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Applications of integral operators containing an extended generalized Mittag-Leffler function in the kernel

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ELEMENT

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Preface

Fractional calculus is a theory of differential and integral operators of non-integer order. In recent years, considerable interest in this theory has been stimulated due to its many applications in almost all applied science, especially in numerical analysis and various fields of physics and engineering. Fractional calculus enabled the adoption of a theoretical model based on experimental data.

Inequalities which involve integrals of functions and their derivatives, whose study has a history of about a century, are of great importance in mathematics, with far-reaching applications in the theory of differential equations, approximations and probability, among others. They occupy a central place in mathematical analysis and its applications.

Fractional differentiation inequalities have applications to fractional differential equations; the most important ones are in establishing uniqueness of the solution of initial problems and giving upper bounds to their solutions. These applications have motivated many researchers in the field of integral inequalities to investigate certain extensions and generalizations using different fractional differential and integral operators. There are several well-known forms of fractional operators: Riemann-Liouville, Weyl, Erdélyi-Kober, Hadamard, Katugampola are just a few. All these forms of fractional operators in a special case are reduced to *the Riemann-Liouville fractional integrals* $J_{a+}^{\sigma} f$ *of order* σ defined as in [92, 135] for $f \in L_1[a,b]$.

As a solution of fractional order differential or integral equations, the Mittag-Leffler function with its generalizations appears. It is a function of one parameter defined by the power series using the gamma function

$$E_{\rho}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + 1)}, \quad (z, \rho \in \mathbb{C}, \Re(\rho) > 0)$$

and it is a natural extension of the exponential, hyperbolic and trigonometric functions. Extensions and generalizations of the Mittag-Leffler function enabled researchers to obtain fractional integral inequalities of different types, for example inequalities of the Opial, Pólya-Szegő, Chebyshev, Hermite-Hadamard and Fejér types, etc. Consequently, new results are produced for more generalized fractional integral operator containing the Mittag-Leffler function in their kernels.

The book is divided in nine chapters. The first chapter presents notation, terms and some important results for continuous and absolutely continuous functions. Definitions and properties of different types of convex functions that will be used in the book are given, as well as an overview of fractional calculus.

In Chapter 2 we define the extended and generalized Mittag-Leffler function and give its properties. For different parameter choices, corresponding known generalizations of the Mittag-Leffler function can be deduced: the Wiman generalization, also known as Mittag-Leffler function of two parameters, Prabhakar's function, Shukla-Prajapati's function, Salim-Faraj's function or recent extension defined by Rahman et al. We also present the corresponding generalized fractional integral operators containing our extended generalized Mittag-Leffler function in the kernel.

Motivated by Opial type inequalities, in Chapter 3 we use the extended generalized Mittag-Leffler function with the corresponding fractional integral operator (in real domain) to obtain fractional generalizations of Opial type inequalities due to Mitrinović and Pečarić. For such inequalities we construct functionals and give their mean value theorems.

In Chapter 4, we present improved and generalized Pólya-Szegő and Chebyshev types fractional integral inequalities that are related to the Mittag-Leffler function. We also use Karamata's estimations of the Chebyshev quotient to obtain even better upper and lower estimations.

Chapter 5 is dedicated to Minkowsky type inequalities. We present fractional generalizations of integral inequality and its reverse versions for generalized fractional integral operators containing our extended generalized Mittag-Leffler function.

Certain classical integral inequalities are presented in Chapter 6. We apply the results from Chapter 2 to obtain improvements, extensions and generalizations of known inequalities.

Famous Hadamard and Fejér-Hadamard inequalities are the main objects of Chapter 7, where we give Hadamard and Fejér- Hadamard types fractional inequalities for convex, relative convex, m-convex, $(h-m)$-convex, harmonically convex and harmonically $(\alpha,h-m)$-convex functions, which include our extended generalized Mittag-Leffler function.

In Chapter 8, fractional integral inequalities are given that provide the bounds of various kinds of fractional integral operators containing our extended generalized Mittag-Leffler function. We also give estimations of these inequalities for different kinds of convex functions. Results are applied on a certain function to establish recurrence relations among Mittag-Leffler functions.

The last chapter begins with the bounds of unified integral operators given for convex functions, continuing with results for the exponentially (s,m)-convex, strongly (s,m)-convex and (α,m)-convex functions, which contain the Mittag-Leffler function.

Authors

Contents

Chapter 1

Preliminaries

1.1 Continuous and Absolutely Continuous Functions

We start with definitions and properties of integrable functions, continuous functions, absolutely continuous functions, and give required notation, terms and overview of some important results (more details could be found in monographs [106, 133]).

L_p spaces

Let $[a,b]$ be a finite interval in $\mathbb{R}$, where $-\infty \le a < b \le \infty$. We denote by $L_p[a,b]$, $1 \le p < \infty$, the space of all Lebesgue measurable functions f for which $\int_a^b |f(t)|^p\,dt < \infty$, with the norm

$$||f||_p = \left(\int_a^b |f(t)|^p \, dt \right)^{\frac{1}{p}}$$

endowed, and by $L_\infty[a,b]$ the set/space of all functions measurable and essentially bounded on $[a,b]$, equipped with the norm

$$||f||_\infty = \operatorname{ess\,sup} \{ |f(x)| : x \in [a,b] \} .$$

Spaces of continuous and absolutely continuous functions

We denote by $C^n[a,b]$, $n \in \mathbb{N}_0$, the space of functions which are n times continuously differentiable on $[a,b]$, that is

$$C^n[a,b] = \left\{ f : [a,b] \to \mathbb{R}\colon\ f^{(k)} \in C[a,b]\,, k = 0, 1, \ldots, n \right\}.$$

In particular, $C^0[a,b] = C[a,b]$ is the space of continuous functions on $[a,b]$ with the norm

$$||f||_{C^n} = \sum_{k=0}^{n} ||f^{(k)}||_C = \sum_{k=0}^{n} \max_{x\in[a,b]} |f^{(k)}(x)|\,,$$

and for $C[a,b]$

$$||f||_C = \max_{x\in[a,b]} |f(x)|\,.$$

Lemma 1.1 *The space $C^n[a,b]$ consists of those and only those functions f which are represented in the form*

$$f(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} \varphi(t)\,dt + \sum_{k=0}^{n-1} c_k (x-a)^k\,, \tag{1.1}$$

where $\varphi \in C[a,b]$ and c_k are arbitrary constants $(k = 0, 1, \ldots, n-1)$.
Moreover,

$$\varphi(t) = f^{(n)}(t)\,, \quad c_k = \frac{f^{(k)}(a)}{k!}\ (k = 0, 1, \ldots, n-1)\,. \tag{1.2}$$

The space of absolutely continuous functions on a finite interval $[a,b]$ is denoted by $AC[a,b]$. It is known that $AC[a,b]$ coincides with the space of primitives of Lebesgue integrable functions $L_1[a,b]$ (see Kolmogorov and Fomin [93, Chapter 33.2]):

$$f \in AC[a,b] \quad \Leftrightarrow \quad f(x) = f(a) + \int_a^x \varphi(t)\,dt\,, \quad \varphi \in L_1[a,b]\,,$$

and therefore an absolutely continuous function f has an integrable derivative $f'(x) = \varphi(x)$ almost everywhere on $[a,b]$. We denote by $AC^n[a,b]$, $n \in \mathbb{N}$, the space

$$AC^n[a,b] = \left\{ f \in C^{n-1}[a,b]\colon\ f^{(n-1)} \in AC[a,b] \right\}.$$

In particular, $AC^1[a,b] = AC[a,b]$.

Lemma 1.2 *The space $AC^n[a,b]$ consists of those and only those functions which can be represented in the form* (1.1)*, where $\varphi \in L_1[a,b]$ and c_k are arbitrary constants $(k = 0, 1, \ldots, n-1)$.*
Moreover, (1.2) *holds.*

Theorem 1.1 (FUBINI'S THEOREM) *Let* $(X,\mathscr{M},\mu)$ *and* $(Y,\mathscr{N},\nu)$ *be* σ*-finite measure spaces and* f $\mu\times\nu$*-measurable function on* $X\times Y$*. If* $f\geq 0$*, then next integrals are equal*

$$\int_{X\times Y} f(x,y)\,d(\mu\times\nu)(x,y),$$

$$\int_X\left(\int_Y f(x,y)\,d\nu(y)\right)d\mu(x)$$

and

$$\int_Y\left(\int_X f(x,y)\,d\mu(x)\right)d\nu(y).$$

If f *is a complex function, then above equalities hold with additional requirement*

$$\int_{X\times Y}|f(x,y)|\,d(\mu\times\nu)(x,y)<\infty.$$

Fubini's theorem and its consequences below have numerous applications involving multiple integrals:

$$\int_a^b dx\int_c^d f(x,y)\,dy=\int_c^d dy\int_a^b f(x,y)\,dx;$$

$$\int_a^b dx\int_a^x f(x,y)\,dy=\int_a^b dy\int_y^b f(x,y)\,dx. \tag{1.3}$$

1.2 The Gamma and Beta Functions

The *gamma function* Γ is the function of complex variable defined by Euler's integral of second kind

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,dt\,,\quad \Re(z)>0\,. \tag{1.4}$$

This integral is convergent for each $z\in\mathbb{C}$ such that $\Re(z)>0$. It has next property

$$\Gamma(z+1)=z\Gamma(z)\,,\quad \Re(z)>0\,,$$

from which follows

$$\Gamma(n+1)=n!\,,\quad n\in\mathbb{N}_0\,.$$

For domain $\Re(z)\leq 0$ we have

$$\Gamma(z)=\frac{\Gamma(z+n)}{(z)_n}\,,\quad \Re(z)>-n;\ n\in\mathbb{N};\ z\notin\mathbb{Z}_0^-=\{0,-1,-2,\ldots\}\,, \tag{1.5}$$

where $(z)_n$ is the *Pochhammer's symbol* defined for $z\in\mathbb{C}$ and $n\in\mathbb{N}_0$ by

$$(z)_0=1;$$

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)} = z(z+1)\cdots(z+n-1),\; n \in \mathbb{N}. \tag{1.6}$$

The *generalized Pochhammer's symbol* is defined for $z, \nu \in \mathbb{C}$ by

$$(z)_\nu = \frac{\Gamma(z+\nu)}{\Gamma(z)}. \tag{1.7}$$

The gamma function is analytic in complex plane except in $0, -1, -2, \ldots$ which are simple poles. Another interesting equality holds:

$$(z)_{m+n} = (z+m)_n\,(z)_m, \quad n, m \in \mathbb{N}.$$

The *beta function* is the function of two complex variables defined by Euler's integral of the first kind

$$\mathrm{B}(z,w) = \int_0^1 t^{z-1}\,(1-t)^{w-1}\,dt, \quad \Re(z), \Re(w) > 0. \tag{1.8}$$

It is related to the gamma function with

$$\mathrm{B}(z,w) = \frac{\Gamma(z)\,\Gamma(w)}{\Gamma(z+w)}, \quad z, w \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\},$$

which gives

$$\mathrm{B}(z+1,w) = \frac{z}{z+w}\mathrm{B}(z,w).$$

Further, we have an extension of the beta function (for more details see [31])

$$\mathrm{B}_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{-\frac{p}{t(1-t)}}\,dt, \quad \Re(x), \Re(y), \Re(p) > 0. \tag{1.9}$$

Here we emphasize two equalities for the extended beta function:

$$\mathrm{B}_p(x,y+1) + \mathrm{B}_p(x+1,y) = \mathrm{B}_p(x,y),$$

$$\int_0^\infty \mathrm{B}_p(x,y)\,dp = \mathrm{B}(x+1,y+1), \quad \Re(x), \Re(y) > -1.$$

Following examples of integrals will be often used in proofs and calculations in this book.

Example 1.1 Let $\alpha, \beta > 0$ and $x \in [a,b]$. Then by substitution $t = x - s(x-a)$ we have

$$\begin{aligned} \int_a^x (x-t)^{\alpha-1}(t-a)^{\beta-1}\,dt &= \int_0^1 (x-a)^{\alpha+\beta-1} s^{\alpha-1}(1-s)^{\beta-1}\,ds \\ &= B(\alpha,\beta)(x-a)^{\alpha+\beta-1}. \end{aligned}$$

Analogously, by substitution $t = x + s(b-x)$, it follows

$$\int_x^b (t-x)^{\alpha-1}(b-t)^{\beta-1}\,dt = B(\alpha,\beta)\,(b-x)^{\alpha+\beta-1}.$$

Example 1.2 Let $\alpha,\beta > 0$, $f \in L_1[a,b]$ and $x \in [a,b]$. Then interchanging the order of integration and evaluating the inner integral we obtain

$$\begin{aligned} \int_a^x (x-t)^{\alpha-1} \int_a^t (t-s)^{\beta-1} f(s)\,ds\,dt &= \int_{s=a}^x f(s) \int_{t=s}^x (x-t)^{\alpha-1}(t-s)^{\beta-1}\,dt\,ds \\ &= B(\alpha,\beta) \int_a^x (x-s)^{\alpha+\beta-1} f(s)\,ds\,. \end{aligned}$$

Analogously,

$$\int_x^b (t-x)^{\alpha-1} \int_t^b (s-t)^{\beta-1} f(s)\,ds\,dt = B(\alpha,\beta) \int_x^b (s-x)^{\alpha+\beta-1} f(s)\,ds\,.$$

1.3 Convex Functions and Classes of Convexity

Definitions and properties of convex functions, with more details, could be found in monographs [107, 110, 121].

Let I be an interval in $\mathbb{R}$.

Definition 1.1 *A function $f : I \to \mathbb{R}$ is called convex if*

$$f((1-\lambda)x + \lambda y) \le (1-\lambda) f(x) + \lambda f(y) \tag{1.10}$$

for all points x and y in I and all $\lambda \in [0,1]$. It is called strictly convex if the inequality (1.10) *holds strictly whenever x and y are distinct points and $\lambda \in (0,1)$. If $-f$ is convex (respectively, strictly convex) then we say that f is concave (respectively, strictly concave). If f is both convex and concave, then f is said to be affine.*

Lemma 1.3 (THE DISCRETE CASE OF JENSEN'S INEQUALITY) *A real-valued function f defined on an interval I is convex if and only if for all $x_1,\ldots,x_n$ in I and all scalars $\lambda_1,\ldots,\lambda_n$ in $[0,1]$ with $\sum_{k=1}^n \lambda_k = 1$ we have*

$$f\left(\sum_{k=1}^n \lambda_k x_k\right) \le \sum_{k=1}^n \lambda_k f(x_k)\,. \tag{1.11}$$

The above inequality is strict if f is strictly convex, all the points x_k are distinct and all scalars λ_k are positive.

Theorem 1.2 (JENSEN) *Let $f : I \to \mathbb{R}$ be a continuous function. Then f is convex if and only if f is midpoint convex, that is,*

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} \tag{1.12}$$

for all $x,y \in I$.

Corollary 1.1 *Let $f : I \to \mathbb{R}$ be a continuous function. Then f is convex if and only if*

$$f(x+h)+f(x-h)-2f(x) \geq 0 \tag{1.13}$$

for all $x \in I$ and all $h > 0$ such that both $x+h$ and $x-h$ are in I.

Proposition 1.1 (THE OPERATIONS WITH CONVEX FUNCTIONS)

(i) The addition of two convex functions (defined on the same interval) is a convex function; if one of them is strictly convex, then the sum is also strictly convex.

(ii) The multiplication of a (strictly) convex function with a positive scalar is also a (strictly) convex function.

(iii) The restriction of every (strictly) convex function to a subinterval of its domain is also a (strictly) convex function.

(iv) If $f : I \to \mathbb{R}$ is a convex (respectively a strictly convex) function and $g : \mathbb{R} \to \mathbb{R}$ is a nondecreasing (respectively an increasing) convex function, then $g \circ f$ is convex (respectively strictly convex)

(v) Suppose that f is a bijection between two intervals I and J. If f is increasing, then f is (strictly) convex if and only if f^{-1} is (strictly) concave. If f is a decreasing bijection, then f and f^{-1} are of the same type of convexity.

Definition 1.2 *If g is strictly monotonic, then f is said to be (strictly) convex with respect to g if $f \circ g^{-1}$ is (strictly) convex.*

Proposition 1.2 *If $x_1, x_2, x_3 \in I$ are such that $x_1 < x_2 < x_3$, then the function $f : I \to \mathbb{R}$ is convex if and only if the inequality*

$$(x_3 - x_2)\,f(x_1) + (x_1 - x_3)\,f(x_2) + (x_2 - x_1)\,f(x_3) \;\geq\; 0$$

holds.

Proposition 1.3 *If f is a convex function on an interval I and if $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \;\leq\; \frac{f(y_2) - f(y_1)}{y_2 - y_1}\,.$$

If the function f is concave, then the inequality reverses.

The following theorems concern derivatives of convex functions.

Theorem 1.3 *Let $f : I \to \mathbb{R}$ be convex. Then*

(i) f is Lipschitz on any closed interval in I (recall, a function f such that $|f(x) - f(y)| \leq C|x-y|$ for all x and y, where C is a constant independent of x and y, is called a Lipshitz function);

(*ii*) *f'_+ and f'_- exist and are increasing in I, and $f'_- \le f'_+$ (if f is strictly convex, then these derivatives are strictly increasing);*

(*iii*) *f' exists, except possibly on a countable set, and on the complement of which it is continuous.*

Proposition 1.4 *Suppose that $f : I \to \mathbb{R}$ is a twice differentiable function. Then*

(*i*) *f is convex if and only if $f'' \ge 0$;*

(*ii*) *f is strictly convex if and only if $f'' \ge 0$ and the set of points where f'' vanishes does not include intervals of positive length.*

Next we need divided differences, commonly used when dealing with functions that have different degree of smoothness.

Definition 1.3 *Let $f : I \to \mathbb{R}$, $n \in \mathbb{N}_0$ and let $x_0, x_1, \ldots, x_n \in I$ be mutually different points. The n-th order divided difference of a function at $x_0, \ldots, x_n$ is defined recursively by*

$$\begin{aligned}
[x_i; f] &= f(x_i), \quad i = 0, 1, \ldots, n, \\
[x_0, x_1; f] &= \frac{[x_0; f] - [x_1; f]}{x_0 - x_1} = \frac{f(x_0) - f(x_1)}{x_0 - x_1}, \\
[x_0, x_1, x_2; f] &= \frac{[x_0, x_1; f] - [x_1, x_2; f]}{x_0 - x_2}, \\
&\vdots \\
[x_0, \ldots, x_n; f] &= \frac{[x_0, \ldots, x_{n-1}; f] - [x_1, \ldots, x_n; f]}{x_0 - x_n}.
\end{aligned} \tag{1.14}$$

Remark 1.1 The value $[x_0, x_1, x_2; f]$ is independent of the order of the points x_0, x_1 and x_2. This definition may be extended to include the case in which some or all the points coincide. Namely, taking the limit $x_1 \to x_0$ in (1.14), we get

$$\lim_{x_1 \to x_0} [x_0, x_1, x_2; f] = [x_0, x_0, x_2; f] = \frac{f(x_0) - f(x_2) - f'(x_0)(x_0 - x_2)}{(x_0 - x_2)^2}, \; x_2 \neq x_0$$

provided that f' exists, and furthermore, taking the limits $x_i \to x_0$, $i = 1, 2$ in (1.14), we get

$$\lim_{x_2 \to x_0} \lim_{x_1 \to x_0} [x_0, x_1, x_2; f] = [x_0, x_0, x_0; f] = \frac{f''(x_0)}{2}$$

provided that f'' exists.

Definition 1.4 *A function $f : I \to \mathbb{R}$ is said to be n-convex ($n \in \mathbb{N}_0$) if for all choices of $n+1$ distinct points $x_0, \ldots, x_n \in I$, the n-th order divided difference of f satisfies*

$$[x_0, \ldots, x_n; f] \ge 0. \tag{1.15}$$

Thus the 1-convex functions are the nondecreasing functions, while the 2-convex functions are precisely the classical convex functions.

Definition 1.5 *A function $f : I \to (0,\infty)$ is called log-convex if*

$$f((1-\lambda)x+\lambda y) \leq f(x)^{1-\lambda} f(y)^{\lambda} \tag{1.16}$$

for all points x and y in I and all $\lambda \in [0,1]$.

If a function $f : I \to \mathbb{R}$ is log-convex, then it is also convex, which is a consequence of the weighted AG-inequality.

We continue with definitions and properties of other types of convex functions that will be used in the book.

Definition 1.6 [112] *Let T_g be a set of real numbers. This set T_g is said to be relative convex with respect to an arbitrary function $g : \mathbb{R} \to \mathbb{R}$ if*

$$(1-t)x+tg(y) \in T_g$$

where $x,y \in \mathbb{R}$ such that $x,g(y) \in T_g$, $t \in [0,1]$.

Definition 1.7 [112] *A function $f : T_g \to \mathbb{R}$ is said to be relative convex, if there exists an arbitrary function $g : \mathbb{R} \to \mathbb{R}$ such that*

$$f((1-t)x+tg(y)) \leq (1-t)f(x)+tf(g(y)),$$

holds, where $x,y \in \mathbb{R}$ such that $x,g(y) \in T_g$, $t \in [0,1]$.

Note that if g is identity function, then convex set and convex function are reproduced from relative convex set and relative convex function.

Definition 1.8 [73] *Let I be an interval of real numbers. Then a function $f : I \to \mathbb{R}$ is said to be quasi-convex function, if for all $a,b \in I$ and $0 \leq t \leq 1$ the following inequality holds*

$$f(ta+(1-t)b) \leq max\{f(a),f(b)\}. \tag{1.17}$$

Example 1.3 [80] The function $f : [-2,2] \to \mathbb{R}$, given by

$$f(x) = \begin{cases} 1 & x \in [-2,-1] \\ x^2 & x \in (-1,2] \end{cases}$$

is not a convex function on $[-2,2]$, but it is quasi-convex function on $[-2,2]$.

Definition 1.9 *Let I be an interval of non-zero real numbers. A function $f : I \to \mathbb{R}$ is said to be harmonically convex function, if*

$$f\left(\frac{ab}{ta+(1-t)b}\right) \leq tf(b)+(1-t)f(a) \tag{1.18}$$

holds for all $a,b \in I$ and $t \in [0,1]$. If inequality in (1.18) is reversed, then f is said to be harmonically concave function.

Example 1.4 [76] Let $f:(0,\infty)\to\mathbb{R}$ be defined by $f(t)=t$ and $g:(-\infty,0)\to\mathbb{R}$ defined by $g(t)=t$. Then f is harmonically convex function and g is harmonically concave function.

Following results are obvious from above example.

(i) If $I\subset(0,\infty)$ and f is non-decreasing convex function, then f is harmonically convex.

(ii) If $I\subset(0,\infty)$ and f is non-increasing harmonically convex function, then f is convex.

(iii) If $I\subset(-\infty,0)$ and f is non-decreasing harmonically convex function, then f is convex.

(iv) If $I\subset(-\infty,0)$ and f is non-increasing convex function, then f is harmonically convex.

Definition 1.10 [95] *A function* $h:[a,b]\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ *is said to be harmonically symmetric about* $\frac{a+b}{2ab}$ *if for all* $x\in[a,b]$

$$h\left(\frac{1}{x}\right)=h\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right).$$

We give another new notion of harmonically $(h-m)$-convex function by setting $\alpha=1$ as follows:

Definition 1.11 *Let* $h:[0,1]\subseteq J\to\mathbb{R}$ *be a nonnegative function. A function* $f:J\subseteq\mathbb{R}_+\to\mathbb{R}$ *is said to be harmonically* $(h-m)$*-convex if*

$$f\left(\frac{mxy}{mty+(1-t)x}\right)\le h(t)f(x)+mh(1-t)f(y)$$

holds for all $x,y\in J$, $t\in[0,1]$ *and* $m\in(0,1]$.

Next is a harmonically $(\alpha,h-m)$-convex function:

Definition 1.12 *Let* $h:[0,1]\subseteq J\to\mathbb{R}$ *be a nonnegative function. A function* $f:J\subseteq\mathbb{R}_+\to\mathbb{R}$ *is said to be harmonically* $(\alpha,h-m)$*-convex if*

$$f\left(\frac{mxy}{mty+(1-t)x}\right)\le h(t^{\alpha})f(x)+mh(1-t^{\alpha})f(y),$$

holds for all $x,y\in J$, t, $\alpha\in[0,1]$ *and* $m\in(0,1]$.

This unifies the definitions of harmonically (α,m)-convexity and harmonically h-convexity of functions. For different specific choices of α,h,m, almost all kinds of well-known harmonically convex functions can be obtained:

Remark 1.2 (i) If $h(t) = t$, then harmonically (α,m)-convex function can be obtained [74].

(ii) If $\alpha = 1$ and $h(t) = t^s$, then harmonically (s,m)-convex function can be obtained [27].

(iii) If $\alpha = 1$ and $h(t) = t$, then harmonically m-convex function can be obtained [27].

(iv) If $\alpha = h(t) = m = 1$, then harmonically P-function can be obtained [113].

(v) If $\alpha = 1$, $h(t) = t^s$ and $m = 1$, then harmonically s-convex function can be obtained [113].

(vi) If $\alpha = 1$, $h(t) = \frac{1}{t}$ and $m = 1$, then harmonically Godunova-Levin function can be obtained [113].

(vii) If $\alpha = 1$, $h(t) = \frac{1}{t^s}$ and $m = 1$, then harmonically s-Godunova-Levin function can be obtained [113].

(viii) If we set $m = 1$ and $\alpha = 1$, then harmonically h-convex function can be achieved [113].

(ix) By putting $\alpha = 1$, $h(t) = t$ and $m = 1$, then harmonically-convex function can be obtained [76].

Definition 1.13 [147] *A function $f : [0,b] \to \mathbb{R}$, $b > 0$ is said to be m-convex function if for all $x,y \in [0,b]$ and $t \in [0,1]$*

$$f(tx+m(1-t)y) \leq tf(x)+m(1-t)f(y)$$

holds where $m \in [0,1]$.

Example 1.5 [105] A function $f : [0,\infty] \to \mathbb{R}$ given by

$$f(x) = \frac{1}{12}(x^3 - 5x^2 + 9x - 5x)$$

is $\frac{16}{17}$-convex function. If $m \in (\frac{16}{17}, 1]$, then f is not m-convex.

For $m = 1$ the m-convex function reduces to convex function and for $m = 0$ it gives star-shaped function. If set of m-convex functions on $[0,b]$ for which $f(0) < 0$ is denoted by $K_m(b)$, then we have

$$K_1(b) \subset K_m(b) \subset K_0(b)$$

whenever $m \in (0,1)$. In the class $K_1(b)$ there are convex functions $f : [0,b] \to \mathbb{R}$ for which $f(0) \leq 0$ (see, [40]).

Definition 1.14 [130] *A function $f : [a,b] \to \mathbb{R}$ is said to be exponentially m-convex if*

$$e^{f(zx+m(1-z)y)} \leq ze^{f(x)} + m(1-z)e^{f(y)}. \tag{1.19}$$

for all $x,y \in [a,b]$ and $z \in [0,1]$ where $m \in (0,1]$.

Definition 1.15 *Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $h : J \to \mathbb{R}$ be a nonnegative function. We say that $f : [0,b] \to \mathbb{R}$ is a $(h-m)-$convex function, if f is nonnegative and for all $x,y \in [0,b]$, $m \in [0,1]$ and $\alpha \in (0,1)$, one has*

$$f(\alpha x + m(1-\alpha)y) \leq h(\alpha)f(x) + mh(1-\alpha)f(y).$$

For suitable choices of h and m, class of $(h-m)-$convex functions is reduces to different known classes of convex and related functions defined on $[0,b]$ given in the following remark:

Remark 1.3

(i) If $m = 1$, then we get $h-$convex function.

(ii) If $h(\alpha) = \alpha$, then we get $m-$convex function.

(iii) If $h(\alpha) = \alpha$ and $m = 1$, then we get convex function.

(iv) If $h(\alpha) = 1$ and $m = 1$, then we get $p-$function.

(v) If $h(\alpha) = \alpha^s$ and $m = 1$, then we get $s-$convex function of second sense.

(vi) If $h(\alpha) = \frac{1}{\alpha}$ and $m = 1$, then we get Godunova-Levin function.

(vii) If $h(\alpha) = \frac{1}{\alpha^s}$ and $m = 1$, then we get $s-$Godunova-Levin function of second kind.

Definition 1.16 [103] *A function $f : [0,b] \to \mathbb{R}$, $b > 0$ is said to be (α,m)-convex, where $(\alpha,m) \in [0,1]^2$ if*

$$f(tx + m(1-t)y) \leq t^{\alpha} f(x) + m(1-t^{\alpha})f(y) \tag{1.20}$$

holds for all $x,y \in [0,b]$ and $t \in [0,1]$.

Remark 1.4 (i) If we set $\alpha = 1$, then (1.20) gives the definition of m-convex function.

(ii) If we put $(\alpha,m) = (1,1)$, then (1.20) gives the definition of convex function.

(iii) If we put $(\alpha,m) = (1,0)$, then (1.20) gives the definition of starshaped function.

Definition 1.17 [71] *Let $s \in [0,1]$. A function $f : [0,\infty) \to \mathbb{R}$ is said to be s-convex function in the second sense if*

$$f(ta + (1-t)b) \leq t^s f(a) + (1-t)^s f(b),$$

holds for all $a,b \in [0,\infty)$ and $t \in [0,1]$.

Definition 1.18 [102] *Let $s \in (0,1]$ and $I \subseteq [0,\infty)$ be an interval. A function $f : I \to \mathbb{R}$ is said to be exponentially s-convex in the second sense if*

$$f(ta + (1-t)b) \leq t^s \frac{f(a)}{e^{\alpha a}} + (1-t)^s \frac{f(b)}{e^{\alpha b}}, \tag{1.21}$$

holds for all $a,b \in I$, $t \in [0,1]$ and $\alpha \in \mathbb{R}$. If the inequality in (1.21) is reversed, then f is called exponentially s-concave.

Definition 1.19 [7] *A function $f:[0,b]\to\mathbb{R}$, $b>0$ is said to be (s,m)-convex function where $(s,m)\in[0,1]^2$, if*

$$f(ta+m(1-t)b)\le t^s f(a)+m(1-t^s)f(b)$$

for all $x,y\in[0,b]$ and $t\in[0,1]$.

Definition 1.20 [125] *Let $s\in[0,1]$ and $I\subseteq[0,\infty)$ be an interval. A function $f:I\to\mathbb{R}$ is said to be exponentially (s,m)-convex in second sense, if*

$$f(tx+m(1-t)y)\le t^s\frac{f(x)}{e^{\alpha x}}+m(1-t)^s\frac{f(y)}{e^{\alpha y}} \tag{1.22}$$

for all $m\in(0,1]$ and $\alpha\in\mathbb{R}$.

Remark 1.5

(i) For $m=1$, (1.22) produces the definition of exponentially s-convex function.

(ii) For $\alpha=0$, (1.22) produces the definition of (s,m)-convex function.

(iii) For $\alpha=0$ and $m=1$, (1.22) produces the definition of s-convex function.

(iv) For $\alpha=0$ and $m=1$, (1.22) produces the definition of convex function.

(v) For $\alpha=0$ and $s=1$, (1.22) produces the definition of m-convex function.

Definition 1.21 [75] *Let $I\subset(0,\infty)$ be a real interval and $p\in\mathbb{R}\setminus\{0\}$. Then a function $f:I\to\mathbb{R}$ is said to be p-convex, if*

$$f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\le tf(a)+(1-t)f(b)$$

for all $a,b\in I$ and $t\in[0,1]$. Note that for $p=1$ and $p=-1$, p-convexity reduces to ordinary convexity and harmonically convexity, respectively.

Definition 1.22 [94] *Let $p\in\mathbb{R}\setminus\{0\}$. Then a function $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ is said to be p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}$ if*

$$f(t^{\frac{1}{p}})=f\left([a^p+b^p-t]^{\frac{1}{p}}\right)$$

holds for $t\in[a,b]$.

1.4 Fractional Calculus

Fractional calculus is a theory of differential and integral operators of non-integer order. In recent years considerable interest in this theory has been stimulated by the applications that this calculus finds in numerical analysis and different areas of physics and engineering. Fractional calculus made it possible to adopt a theoretical model on experimental data. There are several well known forms of the fractional operators (meaning fractional integral and fractional derivative) that have been studied extensively for their applications: Riemann-Liouville, Weyl, Erdély-Kober, Hadamard, Katugampola are just a few.

Fractional integrals and fractional derivatives will be observed in the real domain. Let $[a,b] \subset \mathbb{R}$ be a finite interval, that is $-\infty < a < b < \infty$. For the integral part of a real number α we use notation $[\alpha]$.

1.4.1 The Riemann-Liouville Fractional Integrals and Derivatives

More on the Riemann-Liouville fractional integrals and derivatives can be found in monographs [21, 92, 135].

Definition 1.23 ([92]) *Let $\alpha > 0$ and $f \in L_1[a,b]$. The left-sided and the right-sided Riemann-Liouville fractional integrals $J^\alpha_{a+}f$ and $J^\alpha_{b-}f$ of order α are defined by*

$$J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt\,, \quad x\in[a,b]\,, \tag{1.23}$$

$$J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\,dt\,, \quad x\in[a,b]\,. \tag{1.24}$$

For $\alpha = n \in \mathbb{N}$ fractional integrals are actually n-fold integrals, that is

$$\begin{aligned} J^n_{a+}f(x) &= \int_a^x dt_1 \int_a^{t_1} dt_2 \cdots \int_a^{t_{n-1}} f(t_n)\,dt_n \\ &= \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)\,dt\,, \end{aligned} \tag{1.25}$$

$$\begin{aligned} J^n_{b-}f(x) &= \int_x^b dt_1 \int_{t_1}^{b} dt_2 \cdots \int_{t_{n-1}}^{b} f(t_n)\,dt_n \\ &= \frac{1}{(n-1)!}\int_x^b (t-x)^{n-1} f(t)\,dt\,. \end{aligned} \tag{1.26}$$

Lemma 1.4 *Let $\alpha,\beta > 0$ and $f \in L_p[a,b]$, $1 \le p \le \infty$. Then for almost every $x \in [a,b]$ we have*

$$J^\alpha_{a+}J^\beta_{a+}f(x) = J^{\alpha+\beta}_{a+}f(x)\,, \qquad J^\alpha_{b-}J^\beta_{b-}f(x) = J^{\alpha+\beta}_{b-}f(x)\,. \tag{1.27}$$

If $f \in C[a,b]$ or $\alpha+\beta > 1$, then equalities (1.27) hold for each x in $[a,b]$.

The homogeneous Abel integral equation has only trivial solution (see Samko et al. [135], Section 2.4).

Lemma 1.5 *Let $\alpha > 0$ and $f \in L_1[a,b]$. Then integral equations $J^{\alpha}_{a+}f = 0$ and $J^{\alpha}_{b-}f = 0$ have only trivial solution $f = 0$ (almost everywhere).*

Next result shows that the Riemann-Liouville fractional integral is bounded operator on $L_p[a,b]$.

Lemma 1.6 *Let $\alpha > 0$ and $1 \le p \le \infty$. Then the Riemann-Liouville fractional integrals are bounded on $L_p[a,b]$, that is*

$$||J^{\alpha}_{a+}f||_p \le K||f||_p\,, \quad ||J^{\alpha}_{b-}f||_p \le K||f||_p\,, \tag{1.28}$$

where

$$K = \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\,.$$

For $C[a,b]$ we have

$$||J^{\alpha}_{a+}f||_C \le K||f||_C\,, \quad ||J^{\alpha}_{b-}f||_C \le K||f||_C\,. \tag{1.29}$$

We continue with definition and properties of the Riemann-Liouville fractional derivatives.

Definition 1.24 *Let $\alpha > 0$, $n = [\alpha]+1$ and $f : [a,b] \to \mathbb{R}$. The left-sided and the right-sided Riemann-Liouville fractional derivatives $D^{\alpha}_{a+}f$ and $D^{\alpha}_{b-}f$ of order α are defined by*

$$\begin{aligned} D^{\alpha}_{a+}f(x) &= \frac{d^n}{dx^n} J^{n-\alpha}_{a+} f(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} f(t)\,dt\,, \end{aligned} \tag{1.30}$$

$$\begin{aligned} D^{\alpha}_{b-}f(x) &= (-1)^n \frac{d^n}{dx^n} J^{n-\alpha}_{b-} f(x) \\ &= \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b (t-x)^{n-\alpha-1} f(t)\,dt\,. \end{aligned} \tag{1.31}$$

In particular, if $0 < \alpha < 1$, then

$$D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} f(t)\,dt\,, \tag{1.32}$$

$$D^{\alpha}_{b-}f(x) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (t-x)^{-\alpha} f(t)\,dt\,. \tag{1.33}$$

For $\alpha = n \in \mathbb{N}$ we have

$$D^{n}_{a+}f(x) = f^{(n)}(x)\,, \quad D^{n}_{b-}f(x) = (-1)^n f^{(n)}(x)\,, \tag{1.34}$$

and for $\alpha = 0$

$$D^0_{a+}f(x) = D^0_{b-}f(x) = f(x). \tag{1.35}$$

The following result indicates that the functions $(x-a)^{\alpha-j}$ and $(b-x)^{\alpha-j}$, play the same role for the Riemann-Liouville fractional derivatives as the constants do in usual differentiation.

Lemma 1.7 *Let $\alpha > 0$ and $n = [\alpha]+1$.*

(*i*) *The equality $D^\alpha_{a+}f(x) = 0$ is valid if and only if $f(x) = \sum_{j=1}^{n} c_j(x-a)^{\alpha-j}$, where $c_j \in \mathbb{R}$ $(j = 1,\ldots,n)$ are arbitrary constants.*
In particular, when $0 < \alpha \le 1$, the relation $D^\alpha_{a+}f(x) = 0$ holds if and only if $f(x) = c(x-a)^{\alpha-1}$ with every $c \in \mathbb{R}$.

(*ii*) *The equality $D^\alpha_{b-}f(x) = 0$ is valid if and only if $f(x) = \sum_{j=1}^{n} d_j(b-x)^{\alpha-j}$, where $d_j \in \mathbb{R}$ $(j = 1,\ldots,n)$ are arbitrary constants.*
In particular, when $0 < \alpha \le 1$, the relation $D^\alpha_{b-}f(x) = 0$ holds if and only if $f(x) = d(b-x)^{\alpha-1}$ with every $d \in \mathbb{R}$.

We end with conditions for the existence of fractional derivatives in the space $AC^n[a,b]$.

Theorem 1.4 *Let $\alpha \ge 0$ and $n = [\alpha]+1$. If $f \in AC^n[a,b]$, then the Riemann-Liouville fractional derivatives $D^\alpha_{a+}f$ and $D^\alpha_{b-}f$ exist almost everywhere on $[a,b]$ and can be represented in the forms*

$$D^\alpha_{a+}f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha} + \frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)\,dt, \tag{1.36}$$

$$D^\alpha_{b-}f(x) = \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(b)}{\Gamma(k-\alpha+1)}(b-x)^{k-\alpha} + \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b (t-x)^{n-\alpha-1} f^{(n)}(t)\,dt. \tag{1.37}$$

Chapter 2

An Extended Generalized Mittag-Leffler Function

The Mittag-Leffler function of one parameter is defined by the power series using the gamma function Γ

$$E_\rho(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + 1)}, \quad (z, \rho \in \mathbb{C}, \Re(\rho) > 0) \tag{2.1}$$

and it is a natural extension of the exponential, hyperbolic and trigonometric functions. For instance:

$$\begin{aligned}
E_0(z) &= \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \\
E_1(z) &= \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = \exp(z), \\
E_2(z) &= \cosh(\sqrt{z}), \\
E_3(z) &= \frac{1}{3}\left[\exp\left(z^{\frac{1}{3}}\right) + 2\exp\left(\frac{-z^{\frac{1}{3}}}{2}\right)\cos\left(\frac{1}{2}\sqrt{3}z^{\frac{1}{3}}\right)\right], \\
E_4(z) &= \frac{1}{2}\left[\cos(z^{\frac{1}{4}}) + \cosh(z^{\frac{1}{4}})\right], \\
E_2(-z^2) &= \cos z.
\end{aligned}$$

This function and its generalizations appear as solutions of fractional order differential or integral equations. First is the well known Wiman's generalization, also known as Mittag-

Leffler function of two parameters:

$$E_{\rho,\sigma}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + \sigma)}, \quad (z, \rho, \sigma \in \mathbb{C}, \Re(\rho) > 0), \tag{2.2}$$

followed by Prabhakar's function [123], i.e. Mittag-Leffler function of three parameters:

$$E_{\rho,\sigma}^{\delta}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_n}{\Gamma(\rho n + \sigma)} \frac{z^n}{n!}, \quad (z, \rho, \sigma, \delta \in \mathbb{C}, \Re(\rho) > 0). \tag{2.3}$$

Next extension was introduced by Shukla and Prajapati in [143]:

$$E_{\rho,\sigma}^{\delta,q}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_{nq}}{\Gamma(\rho n + \sigma)} \frac{z^n}{n!}, \tag{2.4}$$

where $z, \rho, \sigma, \delta, q \in \mathbb{C}$, $\Re(\rho) > \max\{0, \Re(q) - 1\}, \Re(q) > 0$, and $(\delta)_{nq}$ denotes the generalized Pochhammer symbol (1.6).
Further, Salim and Faraj presented in [134] the function

$$E_{\rho,\sigma,\tau}^{\delta,q,r}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_{nq}}{\Gamma(\rho n + \sigma)} \frac{z^n}{(\tau)_{nr}}, \tag{2.5}$$

where $z, \rho, \sigma, \delta, \tau \in \mathbb{C}$, $\min\{\Re(\rho), \Re(\sigma), \Re(\delta), \Re(\tau)\} > 0$, $q, r > 0$ and $q \le \Re(\rho) + r$. Another recent extension is defined by Rahman et al. in [127]

$$E_{\rho,\sigma}^{\delta,q,c}(z;p) = \sum_{n=0}^{\infty} \frac{\mathrm{B}_p(\delta + nq, c - \delta)}{\mathrm{B}(\delta, c - \delta)} \frac{(c)_{nq}}{\Gamma(\rho n + \sigma)} \frac{z^n}{n!}, \tag{2.6}$$

where $z, \rho, \sigma, \delta, c \in \mathbb{C}$, $\Re(\rho), \Re(\delta), \Re(c) > 0$, $p \ge 0$, $q > 0$ with B_p as an extension of the beta function (1.9).

This chapter is based on our results from [15].

2.1 A Further Extension of the Mittag-Leffler Function

We define more extended and generalized Mittag-Leffler function $E_{\rho,\sigma,\tau}^{\delta,c,q,r}(z;p)$ as follows:

Definition 2.1 *Let $\rho, \sigma, \tau, \delta, c \in \mathbb{C}$, $\Re(\rho), \Re(\sigma), \Re(\tau) > 0$, $\Re(c) > \Re(\delta) > 0$ with $p \ge 0$, $r > 0$ and $0 < q \le r + \Re(\rho)$. Then the extended generalized Mittag-Leffler function $E_{\rho,\sigma,\tau}^{\delta,c,q,r}(z;p)$ is defined by*

$$E_{\rho,\sigma,\tau}^{\delta,c,q,r}(z;p) = \sum_{n=0}^{\infty} \frac{\mathrm{B}_p(\delta + nq, c - \delta)}{\mathrm{B}(\delta, c - \delta)} \frac{(c)_{nq}}{\Gamma(\rho n + \sigma)} \frac{z^n}{(\tau)_{nr}}. \tag{2.7}$$

For different parameter choices, the corresponding known generalizations of Mittag-Leffler function can be deduced, as described below.

Remark 2.1 Evidently, (2.7) is a generalization of the following functions:

(i) setting $p = 0$, it reduces to (2.5), which is the Salim-Faraj function $E^{\delta,\tau,q}_{\rho,\sigma,r}(z)$ defined in [134],

(ii) setting $\tau = r = 1$, it reduces to (2.6), which is the function $E^{\delta,q,c}_{\rho,\sigma}(z;p)$ defined by Rahman et al. in [127],

(iii) setting $p = 0$ and $\tau = r = 1$, it reduces to (2.4), which is the Shukla-Prajapati function $E^{\delta,q}_{\rho,\sigma}(z)$ defined in [143],

(iv) setting $p = 0$ and $\tau = r = q = 1$, it reduces to (2.3), which is the Prabhakar function $E^{\delta}_{\rho,\sigma}(z)$ defined in [123],

(v) setting $p = 0$ and $\tau = r = q = \delta = 1$, it reduces to (2.2), i.e. the Wiman function $E_{\rho,\sigma}(z)$, which for $\sigma = 1$ results with the Mittag-Leffler function $E_\rho(z)$, the equation (2.1).

Theorem 2.1 *The series in* (2.7) *is absolutely convergent for all values of z provided that $q < r + \Re(\rho)$. Moreover, if $q = r + \Re(\rho)$, then $E^{\delta,c,q,r}_{\rho,\sigma,\tau}(z;p)$ converges for $|z| < \frac{r^r\,\Re(\rho)^{\Re(\rho)}}{q^q}$.*

Proof. Let $E^{\delta,c,q,r}_{\rho,\sigma,\tau}(z;p) = \sum_{n=0}^{\infty} a_n z^n$, where

$$a_n = \frac{\mathrm{B}_p(\delta + nq, c-\delta)}{\mathrm{B}(\delta, c-\delta)} \frac{(c)_{nq}}{\Gamma(\rho n + \sigma)(\tau)_{nr}}.$$

Using the Cauchy-Hadamard formula for the radius of convergence

$$R = \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}}$$

and its alternative formula

$$R = \limsup_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

the asymptotic formula for gamma function

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left[1 + \frac{(a-b)(a+b-1)}{2z} + O\left(\frac{1}{z^2}\right)\right], \quad |z| \to \infty, |\arg z| < \pi$$

and the asymptotic behaviour of the generalized beta function for large x (with y and p finite) by Chaudhry et al. ([30])

$$\mathrm{B}_p(x,y) \sim \frac{\pi^{\frac{1}{2}} p^{\frac{1}{4}}}{x^{\frac{3}{4}}} \left(\frac{p}{x}\right)^{\frac{y-1}{2}} \exp\left[-2(px)^{\frac{1}{2}} - \frac{3p}{2}\right] \quad (x \to \infty),$$

we obtain $\lim\limits_{n\to\infty} a_n = 0$ and

$$\left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}_p(\delta+(n+1)q,c-\delta)}\cdot\frac{(c)_{nq}}{(c)_{(n+1)q}}\cdot\frac{(\tau)_{(n+1)r}}{(\tau)_{nr}}\cdot\frac{\Gamma(\rho(n+1)+\sigma)}{\Gamma(\rho n+\sigma)}\right|$$
$$\sim \left(1+\frac{q}{\delta+nq}\right)^{\frac{2(c-\delta)+1}{4}}\exp\left[2\sqrt{p(\delta+nq+q)}-2\sqrt{p(\delta+nq)}\right]$$
$$\times (nq)^{-q}\left[1+\frac{-q(2c+q-1)}{2nq}+O\left(\frac{1}{(nq)^2}\right)\right]$$
$$\times (nr)^{r}\left[1+\frac{r(2\tau+r-1)}{2nr}+O\left(\frac{1}{(nr)^2}\right)\right]$$
$$\times (n\rho)^{\rho}\left[1+\frac{\rho(2\sigma+\rho-1)}{2n\rho}+O\left(\frac{1}{(n\rho)^2}\right)\right], \quad n\to\infty.$$

Hence,

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{r^r\rho^\rho}{q^q}n^{r+\rho-q} = R.$$

This means that the function $E^{\delta,c,q,r}_{\rho,\sigma,\tau}(z;p)$ converges for all z provided that $q<r+\Re(\rho)$, and it is an entire function. Moreover if $q=r+\Re(\rho)$, then $E^{\delta,c,q,r}_{\rho,\sigma,\tau}(z;p)$ converges for $|z|<\frac{r^r\,\Re(\rho)^{\Re(\rho)}}{q^q}$. □

The following theorems list some basic properties of this function. First we have some recurrence relations.

Theorem 2.2 *If* $\rho,\sigma,\tau,\delta,c\in\mathbb{C}$, $\Re(\rho),\Re(\sigma),\Re(\tau)>0$, $\Re(c)>\Re(\delta)>0$ *with* $p\geq 0$, $r>0$ *and* $0<q<r+\Re(\rho)$, *then*

$$E^{\delta,c,q,r}_{\rho,\sigma,\tau}(z;p)-E^{\delta,c,q,r}_{\rho,\sigma,\tau-1}(z;p)=\frac{zr}{1-\tau}\frac{d}{dz}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(z;p), \quad \Re(\tau)>1; \tag{2.8}$$

$$E^{\delta,c,q,r}_{\rho,\sigma,\tau}(z;p)=\sigma E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(z;p)+\rho z\frac{d}{dz}E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(z;p). \tag{2.9}$$

Proof.

$$E^{\delta,c,q,r}_{\rho,\sigma,\tau}(z;p)-E^{\delta,c,q,r}_{\rho,\sigma,\tau-1}(z;p)=\sum_{n=0}^{\infty}\frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}(\delta,c-\delta)}\frac{(c)_{nq}}{\Gamma(\rho n+\sigma)}\left[\frac{1}{(\tau)_{nr}}-\frac{1}{(\tau-1)_{nr}}\right]z^n$$
$$=\sum_{n=0}^{\infty}\frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}(\delta,c-\delta)}\frac{(c)_{nq}}{\Gamma(\rho n+\sigma)}\frac{\Gamma(\tau)}{\Gamma(\tau+nr)}\left[\frac{nr}{1-\tau}\right]z^n$$
$$=\frac{zr}{1-\tau}\sum_{n=1}^{\infty}\frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}(\delta,c-\delta)}\frac{(c)_{nq}}{\Gamma(\rho n+\sigma)}\frac{nz^{n-1}}{(\tau)_{nr}}$$
$$=\frac{zr}{1-\tau}\frac{d}{dz}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(z;p)$$

which proves (2.8).

Further,

$$\begin{aligned}
E_{\rho,\sigma,\tau}^{\delta,c,q,r}(z;p) &= \sum_{n=0}^{\infty} \frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}(\delta,c-\delta)} \frac{(c)_{nq}}{\Gamma(\rho n+\sigma)} \frac{z^n}{(\tau)_{nr}} \\
&= (\rho n+\sigma) \sum_{n=0}^{\infty} \frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}(\delta,c-\delta)} \frac{(c)_{nq}}{\Gamma(\rho n+\sigma+1)} \frac{z^n}{(\tau)_{nr}} \\
&= (\rho n+\sigma) E_{\rho,\sigma+1,\tau}^{\delta,c,q,r}(z;p) \\
&= \sigma E_{\rho,\sigma+1,\tau}^{\delta,c,q,r}(z;p) + \rho z \frac{d}{dz} E_{\rho,\sigma+1,\tau}^{\delta,c,q,r}(z;p),
\end{aligned}$$

hence (2.9) is proved. □

Next are some differential relations.

Theorem 2.3 *If $m \in \mathbb{N}$, $w,\rho,\sigma,\tau,\delta,c \in \mathbb{C}$, $\Re(\rho),\Re(\sigma),\Re(\tau) > 0$, $\Re(c) > \Re(\delta) > 0$ with $p \geq 0$, $r > 0$ and $0 < q < r+\Re(\rho)$, then*

$$\left(\frac{d}{dz}\right)^m E_{\rho,\sigma,\tau}^{\delta,c,q,r}(z;p) = \frac{(c)_{mq}}{(\tau)_{mr}} \sum_{n=0}^{\infty} \frac{\mathrm{B}_p(\delta+(n+m)q,c-\delta)}{\mathrm{B}(\delta,c-\delta)} \frac{(c+mq)_{nq}}{\Gamma(\rho(n+m)+\sigma)} \frac{(n+1)_m z^n}{(\tau+mr)_{nr}}; \tag{2.10}$$

$$\left(\frac{d}{dz}\right)^m \left[z^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wz^\rho;p)\right] = z^{\sigma-m-1} E_{\rho,\sigma-m,\tau}^{\delta,c,q,r}(wz^\rho;p), \quad \Re(\sigma) > m. \tag{2.11}$$

Proof. We have

$$\begin{aligned}
&\left(\frac{d}{dz}\right)^m E_{\rho,\sigma,\tau}^{\delta,c,q,r}(z;p) \\
&= \sum_{n=m}^{\infty} \frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}(\delta,c-\delta)} \frac{(c)_{nq}}{\Gamma(\rho n+\sigma)} \frac{[n(n-1)\cdots(n-(m-1))]z^{n-m}}{(\tau)_{nr}} \\
&= \sum_{n=0}^{\infty} \frac{\mathrm{B}_p(\delta+(n+m)q,c-\delta)}{\mathrm{B}(\delta,c-\delta)} \frac{(c)_{(n+m)q}}{\Gamma(\rho(n+m)+\sigma)} \\
&\quad \times \frac{[(n+m)(n+m-1)\cdots(n+1)]z^n}{(\tau)_{(n+m)r}}.
\end{aligned}$$

From $(x)_{a+b} = (x+a)_b(x)_a$ follows (2.10).
Further,

$$\begin{aligned}
&\left(\frac{d}{dz}\right)^m \left[z^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wz^\rho;p)\right] \\
&= \sum_{n=0}^{\infty} \frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}(\delta,c-\delta)} \frac{(c)_{nq}}{\Gamma(\rho n+\sigma)} \\
&\quad \times \frac{[(\rho n+\sigma-1)(\rho n+\sigma-2)\cdots(\rho n+\sigma-m)]w^n z^{\rho n+\sigma-m-1}}{(\tau)_{nr}} \\
&= \sum_{n=0}^{\infty} \frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}(\delta,c-\delta)} \frac{(c)_{nq}}{\Gamma(\rho n+\sigma-m)} \frac{w^n z^{\rho n+\sigma-m-1}}{(\tau)_{nr}}.
\end{aligned}$$

□

Finally, we observe some special cases of the extended generalized Mittag-Leffler function that we will use to prove our inequalities in the forthcoming chapters.

- if $z = 0$, then
$$E_{\rho,\sigma,\tau}^{\delta,c,q,r}(0;p) = \frac{B_p(\delta, c-\delta)}{B(\delta, c-\delta)} \frac{1}{\Gamma(\sigma)},$$
- if $p = z = 0$, then
$$E_{\rho,\sigma,\tau}^{\delta,c,q,r}(0;0) = \frac{1}{\Gamma(\sigma)}.$$

2.2 Fractional Integral Operators Associated with the Mittag-Leffler Function

The corresponding generalized fractional integral operators, the left-sided $\varepsilon_{a^+,\rho,\sigma,\tau}^{w,\delta,c,q,r} f$ and the right-sided $\varepsilon_{b^-,\rho,\sigma,\tau}^{w,\delta,c,q,r} f$, which contain the extended generalized Mittag-Leffler function as its kernel, we define by:

Definition 2.2 *Let $w, \rho, \sigma, \tau, \delta, c \in \mathbb{C}$, $\Re(\rho), \Re(\sigma), \Re(\tau) > 0$, $\Re(c) > \Re(\delta) > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \Re(\rho)$. Let $f \in L_1[a,b]$ and $x \in [a,b]$. Then the left-sided and the right-sided generalized fractional integral operator $\varepsilon_{a^+,\rho,\sigma,\tau}^{w,\delta,c,q,r} f$ and $\varepsilon_{b^-,\rho,\sigma,\tau}^{w,\delta,c,q,r} f$ are defined by*

$$\left(\varepsilon_{a^+,\rho,\sigma,\tau}^{w,\delta,c,q,r} f\right)(x;p) = \int_a^x (x-t)^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w(x-t)^{\rho};p) f(t)dt, \tag{2.12}$$

$$\left(\varepsilon_{b^-,\rho,\sigma,\tau}^{w,\delta,c,q,r} f\right)(x;p) = \int_x^b (t-x)^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w(t-x)^{\rho};p) f(t)dt. \tag{2.13}$$

If we apply different parameter choices, then the corresponding known generalizations of fractional integral operators can be deduced. We list those associated with the left-sided fractional integral operator. Analogously holds for the right-sided.

Remark 2.2 (2.12) is a generalization of the following fractional integral operators:

(i) setting $p = 0$, it reduces to the Salim-Faraj fractional integral operator $\varepsilon_{a^+,\rho,\sigma,\tau}^{w,\delta,q,r} f(x)$ defined in [134],

(ii) setting $\tau = r = 1$, it reduces to the fractional integral operator $\varepsilon_{a^+,\rho,\sigma}^{w,\delta,q,c} f(x;p)$ defined by Rahman et al. in [127],

(iii) setting $p = 0$ and $\tau = r = 1$, it reduces to the Srivastava-Tomovski fractional integral operator $\varepsilon_{a^+,\rho,\sigma}^{w,\delta,q} f(x)$ defined in [144], (*iv*) setting $p = 0$ and $\tau = r = q = 1$, it reduces to the Prabhakar fractional integral operator $\varepsilon(\rho,\sigma;\delta;w)f(x)$ defined in [123],

(*v*) setting $p = w = 0$, it reduces to the left-sided Riemann-Liouville fractional integral $J^{\sigma}_{a+}f$ of order σ as in (1.23).

We follow with some properties of these operators.

Theorem 2.4 *If $w,\rho,\sigma,\tau,\delta,c \in \mathbb{C}$, $\Re(\rho),\Re(\sigma),\Re(\tau) > 0$, $\Re(c) > \Re(\delta) > 0$, $p \geq 0$, $r > 0$, $0 < q \leq r+\Re(\rho)$ with $f \in L_1[a,b]$, $x \in [a,b]$, then the left-sided fractional integral operator $\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}$ and the right-sided $\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}$ are bounded on $L_1[a,b]$ and*

$$\left\|\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right\|_1 \leq C\,\|f\|_1 \tag{2.14}$$

and

$$\left\|\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right\|_1 \leq C\,\|f\|_1\,, \tag{2.15}$$

where the constant $C\,(0 < C < \infty)$ is given by

$$C=(b-a)^{\Re(\sigma)}\sum_{n=0}^{\infty}\frac{\left|\mathrm{B}_p(\delta+nq,c-\delta)\right|}{|\mathrm{B}(\delta,c-\delta)|}\,\frac{\left|(c)_{nq}\right|}{(\Re(\rho)n+\Re(\sigma))\,|\Gamma(\rho n+\sigma)|}\,\frac{\left|w(b-a)^{\Re(\rho)}\right|^n}{|(\tau)_{nr}|}. \tag{2.16}$$

Proof. Using Fubini's theorem we obtain

$$\begin{aligned}
\left\|\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right\|_1 &= \int_a^b\left|\int_a^x (x-t)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-t)^{\rho};p)f(t)dt\right|dx\\
&\leq \int_a^b |f(t)|\left[\int_t^b (x-t)^{\Re(\sigma)-1}\left|E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-t)^{\rho};p)\right|dx\right]dt\\
&= \int_a^b |f(t)|\left[\int_0^{b-t} u^{\Re(\sigma)-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wu^{\rho};p)|du\right]dt\\
&\leq \int_a^b |f(t)|\left[\int_0^{b-a} u^{\Re(\sigma)-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wu^{\rho};p)|du\right]dt\\
&\leq \sum_{n=0}^{\infty}\frac{\left|\mathrm{B}_p(\delta+nq,c-\delta)\right|}{|\mathrm{B}(\delta,c-\delta)|}\,\frac{\left|(c)_{nq}\right|}{|\Gamma(\rho n+\sigma)|}\,\frac{|w^n|}{|(\tau)_{nr}|}\\
&\quad\times\int_0^{b-a} u^{\Re(\rho)n+\Re(\sigma)-1}du\,\|f\|_1\\
&= C\,\|f\|_1\,.
\end{aligned}$$

We can see that the constant C, defined by (2.16), is finite.
For the right-sided fractional integral operator, inequality (2.15) can be proved analogously. □

Next we have several results for a fractional integral operator applied on a power function.

Theorem 2.5 *If* $\mu, w, \rho, \sigma, \tau, \delta, c \in \mathbb{C}$, $\Re(\rho), \Re(\sigma), \Re(\tau) > 0$, $\Re(c) > \Re(\delta) > 0$ *with* $p \geq 0$, $r > 0$ *and* $0 < q \leq r + \Re(\rho)$, *then for* $f(t) = (t-a)^{\mu-1}$ *follows*

$$\left(\varepsilon_{a^+,\rho,\sigma,\tau}^{w,\delta,c,q,r}(t-a)^{\mu-1}\right)(x;p) = \Gamma(\mu)(x-a)^{\mu+\sigma-1}E_{\rho,\sigma+\mu,\tau}^{\delta,c,q,r}(w(x-a)^{\rho};p). \tag{2.17}$$

Proof. By definition for the left-sided fractional integral operator, we have

$$\begin{aligned}
&\left(\varepsilon_{a^+,\rho,\sigma,\tau}^{w,\delta,c,q,r}(t-a)^{\mu-1}\right)(x;p)\\
&= \int_a^x (x-t)^{\sigma-1}E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w(x-t)^{\rho};p)(t-a)^{\mu-1}dt\\
&= \int_a^x (x-t)^{\sigma-1}\sum_{n=0}^{\infty}\frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}(\delta,c-\delta)}\frac{(c)_{nq}}{\Gamma(\rho n+\sigma)}\frac{w^n(x-t)^{\rho n}}{(\tau)_{nr}}(t-a)^{\mu-1}dt\\
&= \sum_{n=0}^{\infty}\frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}(\delta,c-\delta)}\frac{(c)_{nq}}{\Gamma(\rho n+\sigma)}\frac{w^n}{(\tau)_{nr}}\int_a^x (x-t)^{\rho n+\sigma-1}(t-a)^{\mu-1}dt
\end{aligned}$$

Now we use the substitution $t = x - s(x-a)$, as in Example 1.1, to obtain

$$\int_a^x (x-t)^{\rho n+\sigma-1}(t-a)^{\mu-1}dt = (x-a)^{\rho n+\sigma+\mu-1}\mathrm{B}(\mu,\rho n+\sigma),$$

from which follows

$$\begin{aligned}
&\left(\varepsilon_{a^+,\rho,\sigma,\tau}^{w,\delta,c,q,r}(t-a)^{\mu-1}\right)(x;p)\\
&= (x-a)^{\sigma+\mu-1}\sum_{n=0}^{\infty}\frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}(\delta,c-\delta)}\frac{(c)_{nq}}{\Gamma(\rho n+\sigma)}\frac{w^n}{(\tau)_{nr}}(x-a)^{\rho n}\frac{\Gamma(\mu)\Gamma(\rho n+\sigma)}{\Gamma(\rho n+\sigma+\mu)}\\
&= \Gamma(\mu)(x-a)^{\mu+\sigma-1}E_{\rho,\sigma+\mu,\tau}^{\delta,c,q,r}(w(x-a)^{\rho};p).
\end{aligned}$$

□

If we set $a = 0$ and $x = 1$ in the previous theorem, then we obtain a following result.

Corollary 2.1 *If* $\mu, w, \rho, \sigma, \tau, \delta, c \in \mathbb{C}$, $\Re(\rho), \Re(\sigma), \Re(\tau) > 0$, $\Re(c) > \Re(\delta) > 0$ *with* $p \geq 0$, $r > 0$ *and* $0 < q \leq r + \Re(\rho)$, *then*

$$\frac{1}{\Gamma(\mu)}\int_0^1 t^{\mu-1}(1-t)^{\sigma-1}E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w(1-t)^{\rho};p)dt = E_{\rho,\sigma+\mu,\tau}^{\delta,c,q,r}(w;p).$$

Setting $\mu = 1$, we get following identity for the constant function.

Corollary 2.2 *If* $w, \rho, \sigma, \tau, \delta, c \in \mathbb{C}$, $\Re(\rho), \Re(\sigma), \Re(\tau) > 0$, $\Re(c) > \Re(\delta) > 0$ *with* $p \geq 0$, $r > 0$ *and* $0 < q \leq r + \Re(\rho)$, *then*

$$\left(\varepsilon_{a^+,\rho,\sigma,\tau}^{w,\delta,c,q,r}1\right)(x;p) = (x-a)^{\sigma}E_{\rho,\sigma+1,\tau}^{\delta,c,q,r}(w(x-a)^{\rho};p). \tag{2.18}$$

Proof. Identity (2.18) is a direct consequence of the Theorem 2.5, but it can also be easily obtain as follows:

$$\begin{aligned}\left(\varepsilon_{a^+,\rho,\sigma,\tau}^{w,\delta,c,q,r}1\right)(x;p) &= \int_a^x (x-t)^{\sigma-1}E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w(x-t)^\rho;p)dt \\ &= \int_a^x (x-t)^{\sigma-1}\sum_{n=0}^{\infty}\frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}(\delta,c-\delta)}\frac{(c)_{nq}}{\Gamma(\rho n+\sigma)}\frac{w^n(x-t)^{\rho n}}{(\tau)_{n\rho}}dt \\ &= \sum_{n=0}^{\infty}\frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}(\delta,c-\delta)}\frac{(c)_{nq}}{\Gamma(\rho n+\sigma)}\frac{w^n}{(\tau)_{n\rho}}\int_a^x (x-t)^{\rho n+\sigma-1}dt \\ &= (x-a)^\sigma\sum_{n=0}^{\infty}\frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}(\delta,c-\delta)}\frac{(c)_{nq}}{\Gamma(\rho n+\sigma)}\frac{w^n}{(\tau)_{n\rho}}\frac{(x-a)^{\rho n}}{\rho n+\sigma} \\ &= (x-a)^\sigma E_{\rho,\sigma+1,\tau}^{\delta,c,q,r}(w(x-a)^\rho;p).\end{aligned}$$

□

Similarly, for the right-sided fractional integral operator follows:

Theorem 2.6 *If* $\mu,w,\rho,\sigma,\tau,\delta,c\in\mathbb{C}$, $\Re(\rho),\Re(\sigma),\Re(\tau)>0$, $\Re(c)>\Re(\delta)>0$ *with* $p\geq 0$, $r>0$ *and* $0<q\leq r+\Re(\rho)$, *then for* $g(t)=(b-t)^{\mu-1}$ *follows*

$$\left(\varepsilon_{b^-,\rho,\sigma,\tau}^{w,\delta,c,q,r}(b-t)^{\mu-1}\right)(x;p)=\Gamma(\mu)(b-x)^{\mu+\sigma-1}E_{\rho,\sigma+\mu,\tau}^{\delta,c,q,r}(w(b-x)^\rho;p). \qquad (2.19)$$

Proof. By definition for the right-sided fractional integral operator, we have

$$\begin{aligned}&\left(\varepsilon_{b^-,\rho,\sigma,\tau}^{w,\delta,c,q,r}(b-t)^{\mu-1}\right)(x;p) \\ &= \int_x^b (t-x)^{\sigma-1}E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w(t-x)^\rho;p)(b-t)^{\mu-1}dt \\ &= \int_x^b (t-x)^{\sigma-1}\sum_{n=0}^{\infty}\frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}(\delta,c-\delta)}\frac{(c)_{nq}}{\Gamma(\rho n+\sigma)}\frac{w^n(t-x)^{\rho n}}{(\tau)_{nr}}(b-t)^{\mu-1}dt \\ &= \sum_{n=0}^{\infty}\frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}(\delta,c-\delta)}\frac{(c)_{nq}}{\Gamma(\rho n+\sigma)}\frac{w^n}{(\tau)_{nr}}\int_x^b (t-x)^{\rho n+\sigma-1}(b-t)^{\mu-1}dt\end{aligned}$$

For the right-sided integral operator we use the substitution $t=x+s(b-x)$ (see Example 1.1), to obtain

$$\int_x^b (t-x)^{\rho n+\sigma-1}(b-t)^{\mu-1}dt=(b-x)^{\rho n+\sigma+\mu-1}\mathrm{B}(\mu,\rho n+\sigma),$$

from which we get

$$\begin{aligned}&\left(\varepsilon_{b^-,\rho,\sigma,\tau}^{w,\delta,c,q,r}(b-t)^{\mu-1}\right)(x;p) \\ &= (b-x)^{\sigma+\mu-1}\sum_{n=0}^{\infty}\frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}(\delta,c-\delta)}\frac{(c)_{nq}}{\Gamma(\rho n+\sigma)}\frac{w^n}{(\tau)_{nr}}(b-x)^{\rho n}\frac{\Gamma(\mu)\Gamma(\rho n+\sigma)}{\Gamma(\rho n+\sigma+\mu)} \\ &= \Gamma(\mu)(b-x)^{\mu+\sigma-1}E_{\rho,\sigma+\mu,\tau}^{\delta,c,q,r}(w(b-x)^\rho;p).\end{aligned}$$

□

Setting $b=1$ and $x=0$ in previous theorem we have the next result.

Corollary 2.3 *If $\mu, w, \rho, \sigma, \tau, \delta, c \in \mathbb{C}$, $\Re(\rho), \Re(\sigma), \Re(\tau) > 0$, $\Re(c) > \Re(\delta) > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \Re(\rho)$, then*

$$\frac{1}{\Gamma(\mu)} \int_0^1 t^{\sigma-1}(1-t)^{\mu-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^{\rho};p)dt = E_{\rho,\sigma+\mu,\tau}^{\delta,c,q,r}(w;p).$$

For the constant function, we obtain following identity.

Corollary 2.4 *If $w, \rho, \sigma, \tau, \delta, c \in \mathbb{C}$, $\Re(\rho), \Re(\sigma), \Re(\tau) > 0$, $\Re(c) > \Re(\delta) > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \Re(\rho)$, then*

$$\left(\varepsilon_{b^-,\rho,\sigma,\tau}^{w,\delta,c,q,r} 1\right)(x;p) = (b-x)^{\sigma} E_{\rho,\sigma+1,\tau}^{\delta,c,q,r}(w(b-x)^{\rho};p). \tag{2.20}$$

2.3 Unified Integral Operators

In recent years considerable interest in fractional calculus has been stimulated by the applications that this calculus finds in numerical analysis and different areas of physics and engineering. All forms of the fractional operators that have been studied extensively for their applications, in a special case are reduced to *the left-sided and the right-sided Riemann-Liouville fractional integrals $J_{a+}^{\sigma} f$ and $J_{b-}^{\sigma} f$ of order σ* defined as in [92, 135] for $f \in L_1[a,b]$ with (1.23) and (1.24).

Further generalization of the fractional integral operator we give in [46, 101] as follows:

Definition 2.3 [46] *Let $w, \sigma, \tau, \delta, c \in \mathbb{C}$, $\Re(\sigma), \Re(\tau) > 0$, $\Re(c) > \Re(\delta) > 0$ with $p \geq 0$, $\rho, r > 0$ and $0 < q \leq r + \rho$. Let $f \in L_1[a,b]$, $0 < a < b < \infty$, be a positive function. Let $h : [a,b] \to \mathbb{R}$ be a differentiable function, strictly increasing. Also let $\frac{\phi}{x}$ be an increasing function on $[a,\infty)$ and $x \in [a,b]$. Then the left and the right generalized fractional integral operators ${}_h^{\phi}F_{a^+,\rho,\sigma,\tau}^{w,\delta,c,q,r} f$ and ${}_h^{\phi}F_{b^-,\rho,\sigma,\tau}^{w,\delta,c,q,r} f$ are defined by*

$$\begin{aligned}&\left({}_h^{\phi}F_{a^+,\rho,\sigma,\tau}^{w,\delta,c,q,r} f\right)(x;p) \\ &= \int_a^x \frac{\phi(h(x)-h(t))}{h(x)-h(t)} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w(h(x)-h(t))^{\rho};p)h'(t)f(t)dt\end{aligned} \tag{2.21}$$

and

$$\begin{aligned}&\left({}_h^{\phi}F_{b^-,\rho,\sigma,\tau}^{w,\delta,c,q,r} f\right)(x;p) \\ &= \int_x^b \frac{\phi(h(t)-h(x))}{h(t)-h(x)} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w(h(t)-h(x))^{\rho};p)h'(t)f(t)dt\end{aligned} \tag{2.22}$$

Next is a special case of the above operator, setting $\phi(x) = x^{\sigma}$, $\sigma > 0$. This operator we denote ${}_h\Upsilon_{a^+,\rho,\sigma,\tau}^{w,\delta,c,q,r} f$ (analogously ${}_h\Upsilon_{b^-,\rho,\sigma,\tau}^{w,\delta,c,q,r} f$):

Definition 2.4 [101] *Let* $w, \sigma, \tau, \delta, c \in \mathbb{C}$, $\Re(\sigma), \Re(\tau) > 0$, $\Re(c) > \Re(\delta) > 0$ *with* $p \geq 0$, $\rho, r > 0$ *and* $0 < q \leq r + \rho$. *Let* $f \in L_1[a,b]$, $0 < a < b < \infty$, *be a positive function. Let* $h : [a,b] \to \mathbb{R}$ *be a differentiable function, strictly increasing and let* $x \in [a,b]$. *Then the left and the right generalized fractional integral operators* ${}_h\Upsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f$ *and* ${}_h\Upsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f$ *are defined by*

$$\left({}_h\Upsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p) = \int_a^x (h(x)-h(t))^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(h(x)-h(t))^{\rho};p)h'(t)f(t)dt \tag{2.23}$$

and

$$\left({}_h\Upsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(x;p) = \int_x^b (h(t)-h(x))^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(h(t)-h(x))^{\rho};p)h'(t)f(t)dt. \tag{2.24}$$

If we set $p = \omega = 0$ in this definition, then (2.23) reduces to *the left-sided Riemann-Liouville fractional integral of a function* f *with respect to another function* h *of order* σ ([92, 135]):

$$J^{\sigma}_{a+;h}f(x) = \frac{1}{\Gamma(\sigma)} \int_a^x (h(x)-h(t))^{\sigma-1} h'(t)f(t)\,dt, \quad x \in (a,b].$$

For the constant function we have following identities.

Proposition 2.1 *Let* $w, \sigma, \tau, \delta, c \in \mathbb{C}$, $\Re(\sigma), \Re(\tau) > 0$, $\Re(c) > \Re(\delta) > 0$ *with* $p \geq 0$, $\rho, r > 0$ *and* $0 < q \leq r + \rho$. *Let* $0 < a < b < \infty$ *and let* $h : [a,b] \to \mathbb{R}$ *be a differentiable function, strictly increasing. Then*

$$\left({}_h\Upsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}1\right)(x;p) = (h(x)-h(a))^{\sigma} E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w(h(x)-h(a))^{\rho};p)$$

and

$$\left({}_h\Upsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}1\right)(x;p) = (h(b)-h(x))^{\sigma} E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w(h(b)-h(x))^{\rho};p).$$

Proof. Analogously as in Corollary 2.2. □

Chapter 3

Opial Type Fractional Integral Inequalities Associated with the Mittag-Leffler Function

Inequalities which involve integrals of functions and their derivatives, whose study has a history of about one century, are of great importance in mathematics, with far-reaching applications in the theory of differential equations, approximations and probability, among others. They occupy a central position in mathematical analysis and its applications.

In 1960. Opial published an inequality involving integrals of a function and its derivative, which now bears his name ([117]):

Theorem 3.1 (OPIAL'S INEQUALITY) *Let $f \in C^1[0,h]$ be such that $f(0) = f(h) = 0$ and $f(x) > 0$ for $x \in (0,h)$. then*

$$\int_0^h \left| f(x) f'(x) \right| dx \le \frac{h}{4} \int_0^h \left[f'(x) \right]^2 dx, \tag{3.1}$$

where constant $h/4$ is the best possible.

A monograph by Agarwal and Pang [6] is dedicated to the theory of Opial type inequalities and its applications. We also did several papers involving various extensions, generalizations and discrete analogues (see [8]-[25], [60]-[63]). To present Mitrinović-Pečarić generalizations of Opial type inequalities given in [121], we need the next characterization: We say that a function $u : [a,b] \longrightarrow \mathbb{R}$ belongs to the class $U(v,K)$ if it admits the representation

$$u(x) = \int_a^x K(x,t) v(t)\, dt,$$

where v is a continuous function and K is an arbitrary nonnegative kernel such that $v(x)>0$ implies $u(x)>0$ for every $x\in[a,b]$. We also assume that all integrals under consideration exist and converge. In the following theorems we have Opial type inequalities.

Theorem 3.2 ([121, p. 236]) *Let $u_1\in U(v_1,K)$, $u_2\in U(v_2,K)$ and $v_2(x)>0$ for every $x\in[a,b]$. Further, let $\phi(u)$ and $f(u)$ be convex and increasing for $u\geq 0$ and $f(0)=0$. If f is a differentiable function and $M=\max K(x,t)$, then*

$$M\int_a^b v_2(t)\,\phi\left(\left|\frac{v_1(t)}{v_2(t)}\right|\right) f'\left(u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right)\right)dt$$
$$\leq f\left(M\int_a^b v_2(t)\,\phi\left(\left|\frac{v_1(t)}{v_2(t)}\right|\right)dt\right).$$

Theorem 3.3 ([121, p. 238]) *Let $\phi:[0,\infty)\longrightarrow\mathbb{R}$ be a differentiable function such that for $m>1$ the function $\phi(x^{\frac{1}{m}})$ is convex and $\phi(0)=0$. Let $u\in U(v,K)$ where $\left(\int_a^x(K(x,t))^l\,dt\right)^{\frac{1}{l}}\leq M$, $l^{-1}+m^{-1}=1$. Then*

$$\int_a^b |\,u(x)\,|^{1-m}\,\phi'(|\,u(x)\,|)\,|\,v(x)\,|^m\,dx\leq\frac{m}{M^m}\phi\left(M\left(\int_a^b |\,v(x)\,|^m\,dx\right)^{\frac{1}{m}}\right).$$

If the function $\phi(x^{\frac{1}{m}})$ is concave, then the reverse inequality holds.

In [8, 10], also [21], we gave the following generalizations of the above inequalities.

Theorem 3.4 ([21, p. 86]) *Let $u_1\in U(v_1,K)$, $u_2\in U(v_2,K)$ and $v_2(x)>0$ for every $x\in[a,b]$. Further, let $\phi(u)$ be convex, nonnegative and increasing for $u\geq 0$, $f(u)$ be convex for $u\geq 0$ and $f(0)=0$. If f is a differentiable function and $M=\max K(x,t)$, then*

$$M\int_a^b v_2(t)\phi\left(\left|\frac{v_1(t)}{v_2(t)}\right|\right) f'\left(u_2(t)\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right)\right)dt$$
$$\leq f\left(M\int_a^b v_2(t)\phi\left(\left|\frac{v_1(t)}{v_2(t)}\right|\right)dt\right)$$
$$\leq\frac{1}{b-a}\int_a^b f\left(M(b-a)v_2(t)\phi\left(\left|\frac{v_1(t)}{v_2(t)}\right|\right)\right)dt.$$

Theorem 3.5 ([21, p. 101]) *Let $\phi:[0,\infty)\to\mathbb{R}$ be a differentiable function such that for $m>1$ the function $\phi(x^{\frac{1}{m}})$ is convex and $\phi(0)=0$. Let $u\in U(v,K)$, where $\left(\int_a^x(K(x,t))^l dt\right)^{\frac{1}{l}}\leq M$ and $l^{-1}+m^{-1}=1$. Then*

$$\int_a^b |u(x)|^{1-m}\phi'\left(|u(x)|\right)|v(x)|^m dx\leq\frac{m}{M^m}\phi\left(M\left(\int_a^b |v(x)|^m dx\right)^{\frac{1}{m}}\right)$$
$$\leq\frac{m}{M^m(b-a)}\int_a^b\phi\left((b-a)^{\frac{1}{m}}M|v(x)|\right)dx. \tag{3.2}$$

If the function $\phi(x^{\frac{1}{m}})$ is concave, then the reverse inequalities hold.

Fractional differentiation inequalities have applications to fractional differential equations; the most important ones are in establishing uniqueness of the solution of initial problems and giving upper bounds to their solutions. These applications have motivated many researchers in the field of integral inequalities to explore certain extensions and generalizations by using different fractional differential and integral operators.

The right-sided versions of following inequalities can be established and proved analogously by using the right-sided fractional integral operator $\varepsilon_{b^-,\rho,\sigma,\tau}^{w,\delta,c,q,r} f$ defined with (2.13).

This chapter is based on our results from [15].

3.1 Opial Type Fractional Integral Inequalities and an Extended Generalized Mittag-Leffler Function

In this section we use the extended generalized Mittag-Leffler function $E_{\rho,\sigma,\tau}^{\delta,c,q,r}$ with the corresponding fractional integral operator $\varepsilon_{a^+,\rho,\sigma,\tau}^{w,\delta,c,q,r} f$ (in real domain) to obtain fractional generalizations of Opial type inequalities due to Mitrinović and Pečarić.

Here, for the reader's convenience we will use a simplified notation

$$\begin{aligned} \boldsymbol{E}(z;p) &:= E_{\rho,\sigma,\tau}^{\delta,c,q,r}(z;p) \\ &= \sum_{n=0}^{\infty} \frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}(\delta,c-\delta)} \frac{(c)_{nq}}{\Gamma(\rho n+\sigma)} \frac{z^n}{(\tau)_{nr}}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} (\boldsymbol{\varepsilon} f)(x;p) &:= \left(\varepsilon_{a^+,\rho,\sigma,\tau}^{w,\delta,c,q,r} f\right)(x;p) \\ &= \int_a^x (x-t)^{\sigma-1} \boldsymbol{E}(w(x-t)^{\rho};p) f(t)dt. \end{aligned} \tag{3.4}$$

The first result is a generalization of Theorem 3.4. As mentioned before, from Theorem 3.4 immediately follows Theorem 3.2.

Theorem 3.6 *Let $w \in \mathbb{R}$, $r,\rho,\tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let u_1, u_2, ϕ and f be the same as in Theorem 3.4. Then for $\sigma > 1$ following inequalities hold*

$$\begin{aligned} &\boldsymbol{E}(w(b-a)^{\rho};p)(b-a)^{\sigma-1} \\ &\times \int_a^b v_2(x)\,\phi\left(\left|\frac{v_1(x)}{v_2(x)}\right|\right) f'\left((\boldsymbol{\varepsilon} v_2)(x;p)\,\phi\left(\left|\frac{(\boldsymbol{\varepsilon} v_1)(x;p)}{(\boldsymbol{\varepsilon} v_2)(x;p)}\right|\right)\right) dx \\ &\leq f\left(\boldsymbol{E}(w(b-a)^{\rho};p)(b-a)^{\sigma-1} \int_a^b v_2(x)\,\phi\left(\left|\frac{v_1(x)}{v_2(x)}\right|\right) dx\right) \qquad (3.5) \\ &\leq \frac{1}{b-a} \int_a^b f\left(\boldsymbol{E}(w(b-a)^{\rho};p)(b-a)^{\sigma} v_2(x)\,\phi\left(\left|\frac{v_1(x)}{v_2(x)}\right|\right)\right) dx. \qquad (3.6) \end{aligned}$$

Proof. Let us define the following kernel

$$K(x,t) = \begin{cases} (x-t)^{\sigma-1}\boldsymbol{E}(w(x-t)^{\rho};p) & a \le t \le x, \\ 0 & x < t \le b, \end{cases} \tag{3.7}$$

and denote the extended generalized fractional integral operators $\boldsymbol{\varepsilon} v_1$ and $\boldsymbol{\varepsilon} v_2$ by u_1 and u_2 as

$$u_i(x) = (\boldsymbol{\varepsilon} v_i)(x;p) = \int_a^x (x-t)^{\sigma-1}\boldsymbol{E}(w(x-t)^{\rho};p)v_i(t)dt, \tag{3.8}$$

for $i = 1,2$. Next we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\beta_p(\delta+n,c-\delta)}{B(\delta,c-\delta)} \frac{(c)_{nq}w^n(x-t)^{n\rho}}{\Gamma(\rho n+\sigma)(\tau)_{nr}} &\le \sum_{n=0}^{\infty} \frac{\beta_p(\delta+n,c-\delta)}{B(\delta,c-\delta)} \frac{(c)_{nq}w^n(b-a)^{n\rho}}{\Gamma(\rho n+\sigma)(\tau)_{nr}} \\ &= \boldsymbol{E}(w(b-a)^{\rho};p),\end{aligned}$$

where convergence of $\boldsymbol{E}(w(b-a)^{\rho};p)$ follows by Theorem 2.1. Using

$$(x-t)^{\sigma-1} \le (x-a)^{\sigma-1} \le (b-a)^{\sigma-1}, \quad \sigma > 1,$$

we obtain

$$K(x,t) \le \boldsymbol{E}(w(b-a)^{\rho};p)(b-a)^{\sigma-1}, \quad \sigma > 1, q < r + \Re(\rho).$$

Finally, if we set

$$M = \boldsymbol{E}(w(b-a)^{\rho};p)(b-a)^{\sigma-1}$$

and use here the functions u_1, u_2, then respectively, by Theorem 3.4 inequalities (3.5) and (3.6) follow. □

Remark 3.1 For different choices of parameters we can obtain the corresponding fractional integral inequalities, such as:

(i) setting $p = 0$ in (3.5), we get [62, Theorem 2.2],

(ii) setting $p = 0$ in (3.6), we get [62, Theorem 2.3] (*there is a misprint in the* (2.5) *in [62, Theorem 2.3]: instead of* $(b-a)^{\beta-1}$ *it should be* $(b-a)^{\beta}$),

(iii) setting $p = 0$ and $\tau = r = q = 1$, we get Opial type inequalities for Prabhakar fractional integral operator defined in [123],

(iv) setting $p = 0$, $\tau = r = q = 1$ and $w = 0$, we get the result for left-sided Riemann-Liouville fractional integral in [10, Corollary 3.2].

If we set $\phi(x) = x^{l+m}$ in Theorem 3.6, then we have the following result.

Corollary 3.1 *Let* $w \in \mathbb{R}$, $r,\rho,\tau > 0$, $c > \delta > 0$ *with* $p \ge 0$ *and* $0 < q \le r+\rho$. *Let* u_1, u_2 *and* f *be the same as in Theorem 3.4. Let* $m, l > 1$ *with* $l^{-1} + m^{-1} = 1$. *Then for* $\sigma > 1$ *following inequalities hold*

$$\boldsymbol{E}(w(b-a)^{\rho};p)(b-a)^{\sigma-1} \\ \times \int_a^b v_2(x)\left|\frac{v_1(x)}{v_2(x)}\right|^{l+m} f'\left((\boldsymbol{\varepsilon} v_2)(x;p)\left|\frac{(\boldsymbol{\varepsilon} v_1)(x;p)}{(\boldsymbol{\varepsilon} v_2)(x;p)}\right|^{l+m}\right)dx$$

$$\leq f\left(\boldsymbol{E}(w(b-a)^{\rho};p)(b-a)^{\sigma-1}\int_a^b v_2(x)\left|\frac{v_1(x)}{v_2(x)}\right|^{l+m}dx\right)$$
$$\leq \frac{1}{b-a}\int_a^b f\left(\boldsymbol{E}(w(b-a)^{\rho};p)(b-a)^{\sigma}v_2(x)\left|\frac{v_1(x)}{v_2(x)}\right|^{l+m}\right)dx.$$

The next result is a generalization of Theorem 3.5 from which immediately follows Theorem 3.3.

Theorem 3.7 *Let $w\in\mathbb{R}$, $r,\rho,\tau>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let l,m,u and ϕ be the same as in Theorem 3.5. Then for $\sigma>\frac{1}{m}$ following inequalities hold*

$$\begin{aligned}&\int_a^b |(\boldsymbol{\varepsilon}v)(x;p)|^{1-m}\phi'(|(\boldsymbol{\varepsilon}v)(x;p)|)|v(x)|^m dx\\ &\leq \frac{m l^{\frac{m}{l}}\left(\sigma-\frac{1}{m}\right)^{\frac{m}{l}}}{(\boldsymbol{E}(w(b-a)^{\rho};p)^m(b-a)^{m\sigma-1}}\\ &\quad\times\phi\left(\boldsymbol{E}(w(b-a)^{\rho};p)\frac{(b-a)^{\sigma-\frac{1}{m}}}{l^{\frac{1}{l}}\left(\sigma-\frac{1}{m}\right)^{\frac{1}{l}}}\left(\int_a^b|v(x)|^m dx\right)^{\frac{1}{m}}\right) \qquad (3.9)\\ &\leq \frac{m l^{\frac{m}{l}}\left(\sigma-\frac{1}{m}\right)^{\frac{m}{l}}}{(\boldsymbol{E}(w(b-a)^{\rho};p))^m(b-a)^{m\sigma}}\\ &\quad\times\int_a^b\phi\left(\boldsymbol{E}(w(b-a)^{\rho};p)\frac{(b-a)^{\sigma}}{l^{\frac{1}{l}}\left(\sigma-\frac{1}{m}\right)^{\frac{1}{l}}}|v(x)|\right)dx. \qquad (3.10)\end{aligned}$$

Proof. Let us define the kernel $K(x,t)$ as (3.7) and the extended generalized fractional integral operators $\boldsymbol{\varepsilon}v$ by u as (3.8). Using the same argument as in the proof of Theorem 3.6 we obtain

$$\boldsymbol{E}(w(x-t)^{\rho};p)\leq\boldsymbol{E}(w(b-a)^{\rho};p)$$

from which follows

$$\begin{aligned}\left(\int_a^x (K(x,t))^l dt\right)^{\frac{1}{l}} &\leq \boldsymbol{E}(w(b-a)^{\rho};p)\left(\int_a^x (x-t)^{l(\sigma-1)}dt\right)^{\frac{1}{l}}\\ &= \boldsymbol{E}(w(b-a)^{\rho};p)\frac{(x-a)^{\sigma-\frac{1}{m}}}{l^{\frac{1}{l}}\left(\sigma-\frac{1}{m}\right)^{\frac{1}{l}}}.\end{aligned}$$

Since $\sigma>\frac{1}{m}$, if we use

$$M=\boldsymbol{E}(w(b-a)^{\rho};p)\frac{(b-a)^{\sigma-\frac{1}{m}}}{l^{\frac{1}{l}}\left(\sigma-\frac{1}{m}\right)^{\frac{1}{l}}} \qquad (3.11)$$

and $u(x)=(\boldsymbol{\varepsilon}v)(x;p)$, then by Theorem 3.5 inequalities (3.9) and (3.10) follow, respectively. □

Remark 3.2 Let the assumptions of Theorem 3.7 hold. For $\sigma > 1$ we have

$$\begin{aligned}
&\int_a^b \left|(\boldsymbol{\varepsilon} v)(x;p)\right|^{1-m} \phi'\left(\left|(\boldsymbol{\varepsilon} v)(x;p)\right|\right) |v(x)|^m \, dx \\
&\leq \frac{m}{(\boldsymbol{E}(w(b-a)^{\rho};p)^m (b-a)^{m\sigma-1}} \\
&\quad \times \phi\left(\boldsymbol{E}(w(b-a)^{\rho};p)(b-a)^{\sigma-\frac{1}{m}}\left(\int_a^b |v(x)|^m \, dx\right)^{\frac{1}{m}}\right) \\
&\leq \frac{m}{(\boldsymbol{E}(w(b-a)^{\rho};p))^m (b-a)^{m\sigma}} \\
&\quad \times \int_a^b \phi\left(\boldsymbol{E}(w(b-a)^{\rho};p)(b-a)^{\sigma} |v(x)|\right) dx.
\end{aligned}$$

These inequalities will follow if we use the next estimate

$$K(x,t) \leq \boldsymbol{E}(w(b-a)^{\rho};p)(b-a)^{\sigma-1}, \quad \sigma > 1, q < r + \Re(\rho),$$

and set M to be

$$\begin{aligned}
\left(\int_a^x (K(x,t))^l \, dt\right)^{\frac{1}{l}} &\leq \boldsymbol{E}(w(b-a)^{\rho};p)(b-a)^{\sigma-1}\left(\int_a^x dt\right)^{\frac{1}{l}} \\
&\leq \boldsymbol{E}(w(b-a)^{\rho};p)(b-a)^{\sigma-\frac{1}{m}} = M.
\end{aligned}$$

Setting $p = 0$, we get [62, Theorem 2.5], hence Theorem 3.7 is a generalization and an improvement of [62, Theorem 2.5].

Also, others corresponding fractional integral inequalities can be obtain by fixing certain parameters in Theorem 3.7, such as:

(i) setting $p = 0$ and $\tau = r = q = 1$, we get Opial type inequalities for Prabhakar fractional integral operator defined in [123],

(ii) setting $p = 0$, $\tau = r = q = 1$ and $w = 0$, we get the result for left-sided Riemann-Liouville fractional integral in [8, Theorem 3.1].

If we consider $\phi(x) = x^{l+m}$ in Theorem 3.7, then we have the following.

Corollary 3.2 *Let $w \in \mathbb{R}$, $r, \rho, \tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let l, m and u be the same as in Theorem 3.5. Then for $\sigma > \frac{1}{m}$ following inequalities hold*

$$\begin{aligned}
&\int_a^b \left|(\boldsymbol{\varepsilon} v)(x;p)\right|^l |v(x)|^m \, dx \\
&\leq \frac{(\boldsymbol{E}(w(b-a)^{\rho};p))^l (b-a)^{l\left(\sigma-\frac{1}{m}\right)}}{l^2\left(\sigma-\frac{1}{m}\right)}\left(\int_a^b |v(x)|^m \, dx\right)^l \\
&\leq \frac{(\boldsymbol{E}(w(b-a)^{\rho};p))^l (b-a)^{l\sigma}}{l^2\left(\sigma-\frac{1}{m}\right)} \int_a^b |v(x)|^{l+m} \, dx.
\end{aligned}$$

3.2 Properties of Associated Fractional Linear Functional

Motivated by the inequalities given in Theorems 3.4 and 3.5, we have studied functionals derived from them, and we applied them with respect to several types of fractional integrals and fractional derivatives (see [9, 10], also [21]). Here we consider one of them, a nonnegative difference in (3.2) define as follows

$$\Psi_\phi(u,v) = \frac{m}{M^m(b-a)} \int_a^b \phi\left((b-a)^{\frac{1}{m}} M \left|v(x)\right|\right) dx - \int_a^b \left|u(x)\right|^{1-m} \phi'\left(\left|u(x)\right|\right) \left|v(x)\right|^m dx.$$

For this functional we have the following mean value theorems (given in [9], also [21]):

Theorem 3.8 ([21, p. 103]) *Let $\phi : [0,\infty) \to \mathbb{R}$ be a differentiable function such that for $m > 1$ the function $\phi(x^{\frac{1}{m}})$ is convex and $\phi(0) = 0$. Let $u \in U(v,K)$ where $(\int_a^x (K(x,t))^l dt)^{\frac{1}{l}} \leq M$ and $l^{-1} + m^{-1} = 1$. If $\phi \in C^2(I)$, where $I \subseteq [0,\infty)$ is closed interval, then there exists $\xi \in I$ such that the following equality holds*

$$\psi_\phi(u,v) = \frac{\xi\phi''(\xi) - (m-1)\phi'(\xi)}{2m\xi^{2m-1}} \times \left((b-a)M^m \int_a^b \left|v(x)\right|^{2m} dx - 2\int_a^b \left|u(x)\right|^m \left|v(x)\right|^m dx\right).$$

Theorem 3.9 ([21, p. 104]) *Let $\phi_1, \phi_2 : [0,\infty) \to \mathbb{R}$ be differentiable functions such that for $m > 1$ the function $\phi_i(x^{\frac{1}{m}})$ is convex and $\phi_i(0) = 0$, $i = 1,2$. Let $u \in U(v,K)$, where $(\int_a^x (K(x,t))^l dt)^{\frac{1}{l}} \leq M$ and $l^{-1} + m^{-1} = 1$. If $\phi_1, \phi_2 \in C^2(I)$, where $I \subseteq [0,\infty)$ is closed interval and*

$$(b-a)M^m \int_a^b |v(x)|^{2m} dx - 2\int_a^b |u(x)|^q |v(x)|^m dx \neq 0,$$

then there exists $\xi \in I$ such that we have

$$\frac{\psi_{\phi_1}(u,v)}{\psi_{\phi_2}(u,v)} = \frac{\xi\phi_1''(\xi) - (m-1)\phi_1'(\xi)}{\xi\phi_2''(\xi) - (m-1)\phi_2'(\xi)},$$

provided that denominators are not equal to zero.

Hence, we study a functional derived from the inequality (3.10), i.e. its nonnegative difference, which we denote by Ψ_ϕ:

$$\Psi_\phi(u,v) = \frac{m l^{\frac{m}{l}} \left(\sigma - \frac{1}{m}\right)^{\frac{m}{l}}}{(\boldsymbol{E}(w(b-a)^\rho;p))^m (b-a)^{m\sigma}} \times \int_a^b \phi\left(\boldsymbol{E}(w(b-a)^\rho;p) \frac{(b-a)^\sigma}{l^{\frac{1}{l}} \left(\sigma - \frac{1}{m}\right)^{\frac{1}{l}}} |v(x)|\right) dx - \int_a^b |(\boldsymbol{\varepsilon}v)(x;p)|^{1-m} \phi'\left(|(\boldsymbol{\varepsilon}v)(x;p)|\right) |v(x)|^m dx.$$

A simplified notation $\boldsymbol{E}$ and $\boldsymbol{\varepsilon} f$ as in (3.3) and (3.4) is used. For this functional we present following mean value theorems.

Theorem 3.10 *Let $w \in \mathbb{R}$, $r, \rho, \tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let $\phi : [0, \infty) \to \mathbb{R}$ be a differentiable function such that for $m > 1$ the function $\phi(x^{\frac{1}{m}})$ is convex and $\phi(0) = 0$. Let $l^{-1} + m^{-1} = 1$, $\sigma > \frac{1}{m}$ and $v \in L_1[a,b]$. If $\phi \in C^2(I)$, where $I \subseteq [0, \infty)$ is closed interval, then there exists $\xi \in I$ such that the following equality holds*

$$\Psi_\phi\left(\left(\boldsymbol{\varepsilon} v\right)(x;p), v(x)\right) = \frac{\xi \phi''(\xi) - (m-1)\phi'(\xi)}{2m\xi^{2m-1}}$$

$$\times \left[\frac{\left(\boldsymbol{E}(w(b-a)^\rho; p)\right)^m (b-a)^{m\sigma}}{l^{\frac{m}{l}} \left(\sigma - \frac{1}{m}\right)^{\frac{m}{l}}} \int_a^b |v(x)|^{2m} dx \right.$$
$$\left. -2 \int_a^b \left|\left(\boldsymbol{\varepsilon} v\right)(x;p)\right|^m |v(x)|^m dx \right]. \tag{3.12}$$

Proof. It follows directly for M defined by (3.11), function $u(x) = (\boldsymbol{\varepsilon} v)(x;p)$ and Theorem 3.8. □

Theorem 3.11 *Let $w \in \mathbb{R}$, $r, \rho, \tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let $\phi_1, \phi_2 : [0, \infty) \to \mathbb{R}$ be differentiable functions such that for $m > 1$ the function $\phi_i(x^{\frac{1}{m}})$ is convex and $\phi_i(0) = 0$, $i = 1, 2$. Let $l^{-1} + m^{-1} = 1$, $\sigma > \frac{1}{m}$ and $v \in L_1[a,b]$. If $\phi_1, \phi_2 \in C^2(I)$, where $I \subseteq [0, \infty)$ is closed interval, then there exists $\xi \in I$ such that the following equality holds*

$$\frac{\psi_{\phi_1}\left(\left(\boldsymbol{\varepsilon} v\right)(x;p), v(x)\right)}{\psi_{\phi_2}\left(\left(\left(\boldsymbol{\varepsilon} v\right)(x;p), v(x)\right)\right.} = \frac{\xi \phi_1''(\xi) - (m-1)\phi_1'(\xi)}{\xi \phi_2''(\xi) - (m-1)\phi_2'(\xi)}, \tag{3.13}$$

provided that denominators are not equal to zero.

Proof. It follows directly for the function $u(x) = (\boldsymbol{\varepsilon} v)(x;p)$ and Theorem 3.9. □

Remark 3.3 For different choices of parameters we can get corresponding fractional integral inequalities, such as:

(i) setting $p = 0$ in (3.12) and (3.13), using the same arguments as in Remark 3.2 we get improvements of Theorem 2.11 and Theorem 2.12 in [62], respectively,

(ii) setting $p = 0$, $\tau = r = q = 1$ and $w = 0$ in (3.12) and (3.13), we get results for left-sided Riemann-Liouville fractional integral in Theorem 4.1 and Theorem 4.2 in [9], respectively.

Chapter 4

Pólya-Szegő and Chebyshev Types Fractional Integral Inequalities Associated with the Mittag-Leffler Function

The Chebyshev functional $T(f,g)$ for two Lebesgue integrable functions f and g on interval $[a,b]$ is defined by

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right). \qquad (4.1)$$

Majority of problems involving Chebyshev functional are to give a lower bound or an upper bound for T, under various assumptions. For instance, if f and g are monotonic in the same sense (in the opposite sense) then we obtain a well-known Chebyshev inequality ([32])

$$T(f,g) \geq 0 \quad (\leq 0). \qquad (4.2)$$

Also, if we have constants $m, M, n, N \in \mathbb{R}$ such that for $x \in [a,b]$

$$m \leq f(x) \leq M, \quad n \leq g(x) \leq N,$$

then the Grüss inequality ([68]) states

$$|T(f,g)| \leq \frac{(M-m)(N-n)}{4}. \qquad (4.3)$$

For more recent inequalities see [28, 39, 41, 84, 114, 115, 120]. Following inequalities are the subject of our research: the first one was introduced by Pólya and Szegő ([122])

$$\frac{\left(\int_a^b f^2(x)dx\right)\left(\int_a^b g^2(x)dx\right)}{\left(\int_a^b f(x)g(x)dx\right)^2} \le \frac{1}{4}\left(\sqrt{\frac{mn}{MN}}+\sqrt{\frac{MN}{mn}}\right)^2. \tag{4.4}$$

Next is the inequality by Dragomir and Diamond ([43])

$$|T(f,g)| \le \frac{(M-m)(N-n)}{4\sqrt{mMnN}}\frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx. \tag{4.5}$$

Using Karamata's estimations of the Chebyshev quotient ([89]), Pečarić and Perić give generalized and improved inequality of (4.5) for positive normalized functional Φ in ([120])

$$\begin{aligned}
-\frac{(M-m)(N-n)}{\left(\sqrt{mN}+\sqrt{Mn}\right)^2}\Phi(fg) \ &\le \ -\frac{(M-m)(N-n)}{\left(\sqrt{mn}+\sqrt{MN}\right)^2}\Phi(f)\Phi(g) \le \ \Phi(fg)-\Phi(f)\Phi(g) \\
\le \ \frac{(M-m)(N-n)}{\left(\sqrt{mn}+\sqrt{MN}\right)^2}\Phi(fg) \ &\le \ (M-m)(N-n)\left(\sqrt{mN}+\sqrt{Mn}\right)^2\Phi(f)\Phi(9).
\end{aligned} \tag{4.6}$$

Motivated by the paper [114], where authors have proved Pólya-Szegő and Chebyshev types fractional integral inequalities for the Riemann-Liouville fractional integral operator, we presents improved and generalized corresponding results using our extended generalized Mittag-Leffler function with its fractional integral operator.

