MONOGRAPHS IN INEQUALITIES 1

Mond-Pečarić Method in Operator Inequalities

Inequalities for bounded selfadjoint operators on a Hilbert space

Josip Pečarić, Takayuki Furuta, Jadranka Mićić Hot and Yuki Seo

ELEMENT

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Josip Pečarić
Faculty of Textile Technology
University of Zagreb
Zagreb, Croatia

Takayuki Furuta
Department of Mathematical Information Science
Faculty of Science, Tokyo University of Science
Tokyo, Japan

Jadranka Mićić Hot
Electrical Engineering Department
Polytechnic of Zagreb
Zagreb, Croatia

Yuki Seo
Tennoji Branch, Senior Highschool
Osaka Kyoiku University
Osaka, Japan

ELEMENT

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"Do you mean to say that the story has a moral?"
"Certainly," said the Linnet.
"Well, really," said the Water-rat, in a very angry manner, "I think you should have told me that before you *began*. If you had done so, I certainly would not have listened to you; in fact, I should have said 'Pooh,' like the critic. However, I can say it now;" so he *shouted out* "Pooh," *at the top of his voice, gave a whisk* with his tail, and went back into his hole.

"The devoted Friend" by OSCAR WILDE

Who contributed to this book

This book is the results of several years of development of the Mond-Pečarić method and its applications in the theory of matrices and bounded linear operators on a Hilbert space. We select the most important and interesting topics, which have been introduced in many mathematical journals, in books for operator and matrix theory and at several international conferences.

Preface

Jensen's inequality for normalized positive linear maps between the algebras of bounded linear operators on a Hilbert space is one of the most important inequalities in the functional analysis as follows:

$$f(\Phi(A)) \le \Phi(f(A)) \quad \text{for every bounded selfadjoint operator } A, \qquad \text{(i)}$$

where f is an operator convex function and Φ is a normalized positive linear map. As a special case there is Jensen's operator inequality:

$$f(X^*AX) \le X^*f(A)X \quad \text{for every bounded selfadjoint operator } A, \qquad \text{(ii)}$$

where X is an isometry and f is an operator convex function.

In this book the converses of Jensen's inequality for bounded selfadjoint operators were considered.

Mond and Pečarić showed several extensions of the Kantorovich type operator inequalities on normalized positive linear maps and pointed out that the problem of determining the upper estimates of the difference and the ratio in Jensen's inequality is reduced to solving a single variable maximization or minimization problem by using the concavity of a real valued function f. Based on the method, they showed the complementary inequalities to the Hölder-McCarthy inequality and Kantorovich type one, gave the estimation of the difference and ratio of means of operators, and discussed various converses of Jensen's inequality for normalized positive linear maps. In the concave case of f they obtained the dual problem. The principle yields a rich harvest in the field of operator inequalities. We call it the Mond-Pečarić method.

This book consists of eight chapters:

In Chapter 1 a very brief and rapid review of some basic topics in Jensen's inequality for positive linear maps and Kantorovich inequality for several types are given. Some basic ideas and the viewpoints of the Mond-Pečarić method are given.

In Chapter 2 general converses of Jensen's inequality (i) are considered. The Mond-Pečarić method is used to obtain the bounds. Many interesting inequalities are particularly considered.

In Chapter 3 a generalization of a theorem of Li-Mathias for the normalized positive linear maps as an application of Mond-Pečarić method is considered. Lower and upper bounds in converses of Jensen's type inequalities are given. The cases of the sharp inequalities are investigated. The conversions of Jensen's inequality and other inequalities are particularly considered.

In Chapter 4 the previous results and the same methods are applied to obtain the inequalities for the means. Reverse inequalities of power operator means on positive linear maps are studied. Several properties of power operator means under the chaotic order are considered. New bounds in inequalities for power operator means are given.

In chapter 5 the theory of operator means established by Kubo and Ando assocaiated with the operator monotone functions is introduced. Based on complementary inequalities to Jensen's inequalities on positive linear maps, complementary inequalities to Ando's inequalities assocaiated with operator means are studied.

In Chapter 6 the results and the same methods in the chapter 2 are applied to obtain the inequalities for the Hadamard product. Then the reverses inequalities on the Hadamard product of operators and operator means are considered. General inequalities for the Hadamard product of operators are observed.

In chapter 7 a brief survey of several applications of both Furuta inequality and generalized Furuta inequality is given.

In Chapter 8 the claims preserving the operator order and the chaotic order are considered as an application of the Mond-Pečarić method. The overall results on the functions which preserve the operator order and the chaotic order are particularly considered.

Notation

$\mathbb{R}$	the real numbers
$\mathbb{C}$	the complex numbers
λ, μ, ν, etc.	scalars
$\mathbf{F}$	a field (usually $\mathbb{R}$ or $\mathbb{C}$)
H, K, etc.	Hilbert spaces over $\mathbb{C}$
x, y, z, etc.	vectors in H
(x, y)	inner product of the vector x and the vector y
$\|x\|$	norm of the vector x
$(H \to H)$	algebra of all linear operators on Hilbert space H to H with the operator norm
$\mathscr{B}(H)$	semi-algebra of all bounded linear operators on a Hilbert space H to H
$\mathscr{B}_h(H)$	semi-space of all selfadjoint bounded operators from $\mathscr{B}(H)$
A, B, C, etc.	linear operators in $(H \to H)$
$\|A\|$	operator norm of A
1_H	identity operator in $\mathscr{B}(H)$
1_k	identity matrix in $\mathscr{M}_k$
0	zero scalar, vector or operator
$\mathsf{Sp}(A)$	spectrum of an operator A
$A \geq 0$	positive operator, $(Ax, x) \geq 0$ for all $x \in H$

$\mathscr{B}^+(H)$	set of all positive operators in $\mathscr{B}_h(H)$
$A > 0$	strictly positive operator, exists $m \in \mathbb{R}$, $m > 0$ such that $(Ax,x) \geq m(x,x)$ for all $x \in H$
$\mathscr{B}^{++}(H)$	set of all strictly positive operators in $\mathscr{B}_h(H)$
Φ,Ψ,Ω, etc.	linear maps on $\mathscr{B}$ to $\mathscr{K}$
$P[\mathscr{B}(H),\mathscr{B}(K)]$	set of all positive linear maps, $\Phi : \mathscr{B}(H) \to \mathscr{B}(K)$ such that $A \in \mathscr{B}^+(H) \mapsto \Phi(A) \in \mathscr{B}^+(K)$
$P_N[\mathscr{B}(H),\mathscr{B}(K)]$	set of all normalized positive linear maps in $P[\mathscr{B}(H),\mathscr{B}(K)]$ such that $\Phi(1_H) = 1_K$
$\mathscr{M}_n$	algebra of all n-range complex square matrices with matrix norm
$\mathscr{H}_n$	space of all $n \times n$ Hermitian matrices
$\mathscr{H}_n^+$	set of all positive semi-definite matrices from $\mathscr{H}_n$
$\mathscr{H}_n^{++}$	set of all positive definite matrices from $\mathscr{H}_n$
σ,τ, etc.	operator means
σ_p,τ_q, etc.	weighted operator means
σ^0	transpose of mean σ, if $A\ \sigma^0\ B = B\ \sigma\ A$
σ^*	adjoint to mean σ, if $A\ \sigma^*\ B = \left(A^{-1}\ \sigma\ B^{-1}\right)^{-1}$ for every invertible A,B.
∇	arithmetic means
$!$	harmonic means
$\sharp$	geometric means
$\circ$	Hadamard product
$\otimes$	Kronecker product or tensor product
$\gg$	chaotic order, $A \gg B$ if $\log A \geq \log B$ for all $A,B > 0$

Contents

Chapter 1

Fundamental inequalities and Mond-Pečarić method

In this chapter, we have given a very brief and rapid review of some basic topics in Jensen's inequality for positive linear maps and the Kantorovich inequality for several types. We present some basic ideas and the viewpoints of the Mond-Pečarić method for convex functions.

1.1 Classical Jensen's inequality

In this section, we introduce a classical Jensen's inequality associated with a convex function, and naturally extend it to an operator version. First we introduce some notations.

If a complex vector space H having the inner product is complete with respect to the distance $d(x,y) = \|x-y\|$ defined by the norm $\|x\| := (x,x)^{1/2}$, then H is called a Hilbert space. A linear operator A on a Hilbert space H is said to be bounded if

$$\|A\| := \sup\{\|Ax\| : \|x\| \leq 1, x \in H\} < \infty.$$

Then $\|A\|$ is said to be the operator norm of A. The adjoint operator A^* of A is defined by $(Ax,y) = (x,A^*y)$ for $x,y \in H$. Then it follows that $\|A\| = \|A^*\| = \|A^*A\|^{1/2}$. In an algebra of all linear operators $(H \to H)$ on a Hilbert space H with the operator norm, we denote by

$\mathscr{B}(H)$ a semi-algebra of all bounded (i.e., continuous) linear operators on H. The spectrum of an operator A is the set

$$\mathsf{Sp}(A) = \{\lambda \in \mathbb{C} : A - \lambda 1_H \text{ is not invertible in } \mathscr{B}(H)\}.$$

The spectrum $\mathsf{Sp}(A)$ is nonempty and compact. A bounded linear operator A on a Hilbert space H is said to be selfadjoint if $A = A^*$. An operator $A \in \mathscr{B}(H)$ is selfadjoint if and only if $(Ax, x) \in \mathbb{R}$ for every vector $x \in H$. We denote by $\mathscr{B}_h(H)$ a semi-space of all selfadjoint operators in $\mathscr{B}(H)$.

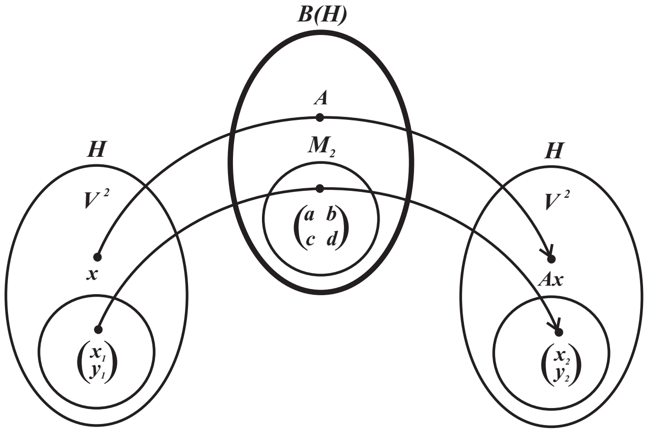

Figure 1.1: Graphic chart of space $\mathscr{B}(H)$

We introduce a partial order in $\mathscr{B}_h(H)$ as follows:

Definition 1.1 *An operator $A \in \mathscr{B}_h(H)$ is* **positive semi-definite** , (*simply,* **positive**) *and we write $A \geq 0$, if $(Ax,x) \geq 0$ for every vector $x \in H$. An operator $A \in \mathscr{B}(H)$ is positive if and only if $A = B^*B$ for some operator $B \in \mathscr{B}(H)$.*

For operators $A, B \in \mathscr{B}_h(H)$ we write $A \leq B$ (or $B \geq A$) if $B - A \geq 0$, i.e., $(Bx,x) \geq (Ax,x)$ for every vector $x \in H$. We call it the **operator order**. *In particular, for some scalars m and M, we write $m1_H \leq A \leq M1_H$ if $m \leq (Ax,x) \leq M$ for every unit vector $x \in H$. Notice that for a selfadjoint operator A, $\mathsf{Sp}(A) \subset [m,M]$ implies $m1_H \leq A \leq M1_H$.*

A positive semi-definite operator $A \in \mathscr{B}_h(H)$ is **positive definite** (**strictly positive**) *and we write $A > 0$, if there is a real number $m > 0$ such that $A \geq m1_H$.*

We denote by $\mathscr{B}^+(H)$ the set of all positive operators and $\mathscr{B}^{++}(H)$ the set of all strictly positive operators (or positive invertible operators) in $\mathscr{B}_h(H)$. The set $\mathscr{B}^+(H)$ is the convex cone contained in $\mathscr{B}_h(H)$.

Now, we review the continuous functional calculus. A rudimentary functional calculus for an operator A can be defined as follows: For a polynomial $p(t) = \sum_{j=0}^{k} \alpha_j t^j$, define

$$p(A) = \alpha_0 1_H + \alpha_1 A + \alpha_2 A^2 + \cdots + \alpha_k A^k.$$

The mapping $p \to p(A)$ is a homomorphism from the algebra of polynomials to the algebra of operators. The extension of this map to larger algebras of functions is really significant in operator theory.

Let A be a selfadjoint operator on a Hilbert space H. Then the Gelfand map establishes a $*$-isometrically isomorphism Φ between the set $C(\mathsf{Sp}(A))$ of all continuous functions on $\mathsf{Sp}(A)$ and C*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows: For $f,g \in C(\mathsf{Sp}(A))$ and $\alpha, \beta \in \mathbb{C}$

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$.

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\overline{f}) = \Phi(f)^*$.

(iii) $\|\Phi(f)\| = \|f\| (:= \sup_{t \in \mathsf{Sp}(A)} |f(t)|)$.

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$.

With this notation, we define

$$f(A) = \Phi(f)$$

for all $f \in C(\mathsf{Sp}(A))$ and we call it the continuous functional calculus for a selfadjoint operator A. This map is an extension of $p(A)$ for a polynomial p. The continuous functional calculus is applicable. For example, if A is a positive operator and $f_{1/2}(t) = \sqrt{t}$, then $A^{1/2} = f_{1/2}(A)$. If A is a selfadjoint operator and $f(t)$ is a real valued continuous function on $\mathsf{Sp}(A)$ such that $f(t) \geq 0$ on $\mathsf{Sp}(A)$, then $f(A) \geq 0$, i.e., $f(A)$ is a positive operator. Moreover, if $g(t)$ is a real valued continuous function on $\mathsf{Sp}(A)$ such that $f(t) \geq g(t)$ on $\mathsf{Sp}(A)$, then $f(A) \geq g(A)$.

Next, we shall introduce a spectral decomposition theorem for selfadjoint, bounded linear operators on a Hilbert space H. For the sake of convenience, we recall the following well known diagonalization of Hermitian matrices in matrix theory.

If A is a Hermitian $k \times k$ matrix, then there exists a unitary matrix U (i.e., $U^*U = UU^* = 1_k$) such that

$$A = U^* \Lambda U, \tag{1.1}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_k)$ and the $\lambda_i (\in \mathbb{R})$ are the eigenvalues of A. If we put

$$\begin{aligned} E_1 &= U^* \mathrm{diag}(1,0,\cdots,0)U, \quad E_2 = U^* \mathrm{diag}(1,1,0,\cdots,0)U \\ &\cdots \\ E_k &= U^* \mathrm{diag}(1,1,\cdots,1)U, \end{aligned}$$

then (1.1) can be rewritten as follows:

$$A = \lambda_1 E_1 + \lambda_2 (E_2 - E_1) + \cdots + \lambda_k (E_k - E_{k-1}) = \sum_{j=1}^{k} \lambda_j \Delta E_j, \tag{1.2}$$

where $\Delta E_j = E_j - E_{j-1}$ and $E_0 = 0$. If $f(t)$ is a real valued continuous function on the spectrum $\mathsf{Sp}(A)$, then $f(A)$ may be defined by

$$f(A) = \sum_{j=1}^{k} f(\lambda_j) \Delta E_j. \tag{1.3}$$

This result can be generalized to selfadjoint operators on a Hilbert space H.

Let A be a selfadjoint operator on a Hilbert space H and $f(t)$ a real valued continuous function defined on an interval $[m,M]$, where $m = \inf_{\|x\|=1}(Ax,x)$ and $M = \max_{\|x\|=1}(Ax,x)$. Then A can be expressed as follows:

$$A = \int_{m-0}^{M} \lambda dE_\lambda \tag{1.4}$$

where $\{E_\lambda : \lambda \in R\}$ is a family of projections such that $E_\lambda \le E_\mu$ if $\lambda \le \mu$, $E_{\lambda+0} = E_\lambda$, $E_{-\infty} = 0$ and $E_\infty = 1_H$. Since a selfadjoint operator A on a Hilbert space H is an extension of a selfadjoint matrix, (1.4) can be naturally considered as an extension of (1.2). Therefore, we have an extension of (1.3) under the above situation as follows:

$$f(A) = \int_{m-0}^{M} f(\lambda) dE_\lambda. \tag{1.5}$$

Next, we shall introduce a classical Jensen's inequality as an inequality associated with a convex function:

Theorem 1.1 (CLASSICAL JENSEN'S INEQUALITY) *If $f(t)$ is a convex function on an interval $[m,M]$ for some scalars $m < M$, then for every $x_1, x_2, \cdots, x_k \in [m,M]$ and every positive real numbers $t_1, t_2, \cdots, t_k$ with $\sum_{j=1}^k t_j = 1$,*

$$f\left(\sum_{j=1}^k t_j x_j\right) \le \sum_{j=1}^k t_j f(x_j). \tag{1.6}$$

Proof. Since $f(t)$ is convex, then for each point $(s, f(s))$ there exists a real number l such that

$$l(x-s) + f(s) \le f(x) \quad \text{for all} \quad x \in [m,M]. \tag{1.7}$$

Put $s_0 = \sum_{j=1}^k t_j x_j \in [m,M]$, then it follows from (1.7) that

$$l(x_j - s_0) + f(s_0) \le f(x_j) \quad \text{for} \quad j = 1, 2, \cdots, k.$$

Multiplying this inequality with $t_j \in \mathbb{R}_+$ and summing of j we have

$$\sum_{j=1}^k t_j (l(x_j - s_0) + f(s_0)) \le \sum_{j=1}^k t_j f(x_j).$$

Since

$$\sum_{j=1}^k t_j (l(x_j - s_0) + f(s_0)) = l\left(\sum_{j=1}^k t_j x_j - s_0 \sum_{j=1}^k t_j\right) + f(s_0) = f(s_0),$$

we have a desired inequality. □

We rephrase it under matrix situation. If we put

$$A = \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} \sqrt{t_1} \\ \vdots \\ \sqrt{t_n} \end{pmatrix},$$

then a classical Jensen's inequality (1.6) in Theorem 1.1 is expressed as

$$f((Ax,x)) \le (f(A)x,x) \qquad \text{for every unit vector } x.$$

The following theorem is an operator version of Theorem 1.1 (classical Jensen's inequality).

Theorem 1.2 *Let $A \in \mathscr{B}_h(H)$ be a selfadjoint operator with $\mathsf{Sp}(A) \subset [m,M]$ for some scalars $m < M$. If $f(t)$ is a convex function on $[m,M]$, then*

$$f((Ax,x)) \le (f(A)x,x) \tag{1.8}$$

for every unit vector $x \in H$.

Proof. If we put $s = (Ax,x)$, then $m \le s \le M$. For a given $\varepsilon > 0$, there exist a straight line $l(t)$ such that (i) $l(t) \le f(t)$ for all $t \in [m,M]$ and (ii) $l(s) \ge f(s) - \varepsilon$. Then (i) implies $l(A) \le f(A)$. Hence we have

$$(f(A)x,x) \ge (l(A)x,x) = l(s) \ge f(s) - \varepsilon$$

for every unit vector $x \in H$. Since ε is an arbitrary, we have $(f(A)x,x) \ge f((Ax,x))$. □

The following theorem is a multiple operator version of Theorem 1.2:

Theorem 1.3 *Let $A_j \in \mathscr{B}_h(H)$ be selfadjoint operators with $\mathsf{Sp}(A_j) \subset [m,M]$ $(j = 1,2,\cdots,k)$ for some scalars $m < M$. Let $x_1, x_2, \cdots, x_k \in H$ be any finite number of vectors such that $\sum_{j=1}^k \|x_j\|^2 = 1$. If $f(t)$ is a convex function on $[m,M]$, then*

$$f\left(\sum_{j=1}^k (A_j x_j, x_j)\right) \le \sum_{j=1}^k (f(A_j)x_j, x_j). \tag{1.9}$$

Proof. If we put

$$\tilde{A} = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_k \end{pmatrix} \quad \text{and} \quad \tilde{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix},$$

then we have $\mathsf{Sp}(\tilde{A}) \subset [m,M]$, $\|\tilde{x}\| = 1$ and $\sum_{j=1}^k (A_j x_j, x_j) = (\tilde{A}\tilde{x}, \tilde{x})$. It follows from Theorem 1.2 that $f((\tilde{A}\tilde{x},\tilde{x})) \le (f(\tilde{A})\tilde{x},\tilde{x})$ and hence we have (1.9). □

As a special case of Theorem 1.2, we have the following Hölder-McCarthy inequality.

Theorem 1.4 (HÖLDER-MCCARTHY INEQUALITY) *Let $A \in \mathscr{B}_h(H)$ be a positive operator on a Hilbert space H. Then*

(i) $(A^r x,x) \ge (Ax,x)^r$ for all $r > 1$ and every unit vector $x \in H$.

(ii) $(A^r x,x) \le (Ax,x)^r$ for all $0 < r < 1$ and every unit vector $x \in H$.

(iii) If A is invertible, then $(A^r x,x) \ge (Ax,x)^r$ for all $r < 0$ and every unit vector $x \in H$.

Proof. Since the power function $f(t) = t^r$ is convex for $r > 1$ or $r < 0$, and concave for $0 < r < 1$, this theorem follows from Theorem 1.2. □

1.2 Operator convexity

In this section, we consider another operator version of a classical Jensen's inequality (1.6) in Theorem 1.1. We rephrase it under another matrix situation. If we put

$$A = \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} \sqrt{t_1} & 0 & \cdots & 0 \\ \vdots & & & \\ \sqrt{t_n} & 0 & \cdots & 0 \end{pmatrix},$$

then a classic Jensen's inequality is expressed as

$$f(V^*AV) \le V^* f(A) V.$$

The formulation offers a fresh insight into the noncommutative case. Its noncommutative version is considered in various way. We shall start with the following definition.

Definition 1.2 *A real valued continuous function $f(t)$ on an interval I is said to be* **operator convex** *(resp.* **operator concave***) if*

$$f((1-\lambda)A+\lambda B) \le (1-\lambda)f(A)+\lambda f(B) \tag{1.10}$$

(resp.

$$f((1-\lambda)A+\lambda B) \ge (1-\lambda)f(A)+\lambda f(B)) \tag{1.11}$$

for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Also, the condition (1.10) can be replaced by the more special condition

$$f\left(\frac{A+B}{2}\right) \le \frac{f(A)+f(B)}{2}. \tag{1.12}$$

Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function $f(t)$ on an interval I is said to be **operator monotone** *if it is monotone with respect to the operator order, i.e.,*

$$A \le B \quad \textit{with } \mathsf{Sp}(A), \mathsf{Sp}(B) \subset I \quad \textit{implies} \quad f(A) \le f(B).$$

Before we present basic examples of such functions, we prove some lemmas needed later.

Lemma 1.5 *If $A \in \mathscr{B}^+(H)$ is positive, then $X^*AX \ge 0$ for every $X \in \mathscr{B}(H)$.*

Proof. For every vector $x \in H$, $(X^*AXx, x) = (AXx, Xx) \ge 0$. □

Lemma 1.6 *If $A \in \mathscr{B}_h(H)$ is selfadjoint and U is unitary, i.e. $U^*U = UU^* = 1_H$, then $f(U^*AU) = U^*f(A)U$ for every $f \in C(\mathsf{Sp}(A))$.*

Proof. Put $B = U^*AU$, then B is selfadjoint and $\mathsf{Sp}(B) = \mathsf{Sp}(A)$. Since $B^m = U^*A^mU$ for every integer $m \geq 0$, we have $p(B) = U^*p(A)U$ for every polynomial $p(t)$. Since there exist polynomials $\{p_j\}$ such that $\|f - p_j\| \mapsto 0$ as $j \to \infty$ for a given $f \in C(\mathsf{Sp}(A))$, we have

$$\begin{aligned}\|f(U^*AU) \ - \ U^*f(A)U\| &\leq \|f(U^*AU) - p_j(U^*AU)\| \\ &+ \ \|p_j(U^*AU) - U^*p_j(A)U\| + \|U^*p_j(A)U - U^*f(A)U\| \mapsto 0\end{aligned}$$

as $j \to \infty$ and so $f(U^*AU) = U^*f(A)U$. □

Lemma 1.7 *If $A \in \mathscr{B}(H)$ and $f \in C([0, \|A\|^2])$, then $Af(A^*A) = f(AA^*)A$.*

Proof. Since $A(A^*A)^n = (AA^*)^nA$ for every integer $n \geq 0$, we have $Ap(A^*A) = p(AA^*)A$ for every polynomial $p(t)$. Since there exist polynomials $\{p_j\}$ such that $\|f - p_j\| \mapsto 0$ as $j \to \infty$ for a given $f \in C([0, \|A\|^2])$, we obtain $Af(A^*A) = f(AA^*)A$. □

Now, we study basic examples of such functions.

Example 1.1 *The function $f(t) = \alpha + \beta t$ is operator monotone on every interval for all $\alpha \in \mathbb{R}$ and $\beta \geq 0$. It is operator convex for all $\alpha, \beta \in R$.*

Example 1.2 *If f, g are operator monotone, and if α, β are positive real numbers, then $\alpha f + \beta g$ is also operator monotone. If f_n are operator monotone and $f_n(t) \to f(t)$ as $n \to \infty$, then f is also operator monotone.*

Example 1.3 *The function $f(t) = t^2$ on $[0, \infty)$ is not operator monotone though it is monotone increasing. As a matter of fact, if we put*

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

then $A \geq B$ and $A^2 \not\geq B^2$ since

$$A^2 - B^2 = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix} \not\geq 0$$

Example 1.4 *The function $f(t) = t^2$ is operator convex on every interval. To see it, for any selfadjoint operators A and B,*

$$\frac{A^2 + B^2}{2} - \left(\frac{A + B}{2}\right)^2 = \frac{1}{4}(A^2 + B^2 - AB - BA) = \frac{1}{4}(A - B)^2 \geq 0.$$

This shows that the function $f(t) = \alpha t^2 + \beta t + \gamma$ is operator convex for all $\beta, \gamma \in R$, $\alpha \geq 0$.

Example 1.5 *The function $f(t) = t^3$ on $[0,\infty)$ is not operator convex though it is convex on $[0,\infty)$. In fact, if we put*

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

then we have

$$\frac{A^3+B^3}{2} - \left(\frac{A+B}{2}\right)^3 = \frac{1}{4}\begin{pmatrix} 11 & 9 \\ 9 & 7 \end{pmatrix} \ngeq 0.$$

Example 1.6 *The function $f(t) = \frac{1}{t}$ is operator convex on $(0,\infty)$ and $g(t) = -\frac{1}{t}$ is operator monotone on $(0,\infty)$. In fact, for any positive invertible operators A and B*

$$\begin{aligned}
&\frac{A^{-1}+B^{-1}}{2} - \left(\frac{A+B}{2}\right)^{-1} \\
&= \frac{A^{-1}+B^{-1}-4(A(A^{-1}+B^{-1})B)^{-1}}{2} \\
&= \frac{A^{-1}+B^{-1}-4B^{-1}(A^{-1}+B^{-1})^{-1}A^{-1}}{2} \\
&= \frac{(A^{-1}+B^{-1}-2B^{-1})(A^{-1}+B^{-1})^{-1}(2A^{-1}-(A^{-1}+B^{-1}))}{2} \\
&= \frac{(A^{-1}-B^{-1})(A^{-1}+B^{-1})^{-1}(A^{-1}-B^{-1})}{2} \geq 0.
\end{aligned}$$

The last inequality holds by Lemma 1.5.

This fact shows that $f(t) = \frac{1}{t}$ is operator convex.

Next, let $A \geq B \geq 0$. Then $1_H \geq A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. Taking inverse both sides, we have $I_H \leq A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}$ and hence $A^{-1} \leq B^{-1}$. Therefore it follows that $-A^{-1} \geq -B^{-1}$ and hence $g(t) = -\frac{1}{t}$ is operator monotone on $(0,\infty)$.

To relate this, we introduce the following famous Löwner-Heinz inequality established in 1934.

Theorem 1.8 (LÖWNER-HEINZ INEQUALITY) *Let A and B be positive operators on a Hilbert space H. If $A \geq B \geq 0$, then $A^r \geq B^r$ for all $r \in [0,1]$.*

We need some elementary results in operator theory in order to prove it. The spectral radius of an operator A is defined as

$$r(A) = \max\{|\lambda| : \lambda \in \mathsf{Sp}(A)\}.$$

Notice that $r(A) \leq \|A\|$ and $r(A) = \|A\|$ if A is a selfadjoint operator. Moreover, it follows that $r(AB) = r(BA)$ for all $A, B \in B(H)$, since $\mathsf{Sp}(AB)\backslash\{0\} = \mathsf{Sp}(BA)\backslash\{0\}$. Also, if A is positive, then $A \leq 1_H$ if and only if $r(A) \leq 1$. An operator A is a contraction ($\|A\| \leq 1$) if and only if $A^*A \leq 1_H$.

Proof of Theorem 1.8. Let $A \geq B \geq 0$. Suppose that A is invertible. Put

$$\Delta = \{r \in R : A^r \geq B^r\}.$$

Then the set Δ is closed since $r \to A^r, B^r$ are norm continuous and $0 \in \Delta$ obviously. The hypothesis $A \geq B \geq 0$ ensures $1 \in \Delta$. Therefore, to prove $[0,1] \subset \Delta$ is sufficient to show that $r, s \in \Delta$ implies $\frac{r+s}{2} \in \Delta$.

If $r \in \Delta$, then $1_H \geq A^{-\frac{r}{2}} B^r A^{-\frac{r}{2}} = \left(B^{\frac{r}{2}} A^{-\frac{r}{2}}\right)^* \left(B^{\frac{r}{2}} A^{-\frac{r}{2}}\right)$ and hence

$$\left\| B^{\frac{r}{2}} A^{-\frac{r}{2}} \right\| \leq 1.$$

By the same argument, if $s \in \Delta$, then $\left\| B^{\frac{s}{2}} A^{-\frac{s}{2}} \right\| \leq 1$.

So, we have

$$\begin{aligned}
&\left\| A^{\frac{-(r+s)}{4}} B^{\frac{r+s}{2}} A^{\frac{-(r+s)}{4}} \right\| \\
&= r\left(A^{\frac{-(r+s)}{4}} B^{\frac{r+s}{2}} A^{\frac{-(r+s)}{4}}\right) \qquad \text{by} \quad A^{\frac{-(r+s)}{4}} B^{\frac{r+s}{2}} A^{\frac{-(r+s)}{4}} \quad \text{is positive} \\
&= r\left(A^{\frac{r-s}{4}} A^{\frac{-(r+s)}{4}} B^{\frac{r+s}{2}} A^{\frac{-(r+s)}{4}} A^{\frac{s-r}{4}}\right) \qquad \text{by} \quad r(ST) = r(TS) \\
&= r\left(A^{\frac{-s}{2}} B^{\frac{r+s}{2}} A^{\frac{-r}{2}}\right) \\
&\leq \left\| A^{\frac{-s}{2}} B^{\frac{r+s}{2}} A^{\frac{-r}{2}} \right\| \qquad \text{by} \quad r(X) \leq \|X\| \\
&\leq \left\| B^{\frac{r}{2}} A^{-\frac{r}{2}} \right\| \left\| B^{\frac{s}{2}} A^{-\frac{s}{2}} \right\| \leq 1.
\end{aligned}$$

Therefore we have

$$A^{\frac{-(r+s)}{4}} B^{\frac{r+s}{2}} A^{\frac{-(r+s)}{4}} \leq 1_H$$

and hence

$$A^{\frac{r+s}{2}} \geq B^{\frac{r+s}{2}}, \quad \text{i.e.,} \quad \frac{r+s}{2} \in \Delta.$$

This fact shows the theorem under the assumption that A is invertible.

Suppose that A is not invertible. For each $\varepsilon > 0$, $A + \varepsilon 1_H$ is invertible and $A + \varepsilon 1_H \geq B$. Therefore it follows from above argument that

$$(A + \varepsilon 1_H)^r \geq B^r \qquad \text{for all} \quad 0 \leq r \leq 1.$$

By letting $\varepsilon \to 0$, we have the desired inequality $A^r \geq B^r$. $\square$

Now, we go back to Jensen's inequality. We show some characterizations of operator convexity and operator monotonicity based on the ideas due to Hansen-Pedersen. This leads to some conditions equivalent to Jensen's inequality.

Theorem 1.9 (JENSEN'S OPERATOR INEQUALITY) *Let H and K be Hilbert space. Let f be a real valued continuous function on an interval I. Let A and A_j be selfadjoint operators on H with spectra contained in I ($j = 1,2,\cdots,k$). Then the following conditions are mutually equivalent:*

(i) f *is operator convex on* I.

(ii) $f(C^*AC) \le C^*f(A)C$ *for every* $A \in \mathscr{B}_h(H)$ *and isometry* $C \in \mathscr{B}(K,H)$, *i.e.,* $C^*C = 1_K$.

(iii) $f(C^*AC) \le C^*f(A)C$ *for every* $A \in \mathscr{B}_h(H)$ *and isometry* $C \in \mathscr{B}(H)$.

(iv) $f\left(\sum_{j=1}^k C_j^*A_jC_j\right) \le \sum_{j=1}^k C_j^*f(A_j)C_j$ *for every* $A_j \in \mathscr{B}_h(H)$ *and* $C_j \in \mathscr{B}(K,H)$ *with* $\sum_{j=1}^k C_j^*C_j = 1_K$ $(j = 1,\cdots,k)$.

(v) $f\left(\sum_{j=1}^k C_j^*A_jC_j\right) \le \sum_{j=1}^k C_j^*f(A_j)C_j$ *for every* $A_j \in \mathscr{B}_h(H)$ *and* $C_j \in \mathscr{B}(H)$ *with* $\sum_{j=1}^k C_j^*C_j = 1_H$ $(j = 1,\cdots,k)$.

(vi) $f\left(\sum_{j=1}^k P_jA_jP_j\right) \le \sum_{j=1}^k P_jf(A_j)P_j$ *for every* $A_j \in \mathscr{B}_h(H)$ *and projection* $P_j \in \mathscr{B}_h(H)$ *with* $\sum_{j=1}^k P_j = 1_H$ $(j = 1,\cdots,k)$.

Proof. $(i) \Rightarrow (ii)$: Let $X = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathscr{B}_h(H \oplus K)$ for some selfadjoint operator $B \in \mathscr{B}_h(K)$ with $\sigma(B) \subset I$ and

$$U = \begin{pmatrix} C & D \\ 0 & -C^* \end{pmatrix}, V = \begin{pmatrix} C & -D \\ 0 & C^* \end{pmatrix} \in \mathscr{B}(K \oplus H, H \oplus K),$$

where $D = \sqrt{1_H - CC^*}$. Since $C^*D = \sqrt{1_K - C^*C}C^* = 0 \in \mathscr{B}_h(H,K)$ and $DC = C\sqrt{1_K - C^*C} = 0 \in \mathscr{B}_h(K,H)$, it follows that both U and V are unitary operators of $K \oplus H$ onto $H \oplus K$. Then

$$U^*XU = \begin{pmatrix} C^*AC & C^*AD \\ DAC & DAD + CBC^* \end{pmatrix}$$

and

$$V^*XV = \begin{pmatrix} C^*AC & -C^*AD \\ -DAC & DAD + CBC^* \end{pmatrix}.$$

So, we have

$$\begin{pmatrix} C^*AC & 0 \\ 0 & D^*AD + CBC^* \end{pmatrix} = \frac{U^*XU + V^*XV}{2}.$$

Hence, it follows from the operator convexity of f and Lemma 1.6 that

$$\begin{aligned} \begin{pmatrix} f(C^*AC) & 0 \\ 0 & f(D^*AD + CBC^*) \end{pmatrix} &= f\begin{pmatrix} C^*AC & 0 \\ 0 & D^*AD + CBC^* \end{pmatrix} \\ &= f\left(\frac{U^*XU + V^*XV}{2}\right) \\ &\le \frac{f(U^*XU) + f(V^*XV)}{2} = \frac{U^*f(X)U + V^*f(X)V}{2} \\ &= \begin{pmatrix} C^*f(A)C & 0 \\ 0 & D^*f(A)D + Cf(B)C^* \end{pmatrix}. \end{aligned}$$

Thus we have $f(C^*AC) \le C^*f(A)C$ by seeing the (1,1)-components.

$(ii) \Rightarrow (iv)$: Let

$$X = \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_k \end{pmatrix} \in \mathscr{B}_h(H \oplus \cdots \oplus H), \tilde{C} = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_k \end{pmatrix} \in \mathscr{B}(K, H \oplus \cdots \oplus H).$$

Then $\tilde{C}^*\tilde{C} = 1_K$ and hence it follow from (ii) that

$$f(\sum_{j=1}^{k} C_j^* A_j C_j) = f(\tilde{C}^* X \tilde{C}) \le \tilde{C}^* f(X)\tilde{C} = \sum_{j=1}^{k} C_j^* f(A_j) C_j.$$

$(iv) \Rightarrow (vi)$: Obviously.

$(vi) \Rightarrow (i)$: Let A and B be selfadjoint operators with spectrum in I and let $0 \le t \le 1$. Let $X = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, $P = \begin{pmatrix} 1_H & 0 \\ 0 & 0 \end{pmatrix}$ and $U = \begin{pmatrix} \sqrt{1-t} & -\sqrt{t} \\ \sqrt{t} & \sqrt{1-t} \end{pmatrix}$. Then U is a unitary operator on $H \oplus H$. Thus we have

$$\begin{aligned} \begin{pmatrix} f((1-t)A+tB) & 0 \\ 0 & f(tA+(1-t)B) \end{pmatrix} &= f(PU^*XUP + (1_{H\otimes H} - P)U^*XU(1_{H\otimes H} - P)) \\ &\le Pf(U^*XU)P + (1_{H\otimes H} - P)f(U^*XU)(1_{H\otimes H} - P) \\ &= PU^*f(X)UP + (1_{H\otimes H} - P)U^*f(X)U(1_{H\otimes H} - P) \\ &= \begin{pmatrix} (1-t)f(A)+tf(B) & 0 \\ 0 & tf(A)+(1-t)f(B) \end{pmatrix}. \end{aligned}$$

Hence f is operator convex on I by seeing the (1,1)-components.

Therefore, we proved the implications $(i) \Rightarrow (ii) \Rightarrow (iv) \Rightarrow (vi) \Rightarrow (i)$.

To complete the proof, we need the implication $(iii) \Rightarrow (v)$ because it is non-trivial in $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (v) \Rightarrow (vi) \Rightarrow (i)$.

$(iii) \Rightarrow (v)$: We only show the case of $k = 2$, which is essential. Let

$$X = \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & A_2 & \\ 0 & & & \ddots \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} C_1 & 0 & \cdots & 0 \\ C_2 & 0 & \cdots & \\ 0 & 1_H & 0 & \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Then C is isometry in $\mathscr{B}(H \oplus H \oplus \cdots)$, i.e., $C^*C = 1_{H\oplus H\oplus\cdots}$. Hence it follows from (iii) that

$$\begin{aligned} &\begin{pmatrix} f(C_1^*A_1C_1 + C_2^*A_2C_2) & & \\ & f(A_2) & \\ & & \ddots \end{pmatrix} \\ &= f(C^*XC) \le C^*f(X)C \\ &= \begin{pmatrix} C_1^*f(A_1)C_1 + C_2^*f(A_2)C_2 & & \\ & f(A_2) & \\ & & \ddots \end{pmatrix}. \end{aligned}$$

Thus we have $f(C_1^*A_1C_1+C_2^*A_2C_2)\le C_1^*f(A_1)C_1+C_2^*f(A_2)C_2$ by seeing the (1,1)-components. $\square$

By Theorem 1.9, we show the following Hansen-Pedersen type Jensen's inequality.

Theorem 1.10 (HANSEN-PEDERSEN-JENSEN'S INEQUALITY) *Let I be an interval containing 0 and let f be a real valued continuous function defined on I. Let A and A_j be selfadjoint operators on H with spectra contained in I ($j=1,2,\cdots,k$). Then the following conditions are mutually equivalent:*

(i) f *is operator convex on I and* $f(0)\le 0$.

(ii) $f(C^*AC)\le C^*f(A)C$ *for every* $A\in\mathscr{B}_h(H)$ *and contraction* $C\in\mathscr{B}(H)$, *i.e.,* $C^*C\le 1_H$.

(iii) $f(\sum_{j=1}^k C_j^*A_jC_j)\le\sum_{j=1}^k C_j^*f(A_j)C_j$ *for every* $A\in\mathscr{B}_h(H)$ *and* $C_j\in\mathscr{B}(H)$ *with* $\sum_{j=1}^k C_j^*C_j\le 1_H$

(iv) $f(PAP)\le Pf(A)P$ *for every* $A\in\mathscr{B}_h(H)$ *and projection* P.

Proof. $(i)\Rightarrow(ii)$: Suppose that f is operator convex and $f(0)\le 0$. For every contraction C, put $D=\sqrt{1_H-C^*C}$. Since $C^*C+D^*D=1_H$, it follows from (v) of Theorem 1.9 that

$$\begin{aligned} f(C^*AC) &= f(C^*AC+D^*0D)\\ &\le C^*f(A)C+D^*f(0)D=C^*f(A)C \quad \text{by } f(0)\le 0 \end{aligned}$$

and hence we have (ii).

$(ii)\Rightarrow(iii)$: Put X and $\tilde{C}$ as in the proof $(ii)\Rightarrow(iv)$ of Theorem 1.9, then $\tilde{C}^*\tilde{C}\le 1_H$ and hence we have (iii).

$(iii)\Rightarrow(iv)$: obviously.

$(iv)\Rightarrow(i)$: Under the same situation in the proof $(iv)\Rightarrow(i)$ of Theorem 1.9, we have

$$\begin{aligned} &\begin{pmatrix} f((1-t)A+tB) & 0\\ 0 & f(0)\end{pmatrix} = f(PU^*XUP)\\ &\le PU^*f(X)UP=\begin{pmatrix}(1-t)f(A)+tf(B) & 0\\ 0 & 0\end{pmatrix}. \end{aligned}$$

Hence f is operator convex and $f(0)\le 0$. $\square$

Theorem 1.11 *Let $f\in\mathscr{C}([0,\infty)$. If $f(t)\le 0$ for all $t\in[0,\infty)$, then conditions (i)–(vi) in Theorems 1.9 are again equivalent to the following condition*

(vii) $-f$ *is an operator monotone function.*

Proof. Suppose that f is operator convex. Let $A,B \in \mathscr{B}(H)$, $0 \le A \le B$. Then for any $0 < \lambda < 1$ we can write

$$\lambda B = \lambda A + (1-\lambda)\frac{\lambda}{1-\lambda}(B-A).$$

Since f is operator convex, we have

$$f(\lambda B) \le \lambda f(A) + (1-\lambda) f\left(\frac{\lambda}{1-\lambda}(B-A)\right).$$

Since $-f(X)$ is positive for every positive operator X, it follows that $f(\lambda B) \le \lambda f(A)$. Letting λ tend to 1, we have $f(B) \le f(A)$. Hence $-f$ is operator monotone.

Conversely, suppose that $-f$ is operator monotone. Let $C \in \mathscr{B}(H)$ be an isometry. Consider the unitary operator U on $H \oplus H$ given by

$$U = \begin{pmatrix} C & -D \\ 0 & C^* \end{pmatrix}$$

where $D = \sqrt{1_H - CC^*}$. We put

$$X = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in \mathscr{B}_h(H \oplus H)$$

and note that

$$U^*XU = \begin{pmatrix} C^*AC & -C^*AD \\ -DAC & DAD \end{pmatrix}.$$

Choose now a constant $\varepsilon > 0$ and set

$$Y = \begin{pmatrix} C^*AC + \varepsilon 1_H & 0 \\ 0 & 2\lambda 1_H \end{pmatrix},$$

where λ is a positive constant to be fixed later. We observe that

$$\begin{aligned} Y - U^*XU \ &= \begin{pmatrix} \varepsilon 1_H & C^*AD \\ DAC & 2\lambda 1_H - DAD \end{pmatrix} \\ &\ge \begin{pmatrix} \varepsilon 1_H & F \\ F^* & \lambda 1_H \end{pmatrix} \qquad \text{for} \quad \lambda 1_H \ge DAD, \end{aligned}$$

where $F = C^*AD$. Furthermore let $\xi,\, \eta \in H$, then

$$\begin{aligned} &\left(\begin{pmatrix} \varepsilon 1_H & F \\ F^* & \lambda 1_H \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right) \\ &= \varepsilon \|\xi\|^2 + (F\xi, \eta) + (F^*\xi, \eta) + \lambda \|\eta\|^2 \\ &\ge \varepsilon \|\xi\|^2 - 2\|F\|\|\xi\|\|\eta\| + \lambda \|\eta\|^2 \\ &\ge 0 \qquad \text{for} \quad \lambda \ge \frac{\|F\|^2}{\varepsilon}. \end{aligned}$$

For a sufficiently large λ we thus obtain

$$U^*XU \le Y$$

and consequently the operator monotonicity of $-f$ implies

$$U^* f(X)U = f(U^* XU) \geq f(Y)$$

or written as matrices

$$\begin{pmatrix} C^* f(A)C & -C^* f(A)D \\ -Df(A)C^* & Df(A)D \end{pmatrix} \geq \begin{pmatrix} f(C^* AC + \varepsilon 1_H) & 0 \\ 0 & f(2\lambda 1_H) \end{pmatrix}.$$

In particular we have $C^* f(A)C \geq f(C^* AC + \varepsilon 1_H)$. Letting ε tend to 0, we get the conclusion of the theorem. □

Corollary 1.12 *Let f be a real valued continuous function mapping the positive half line $[0,\infty)$ into itself. Then f is operator monotone if and only if f is operator concave.*

Theorem 1.13 *Let $f \in \mathscr{C}([0,r))$ and $r \leq \infty$. Then the following conditions are mutually equivalent.*

(i) f is operator convex and $f(0) \leq 0$.

(ii) The function $t \mapsto \frac{f(t)}{t}$ is operator monotone on $(0,r)$.

Proof. Suppose that f is operator convex. Let $A, B \in \mathscr{B}_h(H)$ be selfadjoint operators with $\mathsf{Sp}(A), \mathsf{Sp}(B) \subset (0,r)$ and $A \leq B$. Then A and B are invertible. If we put $C = B^{-1/2}A^{1/2}$, then $CC^* = B^{-1/2}AB^{-1/2} \leq 1_H$ and $\|C\| \leq 1$. Since $A = C^* BC$, it follows from (ii) in Theorem 1.10 that

$$f(A) = f(C^* BC) \leq C^* f(B)C = A^{1/2}B^{-1/2}f(B)B^{-1/2}A^{1/2}$$

and hence $A^{-1/2}f(A)A^{-1/2} \leq B^{-1/2}f(B)B^{-1/2}$. Therefore we have $A^{-1}f(A) \leq B^{-1}f(B)$ and $f(t)/t$ is operator monotone.

Conversely, suppose that $f(t)/t$ is operator monotone on $(0,r)$. Since $f(t)/t \leq f(\beta)/\beta$ for $0 < t < \beta \leq r$, we have $f(t) \leq (f(\beta)/\beta)t$. Letting $t \mapsto 0$, we have $f(0) \leq 0$. We will show that f satisfies the condition (iv) of Theorem 1.10. Let P be any projection and let A be any positive operator with spectrum in $(0,r)$ and $\mathsf{Sp}((1+\varepsilon)A) \subset (0,r)$ for a sufficiently small $\varepsilon > 0$. Put $P_\varepsilon = P + \varepsilon 1_H$ and $X_\varepsilon = P_\varepsilon^{\frac{1}{2}} A^{\frac{1}{2}}$. Since $P_\varepsilon \leq (1+\varepsilon)1_H$, we have $A^{\frac{1}{2}} P_\varepsilon A^{\frac{1}{2}} \leq (1+\varepsilon)A$. Since

$$\begin{aligned} \left(\frac{f}{t}\right)\left(A^{\frac{1}{2}} P_\varepsilon A^{\frac{1}{2}}\right) &= f\left(A^{\frac{1}{2}} P_\varepsilon A^{\frac{1}{2}}\right)\left(A^{\frac{1}{2}} P_\varepsilon A^{\frac{1}{2}}\right)^{-1} \\ &= f(X_\varepsilon^* X_\varepsilon)(X_\varepsilon^* X_\varepsilon)^{-1} \\ &= X_\varepsilon^{-1} f(X_\varepsilon X_\varepsilon^*) X_\varepsilon X_\varepsilon^{-1} X_\varepsilon^{*-1} \\ &= X_\varepsilon^{-1} f(X_\varepsilon X_\varepsilon^*) X_\varepsilon^{*-1}, \end{aligned}$$

it follows from the operator monotonicity of $f(t)/t$ that

$$\begin{aligned} X_\varepsilon^{-1} f(X_\varepsilon X_\varepsilon^*) X_\varepsilon^{*-1} &= \left(\frac{f}{t}\right)\left(A^{\frac{1}{2}} P_\varepsilon A^{\frac{1}{2}}\right) \\ &\leq \left(\frac{f}{t}\right)((1+\varepsilon)A) \\ &= (1+\varepsilon)^{-1} A^{\frac{1}{2}} f((1+\varepsilon)A) A^{\frac{1}{2}}. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} f(X_\varepsilon X_\varepsilon^*) &\le (1+\varepsilon)^{-1} X_\varepsilon A^{\frac{1}{2}} f((1+\varepsilon)A) A^{\frac{1}{2}} X_\varepsilon^* \\ &= (1+\varepsilon)^{-1} P_\varepsilon f((1+\varepsilon)A) P_\varepsilon. \end{aligned}$$

Let $\varepsilon \to 0$. This gives $X_\varepsilon X_\varepsilon^* \to A^{\frac{1}{2}} P A^{\frac{1}{2}}$ and hence we have $f(APA) \le Pf(A)P$ as desired. □

From the previous theorem we obtain the following corollary.

Corollary 1.14 *Let $f \in \mathscr{C}([0,\infty))$ and $f > 0$. The function f is operator monotone if and only if the function $t/f(t)$ is operator monotone.*

Proof. Suppose that f is operator monotone. Since $-f$ is operator convex, it follows from Theorem 1.13 that $-f(t)/t$ is operator monotone on $(0,\infty)$. Hence $t/f(t) = -(-f(t)/t)^{-1}$ is operator monotone on $(0,\infty)$. By the continuity of f, we have the desired result.

Conversely, suppose that $t/f(t)$ is operator monotone. If we put $g(t) = -t/f(t)$, then $g(t) \ge 0$ and by Theorem 1.11 the operator monotonicity of $-g(t)$ implies the operator convexity of $g(t)$ and $g(0) \le 0$. It follows from Theorem 1.13 that $g(t)/t = -1/f(t)$ is operator monotone on $(0,\infty)$ and this fact is equivalent to the operator monotonicity of $f(t)$. □

In general, for a finite interval I, it seems that there is no conjunction between operator concavity and operator monotonicity. For example, $f(t) = \tan t$ is not operator concave (or convex) on $(-\pi/2, \pi/2)$ while it is operator monotone. However, they coincide if the interval is infinite. The following theorem is a slight extension of Theorem 1.11.

Theorem 1.15 *Let f be a real valued continuous function on an interval $I = [\alpha,\infty)$ and bounded below, i.e., there exists $m \in \mathbb{R}$ such that $m \le f(t)$ for all $t \in I$. Then the following conditions are mutually equivalent:*

(i) f is operator concave on I

(ii) f is operator monotone on I

Proof. $(i) \Rightarrow (ii)$: Let $\alpha 1_H \le A \le B$. For every $0 < t < 1$, we have

$$t(B - \alpha 1_H) + \alpha 1_H = tA + (1-t)\left(\frac{t}{1-t}(B-A) + \alpha 1_H\right)$$

and

$$\alpha 1_H \le t(B - \alpha 1_H) + \alpha 1_H, \quad \frac{t}{1-t}(B-A) + \alpha 1_H.$$

Therefore, it follows from the operator concavity of f that

$$\begin{aligned} f(t(B - \alpha 1_H) + \alpha 1_H) &\ge t f(A) + (1-t) f\left(\frac{t}{1-t}(B-A) + \alpha 1_H\right) \\ &\ge t f(A) + (1-t)m. \end{aligned}$$

Let $t \to 1$ and hence $f(B) \geq f(A)$.

$(ii) \Rightarrow (i)$: For every isometry C, put $D = \sqrt{1_H - CC^*}$ and let

$$X = \begin{pmatrix} A & 0 \\ 0 & \alpha 1_H \end{pmatrix}, \quad \text{and} \quad U = \begin{pmatrix} C & D \\ 0 & -C^* \end{pmatrix}.$$

Then U is a unitary operator. For sufficiently large $M > \alpha$ and small $\varepsilon > 0$, we have

$$U^*XU = \begin{pmatrix} C^*AC & C^*AD \\ DAC & DAD + \alpha CC^* \end{pmatrix} \leq \begin{pmatrix} C^*AC + \varepsilon 1_H & 0 \\ 0 & M1_H \end{pmatrix} \equiv X_{M,\varepsilon}.$$

Therefore it follows from an operator monotonicity of f that

$$\begin{aligned} \begin{pmatrix} C^*f(A)C & C^*f(A)D \\ Df(A)C & Df(A)D + f(\alpha)CC^* \end{pmatrix} &= U^*f(X)U = f(U^*XU) \\ \leq f(X_{M,\varepsilon}) = \begin{pmatrix} f(C^*AC + \varepsilon 1_H) & 0 \\ 0 & f(M1_H) \end{pmatrix} & \end{aligned}$$

and hence $C^*f(A)C \leq f(C^*(A + \varepsilon 1_H)C)$. Letting $\varepsilon \to 0$, we have $C^*f(A)C \leq f(C^*AC)$, namely f is operator concave by Theorem 1.9. □

Corollary 1.16 *The function $f(t) = t^r$ is operator monotone on $[0,\infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0,\infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0,\infty)$ if $0 \leq r \leq 1$.*

Proof. If $0 \leq r \leq 1$, then $f(t) = t^r$ is operator monotone by Theorem 1.8. If r is not in $[0,1]$, then $f(t) = t^r$ is not concave on $(0,\infty)$. Therefore, it cannot be operator monotone by Corollary 1.12. Also, we can show directly that for each $r > 1$, there exist $A \geq B \geq 0$ such that $A^r \not\geq B^r$. In fact, if we put

$$A = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{4} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

then $A \geq B \geq 0$ and

$$A^r - B^r = \begin{pmatrix} (\frac{3}{2})^r - \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & (\frac{3}{4})^r - \frac{1}{2} \end{pmatrix}$$

$$\det(A^r - B^r) = \left(\frac{3}{8}\right)^r \left(3^r - \frac{2^r + 4^r}{2}\right) \not\geq 0.$$

Therefore we have $A^r \not\geq B^r$.

Next, if $1 \leq r \leq 2$, then it follows that $t^r/t = t^{r-1}$ is operator monotone on $(0,\infty)$ and hence t^r is operator convex on $(0,\infty)$ by Theorem 1.13. If $-1 \leq r \leq 0$, then $t^r = 1/t^{-r}$ is operator convex. □

Example 1.7 *The logarithm function $f(t) = \log t$ is operator monotone on $(0,\infty)$. In fact, by Löwner-Heinz inequality, it follows that for positive invertible operators A and B such that $A \ge B > 0$,*

$$\frac{A^r - 1_H}{r} \ge \frac{B^r - 1_H}{r} \quad \text{for} \quad 0 < r < 1.$$

Since $\lim_{r\to+0} \frac{x^r-1}{r} = \log x$, we have $\log A \ge \log B$.

Moreover, the function $f(t) = \log t$ is operator concave on $(0,\infty)$. In fact, since t^r is operator concave for $0 < r < 1$, we have $\left(\frac{A+B}{2}\right)^r \ge \frac{A^r+B^r}{2}$ for $0 < r < 1$ and hence

$$\frac{\left(\frac{A+B}{2}\right)^r - 1_H}{r} \ge \frac{\frac{A^r-1_H}{r} + \frac{B^r-1_H}{r}}{2}.$$

By letting $r \to 0$, it follows that $\log\left(\frac{A+B}{2}\right) \ge \frac{1}{2}(\log A + \log B)$.

Example 1.8 *The exponential function $f(t) = e^t$ is neither operator convex nor operator monotone. In fact, since $f(t)$ is strictly convex, it follows from Corollary 1.12 that $f(t)$ is not operator monotone.*

Example 1.9 *The entropy function $\eta(t) = -t\log t$ is operator concave on $(0,\infty)$. Firstly we recall the following result*

$$\lim_{n\to\infty}(A^{-\frac{1}{n}} - 1_H)n = -\log A \qquad \textit{for all } A > 0.$$

Since t^r is operator concave for $r \in [0,1]$, then for $A > 0$, $B > 0$ and $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$

$$(\alpha A + \beta B)^{1-\frac{1}{n}} \ge \alpha A^{1-\frac{1}{n}} + \beta B^{1-\frac{1}{n}} \quad \textit{for any natural number } n$$

and hence we obtain

$$(\alpha A + \beta B)\left((\alpha A + \beta B)^{-\frac{1}{n}} - 1_H\right)n \ge \alpha A\left(A^{-\frac{1}{n}} - 1_H\right)n + \beta B\left(B^{-\frac{1}{n}} - 1_H\right)n.$$

Letting n tend to ∞, we have

$$-(\alpha A + \beta B)\log(\alpha A + \beta B) \ge (-\alpha A\log A - \beta B\log B)$$

and hence

$$\eta(\alpha A + \beta B) \ge \alpha\eta(A) + \beta\eta(B).$$

Therefore, $\eta(t)$ is operator concave.

1.3 Positive linear maps

In this section, we state fundamental properties of positive linear maps and introduce a non-commutative Davis-Choi type Jensen's inequality which extends matrix version mentioned in § 1.2.

The definition of a normalized positive linear map [24] is as follows:

Definition 1.3 *A map $\Phi : \mathscr{B}(H) \to \mathscr{B}(K)$ is* **linear** *if it is additive and homogeneous, i.e. $\Phi(\lambda X + \mu Y) = \lambda \Phi(X) + \mu \Phi(Y)$ for any $\lambda, \mu \in \mathbb{C}$ and for any $X, Y \in \mathscr{B}(H)$.*

A linear map $\Phi : \mathscr{B}(H) \to \mathscr{B}(K)$ is **positive** *if it preserves the operator order $\geq$, i.e. $A \in \mathscr{B}^+(H)$ implies $\Phi(A) \in \mathscr{B}^+(K)$.*

A linear map $\Phi : \mathscr{B}(H) \to \mathscr{B}(K)$ is **normalized** *if it preserves the identity operator, i.e. $\Phi(1_H) = 1_K$.*

We denote $\mathbf{P}[\mathscr{B}(H), \mathscr{B}(K)]$ as the set of all positive linear maps $\Phi : \mathscr{B}(H) \to \mathscr{B}(K)$ and $\mathbf{P}_N[\mathscr{B}(H), \mathscr{B}(K)]$ as the set of all normalized positive linear maps $\Phi \in \mathbf{P}[\mathscr{B}(H), \mathscr{B}(K)]$.

A positive linear map $\Phi \in \mathbf{P}[\mathscr{B}(H), \mathscr{B}(K)]$ preserves order relation, that is, $A \leq B$ implies $\Phi(A) \leq \Phi(B)$, and preserves adjoint operation, that is, $\Phi(A^*) = \Phi(A)^*$. If $\Phi \in \mathbf{P}[\mathscr{B}(H), \mathscr{B}(K)]$ is normalized, then $\alpha 1_H \leq A \leq \beta 1_H$ implies $\alpha 1_K \leq \Phi(A) \leq \beta 1_K$.

Example 1.10 (I) An affine map $\Phi : \mathscr{B}(H) \to \mathscr{B}(H)$,

$$\Phi(A) = \alpha A + \beta 1_H \qquad \text{for} \quad \alpha, \beta \in \mathbb{R}$$

is not a linear map generally. A map

$$\Phi(A) = \alpha A \qquad \text{for} \quad \alpha \in \mathbb{R}_+$$

is a positive linear map, but is not normalized positive linear map if $\alpha \neq 1$.

(II) Let $P_j \in \mathscr{B}(H)$, $j = 1, \ldots, k$ be contractions with

$$\sum_{j=1}^{k} P_i^* P_i = 1_H.$$

A map $\Phi : \mathscr{B}(H) \to \mathscr{B}(H)$

$$\Phi(A) = \sum_{j=1}^{k} P_j^* A P_j$$

is a normalized positive linear map. In particular, if V is isometry in $B(H)$, i.e., $V^*V = 1_H$, then so is $\Phi(A) = V^*AV$.

(III) We denote by $\mathscr{M}_n$ the set of all $n \times n$ square matrices, $\mathscr{H}_n^+$ the set of all positive semi-definite hermitian matrices in $\mathscr{M}_n$ and $\mathscr{H}_n^{++}$ the set of all positive definite matrices in $\mathscr{M}_n$. Compression map $\Phi : \mathscr{M}_n \to \mathscr{M}_k$, $k < n$,

$$\Phi((a_{ij})_{1\leq i,j\leq n}) = (a_{ij})_{1\leq i,j\leq k}$$

is a normalized positive linear map.

(IV) Let K be a correlation matrix of $X \in \mathscr{H}_n^{++}$, i.e.

$$K = (x_{ij}/(x_{ii}x_{jj})^{1/2}) \qquad \text{for } X = (x_{ij}).$$

The matrix K is positive definite [107, Problem 5, p. 400]. We define a map $\Phi : \mathscr{M}_n \to \mathscr{M}_n$, $\Phi(A) = K \circ A$, where $\circ$ denotes the Hadamard product matrices. Then Φ is a normalized positive linear map.

(V) We denote by $\mathscr{M}_{n,k}$ the space of all $n \times k$ complex matrices. Let $P_i, Q_j \in \mathscr{M}_{n,k}$, $1 \leq i \leq p$, $1 \leq j \leq q$, such that

$$\sum_{i=1}^{p} P_i^* P_i + \sum_{j=1}^{q} Q_j^* Q_j = 1_k.$$

A map $\Phi : \mathscr{M}_n \to \mathscr{M}_k$ defined as

$$\Phi(A) = \sum_{i=1}^{p} P_i^* A P_i + \sum_{j=1}^{q} Q_j^* A^T Q_j$$

is a normalized positive linear map [26]. In fact, the maps in the above two examples (III) and (IV) are special cases of this map (V).

The normalized positive linear map $\Phi : \mathscr{M}_n \to \mathscr{M}_k$ is **decomposable** if exist matrices $P_i, Q_j \in \mathscr{M}_{n,k}$ such that $\Phi(A) = \sum P_i^* A P_i + Q_j^* A^T Q_j$, for all $A \in \mathscr{M}_n$. There arises a natural question: *Must every positive linear map be decomposable?* The answer is negative. Man-Duen Choi [26] made the following example: a map $\Phi : \mathscr{M}_3 \to \mathscr{M}_3$ such that

$$\Phi\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right) = \begin{bmatrix} a_{11} & -a_{12} & -a_{13} \\ -a_{21} & a_{22} & -a_{23} \\ -a_{31} & -a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} a_{33} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{22} \end{bmatrix}$$

is the simplest example of the positive linear map which is not decomposable. More about matrix maps can be seen in [24, 25, 26].

We show Kadison's Schwarz inequalities on a positive linear map.

Theorem 1.17 (KADISON'S SCHWARZ INEQUALITY) *Let Φ be a normalized positive linear map in $\mathbf{P}_N[\mathscr{B}(H), \mathscr{B}(K)]$. Then Φ has the following properties.*

(i) $\Phi(A^2) \geq \Phi(A)^2$ *for all selfadjoint operators* $A \in \mathscr{B}_h(H)$.

(ii) $\Phi(A^{-1}) \geq \Phi(A)^{-1}$ *for all positive invertible operators* $A \in \mathscr{B}^{++}(H)$.

To prove Theorem 1.17, we need the following Lemma:

Lemma 1.18 *If $A \in \mathscr{B}^+(H)$ is a positive operator and $B \in \mathscr{B}_h(H)$ is a selfadjoint operator, then*

$$\begin{pmatrix} A & B \\ B & 1_H \end{pmatrix} \geq 0 \qquad \Longrightarrow \qquad A \geq B^2.$$

Also, if $A, B \in \mathscr{B}^{++}(H)$ are positive invertible operators, then

$$\begin{pmatrix} A & 1_H \\ 1_H & B \end{pmatrix} \geq 0 \qquad \Longrightarrow \qquad B \geq A^{-1}.$$

Proof. Since

$$\left(\begin{pmatrix} A & B \\ B & 1_H \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right) \geq 0 \qquad \text{for all } x, y \in H,$$

we have

$$(Ax,x) + 2\mathrm{Re}(Bx,y) + (y,y) \geq 0,$$

where $\mathrm{Re}z = (z+\overline{z})/2$ is a real part of a complex number z. Replacing y by $-Bx$, since $(B^2x,x) = \mathrm{Re}(Bx,Bx)$, we have

$$(Ax,x) - 2\mathrm{Re}(Bx,Bx) + (Bx,Bx) = (Ax,x) - (B^2x,x) \geq 0$$

and hence $A \geq B^2$.

Also, the latter follows as in the proof above. □

Proof of Theorem 1.17. (i): A selfadjoint operator A can be approximated uniformly by a simple function $A' = \sum_j t_j E_j$ where $\{E_j\}$ is a decomposition of the unit 1_H. Since Φ is normalized, we have $\sum_j \Phi(E_j) = 1_K$. Therefore, by Lemma 1.18 and the continuity of Φ, it suffices to prove the positivity of $\begin{pmatrix} \Phi(A'^2) & \Phi(A') \\ \Phi(A') & \Phi(1_H) \end{pmatrix}$. We have

$$\begin{aligned} \begin{pmatrix} \Phi(A'^2) & \Phi(A') \\ \Phi(A') & \Phi(1_H) \end{pmatrix} &= \sum_j \begin{pmatrix} t_j^2\Phi(E_j) & t_j\Phi(E_j) \\ t_j\Phi(E_j) & \Phi(E_j) \end{pmatrix} \\ &= \sum_j \begin{pmatrix} t_j & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Phi(E_j) & \Phi(E_j) \\ \Phi(E_j) & \Phi(E_j) \end{pmatrix} \begin{pmatrix} t_j & 0 \\ 0 & 1 \end{pmatrix} \geq 0, \end{aligned}$$

because

$$\begin{pmatrix} \Phi(E_j) & \Phi(E_j) \\ \Phi(E_j) & \Phi(E_j) \end{pmatrix} = \begin{pmatrix} \Phi(E_j)^{1/2} & 0 \\ \Phi(E_j)^{1/2} & 0 \end{pmatrix} \begin{pmatrix} \Phi(E_j)^{1/2} & \Phi(E_j)^{1/2} \\ 0 & 0 \end{pmatrix} \geq 0.$$

Hence we have (i).

(ii): By Lemma 1.18 and the continuity of Φ, it suffices to prove the positivity of $\begin{pmatrix} \Phi(A') & \Phi(1_H) \\ \Phi(1_H) & \Phi(A'^{-1}) \end{pmatrix}$. We have

$$\begin{aligned} \begin{pmatrix} \Phi(A') & \Phi(1_H) \\ \Phi(1_H) & \Phi(A'^{-1}) \end{pmatrix} &= \sum_j \begin{pmatrix} t_j\Phi(E_j) & \Phi(E_j) \\ \Phi(E_j) & t_j^{-1}\Phi(E_j) \end{pmatrix} \\ &= \sum_j \begin{pmatrix} t_j^{1/2} & 0 \\ 0 & t_j^{-1/2} \end{pmatrix} \begin{pmatrix} \Phi(E_j) & \Phi(E_j) \\ \Phi(E_j) & \Phi(E_j) \end{pmatrix} \begin{pmatrix} t_j^{1/2} & 0 \\ 0 & t_j^{-1/2} \end{pmatrix} \geq 0 \end{aligned}$$

Hence we have (ii). □

Remark 1.1 *Inequality (i) in Theorem 1.17 implies the following inequality:*

$$\Phi\left(A^{\frac{1}{2}}\right) \leq \Phi(A)^{\frac{1}{2}} \quad \text{for all positive operators } A \in \mathscr{B}^{+}(H). \tag{1.13}$$

In fact, if we replace A by $A^{1/2}$ in (i), then we have $\Phi\left(A^{\frac{1}{2}}\right)^2 \leq \Phi(A)$ and hence by Theorem 1.8 (Löwner-Heinz inequality) we have the desired inequality.

The following theorem unifies Kadison's Schwarz inequalities (Theorem 1.17) into a single form without the presence of normalization.

Theorem 1.19 *Let Φ be a positive linear map. Then for any positive invertible operators A and B*

$$\Phi(B)\Phi(A)^{-1}\Phi(B) \leq \Phi(BA^{-1}B) \tag{1.14}$$

holds.

Proof. Consider the map Ψ defined by

$$\Psi(X) = \Phi(B)^{-\frac{1}{2}}\Phi(B^{\frac{1}{2}}XB^{\frac{1}{2}})\Phi(B)^{-\frac{1}{2}}.$$

Then $\Psi(1_H) = 1_K$ and Ψ is a positive linear map as Φ is so. By Theorem 1.17, we have

$$\begin{aligned}\Phi(B)^{\frac{1}{2}}\Phi(A)^{-1}\Phi(B)^{\frac{1}{2}} &= \Psi\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)^{-1} \\ &\leq \Psi\left(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\right) \\ &= \Phi(B)^{-\frac{1}{2}}\Phi(BA^{-1}B)\Phi(B)^{-\frac{1}{2}}.\end{aligned}$$

□

The following theorem is the Davis-Choi-Jensen's inequality for operator convex functions. We present a proof by means of Kadison's Schwarz inequalities and the integral representation of the operator convex function.

Theorem 1.20 (DAVIS-CHOI-JENSEN'S INEQUALITY) *If Φ is a normalized positive linear map in $\mathbf{P}_N[\mathscr{B}(H),\mathscr{B}(K)]$, and f is an operator convex function on an interval I, then*

$$\Phi(f(A)) \geq f(\Phi(A))$$

for every selfadjoint operator A on H whose spectrum is contained in I.

Proof. It suffices to consider the case $I = (-1,1)$. Then f admits a representation

$$f(\lambda) = a + b\lambda + \int_{-1}^{1} \frac{\lambda^2}{1-\lambda t} dm(t)$$

where $b \geq 0$, a is a real number and m is a positive finite measure. For A with $-1_H \leq A \leq 1_H$,

$$\Phi(f(A)) = a1_K + b\Phi(A) + \int_{-1}^{1} \Phi(A^2(1_H - tA)^{-1}) dm(t)$$

and

$$f(\Phi(A)) = a1_K + b\Phi(A) + \int_{-1}^{1} \Phi(A)^2(1_H - t\Phi(A))^{-1} dm(t).$$

By Theorem 1.17

$$\begin{aligned}
\Phi\left(A^2(1_H - tA)^{-1}\right) &= \Phi\left(-\frac{1}{t^2}1_H - \frac{1}{t}A + \frac{1}{t^2}(1_H - tA)^{-1}\right) \\
&= -\frac{1}{t^2}1_K - \frac{1}{t}\Phi(A) + \frac{1}{t^2}\Phi\left((1_H - tA)^{-1}\right) \\
&\geq -\frac{1}{t^2}1_K - \frac{1}{t}\Phi(A) + \frac{1}{t^2}(1_K - t\Phi(A))^{-1} \\
&= \Phi(A)^2(1_K - t\Phi(A))^{-1}.
\end{aligned}$$

This fact induces $\Phi(f(A)) \geq f(\Phi(A))$. □

We show an alternative proof of Theorem 1.20 by means of the characterizations of the operator convexity in Theorem 1.9.

Theorem 1.21 *Let f be a real valued continuous function defined on an interval I. Then the following conditions are equivalent:*

(i) f is operator convex on I

(ii) $f(\Phi(A)) \leq \Phi(f(A))$ for every normalized positive linear map $\Phi \in \mathbf{P}_N[\mathscr{B}(H), \mathscr{B}(K)]$ and every selfadjoint operator A with spectrum in I.

Proof. $(i) \Rightarrow (ii)$: A selfadjoint operator A can be approximated uniformly by a simple function $A' = \sum_j t_j E_j$ where $\{E_j\}$ is a decomposition of the unit 1_H. Since Φ is normalized, we have $\sum_j \Phi(E_j) = 1_K$. Then applying (iv) of Theorem 1.9 to $C_j = \sqrt{\Phi(E_j)}$, it follows that

$$\begin{aligned}
f(\Phi(A')) &= f\left(\sum_j t_j \Phi(E_j)\right) = f\left(\sum_j C_j t_j C_j\right) \leq \sum_j C_j f(t_j) C_j \\
&= \sum_j f(t_j)\Phi(E_j) = \Phi\left(\sum_j f(t_j)E_j\right) = \Phi(f(A')).
\end{aligned}$$

Therefore, we have (ii) by the continuity of Φ.

$(ii) \Rightarrow (i)$: For every isometry C, putting $\Phi(X) = C^*XC$, then it follows that Φ is a normalized positive linear map. Hence we have (i) by (ii) in Theorem 1.9. □

Remark 1.2 *By Stinespring decomposition theorem, we have another proof of Theorem 1.20. In fact, let A be a selfadjoint operator on a Hilbert space H. Then a C^*-algebra $C^*(A)$ generated by A and 1_H is a commutative C^*-algebra. Then Φ restricted to $C^*(A)$ admits a decomposition $\Phi(X) = V^*\phi(X)V$ for all $X \in C^*(A)$, where ϕ is a representation of $C^*(A) \subset B(H)$, and V is an isometry from H into H. Hence it follows from Theorem 1.9 that*

$$\begin{aligned} f(\Phi(A)) &= f(V^*\phi(A)V) \le V^*f(\phi(A))V \\ &= V^*\phi(f(A))V = \Phi(f(A)). \end{aligned}$$

Corollary 1.22 *Let Φ be a normalized positive linear map in $\mathbf{P}_N[\mathscr{B}(H),\mathscr{B}(K)]$ and $A \in \mathscr{B}^{++}(H)$ a positive invertible operator. Then*

(i) $\Phi(A^r) \le \Phi(A)^r$ *for* $0 \le r \le 1$.

(ii) $\Phi(A)^r \le \Phi(A^r)$ *for* $1 \le r \le 2$.

(iii) $\Phi(A) \le \Phi(A^r)^{\frac{1}{r}}$ *for* $1 \le r < \infty$.

(iv) $\Phi(A^r)^{\frac{1}{r}} \le \Phi(A)$ *for* $\frac{1}{2} \le r \le 1$.

(v) $\Phi(\log A) \le \log \Phi(A)$.

(vi) $\Phi(\eta(A)) \le \eta(\Phi(A))$.

Proof. By Corollary 1.16 and Theorem 1.20, we have (i) and (ii). Since $t^{\frac{1}{r}}$ is operator concave for $1 \le r < \infty$, it follows from Theorem 1.20 that $\Phi(A^{\frac{1}{r}}) \le \Phi(A)^{\frac{1}{r}}$ and replacing A by A^r, we have (iii):

$$\Phi(A) \le \Phi(A^r)^{\frac{1}{r}}.$$

Since $t^{\frac{1}{r}}$ is operator convex for $\frac{1}{2} \le r < 1$, we have (iv) analogously. Finally (v) and (vi) follow from Theorem 1.20 because of the operator concavity of $\log t$ and the entropy function $\eta(t)$. □

An operator convex function plays an essential role in the above result. The following example shows that Theorem 1.21 would be false if we replace an operator convex function by a general convex function: The function $f(t) = t^4$ is convex but not operator convex. It is sufficient to put $\dim X = 3$ and in this case we have the following matrix case: For $\Phi : \mathscr{M}_3 \to \mathscr{M}_2$ the contraction map $\Phi((a_{ij})_{1\le i,j\le 3}) = (a_{ij})_{1\le i,j\le 2}$ and $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ we have $\Phi(A)^4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \not\le \begin{bmatrix} 9 & 5 \\ 5 & 3 \end{bmatrix} = \Phi(A^4)$.

1.4 Kantorovich inequality

In this section, we shall introduce the celebrated Kantorovich inequality which is a start point in our book. The Kantorovich inequality enables us to take another approach to study Jensen's inequalities associated with convex functions.

Theorem 1.23 (KANTOROVICH INEQUALITY) *Let A be a positive operator on a Hilbert space H satisfying $M1_H \geq A \geq m1_H > 0$ for some scalars $m < M$. Then*

$$(Ax,x)(A^{-1}x,x) \leq \frac{(M+m)^2}{4Mm} \tag{1.15}$$

for every unit vector $x \in H$.

To prove it, we need the following lemma:

Lemma 1.24 *Let A be a positive operator on H satisfying $M1_H \geq A \geq m1_H > 0$ for some scalars $m < M$. Then*

$$(M+m)1_H \geq MmA^{-1} + A.$$

Proof. Since $M1_H - A \geq 0$, $\frac{1}{m}1_H - A^{-1} \geq 0$ by the hypothesis and $M1_H - A$ and $\frac{1}{m}1_H - A^{-1}$ commute, it follows that

$$(M1_H - A)\left(\frac{1}{m}1_H - A^{-1}\right) \geq 0$$

from which we find that

$$(M+m)1_H \geq MmA^{-1} + A.$$

□

Proof of Theorem 1.23. By Lemma 1.24, we have

$$(M+m)1_H \geq MmA^{-1} + A$$

and hence

$$M + m \geq Mm(A^{-1}x,x) + (Ax,x)$$

for every unit vector $x \in H$. By using the arithmetic-geometric mean inequality, it follows that

$$M + m \geq Mm(A^{-1}x,x) + (Ax,x) \geq 2\sqrt{Mm(A^{-1}x,x)(Ax,x)}.$$

Square both sides, we obtain the desired inequality

$$(Ax,x)(A^{-1}x,x) \leq \frac{(M+m)^2}{4Mm}.$$

□

The following theorem is a multiple version of the Kantorovich inequality.

