(x)h\left(\frac{b-x}{b-a}\right)dx \\ & + mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\int_a^b p(x)h\left(\frac{x-a}{b-a}\right)dx.\end{aligned}$$

From symmetric property of p and with suitable substitution we have

$$\begin{aligned}\int_a^b p(x)h\left(\frac{b-x}{b-a}\right)dx &= \int_a^b p(a+b-x)h\left(\frac{b-x}{b-a}\right)dx \\ &= \int_a^b p(x)h\left(\frac{x-a}{b-a}\right)dx, \end{aligned} \tag{3.33}$$

which gives us the inequality (3.32). □

Remark 3.10 The previous result, as well as the similar one

$$\int_a^b p(x)f(x)dx \le \left[f(b)g(b) + mf\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\right]\int_a^b p(x)h\left(\frac{b-x}{b-a}\right)dx, \tag{3.34}$$

are proved in Theorems 3.7 and 3.8. If we use weight's symmetry (3.33) and add inequalities (3.32) and (3.34), then we obtain (3.18) in Theorem 3.6.

3.4.2 Inequalities of Giaccardi, Popoviciu and Petrović

We present applications of previous results from this section to inequalities of Giaccardi, Popoviciu and Petrović.

Let f be a convex function on an interval I, $\mathbf{p}$ a nonnegative n-tuple with

$$\sum_{i=1}^{n} p_i \neq 0$$

and $\mathbf{x}$ a real n-tuple. If $\mathbf{x} \in I^n$ and $x_0 \in I$ are such that

$$\sum_{i=1}^{n} p_i x_i = \tilde{x} \in I, \quad \tilde{x} \neq x_0$$

and

$$(x_i - x_0)(\tilde{x} - x_i) \geq 0, \quad i = 1, \ldots, n,$$

then the Lah-Ribarič inequality (3.25) with $m = x_0$ and $M = \tilde{x}$ yields the Giaccardi inequality

$$\sum_{i=1}^{n} p_i f(x_i) \leq A f(\tilde{x}) + B \left(\sum_{i=1}^{n} p_i - 1 \right) f(x_0),$$

where

$$A = \frac{\sum_{i=1}^{n} p_i (x_i - x_0)}{\sum_{i=1}^{n} p_i x_i - x_0}, \quad B = \frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i x_i - x_0}.$$

When $x_i \geq 0$ $(i = 1, \ldots, n)$ are such that they satisfy

$$x_j \leq \sum_{i=1}^{n} p_i x_i, \quad j = 1, \ldots, n,$$

then from the Lah-Ribarič inequality (3.25) with $m = 0$ and $M = \tilde{x}$ follows the Popoviciu inequality

$$\sum_{i=1}^{n} p_i f(x_i) \leq f(\tilde{x}) + \left(\sum_{i=1}^{n} p_i - 1 \right) f(0).$$

If $x_i \geq 0$ $(i = 1, \ldots, n)$ and f is a convex function of $[0, \sum_{i=1}^{n} x_i]$, then we obtain the Petrović inequality from the Lah-Ribarič inequality (3.25) by setting $m = 0$, $M = \sum_{i=1}^{n} x_i$ and $p_i = 1$ for $i = 1, \ldots, n$, i. e.

$$\sum_{i=1}^{n} f(x_i) \leq f\left(\sum_{i=1}^{n} x_i \right) + (n-1) f(0).$$

More on the above inequalities can be found in [21, 25].

Now we present these inequalities for an $(h,g;m)$-convex function using Theorem 3.13 as follows.

Theorem 3.17 (THE GIACCARDI INEQUALITY FOR $(h,g;m)$-CONVEX FUNCTIONS) *Let $\mathbf{p}$ be a nonnegative n-tuple with $\sum_{i=1}^{n} p_i \neq 0$ and $\mathbf{x}$ be a real n-tuple. Let $\mathbf{x} \in I^n$ and $x_0 \in I$ be such that*

$$\sum_{i=1}^{n} p_i x_i = \tilde{x} \in I, \quad \tilde{x} \neq x_0$$

and

$$(x_i - x_0)(\tilde{x} - x_i) \geq 0, \quad i = 1, \ldots, n.$$

Let f be a nonnegative $(h,g;m)$-convex function on $[0,\infty)$ such that $I \subseteq [0,\infty)$, where h is a nonnegative function on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$, g is a positive function on $[0,\infty)$ and $m \in (0,1]$. Then the following inequality holds

$$\sum_{i=1}^{n} p_i f(x_i) \leq f(x_0)g(x_0)\sum_{i=1}^{n} p_i h\left(\frac{\tilde{x}-x_i}{\tilde{x}-x_0}\right) + mf\left(\frac{\tilde{x}}{m}\right)g\left(\frac{\tilde{x}}{m}\right)\sum_{i=1}^{n} p_i h\left(\frac{x_i-x_0}{\tilde{x}-x_0}\right). \tag{3.35}$$

Proof. Proof follows directly from Theorem 3.13 for $\mu = x_0$ and $M = \tilde{x}$. □

Corollary 3.12 (THE POPOVICIU INEQUALITY FOR $(h,g;m)$-CONVEX FUNCTIONS) *Let* $\mathbf{p}$ *and* $\mathbf{x}$ *be nonnegative n-tuples with* $\sum_{i=1}^{n} p_i \neq 0$. *Let* $\mathbf{x} \in [0,a]^n$, $0 < a < \infty$, *be such that*

$$\sum_{i=1}^{n} p_i x_i = \tilde{x} \in (0,a]$$

and

$$\tilde{x} - x_i \geq 0, \quad i = 1,\ldots,n.$$

Let f *be a nonnegative* $(h,g;m)$*-convex function on* $[0,\infty)$*, where* h *is a nonnegative function on* $J \subseteq \mathbb{R}$*,* $(0,1) \subseteq J$*,* $h \not\equiv 0$*,* g *is a positive function on* $[0,\infty)$ *and* $m \in (0,1]$*. Then the following inequality holds*

$$\sum_{i=1}^{n} p_i f(x_i) \leq f(0)g(0)\sum_{i=1}^{n} p_i h\left(\frac{\tilde{x}-x_i}{\tilde{x}}\right) + mf\left(\frac{\tilde{x}}{m}\right)g\left(\frac{\tilde{x}}{m}\right)\sum_{i=1}^{n} p_i h\left(\frac{x_i}{\tilde{x}}\right). \tag{3.36}$$

Corollary 3.13 (THE PETROVIĆ INEQUALITY FOR $(h,g;m)$-CONVEX FUNCTIONS) *Let* $\mathbf{x}$ *be a nonnegative n-tuple. Let* f *be a nonnegative* $(h,g;m)$*-convex function on* $[0,\infty)$ *such that* $[0,\sum_{i=1}^{n} x_i] \subseteq [0,\infty)$*, where* h *is a nonnegative function on* $J \subseteq \mathbb{R}$*,* $(0,1) \subseteq J$*,* $h \not\equiv 0$*,* g *is a positive function on* $[0,\infty)$ *and* $m \in (0,1]$*. Then the following inequality holds*

$$\sum_{i=1}^{n} f(x_i) \leq f(0)g(0)\sum_{i=1}^{n} h\left(\frac{\sum_{j=1}^{n} x_j - x_i}{\sum_{j=1}^{n} x_j}\right) + mf\left(\frac{1}{m}\sum_{i=1}^{n} x_i\right)g\left(\frac{1}{m}\sum_{i=1}^{n} x_i\right)\sum_{i=1}^{n} h\left(\frac{x_i}{\sum_{j=1}^{n} x_j}\right). \tag{3.37}$$

Next we observe Giaccardi's inequality for an $(h,g;m)$-convex function, where h is a supermultiplicative function.

Theorem 3.18 *Suppose that assumptions of Theorem 3.17 hold and let* h *be a supermultiplicative function with* $\tilde{x} - x_0 \in J$ *and* $\tilde{x} - x_i$*,* $x_i - x_0 \in J$ *for* $i =, 1,\ldots,n$*. Then*

$$\sum_{i=1}^{n} p_i f(x_i) \leq \frac{f(x_0)g(x_0)}{h(\tilde{x}-x_0)}\sum_{i=1}^{n} p_i h(\tilde{x}-x_i) + \frac{m}{h(\tilde{x}-x_0)} f\left(\frac{\tilde{x}}{m}\right)g\left(\frac{\tilde{x}}{m}\right)\sum_{i=1}^{n} p_i h(x_i - x_0). \tag{3.38}$$

If additionally (3.29) *holds, then*

$$\sum_{i=1}^{n} p_i f(x_i) \leq f(x_0)g(x_0)\left[\sum_{i=1}^{n} p_i - \frac{\sum_{i=1}^{n} p_i h(x_i - x_0)}{h(\tilde{x} - x_0)}\right] + \frac{m}{h(\tilde{x} - x_0)} f\left(\frac{\tilde{x}}{m}\right) g\left(\frac{\tilde{x}}{m}\right) \sum_{i=1}^{n} p_i h(x_i - x_0). \tag{3.39}$$

Proof. Proof follows from Theorem 3.14 for $\mu = x_0$ and $M = \tilde{x}$. Here, for $i = 1,\ldots,n$ we have

$$\frac{\tilde{x} - x_i}{\tilde{x} - x_0} \in (0,1) \subseteq J, \quad \frac{x_i - x_0}{\tilde{x} - x_0} \in (0,1) \subseteq J.$$

Since h is a supermultiplicative function, with assumptions that $\tilde{x} - x_0$, $\tilde{x} - x_i$, $x_i - x_0$ for all $i = 1,\ldots,n$ are in J, we obtain (3.38).

The second inequality (3.39) follows from

$$h\left(\frac{\tilde{x} - x_i}{\tilde{x} - x_0}\right) \leq 1 - \frac{h(x_i - x_0)}{h(\tilde{x} - x_0)}.$$

□

Remark 3.11 Setting $g \equiv 1$ and $m = 1$ in (3.39) we obtain the result for h-convex function given in [29, Theorem 2.1]:

$$\sum_{i=1}^{n} p_i f(x_i) \leq f(x_0)\left[\sum_{i=1}^{n} p_i - \frac{\sum_{i=1}^{n} p_i h(x_i - x_0)}{h(\tilde{x} - x_0)}\right] + \frac{f(\tilde{x})}{h(\tilde{x} - x_0)} \sum_{i=1}^{n} p_i h(x_i - x_0).$$

Taking $x_0 = 0$ gives us [29, Theorem 2.2]:

$$\sum_{i=1}^{n} p_i f(x_i) \leq f(0)\left[\sum_{i=1}^{n} p_i - \frac{\sum_{i=1}^{n} p_i h(x_i)}{h(\tilde{x})}\right] + \frac{f(\tilde{x})}{h(\tilde{x})} \sum_{i=1}^{n} p_i h(x_i).$$

Corresponding inequalities can be stated using special functions for h and/or g, and choosing a fixed parameter for m. However, the details are omitted.