The right-sided versions of following inequalities can be established and proved analogously by using the right-sided fractional integral operator $\varepsilon_{b^-,\rho,\sigma,\tau}^{w,\delta,c,q,r}f$ defined with (2.13).

This chapter is based on our results from [14].

4.1 Pólya-Szegő Type Fractional Integral Inequalities and an Extended Generalized Mittag-Leffler Function

In this section we use extended generalized Mittag-Leffler function $E_{\rho,\sigma,\tau}^{\delta,c,q,r}$ with the corresponding fractional integral operator $\varepsilon_{a^+,\rho,\sigma,\tau}^{w,\delta,c,q,r}f$ (in real domain) to obtain fractional generalizations of inequalities due to Pólya and Szegő. Following theorems are based on [114] where this was done for the Riemann-Liouville fractional integral operator. The role of the parameter $\sigma > 0$ will be of great significance and for the reader's convenience we will use a simplified notation

$$\begin{aligned}
\boldsymbol{E}_\sigma(z;p) \ &:= \ E_{\rho,\sigma,\tau}^{\delta,c,q,r}(z;p) \\
&= \sum_{n=0}^{\infty}\frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}(\delta,c-\delta)}\frac{(c)_{nq}}{\Gamma(\rho n+\sigma)}\frac{z^n}{(\tau)_{nr}},
\end{aligned} \tag{4.7}$$

$$(\boldsymbol{\varepsilon}_\sigma f)(x;p) := \left(\varepsilon_{a^+,\rho,\sigma,\tau}^{w,\delta,c,q,r} f\right)(x;p) \tag{4.8}$$
$$= \int_a^x (x-t)^{\sigma-1} \boldsymbol{E}_\sigma(w(x-t)^\rho;p) f(t)dt.$$

Theorem 4.1 *Let $w \in \mathbb{R}$, $r, \rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let $f, g, \varphi_1, \varphi_2, \psi_1$ and ψ_2 be positive integrable functions on $[0,\infty)$ satisfying*

$$0 < \varphi_1(u) \leq f(u) \leq \varphi_2(u), \quad 0 < \psi_1(u) \leq g(u) \leq \psi_2(u), \quad u \in [a,x]. \tag{4.9}$$

Then the following inequality holds

$$\frac{\left(\boldsymbol{\varepsilon}_\sigma \psi_1 \psi_2 f^2\right)(x;p) \left(\boldsymbol{\varepsilon}_\sigma \varphi_1 \varphi_2 g^2\right)(x;p)}{\left[\left(\boldsymbol{\varepsilon}_\sigma(\varphi_1\psi_1 + \varphi_2\psi_2) fg\right)(x;p)\right]^2} \leq \frac{1}{4}. \tag{4.10}$$

Proof. From the given conditions follows

$$\left(\frac{\varphi_2(u)}{\psi_1(u)} - \frac{f(u)}{g(u)}\right)\left(\frac{f(u)}{g(u)} - \frac{\varphi_1(u)}{\psi_2(u)}\right) \geq 0,$$

that is

$$(\varphi_1(u)\psi_1(u) + \varphi_2(u)\psi_2(u)) f(u)g(u) \geq \psi_1(u)\psi_2(u)(f(u))^2 + \varphi_1(u)\varphi_2(u)(g(u))^2.$$

Multiplying above inequality by $(x-u)^{\sigma-1}\boldsymbol{E}_\sigma(w(x-u)^\rho;p)$ and integrating on $[a,x]$ we obtain

$$\int_a^x (x-u)^{\sigma-1}\boldsymbol{E}_\sigma(w(x-u)^\rho;p)(\varphi_1(u)\psi_1(u) + \varphi_2(u)\psi_2(u)) f(u)g(u)du$$
$$\geq \int_a^x (x-u)^{\sigma-1}\boldsymbol{E}_\sigma(w(x-u)^\rho;p)\psi_1(u)\psi_2(u)(f(u))^2 du$$
$$+ \int_a^x (x-u)^{\sigma-1}\boldsymbol{E}_\sigma(w(x-u)^\rho;p)\varphi_1(u)\varphi_2(u)(g(u))^2 du,$$

that is

$$\left(\boldsymbol{\varepsilon}_\sigma(\varphi_1\psi_1 + \varphi_2\psi_2) fg\right)(x;p)$$
$$\geq \left(\boldsymbol{\varepsilon}_\sigma \psi_1\psi_2 f^2\right)(x;p) + \left(\boldsymbol{\varepsilon}_\sigma \varphi_1\varphi_2 g^2\right)(x;p).$$

Since $a + b \geq 2\sqrt{ab}$ for $a, b \in \mathbb{R}$ (the AM-GM inequality), we have

$$\left(\boldsymbol{\varepsilon}_\sigma(\varphi_1\psi_1 + \varphi_2\psi_2) fg\right)(x;p)$$
$$\geq 2\sqrt{\left(\boldsymbol{\varepsilon}_\sigma \psi_1\psi_2 f^2\right)(x;p)\left(\boldsymbol{\varepsilon}_\sigma \varphi_1\varphi_2 g^2\right)(x;p)},$$

which leads to the inequality (4.10) as required. □

Fixing the bounds on functions f and g we get the following special case of Theorem 4.1.

Corollary 4.1 *Let $w \in \mathbb{R}$, $r, \rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let f and g be two positive integrable functions on $[0, \infty)$ satisfying*

$$0 < m \leq f(u) \leq M < \infty, \quad 0 < n \leq g(u) \leq N < \infty, \quad u \in [a, x]. \tag{4.11}$$

Then the following inequality holds

$$\frac{\left(\boldsymbol{\varepsilon}_\sigma f^2\right)(x;p)\left(\boldsymbol{\varepsilon}_\sigma g^2\right)(x;p)}{\left[\left(\boldsymbol{\varepsilon}_\sigma fg\right)(x;p)\right]^2} \leq \frac{1}{4}\left(\sqrt{\frac{mn}{MN}} + \sqrt{\frac{MN}{mn}}\right)^2.$$

Remark 4.1 Choosing particular values of parameters in Theorem 4.1, known Pólya-Szegő type inequalities for several fractional integral operators can be deduced (for more details on fractional integral operators see [15] and references therein). For example, setting $w = p = 0$ (and $a = 0$), we get Pólya-Szegő inequality for the Riemann-Liouville fractional integral operator given in [114, Lemma 3.1].

Now we prove the next Pólya-Szegő type inequality.

Theorem 4.2 *Suppose that the assumptions of Theorem 4.1 hold with $\beta > 0$. Then*

$$\frac{\left(\boldsymbol{\varepsilon}_\sigma \varphi_1\varphi_2\right)(x;p)\left(\boldsymbol{\varepsilon}_\beta \psi_1\psi_2\right)(x;p)\left(\boldsymbol{\varepsilon}_\sigma f^2\right)(x;p)\left(\boldsymbol{\varepsilon}_\beta g^2\right)(x;p)}{\left[\left(\boldsymbol{\varepsilon}_\sigma \varphi_1 f\right)(x;p)\left(\boldsymbol{\varepsilon}_\beta \psi_1 g\right)(x;p) + \left(\boldsymbol{\varepsilon}_\sigma \varphi_2 f\right)(x;p)\left(\boldsymbol{\varepsilon}_\beta \psi_2 g\right)(x;p)\right]^2} \leq \frac{1}{4}. \tag{4.12}$$

Proof. Under given conditions on f, g and $\varphi_i, \psi_i (i = 1, 2)$ in (4.9), for $u, v \in [a, t]$ we have

$$\frac{\varphi_2(u)}{\psi_1(v)} - \frac{f(u)}{g(v)} \geq 0 \quad \text{and} \quad \frac{f(u)}{g(v)} - \frac{\varphi_1(u)}{\psi_2(v)} \geq 0,$$

which imply

$$\left(\frac{\varphi_1(u)}{\psi_2(v)} + \frac{\varphi_2(u)}{\psi_1(v)}\right)\frac{f(u)}{g(v)} \geq \frac{(f(u))^2}{(g(v))^2} + \frac{\varphi_1(u)\varphi_2(u)}{\psi_1(v)\psi_2(v)}$$

that is

$$\varphi_1(u)f(u)\psi_1(v)g(v) + \varphi_2(u)f(u)\psi_2(v)g(v) \geq \psi_1(v)\psi_2(v)(f(u))^2 + \varphi_1(u)\varphi_2(u)(g(v))^2.$$

Multiplying above inequality by

$$(x-u)^{\sigma-1}(x-v)^{\beta-1}\boldsymbol{E}_\sigma(w(x-u)^\rho;p)\boldsymbol{E}_\beta(w(x-v)^\rho;p)$$

and integrating, we obtain

$$\begin{aligned}
&\int_a^x\int_a^x (x-u)^{\sigma-1}(x-v)^{\beta-1}\boldsymbol{E}_\sigma(w(x-u)^\rho;p) \\
&\quad\times\boldsymbol{E}_\beta(w(x-v)^\rho;p)\varphi_1(u)f(u)\psi_1(v)g(v)\,du\,dv \\
&+\int_a^x\int_a^x (x-u)^{\sigma-1}(x-v)^{\beta-1}\boldsymbol{E}_\sigma(w(x-u)^\rho;p) \\
&\quad\times\boldsymbol{E}_\beta(w(x-v)^\rho;p)\varphi_2(u)f(u)\psi_2(v)g(v)\,du\,dv
\end{aligned}$$

$$\begin{aligned}
&\geq \\
&\int_a^x\int_a^x (x-u)^{\sigma-1}(x-v)^{\beta-1}\boldsymbol{E}_\sigma(w(x-u)^\rho;p) \\
&\times\boldsymbol{E}_\beta(w(x-v)^\rho;p)\psi_1(v)\psi_2(v)(f(u))^2dudv \\
&+\int_a^x\int_a^x (x-u)^{\sigma-1}(x-v)^{\beta-1}\boldsymbol{E}_\sigma(w(x-u)^\rho;p) \\
&\times\boldsymbol{E}_\beta(w(x-v)^\rho;p)\varphi_1(u)\varphi_2(u)(g(v))^2dudv,
\end{aligned}$$

that is

$$\begin{aligned}
&(\boldsymbol{\varepsilon}_\sigma\varphi_1 f)(x;p)\left(\boldsymbol{\varepsilon}_\beta\psi_1 g\right)(x;p)+(\boldsymbol{\varepsilon}_\sigma\varphi_2 f)(x;p)\left(\boldsymbol{\varepsilon}_\beta\psi_2 g(x;p)\right) \\
&\geq \left(\boldsymbol{\varepsilon}_\sigma f^2\right)(x;p)\left(\boldsymbol{\varepsilon}_\beta\psi_1\psi_2\right)(x;p)+(\boldsymbol{\varepsilon}_\sigma\varphi_1\varphi_2)(x;p)\left(\boldsymbol{\varepsilon}_\beta g^2\right)(x;p).
\end{aligned}$$

Now if we apply the AM-GM inequality we get

$$\begin{aligned}
&(\boldsymbol{\varepsilon}_\sigma\varphi_1 f)(x;p)\left(\boldsymbol{\varepsilon}_\beta\psi_1 g\right)(x;p)+(\boldsymbol{\varepsilon}_\sigma\varphi_2 f)(x;p)\left(\boldsymbol{\varepsilon}_\beta\psi_2 g\right)(x;p) \\
&\geq 2\sqrt{(\boldsymbol{\varepsilon}_\sigma f^2)(x;p)\left(\boldsymbol{\varepsilon}_\beta\psi_1\psi_2\right)(x;p)(\boldsymbol{\varepsilon}_\sigma\varphi_1\varphi_2)(x;p)\left(\boldsymbol{\varepsilon}_\beta g^2\right)(x;p)}
\end{aligned}$$

which leads to the inequality (4.12). □

In the results that follow, we need next equality from Corollary 2.2:

$$(\boldsymbol{\varepsilon}_\sigma 1)(x;p)=(x-a)^\sigma\boldsymbol{E}_{\sigma+1}(w(x-a)^\rho;p). \tag{4.13}$$

We continue with a special case of Theorem 4.2.

Corollary 4.2 *Suppose that the assumptions of Corollary 4.1 hold with $\beta>0$. Then*

$$\frac{(\boldsymbol{\varepsilon}_\sigma 1)(x;p)\left(\boldsymbol{\varepsilon}_\beta 1\right)(x;p)\left(\boldsymbol{\varepsilon}_\sigma f^2\right)(x;p)\left(\boldsymbol{\varepsilon}_\beta g^2\right)(x;p)}{\left[(\boldsymbol{\varepsilon}_\sigma f)(x;p)\left(\boldsymbol{\varepsilon}_\beta g\right)(x;p)\right]^2}\leq\frac{1}{4}\left(\sqrt{\frac{mn}{MN}}+\sqrt{\frac{MN}{mn}}\right)^2.$$

Theorem 4.3 *Suppose that the assumptions of Theorem 4.1 hold with $\beta>0$. Then*

$$\begin{aligned}
&\left(\boldsymbol{\varepsilon}_\sigma f^2\right)(x;p)\left(\boldsymbol{\varepsilon}_\beta g^2\right)(x;p) \\
&\leq\left(\boldsymbol{\varepsilon}_\sigma\left(\varphi_2 fg/\psi_1\right)\right)(x;p)\left(\boldsymbol{\varepsilon}_\beta(\psi_2 fg/\varphi_1)\right)(x;p).
\end{aligned}\tag{4.14}$$

Proof. Under given conditions on f,g and $\varphi_i,\psi_i (i=1,2)$ in (4.9), for $u,v\in[a,t]$ we have

$$\frac{\varphi_2(u)f(u)g(u)}{\psi_1(u)}-(f(u))^2\geq 0 \quad\text{and}\quad \frac{\psi_2(v)f(v)g(v)}{\varphi_1(v)}-(g(v))^2\geq 0,$$

hence

$$\begin{aligned}
&\int_a^x (x-u)^{\sigma-1}\boldsymbol{E}_\sigma(w(x-u)^\rho;p)(f(u))^2du \\
&\leq\int_a^x (x-u)^{\sigma-1}\boldsymbol{E}_\sigma(w(x-u)^\rho;p)\frac{\varphi_2(u)}{\psi_1(u)}f(u)g(u)du,
\end{aligned}$$

and

$$\int_a^x (x-v)^{\beta-1}\boldsymbol{E}_\beta(w(x-v)^\rho;p)(g(v))^2dv$$
$$\leq \int_a^x (x-v)^{\beta-1}\boldsymbol{E}_\beta(w(x-v)^\rho;p)\frac{\psi_2(v)}{\varphi_1(v)}f(v)g(v)dv,$$

which imply

$$\left(\boldsymbol{\varepsilon}_\sigma f^2\right)(x;p) \leq \left(\boldsymbol{\varepsilon}_\sigma(\varphi_2 fg/\psi_1)\right)(x;p),$$
$$\left(\boldsymbol{\varepsilon}_\beta g^2\right)(x;p) \leq \left(\boldsymbol{\varepsilon}_\beta(\psi_2 fg/\varphi_1)\right)(x;p).$$

Multiplying above inequalities we obtain (4.14). □

Corollary 4.3 *Suppose that the assumptions of Corollary 4.1 hold with $\beta > 0$. Then*

$$\frac{\left(\boldsymbol{\varepsilon}_\sigma f^2\right)(x;p)\left(\boldsymbol{\varepsilon}_\beta g^2\right)(x;p)}{\left(\boldsymbol{\varepsilon}_\sigma fg\right)(x;p)\left(\boldsymbol{\varepsilon}_\beta fg\right)(x;p)} \leq \frac{MN}{mn}.$$

Remark 4.2 As before, choosing particular values of parameters in Theorem 4.2 and Theorem 4.3, known Pólya-Szegő type inequalities for several fractional integral operators can be deduced, such as inequalities for the Riemann-Liouville fractional integral operator in [114, Lemma 3.3, Lemma 3.4] if we set $w = p = 0$.

4.2 Chebyshev Type Fractional Integral Inequalities and an Extended Generalized Mittag-Leffler Function

Using Pólya-Szegő type inequality in Theorem 4.1, we obtain following Chebyshev inequalities based on [120] and [114].

A simplified notation $\boldsymbol{E}_\sigma$ and $\boldsymbol{\varepsilon}_\sigma f$ as in (4.7) and (4.8) is used here.

Theorem 4.4 *Let $w \in \mathbb{R}$, $r, \rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let $f, g, \varphi_1, \varphi_2, \psi_1$ and ψ_2 be positive integrable functions on $[0,\infty)$ functions satisfying*

$$0 < \varphi_1(u) \leq f(u) \leq \varphi_2(u), \quad 0 < \psi_1(u) \leq g(u) \leq \psi_2(u), \quad u \in [a,x].$$

Suppose also $\beta > 0$. Then

$$\begin{aligned}
&\big|\left(\boldsymbol{\varepsilon}_\sigma 1\right)(x;p)\left(\boldsymbol{\varepsilon}_\beta fg\right)(x;p) + \left(\boldsymbol{\varepsilon}_\beta 1\right)(x;p)\left(\boldsymbol{\varepsilon}_\sigma fg\right)(x;p) \\
&\quad - \left(\boldsymbol{\varepsilon}_\sigma f\right)(x;p)\left(\boldsymbol{\varepsilon}_\beta g\right)(x;p) - \left(\boldsymbol{\varepsilon}_\sigma g\right)(x;p)\left(\boldsymbol{\varepsilon}_\beta f\right)(x;p)\big| \\
&\leq \left|G_{\sigma,\beta}(f,\varphi_1,\varphi_2)(x;p) + G_{\beta,\sigma}(f,\varphi_1,\varphi_2)(x;p)\right|^{\frac{1}{2}} \\
&\quad \times \left|G_{\sigma,\beta}(g,\psi_1,\psi_2)(x;p) + G_{\beta,\sigma}(g,\psi_1,\psi_2)(x;p)\right|^{\frac{1}{2}},
\end{aligned} \tag{4.15}$$

where

$$G_{\sigma,\beta}(u,v,w)(x;p) = \frac{\left(\boldsymbol{\varepsilon}_{\beta}1\right)(x;p)\left[\left(\boldsymbol{\varepsilon}_{\sigma}(v+w)u\right)(x;p)\right]^2}{4\left(\boldsymbol{\varepsilon}_{\sigma}vw\right)(x;p)} - \left(\boldsymbol{\varepsilon}_{\sigma}u\right)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}u\right)(x;p). \tag{4.16}$$

Proof. Let f and g be two positive integrable function on $[0,\infty)$. For $u,v \in [a,t]$ we define $A(u,v)$ as

$$A(u,v) = (f(u)-f(v))(g(u)-g(v)),$$

that is

$$A(u,v) = f(u)g(u)+f(v)g(v)-f(u)g(v)-f(v)g(u).$$

Multiplying above equality by

$$(x-u)^{\sigma-1}(x-v)^{\beta-1}\boldsymbol{E}_{\sigma}(w(x-u)^{\rho};p)\boldsymbol{E}_{\beta}(w(x-v)^{\rho};p)$$

and integrating, we obtain

$$\begin{aligned}\int_a^x\int_a^x &(x-u)^{\sigma-1}(x-v)^{\beta-1}\boldsymbol{E}_{\sigma}(w(x-u)^{\rho};p)\boldsymbol{E}_{\beta}(w(x-v)^{\rho};p)A(u,v)dudv\\ &= \left(\boldsymbol{\varepsilon}_{\beta}1\right)(x;p)\left(\boldsymbol{\varepsilon}_{\sigma}fg\right)(x;p)+\left(\boldsymbol{\varepsilon}_{\sigma}1\right)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}fg\right)(x;p)\\ &\quad -\left(\boldsymbol{\varepsilon}_{\sigma}f\right)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}g\right)(x;p)-\left(\boldsymbol{\varepsilon}_{\beta}f\right)(x;p)\left(\boldsymbol{\varepsilon}_{\sigma}g\right)(x;p).\end{aligned} \tag{4.17}$$

By the Cauchy-Schwartz inequality for double integrals we have

$$\begin{aligned}&\left|\int_a^x\int_a^x(x-u)^{\sigma-1}(x-v)^{\beta-1}\boldsymbol{E}_{\sigma}(w(x-u)^{\rho};p)\boldsymbol{E}_{\beta}(w(x-v)^{\rho};p)A(u,v)dudv\right|\\ &\leq \left[\int_a^x\int_a^x(x-u)^{\sigma-1}(x-v)^{\beta-1}\boldsymbol{E}_{\sigma}(w(x-u)^{\rho};p)\boldsymbol{E}_{\beta}(w(x-v)^{\rho};p)(f(u))^2dudv\right.\\ &\quad +\int_a^x\int_a^x(x-u)^{\sigma-1}(x-v)^{\beta-1}\boldsymbol{E}_{\sigma}(w(x-u)^{\rho};p)\boldsymbol{E}_{\beta}(w(x-v)^{\rho};p)(f(v))^2dudv\\ &\quad \left.-2\int_a^x\int_a^x(x-u)^{\sigma-1}(x-v)^{\beta-1}\boldsymbol{E}_{\sigma}(w(x-u)^{\rho};p)\boldsymbol{E}_{\beta}(w(x-v)^{\rho};p)f(u)f(v)dudv\right]^{\frac{1}{2}}\\ &\quad \times\left[\int_a^x\int_a^x(x-u)^{\sigma-1}(x-v)^{\beta-1}\boldsymbol{E}_{\sigma}(w(x-u)^{\rho};p)\boldsymbol{E}_{\beta}(w(x-v)^{\rho};p)(g(u))^2dudv\right.\\ &\quad +\int_a^x\int_a^x(x-u)^{\sigma-1}(x-v)^{\beta-1}\boldsymbol{E}_{\sigma}(w(x-u)^{\rho};p)\boldsymbol{E}_{\beta}(w(x-v)^{\rho};p)(g(v))^2dudv\\ &\quad \left.-2\int_a^x\int_a^x(x-u)^{\sigma-1}(x-v)^{\beta-1}\boldsymbol{E}_{\sigma}(w(x-u)^{\rho};p)\boldsymbol{E}_{\beta}(w(x-v)^{\rho};p)g(u)g(v)dudv\right]^{\frac{1}{2}}\\ &= \left[\left(\boldsymbol{\varepsilon}_{\beta}1\right)(x;p)\left(\boldsymbol{\varepsilon}_{\sigma}f^2\right)(x;p)+\left(\boldsymbol{\varepsilon}_{\sigma}1\right)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}f^2\right)(x;p)\right.\\ &\quad \left.-2\left(\boldsymbol{\varepsilon}_{\sigma}f\right)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}f\right)(x;p)\right]^{\frac{1}{2}}\\ &\quad \times\left[\left(\boldsymbol{\varepsilon}_{\beta}1\right)(x;p)\left(\boldsymbol{\varepsilon}_{\sigma}g^2\right)(x;p)+\left(\boldsymbol{\varepsilon}_{\sigma}1\right)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}g^2\right)(x;p)\right.\\ &\quad \left.-2\left(\boldsymbol{\varepsilon}_{\sigma}g\right)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}g\right)(x;p)\right]^{\frac{1}{2}}\end{aligned} \tag{4.18}$$

For $\psi_1(t) = \psi_2(t) = g(t) = 1$ by Theorem 4.1 follows

$$\left(\boldsymbol{\varepsilon}_\sigma f^2\right)(x;p) \leq \frac{\left[\left(\boldsymbol{\varepsilon}_\sigma (\varphi_1+\varphi_2) f\right)(x;p)\right]^2}{4\left(\boldsymbol{\varepsilon}_\sigma \varphi_1\varphi_2\right)(x;p)}.$$

This implies

$$\begin{aligned}
&\left(\boldsymbol{\varepsilon}_\beta 1\right)(x;p)\left(\boldsymbol{\varepsilon}_\sigma f^2\right)(x;p) - \left(\boldsymbol{\varepsilon}_\sigma f\right)(x;p)\left(\boldsymbol{\varepsilon}_\beta f\right)(x;p) \\
&\leq \frac{\left(\boldsymbol{\varepsilon}_\beta 1\right)(x;p)\left[\left(\boldsymbol{\varepsilon}_\sigma (\varphi_1+\varphi_2) f\right)(x;p)\right]^2}{4\left(\boldsymbol{\varepsilon}_\sigma \varphi_1\varphi_2\right)(x;p)} - \left(\boldsymbol{\varepsilon}_\sigma f\right)(x;p)\left(\boldsymbol{\varepsilon}_\beta f\right)(x;p) \\
&= G_{\sigma,\beta}(f,\varphi_1,\varphi_2)(x;p)
\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}
&\left(\boldsymbol{\varepsilon}_\sigma 1\right)(x;p)\left(\boldsymbol{\varepsilon}_\beta f^2\right)(x;p) - \left(\boldsymbol{\varepsilon}_\sigma f\right)(x;p)\left(\boldsymbol{\varepsilon}_\beta f\right)(x;p) \\
&\leq \frac{\left(\boldsymbol{\varepsilon}_\sigma 1\right)(x;p)\left[\left(\boldsymbol{\varepsilon}_\beta (\varphi_1+\varphi_2) f\right)(x;p)\right]^2}{4\left(\boldsymbol{\varepsilon}_\beta \varphi_1\varphi_2\right)(x;p)} - \left(\boldsymbol{\varepsilon}_\sigma f\right)(x;p)\left(\boldsymbol{\varepsilon}_\beta f\right)(x;p) \\
&= G_{\beta,\sigma}(f,\varphi_1,\varphi_2)(x;p)
\end{aligned} \tag{4.20}$$

Applying the same procedure for $\varphi_1(t) = \varphi_2(t) = f(t) = 1$, we get

$$\begin{aligned}
&\left(\boldsymbol{\varepsilon}_\beta 1\right)(x;p)\left(\boldsymbol{\varepsilon}_\sigma g^2\right)(x;p) - \left(\boldsymbol{\varepsilon}_\sigma g\right)(x;p)\left(\boldsymbol{\varepsilon}_\beta g\right)(x;p) \\
&\qquad \leq G_{\sigma,\beta}(g,\psi_1,\psi_2)(x;p)
\end{aligned} \tag{4.21}$$

$$\begin{aligned}
&\left(\boldsymbol{\varepsilon}_\sigma 1\right)(x;p)\left(\boldsymbol{\varepsilon}_\beta g^2\right)(x;p) - \left(\boldsymbol{\varepsilon}_\sigma g\right)(x;p)\left(\boldsymbol{\varepsilon}_\beta g\right)(x;p) \\
&\qquad \leq G_{\beta,\sigma}(g,\psi_1,\psi_2)(x;p)
\end{aligned} \tag{4.22}$$

Finally, considering (4.17) to (4.22), we arrive at the desired result in (4.15). This completes the proof. □

Setting $\sigma = \beta$ in Theorem 4.4, next inequality follows.

Corollary 4.4 *Suppose that the assumptions of Theorem 4.4 hold. Then*

$$\begin{aligned}
&\left|\left(\boldsymbol{\varepsilon}_\sigma 1\right)(x;p)\left(\boldsymbol{\varepsilon}_\sigma fg\right)(x;p) - \left(\boldsymbol{\varepsilon}_\sigma f\right)(x;p)\left(\boldsymbol{\varepsilon}_\sigma g\right)(x;p)\right| \\
&\leq \left|G_{\sigma,\sigma}(f,\varphi_1,\varphi_2)(x;p)\, G_{\sigma,\sigma}(g,\psi_1,\psi_2)(x;p)\right|^{\frac{1}{2}}.
\end{aligned}$$

If we set $\varphi_1 = m$, $\varphi_2 = M$, $\psi_1 = n$ and $\psi_2 = N$ in the previous corollary, then we obtain

$$G_{\sigma,\sigma}(f,m,M)(x;p) = \frac{(M-m)^2}{4mM}\left[\left(\boldsymbol{\varepsilon}_\sigma f\right)(x;p)\right]^2,$$

$$G_{\sigma,\sigma}(g,n,N)(x;p) = \frac{(N-n)^2}{4nN}\left[\left(\boldsymbol{\varepsilon}_\sigma g\right)(x;p)\right]^2.$$

Corollary 4.5 *Let $w \in \mathbb{R}$, $r, \rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let f and g be two positive integrable functions on $[0, \infty)$ satisfying*

$$0 < m \leq f(u) \leq M < \infty, \quad 0 < n \leq g(u) \leq N < \infty, \quad u \in [a, x].$$

Then

$$\begin{aligned} &\left| (\boldsymbol{\varepsilon}_{\sigma} 1)(x;p)(\boldsymbol{\varepsilon}_{\sigma} fg)(x;p) - (\boldsymbol{\varepsilon}_{\sigma} f)(x;p)(\boldsymbol{\varepsilon}_{\sigma} g)(x;p) \right| \\ &\quad \leq \frac{(M-m)(N-n)}{4\sqrt{mMnN}} (\boldsymbol{\varepsilon}_{\sigma} f)(x;p)(\boldsymbol{\varepsilon}_{\sigma} g)(x;p). \end{aligned}$$

Remark 4.3 Setting $w = p = 0$ (and $a = 0$) in Theorem 4.4, Corollary 4.4 and Corollary 4.5 we get Pólya-Szegő type inequalities for the Riemann-Liouville fractional integral operator given in [114, Theorem 3.6, Theorem 3.7, Corollary 3.4].

Recently, in [111] Nikolova and Varošanec generalized results from [114] for any two linear isotonic functionals. Here, we will give another approach. In the next theorem we will use Karamata's estimations of the Chebyshev quotient ([89]),

$$\frac{1}{K^2} \leq \frac{\left(\frac{1}{b-a} \int_a^b f(x)dx \right) \left(\frac{1}{b-a} \int_a^b g(x)dx \right)}{\frac{1}{b-a} \int_a^b f(x)g(x)dx} \leq K^2, \tag{4.23}$$

where

$$K = \frac{\sqrt{mn} + \sqrt{MN}}{\sqrt{mN} + \sqrt{Mn}}, \tag{4.24}$$

and the result (4.6) by Pečarić and Perić ([120]). In this way we will obtain even better upper and lower estimations than those in Corollary 4.5.

Theorem 4.5 *Suppose that the assumptions of Corollary 4.5 hold. Then*

$$\frac{1}{K^2} \leq \frac{(\boldsymbol{\varepsilon}_{\sigma} 1)(x;p)(\boldsymbol{\varepsilon}_{\sigma} fg)(x;p)}{(\boldsymbol{\varepsilon}_{\sigma} f)(x;p)(\boldsymbol{\varepsilon}_{\sigma} g)(x;p)} \leq K^2 \tag{4.25}$$

where K is given by (4.24).

Proof. Without loss of generality we can assume

$$1 \leq f(u) \leq \mu_1, \quad 1 \leq g(u) \leq \mu_2$$

for every $u \in [a, t]$ and some $\mu_1, \mu_2 \geq 1$. From the obvious inequality

$$[\mu_1 - f(u)][f(v) - 1][\mu_2 g(v) - g(u)] \geq 0$$

we obtain

$$\begin{aligned} &\mu_1 \mu_2 f(v)g(v) - \mu_1 \mu_2 g(v) - \mu_1 f(v)g(u) + \mu_1 g(u) \\ &- \mu_2 f(u)f(v)g(v) + f(u)f(v)g(u) + \mu_2 f(u)g(v) - f(u)g(u) \geq 0. \end{aligned}$$

Multiplying above inequality by

$$(x-u)^{\sigma-1}(x-v)^{\sigma-1} \boldsymbol{E}_{\sigma}(w(x-u)^{\rho}; p) \boldsymbol{E}_{\sigma}(w(x-v)^{\rho}; p)$$

and then integrating, we have

$$\begin{aligned}&\mu_1\mu_2\left(\boldsymbol{\varepsilon}_\sigma 1\right)(x;p)\left(\boldsymbol{\varepsilon}_\sigma fg\right)(x;p)-\mu_1\mu_2\left(\boldsymbol{\varepsilon}_\sigma 1\right)(x;p)\left(\boldsymbol{\varepsilon}_\sigma g\right)(x;p)\\&\quad-\mu_1\left(\boldsymbol{\varepsilon}_\sigma f\right)(x;p)\left(\boldsymbol{\varepsilon}_\sigma g\right)(x;p)+\mu_1\left(\boldsymbol{\varepsilon}_\sigma 1\right)(x;p)\left(\boldsymbol{\varepsilon}_\sigma g\right)(x;p)\\&\quad-\mu_2\left(\boldsymbol{\varepsilon}_\sigma f\right)(x;p)\left(\boldsymbol{\varepsilon}_\sigma fg\right)(x;p)+\left(\boldsymbol{\varepsilon}_\sigma f\right)(x;p)\left(\boldsymbol{\varepsilon}_\sigma fg\right)(x;p)\\&\quad+\mu_2\left(\boldsymbol{\varepsilon}_\sigma f\right)(x;p)\left(\boldsymbol{\varepsilon}_\sigma g\right)(x;p)-\left(\boldsymbol{\varepsilon}_\sigma 1\right)(x;p)\left(\boldsymbol{\varepsilon}_\sigma fg\right)(x;p))\;\geq 0\end{aligned}$$

from which follows

$$\begin{aligned}&\frac{\mu_2\left[\mu_1\left(\boldsymbol{\varepsilon}_\sigma 1\right)(x;p)-\left(\boldsymbol{\varepsilon}_\sigma f\right)(x;p)\right]+\mu_1\left[\left(\boldsymbol{\varepsilon}_\sigma f\right)(x;p)-\left(\boldsymbol{\varepsilon}_\sigma 1\right)(x;p)\right]}{\mu_2\left[\mu_1\left(\boldsymbol{\varepsilon}_\sigma 1\right)(x;p)-\left(\boldsymbol{\varepsilon}_\sigma f\right)(x;p)\right]+\left(\boldsymbol{\varepsilon}_\sigma f\right)(x;p)-\left(\boldsymbol{\varepsilon}_\sigma 1\right)(x;p)}\\&\leq\frac{\left(\boldsymbol{\varepsilon}_\sigma fg\right)(x;p)}{\left(\boldsymbol{\varepsilon}_\sigma g\right)(x;p)}.\end{aligned}\tag{4.26}$$

Similarly, from

$$[\mu_1-f(u)]\,[f(v)-1]\,[\mu_2 g(u)-g(v)]\geq 0$$

follows

$$\begin{aligned}&\frac{\left(\boldsymbol{\varepsilon}_\sigma fg\right)(x;p)}{\left(\boldsymbol{\varepsilon}_\sigma g\right)(x;p)}\leq\\&\frac{\mu_1\left(\boldsymbol{\varepsilon}_\sigma 1\right)(x;p)-\left(\boldsymbol{\varepsilon}_\sigma f\right)(x;p)+\mu_1\mu_2\left[\left(\boldsymbol{\varepsilon}_\sigma f\right)(x;p)-\left(\boldsymbol{\varepsilon}_\sigma 1\right)(x;p)\right]}{\mu_1\left(\boldsymbol{\varepsilon}_\sigma 1\right)(x;p)-\left(\boldsymbol{\varepsilon}_\sigma f\right)(x;p)+\mu_2\left[\left(\boldsymbol{\varepsilon}_\sigma f\right)(x;p)-\left(\boldsymbol{\varepsilon}_\sigma 1\right)(x;p)\right]}.\end{aligned}\tag{4.27}$$

Hence, from (4.26) and (4.27) we have

$$\begin{aligned}&\frac{\mu_2\left[\mu_1-\frac{(\boldsymbol{\varepsilon}_\sigma f)(x;p)}{(\boldsymbol{\varepsilon}_\sigma 1)(x;p)}\right]+\mu_1\left[\frac{(\boldsymbol{\varepsilon}_\sigma f)(x;p)}{(\boldsymbol{\varepsilon}_\sigma 1)(x;p)}-1\right]}{\mu_2\left[\mu_1-\frac{(\boldsymbol{\varepsilon}_\sigma f)(x;p)}{(\boldsymbol{\varepsilon}_\sigma 1)(x;p)}\right]+\frac{(\boldsymbol{\varepsilon}_\sigma f)(x;p)}{(\boldsymbol{\varepsilon}_\sigma 1)(x;p)}-1}\\&\quad\leq\frac{\left(\boldsymbol{\varepsilon}_\sigma fg\right)(x;p)}{\left(\boldsymbol{\varepsilon}_\sigma g\right)(x;p)}\leq\frac{\mu_1-\frac{(\boldsymbol{\varepsilon}_\sigma f)(x;p)}{(\boldsymbol{\varepsilon}_\sigma 1)(x;p)}+\mu_1\mu_2\left[\frac{(\boldsymbol{\varepsilon}_\sigma f)(x;p)}{(\boldsymbol{\varepsilon}_\sigma 1)(x;p)}-1\right]}{\mu_1-\frac{(\boldsymbol{\varepsilon}_\sigma f)(x;p)}{(\boldsymbol{\varepsilon}_\sigma 1)(x;p)}+\mu_2\left[\frac{(\boldsymbol{\varepsilon}_\sigma f)(x;p)}{(\boldsymbol{\varepsilon}_\sigma 1)(x;p)}-1\right]}.\end{aligned}$$

Next, we define functions $h,H:[1,\mu_1]\to(0,\infty)$ by

$$H(t)=\frac{1}{t}\frac{\mu_1-t+\mu_1\mu_2(t-1)}{\mu_1-t+\mu_2(t-1)},\quad h(t)=\frac{1}{H(\mu_1/t)}.$$

For $t_1=\frac{\sqrt{\mu_1}(\sqrt{\mu_1}+\sqrt{\mu_2})}{\sqrt{\mu_1\mu_2}+1}$ it is straightforward to check that

$$\max_{t\in[1,\mu_1]}H(t)=H(t_1)=\left(\frac{1+\sqrt{\mu_1\mu_2}}{\sqrt{\mu_1}+\sqrt{\mu_2}}\right)^2=K^2$$

and

$$\min_{t\in[1,\mu_1]}h(t)=h(\mu_1/t_1)=1/H(t_1)=1/K^2.$$

Using (4.28) we obtain

$$h\left(\frac{(\boldsymbol{\varepsilon}_\sigma f)(x;p)}{(\boldsymbol{\varepsilon}_\sigma 1)(x;p)}\right) \le \frac{(\boldsymbol{\varepsilon}_\sigma 1)(x;p)(\boldsymbol{\varepsilon}_\sigma fg)(x;p)}{(\boldsymbol{\varepsilon}_\sigma f)(x;p)(\boldsymbol{\varepsilon}_\sigma g)(x;p)} \le H\left(\frac{(\boldsymbol{\varepsilon}_\sigma f)(x;p)}{(\boldsymbol{\varepsilon}_\sigma 1)(x;p)}\right)$$

from which follows (4.25). □

Corollary 4.6 *If the assumptions of Theorem 4.5 hold, then*

$$\begin{aligned}
&-\frac{(M-m)(N-n)}{\left(\sqrt{mN}+\sqrt{Mn}\right)^2}(\boldsymbol{\varepsilon}_\sigma 1)(x;p)(\boldsymbol{\varepsilon}_\sigma fg)(x;p)\\
&\le -\frac{(M-m)(N-n)}{\left(\sqrt{mn}+\sqrt{MN}\right)^2}(\boldsymbol{\varepsilon}_\sigma f)(x;p)(\boldsymbol{\varepsilon}_\sigma g)(x;p)\\
&\le (\boldsymbol{\varepsilon}_\sigma 1)(x;p)(\boldsymbol{\varepsilon}_\sigma fg)(x;p)-(\boldsymbol{\varepsilon}_\sigma f)(x;p)(\boldsymbol{\varepsilon}_\sigma g)(x;p)\\
&\le \frac{(M-m)(N-n)}{\left(\sqrt{mn}+\sqrt{MN}\right)^2}(\boldsymbol{\varepsilon}_\sigma 1)(x;p)(\boldsymbol{\varepsilon}_\sigma fg)(x;p)\\
&\le \frac{(M-m)(N-n)}{\left(\sqrt{mN}+\sqrt{Mn}\right)^2}(\boldsymbol{\varepsilon}_\sigma f)(x;p)(\boldsymbol{\varepsilon}_\sigma g)(x;p).
\end{aligned} \tag{4.28}$$

Proof. As in [120, Corrolary 1], we see that direct consequences of (4.25) are the first and the last inequality in (4.28). From the lower bound in (4.25) we have

$$\begin{aligned}
&\left[\frac{1}{K^2}-1\right](\boldsymbol{\varepsilon}_\sigma 1)(x;p)(\boldsymbol{\varepsilon}_\sigma fg)(x;p)\\
&\le (\boldsymbol{\varepsilon}_\sigma 1)(x;p)(\boldsymbol{\varepsilon}_\sigma fg)(x;p)-(\boldsymbol{\varepsilon}_\sigma f)(x;p)(\boldsymbol{\varepsilon}_\sigma g)(x;p)
\end{aligned}$$

from which follows the second inequality in (4.28). Analogously, from the upper bound in (4.25) follows the third inequality in (4.28). □

Remark 4.4 If we observe results from Corollary 4.5 and Corollary 4.6 we can see that the upper estimate from Corollary 4.6 is better, i.e. inequality

$$\frac{(M-m)(N-n)}{\left(\sqrt{mN}+\sqrt{Mn}\right)^2} \le \frac{(M-m)(N-n)}{4\sqrt{mMnN}}$$

is equivalent with inequality

$$0 \le \left(\sqrt{Mn}-\sqrt{mN}\right)^2.$$

The upper estimates are equal if and only if $M/m = N/n$.
The lower estimate from Corollary 4.6 is also better, i.e. inequality

$$-\frac{(M-m)(N-n)}{4\sqrt{mMnN}} \le -\frac{(M-m)(N-n)}{\left(\sqrt{mn}+\sqrt{MN}\right)^2}$$

is equivalent with inequality

$$0 \le \left(\sqrt{mn}-\sqrt{MN}\right)^2.$$

The lower estimates are equal if and only if $m = M$ and $n = N$.

Chapter 5

Minkowski Type Fractional Integral Inequalities Associated with the Mittag-Leffler Function

Motivated by the papers [29, 142], where authors have proved Minkowski type integral inequalities, we present generalized corresponding results using our extended generalized Mittag-Leffler function with its fractional integral operators. We will need the Minkowski integral inequality and its reverse versions:

Theorem 5.1 (THE MINKOWSKI INTEGRAL INEQUALITY) *Let $p \geq 1$ and let $f,g \in L_p[a,b]$. Then*

$$\left(\int_a^b |f(x)+g(x)|^p dx\right)^{\frac{1}{p}} \leq \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_a^b |g(x)|^p dx\right)^{\frac{1}{p}}. \tag{5.1}$$

Theorem 5.2 [29] *Let $p \geq 1$ and let $f,g \in L_p[a,b]$ be positive functions satisfying $0 < m \leq \frac{f(x)}{g(x)} \leq M$ for $x \in [a,b]$. Then*

$$\left(\int_a^b (f(x))^p dx\right)^{\frac{1}{p}} + \left(\int_a^b (g(x))^p dx\right)^{\frac{1}{p}} \leq \frac{M(m+2)+1}{(m+1)(M+1)} \left(\int_a^b (f(x)+g(x))^p dx\right)^{\frac{1}{p}}.$$

Theorem 5.3 [142] *Let $p \geq 1$ and let $f,g \in L_p[a,b]$ be positive functions satisfying $0 < m \leq \frac{f(x)}{g(x)} \leq M$ for $x \in [a,b]$. Then*

$$\left(\int_a^b (f(x))^p dx\right)^{\frac{2}{p}} + \left(\int_a^b (g(x))^p dx\right)^{\frac{2}{p}}$$
$$\geq \frac{M(m-1)+m+1}{M} \left(\int_a^b (f(x))^p dx\right)^{\frac{1}{p}} \left(\int_a^b (g(x))^p dx\right)^{\frac{1}{p}}.$$

The right-sided versions of following inequalities can be established and proved analogously by using the right-sided fractional integral operators $\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f$ and ${}_h\Upsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f$ defined with (2.13) and (2.24) respectively.

This chapter is based on our results from [16, 17].

5.1 Reverse Minkowski Type Inequalities Involving an Extended Generalized Mittag-Leffler Function

We start this section by presenting the reverse fractional Minkowski integral inequality using extended generalized Mittag-Leffler function $E^{\delta,c,v,r}_{\rho,\sigma,\tau}$ with the corresponding fractional integral operator $\varepsilon^{w,\delta,c,v,r}_{a^+,\rho,\sigma,\tau}f$ (in real domain).

Here, for the reader's convenience we will use a simplified notation

$$\begin{aligned} \boldsymbol{E}(z;u) &:= E^{\delta,c,v,r}_{\rho,\sigma,\tau}(z;u) \\ &= \sum_{n=0}^{\infty} \frac{\mathrm{B}_u(\delta+nv,c-\delta)}{\mathrm{B}(\delta,c-\delta)} \frac{(c)_{nv}}{\Gamma(\rho n+\sigma)} \frac{z^n}{(\tau)_{nr}}, \end{aligned} \tag{5.2}$$

$$\begin{aligned} (\boldsymbol{\varepsilon} f)(x;u) &:= \left(\varepsilon^{w,\delta,c,v,r}_{a^+,\rho,\sigma,\tau}f\right)(x;u) \\ &= \int_a^x (x-t)^{\sigma-1}\boldsymbol{E}(w(x-t)^{\rho};u)f(t)dt. \end{aligned} \tag{5.3}$$

For proving our inequality, we follow methods as in the paper by L. Bougoffa ([29]), which we supplement with the necessary steps to generalize Theorem 5.2.

Theorem 5.4 *Let $w \in \mathbb{R}$, $r,\rho,\sigma,\tau > 0$, $c > \delta > 0$ with $u \geq 0$ and $0 < v \leq r+\rho$. Let $p \geq 1$ and let $f,g \in L_p[a,b]$ be positive functions satisfying*

$$0 < m \leq \frac{f(x)}{g(x)} \leq M, \quad x \in [a,b]. \tag{5.4}$$

Then

$$[(\boldsymbol{\varepsilon} f^p)(x;u)]^{\frac{1}{p}} + [(\boldsymbol{\varepsilon} g^p)(x;u)]^{\frac{1}{p}} \le c_1 \left[(\boldsymbol{\varepsilon}(f+g)^p)(x;u)\right]^{\frac{1}{p}}, \tag{5.5}$$

where

$$c_1 = \frac{M(m+2)+1}{(m+1)(M+1)}. \tag{5.6}$$

Proof. From the hypothesis $\frac{f(t)}{g(t)} \le M$ we have

$$f(t) \le M[f(t)+g(t)] - Mf(t), \quad t \in [a,b],$$

from which follows the inequality

$$(M+1)^p (f(t))^p \le M^p \left[f(t)+g(t)\right]^p, \quad p \ge 1,\ t \in [a,b].$$

Multiplying both sides of the above inequality by

$$(x-t)^{\sigma-1} \boldsymbol{E}(w(x-t)^\rho; u)$$

and integrating on $[a,x]$ with respect to the variable t, we obtain

$$(M+1)^p (\boldsymbol{\varepsilon} f^p)(x;u) \le M^p (\boldsymbol{\varepsilon}(f+g)^p)(x;u),$$

from which we get

$$[(\boldsymbol{\varepsilon} f^p)(x;u)]^{\frac{1}{p}} \le \frac{M}{M+1} \left[(\boldsymbol{\varepsilon}(f+g)^p)(x;u)\right]^{\frac{1}{p}}. \tag{5.7}$$

Further, for the lower bound $\frac{f(t)}{g(t)} \ge m$ we have

$$g(t) \le \frac{1}{m}[f(t)+g(t)] - \frac{1}{m} g(t), \quad t \in [a,b],$$

and

$$\left(1+\frac{1}{m}\right)^p (g(t))^p \le \left(\frac{1}{m}\right)^p [f(t)+g(t)]^p, \quad p \ge 1,\ t \in [a,b].$$

Similarly, if we multiply above inequality by $(x-t)^{\sigma-1} \boldsymbol{E}(w(x-t)^\rho; u)$ and integrate on $[a,x]$, then we get

$$\left(1+\frac{1}{m}\right)^p (\boldsymbol{\varepsilon} g^p)(x;u) \le \left(\frac{1}{m}\right)^p (\boldsymbol{\varepsilon}(f+g)^p)(x;u),$$

from which follows

$$[(\boldsymbol{\varepsilon} g^p)(x;u)]^{\frac{1}{p}} \le \frac{1}{m+1} \left[(\boldsymbol{\varepsilon}(f+g)^p)(x;u)\right]^{\frac{1}{p}}. \tag{5.8}$$

The resulting inequality (5.5) now follows by adding (5.7) and (5.8). □

Next theorem is a consequence of the Minkowski integral inequality (as shown by Set et al. in [142]) and it is a generalization of Theorem 5.3.

Theorem 5.5 *Suppose that assumptions of Theorem 5.4 hold. Then*

$$[(\boldsymbol{\varepsilon} f^p)(x;u)]^{\frac{2}{p}} + [(\boldsymbol{\varepsilon} g^p)(x;u)]^{\frac{2}{p}} \geq c_2 [(\boldsymbol{\varepsilon} f^p)(x;u)]^{\frac{1}{p}} [(\boldsymbol{\varepsilon} g^p)(x;u)]^{\frac{1}{p}}, \tag{5.9}$$

where

$$c_2 = \frac{M(m-1)+m+1}{M}. \tag{5.10}$$

Proof. Taking the product of the inequalities (5.7) and (5.8) we obtain

$$\frac{(M+1)(m+1)}{M} [(\boldsymbol{\varepsilon} f^p)(x;u)]^{\frac{1}{p}} [(\boldsymbol{\varepsilon} g^p)(x;u)]^{\frac{1}{p}} \leq [(\boldsymbol{\varepsilon}(f+g)^p)(x;u)]^{\frac{2}{p}}.$$

If we apply Minkowski's inequality (5.1) on the right hand side, we get

$$\begin{aligned}
&\frac{(M+1)(m+1)}{M} [(\boldsymbol{\varepsilon} f^p)(x;u)]^{\frac{1}{p}} [(\boldsymbol{\varepsilon} g^p)(x;u)]^{\frac{1}{p}} \\
&\leq \left[[(\boldsymbol{\varepsilon} f^p)(x;u)]^{\frac{1}{p}} + [(\boldsymbol{\varepsilon} g^p)(x;u)]^{\frac{1}{p}} \right]^2 \\
&= [(\boldsymbol{\varepsilon} f^p)(x;u)]^{\frac{2}{p}} + 2 [(\boldsymbol{\varepsilon} f^p)(x;u)]^{\frac{1}{p}} [(\boldsymbol{\varepsilon} g^p)(x;u)]^{\frac{1}{p}} + [(\boldsymbol{\varepsilon} g^p)(x;u)]^{\frac{2}{p}}.
\end{aligned}$$

From this we can easily obtain the inequality (5.9). □

5.2 Related Minkowski Type Inequalities

In the following, we continue with the generalizations of the reverse Minkowski type integral inequalities. For more similar results related to the Minkowski inequality in fractional calculus operators point of view, we refer the readers to see [37, 90, 91, 128, 149, 145, 146]. A simplified notation $\boldsymbol{E}$ and $\boldsymbol{\varepsilon} f$ as in (5.2) and (5.3) is used here.

We start with two theorems involving parameters $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 5.6 *Suppose that assumptions of Theorem 5.4 hold. Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$[(\boldsymbol{\varepsilon} f)(x;u)]^{\frac{1}{p}} [(\boldsymbol{\varepsilon} g)(x;u)]^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} (\boldsymbol{\varepsilon}(f^{\frac{1}{p}} g^{\frac{1}{q}}))(x;u). \tag{5.11}$$

Proof. From the hypothesis $\frac{f(t)}{g(t)} \leq M$ we obtain

$$(f(t))^{\frac{1}{q}} \leq M^{\frac{1}{q}} (g(t))^{\frac{1}{q}},$$

and after multiplication by $(f(t))^{\frac{1}{p}}$ we get

$$f(t) \leq M^{\frac{1}{q}} (f(t))^{\frac{1}{p}} (g(t))^{\frac{1}{q}}, \quad t \in [a,b].$$

Multiplying both sides of the above inequality by $(x-t)^{\sigma-1}\boldsymbol{E}(w(x-t)^{\rho};u)$ and integrating on $[a,x]$ with respect to the variable t, we derive

$$(\boldsymbol{\varepsilon} f)(x;u) \le M^{\frac{1}{q}}(\boldsymbol{\varepsilon}(f^{\frac{1}{p}}g^{\frac{1}{q}}))(x;u),$$

and

$$[(\boldsymbol{\varepsilon} f)(x;u)]^{\frac{1}{p}} \le M^{\frac{1}{pq}}\left[(\boldsymbol{\varepsilon}(f^{\frac{1}{p}}g^{\frac{1}{q}}))(x;u)\right]^{\frac{1}{p}}. \tag{5.12}$$

Further, we obtain

$$g(t) \le m^{-\frac{1}{p}}(f(t))^{\frac{1}{p}}(g(t))^{\frac{1}{q}}$$

by using lower bound $m \le \frac{f(t)}{g(t)}$ and multiplication by $(g(t))^{\frac{1}{q}}$. Multiplying above inequality by $(x-t)^{\sigma-1}\boldsymbol{E}(w(x-t)^{\rho};u)$ and integrating on $[a,x]$ we have

$$(\boldsymbol{\varepsilon} g)(x;u) \le m^{-\frac{1}{p}}(\boldsymbol{\varepsilon}(f^{\frac{1}{p}}g^{\frac{1}{q}}))(x;u)$$

from which we get

$$[(\boldsymbol{\varepsilon} g)(x;u)]^{\frac{1}{q}} \le m^{-\frac{1}{pq}}\left[(\boldsymbol{\varepsilon}(f^{\frac{1}{p}}g^{\frac{1}{q}}))(x;u)\right]^{\frac{1}{q}}. \tag{5.13}$$

The inequality (5.11) now follows from the product of inequalities (5.12) and (5.13). □

In the next theorem we will need the well known Young's inequality for products of $x,y \ge 0$ with $p,q > 1$ such that $\frac{1}{p}+\frac{1}{q}=1$:

$$xy \le \frac{x^p}{p}+\frac{y^q}{q}, \tag{5.14}$$

and following elementary inequality for $x,y \ge 0$ and $p>1$:

$$(x+y)^p \le 2^{p-1}(x^p+y^p). \tag{5.15}$$

Theorem 5.7 *Suppose that assumptions of Theorem 5.4 hold. Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\begin{aligned}(\boldsymbol{\varepsilon}(fg))(x;u) \;\le\; & \frac{2^{p-1}}{p}\left(\frac{M}{M+1}\right)^p(\boldsymbol{\varepsilon}(f^p+g^p))(x;u)\\ &+\frac{2^{q-1}}{q}\left(\frac{1}{m+1}\right)^q(\boldsymbol{\varepsilon}(f^q+g^q))(x;u).\end{aligned} \tag{5.16}$$

Proof. We follow the same steps as in the proof of Theorem 5.4 to obtain (5.7) and (5.8) from which follow

$$\frac{1}{p}(\boldsymbol{\varepsilon} f^p)(x;u) \le \frac{M^p}{p(M+1)^p}(\boldsymbol{\varepsilon}(f+g)^p)(x;u) \tag{5.17}$$

and

$$\frac{1}{q}(\boldsymbol{\varepsilon} g^q)(x;u) \le \frac{1}{q(m+1)^q}(\boldsymbol{\varepsilon}(f+g)^q)(x;u). \tag{5.18}$$

Using Young's inequality (5.14) we have

$$f(t)g(t) \le \frac{(f(t))^p}{p} + \frac{(g(t))^q}{q}.$$

Multiplying both sides of the above inequality by $(x-t)^{\sigma-1}\boldsymbol{E}(w(x-t)^{\rho};u)$ and integrating on $[a,x]$ we get

$$(\boldsymbol{\varepsilon}(fg))(x;u) \le \frac{1}{p}(\boldsymbol{\varepsilon} f^p)(x;u) + \frac{1}{q}(\boldsymbol{\varepsilon} g^q)(x;u). \tag{5.19}$$

From (5.17), (5.18) and (5.19) follows

$$(\boldsymbol{\varepsilon}(fg))(x;u) \le \frac{M^p}{p(M+1)^p}(\boldsymbol{\varepsilon}(f+g)^p)(x;u) + \frac{1}{q(m+1)^q}(\boldsymbol{\varepsilon}(f+g)^q)(x;u). \tag{5.20}$$

Using elementary inequality (5.15) we obtain

$$(\boldsymbol{\varepsilon}(f+g)^p)(x;u) \le 2^{p-1}(\boldsymbol{\varepsilon}(f^p+g^p))(x;u) \tag{5.21}$$

and

$$(\boldsymbol{\varepsilon}(f+g)^q)(x;u) \le 2^{q-1}(\boldsymbol{\varepsilon}(f^q+g^q))(x;u). \tag{5.22}$$

Hence, from (5.20), (5.21) and (5.22) follows (5.16). □

Theorem 5.8 *Suppose that assumptions of Theorem 5.4 hold. Then*

$$\begin{aligned} \frac{1}{M}(\boldsymbol{\varepsilon}(fg))(x;u) \ &\le \ \frac{1}{(m+1)(M+1)}(\boldsymbol{\varepsilon}(f+g)^2)(x;u) \\ &\le \ \frac{1}{m}(\boldsymbol{\varepsilon}(fg))(x;u). \end{aligned} \tag{5.23}$$

Proof. From $0 < m \le \frac{f(t)}{g(t)} \le M$ and $\frac{1}{M} \le \frac{g(t)}{f(t)} \le \frac{1}{m}$ we obatin

$$(m+1)g(t) \le f(t)+g(t) \le (M+1)g(t) \tag{5.24}$$

and

$$\left(\frac{M+1}{M}\right) f(t) \le f(t)+g(t) \le \left(\frac{m+1}{m}\right) f(t). \tag{5.25}$$

Multiplying inequalities (5.24) and (5.25) we get

$$\frac{1}{M} f(t)g(t) \le \frac{\big(f(t)+g(t)\big)^2}{(M+1)(m+1)} \le \frac{1}{m} f(t)g(t).$$

Finally, multiplying above inequalities by $(x-t)^{\sigma-1}\boldsymbol{E}(w(x-t)^{\rho};u)$ and integrating on $[a,x]$ with respect to the variable t, we obtain inequalities (5.23). □

In the last theorem of this section, we will add a positive parameter μ and assume $\mu < m$, i.e.

$$0 < \mu < m \le \frac{f(x)}{g(x)} \le M, \quad x \in [a,b].$$

Theorem 5.9 *Suppose that assumptions of Theorem 5.4 hold. Let $\mu > 0$ be such that $\mu < m$ in (5.4). Then*

$$\frac{M+1}{M-\mu}(\boldsymbol{\varepsilon}(f-\mu g)^{\frac{1}{p}})(x;u) \le [(\boldsymbol{\varepsilon} f^p)(x;u)]^{\frac{1}{p}} + [(\boldsymbol{\varepsilon} g^p)(x;u)]^{\frac{1}{p}} \le \frac{m+1}{m-\mu}(\boldsymbol{\varepsilon}(f-\mu g)^{\frac{1}{p}})(x;u). \tag{5.26}$$

Proof. From the given condition $0 < \mu < m \le M$, we have

$$m\mu \le M\mu \Rightarrow m\mu + m \le m\mu + M \le M\mu + M,$$

that is

$$m - M\mu \le M - m\mu.$$

Now we have

$$m - M\mu + Mm - \mu \le M - m\mu + Mm - \mu$$

and

$$\frac{M+1}{M-\mu} \le \frac{m+1}{m-\mu}.$$

Easily we get

$$m-\mu \le \frac{f(t)-\mu g(t)}{g(t)} \le M-\mu$$

from which we can obtain

$$\frac{[f(t)-\mu g(t)]^p}{(M-\mu)^p} \le (g(t))^p \le \frac{[f(t)-\mu g(t)]^p}{(m-\mu)^p}.$$

Multiplying above inequalities by $(x-t)^{\sigma-1}\boldsymbol{E}(w(x-t)^\rho;u)$ and integrating on $[a,x]$ with respect to the variable t, we obtain

$$\frac{1}{(M-\mu)^p}(\boldsymbol{\varepsilon}(f-\mu g)^p)(x;u) \le (\boldsymbol{\varepsilon} g^p)(x;u) \le \frac{1}{(m-\mu)^p}(\boldsymbol{\varepsilon}(f-\mu g)^p)(x;u),$$

followed by

$$\frac{1}{M-\mu}[(\boldsymbol{\varepsilon}(f-\mu g)^p)(x;u)]^{\frac{1}{p}} \le [(\boldsymbol{\varepsilon} g^p)(x;u)]^{\frac{1}{p}} \le \frac{1}{m-\mu}[(\boldsymbol{\varepsilon}(f-\mu g)^p)(x;u)]^{\frac{1}{p}}. \tag{5.27}$$

Further, from $\frac{1}{M} \le \frac{g(t)}{f(t)} \le \frac{1}{m}$ we get

$$\frac{m-\mu}{m} \le \frac{f(t)-\mu g(t)}{f(t)} \le \frac{M-\mu}{M},$$

from which we have

$$\frac{M^p\,[f(t)-\mu g(t)]^p}{(M-\mu)^p} \le (f(t))^p \le \frac{m^p\,[f(t)-\mu g(t)]^p}{(m-\mu)^p}.$$

Again, multiplying above inequalities by $(x-t)^{\sigma-1}\boldsymbol{E}(w(x-t)^{\rho};u)$ and integrating on $[a,x]$ we obtain

$$\begin{aligned}\frac{M^p}{(M-\mu)^p}(\boldsymbol{\varepsilon}(f-\mu g)^p)(x;u) &\le (\boldsymbol{\varepsilon} f^p)(x;u)\\ &\le \frac{m^p}{(m-\mu)^p}(\boldsymbol{\varepsilon}(f-\mu g)^p)(x;u),\end{aligned}$$

followed by

$$\begin{aligned}\frac{M}{M-\mu}\left[(\boldsymbol{\varepsilon}(f-\mu g)^p)(x;u)\right]^{\frac{1}{p}} &\le \left[(\boldsymbol{\varepsilon} f^p)(x;u)\right]^{\frac{1}{p}}\\ &\le \frac{m}{m-\mu}\left[(\boldsymbol{\varepsilon}(f-\mu g)^p)(x;u)\right]^{\frac{1}{p}}.\end{aligned} \tag{5.28}$$

By adding the inequalities (5.27) and (5.28) follows the result (5.26). □

5.3 Further Generalizations of Minkowski Type Inequalities

In this section we give further generalizations of reverse Minkowski type integral inequalities for a generalized fractional integral operator ${}_h\Upsilon^{w,\delta,c,v,r}_{a^+,\rho,\sigma,\tau}f$ (2.23) containing an extended Mittag-Leffler function $E^{\delta,c,v,r}_{\rho,\sigma,\tau}$ in the kernel and then we prove some related fractional Minkowski type integral inequalities.

For the reader's convenience we will use a simplified notation

$$\begin{aligned}\boldsymbol{E}(z;u) &:= E^{\delta,c,v,r}_{\rho,\sigma,\tau}(z;u),\\ ({}_{\boldsymbol{h}}\boldsymbol{\Upsilon} f)(x;u) &:= \left({}_h\Upsilon^{w,\delta,c,v,r}_{a^+,\rho,\sigma,\tau}f\right)(x;u)\\ &= \int_a^x (h(x)-h(t))^{\sigma-1}E^{\delta,c,v,r}_{\rho,\sigma,\tau}(w(h(x)-h(t))^{\rho};u)h'(t)f(t)dt.\end{aligned}$$

Theorem 5.10 *Let $w\in\mathbb{R}$, $\rho,\sigma,\tau,r>0$, $c>\delta>0$ with $u\ge 0$ and $0<v\le r+\rho$. Let $h:[a,b]\to\mathbb{R}$ be a differentiable, strictly increasing function and let $f,g,\varphi_1,\varphi_2\in L_p[a,b]$ be positive functions satisfying*

$$0<\varphi_1(x)\le\frac{f(x)}{g(x)}\le\varphi_2(x),\quad x\in[a,b].$$

Then for $p \geq 1$ the following inequality holds

$$[({}_{\boldsymbol{h}}\boldsymbol{\Upsilon} f^p)(x;u)]^{\frac{1}{p}} + [({}_{\boldsymbol{h}}\boldsymbol{\Upsilon} g^p)(x;u)]^{\frac{1}{p}}$$
$$\leq \left[\left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}\left(\frac{\varphi_2}{1+\varphi_2}\right)^p (f+g)^p\right)(x;u)\right]^{\frac{1}{p}} + \left[\left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}\left(\frac{1}{1+\varphi_1}\right)^p (f+g)^p\right)(x;u)\right]^{\frac{1}{p}}. \quad (5.29)$$

Proof. Let $t \in [a,b]$. From $\frac{f(t)}{g(t)} \leq \varphi_2(t)$ we have

$$f(t) \leq \varphi_2(t)[f(t)+g(t)] - \varphi_2(t)f(t),$$

and for $p \geq 1$

$$(f(t))^p \leq \left(\frac{\varphi_2(t)}{1+\varphi_2(t)}\right)^p [f(t)+g(t)]^p. \quad (5.30)$$

Multiplying both sides of the above inequality by

$$(h(x)-h(t))^{\sigma-1}\boldsymbol{E}\left(w(h(x)-h(t))^{\rho};u\right)h'(t) \quad (5.31)$$

and integrating on $[a,x]$ with respect to the variable t, we obtain

$$({}_{\boldsymbol{h}}\boldsymbol{\Upsilon} f^p)(x;u) \leq \left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}\left(\left(\frac{\varphi_2}{1+\varphi_2}\right)^p (f+g)^p\right)\right)(x;u).$$

from which we get

$$[({}_{\boldsymbol{h}}\boldsymbol{\Upsilon} f^p)(x;u)]^{\frac{1}{p}} \leq \left[\left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}\left(\left(\frac{\varphi_2}{1+\varphi_2}\right)^p (f+g)^p\right)\right)(x;u)\right]^{\frac{1}{p}}. \quad (5.32)$$

Further, from $\frac{f(t)}{g(t)} \geq \varphi_1(t)$ follows

$$g(t) \leq \frac{1}{\varphi_1(t)}[f(t)+g(t)] - \frac{1}{\varphi_1(t)}g(t),$$

and if $p \geq 1$, then

$$(g(t))^p \leq \left(\frac{1}{1+\varphi_1(t)}\right)^p [f(t)+g(t)]^p. \quad (5.33)$$

Similarly, if we multiply above inequality by (5.31) and integrate on $[a,x]$, then we get

$$({}_{\boldsymbol{h}}\boldsymbol{\Upsilon} g^p)(x;u) \leq \left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}\left(\left(\frac{1}{1+\varphi_1}\right)^p (f+g)^p\right)\right)(x;u)$$

and also

$$[({}_{\boldsymbol{h}}\boldsymbol{\Upsilon} g^p)(x;u)]^{\frac{1}{p}} \leq \left[\left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}\left(\left(\frac{1}{1+\varphi_1}\right)^p (f+g)^p\right)\right)(x;u)\right]^{\frac{1}{p}}. \quad (5.34)$$

By adding (5.32) and (5.34), the resulting inequality (5.29) follows. □

Setting φ_1 and φ_2 to be constant functions, i.e. $\varphi_1(x) = m$ and $\varphi_2(x) = M$ for all $x \in [a,b]$, we obtain the following result.

Corollary 5.1 *Let $w \in \mathbb{R}$, $\rho, \sigma, \tau, r > 0$, $c > \delta > 0$ with $u \geq 0$ and $0 < v \leq r + \rho$. Let $h : [a,b] \to \mathbb{R}$ be a differentiable, strictly increasing function and let $f, g \in L_p[a,b]$ be positive functions satisfying the condition* (5.4)*. Then for $p \geq 1$ the following inequality holds*

$$\begin{aligned}&[({}_h\Upsilon f^p)(x;u)]^{\frac{1}{p}} + [({}_h\Upsilon g^p)(x;u)]^{\frac{1}{p}}\\ &\leq c_1 \left[({}_h\Upsilon (f+g)^p)(x;u)\right]^{\frac{1}{p}},\end{aligned}$$

where c_1 is defined by (5.6)*.*

Remark 5.1 If the function h is the identity function, then we obtain an inequality from Theorem 5.4 for the generalized fractional operator $\varepsilon^{w,\delta,c,v,r}_{a^+,\rho,\sigma,\tau} f$.

Next theorem is a fractional generalization of Theorem 5.3. It follows by the use of the Minkowski integral inequality (5.1).

Theorem 5.11 *Suppose the assumptions of Theorem 5.10 hold. Then*

$$\begin{aligned}&[({}_h\Upsilon f^p)(x;u)]^{\frac{2}{p}} + [({}_h\Upsilon g^p)(x;u)]^{\frac{2}{p}}\\ &\geq \left[\left({}_h\Upsilon \left(\frac{1+\varphi_2}{\varphi_2}\right)^p f^p\right)(x;u)\right]^{\frac{1}{p}} \left[({}_h\Upsilon (1+\varphi_1)^p g^p)(x;u)\right]^{\frac{1}{p}}\\ &\quad -2\left[({}_h\Upsilon f^p)(x;u)\right]^{\frac{1}{p}}\left[({}_h\Upsilon g^p)(x;u)\right]^{\frac{1}{p}}.\end{aligned} \tag{5.35}$$

Proof. For $p \geq 1$ and $t \in [a,b]$, inequalities (5.30) and (5.33) can also be written as

$$\left(\frac{1+\varphi_2(t)}{\varphi_2(t)}\right)^p ((f(t))^p \leq [f(t)+g(t)]^p$$

and

$$(1+\varphi_1(t))^p (g(t))^p \leq [f(t)+g(t)]^p.$$

If we multiply both sides of each inequality by (5.31), integrate on $[a,x]$ with respect to the variable t and use power to the $\frac{1}{p}$, then we obtain

$$\left[\left({}_h\Upsilon \left(\left(\frac{1+\varphi_2}{\varphi_2}\right)^p f^p\right)\right)(x;u)\right]^{\frac{1}{p}} \leq \left[({}_h\Upsilon (f+g)^p)(x;u)\right]^{\frac{1}{p}} \tag{5.36}$$

and

$$\left[({}_h\Upsilon ((1+\varphi_1)^p g^p))(x;u)\right]^{\frac{1}{p}} \leq \left[({}_h\Upsilon (f+g)^p)(x;u)\right]^{\frac{1}{p}}. \tag{5.37}$$

Taking the product of the inequalities (5.36) and (5.37) we obtain

$$\begin{aligned}&\left[\left({}_h\Upsilon \left(\left(\frac{1+\varphi_2}{\varphi_2}\right)^p f^p\right)\right)(x;u)\right]^{\frac{1}{p}} \left[({}_h\Upsilon ((1+\varphi_1)^p g^p))(x;u)\right]^{\frac{1}{p}}\\ &\leq \left[({}_h\Upsilon (f+g)^p)(x;u)\right]^{\frac{2}{p}}.\end{aligned}$$

If we apply Minkowski's inequality on right hand side, we get

$$\left[\left({}_{h}\Upsilon\left(\left(\frac{1+\varphi_2}{\varphi_2}\right)^p f^p\right)\right)(x;u)\right]^{\frac{1}{p}} \left[\left({}_{h}\Upsilon\left((1+\varphi_1)^p g^p\right)\right)(x;u)\right]^{\frac{1}{p}}$$
$$\leq \left[\left(\left({}_{h}\Upsilon f^p\right)(x;u)\right)^{\frac{1}{p}} + \left(\left({}_{h}\Upsilon g^p\right)(x;u)\right)^{\frac{1}{p}}\right]^2.$$

From this we can easily obtain the inequality (5.35). □

If $\varphi_1(x) = m$ and $\varphi_2(x) = M$ for all $x \in [a,b]$, then the next inequality follows.

Corollary 5.2 *Suppose the assumptions of Corollary 5.1 hold. Then*

$$\left[\left({}_{h}\Upsilon f^p\right)(x;u)\right]^{\frac{2}{p}} + \left[\left({}_{h}\Upsilon g^p\right)(x;u)\right]^{\frac{2}{p}}$$
$$\geq c_2 \left[\left({}_{h}\Upsilon f^p\right)(x;u)\right]^{\frac{1}{p}} \left[\left({}_{h}\Upsilon g^p\right)(x;u)\right]^{\frac{1}{p}},$$

where c_2 is defined by (5.10).

Remark 5.2 If the function h is the identity function, then we obtain Theorem 5.5, an inequality for the generalized fractional operator $\varepsilon^{w,\delta,c,v,r}_{a^+,\rho,\sigma,\tau} f$.

We continue with the generalizations of the reverse Minkowski type integral inequalities. Starting conditions that we will need in this section are those given in Corollary 5.1, where we have $0 < m \leq \frac{f(x)}{g(x)} \leq M$.

Theorem 5.12 *Suppose that assumptions of Corollary 5.1 hold. Let $p,q > 1$ with $\frac{1}{p} + \frac{1}{q} =$ 1. Then*

$$\left[\left({}_{h}\Upsilon f\right)(x;u)\right]^{\frac{1}{p}} \left[\left({}_{h}\Upsilon g\right)(x;u)\right]^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} \left({}_{h}\Upsilon\left(f^{\frac{1}{p}} g^{\frac{1}{q}}\right)\right)(x;u). \tag{5.38}$$

Proof. Let $t \in [a,b]$. From $\frac{f(t)}{g(t)} \leq M$ we obtain

$$(f(t))^{\frac{1}{q}} \leq M^{\frac{1}{q}} (g(t))^{\frac{1}{q}},$$

and after multiplication by $(f(t))^{\frac{1}{p}}$ we get

$$f(t) \leq M^{\frac{1}{q}} (f(t))^{\frac{1}{p}} (g(t))^{\frac{1}{q}}.$$

If we multiply both sides of the above inequality by (5.31), integrate on $[a,x]$ with respect to the variable t and use power to the $\frac{1}{p}$, then we obtain

$$\left[\left({}_{h}\Upsilon f\right)(x;u)\right]^{\frac{1}{p}} \leq M^{\frac{1}{pq}} \left[\left({}_{h}\Upsilon\left(f^{\frac{1}{p}} g^{\frac{1}{q}}\right)\right)(x;u)\right]^{\frac{1}{p}}. \tag{5.39}$$

Next from the lower bound $m \leq \frac{f(t)}{g(t)}$ we have

$$g(t) \leq m^{-\frac{1}{p}} (f(t))^{\frac{1}{p}} (g(t))^{\frac{1}{q}}.$$

Again, if we multiply both sides of the above inequality by (5.31), integrate on $[a,x]$ with respect to the variable t and use power to the $\frac{1}{p}$, then we get

$$[({}_{h}\Upsilon g)(x;u)]^{\frac{1}{q}} \leq m^{-\frac{1}{pq}} \left[\left({}_{h}\Upsilon\left(f^{\frac{1}{p}} g^{\frac{1}{q}}\right)\right)(x;u)\right]^{\frac{1}{q}}. \tag{5.40}$$

The inequality (5.38) now follows from the product of inequalities (5.39) and (5.40). □

In the next theorem we will need the inequality (5.15) along with Young's inequality (5.14).

Theorem 5.13 *Suppose that assumptions of Corollary 5.1 hold. Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\begin{aligned}({}_{h}\Upsilon(fg))(x;u) \leq\ & \frac{2^{p-1}}{p}\left(\frac{M}{M+1}\right)^{p}({}_{h}\Upsilon(f^{p}+g^{p}))(x;u)\\ &+\frac{2^{q-1}}{q}\left(\frac{1}{m+1}\right)^{q}({}_{h}\Upsilon(f^{q}+g^{q}))(x;u).\end{aligned} \tag{5.41}$$

Proof. Let $t\in[a,b]$, $p,q>1$ and $\frac{1}{p}+\frac{1}{q}=1$. From $\frac{f(t)}{g(t)}\leq M$ we get

$$(f(t))^{p} \leq \left(\frac{M}{M+1}\right)^{p}[f(t)+g(t)]^{p}.$$

If we multiply both sides of the above inequality by (5.31) and integrate on $[a,x]$ with respect to the variable t, then we obtain

$$\frac{1}{p}({}_{h}\Upsilon f^{p})(x;u) \leq \frac{1}{p}\left(\frac{M}{M+1}\right)^{p}({}_{h}\Upsilon(f+g)^{p})(x;u) \tag{5.42}$$

Next from $m\leq\frac{f(t)}{g(t)}$ we obtain

$$(g(t))^{q} \leq \left(\frac{1}{m+1}\right)^{q}[f(t)+g(t)]^{q}.$$

Again, if we multiply both sides of the above inequality by (5.31) and integrate on $[a,x]$ with respect to the variable t, then we get

$$\frac{1}{q}({}_{h}\Upsilon g^{q})(x;u) \leq \frac{1}{q}\left(\frac{1}{m+1}\right)^{q}({}_{h}\Upsilon(f+g)^{q})(x;u). \tag{5.43}$$

Using Young's inequality (5.14) we have

$$f(t)g(t) \leq \frac{(f(t))^{p}}{p}+\frac{(g(t))^{q}}{q}.$$

Multiplying both sides of the above inequality by (5.31) and integrating on $[a,x]$ we get

$$({}_{h}\Upsilon(fg))(x;u) \leq \frac{1}{p}({}_{h}\Upsilon f^{p})(x;u)+\frac{1}{q}({}_{h}\Upsilon g^{q})(x;u). \tag{5.44}$$

From (5.42), (5.43) and (5.44) we obtain

$$\begin{aligned}({}_{h}\Upsilon(fg))\,(x;u) \;\leq\; & \frac{1}{p}\left(\frac{M}{M+1}\right)^{p}({}_{h}\Upsilon(f+g)^{p})\,(x;u)\\ & +\frac{1}{q}\left(\frac{1}{m+1}\right)^{q}({}_{h}\Upsilon(f+g)^{q})\,(x;u).\end{aligned} \tag{5.45}$$

Using elementary inequality (5.15) we obtain

$$({}_{h}\Upsilon\,(f+g)^{p})\,(x;u) \leq 2^{p-1}\,({}_{h}\Upsilon\,(f^{p}+g^{p}))\,(x;u) \tag{5.46}$$

and

$$({}_{h}\Upsilon(f+g)^{q})\,(x;u) \leq 2^{q-1}\,({}_{h}\Upsilon\,(f^{q}+g^{q}))\,(x;u). \tag{5.47}$$

Hence, from (5.45), (5.46) and (5.47) we obtain (5.41). □

Next theorem needs a simple application of the given condition (5.4).

Theorem 5.14 *Suppose that assumptions of Corollary 5.1 hold. Then*

$$\begin{aligned}\frac{1}{M}\,({}_{h}\Upsilon(fg))\,(x;u) \;\leq\; & \frac{1}{(m+1)(M+1)}\left({}_{h}\Upsilon(f+g)^{2}\right)(x;u)\\ \leq\; & \frac{1}{m}\,({}_{h}\Upsilon(fg))\,(x;u).\end{aligned} \tag{5.48}$$

Proof. From $0 < m \leq \frac{f(t)}{g(t)} \leq M$ and $\frac{1}{M} \leq \frac{g(t)}{f(t)} \leq \frac{1}{m}$ we obtain

$$(m+1)g(t) \leq f(t)+g(t) \leq (M+1)g(t) \tag{5.49}$$

and

$$\left(\frac{M+1}{M}\right)f(t) \leq f(t)+g(t) \leq \left(\frac{m+1}{m}\right)f(t). \tag{5.50}$$

Multiplying inequalities (5.49) and (5.50) we get

$$\frac{1}{M}f(t)g(t) \leq \frac{(f(t)+g(t))^{2}}{(M+1)(m+1)} \leq \frac{1}{m}f(t)g(t).$$

Inequalities (5.48) now follow if we multiply the above by (5.31) and integrate on $[a,x]$ with respect to the variable t. □

Theorem 5.15 *Suppose that assumptions of Corollary 5.1 hold. Let $\vartheta > 0$ be such that $\vartheta < m$ in (5.4). Then*

$$\begin{aligned}&\frac{M+1}{M-\vartheta}\left({}_{h}\Upsilon(f-\vartheta g)^{\frac{1}{p}}\right)(x;u)\\ &\leq [({}_{h}\Upsilon f^{p})\,(x;u)]^{\frac{1}{p}}+[({}_{h}\Upsilon g^{p})\,(x;u)]^{\frac{1}{p}}\\ &\leq \frac{m+1}{m-\vartheta}\left({}_{h}\Upsilon(f-\vartheta g)^{\frac{1}{p}}\right)(x;u).\end{aligned} \tag{5.51}$$

Proof. From the given condition $0 < \vartheta < m \le M$, we have

$$m - M\vartheta \le M - m\vartheta,$$

from which follow

$$\frac{M+1}{M-\vartheta} \le \frac{m+1}{m-\vartheta},$$

$$m - \vartheta \le \frac{f(t) - \vartheta g(t)}{g(t)} \le M - \vartheta$$

and

$$\frac{[f(t) - \vartheta g(t)]^p}{(M-\vartheta)^p} \le (g(t))^p \le \frac{[f(t) - \vartheta g(t)]^p}{(m-\vartheta)^p}.$$

If we multiply above inequalities by (5.31), integrate on $[a,x]$ with respect to the variable t and use power to the $\frac{1}{p}$, then we get

$$\begin{aligned} &\frac{1}{M-\vartheta}\left[\left({}_h\Upsilon(f-\vartheta g)^p\right)(x;u)\right]^{\frac{1}{p}} \\ &\le \left[\left({}_h\Upsilon g^p\right)(x;u)\right]^{\frac{1}{p}} \\ &\le \frac{1}{m-\vartheta}\left[\left({}_h\Upsilon(f-\vartheta g)^p\right)(x;u)\right]^{\frac{1}{p}}. \end{aligned} \tag{5.52}$$

Further, from $\frac{1}{M} \le \frac{g(t)}{f(t)} \le \frac{1}{m}$ we get

$$\frac{m-\vartheta}{m} \le \frac{f(t) - \vartheta g(t)}{f(t)} \le \frac{M-\vartheta}{M},$$

from which we have

$$\frac{M^p\,[f(t) - \vartheta g(t)]^p}{(M-\vartheta)^p} \le (f(t))^p \le \frac{m^p\,[f(t) - \vartheta g(t)]^p}{(m-\vartheta)^p}.$$

Again, multiplying above inequalities by (5.31), integrating on $[a,x]$ and using power to the $\frac{1}{p}$, we obtain

$$\begin{aligned} &\frac{M}{M-\vartheta}\left[\left({}_h\Upsilon(f-\vartheta g)^p\right)(x;u)\right]^{\frac{1}{p}} \\ &\le \left[\left({}_h\Upsilon f^p\right)(x;u)\right]^{\frac{1}{p}} \\ &\le \frac{m}{m-\vartheta}\left[\left({}_h\Upsilon(f-\vartheta g)^p\right)(x;u)\right]^{\frac{1}{p}}. \end{aligned} \tag{5.53}$$

By adding the inequalities (5.52) and (5.53), we get (5.51). □

Remark 5.3 If in the obtained results we use the identity function for the function h, then we obtain Theorem 5.6 - Theorem 5.9 .

Chapter 6

Classical Integral Inequalities and the Mittag-Leffler Function

This chapter is motivated with researches of classical integral inequalities by W. Liu et al. [99] and by Z. Dahmani [38], such as:

Theorem 6.1 [99, Theorem 4] *Let f, g be positive continuous functions on $[a,b]$ such that f is decreasing and g is increasing. Then the following inequality*

$$\frac{\int_a^x (f(t))^\beta \, dt}{\int_a^x (f(t))^\gamma \, dt} \geq \frac{\int_a^x (g(t))^\alpha (f(t))^\beta \, dt}{\int_a^x (g(t))^\alpha (f(t))^\gamma \, dt} \tag{6.1}$$

holds for every $\alpha > 0$ and $\beta \geq \gamma > 0$. If f is increasing, then (6.1) *is reversed.*

Theorem 6.2 [38, Theorem 3.6] *Let $(f_i)_{i=1,2,\ldots,n}$ and g be positive continuous functions on $[a,b]$ such that $(f_i)_{i=1,2,\ldots,n}$ are decreasing and g is increasing. Then the following inequality*

$$\frac{J_{a+}^\sigma \left[\prod_{i\neq s}^n f_i^{\gamma_i} f_s^\beta (x)\right] J_{a+}^\sigma \left[g^\alpha (x) \prod_{i=1}^n f_i^{\gamma_i}(x)\right]}{J_{a+}^\sigma \left[g^\alpha (x) \prod_{i\neq s}^n f_i^{\gamma_i} f_s^\beta (x)\right] J_{a+}^\sigma \left[\prod_{i=1}^n f_i^{\gamma_i}(x)\right]} \geq 1 \tag{6.2}$$

holds for every $a < x \leq b$, $\sigma > 0$, $\alpha > 0$, $\beta \geq \gamma_s > 0$, where s is a fixed integer in $\{1,2,\ldots,n\}$.

The aim is to present corresponding results using our extended generalized Mittag-Leffler function with its fractional integral operators.

The right-sided versions of following inequalities can be established and proved analogously by using the right-sided fractional integral operators $\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f$ and ${}_h\Upsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f$ defined with (2.13) and (2.24) respectively.

This chapter is based on our results from [18, 19].

6.1 Generalizations of Classical Integral Inequalities Involving an Extended Generalized Mittag-Leffler Function

In this section we present certain classical integral inequalities using our extended generalized Mittag-Leffler function $E^{\delta,c,q,r}_{\rho,\sigma,\tau}$ with the corresponding fractional integral operator $\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f$, in real domain.

Here, for the reader's convenience we will use a simplified notation

$$\begin{aligned} \boldsymbol{E}(z;p) &:= E^{\delta,c,q,r}_{\rho,\sigma,\tau}(z;p) \qquad (6.3)\\ &= \sum_{n=0}^{\infty} \frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}(\delta,c-\delta)} \frac{(c)_{nq}}{\Gamma(\rho n+\sigma)} \frac{z^n}{(\tau)_{nr}}, \\ (\boldsymbol{\varepsilon} f)(x;p) &:= \left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p) \qquad (6.4)\\ &= \int_a^x (x-t)^{\sigma-1}\boldsymbol{E}(w(x-t)^{\rho};p)f(t)dt. \end{aligned}$$

For proving our inequalities, we follow similar methods as in the paper by W. Liu et al. ([99]), which we supplement with the necessary steps to obtain generalized results.

Further extensions of these results are given in the following section.

Theorem 6.3 *Let $w\in\mathbb{R}$, $r,\rho,\sigma,\tau>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $\alpha>0$, $\beta\geq\gamma>0$ and $x\in[a,b]$. Let f,g be positive continuous functions, monotonic in the opposite sense with $f\in L_\beta[a,b]$ and $g\in L_\alpha[a,b]$. Then the following inequality holds*

$$\frac{(\boldsymbol{\varepsilon} f^{\beta})(x;p)}{(\boldsymbol{\varepsilon} f^{\gamma})(x;p)} \geq \frac{(\boldsymbol{\varepsilon}(g^{\alpha}f^{\beta}))(x;p)}{(\boldsymbol{\varepsilon}(g^{\alpha}f^{\gamma}))(x;p)}. \qquad (6.5)$$

If f and g are monotonic functions in the same sense, then the inequality (6.5) *is reversed.*

Proof. Let f, g be monotonic functions in the opposite sense, both positive and continuous. Then for $u, v \in [a, x]$ we obtain

$$[(g(u))^{\alpha} - (g(v))^{\alpha}]\left[(f(v))^{\beta-\gamma} - (f(u))^{\beta-\gamma}\right] \geq 0, \tag{6.6}$$

that is

$$(g(u))^{\alpha}(f(v))^{\beta-\gamma} + (g(v))^{\alpha}(f(u))^{\beta-\gamma}$$
$$\geq (g(u))^{\alpha}(f(u))^{\beta-\gamma} + (g(v))^{\alpha}(f(v))^{\beta-\gamma}.$$

Multiplying both sides of the above inequality by

$$(x-v)^{\sigma-1}\boldsymbol{E}(\omega(x-v)^{\rho}; p)(f(v))^{\gamma}$$

and integrating on $[a, x]$ with respect to the variable v, we get

$$(g(u))^{\alpha}(\boldsymbol{\varepsilon} f^{\beta})(x; p) + (f(u))^{\beta-\gamma}(\boldsymbol{\varepsilon}(g^{\alpha} f^{\gamma}))(x; p)$$
$$\geq (g(u))^{\alpha}(f(u))^{\beta-\gamma}(\boldsymbol{\varepsilon} f^{\gamma})(x; p) + (\boldsymbol{\varepsilon}(g^{\alpha} f^{\beta}))(x; p).$$

Further multiplying by

$$(x-u)^{\sigma-1}\boldsymbol{E}(\omega(x-u)^{\rho}; p)(f(u))^{\gamma}$$

and then integrating on $[a, x]$ with respect to the variable u, we have

$$(\boldsymbol{\varepsilon}(g^{\alpha} f^{\gamma}))(x; p)(\boldsymbol{\varepsilon} f^{\beta})(x; p) \geq (\boldsymbol{\varepsilon}(g^{\alpha} f^{\beta}))(x; p)(\boldsymbol{\varepsilon} f^{\gamma})(x; p)$$

from which follows (6.5).
If f and g are monotonic in the same sense, then the reverse inequality of (6.5) can be proved analogously. □

For a special case of an increasing function on $[a, b]$, $g(x) = x - a$, we have the following corollary.

Corollary 6.1 *Let $w \in \mathbb{R}$, $r, \rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let $\alpha > 0$, $\beta \geq \gamma > 0$ and $x \in (a, b]$. Let $f \in L_{\beta}[a, b]$ be a positive continuous decreasing function. Then the following inequality holds*

$$\frac{(\boldsymbol{\varepsilon} f^{\beta})(x; p)}{(\boldsymbol{\varepsilon} f^{\gamma})(x; p)} \geq \frac{(\boldsymbol{\varepsilon}((x-a)^{\alpha} f^{\beta}))(x; p)}{(\boldsymbol{\varepsilon}((x-a)^{\alpha} f^{\gamma}))(x; p)}. \tag{6.7}$$

If f is increasing, then the inequality (6.7) *is reversed.*

Remark 6.1 If we consider special case of Mittag-Leffler function and its corresponding generalized fractional integral operator for $p = w = 0$, given in Remark 2.2, we obtain the left-sided Riemann-Liouville fractional integral

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt.$$

If we also set $\sigma = 1$, then inequality (6.5) implies Theorem 6.1. By verifying the condition (6.6), it is easy to see that although Theorem 6.1 is stated only for the case of decreasing function f and increasing function g, inequality (6.1) remains valid even if f is increasing and g is decreasing function, hence monotone in the opposite sense.
Similarly, for $p = w = 0$ and $\sigma = 1$, the inequality (6.7) implies [99, Theorem 3].

Theorem 6.4 *Let $w \in \mathbb{R}$, $r,\rho,\sigma,\tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $\alpha > 0$, $\beta \geq \gamma > 0$ and $x \in [a,b]$. Let f,g be positive continuous functions, $f \in L_{\alpha+\beta}[a,b]$, $g \in L_\alpha[a,b]$, such that for $u,v \in [a,x]$*

$$[(g(u))^\alpha (f(v))^\alpha - (g(v))^\alpha (f(u))^\alpha]\left[(f(v))^{\beta-\gamma} - (f(u))^{\beta-\gamma}\right] \geq 0. \tag{6.8}$$

Then the following inequality holds

$$\frac{(\boldsymbol{\varepsilon} f^{\alpha+\beta})(x;p)}{(\boldsymbol{\varepsilon} f^{\alpha+\gamma})(x;p)} \geq \frac{(\boldsymbol{\varepsilon}(g^\alpha f^\beta))(x;p)}{(\boldsymbol{\varepsilon}(g^\alpha f^\gamma))(x;p)}. \tag{6.9}$$

If the condition (6.8) *is reversed, then the inequality* (6.9) *is reversed.*

Proof. According to condition (6.8) we arrive at

$$\begin{aligned}
&(g(u))^\alpha (f(v))^{\alpha+\beta-\gamma} + (g(v))^\alpha (f(u))^{\alpha+\beta-\gamma} \\
&\geq (g(u))^\alpha (f(v))^\alpha (f(u))^{\beta-\gamma} + (g(v))^\alpha (f(u))^\alpha (f(v))^{\beta-\gamma}.
\end{aligned}$$

Multiplying the above by

$$(x-v)^{\sigma-1} \boldsymbol{E}(\omega(x-v)^\rho; p)(f(v))^\gamma$$

and integrating on $[a,x]$ with respect to the variable v, we get

$$\begin{aligned}
&(g(u))^\alpha (\boldsymbol{\varepsilon} f^{\alpha+\beta})(x;p) + (f(u))^{\alpha+\beta-\gamma}(\boldsymbol{\varepsilon}(g^\alpha f^\gamma))(x;p) \\
&\geq (g(u))^\alpha (f(u))^{\beta-\gamma}(\boldsymbol{\varepsilon} f^{\alpha+\gamma})(x;p) + (f(u))^\alpha (\boldsymbol{\varepsilon}(g^\alpha f^\beta))(x;p).
\end{aligned}$$

Once more, multiplying the above by

$$(x-u)^{\sigma-1} \boldsymbol{E}(\omega(x-u)^\rho; p)(f(u))^\gamma$$

and then integrating on $[a,x]$ with respect to the variable u, we obtain

$$(\boldsymbol{\varepsilon}(g^\alpha f^\gamma))(x;p)(\boldsymbol{\varepsilon} f^{\alpha+\beta})(x;p) \geq (\boldsymbol{\varepsilon}(g^\alpha f^\beta))(x;p)(\boldsymbol{\varepsilon} f^{\alpha+\gamma})(x;p)$$

from which follows (6.9).
If the condition (6.8) is reversed, then the reverse inequality of (6.9) can be proved analogously. □

Again, we have the following corollary for a special case $g(x) = x - a$.

Corollary 6.2 *Let $w \in \mathbb{R}$, $r,\rho,\sigma,\tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $\alpha > 0$, $\beta \geq \gamma > 0$ and $x \in [a,b]$. Let $f \in L_{\alpha+\beta}[a,b]$ be a positive continuous function such that for $u,v \in [a,x]$*

$$[(u-a)^{\alpha}(f(v))^{\alpha} - (v-a)^{\alpha}(f(u))^{\alpha}]\left[(f(v))^{\beta-\gamma} - (f(u))^{\beta-\gamma}\right] \geq 0. \tag{6.10}$$

Then the following inequality holds

$$\frac{(\boldsymbol{\varepsilon} f^{\alpha+\beta})(x;p)}{(\boldsymbol{\varepsilon} f^{\alpha+\gamma})(x;p)} \geq \frac{(\boldsymbol{\varepsilon}((x-a)^{\alpha} f^{\beta}))(x;p)}{(\boldsymbol{\varepsilon}((x-a)^{\alpha} f^{\gamma}))(x;p)}. \tag{6.11}$$

If the condition (6.10) *is reversed, then the inequality* (6.11) *is reversed.*

Remark 6.2 For $p = w = 0$ and $\sigma = 1$, inequalities (6.9) and (6.11) imply [99, Theorem 6] and [99, Theorem 5], respectively.

Next we present an essential integral inequality that we need in order to easily obtain Theorem 6.7.

Theorem 6.5 *Let $w \in \mathbb{R}$, $r,\rho,\sigma,\tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $x \in [a,b]$. Let $f,g,h \in L_1[a,b]$ be positive continuous functions such that f/h and g are monotonic in the opposite sense. Then the following inequality holds*

$$\frac{(\boldsymbol{\varepsilon} f)(x;p)}{(\boldsymbol{\varepsilon} h)(x;p)} \geq \frac{(\boldsymbol{\varepsilon}(fg))(x;p)}{(\boldsymbol{\varepsilon}(hg))(x;p)}. \tag{6.12}$$

If f/h and g are monotonic in the same sense, then the inequality (6.12) *is reversed.*

Proof. From hypotheses on functions, for $u,v \in [a,x]$ we have

$$[g(u) - g(v)]\left[\frac{f(v)}{h(v)} - \frac{f(u)}{h(u)}\right] \geq 0,$$

that is

$$g(u)\frac{f(v)}{h(v)} + g(v)\frac{f(u)}{h(u)} \geq g(u)\frac{f(u)}{h(u)} + g(v)\frac{f(v)}{h(v)}.$$

Multiplying both sides of the above inequality by

$$(x-v)^{\sigma-1}\boldsymbol{E}(\omega(x-v)^{\rho};p)h(v)$$

and integrating on $[a,x]$ with respect to the variable v, we get

$$g(u)(\boldsymbol{\varepsilon} f)(x;p) + \frac{f(u)}{h(u)}(\boldsymbol{\varepsilon}(gh))(x;p) \geq g(u)\frac{f(u)}{h(u)}(\boldsymbol{\varepsilon} h)(x;p) + (\boldsymbol{\varepsilon}(gf))(x;p).$$

Again, multiplying the above by

$$(x-u)^{\sigma-1}\boldsymbol{E}(\omega(x-u)^{\rho};p)h(u)$$

and then integrating on $[a,x]$ with respect to the variable u, we arrive at

$$(\boldsymbol{\varepsilon}(gh)(x;p)(\boldsymbol{\varepsilon}f)(x;p) \ \geq \ (\boldsymbol{\varepsilon}(gf))(x;p)(\boldsymbol{\varepsilon}h)(x;p)$$

from which follows (6.12).
If f/h and g are monotonic functions in the same sense, then the reverse inequality of (6.12) can be proved analogously. □

The counterpart of the previous result follows, where we assume $f(x) \leq h(x)$. Hence, inequality (6.12) remains satisfied if g is replaced by $f^{\alpha-1}$.

Theorem 6.6 *Let $w \in \mathbb{R}$, $r, \rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $\alpha \geq 1$ and $x \in [a,b]$. Let $f, h \in L_\alpha[a,b]$ be positive continuous functions such that f/h and f are monotonic in the opposite sense, with $f(x) \leq h(x)$ on $[a,b]$. Then the following inequality holds*

$$\frac{(\boldsymbol{\varepsilon}f)(x;p)}{(\boldsymbol{\varepsilon}h)(x;p)} \geq \frac{(\boldsymbol{\varepsilon}f^\alpha)(x;p)}{(\boldsymbol{\varepsilon}h^\alpha)(x;p)}. \tag{6.13}$$

If f/h and f are monotonic functions in the same sense, then the inequality (6.13) *is reversed.*

Proof. Assume that f/h is a decreasing function and f an increasing one. Then for $\alpha \geq 1$ function $f^{\alpha-1}$ is also increasing. By applying Theorem 6.5 we obtain

$$\frac{(\boldsymbol{\varepsilon}f)(x;p)}{(\boldsymbol{\varepsilon}h)(x;p)} \geq \frac{(\boldsymbol{\varepsilon}(f^\alpha))(x;p)}{(\boldsymbol{\varepsilon}(hf^{\alpha-1}))(x;p)}.$$

This together with the assumption $f(x) \leq h(x)$ lead to (6.13). Analogously we can prove the case when f/h is increasing and f decreasing, and obtain reversed inequality if f/h and f are monotonic in the same sense. □

In the last theorem of this section we involve a convex function in the inequality.

Theorem 6.7 *Let $w \in \mathbb{R}$, $r, \rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $x \in [a,b]$. Let $f, g, h \in L_1[a,b]$ be positive continuous functions such that f/h is decreasing function and f, g are increasing, with $f(x) \leq h(x)$ on $[a,b]$. Let ϕ be a convex function on $[0,\infty]$ with $\phi(0) = 0$. Then the following inequality holds*

$$\frac{(\boldsymbol{\varepsilon}f)(x;p)}{(\boldsymbol{\varepsilon}h)(x;p)} \geq \frac{(\boldsymbol{\varepsilon}(\phi(f)g))(x;p)}{(\boldsymbol{\varepsilon}(\phi(h)g))(x;p)}. \tag{6.14}$$

Proof. The function $\frac{\phi(x)}{x}$ is increasing since ϕ is a convex function on $[0,\infty]$ with $\phi(0) = 0$. From the assumption $f(x) \leq h(x)$ with the positivity of f and h, we get

$$\frac{\phi(f(x))}{f(x)} \leq \frac{\phi(h(x))}{h(x)}.$$

Further, since f, g and $\frac{\phi(x)}{x}$ are increasing then the following function

$$\frac{\phi(f(x))}{f(x)} g(x)$$

is also increasing. Hence

$$\frac{(\boldsymbol{\varepsilon}(\phi(f)g))(x;p)}{(\boldsymbol{\varepsilon}(\phi(h)g))(x;p)} = \frac{(\boldsymbol{\varepsilon}(\frac{\phi(f)}{f}fg))(x;p)}{(\boldsymbol{\varepsilon}(\frac{\phi(h)}{h}hg))(x;p)} \le \frac{(\boldsymbol{\varepsilon}(\frac{\phi(f)}{f}fg))(x;p)}{(\boldsymbol{\varepsilon}(\frac{\phi(f)}{f}hg))(x;p)},$$

and by applying Theorem 6.5 for f, h, $\frac{\phi(f)}{f}g$ we obtain

$$\frac{(\boldsymbol{\varepsilon}(\phi(f)g))(x;p)}{(\boldsymbol{\varepsilon}(\phi(h)g))(x;p)} \le \frac{(\boldsymbol{\varepsilon}(f))(x;p)}{(\boldsymbol{\varepsilon}(h))(x;p)}.$$

□

Corollary 6.3 *Let $w \in \mathbb{R}$, $r,\rho,\sigma,\tau > 0$, $c > \delta > 0$ with $p \ge 0$ and $0 < q \le r+\rho$. Let $x \in [a,b]$. Let $f,h \in L_1[a,b]$ be positive continuous functions such that f/h is decreasing function and f is increasing, with $f(x) \le h(x)$ on $[a,b]$. Let ϕ be a convex function on $[0,\infty]$ with $\phi(0) = 0$. Then the following inequality holds*

$$\frac{(\boldsymbol{\varepsilon}f)(x;p)}{(\boldsymbol{\varepsilon}h)(x;p)} \ge \frac{(\boldsymbol{\varepsilon}(\phi(f))(x;p)}{(\boldsymbol{\varepsilon}(\phi(h))(x;p)}. \tag{6.15}$$

Remark 6.3 If we set $p = w = 0$ and $\sigma = 1$, then Theorem 6.5, Theorem 6.6, Theorem 6.7 and Corollary 6.3 generalize Theorem 7, Theorem 8, Theorem 10 and Theorem 9 from [99], respectively.