Theorem 1.25 *Let A_j be positive operators on H satisfying $M1_H \geq A_j \geq m1_H > 0$ for some scalars $m < M$ ($j = 1, 2, \cdots, k$). Let $x_1, x_2, \cdots, x_k$ be any finite number of vectors in H such that $\sum_{j=1}^{k} \|x_j\|^2 = 1$. Then*

$$\left(\sum_{j=1}^{k} (A_j x_j, x_j)\right)\left(\sum_{j=1}^{k} (A_j^{-1} x_j, x_j)\right) \leq \frac{(M+m)^2}{4Mm}. \tag{1.16}$$

Proof. If we put

$$\tilde{A} = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_k \end{pmatrix} \quad \text{and} \quad \tilde{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix},$$

then we have $\mathrm{Sp}(\tilde{A}) \subset [m, M]$, $\|\tilde{x}\| = 1$ and $\sum_{j=1}^{k} (A_j x_j, x_j) = (\tilde{A}\tilde{x}, \tilde{x})$. It follows from Theorem 1.23 that $(\tilde{A}\tilde{x}, \tilde{x})(\tilde{A}^{-1}\tilde{x}, \tilde{x}) \leq \frac{(M+m)^2}{4Mm}$ and hence we have the desired inequality (1.16). □

Next, finding the square roots of both sides of the Kantorovich inequality, we have

$$\{(Ax, x)(A^{-1}x, x)\}^{\frac{1}{2}} \leq \frac{M+m}{2\sqrt{Mm}} \tag{1.17}$$

for every unit vector $x \in H$. We show an extension of the form associated with a positive linear map.

Theorem 1.26 *Let Φ be a normalized positive linear map in $\mathbf{P}_N[\mathscr{B}(H), \mathscr{B}(K)]$. If A is a positive operator on H satisfying $M1_H \geq A \geq m1_H > 0$ for some scalars $m < M$, then*

$$\Phi(A) \sharp \Phi(A^{-1}) \leq \frac{M+m}{2\sqrt{Mm}}. \tag{1.18}$$

To enter the proof, we need some explanations. The geometric mean $A \sharp B$ of positive operators A and B is defined as

$$A \sharp B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{\frac{1}{2}} A^{\frac{1}{2}}$$

if A is invertible. If A and B commute, then $A \sharp B = \sqrt{AB}$, i.e., $A \sharp B$ is the usual geometric mean. The geometric mean is symmetric in the sense that $A \sharp B = B \sharp A$. In fact, let $C = A^{-1/2} B^{1/2} = U|C|$ be the polar decomposition of $A^{-1/2} B^{1/2}$, where U is unitary. Then we have

$$C(C^*C)^{-\frac{1}{2}} C^* = U|C||C|^{-1}|C|U = U|C|U = (CC^*)^{\frac{1}{2}}$$

and hence

$$A \sharp B = A^{\frac{1}{2}} (CC^*)^{\frac{1}{2}} A^{\frac{1}{2}} = B^{\frac{1}{2}} (C^*C)^{-\frac{1}{2}} B^{\frac{1}{2}} = B \sharp A.$$

Then the noncommutative arithmetic-geometric mean inequality holds.

Theorem 1.27 *The geometric mean is not greater than the arithmetic mean;*

$$A \sharp B \le \frac{A+B}{2}$$

for every positive operator A and B.

Proof. Since $\sqrt{t} \le \frac{1+t}{2}$ for all $t \ge 0$, it follows that

$$\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}} \le \frac{1_H + A^{-\frac{1}{2}}BA^{-\frac{1}{2}}}{2}$$

and multiplying both sides by $A^{\frac{1}{2}}$ we have

$$A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}} \le \frac{A+B}{2}.$$

Therefore we have $A \sharp B \le \frac{A+B}{2}$. □

Proof of Theorem 1.26. Since $M1_H \ge A \ge m1_H > 0$, it follows from Lemma 1.24 that

$$(M+m)1_H \ge MmA^{-1} + A.$$

Since Φ is a normalized positive linear map, we have

$$\Phi((M+m)1_H) \ge \Phi(MmA^{-1}) + \Phi(A)$$

and hence

$$(M+m)1_K \ge Mm\Phi(A^{-1}) + \Phi(A).$$

By using Theorem 1.27, we have

$$(M+m)1_K \ge Mm\Phi(A^{-1}) + \Phi(A) \ge 2\sqrt{Mm}\left(\Phi(A^{-1}) \sharp \Phi(A)\right).$$

Therefore it follows that

$$\Phi(A^{-1}) \sharp \Phi(A) = \Phi(A) \sharp \Phi(A^{-1}) \le \frac{M+m}{2\sqrt{Mm}},$$

since the geometric mean is symmetric, cf., Definition 5.2 in § 5.1. □

If we put $\Phi(X) = \sum_{j=1}^k U_j^* X U_j$ for contractions U_j with $\sum_{j=1}^k U_j^* U_j = 1_H$, then Φ is a normalized positive linear map. Therefore, Theorem 1.26 implies the following corollary which is another extension of the Kantorovich inequality.

Corollary 1.28 *Let U_j be contractions with $\sum_{j=1}^k U_j^* U_j = 1_H$ ($j = 1, 2, \cdots, k$). If A is a positive operator on H satisfying $M1_H \ge A \ge m1_H > 0$ for some scalars $m < M$, then*

$$\left(\sum_{j=1}^k U_j^* A U_j\right) \sharp \left(\sum_{j=1}^k U_j^* A^{-1} U_j\right) \le \frac{M+m}{2\sqrt{Mm}}.$$

We investigate several operator inequalities obtained by a view of the Kantorovich inequalities. Let A be a positive operator on H satisfying $M1_H \geq A \geq m1_H > 0$ for some scalars m and M. By Cauchy-Scwarz inequality, we have

$$1 = (x,x)^2 = \left(A^{\frac{1}{2}}x, A^{-\frac{1}{2}}x\right)^2 \leq \left\|A^{\frac{1}{2}}x\right\|^2 \left\|A^{-\frac{1}{2}}x\right\|^2 = (Ax,x)(A^{-1}x,x)$$

for every unit vector $x \in H$. We can realize that the Kantorovich inequality estimates the upper boundary of $(Ax,x)(A^{-1}x,x)$ by means of the spectrum of A. The Hölder-McCarthy inequality implies

$$(Ax,x)^2 \leq (A^2x,x)$$

for every unit vector $x \in H$. We show the following result by the Kantorovich inequlaity.

Theorem 1.29 *Let A be a positive operator on H satisfying $M1_H \geq A \geq m1_H > 0$ for some scalars $m < M$. Then*

$$(A^2x,x) \leq \frac{(M+m)^2}{4Mm}(Ax,x)^2$$

for every unit vector $x \in H$.

Proof. Substituting $\frac{A^{\frac{1}{2}}x}{\|A^{\frac{1}{2}}x\|}$ for a unit vector x in the Kantorovich inequality, we have

$$\frac{\left(AA^{\frac{1}{2}}x, A^{\frac{1}{2}}x\right)}{\left\|A^{\frac{1}{2}}x\right\|^2} \frac{\left(A^{-1}A^{\frac{1}{2}}x, A^{\frac{1}{2}}x\right)}{\left\|A^{\frac{1}{2}}x\right\|^2} \leq \frac{(M+m)^2}{4Mm}$$

and hence

$$(A^2x,x) \leq \frac{(M+M)^2}{4Mm}(Ax,x)^2.$$

□

Next, we investigate the estimations of the upper boundary of the difference $(A^2x,x) - (Ax,x)^2$ by means of the spectrum of A.

Theorem 1.30 *Let A be a selfadjoint operator on H satisfying $M1_H \geq A \geq m1_H$ for some scalars $m < M$. Then*

$$(A^2x,x) - (Ax,x)^2 \leq \frac{(M-m)^2}{4}$$

for every unit vector $x \in H$.

Proof. We first note that $(M-t)(t-m) \leq \left(\frac{M-m}{2}\right)^2$ for all real numbers t. Hence it follows from $(M1_H - A)(A - m1_H) \geq 0$ that

$$\begin{aligned}
&(A^2x,x) - (Ax,x)^2 \\
&\quad = (M - (Ax,x))((Ax,x) - m) - ((M1_H - A)(A - m1_H)x,x) \\
&\quad \leq (M - (Ax,x))((Ax,x) - m) \\
&\quad \leq \frac{(M-m)^2}{4}
\end{aligned}$$

for every unit vector $x \in H$. □

Theorem 1.31 *Let A be a positive operator on H satisfying $M1_H \geq A \geq m1_H > 0$ for some scalars $m < M$. Then*

$$(A^{-1}x,x) - (Ax,x)^{-1} \leq \frac{(\sqrt{M}-\sqrt{m})^2}{Mm}$$

for every unit vector $x \in H$.

Proof. By Lemma 1.24, we have

$$(M+m)1_H \geq MmA^{-1} + A$$

and hence

$$(A^{-1}x,x) \leq \frac{M+m}{Mm} - \frac{1}{Mm}(Ax,x)$$

for every unit vector $x \in H$. Then it follows that

$$\begin{aligned}
&(A^{-1}x,x) - (Ax,x)^{-1} \\
&\leq (\frac{1}{m} + \frac{1}{M}) - \frac{1}{Mm}(Ax,x) - (Ax,x)^{-1} \\
&= \left(\frac{1}{\sqrt{m}} - \frac{1}{\sqrt{M}}\right)^2 - \left(\frac{1}{\sqrt{Mm}}(Ax,x)^{\frac{1}{2}} - (Ax,x)^{-\frac{1}{2}}\right)^2 \\
&\leq \left(\frac{1}{\sqrt{m}} - \frac{1}{\sqrt{M}}\right)^2
\end{aligned}$$

and hence we have

$$(A^{-1}x,x) - (Ax,x)^{-1} \leq \frac{(\sqrt{M}-\sqrt{m})^2}{Mm}.$$

□

Similarly we have the following Kantorovich type inequalities for positive linear maps.

Theorem 1.32 *Let Φ be a normalized positive linear map in $\mathbf{P}_N[\mathscr{B}(H),\mathscr{B}(K)]$. If A is a positive operator on H satisfying $M1_H \geq A \geq m1_H > 0$ for some scalars $m < M$, then*

(i) $\Phi(A^2) - \Phi(A)^2 \leq \frac{(M-m)^2}{4} 1_K$.

(ii) $\Phi(A^{-1}) - \Phi(A)^{-1} \leq \frac{(\sqrt{M}-\sqrt{m})^2}{Mm} 1_K$.

(iii) $\Phi(A^2) \leq \frac{(M+m)^2}{4Mm} \Phi(A)^2$.

(iv) $\Phi(A^{-1}) \leq \frac{(M+m)^2}{4Mm} \Phi(A)^{-1}$.

Proof. Since $(M1_K - \Phi(A))(\Phi(A) - m1_K) \geq 0$, it follows that

$$\begin{aligned}
\Phi(A^2) - \Phi(A)^2 &= (M1_K - \Phi(A))(\Phi(A) - m1_K) - \Phi((M1_H - A)(A - m1_H)) \\
&\leq (M1_K - \Phi(A))(\Phi(A) - m1_K) \\
&\leq \frac{(M-m)^2}{4} 1_K
\end{aligned}$$

and so we have (i).

We have the proof of (ii) by the same method as in Theorem 1.31.

For (iii), since $(M1_H - A)(A - m1_H) \geq 0$, we have

$$(M+m)A - A^2 - Mm1_H \geq 0$$

and so

$$(M+m)\Phi(A) - \Phi(A^2) - Mm1_K \geq 0.$$

Also, $((M+m)\Phi(A) - 2Mm1_K)^2 \geq 0$ implies

$$(M+m)^2\Phi(A)^2 - 4Mm(M+m)\Phi(A) + 4M^2m^2 1_K \geq 0.$$

Combined with two inequalities above, we have

$$\begin{aligned}
\Phi(A^2) &\leq (M+m)\Phi(A) - Mm1_K \\
&\leq \frac{(M+m)^2}{4Mm}\Phi(A)^2
\end{aligned}$$

and so we have (iii).

Similarly we have (iv). □

1.5 Mond-Pečarić method

In this section, we shall introduce the Mond-Pečarić method which gives complementary inequalities to Jensen's type inequalities associated with convex functions.

Let $f(t)$ be a real valued continuous convex function and A a selfadjoint operator on a Hilbert space H. Then Jensen's inequality for a vector state asserts that

$$f((Ax,x)) \leq (f(A)x,x) \tag{1.19}$$

for every unit vector $x \in H$. In particular, if $f(t) = 1/t$ (resp. t^2), then we have

$$(Ax,x)^{-1} \leq (A^{-1}x,x) \qquad \text{(resp.} \quad (Ax,x)^2 \leq (A^2x,x)) \tag{1.20}$$

for every unit vector $x \in H$ since $f(t) = 1/t$ (resp. t^2) is convex.

The Kantorovich inequality asserts that if A is a positive operator on H satisfying $M1_H \geq A \geq m1_H > 0$ for some scalars m and M, then

$$(Ax,x)(A^{-1}x,x) \leq \frac{(M+m)^2}{4Mm} \tag{1.21}$$

for every unit vector $x \in H$. If we rephrase it by

$$(A^{-1}x,x) \leq \frac{(M+m)^2}{4Mm}(Ax,x)^{-1}, \tag{1.22}$$

then it can be recognized as a complementary inequality to Jensen's inequality for the convex function $f(t) = 1/t$. Namely, it estimates the upper boundary of the ratio in Jensen's inequality. Moreover, Theorem 1.31 says that

$$(A^{-1}x,x) - (Ax,x)^{-1} \leq \frac{(\sqrt{M}-\sqrt{m})^2}{Mm}. \tag{1.23}$$

From this point of view, Theorem 1.31 can be recognized as an estimates of the upper boundary of the difference in Jensen'inequality.

Many authors have been investigated on extensions of the Kantorovich inequality. Among others, we pay our attentions to a long research series of Mond and Pečarić [141, 143, 144, 145, 146]. They established the method which gives complementary inequalities to Jensen's type inequalities associated with convex functions. Consequently they gave complementary inequalities to the Hölder-McCarthy inequality and extensions of the Kantorovich type one. Furuta [74, 76] moreover gave extensions of Ky Fan [30] and Mond-Pečarić generalizations of the Kantorovich one by applying both ideas of Ky Fan, Mond and Pečarić. On the other hand, in the integral expression, S.-E.Takahasi et al. [182] gave another formula for a complementary inequality to Jensen's inequality which includes the Kantorovich inequality as a special case. By reconstructing both ideas of Furuta and Takahasi, we discover new merits in the method established by Mond and Pečarić and apply it to obtain complementary inequalities to Jensen's inequality for convex functions.

We consider complementary inequalities to Jensen's type inequalities associated with convex functions in a general setting. More precisely, if a selfadjoint operator A on H satisfies $M1_H \geq A \geq m1_H > 0$ for some scalars m and M and a real valued continuous function $f(t)$ is convex on $[m,M]$, then there exists the most suitable real number β such that for a given real number α and a real valued continuous function $g(t)$

$$(f(A)x,x) \leq \alpha g((Ax,x)) + \beta \tag{1.24}$$

holds for every unit vector $x \in H$. The generalization gives us a unified view to the operator inequalities (1.22) and (1.23). Plainly speaking, if we put $\alpha = 1$ and $g = f$ in (1.24), then the upper estimation of the difference in Jensen's inequality is given by

$$(f(A)x,x) - f((Ax,x)) \leq \beta. \tag{1.25}$$

If we choose α such that $\beta = 0$ and $g = f$ in (1.24), then the upper estimation of the ratio in Jensen's inequality is given by

$$(f(A)x,x) \leq \alpha f((Ax,x)). \tag{1.26}$$

Thus, we consider the problem of determining the upper estimates of such a β in complementary inequalities to Jensen's inequalities.

Since $f(t)$ is convex on $[m,M]$, then we have

$$f(t) \leq l(t) \equiv \mu_f t + \nu_f \qquad \text{on} \quad [m,M] \tag{1.27}$$

where

$$\mu_f = \frac{f(M)-f(m)}{M-m} \qquad \text{and} \qquad \nu_f = \frac{Mf(m)-mf(M)}{M-m} \tag{1.28}$$

that is, a straight line $l(t)$ is a linear function limiting $f(t)$ from above. Using the operator calculus it follows that

$$f(A) \leq \mu_f A + \nu_f 1_H$$

and

$$f(m)1_H \leq \mu_f A + \nu_f 1_H \leq f(M)1_H$$

Then such a β is obtained as follows: The hypothesis ensures the inequality $m \leq (Ax,x) \leq M$. Then it follows that

$$\begin{aligned}(f(A)x,x) - \alpha g((Ax,x)) &\leq ((\mu_f A + \nu_f)x,x) - \alpha g((Ax,x)) \\ &= \mu_f(Ax,x) + \nu_f - \alpha g((Ax,x)) \\ &\leq \max_{m\leq t\leq M}\{f(m) + \mu_f(t-m) - \alpha g(t)\}.\end{aligned}$$

Therefore, if we put $\beta = \max_{m\leq t\leq M}\{f(m)+\mu_f(t-m)-\alpha g(t)\}$, then we have the desired inequality.

By this view, we can realize that the problem of determining such a β is reduced to solving a single variable maximization or minimization problem by using the convexity of $f(t)$. Based on the method, we shall deal with general complementary inequalities to Jensen's inequalities for convex functions. Under this formulation, the concept of complementary inequalities is simplified and, notions and proofs become clearer. This point of view is quite available for the study of the Hadamard product, positive linear maps, operator means and order preserving operator inequalities. The principle yields a rich harvest in the field of operator inequalities. We call it the **Mond-Pečarić method**.

1.6 Notes

Theorem 1.2 is due to Mond and Pečarić [141] and [148]. The original proof of Theorem 1.4 is due to McCarthy [129] and another proof of Theorem 1.4 (ii) appeared in Kitamura and Seo [119]. For fundamental results associated with C*-algebras we refer to Arveson [12].

The examples in Section 1.2 are from a work of Bhatia [19] in Chapter V. For the Löwner-Heinz inequality we refer a simplified proof in Pedersen [170]. A proof of Theorem 1.8 is given in Heinz [102] and more general form of the Löwner-Heinz inequality had been given in Löwner [125]. Theorem 1.9 and Theorem 1.15 are due to J.I.Fujii and M.Fujii [39], Hansen and Pedersen [99]. The implication $(iii) \Rightarrow (v)$ of Theorem 1.9 is due to M.Fujii [53]. The implication $(ii) \Rightarrow (i)$ of Theorem 1.13 is due to M.Fujii, T.Furuta and R.Nakamoto [56]. The main results concerning some characterizations of operator concavity and operator monotonicity are due to Hansen and Pedersen [98]. Example 1.9 is due to Furuta [82].

The study of positive linear maps on an algebra of bounded linear operators on a Hilbert space has been developed by many authors (T.Ando, W.B.Arveson, M.D.Choi, T.Y.Lam, E.G.Effres, C.Davis, R.V.Kadison, E.H.Lieb, M.B.Ruskai, W.E.Stinespring, E.Størmer, S.L.Woronowicz). Theorem 1.17 and Theorem 1.20 are due to Ando [3]. Theorem 1.21 is due to J.I.Fujii and M.Fujii [39]. A proof of Theorem 1.20 in Remark 1.2 by using the Stinespring decomposition theorem [180] is due to Davis [27] and Choi [25]. The counterexample in Theorem 1.21 is due to Choi [25].

Kantorovich [115] firstly showed Theorem 1.23 in the case of sequences. Greub and Rheinboldt [96] formulated an operator version (Theorem 1.23) of the inequality due to Kantorovich and proved it by a somewhat different way. Nakamura [161] gave a simple proof of Theorem 1.23 by using a convexity of $f(t) = t^{-1}$, which is based on the idea of Mond-Pečarić method. Equality problems on the Kantorovich inequality are considered by Henrici [103] and Tsukada and Takahasi [188]. For another proof of the Kantorovich inequality, we refer to M.Fujii, Furuta, Nakamoto and Takahasi [57]. An extension (Theorem 1.26) of the Kantorovich inequality associated with a positive linear map is due to Nakamoto and Nakamura [160]. Theorem 1.32 is due to Mond and Pečarić [140] and [145]. Theorem 1.29 appeared in Krasnoselskii and Krein [120] (see, e.g., Mond [138]). Theorem 1.30 is due to J.I.Fujii, M.Fujii, Nakamoto and Takahasi [57].

Chapter 2

Converses of Jensen's inequalities

In this chapter, we study complementary inequalities to Jensen's inequalities for normalized positive linear maps in a more general setting. Under this formulation, the concept of complementary inequalities is made clear and proofs are unified and so become clearer.

2.1 Converses of Jensen's inequalities for positive linear maps

First, we give a generalization of Jensen's inequality for normalized positive linear maps. For convenience, we denote by $\mathscr{C}([m,M])$ the set of all real valued continuous functions on an interval $[m,M]$.

Lemma 2.1 *Let $A_j \in \mathscr{B}_h(H)$ be selfadjoint operators with $\mathsf{Sp}(A_j) \subseteq [m,M]$ for some scalars $m < M$ and $\Phi_j \in \mathbf{P}_N[\mathscr{B}(H),\mathscr{B}(K)]$ normalized positive linear maps $(j = 1,\dots,k)$. Let $\omega_1,\omega_2,\cdots,\omega_k \in \mathbb{R}_+$ be any finite number of positive real numbers such that $\sum_{j=1}^{k}\omega_j = 1$. If $f \in \mathscr{C}([m,M])$ is operator convex on $[m,M]$, then*

$$f\left(\sum_{j=1}^{k}\omega_j\Phi_j(A_j)\right) \leq \sum_{j=1}^{k}\omega_j\Phi_j\left(f(A_j)\right). \tag{2.1}$$

Proof. Since f is operator convex, using mathematical induction, if $\omega_j > 0$ and $\sum_{j=1}^{k}\omega_j = 1$, then we have

$$f\left(\sum_{j=1}^{k}\omega_j A_j\right) \leq \sum_{j=1}^{k}\omega_j f(A_j).$$

Moreover, Davis-Choi-Jensen's inequality (Theorem 1.20) says that

$$f\left(\Phi_j(A_j)\right) \leq \Phi_j\left(f(A_j)\right) \qquad \text{for} \quad j = 1,2,\cdots,k.$$

Using two inequalities above, we have

$$f\left(\sum_{j=1}^{k}\omega_j\Phi_j(A_j)\right) \leq \sum_{j=1}^{k}\omega_j f\left(\Phi_j(A_j)\right) \leq \sum_{j=1}^{k}\omega_j\Phi_j\left(f(A_j)\right).$$

□

By using Lemma 2.1, we have the following Jensen's type inequality associated with two functions.

Lemma 2.2 *Let A_j, Φ_j and ω_j, $j = 1,\ldots,k$, be as in Lemma 2.1. Let $f,g \in \mathscr{C}([m,M])$ and $f \leq g$ on $[m,M]$. If f is operator convex on $[m,M]$, then*

$$f\left(\sum_{j=1}^{k}\omega_j\Phi_j(A_j)\right) \leq \sum_{j=1}^{k}\omega_j\Phi_j\left(g(A_j)\right). \tag{2.2}$$

Proof. It follows from the spectral theorem and the map positivity of Φ_j that $f \leq g$ on $[m,M]$ implies $\Phi_j\left(f(A_j)\right) \leq \Phi_j\left(g(A_j)\right)$, $j = 1,\ldots,k$. Multiplying this inequality with $\omega_j \in \mathbb{R}_+$ and summing of j we have

$$\sum_{j=1}^{k}\omega_j\Phi_j\left(f(A_j)\right) \leq \sum_{j=1}^{k}\omega_j\Phi_j\left(g(A_j)\right).$$

Since f is operator convex, it follows from Lemma 2.1 that

$$f\left(\sum_{j=1}^{k}\omega_j\Phi_j(A_j)\right) \leq \sum_{j=1}^{k}\omega_j\Phi_j\left(f(A_j)\right).$$

Using two inequalities above, we have the desired inequality (2.2) as in Lemma 2.1. □

Here, we present converses of Jensen's inequality for positive linear maps in general form. This extremely shows the basic idea of Mond-Pečarić method. As a special case, we obtain many applications as mentioned after chapters. Notice that we don't assume the operator convexity of f.

For convenience, let $k(t)$ be a real valued continuous function on the interval $[m,M]$. We define:

$$\mu_k = \frac{k(M)-k(m)}{M-m} \qquad \text{and} \qquad \nu_k = \frac{Mk(m)-mk(M)}{M-m}.$$

We remark that a straight line $l(t) = \mu_k t + \nu_k$ is a line thought two points $(m,k(m))$ and $(M,k(M))$.

Theorem 2.3 *Let $A_j \in \mathscr{B}_h(H)$ be selfadjoint operators with $\mathsf{Sp}(A_j) \subseteq [m,M]$ for some scalars $m < M$, $\Phi_j \in \mathbf{P}_N[\mathscr{B}(H),\ \mathscr{B}(K)]$ normalized positive linear maps $(j = 1,\ldots,k)$. Let $\omega_1,\cdots,\omega_k \in \mathbb{R}_+$ be any finite number of positive real numbers such that $\sum_{j=1}^k \omega_j = 1$. Let $f,g \in \mathscr{C}([m,M])$ and $F(u,v)$ be a real valued continuous function defined on $U \times V$, where $U \supset f[m,M]$, $V \supset g[m,M]$. If $F(u,v)$ is operator monotone on a first variable u and f is convex on $[m,M]$, then*

$$F\left[\sum_{j=1}^k \omega_j \Phi_j\left(f(A_j)\right), g\left(\sum_{j=1}^k \omega_j \Phi_j(A_j)\right)\right] \le \left\{\max_{m\le t\le M} F\left[\mu_f t + \nu_f, g(t)\right]\right\} 1_K. \tag{2.3}$$

In the dual case (when f is concave) we have the opposite inequality with dual extreme (min *instead of* max).

Proof. We only prove the case where f is convex on $[m,M]$. Since $f(t) \le \mu_f t + \nu_f$ for every $t \in [m,M]$, it follows that $f(A_j) \le \mu_f A_j + \nu_f 1_H$ for all $j = 1,\cdots,k$. Since Φ_j is a normalized positive linear map, we have

$$\begin{aligned} \Phi_j\left(f(A_j)\right) &\le \Phi_j(\mu_f A_j + \nu_f 1_H) \\ &= \mu_f \Phi_j(A_j) + \nu_f \Phi_j(1_H) \\ &= \mu_f \Phi_j(A_j) + \nu_f 1_K \quad \text{for } j = 1,\cdots,k. \end{aligned}$$

Further, multiplying them with $\omega_j \in \mathbb{R}_+$, summing of all $j = 1,\ldots,k$, and using $\sum_{j=1}^k \omega_j = 1$ we have

$$\sum_{j=1}^k \omega_j \Phi_j\left(f(A_j)\right) \le \mu_f \sum_{j=1}^k \omega_j \Phi_j(A_j) + \nu_f 1_K. \tag{2.4}$$

Since $m1_H \le A_j \le M1_H$ we have $m1_K \le \sum_{j=1}^k \omega_j \Phi_j(A_j) \le M1_K$, i.e. $\mathsf{Sp}\left(\sum_{j=1}^k \omega_j \Phi_j(A_j)\right) \subseteq [m,M]$. Now, using an operator monotonicity of $F(\cdot,v)$, we obtain

$$\begin{aligned} & F\left[\sum_{j=1}^k \omega_j \Phi_j\left(f(A_j)\right), g\left(\sum_{j=1}^k \omega_j \Phi_j(A_j)\right)\right] \le \\ & \le F\left[\mu_f \sum_{j=1}^k \omega_j \Phi_j(A_j) + \nu_f 1_K, g\left(\sum_{j=1}^k \omega_j \Phi_j(A_j)\right)\right] \\ & \le \left\{\max_{m\le t\le M} F\left[\mu_f t + \nu_f, g(t)\right]\right\} 1_K, \end{aligned}$$

which is a desired inequality. □

We consider complementary problems to Jensen's type inequality (2.2) in Lemma 2.2. We attempt to determine upper estimates for $\sum_{j=1}^{k}\omega_j\Phi_j\left(f(A_j)\right) - g\left(\sum_{j=1}^{k}\omega_j\Phi_j(A_j)\right)$ by means of scalar multiples of the identity operator 1_K, that is,

$$\sum_{j=1}^{k}\omega_j\Phi_j\left(f(A_j)\right) - \alpha g\left(\sum_{j=1}^{k}\omega_j\Phi_j(A_j)\right) \leq \beta 1_K$$

and upper estimates for $\sum_{j=1}^{k}\omega_j\Phi_j\left(f(A_j)\right)$ by means of scalar multiples of $g\left(\sum_{j=1}^{k}\omega_j\Phi_j(A_j)\right)$, that is,

$$\sum_{j=1}^{k}\omega_j\Phi_j\left(f(A_j)\right) \leq \alpha g\left(\sum_{j=1}^{k}\omega_j\Phi_j(A_j)\right).$$

To this goal, a particular choice of the function F in Theorem 2.3 implies the following complementary inequality to Jensen's inequality, by which is given the unified view of upper estimates in two expressions above.

Theorem 2.4 *Let $A_j,\Phi_j,\omega_j,\ j=1,\ldots,k$ be as in Theorem 2.3 and $f,g\in\mathscr{C}([m,M])$. If f is convex on $[m,M]$, then for any real numbers $\alpha\in\mathbb{R}$*

$$\sum_{j=1}^{k}\omega_j\Phi_j\left(f(A_j)\right) \leq \alpha g\left(\sum_{j=1}^{k}\omega_j\Phi_j(A_j)\right) + \beta 1_K, \tag{2.5}$$

where

$$\beta = \max_{m\leq t\leq M}\left\{\mu_f t + \nu_f - \alpha g(t)\right\}. \tag{2.6}$$

Further, suppose that the function g satisfies either of the following conditions:

(i) αg is concave.

(ii) αg is strictly convex differentiable.

Then for the boundary β we have

$$\beta = \max_{s\in\{m,M\}}\{f(s)-\alpha g(s)\}$$

in the case (i) and

$$\begin{aligned}\max_{s\in\{m,M\}}\{f(s)-\alpha g(s)\} &\leq \beta \\ &\leq \min_{s\in\{m,M\}}\left\{f(s)-\alpha g(s)+\left|\mu_f-\alpha g'(s)\right|\ (M-m)\right\}\end{aligned}$$

in the case (ii).

We can determine more precisely the value $\beta \equiv \beta(m,M,f,g,\alpha)$ in (2.6) as follows:

$$\beta = \mu_f t_o + \nu_f - \alpha g(t_o),$$

where

$$t_o = \begin{cases} M \text{ if } & \mu_f \geq \alpha\mu_g, \\ m \text{ if } & \mu_f < \alpha\mu_g, \end{cases} \qquad \textit{in the case } (i),$$

and

$$t_o = \begin{cases} g'^{-1}(\mu_f/\alpha) & \text{if } \quad \alpha g'(m) < \mu_f < \alpha g'(M), \\ m & \text{if } \quad \alpha g'(m) \geq \mu_f, \\ M & \text{if } \quad \alpha g'(M) \leq \mu_f, \end{cases} \qquad \textit{in the case } (ii).$$

In the dual case we have the opposite inequality with dual extreme, with the dual estimation for β and the opposite condition while determining t_o.

Proof. We only prove the convex case. Put $T_0 = \sum_{j=1}^{k} \omega_j \Phi_j(A_j)$, then the hypothesis ensures the inequality $m1_K \leq T_0 \leq M1_K$. Put

$$F(u,v) = u - \alpha v.$$

Then F is operator monotone on u and hence it follows from Theorem 2.3 that

$$\begin{aligned} \sum_{j=1}^{k} \omega_j \Phi_j(f(A_j)) - \alpha g\left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j)\right) &\leq \max_{m \leq t \leq M} F(\mu_f t + \nu_f, g(t)) 1_K \\ &= \max_{m \leq t \leq M} \{\mu_f t + \nu_f - \alpha g(t)\} 1_K, \end{aligned}$$

which gives the desired inequality (2.5).

Put $h(t) = \mu_f t + \nu_f - \alpha g(t)$. Further, suppose (i), i.e., αg is a concave function on $[m,M]$. Then $h(t)$ is a convex function and hence $\beta = \max_{m \leq t \leq M} h(t) = \max\{h(m), h(M)\}$.

Next, suppose (ii), i.e., αg is a strictly convex differentiable function on $[m,M]$. Then $\alpha g(t) - \alpha g(s) > \alpha g'(s)(t-s)$ for all $t \neq s$, $t,s \in [m,M]$. Hence for $t = t_o$ and $s = m,M$ we have

$$\begin{aligned} \beta &= f(s) + \mu_f(t_o - s) - \alpha g(t_o) = f(s) - \alpha g(s) + [\mu_f(t_o - s) - \alpha g(t_o) + \alpha g(s)] \\ &\leq f(s) - \alpha g(s) + [\mu_f - \alpha g'(s)](t_o - s) \leq f(s) - \alpha g(s) + \left|\mu_f - \alpha g'(s)\right| (M - m), \end{aligned}$$

so that we have an upper estimate of β. The lower estimate of β is evident.

More precisely, since $h(t)$ is concave, $h'(t)$ is evidently a strictly decreasing function on $[m,M]$. Then we have one of three possibilities. If $h'(m) > 0$ and $h'(M) < 0$, in other words, $\alpha g'(m) < \mu_f < \alpha g'(M)$, then the equation $h'(t) = 0$ has exactly one solution $t_o \in (m,M)$ where the function h attains the maximum value for $t_0 = g'^{-1}(\mu_f/\alpha)$. If $h'(m) \leq 0$, then $h' \leq 0$ on $[m,M]$, since h is a decreasing function on $[m,M]$ and the maximum value is attained for $t = m$. If $h'(M) \geq 0$, then $h' \geq 0$ on $[m,M]$, i.e., h is an increasing function on $[m,M]$ and the function h attains the maximum value for $t = M$. □

Remark 2.1 *Notice that the operator convexity of f and the condition $f \leq g$ on $[m,M]$ are not assumed in Theorem 2.4.*

If we put $g = f$ in Theorem 2.4, then we obtain the following complementary inequality to Jensen's inequality in Lemma 2.1.

Theorem 2.5 *Let A_j, Φ_j, ω_j, $j = 1,\dots,k$ be as in Theorem 2.3. Let $f \in \mathscr{C}([m,M])$ be a nonnegative real valued continuous strictly convex twice differentiable function on $[m,M]$. Then for any positive real numbers $\alpha(>0) \in \mathbb{R}$*

$$\sum_{j=1}^{k} \omega_j \Phi_j\left(f(A_j)\right) \leq \alpha f\left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j)\right) + \beta 1_K, \tag{2.7}$$

where $\beta = \mu_f t_o + \nu_f - \alpha f(t_o)$ and

$$t_o = \begin{cases} f'^{-1}(\mu_f/\alpha) & \text{if} \quad \alpha f'(m) < \mu_f < \alpha f'(M), \\ m & \text{if} \quad \alpha f'(m) \geq \mu_f, \\ M & \text{if} \quad \alpha f'(M) \leq \mu_f. \end{cases}.$$

In the dual case we have the opposite inequality with dual extreme, with the dual estimation for β and the opposite condition while determining t_o.

Proof. Since αf is strictly convex twice differentiable, this theorem follows from Theorem 2.4. □

Corollary 2.6 *Let $A_j \in \mathscr{B}^+(H)$ be positive operators with $\mathsf{Sp}(A_j) \subseteq [m,M]$ for some scalars $0 < m < M$, $\Phi_j \in \mathbf{P}_N[\mathscr{B}(H), \mathscr{B}(K)]$ normalized positive linear maps $(j = 1,\dots,k)$. Let $\omega_1,\cdots,\omega_k \in \mathbb{R}_+$ be any finite number of positive real numbers such that $\sum_{j=1}^{k} \omega_j = 1$. Let $f \in \mathscr{C}([m,M])$ and $q, \alpha \in \mathbb{R}$. If f is convex, then*

$$\sum_{j=1}^{k} \omega_j \Phi_j\left(f(A_j)\right) \leq \alpha \left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j)\right)^q + \beta\, 1_K \tag{2.8}$$

holds for

$$\beta = \begin{cases} \alpha(q-1)\left(\frac{\mu_f}{\alpha q}\right)^{\frac{q}{q-1}} + \nu_f & \text{if } m < \left(\frac{\mu_f}{\alpha q}\right)^{\frac{1}{q-1}} < M \text{ and } \alpha q(q-1) > 0, \\ \max\{f(m) - \alpha m^q, f(M) - \alpha M^q\} & \text{otherwise.} \end{cases}$$

If f is concave, then

$$\sum_{j=1}^{k} \omega_j \Phi_j\left(f(A_j)\right) \geq \alpha \left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j)\right)^q + \beta\, 1_K \tag{2.9}$$

holds for

$$\beta = \begin{cases} \alpha(q-1)\left(\frac{\mu_f}{\alpha q}\right)^{\frac{q}{q-1}} + \nu_f & \textit{if } m < \left(\frac{\mu_f}{\alpha q}\right)^{\frac{1}{q-1}} < M \textit{ and } \alpha q(q-1) < 0, \\ \min\{f(m) - \alpha m^q, f(M) - \alpha M^q\} & \textit{otherwise.} \end{cases}$$

In particular, if $p \in \mathbb{R}\backslash[0,1]$, *then*

$$\sum_{j=1}^{k} \omega_j \Phi_j(A_j^p) \leq \alpha \left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j) \right)^q + \beta_1 1_K, \tag{2.10}$$

where

$$\beta_1 = \begin{cases} \alpha(q-1)\left(\frac{\mu_{t^p}}{\alpha q}\right)^{\frac{q}{q-1}} + \nu_{t^p} & \textit{if } m < \left(\frac{\mu_{t^p}}{\alpha q}\right)^{\frac{1}{q-1}} < M \textit{ and } \alpha q(q-1) > 0, \\ \max\{f(m) - \alpha m^q, f(M) - \alpha M^q\} & \textit{otherwise.} \end{cases}$$

If $p \in (0,1]$, *then*

$$\sum_{j=1}^{k} \omega_j \Phi_j(A_j^p) \geq \alpha \left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j) \right)^q + \beta_1 1_K \tag{2.11}$$

where

$$\beta_1 = \begin{cases} \alpha(q-1)\left(\frac{\mu_{t^p}}{\alpha q}\right)^{\frac{q}{q-1}} + \nu_{t^p} & \textit{if } m < \left(\frac{\mu_{t^p}}{\alpha q}\right)^{\frac{1}{q-1}} < M \textit{ and } \alpha q(q-1) < 0, \\ \min\{f(m) - \alpha m^q, f(M) - \alpha M^q\} & \textit{otherwise.} \end{cases}$$

Proof. We only prove the case where f is convex on $[m,M]$. If we put $g(t) = t^q$ in Theorem 2.4, then we obtain the boundary $\beta = \mu_f t_o + \nu_f - \alpha t_o^q$, where $t_o = \left(\frac{\mu_f}{\alpha q}\right)^{\frac{1}{q-1}}$ if $\alpha q(q-1) > 0$ i.e., αt^q is convex, and $m < t_o < M$; $t_o = m, M$ if otherwise . Further, if we put $f(t) = t^p$, $p \in \mathbb{R}\backslash[0,1)$ in (2.8), then we have (2.10). □

Corollary 2.7 *Let* A_j, Φ_j, ω_j, $j = 1,\ldots,k$ *be as in Corollary 2.6. Let* $f \in \mathscr{C}([m,M])$ *and* $\alpha \in \mathbb{R}$. *If* f *is convex, then*

$$\sum_{j=1}^{k} \omega_j \Phi_j\left(f(A_j)\right) \leq \alpha \log \left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j) \right) + \beta_1 1_K \tag{2.12}$$

where

$$\beta_1 = \begin{cases} \alpha + \nu_f + \log(\frac{\mu_f}{\alpha})^\alpha & \textit{if } \ M\mu_f < \alpha < m\mu_f < 0, \\ \max\{f(m) - \log m^\alpha, f(M) - \log M^\alpha\} & \textit{otherwise} \end{cases}$$

and

$$\sum_{j=1}^{k} \omega_j \Phi_j\left(f(A_j)\right) \leq \alpha \exp \left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j) \right) + \beta_2 1_K \tag{2.13}$$

where

$$\beta_2 = \begin{cases} \nu_f - \mu_f + \log\left(\frac{\mu_f}{\alpha}\right)^{\mu_f} & \text{if} \quad 0 < \alpha e^m < \mu_f < \alpha e^M, \\ \max\left\{f(m) - \alpha\, e^m, f(M) - \alpha\, e^M\right\} & \text{otherwise.} \end{cases}$$

If f is concave, then we obtain the opposite inequalities with dual value of constants β_1 and β_2.

Proof. This corollary follows from the Theorem 2.4 if we put $g(t) = \log t$ and $g(t) = e^t$. □

The following corollary is complementary inequalities to the logarithmic function, the exponential function and the power function.

Recall that the logarithmic mean $L(m,M)$ is defined for $0 < m < M$ as

$$L(m,M) = \frac{M-m}{\log M - \log m} \quad (M > m) \quad \text{and} \quad L(m,m) = m.$$

It is easy to see that $m \le L(m,M) \le M$.

Corollary 2.8 *Let A_j, Φ_j, ω_j, $j = 1,\ldots,k$ be as in Corollary 2.6. Let $\alpha \in \mathbb{R}$ be a given positive real number. Then*

$$\sum_{j=1}^{k} \omega_j \Phi_j\left(\log(A_j)\right) \ge \alpha \log\left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j)\right) + \beta_1 1_K, \tag{2.14}$$

where

$$\beta_1 = \begin{cases} \alpha - \alpha\log(\alpha L(m,M)) + \frac{M\log m - m\log M}{M-m} & \text{if} \quad m < \alpha L(m,M) < M, \\ (1-\alpha)\log M & \text{if} \quad M \le \alpha L(m,M), \\ (1-\alpha)\log m & \text{if} \quad \alpha L(m,M) \le m \end{cases}$$

and

$$\sum_{j=1}^{k} \omega_j \Phi_j\left(\exp(A_j)\right) \le \alpha \exp\left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j)\right) + \beta_2 1_K, \tag{2.15}$$

where

$$\beta_2 = \begin{cases} \frac{e^M - e^m}{M-m} \log \frac{e^M - e^m}{\alpha(M-m)} + \frac{(M+1)e^m - (m+1)e^M}{M-m} & \text{if} \quad m < \log \frac{e^M - e^m}{\alpha(M-m)} < M, \\ (1-\alpha)e^M & \text{if} \quad M \le \log \frac{e^M - e^m}{\alpha(M-m)}, \\ (1-\alpha)e^m & \text{if} \quad \log \frac{e^M - e^m}{\alpha(M-m)} \le m. \end{cases}$$

If $p \in \mathbb{R}\setminus[0,1]$, then

$$\sum_{j=1}^{k} \omega_j \Phi_j(A_j^p) \le \alpha \left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j)\right)^p + \beta_3 1_K. \tag{2.16}$$

If $p \in (0,1]$, *then*

$$\sum_{j=1}^{k} \omega_j \Phi_j(A_j^p) \geq \alpha \left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j) \right)^p + \beta_3 1_K, \tag{2.17}$$

where

$$\beta_3 = \begin{cases} \alpha(p-1)\left(\frac{\mu_{t^p}}{\alpha p}\right)^{\frac{p}{p-1}} + \nu_{t^p} & \text{if } m < \left(\frac{\mu_{t^p}}{\alpha p}\right)^{\frac{1}{p-1}} < M, \\ (1-\alpha)M^p & \text{if } M \leq \left(\frac{\mu_{t^p}}{\alpha p}\right)^{\frac{1}{p-1}}, \\ (1-\alpha)m^p & \text{if } \left(\frac{\mu_{t^p}}{\alpha p}\right)^{\frac{1}{p-1}} \leq m. \end{cases}$$

2.2 Ratio type reverse inequalities

In this section, as applications of our general theorem (Theorem 2.3), we show ratio type reverse inequalities to Jensen's inequalities and give the explicit expressions in the estimations of the ratio.

If we choose the constant α such that $\beta = 0$ in Theorem 2.4, then we obtain the following ratio type reverse inequality as a complementary inequalities to Jensen's type inequality.

Theorem 2.9 *Let* $A_j \in \mathscr{B}_h(H)$ *be selfadjoint operators with* $\mathrm{Sp}(A_j) \subseteq [m,M]$ *for some scalars* $m < M$, $\Phi_j \in \mathbf{P}_N[\mathscr{B}(H), \mathscr{B}(K)]$ *normalized positive linear maps* $(j = 1,\ldots,k)$. *Let* $\omega_1,\cdots,\omega_k \in \mathbb{R}_+$ *be any finite number of positive real numbers such that* $\sum_{j=1}^{k}\omega_j = 1$. *Let* $f, g \in \mathscr{C}([m,M])$ *and suppose that either of the following conditions holds:*

(i) $g(t) > 0$ *for all* $t \in [m,M]$,

(ii) $g(t) < 0$ *for all* $t \in [m,M]$.

If f *is a convex function on* $[m,M]$, *then*

$$\sum_{j=1}^{k} \omega_j \Phi_j\left(f(A_j)\right) \leq \alpha_o\, g\left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j) \right), \tag{2.18}$$

where

$$\alpha_o = \max_{m \leq t \leq M} \left\{ \frac{1}{g(t)}(\mu_f t + \nu_f) \right\} \quad \text{in the case } (i),$$
$$\text{or } \alpha_o = \min_{m \leq t \leq M} \left\{ \frac{1}{g(t)}(\mu_f t + \nu_f) \right\} \quad \text{in the case } (ii)$$

and

$$\mu_f = \frac{f(M)-f(m)}{M-m} \quad \text{and} \quad \nu_f = \frac{Mf(m)-mf(M)}{M-m}.$$

Furthermore, suppose that either of the additional conditions holds:

(iii) $f(m) > 0, f(M) > 0$ *and* $g(t)$ *is a strictly concave differentiable function in the case of (i),*

(iv) $f(m) < 0, f(M) < 0$ *and* $g(t)$ *is a strictly convex twice differentiable function in the case of (ii).*

Then the boundary α_o *satisfies the following conditions:*

$$\alpha_o \geq \max_{s\in\{m,M\}} \left\{ \frac{f(s)}{g(s)} \right\} > 0 \quad \text{in the case } (iii),$$

$$\min_{s\in\{m,M\}} \left\{ \frac{f(s)}{g(s)} \right\} \geq \alpha_o > 0 \quad \text{in the case } (iv).$$

We can determine more precisely the value $\alpha_o \equiv \alpha_o(m,M,f,g)$ *in (2.18) as follows:*

$$\alpha_o = \frac{\mu_f t_o + \nu_f}{g(t_o)},$$

where

$$t_o = \begin{cases} M \text{ if } & \frac{\mu_f}{\mu_g}\nu_g \geq \nu_f, \\ m \text{ if } & \frac{\mu_f}{\mu_g}\nu_g < \nu_f, \end{cases} \qquad \text{in the case } (iii),$$

or

$$t_o = \begin{cases} \left.\begin{array}{c} \text{the solution of} \\ \mu_f g(t) = \left(\mu_f t + \nu_f\right) g'(t) \end{array}\right\} & \text{if} \quad f(m)\frac{g'(m)}{g(m)} < \mu_f < f(M)\frac{g'(M)}{g(M)}, \\ M & \text{if} \quad \mu_f \geq f(M)\frac{g'(M)}{g(M)}, \\ m & \text{if} \quad \mu_f \leq f(m)\frac{g'(m)}{g(m)}, \end{cases}$$

in the case (iv).

In the dual case $(f$ *concave,* g *strictly convex or strictly concave$)$ we have the opposite inequality with dual extreme, with the dual estimation for* α_o *and the opposite condition while determining* t_o.

Proof. Suppose that (i). If we put $F(u,v) = v^{-1/2}uv^{-1/2}$ in Theorem 2.3, then we have

$$\sum_{j=1}^{k} \omega_j \Phi_j\left(f(A_j)\right) \leq \max_{m\leq t\leq M} h(t,m,M,f,g)\, g\left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j)\right),$$

where

$$h(t) \equiv h(t,m,M,f,g) = \frac{\mu_f t + \nu_f}{g(t)}.$$

Moreover, suppose that (iii). We have $h'(t) = H(t)/g(t)^2$, where

$$H(t) = \mu_f g(t) - (\mu_f t + \nu_f) g'(t).$$

Since $f(m) > 0$ and $f(M) > 0$, we have $\mu_f t + \nu_f = \frac{f(m)(M-t)+f(M)(t-m)}{M-m} > 0$ for all $t \in [m,M]$. Since $g(t)$ is a strictly concave twice differentiable function on $[m,M]$, i.e. $g''(t) <$

0, it follows that $H'(t) = -(\mu_f t + \nu_f)g''(t) > 0$, so that $H(t)$ is an increasing function on $[m,M]$. If $H(m) > 0$, then $h'(t) > 0$ and hence the maximum value of $h(t)$ is attained for $t = M$. If $H(M) < 0$, then $h'(t) < 0$ and hence the maximum value of $h(t)$ is attained for $t = m$. If $H(m) < 0$ and $H(M) > 0$, then the maximum value of $h(t)$ is attained for $t = m$ or $t = M$ since $H(t)$ is increasing. Since $h(m) \le h(M)$ is equivalent to $\nu_f \mu_g \le \mu_f \nu_g$, the proof in the case (i) and (iii) is complete.

Next, suppose that (ii). If we put $g_1(t) = -g(t) > 0$ for all $t \in [m,M]$, then as proved above, we have

$$\begin{aligned}
\sum_{j=1}^{k} \omega_j \Phi_j(f(A_j)) &\le \max_{m \le t \le M} h(t,m,M,f,g_1)\, g_1\left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j)\right) \\
&= -\max_{m \le t \le M} h(t,m,M,f,-g)\, g\left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j)\right) \\
&= \min_{m \le t \le M} h(t,m,M,f,g)\, g\left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j)\right).
\end{aligned}$$

Moreover, suppose that (iv). Since $f(m) < 0$ and $f(M) < 0$, we have $\mu_f t + \nu_f = \frac{f(m)(M-t)+f(M)(t-m)}{M-m} < 0$ for all $t \in [m,M]$. Since $g(t)$ is a strictly convex twice differentiable function on $[m,M]$, i.e. $g''(t) > 0$, it follows that $H'(t) = -(\mu_f t + \nu_f)g''(t) > 0$, so that $H(t)$ is an increasing function on $[m,M]$. If $H(m) \ge 0$, then $\mu_f \le f(m)\frac{g'(m)}{g(m)}$ and $h'(t) > 0$ and hence the minimum value of $h(t)$ is attained for $t = m$. If $H(M) \le 0$, then $\mu_f \ge f(M)\frac{g'(M)}{g(M)}$ and $h'(t) < 0$ and hence the minimum value of $h(t)$ is attained for $t = M$. If $H(m) < 0$ and $H(M) > 0$, then the equation $H(t) = 0$ has exactly one solution $\bar{t} \in [m,M]$. Hence the minimum value of $h(t)$ is attained for $t = \bar{t}$, since $h'(t) < 0$ $(t < \bar{t})$, $h'(t) > 0$ $(t > \bar{t})$ and $h'(\bar{t}) = 0$. Thus the proof in the case (ii) and (iv) is complete. □

Remark 2.2 *For α_o in above theorem we have also the following estimation: $\alpha_o g'(M) \le \mu_f \le \alpha_o g'(m)$ if g is strictly concave differentiable function or $\alpha_o g'(m) \le \mu_f \le \alpha_o g'(M)$ if g is strictly convex differentiable.*

If we put $g = f$ in Theorem 2.9, then we have the following corollary.

Corollary 2.10 *Let A_j, Φ_j, ω_j, $j = 1,\dots,k$ be as in Theorem 2.9. Let $f \in \mathscr{C}([m,M])$ be a strictly convex twice differentiable function on $[m,M]$. Suppose that either of the following conditions holds*

(i) $f(t) > 0$ for all $t \in [m,M]$,

(ii) $f(t) < 0$ for all $t \in [m,M]$.

Then

$$\sum_{j=1}^{k} \omega_j \Phi_j\left(f(A_j)\right) \le \alpha_o f\left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j)\right),$$

where the boundary α_o *satisfies the conditions* $\alpha_o > 1$ *in the case (i) and* $1 > \alpha_o > 0$ *in the case (ii).*

More precisely the value α_o *is given by*

$$\alpha_o = \frac{\mu_f t_o + \nu_f}{f(t_o)}$$

for the unique solution t_o *of the equation* $\mu_f f(t) = f'(t)(\mu_f t + \nu_f)$.

Proof. Suppose that (i). Then we have

$$\sum_{j=1}^{k} \omega_j \Phi_j\left(f(A_j)\right) \le \max_{m \le t \le M} h(t,m,M,f)\, f\left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j)\right),$$

where

$$h(t) \equiv h(t,m,M,f) = \frac{\mu_f t + \nu_f}{f(t)}.$$

Now, $h'(t) = H(t)/f(t)^2$, where

$$H(t) = \mu_f f(t) - (\mu_f t + \nu_f) f'(t).$$

Since $f(m) > 0$ and $f(M) > 0$, we have $\mu_f t + \nu_f = \frac{f(m)(M-t)+f(M)(t-m)}{M-m} > 0$ for all $t \in [m,M]$. Since $f(t)$ is a strictly convex twice differentiable function on $[m,M]$, i.e. $f''(t) > 0$, it follows that $H'(t) = -(\mu_f t + \nu_f) f''(t) < 0$, so that $H(t)$ is a decreasing function on $[m,M]$. Since $f'(m) \le \mu_f \le f'(M)$, the condition $H(m)H(M) < 0$ automatically holds. Therefore the equation $H(t) = 0$ has exactly one solution $t_o \in [m,M]$ and hence the maximum value of $h(t)$ is attained for $t = t_o$.

Suppose that (ii). Then we have this corollary by replacing g by f in Theorem 2.9. □

For the sake of convenience, we prepare some notations. Let $f(t)$ be a real valued continuous function on an interval $[m,M]$. We introduce the following constants:

$$K(m,M,f,q) = \frac{mf(M) - Mf(m)}{(q-1)(M-m)} \left(\frac{(q-1)(f(M)-f(m))}{q(mf(M)-Mf(m))}\right)^q \tag{2.19}$$

where q is a real number.

In particular, if we put $f(t) = t^p$, then

$$K(m,M,t^p,q) = \frac{mM^p - Mm^p}{(q-1)(M-m)} \left(\frac{(q-1)(M^p - m^p)}{q(mM^p - Mm^p)}\right)^q. \tag{2.20}$$

Moreover, if we put $q = p$, then

$$K(m,M,t^p,p) = \frac{mM^p - Mm^p}{(p-1)(M-m)} \left(\frac{(p-1)(M^p - m^p)}{p(mM^p - Mm^p)} \right)^p. \tag{2.21}$$

The constant $K(m,M,t^p,q)$ (resp. $K(m,M,t^p,p)$) is denoted simply by $K(m,M,p,q)$ (resp. $K(m,M,p)$).

Corollary 2.11 *Let $A_j \in \mathscr{B}^+(H)$ be positive operators with $\mathsf{Sp}(A_j) \subseteq [m,M]$ for some scalars $0 < m < M$, $\Phi_j \in \mathbf{P}_N[\mathscr{B}(H), \mathscr{B}(K)]$ normalized positive linear maps $(j = 1,\ldots,k)$. Let $\omega_1,\cdots,\omega_k \in \mathbb{R}_+$ be any finite number of positive real numbers such that $\sum_{j=1}^k \omega_j = 1$.*

If $f \in \mathscr{C}([m,M])$ is convex, then

$$\sum_{j=1}^k \omega_j \Phi_j\left(f(A_j)\right) \leq \alpha_1 \left(\sum_{j=1}^k \omega_j \Phi_j(A_j) \right)^q \tag{2.22}$$

holds for

$$\alpha_1 = \begin{cases} K(m,M,f,q) & \text{if} \quad \frac{f(m)}{m} q < \mu_f < \frac{f(M)}{M} q \text{ and } \quad \mu_f(q-1) > 0, \\ \max\left\{ \frac{f(m)}{m^q}, \frac{f(M)}{M^q} \right\} & \text{otherwise.} \end{cases}$$

If $f \in \mathscr{C}([m,M])$ is concave, then

$$\sum_{j=1}^k \omega_j \Phi_j\left(f(A_j)\right) \geq \alpha_1 \left(\sum_{j=1}^k \omega_j \Phi_j(A_j) \right)^q \tag{2.23}$$

holds for

$$\alpha_1 = \begin{cases} K(m,M,f,q) & \text{if} \quad \frac{f(m)}{m} q > \mu_f > \frac{f(M)}{M} q \text{ and } \quad \mu_f(q-1) < 0, \\ \min\left\{ \frac{f(m)}{m^q}, \frac{f(m)}{m^q} \right\} & \text{otherwise.} \end{cases}$$

In particular, if $p \in \mathbb{R}\backslash[0,1)$, then

$$\sum_{j=1}^k \omega_j \Phi_j(A_j^p) \leq \alpha_2 \left(\sum_{j=1}^k \omega_j \Phi_j(A_j) \right)^q \tag{2.24}$$

holds for

$$\alpha_2 = \begin{cases} K(m,M,p,q) & \text{if} \quad qm^{p-1} < \mu_{t^p} < qM^{p-1} \text{ and } \quad q \in \mathbb{R}\backslash[0,1),\, pq > 0, \\ \max\{m^{p-q}, M^{p-q}\} & \text{otherwise,} \end{cases}$$

and if $p \in (0,1]$, then

$$\sum_{j=1}^k \omega_j \Phi_j(A_j^p) \geq \alpha_2 \left(\sum_{j=1}^k \omega_j \Phi_j(A_j) \right)^q \tag{2.25}$$

holds for

$$\alpha_2 = \begin{cases} K(m,M,p,q) & \text{if} \quad qM^{p-1} < \mu_{t^p} < qm^{p-1} \ \text{and} \quad q \in (0,1], \\ \min\{m^{p-q}, M^{p-q}\} & \text{otherwise.} \end{cases}$$

Proof. We only prove the case where f is convex. If we put $h(t) = \frac{\mu_f t + \nu_f}{t^q}$, then we have $h'(t) = \frac{(1-q)\mu_f t - q\nu_f}{t^{q+1}}$ and $h'(t_o) = 0$ if and only if $t_o = \frac{q}{1-q}\frac{\nu_f}{\mu_f}$. Suppose that $0 < m \leq t_0 \leq M$ and $\mu_f(q-1) > 0$. We remark that the condition $0 < m \leq t_0 \leq M$ is equivalent to $\frac{f(m)}{m}q < \mu_f < \frac{f(M)}{M}q$. Then we have $h''(t_0) = \frac{q\nu_f}{t_0^{q+2}} < 0$ and hence the maximum value of $h(t)$ is attained for $t = t_o$. If $t_0 < m$ or $M < t_0$, then $h(t)$ is monotone on $[m,M]$ and its extreme occurs at m or M. If $\mu_f(q-1) < 0$, then $\mu_f(1-q) > 0$ and hence $h'(t)$ is increasing on $[m,M]$. Therefore its extreme occurs at m or M. If $\mu_f = 0$ or $q = 1$, then $h(t)$ is evidently nonincreasing or nondecreasing on $[m,M]$.

Next, we show (2.24). Suppose that $p \in \mathbb{R}\backslash[0,1)$. Note that the condition $\mu_{t^p}(q-1) > 0$ is equivalent to $p,q > 1$ or $p,q < 0$ since we have $t_0 < 0$ for $0 < q < 1$. Therefore, if we put $f(t) = t^p$ in (2.22), then we have (2.24). □

If we put $p = q$ in inequalities (2.24) and (2.25), then since $pm^{p-1} < \mu_{t^p} < pM^{p-1}$, we have the following corollary.

Corollary 2.12 *Let A_j, Φ_j, ω_j, $j = 1,\dots,k$ be as in Corollary 2.11. If $p \in \mathbb{R}\backslash[0,1)$, then*

$$\sum_{j=1}^{k} \omega_j \Phi_j(A_j^p) \leq K(m,M,p) \left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j) \right)^p \tag{2.26}$$

and if $p \in (0,1]$, then

$$\sum_{j=1}^{k} \omega_j \Phi_j(A_j^p) \geq K(m,M,p) \left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j) \right)^p. \tag{2.27}$$

Corollary 2.13 *Let A_j, Φ_j, ω_j, $j = 1,\dots,k$ be as in Corollary 2.11 and $p \in \mathbb{R}$. Then*

$$\sum_{j=1}^{k} \omega_j \Phi_j(A_j^{-1}) \leq \alpha_1 \left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j) \right)^{-p} \tag{2.28}$$

holds for

$$\alpha_1 = \begin{cases} \frac{p^p}{(p+1)^{p+1}} \frac{(m+M)^{p+1}}{mM} & \text{if} \quad \frac{m}{M} < p < \frac{M}{m} \quad \text{and} \quad p > 0, \\ \max\{m^{p-1}, M^{p-1}\} & \text{otherwise} \end{cases}$$

and

$$\sum_{j=1}^{k} \omega_j \Phi_j(A_j^2) \leq \alpha_2 \left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j) \right)^{p+1} \tag{2.29}$$

holds for

$$\alpha_2 = \begin{cases} \frac{p^p}{(p+1)^{p+1}} \frac{(m+M)^{p+1}}{(mM)^p}, & \text{if} \quad \frac{m}{M} < p < \frac{M}{m} \quad \text{and} \quad p > 0, \\ \max\{m^{1-p}, M^{1-p}\} & \text{otherwise.} \end{cases}$$

Proof. The inequality (2.28) follows from (2.24) in Corollary 2.11 if we put $f(t) = t^{-1}$ and replace q by $-p$. The inequality (2.29) follows from (2.24) if we put $f(t) = t^2$ and replace q by $p+1$ for $p > 0$. □

Remark 2.3 *If we put $p = 1$ in Corollary 2.13, then we have a variant of Kantorovich inequality:*

(i) $\sum_{j=1}^k \omega_j \Phi_j(A_j^{-1}) \leq \frac{(M+m)^2}{4Mm} \left(\sum_{j=1}^k \omega_j \Phi_j(A_j)\right)^{-1}$.

(ii) $\sum_{j=1}^k \omega_j \Phi_j(A_j^2) \leq \frac{(M+m)^2}{4Mm} \left(\sum_{j=1}^k \omega_j \Phi_j(A_j)\right)^{2}$.

In the next corollary we show ratio type reverse inequalities to Jensen's type inequalities for the exponential function analogous to the inequality (2.13).

Corollary 2.14 *Let A_j, Φ_j, ω_j, $j = 1, \dots, k$ be as in Theorem 2.9 and $\lambda \in \mathbb{R}\backslash\{0\}$.*
If $f \in \mathscr{C}([m,M])$ is convex, then

$$\sum_{j=1}^k \omega_j \Phi_j\left(f(A_j)\right) \leq \alpha_1 \exp\left(\lambda \sum_{j=1}^k \omega_j \Phi_j(A_j)\right) \tag{2.30}$$

holds for

$$\alpha_1 = \begin{cases} \frac{\mu_f}{\lambda e} \exp\left(\frac{\lambda \nu_f}{\mu_f}\right), & \text{if} \quad m < \frac{\lambda}{1-\lambda}\frac{\nu_f}{\mu_f} < M \quad \text{and} \quad \mu_f(\lambda - 1) > 0, \\ \max\left\{\frac{f(m)}{e^{\lambda m}}, \frac{f(M)}{e^{\lambda M}}\right\} & \text{otherwise.} \end{cases}$$

If f is concave, then we obtain opposite inequality with dual value of the constant α_1.
Additionally, if $\beta \in \mathbb{R}$ is such that $\lambda\beta > 0$, then

$$\sum_{j=1}^k \omega_j \Phi_j\left(\exp(\beta A_j)\right) \leq \alpha_2 \exp\left(\lambda \sum_{j=1}^k \omega_j \Phi_j(A_j)\right) \tag{2.31}$$

holds for

$$\alpha_2 = \begin{cases} \frac{\bar{\mu}}{\lambda e} \exp\left(\frac{\lambda \bar{\nu}}{\mu}\right) & \text{if} \quad \lambda e^{\beta m} < \bar{\mu} < \lambda e^{\beta M}, \\ \max\left\{e^{(\beta-\lambda)m}, e^{(\beta-\lambda)M}\right\} & \text{otherwise,} \end{cases}$$

where

$$\bar{\mu} = \frac{e^{\beta M} - e^{\beta m}}{M - m} \qquad \text{and} \qquad \bar{\nu} = \frac{M e^{\beta m} - m e^{\beta M}}{M - m}.$$

In particular, if $A \in \mathscr{B}_h(H)$ is a selfadjoint operator with $\mathrm{Sp}(A) \subseteq [m,M]$, then

$$\Phi\left(e^A\right) \leq \left\{\frac{e^M - e^m}{e(M-m)} \exp\left(\frac{Me^m - me^M}{e^M - e^m}\right)\right\} e^{\Phi(A)}. \tag{2.32}$$

Proof. We can prove this corollary by a similar method as Corollary 2.11. □

2.3 Difference type reverse inequalities

In this section, as applications of our general theorem (Theorem 2.3), we show difference type reverse inequalities to Jensen's inequalities and give the explicit expressions in the estimations of the difference.

If we put $\alpha = 1$ in Theorem 2.4 then we obtain the following difference type reverse inequalities as a complementary inequality to Jensen's inequality.

Corollary 2.15 *Let $A_j \in \mathscr{B}_h(H)$ be selfadjoint operators with $\mathsf{Sp}(A_j) \subseteq [m,M]$ for some scalars $m < M$ and $\Phi_j \in \mathbf{P}_N[\mathscr{B}(H),\mathscr{B}(K)]$ normalized positive linear maps $(j = 1,\ldots,k)$. Let $\omega_1,\omega_2,\cdots,\omega_k \in \mathbb{R}_+$ be any finite number of positive real numbers such that $\sum_{j=1}^k \omega_j = 1$. If $f,g \in \mathscr{C}([m,M])$ and f is a convex function on $[m,M]$, then*

$$\sum_{j=1}^{k} \omega_j \Phi_j(f(A_j)) - g\left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j)\right) \leq \beta 1_K, \tag{2.33}$$

where

$$\beta = \max_{m \leq t \leq M} \{\mu_f t + \nu_f - g(t)\} \tag{2.34}$$

and

$$\mu_f = \frac{f(M)-f(m)}{M-m} \quad \text{and} \quad \nu_f = \frac{Mf(m)-mf(M)}{M-m}.$$

Furthermore, if g is a strictly convex differentiable function on $[m,M]$, then the constant β satisfies the following condition:

$$f(m)-g(m) \leq \beta \leq f(m)-g(m)+(\mu_f - g'(m))(M-m).$$

We can determine more precisely the value $\beta \equiv \beta(m,M,f,g)$ in (2.34) as follows:

$$\beta = \mu_f t_0 + \nu_f - g(t_0),$$

where

$$t_o = \begin{cases} \left.\begin{array}{c}\text{the solution of}\\ g'(t) = \mu_f\end{array}\right\} & \text{if} \quad g'(m) \leq \mu_f \leq g'(M), \\ M & \text{if} \quad g'(M) < \mu_f, \\ m & \text{if} \quad \mu_f < g'(m). \end{cases}$$

Proof. If we put $F(u,v) = u - v$ in Theorem 2.3, then we have

$$\sum_{j=1}^{k} \omega_j \Phi_j(f(A_j)) - g\left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j)\right) \leq \max_{m \leq t \leq M} h(t;m,M,f,g) 1_K,$$

where

$$h(t) \equiv h(t;m,M,f,g) = \mu_f t + \nu_f - g(t).$$

Since $g(t)$ is a differentiable function and $g'(t)$ is strictly increasing on $[m,M]$, it follows that $h'(t) = \mu_f - g'(t)$ is strictly decreasing on $[m,M]$. If $g'(m) \le \mu_f \le g'(M)$, then the equation $g'(t) - \mu_f = 0$ has exactly one solution $t_0 \in [m,M]$ and the maximum value of $h(t)$ is attained for $t = t_0$. If $\mu_f < g'(m)$, then we have $h'(t) < 0$ on $[m,M]$ because $h'(t)$ is a decreasing function. Therefore $h(t)$ is a decreasing function and hence the maximum value of $h(t)$ is attained for $t_0 = m$. Similarly, we have $t_0 = M$ if $\mu_f > g'(M)$.

Next, since $g(t)$ is a strictly convex function, it follows that

$$g(m) - g(t_0) \le g'(m)(m - t_0) \quad \text{if} \quad m \le t_0 \le M$$

Then we have

$$\begin{aligned} \beta &= \mu_f t_0 + \nu_f - g(t_0) = f(m) - g(m) + (g(m) - g(t_0) + \mu_f(t_0 - m)) \\ &\le f(m) - g(m) + (-g'(m) + \mu_f)(t_0 - m) \\ &\le f(m) - g(m) + (-g'(m) + \mu_f)(M - m). \end{aligned}$$

Hence we have the upper boundary for β. The lower boundary is evident. $\square$

If $g \equiv f$ in Corollary 2.15, then we have the following corollary.

Corollary 2.16 *Let A_j, Φ_j, ω_j, $j = 1, \ldots, k$ be as in Corollary 2.15. If $f \in \mathscr{C}([m,M])$ is a strictly convex differentiable convex function on $[m,M]$, then*

$$\sum_{j=1}^{k} \omega_j \Phi_j(f(A_j)) - f\left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j)\right) \le \beta 1_K, \tag{2.35}$$

where

$$\beta = \max_{m \le t \le M} \{\mu_f t + \nu_f - f(t)\}. \tag{2.36}$$

The constant β satisfies the condition $0 < \beta < (M - m)(\mu_f - f'(m))$.

More precisely, the constant β for (2.36) may be determined as follows: let $t = t_0$ be the unique solution in $[m,M]$ of the equation $f'(t) = \mu_f$. Then $\beta = \mu_f t_0 + \nu_f - f(t_0)$.

For the sake of convenience, we prepare some notations. Let $f(t)$ be a real valued continuous function on an interval $[m,M]$. We introduce the following constants:

$$C(m,M,f,q) = \frac{Mf(m) - mf(M)}{M - m} + (q-1)\left(\frac{f(M) - f(m)}{q(M-m)}\right)^{\frac{q}{q-1}}, \tag{2.37}$$

where q is a real number.

In particular, if $f(t) = t^p$, then

$$C(m,M,t^p,q) = \frac{Mm^p - mM^p}{M - m} + (q-1)\left(\frac{M^p - m^p}{q(M-m)}\right)^{\frac{q}{q-1}}. \tag{2.38}$$

Moreover, if we put $q = p$, then

$$C(m,M,t^p,p) = \frac{Mm^p - mM^p}{M-m} + (p-1)\left(\frac{M^p - m^p}{p(M-m)}\right)^{\frac{p}{p-1}}. \tag{2.39}$$

The constant $C(m,M,t^p,q)$ (resp. $C(m,M,t^p,p)$) is denoted simply by $C(m,M,p,q)$ (resp. $C(m,M,p)$).

If we put $\alpha = 1$ in Corollary 2.6, then we have the following corollary.

Corollary 2.17 *Let $A_j \in \mathscr{B}^+(H)$ be positive operators with $\mathsf{Sp}(A_j) \subseteq [m,M]$ for some scalars $0 < m < M$, $\Phi_j \in \mathbf{P}_N[\mathscr{B}(H), \mathscr{B}(K)]$ normalized positive linear maps ($j = 1,\dots,k$). Let $\omega_1,\cdots,\omega_k \in \mathbb{R}_+$ be any finite number of positive real numbers such that $\sum_{j=1}^k \omega_j = 1$. Let $f \in \mathscr{C}([m,M])$ and $q \in \mathbb{R}$.*

If f is convex, then

$$\sum_{j=1}^k \omega_j \Phi_j(f(A_j)) - \left(\sum_{j=1}^k \omega_j \Phi_j(A_j)\right)^q \le \beta 1_K, \tag{2.40}$$

where

$$\beta = \begin{cases} C(m,M,f,q) & \text{if } qm^{q-1} \le \mu_f \le qM^{q-1} \text{ and } q(q-1) > 0, \\ \max\{f(m) - m^q, f(M) - M^q\} & \text{otherwise.} \end{cases}$$

If f is concave, then

$$\sum_{j=1}^k \omega_j \Phi_j(f(A_j)) - \left(\sum_{j=1}^k \omega_j \Phi_j(A_j)\right)^q \ge \beta 1_K, \tag{2.41}$$

where

$$\beta = \begin{cases} C(m,M,f,q) & \text{if } qm^{q-1} \le \mu_f \le qM^{q-1} \text{ and } q(q-1) < 0, \\ \min\{f(m) - m^q, f(M) - M^q\} & \text{otherwise,} \end{cases}$$

and $C(m,M,f,g)$ is defined as (2.37).

If we put $f(t) = t^p$ and $q = p$ in Corollary 2.17, then the conditions in Corollary 2.17 automatically satisfies because $m < (\frac{1}{p}\mu_{t^p})^{1/p-1} < M$ holds. Therefore we have the following corollary.

Corollary 2.18 *Let A_j, Φ_j, ω_j, $j = 1,\dots,k$ be as in Corollary 2.17.*

If $p < 0$ or $p \ge 1$, then

$$\sum_{j=1}^k \omega_j \Phi_j\left(A_j^p\right) \le \left(\sum_{j=1}^k \omega_j \Phi_j(A_j)\right)^p + C(m,M,p)\, 1_K.$$

If $0 < p \le 1$, then

$$\sum_{j=1}^k \omega_j \Phi_j\left(A_j^p\right) \ge \left(\sum_{j=1}^k \omega_j \Phi_j(A_j)\right)^p + C(m,M,p)\, 1_K,$$

where $C(m,M,p)$ is defined as (2.39).

Moreover, we show some deformations of the Kantorovich inequality by Corollary 2.17.