3.4.3 Applications: Converses of the Jensen inequality

We consider converses of Jensen's inequality, using an analogue of the Mond-Pečarić method in operator inequalities, to obtain the bounds. Numerous applications of the following theorem can be given as in monographs [13, 14] involving operators, or for real variables as in [25].

Theorem 3.19 *Let* $w : [a,b] \to \mathbb{R}$ *be an integrable function such that* $0 \leq a < b < \infty$, $\mu \leq w(x) \leq M$ *for* $x \in [a,b]$, $\mu < M$, *and let* $p : [a,b] \to \mathbb{R}$ *be a nonnegative function. Let* f *be a nonnegative* $(h,g;m)$*-convex function on* $[0,\infty)$ *such that* $[\mu,M] \subseteq [0,\infty)$, *where* h *is a nonnegative function on* $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$, g *is a positive function on* $[0,\infty)$,

$m \in (0,1]$ and $f,g,h \in L_1[a,b]$. Furthermore, let $F : U \times U \to \mathbb{R}$ be a function such that $f([\mu,M]) \subset U$. If F is increasing in the first variable and h is a concave function, then the following inequality holds

$$\begin{aligned} &F\left[\frac{1}{\int_a^b p(x)dx}\int_a^b p(x)f(w(x))dx, f\left(\frac{1}{\int_a^b p(x)dx}\int_a^b p(x)w(x)dx\right)\right] \\ &\quad \le \max_{x\in[\mu,M]} F\left[\frac{h(M-x)}{M-\mu}f(\mu)g(\mu)+m\frac{h(x-\mu)}{M-\mu}f\left(\frac{M}{m}\right)g\left(\frac{M}{m}\right), f(x)\right]. \end{aligned} \tag{3.40}$$

Proof. Since h is a concave function, then from Theorem 3.13 we obtain

$$\begin{aligned} &\int_a^b p(x)f(w(x))dx \\ &\quad \le f(\mu)g(\mu)\int_a^b p(x)h\left(\frac{M-w(x)}{M-\mu}\right)dx + mf\left(\frac{M}{m}\right)g\left(\frac{M}{m}\right)\int_a^b p(x)h\left(\frac{w(x)-\mu}{M-\mu}\right)dx \\ &\quad \le f(\mu)g(\mu)\left(\int_a^b p(x)dx\right)h\left(\frac{1}{\int_a^b p(x)dx}\int_a^b p(x)\frac{M-w(x)}{M-\mu}dx\right) \\ &\qquad + mf\left(\frac{M}{m}\right)g\left(\frac{M}{m}\right)\left(\int_a^b p(x)dx\right)h\left(\frac{1}{\int_a^b p(x)dx}\int_a^b p(x)\frac{w(x)-\mu}{M-\mu}dx\right) \end{aligned}$$

that is

$$\frac{\int_a^b p(x)f(w(x))dx}{\int_a^b p(x)dx} \le \frac{h(M-\bar{w})}{M-\mu}f(\mu)g(\mu)+m\frac{h(\bar{w}-\mu)}{M-\mu}f\left(\frac{M}{m}\right)g\left(\frac{M}{m}\right)$$

where

$$\bar{w} = \frac{\int_a^b p(x)w(x)dx}{\int_a^b p(x)dx}.$$

Using the increasing property of $F(\cdot,y)$ we have

$$\begin{aligned} &F\left[\frac{\int_a^b p(x)f(w(x))dx}{\int_a^b p(x)dx}, f(\bar{w})\right] \\ &\quad \le F\left[\frac{h(M-\bar{w})}{M-\mu}f(\mu)g(\mu)+m\frac{h(\bar{w}-\mu)}{M-\mu}f\left(\frac{M}{m}\right)g\left(\frac{M}{m}\right), f(\bar{w})\right] \\ &\quad \le \max_{x\in[\mu,M]} F\left[\frac{h(M-x)}{M-\mu}f(\mu)g(\mu)+m\frac{h(x-\mu)}{M-\mu}f\left(\frac{M}{m}\right)g\left(\frac{M}{m}\right), f(x)\right], \end{aligned}$$

which gives us the inequality (3.40). □

Remark 3.12 If we apply Theorem 3.19 on the function $F(x,y) = x - y$, then a difference type of converse of Jensen's inequality for $(h,g;m)$-convex functions follows

$$\frac{1}{\int_a^b p(x)dx}\int_a^b p(x)f(w(x))dx - f\left(\frac{1}{\int_a^b p(x)dx}\int_a^b p(x)w(x)dx\right)$$
$$\leq \max_{x\in[\mu,M]}\left[\frac{h(M-x)}{M-\mu}f(\mu)g(\mu) + m\frac{h(x-\mu)}{M-\mu}f\left(\frac{M}{m}\right)g\left(\frac{M}{m}\right) - f(x)\right].$$

Setting $U = (0,\infty)$ and $F(x,y) = x/y$ in Theorem 3.19, we obtain a ratio type converse of Jensen's inequality for $(h,g;m)$-convex functions

$$\frac{\frac{1}{\int_a^b p(x)dx}\int_a^b p(x)f(w(x))dx}{f\left(\frac{1}{\int_a^b p(x)dx}\int_a^b p(x)w(x)dx\right)}$$
$$\leq \max_{x\in[\mu,M]}\left[\frac{h(M-x)}{f(x)(M-\mu)}f(\mu)g(\mu) + m\frac{h(x-\mu)}{f(x)(M-\mu)}f\left(\frac{M}{m}\right)g\left(\frac{M}{m}\right)\right].$$

Therefore, a particular choice of the function F in Theorem 3.19 implies complementary inequality to Jensen's inequality, by which is given the unified view of upper estimates. Thus the problem of determining the upper estimates is reduced to solving a single variable maximization problem.

3.5 Jensen type inequalities for $(h,g;m)$-convex functions

The following lemma is equivalent to the definition of a convex function (3.1).

Lemma 3.3 ([25]) *Let $x_1,x_2,x_3 \in I$ be such that $x_1 < x_2 < x_3$. The function $f : I \to \mathbb{R}$ is convex if and only if the following inequality holds*

$$(x_3 - x_2)f(x_1) + (x_1 - x_3)f(x_2) + (x_2 - x_1)f(x_3) \geq 0.$$

By mathematical induction, we can extend the inequality (3.1) to the convex combinations of finitely many points in I and next to random variables associated to arbitrary probability spaces. These extensions are known as the discrete Jensen inequality and the integral Jensen inequality, respectively.

Theorem 3.20 (THE DISCRETE JENSEN INEQUALITY) *A real-valued function f defined on an interval I is convex if and only if for all $x_1,\ldots,x_n$ in I and all scalars $\lambda_1,\ldots,\lambda_n$ in $[0,1]$ with $\sum_{i=1}^n \lambda_i = 1$ we have*

$$f\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i f(x_i).$$

The above inequality is strict if f is strictly convex, all the points x_i are distinct and all scalars λ_i are positive.

Here we will obtain Schur and Jensen type inequalities for $(h,g;m)$-convex functions, which will generalize and extend corresponding inequalities for the classes of convex functions that already exist in literature. We will use super(sub)multiplicative functions as in Definition 3.2.

3.5.1 Schur type inequalities

We start with a result related to the definition of $(h,g;m)$-convex functions.

Proposition 3.6 *Let f be a nonnegative $(h,g;m)$-convex function on $I \subseteq \mathbb{R}$, where h is a nonnegative supermultiplicative function on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$, g is a positive function on I and $m \in (0,1]$. Then for $x_1, x_2, x_3 \in I$, $x_1 < x_2 < x_3$ with $x_3 - x_2, x_2 - x_1, x_3 - x_1 \in J$ the following inequality holds*

$$h(x_3 - x_2)f(x_1)g(x_1) - h(x_3 - x_1)f(x_2) + mh(x_2 - x_1)f\left(\frac{x_3}{m}\right)g\left(\frac{x_3}{m}\right) \geq 0. \quad (3.41)$$

If f is an $(h,g;m)$-concave function where h is a submultiplicative function, then inequality (3.41) *is reversed.*

Proof. Let f be an $(h,g;m)$-convex function and $x_1, x_2, x_3 \in I$. From the assumptions, we have

$$\frac{x_3 - x_2}{x_3 - x_1} \in (0,1) \subseteq J, \quad \frac{x_2 - x_1}{x_3 - x_1} \in (0,1) \subseteq J$$

and

$$\frac{x_3 - x_2}{x_3 - x_1} + \frac{x_2 - x_1}{x_3 - x_1} = 1.$$

Since h is a supermultiplicative function and $x_3 - x_2, x_2 - x_1, x_3 - x_1 \in J$, we obtain

$$h(x_3 - x_2) = h\left(\frac{x_3 - x_2}{x_3 - x_1} \cdot (x_3 - x_1)\right) \geq h\left(\frac{x_3 - x_2}{x_3 - x_1}\right) h(x_3 - x_1)$$

and also

$$h(x_2 - x_1) \geq h\left(\frac{x_2 - x_1}{x_3 - x_1}\right) h(x_3 - x_1).$$

Assume $h(x_3 - x_1) > 0$. If we set in (3.3) $\lambda = \frac{x_3 - x_2}{x_3 - x_1}$, $x = x_1$, $y = x_3$, then we obtain

$$\begin{aligned} f(x_2) &= f\left(\lambda x + m(1-\lambda)\frac{y}{m}\right) \\ &\leq h(\lambda)f(x)g(x) + mh(1-\lambda)f\left(\frac{y}{m}\right)g\left(\frac{y}{m}\right) \\ &= h\left(\frac{x_3 - x_2}{x_3 - x_1}\right)f(x_1)g(x_1) + mh\left(\frac{x_2 - x_1}{x_3 - x_1}\right)f\left(\frac{x_3}{m}\right)g\left(\frac{x_3}{m}\right) \quad (3.42) \\ &\leq \frac{h(x_3 - x_2)}{h(x_3 - x_1)}f(x_1)g(x_1) + m\frac{h(x_2 - x_1)}{h(x_3 - x_1)}f\left(\frac{x_3}{m}\right)g\left(\frac{x_3}{m}\right). \end{aligned}$$

Hence, (3.41) is proven.

Analogously follows reversed inequality (3.41) if f is an $(h,g;m)$-concave function where h is a submultiplicative function. □

Recall the Schur inequality:

If x,y,z are positive numbers and if λ is real, then

$$x^{\lambda}(x-y)(x-z)+y^{\lambda}(y-z)(y-x)+z^{\lambda}(z-x)(z-y)\geq 0$$

with equality if and only if $x=y=z$.

This inequality follows from (3.41) for $f(x)=x^{\lambda}$, $\lambda\in\mathbb{R}$, $h(x)=\frac{1}{x}$, $g\equiv 1$ and $m=1$.