6.2 Extensions of Classical Integral Inequalities Involving an Extended Generalized Mittag-Leffler Function

We continue to further extend the previously presented integral inequalities. Again we use our extended generalized Mittag-Leffler function with the corresponding fractional integral operator (in real domain) applied on $(f_i)_{i=1,2,\dots,n}$. A simplified notation $\boldsymbol{E}$ and $\boldsymbol{\varepsilon}f$ as in (6.3) and (6.4) is used here.

First theorem is an extension of Theorem 6.3.

Theorem 6.8 *Let $w \in \mathbb{R}$, $r,\rho,\sigma,\tau > 0$, $c > \delta > 0$ with $p \ge 0$ and $0 < q \le r+\rho$. Let $\alpha > 0$, $\beta \ge \gamma_i > 0$ for $i = 1,2,\dots,n$ and let $x \in [a,b]$. Let $(f_i)_{i=1,2,\dots,n}$ and g be positive continuous functions, such that $(f_i)_{i=1,2,\dots,n}$ are decreasing and g is increasing with $(f_i)_{i=1,2,\dots,n} \in L_\beta[a,b]$ and $g \in L_\alpha[a,b]$. Then for the fixed integer $s \in \{1,2,\dots,n\}$ the following inequality holds*

$$\frac{\left(\boldsymbol{\varepsilon}\left(\prod_{i\ne s}^{n} f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)}{\left(\boldsymbol{\varepsilon}\left(\prod_{i=1}^{n} f_i^{\gamma_i}\right)\right)(x;p)} \ge \frac{\left(\boldsymbol{\varepsilon}\left(g^{\alpha}\prod_{i\ne s}^{n} f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)}{\left(\boldsymbol{\varepsilon}\left(g^{\alpha}\prod_{i=1}^{n} f_i^{\gamma_i}\right)\right)(x;p)}. \tag{6.16}$$

If $(f_i)_{i=1,2,\dots,n}$ are increasing and g is decreasing, then the inequality (6.16) *also holds. If all functions are monotonic in the same sense, then the inequality* (6.16) *is reversed.*

Proof. Let $(f_i)_{i=1,2,\dots,n}$ be decreasing and g increasing, all positive and continuous. Let $s \in \{1,2,3,\dots,n\}$. Then for $u, v \in [a,x]$ we obtain

$$[(g(u))^{\alpha} - (g(v))^{\alpha}]\left[(f_s(v))^{\beta-\gamma_s} - (f_s(u))^{\beta-\gamma_s}\right] \geq 0,$$

that is

$$(g(u))^{\alpha}(f_s(v))^{\beta-\gamma_s} + (g(v))^{\alpha}(f_s(u))^{\beta-\gamma_s} \geq (g(u))^{\alpha}(f_s(u))^{\beta-\gamma_s} + (g(v))^{\alpha}(f_s(v))^{\beta-\gamma_s}.$$

Multiplying both sides of the above inequality by

$$(x-v)^{\sigma-1}\boldsymbol{E}(\omega(x-v)^{\rho};p)\prod_{i=1}^{n}(f_i(v))^{\gamma_i}$$

and integrating on $[a,x]$ with respect to the variable v, we get

$$\begin{aligned}&(g(u))^{\alpha}\left(\boldsymbol{\varepsilon}\left(\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p) + (f_s(u))^{\beta-\gamma_s}\left(\boldsymbol{\varepsilon}\left(g^{\alpha}\prod_{i=1}^{n} f_i^{\gamma_i}\right)\right)(x;p)\\ &\geq (g(u))^{\alpha}(f_s(u))^{\beta-\gamma_s}\left(\boldsymbol{\varepsilon}\left(\prod_{i=1}^{n} f_i^{\gamma_i}\right)\right)(x;p) + \left(\boldsymbol{\varepsilon}\left(g^{\alpha}\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p).\end{aligned}$$

Further multiplying by

$$(x-u)^{\sigma-1}\boldsymbol{E}(\omega(x-u)^{\rho};p)\prod_{i=1}^{n}(f_i(u))^{\gamma_i}$$

and then integrating on $[a,x]$ with respect to the variable u, we have

$$\begin{aligned}&\left(\boldsymbol{\varepsilon}\left(g^{\alpha}\prod_{i=1}^{n} f_i^{\gamma_i}\right)\right)(x;p)\left(\boldsymbol{\varepsilon}\left(\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)\\ &\geq \left(\boldsymbol{\varepsilon}\left(g^{\alpha}\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)\left(\boldsymbol{\varepsilon}\left(\prod_{i=1}^{n} f_i^{\gamma_i}\right)\right)(x;p).\end{aligned}$$

from which follows (6.16).
Analogously we can prove the case when $(f_i)_{i=1,2,\dots,n}$ are increasing and g is decreasing, and obtain reversed inequality if all functions are monotonic in the same sense. □

Remark 6.4 For $p = w = 0$ we obtain the left-sided Riemann-Liouville fraction integral J_{a+}^{σ} of order σ, as a special case of Mittag-Leffler function and its corresponding generalized fractional integral operator. Therefore, Theorem 6.8 generalizes Theorem 6.2.
On the other hand, if we set $n = 1$, then $s = 1$ which implies Theorem 6.3.

For $g(x) = x - a$, which is an increasing function on $[a,b]$, we have the following corollary. In this case, the inequality (6.17) implies [38, Theorem 3.1].

Corollary 6.4 *Let $w \in \mathbb{R}$, $r, \rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $\alpha > 0$, $\beta \geq \gamma_i > 0$ for $i = 1,2,\dots,n$ and let $x \in (a,b]$. Let $(f_i)_{i=1,2,\dots,n}$ be positive continuous decreasing functions with $(f_i)_{i=1,2,\dots,n} \in L_\beta[a,b]$. Then for the fixed integer $s \in \{1,2,\dots,n\}$ the following inequality holds*

$$\frac{\left(\boldsymbol{\varepsilon}\left(\prod_{i\neq s}^n f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)}{\left(\boldsymbol{\varepsilon}\left(\prod_{i=1}^n f_i^{\gamma_i}\right)\right)(x;p)} \geq \frac{\left(\boldsymbol{\varepsilon}\left((x-a)^{\alpha}\prod_{i\neq s}^n f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)}{\left(\boldsymbol{\varepsilon}\left((x-a)^{\alpha}\prod_{i=1}^n f_i^{\gamma_i}\right)\right)(x;p)}. \tag{6.17}$$

If $(f_i)_{i=1,2,\dots,n}$ are increasing, then the inequality (6.17) *is reversed.*

Theorem 6.9 *Let $w \in \mathbb{R}$, $r, \rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $\alpha > 0$, $\beta \geq \gamma_i > 0$ for $i = 1,2,\dots,n$ and let $x \in [a,b]$. Let $(f_i)_{i=1,2,\dots,n}$ and g be positive continuous functions, $(f_i)_{i=1,2,\dots,n} \in L_{\alpha+\beta}[a,b]$ and $g \in L_\alpha[a,b]$. Let $s \in \{1,2,\dots,n\}$ be fixed integer and for $u,v \in [a,x]$*

$$[(g(u))^{\alpha}(f_s(v))^{\alpha} - (g(v))^{\alpha}(f_s(u))^{\alpha}]\left[(f_s(v))^{\beta-\gamma_s} - (f_s(u))^{\beta-\gamma_s}\right] \geq 0. \tag{6.18}$$

Then the following inequality holds

$$\frac{\left(\boldsymbol{\varepsilon}\left(\prod_{i\neq s}^n f_i^{\gamma_i} f_s^{\alpha+\beta}\right)\right)(x;p)}{\left(\boldsymbol{\varepsilon}\left(\prod_{i\neq s}^n f_i^{\gamma_i} f_s^{\alpha+\gamma_s}\right)\right)(x;p)} \geq \frac{\left(\boldsymbol{\varepsilon}\left(g^{\alpha}\prod_{i\neq s}^n f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)}{\left(\boldsymbol{\varepsilon}\left(g^{\alpha}\prod_{i=1}^n f_i^{\gamma_i}\right)\right)(x;p)}. \tag{6.19}$$

If the condition (6.18) *is reversed, then the inequality* (6.19) *is reversed.*

Proof. From the (6.18) we obtain

$$\begin{aligned}&(g(u))^{\alpha}(f_s(v))^{\alpha+\beta-\gamma_s} + (g(v))^{\alpha}(f_s(u))^{\alpha+\beta-\gamma_s}\\ &\geq (g(u))^{\alpha}(f_s(v))^{\alpha}(f_s(u))^{\beta-\gamma_s} + (g(v))^{\alpha}(f_s(u))^{\alpha}(f_s(v))^{\beta-\gamma_s}.\end{aligned}$$

Multiplying both sides of the above inequality by

$$(x-v)^{\sigma-1}\boldsymbol{E}(\omega(x-v)^{\rho};p)\prod_{i=1}^n (f_i(v))^{\gamma_i}$$

and integrating on $[a,x]$ with respect to the variable v, we get

$$\begin{aligned}&(g(u))^{\alpha}\left(\boldsymbol{\varepsilon}\left(\prod_{i\neq s}^n f_i^{\gamma_i} f_s^{\alpha+\beta}\right)\right)(x;p) + (f_s(u))^{\alpha+\beta-\gamma_s}\left(\boldsymbol{\varepsilon}\left(g^{\alpha}\prod_{i=1}^n f_i^{\gamma_i}\right)\right)(x;p)\\ &\geq (g(u))^{\alpha}(f_s(u))^{\beta-\gamma_s}\left(\boldsymbol{\varepsilon}\left(\prod_{i\neq s}^n f_i^{\gamma_i} f_s^{\alpha+\gamma_s}\right)\right)(x;p)\\ &\quad + (f_s(u))^{\alpha}\left(\boldsymbol{\varepsilon}\left(g^{\alpha}\prod_{i\neq s}^n f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p).\end{aligned}$$

Further multiplying by

$$(x-u)^{\sigma-1}\boldsymbol{E}(\omega(x-u)^{\rho};p)\prod_{i=1}^{n}(f_i(u))^{\gamma_i}$$

and then integrating on $[a,x]$ with respect to the variable u, we have

$$\begin{aligned}&\left(\boldsymbol{\varepsilon}\left(g^{\alpha}\prod_{i=1}^{n}f_i^{\gamma_i}\right)\right)(x;p)\left(\boldsymbol{\varepsilon}\left(\prod_{i\neq s}^{n}f_i^{\gamma_i}f_s^{\alpha+\beta}\right)\right)(x;p)\\ &\geq\left(\boldsymbol{\varepsilon}\left(g^{\alpha}\prod_{i\neq s}^{n}f_i^{\gamma_i}f_s^{\beta}\right)\right)(x;p)\left(\boldsymbol{\varepsilon}\left(\prod_{i\neq s}^{n}f_i^{\gamma_i}f_s^{\alpha+\gamma_s}\right)\right)(x;p).\end{aligned}$$

from which follows (6.19).
If the condition (6.18) is reversed, then the reverse inequality of (6.19) can be proved analogously. □

Remark 6.5 For $p=w=0$, Theorem 6.9 generalizes [38, Theorem 3.10].
Setting $n=1$, Theorem 6.4 follows.

Corollary 6.5 *Let $w\in\mathbb{R}$, $r,\rho,\sigma,\tau>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $\alpha>0$, $\beta\geq\gamma_i>0$ for $i=1,2,\ldots,n$ and let $x\in(a,b]$. Let $(f_i)_{i=1,2,\ldots,n}\in L_{\alpha+\beta}[a,b]$ be positive continuous functions. Let $s\in\{1,2,\ldots,n\}$ be fixed integer and for $u,v\in[a,x]$*

$$[(u-a)^{\alpha}(f_s(v))^{\alpha}-(u-v)^{\alpha}(f_s(u))^{\alpha}]\left[(f_s(v))^{\beta-\gamma_s}-(f_s(u))^{\beta-\gamma_s}\right]\geq 0. \tag{6.20}$$

Then the following inequality holds

$$\frac{\left(\boldsymbol{\varepsilon}\left(\prod_{i\neq s}^{n}f_i^{\gamma_i}f_s^{\alpha+\beta}\right)\right)(x;p)}{\left(\boldsymbol{\varepsilon}\left(\prod_{i\neq s}^{n}f_i^{\gamma_i}f_s^{\alpha+\gamma_s}\right)\right)(x;p)}\geq\frac{\left(\boldsymbol{\varepsilon}\left((x-a)^{\alpha}\prod_{i\neq s}^{n}f_i^{\gamma_i}f_s^{\beta}\right)\right)(x;p)}{\left(\boldsymbol{\varepsilon}\left((x-a)^{\alpha}\prod_{i=1}^{n}f_i^{\gamma_i}\right)\right)(x;p)}. \tag{6.21}$$

If the condition (6.20) *is reversed, then the inequality* (6.21) *is reversed.*

Theorem 6.10 *Let $w\in\mathbb{R}$, $r,\rho,\sigma,\tau>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $\alpha>0$, $\beta\geq\gamma_i>0$ for $i=1,2,\ldots,n$ and let $x\in[a,b]$. Let $f,g,(h_i)_{i=1,2,\ldots,n}\in L_1[a,b]$ be positive continuous functions such that f/h_s and g are monotonic in the opposite sense, for $s\in\{1,2,\ldots,n\}$. Then the following inequality holds*

$$\frac{\left(\boldsymbol{\varepsilon}\left(f\prod_{i\neq s}^{n}h_i\right)\right)(x;p)}{\left(\boldsymbol{\varepsilon}\left(\prod_{i=1}^{n}h_i\right)\right)(x;p)}\geq\frac{\left(\boldsymbol{\varepsilon}\left(gf\prod_{i\neq s}^{n}h_i\right)\right)(x;p)}{\left(\boldsymbol{\varepsilon}\left(g\prod_{i=1}^{n}h_i\right)\right)(x;p)}. \tag{6.22}$$

If f/h_s and g are monotonic in the same sense for $s\in\{1,2,\ldots,n\}$, the inequality (6.22) *is reversed.*

Proof. From hypotheses on functions, for $u,v \in [a,x]$ we have

$$[g(u)-g(v)]\left[\frac{f(v)}{h_s(v)}-\frac{f(u)}{h_s(u)}\right] \geq 0,$$

that is

$$g(u)\frac{f(v)}{h_s(v)}+g(v)\frac{f(u)}{h_s(u)} \geq g(u)\frac{f(u)}{h_s(u)}+g(v)\frac{f(v)}{h_s(v)}.$$

Multiplying both sides of the above inequality by

$$(x-v)^{\sigma-1}\boldsymbol{E}(\omega(x-v)^{\rho};p)\prod_{i=1}^{n}h_i(v)$$

and integrating on $[a,x]$ with respect to the variable v, we get

$$\begin{aligned} g(u)\left(\boldsymbol{\varepsilon}\left(f\prod_{i\neq s}^{n}h_i\right)\right)(x;p)+\frac{f(u)}{h_s(u)}\left(\boldsymbol{\varepsilon}\left(g\prod_{i=1}^{n}h_i\right)\right)(x;p) \\ \geq\ g(u)\frac{f(u)}{h_s(u)}\left(\boldsymbol{\varepsilon}\left(\prod_{i=1}^{n}h_i\right)\right)(x;p)+\left(\boldsymbol{\varepsilon}\left(gf\prod_{i\neq s}^{n}h_i\right)\right)(x;p). \end{aligned}$$

Again, multiplying the above by

$$(x-u)^{\sigma-1}\boldsymbol{E}(\omega(x-u)^{\rho};p)\prod_{i=1}^{n}h_i(u)$$

and then integrating on $[a,x]$ with respect to the variable u, we arrive at

$$\begin{aligned} \left(\boldsymbol{\varepsilon}\left(g\prod_{i=1}^{n}h_i\right)\right)(x;p)\left(\boldsymbol{\varepsilon}\left(f\prod_{i\neq s}^{n}h_i\right)\right)(x;p) \\ \geq \left(\boldsymbol{\varepsilon}\left(\prod_{i=1}^{n}h_i\right)\right)(x;p)\left(\boldsymbol{\varepsilon}\left(gf\prod_{i\neq s}^{n}h_i\right)\right)(x;p). \end{aligned}$$

from which follows (6.22).
If f/h_s and g are monotonic in the same sense for $s \in \{1,2,\ldots,n\}$, then the reverse inequality of (6.22) can be proved analogously. □

Remark 6.6 For $p=w=0$, Theorem 6.10 generalizes [38, Theorem 3.14]. Setting $n=1$, Theorem 6.5 follows.

6.3 Further Generalizations of Some Classical Integral Inequalities

In this section we give further generalizations of several classical integral inequalities for a generalized fractional integral operator ${}_h\Upsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f$ (2.23) containing an extended Mittag-Leffler function $E^{\delta,c,q,r}_{\rho,\sigma,\tau}$ in the kernel.

We use a simplified notation

$$\begin{aligned}\boldsymbol{E}(z;p) &:= E^{\delta,c,q,r}_{\rho,\sigma,\tau}(z;p),\\ (\boldsymbol{{}_h\Upsilon} f)(x;p) &:= \left({}_h\Upsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)\\ &= \int_a^x (h(x)-h(t))^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(h(x)-h(t))^{\rho};p)h'(t)f(t)dt.\end{aligned}$$

Theorem 6.11 *Let $w\in\mathbb{R}$, $\rho,\sigma,\tau,r>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $h:[a,b]\to\mathbb{R}$ be a differentiable, strictly increasing function. Let $\alpha>0$, $\beta\geq\gamma_i>0$ for $i=1,2,\dots,n$ and let $x\in[a,b]$. Let $(f_i)_{i=1,2,\dots,n}$ and g be positive continuous functions, such that $(f_i)_{i=1,2,\dots,n}$ are decreasing and g is increasing with $(f_i)_{i=1,2,\dots,n}\in L_\beta[a,b]$ and $g\in L_\alpha[a,b]$. Then for the fixed integer $s\in\{1,2,\dots,n\}$ the following inequality holds*

$$\frac{\left(\boldsymbol{{}_h\Upsilon}\left(\prod_{i\neq s}^n f_i^{\gamma_i}f_s^{\beta}\right)\right)(x;p)}{\left(\boldsymbol{{}_h\Upsilon}\left(\prod_{i=1}^n f_i^{\gamma_i}\right)\right)(x;p)} \geq \frac{\left(\boldsymbol{{}_h\Upsilon}\left(g^\alpha\prod_{i\neq s}^n f_i^{\gamma_i}f_s^{\beta}\right)\right)(x;p)}{\left(\boldsymbol{{}_h\Upsilon}\left(g^\alpha\prod_{i=1}^n f_i^{\gamma_i}\right)\right)(x;p)}. \tag{6.23}$$

If $(f_i)_{i=1,2,\dots,n}$ are increasing and g is decreasing, then the inequality (6.23) *also holds. If all functions are monotonic in the same sense, then the inequality* (6.23) *is reversed.*

Proof. Let $u,v\in[a,x]$. Let $(f_i)_{i=1,2,\dots,n}$ be decreasing and g increasing, all positive and continuous. Let $s\in\{1,2,3,\dots,n\}$. Then

$$[(g(u))^\alpha-(g(v))^\alpha]\left[(f_s(v))^{\beta-\gamma_s}-(f_s(u))^{\beta-\gamma_s}\right]\geq 0, \tag{6.24}$$

hence

$$(g(u))^\alpha(f_s(v))^{\beta-\gamma_s}+(g(v))^\alpha(f_s(u))^{\beta-\gamma_s}\geq(g(u))^\alpha(f_s(u))^{\beta-\gamma_s}+(g(v))^\alpha(f_s(v))^{\beta-\gamma_s}.$$

Multiplying both sides of the above inequality by

$$(h(x)-h(v))^{\sigma-1}\boldsymbol{E}\left(w(h(x)-h(v))^{\rho};p\right)\prod_{i=1}^n(f_i(v))^{\gamma_i}h'(v) \tag{6.25}$$

and integrating on $[a,x]$ with respect to the variable v, we get

$$(g(u))^{\alpha}\left({}_{h}\Upsilon\left(\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)+(f_s(u))^{\beta-\gamma_s}\left({}_{h}\Upsilon\left(g^{\alpha}\prod_{i=1}^{n} f_i^{\gamma_i}\right)\right)(x;p)$$
$$\geq (g(u))^{\alpha}(f_s(u))^{\beta-\gamma_s}\left({}_{h}\Upsilon\left(\prod_{i=1}^{n} f_i^{\gamma_i}\right)\right)(x;p)+\left({}_{h}\Upsilon\left(g^{\alpha}\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p).$$

Further multiplying by

$$(h(x)-h(u))^{\sigma-1}\boldsymbol{E}\left(w(h(x)-h(u))^{\rho};p\right)\prod_{i=1}^{n}(f_i(u))^{\gamma_i}h'(u) \tag{6.26}$$

and then integrating on $[a,x]$ with respect to the variable u, we have

$$\left({}_{h}\Upsilon\left(g^{\alpha}\prod_{i=1}^{n} f_i^{\gamma_i}\right)\right)(x;p)\left({}_{h}\Upsilon\left(\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)$$
$$\geq \left({}_{h}\Upsilon\left(g^{\alpha}\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)\left({}_{h}\Upsilon\left(\prod_{i=1}^{n} f_i^{\gamma_i}\right)\right)(x;p).$$

from which follows (6.23).
Analogously we can prove the case when $(f_i)_{i=1,2,\dots,n}$ are increasing and g is decreasing, and obtain reversed inequality if all functions are monotonic in the same sense. □

Remark 6.7 If the function h is the identity function, then we obtain an inequality from Theorem 6.8 for the generalized fractional operator $\varepsilon_{a^+,\rho,\sigma,\tau}^{w,\delta,c,v,r}f$.

Also, if the h is the identity function and $p=w=0$, then we obtain the left-sided Riemann-Liouville fraction integral J_{a+}^{σ} of order σ, i.e. a special case of Mittag-Leffler function and its corresponding generalized fractional integral operator. Therefore, Theorem 6.11 generalizes Theorem 6.2.

The conditions under which inequality (6.23) and reverse inequality hold are complemented by the remaining cases of monotonicity of functions in the theorem.

Next inequality follows by setting $g(x)=x-a$ and it is a generalization of [38, Theorem 3.1].

Corollary 6.6 *Let $w\in\mathbb{R}$, $\rho,\sigma,\tau,r>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $h:[a,b]\to\mathbb{R}$ be a differentiable, strictly increasing function. Let $\alpha>0$, $\beta\geq\gamma_i>0$ for $i=1,2,\dots,n$ and let $x\in(a,b]$. Let $(f_i)_{i=1,2,\dots,n}$ be positive continuous decreasing functions with $(f_i)_{i=1,2,\dots,n}\in L_{\beta}[a,b]$. Then for the fixed integer $s\in\{1,2,\dots,n\}$ the following inequality holds*

$$\frac{\left({}_{h}\Upsilon\left(\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)}{\left({}_{h}\Upsilon\left(\prod_{i=1}^{n} f_i^{\gamma_i}\right)\right)(x;p)}\geq\frac{\left({}_{h}\Upsilon\left((x-a)^{\alpha}\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)}{\left({}_{h}\Upsilon\left((x-a)^{\alpha}\prod_{i=1}^{n} f_i^{\gamma_i}\right)\right)(x;p)}. \tag{6.27}$$

If $(f_i)_{i=1,2,\dots,n}$ are increasing, then the inequality (6.27) *is reversed.*

For the next result we set $n = 1$ in Theorem 6.11.

Corollary 6.7 *Let $w \in \mathbb{R}$, $\rho, \sigma, \tau, r > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let $h : [a,b] \to \mathbb{R}$ be a differentiable, strictly increasing function. Let $\alpha > 0$, $\beta \geq \gamma > 0$ and $x \in [a,b]$. Let f, g be positive continuous functions, monotonic in the opposite sense with $f \in L_\beta[a,b]$ and $g \in L_\alpha[a,b]$. Then the following inequality holds*

$$\frac{\left({}_{h}\Upsilon f^{\beta}\right)(x;p)}{\left({}_{h}\Upsilon f^{\gamma}\right)(x;p)} \geq \frac{\left({}_{h}\Upsilon (g^{\alpha} f^{\beta})\right)(x;p)}{\left({}_{h}\Upsilon (g^{\alpha} f^{\gamma})\right)(x;p)}. \tag{6.28}$$

If f and g are monotonic functions in the same sense, then the inequality (6.28) *is reversed.*

If additionally $g(x) = x - a$, then the following corollary holds.

Corollary 6.8 *Let $w \in \mathbb{R}$, $\rho, \sigma, \tau, r > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let $h : [a,b] \to \mathbb{R}$ be a differentiable, strictly increasing function. Let $\alpha > 0$, $\beta \geq \gamma > 0$ and $x \in (a,b]$. Let $f \in L_\beta[a,b]$ be a positive continuous decreasing function. Then the following inequality holds*

$$\frac{\left({}_{h}\Upsilon f^{\beta}\right)(x;p)}{\left({}_{h}\Upsilon f^{\gamma}\right)(x;p)} \geq \frac{\left({}_{h}\Upsilon ((x-a)^{\alpha} f^{\beta})\right)(x;p)}{\left({}_{h}\Upsilon ((x-a)^{\alpha} f^{\gamma})\right)(x;p)}. \tag{6.29}$$

If f is increasing, then the inequality (6.29) *is reversed.*

Remark 6.8 If the function h is the identity function, $p = w = 0$ and $\sigma = 1$, then inequalities (6.28) and (6.29) imply Theorem 6.1 and [99, Theorem 3], respectively.

Further, from the condition (6.24) with $n = 1$ (i.e. we have only one function f) and for $u, v \in [a,x]$ we obtain

$$[(g(u))^{\alpha} - (g(v))^{\alpha}]\left[(f(v))^{\beta-\gamma} - (f(u))^{\beta-\gamma}\right] \geq 0.$$

Now it is easy to see that although Theorem 6.1 is stated only for the case of decreasing function f and increasing function g, inequality (6.1) remains valid even if f is increasing and g is decreasing function g, hence monotone functions in the opposite sense.

Theorem 6.12 *Let $w \in \mathbb{R}$, $\rho, \sigma, \tau, r > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let $h : [a,b] \to \mathbb{R}$ be a differentiable, strictly increasing function. Let $\alpha > 0$, $\beta \geq \gamma_i > 0$ for $i = 1, 2, \ldots, n$ and let $x \in [a,b]$. Let $(f_i)_{i=1,2,\ldots,n}$ and g be positive continuous functions, $(f_i)_{i=1,2,\ldots,n} \in L_{\alpha+\beta}[a,b]$ and $g \in L_\alpha[a,b]$. Let $s \in \{1, 2, \ldots, n\}$ be fixed integer and for $u, v \in [a,x]$ let the condition* (6.18) *holds. Then*

$$\frac{\left({}_{h}\Upsilon \left(\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\alpha+\beta}\right)\right)(x;p)}{\left({}_{h}\Upsilon \left(\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\alpha+\gamma_s}\right)\right)(x;p)} \geq \frac{\left({}_{h}\Upsilon \left(g^{\alpha} \prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)}{\left({}_{h}\Upsilon \left(g^{\alpha} \prod_{i=1}^{n} f_i^{\gamma_i}\right)\right)(x;p)}. \tag{6.30}$$

If the condition (6.18) *is reversed, then the inequality* (6.30) *is reversed.*

Proof. From the (6.18) we obtain

$$(g(u))^{\alpha}(f_s(v))^{\alpha+\beta-\gamma_s}+(g(v))^{\alpha}(f_s(u))^{\alpha+\beta-\gamma_s}$$
$$\geq (g(u))^{\alpha}(f_s(v))^{\alpha}(f_s(u))^{\beta-\gamma_s}+(g(v))^{\alpha}(f_s(u))^{\alpha}(f_s(v))^{\beta-\gamma_s}.$$

Multiplying both sides of the above inequality by (6.25) and integrating on $[a,x]$ with respect to the variable v, we get

$$(g(u))^{\alpha}\left({}_{h}\Upsilon\left(\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\alpha+\beta}\right)\right)(x;p)+(f_s(u))^{\alpha+\beta-\gamma_s}\left({}_{h}\Upsilon\left(g^{\alpha}\prod_{i=1}^{n} f_i^{\gamma_i}\right)\right)(x;p)$$
$$\geq (g(u))^{\alpha}(f_s(u))^{\beta-\gamma_s}\left({}_{h}\Upsilon\left(\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\alpha+\gamma_s}\right)\right)(x;p)+(f_s(u))^{\alpha}\left({}_{h}\Upsilon\left(g^{\alpha}\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p).$$

Further multiplying the above by (6.26) and then integrating on $[a,x]$ with respect to the variable u, we obtain

$$\left({}_{h}\Upsilon\left(g^{\alpha}\prod_{i=1}^{n} f_i^{\gamma_i}\right)\right)(x;p)\left({}_{h}\Upsilon\left(\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\alpha+\beta}\right)\right)(x;p)$$
$$\geq \left({}_{h}\Upsilon\left(g^{\alpha}\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)\left({}_{h}\Upsilon\left(\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\alpha+\gamma_s}\right)\right)(x;p).$$

from which follows (6.30).
If the condition (6.18) is reversed, then the reverse inequality of (6.30) can be proved analogously. □

Remark 6.9 If the function h is the identity function and $p=w=0$, then Theorem 6.12 generalizes [38, Theorem 3.10].

Corollary 6.9 *Let $w\in\mathbb{R}$, $\rho,\sigma,\tau,r>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $h:[a,b]\to\mathbb{R}$ be a differentiable, strictly increasing function. Let $\alpha>0$, $\beta\geq\gamma_i>0$ for $i=1,2,\ldots,n$ and let $x\in(a,b]$. Let $(f_i)_{i=1,2,\ldots,n}\in L_{\alpha+\beta}[a,b]$ be positive continuous functions. Let $s\in\{1,2,\ldots,n\}$ be fixed integer and for $u,v\in[a,x]$ let the condition* (6.20) *holds. Then*

$$\frac{\left({}_{h}\Upsilon\left(\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\alpha+\beta}\right)\right)(x;p)}{\left({}_{h}\Upsilon\left(\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\alpha+\gamma_s}\right)\right)(x;p)}\geq\frac{\left({}_{h}\Upsilon\left((x-a)^{\alpha}\prod_{i\neq s}^{n} f_i^{\gamma_i} f_s^{\beta}\right)\right)(x;p)}{\left({}_{h}\Upsilon\left((x-a)^{\alpha}\prod_{i=1}^{n} f_i^{\gamma_i}\right)\right)(x;p)}. \tag{6.31}$$

If the condition (6.20) *is reversed, then the inequality* (6.31) *is reversed.*

If we set $n=1$ in Theorem 6.12, then we obtain the following inequality.

Corollary 6.10 *Let $w\in\mathbb{R}$, $\rho,\sigma,\tau,r>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $h:[a,b]\to\mathbb{R}$ be a differentiable, strictly increasing function. Let $\alpha>0$, $\beta\geq\gamma>0$ and $x\in[a,b]$. Let f,g be positive continuous functions, $f\in L_{\alpha+\beta}[a,b]$, $g\in L_{\alpha}[a,b]$, such that for $u,v\in[a,x]$ the condition* (6.8) *holds. Then*

$$\frac{\left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon} f^{\alpha+\beta}\right)(x;p)}{\left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon} f^{\alpha+\gamma}\right)(x;p)} \geq \frac{\left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}(g^{\alpha} f^{\beta})\right)(x;p)}{\left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}(g^{\alpha} f^{\gamma})\right)(x;p)}. \tag{6.32}$$

If the condition (6.8) *is reversed, then the inequality* (6.32) *is reversed.*

Corollary 6.11 *Let $w \in \mathbb{R}$, $\rho,\sigma,\tau,r > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $h : [a,b] \to \mathbb{R}$ be a differentiable, strictly increasing function. Let $\alpha > 0$, $\beta \geq \gamma > 0$ and $x \in (a,b]$. Let $f \in L_{\alpha+\beta}[a,b]$ be positive continuous function such that for $u,v \in [a,x]$ the condition* (6.10) *holds. Then*

$$\frac{\left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon} f^{\alpha+\beta}\right)(x;p)}{\left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon} f^{\alpha+\gamma}\right)(x;p)} \geq \frac{\left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}((x-a)^{\alpha} f^{\beta})\right)(x;p)}{\left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}((x-a)^{\alpha} f^{\gamma})\right)(x;p)}. \tag{6.33}$$

If the condition (6.10) *is reversed, then the inequality* (6.33) *is reversed.*

Remark 6.10 If the function h is the identity function, $p = w = 0$ and $\sigma = 1$, then inequalities (6.32) and (6.33) imply [99, Theorem 6] and [99, Theorem 5], respectively.

Theorem 6.13 *Let $w \in \mathbb{R}$, $\rho,\sigma,\tau,r > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $h : [a,b] \to \mathbb{R}$ be a differentiable, strictly increasing function. Let $\alpha > 0$, $\beta \geq \gamma_i > 0$ for $i = 1,2,\ldots,n$ and let $x \in [a,b]$. Let $f,g,(\varphi_i)_{i=1,2,\ldots,n} \in L_1[a,b]$ be positive continuous functions such that f/φ_s and g are monotonic in the opposite sense, for $s \in \{1,2,\ldots,n\}$. Then the following inequality holds*

$$\frac{\left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}\left(f\prod_{i\neq s}^{n}\varphi_i\right)\right)(x;p)}{\left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}\left(\prod_{i=1}^{n}\varphi_i\right)\right)(x;p)} \geq \frac{\left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}\left(gf\prod_{i\neq s}^{n}\varphi_i\right)\right)(x;p)}{\left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}\left(g\prod_{i=1}^{n}\varphi_i\right)\right)(x;p)}. \tag{6.34}$$

If f/φ_s and g are monotonic in the same sense for $s \in \{1,2,\ldots,n\}$, then the inequality (6.34) *is reversed.*

Proof. From hypotheses on functions, for $u,v \in [a,x]$ we have

$$[g(u)-g(v)]\left[\frac{f(v)}{\varphi_s(v)} - \frac{f(u)}{\varphi_s(u)}\right] \geq 0,$$

that is

$$g(u)\frac{f(v)}{\varphi_s(v)} + g(v)\frac{f(u)}{\varphi_s(u)} \geq g(u)\frac{f(u)}{\varphi_s(u)} + g(v)\frac{f(v)}{\varphi_s(v)}.$$

Multiplying both sides of the above inequality by

$$(h(x)-h(v))^{\sigma-1}\boldsymbol{E}(w(h(x)-h(v))^{\rho};p)\prod_{i=1}^{n}\varphi_i(v)h'(v)$$

and integrating on $[a,x]$ with respect to the variable v, we get

$$\begin{aligned} g(u)&\left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}\left(f\prod_{i\neq s}^{n}\varphi_i\right)\right)(x;p) + \frac{f(u)}{\varphi_s(u)}\left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}\left(g\prod_{i=1}^{n}\varphi_i\right)\right)(x;p) \\ \geq\ g(u)&\frac{f(u)}{\varphi_s(u)}\left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}\left(\prod_{i=1}^{n}\varphi_i\right)\right)(x;p) + \left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}\left(gf\prod_{i\neq s}^{n}\varphi_i\right)\right)(x;p). \end{aligned}$$

Again, multiplying the above by

$$(h(x)-h(u))^{\sigma-1}\boldsymbol{E}(w(h(x)-h(u))^{\rho};p)\prod_{i=1}^{n}\varphi_i(u)h'(u)$$

and then integrating on $[a,x]$ with respect to the variable u, we arrive at

$$\left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}\left(g\prod_{i=1}^{n}\varphi_i\right)\right)(x;p)\left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}\left(f\prod_{i\neq s}^{n}\varphi_i\right)\right)(x;p)$$
$$\geq\left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}\left(\prod_{i=1}^{n}\varphi_i\right)\right)(x;p)\left({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}\left(gf\prod_{i\neq s}^{n}\varphi_i\right)\right)(x;p).$$

from which follows (6.34).
If f/φ_s and g are monotonic in the same sense for $s\in\{1,2,\ldots,n\}$, then the reverse inequality of (6.34) can be proved analogously. □

Remark 6.11 If the function h is the identity function and $p=w=0$, then Theorem 6.13 generalizes [38, Theorem 3.14].

If we set $n=1$ in Theorem 6.13, then we obtain the following generalization of [99, Theorem 7].

Corollary 6.12 *Let $w\in\mathbb{R}$, $\rho,\sigma,\tau,r>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $h:[a,b]\to\mathbb{R}$ be a differentiable, strictly increasing function. Let $x\in[a,b]$. Let $f,g,\varphi\in L_1[a,b]$ be positive continuous functions such that f/φ and g are monotonic in the opposite sense. Then the following inequality holds*

$$\frac{({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}f)(x;p)}{({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}\varphi)(x;p)}\geq\frac{({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}(fg))(x;p)}{({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}(\varphi g))(x;p)}.\tag{6.35}$$

If f/φ and g are monotonic in the same sense, then the inequality (6.35) *is reversed.*

Next is a counterpart of the previous result, where we assume $f(x)\leq\varphi(x)$. Hence, inequality (6.35) remains satisfied if g is replaced by $f^{\alpha-1}$.

Theorem 6.14 *Let $w\in\mathbb{R}$, $\rho,\sigma,\tau,r>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $h:[a,b]\to\mathbb{R}$ be a differentiable, strictly increasing function. Let $\alpha\geq 1$ and $x\in[a,b]$. Let $f,\varphi\in L_\alpha[a,b]$ be positive continuous functions such that f/φ and f are monotonic in the opposite sense, with $f(x)\leq\varphi(x)$ on $[a,b]$. Then the following inequality holds*

$$\frac{({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}f)(x;p)}{({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}\varphi)(x;p)}\geq\frac{({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}f^{\alpha})(x;p)}{({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}\varphi^{\alpha})(x;p)}.\tag{6.36}$$

If f/φ and f are monotonic functions in the same sense, then the inequality (6.36) *is reversed.*

Proof. Assume that f/φ is a decreasing function and f an increasing one. Then for $\alpha\geq 1$ function $f^{\alpha-1}$ is also increasing. By applying Corollary 6.12 we obtain

$$\frac{({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}f)(x;p)}{({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}\varphi)(x;p)}\geq\frac{({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}(f^{\alpha}))(x;p)}{({}_{\boldsymbol{h}}\boldsymbol{\Upsilon}(\varphi f^{\alpha-1}))(x;p)}.$$

This together with the assumption $f(x) \leq \varphi(x)$ lead to (6.36). Analogously we can prove the case when f/φ is increasing and f decreasing, and obtain reversed inequality if f/φ and f are monotonic in the same sense. □

For the last theorem we involve a convex function in the inequality.

Theorem 6.15 *Let $w \in \mathbb{R}$, $\rho, \sigma, \tau, r > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let $h : [a,b] \to \mathbb{R}$ be a differentiable, strictly increasing function and $x \in [a,b]$. Let $f, g, \varphi \in L_1[a,b]$ be positive continuous functions such that f/φ is decreasing function and f, g are increasing, with $f(x) \leq \varphi(x)$ on $[a,b]$. Let ϕ be a convex function on $[0,\infty]$ with $\phi(0) = 0$. Then the following inequality holds*

$$\frac{\left({}_h\Upsilon f\right)(x;p)}{\left({}_h\Upsilon \varphi\right)(x;p)} \geq \frac{\left({}_h\Upsilon(\phi(f)g)\right)(x;p)}{\left({}_h\Upsilon(\phi(\varphi)g)\right)(x;p)}. \tag{6.37}$$

Proof. The function $\frac{\phi(x)}{x}$ is increasing since ϕ is a convex function on $[0,\infty]$ with $\phi(0) = 0$. From the assumption $f(x) \leq \varphi(x)$ with the positivity of f and φ, we get

$$\frac{\phi(f(x))}{f(x)} \leq \frac{\phi(\varphi(x))}{\varphi(x)}.$$

Further, since f, g and $\frac{\phi(x)}{x}$ are increasing then the following function

$$\frac{\phi(f(x))}{f(x)} g(x)$$

is also increasing. Hence

$$\frac{\left({}_h\Upsilon(\phi(f)g)\right)(x;p)}{\left({}_h\Upsilon(\phi(\varphi)g)\right)(x;p)} = \frac{\left({}_h\Upsilon(\frac{\phi(f)}{f} fg)\right)(x;p)}{\left({}_h\Upsilon(\frac{\phi(\varphi)}{\varphi} \varphi g)\right)(x;p)} \leq \frac{\left({}_h\Upsilon(\frac{\phi(f)}{f} fg)\right)(x;p)}{\left({}_h\Upsilon(\frac{\phi(f)}{f} \varphi g)\right)(x;p)},$$

and by applying Corollary 6.12 for f, φ, $\frac{\phi(f)}{f} g$ we obtain

$$\frac{\left({}_h\Upsilon(\phi(f)g)\right)(x;p)}{\left({}_h\Upsilon(\phi(\varphi)g)\right)(x;p)} \leq \frac{\left({}_h\Upsilon(f)\right)(x;p)}{\left({}_h\Upsilon(\varphi)\right)(x;p)}.$$

□

Corollary 6.13 *Let $w \in \mathbb{R}$, $\rho, \sigma, \tau, r > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let $h : [a,b] \to \mathbb{R}$ be a differentiable, strictly increasing function and $x \in [a,b]$. Let $f, \varphi \in L_1[a,b]$ be positive continuous functions such that f/φ is decreasing function and f is increasing, with $f(x) \leq \varphi(x)$ on $[a,b]$. Let ϕ be a convex function on $[0,\infty]$ with $\phi(0) = 0$. Then the following inequality holds*

$$\frac{\left({}_h\Upsilon f\right)(x;p)}{\left({}_h\Upsilon \varphi\right)(x;p)} \geq \frac{\left({}_h\Upsilon(\phi(f))\right)(x;p)}{\left({}_h\Upsilon(\phi(\varphi))\right)(x;p)}. \tag{6.38}$$

Remark 6.12 If the function h is the identity function, $p = w = 0$ and $\sigma = 1$, then Theorem 6.14, Theorem 6.15 and Corollary 6.13 generalize Theorem 8, Theorem 10 and Theorem 9 from [99], respectively.

Chapter 7

Hadamard and Fejér-Hadamard Types Fractional Integral Inequalities Associated with the Mittag-Leffler Function

In this chapter Hadamard and Fejér-Hadamard inequalities for convex, relative convex, m-convex, $(h-m)$-convex, harmonically convex and harmonically $(\alpha,h-m)$-convex functions via fractional integrals involving Mittag-Leffler function are given.

This chapter is based on our results from [50, 58, 64, 83, 86, 87].

7.1 Hadamard and Fejér-Hadamard Inequalities for Convex Functions

For a convex function $f : I \to \mathbb{R}$ where I is an interval in $\mathbb{R}$, the following inequality holds

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{7.1}$$

where $a,b \in I$ and $a < b$.
Inequality (7.1) is well known in literature as Hadamard inequality.

Fejér gave generalization of the Hadamard inequality known as Fejér-Hadamard inequality stated as follows [67].
For a convex function $f : I \to \mathbb{R}$ where I is an interval in $\mathbb{R}$, the following inequality holds

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \le \frac{1}{b-a}\int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx, \tag{7.2}$$

where g is a function which is integrable, nonnegative and symmetric about $\frac{a+b}{2}$.
For Riemann-Liouville fractional integrals the Hadamard inequality is given in next results.

Theorem 7.1 *[140] Let $f : [a,b] \to \mathbb{R}$ be a function with $0 \le a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequality holds*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\sigma+1)}{2(b-a)^\sigma}\left[J^\sigma_{a+}f(b)+J^\sigma_{b-}f(a)\right] \le \frac{f(a)+f(b)}{2} \tag{7.3}$$

with $\sigma > 0$.

Another version of the Hadamard inequality is given as follows.

Theorem 7.2 *[141] Let $f : [a,b] \to \mathbb{R}$ be a function with $0 \le a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequality holds*

$$f\left(\frac{b+a}{2}\right) \le \frac{2^{\sigma+1}\Gamma(\sigma+1)}{(b-a)^\sigma}[J^\sigma_{(\frac{a+b}{2})^+}f(b)+J^\sigma_{(\frac{a+b}{2})^-}f(a)] \le \frac{f(a)+f(b)}{2}$$

with $\sigma > 0$.

The generalized Hadamard inequality containing Mittag-Leffler function is given in the theorem. The following notations will be used frequently

$$H^{w'}_{a^+,\sigma}(x;p) := (\varepsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma,\tau}1)(x;p),\; H^{w'}_{b^-,\sigma}(x;p) := (\varepsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau}1)(x;p),$$

while notations of fractional integral operators will be followed as it is.

Theorem 7.3 *Let $w \in \mathbb{R}$, $r,\rho,\sigma,\tau > 0$, $\delta > 0$ with $p \ge 0$ and $0 < q \le r+\rho$. Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequality for extended generalized fractional integral holds*

$$\begin{aligned} &f\left(\frac{a+b}{2}\right)H^{w'}_{a^+,\sigma}(b;p) \\ &\le \frac{(\varepsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f)(b;p)+(\varepsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f)(a;p)}{2} \\ &\le \left(\frac{f(a)+f(b)}{2}\right)H^{w'}_{b^-,\sigma}(a;p), \end{aligned} \tag{7.4}$$

where $w' = \frac{w}{(b-a)^\rho}$.

Proof. Since f is convex function on $[a,b]$, for $t \in [0,1]$ we have

$$f\left(\frac{(ta+(1-t)b)+((1-t)a+tb)}{2}\right) \leq \frac{f(ta+(1-t)b)+f((1-t)a+tb)}{2}. \tag{7.5}$$

Multiplying both sides of above inequality with $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)$ we get

$$\begin{aligned}&2t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f\left(\frac{a+b}{2}\right)\\ &\leq t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)\left(f(ta+(1-t)b)+f((1-t)a+tb)\right).\end{aligned}$$

Integrating with respect to t over $[0,1]$ we have

$$\begin{aligned}&2f\left(\frac{a+b}{2}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)dt\\ &\leq \int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f(ta+(1-t)b)dt\\ &+\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f((1-t)a+tb)dt.\end{aligned}$$

If we put $u = at+(1-t)b$, then $t = \frac{b-u}{b-a}$ and if $v = (1-t)a+tb$, then $t = \frac{v-a}{b-a}$. Therefore by using Definition 2.2 one can have

$$f\left(\frac{a+b}{2}\right)H^{w'}_{a^+,\sigma}(b;p) \leq \frac{(\varepsilon^{w',\delta,q,r}_{a^+,\rho,\sigma,\tau}f)(b;p)+(\varepsilon^{w',\delta,q,r}_{b^-,\rho,\sigma,\tau}f)(a;p)}{2}. \tag{7.6}$$

Again by using that f is convex function on $[a,b]$, and for $t \in [0,1]$ we have

$$f(ta+(1-t)b)+f((1-t)a+tb) \leq tf(a)+(1-t)f(b)+(1-t)f(a)+tf(b). \tag{7.7}$$

Now multiplying with $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)$ and integrating over $[0,1]$ we get

$$\begin{aligned}&\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f(ta+(1-t)b)dt+\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f((1-t)a+tb)dt\\ &\leq [f(a)+f(b)]\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)dt\end{aligned}$$

from which by using change of variables as for (7.6), we get

$$(\varepsilon^{w',\delta,q,r}_{a^+,\rho,\sigma,\tau}f)(b;p)+(\varepsilon^{w',\delta,q,r}_{b^-,\rho,\sigma,\tau}f)(a;p) \leq (f(a)+f(b))H^{w'}_{b^-,\sigma}(a;p). \tag{7.8}$$

From the inequalities (7.6) and (7.8), the inequality (7.4) is obtained. □

Remark 7.1 In Theorem 7.3

(i) if $p = 0$, then we get [51, Theorem 2.1].

(ii) if $w = p = 0$, then we get [140, Theorem 2].

(iii) if $w = p = 0$ and $\sigma = 1$, then we get (7.1).

In the following we give the Fejér-Hadamard inequality for fractional integral operator containing the extended generalized Mittag-Leffler function.

Theorem 7.4 *Let $w \in \mathbb{R}$, $r, \rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let $f : [a,b] \to \mathbb{R}$ be a convex function with $0 \leq a < b$ and $f \in L_1[a,b]$. Also, let $g : [a,b] \to \mathbb{R}$ be a function which is nonnegative, integrable and symmetric about $\frac{a+b}{2}$. Then the following inequality for the extended generalized fractional integral holds*

$$f\left(\frac{a+b}{2}\right)(\varepsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma,\tau} g)(b;p) \tag{7.9}$$
$$\leq \frac{(\varepsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma,\tau} fg)(b;p) + (\varepsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau} fg)(a;p)}{2}$$
$$\leq \frac{f(a)+f(b)}{2}(\varepsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau} g)(a;p),$$

where $w' = \frac{w}{(b-a)^\sigma}$.

Proof. Multiplying (7.5) with $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)g(tb+(1-t)a)$ we get

$$2t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)f\left(\frac{a+b}{2}\right)g(tb+(1-t)a)$$
$$\leq t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)\left(f(ta+(1-t)b)+f((1-t)a+tb)\right)g(tb+(1-t)a).$$

Integrating with respect to t over $[0,1]$ we have

$$2f\left(\frac{a+b}{2}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)g(tb+(1-t)a)dt$$
$$\leq \int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)f(ta+(1-t)b)g(tb+(1-t)a)dt$$
$$+\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)f((1-t)a+tb)g(tb+(1-t)a)dt.$$

If we put $u = at + (1-t)b$, then $t = \frac{b-u}{b-a}$ and if $v = (1-t)a + tb$, then $t = \frac{v-a}{b-a}$. Therefore one can have

$$2f\left(\frac{a+b}{2}\right)\int_a^b (b-u)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w\left(\frac{b-u}{b-a}\right)^\rho : p\right)g(a+b-u)du$$
$$\leq \int_a^b (b-u)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w\left(\frac{b-u}{b-a}\right)^\rho ;p\right)f(u)g(a+b-u)du$$
$$+\int_b^a (v-a)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w\left(\frac{v-a}{b-a}\right)^\rho ;p\right)f(v)g(a+b-v)dv.$$

From symmetry of function g about $\frac{a+b}{2}$ we have $g(a+b-x)=g(x)$, $x\in[a,b]$, therefore using this fact we have

$$f\left(\frac{a+b}{2}\right)(\varepsilon^{w',\delta,q,r}_{a^+,\rho,\sigma,\tau}g)(b;p)\leq\frac{(\varepsilon^{w',\delta,q,r}_{a^+,\rho,\sigma,\tau}fg)(b;p)+(\varepsilon^{w',\delta,q,r}_{b^-,\rho,\sigma,\tau}fg)(a;p)}{2}. \tag{7.10}$$

Now multiplying (7.7) with $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)g(ta+(1-t)b)$ and integrating with respect to t over $[0,1]$ we get

$$\begin{aligned}&\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)f(ta+(1-t)b)g(ta+(1-t)b)dt\\&+\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)f((1-t)a+tb)g(ta+(1-t)b)dt\\&\leq(f(a)+f(b))\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)g(ta+(1-t)b)dt.\end{aligned}$$

By change of variables as for (7.10), we get

$$(\varepsilon^{w',\delta,q,r}_{a^+,\rho,\sigma,\tau}fg)(b;p)+(\varepsilon^{w',\delta,q,r}_{b^-,\rho,\sigma,\tau}fg)(a;p)\leq(f(a)+f(b))(\varepsilon^{w',\delta,q,r}_{b^-,\rho,\sigma,\tau}g)(a;p). \tag{7.11}$$

From inequalities (7.11) and (7.10), we get inequality in (7.9). □

Remark 7.2 In Theorem 7.4

(i) if $g=1$, then we get Theorem 7.3.

(ii) if $p=0$, then we get [51, Theorem 2.2].

(iii) if $w=p=0$, then we get [77, Theorem 2.2].

Another generalized version of the Hadamard inequality is given in next result.

Theorem 7.5 *Let $w\in\mathbb{R}$, $r,\rho,\sigma,\tau>0$, $c>\delta>0$ with $p\geq0$ and $0<q\leq r+\rho$. Let $f:[a,b]\to\mathbb{R}$ be a function such that $f\in L_1[a,b]$ with $a<b$. If f is convex on $[a,b]$, then the following inequalities for extended generalized fractional integral operator holds*

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)H^{w'}_{(\frac{a+b}{2})^+,\sigma}(b;p)\\&\leq\left[\left(\varepsilon^{w',\delta,c,q,r}_{(\frac{a+b}{2})^+,\rho,\sigma,\tau}f\right)(b;p)+\left(\varepsilon^{w',\delta,c,q,r}_{(\frac{a+b}{2})^-,\rho,\sigma,\tau}f\right)(a;p)\right]\\&\leq\frac{f(a)+f(b)}{2}H^{w'}_{(\frac{a+b}{2})^-,\sigma}(a;p)\end{aligned} \tag{7.12}$$

where $w'=\frac{2^\rho w}{(b-a)^\rho}$.

Proof. Since f is convex function, for $t \in [0,1]$ we have

$$2f\left(\frac{a+b}{2}\right) \leq f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)+f\left(\frac{t}{2}a+\frac{2-t}{2}b\right). \tag{7.13}$$

Also from convexity of f we have

$$\begin{aligned} &f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)+f\left(\frac{t}{2}a+\frac{2-t}{2}b\right) \\ &\leq \frac{2-t}{2}f(a)+\frac{t}{2}f(b)+\frac{t}{2}f(a)+\frac{2-t}{2}f(b) \\ &= f(a)+f(b). \end{aligned} \tag{7.14}$$

Multiplying (7.13) by $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)$ on both sides and then integrating on $[0,1]$ we have

$$\begin{aligned} &2f\left(\frac{a+b}{2}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)dt \\ &\leq \int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt \\ &+\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt. \end{aligned} \tag{7.15}$$

Putting $u=\frac{2-t}{2}a+\frac{t}{2}b$ and $v=\frac{t}{2}a+\frac{2-t}{2}b$ in (7.15), we have

$$\begin{aligned} &2f\left(\frac{a+b}{2}\right)\int_{\frac{a+b}{2}}^{b}(b-v)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w'(b-v)^{\rho};p)dv \\ &\leq \int_{\frac{a+b}{2}}^{b}(b-v)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w'(b-v)^{\rho};p)f(v)dv \\ &+\int_a^{\frac{a+b}{2}}(u-a)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w'(u-a)^{\rho};p)f(u)du \end{aligned} \tag{7.16}$$

By simplifying we get the first inequality of (7.12).
Now multiplying (7.14) by $t^{\sigma-1}E^{\delta,c,q}_{\rho,\sigma,\tau}(wt^{\sigma})$ on both sides and then integrating over $[0,1]$ we have

$$\begin{aligned} &\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt \\ &+\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f\left(\frac{t}{2}a+\frac{2-t}{2}b\right) \\ &\leq [f(a)+f(b)]\int_0^1 (t^{\sigma-1})E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)dt. \end{aligned} \tag{7.17}$$

Putting $u = \frac{2-t}{2}a + \frac{t}{2}b$ and $v = \frac{t}{2}a + \frac{2-t}{2}b$ in (7.17), we have

$$\begin{aligned}&\int_a^{\frac{a+b}{2}} (u-a)^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w'(u-a)^{\rho};p) f(u)du \\ &+ \int_{\frac{a+b}{2}}^{b} (b-v)^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w'(b-v)^{\rho};p) f(v)dv \\ &\leq \int_a^{\frac{a+b}{2}} (u-a)^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w'(u-a)^{\rho};p)du,\end{aligned} \tag{7.18}$$

which gives the second inequality of (7.12). □

Remark 7.3 In Theorem 7.5

(i) If $p = 0$, then we get [4, Theorem 3.9].

(ii) If $w' = p = 0$ in Theorem 7.5, we get [141, Theorem 4].

(iii) If $w' = p = 0$ and $\sigma = 1$, then we get the classical Hadamard inequality.

In the next results generalized fractional integral operators by a monotone increasing function are utilized.

Theorem 7.6 *Let $f, g : [a,b] \to \mathbb{R}$, $0 < a < b$ be such that f is positive, $f \in L_1[a,b]$ and convex on $[a,b]$ and g differentiable and strictly increasing. Then the following inequalities for extended generalized fractional integral operator defined in* (2.23) *and* (2.24) *hold:*

$$\begin{aligned}&f\left(\frac{g(a)+g(b)}{2}\right)\left({}_g\Upsilon_{a^+,\rho,\sigma,\tau}^{w',\delta,c,q,r}1\right)(b;p) \\ &\leq \frac{1}{2}\left[\left({}_g\Upsilon_{a^+,\rho,\sigma,\tau}^{w',\delta,c,q,r}f\circ g\right)(b;p) + \left({}_g\Upsilon_{b^-,\rho,\sigma,\tau}^{w',\delta,c,q,r}f\circ g\right)(a;p)\right] \\ &\leq \frac{f(g(a))+f(g(b))}{2}\left({}_g\Upsilon_{b^-,\rho,\sigma,\tau}^{w',\delta,c,q,r}1\right)(a;p);\ w' = \frac{w}{(g(b)-g(a))^{\sigma}}.\end{aligned}$$

Proof. Considering the following identity

$$\frac{g(a)+g(b)}{2} = \frac{1}{2}[tg(a)+(1-t)g(b)] + \frac{1}{2}[(1-t)g(a)+tg(b)].$$

For convex function f we have

$$2f\left(\frac{g(a)+g(b)}{2}\right) \leq f(tg(a)+(1-t)g(b)) + f((1-t)g(a)+tg(b)). \tag{7.19}$$

Further from (7.19) one can obtain the following inequality:

$$2f\left(\frac{g(a)+g(b)}{2}\right)\int_0^1 t^{\sigma-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(wt^{\rho};p)dt \tag{7.20}$$
$$\leq \int_0^1 t^{\sigma-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(wt^{\rho};p)f(tg(a)+(1-t)g(b))dt$$
$$+\int_0^1 t^{\sigma-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(wt^{\rho};p)f((1-t)g(a)+tg(b))dt.$$

Setting $tg(a)+(1-t)g(b)=g(x)$ that is $t=\frac{g(b)-g(x)}{g(b)-g(a)}$ and $(1-t)g(a)+tg(b)=g(y)$ that is $t=\frac{g(y)-g(a)}{g(b)-g(a)}$ in (7.20), we get the following inequality:

$$2f\left(\frac{g(a)+g(b)}{2}\right)\left({}_g\Upsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma,\tau}1\right)(b;p) \tag{7.21}$$
$$\leq\left[\left({}_g\Upsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\circ g\right)(b;p)+\left({}_g\Upsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\circ g\right)(a;p)\right].$$

Further, by using the convexity of f, one can obtain

$$f(tg(a)+(1-t)g(b))+f((1-t)g(a)+tg(b)) \tag{7.22}$$
$$\leq tf(g(a))+(1-t)f(g(b))+(1-t)f(g(a))+tf(g(b)).$$

This leads to the following integral inequality:

$$\int_0^1 t^{\sigma-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(wt^{\alpha};p)f(tg(a)+(1-t)g(b))dt \tag{7.23}$$
$$+\int_0^1 t^{\sigma-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(wt^{\alpha};p)f((1-t)g(a)+tg(b))dt$$
$$\leq\int_0^1 t^{\sigma-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(wt^{\alpha};p)(tf(g(a))+(1-t)f(g(b)))dt$$
$$+\int_0^1 t^{\sigma-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(wt^{\alpha};p)((1-t)f(g(a))+tf(g(b)))dt.$$

Setting $tg(a)+(1-t)g(b)=g(x)$ that is $t=\frac{g(b)-g(x)}{g(b)-g(a)}$ and $(1-t)g(a)+tg(b)=g(y)$ that is $t=\frac{g(y)-g(a)}{g(b)-g(a)}$ in (7.23), and after calculation we get

$$\left({}_g\Upsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\circ g\right)(b;p)+\left({}_g\Upsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\circ g\right)(a;p) \tag{7.24}$$
$$\leq\frac{f(g(a))+f(g(b))}{2}\left({}_g\Upsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau}1\right)(a;p).$$

Combining (7.21) and (7.24), we get the required result. □

The Fejér-Hadamard inequality is given in the following result.

Theorem 7.7 *Let $f,g,h:[a,b]\to\mathbb{R}$, $0<a<b$ be such that f and h are positive, f and $h\in L_1[a,b]$ and f convex on $[a,b]$, while g is a differentiable and strictly increasing. If $f(g(a)+g(b)-g(x))=f(g(x))$, then the following inequalities for extended generalized fractional integral operator defined in (2.23) and (2.24) hold:*

$$\begin{aligned}
& f\left(\frac{g(a)+g(b)}{2}\right)\left({}_g\Upsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma,\tau} h\circ g\right)(b;p)\\
&\leq \frac{1}{2}\left[\left({}_g\Upsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma,\tau}(h\circ g)(f\circ g)\right)(b;p)+\left({}_g\Upsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau}(h\circ g)(f\circ g)\right)(a;p)\right]\\
&\leq \frac{f(g(a))+f(g(b))}{2}\left({}_g\Upsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau} h\circ g\right)(a;p),\ w'=\frac{w}{(g(b)-g(a))^{\sigma}}.
\end{aligned}$$

Proof. Multiplying both sides of (7.19) by $t^{\sigma-1}h(tg(a)+(1-t)g(b))E^{\rho,r,k,c}_{\sigma,\tau,\delta}(wt^{\rho};p)$ and integrating on $[0,1]$, we get

$$\begin{aligned}
& 2f\left(\frac{g(a)+g(b)}{2}\right)\int_0^1 t^{\sigma-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(wt^{\rho};p)h(tg(a)+(1-t)g(b))dt \qquad (7.25)\\
&\leq \int_0^1 t^{\sigma-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(wt^{\rho};p)h(tg(a)+(1-t)g(b))f(tg(a)+(1-t)g(b))dt\\
&+\int_0^1 t^{\sigma-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(wt^{\rho};p)h(tg(a)+(1-t)g(b))f((1-t)g(a)+tg(b))dt.
\end{aligned}$$

Setting $tg(a)+(1-t)g(b)=g(x)$ that is $t=\frac{g(b)-g(x)}{g(b)-g(a)}$ and $(1-t)g(a)+tg(b)$=$g(a)+g(b)-g(x)$ in (7.25), and using $f(g(a)+g(b)-g(x))=f(g(x))$, the following inequality is obtained:

$$\begin{aligned}
& 2f\left(\frac{g(a)+g(b)}{2}\right)\left({}_g\Upsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma,\tau} h\circ g\right)(b;p) \qquad (7.26)\\
&\leq \left[\left({}_g\Upsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma,\tau}(h\circ g)(f\circ g)\right)(b;p)+\left({}_g\Upsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau}(h\circ g)(f\circ g)\right)(a;p)\right].
\end{aligned}$$

Multiplying $t^{\sigma-1}h((tg(a)+(1-t)g(b))E^{\rho,r,k,c}_{\sigma,\tau,\delta}(wt^{\alpha};p)$ on both sides of (7.22) and integrating over $[0,1]$, we have

$$\begin{aligned}
& \int_0^1 t^{\sigma-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(wt^{\alpha};p)h((tg(a)+(1-t)g(b))f(tg(a)+(1-t)g(b))dt \qquad (7.27)\\
&+\int_0^1 t^{\sigma-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(wt^{\alpha};p)h((tg(a)+(1-t)g(b))f((1-t)g(a)+tg(b))dt\\
&\leq \int_0^1 t^{\sigma-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(wt^{\alpha};p)h((tg(a)+(1-t)g(b))(tf(g(a))+(1-t)f(g(b)))dt\\
&+\int_0^1 t^{\sigma-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(wt^{\alpha};p)h((tg(a)+(1-t)g(b))((1-t)f(g(a))+tf(g(b)))dt.
\end{aligned}$$

Setting $tg(a)+(1-t)g(b)=g(x)$ and using $f(g(a)+g(b)-g(x))=f(g(x))$ we get

$$\begin{aligned}&\left({}_g\Upsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma,\tau}(h\circ g)(f\circ g)\right)(b;p)+\left({}_g\Upsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau}(h\circ g)(f\circ g)\right)(a;p) \\ &\leq \frac{f(g(a))+f(g(b))}{2}\left({}_g\Upsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau}(h\circ g)\right)(a;p).\end{aligned} \tag{7.28}$$

Combining (7.26) and (7.28), we get the required result. □

Remark 7.4 The Hadamard and the Fejér Hadamard inequalities given in Theorems 2.1–2.5 are special cases of theorems of this section.

7.2 Hadamard and Fejér-Hadamard Inequalities for Relative Convex Functions

The following Hadamard inequality for relative convex functions via Riemann-Liouville fractional integral operator is given in [112].

Theorem 7.8 *Let f be a positive relative convex function integrable on $[a,g(b)]$. Then the following inequality holds*

$$f\left(\frac{a+g(b)}{2}\right)\leq \frac{\Gamma(\sigma+1)}{2(g(b)-a)^{\sigma}}[J^{\sigma}_{a^+}fg(b)+J^{\sigma}_{b\_}f(a)]\leq \frac{f(a)+f(g(b))}{2}$$

with $\sigma>0$.

The generalized Hadamard inequality containing Mittag-Leffler function is given in the following theorem.

Theorem 7.9 *Let $w\in\mathbb{R}$, $r,\rho,\sigma,\tau>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $f:[a,g(b)]\to\mathbb{R}$ be a function such that $f\in L_1[a,g(b)]$ with $a<b$. If f is relative convex on $[a,g(b)]$, then for extended generalized fractional integral operator the following inequality holds*

$$\begin{aligned}&f\left(\frac{a+g(b)}{2}\right)H^{w'}_{a^+,\sigma}(g(b);p) \\ &\leq \frac{1}{2}\left[\left(\varepsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(g(b);p)+\left(\varepsilon^{w',\delta,c,q,r}_{g(b)^-,\rho,\sigma,\tau}f\right)(a;p)\right] \\ &\leq \frac{f(a)+f(g(b))}{2}H^{w'}_{g(b)^-,\sigma}(a;p),\end{aligned} \tag{7.29}$$

where $w'=\frac{w}{(g(b)-a)^{\rho}}$.

Proof. Since f is relative convex function, for $t \in [0,1]$ we have

$$2f\left(\frac{a+g(b)}{2}\right) \leq f(ta+(1-t)g(b))+f((1-t)a+tg(b)). \tag{7.30}$$

Multiplying (7.36) by $2t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)$ on both sides and then integrating over $[0,1]$ we have

$$\begin{aligned} &2f\left(\frac{a+g(b)}{2}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)dt \\ &\leq \int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f(ta+(1-t)g(b))dt \\ &+\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f((1-t)a+tg(b))dt. \end{aligned} \tag{7.31}$$

Putting $x = ta+(1-t)g(b)$ and $y = (1-t)a+tg(b)$ in the above inequality we have

$$\begin{aligned} &2f\left(\frac{a+g(b)}{2}\right)\int_{g(b)}^{a}\left(\frac{g(b)-x}{g(b)-a}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w\left(\frac{g(b)-x}{g(b)-a}\right)^{\rho};p\right)\left(\frac{-dx}{g(b)-a}\right) \\ &\leq \int_{g(b)}^{a}\left(\frac{g(b)-x}{g(b)-a}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w\left(\frac{g(b)-x}{g(b)-a}\right)^{\rho};p\right)f(x)\left(\frac{-dx}{g(b)-a}\right) \\ &+\int_{a}^{g(b)}\left(\frac{y-a}{g(b)-a}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w\left(\frac{y-a}{g(b)-a}\right)^{\rho};p\right)f(y)\left(\frac{dy}{g(b)-a}\right). \end{aligned} \tag{7.32}$$

From the above inequality we get

$$2f\left(\frac{a+g(b)}{2}\right)H^{w'}_{a^+,\sigma}(g(b);p) \leq \left(\varepsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(g(b);p)+\left(\varepsilon^{w',\delta,c,q,r}_{g(b)^-,\rho,\sigma,\tau}f\right)(a;p). \tag{7.33}$$

Again using the relative convexity of f we have

$$f(ta+(1-t)g(b))+f((1-t)a+tg(b)) \leq tf(a)+(1-t)f(g(b))+(1-t)f(a)+tf(g(b)). \tag{7.34}$$

Multiplying (7.34) by $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)$ on both sides and then integrating over $[0,1]$ we have

$$\begin{aligned} &\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f(ta+(1-t)g(b))dt \\ &+\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f((1-t)a+tg(b))dt \\ &\leq \int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)tf(a)+(1-t)f(g(b))dt \\ &+\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)(1-t)f(a)+tf(g(b))dt. \end{aligned}$$

Putting $x = ta + (1-t)g(b)$ and $y = (1-t)a + tg(b)$ in above we get

$$\left(\varepsilon_{a^+,\rho,\sigma,\tau}^{w',\delta,c,q,r} f\right)(g(b);p) + \left(\varepsilon_{g(b)^-,\rho,\sigma,\tau}^{w',\delta,c,q,r} f\right)(a;p) \leq [f(a) + f(g(b))] H_{g(b)^-,\sigma}^{w'}(a;p). \quad (7.35)$$

From (7.35) and (7.33), the inequality (7.29) is obtained. □

Remark 7.5

(i) If $p = 0$ in above Theorem 7.9, then we get [1, Theorem 2.8].

(ii) If $w = p = 0$ and $q = 1$ in Theorem 7.9, then we get Theorem 7.8.

In the next theorem the generalization of previous result is proved.

Theorem 7.10 *Let $w \in \mathbb{R}$, $r, \rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let $f : [g(a), g(b)] \to \mathbb{R}$ be a function such that $f \in L_1[g(a), g(b)]$ with $a < b$. If f is relative convex on $[g(a), g(b)]$, then for extended generalized fractional integral operator the following inequality holds*

$$\begin{aligned} & f\left(\frac{g(a)+g(b)}{2}\right) H_{g(a)^+,\sigma}^{w'}(g(b);p) \\ & \leq \frac{1}{2}\left[\left(\varepsilon_{g(a)^+,\rho,\sigma,\tau}^{w',\delta,c,q,r} f\right)(g(b);p) + \left(\varepsilon_{g(b)^-,\rho,\sigma,\tau}^{w',\delta,c,q,r} f\right)(g(a);p)\right] \\ & \leq \frac{f(g(a)) + f(g(b))}{2} H_{g(b)^-,\sigma}^{w'}(g(a);p), \end{aligned}$$

where $w' = \frac{w}{(g(b)-g(a))^\rho}$.

Proof. Since f is relative convex function on $[g(a), g(b)]$, for $t \in [0,1]$ we have

$$2f\left(\frac{g(a)+g(b)}{2}\right) \leq f\left(tg(a) + (1-t)g(b)\right) + f\left((1-t)g(a) + tg(b)\right). \quad (7.36)$$

Remaining proof of is on same lines as the proof of above theorem. □

Remark 7.6

(i) If $p = 0$ in above Theorem 7.10, then we get [1, Theorem 2.10].

(ii) If $w = p = 0$ in above Theorem 7.10, then we get [66, Corollary 1].

7.3 Hadamard and Fejér-Hadamard Inequalities for m-convex Functions

In this section we give Hadamard and Fejér-Hadamard inequalities for m-convex functions via extended Mittag-Leffler function.

Theorem 7.11 *Let $w \in \mathbb{R}$, $r,\rho,\sigma,\tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $f : [0,\infty) \to \mathbb{R}$ be a positive function. Let $a,b \in [0,\infty)$ with $0 \leq a < mb$ and $f \in L_1[a,mb]$. If f is m-convex function on $[a,mb]$, then the following inequality for the extended generalized fractional integral holds*

$$\begin{aligned} & f\left(\frac{a+mb}{2}\right) H_{a^+,\sigma}^{w'}(mb;p) \\ & \leq \frac{(\varepsilon_{a^+,\rho,\sigma,\tau}^{w',\delta,c,q,r} f)(mb;p) + m^{\rho+1}(\varepsilon_{b^-,\rho,\sigma,\tau}^{m^\rho w',\delta,c,q,r} f)(\frac{a}{m};p)}{2} \\ & \leq \frac{m^{\rho+1}}{2}\left[\frac{f(a)-m^2 f\left(\frac{a}{m^2}\right)}{mb-a} H_{b^-,\sigma+1}^{m^\rho w'}\left(\frac{a}{m};p\right)\right. \\ & \left. + \left(f(b)+mf\left(\frac{a}{m^2}\right)\right) H_{b^-,\sigma}^{m^\rho w'}\left(\frac{a}{m};p\right)\right], \end{aligned} \tag{7.37}$$

where $w' = \frac{w}{(mb-a)^\rho}$.

Proof. Since f is m-convex function on $[a,mb]$, for $t \in [0,1]$ we have

$$\begin{aligned} & f\left(\frac{(ta+m(1-t)b)+m\left((1-t)\frac{a}{m}+tb\right)}{2}\right) \\ & \leq \frac{f(ta+m(1-t)b)+mf\left((1-t)\frac{a}{m}+tb\right)}{2}. \end{aligned} \tag{7.38}$$

Multiplying with $t^{\sigma-1}E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^\rho;p)$ the both sides of above inequality we get

$$\begin{aligned} & 2t^{\sigma-1}E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^\rho;p) f\left(\frac{a+mb}{2}\right) \\ & \leq E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^\rho;p)\left(f(ta+m(1-t)b)+mf\left((1-t)\frac{a}{m}+tb\right)\right). \end{aligned}$$

Integrating with respect to t over $[0,1]$ we have

$$\begin{aligned} & 2f\left(\frac{a+mb}{2}\right)\int_0^1 t^{\sigma-1}E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^\rho;p)dt \\ & \leq \int_0^1 t^{\sigma-1}E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^\rho;p) f(ta+m(1-t)b)dt \\ & + m\int_0^1 t^{\sigma-1}E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^\rho;p) f\left((1-t)\frac{a}{m}+tb\right)dt. \end{aligned}$$

If we take $u = at + m(1-t)b$, then $t = \frac{mb-u}{mb-a}$ and if $v = (1-t)\frac{a}{m} + tb$, then $t = \frac{v-\frac{a}{m}}{b-\frac{a}{m}}$. Therefore one can have

$$f\left(\frac{a+mb}{2}\right) H^{w'}_{a^+,\sigma}(mb;p) \leq \frac{(\varepsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma,\tau} f)(mb;p) + m^{\rho+1}(\varepsilon^{m^\rho w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau} f)\left(\frac{a}{m};p\right)}{2}. \quad (7.39)$$

Again by using that f is m-convex function we have

$$\begin{aligned} & f(ta + m(1-t)b) + mf\left((1-t)\frac{a}{m} + tb\right) \quad (7.40) \\ & \leq tf(a) + m(1-t)f(b) + m^2(1-t)f\left(\frac{a}{m^2}\right) + mtf(b). \end{aligned}$$

Now multiplying with $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)$ and integrating with respect to t over $[0,1]$ we get

$$\begin{aligned} & \int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)f(ta+m(1-t)b)dt \\ & + m\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)f\left((1-t)\frac{a}{m}+tb\right)dt \\ & \leq \left[f(a) - m^2 f\left(\frac{a}{m^2}\right)\right]\int_0^1 t^\sigma E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)dt \\ & + m\left[f(b) + mf\left(\frac{a}{m^2}\right)\right]\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)dt. \end{aligned}$$

From which by using change of variables as did for (7.39), we get

$$\begin{aligned} & \frac{(\varepsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma,\tau} f)(mb;p) + m^{\rho+1}(\varepsilon^{m^\rho w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau} f)(\frac{a}{m};p)}{2} \quad (7.41) \\ & \leq \frac{m^{\rho+1}}{2}\left[\frac{f(a) - m^2 f\left(\frac{a}{m^2}\right)}{mb-a} H^{m^\rho w'}_{b^-,\sigma+1}\left(\frac{a}{m};p\right)\right. \\ & \left. + \left(f(b) + mf\left(\frac{a}{m^2}\right)\right) H^{m^\rho w'}_{b^-,\sigma}\left(\frac{a}{m};p\right)\right]. \end{aligned}$$

From inequalities (7.39) and (7.41), we get the inequality (7.37). □

Remark 7.7 In Theorem 7.11

(i) if $p = 0$, then we get [50, Theorem 3].

(ii) if $w = p = 0$, $m = 1$, then we get [140, Theorem 2].

(iii) if $m = 1$, then we get Theorem 7.3.

Theorem 7.12 *Let $w \in \mathbb{R}$, $r,\rho,\sigma,\tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $f : [a,mb] \to \mathbb{R}$ be a function such that $f \in L_1[a,mb]$ with $0 \leq a < mb$. If f is m-convex on $[a,mb]$, then the following inequality for extended generalized fractional integral operator holds*

$$\begin{aligned} &2f\left(\frac{a+mb}{2}\right) H^{w'}_{(\frac{a+mb}{2})^+,\sigma}(mb;p) \\ &\leq \left(\varepsilon^{w',\delta,c,q,r}_{(\frac{a+mb}{2})^+,\rho,\sigma,\tau} f\right)(mb;p) + m^{\rho+1}\left(\varepsilon^{m^\rho w',\delta,c,q,r}_{(\frac{a+mb}{2})^-,\rho,\sigma,\tau} f\right)\left(\frac{a}{m};p\right) \\ &\leq \frac{1}{mb-a}\left(f(a) - m^2 f\left(\frac{a}{m^2}\right)\right) H^{w'}_{(\frac{a+mb}{2})^+,\sigma+1}(mb;p) \\ &+ m^{\rho+1}\left(f(b) + mf\left(\frac{a}{m^2}\right)\right) H^{w' m^\rho}_{(\frac{a+mb}{2})^-,\sigma}\left(\frac{a}{m};p\right) \end{aligned} \tag{7.42}$$

where $w' = \frac{2^\rho w}{(mb-a)^\rho}$.

Proof. Since f is m-convex, for $t \in [0,1]$ we have

$$2f\left(\frac{a+mb}{2}\right) \leq f\left(\frac{t}{2}a + \frac{2-t}{2}mb\right) + mf\left(\frac{2-t}{2m}a + \frac{t}{2}b\right). \tag{7.43}$$

Also from m-convexity of f, we have

$$\begin{aligned} &f\left(\frac{t}{2}a + m\frac{2-t}{2}b\right) + mf\left(\frac{2-t}{2m}a + \frac{t}{2}b\right) \\ &\leq \frac{t}{2}\left(f(a) - m^2 f\left(\frac{a}{m^2}\right)\right) + m\left(f(b) + mf\left(\frac{a}{m^2}\right)\right). \end{aligned} \tag{7.44}$$

Multiplying (7.43) by $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)$ on both sides and then integrating on $[0,1]$ we have

$$\begin{aligned} &2f\left(\frac{a+mb}{2}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)dt \\ &\leq \int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p) f\left(\frac{t}{2}a + \frac{2-t}{2}mb\right)dt \\ &+ m\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p) f\left(\frac{2-t}{2m}a + \frac{t}{2}b\right)dt. \end{aligned} \tag{7.45}$$

Putting $u = \frac{t}{2}a + \frac{2-t}{2}mb$ and $v = \frac{2-t}{2m}a + \frac{t}{2}b$ in (7.45), we have

$$\begin{aligned} &2f\left(\frac{a+mb}{2}\right)\int_{\frac{a+mb}{2}}^{mb}(mb-u)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w'(mb-u)^\rho;p)du \\ &\leq \int_{\frac{a+mb}{2}}^{mb}(mb-u)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w'(mb-u)^\rho;p)f(u)du \\ &+ m^{\rho+1}\int_{\frac{a}{m}}^{\frac{a+mb}{2m}}\left(v-\frac{a}{m}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(m^\rho w'(v-\frac{a}{m})^\rho : p\right)f(v)dv. \end{aligned}$$

By using (2.20) and Definition 2.2 we get first inequality of (7.42).
Now multiplying (7.44) by $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)$ on both sides and then integrating with respect to t over $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f\left(\frac{t}{2}a+m\frac{2-t}{2}b\right)dt \\
&+m\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f\left(\frac{2-t}{2m}a+\frac{t}{2}b\right) \\
&\leq \frac{1}{2}\left(f(a)-m^2f(\frac{a}{m^2})\right)\int_0^1 t^{\rho}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)dt \\
&+m\left(f(b)+mf(\frac{a}{m^2})\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)dt.
\end{aligned} \tag{7.46}$$

Putting $u=\frac{t}{2}a+m\frac{2-t}{2}b$ and $v=\frac{2-t}{2m}a+\frac{t}{2}b$ in (7.46), we have

$$\begin{aligned}
&\int_{\frac{a+mb}{2}}^{mb}(mb-u)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w'(mb-u)^{\rho};p)f(u)du \\
&+\int_{\frac{a}{m}}^{\frac{a+mb}{2m}}\left(v-\frac{a}{m}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(m^{\rho}w'\left(v-\frac{a}{m}\right)^{\rho};p)f(v)dv \\
&\leq \frac{1}{2}\left(f(a)-m^2f(\frac{a}{m^2})\right)\int_{\frac{a+mb}{2}}^{mb}(mb-u)^{\sigma}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w'(mb-u)^{\rho};p\right)dt \\
&+m^{\rho+1}\left(f(b)+mf(\frac{a}{m^2})\right)\int_{\frac{a}{m}}^{\frac{a+mb}{2m}}\left(v-\frac{a}{m}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(m^{\rho}w'\left(v-\frac{a}{m}\right)^{\rho};p)dt.
\end{aligned} \tag{7.47}$$

By using (2.18) and Definition 2.2 we get second inequality of (7.42). □

Remark 7.8 In Theorem 7.12.

(i) If $p=0$, then we get [53, Theorem 3.10].

(ii) If $m=1$, then we get Theorem 7.5.

(iii) If $w=p=0$, $m=1$ and $\sigma=1$, then we get the classical Hadamard inequality.

Theorem 7.13 *Let $w\in\mathbb{R}$, $r,\rho,\sigma,\tau>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $f:[0,\infty)\to\mathbb{R}$ be a m-convex function, $a,b\in[0,\infty)$ with $0\leq a<mb$ and $f\in L_1[a,mb]$. Also, let $g:[a,mb]\to\mathbb{R}$ be a function which is nonnegative and integrable on $[a,mb]$. If $f(a+mb-mx)=f(x)$, then the following inequality for extended generalized fractional integral holds*

$$\begin{aligned}
&f\left(\frac{a+mb}{2}\right)(\varepsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau}g)\left(\frac{a}{m};p\right) \\
&\leq \frac{(1+m)}{2}(\varepsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma+1,\tau}fg)\left(\frac{a}{m};p\right)
\end{aligned} \tag{7.48}$$

$$\leq \frac{1}{2}\left[\frac{f(a)-m^2 f\left(\frac{a}{m^2}\right)}{mb-a}(\varepsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma+1,\tau}g)\left(\frac{a}{m};p\right)\right.$$
$$\left.+m\left(f(b)+mf\left(\frac{a}{m^2}\right)\right)(\varepsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau}g)\left(\frac{a}{m};p\right)\right]$$

where $w' = \frac{w}{(b-\frac{a}{m})^\rho}$.

Proof. Proof of this theorem is on same lines as the proof of Theorem 7.18. □

In the next results generalized fractional integral operators by a monotone increasing function are utilized.

Theorem 7.14 *Let* $f,g:[a,b]\to\mathbb{R}$, $0<a<b$, $\mathrm{Range}(g)\subset[a,b]$, *be such that* f *is positive and* $f\in L_1[a,b]$, g *differentiable and strictly increasing. If* f *be m-convex* $m\in(0,1]$ *and* $g(a)<mg(b)$, *then the following inequalities for fractional operators (2.23) and (2.24) hold:*

$$f\left(\frac{g(a)+mg(b)}{2}\right)\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w',a^+}1\right)(g^{-1}(mg(b));p)$$
$$\leq \frac{\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w',a^+}(f\circ g)\right)(g^{-1}(mg(b));p)+m^{\rho+1}\left({}_g\Upsilon^{m^\rho w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau}(f\circ g)\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right)}{2}$$
$$\leq \frac{m^{\rho+1}}{2}\left[\frac{f(g(a))-m^2 f\left(\frac{g(a)}{m^2}\right)}{mg(b)-g(a)}\left({}_g\Upsilon^{m^\rho w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau}1\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right)\right.$$
$$\left.+\left(f(g(b))+mf\left(\frac{g(a)}{m^2}\right)\right)\left({}_g\Upsilon^{m^\rho w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau}1\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right)\right], w'=\frac{w}{(mg(b)-g(a))^\sigma}.$$

Proof. By definition of m-convex function f, we have

$$2f\left(\frac{g(a)+mg(b)}{2}\right)\leq f\left(tg(a)+m(1-t)g(b)\right)+mf\left(tg(b)+(1-t)\frac{g(a)}{m}\right). \tag{7.49}$$

Further, from (7.49), one can obtain the following integral inequality:

$$2f\left(\frac{g(a)+mg(b)}{2}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)dt \tag{7.50}$$
$$\leq \int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)f(tg(a)+m(1-t)g(b))dt$$
$$+m\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)f\left(tg(b)+(1-t)\frac{g(a)}{m}\right)dt.$$

Setting $g(x)=tg(a)+m(1-t)g(b)$ and $g(y)=tg(b)+(1-t)\frac{g(a)}{m}$ in (7.50), we get the following inequality:

$$2f\left(\frac{g(a)+mg(b)}{2}\right)\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w',a^+}1\right)(g^{-1}(mg(b));p) \tag{7.51}$$
$$\leq \left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w',a^+}(f\circ g)\right)(g^{-1}(mg(b));p)+m^{\rho+1}\left({}_g\Upsilon^{m^\rho w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau}(f\circ g)\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right).$$

Also by using the m-convexity of f, we obtain

$$f(tg(a)+m(1-t)g(b))+mf\left(tg(b)+(1-t)\frac{g(a)}{m}\right) \tag{7.52}$$
$$\leq m\left(f(g(b))+mf\left(\frac{g(a)}{m^2}\right)\right)+\left(f(g(a))-m^2f\left(\frac{g(a)}{m^2}\right)\right)t.$$

This leads to the following integral inequality:

$$\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^{\rho};p)f(tg(a)+m(1-t)g(b))dt \tag{7.53}$$
$$+m\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^{\rho};p)f\left(tg(b)+(1-t)\frac{g(a)}{m}\right)dt$$
$$\leq m\left(f(g(b))+mf\left(\frac{g(a)}{m^2}\right)\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^{\rho};p)dt$$
$$+\left(f(g(a))-m^2f\left(\frac{g(a)}{m^2}\right)\right)\int_0^1 t^{\sigma}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^{\rho};p)dt.$$

Again by setting $g(x)=tg(a)+m(1-t)g(b)$, $g(y)=tg(b)+(1-t)\frac{g(a)}{m}$ in (7.53) and after calculation, we get

$$\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w',a^+}(f\circ g)\right)(g^{-1}(mg(b));p)+m^{\rho+1}\left({}_g\Upsilon^{m^{\rho}w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau}(f\circ g)\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right) \tag{7.54}$$
$$\leq m^{\rho+1}\left(\frac{f(g(a))-m^2f\left(\frac{g(a)}{m^2}\right)}{mg(b)-g(a)}\left({}_g\Upsilon^{m^{\rho}w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau}1\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right)\right.$$
$$\left.+\left(f(g(b))+mf\left(\frac{g(a)}{m^2}\right)\right)\left({}_g\Upsilon^{m^{\rho}w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau}1\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right)\right).$$

By combining (7.51) and (7.54), we get the desired result. □

Remark 7.9

(i) Setting $m=1$ in Theorem 7.14, we obtain [132, Theorem 3.1].

(ii) Setting $g=I$ in Theorem 7.14, we obtain [87, Theorem 3.1].

The following theorem gives the Fejér-Hadamard inequality for m-convex functions.

Theorem 7.15 *Let $f,g,h:[a,b]\to\mathbb{R}$, $0<a<b$, Range(g), be such that f is positive and $f\in L_1[a,b]$, g differentiable and strictly increasing and h integrable and nonnegative. If f is m-convex, $m\in(0,1]$, $g(a)<mg(b)$ and $g(a)+mg(b)-mg(x)=g(x)$, then the following inequalities for fractional operator (2.24) hold:*

$$2f\left(\frac{g(a)+mg(b)}{2}\right)\left({}_g\Upsilon^{w',\delta,c,q,r}_{mb_-,\rho,\sigma,\tau}h\circ g\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right)$$
$$\leq(1+m)\left({}_g\Upsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau}(f\circ g)(h\circ g)\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right)$$

$$\leq \frac{f(g(a)) - m^2 f\left(\frac{g(a)}{m^2}\right)}{(g(b) - \frac{g(a)}{m})} \left({}_g\Upsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau} h \circ g\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right)$$
$$+m\left(f(g(b)) + mf\left(\frac{g(a)}{m^2}\right)\right)\left({}_g\Upsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau} h \circ g\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right), w' = \frac{w}{(g(b) - \frac{g(a)}{m})^\rho}.$$

Proof. Multiplying both sides of (7.49) by $2t^{\sigma-1}h\left(tg(b) + (1-t)\frac{g(a)}{m}\right)E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)$ and integrating on $[0,1]$, we get

$$2f\left(\frac{g(a)+mg(b)}{2}\right)\int_0^1 t^{\sigma-1}h\left(tg(b) + (1-t)\frac{g(a)}{m}\right)E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)dt \tag{7.55}$$
$$\leq \int_0^1 t^{\sigma-1}h\left(tg(b) + (1-t)\frac{g(a)}{m}\right)E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)f(tg(a)+m(1-t)g(b))dt$$
$$+m\int_0^1 t^{\sigma-1}h\left(tg(b) + (1-t)\frac{g(a)}{m}\right)E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)f\left(tg(b) + (1-t)\frac{g(a)}{m}\right)dt.$$

Setting $g(x) = tg(b) + (1-t)\frac{g(a)}{m}$ and also using $g(a) + mg(b) - mg(x) = g(x)$ the following inequality is obtained:

$$2f\left(\frac{g(a)+mg(b)}{2}\right)\left({}_g\Upsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau} h \circ g\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right) \tag{7.56}$$
$$\leq (1+m)\left({}_g\Upsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau}(f\circ g)(h\circ g)\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right).$$

Multiplying both sides of inequality (7.52) with $t^{\sigma-1}h\left(tg(b) + (1-t)\frac{g(a)}{m}\right)E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)$ and integrating on $[0,1]$, then setting $g(x) = tg(b) + (1-t)\frac{g(a)}{m}$ and also using $g(a) + mg(b) - mg(x) = g(x)$ we have

$$(1+m)\left({}_g\Upsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau}(f\circ g)(h\circ g)\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right) \tag{7.57}$$
$$\leq \frac{(f(g(a)) - m^2 f\left(\frac{g(a)}{m^2}\right))}{g((b) - \frac{g(a)}{m})}\left({}_g\Upsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau} h \circ g\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right)$$
$$+m\left(f(g(b)) + mf\left(\frac{g(a)}{m^2}\right)\right)\left({}_g\Upsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau} h \circ g\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right).$$

By combining (7.56) and (7.57), we get the desired result. □

Remark 7.10

(i) In the Theorem 7.15, if we put $m = 1$, then we get [132, Theorem 3.2].

(ii) In the Theorem 7.15, if we put $g = I$ and $p = 0$, then we get [3, Theorem].

(iii) In the Theorem 7.15, if we put $g = I$, then we get [57, Theorem].

7.4 Hadamard and Fejér-Hadamard Inequalities for $(h-m)$-convex Functions

In this section the results proved in previous sections are generalized via $(h-m)$-convex functions.

Theorem 7.16 *Let $w \in \mathbb{R}$, $r, \rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $f : [0,\infty) \to \mathbb{R}$ be an integrable and $(h-m)$-convex function with $m \in (0,1]$. Then the following inequality for extended generalized fractional integral holds*

$$\begin{aligned}
& f\left(\frac{bm+a}{2}\right) H^{w'}_{a^+,\sigma}(mb;p) \\
& \leq h\left(\frac{1}{2}\right)\left[m^{\rho+1}(\varepsilon^{w'm^{\rho},\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f)\left(\frac{a}{m};p\right) + (\varepsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f)(mb;p)\right] \\
& \leq h\left(\frac{1}{2}\right)(mb-a)^{\sigma}\left\{\left[m^2 f\left(\frac{a}{m^2}\right) + mf(b)\right](\varepsilon^{w',\delta,c,q,r}_{0^+,\rho,\sigma,\tau}h)(1;p)\right. \\
& \left. + [mf(b)+f(a)](\varepsilon^{w',\delta,c,q,r}_{1^-,\rho,\sigma,\tau}h)(0;p)\right\}
\end{aligned} \tag{7.58}$$

where $w' = \frac{w}{(bm-a)^{\rho}}$.

Proof. Since f is $(h-m)$-convex function, we have

$$f\left(\frac{xm+y}{2}\right) \leq h\left(\frac{1}{2}\right)(mf(x)+f(y)). \tag{7.59}$$

Setting $x = (1-t)\frac{a}{m} + tb$ and $y = m(1-t)b + ta$, $t \in [0,1]$, then integrating over $[0,1]$ after multiply with $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)$ we get

$$\begin{aligned}
& f\left(\frac{bm+a}{2}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)dt \\
& \leq h\left(\frac{1}{2}\right)\left[\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)mf\left((1-t)\frac{a}{m}+tb\right)dt\right. \\
& \left. + \int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f(m(1-t)b+ta)dt\right].
\end{aligned}$$

By substituting $s = (1-t)\frac{a}{m} + tb$ and $z = m(1-t)b + ta$ we have

$$\begin{aligned}
& f\left(\frac{bm+a}{2}\right) H^{w'}_{a^+,\sigma}(mb;p) \\
& \leq h\left(\frac{1}{2}\right)\left[m^{\rho+1}(\varepsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f)\left(\frac{a}{m};p\right) + (\varepsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f)(mb;p)\right].
\end{aligned} \tag{7.60}$$

This completes the proof of first inequality in (7.58). For the second inequality $(h-m)$-convexity of f also gives

$$
\begin{aligned}
& mf\left((1-t)\frac{a}{m}+tb\right)+f(m(1-t)b+ta) \\
& \leq m^2h(1-t)f\left(\frac{a}{m^2}\right)+mh(t)f(b)+mh(1-t)f(b)+h(t)f(a).
\end{aligned}
$$

Multiplying both sides of above inequality with $h\left(\frac{1}{2}\right)t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)$ and integrating over $[0,1]$, then by change of variables we have

$$
\begin{aligned}
& h\left(\frac{1}{2}\right)\left[m^{\rho+1}(\varepsilon^{w'm^{\rho},\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f)\left(\frac{a}{m};p\right)+(\varepsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f)(mb;p)\right]. \\
& \leq h\left(\frac{1}{2}\right)(mb-a)^{\sigma}\left\{\left[m^2f\left(\frac{a}{m^2}\right)+mf(b)\right]\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)h(1-t)dt\right. \\
& \left.+[mf(b)+f(a)]\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)h(t)dt\right\}.
\end{aligned}
$$

This gives us the second inequality in (7.58). □

Several known results are special cases of the above generalized fractional Hadamard inequality shown in the following remark.

Remark 7.11

(i) If $p=0$, then we get [131, Theorem 2.1].

(ii) If $h(t)=t$, $p=0$ and $m=1$, then we get [51, Theorem 2.1].

(iii) If $h(t)=t$, $p=0$, then we get [50, Theorem 3].

(iv) If $h(t)=t$, $p=0$ and $w=0$, then we get [65, Theorem 2.1].

(v) If $h(t)=t$, $p=0$, $m=1$ and $w=0$, then we get [140, Theorem 2].

(vi) If $h(t)=t$, $p=0$, $m=1$, $\sigma=1$ and $w=0$, then we get the Hadamard inequality.

Another generalized Hadamard inequality containing Mittag-Leffler function is given in the following theorem.

Theorem 7.17 *Let $w\in\mathbb{R}$, $r,\rho,\sigma,\tau>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $f:[0,\infty)\to\mathbb{R}$ be an integrable and $(h-m)$-convex function with $m\in(0,1]$. Then the following inequality for extended generalized fractional integral holds*

$$
\begin{aligned}
& f\left(\frac{a+bm}{2}\right)H^{w'2^{\rho}}_{\left(\frac{a+bm}{2}\right)^+,\sigma}(mb;p) \qquad (7.61)\\
& \leq h\left(\frac{1}{2}\right)\left[(\varepsilon^{w'2^{\rho},\delta,c,q,r}_{\left(\frac{a+bm}{2}\right)^+,\rho,\sigma,\tau}f)(mb;p)+m^{\rho+1}(\varepsilon^{w'(2m)^{\rho},\delta,c,q,r}_{\left(\frac{a+bm}{2m}\right)^-,\rho,\sigma,\tau}f)\left(\frac{a}{m};p\right)\right] \\
& \leq h\left(\frac{1}{2}\right)\frac{(mb-a)^{\sigma}}{2^{\sigma}}\left\{\left(m^2f\left(\frac{a}{m^2}\right)+mf(b)\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)h\left(\frac{2-t}{2}\right)dt\right. \\
& \left.+[mf(b)+f(a)]\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)h\left(\frac{t}{2}\right)dt\right\}.
\end{aligned}
$$

Proof. Putting $x=\frac{t}{2}b+\frac{(2-t)}{2}\frac{a}{m}$ and $y=\frac{t}{2}a+m\frac{(2-t)}{2}b$ in (7.59), we get

$$f\left(\frac{a+bm}{2}\right)\leq h\left(\frac{1}{2}\right)\left[mf\left(\frac{t}{2}b+\frac{(2-t)}{2}\frac{a}{m}\right)+f\left(\frac{t}{2}a+m\frac{(2-t)}{2}b\right)\right]. \tag{7.62}$$

Multiplying (7.62) by $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)$ on both sides, then integrating over $[0,1]$, we have

$$\begin{aligned}&f\left(\frac{a+bm}{2}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)dt\\&\leq h\left(\frac{1}{2}\right)\left[\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)mf\left(\frac{t}{2}a+m\frac{(2-t)}{2}b\right)dt\right.\\&\left.+\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f\left(\frac{t}{2}b+\frac{(2-t)}{2}\frac{a}{m}\right)dt\right].\end{aligned}$$

By substituting $x=\frac{t}{2}b+\frac{(2-t)}{2}\frac{a}{m}$ and $y=\frac{t}{2}a+m\frac{(2-t)}{2}b$, one gets

$$\begin{aligned}&f\left(\frac{a+bm}{2}\right)H^{w'2^{\rho}}_{\left(\frac{a+bm}{2}\right)^{+},\sigma}(mb;p) \qquad (7.63)\\&\leq h\left(\frac{1}{2}\right)\left[(\varepsilon^{w'2^{\rho},\delta,c,q,r}_{\left(\frac{a+bm}{2}\right)^{+},\rho,\sigma,\tau}f)(mb;p)++m^{\rho+1}(\varepsilon^{w'(2m)^{\rho},\delta,c,q,r}_{\left(\frac{a+bm}{2m}\right)^{-},\rho,\sigma,\tau}f)\left(\frac{a}{m};p\right)\right].\end{aligned}$$

Again by using $(h-m)-$convexity of f, we have

$$\begin{aligned}&f\left(\frac{t}{2}a+m\frac{(2-t)}{2}b\right)+mf\left(\frac{t}{2}b+\frac{(2-t)}{2}\frac{a}{m}\right) \qquad (7.64)\\&\leq h\left(\frac{t}{2}\right)f(a)+mh\left(\frac{2-t}{2}\right)f(b)+mh\left(\frac{t}{2}\right)f(b)+m^2h\left(\frac{2-t}{2}\right)f\left(\frac{a}{m^2}\right)\\&=m\left[mf\left(\frac{a}{m^2}\right)+f(b)\right]h\left(\frac{2-t}{2}\right)+[mf(b)+f(a)]h\left(\frac{t}{2}\right).\end{aligned}$$

Multiplying (7.64) by $h\left(\frac{1}{2}\right)t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)$ on both sides, then integrating over $[0,1]$, we have

$$\begin{aligned}&h\left(\frac{1}{2}\right)\left[\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f\left(\frac{t}{2}a+m\frac{(2-t)}{2}b\right)dt\right.\\&\left.+\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)mf\left(\frac{t}{2}b+\frac{(2-t)}{2}\frac{a}{m}\right)dt\right]\\&\leq h\left(\frac{1}{2}\right)\left\{m\left[mf\left(\frac{a}{m^2}\right)+f(b)\right]\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)h\left(\frac{2-t}{2}\right)dt\right.\\&\left.+[mf(b)+f(a)]\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)h\left(\frac{t}{2}\right)dt\right\}.\end{aligned}$$

By using change of variables we conclude

$$h\left(\frac{1}{2}\right)\left[(\varepsilon^{w'2^{\rho},\delta,c,q,r}_{\left(\frac{a+bm}{2}\right)^{+},\rho,\sigma,\tau}f)(mb;p)++m^{\rho+1}(\varepsilon^{w'(2m)^{\rho},\delta,c,q,r}_{\left(\frac{a+bm}{2m}\right)^{-},\rho,\sigma,\tau}f)\left(\frac{a}{m};p\right)\right]$$

$$\leq h\left(\frac{1}{2}\right)\frac{(bm-a)^{\sigma}}{2^{\sigma}}\left\{m\left[mf\left(\frac{a}{m^2}\right)+f(b)\right]\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)h\left(\frac{2-t}{2}\right)dt\right.$$
$$\left.+[mf(b)+f(a)]\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)h\left(\frac{t}{2}\right)dt\right\}.$$

From above inequality and (7.63), we get the required inequality (7.61). □

Remark 7.12

(i) If we put $p=0$ in (7.61), then [131, Theorem 2.2] is obtained.

(ii) If $h(t)=t$, $p=0$ and $m=1$, then we get [131, Corollary 2.3].

(iii) If $h(t)=t$, then we get Theorem 7.12.