Corollary 2.19 *Let $A_j, \Phi_j, \omega_j,\ j = 1,\ldots,k$ be as in Corollary 2.17 and $p \in \mathbb{R}$. Then*

$$\sum_{j=1}^{k} \omega_j \Phi_j(A_j^{-1}) - \left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j)\right)^{-p} \le \beta_1\, 1_K \tag{2.42}$$

where

$$\beta_1 = \begin{cases} \frac{M+m}{mM} - \frac{p+1}{(pMm)^{p/(p+1)}} & \text{if } \frac{m^p}{M} < p < \frac{M^p}{m} \text{ and } p(p+1) > 0, \\ \max\left\{\frac{1}{m} - \frac{1}{m^p}, \frac{1}{M} - \frac{1}{M^p}\right\} & \text{otherwise,} \end{cases}$$

and

$$\sum_{j=1}^{k} \omega_j \Phi_j(A_j^{2}) - \left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j)\right)^{p+1} \le \beta_2\, 1_K \tag{2.43}$$

where

$$\beta_2 = \begin{cases} p\left(\frac{M+m}{p+1}\right)^{(p+1)/p} - mM & \text{if } \frac{M+m}{M^p} < p+1 < \frac{M+m}{m^p} \text{ and } p(p+1) > 0, \\ \max\{m^2 - m^{1+p}, M^2 - M^{1+p}\} & \text{otherwise.} \end{cases}$$

Additionally, if $f \in \mathscr{C}([m,M])$ is convex and $\alpha \in \mathbb{R}$, then

$$\sum_{j=1}^{k} \omega_j \Phi_j(f(A_j)) + \alpha \sum_{j=1}^{k} \omega_j \Phi_j(A_j) \le \beta_3\, 1_K \tag{2.44}$$

holds for

$$\beta_3 = \begin{cases} \nu_f & \text{if } \alpha = -\mu_f, \\ \max\{f(m) + \alpha m, f(M) + \alpha M\} & \text{otherwise.} \end{cases}$$

Proof. The inequality (2.42) follows from (2.40) in Corollary 2.17 if we put $f(t) = t^{-1}$ and $q = -p$. The inequality (2.43) follows from (2.40) if we put $f(t) = t^2$ and $q = p+1$ in Corollary 2.17. The inequality (2.44) follows from Theorem 2.4, if we put $g(t) = -\alpha t$. The constant β_3 is the maximum value on $[m,M]$ of the linear function $h(t) = (\alpha + \mu_f)t + \nu_f$. □

Remark 2.4 *If we put $p = 1$ in (2.42) and in (2.43), then we have a variant of the Kantorovich inequality (Theorem 1.32) :*

(i) $\sum_{j=1}^{k} \omega_j \Phi_j(A_j^{-1}) - \left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j)\right)^{-1} \le \frac{(\sqrt{M}-\sqrt{m})^2}{Mm} 1_K,$

(ii) $\sum_{j=1}^{k} \omega_j \Phi_j(A_j^{2}) - \left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j)\right)^{2} \le \frac{(M-m)^2}{4} 1_K.$

Corollary 2.20 *Let A_j, Φ_j, ω_j, $j = 1,\dots,k$ be as in Corollary 2.17, $f \in \mathscr{C}([m,M])$ be convex, $f > 0$ on $[m,M]$ and $q \in \mathbb{R}$. Suppose that either of the following conditions holds: (i) $f(m) > f(M)$ if $q < 0$ or (ii) $f(m) < f(M)$ if $0 < q \le 1$. Let $f_{\min} = \min_{m\le t\le M} f(t)$, $f_{\max} = \max_{m\le t\le M} f(t)$. Then the following inequality*

$$\sum_{j=1}^{k} \omega_j \Phi_j(A_j) - \left(\sum_{j=1}^{k} \omega_j \Phi_j(f(A_j)) \right)^q \le \beta_1 1_K \tag{2.45}$$

holds for

$$\begin{aligned}\beta_1 &= \begin{cases} -\frac{\nu_f}{\mu_f} + (q-1)(\mu_f q)^{q/(1-q)} & \text{if } \ \frac{1}{q} f_{\max}^{1-q} < \mu_f < \frac{1}{q} f_{\min}^{1-q}, \\ -\frac{\nu_f}{\mu_f} + \frac{1}{\mu_f} f_{\min} - f_{\min}^q & \text{if } \ \mu_f \ge \frac{1}{q} f_{\min}^{1-q}, \\ -\frac{\nu_f}{\mu_f} + \frac{1}{\mu_f} f_{\max} - f_{\max}^q & \text{if } \ \mu_f \le \frac{1}{q} f_{\max}^{1-q}, \end{cases} \\ &\le -\frac{\nu_f}{\mu_f} + (q-1)(\mu_f q)^{q/(1-q)},\end{aligned}$$

in the case (i), and the following inequality

$$\left(\sum_{j=1}^{k} \omega_j \Phi_j(f(A_j)) \right)^q - \sum_{j=1}^{k} \omega_j \Phi_j(A_j) \le \beta_2 1_K \tag{2.46}$$

holds for

$$\begin{aligned}\beta_2 &= \begin{cases} \frac{\nu_f}{\mu_f} + (1-q)(\mu_f q)^{q/(1-q)} & \text{if } \ \frac{1}{q} f_{\min}^{1-q} < \mu_f < \frac{1}{q} f_{\max}^{1-q}, \\ \frac{\nu_f}{\mu_f} - \frac{1}{\mu_f} f_{\min} + f_{\min}^q & \text{if } \ \mu_f \le \frac{1}{q} f_{\min}^{1-q}, \\ \frac{\nu_f}{\mu_f} - \frac{1}{\mu_f} f_{\max} + f_{\max}^q & \text{if } \ \mu_f \ge \frac{1}{q} f_{\max}^{1-q}, \end{cases} \\ &\le \frac{\nu_f}{\mu_f} + (1-q)(\mu_f q)^{q/(1-q)},\end{aligned}$$

in the case (ii).
In particular, if $p > 0$, then

$$\sum_{j=1}^{k} \omega_j \Phi_j(A_j) - \left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j^{-1}) \right)^{-p} \le \beta_3 1_K \tag{2.47}$$

holds for

$$\begin{aligned}\beta_3 &= \begin{cases} M + m - (1+p)\left(\frac{mM}{p}\right)^{p/(1+p)} & \text{if } \ \frac{m}{M^p} < p < \frac{M}{m^p}, \\ M - M^p & \text{if } \ p \le \frac{m}{M^p}, \\ m - m^p & \text{if } \ p \ge \frac{M}{m^p}, \end{cases} \\ &\le M + m - (1+p)\left(\frac{mM}{p}\right)^{p/(1+p)},\end{aligned}$$

and if $p > 1$, *then*

$$\left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j^2)\right)^{1/p} - \sum_{j=1}^{k} \omega_j \Phi_j(A_j) \le \beta_4 1_K \tag{2.48}$$

holds for

$$\beta_4 = \begin{cases} -\frac{Mm}{M+m} + \frac{p-1}{p}\left(\frac{M+m}{p}\right)^{1/(p-1)} & \text{if } \ m^2 < \left(\frac{M+m}{p}\right)^{p/(p-1)} < M^2 \\ m^{2/p} - m \quad \text{if } \ \left(\frac{M+m}{p}\right)^{p/(p-1)} \le m^2, \\ M^{2/p} - M \quad \text{if } \ \left(\frac{M+m}{p}\right)^{p/(p-1)} \ge M^2, \end{cases}$$
$$\le -\frac{Mm}{M+m} + \frac{p-1}{p}\left(\frac{M+m}{p}\right)^{1/(p-1)}.$$

Proof. Let (i) be satisfied. Since $\mu_f < 0$ and $\sum_{j=1}^{k} \omega_j \Phi_j(f(A_j)) \le \mu_f \sum_{j=1}^{k} \omega_j \Phi_j(A_j) + \nu_f 1_K$, we have

$$\sum_{j=1}^{k} \omega_j \Phi_j(A_j) \le -\frac{\nu_f}{\mu_f} 1_K + \frac{1}{\mu_f} \sum_{j=1}^{k} \omega_j \Phi_j(f(A_j)),$$

and hence

$$\begin{aligned} &\sum_{j=1}^{k} \omega_j \Phi_j(A_j) - \left(\sum_{j=1}^{k} \omega_j \Phi_j(f(A_j))\right)^q \\ &\quad \le -\frac{\nu_f}{\mu_f} 1_K + \frac{1}{\mu_f} \sum_{j=1}^{k} \omega_j \Phi_j(f(A_j)) - \left(\sum_{j=1}^{k} \omega_j \Phi_j(f(A_j))\right)^q. \end{aligned} \tag{2.49}$$

Because $0 < f_{\min} \le f(t) \le f_{\max}$ holds, then the operator calculus give $f_{\min} 1_H \le f(A_j) \le f_{\max} 1_H$, $j = 1, \dots, k$. It follows $f_{\min} 1_K \le \Phi_j(f(A_j)) \le f_{\max} 1_K$, $j = 1, \dots, k$. Multiplying with ω_j and summing, we have $\mathsf{Sp}\left(\sum_{j=1}^{k} \omega_j \Phi_j(f(A_j))\right) \subseteq [f_{\min}, f_{\max}]$. From (2.49) it follows that

$$\sum_{j=1}^{k} \omega_j \Phi_j(A_j) - \left(\sum_{j=1}^{k} \omega_j \Phi_j(f(A_j))\right)^q \le \beta_1 1_K$$

where $\beta_1 = \max_{t \in [f_{\min}, f_{\max}]} \left\{-\frac{\nu_f}{\mu_f} + \frac{1}{\mu_f} t - t^q\right\}$. Further we obtain the bound β_1 by common differential calculus.

We prove the case (ii) in the same way. We obtain the inequality (2.47) if we put $f(t) = t^{-1}$ in (2.45) and replace q by $-p$, and the inequality (2.48) we obtain if we put $f(t) = t^2$ in (2.46) and replace q by $\frac{1}{p}$. □

Corollary 2.21 *Let* A_j, Φ_j, ω_j, $j = 1, \dots, k$ *be as in Corollary 2.15,* $f \in \mathscr{C}([m, M])$ *be convex. Suppose that either of the following conditions holds:*

(i) $f(m) < f(M)$ *and* $\alpha > 0$ *is a positive real number.*

(ii) $f(m) > f(M)$ *and* $\alpha < 0$ *is a negative real number.*

Then

$$\sum_{j=1}^{k} \omega_j \Phi_j(f(A_j)) - \exp\left(\sum_{j=1}^{k} \alpha\omega_j \Phi_j(A_j)\right) \le \beta_1 1_K, \tag{2.50}$$

where

$$\beta = \begin{cases} \frac{Mf(m)-mf(M)}{M-m} + \frac{\mu_f}{\alpha}\left(\log\frac{\mu_f}{\alpha} - 1\right) & \textit{if} \quad \alpha e^{\alpha m} \le \mu_f \le \alpha e^{\alpha M} \\ \max\{f(m) - e^{\alpha m}, f(M) - e^{\alpha M}\} & \textit{otherwise.} \end{cases}$$

Corollary 2.22 *Let* A_j, Φ_j, ω_j, $j = 1,\dots,k$ *be as in Corollary 2.15. Then for any real number* $\alpha \neq 0$

$$\sum_{j=1}^{k} \omega_j \Phi_j(e^{\alpha A_j}) - \exp\left(\sum_{j=1}^{k} \alpha\omega_j \Phi_j(A_j)\right) \le \left(\frac{Me^{\alpha m} - me^{\alpha M}}{M-m} + \frac{e^{\alpha M} - e^{\alpha m}}{\alpha(M-m)} \log\left(\frac{e^{\alpha M} - e^{\alpha m}}{\alpha e(M-m)}\right)\right) 1_K.$$

In particular, if $A \in \mathscr{B}_h(H)$ *is a selfadjoint operator with* $\mathsf{Sp}(A) \subseteq [m,M]$*, then*

$$\Phi(e^A) - e^{\Phi(A)} \le \left(\frac{Me^m - me^M}{M-m} + \frac{e^M - e^m}{M-m} \log\left(\frac{e^M - e^m}{e(M-m)}\right)\right) 1_K. \tag{2.51}$$

2.4 A generalization of Ky Fan type inequalities

In this section we choose the map $\Phi_j : \mathscr{B}(H) \to \mathscr{B}(K)$ as follows: Let $x_1,\dots,x_k$ be any finite number of vectors in a Hilbert space H such that $\sum_{j=1}^{k} \|x_j\|^2 = 1$. For every $A_j \in \mathscr{B}(H)$, $j = 1,\dots,k$, define $\Phi_j(A_j) = (A_j x_j, x_j)/(x_j, x_j)$. Then $\Phi_j : \mathscr{B}(H) \to \mathbb{R}$ is a normalized positive linear functional. Further, if we get $\omega_j \in \mathbb{R}_+$ such that $\omega_j = (x_j, x_j)$, $j = 1,\dots,k$, then we have $\sum_{j=1}^{k} \omega_j \Phi_j(A_j) = \sum_{j=1}^{k} (A_j x_j, x_j)$, so all statements from Sections 2.1 and 2.3 hold, when we replace $\sum_{j=1}^{k} \omega_j \Phi_j(A_j)$ by $\sum_{j=1}^{k} (A_j x_j, x_j)$.

Because, in this case $\Phi_j(A_j) \in \mathbb{R}$, then in statements before where we demanded functions be operator convex or monotone, now we request some weaker conditions, i.e. we demand (real) convex or monotone functions. Additionally, in this case we consider necessary and sufficient conditions at which the equality holds. We cite more interesting results. All of results from this section and many another results, besides the latest theorem, were proved directly in [136].

First we restate the Jensen's inequality for a convex function and its opposite inequality.

Lemma 2.23 *Let $A_j \in \mathscr{B}_h(H)$ be selfadjoint operators with $\mathsf{Sp}(A_j) \subseteq [m,M]$ ($j = 1,\dots,k$) for some scalars $m < M$. Let $x_1, x_2, \cdots, x_k \in H$ be any finite number of vectors such that $\sum_{j=1}^k \|x_j\|^2 = 1$. If $f \in \mathscr{C}([m,M])$ is a convex function on $[m,M]$, then*

$$
\begin{aligned}
f\left(\sum_{j=1}^{k}(A_jx_j,x_j)\right) &\leq \sum_{j=1}^{k}(f(A_j)x_j,x_j) \\
&\leq \frac{f(M)-f(m)}{M-m}\left(\sum_{j=1}^{k}(A_jx_j,x_j)-m\right)+f(m).
\end{aligned}
\tag{2.52}
$$

Proof. The first inequality follows from Theorem 1.3 and the second inequality follows from the convexity of f. In fact, since $f(t) \leq \frac{f(M)-f(m)}{M-m}(t-m)+f(m)$ for all $t \in [m,M]$, we have $f(A_j) \leq \frac{f(M)-f(m)}{M-m}(A_j - m_j 1_H) + f(m)1_H$ for all $j = 1,\cdots,k$. Therefore we have the desired inequality. □

The following theorem is a consequence of the main Theorem 2.3.

Theorem 2.24 *Let $A_j \in \mathscr{B}_h(H)$ be selfadjoint operators with $\mathsf{Sp}(A_j) \subseteq [m,M]$ for some scalars $m < M$ ($j = 1,\dots,k$), and $x_1, x_2, \cdots x_k \in H$ any finite number of vectors such that $\sum_{j=1}^k \|x_j\|^2 = 1$. Let $f, g \in \mathscr{C}([m,M])$ and $F(u,v)$ be a real valued function defined on $U \times V$, where $U \supset f[m,M]$, $V \supset g[m,M]$. If $F(u,v)$ is non-decreasing in u and f is a convex function on $[m,M]$, then*

$$
\begin{aligned}
& F\left[\sum_{j=1}^{k}(f(A_j)x_j,x_j), g\left(\sum_{j=1}^{k}(A_jx_j,x_j)\right)\right] \\
&\leq \max_{m\leq t\leq M} F\left[\frac{f(M)-f(m)}{M-m}(t-m)+f(m), g(t)\right] \\
&= \max_{0\leq\theta\leq 1} F\left[\theta f(m)+(1-\theta)f(M), g(\theta m+(1-\theta)M)\right].
\end{aligned}
\tag{2.53}
$$

Proof. The second expression on the right side of (2.53) follows from the change of variable $\theta = \frac{M-t}{M-m}$, so $t = \theta m + (1-\theta)M$ with $0 \leq \theta \leq 1$. □

Remark 2.5 *In Theorem 2.24, if F is non-increasing in u and f is concave, then we obtain the inequality (2.53) again. If F is non-increasing in u and f is convex or if F is non-decreasing in u and f is concave, then we obtain the opposite inequality with dual extreme.*

As an application of Theorem 2.24, we discuss complementary inequalities to Jensen's type inequalities for convex functions, which gives us a unified view to several inequalities due to Ky Fan, Furuta and Mond-Pečaić. Moreover we shall consider the conditions under which the equality holds. For convenience, we define

$$
\mu_k = \frac{k(M)-k(m)}{M-m} \quad \text{and} \quad \nu_k = \frac{Mk(m)-mk(M)}{M-m}
$$

for a real valued function k on an interval $[m,M]$.

Theorem 2.25 *Let A_j, x_j, $j = 1,\dots,k$, as in Theorem 2.24 and $f,g \in \mathscr{C}([m,M])$. If f is convex, then for any real number $\alpha \in \mathbb{R}$*

$$\sum_{j=1}^{k}(f(A_j)x_j,x_j) \le \alpha g\left(\sum_{j=1}^{k}(A_jx_j,x_j)\right)+\beta \tag{2.54}$$

where

$$\beta = \max_{m\le t\le M}\{\mu_f t+\nu_f-\alpha g(t)\}.$$

Further, suppose in addition that the function g satisfies either of the following conditions:

(i) αg is concave

(ii) αg is strictly convex differentiable.

Then we can determine more precisely the value of $\beta \equiv \beta(m,M,f,g,\alpha)$ in (2.25) as follows:

$$\beta = \mu_f t_o+\nu_f-\alpha g(t_o),$$

where

$$t_o = \begin{cases} M \text{ if } & \mu_f \ge \alpha\mu_g, \\ m \text{ if } & \mu_f < \alpha\mu_g, \end{cases} \qquad \text{in the case } (i),$$

or

$$t_o = \begin{cases} g'^{-1}(\mu_f/\alpha) & \text{if } \alpha g'(m) < \mu_f < \alpha g'(M), \\ m & \text{if } \alpha g'(m) \ge \mu_f, \\ M & \text{if } \alpha g'(M) \le \mu_f, \end{cases} \qquad \text{in the case } (ii).$$

Moreover, suppose that $\beta = \mu_f\sum_{j=1}^{k}(A_jx_j,x_j)+\nu_f-\alpha g(\sum_{j=1}^{k}(A_jx_j,x_j))$ for some vectors x_j in H such that $\sum_{j=1}^{k}\|x_j\|^2 = 1$. Then the equality is attained in (2.54) if and only if there exist orthogonal vectors y_j and z_j such that

$$x_j = y_j+z_j, \quad A_jy_j = my_j, \quad A_jz_j = Mz_j. \tag{2.55}$$

Proof. This theorem follows from Theorem 2.4, with exception of conditions under which the equality holds. We investigate this conditions. Put $t_o = \sum_{j=1}^{k}(A_jx_j,x_j)$, then the hypothesis $\mathsf{Sp}(A_j) \subseteq [m,M]$ ensures the inequality $m \le t_o \le M$.

Suppose that the equality $\sum_{j=1}^{k}(f(A_j)x_j,x_j) = \alpha g(t_0)+\beta$ holds. By definition of β in (2.54), notice that the equality $\sum_{j=1}^{k}(f(A_j)x_j,x_j) = \alpha g(t_0)+\beta$ holds if and only if the equality $\sum_{j=1}^{k}(f(A_j)x_j,x_j) = \mu_f t_o+\nu_f$ holds. Let $E_j(t)$ be the spectral resolution of the identity of A_j, that is, $A_j = \int_{m-0}^{M} t\,dE_j(t)$. Put $P_j = E_j(M)-E_j(M-0)$, $Q_j = E_j(M-0)-E_j(m)$ and $R_j = E_j(m)-E_j(m-0)$. Then $(A_jP_jx_j,x_j) = M(P_jx_j,x_j)$ and $(A_jR_jx_j,x_j) = m(R_jx_j,x_j)$. Notice also that

$$\begin{aligned}(f(A_j)P_jx_j,x_j) &= \textstyle\int_{m-0}^{M} f(t)d(E_j(t)P_jx_j,x_j) = f(M)(P_jx_j,x_j) \\ &= ((\mu_f M+\nu_f)P_jx_j,x_j)\end{aligned}$$

and

$$\begin{aligned}(f(A_j)R_jx_j,x_j) &= \int_{m-0}^{M} f(t)d(E_j(t)R_jx_j,x_j) = f(m)(R_jx_j,x_j)\\ &= ((\mu_f m+\nu_f)R_jx_j,x_j).\end{aligned}$$

Since $\sum_{j=1}^{k}(f(A_j)x_j,x_j) = \mu_f t_o+\nu_f$, it follows that $\sum_{j=1}^{k}((\mu_f A_j+\nu_f - f(A_j))Q_jx_j,x_j) = 0$ and hence $Q_jx_j = 0$ for every j because $\mu_f s+\nu_f - f(s) > 0$ for $s \in (m,M)$. Thus we obtain the desired decomposition of x_j setting $y_j = R_jx_j$ and $z_j = P_jx_j$.

Assume conversely (2.55). Then it follows that

$$\begin{aligned}\mu_f t_0+\nu_f &= \mu_f \textstyle\sum_{j=1}^{k}(m\|y_j\|^2+M\|z_j\|^2)+\nu_f \sum_{j=1}^{k}(\|y_j\|^2+\|z_j\|^2)\\ &= (\mu_f m+\nu_f)\textstyle\sum_{j=1}^{k}\|y_j\|^2+(\mu_f M+\nu_f)\sum_{j=1}^{k}\|z_j\|^2\\ &= \textstyle\sum_{j=1}^{k}(f(m)y_j,y_j)+\sum_{j=1}^{k}(f(M)z_j,z_j) = \sum_{j=1}^{k}(f(A_j)x_j,x_j),\end{aligned}$$

which is the desired equality. □

Remark 2.6 *If we put $\alpha = 1$ in Theorem 2.25 and $g(t)$ is a real valued strictly convex twice differentiable function on $[m,M]$, then*

$$\sum_{j=1}^{k}(f(A_j)x_j,x_j) \le g\left(\sum_{j=1}^{k}(A_jx_j,x_j)\right)+\beta \tag{2.56}$$

where $\beta = \mu_f t_0+\nu_f - g(t_0)$ and

$$t_o = \begin{cases} g'^{-1}(\mu_f) & \text{if} \quad g'(m) < \mu_f < g'(M),\\ m & \text{if} \quad g'(m) \ge \mu_f,\\ M & \text{if} \quad g'(M) \le \mu_f.\end{cases}$$

If we choose α such that $\beta = 0$ in Theorem 2.25, then we have the following corollary.

Corollary 2.26 *Let A_j, x_j, $j = 1,\dots,k$, as in Theorem 2.24. Let $f,g \in \mathscr{C}([m,M])$ and suppose that either of the following conditions holds:*

(i) $g(t) > 0$ for all $t \in [m,M]$

(ii) $g(t) < 0$ for all $t \in [m,M]$.

If $f(t)$ is convex, then

$$\sum_{j=1}^{k}(f(A_j)x_j,x_j) \le \alpha_o g\left(\sum_{j=1}^{k}(A_jx_j,x_j)\right) \tag{2.57}$$

holds for

$$\alpha_o = \max_{m\le t\le M}\left\{(\mu_f t+\nu_f)/g(t)\right\} \qquad \text{in the case } (i),$$

$$\text{or}\ \ \alpha_o = \min_{m\le t\le M}\left\{(\mu_f t+\nu_f)/g(t)\right\} \qquad \text{in the case } (ii).$$

Further, suppose that with (i) or (ii) the addition condition holds: (i_+) $f(m) > 0$, $f(M) > 0$ if $g > 0$ or (ii_+) $f(m) < 0$, $f(M) < 0$ if $g < 0$, and additionally let either of following conditions be valid: (iii) g is a strictly concave differentiable or (iv) g is a strictly convex twice differentiable function. Then the value of $\alpha_o \equiv \alpha_o(m,M,f,g)$ in (2.57) we can determine as follows: $\alpha_o = (\mu_f t_o + \nu_f)/g(t_o)$, where

$$t_o = \begin{cases} M \text{ if } & \frac{\mu_f}{\mu_g}\nu_g \geq \nu_f, \\ m \text{ if } & \frac{\mu_f}{\mu_g}\nu_g < \nu_f, \end{cases} \qquad \text{in the case } (iii),$$

or

$$t_o = \begin{cases} \left.\begin{array}{c}\text{the solution of} \\ \mu_f g(t) = \left(\mu_f t + \nu_f\right) g'(t)\end{array}\right\} & \text{if} \quad f(m)\frac{g'(m)}{g(m)} < \mu_f < f(M)\frac{g'(M)}{g(M)}, \\ M & \text{if} \quad \mu_f \geq f(M)\frac{g'(M)}{g(M)}, \\ m & \text{if} \quad \mu_f \leq f(m)\frac{g'(m)}{g(m)}, \end{cases}$$

in the case (iv).

In the dual case we have the opposite inequality with dual extreme, with the dual estimation for α_o and the opposite condition while determining t_o.

If we put $g \equiv f$ in Theorem 2.25, then we have the following theorem:

Theorem 2.27 *Let A_j, x_j, $j = 1,\ldots,k$, as in Theorem 2.24. Let $f \in \mathscr{C}([m,M])$ be a strictly convex twice differentiable function on $[m,M]$. Then for any positive real number $\alpha \in \mathbb{R}_+$*

$$\sum_{j=1}^{k}(f(A_j)x_j,x_j) \leq \alpha f\left(\sum_{j=1}^{k}(A_j x_j,x_j)\right) + \beta, \tag{2.58}$$

where

$$\beta = \mu_f t_o + \nu_f - \alpha f(t_o)$$

and

$$t_o = \begin{cases} f'^{-1}(\mu_f/\alpha) & \text{if} \quad m < f'^{-1}\left(\frac{\mu_f}{\alpha}\right) < M, \\ M & \text{if} \quad M \leq f'^{-1}\left(\frac{\mu_f}{\alpha}\right), \\ m & \text{if} \quad f'^{-1}\left(\frac{\mu_f}{\alpha}\right) \leq m. \end{cases}$$

The equality is attained in (2.58) if and only if there exist orthogonal vectors y_j and z_j in H such that

$$\begin{gathered} x_j = y_j + z_j, \quad A_j y_j = m y_j, \\ A_j z_j = M z_j \quad \text{and} \quad t_o = m\textstyle\sum_{j=1}^{k}\|y_j\|^2 + M\sum_{j=1}^{k}\|z_j\|^2. \end{gathered} \tag{2.59}$$

Proof. Since the graph of $\alpha f(t) + \beta$ touches the line of $f(m) + \mu(t_o - m)$ at the point t_o, it follows that the equality

$$\sum_{j=1}^{k}(f(A_j)x_j,x_j) = \alpha f\left(\sum_{j=1}^{k}(A_j x_j,x_j)\right) + \beta$$

holds if and only if two equalities $t_o = \sum_{j=1}^{k}(A_jx_j,x_j)$ and $\sum_{j=1}^{k}(f(A_j)x_j,x_j) = f(m) + \mu(t_o - m)$ hold. Therefore we obtain Theorem 2.27 by the same proof as Theorem 2.25. □

Remark 2.7 *If we put $\alpha = 1$ in Theorem 2.27 and f is a strictly convex twice differentiable function on $[m,M]$, then*

$$\sum_{j=1}^{k}(f(A_j)x_j,x_j) \leq f\left(\sum_{j=1}^{k}(A_jx_j,x_j)\right) + \beta, \tag{2.60}$$

where

$$\beta = \mu_f t_o + \nu_f - f(t_o)$$

and $t_o \in (m,M)$ is the unique solution of the equation $f'(t) = \mu_f$.

Corollary 2.28 *Let $A_j, x_j, j = 1,\ldots,k$, as in Theorem 2.24. Let $f \in \mathscr{C}([m,M])$ be a strictly convex twice differentiable function on $[m,M]$ and suppose that either of the following conditions holds:*

(i) $f(t) > 0$ for all $t \in [m,M]$.

(ii) $f(t) < 0$ for all $t \in [m,M]$.

Then

$$\sum_{j=1}^{k}(f(A_j)x_j,x_j) \leq \alpha_o f\left(\sum_{j=1}^{k}(A_jx_j,x_j)\right) \tag{2.61}$$

holds for $\alpha_o > 1$ in the case (i), or $0 < \alpha_o < 1$ in the case (ii). More precisely the value α_o may be determined as follows:

$$\alpha_o = \frac{\mu_f t_o + \nu_f}{f(t_o)}$$

and t_o is the unique solution of the equation $\mu_f f(t) = f'(t)(\mu_f t + \nu_f)$.

If we put a power function $f(t) = t^p$ and $g(t) = t^q$ in Theorems 2.25 and 2.27, then we have the following three corollaries.

Corollary 2.29 *Let $A_j \in \mathscr{B}^+(H)$ be positive operators with $\mathsf{Sp}(A_j) \subseteq [m,M]$ for some scalars $0 < m < M$, $x_j \in H$ such that $\sum_{j=1}^{k}\|x_j\|^2 = 1$ $(j = 1,\ldots,k)$. If $p \in \mathbb{R}\backslash[0,1]$ (resp. $p \in (0,1)$), then for any real number $\alpha \in \mathbb{R}$*

$$\sum_{j=1}^{k}\left(A_j^p x_j,x_j\right) \leq \alpha\left(\sum_{j=1}^{k}(A_jx_j,x_j)\right)^q + \beta_1, \tag{2.62}$$

(resp.

$$\sum_{j=1}^{k}\left(A_j^p x_j,x_j\right) \geq \alpha\left(\sum_{j=1}^{k}(A_jx_j,x_j)\right)^q + \beta_2),$$

where

$$\beta_1 = \begin{cases} \alpha(q-1)\left(\frac{1}{\alpha q}\mu_{tp}\right)^{\frac{q}{q-1}} + \nu_{tp} & \textit{if} \quad m < \left(\frac{\mu_{tp}}{\alpha q}\right)^{\frac{1}{q-1}} < M \textit{ and } \alpha q(q-1) > 0, \\ \max\{m^p - \alpha m^q, M^p - \alpha M^q\} & \textit{otherwise.} \end{cases}$$

(resp.

$$\beta_2 = \begin{cases} \alpha(q-1)\left(\frac{1}{\alpha q}\mu_{tp}\right)^{\frac{q}{q-1}} + \nu_{tp} & \textit{if} \quad m < \left(\frac{\mu_{tp}}{\alpha q}\right)^{\frac{1}{q-1}} < M \textit{ and } \alpha q(q-1) < 0, \\ \min\{m^p - \alpha m^q, M^p - \alpha M^q\} & \textit{otherwise}). \end{cases}$$

Corollary 2.30 *Let A_j, x_j, $j = 1,\ldots,k$, be as in Corollary 2.29. Let the constant $C(m,M,p,q)$ be defined by (2.38) and $K(m,M,p,q)$ defined by (2.20).*

If $p \in \mathbb{R}\backslash[0,1)$, then

$$\sum_{j=1}^{k}\left(A_j^p x_j, x_j\right) \le \left(\sum_{j=1}^{k}(A_j x_j, x_j)\right)^q + \beta, \tag{2.63}$$

where

$$\beta = \begin{cases} C(m,M,p,q) & \textit{if} \quad m < \left(\frac{1}{q}\mu_{tp}\right)^{\frac{1}{q-1}} < M \textit{ and } q(q-1) > 0 \\ \max\{m^p - m^q, M^p - M^q\} & \textit{otherwise} \end{cases}$$

and

$$\sum_{j=1}^{k}\left(A_j^p x_j, x_j\right) \le \alpha\left(\sum_{j=1}^{k}(A_j x_j, x_j)\right)^q, \tag{2.64}$$

where

$$\alpha = \begin{cases} K(m,M,p,q) & \textit{if} \quad m < \frac{q}{1-q}\frac{\nu_{tp}}{\mu_{tp}} < M \textit{ and } q \in \mathbb{R}\backslash[0,1),\ pq > 0 \\ \max\{m^{p-q}, M^{p-q}\} & \textit{otherwise.} \end{cases}$$

But if $p \in (0,1)$, then

$$\sum_{j=1}^{k}\left(A_j^p x_j, x_j\right) \ge \left(\sum_{j=1}^{k}(A_j x_j, x_j)\right)^q + \beta, \tag{2.65}$$

where

$$\beta = \begin{cases} C(m,M,p,q) & \textit{if} \quad m < \left(\frac{1}{q}\mu_{tp}\right)^{\frac{1}{q-1}} < M \textit{ and } q(q-1) < 0 \\ \min\{m^p - m^q, M^p - M^q\} & \textit{otherwise} \end{cases}$$

and

$$\sum_{j=1}^{k}\left(A_j^p x_j, x_j\right) \ge \alpha\left(\sum_{j=1}^{k}(A_j x_j, x_j)\right)^q, \tag{2.66}$$

where

$$\alpha = \begin{cases} K(m,M,p,q) & \textit{if} \quad m < \frac{q}{1-q}\frac{\nu_{tp}}{\mu_{tp}} < M \quad \textit{and} \quad q \in (0,1) \\ \min\{m^{p-q}, M^{p-q}\} & \textit{otherwise.} \end{cases}$$

If we put $q = p$ in Corollary 2.30, then we have the following corollary.

Corollary 2.31 *Let A_j, x_j, $j = 1, \ldots, k$, be as in Corollary 2.29. Let the constant $C(m, M, p)$ be defined by (2.39) and $K(m, M, p)$ defined by (2.21). If $p \notin [0, 1]$, then*

$$\sum_{j=1}^{k} \left(A_j^p x_j, x_j\right) \leq \left(\sum_{j=1}^{k} (A_j x_j, x_j)\right)^p + C(m, M, p) \tag{2.67}$$

and

$$\sum_{j=1}^{k} \left(A_j^p x_j, x_j\right) \leq K(m, M, p) \left(\sum_{j=1}^{k} (A_j x_j, x_j)\right)^p. \tag{2.68}$$

If $p \in (0, 1)$, then we have the opposite inequalities.

If we put an exponential function $f(t) = e^{\alpha t}$ in Theorems 2.27 and 2.28, then we have the following two corollaries.

Corollary 2.32 *Let $A_j \in \mathscr{B}_h(H)$ be selfadjoint operators with $\mathsf{Sp}(A_j) \subseteq [m, M]$ for some scalars $m < M$ ($j = 1, \ldots, k$) and $x_1, x_2, \cdots, x_k \in H$ any finite number of vectors such that $\sum_{j=1}^{k} \|x_j\|^2 = 1$. Then for any real number $\alpha \neq 0$*

$$\sum_{j=1}^{k} (e^{\alpha A_j} x_j, x_j) \leq \frac{e^{\alpha M} - e^{\alpha m}}{\alpha e(M - m)} \exp\left(\frac{\alpha(M e^{\alpha m} - m e^{\alpha M})}{e^{\alpha M} - e^{\alpha m}}\right) \exp\left(\alpha \sum_{j=1}^{k} (A_j x_j, x_j)\right).$$

Corollary 2.33 *Let A_j, x_j, ($j = 1, \ldots, k$) as in Corollary 2.32. Then for any real number $\alpha \neq 0$*

$$\begin{aligned} &\sum_{j=1}^{k} (e^{\alpha A_j} x_j, x_j) - \exp\left(\alpha \sum_{j=1}^{k} (A_j x_j, x_j)\right) \\ &\quad \leq \frac{M e^{\alpha m} - m e^{\alpha M}}{M - m} + \frac{e^{\alpha M} - e^{\alpha m}}{\alpha(M - m)} \log\left(\frac{e^{\alpha M} - e^{\alpha m}}{\alpha e(M - m)}\right). \end{aligned}$$

Corollary 2.34 *Let A_j, x_j, $j = 1, \ldots, k$ be as in Corollary 2.29. Then*

$$\begin{aligned} &\sum_{j=1}^{k} \omega_j \Phi_j (\log A_j) - \log\left(\sum_{j=1}^{k} \omega_j \Phi_j (A_j)\right) \\ &\quad \geq 1 - \log(L(m, M)) + \frac{M \log m - m \log M}{M - m}. \end{aligned}$$

2.5 Converses of Jensen's inequality for selfadjoint operators

In this section we choose the identical map $\Phi_j \in \mathbf{P}_N[\mathscr{B}(H), \mathscr{B}(K)]$, $j = 1, \ldots, k$, in Section 2.1, i.e. $H = K$ and $\Phi_j(A_j) = A_j$ for every $A_j \in \mathscr{B}_h(H)$, $j = 1, \ldots, k$. Then Jensen's inequality for many operator maps from Lemma 2.1 becomes Jensen's inequality for sum of operators:

$$f\left(\sum_{j=1}^{k} \omega_j A_j\right) \leq \sum_{j=1}^{k} \omega_j f(A_j),$$

where $A_j \in \mathscr{B}_h(H)$ with $\mathsf{Sp}(A_j) \subseteq [m, M]$ for some scalars $m < M$, $\omega_j \in \mathbb{R}_+$ such that $\sum_{j=1}^{k} \omega_j = 1$ $(j = 1, \ldots, k)$ and f is an operator convex function on $[m, M]$.

From the main Theorem 2.3 we obtain immediately the following corollary:

Corollary 2.35 *Let $A_j \in \mathscr{B}_h(H)$ be selfadjoint operators with $\mathsf{Sp}(A_j) \subseteq [m, M]$ for some scalars $m < M$, $\omega_j \in \mathbb{R}_+$ such that $\sum_{j=1}^{k} \omega_j = 1$ $(j = 1, \ldots, k)$. Let $f, g \in \mathscr{C}([m, M])$ and $F(u, v)$ be a real function define on $U \times V$, where $U \supset f[m, M]$, $V \supset g[m, M]$. If $F(u, v)$ is an operator monotone on u and f is convex, then*

$$F\left[\sum_{j=1}^{k} \omega_j f(A_j), g\left(\sum_{j=1}^{k} \omega_j A_j\right)\right] \leq \left\{\max_{m \leq t \leq M} F\left[\mu_f t + \nu_f, g(t)\right]\right\} 1_H. \tag{2.69}$$

If f is a concave function, then we have opposite inequality with dual extreme.

By Theorems 2.4 and Theorem 2.5, we have converses of Jensen's inequality for sum of operators as follows:

Corollary 2.36 *Let A_j, x_j, $j = 1, \ldots, k$ be as in Corollary 2.35 and $f, g \in \mathscr{C}([m, M])$. If f is convex on $[m, M]$, then for $\alpha \in \mathbb{R}$ we have*

$$\sum_{j=1}^{k} \omega_j f(A_j) \leq \alpha g\left(\sum_{j=1}^{k} \omega_j A_j\right) + \beta\, 1_H,$$

where

$$\beta = \max_{m \leq t \leq M} \left\{\mu_f t + \nu_f - \alpha g(t)\right\}.$$

Additionally, let either of the following conditions be valid: (i) $g > 0$ on $[m, M]$ or (ii) $g < 0$ on $[m, M]$. Then

$$\sum_{j=1}^{k} \omega_j f(A_j) \leq \alpha_o\, g\left(\sum_{j=1}^{k} \omega_j A_j\right),$$

where

$$\alpha_o = \max_{m \le t \le M} \{(\mu_f t + \nu_f)/g(t)\} \qquad \textit{in the case } (i),$$
$$\textit{or } \alpha_o = \min_{m \le t \le M} \{(\mu_f t + \nu_f)/g(t)\} \qquad \textit{in the case } (ii).$$

If f is concave on $[m,M]$, then opposite inequalities hold with dual extreme in boundary.

As an application, we estimate the boundary of the operator convexity for convex functions.

Theorem 2.37 *Let A_j, x_j, $j = 1,\dots,k$ be as in Corollary 2.35 and $f \in \mathscr{C}([m,M])$. If f is convex on $[m,M]$ such that $f(t) > 0$ on $[m,M]$, then*

$$\frac{1}{\alpha_0} f\left(\sum_{j=1}^{k} \omega_j A_j\right) \le \sum_{j=1}^{k} \omega_j f(A_j) \le \alpha_0 f\left(\sum_{j=1}^{k} \omega_j A_j\right),$$

where $\alpha_0 = \max\{\frac{1}{f(t)}(\mu_f t + \nu_f) : t \in [m,M]\}$.

Proof. For a unit vector $x \in H$, put $x_j = \sqrt{\omega_j}\, x$ in Corollary 2.28, then we have

$$\sum_{j=1}^{k} \omega_j (f(A_j)x,x) \le \alpha_o f\left(\sum_{j=1}^{k} \omega_j (A_j x,x)\right).$$

Hence it follows that

$$\left(\sum_{j=1}^{k} \omega_j f(A_j)x,x\right) \le \alpha_o f\left(\sum_{j=1}^{k} \omega_j (A_j x,x)\right) \le \alpha_o \left(f\left(\sum_{j=1}^{k} \omega_j A_j\right)x,x\right)$$

and the last inequality holds by the convexity of f. Therefore we have

$$\sum_{j=1}^{k} \omega_j f(A_j) \le \alpha_0 f\left(\sum_{j=1}^{k} \omega_j A_j\right).$$

Next, since f is convex, it follows from Thereom 1.3 that

$$\left(\sum_{j=1}^{k} \omega_j f(A_j)x,x\right) = \sum_{j=1}^{k} \omega_j (f(A_j)x,x) \ge f\left(\sum_{j=1}^{k} \omega_j (A_j x,x)\right).$$

Since $m \le \sum_{j=1}^{k} \omega_j A_j \le M$, it follows from Corollary 2.28 that

$$f\left(\sum_{j=1}^{k} \omega_j (A_j x,x)\right) = f\left(\left(\left(\sum_{j=1}^{k} \omega_j A_j\right)x,x\right)\right) \ge \frac{1}{\alpha_0}\left(f\left(\sum_{j=1}^{k} \omega_j A_j\right)x,x\right).$$

Therefore we have

$$\frac{1}{\alpha_0} f\left(\sum_{j=1}^{k} \omega_j A_j\right) \le \sum_{j=1}^{k} \omega_j f(A_j).$$

□

We have the following complementary result of Theorem 2.37 for concave functions.

Theorem 2.38 *Let A_j, x_j, $j = 1,\ldots,k$ be as in Corollary 2.35 and $f \in \mathscr{C}([m,M])$. If f is concave on $[m,M]$ such that $f(t) > 0$ on $[m,M]$, then*

$$\frac{1}{\overline{\alpha_0}} f\left(\sum_{j=1}^{k} \omega_j A_j\right) \ge \sum_{j=1}^{k} \omega_j f(A_j) \ge \overline{\alpha_0}\, f\left(\sum_{j=1}^{k} \omega_j A_j\right),$$

where $\overline{\alpha_0} = \min\{\frac{1}{f(t)}(\mu_f t + \nu_f) : t \in [m,M]\}$.

2.6 Determinant for positive operators

In this section, we shall extend the notion of the determinant to vector states in the manner of Fuglede and Kadison. We discuss it as a continuous (weighted) geometric mean (with the weighted x) and observe some inequalities around the determinant from this point of view. By using the Mond-Pečarić method, we show an operator version of Specht's theorem which gave the ratio of the arithmetic mean to the geometric one.

There are some attempts to extend the notion of the determinant for matrices. In 1950s, Fuglede and Kadison defined the determinant on invertible operators A in II_1-factor M with the canonical (normalized) trace Tr as

$$\Delta(A) = \exp Tr(\log|A|)$$

and discussed the properties of this determinant. Afterwards, Arveson developed it in general von Neumann algebras and investigated some additional properties.

Here, note that the determinant of a positive definite matrix is the product of all eigenvalues, which contrasts with the fact that the trace of it is the sum of them. The normalization of the trace in $\Delta(A)$ yields another view for the determinants. For a positive definite $n \times n$ matrix A with the spectrum $\mathsf{Sp}(A) = \{t_1,\cdots,t_k\}$, the determinant in their sense is just the geometric mean

$$\prod_{i=1}^{n} t_i^{1/n}$$

while the normalized trace is the arithmetic mean $\frac{1}{k}\sum_{i=1}^{k} t_i$. So their determinant for positive operators is considered as the the continuous geometric mean, which reminds us of the product integral introduced by G.Birkhoff [20].

Definition 2.1 *Let A be a positive invertible operator on a Hilbert space H and x a unit vector in H. The determinant $\Delta_x(A)$ for A at x is defined as*

$$\Delta_x(A) = \exp(\log Ax, x).$$

Note that this definition is easily extended to that at a state on a suitable operator algebra via the GNS representation.

It immediately follows by definition that

$$\Delta_x(tA) = t\Delta_x(A) \quad \text{and} \quad \Delta_x(1_H) = 1$$

for all positive numbers t. We also have the norm continuity for the maps $x \to \Delta_x(A)$ and $A \to \Delta_x(A)$. Moreover the latter map is monotone by the operator monotonicity of the logarithm function.

Theorem 2.39 *The map $A \to \Delta_x(A)$ is monotone:*

$$A \leq B \quad \textit{implies} \quad \Delta_x(A) \leq \Delta_x(B).$$

It is easy to see

$$\Delta_x\left(\sum_{i=1}^n t_i E_i\right) = \prod_{i=1}^n t_i^{(E_i x, x)} \tag{2.70}$$

for the projections E_i with $\sum_i E_i = 1_H$. Then the equation (2.70) prompts us to consider another 'product' integral for a positive operator A after G.Birkhoff [20]. By the simple function $A_n = \sum_{i=1}^n t_i^{(n)} E_i^{(n)}$ of A converging uniformly to $A = \int_m^M t dE_t$, we define

$$\Pi \int_m^M t d(E_t x, x) := \lim_{n\to\infty} \prod_{i=1}^n t_i^{(n)(E_i^{(n)} x, x)}.$$

This definition makes sense by the above properties and it also shows

$$\Pi \int_m^M t d(E_t x, x) = \Delta_x(A).$$

Thus we may say that the determinant for positive operators is a continuous weighted geometric mean with the weighted x. Similarly we may consider $(A^{-1}x,x)^{-1}$ as a continuous harmonic mean. Thereby a continuous version of the arithmetic-geometric-harmonic mean inequality is the following basic one:

Theorem 2.40 *The determinant $\Delta_x(A)$ is not greater (resp. smaller) than (Ax,x) (resp. $(A^{-1}x,x)^{-1}$).*

$$(A^{-1}x,x)^{-1} \leq \Delta_x(A) \leq (Ax,x)$$

for every unit vector $x \in H$.

We immediately have the following inequalities:

Corollary 2.41 *If $A \in \mathscr{B}(H)^{++}$ is a positive invertible operator, then*

$$\|A^{-1}\|^{-1} \le \Delta_x(A) \le r(A) = \|A\|,$$

where $r(A)$ is the spectral radius of A.

Moreover it is well-known that the power (arithmetic) means

$$M_r(x_1, \cdots, x_n) = \left(\frac{x_1^r + \cdots + x_k^n}{n}\right)^{1/r}$$

make a path of means from the harmonic one at $r = -1$ to the arithmetic one at $r = 1$ via the geometric one at $r = 0$ (precisely the limit as $r \to 0$). Since $(A^r x, x)^{1/r}$ is considered as a continuous power mean from the above viewpoint, we have an extension of Theorem 2.40:

Theorem 2.42 *For a positive invertible operator A, the continuous power mean $(A^r x, x)^{1/r}$ converges monotone decreasingly (resp. increasingly) to $\Delta_x(A)$ as $r \downarrow 0$ (resp. $r \uparrow 0$).*

Proof. For the case of $r \downarrow 0$, the monotonicity follows from Jensen's inequality: If $0 < r < s$, then

$$1 \le \frac{s}{r} \quad \text{implies} \quad (A^r x, x)^{s/r} \le (A^{r(s/r)} x, x) = (A^s x, x),$$

therefore it follows that $(A^r x, x)^{1/r} \le (A^s x, x)^{1/s}$.

As for the convergence, the l'Hospital theorem shows

$$\lim_{t \downarrow 0} \frac{\log(A^t x, x)}{t} = \lim_{t \downarrow 0} \frac{d(A^t x, x)/dt}{(A^t x, x)} = \lim_{t \downarrow 0} \frac{(A^t \log A x, x)}{(A^t x, x)} = (\log A x, x),$$

so that we have the required convergence. Similarly we have the proof of the case of $r \uparrow 0$. □

In the Kubo-Ando theory of operator means in chapter 5, the notion of duality was introduced. In terms of the power mean, M_r is the dual of M_{-r};

$$M_r(x_1^{-1}, \cdots, x_n^{-1}) = M_{-r}(x_1, \cdots, x_n)^{-1}.$$

In particular, the geometric mean is selfdual, which reflects the following property:

Corollary 2.43 *For a positive invertible operator A,*

$$\Delta_x(A^{-1}) = \Delta_x(A)^{-1}$$

for every unit vector $x \in H$.

We observe various inequalities around the determinant. First we pose the well-known Ky Fan inequality, which follows from the operator concavity of the logarithm:

Theorem 2.44 *If A and B are positive invertible operators and* $\alpha > 0$, $\beta > 0$ *such that* $\alpha + \beta = 1$, *then*

$$\Delta_x(\alpha A + \beta B) \geq \Delta_x(A)^{\alpha} \Delta_x(B)^{\beta}$$

for every unit vector $x \in H$.

Next we discuss Arveson's inequality, which is an extension of the Ky Fan inequality. Note that $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B.

Theorem 2.45 *The determinant* $\Delta_x(A)$ *is the infimum of the set*

$$\{(ABx,x) \mid \Delta_x(B) \geq 1, B \in \{A\}'\}.$$

Proof. Since a positive operator B commutes with A and $\Delta_x(B) \geq 1$, we have

$$\Delta_x(AB) = \Delta_x(A)\Delta_x(B) \geq \Delta_x(A).$$

On the other hand, consider $B = \Delta_x(A)A^{-1}$. Then $\Delta_x(B) = 1$ and

$$(ABx,x) = \Delta_x(A)(AA^{-1}x,x) = \Delta_x(A),$$

which shows the above formula. □

Corollary 2.46 *If A and B commutes, then* $\Delta_x(A+B) \geq \Delta_x(A) + \Delta_x(B)$.

Following Turing, **the condition number** $h = h(A)$ of an invertible operator A on H is defined by $h(A) = \|A\|\|A^{-1}\|$. If a positive operator A satisfies the condition $M1_H \geq A \geq m1_H > 0$, i.e., $\mathrm{Sp}(A) \subseteq [m,M]$, then it may be thought as $M = \|A\|$ and $m = \|A^{-1}\|^{-1}$, so that $h = h(A) = \frac{M}{m}$.

Also, Specht [178] estimated the upper boundary of the arithmetic mean by the geometric one for positive numbers: For $x_1, \cdots, x_n \in [m,M]$ with $M \geq m > 0$,

$$\frac{(h-1)h^{\frac{1}{h-1}}}{e\log h} \sqrt[n]{x_1 \cdots x_n} \geq \frac{x_1 + \cdots x_n}{n}, \tag{2.71}$$

where $h = \frac{M}{m} (\geq 1)$. It is well known that

$$\frac{x_1 + \cdots x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n} \tag{2.72}$$

holds for positive numbers $x_1, x_2, \cdots, x_k$. Therefore, the Specht theorem (2.71) means a ratio type reverse inequality of the arithmetic-geometric mean inequality (2.72).

So we define the following constant:

$$S(t,p) = \frac{(t^p - 1)t^{\frac{p}{t^p-1}}}{pe\log t} \tag{2.73}$$

for all $p \in \mathbb{R}$ and all positive numbers t. We call $S(t,p)$ the **Specht ratio**. If we put $t = h$ and $p = 1$ in (2.73), then we have

$$S(h,1) = \frac{(h-1)h^{\frac{1}{h-1}}}{e \log h}. \tag{2.74}$$

We collect basic properties of the Specht ratio:

Lemma 2.47 *Let $t > 0$ and $p \in \mathbb{R}$.*

(i) $S(t,p) = S(t^p,1)$ *for all* $p \in \mathbb{R}$ *and* $t > 0$.

(ii) $S(1,1) = 1$.

(iii) $S(t,p) = S(t^{-1},p)$ *for all* $p \in \mathbb{R}$ *and* $t > 0$.

(iv) A function $S(t,1)$ *is strictly decreasing for* $0 < t < 1$ *and strictly increasing for* $t > 1$.

(v) $\lim_{p \to 0} S(t,p)^{\frac{1}{p}} = 1$.

(vi) $\lim_{p \to \infty} S(t,p)^{\frac{1}{p}} = t$ *for* $t > 1$ *and* $\lim_{p \to \infty} S(t,p)^{\frac{1}{p}} = t^{-1}$ *for* $1 > t > 0$.

Proof. We have (i) by definition. We have by L'Hospital's theorem

$$\begin{aligned}
\lim_{t \to 1} \log S(t,1) &= \lim_{t \to 1} \log \frac{t^{\frac{1}{t-1}}}{e \log t^{\frac{1}{t-1}}} = \lim_{t \to 1} \left(\frac{\log t}{t-1} - 1 - \log \frac{\log t}{t-1} \right) \\
&= \lim_{t \to 1} \left(\frac{1}{t} - 1 - \log \frac{1}{t} \right) = 0,
\end{aligned}$$

and so $S(1,1) = 1$.

For (iii), we may assume that $p = 1$ by (i). Then the equation $\left(\frac{1}{t}\right)^{\frac{1}{1/t-1}} = t^{\frac{t}{t-1}} = t \cdot t^{\frac{1}{t-1}}$ implies the equation

$$S(t^{-1},1) = \frac{(t^{-1})^{\frac{1}{t^{-1}-1}}}{e \log (t^{-1})^{\frac{1}{t^{-1}-1}}} = \frac{t \cdot t^{\frac{1}{t-1}}}{e \log t^{\frac{1}{t-1}}} = S(t,1).$$

Furthermore we have by a differential calculation

$$\begin{aligned}
\frac{d}{dt} \log S(t,1) &= \frac{d}{dt} \left(\frac{\log t}{t-1} - 1 - \log \frac{\log t}{t-1} \right) \\
&= \frac{t^{-1}(t-1) - \log t}{(t-1)^2} - \frac{t-1}{\log t} \frac{t^{-1}(t-1) - \log t}{(t-1)^2} \\
&= \frac{(\log t - t + 1)(1 - t^{-1} - \log t)}{(t-1)^2 \log t}.
\end{aligned}$$

So for all $t > 1$, the function $S(t,1)$ is strictly increasing from $\frac{d}{dt}\log S(t,1) > 0$. On the other hand, for $0 < t < 1$ we see that a function $S(t,1)$ is strictly decreasing.

For (v), put $g(p) = t^{\frac{p}{t^p-1}}$, then we have $S(t,p) = \frac{g(p)}{e\log g(p)}$. It is easily obtained that

$$\lim_{p\to+0} g(p) = t^{\frac{1}{\log t}} = e$$

and

$$g'(p) = \left\{\frac{t^p - 1 - pt^p\log t}{(t^p-1)^2}\right\} t^{\frac{p}{t^p-1}} \log t.$$

Then $g'(p)$ is bounded as $p \to +0$ since

$$\lim_{p\to+0} \frac{t^p - 1 - pt^p\log t}{(t^p-1)^2} = -\frac{1}{2}.$$

Then we have

$$\begin{aligned}\lim_{p\to+0} \log S(t,p)^{\frac{1}{p}} &= \lim_{p\to+0} \frac{\log g(p) - \log(\log g(p)) - 1}{p} \\ &= \lim_{p\to+0} \frac{g'(p)}{g(p)}\left\{1 - \frac{1}{\log g(p)}\right\} \\ &= 0,\end{aligned}$$

so that $\lim_{p\to+0} S(t,p)^{\frac{1}{p}} = 1$.

For (vi), we mat assume that $t > 1$ by (iii). Since

$$t^{\frac{p}{t^p-1}\frac{1}{p}} \to t^0 = 1 \quad \text{and} \quad \left(\frac{t^p-1}{pe\log t}\right)^{\frac{1}{p}} \to t \quad \text{as} \quad p \to \infty,$$

it follows that $S(t,p)^{1/p} \to t$ as $p \to \infty$.

$\square$

We rephrase the Specht theorem (2.71) under matrix situation. If we put

$$A = \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{pmatrix} \quad \text{and} \quad x = \frac{1}{\sqrt{n}}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

then the Specht theorem implies $S(h,1)\Delta_x(A) \geq (Ax,x)$.

Then the following lemma is a geometric representation of the Specht theorem:

Lemma 2.48 *Let A be a positive operator on H with $0 < m1_H \leq A \leq M1_H$. If $a = L(m,M)$ and $b = \frac{m\log M - M\log m}{\log M - \log m}$, then*

$$(Ax,x) \leq ae^{\frac{b-a}{a}}\Delta_x(A)$$

and the equality holds if and only if $((\log A)x,x) = (a-b)/a$ and a unit vector x is a linear combination of eigenvectors corresponding to m and M.

Proof. Since $y = e^t$ is a convex function, then, for the line $at+b$ crossing e^t at $t = \log m$ and $t = \log M$, we have

$$e^t \le at+b \le ae^{\frac{b-a}{a}} e^t$$

on $[\log m, \log M]$. In fact, $F(t) = ae^{\frac{b-a}{a}} e^t - at - b$ is also a convex function with the minimum zero at $t = (a-b)/a$ by $F'((a-b)/a) = 0$ and $(a-b)/a \in [\log m, \log M]$ is guaranteed by the inequalities $m \le L(m,M) \le M$. Thus we have the latter inequality. Putting $S = \log A$, we have

$$(e^S x, x) \le ((aS+b)x, x) = a(Sx,x) + b \le ae^{\frac{b-a}{a}} e^{(Sx,x)},$$

so that, we have the required inequality. Since $e^t < at+b$ for $t \in (\log m, \log M)$, the equality $(e^S x, x) = ((aS+b)x, x)$ holds if and only if x is a linear combination of eigenvectors corresponding to m and M. Moreover the only zero of F is $(a-b)/a$, and hence the equality $a(Sx,x)+b = ae^{\frac{b-a}{a}} e^{(Sx,x)}$ holds if and only if $(Sx,x) = (a-b)/a$. □

Then the above inequality itself is an estimation of Specht's type:

Theorem 2.49 *Let A be a positive operator with $0 < m1_H \le A \le M1_H$, x a unit vector and $h = M/m$ the condition number. Then the ratio of (Ax,x) to the determinant for A at x is not greater than the Specht ratio:*

$$(Ax,x) \le S(h,1)\Delta_x(A)$$

and the equality holds if and only if both m and M are eigenvalues of A and

$$x = \sqrt{\frac{h}{h-1} - \frac{1}{\log h}}\, e_m + \sqrt{\frac{1}{\log h} - \frac{1}{h-1}}\, e_M,$$

where e_m and e_M are corresponding unit eigenvectors to m and M respectively.

Proof. The number $ae^{\frac{b-a}{a}}$ in Lemma 2.48 is exactly the Specht ratio. In fact, $a = \frac{(h-1)m}{\log h}$ and

$$\frac{b}{a} = \frac{m\log M - M\log m}{M-m} = \frac{\log M - h\log m}{h-1} = \frac{\log(hm^{1-h})}{h-1},$$

and hence we have

$$ae^{\frac{b-a}{a}} = \frac{(h-1)m}{\log h}(hm^{1-h})^{1/(h-1)}e^{-1} = \frac{(h-1)h^{1/(h-1)}}{e\log h}.$$

To verify the equality condition, we can put $x = \sqrt{1-t^2}e_m + te_M$ for a number $0 < t < 1$ by Lemma 2.48. Then it follows from Lemma 2.48 again that

$$\log\left(m^{1-t^2}M^{t^2}\right) = (Sx,x) = \frac{a-b}{a} = 1 - \frac{\log(hm^{1-h})}{h-1} = 1 + \log(h^{1/(1-h)}m),$$

or $t^2 \log h = 1 + \log h^{1/(1-h)}$. Thus

$$t^2 = \frac{1 + 1/(1-h)\log h}{\log h} = \frac{1}{\log h} - \frac{1}{h-1}.$$

□

Incidentally we have a variation of our theorem:

Corollary 2.50 *In the notations in the above theorem,*

$$(A^p x,x) \le S(h,p)\Delta_x(A^p)$$

for all real numbers $p \in \mathbb{R}$.

Proof. Since the condition number of A^p is h^p for $p \ge 0$, we have the above inequality immediately for $p \ge 0$ by Theorem 2.49. Suppose $p < 0$. Then the condition number of A^p is h^{-p} and hence

$$(A^p x,x) \le S(h^{-p},1)\Delta_x(A^p).$$

However we have $S(h^{-p},1) = S(h^{-1},p) = S(h,p)$ by Lemma 2.47. □

Remark 2.8 *The above corollary is obtained by Corollary 2.32:*

We can easily modify into the conditions $m > 0$ and all real nnumbers t. In fact, putting $B = \log A$, $l = \log m$ and $L = \log M$ in the above inequality, we have

$$(e^{tB}x,x) = (A^t x,x) \quad \textit{and} \quad \exp(t(Bx,x)) = \Delta_x(A^t)$$

and

$$\begin{aligned}
\frac{e^{tL}-e^{tl}}{te(L-l)}\exp\left(\frac{t(Le^{tl}-le^{tL})}{e^{tL}-e^{tl}}\right) &= \frac{M^t-m^t}{e\log h^t}\exp\left(\frac{m^t\log M^t - M^t\log m^t}{M^t-m^t}\right)\\
&= m^t\frac{h^t-1}{e\log h^t}\exp\left(\frac{\log M^t - h^t\log m^t}{h^t-1}\right)\\
&= m^t\frac{h^t-1}{e\log h^t}\exp\left(\frac{t\log h}{h^t-1}-\log m^t\right)\\
&= \frac{(h^t-1)h^{t/(h^t-1)}}{e\log h^t} = S(h,t).
\end{aligned}$$

On the other hand, Mond and Shisha gave an estimate of the difference between the arithmetic mean and the geometric one: For positive numbers $x_1,\cdots,x_n \in [m,M]$ with $M > m > 0$ and $h = \frac{M}{m}$,

$$\sqrt[n]{x_1x_2\cdots x_n} + D(m,M) \ge \frac{x_1+x_2+\cdots x_n}{n} \tag{2.75}$$

where

$$D(m,M) = \theta M + (1-\theta)m - M^\theta m^{1-\theta} \quad \text{and} \quad \theta = \log\left(\frac{h-1}{\log h}\right)\frac{1}{\log h}. \tag{2.76}$$

We call $D(m,M)$ the **Mond-Shisha difference**. Notice that (2.75) means a difference type reverse inequality of the arithmetic-geometric mean inequality.

Lemma 2.51 *The Mond-Shisha difference coincides with the following constant via the Specht ratio: If $M > m > 0$ and $h = \frac{M}{m} > 1$, then*

$$D(m^p, M^p) = L(m^p, M^p) \log S(h, p) \tag{2.77}$$

for all $p \in \mathbb{R}$.

Proof. If we put $\theta = \log\left(\frac{h^p - 1}{p \log h}\right) \frac{1}{p \log h}$, then we have

$$\begin{aligned} L(m^p, M^p) \log S(h, p) &= \frac{m^p(h^p - 1)}{p \log h}\left(\log\left(\frac{h^p - 1}{p \log h}\right) + \frac{p \log h}{h^p - 1} - 1\right) \\ &= m^p\left(\log\left(\frac{h^p - 1}{p \log h}\right)\frac{h^p - 1}{p \log h} + 1 - \frac{h^p - 1}{p \log h}\right) \\ &= m^p\left(\theta(h^p - 1) + 1 - h^{p\theta}\right) \\ &= D(m^p, M^p). \end{aligned}$$

□

We show the following result, which is considered as a continuous version of the Mond-Shisha result (2.75):

Theorem 2.52 *Let A be a positive operator on H satisfying $M1_H \geq A \geq m1_H > 0$. Put $h = \frac{M}{m}$. Then the difference between (Ax, x) and the determinant $\Delta_x(A)$ for A at a unit vector $x \in H$ is not greater than the Mond-Shisha difference*:

$$(Ax, x) - \Delta_x(A) \leq D(m, M),$$

where $D(m, M)$ is defined in (2.76) and the equality holds if and only if both m and M are eigenvalues of A and

$$x = \sqrt{1 - \log\left(\frac{h-1}{\log h}\right)\frac{1}{\log h}}\, e_m + \sqrt{\log\left(\frac{h-1}{\log h}\right)\frac{1}{\log h}}\, e_M,$$

where e_m and e_M are corresponding unit eigenvectors to m and M respectively.

Proof. Put $S = \log A$, $a = L(m, M)$ and $b = \frac{m \log M - M \log m}{\log M - \log m}$, then we have

$$(e^S x, x) \leq a(Sx, x) + b \leq e^{(Sx, x)} + a \log a + b - a.$$

The number $a \log a + b - a$ is exactly the Mond-Shisha difference. In fact, $a = \frac{m(h-1)}{\log h}$ and hence we have

$$\begin{aligned} a \log a + b - a &= a\left(\log a + \frac{m(\log M - \log m) - (M - m)\log m}{M - m} - 1\right) \\ &= \frac{m(h-1)}{\log h}\left(\log(h-1) - \log(\log h) + \frac{\log h}{h-1} - 1\right) \\ &= m\left((h-1)\log\left(\frac{h-1}{\log h}\right)\frac{1}{\log h} + 1 - \frac{h-1}{\log h}\right) \\ &= m((h-1)\theta + 1 - h^{\theta}) \\ &= D(m, M), \end{aligned}$$

where $\theta = \log\left(\frac{h-1}{\log h}\right)\frac{1}{\log h}$.

To verify the equality condition, we can put $x = \sqrt{1-t^2}e_m + te_M$ for a number $0 < t < 1$. Then it follows that

$$\log m^{1-t^2}M^{t^2} = (Sx,x) = \log a = \log \frac{m(h-1)}{\log h},$$

and so we have

$$t^2 = \log\left(\frac{h-1}{\log h}\right)\frac{1}{\log h}(>0).$$

□

2.7 A generalized Kantorovich constant

In this section, we investigate basic properties of a generalized Kantorovich constant. First we recall the celebrated Kantorovich inequality: Let A be a positive operator on a Hilbert space H satisfying $M1_H \geq A \geq m1_H$ for some scalars $0 < m < M$. Then

$$(Ax,x)^{-1} \leq (A^{-1}x,x) \leq \frac{(M+m)^2}{4Mm}(Ax,x)^{-1}$$

for every unit vector $x \in H$ and this inequality is just equivalent to the following one

$$(Ax,x)^2 \leq (A^2x,x) \leq \frac{(M+m)^2}{4Mm}(Ax,x)^2$$

for every unit vector $x \in H$. We remark that the constant $\frac{(M+m)^2}{4Mm}$ can be expressed as follows:

$$\frac{(M+m)^2}{4Mm} = \left(\frac{\frac{M+m}{2}}{\sqrt{Mm}}\right)^2,$$

that is, inside the bracket ( ), the numerator is the arithmetic mean and the denominator is the geometric one of m and M, respectively. This constant is said to be the **Kantorovich constant**.

By Corollary 2.31, we have the converse of Hölder-McCarthy inequality as an extension of the Kantorovich inequality.

Theorem 2.53 *Let A be a positive operator on a Hilbert space H satisfying $M1_H \geq A \geq m1_H$ for some scalars $0 < m < M$. Then*

$$(Ax,x)^p \leq (A^px,x) \leq K(m,M,p)(Ax,x)^p \quad \textit{for } p \notin [0,1]$$

and

$$(Ax,x)^p \geq (A^p x,x) \geq K(m,M,p)(Ax,x)^p \quad \text{for } p \in [0,1]$$

for every unit vector $x \in H$, where

$$K(m,M,p) = \frac{(mM^p - Mm^p)}{(p-1)(M-m)} \left(\frac{p-1}{p} \frac{M^p - m^p}{mM^p - Mm^p} \right)^p \tag{2.78}$$

for any real number $p \in \mathbb{R}$.

Definition 2.2 *Let $h > 0$. A generalized Kantorovich constant $K(h,p)$ is defined by*

$$K(h,p) = \frac{(h^p - h)}{(p-1)(h-1)} \left(\frac{p-1}{p} \frac{h^p - 1}{h^p - h} \right)^p \tag{2.79}$$

for any real number $p \in \mathbb{R}$ and $K(h,p)$ is sometimes denoted by $K(p)$ briefly.

We remark that $K(m,M,p)$ just coincides with $K(h,p)$ by putting $h = \frac{M}{m} > 1$ and $K(m,M,p)$ is an extension of the Kantorovich constant $\frac{(M+m)^2}{4Mm}$, in fact, $K(m,M,2) = K(m,M,-1) = \frac{(M+m)^2}{4Mm}$. We mention some important properties of $K(h,p)$.

Theorem 2.54 *Let $h > 0$ be given. Then a generalized Kantorovich constant $K(h,p)$ has the following properties.*

(i) $K(h,p) = K(\frac{1}{h},p)$ *for all* $p \in \mathbb{R}$.

(ii) $K(h,p) = K(h,1-p)$ *for all* $p \in \mathbb{R}$.

(iii) $K(h,0) = K(h,1) = 1$ *and* $K(1,p) = 1$ *for all* $p \in \mathbb{R}$.

(iv) $K(h,p)$ *is increasing for* $p > \frac{1}{2}$ *and decreasing for* $p < \frac{1}{2}$.

(v) $K\left(h^r, \frac{p}{r}\right)^{\frac{1}{p}} = K\left(h^p, \frac{r}{p}\right)^{-\frac{1}{r}}$ *for* $pr \neq 0$.

(vi) $K(h,p) \leq h^{p-1}$ *for all* $p \geq 1$ *and* $h > 1$.

We need the following lemma to give a proof of Theorem 2.54.

Lemma 2.55 *Let $h > 1$ and $p \geq \frac{1}{2}$. Then the following* (i) *and* (ii) *hold:*

(i) $\delta \frac{h^p(h^p - 1 + p - ph)}{(h^p-1)(h^p-h)} \geq \frac{1}{2}$.

(ii) $\delta \frac{h^{\frac{1}{2}}(p-1)(h^p-1)}{p(h^p-h)} \geq 1$.

Proof. Recall $h^p - h \geq 0$ for $p \geq 1$ and $h^p - h \leq 0$ for $1 \geq p \geq \frac{1}{2}$, so that we can divide the case $p \geq \frac{1}{2}$ into the case $p \geq 1$ and $1 \geq p \geq \frac{1}{2}$ to prove (i) and (ii).

(i). (i-a) in the case $p \geq 1$. To prove (i) in this case, it suffices to show

$$\begin{aligned}&2h^p(h^p - 1 + p - ph) - (h^p - 1)(h^p - h)\\ &= h^p(h^p - h^{1-p} - (2p-1)(h-1)) \geq 0\end{aligned}$$

and we have only to prove the following

$$f_1(h) = h^p - h^{1-p} - (2p-1)(h-1) \geq 0 \quad \text{for } h > 1 \text{ and } p \geq 1. \tag{2.80}$$

In fact, $f_1(1) = 0$ and $f_1'(h) = ph^{p-1} - (1-p)h^{-p} - (2p-1)$, so that $f_1'(1) = 0$ and

$$f_1''(h) = p(p-1)(h^{p-2} - h^{-p-1}) \geq 0 \quad \text{holds for } p \geq 1 \tag{2.81}$$

therefore $f_1'(h) \geq 0$ by $f_1'(1) = 0$ and (2.81), so that $f_1(h) \geq 0$ by $f_1(1) = 0$ and $f_1'(h) \geq 0$, that is, we have (2.80).

(i-b) in the case $1 \geq p \geq \frac{1}{2}$. To prove (i) in this case, it suffices to show by the same way as (i-a)

$$\begin{aligned}&2h^p(-h^p + 1 - p + ph) - (h^p - 1)(h - h^p)\\ &= h^p(-h^p + h^{1-p} + (2p-1)(h-1)) \geq 0\end{aligned}$$

and we have only to prove the following

$$f_2(h) = -h^p + h^{1-p} + (2p-1)(h-1) \geq 0 \quad \text{for } h > 1 \text{ and } 1 \geq p \geq \tfrac{1}{2}. \tag{2.82}$$

In fact, $f_2(1) = 0$ and $f_2'(h) = -ph^{p-1} + (1-p)h^{-p} + (2p-1)$, so that $f_2'(1) = 0$ and

$$f_2''(h) = p(1-p)(h^{p-2} - h^{-p-1}) \geq 0 \quad \text{for } 1 \geq p \geq \tfrac{1}{2} \tag{2.83}$$

therefore $f_2'(h) \geq 0$ by $f_2'(1) = 0$ and (2.83), so that $f_2(h) \geq 0$ by $f_2(1) = 0$ and $f_2'(h) \geq 0$, that is, we have (2.82). Whence the proof of (i) is complete by (i-a) and (i-b).

(ii). (ii-a) in the case $p \geq 1$. To prove (ii) in this case, it suffices to show

$$\begin{aligned}&h^{\frac{1}{2}}(p-1)(h^p - 1) - p(h^p - h)\\ &= \left\{(p-1)(h^p - 1) - p\left(h^{p-\frac{1}{2}} - h^{\frac{1}{2}}\right)\right\} h^{\frac{1}{2}} \geq 0\end{aligned}$$

and we have only to prove the following

$$f_3(h) = (p-1)(h^p - 1) - p\left(h^{p-\frac{1}{2}} - h^{\frac{1}{2}}\right) \geq 0 \quad \text{for } p \geq 1. \tag{2.84}$$

In fact, $f_3(1) = 0$ and

$$f_3'(h) = \left\{(p-1)ph^{p-\frac{1}{2}} - p\left(p - \frac{1}{2}\right)h^{p-1} + \frac{p}{2}\right\} h^{\frac{-1}{2}} = g_3(h)h^{\frac{-1}{2}} \tag{2.85}$$

where $g_3(h)$ in (2.85) is defined by

$$g_3(h) = (p-1)ph^{p-\frac{1}{2}} - p(p-\frac{1}{2})h^{p-1} + \frac{p}{2}.$$

We have $g_3(1) = 0$ and

$$g_3'(h) = p(p-1)\left(p-\frac{1}{2}\right)\left(h^{p-\frac{3}{2}} - h^{p-2}\right) \geq 0 \quad \text{holds for } p \geq 1 \tag{2.86}$$

so that $g_3(h) \geq 0$ by $g_3(1) = 0$ and (2.86), so that $f_3'(h) \geq 0$ by (2.85), therefore $f_3(h) \geq 0$ by $f_3(1) = 0$ and $f_3'(h) \geq 0$, so we have (2.84).

(ii-b) in the case $1 \geq p \geq \frac{1}{2}$. To prove (ii) in this case, it suffices to show by the same way as (ii-a)

$$\begin{aligned} & h^{\frac{1}{2}}(1-p)(h^p-1) - p(h-h^p) \\ = & \left\{(1-p)(h^p-1) - p\left(h^{\frac{1}{2}} - h^{p-\frac{1}{2}}\right)\right\} h^{\frac{1}{2}} \geq 0 \end{aligned}$$

and we have only to prove the following

$$f_4(h) = (1-p)(h^p-1) - p\left(h^{\frac{1}{2}} - h^{p-\frac{1}{2}}\right) \geq 0 \quad \text{for } 1 \geq p \geq \tfrac{1}{2}. \tag{2.87}$$

In fact, $f_4(1) = 0$ and

$$f_4'(h) = \left\{(1-p)ph^{p-\frac{1}{2}} + p\left(p-\frac{1}{2}\right)h^{p-1} - \frac{p}{2}\right\} h^{\frac{-1}{2}} = g_4(h)h^{\frac{-1}{2}} \tag{2.88}$$

where $g_4(h)$ in (2.88) is defined by

$$g_4(h) = (1-p)ph^{p-\frac{1}{2}} + p(p-\frac{1}{2})h^{p-1} - \frac{p}{2}.$$

We have $g_4(1) = 0$ and

$$g_4'(h) = p(1-p)(p-\frac{1}{2})(h^{p-\frac{3}{2}} - h^{p-2}) \geq 0 \quad \text{holds for } 1 \geq p \geq \tfrac{1}{2} \tag{2.89}$$

so that $g_4(h) \geq 0$ by $g_4(1) = 0$ and (2.89), so that $f_4'(h) \geq 0$ by (2.88), therefore $f_4(h) \geq 0$ by $f_4(1) = 0$ and $f_4'(h) \geq 0$, so we have (2.87). Therefore the proof of (ii) is complete by (ii-a) and (ii-b). Whence we have finished a proof of Lemma 2.55. □

Proof of Theorem 2.54. (i) By an easy calculation, we have

$$\begin{aligned} K\left(\frac{1}{h},p\right) &= \frac{(h^{-p}-h^{-1})}{(p-1)(h^{-1}-1)}\left(\frac{(p-1)(h^{-p}-1)}{p(h^{-p}-h^{-1})}\right)^p \\ &= \frac{(h^{1-p}-1)}{(p-1)(1-h)}\left(\frac{(p-1)(1-h^p)}{p(1-h^{p-1})}\right)^p \\ &= \frac{(h^p-h)}{(p-1)(h-1)}\left(\frac{(p-1)(h^p-1)}{p(h^p-h)}\right)^p \\ &= K(h,p). \end{aligned}$$

(ii) For all $p \in \mathbb{R}$, we have

$$\begin{aligned} K(h,1-p) &= \frac{(h^{1-p}-h)}{((1-p)-1)(h-1)}\left(\frac{((1-p)-1)(h^{1-p}-1)}{(1-p)(h^{1-p}-h)}\right)^{1-p} \\ &= \frac{h^{1-p}-h}{-p(h-1)}\left(\frac{-p(h^{1-p}-1)}{(1-p)(h^{1-p}-h)}\right)^{1-p} \\ &= \frac{h^{1-p}-1}{(h-1)(1-p)}\left(\frac{(p-1)(h-h^{p+1})}{p(h-h^{p})}\right)^{p} \\ &= K(h,p). \end{aligned}$$

(iii) By (i) and (ii), it suffices to prove the case of $h > 1$. Then we have

$$\begin{aligned} \log K(h,p) &= \log\frac{h^p-h}{(p-1)(h-1)} + p\log\frac{p-1}{h^p-h} + p\log\frac{h^p-1}{p} \\ &\to \log\frac{1-h}{(-1)(h-1)} + 0\times\log\frac{-1}{1-h} + 0\times\log\log h = 0, \end{aligned}$$

as $p \to +0$. Therefore we have $K(h,0) = \lim_{p\to 0} K(h,p) = 1$ and we obtain $K(h,1) = 1$ by (ii). Since

$$\lim_{h\to 1}\frac{h^p-h}{h-1} = \lim_{h\to 1}(ph^{p-1}-1) = p-1$$

and

$$\lim_{h\to 1}\frac{h^p-1}{h^p-h} = \lim_{h\to 1}\frac{ph^{p-1}}{ph^{p-1}-1} = \frac{p}{p-1}$$

by l'Hospital's theorem, we have $K(1,p) = \lim_{h\to 1} K(h,p) = 1$.

(iv) We may assume $h > 1$ by (i). Let $p \geq \frac{1}{2}$. Since $\frac{h^p-h}{p-1} \geq 0$ for $1 \geq p \geq \frac{1}{2}$ and $\frac{h^p-h}{p-1} \geq 0$ for $p \geq 1$, we have

$$\frac{h^p-h}{p-1} \geq 0 \quad \text{for any } p \geq \tfrac{1}{2}. \tag{2.90}$$

Differentiate $K(p)$ by p, we obtain $K'(p)$ as follows:

$$\begin{aligned}
K'(p) &= \frac{\left(\frac{(p-1)}{p}\frac{(h^p-1)}{(h^p-h)}\right)^p}{(h-1)(h^p-1)(p-1)}\Big\{h^p(h^p-1+p-hp)\log h \\
&\qquad\qquad +(h^p-1)(h^p-h)\log\frac{(p-1)(h^p-1)}{p(h^p-h)}\Big\} \\
&= \frac{\left(\frac{(p-1)}{p}\frac{(h^p-1)}{(h^p-h)}\right)^p}{(h-1)}\frac{(h^p-h)}{(p-1)}\left\{\frac{h^p(h^p-1+p-hp)}{(h^p-1)(h^p-h)}\log h+\log\frac{(p-1)(h^p-1)}{p(h^p-h)}\right\} \\
&\geq \frac{\left(\frac{(p-1)}{p}\frac{(h^p-1)}{(h^p-h)}\right)^p}{(h-1)}\frac{(h^p-h)}{(p-1)}\left\{\frac{1}{2}\log h+\log\frac{(p-1)(h^p-1)}{p(h^p-h)}\right\} \\
&\qquad\qquad \text{by (i) of Lemma 2.55, (2.90) and } \log h>0 \\
&= \frac{\left(\frac{(p-1)}{p}\frac{(h^p-1)}{(h^p-h)}\right)^p}{(h-1)}\frac{(h^p-h)}{(p-1)}\left\{\log\frac{h^{\frac{1}{2}}(p-1)(h^p-1)}{p(h^p-h)}\right\} \\
&\geq 0 \quad \text{by (ii) of Lemma 2.55 and (2.90),}
\end{aligned}$$

so that $K(p)$ is an increasing function of p for $p \geq \frac{1}{2}$, and this result implies that $K(p)$ is a decreasing function of p for $p \leq \frac{1}{2}$ by (ii).

