Chapter 1

# Refinements of Jensen's and the Lah-Ribarič inequalities and applications to the Csiszár divergence

Research of the classical inequalities, such as the Jensen, the Hölder and similar, has experienced great expansion. These inequalities first appeared in discrete and integral forms, and then many generalizations and improvements have been proved. Lately, they are proven to be very useful in information theory.

Since all of these inequalities are related to the class of convex functions, we start with the definition of convex functions.

**Definition 1.1** *Let $I$ be an interval in $\mathbb{R}$. Function $f: I \to \mathbb{R}$ is said to be a convex function on $I$ if for all $x, y \in I$ and all $\lambda \in [0,1]$*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

*holds. If inequality is strict for all $x, y \in I$, $x \neq y$ and for all $\lambda \in (0,1)$, then $f$ is said to be strictly convex. If the inequality is reversed, then $f$ is said to be concave.*

Jensen's inequality is one of the most famous inequalities in convex analysis, which special cases are other well-known inequalities (such as Hölder's inequality, A-G-H inequality, etc.). Beside mathematics, it has many applications in statistics, information theory and engineering.

**Theorem 1.1** (JENSEN'S INEQUALITY) *Let $I$ be an interval in $\mathbb{R}$ and $f\colon I \to \mathbb{R}$ a convex function. If $\boldsymbol{x} = (x_1, \ldots, x_n)$ is any n-tuple in $I^n$ and $\boldsymbol{p} = (p_1, \ldots, p_n)$ a nonnegative n-tuple such that $P_n = \sum_{i=1}^n p_i > 0$, then the following inequality holds:*

$$f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) \le \frac{1}{P_n}\sum_{i=1}^n p_i f(x_i). \tag{1.1}$$

*If $f$ is strictly convex then (1.1) is strict unless $x_i = c$ for all $i \in \{j\colon p_j > 0\}$. If $f$ is concave, then (1.1) is reversed.*

Strongly related to Jensen's inequality is the converse Jensen inequality. One of the most famous variants of the converse inequality is the Lah-Ribarič inequality (see [11]).

**Theorem 1.2** (LAH-RIBARIČ INEQUALITY) *Let $f\colon I \to \mathbb{R}$ be a convex function on $I$, $[m,M] \subset I$, $-\infty < m < M < +\infty$. Let $\boldsymbol{p}$ be as in Theorem 1.1, $\boldsymbol{x} = (x_1, \ldots, x_n)$ is any n-tuple in $[m,M]^n$ and $\overline{x} = \frac{1}{P_n}\sum_{i=1}^n p_i x_i$. Then the following inequality holds:*

$$\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i) \le \frac{M-\overline{x}}{M-m} f(m) + \frac{\overline{x}-m}{M-m} f(M). \tag{1.2}$$

*If $f$ is strictly convex then (1.2) is strict unless $x_i \in \{m, M\}$ for all $i \in \{j\colon p_j > 0\}$.*

The Lah-Ribarič inequality has been largely investigated and the interested reader can find many related results in the recent literature as well as in monographs such as [13] and [16]. It is interesting to find further refinements of the above inequality.

Integral form of the Jensen inequality is given in the following theorem (see [2], [7], or for example [8]).

**Theorem 1.3** (INTEGRAL FORM OF JENSEN'S INEQUALITY) *Let $g\colon [a,b] \to \mathbb{R}$ be an integrable function and let $p\colon [a,b] \to \mathbb{R}$ be a nonnegative function. If $f$ is a convex function given on an interval $I$ that includes the image of $g$, then the following inequality holds*

$$f\left(\frac{1}{P(b)}\int_a^b p(t)g(t)dt\right) \le \frac{1}{P(b)}\int_a^b p(t)f(g(t))dt, \tag{1.3}$$

*where $P(t)$ is defined as*

$$P(t) = \int_a^t p(x)dx.$$

Integral form of the Lah-Ribarič inequality is given in the following theorem.