(iv) If $h(t)=t$, $p=0$ and $w=0$, then we get Theorem 7.2.

(v) If $m=1$, then we get the inequality for h-convex function.

Theorem 7.18 *Let $w\in\mathbb{R}$, $r,\rho,\sigma,\tau>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $f:[a,b]\to\mathbb{R}$ be a $(h-m)$-convex function with $0\leq a<b$ and $f\in L_1[a,b]$. Also, let $g:[a,b]\to\mathbb{R}$ be a function which is nonnegative, integrable and symmetric about $\frac{a+mb}{2}$. If $f(mb+a-mx)=f(x)$, then the following inequalities for extended generalized fractional integrals hold*

$$f\left(\frac{bm+a}{2}\right)(\varepsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau}g)\left(\frac{a}{m};p\right)\leq h\left(\frac{1}{2}\right)(m+1)(\varepsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau}fg)\left(\frac{a}{m};p\right) \quad (7.65)$$
$$\leq h\left(\frac{1}{2}\right)\left\{\left[m^2f\left(\frac{a}{m^2}\right)+mf(b)\right](\varepsilon^{w',\delta,c,q,r}_{0^+,\rho,\sigma,\tau}h)(1;p)\right.$$
$$\left.+[mf(b)+f(a)](\varepsilon^{w',\delta,c,q,r}_{1^-,\rho,\sigma,\tau}h)(0;p)\right\},$$

where $w'=\frac{w}{(bm-a)^{\rho}}$.

Proof. Since f is $(h-m)$-convex function, for $t\in[a,b]$ we have

$$f\left(\frac{bm+a}{2}\right)\leq h\left(\frac{1}{2}\right)\left[mf\left((1-t)\frac{a}{m}+tb\right)+f(m(1-t)b+ta)\right].$$

Multiplying both sides of above inequality with $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)g\left(tb+(1-t)\frac{a}{m}\right)$ and integrating over $[0,1]$, we have

$$f\left(\frac{bm+a}{2}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,,r}_{\rho,\sigma,\tau}(wt^{\rho};p)g\left(tb+(1-t)\frac{a}{m}\right)dt$$
$$\leq h\left(\frac{1}{2}\right)\left[\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)g\left(tb+(1-t)\frac{a}{m}\right)mf\left((1-t)\frac{a}{m}+tb\right)dt\right.$$
$$\left.+\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)g\left(tb+(1-t)\frac{a}{m}\right)f(m(1-t)b+ta)dt\right].$$

If we set $x=(1-t)\frac{a}{m}+tb$ and use the given condition $f(mb+a-mx)=f(x)$, we conclude

$$f\left(\frac{bm+a}{2}\right)(\varepsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau}g)\left(\frac{a}{m};p\right)\leq h\left(\frac{1}{2}\right)(m+1)(\varepsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau}fg)\left(\frac{a}{m};p\right). \tag{7.66}$$

This completes the proof of first inequality in (7.65). For the second inequality using $(h-m)$-convexity of f we have

$$\begin{aligned}&mf\left((1-t)\frac{a}{m}+tb\right)+f(m(1-t)b+ta)\\&\leq m^2h(1-t)f\left(\frac{a}{m^2}\right)+mh(t)f(b)+mh(1-t)f(b)+h(t)f(a).\end{aligned}$$

Multiplying both sides with $h\left(\frac{1}{2}\right)t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)g\left(tb+(1-t)\frac{a}{m}\right)$ and integrating over $[0,1]$, we have

$$\begin{aligned}&h\left(\frac{1}{2}\right)(m+1)(\varepsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau}fg)\left(\frac{a}{m};p\right)\\&\leq h\left(\frac{1}{2}\right)\left\{\left[m^2f\left(\frac{a}{m^2}\right)+mf(b)\right]\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)h(1-t)g\left(tb+(1-t)\frac{a}{m}\right)dt\right.\\&\left.+[mf(b)+f(a)]\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)h(t)g\left(tb+(1-t)\frac{a}{m}\right)dt\right\}.\end{aligned}$$

By change of variables the second inequality in (7.65) can be obtained. □

Remark 7.13

(i) If we put $p=0$ in (7.65), then [131, Theorem 2.5] is obtained.

(ii) If we put $h(t)=t$, $m=1$ and $p=0$ in (7.65), then [51, Theorem 2.2] is obtained.

7.5 Hadamard and Fejér-Hadamard inequalities for Harmonically Convex Functions

In the following we give the Hadamard inequality for harmonically convex functions.

Theorem 7.19 *[76] Let $f:I\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be harmonically convex function and $a,b\in I$ with $a<b$. If $f\in L[a,b]$ then the following inequality hold:*

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\leq\frac{f(a)+f(b)}{2}. \tag{7.67}$$

A Fejér-Hadamard inequality for harmonically convex functions is stated as follows.

Theorem 7.20 *[33] Let $f : I \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a harmonically convex function and $a,b \in I$ with $a < b$. If $f \in L[a,b]$ and Let $u : [a,b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is non negative integrable and harmonically symmetric with respect to $\frac{2ab}{a+b}$ then the following inequality for fractional integral operator hold:*

$$f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{u(x)}{x^2}dx \le \int_a^b \frac{f(x)u(x)}{x^2}dx \le \frac{f(a)+f(b)}{2}\int_a^b \frac{u(x)}{x^2}. \tag{7.68}$$

A version of the Fejér-Hadamard inequality for harmonically convex functions via Riemann-Liouville fractional integrals is stated as follows.

Theorem 7.21 *[79] Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a function such that $a,b \in I$ with $a < b$. If $f \in L[a,b]$ and f is harmonically convex function then the following inequality for fractional integral operator hold:*

$$f\left(\frac{2ab}{a+b}\right) \le \frac{\Gamma(\nu+1)}{2}\left(\frac{ab}{b-a}\right)^{\sigma}\left[J^{\nu}_{\frac{1}{b}+}(f\circ h)(\frac{1}{a}) + J^{\nu}_{\frac{1}{a}-}(f\circ h)(\frac{1}{b})\right] \le \frac{f(a)+f(b)}{2} \tag{7.69}$$

where $h(x) = \frac{1}{x}$ and $x \in [a,b]$.

Another version of the Fejér-Hadamard inequality for harmonically convex functions via Riemann-Liouville fractional integrals is stated as follows.

Theorem 7.22 *[95] Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a harmonically convex function on $[a,b]$ for $a,b \in I$ with $a < b$. If $f \in L[a,b]$ then the following inequalities for fractional integral hold*

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\le \frac{\Gamma(\sigma+1)}{2^{1-\sigma}}\left(\frac{ab}{b-a}\right)^{\sigma}\left(J^{\sigma}_{\frac{a+b}{2ab}-} f\circ g\left(\frac{1}{b}\right) + J^{\sigma}_{\frac{a+b}{2ab}+} f\circ g\left(\frac{1}{a}\right)\right) \\ &\le \frac{f(a)+f(b)}{2} \end{aligned}$$

where $g(t) = \frac{1}{t}$ for $t \in [\frac{1}{b},\frac{1}{a}]$.

The following Fejér-Hadamard inequality for harmonically convex functions via extended Mittag-Leffler functions is the generalized version.

Theorem 7.23 *Let $w \in \mathbb{R}$, $r,\rho,\sigma,\tau > 0$, $c > \delta > 0$ with $p \ge 0$ and $0 < q \le r+\rho$. Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be such that $f \in L_1[a,b]$ with $a < b$. If f is a harmonically convex function on $[a,b]$, then the following inequalities for extended generalized fractional integral operators hold*

$$\begin{aligned} &f\left(\frac{2ab}{a+b}\right) H^{w'}_{\frac{1}{a}-,\sigma}\left(\frac{1}{b};p\right) \\ &\le \frac{1}{2}\left(\left(\varepsilon^{w',\delta,c,q,r}_{\frac{1}{a}-,\rho,\sigma,\tau} f\circ g\right)\left(\frac{1}{b};p\right) + \left(\varepsilon^{w',\delta,c,q,r}_{\frac{1}{b}+,\rho,\sigma,\tau} f\circ g\right)\left(\frac{1}{a};p\right)\right) \\ &\le \frac{f(a)+f(b)}{2} H^{w'}_{\frac{1}{b}+,\sigma}\left(\frac{1}{a};p\right), \end{aligned}$$

where $w' = w(\frac{ab}{b-a})^{\rho}$ and $g(t) = \frac{1}{t}$ for $t \in [\frac{1}{b},\frac{1}{a}]$.

Proof. Since f is harmonically convex function, for all $x, y \in [a,b]$, $f\left(\frac{2xy}{x+y}\right) \leq \frac{f(x)+f(y)}{2}$. For $x = \frac{ab}{tb+(1-t)a}$ and $y = \frac{ab}{ta+(1-t)b}$ we have

$$2f\left(\frac{2ab}{a+b}\right) \leq f\left(\frac{ab}{tb+(1-t)a}\right) + f\left(\frac{ab}{ta+(1-t)b}\right). \tag{7.70}$$

Multiplying (7.70) by $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)$ on both sides, and integrating over $[0,1]$ we have

$$2f\left(\frac{2ab}{a+b}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)dt \leq \int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f\left(\frac{ab}{tb+(1-t)a}\right)dt$$
$$+\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f\left(\frac{ab}{ta+(1-t)b}\right)dt.$$

If we put in above $x = \frac{tb+(1-t)a}{ab}$ and $y = \frac{ta+(1-t)b}{ab}$, then we have the following inequality

$$2f\left(\frac{2ab}{a+b}\right)\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{ab}{b-a}\right)^{\sigma-1}\left(x-\frac{1}{b}\right)^{\sigma-1} \tag{7.71}$$
$$E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w\left(\frac{ab}{b-a}\right)^{\rho}\left(x-\frac{1}{b}\right)^{\rho};p\right)\left(\frac{ab}{b-a}\right)dx$$
$$\leq \int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{ab}{b-a}\right)^{\sigma-1}\left(x-\frac{1}{b}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w\left(\frac{ab}{b-a}\right)^{\rho}\left(x-\frac{1}{b}\right)^{\rho};p\right)$$
$$f\left(\frac{1}{x}\right)\left(\frac{ab}{b-a}\right)dx + \int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{ab}{b-a}\right)^{\sigma-1}\left(\frac{1}{a}-y\right)^{\sigma-1}$$
$$E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w\left(\frac{ab}{b-a}\right)^{\rho}\left(\frac{1}{a}-y\right)^{\rho};p\right)f\left(\frac{1}{y}\right)\left(\frac{ab}{b-a}\right)dy.$$

After simplification we get

$$2f\left(\frac{2ab}{a+b}\right)\int_{\frac{1}{b}}^{\frac{1}{a}}\left(x-\frac{1}{b}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w'\left(x-\frac{1}{b}\right)^{\rho};p\right)dx$$
$$\leq \int_{\frac{1}{b}}^{\frac{1}{a}}\left(x-\frac{1}{b}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w'\left(x-\frac{1}{b}\right)^{\rho};p\right)f\left(\frac{1}{x}\right)dx$$
$$+\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{1}{a}-y\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w'\left(\frac{1}{a}-y\right)^{\rho};p\right)f\left(\frac{1}{y}\right)dy.$$

By using Definition 2.2 we get

$$2f\left(\frac{2ab}{a+b}\right)\left(\varepsilon^{w',\delta,c,q,r}_{\frac{1}{a}-,\rho,\sigma,\tau}1\right)\left(\frac{1}{b};p\right) \leq \left(\varepsilon^{w',\delta,c,q,r}_{\frac{1}{a}-,\rho,\sigma,\tau}f\circ g\right)\left(\frac{1}{b};p\right)$$
$$+\left(\varepsilon^{w',\delta,c,q,r}_{\frac{1}{b}+,\rho,\sigma,\tau}f\circ g\right)\left(\frac{1}{a};p\right). \tag{7.72}$$

Again by using that harmonically convexity of f for $t\in[0,1]$ one can have

$$f\left(\frac{ab}{tb+(1-t)a}\right)+f\left(\frac{ab}{ta+(1-t)b}\right)\le f(a)+f(b). \tag{7.73}$$

Multiplying (7.73) by $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)$ on both sides, and integrating over $[0,1]$ we have

$$\begin{aligned}&\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f\left(\frac{ab}{tb+(1-t)a}\right)dt \\ &+\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f\left(\frac{ab}{ta+(1-t)b}\right)dt \\ &\le (f(a)+f(b))\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)dt.\end{aligned} \tag{7.74}$$

By putting in above $x=\frac{tb+(1-t)a}{ab}$ and $y=\frac{ta+(1-t)b}{ab}$ then after simplifications, we have

$$\begin{aligned}&\left(\varepsilon^{w',\delta,c,q,r}_{\frac{1}{a}-,\rho,\sigma,\tau}f\circ g\right)\left(\frac{1}{b};p\right)+\left(\varepsilon^{w',\delta,c,q,r}_{\frac{1}{b}+,\rho,\sigma,\tau}f\circ g\right)\left(\frac{1}{a};p\right) \\ &\le (f(a)+f(b))\left(\varepsilon^{w',\delta,c,q,r}_{\frac{1}{b}+,\rho,\sigma,\tau}1\right)\left(\frac{1}{a};p\right).\end{aligned} \tag{7.75}$$

Inequalities (7.72) and (7.75) provide the required inequality. □

Remark 7.14 In Theorem 7.23,

(i) if we put $p=0$, then we get [2, Theorem 3.1],

(ii) if we put $w=p=0$, then we get [78, Theorem 4].

Another version is given in the next result.

Theorem 7.24 *Let $w\in\mathbb{R}$, $r,\rho,\sigma,\tau>0$, $c>\delta>0$ with $p\ge 0$ and $0<q\le r+\rho$. Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a harmonically convex function such that $f\in L_1[a,b]$ with $a<b$. If f is a harmonically symmetric about $\frac{a+b}{2ab}$, then the following inequalities for extended generalized fractional integral operators hold*

$$\begin{aligned}&f\left(\frac{2ab}{a+b}\right)H^{w'}_{\frac{a+b}{2ab}-,\sigma}\left(\frac{1}{b};p\right) \\ &\le\frac{1}{2}\left(\left(\varepsilon^{w',\delta,c,q,r}_{\frac{a+b}{2ab}+,\rho,\sigma,\tau}f\circ g\right)\left(\frac{1}{a};p\right)+\left(\varepsilon^{w',\delta,c,q,r}_{\frac{a+b}{2ab}-,\rho,\sigma,\tau}f\circ g\right)\left(\frac{1}{b};p\right)\right) \\ &\le\frac{f(a)+f(b)}{2}H^{w'}_{\frac{a+b}{2ab}+,\sigma}\left(\frac{1}{a};p\right),\end{aligned}$$

where $w'=w(\frac{ab}{b-a})^{\rho}$ and $g(t)=\frac{1}{t}$, $t\in[\frac{1}{b},\frac{1}{a}]$.

Proof. Multiplying (7.70) by $2t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)$ on both sides, and integrating over $[0,\frac{1}{2}]$ we have

$$2f\left(\frac{2ab}{a+b}\right)\int_0^{\frac{1}{2}} t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)dt \leq \int_0^{\frac{1}{2}} t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f\left(\frac{ab}{ta+(1-t)b}\right)dt \tag{7.76}$$

$$+\int_0^{\frac{1}{2}} t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f\left(\frac{ab}{tb+(1-t)a}\right)dt.$$

Putting in above $x=\frac{tb+(1-t)a}{ab}$ that is $\frac{ab}{ta+(1-t)b}=\frac{1}{\frac{1}{a}+\frac{1}{b}-x}$ then we have

$$2f\left(\frac{2ab}{a+b}\right)\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\frac{ab}{b-a}\right)^{\sigma}\left(x-\frac{1}{b}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w\left(\frac{ab}{b-a}\right)^{\rho}\left(x-\frac{1}{b}\right)^{\rho};p\right)dx$$
$$\leq \int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\frac{ab}{b-a}\right)^{\sigma}\left(x-\frac{1}{b}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w\left(\frac{ab}{b-a}\right)^{\rho}\left(x-\frac{1}{b}\right)^{\rho};p\right)$$
$$f\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right)dx+\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\frac{ab}{b-a}\right)^{\sigma}\left(x-\frac{1}{b}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w\left(\frac{ab}{b-a}\right)^{\rho}\right)$$
$$\left(\left(x-\frac{1}{b}\right)^{\rho};p\right)f\left(\frac{1}{x}\right)dx.$$

Since f is harmonically symmetric about $\frac{a+b}{2ab}$, we replace $\frac{1}{a}+\frac{1}{b}-x$ by x in first term on the right hand side of the above inequality and after simplification we have

$$2f\left(\frac{2ab}{a+b}\right)\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(x-\frac{1}{b}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w'\left(x-\frac{1}{b}\right)^{\rho};p\right)dx$$
$$\leq \int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\frac{1}{a}-x\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w'\left(\frac{1}{a}-x\right)^{\rho};p\right)f\left(\frac{1}{x}\right)dx$$
$$+\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(x-\frac{1}{b}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w'\left(x-\frac{1}{b}\right)^{\rho};p\right)f\left(\frac{1}{x}\right)dx.$$

By using Definition 2.2 we get

$$2f\left(\frac{2ab}{a+b}\right)\left(\varepsilon^{w',\delta,c,q,r}_{\frac{a+b}{2ab}-,\rho,\sigma,\tau}1\right)\left(\frac{1}{b};p\right) \tag{7.77}$$
$$\leq \left(\varepsilon^{w',\delta,c,q,r}_{\frac{a+b}{2ab}+,\rho,\sigma,\tau}f\circ g\right)\left(\frac{1}{a};p\right)+\left(\varepsilon^{w',\delta,c,q,r}_{\frac{a+b}{2ab}-,\rho,\sigma,\tau}f\circ g\right)\left(\frac{1}{b};p\right).$$

Now multiplying (7.73) by $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)$ on both sides, and integrating over $[0,\frac{1}{2}]$ we have

$$\int_0^{\frac{1}{2}} t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f\left(\frac{ab}{tb+(1-t)a}\right)dt$$
$$+\int_0^{\frac{1}{2}} t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f\left(\frac{ab}{ta+(1-t)b}\right)dt \tag{7.78}$$

$$\leq (f(a)+f(b))\int_0^{\frac{1}{2}} t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)dt.$$

Putting in above $x=\frac{tb+(1-t)a}{ab}$ that is $\frac{ab}{ta+(1-t)b}=\frac{1}{\frac{1}{a}+\frac{1}{b}-x}$ then we have

$$\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\frac{ab}{b-a}\right)^{\sigma}\left(x-\frac{1}{b}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w\left(\frac{ab}{b-a}\right)^{\rho}\left(x-\frac{1}{b}\right)^{\rho};p\right)f\left(\frac{1}{x}\right)dx$$
$$+\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\frac{ab}{b-a}\right)^{\sigma}\left(x-\frac{1}{b}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w\left(\frac{ab}{b-a}\right)^{\rho}\left(x-\frac{1}{b}\right)^{\rho};p\right)$$
$$f\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right)dx\leq (f(a)+f(b))\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\frac{ab}{b-a}\right)^{\sigma}\left(x-\frac{1}{b}\right)^{\sigma-1}$$
$$E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w\left(\frac{ab}{b-a}\right)^{\rho}\left(x-\frac{1}{b}\right)^{\rho};p\right)dx.$$

Since f is harmonically symmetric about $\frac{a+b}{2ab}$, using this fact by replacing $\frac{1}{a}+\frac{1}{b}-x$ with x in first term of the left hand side of above inequality and after simple calculation, we have

$$\left(\varepsilon^{w',\delta,c,q,r}_{\frac{a+b}{2ab}^{+},\rho,\sigma,\tau}f\circ g\right)\left(\frac{1}{a};p\right)+\left(\varepsilon^{w',\delta,c,q,r}_{\frac{a+b}{2ab}_{-},\rho,\sigma,\tau}f\circ g\right)\left(\frac{1}{b};p\right)\tag{7.79}$$
$$\leq (f(a)+f(b))\left(H^{w'}_{\frac{a+b}{2ab}^{+},\sigma}\right)\left(\frac{1}{a};p\right).$$

Inequalities (7.77) and (7.79) provide the required inequality. □

Remark 7.15 In Theorem 7.24,

(i) if $p=0$, then we get [2, Theorem 3.3],

(ii) if $w=p=0$, then we get Theorem 7.22.

The following lemma is needed to prove the next result.

Lemma 7.1 *Let $f:[a,b]\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be integrable and harmonically symmetric about $\frac{a+b}{2ab}$. Then we have the following equality for extended generalized fractional integral operators holds*

$$\left(\varepsilon^{w,\delta,c,q,r}_{\frac{a+b}{2ab}^{+},\rho,\sigma,\tau}f\circ g\right)\left(\frac{1}{a};p\right)=\left(\varepsilon^{w,\delta,c,q,r}_{\frac{a+b}{2ab}^{-},\rho,\sigma,\tau}f\circ g\right)\left(\frac{1}{b};p\right)\tag{7.80}$$
$$=\frac{1}{2}\left(\left(\varepsilon^{w,\delta,c,q,r}_{\frac{a+b}{2ab}^{+},\rho,\sigma,\tau}f\circ g\right)\left(\frac{1}{a};p\right)+\left(\varepsilon^{w,\delta,c,q,r}_{\frac{a+b}{2ab}^{-},\rho,\sigma,\tau}f\circ g\right)\left(\frac{1}{b};p\right)\right)$$

where $g(t)=\frac{1}{t}$ for $t\in[\frac{1}{b},\frac{1}{a}]$.

Proof. Since f is harmonically symmetric about $\frac{a+b}{2ab}$, we have $f(\frac{1}{x}) = f\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right)$. By the definition of extended generalized fractional integral operator,

$$\left(\varepsilon^{w,\delta,c,q,r}_{\frac{a+b}{2ab}^{+},\rho,\sigma,\tau} f\circ g\right)\left(\frac{1}{a};p\right) = \int_{\frac{a+b}{2ab}}^{\frac{1}{a}} \left(\frac{1}{a}-t\right)^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w\left(\frac{1}{a}-t\right)^{\rho};p\right) f\left(\frac{1}{t}\right)dt, \tag{7.81}$$

replace t by $\frac{1}{a}+\frac{1}{b}-x$ in above we have

$$\begin{aligned}
&\left(\varepsilon^{w,\delta,c,q,r}_{\frac{a+b}{2ab}^{+},\rho,\sigma,\tau} f\circ g\right)\left(\frac{1}{a};p\right)\\
&= \int_{\frac{1}{b}}^{\frac{a+b}{2ab}} \left(x-\frac{1}{b}\right)^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w\left(x-\frac{1}{b}\right)^{\rho};p\right) f\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right)dx\\
&= \int_{\frac{1}{b}}^{\frac{a+b}{2ab}} \left(x-\frac{1}{b}\right)^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w\left(x-\frac{1}{b}\right)^{\rho};p\right) f\left(\frac{1}{x}\right)dx.
\end{aligned}$$

By using Definition 2.2 we get

$$\left(\varepsilon^{w,\delta,c,q,r}_{\frac{a+b}{2ab}^{+},\rho,\sigma,\tau} f\circ g\right)\left(\frac{1}{a};p\right) = \left(\varepsilon^{w,\delta,c,q,r}_{\frac{a+b}{2ab}^{-},\rho,\sigma,\tau} f\circ g\right)\left(\frac{1}{b};p\right). \tag{7.82}$$

By adding $\left(\varepsilon^{w,\delta,c,q,r}_{\frac{a+b}{2ab}^{+},\rho,\sigma,\tau} f\circ g\right)(\frac{1}{a};p)$ to both sides of above we have

$$\begin{aligned}
2\left(\varepsilon^{w,\delta,c,q,r}_{\frac{a+b}{2ab}^{+},\rho,\sigma,\tau} f\circ g\right)\left(\frac{1}{a};p\right) &= \left(\varepsilon^{w,\delta,c,q,r}_{\frac{a+b}{2ab}^{+},\rho,\sigma,\tau} f\circ g\right)\left(\frac{1}{a};p\right)\\
&+ \left(\varepsilon^{w,\delta,c,q,r}_{\frac{a+b}{2ab}^{-},\rho,\sigma,\tau} f\circ g\right)\left(\frac{1}{b};p\right)
\end{aligned} \tag{7.83}$$

which is required. □

The next result provides the generalized version of Fejér-Hadamard inequality for harmonically convex functions.

Theorem 7.25 *Let $w \in \mathbb{R}$, $r,\rho,\sigma,\tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be a harmonically convex function with $a < b$. Let $f \in L_1[a,b]$ and also let $g : [a,b] \to \mathbb{R}$ be a nonnegative, integrable and harmonically symmetric about $\frac{a+b}{2ab}$. Then the following inequalities for extended generalized fractional integral operators hold*

$$\begin{aligned}
&f\left(\frac{2ab}{a+b}\right)\left[\left(\varepsilon^{w',\delta,c,q,r}_{\frac{a+b}{2ab}^{+},\rho,\sigma,\tau} g\circ h\right)\left(\frac{1}{a};p\right) + \left(\varepsilon^{w',\delta,c,q,r}_{\frac{a+b}{2ab}_{-},\rho,\sigma,\tau} g\circ h\right)\left(\frac{1}{b};p\right)\right]\\
&\leq \left(\varepsilon^{w',\delta,c,q,r}_{\frac{a+b}{2ab}^{+},\rho,\sigma,\tau} fg\circ h\right)\left(\frac{1}{a};p\right) + \left(\varepsilon^{w',\delta,c,q,r}_{\frac{a+b}{2ab}_{-},\rho,\sigma,\tau} fg\circ h\right)\left(\frac{1}{b};p\right)\\
&\leq \frac{f(a)+f(b)}{2}\left[\left(\varepsilon^{w',\delta,c,q,r}_{\frac{a+b}{2ab}^{+},\rho,\sigma,\tau} g\circ h\right)\left(\frac{1}{a};p\right) + \left(\varepsilon^{w',\delta,c,q,r}_{\frac{a+b}{2ab}_{-},\rho,\sigma,\tau} g\circ h\right)\left(\frac{1}{b};p\right)\right],
\end{aligned} \tag{7.84}$$

where $w' = w(\frac{ab}{b-a})^{\rho}$ and $h(t) = \frac{1}{t}$ for $t \in [\frac{1}{b},\frac{1}{a}]$.

Proof. Multiplying (7.70) by $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)g\left(\frac{ab}{tb+(1-t)a}\right)$ on both sides, and integrating over $[0,\frac{1}{2}]$ we have

$$2f\left(\frac{2ab}{a+b}\right)\int_0^{\frac{1}{2}} t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)g\left(\frac{ab}{tb+(1-t)a}\right)dt \tag{7.85}$$
$$\leq \int_0^{\frac{1}{2}} t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{tb+(1-t)a}\right)dt$$
$$+\int_0^{\frac{1}{2}} t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f\left(\frac{ab}{tb+(1-t)a}\right)g\left(\frac{ab}{tb+(1-t)a}\right)dt.$$

Putting in above $x=\frac{tb+(1-t)a}{ab}$ that is $\frac{ab}{ta+(1-t)b}=\frac{1}{\frac{1}{a}+\frac{1}{b}-x}$ then we have

$$2f\left(\frac{2ab}{a+b}\right)\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\frac{ab}{b-a}\right)^{\sigma}\left(x-\frac{1}{b}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w\left(\frac{ab}{b-a}\right)^{\rho}\left(x-\frac{1}{b}\right)^{\rho};p\right) \tag{7.86}$$
$$g\left(\frac{1}{x}\right)dx\leq\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\frac{ab}{b-a}\right)^{\sigma}\left(x-\frac{1}{b}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w\left(\frac{ab}{b-a}\right)^{\rho}\left(x-\frac{1}{b}\right)^{\rho};p\right)$$
$$f\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right)g\left(\frac{1}{x}\right)dx+\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\frac{ab}{b-a}\right)^{\sigma}\left(x-\frac{1}{b}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w\left(\frac{ab}{b-a}\right)^{\rho}\right)$$
$$\left(\left(x-\frac{1}{b}\right)^{\rho};p\right)f\left(\frac{1}{x}\right)g\left(\frac{1}{x}\right)dx.$$

Since f is harmonically symmetric about $\frac{a+b}{2ab}$, after simplification (7.86) becomes

$$2f\left(\frac{2ab}{a+b}\right)\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(x-\frac{1}{b}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w'\left(x-\frac{1}{b}\right)^{\rho};p\right)g\left(\frac{1}{x}\right)dx \tag{7.87}$$
$$\leq\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\frac{1}{a}-x\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w'\left(\frac{1}{a}-x\right)^{\rho};p\right)f\left(\frac{1}{x}\right)g\left(\frac{1}{x}\right)dx$$
$$+\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(x-\frac{1}{b}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w'\left(x-\frac{1}{b}\right)^{\rho};p\right)f\left(\frac{1}{x}\right)g\left(\frac{1}{x}\right)dx.$$

By using Definition 2.2 we get

$$2f\left(\frac{2ab}{a+b}\right)\left(\varepsilon^{w',\delta,c,q,r}_{\frac{a+b}{2ab}-,\rho,\sigma,\tau}g\circ h\right)\left(\frac{1}{b};p\right)$$
$$\leq\left(\varepsilon^{w',\delta,c,q,r}_{\frac{a+b}{2ab}+,\rho,\sigma,\tau}fg\circ h\right)\left(\frac{1}{a};p\right)+\left(\varepsilon^{w',\delta,c,q,r}_{\frac{a+b}{2ab}-,\rho,\sigma,\tau}fg\circ h\right)\left(\frac{1}{b};p\right).$$

Using Lemma 7.1 in above inequality, we have

$$f\left(\frac{2ab}{a+b}\right)\left[\left(\varepsilon^{w',\delta,c,q,r}_{\frac{a+b}{2ab}-,\rho,\sigma,\tau}g\circ h\right)\left(\frac{1}{b};p\right)+\left(\varepsilon^{w',\delta,c,q,r}_{\frac{a+b}{2ab}+,\rho,\sigma,\tau}g\circ h\right)\left(\frac{1}{a};p\right)\right] \tag{7.88}$$
$$\leq\left(\varepsilon^{w',\delta,c,q,r}_{\frac{a+b}{2ab}+,\rho,\sigma,\tau}fg\circ h\right)\left(\frac{1}{a};p\right)+\left(\varepsilon^{w',\delta,c,q,r}_{\frac{a+b}{2ab}-,\rho,\sigma,\tau}fg\circ h\right)\left(\frac{1}{b};p\right).$$

Now multiplying (7.73) by $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)g\left(\frac{ab}{tb+(1-t)a}\right)$ on both sides, and integrating over $[0,\frac{1}{2}]$ we have

$$\begin{aligned}&\int_0^{\frac{1}{2}} t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f\left(\frac{ab}{tb+(1-t)a}\right)g\left(\frac{ab}{tb+(1-t)a}\right)dt\\&+\int_0^{\frac{1}{2}} t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{tb+(1-t)a}\right)dt\\&\leq (f(a)+f(b))\int_0^{\frac{1}{2}} t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)g\left(\frac{ab}{tb+(1-t)a}\right)dt.\end{aligned} \tag{7.89}$$

By putting $x=\frac{tb+(1-t)a}{ab}$ and using harmonically symmetry of f with respect to $\frac{a+b}{2ab}$ in above then after simplification we have

$$\begin{aligned}&\left(\epsilon^{w',\delta,c,q,r}_{\frac{a+b}{2ab}^+,\rho,\sigma,\tau}fg\circ h\right)\left(\frac{1}{a};p\right)+\left(\epsilon^{w',\delta,c,q,r}_{\frac{a+b}{2ab}^-,\rho,\sigma,\tau}fg\circ h\right)\left(\frac{1}{b};p\right)\\&\leq (f(a)+f(b))\left(\epsilon^{w',\delta,c,q,r}_{\frac{a+b}{2ab}^+,\rho,\sigma,\tau}g\circ h\right)\left(\frac{1}{a};p\right).\end{aligned} \tag{7.90}$$

Using Lemma 7.1 in (7.90) we have

$$\begin{aligned}&\left(\epsilon^{w',\delta,c,q,r}_{\frac{a+b}{2ab}^+,\rho,\sigma,\tau}fg\circ h\right)\left(\frac{1}{a};p\right)+\left(\epsilon^{w',\delta,c,q,r}_{\frac{a+b}{2ab}^-,\rho,\sigma,\tau}fg\circ h\right)\left(\frac{1}{b};p\right)\\&\leq \frac{(f(a)+f(b))}{2}\left(\epsilon^{w',\delta,c,q,r}_{\frac{a+b}{2ab}^-,\rho,\sigma,\tau}g\circ h\right)\left(\frac{1}{b};p\right)+\left(\epsilon^{w',\delta,c,q,r}_{\frac{a+b}{2ab}^+,\rho,\sigma,\tau}g\circ h\right)\left(\frac{1}{a};p\right).\end{aligned} \tag{7.91}$$

Inequalities (7.88) and (7.91) provide the required inequality. □

Remark 7.16 In Theorem 7.25,

(i) if $p=0$, then we get [2, Theorem 3.6].

(ii) if $w=p=0$ and $g(x)=1$, then we get Theorem 7.22.

Corollary 7.1 *In Theorem 7.25, if we put $w=p=0$, then we get the following inequalities via Riemann-Liouville fractional integral operator*

$$\begin{aligned}&f\left(\frac{2ab}{a+b}\right)\left[\left(J^{\sigma}_{\frac{a+b}{2ab}^+}g\circ h\right)\left(\frac{1}{a}\right)+\left(J^{\sigma}_{\frac{a+b}{2ab}^-}g\circ h\right)\left(\frac{1}{b}\right)\right]\\&\leq \left(J^{\sigma}_{\frac{a+b}{2ab}^+}fg\circ h\right)\left(\frac{1}{a}\right)+\left(J^{\sigma}_{\frac{a+b}{2ab}^-}fg\circ h\right)\left(\frac{1}{b}\right)\\&\leq \frac{f(a)+f(b)}{2}\left[\left(J^{\sigma}_{\frac{a+b}{2abb}^+}g\circ h\right)\left(\frac{1}{a}\right)+\left(J^{\sigma}_{\frac{a+b}{2ab}^-}g\circ h\right)\left(\frac{1}{b}\right)\right].\end{aligned}$$

In the next results generalized fractional integral operators by a monotone increasing function are utilized.

Theorem 7.26 *Let $f, g : [a,b] \to \mathbb{R}$, $0 < a < b$, $\mathrm{Range}(g) \subset [a,b]$ be such that f is positive and $f \in L_1[a,b]$ and g differentiable and strictly increasing. If f is a harmonically convex function on $[a,b]$, then for fractional integral operators (2.23) and (2.24) we have:*

$$
\begin{aligned}
& f\left(\frac{2g(a)g(b)}{g(a)+g(b)}\right)\left({}_g\Upsilon^{w',\delta,c,q,r}_{\left(g^{-1}\left(\frac{1}{g(a)}\right)\right)^-,\rho,\sigma,\tau}1\right)\left(g^{-1}\left(\frac{1}{g(b)}\right);p\right) \\
& \leq \frac{1}{2}\Bigg(\left({}_g\Upsilon^{w',\delta,c,q,r}_{\left(g^{-1}\left(\frac{1}{g(a)}\right)\right)^-,\rho,\sigma,\tau}f\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(b)}\right);p\right) \\
& + \left({}_g\Upsilon^{w',\delta,c,q,r}_{\left(g^{-1}\left(\frac{1}{g(b)}\right)\right)^+,\rho,\sigma,\tau}f\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right)\Bigg) \\
& \leq \frac{f(g(a))+f(g(b))}{2}\left({}_g\Upsilon^{w',\delta,c,q,r}_{\left(g^{-1}\left(\frac{1}{g(b)}\right)\right)^+,\rho,\sigma,\tau}1\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right),
\end{aligned}
\tag{7.92}
$$

where $\psi(t) = \frac{1}{g(t)}$ for all $t \in [\frac{1}{b}, \frac{1}{a}]$ and $w' = w\left(\frac{g(a)(b)}{g(b)-g(a)}\right)^{\sigma}$.

Proof. Since f is harmonically convex on $[a,b]$, for $x, y \in [a,b]$, the following inequality holds:

$$
f\left(\frac{2g(x)g(y)}{g(x)+g(y)}\right) \leq \frac{f(g(x))+f(g(y))}{2}. \tag{7.93}
$$

By taking $g(x) = \frac{g(a)g(b)}{tg(b)+(1-t)g(a)}$ and $g(y) = \frac{g(a)g(b)}{tg(a)+(1-t)g(b)}$ in (7.93), we have

$$
2f\left(\frac{2g(a)g(b)}{g(a)+g(b)}\right) \leq f\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right) + f\left(\frac{g(a)g(b)}{tg(a)+(1-t)g(b)}\right). \tag{7.94}
$$

Multiplying (7.94) by $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(w(t^{\rho});p)$, and integrating over $[0,1]$ we get

$$
\begin{aligned}
& 2f\left(\frac{2g(a)g(b)}{g(a)+g(b)}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^{\rho};p)dt \\
& \leq \int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^{\rho};p)f\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)dt \\
& + \int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^{\rho};p)f\left(\frac{g(a)g(b)}{tg(a)+(1-t)g(b)}\right)dt.
\end{aligned}
\tag{7.95}
$$

By setting $g(x) = \frac{tg(b)+(1-t)g(a)}{g(a)g(b)}$ and $g(y) = \frac{tg(a)+(1-t)g(b)}{g(a)g(b)}$ in (7.95) and using (2.23), (2.24), the first inequality of (7.92) can be obtained. On the other hand from harmonically convexity of f we have

$$
f\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right) + f\left(\frac{g(a)g(b)}{tg(a)+(1-t)g(b)}\right) \leq f(g(a))+f(g(b)). \tag{7.96}
$$

Multiplying (7.96) by $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(w(t^{\rho});p)$ and then integrating over $[0,1]$ we get

$$
\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^{\rho};p)f\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)dt \tag{7.97}
$$

$$+\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)f\left(\frac{g(a)g(b)}{tg(a)+(1-t)g(b)}\right)dt$$
$$\leq (f(g(a))+f(g(b)))\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)dt.$$

By setting $g(x)=\frac{tg(b)+(1-t)g(a)}{g(a)g(b)}$ and $g(y)=\frac{tg(a)+(1-t)g(b)}{g(a)g(b)}$ in (7.97), and using (2.23), (2.24), the second inequality of (7.92) can be obtained. □

Remark 7.17

(i) By setting $p=0$ and $g=I$, [2, Theorem 3.1] is obtained.

(ii) By setting $g=I$, [64, Theorem 2.1] is obtained.

(iii) By setting $w=p=0$, $g=I$, [78, Theorem 4] is obtained.

Corollary 7.2 *If we take $\psi(x)=x$ in Theorem 7.26, then we get the following inequalities.*

$$f\left(\frac{2}{a+b}\right)\left({}_g\Upsilon^{w'.\delta,c,q,r}_{\left(g^{-1}(\frac{1}{a})\right)^-,\rho,\sigma,\tau}1\right)\left(\frac{1}{b};p\right)$$
$$\leq \frac{1}{2}\left(\left({}_g\Upsilon^{w'.\delta,c,q,r}_{\left(g^{-1}(\frac{1}{a})\right)^-,\rho,\sigma,\tau}f\right)\left(\frac{1}{b};p\right)+\left({}_g\Upsilon^{w'.\delta,c,q,r}_{\left(g^{-1}(\frac{1}{b})\right)^+,\rho,\sigma,\tau}f\right)\left(\frac{1}{a};p\right)\right)$$
$$\leq \frac{f(\frac{1}{a})+f(\frac{1}{b})}{2}\left({}_g\Upsilon^{w'.\delta,c,q,r}_{\left(g^{-1}(\frac{1}{b})\right)^+,\rho,\sigma,\tau}1\right)\left(\frac{1}{a};p\right),$$

where g is reciprocal function.

The following lemma is useful to give the next result.

Lemma 7.2 *Let $f,g:[a,b]\to\mathbb{R}$, $0<a<b$, $\mathrm{Range}(g)\subset[a,b]$ be the functions such that f be positive and $f\in L_1[a,b]$ and g be a differentiable and strictly increasing. If $f\left(\frac{1}{g(x)}\right)=f\left(\frac{1}{\frac{1}{g(a)}+\frac{1}{g(b)}-g(x)}\right)$, then for operators (2.23) and (2.24) we have:*

$$\left({}_g\Upsilon^{w.\delta,c,q,r}_{\left(g^{-1}(\frac{1}{g(b)})\right)^+,\rho,\sigma,\tau}f\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right) \tag{7.98}$$
$$=\left({}_g\Upsilon^{w.\delta,c,q,r}_{\left(g^{-1}(\frac{1}{g(a)})\right)^-,\rho,\sigma,\tau}f\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(b)}\right);p\right)$$
$$=\frac{1}{2}\left(\left({}_g\Upsilon^{w.\delta,c,q,r}_{\left(g^{-1}(\frac{1}{g(b)})\right)^+,\rho,\sigma,\tau}f\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right)\right.$$
$$\left.+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,\left(g^{-1}(\frac{1}{g(a)})\right)^-}f\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(b)}\right);p\right)\right),$$

where $\psi(t)=\frac{1}{g(t)}$ for all $t\in[\frac{1}{b},\frac{1}{a}]$.

Proof. By operators (2.23) and (2.24), we can write

$$\left({}_{g}\Upsilon^{w,\delta,c,q,r}_{\left(g^{-1}(\frac{1}{g(b)})\right)^{+},\rho,\sigma,\tau}f\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right) \tag{7.99}$$
$$=\int_{g^{-1}(\frac{1}{g(b)})}^{g^{-1}(\frac{1}{g(a)})}\left(\frac{1}{g(a)}-g(t)\right)^{\tau-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}\left(w\left(\frac{1}{g(a)}-g(t)\right)^{\sigma};p\right)f\left(\frac{1}{g(t)}\right)d(g(t)).$$

Replacing $g(t)$ by $\frac{1}{g(a)}+\frac{1}{g(b)}-g(x)$ in equation (7.99) and then using $f\left(\frac{1}{g(x)}\right)$ $=f\left(\frac{1}{\frac{1}{g(a)}+\frac{1}{g(b)}-g(x)}\right)$, we have

$$\left({}_{g}\Upsilon^{w,\delta,c,q,r}_{\left(g^{-1}(\frac{1}{g(b)})\right)^{+},\rho,\sigma,\tau}f\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right) \tag{7.100}$$
$$=\left({}_{g}\Upsilon^{w,\delta,c,q,r}_{\left(g^{-1}(\frac{1}{g(a)})\right)^{-},\rho,\sigma,\tau}f\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(b)}\right);p\right).$$

By adding $\left({}_{g}\Upsilon^{w,\delta,c,q,r}_{\left(g^{-1}(\frac{1}{g(b)})\right)^{+},\rho,\sigma,\tau}f\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right)$ on both sides of (7.100), we have

$$2\left({}_{g}\Upsilon^{w,\delta,c,q,r}_{\left(g^{-1}(\frac{1}{g(b)})\right)^{+},\rho,\sigma,\tau}f\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right) \tag{7.101}$$
$$=\left({}_{g}\Upsilon^{w,\delta,c,q,r}_{\left(g^{-1}(\frac{1}{g(b)})\right)^{+},\rho,\sigma,\tau}f\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right)$$
$$+\left({}_{g}\Upsilon^{w,\delta,c,q,r}_{\left(g^{-1}(\frac{1}{g(a)})\right)^{-},\rho,\sigma,\tau}f\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(b)}\right);p\right)$$

The equations (7.100) and (7.101) give required result. □

Theorem 7.27 *Let $f,g,h:[a,b]\to\mathbb{R}$, $0<a<b$, $\mathrm{Range}(g),\mathrm{Range}(h)\subset[a,b]$ be such that f is positive and $f\in L_1[a,b]$, g differentiable, strictly increasing and h nonnegative and integrable. If f is harmonically convex and $f\left(\frac{1}{g(x)}\right)=f\left(\frac{1}{\frac{1}{g(a)}+\frac{1}{g(b)}-g(x)}\right)$, then for fractional integral operators (2.23) and (2.24) we have:*

$$\begin{aligned}
& f\left(\frac{2g(a)g(b)}{g(a)+g(b)}\right)\left(\left({}_g\Upsilon^{w.\delta,c,q,r}_{\left(g^{-1}\left(\frac{1}{g(b)}\right)\right)^+,\rho,\sigma,\tau} h\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right)\right. \\
&\quad \left. + \left({}_g\Upsilon^{w.\delta,c,q,r}_{\left(g^{-1}\left(\frac{1}{g(a)}\right)\right)^-,\rho,\sigma,\tau} h\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(b)}\right);p\right)\right) \\
& \le \left({}_g\Upsilon^{w.\delta,c,q,r}_{\left(g^{-1}\left(\frac{1}{g(b)}\right)\right)^+,\rho,\sigma,\tau} fh\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right) \\
&\quad + \left({}_g\Upsilon^{w.\delta,c,q,r}_{\left(g^{-1}\left(\frac{1}{g(a)}\right)\right)^-,\rho,\sigma,\tau} fh\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(b)}\right);p\right) \\
& \le \frac{f(g(a))+f(g(b))}{2}\left(\left({}_g\Upsilon^{w.\delta,c,q,r}_{\left(g^{-1}\left(\frac{1}{g(b)}\right)\right)^+,\rho,\sigma,\tau} h\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right)\right. \\
&\quad \left. + \left({}_g\Upsilon^{w.\delta,c,q,r}_{\left(g^{-1}\left(\frac{1}{g(a)}\right)\right)^-,\rho,\sigma,\tau} h\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(b)}\right);p\right)\right),
\end{aligned} \tag{7.102}$$

where $\psi(t)=\frac{1}{g(t)}$ *for all* $t\in[\frac{1}{b},\frac{1}{a}]$, $fh\circ\psi=(f\circ\psi)(h\circ\psi)$ *and* $w'=w\left(\frac{g(a)g(b)}{g(b)-g(a)}\right)^{\sigma}$.

Proof. Multiplying (7.94) by $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^{\rho};p)h\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)$, then integrating over $[0,1]$ we get

$$\begin{aligned}
& 2f\left(\frac{2g(a)g(b)}{g(a)+g(b)}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^{\rho};p)h\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)dt \\
& \le \int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^{\rho};p)f\left(\frac{g(a)g(b)}{tg(a)+(1-t)g(b)}\right)h\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)dt \\
& + \int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^{\rho};p)f\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)h\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)dt.
\end{aligned} \tag{7.103}$$

By setting $g(x)=\frac{tg(b)+(1-t)g(a)}{g(a)g(b)}$ in (7.103) and using (2.23), (2.24), and the condition $f\left(\frac{1}{g(x)}\right)=f\left(\frac{1}{\frac{1}{g(a)}+\frac{1}{g(b)}-g(x)}\right)$, we have

$$\begin{aligned}
& 2f\left(\frac{2g(a)g(b)}{g(a)+g(b)}\right)\left({}_g\Upsilon^{w.\delta,c,q,r}_{\left(g^{-1}\left(\frac{1}{g(a)}\right)\right)^-,\rho,\sigma,\tau} h\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(b)}\right);p\right) \\
& \le \left({}_g\Upsilon^{w.\delta,c,q,r}_{\left(g^{-1}\left(\frac{1}{g(b)}\right)\right)^+,\rho,\sigma,\tau} fh\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right) \\
& + \left({}_g\Upsilon^{w.\delta,c,q,r}_{\left(g^{-1}\left(\frac{1}{g(a)}\right)\right)^-,\rho,\sigma,\tau} fh\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(b)}\right);p\right).
\end{aligned} \tag{7.104}$$

By using Lemma 7.2 in above inequality, we get first inequality in (7.102). For second inequality of (7.102), multiplying (7.96) by $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^{\rho};p)h\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)$, then integrating over $[0,1]$ we get

$$\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)f\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)h\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)dt \tag{7.105}$$
$$+\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)f\left(\frac{g(a)g(b)}{tg(a)+(1-t)g(b)}\right)h\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)dt$$
$$\leq (f(g(a))+f(g(b)))\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)h\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)dt.$$

Setting $g(x)=\frac{tg(b)+(1-t)g(a)}{g(a)g(b)}$ in (7.105) and using (2.23), (2.24), and the condition $f\left(\frac{1}{g(x)}\right)=f\left(\frac{1}{\frac{1}{g(a)}+\frac{1}{g(b)}-g(x)}\right)$, we have

$$\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w',\left(g^{-1}\left(\frac{1}{g(b)}\right)\right)^+}fh\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right) \tag{7.106}$$
$$+\left({}_g\Upsilon^{w.\delta,c,q,r}_{\left(g^{-1}\left(\frac{1}{g(a)}\right)\right)^-,\rho,\sigma,\tau}fh\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(b)}\right);p\right)$$
$$\leq (f(g(a))+f(g(b)))\left({}_g\Upsilon^{w.\delta,c,q,r}_{\left(g^{-1}\left(\frac{1}{g(b)}\right)\right)^+,\rho,\sigma,\tau}h\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right).$$

Again using Lemma 7.2 in (7.106), we get the second inequality of (7.102). □

Remark 7.18

(i) By setting $p=0$, $h(x)=1$ and $g=I$, [2, Theorem 3.1] is obtained.

(ii) By setting $g=I$ and $h(x)=1$, [64, Theorem 2.1] is obtained.

(iii) By setting $w=p=0$, $h(x)=1$ and $g=I$, [78, Theorem 4] is obtained.

(iv) By setting $w=p=0$, $\sigma=1$ and $g=I$, [33, Theorem 8] is obtained.

(v) By setting $w=p=0$, $\sigma=1$, $h(x)=1$ and $g=I$, [95, Theorem 2.4] is obtained.

Theorem 7.28 *Let $f,g:[a,b]\to\mathbb{R}$, $0<a<b$, $\mathrm{Range}(g)\subset[a,b]$ be such that f is positive and $f\in L_1[a,b]$ and g differentiable and strictly increasing. If f is harmonically convex on $[a,b]$ and $f\left(\frac{1}{g(x)}\right)=f\left(\frac{1}{\frac{1}{g(a)}+\frac{1}{g(b)}-g(x)}\right)$, then for operators (2.23) and (2.24) we have:*

$$f\left(\frac{2g(a)g(b)}{g(a)+g(b)}\right)\left({}_g\Upsilon^{w.\delta,c,q,r}_{\left(g^{-1}\left(\frac{g(a)+g(b)}{2g(a)g(b)}\right)\right)^-,\rho,\sigma,\tau}1\right)\left(g^{-1}\left(\frac{1}{g(b)}\right);p\right) \tag{7.107}$$
$$\leq \frac{1}{2}\left(\left({}_g\Upsilon^{w',\delta,c,q,r}_{\left(g^{-1}\left(\frac{g(a)+g(b)}{2g(a)g(b)}\right)\right)^+,\rho,\sigma,\tau}f\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right)\right.$$

$$+\left({}_g\Upsilon^{w',\delta,c,q,r}_{\left(g^{-1}\left(\frac{g(a)+g(b)}{2g(a)g(b)}\right)\right)^-,\rho,\sigma,\tau} f\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(b)}\right);p\right)\Bigg)$$
$$\leq \frac{f(g(a))+f(g(b))}{2}\left({}_g\Upsilon^{w',\delta,c,q,r}_{\left(g^{-1}\left(\frac{g(a)+g(b)}{2g(a)g(b)}\right)\right)^+,\rho,\sigma,\tau} 1\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right),$$

where $\psi(t)=\frac{1}{g(t)}$ *for* $t\in[\frac{1}{b},\frac{1}{a}]$ *and* $w'=w\left(\frac{g(a)g(b)}{g(b)-g(a)}\right)^\sigma$.

Proof. Multiplying (7.94) by $2t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)$ then integrating over $[0,\frac{1}{2}]$ we have

$$2f\left(\frac{2g(a)g(b)}{g(a)+g(b)}\right)\int_0^{\frac{1}{2}} t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)dt \tag{7.108}$$
$$\leq \int_0^{\frac{1}{2}} t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)f\left(\frac{g(a)g(b)}{tg(a)+(1-t)g(b)}\right)dt$$
$$+\int_0^{\frac{1}{2}} t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)f\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)dt.$$

Setting $g(x)=\frac{tg(b)+(1-t)g(a)}{g(a)g(b)}$ in (7.108) and using $f\left(\frac{1}{g(x)}\right)=f\left(\frac{1}{\frac{1}{g(a)}+\frac{1}{g(b)}-g(x)}\right)$, (2.23) and (2.24) the first inequality of (7.107) can be obtained.

For second inequality multiplying (7.96) by $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)$ then integrating over $[0,\frac{1}{2}]$, we get

$$\int_0^{\frac{1}{2}} t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)f\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)dt \tag{7.109}$$
$$+\int_0^{\frac{1}{2}} t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)f\left(\frac{g(a)g(b)}{tg(a)+(1-t)g(b)}\right)dt$$
$$\leq (f(g(a))+f(g(b)))\int_0^{\frac{1}{2}} t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)dt.$$

Setting $g(x)=\frac{tg(b)+(1-t)g(a)}{g(a)g(b)}$ in (7.109) and using $f\left(\frac{1}{g(x)}\right)=f\left(\frac{1}{\frac{1}{g(a)}+\frac{1}{g(b)}-g(x)}\right)$, (2.23) and (2.24), the second inequality of (7.107) can be obtained. □

Remark 7.19

(i) By setting $p=0$ and $g=I$, [2, Theorem 3.3] is obtained.

(ii) By setting $g=I$, [64, Theorem 2.3] is obtained.

(iii) By setting $w=p=0$ and $g=I$, [95, Theorem 4] is obtained.

To prove the next result the following lemma is needed:

Lemma 7.3 *Let $f,g:[a,b]\to\mathbb{R}$, $0<a<b$, $\mathrm{Range}(g)\subset[a,b]$ be the functions such that f be positive and $f\in L_1[a,b]$ and g be a differentiable and strictly increasing. If f is a harmonically convex and $f\left(\frac{1}{g(x)}\right)=f\left(\frac{1}{\frac{1}{g(a)}+\frac{1}{g(b)}-g(x)}\right)$, then for fractional integral operators (2.23) and (2.24) we have:*

$$\begin{aligned}
&\left({}_g\Upsilon^{w,\delta,c,q,r}_{\left(g^{-1}\left(\frac{g(a)+g(b)}{2g(a)g(b)}\right)\right)^+,\rho,\sigma,\tau} f\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right) \qquad (7.110)\\
&=\left({}_g\Upsilon^{w,\delta,c,q,r}_{\left(g^{-1}\left(\frac{g(a)+g(b)}{2g(a)g(b)}\right)\right)^-,\rho,\sigma,\tau} f\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(b)}\right);p\right)\\
&=\frac{1}{2}\left(\left({}_g\Upsilon^{w,\delta,c,q,r}_{\left(g^{-1}\left(\frac{g(a)+g(b)}{2g(a)g(b)}\right)\right)^+,\rho,\sigma,\tau} f\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right)\right.\\
&\left.+\left({}_g\Upsilon^{w,\delta,c,q,r}_{\left(g^{-1}\left(\frac{g(a)+g(b)}{2g(a)g(b)}\right)\right)^-,\rho,\sigma,\tau} f\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(b)}\right);p\right)\right),\ \ \psi(t)=\frac{1}{g(t)}, t\in\left[\frac{1}{b},\frac{1}{a}\right].
\end{aligned}$$

Proof. By using Definition 7.65, we can write

$$\begin{aligned}
&\left({}_g\Upsilon^{w,\delta,c,q,r}_{\left(g^{-1}\left(\frac{g(a)+g(b)}{2g(a)g(b)}\right)\right)^+,\rho,\sigma,\tau} f\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right) \qquad (7.111)\\
&=\int_{g^{-1}\left(\frac{g(a)+g(b)}{2g(a)g(b)}\right)}^{g^{-1}\left(\frac{1}{g(a)}\right)}\left(\frac{1}{g(a)}-g(t)\right)^{\tau-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}\left(w\left(\frac{1}{g(a)}-g(t)\right)^{\sigma};p\right)f\left(\frac{1}{g(t)}\right)d(g(t)).
\end{aligned}$$

By replacing $g(t)$ by $\frac{1}{g(a)}+\frac{1}{g(b)}-g(x)$ in equation (7.111) and using the condition $f\left(\frac{1}{g(x)}\right)=f\left(\frac{1}{\frac{1}{g(a)}+\frac{1}{g(b)}-g(x)}\right)$, we have

$$\begin{aligned}
&\left({}_g\Upsilon^{w,\delta,c,q,r}_{\left(g^{-1}\left(\frac{g(a)+g(b)}{2g(a)g(b)}\right)\right)^+,\rho,\sigma,\tau} f\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right) \qquad (7.112)\\
&\qquad=\left({}_g\Upsilon^{w,\delta,c,q,r}_{\left(g^{-1}\left(\frac{g(a)+g(b)}{2g(a)g(b)}\right)\right)^-,\rho,\sigma,\tau} f\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(b)}\right);p\right).
\end{aligned}$$

By adding $\left({}_g\Upsilon^{w,\delta,c,q,r}_{\left(g^{-1}\left(\frac{g(a)+g(b)}{2g(a)g(b)}\right)\right)^+,\rho,\sigma,\tau} f\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right)$ on both sides of (7.112), we have

$$2\left({}_g\Upsilon^{w,\delta,c,q,r}_{\left(g^{-1}\left(\frac{g(a)+g(b)}{2g(a)g(b)}\right)\right)^+,\rho,\sigma,\tau} f\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right) \qquad (7.113)$$

$$=\left({}_g\Upsilon^{w,\delta,c,q,r}_{\left(g^{-1}\left(\frac{g(a)+g(b)}{2g(a)g(b)}\right)\right)^+,\rho,\sigma,\tau} f\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right)$$
$$+\left({}_g\Upsilon^{w,\delta,c,q,r}_{\left(g^{-1}\left(\frac{g(a)+g(b)}{2g(a)g(b)}\right)\right)^-,\rho,\sigma,\tau} f\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(b)}\right);p\right).$$

The equations (7.112) and (7.113) give the required result. □

Theorem 7.29 *Let $f,g,h:[a,b]\to\mathbb{R}$, $0<a<b$, $\mathrm{Range}(g),\mathrm{Range}(h)\subset[a,b]$ be such that f is positive and $f\in L_1[a,b]$, g differentiable, strictly increasing and h nonnegative and integrable. If f is harmonically convex and $f\left(\frac{1}{g(x)}\right)=f\left(\frac{1}{\frac{1}{g(a)}+\frac{1}{g(b)}-g(x)}\right)$, then for fractional integral operators (2.23) and (2.24) we have*

$$f\left(\frac{2g(a)g(b)}{g(a)+g(b)}\right)\left(\left({}_g\Upsilon^{w',\delta,c,q,r}_{\left(g^{-1}\left(\frac{g(a)+g(b)}{2g(a)g(b)}\right)\right)^+,\rho,\sigma,\tau} h\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right)\right. \tag{7.114}$$
$$\left.+\left({}_g\Upsilon^{w',\delta,c,q,r}_{\left(g^{-1}\left(\frac{g(a)+g(b)}{2g(a)g(b)}\right)\right)^-,\rho,\sigma,\tau} h\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(b)}\right);p\right)\right)$$
$$\leq\left({}_g\Upsilon^{w',\delta,c,q,r}_{\left(g^{-1}\left(\frac{g(a)+g(b)}{2g(a)g(b)}\right)\right)^+,\rho,\sigma,\tau} fh\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right)$$
$$+\left({}_g\Upsilon^{w',\delta,c,q,r}_{\left(g^{-1}\left(\frac{g(a)+g(b)}{2g(a)g(b)}\right)\right)^-,\rho,\sigma,\tau} fh\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(b)}\right);p\right)$$
$$\leq\frac{f(g(a))+f(g(b))}{2}\left(\left({}_g\Upsilon^{w',\delta,c,q,r}_{\left(g^{-1}\left(\frac{g(a)+g(b)}{2g(a)g(b)}\right)\right)^+,\rho,\sigma,\tau} h\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right)\right.$$
$$\left.+\left({}_g\Upsilon^{w',\delta,c,q,r}_{\left(g^{-1}\left(\frac{g(a)+g(b)}{2g(a)g(b)}\right)\right)^-,\rho,\sigma,\tau} h\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(b)}\right);p\right)\right),$$

where $\psi(t)=\frac{1}{g(t)}$ for $t\in[\frac{1}{b},\frac{1}{a}]$, $fh\circ\psi=(f\circ\psi)(h\circ\psi)$ and $w'=w\left(\frac{g(a)g(b)}{g(b)-g(a)}\right)^\sigma$.

Proof. Multiplying (7.94) by $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)h\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)$ then integrating over $[0,\frac{1}{2}]$, we have

$$2f\left(\frac{2g(a)g(b)}{g(a)+g(b)}\right)\int_0^{\frac{1}{2}}t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)h\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)dt \tag{7.115}$$
$$\leq\int_0^{\frac{1}{2}}t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)f\left(\frac{g(a)g(b)}{tg(a)+(1-t)g(b)}\right)h\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)dt$$
$$+\int_0^{\frac{1}{2}}t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)f\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)h\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)dt.$$

By choosing $g(x)=\frac{tg(b)+(1-t)g(a)}{g(a)g(b)}$ and using the condition $f\left(\frac{1}{g(x)}\right)=f\left(\frac{1}{\frac{1}{g(a)}+\frac{1}{g(b)}-g(x)}\right)$, in (7.115) we have

$$\begin{aligned}&2f\left(\frac{2g(a)g(b)}{g(a)+g(b)}\right)\left({}_{g}\Upsilon^{w',\delta,c,q,r}_{\left(g^{-1}\left(\frac{g(a)+g(b)}{2g(a)g(b)}\right)\right)^{-},\rho,\sigma,\tau}h\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(b)}\right);p\right) \qquad (7.116)\\&\leq\left({}_{g}\Upsilon^{w',\delta,c,q,r}_{\left(g^{-1}\left(\frac{g(a)+g(b)}{2g(a)g(b)}\right)\right)^{+},\rho,\sigma,\tau}fh\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right)\\&+\left({}_{g}\Upsilon^{w',\delta,c,q,r}_{\left(g^{-1}\left(\frac{g(a)+g(b)}{2g(a)g(b)}\right)\right)^{-},\rho,\sigma,\tau}fh\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(b)}\right);p\right).\end{aligned}$$

Using Lemma 7.3 in above inequality we obtain the first inequality of (7.114).

For second inequality of (7.114), multiplying (7.96) by $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^{\rho};p)h\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)$ then integrating over $[0,\frac{1}{2}]$, we have

$$\begin{aligned}&\int_0^{\frac{1}{2}}t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^{\rho};p)f\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)h\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)dt \qquad (7.117)\\&+\int_0^{\frac{1}{2}}t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^{\rho};p)f\left(\frac{g(a)g(b)}{tg(a)+(1-t)g(b)}\right)h\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)dt\\&\leq(f(g(a))+f(g(b)))\int_0^{\frac{1}{2}}t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^{\rho};p)h\left(\frac{g(a)g(b)}{tg(b)+(1-t)g(a)}\right)dt.\end{aligned}$$

Setting $g(x)=\frac{tg(b)+(1-t)g(a)}{g(a)g(b)}$ in (7.117) and using (2.23), (2.24) and condition $f\left(\frac{1}{g(x)}\right)=f\left(\frac{1}{\frac{1}{g(a)}+\frac{1}{g(b)}-g(x)}\right)$, we have

$$\begin{aligned}&\left({}_{g}\Upsilon^{w',\delta,c,q,r}_{\left(g^{-1}\left(\frac{g(a)+g(b)}{2g(a)g(b)}\right)\right)^{+},\rho,\sigma,\tau}fh\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right) \qquad (7.118)\\&+\left({}_{g}\Upsilon^{w',\delta,c,q,r}_{\left(g^{-1}\left(\frac{g(a)+g(b)}{2g(a)g(b)}\right)\right)^{-},\rho,\sigma,\tau}fh\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(b)}\right);p\right)\\&\leq(f(g(a))+f(g(b)))\left({}_{g}\Upsilon^{w',\delta,c,q,r}_{\left(g^{-1}\left(\frac{g(a)+g(b)}{2g(a)g(b)}\right)\right)^{+},\rho,\sigma,\tau}h\circ\psi\right)\left(g^{-1}\left(\frac{1}{g(a)}\right);p\right).\end{aligned}$$

Again using Lemma 7.3 in (7.118), the second inequality of (7.114) can be obtained. □

Remark 7.20

(i) By setting $p=0$ and $g=I$, [2, Theorem 3.6] is obtained.

(ii) By setting $g=I$, [64, Theorem 2.6] is obtained.

(iii) By setting $w=p=0$, $g=I$ and $\sigma=1$, [33, Theorem 8], is obtained.

Corollary 7.3 *When we set $w = p = 0$ and $g = I$ in Theorem 7.29, then we get the following inequalities via Riemann-Liouville fractional integrals:*

$$f\left(\frac{2ab}{a+b}\right)\left(\left(I^{\tau}_{\frac{a+b}{2ab}+}h\circ\psi\right)\left(\frac{1}{a}\right)+\left(I^{\tau}_{\frac{a+b}{2ab}-}h\circ\psi\right)\left(\frac{1}{b}\right)\right)$$
$$\leq\left(I^{\tau}_{\frac{a+b}{2ab}+}fh\circ\psi\right)\left(\frac{1}{a}\right)+\left(I^{\tau}_{\frac{a+b}{2ab}-}fh\circ\psi\right)\left(\frac{1}{b}\right)$$
$$\leq\frac{f(a)+f(b)}{2}\left(\left(I^{\tau}_{\frac{a+b}{2ab}+}h\circ\psi\right)\left(\frac{1}{a}\right)+\left(I^{\tau}_{\frac{a+b}{2ab}-}h\circ\psi\right)\left(\frac{1}{b}\right)\right).$$

7.6 Hadamard and Fejér-Hadamard Inequalities for Harmonically $(\alpha, h-m)$-convex Functions

In this whole section the following notations are frequently used:
$\left(\mathscr{F}^{a^+}_{b,\tau}\right)(w,f)=\left({}_g\Upsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(b;p)$, $\left(\mathscr{F}^{b^-}_{a,\tau}\right)(w,f)=\left({}_g\Upsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(a;p)$.

Theorem 7.30 *Let $f,g:[a,b]\to\mathbb{R}$, $0<a<b$, $\mathrm{Range}(g)\subset[a,b]$ be such that f is positive and $f\in L_1[a,b]$ and g differentiable and strictly increasing. If f is harmonically $(\alpha,h-m)$-convex on $[a,b]$, then for operators (2.23) and (2.24) we have:*

$$f\left(\frac{2m^2g(a)g(b)}{g(a)+mg(b)}\right)\left(\mathscr{F}^{g^{-1}\left(\frac{1}{mg(a)}\right)^-}_{g^{-1}\left(\frac{1}{m^2g(b)}\right),\tau}\right)(m^{\rho}w',1) \tag{7.119}$$
$$\leq h\left(\frac{1}{2^{\alpha}}\right)\left(\mathscr{F}^{g^{-1}\left(\frac{1}{mg(a)}\right)^-}_{g^{-1}\left(\frac{1}{m^2g(b)}\right),\tau}\right)(m^{\rho}w',f\circ\psi)$$
$$+m^{1-2\tau}h\left(1-\frac{1}{2^{\alpha}}\right)\left(\mathscr{F}^{g^{-1}\left(\frac{1}{mg(b)}\right)^+}_{g^{-1}\left(\frac{1}{g(a)}\right),\tau}\right)(w',f\circ\psi)\leq\left(\frac{mg(b)-g(a)}{m^2g(a)g(b)}\right)^{\tau}$$
$$\left\{m\left(h\left(\frac{1}{2^{\alpha}}\right)f\left(\frac{g(a)}{m^2}\right)+mh\left(1-\frac{1}{2^{\alpha}}\right)f(g(b))\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(w(t^{\rho});p)h(t^{\alpha})dt\right.$$
$$\left.+\left(h\left(\frac{1}{2^{\alpha}}\right)f(g(b))+mh\left(1-\frac{1}{2^{\alpha}}\right)f(g(a))\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(w(t^{\rho});p)h(1-t^{\alpha})dt\right\},$$

where $w'=w\left(\frac{mg(a)(b)}{mg(b)-g(a)}\right)^{\sigma}$ and $\psi(t)=\frac{1}{g(t)}$ for all $t\in[\frac{1}{b},\frac{1}{a}]$.

Proof. Since f is harmonically $(\alpha,h-m)$-convex, for all $x,y\in[a,b]$, the following inequality holds:

$$f\left(\frac{2mg(x)g(y)}{mg(y)+g(x)}\right)\leq h\left(\frac{1}{2^{\alpha}}\right)f(g(x))+mh\left(1-\frac{1}{2^{\alpha}}\right)f(g(y)). \tag{7.120}$$

By setting $g(x)=\frac{mg(a)g(b)}{tg(b)+(1-t)\frac{g(a)}{m}}$ and $g(y)=\frac{mg(a)g(b)}{tg(a)+m(1-t)g(b)}$ in (7.120), we have

$$f\left(\frac{2m^2g(a)g(b)}{g(a)+mg(b)}\right)\leq h\left(\frac{1}{2^\alpha}\right)f\left(\frac{mg(a)g(b)}{tg(b)+(1-t)\frac{g(a)}{m}}\right) \tag{7.121}$$
$$+mh\left(1-\frac{1}{2^\alpha}\right)f\left(\frac{mg(a)g(b)}{tg(a)+m(1-t)g(b)}\right).$$

Multiplying (7.121) by $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(w(t^\rho);p)$, and integrating over $[0,1]$ we get

$$f\left(\frac{2m^2g(a)g(b)}{g(a)+mg(b)}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)dt \tag{7.122}$$
$$\leq h\left(\frac{1}{2^\alpha}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)f\left(\frac{mg(a)g(b)}{tg(b)+(1-t)\frac{g(a)}{m}}\right)dt$$
$$+mh\left(1-\frac{1}{2^\alpha}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)f\left(\frac{mg(a)g(b)}{tg(a)+m(1-t)g(b)}\right)dt.$$

By setting $g(x)=\frac{tg(b)+(1-t)\frac{g(a)}{m}}{mg(a)g(b)}$ and $g(y)=\frac{tg(a)+m(1-t)g(b)}{mg(a)g(b)}$ in (7.122) and using (2.23) and (2.24), the first inequality of (7.119) can be obtained. On the other hand from harmonically $(\alpha,h-m)$-convexity of f we have

$$h\left(\frac{1}{2^\alpha}\right)f\left(\frac{mg(a)g(b)}{tg(b)+(1-t)\frac{g(a)}{m}}\right)+mh\left(1-\frac{1}{2^\alpha}\right)f\left(\frac{mg(a)g(b)}{tg(a)+m(1-t)g(b)}\right) \tag{7.123}$$
$$\leq h\left(\frac{1}{2^\alpha}\right)\left(h(1-t^\alpha)f(g(b))+mh(t^\alpha)f\left(\frac{g(a)}{m^2}\right)\right)$$
$$+mh\left(1-\frac{1}{2^\alpha}\right)\left(h(1-t^\alpha)f(g(a))+mh(t^\alpha)f(g(b))\right)$$
$$=m\left(h\left(\frac{1}{2^\alpha}\right)f\left(\frac{g(a)}{m^2}\right)+mh\left(1-\frac{1}{2^\alpha}\right)f(g(b))\right)h(t^\alpha)$$
$$+\left(h\left(\frac{1}{2^\alpha}\right)f(g(b))+mh\left(1-\frac{1}{2^\alpha}\right)f(g(a))\right)h(1-t^\alpha).$$

Multiplying (7.123) by $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(w(t^\rho);p)$ and then integrating over $[0,1]$ we get

$$h\left(\frac{1}{2^\alpha}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)f\left(\frac{mg(a)g(b)}{tg(b)+(1-t)\frac{g(a)}{m}}\right)dt \tag{7.124}$$
$$+mh\left(1-\frac{1}{2^\alpha}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)f\left(\frac{mg(a)g(b)}{tg(a)+m(1-t)g(b)}\right)dt$$
$$\leq m\left(h\left(\frac{1}{2^\alpha}\right)f\left(\frac{g(a)}{m^2}\right)+mh\left(1-\frac{1}{2^\alpha}\right)f(g(b))\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)h(t^\alpha)dt$$
$$+\left(h\left(\frac{1}{2^\alpha}\right)f(g(b))+mh\left(1-\frac{1}{2^\alpha}\right)f(g(a))\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)h(t^\alpha)h(1-t^\alpha)dt.$$

By setting $g(x)=\frac{tg(b)+(1-t)\frac{g(a)}{m}}{mg(a)g(b)}$ and $g(y)=\frac{tg(a)+m(1-t)g(b)}{mg(a)g(b)}$ in (7.124) and using (2.23) and (2.24), the second inequality of (7.119) can be obtained. □

Remark 7.21

(i) By setting $p=0$, $\alpha=m=1$, $h(t)=t$ and $g=I$, [2, Theorem 3.1] is obtained

(ii) By setting $\alpha=m=1$, $h(t)=t$ and $g=I$, [64, Theorem 2.1] is obtained.

(iii) By setting $w=p=0$, $\alpha=m=1$, $h(t)=t$ and $g=I$, [78, Theorem 4] is obtained.

Theorem 7.31 *Let $f,g,\overline{h}:[a,b]\to\mathbb{R}$, $0<a<b$, $\mathrm{Range}(g),\mathrm{Range}(\overline{h})\subset[a,b]$ be such that f is positive and $f\in L_1[a,b]$, g differentiable, strictly increasing and $\overline{h}$ nonnegative and integrable. If f is harmonically $(\alpha,h-m)$-convex and $f\left(\frac{1}{g(x)}\right)=f\left(\frac{1}{\frac{1}{g(a)}+\frac{1}{mg(b)}-mg(x)}\right)$, then for operators (2.23) and (2.24) we have:*

$$f\left(\frac{2m^2g(a)g(b)}{g(a)+mg(b)}\right)\left(\mathscr{F}^{g^{-1}\left(\frac{1}{mg(a)}\right)^-}_{g^{-1}\left(\frac{1}{m^2g(b)}\right),\tau}\right)(m^\rho w',\overline{h}\circ\psi) \tag{7.125}$$
$$\leq\left(h\left(\frac{1}{2^\alpha}\right)+mh\left(1-\frac{1}{2^\alpha}\right)\right)\left(\mathscr{F}^{g^{-1}\left(\frac{1}{mg(a)}\right)^-}_{g^{-1}\left(\frac{1}{m^2g(b)}\right),\tau}\right)(m^\rho w',(\overline{h}\circ\psi)(f\circ\psi))$$
$$\leq\left(\frac{mg(b)-g(a)}{m^2g(a)g(b)}\right)^\tau\left\{m\left(h\left(\frac{1}{2^\alpha}\right)f\left(\frac{g(a)}{m^2}\right)+mh\left(1-\frac{1}{2^\alpha}\right)f(g(b))\right)\right.$$
$$\times\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(w(t^\rho);p)\overline{h}\left(\frac{mg(a)g(b)}{tg(b)+(1-t)\frac{g(a)}{m}}\right)h(t^\alpha)dt$$
$$+\left(h\left(\frac{1}{2^\alpha}\right)f(g(b))+mh\left(1-\frac{1}{2^\alpha}\right)f(g(a))\right)$$
$$\left.\times\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(w(t^\rho);p)\overline{h}\left(\frac{mg(a)g(b)}{tg(b)+(1-t)\frac{g(a)}{m}}\right)h(1-t^\alpha)dt\right\},$$

where $w'=w\left(\frac{mg(a)(b)}{mg(b)-g(a)}\right)^\sigma$, $\psi(t)=\frac{1}{g(t)}$ for all $t\in[\frac{1}{b},\frac{1}{a}]$.

Proof. Multiplying (7.121) by $t^{\sigma-1}\overline{h}\left(\frac{mg(a)g(b)}{tg(b)+(1-t)\frac{g(a)}{m}}\right)E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(w(t^\rho);p)$, and then integrating over $[0,1]$ we get

$$f\left(\frac{2m^2g(a)g(b)}{g(a)+mg(b)}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)\overline{h}\left(\frac{mg(a)g(b)}{tg(b)+(1-t)\frac{g(a)}{m}}\right)dt \tag{7.126}$$
$$\leq h\left(\frac{1}{2^\alpha}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)\overline{h}\left(\frac{mg(a)g(b)}{tg(b)+(1-t)\frac{g(a)}{m}}\right)f\left(\frac{mg(a)g(b)}{tg(b)+(1-t)\frac{g(a)}{m}}\right)dt$$
$$+mh\left(1-\frac{1}{2^\alpha}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)\overline{h}\left(\frac{mg(a)g(b)}{tg(b)+(1-t)\frac{g(a)}{m}}\right)f\left(\frac{mg(a)g(b)}{tg(a)+m(1-t)g(b)}\right)dt.$$

By choosing $g(x)=\frac{tg(b)+(1-t)\frac{g(a)}{m}}{mg(a)g(b)}$ that is $\frac{mg(a)g(b)}{tg(a)+m(1-t)g(b)}=\frac{1}{\frac{1}{g(a)}+\frac{1}{mg(b)}-mg(x)}$ in (7.126) and using (2.23) and (2.24), and the condition $f\left(\frac{1}{g(x)}\right)=f\left(\frac{1}{\frac{1}{g(a)}+\frac{1}{mg(b)}-mg(x)}\right)$, the first inequality of (7.125) can be obtained.

For the second inequality of (7.125), multiplying (7.123) by $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^{\rho};p)\overline{h}\left(\frac{mg(a)g(b)}{tg(b)+(1-t)\frac{g(a)}{m}}\right)$, then integrating over $[0,1]$ we get

$$h\left(\frac{1}{2^{\alpha}}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^{\rho};p)\overline{h}\left(\frac{mg(a)g(b)}{tg(b)+(1-t)\frac{g(a)}{m}}\right)f\left(\frac{mg(a)g(b)}{tg(b)+(1-t)\frac{g(a)}{m}}\right)dt \tag{7.127}$$

$$+mh\left(1-\frac{1}{2^{\alpha}}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^{\rho};p)\overline{h}\left(\frac{mg(a)g(b)}{tg(b)+(1-t)\frac{g(a)}{m}}\right)f\left(\frac{mg(a)g(b)}{tg(a)+m(1-t)g(b)}\right)dt$$

$$\leq m\left(h\left(\frac{1}{2^{\alpha}}\right)f\left(\frac{g(a)}{m^2}\right)+mh\left(1-\frac{1}{2^{\alpha}}\right)f(g(b))\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^{\rho};p)$$

$$\times\overline{h}\left(\frac{mg(a)g(b)}{tg(b)+(1-t)\frac{g(a)}{m}}\right)h(t^{\alpha})dt+\left(h\left(\frac{1}{2^{\alpha}}\right)f(g(b))+mh\left(1-\frac{1}{2^{\alpha}}\right)f(g(a))\right)$$

$$\times\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^{\rho};p)h(t^{\alpha})\overline{h}\left(\frac{mg(a)g(b)}{tg(b)+(1-t)\frac{g(a)}{m}}\right)h(1-t^{\alpha})dt.$$

By setting $g(x)=\frac{tg(b)+(1-t)\frac{g(a)}{m}}{mg(a)g(b)}$ in (7.127) and using (2.23) and (2.24), and the condition $f\left(\frac{1}{g(x)}\right)=f\left(\frac{1}{\frac{1}{g(a)}+\frac{1}{mg(b)}-mg(x)}\right)$, the second inequality of (7.125) can be obtained. □

Remark 7.22

(i) By setting $p=0$, $\alpha=m=1$, $h(t)=t$ $\overline{h}(x)=1$ and $g=I$, [2, Theorem 3.1] is obtained.

(ii) By setting $g=I$, $\alpha=m=1$, $h(t)=t$ and $\overline{h}(x)=1$, [64, Theorem 2.1] is obtained.

(iii) By setting $w=p=0$, $\overline{h}(x)=1$, $\alpha=m=1$, $h(t)=t$ and $g=I$, [78, Theorem 4] is obtained.

(iv) By setting $w=p=0$, $\sigma=1$, $\alpha=m=1$, $h(t)=t$ and $g=I$, [33, Theorem 8] is obtained.

(v) By setting $w=p=0$, $\sigma=1$, $\overline{h}(x)=1$, $\alpha=m=1$, $h(t)=t$ and $g=I$, [95, Theorem 2.4] is obtained.

Theorem 7.32 *Let $f,g:[a,b]\to\mathbb{R}$, $0<a<b$, $\mathrm{Range}(g)\subset[a,b]$ be such that f is positive and $f\in L_1[a,b]$ and g differentiable and strictly increasing. If f is harmonically $(\alpha,h-m)$-convex on $[a,b]$, then for operators (2.23) and (2.24) we have:*

$$
\begin{aligned}
&f\left(\frac{2m^2g(a)g(b)}{g(a)+mg(b)}\right)\left(\mathscr{F}^{g^{-1}\left(\frac{g(a)+mg(b)}{2m^2g(a)g(b)}\right)^-}_{g^{-1}\left(\frac{1}{m^2g(b)}\right),\tau}\right)((2m)^\sigma w',1) \qquad (7.128)\\
&\le h\left(\frac{1}{2^\alpha}\right)\left(\mathscr{F}^{g^{-1}\left(\frac{g(a)+mg(b)}{2m^2g(a)g(b)}\right)^-}_{g^{-1}\left(\frac{1}{m^2g(b)}\right),\tau}\right)((2m)^\sigma w',f\circ\psi)\\
&+m^{1-2\tau}h\left(1-\frac{1}{2^\alpha}\right)\left(\mathscr{F}^{g^{-1}\left(\frac{g(a)+mg(b)}{2mg(a)g(b)}\right)^+}_{g^{-1}\left(\frac{1}{g(a)}\right),\tau}\right)(2^\sigma w',f\circ\psi)\\
&\le\left(\frac{mg(b)-g(a)}{2m^2g(a)g(b)}\right)^\tau\left\{m\left(h\left(\frac{1}{2^\alpha}\right)f\left(\frac{g(a)}{m^2}\right)+mh\left(1-\frac{1}{2^\alpha}\right)f(g(b))\right)\right.\\
&\times\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(w(t^\rho);p)h\left(\left(\frac{t}{2}\right)^\alpha\right)dt+\left(h\left(\frac{1}{2^\alpha}\right)f(g(b))+mh\left(1-\frac{1}{2^\alpha}\right)f(g(a))\right)\\
&\left.\times\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(w(t^\rho);p)h\left(1-\left(\frac{t}{2}\right)^\alpha\right)dt\right\},\ w'=w\left(\frac{mg(a)(b)}{mg(b)-g(a)}\right)^\sigma,
\end{aligned}
$$

where $\psi(t)=\frac{1}{g(t)}$ and $t\in[\frac{1}{b},\frac{1}{a}]$.

Proof. By setting $g(x)=\frac{mg(a)g(b)}{\frac{t}{2}g(b)+(1-\frac{t}{2})\frac{g(a)}{m}}$ and $g(y)=\frac{mg(a)g(b)}{\frac{t}{2}g(a)+m(1-\frac{t}{2})g(b)}$ in (7.120), we have

$$
\begin{aligned}
f\left(\frac{2m^2g(a)g(b)}{g(a)+mg(b)}\right)&\le h\left(\frac{1}{2^\alpha}\right)f\left(\frac{mg(a)g(b)}{\frac{t}{2}g(b)+(1-\frac{t}{2})\frac{g(a)}{m}}\right) \qquad (7.129)\\
&+mh\left(1-\frac{1}{2^\alpha}\right)f\left(\frac{mg(a)g(b)}{\frac{t}{2}g(a)+m(1-\frac{t}{2})g(b)}\right).
\end{aligned}
$$

Multiplying (7.129) by $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)$ and then integrating over $[0,1]$, we get

$$
\begin{aligned}
&f\left(\frac{2m^2g(a)g(b)}{g(a)+mg(b)}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)dt \qquad (7.130)\\
&\le h\left(\frac{1}{2^\alpha}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)f\left(\frac{mg(a)g(b)}{\frac{t}{2}g(b)+(1-\frac{t}{2})\frac{g(a)}{m}}\right)dt\\
&+mh\left(1-\frac{1}{2^\alpha}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)f\left(\frac{mg(a)g(b)}{\frac{t}{2}g(a)+m(1-\frac{t}{2})g(b)}\right)dt.
\end{aligned}
$$

By setting $g(x)=\frac{\frac{t}{2}g(b)+(1-\frac{t}{2})\frac{g(a)}{m}}{mg(a)g(b)}$ and $g(y)=\frac{\frac{t}{2}g(a)+m(1-\frac{t}{2})g(b)}{mg(a)g(b)}$ in (7.130) and using (2.23) and (2.24), first inequality of (7.128) can be obtained. On the other hand from harmonically

$(\alpha,h-m)$-convexity of f we have

$$h\left(\frac{1}{2^\alpha}\right)f\left(\frac{mg(a)g(b)}{\frac{t}{2}g(b)+(1-\frac{t}{2})\frac{g(a)}{m}}\right)+mh\left(1-\frac{1}{2^\alpha}\right)f\left(\frac{mg(a)g(b)}{\frac{t}{2}g(a)+m(1-\frac{t}{2})g(b)}\right) \tag{7.131}$$
$$\leq h\left(\frac{1}{2^\alpha}\right)\left(h\left(1-\left(\frac{t}{2}\right)^\alpha\right)f(g(b))+mh\left(\left(\frac{t}{2}\right)^\alpha\right)f\left(\frac{g(a)}{m^2}\right)\right)$$
$$+mh\left(1-\frac{1}{2^\alpha}\right)\left(h\left(1-\left(\frac{t}{2}\right)^\alpha\right)f(g(a))+mh\left(\left(\frac{t}{2}\right)^\alpha\right)f(g(b))\right)$$
$$=m\left(h\left(\frac{1}{2^\alpha}\right)f\left(\frac{g(a)}{m^2}\right)+mh\left(1-\frac{1}{2^\alpha}\right)f(g(b))\right)h\left(\left(\frac{t}{2}\right)^\alpha\right)$$
$$+\left(h\left(\frac{1}{2^\alpha}\right)f(g(b))+mh\left(1-\frac{1}{2^\alpha}\right)f(g(a))\right)h\left(1-\left(\frac{t}{2}\right)^\alpha\right).$$

Multiplying (7.131) by $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)$ and integrating over $[0,1]$, we get

$$h\left(\frac{1}{2^\alpha}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)f\left(\frac{mg(a)g(b)}{\frac{t}{2}g(b)+(1-\frac{t}{2})\frac{g(a)}{m}}\right)dt \tag{7.132}$$
$$+mh\left(1-\frac{1}{2^\alpha}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)f\left(\frac{mg(a)g(b)}{\frac{t}{2}g(a)+m(1-\frac{t}{2})g(b)}\right)dt$$
$$\leq m\left(h\left(\frac{1}{2^\alpha}\right)f\left(\frac{g(a)}{m^2}\right)+mh\left(1-\frac{1}{2^\alpha}\right)f(g(b))\right)$$
$$\times\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)h\left(\left(\frac{t}{2}\right)^\alpha\right)dt+\left(h\left(\frac{1}{2^\alpha}\right)f(g(b))+mh\left(1-\frac{1}{2^\alpha}\right)f(g(a))\right)$$
$$\times\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)h\left(1-\left(\frac{t}{2}\right)^\alpha\right)dt.$$

By setting $g(x)=\frac{\frac{t}{2}g(b)+(1-\frac{t}{2})\frac{g(a)}{m}}{mg(a)g(b)}$ and $g(y)=\frac{\frac{t}{2}g(a)+m(1-\frac{t}{2})g(b)}{mg(a)g(b)}$ in (7.132) and using (2.23) and (2.24), second inequality of (7.128) can be obtained. □

Remark 7.23

(i) By setting $p=0$, $\alpha=m=1$, $h(t)=t$ and $g=I$, [2, Theorem 3.3] is obtained.

(ii) By setting $\alpha=m=1$, $h(t)=t$ and $g=I$, [64, Theorem 2.3] is obtained.

Theorem 7.33 *Let $f,g,\overline{h}:[a,b]\to\mathbb{R}$, $0<a<mb$, $\mathrm{Range}(g),\mathrm{Range}(\overline{h})\subset[a,b]$ be such that f is positive and $f\in L_1[a,b]$, g differentiable, strictly increasing and $\overline{h}$ nonnegative and integrable. If f is harmonically $(\alpha$,h-m$)$-convex and $f\left(\frac{1}{g(x)}\right)=f\left(\frac{1}{\frac{1}{g(a)}+\frac{1}{mg(b)}-mg(x)}\right)$, then for operators (2.23) and (2.24) we have:*

$$f\left(\frac{2m^2g(a)g(b)}{g(a)+mg(b)}\right)\left(\mathscr{F}^{g^{-1}\left(\frac{g(a)+mg(b)}{2m^2g(a)g(b)}\right)^-}_{g^{-1}\left(\frac{1}{m^2g(b)}\right),\tau}\right)((2m)^\sigma w',\overline{h}\circ\psi) \tag{7.133}$$
$$\leq\left(h\left(\frac{1}{2^\alpha}\right)+mh\left(1-\frac{1}{2^\alpha}\right)\right)\left(\mathscr{F}^{g^{-1}\left(\frac{g(a)+mg(b)}{2m^2g(a)g(b)}\right)^-}_{g^{-1}\left(\frac{1}{m^2g(b)}\right),\tau}\right)((2m)^\sigma w',(\overline{h}\circ\psi)(f\circ\psi))$$
$$\leq\left(\frac{mg(b)-g(a)}{2m^2g(a)g(b)}\right)^\tau\left\{m\left(h\left(\frac{1}{2^\alpha}\right)f\left(\frac{g(a)}{m^2}\right)+mh\left(1-\frac{1}{2^\alpha}\right)f(g(b))\right)\right.$$
$$\times\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(w(t^\rho);p)\overline{h}\left(\frac{mg(a)g(b)}{\frac{t}{2}g(b)+(1-\frac{t}{2})\frac{g(a)}{m}}\right)h\left(\left(\frac{t}{2}\right)^\alpha\right)dt$$
$$+\left(h\left(\frac{1}{2^\alpha}\right)f(g(b))+mh\left(1-\frac{1}{2^\alpha}\right)f(g(a))\right)$$
$$\left.\times\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(w(t^\rho);p)\overline{h}\left(\frac{mg(a)g(b)}{\frac{t}{2}g(b)+(1-\frac{t}{2})\frac{g(a)}{m}}\right)h\left(1-\left(\frac{t}{2}\right)^\alpha\right)dt\right\},$$

where $w'=w\left(\frac{mg(a)(b)}{mg(b)-g(a)}\right)^\sigma$, $\psi(t)=\frac{1}{g(t)}$ *for all* $t\in[\frac{1}{b},\frac{1}{a}]$.

Proof. Multiplying (7.129) by $t^{\sigma-1}\overline{h}\left(\frac{mg(a)g(b)}{\frac{t}{2}g(b)+(1-\frac{t}{2})\frac{g(a)}{m}}\right)E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)$ and integrating over $[0,1]$ we get

$$f\left(\frac{2m^2g(a)g(b)}{g(a)+mg(b)}\right)\int_0^1 t^{\sigma-1}\overline{h}\left(\frac{mg(a)g(b)}{\frac{t}{2}g(b)+(1-\frac{t}{2})\frac{g(a)}{m}}\right)E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)dt\leq h\left(\frac{1}{2^\alpha}\right) \tag{7.134}$$
$$\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)\overline{h}\left(\frac{mg(a)g(b)}{\frac{t}{2}g(b)+(1-\frac{t}{2})\frac{g(a)}{m}}\right)f\left(\frac{mg(a)g(b)}{\frac{t}{2}g(b)+(1-\frac{t}{2})\frac{g(a)}{m}}\right)dt+m$$
$$h\left(1-\frac{1}{2^\alpha}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)\overline{h}\left(\frac{mg(a)g(b)}{\frac{t}{2}g(b)+(1-\frac{t}{2})\frac{g(a)}{m}}\right)f\left(\frac{mg(a)g(b)}{\frac{t}{2}g(a)+m(1-\frac{t}{2})g(b)}\right)dt.$$

By choosing $g(x)=\frac{mg(a)g(b)}{\frac{t}{2}g(b)+(1-\frac{t}{2})\frac{g(a)}{m}}$ in (7.134) and using (2.23) and (2.24), and the condition $f\left(\frac{1}{g(x)}\right)=f\left(\frac{1}{\frac{1}{g(a)}+\frac{1}{mg(b)}-mg(x)}\right)$, then first inequality of (7.133) can be obtained.

On the other hand by harmonically (α,h-m)-convexity of f, multiplying (7.131) by $t^{\sigma-1}\overline{h}\left(\frac{mg(a)g(b)}{\frac{t}{2}g(b)+(1-\frac{t}{2})\frac{g(a)}{m}}\right)E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)$ and integrating over $[0,1]$, we get

$$h\left(\frac{1}{2^\alpha}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)\overline{h}\left(\frac{mg(a)g(b)}{\frac{t}{2}g(b)+(1-\frac{t}{2})\frac{g(a)}{m}}\right)f\left(\frac{mg(a)g(b)}{\frac{t}{2}g(b)+(1-\frac{t}{2})\frac{g(a)}{m}}\right)dt \tag{7.135}$$

$$+mh\left(1-\frac{1}{2^\alpha}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)\overline{h}\left(\frac{mg(a)g(b)}{\frac{t}{2}g(b)+(1-\frac{t}{2})\frac{g(a)}{m}}\right)$$

$$\times f\left(\frac{mg(a)g(b)}{\frac{t}{2}g(a)+m(1-\frac{t}{2})g(b)}\right)dt \le m\left(h\left(\frac{1}{2^\alpha}\right)f\left(\frac{g(a)}{m^2}\right)+mh\left(1-\frac{1}{2^\alpha}\right)f(g(b))\right)$$

$$\times \int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)\overline{h}\left(\frac{mg(a)g(b)}{\frac{t}{2}g(b)+(1-\frac{t}{2})\frac{g(a)}{m}}\right)h\left(\left(\frac{t}{2}\right)^\alpha\right)dt$$

$$+\left(h\left(\frac{1}{2^\alpha}\right)f(g(b))+mh\left(1-\frac{1}{2^\alpha}\right)f(g(a))\right)$$

$$\times \int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(wt^\rho;p)\overline{h}\left(\frac{mg(a)g(b)}{\frac{t}{2}g(b)+(1-\frac{t}{2})\frac{g(a)}{m}}\right)h\left(1-\left(\frac{t}{2}\right)^\alpha\right)dt.$$

By choosing $g(x)=\frac{mg(a)g(b)}{\frac{t}{2}g(b)+(1-\frac{t}{2})\frac{g(a)}{m}}$ in (7.135) and using (2.23) and (2.24), and the condition $f\left(\frac{1}{g(x)}\right)=f\left(\frac{1}{\frac{1}{g(a)}+\frac{1}{mg(b)}-mg(x)}\right)$ second inequality of (7.133) can be obtained. □

Remark 7.24

(i) By setting $p=0$, $\alpha=m=1$, $h(t)=t$ and $g=I$, [2, Theorem 3.6] is obtained.

(ii) By setting $\alpha=m=1$, $h(t)=t$ and $g=I$, [64, Theorem 2.6] is obtained.