(v) By a simple calculation, we have

$$\begin{aligned}
K\left(h^r,\frac{p}{r}\right)^{\frac{1}{p}} &= \left(\frac{h^p-h^r}{(\frac{p}{r}-1)(h^r-1)}\right)^{\frac{1}{p}}\left(\frac{(\frac{p}{r}-1)(h^p-1)}{\frac{p}{r}(h^p-h^r)}\right)^{\frac{1}{r}} \\
&= \left(\frac{h^r-h^p}{(\frac{r}{p}-1)(h^p-1)}\right)^{-\frac{1}{r}}\left(\frac{(\frac{r}{p}-1)(h^r-1)}{\frac{r}{p}(h^r-h^p)}\right)^{-\frac{1}{p}} \\
&= K\left(h^p,\frac{r}{p}\right)^{-\frac{1}{r}}.
\end{aligned}$$

(vi) Let $p>1$ and $q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$. Then the inequality

$$(p-1)t-pt^{\frac{1}{q}}+1\geq 0 \tag{2.91}$$

holds for $t\geq 1$. Multiplying (2.91) by $t^{\frac{1}{p}}$, then

$$0\leq (p-1)tt^{\frac{1}{p}}-pt+t^{\frac{1}{p}},$$

that is,

$$\frac{t^{\frac{1}{p}}}{t}\frac{t-1}{t^{\frac{1}{p}}-1}\leq p \qquad \text{for } t\geq 1. \tag{2.92}$$

Taking exponent $\frac{1}{p}$ in (2.92) and taking exponent $\frac{1}{q}$ in (2.92), respectively, we have

$$\left(\frac{t^{\frac{1}{p}}}{t}\frac{t-1}{t^{\frac{1}{p}}-1}\right)^{\frac{1}{p}} \le p^{\frac{1}{p}} \quad \text{and} \quad \left(\frac{t^{\frac{1}{q}}}{t}\frac{t-1}{t^{\frac{1}{q}}-1}\right)^{\frac{1}{q}} \le q^{\frac{1}{q}}. \tag{2.93}$$

Modifying (2.93), we have

$$\frac{t-1}{(t^{1/p}-1)^{1/p}(t^{1/q}-1)^{1/q}t^{2/pq}} \le \frac{p}{(p-1)^{(p-1)/p}} \qquad \text{for } t \ge 1.$$

Taking exponent p of both sides,

$$\frac{(t-1)^p}{(t^{1/p}-1)(t^{1/q}-1)^p t^{2/q}} \le \frac{p^p}{(p-1)^{p-1}} \qquad \text{for } t \ge 1. \tag{2.94}$$

Putting $t = h^p$ in (2.94), we have

$$\frac{(h^p-1)^p}{(h^{p-1}-1)^{p-1}(h-1)h^{2p-2}} \le \frac{p^p}{(p-1)^{p-1}}$$

and so $K(h,p) \le h^{p-1}$ for $p \ge 1$ and $h > 1$. □

We have the representation of the Specht ratio by the limit of Kantorovich constant.

Theorem 2.56 *Let $h > 0$ be given. Then*

(i) $\lim_{r\to 0} K\left(h^r, \frac{p}{r}\right) = S(h,p)$.

(ii) $\lim_{r\to 0} K\left(h^r, \frac{r+p}{r}\right) = S(h,p)$.

Proof. For (i), it follows that

$$\begin{aligned} K\left(h^r, \frac{p}{r}\right) &= \frac{h^p-h^r}{p-r}\frac{r}{h^r-1}\left(\frac{h^p-1}{p}\frac{p-r}{h^p-h^r}\right)^{\frac{p}{r}} \\ &\to \frac{h^p-1}{p}\frac{1}{\log h}\frac{1}{e}h^{\frac{p}{h^p-1}} = S(h,p) \quad \text{as } r \to 0, \end{aligned}$$

where the convergence of the final term is assured by l'Hospital's theorem as follows:

$$\begin{aligned} \lim_{r\to 0}\frac{p\log\left(\frac{h^p-1}{p}\frac{p-r}{h^p-h^r}\right)}{r} &= \lim_{r\to 0}\frac{p}{p-r}\frac{-(h^p-h^r)+(p-r)h^r\log h}{h^p-h^r} \\ &= -1+\frac{p}{h^p-1}\log h = \log\left(\frac{1}{e}h^{\frac{p}{h^p-1}}\right). \end{aligned}$$

For (ii), it follows that

$$\begin{aligned} K\left(h^r, \frac{p+r}{r}\right) &= K\left(h^r, 1-\frac{p+r}{r}\right) = K\left(\left(\frac{1}{h}\right)^{-r}, \frac{p}{-r}\right) \\ &\mapsto S(h^{-1},p) = S(h,p) \quad \text{as } r \mapsto 0. \end{aligned}$$

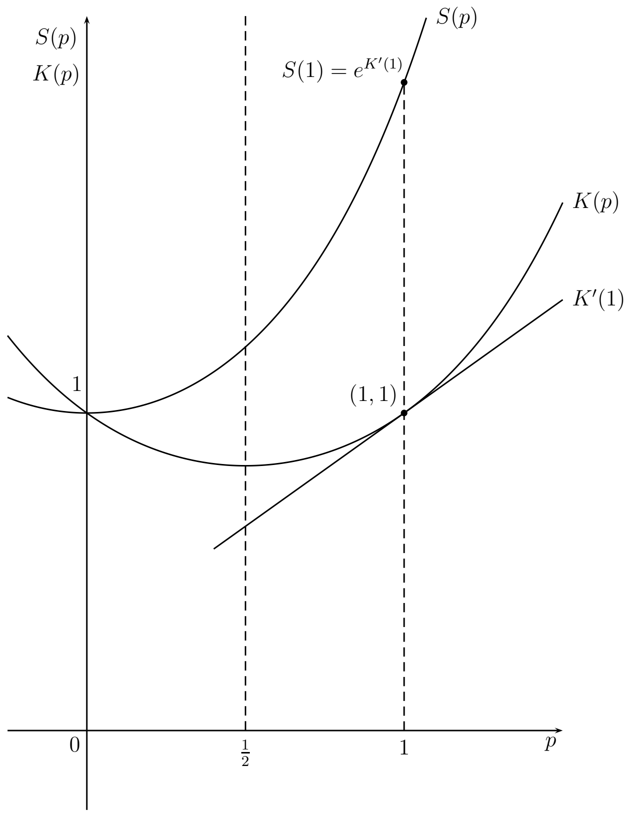

Figure 2.1: Relation between $K(p)$ and $S(p)$

□

Moreover, we have the following most crucial result on the Kantorovich constant.

Theorem 2.57 *Let* $h > 1$. *Then*

$$S(h,1) = e^{-K'(0)} = e^{K'(1)}.$$

Proof. Differentiate $K(p)$ by p, we obtain $K'(p)$ as follows:

$$\delta K'(p) = \tag{2.95}$$

$$\delta \frac{\left(\frac{(p-1)}{p}\frac{(h^p-1)}{(h^p-h)}\right)^p}{(h-1)(h^p-1)} \left\{ \frac{h^p(h^p-1+p-hp)\log h + (h^p-1)(h^p-h)\log \frac{(p-1)(h^p-1)}{p(h^p-h)}}{p-1} \right\}.$$

By using l'Hospital's theorem to (2.95), we have

$$\begin{aligned}\lim_{p\to 1} K'(p) &= \frac{h-1}{h\log h}\frac{1}{(h-1)^2}\left\{h\log h(h\log h+1-h)+(h-1)h\log h\log\left[\frac{h-1}{h\log h}\right]\right\} \\ &= \delta\frac{h}{h-1}\log h - 1 + \log\left[\frac{h-1}{h\log h}\right] = \delta\log\left[\frac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}}\right] = \delta\log S(h,1)\end{aligned}$$

so that we have $S(h,1) = e^{K'(1)}$ and also $S(h,1) = e^{-K'(0)}$ by the same way. □

Next, we observe the difference type of Kantorovich inequality. Let A be a positive operator on a Hilbert space H satisfying $M1_H \geq A \geq m1_H$ for some scalars $0 < m < M$. By Theorem 1.30 and Theorem 1.31, we have

$$0 \leq (A^{-1}x,x) - (Ax,x)^{-1} \leq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{Mm}$$

and

$$0 \leq (A^2x,x) - (Ax,x)^2 \leq \frac{(M-m)^2}{4}$$

for every unit vector $x \in H$.

By using the Mond-Pečarić method, we have the difference type converse of Hölder-McCarthy inequality by Corollary 2.31.

Theorem 2.58 *Let A be a positive operator on a Hilbert space H satisfying $M1_H \geq A \geq m1_H$ for some scalars m and M. Then*

$$0 \leq (A^p x,x) - (Ax,x)^p \leq C(m,M,p) \qquad \text{for } p \notin [0,1]$$

and

$$0 \geq (A^p x,x) - (Ax,x)^p \geq C(m,M,p) \qquad \text{for } p \in [0,1]$$

for every unit vector $x \in H$, where

$$C(m,M,p) = (p-1)\left(\frac{M^p-m^p}{p(M-m)}\right)^{\frac{p}{p-1}} + \frac{Mm^p - mM^p}{M-m} \tag{2.96}$$

for any real number $p \in \mathbb{R}$.

We collect basic properties of $C(m,M,p)$:

Lemma 2.59 *Let $M > m > 0$ and $p \in \mathbb{R}$.*

(i) $C(m,M,p) = \frac{mM^p-Mm^p}{M-m}\left\{K(m,M,p)^{\frac{1}{p-1}} - 1\right\}$.

(ii) $0 \leq C(m,M,p) \leq M(M^{p-1}-m^{p-1})$ *for all* $p > 1$.

(iii) $C(m,M,1) = 0$.

(iv) $\lim_{r\to 0} C\left(m^r, M^r, \frac{p}{r}\right) = L(m^p, M^p)\log S(h,p)$ *for all* $p \in \mathbb{R}$ *and* $h = \frac{M}{m}$.

Proof. (i) By a simple calculation, we have (i).

(ii) Since $K(m,M,p) \geq 1$ for $p \geq 1$, we have $C(m,M,p) = \frac{mM^p - Mm^p}{M-m} \left\{K(m,M,p)^{\frac{1}{p-1}} - 1\right\} \geq 0$. By Theorem 2.54, we have

$$\begin{aligned} C(m,M,p) &\leq \frac{mM^p - Mm^p}{M-m}\left(\frac{M}{m} - 1\right) \\ &= M(M^{p-1} - m^{p-1}). \end{aligned}$$

(iii) We have to only put $p = 1$ in (ii).

(iv) For all $p \in \mathbb{R}$, we have

$$\begin{aligned} C(m^r, M^r, \frac{p}{r}) &= \frac{m^r M^p - M^r m^p}{M^r - m^r}\left\{K\left(m^r, M^r, \frac{p}{r}\right)^{\frac{r}{p-r}} - 1\right\} \\ &= \frac{r}{h^r - 1} m^p (h^p - h^r) \frac{K\left(m^r, M^r, \frac{p}{r}\right)^{\frac{r}{p-r}} - 1}{r} \\ &\to \frac{1}{\log h}(M^p - m^p)\log S(h,p)^{\frac{1}{p}} \qquad (\text{as } r \to 0) \\ &= L(m^p, M^p)\log S(h,p). \end{aligned}$$

□

Finally, we study several inequalities on the continuous power mean.

Theorem 2.60 *Let* A *be a positive invertible operator on a Hilbert space* H *and* x *a unit vector in* H. *Then the continuous power mean* $(A^r x, x)^{1/r}$ *is monotone increasing for* $r \in \mathbb{R}$, *that is,*

$$(A^r x, x)^{1/r} \leq (A^s x, x)^{1/s} \qquad \text{for } r \leq s.$$

Proof. The case of $0 < r < s$ and $r < s < 0$ are shown in Theorem 2.42. For the case of $r < 0 < s$, by Theorem 1.4 (Hölder-McCarthy inequality)

$$\frac{s}{r} < 0 \quad \text{implies} \quad (A^r x, x)^{s/r} \leq (A^{r(s/r)} x, x) = (A^s x, x).$$

By raising both sides to the power $\frac{1}{s} > 0$, we have the desired inequality. □

By virtue of Theorem 2.53, we show a ratio type reverse inequality to Theorem 2.60.

Theorem 2.61 *Let* A *be a positive invertible operator on a Hilbert space* H *such that* $M1_H \geq A \geq m1_H > 0$ *for some scalars* m *and* M. *Then for* $r \leq s$

$$(A^s x, x)^{1/s} \leq \Delta(h,r,s)(A^r x, x)^{1/r}$$

for every unit vector $x \in H$, where

$$\Delta(h,r,s) = \begin{cases} \left(\dfrac{r}{s-r}\dfrac{h^s-h^r}{h^r-1}\right)^{1/s}\left(\dfrac{s}{r-s}\dfrac{h^r-h^s}{h^s-1}\right)^{-1/r} & \text{if } rs \neq 0, \\ \left(\dfrac{h^{\frac{s}{h^s-1}}}{e\log h^{\frac{s}{h^s-1}}}\right)^{1/s} & \text{if } r=0, \\ \left(\dfrac{h^{\frac{r}{h^r-1}}}{e\log h^{\frac{r}{h^r-1}}}\right)^{-1/r} & \text{if } s=0. \end{cases}$$

Proof. Suppose that $s \geq r > 0$. By using Theorem 2.53 to $M^r 1_H \geq A^r \geq m^r 1_H > 0$, we have

$$K\left(m^r, M^r, \frac{s}{r}\right)(A^r x, x)^{\frac{s}{r}} \geq (A^s x, x),$$

since $\frac{s}{r} > 1$. Therefore it follows that

$$K\left(m^r, M^r, \frac{s}{r}\right)^{\frac{1}{s}}(A^r x, x)^{\frac{1}{r}} \geq (A^s x, x)^{\frac{1}{s}},$$

since $\frac{1}{s} > 0$.

Suppose that $0 > s \geq r$. By using Theorem 2.53 to $m^s 1_H \geq A^s \geq M^s 1_H > 0$, we have

$$K\left(M^s, m^s, \frac{r}{s}\right)(A^s x, x)^{\frac{r}{s}} \geq (A^r x, x),$$

since $\frac{r}{s} > 1$. Therefore it follows that

$$K\left(M^s, m^s, \frac{r}{s}\right)^{\frac{1}{r}}(A^s x, x)^{\frac{1}{s}} \leq (A^r x, x)^{\frac{1}{r}},$$

since $\frac{1}{r} < 0$ and hence we have

$$\begin{aligned} K\left(m^r, M^r, \frac{s}{r}\right)^{\frac{1}{s}}(A^r x, x)^{\frac{1}{r}} &= K\left(M^s, m^s, \frac{r}{s}\right)^{-\frac{1}{r}}(A^r x, x)^{\frac{1}{r}} \\ &\geq (A^s x, x)^{\frac{1}{s}}. \end{aligned}$$

In the case $s > 0 > r$, we can show this theorem similarly. If we put $r \to 0$, then notice that $(A^r x, x)^{\frac{1}{r}} \to \Delta_x(A)$ by Theorem 2.42. In the case $r = 0 < s$, it follows from Corollary 2.50 that

$$(A^s x, x) \leq S(h,s)\Delta_x(A^s).$$

Since $s > 0$, we have

$$(A^s x, x)^{\frac{1}{s}} \leq S(h,s)^{\frac{1}{s}}\Delta_x(A^s)^{\frac{1}{s}} = S(h,s)^{\frac{1}{s}}\Delta_x(A).$$

Similarly, in the case $r < 0 = s$, we have

$$\Delta_x(A) \leq S(h,r)^{-\frac{1}{r}}(A^r x, x)^{\frac{1}{r}}.$$

$\square$

Remark 2.9 *If we put $s = 1$ and $r \to 0$ in Theorem 2.61, then we obtain a continuous version of the Specht theorem (Theorem 2.49):*

$$(Ax,x) \le S(h,1)\exp(\log Ax,x) = S(h,1)\Delta_x(A)$$

for every unit vector $x \in H$.

Therefore we have the following expression: Let $h > 0$ and $r,s \in \mathbb{R}$. Then

$$\Delta(h,r,s) = \begin{cases} K(h^r, \frac{s}{r})^{\frac{1}{s}} & \text{if} \quad rs \neq 0, \\ \Delta(h,0,s) = S(h^s)^{\frac{1}{s}} & \text{if} \quad r = 0, \\ \Delta(h,r,0) = S(h^r)^{-\frac{1}{r}} & \text{if} \quad s = 0. \end{cases} \tag{2.97}$$

In particular, if we put $r = 0$ and $s = 1$, then $\Delta(h,0,1) = S(h,1)$. Thus we call $\Delta(h,r,s)$ **a generalized Specht ratio**. We investigate basic properties of a generalized Specht ratio, also see Lemma 2.47.

Theorem 2.62 *For given $r \le s$, a generalized Specht ratio have the following properties.*

(i) For given $r < s$, a function $\Delta(h,r,s)$ is strictly decreasing for $0 < h < 1$ and strictly increasing for $h > 1$.

(ii) $\Delta(1,r,s) = 1$ and $\Delta(h,r,s) = \Delta(h^{-1},r,s)$ for all $h > 0$.

(iii) For $h > 1$, $\Delta(h,r,s) \to h$ as $s \to \infty$.

(iv) For $0 < h < 1$, $\Delta(h,r,s) \to \frac{1}{h}$ as $s \to \infty$.

(v) $\Delta(h,r,s) = \Delta(h,s,r)^{-1}$.

(vi) $\Delta(h,-s,-r) = \Delta(h,s,r)$.

(vii) $\Delta(h,r-s,s) = \Delta(h,s,r)^{\frac{-s}{r-s}}$ for $s > 0$.

Proof. Let $\Delta(h) = \Delta(h,r,s)$. Since

$$\log\Delta(h) = \frac{1}{s}\log\left(\frac{r}{s-r}\frac{h^s-h^r}{h^r-1}\right) - \frac{1}{r}\log\left(\frac{s}{s-r}\frac{h^s-h^r}{h^s-1}\right),$$

it follows from L'Hospital's theorem that

$$\begin{aligned} \lim_{h\to 1}\log\Delta(h) &= \lim_{h\to 1}\frac{1}{s}\log\left(\frac{r}{s-r}\frac{sh^{s-1}-rh^{r-1}}{rh^{r-1}}\right) - \frac{1}{r}\log\left(\frac{s}{s-r}\frac{sh^{s-1}-rh^{r-1}}{sh^{s-1}}\right) \\ &= \frac{1}{s}\log 1 - \frac{1}{r}\log 1 \\ &= 0 \end{aligned}$$

and so $\Delta(1) = 1$. Also, we have $\Delta(h) = \Delta\left(\frac{1}{h}\right)$ by a direct computation. Therefore, we have (ii).

Furthermore we have by a differential calculation

$$\frac{d}{dh}\log\Delta(h) = h^{r-1}k(h)\frac{r(h^s-1)-s(h^r-1)}{sr(h^r-1)(h^s-1)(h^s-h^r)}$$

where

$$k(h) = (s-r)h^s - sh^{s-r} + r.$$

Suppose that $0 < r < s$. Then we have $k(h) > 0$ since $k'(h) > 0$ and $k(1) = 0$. Since $\frac{h^r-1}{r}$ is strictly increasing for $r \in \mathbb{R}$, it follows that $0 < r < s$ implies $s(h^r-1) < r(h^s-1)$. Therefore we have $\frac{d}{dh}\log\Delta(h) > 0$. Similarly we have $\frac{d}{dh}\log\Delta(h) > 0$ in the case of $r < 0 < s$ or $r < s < 0$ and hence a function $\Delta(h)$ is strictly increasing for all $h > 1$. On the other hand, we see that a function $\Delta(h)$ is strictly decreasing for all $0 < h < 1$. Therefore we have (i).

Suppose that $h > 1$. Then

$$\left(\frac{s}{s-r}\frac{h^s-h^r}{h^s-1}\right)^{-1/r} = \left(\frac{1}{1-\frac{r}{s}}\frac{1-h^{r-s}}{1-h^{-s}}\right)^{-1/r} \to 1$$

as $s \to \infty$ and by L'Hospital's theorem

$$\begin{aligned}\lim_{s\to\infty}\log\left(\frac{r}{s-r}\frac{h^s-h^r}{h^r-1}\right)^{1/s} &= \lim_{s\to\infty}\frac{\log\left(\frac{r}{s-r}\frac{h^s-h^r}{h^r-1}\right)}{s} \\ &= \lim_{s\to\infty}\left(\frac{h^s}{h^s-h^r}\log h - \frac{1}{s-r}\right) \to \log h.\end{aligned}$$

Therefore we have $\Delta(h,r,s) \to h$ as $s \to \infty$. If $0 < h < 1$, then $1/h > 1$ and as we proved above, we have $\Delta(h,r,s) = \Delta(h^{-1},r,s) \to h^{-1}$ as $s \to \infty$ by (ii). Therefore we have (iii) and (iv).

By definition, we have (v),(vi) and (vii). □

Next, we show a difference type reverse inequalities to Theorem 2.60.

Theorem 2.63 *Let A be a positive invertible operator on a Hilbert space H such that $M1_H \geq A \geq m1_H > 0$ for some scalars m and M. Then for $r \leq s$*

$$(A^sx,x)^{1/s} - (A^rx,x)^{1/r} \leq \max_{\theta\in[0,1]}\left\{(\theta M^s + (1-\theta)m^s)^{\frac{1}{s}} - (\theta M^r + (1-\theta)m^r)^{\frac{1}{r}}\right\}$$

for every unit vector $x \in H$.

Proof. Suppose that $0 < r \leq s$. Since $0 < \frac{r}{s} < 1$, we have

$$A^{\frac{r}{s}} \geq \frac{M^{\frac{r}{s}} - m^{\frac{r}{s}}}{M-m}A + \frac{Mm^{\frac{r}{s}} - mM^{\frac{r}{s}}}{M-m}1_H.$$

Replacing A by A^s, we have

$$(\overline{\mu}(A^sx,x) + \overline{\nu})^{\frac{1}{r}} \leq (A^rx,x)^{\frac{1}{r}}$$

for every unit vector $x \in H$, where

$$\overline{\mu} = \frac{M^r - m^r}{M^s - m^s} \qquad \text{and} \qquad \overline{\nu} = \frac{M^s m^r - m^s M^r}{M^s - m^s}.$$

Therefore, it follows that

$$\begin{aligned}(A^s x,x)^{1/s} - (A^r x,x)^{1/r} &\le (A^s x,x)^{1/s} - (\overline{\mu}(A^s x,x) + \overline{\nu})^{\frac{1}{r}} \\ &= \max_{t \in \overline{T}} \left\{ t^{\frac{1}{s}} - (\overline{\mu} t + \overline{\nu})^{\frac{1}{r}} \right\},\end{aligned}$$

where T denotes the open interval joining m^s to M^s and $\overline{T}$ is the closure of T. We set $\theta = (t - m^s)/(M^s - m^s)$. Then a simple calculation implies $\overline{\mu} t + \overline{\nu} = \theta M^r + (1-\theta) m^r$. □

Remark 2.10 *If we put $s = 1$ and $r \to 0$ in Theorem 2.63, then we obtain a continuous version of the Mond-Shisha theorem (Theorem 2.52):*

$$(Ax,x) - \Delta_x(A) = (Ax,x) - \exp(\log Ax,x) \le D(m,M)$$

for every unit vector $x \in H$, because

$$\max_{\theta \in [0,1]} \{\theta M + (1-\theta) m - M^{\theta} m^{1-\theta}\} = D(m,M).$$

Finally, we show an alternative proof of Theorem 2.57 by means of a generalized Specht ratio:

Theorem 2.64 *The following property on $K(p) = K(h,p)$ and $S(h) = S(h,1)$ hold:*

$$S(h) = e^{K'(1)} = e^{-K'(0)}.$$

Proof. Since $\Delta(h,r,1) = K\left(h^r, \frac{1}{r}\right) = K(h,r)^{-1/r}$, we have

$$\begin{aligned}\log S(h) &= \lim_{r \to 0} \log \Delta(h,r,1) = \lim_{r \to 0} -\frac{\log K(h,r)}{r} \\ &= \lim_{r \to 0} -\frac{\log K(h,r) - \log K(h,0)}{r - 0} = -\frac{K'(0)}{K(0)} = -K'(0)\end{aligned}$$

and hence $\log S(h) = -K'(0)$. On the other hand,

$$\Delta(h,p,p+1)^{p+1} \to \Delta(h,0,1) = S(h) \quad \text{as } p \to 0$$

and hence

$$\begin{aligned}\log \Delta(h,p,p+1)^{p+1} &= \log K\left(h^p, \frac{p+1}{p}\right) = \log K(h,p+1)^{\frac{1}{p}} \\ &= \frac{\log K(h,p+1) - \log K(h,1)}{p+1-1} \to \frac{K'(h,1)}{K(h,1)} = K'(h,1)\end{aligned}$$

as $p \to 0$. Therefore we have $\log S(h) = K'(h,1)$. □

2.8 Notes

A generalization of the Kantorovich inequality is firstly initiated by Ky Fan [30], a power mean version generalization by Mond [138] and a matrix version generalization by Mond and Pečarić [151] and [146]. Mond and Pečarić established the method which gives complementary inequalities to Jensen's type inequalities associated with convex functions in [141] and [150]. Elementary proofs of both extensions of Ky Fan and Mond and Pečarić generalizations of the Kantorovich type inequalities are given in Furuta [71, 72, 78]. On the other hand, S.-E.Takahasi, Tsukada, Tanahashi and Ogiwara [182] discussed an inverse type of Jensen's inequality for convex functions in the framework of integral theory. By reconstructing both ideas of Furuta and Takahasi, we rediscover new merits for the Mond-Pečarić method and have many applications in operator inequalities.

The results included in Sections 2.1, 2.2 and 2.3 are essentially due to Mond and Pečarić [144, 145]. For our exposition in Section 2.4 we have used [136]. The determinant for invertible operators to the canonical trace was introduced by Fuglede and Kadison [32, 33] and afterwards Arveson [11] developed it in general von Neumann algebras. An extension of the notion of the determinant to vector states is due to J.I.Fujii and Seo [49], and J.I.Fujii, Izumino and Seo [45]. The results in Section 2.6 are due to [52, 186, 49, 45, 51]. Theorem 2.45 is due to Arveson [11]. The Specht ratio is due to Specht [178] and rediscovered by Izumino [45]. Properties of the Specht ratio are due to Tominaga [186] and J.I.Fujii, Seo and Tominaga [52]. Corollary 2.50 is essentially due to [72]. Theorem 2.54 and Lemma 2.55 are due to Furuta [87] and J.I.Fujii, M.Fujii, Seo and Tominaga [43]. Theorem 2.56 is due to Yamazaki and Yanagida [199]. Theorem 2.57 is due to Furuta [88]. Lemma 2.59 is due to Yamazaki [197]. Difference and ratio inequalities in power means are due to Mond [138], Mond and Shisha [158, 159, 177] and Yamazaki [197]. The condition number is introduced by Turing [189].

Chapter 3

Li-Mathias type inequality

In this chapter we develope a generalization of a theorem of Li-Mathias as an application of Mond-Pečarić method. We study complementary inequalities to Jensen's type inequalities under a more general setting without the assumption of the convexity and the concavity. Lower and upper boundaries in converses of Jensen's type inequalities are given. In the finite dimensional case we see that boundaries on complementary inequalities are the optimum estimate in the sense that a non trivial positive linear map attaining the equality is given.

3.1 Preliminary and Li-Mathias inequality

In this section we first introduce results due to Li and Mathias. Let $\mathscr{M}_n$ be an algebra of all $n \times n$ complex square matrices with matrix norm and $\mathscr{H}_n$ subspace of all $n \times n$ Hermitian matrices with partial order. Li and Mathias showed the following complementary inequalities to Jensen's type inequalities under a more general setting:

Theorem 3.1 *Let $f \in \mathscr{C}([m,M])$ be a real valued continuous function on an interval $[m,M]$ and $A \in \mathscr{H}_n$ a $n \times n$ Hermitian matrix with $\mathsf{Sp}(A) \subseteq [m,M]$ for some scalars $m < M$. Let $\Phi \in \mathbf{P}_N[\mathscr{M}_n, \mathscr{M}_k]$ be a normalized positive linear map. Then*

$$\begin{aligned}
&\left[\max_{\substack{\varphi\in\{conc.\}\\ f\leq\varphi}}\min_{m\leq t\leq M}\{f(t)-\varphi(t)\}\right]1_k\\
&\leq f(\Phi(A))-\Phi(f(A)) \qquad (3.1)\\
&\leq\left[\min_{\substack{\varphi\in\{conx.\}\\ f\geq\varphi}}\max_{m\leq t\leq M}\{f(t)-\varphi(t)\}\right]1_k.
\end{aligned}$$

Additionally, if $f(t)>0$ for all $t\in[m,M]$, then

$$\begin{aligned}
&\left[\max_{\substack{\varphi\in\{conc.\}\\ f\leq\varphi}}\min_{m\leq t\leq M}\{f(t)/\varphi(t)\}\right]\Phi(f(A))\\
&\leq f(\Phi(A)) \qquad (3.2)\\
&\leq\left[\min_{\substack{f\in\{conx.\}\\ f\geq\varphi>0}}\max_{m\leq t\leq M}\{f(t)/\varphi(t)\}\right]\Phi(f(A)),
\end{aligned}$$

where $\{conx.\}$ (resp. $\{conc.\}$) is the set of all matrix convex (resp. matrix concave) functions on $[m,M]$.

Furthermore, if f is concave or convex, then they showed that boundaries in Theorem 3.1 are the optimum estimate. We denote by $\lambda_{min}(A)=\min\{\lambda|\lambda\in\mathsf{Sp}(A)\}$ and $\lambda_{max}(A)=\max\{\lambda|\lambda\in\mathsf{Sp}(A)\}$ for a Hermitian matrix $A\in\mathscr{H}_n$.

Theorem 3.2 *Let $A\in\mathscr{H}_n$ be a $n\times n$ Hermitian matrix with $\lambda_{min}(A)=m$ and $\lambda_{max}(A)=M$. Let $f\in\mathscr{C}([m,M])$ and put*

$$h(t)=\frac{f(M)-f(m)}{M-m}(t-m)+f(m).$$

If f is concave (resp. convex) on $[m,M]$, then the minimum (resp. maximum) in the upper (resp. lower) boundary in (3.1) in Theorem 3.1 are attained at the linear function h and in each case there is a normalized positive linear map Φ for which $f(\Phi(A))-\Phi(f(A))$ is equal to the upper (resp. lower) boundary.

Li and Mathias applied their inequalities to the power functions and obtained the following converses of Jensen's type inequalities in Corollary 1.22 (i) and (ii): Let $A\in\mathscr{H}_n$ be a Hermitian matrix with $\mathsf{Sp}(A)\subseteq[m,M]$ for some scalars $m<M$ and $\Phi\in P_N[\mathscr{M}_n,\mathscr{M}_k]$ be a normalized positive linear map. Then real constants α_j and β_j $(j=1,2)$ such that for $p\in\mathbb{R}\backslash\{0\}$

$$\beta_2 1_k\leq\Phi(A)^p-\Phi(A^p)\leq\beta_1 1_k, \qquad (3.3)$$

$$\alpha_2\Phi(A^p)\leq\Phi(A)^p\leq\alpha_1\Phi(A^p), \qquad (3.4)$$

are explicitly determined, which depends on m,M,p and $h=M/m$.

Remark 3.1 The quantity

$$h(A) := \begin{cases} \|A^{-1}\|\|A\| & \text{if } A \text{ is nonsingular,} \\ \infty & \text{if } A \text{ is singular.} \end{cases}$$

is called **the condition number** for matrix inversion with respect to the matrix norm $\|\cdot\|$ [107, pp. 336-340]. If $A \in \mathscr{H}_n$ is a Hermitian positive definite matrix with $\lambda_{min}(A) = m$ and $\lambda_{max}(A) = M$, then it follows that $h(A) = M/m$.

Moreover, Li and Mathias cited that the following two problems for continuous functions f and g could be studied.

Problem 1. Determine β_1, β_2 such that for a given Hermitian matrix A with $\mathsf{Sp}(A) \subseteq [m,M]$,

$$\beta_1 1_k \le \Phi(f(A)) - g(\Phi(A)) \le \beta_2 1_k.$$

For solution of this problem they said that we can let

$$\begin{aligned} \beta_1 &= \max_{g(t) \le ct+d} \min_{m \le t \le M} \left\{ t - (cf^{-1}(t) + d) \right\}, \\ \beta_2 &= \min_{g(t) \ge ct+d} \max_{m \le t \le M} \left\{ t - (cf^{-1}(t) + d) \right\} \end{aligned}$$

if f^{-1} exists.

Problem 2. Determine α_1, α_2 such that for a given Hermitian matrix A with $\mathsf{Sp}(A) \subseteq [m,M]$,

$$\alpha_1 \Phi(f(A)) \le g(\Phi(A)) \le \alpha_2 \Phi(f(A)).$$

In the following section, we show converses of Jensen's type inequalities as the solutions of problems 1 and 2 due to Li and Mathias.

Though it is just a repetition of the previous chapter, we observe converses of Jensen's inequality in the framework of matrix theory as a special case of (3.1) in the remainder of this section. We recall the famous Kantorovich inequality in the framework of matrix theory, which is given by Nobel prize winner Leonid Vitalevič Kantorovič [2, 195] in 1948: *Let $A \in \mathscr{H}_n$ be a $n \times n$ Hermitian matrix with eigenvalues $0 < \lambda_1 \le \ldots \le \lambda_n$ and $x \in \mathbb{C}^n$. Then*

$$\frac{x^*Ax \cdot x^*A^{-1}x}{(x^*x)^2} \le \frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1\lambda_n}. \tag{3.5}$$

The equality is attained if and only if

$$\lambda_1 = \ldots = \lambda_l < \lambda_{l+1} \le \ldots \le \lambda_{n-h} < \lambda_{n-h+1} = \ldots = \lambda_n, \tag{3.6}$$

where $l \ge 1$ and $h \ge 1$ are multiple of λ_1 and λ_n respectively and

$$x = \frac{1}{\sqrt{2}}(E_1u_1 \pm E_2u_2).$$

Here $E_1 \in \mathscr{M}_{n,l}$ and $E_2 \in \mathscr{M}_{n,h}$ are matrices which columns contain orthogonal eigenvectors of matrix A corresponding to λ_1 and λ_n respectively, and u_1 and u_2 are unit vectors. A converse of the inequality (3.5) is: $1 \le x^*Ax \cdot x^*A^{-1}x/(x^*x)^2$. This is the well know version of Cauchy-Schwartz inequality: $(x^*y)^2 \le x^*x \cdot y^*y$, where $x, y \neq \mathbf{0}$ are vectors.

In 1966 Ky Fan [30] proved the following inequality. *Let* $A_j \in \mathscr{H}_n$ *with* $\mathsf{Sp}(A_j) \subseteq [m, M]$, $j = 1, \ldots, k$, $x_j \in \mathbb{C}^n$ *such that* $\sum_{j=1}^k x_j^* x_j = 1$ *and* $p \neq 0, 1$ *arbitrary integer number* (*not necessarily positive*). *Then*

$$\frac{\sum_{j=1}^k x_j^* A^p x_j}{\left(\sum_{j=1}^k x_j^* A x_j\right)^p} \le \frac{(p-1)^{p-1}}{p^p} \frac{(M^p - m^p)^p}{(M-m)(mM^p - Mm^p)^{p-1}}. \tag{3.7}$$

If we put $p = -1$ in (3.7) we obtain a generalized Kantorovich inequality (3.5):

$$\sum_{j=1}^k x_j^* A x_j \cdot \sum_{j=1}^k x_j^* A^{-1} x_j \le \frac{(M+m)^2}{4Mm},$$

which was proved by Ky Fan in 1959.

In 1997 Mond and Pečarić [154, Theorems 3, 4] gave the following two generalization of Ky Fan inequality (3.7) :

Theorem 3.3 *Let* $A_j \in \mathscr{H}_n$ *with* $\mathsf{Sp}(A_j) \subseteq [m, M]$, $j = 1, \cdots, k$ *and* $U_j \in \mathscr{M}_{r,n}$ *such that* $\sum_{j=1}^k U_j U_j^* = 1_r$. *Let* f *be a strictly convex differentiable function on* $[m, M]$. *Then the inequality*

$$\sum_{j=1}^k U_j f(A_j) U_j^* \le \beta\, 1_r + \sum_{j=1}^k f\left(U_j A_j U_j^*\right) \tag{3.8}$$

holds for some β *such that* $0 < \beta < (M-m)(\mu_f - f'(m))$. *The value of* β *(which depends on m, M, f) in (3.8) can be determined more precisely as follows: Let* $t = \overline{t}$ *be the unique solution of* $f'(t) = \mu$, $(m < \overline{t} < M)$; *then* $\beta = f(m) - f(\overline{t}) + \mu(t - m)$ *is sufficient for the inequality (3.8).*

Theorem 3.4 *Let* A_j, U_j, $j = 1, \cdots, k$ *be as in Theorem 3.3. Let* f *be a strictly convex two differentiable function on* $[m, M]$. *Assume that, additionally, either of the following conditions holds:* (i) $f > 0$ *on* $[m, M]$ *or* (ii) $f < 0$ *on* $[m, M]$. *Then the inequality*

$$\sum_{j=1}^k U_j f(A_j) U_j^* \le \alpha_o \sum_{j=1}^k f\left(U_j A_j U_j^*\right) \tag{3.9}$$

holds for $\alpha_o > 1$ *in case* (i); *or* $0 < \alpha_o < 1$ *in case* (ii). *The value of* α_o *in (3.9) can be determined more precisely as follows: If* $\mu_f = 0$, *let* $t = \overline{t}$ *be the unique solution of* $f'(t) = 0$, $(m < \overline{t} < M)$; *then* $\alpha_o = f(m)/f(\overline{t})$ *is sufficient for the inequality (3.9). If* $\mu_f \neq 0$, *let* $t = \overline{t}$ *be the unique solution of* $\mu_f f(t) - f'(t)(f(m) + \mu_f(t-m)) = 0$, $(m < \overline{t} < M)$; *then* $\alpha_o = \mu_f / f'(\overline{t})$ *is sufficient for the inequality (3.9).*

In the following section we give a generalization of the inequalities (3.1) and (3.2) due to Li and Mathias in the framework of operator theory. In the last section we apply these inequalities to power functions and obtain a generalization of Ky Fan type inequalities (3.8) and (3.9). We moreover devote special attention to consideration the cases where the equality holds.

3.2 Generalization of Li-Mathias inequality

We first cite the following Jensen's type inequality for two functions as a special case of Lemma 2.2.

Lemma 3.5 *Let $A \in \mathscr{B}_h(H)$ be a selfadjoint operator with $\mathsf{Sp}(A) \subseteq [m,M]$ for some scalars $m < M$ and $\Phi \in \mathbf{P}_N[\mathscr{B}(H),\mathscr{B}(K)]$ a normalized positive linear map. Let $f,g \in \mathscr{C}([m,M])$ such that $f \leq g$ on $[m,M]$. If $f \in \mathscr{C}([m,M])$ is operator convex on $[m,M]$, then*

$$f(\Phi(A)) \leq \Phi(g(A)). \tag{3.10}$$

In the case where f is operator concave such that $f \geq g$ on $[m,M]$, we have the opposite inequality.

We shall generalize a theorem of Li-Mathias (Theorem 3.1). Notice that the convexity of f is not assumed in Theorem 3.6. We denote by $\{conx.\}$ (resp. $\{conc.\}$) the set of all operator convex (resp. operator concave) functions on $[m,M]$.

Theorem 3.6 *Let $A \in \mathscr{B}_h(H)$ be a selfadjoint operator with $\mathsf{Sp}(A) \subseteq [m,M]$ for some scalars $m < M$ and $\Phi \in \mathbf{P}_N[\mathscr{B}(H),\mathscr{B}(K)]$ a normalized positive linear map. Let $f,g \in \mathscr{C}([m,M])$ and $F(u,v)$ a real valued function defined on $U \times V$, where U and V are intervals such that $U \supset f[m,M]$, $V \supset g[m,M]$. If $F(u,v)$ is operator monotone in the first variable u, then*

$$\begin{aligned} &\left\{ \max_{\substack{\varphi \in \{conx.\} \\ \varphi \leq f}} \min_{m \leq t \leq M} F\left[\varphi(t), g(t)\right] \right\} 1_K \\ &\leq F\left[\Phi\left(f(A)\right), g\left(\Phi(A)\right)\right] \qquad (3.11) \\ &\leq \left\{ \min_{\substack{\varphi \in \{conc.\} \\ \varphi \geq f}} \max_{m \leq t \leq M} F\left[\varphi(t), g(t)\right] \right\} 1_K. \end{aligned}$$

Proof. We prove the right hand inequality of (3.11). Let φ be an operator concave function on $[m,M]$ such that $f \leq \varphi$ on $[m,M]$. It follows from Lemma 3.5 that $\Phi(f(A)) \leq \varphi(\Phi(A))$. Using the operator non-decreasing character of $F(\cdot,v)$, we have

$$\begin{aligned} &F\left[\Phi\left(f(A)\right), g\left(\Phi(A)\right)\right] \leq F\left[\varphi\left(\Phi(A)\right), g\left(\Phi(A)\right)\right] \\ &\leq \left\{ \max_{t \in \mathsf{Sp}(\Phi(A))} F\left[\varphi(t), g(t)\right] \right\} 1_K \leq \left\{ \max_{m \leq t \leq M} F\left[\varphi(t), g(t)\right] \right\} 1_K. \end{aligned}$$

Therefore, we minimize this boundary over all operator concave functions φ on $[m,M]$ such that $\varphi \geq f$ to obtain the upper boundary in (3.11). We prove the left hand inequality in the same way. □

As a complementary result, we cite the following theorem:

Theorem 3.7 *Under the same hypothesis as in Theorem 3.6, except that F is operator non-increasing in its first variable, the following inequalities hold*

$$\begin{aligned} &\left\{ \max_{\substack{\varphi \in \{conc.\} \\ \varphi \geq f}} \min_{m \leq t \leq M} F\left[\varphi(t), g(t)\right] \right\} 1_K \\ &\leq F\left[\Phi\left(f(A)\right), g\left(\Phi(A)\right)\right] \qquad (3.12) \\ &\leq \left\{ \min_{\substack{\varphi \in \{conx.\} \\ \varphi \leq f}} \max_{m \leq t \leq M} F\left[\varphi(t), g(t)\right] \right\} 1_K. \end{aligned}$$

Proof. We prove this theorem in the same way as Theorem 3.6. □

If we put $g \equiv f$ in Theorem 3.6, then we have the following corollary:

Corollary 3.8 *Let A, Φ and F be as in Theorem 3.6. If $f \in \mathscr{C}([m,M])$, then*

$$\begin{aligned} &\left\{ \max_{\substack{\varphi \in \{conx.\} \\ \varphi \leq f}} \min_{m \leq t \leq M} F\left[\varphi(t), f(t)\right] \right\} 1_K \\ &\leq F\left[\Phi\left(f(A)\right), f\left(\Phi(A)\right)\right] \qquad (3.13) \\ &\leq \left\{ \min_{\substack{\varphi \in \{conc.\} \\ \varphi \geq f}} \max_{m \leq t \leq M} F\left[\varphi(t), f(t)\right] \right\} 1_K. \end{aligned}$$

Remark 3.2 Notice that the set $\{\varphi \mid \varphi \in \{conc.\},\ \varphi \geq f \text{ on } [m,M]\}$ is partial ordered with relation $\geq$. Because the constant function $\varphi(t) = \max_{m \leq s \leq M} f(s)$ and $\psi(t) = \min_{m \leq s \leq M} f(s)$ for all $t \in [m,M]$ are matrix concave functions, this set is not-empty and is bounded from above. We can show that there indeed is a function φ that attains the minimum at the boundary of the right-hand side of (3.11) over all real valued continuous concave functions. Then φ is a linear function which is equal to f at m and M.

If f is a convex function, then we can determine explicitly an operator concave function φ on $[m,M]$ in the right hand inequality of (3.11) which bounded f at upper side and for which the minimum is attained. For the left hand inequality of (3.11) we similarly have the dual result. This is contents of the following theorem.

Theorem 3.9 *Let A, Φ, F, f and g be as in Theorem 3.6. If f is convex, then*

$$F\left[\Phi\left(f(A)\right), g\left(\Phi(A)\right)\right] \leq \left\{ \max_{m \leq t \leq M} F\left[\mu_f t + \nu_f, g(t)\right] \right\} 1_K. \qquad (3.14)$$

If f is concave, then

$$F\left[\Phi\left(f(A)\right), g\left(\Phi(A)\right)\right] \geq \left\{ \min_{m \leq t \leq M} F\left[\mu_f t + \nu_f, g(t)\right] \right\} 1_K, \qquad (3.15)$$

where

$$\mu_f = \frac{f(M)-f(m)}{M-m} \quad \text{and} \quad \nu_f = \frac{Mf(m)-mf(M)}{M-m}.$$

Proof. If we put $\big(h(t) = \mu_f t + \nu_f\big)$, then h is operator concave. The convexity of f ensures that $f(t) \le h(t)$ for all $t \in [m,M]$. If $\varphi \in \{conc.\}$ is an operator concave function and $f(t) \le \varphi(t)$ for all $t \in [m,M]$ then $h(m) = f(m) \le \varphi(m)$ and $h(M) = f(M) \le \varphi(M)$. Since an operator concave function is necessarily (real) concave, we have $h(t) \le \varphi(t)$ for all $t \in [m,M]$. Using the operator non-decreasing character of $F(\cdot,v)$, we have

$$F\left[h(t),g(t)\right] \le F\left[\varphi(t),g(t)\right] \quad \text{for all } t \in [m,M].$$

It follows that the minimum in the right hand of (3.14) is attained at h. Thus we proved the inequality (3.14). The inequality (3.15) is proved in the same way. □

Remark 3.3 Notice that in hypothesis of Theorem 3.9 we do not need the condition that f is operator concave (resp. operator convex) .

3.3 Li-Mathias type complementary inequalities

As applications of Theorem 3.6, we discuss an extension of Theorem 3.1, which give us a unified view to boundaries in two Li-Mathias inequalities (3.1) and (3.2). As a matter of fact, if we choose an appropriate value of the constant α, then we obtain two converses of Jensen's type inequality. Moreover, we shall consider the optimality of our results.

Theorem 3.10 *Let $A \in \mathscr{B}_h(H)$ be a selfadjoint operator with $\mathsf{Sp}(A) \subseteq [m,M]$ for some scalars $m < M$ and $\Phi \in \mathbf{P}_N[\mathscr{B}(H),\mathscr{B}(K)]$ a normalized positive linear map. Let $f,g \in \mathscr{C}([m,M])$. Then for any real numbers $\alpha \in \mathbb{R}$*

$$\alpha g(\Phi(A)) + \beta_1 1_K \le \Phi(f(A)) \le \alpha g(\Phi(A)) + \beta_2 1_K, \tag{3.16}$$

where

$$\beta_1 = \max_{\substack{\varphi \in \{conx.\} \\ \varphi \le f}} \min_{m \le t \le M} \{\varphi(t) - \alpha g(t)\},$$

$$\beta_2 = \min_{\substack{\varphi \in \{conc.\} \\ \varphi \ge f}} \max_{m \le t \le M} \{\varphi(t) - \alpha g(t)\}.$$

Proof. Let us put $F(u,v) = u - \alpha v$ in Theorem 3.6. Then it follows from the right-hand side of (3.11) that

$$\begin{aligned} \Phi(f(A)) - \alpha g(\Phi(A)) &\le \min_{\substack{\varphi \in \{conc.\} \\ \varphi \ge f}} \max_{m \le t \le M} F[\varphi(t),g(t)]1_K \\ &= \min_{\substack{\varphi \in \{conc.\} \\ \varphi \ge f}} \max_{m \le t \le M} \{\varphi(t) - \alpha g(t)\} 1_K. \end{aligned}$$

We prove the left hand side inequality in (3.16) in the same way. □

Remark 3.4 Boundaries in Theorem 3.10 are rather hard to be evaluated in a general way. One may consider only linear functions φ instead of all operator concave or operator convex functions. This simplifies the evaluation of boundaries at the cost of the possibility to get lower accuracy. For example, we observe the right hand inequality of (3.16) in a special case when $\alpha \in \mathbb{R}_+$ and $f \equiv g$ is a concave function on $[m,M]$. Then we can consider the graph of the linear function φ_r which satisfies $\varphi_r \geq f$ and

$$\min_{\substack{\varphi \in \{conc.\} \\ \varphi \geq f}} \max_{m \leq t \leq M} \{\varphi(t) - \alpha f(t)\} \leq \max_{m \leq t \leq M} \{\varphi_r(t) - \alpha f(t)\},$$

being a tangent to the graph of $y = f(t)$ passing through a point $(r, f(r))$ with $m \leq r \leq M$, that is,

$$\varphi_r(t) = f(r) + f'(r)(t-r).$$

The maximum value $(\max_{m \leq t \leq M}\{\varphi_r(t) - \alpha f(t)\})$ occurs at $t = m$ or M. It follows that the optimal solution of this maximization problem occurs at the function φ_r such that

$$\varphi_r(m) - \alpha f(m) = \varphi_r(M) - \alpha f(M).$$

We are not sure whether the obtained inequality

$$\Phi(f(A)) - \alpha f(\Phi(A)) \leq \left[\max_{m \leq t \leq M}\{\varphi_r(t) - \alpha f(t)\}\right] 1_K = [\varphi_r(m) - \alpha f(m)]\, 1_K$$

is sharp or not in the sense of the following definition.

Definition 3.1 *A right hand (a left hand) inequality of (3.16) is said to be* **sharp** *if for any selfadjoint operator $A \in \mathscr{B}_h(H)$ with $\mathsf{Sp}(A) \subseteq [m,M]$ for some scalars $m < M$ and any two function $f, g \in \mathscr{C}([m,M])$ there is a non-trivial normalized positive linear map $\Phi_0 \in \mathbf{P}_N[\mathscr{B}(H), \mathscr{B}(K)]$ for which the boundary is attained:*

$$\Phi_0(f(A)) = \alpha g(\Phi_0(A)) + \beta 1_K.$$

If we put $\alpha = 1$ in Theorem 3.10, then we obtain a generalization of the inequality (3.1) in Theorem 3.1. Furthermore, a simple differential calculus determines the point for which the extreme value of function $\varphi - g$ needed for Section 3.4 is attained.

Corollary 3.11 *Let A, Φ, f and g as in Theorem 3.10. Then*

$$\begin{aligned} &\left[\max_{\substack{\varphi \in \{conx.\} \\ \varphi \leq f}} \min_{m \leq t \leq M} \{\varphi(t) - g(t)\}\right] 1_K \\ &\leq \Phi(f(A)) - g(\Phi(A)) \qquad (3.17) \\ &\leq \left[\min_{\substack{\varphi \in \{conc.\} \\ \varphi \geq f}} \max_{m \leq t \leq M} \{\varphi(t) - g(t)\}\right] 1_K. \end{aligned}$$

Moreover if g is a strictly convex differentiable function on $[m,M]$, then for any differentiable function $\varphi \in \{conc.\}$ such that $\varphi \geq f$ on $[m,M]$

$$\max_{m \leq t \leq M} \{\varphi(t) - g(t)\} = \varphi(t_o) - g(t_o),$$

where $t_o \in (m,M)$ is defined as the unique solution of $\varphi'(t) = g'(t)$ if $\varphi'(m) > g'(m)$ and $\varphi'(M) < g'(M)$, otherwise t_o is defined as m or M according as $\varphi'(m) \leq g'(m)$ or $\varphi'(M) \geq g'(M)$.

In the dual case, i.e. if g is strictly concave and φ is operator convex on $[m,M]$, the conditions determining t_o change their order.

Proof. The inequality (3.17) follows from Theorem 3.10 if we put $\alpha = 1$. Next, let g be strictly convex and φ (operator) concave and both differentiable. Let $h(t) = \varphi(t) - g(t)$. Obviously $h'(t)$ is strictly decreasing on $[m,M]$. Then we have one of three possibilities. If $h'(m) > 0$ and $h'(M) < 0$, then the equation $h'(t) = 0$ has exactly one solution $t_o \in (m,M)$ and hence the function h has the maximum value on $[m,M]$ which is attained for $t = t_o$. If $h'(m) \leq 0$, then $h' \leq 0$ on $[m,M]$. It follows that h is decreasing on $[m,M]$ and its maximum (on $[m,M]$) is attained for $t = m$. If $h'(M) \geq 0$ then $h' \geq 0$ on $[m,M]$, i.e. h is increasing on $[m,M]$ and it has the maximum value for $t = M$. The case when g is a strictly concave and φ a strictly convex both differentiable function is proved in the same way. □

Remark 3.5 If we put $g \equiv f$ and $F(u,v) = u - v$ in Theorem 3.6, then we obtain the first inequality (3.1) in Theorem 3.1 due to Li-Mathias.

If we choose the constant α such that $\beta = 0$ in Theorem 3.10, then we obtain a generalization of an inequality (3.2) in Theorem 3.1. Furthermore, we determine the point where we have the extreme value of function φ/g that is needed for Section 3.4.

Corollary 3.12 *Let A, Φ, f and g as in Theorem 3.10. Suppose in addition that either of the following conditions holds:*
(i) $g(t) > 0$ for all $t \in [m,M]$
or
(ii) $g(t) < 0$ for all $t \in [m,M]$.

Then

$$\begin{aligned} &\left[\max_{\substack{\varphi \in \{conx.\} \\ \varphi \leq f}} \min_{m \leq t \leq M} \left\{ \frac{\varphi(t)}{g(t)} \right\} \right] g\left(\Phi(A)\right) \\ &\leq \Phi\left(f(A)\right) \qquad (3.18) \\ &\leq \left[\min_{\substack{\varphi \in \{conc.\} \\ \varphi \geq f}} \max_{m \leq t \leq M} \left\{ \frac{\varphi(t)}{g(t)} \right\} \right] g\left(\Phi(A)\right) \end{aligned}$$

in the case (i), *or*

$$\left[\max_{\substack{\varphi\in\{conx.\}\\ \varphi\le f}}\max_{m\le t\le M}\left\{\frac{\varphi(t)}{g(t)}\right\}\right]g(\Phi(A))$$

$$\le\Phi(f(A)) \tag{3.19}$$

$$\le\left[\min_{\substack{\varphi\in\{conc.\}\\ \varphi\ge f}}\min_{m\le t\le M}\left\{\frac{\varphi(t)}{g(t)}\right\}\right]g(\Phi(A))$$

in the case (ii).

Furthermore, if g *is a strictly convex twice differentiable function on* $[m,M]$ *and if* $f/g>0$ *on* $[m,M]$ *under* (i) *or* (ii), *then for any strictly operator concave twice differentiable function* $\varphi\in\{conc.\}$ *such that* $\varphi\ge f$ *on* $[m,M]$

$$\max_{m\le t\le M}\left\{\frac{\varphi(t)}{g(t)}\right\}=\frac{\varphi(t_o)}{g(t_o)}\quad\left(resp.\ \min_{m\le t\le M}\left\{\frac{\varphi(t)}{g(t)}\right\}=\frac{\varphi(t_o)}{g(t_o)}\right),$$

where $t_o\in[m,M]$ *is defined as the unique solution of* $\varphi'(t)g(t)=\varphi(t)g'(t)$ *if* $\varphi'(m)>\varphi(m)\frac{g'(m)}{g(m)}$ *and* $\varphi'(M)<\varphi(M)\frac{g'(M)}{g(M)}$, *otherwise* t_o *is defined as* m *or* M *according to* $\varphi'(m)\le\varphi(m)\frac{g'(m)}{g(m)}$ *or* $\varphi'(M)\ge\varphi(M)\frac{g'(M)}{g(M)}$.

In the dual case, i.e. if g *is strictly concave,* $f/g<0$ *on* $[m,M]$ *and* φ *is operator convex on* $[m,M]$, *the conditions determining* t_o *change their order.*

Proof. Inequalities (3.18) and (3.19) follow from Theorem 3.10 if we put the value of constant α_j such that $\beta_j=0$ $(j=1,2)$. By the simple differential calculus, we obtain the corollary. □

Remark 3.6 If we put $g\equiv f>0$ in Corollary 3.12 and we take into consideration that $t\mapsto -t^{-1}$, $t>0$, is an operator monotone function, then we obtain the second inequality (3.2) in Theorem 3.1 due to Li and Mathias.

When we add new hypothesis about convexity or concavity of f in Theorem 3.10 we obtain sharp inequalities in the sense of Definition 3.1 in the matrix case.

To prove that complementary inequalities are sharp, we need the following lemma.

We denote by

$$\mu_k=\frac{k(M)-k(m)}{M-m}\quad\text{and}\quad\nu_k=\frac{Mk(m)-mk(M)}{M-m}$$

for a real valued function k on an interval $[m,M]$.

Lemma 3.13 *Let* $A\in\mathscr{H}_n$ *be a* $n\times n$ *Hermitian matrix with* $\lambda_{min}(A)=m$ *and* $\lambda_{max}(A)=M$. *Let* $f,g\in\mathscr{C}([m,M])$ *and* $F(u,v)$ *a real function defined on* $U\times V$, *where* $U\supset f[m,M]$ *and* $V\supset g[m,M]$. *Then for any* $t^*\in[m,M]$ *there is a real valued normalized positive linear map* $\mathbf{\Phi}\in\mathbf{P}_N[\mathscr{M}_n,\mathbb{C}]$ *such that*

$$F[\Phi(f(A)),g(\Phi(A))]=F\left[\mu_f t^*+\nu_f,g(t^*)\right].$$

Proof. Let U be a unitary matrix such that

$$U^*AU = diag(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

where $(\lambda_1 = m)$ and $(\lambda_2 = M.)$ For $t^* \in [m, M]$, we denote $\theta = (M - t^*)/(M - m)$. We define a map $\Phi : \mathscr{H}_n \to \mathbb{C}$ by

$$\Phi(X) = \left(\sqrt{\theta}e_1 + \sqrt{1-\theta}e_2\right)^* X \left(\sqrt{\theta}e_1 + \sqrt{1-\theta}e_2\right), \tag{3.20}$$

where e_1 and e_2 are unit eigenvectors of A corresponding to λ_1 and λ_2, respectively. Then it follows that the map Φ is a normalized positive linear map (see Example 1.10-V). Now we have

$$\begin{aligned} g(\Phi(A)) &= g\left(\left(\sqrt{\theta}e_1 + \sqrt{1-\theta}e_2\right)^* A \left(\sqrt{\theta}e_1 + \sqrt{1-\theta}e_2\right)\right) \\ &= g(\theta\lambda_1 + (1-\theta)\lambda_2) = g(\theta m + (1-\theta)M) = g(t^*) \end{aligned}$$

and

$$\begin{aligned} \Phi(f(A)) &= \left(\sqrt{\theta}e_1 + \sqrt{1-\theta}e_2\right)^* f(A) \left(\sqrt{\theta}e_1 + \sqrt{1-\theta}e_2\right) \\ &= \theta f(m) + (1-\theta) f(M) = \frac{M-t^*}{M-m} f(m) + \frac{t^*-m}{M-m} f(M) = \mu_f t^* + \nu_f. \end{aligned}$$

Thus we have

$$F[\Phi(f(A)), g(\Phi(A))] = F\left[\mu_f t^* + \nu_f, g(t^*)\right],$$

as required. □

By using Lemma 3.13, we show that the upper boundary of complementary inequalities are the optimum estimate in the matrix case. We cite Theorem 2.4 in § 2.1 for the case of $k = 1$ as follows:

Theorem 3.14 *Let $A \in \mathscr{B}_h(H)$ be a selfadjoint operator with $\mathsf{Sp}(A) \subseteq [m, M]$ for some scalars $m < M$, $\Phi \in \mathbf{P}_N[\mathscr{B}(H), \mathscr{B}(K)]$ a normalized positive linear map and $f, g \in \mathscr{C}([m, M])$. If f is convex (resp. concave) on $[m, M]$, then for any real numbers $\alpha \in \mathbb{R}$*

$$\Phi(f(A)) \leq \alpha g(\Phi(A)) + \beta 1_K \ \ (\textit{resp.}\ \ \Phi(f(A)) \geq \alpha g(\Phi(A)) + \beta 1_K), \tag{3.21}$$

where

$$\beta = \max_{m \leq t \leq M} \{\mu_f t + \nu_f - \alpha g(t)\} \ \left(\textit{resp.}\ \ \beta = \min_{m \leq t \leq M} \{\mu_f t + \nu_f - \alpha g(t)\}\right).$$

Suppose in addition that either of the following conditions holds (i) αg is concave (resp. convex), or (ii) αg is strictly convex (resp. strictly concave) differentiable. Then

$$\beta = \mu_f t_o + \nu_f - \alpha g(t_o),$$

where

$$t_o = \begin{cases} M \text{ if } & \mu_f \geq \alpha\mu_g, \quad (\text{resp. } \mu_f \leq \alpha\mu_g), \\ m \text{ if } & \mu_f < \alpha\mu_g, \quad (\text{resp. } \mu_f > \alpha\mu_g), \end{cases} \qquad \text{in the case } (i),$$

or

$$t_o = \begin{cases} g'^{-1}(\mu_f/\alpha) & \text{if } \alpha g'(m) < \mu_f < \alpha g'(M), \ (\text{resp.} \alpha g'(M) < \mu_f < \alpha g'(m)), \\ m & \text{if } \alpha g'(m) \geq \mu_f, \quad (\text{resp. } \alpha g'(m) \leq \mu_f), \\ M & \text{if } \alpha g'(M) \leq \mu_f, \quad (\text{resp. } \alpha g'(M) \geq \mu_f), \end{cases}$$

in the case (ii).

The inequality (3.21) is sharp in the sense of Definition 3.1, that is, for any Hermitian matrix A with $\lambda_{min}(A) = m$ *and* $\lambda_{max}(A) = M$ *there is a real valued normalized positive linear map* Φ *such that*

$$\Phi(f(A)) - \alpha g(\Phi(A)) = \beta.$$

Proof. If we put $F(u,v) = u - \alpha v$ in Theorem 3.9, then the inequality (3.21) follows from (3.14) in the case of convexity of f and from (3.15) in the case of concavity.

To prove that the inequality (3.21) is sharp in the matrix case, let $t_o \in [m,M]$ be the point at which an extreme value is reached

$$\mu_f t_o + \nu_f - \alpha g(t_o) = \max_{m \leq t \leq M} \{\mu_f t + \nu_f - \alpha g(t)\}$$

$$(\text{resp. } \mu_f t_o + \nu_f - \alpha g(t_o) = \min_{m \leq t \leq M} \{\mu_f t + \nu_f - \alpha g(t)\}).$$

Denote $\Phi : \mathscr{M}_n \to \mathbb{R}$ a normalized positive linear map defined by

$$\Phi(X) = \left(\sqrt{\theta}e_1 + \sqrt{1-\theta}e_2\right)^* X \left(\sqrt{\theta}e_1 + \sqrt{1-\theta}e_2\right),$$

where $\theta = (M - t_o)/(M - m)$, e_1 and e_2 are unit eigenvectors of A corresponding to $\lambda_{min}(A) = m$ and $\lambda_{max}(A) = M$ respectively. Applying Lemma 3.13 for $t^* = t_o$, we obtain $\Phi(f(A)) - \alpha g(\Phi(A)) = \mu_f t_o + \nu_f - \alpha g(t_o)$. This implies the inequality (3.21) is sharp. □

Remark 3.7 Considering in the same way as in Remark 3.4, we can obtain the inequality opposite to the first inequality (3.21). Let Φ and A be as in Theorem 3.14, $f,g \in \mathscr{C}([m,M])$ be differentiable and f be convex. For the sake of convenience let $\alpha = 1$. Then from the left hand inequality of (3.16) in Theorem 3.10 we have $\beta\, 1_K \leq \Phi(f(A)) - g(\Phi(A))$ for $\beta = \max_{\substack{\varphi \in \{conx.\} \\ \varphi \leq f}} \min_{m \leq t \leq M} \{\varphi(t) - g(t)\}$. Instead of maximizing over all operator convex functions we took an easier route over the favorable chosen linear functions. If $f'(m) \leq \mu_g \leq f'(M)$, then

$$\{f(r) - \mu_g r - \nu_g\} 1_K \leq \Phi(f(A)) - g(\Phi(A)),$$

when g is strictly convex or

$$\{f(r) - g(t_o) + \mu_g(t_o - r)\} 1_K \leq \Phi(f(A)) - g(\Phi(A)),$$

when g is strictly concave, where $r = f'^{-1}(\mu_g)$ and $t_o = g'^{-1}(\mu_g)$. Otherwise, if $\mu_g < f'(m)$ or $f'(M) < \mu_g$, then

$$\left[\max_{s\in\{m,M\}} \min_{m\le t\le M} \left\{f(s)+f'(s)(t-s)-g(t)\right\}\right] 1_K \le \Phi(f(A)) - g(\Phi(A)).$$

Indeed, because f is convex, we have

$$\begin{aligned}\left[\min_{m\le t\le M}\{h_r(t)\}\right] 1_K &\le \left[\max_{\substack{\varphi\in\{conx.\}\\ \varphi\le f}} \min_{m\le t\le M}\{\varphi(t)-g(t)\}\right] 1_K \\ &\le \Phi(f(A)) - g(\Phi(A)),\end{aligned}$$

where $h_r(t) = f(r)+f'(r)(t-r)-g(t)$, $r\in[m,M]$. We choose $r = f'^{-1}(\mu_g)$, when $f'(m)\le \mu_g \le f'(M)$; $r = m$ when $\mu_g < f'(m)$ or $r = M$ when $f'(M) < \mu_g$. In the case of convexity of g the function h_r is concave and its minimum is attained at m or M. (Particularly, we have $h_r(m) = h_r(M)$ when $r = f'^{-1}(\mu_g)$). In the case of concavity of g the function h_r is convex and so its minimum is attained at $t_o \in [m,M]$.

We can obtain the opposite inequality to the second inequality (3.21) in the same way.

Remark 3.8 If we put $\alpha = 1$ in Theorem 3.14, then we obtain a generalization of Theorem 3.2 due to Li and Mathias.

Further, in the case when $\alpha = 1$, we have the next estimate for the boundary β from the inequality (3.21) (see Theorem 2.4): $\beta = \max_{s\in\{m,M\}}\{f(s)-\alpha g(s)\}$ in the case (i) and $f(m)-g(m) \le \beta \le f(m)-g(m)+\left[\mu_f - g'(m)\right](M-m)$ in the case (ii); or if $g \equiv f$ then $\left(0<\beta<\left[\mu_f - f'(m)\right](M-m).\right)$

By Theorem 3.14, we show that boundaries of the ratio type reverse inequalities in Theorem 2.9 of § 2.2 are optimal in the matrix case as follows:

Corollary 3.15 *Let A and Φ as in Theorem 3.14. Let $f,g\in\mathscr{C}([m,M])$ and either of the following conditions holds:*
(i) $g(t)>0$ for all $t\in[m,M]$
or
(ii) $g(t)<0$ for all $t\in[m,M]$.

If f is convex on $[m,M]$, then

$$\Phi(f(A)) \le \alpha_o\, g(\Phi(A)), \tag{3.22}$$

where

$$\alpha_o = \max_{m\le t\le M}\left\{\frac{\mu_f t+\nu_f}{g(t)}\right\} \qquad \text{in the case } (i),$$
$$\text{or } \alpha_o = \min_{m\le t\le M}\left\{\frac{\mu_f t+\nu_f}{g(t)}\right\} \qquad \text{in the case } (ii).$$

Suppose in addition that either of the following conditions holds: $f(m)>0$, $f(M)>0$ in the case of (i) or $f(m)<0$, $f(M)<0$ in the case of (ii) and g is a strictly convex twice *differentiable function on $[m,M]$. Then the inequality (3.22) is sharp in the matrix case, that is, for any Hermitian matrix A with $\lambda_{min}(A)=m$ and $\lambda_{max}(A)=M$ the equality is attained for a real valued normalized positive linear map Φ defined by (3.20) and*

$$t_o=\begin{cases}\left.\begin{array}{c}\text{the solution of}\\ \mu_f g(t)=\left(\mu_f t+\nu_f\right)g'(t)\end{array}\right\} & \text{if} \quad f(m)\frac{g'(m)}{g(m)}<\mu_f<f(M)\frac{g'(M)}{g(M)},\\ M & \text{if} \quad \mu_f\geq f(M)\frac{g'(M)}{g(M)},\\ m & \text{if} \quad \mu_f\leq f(m)\frac{g'(m)}{g(m)}.\end{cases} \tag{3.23}$$

In the dual case (f concave, g strictly concave) we have the sharp opposite inequality with dual extreme, with the dual estimation and the opposite condition while determining t_o.

Proof. To prove that the inequality (3.22) is sharp for a convex function f and a strictly convex function g, we only proceed with the case (i) since the proof in the case (ii) is essentially the same.

Since $f(m)>0$, $f(M)>0$ and $g(t)>0$, we have $(\mu_f t+\nu_f)/g(t)>0$. It follows from Corollary 3.12 that $\max_{m\leq t\leq M}\{\frac{\mu_f t+\nu_f}{g(t)}\}=\frac{\mu_f t_0+\nu_f}{g(t_0)}$ for $t_0\in[m,M]$ determined by (3.23). Using Lemma 3.13 for $t^*=t_0$ and the map Φ defined by (3.20) in Lemma 3.13, we have

$$\Phi(f(A))=\frac{\mu_f t_0+\nu_f}{g(t_0)}g(\Phi(A))=\left[\max_{m\leq t\leq M}\{\frac{\mu_f t+\nu_f}{g(t)}\right]g(\Phi(A)).$$

□

Remark 3.9 Similarly to Remark 3.7, we obtain the opposite inequality of (3.22). Let Φ and A be as in Corollary 3.15, $f,g\in\mathscr{C}([m,M])$ be twice differentiable, $f>0$, $g>0$ on $[m,M]$ and f be convex. If $g(m)\frac{f'(m)}{f(m)}\leq\mu_g\leq g(M)\frac{f'(M)}{f(M)}$, then

$$\frac{f(r)}{\mu_g r+\nu_g}g(\Phi(A))\leq\Phi(f(A)),$$

when g is a strictly convex function or

$$\frac{f(r)}{\mu_g r+\nu_g}\frac{\mu_g t_o+\nu_g}{f(t_o)}g(\Phi(A))\leq\Phi(f(A)),$$

when g is strictly concave, where $r\in[m,M]$ is the unique solution of $\frac{f'(r)}{f(r)}=\frac{\mu_g}{\mu_g r+\nu_g}$ and $t_o\in[m,M]$ is the unique solution of $\frac{g'(t)}{g(t)}=\frac{\mu_g}{\mu_g t+\nu_g}$. Otherwise, if $\frac{\mu_g}{g(m)}<\frac{f'(m)}{f(m)}$ or $\frac{f'(M)}{f(M)}<\frac{\mu_g}{g(M)}$, then

$$\max_{s\in\{m,M\}}\min_{m\leq t\leq M}\left\{\frac{f(s)+f'(s)(t-s)}{g(t)}\right\}g(\Phi(A))\leq\Phi(f(A)).$$

Similarly, we can obtain inequality opposite to (3.22) when $f,g<0$.

Remark 3.10 Further, we have the following estimate for the boundary α_o in the inequality (3.22) (see Theorem 2.9): $\max\left\{\frac{f(m)}{g(m)},\frac{f(M)}{g(M)}\right\} \leq \alpha_o$ in the case (i) or $0 \leq \alpha_o \leq \min\left\{\frac{f(m)}{g(m)},\frac{f(M)}{g(M)}\right\}$ in the case (ii); but if $g \equiv f$ then $1 < \alpha_o$ in the case (i) or $0 < \alpha_o < 1$ in the case (ii).

3.4 Application to power functions

In this section we shall consider the upper and lower boundary of the difference and ratio of Jensen's type inequalities in the power function. Firstly, we give a generalization of the inequalities (3.3) and (3.4) due to Li and Mathias.

Theorem 3.16 *Let $A \in \mathscr{B}^+(H)$ be a positive operator with $\mathsf{Sp}(A) \subseteq [m,M]$ for some scalars $0 < m < M$ and $\Phi \in \mathbf{P}_N[\mathscr{B}(H),\mathscr{B}(K)]$ a normalized positive linear map and $\alpha, q \in \mathbb{R}$. If f is convex on $[m,M]$, then*

$$\Phi(f(A)) \leq \alpha\Phi(A)^q + \beta\, 1_K \tag{3.24}$$

where

$$\beta = \begin{cases} \alpha(q-1)\left(\frac{\mu_f}{\alpha q}\right)^{\frac{q}{q-1}} + \nu_f & \text{if} \quad m < \left(\frac{\mu_f}{\alpha q}\right)^{\frac{1}{q-1}} < M \text{ and } \alpha q(q-1) > 0, \\ \max\{f(M)-\alpha M^q, f(m)-\alpha m^q\} & \text{otherwise.} \end{cases}$$

If f is concave on $[m,M]$, then

$$\Phi(f(A)) \geq \alpha\Phi(A)^q + \beta\, 1_K \tag{3.25}$$

where

$$\beta = \begin{cases} \alpha(q-1)\left(\frac{\mu_f}{\alpha q}\right)^{\frac{q}{q-1}} + \nu_f & \text{if} \quad m < \left(\frac{\mu_f}{\alpha q}\right)^{\frac{1}{q-1}} < M \text{ and } \alpha q(q-1) < 0, \\ \min\{f(M)-\alpha M^q, f(m)-\alpha m^q\} & \text{otherwise.} \end{cases}$$

Moreover, if $p \in \mathbb{R}\backslash[0,1)$ (resp. $p \in (0,1]$), then

$$\Phi(A^p) \leq \alpha\Phi(A)^q + \beta_1\, 1_K \quad (\textit{resp. } \Phi(A^p) \geq \alpha\Phi(A)^q + \beta_2\, 1_K), \tag{3.26}$$

where

$$\beta_1 = \begin{cases} \alpha(q-1)\left(\frac{1}{\alpha q}\mu_{t^p}\right)^{\frac{q}{q-1}} + \nu_{t^p} & \\ \qquad\qquad \text{if} \quad m < \left(\frac{1}{\alpha q}\mu_{t^p},\right)^{\frac{1}{q-1}} < M \text{ and } \alpha q(q-1) > 0, & \\ \max\{m^p-\alpha m^q, M^p-\alpha M^q\} & \text{otherwise.} \end{cases}$$

(*resp.*

$$\beta_2 = \begin{cases} \alpha(q-1)\left(\frac{1}{\alpha q}\mu_{t^p}\right)^{\frac{q}{q-1}} + \nu_{t^p} \\ \qquad\qquad \text{if} \quad m < \left(\frac{1}{\alpha q}\mu_{t^p}\right)^{\frac{1}{q-1}} < M \text{ and } \alpha q(q-1) < 0, \\ \min\{m^p - \alpha m^q, M^p - \alpha M^q\} \qquad \text{otherwise.}) \end{cases}$$

Proof. Suppose that f is convex. If we put $g(t) = t^q$ in Theorem 3.14, then we obtain the inequality (3.24). In fact, if $\alpha q(q-1) > 0$, then $\alpha g(t)$ is strictly convex and we have $\beta = \mu_f t_o + \nu_f - \alpha g(t_o)$ for $t_o = \left(\frac{\mu_f}{\alpha q}\right)^{\frac{q}{q-1}}$ if $m < t_o < M$, otherwise $t_o = m$ or $t_o = M$. If f is concave, then we apply Theorem 3.14 in the dual case. Moreover, if we put $f(t) = t^p$ in these inequalities, then we obtain (3.26). □

Remark 3.11 *All inequalities in Theorem 3.16 are sharp in the matrix case.*

Next, we shall show the following two theorems, which are extensions of the inequalities (3.3) and (3.4) due to Li and Mathias:

Theorem 3.17 *Let $A \in \mathscr{B}^+(H)$ be a positive operator with $\mathrm{Sp}(A) \subseteq [m,M]$ for some scalars $0 < m < M$ and $\Phi \in \mathbf{P}_N[\mathscr{B}(H), \mathscr{B}(K)]$ a normalized positive linear map. If $p \in \mathbb{R}\backslash\{0\}$ and $q \in \mathbf{R}$, then*

$$\beta_2 1_K \le \Phi(A^p) - \Phi(A)^q \le \beta_1 1_K \tag{3.27}$$

where

$$\beta_1 = \begin{cases} C(m,M,p,q) \quad \text{if} \quad m < \left(\frac{1}{q}\mu_{t^p}\right)^{\frac{1}{q-1}} < M \text{ and } q(q-1) > 0, \\ \max\{m^p - m^q, M^p - M^q\} \qquad \text{otherwise,} \end{cases}$$

if $p \in \mathbb{R}\backslash[0,1]$,

$$\beta_1 = \begin{cases} \left(\frac{q}{p}\right)^{\frac{p}{p-q}} - \left(\frac{q}{p}\right)^{\frac{q}{p-q}} \quad \text{if} \quad m < \left(\frac{q}{p}\right)^{\frac{1}{p-q}} < M \text{ and } 0 < p < q, \\ \max\{m^p - m^q, M^p - M^q\} \qquad \text{otherwise,} \end{cases}$$

if $p \in (0,1]$

and

$$\beta_2 = \begin{cases} C(m,M,p,q) \quad \text{if} \quad m < \left(\frac{1}{q}\mu_{t^p}\right)^{\frac{1}{q-1}} < M \text{ and } q(q-1) < 0, \\ \min\{m^p - m^q, M^p - M^q\} \qquad \text{otherwise,} \end{cases}$$

if $p \in (0,1)$,

$$\beta_2 = \begin{cases} \left(\frac{q}{p}\right)^{\frac{p}{p-q}} - \left(\frac{q}{p}\right)^{\frac{q}{p-q}} \quad \text{if} \quad m < \left(\frac{q}{p}\right)^{\frac{1}{p-q}} < M \text{ and } q(p-q) > 0, \\ \min\{m^p - m^q, M^p - M^q\} \qquad \text{otherwise,} \end{cases}$$

if $p \in [-1,0)$ or $p \in [1,2]$,

$$\beta_2 = \begin{cases} -C(m,M,q,p) \quad \textit{if} \quad m \le \left(\frac{1}{p}\mu_{t^q}\right)^{\frac{1}{p-1}} \le M \textit{ and } q(q-1) > 0, \\ (1-p)\left(\frac{1}{p}\mu_{t^q}\right)^{\frac{p}{p-1}} + (q-1)\left(\frac{1}{q}\mu_{t^q}\right)^{\frac{q}{q-1}} \\ \qquad\qquad \textit{if} \quad m \le \left(\frac{1}{p}\mu_{t^q}\right)^{\frac{1}{p-1}} \le M \textit{ and } q(q-1) < 0, \\ \max_{s\in\{m,M\}} \min_{m\le t\le M} \left\{(1-p)s^p + ps^{p-1}t - t^q\right\} \qquad \textit{otherwise}, \end{cases}$$

if $p < -1$ or $p > 2$,

where $C(m,M,p,q)$ is defined as (2.38) in Section 2.3:

$$\begin{aligned} C(m,M,p,q) &= (q-1)\left(\frac{\mu_{t^p}}{q}\right)^{\frac{q}{q-1}} + \nu_{t^p} \\ &= \frac{Mm^p - mM^p}{M-m} + (q-1)\left(\frac{M^p - m^p}{q(M-m)}\right)^{\frac{q}{q-1}}. \end{aligned}$$

Proof. First we consider β_1.
Case 1. Suppose $p > 1$ or $p < 0$. Put $\alpha = 1$ in the first inequality (3.26) in Theorem 3.16 and we obtain β_1.
Case 2. Suppose $0 < p \le 1$. Then $f(t) = t^p$ is operator concave, so we can take that $\varphi \equiv f$ at the right hand inequality of (3.17) in Corollary 3.11. Hence we determine β_1 in the usual way.