**Theorem 1.4** (INTEGRAL FORM OF THE LAH-RIBARIČ INEQUALITY) *Let $g\colon [a,b] \to \mathbb{R}$ be an integrable function such that $m \le g(t) \le M$, for all $t \in [a,b]$, $m < M$, and let $p\colon [a,b] \to \mathbb{R}$ be a nonnegative function. If $f$ is a convex function given on an interval $I$ such that $[m,M] \subseteq I$, then the following inequality holds*

$$\frac{1}{P(b)} \int_a^b p(t) f(g(t))dt \le \frac{M-\bar{g}}{M-m} f(m) + \frac{\bar{g}-m}{M-m} f(M), \tag{1.4}$$

*where $P$ is defined as*

$$P(t) = \int_a^t p(x)dx,$$

*and $\bar{g}$ is defined as*

$$\bar{g} = \frac{\int_a^b p(t)g(t)dt}{P(b)}.$$

We give a new refinement of the Lah-Ribarič inequality (1.2), and using the same technique we will give a refinement of the Jensen inequality (1.1) (see [17]).

We also give refinements of the integral form of Jensen's inequality (1.3) and the Lah-Ribarič inequality (1.4).

The idea for proving can be also found in a well known result (see [16, pages 55 - 60]). Refinement of the inequality on the interval is obtained by applying the same inequality on subintervals.

Using obtained results we give a refinement of the famous Hölder inequality and some new refinements for the weighted power means and quasi arithmetic means.

Also, we give a historical remark about the Jensen-Boas inequality.

In the last section, we deal with the notion of $f$-divergences, the Csiszár $f$-divergences in the first place, where by varying the generating functions we distinguish e.g. Jeffrey's distance, the KullbackLeibler divergence, the Hellinger distance, the Bhattacharyya distance. We deduce the relations for the mentioned $f$-divergences. In the discrete case, these results are further examined for the Zipf-Mandelbrot law.

## 1.1 New refinements

The starting point for this consideration is the following lemma.

**Lemma 1.1** *Let $f$ be a convex function on an interval $I$. If $a,b,c,d \in I$ such that $a \le b < c \le d$, then the inequality*

$$\frac{c-u}{c-b} f(b) + \frac{u-b}{c-b} f(c) \le \frac{d-u}{d-a} f(a) + \frac{u-a}{d-a} f(d)$$

*holds for any $u \in [b,c]$.*

*Proof.* We can write

$$b = \frac{d-b}{d-a}a + \frac{b-a}{d-a}d$$
$$c = \frac{d-c}{d-a}a + \frac{c-a}{d-a}d$$

and since $f$ is convex, it follows that

$$f(b) \leq \frac{d-b}{d-a}f(a) + \frac{b-a}{d-a}f(d)$$
$$f(c) \leq \frac{d-c}{d-a}f(a) + \frac{c-a}{d-a}f(d).$$

Now we have

$$\begin{aligned}
&\frac{c-u}{c-b}f(b) + \frac{u-b}{c-b}f(c) \\
&\quad \leq \frac{c-u}{c-b}\left[\frac{d-b}{d-a}f(a) + \frac{b-a}{d-a}f(d)\right] + \frac{u-b}{c-b}\left[\frac{d-c}{d-a}f(a) + \frac{c-a}{d-a}f(d)\right] \\
&\quad = \frac{d-u}{d-a}f(a) + \frac{u-a}{d-a}f(d).
\end{aligned}$$

□

First main result is a refinement of the Lah-Ribarič inequality (1.2). As we will see, its proof is based on the idea from the proof of the Jensen-Boas inequality.

**Theorem 1.5** *Let $f\colon I \to \mathbb{R}$ be a convex function on $I$, $[m,M] \subset I$, $-\infty < m < M < +\infty$, $\boldsymbol{p}$ is as in Theorem 1.1, $\boldsymbol{x} = (x_1,\dots,x_n)$ be any n-tuple in $[m,M]^n$ and $\bar{x} = \frac{1}{P_n}\sum_{i=1}^n p_i x_i$. Let $N_i \subseteq \{1,2,\dots,n\}$, $i = 1,\dots,m$ where $N_i \cap N_j = \emptyset$ for $i \neq j$, $\cup_{i=1}^m N_i = \{1,2,\dots,n\}$, $\sum_{j\in N_i} p_j > 0$, for $i = 1,\dots,m$ and $m_i = \min\{x_j\colon j \in N_i\}$, $M_i = \max\{x_j\colon j \in N_i\}$, for $i = 1,\dots,m$. Then*

$$\begin{aligned}
\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i) &\leq \frac{1}{P_n}\sum_{i=1}^m \left(\sum_{j\in N_i} p_j\right)\left[\frac{M_i - \bar{x}_i}{M_i - m_i}f(m_i) + \frac{\bar{x}_i - m_i}{M_i - m_i}f(M_i)\right] \\
&\leq \frac{M - \bar{x}}{M - m}f(m) + \frac{\bar{x} - m}{M - m}f(M)
\end{aligned} \tag{1.5}$$