A related inequality was proved in [22] by Mitrinović and Pečarić:

$$(x_1-x_2)(x_1-x_3)f(x_1)+(x_2-x_1)(x_2-x_3)f(x_2)+(x_3-x_1)(x_3-x_2)f(x_3)\geq 0$$

where f is a Godunova-Levin function, that is an $(h,g;m)\equiv(x^{-1},1;1)$-convex function:

$$f(\lambda x+(1-\lambda)y)\leq\frac{f(x)}{\lambda}+\frac{f(y)}{1-\lambda}.$$

Next inequality is of Schur type for $(x^{-k},g;m)$-convex (and concave) functions, obtained for $h(x)=\frac{1}{x^k}$, $k\in\mathbb{R}$:

Corollary 3.14 *Let f be a positive $(x^{-k},g;m)$-convex function on $I\subseteq\mathbb{R}$, where $k\in\mathbb{R}$, g is a positive function on I and $m\in(0,1]$. Then for $x_1,x_2,x_3\in I$, $x_1<x_2<x_3$ the following inequality holds*

$$\begin{aligned}&f(x_1)g(x_1)(x_3-x_1)^k(x_2-x_1)^k-f(x_2)(x_3-x_2)^k(x_2-x_1)^k\\&\quad+mf\left(\frac{x_3}{m}\right)g\left(\frac{x_3}{m}\right)(x_3-x_1)^k(x_3-x_2)^k\geq 0.\end{aligned}\tag{3.43}$$

If the function f is a positive $(x^{-k},g;m)$-concave function, then inequality (3.43) *is reversed.*

As an example of a special case, if we set $h(x)=x^s$, $s\in(0,1]$, $g(x)=e^{-\alpha x}$, $\alpha\in\mathbb{R}$, then we obtain following Schur type inequality for convexity (3.2), i.e., exponentially (s,m)-convex functions in the second sense.

Corollary 3.15 *Let f be an exponentially (s,m)-convex function in the second sense on $I\subseteq\mathbb{R}$, where $s,m\in(0,1]$. Then, for $x_1,x_2,x_3\in I$, $x_1<x_2<x_3$, the following inequality holds*

$$\frac{(x_3-x_2)^s}{e^{\alpha x_1}}f(x_1)-(x_3-x_1)^sf(x_2)+\frac{m(x_2-x_1)^s}{e^{\frac{\alpha}{m}x_3}}f\left(\frac{x_3}{m}\right)\geq 0.\tag{3.44}$$

If the function f is an exponentially (s,m)-concave function in the second sense, then inequality (3.44) *is reversed.*

Remark 3.13 Using special functions for h and/or g, as well as choosing a fixed parameter for m, Schur type inequalities for different types of convexity can be derived. For instance, setting $g\equiv 1$ and $m=1$ in (3.41) and (3.43), we obtain results for h-convex functions given in [36].

3.5.2 Jensen type inequalities

We continue with Jensen type inequalities for $(h,g;m)$-convex functions, where h is supermultiplicative function. In the following, for $n \in \mathbb{N}$, let

$$P_n = \sum_{i=1}^{n} p_i, \tag{3.45}$$

$$X_n = \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i, \tag{3.46}$$

$$G_i^n = \prod_{j=i}^{n} g(X_j), \quad i \geq 1. \tag{3.47}$$

We will set empty products equal to 1, for example $G_{n+1}^n = \prod_{j=n+1}^{n} g(X_j) \equiv 1$.

Notice that $P_1 = p_1$, $X_1 = x_1$ and $G_n^n = g(X_n)$. Furthermore, the following recursive formulas hold

$$G_i^n = g(X_i) \cdot G_{i+1}^n, \quad i = 1, \ldots, n, \tag{3.48}$$

$$G_i^n = G_i^{n-1} \cdot g(X_n), \quad i = 1, \ldots, n. \tag{3.49}$$

Theorem 3.21 (THE JENSEN INEQUALITY FOR $(h,g;m)$-CONVEX FUNCTIONS) *Let $p_1, \ldots, p_n$ be positive real numbers. Let f be a nonnegative $(h,g;m)$-convex function on $[0,\infty)$ such that $I \subseteq [0,\infty)$, where h is a nonnegative supermultiplicative function on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$, g is a positive function on $[0,\infty)$ and $m \in (0,1]$. Then, for $x_1, \ldots, x_n \in I$, the following inequality holds*

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq h\left(\frac{p_1}{P_n}\right) f(x_1) G_1^{n-1} + m \sum_{i=2}^{n} h\left(\frac{p_i}{P_n}\right) f\left(\frac{x_i}{m}\right) g\left(\frac{x_i}{m}\right) G_i^{n-1}. \tag{3.50}$$

If f is an $(h,g;m)$-concave function where h is a submultiplicative function, then inequality (3.50) *is reversed.*

Proof. We will prove the theorem by the mathematical induction.

If $n = 2$, then (3.50) is equivalent to (3.3) with $\lambda = \frac{p_1}{P_2}$, $1 - \lambda = \frac{p_2}{P_2}$, $x = x_1$ and $y = \frac{x_2}{m}$ (notice, $G_1^1 = g(X_1) = g(x_1)$ and $G_2^1 \equiv 1$).

Assume that (3.50) holds for $n-1$. Then, for $p_1, \ldots, p_n$ and $x_1, \ldots, x_n$ we have

$$\begin{aligned} f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) &= f\left(m\frac{p_n}{P_n}\frac{x_n}{m} + \frac{P_{n-1}}{P_n}\sum_{i=1}^{n-1}\frac{p_i}{P_{n-1}}x_i\right) \\ &\leq m h\left(\frac{p_n}{P_n}\right) f\left(\frac{x_n}{m}\right) g\left(\frac{x_n}{m}\right) \\ &\quad + h\left(\frac{P_{n-1}}{P_n}\right) f\left(\sum_{i=1}^{n-1}\frac{p_i}{P_{n-1}}x_i\right) g\left(\sum_{i=1}^{n-1}\frac{p_i}{P_{n-1}}x_i\right) \end{aligned}$$

$$\leq mh\left(\frac{p_n}{P_n}\right)f\left(\frac{x_n}{m}\right)g\left(\frac{x_n}{m}\right)$$
$$+h\left(\frac{P_{n-1}}{P_n}\right)\left[h\left(\frac{p_1}{P_{n-1}}\right)f(x_1)G_1^{n-2}\right.$$
$$\left.+m\sum_{i=2}^{n-1}h\left(\frac{p_i}{P_{n-1}}\right)f\left(\frac{x_i}{m}\right)g\left(\frac{x_i}{m}\right)G_i^{n-2}\right]g(X_{n-1}).$$

Since h is a supermultiplicative function, we obtain

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right)\leq mh\left(\frac{p_n}{P_n}\right)f\left(\frac{x_n}{m}\right)g\left(\frac{x_n}{m}\right)+h\left(\frac{p_1}{P_n}\right)f(x_1)g(X_{n-1})G_1^{n-2}$$
$$+m\sum_{i=2}^{n-1}h\left(\frac{p_i}{P_n}\right)f\left(\frac{x_i}{m}\right)g\left(\frac{x_i}{m}\right)g(X_{n-1})G_i^{n-2}.$$

Now, we apply the recursive formula (3.49) to find inequality (3.50). □

Remark 3.14 As before, if we use special h, g and m in (3.50), then we obtain Jensen type inequalities for different types of convexity. Hence, Theorem 3.21 is a generalization of Jensen's inequality for h-convex functions given in [36].

The last result is a conversion of Jensen's inequality.

Theorem 3.22 *Let $p_1,\ldots,p_n$ be a positive real numbers and $[\mu,M]\subseteq[0,\infty)$. Let f be a nonnegative $(h,g;m)$-convex function on $[0,\infty)$, where h is a nonnegative supermultiplicative function on $(0,\infty)$, g is a positive function on $[0,\infty)$ and $m\in(0,1]$. Then for $x_i\in(\mu,M)$ and $M-\frac{x_i}{m}>0$ $(i=1,\ldots,n)$, the following inequalities hold*

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right)-h\left(\frac{p_1}{P_n}\right)f(x_1)G_1^{n-1}$$
$$\leq m\sum_{i=2}^{n}h\left(\frac{p_i}{P_n}\right)f\left(\frac{x_i}{m}\right)g\left(\frac{x_i}{m}\right)G_i^{n-1}$$
$$\leq m\sum_{i=2}^{n}h\left(\frac{p_i}{P_n}\right)\left[h\left(\frac{M-\frac{x_i}{m}}{M-\mu}\right)f(\mu)g(\mu)\right.$$
$$\left.+mh\left(\frac{\frac{x_i}{m}-\mu}{M-\mu}\right)f\left(\frac{M}{m}\right)g\left(\frac{M}{m}\right)\right]g\left(\frac{x_i}{m}\right)G_i^{n-1}. \tag{3.51}$$

If f is an $(h,g;m)$-concave function where h is a submultiplicative function, then inequality (3.51) *is reversed.*

Proof. From (3.42) in Proposition 3.6 we have

$$f(x_2)\leq h\left(\frac{x_3-x_2}{x_3-x_1}\right)f(x_1)g(x_1)+mh\left(\frac{x_2-x_1}{x_3-x_1}\right)f\left(\frac{x_3}{m}\right)g\left(\frac{x_3}{m}\right),$$

which gives us for $x_1 = \mu$, $x_2 = \frac{x_i}{m}$ and $x_3 = M$ for $i = 1, \ldots, n$

$$f\left(\frac{x_i}{m}\right) \leq h\left(\frac{M - \frac{x_i}{m}}{M-\mu}\right) f(\mu)g(\mu) + mh\left(\frac{\frac{x_i}{m} - \mu}{M-\mu}\right) f\left(\frac{M}{m}\right) g\left(\frac{M}{m}\right).$$

Notice, since $m \leq 1$, then $\mu < \frac{x_i}{m}$, and by the assumption we have $\frac{x_i}{m} < M$. With this, h function can by applied.

If we multiply the above with

$$mh\left(\frac{p_i}{P_n}\right) g\left(\frac{x_i}{m}\right) G_i^{n-1},$$

then, after adding all inequalities, from Theorem 3.21, (3.51) follows. □

Remark 3.15 Corresponding conversions of Jensen's inequality for different types of convexity can be stated. However, the details are omitted.

If, in (3.42), we let $x_1 = \mu$, $x_2 = x_i$, $x_3 = M$ and if we multiply such inequality with p_i, then after adding all inequalities for $i = 1, \ldots, n$ we obtain the discrete Lah-Ribarič inequality for an $(h,g;m)$ convex function

$$\sum_{i=1}^{n} p_i f(x_i) \leq \frac{f(\mu)g(\mu)}{h(M-\mu)} \sum_{i=1}^{n} p_i h(M - x_i) + \frac{m}{h(M-\mu)} f\left(\frac{M}{m}\right) g\left(\frac{M}{m}\right) \sum_{i=1}^{n} p_i h(x_i - \mu).$$

Integral version of this inequality is given in Theorem 3.14.