Next, we consider β_2.
Case 1. Suppose $0 < p < 1$. Put $\alpha = 1$ in the second inequality (3.26) in Theorem 3.16 and we obtain β_2.
Case 2. Suppose $-1 \le p < 0$ or $1 \le p \le 2$. Then $f(t) = t^p$ is operator concave, so we can take that $\varphi \equiv f$ in the left hand inequality of (3.17) in Corollary 3.11. Hence we determine β_2 in the usual way.
Case 3. Suppose $p > 2$ or $p < -1$. Then $f(t) = t^p$ is convex and we can applied Remark 3.7 for determination of β_2. □

Remark 3.12 *The right hand inequality of (3.27) in Theorem 3.17 is sharp for all values of p and the left hand inequality is sharp when $p \in [-1,2]$ in the matrix case.*

Remark 3.13 If we put $p = q$ in Theorem 3.17, then we obtain the inequality (3.3) due to Li and Mathias for

$$\beta_1 = \begin{cases} C(m,M,p) & \text{if} \quad p > 2 \text{ or } p < -1, \\ 0 & \text{if} \quad -1 \le p < 0 \text{ or } 1 \le p \le 2, \\ -C(m,M,p) & \text{if} \quad 0 < p < 1, \end{cases}$$

$$\beta_2 = \begin{cases} -C(m,M,p) & \text{if} \quad p > 1 \text{ or } p < 0, \\ 0 & \text{if} \quad 0 < p \le 1, \end{cases}$$

where $C(m,M,p)$ is defined as (2.39) in § 2.3:

$$\begin{aligned} C(m,M,p) &= (p-1)\left(\frac{\mu_{tp}}{p}\right)^{\frac{p}{p-1}} + \nu_{tp} \\ &= M^p\ \frac{1-h^{1-p}}{1-h} + m^p(p-1)\left\{\frac{p(h-1)}{h^p-1}\right\}^{\frac{p}{1-p}}, \qquad h=\frac{M}{m}. \end{aligned}$$

Theorem 3.18 *Let $A \in \mathscr{B}^+(H)$ be a positive operator with $\mathsf{Sp}(A) \subseteq [m,M]$ for some scalars $0<m<M$ and $\Phi \in \mathbf{P}_N[\mathscr{B}(H),\mathscr{B}(K)]$ a normalized positive linear map. If $p \in \mathbb{R}\backslash\{0\}$ and $q \in \mathbf{R}$, then*

$$\alpha_2 \Phi(A)^q \le \Phi(A^p) \le \alpha_1 \Phi(A)^q, \tag{3.28}$$

where

$$\alpha_1 = \begin{cases} K(m,M,p,q) \\ \qquad\quad \text{if} \quad m < \frac{q}{1-q}\frac{\nu_{tp}}{\mu_{tp}} < M \text{ and } q(q-1)>0 \text{ and } pq>0, \\ \max\{m^{p-q}, M^{p-q}\} \qquad \text{otherwise}, \end{cases}$$

if $p \in \mathbb{R}\backslash[0,1]$,

$$\alpha_1 = \begin{cases} m^{p-q} & \text{if} \quad p<q, \\ M^{p-q} & \text{if} \quad p \ge q, \end{cases}$$

if $p \in (0,1]$,

and

$$\alpha_2 = \begin{cases} K(m,M,p,q) & \text{if} \quad m < \frac{q}{1-q}\frac{\nu_{tp}}{\mu_{tp}} < M \text{ and } 0<q<1, \\ \min\{m^{p-q}, M^{p-q}\} & \text{otherwise}, \end{cases}$$

if $p \in (0,1)$,

$$\alpha_2 = \begin{cases} m^{p-q} & \text{if} \quad p>q, \\ M^{p-q} & \text{if} \quad p \le q, \end{cases}$$

if $p \in [-1,0)$ or $p \in [1,2]$,

$$\alpha_2 = \begin{cases} K(m,M,q,p)^{-1} \quad \text{if} \quad pm^{q-1} \le \mu_{tq} \le pM^{q-1} \text{ and } q(q-1)>0, \\ \frac{1-p}{1-q}K(m,M,q,p)^{-1}K(m,M,q) \\ \qquad\qquad \text{if} \quad pm^{q-1} \le \mu_{tq} \le pM^{q-1} \text{ and } 0<q<1, \\ \max_{s\in\{m,M\}} \min_{m\le t\le M}\left\{\frac{(1-p)s^p+ps^{p-1}t}{t^q}\right\} \qquad \text{otherwise}, \end{cases}$$

if $p<-1$ or $p>2$,

where $K(m,M,p,q)$ is defined as (2.20) in Section 2.2:

$$K(m,M,p,q) = \frac{mM^p - Mm^p}{(q-1)(M-m)}\left(\frac{(q-1)(M^p-m^p)}{q(mM^p-Mm^p)}\right)^q.$$

Proof. The proof is similar to the proof of Theorem 3.17. □

Remark 3.14 *The right hand inequality (3.28) in Theorem 3.18 is sharp for all values of p and the left hand inequality is sharp when $p \in [-1,2]$ in the matrix case.*

Remark 3.15 If we put $p = q$ in Theorem 3.18, then we obtain Li-Mathias inequality (3.4) for

$$\alpha_1 = \begin{cases} K(m,M,p)^{-1} & \text{if} \quad p > 2 \ \text{or} \ p < -1, \\ 1 & \text{if} \quad -1 \le p < 0 \ \text{or} \ 1 \le p \le 2, \\ K(m,M,p) & \text{if} \quad 0 < p < 1, \end{cases}$$

$$\alpha_2 = \begin{cases} K(m,M,p) & \text{if} \quad p > 1 \ \text{or} \ p < 0, \\ 1 & \text{if} \quad 0 < p \le 1, \end{cases}$$

where $K(m,M,p)$ is defined as (2.21) in § 2.2:

$$\begin{aligned} K(m,M,p) &= \frac{mM^p - Mm^p}{(p-1)(M-m)} \left(\frac{(p-1)(M^p - m^p)}{p(mM^p - Mm^p)} \right)^p \\ &= \frac{h^p - h}{(p-1)(h-1)} \left(\frac{p-1}{p} \frac{h^p - 1}{h^p - h} \right)^p, \qquad h = \frac{M}{m}. \end{aligned}$$

3.5 Matrix inequalities of Ky Fan type

In this section we give a generalization of Mond-Pečarić matrix inequalities (3.8) and (3.9) of Ky Fan type in the matrix case. It follows from Theorem 3.6 for a special choice of maps. Throughout this section, it assumes that H is a finite dimensional Hilbert space.

Corollary 3.19 *Let $A_j \in \mathscr{H}_n$ with $\mathrm{Sp}(A_j) \subseteq [m,M]$, $j = 1, \cdots, k$ and $U_j \in \mathscr{M}_{t,n}$ such that $\sum_{j=1}^{k} U_j U_j^* = 1_t$. Let $f, g \in \mathscr{C}([m,M])$ and $F(u,v)$ be real function defined on $U \times V$, where $U \supset f[m,M]$, $V \supset g[m,M]$. If $F(u,v)$ is matrix monotone in u, then*

$$\begin{aligned} &\left\{ \max_{\substack{\varphi \in \{conx.\} \\ \varphi \le f}} \min_{m \le t \le M} F\left[\varphi(t), g(t)\right] \right\} 1_t \\ &\le F[\sum_{j=1}^{k} U_j f(A_j) U_j^*, g(\sum_{j=1}^{k} U_j A_j U_j^*)] \qquad (3.29) \\ &\le \left\{ \min_{\substack{\varphi \in \{conc.\} \\ \varphi \ge f}} \max_{m \le t \le M} F\left[\varphi(t), g(t)\right] \right\} 1_t. \end{aligned}$$

Proof. Let $U \in \mathscr{M}_{t,n\cdot k}$ be an unitary matrix. Let a map $\Phi : \mathscr{M}_{n\cdot k} \to \mathscr{M}_t$ be defined by $\Phi(A) = UAU^*$. Obviously, Φ is normalized positive linear map (see Example 1.10-II). It follows from Theorem 3.6 that an inequality

$$\begin{aligned} &\left\{ \max_{\substack{\varphi \in \{conx.\} \\ \varphi \le f}} \min_{m\le t\le M} F\left[\varphi(t), g(t)\right] \right\} 1_t \\ &\le F[Uf(A)U^*, g(UAU^*)] \qquad (3.30) \\ &\le \left\{ \min_{\substack{\varphi \in \{conc.\} \\ \varphi \ge f}} \max_{m\le t\le M} F\left[\varphi(t), g(t)\right] \right\} 1_t, \end{aligned}$$

holds for any matrix $A \in \mathscr{H}_{n\cdot k}$ with $\mathsf{Sp}(A) \subseteq [m,M]$. For A_j and U_j, $j = 1,\cdots,k$ from the hypothesis of this corollary we have $\sum_{j=1}^k U_jA_jU_j^* = UAU^*$, where $A = A_1 \dot{+} A_2 \dot{+} \cdots \dot{+} A_k$, $U = [U_1U_2\cdots U_k]$ and $\sum_{j=1}^k U_jf(A_j)U_j^* = Uf(A)U^*$. If we put A and U in (3.30), then we obtain the desired inequality (3.29). □

In the same way, applying theorem 3.9 we have the next corollary:

Corollary 3.20 *Let A_j, U_j, , $j = 1,\cdots,k$, f, g and $F(u,v)$ as in Corollary 3.19. If f is convex, then*

$$F\left[\sum_{j=1}^k U_jf(A_j)U_j^*, g\left(\sum_{j=1}^k U_jA_jU_j^*\right)\right] \le \beta\, 1_t,$$

for $\beta = \max_{m\le t\le M} F\left[\mu_f t + \nu_f, g(t)\right]$. If f is concave, then the opposite inequality holds with $\beta = \min_{m\le t\le M} F\left[\mu_f t + \nu_f, g(t)\right]$.

If we put $F(u,v) = u - \alpha v$, $\alpha \in \mathbb{R}$ in Corollaries 3.19 and 3.20 we have the next two corollaries:

Corollary 3.21 *Let $A_j \in \mathscr{H}_n$ with $\mathsf{Sp}(A_j) \subseteq [m,M]$, $j = 1,\cdots,k$ and $U_j \in \mathscr{M}_{t,n}$ such that $\sum_{j=1}^k U_jU_j^* = 1_t$. If $f, g \in \mathscr{C}([m,M])$ and $\alpha \in \mathbb{R}$, then*

$$\alpha\, g\left(\sum_{j=1}^k U_jA_jU_j^*\right) + \beta_2\, 1_t \le \sum_{j=1}^k U_jf(A_j)U_j^* \le \alpha\, g\left(\sum_{j=1}^k U_jA_jU_j^*\right) + \beta_1\, 1_t,$$

holds for

$$\begin{aligned} \beta_1 &= \min_{\substack{\varphi \in \{conc.\} \\ \varphi \ge f}} \max_{m\le t\le M} \{\varphi(t) - \alpha g(t)\}, \\ \beta_2 &= \max_{\substack{\varphi \in \{conx.\} \\ \varphi \le f}} \min_{m\le t\le M} \{\varphi(t) - \alpha g(t)\}. \end{aligned}$$

Corollary 3.22 *Let A_j, U_j, $j = 1,\cdots,k$, f and g be as in Corollary 3.21. If f is convex and $\alpha \in \mathbb{R}$, then*

$$\sum_{j=1}^k U_jf(A_j)U_j^* \le \alpha\, g\left(\sum_{j=1}^k U_jA_jU_j^*\right) + \beta\, 1_t,$$

holds for $\beta = \max_{m \le t \le M} \left\{\mu_f t + \nu_f - \alpha g(t)\right\}$. *If* f *is concave, then the opposite inequality holds with* $\beta = \min_{m \le t \le M} \left\{\mu_f t + \nu_f - \alpha g(t)\right\}$.

Remark 3.16 We put $\alpha = 1$ in Corollary 3.22 and obtain a generalization of Theorem 3.3 with boundary as in Theorem 3.14. If we choose the value of constant α such that $\beta = 0$ in Corollary 3.22, then we obtain a generalization of Theorem 3.4 with boundary as in Corollary 3.15.

In the same way we can apply map $\Phi : A \mapsto UAU^*$ $(A \in \mathscr{M}_{n \cdot k})$ to the remainder of results in § 3.3 and in particular on power functions in § 3.4.

3.6 Notes

Theorem 3.1 and Theorem 3.2 are due to Li and Mathias [122].

For our exposition we have used [135]. Mićić, Pečarić, Seo and Tominaga discussed Li-Mathias type inequalities in the framework of matrix theory. However, we discuss in the framework of operator theory for the sake of convenience.

Chapter 4

Power mean

In this chapter we study reverse inequalities of power operator means on positive linear maps. We investigate several properties of power operator means under the chaotic order.

4.1 Preliminary

For positive numbers $x_j \in \mathbb{R}_+$, the power (arithmetic) means

$$M_r(x_1, \cdots, x_k) = \left(\frac{x_1^r + \cdots + x_k^r}{k}\right)^{1/r}$$

make a path of means from the harmonic one at $r = -1$ to the arithmetic one ar $r = 1$ via the geometric one at $r = 0$ (precisely the limit as $r \to 0$). We consider the traditional averaging operation which is a natural noncommutative operator version of the power arithmetic mean: For positive invertible operators $A_j \in \mathscr{B}^{++}(H)$ $(j = 1, \cdots, k)$

$$\mathbf{A} = (A_1, \cdots, A_k) \mapsto M_r(\mathbf{A}) := \left(\frac{1}{k}\sum_{j=1}^{k} A_j^r\right)^{1/r} \quad \text{with } r \in \mathbb{R}\backslash\{0\}.$$

This operation $M_r(\mathbf{A})$ is not an operator mean except for $r = 1$ in the sense of Definition 5.1. In fact, this mean does not satisfy the monotonicity condition $(S1)$ and nor the transformer inequality $(S3)$ in Definition 5.1. However, by the operator monotonicity of the function t^r for $0 < r \leq 1$ it follows that

$$M_s(\mathbf{A}) \geq M_r(\mathbf{A}) \quad \text{whenever } s \geq r \geq 1$$

as observed in Bhagwat and Subramanian [18]. The limit of $M_r(\mathbf{A})$ as $r \mapsto 0$ exists and equals $\exp\left(\frac{1}{k}\sum_{j=1}^{k}\log A_j\right)$ as mentioned later in Theorem 4.19. Contrary to the scalar case or the case of commuting A and B, this limit does not coincide with the geometric mean $A \sharp B$ in general. The map $\mathbf{A} = (A_1, \cdots, A_k) \mapsto \exp\left(\frac{1}{k}\sum_{j=1}^{k}\log A_j\right)$ is not operator monotone in $\mathbf{A}$.

In this chapter, we study the (weighted) power operator mean on positive linear maps in a more general setting: Let $A_j \in \mathscr{B}_h(H)$ be positive invertible operators with $\mathsf{Sp}(A_j) \subseteq (0,\infty)$, $\Phi_j \in \mathbf{P}_N[\mathscr{B}(H), \mathscr{B}(K)]$ normalized positive linear maps and $\omega_j \in \mathbb{R}_+$ such that $\sum_{j=1}^{k}\omega_j = 1$ $(j = 1, \ldots, k)$. Then the power operator mean is defined as

$$\mathbf{A} = (A_1, \cdots, A_k) \mapsto M_k^{[r]}(\mathbf{A}; \mathbf{\Phi}, w) := \left(\sum_{j=1}^{k}\omega_j\, \Phi_j\left(A_j^r\right)\right)^{1/r} \quad \text{with } r \in \mathbb{R}\backslash\{0\}. \tag{4.1}$$

The power operator mean has the following monotonicity. More detailed consideration is given in after theorem (Theorem 4.4 and Theorem 4.7).

Theorem 4.1

$$M_k^{[r]}(\mathbf{A}; \mathbf{\Phi}, w) \leq M_k^{[s]}(\mathbf{A}; \mathbf{\Phi}, w) \tag{4.2}$$

holds if either $r \leq s, r \notin (-1,1), s \notin (-1,1)$ or $\frac{1}{2} \leq r \leq 1 \leq s$ or $r \leq -1 \leq s \leq -\frac{1}{2}$.

Proof. Suppose that $1 \leq r \leq s$. Since $0 < \frac{r}{s} \leq 1$, it follows from the operator concavity of $t^{r/s}$ that $\Phi_j(A_j^{\frac{r}{s}}) \leq \Phi_j(A_j)^{\frac{r}{s}}$ by Davis-Choi-Jensen's inequality (Theorem 1.20). Multiplying them with $\omega_j \in \mathbb{R}_+$ and summing of all $j = 1, \cdots, k$, we have

$$\sum_{j=1}^{k}\omega_j\, \Phi_j\left(A_j^{\frac{r}{s}}\right) \leq \sum_{j=1}^{k}\omega_j\, \Phi_j\left(A_j\right)^{\frac{r}{s}} \leq \left(\sum_{j=1}^{k}\omega_j\, \Phi_j\left(A_j\right)\right)^{\frac{r}{s}}.$$

Replacing A_j by A_j^s and raising both sides to the power $\frac{1}{r}(\leq 1)$, it follows from Theorem 1.8 (Löwner-Heinz inequality) that

$$\left(\sum_{j=1}^{k}\omega_j\, \Phi_j\left(A_j^r\right)\right)^{\frac{1}{r}} \leq \left(\sum_{j=1}^{k}\omega_j\, \Phi_j\left(A_j^s\right)\right)^{\frac{1}{s}} \qquad \text{for } 1 \leq r \leq s. \tag{4.3}$$

Suppose that $r \leq s \leq -1$. Since $0 < \frac{s}{r} \leq 1$, as we prove above, it follows that

$$\sum_{j=1}^{k}\omega_j\, \Phi_j\left(A_j^s\right) \leq \left(\sum_{j=1}^{k}\omega_j\, \Phi_j\left(A_j^r\right)\right)^{\frac{s}{r}}.$$

Raising both sides to the power $-1 \le \frac{1}{s} < 0$, we have

$$\left(\sum_{j=1}^{k} \omega_j \, \Phi_j\left(A_j^r\right)\right)^{\frac{1}{r}} \le \left(\sum_{j=1}^{k} \omega_j \, \Phi_j\left(A_j^s\right)\right)^{\frac{1}{s}} \qquad \text{for } r \le s \le -1. \tag{4.4}$$

Suppose that $r \le -1, 1 \le s$. By the operator convexity of t^{-1}, we have

$$\sum_{j=1}^{k} \omega_j \, \Phi_j\left(A_j^{-1}\right) \ge \left(\sum_{j=1}^{k} \omega_j \, \Phi_j\left(A_j\right)\right)^{-1}$$

and hence

$$\left(\sum_{j=1}^{k} \omega_j \, \Phi_j\left(A_j^{-1}\right)\right)^{-1} \le \sum_{j=1}^{k} \omega_j \, \Phi_j\left(A_j\right).$$

If we put $r = 1$ in (4.3) and $s = -1$ in (4.4), then it follows that

$$\begin{aligned}\left(\sum_{j=1}^{k} \omega_j \, \Phi_j\left(A_j^r\right)\right)^{\frac{1}{r}} &\le \left(\sum_{j=1}^{k} \omega_j \, \Phi_j\left(A_j^{-1}\right)\right)^{-1} \le \sum_{j=1}^{k} \omega_j \, \Phi_j\left(A_j\right) \\ &\le \left(\sum_{j=1}^{k} \omega_j \, \Phi_j\left(A_j^s\right)\right)^{\frac{1}{s}}.\end{aligned}$$

Therefore the desired inequality holds for $r \le -1, 1 \le s$.

Suppose that $\frac{1}{2} \le r \le 1 \le s$. Since $1 \le \frac{1}{r} \le 2$, it follows from the operator convexity of $t^{\frac{1}{r}}$ that

$$\left(\sum_{j=1}^{k} \omega_j \, \Phi_j\left(A_j\right)\right)^{\frac{1}{r}} \le \sum_{j=1}^{k} \omega_j \, \Phi_j\left(A_j\right)^{\frac{1}{r}} \le \sum_{j=1}^{k} \omega_j \, \Phi_j\left(A_j^{\frac{1}{r}}\right).$$

Replacing A_j by A_j^r, we have

$$\left(\sum_{j=1}^{k} \omega_j \, \Phi_j\left(A_j^r\right)\right)^{\frac{1}{r}} \le \sum_{j=1}^{k} \omega_j \, \Phi_j\left(A_j\right).$$

The assumption $s \ge 1$ implies

$$\left(\sum_{j=1}^{k} \omega_j \, \Phi_j\left(A_j^r\right)\right)^{\frac{1}{r}} \le \sum_{j=1}^{k} \omega_j \, \Phi_j\left(A_j\right) \le \left(\sum_{j=1}^{k} \omega_j \, \Phi_j\left(A_j^s\right)\right)^{\frac{1}{s}}.$$

Therefore the desired inequality holds for $\frac{1}{2} \le r \le 1 \le s$.

The case of $r \le -1 \le s \le -\frac{1}{2}$ is proved in the same way. $\square$

4.2 Complementary inequality to power means

In this section, we investigate the lower and upper estimates of the difference and ratio in the power operator means on a positive linear map: For a positive invertible operator $A \in \mathscr{B}^{++}(H)$ and a normalized positive linear map $\Phi \in \mathbf{P}_N[\mathscr{B}(H), \mathscr{B}(K)]$ consider

$$\Phi(A^r)^{\frac{1}{r}} \qquad \text{for } r \in \mathbb{R}\backslash\{0\}$$

in the case of $k = 1$ in (4.1).

By using Theorem 4.1 for the case of $k = 1$, it follows that this mean has the following monotonicity:

Lemma 4.2

$$\Phi(A^r)^{1/r} \leq \Phi(A^s)^{1/s} \tag{4.5}$$

holds if either $r \leq s, r \notin (-1,1), s \notin (-1,1)$ *or* $\frac{1}{2} \leq r \leq 1 \leq s$ *or* $r \leq -1 \leq s \leq -\frac{1}{2}$.

First we prepare the following intervals given in Figure 4.1 and Table 4.1.

Based on the theory of extended operator inequalities displayed in Chapter 2, we show the complementary inequalities to (4.5).

First, we recall the following result (see Remark 3.15):

Lemma 4.3 *Let* $\Phi \in \mathbf{P}_N[\mathscr{B}(H), \mathscr{B}(K)]$ *be a normalized positive linear map and* $A \in \mathscr{B}^{++}(H)$ *a positive invertible operator with* $\mathsf{Sp}(A) \subseteq [m,M]$ *for some scalars* $0 < m < M$. *Then*

$$\alpha_2 \Phi(A)^p \leq \Phi(A^p) \leq \alpha_1 \Phi(A)^p \tag{4.6}$$

for

$$\alpha_2 = \begin{cases} K(m,M,p)^{-1} & \text{if} \quad p < -1 \ \text{or} \ 2 < p, \\ 1 & \text{if} \quad -1 \leq p < 0 \ \text{or} \ 1 \leq p \leq 2, \\ K(m,M,p) & \text{if} \quad 0 < p < 1, \end{cases}$$

$$\alpha_1 = \begin{cases} K(m,M,p) & \text{if} \quad p < 0 \ \text{or} \ 1 < p, \\ 1 & \text{if} \quad 0 < p \leq 1, \end{cases}$$

where a generalized Kantorovich constant $K(m,M,p)$ *is defined as (2.21) in § 2.2:*

$$\begin{aligned} K(m,M,p) &= \frac{mM^p - Mm^p}{(p-1)(M-m)} \left(\frac{p-1}{p} \frac{M^p - m^p}{mM^p - Mm^p} \right)^p \\ &= \frac{h^p - h}{(p-1)(h-1)} \left(\frac{p-1}{p} \frac{h^p - 1}{h^p - h} \right)^p, \qquad h = \frac{M}{m}. \end{aligned}$$

If we put $p = s/r$ or $p = r/s$ in (4.6) of Lemma 4.3 and replace A by A^r or A by A^s respectively we obtain the following ratio type inequalities as complementary inequalities to the power means given in Theorem 4.4.

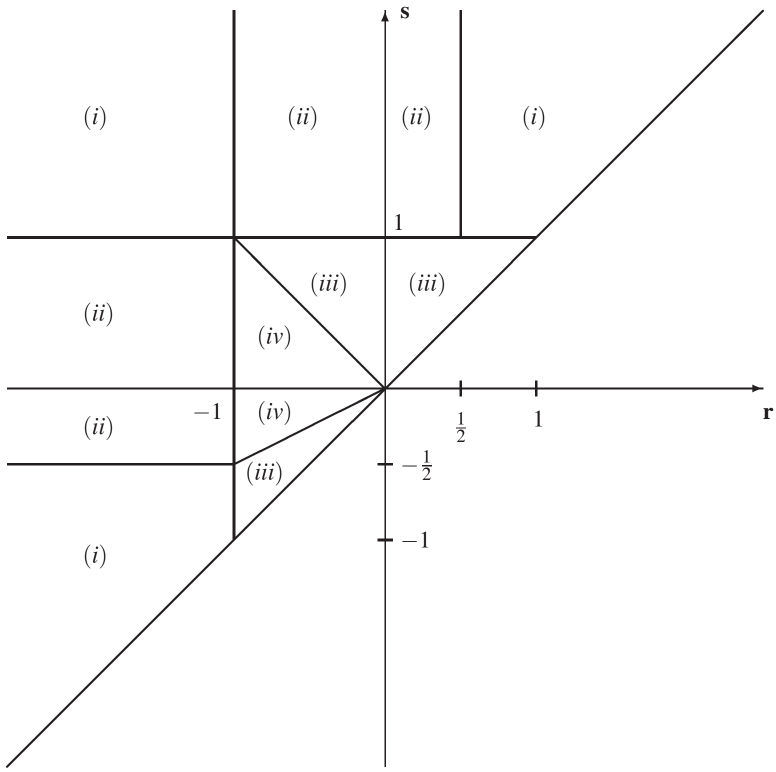

Figure 4.1: *Intervals (i)–(iv)*

(i)	$r \leq s,\ s \notin (-1,1),\ r \notin (-1,1)$ or $1/2 \leq r \leq 1 \leq s$ or $r \leq -1 \leq s \leq -1/2,$
(ii)	$s \geq 1,\ -1 < r < 1/2,\ r \neq 0$ or $r \leq -1,\ -1/2 < s < 1,\ s \neq 0,$
(iii)	$-1 \leq -s \leq r \leq s \leq 1,\ r \neq 0$ or $-1 \leq r \leq s \leq r/2 < 0,$
(iv)	$-1/2 \leq r/2 < s < -r \leq 1,\ s \neq 0.$

Table 4.1: *Intervals from (i) to (iv)*

Theorem 4.4 *Let $\Phi \in \mathbf{P}_N[\mathscr{B}(H),\mathscr{B}(K)]$ be a normalized positive linear map and $A \in \mathscr{B}^{++}(H)$ a positive invertible operator with $\mathsf{Sp}(A) \subseteq [m,M]$ for some scalars $0 < m < M$. Let $r,s \in \mathbb{R}$, $r \leq s$ and $rs \neq 0$.*

(i) If $r \leq s,\ s \notin (-1,1),\ r \notin (-1,1)$ or $1/2 \leq r \leq 1 \leq s$ or $r \leq -1 \leq s \leq -1/2$ then

$$\Delta(h,r,s)^{-1}\Phi(A^s)^{1/s} \leq \Phi(A^r)^{1/r} \leq \Phi(A^s)^{1/s}.$$

(ii) If $1 \le s,\ -1 < r < 1/2,\ r \neq 0$ *or* $r \le -1,\ -1/2 < s < 1,\ s \neq 0$ *then*

$$\Delta(h,r,s)^{-1}\Phi(A^s)^{1/s} \le \Phi(A^r)^{1/r} \le \Delta(h,r,s)\Phi(A^s)^{1/s}.$$

(iii) If $-1 \le -s \le r \le s \le 1,\ r \neq 0$ *or* $-1 \le r \le s \le r/2 < 0$ *then*

$$\Delta(h,r,1)^{-1}\Delta(h,r,s)^{-1}\Phi(A^s)^{1/s} \le \Phi(A^r)^{1/r} \le \Delta(h,r,1)\Phi(A^s)^{1/s}.$$

(iv) If $-1/2 \le r/2 < s < -r \le 1,\ s \neq 0$ *then*

$$\Delta(h,s,1)^{-1}\Delta(h,r,s)^{-1}\Phi(A^s)^{1/s} \le \Phi(A^r)^{1/r} \le \Delta(h,s,1)\Phi(A^s)^{1/s},$$

where a generalized Specht ratio $\Delta(h,r,s)$ *is defined as (2.97) in § 2.7:*

$$\Delta(h,r,s) = \left\{\frac{r(h^s - h^r)}{(s-r)(h^r - 1)}\right\}^{\frac{1}{s}} \left\{\frac{s(h^r - h^s)}{(r-s)(h^s - 1)}\right\}^{-\frac{1}{r}} \quad \text{and} \quad h = \frac{M}{m}.$$

The left hand inequality is sharp when r and s satisfy (i) *or* (ii) *and the right hand inequality when r and s satisfy* (i) *in the matrix case.*

Proof. Suppose that $s \ge 1$ and $r < 1$. We put $p = \frac{s}{r}$. If $0 < r < 1$ then Lemma 4.3 for $1 \le p \le 2$ or $p > 2$ gives

$$\Phi(A)^{s/r} \le \Phi(A^{s/r}) \le K\left(m,M,\tfrac{s}{r}\right)\Phi(A)^{s/r} \qquad \text{if } s/2 \le r \le s,$$

$$K\left(m,M,\tfrac{s}{r}\right)^{-1}\Phi(A)^{s/r} \le \Phi(A^{s/r}) \le K\left(m,M,\tfrac{s}{r}\right)\Phi(A)^{s/r} \quad \text{if } 0 < r < s/2.$$

Replacing A by A^r we have

$$\Phi(A^r)^{s/r} \le \Phi(A^s) \le K\left(m^r,M^r,\tfrac{s}{r}\right)\Phi(A^r)^{s/r} \qquad \text{if } s/2 \le r \le s,$$

$$K\left(m^r,M^r,\tfrac{s}{r}\right)^{-1}\Phi(A^r)^{s/r} \le \Phi(A^s) \le K\left(m^r,M^r,\tfrac{s}{r}\right)\Phi(A^r)^{s/r} \quad \text{if } 0 < r < s/2.$$

By raising above inequalities to the power $0 < 1/s \le 1$, it follows from the Löwner-Heinz theorem that

$$\Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s} \le K\left(m^r,M^r,\frac{s}{r}\right)^{1/s}\Phi(A^r)^{1/r}$$

if $s/2 \le r \le s$, and

$$K\left(m^r,M^r,\frac{s}{r}\right)^{-1/s}\Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s} \le K\left(m^r,M^r,\frac{s}{r}\right)^{1/s}\Phi(A^r)^{1/r}$$

if $0 < r < s/2$. Notice that

$$\begin{aligned} K\left(m^r,M^r,\frac{s}{r}\right)^{1/s} &= K(h^r,\frac{s}{r})^{1/s} \\ &= \left\{\frac{r(h^s - h^r)}{(s-r)(h^r - 1)}\right\}^{1/s} \left\{\frac{s(h^r - h^s)}{(r-s)(h^s - 1)}\right\}^{-1/r} = \Delta(h,r,s) \end{aligned}$$

(see Theorem 2.61 and (2.97)). If we put $s=1$ or $r=1$ in (4.5), then we have

$$\Phi(A^r)^{1/r} \le \Phi(A) \le \Phi(A^s)^{1/s} \quad \text{for } 1/2 \le r \le 1 \text{ and } s \ge 1.$$

Therefore for $s>1$ we have

$$\begin{aligned} &\Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s} \le \tilde{\Delta}(h,r,s)\Phi(A^r)^{1/r} && \text{if } 1/2 \le r \le 1, \\ &\tilde{\Delta}(h,r,s)^{-1}\Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s} \le \tilde{\Delta}(h,r,s)\Phi(A^r)^{1/r} && \text{if } 0<r<1/2. \end{aligned}$$

So, we obtain

$$\begin{aligned} &\tilde{\Delta}(h,r,s)^{-1}\Phi(A^s)^{1/s} \le \Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s} && \text{if } 1/2 \le r \le 1,\ 1 \le s, \\ &\tilde{\Delta}(h,r,s)^{-1}\Phi(A^s)^{1/s} \le \Phi(A^r)^{1/r} \le \tilde{\Delta}(h,r,s)\Phi(A^s)^{1/s} && \text{if } 0<r<1/2,\ 1 \le s. \end{aligned}$$

If $r<0$ then Lemma 4.3 for $-1 \le p < 0$ or $p<-1$ with the Löwner-Heinz theorem gives

$$\Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s} \le K(M^r,m^r,\tfrac{s}{r})^{1/s}\Phi(A^r)^{1/r} \quad \text{if } r \le -s,$$

$$K\left(M^r,m^r,\tfrac{s}{r}\right)^{-1/s}\Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s} \le K\left(M^r,m^r,\tfrac{s}{r}\right)^{1/s}\Phi(A^r)^{1/r}$$

if $-s<r<0$, where $K\left(M^r,m^r,\tfrac{s}{r}\right)^{1/s} = K\left(m^r,M^r,\tfrac{s}{r}\right)^{1/s} = \tilde{\Delta}(h,r,s)$ by Theorem 2.54 (i) and Theorem 2.62 (ii). Therefore, similarly to above we have

$$\begin{aligned} &\Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s} \le \tilde{\Delta}(h,r,s)\Phi(A^r)^{1/r} && \text{if } r \le -1, \\ &\tilde{\Delta}(h,r,s)^{-1}\Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s} \le \tilde{\Delta}(h,r,s)\Phi(A^r)^{1/r} && \text{if } -1<r<0. \end{aligned}$$

So, we obtain

$$\begin{aligned} &\Delta(h,r,s)^{-1}\Phi(A^s)^{1/s} \le \Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s} && \text{if } r \le -1,\ 1 \le s, \\ &\Delta(h,r,s)^{-1}\Phi(A^s)^{1/s} \le \Phi(A^r)^{1/r} \le \Delta(h,r,s)\Phi(A^s)^{1/s} && \text{if } -1<r<0,\ 1 \le s. \end{aligned}$$

Now, suppose that $1 \le r \le s$. We put $p=\frac{r}{s}$. Then Lemma 4.3 for $0<p\le 1$ with the Löwner-Heinz theorem gives

$$K\left(m^s,M^s,\frac{r}{s}\right)^{1/r}\Phi(A^s)^{1/s} \le \Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s}.$$

Since $K\left(m^s,M^s,\tfrac{r}{s}\right)^{1/r} = K\left(m^r,M^r,\tfrac{s}{r}\right)^{-1/s} = \Delta(h,r,s)^{-1}$ by Theorem 2.54 (v), we obtain

$$\Delta(h,r,s)^{-1}\Phi(A^s)^{1/s} \le \Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s} \quad \text{if } 1 \le r \le s.$$

Therefore, we have the desired results in the case (i) and (ii) for $1 \le s$.

Next we shall prove the desired results in the case (i) and (ii) for $r \le -1$.

If $-1 < s < 1$ we put $p = \frac{r}{s}$. If $0 < s < 1$ then Lemma 4.3 for $p < -1$ gives

$$K\left(m^s, M^s, \frac{r}{s}\right)^{-1} (\Phi(A^s))^{r/s} \leq \Phi(A^r) \leq K\left(m^s, M^s, \frac{r}{s}\right) (\Phi(A^s))^{r/s}.$$

Since the function $f(t) = t^{1/r}$ is operator decreasing for $r \leq -1$, we obtain

$$K\left(m^s, M^s, \frac{r}{s}\right)^{-1/r} \Phi(A^s)^{1/s} \geq \Phi(A^r)^{1/r} \geq K\left(m^s, M^s, \frac{r}{s}\right)^{1/r} \Phi(A^s)^{1/s},$$

so we have

$$\Delta(h,r,s)^{-1}\Phi(A^s)^{1/s} \leq \Phi(A^r)^{1/r} \leq \Delta(h,r,s)\Phi(A^s)^{1/s} \quad \text{if } 0 < s < 1,\ r \leq -1.$$

If $-1 < s < 0$ then Lemma 4.3 for $1 \leq p \leq 2$ or $p > 2$, with the fact that the function $f(t) = t^{1/r}$ is operator decreasing for $r \leq -1$, gives

$$\Phi(A^s)^{1/s} \geq \Phi(A^r)^{1/r} \geq K\left(M^s, m^s, \tfrac{r}{s}\right)^{1/r} \Phi(A^s)^{1/s} \quad \text{if } -1 < s \leq r/2,$$
$$K\left(M^s, m^s, \tfrac{r}{s}\right)^{-1/r} \Phi(A^s)^{1/s} \geq \Phi(A^r)^{1/r} \geq K\left(M^s, m^s, \tfrac{r}{s}\right)^{1/r} \Phi(A^s)^{1/s}$$

if $r/2 < s < 0$, where $K\left(M^s, m^s, \frac{r}{s}\right)^{1/r} = K\left(m^s, M^s, \frac{r}{s}\right)^{1/r} = \Delta(h,r,s)^{-1}$ by Theorem 2.54 (i). If we put $s = -1$ or $r = -1$ in (4.5), then we have

$$\Phi(A^r)^{1/r} \leq \Phi(A^{-1})^{-1} \leq \Phi(A^s)^{1/s} \qquad \text{for } r \leq -1 \text{ and } -1 < s \leq -1/2,$$

so we have

$$\Delta(h,r,s)^{-1}\Phi(A^s)^{1/s} \leq \Phi(A^r)^{1/r} \leq \Phi(A^s)^{1/s}$$

if $-1 < s \leq -1/2,\ \ r \leq -1$ and

$$\Delta(h,r,s)^{-1}\Phi(A^s)^{1/s} \leq \Phi(A^r)^{1/r} \leq \Delta(h,r,s)\Phi(A^s)^{1/s}$$

if $-1/2 < s < 0,\ \ r \leq -1$.

If $r \leq s \leq -1$ then we put $p = \frac{s}{r}$. Lemma 4.3 for $0 < p \leq 1$, with the fact that the function $f(t) = t^{1/s}$ is operator decreasing for $s \leq -1$, gives

$$K\left(M^r, m^r, \frac{s}{r}\right)^{1/s} \Phi(A^r)^{1/r} \geq \Phi(A^s)^{1/s} \geq \Phi(A^r)^{1/r},$$

so we have

$$\Delta(h,r,s)^{-1}\Phi(A^s)^{1/s} \leq \Phi(A^r)^{1/r} \leq \Phi(A^s)^{1/s} \quad \text{if } r \leq s \leq -1.$$

We have the desired results in the case (i) and (ii) for $r \leq -1$.

(iii) If $0 < r \leq s \leq 1$ then $0 < \frac{r}{s} \leq 1$. If we put $p = \frac{r}{s}$ in Lemma 4.3 for $0 < p \leq 1$ and replace A by A^s, then we obtain

$$K\left(m^s, M^s, \frac{r}{s}\right) \Phi(A^s)^{r/s} \leq \Phi(A^r) \leq \Phi(A^s)^{r/s}.$$

By raising above inequality to the power $1/r(\geq 1)$ it follows from Theorem 8.3 and $m^r 1_K \leq \Phi(A^r) \leq M^r 1_K$ that

$$K\left(m^r, M^r, \frac{1}{r}\right)^{-1} K\left(m^s, M^s, \frac{r}{s}\right)^{1/r} \Phi(A^s)^{1/s} \leq \Phi(A^r)^{1/r} \leq K\left(m^r, M^r, \frac{1}{r}\right) \Phi(A^s)^{1/s}.$$

So, we have

$$\Delta(h,r,1)^{-1}\Delta(h,r,s)^{-1}\Phi(A^s)^{1/s} \leq \Phi(A^r)^{1/r} \leq \Delta(h,r,1)\Phi(A^s)^{1/s}.$$

If $-1 \leq -s \leq r < 0$ then $-1 \leq \frac{r}{s} < 0$, but if $-1 \leq r \leq s \leq r/2 < 0$ then $1 \leq \frac{r}{s} \leq 2$. If we put $p = \frac{r}{s}$ in Lemma 4.3 for $-1 \leq p < 0$ or $1 \leq p \leq 2$ and replace A by A^s, then we obtain

$$\Phi(A^s)^{r/s} \leq \Phi(A^r) \leq K\left(m^s, M^s, \tfrac{r}{s}\right) \Phi(A^s)^{r/s} \text{ if } -1 \leq -s \leq r < 0,$$

$$\Phi(A^s)^{r/s} \leq \Phi(A^r) \leq K\left(M^s, m^s, \tfrac{r}{s}\right) \Phi(A^s)^{r/s} \text{ if } -1 \leq r \leq s \leq r/2 < 0.$$

Using that $K\left(M^s, m^s, \frac{r}{s}\right) = K\left(m^s, M^s, \frac{r}{s}\right)$ by Theorem 2.54 (i) and by raising above inequalities to the power $1/r \leq -1$, then it follows from Corollary 8.51 and $M^r 1_K \leq \Phi(A^r) \leq m^r 1_K$ that

$$K\left(m^r, M^r, \frac{1}{r}\right) \Phi(A^s)^{1/s} \geq \Phi(A^r)^{1/r} \geq K\left(m^r, M^r, \frac{1}{r}\right)^{-1} K\left(m^s, M^s, \frac{r}{s}\right)^{1/r} \Phi(A^s)^{1/s}$$

if $-1 \leq -s \leq r < 0$ or $-1 \leq r \leq s \leq r/2 < 0$. So, we have

$$\Delta(h,r,1)^{-1}\Delta(h,r,s)^{-1}\Phi(A^s)^{1/s} \leq \Phi(A^r)^{1/r} \leq \Delta(h,r,1)\Phi(A^s)^{1/s}$$

if $-1 \leq -s \leq r \leq s \leq 1$, $r \neq 0$ or $-1 \leq r < s \leq r/2 < 0$ and hence we prove (iii).

(iv) Next, let $-1 \leq r < -s < 0$ or $-1/2 \leq r/2 < s < 0$. Then $-1 < \frac{s}{r} < 0$ or $0 < \frac{s}{r} < \frac{1}{2}$. If we put $p = \frac{s}{r}$ in Lemma 4.3 for $-1 \leq p < 0$ or $0 < p \leq 1$ and replace A by A^r, then we obtain

$$\Phi(A^r)^{s/r} \leq \Phi(A^s) \leq K\left(M^r, m^r, \tfrac{s}{r}\right) \Phi(A^r)^{s/r} \text{ if } -1 \leq r < -s < 0,$$

$$K\left(M^r, m^r, \tfrac{s}{r}\right) \Phi(A^r)^{s/r} \leq \Phi(A^s) \leq \Phi(A^r)^{s/r} \text{ if } -1/2 \leq r/2 < s < 0.$$

By raising above inequalities to the power $1/s$, it follows from Theorem 8.3 and Corollary 8.51 that

$$K\left(m^s, M^s, \frac{1}{s}\right)^{-1} \Phi(A^r)^{1/r} \leq \Phi(A^s)^{1/s} \leq K\left(m^s, M^s, \frac{1}{s}\right) K\left(M^r, m^r, \frac{s}{r}\right)^{1/s} \Phi(A^r)^{1/r}$$

if $-1 \leq r < -s < 0$ and

$$K\left(M^s, m^s, \frac{1}{s}\right) K\left(M^r, m^r, \frac{s}{r}\right)^{1/s} \Phi(A^r)^{1/r} \geq \Phi(A^s)^{1/s} \geq K\left(M^s, m^s, \frac{1}{s}\right)^{-1} \Phi(A^r)^{1/r}$$

if $-1/2 \le r/2 < s < 0$. Since $K\left(M^s, m^s, \frac{1}{s}\right) = K\left(m^s, M^s, \frac{1}{s}\right) = \Delta(h,1,s)^{-1} = \Delta(h,s,1)$ by Theorem 2.62 (v), we have

$$\Delta(h,s,1)^{-1}\Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s} \le \Delta(h,s,1)\Delta(h,r,s)\Phi(A^r)^{1/r}$$

if $-1/2 \le r/2 < s < -r \le 1$, $s \ne 0$. So, we have

$$\Delta(h,s,1)^{-1}\Delta(h,r,s)^{-1}\Phi(A^s)^{1/s} \le \Phi(A^r)^{1/r} \le \Delta(h,s,1)\Phi(A^s)^{1/s}$$

if $-1/2 \le r/2 < s < -r \le 1$, $s \ne 0$.

Hence the proof of $(i) \sim (iv)$ in Theorem 4.4 is now complete.

In the matrix case, by Theorem 3.18 all inequalities in the case (i) and (ii) are sharp except when the right hand boundary is Δ. □

Similarly to above, we shall give the estimate of the difference $\Phi(A^s)^{1/s} - \Phi(A^r)^{1/r}$ for $r \le s$. We recall the following result (see Remark 3.13 in § 3.4):

Lemma 4.5 *Let $\Phi \in \mathbf{P}_N[\mathscr{B}(H), \mathscr{B}(K)]$ be a normalized positive linear map and $A \in \mathscr{B}^{++}(H)$ a positive invertible operator with $\mathsf{Sp}(A) \subseteq [m,M]$ for some scalars $0 < m < M$. Then*

$$\beta_2 1_K \le \Phi(A^p) - \Phi(A)^p \le \beta_1 1_K, \tag{4.7}$$

where

$$\begin{aligned}
\beta_1 &= \begin{cases} C(m,M,p) & \text{if} \quad p < 0 \ \text{or} \ 1 < p, \\ 0 & \text{if} \quad 0 < p \le 1, \end{cases} \\
\beta_2 &= \begin{cases} -C(m,M,p) & \text{if} \quad p < -1 \ \text{or} \ 2 < p, \\ 0 & \text{if} \quad -1 \le p < 0 \ \text{or} \ 1 \le p \le 2, \\ C(m,M,p) & \text{if} \quad 0 < p < 1, \end{cases}
\end{aligned}$$

and $C(m,M,p)$ is defined as (2.39):

$$\begin{aligned}
C(m,M,p) &= \frac{Mm^p - mM^p}{M-m} + (p-1)\left(\frac{1}{p}\frac{M^p - m^p}{M-m}\right)^{\frac{p}{p-1}} \\
&= M^p \frac{1-h^{1-p}}{1-h} + m^p(p-1)\left\{\frac{p(h-1)}{h^p - 1}\right\}^{\frac{p}{1-p}}, \quad \text{and} \quad h = \frac{M}{m}.
\end{aligned}$$

Lemma 4.6 *Let the hypothesis of Theorem 4.4 be satisfied.*

(a) If $\ 1 \le r \le s \ $ or $\ r \le -1 \le s$, then

$$\begin{aligned}
&[\bar{\mu}\Phi(A^s) + \bar{\nu}1_K]^{1/r} \le \Phi(A^r)^{1/r} \\
&\le \begin{cases} \left[\bar{\mu}\Phi(A^s) + \left(1 - \frac{r}{s}\right)\left(\frac{s}{r}\bar{\mu}\right)^{\frac{r}{r-s}} 1_K\right]^{1/r} & \text{if} \quad -1/2 < s < 1, s \ne 0, \\ \Phi(A^s)^{1/s} & \text{otherwise}. \end{cases}
\end{aligned} \tag{4.8}$$

(b) If $r \le s \le -1$ *or* $r \le 1 \le s$, *then*

$$\left[\tfrac{1}{\bar{\mu}}\Phi(A^r) - \tfrac{\bar{\nu}}{\bar{\mu}}1_K\right]^{1/s} \ge \Phi(A^s)^{1/s}$$
$$\ge \begin{cases} \left[\frac{1}{\bar{\mu}}\Phi(A^r) - \frac{1}{\bar{\mu}}\left(1-\frac{r}{s}\right)\left(\frac{s}{r}\bar{\mu}\right)^{\frac{r}{r-s}}1_K\right]^{1/s} & \textit{if} \quad -1 < r < 1/2, r \ne 0, \\ \Phi(A^r)^{1/r}, & \textit{otherwise}. \end{cases} \tag{4.9}$$

(c) If $-1 \le -s \le r \le s \le 1$, $r \ne 0$ *or* $-1 \le r \le s \le r/2 < 0$ *then*

$$[\bar{\mu}\Phi(A^s) + \bar{\nu}1_K]^{1/r} - C\left(m^r, M^r, \tfrac{1}{r}\right)1_K \le \Phi(A^r)^{1/r}$$
$$\le \Phi(A^s)^{1/s} + C\left(m^r, M^r, \tfrac{1}{r}\right)1_K.$$

(d) If $-1/2 \le r/2 < s < -r \le 1$, $s \ne 0$ *then*

$$[\bar{\mu}\Phi(A^s) + \bar{\nu}1_K]^{1/r} - C\left(m^r, M^r, \tfrac{1}{r}\right)1_K \le \Phi(A^r)^{1/r}$$
$$\le \left[\bar{\mu}\Phi(A^s) + \left(1-\tfrac{r}{s}\right)\left(\tfrac{s}{r}\bar{\mu}\right)^{\frac{r}{r-s}}1_K\right]^{1/r} + C\left(m^r, M^r, \tfrac{1}{r}\right)1_K,$$

where $\left(\bar{\mu} = \frac{M^r - m^r}{M^s - m^s}\right)$ *and* $\left(\bar{\nu} = \frac{M^s m^r - M^r m^s}{M^s - m^s}\right)$.

Proof. The following two inequalities hold: If $0 < p \le 1$, then

$$\mu_{t^p}\Phi(A) + \nu_{t^p}1_K \le \Phi(A^p) \le \Phi(A)^p. \tag{4.10}$$

If $p < 0$ or $1 < p$, then

$$\mu_{t^p}\Phi(A) + \nu_{t^p}1_K \ge \Phi(A^p) \ge \begin{cases} \Phi(A)^p & \text{if } -1 \le p < 0 \text{ or } 1 \le p \le 2, \\ \mu_{t^p}\Phi(A) + \nu^*_{t^p}1_K & \text{if } p < -1 \text{ or } 2 < p, \end{cases} \tag{4.11}$$

where $\left(\nu^*_{t^p} = (1-p)(\mu_{t^p}/p)^{p/(p-1)}.\right)$

Indeed, the right hand inequalities of (4.10) for $0 < p \le 1$ and (4.11) for $-1 \le p < 0$ or $1 \le p \le 2$ follow directly from (4.7) in Lemma 4.5. The right hand inequality of (4.11) for $p < -1$ or $2 < p$ follows from Remark 3.7 for functions $f(t) = t^p$ and $g(t) = \mu_{t^p}t$. The left hand inequalities of (4.10) and (4.11) follow from Theorem 3.14 for $f(t) = t^p$, $\alpha = \mu_{t^p}$ and $g(t) = t$.

Firstly we prove (a). Let $r \notin (-1,1)$. We put $p = \frac{r}{s}$ in (4.10)–(4.11) and replace A by A^s. Then

$$\begin{array}{ll} \Phi(A^s)^{r/s} \le \Phi(A^r) \le \bar{\mu}\Phi(A^s) + \bar{\nu}1_K & \text{if } r \le -1 \text{ and } (r \le s \le r/2 \text{ or } -r \le s), \\ \bar{\mu}\Phi(A^s) + \nu^*1_K \le \Phi(A^r) \le \bar{\mu}\Phi(A^s) + \bar{\nu}1_K & \text{if } r \le -1,\ r/2 < s < -r,\ s \ne 0, \\ \bar{\mu}\Phi(A^s) + \bar{\nu}1_K \le \Phi(A^r) \le \Phi(A^s)^{r/s} & \text{if } 1 \le r \le s, \end{array}$$

where $\nu^* = \left(1-\frac{r}{s}\right)\left(\frac{s}{r}\bar{\mu}\right)^{r/(r-s)}$. Using the fact that the function $f(t) = t^{\frac{1}{r}}$ is operator increasing for $r \ge 1$ and operator decreasing for $r \le -1$, we have

$$\Phi(A^s)^{1/s} \ge \Phi(A^r)^{1/r} \ge [\bar{\mu}\Phi(A^s) + \bar{\nu}1_K]^{1/r}$$

if $r \le -1$ and ($r \le s \le r/2$ or $-r \le s$),

$$[\bar{\mu}\Phi(A^s) + \nu^* 1_K]^{1/r} \ge \Phi(A^r)^{1/r} \ge [\bar{\mu}\Phi(A^s) + \bar{\nu} 1_K]^{1/r}$$

if $r \le -1$, $r/2 < s < -r$, $s \ne 0$,

$$[\bar{\mu}\Phi(A^s) + \bar{\nu} 1_K]^{1/r} \le \Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s} \quad \text{if } 1 \le r \le s.$$

If we put $s = -1$ or $r = -1$ in (4.5), then we have

$$\Phi(A^r)^{1/r} \le \Phi(A^{-1})^{-1} \le \Phi(A^s)^{1/s} \quad \text{ for } r \le -1 \text{ and } -1 < s \le -1/2.$$

So, we obtain

$$[\bar{\mu}\Phi(A^s) + \bar{\nu} 1_K]^{1/r} \le \Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s}$$

if $r \le -1 \le s \le -1/2$ or ($r \le -1$, $1 \le s$) or $1 \le r \le s$,

$$[\bar{\mu}\Phi(A^s) + \bar{\nu} 1_K]^{1/r} \le \Phi(A^r)^{1/r} \le [\bar{\mu}\Phi(A^s) + \nu^* 1_K]^{1/r}$$

if $r \le -1$, $-1/2 < s < 1$, $s \ne 0$. Therefore we have (a).

Next we prove (b). Let $s \notin (-1,1)$. We put $p = \frac{s}{r}$ in (4.10)–(4.11) and replace A by A^r. Then

$$\begin{array}{ll} \Phi(A^r)^{s/r} \le \Phi(A^s) \le \tilde{\mu}\Phi(A^r) + \tilde{\nu} 1_K & \text{if } s \ge 1 \text{ and } (s/2 \le r \le s \text{ or } r \le -s), \\ \tilde{\mu}\Phi(A^r) + \tilde{\nu}^* 1_K \le \Phi(A^s) \le \tilde{\mu}\Phi(A^r) + \tilde{\nu} 1_K & \text{if } s \ge 1,\ -s < r < s/2,\ r \ne 0, \\ \tilde{\mu}\Phi(A^r) + \tilde{\nu} 1_K \le \Phi(A^s) \le \Phi(A^r)^{s/r} & \text{if } r \le s \le -1, \end{array}$$

where $\tilde{\mu} = \frac{M^s - m^s}{M^r - m^r} = \frac{1}{\mu}$, $\quad \tilde{\nu} = \frac{M^r m^s - M^s m^r}{M^r - m^r} = -\frac{\bar{\nu}}{\mu}$, $\quad \tilde{\nu}^* = \left(1 - \frac{s}{r}\right)\left(\frac{r}{s}\tilde{\mu}\right)^{\frac{s}{s-r}} = -\frac{\nu^*}{\mu}$. By raising above inequalities to the power $\frac{1}{s}$ we obtain

$$\Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s} \le \left[\frac{1}{\mu}\Phi(A^r) - \frac{\bar{\nu}}{\mu} 1_K\right]^{1/s}$$

if $s \ge 1$ and ($s/2 \le r \le s$ or $r \le -s$),

$$\left[\frac{1}{\mu}\Phi(A^r) - \frac{\nu^*}{\mu} 1_K\right]^{1/s} \le \Phi(A^s)^{1/s} \le \left[\frac{1}{\mu}\Phi(A^r) - \frac{\bar{\nu}}{\mu} 1_K\right]^{1/s}$$

if $s \ge 1$, $-s < r < s/2$, $r \ne 0$,

$$\Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s} \le \left[\frac{1}{\mu}\Phi(A^r) - \frac{\bar{\nu}}{\mu} 1_K\right]^{1/s} \qquad \text{if } r < s \le -1.$$

If we put $s = 1$ or $r = 1$ in (4.5), then we have

$$\Phi(A^r)^{1/r} \le \Phi(A) \le \Phi(A^s)^{1/s} \qquad \text{for } s \ge 1 \text{ and } 1/2 \le r \le 1.$$

So, we obtain

$$\Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s} \le \left[\frac{1}{\bar{\mu}}\Phi(A^r) - \frac{\bar{\nu}}{\bar{\mu}}1_K\right]^{1/s}$$

if $1/2 \le r \le 1 \le s$ or $(s \ge 1, r \le -1)$ or $r \le s \le -1$,

$$\left[\frac{1}{\bar{\mu}}\sum_{j=1}^{k}\omega_j\Phi(A^r) - \frac{\nu^*}{\bar{\mu}}1_K\right]^{1/s} \le \Phi(A^s)^{1/s} \le \left[\frac{1}{\bar{\mu}}\Phi(A^r) - \frac{\bar{\nu}}{\bar{\mu}}1_K\right]^{1/s}$$

if $s \ge 1$, $-1 < r < 1/2$, $r \ne 0$. Therefore we have (b).

Next we prove (c). If $0 < r \le s \le 1$ then $0 < \frac{r}{s} \le 1$ and by (4.10) we obtain

$$\bar{\mu}\sum_{j=1}^{k}\omega_j\Phi(A^s) + \bar{\nu}1_K \le \Phi(A^r) \le \Phi(A^s)^{r/s}.$$

Using Theorem 8.3 for $p = \frac{1}{r}$, and since $m^r 1_K \le \bar{\mu}\Phi(A^s) + \bar{\nu}1_K \le M^r 1_K$ and $m^r 1_K \le \Phi(A^r) \le M^r 1_K$ we obtain

$$[\bar{\mu}\Phi(A^s) + \bar{\nu}1_K]^{1/r} - C\left(m^r, M^r, \frac{1}{r}\right)1_K \le \Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s} + C\left(m^r, M^r, \frac{1}{r}\right)1_K.$$

If $(-1 \le r < 0 < s \le 1$ and $-1 \le \frac{r}{s} < 0)$ or $(-1 \le r \le s < 0$ and $1 \le \frac{r}{s} \le 2)$ then by (4.11) we obtain

$$\Phi(A^s)^{r/s} \le \Phi(A^r) \le \bar{\mu}\Phi(A^s) + \bar{\nu}1_K.$$

Using Corollary 8.51 for $p = \frac{1}{r} < -1$, and since $m^r 1_K \le \bar{\mu}\Phi(A^s) + \bar{\nu}1_K \le M^r 1_K$ and $m^r 1_K \le \Phi(A^r) \le M^r 1_K$ we obtain

$$\Phi(A^s)^{1/s} + C\left(M^r, m^r, \frac{1}{r}\right)I \ge \Phi(A^r)^{1/r} \ge [\bar{\mu}\Phi(A^s) + \bar{\nu}1_K]^{1/r} - C\left(M^r, m^r, \frac{1}{r}\right)1_K.$$

Since $C\left(M^r, m^r, \frac{1}{r}\right) = C\left(m^r, M^r, \frac{1}{r}\right)$, it follows that

$$[\bar{\mu}\Phi(A^s) + \bar{\nu}1_K]^{1/r} - C\left(m^r, M^r, \frac{1}{r}\right)1_K \le \Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s} + C\left(m^r, M^r, \frac{1}{r}\right)1_K$$

holds if $-1 \le -s \le r \le s \le 1$, $r \ne 0$ or $-1 \le r \le s \le r/2 < 0$. Therefore we have (c).

Next we prove (d). If $(-1 \le r < 0 < s \le 1$ and $\frac{r}{s} < -1)$ or $(-1 \le r < s < 0$ and $\frac{r}{s} > 2)$ then from (4.11) we obtain

$$\bar{\mu}\Phi(A^s) + \nu^* 1_K \le \Phi(A^r) \le \bar{\mu}\Phi(A^s) + \bar{\nu}1_K.$$

Using Corollary 8.51 for $p = \frac{1}{r} < -1$ we have that

$$\begin{aligned}[\bar{\mu}\Phi(A^s) + \bar{\nu}1_K]^{1/r} - C\left(m^r, M^r, \tfrac{1}{r}\right)I &\le \Phi(A^r)^{1/r} \\ &\le [\bar{\mu}\Phi(A^s) + \nu^* 1_K]^{1/r} + C\left(m^r, M^r, \tfrac{1}{r}\right)1_K\end{aligned}$$

holds if $-1/2 \le r/2 < s < -r \le 1$, $s \ne 0$. Therefore we have (d). $\square$

Remark 4.1 *Let $\Phi \in \mathbf{P}_N[\mathscr{M}_n,\mathscr{M}_k]$ be a normalized positive linear map and $A \in \mathscr{H}_n^{++}$ strictly positive Hermitian matrix with $\mathsf{Sp}(A) \subseteq [m,M]$.*

In (4.8), the left hand inequality is sharp for all values of r,s and the right hand inequality for $1 \le r \le s$ or $r \le -1 \le s \le -1/2$ or $r \le -1, 1 \le s$.

In (4.9), the left hand inequality is sharp for all values of r,s and the right hand inequality when $r \le s \le -1$ or $1/2 \le r \le 1 \le s$ or $r \le -1, 1 \le s$.

By Lemma 4.6, we obtain the following difference type inequalities as a complementary inequality to the inequality (4.5) in Lemma 4.2.

Theorem 4.7 *Let $\Phi_j \in \mathbf{P}_N[\mathscr{B}(H),\mathscr{B}(K)]$ be a normalized positive linear map and $A \in \mathscr{B}^{++}(H)$ a positive invertible operator with $\mathsf{Sp}(A) \subseteq [m,M]$ for some scalars $0 < m < M$.*

(i) If $r \le s$, $s \notin (-1,1)$, $r \notin (-1,1)$ or $1/2 \le r \le 1 \le s$ or $r \le -1 \le s \le -1/2$ then

$$0 \le \Phi(A^s)^{1/s} - \Phi(A^r)^{1/r} \le \tilde{\Delta} 1_K. \tag{4.12}$$

(ii) If $s \ge 1$, $-1 < r < 1/2$, $r \neq 0$ or $r \le -1$, $-1/2 < s < 1$, $s \neq 0$ then

$$\tilde{\Delta}^* 1_K \le \Phi(A^s)^{1/s} - \Phi(A^r)^{1/r} \le \tilde{\Delta} 1_K. \tag{4.13}$$

(iii) If $-1 \le -s \le r \le s \le 1$, $r \neq 0$ or $-1 \le r \le s \le r/2 < 0$ then

$$-C\left(m^r, M^r, \frac{1}{r}\right) 1_K \le \Phi(A^s)^{1/s} - \Phi(A^r)^{1/r} \le \tilde{\Delta} 1_K + C\left(m^r, M^r, \frac{1}{r}\right) 1_K.$$

(iv) If $-1/2 \le r/2 < s < -r \le 1$, $s \neq 0$ then

$$\tilde{\Delta}^* 1_K - C\left(m^r, M^r, \frac{1}{r}\right) 1_K \le \Phi(A^s)^{1/s} - \Phi(A^r)^{1/r} \le \tilde{\Delta} 1_K + C\left(m^r, M^r, \frac{1}{r}\right) 1_K,$$

where

$$\begin{aligned}
\tilde{\Delta} &= \max_{\theta \in [0,1]} \left\{ [\theta M^s + (1-\theta) m^s]^{\frac{1}{s}} - [\theta M^r + (1-\theta) m^r]^{\frac{1}{r}} \right\}, \\
\tilde{\Delta}^* &= \min_{\theta \in [0,1] \cup [\frac{d}{M^r - m^r}, \frac{d}{M^r - m^r} + 1]} \left\{ [\theta M^s + (1-\theta) m^s]^{\frac{1}{s}} \right. \\
&- \left. [\theta M^r + (1-\theta) m^r - d]^{\frac{1}{r}} \right\}, \\
d &= \frac{M^s m^r - M^r m^s}{M^s - m^s} - \left(1 - \frac{r}{s}\right) \left(\frac{s}{r} \frac{M^r - m^r}{M^s - m^s} \right)^{\frac{r}{r-s}}.
\end{aligned}$$

Proof. By Lemma 4.6 (a) we obtain that

$$\begin{aligned}
\Phi(A^s)^{1/s} - [\bar{\mu}\Phi(A^s) + \nu^* 1_K]^{1/r} &\le \Phi(A^s)^{1/s} - \Phi(A^r)^{1/r} \\
&\le \Phi(A^s)^{1/s} - [\bar{\mu}\Phi(A^s) + \bar{\nu} 1_K]^{1/r}
\end{aligned} \tag{4.14}$$

holds if $-1/2 < s < 1$, $s \neq 0$, $r \le -1$ and

$$\begin{aligned}
0 = \Phi(A^s)^{1/s} - \Phi(A^s)^{1/s} &\le \Phi(A^s)^{1/s} - \Phi(A^r)^{1/r} \\
&\le \Phi(A^s)^{1/s} - [\bar{\mu}\Phi(A^s) + \bar{\nu} 1_K]^{1/r}
\end{aligned} \tag{4.15}$$

holds if $1 \leq r \leq s$ or ($r \leq -1$, $s \geq 1$) or $r \leq -1 \leq s \leq -1/2$.

By Lemma 4.6 (b) we obtain that

$$\left[\frac{1}{\mu}\Phi(A^r) - \frac{\nu^*}{\mu}1_K\right]^{1/s} - \Phi(A^r)^{1/r} \leq \Phi(A^s)^{1/s} - \Phi(A^r)^{1/r}$$
$$\leq \left[\frac{1}{\mu}\Phi(A^r) - \frac{\bar{\nu}}{\mu}1_K\right]^{1/s} - \Phi(A^r)^{1/r} \tag{4.16}$$

holds if $-1 < r < 1/2$, $r \neq 0$, $s \geq 1$ and

$$0 = \Phi(A^r)^{1/r} - (A^r)^{1/r} \leq \Phi(A^s)^{1/s} - \Phi(A^r)^{1/r}$$
$$\leq \left[\frac{1}{\mu}\Phi(A^r) - \frac{\bar{\nu}}{\mu}1_K\right]^{1/s} - \Phi(A^r)^{1/r} \tag{4.17}$$

holds if $r \leq s \leq -1$ or ($r \leq -1$, $s \geq 1$) or $1/2 \leq r \leq 1 \leq s$.

It follows from the right hand inequalities of (4.14) and (4.15) that

$$\Phi(A^s)^{1/s} - \Phi(A^r)^{1/r} \leq \Phi(A^s)^{1/s} - [\bar{\mu}\Phi(A^s) + \bar{\nu}1_K]^{1/r}$$
$$\leq \max_{t\in\bar{T}}\left\{t^{1/s} - [\bar{\mu}\,t + \bar{\nu}]^{1/r}\right\}1_K$$

holds, where $\bar{T}$ denotes the close interval joining m^s to M^s. We set $t = \theta M^s + (1-\theta)m^s$ for some $\theta \in [0,1]$. Then we have $\bar{\mu}\cdot t + \bar{\nu} = \theta M^r + (1-\theta)m^r$ and hence $\max_{t\in\bar{T}}\left\{t^{1/s} - [\bar{\mu}\,t + \bar{\nu}]^{1/r}\right\} = \tilde{\Delta}$. Therefore, we obtain $\Phi(A^s)^{1/s} - \Phi(A^r)^{1/r} \leq \tilde{\Delta}1_K$ if $1 \leq r \leq s$ or $r \leq -1 \leq s$.

It follows from the right hand inequalities of (4.16) and (4.17) that

$$\Phi(A^s)^{1/s} - \Phi(A^r)^{1/r} \leq \left[\frac{1}{\mu}\Phi(A^r) - \frac{\bar{\nu}}{\mu}1_K\right]^{1/s} - \Phi(A^r)^{1/r}$$
$$\leq \max_{t\in\bar{T_1}}\left\{\left[\frac{1}{\mu}\,t - \frac{\bar{\nu}}{\mu}\right]^{1/s} - t^{1/r}\right\}1_K$$

holds, where $\bar{T_1}$ denotes the close interval joining m^r to M^r. We set $t = \theta M^r + (1-\theta)m^r$ for some $\theta \in [0,1]$. Then we have $\frac{1}{\mu}\cdot t - \frac{\bar{\nu}}{\mu} = \theta M^s + (1-\theta)m^s$ and hence $\max_{t\in\bar{T_1}}\left\{\left[\frac{1}{\mu}\,t - \frac{\bar{\nu}}{\mu}\right]^{1/s} - t^{1/r}\right\} = \tilde{\Delta}$. Therefore, we obtain $\Phi(A^s)^{1/s} - \Phi(A^r)^{1/r} \leq \tilde{\Delta}1_K$ if $r \leq s \leq -1$ and $r \leq 1 \leq s$.

Then we have the right hand inequalities of (i) and (ii) in this theorem.

By the left hand inequalities of (4.15) and (4.17) we have the left hand inequality of (i).

By the left hand inequality of (4.14) we obtain that

$$\Phi(A^s)^{1/s} - \Phi(A^r)^{1/r} \geq \Phi(A^s)^{1/s} - [\bar{\mu}\Phi(A^s) + \nu^*1_K]^{1/r}$$
$$\geq \min_{t\in\bar{T}}\left\{t^{1/s} - [\bar{\mu}\,t + \nu^*]^{1/r}\right\}1_K = \min_{t\in\bar{T}}\left\{t^{1/s} - [\bar{\mu}\,t + \bar{\nu} - d]^{1/r}\right\}1_K$$
$$= \min_{\theta\in[0,1]}\left\{[\theta M^s + (1-\theta)m^s]^{1/s} - [\theta M^r + (1-\theta)m^r - d]^{1/r}\right\}1_K$$
$$\geq \tilde{\Delta}^*1_K \tag{4.18}$$

holds if $-1/2 < s < 1$, $s \neq 0$, $r \leq -1$.

By the left hand inequality of (4.16) we obtain that

$$\begin{aligned}
\Phi(A^s)^{1/s}-\Phi(A^r)^{1/r} &\geq \left[\tfrac{1}{\mu}\Phi(A^r)-\tfrac{v^*}{\mu}1_K\right]^{1/s}-\Phi(A^r)^{1/r}\\
&\geq \min_{t\in\bar{T}_1}\left\{\left[\tfrac{1}{\mu}t-\tfrac{v^*}{\mu}\right]^{1/s}-t^{1/r}\right\}1_K\\
&= \min_{\theta\in[0,1]}\left\{[\theta M^s+(1-\theta)m^s+\tfrac{d}{\mu}]^{1/s}-[\theta M^r+(1-\theta)m^r]^{1/r}\right\}1_K\\
&= \min_{\theta\in[\frac{d}{M^r-m^r},\frac{d}{M^r-m^r}+1]}\{[\theta M^s+(1-\theta)m^s]^{1/s}-[\theta M^r+(1-\theta)m^r-d]^{1/r}\}1_K\\
&\geq \tilde{\Delta}^*1_K
\end{aligned} \tag{4.19}$$

holds if $-1<r<1/2$, $r\neq 0$, $s\geq 1$.

Combined with two inequalities (4.18) and (4.19), we have the left hand inequality of (ii) in this theorem. Therefore we have (i) and (ii) in this theorem.

By Lemma 4.6 (c) we obtain

$$\begin{aligned}
-C\left(m^r,M^r,\tfrac{1}{r}\right)1_K &\leq \Phi(A^s)^{1/s}-\Phi(A^r)^{1/r}\\
&\leq \Phi(A^s)^{1/s}-[\bar{\mu}\Phi(A^s)+\bar{v}1_K]^{1/r}+C(m^r,M^r,\tfrac{1}{r})1_K\\
&\leq \max_{t\in\bar{T}}\left\{t^{1/s}-[\bar{\mu}\,t+\bar{v}]^{1/r}\right\}1_K+C\left(m^r,M^r,\tfrac{1}{r}\right)1_K\\
&= \tilde{\Delta}1_K+C\left(m^r,M^r,\tfrac{1}{r}\right)1_K
\end{aligned}$$

if $-1\leq -s\leq r\leq s\leq 1$, $r\neq 0$ or $-1\leq r\leq s\leq r/2<0$. Then we have (iii) in this theorem.

By Lemma 4.6 (d) we obtain

$$\begin{aligned}
\Phi(A^s)^{1/s}-[\bar{\mu}\Phi(A^s)+v^*1_K]^{1/r}-C\left(m^r,M^r,\tfrac{1}{r}\right)1_K\\
\leq \Phi(A^s)^{1/s}-\Phi(A^r)^{1/r}\\
\leq \Phi(A^s)^{1/s}-[\bar{\mu}\Phi(A^s)+\bar{v}1_K]^{1/r}+C\left(m^r,M^r,\tfrac{1}{r}\right)1_K
\end{aligned}$$

if $-1/2\leq r/2<s<-r\leq 1$, $s\neq 0$. Then

$$\begin{aligned}
\tilde{\Delta}^*1_K-C\left(m^r,M^r,\tfrac{1}{r}\right)1_K &\leq \min_{t\in\bar{T}}\left\{t^{1/s}-[\bar{\mu}\,t+\bar{v}-d]^{1/r}\right\}1_K\\
-C\left(m^r,M^r,\tfrac{1}{r}\right)1_K \leq \Phi(A^s)^{1/s}-\Phi(A^r)^{1/r} &\leq \max_{t\in\bar{T}}\left\{t^{1/s}-[\bar{\mu}\,t+\bar{v}]^{1/r}\right\}1_K\\
+C\left(m^r,M^r,\tfrac{1}{r}\right)1_K &= \tilde{\Delta}1_K+C\left(m^r,M^r,\tfrac{1}{r}\right)1_K
\end{aligned}$$

and we have (iv) in this theorem. □

Remark 4.2 *In the matrix case, let $\Phi\in\mathbf{P}_N[\mathscr{M}_n,\mathscr{M}_k]$ be a normalized positive linear map and $A\in\mathscr{H}_n^{++}$ a positive definite Hermitian matrix with $\mathsf{Sp}(A)\subseteq[m,M]$. Then the inequalities (4.12) are sharp and the right hand inequality of (4.13) is sharp.*

If we put $s = 1$ and $r = p$ in Theorems 4.4 and 4.7 we obtain the following two corollaries, which are complementary inequalities to (iii) and (iv) of Corollary 1.22.

Corollary 4.8 *Let $\Phi_j \in \mathbf{P}_N[\mathscr{B}(H),\mathscr{B}(K)]$ and $A \in \mathscr{B}^{++}(H)$ with $\mathsf{Sp}(A) \subseteq [m,M]$ for some scalars $0 < m < M$. If $p \in \mathbb{R}\backslash\{0\}$, then*

$$\alpha_2 \Phi(A^p)^{1/p} \le \Phi(A) \le \alpha_1 \Phi(A^p)^{1/p}$$

holds for

$$\alpha_2 = \begin{cases} \Delta_1^{-1} & \textit{if} \quad -1 < p < 0 \ \textit{or} \ 0 < p < 1/2, \\ 1 & \textit{if} \quad p \le -1 \ \textit{or} \ 1/2 \le p \le 1, \\ \Delta_1 & \textit{if} \quad 1 < p, \end{cases}$$
$$\alpha_1 = \begin{cases} \Delta_1 & \textit{if} \quad p < 0 \ \textit{or} \ 0 < p < 1, \\ 1 & \textit{if} \quad 1 \le p, \end{cases}$$

where

$$\Delta_1 = (h-1)^{\frac{1}{p}} \frac{p}{h^p - 1} \left\{ \frac{p-1}{h^p - h} \right\}^{\frac{1-p}{p}} \quad \textit{and} \quad h = \frac{M}{m}.$$

Corollary 4.9 *Let the hypothesis of Corollary 4.8 be satisfied. Then*

$$\beta_2 1_K \le \Phi(A) - \Phi(A^p)^{1/p} \le \beta_1 1_K$$

holds for

$$\beta_2 = \begin{cases} -\Delta_2 & \textit{if} \quad -1 < p < 0 \ \textit{or} \ 0 < p < 1/2, \\ 0 & \textit{if} \quad p \le -1 \ \textit{or} \ 1/2 \le p \le 1, \\ \Delta_2 & \textit{if} \quad 1 < p, \end{cases}$$
$$\beta_1 = \begin{cases} \Delta_2 & \textit{if} \quad p < 0 \ \textit{or} \ 0 < p < 1, \\ 0 & \textit{if} \quad 1 \le p, \end{cases}$$

where

$$\Delta_2 = M \frac{1 - h^{p-1}}{1 - h^p} + m \left(\frac{1}{p} - 1 \right) \left\{ \frac{p(h-1)}{h^p - 1} \right\}^{\frac{1}{1-p}} \quad \textit{and} \quad h = \frac{M}{m}.$$

4.3 Ky Fan type inequalities

Now, in the matrix case, we observe the matrix power mean: Let $A_j \in \mathscr{H}_n^{++}$ be a positive definite Hermitian matrices with $\mathsf{Sp}(A_j) \subseteq [m,M]$ and $U_j \in \mathscr{M}_{t,n}$ be such that $\sum_{j=1}^k U_j U_j^* = 1_t$ $(j = 1,\ldots,k)$. Then we denote by

$$M_k^{[r]}(\mathbf{A};\mathbf{U}) := \left(\sum_{j=1}^k U_j^* A_j^r U_j \right)^{1/r} \quad \text{for} \quad r \in \mathbb{R}\backslash\{0\}.$$

As applications of Theorems 4.4 and 4.7 we obtain the following results for matrix power mean.