7.6.1 Results for Harmonically $(h-m)$-convex Functions

By setting $\alpha=1$ in Theorem 7.30-Theorem 7.33, the results for harmonically $(h-m)$-convex functions are obtained as follows:

Theorem 7.34 *Under the assumptions of Theorem 7.30, the following inequality holds for harmonically $(h-m)$-convex functions:*

$$f\left(\frac{2m^2g(a)g(b)}{g(a)+mg(b)}\right)\left(\mathscr{F}^{g^{-1}\left(\frac{1}{mg(a)}\right)^-}_{g^{-1}\left(\frac{1}{m^2g(b)}\right),\tau}\right)(m^\rho w',1) \tag{7.136}$$

$$\le h\left(\frac{1}{2}\right)\left\{\left(\mathscr{F}^{g^{-1}\left(\frac{1}{mg(a)}\right)^-}_{g^{-1}\left(\frac{1}{m^2g(b)}\right),\tau}\right)(m^\rho w',f\circ\psi)+m^{1-2\tau}\left(\mathscr{F}^{g^{-1}\left(\frac{1}{mg(b)}\right)^+}_{g^{-1}\left(\frac{1}{g(a)}\right),\tau}\right)(w',f\circ\psi)\right\}$$

$$\le h\left(\frac{1}{2}\right)\left(\frac{mg(b)-g(a)}{m^2g(a)g(b)}\right)^\tau\left\{m\left(f\left(\frac{g(a)}{m^2}\right)+mf(g(b))\right)\right.$$

$$\left.\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(w(t^\rho);p)h(t)dt+(f(g(b))+mf(g(a)))\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(w(t^\rho);p)h(1-t)dt\right\}.$$

Theorem 7.35 *Under the assumptions of Theorem 7.31, the following inequality holds for harmonically $(h-m)$-convex functions:*

$$f\left(\frac{2m^2g(a)g(b)}{g(a)+mg(b)}\right)\left(\mathscr{F}^{g^{-1}\left(\frac{1}{mg(a)}\right)^-}_{g^{-1}\left(\frac{1}{m^2g(b)}\right),\tau}\right)(m^\rho w',\overline{h}\circ\psi) \tag{7.137}$$
$$\leq h\left(\frac{1}{2}\right)(1+m)\left(\mathscr{F}^{g^{-1}\left(\frac{1}{mg(a)}\right)^-}_{g^{-1}\left(\frac{1}{m^2g(b)}\right),\tau}\right)(m^\rho w',(\overline{h}\circ\psi)(f\circ\psi))$$
$$\leq h\left(\frac{1}{2}\right)\left(\frac{mg(b)-g(a)}{m^2g(a)g(b)}\right)^\tau\left\{m\left(f\left(\frac{g(a)}{m^2}\right)+mf(g(b))\right)\right.$$
$$\times\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(w(t^\rho);p)\overline{h}\left(\frac{mg(a)g(b)}{tg(b)+(1-t)\frac{g(a)}{m}}\right)h(t)dt$$
$$\left.+(f(g(b))+mf(g(a)))\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(w(t^\rho);p)\overline{h}\left(\frac{mg(a)g(b)}{tg(b)+(1-t)\frac{g(a)}{m}}\right)h(1-t)dt\right\}.$$

Theorem 7.36 *Under the assumptions of Theorem 7.32, the following inequality holds for harmonically $(h-m)$-convex functions:*

$$f\left(\frac{2m^2g(a)g(b)}{g(a)+mg(b)}\right)\left(\mathscr{F}^{g^{-1}\left(\frac{g(a)+mg(b)}{2m^2g(a)g(b)}\right)^-}_{g^{-1}\left(\frac{1}{m^2g(b)}\right),\tau}\right)((2m)^\sigma w',1)\leq h\left(\frac{1}{2}\right) \tag{7.138}$$
$$\left\{\left(\mathscr{F}^{g^{-1}\left(\frac{g(a)+mg(b)}{2m^2g(a)g(b)}\right)^-}_{g^{-1}\left(\frac{1}{m^2g(b)}\right),\tau}\right)((2m)^\sigma w',f\circ\psi)+m^{1-2\tau}\left(\mathscr{F}^{g^{-1}\left(\frac{g(a)+mg(b)}{2mg(a)g(b)}\right)^+}_{g^{-1}\left(\frac{1}{g(a)}\right),\tau}\right)(2^\sigma w',f\circ\psi)\right\}$$
$$\leq h\left(\frac{1}{2}\right)\left(\frac{mg(b)-g(a)}{2m^2g(a)g(b)}\right)^\tau\left\{m\left(f\left(\frac{g(a)}{m^2}\right)+mf(g(b))\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(w(t^\rho);p)h\left(\frac{t}{2}\right)dt\right.$$
$$\left.+(f(g(b))+mf(g(a)))\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(w(t^\rho);p)h\left(1-\frac{t}{2}\right)dt\right\}.$$

Theorem 7.37 *Under the assumptions of Theorem 7.33, the following inequality holds for harmonically $(h-m)$-convex functions:*

$$f\left(\frac{2m^2g(a)g(b)}{g(a)+mg(b)}\right)\left(\mathscr{F}^{g^{-1}\left(\frac{g(a)+mg(b)}{2m^2g(a)g(b)}\right)^-}_{g^{-1}\left(\frac{1}{m^2g(b)}\right),\tau}\right)((2m)^\sigma w',\overline{h}\circ\psi) \tag{7.139}$$
$$\leq h\left(\frac{1}{2}\right)(1+m)\left(\mathscr{F}^{g^{-1}\left(\frac{g(a)+mg(b)}{2m^2g(a)g(b)}\right)^-}_{g^{-1}\left(\frac{1}{m^2g(b)}\right),\tau}\right)((2m)^\sigma w',(\overline{h}\circ\psi)(f\circ\psi))$$
$$\leq h\left(\frac{1}{2}\right)\left(\frac{mg(b)-g(a)}{2m^2g(a)g(b)}\right)^\tau\left\{m\left(f\left(\frac{g(a)}{m^2}\right)+mf(g(b))\right)\right.$$

$$\int_0^1 t^{\sigma-1} E_{\rho,\sigma,\tau,}^{\delta,c,q,r}(w(t^\rho);p)\overline{h}\left(\frac{mg(a)g(b)}{\frac{t}{2}g(b)+(1-\frac{t}{2})\frac{g(a)}{m}}\right) h\left(\frac{t}{2}\right) dt + (f(g(b))+mf(g(a)))$$
$$\int_0^1 t^{\sigma-1} E_{\rho,\sigma,\tau,}^{\delta,c,q,r}(w(t^\rho);p)\overline{h}\left(\frac{mg(a)g(b)}{\frac{t}{2}g(b)+(1-\frac{t}{2})\frac{g(a)}{m}}\right) h\left(1-\frac{t}{2}\right) dt\bigg\}.$$

7.6.2 Results for Harmonically (α, m)-convex Functions

By setting $h(t) = t$ in Theorem 7.30-Theorem 7.33, the results for harmonically (α-m)-convex functions are obtained as follows:

Theorem 7.38 *Under the assumptions of Theorem 7.30, the following inequality holds for harmonically (α,m)-convex functions:*

$$f\left(\frac{2m^2 g(a)g(b)}{g(a)+mg(b)}\right)\left(\mathscr{F}^{g^{-1}\left(\frac{1}{mg(a)}\right)^-}_{g^{-1}\left(\frac{1}{m^2 g(b)}\right),\tau}\right)(m^\rho w',1) \tag{7.140}$$
$$\leq \left(\frac{1}{2^\alpha}\right)\left\{\left(\mathscr{F}^{g^{-1}\left(\frac{1}{mg(a)}\right)^-}_{g^{-1}\left(\frac{1}{m^2 g(b)}\right),\tau}\right)(m^\rho w', f\circ\psi) + m^{1-2\tau}(2^\alpha-1)\left(\mathscr{F}^{g^{-1}\left(\frac{1}{mg(b)}\right)^+}_{g^{-1}\left(\frac{1}{g(a)}\right),\tau}\right)(w', f\circ\psi)\right\}$$
$$\leq \left(\frac{1}{2^\alpha}\right)\left(\frac{mg(b)-g(a)}{m^2 g(a)g(b)}\right)^\tau \left\{m\left(f\left(\frac{g(a)}{m^2}\right) + m(2^\alpha-1)f(g(b))\right)\right.$$
$$\times \left(\mathscr{F}^{g^{-1}\left(\frac{1}{mg(a)}\right)^-}_{g^{-1}\left(\frac{1}{m^2 g(b)}\right),\tau}\right)(m^\rho w',1) + (f(g(b))+m(2^\alpha-1)f(g(a)))$$
$$\times \left.\left(\left(\mathscr{F}^{g^{-1}\left(\frac{1}{mg(a)}\right)^-}_{g^{-1}\left(\frac{1}{m^2 g(b)}\right),\tau}\right)(m^\rho w',1) - \left(\mathscr{F}^{g^{-1}\left(\frac{1}{mg(a)}\right)^-}_{g^{-1}\left(\frac{1}{m^2 g(b)}\right),\tau+\alpha}\right)(m^\rho w',1)\right)\right\}.$$

Theorem 7.39 *Under the assumptions of Theorem 7.31, the following inequality holds for harmonically (α,m)-convex functions:*

$$f\left(\frac{2m^2 g(a)g(b)}{g(a)+mg(b)}\right)\left(\mathscr{F}^{g^{-1}\left(\frac{1}{mg(a)}\right)^-}_{g^{-1}\left(\frac{1}{m^2 g(b)}\right),\tau}\right)(m^\rho w', \overline{h}\circ\psi) \leq \left(\frac{1}{2^\alpha}\right)(1+m) \tag{7.141}$$
$$\left(\mathscr{F}^{g^{-1}\left(\frac{1}{mg(a)}\right)^-}_{g^{-1}\left(\frac{1}{m^2 g(b)}\right),\tau}\right)(m^\rho w', (\overline{h}\circ\psi)(f\circ\psi)) \leq \left(\frac{1}{2^\alpha}\right)\left(\frac{mg(b)-g(a)}{m^2 g(a)g(b)}\right)^\tau$$
$$\left\{m\left(f\left(\frac{g(a)}{m^2}\right) + m(2^\alpha-1)f(g(b))\right)\int_0^1 t^{\sigma+\alpha-1} E_{\rho,\sigma,\tau,}^{\delta,c,q,r}(w(t^\rho);p)\right.$$
$$\overline{h}\left(\frac{mg(a)g(b)}{tg(b)+(1-t)\frac{g(a)}{m}}\right) dt + (f(g(b))+m(2^\alpha-1)f(g(a)))$$
$$\times \left.\int_0^1 (t^{\sigma-1}-t^{\sigma+\alpha-1}) E_{\rho,\sigma,\tau,}^{\delta,c,q,r}(w(t^\rho);p)\overline{h}\left(\frac{mg(a)g(b)}{tg(b)+(1-t)\frac{g(a)}{m}}\right) dt\right\}.$$

Theorem 7.40 *Under the assumptions of Theorem 7.32, the following inequality holds for harmonically* (α,m)*-convex functions:*

$$f\left(\frac{2m^2g(a)g(b)}{g(a)+mg(b)}\right)\left(\mathscr{F}^{g^{-1}\left(\frac{g(a)+mg(b)}{2m^2g(a)g(b)}\right)^-}_{g^{-1}\left(\frac{1}{m^2g(b)}\right),\tau}\right)((2m)^\sigma w',1)\leq\left(\frac{1}{2^\alpha}\right) \tag{7.142}$$

$$\left\{\left(\mathscr{F}^{g^{-1}\left(\frac{g(a)+mg(b)}{2m^2g(a)g(b)}\right)^-}_{g^{-1}\left(\frac{1}{m^2g(b)}\right),\tau}\right)((2m)^\sigma w',f\circ\psi)+m^{1-2\tau}\left(\mathscr{F}^{g^{-1}\left(\frac{g(a)+mg(b)}{2mg(a)g(b)}\right)^+}_{g^{-1}\left(\frac{1}{g(a)}\right),\tau}\right)(2^\sigma w',f\circ\psi)\right\}$$

$$\leq\left(\frac{1}{2^\alpha}\right)\left(\frac{mg(b)-g(a)}{2m^2g(a)g(b)}\right)^\tau\left\{m\left(f\left(\frac{g(a)}{m^2}\right)+m(2^\alpha-1)f(g(b))\right)\right.$$

$$\times\left(\frac{1}{2^\alpha}\right)\left(\mathscr{F}^{g^{-1}\left(\frac{g(a)+mg(b)}{2m^2g(a)g(b)}\right)^-}_{g^{-1}\left(\frac{1}{m^2g(b)}\right),\tau+\alpha}\right)((2m)^\sigma w',1)+(f(g(b))+m(2^\alpha-1)f(g(a)))$$

$$\left.\times\left(\left(\mathscr{F}^{g^{-1}\left(\frac{g(a)+mg(b)}{2m^2g(a)g(b)}\right)^-}_{g^{-1}\left(\frac{1}{m^2g(b)}\right),\tau}\right)((2m)^\sigma w',1)-\left(\frac{1}{2^\alpha}\right)\left(\mathscr{F}^{g^{-1}\left(\frac{g(a)+mg(b)}{2m^2g(a)g(b)}\right)^-}_{g^{-1}\left(\frac{1}{m^2g(b)}\right),\tau+\alpha}\right)((2m)^\sigma w',1)\right)\right\}.$$

Theorem 7.41 *Under the assumptions of Theorem 7.33, the following inequality holds for harmonically* (α,m)*-convex functions:*

$$f\left(\frac{2m^2g(a)g(b)}{g(a)+mg(b)}\right)\left(\mathscr{F}^{g^{-1}\left(\frac{g(a)+mg(b)}{2m^2g(a)g(b)}\right)^-}_{g^{-1}\left(\frac{1}{m^2g(b)}\right),\tau}\right)((2m)^\sigma w',\overline{h}\circ\psi) \tag{7.143}$$

$$\leq\left(\frac{1}{2^\alpha}\right)(1+m)\left(\mathscr{F}^{g^{-1}\left(\frac{g(a)+mg(b)}{2m^2g(a)g(b)}\right)^-}_{g^{-1}\left(\frac{1}{m^2g(b)}\right),\tau}\right)((2m)^\sigma w',(\overline{h}\circ\psi)(f\circ\psi))$$

$$\leq\left(\frac{1}{2^\alpha}\right)\left(\frac{mg(b)-g(a)}{2m^2g(a)g(b)}\right)^\tau\left\{m\left(f\left(\frac{g(a)}{m^2}\right)+m(2^\alpha-1)f(g(b))\right)\right.$$

$$\times\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(w(t^\rho);p)\overline{h}\left(\frac{mg(a)g(b)}{\frac{t}{2}g(b)+(1-\frac{t}{2})\frac{g(a)}{m}}\right)\left(\frac{t}{2}\right)^\alpha dt$$

$$+(f(g(b))+m(2^\alpha-1)f(g(a)))$$

$$\left.\times\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau,}(w(t^\rho);p)\overline{h}\left(\frac{mg(a)g(b)}{\frac{t}{2}g(b)+(1-\frac{t}{2})\frac{g(a)}{m}}\right)\left(1-\left(\frac{t}{2}\right)^\alpha\right)dt\right\}.$$

Chapter 8

Error Bounds of Hadamard and Fejér-Hadamard Inequalities and Bounds of Fractional Integral Operators Associated with Mittag-Leffler Function

In this chapter fractional integral inequalities are given which provide bounds of various kinds of fractional integral operators containing extended Mittag-Leffler functions. Also estimations of Hadamard and Fejér-Hadamard inequalities are given for different kinds of convex functions.

This chapter is based on our results from [35, 36, 55, 57, 54, 70, 101, 125, 132, 148, 150].

8.1 Error Bounds Associated with Fractional Integral Inequalities for Convex Functions

The following lemma is needed to prove the results of this section.

Lemma 8.1 *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function such that $f'\in L_1[a,b]$ with $a<b$. Also let $g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$, then the following identity for extended generalized fractional integral operators holds*

$$\begin{aligned}
&\left(\int_a^b g(s)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(ws^{\rho};p)ds\right)^{\sigma}[f(a)+f(b)] \qquad (8.1)\\
&-\sigma\int_a^b\left(\int_a^t g(s)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(ws^{\rho};p)ds\right)^{\sigma-1}g(t)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f(t)dt\\
&-\sigma\int_a^b\left(\int_t^b g(s)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(ws^{\rho};p)ds\right)^{\sigma-1}g(t)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f(t)dt\\
&=\int_a^b\left(\int_a^t g(s)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(ws^{\rho};p)ds\right)^{\sigma}f'(t)dt-\int_a^b\left(\int_t^b g(s)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(ws^{\rho};p)ds\right)^{\sigma}f'(t)dt.
\end{aligned}$$

Proof. On integrating by parts we have

$$\begin{aligned}
&\int_a^b\left(\int_a^t g(s)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(ws^{\rho};p)ds\right)^{\sigma}f'(t)dt \qquad (8.2)\\
&=\left(\int_a^b g(s)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(ws^{\rho};p)ds\right)^{\sigma}f(b)-\sigma\int_a^b\left(\int_a^t g(s)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(ws^{\rho};p)ds\right)^{\sigma-1}\\
&g(t)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f(t)dt,
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b\left(\int_t^b g(s)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(ws^{\rho};p)ds\right)^{\sigma}f'(t)dt \qquad (8.3)\\
&=-\left(\int_a^b g(s)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(ws^{\rho};p)ds\right)^{\sigma}f(a)+\sigma\int_a^b\left(\int_t^b g(s)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(ws^{\rho};p)ds\right)^{\sigma-1}\\
&g(t)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f(t)dt.
\end{aligned}$$

Subtracting (8.3) from (8.2) we get (8.1) which is required identity. □

In [15] Andrić et al. proved the absolute convergence of the function $E^{\rho,r,k,c}_{\sigma,\tau,\delta}(t;p)$. If we set

$$M:=\sum_{n=0}^{\infty}\left|\frac{\beta_p(\rho+nk,c-\rho)(c)_{nk}t^n}{\beta(\rho,c-\rho)\Gamma(\sigma n+\tau)(\delta)_{nr}}\right|,$$

then $|E^{\rho,r,k,c}_{\sigma,\tau,\delta}(t;p)|\leq M$. We use this and the the identity (8.1) to prove the next results.

Theorem 8.1 *Let $w \in \mathbb{R}$, $r,\rho,\sigma,\tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $f : [a,b] \to \mathbb{R}$ be a differentiable function such that $f' \in L_1[a,b]$ with $a < b$. Also let $g : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$. If $|f'|$ is convex function on $[a,b]$, then the following inequality for extended generalized fractional integral operators holds*

$$\begin{aligned}
&\Bigg|\left(\int_a^b g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^\rho;p)ds\right)^\sigma [f(a)+f(b)] \\
&-\sigma\int_a^b \left(\int_a^t g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^\rho;p)ds\right)^{\sigma-1} g(t)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^\rho;p)f(t)dt \\
&-\sigma\int_a^b \left(\int_t^b g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^\rho;p)ds\right)^{\sigma-1} g(t)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^\rho;p)f(t)dt\Bigg| \\
&\leq \frac{(b-a)^{\sigma+1}\|g\|_\infty^\sigma M^\sigma}{(\sigma+1)}[|f'(a)|+|f'(b)|]
\end{aligned} \tag{8.4}$$

for $q < r+\Re(\rho)$ and $\| g \|_\infty = \sup_{t\in[a,b]} |g(t)|$.

Proof. From Lemma 8.1, we have

$$\begin{aligned}
&\Bigg|\left(\int_a^b g(s)E_{\rho,\sigma,r}^{\delta,\tau,q}(ws^\rho;p)ds\right)^\sigma [f(a)+f(b)] \\
&-\sigma\int_a^b \left(\int_a^t g(s)E_{\rho,\sigma,r}^{\delta,\tau,q}(ws^\rho;p)ds\right)^{\sigma-1} g(t)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^\rho;p)f(t)dt \\
&-\sigma\int_a^b \left(\int_t^b g(s)E_{\rho,\sigma,r}^{\delta,\tau,q}(ws^\rho;p)ds\right)^{\sigma-1} g(t)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^\rho;p)f(t)dt\Bigg| \\
&\leq \int_a^b \left|\int_a^t g(s)E_{\rho,\sigma,r}^{\delta,\tau,q}(ws^\rho;p)ds\right|^\sigma |f'(t)|dt + \int_a^b \left|\int_t^b g(s)E_{\rho,\sigma,r}^{\delta,\tau,q}(ws^\rho;p)ds\right|^\sigma |f'(t)|dt
\end{aligned} \tag{8.5}$$

Using absolute convergence of Mittag-Leffler function and $\| g \|_\infty = \sup_{t\in[a,b]} |g(t)|$, we have

$$\begin{aligned}
&\Bigg|\left(\int_a^b g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^\rho;p)ds\right)^\sigma [f(a)+f(b)] \\
&-\sigma\int_a^b \left(\int_a^t g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^\rho;p)ds\right)^{\sigma-1} g(t)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^\rho;p)f(t)dt \\
&-\sigma\int_a^b \left(\int_t^b g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^\rho;p)ds\right)^{\sigma-1} g(t)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^\rho;p)f(t)dt\Bigg| \\
&\leq \|g\|_\infty^\sigma M^\sigma \left[\int_a^b (t-a)^\sigma |f'(t)|dt + \int_a^b (b-t)^\sigma |f'(t)|dt\right].
\end{aligned} \tag{8.6}$$

As $|f'|$ is convex function, for $t \in [a,b]$ we have

$$|f'(t)| \leq \frac{b-t}{b-a}|f'(a)| + \frac{t-a}{b-a}|f'(b)|. \tag{8.7}$$

Using (8.7) in (8.6), we have

$$\begin{aligned}
&\Bigg|\left(\int_a^b g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^{\rho};p)ds\right)^{\sigma}[f(a)+f(b)] \\
&-\sigma\int_a^b\left(\int_a^t g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^{\rho};p)ds\right)^{\sigma-1} g(t)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^{\rho};p)f(t)dt \\
&-\sigma\int_a^b\left(\int_t^b g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^{\rho};p)ds\right)^{\rho-1} g(t)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^{\rho};p)f(t)dt\Bigg| \\
&\leq \|g\|_{\infty}^{\sigma}M^{\sigma}\left[\int_a^b (t-a)^{\sigma}\left(\frac{b-t}{b-a}|f'(a)|+\frac{t-a}{b-a}|f'(b)|\right)dt\right. \\
&\left.+\int_a^b (b-t)^{\sigma}\left(\frac{b-t}{b-a}|f'(a)|+\frac{t-a}{b-a}|f'(b)|\right)dt\right].
\end{aligned} \tag{8.8}$$

After simple calculation of the above inequality, (8.4) can be obtained. □

Remark 8.1 In Theorem 8.1.

(i) If $p=0$, then we get [4, Theorem 3.2].

(ii) If $w=p=0$, then we get [137, Theorem 6].

(iii) If $g(s)=1$ along with $w=p=0$, then we get [137, Corollary 2].

Corollary 8.1 *In Theorem 8.1, for $w=0$, $\sigma=\mu$ and $g(s)=1$ we have the following inequality for Riemann-Liouville fractional integral operator*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\mu+1)}{2(b-a)^{\rho}}[J_{a^+}^{\mu}f(b)+J_{b_-}^{\mu}f(a)]\right| \leq \frac{b-a}{2(\mu+1)}[|f'(a)|+|f'(b)|]$$

$\mu>0$.

Theorem 8.2 *Let $w\in\mathbb{R}$, $r,\rho,\sigma,\tau>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $f:[a,b]\to\mathbb{R}$ be a differentiable function such that $f'\in L_1[a,b]$ with $a<b$. Also let $g:[a,b]\to\mathbb{R}$ be a continuous function on $[a,b]$. If $|f'|^q$ is convex function on $[a,b]$, then for $q>0$ the following inequality for extended generalized fractional integral operators holds*

$$\begin{aligned}
&\Bigg|\left(\int_a^b g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^{\rho};p)ds\right)^{\sigma}[f(a)+f(b)] \\
&-\sigma\int_a^b\left(\int_a^t g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^{\rho};p)ds\right)^{\sigma-1} g(t)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^{\rho};p)f(t)dt \\
&-\sigma\int_a^b\left(\int_t^b g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^{\rho};p)ds\right)^{\sigma-1} g(t)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^{\rho};p)f(t)dt\Bigg| \\
&\leq \frac{2(b-a)^{\sigma+1}\|g\|_{\infty}^{\sigma}M^{\sigma}}{(\sigma p+1)^{\frac{1}{q}}}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}}
\end{aligned} \tag{8.9}$$

for $q < r + \Re(\rho)$ *and* $\| g \|_\infty = \sup_{t\in[a,b]} |g(t)|$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. The proof is on the same lines as the proof of Theorem 8.1. □

Remark 8.2 In Theorem 8.2.

(i) If $p = 0$, then we get [4, Theorem 3.5].

(ii) If $w = p = 0$, then we get [137, Theorem 7].

(iii) If $w = p = 0, \sigma = 1$, then we get [137, Corollary 3].

(iv) If $g(s) = 1$ and $w = p = 0$, then we get [42, Theorem 2.3].

Corollary 8.2 *In Theorem 8.2, if we take* $w = 0$ *and* $g(s) = 1$*, then we have the following inequality of Riemann-Liouville integral for fractional integral operator*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\mu+1)}{2(b-a)^\rho}[J^\mu_{a^+} f(b) + J^\mu_{b_-} f(a)]\right| \leq \frac{b-a}{(\mu p+1)^{\frac{1}{p}}}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{\frac{1}{q}}$$

$\mu > 0$.

Lemma 8.2 *Let* $f, g : [a,b] \to \mathbb{R}$*,* $0 < a < b$ *be the functions such that* f *be positive and* $f \in L_1[a,b]$ *and* g *be a differentiable and strictly increasing. If the function* g *is symmetric about* $\frac{a+b}{2}$*, then we have*

$$\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+} f\circ g\right)(b;p) = \left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,b^-} f\circ g\right)(a;p) \tag{8.10}$$
$$= \frac{1}{2}\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+} f\circ g\right)(b;p) + \left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,b^-} f\circ g\right)(a;p)\right].$$

Proof. Since g is symmetric about $\frac{a+b}{2}$, by Definition 2.4 of extended generalized fractional integral operator, we have

$$\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+} f\circ g\right)(b;p) = \int_a^b (g(b)-g(t))^{\tau-1} E^{\rho,r,k,c}_{\sigma,\tau,\delta}(w(g(b)-g(t))^\sigma;p) f(g(t))d(g(t)). \tag{8.11}$$

If we replace t by $a+b-t$ in (8.11), then we get

$$\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+} f\circ g\right)(b;p) = \int_a^b (g(t)-g(a))^{\tau-1} E^{\rho,r,q,c}_{\sigma,\tau,\delta}(w(g(t)-g(a))^\sigma;p) f(g(t))d(g(t)).$$

This implies

$$\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+} f\circ g\right)(b;p) = \left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,b^-} f\circ g\right)(a;p). \tag{8.12}$$

By adding equations (8.11) and (8.12), we get (8.10). □

Lemma 8.3 *Let $f,g,h:[a,b]\to\mathbb{R}$, $0<a<b$ be the functions such that f be positive and $f\circ g\in L_1[a,b]$, where g be a differentiable and strictly increasing and h be continuous. For $f'\circ g\in L_1[a,b]$, the following equality for extended generalized fractional integral operators* (2.23) *and* (2.24) *holds:*

$$\begin{aligned}
&\left(\frac{f(g(a))+f(g(b))}{2}\right)\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+}h\right)(b;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,b^-}h\right)(a;p)\right] \qquad (8.13)\\
&-\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+}h(f\circ g)\right)(b;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,b^-}h(f\circ g)\right)(a;p)\right]\\
&=\int_a^b\left[\int_a^t (g(b)-g(s))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(w(g(b)-g(s))^{\sigma};p)h(s)d(g(s))\right.\\
&\left.-\int_t^b (g(s)-g(a))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(w(g(s)-g(a))^{\sigma};p)h(s)d(g(s))\right]f'(g(t))d(g(t)).
\end{aligned}$$

Proof. To prove this lemma, we take its right hand side. On integrating by parts and after simplification, we have

$$\begin{aligned}
&\int_a^b\left[\int_a^t (g(b)-g(s))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(w(g(b)-g(s))^{\sigma};p)h(s)d(g(s))\right]f'(g(t))d(g(t))\\
&=f(g(b))\left(\int_a^b (g(b)-g(s))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(w(g(b)-g(s))^{\sigma};p)h(s)d(g(s))\right)\\
&-\int_a^b\left((g(b)-g(t))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(w(g(b)-g(t))^{\sigma};p)\right)h(t)f(g(t))d(g(t))\\
&=f(g(b))\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+}h\right)(b;p)-\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+}h(f\circ g)\right)(b;p).
\end{aligned}$$

By using Lemma 8.2, we have

$$\begin{aligned}
&\int_a^b\left[\int_a^t (g(b)-g(s))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(w(g(b)-g(s))^{\sigma};p)h(s)d(g(s))\right]f'(g(t))d(g(t)) \qquad (8.14)\\
&=\frac{f(g(b))}{2}\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+}h\right)(b;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,b^-}h\right)(a;p)\right]-\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+}h(f\circ g)\right)(b;p).
\end{aligned}$$

Similarly

$$\begin{aligned}
&\int_a^b\left[-\int_t^b (g(s)-g(a))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(w(g(s)-g(a))^{\sigma};p)h(s)d(g(s))\right]f'(g(t))d(g(t))\\
&\qquad (8.15)\\
&=\frac{f(g(a))}{2}\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+}h\right)(b;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,b^-}h\right)(a;p)\right]-\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,b^-}h(f\circ g)\right)(a;p).
\end{aligned}$$

Summing (8.14) and (8.15), we get (8.13). □

Theorem 8.3 *Let $w\in\mathbb{R}$, $r,\rho,\sigma,\tau>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $f,g,h:[a,b]\to\mathbb{R}$, $0<a<b$ be the functions such that f be positive and $(f\circ g)'\in L_1[a,b]$, where g be a differentiable and strictly increasing and h be continuous and symmetric*

about $\frac{a+b}{2}$. Also let $f \circ g$ be symmetric about $\frac{a+b}{2}$ and if $|(f \circ g)'|$ is convex. Then for $k < r + \Re(\sigma)$, the following inequality holds:

$$\begin{aligned}
&\Bigg|\left(\frac{f(g(a))+f(g(b))}{2}\right)\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+}h\right)(b;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,b^-}h\right)(a;p)\right]\\
&-\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+}(f\circ g)h\right)(b;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,b^-}(f\circ g)h\right)(a;p)\right]\Bigg|\\
&\leq \frac{\| h \|_\infty M(g(b)-g(a))^{\tau+1}}{\tau(\tau+1)}(1-\Phi)\left[|f'(g(a))+f'(g(b))|\right],
\end{aligned}$$

where $\| h \|_\infty = \sup_{t\in[a,b]} |h(t)|$ and

$$\begin{aligned}
\Phi &= \frac{1}{\tau+2}\left[\left(\frac{g(\frac{a+b}{2})-g(a)}{g(b)-g(a)}\right)^{\tau+2}+\left(\frac{g(b)-g(\frac{a+b}{2})}{g(b)-g(a)}\right)^{\tau+2}\right]\\
&-\frac{\tau+1}{\tau+2}\left[\left(\frac{g(\frac{a+b}{2})-g(a)}{g(b)-g(a)}\right)^{\tau+2}+\left(\frac{g(b)-g(\frac{a+b}{2})}{g(b)-g(a)}\right)^{\tau+2}\right]\\
&-\left(\frac{g(\frac{a+b}{2})-g(a)}{g(b)-g(a)}\right)^{\tau+1}\left(\frac{g(b)-g(\frac{a+b}{2})}{g(b)-g(a)}\right)+\left(\frac{g(\frac{a+b}{2})-g(a)}{g(b)-g(a)}\right)\left(\frac{g(b)-g(\frac{a+b}{2})}{g(b)-g(a)}\right)^{\tau+1}.
\end{aligned}$$

Proof. By using Lemma 8.3, we have

$$\begin{aligned}
&\Bigg|\left(\frac{f(g(a))+f(g(b))}{2}\right)\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+}h\right)(b;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,b^-}h\right)(a;p)\right]\\
&-\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+}(f\circ g)h\right)(b;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,b^-}(f\circ g)h\right)(a;p)\right]\Bigg|\\
&\leq \int_a^b\Bigg|\Bigg[\int_a^t (g(b)-g(s))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(w(g(b)-g(s))^\sigma;p)h(s)d(g(s))\\
&-\int_t^b (g(s)-g(a))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(w(g(s)-g(a))^\sigma;p)h(s)d(g(s))\Bigg]\Bigg||f'(g(t))|d(g(t)).
\end{aligned} \tag{8.16}$$

Using the convexity of $|f'(g)|$ on $[a,b]$, we have

$$|f'(g(t))| \leq \frac{g(b)-g(t)}{g(b)-g(a)}|f'(g(a))|+\frac{g(t)-g(a)}{g(b)-g(a)}|f'(g(b))|, t\in[a,b]. \tag{8.17}$$

The symmetry of h implies $h(t)=h(a+b-t)$, $h(s)=h(a+b-s)$ and replacing t by $a+b-t$, s by $a+b-s$ in second integral, we get

$$\begin{aligned}
&\Big|\int_a^t (g(b)-g(s))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(w(g(b)-g(s))^{\sigma};p)h(s)d(g(s))\\
&-\int_t^b (g(s)-g(a))^{\tau-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(s)-g(a))^{\sigma};p)h(s)d(g(s))\Big|\\
=&\Big|-\int_t^a (g(b)-g(s))^{\tau-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(b)-g(s))^{\sigma};p)h(s)d(g(s))\\
&-\int_a^{a+b-t} (g(b)-g(s))^{\tau-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(b)-g(s))^{\sigma};p)h(s)d(g(s))\Big|\\
=&\Big|\int_t^{a+b-t} (g(b)-g(s))^{\tau-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(b)-g(s))^{\sigma};p)h(s)d(g(s))\Big|
\end{aligned}$$

$$\leq\begin{cases}\int_t^{a+b-t}|(g(b)-g(s))^{\tau-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(b)-g(s))^{\sigma};p)h(s)|d(g(s)),\ t\in[a,\frac{a+b}{2}]\\[2ex]
\int_{a+b-t}^{t}|(g(b)-g(s))^{\tau-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(b)-g(s))^{\sigma};p)h(s)|d(g(s)),\ \ t\in[\frac{a+b}{2},b].\end{cases}\tag{8.18}$$

By (8.16), (8.17), (8.18) and absolute convergence of Mittag-Leffler function, we have

$$\begin{aligned}
&\Big|\left(\frac{f(g(a))+f(g(b))}{2}\right)\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+}h\right)(b;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,b^-}h\right)(a;p)\right]\\
&-\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+}(f\circ g)h\right)(b;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,b^-}(f\circ g)h\right)(a;p)\right]\Big|\\
&\leq\int_a^{\frac{a+b}{2}}\left(\int_a^{a+b-t}|(g(b)-g(s))^{\tau-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(b)-g(s))^{\sigma};p)h(s)|d(g(s))\right)\\
&\times\left(\frac{g(b)-g(t)}{g(b)-g(a)}|f'(g(a))|+\frac{g(t)-g(a)}{g(b)-g(a)}|f'(g(b))|\right)d(g(t))\\
&+\int_{\frac{a+b}{2}}^b\left(\int_{a+b-t}^t|(g(b)-g(s))^{\tau-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(b)-g(s))^{\sigma};p)h(s)|d(g(s))\right)\\
&\times\left(\frac{g(b)-g(t)}{g(b)-g(a)}|f'(g(a))|+\frac{g(t)-g(a)}{g(b)-g(a)}|f'(g(b))|\right)d(g(t)).\\
&\leq\frac{\|h\|_\infty M}{\tau(g(b)-g(a))}\Bigg[\int_a^{\frac{a+b}{2}}((g(b)-g(t))^{\tau}-(g(t)-g(a))^{\tau}(g(b)-g(t))|f'(g(a))|)d(g(t))\\
&+\int_a^{\frac{a+b}{2}}((g(b)-g(t))^{\tau}-(g(t)-g(a))^{\tau}(g(t)-g(a))|f'(g(b))|)d(g(t))\\
&+\int_{\frac{a+b}{2}}^b((g(t)-g(a))^{\tau}-(g(b)-g(t))^{\tau}(g(b)-g(t))|f'(g(a))|)d(g(t))\\
&+\int_{\frac{a+b}{2}}^b((g(t)-g(a))^{\tau}-(g(b)-g(t))^{\tau}(g(t)-g(a))|f'(g(b))|)d(g(t))\Bigg].
\end{aligned}\tag{8.19}$$

After solving the terms of above inequality, we have the following values

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}} \left((g(b)-g(t))^{\tau}-(g(t)-g(a))^{\tau}\right)(g(b)-g(t))d(g(t))\\
&=\int_{\frac{a+b}{2}}^{b} \left((g(t)-g(a))^{\tau}-(g(b)-g(t))^{\tau}\right)(g(t)-g(a))d(g(t))\\
&=\frac{(g(b)-g(a))^{\tau+2}}{\tau+2}-\frac{\left(g(b)-g(\frac{a+b}{2})\right)^{\tau+2}}{\tau+2}\\
&-\frac{\left(g(\frac{a+b}{2})-g(a)\right)^{\tau+1}}{\tau+1}\left(g(b)-g(\frac{a+b}{2})\right)-\frac{\left(g(\frac{a+b}{2})-g(a)\right)^{\tau+2}}{(\tau+1)(\tau+2)}
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}} \left((g(b)-g(t))^{\tau}-(g(t)-g(a))^{\tau}\right)(g(t)-g(a))d(g(t))\\
&=\int_{\frac{a+b}{2}}^{b} \left((g(t)-g(a))^{\tau}-(g(b)-g(t))^{\tau}\right)(g(b)-g(t))d(g(t))\\
&=-\frac{\left(g(\frac{a+b}{2})-g(a)\right)^{\tau+1}}{\tau+1}\left(g(b)-g(\frac{a+b}{2})\right)\\
&+\frac{(g(b)-g(a))^{\tau+2}}{(\tau+1)(\tau+2)}-\frac{\left(g(\frac{a+b}{2})-g(a)\right)^{\tau+2}}{(\tau+1)(\tau+2)}-\frac{\left(g(b)-g(\frac{a+b}{2})\right)^{\tau+2}}{\tau+2}.
\end{aligned}$$

Using the above calculations of integrals in (8.19), we get the required inequality. □

Remark 8.3

(i) In Theorem 8.3 if we put $g=I$, we get [33, Theorem 2.3],

(ii) In Theorem 8.3 if we put $p=0$ and $g=I$, we get [1, Theorem 2.3].

(iii) In Theorem 8.3 if we put $w=p=0$ and $g=I$, we get [94, Theorem 2.36].

Theorem 8.4 *Let $w\in\mathbb{R}$, $r,\rho,\sigma,\tau>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $f,g,h:[a,b]\to\mathbb{R}$, $0<a<b$ be the functions such that f be positive and $(f\circ g)'\in L_1[a,b]$, where g be a differentiable and strictly increasing and h be continuous and symmetric about $\frac{a+b}{2}$. Also let $f\circ g$ be symmetric about $\frac{a+b}{2}$ and if $|(f\circ g)'|^q$, $q\geq 1$ is convex on $[a,b]$. Then for $k<r+\Re(\sigma)$, the following inequality holds:*

$$\begin{aligned}
&\Bigg|\left(\frac{f(g(a))+f(g(b))}{2}\right)\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+}h\right)(b;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,b^-}h\right)(a;p)\right]\\
&-\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+}(f\circ g)h\right)(b;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,b^-}(f\circ g)h\right)(a;p)\right]\Bigg|\\
&\leq\frac{2\parallel h\parallel_\infty M(g(b)-g(a))^{\tau+1}}{\tau(\tau+1)}\left[(1-\Psi)^{1-\frac{1}{q}}(1-\Phi)^{\frac{1}{q}}\right]\left(\frac{|f'(g(a))|^q+|f'(g(b))|^q}{2}\right)^{\frac{1}{q}},
\end{aligned}\tag{8.20}$$

where $\parallel h\parallel_\infty=\sup_{t\in[a,b]}|h(t)|$, $\Psi=\left(\frac{g(b)-g(\frac{a+b}{2})}{g(b)-g(a)}\right)^{\tau+1}+\left(\frac{g(\frac{a+b}{2})-g(a)}{g(b)-g(a)}\right)^{\tau+1}$ *and*

$$\Phi = \tfrac{1}{\tau+2}\left[\left(\tfrac{g(\frac{a+b}{2})-g(a)}{g(b)-g(a)}\right)^{\tau+2}+\left(\tfrac{g(b)-g(\frac{a+b}{2})}{g(b)-g(a)}\right)^{\tau+2}\right]$$
$$-\tfrac{\tau+1}{\tau+2}\left[\left(\tfrac{g(\frac{a+b}{2})-g(a)}{g(b)-g(a)}\right)^{\tau+2}+\left(\tfrac{g(b)-g(\frac{a+b}{2})}{g(b)-g(a)}\right)^{\tau+2}\right]$$
$$-\left(\tfrac{g(\frac{a+b}{2})-g(a)}{g(b)-g(a)}\right)^{\tau+1}\left(\tfrac{g(b)-g(\frac{a+b}{2})}{g(b)-g(a)}\right)+\left(\tfrac{g(\frac{a+b}{2})-g(a)}{g(b)-g(a)}\right)\left(\tfrac{g(b)-g(\frac{a+b}{2})}{g(b)-g(a)}\right)^{\tau+1}.$$

Proof. Using Lemma 8.3, Hölder inequality, (8.18) and convexity of $|f'(g)|^q$ respectively, we get

$$\begin{aligned}&\left|\left(\frac{f(g(a))+f(g(b))}{2}\right)\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+}h\right)(b;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,b^-}h\right)(a;p)\right]\right.\\&\left.-\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+}(f\circ g)h\right)(b;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,b^-}(f\circ g)h\right)(a;p)\right]\right|\\&\le\left[\left|\int_a^b\left(\int_t^{a+b-t}(g(b)-g(s))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(w(g(b)-g(s))^{\sigma})\right)h(s)d(g(s))\right|d(g(t))\right]^{1-\frac{1}{q}}\\&\left[\left|\int_a^b\left(\int_t^{a+b-t}(g(b)-g(s))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(w(g(b)-g(s))^{\sigma})\right)h(s)d(g(s))\right||f'(g(t))|^q d(g(t))\right]^{\frac{1}{q}}.\end{aligned}\tag{8.21}$$

Since $|f'(g)|^q$ is convex on $[a,b]$, we have

$$|f'(g(t))|^q\le\frac{g(b)-g(t)}{g(b)-g(a)}|f'(g(a))|^q+\frac{g(t)-g(a)}{g(b)-g(a)}|f'(g(b))|^q.\tag{8.22}$$

Using $\| h\|_\infty=\sup_{t\in[a,b]}|h(t)|$, and absolute convergence of Mittag-Leffler function, inequality (8.21) becomes

$$\begin{aligned}&\left|\frac{f(g(a))+f(g(b))}{2}\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+}h\right)(b;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,b^-}h\right)(a;p)\right]\right.\\&\left.-\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+}(f\circ g)h\right)(b;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,b^-}(f\circ g)h\right)(a;p)\right]\right|\\&\le\frac{\| h\|_\infty M}{\tau}\left[\int_a^{\frac{a+b}{2}}\{(g(b)-g(t))^{\tau}-(g(t)-g(b))^{\tau}\}d(g(t))\right.\\&\left.+\int_{\frac{a+b}{2}}^b\{(g(t)-g(a))^{\tau}-(g(b)-g(t))^{\tau}\}d(g(t))\right]^{1-\frac{1}{q}}\\&\times\left[\int_a^{\frac{a+b}{2}}\{(g(b)-g(t))^{\tau}-(g(t)-g(b))^{\tau}\}\right.\\&\times\left(\frac{g(b)-g(t)}{g(b)-g(a)}|f'(g(a))|^q+\frac{g(t)-g(a)}{g(b)-g(a)}|f'(g(b))|^q\right)d(g(t))\end{aligned}$$

$$+\int_{\frac{a+b}{2}}^{b}\{(g(t)-g(a))^{\tau}-(g(b)-g(t))^{\tau}\}$$
$$\times\left(\frac{g(b)-g(t)}{g(b)-g(a)}|f'(g(a))|^{q}+\frac{g(t)-g(a)}{g(b)-g(a)}|f'(g(b))|^{q}\right)d(g(t))\Bigg]^{\frac{1}{q}}.$$

After integrating and simplifying above inequality, we get (8.20). □

Remark 8.4

(i) In Theorem 8.4 if we put $g=I$, we get [33, Theorem 2.5].

(ii) In Theorem 8.4 if we put $p=0$ and $g=I$, we get [1, Theorem 2.6].

(iii) In Theorem 8.4 if we put $w=p=0$ and $g=I$, we get [94, Theorem 2.8].

8.2 Error Bounds Associated with Fractional Integral Inequalities for m-convex Functions

The following lemma is needed to prove results of this section.

Lemma 8.4 *Let $f:[a,mb]\to\mathbb{R}$ be a differentiable function such that $f'\in L_1[a,mb]$ with $0\leq a<mb$. Also let $g:[a,mb]\to\mathbb{R}$ be a continuous function on $[a,mb]$, then the following identity for extended generalized fractional integral operators holds*

$$\left(\int_{a}^{mb}g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^{\rho};p)ds\right)^{\sigma}[f(a)+f(mb)] \tag{8.23}$$
$$-\sigma\int_{a}^{mb}\left(\int_{a}^{t}g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^{\rho};p)ds\right)^{\sigma-1}g(t)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^{\rho};p)f(t)dt$$
$$-\sigma\int_{a}^{mb}\left(\int_{t}^{mb}g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^{\rho};p)ds\right)^{\sigma-1}g(t)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^{\rho};p)f(t)dt$$
$$=\int_{a}^{mb}\left(\int_{a}^{t}g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^{\rho};p)ds\right)^{\sigma}f'(t)dt-\int_{a}^{mb}\left(\int_{t}^{mb}g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^{\rho};p)ds\right)^{\sigma}f'(t)dt.$$

Proof. Proof is similar to the proof of Lemma 8.1. □

Theorem 8.5 *Let $w\in\mathbb{R}$, $r,\rho,\sigma,\tau>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $f:[a,mb]\to\mathbb{R}$ be a differentiable function such that $f'\in L_1[a,mb]$ with $0\leq a<mb$. Also let $g:[a,mb]\to\mathbb{R}$ be a continuous function on $[a,mb]$. If $|f'|$ is m-convex function on $[a,mb]$, then the following inequality for extended generalized fractional integral operators holds*

$$\left|\left(\int_a^{mb} g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^\rho;p)ds\right)^\sigma (f(a)+f(mb))\right. \tag{8.24}$$
$$-\sigma\int_a^{mb}\left(\int_a^t g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^\rho;p)ds\right)^{\sigma-1} g(t)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^\rho;p)f(t)dt$$
$$\left.-\sigma\int_a^{mb}\left(\int_t^{mb} g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^\rho;p)ds\right)^{\sigma-1} g(t)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^\rho;p)f(t)dt\right|$$
$$\leq \frac{(mb-a)^{\sigma+1}\|g\|_\infty^\sigma M^\sigma}{(\sigma+1)}(|f'(a)|+m|f'(b)|)$$

for $q<r+\Re(\rho)$ *and* $\| g \|_\infty = \sup_{t\in[a,mb]}|g(t)|$.

Proof. From Lemma 2.22, we have

$$\left|\left(\int_a^{mb} g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^\rho;p)ds\right)^\sigma (f(a)+f(mb))\right. \tag{8.25}$$
$$-\sigma\int_a^{mb}\left(\int_a^t g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^\rho;p)ds\right)^{\sigma-1} g(t)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^\rho;p)f(t)dt$$
$$\left.-\sigma\int_a^{mb}\left(\int_t^{mb} g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^\rho;p)ds\right)^{\sigma-1} g(t)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^\rho;p)f(t)dt\right|$$
$$\leq \int_a^{mb}\left|\int_a^t g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^\rho;p)ds\right|^\sigma |f'(t)|dt+\int_a^{mb}\left|\int_t^{mb} g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^\rho;p)ds\right|^\sigma |f'(t)|dt.$$

Using absolute convergence of Mittag-Leffler function and $\| g \|_\infty = \sup_{t\in[a,b]}|g(t)|$, we have

$$\left|\left(\int_a^{mb} g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^\rho;p)ds\right)^\sigma (f(a)+f(mb))\right. \tag{8.26}$$
$$-\sigma\int_a^{mb}\left(\int_a^t g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^\rho;p)ds\right)^{\sigma-1} g(t)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^\rho;p)f(t)dt$$
$$\left.-\sigma\int_a^{mb}\left(\int_t^{mb} g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^\rho;p)ds\right)^{\sigma-1} g(t)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^\rho;p)f(t)dt\right|$$
$$\leq \|g\|_\infty^\sigma M^\sigma\left(\int_a^{mb}(t-a)^\sigma|f'(t)|dt+\int_a^{mb}(mb-t)^\sigma|f'(t)|dt\right).$$

As $|f'|$ is convex function, for $t\in[a,mb]$ we have

$$|f'(t)|\leq \frac{mb-t}{mb-a}|f'(a)|+m\frac{t-a}{mb-a}|f'(b)|. \tag{8.27}$$

Using (8.27) in (8.26), we have

$$\begin{aligned}
&\left|\left(\int_a^{mb} g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^\rho;p)ds\right)^\sigma (f(a)+f(mb))\right. \\
&-\sigma\int_a^{mb}\left(\int_a^t g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^\rho;p)ds\right)^{\sigma-1} g(t)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^\rho;p)f(t)dt \\
&\left.-\sigma\int_a^{mb}\left(\int_t^{mb} g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^\rho;p)ds\right)^{\sigma-1} g(t)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^\rho;p)f(t)dt\right| \\
&\leq \|g\|_\infty^\sigma M^\sigma\left(\int_a^{mb}(t-a)^\sigma\left(\frac{mb-t}{mb-a}|f'(a)|+m\frac{t-a}{mb-a}|f'(b)|\right)dt\right. \\
&\left.+\int_a^{mb}(mb-t)^\sigma\left(\frac{mb-t}{mb-a}|f'(a)|+m\frac{t-a}{mb-a}|f'(b)|\right)dt\right).
\end{aligned} \tag{8.28}$$

After simple calculation of above inequality we get (8.24) which is required. □

Remark 8.5

(i) If $p=0$ in Theorem 8.5, then we get [53, Theorem 3.2]. For $m=1$, Theorem 8.1 is obtained. Also for $m=1$.

(ii) If $w=p=0$, then we get [137, Theorem 6].

(iii) If $g(s)=1$ along with $w=p=0$ and $\sigma=\mu$, then we get [137, Corollary 2].

Next we give another fractional integral inequality.

Theorem 8.6 *Let $w\in\mathbb{R}$, $r,\rho,\sigma,\tau>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $f:[a,mb]\to\mathbb{R}$ be a differentiable function such that $f\in L_1[a,mb]$ with $0\leq a<mb$. Also let $g:[a,mb]\to\mathbb{R}$ be a continuous function on $[a,mb]$. If $|f'|^q$ is convex function on $[a,mb]$, then for $q>0$ the following inequality for extended generalized fractional integral operators holds*

$$\begin{aligned}
&\left|\left(\int_a^{mb} g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^\rho;p)ds\right)^\sigma (f(a)+f(mb))\right. \\
&-\sigma\int_a^{mb}\left(\int_a^t g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^\rho;p)ds\right)^{\sigma-1} g(t)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^\rho;p)f(t)dt \\
&\left.-\sigma\int_a^{mb}\left(\int_t^{mb} g(s)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(ws^\rho;p)ds\right)^{\sigma-1} g(t)E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^\rho;p)f(t)dt\right| \\
&\leq \frac{2(mb-a)^{\sigma+1}\|g\|_\infty^\sigma M^\sigma}{(\sigma p+1)^{\frac{1}{q}}}\left(\frac{|f'(a)|^q+m|f'(b)|^q}{2}\right)^{\frac{1}{q}}
\end{aligned} \tag{8.29}$$

for $q<r+\Re(\rho)$ and $\| g \|_\infty=\sup_{t\in[a,mb]}|g(t)|$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.22, we have

$$\begin{aligned}&\Bigg|\left(\int_a^{mb} g(s)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(ws^{\rho};p)ds\right)^{\sigma}(f(a)+f(mb)) \\ &-\sigma\int_a^{mb}\left(\int_a^{t} g(s)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(ws^{\rho};p)ds\right)^{\sigma-1} g(t)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f(t)dt \\ &-\sigma\int_a^{mb}\left(\int_t^{mb} g(s)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(ws^{\rho};p)ds\right)^{\sigma-1} g(t)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f(t)dt\Bigg| \\ &\leq \int_a^{mb}\left|\int_a^{t} g(s)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(ws^{\rho};p)ds\right|^{\sigma}|f'(t)|dt \\ &+\int_a^{mb}\left|\int_t^{mb} g(s)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(ws^{\rho};p)ds\right|^{\sigma}|f'(t)|dt.\end{aligned} \tag{8.30}$$

By using Hölder inequality we have

$$\begin{aligned}&\Bigg|\left(\int_a^{mb} g(s)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(ws^{\rho};p)ds\right)^{\sigma}(f(a)+f(mb)) \\ &-\sigma\int_a^{mb}\left(\int_a^{t} g(s)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(ws^{\rho};p)ds\right)^{\sigma-1} g(t)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f(t)dt \\ &-\sigma\int_a^{mb}\left(\int_t^{mb} g(s)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(ws^{\rho};p)ds\right)^{\sigma-1} g(t)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f(t)dt\Bigg| \\ &\leq \left(\int_a^{mb}\left|\int_a^{t} g(s)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(ws^{\rho};p)ds\right|^{\sigma p}dt\right)^{\frac{1}{p}}\left(\int_a^{mb}|f'(t)|^q dt\right)^{\frac{1}{q}} \\ &+\left(\int_a^{mb}\left|\int_t^{mb} g(s)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(ws^{\rho};p)ds\right|^{\sigma p}dt\right)^{\frac{1}{p}}\left(\int_a^{mb}|f'(t)|^q dt\right)^{\frac{1}{q}}.\end{aligned} \tag{8.31}$$

Using absolute convergence of Mittag-Leffler function and $\| g \|_{\infty}= \sup_{t\in[a,b]}|g(t)|$, we have

$$\begin{aligned}&\Bigg|\left(\int_a^{mb} g(s)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(ws^{\rho};p)ds\right)^{\sigma}(f(a)+f(mb)) \\ &-\sigma\int_a^{mb}\left(\int_a^{t} g(s)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(ws^{\rho};p)ds\right)^{\sigma-1} g(t)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f(t)dt \\ &-\sigma\int_a^{mb}\left(\int_t^{mb} g(s)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(ws^{\rho};p)ds\right)^{\sigma-1} g(t)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f(t)dt\Bigg| \\ &\leq \|g\|_{\infty}^{\sigma}M^{\sigma}\left(\left(\int_a^{mb}|t-a|^{\sigma p}dt\right)^{\frac{1}{p}}+\left(\int_a^{mb}|mb-t|^{\sigma p}dt\right)^{\frac{1}{p}}\right)\left(\int_a^{mb}|f'(t)|^q dt\right)^{\frac{1}{q}}.\end{aligned} \tag{8.32}$$

As $|f'(t)|^q$ is m-convex, for $t\in[a,mb]$ we have

$$|f'(t)|^q \leq \frac{mb-t}{mb-a}|f'(a)|^q + m\frac{t-a}{mb-a}|f'(b)|^q. \tag{8.33}$$

Using (8.33) in (8.32), we have

$$\begin{aligned}
&\Bigg|\left(\int_a^{mb} g(s)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(ws^\rho;p)ds\right)^\sigma (f(a)+f(mb)) \qquad (8.34)\\
&-\sigma\int_a^{mb}\left(\int_a^t g(s)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(ws^\rho;p)ds\right)^{\sigma-1} g(t)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)f(t)dt\\
&-\sigma\int_a^{mb}\left(\int_t^{mb} g(s)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(ws^\rho;p)ds\right)^{\sigma-1} g(t)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)f(t)dt\Bigg|\\
&\leq \|g\|_\infty^\sigma M^\sigma\left(\left(\int_a^{mb}|t-a|^{\sigma p}dt\right)^{\frac{1}{p}}+\left(\int_a^{mb}|mb-t|^{\sigma p}dt\right)^{\frac{1}{p}}\right)\\
&\times\left(\int_a^{mb}\frac{mb-t}{mb-a}|f'(a)|^q+m\frac{t-a}{mb-a}|f'(b)|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

After simple calculation of above inequality, the inequality (8.29) which is obtained. □

Remark 8.6

(i) If $p=0$ in Theorem 8.6, then we get [53, Theorem 3.6]. For $m=1$, we get Theorem 8.2. Also for $m=1$.

(ii) If $w=p=0$, then we get [137, Theorem 7].

(iii) If $g(s)=1$ and $w=p=0$, then we get [42, Theorem 2.3].

(iv) If $w=p=0$ and $\sigma=1$, then we get [42, Corollary 3].

To find error estimates first we prove the following two lemmas.

Lemma 8.5 *Let $f,g:[a,mb]\to\mathbb{R}$, $0<a<mb$, Range $(g)\subset[a,mb]$ be the functions such that f be positive and $f\circ g\in L_1[a,mb]$ and g be a differentiable and strictly increasing. Also if $f(g(x))=f(g(a)+g(mb)-g(x))$, then the following equality for fractional operators (2.23) and (2.24) holds:*

$$\begin{aligned}
&\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+}f\circ g\right)(mb;p)=\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,mb_-}f\circ g\right)(a;p)\\
&=\frac{\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+}f\circ g\right)(mb;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,mb_-}f\circ g\right)(a;p)}{2}. \qquad (8.35)
\end{aligned}$$

Proof. By the definition of generalized fractional integral operator containing extended Mittag-Leffler function by a monotone function, we have

$$\begin{aligned}
&\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+}f\circ g\right)(mb;p) \qquad (8.36)\\
&=\int_a^{mb}(g(mb)-g(x))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(w(g(mb)-g(x))^\sigma;p)f(g(x))d(g(x)),
\end{aligned}$$

Replacing $g(x)$ by $g(a)+g(mb)-g(x)$ in (8.36) and using $f(g(x)) = f(g(a)+g(mb)-g(x))$, we have

$$\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+} f\circ g\right)(mb;p) = \int_a^{mb} (g(x)-g(a))^{\tau-1} E^{\rho,r,k,c}_{\sigma,\tau,\delta}(w(g(x)-g(a))^{\sigma};p) f(g(x))d(g(x)).$$

This implies

$$\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+} f\circ g\right)(mb;p) = \left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,mb_-} f\circ g\right)(a;p). \tag{8.37}$$

By adding (8.36) and (8.53), we get (8.35). □

Lemma 8.6 *Let $f,g,h : [a,mb] \to \mathbb{R}$, $0 < a < mb$, Range (g), Range $(h) \subset [a,mb]$ be the functions such that f be positive and $f\circ g, h\circ g \in L_1[a,mb]$, g be a differentiable and strictly increasing and h be nonnegative and continuous. If $f'\circ g \in L_1[a,mb]$ and $h(g(t)) = h(g(a)+g(mb)-g(t))$, then the following equality for the generalized fractional integral operators (2.23) and (2.24) holds:*

$$\begin{aligned} &\frac{f(g(a))+f(g(mb))}{2}\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+} h\circ g\right)(mb;p) + \left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,mb_-} h\circ g\right)(a;p)\right] \\ &-\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+} (f\circ g)(h\circ g)\right)(mb;p) + \left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,mb_-} (f\circ g)(h\circ g)\right)(a;p)\right] \\ &= \int_a^{mb}\left[\int_a^t (g(mb)-g(s))^{\tau-1} E^{\rho,r,k,c}_{\sigma,\tau,\delta}(w(g(mb)-g(s))^{\sigma};p)h(g(s))d(g(s)) \right. \\ &\left. - \int_t^{mb} (g(s)-g(a))^{\tau-1} E^{\rho,r,k,c}_{\sigma,\tau,\delta}(w(g(s)-g(a))^{\sigma};p)h(g(s))d(g(s))\right] f'(g(t))d(g(t)). \end{aligned} \tag{8.38}$$

Proof. To prove the lemma, we have

$$\begin{aligned} &\int_a^{mb}\left[\int_a^t (g(mb)-g(s))^{\tau-1} E^{\rho,r,k,c}_{\sigma,\tau,\delta}(w(g(mb)-g(s))^{\sigma};p)h(g(s))d(g(s))\right] f'(g(t))d(g(t)) \\ &= f(g(mb))\int_a^{mb} (g(mb)-g(s))^{\tau-1} E^{\rho,r,k,c}_{\sigma,\tau,\delta}(w(g(mb)-g(s))^{\sigma};p)\, h(g(s))d(g(s)) \\ &- \int_a^{mb}\left((g(mb)-g(t))^{\tau-1} E^{\rho,r,k,c}_{\sigma,\tau,\delta}(w(g(mb)-g(t))^{\sigma};p)\right) f(g(t))h(g(t))d(g(t)) \\ &= f(g(mb))\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+} h\circ g\right)(mb;p) - \left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+} (f\circ g)(h\circ g)\right)(mb;p). \end{aligned}$$

By using Lemma 8.5, we have

$$\begin{aligned} &\int_a^{mb}\left[\int_a^t (g(mb)-g(s))^{\tau-1} E^{\rho,r,k,c}_{\sigma,\tau,\delta}(w(g(mb)-g(s))^{\sigma};p)h(g(s))d(g(s))\right] f'(g(t))d(g(t)) \\ &= \frac{f(g(mb))}{2}\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+} h\circ g\right)(mb;p) + \left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,mb_-} h\circ g\right)(a;p)\right] \\ &\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+} (f\circ g)(h\circ g)\right)(mb;p). \end{aligned}$$

In the same way we have

$$\int_a^{mb}\left[-\int_t^{mb}(g(s)-g(a))^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(w(g(s)-g(a))^{\sigma};p)h(g(s))d(g(s))\right]f'(g(t))d(g(t))$$
$$=\frac{f(g(a))}{2}\left[\left({}_g\Upsilon_{\sigma,\tau,\delta,w,a^+}^{\rho,r,k,c}h\circ g\right)(mb;p)+\left({}_g\Upsilon_{\sigma,\tau,\delta,w,mb_-}^{\rho,r,k,c}h\circ g\right)(a;p)\right]$$
$$\left({}_g\Upsilon_{\sigma,\tau,\delta,w,mb_-}^{\rho,r,k,c}(f\circ g)(h\circ g)\right)(a;p).$$

By adding (8.39) and (8.39), we get (8.38). □

By using Lemma 8.6, we prove the following theorem.

Theorem 8.7 *Let $w\in\mathbb{R}$, $r,\rho,\sigma,\tau>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $f,g,h:[a,mb]\to\mathbb{R}$, $0<a<mb$, Range (g), Range $(h)\subset[a,mb]$ be the functions such that f be positive and $(f\circ g)'\in L_1[a,mb]$, where g be a differentiable and strictly increasing and h be nonnegative and continuous. Also let $h(g(t))=h(g(a)+g(mb)-g(t))$ and $|(f\circ g)'|$ is m-convex on $[a,b]$. Then for $k<r+\Re(\sigma)$, the following inequality for fractional integral operators (2.23) and (2.24) holds:*

$$\Bigg|\frac{f(g(a))+f(g(mb))}{2}\left[\left({}_g\Upsilon_{\sigma,\tau,\delta,w,a^+}^{\rho,r,k,c}h\circ g\right)(mb;p)+\left({}_g\Upsilon_{\sigma,\tau,\delta,w,mb_-}^{\rho,r,k,c}h\circ g\right)(a;p)\right] \tag{8.39}$$
$$-\left[\left({}_g\Upsilon_{\sigma,\tau,\delta,w,a^+}^{\rho,r,k,c}(f\circ g)(h\circ g)\right)(mb;p)+\left({}_g\Upsilon_{\sigma,\tau,\delta,w,mb_-}^{\rho,r,k,c}(f\circ g)(h\circ g)\right)(a;p)\right]\Bigg|$$
$$\leq\frac{\|h\|_\infty M(g(mb)-g(a))^{\tau+1}}{\tau(\tau+1)}(1-\Omega)\left[|f'(g(a))+mf'(g(b))|\right],$$

where $\|h\|_\infty=\sup_{t\in[a,mb]}|h(t)|$ *and*

$$\Omega=\frac{1}{\tau+2}\left[\left\{\left(\frac{g(\frac{a+mb}{2})-g(a)}{g(mb)-g(a)}\right)^{\tau+2}\right\}+\left\{\left(\frac{g(mb)-g(\frac{a+mb}{2})}{g(mb)-g(a)}\right)^{\tau+2}\right\}\right]$$
$$-\frac{\tau+1}{\tau+2}\left[\left\{\left(\frac{g(\frac{a+mb}{2})-g(a)}{g(mb)-g(a)}\right)^{\tau+2}\right\}+\left\{\left(\frac{g(mb)-g(\frac{a+mb}{2})}{g(mb)-g(a)}\right)^{\tau+2}\right\}\right]$$
$$-\left(\frac{g(\frac{a+mb}{2})-g(a)}{g(mb)-g(a)}\right)^{\tau+1}\left(\frac{g(mb)-g(\frac{a+mb}{2})}{g(mb)-g(a)}\right)+\left(\frac{g(\frac{a+mb}{2})-g(a)}{g(mb)-g(a)}\right)\left(\frac{g(mb)-g(\frac{a+mb}{2})}{g(mb)-g(a)}\right)^{\tau+1}.$$

Proof. Using Lemma 8.6, we have

$$\Bigg|\left(\frac{f(g(a))+f(g(mb))}{2}\right)\left[\left({}_g\Upsilon_{\sigma,\tau,\delta,w,a^+}^{\rho,r,k,c}h\circ g\right)(mb;p)+\left({}_g\Upsilon_{\sigma,\tau,\delta,w,mb_-}^{\rho,r,k,c}h\circ g\right)(a;p)\right]$$
$$-\left[\left({}_g\Upsilon_{\sigma,\tau,\delta,w,a^+}^{\rho,r,k,c}(f\circ g)(h\circ g)\right)(mb;p)+\left({}_g\Upsilon_{\sigma,\tau,\delta,w,mb_-}^{\rho,r,k,c}(f\circ g)(h\circ g)\right)(a;p)\right]\Bigg|$$
$$\leq\int_a^{mb}\Bigg|\Bigg[\int_a^t(g(mb)-g(s))^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(w(g(mb)-g(s))^{\sigma};p)h(g(s))d(g(s))$$
$$-\int_t^{mb}(g(s)-g(a))^{\tau-1}E_{\sigma,\tau,\delta}^{\rho,r,k,c}(w(g(s)-g(a))^{\sigma};p)h(g(s))d(g(s))\Bigg]\Bigg||f'(g(t))|d(g(t)). \tag{8.40}$$

Using the m-convexity of $|(f\circ g)'|$ on $[a,b]$, we have

$$|f'(g(t))| \leq \frac{g(mb)-g(t)}{g(mb)-g(a)}|f'(g(a))| + m\frac{g(t)-g(a)}{g(mb)-g(a)}|f'(g(b))|, t\in[a,b]. \tag{8.41}$$

If we replace $g(s)$ by $g(a)+g(mb)-g(s)$ and using $h(g(s)) = h(g(a)+g(mb)-g(s))$, $t' = g^{-1}(g(a)+g(mb)-g(t))$, in second integral in the followings, we get

$$\begin{aligned}
&\left|\int_a^t (g(mb)-g(s))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(w(g(mb)-g(s))^{\sigma};p)h(g(s))d(g(s))\right.\\
&\left.-\int_t^{mb}(g(s)-g(a))^{\tau-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(s)-g(a))^{\sigma};p)h(g(s))d(g(s))\right|\\
=&\left|-\int_t^a (g(mb)-g(s))^{\tau-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(mb)-g(s))^{\sigma};p)h(g(s))d(g(s))\right.\\
&\left.-\int_a^{t'}(g(mb)-g(s))^{\tau-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(mb)-g(s))^{\sigma};p)h(g(s))d(g(s))\right|\\
=&\left|\int_t^{t'}(g(mb)-g(s))^{\tau-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(mb)-g(s))^{\sigma};p)h(g(s))d(g(s))\right|
\end{aligned}$$

$$\leq \begin{cases} \int_t^{t'}|(g(mb)-g(s))^{\tau-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(mb)-g(s))^{\sigma};p)h(g(s))|d(g(s)), & t\in[a,\frac{a+mb}{2}] \\ \int_{t'}^{t}|(g(mb)-g(s))^{\tau-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(mb)-g(s))^{\sigma};p)h(g(s))|d(g(s)), & t\in[\frac{a+mb}{2},mb]. \end{cases} \tag{8.42}$$

By (8.40), (8.41), (8.42) and using absolute convergence of extended Mittag-Leffler function, we have

$$\begin{aligned}
&\left|\frac{f(g(a))+f(g(mb))}{2}\left(\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+}h\circ g\right)(mb;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,mb_-}h\circ g\right)(a;p)\right)\right.\\
&\left.-\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+}(f\circ g)(h\circ g)\right)(mb;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,mb_-}(f\circ g)(h\circ g)\right)(a;p)\right]\right|\\
&\leq \int_a^{\frac{a+mb}{2}}\left(\int_a^{a+mb-t}|(g(mb)-g(s))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(w(g(mb)-g(s))^{\sigma};p)h(g(s))|d(g(s))\right)\\
&\times\left(\frac{g(mb)-g(t)}{g(mb)-g(a)}|f'(g(a))|+m\frac{g(t)-g(a)}{g(mb)-g(a)}|f'(g(b))|\right)d(g(t))\\
&+\int_{\frac{a+mb}{2}}^{mb}\left(\int_{a+mb-t}^{t}|(g(mb)-g(s))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(w(g(mb)-g(s))^{\sigma};p)h(g(s))|d(g(s))\right)\\
&\times\left(\frac{g(mb)-g(t)}{g(mb)-g(a)}|f'(g(a))|+m\frac{g(t)-g(a)}{g(mb)-g(a)}|f'(g(b))|\right)d(g(t)).\\
&\leq \frac{\|h\|_\infty M}{\tau(g(mb)-g(a))}\left[\int_a^{\frac{a+mb}{2}}((g(mb)-g(t))^{\tau}-(g(t)-g(a))^{\tau})(g(mb)-g(t))|f'(g(a))|d(g(t))\right.\\
&+m\int_a^{\frac{a+mb}{2}}((g(mb)-g(t))^{\tau}-(g(t)-g(a))^{\tau})m(g(t)-g(a))|f'(g(b))|d(g(t))
\end{aligned}$$

$$+\int_{\frac{a+mb}{2}}^{mb}((g(t)-g(a))^{\tau}-(g(mb)-g(t))^{\tau})(g(mb)-g(t))|f'(g(a))|d(g(t))$$
$$+m\int_{\frac{a+mb}{2}}^{mb}((g(t)-g(a))^{\tau}-(g(mb)-g(t))^{\tau})m(g(t)-g(a))|f'(g(b))|d(g(t))\Bigg]. \quad (8.43)$$

After some calculations, we get

$$\int_{a}^{\frac{a+mb}{2}}((g(mb)-g(t))^{\tau}-(g(t)-g(a))^{\tau})(g(mb)-g(t))d(g(t))$$
$$=\int_{\frac{a+mb}{2}}^{mb}((g(t)-g(a))^{\tau}-(g(mb)-g(t))^{\tau})(g(t)-g(a))d(g(t))$$
$$=\frac{(g(mb)-g(a))^{\tau+2}}{\tau+2}-\frac{\left(g(mb)-g(\frac{a+mb}{2})\right)^{\tau+2}}{\tau+2}-\frac{\left(g(\frac{a+mb}{2})-g(a)\right)^{\tau+1}}{\tau+1}$$
$$\left(g(mb)-g(\frac{a+mb}{2})\right)-\frac{\left(g(\frac{a+mb}{2})-g(a)\right)^{\tau+2}}{(\tau+1)(\tau+2)},$$

and

$$\int_{a}^{\frac{a+mb}{2}}((g(mb)-g(t))^{\tau}-(g(t)-g(a))^{\tau})(g(t)-g(a))d(g(t))$$
$$=\int_{\frac{a+mb}{2}}^{mb}((g(t)-g(a))^{\tau}-(g(mb)-g(t))^{\tau})(g(mb)-g(t))d(g(t))$$
$$=-\frac{\left(g(\frac{a+mb}{2})-g(a)\right)^{\tau+1}}{\tau+1}\left(g(mb)-g(\frac{a+mb}{2})\right)+\frac{(g(mb)-g(a))^{\tau+2}}{(\tau+1)(\tau+2)}$$
$$-\frac{\left(g(\frac{a+mb}{2})-g(a)\right)^{\tau+2}}{(\tau+1)(\tau+2)}-\frac{\left(g(mb)-g(\frac{a+mb}{2})\right)^{\tau+2}}{\tau+2}.$$

Using the above evaluations of integrals in (8.43), we get the required inequality (8.39).

□

Remark 8.7

(i) In Theorem 8.7, if we put $m=1$, we get [132, Theorem]

(ii) In Theorem 8.7, if we put $g=I$ and $p=0$, we get [3, Theorem 2.3].

(iii) In Theorem 8.7, if we put $g=I$, $p=0$ and $m=1$, we get [1, Theorem 2.3].

(iv) In Theorem 8.7, if we put $g=I$, we get [57, Theorem].

(v) In Theorem 8.7, if we put $g=I$, $m=1$, we get [33, Theorem 2.3].

(vi) In Theorem 8.7, for $w=p=0$, $g=I$ and $h=1$ along with $\tau=m=1$, we get [42, Theorem 2.2].

(vii) In Theorem 8.7, if we put $w=p=0$, $g=I$ and $h=1$ with $m=1$, then we get [140, Theorem 3].

Theorem 8.8 *Let $w \in \mathbb{R}$, $r,\rho,\sigma,\tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $f,g,h : [a,mb] \to \mathbb{R}$, $0 < a < mb$, Range (g), Range $(h) \subset [a,mb]$ be the functions such that f be positive, $(f \circ g)' \in L_1[a,mb]$, g be a differentiable and strictly increasing and h be continuous. Also let $h(g(t)) = h(g(a)+g(mb)-g(t))$ and $|(f\circ g)'|^{q_1}$, $q_1 \geq 1$ is m-convex. Then for $k < r+\Re(\sigma)$, the following inequality for fractional integral operators (2.23) and (2.24) holds:*

$$\begin{aligned}&\Bigg|\left(\frac{f(g(a))+f(g(mb))}{2}\right)\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+}h\circ g\right)(mb;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,mb_-}h\circ g\right)(a;p)\right]\\&-\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+}(f\circ g)(h\circ g)\right)(mb;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,mb_-}(f\circ g)(h\circ g)\right)(a;p)\right]\Bigg|\\&\leq \frac{\| h\|_\infty M(g(mb)-g(a))^{\tau+1}}{\tau(\tau+1)}\left((1-\Psi)^{\frac{1}{p_1}}(1-\Omega)^{\frac{1}{q_1}}\right)\left(\frac{|f'(g(a))|^{q_1}+m|f'(g(b))|^{q_1}}{2}\right)^{\frac{1}{q_1}},\end{aligned} \tag{8.44}$$

where $\| h \|_\infty = \sup_{t\in[a,mb]} |h(t)|$, $\frac{1}{p_1}+\frac{1}{q_1}=1$,

$$\Psi = \left(\frac{g(mb)-g(\frac{a+mb}{2})}{g(mb)-g(a)}\right)^{\tau+1}+\left(\frac{g(\frac{a+mb}{2})-g(a)}{g(mb)-g(a)}\right)^{\tau+1} \text{ and}$$

$$\begin{aligned}\Omega =& \frac{1}{\tau+2}\left[\left\{\left(\frac{g(\frac{a+mb}{2})-g(a)}{g(mb)-g(a)}\right)^{\tau+2}\right\}+\left\{\left(\frac{g(mb)-g(\frac{a+mb}{2})}{g(mb)-g(a)}\right)^{\tau+2}\right\}\right]\\&-\frac{\tau+1}{\tau+2}\left[\left\{\left(\frac{g(\frac{a+mb}{2})-g(a)}{g(mb)-g(a)}\right)^{\tau+2}\right\}+\left\{\left(\frac{g(mb)-g(\frac{a+mb}{2})}{g(mb)-g(a)}\right)^{\tau+2}\right\}\right]\\&-\left(\frac{g(\frac{a+mb}{2})-g(a)}{g(mb)-g(a)}\right)^{\tau+1}\left(\frac{g(mb)-g(\frac{a+mb}{2})}{g(mb)-g(a)}\right)+\left(\frac{g(\frac{a+mb}{2})-g(a)}{g(mb)-g(a)}\right)\left(\frac{g(mb)-g(\frac{a+mb}{2})}{g(mb)-g(a)}\right)^{\tau+1}.\end{aligned}$$

Proof. Using Lemma 8.6, power mean inequality, (8.42) and m-convexity of $|(f\circ g)'|^{q_1}$ respectively, we have

$$\begin{aligned}&\Bigg|\left(\frac{f(g(a))+f(g(mb))}{2}\right)\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+}h\circ g\right)(mb;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,mb_-}h\circ g\right)(a;p)\right]\\&-\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+}(f\circ g)(h\circ g)\right)(mb;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,mb_-}(f\circ g)(h\circ g)\right)(a;p)\right]\Bigg|\\&\leq\left[\int_a^{mb}\left|\int_t^{a+mb-t}(g(mb)-g(s))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(w(g(mb)-g(s))^\sigma;p)h(g(s))d(g(s))\right|d(g(t))\right]^{1-\frac{1}{q_1}}\\&\left[\int_a^{mb}\left|\int_t^{a+mb-t}(g(mb)-g(s))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(w(g(mb)-g(s))^\sigma;p)h(g(s))d(g(s))\right||f'(g(t))|^{q_1}\right]^{\frac{1}{q_1}}.\end{aligned} \tag{8.45}$$

Since $|(f\circ g)'|^{q_1}$ is m-convex, we have

$$|f'(g(t))|^{q_1} \leq \frac{g(mb)-g(t)}{g(mb)-g(a)}|f'(g(a))|^{q_1}+m\frac{g(t)-g(a)}{g(mb)-g(a)}|f'(g(b))|^{q_1}. \tag{8.46}$$

Using $\| h \|_\infty = \sup_{t\in[a,mb]}|h(t)|$, and absolute convergence of extended Mittag-Leffler function, inequality (8.45) becomes

$$\left|\left(\frac{f(g(a))+f(g(mb))}{2}\right)\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+}h\circ g\right)(mb;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,mb^-}h\circ g\right)(a;p)\right]\right.$$
$$\left.-\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,a^+}(f\circ g)(h\circ g)\right)(mb;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,w,mb^-}(f\circ g)(h\circ g)\right)(a;p)\right]\right|$$
$$\leq \| h \|_\infty^{1-\frac{1}{q_1}} M^{1-\frac{1}{q_1}}\left[\int_a^{\frac{a+mb}{2}}\left(\int_t^{a+mb-t}(g(mb)-g(s))^{\tau-1}d(g(s))\right)d(g(t))\right.$$
$$\left.+\int_{\frac{a+mb}{2}}^{b}\left(\int_{a+mb-t}^{t}(g(mb)-g(s))^{\tau-1}d(g(s))\right)d(g(t))\right]^{1-\frac{1}{q_1}}$$
$$\times \| h \|_\infty^{\frac{1}{q_1}} M^{\frac{1}{q_1}}\left[\int_a^{\frac{a+mb}{2}}\left(\int_t^{a+mb-t}(g(mb)-g(s))^{\tau-1}d(g(s))\right)\right.$$
$$\times\left(\frac{g(mb)-g(t)}{g(mb)-g(a)}|f'(g(a))|^{q_1}+m\frac{g(t)-g(a)}{g(mb)-g(a)}|f'(g(b))|^{q_1}\right)d(g(t))$$
$$+\int_{\frac{a+mb}{2}}^{b}\left(\int_{a+mb-t}^{t}(g(mb)-g(s))^{\tau-1}d(g(s))\right)$$
$$\left.\times\left(\frac{g(mb)-g(t)}{g(mb)-g(a)}|f'(g(a)))|^{q_1}+m\frac{g(t)-g(a)}{g(mb)-g(a)}|f'(g(b))|^{q_1}\right)d(g(t))\right]^{\frac{1}{q_1}}.$$

After integrating and simplifying above inequality, we get (8.44). □

Remark 8.8

(i) In Theorem 8.8, if we put $m=1$, we get [132, Theorem].

(ii) In Theorem 8.8, if we put $g=I$ and $p=0$, we get [3, Theorem 2.6].

(iii) In Theorem 8.8, if we put $g=I$, $p=0$ and $m=1$ we get [1, Theorem 2.6].

(iv) In Theorem 8.8, if we put $g=I$, we get [57, Theorem].

(v) In Theorem 8.8, if we put $g=I$,$m=1$ we get [33, Theorem 2.5].

8.3 Bounds of Fractional Integral Operators for $(h-m)$-convex Functions

In this section the bounds of extended generalized fractional integral operators for $(h-m)$-convex functions are given.

Theorem 8.9 *Let $w \in \mathbb{R}$, $r,\rho,\sigma,\tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $f : [a,b] \to \mathbb{R}$, $0 \leq a < mb$, be a real valued function. If f is positive and $(h-m)$-convex, then for $\sigma, \sigma' \geq 1$, the following inequality for extended generalized fractional integral operators holds*

$$\begin{aligned}
&\left(\varepsilon_{a^+,\rho,\sigma,\tau}^{w,\delta,c,q,r} f\right)(x;p) + \left(\varepsilon_{b^-,\rho,\sigma',\tau}^{w,\delta,c,q,r} f\right)(x;p) \qquad (8.47)\\
&\leq \Big((x-a)f(a)H_{a^+,\sigma-1}^{w}(x;p) + (b-x)f(b)H_{b^-,\sigma'-1}^{w}(x;p)\\
&+mf\left(\frac{x}{m}\right)\left((x-a)H_{a^+,\sigma-1}^{w}(x;p) + (b-x)H_{b^-,\sigma'-1}^{w}(x;p)\right)\Big)\int_0^1 h(z)dz.
\end{aligned}$$

Proof. Let $x \in [a,b]$. Then first we observe the function f on the interval $[a,x]$; for $t \in [a,x]$ and $\sigma \geq 1$, we have the following inequality:

$$(x-t)^{\sigma-1}E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w(x-t)^{\rho};p) \leq (x-a)^{\sigma-1}E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w(x-a)^{\rho};p). \qquad (8.48)$$

As f is $(h-m)$-convex, for $t \in [a,x]$, we have

$$f(t) \leq h\left(\frac{x-t}{x-a}\right)f(a) + mh\left(\frac{t-a}{x-a}\right)f\left(\frac{x}{m}\right). \qquad (8.49)$$

Multiplying (8.48) and (8.49), then integrating over $[a,x]$, we get

$$\begin{aligned}
&\int_a^x (x-t)^{\sigma-1}E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w(x-t)^{\rho};p)f(t)dt \qquad (8.50)\\
&\leq f(a)(x-a)^{\sigma-1}E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w(x-a)^{\rho};p)\int_a^x h\left(\frac{x-t}{x-a}\right)dt\\
&+mf\left(\frac{x}{m}\right)(x-a)^{\sigma-1}E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w(x-a)^{\rho};p)\int_a^x h\left(\frac{t-a}{x-a}\right)dt.
\end{aligned}$$

By using (2.2), we have

$$\left(\varepsilon_{a^+,\rho,\sigma,\tau}^{w,\delta,c,q,r} f\right)(x;p) \leq (x-a)H_{a^+,\sigma-1}^{w}(x;p)\left(f(a) + mf\left(\frac{x}{m}\right)\right)\int_0^1 h(z)dz. \qquad (8.51)$$

Now on the other hand we observe the function f on the interval $[x,b]$; for $t \in [x,b]$ and $\sigma' \geq 1$, we get the following inequality:

$$(t-x)^{\sigma'-1}E_{\rho,\sigma',\tau}^{\delta,c,q,r}(w(t-x)^{\rho};p) \leq (b-x)^{\sigma'-1}E_{\rho,\sigma',\tau}^{\delta,c,q,r}(w(b-x)^{\rho};p). \qquad (8.52)$$

Again from $(h-m)$-convexity of f for $t \in [x,b]$, we have

$$f(t) \leq h\left(\frac{t-x}{b-x}\right) f(b) + mh\left(\frac{b-t}{b-x}\right) f\left(\frac{x}{m}\right). \tag{8.53}$$

Similarly multiplying (8.52) and (8.53), then integrating over $[x,b]$, we get

$$\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau} f\right)(x;p) \leq (b-x) H^{w}_{b^-,\sigma'-1}(x;p)\left(f(b) + mf\left(\frac{x}{m}\right)\right)\int_0^1 h(z)dz. \tag{8.54}$$

Adding (8.51) and (8.54), inequality (8.47) is obtained. □

If $m=1$ and $h(z)=z$ in (8.47), then the following result holds for convex functions:

Corollary 8.3 *Let $f : [a,b] \to \mathbb{R}$, $a<b$, be a real valued function. If f is positive and convex, then for $\sigma, \sigma' \geq 1$, we have*

$$\begin{aligned}
&\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau} f\right)(x;p) + \left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau} f\right)(x;p) \\
&\leq \frac{(x-a)f(a)H^{w}_{a^+,\sigma-1}(x;p) + (b-x)f(b)H^{w}_{b^-,\sigma'-1}(x;p)}{2} \\
&+ f(x)\left[\frac{(x-a)H^{w}_{a^+,\sigma-1}(x;p) + (b-x)H^{w}_{b^-,\sigma'-1}(x;p)}{2}\right].
\end{aligned} \tag{8.55}$$

Remark 8.9 If $w=p=0$ in (8.55), then [47, Theorem 1] is obtained.

Lemma 8.7 *[45] Let $f : [0,\infty) \to \mathbb{R}$ be a $(h-m)$-convex function. If $0 \leq a < mb$ and $f(x) = f\left(\frac{a+b-x}{m}\right)$, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq (m+1)h\left(\frac{1}{2}\right) f(x), \quad x \in [a,b]. \tag{8.56}$$

Theorem 8.10 *Let $w \in \mathbb{R}$, $r,\rho,\sigma,\tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $f : [a,b] \to \mathbb{R}$, $0 \leq a < b$, be a real valued function. If f is positive, $(h-m)$-convex and $f(x) = f\left(\frac{a+b-x}{m}\right)$, then for $\sigma, \sigma' > 0$,the following inequality for extended generalized fractional integral operators holds*

$$\begin{aligned}
&\frac{f\left(\frac{a+b}{2}\right)}{(m+1)h\left(\frac{1}{2}\right)}\left[H^{w}_{b^-,\sigma'+1}(a;p) + H^{w}_{a^+,\sigma+1}(b;p)\right] \\
&\leq \left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau} f\right)(a;p) + \left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau} f\right)(b;p) \\
&\leq (b-a)^2\left[H^{w}_{b^-,\sigma'-1}(a;p) + H^{w}_{a^+,\sigma-1}(b;p)\right]\left(f(a) + mf\left(\frac{b}{m}\right)\right)\int_0^1 h(z)dz.
\end{aligned} \tag{8.57}$$

Proof. For $x \in [a,b]$, we have

$$(x-a)^{\sigma'} E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(x-a)^{\rho};p) \leq (b-a)^{\sigma'} E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(b-a)^{\rho};p), \quad \sigma' > 0. \tag{8.58}$$

As f is $(h-m)$-convex, for $x\in[a,b]$ we have

$$f(x)\leq mh\left(\frac{x-a}{b-a}\right)f\left(\frac{b}{m}\right)+h\left(\frac{b-x}{b-a}\right)f(a). \tag{8.59}$$

Multiplying (8.58) and (8.59), then integrating over $[a,b]$, we get

$$\begin{aligned}&\int_a^b (x-a)^{\sigma'}E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(x-a)^{\rho};p)f(x)dx\\ &\leq m(b-a)^{\sigma'}E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(b-a)^{\rho};p)f\left(\frac{b}{m}\right)\int_a^b h\left(\frac{x-a}{b-a}\right)dx\\ &+(b-a)^{\sigma'}E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(b-a)^{\rho};p)f(a)\int_a^b h\left(\frac{b-x}{b-a}\right)dx.\end{aligned}$$

From which we have

$$\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p)\leq (b-a)^2H^{w}_{b^-,\sigma'-1}(a;p)\left(f(a)+mf\left(\frac{b}{m}\right)\right)\int_0^1 h(z)dz. \tag{8.60}$$

On the other hand for $x\in[a,b]$, we have

$$(b-x)^{\sigma}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(b-x)^{\rho};p)\leq (b-a)^{\sigma}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(b-a)^{\rho};p)\,\sigma>0. \tag{8.61}$$

Similarly multiplying (8.59) and (8.61), then integrating over $[a,b]$, we get

$$\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p)\leq (b-a)^2H^{w}_{a^+,\sigma-1}(b;p)\left(f(a)+mf\left(\frac{b}{m}\right)\right)\int_0^1 h(z)dz. \tag{8.62}$$

Adding (8.60) and (8.62), we get

$$\begin{aligned}&\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p)+\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p)\\ &\leq (b-a)^2\left(H^{w}_{b^-,\sigma'-1}(a;p)+H^{w}_{a^+,\sigma-1}(b;p)\right)\left(f(a)+mf\left(\frac{b}{m}\right)\right)\int_0^1 h(z)dz.\end{aligned}$$

Multiplying (8.56) with $(x-a)^{\sigma'}E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(x-a)^{\rho};p)$, then integrating over $[a,b]$, we get

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\int_a^b (x-a)^{\sigma'}E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(x-a)^{\rho};p)dx\\ &\leq (m+1)h\left(\frac{1}{2}\right)\int_a^b (x-a)^{\sigma'}E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(x-a)^{\rho};p)f(x)dx,\end{aligned}$$

by using (2.2), we get

$$\frac{f\left(\frac{a+b}{2}\right)}{(m+1)h\left(\frac{1}{2}\right)}H^{w}_{b^-,\sigma'+1}(a;p)\leq \left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p). \tag{8.63}$$

Similarly multiplying (8.56) with $(b-x)^{\sigma}E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w(b-x)^{\rho};p)$, then integrating over $[a,b]$ and using (2.2), we get

$$\frac{f\left(\frac{a+b}{2}\right)}{(m+1)h\left(\frac{1}{2}\right)}H_{a^+,\sigma+1}^{w}(b;p) \leq \left(\varepsilon_{a^+,\rho,\sigma+1,\tau}^{w,\delta,c,q,r}f\right)(b;p). \tag{8.64}$$

Adding (8.63) and (8.64), we get

$$\begin{aligned}&\frac{f\left(\frac{a+b}{2}\right)}{(m+1)h\left(\frac{1}{2}\right)}\left[H_{b^-,\sigma'+1}^{w}(a;p)+H_{a^+,\sigma+1}^{w}(b;p)\right] \\ &\leq \left(\varepsilon_{b^-,\rho,\sigma'+1,\tau}^{w,\delta,c,q,r}f\right)(a;p)+\left(\varepsilon_{a^+,\rho,\sigma+1,\tau}^{w,\delta,c,q,r}f\right)(b;p).\end{aligned} \tag{8.65}$$

From inequalities (8.63) and (8.65), inequality (8.57) is obtained. □

If $m=1$ and $h(z)=z$ in (8.57), then the following result holds for convex function:

Corollary 8.4 *Let $f:[a,b]\longrightarrow\mathbb{R}$, $a<b$, be a real valued function. If f is positive, convex and symmetric about $\frac{a+b}{2}$, then for $\sigma,\sigma'>0$, we have*

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\left[H_{b^-,\sigma'+1}^{w}(a;p)+H_{a^+,\sigma+1}^{w}(b;p)\right] \\ &\leq \left(\varepsilon_{b^-,\rho,\sigma'+1,\tau}^{w,\delta,c,q,r}f\right)(a;p)+\left(\varepsilon_{a^+,\rho,\sigma+1,\tau}^{w,\delta,c,q,r}f\right)(b;p) \\ &\leq (b-a)^2\left[H_{b^-,\sigma'-1}^{w}(a;p)+H_{a^+,\sigma-1}^{w}(b;p)\right]\left[\frac{f(a)+f(b)}{2}\right].\end{aligned} \tag{8.66}$$

Remark 8.10

(i) If $w=p=0$ in (8.66), then [47, Theorem 3] is obtained.

(ii) If $\sigma,\sigma'\to 0$ and $w=p=0$, then from above inequality, we get the Hadamard inequality.

Theorem 8.11 *Let $w\in\mathbb{R}$, $r,\rho,\sigma,\tau>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $f:[a,b]\to\mathbb{R}$, $0\leq a<mb$, be a real valued function. If f is differentiable and $|f'|$ is $(h-m)$-convex, then for $\sigma,\sigma'\geq 1$, the following inequality for extended generalized fractional integral operators holds*

$$\begin{aligned}&\left|\left(\varepsilon_{a^+,\rho,\sigma-1,\tau}^{w,\delta,c,q,r}f\right)(x;p)+\left(\varepsilon_{b^-,\rho,\sigma'-1,\tau}^{w,\delta,c,q,r}f\right)(x;p)\right. \\ &\left.-\left(f(a)H_{a^+,\sigma-1}^{w}(x;p)+f(b)H_{b^-,\sigma'-1}^{w}(x;p)\right)\right| \\ &\leq \left((x-a)|f'(a)|H_{a^+,\sigma-1}^{w}(x;p)+(b-x)|f'(b)|H_{b^-,\sigma'-1}^{w}(x;p)\right. \\ &\left.+m\left|f'\left(\frac{x}{m}\right)\right|\left((x-a)H_{a^+,\sigma-1}^{w}(x;p)+(b-x)H_{b^-,\sigma'-1}^{w}(x;p)\right)\right)\int_0^1 h(z)dz.\end{aligned} \tag{8.67}$$

Proof. Let $x \in [a,b]$ and $t \in [a,x]$. Then using $(h-m)$-convexity of $|f'|$, we have

$$|f'(t)| \leq h\left(\frac{x-t}{x-a}\right)|f'(a)| + mh\left(\frac{t-a}{x-a}\right)\left|f'\left(\frac{x}{m}\right)\right|. \tag{8.68}$$

From (8.68) follows

$$f'(t) \leq h\left(\frac{x-t}{x-a}\right)|f'(a)| + mh\left(\frac{t-a}{x-a}\right)\left|f'\left(\frac{x}{m}\right)\right|. \tag{8.69}$$

Multiplying (8.48) and (8.69), then integrating over $[a,x]$, we get

$$\begin{aligned}
&\int_a^x (x-t)^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w(x-t)^{\rho};p) f'(t)dt \\
&\leq (x-a)^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w(x-a)^{\rho};p) \\
&\times \left[|f'(a)|\int_a^x h\left(\frac{x-t}{x-a}\right)dt + m\left|f'\left(\frac{x}{m}\right)\right|\int_a^x h\left(\frac{t-a}{x-a}\right)dt\right].
\end{aligned} \tag{8.70}$$

The left hand side is calculated as:

$$\int_a^x (x-t)^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w(x-t)^{\rho};p) f'(t)dt, \tag{8.71}$$

put $x-t=z$ that is $t=x-z$, also using the derivative property of Mittag-Leffler function, we have

$$\begin{aligned}
&\int_0^{x-a} z^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wz^{\rho};p) f'(x-z)dz \\
&= (x-a)^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w(x-a)^{\rho};p) f(a) - \int_0^{x-a} z^{\sigma-2} E_{\rho,\sigma-1,\tau}^{\delta,c,q,r}(wz^{\rho};p) f(x-z)dz,
\end{aligned}$$

now put $x-z=t$ in second term of the right hand side of the above equation, we get

$$\begin{aligned}
&\int_a^x (x-t)^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w(x-t)^{\rho};p) f'(t)dt \\
&= \int_0^{x-a} z^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wz^{\rho};p) f'(x-z)dz \\
&= (x-a)^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w(x-a)^{\rho};p) f(a) - \left(\varepsilon_{a^+,\rho,\sigma+1,\tau}^{w,\delta,c,q,r} f\right)(x;p) \\
&H_{a^+,\sigma-1}^{w}(x;p) f(a) - \left(\varepsilon_{a^+,\rho,\sigma+1,\tau}^{w,\delta,c,q,r} f\right)(x;p).
\end{aligned} \tag{8.72}$$

Therefore, (8.70) takes the form as follows:

$$\begin{aligned}
&f(a) H_{a^+,\sigma-1}^{w}(x;p) - \left(\varepsilon_{a^+,\rho,\sigma-1,\tau}^{w,\delta,c,q,r} f\right)(x;p) \\
&\leq (x-a) H_{a^+,\sigma-1}^{w}(x;p)\left(|f'(a)| + m\left|f'\left(\frac{x}{m}\right)\right|\right)\int_0^1 h(z)dz.
\end{aligned} \tag{8.73}$$

Also, from (8.68) we obtain

$$f'(t) \geq -\left(h\left(\frac{x-t}{x-a}\right)|f'(a)| + mh\left(\frac{t-a}{x-a}\right)\left|f'\left(\frac{x}{m}\right)\right|\right). \tag{8.74}$$

Following the same procedure as did for (8.69), we also have

$$\begin{aligned}&\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma-1,\tau}f\right)(x;p) - f(a)H^w_{a^+,\sigma-1}(x;p) \\ &\leq (x-a)H^w_{a^+,\sigma-1}(x;p)\left(|f'(a)| + m\left|f'\left(\frac{x}{m}\right)\right|\right)\int_0^1 h(z)dz.\end{aligned} \tag{8.75}$$

From (8.73) and (8.75), we get

$$\begin{aligned}&\left|\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma-1,\tau}f\right)(x;p) - f(a)H^w_{a^+,\sigma-1}(x;p)\right| \\ &\leq (x-a)H^w_{a^+,\sigma-1}(x;p)\left(|f'(a)| + m\left|f'\left(\frac{x}{m}\right)\right|\right)\int_0^1 h(z)dz.\end{aligned} \tag{8.76}$$

Now let $t \in [x,b]$. Then using $(h-m)$-convexity of $|f'|$, we have

$$|f'(t)| \leq h\left(\frac{t-x}{b-x}\right)|f'(b)| + mh\left(\frac{b-t}{b-x}\right)\left|f'\left(\frac{x}{m}\right)\right|. \tag{8.77}$$

On the same lines as we have done for (8.48), (8.69) and (8.74), from (8.52) and (8.77) we have the following inequality:

$$\begin{aligned}&\left|\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'-1,\tau}f\right)(x;p) - f(b)H^w_{b^-,\sigma'-1}(x;p)\right| \\ &\leq (b-x)H^w_{b^-,\sigma'-1}(x;p)\left(|f'(b)| + m\left|f'\left(\frac{x}{m}\right)\right|\right)\int_0^1 h(z)dz.\end{aligned} \tag{8.78}$$

From (8.76) and (8.78) via triangular inequality, inequality (8.67) is obtained. □

If $m=1$ and $h(z)=z$ in (8.67), then the following result holds for convex function:

Corollary 8.5 *Let $f:[a,b]\longrightarrow\mathbb{R}$, $a<b$, be a real valued function. If f is differentiable and $|f'|$ is convex, then for $\sigma,\sigma'\geq 1$, we have*

$$\begin{aligned}&\left|\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma-1,\tau}f\right)(x;p) + \left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'-1,\tau}f\right)(x;p)\right. \\ &\left. -\left(f(a)H^w_{a^+,\sigma-1}(x;p) + f(b)H^w_{b^-,\sigma'-1}(x;p)\right)\right| \\ &\leq \frac{(x-a)|f'(a)|H^w_{a^+,\sigma-1}(x;p) + (b-x)|f'(b)|H^w_{b^-,\sigma'-1}(x;p)}{2} \\ &+|f'(x)|\left(\frac{(x-a)H^w_{a^+,\sigma-1}(x;p) + (b-x)H^w_{b^-,\sigma'-1}(x;p)}{2}\right).\end{aligned} \tag{8.79}$$

Remark 8.11

(i) If $w=p=0$ and replace α by $\alpha+1$ in (8.79), then [47, Theorem 2] is obtained.

(ii) If $w=p=0$, $\alpha=\beta=1$ and f' passes through $x=\frac{a+b}{2}$, then from (8.79) [42, Theorem 2.2] is obtained.

8.4 Inequalities for the Extended Generalized Mittag-Leffler Function

In this section, results of previous section are applied for the function $f(x)=x^2$. Function f is convex and $|f'(x)|=2|x|$ which is also convex. By virtue of this function we succeeded to establish recurrence relations among Mittag-Leffler functions useful in the solutions of fractional boundary value problems and fractional differential equations. Ullah et al. computed generalized fractional integral operators for the function $f(x)=x^2$, in [148], as follows:

$$\left(\epsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)=(x-a)^{\sigma}\Big[a^2E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w(x-a)^{\rho};p)$$
$$+2a(x-a)E^{\delta,c,q,r}_{\rho,\sigma+2,\tau}(w(x-a)^{\rho};p)+2(x-a)^2E^{\delta,c,q,r}_{\rho,\sigma+3,\tau}(w(x-a)^{\rho};p)\Big]. \tag{8.80}$$

$$\left(\epsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(x;p)=(b-x)^{\sigma}\Big[b^2E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w(b-x)^{\rho};p)$$
$$-2b(b-x)E^{\delta,c,q,r}_{\rho,\sigma+2,\tau}(w(b-x)^{\rho};p)+2(b-x)^2E^{\delta,c,q,r}_{\rho,\sigma+3,\tau}(w(b-x)^{\rho};p)\Big]. \tag{8.81}$$

Theorem 8.12 *Mittag-Leffler functions satisfy the following recurrence relation:*

$$\frac{(a^2+b^2)}{(b-a)^2}E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w'(b-a)^{\rho};p)+E^{\delta,c,q,r}_{\rho,\sigma+3,\tau}(w'(b-a)^{\rho};p) \tag{8.82}$$
$$\leq\frac{((2m+1)(a^2+b^2)+2ab)}{2m(b-a)^2}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w'(b-a)^{\rho};p)\int_0^1h(z)dz+E^{\delta,c,q,r}_{\rho,\sigma+2,\tau}(w'(b-a)^{\rho};p)$$

where $w'=\frac{w}{2^{\rho}}$.

Proof. By using (8.80), (8.81) and $f(x)=x^2$ in (8.47) of Theorem 8.9, we have

$$(x-a)^{\sigma}\Big[a^2E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w(x-a)^{\rho};p)+2a(x-a)E^{\delta,c,q,r}_{\rho,\sigma+2,\tau}(w(x-a)^{\rho};p) \tag{8.83}$$
$$+2(x-a)^2E^{\delta,c,q,r}_{\rho,\sigma+3,\tau}(w(x-a)^{\rho};p)\Big]+(b-x)^{\sigma'}\Big[b^2E^{\delta,c,q,r}_{\rho,\sigma'+1,\tau}(w(b-x)^{\rho};p)$$
$$-2b(b-x)E^{\delta,c,q,r}_{\rho,\sigma'+2,\tau}(w(b-x)^{\rho};p)+2(b-x)^2E^{\delta,c,q,r}_{\rho,\sigma'+3,\tau}(w(b-x)^{\rho};p)\Big]$$
$$\leq\left(\left(a^2+\frac{x^2}{m}\right)(x-a)^{\sigma}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-a)^{\rho};p)\right.$$
$$\left.+\left(b^2+\frac{x^2}{m}\right)(b-x)^{\sigma'}E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(b-x)^{\rho};p)\right)\int_0^1h(z)dz.$$

Now by putting $x=\frac{a+b}{2}$ and $\sigma=\sigma'$ in (8.83), then after simplification, inequality (8.82) is obtained. □

Corollary 8.6 *If $m = 1$ and $h(z) = z$ in (8.82), then we have*

$$\frac{(a^2+b^2)}{(b-a)^2} E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w'(b-a)^\rho;p) + E^{\delta,c,q,r}_{\rho,\sigma+3,\tau}(w'(b-a)^\rho;p) \tag{8.84}$$
$$\leq \frac{(3a^2+3b^2+2ab)}{4(b-a)^2} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w'(b-a)^\rho;p) + E^{\delta,c,q,r}_{\rho,\sigma+2,\tau}(w'(b-a)^\rho;p).$$

Theorem 8.13 *Mittag-Leffler functions satisfy the following recurrence relation:*

$$\left| E^{\delta,c,q,r}_{\rho,\sigma+2,\tau}(w'(b-a)^\rho;p) - E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w'(b-a)^\rho;p) \right| \tag{8.85}$$
$$\leq \frac{1}{m(b-a)} (ma+mb+(a+b)) E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w'(b-a)^\rho;p) \int_0^1 h(z)dz$$

where $w' = \frac{w}{2^\rho}$.

Proof. By using (8.80), (8.81) and $|f'(x)| = 2|x|$ in (8.67) of Theorem 8.11, we have

$$\Big|(x-a)^\sigma \Big[a^2(x-a)^{-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-a)^\rho;p) + 2aE^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w(x-a)^\rho;p) \tag{8.86}$$
$$+2(x-a)E^{\delta,c,q,r}_{\rho,\sigma+2,\tau}(w(x-a)^\rho;p)\Big] + (b-x)^{\sigma'}\Big[b^2(b-x)^{-1}E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(b-x)^\rho;p)$$
$$-2bE^{\delta,c,q,r}_{\rho,\sigma'+1,\tau}(w(b-x)^\rho;p) + 2(b-x)E^{\delta,c,q,r}_{\rho,\sigma'+2,\tau}(w(b-x)^\rho;p)\Big]$$
$$-\left(a^2(x-a)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-a)^\rho;p) + b^2(b-x)^{\sigma'-1}E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(b-x)^\rho;p)\right)\Big|$$
$$\leq \left(2\left(|a| + \frac{|x|}{m}\right)(x-a)^\sigma E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-a)^\rho;p)\right.$$
$$\left.+2\left(|b| + \frac{|x|}{m}\right)(b-x)^{\sigma'} E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(b-x)^\rho;p)\right) \int_0^1 h(z)dz.$$

Now by putting $x = \frac{a+b}{2}$ and $\sigma = \sigma'$ in (8.86), then after simplification, inequality (8.85) is obtained. □

Corollary 8.7 *If $m = 1$ and $h(z) = z$ in (8.85), then we have*

$$\left| E^{\delta,c,q,r}_{\rho,\sigma+2,\tau}(w'(b-a)^\rho;p) - E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w'(b-a)^\rho;p) \right| \tag{8.87}$$
$$\leq \frac{1}{b-a}(a+b)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w'(b-a)^\rho;p).$$

Theorem 8.14 *Mittag-Leffler functions satisfy the following recurrence relation:*

$$E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w(b-a)^\rho;p) - (1+\frac{1}{m})E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(b-a)^\rho;p) \int_0^1 h(z)dz \tag{8.88}$$
$$\leq \frac{2(b-a)^2}{(a^2+b^2)} \left(E^{\delta,c,q,r}_{\rho,\sigma+2,\tau}(w(b-a)^\rho;p) - 2E^{\delta,c,q,r}_{\rho,\sigma+3,\tau}(w(b-a)^\rho;p)\right).$$

Proof. In (8.83), putting $x = a$ and $x = b$, then adding for $\sigma = \sigma'$, inequality (8.88) is obtained. □

Corollary 8.8 *If $m = 1$ and $h(z) = z$ in (8.88), then we have*

$$E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w(b-a)^{\rho};p) - E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(b-a)^{\rho};p) \tag{8.89}$$
$$\leq \frac{2(b-a)^2}{(a^2+b^2)}\left(E^{\delta,c,q,r}_{\rho,\sigma+2,\tau}(w(b-a)^{\rho};p) - 2E^{\delta,c,q,r}_{\rho,\sigma+3,\tau}(w(b-a)^{\rho};p)\right).$$

Theorem 8.15 *Mittag-Leffler functions satisfy the following recurrence relation:*

$$\left|E^{\delta,c,q,r}_{\rho,\sigma+2,\tau}(w(b-a)^{\rho};p) - \frac{1}{2}E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w(b-a)^{\rho};p)\right| \tag{8.90}$$
$$\leq \frac{(1+\frac{1}{m})}{2(b-a)}(a+b)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(b-a)^{\rho};p)\int_0^1 h(z)dz.$$

Proof. In (8.86), putting $x = a$ and $x = b$, then adding for $\sigma = \sigma'$, inequality (8.90) is obtained. □

Corollary 8.9 *If $m = 1$ and $h(z) = z$ in (8.90), then we have*

$$\left|E^{\delta,c,q,r}_{\rho,\sigma+2,\tau}(w(b-a)^{\rho};p) - \frac{1}{2}E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w(b-a)^{\rho};p)\right| \tag{8.91}$$
$$\leq \frac{1}{2(b-a)}(a+b)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(b-a)^{\rho};p).$$

Theorem 8.16 *Mittag-Leffler functions satisfy the following recurrence relation:*

$$E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w;p) - (1+\frac{1}{m})E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)\int_0^1 h(z)dz \tag{8.92}$$
$$\leq 2\left(E^{\delta,c,q,r}_{\rho,\sigma+2,\tau}(w;p) - 2E^{\delta,c,q,r}_{\rho,\sigma+3,\tau}(w;p)\right).$$

Proof. In (8.88), putting $a = 0$ and $b = 1$, then inequality (8.92) is obtained. □

Corollary 8.10 *If $m = 1$ and $h(z) = z$ in (8.92), then we have*

$$E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w;p) - E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p) \leq 2\left(E^{\delta,c,q,r}_{\rho,\sigma+2,\tau}(w;p) - 2E^{\delta,c,q,r}_{\rho,\sigma+3,\tau}(w;p)\right). \tag{8.93}$$

Theorem 8.17 *Mittag-Leffler functions satisfy the following recurrence relation:*

$$\left|2E^{\delta,c,q,r}_{\rho,\sigma+2,\tau}(w;p) - E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w;p)\right| \leq (1+\frac{1}{m})E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)\int_0^1 h(z)dz. \tag{8.94}$$

Proof. In (8.90), putting $a = 0$ and $b = 1$, then inequality (8.94) is obtained. □

Corollary 8.11 *If $m = 1$ and $h(z) = z$ in (8.94), then we have*

$$\left|2E^{\delta,c,q,r}_{\rho,\sigma+2,\tau}(w;p) - E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w;p)\right| \leq E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p). \tag{8.95}$$

By applying Theorem 8.10 similar relations can be established which we leave for the reader.