3.6 Fractional inequalities of the Hermite-Hadamard type for $(h,g;m)$-convex functions

In recent years, in the field of applied sciences, fractional calculus has been used with different boundary conditions to develop mathematical models relating to real-world problems. This significant interest in the theory of fractional calculus has been stimulated by many of its applications, especially in the various fields of physics and engineering.

Inequalities involving integrals of functions and their derivatives are of great importance in mathematical analysis and its applications. Inequalities containing fractional derivatives have applications in regard to fractional differential equations, especially in establishing the uniqueness of the solutions of initial value problems and their upper bounds. This kind of application motivated the researchers towards the theory of integral inequalities, with the aim of extending and generalizing classical inequalities using different fractional integral operators.

The motivation for this research on Hermite-Hadamard-type integral inequalities was provided by recent studies on these inequalities for different types of integral operators (see [4, 5, 26, 28, 30, 33, 34, 37]) and different classes of convexity (see [2, 6, 12, 20, 23, 24, 27, 35, 36]).

We observe the Hermite-Hadamard inequality given in Theorem 3.1, and its fractional version, involving Riemann-Liouville fractional integrals, given in [32]:

Theorem 3.23 ([32]) *Let $f:[a,b]\to\mathbb{R}$ be a convex function with $f\in L_1[a,b]$. Then for $\sigma>0$*

$$f\left(\frac{a+b}{2}\right)\le\frac{\Gamma(\sigma+1)}{2(b-a)^{\sigma}}\left[J^{\sigma}_{a+}f(b)+J^{\sigma}_{b-}f(a)\right]\le\frac{f(a)+f(b)}{2}.$$

Recall that the left-sided and the right-sided Riemann-Liouville fractional integrals of order $\sigma>0$ are defined as in [18] for $f\in L_1[a,b]$ with

$$J^{\sigma}_{a+}f(x)=\frac{1}{\Gamma(\sigma)}\int_a^x(x-t)^{\sigma-1}f(t)\,dt,\quad x\in(a,b],\tag{3.52}$$

$$J^{\sigma}_{b-}f(x)=\frac{1}{\Gamma(\sigma)}\int_x^b(t-x)^{\sigma-1}f(t)\,dt,\quad x\in[a,b).\tag{3.53}$$

Our aim is to prove Hermite-Hadamard's inequality in more general settings, and for this we need an extended generalized Mittag-Leffler function with its fractional integral operators and a class of $(h,g;m)$-convex functions.

An extended generalized form of the Mittag-Leffler function

The Mittag-Leffler function

$$E_{\rho}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\rho n+1)}\qquad(z\in\mathbb{C},\ \Re(\rho)>0)$$

with its generalizations appears as a solution of fractional differential or integral equations. The first generalization for two parameters was carried out by Wiman [37]:

$$E_{\rho,\sigma}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\rho n+\sigma)},\quad z,\rho,\sigma\in\mathbb{C},\ \Re(\rho)>0,\tag{3.54}$$

after which Prabhakar defined the Mittag-Leffler function of three parameters [26]:

$$E^{\delta}_{\rho,\sigma}(z)=\sum_{n=0}^{\infty}\frac{(\delta)_n}{\Gamma(\rho n+\sigma)}\frac{z^n}{n!},\quad z,\rho,\sigma,\delta\in\mathbb{C},\ \Re(\rho)>0.\tag{3.55}$$

Recently we presented in [4] (see also [5]) an extended generalized form of the Mittag-Leffler function $E^{\delta,c,v,r}_{\rho,\sigma,\tau}(z;p)$:

Definition 3.3 ([4]) *Let $\rho,\sigma,\tau,\delta,c\in\mathbb{C}$, $\Re(\rho),\Re(\sigma),\Re(\tau)>0$, $\Re(c)>\Re(\delta)>0$ with $p\ge 0$, $r>0$ and $0<q\le r+\Re(\rho)$. Then the extended generalized Mittag-Leffler function $E^{\delta,c,q,r}_{\rho,\sigma,\tau}(z;p)$ is defined by*

$$E^{\delta,c,q,r}_{\rho,\sigma,\tau}(z;p)=\sum_{n=0}^{\infty}\frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}(\delta,c-\delta)}\frac{(c)_{nq}}{\Gamma(\rho n+\sigma)}\frac{z^n}{(\tau)_{nr}}.\tag{3.56}$$

Note, we use the generalized Pochhammer symbol $(c)_{nq} = \frac{\Gamma(c+nq)}{\Gamma(c)}$ and an extended beta function $B_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}e^{-\frac{p}{t(1-t)}}dt$, where $\Re(x), \Re(y), \Re(p) > 0$.

Remark 3.16 Several generalizations of the Mittag-Leffler function can be obtained for different parameter choices. For instance, the function (3.56) is reduced to

(i) the Salim-Faraj function $E^{\delta,\tau,q}_{\rho,\sigma,r}(z)$ for $p = 0$ [30],

(ii) the Rahman function $E^{\delta,q,c}_{\rho,\sigma}(z;p)$ for $\tau = r = 1$ [28],

(iii) the Shukla-Prajapati function $E^{\delta,q}_{\rho,\sigma}(z)$ for $p = 0$ and $\tau = r = 1$ [33],

(iv) the Prabhakar function $E^{\delta}_{\rho,\sigma}(z)$ for $p = 0$ and $\tau = r = q = 1$ [26],

(v) the Wiman function $E_{\rho,\sigma}(z)$ for $p = 0$ and $\tau = r = q = \delta = 1$ [37],

(vi) the Mittag-Leffler function $E_{\rho}(z)$ for $p = 0$, $\tau = r = q = \delta = 1$ and $\sigma = 1$.

Next we have corresponding fractional integral operators, the left-sided $\varepsilon^{\omega,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f$ and the right-sided $\varepsilon^{\omega,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f$, where the kernel is a function $E^{\delta,c,q,r}_{\rho,\sigma,\tau}(z;p)$:

Definition 3.4 ([4]) *Let $\omega,\rho,\sigma,\tau,\delta,c \in \mathbb{C}$, $\Re(\rho),\Re(\sigma),\Re(\tau) > 0$, $\Re(c) > \Re(\delta) > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \Re(\rho)$. Let $f \in L_1[a,b]$ and $x \in [a,b]$. Then the left-sided and the right-sided generalized fractional integral operators $\varepsilon^{\omega,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f$ and $\varepsilon^{\omega,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f$ are defined by*

$$\left(\varepsilon^{\omega,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p) = \int_a^x (x-t)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(\omega(x-t)^{\rho};p)f(t)dt, \tag{3.57}$$

$$\left(\varepsilon^{\omega,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(x;p) = \int_x^b (t-x)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(\omega(t-x)^{\rho};p)f(t)dt. \tag{3.58}$$

Remark 3.17 If we apply different parameter choices, then (3.57) is a generalization of

(i) the Salim-Faraj fractional integral operator $\varepsilon^{\omega,\delta,q,r}_{a^+,\rho,\sigma,\tau}f(x)$ for $p = 0$ [30],

(ii) the Rahman fractional integral operator $\varepsilon^{\omega,\delta,q,c}_{a^+,\rho,\sigma}f(x;p)$ for $\tau = r = 1$ [28],

(iii) the Srivastava-Tomovski fractional integral operator $\varepsilon^{\omega,\delta,q}_{a^+,\rho,\sigma}f(x)$ for $p = 0$ and $\tau = r = 1$ [34],

(iv) the Prabhakar fractional integral operator $\varepsilon(\rho,\sigma;\delta;\omega)f(x)$ for $p = 0$ and $\tau = r = q = 1$ [26],

(v) the left-sided Riemann-Liouville fractional integral $J^{\sigma}_{a^+}f(x)$ for $p = \omega = 0$, that is, (3.52).

We listed reductions for the left-sided fractional integral operator, whereas the analogs are valid for the right-sided.

More details on this generalized form of the Mittag-Leffler function and its fractional integral operators can be found in [4, 5]. Here are some results we will use in this study:

Theorem 3.24 ([4]) *If* $\alpha,\omega,\rho,\sigma,\tau,\delta,c\in\mathbb{C}$, $\Re(\rho),\Re(\sigma),\Re(\tau)>0$, $\Re(c)>\Re(\delta)>0$ *with* $p\geq 0$, $r>0$ *and* $0<q\leq r+\Re(\rho)$, *then for power functions* $(t-a)^{\alpha-1}$ *and* $(b-t)^{\alpha-1}$ *follow*

$$\left(\varepsilon^{\omega,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}(t-a)^{\alpha-1}\right)(x;p)=\Gamma(\alpha)(x-a)^{\alpha+\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma+\alpha,\tau}(\omega(x-a)^{\rho};p), \tag{3.59}$$

$$\left(\varepsilon^{\omega,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}(b-t)^{\alpha-1}\right)(x;p)=\Gamma(\alpha)(b-x)^{\alpha+\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma+\alpha,\tau}(\omega(b-x)^{\rho};p). \tag{3.60}$$

If we set $a=0$ and $x=1$ in (3.59), or $b=1$ and $x=0$ in (3.60), then we obtain the following corollary.

Corollary 3.16 ([4]) *If* $\alpha,\omega,\rho,\sigma,\tau,\delta,c\in\mathbb{C}$, $\Re(\rho),\Re(\sigma),\Re(\tau)>0$, $\Re(c)>\Re(\delta)>0$ *with* $p\geq 0$, $r>0$ *and* $0<q\leq r+\Re(\rho)$, *then*

$$\frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(\omega(1-t)^{\rho};p)dt=E^{\delta,c,q,r}_{\rho,\sigma+\alpha,\tau}(\omega;p).$$

Setting $\alpha=1$ in theorem 3.24, we obtain following identities for the constant function:

Corollary 3.17 ([5]) *Let the assumptions of Theorem 3.24 hold with* $\alpha=1$. *Then*

$$\left(\varepsilon^{\omega,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}1\right)(x;p)=(x-a)^{\sigma}E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(\omega(x-a)^{\rho};p), \tag{3.61}$$

$$\left(\varepsilon^{\omega,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}1\right)(x;p)=(b-x)^{\sigma}E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(\omega(b-x)^{\rho};p). \tag{3.62}$$

Here we will use simplified notation to avoid a complicated manuscript form:

$$\boldsymbol{E}(z;p)\ :=\ E^{\delta,c,q,r}_{\rho,\sigma,\tau}(z;p)$$

and

$$(\boldsymbol{\varepsilon}^{\omega}_{a^+}f)(x;p)\ :=\ \left(\varepsilon^{\omega,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p),$$

$$(\boldsymbol{\varepsilon}^{\omega}_{b^-}f)(x;p)\ :=\ \left(\varepsilon^{\omega,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(x;p).$$

Of course, the conditions on all parameters $\rho,\sigma,\tau,\omega,\delta,c,q,r$ are essential and will be added to all theorems.