Corollary 4.10 *Let $A_j \in \mathscr{H}_n^{++}$ with $\mathsf{Sp}(A_j) \subseteq [m,M]$ and $U_j \in \mathscr{M}_{t,n}$ be such that $\sum_{j=1}^k U_j U_j^* = 1_t$ ($j = 1,\ldots,k$). If $r,s \in \mathbb{R}$, $r \leq s$, then*

$$\alpha_2 M_k^{[s]}(\mathbf{A};\mathbf{U}) \leq M_k^{[r]}(\mathbf{A};\mathbf{U}) \leq \alpha_1 M_k^{[s]}(\mathbf{A};\mathbf{U})$$

holds for same boundaries α_1 and α_2 as in Theorems 4.4 (i)–(iv).

Proof. For A_j and U_j ($j = 1,\ldots,k$) in the hypothesis of the corollary we denote by $A = A_1 \dotplus A_2 \dotplus \cdots \dotplus A_k$ and $U = [U_1 U_2 \cdots U_k]$. Then $UAU^* = \sum_{j=1}^k U_j A_j U_j^*$. In the same way as in proof of Corollary 3.19, we define the map $\Phi \in \mathbf{P}_N[\mathscr{M}_{n\cdot k}, \mathscr{M}_t]$ whit $\Phi(A) = UAU^*$, where $U \in \mathscr{M}_{t,k\cdot n}$ is unitary matrix. It follow from Theorem 4.4 that

$$\alpha_2 (UA^r U^*)^{1/r} \leq (UA^s U^*)^{1/s} \leq \alpha_1 (UA^r U^*)^{1/r}.$$

Taking into account that $(UA^r U^*)^{1/r} = M_k^{[r]}(\mathbf{A};\mathbf{U})$ we obtain the desired inequality. □

Corollary 4.11 *Let A_j, U_j ($j = 1,\ldots,k$) and r, s be as in Corollary 4.10. Then*

$$\beta_2 1_t \leq M_k^{[s]}(\mathbf{A};\mathbf{U}) - M_k^{[r]}(\mathbf{A};\mathbf{U}) \leq \beta_1 1_t$$

holds for same boundaries β_1 and β_2 as in Theorems 4.7 (i)–(iv).

Proof. We obtain the desired inequality by virtue of Theorem 4.7 when we take the map $\Phi \in \mathbf{P}_N[\mathscr{M}_{n\cdot k}, \mathscr{M}_t]$ as we did in proof of Corollary 4.10. □

4.4 Inequalities for power means

In this section we shall generalize Theorems 4.4 and 4.7 to obtain complementary inequalities to inequalities for the power operator mean on positive linear maps

$$M_k^{[r]}(\mathbf{A};\mathbf{\Phi},w) = \left(\sum_{j=1}^k \omega_j \, \Phi_j \left(A_j^r\right) \right)^{1/r} \quad \text{for } r \in \mathbb{R}\backslash\{0\}.$$

As we mentioned in Remarks 3.8 and 3.10, we can obtain analogous statements as in Theorems 4.4 and 4.7 if we replace $\Phi(A)$ by $\sum_{j=1}^k \omega_j \Phi_j(A_j)$. First we give the following generalization of Theorem 3.10:

Theorem 4.12 *Let $\Phi_j \in \mathbf{P}_N[\mathscr{B}(H), \mathscr{B}(K)]$, $A_j \in \mathscr{B}^{++}(H)$ with $\mathsf{Sp}(A_j) \subseteq [m,M]$ and $\omega_j \in \mathbb{R}_+$ be such that $\sum_{j=1}^k \omega_j = 1$ ($j = 1,\ldots,k$). If $f,g \in \mathscr{C}([m,M])$ and $\alpha \in \mathbb{R}$ then*

$$\begin{aligned} \alpha\, g\left(\textstyle\sum_{j=1}^k \omega_j \Phi_j(A_j)\right) &+ \max_{\substack{\varphi \in \{conx.\} \\ \varphi \leq f}} \min_{m \leq t \leq M} \{\varphi(t) - \alpha g(t)\}\, 1_K \\ &\leq \textstyle\sum_{j=1}^k \omega_j \Phi_j\left(f(A_j)\right) \qquad (4.20) \\ \leq \alpha\, g\left(\textstyle\sum_{j=1}^k \omega_j \Phi_j(A_j)\right) &+ \min_{\substack{\varphi \in \{conc.\} \\ \varphi \geq f}} \max_{m \leq t \leq M} \{\varphi(t) - \alpha g(t)\}\, 1_K. \end{aligned}$$

Proof. We only prove the right hand inequality of (4.20). Let φ be operator concave function on $[m,M]$ such that $f(t) \le \varphi(t)$ for every $t \in [m,M]$. Then $\Phi_j(f(A_j)) \le \Phi_j(\varphi(A_j))$ holds. Multiplying this inequality by $\omega_j \in \mathbb{R}_+$ and summing over $j = 1,\ldots,k$, we have

$$\sum_{j=1}^{k} \omega_j \Phi_j(f(A_j)) \le \sum_{j=1}^{k} \omega_j \Phi_j(\varphi(A_j)).$$

It follows from Jensen's inequality for many operator maps (Lemma 2.1) that

$$\sum_{j=1}^{k} \omega_j \Phi_j(\varphi(A_j)) \le \varphi\left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j)\right).$$

Combined with the two inequalities above we have

$$\sum_{j=1}^{k} \omega_j \Phi_j(f(A_j)) \le \varphi\left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j)\right).$$

Since $m1_H \le A_j \le M1_H$ we have $m1_K \le \sum_{j=1}^{k} \omega_j \Phi_j(A_j) \le M1_K$ and

$$\begin{aligned}
&\textstyle\sum_{j=1}^{k} \omega_j \Phi_j(f(A_j)) - \alpha g\left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j)\right) \\
&\le \varphi\left(\textstyle\sum_{j=1}^{k} \omega_j \Phi_j(A_j)\right) - \alpha g\left(\textstyle\sum_{j=1}^{k} \omega_j \Phi_j(A_j)\right) \le \max_{m\le t\le M}\{\varphi(t) - \alpha g(t)\}\, 1_K.
\end{aligned}$$

When we minimize this boundary over all operator concave function on $[m,M]$ such that $\varphi \ge f$, we obtain the upper boundary in (4.20). □

Applying (4.20) to the power functions we obtain a generalization of (4.6) and (4.7) in the same way we made it for $k = 1$ using Theorems 3.18 and 3.17.

Lemma 4.13 *Let $\Phi_j \in \mathbf{P}_N[\mathscr{B}(H),\mathscr{B}(K)]$, $A_j \in \mathscr{B}^{++}(H)$ with $\mathsf{Sp}(A_j) \subseteq [m,M]$ and $\omega_j \in \mathbb{R}_+$ be such that $\sum_{j=1}^{k} \omega_j = 1$ ($j = 1,\ldots,k$). If $p \in \mathbb{R}$ then*

$$\alpha_2 \left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j)\right)^p \le \sum_{j=1}^{k} \omega_j \Phi_j(A_j^p) \le \alpha_1 \left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j)\right)^p \tag{4.21}$$

holds for

$$\alpha_1 = \begin{cases} K(h,p)^{-1} & \text{if} \quad p < -1 \ \text{or} \ 2 < p, \\ 1 & \text{if} \quad -1 \le p < 0 \ \text{or} \ 1 \le p \le 2, \\ K(h,p) & \text{if} \quad 0 < p < 1, \end{cases}$$

$$\alpha_2 = \begin{cases} K(h,p) & \text{if} \quad p < 0 \ \text{or} \ 1 < p, \\ 1 & \text{if} \quad 0 < p \le 1, \end{cases}$$

where a generalized Kantorovich constant $K(h,p)$ is defined as (2.79) in Definition 2.2:

$$K(h,p) = \frac{h^p - h}{(p-1)(h-1)}\left(\frac{(p-1)(h^p-1)}{p(h^p-h)}\right)^p, \qquad h = \frac{M}{m}.$$

Lemma 4.14 *Let the hypothesis of Theorem 4.13 be satisfied. Then*

$$\beta_2 1_K \le \sum_{j=1}^{k} \omega_j \Phi_j(A_j^p) - \left(\sum_{j=1}^{k} \omega_j \Phi_j(A_j) \right)^p \le \beta_1 1_K \tag{4.22}$$

holds for

$$\beta_2 = \begin{cases} -C(m,M,p) & \text{if} \quad p < -1 \text{ or } 2 < p, \\ 0 & \text{if} \quad -1 \le p < 0 \text{ or } 1 \le p \le 2, \\ C(m,M,p) & \text{if} \quad 0 < p < 1, \end{cases}$$

$$\beta_1 = \begin{cases} C(m,M,p) & \text{if} \quad p < 0 \text{ or } 1 < p, \\ 0 & \text{if} \quad 0 < p \le 1, \end{cases}$$

where $C(m,M,p)$ is defined as (2.96):

$$C(m,M,p) = M^p \frac{1-h^{1-p}}{1-h} + m^p(p-1) \left\{ \frac{p(h-1)}{h^p - 1} \right\}^{\frac{p}{1-p}}, \qquad h = \frac{M}{m}.$$

We have the following inequalities for power operator means on positive linear maps.

Theorem 4.15 *Let $\Phi_j \in \mathbf{P}_N[\mathscr{M}_n, \mathscr{M}_t]$, $A_j \in \mathscr{H}_n$ with $\mathsf{Sp}(A_j) \subseteq [m,M]$ and $\omega_j \in \mathbb{R}_+$ be such that $\sum_{j=1}^k \omega_j = 1$ $(j = 1,\ldots,k)$. If $r,s \in \mathbb{R}$, $r \le s$, then*

$$\alpha_2 M_k^{[s]}(\mathbf{A};\mathbf{\Phi},w) \le M_k^{[r]}(\mathbf{A};\mathbf{\Phi},w) \le \alpha_1 M_k^{[s]}(\mathbf{A};\mathbf{\Phi},w)$$

holds for same boundaries α_1 and α_2 as in Theorems 4.4 (i)–(iv).

Proof. Desired inequalities follows from (4.21) in the same way as inequalities in Theorems 4.4 follows from (4.6) for $k = 1$. □

Theorem 4.16 *Let the hypothesis of Theorem 4.15 be satisfied. Then*

$$\beta_2 1_K \le M_k^{[s]}(\mathbf{A};\mathbf{\Phi},w) - M_k^{[r]}(\mathbf{A};\mathbf{\Phi},w) \le \beta_1 1_K$$

holds for same boundaries β_1 and β_2 as in Theorems 4.7 (i)–(iv).

Proof. First we obtain the following inequalities in the same way we obtain inequalities in Lemma 4.6 (a)–(d) for $k = 1$.

(a) If $1 \le r \le s$ or $r \le -1 \le s$ then

$$\left[\bar{\mu} \sum_{j=1}^k \omega_j \Phi_j(A_j^s) + \bar{\nu} 1_K \right]^{1/r} \le \left(\sum_{j=1}^k \omega_j \Phi_j(A_j^r) \right)^{1/r}$$

$$\le \begin{cases} \left[\bar{\mu} \sum_{j=1}^k \omega_j \Phi_j(A_j^s) + \left(1 - \frac{r}{s}\right) \left(\frac{s}{r} \bar{\mu}\right)^{\frac{r}{r-s}} 1_K \right]^{1/r} & \text{if } -1/2 < s < 1, s \ne 0, \\ \left(\sum_{j=1}^k \omega_j \Phi_j(A_j^s) \right)^{1/s} & \text{otherwise.} \end{cases}$$

(b) If $\;r \leq s \leq -1\;$ or $\;r \leq 1 \leq s\;$ then

$$\left[\tfrac{1}{\bar{\mu}}\sum_{j=1}^{k}\omega_j\Phi_j(A_j^r)-\tfrac{\bar{\nu}}{\bar{\mu}}1_K\right]^{1/s} \geq \left(\sum_{j=1}^{k}\omega_j\Phi_j(A_j^s)\right)^{1/s}$$
$$\geq \begin{cases} \left[\tfrac{1}{\bar{\mu}}\sum_{j=1}^{k}\omega_j\Phi_j(A_j^r)-\tfrac{1}{\bar{\mu}}\left(1-\tfrac{r}{s}\right)\left(\tfrac{s}{r}\bar{\mu}\right)^{\frac{r}{r-s}}1_K\right]^{1/s} & \text{if } -1<r<1/2, r\neq 0,\\ \left(\sum_{j=1}^{k}\omega_j\Phi_j(A_j^r)\right)^{1/r} & \text{otherwise.}\end{cases}$$

(c) If $\;-1 \leq -s \leq r \leq s \leq 1, r \neq 0\;$ or $\;-1 \leq r \leq s \leq r/2 < 0\;$ then

$$\left[\bar{\mu}\sum_{j=1}^{k}\omega_j\Phi_j(A_j^s)+\bar{\nu}1_K\right]^{1/r}-C\left(m^r,M^r,\tfrac{1}{r}\right)1_K \leq \left(\sum_{j=1}^{k}\omega_j\Phi_j(A_j^r)\right)^{1/r}$$
$$\leq \left(\sum_{j=1}^{k}\omega_j\Phi_j(A_j^s)\right)^{1/s}+C\left(m^r,M^r,\tfrac{1}{r}\right)1_K.$$

(d) If $\;-1/2 \leq r/2 < s < -r \leq 1, s \neq 0\;$ then

$$\left[\bar{\mu}\sum_{j=1}^{k}\omega_j\Phi_j(A_j^s)+\bar{\nu}1_K\right]^{1/r}-C\left(m^r,M^r,\tfrac{1}{r}\right)1_K \leq \left(\sum_{j=1}^{k}\omega_j\Phi_j(A_j^r)\right)^{1/r}$$
$$\leq \left[\bar{\mu}\sum_{j=1}^{k}\omega_j\Phi_j(A_j^s)+\left(1-\tfrac{r}{s}\right)\left(\tfrac{s}{r}\bar{\mu}\right)^{\frac{r}{r-s}}1_K\right]^{1/r}+C\left(m^r,M^r,\tfrac{1}{r}\right)1_K.$$

In the above inequalities we denote $\left(\bar{\mu}=\frac{M^r-m^r}{M^s-m^s}\right)$ and $\left(\bar{\nu}=\frac{M^s m^r-M^r m^s}{M^s-m^s}\right)$.

Further we obtain the desired inequality from these inequalities in the same way we obtained inequalities in Theorems 4.7 from inequalities in Lemma 4.6 for $k=1$.

□

Remark 4.3 In the case when Φ_j, $j=1,\dots,k$, are identity maps, by Theorems 4.15 and 4.16 we have inequalities for power matrix means $M_k^{[r]}(\mathbf{A};w) := \left(\sum_{j=1}^{k}\omega_j A_j^r\right)^{1/r}$, $r \in \mathbb{R}\backslash\{0\}$, where $A_j \in \mathscr{H}_n^{++}$ and $\omega_j \in \mathbb{R}_+$ such that $\sum_{j=1}^{k}\omega_j = 1$ $(j=1,\dots,k)$. These results are an extension of results from [168].

4.5 Chaotic order among power means

In this section, we study several properties of the weighted power mean of positive invertible operators as an application.

Definition 4.1 *For positive invertible operators A and B in $\mathscr{B}^{++}(H)$, we denote by $A \gg B$ if $\log A \geq \log B$ and we call it* **the chaotic order**.

The chaotic order is based on the fact that $\log t$ is operator monotone on $(0,\infty)$, by which it is weaker than the usual operator order $\geq$.

Let $A_j \in \mathscr{B}^{++}(H)$ be positive invertible operators on H $(j=1,\dots,k)$ and $\omega_j \in \mathbb{R}_+$ be such that $\sum_{j=1}^k \omega_j = 1$. We define

$$F(r) := \begin{cases} \left(\sum_{j=1}^k \omega_j A_j^r\right)^{1/r} & \text{if } r \in \mathbb{R}\backslash\{0\}, \\ \exp\left(\sum_{j=1}^k \omega_j \log A_j\right) & \text{if } r = 0. \end{cases} \tag{4.23}$$

First of all, we discuss the monotonicity of the operator function $F(r)$.

Lemma 4.17 *The operator function $F(r)$ is monotone increasing on the following intervals, i.e. $F(r) \leq F(s)$ for $r \leq s$ with (i) $r,s \notin (-1,1)$, (ii) $1/2 \leq r \leq 1 \leq s$ and (iii) $r \leq -1 \leq s \leq -1/2$. In addition $F(r)$ is not monotone increasing on $(0,1]$ generally.*

Proof. The first assertion follows if we put the identity map Φ_j for all $j = 1,\cdots,k$ in Theorem 4.15.

We give a simple counterexample to the second one as follows: Put

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^3 \quad \text{and} \quad B = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}^3.$$

Then

$$F(1) = \frac{1}{2}(A+B) = \begin{pmatrix} 14 & 14 \\ 14 & 20 \end{pmatrix}$$

and

$$F\left(\frac{1}{3}\right) = \frac{1}{8}\left(A^{\frac{1}{3}} + B^{\frac{1}{3}}\right)^3 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^3 = \begin{pmatrix} 14 & 13 \\ 13 & 14 \end{pmatrix}$$

so that

$$F(1) - F\left(\frac{1}{3}\right) = \begin{pmatrix} 0 & 1 \\ 1 & 6 \end{pmatrix} \ngeq 0$$

□

Moreover, if we put the identity map Φ_j for all $j = 1,\cdots,k$ in Theorem 4.15, then we have the following complementary inequalities to Lemma 4.17:

Lemma 4.18 *If $r,s \in \mathbb{R}$, $r \leq s$, then*

$$\alpha_2 F(s) \leq F(r) \leq \alpha_1 F(s)$$

holds for boundaries α_1 and α_2 given in Theorem 4.15.

Next, we discuss it under the chaotic order.

Theorem 4.19 *The operator function $F(r)$ is monotone increasing under the chaotic order, i.e. $F(r) \ll F(s)$ if $r \leq s$. In particular,*

$$\mathbf{s}-\lim_{r\to 0} F(r) = \exp\left(\sum_{j=1}^{k}\omega_j \log A_j\right)$$

Proof. It suffices to show that for $r < s$ with $r, s \neq 0$

$$\frac{1}{r}\log\left(\sum_{j=1}^{k}\omega_j A_j^r\right) \leq \frac{1}{s}\log\left(\sum_{j=1}^{k}\omega_j A_j^s\right).$$

To prove this, the operator concavity of t^r for $r \in [0,1]$ is available. We first assume $0 < r < s$. Then

$$\log\left(\sum_{j=1}^{k}\omega_j A_j^s\right)^{r/s} \geq \log\left(\sum_{j=1}^{k}\omega_j A_j^r\right),$$

and so $\log F(s) \geq \log F(r)$. Next, if $r < s < 0$, then $s/r \in (0,1)$ and hence

$$\log\left(\sum_{j=1}^{k}\omega_j A_j^r\right)^{s/r} \geq \log\left(\sum_{j=1}^{k}\omega_j A_j^s\right).$$

Noting $s < 0$, we have $\log F(r) \leq \log F(s)$.

Now, we prove the second assertion. By the operator concavity of $\log t$ and the Krein inequality $t - 1 \geq \log t$, it implies that for any $r > 0$

$$\begin{aligned}
\left(\sum_{j=1}^{k}\omega_j \log A_j\right) &= \frac{1}{r}\left(\sum_{j=1}^{k}\omega_j \log A_j^r\right) \leq \frac{1}{r}\log\left(\sum_{j=1}^{k}\omega_j A_j^r\right) \\
&\leq \frac{1}{r}\left(\sum_{j=1}^{k}\omega_j A_j^r - 1\right) = \left(\sum_{j=1}^{k}\omega_j \frac{A_j^r - 1}{r}\right) \\
&\to \left(\sum_{j=1}^{k}\omega_j \log A_j\right)
\end{aligned}$$

as $r \to +0$. Therefore it follows that

$$s-\lim_{r\to +0}\log\left(\sum_{j=1}^{k}\omega_j A_j^r\right)^{1/r} = \sum_{j=1}^{k}\omega_j \log A_j,$$

so that

$$s-\lim_{r\to +0}\left(\sum_{j=1}^{k}\omega_j A_j^r\right)^{1/r} = \exp\left(\sum_{j=1}^{k}\omega_j \log A_j\right).$$

On the other hand, it follows from the expression obtained above that for $r > 0$

$$\begin{aligned} F(-r) &= \left(\sum_{j=1}^{k} \omega_j A_j^{-r}\right)^{-1/r} \to \exp\left(\sum_{j=1}^{k} \omega_j \log A_j^{-1}\right)^{-1} \\ &= \exp\left(\sum_{j=1}^{k} \omega_j \log A_j\right). \end{aligned}$$

Hence we have the second assertion, which says that $s - \lim_{h\to 0} F(h)$ can be regarded as $F(0)$. Therefore, if $r < 0 < s$, then

$$F(r) \ll F(0) \ll F(s).$$

Consequently we have the monotonicity of $F(r)$ under the chaotic order. □

Theorem 4.20 *Let $A_j \in \mathscr{B}^{++}(H)$ be positive invertible operators with $\mathsf{Sp}(A_j) \subseteq [m,M]$ for some scalars $0 < m < M$ and $\omega_j \in \mathbb{R}_+$ such that $\sum_{j=1}^{k} \omega_j = 1$ ($j = 1,\ldots,k$). Denote $h = \frac{M}{m}$. If $r \leq s$, $r,s \in \mathbb{R}$ then*

$$\Delta(h,r,s)^{-1} F(s) \ll F(r) \tag{4.24}$$

where a generalized Specht ratio $\Delta(h,r,s)$ is defined as (2.97):

$$\Delta(h,r,s) = \begin{cases} K\left(h^r, \frac{s}{r}\right)^{1/s} & \text{if} \quad r < s, \ r,s \neq 0, \\ \left(\dfrac{e \log h^{\frac{p}{h^p-1}}}{h^{\frac{p}{h^p-1}}}\right)^{\frac{sign(p)}{p}} & \text{if} \quad r = 0 < s = p \ \text{or} \ r = p < s = 0. \end{cases} \tag{4.25}$$

Proof. We first show that for $r,s \in \mathbf{R} \setminus \{0\}$, $r < s$,

$$\log\left(\Delta(h,r,s)^{-1} F(s)\right) \leq \log F(r).$$

We assume $0 < r < s$. Then $m1_H \leq A_j \leq M1_H$ ($j = 1,\ldots,k$) implies $m^s 1_H \leq \sum_{j=1}^{k} \omega_j A_j^s \leq M^s 1_H$. By putting $p = \frac{r}{s}$ ($0 < p < 1$) in Lemma 4.3 and replacing A_j by A_j^s, we have

$$K\left(h^s, \frac{r}{s}\right)\left(\sum_{j=1}^{k} \omega_j A_j^s\right)^{r/s} \leq \sum_{j=1}^{k} \omega_j A_j^r.$$

As the function $f(t) = \log t$ is operator monotone on $(0,\infty)$ we have

$$r \log\left(K\left(h^s, \frac{r}{s}\right)^{1/r}\left(\sum_{j=1}^{k} \omega_j A_j^s\right)^{1/s}\right) \leq \log\left(\sum_{j=1}^{k} \omega_j A_j^r\right)$$

and so

$$\log\left(\Delta(h,r,s)^{-1} F(s)\right) \leq \log F(r). \tag{4.26}$$

where $K\left(h^s, \frac{r}{s}\right)^{1/r} = \Delta(h;r,s)^{-1}$.

Next, we assume $r < s < 0$. Then $M^r 1_H \leq A_j^r \leq m^r 1_H$, $(j = 1,\dots,k)$ and so $M^r 1_H \leq \sum_{j=1}^k \omega_j A_j^r \leq m^r 1_H$. By putting $p = \frac{s}{r}$ $(0 < p < 1)$ in Lemma 4.3 and replacing A_j by A_j^r, we have

$$K\left(h^r, \frac{s}{r}\right)\left(\sum_{j=1}^k \omega_j A_j^r\right)^{s/r} \leq \sum_{j=1}^k \omega_j A_j^s,$$

and so

$$\log\left(K\left(h^r, \frac{s}{r}\right)^{1/s} F(r)\right) \geq \log F(s), \tag{4.27}$$

where $K\left(h^r, \frac{s}{r}\right)^{1/s} = \Delta(h,r,s)$.

Next, we assume $r < 0 < s$. If $0 < -r < s$ or $0 < s < -r$, we put $p = \frac{r}{s}$ or $p = \frac{s}{r}$ in Lemma 4.3 $(-1 \leq p < 0)$, respectively. Then we have

$$\sum_{j=1}^k \omega_j A_j^r \leq K\left(h^s, \frac{r}{s}\right)\left(\sum_{j=1}^k \omega_j A_j^s\right)^{r/s}$$

or

$$\sum_{j=1}^k \omega_j A_j^s \leq K\left(h^r, \frac{s}{r}\right)\left(\sum_{j=1}^k \omega_j A_j^r\right)^{s/r}.$$

So we obtain

$$\log F(r) \geq \log\left(K\left(h^s, \frac{r}{s}\right)^{1/r} F(s)\right), \tag{4.28}$$

with $K\left(h^s, \frac{r}{s}\right)^{1/r} = \Delta(h,r,s)^{-1}$, or

$$\log F(s) \leq \log\left(K\left(h^r, \frac{s}{r}\right)^{1/s} F(r)\right), \tag{4.29}$$

with $K\left(h^r, \frac{s}{r}\right)^{1/s} = \Delta(h,r,s)$. Then the inequality (4.24) holds when $r < s$, $r,s \neq 0$.

At the end, if $r \to 0$ in (4.26), then

$$\Delta(h,0,s)^{-1}\ F(s) \ll F(0).$$

Similarly, if $s \to 0$ in (4.27), then

$$F(0) \ll \Delta(h,r,0)\ F(r).$$

Then the inequality (4.24) holds when $r = 0 < s$ or $r < s = 0$. □

4.6 Notes

Alić, Mond, Pečarić and Volenec [1, 168] studied the power matrix mean: $M_k^{[r]}(\mathbf{A};w) := \left(\sum_{j=1}^k \omega_j C_j^r\right)^{1/r}$, $r \in \mathbb{R}\backslash\{0\}$, where $C_j \in \mathscr{H}_n^{++}$, $j=1,\ldots,k$ and $\omega_j \in \mathbb{R}_+$ such that $\sum_{j=1}^k \omega_j = 1$ and they obtained the complementary inequalities to the power matrix mean.

Additionally, Mond and Pečarić [149] observed the power operator mean: $M_k^{[r]}(\mathbf{A};\mathbf{X}) := \left(\sum_{i=1}^k X_j^* A_j^r X_j\right)^{1/r}$, $r \in \mathbb{R}\backslash\{0\}$, where $A_j \in \mathscr{B}_h(H)$ with $\mathsf{Sp}(A_j) \subseteq (0,\infty)$, $j=1,\ldots,k$ and $X_j \in \mathscr{B}(H)$ are contractions, such that $\sum_{j=1}^k X_j^* X_j = 1_H$. They proved the monotonicity of the power operator mean in an interval (i) [149, Theorem 2], and its converses in an interval (ii) [149, Theorem 4]. They [168] obtained same results for matrix means same type.

The results in Section 4.2 are due to [135] and the results in Section 4.5 are due to [68, 166].

M.Fujii and Nakamoto [68] call $F(0) = \mathrm{s}-\lim_{r\to 0} F(r)$ the chaotically geometric mean. Further topics related to it are contained in [64] and [66].

Chapter 5

Operator means

In this chapter, we introduce the theory of operator means established by Kubo and Ando associated with the operator monotone functions. Based on several complementary inequalities to Jensen's inequalities on positive linear maps, we study complementary inequalities to Ando's inequalities associated with operator means.

5.1 Operator means

The theory of operator means for positive (bounded linear) operators on a Hilbert space is initiated by T.Ando and established by F.Kubo and T.Ando in connection with Lowner's theory for the operator monotone functions. Throughout this chapter, we use the capital letter A,B,C,D as positive (bounded linear) operators on a Hilbert space.

Definition 5.1 *A binary operation* $(A,B) \in \mathscr{B}^+(H) \times \mathscr{B}^+(H) \to A \, \sigma \, B \in \mathscr{B}^+(H)$ *in the cone of positive operators on a Hilbert space* H *is called a* **connection** *if the following conditions are satisfied:*

$(\mathbf{S1})$ **monotonicity**: $A \le C$ *and* $B \le D$ *imply* $A \, \sigma \, B \le C \, \sigma \, D$,

$(\mathbf{S2})$ **upper continuity**: $A_n \downarrow A$ *and* $B_n \downarrow B$ *imply* $A_n \, \sigma \, B_n \downarrow A \, \sigma \, B$,

$(\mathbf{S3})$ **transformer inequality**: $T^*(A \, \sigma \, B)T \le (T^*AT) \, \sigma \, (T^*BT)$ *for every operator* T.

An **operator mean** *is a connection with normalized condition*

(S4) **normalized condition***:* $1_H \ \sigma \ 1_H = 1_H$.

Here we denote by $A_n \downarrow A$ a series of operators $\{A_n\}$, $A_n \in \mathscr{B}_h(H)$ such that $A_1 \geq A_2 \geq \dots$ and $A_n \to A$ in the strong operator topology for $A \in \mathscr{B}_h(H)$.

Lemma 5.1 *Let σ be a connection. If T is invertible, then σ satisfies the transformer equality:*

$$T^*(A \ \sigma \ B)T = (T^*AT) \ \sigma \ (T^*BT). \tag{5.1}$$

In particular, σ is positively homogeneous in the sense:

$$\alpha(A \ \sigma \ B) = (\alpha A) \ \sigma \ (\alpha B) \tag{5.2}$$

for all $\alpha > 0$.

Proof. It follows from the transformer inequality $(S3)$ that

$$T^{*-1}\{(T^*AT) \ \sigma \ (T^*BT)\}T^{-1} \leq (T^{*-1}T^*ATT^{-1}) \ \sigma \ (T^{*-1}T^*BTT^{-1}) = A \ \sigma \ B,$$

and hence $(T^*AT) \ \sigma \ (T^*BT) \leq T^*(A \ \sigma \ B)T$. Therefore we have (5.1). Also, if we put $T = \alpha^{1/2}1_H$ for $\alpha > 0$, then we have (5.2). □

Simple examples of operator means are **the arithmetic mean**, in symbol ∇,

$$A \ \nabla \ B := \frac{1}{2}(A+B).$$

Left trivial mean ω_l and right trivial mean ω_r are by definition

$$A \ \omega_l \ B = A \qquad \text{and} \qquad A \ \omega_r \ B = B.$$

For invertible A, B, **the parallel sum** $A \ : \ B$ is given by

$$A \ : \ B = (A^{-1} + B^{-1})^{-1}.$$

The normalized parallel sum is called **the harmonic mean**, in symbol !

$$A \ ! \ B := 2(A \ : \ B) = \left(\frac{1}{2}(A^{-1} + B^{-1})\right)^{-1}.$$

For invertible A, B, **the geometric mean** $A \ \sharp \ B$ is given by

$$A \ \sharp \ B := A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}.$$

A positive linear combination of two connections is defined in a natural way. If σ, τ are connections and a, b are positive numbers, then the connection $a\sigma + b\tau$ is defined by

$$A \ (a\sigma + b\tau) \ B = a(A \ \sigma \ B) + b(A \ \tau \ B).$$

Then the class of means becomes a convex set.

A partial order $\geq$ between two connections is introduced in a natural way. $\sigma \geq \tau$ means by definition that $A \ \sigma \ B \geq A \ \tau \ B$ for all positive operators A and B. Then an important inequality is $! \leq \sharp \leq \nabla$ (see Theorem 1.27).

We investigate some properties of the parallel sum.

Lemma 5.2 *The parallel sum has the following properties.*

(*i*) *If* $A,B,C,D \in \mathscr{B}^+(H)$ *are invertible and* $A \le C$, $B \le D$, *then* $A:B \le C:D$.

(*ii*) *Let* $A,B \in \mathscr{B}^+(H)$. *If* $A_n, B_n \in \mathscr{B}^+(H)$ *are invertible and* $A_n \downarrow A$ *and* $B_n \downarrow B$, *then* $\mathrm{s}-\lim_{n\to\infty} A_n : B_n$ *exists. The strong limit does not depend on the choice of* $\{A_n\}$ *and* $\{B_n\}$.

Proof. (i) Since $A \le C$ and $B \le D$, we have $A^{-1} \ge C^{-1}$ and $B^{-1} \ge D^{-1}$, and hence $(A^{-1}+B^{-1})^{-1} \le (C^{-1}+D^{-1})^{-1}$. Therefore it follows that $A:B \le C:D$.

(ii) Suppose that A and B are invertible and $A_n \downarrow A$ and $B_n \downarrow B$. Since $\mathsf{Sp}(A)$, $\mathsf{Sp}(A_n)$, $\mathsf{Sp}(B)$, $\mathsf{Sp}(B_n) \subseteq [\alpha,\beta]$ where $0 < \alpha < \beta < \infty$, we have $A_n^{-1} \mapsto A^{-1}, B_n^{-1} \mapsto B^{-1}$ and $A_n^{-1} + B_n^{-1} \mapsto A^{-1} + B^{-1}$ in the strong operator topology. Moreover, since $\mathsf{Sp}(A_n^{-1} + B_n^{-1}), \mathsf{Sp}(A^{-1}+B^{-1}) \subseteq [2\beta^{-1}, 2\alpha^{-1}]$, it follows that $A_n : B_n \mapsto A:B$ in the strong operator topology.

Suppose that A and B are noninvertible. If $A_n, B_n \in \mathscr{B}^+(H)$ are invertible and $A_n \downarrow A$ and $B_n \downarrow B$, then $\{A_n : B_n\}$ is monotone decreasing by (i) and $\{A_n : B_n\}$ is bounded below by 0. Hence it follows that s-$\lim_{n\to\infty} A_n : B_n$ exists.

Next, we show that the strong limit does not depend on the choice of $\{A_n\}$ and $\{B_n\}$. Suppose that $A_n^{'}, B_n^{'} \in \mathscr{B}^+(H)$ are invertible and $A_n^{'} \downarrow A$ and $B_n^{'} \downarrow B$. Since $A_n \le A_n + A_m^{'} - A$ and $B_n \le B_n + B_m^{'} - B$ for all $n,m \in \mathbb{N}$, we have

$$A_n : B_n \le (A_n + A_m^{'} - A) : (B_n + B_m^{'} - B).$$

Since $A_n + A_m^{'} - A \downarrow A_m^{'}$ and $B_n + B_m^{'} - B \downarrow B_m^{'}$ as $n \mapsto \infty$, as we see above, we have

$$(A_n + A_m^{'} - A) : (B_n + B_m^{'} - B) \downarrow A_m^{'} : B_m^{'}.$$

Therefore it follows that $\mathrm{s}-\lim_{n\to\infty} A_n : B_n \le A_m^{'} : B_m^{'}$ and hence

$$\mathrm{s}-\lim_{n\to\infty} A_n : B_n \le \mathrm{s}-\lim_{n\to\infty} A_n^{'} : B_n^{'}.$$

By symmetry, we have

$$\mathrm{s}-\lim_{n\to\infty} A_n : B_n = \mathrm{s}-\lim_{n\to\infty} A_n^{'} : B_n^{'}.$$

□

By Lemma 5.2, for positive $A,B \in \mathscr{B}^+(H)$ the parallel sum is given by

$$A:B = \mathrm{s}-\lim_{\varepsilon\downarrow 0}(A+\varepsilon 1_H) : (B+\varepsilon 1_H).$$

The parallel sum for positive operators is characterized as follows.

Lemma 5.3

$$((A:B)x,x) = \inf\{(Ay,y)+(Bz,z) : y,z \in H, y+z = x\}$$

for ever vector $x \in H$.

Proof. If A and B are invertible, then

$$\begin{aligned} A:B &= \{B^{-1}(A+B)A^{-1}\}^{-1} = \{(A+B)-B\}(A+B)^{-1}B \\ &= B - B(A+B)^{-1}B. \end{aligned}$$

Therefore we have

$$\begin{aligned} (Ay,y) &+ (B(x-y),x-y) - ((A:B)x,x) \\ &= (Bx,x) + ((A+B)y,y) - 2\mathrm{Re}(Bx,y) - ((A:B)x,x) \\ &= (B(A+B)^{-1}Bx,x) + ((A+B)y,y) - 2\mathrm{Re}(Bx,y) \\ &= \|(A+B)^{-1/2}Bx\|^2 + \|(A+B)^{1/2}y\|^2 \\ &\qquad -2\mathrm{Re}((A+B)^{-1/2}Bx,(A+B)^{1/2}y) \\ &\geq 0 \end{aligned}$$

for every vector $x,y \in H$, where $\mathrm{Re}z = (z+\overline{z})/2$ is a real part of a complex number z. If we put $y = (A+B)^{-1}Bx$, then the above expression is equal to 0 and hence we obtain this Lemma in the case that A and B are invertible. Next, for positive A,B, we have

$$\begin{aligned} ((A:B)x,x) &= \inf_{\varepsilon>0}(((A+\varepsilon 1_H):(B+\varepsilon 1_H))x,x) \\ &= \inf_{\varepsilon>0}\inf_{y}\{((A+\varepsilon 1_H)y,y) + ((B+\varepsilon 1_H)(x-y),x-y)\} \\ &= \inf_{y}\{(Ay,y)+(B(x-y),x-y)\}. \end{aligned}$$

□

Lemma 5.4 *The parallel sum $A:B$ is a connection.*

Proof. (S1) of Definition 5.1 follows from (i) of Lemma 5.2. For (S2), suppose that $A_n \downarrow A$ and $B_n \downarrow B$. Since $A:B \leq A_n : B_n$, we have $A:B \leq \mathrm{s-}\lim A_n : B_n$. Also, since $A_n : B_n \leq (A_n+\varepsilon 1_H):(B_n+\varepsilon 1_H)$ for all $\varepsilon > 0$, it follows from (ii) of Lemma 5.2 that $\mathrm{s-}\lim A_n : B_n \leq (A+\varepsilon 1_H):(B+\varepsilon 1_H)$ and hence $\mathrm{s-}\lim A_n : B_n \leq A:B$. Therefore we have $A_n : B_n \downarrow A:B$.

Finally we show (S3). If $y+z=x$, then it follows from Lemma 5.3 that

$$\begin{aligned} (T^*(A:B)Tx,x) &= ((A:B)Tx,Tx) \\ &\leq (ATy,Ty)+(BTz,Tz) \\ &= (T^*ATy,y)+(T^*BTz,z) \end{aligned}$$

and hence $T^*(A:B)T \leq (T^*AT):(T^*BT)$ for every operator T. Therefore the parallel sum satisfies the transformer inequality. Thus, the parallel sum is a connection. □

Now, we state the principal result in the Kubo-Ando theory.

Theorem 5.5 (KUBO-ANDO THEOREM) *For each connection σ, if we put*

$$f(t)1_H = 1_H \ \sigma \ (t1_H) \qquad (t \geq 0), \tag{5.3}$$

then $f(t)$ is a nonnegative number and $f(t)$ is operator monotone on $[0,\infty)$. Then the following assertions hold.

(i) A map $\sigma \mapsto f$ establishes a one-to-one afine isomorphism between the class of connections and the class of nonnegative operator monotone functions on $[0,\infty)$. Moreover, a map $\sigma \mapsto f$ preserves the order in the sense

$$A \ \sigma_1 \ B \leq A \ \sigma_2 \ B \quad (A,B \in \mathscr{B}^+(H)) \quad \Longleftrightarrow \quad f_1(t) \leq f_2(t) \quad (t \geq 0).$$

(ii) If A is invertible, then

$$A \ \sigma \ B = A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}}. \tag{5.4}$$

(iii) A connection σ is an operator mean if and only if f is normalized in the sense $f(1) = 1$.

Proof. Let σ be a connection. Firstly we show that if a projection P commutes with positive operators A and B, then it commutes with $A \ \sigma \ B$ and

$$(AP \ \sigma \ BP)P = (A \ \sigma \ B)P. \tag{5.5}$$

Since commutativity implies

$$PAP = AP \leq A \qquad \text{and} \qquad PBP = BP \leq B,$$

it follows from (S1) and (S3) that

$$\begin{aligned} P(A \ \sigma \ B)P &\leq (PAP) \ \sigma \ (PBP) \\ &= (AP) \ \sigma \ (BP) \\ &\leq A \ \sigma \ B. \end{aligned}$$

Then the operator $A \ \sigma \ B - P(A \ \sigma \ B)P$ is positive and

$$\left(\{A \ \sigma \ B - P(A \ \sigma \ B)P\}^{\frac{1}{2}} P\right)^2 = P(A \ \sigma \ B - P(A \ \sigma \ B)P)P = 0.$$

Therefore we have

$$\{A \ \sigma \ B - P(A \ \sigma \ B)P\}^{\frac{1}{2}} P = 0$$

and hence

$$(A \ \sigma \ B)P = P(A \ \sigma \ B)P,$$

which implies the commutativity of P and $A \ \sigma \ B$. Analogously P commutes with $(AP) \ \sigma \ (BP)$. These together prove (5.5).

Now, since 1_H and $t1_H$ commutes with all projections for every $t \geq 0$, so does the operator $1_H \ \sigma \ (t1_H)$, and hence $1_H \ \sigma \ (t1_H)$ is a scalar. Thus (5.3) determines a nonnegative function f on $[0,\infty)$. It follows from (S2) that $f(t)$ is right continuous on $[0,\infty)$. By the positively homogenity, we have $t^{-1}f(t)1_H = (t^{-1}1_H)\ \sigma\ 1_H\ (t>0)$. Hence $f(t)$ is left continuous on $(0,\infty)$ by (S2). Therefore it follows that $f(t)$ is continuous on $[0,\infty)$.

Let us show that f is operator monotone. In the case of

$$A = \sum_j t_j E_j \le B = \sum_i s_i F_i,$$

where $\{E_j\}$ and $\{F_i\}$ are decomposition of the unit 1_H and $t_j, s_i > 0$, it follows from (5.5) that

$$\begin{aligned} 1_H\ \sigma\ A &= \sum_j (1_H\ \sigma A)E_j = \sum_j (E_j\ \sigma A E_j)E_j \\ &= \sum_j (E_j\ \sigma\ (t_j E_j))E_j = \sum_j (1_H\ \sigma\ t_j 1_H)E_j \\ &= \sum_j f(t_j)E_j = f(A) \end{aligned}$$

and similarly $1_H\ \sigma\ B = f(B)$. Since every positive operator A can be approximated uniformly by simple functions A_n with $A_n \downarrow A$, it follows from (S2) that

$$1_H\ \sigma\ A = \text{s}-\lim_{n\to\infty} 1_H\ \sigma\ A_n = \text{s}-\lim_{n\to\infty} f(A_n) = f(A).$$

Therefore $A \le B$ implies

$$f(A) = 1_H\ \sigma\ A \le 1_H\ \sigma\ B = f(B)$$

and hence f is operator monotone.

For invertible A, we have

$$\begin{aligned} A\ \sigma\ B &= A^{\frac{1}{2}}\left(1_H\ \sigma A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}} \\ &= A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}. \end{aligned}$$

Also, $1_H\ \sigma\ 1_H = 1_H$ implies $f(1)=1$ and so we have (ii) and (iii).

It remains only to prove that every operator monotone function is obtained in the form (5.3). Take an operator monotone function f with integral representation

$$f(t) = a + bt + \int_{(0,\infty)} \frac{t(1+\lambda)}{t+\lambda}\, dm(\lambda),$$

where $a = m(\{0\})$ and $b = m(\{\infty\})$. Recall that $A\ :\ B$ is the parallel sum of A and B. Define a binary operation σ by

$$A\ \sigma\ B = aA + bB + \int_{(0,\infty)} \frac{1+\lambda}{\lambda}\{(\lambda A) : B\}\, dm(\lambda).$$

In fact,

$$(\lambda A) : B \le (\lambda \|A\| 1_H) : (\|B\| 1_H) = \frac{\|A\|\|B\|\lambda}{\|A\|\lambda + \|B\|} 1_H$$

implies

$$\frac{1+\lambda}{\lambda}\|(\lambda A):B\| \le \frac{\|A\|\|B\|(1+\lambda)}{\|A\|\lambda+\|B\|}$$

and hence $\lambda^{-1}(1+\lambda)\{(\lambda A):B\}$ is uniformly bounded for $\lambda > 0$. Therefore $A\ \sigma\ B \in \mathscr{B}^+(H)$. Since the parallel sum and trivial means satisfy conditions (S1), (S2) and (S3) by Lemma 5.4, the operation σ satisfies (S1) and (S3). For (S2), if $A_n \downarrow A$ and $B_n \downarrow B$, then $(\lambda A_n):B_n \downarrow (\lambda A):B$ and by the monotone convergence theorem in the integral theory

$$\begin{aligned}
&\lim_{n\to\infty}((A_n\ \sigma\ B_n)x,x) \\
&= \lim_{n\to\infty}\left(a(A_nx,x)+b(B_nx,x)+\int_{(0,\infty)}(((\lambda A_n):B_n)x,x)dm(\lambda)\right) \\
&= a(Ax,x)+b(Bx,x)+\int_{(0,\infty)}(((\lambda A)\ :\ B)x,x)dm(\lambda) \\
&= ((A\ \sigma\ B)x,x).
\end{aligned}$$

Hence we have $A_n\ \sigma\ B_n \downarrow A\ \sigma\ B$ and σ is a connection. Finally for $t > 0$

$$1_H\ \sigma\ t1_H = a+bt+\int_{(0,\infty)}\frac{t(1+\lambda)}{t+\lambda}dm(\lambda) = f(t).$$

Thus the function f is obtained from the connection σ. This completes the proof. □

Here the operator monotone function f produced from a connection σ by (5.3) is called **the representing function** for σ. In this case, notice that a function $f > 0$ on $[0,\infty)$ is operator monotone if and only if it is operator concave (see Corollary 1.14).

The representing functions of left trivial mean ω_l and right trivial mean ω_r are 1 and t, respectively. The representing functions of the arithmetic mean ∇, the harmonic mean ! and the geometric mean $\sharp$ are as follows:

$$\begin{aligned}
f_\nabla(t) &= \frac{1+t}{2} && \text{for the arithmetic mean } \nabla. \\
f_!(t) &= \frac{2t}{1+t} && \text{for the harmonic mean } !. \\
f_\sharp(t) &= \sqrt{t} && \text{for the geometric mean } \sharp.
\end{aligned}$$

Then $f_\nabla(t) \ge f_\sharp(t) \ge f_!(t)$ implies the arithmetic-geometric-harmonic mean inequality $\nabla \ge \sharp \ge !$.

Notice that every representing function f of an operator mean σ satisfies

$$\begin{aligned}
t &\le f(t) \le 1 && \text{for } 0 \le t \le 1, \\
1 &\le f(t) \le t && \text{for } t \ge 1.
\end{aligned}$$

In fact, the derivative $f'(1)$ is not greater than 1 since f is a nonnegative and concave function with $f(1) = 1$. Moreover $f'(1) \ge 0$ since f is monotone nondecreasing. Then the required inequalities are obtained by using nonnegativity, monotonicity and concavity again.

By the theory of the operator mean, we can show the following Hansen's theorem (see Theorem 1.9):

Theorem 5.6 (HANSEN'S THEOREM) *If f is a nonnegative operator monotone function on $[0,\infty)$, then*

$$C^* f(A)C \le f(C^*AC) \tag{5.6}$$

for every contraction C and every positive operator A.

Proof. By the Kubo-Ando theory, for a nonnegative operator monotone function f, there exists an operator mean σ such that $f(t) = 1_H \, \sigma \, t1_H$. Then the transformer inequality (S3) implies

$$C^* f(A)C = C^*(1_H \ \sigma \ A)C \le (C^*C) \ \sigma \ (C^*AC) \le 1_H \ \sigma \ C^*AC = f(C^*AC).$$

□

Here we state some properties of operator means:

Theorem 5.7 *Every operator mean σ is subadditive:*

$$A \ \sigma \ C + B \ \sigma \ D \le (A+B) \ \sigma \ (C+D)$$

and jointly concave:

$$\lambda(A \ \sigma \ C) + (1-\lambda)(B \ \sigma \ D) \le (\lambda A + (1-\lambda)B) \ \sigma \ (\lambda C + (1-\lambda)D)$$

for $0 \le \lambda \le 1$.

Proof. By the upper continuity of σ, we may assume that the above positive operators are invertible. Put

$$\begin{aligned} X &= A^{1/2}(A+B)^{-1/2} \quad \text{and} \quad Y = B^{1/2}(A+B)^{-1/2} \\ V &= A^{-1/2}CA^{-1/2} \quad \text{and} \quad W = B^{-1/2}DB^{-1/2}. \end{aligned}$$

Since $X^*X + Y^*Y = 1_H$, it follows that $\begin{pmatrix} X & 0 \\ Y & 0 \end{pmatrix}$ is a contraction. For the representing function f for an operator mean σ, it follows from Theorem 5.6 (Hansen's theorem) that

$$\begin{aligned} \begin{pmatrix} X^*f(V)X + Y^*f(W)Y & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} X & 0 \\ Y & 0 \end{pmatrix}^* f\left(\begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix}\right) \begin{pmatrix} X & 0 \\ Y & 0 \end{pmatrix} \\ &\le f\left(\begin{pmatrix} X & 0 \\ Y & 0 \end{pmatrix}^* \begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix} \begin{pmatrix} X & 0 \\ Y & 0 \end{pmatrix}\right) \\ &= \begin{pmatrix} f(X^*VX + Y^*WY) & 0 \\ 0 & f(0) \end{pmatrix}. \end{aligned}$$

Thus we have $X^*f(V)X + Y^*f(W)Y \le f(X^*VX + Y^*WY)$ and consequently

$$\begin{aligned} A\,\sigma\,C + B\,\sigma\,D &= (A+B)^{1/2}\{(X^*X\sigma X^*VX) + (Y^*Y\sigma Y^*WY)\}(A+B)^{1/2} \\ &= (A+B)^{1/2}(X^*f(V)X + Y^*f(W)Y)(A+B)^{1/2} \\ &\le (A+B)^{1/2}f(X^*VX + Y^*WY)(A+B)^{1/2} \\ &= (A+B)^{1/2}(1_H\,\sigma\,(X^*VX + Y^*WY)(A+B)^{1/2} \\ &= (A+B)\,\sigma\,(A^{1/2}VA^{1/2} + B^{1/2}WB^{1/2}) \\ &= (A+B)\,\sigma\,(C+D) \end{aligned}$$

Since σ is positively homogeneous, we have the jointly concavity:

$$\begin{aligned} \lambda(A\,\sigma\,C) + (1-\lambda)(B\,\sigma\,D) &= (\lambda A)\,\sigma\,(\lambda C) + (1-\lambda)B\,\sigma\,(1-\lambda)D \\ &\le (\lambda A + (1-\lambda)B)\,\sigma\,(\lambda C + (1-\lambda)D). \end{aligned}$$

□

Ando showed the following property of a positive linear map in connection with an operator mean. If Φ is a normalized positive linear map in $\mathbf{P}_N[\mathscr{B}(H),\mathscr{B}(K)]$, then for any positive operators A and B

$$\Phi(A \sharp B) \le \Phi(A) \sharp \Phi(B) \quad \text{and} \quad \Phi(A\,!\,B) \le \Phi(A)\,!\,\Phi(B). \tag{5.7}$$

It is considered as a natural extension of Remark 1.1 and (i) in Theorem 1.17

$$\Phi(A^{\frac{1}{2}}) \le \Phi(A)^{\frac{1}{2}} \quad \text{and} \quad \Phi(A)^{-1} \le \Phi(A^{-1}) \tag{5.8}$$

by putting $B = 1_H$ in (5.7). The inequality (5.7) is extended to an operator mean as follows:

Theorem 5.8 *If Φ is a normalized positive linear map in $\mathbf{P}_N[\mathscr{B}(H),\mathscr{B}(K)]$, then for every operator mean σ*

$$\Phi(A\,\sigma\,B) \le \Phi(A)\,\sigma\,\Phi(B). \tag{5.9}$$

Proof. Suppose that A is invertible. Then so does $\Phi(A)$. Define a map

$$\Psi(X) = \Phi(A)^{-\frac{1}{2}}\Phi(A^{\frac{1}{2}}XA^{\frac{1}{2}})\Phi(A)^{-\frac{1}{2}}.$$

Then Ψ is a normalized positive linear map. So we have by Theorem 1.20 (Davis-Choi-Jensen's inequality)

$$\Psi(f(X)) \le f(\Psi(X))$$

for every operator concave function f on $[0,\infty)$. Let f be the representing function for σ and by Corollary 1.12 f is operator concave. Therefore it follows that

$$\begin{aligned} \Phi(A\,\sigma\,B) &= \Phi\left(A^{\frac{1}{2}}f(X)A^{\frac{1}{2}}\right) \\ &= \Phi(A)^{\frac{1}{2}}\Psi(f(X))\Phi(A)^{\frac{1}{2}} \\ &\le \Phi(A)^{\frac{1}{2}}f(\Psi(X))\Phi(A)^{\frac{1}{2}} \\ &= \Phi(A)\,\sigma\,\Phi(B) \end{aligned}$$

for $X = A^{-1/2}BA^{-1/2}$.

Suppose that A is not invertible. Since $\|\Phi(A+\varepsilon 1_H) - \Phi(A)\| = \|\varepsilon\Phi(1_H)\| = \varepsilon \to 0$ as $\varepsilon \downarrow 0$, we have $\Phi(A+\varepsilon 1_H) \downarrow \Phi(A)$ as $\varepsilon \downarrow 0$. As we proved in the preceding paragraph,

$$\begin{aligned}\Phi(A\ \sigma\ B) &\le \Phi((A+\varepsilon 1_H)\ \sigma\ B)\\ &\le \Phi(A+\varepsilon 1_H)\ \sigma\ \Phi(B),\end{aligned}$$

and let $\varepsilon \to 0$, we have (5.9). □

Let σ be an operator mean with representing function f. Then it follows that $f > 0$ on $(0,\infty)$. In fact, suppose that there is $\alpha > 0$ such that $f(\alpha) = 0$. Since f is monotone increasing, we have $f(t) = 0$ for $0 \le t \le \alpha$ and hence $f \equiv 0$ because f is concave. This contradicts the fact $f(1) = 1$. Therefore, the functions $f(t^{-1})^{-1}$, $t/f(t)$ and $tf(t^{-1})$ are operator monotone on $[0,\infty)$. Hence we can define the adjoint, transpose and dual of a given operator mean as follows:

Definition 5.2 *Let σ be an operator mean with representing function f.*

(i) The operator mean with representing function $f(t^{-1})^{-1}$ is called **the adjoint** *of σ and denoted by σ^*. Formula (5.4) gives an explicit form to the adjoint*

$$A\ \sigma^*\ B = (A^{-1}\ \sigma\ B^{-1})^{-1} \qquad \text{for invertible } A \text{ and } B.$$

(ii) The operator mean with representing function $tf(t^{-1})$ is called **the transpose** *of σ and denoted by σ^0. Formula (5.4) gives an explicit form to the transpose*

$$A\ \sigma^0\ B = B\ \sigma\ A \qquad \text{for every } A \text{ and } B.$$

An operator mean σ is called **symmetric** *if $\sigma = \sigma^0$.*

(iii) The operator mean with representing function $t/f(t)$ is called **the dual** *of σ and denoted by $\sigma^\perp$.*

The adjoint formation is involutive, $(\sigma^*)^* = \sigma$. The adjoint mean of the arithmetic mean is the harmonic mean, i.e. $\nabla^* = !$ and the geometric mean are selfadjoint, i.e. $(\sharp)^* = \sharp$. The dual formation is involutive, $(\sigma^\perp)^\perp = \sigma$, and it follows that

$$\sigma^\perp = (\sigma^0)^* = (\sigma^*)^0 \qquad \text{and} \qquad \sigma^0 = (\sigma^*)^\perp = (\sigma^\perp)^*.$$

We have the weighted means correspondent to above means: **the weighted arithmetic mean** ∇_p, **the weighted harmonic mean** $!_p$ and **the p-power mean** (**the weighted geometric mean**) defined for $0 < p < 1$:

$$\begin{aligned}A\ \nabla_p\ B &:= (1-p)A + pB,\\ A\ !_p\ B &:= \left((1-p)A^{-1} + pB^{-1}\right)^{-1},\\ A\ \sharp_p\ B &:= A^{\frac12}\left(A^{-\frac12}BA^{-\frac12}\right)^p A^{\frac12}.\end{aligned}$$

Then it follows that $(\sharp_p)^0 = \sharp_{1-p}$ and $(\sharp_p)^* = \sharp_p$.

Finally, Mond et al. studied inequalities for mixed operator means and mixed matrix means in 1996-1997 (for example see [153], [157], [1]). A simple inequalities of this type are:

$$\begin{array}{l} A \sharp_{\mu} \ (A \nabla_{\lambda} \ B) \geq A \nabla_{\lambda} \ \left(A \sharp_{\mu} \ B\right), \\ A \, !_{\lambda} \ \left(A \sharp_{\mu} \ B\right) \geq A \sharp_{\mu} \ (A \, !_{\lambda} \ B), \\ A \, !_{\mu} \ (A \nabla_{\lambda} \ B) \geq A \nabla_{\lambda} \ \left(A \, !_{\mu} \ B\right), \end{array} \tag{5.10}$$

where $A, B \in \mathscr{B}^{+}(H)$ are invertible and $\lambda, \mu \in (0,1)$.

5.2 Relative operator entropy

In this section, we introduce the relative operator entropy defined by Fujii-Kamei and show the entropy-like properties.

Nakamura and Umegaki extended the notion of the entropy formulated by J.von Neumann and gave the operator entropy by $-A\log A$ for a positive operator A on a Hilbert space H. Also, Umegaki introduced the relative entropy as a noncommutative version of the Kullback-Leibler entropy, which is given by the trace of $A\log A - A\log B$, i.e.

$$\tau(A\log A - A\log B)$$

for positive operators A, B affiliated with a semifinite von Neumann algebra.

J.I.Fujii and Kamei introduced the relative operator entropy which is a relative version of the operator entropy defined by Nakamura-Umegaki:

Definition 5.3 *For positive invertible operators A and B, then* **the relative operator entropy** *is defined by*

$$S(A|B) = A^{\frac{1}{2}} \left(\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}}. \tag{5.11}$$

Generally $S(A|B) = \mathrm{s} - \lim_{\varepsilon \to +0} S(A + \varepsilon 1_H|B)$ *if it exists.*

For the entropy function $\eta(t) = -t\log t$, the operator entropy has the following expression:

$$\eta(A) = -A\log A = S(A|1_H) \geq 0 \tag{5.12}$$

for positive contraction A. This shows that the relative operator entropy (5.11) is a relative version of the operator entropy.

Now, we give variational forms of (5.11).

Lemma 5.9 *If A and B are positive invertible, then*

$$S(A|B) = -A^{\frac{1}{2}} \left(\log A^{\frac{1}{2}} B^{-1} A^{\frac{1}{2}}\right) A^{\frac{1}{2}}, \tag{5.13}$$

and

$$S(A|B) = B^{\frac{1}{2}} \eta \left(B^{-\frac{1}{2}} A B^{-\frac{1}{2}}\right) B^{\frac{1}{2}}. \tag{5.14}$$

Proof. The formula (5.13) is obtained by $\log(1/t) = -\log t$. Since $Xf(X^*X) = f(XX^*)X$ by Lemma 1.7, applying it for $X = A^{\frac{1}{2}}B^{-\frac{1}{2}}$, we have

$$\begin{aligned} S(A|B) &= -A^{\frac{1}{2}}\left(\log A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}\right)A^{\frac{1}{2}} \\ &= -A^{\frac{1}{2}}\log(XX^*)XB^{\frac{1}{2}} = -A^{\frac{1}{2}}X\log(X^*X)B^{\frac{1}{2}} \\ &= -B^{\frac{1}{2}}X^*X\log(X^*X)B^{\frac{1}{2}} = B^{\frac{1}{2}}\eta(X^*X)B^{\frac{1}{2}} \\ &= B^{\frac{1}{2}}\eta\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)B^{\frac{1}{2}}, \end{aligned}$$

as desired. □

The above lemma says that if B is invertible, then one can define $S(A|B)$ by (5.14) even if A is not invertible. Thus, considering the operator monotonicity with respect to B, we redefine the relative operator entropy as follows.

Definition 5.4 *For positive operators A and B, then the relative operator entropy is defined by*

$$S(A|B) = -\,\mathrm{s}-\lim_{\varepsilon\to 0} A^{\frac{1}{2}}\left(\log A^{\frac{1}{2}}(B+\varepsilon 1_H)^{-1}A^{\frac{1}{2}}\right)A^{\frac{1}{2}} \tag{5.15}$$

if the strong limit exists.

Furthermore, if A and B commute, then

$$S(A|B) = -A\log A + A\log B$$

if the strong limit exists. The relative operator entropy for noninvertible positive operators does not always exist. In fact, $S(1_H|\varepsilon 1_H) = (\log\varepsilon)1_H$ is not bounded below and hence $S(1_H|0)$ does not make sense. However, if B majorizes A in the sense of Douglas, i.e., $\lambda A \le B$ for some positive number λ, then $S(A|B)$ always exists. But the majorization is only a sufficient condition to the existence. For example, even in a commutative case $B = A^2$, B cannot majorize A but $S(A|B)$ exists:

$$S(A|B) = -A\log A + A\log B = -A\log A + 2A\log A = A\log A.$$

Now, we characterize the domain of the relative operator entropy.

Lemma 5.10 *The strong limit of $S(A|B+\varepsilon 1_H)$ as $\varepsilon \downarrow 0$ exists if and only if $H(\alpha) = \alpha B - (\log\alpha)A$ is bounded below for $\alpha > 0$.*

Proof. For $\varepsilon > 0$, put $X = (B+\varepsilon 1_H)^{-1/2}A(B+\varepsilon 1_H)^{-1/2}$. Suppose that $S(A|B)$ exists. Since $\eta(t) = -t\log t$ is concave, the tangent line G_α of $\eta(t)$ at α is

$$G_\alpha(t) = -(\log\alpha+1)(t-\alpha)+\eta(\alpha) = -(\log\alpha+1)t+\alpha \ge \eta(t).$$

It follows that

$$\begin{aligned} S(A|B) &\le S(A|B+\varepsilon 1_H) = (B+\varepsilon 1_H)^{1/2}\eta(X)(B+\varepsilon 1_H)^{1/2} \\ &\le (B+\varepsilon 1_H)^{1/2}G_\alpha(X)(B+\varepsilon 1_H)^{1/2} = -(\log\alpha+1)A+\alpha(B+\varepsilon 1_H), \end{aligned}$$

hence we have $S(A|B) \leq \alpha B - (\log \alpha)A$ for all $\alpha > 0$.

Conversely, suppose that $\alpha B - (\log \alpha)A$ is bounded below for $\alpha > 0$. Then there exists a negative number c with

$$c(B+\varepsilon 1_H) \leq -(\log \alpha + 1)A + \alpha(B+\varepsilon 1_H) = (B+\varepsilon 1_H)^{1/2} G_\alpha(X)(B+\varepsilon 1_H)^{1/2}.$$

Therefore, $c1_H \leq G_\alpha(X)$. Then the inequality $c1_H \leq \inf_{\alpha>0} G_\alpha(X) = \eta(X)$ implies that $c(B+\varepsilon 1_H) \leq S(A|B+\varepsilon 1_H)$, hence $S(A|B+\varepsilon 1_H)$ is bounded below. □

Now we show that the relative operator entropy has many desirable properties like operator means under the existence.

Theorem 5.11 *The relative operator entropy $S(A|B)$ has the following properties.*

$$\begin{array}{l}
(\textit{right monotonicity}): \quad B \leq C \quad \textit{implies} \quad S(A|B) \leq S(A|C). \\
(\textit{right lower continuity}): \quad B_n \downarrow B \quad \textit{implies} \quad S(A|B_n) \downarrow S(A|B). \\
(\textit{homogenity}): \quad S(\alpha A|\alpha B) = \alpha S(A|B) \quad \textit{for } \alpha > 0. \\
(\textit{transformer inequality}): \ T^* S(A|B) T \leq S(T^*AT|T^*BT) \textit{ for every operator } T.
\end{array}$$

Proof. Since $f(t) = \log t$ is operator monotone, the formula (5.11) implies the required monotonicity. The homogenity is clear by (5.11).

Next, we show the transformer inequality. Since $S(A|B)$ exists, as we show in Lemma 5.10, we have

$$S(A|B) \leq \alpha B - (\log \alpha)A \qquad \text{for all } \alpha > 0.$$

Raising both sides to operators T^* and T, we have

$$T^* S(A|B) T \leq \alpha T^*BT - (\log \alpha) T^*AT$$

for $\alpha > 0$ and hence it follows from Lemma 5.10 that $S(T^*AT|T^*BT)$ exists. By using Theorem 5.18, we have

$$\begin{aligned}
S(T^*AT|T^*BT) &= \text{s}-\lim_{\alpha\to 0} \frac{T^*AT \sharp_\alpha T^*BT - T^*AT}{\alpha} \\
&\geq \text{s}-\lim_{\alpha\to 0} \frac{T^*(A \sharp_\alpha B - A)T}{\alpha} \\
&= T^* S(A|B) T,
\end{aligned}$$

since the geometric mean $\sharp_\alpha$ satisfies the transformer inequality (S3).

Finally, we show the right lower continuity. Suppose that B is invertible. Then $B_n^{\frac{1}{2}} \downarrow B^{\frac{1}{2}}$ and $C_n = B_n^{-\frac{1}{2}} A B_n^{-\frac{1}{2}} \mapsto C = B^{-\frac{1}{2}} A B^{-\frac{1}{2}}$ strongyly, so that

$$S(A|B_n) = B_n^{\frac{1}{2}} \eta(C_n) B_n^{\frac{1}{2}} \downarrow B^{\frac{1}{2}} \eta(C) B^{\frac{1}{2}} = S(A|B)$$

by the continuity of η and the right monotonicity of S.

Next, suppose that B is not invertible. Since $S(A|B) \leq S(A|B_{n+1}) \leq S(A|B_n)$, we have

$$\begin{aligned}
0 &\leq S(A|B_n) - S(A|B) \\
&= S(A|B_n) - S(A|B_n + \delta 1_H) + S(A|B_n + \delta 1_H) - S(A|B + \delta 1_H) \\
&\qquad + S(A|B + \delta 1_H) - S(A|B) \\
&\leq S(A|B_n + \delta 1_H) - S(A|B + \delta 1_H) + S(A|B + \delta 1_H) - S(A|B).
\end{aligned}$$

Since $S(A|B + \delta 1_H) \downarrow S(A|B)$ as $\delta \downarrow 0$ by definition, for $\varepsilon > 0$ and a unit vector $x \in H$, we can choose $\delta > 0$ such that $0 \leq ((S(A|B + \delta 1_H) - S(A|B))x,x) \leq \frac{\varepsilon}{2}$. Moreover, for this δ, there exists n_0 such that $n \geq n_0$ implies $0 \leq ((S(A|B_n + \delta 1_H) - S(A|B + \delta 1_H))x,x) < \frac{\varepsilon}{2}$ by the above invertible case. So we have

$$\begin{aligned}
0 &\leq ((S(A|B_n) - S(A|B))x,x) \\
&\leq ((S(A|B_n + \delta 1_H) - S(A|B + \delta 1_H))x,x) + ((S(A|B + \delta 1_H) - S(A|B))x,x) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Therefore we have $S(A|B_n) \downarrow S(A|B)$. □

The upper boundaries of the relative operator entropy is always guaranteed.

Theorem 5.12 *The relative operator entropy is upper bounded:*

$$\begin{aligned}
S(A|B) &\leq -A\log A + A\log \|B\|, \\
S(A|B) &\leq B - A.
\end{aligned}$$

Proof. By the operator monotonicity, we have

$$S(A|B) \leq S(A|\|B\|) = -A\log A + A\log \|B\|.$$

It follows from the Klein inequality $\log t \leq t - 1$ that

$$\begin{aligned}
S(A|B) &= A^{\frac{1}{2}} \left(\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}} \\
&\leq A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} - 1_H \right) A^{\frac{1}{2}} = B - A.
\end{aligned}$$

□

By Theorem 5.12, we have a simple condition that $S(A|B)$ is negative.

Corollary 5.13 *If $A \geq B$, then $S(A|B) \leq 0$.*

Corollary 5.14 *For positive operators A, B with $A \geq B$, $S(A|B) = 0$ if and only if $A = B$.*

Proof. Suppose that $S(A|B) = 0$. Then it follows from Theorem 5.12 that $0 = S(A|B) \leq B - A \leq 0$, which implies $A = B$. Conversely, we have $S(A|A) = A\log \mathrm{supp}A = 0$, where $\mathrm{supp}C$ is the support projection of C. □

In addition, the relative operator entropy has entropy like properties.

Theorem 5.15 *The relative operator entropy is subadditive.*

$$S(A+B|C+D) \geq S(A|C)+S(B|D).$$

Proof. We may assume that both C and D are invertible. Put $X = C^{1/2}(C+D)^{-1/2}$ and $Y = D^{1/2}(C+D)^{-1/2}$. Since $X^*X+Y^*Y = 1_H$, it follows from Theorem 1.9 that

$$\begin{aligned}
S(A+B|C+D) &= (C+D)^{1/2}\eta((C+D)^{-1/2}(A+B)(C+D)^{-1/2})(C+D)^{1/2}\\
&= (C+D)^{1/2}\eta(X^*C^{-1/2}AC^{-1/2}X+Y^*D^{-1/2}BD^{-1/2}Y)(C+D)^{1/2}\\
&\geq (C+D)^{1/2}\left(X^*\eta\left(C^{-1/2}AC^{-1/2}\right)X+Y^*\eta\left(D^{-1/2}BD^{-1/2}\right)Y\right)(C+D)^{1/2}\\
&= C^{1/2}\eta\left(C^{-1/2}AC^{-1/2}\right)C^{1/2}+D^{1/2}\eta\left(D^{-1/2}BD^{-1/2}\right)D^{1/2}\\
&= S(A|C)+S(B|D).
\end{aligned}$$

□

Theorem 5.16 *The relative operator entropy is jointly concave. If $A = tA_1+(1-t)A_2$ and $B = tB_1+(1-t)B_2$ for $0 \leq t \leq 1$, then*

$$S(A|B) \geq tS(A_1|B_1)+(1-t)S(A_2|B_2).$$

Proof. By subadditivity and homogenity of the relative operator entropy, we have

$$\begin{aligned}
S(A|B) &= S(tA_1+(1-t)A_2|tB_1+(1-t)B_2)\\
&\geq S(tA_1|tB_1)+S((1-t)A_2|(1-t)B_2)\\
&= tS(A_1|B_1)+(1-t)S(A_2|B_2).
\end{aligned}$$

□

The relative operator entropy has informational monotonicity.

Theorem 5.17 *Let $\Phi \in \mathbf{P}_N[\mathscr{B}(H),\mathscr{B}(K)]$ be a normalized positive linear map. Then*

$$\Phi(S(A|B)) \leq S(\Phi(A)|\Phi(B)). \tag{5.16}$$

Proof. Suppose that B is invertible. Then so does $\Phi(B)$. Define a normalized positive linear map by

$$\Phi_B(X) = \Phi(B)^{-1/2}\Phi\left(B^{1/2}XB^{1/2}\right)\Phi(B)^{-1/2}.$$

So we have by Davis-Choi-Jensen's inequality (Theorem 1.20)

$$\Phi_B(F(X)) \leq F(\Phi_B(X))$$

for every operator concave function F on $(0,\infty)$. Then it follows from Lemma 5.9 that

$$\begin{aligned}
\Phi(S(A|B)) &= \Phi\left(B^{1/2}\eta\left(B^{-1/2}AB^{-1/2}\right)B^{1/2}\right)\\
&= \Phi(B)^{1/2}\Phi_B\left(\eta\left(B^{-1/2}AB^{-1/2}\right)\right)\Phi\left(B^{1/2}\right)\\
&\leq \Phi(B)^{1/2}\eta\left(\Phi_B\left(B^{-1/2}AB^{-1/2}\right)\right)\Phi\left(B^{1/2}\right)\\
&= S(\Phi(A)|\Phi(B)).
\end{aligned}$$

We prove (5.16) in the same way in the proof of Theorem 5.8 if B is not invertible. $\square$

Finally, we discuss the relation between the relative operator entropy and the relative entropy which introduced by Umegaki and developed by Araki, Uhlmann and Pusz-Woronowicz.

Now we recall the relative entropy $S(\varphi|\psi)$ for states φ, ψ on an operator algebra. Derived from the Kullback-Leibler information (divergence):

$$\sum_{j=1}^{k} p_j \log \frac{p_j}{q_j} \qquad \text{for probability vectors } p = \begin{pmatrix} p_1 \\ \vdots \\ p_k \end{pmatrix} \text{ and } q = \begin{pmatrix} q_1 \\ \vdots \\ q_k \end{pmatrix}.$$

Umegaki introduced a relative entropy $S(\varphi|\psi)$ for states φ, ψ on a semifinite von Neumann algebra, which is defined as

$$S(\varphi|\psi) = \tau(A\log A - A\log B)$$

where A and B are density operators of φ and ψ respectively, i.e.,

$$\varphi(X) = \tau(AX) \quad \text{and} \quad \psi(X) = \tau(BX).$$

Araki generalized it by making use of the Tomita-Takesaki theory, Uhlmann by the quadratic interpolation, Pusz-Woronoxicz by their functional calculus. These generalization are all equivalent. The quadratic interpolation $QI_t(p,q)$ for seminorms $p(x) = (Ax,x)^{1/2}$ and $q(x) = (Bx,x)^{1/2}$ is the seminorm defined by $A \natural_t B$ for commuting A and B:

$$QI_t(p,q)(x) = (A \natural_t Bx,x)^{\frac{1}{2}}. \tag{5.17}$$

Uhlmann's relative entropy is based on the interpolation theory of positive linear forms. For positive linear forms φ, ψ on a unital $*-$algebra $\mathscr{A}$, put a sesquilinear form

$$(x,y) = \varphi(xy^*) + \psi(y^*x). \tag{5.18}$$

Let $x \mapsto x^o$ be the usual map from $\mathscr{A}$ to $\mathscr{H}$ which is the Hilbert space with the linear product corresponding to (5.18). Then there exists derivatives A, B on $\mathscr{H}$ with

$$(Ax^o,y^o) = \varphi(xy^*) \quad \text{and} \quad (Bx^o,y^o) = \psi(y^*x).$$

It follows from (5.17) that A and B are commuting positive contraction with $A+B = 1_H$. In this situation, Uhlmann's relative entropy $S(\varphi|\psi)_U$ is expressed by

$$S(\varphi|\psi)_U = -\liminf_{t\to 0}\left(\frac{A \natural_t B - A}{t} 1^o, 1^o\right).$$

According to this definition, we can construct $S(A|B)$ by a similar way.

Theorem 5.18 *The relative operator entropy $S(A|B)$ is constructed by Uhlmann's way.*

$$S(A|B) = \text{s}-\lim_{t\to 0}\frac{A \natural_t B - A}{t}.$$

Proof. For invertible A, put $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. Since $\lim_{t\to 0}\frac{x^t-1}{t} = \log x$, we have

$$A^{\frac{1}{2}}\left(\text{s}-\lim_{t\to 0}\frac{X^t-1_H}{t}\right)A^{\frac{1}{2}} = A^{\frac{1}{2}}(\log X)A^{\frac{1}{2}} = S(A|B).$$

□

Hiai and Petz discussed a bridge between the relative entropy and the relative operator entropy. Note that if the density operators A and B commute, then

$$S(\varphi|\psi) = S(\varphi|\psi)_U = -\tau(S(A|B)).$$

They showed that

$$S(\varphi|\psi) \geq -\tau(S(A|B))$$

for states on a finite dimensional C*-algebra.