*holds, where*

$$\bar{x}_i = \frac{1}{\sum_{j\in N_i} p_j}\sum_{j\in N_i} p_j x_j.$$

*If $f$ is concave on $I$, then the inequalities in* (1.5) *are reversed.*

*Proof.* We have

$$\begin{aligned}
\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i) &= \frac{1}{P_n}\left[\sum_{i=1}^m \sum_{j\in N_i} p_j f(x_j)\right] \\
&= \frac{1}{P_n}\sum_{i=1}^m \left(\sum_{j\in N_i} p_j\right)\left[\frac{1}{\sum_{j\in N_i} p_j}\sum_{j\in N_i} p_j f(x_j)\right].
\end{aligned}$$

Using the Lah-Ribarič inequality (1.2) for each of the subsets $N_i$, we get

$$\begin{aligned}
&\frac{1}{P_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i}p_j\right)\left[\frac{1}{\sum_{j\in N_i}p_j}\sum_{j\in N_i}p_jf(x_j)\right]\\
&\quad\le\frac{1}{P_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i}p_j\right)\left[\frac{M_i-\frac{1}{\sum_{j\in N_i}p_j}\sum_{j\in N_i}p_jx_j}{M_i-m_i}f(m_i)+\frac{\frac{1}{\sum_{j\in N_i}p_j}\sum_{j\in N_i}p_jx_j-m_i}{M_i-m_i}f(M_i)\right]\\
&\quad=\frac{1}{P_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i}p_j\right)\left[\frac{M_i-\bar{x}_i}{M_i-m_i}f(m_i)+\frac{\bar{x}_i-m_i}{M_i-m_i}f(M_i)\right].
\end{aligned}$$

Using $m\le m_i\le\bar{x}_i\le M_i\le M$, $m<M, m_i<M_i$ and Lemma 1.1, we get

$$\begin{aligned}
&\frac{1}{P_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i}p_j\right)\left[\frac{M_i-\bar{x}_i}{M_i-m_i}f(m_i)+\frac{\bar{x}_i-m_i}{M_i-m_i}f(M_i)\right]\\
&\quad\le\frac{1}{P_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i}p_j\right)\left[\frac{M-\bar{x}_i}{M-m}f(m)+\frac{\bar{x}_i-m}{M-m}f(M)\right]\\
&\quad=\frac{M-\bar{x}}{M-m}f(m)+\frac{\bar{x}-m}{M-m}f(M).
\end{aligned}$$

□

**Remark 1.1** If $N_i=\{x_j\}$ $(|N_i|=1)$, the related term in the sum on the right-hand side of the first inequality in the proof of Theorem 1.5 remains unaltered (i.e. is equal to $f(x_j)$).

Using the same technique, we obtain the following refinement of the Jensen inequality (1.1).

**Theorem 1.6** *Let $I$ be an interval in $\mathbb{R}$ and $f\colon I\to\mathbb{R}$ a convex function. Let $\boldsymbol{x}=(x_1,\dots,x_n)$ be any $n$-tuple in $I^n$ and $\boldsymbol{p}=(p_1,\dots,p_n)$ a nonnegative $n$-tuple such that $P_n=\sum_{i=1}^n p_i>0$. Let $N_i\subseteq\{1,2,\dots,n\}$, $i=1,\dots,m$ where $N_i\cap N_j=\emptyset$ for $i\ne j$, $\cup_{i=1}^m N_i=\{1,2,\dots,n\}$ and $\sum_{j\in N_i}p_j>0$, $i=1,\dots,m$. Then*

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right)\le\frac{1}{P_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i}p_j\right)f\left(\frac{\sum_{j\in N_i}p_jx_j}{\sum_{j\in N_i}p_j}\right)\le\frac{1}{P_n}\sum_{i=1}^{n}p_if(x_i)\tag{1.6}$$

*holds.*

*If $f$ is concave on $I$, then the inequalities in* (1.6) *are reversed.*

*Proof.* We have

$$\begin{aligned}
f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right)&=f\left(\frac{1}{P_n}\sum_{i=1}^{m}\left[\sum_{j\in N_i}p_jx_j\right]\right)\\
&=f\left(\frac{1}{P_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i}p_j\right)\frac{\sum_{j\in N_i}p_jx_j}{\sum_{j\in N_i}p_j}\right).
\end{aligned}$$