Another direction for the generalization of the Hermite-Hadamard inequality is the use of different classes of convexity. For this we need a class of $(h,g;m)$-convex functions.

3.6.1 Fractional Hermite-Hadamard inequalities

Hermite-Hadamard type inequalities for $(h,g;m)$-convex functions are obtained in Section 3.2, where some special results are pointed out and several known inequalities are improved upon. Here we will obtain their fractional generalizations, using (3.56)–(3.58), that is, the extended generalized Mittag-Leffler function $\boldsymbol{E}$ with fractional integral operators $\boldsymbol{\varepsilon}^{\omega}_{a+}f$ and $\boldsymbol{\varepsilon}^{\omega}_{b-}f$ in the real domain.

In this section, it is necessary to introduce the following conditions on the parameters and the interval $[a,b]$:

Assumption 3.1 *Let* $\omega \in \mathbb{R}$, $\rho,\sigma,\tau > 0$, $c > \delta > 0$ *with* $p \geq 0$ *and* $0 < q \leq r+\rho$. *Furthermore, let* $0 \leq a < b < \infty$.

We start with the left side, i.e., the first Hermite-Hadamard fractional integral inequality for $(h,g;m)$-convex functions involving the extended generalized Mittag-Leffler function.

Theorem 3.25 *Let Assumption 3.1 hold. Let f be a nonnegative $(h,g;m)$-convex function on $[0,\infty)$, where h is a nonnegative function on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$, g is a positive function on $[0,\infty)$ and $m \in (0,1]$. If $f,g \in L_1[a,\frac{b}{m}]$, then the following inequality holds*

$$f\left(\frac{a+b}{2}\right)(\boldsymbol{\varepsilon}^{\overline{\omega}}_{a+}1)(b;p) \leq h\left(\frac{1}{2}\right)\left[(\boldsymbol{\varepsilon}^{\overline{\omega}}_{a+}fg)(b;p)+m^{\sigma+1}\left(\boldsymbol{\varepsilon}^{\overline{\overline{\omega}}}_{\frac{b}{m}-}fg\right)\left(\frac{a}{m};p\right)\right], \quad (3.63)$$

where

$$\overline{\omega} = \frac{\omega}{(b-a)^{\rho}}, \quad \overline{\overline{\omega}} = \frac{m^{\rho}\omega}{(b-a)^{\rho}}. \quad (3.64)$$

Proof. Let f be an $(h,g;m)$-convex function on $[0,\infty)$, $m \in (0,1]$. Then for $t=\frac{1}{2}$ we have

$$f\left(\frac{x+my}{2}\right) \leq h\left(\frac{1}{2}\right)f(x)g(x)+mh\left(\frac{1}{2}\right)f(y)g(y).$$

Choosing $y \equiv \frac{y}{m}$ we obtain

$$f\left(\frac{x+y}{2}\right) \leq h\left(\frac{1}{2}\right)\left[f(x)g(x)+mf\left(\frac{y}{m}\right)g\left(\frac{y}{m}\right)\right].$$

Let $x = ta+(1-t)b$ and $y=(1-t)a+tb$. Then

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq h\left(\frac{1}{2}\right)\Big[f(ta+(1-t)b)g(ta+(1-t)b) \\ &\quad +mf\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)g\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)\Big]. \end{aligned}$$

In the following step we will need to multiply both sides of the above inequality by $t^{\sigma-1}\boldsymbol{E}(\omega t^{\rho};p)$ and integrate on $[0,1]$ with respect to the variable t, which gives us

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)\int_0^1 t^{\sigma-1}\boldsymbol{E}(\omega t^{\rho};p)dt\\
&\quad\le h\left(\frac{1}{2}\right)\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)t^{\sigma-1}\boldsymbol{E}(\omega t^{\rho};p)dt\\
&\qquad+mh\left(\frac{1}{2}\right)\int_0^1 f\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)g\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)t^{\sigma-1}\boldsymbol{E}(\omega t^{\rho};p)dt.
\end{aligned}$$

With substitutions $u=ta+(1-t)b$ and $v=(1-t)\frac{a}{m}+t\frac{b}{m}$ we obtain

$$\begin{aligned}
&\frac{1}{(b-a)^{\sigma}}f\left(\frac{a+b}{2}\right)\int_a^b (b-u)^{\sigma-1}\boldsymbol{E}(\overline{\omega}(b-u)^{\rho};p)du\\
&\quad\le\frac{h\left(\frac{1}{2}\right)}{(b-a)^{\sigma}}\int_a^b f(u)g(u)(b-u)^{\sigma-1}\boldsymbol{E}(\overline{\omega}(b-u)^{\rho};p)du\\
&\qquad+\frac{m^{\sigma+1}h\left(\frac{1}{2}\right)}{(b-a)^{\sigma}}\int_{\frac{a}{m}}^{\frac{b}{m}} f(v)g(v)\left(v-\frac{a}{m}\right)^{\sigma-1}\boldsymbol{E}\left(\overline{\overline{\omega}}\left(v-\frac{a}{m}\right)^{\rho};p\right)dv.
\end{aligned}$$

Since $m\in(0,1]$, then $a\le a/m$, $b\le b/m$ and $[a,b]\subset[a,\frac{b}{m}]$. Therefore, the condition $f,g\in L_1[a,\frac{b}{m}]$ is stated in this theorem. The above inequality can be written as

$$\begin{aligned}
&\frac{1}{(b-a)^{\sigma}}f\left(\frac{a+b}{2}\right)(\boldsymbol{\varepsilon}_{a+}^{\overline{\omega}}1)(b;p)\\
&\quad\le\frac{h\left(\frac{1}{2}\right)}{(b-a)^{\sigma}}\left[(\boldsymbol{\varepsilon}_{a+}^{\overline{\omega}}fg)(b;p)+m^{\sigma+1}\left(\boldsymbol{\varepsilon}_{\frac{b}{m}-}^{\overline{\overline{\omega}}}fg\right)\left(\frac{a}{m};p\right)\right].
\end{aligned}$$

Note that with Corollary 3.17 we can obtain the constant $(\boldsymbol{\varepsilon}_{a+}^{\overline{\omega}}1)(b;p)$. This completes the proof. □

Next we have the second Hermite-Hadamard fractional integral inequality.

Theorem 3.26 *Let the assumptions of Theorem 3.25 hold with $f,g,h\in L_1[a,\frac{b}{m}]$. Then*

$$\begin{aligned}
&(\boldsymbol{\varepsilon}_{a+}^{\overline{\omega}}fg)(b;p)+m^{\sigma+1}\left(\boldsymbol{\varepsilon}_{\frac{b}{m}-}^{\overline{\overline{\omega}}}fg\right)\left(\frac{a}{m};p\right)\\
&\quad\le f(a)g(a)\int_a^b h\left(\frac{b-x}{b-a}\right)g(x)(b-x)^{\sigma-1}\boldsymbol{E}(\overline{\omega}(b-x)^{\rho};p)dx\\
&\qquad+mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\int_a^b h\left(\frac{x-a}{b-a}\right)g(x)(b-x)^{\sigma-1}\boldsymbol{E}(\overline{\omega}(b-x)^{\rho};p)dx\\
&\qquad+m^{\sigma+1}f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\int_{\frac{a}{m}}^{\frac{b}{m}} h\left(\frac{b-mx}{b-a}\right)g(x)\left(x-\frac{a}{m}\right)^{\sigma-1}\boldsymbol{E}\left(\overline{\overline{\omega}}\left(x-\frac{a}{m}\right)^{\rho};p\right)dx\\
&\qquad+m^{\sigma+2}f\left(\frac{b}{m^2}\right)g\left(\frac{b}{m^2}\right)\int_{\frac{a}{m}}^{\frac{b}{m}} h\left(\frac{mx-a}{b-a}\right)g(x)\left(x-\frac{a}{m}\right)^{\sigma-1}\boldsymbol{E}\left(\overline{\overline{\omega}}\left(x-\frac{a}{m}\right)^{\rho};p\right)dx,
\end{aligned}\tag{3.65}$$

where $\overline{\varpi}$ *and* $\overline{\overline{\varpi}}$ *are defined by* (3.64).

Proof. Due to the $(h,g;m)$-convexity of f we have

$$f(ta+(1-t)b) \leq h(t)f(a)g(a)+mh(1-t)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right).$$

Multiplying both sides of above inequality by $g(ta+(1-t)b)t^{\sigma-1}\boldsymbol{E}(\omega t^{\rho};p)$ and integrating on $[0,1]$ with respect to the variable t, we obtain

$$\begin{aligned}
&\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)t^{\sigma-1}\boldsymbol{E}(\omega t^{\rho};p)dt \\
&\leq f(a)g(a)\int_0^1 h(t)\,g(ta+(1-t)b)t^{\sigma-1}\boldsymbol{E}(\omega t^{\rho};p)dt \\
&\quad +mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\int_0^1 h(1-t)\,g(ta+(1-t)b)t^{\sigma-1}\boldsymbol{E}(\omega t^{\rho};p)dt.
\end{aligned}$$

With the substitution $u = ta+(1-t)b$ we obtain

$$\begin{aligned}
&\frac{1}{(b-a)^{\sigma}}\int_a^b f(u)\,g(u)\,(b-u)^{\sigma-1}\boldsymbol{E}(\overline{\varpi}(b-u)^{\rho};p)du \\
&\leq \frac{f(a)\,g(a)}{(b-a)^{\sigma}}\int_a^b h\left(\frac{b-u}{b-a}\right)g(u)(b-u)^{\sigma-1}\boldsymbol{E}(\overline{\varpi}(b-u)^{\rho};p)du \\
&\quad +\frac{mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)}{(b-a)^{\sigma}}\int_a^b h\left(\frac{u-a}{b-a}\right)g(u)(b-u)^{\sigma-1}\boldsymbol{E}(\overline{\varpi}(b-u)^{\rho};p)du,
\end{aligned}$$

that is

$$\begin{aligned}
&(\boldsymbol{\varepsilon}_{a^+}^{\overline{\varpi}}fg)(b;p) \\
&\leq f(a)\,g(a)\int_a^b h\left(\frac{b-u}{b-a}\right)g(u)(b-u)^{\sigma-1}\boldsymbol{E}(\overline{\varpi}(b-u)^{\rho};p)du \\
&\quad +mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\int_a^b h\left(\frac{u-a}{b-a}\right)g(u)(b-u)^{\sigma-1}\boldsymbol{E}(\overline{\varpi}(b-u)^{\rho};p)du.
\end{aligned} \tag{3.66}$$