5.3 Interpolational path

In this section, we study an operator version of Uhlmann's interpolation. We recall that an operator mean σ is symmetric if $A\ \sigma\ B = B\ \sigma\ A$ for all positive operators A and B. For a symmetric operator mean σ, a parameterized operator mean σ_t is called an interpolational path for σ if it satisfies

(1) $A\ \sigma_0\ B = A,\quad A\ \sigma_{1/2}\ B = A\ \sigma\ B\quad$ and $\quad A\ \sigma_1\ B = B$
(2) $(A\ \sigma_p\ B)\ \sigma\ (A\ \sigma_q\ B) = A\ \sigma_{\frac{p+q}{2}}\ B$ for all $p,q \in [0,1]$
(3) the map $t \in [0,1] \mapsto A\ \sigma_t\ B$ is norm continuous for each A and B.

Typical interpolational means are so-called power means;

$$A\ m_r\ B = A^{\frac{1}{2}}\left(\frac{1+\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^r}{2}\right)^{\frac{1}{r}}A^{\frac{1}{2}}\quad \text{for } r \in [-1,1]$$

and their interpolational paths are

$$A\ m_{r,t}\ B = A^{\frac{1}{2}}\left(1-t+t\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^r\right)^{\frac{1}{r}}A^{\frac{1}{2}}\quad \text{for } t \in [0,1].$$

For each $r \in [-1,1]$, $A\ m_{r,t}\ B$ ($t \in [0,1]$) is a path from A to B via $A\ m_r\ B$. In particular,

$$\begin{aligned}
A\ m_{1,t}\ B &= A\ \nabla_t\ B = (1-t)A + tB,\\
A\ m_{0,t}\ B &= A\ \sharp_t\ B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}},\\
A\ m_{-1,t}\ B &= A\ !_t\ B = ((1-t)A^{-1} + tB^{-1})^{-1}.
\end{aligned}$$

They are called the arithmetic, geometric and harmonic interpolations respectively.

Here generally, for positive invertible operators A and B on a Hilbert space H, we define a norm continuous path of positive invertible operators $A\ m_{r,t}\ B$ by

$$A\ m_{r,t}\ B = A^{\frac{1}{2}}\left(1-t+t\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^r\right)^{\frac{1}{r}}A^{\frac{1}{2}}$$

for all real numbers $r \in \mathbb{R}$ and t with $0 \le t \le 1$ as an extension of $m_{r,t}$. We also define the representing function $F_{r,t}$ for $m_{r,t}$ by

$$F_{r,t}(x) = 1\ m_{r,t}\ x = (1-t+tx^r)^{\frac{1}{r}} \quad \text{for all } x > 0.$$

Then we have

$$Am_{r,t}B = A^{\frac{1}{2}}F_{r,t}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}.$$

First of all, we see the convexity for representing functions.

Lemma 5.19 *Every function $F_{r,t}(x)$ is strictly increasing and strictly convex (resp. strictly concave) for $r > 1$ (resp. $r < 1$).*

Proof. It is increasing since

$$\frac{d}{dx}F_{r,t}(x) = tx^{r-1}(1-t+tx^r)^{\frac{1-r}{r}} > 0 \quad \text{for } t \in (0,1)$$

Moreover the latter part is shown by

$$\frac{d^2}{dx^2}F_{r,t}(x) = tx^{r-2}(1-t+tx^r)^{\frac{1-2r}{r}}(r-1)(1-t+2tx^r).$$

□

The adjoint for $m_{r,t}$ and $F_{r,t}$ are as follows.

$$A\ m_{r,t}^*\ B = (A^{-1}\ m_{r,t}\ B^{-1})^{-1} = A^{\frac{1}{2}}F_{r,t}\left(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}\right)^{-1}A^{\frac{1}{2}}$$

and

$$F_{r,t}^*(x) = \frac{1}{F_{r,t}(\frac{1}{x})}.$$

Since $F_{r,t}^* = F_{-r,t}$, it follows that this operation preserves the operator monotonicity, so that the above lemma shows that $F_{r,t}$ cannot be operator monotone for $|r| > 1$.

Theorem 5.20 *The inequality $|r| \le 1$ is the equivalent condition that $m_{r,t}$ is operator mean, or equivalently $F_{r,t}$ is operator monotone.*

Proof. Every function $F_{r,t}$ for $r > 1$ cannot be operator monotone since it is strictly convex. Though $F_{r,t}$ is concave for $r < -1$, it cannot be either. In fact, if it is operator monotone, then so is a convex function $F_{r,t}^* = F_{-r,t}$, which is a contradiction. □

Theorem 5.21 *For each $t \in (0,1)$ a path $A\ m_{r,t}\ B$ is nondecreasing and norm continuous for $r \in \mathbb{R}$: For $r \le s$*

$$A\ m_{r,t}\ B \le A\ m_{s,t}\ B.$$

Proof. By Lemma 5.19, for $r \le s$ $F_{r,t}(x) \le F_{s,t}(x)$ implies

$$A\ m_{r,t}\ B = A^{\frac{1}{2}} F_{r,t}\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}} \le A^{\frac{1}{2}} F_{s,t}\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}} = A\ m_{s,t}\ B.$$

□

The transformer equality makes operator means easy to handle. Though $A\ m_{r,t}\ B$ is no longer an operator mean in general, the transformer equality also holds unexpectedly.

Lemma 5.22 *If X is an invertible operator, then*

$$X^* F_{r,t}(Y) X = X^*X\ m_{r,t}\ X^*YX$$

for all real numbers $r \in \mathbb{R}$.

Proof. For the unitary U in the polar decomposition of $X = U|X|$, we have

$$\begin{aligned} X^*X\ m_{r,t}\ X^*YX &= |X| F_{r,t}(|X|^{-1} X^*YX |X|^{-1}) |X| = |X| F_{r,t}(U^*YU) |X| \\ &= |X| U^* F_{r,t}(Y) U |X| = X^* F_{r,t}(Y) X. \end{aligned}$$

□

Theorem 5.23 *The transformer equality holds for $m_{r,t}$ for all real numbers $r \in \mathbb{R}$.*

Proof. For invertible X, the above lemma implies

$$X^*(A\ m_{r,t}\ B)X = X^* A^{\frac{1}{2}} F_{r,t}\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}} X = X^*AX\ m_{r,t}\ X^*BX.$$

□

Theorem 5.24 *A path $A\ m_{r,t}\ B$ is interpolational that*

$$(A\ m_{r,p}\ B)\ m_{r,t}\ (A\ m_{r,q}\ B) = A\ m_{r,(1-t)p+tq}\ B$$

for all real numbers $r \in \mathbb{R}$ and $0 \le p,q,t \le 1$. In particular, the transposition formula holds:

$$B\ m_{r,t}\ A = A\ m_{r,1-t}\ B.$$

Proof. Since

$$\begin{aligned} F_{r,p}(x) F_{r,t}\left(\frac{F_{r,q}(x)}{F_{r,p}(x)}\right) &= (1-p+px^r)^{1/r}\left(1-t+t\left(\frac{1-q+qx^r}{1-p+px^r}\right)\right)^{1/r} \\ &= ((1-t)(1-p+px^r)+t(1-q+qx^r))^{1/r} = F_{r,(1-t)p+tq}(x), \end{aligned}$$

we have the required result by the transformer equality. The transposition formula is the case for $p = 1$ and $q = 0$. □

We investigate estimates of the upper boundary for the ratio between extended interpolational paths $m_{r,t}$ by terms of a generalized Specht ratio.

Theorem 5.25 *Let A and B be positive invertible operators on H such that $M1_H \geq A, B \geq m1_H > 0$ for some scalars $M > m > 0$. Put $h = \frac{M}{m}$. Then for $r \leq s$ and $t \in (0,1)$*

$$A\ m_{s,t}\ B \leq \Delta(h,r,s)\ A\ m_{r,t}\ B,$$

where a generalized Specht ratio $\Delta(h,r,s)$ is defined as (2.97) in § 2.7.

Proof. Let C be a positive invertible operator on H satisfying $M1_H \geq C \geq m1_H > 0$. Then it follow from Theorem 2.61 that

$$(1-t+tC^s)^{1/s} \leq \max_{m \leq x \leq M} \Delta(x,r,s)(1-t+tC^r)^{1/r}$$

for all $r \leq s$ and $t \in [0,1]$. Since the maximum of $\Delta(x,r,s)$ in $x \in [m,M]$ is given by $\max\{\Delta(m,r,s), \Delta(M,r,s)\}$ by Theorem 2.62, we have

$$(1-t+tC^s)^{1/s} \leq \max\{\Delta(m,r,s), \Delta(M,r,s)\}(1-t+tC^r)^{1/r}$$

Since $0 < m1_H \leq A, B \leq M1_H$, we obtain $\frac{1}{h}1_H \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq h1_H$. Replacing C by $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in above inequality, we have for $t \in [0,1]$

$$\left(1-t+t\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^s\right)^{1/s} \leq \Delta(h,r,s)\left(1-t+t\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^r\right)^{1/r}$$

since $\Delta\left(\frac{1}{h},r,s\right) = \Delta(h,r,s)$ by Theorem 2.62. Multiplying both sides by $A^{1/2}$, we have

$$A\ m_{s,t}\ B \leq \Delta(h,r,s) A\ m_{r,t}\ B$$

for $r \leq s$. □

We investigate the order relation between the arithmetic mean, the geometric one and the harmonic one:

Corollary 5.26 *Let A and B be positive operators on H such that $M1_H \geq A, B \geq m1_H > 0$ for some scalars $M > m > 0$. Put $h = \frac{M}{m}$. Then for $r < 0 < s$ and $t \in (0,1)$*

$$S(h^s)^{-1/s} A\ m_{s,t}\ B \leq A \sharp_t B \leq S(h^r)^{-1/r} A\ m_{r,t}\ B,$$

where the Specht ratio $S(h) = S(h,1)$ is defined as (2.74) in § 2.6.

In particular,

$$S(h)^{-1} A\ \nabla_t\ B \leq A \sharp_t B \leq S(h) A\ !_t\ B.$$

5.4 Converses of Ando type operator means inequalities

In this section, applying the Mond-Pečarić method to normalized positive linear maps, we show several complementary inequalities to Jensen's inequality on positive linear maps and consequently obtain complementary inequalities to Ando's inequality (Theorem 5.8) associated with an operator mean.

Throughout this section, we assume that $0 < m_1 < M_1$ and $0 < m_2 < M_2$. First we start with an extension of Theorem 5.8:

Lemma 5.27 *Let $\Phi \in P[\mathscr{B}(H), \mathscr{B}(K)]$. Suppose that operator means σ and τ have representing functions f and g respectively. Then the following statements are mutually equivalent:*

$$\text{(i)} \qquad \Phi(A\ \sigma\ B) \le \Phi(A)\ \tau\ \Phi(B) \quad \textit{for every} \quad A, B \in \mathscr{B}^+(H).$$

$$\text{(ii)} \qquad \Phi(f(A)) \le g(\Phi(A)) \quad \textit{for every} \quad A \in \mathscr{B}^+(H).$$

$$\text{(iii)} \qquad f \le g \quad \textit{on } [0, \infty).$$

Proof. It is sufficient to prove that (ii) implies (i). We consider the map Ψ by

$$\Psi(X) = \Phi(A)^{-\frac{1}{2}} \Phi(A^{\frac{1}{2}} X A^{\frac{1}{2}}) \Phi(A)^{-\frac{1}{2}}.$$

Since $\Psi \in \mathbf{P}_N[\mathscr{B}(H), \mathscr{B}(K)]$, it follows from the assumption of (ii) that $\Psi(f(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})) \le g(\Psi(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}))$. Then we have

$$\begin{aligned} \Phi(A\ \sigma\ B) &= \Phi(A)^{\frac{1}{2}} \Psi\left(f\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)\right) \Phi(A)^{\frac{1}{2}} \\ &\le \Phi(A)^{\frac{1}{2}} \left(g\left(\Psi\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)\right)\right) \Phi(A)^{\frac{1}{2}} \\ &= \Phi(A)\ \tau\ \Phi(B). \end{aligned}$$

□

Remark 5.1 If we put $\sigma = \tau$ in Lemma 5.27, then we have Theorem 5.8, because the representing function $f = g$ is operator concave.

Since the representing functions f and g have no order relation, it follows that $\Phi(A\ \sigma\ B)$ and $\Phi(A)\ \tau\ \Phi(B)$ have no relation to the operator order generally. Thus we apply Lemma 5.27 to consider the following complementary theorem:

Theorem 5.28 *Suppose that operator means σ and τ have representing functions f and g respectively, which are not affine. Let $\Phi \in P[\mathscr{B}(H), \mathscr{B}(K)]$, $A, B \in \mathscr{B}^+(H)$ with $\mathsf{Sp}(A) \subseteq [m_1, M_1]$, $\mathsf{Sp}(B) \subseteq [m_2, M_2]$ and $m = m_2/M_1$, $M = M_2/m_1$. Then for a given $\alpha \in \mathbb{R}_+$*

$$\Phi(A\ \sigma\ B) \ge \alpha \Phi(A)\ \tau\ \Phi(B) + \beta \Phi(A), \tag{5.19}$$

where $\beta = \beta(m,M,f,g,\alpha) = \mu_f t_o + \nu_f - \alpha g(t_o)$ and $t_o \in [m,M]$ is defined as the unique solution of $g'(t) = \mu_f/\alpha$ when $g'(M) < \mu_f/\alpha < g'(m)$, otherwise t_o is defined as M or m according to $\mu_f/\alpha \le g'(M)$ or $g'(m) \le \mu_f/\alpha$.

Proof. As in Lemma 5.27, we consider a map $\Psi \in \mathbf{P}_N[\mathscr{B}(H),\mathscr{B}(K)]$ given by $\Psi(X) = \Phi(A)^{-\frac{1}{2}}\Phi\left(A^{\frac{1}{2}}XA^{\frac{1}{2}}\right)\Phi(A)^{-\frac{1}{2}}$. Since the representing functions f, g are nonnegative operator concave functions, it follows from Theorem 2.4 that for $k=1$ and a given $\alpha > 0$

$$\Psi\left(f\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right) \ge \alpha g\left(\Psi\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right) + \beta 1_H$$

holds for $\beta = \beta(\frac{m_2}{M_1}, \frac{M_2}{m_1}, f, g, \alpha)$ defined as it was in Theorem 2.4. Therefore we have

$$\begin{aligned}
\Phi(A\ \sigma\ B) &= \Phi(A)^{\frac{1}{2}}\Psi\left(f\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)\Phi(A)^{\frac{1}{2}} \\
&\ge \Phi(A)^{\frac{1}{2}}\left(\alpha g\left(\Psi(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\right) + \beta 1_H\right)\Phi(A)^{\frac{1}{2}} \\
&= \alpha\Phi(A)\ \tau\ \Phi(B) + \beta\Phi(A).
\end{aligned}$$

□

Remark 5.2 If we put $\alpha = 1$ in Theorem 5.28 we obtain the following inequality:

$$-\beta\Phi(A) \ge \Phi(A)\ \tau\ \Phi(B) - \Phi(A\ \sigma\ B),$$

where

$$\beta = \min_{\frac{m_2}{M_1} \le t \le \frac{M_2}{m_1}} \{\mu_f t + \nu_f - g(t)\}.$$

If we choose a value of constant α such that $\beta = 0$ in Theorem 5.28, then we obtain the following corollary.

Corollary 5.29 *Let the hypothesis of Theorem 5.28 be satisfied. Then*

$$\Phi(A\ \sigma\ B) \ge \alpha_1\ \Phi(A)\ \tau\ \Phi(B),$$

where $\alpha_1 = \left(\mu_f t_o + \nu_f\right)/g(t_o)$ and $t_o \in [m,M]$ is defined as the unique solution of $\mu_f g(t) = g'(t)(\mu_f t + \nu_f)$ when $f(M)g'(M)/g(M) < \mu_f < f(m)g'(m)/g(m)$, otherwise t_o is defined as M or m according to $\mu_f \le f(M)g'(M)/g(M)$ or $f(m)g'(m)/g(m) \le \mu_f$.

By virtue of Theorem 5.28, we obtain lower estimates for complementary inequalities to Jensen's type inequalities on a positive linear map under a general setting.

Theorem 5.30 *Suppose that two operator means σ and τ have representing functions f and g respectively, which are not affine. Let $\Phi \in P[\mathscr{B}(H),\mathscr{B}(K)]$, $A,B \in \mathscr{B}^+(H)$ with* $\mathsf{Sp}(A) \subseteq [m_1,M_1]$, $\mathsf{Sp}(B) \subseteq [m_2,M_2]$. *For a given $\alpha > 0$, put $\beta = \beta\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}, f, g, \alpha\right)$ and*

$\beta^0 = \beta\left(\frac{m_1}{M_2}, \frac{M_1}{m_2}, f^0, g^0, \alpha\right)$ *defined as it was in Theorem 5.28.*
(i) *If* $\beta \geq 0$ *and* $\beta^0 \geq 0$, *then for every operator mean* ρ

$$\Phi(A\ \sigma\ B) - \alpha\Phi(A)\ \tau\ \Phi(B) \geq (\beta\Phi(A))\ \rho\ \left(\beta^0\Phi(B)\right). \tag{5.20}$$

(ii) *If* $\beta < 0$ *and* $\beta^0 < 0$, *then for every operator mean* ρ

$$(\Phi(A\ \sigma\ B) - \beta\Phi(A))\ \rho\ \left(\Phi(A\ \sigma\ B) - \beta^0\Phi(B)\right) \geq \alpha\ \Phi(A)\ \tau\ \Phi(B). \tag{5.21}$$

(iii) *If* $\beta\beta^0 < 0$, *then*

$$\Phi(A\ \sigma\ B) - \alpha\Phi(A)\ \tau\ \Phi(B) \geq \max\{\beta\ \Phi(A), \beta^0\ \Phi(B)\}. \tag{5.22}$$

Proof. By Theorem 5.28, we have that for a given $\alpha > 0$

$$\Phi(A\ \ \sigma\ B) \geq \alpha\Phi(A)\ \tau\ \Phi(B) + \beta\Phi(A)$$

holds for $\beta = \beta\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}, f, g, \alpha\right)$. By applying Theorem 5.28 to the transpose σ^o and τ^o, we have that

$$\Phi(B\ \sigma^o\ A) \geq \alpha\Phi(B)\ \tau^o\ \Phi(A) + \beta^o\Phi(B) \tag{5.23}$$

holds for $\beta^0 = \beta\left(\frac{m_1}{M_2}, \frac{M_1}{m_2}, f^0, g^0, \alpha\right)$, where f^0 and g^0 are the transpose representing functions for f and g respectively. Therefore we have

$$\Phi(A\ \sigma\ B) \geq \alpha\Phi(A)\ \tau\ \Phi(B) + \beta^o\Phi(B).$$

Suppose that $\beta \geq 0$ and $\beta^0 \geq 0$. Then we have

$$\Phi(A\ \sigma\ B) - \alpha\Phi(A)\ \tau\ \Phi(B) \geq \beta\ \Phi(A) \geq 0 \tag{5.24}$$

and

$$\Phi(A\ \sigma\ B) - \alpha\Phi(A)\ \tau\ \Phi(B) \geq \beta^0\ \Phi(B) \geq 0. \tag{5.25}$$

Combining (5.24) and (5.25), it follows from the normalization of ρ that

$$\begin{aligned}
&\Phi(A\ \sigma\ B) - \alpha\Phi(A)\ \tau\ \Phi(B) \\
&\quad = (\Phi(A\ \sigma\ B) - \alpha\Phi(A)\ \tau\ \Phi(B))\ \rho\ (\Phi(A\ \sigma\ B) - \alpha\Phi(A)\ \tau\ \Phi(B)) \\
&\quad \geq (\beta\Phi(A))\ \rho\ \left(\beta^0\Phi(B)\right),
\end{aligned}$$

which implies (5.20).

Suppose that $\beta < 0$ and $\beta^0 < 0$. Then we have

$$\Phi(A\ \sigma\ B) - \beta\Phi(A) \geq \alpha\Phi(A)\ \tau\ \Phi(B) \geq 0 \tag{5.26}$$

and

$$\Phi(A\ \sigma\ B) - \beta^0\ \Phi(B) \geq \alpha\Phi(A)\ \tau\ \Phi(B) \geq 0. \tag{5.27}$$

Combining (5.26) and (5.27), we have

$$(\Phi(A\ \sigma\ B) - \beta\Phi(A))\ \rho\ \left(\Phi(A\ \sigma\ B) - \beta^0\Phi(B)\right) \geq \alpha\ \Phi(A)\ \tau\ \Phi(B), \tag{5.28}$$

which implies (5.21).

Finally, if $\beta\beta^0 < 0$, then $\beta\Phi(A) \geq 0 \geq \beta^0\Phi(B)$ or $\beta^0\Phi(B) \geq 0 \geq \beta\Phi(A)$. Hence we have the desired result (5.22). □

Corollary 5.31 *Assume that the conditions of Theorem 5.28 hold. If* $\sigma\ \leq \tau$, *then for every operator mean* ρ

$$(-\beta\Phi(A))\ \rho\ (-\beta^o\Phi(B)) \geq \Phi(A)\ \tau\ \Phi(B) - \Phi(A\ \ \sigma\ \ B) \geq 0$$

holds for $\beta = \beta\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}, f, g, \alpha = 1\right)$ *and* $\beta^0 = \beta\left(\frac{m_1}{M_2}, \frac{M_1}{m_2}, f^0, g^0, \alpha = 1\right)$ *defined as it was in Theorem 5.28.*

Proof. If we put $\alpha = 1$ in Theorem 5.28, then it follows from $\sigma\ \leq\ \tau$ that

$$\mu_f t_0 + \nu_f \leq f(t_0) \leq g(t_0),$$

that is, $\beta < 0$. Similarly we have $\beta^0 < 0$ since $\sigma^0 \leq \tau^0$. □

Further, if we choose α such that $\beta = 0$ in (5.19) of Theorem 5.28, then we have the following corollary:

Corollary 5.32 *Assume that the conditions of Theorem 5.28 hold. Then*

$$\Phi(A\ \ \sigma\ \ B) \geq \max\left\{ \min_{\frac{m_2}{M_1} \leq t \leq \frac{M_2}{m_1}} \left\{ \frac{\mu_f t + \nu_f}{g(t)} \right\}, \min_{\frac{m_1}{M_2} \leq s \leq \frac{M_1}{m_2}} \left\{ \frac{a_{f^o} s + b_{f^o}}{g^o(s)} \right\} \right\} \Phi(A)\ \tau\ \Phi(B).$$

Proof. Since the representing function of an operator mean is a not affine and a non-negative operator concave function, Corollary 5.32 follows from Corollary 5.29. In fact, it follows from Corollary 5.29 that

$$\Phi(A\ \ \sigma\ \ B) \geq \left(\min_{\frac{m_2}{M_1} \leq t \leq \frac{M_2}{m_1}} \left\{ \frac{\mu_f t + \nu_f}{g(t)} \right\} \right) \Phi(A)\ \tau\ \Phi(B).$$

In a similar way we obtain

$$\Phi(A\ \ \sigma\ \ B) \geq \left(\min_{\frac{m_1}{M_2} \leq s \leq \frac{M_1}{m_2}} \left\{ \frac{a_{f^o} s + b_{f^o}}{g^o(s)} \right\} \right) \Phi(A)\ \ \tau\ \ \Phi(B).$$

Since $(\alpha_1 X)\ \rho\ (\alpha_2 X) = (\alpha_1\ \rho\ \alpha_2)X$ for $\alpha_1, \alpha_2 > 0$ and $X \geq 0$, we have that for every operator mean ρ

$$\Phi(A\ \ \sigma\ \ B) \geq \left(\min_{\frac{m_2}{M_1} \leq t \leq \frac{M_2}{m_1}} \left\{ \frac{\mu_f t + \nu_f}{g(t)} \right\} \right) \rho \left(\min_{\frac{m_1}{M_2} \leq s \leq \frac{M_1}{m_2}} \left\{ \frac{a_{f^o} s + b_{f^o}}{g^o(s)} \right\} \right) \Phi(A)\ \ \tau\ \ \Phi(B).$$

Therefore, we obtain the desired result. □

Next, we shall consider how the weighted geometric mean modifies when filtered through a positive linear map.

Corollary 5.33 *Let $\Phi \in P[\mathscr{B}(H), \mathscr{B}(K)]$, $A, B \in \mathscr{B}^+(H)$ with $\mathrm{Sp}(A) \subseteq [m_1, M_1]$, $\mathrm{Sp}(B) \subseteq [m_2, M_2]$. Let $p, q \in (0,1)$ be given. Then for a given $\alpha > 0$*

$$\Phi(A \sharp_p B) - \alpha \Phi(A) \sharp_q \Phi(B) \geq \beta \Phi(A)$$

holds for $\beta = \beta(m, M, t^p, t^q, \alpha) =$

$$\begin{cases} \alpha(q-1)\left(\frac{1}{\alpha q}\frac{M^p - m^p}{M-m}\right)^{\frac{q}{q-1}} + \frac{Mm^p - mM^p}{M-m} & \text{if} \quad qm^{q-1} \geq \frac{1}{\alpha}\frac{M^p - m^p}{M-m} \geq qM^{q-1}, \\ \min\{M^p - \alpha M^q, m^p - \alpha m^q\} & \text{otherwise,} \end{cases}$$

where $m = \frac{m_2}{M_1}$ and $M = \frac{M_2}{m_1}$, and

$$\Phi(A \sharp_p B) - \alpha \Phi(A) \sharp_q \Phi(B) \geq \beta^0 \Phi(B)$$

holds for $\beta^0 = \beta\left(\frac{m_1}{M_2}, \frac{M_1}{m_2}, t^{1-p}, t^{1-q}, \alpha\right)$ which is defined just as above.

Proof. This corollary follows from Theorem 5.28 since the representing function of the p-power mean $\sharp_p$ and the q-power mean $\sharp_q$ are $f(t) = t^p$ and $g(t) = t^q$ respectively. □

Corollary 5.34 *Let $\Phi \in P[\mathscr{B}(H), \mathscr{B}(K)]$, $A, B \in \mathscr{B}^+(H)$ with $\mathrm{Sp}(A) \subseteq [m_1, M_1]$, $\mathrm{Sp}(B) \subseteq [m_2, M_2]$. Let $p, q \in (0,1)$ be given. Then*

$$\Phi(A \sharp_p B) \geq \max\{\alpha_1, \ \alpha_2\} \ \Phi(A) \sharp_q \Phi(B)$$

holds for

$$\alpha_1 = \begin{cases} \frac{1}{1-q}\frac{Mm^p - mM^p}{M-m}\left(\frac{1-q}{q}\frac{M^p - m^p}{Mm^p - mM^p}\right)^q & \text{if} \quad m \leq \frac{q}{1-q}\frac{Mm^p - mM^p}{M^p - m^p} \leq M, \\ \min\{m^{p-q}, M^{p-q}\} & \text{otherwise,} \end{cases}$$

where $m = \frac{m_2}{M_1}$ and $M = \frac{M_2}{m_1}$,
and

$$\alpha_2 = \begin{cases} \frac{1}{q}\frac{Mm^{1-p} - mM^{1-p}}{M-m}\left(\frac{q}{1-q}\frac{M^{1-p} - m^{1-p}}{Mm^{1-p} - mM^{1-p}}\right)^{1-q} & \text{if} \quad m \leq \frac{1-q}{q}\frac{Mm^{1-p} - mM^{1-p}}{M^{1-p} - m^{1-p}} \leq M, \\ \min\{m^{q-p}, M^{q-p}\} & \text{otherwise,} \end{cases}$$

where $m = \frac{m_1}{M_2}$ and $M = \frac{M_1}{m_2}$.

Remark 5.3 If we put $p = q$ in Corollary 5.33 we obtain converses of the inequality $\Phi(A \sharp_p B) \leq \Phi(A) \sharp_p \Phi(B)$. In particular, for $p = \frac{1}{2}$ we obtain the following converses of Ando inequality $\Phi(A \sharp B) \leq \Phi(A) \sharp \Phi(B)$:

$$\Phi(A) \sharp \Phi(B) \leq \frac{\sqrt{M_1 M_2} + \sqrt{m_1 m_2}}{2\sqrt[4]{m_1 m_2 M_1 M_2}} \Phi(A \sharp B),$$

$$\Phi(A) \sharp \Phi(B) - \Phi(A \sharp B) \leq \min\left\{\sqrt{\frac{M_1}{m_1}}, \sqrt{\frac{M_2}{m_2}}\right\} \frac{(\sqrt{M_1 M_2} - \sqrt{m_1 m_2})^2}{4(\sqrt{M_1 M_2} + \sqrt{m_1 m_2})} 1_K.$$

In the case when Φ is the identity map in Corollary 5.33, we obtain the estimation of the difference of the geometric interpolation.

Corollary 5.35 *Let $A, B \in \mathscr{B}^+(H)$ with $\mathsf{Sp}(A) \subseteq [m_1, M_1]$, $\mathsf{Sp}(B) \subseteq [m_2, M_2]$ and $m = m_2/M_1$, $M = M_2/m_1$. If $p, q \in (0,1)$, then*

$$-\beta' A \geq A \sharp_p B - A \sharp_q B \geq \beta A,$$

where $\beta = \beta(m, M, t^p, t^q, \alpha = 1)$ and $\beta' = \beta(m, M, t^q, t^p, \alpha = 1)$ are defined as in Corollary 5.33.

In the next corollary we give the estimation of the difference of two path $A \nabla_p B$ and $A \sharp_p B$:

Corollary 5.36 *Let A, B, M and m be as in Corollary 5.35. If $p \in (0,1)$, then*

$$\max\{1 - p + pm - m^p, 1 - p + pM - M^p\} A \geq A \nabla_p B - A \sharp_p B \geq 0.$$

Proof. It is obvious

$$x^p - (1 - p + px) \geq \begin{cases} m^p - (1 - p + pm), & \text{if} \quad p \leq \frac{M^p - m^p}{M - m}, \\ M^p - (1 - p + pM), & \text{if} \quad p \geq \frac{M^p - m^p}{M - m}. \end{cases}$$

If $\beta = \max\{1 - p + pm - m^p, 1 - p + pM - M^p\}$, then

$$\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^p - \left((1-p) + pA^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) \geq -\beta \, 1_H.$$

Now we have $A \sharp_p B - A \nabla_p B \geq -\beta A$. □

Remark 5.4 In the same way as above we have that for $\alpha \in \mathbb{R}_+$

$$\Phi(A \sharp B) - \alpha \Phi(A) \nabla \Phi(B) \geq \min left \sqrt{m} - \frac{\alpha(m+1)}{2}, \sqrt{M} - \frac{\alpha(M+1)}{2} right \Phi(A)$$

holds if $\Phi \in P[\mathscr{B}(H), \mathscr{B}(K)]$, $A, B \in \mathscr{B}^+(H)$ with $\mathsf{Sp}(A) \subseteq [m_1, M_1]$, $\mathsf{Sp}(B) \subseteq [m_2, M_2]$ and $m = m_2/M_1$, $M = M_2/m_1$.

Next, if we put $q = \frac{1}{2}$ in Corollary 5.33 and use the fact that the geometric mean is symmetric then we have the following corollary:

Corollary 5.37 *Let $\Phi \in P[\mathscr{B}(H), \mathscr{B}(K)]$, $A, B \in \mathscr{B}^+(H)$ with $\mathrm{Sp}(A) \subseteq [m_1, M_1]$, $\mathrm{Sp}(B) \subseteq [m_2, M_2]$ and $m = m_2/M_1$, $M = M_2/m_1$. If $p \in (0,1)$, then*

$$\Phi(A \sharp_p B) - \Phi(A) \sharp \Phi(B) \geq \beta \Phi(A),$$
$$\Phi(A \sharp_p B) \geq \alpha \Phi(A) \sharp \Phi(B)$$

hold for

$$\beta \equiv \beta(m,M,p) = \begin{cases} -\frac{1}{4}\frac{M-m}{M^p-m^p} + \frac{Mm^p-mM^p}{M-m}, & \text{if} \quad 2\sqrt{m} < \frac{M-m}{M^p-m^p} < 2\sqrt{M}, \\ \min\{M^p - \sqrt{M}, m^p - \sqrt{m}\} & \text{otherwise,} \end{cases}$$

$$\alpha = \begin{cases} 2\frac{\sqrt{(M^p-m^p)(Mm^p-mM^p)}}{M-m}, & \text{if} \quad m < \frac{Mm^p-mM^p}{M^p-m^p} < M, \\ \min\{m^{p-1/2}, M^{p-1/2}\} & \text{otherwise.} \end{cases}$$

Also

$$\Phi(B \sharp_p A) - \Phi(A) \sharp \Phi(B) \geq \beta \Phi(B),$$
$$\Phi(B \sharp_p A) \geq \alpha \Phi(A) \sharp \Phi(B)$$

hold for β and α which are defined just as above with $m = \frac{m_1}{M_2}$ and $M = \frac{M_1}{m_2}$.

Remark 5.5 It can be easily checked that for $\beta(m,M,p)$ from Corollary 5.37 holds: $\beta(m,M,p) < 0$ if $0 < p < \frac{1}{2}$ and $\beta(m,M,p) > 0$ if $\frac{1}{2} < p < 1$.

Corollary 5.38 *Let Φ, A, B, M and m be as in Corollary 5.37. If $q \in (0,1)$, then*

$$\Phi(A \,!\, B) - \Phi(A) \sharp_q \Phi(B) \geq \beta \Phi(A),$$
$$\Phi(A \,!\, B) \geq \alpha_1 \Phi(A) \sharp_q \Phi(B)$$

hold for

$$\beta = \begin{cases} \alpha(q-1)\left[\frac{q(1+m)(1+M)}{2}\right]^{\frac{q}{1-q}} + \frac{2Mm}{(1+m)(1+M)} \\ \qquad\qquad \text{if} \quad m^{1-q}/q < \frac{\alpha(1+m)(1+M)}{2} < M^{1-q}/q, \\ \min\{\frac{2M}{1+M} - M^q, \frac{2m}{1+m} - m^q\} \quad \text{otherwise,} \end{cases}$$

$$\alpha_1 = \begin{cases} \frac{2}{q(1+M)(1+m)}\left(\frac{q}{1-q}Mm\right)^{1-q} & \text{if} \quad m < \frac{1-q}{q} < M, \\ \min\{\frac{2m^{1-q}}{1+m}, \frac{2M^{1-q}}{1+M}\} & \text{otherwise.} \end{cases}$$

Also

$$\Phi(A \,!\, B) - \alpha \Phi(B) \sharp_q \Phi(A) \geq \beta \Phi(B),$$
$$\Phi(A \,!\, B) \geq \alpha_1 \Phi(B) \sharp_q \Phi(A)$$

hold for β and α_1 which are defined just as above with $m = \frac{m_1}{M_2}$ and $M = \frac{M_1}{m_2}$.

Proof. If we put $\sigma = !$ and $\tau = \sharp_q$ in Theorem 5.28 and Corollary 5.29, then we have this corollary, since the representing functions of the harmonic mean ! is $f(t) = 2t/(1+t)$ and the harmonic mean is symmetric. □

By virtue of Corollary 5.38, we can obtain the converse of $\Phi(A!B) \leq \Phi(A\sharp B) \leq \Phi(A)\sharp\Phi(B)$.

Corollary 5.39 *Let $\Phi \in P[\mathscr{B}(H), \mathscr{B}(K)]$, $A, B \in \mathscr{B}^+(H)$ with $\mathsf{Sp}(A) \subseteq [m_1, M_1]$, $\mathsf{Sp}(B) \subseteq [m_2, M_2]$ and $m = m_2/M_1$, $M = M_2/m_1$. If $\alpha \in \mathbb{R}_+$, then*

$$\Phi(A \, ! \, B) - \alpha\Phi(A) \sharp \Phi(B) \geq \beta\Phi(A)$$

holds for $\beta = \beta(m, M, \alpha) =$

$$\begin{cases} -\alpha^2 \frac{(1+m)(1+M)}{8} + \frac{2Mm}{(1+m)(1+M)} & \text{if} \quad \sqrt{m} < \frac{\alpha(1+m)(1+M)}{4} < \sqrt{M}, \\ \min\left\{ \frac{2M}{1+M} - \alpha\sqrt{M}, \frac{2m}{1+m} - \alpha\sqrt{m} \right\} & \text{otherwise.} \end{cases}$$

In particular,

$$\Phi(A \, ! \, B) \geq \alpha_1 \Phi(A) \sharp \Phi(B)$$

holds for

$$\alpha_1 = \begin{cases} \frac{4\sqrt{m_1 m_2 M_1 M_2}}{(M_2+m_1)(M_1+m_2)} & \text{if} \quad m_1 < M_2 \quad \text{and} \quad m_2 < M_1, \\ \min\left\{ \frac{2\sqrt{m_2 M_1}}{m_2+M_1}, \frac{2\sqrt{m_1 M_2}}{m_1+M_2} \right\} & \text{otherwise,} \end{cases}$$

and

$$\Phi(A \, ! \, B) - \Phi(A) \sharp \Phi(B) \geq \beta_1 1_K \tag{5.29}$$

holds for

$$\beta_1 = \max\{\frac{1}{m_1}, \frac{1}{m_2}\} \left[-\frac{(M_2+m_1)(M_1+m_2)}{8} + \frac{2M_1 m_1 M_2 m_2}{(M_2+m_1)(M_1+m_2)} \right].$$

Proof. If we put $q = 1/2$ in Corollary 5.38 then we have the first two inequalities. Now, we prove the inequality (5.29). If we put $\alpha = 1$ in the first inequality then we have $\beta(m, M, 1) \leq \mu_f m + \nu_f - g(m) = 2m/(1+m) - \sqrt{m} < 0$ where $f(t) = 2t/(1+t)$ and $g(t) = \sqrt{t}$. We denote $\beta_2(m, M) = -\frac{(1+m)(1+M)}{8} + \frac{2}{(1+m)(1+M)}$. Then

$$\beta_2 \equiv \beta_2(m_2/M_1, M_2/m_1) = \frac{1}{m_1 M_1} \left(-\frac{(M_2+m_1)(M_1+m_2)}{8} + \frac{2M_1 m_1 M_2 m_2}{(M_2+m_1)(M_1+m_2)} \right)$$

and $0 > \beta(m, M, 1) \geq \beta_2$. As following we have $\Phi(A \, ! \, B) - \Phi(A) \sharp \Phi(B) \geq \beta_2 \Phi(A) \geq \beta_2 M_1$. Similarly we have $\Phi(A \, ! \, B) - \Phi(A) \sharp \Phi(B) \geq \beta_3 \Phi(B) \geq \beta_3 M_2$, for $\beta_3 = \beta_2(m_1/M_2, M_1/m_2) = \frac{1}{m_2 M_2} \left(-\frac{(M_2+m_1)(M_1+m_2)}{8} + \frac{2M_1 m_1 M_2 m_2}{(M_2+m_1)(M_1+m_2)} \right)$. Combining these two inequalities we obtain the desired inequality:

$$\Phi(A \, ! \, B) - \Phi(A) \sharp \Phi(B) \geq max_{\{}\beta_2 M_1, \beta_3 M_2\} 1_H = \beta_1 \, 1_K.$$

□

Remark 5.6 If we put $\sigma = \tau = \,!$ in Theorem 5.28 then we obtain converses of Ando inequality $\Phi(A\,!\,B) \le \Phi(A)\,!\,\Phi(B)$, which are proved directly in [130, Corollary 3.8].

In the same way we can obtain inequalities for the weighted harmonic mean $!_\lambda$, except the case when we need the condition of symmetric mean.

5.5 Mixed operator means

In this section we shall give inequalities for mixed operator means based on the inequality (5.19) in Theorem 5.28 and on the following simple inequalities for mixed operator means of type (5.10) in § 5.1.

Lemma 5.40 *Let $\Phi \in \mathbf{P}[\mathscr{B}(H),\mathscr{B}(K)]$, $A,B \in \mathscr{B}^{++}(H)$ and σ be an operator mean. If $\lambda,\mu \in (0,1)$, then*

$$\Phi(A)\;\sigma\;(\Phi(A)\;\nabla_\lambda\;\Phi(B)) \;\ge\; \Phi(A)\;\nabla_\lambda\;(\Phi(A)\;\sigma\;\Phi(B)), \tag{5.30}$$

$$\Phi(A)^{-1}\;\sigma\;\left(\Phi(A)^{-1}\;!_\lambda\;\Phi(B)^{-1}\right) \;\le\; \Phi(A)^{-1}\;!_\lambda\;\left(\Phi(A)^{-1}\;\sigma\;\Phi(B)^{-1}\right). \tag{5.31}$$

In particular,

$$\begin{aligned}
\Phi(A)\;\sharp_\mu\;(\Phi(A)\;\nabla_\lambda\;\Phi(B)) &\ge \Phi(A)\;\nabla_\lambda\;\left(\Phi(A)\;\sharp_\mu\;\Phi(B)\right),\\
\Phi(A)^{-1}\;\sharp_\mu\;\left(\Phi(A)^{-1}\;!_\lambda\;\Phi(B)^{-1}\right) &\le \Phi(A)^{-1}\;!_\lambda\;\left(\Phi(A)^{-1}\;\sharp_\mu\;\Phi(B)^{-1}\right),\\
\Phi(A)\;!_\mu\;(\Phi(A)\;\nabla_\lambda\;\Phi(B)) &\ge \Phi(A)\;\nabla_\lambda\;\left(\Phi(A)\;!_\mu\;\Phi(B)\right).
\end{aligned}$$

Proof. By homogenity and subadditivity of the operator mean, we have

$$\begin{aligned}
\Phi(A)\;\sigma\;(\Phi(A)\;\nabla_\lambda\;\Phi(B)) &= (\lambda\Phi(A)+(1-\lambda)\Phi(A))\;\sigma\;(\lambda\Phi(A)+(1-\lambda)\Phi(B))\\
&\ge \lambda\,(\Phi(A)\;\sigma\;\Phi(A))+(1-\lambda)\,(\Phi(A)\;\sigma\;\Phi(B)) = \Phi(A)\;\nabla_\lambda\;(\Phi(A)\;\sigma\;\Phi(B)),
\end{aligned}$$

that proves (5.30). If we replace σ by the adjoint mean σ^* in (5.30) then

$$\Phi(A)\;\sigma^*\;(\Phi(A)\;\nabla_\lambda\;\Phi(B)) \ge \Phi(A)\;\nabla_\lambda\;(\Phi(A)\;\sigma^*\;\Phi(B)).$$

By using the fact that the function $t \to -t^{-1}$ is operator monotone on $(0,\infty)$, we have

$$\begin{aligned}
\left(\Phi(A)^{-1}\;\sigma\;(\Phi(A)\;\nabla_\lambda\;\Phi(B))^{-1}\right)^{-1} &\ge \Phi(A)\;\nabla_\lambda\;\left(\Phi(A)^{-1}\;\sigma\;\Phi(B)^{-1}\right)^{-1}\\
&= \left(\Phi(A)^{-1}\;!_\lambda\;\left(\Phi(A)^{-1}\;\sigma\;\Phi(B)^{-1}\right)\right)^{-1},
\end{aligned}$$

that proves (5.31).

If we replace σ by $\sharp_\mu$ in (5.30) and (5.31), and we replace σ by $!_\mu$ in (5.30), then we obtain the remainder. □

In the next theorem we show the converse of the inequality (5.30).

Theorem 5.41 *Suppose that any two operator means σ and τ have representing functions f and g respectively. Let $\Phi \in P[\mathscr{B}(H), \mathscr{B}(K)]$, $A, B \in \mathscr{B}^+(H)$ with $\mathsf{Sp}(A) \subseteq [m_1, M_1]$, $\mathsf{Sp}(B) \subseteq [m_2, M_2]$. Let $\lambda \in (0,1)$. Then for a given $\alpha \in \mathbb{R}_+$*

$$\Phi(A\ \sigma\ (A\ \nabla_\lambda\ B)) \geq \alpha\ \Phi(A)\ \nabla_\lambda\ (\Phi(A)\ \tau\ \Phi(B)) + \beta\ \Phi(A), \tag{5.32}$$

holds for $\beta = \mu_f t_o + \nu_f - \alpha g(t_o)$ and $t_o \in [m_\lambda, M_\lambda]$ is defined as the unique solution of $g'(t) = \mu_f/\alpha$ when $g'(M_\lambda) < \mu_f/\alpha < g'(m_\lambda)$, otherwise t_0 is defined as M_λ or m_λ according to $\mu_f/\alpha \leq g'(M_\lambda)$ or $g'(m_\lambda) \leq \mu_f/\alpha$, where $m_\lambda = \frac{\lambda m_1 + (1-\lambda)m_2}{M_1}$ and $M_\lambda = \frac{\lambda M_1 + (1-\lambda)M_2}{m_1}$ and

$$\Phi(A\ \sigma\ (A\ \nabla_\lambda\ B)) \geq \alpha\Phi(A)\ \nabla_\lambda\ (\Phi(A)\ \tau\ \Phi(B)) + \beta^0\Phi(A\ \nabla_\lambda\ B)$$

holds for β^0 which is defined just as above with $m_\lambda = \frac{m_1}{\lambda M_1 + (1-\lambda)M_2}$, $M_\lambda = \frac{M_1}{\lambda m_1 + (1-\lambda)m_2}$ and f^0, g^0.

Proof. If we replace B by $(A\ \nabla_\lambda\ B)$ in Theorem 5.28, then

$$\begin{aligned} \Phi(A\ \sigma\ (A\ \nabla_\lambda\ B)) &\geq \alpha\ \Phi(A)\ \tau\ \Phi(A \nabla_\lambda B) + \beta\Phi(A) \\ &= \alpha\ \Phi(A)\ \tau\ (\Phi(A) \nabla_\lambda \Phi(B)) + \beta\Phi(A), \end{aligned}$$

holds for β as above. It implies by (5.30) that

$$\alpha\Phi(A)\ \tau\ (\Phi(A)\nabla_\lambda\Phi(B)) + \beta\Phi(A) \geq \alpha\Phi(A)\ \nabla_\lambda\ (\Phi(A)\ \tau\ \Phi(B)) + \beta\Phi(A).$$

Combining these two inequalities we obtain

$$\Phi(A\ \sigma\ (A\ \nabla_\lambda\ B)) \geq \alpha\Phi(A)\ \nabla_\lambda\ (\Phi(A)\ \tau\ \Phi(B)) + \beta\Phi(A).$$

On the other hand, if we replace σ, τ by the transpose σ^0, τ^0, then we have that

$$\Phi\left((A\ \nabla_\lambda\ B)\,\sigma^0\ A\right) \geq \alpha\Phi(A\ \nabla_\lambda\ B)\ \tau^0\ \Phi(A) + \beta^0\Phi(A\nabla_\lambda\ B)$$

holds for β^0 as above. By Lemma 5.40, we have

$$\Phi(A\ \sigma\ (A\ \nabla_\lambda\ B)) \geq \alpha\Phi(A)\ \nabla_\lambda\ (\Phi(A)\ \tau\ \Phi(B)) + \beta^0\Phi(A\ \nabla_\lambda\ B).$$

□

Corollary 5.42 *Let $\Phi, \sigma, \tau, A, B, m_1, M_1, m_2, M_2$ and λ be as in Theorem 5.41. Then* $\Phi(A\ \sigma\,(A\ \nabla_\lambda B)) \geq$

$$\max\left\{ \min_{m_\lambda \leq t \leq M_\lambda} \left\{ \frac{\mu_f t + \nu_f}{g(t)} \right\},\ \min_{m_\lambda^0 \leq t \leq M_\lambda^0} \left\{ \frac{a_{f^0} t + b_{f^0}}{g^0(t)} \right\} \right\} \Phi(A)\ \nabla_\lambda\ (\Phi(A)\ \tau\ \Phi(B)),$$

where $m_\lambda = \frac{\lambda m_1 + (1-\lambda)m_2}{M_1}$, $M_\lambda = \frac{\lambda M_1 + (1-\lambda)M_2}{m_1}$, $m_\lambda^0 = \frac{m_1}{\lambda M_1 + (1-\lambda)M_2}$ and $M_\lambda^0 = \frac{M_1}{\lambda m_1 + (1-\lambda)m_2}$.

A value of $\alpha = \min_{m_\lambda \le t \le M_\lambda}\{\frac{\mu_f t + \nu_f}{g(t)}\}$ *is given by*

$$\alpha = \frac{\mu_f t_0 + \nu_f}{g(t_0)},$$

where $t_0 \in [m_\lambda, M_\lambda]$ *is defined as the unique solution of* $\mu_f g(t) = g'(t)(\mu_f t + \nu_f)$ *when* $f(M_\lambda)g'(M_\lambda)/g(M_\lambda) \le \mu_f \le f(m_\lambda)g'(m_\lambda)/g(m_\lambda)$*, otherwise* t_0 *is defined as* M_λ *or* m_λ *according to* $\mu_f \le f(M)g'(M)/g(M)$ *or* $f(m)g'(m)/g(m) \le \mu_f$*.*

Proof. This corollary follows from Corollary 5.32 and the inequality (5.30). □

If we put $\sigma = \sharp_p$ or $\sigma = \,!$ and $\tau = \sharp_p$ in Theorem 5.41 and Corollary 5.42, then we have the next corollary:

Corollary 5.43 *Let* $\Phi, A, B, m_1, M_1, m_2, M_2$ *and* λ *be as in Theorem 5.41. Let* $q, p \in (0,1)$*. Then for a given* $\alpha \in \mathbb{R}_+$

$$\Phi(A \sharp_p (A \nabla_\lambda B)) \ge \alpha\, \Phi(A) \nabla_\lambda (\Phi(A) \sharp_q \Phi(B)) + \beta_1 \Phi(A),$$
$$\Phi(A \,!\, (A \nabla_\lambda B)) \ge \alpha\, \Phi(A) \nabla_\lambda (\Phi(A) \sharp_q \Phi(B)) + \beta_2 \Phi(A)$$

hold for

$$\beta_1 = \begin{cases} \alpha(q-1)\,(\alpha q/\mu_{t^p})^{\frac{q}{1-q}} + \nu_{t^p} & \text{if } \; m^{1-q}/q \le \alpha/\mu_{t^p} \le M^{1-q}/q, \\ \min\{M^p - \alpha M^q, m^p - \alpha m^q\} & \text{otherwise,} \end{cases}$$

$$\beta_2 = \begin{cases} \alpha^{\frac{1}{1-q}}(q-1)\left[\frac{q(1+m)(1+M)}{2}\right]^{\frac{q}{1-q}} + \frac{2Mm}{(1+m)(1+M)} \\ \qquad\qquad \text{if } \; \frac{m^{1-q}}{q} \le \frac{\alpha(1+m)(1+M)}{2} \le \frac{M^{1-q}}{q}, \\ \min\left\{\frac{2M}{1+M} - \alpha M^q, \frac{2m}{1+m} - \alpha m^q\right\} \quad \text{otherwise,} \end{cases}$$

where $m = m_\lambda = \frac{\lambda m_1 + (1-\lambda)m_2}{M_1}$ *and* $M = M_\lambda = \frac{\lambda M_1 + (1-\lambda)M_2}{m_1}$.

In particular,

$$\Phi(A \sharp_p (A \nabla_\lambda B)) \ge \alpha_1 \Phi(A) \nabla_\lambda (\Phi(A) \sharp_q \Phi(B)),$$
$$\Phi(A \,!\, (A \nabla_\lambda B)) \ge \alpha_2 \Phi(A) \nabla_\lambda (\Phi(A) \sharp_q \Phi(B))$$

hold for

$$\alpha_1 = \begin{cases} K(m,M,p,q) & \text{if } \; m < \frac{q}{1-q}\nu_{t^p}/\mu_{t^p} < M, \\ \min\{m^{p-q}, M^{p-q}\} & \text{otherwise,} \end{cases}$$

$$\alpha_2 = \begin{cases} \frac{2}{q(1+M)(1+m)}\left(\frac{q}{1-q}Mm\right)^{1-q} & \text{if } \; m \le \frac{1-q}{q} \le M, \\ \min\left\{\frac{2m^{1-q}}{1+m}, \frac{2M^{1-q}}{1+M}\right\} & \text{otherwise.} \end{cases}$$

5.6 Notes

The theory of operator means for positive operators on a Hilbert space is established by Kubo and Ando [121] and we refer to Hiai and Yanagi [106] and J.I.Fujii [36].

The relative operator entropy as a generalization of operator means which is called solidarities, appeared in the works of J.I.Fujii and Kamei [46], J.I.Fujii [36] and Kamei [112]. Further topics associated with the relative operator entropy are [35], [41], [79], [86] and [9].

Topics associated with the relative entropy are Araki [10], Pusz-Woronowicz [172], Hiai-Petz [104] and Ohya-Petz [164]. Uhlmann [191] discussed the quadratic interpolation and introduced the relative entropy for states on an operator algebra. His quadratic interpolation is reduced to a path generated by the geometric mean and the relative entropy is the derivative of this path. J.I.Fujii and Kamei [47, 48] introduced interpolational paths generated by an operator mean based on Uhlman's method. Further topics are [113], [42] and [38].

Chapter 6

Inequalities on the Hadamard product

In this chapter, we discuss complemetary results to Jensen's type inequalities on the Hadamard product of positive operators on a Hilbert space, which is based on the Mond-Pečarić method. As a result, we extend a theorem by Liu and Neudecker and moreover show Hadamard product versions of operator inequalities associated with extensions of Hölder-McCarthy and Kantorovich inequalities.

6.1 Preliminaries

We discuss several fundamental inequalities on the Hadamard product. The Hadamard product is expressed as the deformation of the tensor product, which is one of the most powerful tools for the study of the Hadamard product of operators on a separable Hilbert space.

Definition 6.1 *Let $\{e_j\}$ be an orthogonal basis of a Hilbert space H and $A \otimes B$ be tensor product of operators A and B on H regarding to $\{e_j\}$. Let $U : H \to H \otimes H$ be the isometry such that $Ue_j = e_j \otimes e_j$.* **The Hadamard product** *$A \circ B$ regarding to $\{e_j\}$ is expressed as*

$$A \circ B = U^*(A \otimes B)U. \tag{6.1}$$

In the finite dimension case if operators A and B have the matrices $\mathbf{A} = [a_{ij}] \in \mathscr{M}_n$ and $\mathbf{B} = [b_{ij}] \in \mathscr{M}_n$ regarding to same basis, then the Hadamard product $A \circ B$ has an associated matrix $\mathbf{A} \circ \mathbf{B} = [a_{ij} b_{ij}] \in \mathscr{M}_n$ and tensor product $A \otimes B$ has an associated matrix $\mathbf{A} \otimes \mathbf{B} = [a_{ij}\mathbf{B}] \in \mathscr{M}_{n^2}$ (sometimes called the Kronecker product).

The following formulas for tensor products are well known:

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$$

and

$$(A \otimes B)^* = A^* \otimes B^*.$$

Since $1_H \otimes 1_H = 1_H$,

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$

whenever both A and B are invertible. They imply that if both A and B are positive operators, so is their tensor product. More generally, $A_1 \geq A_2 \geq 0$ and $B_1 \geq B_2 \geq 0$ imply $A_1 \otimes B_1 \geq A_2 \otimes B_2 \geq 0$. Moreover, we obtain that if A_j and B_j are positive invertible operators ($j = 1, 2$), then

$$(A_1 \otimes B_1) \sharp_\alpha (A_2 \otimes B_2) = (A_1 \sharp_\alpha A_2) \otimes (B_1 \sharp_\alpha B_2)$$

for all $\alpha \in [0,1]$.

Now, the Hadamard product differs from the usual product in many ways. The most important is commutativity of Hadamard multiplication:

$$A \circ B = B \circ A.$$

The diagonal operator formed from an operator A can be obtained by Hadamard multiplication with the identity operator and the following holds

$$(A \circ B) \circ 1_H = (A \circ 1_H)(B \circ 1_H).$$

The first application is the following theorem of Schur.

Theorem 6.1 (SCHUR) *If A and B are positive operators on a Hilbert space, then the Hadamard product $A \circ B$ is also positive. More generally, $A_1 \geq A_2 \geq 0$ and $B_1 \geq B_2 \geq 0$ imply $A_1 \circ B_1 \geq A_2 \circ B_2 \geq 0$.*

Proof. It easily follows that if X and Y are positive and commutes, then XY is positive. Since $A \otimes 1_H$ and $1_H \otimes B$ commutes, it follows that $A \otimes B = (A \otimes 1_H)(1_H \otimes B)$ is positive and hence by Definition 6.1 the Hadamard product $A \circ B$ is positive. Also, the monotonicity of the Hadamard product follows from the monotonicity of the tensor product and Definition 6.1. $\square$

Definition 6.2 *Let I be an interval in $\mathbb{R}$. A function $f \in \mathscr{C}(I)$ is* **super-multiplicative** *on I if $f(xy) \geq f(x)f(y)$ for every $x, y \in I$. If the inequality is opposite then f is* **sub-multiplicative** *on I.*

Lemma 6.2 *If a function $f \in \mathscr{C}(I)$ is super-multiplicative (resp. sub-multiplicative) on $[0,\infty)$, then*

$$f(A \otimes B) \geq f(A) \otimes f(B) \quad (resp. \quad f(A \otimes B) \leq f(A) \otimes f(B))$$

for every positive operator A and B.

Proof. Let $A = \int \lambda dE_\lambda$ and $B = \int \mu dF_\mu$ be the spectral decompositions. Then it follows that

$$\begin{aligned} f(A \otimes B) &= \int\int f(\lambda\mu) dE_\lambda \otimes F_\mu \\ &\geq \left(\int f(\lambda) dE_\lambda\right) \otimes \left(\int f(\mu) dF_\mu\right) = f(A) \otimes f(B). \end{aligned}$$

□

We have the following inequality of Jensen's type for the Hadamard product.

Theorem 6.3 *Let A, $B \in \mathscr{B}^+(H)$ be positive operators and $\Phi \in \mathbf{P}_N[\mathscr{B}(H),\mathscr{B}(K)]$ a normalized positive linear map. If f is a sub-multiplicative operator convex function (resp. a super-multiplicative operator concave function) on $(0,\infty)$, then*

$$f(\Phi(A \circ B)) \leq \Phi(f(A) \circ f(B)) \quad (resp.\ f(\Phi(A \circ B)) \geq \Phi(f(A) \circ f(B))). \tag{6.2}$$

In particular,

$$f(A \circ B) \leq f(A) \circ f(B) \quad (resp.\ f(A \circ B) \geq f(A) \circ f(B)). \tag{6.3}$$

Proof. We show the sub-multiplicative operator convex case only. By Lemma 6.2 and Theorem 1.20 (Davis-Choi-Jensen's inequality), we have

$$\begin{aligned} f(A \circ B) &= f(U^*(A \otimes B)U) \leq U^* f(A \otimes B) U \\ &\leq U^*(f(A) \otimes f(B))U = f(A) \circ f(B). \end{aligned}$$

It follows from Davis-Choi-Jensen's inequality again that

$$f(\Phi(A \circ B)) \leq \Phi(f(A \circ B)) \leq \Phi(f(A) \circ f(B)).$$

□

Since the power function is sub-multiplicative and super-multiplicative, we have the following Hölder-McCarthy type inequality on the Hadamard product.

Corollary 6.4 *Let A, $B \in \mathscr{B}^{++}(H)$ be positive invertible operators and $\Phi \in \mathbf{P}_N[\mathscr{B}(H),\mathscr{B}(K)]$ a normalized positive linear map. Then*

(i) $(\Phi(A \circ B))^p \leq \Phi(A^p \circ B^p)$ *if* $-1 \leq p \leq 0$ *or* $1 \leq p \leq 2$.

(ii) $(\Phi(A \circ B))^p \geq \Phi(A^p \circ B^p)$ *if* $0 \leq p \leq 1$.

Proof. By Theorem 6.3, the proof depends upon the facts that the function $f(t) = t^p$ is operator convex on $(0,\infty)$ for $-1 \le p \le 0$ or $1 \le p \le 2$ and is operator concave on $(0,\infty)$ for $0 \le p \le 1$ by Corollary 1.16 in § 1.2. □

Corollary 6.5 *Let A, B and Φ be as in Corollary 6.4. Then*

$$(\Phi(A^r \circ B^r))^{\frac{1}{r}} \le (\Phi(A^s \circ B^s))^{\frac{1}{s}}$$

for $r \le s$ with (i) $r, s \notin (-1,1)$, (ii) $1/2 \le r \le 1 \le s$ and (iii) $r \le -1 \le s \le -1/2$.

Proof. Assume that $1 \le r \le s$. Then by (ii) of Corollary 6.4, we have

$$\Phi(A^r \circ B^r) \le (\Phi(A^s \circ B^s))^{\frac{r}{s}}.$$

Since $1 \le r$, the above inequality implies

$$(\Phi(A^r \circ B^r))^{\frac{1}{r}} \le (\Phi(A^s \circ B^s))^{\frac{1}{s}}$$

by the Löwner-Heinz inequality. The remainders of the proof are similar to that of Theorem 4.1. □

Next, we show several inequalities for the Hadamard product involving operator means.

Theorem 6.6 *If A and B are positive operators, then*

$$A \circ B \ge (A \sharp_\alpha B) \circ (A \sharp_{1-\alpha} B) \quad \text{and} \quad A \circ B \ge (A \,!_{1-\alpha}\, B) \circ (A \nabla_\alpha B) \tag{6.4}$$

for all $\alpha \in [0,1]$.

Proof. By the commutativity of the Hadamard product, we have $A \circ B = (A \circ B) \sharp_\alpha (B \circ A)$. Then it follows from the transformer inequality that

$$\begin{aligned} A \circ B &= U^*(A \otimes B)U \sharp_\alpha U^*(B \otimes A)U \ge U^*(A \otimes B) \sharp_\alpha (B \otimes A)U \\ &= U^*(A \sharp_\alpha B) \otimes (B \sharp_\alpha A)U = (A \sharp_\alpha B) \circ (B \sharp_\alpha A) \\ &= (A \sharp_\alpha B) \circ (A \sharp_{1-\alpha} B). \end{aligned}$$

Next, since it is easily seen from the definition of the geometric mean that $(XY^{-1}X) \sharp Y = X$ for any positive operators X and Y, we have

$$\begin{aligned} & A\{(1-\alpha)A + \alpha B\}A \circ \{(1-\alpha)A + \alpha B\} \\ \ge & \left(A\{(1-\alpha)A + \alpha B\}^{-1}A \sharp \{(1-\alpha)A + \alpha B\}\right) \\ & \qquad \circ \left(A\{(1-\alpha)A + \alpha B\}^{-1}A \sharp \{(1-\alpha)A + \alpha B\}\right) \\ = & A \circ A. \end{aligned}$$

Since $\left(\alpha A^{-1}+\alpha^{-1}(1-\alpha)B^{-1}\right)\left(A-(1-\alpha)A\{(1-\alpha)A+\alpha B\}^{-1}A\right)=1_H$, we have

$$\begin{aligned}
&\{\alpha A^{-1}+(1-\alpha)B^{-1}\}^{-1}\circ\{(1-\alpha)A+\alpha B\}\\
=\ &\alpha^{-1}\{A-(1-\alpha)A\{(1-\alpha)A+\alpha B\}^{-1}A\}\circ\{(1-\alpha)A+\alpha B\}\\
=\ &\alpha^{-1}(1-\alpha)A\circ A+A\circ B\\
&\qquad\qquad -\alpha^{-1}(1-\alpha)A\{(1-\alpha)A+\alpha B\}^{-1}A\circ\{(1-\alpha)A+\alpha B\}\\
\le\ &\alpha^{-1}(1-\alpha)A\circ A+A\circ B-\alpha^{-1}(1-\alpha)A\circ A=A\circ B
\end{aligned}$$

as desired. □

Recall that the transpose σ^0 with f^0 is defined by

$$A\ \sigma^0\ B=B\ \sigma\ A \qquad \text{and} \qquad f^0(t)=tf\left(\frac{1}{t}\right)$$

for an operator mean σ with the representing function f.

Remark 6.1 *The first expression of (6.4) in Theorem 6.6 is restated as follows.*

$$A\circ B\ge(A\ \sharp_\alpha\ B)\circ(A\ (\sharp_\alpha)^0\ B).$$

We have the following extension by virtue of the transpose of the operator mean. If σ is an operator mean with a supermultiplicative representing function f, then

$$A\circ B\ge(A\ \sigma\ B)\circ(A\ \sigma^0\ B)$$

holds for positive operators A and B. As a matter of fact, it follows from the transformer inequality that

$$\begin{aligned}
A\circ B &= (A\circ B)\ \sigma\ (B\circ A)=U^*(A\otimes B)U\ \sigma\ U^*(B\otimes A)U\\
&\ge U^*(A\otimes B\ \sigma\ B\otimes A)U.
\end{aligned}$$

Put $X=A^{-1/2}BA^{-1/2}$ and $Y=B^{-1/2}AB^{-1/2}$, then we have

$$\begin{aligned}
(A\otimes B)\ \sigma\ (B\otimes A) &= (A\otimes B)^{\frac12}f(X\otimes Y)(A\otimes B)^{\frac12}\\
&\ge (A\otimes B)^{\frac12}f(X)\otimes f(Y)(A\otimes B)^{\frac12}\\
&= \left(A^{\frac12}f(X)A^{\frac12}\right)\otimes\left(B^{\frac12}f(Y)B^{\frac12}\right)\\
&= (A\ \sigma\ B)\otimes(B\ \sigma\ A)
\end{aligned}$$

by the super-multiplicativity of f. Hence we have

$$\begin{aligned}
A\circ B &\ge U^*(A\ \sigma\ B)\otimes(B\ \sigma\ A)U\\
&= (A\ \sigma B)\circ(B\ \sigma\ A)=(A\ \sigma B)\circ(A\ \sigma^0\ B).
\end{aligned}$$

Aujla and Vasudeva show the following inequality involving the Hadamard product and the operator mean, which is an extension of results due to Ando and Fiedler.

$$(A \circ B) \sharp (C \circ D) \geq (A \sharp C) \circ (B \sharp D). \tag{6.5}$$

The following theorem is a generalization of (6.5).

Theorem 6.7 *If σ is an operator mean with a super-multiplicative representing function f, then*

$$(A \,\sigma\, C) \circ (B \,\sigma\, D) \leq (A \circ B) \,\sigma\, (C \circ D) \tag{6.6}$$

holds for $A, B, C, D \in \mathscr{B}^+(H)$.

Proof. Putting $X = A^{-1/2}CA^{-1/2}$ and $Y = B^{-1/2}DB^{-1/2}$, it follows from Lemma 6.2 that

$$\begin{aligned}
(A \,\sigma\, C) \otimes (B \,\sigma\, D) &= (A \otimes B)^{1/2}(f(X) \otimes f(Y))(A \otimes B)^{1/2} \\
&\leq (A \otimes B)^{1/2}(f(X \otimes Y))(A \otimes B)^{1/2} \\
&= (A \otimes B) \,\sigma\, (C \otimes D).
\end{aligned}$$

So the transformer inequality shows

$$\begin{aligned}
(A \,\sigma\, C) \circ (B \,\sigma\, D) &= U^*((A \,\sigma\, C) \otimes (B \,\sigma\, D))U = U^*((A \otimes B) \,\sigma\, (C \otimes D))U \\
&\leq U^*(A \otimes B)U \,\sigma\, U^*(C \otimes D)U = (A \circ B) \,\sigma\, (C \circ D).
\end{aligned}$$

□

As an application of Theorem 6.6, we have the following Fiedler inequality .

Theorem 6.8 (FIEDLER INEQUALITY) *If A is a positive invertible operator on a Hilbert space H, then*

$$A \circ A^{-1} \geq 1_H$$

Proof. By Theorem 6.6, we have

$$A \circ A^{-1} \geq (A \sharp A^{-1}) \circ (A \sharp A^{-1}) = 1_H \circ 1_H = 1_H.$$

□

By virture of Theorem 6.7, we show an extension of the Fiedler inequality for operators.

Corollary 6.9 *If $a, b \in \mathbb{R}$ and $t + s = 1$ for nonnegative numbers t and s, then*

$$A^a \circ A^b \geq A^{ta+sb} \circ A^{sa+tb}$$

for every positive operator A. In particular,

$$A \circ A^{-1} \geq 1_H.$$

Proof. In Theorem 6.7, replacing both A and D by A^a, both B and C by A^b and applying the operator mean with the representing function $f(t) = t^s$, we have

$$\begin{aligned} A^a \circ A^b &= (A^a \circ A^b) \natural_s (A^b \circ A^a) \\ &\geq (A^a \natural_s A^b) \circ (A^b \natural_s A^a) \\ &= A^{ta+sb} \circ A^{sa+tb}. \end{aligned}$$

If we put $a = 1$, $b = -1$ and $t = s = \frac{1}{2}$ in the expression above, then we have the Fiedler inequality. □

Theorem 6.10 *If f is a super-multiplicative nonnegative operator monotone function on $(0,\infty)$, then*

$$f(A) \circ f^0(B) \leq (B \circ 1_H) f\left((A \circ 1_H)(B \circ 1_H)^{-1}\right) \tag{6.7}$$

for every positive invertible operators A and B.

Proof. Let σ be the operator mean corresponding to f. Since both $A \circ 1_H$ and $B \circ 1_H$ are diagonal operators and hence commutes, it follows from Theorem 6.7 that

$$\begin{aligned} f(A) \circ f^0(B) &= (1_H \ \sigma \ A) \circ (B \ \sigma \ 1_H) \leq (1_H \circ B) \ \sigma \ (A \circ 1_H) \\ &= (B \circ 1_H) \ \sigma \ (A \circ 1_H) = (B \circ 1_H) f\left((A \circ 1_H)(B \circ 1_H)^{-1}\right). \end{aligned}$$

□

Corollary 6.11 *If $r + s = 1$ for nonnegative numbers r and s, then*

$$A^r \circ B^s \leq (A \circ 1_H)^r (B \circ 1_H)^s$$

for every positive operator A and B.

Proof. If we put $f(t) = t^r$ in Theorem 6.10, then we have this corollary. □

By Corollary 6.11, we have the following Hölder's inequality for the Hadamard product, which gives an estimate by the diagonal operators.

Corollary 6.12

$$A \circ B \leq (A^r \circ 1_H)^{\frac{1}{r}} (B^s \circ 1_H)^{\frac{1}{s}} \tag{6.8}$$

for $r, s \geq 1$ with $\frac{1}{r} + \frac{1}{s} = 1$.

Next, we consider the Kantorovich inequality on the Hadamard product.

Lemma 6.13 *Let A and B be positive operators such that $0 < m_1 1_H \leq A \leq M_1 1_H$ and $0 < m_2 1_H \leq B \leq M_2 1_H$. Then*

$$\frac{M_2}{m_1}(A \otimes B^{-1}) + \frac{M_1}{m_2}(A^{-1} \otimes B) \leq \left(1 + \frac{M_1 M_2}{m_1 m_2}\right) 1_{H \otimes H}.$$

Proof. Since $\frac{m_1}{M_2}1_{H\otimes H} \le A\otimes B^{-1} \le \frac{M_1}{m_2}1_{H\otimes H}$ and $\frac{m_2}{M_1}1_{H\otimes H} \le A^{-1}\otimes B \le \frac{M_2}{m_1}1_{H\otimes H}$, we have

$$\left(\frac{M_2}{m_1}1_{H\otimes H} - A^{-1}\otimes B\right)\left(\frac{M_1}{m_2}1_{H\otimes H} - A\otimes B^{-1}\right) \ge 0,$$

which is equivalent to the desired inequality. □

The following theorem gives an estimate from above to the Fiedler inequality.

Theorem 6.14 *If A is a positive operator on a Hilbert space H such that $0 < m1_H \le A \le M1_H$, then*

$$A\circ A^{-1} \le (A^2\circ 1_H)^{\frac{1}{2}}(A^{-2}\circ 1_H)^{\frac{1}{2}} \le \frac{M^2+m^2}{2Mm}1_H = \frac{1}{2}(h+h^{-1})1_H,$$

where $h = \frac{M}{m}$.

Proof. Applying Lemma 6.13 for $B = 1_H$, we have

$$\frac{1}{m^2}(A^2\otimes 1_H) + M^2(A^{-2}\otimes 1_H) \le \left(1+\frac{M^2}{m^2}\right)1_{H\otimes H},$$

so that Definition 6.1 implies

$$\frac{1}{m^2}(A^2\circ 1_H) + M^2(A^{-2}\circ 1_H) \le \left(1+\frac{M^2}{m^2}\right)1_H.$$

Since $A^2\circ 1_H$ and $A^{-2}\circ 1_H$ commute, the arithmetic-geometric mean inequality ensures

$$\begin{aligned}\frac{M}{m}(A^2\circ 1_H)^{\frac{1}{2}}(A^{-2}\circ 1_H)^{\frac{1}{2}} &= \frac{1}{m^2}(A^2\circ 1_H)\,\sharp\, M^2(A^{-2}\circ 1_H)\\ &\le \frac{1}{2}\left(\frac{1}{m^2}(A^2\circ 1_H) + M^2(A^{-2}\circ 1_H)\right)\\ &\le \frac{M^2+m^2}{2m^2}1_H,\end{aligned}$$

which is the desired inequality. The former inequality follows from Corollary 6.12. □

The following corollary is the Kantorovich inequality on the Hadamard product.

Corollary 6.15 *If A is a positive operator on H such that $0 < m1_H \le A \le M1_H$, then*

$$A\circ A^{-1}\circ 1_H \le \frac{(M+m)^2}{4Mm}1_H.$$

Proof. By Theorem 6.14, we have

$$(A^2\circ A^{-2})\circ 1_H = (A^2\circ 1_H)\circ(A^{-2}\circ 1_H) = (A^2\circ 1_H)(A^{-2}\circ 1_H) \le \left(\frac{M^2+m^2}{2Mm}\right)^2,$$

which is equivalent to the desired inequality, replacing A by $A^{\frac{1}{2}}$. □

Remark 6.2 *It follows that Corollary 6.15 implies the Kantorovich inequality. In fact, for a given unit vector x, we take a complete orthogonal basis $\{e_j\}$ with $e_1 = x$, then Corollary 6.15 ensures*

$$\begin{aligned}(Ax,x)(A^{-1}x,x) &= ((A\otimes A^{-1})Ux,Ux) = (A\circ A^{-1}x,x)(x,x)\\ &= ((A\circ A^{-1})\otimes 1_H)Ux,Ux) = ((A\circ A^{-1}\circ 1_H)x,x)\\ &\le \frac{(M+m)^2}{4Mm}.\end{aligned}$$

Thus we may call Corollary 6.15 the Kantorovich inequality on the Hadamard product.

Also, it follows that under the hypothesis of Theorem 6.14

$$((A\circ A^{-1})e_i,e_i) \le \frac{(M+m)^2}{4Mm}$$

whereas

$$A\circ A^{-1} \le \frac{(M+m)^2}{4Mm}1_H$$

does not hold in general: If $A = \begin{pmatrix} 2 & 1\\ 1 & 1\end{pmatrix}$, then $m = \frac{3-\sqrt{5}}{2} \le A \le \frac{3+\sqrt{5}}{2} = M$ and so $\frac{(M+m)^2}{4Mm} = \frac{9}{4}$. On the other hand, we have $\|A\circ A^{-1}\| = 3 > \frac{9}{4}$. This example might clarify the meaning of Corollary 6.15 and consequently Theorem 6.14 .

Finally we present Kantorovich type inequalities on the Hadamard product, which is initiated by Liu and Neudecker in the matrix case.

Theorem 6.16 *Let A and B be positive operators such that $0 < m1_{H\otimes H} \le A\otimes B \le M1_{H\otimes H}$. Then*

(i) $A^2\circ B^2 - (A\circ B)^2 \le \frac{1}{4}(M-m)^2 1_H$.

(ii) $A\circ B - (A^{-1}\circ B^{-1})^{-1} \le (\sqrt{M}-\sqrt{m})^2 1_H$.

(iii) $(A^2\circ B^2)^{\frac{1}{2}} \le \frac{M+m}{2\sqrt{Mm}} A\circ B$.

(iv) $(A^2\circ B^2)^{\frac{1}{2}} - A\circ B \le \frac{(M-m)^2}{4(M+m)} 1_H$.

Proof. By Definition 6.1, we have (i):

$$\begin{aligned}A^2\circ B^2 - (A\circ B)^2 &= U^*(A^2\otimes B^2)U - (U^*(A\otimes B)U)^2\\ &= U^*(A\otimes B)^2U - (U^*(A\otimes B)U)^2\\ &\le \frac{1}{4}(M-m)^2 1_H\end{aligned}$$

and the last inequality holds by (i) of Theorem 1.32.

For (iii), we have

$$\begin{aligned} A^2 \circ B^2 &= U^*(A\otimes B)^2 U \\ &\leq \frac{(M+m)^2}{4Mm}(U^*(A\otimes B)U)^2 \\ &= \frac{(M+m)^2}{4Mm}(A\circ B)^2 \end{aligned}$$

by (iii) of Theorem 1.32. Rasing both sides to the power $1/2$, it follows from Theorem 1.8 (the Löwner-Heinz inequality) that

$$(A^2\circ B^2)^{\frac{1}{2}} \leq \frac{M+m}{2\sqrt{Mm}} A\circ B.$$

Similarly (ii) follows from Theorem 1.32 and (iv) from Corollary 2.20. □

6.2 Converses of Jensen's type inequalities

In this section, we discuss complementary results to Jensen's type inequalities on the Hadamard product of positive operators (Theorem 6.3). We show Hadamard product versions of operator inequalities associated with extensions of Hölder-McCarthy and Kantorovich inequalities.