Using Jensen's inequality (1.1), we get

$$
\begin{aligned}
f\left(\frac{1}{P_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i}p_j\right)\frac{\sum_{j\in N_i}p_jx_j}{\sum_{j\in N_i}p_j}\right) &\le \frac{1}{P_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i}p_j\right)f\left(\frac{\sum_{j\in N_i}p_jx_j}{\sum_{j\in N_i}p_j}\right)\\
&\le \frac{1}{P_n}\sum_{i=1}^{m}\left(\sum_{j\in N_i}p_j\right)\left[\frac{1}{\sum_{j\in N_i}p_j}\sum_{j\in N_i}p_jf(x_j)\right]\\
&= \frac{1}{P_n}\sum_{i=1}^{m}\sum_{j\in N_i}p_jf(x_j),
\end{aligned}
$$

which is (1.6). □

We can find this idea for proving the refinement of our main results (and the refinement of the Jensen inequality) in one other well-known result (see [16, pages 55–60]).

In Jensen's inequality there is a condition "$\boldsymbol{p}=(p_1,\dots,p_n)$ a nonnegative $n$-tuple such that $P_n=\sum_{i=1}^{n}p_i>0$". In 1919. Steffensen gave the same inequality (1.1) with a slightly relaxed conditions.

**Theorem 1.7** (JENSEN-STEFFENSEN) *If $f\colon I\to\mathbb{R}$ is a convex function, $\mathbf{x}$ is a real monotonic n-tuple such that $x_i\in I$, $i=1,\dots,n$, and $\mathbf{p}$ is a real n-tuple such that*

$$0\le P_k\le P_n,\quad k=1,\dots,n,\quad P_n>0.$$

*Then* (1.1) *holds. If f is strictly convex, then inequality* (1.1) *is strict unless $x_1=x_2=\cdots=x_n$.*

One of many generalizations of the Jensen inequality is the Riemann-Stieltjes integral form of the Jensen inequality.

**Theorem 1.8** (THE RIEMANN-STIELTJES FORM OF JENSEN'S INEQUALITY)
*Let $\phi\colon I\to\mathbb{R}$ be a continuous convex function where I is a range of a continuous function $f\colon[a,b]\to\mathbb{R}$. The inequality*

$$\phi\left(\frac{\int_a^b f(x)d\lambda(x)}{\int_a^b d\lambda(x)}\right)\le\frac{\int_a^b\phi(f(x))\,d\lambda(x)}{\int_a^b d\lambda(x)} \tag{1.7}$$

*holds, providing that $\lambda\colon[a,b]\to\mathbb{R}$ is increasing, bounded and $\lambda(a)\neq\lambda(b)$.*

Analogously, integral form of the Jensen-Steffensen inequality is given.

**Theorem 1.9** (THE JENSEN-STEFFENSEN) *If f is continuous and monotonic (either increasing or decreasing) and $\lambda$ is either continuous or of bounded variation satisfying*

$$\lambda(a)\le\lambda(x)\le\lambda(b)\ \text{ for all }\ x\in[a,b],\quad \lambda(a)<\lambda(b),$$

*then* (1.7) *holds.*

In 1970. Boas gave the integral analogue of Jensen-Steffensen's inequality with slightly different conditions.

**Theorem 1.10** (THE JENSEN-BOAS INEQUALITY) *If $\lambda$ is continuous or of bounded variation satisfying*

$$\lambda(a) \leq \lambda(x_1) \leq \lambda(y_1) \leq \lambda(x_2) \leq \cdots \leq \lambda(y_{n-1}) \leq \lambda(x_n) \leq \lambda(b)$$

*for all $x_k \in (y_{k-1}, y_k)$, and $\lambda(b) > \lambda(a)$, and if $f\colon [a,b] \to \mathbb{R}$ is continuous and monotonic (either increasing or decreasing) in each of the $n-1$ intervals $(y_{k-1}, y_k)$, then inequality* (1.7) *holds for a continuous convex function $\phi : I \to \mathbb{R}$, where $I$ is the range of the function $f$.*

In 1982. J. Pečarić gave the following proof of the Jensen-Boas inequality.