Again, due to the $(h,g;m)$-convexity of f we have

$$f\left((1-t)\frac{a}{m}+t\frac{b}{m}\right) \leq h(1-t)f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)+mh(t)f\left(\frac{b}{m^2}\right)g\left(\frac{b}{m^2}\right).$$

Multiplying both sides of above inequality by $g\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)t^{\sigma-1}\boldsymbol{E}(\omega t^{\rho};p)$ and integrating on $[0,1]$ with respect to the variable t, we obtain

$$\begin{aligned}
&\int_0^1 f\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)g\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)t^{\sigma-1}\boldsymbol{E}(\omega t^{\rho};p)dt \\
&\leq f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\int_0^1 h(1-t)\,g\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)t^{\sigma-1}\boldsymbol{E}(\omega t^{\rho};p)dt \\
&\quad +mf\left(\frac{b}{m^2}\right)g\left(\frac{b}{m^2}\right)\int_0^1 h(t)\,g\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)t^{\sigma-1}\boldsymbol{E}(\omega t^{\rho};p)dt.
\end{aligned}$$

With the substitution $v = (1-t)\frac{a}{m} + t\frac{b}{m}$ we obtain

$$\begin{aligned}
&\frac{m^{\sigma}}{(b-a)^{\sigma}} \int_{\frac{a}{m}}^{\frac{b}{m}} f(v)g(v)\left(v-\frac{a}{m}\right)^{\sigma-1} \boldsymbol{E}\left(\overline{\overline{\varpi}}\left(v-\frac{a}{m}\right)^{\rho};p\right)dv \\
&\le \frac{m^{\sigma} f\left(\frac{a}{m}\right) g\left(\frac{a}{m}\right)}{(b-a)^{\sigma}} \int_{\frac{a}{m}}^{\frac{b}{m}} h\left(\frac{b-mv}{b-a}\right) g(v)\left(v-\frac{a}{m}\right)^{\sigma-1} \boldsymbol{E}\left(\overline{\overline{\varpi}}\left(v-\frac{a}{m}\right)^{\rho};p\right)dv \\
&\quad + \frac{m^{\sigma+1} f\left(\frac{b}{m^2}\right) g\left(\frac{b}{m^2}\right)}{(b-a)^{\sigma}} \int_{\frac{a}{m}}^{\frac{b}{m}} h\left(\frac{vm-a}{b-a}\right) g(v)\left(v-\frac{a}{m}\right)^{\sigma-1} \boldsymbol{E}\left(\overline{\overline{\varpi}}\left(v-\frac{a}{m}\right)^{\rho};p\right)dv,
\end{aligned}$$

that is

$$\begin{aligned}
&\left(\boldsymbol{\varepsilon}^{\overline{\overline{\varpi}}}_{\frac{b}{m}-} fg\right)\left(\frac{a}{m};p\right) \\
&\le f\left(\frac{a}{m}\right) g\left(\frac{a}{m}\right) \int_{\frac{a}{m}}^{\frac{b}{m}} h\left(\frac{b-mv}{b-a}\right) g(v)\left(v-\frac{a}{m}\right)^{\sigma-1} \boldsymbol{E}\left(\overline{\overline{\varpi}}\left(v-\frac{a}{m}\right)^{\rho};p\right)dv \\
&\quad + m f\left(\frac{b}{m^2}\right) g\left(\frac{b}{m^2}\right) \int_{\frac{a}{m}}^{\frac{b}{m}} h\left(\frac{vm-a}{b-a}\right) g(v)\left(v-\frac{a}{m}\right)^{\sigma-1} \boldsymbol{E}\left(\overline{\overline{\varpi}}\left(v-\frac{a}{m}\right)^{\rho};p\right)dv.
\end{aligned} \tag{3.67}$$

Inequality (3.65) now follows from (3.66) and (3.67). □

In the following we derive fractional integral inequalities of Hermite-Hadamard type for different types of convexity, and state several corollaries, using special functions for h and/or g, and the parameter m. The first consequence of Theorems 3.25 and 3.26 obtained via the setting $g \equiv 1$ (i.e., $g(x) = 1$) is the Hermite-Hadamard fractional integral inequality for $(h-m)$-convex functions given in ([16], Theorem 2.1):

Corollary 3.18 *Let Assumption 3.1 hold. Let f be a nonnegative $(h-m)$-convex function on $[0,\infty)$ where h is a nonnegative function on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$ and $m \in (0,1]$. If $f \in L_1[a,\frac{b}{m}]$ and $h \in L_1[0,1]$, then the following inequalities hold*

$$\begin{aligned}
f\left(\frac{a+b}{2}\right)(\boldsymbol{\varepsilon}^{\overline{\varpi}}_{a+}1)(b;p) &\le h\left(\frac{1}{2}\right)\left[(\boldsymbol{\varepsilon}^{\overline{\varpi}}_{a+}f)(b;p) + m^{\sigma+1}\left(\boldsymbol{\varepsilon}^{\overline{\overline{\varpi}}}_{\frac{b}{m}-} f\right)\left(\frac{a}{m};p\right)\right] \\
&\le h\left(\frac{1}{2}\right)(b-a)^{\sigma}\left\{\left[f(a) + m^2 f\left(\frac{b}{m^2}\right)\right](\boldsymbol{\varepsilon}^{\omega}_{1-}h)(0;p)\right. \\
&\quad \left. + \left[mf\left(\frac{a}{m}\right) + mf\left(\frac{b}{m}\right)\right](\boldsymbol{\varepsilon}^{\omega}_{0+}h)(1;p)\right\},
\end{aligned} \tag{3.68}$$

where $\overline{\varpi}$ and $\overline{\overline{\varpi}}$ are defined by (3.64).

Proof. First we use substitutions $t = \frac{b-x}{b-a}$ and $z = \frac{mx-a}{b-a}$ in Theorem 3.26, after which we apply identities

$$\int_0^1 h(t)t^{\sigma-1}\boldsymbol{E}(\omega t^{\rho};p)dt = (\boldsymbol{\varepsilon}^{\omega}_{1-}h)(0;p) \tag{3.69}$$

and

$$\int_0^1 h(1-t)t^{\sigma-1}\boldsymbol{E}(\omega t^\rho;p)dt$$
$$= \int_0^1 h(z)(1-z)^{\sigma-1}\boldsymbol{E}(\omega(1-z)^\rho;p)dz = (\boldsymbol{\varepsilon}_{0+}^{\omega}h)(1;p). \tag{3.70}$$

The result now follows from the above and Theorem 3.25. □

By setting the function $g \equiv 1$ and the parameter $m = 1$, the previous result is reduced to the Hermite-Hadamard fractional integral inequality for h-convex functions:

Corollary 3.19 *Let Assumption 3.1 hold. Let f be a nonnegative h-convex function on $[0,\infty)$ where h is a nonnegative function on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$. If $f \in L_1[a,\frac{b}{m}]$ and $h \in L_1[0,1]$, then the following inequalities hold*

$$f\left(\frac{a+b}{2}\right)(\boldsymbol{\varepsilon}_{a+}^{\overline{\omega}}1)(b;p)$$
$$\le h\left(\frac{1}{2}\right)\left[(\boldsymbol{\varepsilon}_{a+}^{\overline{\omega}}f)(b;p) + (\boldsymbol{\varepsilon}_{b-}^{\overline{\omega}}f)(a;p)\right]$$
$$\le h\left(\frac{1}{2}\right)(b-a)^\sigma[f(a)+f(b)]\left[(\boldsymbol{\varepsilon}_{1-}^{\omega}h)(0;p) + (\boldsymbol{\varepsilon}_{0+}^{\omega}h)(1;p)\right], \tag{3.71}$$

where $\overline{\omega}$ is defined by (3.64).

In the following, we set the function $h \equiv \mathrm{id}$, the identity function. With $g \equiv 1$ we obtain the Hermite-Hadamard fractional integral inequality for m-convex functions from ([17], Theorem 3.1):

Corollary 3.20 *Let Assumption 3.1 hold. Let f be a nonnegative m-convex function on $[0,\infty)$ with $m \in (0,1]$. If $f \in L_1[a,\frac{b}{m}]$, then the following inequalities hold*

$$f\left(\frac{a+b}{2}\right)(\boldsymbol{\varepsilon}_{a+}^{\overline{\omega}}1)(b;p) \le \frac{1}{2}\left[(\boldsymbol{\varepsilon}_{a+}^{\overline{\omega}}f)(b;p) + m^{\sigma+1}\left(\boldsymbol{\varepsilon}_{\frac{b}{m}-}^{\overline{\overline{\omega}}}f\right)\left(\frac{a}{m};p\right)\right]$$
$$\le \frac{(b-a)^\sigma}{2}\left\{\left[f(a) + m^2 f\left(\frac{b}{m^2}\right)\right](\boldsymbol{\varepsilon}_{1-}^{\omega}\mathrm{id})(0;p)\right.$$
$$\left. + \left[mf\left(\frac{a}{m}\right) + mf\left(\frac{b}{m}\right)\right](\boldsymbol{\varepsilon}_{0+}^{\omega}\mathrm{id})(1;p)\right\}, \tag{3.72}$$

where $\overline{\omega}$ and $\overline{\overline{\omega}}$ are defined by (3.64).

The Hermite-Hadamard fractional integral inequality for convex functions is given in ([17], Theorem 2.1). Here it is a merely a consequence for $h \equiv \mathrm{id}$, $g \equiv 1$ and $m = 1$:

Corollary 3.21 *Let Assumption 3.1 hold. Let f be a nonnegative convex function on $[0,\infty)$. If $f \in L_1[a,b]$, then the following inequalities hold*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)(\boldsymbol{\varepsilon}_{a+}^{\overline{\omega}}1)(b;p) &\leq \frac{1}{2}\left[(\boldsymbol{\varepsilon}_{a+}^{\overline{\omega}}f)(b;p)+(\boldsymbol{\varepsilon}_{b-}^{\overline{\omega}}f)(a;p)\right] \\ &\leq \frac{f(a)+f(b)}{2}(\boldsymbol{\varepsilon}_{a+}^{\overline{\omega}}1)(b;p), \end{aligned} \tag{3.73}$$

where $\overline{\omega}$ is defined by (3.64)*.*

Proof. Here we use

$$\begin{aligned} &(\boldsymbol{\varepsilon}_{0+}^{\omega}\,\mathrm{id})(1;p)+(\boldsymbol{\varepsilon}_{1-}^{\omega}\,\mathrm{id})(0;p) \\ &\quad= \int_0^1 t^{\sigma}\boldsymbol{E}(\omega t^{\rho};p)dt + \int_0^1 t(1-t)^{\sigma-1}\boldsymbol{E}(\omega(1-t)^{\rho};p)dt \\ &\quad= \int_0^1 (1-t)^{\sigma}\boldsymbol{E}(\omega(1-t)^{\rho};p)dt + \int_0^1 t(1-t)^{\sigma-1}\boldsymbol{E}(\omega(1-t)^{\rho};p)dt \\ &\quad= \int_0^1 (1-t)^{\sigma-1}\boldsymbol{E}(\omega(1-t)^{\rho};p)dt \\ &\quad= (\boldsymbol{\varepsilon}_{0+}^{\omega}1)(1;p) \\ &\quad= \frac{1}{(b-a)^{\sigma}}(\boldsymbol{\varepsilon}_{a+}^{\overline{\omega}}1)(b;p). \end{aligned}$$

□

We have presented several Hermite-Hadamard-type inequalities for the $(h,g;m)$-convex function using fractional integral operators, where the kernel is an extended generalized Mittag-Leffler function. If we apply different parameter choices, as in Remark 3.17, then we obtain corresponding inequalities for different fractional operators.