8.5 Error Bounds of Fractional Integral Operators for Quasi-convex Functions

The very first result provides the upper bound of the sum of the left sided and right sided fractional integrals via quasi-convex functions.

Theorem 8.18 *Let $w \in \mathbb{R}$, $r, \rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let $f : [a,b] \to \mathbb{R}$ be a function such that $f \in L_1[a,b]$ with $a < b$. If f is quasi-convex on $[a,b]$, then the following inequality holds*

$$\left(\varepsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma,\tau} f\right)(b;p) + \left(\varepsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma',\tau} f\right)(a;p) \leq \max\{f(a), f(b)\} \left(H^{w'}_{a^+,\sigma}(b;p) + H^{w'}_{b^-,\sigma'}(a;p)\right) \tag{8.96}$$

where $w' = \frac{w}{(b-a)^\rho}$.

Proof. By using quasi-convexity in the form $f((1-t)a+tb) \leq max\{f(a), f(b)\}$ the following inequality can be obtained

$$\int_0^1 t^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p) f(ta+(1-t)b)dt \leq max\{f(a), f(b)\} \int_0^1 t^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)dt. \tag{8.97}$$

Making substitution $x = ta + (1-t)b$ that is $t = \frac{b-x}{b-a}$ in the inequality (8.97). Then it takes the form

$$\int_a^b \left(\frac{b-x}{b-a}\right)^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w\left(\frac{b-x}{b-a}\right)^\rho;p\right) f(x)\frac{dx}{b-a} \tag{8.98}$$
$$\leq max\{f(a), f(b)\} \int_a^b \left(\frac{b-x}{b-a}\right)^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w\left(\frac{b-x}{b-a}\right)^\rho;p\right)\frac{dx}{b-a}.$$

From above inequality follows

$$\left(\varepsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma,\tau} f\right)(b;p) \leq H^{w'}_{a^+,\sigma}(b;p) \max\{f(a), f(b)\}. \tag{8.99}$$

Also by using quasi-convexity in the form $f((1-t)a+tb) \leq max\{f(a), f(b)\}$ the following inequality can be obtained

$$\int_0^1 t^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p) f((1-t)a+tb)dt \leq max\{f(a), f(b)\} \int_0^1 t^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)dt. \tag{8.100}$$

Now making substitution of $y = (1-t)a + tb$ that is $t = \frac{y-a}{b-a}$ in the inequality (8.100). Then it takes the form

$$\int_a^b \left(\frac{y-a}{b-a}\right)^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w\left(\frac{y-a}{b-a}\right)^\rho;p\right) f(y)\frac{dy}{b-a} \tag{8.101}$$
$$\leq max\{f(a), f(b)\} \int_a^b \left(\frac{y-a}{b-a}\right)^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(w\left(\frac{y-a}{b-a}\right)^\rho;p\right)\frac{dy}{b-a}.$$

From above inequality we obtain

$$\left(\varepsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(a;p) \leq H^{w'}_{b^-,\sigma'}(a;p)\max\{f(a),f(b)\}. \tag{8.102}$$

Adding (8.99) and (8.102), we get (8.96) which is required. □

Corollary 8.12 *Setting $\sigma = \sigma'$ in (8.96), then we get the following inequality*

$$\left(\varepsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(b;p)+\left(\varepsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(a;p)\leq \max\{f(a),f(b)\}\left(H^{w'}_{a^+,\sigma}(b;p)+H^{w'}_{b^-,\sigma}(a;p)\right). \tag{8.103}$$

Corollary 8.13 *[118] Setting $w = p = 0$ in (8.103), then we get the following inequality for Riemann-Liouville fractional integrals*

$$J^{\sigma}_{a^+}f(b)+J^{\sigma}_{b^-}f(a) \leq \frac{2(b-a)^{\sigma}}{\Gamma(\sigma+1)}max\{f(a),f(b)\}. \tag{8.104}$$

Remark 8.12 If we take $\sigma = 1$ in (8.104), then we get the following inequality for quasi-convex function which is related to the Hadamard inequality given by Dragomir and Pearce in [44]

$$\frac{1}{b-a}\int_a^b f(t)dt \leq max\{f(a),f(b)\}. \tag{8.105}$$

8.5.1 Recurrence Inequalities for Mittag-Leffler Functions

Let us consider the function $f(x) = x^2$. The function f is convex on $[a,b]$ and $|f'(x)| = 2|x|$ which is again convex function on $[a,b]$. Since f and $|f'|$ are convex and finite on $[a,b]$, therefore are quasi-convex. Results of previous section are applied for this function and inequalities among the generalized extended Mittag-Leffler function are established.

Theorem 8.19 *The Mittag-Leffler function satisfies the following recurrence inequality*

$$\begin{aligned}&2E^{\delta,c,q,r}_{\rho,\sigma+3,\tau}(w'(b-a)^{\rho};p)-E^{\delta,c,q,r}_{\rho,\sigma+2,\tau}(w'(b-a)^{\rho};p)\\&\leq E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w'(b-a)^{\rho};p)\left[\frac{max\{a^2,b^2\}-(a^2+b^2)}{2(b-a)^2}\right]\end{aligned} \tag{8.106}$$

where $w' = \frac{w}{(b-a)^{\rho}}$.

Proof. For the function $f(t) = t^2$ the generalized fractional integral operator is evaluated as follows

$$\begin{aligned}&\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)\\&=\int_a^x (x-t)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-t)^{\rho};p)t^2dt\\&=\int_a^x (x-t)^{\sigma-1}\sum_{n=0}^{\infty}\frac{\beta_p(\delta+nq,r-\delta)}{\beta(\delta,r-\delta)}\frac{(r)_{nq}}{\Gamma(\rho n+\sigma)}\frac{w^n(x-t)^{\sigma n}}{(\tau)_{nc}}t^2dt\end{aligned} \tag{8.107}$$

$$= \sum_{n=0}^{\infty} \frac{\beta_p(\delta+nq, r-\delta)}{\beta(\delta, r-\delta)} \frac{(r)_{nq}}{\Gamma(\rho n+\sigma)} \frac{w^n}{(\tau)_{nc}} \int_a^x (x-t)^{\rho n+\sigma-1} t^2 dt$$
$$= (x-a)^{\sigma} \Big[a^2 E_{\rho,\sigma+1,\tau}^{\delta,c,q,r}(w(x-a)^{\rho};p) + 2a(x-a) E_{\rho,\sigma+2,\tau}^{\delta,c,q,r}(w(x-a)^{\rho};p)$$
$$+2(x-a)^2 E_{\rho,\sigma+3,\tau}^{\delta,c,q,r}(w(x-a)^{\rho};p) \Big],$$

Similarly

$$\left(\varepsilon_{b^-,\rho,\sigma,\tau,}^{w,\delta,c,q,r} f\right)(x;p) = (b-x)^{\sigma} \Big[b^2 E_{\rho,\sigma+1,\tau}^{\delta,c,q,r}(w(b-x)^{\rho};p) - 2b(b-x) \tag{8.108}$$
$$E_{\rho,\sigma+2,\tau}^{\delta,c,q,r}(w(b-x)^{\rho};p) + 2(b-x)^2 E_{\rho,\sigma+3,\tau}^{\delta,c,q,r}(w(b-x)^{\rho};p) \Big].$$

Using (8.96) of Theorem 8.18 for the function t^2 takes the form

$$(b-a)^{\sigma} \Big[a^2 E_{\rho,\sigma+1,\tau}^{\delta,c,q,r}(w'(b-a)^{\rho};p) + 2a(b-a) E_{\rho,\sigma+2,\tau}^{\delta,c,q,r}(w'(b-a)^{\rho};p) \tag{8.109}$$
$$+2(b-a)^2 E_{\rho,\sigma+3,\tau}^{\delta,c,q,r}(w'(b-a)^{\rho};p) \Big] + (b-a)^{\sigma'} \Big[b^2 E_{\rho,\sigma'+1,\tau}^{\delta,c,q,r}(w'(b-a)^{\rho};p)$$
$$-2b(b-a) E_{\rho,\sigma'+2,\tau}^{\delta,c,q,r}(w'(b-a)^{\rho};p) + 2(b-a)^2 E_{\rho,\sigma'+3,\tau}^{\delta,c,q,r}(w'(b-a)^{\rho};p) \Big]$$
$$\leq 2max\{a^2, b^2\} \Big((b-a)^{\sigma} E_{\rho,\sigma+1,\tau}^{\delta,c,q,r}(w'(b-a)^{\rho};p)$$
$$+(b-a)^{\sigma'} E_{\rho,\sigma'+1,\tau}^{\delta,c,q,r}(w'(b-a)^{\rho};p) \Big).$$

Now taking $\sigma = \sigma'$ in (8.109), then after simplification we get (8.106). □

Theorem 8.20 *The Mittag-Leffler function satisfies the following recurrence inequality*

$$\Bigg| \frac{a^2+b^2}{2} - \frac{1}{2E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w;p)} \Big[(a^2+b^2) E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w'(b-a)^{\rho};p) \tag{8.110}$$
$$+2(b-a)^2 (2E_{\rho,\sigma+2,\tau}^{\delta,c,q,r}(w'(b-a)^{\rho};p) - E_{\rho,\sigma+1,\tau}^{\delta,c,q,r}(w'(b-a)^{\rho};p)) \Big] \Bigg|$$
$$\leq \frac{2(b-a)M}{\sigma E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w;p)} max\{|a|, |b|\}$$

where $w' = \frac{w}{(b-a)^{\rho}}$.

Proof. By using (8.107), (8.108) and $f(t) = t^2$, $|f'(t)| = 2|t|$ in (8.119) of Theorem 8.23, we have

$$\Bigg| \frac{a^2+b^2}{2} - \frac{1}{2(b-a)^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w;p)} \tag{8.111}$$
$$\times \Big[(b-a)^{\sigma} \Big((b-a)^{-1} (a^2+b^2) E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w'(b-a)^{\rho};p)$$

$$-2(b-a)E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w'(b-a)^{\rho};p)+4(b-a)E^{\delta,c,q,r}_{\rho,\sigma+2,\tau}(w'(b-a)^{\rho};p)\Big)\Big]\Big|$$
$$\leq\frac{(b-a)M}{\sigma E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}max\{2|a|,2|b|\}.$$

After simplification we get (8.110). □

Theorem 8.21 *The Mittag-Leffler function satisfies the following recurrence inequality*

$$\Big|\frac{a^2+b^2}{2}-\frac{1}{2E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}\Big[(a^2+b^2)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w'(b-a)^{\rho};p) \tag{8.112}$$
$$+2(b-a)^2(2E^{\delta,c,q,r}_{\rho,\sigma+2,\tau}(w'(b-a)^{\rho};p)-E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w'(b-a)^{\rho};p))\Big]\Big|$$
$$\leq\frac{2(b-a)M}{E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}\frac{1}{((\sigma-1)p+1)^{\frac{1}{p}}}(max\{|a|^q,|b|^q\})^{\frac{1}{q}}$$

where $w'=\frac{w}{(b-a)^{\rho}}$.

Proof.
By using (8.107), (8.108) and $f(t)=t^2$, $|f'(t)|=2|t|$ in (8.123) of Theorem 8.24, we have

$$\Big|\frac{a^2+b^2}{2}-\frac{1}{2(b-a)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)} \tag{8.113}$$
$$\times\Big[(b-a)^{\sigma}\Big((b-a)^{-1}(a^2+b^2)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w'(b-a)^{\rho};p)$$
$$-2(b-a)E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w'(b-a)^{\rho};p)+4(b-a)E^{\delta,c,q,r}_{\rho,\sigma+2,\tau}(w'(b-a)^{\rho};p)\Big)\Big]\Big|$$
$$\leq\frac{(b-a)M}{E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}\frac{1}{((\sigma-1)p+1)^{\frac{1}{p}}}(max\{(2|a|)^q,(2|b|)^q\})^{\frac{1}{q}}.$$

After simplification we get (8.112). □

Theorem 8.22 *The Mittag-Leffler function satisfies the following recurrence inequality*

$$\Big|\frac{a^2+b^2}{2}-\frac{1}{2E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}\Big[(a^2+b^2)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w'(b-a)^{\rho};p) \tag{8.114}$$
$$+2(b-a)^2(2E^{\delta,c,q,r}_{\rho,\sigma+2,\tau}(w'(b-a)^{\rho};p)-E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w'(b-a)^{\rho};p))\Big]\Big|$$
$$\leq\frac{2(b-a)M}{\sigma E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}(max\{|a|^q,|b|^q\})^{\frac{1}{q}}$$

where $w'=\frac{w}{(b-a)^{\rho}}$.

Proof. By using (8.107), (8.108) and $f(t)=t^2$, $|f'(t)|=2|t|$ in (7.29) of Theorem 8.25, we have

$$\left|\frac{a^2+b^2}{2}-\frac{1}{2(b-a)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}\right. \tag{8.115}$$
$$\times\Big[(b-a)^{\sigma}\Big((b-a)^{-1}(a^2+b^2)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w'(b-a)^{\rho};p)$$
$$\left.-2(b-a)E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w'(b-a)^{\rho};p)+4(b-a)E^{\delta,c,q,r}_{\rho,\sigma+2,\tau}(w'(b-a)^{\rho};p)\Big)\Big]\right|$$
$$\leq\frac{2(b-a)M}{\sigma E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}(max\{|a|^q,|b|^q\})^{\frac{1}{q}}.$$

After simplification we get (8.114). □

8.5.2 Error Bounds of Hadamard and Fejér-Hadamard Inequalities

The following identity is very important to give the Hadamard type inequalities.

Lemma 8.8 *Let $f:[a,b]\to\mathbb{R}$ be a function such that $f'\in L_1[a,b]$ with $a<b$, then for generalized fractional integral operators the following identity holds*

$$\frac{f(a)+f(b)}{2}-\frac{\left(\varepsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma-1,\tau}f\right)(b;p)+\left(\varepsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma-1,\tau}f\right)(a;p)}{2(b-a)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)} \tag{8.116}$$
$$=\frac{b-a}{2E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}\int_0^1\left((1-t)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(1-t)^{\rho};p)-t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)\right)$$
$$\times f'(ta+(1-t)b)dt$$

where $w'=\frac{w}{(b-a)^{\rho}}$.

Proof. We have

$$\frac{b-a}{2E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}\int_0^1\left((1-t)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(1-t)^{\rho};p)-t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)\right) \tag{8.117}$$
$$\times f'(ta+(1-t)b)dt$$
$$=\frac{b-a}{2E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}\int_0^1(1-t)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(1-t)^{\rho};p)f'(ta+(1-t)b)dt$$
$$-\frac{b-a}{2E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f'(ta+(1-t)b)dt.$$

We first consider the first term of right hand side of (8.117): putting $z=1-t$ that is $t=1-z$ and using the derivative property of Mittag-Leffler function, it takes form

$$\frac{b-a}{2E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}\int_0^1 z^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wz^{\rho};p)f'((1-z)a+zb)dz=\frac{b-a}{2E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}$$
$$\times\left[\frac{E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)f(b)}{b-a}-\frac{1}{b-a}\int_0^1 z^{\sigma-2}E^{\gamma,\delta,k,c}_{\rho,\sigma-1,\tau}(wz^{\rho};p)f((1-z)a+zb)dz\right].$$

Making substitution $x=(1-z)a+zb$ in above we get

$$\frac{b-a}{2E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}\int_0^1 z^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wz^{\rho};p)f'((1-z)a+zb)dz \tag{8.118}$$
$$=\frac{b-a}{2E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}\left[\frac{E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)f(b)}{b-a}-\frac{1}{(b-a)^{\sigma}}\left(\varepsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma-1,\tau}f\right)(a;p)\right].$$

Similarly consider the second term of right hand side of (8.117), we get

$$-\frac{b-a}{2E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)f'(ta+(1-t)b)dt$$
$$=-\frac{b-a}{2E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}\left[-\frac{E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)f(a)}{b-a}+\frac{1}{(b-a)^{\sigma}}\left(\varepsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma-1,\tau}f\right)(b;p)\right],$$

here we use substitution $x=ta+(1-t)b$.

Now by using final form of both terms in (8.117), identity (8.116) is established. □

In the following we give Hadamard type inequality by using the above lemma.

Theorem 8.23 *Let $w\in\mathbb{R}$, $r,\rho,\sigma,\tau>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $f:[a,b]\to\mathbb{R}$ be a differentiable function such that $f'\in L_1[a,b]$ with $a<b$. If $|f'|$ is quasi-convex on $[a,b]$, then for generalized fractional integral operators the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\left(\varepsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma-1,\tau}f\right)(b;p)+\left(\varepsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma-1,\tau}f\right)(a;p)}{2(b-a)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}\right| \tag{8.119}$$
$$\leq\frac{(b-a)M}{\sigma E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}max\{|f'(a)|,|f'(b)|\}$$

for $q<r+\Re(\rho)$, where $w'=\frac{w}{(b-a)^{\rho}}$.

Proof. Using Lemma 8.8 and properties of modulus, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\left(\varepsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma-1,\tau}f\right)(b;p)+\left(\varepsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma-1,\tau}f\right)(a;p)}{2(b-a)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}\right| \tag{8.120}$$
$$\leq\frac{b-a}{2E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}\int_0^1\left|(1-t)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(1-t)^{\rho};p)-t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)\right|$$
$$\times|f'(ta+(1-t)b|dt$$

$$\leq \frac{b-a}{2E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)} \left(\int_0^1 \left|(1-t)^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(1-t)^{\rho};p)\right| \right.$$
$$\left. + \int_0^1 \left|t^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)\right| \right) |f'(ta+(1-t)b|dt.$$

Since $|f'|$ is quasi-convex, also using absolute convergence of Mittag-Leffler function, we have

$$\left| \frac{f(a)+f(b)}{2} - \frac{\left(\varepsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma-1,\tau} f\right)(b;p) + \left(\varepsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma-1,\tau} f\right)(a;p)}{2(b-a)^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)} \right| \tag{8.121}$$
$$\leq \frac{(b-a)M}{2E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)} \left(\int_0^1 |(1-t)^{\sigma-1}|dt + \int_0^1 |t^{\sigma-1}|dt \right) max\{|f'(a)|,|f'(b)|\}.$$

After simple calculation we get (8.119) which is required. □

Corollary 8.14 *Setting $w=p=0$ in (8.119), then we get the following inequality for Riemann-Liouville fractional integrals*

$$\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\sigma)}{2(b-a)^{\sigma-1}} \left[J^{\sigma-1}_{a^+} f(b) + J^{\sigma-1}_{b^-} f(a) \right] \right| \leq \frac{b-a}{\sigma} max\{|f'(a)|,|f'(b)|\}. \tag{8.122}$$

Corollary 8.15 *If we take $\sigma = 2$ in (8.122), then we get the following inequality for quasi-convex function*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{b-a}{2} max\{|f'(a)|,|f'(b)|\}.$$

In the following we give the Hadamard type inequality by using Lemma 8.8, Hölder's inequality and quasi-convexity of $|f'|^q$.

Theorem 8.24 *Let $w \in \mathbb{R}$, $r,\rho,\sigma,\tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $f:[a,b] \to \mathbb{R}$ be a differentiable function such that $f' \in L_1[a,b]$ with $a<b$. If $|f'|^q$, $q>1$ is quasi-convex on $[a,b]$, then for generalized fractional integral operators the following inequality holds*

$$\left| \frac{f(a)+f(b)}{2} - \frac{\left(\varepsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma-1,\tau} f\right)(b;p) + \left(\varepsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma-1,\tau} f\right)(a;p)}{2(b-a)^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)} \right| \tag{8.123}$$
$$\leq \frac{(b-a)M}{E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)} \frac{1}{((\sigma-1)p+1)^{\frac{1}{p}}} \left(max\{|f'(a)|^q,|f'(b)|^q\} \right)^{\frac{1}{q}}$$

for $q < r+\Re(\rho)$ and $\frac{1}{p}+\frac{1}{q}=1$, where $w' = \frac{w}{(b-a)^{\rho}}$.

Proof. From Lemma 8.8, properties of modulus and Hölder's inequality, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\left(\varepsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma-1,\tau}f\right)(b;p)+\left(\varepsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma-1,\tau}f\right)(a;p)}{2(b-a)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}\right| \tag{8.124}$$
$$\leq \frac{b-a}{2E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}\int_0^1\left|(1-t)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(1-t)^{\rho};p)-t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)\right|$$
$$\times|f'(ta+(1-t)b|dt$$
$$\leq \frac{b-a}{2E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}\left(\int_0^1\left|(1-t)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(1-t)^{\rho};p)\right|\right.$$
$$\left.+\int_0^1\left|t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)\right|\right)|f'(ta+(1-t)b|dt$$
$$\leq \frac{b-a}{2E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}\left(\left(\int_0^1\left|(1-t)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(1-t)^{\rho};p)\right|^p dt\right)^{\frac{1}{p}}\right.$$
$$\left.+\left(\int_0^1\left|t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)\right|^p dt\right)^{\frac{1}{p}}\right)\left(\int_0^1|f'(ta+(1-t)b|^q dt\right)^{\frac{1}{q}}.$$

Since $|f'|^q$ is quasi-convex, also using absolute convergence of Mittag-Leffler function, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\left(\varepsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma-1,\tau}f\right)(b;p)+\left(\varepsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma-1,\tau}f\right)(a;p)}{2(b-a)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}\right| \tag{8.125}$$
$$\leq \frac{(b-a)M}{2E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}\left(\left(\int_0^1\left|(1-t)^{\sigma-1}\right|^p dt\right)^{\frac{1}{p}}+\left(\int_0^1\left|t^{\sigma-1}\right|^p dt\right)^{\frac{1}{p}}\right)$$
$$\times\left(max\{|f'(a)|^q,|f'(b)|^q\right)^{\frac{1}{q}}.$$

After simple calculation we get (8.123) which is required. □

Corollary 8.16 *Setting $w=p=0$ in (8.123), then we get the following inequality for Riemann-Liouville fractional integrals*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\sigma)}{2(b-a)^{\sigma-1}}\left[J^{\sigma-1}_{a^+}f(b)+J^{\sigma-1}_{b^-}f(a)\right]\right| \tag{8.126}$$
$$\leq \frac{b-a}{((\sigma-1)p+1)^{\frac{1}{p}}}\left(max\{|f'(a)|^q,|f'(b)|^q\}\right)^{\frac{1}{q}}.$$

Corollary 8.17 *If we take $\sigma=2$ in (8.126), then we get the following inequality for quasi-convex function*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{b-a}{(p+1)^{\frac{1}{p}}}\left(max\{|f'(a)|^q,|f'(b)|^q\}\right)^{\frac{1}{q}}.$$

Theorem 8.25 *Let $w \in \mathbb{R}$, $r,\rho,\sigma,\tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $f : [a,b] \to \mathbb{R}$ be a differentiable function such that $f' \in L_1[a,b]$ with $a < b$. If $|f'|^q$, $q \geq 1$ is quasi-convex on $[a,b]$, then for generalized fractional integral operators the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\left(\varepsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma-1,\tau}f\right)(b;p)+\left(\varepsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma-1,\tau}f\right)(a;p)}{2(b-a)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma-1,\tau}(w;p)}\right| \tag{8.127}$$
$$\leq \frac{(b-a)M}{\sigma E^{\delta,c,q,r}_{\rho,\sigma-1,\tau}(w;p)}\left(max\{|f'(a)|^q,|f'(b)|^q\}\right)^{\frac{1}{q}}$$

for $q < r+\Re(\rho)$, where $w' = \frac{w}{(b-a)^\rho}$.

Proof. From Lemma 8.8, properties of modulus and power mean inequality, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{\left(\varepsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma-1,\tau}f\right)(b;p)+\left(\varepsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma-1,\tau}f\right)(a;p)}{2(b-a)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}\right| \tag{8.128}$$
$$\leq \frac{b-a}{2E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}\int_0^1\left|(1-t)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(1-t)^\rho;p)-t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)\right|$$
$$\times|f'(ta+(1-t)b|dt$$
$$\leq \frac{b-a}{2E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}\left(\int_0^1|(1-t)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(1-t)^\rho;p)\right.$$
$$\left.-t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)|dt\right)^{1-\frac{1}{q}}\times\left(\int_0^1|(1-t)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(1-t)^\rho;p)\right.$$
$$\left.-t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)||f'(ta+(1-t)b|^q dt\right)^{\frac{1}{q}}.$$

Since $|f'|^q$ is quasi-convex, also using absolute convergence of Mittag-Leffler function, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{\left(\varepsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma-1,\tau}f\right)(b;p)+\left(\varepsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma-1,\tau}f\right)(a;p)}{2(b-a)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}\right| \tag{8.129}$$
$$\leq \frac{(b-a)M}{2E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w;p)}\left(\int_0^1|(1-t)^{\sigma-1}|dt+\int_0^1|t^{\sigma-1}|dt\right)\left(max\{|f'(a)|^q,|f'(b)|^q\}\right)^{\frac{1}{q}}.$$

After simple calculation we get (8.127) which is required. □

Corollary 8.18 *Setting $w = p = 0$ in (8.127), then we get the following inequality for Riemann-Liouville fractional integrals*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\sigma)}{2(b-a)^{\sigma-1}}\left[J^{\sigma-1}_{a^+}f(b)+J^{\sigma-1}_{b^-}f(a)\right]\right|\leq\frac{b-a}{\sigma}\left(max\{|f'(a)|^q,|f'(b)|^q\}\right)^{\frac{1}{q}}. \tag{8.130}$$

Corollary 8.19 *If we take $\sigma = 2$ in (8.130), then we get the following inequality for quasi-convex function*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{2}\left(max\{|f'(a)|^q, |f'(b)|^q\}\right)^{\frac{1}{q}}.$$

8.6 Bounds of Fractional Integral Operators for s-convex Functions

In this section bounds of fractional integral operators are proved for s-convex functions.

Theorem 8.26 *Let $w \in \mathbb{R}$, $r, \rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let $f : [a,b] \longrightarrow \mathbb{R}$ be a real valued function. If f is positive and s-convex, then for $\sigma, \sigma' \geq 1$, the following fractional integral inequality for generalized integral operators holds:*

$$\begin{aligned}
&\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau} f\right)(x;p) + \left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau} f\right)(x;p) \\
&\leq \left(\frac{f(a)+f(x)}{s+1}\right)(x-a)H^w_{a^+,\sigma-1}(x;p) \\
&+ \left(\frac{f(b)+f(x)}{s+1}\right)(b-x)H^w_{b^-,\sigma'-1}(x;p), x \in [a,b].
\end{aligned} \tag{8.131}$$

Proof. As f is s-convex so for $t \in [a,x]$, we have

$$f(t) \leq \left(\frac{x-t}{x-a}\right)^s f(a) + \left(\frac{t-a}{x-a}\right)^s f(x). \tag{8.132}$$

First multiplying (8.48) and (8.132). Then integrating over $[a,x]$, we get

$$\begin{aligned}
&\int_a^x (x-t)^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-t)^\rho;p) f(t)dt \\
&\leq \frac{(x-a)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-a)^\rho;p)}{(x-a)^s}\{f(a)\int_a^x (x-t)^s dt + f(x)\int_a^x (t-a)^s dt\},
\end{aligned}$$

and then we have

$$\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau} f\right)(x;p) \leq (x-a)H^w_{a^+,\sigma-1}(x;p)\left(\frac{f(a)+f(x)}{s+1}\right). \tag{8.133}$$

Again for $t \in [x,b]$ using convexity of f we have

$$f(t) \leq \left(\frac{t-x}{b-x}\right)^s f(b) + \left(\frac{b-t}{b-x}\right)^s f(x). \tag{8.134}$$

Multiplying (8.52) and (8.134), then integrating over $[x,b]$, we get

$$\int_x^b (t-x)^{\sigma'-1} E_{\rho,\sigma',\tau}^{\delta,c,q,r}(w(t-x)^{\rho};p) f(t)dt$$
$$\leq \frac{(b-x)^{\sigma'-1} E_{\rho,\sigma',\tau}^{\delta,c,q,r}(w(b-x)^{\rho};p)}{(b-x)^s} \{f(b)\int_a^x (t-x)^s dt + f(x)\int_a^x (b-t)^s dt\},$$

and then we have

$$\left(\varepsilon_{b^-,\rho,\sigma',\tau}^{w,\delta,c,q,r} f\right)(x;p) \leq (b-x) H_{b^-,\sigma'-1}^{w}(x;p)\left(\frac{f(b)+f(x)}{s+1}\right). \tag{8.135}$$

Adding (8.133) and (8.135), the required inequality (8.131) is obtained. □

Some special cases are studied in the following corollaries.

Corollary 8.20 *If we set $\sigma = \sigma'$ in (8.131), then we get following inequality:*

$$\begin{aligned} &\left(\varepsilon_{a^+,\rho,\sigma,\tau}^{w,\delta,c,q,r} f\right)(x;p) + \left(\varepsilon_{b^-,\rho,\sigma,\tau}^{w,\delta,c,q,r} f\right)(x;p) \\ &\leq \left(\frac{f(a)+f(x)}{s+1}\right)(x-a) H_{a^+,\sigma-1}^{w}(x;p) \\ &+ \left(\frac{f(b)+f(x)}{s+1}\right)(b-x) H_{b^-,\sigma-1}^{w}(x;p), x \in [a,b]. \end{aligned} \tag{8.136}$$

Corollary 8.21 *Along with assumption of Theorem 1, if $f \in L_\infty[a,b]$, then we get following inequality:*

$$\begin{aligned} &\left(\varepsilon_{a^+,\rho,\sigma,\tau}^{w,\delta,c,q,r} f\right)(x;p) + \left(\varepsilon_{b^-,\rho,\sigma',\tau}^{w,\delta,c,q,r} f\right)(x;p) \\ &\leq \frac{||f||_\infty}{s+1}\left[(x-a) H_{a^+,\sigma-1}^{w}(x;p) + (b-x) H_{b^-,\sigma'-1}^{w}(x;p)\right]. \end{aligned} \tag{8.137}$$

Corollary 8.22 *If $\sigma = \sigma'$ in (8.137), then we get following result:*

$$\begin{aligned} &\left(\varepsilon_{a^+,\rho,\sigma,\tau}^{w,\delta,c,q,r} f\right)(x;p) + \left(\varepsilon_{b^-,\rho,\sigma,\tau}^{w,\delta,c,q,r} f\right)(x;p) \\ &\leq \frac{||f||_\infty}{s+1}\left[(x-a) H_{a^+,\sigma-1}^{w}(x;p) + (b-x) H_{b^-,\sigma-1}^{w}(x;p)\right]. \end{aligned} \tag{8.138}$$

Corollary 8.23 *If $s = 1$ in (8.137), then we get following result:*

$$\begin{aligned} &\left(\varepsilon_{a^+,\rho,\sigma,\tau}^{w,\delta,c,q,r} f\right)(x;p) + \left(\varepsilon_{b^-,\rho,\sigma',\tau}^{w,\delta,c,q,r} f\right)(x;p) \\ &\leq \frac{||f||_\infty}{2}\left[(x-a) H_{a^+,\sigma-1}^{w}(x;p) + (b-x) H_{b^-,\sigma'-1}^{w}(x;p)\right]. \end{aligned} \tag{8.139}$$

Theorem 8.27 *With the assumption of Theorem 8.26 if $f \in L_\infty[a,b]$, then operators defined in (2.12) and (2.13) are bounded and continuous.*

Proof. Let $f \in L_\infty[a,b]$. Then from (8.133) we have

$$\left|\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)\right| \leq 2||f||_\infty|x-a|H^w_{\sigma-1,a^+}(x;p) \leq \frac{2(b-a)H^w_{\sigma-1,a^+}(b;p)||f||_\infty}{s+1}. \tag{8.140}$$

That is

$$\left|\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)\right| \leq M||f||_\infty,$$

where $M = \frac{2(b-a)H^w_{\sigma-1,a^+}(b;p)}{s+1}$.

Therefore $\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)$ is bounded also it is easy to see that it is linear, hence this is continuous operator. On the other hand, from (8.135) we obtain

$$\left|\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p)\right| \leq K||f||_\infty,$$

where $K = \frac{2(b-a)H^w_{\sigma'-1,b^-}(a;p)}{s+1}$. Therefore $\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p)$ is bounded and linear, hence it is continuous operator. □

Lemma 8.9 *Let $f:[a,b]\to\mathbb{R}$ be a s-convex function. If f is symmetric about $\frac{a+b}{2}$, then the following inequality holds*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2^{s-1}}f(x). \tag{8.140}$$

Theorem 8.28 *Let $w\in\mathbb{R}$, $r,\rho,\sigma,\tau>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $f:[a,b]\longrightarrow\mathbb{R}$, $a>b$, be a real valued function. If f is positive, s-convex and symmetric about $\frac{a+b}{2}$, then for $\sigma,\sigma'>0$, the following fractional integral inequality for generalized integral operators holds:*

$$\begin{aligned} &2^{s-1}f\left(\frac{a+b}{2}\right)\left[H^w_{b^-,\sigma'+1}(a;p)+H^w_{a^+,\sigma+1}(b;p)\right] \\ &\leq \left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p)+\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p) \\ &\leq \left[H^w_{b^-,\sigma'-1}(a;p)+H^w_{a^+,\sigma-1}(b;p)\right](b-a)^2\left(\frac{f(a)+f(b)}{s+1}\right). \end{aligned} \tag{8.141}$$

Proof. As f is s-convex so for $x\in[a,b]$, we have:

$$f(x) \leq \left(\frac{x-a}{b-a}\right)^s f(b)+\left(\frac{b-x}{b-a}\right)^s f(a). \tag{8.142}$$

Multiplying (8.58) and (8.142), then integrating over $[a,b]$, we get

$$\begin{aligned} &\int_a^b (x-a)^{\sigma'}E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(x-a)^\rho;p)f(x)dx \\ &\leq \frac{(b-a)^{\sigma'}E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(b-a)^\rho;p)}{(b-a)^s}\left[f(b)\int_a^b(x-a)^s dx+f(a)\int_a^b(b-x)^s dx\right]. \end{aligned}$$

From which we have

$$\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p) \leq (b-a)^{\sigma'+1}E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(b-a)^{\rho};p)\left(\frac{f(a)+f(b)}{s+1}\right) \tag{8.143}$$

$$\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p) \leq (b-a)^{2}H^{w}_{b^-,\sigma'-1}(a;p)\left(\frac{f(a)+f(b)}{s+1}\right). \tag{8.144}$$

Multiplying (8.61) and (8.142), then integrating over $[a,b]$, we get

$$\begin{aligned}&\int_a^b (b-x)^{\sigma}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(b-x)^{\rho};p)f(x)dx\\ &\leq \frac{(b-a)^{\sigma}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(b-a)^{\rho};p)}{(b-a)^s}\left[f(b)\int_a^b (x-a)^s dx + f(a)\int_a^b (b-x)^s dx\right].\end{aligned}$$

From which we have

$$\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p) \leq (b-a)^{\sigma+1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(b-a)^{\rho};p)\left(\frac{f(a)+f(b)}{s+1}\right) \tag{8.145}$$

$$\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p) \leq (b-a)^{2}H^{w}_{a^+,\sigma-1}(b;p)\left(\frac{f(a)+f(b)}{s+1}\right). \tag{8.146}$$

Adding (8.144) and (8.146), we get

$$\begin{aligned}&\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p) + \left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p)\\ &\leq \left[H^{w}_{b^-,\sigma'-1}(a;p) + H^{w}_{a^+,\sigma-1}(b;p)\right](b-a)^2\left(\frac{f(a)+f(b)}{s+1}\right).\end{aligned} \tag{8.147}$$

Multiplying (8.140) by $(x-a)^{\sigma'}E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(x-a)^{\rho};p)$ and integrating over $[a,b]$

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\int_a^b (x-a)^{\sigma'}E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(x-a)^{\rho};p)dx\\ &\leq \frac{1}{2^{s-1}}\int_a^b (x-a)^{\sigma'}E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(x-a)^{\rho};p)f(x)dx.\end{aligned} \tag{8.148}$$

By using (2.2), we get

$$f\left(\frac{a+b}{2}\right)H^{w}_{b^-,\sigma'+1}(a;p) \leq \frac{1}{2^{s-1}}\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p). \tag{8.149}$$

Multiplying (8.140) with $(b-x)^{\sigma}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(b-x)^{\rho};p)$ and integrating over $[a,b]$, also using (2.2), we get

$$f\left(\frac{a+b}{2}\right)H^{w}_{a^+,\sigma+1}(b;p) \leq \frac{1}{2^{s-1}}\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p). \tag{8.150}$$

By adding (8.149) and (8.150), we get;

$$2^{s-1} f\left(\frac{a+b}{2}\right)\left[H^{w}_{b^-,\sigma'+1}(a;p)+H^{w}_{a^+,\sigma+1}(b;p)\right] \tag{8.151}$$
$$\leq \left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau} f\right)(a;p)+\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau} f\right)(b;p).$$

By adding (8.147) and (8.151), inequality (8.141) is obtained. □

Corollary 8.24 *If we take $\sigma=\sigma'$ in (8.141), then we get*

$$2^{s-1} f\left(\frac{a+b}{2}\right)\left[H^{w}_{b^-,\sigma+1}(a;p)+H^{w}_{a^+,\sigma+1}(b;p)\right] \tag{8.152}$$
$$\leq \left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma+1,\tau} f\right)(a;p)+\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau} f\right)(b;p)$$
$$\leq \left[H^{w}_{b^-,\sigma-1}(a;p)+H^{w}_{a^+,\sigma-1}(b;p)\right](b-a)^2\left(\frac{f(a)+f(b)}{s+1}\right).$$

Theorem 8.29 *Let $w\in\mathbb{R}$, $r,\rho,\sigma,\tau>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $f:[a,b]\longrightarrow\mathbb{R}$ be a real valued function. If f is differentiable and $|f'|$ is s-convex, then for $\sigma,\sigma'\geq 1$, the following fractional integral inequality for generalized integral operators holds:*

$$\left|\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau} f\right)(x;p)+\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau} f\right)(x;p)\right. \tag{8.153}$$
$$\left.-\left(H^{w}_{a^+,\sigma-1}(x;p)f(a)+H^{w}_{b^-,\sigma'-1}(x;p)f(b)\right)\right|$$
$$\leq \left(\frac{|f'(a)|+|f'(x)|}{s+1}\right)(x-a)H^{w}_{a^+,\sigma-1}(x;p)$$
$$+\left(\frac{|f'(b)|+|f'(x)|}{s+1}\right)(b-x)H^{w}_{b^-,\sigma'-1}(x;p), x\in[a,b].$$

Proof. Let $x\in[a,b]$ and $t\in[a,x]$, by using s-convexity of $|f'|$, we have

$$|f'(t)|\leq\left(\frac{x-t}{x-a}\right)^s|f'(a)|+\left(\frac{t-a}{x-a}\right)^s|f'(x)|. \tag{8.154}$$

From (8.154) we have

$$f'(t)\leq\left(\frac{x-t}{x-a}\right)^s|f'(a)|+\left(\frac{t-a}{x-a}\right)^s|f'(x)|. \tag{8.155}$$

The product of (8.48) and (8.155), gives the following inequality

$$(x-t)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-t)^{\rho};p)f'(t)dt \tag{8.156}$$
$$\leq (x-a)^{\sigma-2}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-a)^{\rho};p)\left(|f'(a)|(x-t)+|f'(x)|(t-a)\right).$$

After integrating above inequality over $[a,x]$, we get

$$\int_a^x (x-t)^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w(x-t)^\rho;p) f'(t)dt \tag{8.157}$$
$$\leq (x-a)^{\sigma-2} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w(x-a)^\rho;p)\{|f'(a)|\int_a^x (x-t)dt + |f'(x)|\int_a^x (t-a)dt\}$$
$$= (x-a)^{\sigma} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w(x-t)^\rho;p)\left(\frac{|f'(a)|+|f'(x)|}{s+1}\right).$$

By using (8.72), (8.157) takes the following form

$$\left(H_{\sigma-1,a^+}^w(x;p)\right) f(a) - \left(\varepsilon_{a^+,\rho,\sigma+1,\tau}^{w,\delta,c,q,r} f\right)(x;p) \leq (x-a) H_{\sigma-1,a^+}^w(x;p)\left(\frac{|f'(a)|+|f'(x)|}{s+1}\right). \tag{8.158}$$

Also, from (8.154) we get

$$f'(t) \geq -\left(\left(\frac{x-t}{x-a}\right)^s |f'(a)| + \left(\frac{t-a}{x-a}\right)^s |f'(x)|\right). \tag{8.159}$$

Following the same procedure as we did for (8.155), we also have

$$\left(\varepsilon_{a^+,\rho,\sigma+1,\tau}^{w,\delta,c,q,r} f\right)(x;p) - H_{\sigma-1,a^+}^w(x;p) f(a) \tag{8.160}$$
$$\leq (x-a) H_{\sigma-1,a^+}^w(x;p)\left(\frac{|f'(a)|+|f'(x)|}{s+1}\right).$$

From (8.158) and (8.160), we get

$$\left|\left(\varepsilon_{a^+,\rho,\sigma+1,\tau}^{w,\delta,c,q,r} f\right)(x;p) - H_{\sigma-1,a^+}^w(x;p) f(a)\right| \tag{8.161}$$
$$\leq (x-a) H_{\sigma-1,a^+}^w(x;p)\left(\frac{|f'(a)|+|f'(x)|}{s+1}\right).$$

Now we let $x \in [a,b]$ and $t \in [x,b]$. Then by using s-convexity of $|f'|$ we have

$$|f'(t)| \leq \left(\frac{t-x}{b-x}\right)^s |f'(b)| + \left(\frac{b-t}{b-x}\right)^s |f'(x)|. \tag{8.162}$$

On the same lines as we have done for (8.48), (8.155) and (8.159) one can get from (8.52) and (8.162), the following inequality:

$$\left|\left(\varepsilon_{b^-,\rho,\sigma'+1,\tau}^{w,\delta,c,q,r} f\right)(x;p) - H_{\sigma'-1,b^-}^w(x;p) f(b)\right| \tag{8.163}$$
$$\leq (b-x) H_{\sigma'-1,b^-}^w(x;p)\left(\frac{|f'(b)|+|f'(x)|}{s+1}\right).$$

From inequalities (8.161) and (8.163) via triangular inequality (8.153) is obtained. □

Corollary 8.25 *If we take $\sigma = \sigma'$ in (8.153), then we get*

$$\left|\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(x;p) + \left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma+1,\tau}f\right)(x;p) - \left(H^w_{a^+,\sigma-1}(x;p)f(a) + H^w_{b^-,\sigma-1}(x;p)f(b)\right)\right| \leq \left(\frac{|f'(a)|+|f'(x)|}{s+1}\right)(x-a)H^w_{a^+,\sigma-1}(x;p) + \left(\frac{|f'(b)|+|f'(x)|}{s+1}\right)(b-x)H^w_{b^-,\sigma-1}(x;p), x \in [a,b]. \tag{8.164}$$

8.7 Bounds of Fractional Integral Inequalities for (s,m)-convex Functions

The results of this section are generalizations of results of previous section.

Theorem 8.30 *Let $w \in \mathbb{R}$, $r,\rho,\sigma,\tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $f : [a,b] \longrightarrow \mathbb{R}$ be a real valued function. If f is positive and (s,m)-convex, then for $\sigma,\sigma' \geq 1$, the following fractional integral inequality holds:*

$$\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p) + \left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p) \leq \left(\frac{f(a)+mf(x)}{s+1}\right)(x-a)H^w_{a^+,\sigma-1}(x;p) + \left(\frac{f(b)+mf(x)}{s+1}\right)(b-x)H^w_{b^-,\sigma'-1}(x;p), x \in [a,b]. \tag{8.165}$$

Proof. Since f is (s,m)-convex, we obtain

$$f(t) \leq \left(\frac{x-t}{x-a}\right)^s f(a) + m\left(\frac{t-a}{x-a}\right)^s f(x). \tag{8.166}$$

By multiplying (8.48) and (8.166) and then integrating over $[a,x]$, we get

$$\int_a^x (x-t)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-t)^\rho;p)f(t)dt \leq (x-a)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-a)^\rho;p)\left(\frac{f(a)}{(x-a)^s}\int_a^x (x-t)^s dt + mf(x)\int_a^x \left(\frac{t-a}{x-a}\right)^s dt\right),$$

that is, the left integral operator satisfies the following inequality:

$$\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p) \leq (x-a)H^w_{a^+,\sigma-1}(x;p)\left(\frac{f(a)+mf(x)}{s+1}\right). \tag{8.167}$$

Again from (s,m)-convexity of f, we have

$$f(t) \le \left(\frac{t-x}{b-x}\right)^s f(b) + m\left(\frac{b-t}{b-x}\right)^s f(x). \tag{8.168}$$

By multiplying (8.52) and (8.168) and then integrating over $[x,b]$, we have

$$\int_x^b (t-x)^{\sigma'-1} E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(t-x)^\rho;p)f(t)dt$$
$$\le (b-x)^{\sigma'-1} E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(b-x)^\rho;p)\left(\frac{f(a)}{(b-x)^s}\int_x^b (t-x)^s dt\right.$$
$$\left. + mf(x)\int_x^b \left(\frac{b-t}{b-x}\right)^s dt\right),$$

that is, the right integral operator satisfies the following inequality:

$$\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau} f\right)(x;p) \le (b-x)H^w_{b^-,\sigma'-1}(x;p)\left(\frac{f(b)+mf(x)}{s+1}\right). \tag{8.169}$$

By Adding (8.167) and (8.169), the required inequality (8.165) is established. □

Some particular results are stated in the following corollaries.

Corollary 8.26 *If we set $\sigma = \sigma'$ in (8.165), then the following inequality is obtained:*

$$\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau} f\right)(x;p) + \left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma,\tau} f\right)(x;p) \tag{8.170}$$
$$\le \left(\frac{f(a)+mf(x)}{s+1}\right)(x-a)H^w_{a^+,\sigma-1}(x;p)$$
$$+ \left(\frac{f(b)+mf(x)}{s+1}\right)(b-x)H^w_{b^-,\sigma-1}(x;p), x \in [a,b].$$

Corollary 8.27 *Along with assumptions of Theorem 1, if $f \in L_\infty[a,b]$, then the following inequality is obtained:*

$$\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau} f\right)(x;p) + \left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau} f\right)(x;p) \tag{8.171}$$
$$\le \frac{||f||_\infty(1+m)}{s+1}\left[(x-a)H^w_{a^+,\sigma-1}(x;p) + (b-x)H^w_{b^-,\sigma'-1}(x;p)\right].$$

Corollary 8.28 *If we take $\sigma = \sigma'$ in (8.171), then we get the following result:*

$$\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau} f\right)(x;p) + \left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau} f\right)(x;p) \tag{8.172}$$
$$\le \frac{||f||_\infty(1+m)}{s+1}\left[(x-a)H^w_{a^+,\sigma-1}(x;p) + (b-x)H^w_{b^-,\sigma-1}(x;p)\right].$$

Corollary 8.29 *If we take $s=1$ in (8.171), then we get the following result:*

$$\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)+\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p) \tag{8.173}$$
$$\leq \frac{||f||_\infty(1+m)}{2}\left[(x-a)H^w_{a^+,\sigma-1}(x;p)+(b-x)H^w_{b^-,\sigma'-1}(x;p)\right].$$

Theorem 8.31 *With the assumptions of Theorem 1 if $f\in L_\infty[a,b]$, then operators defined in (2.12) and (2.13) are bounded and continuous.*

Proof. If $f\in L_\infty[a,b]$, then from (8.167) we have

$$\left|\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)\right| \leq \frac{2||f||_\infty(1+m)|x-a|H^w_{a^+,\sigma-1}(x;p)}{s+1} \tag{8.174}$$
$$\leq \frac{2||f||_\infty(b-a)H^w_{a^+,\sigma-1}(b;p)(1+m)}{s+1},$$

that is

$$\left|\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)\right| \leq M||f||_\infty,$$

where $M=\frac{2(b-a)H^w_{a^+,\sigma-1}(b;p)(1+m)}{s+1}$. Therefore $\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)$ is bounded also it is easy to see that it is linear, hence this is continuous operator. Also on the other hand from (8.169) we can obtain:

$$\left|\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p)\right| \leq K||f||_\infty,$$

where $K=\frac{2(b-a)H^w_{b^-,\sigma'-1}(a;p)(1+m)}{s+1}$. Therefore $\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p)$ is bounded also it is linear, hence continuous. □

The following lemma is needed to prove the next result.

Lemma 8.10 *Let $f:[a,mb]\to\mathbb{R}$ be (s,m)-convex function. If f is $f(a+mb-x)=f(x)$ and $(s,m)\in[0,1]^2$, then the following inequality holds:*

$$f\left(\frac{a+mb}{2}\right)\leq\frac{(1+m)f(x)}{2^s}. \tag{8.174}$$

Proof. As f is (s,m)-convex function, we have

$$f\left(\frac{a+mb}{2}\right)\leq\frac{f((1-t)a+mtb)}{2^s}+\frac{mf(ta+m(1-t)b)}{2^s}. \tag{8.175}$$

Let $x=a(1-t)+mtb$. Then we have $a+mb-x=ta+m(1-t)b$.

$$f\left(\frac{a+mb}{2}\right)\leq\frac{f(x)}{2^s}+m\frac{f(a+mb-x)}{2^s}. \tag{8.176}$$

Hence by using $f(a+mb-x)=f(x)$, the inequality (8.174) can be obtained. □

Theorem 8.32 *Let $w\in\mathbb{R}$, $r,\rho,\sigma,\tau>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $f:[a,b]\longrightarrow\mathbb{R}$, $a>b$, be a real valued function. If f is positive, (s,m)-convex and $f(a+mb-x)=f(x)$, then for $\sigma,\sigma'>0$, the following fractional integral inequality holds:*

$$\begin{aligned}&\frac{2^s(b-a)}{1+m}f\left(\frac{a+mb}{2}\right)\left[H^w_{b^-,\sigma'}(a;p)+H^w_{a^+,\sigma}(b;p)\right] \\ &\leq\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p)+\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p) \\ &\leq\left[H^w_{b^-,\sigma'}(a;p)+H^w_{a^+,\sigma}(b;p)\right](b-a)\left(\frac{f(b)+mf(a)}{s+1}\right).\end{aligned} \tag{8.177}$$

Proof. As f is (s,m)-convex so for $x\in[a,b]$, we have:

$$f(x)\leq\left(\frac{x-a}{b-a}\right)^s f(b)+m\left(\frac{b-x}{b-a}\right)^s f(a). \tag{8.178}$$

By multiplying (8.58) and (8.178) and then integrating over $[a,b]$, we get

$$\begin{aligned}&\int_a^b (x-a)^{\sigma'}E^{\delta,c,q,r}_{\rho,\sigma'+1,\tau}(w(x-a)^\rho;p)f(x)dx \\ &\leq(b-a)^{\sigma'}E^{\delta,c,q,r}_{\rho,\sigma'+1,\tau}(w(b-a)^\rho;p)\left(\frac{f(b)}{(b-a)^s}\int_a^b(x-a)^s dx\right. \\ &\left.+mf(a)\int_a^b\left(\frac{b-x}{b-a}\right)^s dx\right).\end{aligned}$$

from which we have

$$\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p)\leq(b-a)^{\sigma'+1}E^{\delta,c,q,r}_{\rho,\sigma'+1,\tau}(w(b-a)^\rho;p)\left(\frac{f(b)+msf(a)}{s+1}\right), \tag{8.179}$$

that is

$$\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p)\leq(b-a)H^w_{b^-,\sigma'}(a;p)\left(\frac{f(b)+mf(a)}{s+1}\right). \tag{8.180}$$

By multiplying (8.178) and (8.61)and then integrating over $[a,b]$, we get

$$\begin{aligned}&\int_a^b (b-x)^{\sigma}E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w(b-x)^\rho;p)f(x)dx \\ &\leq(b-a)^{\sigma}E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w(b-a)^\rho;p)\left(\frac{f(b)}{(b-a)^s}\int_a^b(x-a)^s dx\right. \\ &\left.+mf(a)\int_a^b\left(\frac{b-x}{b-a}\right)^s dx\right).\end{aligned}$$

from which we have

$$\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p)\leq(b-a)^{\sigma+1}E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w(b-a)^\rho;p)\left(\frac{f(b)+mf(a)}{s+1}\right), \tag{8.181}$$

that is

$$\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p) \leq (b-a)H^w_{a^+,\sigma}(b;p)\left(\frac{f(b)+msf(a)}{s+1}\right). \tag{8.182}$$

Adding (8.180) and (8.182), we get;

$$\begin{aligned}&\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p)+\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p) \\ &\leq \left[H^w_{b^-,\sigma'}(a;p)+H^w_{a^+,\sigma}(b;p)\right](b-a)\left(\frac{f(b)+mf(a)}{s+1}\right).\end{aligned} \tag{8.183}$$

Multiplying (8.174) with $(x-a)^{\sigma'}E^{\delta,c,q,r}_{\rho,\sigma'+1,\tau}(w(x-a)^{\rho};p)$ and integrating over $[a,b]$, we get

$$\begin{aligned}&f\left(\frac{a+mb}{2}\right)\int_a^b (x-a)^{\sigma'}E^{\delta,c,q,r}_{\rho,\sigma'+1,\tau}(w(x-a)^{\rho};p)dx \\ &\leq \frac{1+m}{2^s}\int_a^b (x-a)^{\sigma'}E^{\delta,c,q,r}_{\rho,\sigma'+1,\tau}(w(x-a)^{\rho};p)f(x)dx,\end{aligned} \tag{8.184}$$

$$f\left(\frac{a+mb}{2}\right)(b-a)H^w_{b^-,\sigma'}(a;p) \leq \frac{1+m}{2^s}\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p). \tag{8.185}$$

By multiplying (8.174) with $(b-x)^{\sigma}E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w(b-x)^{\rho};p)$ and integrating over $[a,b]$, and after simplification we get

$$f\left(\frac{a+mb}{2}\right)(b-a)H^w_{a^+,\sigma}(b;p) \leq \frac{1+m}{2^s}\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p). \tag{8.186}$$

By adding (8.185) and (8.186), we get;

$$\begin{aligned}&\frac{2^s}{1+m}f\left(\frac{a+mb}{2}\right)(b-a)\left[H^w_{b^-,\sigma'}(a;p)+H^w_{a^+,\sigma}(b;p)\right] \\ &\leq \left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p)+\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p).\end{aligned} \tag{8.187}$$

By combining (8.183) and (8.187), inequality (8.177) can be obtained. □

Corollary 8.30 *If we take $\sigma=\sigma'$ in (8.177), then the following inequality is obtained:*

$$\begin{aligned}&\frac{2^s(b-a)}{1+m}f\left(\frac{a+mb}{2}\right)\left[H^w_{b^-,\sigma}(a;p)+H^w_{a^+,\sigma}(b;p)\right] \\ &\leq \left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma+1,\tau}f\right)(a;p)+\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p) \\ &\leq \left[H^w_{b^-,\sigma}(a;p)+H^w_{a^+,\sigma}(b;p)\right](b-a)\left(\frac{f(b)+mf(a)}{s+1}\right).\end{aligned} \tag{8.188}$$

Theorem 8.33 *Let $w \in \mathbb{R}$, $r,\rho,\sigma,\tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $f : [a,b] \longrightarrow \mathbb{R}$ be a real valued function. If f is differentiable and $|f'|$ is (s,m)-convex, then for $\sigma,\sigma' \geq 1$, the following fractional integral inequality for generalized integral operators holds:*

$$\begin{aligned}
&\left|\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(x;p) + \left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(x;p)\right. \\
&\left. - \left(H^w_{a^+,\sigma-1}(x;p)f(a) + H^w_{b^-,\sigma'-1}(x;p)f(b)\right)\right| \\
&\leq \left(\frac{|f'(a)|+ms|f'(x)|}{s+1}\right)(x-a)H^w_{a^+,\sigma-1}(x;p) \\
&+ \left(\frac{|f'(b)|+ms|f'(x)|}{s+1}\right)(b-x)H^w_{b^-,\sigma'-1}(x;p), x \in [a,b].
\end{aligned} \tag{8.189}$$

Proof. Let $x \in [a,b]$ and $t \in [a,x)$, by using (s,m)-convexity of $|f'|$, we have

$$|f'(t)| \leq \left(\frac{x-t}{x-a}\right)^s |f'(a)| + m\left(\frac{t-a}{x-a}\right)^s |f'(x)|. \tag{8.190}$$

From (8.190) follows

$$f'(t) \leq \left(\frac{x-t}{x-a}\right)^s |f'(a)| + m\left(\frac{t-a}{x-a}\right)^s |f'(x)|. \tag{8.191}$$

The product of (8.48) and (8.191), gives the following inequality:

$$\begin{aligned}
&(x-t)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-t)^\rho;p)f'(t)dt \\
&\leq (x-a)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-a)^\rho;p)\left(\left(\frac{x-t}{x-a}\right)^s |f'(a)| + m\left(\frac{t-a}{x-a}\right)^s |f'(x)|\right).
\end{aligned} \tag{8.192}$$

After integrating above inequality over $[a,x]$, we get

$$\begin{aligned}
&\int_a^x (x-t)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-t)^\rho;p)f'(t)dt \\
&\leq (x-a)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-a)^\rho;p)\left(\frac{|f'(a)|}{(x-a)^s}\int_a^x (x-t)^s dt\right. \\
&\left. + m|f'(x)|\int_a^x \left(\frac{t-a}{x-a}\right)^s dt\right) \\
&= (x-a)^\sigma E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-a)^\rho;p)\left(\frac{|f'(a)|+ms|f'(x)|}{s+1}\right).
\end{aligned} \tag{8.193}$$

By using (8.72), (8.193) takes the following form:

$$\begin{aligned}
&H^w_{a^+,\sigma-1}(x;p)f(a) - \left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(x;p) \\
&\leq (x-a)H^w_{a^+,\sigma-1}(x;p)\left(\frac{|f'(a)|+ms|f'(x)|}{s+1}\right).
\end{aligned} \tag{8.194}$$

Also, from (8.190) we have

$$f'(t) \geq -\left(\left(\frac{x-t}{x-a}\right)^s |f'(a)| + m\left(\frac{t-a}{x-a}\right)^s |f'(x)|\right). \tag{8.195}$$

Following the same procedure as we did for (8.191), we obtain:

$$\begin{aligned}&\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(x;p) - H^w_{a^+,\sigma-1}(x;p)f(a) \\ &\leq (x-a)H^w_{a^+,\sigma-1}(x;p)\left(\frac{|f'(a)|+ms|f'(x)|}{s+1}\right).\end{aligned} \tag{8.196}$$

From (8.194) and (8.196), we get

$$\begin{aligned}&\left|\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(x;p) - H^w_{a^+,\sigma-1}(x;p)f(a)\right| \\ &\leq (x-a)H^w_{a^+,\sigma-1}(x;p)\left(\frac{|f'(a)|+ms|f'(x)|}{s+1}\right).\end{aligned} \tag{8.197}$$

Now we let $x \in [a,b]$ and $t \in (x,b]$. Then by using (s,m)-convexity of $|f'|$ we have

$$|f'(t)| \leq \left(\frac{t-x}{b-x}\right)^s |f'(b)| + m\left(\frac{b-t}{b-x}\right)^s |f'(x)|. \tag{8.198}$$

On the same lines as we have done for (8.48), (8.191) and (8.195) one can get from (8.52) and (8.198), the following inequality:

$$\begin{aligned}&\left|\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(x;p) - H^w_{b^-,\sigma'-1}(x;p)f(b)\right| \\ &\leq (b-x)H^w_{b^-,\sigma'-1}(x;p)\left(\frac{|f'(b)|+ms|f'(x)|}{s+1}\right).\end{aligned} \tag{8.199}$$

From inequalities (8.197) and (8.199) via triangular inequality (8.189) is obtained. □

Corollary 8.31 *If we take $\sigma = \sigma'$ in (8.189), then the following inequality is obtained:*

$$\begin{aligned}&\left|\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(x;p) + \left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma+1,\tau}f\right)(x;p)\right. \\ &\left. - \left(H^w_{a^+,\sigma-1}(x;p)f(a) + H^w_{b^-,\sigma-1}(x;p)f(b)\right)\right| \\ &\leq \left(\frac{|f'(a)|+ms|f'(x)|}{s+1}\right)(x-a)H^w_{a^+,\sigma-1}(x;p) \\ &+ \left(\frac{|f'(b)|+ms|f'(x)|}{s+1}\right)(b-x)H^w_{b^-,\sigma-1}(x;p), x \in [a,b].\end{aligned} \tag{8.200}$$

8.8 Bounds of Fractional Integral Operators for Exponentially s-convex Functions

The generalizations of results proved for s-convex functions are given in this section.

Theorem 8.34 *Let $w \in \mathbb{R}$, $r,\rho,\sigma,\tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $f : [a,b] \longrightarrow \mathbb{R}$ be a real valued function. If f is positive and exponentially s-convex, then for $\sigma,\sigma' \geq 1$, the following upper bound for generalized integral operators holds:*

$$\begin{aligned}
&\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p) + \left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p) \\
&\leq \left(\frac{f(a)}{e^{\alpha a}} + \frac{f(x)}{e^{\alpha x}}\right)\frac{(x-a)H^w_{a^+,\sigma-1}(x;p)}{s+1} \\
&+ \left(\frac{f(b)}{e^{\beta b}} + \frac{f(x)}{e^{\beta x}}\right)\frac{(b-x)H^w_{b^-,\sigma'-1}(x;p)}{s+1}, x \in [a,b],\ \alpha,\beta \in \mathbb{R}.
\end{aligned} \tag{8.201}$$

Proof. Since f is exponentially s-convex, we obtain

$$f(t) \leq \left(\frac{x-t}{x-a}\right)^s \frac{f(a)}{e^{\alpha a}} + \left(\frac{t-a}{x-a}\right)^s \frac{f(x)}{e^{\alpha x}}. \tag{8.202}$$

By multiplying (8.48) and (8.202) and then integrating over $[a,x]$, we get

$$\begin{aligned}
&\int_a^x (x-t)^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-t)^\rho;p) f(t)dt \\
&\leq \frac{(x-a)^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-a)^\rho;p)}{(x-a)^s}\left\{\frac{f(a)}{e^{\alpha a}}\int_a^x (x-t)^s dt + \frac{f(x)}{e^{\alpha x}}\int_a^x (t-a)^s dt\right\},
\end{aligned}$$

that is, the left integral operator follow the upcoming inequality:

$$\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p) \leq \frac{(x-a)H^w_{\sigma-1,a^+}(x;p)}{s+1}\left(\frac{f(a)}{e^{\alpha a}} + \frac{f(x)}{e^{\alpha x}}\right). \tag{8.203}$$

Again from exponentially s-convexity of f, we have

$$f(t) \leq \left(\frac{t-x}{b-x}\right)^s \frac{f(b)}{e^{\beta b}} + \left(\frac{b-t}{b-x}\right)^s \frac{f(x)}{e^{\beta x}}. \tag{8.204}$$

By multiplying (8.52) and (8.204) and then integrating over $[x,b]$, we get

$$\begin{aligned}
&\int_x^b (t-x)^{\sigma'-1} E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(t-x)^\rho;p) f(t)dt \\
&\leq \frac{(b-x)^{\sigma'-1} E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(b-x)^\rho;p)}{(b-x)^s}\left\{\frac{f(b)}{e^{\beta b}}\int_x^b (t-x)^s dt + \frac{f(x)}{e^{\beta x}}\int_x^b (b-t)^s dt\right\}
\end{aligned}$$

that is, the right integral operator satisfies the following inequality:

$$\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p) \leq \frac{(b-x)H^{w}_{\sigma'-1,b^-}(x;p)}{s+1}\left(\frac{f(b)}{e^{\beta b}}+\frac{f(x)}{e^{\beta x}}\right). \tag{8.205}$$

By adding (8.203) and (8.205), the required inequality (8.201) can be obtained. □

Some particular results are stated in the following corollaries.

Corollary 8.32 *If we set $\sigma = \sigma'$ in (8.201), then the following inequality is obtained:*

$$\begin{aligned}&\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)+\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(x;p) \\ &\leq \left(\frac{f(a)}{e^{\alpha a}}+\frac{f(x)}{e^{\alpha x}}\right)\frac{(x-a)H^{w}_{a^+,\sigma-1}(x;p)}{s+1} \\ &+\left(\frac{f(b)}{e^{\beta b}}+\frac{f(x)}{e^{\beta x}}\right)\frac{(b-x)H^{w}_{b^-,\sigma-1}(x;p)}{s+1}, x\in[a,b].\end{aligned} \tag{8.206}$$

Corollary 8.33 *Along with assumption of Theorem 1, if $f \in L_\infty[a,b]$, then the following inequality is obtained:*

$$\begin{aligned}&\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)+\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p) \\ &\leq \frac{||f||_\infty}{s+1}\left[\left(\frac{1}{e^{\alpha a}}+\frac{1}{e^{\alpha x}}\right)(x-a)H^{w}_{a^+,\sigma-1}(x;p)\right. \\ &\left.+\left(\frac{1}{e^{\beta b}}+\frac{1}{e^{\beta x}}\right)(b-x)H^{w}_{b^-,\sigma'-1}(x;p)\right].\end{aligned} \tag{8.207}$$

Corollary 8.34 *If we take $\sigma = \sigma'$ in (8.207), then we get following result:*

$$\begin{aligned}&\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)+\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(x;p) \\ &\leq \frac{||f||_\infty}{s+1}\left[\left(\frac{1}{e^{\alpha a}}+\frac{1}{e^{\alpha x}}\right)(x-a)H^{w}_{a^+,\sigma-1}(x;p)\right. \\ &\left.+\left(\frac{1}{e^{\beta b}}+\frac{1}{e^{\beta x}}\right)(b-x)H^{w}_{b^-,\sigma-1}(x;p)\right].\end{aligned} \tag{8.208}$$

Corollary 8.35 *If we take $s = 1$ in (8.207), then following result for exponentially convex functions hold:*

$$\begin{aligned}&\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)+\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p) \\ &\leq \frac{||f||_\infty}{2}\left[\left(\frac{1}{e^{\alpha a}}+\frac{1}{e^{\alpha x}}\right)(x-a)H^{w}_{a^+,\sigma-1}(x;p)\right. \\ &\left.+\left(\frac{1}{e^{\beta b}}+\frac{1}{e^{\beta x}}\right)(b-x)H^{w}_{b^-,\sigma'-1}(x;p)\right].\end{aligned} \tag{8.209}$$

Theorem 8.35 *With the assumptions of Theorem 1 if $f \in L_\infty[a,b]$, then operators defined in (2.12) and (2.13) are linear, bounded and continuous.*

Proof. If $f \in L_\infty[a,b]$, then from (8.167) we have

$$\left|\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)\right| \leq \frac{2||f||_\infty (x-a)H^w_{a^+,\sigma-1}(x;p)}{s+1}\left(\frac{1}{e^{\alpha a}}+\frac{1}{e^{\alpha x}}\right) \tag{8.210}$$
$$\leq \frac{2(b-a)H^w_{a^+,\sigma-1}(b;p)}{s+1}\left(\frac{1}{e^{\alpha a}}+\frac{1}{e^{\alpha x}}\right)||f||_\infty,$$

that is $\left|\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)\right| \leq M||f||_\infty$, where $M = \frac{2(b-a)H^w_{a^+,\sigma-1}(b;p)}{2e^{\alpha a}(s+1)}$, $\alpha \geq 0$ and for $\alpha < 0, M = \frac{2(b-a)H^w_{a^+,\sigma-1}(b;p)}{s+1}\left(\frac{1}{e^{\alpha a}}+\frac{1}{e^{\alpha b}}\right)$. Therefore $\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)$ is bounded also it is easy to see that it is linear, hence this is continuous operator. On the other hand, from (8.169) we obtain:

$$\left|\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma',\tau}f\right)(x;p)\right| \leq K||f||_\infty,$$

where $K = \frac{2(b-a)H^w_{b^-,\sigma'-1}(a;p)}{2e^{\beta a}(s+1)}$, $\beta \geq 0$ and for $\beta < 0, K = \frac{2(b-a)H^w_{b^-,\sigma'-1}(a;p)}{s+1}\left(\frac{1}{e^{\beta a}}+\frac{1}{e^{\beta b}}\right)$.
Therefore $\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma',\tau}f\right)(x;p)$ is bounded also it is linear, hence continuous. □

Definition 8.1 *Let $f:[a,b]\to\mathbb{R}$ be a function, we say f is exponentially symmetric about $\frac{a+b}{2}$, if*

$$\frac{f(x)}{e^{\alpha x}} = \frac{f(a+b-x)}{e^{\alpha(a+b-x)}}, \alpha \in \mathbb{R}. \tag{8.211}$$

It is required to give the following lemma which will be helpful to produce Hadamard type estimations.

Lemma 8.11 *Let $f:[a,b]\to\mathbb{R}$ be an exponentially s-convex function. If f is exponentially symmetric, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(x)}{2^{s-1}e^{\alpha x}}. \tag{8.212}$$

Proof. For $[a,b]\subset\mathbb{R}$ be a closed interval, $t\in[0,1]$ and $\alpha\in\mathbb{R}$, we have
$f\left(\frac{a+b}{2}\right) = f\left(\frac{at+(1-t)b}{2}+\frac{a(1-t)+bt}{2}\right)$
Since f is exponentially s-convex, so

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(at+(1-t)b)}{2^s e^{\alpha(at+(1-t)b)}} + \frac{f(a(1-t)+bt)}{2^s e^{\alpha(a(1-t)+bt)}}. \tag{8.213}$$

Let $x = at+(1-t)b$, where $x\in[a,b]$. Then we have $a+b-x = bt+(1-t)a$ and we get

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(x)}{2^s e^{\alpha(x)}} + \frac{f(a+b-x)}{2^s e^{\alpha(a+b-x)}}. \tag{8.214}$$

Now using the fact of exponentially symmetric we will get (8.212). □

Theorem 8.36 *Let $w \in \mathbb{R}$, $r, \rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let $f : [a,b] \longrightarrow \mathbb{R}$, $a > b$, be a real valued function. If f is positive, exponentially s-convex and symmetric about $\frac{a+b}{2}$, then for $\sigma, \tau > 0$, the following fractional integral inequality for generalized integral operators holds:*

$$
\begin{aligned}
&2^{s-1} e^{\alpha x} f\left(\frac{a+b}{2}\right)\left[H^{w}_{b^-,\sigma'+1}(a;p) + H^{w}_{a^+,\sigma+1}(b;p)\right] \\
&\leq \left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau} f\right)(a;p) + \left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau} f\right)(b;p) \\
&\leq \left[H^{w}_{a^+,\sigma-1}(a;p) + H^{w}_{a^+,\sigma-1}(b;p)\right] \frac{(b-a)^2}{s+1}\left(\frac{f(a)}{e^{\alpha a}} + \frac{f(b)}{e^{\beta b}}\right).
\end{aligned} \tag{8.215}
$$

Proof. As f is exponentially s-convex, for $x \in [a,b]$, we have:

$$
f(x) \leq \left(\frac{x-a}{b-a}\right)^s \frac{f(b)}{e^{\beta b}} + \left(\frac{b-x}{b-a}\right)^s \frac{f(a)}{e^{\alpha a}}. \tag{8.216}
$$

By multiplying (8.58) and (8.216) and then integrating over $[a,b]$, we get

$$
\begin{aligned}
&\int_a^b (x-a)^t E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(x-a)^\rho;p) f(x)dx \\
&\leq (b-a)^{\sigma'-s} E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(b-a)^\rho;p)\left[\frac{f(b)}{e^{\beta b}}\int_a^b (x-a)^s dx + \frac{f(a)}{e^{\alpha a}}\int_a^b (b-x)^s dx\right].
\end{aligned}
$$

From which we have

$$
\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau} f\right)(a;p) \leq \frac{(b-a)^{\sigma'+1} E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(b-a)^\rho;p)}{s+1}\left(\frac{f(a)}{e^{\alpha a}} + \frac{f(b)}{e^{\beta b}}\right), \tag{8.217}
$$

$$
\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau} f\right)(a;p) \leq \frac{(b-a)^2}{s+1} H^{w}_{\sigma'-1,b^-}(a;p)\left(\frac{f(a)}{e^{\alpha a}} + \frac{f(b)}{e^{\beta b}}\right). \tag{8.218}
$$

By multiplying (8.216) and (8.61) and then integrating over $[a,b]$, we get

$$
\begin{aligned}
&\int_a^b (b-x)^\sigma E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(b-x)^\rho;p) f(x)dx \\
&\leq (b-a)^{\sigma-s} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(b-a)^\rho;p)\left[\frac{f(b)}{e^{\beta b}}\int_a^b (x-a)^s dx + \frac{f(a)}{e^{\alpha a}}\int_a^b (b-x)^s dx\right].
\end{aligned}
$$

From which we have

$$
\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau} f\right)(b;p) \leq \frac{(b-a)^{\sigma+1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(b-a)^\rho;p)}{s+1}\left(\frac{f(a)}{e^{\alpha a}} + \frac{f(b)}{e^{\beta b}}\right), \tag{8.219}
$$

$$
\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau} f\right)(b;p) \leq \frac{(b-a)^2}{s+1} H^{w}_{\sigma-1,a^+}(b;p)\left(\frac{f(a)}{e^{\alpha a}} + \frac{f(b)}{e^{\beta b}}\right). \tag{8.220}
$$

Adding (8.218) and (8.220), we get;

$$\begin{aligned}&\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p)+\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p)\\&\leq\left[H^{w}_{\sigma'-1,b^-}(a;p)+H^{w}_{\sigma-1,a^+}(b;p)\right]\frac{(b-a)^2}{s+1}\left(\frac{f(a)}{e^{\alpha a}}+\frac{f(b)}{e^{\beta b}}\right).\end{aligned}\tag{8.221}$$

Multiplying (8.212) with $(x-a)^t E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(x-a)^{\rho};p)$ and integrating over $[a,b]$, we get

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\int_a^b (x-a)^t E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(x-a)^{\rho};p)dx\\&\leq\frac{1}{2^{s-1}e^{\alpha x}}\int_a^b (x-a)^t E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(x-a)^{\rho};p)f(x)dx.\end{aligned}\tag{8.222}$$

By using Definition (2.2), we get

$$f\left(\frac{a+b}{2}\right)H^{w}_{\sigma'+1,b^-}(a;p)\leq\frac{1}{2^{s-1}e^{\alpha x}}\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p).\tag{8.223}$$

Multiplying (8.212) with $(b-x)^{\sigma}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(b-x)^{\rho};p)$ and integrating over $[a,b]$, we get

$$f\left(\frac{a+b}{2}\right)H^{w}_{\sigma+1,a^+}(b;p)\leq\frac{1}{2^{s-1}e^{\alpha x}}\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p).\tag{8.224}$$

By adding (8.223) and (8.224), we get;

$$\begin{aligned}&2^{s-1}e^{\alpha x}f\left(\frac{a+b}{2}\right)\left[H^{w}_{\sigma'+1,b^-}(a;p)+H^{w}_{\sigma+1,a^+}(b;p)\right]\\&\leq\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p)+\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p).\end{aligned}\tag{8.225}$$

By combining (8.221) and (8.225), inequality (8.215) can be obtained. □

Corollary 8.36 *If we take $\sigma=\sigma'$ in (8.215), then the following inequality is obtained:*

$$\begin{aligned}&2^{s-1}e^{\alpha x}f\left(\frac{a+b}{2}\right)\left[H^{w}_{b^-,\sigma+1}(a;p)+H^{w}_{a^+,\sigma+1}(b;p)\right]\\&\leq\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma+1,\tau}f\right)(a;p)+\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p)\\&\leq\left[H^{w}_{b^-,\sigma-1}(a;p)+H^{w}_{a^+,\sigma-1}(b;p)\right]\frac{(b-a)^2}{s+1}\left(\frac{f(a)}{e^{\alpha a}}+\frac{f(b)}{e^{\beta b}}\right).\end{aligned}\tag{8.226}$$

Theorem 8.37 *Let $w \in \mathbb{R}$, $r, \rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let $f : [a,b] \longrightarrow \mathbb{R}$, be a real valued function. If f is differentiable and $|f'|$ is exponentially s-convex, then for $\sigma, \sigma' \geq 1$, the following fractional integral inequality for generalized integral operators holds:*

$$\begin{aligned}
&\left|\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(x;p) + \left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(x;p)\right. \\
&\left. - \left(H^w_{a^+,\sigma-1}(x;p)f(a) + H^w_{a^+,\sigma-1}(x;p)f(b)\right)\right| \\
&\leq \left(\frac{|f'(a)|}{e^{\alpha a}} + \frac{|f'(x)|}{e^{\alpha x}}\right)\frac{(x-a)H^w_{a^+,\sigma-1}(x;p)}{s+1} \\
&+ \left(\frac{|f'(b)|}{e^{\beta b}} + \frac{|f'(x)|}{e^{\beta x}}\right)\frac{(b-x)H^w_{a^+,\sigma-1}(x;p)}{s+1}, x \in [a,b].
\end{aligned} \tag{8.227}$$

Proof. Let $x \in [a,b]$ and $t \in [a,x]$, by using exponentially s-convexity of $|f'|$, we have

$$|f'(t)| \leq \left(\frac{x-t}{x-a}\right)^s \frac{|f'(a)|}{e^{\alpha a}} + \left(\frac{t-a}{x-a}\right)^s \frac{|f'(x)|}{e^{\alpha x}}. \tag{8.228}$$

From (8.228) follows

$$f'(t) \leq \left(\frac{x-t}{x-a}\right)^s \frac{|f'(a)|}{e^{\alpha a}} + \left(\frac{t-a}{x-a}\right)^s \frac{|f'(x)|}{e^{\alpha x}}. \tag{8.229}$$

The product of (8.48) and (8.229), gives the following inequality

$$\begin{aligned}
&(x-t)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-t)^\rho;p)f'(t)dt \\
&\leq (x-a)^{\sigma-1-s}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-a)^\rho;p)\left(\frac{|f'(a)|}{e^{\alpha a}}(x-t)^s + \frac{|f'(x)|}{e^{\alpha x}}(t-a)^s\right).
\end{aligned} \tag{8.230}$$

After integrating above inequality over $[a,x]$, we get

$$\begin{aligned}
&\int_a^x (x-t)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-t)^\rho;p)f'(t)dt \\
&\leq (x-a)^{\sigma-1-s}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-a)^\rho;p)\left\{\frac{|f'(a)|}{e^{\alpha a}}\int_a^x (x-t)^s dt + \frac{|f'(x)|}{e^{\alpha x}}\int_a^x (t-a)^s dt\right\} \\
&= \frac{(x-a)^\sigma E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-t)^\rho;p)}{s+1}\left(\frac{|f'(a)|}{e^{\alpha a}} + \frac{|f'(x)|}{e^{\alpha x}}\right).
\end{aligned} \tag{8.231}$$

By using (8.72), (8.231) takes the following form:

$$\begin{aligned}
&\left(H^w_{\sigma-1,a^+}(x;p)\right)f(a) - \left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(x;p) \\
&\leq \frac{(x-a)H^w_{\sigma-1,a^+}(x;p)}{s+1}\left(\frac{|f'(a)|}{e^{\alpha a}} + \frac{|f'(x)|}{e^{\alpha x}}\right).
\end{aligned} \tag{8.232}$$

Also from (8.228) we have

$$f'(t) \geq -\left(\left(\frac{x-t}{x-a}\right)^s \frac{|f'(a)|}{e^{\alpha a}} + \left(\frac{t-a}{x-a}\right)^s \frac{|f'(x)|}{e^{\alpha x}}\right). \tag{8.233}$$

Following the same procedure as we did for (8.229), one can obtain:

$$\begin{aligned}&\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(x;p) - H^w_{\sigma-1,a^+}(x;p)f(a)\\ &\leq \frac{(x-a)H^w_{\sigma-1,a^+}(x;p)}{s+1}\left(\frac{|f'(a)|}{e^{\alpha a}} + \frac{|f'(x)|}{e^{\alpha x}}\right).\end{aligned} \tag{8.234}$$

From (8.232) and (8.234), we get

$$\begin{aligned}&\left|\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(x;p) - H^w_{\sigma-1,a^+}(x;p)f(a)\right|\\ &\leq \frac{(x-a)H^w_{\sigma-1,a^+}(x;p)}{s+1}\left(\frac{|f'(a)|}{e^{\alpha a}} + \frac{|f'(x)|}{e^{\alpha x}}\right).\end{aligned} \tag{8.235}$$

Now we let $x \in [a,b]$ and $t \in [x,b]$. Then by exponentially s-convexity of $|f'|$ we have

$$|f'(t)| \leq \left(\frac{t-x}{b-x}\right)^s \frac{|f'(b)|}{e^{\beta b}} + \left(\frac{b-t}{b-x}\right)^s \frac{|f'(x)|}{e^{\beta x}}. \tag{8.236}$$

On the same lines as we have done for (8.48), (8.229) and (8.233) one can get from (8.52) and (8.236), the following inequality:

$$\left|\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(x;p) - H^w_{\sigma'-1,b^-}(x;p)f(b)\right| \leq \frac{(b-x)H^w_{\sigma'-1,b^-}(x;p)}{s+1}\left(\frac{|f'(b)|}{e^{\beta b}} + \frac{|f'(x)|}{e^{\beta x}}\right). \tag{8.237}$$

From inequalities (8.235) and (8.237) via triangular inequality (8.227) can be obtained. □

Corollary 8.37 *If we take $\sigma = \sigma'$ in (8.227), then the following inequality is obtained:*

$$\begin{aligned}&\left|\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(x;p) + \left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma+1,\tau}f\right)(x;p)\right.\\ &\left.-\left(H^w_{a^+,\sigma-1}(x;p)f(a) + H^w_{b^-,\sigma-1}(x;p)f(b)\right)\right|\\ &\leq \left(\frac{|f'(a)|}{e^{\alpha a}} + \frac{|f'(x)|}{e^{\alpha x}}\right)\frac{(x-a)H^w_{a^+,\sigma-1}(x;p)}{s+1}\\ &+\left(\frac{|f'(b)|}{e^{\beta b}} + \frac{|f'(x)|}{e^{\beta x}}\right)\frac{(b-x)H^w_{b^-,\sigma-1}(x;p)}{s+1}, x \in [a,b].\end{aligned} \tag{8.238}$$

8.9 Bounds of Fractional Integral Operators for Strongly (s,m)-convex Functions

The first result provides an upper bound of sum of left and right fractional integrals for strongly (s,m)-convex functions.

Theorem 8.38 *Let $w \in \mathbb{R}$, $r,\rho,\tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $f \in L_1[a,b], 0 \leq a < b$. If f is a positive and strongly (s,m)-convex function on $[a,mb]$ with modulus $\lambda > 0$, $m \in (0,1]$, then for $\sigma,\sigma' \geq 1$, the following fractional integral inequality for generalized integral operators holds:*

$$\begin{aligned}
&\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p) + \left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p) \\
&\leq \left(\frac{f(a)+mf\left(\frac{x}{m}\right)}{s+1} - \lambda\frac{(x-ma)^2}{6m}\right)(x-a)H^w_{a^+,\sigma-1}(x;p) \\
&+ \left(\frac{f(b)+mf\left(\frac{x}{m}\right)}{s+1} - \lambda\frac{(mb-x)^2}{6m}\right)(b-x)H^w_{b^-,\sigma'-1}(x;p), \quad x \in [a,b].
\end{aligned} \tag{8.239}$$

Proof. The function f is strongly (s,m)-convex function with modulus λ, therefore one can obtain

$$f(t) \leq \left(\frac{x-t}{x-a}\right)^s f(a) + m\left(\frac{t-a}{x-a}\right)^s f\left(\frac{x}{m}\right) - \lambda\frac{(x-t)(t-a)(x-ma)^2}{m(x-a)^2}. \tag{8.240}$$

By multiplying (8.48) and (8.240) and then integrating over $[a,x]$, we get

$$\begin{aligned}
&\int_a^x (x-t)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-t)^\rho;p)f(t)dt \\
&\leq (x-a)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-a)^\rho;p)\left(\frac{f(a)}{(x-a)^s}\int_a^x (x-t)^s dt\right. \\
&\left.+mf\left(\frac{x}{m}\right)\int_a^x\left(\frac{t-a}{x-a}\right)^s dt - \lambda\frac{(x-ma)^2}{m(x-a)^2}\int_a^x (x-t)(t-a)dt\right).
\end{aligned}$$

Therefore, the left fractional integral operator satisfies the following inequality:

$$\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p) \leq (x-a)H^w_{a^+,\sigma-1}(x;p)\left(\frac{f(a)+mf\left(\frac{x}{m}\right)}{s+1} - \lambda\frac{(x-ma)^2}{6m}\right). \tag{8.241}$$

Again, from strongly (s,m)-convexity of f, we have

$$f(t) \leq \left(\frac{t-x}{b-x}\right)^s f(b) + m\left(\frac{b-t}{b-x}\right)^s f\left(\frac{x}{m}\right) - \lambda\frac{(t-x)(b-t)(mb-x)^2}{m(b-x)^2}. \tag{8.242}$$

By multiplying (8.52) and (8.242) and then integrating over $[x,b]$, we have

$$\begin{aligned}
&\int_x^b (t-x)^{\sigma'-1} E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(t-x)^{\rho};p)f(t)dt \\
&\leq (b-x)^{\sigma'-1} E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(b-x)^{\rho};p)\left(\frac{f(a)}{(b-x)^s}\int_x^b (t-x)^s dt\right. \\
&\left. +mf\left(\frac{x}{m}\right)\int_x^b \left(\frac{b-t}{b-x}\right)^s dt - \lambda\frac{(mb-x)^2}{m(b-x)^2}\int_x^b (t-x)(b-t)dt\right).
\end{aligned}$$

Therefore, we have that, the right integral operator satisfies the following inequality:

$$\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p) \leq (b-x)H^w_{b^-,\sigma'-1}(x;p)\left(\frac{f(b)+mf\left(\frac{x}{m}\right)}{s+1} - \lambda\frac{(mb-x)^2}{6m}\right). \quad (8.243)$$

By adding (8.241) and (8.243), the required inequality (8.239) can be obtained. □

Some particular cases are given in the following results:

Theorem 8.39 *Let $w \in \mathbb{R}$, $r,\rho,\tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $f \in L_1[a,b], 0 \leq a < b$. If f is a positive and (s,m)-convex function on $[a,mb]$, $m \in (0,1]$, then for $\sigma,\sigma' \geq 1$, the following fractional integral inequality for generalized integral operators holds:*

$$\begin{aligned}
&\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p) + \left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p) \qquad (8.244)\\
&\leq \left(\frac{f(a)+mf\left(\frac{x}{m}\right)}{s+1}\right)(x-a)H^w_{a^+,\sigma-1}(x;p) \\
&+ \left(\frac{f(b)+mf\left(\frac{x}{m}\right)}{s+1}\right)(b-x)H^w_{b^-,\sigma'-1}(x;p), \;\; x \in [a,b].
\end{aligned}$$

Proof. For (s,m)-convex functions the inequality (8.241) holds as follows (by setting $\lambda = 0$):

$$\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p) \leq (x-a)H^w_{a^+,\sigma-1}(x;p)\left(\frac{f(a)+mf\left(\frac{x}{m}\right)}{s+1}\right). \quad (8.245)$$

Also, for (s,m)-convex functions the inequality (8.243) holds as follows (by setting $\lambda = 0$):

$$\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p) \leq (b-x)H^w_{b^-,\sigma'-1}(x;p)\left(\frac{f(b)+mf\left(\frac{x}{m}\right)}{s+1}\right). \quad (8.246)$$

By adding (8.245) and (8.246) the inequality (8.244) can be obtained. □

Corollary 8.38 *The following inequality holds for strongly convex functions by taking $s=m=1$ in (8.239):*

$$\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)+\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p) \tag{8.247}$$
$$\leq\left(\frac{f(a)+f(x)}{2}-\lambda\frac{(x-a)^2}{6}\right)(x-a)H^{w}_{a^+,\sigma-1}(x;p)$$
$$+\left(\frac{f(b)+f(x)}{2}-\lambda\frac{(b-x)^2}{6}\right)(b-x)J_{\alpha-1,b^-}(x;p),x\in[a,b].$$

Corollary 8.39 *The following inequality holds for convex functions by taking $s=m=1,\lambda=0$ in (8.239) which is proved in [36, Corollary 1]:*

$$\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)+\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p) \tag{8.248}$$
$$\leq\left(\frac{f(a)+f(x)}{2}\right)(x-a)H^{w}_{a^+,\sigma-1}(x;p)$$
$$+\left(\frac{f(b)+f(x)}{2}\right)(b-x)J_{\alpha-1,b^-}(x;p),x\in[a,b].$$

Remark 8.13 The inequality (8.247) provides refinement of inequality (8.248).

Corollary 8.40 *If we set $\sigma=\sigma'$ in (8.239), then the following inequality is obtained:*

$$\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)+\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p) \tag{8.249}$$
$$\leq\left(\frac{f(a)+mf\left(\frac{x}{m}\right)}{s+1}-\lambda\frac{(x-ma)^2}{6m}\right)(x-a)H^{w}_{a^+,\sigma-1}(x;p)$$
$$+\left(\frac{f(b)+mf\left(\frac{x}{m}\right)}{s+1}-\lambda\frac{(mb-x)^2}{6m}\right)(b-x)J_{\alpha-1,b^-}(x;p),x\in[a,b].$$

Corollary 8.41 *If we set $\sigma=\sigma'$ and $\lambda=0$ in (8.239), then the following inequality is obtained for (s,m)-convex function:*

$$\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)+\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p) \tag{8.250}$$
$$\leq\left(\frac{f(a)+mf\left(\frac{x}{m}\right)}{s+1}\right)(x-a)H^{w}_{a^+,\sigma-1}(x;p)$$
$$+\left(\frac{f(b)+mf\left(\frac{x}{m}\right)}{s+1}\right)(b-x)J_{\alpha-1,b^-}(x;p),x\in[a,b].$$

Corollary 8.42 *Along with assumptions of Theorem 1, if $f \in L_\infty[a,b]$, then the following inequality is obtained:*

$$\begin{aligned} &\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)+\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p) \\ &\le \frac{||f||_\infty(1+m)}{s+1}\left[(x-a)H^w_{a^+,\sigma-1}(x;p)+(b-x)H^w_{b^-,\sigma'-1}(x;p)\right] \\ &-\left[\lambda\frac{(x-ma)^2(x-a)}{6m}H^w_{a^+,\sigma-1}(x;p)+\lambda\frac{(mb-x)^2(b-x)}{6m}H^w_{b^-,\sigma'-1}(x;p)\right]. \end{aligned} \tag{8.251}$$

Corollary 8.43 *Along with assumptions of Theorem 1, if $f \in L_\infty[a,b]$, $\lambda = 0$, then the following inequality is obtained for (s,m)-convex function:*

$$\begin{aligned} &\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)+\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p) \\ &\le \frac{||f||_\infty(1+m)}{s+1}\left[(x-a)H^w_{a^+,\sigma-1}(x;p)+(b-x)H^w_{b^-,\sigma'-1}(x;p)\right] \end{aligned} \tag{8.252}$$

Corollary 8.44 *For $\sigma = \sigma'$ in (8.251), we get the following result:*

$$\begin{aligned} &\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)+\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p) \\ &\le \frac{||f||_\infty(1+m)}{s+1}\left[(x-a)H^w_{a^+,\sigma-1}(x;p)+(b-x)J_{\alpha-1,b^-}(x;p)\right] \\ &-\left[\lambda\frac{(x-ma)^2(x-a)}{6m}H^w_{a^+,\sigma-1}(x;p)+\lambda\frac{(mb-x)^2(b-x)}{6m}J_{\alpha-1,b^-}(x;p)\right]. \end{aligned} \tag{8.253}$$

Corollary 8.45 *For $\sigma = \sigma'$, $\lambda = 0$ in (8.251), we get the following result for (s,m)-convex function:*

$$\begin{aligned} &\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)+\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p) \\ &\le \frac{||f||_\infty(1+m)}{s+1}\left[(x-a)H^w_{a^+,\sigma-1}(x;p)+(b-x)J_{\alpha-1,b^-}(x;p)\right] \end{aligned} \tag{8.254}$$

Corollary 8.46 *For $s = 1$ in (8.251), we get the following result:*

$$\begin{aligned} &\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)+\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p) \\ &\le \frac{||f||_\infty(1+m)}{2}\left[(x-a)H^w_{a^+,\sigma-1}(x;p)+(b-x)H^w_{b^-,\sigma'-1}(x;p)\right] \\ &-\left[\lambda\frac{(x-ma)^2(x-a)}{6m}H^w_{a^+,\sigma-1}(x;p)+\lambda\frac{(mb-x)^2(b-x)}{6m}H^w_{b^-,\sigma'-1}(x;p)\right]. \end{aligned} \tag{8.255}$$

Corollary 8.47 *For $s=1,\lambda=0$ in (8.251), we get the following result for (s,m)-convex function:*

$$\begin{aligned}&\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)+\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p)\\&\leq\frac{||f||_\infty(1+m)}{2}\left[(x-a)H^w_{a^+,\sigma-1}(x;p)+(b-x)H^w_{b^-,\sigma'-1}(x;p)\right].\end{aligned}\tag{8.256}$$

Remark 8.14 The inequality (8.255) provides refinement of inequality (8.256).

Theorem 8.40 *Let $w\in\mathbb{R}$, $r,\rho,\tau>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $f\in L_1[a,b], 0\leq a<b$. If f is a differentiable and $|f'|$ is a strongly (s,m)-convex function on $[a,mb]$ with modulus $\lambda>0$, $m\in(0,1]$, then for $\sigma,\sigma'\geq 1$, the following fractional integral inequality for generalized integral operators holds:*

$$\begin{aligned}&\left|\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(x;p)+\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(x;p)\right.\\&\left.-\left(H^w_{a^+,\sigma-1}(x;p)f(a)+H^w_{b^-,\sigma'-1}(x;p)f(b)\right)\right|\\&\leq\left(\frac{|f'(a)|+m\left|f'\left(\frac{x}{m}\right)\right|}{s+1}-\lambda\frac{(x-ma)^2}{6m}\right)(x-a)H^w_{a^+,\sigma-1}(x;p)\\&+\left(\frac{|f'(b)|+m\left|f'\left(\frac{x}{m}\right)\right|}{s+1}-\lambda\frac{(mb-x)^2}{6m}\right)(b-x)H^w_{b^-,\sigma'-1}(x;p),\quad x\in[a,b].\end{aligned}\tag{8.257}$$

Proof. As $x\in[a,b]$ and $t\in[a,x)$, by using strongly (s,m)-convexity of $|f'|$, we have

$$|f'(t)|\leq\left(\frac{x-t}{x-a}\right)^s|f'(a)|+m\left(\frac{t-a}{x-a}\right)^s\left|f'\left(\frac{x}{m}\right)\right|-\lambda\frac{(x-t)(t-a)(x-ma)^2}{m(x-a)^2}.\tag{8.258}$$

From (8.258) we get

$$f'(t)\leq\left(\frac{x-t}{x-a}\right)^s|f'(a)|+m\left(\frac{t-a}{x-a}\right)^s\left|f'\left(\frac{x}{m}\right)\right|-\lambda\frac{(x-t)(t-a)(x-ma)^2}{m(x-a)^2}.\tag{8.259}$$

The product of (8.48) and (8.259), gives the following inequality:

$$\begin{aligned}&(x-t)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-t)^\rho;p)f'(t)dt\\&\leq(x-a)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-a)^\rho;p)\\&\left(\left(\frac{x-t}{x-a}\right)^s|f'(a)|+m\left(\frac{t-a}{x-a}\right)^s\left|f'\left(\frac{x}{m}\right)\right|-\lambda\frac{(x-t)(t-a)(x-ma)^2}{m(x-a)^2}\right).\end{aligned}\tag{8.260}$$

After integrating the above inequality over $[a,x]$, we get

$$\int_a^x (x-t)^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w(x-t)^\rho;p) f'(t)dt \tag{8.261}$$
$$\leq (x-a)^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w(x-a)^\rho;p) \left(\frac{|f'(a)|}{(x-a)^s} \int_a^x (x-t)^s dt \right.$$
$$\left. +m\left|f'\left(\frac{x}{m}\right)\right| \int_a^x \left(\frac{t-a}{x-a}\right)^s dt - \lambda \frac{(x-ma)^2}{m(x-a)^2} \int_a^x (x-t)(t-a)dx \right)$$
$$= (x-a)^\sigma E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w(x-a)^\rho;p) \left(\frac{|f'(a)|+m\left|f'\left(\frac{x}{m}\right)\right|}{s+1} - \lambda \frac{(x-ma)^2}{6m} \right).$$

By using (8.72), (8.261) takes the following form:

$$\left(H_{a^+,\sigma-1}^w(x;p)\right) f(a) - \left(\varepsilon_{a^+,\rho,\sigma+1,\tau}^{w,\delta,c,q,r} f\right)(x;p) \tag{8.262}$$
$$\leq (x-a) H_{a^+,\sigma-1}^w(x;p) \left(\frac{|f'(a)|+m\left|f'\left(\frac{x}{m}\right)\right|}{s+1} - \lambda \frac{(x-ma)^2}{6m} \right).$$

Also, from (8.258) follows

$$f'(t) \geq -\left(\left(\frac{x-t}{x-a}\right)^s |f'(a)| + m\left(\frac{t-a}{x-a}\right)^s \left|f'\left(\frac{x}{m}\right)\right| - \lambda \frac{(x-t)(t-a)(x-ma)^2}{m(x-a)^2} \right). \tag{8.263}$$

Following the same procedure as we did for (8.259), one can obtain:

$$\left(\varepsilon_{a^+,\rho,\sigma+1,\tau}^{w,\delta,c,q,r} f\right)(x;p) - H_{a^+,\sigma-1}^w(x;p) f(a) \tag{8.264}$$
$$\leq (x-a) H_{a^+,\sigma-1}^w(x;p) \left(\frac{|f'(a)|+m\left|f'\left(\frac{x}{m}\right)\right|}{s+1} - \lambda \frac{(x-ma)^2}{6m} \right).$$

From (8.262) and (8.264), we get

$$\left| \left(\varepsilon_{a^+,\rho,\sigma+1,\tau}^{w,\delta,c,q,r} f\right)(x;p) - H_{a^+,\sigma-1}^w(x;p) f(a) \right| \tag{8.265}$$
$$\leq (x-a) H_{a^+,\sigma-1}^w(x;p) \left(\frac{|f'(a)|+m\left|f'\left(\frac{x}{m}\right)\right|}{s+1} - \lambda \frac{(x-ma)^2}{6m} \right).$$

Now, we let $x \in [a,b]$ and $t \in (x,b]$. Then by using strongly (s,m)-convexity of $|f'|$ we have

$$|f'(t)| \leq \left(\frac{t-x}{b-x}\right)^s |f'(b)| + m\left(\frac{b-t}{b-x}\right)^s \left|f'\left(\frac{x}{m}\right)\right| - \lambda \frac{(t-x)(b-t)(mb-x)^2}{m(b-x)^2}. \tag{8.266}$$

Similarly, from (8.52) and (8.266) the following inequality can be obtained:

$$\left|\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(x;p)-H^w_{b^-,\sigma'-1}(x;p)f(b)\right| \tag{8.267}$$
$$\leq (b-x)H^w_{b^-,\sigma'-1}(x;p)\left(\frac{|f'(b)|+m\left|f'\left(\frac{x}{m}\right)\right|}{s+1}-\lambda\frac{(mb-x)^2}{6m}\right).$$

From inequalities (8.265) and (8.267) via triangular inequality, (8.257) is obtained. □

Theorem 8.41 *Let $w\in\mathbb{R}$, $r,\rho,\tau>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $f\in L_1[a,b], 0\leq a<b$. If f is a differentiable and $|f'|$ is a (s,m)-convex function on $[a,mb]$, $m\in(0,1]$, then for $\sigma,\sigma'\geq 1$, the following fractional integral inequality for generalized integral operators holds:*

$$\left|\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(x;p)+\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(x;p)\right. \tag{8.268}$$
$$\left.-\left(H^w_{a^+,\sigma-1}(x;p)f(a)+H^w_{b^-,\sigma'-1}(x;p)f(b)\right)\right|$$
$$\leq\left(\frac{|f'(a)|+m\left|f'\left(\frac{x}{m}\right)\right|}{s+1}\right)(x-a)H^w_{a^+,\sigma-1}(x;p)$$
$$+\left(\frac{|f'(b)|+m\left|f'\left(\frac{x}{m}\right)\right|}{s+1}\right)(b-x)H^w_{b^-,\sigma'-1}(x;p),\ \ x\in[a,b].$$

Proof. For the (s,m)-convex function $|f'|$ the inequality (8.265) holds as follows (by setting $\lambda=0$):

$$\left|\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(x;p)-H^w_{a^+,\sigma-1}(x;p)f(a)\right| \tag{8.269}$$
$$\leq (x-a)H^w_{a^+,\sigma-1}(x;p)\left(\frac{|f'(a)|+m\left|f'\left(\frac{x}{m}\right)\right|}{s+1}\right).$$

Also, for the (s,m)-convex function $|f'|$ the inequality (8.267) holds as follows (by setting $\lambda=0$):

$$\left|\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(x;p)-H^w_{b^-,\sigma'-1}(x;p)f(b)\right| \tag{8.270}$$
$$\leq (b-x)H^w_{b^-,\sigma'-1}(x;p)\left(\frac{|f'(b)|+m\left|f'\left(\frac{x}{m}\right)\right|}{s+1}\right).$$

By adding (8.269) and (8.270), the inequality (8.268) can be obtained. □

Remark 8.15 The inequality (8.257) provides refinement of inequality (8.268).