*Proof.* If $\lambda(a) < \lambda(x_1) < \lambda(y_1) < \lambda(x_2) < \cdots < \lambda(y_{n-1}) < \lambda(x_n) < \lambda(b)$ with the notation

$$p_k = \int_{y_{k-1}}^{y_k} d\lambda(x), \quad t_k = \frac{\int_{y_{k-1}}^{y_k} f(x)d\lambda(x)}{\int_{y_{k-1}}^{y_k} d\lambda(x)}, \quad k = 1, \ldots, n,$$

we have

$$\phi\left(\frac{\int_a^b f(x)d\lambda(x)}{\int_a^b d\lambda(x)}\right) = \phi\left(\frac{\sum_{k=1}^n \int_{y_{k-1}}^{y_k} f(x)d\lambda(x)}{\sum_{k=1}^n \int_{y_{k-1}}^{y_k} d\lambda(x)}\right) = \phi\left(\frac{\sum_{k=1}^n p_k t_k}{\sum_{k=1}^n p_k}\right).$$

Using Jensen's inequality (1.1), we get

$$\begin{aligned}\phi\left(\frac{\sum_{k=1}^n p_k t_k}{\sum_{k=1}^n p_k}\right) &\leq \frac{1}{\sum_{k=1}^n p_k} \sum_{k=1}^n p_k \phi(t_k) \\ &= \frac{1}{\sum_{k=1}^n p_k} \left[\sum_{k=1}^n p_k \phi\left(\frac{\int_{y_{k-1}}^{y_k} f(x)d\lambda(x)}{\int_{y_{k-1}}^{y_k} d\lambda(x)}\right)\right].\end{aligned}$$

Using the Jensen-Steffensen's inequality (1.7) on each subinterval $[y_{k-1}, y_k]$, $k = 1, \ldots, n$, we get

$$\begin{aligned}&\frac{1}{\sum_{k=1}^n p_k} \left[\sum_{k=1}^n p_k \phi\left(\frac{\int_{y_{k-1}}^{y_k} f(x)d\lambda(x)}{\int_{y_{k-1}}^{y_k} d\lambda(x)}\right)\right] \\ &\quad \leq \frac{1}{\sum_{k=1}^n p_k} \left[\sum_{k=1}^n p_k \frac{1}{\int_{y_{k-1}}^{y_k} d\lambda(x)} \int_{y_{k-1}}^{y_k} \phi(f(x))\, d\lambda(x)\right] \\ &\quad = \frac{1}{\sum_{k=1}^n \int_{y_{k-1}}^{y_k} d\lambda(x)} \sum_{k=1}^n \int_{y_{k-1}}^{y_k} \phi(f(x))\, d\lambda(x) \\ &\quad = \frac{\int_a^b \phi(f(x))\, d\lambda(x)}{\int_a^b d\lambda(x)}.\end{aligned}$$

If $\lambda(y_{j-1}) = \lambda(y_j)$, for some $j$, then $d\lambda(x) = 0$ on $[y_{j-1}, y_j]$ and we can easily prove that the Jensen-Boas inequality is valid. □

If we look at the previous proof, we see that the technique is the same as for our main results and the refinement of the Jensen inequality.

Our next main result will be a refinement of the integral form of the Jensen inequality (1.3).

**Theorem 1.11** *Let $g$ be an integrable function defined on an interval $[a,b]$, let $a_0, a_1, \ldots, a_{n-1}, a_n$ be arbitrary such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$. If $f$ is a convex function given on an interval $I$ that includes the image of $g$, then*

$$f\left(\frac{1}{P(b)}\int_a^b p(t)g(t)dt\right) \leq \frac{1}{P(b)}\sum_{i=1}^{n}\left(\int_{a_{i-1}}^{a_i} p(t)dt\right) f\left(\frac{\int_{a_{i-1}}^{a_i} p(t)g(t)dt}{\int_{a_{i-1}}^{a_i} p(t)dt}\right) \leq \frac{1}{P(b)}\int_a^b p(t)f(g(t))dt \tag{1.8}$$