Several properties of fractional integral operators $\boldsymbol{\varepsilon}_{a+}^{\omega}f$ and $\boldsymbol{\varepsilon}_{b-}^{\omega}f$

At the end of this section we give several results for fractional integral operators.

Proposition 3.7 *Let $\omega,\rho,\sigma,\tau,\delta,c \in \mathbb{C}$, $\Re(\rho),\Re(\sigma),\Re(\tau)>0$, $\Re(c)>\Re(\delta)>0$ with $p\geq 0$, $r>0$ and $0<q\leq r+\Re(\rho)$.*

(i) If the function $f \in L_1[a,b]$ is symmetric about $\frac{a+b}{2}$, then

$$(\boldsymbol{\varepsilon}_{a+}^{\omega}f)(b;p) = (\boldsymbol{\varepsilon}_{b-}^{\omega}f)(a;p). \tag{3.74}$$

In particular,

$$(\boldsymbol{\varepsilon}_{a+}^{\omega}1)(b;p) = (\boldsymbol{\varepsilon}_{b-}^{\omega}1)(a;p). \tag{3.75}$$

(ii) Furthermore,

$$\left(\boldsymbol{\varepsilon}_{a+}^{\omega}(t-a)^{\alpha-1}\right)(b;p) = (\boldsymbol{\varepsilon}_{b-}^{\omega}(b-t)^{\alpha-1})(a;p), \tag{3.76}$$

$$\left(\boldsymbol{\varepsilon}_{a+}^{\omega}(b-t)^{\alpha-1}\right)(b;p) = (\boldsymbol{\varepsilon}_{b-}^{\omega}(t-a)^{\alpha-1})(a;p). \tag{3.77}$$

In particular,

$$\left(\boldsymbol{\varepsilon}_{0+}^{\omega}t^{\alpha-1}\right)(1;p) = (\boldsymbol{\varepsilon}_{1-}^{\omega}(1-t)^{\alpha-1})(0;p), \tag{3.78}$$

$$\left(\boldsymbol{\varepsilon}_{0+}^{\omega}(1-t)^{\alpha-1}\right)(1;p) = (\boldsymbol{\varepsilon}_{1-}^{\omega}t^{\alpha-1})(0;p). \tag{3.79}$$

Proof. (i) If the function f is symmetric about $\frac{a+b}{2}$, i.e., $f(t) = f(a+b-t)$ for all $t \in [a,b]$, then, substituting $z = a+b-t$, (3.74) easily follows:

$$\begin{aligned}
(\boldsymbol{\varepsilon}_{a+}^{\omega}f)(b;p) &= \int_a^b (b-t)^{\sigma-1}\boldsymbol{E}(\omega(b-t)^{\rho};p)f(t)dt \\
&= \int_a^b (z-a)^{\sigma-1}\boldsymbol{E}(\omega(z-a)^{\rho};p)f(a+b-z)dz \\
&= \int_a^b (z-a)^{\sigma-1}\boldsymbol{E}(\omega(z-a)^{\rho};p)f(z)dz \\
&= (\boldsymbol{\varepsilon}_{b-}^{\omega}f)(a;p).
\end{aligned}$$

Note that (3.75) also follows directly from Corollary 3.17 if we set $x = b$ in (3.61) and $x = a$ in (3.62).

(ii) Equations (3.76) and (3.77) follow with the substitution $z = a+b-t$. Furthermore, (3.76) follows directly from Theorem 3.24 if we set $x = b$ in (3.59) and $x = a$ in (3.60). The final two equations are obtained for $a = 0$ and $b = 1$. □

Remark 3.18 To obtain the Hermite-Hadamard inequality for convex functions involving Riemann-Liouville fractional integrals, given in Theorem 3.23, first we need to set $p = \omega = 0$ in (3.56)

$$\boldsymbol{E}(z;0) = \sum_{n=0}^{\infty} \frac{(\delta)_{nq}}{\Gamma(\rho n+\sigma)} \frac{z^n}{(\tau)_{nr}}.$$

Since $\boldsymbol{E}(0;0) = E_{\rho,\sigma,\tau}^{\delta,c,q,r}(0;0) = \frac{1}{\Gamma(\sigma)}$, setting $p = \omega = 0$ in (3.57) we obtain Riemann-Liouville fractional integrals

$$\begin{aligned}
(\boldsymbol{\varepsilon}_{a+}^{0}f)(x;0) &= \frac{1}{\Gamma(\sigma)}\int_a^x (x-t)^{\sigma-1}f(t)\,dt = J_{a+}^{\sigma}f(x), \\
(\boldsymbol{\varepsilon}_{b-}^{0}f)(x;0) &= \frac{1}{\Gamma(\sigma)}\int_x^b (t-x)^{\sigma-1}f(t)\,dt = J_{b-}^{\sigma}f(x).
\end{aligned}$$

Note that a direct consequence of Theorem 3.24 is

$$\left(\boldsymbol{\varepsilon}_{0+}^{\omega}\ \mathrm{id}\right)(1;p) = E_{\rho,\sigma+2,\tau}^{\delta,c,q,r}(\omega;p). \tag{3.80}$$

For the reader's convenience, we will directly prove this:

$$\begin{aligned}
(\boldsymbol{\varepsilon}_{0+}^{\omega}\ \mathrm{id})(1;p) &= \int_0^1 t(1-t)^{\sigma-1}\boldsymbol{E}(\omega(1-t)^{\rho};p)dt \\
&= \int_0^1 t(1-t)^{\sigma-1}\sum_{n=0}^{\infty}\frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}(\delta,c-\delta)}\frac{(c)_{nq}}{\Gamma(\rho n+\sigma)}\frac{\omega^n(1-t)^{n\rho}}{(\tau)_{nr}}dt
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}(\delta,c-\delta)} \frac{(c)_{nq}}{\Gamma(\rho n+\sigma)} \frac{\omega^n}{(\tau)_{nr}} \int_0^1 t(1-t)^{n\rho+\sigma-1}dt$$
$$= \sum_{n=0}^{\infty} \frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}(\delta,c-\delta)} \frac{(c)_{nq}}{\Gamma(\rho n+\sigma)} \frac{\omega^n}{(\tau)_{nr}} \mathrm{B}(2,n\rho+\sigma)$$
$$= \sum_{n=0}^{\infty} \frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}(\delta,c-\delta)} \frac{(c)_{nq}}{\Gamma(\rho n+\sigma)} \frac{\omega^n}{(\tau)_{nr}} \frac{\Gamma(2)\Gamma(n\rho+\sigma)}{\Gamma(2+n\rho+\sigma)}$$
$$= \sum_{n=0}^{\infty} \frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}(\delta,c-\delta)} \frac{(c)_{nq}}{\Gamma(\rho n+(\sigma+2))} \frac{\omega^n}{(\tau)_{nr}}$$
$$= E^{\delta,c,q,r}_{\rho,\sigma+2,\tau}(\omega;p).$$

Hence,

$$\left(\boldsymbol{\varepsilon}^0_{0+}\ \mathrm{id}\right)(1;0) = \frac{1}{\Gamma(\sigma+2)}$$

and

$$(\boldsymbol{\varepsilon}^0_{1-}\ \mathrm{id})(0;0) = \int_0^1 t^{\sigma}\boldsymbol{E}(0;p)dt = \frac{1}{(\sigma+1)\Gamma(\sigma)}, \tag{3.81}$$

from which follows

$$\left(\boldsymbol{\varepsilon}^0_{0+}\ \mathrm{id}\right)(1;0) + (\boldsymbol{\varepsilon}^0_{1-}\ \mathrm{id})(0;0) = \frac{1}{\Gamma(\sigma+1)}.$$

Finally, if we set $h(x)=x$, $g\equiv 1$, $m=1$ and $p=\omega=0$, then Theorems 3.25 and 3.26 are reduced to Theorem 3.23.

3.6.2 Applications: Bounds of fractional integral operators for $(h,g;m)$-convex functions

As an application, we obtain the upper bounds of fractional integral operators for $(h,g;m)$-convex functions.

Assumption 3.2 *Let $\omega\in\mathbb{R}$, $\rho,\sigma,\tau>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let f be a nonnegative $(h,g;m)$-convex function on $[0,\infty)$ where h is a nonnegative function on $J\subseteq\mathbb{R}$, $(0,1)\subseteq J$, $h\not\equiv 0$, g is a positive function on $[0,\infty)$, and $m\in(0,1]$. Furthermore, let $0\leq a<b<\infty$.*

Theorem 3.27 *Let Assumption 3.2 hold. If $f,g\in L_1[a,b]$ and $h\in L_1[0,1]$, then for $x\in[a,b]$ the following inequality holds*

$$\frac{1}{(x-a)^{\sigma}}(\boldsymbol{\varepsilon}^{\omega_a}_{a+}f)(x;p)$$
$$\leq f(a)\,g(a)(\boldsymbol{\varepsilon}^{\omega}_{1-}h)(0;p) + mf\left(\frac{x}{m}\right)g\left(\frac{x}{m}\right)(\boldsymbol{\varepsilon}^{\omega}_{0+}h)(1;p). \tag{3.82}$$

where

$$\omega_a = \frac{\omega}{(x-a)^{\rho}}. \tag{3.83}$$

Proof. Let f be an $(h,g;m)$-convex function on $[0,\infty)$, $x \in [a,b]$, $m \in (0,1]$ and $t \in (0,1)$. Then, similarly to Theorem 3.26, we use

$$f(ta+(1-t)x) \leq h(t)f(a)g(a)+mh(1-t)f\left(\frac{x}{m}\right)g\left(\frac{x}{m}\right).$$