Corollary 8.48 *If we put $\sigma=\sigma'$ in (8.257), then the following inequality is obtained:*

$$\left|\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(x;p)+\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma+1,\tau}f\right)(x;p)\right. \tag{8.271}$$

$$
\begin{aligned}
&-\left(H^{w}_{a^+,\sigma-1}(x;p)f(a)+J_{\alpha-1,b^-}(x;p)f(b)\right)\Big| \\
&\leq \left(\frac{|f'(a)|+m\left|f'\left(\frac{x}{m}\right)\right|}{s+1}-\lambda\frac{(x-ma)^2}{6m}\right)(x-a)H^{w}_{a^+,\sigma-1}(x;p) \\
&+\left(\frac{|f'(b)|+m\left|f'\left(\frac{x}{m}\right)\right|}{s+1}-\lambda\frac{(mb-x)^2}{6m}\right)(b-x)J_{\alpha-1,b^-}(x;p), x\in[a,b].
\end{aligned}
$$

Corollary 8.49 *If we put $\sigma=\sigma'$ and $\lambda=0$ in (8.257), then the following inequality is obtained for (s,m)-convex function:*

$$
\begin{aligned}
&\left|\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(x;p)+\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma+1,\tau}f\right)(x;p)\right. \\
&-\left(H^{w}_{a^+,\sigma-1}(x;p)f(a)+J_{\alpha-1,b^-}(x;p)f(b)\right)\Big| \\
&\leq \left(\frac{|f'(a)|+m\left|f'\left(\frac{x}{m}\right)\right|}{s+1}\right)(x-a)H^{w}_{a^+,\sigma-1}(x;p) \\
&+\left(\frac{|f'(b)|+m\left|f'\left(\frac{x}{m}\right)\right|}{s+1}\right)(b-x)J_{\alpha-1,b^-}(x;p), x\in[a,b].
\end{aligned}
\tag{8.272}
$$

Remark 8.16 The inequality (8.271) provides refinement of the inequality (8.272).

Corollary 8.50 *If we put $s=m=1$ in (8.257), then the following inequality is obtained for strongly convex function:*

$$
\begin{aligned}
&\left|\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(x;p)+\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(x;p)\right. \\
&-\left(H^{w}_{a^+,\sigma-1}(x;p)f(a)+H^{w}_{b^-,\sigma'-1}(x;p)f(b)\right)\Big| \\
&\leq \left(\frac{|f'(a)|+|f'(x)|}{2}-\lambda\frac{(x-a)^2}{6}\right)(x-a)H^{w}_{a^+,\sigma-1}(x;p) \\
&+\left(\frac{|f'(b)|+|f'(x)|}{2}-\lambda\frac{(b-x)^2}{6}\right)(b-x)H^{w}_{b^-,\sigma'-1}(x;p),\ \ x\in[a,b].
\end{aligned}
\tag{8.273}
$$

Corollary 8.51 *If we put $s=m=1$ and $\lambda=0$ in (8.257), then the following inequality is obtained for convex function which is proved in [36, Corollary 2]:*

$$
\begin{aligned}
&\left|\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(x;p)+\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(x;p)\right. \\
&-\left(H^{w}_{a^+,\sigma-1}(x;p)f(a)+H^{w}_{b^-,\sigma'-1}(x;p)f(b)\right)\Big| \\
&\leq \left(\frac{|f'(a)|+|f'(x)|}{2}\right)(x-a)H^{w}_{a^+,\sigma-1}(x;p) \\
&+\left(\frac{|f'(b)|+|f'(x)|}{2}\right)(b-x)H^{w}_{b^-,\sigma'-1}(x;p),\ \ x\in[a,b].
\end{aligned}
\tag{8.274}
$$

The following lemma is useful to prove the next result.

Lemma 8.12 *Let $f:[a,mb]\to\mathbb{R}$ be strongly (s,m)-convex function with modulus λ. If $f(\frac{a+mb-x}{m})=f(x)$ and $(s,m)\in[0,1]^2$, $m\neq 0$, then the following inequality holds:*

$$f\left(\frac{a+mb}{2}\right)\leq\frac{(1+m)f(x)}{2^s}-\lambda\frac{1}{4m}(a+mb-x-mx)^2. \tag{8.275}$$

Proof. As f is strongly (s,m)-convex function, we have

$$f\left(\frac{a+mb}{2}\right)\leq\frac{1}{2^s}\left(f((1-t)a+mtb)+mf\left(\frac{ta+m(1-t)b}{m}\right)\right) \tag{8.276}$$
$$-\frac{\lambda}{4m}(t(1+m)(a-mb)+mb-ma)^2.$$

Let $x=a(1-t)+mtb$. Then we have $a+mb-x=ta+m(1-t)b$.

$$f\left(\frac{a+mb}{2}\right)\leq\frac{f(x)}{2^s}+m\frac{f\left(\frac{a+mb-x}{m}\right)}{2^s}-\lambda\frac{1}{4m}(a+mb-x-mx)^2. \tag{8.277}$$

Hence, by using $f(\frac{a+mb-x}{m})=f(x)$, the inequality (8.275) can be obtained. □

Theorem 8.42 *Let $w\in\mathbb{R}$, $r,\rho,\tau>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $f\in L_1[a,b], 0\leq a<b$. If f is a positive and strongly (s,m)-convex function on $[a,mb]$ with modulus $\lambda>0$, $m\in(0,1]$ and $f(\frac{a+mb-x}{m})=f(x)$, then for σ, $\sigma'\geq 0$, the following fractional integral inequality for generalized integral operators holds:*

$$\frac{2^s}{1+m}\left(f\left(\frac{a+mb}{2}\right)(H^w_{a^+,\sigma+1}(b;p)+H^w_{b^-,\sigma'+1}(a;p))+\frac{\lambda}{4m}(K_1+K_2)\right) \tag{8.278}$$
$$\leq\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p)+\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p)$$
$$\leq\left(H^w_{b^-,\sigma'}(a;p)+H^w_{a^+,\sigma}(b;p)\right)(b-a)\left(\frac{f(b)+mf\left(\frac{a}{m}\right)}{s+1}-\lambda\frac{(mb-a)^2}{6m}\right).$$

where

$$K_1=(b-a)^{\sigma'+2}H^w_{b^-,\sigma'+1}(a;p)-2(1+m)(b-a)^{\sigma'+1}H^w_{b^-,\sigma'+2}(a;p)+2(1+m)^2H^w_{b^-,\sigma'+3}(a;p),$$
$$K_2=(b-a)^{\sigma+2}H^w_{a^+,\sigma'+1}(b;p)-2(1+m)(b-a)^{\sigma+1}H^w_{a^+,\sigma+2}(b;p)+2(1+m)^2H^w_{a^+,\sigma+3}(b;p).$$

Proof. As f is strongly (s,m)-convex so for $x\in[a,b]$, we have:

$$f(x)\leq\left(\frac{x-a}{b-a}\right)^s f(b)+m\left(\frac{b-x}{b-a}\right)^s f\left(\frac{a}{m}\right)-\lambda\frac{(x-a)(b-x)(mb-a)^2}{m(b-a)^2}. \tag{8.279}$$

By multiplying (8.58) and (8.279) and then integrating over $[a,b]$, we get

$$\int_a^b(x-a)^\beta E^{\delta,c,q,r}_{\rho,\sigma'+1,\tau}(w(x-a)^\rho;p)f(x)dx$$

$$\leq (b-a)^{\sigma'} E^{\delta,c,q,r}_{\rho,\sigma'+1,\tau}(w(b-a)^{\rho};p)\left(\frac{f(b)}{(b-a)^s}\int_a^b (x-a)^s dx\right.$$
$$\left.+mf\left(\frac{a}{m}\right)\int_a^b \left(\frac{b-x}{b-a}\right)^s dx - \lambda\frac{(mb-a)^2}{m(b-a)^2}\int_a^b (x-a)(b-x)dx\right).$$

From which we have

$$\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p) \tag{8.280}$$
$$\leq (b-a)^{\sigma'+1} E^{\delta,c,q,r}_{\rho,\sigma'+1,\tau}(w(b-a)^{\rho};p)\left(\frac{f(b)+mf\left(\frac{a}{m}\right)}{s+1} - \lambda\frac{(mb-a)^2}{6m}\right),$$

that is

$$\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p) \leq (b-a)H^{w}_{b^-,\sigma'}(a;p)\left(\frac{f(b)+mf\left(\frac{a}{m}\right)}{s+1} - \lambda\frac{(mb-a)^2}{6m}\right). \tag{8.281}$$

Now, on the other hand by multiplying (8.61) and (8.279)and then integrating over $[a,b]$, we get

$$\int_a^b (b-x)^{\alpha} E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w(b-x)^{\rho};p)f(x)dx$$
$$\leq (b-a)^{\sigma} E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w(b-a)^{\rho};p)\left(\frac{f(b)}{(b-a)^s}\int_a^b (x-a)^s dx\right.$$
$$\left.+mf\left(\frac{a}{m}\right)\int_a^b \left(\frac{b-x}{b-a}\right)^s dx - \lambda\frac{(mb-a)^2}{m(b-a)^2}\int_a^b (x-a)(b-x)dx\right).$$

From which we have

$$\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p) \tag{8.282}$$
$$\leq (b-a)^{\sigma+1} E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w(b-a)^{\rho};p)\left(\frac{f(b)+mf\left(\frac{a}{m}\right)}{s+1} - \lambda\frac{(mb-a)^2}{6m}\right),$$

that is

$$\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p) \leq (b-a)H^{w}_{a^+,\sigma}(b;p)\left(\frac{f(b)+mf\left(\frac{a}{m}\right)}{s+1} - \lambda\frac{(mb-a)^2}{6m}\right). \tag{8.283}$$

By adding (8.281) and (8.283), the second inequality of (8.278) is obtained.

To prove the first inequality; multiplying (8.275) with $(x-a)^{\beta} E^{\delta,c,q,r}_{\rho,\sigma'+1,\tau}(w(x-a)^{\rho};p)$ and integrating over $[a,b]$, we get

$$f\left(\frac{a+mb}{2}\right)\int_a^b (x-a)^{\beta} E^{\delta,c,q,r}_{\rho,\sigma'+1,\tau}(w(x-a)^{\rho};p)dx \tag{8.284}$$
$$\leq \frac{1+m}{2^s}\int_a^b (x-a)^{\beta} E^{\delta,c,q,r}_{\rho,\sigma'+1,\tau}(w(x-a)^{\rho};p)f(x)dx$$

$$-\frac{\lambda}{4m}\int_a^b (x-a)^{\beta} E^{\delta,c,q,r}_{\rho,\sigma'+1,\tau}(w(x-a)^{\rho};p)(a+mb-x-mx)^2dx.$$

By using (2.13) and integrating by parts, we get

$$\begin{aligned}&f\left(\frac{a+mb}{2}\right)H^{w}_{b^-,\sigma'+1}(a;p)\\&\leq\frac{1+m}{2^s}\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p)-\frac{\lambda}{4m}\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}(a+mb-x-mx)^2\right)(a;p).\end{aligned}$$

The integral operator appearing in the last term of the right hand side is calculated as follows:

$$\begin{aligned}&\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}(a+mb-x-mx)^2\right)(a;p)\\&=\sum_{n=0}^{\infty}\frac{\beta_p(\delta+nq,c-\delta)}{\beta(\delta,c-\delta)}\frac{(c)_{nq}}{\Gamma(\rho n+\sigma'+1)}\frac{1}{(\tau)_{nr}}\int_a^b(x-a)^{\sigma'+\rho n}(a+mb-x-mx)^2dx\\&=\sum_{n=0}^{\infty}\frac{\beta_p(\delta+nq,c-\delta)}{\beta(\delta,c-\delta)}\frac{(c)_{nq}}{\Gamma(\rho n+\sigma'+1)}\frac{1}{(\tau)_{nr}}\left(\frac{(b-a)^{\sigma'+\rho n+3}}{\sigma'+\rho n+1}\right.\\&\quad\left.-\frac{2(1+m)(b-a)^{\sigma'+\rho n+3}}{(\sigma'+\rho n+1)(\sigma'+\rho n+2)}+\frac{2(1+m)^2(b-a)^{\sigma'+\rho n+2}}{(\sigma'+\rho n+1)(\sigma'+\rho n+2)(\sigma'+\rho n+3)}\right)\\&=(b-a)^{\sigma'+2}H^{w}_{b^-,\sigma'+1}(a;p)-2(1+m)(b-a)^{\sigma'+1}H^{w}_{b^-,\sigma'+2}(a;p)+2(1+m)^2H^{w}_{b^-,\sigma'+3}(a;p)\end{aligned}$$

i.e

$$\begin{aligned}&f\left(\frac{a+mb}{2}\right)H^{w}_{b^-,\sigma'+1}(a;p)\leq\frac{1+m}{2^s}\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p)-\frac{\lambda}{4m}((b-a)^{\sigma'+2}\\&\times H^{w}_{b^-,\sigma'+1}(a;p)-2(1+m)(b-a)^{\sigma'+1}H^{w}_{b^-,\sigma'+2}(a;p)+2(1+m)^2H^{w}_{b^-,\sigma'+3}(a;p)).\end{aligned}\tag{8.285}$$

By multiplying (8.275) with $(b-x)^{\alpha}E^{\delta,c,q,r}_{\rho,\sigma+1,\tau}(w(b-x)^{\rho};p)$ and integrating over $[a,b]$, we get

$$\begin{aligned}&f\left(\frac{a+mb}{2}\right)H^{w}_{a^+,\sigma'+1}(b;p)\\&\leq\frac{1+m}{2^s}\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p)-\frac{\lambda}{4m}\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}(a+mb-x-mx)^2\right)(b;p).\end{aligned}\tag{8.286}$$

The integral operator appearing in last term of right hand side is calculated as follows:

$$\begin{aligned}&\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}(a+mb-x-mx)^2\right)(b;p)\\&=(b-a)^{\sigma+2}H^{w}_{a^+,\sigma'+1}(b;p)-2(1+m)(b-a)^{\sigma+1}H^{w}_{a^+,\sigma+2}(b;p)+2(1+m)^2H^{w}_{a^+,\sigma+3}(b;p).\end{aligned}$$

By using it in (8.286) then adding resulting inequality in (8.285), the first inequality of (8.278) can be obtained. □

Theorem 8.43 *Let $w \in \mathbb{R}$, $r, \rho, \tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let $f \in L_1[a,b], 0 \leq a < b$. If f is a positive and (s,m)-convex function on $[a,mb]$, $m \in (0,1]$ and $f(\frac{a+mb-x}{m}) = f(x)$, then for $\sigma, \sigma' \geq 0$, the following fractional integral inequality for generalized integral operators holds:*

$$\frac{2^s}{1+m}\left(f\left(\frac{a+mb}{2}\right)(H^w_{a^+,\sigma'+1}(b;p)+H^w_{b^-,\sigma'+1}(a;p))\right) \tag{8.287}$$
$$\leq \left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p)+\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p)$$
$$\leq \left(H^w_{b^-,\sigma'}(a;p)+H^w_{a^+,\sigma}(b;p)\right)(b-a)\left(\frac{f(b)+mf\left(\frac{a}{m}\right)}{s+1}\right).$$

Proof. For the (s,m)-convex function f the inequality (8.281) holds as follows (by setting $\lambda = 0$):

$$\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p) \leq (b-a)H^w_{b^-,\sigma'}(a;p)\left(\frac{f(b)+mf\left(\frac{a}{m}\right)}{s+1}\right). \tag{8.288}$$

Also, for the (s,m)-convex function f the inequality (8.283) holds as follows (by setting $\lambda = 0$):

$$\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p) \leq (b-a)H^w_{a^+,\sigma}(b;p)\left(\frac{f(b)+mf\left(\frac{a}{m}\right)}{s+1}\right). \tag{8.289}$$

By adding (8.288) and (8.289), the second inequality in (8.287) can be obtained. For first inequality using (8.285) for $\lambda = 0$ we have

$$f\left(\frac{a+mb}{2}\right)H^w_{b^-,\sigma'+1}(a;p) \leq \frac{1+m}{2^s}\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p). \tag{8.290}$$

Also, from (8.286) for $\lambda = 0$ we have

$$f\left(\frac{a+mb}{2}\right)H^w_{a^+,\sigma'+1}(b;p) \leq \frac{1+m}{2^s}\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p). \tag{8.291}$$

From inequalities (8.290) and (8.291), the first inequality in (8.287) can be obtained. □

Remark 8.17 The inequality (8.278) provides refinement of inequality (8.287).

If $s = m = 1$ in (8.278), then the following result obtained for strongly convex function.

Corollary 8.52 *Let $f : [a,b] \longrightarrow \mathbb{R}$, $0 \leq a < b$, be a real valued function. If f is a positive, strongly convex and $f(a+b-x) = f(x)$, then for $\sigma, \sigma' > 0$, the following fractional integral inequality holds:*

$$f\left(\frac{a+b}{2}\right)(H^w_{a^+,\sigma'+1}(b;p)+H^w_{b^-,\sigma'+1}(a;p))+\frac{\lambda}{4}\left((b-a)^{\sigma'+2}H^w_{b^-,\sigma'+1}(a;p)\right. \tag{8.292}$$

$$-4(b-a)^{\sigma'+1}H^{w}_{b^-,\sigma'+2}(a;p)+8H^{w}_{b^-,\sigma'+3}(a;p)+(b-a)^{\sigma+2}$$
$$\times H^{w}_{a^+,\sigma'+1}(b;p)-4(b-a)^{\sigma+1}H^{w}_{a^+,\sigma+2}(b;p)+8H^{w}_{a^+,\sigma+3}(b;p)\Big)$$
$$\leq\left(\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p)+\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p)\right)$$
$$\leq\left(H^{w}_{b^-,\sigma'}(a;p)+H^{w}_{a^+,\sigma}(b;p)\right)(b-a)\left(\frac{f(b)+f(a)}{2}-\lambda\frac{(b-a)^2}{6}\right).$$

Remark 8.18 For $\lambda=0$ in (8.292) we get [36, Corollary 3]. Therefore (8.292) is the refinement of [36, Corollary 3].

8.10 Bounds of Fractional Integral Operators for Exponentially (s,m)-convex Functions

In this section the generalizations of results given in aforementioned sections are proved.

Theorem 8.44 *Let $w\in\mathbb{R}$, $r,\rho,\tau>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $f:K\subseteq[0,\infty)\longrightarrow\mathbb{R}$ be a real valued function. If f is positive and exponentially (s,m)-convex, then for $a,b\in K, a<b$ and $\sigma,\sigma'\geq 1$, the following fractional integral inequality for generalized integral operators holds:*

$$\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)+\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p) \tag{8.293}$$
$$\leq\left(\frac{f(a)}{e^{\alpha a}}+\frac{mf(\frac{x}{m})}{e^{\frac{\alpha x}{m}}}\right)\frac{(x-a)H^{w}_{a^+,\sigma-1}(x;p)}{s+1}$$
$$+\left(\frac{f(b)}{e^{\beta b}}+\frac{mf(\frac{x}{m})}{e^{\frac{\beta x}{m}}}\right)\frac{(b-x)H^{w}_{b^-,\sigma'-1}(x;p)}{s+1}, x\in[a,b],\ \alpha,\beta\in\mathbb{R}.$$

Proof. Since f is exponentially (s,m)-convex, we obtain

$$f(t)\leq\left(\frac{x-t}{x-a}\right)^{s}\frac{f(a)}{e^{\alpha a}}+m\left(\frac{t-a}{x-a}\right)^{s}\frac{f(\frac{x}{m})}{e^{\frac{\alpha x}{m}}},\ \alpha\in\mathbb{R}. \tag{8.294}$$

By multiplying (8.48) and (8.294) and then integrating over $[a,x]$, we get

$$\int_a^x (x-t)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-t)^{\rho};p)f(t)dt$$
$$\leq\frac{(x-a)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(x-a)^{\rho};p)}{(x-a)^{s}}$$
$$\times\left(\frac{f(a)}{e^{\alpha a}}\int_a^x(x-t)^{s}dt+\frac{mf(\frac{x}{m})}{e^{\frac{\alpha x}{m}}}\int_a^x(t-a)^{s}dt\right),$$

that is, the left integral operator satisfies the following inequality:

$$\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p) \leq \frac{(x-a)H^{w}_{\sigma-1,a^+}(x;p)}{s+1}\left(\frac{f(a)}{e^{\alpha a}}+m\frac{f(\frac{x}{m})}{e^{\frac{\alpha x}{m}}}\right). \tag{8.295}$$

Again from exponentially (s,m)-convexity of f, we have

$$f(t) \leq \left(\frac{t-x}{b-x}\right)^s \frac{f(b)}{e^{\beta b}} + m\left(\frac{b-t}{b-x}\right)^s \frac{f(\frac{x}{m})}{e^{\frac{\beta x}{m}}}, \beta \in \mathbb{R}. \tag{8.296}$$

By multiplying (8.52) and (8.296) and then integrating over $[x,b]$, we get

$$\begin{aligned}
&\int_x^b (t-x)^{\sigma'-1}E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(t-x)^{\rho};p)f(t)dt \\
&\leq \frac{(b-x)^{\sigma'-1}E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(b-x)^{\rho};p)}{(b-x)^s} \\
&\times\left(\frac{f(b)}{e^{\beta b}}\int_x^b (t-x)^s dt + \frac{mf(\frac{x}{m})}{e^{\frac{\beta x}{m}}}\int_x^b (b-t)^s dt\right)
\end{aligned}$$

that is, the right integral operator satisfies the following inequality:

$$\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p) \leq \frac{(b-x)H^{w}_{\sigma'-1,b^-}(x;p)}{s+1}\left(\frac{f(b)}{e^{\beta b}}+\frac{mf(\frac{x}{m})}{e^{\frac{\beta x}{m}}}\right). \tag{8.297}$$

By adding (8.295) and (8.297), the required inequality (8.293) can be obtained. □

Some special cases are given in the following corollaries.

Corollary 8.53 *If we set $\sigma = \sigma'$ in (8.293), then the following inequality is obtained:*

$$\begin{aligned}
&\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p) + \left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p) \\
&\leq \left(\frac{f(a)}{e^{\alpha a}}+\frac{mf(\frac{x}{m})}{e^{\frac{\alpha x}{m}}}\right)\frac{(x-a)H^{w}_{a^+,\sigma-1}(x;p)}{s+1} \\
&+\left(\frac{f(b)}{e^{\beta b}}+\frac{mf(\frac{x}{m})}{e^{\frac{\beta x}{m}}}\right)\frac{(b-x)H^{w}_{b^-,\sigma-1}(x;p)}{s+1}, x \in [a,b].
\end{aligned} \tag{8.298}$$

Corollary 8.54 *Along with assumption of Theorem 1, if $f \in L_\infty[a,b]$, then the following inequality is obtained:*

$$\begin{aligned}
&\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p) + \left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p) \\
&\leq \frac{||f||_\infty}{s+1}\left(\left(\frac{1}{e^{\alpha a}}+\frac{m}{e^{\frac{\alpha x}{m}}}\right)(x-a)H^{w}_{a^+,\sigma-1}(x;p)\right. \\
&\left.+\left(\frac{1}{e^{\beta b}}+\frac{m}{e^{\frac{\beta x}{m}}}\right)(b-x)H^{w}_{b^-,\sigma'-1}(x;p)\right).
\end{aligned} \tag{8.299}$$

Corollary 8.55 *If we take $\sigma = \sigma'$ in (8.299), then we get the following result:*

$$\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p) + \left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(x;p) \tag{8.300}$$
$$\leq \frac{||f||_\infty}{s+1}\left(\left(\frac{1}{e^{\alpha a}} + \frac{m}{e^{\frac{\alpha x}{m}}}\right)(x-a)H^w_{a^+,\sigma-1}(x;p)\right.$$
$$\left. + \left(\frac{1}{e^{\beta b}} + \frac{m}{e^{\frac{\beta x}{m}}}\right)(b-x)H^w_{b^-,\sigma-1}(x;p)\right).$$

Corollary 8.56 *If we take $s = 1$ in (8.299), then we get the following result:*

$$\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p) + \left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p) \tag{8.301}$$
$$\leq \frac{||f||_\infty}{2}\left(\left(\frac{1}{e^{\alpha a}} + \frac{m}{e^{\frac{\alpha x}{m}}}\right)(x-a)H^w_{a^+,\sigma-1}(x;p)\right.$$
$$\left. + \left(\frac{1}{e^{\beta b}} + \frac{m}{e^{\frac{\beta x}{m}}}\right)(b-x)H^w_{b^-,\sigma'-1}(x;p)\right).$$

Theorem 8.45 *With the assumptions of Theorem 1 if $f \in L_\infty[a,b]$, then operators defined in (2.12) and (2.13) are continuous.*

Proof. If $f \in L_\infty[a,b]$, then from (8.295) we have

$$\left|\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)\right| \leq \frac{2||f||_\infty (x-a)H^w_{\sigma-1,a^+}(x;p)}{s+1}\left(\frac{1}{e^{\alpha a}} + \frac{m}{e^{\frac{\alpha x}{m}}}\right) \tag{8.302}$$
$$\leq \frac{2(b-a)H^w_{\sigma-1,a^+}(b;p)}{s+1}\left(\frac{1}{e^{\alpha a}} + \frac{m}{e^{\frac{\alpha x}{m}}}\right)||f||_\infty,$$

that is $\left|\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)\right| \leq M||f||_\infty$, where $M = \frac{2(b-a)H^w_{\sigma-1,a^+}(b;p)}{s+1}\left(\frac{1}{e^{\alpha a}} + \frac{m}{e^{\frac{\alpha x}{m}}}\right)$. Therefore $\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x;p)$ is bounded also it is easy to see that it is linear, hence this is continuous operator. On the other hand, from (8.297) we obtain:

$$\left|\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p)\right| \leq K||f||_\infty,$$

where $K = \frac{2(b-a)H^w_{\sigma'-1,b^-}(a;p)}{s+1}\left(\frac{1}{e^{\beta a}} + \frac{m}{e^{\frac{\beta x}{m}}}\right)$. Therefore $\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x;p)$ is bounded also it is linear, hence continuous. □

Definition 8.2 *Let $f : [a,b] \to \mathbb{R}$ be a function, we will say f is exponentially m-symmetric about $\frac{a+b}{2}$, if*

$$\frac{f(x)}{e^{\alpha x}} = \frac{f(\frac{a+b-x}{m})}{e^{\alpha(\frac{a+b-x}{m})}}, \alpha \in \mathbb{R}. \tag{8.302}$$

It is required to give the following lemma which will be helpful to produce Hadamard type estimations for the generalized fractional integral operators.

Lemma 8.13 *Let $f : K \subseteq [0,\infty) \longrightarrow \mathbb{R}$, $a,b \in K, a < mb$ be an exponentially (s,m)-convex function. If f is exponentially m-symmetric about $\frac{a+b}{2}$, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \le (1+m)\frac{f(x)}{2^s e^{\alpha x}}, \alpha \in \mathbb{R}. \tag{8.303}$$

Proof. Since f is exponentially (s,m)-convex, we have

$$f\left(\frac{a+b}{2}\right) \le \frac{f(at+(1-t)b)}{2^s e^{\alpha(at+(1-t)b)}} + \frac{mf(\frac{a(1-t)+bt}{m})}{2^s e^{\alpha(\frac{a(1-t)+bt}{m})}}, t \in [0,1]. \tag{8.304}$$

Let $x = at+(1-t)b$, where $x \in [a,b]$. Then we have $a+b-x = bt+(1-t)a$ and we get

$$f\left(\frac{a+b}{2}\right) \le \frac{f(x)}{2^s e^{\alpha x}} + m\frac{f(\frac{a+b-x}{m})}{2^s e^{\alpha(\frac{a+b-x}{m})}}. \tag{8.305}$$

Now using that f is exponentially m-symmetric we will get (8.303). □

Theorem 8.46 *Let $w \in \mathbb{R}$, $r,\rho,\tau > 0$, $c > \delta > 0$ with $p \ge 0$ and $0 < q \le r+\rho$. Let $f : K \subseteq [0,\infty) \longrightarrow \mathbb{R}$, $a,b \in K, a < b$, be a real valued function. If f is positive, exponentially (s,m)-convex and exponentially m-symmetric about $\frac{a+b}{2}$, then for $\sigma,\sigma' > 0$, the following fractional integral inequality for generalized integral operators holds:*

$$\begin{aligned}
&\frac{2^s h(\alpha)}{1+m} f\left(\frac{a+b}{2}\right)\left[H^w_{b^-,\sigma'+1}(a;p) + H^w_{a^+,\sigma+1}(b;p)\right] \\
&\le \left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau} f\right)(a;p) + \left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau} f\right)(b;p) \\
&\le \left[H^w_{b^-,\sigma'-1}(a;p) + H^w_{a^+,\sigma-1}(b;p)\right]\frac{(b-a)^2}{s+1}\left(\frac{f(\frac{a}{m})}{e^{\frac{\alpha a}{m}}} + \frac{f(b)}{e^{\beta b}}\right), \alpha,\beta \in \mathbb{R},
\end{aligned} \tag{8.306}$$

where $h(\alpha) = e^{\alpha b}$ for $\alpha < 0$ and $h(\alpha) = e^{\alpha a}$ for $\alpha \ge 0$.

Proof. As f is exponentially (s,m)-convex, for $x \in [a,b]$, we have:

$$f(x) \le \left(\frac{x-a}{b-a}\right)^s \frac{f(b)}{e^{\alpha b}} + m\left(\frac{b-x}{b-a}\right)^s \frac{f(\frac{a}{m})}{e^{\frac{\alpha a}{m}}}, \alpha \in \mathbb{R}. \tag{8.307}$$

By multiplying (8.58) and (8.307) and then integrating over $[a,b]$, we get

$$\begin{aligned}
&\int_a^b (x-a)^\sigma E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(x-a)^\rho;p)f(x)dx \\
&\le (b-a)^{\sigma'-s} E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(b-a)^\rho;p)\left(\frac{f(b)}{e^{\alpha b}}\int_a^b (x-a)^s dx + \frac{mf(\frac{a}{m})}{e^{\frac{\alpha a}{m}}}\int_a^b (b-x)^s dx\right),
\end{aligned}$$

from which we have

$$\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p) \leq \frac{(b-a)^{\sigma'+1}E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(b-a)^{\rho};p)}{s+1}\left(\frac{f(b)}{e^{\alpha b}}+\frac{mf(\frac{a}{m})}{e^{\frac{\alpha a}{m}}}\right), \quad (8.308)$$

$$\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p) \leq \frac{(b-a)^2}{s+1}H^{w}_{\sigma'-1,b^-}(a;p)\left(\frac{f(b)}{e^{\alpha b}}+\frac{mf(\frac{a}{m})}{e^{\frac{\alpha a}{m}}}\right). \quad (8.309)$$

By multiplying (8.307) and (8.61) and then integrating over $[a,b]$, we get

$$\int_a^b (b-x)^{\sigma}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(b-x)^{\rho};p)f(x)dx$$
$$\leq (b-a)^{\sigma-s}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(b-a)^{\rho};p)\left(\frac{f(b)}{e^{\alpha b}}\int_a^b (x-a)^s dx+\frac{mf(\frac{a}{m})}{e^{\frac{\alpha a}{m}}}\int_a^b (b-x)^s dx\right).$$

From which we have

$$\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p) \leq \frac{(b-a)^{\sigma+1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(b-a)^{\rho};p)}{s+1}\left(\frac{f(b)}{e^{\alpha b}}+\frac{mf(\frac{a}{m})}{e^{\frac{\alpha a}{m}}}\right), \quad (8.310)$$

$$\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p) \leq \frac{(b-a)^2}{s+1}H^{w}_{\sigma-1,a^+}(b;p)\left(\frac{f(b)}{e^{\alpha b}}+\frac{mf(\frac{a}{m})}{e^{\frac{\alpha a}{m}}}\right). \quad (8.311)$$

Adding (8.309) and (8.311), we get;

$$\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p)+\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p) \quad (8.312)$$
$$\leq \left[H^{w}_{\sigma'-1,b^-}(a;p)+H^{w}_{\sigma-1,a^+}(b;p)\right]\frac{(b-a)^2}{s+1}\left(\frac{f(b)}{e^{\alpha b}}+\frac{mf(\frac{a}{m})}{e^{\frac{\alpha a}{m}}}\right).$$

Multiplying (8.303) with $(x-a)^{t}E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(x-a)^{\rho};p)$ and integrating over $[a,b]$, we get

$$f\left(\frac{a+b}{2}\right)\int_a^b (x-a)^{t}E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(x-a)^{\rho};p)dx \quad (8.313)$$
$$\leq \frac{m+1}{2^s}\int_a^b (x-a)^{t}E^{\delta,c,q,r}_{\rho,\sigma',\tau}(w(x-a)^{\rho};p)\frac{f(x)}{e^{\alpha x}}dx,$$

$$f\left(\frac{a+b}{2}\right)H^{w}_{\sigma'+1,b^-}(a;p) \leq \frac{m+1}{2^s e^{\alpha x}}\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p). \quad (8.314)$$

Multiplying (8.303) with $(b-x)^{\sigma}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(b-x)^{\rho};p)$ and integrating over $[a,b]$, we get

$$f\left(\frac{a+b}{2}\right)H^{w}_{\sigma+1,a^+}(b;p) \leq \frac{m+1}{2^s h(\alpha)}\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p). \quad (8.315)$$

By adding (8.314) and (8.315), we get;

$$\frac{2^s h(\alpha)}{1+m} f\left(\frac{a+b}{2}\right)\left[H^w_{\sigma'+1,b^-}(a;p)+H^w_{\sigma+1,a^+}(b;p)\right] \tag{8.316}$$
$$\leq \left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(a;p)+\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p).$$

By combining (8.312) and (8.316), inequality (8.306) can be obtained. □

Corollary 8.57 *If we take $\sigma=\sigma'$ in (8.306), then the following inequality is obtained:*

$$\frac{2^s e^{\alpha x}}{1+m} f\left(\frac{a+b}{2}\right)\left[H^w_{b^-,\sigma+1}(a;p)+H^w_{a^+,\sigma+1}(b;p)\right] \tag{8.317}$$
$$\leq \left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma+1,\tau}f\right)(a;p)+\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(b;p)$$
$$\leq \left(H^w_{b^-,\sigma-1}(a;p)+H^w_{a^+,\sigma-1}(b;p)\right)\frac{(b-a)^2}{s+1}\left(\frac{f(b)}{e^{\alpha b}}+\frac{mf(\frac{a}{m})}{e^{\frac{\alpha a}{m}}}\right).$$

Next result provides boundedness of sum of left and right integrals at an arbitrary point for functions whose derivatives in absolute values are exponentially (s,m)-convex.

Theorem 8.47 *Let $w\in\mathbb{R}$, $r,\rho,\tau>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $f:K\subseteq[0,\infty)\longrightarrow\mathbb{R}$ be a real valued function. If f is differentiable and $|f'|$ is exponentially (s,m)-convex, then for $a,b\in K, a<b$ and $\sigma,\sigma'\geq 1$, the following fractional integral inequality for generalized integral operators holds:*

$$\left|\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(x;p)+\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma'+1,\tau}f\right)(x;p)\right. \tag{8.318}$$
$$\left.-\left(H^w_{a^+,\sigma-1}(x;p)f(a)+H^w_{b^-,\sigma'-1}(x;p)f(b)\right)\right|$$
$$\leq\left(\frac{|f'(a)|}{e^{\alpha a}}+\frac{m|f'(\frac{x}{m})|}{e^{\frac{\alpha x}{m}}}\right)\frac{(x-a)H^w_{a^+,\sigma-1}(x;p)}{s+1}$$
$$+\left(\frac{|f'(b)|}{e^{\beta b}}+\frac{m|f'(\frac{x}{m})|}{e^{\frac{\beta x}{m}}}\right)\frac{(b-x)H^w_{b^-,\sigma'-1}(x;p)}{s+1}, x\in[a,b],\ \alpha,\beta\in\mathbb{R}.$$

Proof. Proof is on the same lines as the proof of Theorem 8.33. □

Corollary 8.58 *If we take $\sigma=\sigma'$ in (8.318), then the following inequality is obtained:*

$$\left|\left(\varepsilon^{w,\delta,c,q,r}_{a^+,\rho,\sigma+1,\tau}f\right)(x;p)+\left(\varepsilon^{w,\delta,c,q,r}_{b^-,\rho,\sigma+1,\tau}f\right)(x;p)\right. \tag{8.319}$$
$$\left.-\left(H^w_{a^+,\sigma-1}(x;p)f(a)+H^w_{b^-,\sigma-1}(x;p)f(b)\right)\right|$$
$$\leq\left(\frac{|f'(a)|}{e^{\alpha a}}+\frac{m|f'(\frac{x}{m})|}{e^{\frac{\alpha x}{m}}}\right)\frac{(x-a)H^w_{a^+,\sigma-1}(x;p)}{s+1}$$
$$+\left(\frac{|f'(b)|}{e^{\beta b}}+\frac{m|f'(\frac{x}{m})|}{e^{\frac{\beta x}{m}}}\right)\frac{(b-x)H^w_{b^-,\sigma-1}(x;p)}{s+1}, x\in[a,b], \alpha,\beta\in\mathbb{R}.$$

8.11 Bounds of Fractional Integral Operators for Exponentially m-convex Functions

First we give the fractional Hadamard inequality for exponentially m-convex functions via generalized fractional integral operators.

Theorem 8.48 *Let $w \in \mathbb{R}$, $r, \rho, \sigma, \tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r + \rho$. Let $f : [a, mb] \subset \mathbb{R} \to \mathbb{R}$ be a function such that $f \in L_1[a, mb]$ with $a < mb$. If f is exponentially m-convex function, then the following inequalities hold:*

$$\begin{aligned} & e^{f\left(\frac{a+mb}{2}\right)} H^{w'}_{a^+,\sigma}(mb;p) \\ & \leq \frac{\left(\varepsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma,\tau} e^f\right)(mb;p) + m^{\sigma+1}\left(\varepsilon^{w'm^\rho,\delta,c,q,r}_{b^-,\rho,\sigma,\tau} e^f\right)\left(\frac{a}{m};p\right)}{2} \\ & \leq \frac{m^{\sigma+1}}{2(mb-a)}\left[\left(e^{f(a)} - m^2 e^{f\left(\frac{a}{m^2}\right)}\right) H^{w'm^\rho}_{b^-,\sigma+1}\left(\frac{a}{m};p\right)\right. \\ & \left. +(mb-a)\left(e^{f(b)} + m e^{f\left(\frac{a}{m^2}\right)}\right) H^{w'm^\rho}_{b^-,\sigma}\left(\frac{a}{m};p\right)\right] \end{aligned} \tag{8.320}$$

where $m \in (0, 1]$ and $w' = \frac{w}{(mb-a)^\rho}$.

Proof. Since f is exponentially m-convex, we have

$$e^{f\left(\frac{x+my}{2}\right)} \leq \frac{e^{f(x)} + m e^{f(y)}}{2}, \quad x, y \in [a, mb] \text{ and } m \in (0, 1]. \tag{8.321}$$

Putting $x = za + m(1-z)b$ and $y = (1-z)\frac{a}{m} + zb$ in (8.321), we get

$$2e^{f\left(\frac{a+mb}{2}\right)} \leq e^{f(za+m(1-z)b)} + m e^{f\left((1-z)\frac{a}{m}+zb\right)}. \tag{8.322}$$

Also from exponentially m-convexity of f, we have

$$\begin{aligned} & e^{f(za+m(1-z)b)} + m e^{f\left((1-z)\frac{a}{m}+zb\right)} \\ & \leq z e^{f(a)} + m(1-z)e^{f(b)} + m\left(m(1-z)e^{f\left(\frac{a}{m^2}\right)} + z e^{f(b)}\right) \\ & = z\left(e^{f(a)} - m^2 e^{f\left(\frac{a}{m^2}\right)}\right) + m\left(e^{f(b)} + m e^{f\left(\frac{a}{m^2}\right)}\right). \end{aligned} \tag{8.323}$$

Multiplying both sides of (8.334) with $z^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wz^\rho;p)$ and integrating over $[0, 1]$, we have

$$\begin{aligned} & 2e^{f\left(\frac{a+mb}{2}\right)} \int_0^1 z^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wz^\rho;p)dz \\ & \leq \int_0^1 z^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wz^\rho;p) e^{f(za+m(1-z)b)}dz \end{aligned} \tag{8.324}$$

$$+m\int_0^1 z^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wz^{\rho};p)e^{f((1-z)\frac{a}{m}+zb)}dz.$$

Putting $u=za+m(1-z)b$ and $v=(1-z)\frac{a}{m}+zb$ in (8.336), we get

$$\begin{aligned}
&2e^{f\left(\frac{a+mb}{2}\right)}\int_a^{mb}(mb-u)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w'(mb-u)^{\rho};p)du\\
&\leq \int_a^{mb}(mb-u)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w'(mb-u)^{\rho};p)e^{f(u)}du\\
&+m^{\sigma+1}\int_{\frac{a}{m}}^{b}\left(v-\frac{a}{m}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(m^{\rho}w'(v-\frac{a}{m})^{\rho};p\right)e^{f(v)}dv.
\end{aligned}$$

From above, the first inequality of (8.320) is achieved.
Now multiplying both sides of (8.335) with $z^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wz^{\rho};p)$ and integrating over $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 z^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wz^{\rho};p)e^{f(za+m(1-z)b)}dz \qquad (8.325)\\
&+m\int_0^1 z^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wz^{\rho};p)e^{f((1-z)\frac{a}{m}+zb)}dz\\
&\leq \left(e^{f(a)}-m^2e^{f(\frac{a}{m^2})}\right)\int_0^1 z^{\sigma}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wz^{\rho};p)dz\\
&+m\left(e^{f(b)}+me^{f(\frac{a}{m^2})}\right)\int_0^1 z^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wz^{\rho};p)dz.
\end{aligned}$$

Putting $u=za+m(1-z)b$ and $v=(1-z)\frac{a}{m}+zb$ in (8.337), we get

$$\begin{aligned}
&\int_a^{mb}(mb-u)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w'(mb-u)^{\rho};p)e^{f(u)}du\\
&+m^{\sigma+1}\int_{\frac{a}{m}}^{b}\left(v-\frac{a}{m}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(m^{\rho}w'(v-\frac{a}{m})^{\rho};p\right)e^{f(v)}dv\\
&\leq\frac{m^{\sigma+1}}{(mb-a)}\left[\left(e^{f(a)}-m^2e^{f(\frac{a}{m^2})}\right)\int_{\frac{a}{m}}^{b}\left(v-\frac{a}{m}\right)^{\sigma}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(m^{\rho}w'(v-\frac{a}{m})^{\rho};p\right)dv\right.\\
&\left.+(mb-a)\left(e^{f(b)}+me^{f(\frac{a}{m^2})}\right)\int_{\frac{a}{m}}^{b}\left(v-\frac{a}{m}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(m^{\rho}w'(v-\frac{a}{m})^{\rho};p\right)dv\right].
\end{aligned}$$

By using the definition of generalized integral operators, second inequality of (8.320) is achieved. □

Corollary 8.59 *Suppose that assumptions of Theorem 8.48 hold and let $m=1$. Then following inequalities for exponentially convex function hold:*

$$\begin{aligned}
&e^{f\left(\frac{a+b}{2}\right)}H^{w^*}_{a^+,\sigma}(b;p) \qquad (8.326)\\
&\leq\frac{\left(\varepsilon^{w^*,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}e^{f}\right)(b;p)+\left(\varepsilon^{w^*,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}e^{f}\right)(a;p)}{2}
\end{aligned}$$

$$\leq \frac{e^{f(a)}+e^{f(b)}}{2} H^{w^*}_{b^-,\sigma}(a;p)$$

where $w^* = \frac{w}{(b-a)^\rho}$.

The following Hadamard inequality for exponentially m-convex function is proved which have several misprints.

Theorem 8.49 *[129] Let* $f:[a,mb]\subset\mathbb{R}\to\mathbb{R}$ *be a function such that* $f\in L_1[a,mb]$ *with* $a<mb$. *If* f *is exponentially m-convex function, then the following inequalities hold:*

$$2e^{f\left(\frac{a+mb}{2}\right)} H_{\left(\frac{a+mb}{2}\right)^+,\sigma}(mb;p) \tag{8.327}$$
$$\leq \left(\varepsilon^{w'2^\rho,\delta,c,q,r}_{\left(\frac{a+mb}{2}\right)^+,\rho,\sigma,\tau} e^f\right)(mb;p) + m^{\sigma+1}\left(\varepsilon^{w'2^\rho,\delta,c,q,r}_{\left(\frac{a+mb}{2m}\right)^-,\rho,\sigma,\tau} e^f\right)\left(\frac{a}{m};p\right)$$
$$\leq \frac{a}{(mb-a)}\left(e^{f(a)} - m^2 e^{f\left(\frac{a}{m^2}\right)}\right) H_{\left(\frac{a+mb}{2}\right)^+,\sigma+1}(mb;p)$$
$$+m^{\sigma+1}\left(e^{f(b)} + m e^{f\left(\frac{a}{m^2}\right)}\right) H_{\left(\frac{a+mb}{2m}\right)^-,\sigma}.$$

The correct form of the above theorem is stated and proved in the following theorem.

Theorem 8.50 *Let* $w\in\mathbb{R}$, $r,\rho,\sigma,\tau>0$, $c>\delta>0$ *with* $p\geq 0$ *and* $0<q\leq r+\rho$. *Let* $f:[a,mb]\subset\mathbb{R}\to\mathbb{R}$ *be a function such that* $f\in L_1[a,mb]$ *with* $a<mb$. *If* f *is exponentially m-convex function, then the following inequalities hold:*

$$e^{f\left(\frac{a+mb}{2}\right)} H^{w'2^\rho}_{\left(\frac{a+mb}{2}\right)^+,\sigma}(mb;p) \tag{8.328}$$
$$\leq \frac{\left(\varepsilon^{w'2^\rho,\delta,c,q,r}_{\left(\frac{a+mb}{2}\right)^+,\rho,\sigma,\tau} e^f\right)(mb;p) + m^{\sigma+1}\left(\varepsilon^{w'(2m)^\rho,\delta,c,q,r}_{\left(\frac{a+mb}{2m}\right)^-,\rho,\sigma,\tau} e^f\right)\left(\frac{a}{m};p\right)}{2}$$
$$\leq \frac{m^{\sigma+1}}{2(mb-a)}\left[\left(e^{f(a)} - m^2 e^{f\left(\frac{a}{m^2}\right)}\right) H^{w'(2m)^\rho}_{\left(\frac{a+mb}{2m}\right)^-,\sigma+1}\left(\frac{a}{m};p\right)\right.$$
$$\left.+(mb-a)\left(e^{f(b)} + m e^{f\left(\frac{a}{m^2}\right)}\right) H^{w'(2m)^\rho}_{\left(\frac{a+mb}{2m}\right)^-,\sigma}\left(\frac{a}{m};p\right)\right]$$

where $m\in(0,1]$ *and* w' *is defined in (8.320).*

Proof. Putting $x=\frac{z}{2}a+m\frac{(2-z)}{2}b$ and $y=\frac{z}{2}b+\frac{(2-z)}{2}\frac{a}{m}$ in (8.321), we get

$$2e^{f\left(\frac{a+mb}{2}\right)} \leq e^{f\left(\frac{z}{2}a+m\frac{(2-z)}{2}b\right)} + m e^{f\left(\frac{z}{2}b+\frac{(2-z)}{2}\frac{a}{m}\right)}. \tag{8.329}$$

Multiplying both sides of (8.329) with $z^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wz^\rho;p)$ and integrating over $[0,1]$, we have

$$2e^{f\left(\frac{a+mb}{2}\right)} \int_0^1 z^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wz^\rho;p)dz \tag{8.330}$$

$$\leq \int_0^1 z^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wz^\rho;p) e^{f(\frac{z}{2}a+\frac{(2-z)}{2}b)} dz$$
$$+m\int_0^1 z^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wz^\rho;p) e^{f(\frac{z}{2}b+\frac{(2-z)}{2}\frac{a}{m})} dz.$$

Putting $u = \frac{z}{2}a + m\frac{(2-z)}{2}b$ and $v = \frac{z}{2}b + \frac{(2-z)}{2}\frac{a}{m}$ in (8.330), we get

$$2e^{f\left(\frac{a+mb}{2}\right)} \int_{\frac{a+mb}{2}}^{mb} (mb-u)^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(2^\rho w(mb-u)^\rho;p) du$$
$$\leq \int_{\frac{a+mb}{2}}^{mb} (mb-u)^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(2^\rho w(mb-u)^\rho;p) e^{f(u)} du$$
$$+m^{\sigma+1}\int_{\frac{a}{m}}^{\frac{a+mb}{2m}} \left(v-\frac{a}{m}\right)^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left((2m)^\rho w(v-\frac{a}{m})^\rho;p\right) e^{f(v)} dv,$$

first inequality of (8.328) is achieved.
From exponentially m-convexity of f, we have

$$e^{f(\frac{z}{2}a+m\frac{(2-z)}{2}b)} + me^{f(\frac{z}{2}b+\frac{(2-z)}{2}\frac{a}{m})} \tag{8.331}$$
$$\leq \frac{z}{2}e^{f(a)} + m\frac{(2-z)}{2}e^{f(b)} + m\left(\frac{z}{2}e^{f(b)} + m\frac{(2-z)}{2}e^{f(\frac{a}{m^2})}\right)$$
$$= \frac{z}{2}\left(e^{f(a)} - m^2 e^{f(\frac{a}{m^2})}\right) + m\left(e^{f(b)} + me^{f(\frac{a}{m^2})}\right).$$

Multiplying both sides of (8.331) with $z^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wz^\rho;p)$ and integrating over $[0,1]$, we have

$$\int_0^1 z^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wz^\rho;p) e^{f(\frac{z}{2}a+m\frac{(2-z)}{2}b)} dz \tag{8.332}$$
$$+m\int_0^1 z^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wz^\rho;p) e^{f(\frac{z}{2}b+\frac{(2-z)}{2}\frac{a}{m})} dz$$
$$\leq \frac{1}{2}\left(e^{f(a)} - m^2 e^{f(\frac{a}{m^2})}\right)\int_0^1 z^{\sigma} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wz^\rho;p) dz$$
$$+m\left(e^{f(b)} + me^{f(\frac{a}{m^2})}\right)\int_0^1 z^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wz^\rho;p) dz.$$

Putting $u = \frac{z}{2}a + m\frac{(2-z)}{2}b$ and $v = \frac{z}{2}b + \frac{(2-z)}{2}\frac{a}{m}$ in (8.332), we get

$$\int_{\frac{a+mb}{2}}^{mb} (mb-u)^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(2^\rho w(mb-u)^\rho;p) e^{f(u)} du$$
$$+m^{\sigma+1}\int_{\frac{a}{m}}^{\frac{a+mb}{2m}} \left(v-\frac{a}{m}\right)^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left((2m)^\rho w(v-\frac{a}{m})^\rho;p\right) e^{f(v)} dv$$
$$\leq \frac{m^{\sigma+1}}{mb-a}\Bigg[\left(e^{f(a)} - m^2 e^{f(\frac{a}{m^2})}\right)\int_{\frac{a}{m}}^{\frac{a+mb}{2m}} \left(v-\frac{a}{m}\right)^{\sigma} E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left((2m)^\rho w(v-\frac{a}{m})^\rho;p\right) dv$$

$$+(mb-a)\left(e^{f(b)}+me^{f(\frac{a}{m^2})}\right)\int_{\frac{a}{m}}^{\frac{a+mb}{2m}}\left(v-\frac{a}{m}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left((2m)^{\rho}w(v-\frac{a}{m})^{\rho};p\right)dv\Bigg],$$

second inequality of (8.328) is achieved. □

Corollary 8.60 *Suppose that assumptions of Theorem 8.50 hold and let $m=1$. Then following inequalities for exponentially convex function hold:*

$$\begin{aligned}
&e^{f\left(\frac{a+b}{2}\right)}H^{w^*(2)^{\rho}}_{\left(\frac{a+b}{2}\right)^+,\sigma}(b;p)\\
&\le\frac{\left(\varepsilon^{w^*(2)^{\rho},\delta,c,q,r}_{\left(\frac{a+b}{2}\right)^+,\rho,\sigma,\tau}e^{f}\right)(b;p)+\left(\varepsilon^{w^*(2)^{\rho},\delta,c,q,r}_{\left(\frac{a+b}{2}\right)^+,\rho,\sigma,\tau}e^{f}\right)(a;p)}{2}\\
&\le\frac{e^{f(a)}+e^{f(b)}}{2}H^{w^*(2)^{\rho}}_{\left(\frac{a+b}{2}\right)^-,\sigma}(a;p)
\end{aligned}$$

where w^* is defined in (8.59).

Remark 8.19 If we take $w=p=0$ in (8.328), then [130, Theorem 3.3] is obtained.

In the following we give Fejér-Hadamard inequality for exponentially m-convex functions via generalized fractional integral operators.

Theorem 8.51 *Let $w\in\mathbb{R}$, $r,\rho,\sigma,\tau>0$, $c>\delta>0$ with $p\ge 0$ and $0<q\le r+\rho$. Let $f:[a,mb]\subset\mathbb{R}\to\mathbb{R}$ be a function such that $f\in L_1[a,mb]$ with $a<mb$. Also, let $g:[a,mb]\to\mathbb{R}$ be a function which is nonnegative and integrable. If f is exponentially m-convex function and $f(v)=f(a+mb-mv)$, then the following inequalities hold:*

$$\begin{aligned}
&e^{f\left(\frac{a+mb}{2}\right)}\left(\varepsilon^{w'm^{\rho},\delta,c,q,r}_{b^-,\rho,\sigma,\tau}e^{g}\right)\left(\frac{a}{m};p\right)\\
&\le\frac{(1+m)\left(\varepsilon^{w'm^{\rho},\delta,c,q,r}_{b^-,\rho,\sigma,\tau}e^{f}e^{g}\right)\left(\frac{a}{m};p\right)}{2}\\
&\le\frac{m}{2(mb-a)}\left[\left(e^{f(a)}-m^2e^{f\left(\frac{a}{m^2}\right)}\right)\left(\varepsilon^{w'm^{\rho},\delta,c,q,r}_{b^-,\rho,\sigma+1,\tau}e^{g}\right)\left(\frac{a}{m};p\right)\right.\\
&\left.+(mb-a)\left(e^{f(b)}+me^{f\left(\frac{a}{m^2}\right)}\right)\left(\varepsilon^{w'm^{\rho},\delta,c,q,r}_{b^-,\rho,\sigma,\tau}e^{g}\right)\left(\frac{a}{m};p\right)\right]
\end{aligned}\tag{8.333}$$

where $m\in(0,1]$ and w' is defined in (8.320).

Proof. Putting $x=za+m(1-z)b$ and $y=(1-z)\frac{a}{m}+zb$ in (8.321), we get

$$2e^{f\left(\frac{a+mb}{2}\right)}\le e^{f(za+m(1-z)b)}+me^{f((1-z)\frac{a}{m}+zb)}.\tag{8.334}$$

Also from exponentially m-convexity of f, we have

$$e^{f(za+m(1-z)b)}+me^{f((1-z)\frac{a}{m}+zb)}\tag{8.335}$$

$$\leq ze^{f(a)} + m(1-z)e^{f(b)} + m\left(m(1-z)e^{f(\frac{a}{m^2})} + ze^{f(b)}\right)$$
$$= z\left(e^{f(a)} - m^2 e^{f(\frac{a}{m^2})}\right) + m\left(e^{f(b)} + me^{f(\frac{a}{m^2})}\right).$$

Multiplying both sides of (8.334) with $z^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wz^\rho;p)$ and integrating over $[0,1]$, we have

$$2e^{f\left(\frac{a+mb}{2}\right)}\int_0^1 z^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wz^\rho;p)dz \tag{8.336}$$
$$\leq \int_0^1 z^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wz^\rho;p)e^{f(za+m(1-z)b)}dz$$
$$+m\int_0^1 z^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wz^\rho;p)e^{f((1-z)\frac{a}{m}+zb)}dz.$$

Putting $u = za + m(1-z)b$ and $v = (1-z)\frac{a}{m} + zb$ in (8.336), we get

$$2e^{f\left(\frac{a+mb}{2}\right)}\int_a^{mb}(mb-u)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(mb-u)^\rho;p)du$$
$$\leq \int_a^{mb}(mb-u)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(mb-u)^\rho;p)e^{f(u)}du$$
$$+m^{\sigma+1}\int_{\frac{a}{m}}^b\left(v-\frac{a}{m}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(m^\rho w(v-\frac{a}{m})^\rho;p\right)e^{f(v)}dv,$$

first inequality of (8.320) is achieved.

Now multiplying both sides of (8.335) with $z^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wz^\rho;p)$ and integrating over $[0,1]$, we have

$$\int_0^1 z^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wz^\rho;p)e^{f(za+m(1-z)b)}dz \tag{8.337}$$
$$+m\int_0^1 z^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wz^\rho;p)e^{f((1-z)\frac{a}{m}+zb)}dz$$
$$\leq \left(e^{f(a)} - m^2e^{f(\frac{a}{m^2})}\right)\int_0^1 z^\sigma E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wz^\rho;p)dz$$
$$+m\left(e^{f(b)} + me^{f(\frac{a}{m^2})}\right)\int_0^1 z^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wz^\rho;p)dz.$$

Putting $u = za + m(1-z)b$ and $v = (1-z)\frac{a}{m} + zb$ in (8.337), we get

$$\int_a^{mb}(mb-u)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(mb-u)^\rho;p)e^{f(u)}du$$
$$+m^{\sigma+1}\int_{\frac{a}{m}}^b\left(v-\frac{a}{m}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(m^\rho w(v-\frac{a}{m})^\rho;p\right)e^{f(v)}dv$$
$$\leq \frac{m^{\sigma+1}}{(mb-a)}\left[\left(e^{f(a)} - m^2e^{f(\frac{a}{m^2})}\right)\int_{\frac{a}{m}}^b\left(v-\frac{a}{m}\right)^\sigma E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(m^\rho w(v-\frac{a}{m})^\rho;p\right)dv\right.$$

$$+(mb-a)\left(e^{f(b)}+me^{f(\frac{a}{m^2})}\right)\int_{\frac{a}{m}}^{b}\left(v-\frac{a}{m}\right)^{\sigma-1}E_{\rho,\sigma,\tau}^{\delta,c,q,r}\left(m^{\rho}w(v-\frac{a}{m})^{\rho};p\right)dv\Bigg],$$

second inequality of (8.320) is achieved. □

Corollary 8.61 *Suppose that assumptions of Theorem 8.51 hold and let $m=1$. Then following inequalities for exponentially convex function hold:*

$$e^{f\left(\frac{a+b}{2}\right)}\left(\varepsilon_{b^-,\rho,\sigma,\tau}^{w^*,\delta,c,q,r}e^{g}\right)(a;p)\le\left(\varepsilon_{b^-,\rho,\sigma,\tau}^{w^*,\delta,c,q,r}e^{f}e^{g}\right)(a;p)$$
$$\le\frac{e^{f(a)}+e^{f(b)}}{2}\left(\varepsilon_{b^-,\rho,\sigma,\tau}^{w^*,\delta,c,q,r}e^{g}\right)(a;p)$$

where w^* is defined in (8.59).

Theorem 8.52 *Let $w\in\mathbb{R}$, $r,\rho,\sigma,\tau>0$, $c>\delta>0$ with $p\ge0$ and $0<q\le r+\rho$. Let $f,g:[a,mb]\subset\mathbb{R}\to\mathbb{R}$ be the functions such that $f,g\in L_1[a,mb]$ with $a<mb$. If f and g are exponentially m-convex functions, then the following inequality holds:*

$$\left(\varepsilon_{mb^-,\rho,\sigma,\tau}^{w^*,\delta,c,q,r}e^{f}\right)(a;p)+\left(\varepsilon_{a^+,\rho,\sigma,\tau}^{w',\delta,c,q,r}e^{g}\right)(mb;p) \tag{8.338}$$
$$\le\frac{1}{(mb-a)}\Bigg[\left(e^{g(a)}+me^{f(b)}\right)\left(\varepsilon_{a^+,\rho,\sigma+1,\tau}^{w',\delta,c,q,r}\right)(mb;p)$$
$$+\left(e^{f(a)}+me^{g(b)}\right)\left\{(mb-a)H_{a^+,\sigma}^{w'}(mb;p)-H_{a^+,\sigma+1}^{w'}(mb;p)\right\}\Bigg]$$

where $m\in(0,1]$ and w' is defined in (8.320).

Proof. Since f and g are exponentially m-convex, we have

$$e^{f((1-z)a+mzb)}+e^{g(za+m(1-z)b)}\le(1-z)\left(e^{f(a)}+me^{g(b)}\right)+z\left(e^{g(a)}+me^{f(b)}\right). \tag{8.339}$$

Multiplying both sides of (8.339) with $z^{\sigma-1}E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wz^{\rho};p)$ and integrating over $[0,1]$, we have

$$\int_0^1 z^{\sigma-1}E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wz^{\rho};p)e^{f((1-z)a+mzb)}dz \tag{8.340}$$
$$+\int_0^1 z^{\sigma-1}E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wz^{\rho};p)e^{g(za+m(1-z)b)}dz$$
$$\le\left(e^{f(a)}+me^{g(b)}\right)\int_0^1(1-z)z^{\sigma-1}E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wz^{\rho};p)dz$$
$$+\left(e^{g(a)}+me^{f(b)}\right)\int_0^1 z^{\sigma}E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wz^{\rho};p)dz.$$

Putting $u=(1-z)a+mzb$ and $v=za+m(1-z)b$ in (8.340), inequality (8.338) is achieved. □

Corollary 8.62 *Suppose that assumptions of Theorem 8.52 hold and let $m = 1$. Then following inequality for exponentially convex function holds:*

$$\begin{aligned}&\left(\varepsilon^{w^*,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}e^g\right)(a;p)+\left(\varepsilon^{w^*,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}e^g\right)(b;p)\\&\leq\frac{1}{(b-a)}\left[\left(e^{g(a)}+e^{f(b)}\right)H^{w^*}_{a^+,\sigma+1}(b;p)\right.\\&\left.+\left(e^{f(a)}+e^{g(b)}\right)\left\{(b-a)H^{w^*}_{a^+,\sigma}(b;p)-H^{w^*}_{a^+,\sigma+1}(b;p)\right\}\right].\end{aligned}$$

where w^* is defined in (8.59).

Remark 8.20 If we take $w = p = 0$ in (8.338), then [130, Theorem 3.2] is obtained.

Further generalizations of above proved results are given in the forthcoming results.

Theorem 8.53 *Let $w \in \mathbb{R}$, $r,\rho,\sigma,\tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $f,h : [a,mb] \subset \mathbb{R} \to \mathbb{R}$, $0 < a < mb$ be the real valued-functions. If f be a integrable and exponentially m-convex and h be a differentiable and strictly increasing. Then the following inequalities hold:*

$$\begin{aligned}&2e^{f\left(\frac{h(a)+mh(b)}{2}\right)}\left({}_h\Upsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma,\tau}1\right)(h^{-1}(mh(b));p) &(8.341)\\&\leq\left({}_h\Upsilon^{w',\delta,c,q,r}_{a^+,\rho,\sigma,\tau}e^{f\circ h}\right)(h^{-1}(mh(b));p)+m^{\sigma+1}\left({}_h\Upsilon^{w'm^\rho,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}e^{f\circ h}\right)\left(h^{-1}\left(\frac{h(a)}{m}\right);p\right)\\&\leq\frac{m^{\sigma+1}}{(mh(b)-h(a))}\left[\left(e^{f(h(a))}-m^2e^{f\left(\frac{h(a)}{m^2}\right)}\right)\left({}_h\Upsilon^{w'm^\rho,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}1\right)\left(h^{-1}\left(\frac{h(a)}{m}\right);p\right)\right.\\&\left.+\left(e^{f(h(b))}+me^{f\left(\frac{h(a)}{m^2}\right)}\right)(mh(b)-h(a))\left({}_h\Upsilon^{w'm^\rho,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}1\right)\left(h^{-1}\left(\frac{h(a)}{m}\right);p\right)\right]\end{aligned}$$

where $w' = \frac{w}{(mh(b)-h(a))^\rho}$.

Proof. Since f is exponentially m-convex function on $[a,mb]$, for $t \in [0,1]$, we have

$$2e^{f\left(\frac{h(a)+mh(b)}{2}\right)}\leq e^{f(th(a)+m(1-t)h(b))}+me^{f\left((1-t)\frac{h(a)}{m}+th(b)\right)}. \tag{8.342}$$

Also, from exponentially m-convexity, we have

$$\begin{aligned}&e^{f(th(a)+m(1-t)h(b))}+me^{f\left((1-t)\frac{h(a)}{m}+th(b)\right)} &(8.343)\\&\leq t\left(e^{f(h(a))}-m^2e^{f\left(\frac{h(a)}{m^2}\right)}\right)+m\left(e^{f(h(b))}+me^{f\left(\frac{h(a)}{m^2}\right)}\right).\end{aligned}$$

Multiplying both sides of (8.342) with $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)$ and integrating over $[0,1]$, we have

$$2e^{f\left(\frac{h(a)+mh(b)}{2}\right)}\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)dt \tag{8.344}$$

$$\leq \int_0^1 t^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^\rho;p) e^{f(th(a)+m(1-t)h(b))} dt$$
$$+ m \int_0^1 t^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^\rho;p) e^{f((1-t)\frac{h(a)}{m}+th(b))} dt.$$

Putting $h(u) = th(a) + m(1-t)h(b)$ and $h(v) = (1-t)\frac{h(a)}{m} + th(b)$ in (8.344), we get

$$2e^{f\left(\frac{h(a)+mh(b)}{2}\right)} \int_a^{h^{-1}(mh(b))} (mh(b) - h(u))^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w'(mh(b) - h(u))^\rho;p) d(h(u))$$
$$\leq \int_a^{h^{-1}(mh(b))} (mh(b) - h(u))^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w'(mh(b) - h(u))^\rho;p) e^{f(h(u))} d(h(u))$$
$$+ m^{\sigma+1} \int_{h^{-1}\left(\frac{h(a)}{m}\right)}^{b} \left(h(v) - \frac{h(a)}{m}\right)^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}\left(m^\rho w' \left(h(v) - \frac{h(a)}{m}\right)^\rho;p\right) e^{f(h(v))} d(h(v)).$$

By using the definitions of involved fractional integral operators, the first inequality of (8.341) is obtained.

Now multiplying both sides of (8.343) with $t^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^\rho;p)$ and integrating over $[0,1]$, we have

$$\int_0^1 t^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^\rho;p) e^{f(th(a)+m(1-t)h(b))} dt \tag{8.345}$$
$$+ m \int_0^1 t^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^\rho;p) e^{f((1-t)\frac{h(a)}{m}+th(b))} dt$$
$$\leq \left(e^{f(h(a))} - m^2 e^{f\left(\frac{h(a)}{m^2}\right)}\right) \int_0^1 t^\sigma E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^\rho;p) dt$$
$$+ m\left(e^{f(h(b))} + me^{f\left(\frac{h(a)}{m^2}\right)}\right) \int_0^1 t^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,c,q,r}(wt^\rho;p) dt.$$

Putting $h(u) = th(a) + m(1-t)h(b)$ and $h(v) = (1-t)\frac{h(a)}{m} + th(b)$ in (8.345), then by using the definition of involved fractional integral operators, the second inequality of (8.341) is obtained. □

Corollary 8.63 *Under the assumptions of Theorem 8.53 if we take $m = 1$, then we get following inequalities for exponentially convex function:*

$$2e^{f\left(\frac{h(a)+h(b)}{2}\right)} \left({}_h\Upsilon_{a^+,\rho,\sigma,\tau}^{w',\delta,c,q,r} 1\right)(b;p) \tag{8.346}$$
$$\leq \left({}_h\Upsilon_{a^+,\rho,\sigma,\tau}^{w',\delta,c,q,r} e^{f\circ h}\right)(b;p) + \left({}_h\Upsilon_{b^-,\rho,\sigma,\tau}^{w',\delta,c,q,r} e^{f\circ h}\right)(a;p)$$
$$\leq \left(e^{f(h(b))} + e^{f(h(a))}\right) \left({}_h\Upsilon_{b^-,\rho,\sigma,\tau}^{w',\delta,c,q,r} 1\right)(a;p)$$

where $w' = \frac{w}{(h(b)-h(a))^\rho}$.

Remark 8.21

(i) If $h(u)=u$ in (8.341), then Theorem 8.48 is obtained.

(ii) If $h(u)=u$ and $m=1$ in (8.341), then [87, Corollary 2.2] is obtained.

(iii) If $h(u)=u$ in (8.346), then [87, Corollary 2.2] is obtained.

In the following we give another version of the Hadamard inequality for generalized fractional integral operators.

Theorem 8.54 *Let $w\in\mathbb{R}$, $r,\rho,\sigma,\tau>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $f,h:[a,mb]\subset\mathbb{R}\to\mathbb{R}$, $0<a<mb$ be the real-valued functions. If f be a integrable and exponentially m-convex and h be a differentiable and strictly increasing. Then the following inequalities hold:*

$$\begin{aligned}
&2e^{f\left(\frac{h(a)+mh(b)}{2}\right)}\left({}_h\Upsilon^{w'2^\rho,\delta,c,q,r}_{\left(h^{-1}\left(\frac{h(a)+mh(b)}{2}\right)\right)^+,\rho,\sigma,\tau}1\right)(h^{-1}(mh(b));p) \qquad (8.347)\\
&\leq\left({}_h\Upsilon^{w'2^\rho,\delta,c,q,r}_{\left(h^{-1}\left(\frac{h(a)+mh(b)}{2}\right)\right)^+,\rho,\sigma,\tau}e^{f\circ h}\right)(h^{-1}(mh(b));p)\\
&+m^{\sigma+1}\left({}_h\Upsilon^{w'(2m)^\rho,\delta,c,q,r}_{\left(h^{-1}\left(\frac{h(a)+mh(b)}{2m}\right)\right)^-,\rho,\sigma,\tau}e^{f\circ h}\right)\left(h^{-1}\left(\frac{h(a)}{m}\right);p\right)\\
&\leq\frac{m^{\sigma+1}}{(mh(b)-h(a))}\left[\left(e^{f(h(a))}-m^2e^{f\left(\frac{h(a)}{m^2}\right)}\right)\left({}_h\Upsilon^{w'(2m)^\rho,\delta,c,q,r}_{\left(h^{-1}\left(\frac{h(a)+mh(b)}{2m}\right)\right)^-,\rho,\sigma+1,\tau}1\right)\left(h^{-1}\left(\frac{h(a)}{m}\right);p\right)\right.\\
&\left.+\left(e^{f(h(b))}+me^{f\left(\frac{h(a)}{m^2}\right)}\right)(mh(b)-h(a))\left({}_h\Upsilon^{w'(2m)^\rho,\delta,c,q,r}_{\left(h^{-1}\left(\frac{h(a)+mh(b)}{2m}\right)\right)^-,\rho,\sigma,\tau}1\right)\left(h^{-1}\left(\frac{h(a)}{m}\right);p\right)\right],
\end{aligned}$$

where $w'=\frac{w}{(mh(b)-h(a))^\rho}$.

Proof. Since f is exponentially m-convex function on $[a,mb]$, for $t\in[0,1]$, we have

$$2e^{f\left(\frac{h(a)+mh(b)}{2}\right)}\leq e^{f\left(\frac{t}{2}h(a)+m\frac{(2-t)}{2}h(b)\right)}+me^{f\left(\frac{t}{2}h(b)+\frac{(2-t)}{2}\frac{h(a)}{m}\right)}. \qquad (8.348)$$

Also, from exponentially m-convexity, we have

$$\begin{aligned}
&e^{f\left(\frac{t}{2}h(a)+m\frac{(2-t)}{2}h(b)\right)}+me^{f\left(\frac{t}{2}h(b)+\frac{(2-t)}{2}\frac{h(a)}{m}\right)} \qquad (8.349)\\
&\leq\frac{t}{2}\left(e^{f(h(a))}-m^2e^{f\left(\frac{h(a)}{m^2}\right)}\right)+m\left(e^{f(h(b))}+me^{f\left(\frac{h(a)}{m^2}\right)}\right).
\end{aligned}$$

Multiplying both sides of (8.348) with $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)$ and integrating over $[0,1]$, we have

$$2e^{f\left(\frac{h(a)+mh(b)}{2}\right)}\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)dt \qquad (8.350)$$

$$\leq \int_0^1 t^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p) e^{f\left(\frac{t}{2}h(a)+m\frac{(2-t)}{2}h(b)\right)} dt$$
$$+ m\int_0^1 t^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p) e^{f\left(\frac{t}{2}h(b)+\frac{(2-t)}{2}\frac{h(a)}{m}\right)} dt.$$

Putting $h(u) = \frac{t}{2}h(a) + m\frac{(2-t)}{2}h(b)$ and $h(v) = \frac{t}{2}h(b) + \frac{(2-t)}{2}\frac{h(a)}{m}$ in (8.350), we get

$$2e^{f\left(\frac{h(a)+mh(b)}{2}\right)} \int_{h^{-1}\left(\frac{h(a)+mh(b)}{2}\right)}^{h^{-1}(mh(b))} (mh(b)-h(u))^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(2^\rho w(mh(b)-h(u))^\rho;p) d(h(u))$$
$$\leq \int_{h^{-1}\left(\frac{h(a)+mh(b)}{2}\right)}^{h^{-1}(mh(b))} (mh(b)-h(u))^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(2^\rho w(mh(b)-h(u))^\rho;p) e^{f(h(u))} d(h(u))$$
$$+ m^{\sigma+1}\int_{h^{-1}\left(\frac{h(a)}{m}\right)}^{h^{-1}\left(\frac{h(a)+mh(b)}{2m}\right)} \left(h(v)-\frac{h(a)}{m}\right)^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left((2m)^\rho w(h(v)-\frac{h(a)}{m})^\rho;p\right) e^{f(h(v))} d(h(v)).$$

By using (2.4), the first inequality of (8.347) is obtained. Now multiplying both sides of (8.349) with $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)$ and integrating over $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 t^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p) e^{f\left(\frac{t}{2}h(a)+m\frac{(2-t)}{2}h(b)\right)} dt \\
&+ m\int_0^1 t^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p) e^{f\left(\frac{t}{2}h(b)+\frac{(2-t)}{2}\frac{h(a)}{m}\right)} dt \\
&\leq \frac{1}{2}\left(e^{f(h(a))} - m^2 e^{f\left(\frac{h(a)}{m^2}\right)}\right)\int_0^1 t^{\sigma} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)dt \\
&+ m\left(e^{f(h(b))} + me^{f\left(\frac{h(a)}{m^2}\right)}\right)\int_0^1 t^{\sigma-1} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)dt.
\end{aligned} \tag{8.351}$$

Putting $h(u) = \frac{t}{2}h(a) + m\frac{(2-t)}{2}h(b)$ and $h(v) = \frac{t}{2}h(b) + \frac{(2-t)}{2}\frac{h(a)}{m}$ in (8.351), then by using (2.4), the second inequality of (8.347) is obtained. □

Corollary 8.64 *Under the assumptions of Theorem 8.54 if we take $m = 1$, then we get following inequalities for exponentially convex function:*

$$\begin{aligned}
&2e^{f\left(\frac{h(a)+h(b)}{2}\right)} \left({}_h\Upsilon^{w'(2)^\rho,\delta,c,q,r}_{\left(h^{-1}\left(\frac{h(a)+h(b)}{2}\right)\right)^+,\rho,\sigma,\tau}1\right)(b;p) \\
&\leq \left({}_h\Upsilon^{w'(2)^\rho,\delta,c,q,r}_{\left(h^{-1}\left(\frac{h(a)+h(b)}{2}\right)\right)^+,\rho,\sigma,\tau}e^{f\circ h}\right)(b;p) + \left({}_h\Upsilon^{w'(2)^\rho,\delta,c,q,r}_{\left(h^{-1}\left(\frac{h(a)+h(b)}{2}\right)\right)^-,\rho,\sigma,\tau}e^{f\circ h}\right)(a;p) \\
&\leq \left(e^{f(h(b))} + e^{f(h(a))}\right)\left({}_h\Upsilon^{w'(2)^\rho,\delta,c,q,r}_{\left(h^{-1}\left(\frac{h(a)+h(b)}{2}\right)\right)^-,\rho,\sigma,\tau}1\right)(a;p),
\end{aligned} \tag{8.352}$$

where $w' = \frac{w}{(h(b)-h(a))^\rho}$.

Remark 8.22

(i) If $h(u) = u$ in (8.347), then Theorem 8.49 is obtained.

(ii) If $h(u) = u$ and $m = 1$ in (8.347), then [87, Corollary 2.5] is obtained.

(iii) If $h(u) = u$ in (8.352), then [87, Corollary 2.5] is obtained.

Theorem 8.55 *Let $w \in \mathbb{R}$, $r,\rho,\sigma,\tau > 0$, $c > \delta > 0$ with $p \geq 0$ and $0 < q \leq r+\rho$. Let $f,h : [a,mb] \subset \mathbb{R} \to \mathbb{R}$, $0 < a < mb$ be the real-valued functions. If f be a integrable, exponentially m-convex and $f(h(v)) = f(h(a)+mh(b)-mh(v))$ and h be a differentiable and strictly increasing. Also, let $\gamma : [a,mb] \to \mathbb{R}$ be a function which is nonnegative and integrable. Then the following inequalities hold:*

$$2e^{f\left(\frac{h(a)+mh(b)}{2}\right)}\left({}_h\Upsilon^{w'm^\rho,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}e^{\gamma\circ h}\right)\left(h^{-1}\left(\frac{h(a)}{m}\right);p\right) \tag{8.353}$$
$$\leq (1+m)\left({}_h\Upsilon^{w'm^\rho,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}e^{f\circ h}e^{\gamma\circ h}\right)\left(h^{-1}\left(\frac{h(a)}{m}\right);p\right)$$
$$\leq \frac{m}{(mh(b)-h(a))}\left[\left(e^{f(h(a))}-m^2e^{f\left(\frac{h(a)}{m^2}\right)}\right)\left({}_h\Upsilon^{w'm^\rho,\delta,c,q,r}_{b^-,\rho,\sigma+1,\tau}e^{\gamma\circ h}\right)\left(h^{-1}\left(\frac{h(a)}{m}\right);p\right)\right.$$
$$\left.+\left(e^{f(h(b))}+me^{f\left(\frac{h(a)}{m^2}\right)}\right)(mh(b)-h(a))\left({}_h\Upsilon^{w'm^\rho,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}e^{\gamma\circ h}\right)\left(h^{-1}\left(\frac{h(a)}{m}\right);p\right)\right],$$

where $w' = \frac{w}{(mh(b)-h(a))^\rho}$.

Proof. Multiplying both sides of (8.342) with $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)e^{\gamma((1-t)\frac{h(a)}{m}+\tau h(b))}$ and integrating over $[0,1]$, we have

$$2e^{f\left(\frac{h(a)+mh(b)}{2}\right)}\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)e^{\gamma((1-t)\frac{h(a)}{m}+\tau h(b))}dt \tag{8.354}$$
$$\leq \int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)e^{f(\tau h(a)+m(1-t)h(b))}e^{\gamma((1-t)\frac{h(a)}{m}+\tau h(b))}dt$$
$$+m\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)e^{f((1-t)\frac{h(a)}{m}+\tau h(b))}e^{\gamma((1-t)\frac{h(a)}{m}+\tau h(b))}dt.$$

Putting $h(v) = (1-t)\frac{h(a)}{m}+\tau h(b)$ in (8.354), we get

$$2e^{f\left(\frac{h(a)+mh(b)}{2}\right)}\int_{h^{-1}\left(\frac{h(a)}{m}\right)}^b\left(h(v)-\frac{h(a)}{m}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(m^\rho w(h(v)-\frac{h(a)}{m})^\rho;p\right)e^{\gamma(h(v))}d(h(v))$$
$$\leq\int_{h^{-1}\left(\frac{h(a)}{m}\right)}^b\left(h(v)-\frac{h(a)}{m}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(m^\rho w(h(v)-\frac{h(a)}{m})^\rho;p\right)e^{f(h(a)+mh(b)-mh(v))}e^{\gamma(h(v))}d(h(v))$$
$$+m\int_{h^{-1}\left(\frac{h(a)}{m}\right)}^b\left(h(v)-\frac{h(a)}{m}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left(m^\rho w(h(v)-\frac{h(a)}{m})^\rho;p\right)e^{f(h(v))}e^{\gamma(h(v))}d(h(v)).$$

By using (2.4) and given condition $f(h(v)) = f(h(a)+mh(b)-mh(v))$, the first inequality of (8.353) is obtained.

Now multiplying both sides of (8.343) with $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)e^{\gamma((1-t)\frac{h(a)}{m}+\tau h(b))}$ and integrating over $[0,1]$, we have

$$\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)e^{f(\tau h(a)+m(1-t)h(b))}e^{\gamma((1-t)\frac{h(a)}{m}+\tau h(b))}dt$$
$$+m\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)e^{f((1-t)\frac{h(a)}{m}+\tau h(b))}e^{\gamma((1-t)\frac{h(a)}{m}+\tau h(b))}dt.$$
$$\leq \left(e^{f(h(a))}-m^2e^{f\left(\frac{h(a)}{m^2}\right)}\right)\int_0^1 t^{\sigma}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)e^{\gamma((1-t)\frac{h(a)}{m}+\tau h(b))}dt$$
$$+m\left(e^{f(h(b))}+me^{f\left(\frac{h(a)}{m^2}\right)}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^{\rho};p)e^{\gamma((1-t)\frac{h(a)}{m}+\tau h(b))}dt.$$

From above the second inequality of (8.353) is achieved. □

Corollary 8.65 *Under the assumptions of Theorem 8.55 if we take $m=1$, then we get following inequalities for exponentially convex function:*

$$2e^{f\left(\frac{h(a)+h(b)}{2}\right)}\left({}_h\Upsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau}e^{\gamma\circ h}\right)(a;p)\leq 2\left({}_h\Upsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau}e^{f\circ h}e^{\gamma\circ h}\right)(a;p) \tag{8.355}$$
$$\leq\left(e^{f(h(b))}+e^{f(h(a))}\right)\left({}_h\Upsilon^{w',\delta,c,q,r}_{b^-,\rho,\sigma,\tau}e^{\gamma\circ h}\right)(a;p),$$

where $w'=\frac{w}{(h(b)-h(a))^{\rho}}$.

Remark 8.23

(i) If $h(u)=u$ in (8.353), then Theorem 8.51 is obtained.

(ii) If $h(u)=u$ and $m=1$ in (8.353), then [87, Corollary 2.8] is obtained.

(iii) If $h(u)=u$ in (8.355), then [87, Corollary 2.8] is obtained.

In the following we give another generalized fractional version of the Fejér-Hadamard inequality.

Theorem 8.56 *Let $w\in\mathbb{R}$, $r,\rho,\sigma,\tau>0$, $c>\delta>0$ with $p\geq 0$ and $0<q\leq r+\rho$. Let $f,h:[a,mb]\subset\mathbb{R}\to\mathbb{R}$, $0<a<mb$ be the real-valued functions. If f be a integrable, exponentially m-convex and $f(h(v))=f(h(a)+mh(b)-mh(v))$ and h be a differentiable and strictly increasing. Also, let $\gamma:[a,mb]\to\mathbb{R}$ be a function which is nonnegative and integrable. Then the following inequalities hold:*

$$2e^{f\left(\frac{h(a)+mh(b)}{2}\right)}\left({}_h\Upsilon^{w'(2m)^{\rho},\delta,c,q,r}_{\left(h^{-1}\left(\frac{h(a)+mh(b)}{2m}\right)\right)^-,\rho,\sigma,\tau}e^{\gamma\circ h}\right)\left(h^{-1}\left(\frac{h(a)}{m}\right);p\right) \tag{8.356}$$
$$\leq(1+m)\left({}_h\Upsilon^{w'(2m)^{\rho},\delta,c,q,r}_{\left(h^{-1}\left(\frac{h(a)+mh(b)}{2m}\right)\right)^-,\rho,\sigma,\tau}e^{f\circ h}e^{\gamma\circ h}\right)\left(h^{-1}\left(\frac{h(a)}{m}\right);p\right)$$

$$\leq \frac{m}{(mh(b)-h(a))}\Bigg[\left(e^{f(h(a))}-m^2e^{f\left(\frac{h(a)}{m^2}\right)}\right)$$
$$\left({}_h\Upsilon^{w'(2m)^\rho,\delta,c,q,r}_{\left(h^{-1}\left(\frac{h(a)+mh(b)}{2m}\right)\right)^-,\rho,\sigma,\tau}e^{\gamma\circ h}\right)\left(h^{-1}\left(\frac{h(a)}{m}\right);p\right)$$
$$+\left(e^{f(h(b))}+me^{f\left(\frac{h(a)}{m^2}\right)}\right)(mh(b)-h(a))$$
$$\left({}_h\Upsilon^{w'(2m)^\rho,\delta,c,q,r}_{\left(h^{-1}\left(\frac{h(a)+mh(b)}{2m}\right)\right)^-,\rho,\sigma,\tau}e^{\gamma\circ h}\right)\left(h^{-1}\left(\frac{h(a)}{m}\right);p\right)\Bigg],$$

where $w'=\frac{w}{(mh(b)-h(a))^\rho}$.

Proof. Multiplying both sides of (8.348) with $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)e^{\gamma\left(\frac{t}{2}h(b)+\frac{(2-t)}{2}\frac{h(a)}{m}\right)}$ and integrating over $[0,1]$, we have

$$2e^{f\left(\frac{h(a)+mh(b)}{2}\right)}\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)e^{\gamma\left(\frac{t}{2}h(b)+\frac{(2-t)}{2}\frac{h(a)}{m}\right)}dt \tag{8.357}$$
$$\leq\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)e^{f\left(\frac{t}{2}h(a)+m\frac{(2-t)}{2}h(b)\right)}e^{\gamma\left(\frac{t}{2}h(b)+\frac{(2-t)}{2}\frac{h(a)}{m}\right)}dt$$
$$+m\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)e^{f\left(\frac{t}{2}h(b)+\frac{(2-t)}{2}\frac{h(a)}{m}\right)}e^{\gamma\left(\frac{t}{2}h(b)+\frac{(2-t)}{2}\frac{h(a)}{m}\right)}dt.$$

Putting $h(v)=\frac{t}{2}h(b)+\frac{(2-t)}{2}\frac{h(a)}{m}$ in (8.357), we get

$$2e^{f\left(\frac{h(a)+mh(b)}{2}\right)}\int_{h^{-1}\left(\frac{h(a)}{m}\right)}^{\left(h^{-1}\left(\frac{h(a)+mh(b)}{2m}\right)\right)}\left(h(v)-\frac{h(a)}{m}\right)^{\sigma-1}$$
$$\times E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left((2m)^\rho w(h(v)-\frac{h(a)}{m})^\rho;p\right)e^{\gamma(h(v))}d(h(v))$$
$$\leq\int_{h^{-1}\left(\frac{h(a)}{m}\right)}^{\left(h^{-1}\left(\frac{h(a)+mh(b)}{2m}\right)\right)}\left(h(v)-\frac{h(a)}{m}\right)^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left((2m)^\rho w(h(v)-\frac{h(a)}{m})^\rho;p\right)$$
$$\times e^{f(h(a)+mh(b)-mh(v))}e^{\gamma(h(v))}d(h(v))+m\int_{h^{-1}\left(\frac{h(a)}{m}\right)}^{\left(h^{-1}\left(\frac{h(a)+mh(b)}{2m}\right)\right)}\left(h(v)-\frac{h(a)}{m}\right)^{\sigma-1}$$
$$\times E^{\delta,c,q,r}_{\rho,\sigma,\tau}\left((2m)^\rho w(h(v)-\frac{h(a)}{m})^\rho;p\right)e^{f(h(v))}e^{\gamma(h(v))}d(h(v)).$$

By using (2.4) and given condition $f(h(v))=f(h(a)+mh(b)-mh(v))$, the first inequality of (8.356) is obtained.

Now multiplying both sides of (8.349) with $t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)e^{\gamma\left(\frac{t}{2}h(b)+\frac{(2-t)}{2}\frac{h(a)}{m}\right)}$ and integrating over $[0,1]$, we have

$$\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)e^{f\left(\frac{t}{2}h(a)+m\frac{(2-t)}{2}h(b)\right)}e^{\gamma\left(\frac{t}{2}h(b)+\frac{(2-t)}{2}\frac{h(a)}{m}\right)}dt$$
$$+m\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)e^{f\left(\frac{t}{2}h(b)+\frac{(2-t)}{2}\frac{h(a)}{m}\right)}e^{\gamma\left(\frac{t}{2}h(b)+\frac{(2-t)}{2}\frac{h(a)}{m}\right)}dt.$$
$$\leq \left(e^{f(h(a))}-m^2e^{f\left(\frac{h(a)}{m^2}\right)}\right)\int_0^1 t^{\sigma}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)e^{\gamma\left(\frac{t}{2}h(b)+\frac{(2-t)}{2}\frac{h(a)}{m}\right)}dt$$
$$+m\left(e^{f(h(b))}+me^{f\left(\frac{h(a)}{m^2}\right)}\right)\int_0^1 t^{\sigma-1}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(wt^\rho;p)e^{\gamma\left(\frac{t}{2}h(b)+\frac{(2-t)}{2}\frac{h(a)}{m}\right)}dt.$$

From above the second inequality of (8.356) is achieved. □

Corollary 8.66 *Under the assumptions of Theorem 8.56 if we take $m=1$, then we get following inequalities for exponentially convex function:*

$$2e^{f\left(\frac{h(a)+h(b)}{2}\right)}\left({}_h\Upsilon^{w'2^\rho,\delta,c,q,r}_{\left(h^{-1}\left(\frac{h(a)+h(b)}{2}\right)\right)^-,\rho,\sigma,\tau}e^{\gamma\circ h}\right)(a;p) \tag{8.358}$$
$$\leq 2\left({}_h\Upsilon^{w'2^\rho,\delta,c,q,r}_{\left(h^{-1}\left(\frac{h(a)+h(b)}{2}\right)\right)^-,\rho,\sigma,\tau}e^{\gamma\circ h}\right)(a;p)$$
$$\leq \left(e^{f(h(b))}+e^{f(h(a))}\right)\left({}_h\Upsilon^{w'2^\rho,\delta,c,q,r}_{\left(h^{-1}\left(\frac{h(a)+h(b)}{2}\right)\right)^-,\rho,\sigma,\tau}e^{\gamma\circ h}\right)(a;p),$$

where $w'=\frac{w}{(h(b)-h(a))^\rho}$.

Chapter 9

Bounds of Unified Integral Operators Containing Mittag-Leffler Function

In this chapter bonds of unified integral operators are given for different kinds of convex functions. The results are further deducible for various types of fractional integral operators.

This chapter is based on our results from [56, 98, 72, 109].

9.1 Bounds of Unified Integral Operators of Convex Functions

In this section bounds bonds of unified integral operators are given for convex functions.

Theorem 9.1 *Let $f:[a,b]\to\mathbb{R}$ be a positive convex function, $0<a<b$ and $g:[a,b]\to\mathbb{R}$ be differentiable and strictly increasing function. Also let $\frac{\phi}{x}$ be an increasing function on $[a,b]$ and $c,\sigma,\tau,r\in\mathbb{C}$, $p,\rho,c\geq 0$, and $0<q\leq c+\rho$. Then for $x\in[a,b]$ we have*

$$\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x,w;p)\leq E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(a))^{\rho};p)\,(\phi(g(x)-g(a)))\,(f(x)+f(a)) \tag{9.1}$$

and

$$\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(x,w;p) \leq E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(b)-g(x))^{\rho};p)\left(\phi(g(b)-g(x))\right)(f(x)+f(b)) \tag{9.2}$$

hence

$$\begin{aligned} &\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x,w;p)+\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(x,w;p) \\ &\leq E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(a))^{\rho};p)\left(\phi(g(x)-g(a))\right) \\ &(f(x)+f(a))+E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(b)-g(x))^{\rho};p)\left(\phi(g(b)-g(x))\right)(f(x)+f(b)). \end{aligned} \tag{9.3}$$

Proof. The function g is increasing, therefore for $t\in[a,x], x\in(a,b),\ g(x)-g(t) \leq g(x)-g(a)$. The function $\frac{\phi}{x}$ is increasing, therefore one can obtain:

$$\frac{\phi(g(x)-g(t))}{g(x)-g(t)} \leq \frac{\phi(g(x)-g(a))}{g(x)-g(a)}. \tag{9.4}$$

Now by multiplying with $E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(t))^{\rho};p)g'(t)$ the following inequality is yielded:

$$\begin{aligned} &\frac{\phi(g(x)-g(t))}{g(x)-g(t)}g'(t)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(t))^{\rho};p) \\ &\leq \frac{\phi(g(x)-g(a))}{g(x)-g(a)}g'(t)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(t))^{\rho};p). \end{aligned} \tag{9.5}$$

Also $E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(t))^{\rho};p)$ is series of positive terms, therefore $E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(t))^{\rho};p) \leq E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(a))^{\rho};p)$ so the following inequality holds:

$$\begin{aligned} &\frac{\phi(g(x)-g(t))}{g(x)-g(t)}g'(t)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(t))^{\rho};p) \\ &\leq \frac{\phi(g(x)-g(a))}{g(x)-g(a)}g'(t)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(a))^{\rho};p). \end{aligned} \tag{9.6}$$

Using convexity of f on $[a,x]$ for $x\in(a,b)$ we have

$$f(t) \leq \frac{x-t}{x-a}f(a)+\frac{t-a}{x-a}f(x). \tag{9.7}$$

Multiplying (9.6) and (9.7), then integrating with respect to t over $[a,x]$ we have

$$\begin{aligned} &\int_a^x \frac{\phi(g(x)-g(t))}{g(x)-g(t)}g'(t)f(t)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(t))^{\rho};p)dt \\ &\leq \frac{f(a)}{x-a}\frac{\phi(g(x)-g(a))}{g(x)-g(a)}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(a))^{\rho};p)\int_a^x (x-t)g'(t)dt \\ &+\frac{f(x)}{x-a}\frac{\phi(g(x)-g(a))}{g(x)-g(a)}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(a))^{\rho};p)\int_a^x (t-a)g'(t)dt. \end{aligned} \tag{9.8}$$

By using (2.21), and integrating by parts we get

$$\begin{aligned}&\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x,w;p)\leq E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(a))^{\rho};p)\\ &\times\left(\frac{\phi(g(x)-g(a))}{g(x)-g(a)}\right)\left(\frac{f(a)}{x-a}\left(g(a)(a-x)+\int_a^x g(t)dt\right)\right.\\ &\left.+\frac{f(x)}{x-a}\left((x-a)g(x)-\int_a^x g(t)dt\right)\right)\end{aligned}$$

which further simplifies as follows:

$$\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x,w;p)\leq E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(a))^{\rho};p)\left(\phi(g(x)-g(a))\right)\left(f(x)+f(a)\right). \tag{9.9}$$

Now on the other hand for $t\in(x,b]$, $x\in(a,b)$ the following inequality holds true:

$$\begin{aligned}&\frac{\phi(g(t)-g(x))}{g(t)-g(x)}g'(t)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(t)-g(x))^{\rho};p)\\ &\leq\frac{\phi(g(b)-g(x))}{g(b)-g(x)}g'(t)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(t)-g(x))^{\rho};p).\end{aligned} \tag{9.10}$$

Also $E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(t)-g(x))^{\rho};p)$ is series of positive terms, therefore $E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(t)-g(x))^{\rho};p)\leq E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(b)-g(x))^{\rho};p)$, so the following inequality is valid:

$$\begin{aligned}&\frac{\phi(g(t)-g(x))}{g(t)-g(x)}g'(t)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(t)-g(x))^{\rho};p)\\ &\leq\frac{\phi(g(b)-g(x))}{g(b)-g(x)}g'(t)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(b)-g(x))^{\rho};p).\end{aligned} \tag{9.11}$$

The following inequality also holds for convex function f:

$$f(t)\leq\frac{t-x}{b-x}f(b)+\frac{b-t}{b-x}f(x). \tag{9.12}$$

Multiplying (9.11) and (9.12), then integrating with respect to t over $(x,b]$ and adopting the same pattern of simplification as we did for (9.8), the following inequality is obtained:

$$\begin{aligned}&\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(x,w;p)\leq E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(b)-g(x))^{\rho};p)\\ &\times\left(\frac{\phi(g(b)-g(x))}{g(b)-g(x)}\right)\left(\frac{f(b)}{b-x}\left(g(b)(b-x)-\int_x^b g(t)dt\right)\right.\\ &\left.+\frac{f(x)}{b-x}\left((x-b)g(x)+\int_x^b g(t)dt\right)\right)\end{aligned}$$

which further simplifies as follows:

$$\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(x,w;p)\leq E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(b)-g(x))^{\rho};p)\left(\phi(g(b)-g(x))\right)\left(f(x)+f(b)\right). \tag{9.13}$$

By adding (9.9) and (9.13), (9.3) can be achieved.
Henceforth we give consequences of above theorem for fractional calculus and integral operators defined in [47, 34, 81, 92, 108, 139]. □

Proposition 9.1 *Let $\phi(t) = t^\sigma$ and $p = w = 0$. Then (2.21) and (2.22) produce the the following bound for fractional integral operators defined in [92], as follows:*

$$({}_g^\sigma J_{a^+} f)(x) + ({}_g^\sigma J_{b^-} f)(x) \leq \frac{1}{\Gamma(\sigma)} ((g(x)-g(a))^\sigma$$
$$(f(x)+f(a)) + (g(b)-g(x))^\sigma (f(x)+f(b)))\ \sigma \geq 1.$$

Proposition 9.2 *Let $g(x) = I(x) = x$ and $p = w = 0$. Then (2.21) and (2.22) produce the following bound for fractional integral operators defined in [138] as follows:*

$$({}_{a^+}J_\phi f)(x) + ({}_{b^-}J_\phi f)(x)$$
$$\leq \phi(x-a)(f(x)+f(a)) + \phi(b-x)(f(x)+f(b)).$$

Corollary 9.1 *If we take $\phi(t) = \frac{\Gamma(\sigma) t^{\frac{\sigma}{k}}}{k\Gamma_k(\sigma)}$ and $p = w = 0$. Then (2.21) and (2.22) produce the following bound for the fractional integral operators defined in [97] as follows:*

$$({}_g^\alpha J_{a^+}^k f)(x) + ({}_g^\alpha J_{b^-}^k f)(x)$$
$$\leq \frac{1}{k\Gamma_k(\sigma)} ((g(x)-g(a))^{\frac{\sigma}{k}} (f(x)+f(a))$$
$$+ (g(b)-g(x))^{\frac{\sigma}{k}} (f(b)+f(x)).$$

Corollary 9.2 *If we take $\phi(t) = t^\alpha$ and $g(x) = I(x) = x$ with $p = w = 0$. Then (2.21) and (2.22) produce the following bound for left and right Riemann-Liouville fractional integral defined in [92] as follows:*

$$({}^\sigma J_{a^+} f)(x) + ({}^\sigma J_{b^-} f)(x)$$
$$\leq \frac{1}{\Gamma(\sigma)} ((x-a)^\sigma (f(x)+f(a)) + (b-x)^\sigma (f(b)+f(x)))\ \sigma \geq k.$$

Corollary 9.3 *If we take $\phi(t) = \frac{t^{\frac{\sigma}{k}} \Gamma(\sigma)}{k\Gamma_k(\sigma)}$ and $g(x) = I(x) = x$, $p = w = 0$. Then (2.21) and (2.22) produce the following bound for the fractional integral operators define in [108] as follows:*

$$({}^\sigma J_{b^-}^k f)(x) + ({}^\sigma J_{b^-}^k f)(x)$$
$$\leq \frac{1}{k\Gamma_k(\sigma)} ((x-a)^{\frac{\sigma}{k}} (f(x)+f(a))$$
$$+ (b-x)^{\frac{\sigma}{k}} (f(b)+f(x)))\ \sigma \geq k.$$

Corollary 9.4 *If we take $\phi(t) = t^{\sigma}, \sigma > 0$ and $g(x) = \frac{x^{\rho}}{\rho}$, $\rho > 0$ with $p = w = 0$. Then (2.21) and (2.22) produce following bound for the fractional integral operators defined in [34], as follows:*

$$\begin{aligned}&({}^{\rho}J^{\sigma}_{a^+}f)(x) + ({}^{\rho}J^{\sigma}_{b^-}f)(x)\\&\leq \frac{1}{\rho^{\sigma}\Gamma(\sigma)}((x^{\rho} - a^{\rho})^{\sigma}(f(x) + f(a))\\&+ (b^{\rho} - x^{\rho})^{\sigma}(f(b) + f(x))).\end{aligned}$$

Corollary 9.5 *If we take $\phi(t) = t^{\sigma}, \sigma > 0$ and $g(x) = \frac{x^{s+1}}{s+1}$, $s > 0$, $p = w = 0$. Then (2.21) and (2.22) produce following bound for the fractional integral operators define as follows:*

$$\begin{aligned}&({}^{s}J^{\sigma}_{a^+}f)(x) + ({}^{s}J^{\sigma}_{b^-}f)(x)\\&\leq \frac{1}{(s+1)^{\sigma}\Gamma(\sigma)}((x^{s+1} - a^{s+1})^{\sigma}(f(x) + f(a))\\&+ (b^{s+1} - x^{s+1})^{\sigma}(f(b) + f(x))).\end{aligned}$$

Corollary 9.6 *If we take $\phi(t) = \frac{t^{\frac{\sigma}{k}}\Gamma(\sigma)}{k\Gamma_k(\sigma)}$ and $g(x) = \frac{x^{s+1}}{s+1}$, $s > 0$, $p = w = 0$. Then (2.21) and (2.22) produce following bound for the fractional integral operators defined in [139], as follows:*

$$\begin{aligned}&({}^{s}_{k}J^{\sigma}_{a^+}f)(x) + ({}^{s}_{k}J^{\sigma}_{b^-}f)(x)\\&\leq \frac{1}{(s+1)^{\frac{\sigma}{k}}k\Gamma_k(\sigma)}((f(x) + f(a))(b^{s+1} - x^{s+1})^{\frac{\sigma}{k}}\\&+ (x^{s+1} - a^{s+1})^{\frac{\sigma}{k}}(f(b) + f(x)))\,\sigma \geq k.\end{aligned}$$

Corollary 9.7 *If we take $\phi(t) = t^{\sigma}$ and $g(x) = \frac{x^{\sigma+s}}{\sigma+s}$, $\beta, s > 0$, $p = w = 0$. Then (2.21) and (2.22) produce following bound for the fractional integral operators defined as follows:*

$$\begin{aligned}&({}^{s}_{\beta}J^{\sigma}_{a^+}f)(x) + ({}^{s}_{\beta}J^{\sigma}_{b^-}f)(x)\\&\leq \frac{1}{(\sigma+s)^{\sigma}\Gamma(\sigma)}((x^{\sigma+s} - a^{\sigma+s})^{\sigma}(f(x) + f(a))\\&+ (b^{\sigma+s} - x^{\sigma+s})^{\sigma}(f(b) + f(x))).\end{aligned}$$

Corollary 9.8 *If we take $g(x) = \frac{(x-a)^{\rho}}{\rho}$, $\rho > 0$ in (2.21) and $g(x) = \frac{-(b-x)^{\rho}}{\rho}$, $\rho > 0$ in (2.22) with $\phi(t) = t^{\sigma}, \sigma > 0$, $p = w = 0$. Then (2.21) and (2.22) produce the fractional integral operators defined in [81], as follows:*

$$({}^{\rho}J^{\sigma}_{a^+}f)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^x ((x-a)^{\rho} - (t-a)^{\rho})^{\alpha-1}(t-a)^{\rho-1}f(t)dt$$

and

$$({}^{\rho}J^{\sigma}_{b^-}f)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_x^b ((b-x)^{\rho} - (b-t)^{\rho})^{\alpha-1}(b-t)^{\rho-1}f(t)dt.$$

Further they satisfy the following bound:

$$({}^{\rho}J^{\sigma}_{a^+}f)(x)+({}^{\rho}J^{\sigma}_{b^-}f)(x)\leq\frac{1}{\rho^{\sigma}\Gamma(\sigma)}((x-a)^{\rho\alpha}(f(x)+f(a))+(b-x)^{\rho\alpha}(f(b)+f(x))).$$

Corollary 9.9 *If we take $g(x)=\frac{(x-a)^{\rho}}{\rho}$, $\rho>0$ in (2.21) and $g(x)=\frac{-(b-x)^{\rho}}{\rho}$, $\rho>0$ in (2.22) with $\phi(t)=\frac{t^{\frac{\sigma}{k}}\Gamma(\sigma)}{k\Gamma_k(\sigma)}$, $\sigma>k$, $p=w=0$. Then (2.21) and (2.22) produce the fractional integral operators defined as follows:*

$$\begin{aligned}&({}^{\rho}_{k}J^{\sigma}_{a^+}f)(x)+({}^{\rho}_{k}J^{\sigma}_{b^-}f)(x)\\&\leq\frac{1}{\rho^{\frac{\sigma}{k}}k\Gamma_k(\sigma)}((x-a)^{\frac{\rho\alpha}{k}}(f(x)+f(a))+(b-x)^{\frac{\rho\alpha}{k}}(f(b)+f(x))).\end{aligned}$$

Lemma 9.1 *[47] Let $f:[a,b]\to\mathbb{R}$ be a convex function. If f is symmetric about $\frac{a+b}{2}$, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right)\leq f(x),\quad x\in[a,b].\tag{9.14}$$

The following theorem provides the Hadamard type estimation of integral operators (2.21) and (2.22).