*is valid, where $p\colon [a,b] \to \mathbb{R}$ is nonnegative function and $P$ is defined as*

$$P(t) = \int_a^t p(x)dx.$$

*Proof.* Let $a_0, a_1, \ldots, a_{n-1}, a_n$ be arbitrary such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$. Applying Jensen's inequality, we have

$$\begin{aligned} f\left(\frac{1}{P(b)}\int_a^b p(t)g(t)dt\right) &= f\left(\frac{1}{P(b)}\sum_{i=1}^{n}\int_{a_{i-1}}^{a_i} p(t)g(t)dt\right) \\ &= f\left(\frac{1}{P(b)}\sum_{i=1}^{n}\left(\int_{a_{i-1}}^{a_i} p(t)dt\right)\frac{\int_{a_{i-1}}^{a_i} p(t)g(t)dt}{\int_{a_{i-1}}^{a_i} p(t)dt}\right) \\ &\leq \frac{1}{P(b)}\sum_{i=1}^{n}\left(\int_{a_{i-1}}^{a_i} p(t)dt\right) f\left(\frac{\int_{a_{i-1}}^{a_i} p(t)g(t)dt}{\int_{a_{i-1}}^{a_i} p(t)dt}\right), \end{aligned}$$

which is the left-hand side of (1.8).

Now we will use the inequality (1.3) on each of the subintervals $[a_{i-1}, a_i]$.

$$\begin{aligned} &\frac{1}{P(b)}\sum_{i=1}^{n}\left(\int_{a_{i-1}}^{a_i} p(t)dt\right) f\left(\frac{1}{\int_{a_{i-1}}^{a_i} p(t)dt}\int_{a_{i-1}}^{a_i} p(t)g(t)dt\right) \\ &\leq \frac{1}{P(b)}\sum_{i=1}^{n}\left(\int_{a_{i-1}}^{a_i} p(t)dt\right)\frac{1}{\int_{a_{i-1}}^{a_i} p(t)dt}\int_{a_{i-1}}^{a_i} p(t)f(g(t)dt, \end{aligned}$$

which is the right-hand side of (1.8). □

Last main result is a refinement of the integral form of the Lah-Ribarič inequality (1.4).

**Theorem 1.12** *Let $g$ be an integrable function defined on an interval $[a,b]$, let $a_0, a_1, \ldots, a_{n-1}, a_n$ be arbitrary such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$ and $m_i \leq g(t) \leq M_i$, for all $t \in [a_{i-1}, a_i]$, $m_i < M_i$, $i = 1, \ldots, n$, $m = \min_{1 \leq i \leq n} m_i, M = \max_{1 \leq i \leq n} M_i$. If $f$ is a convex function given on an interval $I$ that includes the image of $g$, then*

$$\begin{aligned}
&\frac{1}{P(b)} \int_a^b p(t) f(g(t)) dt \\
&\quad \leq \frac{1}{P(b)} \sum_{i=1}^{n} p_i \left[ \frac{M_i - \overline{g}_i}{M_i - m_i} f(m_i) + \frac{\overline{g}_i - m_i}{M_i - m_i} f(M_i) \right] \\
&\quad \leq \frac{M - \overline{g}}{M - m} f(m) + \frac{\overline{g} - m}{M - m} f(M)
\end{aligned} \tag{1.9}$$

*is valid, where $p\colon [a,b] \to \mathbb{R}$ is nonnegative function, $P$ is defined as*

$$P(t) = \int_a^t p(x) dx$$

*and $\overline{g}, \overline{g}_i, p_i$ are defined as*

$$\overline{g} = \frac{\int_a^b p(t) g(t) dt}{P(b)}, \quad \overline{g}_i = \frac{\int_{a_{i-1}}^{a_i} p(t) g(t) dt}{\int_{a_{i-1}}^{a_i} p(t) dt}, \quad p_i = \int_{a_{i-1}}^{a_i} p(t) dt.$$

*Proof.* We will use (1.4) on each of the subintervals $[a_{i-1}, a_i]$.

$$\begin{aligned}
&\frac{1}{P(b)} \int_a^b p(t) f(g(t)) dt \\
&\quad = \frac{1}{P(b)} \sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} p(t) f(g(t)) dt \\
&\quad \leq \frac{1}{P(b)} \sum_{i=1}^{n} \left( \int_{a_{i-1}}^{a_i} p(t) dt \right) \left[ \frac{M_i - \frac{\int_{a_{i-1}}^{a_i} p(t) g(t) dt}{\int_{a_{i-1}}^{a_i} p(t) dt}}{M_i - m_i} f(m_i) + \frac{\frac{\int_{a_{i-1}}^{a_i} p(t) g(t) dt}{\int_{a_{i-1}}^{a_i} p(t) dt} - m_i}{M_i - m_i} f(M_i) \right],
\end{aligned}$$

which is the left-hand side of inequality (1.9).