Multiplying both sides of the above inequality by $t^{\sigma-1}\boldsymbol{E}(\omega t^{\rho};p)$ and integrating on $[0,1]$ with respect to the variable t, we obtain

$$\begin{aligned}
&\int_0^1 f(ta+(1-t)x)t^{\sigma-1}\boldsymbol{E}(\omega t^{\rho};p)dt \\
&\quad \leq f(a)g(a)\int_0^1 h(t)t^{\sigma-1}\boldsymbol{E}(\omega t^{\rho};p)dt \\
&\quad\quad +mf\left(\frac{x}{m}\right)g\left(\frac{x}{m}\right)\int_0^1 h(1-t)t^{\sigma-1}\boldsymbol{E}(\omega t^{\rho};p)dt.
\end{aligned}$$

With the substitution $u=ta+(1-t)x$ and identities (3.69), (3.70), we obtain the inequality (3.82). □

Theorem 3.28 *Let Assumption 3.2 hold. If $f,g \in L_1[a,b]$ and $h \in L_1[0,1]$, then for $x \in [a,b]$ the following inequality holds*

$$\begin{aligned}
&\frac{1}{(b-x)^{\sigma}}(\boldsymbol{\varepsilon}_{b-}^{\omega_b}f)(x;p) \\
&\quad \leq f(b)g(b)(\boldsymbol{\varepsilon}_{1-}^{\omega}h)(0;p)+mf\left(\frac{x}{m}\right)g\left(\frac{x}{m}\right)(\boldsymbol{\varepsilon}_{0+}^{\omega}h)(1;p),
\end{aligned}\tag{3.84}$$

where

$$\omega_b=\frac{\omega}{(b-x)^{\rho}}.\tag{3.85}$$

Proof. Using

$$f(tb+(1-t)x) \leq h(t)f(b)g(b)+mh(1-t)f\left(\frac{x}{m}\right)g\left(\frac{x}{m}\right),$$

the proof follows analogously to that of Theorem 3.27. □

From the two previous theorems we can directly obtain the following result.

Corollary 3.22 *Let Assumption 3.2 hold. If $f,g \in L_1[a,b]$ and $h \in L_1[0,1]$, then for $x \in [a,b]$ the following inequality holds*

$$\begin{aligned}
&\frac{1}{(x-a)^{\sigma}}(\boldsymbol{\varepsilon}_{a+}^{\omega_a}f)(x;p)+\frac{1}{(b-x)^{\sigma}}(\boldsymbol{\varepsilon}_{b-}^{\omega_b}f)(x;p) \\
&\quad \leq [f(a)g(a)+f(b)g(b)](\boldsymbol{\varepsilon}_{1-}^{\omega}h)(0;p) \\
&\quad\quad +2mf\left(\frac{x}{m}\right)g\left(\frac{x}{m}\right)(\boldsymbol{\varepsilon}_{0+}^{\omega}h)(1;p).
\end{aligned}\tag{3.86}$$

where ω_a and ω_b are defined by (3.83) *and* (3.85).

If we set $x=b$ in Theorem 3.27 and $x=a$ in Theorem 3.28, then we obtain the next fractional integral inequality of the Hermite-Hadamard type.

Theorem 3.29 *Let Assumption 3.2 hold. If $f,g,h \in L_1[a,b]$, then the following inequalities hold*

$$\begin{aligned}\frac{1}{(b-a)^{\sigma}} &\left[(\boldsymbol{\varepsilon}_{a+}^{\overline{\omega}} f)(b;p) + (\boldsymbol{\varepsilon}_{b-}^{\overline{\omega}} f)(a;p)\right] \\ &\le [f(a)g(a)+f(b)g(b)]\,(\boldsymbol{\varepsilon}_{1-}^{\omega} h)(0;p) \\ &\quad + m\left[f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right](\boldsymbol{\varepsilon}_{0+}^{\omega} h)(1;p), \end{aligned} \tag{3.87}$$

where $\overline{\omega}$ is defined by (3.64).

In the following we will extend our interval to $[ma,b]$. Since $m \in (0,1]$, then $ma \le a$, $mb \le b$, and $[a,b] \subset [ma,b]$.

Theorem 3.30 *Let Assumption 3.2 hold. If $f,g \in L_1[ma,b]$ and $h \in L_1[0,1]$, then the following inequality holds*

$$\begin{aligned}\frac{1}{(mb-a)^{\sigma}} &\left[\left(\boldsymbol{\varepsilon}_{a+}^{\omega_1} f\right)(mb;p) + \left(\boldsymbol{\varepsilon}_{mb-}^{\omega_1} f\right)(a;p)\right] \\ &+\frac{1}{(b-ma)^{\sigma}}\left[\left(\boldsymbol{\varepsilon}_{b-}^{\omega_2} f\right)(ma;p) + \left(\boldsymbol{\varepsilon}_{ma+}^{\omega_2} f\right)(b;p)\right] \\ \le\ & (m+1)\,[f(a)g(a)+f(b)g(b)]\left[(\boldsymbol{\varepsilon}_{1-}^{\omega} h)(0;p)+(\boldsymbol{\varepsilon}_{0+}^{\omega} h)(1;p)\right], \end{aligned} \tag{3.88}$$

where

$$\omega_1 = \frac{\omega}{(mb-a)^{\rho}}, \quad \omega_2 = \frac{\omega}{(b-ma)^{\rho}}. \tag{3.89}$$

Proof. Let f be an $(h,g;m)$-convex function on $[0,\infty)$, $m \in (0,1]$ and $t \in (0,1)$. Then

$$\begin{aligned} f(ta+m(1-t)b) &\le h(t)f(a)g(a)+mh(1-t)f(b)g(b), \\ f((1-t)a+mtb) &\le h(1-t)f(a)g(a)+mh(t)f(b)g(b) \end{aligned}$$

and

$$\begin{aligned} f(tb+m(1-t)a) &\le h(t)f(b)g(b)+mh(1-t)f(a)g(a), \\ f((1-t)b+mta) &\le h(1-t)f(b)g(b)+mh(t)f(a)g(a). \end{aligned}$$

First we add the above inequalities, i.e.,

$$\begin{aligned} f(ta+m(1-t)b)&+f((1-t)a+mtb)+f(tb+m(1-t)a)+f((1-t)b+mta) \\ &\le (m+1)\,[f(a)g(a)+f(b)g(b)]\,h(t)+(m+1)\,[f(a)g(a)+f(b)g(b)]\,h(1-t). \end{aligned}$$

Then we use multiplication by $t^{\sigma-1}\boldsymbol{E}(\omega t^{\rho};p)$ and integration on $[0,1]$ with respect to the variable t to obtain

$$\begin{aligned} \int_0^1 & f(ta+m(1-t)b)t^{\sigma-1}\boldsymbol{E}(\omega t^{\rho};p)dt \\ &+\int_0^1 f((1-t)a+mtb)t^{\sigma-1}\boldsymbol{E}(\omega t^{\rho};p)dt \end{aligned}$$

$$
\begin{aligned}
&+\int_0^1 f(tb+m(1-t)a)t^{\sigma-1}\boldsymbol{E}(\omega t^{\rho};p)dt \\
&+\int_0^1 f((1-t)b+mta)t^{\sigma-1}\boldsymbol{E}(\omega t^{\rho};p)dt \\
\le\ & (m+1)\left[f(a)g(a)+f(b)g(b)\right]\int_0^1 h(t)t^{\sigma-1}\boldsymbol{E}(\omega t^{\rho};p)dt \\
&+(m+1)\left[f(a)g(a)+f(b)g(b)\right]\int_0^1 h(1-t)t^{\sigma-1}\boldsymbol{E}(\omega t^{\rho};p)dt.
\end{aligned}
$$

For the left side of the inequality we need several substitutions. For instance, if we set $u=ta+m(1-t)b$, then we get

$$
\begin{aligned}
&\int_0^1 f(ta+m(1-t)b)t^{\sigma-1}\boldsymbol{E}(\omega t^{\rho};p)dt \\
&\quad = \frac{1}{(mb-a)^{\sigma}}\int_a^{mb} f(u)(mb-u)^{\sigma-1}\boldsymbol{E}\left(\frac{\omega}{(mb-a)^{\rho}}(mb-u)^{\rho};p\right)du.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\frac{1}{(mb-a)^{\sigma}}\int_a^{mb} f(u)(mb-u)^{\sigma-1}\boldsymbol{E}\left(\frac{\omega}{(mb-a)^{\rho}}(mb-u)^{\rho};p\right)du \\
&\quad+\frac{1}{(mb-a)^{\sigma}}\int_a^{mb} f(u)(u-a)^{\sigma-1}\boldsymbol{E}\left(\frac{\omega}{(mb-a)^{\rho}}(u-a)^{\rho};p\right)du \\
&\quad+\frac{1}{(b-ma)^{\sigma}}\int_{ma}^{b} f(u)(u-ma)^{\sigma-1}\boldsymbol{E}\left(\frac{\omega}{(b-ma)^{\rho}}(u-ma)^{\rho};p\right)du \\
&\quad+\frac{1}{(b-ma)^{\sigma}}\int_{ma}^{b} f(u)(b-u)^{\sigma-1}\boldsymbol{E}\left(\frac{\omega}{(b-ma)^{\rho}}(b-u)^{\rho};p\right)du \\
&\le (m+1)\left[f(a)g(a)+f(b)g(b)\right](\boldsymbol{\varepsilon}_{1^-}^{\omega}h)(0;p) \\
&\quad+(m+1)\left[f(a)g(a)+f(b)g(b)\right](\boldsymbol{\varepsilon}_{0^+}^{\omega}h)(1;p),
\end{aligned}
$$

that is

$$
\begin{aligned}
&\frac{1}{(mb-a)^{\sigma}}\left(\boldsymbol{\varepsilon}_{a^+}^{\frac{\omega}{(mb-a)^{\rho}}}f\right)(mb;p)+\frac{1}{(mb-a)^{\sigma}}\left(\boldsymbol{\varepsilon}_{mb^-}^{\frac{\omega}{(mb-a)^{\rho}}}f\right)(a;p) \\
&\quad+\frac{1}{(b-ma)^{\sigma}}\left(\boldsymbol{\varepsilon}_{b^-}^{\frac{\omega}{(b-ma)^{\rho}}}f\right)(ma;p)+\frac{1}{(b-ma)^{\sigma}}\left(\boldsymbol{\varepsilon}_{ma^+}^{\frac{\omega}{(b-ma)^{\rho}}}f\right)(b;p) \\
&\le (m+1)\left[f(a)g(a)+f(b)g(b)\right]\left[(\boldsymbol{\varepsilon}_{1^-}^{\omega}h)(0;p)+(\boldsymbol{\varepsilon}_{0^+}^{\omega}h)(1;p)\right].
\end{aligned}
$$

This provides the required inequality. □

Remark 3.19 With an extended generalized Mittag-Leffler function from Definition 3.3 and a class of $(h,g;m)$-convex functions as in Definition 3.1, for different parameters p, τ, r, q, ω and for different choices of functions h, g and parameter m, we obtain corresponding upper bounds of different fractional operators for different classes of convexity.

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