Theorem 9.2 *Let $f:[a,b]\to\mathbb{R}$ be a positive convex function, $0<a<b$ and $g:[a,b]\to\mathbb{R}$ be differentiable and strictly increasing function. Also let $\frac{\phi}{x}$ be an increasing function on $[a,b]$ and $c,\sigma,\tau,r\in\mathbb{C}$, $p,\rho,c\geq0$, and $0<q\leq c+\rho$, if in addition f is symmetric about $\frac{a+b}{2}$. Then for $x\in[a,b]$ we have*

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\left(\left({}_{g}F^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}1\right)(a;p)+(\left({}_{g}F^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}1\right)(b;p)\right)\\&\leq\left(\left({}_{g}F^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(a,w;p)+\left({}_{g}F^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(b,w;p)\right)\\&\leq 2\phi(g(b)-g(a))E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(b)-g(a))^{\rho};p)(f(a)+f(b)).\end{aligned}\tag{9.15}$$

Proof. For $x\in(a,b)$, under the assumption on g and $\frac{\phi}{x}$ the following inequality holds:

$$\begin{aligned}&\frac{\phi(g(x)-g(a))}{g(x)-g(a)}g'(x)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(a))^{\rho};p)\\&\leq\frac{\phi(g(b)-g(a))}{g(b)-g(a)}g'(x)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(b)-g(a))^{\rho};p).\end{aligned}\tag{9.16}$$

Using convexity of f on $[a,b]$ for $x\in(a,b)$ we have

$$f(x)\leq\frac{x-a}{b-a}f(b)+\frac{b-x}{b-a}f(a).\tag{9.17}$$

Multiplying (9.16) and (9.17) and then integrating with respect to x over $[a,b]$, the following inequality is obtained:

$$\begin{aligned}&\int_a^b \frac{\phi(g(x)-g(a))}{g(x)-g(a)} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(a))^{\rho};p)g'(x)f(x)dx\\ &\leq \frac{f(b)}{b-a}\frac{\phi(g(b)-g(a))}{g(b)-g(a)} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(b)-g(a))^{\rho};p)\int_a^b (x-a)g'(x)dx\\ &+\frac{f(a)}{b-a}\frac{\phi(g(b)-g(a))}{g(b)-g(a)} E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(b)-g(a))^{\rho};p)\int_a^b (b-x)g'(x)dx.\end{aligned}$$

By using (2.21) and integrating by parts we get

$$\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(b,w;p) \leq E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(b)-g(a))^{\rho};p)\left(\phi(g(b)-g(a))\right)\left(f(a)+f(b)\right). \tag{9.18}$$

On the other hand the following inequality holds:

$$\begin{aligned}&\frac{\phi(g(b)-g(x))}{g(b)-g(x)} g'(x)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(b)-g(x))^{\rho};p)\\ &\leq \frac{\phi(g(b)-g(a))}{g(b)-g(a)} g'(x)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(b)-g(a))^{\rho};p).\end{aligned} \tag{9.19}$$

Multiplying (9.17) and (9.19) and then integrating with respect to x over $[a,b]$ and simplifying on the same pattern as we did for (9.16) and (9.17), following inequality is obtained:

$$\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(a,w;p) \leq E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(b)-g(a))^{\rho};p)\left(\phi(g(b)-g(a))\right)\left(f(a)+f(b)\right). \tag{9.20}$$

By adding (9.18)) and (9.20), we have

$$\begin{aligned}&\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(b,w;p)+\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(a,w;p)\\ &\leq 2\phi(g(b)-g(a))E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(b)-g(a))^{\rho};p)(f(a)+f(b)).\end{aligned} \tag{9.21}$$

Multiplying both sides of 9.14 by $\frac{\phi(g(x)-g(a))}{g(x)-g(a)} g'(x)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(a))^{\rho};p)$, then integrating over $[a,b]$ we get

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\int_a^b \frac{\phi(g(x)-g(a))}{g(x)-g(a)} g'(x)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(a))^{\rho};p)dx\\ &\leq \int_a^b \frac{\phi(g(x)-g(a))}{g(x)-g(a)} g'(x)f(x)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(a))^{\rho};p)f(x)dx\end{aligned}$$

By using 2.22 we get

$$f\left(\frac{a+b}{2}\right)\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}1\right)(a;p) \leq \left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(a,w;p) \tag{9.22}$$

Multiplying both sides of 9.14 by $\dfrac{\phi(g(b)-g(x))}{g(b)-g(x)}g'(x)\,E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(b)-g(x))^{\rho};p)$ and integrating over $[a,b]$ we have

$$f\left(\frac{a+b}{2}\right)\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}1\right)(b;p) \le \left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(b,w;p) \tag{9.23}$$

by adding 9.22 and 9.23, the following inequality is obtained:

$$\begin{aligned} &f\left(\frac{a+b}{2}\right)\left(\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}1\right)(a;p)+\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}1\right)(b;p)\right). \\ &\le \left(\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(a,w;p)+\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(b,w;p)\right). \end{aligned} \tag{9.24}$$

Combining 9.21 and 9.24, inequality 9.15 can be achieved. □

Theorem 9.3 *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function. If $|f'|$ is convex, $0<a<b$ and $g:[a,b]\to\mathbb{R}$ be differentiable and strictly increasing function. Also let $\frac{\phi}{x}$ be an increasing function and $c,\sigma,\tau,r\in\mathbb{C}$, $p,\rho,c\ge 0$, and $0<q\le c+\rho$. Then for $x\in(a,b)$ we have*

$$\begin{aligned} &\left|\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f*g\right)(x,w;p)+\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f*g\right)(x,w;p)\right| \\ &\le E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(a))^{\rho};p)\phi(g(x)-g(a))(|f'(x)|+|f'(a)|)+ \\ &E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(b)-g(x))^{\rho};p)\phi(g(b)-g(x))(|f'(x)|+|f'(b)|). \end{aligned} \tag{9.25}$$

Where

$$\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f*g\right)(x,w;p) := \int_a^x \frac{\phi(g(x)-g(t))}{g(x)-g(t)}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(t))^{\rho};p)g'(t)f'(t)dt$$

$$\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f*g\right)(x,w;p) := \int_x^b \frac{\phi(g(t)-g(x))}{g(t)-g(x)}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(t))^{\rho};p)g'(t)f'(t)dt.$$

Proof. Using the convexity of $|f'|$ over $[a,b]$ for $t\in[a,x]$ we have

$$|f'(t)| \le \frac{x-t}{x-a}|f'(a)|+\frac{t-a}{x-a}|f'(x)|. \tag{9.26}$$

From which we can write

$$-\left(\frac{x-t}{x-a}|f'(a)|+\frac{t-a}{x-a}|f'(x)|\right) \le f'(t) \le \left(\frac{x-t}{x-a}|f'(a)|+\frac{t-a}{x-a}|f'(x)|\right) \tag{9.27}$$

we consider the right hand side inequality of the above inequality i.e.

$$f'(t) \le \left(\frac{x-t}{x-a}|f'(a)|+\frac{t-a}{x-a}|f'(x)|\right). \tag{9.28}$$

Further the following inequality holds true:

$$\frac{\phi(g(x)-g(t))}{g(x)-g(t)}g'(x)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(t))^{\rho};p) \\ \leq \frac{\phi(g(x)-g(a))}{g(x)-g(a)}g'(t)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(a))^{\rho};p). \tag{9.29}$$

Multiplying (9.28) and (9.29) and integrating with respect to t over $[a,x]$, the following inequality is obtained:

$$\int_a^x \frac{\phi(g(x)-g(t))}{g(x)-g(t)}g'(t)f'(t)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(t))^{\rho};p)dt \\ \leq \frac{|f'(a)|}{x-a}\frac{\phi(g(x)-g(a))}{g(x)-g(a)}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(a))^{\rho};p)\int_a^x (x-t)g'(t)dt \\ + \frac{|f'(x)|}{x-a}\frac{\phi(g(x)-g(a))}{g(x)-g(a)}E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(a))^{\rho};p)\int_a^x (t-a)g'(t)dt$$

which gives

$$\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f*g\right)(x,w;p) \leq E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(a))^{\rho};p) \\ \times \phi(g(x)-g(a))(|f'(x)|+|f'(a)|). \tag{9.30}$$

If we consider the left hand side inequality from the inequality (9.27) and proceed as we did for the right hand side inequality we have

$$\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f*g\right)(x,w;p) \geq -E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(a))^{\rho};p) \\ \times \phi(g(x)-g(a))(|f'(x)|+|f'(a)|) \tag{9.31}$$

□

Combining (9.30) and (9.31), the following inequality is obtained:

$$\left|\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f*g\right)(x,w;p)\right| \leq E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(a))^{\rho};p) \\ \times \phi(g(x)-g(a))(|f'(x)|+|f'(a)|). \tag{9.32}$$

On the other hand using convexity of $|f'(t)|$ over $[a,b]$ for $t \in (x,b]$ we have

$$|f'(t)| \leq \frac{t-x}{b-x}|f'(b)| + \frac{b-t}{b-x}|f'(x)|. \tag{9.33}$$

Further the following inequality holds true:

$$\frac{\phi(g(x)-g(t))}{g(x)-g(t)}g'(x)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(t))^{\rho};p) \\ \leq \frac{\phi(g(b)-g(x))}{g(b)-g(x)}g'(t)E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(x)-g(a))^{\rho};p). \tag{9.34}$$

By adopting the same treatment as we did for (9.26) and (9.29), one can obtain the following inequality from (9.33) and (9.34):

$$\left|\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f*g\right)(x,w;p)\right| \leq E^{\delta,c,q,r}_{\rho,\sigma,\tau}(w(g(b)-g(x))^{\rho};p)$$
$$\times\,\phi(g(b)-g(x))(|f'(x)|+|f'(b)|). \tag{9.35}$$

Combining (9.32) and (9.35), inequality (9.25) can be achieved.

9.2 Bounds of Unified Integral Operators for Exponentially (s,m)-Convex Functions

Theorem 9.4 *Let $f:[a,b]\to\mathbb{R}$, $0\leq a<b$, be a positive exponentially (s,m)-convex function with $m\in(0,1]$ and $g:[a,b]\to\mathbb{R}$ be a differentiable and strictly increasing function. Also let $\frac{\phi}{x}$ be an increasing function on $[a,b]$. If $\sigma,\tau,\delta,c\in\mathbb{C}$, $p,\rho\geq 0$, $c\geq 0$ and $0<q\leq c+\rho$, then for $x\in(a,b)$ we have*

$$\begin{aligned}&\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x,w;p)+\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(x,w;p) \\ &\leq\; K^a_x(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)\left(\left(m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha(\frac{x}{m})}}g(x)-\frac{f(a)}{e^{\alpha a}}g(a)\right)\right. \\ &\left.-\frac{\Gamma(s+1)}{(x-a)^s}\left(m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha(\frac{x}{m})}}\,{}^sJ_{x^-}g(a)-\frac{f(a)}{e^{\alpha a}}\,{}^sJ_{a^+}g(a)\right)\right) \\ &+K^x_b(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)\left(\left(\frac{f(b)}{e^{\alpha b}}g(b)-m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha(\frac{x}{m})}}g(x)\right)\right. \\ &\left.-\frac{\Gamma(s+1)}{(b-x)^s}\left(\frac{f(b)}{e^{\alpha b}}\,{}^sJ_{b^-}g(x)-m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha(\frac{x}{m})}}\,{}^sJ_{x^+}g(b)\right)\right).\end{aligned} \tag{9.36}$$

Proof. Using exponentially (s,m)-convexity of f, we have

$$f(t)\leq\left(\frac{x-t}{x-a}\right)^s\frac{f(a)}{e^{\alpha a}}+m\left(\frac{t-a}{x-a}\right)^s\frac{f\left(\frac{x}{m}\right)}{e^{\alpha(\frac{x}{m})}}. \tag{9.37}$$

Multiplying (9.6) and (9.37) and integrating over $[a,x]$, we can obtain:

$$\int_a^x K^t_x(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)f(t)d(g(t))\leq\frac{f(a)}{e^{\alpha a}}K^a_x(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)\int_a^x\left(\frac{x-t}{x-a}\right)^s d(g(t)) \tag{9.38}$$
$$+m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha(\frac{x}{m})}}K^a_x(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)\int_a^x\left(\frac{t-a}{x-a}\right)^s d(g(t)).$$

By using Definition 2.3 and integrating by part, the following inequality is obtained:

$$\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x,w;p) \le K^a_x(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)\left(\left(m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}g(x) - \frac{f(a)}{e^{\alpha a}}g(a)\right)\right. \tag{9.39}$$
$$\left.-\frac{\Gamma(s+1)}{(x-a)^s}\left(m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}\,{}^sJ_{x^-}g(a) - \frac{f(a)}{e^{\alpha a}}\,{}^sJ_{a^+}g(a)\right)\right).$$

Using exponentially (s,m)-convexity of f, we have

$$f(t) \le \left(\frac{t-x}{b-x}\right)^s\frac{f(b)}{e^{\alpha b}} + m\left(\frac{b-t}{b-x}\right)^s\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}. \tag{9.40}$$

Adopting the same pattern as we did for (9.6) and (9.37), we obtained the following inequality from (9.11) and (9.40):

$$\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(x,w;p) \le K^x_b(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)\left(\left(\frac{f(b)}{e^{\alpha b}}g(b) - m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}g(x)\right)\right. \tag{9.41}$$
$$\left.-\frac{\Gamma(s+1)}{(b-x)^s}\left(\frac{f(b)}{e^{\alpha b}}\,{}^sJ_{b^-}g(x) - m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}\,{}^sJ_{x^+}g(b)\right)\right).$$

By adding (9.39) and (9.41), (9.36) can be obtained. □

Remark 9.1

(i) If we take $p = w = \sigma = 0$ in (9.36), then we get [96, Theorem 1].

(ii) If we take $(s,m) = (1,1)$ and $\sigma = 0$ in (9.36), then we get [98, Theorem 1].

(iii) If we take $(s,m) = (1,1)$ and $p = w = 0$ in (9.36), then we get [155, Theorem 1].

(iv) If we take $(s,m) = (1,1)$, $p = w = 0$ and $\sigma = 0$ in (9.36), then we get [104, Theorem 1].

(v) If we take $\phi(t) = t^\rho, \rho > 0$, $\sigma = 0$, $p = w = 0$ and (s,m) =(1,1) in (9.36), then we get [59, Corollary 1].

(vi) If we take $\phi(t) = t^\rho, \rho > 0$, $g(x) = x$, $\sigma = 0$, $p = w = 0$ and (s,m) =(1,1) in (9.36), then we get [47, Corollary 1].

Proposition 9.3 *Let $\phi(t) = t^\rho$, and $p = w = 0$. Then (9.36) gives the following bound for $\rho \ge 1$ defined in [92]:*

$$\begin{aligned}&({}^\rho_gJ_{a^+}f)(x) + ({}^\rho_gJ_{b^-}f)(x)\\ &\le \frac{(g(x)-g(a))^{\rho-1}}{\Gamma(\rho)}\left(\left(m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}g(x) - \frac{f(a)}{e^{\alpha a}}g(a)\right)\right.\\ &\left.-\frac{\Gamma(s+1)}{(x-a)^s}\left(m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}\,{}^sJ_{x^-}g(a) - \frac{f(a)}{e^{\alpha a}}\,{}^sJ_{a^+}g(x)\right)\right)\end{aligned}$$

$$+\frac{(g(b)-g(x))^{\rho-1}}{\Gamma(\rho)}\left(\left(\frac{f(b)}{e^{\alpha b}}g(b)-m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}g(x)\right)\right.$$
$$\left.-\frac{\Gamma(s+1)}{(b-x)^s}\left(\frac{f(b)}{e^{\alpha b}}\,{}^sJ_{b^-}g(x)-m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}\,{}^sJ_{x^+}g(b)\right)\right).$$

Proposition 9.4 *Let $g(x)=I(x)=x$ and $p=w=0$. Then (9.36) gives the following bound defined in [138]:*

$$({}_{a^+}J_\phi f)(x)+({}_{b^-}J_\phi f)(x)$$
$$\leq\frac{\phi(x-a))}{(x-a)^{s+1}}\left((x-a)^s\left(m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}g(x)-\frac{f(a)}{e^{\alpha a}}g(a)\right)\right.$$
$$\left.-\Gamma(s+1)\left(m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}\,{}^sJ_{x^-}g(a)-\frac{f(a)}{e^{\alpha a}}\,{}^sJ_{a^+}g(x)\right)\right)$$
$$+\frac{\phi(b-x))}{(b-x)^{s+1}}\left((b-x)^s\left(\frac{f(b)}{e^{\alpha b}}g(b)-m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}g(x)\right)\right.$$
$$\left.-\Gamma(s+1)\left(\frac{f(b)}{e^{\alpha b}}\,{}^sJ_{b^-}g(x)-m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}\,{}^sJ_{x^+}g(b)\right)\right).$$

Corollary 9.10 *If we take $\phi(t)=\frac{\Gamma(\rho)t^{\frac{\rho}{k}}}{k\Gamma_k(\rho)}$ and $p=w=0$, then (9.36) gives the following bound holds for $\rho\geq k$ defined in [97]:*

$$({}_g^\rho J_{a^+}^k f)(x)+({}_g^\rho J_{b^-}^k f)(x)$$
$$\leq\frac{(g(x)-g(a))^{\frac{\rho}{k}-1}}{k\Gamma_k(\rho)}\left(\left(m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}g(x)-\frac{f(a)}{e^{\alpha a}}g(a)\right)\right.$$
$$\left.-\frac{\Gamma(s+1)}{(x-a)^s}\left(m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}\,{}^sJ_{x^-}g(a)-\frac{f(a)}{e^{\alpha a}}\,{}^sJ_{a^+}g(x)\right)\right)$$
$$+\frac{(g(b)-g(x))^{\frac{\rho}{k}-1}}{k\Gamma_k(\rho)}\left(\left(\frac{f(b)}{e^{\alpha b}}g(b)-m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}g(x)\right)\right.$$
$$\left.-\frac{\Gamma(s+1)}{(b-x)^s}\left(\frac{f(b)}{e^{\alpha b}}\,{}^sJ_{b^-}g(x)-m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}\,{}^sJ_{x^+}g(b)\right)\right).$$

Corollary 9.11 *If we take $\phi(t)=t^\rho$, $p=w=0$ and $g(x)=I(x)=x$, then (9.36) gives the following bound for $\rho\geq 1$ defined in [92]:*

$$({}^\rho J_{a^+}f)(x)+({}^\rho J_{b^-}f)(x)$$
$$\leq\frac{(x-a)^{\rho-1}}{\Gamma(\rho)}\left(\left(m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}g(x)-\frac{f(a)}{e^{\alpha a}}g(a)\right)\right.$$
$$\left.-\frac{\Gamma(s+1)}{(x-a)^s}\left(m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}\,{}^sJ_{x^-}g(a)-\frac{f(a)}{e^{\alpha a}}\,{}^sJ_{a^+}g(x)\right)\right)$$

$$+\frac{(b-x)^{\rho-1}}{\Gamma(\rho)}\left(\left(\frac{f(b)}{e^{\alpha b}}g(b)-m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}g(x)\right)\right.$$
$$\left.-\frac{\Gamma(s+1)}{(b-x)^s}\left(\frac{f(b)}{e^{\alpha b}}\,{}^sJ_{b^-}g(x)-m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}\,{}^sJ_{x^+}g(b)\right)\right).$$

Corollary 9.12 *If we take $\phi(t)=\frac{\Gamma(\rho)t^{\frac{\rho}{k}}}{k\Gamma_k(\rho)}$, $p=w=0$ and $g(x)=I(x)=x$, then (9.36) gives the following bound for $\rho\geq k$ defined in [108]:*

$$({}^{\rho}J^k_{a^+}f)(x)+({}^{\rho}J^k_{b^-}f)(x)$$
$$\leq\frac{(x-a))^{\frac{\rho}{k}-1}}{k\Gamma_k(\rho)}\left(\left(m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}g(x)-\frac{f(a)}{e^{\alpha a}}g(a)\right)\right.$$
$$\left.-\frac{\Gamma(s+1)}{(x-a)^s}\left(m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}\,{}^sJ_{x^-}g(a)-\frac{f(a)}{e^{\alpha a}}\,{}^sJ_{a^+}g(x)\right)\right)$$
$$+\frac{(b-x))^{\frac{s}{\rho}-1}}{k\Gamma_k(\rho)}\left(\left(\frac{f(b)}{e^{\alpha b}}g(b)-m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}g(x)\right)\right.$$
$$\left.-\frac{\Gamma(s+1)}{(b-x)^s}\left(\frac{f(b)}{e^{\alpha b}}\,{}^sJ_{b^-}g(x)-m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}\,{}^sJ_{x^+}g(b)\right)\right).$$

Corollary 9.13 *If we take $\phi(t)=t^{\rho}$, $p=w=0$ and $g(x)=\frac{x^{\rho}}{\rho}$, $\rho>0$, then (9.36) gives the following bound defined in [34]:*

$$({}^{\rho}J^{\sigma}_{a^+}f)(x)+({}^{\rho}J^{\sigma}_{b^-}f)(x)$$
$$\leq\frac{(x^{\rho}-a^{\rho})^{\rho-1}}{\Gamma(\rho)\rho^{\sigma-1}}\left(\left(m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}g(x)-\frac{f(a)}{e^{\alpha a}}g(a)\right)\right.$$
$$\left.-\frac{\Gamma(s+1)}{(x-a)^s}\left(m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}\,{}^sJ_{x^-}g(a)-\frac{f(a)}{e^{\alpha a}}\,{}^sJ_{a^+}g(x)\right)\right)$$
$$+\frac{(b^{\rho}-x^{\rho})^{\rho-1}}{\Gamma(\rho)\rho^{\sigma-1}}\left(\left(\frac{f(b)}{e^{\alpha b}}g(b)-m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}g(x)\right)\right.$$
$$\left.-\frac{\Gamma(s+1)}{(x-a)^s}\left(\frac{f(b)}{e^{\alpha b}}\,{}^sJ_{b^-}g(x)-m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}\,{}^sJ_{x^+}g(b)\right)\right).$$

Corollary 9.14 *If we take $\phi(t)=t^{\rho}$, $p=w=0$ and $g(x)=\frac{x^{n+1}}{n+1}$, $n>0$, then (9.36) gives the following bound:*

$$({}^{n}J^{\sigma}_{a^+}f)(x)+({}^{n}J^{\sigma}_{b^-}f)(x)$$
$$\leq\frac{(x^{n+1}-a^{n+1})^{\rho-1}}{\Gamma(\rho)(n+1)^{\rho-1}}\left(\left(m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}g(x)-\frac{f(a)}{e^{\alpha a}}g(a)\right)\right.$$
$$\left.-\frac{\Gamma(s+1)}{(x-a)^s}\left(m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha\left(\frac{x}{m}\right)}}\,{}^sJ_{x^-}g(a)-\frac{f(a)}{e^{\alpha a}}\,{}^sJ_{a^+}g(x)\right)\right)$$

$$+\frac{(b^{n+1}-x^{n+1})^{\rho-1}}{\Gamma(\rho)(n+1)^{\rho-1}}\left(\left(\frac{f(b)}{e^{\alpha b}}g(b)-m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha(\frac{x}{m})}}g(x)\right)\right.$$
$$\left.-\frac{\Gamma(s+1)}{(x-a)^s}\left(\frac{f(b)}{e^{\alpha b}}\,{}^sJ_{b^-}g(x)-m\frac{f\left(\frac{x}{m}\right)}{e^{\alpha(\frac{x}{m})}}\,{}^sJ_{x^+}g(b)\right)\right).$$

The following lemma is essential to prove the next result.

Lemma 9.2 *Let $f:[0,\infty]\to\mathbb{R}$, be an exponentially (s,m)-convex function with $m\in(0,1]$. If $0\le a<b$ and $\frac{f(x)}{e^{\alpha x}}=\frac{f(\frac{a+b-x}{m})}{e^{\alpha(\frac{a+b-x}{m})}}$, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{2^s}(1+m)\frac{f(x)}{e^{\alpha x}},\quad x\in[a,b]. \tag{9.42}$$

Theorem 9.5 *Under the assumptions of Theorem 9.4, in addition if $\frac{f(x)}{e^{\alpha x}}=\frac{f(\frac{a+b-x}{m})}{e^{\alpha(\frac{a+b-x}{m})}}$, then we have*

$$\frac{h(\alpha)2^sf\left(\frac{a+b}{2}\right)}{(1+m)}\left(\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}1\right)(a,w;p)+\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}1\right)(b,w;p)\right) \tag{9.43}$$
$$\le\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(a,w;p)+\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(b,w;p)$$
$$\le 2K_b^a(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)\left(\left(\frac{f(b)}{e^{\alpha b}}g(b)-m\frac{f\left(\frac{a}{m}\right)}{e^{\alpha(\frac{a}{m})}}g(a)\right)\right.$$
$$\left.-\frac{\Gamma(s+1)}{(b-a)^s}\left(\frac{f(b)}{e^{\alpha b}}\,{}^sJ_{b^-}g(a)-m\frac{f\left(\frac{a}{m}\right)}{e^{\alpha(\frac{a}{m})}}\,{}^sJ_{a^+}g(b)\right)\right).$$

Proof. Using exponentially (s,m)-convexity of f, we have

$$f(x)\le\left(\frac{x-a}{b-a}\right)^s\frac{f(b)}{e^{\alpha b}}+m\left(\frac{b-x}{b-a}\right)^s\frac{f\left(\frac{a}{m}\right)}{e^{\alpha(\frac{a}{m})}}. \tag{9.44}$$

Multiplying (9.16) and (9.44) and integrating the resulting inequality over $[a,b]$, we can obtain obtain:

$$\int_a^b K_x^a(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)d(g(x))$$
$$\le m\frac{f\left(\frac{a}{m}\right)}{e^{\alpha(\frac{a}{m})}}K_b^a(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)\int_a^b\left(\frac{b-x}{b-a}\right)^s d(g(x))$$
$$+\frac{f(b)}{e^{\alpha b}}K_b^a(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)\int_a^b\left(\frac{x-a}{b-a}\right)^s d(g(x)).$$

By using Definition 2.3 and integrating by parts, the following inequality is obtained:

$$\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(b,w;p) \leq K^a_b(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)\left(\left(\frac{f(b)}{e^{\alpha b}}g(b) - m\frac{f\left(\frac{a}{m}\right)}{e^{\alpha\left(\frac{a}{m}\right)}}g(a)\right)(b-a)^s \right. \tag{9.45}$$
$$\left. - \frac{\Gamma(s+1)}{(b-a)^s}\left(\frac{f(b)}{e^{\alpha b}}\,{}^sJ_{b^-}g(a) - m\frac{f\left(\frac{a}{m}\right)}{e^{\alpha\left(\frac{a}{m}\right)}}\,{}^sJ_{a^+}g(b)\right)\right).$$

Adopting the same pattern of simplification as we did for (9.16) and (9.44), the following inequality can be observed for (9.44) and (9.19)

$$\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(a,w;p) \leq K^a_b(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)\left(\left(\frac{f(b)}{e^{\alpha b}}g(b) - m\frac{f\left(\frac{a}{m}\right)}{e^{\alpha\left(\frac{a}{m}\right)}}g(a)\right) \right. \tag{9.46}$$
$$\left. - \frac{\Gamma(s+1)}{(b-a)^s}\left(\frac{f(b)}{e^{\alpha b}}\,{}^sJ_{b^-}g(a) - m\frac{f\left(\frac{a}{m}\right)}{e^{\alpha\left(\frac{a}{m}\right)}}\,{}^sJ_{a^+}g(b)\right)\right).$$

By adding (9.45) and (9.46), following inequality can be obtained:

$$\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(b,w;p) + \left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(a,w;p) \tag{9.47}$$
$$\leq K^a_b(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)\left(\left(\frac{f(b)}{e^{\alpha b}}g(b) - m\frac{f\left(\frac{a}{m}\right)}{e^{\alpha\left(\frac{a}{m}\right)}}g(a)\right) \right.$$
$$\left. - \frac{\Gamma(\alpha+1)}{(b-a)^s}\left(\frac{f(b)}{e^{\alpha b}}\,{}^sJ_{b^-}g(b) - m\frac{f\left(\frac{a}{m}\right)}{e^{\alpha\left(\frac{a}{m}\right)}}\,{}^sJ_{a^+}g(b)\right)\right).$$

Multiplying both sides of (9.42) by $K^a_x(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)d(g(x))$, and integrating over $[a,b]$ we have

$$f\left(\frac{a+b}{2}\right)\int_a^b K^a_x(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)d(g(x))$$
$$\leq \left(\frac{1}{2^s}\right)(1+m)\int_a^b K^a_x(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)\frac{f(x)}{e^{\alpha x}}d(g(x)).$$

From Definition 2.3, the following inequality is obtained:

$$h(\alpha)f\left(\frac{a+b}{2}\right)\frac{2^s}{(1+m)}\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}1\right)(a,w;p) \leq \left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(a,w;p). \tag{9.48}$$

Similarly multiplying both sides of (9.42) by $K^x_b(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)d(g(x))$, and integrating over $[a,b]$ we have

$$h(\alpha)f\left(\frac{a+b}{2}\right)\frac{2^s}{(1+m)}\left({}_gF^{\phi,\gamma,\delta,k,c}_{\nu,\alpha,l,a^+}1\right)(b,w;p) \leq \left({}_gF^{\phi,\gamma,\delta,k,c}_{\nu,\alpha,l,a^+}f\right)(b,w;p) \tag{9.49}$$

by adding (9.48) and (9.49) following inequality is obtained:

$$h(\alpha)f\left(\frac{a+b}{2}\right)\frac{2^s}{(1+m)}\left(\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}1\right)(a,w;p) + \left({}_gF^{\phi,\gamma,\delta,k,c}_{\nu,\alpha,l,a^+}1\right)(b,w;p)\right) \tag{9.50}$$
$$\leq \left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(a,w;p) + \left({}_gF^{\phi,\gamma,\delta,k,c}_{\nu,\alpha,l,a^+}f\right)(b,w;p).$$

Using (9.47) and (9.50), inequality (9.43) can be achieved. □

Remark 9.2

(i) If $(s,m)=(1,1)$ and $\sigma=0$ in (9.43), then we get [98, Theorem 2].

(ii) If $(s,m)=(1,1)$ and $p=w=0$ in (9.43), then we get [155, Theorem 4].

(iii) If $p=w=\sigma=0$ in (9.43), then we get [96, Theorem 3].

(iv) If $(s,m)=(1,1)$ and $p=w=\alpha=0$ in (9.43), then we get [104, Theorem 3].

Corollary 9.15 *If we take $\phi(t)=\frac{\Gamma(\rho)t^{\frac{\rho}{k}}}{k\Gamma_k(\rho)}$ and $p=w=0$, then the inequality (9.43) produces the following Hadamard inequality defined in [97]:*

$$\frac{h(\alpha)2^{s+1}f\left(\frac{a+b}{2}\right)}{\rho\Gamma_k(\rho)(m+1)}(g(b)-g(a))^{\rho/k}\leq\left({}_g^{\rho}J_{b\_}^{k}f(a)+{}_g^{\rho}J_{a+}^{k}f(b)\right) \tag{9.51}$$
$$\leq\frac{2(g(b)-g(a))^{\frac{\rho}{k}-1}}{k\Gamma_k(\rho)}\left(\left(\frac{f(b)}{e^{\alpha b}}g(b)-m\frac{f\left(\frac{a}{m}\right)}{e^{\alpha(\frac{a}{m})}}g(a)\right)\right.$$
$$\left.-\frac{\Gamma(s+1)}{(b-a)^s}\left(\frac{f(b)}{e^{\alpha b}}{}^sJ_{b^-}g(a)-m\frac{f\left(\frac{a}{m}\right)}{e^{\alpha(\frac{a}{m})}}{}^sJ_{a^+}g(b)\right)\right).$$

Corollary 9.16 *If we take $\phi(t)=t^{\rho}$ and $p=w=0$, then the inequality (9.43) produces the following Hadamard inequality defined in [92]:*

$$\frac{h(\alpha)2^{s+1}f\left(\frac{a+b}{2}\right)}{\mu\Gamma(\rho)(m+1)}(g(b)-g(b))^{\rho}\leq\left({}_g^{\rho}J_{b\_}f(a)+{}_g^{\rho}J_{a^+}f(b)\right) \tag{9.52}$$
$$\leq\frac{2(g(b)-g(a))^{\rho-1}}{\Gamma(\rho)}\left(\left(\frac{f(b)}{e^{\alpha b}}g(b)-m\frac{f\left(\frac{a}{m}\right)}{e^{\alpha(\frac{a}{m})}}g(a)\right)\right.$$
$$\left.-\frac{\Gamma(s+1)}{(b-a)^s}\left(\frac{f(b)}{e^{\alpha b}}{}^sJ_{b^-}g(a)-m\frac{f\left(\frac{a}{m}\right)}{e^{\alpha(\frac{a}{m})}}{}^sJ_{a^+}g(b)\right)\right).$$

Corollary 9.17 *If we take $\phi(t)=\frac{\Gamma(\rho)t^{\frac{\rho}{k}}}{k\Gamma_k(\rho)}$, $p=w=0$ and g as identity function, then the inequality (9.43) produces the following Hadamard inequality defined in [108]:*

$$\frac{h(\alpha)2^{s+1}f\left(\frac{a+b}{2}\right)}{\mu\Gamma_k(\rho)(m+1)}(b-a)^{\rho/k}\leq\left({}^{\rho}J_{b\_}^{k}f(a)+{}^{\rho}J_{a+}^{k}f(b)\right) \tag{9.53}$$
$$\leq\frac{2(b-a)^{\frac{\rho}{k}-1}}{k\Gamma_k(\rho)}\left(\left(\frac{f(b)}{e^{\alpha b}}g(b)-m\frac{f\left(\frac{a}{m}\right)}{e^{\alpha(\frac{a}{m})}}g(a)\right)\right.$$
$$\left.-\frac{\Gamma(s+1)}{(b-a)^s}\left(\frac{f(b)}{e^{\alpha b}}{}^sJ_{b^-}g(a)-m\frac{f\left(\frac{a}{m}\right)}{e^{\alpha(\frac{a}{m})}}{}^sJ_{a^+}g(b)\right)\right).$$

Corollary 9.18 *If we take $\phi(t) = t^\rho$, $p = w = 0$ and g as identity function then the inequality (9.43) produces the following Hadamard inequality defined in [92]:*

$$\begin{aligned}
&\frac{h(\alpha)2^{s+1} f\left(\frac{a+b}{2}\right)}{\mu\Gamma(\rho)(m+1)}(g(b)-g(a))^\rho \leq \left({}^\rho J_{b_-} f(a) +^\rho J_{a^+} f(b)\right) \\
&\leq \frac{2(b-a)^{\rho-1}}{\Gamma(\rho)}\left(\left(\frac{f(b)}{e^{\alpha b}} g(b) - m\frac{f\left(\frac{a}{m}\right)}{e^{\alpha\left(\frac{a}{m}\right)}} g(a)\right)\right. \\
&\left.- \frac{\Gamma(s+1)}{(b-a)^s}\left(\frac{f(b)}{e^{\alpha b}}\, {}^s J_{b^-} g(a) - m\frac{f\left(\frac{a}{m}\right)}{e^{\alpha\left(\frac{a}{m}\right)}}\, {}^s J_{a^+} g(b)\right)\right).
\end{aligned} \tag{9.54}$$

Theorem 9.6 *Let $f : I \to \mathbb{R}$ be a differentiable function. If $|f'|$ is exponentially (s,m)-convex with $m \in (0,1]$ and $g : I \to \mathbb{R}$ be a differentiable and strictly increasing function. Also let $\frac{\phi}{x}$ be an increasing function on I then for $a,b \in I$, $a < b$. If $\sigma,\tau,\delta,c \in \mathbb{C}$, $p,\rho \geq 0$, $c \geq 0$ and $0 < q \leq c+\rho$, then for $x \in (a,b)$ we have*

$$\begin{aligned}
&\left|\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau} f * g\right)(x,w;p) + \left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau} f * g\right)\right|(x,w;p) \\
&\leq K_x^a(E^{\delta,c,q,r}_{\rho,\sigma,\tau}, g;\phi)\left(\left(m\frac{|f'\left(\frac{x}{m}\right)|}{e^{\alpha\left(\frac{x}{m}\right)}} g(x) - \frac{|f'(a)|}{e^{\alpha a}} g(a)\right)\right. \\
&\left.- \frac{\Gamma(s+1)}{(x-a)^s}\left(m\frac{|f'\left(\frac{x}{m}\right)|}{e^{\alpha\left(\frac{x}{m}\right)}}\, {}^s J_{x^-} g(a) - \frac{|f'(a)|}{e^{\alpha a}}\, {}^s J_{a^+} g(x)\right)\right) \\
&+ K_b^x(E^{\delta,c,q,r}_{\rho,\sigma,\tau}, g;\phi)\left(\left(\frac{|f'(b)|}{e^{\alpha b}} g(b) - m\frac{|f'\left(\frac{x}{m}\right)|}{e^{\alpha\left(\frac{x}{m}\right)}} g(x)\right)\right. \\
&\left.- \frac{\Gamma(s+1)}{(x-a)^s}\left(\frac{|f'(b)|}{e^{\alpha b}}\, {}^s J_{b^-} g(x) - m\frac{|f'\left(\frac{x}{m}\right)|}{e^{\alpha\left(\frac{x}{m}\right)}}\, {}^s J_{x^+} g(b)\right)\right),
\end{aligned} \tag{9.55}$$

where

$$\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau} f * g\right)(x,w;p) := \int_a^x K_x^t(E^{\delta,c,q,r}_{\rho,\sigma,\tau}, g;\phi) f'(t) d(g(t)),$$

$$\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau} f * g\right)(x,w;p) := \int_x^b K_t^x(E^{\delta,c,q,r}_{\rho,\sigma,\tau}, g;\phi) f'(t) d(g(t)).$$

Proof. Using exponentially (s,m)-convexity of $|f'|$ we have

$$|f'(t)| \leq \left(\frac{x-t}{x-a}\right)^s \frac{|f'(a)|}{e^{\alpha a}} + m\left(\frac{t-a}{x-a}\right)^s \frac{|f'\left(\frac{x}{m}\right)|}{e^{\alpha\left(\frac{x}{m}\right)}}. \tag{9.56}$$

(9.56) can be written as follows:

$$\begin{aligned}
&-\left(\left(\frac{x-t}{x-a}\right)^s \frac{|f'(a)|}{e^{\alpha a}} + m\left(\frac{t-a}{x-a}\right)^s \frac{|f'\left(\frac{x}{m}\right)|}{e^{\alpha\left(\frac{x}{m}\right)}}\right) \leq f'(t) \\
&\leq \left(\frac{x-t}{x-a}\right)^s \frac{|f'(a)|}{e^{\alpha a}} + m\left(\frac{t-a}{x-a}\right)^s \frac{|f'\left(\frac{x}{m}\right)|}{e^{\alpha\left(\frac{x}{m}\right)}}.
\end{aligned} \tag{9.57}$$

Let we consider the second inequality of (9.57)

$$f'(t) \le \left(\frac{x-t}{x-a}\right)^s \frac{|f'(a)|}{e^{\alpha a}} + m\left(\frac{t-a}{x-a}\right)^s \frac{|f'\left(\frac{x}{m}\right)|}{e^{\alpha\left(\frac{x}{m}\right)}}. \tag{9.58}$$

Multiplying (9.6) and (9.58) and integrating over $[a,x]$, we can obtain:

$$\begin{aligned}
&\int_a^x K_x^t(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi)f(t)d(g(t)) \\
&\le \frac{|f'(a)|}{e^{\alpha a}} K_x^a(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi)\int_a^x \left(\frac{x-t}{x-a}\right)^s d(g(t)) + m\frac{|f'\left(\frac{x}{m}\right)|}{e^{\alpha\left(\frac{x}{m}\right)}} \\
&\times K_x^a(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi)\int_a^x \left(\frac{t-a}{x-a}\right)^s d(g(t)).
\end{aligned}$$

By using Definition 2.3 and integrating by parts, the following inequality is obtained:

$$\begin{aligned}
&\left({}_gF_{a^+,\rho,\sigma,\tau}^{\phi,\delta,c,q,r}f\right)(x,w;p) \le K_x^a(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi) \\
&\times \left(\left(m\frac{|f'\left(\frac{x}{m}\right)|}{e^{\alpha\left(\frac{x}{m}\right)}}g(x) - \frac{|f'(a)|}{e^{\alpha a}}g(a)\right)\right. \\
&\left.- \frac{\Gamma(s+1)}{(x-a)^s}\left(m\frac{|f'\left(\frac{x}{m}\right)|}{e^{\alpha\left(\frac{x}{m}\right)}}\,{}^sJ_{x^-}g(a) - \frac{|f'(a)|}{e^{\alpha a}}\,{}^sJ_{a^+}g(x)\right)\right).
\end{aligned} \tag{9.59}$$

Now we consider the left hand side from the inequality (9.57) and adopting the same pattern as we did for the right hand side inequality we have

$$\begin{aligned}
&\left({}_gF_{a^+,\rho,\sigma,\tau}^{\phi,\delta,c,q,r}f\right)(x,w;p) \ge -K_x^a(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi) \\
&\times \left(\left(m\frac{|f'\left(\frac{x}{m}\right)|}{e^{\alpha\left(\frac{x}{m}\right)}}g(x) - \frac{|f'(a)|}{e^{\alpha a}}g(a)\right)\right. \\
&\left.- \frac{\Gamma(s+1)}{(x-a)^s}\left(m\frac{|f'\left(\frac{x}{m}\right)|}{e^{\alpha\left(\frac{x}{m}\right)}}\,{}^sJ_{x^-}g(a) - \frac{|f'(a)|}{e^{\alpha a}}\,{}^sJ_{a^+}g(x)\right)\right)
\end{aligned} \tag{9.60}$$

From (9.59) and (9.60), following inequality is observed:

$$\begin{aligned}
&\left|\left({}_gF_{a^+,\rho,\sigma,\tau}^{\phi,\delta,c,q,r}(f*g)\right)(x,w;p)\right| \le K_x^a(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi) \\
&\times \left(\left(m\frac{|f'\left(\frac{x}{m}\right)|}{e^{\alpha\left(\frac{x}{m}\right)}}g(x) - \frac{|f'(a)|}{e^{\alpha a}}g(a)\right)\right. \\
&\left.- \frac{\Gamma(s+1)}{(x-a)^s}\left(m\frac{|f'\left(\frac{x}{m}\right)|}{e^{\alpha\left(\frac{x}{m}\right)}}\,{}^sJ_{x^-}g(a) - \frac{|f'(a)|}{e^{\alpha a}}\,{}^sJ_{a^+}g(x)\right)\right).
\end{aligned} \tag{9.61}$$

Now using exponentially (s,m)-convexity of $|f'|$, we have

$$|f'(t)| \le \left(\frac{t-x}{b-x}\right)^s \frac{|f'(b)|}{e^{\alpha b}} + m\left(\frac{b-t}{b-x}\right)^s \frac{|f'\left(\frac{x}{m}\right)|}{e^{\alpha\left(\frac{x}{m}\right)}}. \tag{9.62}$$

On the same pattern as we did for (9.6) and (9.56), one can obtain following inequality from (9.11) and (9.62):

$$\begin{aligned}
&\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\nu,\alpha,l,b^-}(f*g)\right)(x,w;p)\right| \leq K^x_b(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi) \\
&\times\left(\left(\frac{|f'(b)|}{e^{\alpha b}}g(b)-m\frac{|f'\left(\frac{x}{m}\right)|}{e^{\alpha\left(\frac{x}{m}\right)}}g(x)\right)\right. \\
&\left.-\frac{\Gamma(s+1)}{(x-a)^s}\left(\frac{|f'(b)|}{e^{\alpha b}}{}^sJ_{b^-}g(x)-m\frac{|f'\left(\frac{x}{m}\right)|}{e^{\alpha\left(\frac{x}{m}\right)}}{}^sJ_{x^+}g(b)\right)\right).
\end{aligned} \tag{9.63}$$

By adding (9.61) and (9.63), inequality (9.55) can be achieved. □

Remark 9.3

(i) If we take $\sigma=0$ in (9.55), then we get [96, Theorem 2].

(ii) If we take $(s,m)=(1,1)$ in (9.55), then we get [155, Theorem 3].

(iii) If we take $(s,m)=(1,1)$ and $\sigma=0$ in (9.55), then we get [98, Theorem 3].

Theorem 9.7 *Under the assumptions of Theorem 1, the following inequality holds for m-convex functions:*

$$\begin{aligned}
&\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x,w;p)+\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(x,w;p) \\
&\leq K^a_x(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)(g(x)-g(a))\left(mf\left(\frac{x}{m}\right)+f(a)\right) \\
&+K^x_b(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)(g(b)-g(x))\left(mf\left(\frac{x}{m}\right)+f(b)\right).
\end{aligned} \tag{9.64}$$

Proof. If we put $s=1$ in (9.38), we have

$$\begin{aligned}
&\int_a^x K^t_x(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)f(g(t))d(g(t)) \leq f(a)K^a_x(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi) \\
&\times\int_a^x\left(\frac{x-t}{x-a}\right)d(g(t))+mf\left(\frac{x}{m}\right)K^a_x(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)\int_a^x\left(\frac{t-a}{x-a}\right)d(g(t)).
\end{aligned} \tag{9.65}$$

Further simplification of (9.65), the following inequality holds:

$$\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x,w;p)\leq K^a_x(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)(g(x)-g(a))\left(mf\left(\frac{x}{m}\right)+f(a)\right). \tag{9.66}$$

Similarly from (9.41), the following inequality holds:

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\nu,\alpha,l,b^-}f\right)(x,w;p)\leq K^x_b(E^{\gamma,\delta,k,c}_{\nu,\alpha,l},g;\phi)(g(b)-g(x))\left(mf\left(\frac{x}{m}\right)+f(b)\right). \tag{9.67}$$

From (9.66) and (9.67), (9.64) can be obtained. □

Corollary 9.19 *If we take $m = 1$ in Theorem 4, then the following inequality holds for convex functions:*

$$\begin{aligned}
&\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x,w;p) + \left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(x,w;p) \\
&\leq K^a_x(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)(g(x)-g(a))\,(f(x)+f(a)) \\
&+ K^x_b(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)(g(b)-g(x))\,(f(x)+f(b))\,.
\end{aligned} \tag{9.68}$$

Theorem 9.8 *With assumptions of Theorem 4, if $f \in L_\infty[a,b]$, then unified integral operators for m-convex functions are bounded and continuous.*

Proof. From (9.66) we have

$$\left|\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x,w;p)\right| \leq K^a_b(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)(g(b)-g(a))(m+1)\|f\|_\infty,$$

which further gives

$$\left|\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x,w;p)\right| \leq K\|f\|_\infty,$$

where $K = (g(b)-g(a))(m+1)K^a_b(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)$.
Similarly, from (9.67) the following inequality holds:

$$\left|\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(x,w;p)\right| \leq K\|f\|_\infty.$$

□

Corollary 9.20 *If we take $m = 1$ in Theorem 5, then unified integral operators for convex functions are bounded and continuous and following inequalities hold:*

$$\left|\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x,w;p)\right| \leq K\|f\|_\infty,$$

where $K = 2(g(b)-g(a))K^a_b(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)$ and

$$\left|\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(x,w;p)\right| \leq K\|f\|_\infty.$$

9.3 Bounds of Unified Integral Operators for Strongly (s,m)-Convex Functions

In this section the notation $I(a,b,g) =: \frac{1}{b-a}\int_a^b g(t)dt$ is used frequently.

Theorem 9.9 *Let $f : [a,mb] \to \mathbb{R}$, $0 \le a < mb$ be a positive integrable and strongly (s,m)-convex function, $m \neq 0$. Then for unified integral operators, the following inequality holds:*

$$\begin{aligned}
&\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x,w;p) + \left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x,w;p) \\
&\le K_x^a(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)\left(mf\left(\frac{x}{m}\right)g(x) - f(a)g(a)\right. \\
&-\frac{\Gamma(s+1)}{(x-a)^s}\left(mf\left(\frac{x}{m}\right){}^sJ_{x^-}g(a) - f(a){}^sJ_{a^+}g(x)\right) \\
&\left.+\frac{\lambda(x-ma)^2}{(x-a)}(2I(a,x,I_dg) - (a+x)I(a,x,g))\right) \\
&+K_b^x(E^{\delta,c,q,r}_{\rho,\sigma',\tau},g;\phi)\left(f(b)g(b) - mf\left(\frac{x}{m}\right)g(x)\right. \\
&-\frac{\Gamma(s+1)}{(b-x)^s}\left(f(b){}^sJ_{b^-}g(x) - mf\left(\frac{x}{m}\right){}^sJ_{x^+}g(b)\right) \\
&\left.+\frac{\lambda(mb-x)^2}{(b-x)}(2I(x,b,I_dg) - (x+b)I(x,b,g))\right).
\end{aligned} \tag{9.69}$$

Proof. For strongly (s,m)-convex function the following inequalities hold for $a < t < x$ and $x < t < b$ receptively:

$$f(t) \le \left(\frac{x-t}{x-a}\right)^s f(a) + m\left(\frac{t-a}{x-a}\right)^s f\left(\frac{x}{m}\right) - \frac{\lambda(x-t)(t-a)(x-ma)^2}{m^2(x-a)^2}, \tag{9.70}$$

$$f(t) \le \left(\frac{t-x}{b-x}\right)^s f(b) + m\left(\frac{b-t}{b-x}\right)^s f\left(\frac{x}{m}\right) - \frac{\lambda(t-x)(b-t)(mb-x)^2}{m^2(b-x)^2}. \tag{9.71}$$

From (9.6) and (9.70), one can have

$$\begin{aligned}
&\int_a^x K_x^t(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)f(t)d(g(t)) \le f(a)K_x^a(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi) \\
&\times\int_a^x\left(\frac{x-t}{x-a}\right)^s d(g(t)) + mf\left(\frac{x}{m}\right)K_x^a(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)\int_a^x\left(\frac{t-a}{x-a}\right)^s d(g(t)) \\
&-K_x^a(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)\frac{\lambda(x-ma)^2}{(x-a)^2}\int_a^x(x-t)(t-a)d(g(t)),
\end{aligned} \tag{9.72}$$

i.e,

$$\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x,w;p) \le K^a_x(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)\left(mf\left(\frac{x}{m}\right)g(x) - f(a)g(a)\right. \tag{9.73}$$
$$-\frac{\Gamma(s+1)}{(x-a)^s}\left(mf\left(\frac{x}{m}\right){}^sJ_{x^-}g(a) - f(a){}^sJ_{a^+}g(x)\right)$$
$$\left.+\frac{\lambda(x-ma)^2}{(x-a)}(2I(a,x,I_dg) - (a+x)I(a,x,g))\right).$$

On the other hand from (9.11) and (9.71), one can have

$$\int_x^b K^x_t(E^{\delta,c,q,r}_{\rho,\sigma',\tau},g;\phi)f(t)d(g(t)) \le f(b)K^x_b(E^{\delta,c,q,r}_{\rho,\sigma',\tau},g;\phi) \tag{9.74}$$
$$\times\int_x^b\left(\frac{t-x}{b-x}\right)^s d(g(t)) + mf\left(\frac{x}{m}\right)K^b_x(E^{\delta,c,q,r}_{\rho,\sigma',\tau},g;\phi)\int_x^b\left(\frac{b-t}{b-x}\right)^s d(g(t))$$
$$-K^b_x(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)\frac{\lambda(mb-x)^2}{m^2(b-x)^2}\int_x^b(t-x)(b-t)d(g(t)),$$

i.e,

$$\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}f\right)(x,w;p) \le K^x_b(E^{\delta,c,q,r}_{\rho,\sigma',\tau},g;\phi)\left(f(b)g(b) - mf\left(\frac{x}{m}\right)g(x)\right. \tag{9.75}$$
$$-\frac{\Gamma(s+1)}{(b-x)^s}\left(f(b){}^sJ_{b^-}g(x) - mf\left(\frac{x}{m}\right){}^sJ_{x^+}g(b)\right)$$
$$\left.+\frac{\lambda(mb-x)^2}{(b-x)}(2I(x,b,I_dg) - (x+b)I(x,b,g))\right).$$

By adding (9.73) and (9.75), (9.69) can be obtained. □

Corollary 9.21 *Setting $p = w = 0$ in (9.69) we can obtain the following inequality involving fractional integral operators defined in [49]:*

$$\left(F^{\phi}_{\sigma,a^+}f\right)(x;p) + \left(F^{\phi}_{\sigma',b^-}f\right)(x;p) \tag{9.76}$$
$$\le K_g(a,x;\phi)\left(mf\left(\frac{x}{m}\right)g(x) - f(a)g(a)\right.$$
$$-\frac{\Gamma(s+1)}{(x-a)^s}\left(mf\left(\frac{x}{m}\right){}^sJ_{x^-}g(a) - f(a){}^sJ_{a^+}g(x)\right)$$
$$\left.+\frac{\lambda(x-ma)^2}{(x-a)}(2I(a,x,I_dg) - (a+x)I(a,x,g))\right)$$
$$+K_g(x,b;\phi)\left(f(b)g(b) - mf\left(\frac{x}{m}\right)g(x)\right.$$
$$-\frac{\Gamma(s+1)}{(b-x)^s}\left(f(b){}^sJ_{b^-}g(x) - mf\left(\frac{x}{m}\right){}^sJ_{x^+}g(b)\right)$$
$$\left.+\frac{\lambda(mb-x)^2}{(b-x)}(2I(x,b,I_dg) - (x+b)I(x,b,g))\right).$$

Remark 9.4

(i) If we consider $\lambda = 0$ in (9.69), then [69, Theorem 3.1] can be obtained, for $\lambda > 0$ we get its refinement.

(ii) If we consider $\phi(t) = t^{\sigma}$ and $g(x) = x$ in (9.69), then [55, Theorem 1] can be obtained.

(iii) If we consider $s = m = 1$ in the result of (ii), then [55, Corollary 1] can be obtained.

(iv) If we consider $\sigma = \sigma'$ in the result of (ii), then [55, Corollary 3] can be obtained.

(v) If we consider $f \in L_{\infty}[a,b]$ in the result of (ii), then [55, Corollary 5] can be obtained.

(vi) If we consider $\sigma = \sigma'$ in the result of (v), then [55, Corollary 7] can be obtained.

(vii) If we consider $s = 1$ in the result of (ii), then [55, Corollary 5] can be obtained.

(viii) If we consider (s,m)= (1,1) in (9.69), then [82, Theorem 2] is obtained.

(ix) If we consider $\sigma = \sigma'$, $\lambda = 0$ and (s,m)= (1,1) in (9.69), then [98, Theorem 8] is obtained.

(x) If we consider $\lambda = 0$ and $p = w = 0$ in (9.69), then [96, Theorem 1] is obtained.

(xi) If we consider $\lambda = 0$, $\phi(t) = \Gamma(\sigma)t^{\sigma}$, $p = w = 0$ and (s,m)= (1,1) in (9.69), then [59, Theorem 1] is obtained.

(xii) If we consider $\sigma = \sigma'$ in the result of (xi), then [59, Corollary 1] is obtained.

(xiii) If we consider $\lambda = 0$, $\phi(t) = t^{\sigma}$, $g(x) = x$ and $m = 1$ in (9.69), then [35, Theorem 2.1] is obtained.

(xvi) If we consider $\sigma = \sigma'$ in the result of (xiii), then [35, Corollary 2.1] is obtained.

(xv) If we consider $\lambda = 0$, $\phi(t) = \frac{\Gamma(\alpha)t^{\frac{\sigma}{k}}}{k\Gamma_k(\sigma)}$, (s,m)= (1,1), $g(x) = x$ and $p = w = 0$ in (9.69), then [48, Theorem 1] can be obtained.

(xvi) If we consider $\sigma = \sigma'$ in the result of (xv), then [48, Corollary 1] can be obtained.

(xvii) If we consider $\lambda = 0$, $\phi(t) = \Gamma(\sigma)t^{\sigma}$, $g(x) = x$ and $p = w = 0$ and (s,m)= (1,1) in (9.69), then [47, Theorem 1] is obtained.

(xviii) If we consider $\sigma = \sigma'$ in the result of (xvii), then [47, Corollary 1] can be obtained.

(xviii) If we consider $\sigma = \sigma' = 1$ and $x = a \;\; or \;\; x = b$ in the result of (xvii), then [47, Corollary 2] can be obtained.

(xix) If we consider $\sigma = \sigma' = 1$ and $x = \dfrac{a+b}{2}$ in the result of (xvii), then [47, Corollary 3] can be obtained.

The following lemma is very helpful in the proof of upcoming theorem, see [55].

Lemma 9.3 *Let $f : [a,mb] \to \mathbb{R}$ be strongly strongly (s,m)-convex function, $0 \leq a < mb$. If f is $f(\frac{a+mb-x}{m}) = f(x)$, $m \neq 0$, then the following inequality holds:*

$$f\left(\frac{a+mb}{2}\right) \leq \frac{(1+m)f(x)}{2^s} - \frac{\lambda}{4m}(a+mb-x-mx)^2. \tag{9.77}$$

By using above lemma, the following inequality for strongly (s,m)-convex functions is obtained.

Theorem 9.10 *Under the assumptions of Theorem 9.9, in addition if* $f(x) = f\left(\frac{a+mb-x}{m}\right)$, *then the following inequality holds:*

$$f\left(\frac{a+mb}{2}\right)\frac{2^s}{(1+m)}\left(\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma',\tau}1\right)(b,w;p)\right. \tag{9.78}$$
$$+\frac{\lambda}{4m}\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}(a+mb-x-mx)^2\right)(a,w;p)+\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}1\right)(a,w;p)$$
$$\left.+\frac{\lambda}{4m}\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma',\tau}(a+mb-x-mx)^2\right)(b,w;p)\right)$$
$$\leq\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma',\tau}f\right)(b,w;p)+\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(a,w;p)$$
$$\leq\left(K^a_b(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)+K^a_b(E^{\delta,c,q,r}_{\rho,\sigma',\tau},g;\phi)\right)\left(f(b)g(b)-mf\left(\frac{a}{m}\right)g(a)\right.$$
$$-\frac{\Gamma(s+1)}{(b-a)^s}\left(f(b)\,{}^sJ_{b^-}g(a)-mf\left(\frac{a}{m}\right){}^sJ_{a^+}g(b)\right)$$
$$\left.+\frac{\lambda(mb-a)^2}{(b-a)}\left(2I(a,b,I_dg)-(a+b)I(a,b,g)\right)\right).$$

Proof. For strongly (s,m)-convex function satisfies the following inequalities hold for $a<x<b$:

$$f(x)\leq\left(\frac{x-a}{b-a}\right)^s f(b)+m\left(\frac{b-x}{b-a}\right)^s f\left(\frac{a}{m}\right)-\frac{\lambda(b-x)(x-a)(b-ma)^2}{m^2(b-a)^2}. \tag{9.79}$$

From (9.16) and (9.79), one can have

$$\int_a^b K^a_x(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)f(x)d(g(x))$$
$$\leq mf\left(\frac{a}{m}\right)K^a_b(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)\int_a^b\left(\frac{b-x}{b-a}\right)^s d(g(x))$$
$$+f(b)K^a_b(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)\int_a^b\left(\frac{x-a}{b-a}\right)^s d(g(x))$$
$$-K^b_a(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)\frac{\lambda(b-ma)^2}{m^2(b-a)^2}\int_a^b(x-a)(b-x)d(g(x)).$$

Further, the aforementioned inequality takes the form which involves Riemann-Liouville fractional integrals in the right hand side, thus we have upper bound of the unified left sided integral operator as follows:

$$\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(a,w;p)\leq K^a_b(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)\left(f(b)g(b)-mf\left(\frac{a}{m}\right)g(a)\right. \tag{9.80}$$
$$-\frac{\Gamma(s+1)}{(b-a)^s}\left(f(b)\,{}^sJ_{b^-}g(a)-mf\left(\frac{a}{m}\right){}^sJ_{a^+}g(b)\right)$$
$$\left.+\frac{\lambda(mb-a)^2}{(b-a)}\left(2I(a,b,I_dg)-(a+b)I(a,b,g)\right)\right).$$

On the other hand

$$K_b^x(E_{\rho,\sigma',\tau}^{\delta,c,q,r},g;\phi)g'(x) \leq K_b^a(E_{\rho,\sigma',\tau}^{\delta,c,q,r},g;\phi)g'(x) \;\; a<x<b, \tag{9.81}$$

from (9.81) and (9.79), the following inequality holds which involves Riemann-Liouville fractional integrals on the right hand side, gives estimate of the unified right sided integral operator:

$$\begin{aligned}\left({}_gF_{a^+,\rho,\sigma',\tau}^{\phi,\delta,c,q,r}f\right)(b,w;p) &\leq K_b^a(E_{\rho,\sigma',\tau}^{\delta,c,q,r},g;\phi)\Bigg(f(b)g(b)-mf\left(\frac{a}{m}\right)g(a)\\ &-\frac{\Gamma(s+1)}{(b-a)^s}\left(f(b)\,{}^sJ_{b^-}g(a)-mf\left(\frac{a}{m}\right){}^sJ_{a^+}g(b)\right)\\ &+\frac{\lambda(mb-a)^2}{(b-a)}\left(2I(a,b,I_dg)-(a+b)I(a,b,g)\right)\Bigg).\end{aligned} \tag{9.82}$$

By adding (9.80) and (9.82), following inequality can be obtained:

$$\begin{aligned}&\left({}_gF_{b^-,\rho,\sigma,\tau}^{\phi,\delta,c,q,r}f\right)(a,w;p)+\left({}_gF_{a^+,\rho,\sigma',\tau}^{\phi,\delta,c,q,r}f\right)(b,w;p)\\ &\leq \left(K_b^a(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi)+K_b^a(E_{\rho,\sigma',\tau}^{\delta,c,q,r},g;\phi)\right)\Bigg(f(b)g(b)-mf\left(\frac{a}{m}\right)g(a)\\ &-\frac{\Gamma(s+1)}{(b-a)^s}\left(f(b)\,{}^sJ_{b^-}g(a)-mf\left(\frac{a}{m}\right){}^sJ_{a^+}g(b)\right)\\ &+\frac{\lambda(mb-a)^2}{(b-a)}\left(2I(a,b,I_dg)-(a+b)I(a,b,g)\right)\Bigg).\end{aligned} \tag{9.83}$$

Multiplying both sides of (9.77) by $K_x^a(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi)g'(x)$, and integrating over $[a,b]$ we have

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\int_a^b K_x^a(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi)d(g(x))\\ &\leq \left(\frac{1}{2^s}\right)(1+m)\int_a^b K_b^a(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi)f(x)d(g(x))\\ &-\frac{\lambda}{4m}\int_a^b K_x^a(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi)(a+mb-x-mx)^2d(g(x)).\end{aligned}$$

From which the following inequality is obtained:

$$\begin{aligned}&f\left(\frac{a+mb}{2}\right)\frac{2^s}{(1+m)}\left({}_gF_{b^-,\rho,\sigma,\tau}^{\phi,\delta,c,q,r}1\right)(a,w;p)\leq\left({}_gF_{b^-,\rho,\sigma,\tau}^{\phi,\delta,c,q,r}f\right)(a,w;p)\\ &-\frac{\lambda}{4m}\left({}_gF_{b^-,\rho,\sigma,\tau}^{\phi,\delta,c,q,r}(a+mb-x-mx)^2\right)(a,w;p).\end{aligned} \tag{9.84}$$

Similarly multiplying both sides of (9.77) by $K_b^x(E_{\rho,\sigma',\tau}^{\delta,c,q,r},g;\phi)g'(x)$, and integrating over $[a,b]$ we have

$$\begin{aligned}&f\left(\frac{a+mb}{2}\right)\frac{2^s}{(1+m)}\left({}_gF_{a^+,\rho,\sigma',\tau}^{\phi,\delta,c,q,r}1\right)(b,w;p)\leq\left({}_gF_{a^+,\rho,\sigma',\tau}^{\phi,\delta,c,q,r}f\right)(b,w;p)\\ &-\frac{\lambda}{4m}\left({}_gF_{a^+,\rho,\sigma',\tau}^{\phi,\delta,c,q,r}(a+mb-x-mx)^2\right)(b,w;p).\end{aligned} \tag{9.85}$$

By adding (9.84) and (9.85) the following inequality is obtained:

$$\begin{aligned} & f\left(\frac{a+mb}{2}\right)\frac{2^s}{(1+m)}\left(\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma',\tau}1\right)(b,w;p)\right. \\ & +\frac{\lambda}{4m}\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}(a+mb-x-mx)^2\right)(a,w;p)+\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}1\right)(a,w;p) \\ & \left.+\frac{\lambda}{4m}\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma',\tau}(a+mb-x-mx)^2\right)(b,w;p)\right) \\ & \le\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma',\tau}f\right)(b,w;p)+\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(a,w;p). \end{aligned} \tag{9.86}$$

Using (9.83) and (9.86), inequality (9.78) can be obtained. □

Corollary 9.22 *Setting $p=w=0$ in (9.78) we can obtain the following inequality involving fractional integral operators defined in [49]:*

$$\begin{aligned} & f\left(\frac{a+mb}{2}\right)\frac{2^s}{(1+m)}\left(\left(F^{\phi}_{\beta,a^+}1\right)(b;p)\right. \\ & +\frac{\lambda}{4m}\left(F^{\phi}_{\sigma,b^-}(a+mb-x-mx)^2\right)(a;p)+\left(F^{\phi}_{\alpha,b^-}1\right)(a;p) \\ & \left.+\frac{\lambda}{4m}\left(F^{\phi}_{\sigma',a^+}(a+mb-x-mx)^2\right)(b;p)\right) \\ & \le\left(F^{\phi}_{\beta,a^+}f\right)(b;p)+\left(F^{\phi}_{\alpha,b^-}f\right)(a;p) \\ & \le\left(K_g(a,b;\phi)+K_g(a,b;\phi)\right)\left(f(b)g(b)-mf\left(\frac{a}{m}\right)g(a)\right. \\ & -\frac{\Gamma(s+1)}{(b-a)^s}\left(f(b)\,{}^sJ_{b^-}g(a)-mf\left(\frac{a}{m}\right){}^sJ_{a^+}g(b)\right) \\ & \left.+\frac{\lambda(mb-a)^2}{(b-a)}\left(2I(a,b,I_dg)-(a+b)I(a,b,g)\right)\right). \end{aligned} \tag{9.87}$$

Remark 9.5

(i) If we consider $\phi(t)=t^{\sigma}$ and $g(x)=x$ in (9.78), then [55, Theorem 7] can be obtained.

(ii) If we consider $\lambda=0$ in the result of (i), then [55, Theorem 8] can be obtained.

(iii) If we consider $(s,m)=(1,1)$ in (9.78), then [82, Theorem 3] is obtained.

(iv) If we consider $\lambda=0$ and $(s,m)=(1,1)$ in (9.78), then [98, Theorem 22] is obtained.

(v) If we consider $\lambda=0$, $\phi(t)=\Gamma(\sigma)t^{\sigma+1}$, $p=w=0$ and $(s,m)=(1,1)$ in (9.78), then [59, Theorem 3] is obtained.

(vi) If we consider $\sigma=\sigma'$ in the result of (v), then [59, Corollary 3] is obtained.

(vii) If we consider $\lambda=0$, $\phi(t)=t^{\sigma+1}$, $g(x)=x$ and $m=1$ in (9.78), then [35, Theorem 2.4] is obtained.

(viii) If we consider $\sigma = \sigma'$ in the result of (vii), then [35, Corollary 2.6] is obtained.

(ix) If we consider $\lambda = 0$, $\phi(t) = \Gamma(\alpha)t^{\frac{\alpha}{k}+1}$, (s,m)= (1,1), $g(x) = x$ and $p = w = 0$ in (9.78), then [48, Theorem 3] can be obtained.

(x) If we consider $\sigma = \sigma'$ in the result of (ix), then [48, Corollary 6] can be obtained.

(xi) If we consider $\lambda = 0$, $\phi(t) = \Gamma(\sigma)t^{\sigma+1}$, $p = w = 0$, $(s,m) = 1$ and $g(x) = x$ in (9.78), then [47, Theorem 3] can be obtained.

(xii) If we consider $\sigma = \sigma'$ in the result of (xi), then [47, Corrolary 6] can be obtained.

Theorem 9.11 *Let $f : [a,mb] \to \mathbb{R}$, $0 \le a < mb$ be differential function such that $|f'|$ is strongly (s,m)-convex function, $m \neq 0$. Then for unified integral operators, the following inequality holds:*

$$\begin{aligned}
&\left|\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau} f * g\right)(x,w;p) + \left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma',\tau} f * g\right)(x,w;p)\right| \qquad (9.88)\\
&\le K^a_x(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)\left(\left(m\left|f'\left(\frac{x}{m}\right)\right|g(x) - |f'(a)|g(a)\right.\right.\\
&-\frac{\Gamma(s+1)}{(x-a)^s}\left(m\left|f'\left(\frac{x}{m}\right)\right| {}^sJ_{x^-}g(a) - |f'(a)|\,{}^sJ_{a^+}g(x)\right)\\
&\left.+\frac{\lambda(x-am)^2}{(x-a)}(2I(a,x,I_dg) - (a+x)I(a,x,g))\right)\\
&+K^x_b(E^{\delta,c,q,r}_{\rho,\sigma',\tau},g;\phi)\left(\left(|f'(b)|g(b) - m\left|f'\left(\frac{x}{m}\right)\right|g(x)\right)\right.\\
&-\frac{\Gamma(s+1)}{(b-x)^s}\left(|f'(b)|\,{}^sJ_{b^-}g(x) - m\left|f'\left(\frac{x}{m}\right)\right| {}^sJ_{x^+}g(b)\right)\\
&\left.+\frac{\lambda(mb-x)^2}{(b-x)}(2I(x,b,I_dg) - (x+b)I(x,b,g))\right).
\end{aligned}$$

Proof. For strongly (s,m)-convex function $|f'|$, the following inequalities hold for $a < t < x$ and $x < t < b$ receptively:

$$|f'(t)| \le \left(\frac{x-t}{x-a}\right)^s |f'(a)| + m\left(\frac{t-a}{x-a}\right)^s \left|f'\left(\frac{x}{m}\right)\right| - \frac{\lambda(x-t)(t-a)(x-ma)^2}{m^2(x-a)^2}, \quad (9.89)$$

$$|f'(t)| \le \left(\frac{t-x}{b-x}\right)^s |f'(b)| + m\left(\frac{b-t}{b-x}\right)^s \left|f'\left(\frac{x}{m}\right)\right| - \frac{\lambda(t-x)(b-t)(mb-x)^2}{m^2(b-x)^2}. \quad (9.90)$$

From (9.6) and (9.89), the following inequality is obtained:

$$\begin{aligned}
&\left|\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}(f*g)\right)(x,w;p)\right| \le K^a_x(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)\left(\left(m\left|f'\left(\frac{x}{m}\right)\right|g(x) - |f'(a)|g(a)\right.\right. \qquad (9.91)\\
&-\frac{\Gamma(s+1)}{(x-a)^s}\left(m\left|f'\left(\frac{x}{m}\right)\right| {}^sJ_{x^-}g(a) - |f'(a)|\,{}^sJ_{a^+}g(x)\right)\\
&\left.+\frac{\lambda(x-am)^2}{(x-a)}(2I(a,x,I_dg) - (a+x)I(a,x,g))\right).
\end{aligned}$$

Similarly, from (9.11) and (9.90), the following inequality is obtained:

$$\left|\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma',\tau}(f*g)\right)(x,w;p)\right| \leq K^x_b(E^{\delta,c,q,r}_{\rho,\sigma',\tau},g;\phi)\left(\left(|f'(b)|g(b)-m\left|f'\left(\frac{x}{m}\right)\right|g(x)\right)\right. \tag{9.92}$$
$$-\frac{\Gamma(s+1)}{(b-x)^s}\left(|f'(b)|^sJ_{b^-}g(x)-m\left|f'\left(\frac{x}{m}\right)\right|^sJ_{x^+}g(b)\right)$$
$$\left.+\frac{\lambda(mb-x)^2}{(b-x)}(2I(x,b,I_dg)-(x+b)I(x,b,g))\right).$$

By adding (9.91) and (9.92), inequality (9.88) can be achieved. □

Corollary 9.23 *Setting $p=w=0$ in (9.88) we can obtain the following inequality involving fractional integral operators defined in [49]:*

$$\left|\left(F^{\phi}_{\sigma,a^+}f*g\right)(x,p)+\left(F^{\phi}_{\sigma',b^-}f*g\right)(x,p)\right| \tag{9.93}$$
$$\leq K_g(a,x;\phi)\left(\left(m\left|f'\left(\frac{x}{m}\right)\right|g(x)-|f'(a)|g(a)\right)\right.$$
$$-\frac{\Gamma(s+1)}{(x-a)^s}\left(m\left|f'\left(\frac{x}{m}\right)\right|^sJ_{x^-}g(a)-|f'(a)|^sJ_{a^+}g(x)\right)$$
$$\left.+\frac{\lambda(x-am)^2}{(x-a)}(2I(a,x,I_dg)-(a+x)I(a,x,g))\right)$$
$$+K_b(x,b;\phi)\left(\left(|f'(b)|g(b)-m\left|f'\left(\frac{x}{m}\right)\right|g(x)\right)\right.$$
$$-\frac{\Gamma(s+1)}{(b-x)^s}\left(|f'(b)|^sJ_{b^-}g(x)-m\left|f'\left(\frac{x}{m}\right)\right|^sJ_{x^+}g(b)\right)$$
$$\left.+\frac{\lambda(mb-x)^2}{(b-x)}(2I(x,b,I_dg)-(x+b)I(x,b,g))\right).$$

Remark 9.6

(i) If we consider $\lambda=0$ in (9.88), then [69, Theorem 3.4] can be obtained.

(ii) If we consider $\phi(t)=t^{\sigma}$ and $g(x)=x$ in (9.88), then [55, Theorem 6] can be obtained.

(iii) If we consider $s=m=1$ in the result of (ii), then [55, Corollary 13] can be obtained.

(iv) If we consider $\sigma=\sigma'$ in the result of (ii), then [55, Corollary 11] can be obtained.

(v) If we consider (s,m)= (1,1) in (9.88), then [82, Theorem 3] is obtained.

(vi) If we consider $\lambda=0$ and (s,m)= (1,1) in (9.88), then [98, Theorem 25] is obtained.

(vii) If we consider $\lambda=0$ and $p=w=0$ in (9.88), then [96, Theorem 2] is obtained.

(viii) If we consider $\lambda=0$, $\phi(t)=\Gamma(\sigma)t^{\sigma+1}$, $p=w=0$ and (s,m)= (1,1) in (9.88), then [59, Theorem 2] is obtained.

(ix) If we consider $\sigma=\sigma'$ in the result of (viii), then [59, Corollary 2] is obtained.

(x) If we consider $\lambda = 0$, $\phi(t) = t^{\sigma}$, $g(x) = x$ and $m = 1$ in (9.88), then [35, Theorem 2.3] is obtained.

(xi) If we consider $\sigma = \sigma'$ in the result of (x), then [35, Corollary 2.5] is obtained.

(xii) If we consider $\lambda = 0$, $\phi(t) = \Gamma(\alpha)t^{\frac{\alpha}{k}+1}$, (s,m)= (1,1), $g(x) = x$ and $p = w = 0$ in (9.88), then [48, Theorem 2] can be obtained.

(xiii) If we consider $\sigma = \sigma'$ in the result of (xii), then [48, Corollary 4] can be obtained.

(xiv) If we consider $\sigma = \sigma' = k = 1$ and $x = \dfrac{a+b}{2}$, in the result of (xii), then [48, Corollary 5] can be obtained.

(xv) If we consider $\lambda = 0$, $\phi(t) = \Gamma(\sigma)t^{\sigma+1}$, $g(x) = x$ and $p = w = 0$ and (s,m)= (1,1) in (9.88), then [47, Theorem 2] is obtained.

(xvi) If we consider $\sigma = \sigma'$ in the result of (xv), then [47, Corollary 5] can be obtained.

9.4 Bounds of Unified Integral Operators for (α,m)-Convex Functions

In this section bounds of unified integral operators of (α,m)-convex functions are established.

Theorem 9.12 *Let $f : [a,b] \to \mathbb{R}$ be a positive integrable (α,m)-convex function with $m \in (0,1]$. Let $g : [a,b] \to \mathbb{R}$ be differentiable and strictly increasing function, also let $\frac{\phi}{x}$ be an increasing function on $[a,b]$. If $\alpha,l,\gamma,c \in \mathbb{C}$, $\Re(\alpha),\Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$, $p,\mu,\delta \geq 0$ and $0 < k \leq \delta+\mu$, then for $x \in (a,b)$ we have*

$$\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x,w;p) \leq K^a_x(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)\left(\left(mf\left(\frac{x}{m}\right)g(x) - f(a)g(a)\right)\right. \\ \left. -\frac{\Gamma(\alpha+1)}{(x-a)^{\sigma}}\left(mf\left(\frac{x}{m}\right) - f(a)\right){}^{\alpha}J_{a^+}g(x)\right), \tag{9.94}$$

$$\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(x,w;p) \leq K^x_b(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)\left(\left(f(b)g(b) - mf\left(\frac{x}{m}\right)g(x)\right)\right. \\ \left. \frac{\Gamma(\alpha+1)}{(b-x)^{\sigma}}\left(f(b) - mf\left(\frac{x}{m}\right)\right){}^{\alpha}J_{b^-}g(x)\right) \tag{9.95}$$

and hence

$$\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(x,w;p) + \left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(x,w;p) \\ \leq K^a_x(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)\left(\left(mf\left(\frac{x}{m}\right)g(x) - f(a)g(a)\right)\right. \\ \left. -\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}\left(mf\left(\frac{x}{m}\right) - f(a)\right){}^{\alpha}J_{a^+}g(x)\right) \tag{9.96}$$

$$+K_b^x(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi)\left(\left(f(b)g(b)-mf\left(\frac{x}{m}\right)g(x)\right)\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{(b-x)^\alpha}\left(f(b)-mf\left(\frac{x}{m}\right)\right){}^\alpha J_{b^-}g(x)\right).$$

Proof. Using definition of (α,m)-convexity for f the following inequality is valid:

$$f(t)\leq\left(\frac{x-t}{x-a}\right)^\alpha f(a)+m\left(1-\left(\frac{x-t}{x-a}\right)^\alpha\right)f\left(\frac{x}{m}\right). \tag{9.97}$$

Multiplying (9.6) with (9.97) and integrating over $[a,x]$, one can obtain

$$\int_a^x K_x^t(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi)f(t)d(g(t))\leq f(a)K_x^a(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi)\int_a^x\left(\frac{x-t}{x-a}\right)^\alpha d(g(t))$$
$$+mf\left(\frac{x}{m}\right)K_x^a(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi)\int_a^x\left(1-\left(\frac{x-t}{x-a}\right)^\alpha\right)d(g(t)).$$

By using Definition 2.3 and integrating by parts, the following inequality is obtained:

$$\left({}_gF_{a^+,\rho,\sigma,\tau}^{\phi,\delta,c,q,r}f\right)(x,w;p)\leq K_x^a(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi)\left((x-a)^\sigma\left(mf\left(\frac{x}{m}\right)g(x)-f(a)g(a)\right)\right. \tag{9.98}$$
$$\left.\frac{\Gamma(\alpha+1)}{(x-a)^\alpha}\left(mf\left(\frac{x}{m}\right)-f(a)\right){}^\alpha J_{a^+}g(x)\right).$$

Using (α,m)-convexity of f, we have

$$f(t)\leq\left(\frac{t-x}{b-x}\right)^\alpha f(b)+m\left(1-\left(\frac{t-x}{b-x}\right)^\alpha\right)f\left(\frac{x}{m}\right). \tag{9.99}$$

Adopting the same procedure as we did for (9.6) and (9.97), the following inequality from (9.11) and (9.99) can be obtained:

$$\left({}_gF_{b^-,\rho,\sigma,\tau}^{\phi,\delta,c,q,r}f\right)(x,w;p)\leq K_b^x(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi)\left(\left(f(b)g(b)-mf\left(\frac{x}{m}\right)g(x)\right)\right. \tag{9.100}$$
$$\left.-\frac{\Gamma(\alpha+1)}{(b-x)^\alpha}\left(f(b)-mf\left(\frac{x}{m}\right)\right){}^\alpha J_{b^-}g(x)\right).$$

By adding (9.98) and (9.100), (9.96) can be obtained. □

Remark 9.7

(i) If we consider (α,m) =(1,1) in (9.96), [98, Theorem 1] is obtained.

(ii) If we consider $\phi(t)=\frac{\Gamma(\mu)t^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$ for left hand integral and $\phi(t)=\frac{\Gamma(\nu)t^{\frac{\nu}{k}}}{k\Gamma_k(\nu)}$ for right hand integral and $p=w=0$ in (9.96), then [88, Theorem 1] can be obtained.

(iii) If we consider $\mu=\nu$ in the result of (ii), then [88, corollary 1] can be obtained.

(iv) If we consider $\phi(t)=\Gamma(\mu)t^\mu$, $p=w=0$ and (α,m) =(1,1) in (9.96), [59, Theorem 1] is obtained.

(v) If we consider $\mu=\nu$ in the result of (iv), [59, Corollary 1] is obtained.

(vi) If we consider $\phi(t)=\frac{\Gamma(\mu)t^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$ for left hand integral and $\phi(t)=\frac{\Gamma(\nu)t^{\frac{\nu}{k}}}{k\Gamma_k(\nu)}$ for right hand integral, (α,m) =(1,1), $g(x)=x$ and $p=w=0$, then [48, Theorem 1] can be obtained.

(vii) If we consider $\mu=\nu$ in the result of (vi), then [48, Corollary 1] can be obtained.

(viii) If we consider $\phi(t)=\Gamma(\mu)t^{\mu}$ for left hand integral and $\phi(t)=\Gamma(\nu)t^{\nu}$ for right hand integral, $g(x)=x$ and $p=w=0$ and (α,m) =(1,1) in (9.96), then [47, Theorem 1] is obtained.

(ix) By setting $\mu=\nu$ in the result of (viii), [47, Corollary 1] can be obtained.

(x) By setting $\mu=\nu=1$ and $x=a \ or \ x=b$ in the result of (ix), [47, Corollary 2] can be obtained.

(xi) By setting $\mu=\nu=1$ and $x=\dfrac{a+b}{2}$ in the result of (ix), [47, Corollary 3] can be obtained.

To prove the the next result we need the following lemma [88].

Lemma 9.4 *Let $f:[0,\infty]\to\mathbb{R}$, be an (α,m)-convex function with $m\in(0,1]$. If $f(x)=f(\frac{a+b-x}{m}), 0<a<b$, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{2^{\alpha}}(1+m(2^{\alpha}-1))f(x), \tag{9.101}$$

for all $x\in[a,b]$ and $m\in(0,1]$.

Theorem 9.13 *With the assumptions of Theorem 9.12 in addition if $f(x)=f\left(\dfrac{a+b-x}{m}\right)$, then we have*

$$\begin{aligned}
&\frac{2^{\alpha}f\left(\frac{a+b}{2}\right)}{(1+m(2^{\alpha}-1))}\left(\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}1\right)(a,w;p)+\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}1\right)(b,w;p)\right) \\
&\leq\left({}_gF^{\phi,\delta,c,q,r}_{b^-,\rho,\sigma,\tau}f\right)(a,w;p)+\left({}_gF^{\phi,\delta,c,q,r}_{a^+,\rho,\sigma,\tau}f\right)(b,w;p) \\
&\leq 2K^a_b(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi)\left(\left(f(b)g(b)-mf\left(\frac{a}{m}\right)g(a)\right)\right. \\
&\left.-\frac{\Gamma(\alpha+1)}{(b-a)^{\sigma}}\left(f(b)-mf\left(\frac{a}{m}\right)\right)\,{}^{\alpha}J_{b^-}g(a)\right).
\end{aligned} \tag{9.102}$$

Proof. Using (α,m)-convexity of f for $x\in(a,b)$, we have

$$f(x)\leq\left(\frac{x-a}{b-a}\right)^{\alpha}f(b)+m\left(1-\left(\frac{x-a}{b-a}\right)^{\alpha}\right)f\left(\frac{a}{m}\right). \tag{9.103}$$

Multiplying (9.16) and (9.103) and integrating the resulting inequality over $[a,b]$, one can obtain

$$\begin{aligned}&\int_a^b K_x^a(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi)f(x)d(g(x))\\&\leq mf\left(\frac{a}{m}\right)K_b^a(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi)\int_a^b\left(1-\left(\frac{x-a}{b-a}\right)^{\alpha}\right)d(g(x))\\&+f(b)\frac{\phi(g(b)-g(a))}{g(b)-g(a)}E_{\rho,\sigma,\tau}^{\delta,c,q,r}(w(g(b)-g(a))^{\rho};p)\int_a^b\left(\frac{x-a}{b-a}\right)^{\alpha}d(g(x)).\end{aligned}$$

By using Definition 2.3 and integrating by parts, the following inequality is obtained:

$$\begin{aligned}&\left({}_gF_{a^+,\rho,\sigma,\tau}^{\phi,\delta,c,q,r}f\right)(b,w;p)\leq K_b^a(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi)\left(\left(f(b)g(b)-mf\left(\frac{a}{m}\right)g(a)\right)\right.\\&\left.-\frac{\Gamma(\alpha+1)}{(b-a)^{\sigma}}\left(f(b)-mf\left(\frac{a}{m}\right)\right){}^{\alpha}J_{b^-}g(a)\right).\end{aligned}\tag{9.104}$$

Adopting the same pattern of simplification as we did for (9.16) and (9.103), the following inequality can be observed from (9.103) and (9.19)

$$\begin{aligned}&\left({}_gF_{b^-,\rho,\sigma,\tau}^{\phi,\delta,c,q,r}f\right)(a;p)\leq K_b^a(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi)\left(\left(f(b)g(b)-mf\left(\frac{a}{m}\right)g(a)\right)\right.\\&\left.-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left(f(b)-mf\left(\frac{a}{m}\right)\right){}^{\alpha}J_{b^-}g(a)\right).\end{aligned}\tag{9.105}$$

By adding (9.104) and (9.105), following inequality can be obtained:

$$\begin{aligned}&\left({}_gF_{b^-,\rho,\sigma,\tau}^{\phi,\delta,c,q,r}f\right)(a,w;p)+\left({}_gF_{a^+,\rho,\sigma,\tau}^{\phi,\delta,c,q,r}f\right)(b,w;p)\\&\leq K_b^a(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi)\left(\left(f(b)g(b)-mf\left(\frac{a}{m}\right)g(a)\right)\right.\\&\left.-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left(f(b)-mf\left(\frac{a}{m}\right)\right){}^{\alpha}J_{b^-}g(a)\right).\end{aligned}\tag{9.106}$$

Multiplying both sides of (9.101) by $K_x^a(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi)d(g(x))$, and integrating over $[a,b]$ we have

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\int_a^b K_x^a(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi)d(g(x))\leq\left(\frac{1}{2^{\alpha}}\right)(1+m(2^{\alpha}-1))\\&\times\int_a^b K_x^a(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi)f(x)d(g(x)).\end{aligned}$$

From Definition 2.3, the following inequality is obtained:

$$f\left(\frac{a+b}{2}\right)\frac{2^{\alpha}}{(1+m(2^{\alpha}-1))}\left({}_gF_{b^-,\rho,\sigma,\tau}^{\phi,\delta,c,q,r}1\right)(a;p)\leq\left({}_gF_{b^-,\rho,\sigma,\tau}^{\phi,\delta,c,q,r}f\right)(a,w;p).\tag{9.107}$$

Similarly multiplying both sides of (9.101) by $K_b^x(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi)d(g(x))$, and integrating over $[a,b]$ we have

$$f\left(\frac{a+b}{2}\right)\frac{2^{\alpha}}{(1+m(2^{\alpha}-1))}\left({}_gF_{a^+,\rho,\sigma,\tau}^{\phi,\delta,c,q,r}1\right)(b;p)\leq\left({}_gF_{a^+,\rho,\sigma,\tau}^{\phi,\delta,c,q,r}f\right)(b,w;p). \tag{9.108}$$

By adding (9.107) and (9.108) following inequality is obtained:

$$f\left(\frac{a+b}{2}\right)\frac{2^{\alpha}}{(1+m(2^{\alpha}-1))}\left(\left({}_gF_{b^-,\rho,\sigma,\tau}^{\phi,\delta,c,q,r}1\right)(a,w;p)+\left({}_gF_{a^+,\rho,\sigma,\tau}^{\phi,\delta,c,q,r}1\right)(b,w;p)\right) \tag{9.109}$$
$$\leq\left({}_gF_{b^-,\rho,\sigma,\tau}^{\phi,\delta,c,q,r}f\right)(a,w;p)+\left({}_gF_{a^+,\rho,\sigma,\tau}^{\phi,\delta,c,q,r}f\right)(b,w;p).$$

Using (9.106) and (9.109), inequality (9.102) can be achieved. □

Remark 9.8

(i) If we consider (α,m) =(1,1) in (9.102), [98, Theorem 2] is obtained.

(ii) If we consider $\phi(t)=\Gamma(\mu)t^{\frac{\mu}{k}+1}$ for left hand integral and $\phi(t)=\Gamma(\nu)t^{\frac{\nu}{k}+1}$ and $p=w=0$ in (9.102), then [88, Theorem 3] can be obtained.

(iii) If we consider $\mu=\nu$ in the result of (ii), then [88, corollary 3] can be obtained.

(iv) If we consider $\phi(t)=\Gamma(\mu)t^{\mu+1}$ for left hand integral and $\phi(t)=\Gamma(\nu)t^{\nu+1}$for right hand integral in (9.102), $p=w=0$ and (α,m) =(1,1) in (9.102), [59, Theorem 3] is obtained.

(v) If we consider $\mu=\nu$ in the result of (iv), [59, Corollary 3] is obtained.

(vi) If we consider $\phi(t)=\Gamma(\mu)t^{\frac{\mu}{k}+1}$ for left hand integral and $\phi(t)=\Gamma(\nu)t^{\frac{\nu}{k}+1}$for right hand integral, (α,m) =(1,1), $g(x)=x$ and $p=w=0$ in (9.102), then [48, Theorem 3] can be obtained.

(vii) If we consider $\mu=\nu$ in the result of (vi), then [48, Corollary 6] can be obtained.

(viii) By setting $\phi(t)=\Gamma(\mu)t^{\mu+1}$ for left hand integral and $\phi(t)=\Gamma(\nu)t^{\nu+1}$ for right hand integral $p=w=0$, $(\alpha,m)=1$ and $g(t)=t$ in (9.102), [47, Theorem 3] can be obtained.

(ix) By setting $\mu=\nu$ in the result of (viii), [47, Corrolary 6] can be obtained.

Theorem 9.14 *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function. If $|f'|$ is (α,m)-convex with $m\in(0,1]$ and $g:[a,b]\to\mathbb{R}$ be differentiable and strictly increasing function, also let $\frac{\phi}{x}$ be an increasing function on $[a,b]$. If $\alpha,l,\gamma,c\in\mathbb{C}$, $\Re(\alpha),\Re(l)>0$, $\Re(c)>\Re(\gamma)>0$, $p,\mu,\delta\geq 0$ and $0<k\leq\delta+\mu$, then for $x\in(a,b)$ we have*

$$\left|\left({}_gF_{a^+,\rho,\sigma,\tau}^{\phi,\delta,c,q,r}f\right)(x,w;p)+\left({}_gF_{b^-,\rho,\sigma,\tau}^{\phi,\delta,c,q,r}f\right)\right|(x,w;p) \tag{9.110}$$
$$\leq K_x^a(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi)\left(\left(m\left|f'\left(\frac{x}{m}\right)\right|g(x)-|f'(a)|g(a)\right)\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}\left(m\left|f'\left(\frac{x}{m}\right)\right|-|f'(a)|\right)\,{}^{\alpha}J_{a^+}g(x)\right)$$

$$+K_b^x(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi)\Bigg(\left(|f'(b)|g(b)-m\left|f'\left(\frac{x}{m}\right)\right|g(x)\right)$$
$$-\frac{\Gamma(\alpha+1)}{(b-x)^{\alpha}}\left(|f'(b)|-m\left|f'\left(\frac{x}{m}\right)\right|\right)\,{}^{\alpha}J_{b^-}g(x)\Bigg),$$

Proof. Let $x\in(a,b)$ and $t\in[a,x]$. Then using (α,m)-convexity of $|f'|$ we have

$$|f'(t)|\leq\left(\frac{x-t}{x-a}\right)^{\alpha}|f'(a)|+m\left(1-\left(\frac{x-t}{x-a}\right)^{\alpha}\right)\left|f'\left(\frac{x}{m}\right)\right|. \tag{9.111}$$

The inequality (9.111) can be written as follows:

$$-\left(\left(\frac{x-t}{x-a}\right)^{\alpha}|f'(a)|+m\left(1-\left(\frac{x-t}{x-a}\right)^{\alpha}\right)\left|f'\left(\frac{x}{m}\right)\right|\right)\leq f'(t) \tag{9.112}$$
$$\leq\left(\frac{x-t}{x-a}\right)^{\alpha}|f'(a)|+m\left(1-\left(\frac{x-t}{x-a}\right)^{\alpha}\right)\left|f'\left(\frac{x}{m}\right)\right|.$$

Let we consider the second inequality of (9.112)

$$f'(t)\leq\left(\frac{x-t}{x-a}\right)^{\alpha}|f'(a)|+m\left(1-\left(\frac{x-t}{x-a}\right)^{\alpha}\right)\left|f'\left(\frac{x}{m}\right)\right|. \tag{9.113}$$

Multiplying (9.6) and (9.113) and integrating over $[a,x]$, we can obtain:

$$\int_a^x K_x^t(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi)d(g(t))\leq|f(a)|K_x^a(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi)\int_a^x\left(\frac{x-t}{x-a}\right)^{\alpha}d(g(t))$$
$$+m\left|f\left(\frac{x}{m}\right)\right|K_x^a(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi)\int_a^x\left(1-\left(\frac{x-t}{x-a}\right)^{\alpha}\right)d(g(t)).$$

By using Definition 2.3 and integrating by parts, the following inequality is obtained:

$$\left({}_gF_{a^+,\rho,\sigma,\tau}^{\phi,\delta,c,q,r}f\right)(x,w;p)\leq K_x^a(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi) \tag{9.114}$$
$$\times\left(\left(m\left|f'\left(\frac{x}{m}\right)\right|g(x)-|f'(a)|g(a)\right)\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}\left(m\left|f'\left(\frac{x}{m}\right)\right|-|f'(a)|\right)\,{}^{\alpha}J_{a^+}g(x)\right).$$

If we consider the left hand side from the inequality (9.112) and adopt the same pattern as did for the right hand side inequality, then

$$\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}(f*g)\right)(x,w;p)\geq-K_x^a(E_{\rho,\sigma,\tau}^{\delta,c,q,r},g;\phi) \tag{9.115}$$
$$\times\left(\left(m\left|f'\left(\frac{x}{m}\right)\right|g(x)-|f'(a)|g(a)\right)\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}\left(m\left|f'\left(\frac{x}{m}\right)\right|-|f'(a)|\right)\,{}^{\alpha}J_{a^+}g(x)\right).$$

From (9.114) and (9.115), following inequality is observed:

$$\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}(f*g)\right)(x,w;p)\right| \le K^a_x(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi) \tag{9.116}$$
$$\times\left(\left(m\left|f'\left(\frac{x}{m}\right)\right|g(x)-|f'(a)|g(a)\right)\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{(x-a)^\alpha}\left(m\left|f'\left(\frac{x}{m}\right)\right|-|f'(a)|\right){}^\alpha J_{a^+}g(x)\right).$$

Now using (α,m)-convexity of $|f'|$ on $(x,b]$ for $x\in(a,b)$ we have

$$|f'(t)| \le \left(\frac{t-x}{b-x}\right)^\alpha |f'(b)| + m\left(1-\left(\frac{t-x}{b-x}\right)^\alpha\right)\left|f'\left(\frac{x}{m}\right)\right|. \tag{9.117}$$

On the same procedure as we did for (9.6) and (9.111), one can obtain following inequality from (9.11) and (9.117):

$$\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}(f*g)\right)(x,w;p)\right| \le K^x_b(E^{\delta,c,q,r}_{\rho,\sigma,\tau},g;\phi) \tag{9.118}$$
$$\times\left(\left(|f'(b)|g(b)-m\left|f'\left(\frac{x}{m}\right)\right|g(x)\right)\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{(b-x)^\alpha}\left(|f'(b)|-m\left|f'\left(\frac{x}{m}\right)\right|\right){}^\alpha J_{b^-}g(x)\right).$$

By adding (9.116) and (9.118), inequality (9.110) can be achieved. □

Remark 9.9

(i) If we consider (α,m) =(1,1) in (9.110), then [98, Theorem 3] is obtained.

(ii) If we consider $\phi(t)=\Gamma(\mu)t^{\frac{\mu}{k}+1}$ for left hand integral and $\phi(t)=\Gamma(\nu)t^{\frac{\nu}{k}+1}$ for right hand integral and $p=w=0$ in (9.110), then [88, Theorem 2] can be obtained.

(iii) If we consider $\mu=\nu$ in the result of (ii), then [88, Corollary 2] can be obtained.

(iv) If we consider $\phi(t)=\Gamma(\mu)t^{\mu+1}$ for left hand integral and $\phi(t)=\Gamma(\nu)t^{\nu+1}$ for right hand integral, $p=w=0$ and (α,m) =(1,1) in (9.110), then [59, Theorem 2] is obtained.

(v) If we consider $\mu=\nu$ in the result of (iv), then [59, Corollary 2] is obtained.

(vi) If we consider $\phi(t)=\Gamma(\mu)t^{\frac{\mu}{k}+1}$ for left hand integral and $\phi(t)=\Gamma(\nu)t^{\frac{\nu}{k}+1}$ for right hand integral, (α,m) =(1,1), $g(x)=x$ and $p=w=0$ in (9.110), then [48, Theorem 2] can be obtained.

(vii) If we consider $\mu=\nu$ in the result of (vi), then [48, Corollary 4] can be obtained.

(viii) If we consider $\mu=\nu=k=1$ and $x=\dfrac{a+b}{2}$, in the result of (vii)], then [48, Corollary 5] can be obtained.

(ix) If we consider $\phi(t)=\Gamma(\mu)t^{\mu+1}$ for left hand integral and $\phi(t)=\Gamma(\nu)t^{\nu+1}$ for right hand integral, $g(x)=x$ and $p=w=0$ and (α,m) =(1,1) in (9.110), then [47, Theorem 2] is obtained.

(x) By setting $\mu=\nu$ in the result of (ix), then [47, Corollary 5] can be obtained.

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