Since $m \leq m_i \leq \overline{g}_i \leq M_i \leq M, m < M, m_i < M_i$, then by Lemma 1.1 we get

$$\begin{aligned}
&\frac{1}{P(b)} \sum_{i=1}^{n} p_i \left[ \frac{M_i - \overline{g}_i}{M_i - m_i} f(m_i) + \frac{\overline{g}_i - m_i}{M_i - m_i} f(M_i) \right] \\
&\quad \leq \frac{1}{P(b)} \sum_{i=1}^{n} \left[ \frac{p_i M - \int_{a_{i-1}}^{a_i} p(t) g(t) dt}{M - m} f(m) + \frac{\int_{a_{i-1}}^{a_i} p(t) g(t) dt - p_i m}{M - m} f(M) \right] \\
&\quad = \frac{1}{P(b)} \left[ \frac{\sum_{i=1}^{n} p_i M - \sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} p(t) g(t) dt}{M - m} f(m) + \frac{\sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} p(t) g(t) dt - \sum_{i=1}^{n} p_i m}{M - m} f(M) \right] \\
&\quad = \frac{M - \frac{\int_a^b p(t) g(t) dt}{P(b)}}{M - m} f(m) + \frac{\frac{\int_a^b p(t) g(t) dt}{P(b)} - m}{M - m} f(M),
\end{aligned}$$

which is the right-hand side of (1.9). □

### 1.1.1 The Hermite-Hadamard inequality

Another famous inequality established for the class of convex functions is the Hermite-Hadamard inequality.

**Theorem 1.13** (HERMITE-HADAMARD) *Let $f$ be a convex function on $[a,b] \subset \mathbb{R}$, where $a < b$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1.10}$$

This double inequality was first discovered by Hermite in 1881. This result was later incorrectly attributed to Hadamard who apparently was not aware of Hermite's discovery and today, when relating to (1.10), we use both names.

This result can be improved by applying (1.10) on each of the subintervals $[a, \frac{a+b}{2}]$, $[\frac{a+b}{2}, b]$ and the following result is obtained (see [14, p. 37]):

$$f\left(\frac{a+b}{2}\right) \leq l \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq L \leq \frac{f(a)+f(b)}{2}, \tag{1.11}$$

where $l = \frac{1}{2}\left(f\left(\frac{3b+a}{4}\right) + f\left(\frac{b+3a}{2}\right)\right)$ and $L = \frac{1}{2}\left(f\left(\frac{b+a}{2}\right) + \frac{f(a)+f(b)}{2}\right)$.

The following improvement of (1.11) is given in [3].

**Theorem 1.14** *Assume that $f\colon I \to \mathbb{R}$ is a convex function on $I$. Then for all $\lambda \in [0,1]$ and $a, b \in I$, we have*

$$f\left(\frac{a+b}{2}\right) \leq l(\lambda) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq L(\lambda) \leq \frac{f(a)+f(b)}{2}, \tag{1.12}$$

*where*

$$l(\lambda) = \lambda f\left(\frac{\lambda b + (2-\lambda)a}{2}\right) + (1-\lambda) f\left(\frac{(1+\lambda)b + (1-\lambda)a}{2}\right)$$

*and*

$$L(\lambda) = \frac{1}{2}\left(f(\lambda b + (1-\lambda)a) + \lambda f(a) + (1-\lambda) f(b)\right).$$

The inequality (1.12) for $\lambda = \frac{1}{2}$ gives inequality (1.11). Further improvement was given in [4].

**Theorem 1.15** *Let $I \subseteq \mathbb{R}$ be an interval and $f\colon I \to \mathbb{R}$ be a convex function. Let $\Phi\colon [a,b] \to I$ be such that $f \circ \Phi$ is also convex, where $a < b$. Then for $n \in \mathbb{N}$, $\lambda_0 = 0, \lambda_{n+1} = 1$ and arbitrary $0 \leq \lambda_1 \leq \cdots \leq \lambda_n \leq 1$, we have*

$$f\left(\frac{1}{b-a}\int_a^b \Phi(x)dx\right) \leq l(\lambda_1,\ldots,\lambda_n) \leq \frac{1}{b-a}\int_a^b f\circ\Phi(x)dx \tag{1.13}$$

$$\leq L(\lambda_1,\ldots,\lambda_n) \leq \frac{f\circ\Phi(a) + f\circ\Phi(b)}{2}, \tag{1.14}$$