For the sake of convenience, we prepare some notations and definitions. Let $A,B \in \mathscr{B}^+(H)$ with $\mathsf{Sp}(A) \subseteq [m_1,M_1]$, $\mathsf{Sp}(B) \subseteq [m_2,M_2]$. We assume that in the whole chapter $0 < m_1 \leq M_1$ and $0 < m_2 \leq M_2$ and we denote by

$$m = m_1 m_2,\ \ M = M_1 M_2 \ \text{ and } \ I_u = [m_1,M_1]\cup[m_2,M_2]\cup[m,M].$$

Theorem 6.17 *Let $A,B \in \mathscr{B}^+(H)$ be positive operators with $\mathsf{Sp}(A) \subseteq [m_1,M_1]$, $\mathsf{Sp}(B) \subseteq [m_2,M_2]$, $f \in \mathscr{C}(I_u)$ and $g \in \mathscr{C}([m,M])$, where $m = m_1m_2, M = M_1M_2$ and $I_u = [m_1,M_1]\cup[m_2,M_2]\cup[m,M]$. Let $F(u,v)$ be a real valued continuous function defined on $U\times V$, operator monotone in u, where $U \supset \{f(t)f(s) : t \in [m_1,M_1], s \in [m_2,M_2]\}$, $V \supset \{g(t) : t \in [m,M]\}$. If f is a super-multiplicative convex function (resp. a sub-multiplicative concave function) on I_u, then*

$$\begin{gathered} F[f(A)\circ f(B), g(A\circ B)] \leq \left\{ \max_{m\leq t\leq M} F[\mu_f t + \nu_f, g(t)] \right\} 1_H \\ \left(\textit{resp. } F[f(A)\circ f(B), g(A\circ B)] \geq \min_{m\leq t\leq M} F[\mu_f t + \nu_f, g(t)] 1_H \right), \end{gathered} \tag{6.9}$$

where

$$\mu_f = \frac{f(M)-f(m)}{M-m} \qquad \textit{and} \qquad \nu_f = \frac{Mf(m)-mf(M)}{M-m}.$$

Proof. We prove the case when f is a super-multiplicative convex function. Since f is convex we have $f(t) \le \mu_f t + \nu_f$ for every $t \in [m,M]$. Thus we obtain $f(A \otimes B) \le \mu_f A \otimes B + \nu_f 1_{H\otimes H}$ since $m1_{H\otimes H} \le A \otimes B \le M1_{H\otimes H}$, so that it follows from the super-multiplicativity of f that

$$\begin{aligned} f(A) \circ f(B) &= U^*(f(A) \otimes f(B))U \le U^* f(A \otimes B)U \\ &\le U^*(\mu_f A \otimes B + \nu_f 1_{H\otimes H})U = \mu_f A \circ B + \nu_f 1_H. \end{aligned}$$

By the monotonicity of $F(\cdot,v)$ and $m1_H \le A \circ B \le M1_H$ we have

$$\begin{aligned} F[f(A) \circ f(B), g(A \circ B)] &\le F[\mu_f A \circ B + \nu_f 1_H, g(A \circ B)] \\ &\le \left\{ \max_{m \le t \le M} F[\mu_f t + \nu_f, g(t)] \right\} 1_H. \end{aligned}$$

Thus we obtain the desired inequality. The proof in the sub-multiplicative concave case is essentially the same. □

Remark 6.3 Notice that we do not assume the operator convexity or the operator concavity of the function f in Theorem 6.17.

In the following theorem we give a generalization of converses of Theorem 6.3:

Theorem 6.18 *Let the hypothesis of Theorem 6.17 be satisfied and $\Phi \in \mathbf{P}_N[\mathscr{B}(H), \mathscr{B}(K)]$ a normalized positive linear map. If f is a super-multiplicative convex function on I_u, then*

$$F[\Phi(f(A) \circ f(B)), g(\Phi(A \circ B))] \le \left\{ \max_{m \le t \le M} F[\mu_f t + \nu_f, g(t)] \right\} 1_K.$$

In the dual case (*when f is sub-multiplicative concave function on I_u*) *we have the opposite inequality with dual extreme* (min *instead of* max).

Proof. Since f is the super-multiplicative convex function, then it follows from the proof of Theorem 6.17 that $f(A) \circ f(B) \le \mu_f (A \circ B) + \nu_f 1_{H\otimes H}$. Since Φ is a normalized positive linear map we have $\Phi(f(A) \circ f(B)) \le \mu_f \Phi(A \circ B) + \nu_f 1_K$ and $m1_K \le \Phi(A \circ B) \le M1_K$. Using the operator monotonicity of $F(\cdot,v)$ we obtain

$$\begin{aligned} F[\Phi(f(A) \circ f(B)), g(\Phi(A \circ B)] &\le F[\mu_f \Phi(A \circ B) + \nu_f 1_K, g(\Phi(A \circ B)] \\ &\le \left\{ \max_{m \le t \le M} F[\mu_f t + \nu_f, g(t)] \right\} 1_K. \end{aligned}$$

□

If we put $F(u,v) = u - \alpha v$, $\alpha \in \mathbb{R}$, in Theorem 6.17 we obtain the following generalization of converses of (6.3) in Theorem 6.3:

Theorem 6.19 *Let $A, B \in \mathscr{B}^+(H)$ with $\mathsf{Sp}(A) \subseteq [m_1, M_1]$, $\mathsf{Sp}(B) \subseteq [m_2, M_2]$, $f \in \mathscr{C}(I_u)$ be a super-multiplicative convex function and $g \in \mathscr{C}([m,M])$. Then for a given $\alpha \in \mathbb{R}$*

$$f(A) \circ f(B) \le \alpha g(A \circ B) + \left\{ \max_{t \in [m,M]} \{\mu_f t + \nu_f - \alpha g(t)\} \right\} 1_H.$$

In the dual case we have the opposite inequality with dual extreme.

Remark 6.4 Let $f \in \mathscr{C}(I_u)$ be a super-multiplicative convex function (resp. a sub-multiplicative concave function). If we put $\alpha = 1$ in Theorem 6.19, then we have the following:

$$f(A) \circ f(B) - g(A \circ B) \leq \left(\mu_f t_o + \nu_f - g(t_o)\right) 1_H$$
$$\left(\text{resp. } f(A) \circ f(B) - g(A \circ B) \geq \left(\mu_f t_o + \nu_f - g(t_o)\right) 1_H\right),$$

where

$$t_o = \begin{cases} g'^{-1}(\mu_f) & \text{if} \quad g'(m) < \mu_f < g'(M) \quad \left(\text{resp. } g'(m) > \mu_f > g'(M)\right), \\ m & \text{if} \quad g'(m) \geq \mu_f \quad \left(\text{resp. } g'(m) \leq \mu_f,\right), \\ M & \text{if} \quad g'(M) \leq \mu_f \quad \left(\text{resp. } g'(M) \geq \mu_f\right), \end{cases}$$

in the case when $g \in \mathscr{C}([m,M])$ is strictly convex differentiable or

$$t_o = \begin{cases} M & \text{if} \quad \mu_f \geq \mu_g \quad \left(\text{resp. } \mu_f \leq \mu_g\right), \\ m & \text{if} \quad \mu_f < \mu_g \quad \left(\text{resp. } \mu_f > \mu_g\right), \end{cases}$$

in the case when $g \in \mathscr{C}([m,M])$ is strictly concave.

If we put $g \equiv f$ in Theorem 6.19 then we obtain complementary inequalities to Jensen's type inequalities on the Hadamard product (6.3) in Theorem 6.3:.

Corollary 6.20 *Let $A, B \in \mathscr{B}^+(H)$ with $\mathsf{Sp}(A) \subseteq [m_1, M_1]$, $\mathsf{Sp}(B) \subseteq [m_2, M_2]$. If $f \in \mathscr{C}(I_u)$ is a super-multiplicative convex function (resp. a sub-multiplicative concave function), then for a given $\alpha \in \mathbb{R}$*

$$f(A) \circ f(B) \leq \alpha f(A \circ B) + \beta 1_H \quad \left(\textit{resp. } f(A) \circ f(B) \geq \alpha f(A \circ B) + \beta 1_H\right),$$

where $\beta = -\alpha f(t_o) + \mu_f t_o + \nu_f$, and

$$t_o = \begin{cases} M & \textit{if} \quad M \leq f'^{-1}\left(\frac{\mu_f}{\alpha}\right), \\ m & \textit{if} \quad f'^{-1}\left(\frac{\mu_f}{\alpha}\right) \leq m, \\ f'^{-1}\left(\frac{\mu_f}{\alpha}\right) & \quad \textit{otherwise.} \end{cases}$$

Further if we choose α such that $\beta = 0$ in Theorem 6.19 and if g is strictly convex differentiable function or strictly concave function on $[m,M]$ then we have one more generalization of converse of (6.3):

Corollary 6.21 *Let $A, B \in \mathscr{B}^+(H)$ with $\mathsf{Sp}(A) \subseteq [m_1, M_1]$, $\mathsf{Sp}(B) \subseteq [m_2, M_2]$. Let $f \in \mathscr{C}(I_u)$ be a super-multiplicative convex (resp. sub-multiplicative concave) function and $g \in \mathscr{C}([m,M])$. Assume that either of the following conditions holds: (i) $f(m) > 0$, $f(M) > 0$, $g > 0$ on $[m,M]$ or (ii) $f(m) < 0$, $f(M) < 0$, $g < 0$ on $[m,M]$. Then*

$$f(A) \circ f(B) \leq \alpha_1\, g(A \circ B) \quad \left(\textit{resp. } f(A) \circ f(B) \geq \alpha_1\, g(A \circ B)\right),$$

holds for

$$\alpha_1 = \max\left\{\frac{f(m)}{g(m)}, \frac{f(M)}{g(M)}\right\} \quad \left(\textit{resp. } \alpha_1 = \min\left\{\frac{f(m)}{g(m)}, \frac{f(M)}{g(M)}\right\}\right),$$

if g is strictly concave (resp. strictly convex) differentiable or

$$\alpha_1 = \begin{cases} (\mu_f t_o + \nu_f)/g(t_o) & \text{if} \quad f(m)\frac{g'(m)}{g(m)} < \mu_f < f(M)\frac{g'(M)}{g(M)}, \\ \max\left\{\frac{f(m)}{g(m)}, \frac{f(M)}{g(M)}\right\} \left(\textit{resp. } \min\left\{\frac{f(m)}{g(m)}, \frac{f(M)}{g(M)}\right\}\right) & \text{otherwise}, \end{cases}$$

if g is strictly convex (resp. strictly concave) twice differentiable, where t_o is defined as the unique solution of $\mu_f g(t) = \left(\mu_f t + \nu_f\right) g'(t)$.

If we put $g \equiv f$ in Corollary 6.21, then we obtain the following ratio type inequalities.

Corollary 6.22 *Let the hypothesis of Corollary 6.20 be satisfied. If $f > 0$ on I_u, then*

$$f(A) \circ f(B) \le \alpha f(A \circ B) \quad (\textit{resp. } f(A) \circ f(B) \ge \alpha f(A \circ B))$$

and if $f < 0$ on I_u, then

$$f(A) \circ f(B) \ge \alpha f(A \circ B) \quad (\textit{resp. } f(A) \circ f(B) \le \alpha f(A \circ B)),$$

where $\alpha = (\mu_f t_o + \nu_f)/f(t_o)$ and t_o is the unique solution of $\left(\mu_f f(t) = f'(t)(\mu_f t + \nu_f)\right)$.

6.3 Application to some functions

In this section we shall apply Theorem 6.19 and Corollary 6.21 to the power function and the exponential function. We observe that the power function $f(t) = t^p$ is super-multiplicative strictly convex (resp. sub-multiplicative strictly concave) if $p \notin [0,1]$ (resp. $p \in (0,1)$).

If we put $g(t) = t^q$ in Theorem 6.19, then we obtain the following corollary, which is a step between the desired inequalities.

Corollary 6.23 *Let $A, B \in \mathscr{B}^{++}(H)$ be positive invertible operators with $\mathsf{Sp}(A) \subseteq [m_1, M_1]$, $\mathsf{Sp}(B) \subseteq [m_2, M_2]$, $f \in \mathscr{C}(I_u)$ and $q \in \mathbb{R}$. If f is super-multiplicative convex, then for a given $\alpha \in \mathbb{R}$*

$$f(A) \circ f(B) \le \alpha (A \circ B)^q + \beta 1_H$$

holds for

$$\beta = \begin{cases} \alpha(q-1)\left(\frac{\mu_f}{\alpha q}\right)^{\frac{q}{q-1}} + \nu_f \\ \qquad\qquad \text{if} \quad m < \left(\frac{\mu_f}{\alpha q}\right)^{\frac{1}{q-1}} < M \text{ and } \alpha q(q-1) > 0, \\ \max\{f(m) - \alpha m^q, f(M) - \alpha M^q\} \qquad \text{otherwise}. \end{cases}$$

If f is sub-multiplicative concave, then

$$f(A)\circ f(B) \geq \alpha(A\circ B)^q + \beta 1_H$$

holds for

$$\beta = \begin{cases} \alpha(q-1)\left(\frac{\mu_f}{\alpha q}\right)^{\frac{q}{q-1}} + \nu_f & \\ \qquad\qquad \text{if} \quad m < \left(\frac{\mu_f}{\alpha q}\right)^{\frac{1}{q-1}} < M \text{ and } \alpha q(q-1) < 0, \\ \min\{f(m) - \alpha m^q, f(M) - \alpha M^q\} & \text{otherwise.} \end{cases}$$

Proof. The estimation of β is similar to that in Corollary 2.6. Thus we obtain the desired results by virtue of Theorem 6.19. □

Further if we choose α such that $\beta = 0$ in Corollary 6.23, then we have the following corollary.

Corollary 6.24 *Let the hypothesis of Corollary 6.23 be satisfied. If f is super-multiplicative convex (resp. sub-multiplicative concave), then*

$$f(A)\circ f(B) \leq \alpha_1(A\circ B)^q \quad (\textit{resp. } f(A)\circ f(B) \geq \alpha_1(A\circ B)^q)$$

holds for

$$\alpha_1 = \begin{cases} K(m,M,f,q) & \text{if} \quad m < \frac{q}{1-q}\frac{\nu_f}{\mu_f} < M \text{ and } \mu_f(q-1) > 0, \\ \max\{f(m)/m^q, f(M)/M^q\} & \text{otherwise,} \end{cases}$$

(resp.

$$\alpha_2 = \begin{cases} K(m,M,f,q) & \text{if} \quad m < \frac{q}{1-q}\frac{\nu_f}{\mu_f} < M \text{ and } \mu_f(q-1) < 0, \\ \min\{f(m)/m^q, f(M)/M^q\} & \text{otherwise)} \end{cases}$$

where $K(m,M,f,q) = \frac{\mu_f}{q}\left(\frac{\nu_f}{\mu_f}\frac{q}{1-q}\right)^{1-q}$ *is defined in (2.19).*

Proof. This proof is quite similar to one as Corollary 2.11. □

If we put $f(t) = t^p$ in Corollary 6.23 we obtain the following corollary.

Corollary 6.25 *Let $A,B \in \mathscr{B}^{++}(H)$ with $\mathsf{Sp}(A) \subseteq [m_1,M_1]$, $\mathsf{Sp}(B) \subseteq [m_2,M_2]$. If $p \in \mathbb{R}\backslash(0,1)$ (resp. $p \in (0,1)$ and $q \in \mathbb{R}$, then for a given $\alpha \in \mathbb{R}$*

$$A^p\circ B^p \leq \alpha(A\circ B)^q + \beta\ 1_H \quad (\textit{resp. } A^p\circ B^p \geq \alpha(A\circ B)^q + \beta\ 1_H),$$

holds for

$$\beta_1 = \begin{cases} \alpha(q-1)\left(\frac{1}{\alpha q}\mu_{t^p}\right)^{\frac{q}{q-1}} + \nu_{t^p} & \\ \qquad\qquad \text{if} \quad m < \left(\frac{1}{\alpha q}\mu_{t^p}\right)^{\frac{1}{q-1}} < M \text{ and } \alpha q(q-1) > 0, \\ \max\{m^p - \alpha m^q, M^p - \alpha M^q\} & \text{otherwise,} \end{cases}$$

(*resp.*

$$\beta = \begin{cases} \alpha(q-1)\left(\frac{1}{\alpha q}\mu_{t^p}\right)^{\frac{q}{q-1}} + \nu_{t^p} \\ \qquad\qquad \text{if} \quad m < \left(\frac{1}{\alpha q}\mu_{t^p}\right)^{\frac{1}{q-1}} < M \text{ and } \alpha q(q-1) < 0, \\ \min\{m^p - \alpha m^q, M^p - \alpha M^q\} \qquad \text{otherwise.}) \end{cases}$$

If we put $\alpha = 1$ and if we choose α such that $\beta = 0$ in Corollary 6.25, then we have the following corollary.

Corollary 6.26 *Let $A, B \in \mathscr{B}^{++}(H)$ with $\mathsf{Sp}(A) \subseteq [m_1, M_1]$, $\mathsf{Sp}(B) \subseteq [m_2, M_2]$. If $p, q \in \mathbb{R}\backslash[0,1]$, $p \cdot q > 0$, then*

(i) $\qquad A^p \circ B^p \leq (A \circ B)^q + \beta 1_H$

(ii) $\qquad A^p \circ B^p \leq \alpha\,(A \circ B)^q$

hold for

$$\beta = \begin{cases} C(m,M,p,q) & \text{if} \quad m < \left(\frac{1}{q}\mu_{t^p}\right)^{\frac{1}{q-1}} < M, \\ \max\{m^p - m^q, M^p - M^q\} & \text{otherwise,} \end{cases}$$

and

$$\alpha = \begin{cases} K(m,M,p,q) & \text{if} \quad m < \frac{q-1}{q}\mu_{t^p}/\nu_{t^p} < M, \\ \max\{m^p/m^q, M^p/M^q\} & \text{otherwise.} \end{cases}$$

Also, if $p, q \in (0,1)$, then

(iii) $\qquad A^p \circ B^p \geq (A \circ B)^q + \beta 1_H$

(iv) $\qquad A^p \circ B^p \geq \alpha\,(A \circ B)^q$

hold for

$$\beta = \begin{cases} C(m,M,p,q) & \text{if} \quad m < \left(\frac{1}{q}\mu_{t^p}\right)^{\frac{1}{q-1}} < M, \\ \min\{m^p - m^q, M^p - M^q\} & \text{otherwise,} \end{cases}$$

and

$$\alpha = \begin{cases} K(m,M,p,q) & \text{if} \quad m < \frac{q-1}{q}\mu_{t^p}/\nu_{t^p} < M, \\ \min\{m^p/m^q, M^p/M^q\} & \text{otherwise.} \end{cases}$$

Here $C(m,M,p,q) = \frac{Mm^p - mM^p}{M-m} + (q-1)\left(\frac{M^p - m^p}{q(M-m)}\right)^{\frac{q}{q-1}}$ is defined in (2.38) and $K(m,M,p,q) = \frac{mM^p - Mm^p}{(q-1)(M-m)}\left(\frac{(q-1)(M^p - m^p)}{q(mM^p - Mm^p)}\right)^q$ is defined in (2.20).

We have the following converses of Hölder-McCarthy type inequalities on the Hadamard product (Corollary 6.4).

Corollary 6.27 *Let $A, B \in \mathscr{B}^{++}(H)$ with $\mathsf{Sp}(A) \subseteq [m_1, M_1]$, $\mathsf{Sp}(B) \subseteq [m_2, M_2]$. Then for $p \in \mathbb{R}\backslash[0,1]$*

(i) $\quad A^p \circ B^p - (A \circ B)^p \leq C(m,M,p)1_H.$

(ii) $\quad A^p \circ B^p \leq K(m,M,p)(A \circ B)^p.$

If $p \in (0,1)$ we have the opposite inequalities.

Here $C(m,M,p) = (p-1)\left(\frac{M^p-m^p}{p(M-m)}\right)^{\frac{p}{p-1}} + \frac{Mm^p-mM^p}{M-m}$ is defined in (2.39) and $K(m,M,p) = \frac{mM^p-Mm^p}{(p-1)(M-m)}\left(\frac{p-1}{p}\frac{M^p-m^p}{mM^p-Mm^p}\right)^p$ is defined in (2.21).

Proof. (i) If we put $q = p$ in (i) in Corollary 6.26, then we have the desired constant β since $m^{p-1}p \leq \frac{M^p-m^p}{M-m} \leq M^{p-1}p$.

(ii) If we put $q = p$ in (ii) in Corollary 6.26, then the constant α coincides with $K(m,M,p)$. In this case $m \leq \frac{Mm^p-mM^p}{M^p-m^p} \leq M$ holds. □

Remark 6.5 *If we put $p = 2$ in Corollary 6.27, then we have*

$$A^2 \circ B^2 - (A \circ B)^2 \leq \frac{1}{4}(M-m)^2 1_H \qquad \text{and} \qquad (A^2 \circ B^2) \leq \frac{(M+m)^2}{4Mm}(A \circ B)^2.$$

We directly can prove the second inequality by using Kijima's theorem in [119]: Since $(M1_{H\otimes H} - A \otimes B)(A \otimes B - m1_{H\otimes H}) \geq 0$ for $0 < m1_{H\otimes H} \leq A \otimes B \leq M1_{H\otimes H}$, we have

$$A^2 \otimes B^2 = (A \otimes B)^2 \leq (M+m)(A \otimes B) - Mm1_{H\otimes H}.$$

Since $(M+m)^2X^2 - 4Mm(M+m)X + 4M^2m^2 1_H = ((M+m)X - 2Mm1_H)^2 \geq 0$ for any positive operator X, we have

$$A^2 \circ B^2 \leq (M+m)(A \circ B) - Mm1_H \leq \frac{(M+m)^2}{4Mm}(A \circ B)^2.$$

As an application of Theorem 4.4, we have the following theorem, which is a power mean version on the Hadamard product.

Theorem 6.28 *Let $A, B \in \mathscr{B}^{++}(H)$ with $\mathrm{Sp}(A) \subseteq [m_1, M_1]$, $\mathrm{Sp}(B) \subseteq [m_2, M_2]$. Let $r, s \in \mathbb{R}$, $r \leq s$ and $rs \neq 0$.*

(i) If $r \leq s$, $s \notin (-1,1)$, $r \notin (-1,1)$ or $1/2 \leq r \leq 1 \leq s$ or $r \leq -1 \leq s \leq -1/2$ then

$$\Delta(h,r,s)^{-1}(A^s \circ B^s)^{\frac{1}{s}} \leq (A^r \circ B^r)^{\frac{1}{r}} \leq (A^s \circ B^s)^{\frac{1}{s}}.$$

(ii) If $1 \leq s$, $-1 < r < 1/2$, $r \neq 0$ or $r \leq -1$, $-1/2 < s < 1$, $s \neq 0$ then

$$\Delta(h,r,s)^{-1}(A^s \circ B^s)^{\frac{1}{s}} \leq (A^r \circ B^r)^{\frac{1}{r}} \leq \Delta(h,r,s)(A^s \circ B^s)^{\frac{1}{s}}.$$

(iii) If $-1 \leq -s \leq r \leq s \leq 1$, $r \neq 0$ or $-1 \leq r \leq s \leq r/2 < 0$ then

$$\Delta(h,r,1)^{-1}\Delta(h,r,s)^{-1}(A^s \circ B^s)^{\frac{1}{s}} \leq (A^r \circ B^r)^{\frac{1}{r}} \leq \Delta(h,r,1)(A^s \circ B^s)^{\frac{1}{s}}.$$

(iv) If $-1/2 \le r/2 < s < -r \le 1$, $s \neq 0$ *then*

$$\Delta(h,s,1)^{-1}\Delta(h,r,s)^{-1}(A^s \circ B^s)^{\frac{1}{s}} \le (A^r \circ B^r)^{\frac{1}{r}} \le \Delta(h,s,1)(A^s \circ B^s)^{\frac{1}{s}},$$

where a generalized Specht ratio $\Delta(h,r,s)$ *is defined as (2.97) in § 2.7:*

$$\Delta(h,r,s) = K(h^r, \frac{s}{r})^{\frac{1}{r}} \quad \text{and} \quad h = \frac{M}{m}.$$

Proof. The proof follows from Theorem 4.4 and Definition 6.1. □

As an application of Theorem 4.7, we have the following theorem, which is the difference type inequality to the power mean version on the Hadamard product. theorem [155, Theorem 2.4]:

Theorem 6.29 *Let* $A,B \in \mathscr{B}^{++}(H)$ *with* $\mathrm{Sp}(A) \subseteq [m_1,M_1]$, $\mathrm{Sp}(B) \subseteq [m_2,M_2]$.

(i) If $r \le s$, $s \notin (-1,1)$, $r \notin (-1,1)$ *or* $1/2 \le r \le 1 \le s$ *or* $r \le -1 \le s \le -1/2$ *then*

$$0 \le (A^s \circ B^s)^{\frac{1}{s}} - (A^r \circ B^r)^{\frac{1}{r}} \le \tilde{\Delta} 1_H.$$

(ii) If $s \ge 1$, $-1 < r < 1/2$, $r \neq 0$ *or* $r \le -1$, $-1/2 < s < 1$, $s \neq 0$ *then*

$$\tilde{\Delta}^* 1_H \le (A^s \circ B^s)^{\frac{1}{s}} - (A^r \circ B^r)^{\frac{1}{r}} \le \tilde{\Delta} 1_H.$$

(iii) If $-1 \le -s \le r \le s \le 1$, $r \neq 0$ *or* $-1 \le r \le s \le r/2 < 0$ *then*

$$-C\left(m^r, M^r, \frac{1}{r}\right) 1_H \le (A^s \circ B^s)^{\frac{1}{s}} - (A^r \circ B^r)^{\frac{1}{r}} \le \tilde{\Delta} 1_K + C\left(m^r, M^r, \frac{1}{r}\right) 1_H.$$

(iv) If $-1/2 \le r/2 < s < -r \le 1$, $s \neq 0$ *then*

$$\tilde{\Delta}^* 1_H - C\left(m^r, M^r, \frac{1}{r}\right) 1_H \le (A^s \circ B^s)^{\frac{1}{s}} - (A^r \circ B^r)^{\frac{1}{r}} \le \tilde{\Delta} 1_H + C\left(m^r, M^r, \frac{1}{r}\right) 1_H,$$

where

$$\begin{aligned}
\tilde{\Delta} &= \max_{\theta \in [0,1]} \left\{ [\theta M^s + (1-\theta) m^s]^{\frac{1}{s}} - [\theta M^r + (1-\theta) m^r]^{\frac{1}{r}} \right\}, \\
\tilde{\Delta}^* &= \min_{\theta \in [0,1] \cup [\frac{d}{M^r - m^r}, \frac{d}{M^r - m^r} + 1]} \left\{ [\theta M^s + (1-\theta) m^s]^{\frac{1}{s}} \right. \\
&\qquad \left. - [\theta M^r + (1-\theta) m^r - d]^{\frac{1}{r}} \right\}, \\
d &= \frac{M^s m^r - M^r m^s}{M^s - m^s} - \left(1 - \frac{r}{s}\right) \left(\frac{s}{r} \frac{M^r - m^r}{M^s - m^s}\right)^{\frac{r}{r-s}}.
\end{aligned}$$

Proof. The proof follows from Theorem 4.7 and Definition 6.1. □

We state the following corollary obtained by applying $g(t) = e^{\lambda t}$ to Theorem 6.19 (see Remark 6.4) and Corollary 6.21.

Corollary 6.30 *Let $A,B \in \mathscr{B}_h(H)$ selfadjoint operators with $\mathsf{Sp}(A) \subseteq [m_1,M_1]$, $\mathsf{Sp}(B) \subseteq [m_2,M_2]$. If $f \in \mathscr{C}(I_u)$ is a super-multiplicative strictly convex function and $\lambda \in \mathbb{R}$ such that $\lambda\,\mu_f > 0$, then*

$$f(A) \circ f(B) - \exp\{\lambda\; A \circ B\} \le \beta\; 1_H$$

and

$$f(A) \circ f(B) \le \alpha \exp\{\lambda\; A \circ B\},$$

hold for

$$\beta \;=\; \begin{cases} \frac{\mu_f}{\lambda} \log(\frac{\mu_f}{\lambda e}) + \nu_f & \text{if} \quad m < \lambda^{-1}\log(\mu_f/\lambda) < M, \\ \max\{f(m) - e^{\lambda m}, f(M) - e^{\lambda M}\} & \text{otherwise,} \end{cases}$$

$$\alpha \;=\; \begin{cases} \frac{\mu_f}{\lambda \cdot e} \exp\{\lambda\; \nu_f/\mu_f\} & \text{if} \quad [\mu_f - \lambda f(m)]\,[\mu_f - \lambda f(M)] < 0, \\ \max\{f(m)/e^{\lambda m}, f(M)/e^{\lambda M}\} & \text{otherwise.} \end{cases}$$

Corollary 6.31 *Let $A,B \in \mathscr{B}_h(H)$ selfadjoint operators with $\mathsf{Sp}(A) \subseteq [m_1,M_1]$, $\mathsf{Sp}(B) \subseteq [m_2,M_2]$. Let $f \in \mathscr{C}(I_u)$ be a strictly convex function such that $f(xy) \ge f(x) + f(y)$ for every $x,y \in I_u$. If $\lambda \in \mathbb{R}$ such that $\lambda\,\mu_f > 0$, then*

$$\exp\{f(A)\} \circ \exp\{f(B)\} - \exp\{\lambda\; A \circ B\} \le \beta 1_H$$

and

$$\exp\{f(A)\} \circ \exp\{f(B)\} \le \alpha \exp\{\lambda\; A \circ B\},$$

hold for

$$\beta \;=\; \begin{cases} \frac{\bar{\mu}}{\lambda} \log(\frac{\bar{\mu}}{\lambda e}) + \bar{\nu} & \text{if} \quad m < \lambda^{-1}\log(\bar{\mu}/\lambda) < M, \\ \max\{e^{f(m)} - e^{\lambda m}, e^{f(M)} - e^{\lambda M}\} & \text{otherwise,} \end{cases}$$

$$\alpha \;=\; \begin{cases} \frac{\bar{\mu}}{\lambda \cdot e} \exp\{\lambda\; \bar{\nu}/\bar{\mu}\} & \text{if} \quad [\bar{\mu} - \lambda e^{f(m)}]\,[\bar{\mu} - \lambda e^{f(M)}] < 0, \\ \max\{e^{f(m)-\lambda m}, e^{f(M)-\lambda M}\} & \text{otherwise,} \end{cases}$$

where $\bar{\mu} = (e^{f(M)} - e^{f(m)})/(M-m)$ and $\bar{\nu} = (Me^{f(m)} - me^{f(M)})/(M-m)$.

Proof. Since f is a strictly convex function and $f(xy) \ge f(x) + f(y)$, we have that $\exp\{f(x)\}$ is super-multiplicative strictly convex. Replacing $f(x)$ by $\exp\{f(x)\}$ in Corollary 6.30 will give the desired inequalities. □

6.4 Inequalities on Hadamard product and operator means

In this section we study several inequalities on the Hadamard product associated with operator means. As an application of chapter 5 on positive linear maps, we obtain general complementary estimates for the results by Ando, Aujla-Vasudeva and J.I.Fujii on the Hadamard product and operator means.

We begin with the following complementary inequality by virtue of Theorem 5.28 for one mean:

Theorem 6.32 *Let σ be an operator mean with the representing function f which is sub-multiplicative and not affine. Let $A,B,C,D \in \mathscr{B}^+(H)$ such that $\mathsf{Sp}(A\otimes B) \subseteq [m_1, M_1]$, $\mathsf{Sp}(C\otimes D) \subseteq [m_2, M_2]$. Let $m = \frac{m_2}{M_1}$, $M = \frac{M_2}{m_1}$. Then for a given $\alpha(>0) \in \mathbb{R}_+$*

$$(A\ \sigma\ C)\circ(B\ \sigma\ D) \geq \alpha(A\circ B)\ \sigma\ (C\circ D) + \beta(A\circ B) \tag{6.10}$$

holds for $\beta = \mu_f t_o + \nu_f - \alpha f(t_o)$ and $t_o \in [m,M]$ is defined as the unique solution of $f'(t) = \mu_f/\alpha$ when $f'(M) < \mu_f/\alpha < f'(m)$, otherwise t_o is defined as M or m according to $\mu_f/\alpha \leq f'(M)$ or $f'(m) \leq \mu_f/\alpha$ and

$$(C\ \sigma\ A)\circ(D\ \sigma\ B) \geq \alpha(C\circ D)\ \sigma\ (A\circ B) + \beta(C\circ D)$$

holds for above β where $m = \frac{m_1}{M_2}$ and $M = \frac{M_1}{m_2}$.

Proof. By putting $X = A^{-\frac{1}{2}}CA^{-\frac{1}{2}}$ and $Y = B^{-\frac{1}{2}}DB^{-\frac{1}{2}}$, then it follows from the sub-multiplicativity of f that

$$\begin{aligned}(A\ \sigma\ C)\otimes(B\ \sigma\ D) &= (A\otimes B)^{\frac{1}{2}}(f(X)\otimes f(Y))(A\otimes B)^{\frac{1}{2}}\\ &\geq (A\otimes B)^{\frac{1}{2}}(f(X\otimes Y))(A\otimes B)^{\frac{1}{2}}\\ &= (A\otimes B)\ \sigma\ (C\otimes D).\end{aligned}$$

Since the representing function f is not affine, by Theorem 5.28 when $\sigma = \tau$, for a given $\alpha(>0) \in \mathbb{R}_+$ the following inequality

$$\begin{aligned}(A\ \sigma\ C)\circ(B\ \sigma\ D) &= U^*((A\ \sigma\ C)\otimes(B\ \sigma\ D))U\\ &\geq U^*((A\otimes B)\ \sigma\ (C\otimes D))U\\ &\geq \alpha\, U^*(A\otimes B)U\ \sigma\ U^*(C\otimes D)U + \beta\, U^*(A\otimes B)U\\ &= \alpha\,(A\circ B)\ \sigma\ (C\circ D) + \beta\,(A\circ B)\end{aligned}$$

holds for $\beta = \beta\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}, f, \alpha\right) = \mu_f t_o + \nu_f - \alpha f(t_o)$ defined in Theorem 5.28. □

Remark 6.6 If we put $\alpha = 1$ in Theorem 6.32 then

$$\begin{gathered}-\beta(A\circ B) \geq (A\circ B)\ \sigma\ (C\circ D) - (A\ \sigma\ C)\circ(B\ \sigma\ D)\\ (\text{resp. } -\beta(C\circ D) \geq (C\circ D)\ \sigma\ (A\circ B) - (C\ \sigma\ A)\circ(D\ \sigma\ B))\end{gathered}$$

holds for $\beta = \mu_f t_o + \nu_f - f(t_o)$ and $t_o = f'^{-1}(\mu_f)$, where $m = \frac{m_2}{M_1}$ and $M = \frac{M_2}{m_1}$ (resp. $m = \frac{m_1}{M_2}$ and $M = \frac{M_1}{m_2}$).

If we choose α such that $\beta = 0$ in Theorem 6.32 or if we apply Corollary 5.29 when $\sigma = \tau$ then we have the following corollary:

Corollary 6.33 *Let the hypothesis of Theorem 6.32 be satisfied. Then*

$$(A\ \sigma\ C)\circ(B\ \sigma\ D)\geq\alpha_1\ (A\circ B)\ \sigma\ (C\circ D) \tag{6.11}$$

holds for $\alpha_1=\left(\mu_f t_o+\nu_f\right)/f(t_o)$ *and* $t_o\in[m,M]$ *is defined as the unique solution of* $\mu_f f(t)=f'(t)(\mu_f t+\nu_f)$.

As we assume the sub-multiplicativity of f in Theorem 6.32, the inequality (6.10) is not always a converse of the inequality (6.6) in Theorem 6.7. However, since the representing function $f(x)=x^p$ of the p-power mean is sub-multiplicative and super-multiplicative, we have the following complementary inequalities to the inequality (6.5) of the p-power mean by virtue of Theorem 6.32.

Corollary 6.34 *Let* $A,B,C,D\in\mathscr{B}^+(H)$ *be such that* $\mathsf{Sp}(A\otimes B)\subseteq[m_1,M_1]$, $\mathsf{Sp}(C\otimes D)\subseteq[m_2,M_2]$. *Let* $m=\frac{m_2}{M_1}$, $M=\frac{M_2}{m_1}$. *If* $p\in(0,1)$, *then for a given* $\alpha\in\mathbb{R}_+$

$$(A\ \sharp_p\ C)\circ(B\ \sharp_p\ D)\geq\alpha(A\circ B)\ \sharp_p\ (C\circ D)+\beta\ A\circ B \tag{6.12}$$

holds for

$$\beta=\begin{cases}\alpha(p-1)\left(\frac{1}{\alpha p}\frac{M^p-m^p}{M-m}\right)^{\frac{p}{p-1}}+\frac{Mm^p-mM^p}{M-m} \\ \qquad\qquad \text{if}\quad pm^{p-1}>\frac{1}{\alpha}\frac{M^p-m^p}{M-m}>pM^{p-1}, \\ \min\{(1-\alpha)M^p,(1-\alpha)m^p\} \qquad\qquad \text{otherwise.}\end{cases}$$

In particular,

$$\begin{aligned}(A\circ B)\ \sharp_p\ (C\circ D)\ &-\ (A\ \sharp_p\ C)\circ(B\ \sharp_p\ D)\\ &\leq\left((1-p)\left(\frac{1}{p}\frac{M^p-m^p}{M-m}\right)^{\frac{p}{p-1}}-\frac{Mm^p-mM^p}{M-m}\right)A\circ B\end{aligned}$$

and

$$(A\ \sharp_p\ C)\circ(B\ \sharp_p\ D)\geq\frac{Mm^p-mM^p}{(1-p)(M-m)}\left(\frac{1-p}{p}\frac{M^p-m^p}{Mm^p-mM^p}\right)^p(A\circ B)\ \sharp_p\ (C\circ D).$$

We show the following converses of (6.4) in Theorem 6.6 by means of Corollary 6.34:

Corollary 6.35 *Let* $A,B\in\mathscr{B}^+(H)$ *with* $\mathsf{Sp}(A)\subseteq[m_1,M_1]$, $\mathsf{Sp}(B)\subseteq[m_2,M_2]$. *Then the following inequalities hold*

(*i*) $A\circ B-(A\ \sharp\ B)\circ(A\ \sharp\ B)\leq\frac{1}{4}\sqrt{\frac{M_1M_2}{m_1m_2}}\frac{(M_1M_2-m_1m_2)^2}{M_1M_2+m_1m_2}1_H$,

(*ii*) $A\circ B\leq\frac{M_1M_2+m_1m_2}{2\sqrt{M_1M_2m_1m_2}}(A\ \sharp\ B)\circ(A\ \sharp\ B)$,

(*iii*) $A\circ B-(A\ !\ B)\circ(A\ \nabla\ B)\leq\frac{M_1}{4m_1(m_1+m_2)^2}\frac{((M_1+M_2)^2M_1^2-(m_1+m_2)^2m_1^2)^2}{(M_1+M_2)^2M_1^2+(m_1+m_2)^2m_1^2}1_H$.

Proof. Replacing both C and D by B and A in (6.12) and putting $p = \frac{1}{2}$, then for a given $\alpha \in \mathbb{R}_+$ we give

$$(A \sharp B) \circ (B \sharp A) \geq \alpha (A \circ B) \sharp (B \circ A) + \frac{4m_1m_2M_1M_2 - \alpha^2(M_1M_2 + m_1m_2)^2}{4\sqrt{m_1m_2M_1M_2}(M_1M_2 + m_1m_2)} A \circ B.$$

If we put $\alpha = 1$ then we have the first inequality (i). If we choose α such that $\beta = 0$, then we have the second one (ii). Finally, since $(m_1 + m_2)1_H \leq A + B \leq (M_1 + M_2)1_H$, $\frac{m_1^2}{M_1+M_2} 1_H \leq A(A+B)^{-1}A \leq \frac{M_1^2}{m_1+m_2} 1_H$ and $(XY^{-1}X) \sharp Y = X$ for positive operators X and Y, then we have

$$\begin{aligned} (A^{-1} + B^{-1})^{-1} \circ (A + B) &= (A - A(A+B)^{-1}A) \circ (A+B) \\ &= A \circ A + A \circ B - (A(A+B)^{-1}A) \circ (A+B) \\ &\geq A \circ A + A \circ B - (A \circ A + \beta 1_H) \\ &= A \circ B - \beta 1_H, \end{aligned}$$

where $\beta = \frac{M_1}{4m_1(m_1+m_2)^2} \frac{((M_1+M_2)^2M_1^2 - (m_1+m_2)^2m_1^2)^2}{(M_1+M_2)^2M_1^2 + (m_1+m_2)^2m_1^2}$, which implies the desired inequality (iii). □

We show the following complementary inequalities to an extension of Fiedler's type inequality (Corollary 6.9).

Corollary 6.36 *Let $A \in \mathscr{B}^+(H)$ with $\mathrm{Sp}(A) \subseteq [m,M]$. If $a, b \in \mathbb{R}$ and $t + s = 1$ for non-negative numbers t, s, then for a given $\alpha > 0$*

$$A^{ta+sb} \circ A^{sa+tb} \geq (\alpha + \beta)(A^a \circ A^b)$$

holds for

$$\beta = \alpha(s-1)\left(\frac{1}{\alpha s}\frac{h^{2s}-1}{h^{s-1}(h^2-1)}\right)^{\frac{s}{s-1}} + \frac{1-h^{2s-1}}{h^{s-2}(h^2-1)},$$

where $h = \left(\frac{M}{m}\right)^{a+b}$. In particular,

$$A^{ta+sb} \circ A^{sa+tb} \geq \frac{1}{1-s}\frac{1-h^{2s-2}}{h^{s-2}(h^2-1)}\left(\frac{1-s}{s}\frac{h^{2s}-1}{h-h^{2s-1}}\right)^s (A^a \circ A^b),$$

where $h = \left(\frac{M}{m}\right)^{a+b}$.

Proof. In Theorem 6.32, replacing both A and C by A^a, both B and D by A^b and applying the operator mean with the representing function $f(x) = x^s$, we have this corollary. □

Remark 6.7 If we put $a = 1$, $b = -1$, $s = t = \frac{1}{2}$ in Corollary 6.36, then we have an estimate from above to the Fiedler inequality (Theorem 6.14): $A \circ A^{-1} \leq \frac{M^2+m^2}{2mM} 1_H$.

Moreover, we show the following converse inequality of (6.7) in Theorem 6.10

Theorem 6.37 *Let $A,B \in \mathscr{B}^+(H)$ with $\mathsf{Sp}(A) \subseteq [m_1,M_1]$, $\mathsf{Sp}(B) \subseteq [m_2,M_2]$. If f is a sub-multiplicative nonnegative operator monotone strictly concave function on $(0,\infty)$, then for a given $\alpha > 0$*

$$f(A) \circ f^o(B) \geq \alpha(B \circ 1_H) f\left((A \circ 1_H)(B \circ 1_H)^{-1}\right) + \beta(B \circ 1_H)$$

holds for $\beta = \mu_f t_o + \nu_f - \alpha f(t_o)$ and $t_o \in [m,M]$ is defined as the unique solution of $f'(t) = \mu_f/\alpha$ when $f'(M) < \mu_f/\alpha < f'(m)$, otherwise t_o is defined as M or m according to $\mu_f/\alpha \leq f'(M)$ or $f'(m) \leq \mu_f/\alpha$, where $m = \frac{m_1}{M_2}$ and $M = \frac{M_1}{m_2}$.

Proof. Let σ be the operator mean corresponding to f, then it follows from Theorem 6.32 that

$$\begin{aligned} f(A) \circ f^o(B) &= (1_H \ \sigma \ A) \circ (B \ \sigma \ 1_H) \\ &\geq \alpha(1_H \circ B) \ \sigma \ (A \circ 1_H) + \beta(1_H \circ B) \\ &= \alpha(B \circ 1_H) f\left((A \circ 1_H)(B \circ 1_H)^{-1}\right) + \beta(B \circ 1_H), \end{aligned}$$

where $\beta = \beta\left(\frac{m_1}{M_2}, \frac{M_1}{m_2}, f, \alpha\right)$ as in the theorem. □

Remark 6.8 If we put $\alpha = 1$ in Theorem 6.37, then

$$-\beta(B \circ 1_H) \geq (B \circ 1_H) f\left((A \circ 1_H)(B \circ 1_H)^{-1}\right) - f(A) \circ f^o(B)$$

holds for $\beta = \mu_f t_o + \nu_f - f(t_o)(< 0)$ and t_o such that $f'(t_o) = a_f$, where $m = \frac{m_1}{M_2}$ and $M = \frac{M_1}{m_2}$.

Further if we choose α such that $\beta = 0$ in Theorem 6.37, then we have the following corollary:

Corollary 6.38 *Let A,B,f,m and M be as in Theorem 6.37. Then*

$$f(A) \circ f^o(B) \geq \min_{m \leq t \leq M} \left\{ \frac{\mu_f t + \nu_f}{f(t)} \right\} (B \circ 1_H) f\left((A \circ 1_H)(B \circ 1_H)^{-1}\right).$$

We have the following converse inequality of (6.8) in Corollary 6.12, since the power function $f(x) = x^s$ is super-multiplicative and sub-multiplicative:

Corollary 6.39 *Let $A,B \in \mathscr{B}^+(H)$ with $\mathsf{Sp}(A) \subseteq [m_1,M_1]$, $\mathsf{Sp}(B) \subseteq [m_2,M_2]$. If $t+s = 1$ for nonnegative numbers t and s, then for a given $\alpha > 0$*

$$A \circ B \geq \alpha(A^{1/s} \circ 1_H)^s (B^{1/s} \circ 1_H)^t + \beta(B \circ 1_H)$$

holds for

$$\beta = \begin{cases} \alpha(s-1)\left(\frac{1}{\alpha s}\frac{M^s - m^s}{M-m}\right)^{\frac{s}{s-1}} + \frac{Mm^s - mM^s}{M-m} & \text{if} \quad sm^{s-1} \geq \frac{1}{\alpha}\frac{M^s - m^s}{M-m} \geq sM^{s-1} \\ \min\{(1-\alpha)M^s, (1-\alpha)m^s\} & \text{otherwise,} \end{cases}$$

where $m = m_1^{1/s} M_2^{-1/t}$ *and* $M = M_1^{1/s} m_2^{-1/t}$.

In particular,

$$\begin{aligned} &A \circ B - (A^{1/s} \circ 1_H)^s (B^{1/t} \circ 1_H)^t \\ &\geq \left((s-1) \left(\frac{1}{s} \frac{M^s - m^s}{M-m} \right)^{\frac{s}{s-1}} + \frac{Mm^s - mM^s}{M-m} \right) (B \circ 1_H) \end{aligned}$$

and

$$A \circ B \geq \frac{Mm^s - mM^s}{(1-s)(M-m)} \left(\frac{1-s}{s} \frac{M^s - m^s}{Mm^s - mM^s} \right)^s (A^{1/s} \circ 1_H)^s (B^{1/t} \circ 1_H)^t.$$

Proof. Put $f(x) = x^s$ and $f^o(x) = x^t$ in Theorem 6.37. □

Putting $s = t = 1/2$ in Corollary 6.39, we have the next corollary:

Corollary 6.40 *Let* $A, B \in \mathscr{B}^+(H)$ *with* $\mathsf{Sp}(A) \subseteq [m_1, M_1]$, $\mathsf{Sp}(B) \subseteq [m_2, M_2]$. *Then*

$$A \circ B \leq (A^2 \circ 1_H)^{\frac{1}{2}} (B^2 \circ 1_H)^{\frac{1}{2}} \leq \frac{M_1 M_2 + m_1 m_2}{2\sqrt{m_1 m_2 M_1 M_2}} A \circ B,$$

$$(A^2 \circ 1_H)^{\frac{1}{2}} (B^2 \circ 1_H)^{\frac{1}{2}} - (A \circ B) \leq \min \left\{ \frac{1}{m_1}, \frac{1}{m_2} \right\} \frac{(M_1 M_2 - m_1 m_2)^2}{4(M_1 M_2 + m_1 m_2)} 1_H.$$

By using Theorem 1.19, we have the following Ando-Styan inequality , which extends to the result for correlation matrices by Styan.

Theorem 6.41 (ANDO-STYAN INEQUALITY) *If A is a positive invertible operator on H, then*

$$2(A \circ 1_H)(A^{-1} \circ A + 1)^{-1}(A \circ 1_H) \leq A \circ A.$$

Proof. Put $X = A \otimes A$, $Y = A \otimes 1_H + 1_H \otimes A$ and

$$Z = Y X^{-1} Y = 2(A \otimes A^{-1} + 1_H \otimes 1_H).$$

Then since $X = Y Z^{-1} Y$, Theorem 1.19 yield

$$\Phi(X) \geq \Phi(Y) \Phi(Z)^{-1} \Phi(Y)$$

for a positive linear map Φ and hence Definition 6.1 implies

$$\begin{aligned} A \circ A &\geq (A \circ 1_H + 1_H \circ A) \left(2(A \circ A^{-1} + 1_H) \right)^{-1} (A \circ 1_H + 1_H \circ A) \\ &= 2(A \circ 1_H)(A^{-1} \circ A + 1)^{-1}(A \circ 1_H). \end{aligned}$$

□

We give the converse inequality of the Ando-Styan inequality on the Hadamard product. For this proof we need the next two results.

Corollary 6.42 *Let $A \in \mathscr{B}^{++}(H)$ with $\mathsf{Sp}(A) \subseteq [m,M]$ and $\Phi_j \in \mathbf{P}_N[\mathscr{B}(H),\mathscr{B}(K)]$. Then for a given $\alpha > 0$*

$$\Phi(A^{-1}) \leq \alpha\Phi(A)^{-1} + \beta 1_H$$

holds for $\beta = \beta(m,M,x^{-1},\alpha)$

$$= \begin{cases} \frac{M+m}{Mm} - 2\sqrt{\frac{\alpha}{Mm}} & \text{if} \quad \frac{m}{M} \leq \alpha \leq \frac{M}{m} \\ \max\left\{\frac{1-\alpha}{m}, \frac{1-\alpha}{M}\right\} & \text{if either} \quad 0 < \alpha < \frac{m}{M} \quad \text{or} \quad \frac{M}{m} < \alpha. \end{cases}$$

In particular,

$$\begin{aligned} \Phi(A^{-1}) - \Phi(A)^{-1} &\leq \tfrac{(\sqrt{M}-\sqrt{m})^2}{Mm} 1_H, \\ \Phi(A^{-1}) &\leq \tfrac{(M+m)^2}{4Mm} \Phi(A)^{-1}. \end{aligned}$$

Proof. Put $k = 1$ and $p = q = -1$ in (2.10) in Corollary 2.6. □

By using Corollary 6.42, we show the following converse inequality of Theorem 1.19.

Corollary 6.43 *Let $A,B \in \mathscr{B}^{+}(H)$ with $\mathsf{Sp}(A) \subseteq [m_1,M_1]$, $\mathsf{Sp}(B) \subseteq [m_2,M_2]$ and $\Phi_j \in \mathbf{P}_N[\mathscr{B}(H),\mathscr{B}(K)]$. Then for a given $\alpha > 0$*

$$\Phi(BA^{-1}B) \leq \alpha\Phi(B)\Phi(A)^{-1}\Phi(B) + \beta\Phi(B)$$

holds for

$$\beta = \begin{cases} \frac{M+m}{Mm} - 2\sqrt{\frac{\alpha}{Mm}} & \text{if} \quad \frac{m}{M} \leq \alpha \leq \frac{M}{m}, \\ \max\left\{\frac{1-\alpha}{m}, \frac{1-\alpha}{M}\right\} & \text{if either} \quad 0 < \alpha < \frac{m}{M} \quad \text{or} \quad \frac{M}{m} < \alpha, \end{cases}$$

where $m = \frac{m_1}{M_2}$ and $M = \frac{M_1}{m_2}$. In particular,

$$\Phi(BA^{-1}B) \leq \frac{(m_1m_2 + M_1M_2)^2}{4m_1m_2M_1M_2}\Phi(B)\Phi(A)^{-1}\Phi(B),$$

$$\Phi(BA^{-1}B) - \Phi(B)\Phi(A)^{-1}\Phi(B) \leq M_2\frac{\left(\sqrt{M_1M_2} - \sqrt{m_1m_2}\right)^2}{M_1m_1}\Phi(1_H).$$

Proof. By a similar method as in Theorem 5.28, we have from Corollary 6.42 that

$$\begin{aligned} \Phi(BA^{-1}B) &= \Phi(B)^{\frac{1}{2}}\Psi\left(\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)^{-1}\right)\Phi(B)^{\frac{1}{2}} \\ &\leq \Phi(B)^{\frac{1}{2}}\left(\alpha\Psi\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)^{-1} + \beta 1_H\right)\Phi(B)^{\frac{1}{2}} \\ &= \alpha\Phi(B)\Phi(A)^{-1}\Phi(B) + \beta\Phi(B). \end{aligned}$$

□

By using Corollary 6.42 and Corollary 6.43, we obtain the following complementary inequality to the Ando-Styan inequality on the Hadamard product.

Theorem 6.44 *Let $A \in \mathscr{B}^+(H)$ with $\mathrm{Sp}(A) \subseteq [m,M]$ for some scalars $0 < m < M$. Then for a given $\alpha > 0$ the following inequality holds*

$$A \circ A \leq 2\alpha(A \circ 1_H)(A^{-1} \circ A + 1_H)^{-1}(A \circ 1_H) + \frac{2(m^2+M^2-2\sqrt{\alpha}mM)}{m+M}(A \circ 1_H).$$

In particular,

$$A \circ A \leq \frac{1}{2}\left(\frac{M^2+m^2}{Mm}\right)^2 (A \circ 1_H)(A^{-1} \circ A + 1_H)^{-1}(A \circ 1_H)$$

and

$$A \circ A - 2(A \circ 1_H)(A^{-1} \circ A + 1_H)^{-1}(A \circ 1_H) \leq \frac{2(M-m)^2}{M+m}(A \circ 1_H).$$

Proof. Put $X = A \otimes A^{-1} + A^{-1} \otimes A + 2\,1_{H \otimes H}$ and $Y = A \otimes 1_H + 1_H \otimes A$. Then we have $\left(\frac{2m}{M}+2\right)1_{H\otimes H} \leq X \leq \left(\frac{2M}{m}+2\right)1_{H\otimes H}$, $2m1_{H\otimes H} \leq Y \leq 2M1_{H\otimes H}$ and $A \otimes A = YX^{-1}Y$. Consider the map Φ from $B(H\otimes H)$ to $B(H)$ by $\Phi(X) = U^*XU$ for an isometry U, then by Corollary 6.43 we have

$$\Phi(A \otimes A) = \Phi(YX^{-1}Y) \leq \alpha\Phi(Y)\Phi(X)^{-1}\Phi(Y) + \beta\Phi(Y)$$

holds for $\beta = \beta\left(\frac{M+m}{M^2}, \frac{M+m}{m^2}, x^{-1}, \alpha\right)$ in Corollary 6.42. □

Moreover, we show the following converses inequality of the Ando-Styan inequality for two positive operators.

Corollary 6.45 *Let $A,B \in \mathscr{B}^+(H)$ with $\mathrm{Sp}(A) \subseteq [m_1,M_1]$, $\mathrm{Sp}(B) \subseteq [m_2,M_2]$ for some scalars $0 < m_1 < M_1$ and $0 < m_2 < M_2$. Then for a given $\alpha > 0$*

$$\begin{aligned} A \circ B \;\leq\; & \alpha((A+B)\circ 1_H)\{A^{-1}\circ B + A \circ B^{-1} + 2\,1_H\}^{-1} \\ & \times((A+B)\circ 1_H) + \beta((A+B)\circ 1_H) \end{aligned}$$

holds for

$$\begin{aligned} \beta \;=\; & \frac{(M_1+M_2)M_1M_2}{m_1M_1+m_2M_2+2M_1M_2} + \frac{(m_1+m_2)m_1m_2}{m_1M_1+m_2M_2+2m_1m_2} \\ & -2\sqrt{\alpha\frac{m_1M_1+m_2M_2+2M_1M_2}{(M_1+M_2)M_1M_2}\frac{m_1M_1+m_2M_2+2m_1m_2}{(m_1+m_2)m_1m_2}}. \end{aligned}$$

Proof. Put $X = B \otimes A^{-1} + B^{-1} \otimes A + 21_{H\otimes H}$ and $Y = B \otimes 1_H + 1_H \otimes A$. Then we have $\left(\frac{m_1}{M_2}+\frac{m_2}{M_1}+2\right)1_{H\otimes H} \leq X \leq \left(\frac{M_1}{m_2}+\frac{M_2}{m_1}+2\right)1_{H\otimes H}$, $(m_1+m_2)1_{H\otimes H} \leq Y \leq (M_1+M_2)1_{H\otimes H}$ and $X = Y(A\otimes B)^{-1}Y$. Just as in the proof of Theorem 6.44 we have

$$\Phi(A \otimes B) \leq \alpha\Phi(Y)\Phi(X)^{-1}\Phi(Y) + \beta\Phi(Y)$$

for

$$\beta = \beta\left(\frac{m_1M_1+m_2M_2+2M_1M_2}{M_1M_2(M_1+M_2)}, \frac{m_1M_1+m_2M_2+2m_1m_2}{m_1m_2(m_1+m_2)}, x^{-1}, \alpha\right).$$

□

In the rest part of this section we give the inequalities on the Hadamad product and two operator mean which are a generalization of the inequality (6.10). Let σ and τ be operator means with the super-multiplicative representing functions f and g respectively. If $f \le g$, then

$$(A \ \sigma \ C)\circ(B \ \sigma \ D) \le (A\circ B) \ \tau \ (C\circ D) \tag{6.13}$$

for operators $A,B,C,D \ge 0$. Since the inequality (6.13) does not hold in general, we consider the following complementary inequality to (6.13) by virtue of Theorem 5.28:

Theorem 6.46 *Let A,B,C and D be positive operators such that $0 < m_1I \le A\otimes B \le M_1I$ and $0 < m_2I \le C\otimes D \le M_2I$. Let σ and τ be two operator means with the representing functions f and g which are not affine. Moreover, suppose that f is sub-multiplicative. For a given $\alpha > 0$, put $\beta = \beta(\frac{m_2}{M_1}, \frac{M_2}{m_1}, f, g, \alpha)$ and $\beta^0 = \beta(\frac{m_1}{M_2}, \frac{M_1}{m_2}, f^0, g^0, \alpha)$ defined in Theorem 5.28.*

(i) *If $\beta \ge 0$ and $\beta^0 \ge 0$, then for every operator mean ρ*

$$(A\,\sigma\,C)\circ(B\,\sigma\,D) - \alpha(A\circ B)\ \tau\ (C\circ D) \ge (\beta(A\circ B))\ \rho\ \left(\beta^0(C\circ D)\right).$$

(ii) *If $\beta < 0$ and $\beta^0 < 0$, then for every operator mean ρ*

$$\begin{aligned}((A\,\sigma\,C)\circ(B\,\sigma\,D) - \beta(A\circ B))\ \rho\ \left((A\,\sigma\,C)\circ(B\,\sigma\,D) - \beta^0(C\circ D)\right)\\ \ge \alpha(A\circ B)\ \tau\ (C\circ D).\end{aligned}$$

(iii) *If $\beta\beta^0 < 0$, then*

$$(A\,\sigma\,C)\circ(B\,\sigma\,D) \ge \alpha(A\circ B)\ \tau\ (C\circ D) + \max\{\beta(A\circ B), \beta^0(C\circ D)\}.$$

Proof. We put $X = A^{-\frac{1}{2}}CA^{-\frac{1}{2}}$ and $Y = B^{-\frac{1}{2}}DB^{-\frac{1}{2}}$. Then we have from the sub-multiplicativity of f

$$\begin{aligned}(A\,\sigma\,C)\otimes(B\,\sigma\,D) &= (A\otimes B)^{\frac{1}{2}}(f(X)\otimes f(Y))(A\otimes B)^{\frac{1}{2}}\\ &\ge (A\otimes B)^{\frac{1}{2}}(f(X\otimes Y))(A\otimes B)^{\frac{1}{2}}\\ &= (A\otimes B)\ \sigma\ (C\otimes D).\end{aligned}$$

Since the representing function f is not affine, by Theorem 5.28 the following inequality

$$\begin{aligned}(A\ \sigma\ C)\circ(B\ \sigma\ D) &= U^*((A\ \sigma\ C)\otimes(B\ \sigma\ D))U\\ &\ge U^*((A\otimes B)\ \sigma\ (C\otimes D))U\\ &\ge \alpha U^*(A\otimes B)U\ \ \tau\ \ U^*(C\otimes D)U + \beta U^*(A\otimes B)U\\ &= \alpha(A\circ B)\ \ \tau\ \ (C\circ D) + \beta(A\circ B)\end{aligned}$$

holds for $\beta = \beta\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}, f, g, \alpha\right)$ defined as in Theorem 5.28.

Similarly,

$$(A \ \sigma \ C) \circ (B \ \sigma \ D) \geq \alpha(A \circ B) \ \tau \ (C \circ D) + \beta^0(C \circ D)$$

holds for $\beta^0 = \beta\left(\frac{m_1}{M_2}, \frac{M_1}{m_2}, f^0, g^0, \alpha\right)$.

The remainder of the proof is the same as the proof in Theorem 5.30. □

If we put $\alpha = 1$ in Theorem 6.46, then we have the following generalization of Remark 6.6 :

Corollary 6.47 *Assume that the conditions of Theorem 6.46 hold. If $\sigma \leq \tau$, then for every symmetric mean ρ*

$$(-\beta(A \circ B)) \, \rho \, (-\beta^0(C \circ D)) \geq (A \circ B) \ \tau \ (C \circ D) - (A \ \sigma \ C) \circ (B \ \sigma \ D) \geq 0.$$

Proof. Since $\alpha = 1$ and $f \leq g$, we have that $\beta < 0$ and $\beta^0 < 0$. □

Further if we choose α such that $\beta = 0$ in Theorem 6.46, then we have the following generalization of Corollary 6.33.

Corollary 6.48 *Assume that the conditions of Theorem 6.46 hold. Then*

$$(A \ \sigma \ C) \circ (B \ \sigma \ D)$$

$$\geq \max\left\{ \min_{\frac{m_2}{M_1} \leq t \leq \frac{M_2}{m_1}} \left\{ \frac{a_f t + b_f}{g(t)} \right\}, \min_{\frac{m_1}{M_2} \leq s \leq \frac{M_1}{m_2}} \left\{ \frac{a_{f^0} s + b_{f^0}}{g^0(s)} \right\} \right\} (A \circ B) \ \tau \ (C \circ D).$$

Proof. Since the representing function f of σ is a non affine and a nonnegative operator concave function, Corollary 6.48 follows from Corollary 5.32. □

Furthermore, since the representing function $f(x) = x^p$ of the p-power mean is submultiplicative and super-multiplicative, we have the following converse inequality of inequality of the p-power mean by virtue of Theorem 6.46.

Corollary 6.49 *Let A, B, C and D be positive operators such that $0 < m_1 1_{H\otimes H} \leq A \otimes B \leq M_1 1_{H\otimes H}$ and $0 < m_2 1_{H\otimes H} \leq C \otimes D \leq M_2 1_{H\otimes H}$. Let $0 < p, q < 1$. Then for a given $\alpha > 0$*

$$(A \ \sharp_p \ C) \circ (B \ \sharp_p \ D) \geq \alpha(A \circ B) \ \sharp_q \ (C \circ D) + \beta(A \circ B)$$

holds for $\beta = \beta(m, M, t^p, t^q, \alpha) =$

$$\begin{cases} \alpha(q-1)\left(\frac{1}{\alpha q}\frac{M^p - m^p}{M-m}\right)^{\frac{q}{q-1}} + \frac{Mm^p - mM^p}{M-m} & \text{if} \quad qm^{q-1} \geq \frac{1}{\alpha}\frac{M^p - m^p}{M-m} \geq qM^{q-1} \\ \min\{M^p - \alpha M^q, m^p - \alpha m^q\} & \text{otherwise,} \end{cases}$$

where $m = \frac{m_2}{M_1}$, $M = \frac{M_1}{m_2}$ *and*

$$(A \ \sharp_p \ C) \circ (B \ \sharp_p \ D) \geq \alpha(A \circ B) \ \sharp_q \ (C \circ D) + \beta^0(C \circ D)$$

holds for $\beta^0 = \beta\left(\frac{m_1}{M_2}, \frac{M_1}{m_2}, t^{1-p}, t^{1-q}, \alpha\right)$ *which is defined just as above.*

6.5 Generalization of Hadamard product of matrices

In this section we give the Li-Mathias type inequality (3.11) on the Hadamard product of matrices. For the sake of convenience, we denote $\{supcc.\}$ (resp. $\{subcx.\}$) the set of all real continues super-multiplicative function and matrix concave (resp. sub-multiplicative function and matrix convex) on $[m,M]$. We denote again $m = m_1 m_2$, $M = M_1 M_2$ and $I_u = [m_1,M_1] \cup [m_2,M_2] \cup [m,M]$.

Theorem 6.50 *Let* $A,B \in \mathscr{H}_n^+$ *with* $\mathsf{Sp}(A) \subseteq [m_1,M_1]$, $\mathsf{Sp}(B) \subseteq [m_2,M_2]$. *Let* $\Phi_1,\Phi_2 \in \mathbf{P}_N[\mathscr{M}_n,\mathscr{M}_k]$, $f \in \mathscr{C}(I_u)$ *and* $g \in \mathscr{C}([m,M])$. *Let* $F(u,v)$ *real value function defined on* $U \times V$ *matrix monotone in u, where* $U \supset \{f(t)f(s) : t \in [m_1,M_1], s \in [m_2,M_2]\}$ *and* $V \supset \{g(t) : t \in [m,M]\}$. *Then*

$$\left\{ \max_{\substack{\varphi \in \{subcx.\} \\ \varphi \le f}} \min_{m \le t \le M} F[\varphi(t), g(t)] \right\} 1_k$$

$$\le F[\Phi_1(f(A)) \circ \Phi_2(f(B)), g(\Phi_1(A) \circ \Phi_2(B))] \tag{6.14}$$

$$\le \left\{ \min_{\substack{\varphi \in \{supcc.\} \\ \varphi \ge f}} \max_{m \le t \le M} F[\varphi(t), g(t)] \right\} 1_k.$$

Proof. The proof is similar to the proof of Theorem 3.6. We prove only the right hand inequality (6.14). Since φ is a real value continuous super-multiplicative function and matrix concave such that $f(t) \le \varphi(t)$ for all $t \in I_u$, we have $f(A) \le \varphi(A)$. Using the positivity of Φ_1 and Jensen's inequality for a matrix map (Theorem 1.20) we have $\Phi_1(f(A)) \le \Phi_1(\varphi(A)) \le \varphi(\Phi_1(A))$. From the same function φ we have $(\Phi_2(f(B)) \le \varphi(\Phi_2(B)))$. Using monotonity of Kronecker product [4, str. 216]) we obtain

$$\begin{aligned} &\Phi_1(f(A)) \circ \Phi_2(f(B)) = P^T(\Phi_1(f(A)) \otimes \Phi_2(f(B)))P \\ &\le P^T(\varphi(\Phi_1(A)) \otimes \varphi(\Phi_2(B)))P = \varphi(\Phi_1(A)) \circ \varphi(\Phi_2(B)). \end{aligned}$$

From super-multiplicativity and matrix concavity of φ follows

$$\begin{aligned} &\varphi(\Phi_1(A)) \circ \varphi(\Phi_2(B)) = P^T(\varphi(\Phi_1(A)) \otimes \varphi(\Phi_2(B)))P \\ &\le \varphi(P^T((\Phi_1(A)) \otimes (\Phi_2(B)))P) = \varphi(\Phi_1(A) \circ \Phi_2(B)). \end{aligned}$$

Using the matrix non-decreasing character of $F(\cdot,v)$, we have

$$\begin{aligned} &F[\Phi_1(f(A)) \circ \Phi_2(f(B)), g(\Phi_1(A) \circ \Phi_2(B))] \\ &\le F[\varphi(\Phi_1(A) \circ \Phi_2(B)), g(\Phi_1(A) \circ \Phi_2(B))] \le \left\{ \max_{m \le t \le M} F[\varphi(t), g(t)] \right\} 1_k. \end{aligned}$$

Now we minimize this boundary over all continuous super-multiplicative matrix concave function $\varphi \ge f$, to obtained the right hand (6.14). □

A version of Theorem 3.10 with Hadamard product follows from Theorem 6.50:

Theorem 6.51 *Let the hypothesis of Theorem 6.50 be satisfied. If $f \in C(I_u)$ is a convex function and a function*

$$h(t) \equiv h(t;m,M,f) = \mu_f \cdot t + \nu_f$$

is super-multiplicative, then

$$F[\Phi_1(f(A)) \circ \Phi_2(f(B)), g(\Phi_1(A) \circ \Phi_2(B))] \le \left\{ \max_{m \le t \le M} F[h(t), g(t)] \right\} 1_k,$$

but, if $f \in C(I_u)$ is a concave function and a function h is sub-multiplicative, then

$$F[\Phi_1(f(A)) \circ \Phi_2(f(B)), g(\Phi_1(A) \circ \Phi_2(B))] \ge \left\{ \min_{m \le t \le M} F[h(t), g(t)] \right\} 1_k.$$

6.6 Notes

Marcus and Khan [126] and Toyama [187] showed that the Hadamard product for metrices is the image of the tensor product by a positive map. Paulsen [165] and J.I.Fujii [37] extended above fact to the infinite case as in Definition 6.1. Ando [4] showed Jensen's type inequalities on the Hadamard product of positive definite matrices by applying concavity and convexity theorems. Also, Furuta [75], Aujla and Vasudeva [14, 13], J.I.Fujii [37] and Mond and Pečarić [155] showed another Jensen's type inequalities on the Hadamard product. The fundamental results for tensor products are due to Marcus and Minc [127]. Theorem 6.1 is due to Styan [179]. Definition 6.2 is due to J.I.Fujii [37]. Lemma 6.2, Theorem 6.3, Corollary 6.4 and 6.5, Theorem 6.7, Corollary 6.9 are due to Aujla and Vasudeva [14] for the matrix case and J.I.Fujii [37] for the operator case. Theorem 6.6 is essentially due to Ando [4]. Theorem 6.8 is due to Fiedler [31]. Theorem 6.10, Corollary 6.11 and 6.12 are due to J.I.Fujii [37]. Lemma 6.13 is due to Kijima [117]. Theorem 6.14 and Corollary 6.15 are due to Kitamura and Seo [119]. Theorem 6.16 (i) and (iii) are due to Liu and Neudecker [124], and (ii) and (iv) due to Mond and Pečarić [155].

Liu and Neudecker [124] showed several matrix Kantorovich type inequalities on the Hadamard product and Mond and Pečarić [155] moreover extended them. The results in Sections 6.2 and 6.3 are due to [176]. The results in Section 6.4 are due to [130, 133].

Chapter 7

Furuta inequality and its application

The main purpose of this chapter is to give a brief survey of several applications of Furuta inequality. According to remarkable achievements of many mathematicians who have interested with operator inequalities, at present we have been finding a lot of applications of Furuta inequality in operator theory.

7.1 Furuta inequality

Let A and B be positive operators on a Hilbert space H. The Löwner-Heinz theorem (Theorem 1.8) asserts that $A \geq B \geq 0$ ensures $A^p \geq B^p$ for all $p \in [0,1]$. However $A \geq B$ does not always ensure $A^p \geq B^p$ for $p > 1$ in general. In order to consider operator inequalities, the Löwner-Heinz theorem is very useful, but the above fact is inconvenient because the condition "$p \in [0,1]$" is too restrictive to calculate operator inequalities in the process of operator transformations and operator inequalities. The following Theorem F has been obtained from this point of view. Readers may understand its utility of Theorem F throughout this chapter after reading many applications of Theorem F.

Theorem F (Furuta inequality)

If $A \geq B \geq 0$, then for each $r \geq 0$

(i) $\left(B^{\frac{r}{2}}A^pB^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq \left(B^{\frac{r}{2}}B^pB^{\frac{r}{2}}\right)^{\frac{1}{q}}$

and

(ii) $\left(A^{\frac{r}{2}}A^pA^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq \left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{1}{q}}$

hold for $p \geq 0$ and $q \geq 1$ with

$(1+r)q \geq p+r$.

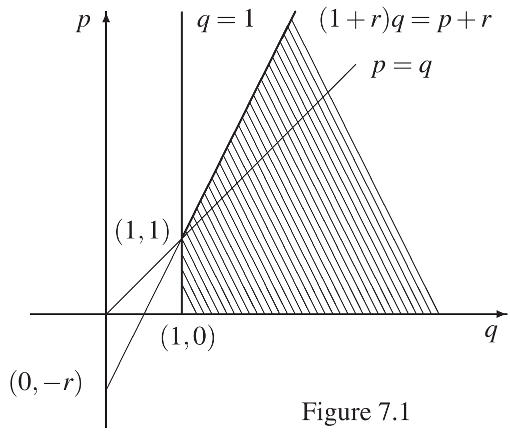

Figure 7.1

In Theorem F, it follows that (i) is equivalent to (ii) which will be shown at the end of the proof of Theorem F. The domain drawn for p,q and r in Figure 7.1 is the best possible one for Theorem F, that is, we can not extend the domain drawn for p,q and r in Figure 7.1 to ensure two inequalities (i) and (ii) in Theorem F.

Theorem F yields the Löwner-Heinz inequality if we put $r = 0$ in (i) or (ii) of Theorem F.

Consider two magic boxes

$$f(\square) = \left(B^{\frac{r}{2}}\square B^{\frac{r}{2}}\right)^{\frac{1}{q}} \quad \text{and} \quad g(\square) = \left(A^{\frac{r}{2}}\square A^{\frac{r}{2}}\right)^{\frac{1}{q}}.$$

Theorem F can be regarded as follows. Although $A \geq B \geq 0$ does not always ensure $A^p \geq B^p$ for $p > 1$ in general, but Theorem F asserts the following two order preserving operator inequalities

$$f(A^p) \geq f(B^p) \quad \text{and} \quad g(A^p) \geq g(B^p)$$

hold whenever $A \geq B \geq 0$ under the condition p , q and r in Figure 7.1.

In order to prove Furuta inequality, we need the following lemma.

Lemma 7.1 *Let X be a positive invertible operator and Y be an invertible operator. For any real number λ,*

$$(YXY^*)^\lambda = YX^{\frac{1}{2}}(X^{\frac{1}{2}}Y^*YX^{\frac{1}{2}})^{\lambda-1}X^{\frac{1}{2}}Y^*.$$

Proof. Let $YX^{\frac{1}{2}} = UH$ be the polar decomposition of $YX^{\frac{1}{2}}$, where U is unitary and $H = \left|YX^{\frac{1}{2}}\right|$. Then we have

$$\begin{aligned}(YXY^*)^\lambda &= (UH^2U^*)^\lambda = YX^{\frac{1}{2}}H^{-1}H^{2\lambda}H^{-1}X^{\frac{1}{2}}Y^* \\ &= YX^{\frac{1}{2}}\left(X^{\frac{1}{2}}Y^*YX^{\frac{1}{2}}\right)^{\lambda-1}X^{\frac{1}{2}}Y^*\end{aligned}$$

for any real bunber λ. $\square$

It easily turns out that we don't require the invertibility of X and Y in the case $\lambda \geq 1$ in Lemma 7.1 which is obviously seen in the proof. Lemma 7.1 is very simple with its proof stated above, but quite useful tool in order to treat operator transformation in operator theory.

Proof of Theorem F. At first we prove (ii). In the case $1 \geq p \geq 0$, the result is obvious by the Löwner-Heinz inequality. We have only to consider $p \geq 1$ and $q = \frac{p+r}{1+r}$ since (ii) of Theorem F for values q larger than $\frac{p+r}{1+r}$ follows by the Löwner-Heinz inequality, that is, we have only to prove the following

$$A^{1+r} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}} \quad \text{for any} \quad p \geq 1 \quad \text{and} \quad r \geq 0. \tag{7.1}$$

We may assume that A and B are *invertible* without loss of generality. In the case $r \in [0,1]$, $A \geq B \geq 0$ ensures $A^r \geq B^r$ holds by the Löwner-Heinz inequality. Then we have

$$\begin{aligned}
\left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}} &= A^{\frac{r}{2}} B^{\frac{p}{2}} \left(B^{\frac{-p}{2}} A^{-r} B^{\frac{-p}{2}}\right)^{\frac{p-1}{p+r}} B^{\frac{p}{2}} A^{\frac{r}{2}} \quad \text{by Lemma 7.1} \\
&\leq A^{\frac{r}{2}} B^{\frac{p}{2}} \left(B^{\frac{-p}{2}} B^{-r} B^{\frac{-p}{2}}\right)^{\frac{p-1}{p+r}} B^{\frac{p}{2}} A^{\frac{r}{2}} \\
&= A^{\frac{r}{2}} B A^{\frac{r}{2}} \leq A^{1+r},
\end{aligned}$$

and the first inequality follows by $B^{-r} \geq A^{-r}$ and the Löwner-Heinz inequality since $\frac{p-1}{p+r} \in [0,1]$ holds, and the last inequality follows by $A \geq B \geq 0$, so we have the following

$$A^{1+r} \geq \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}} \qquad \text{for } p \geq 1 \text{ and } r \in [0,1]. \tag{7.2}$$

Put $A_1 = A^{1+r}$ and $B_1 = \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}}$ in (7.2). Repeating (7.2) again for $A_1 \geq B_1 \geq 0$, $r_1 \in [0,1]$ and $p_1 \geq 1$,

$$A_1^{1+r_1} \geq \left(A_1^{\frac{r_1}{2}} B_1^{p_1} A_1^{\frac{r_1}{2}}\right)^{\frac{1+r_1}{p_1+r_1}}.$$

Put $p_1 = \frac{p+r}{1+r} \geq 1$ and $r_1 = 1$, then

$$A^{2(1+r)} \geq \left(A^{r+\frac{1}{2}} B^p A^{r+\frac{1}{2}}\right)^{\frac{2(1+r)}{p+2r+1}} \qquad \text{for } p \geq 1 \text{ and } r \in [0,1]. \tag{7.3}$$

Put $\frac{s}{2} = r + \frac{1}{2}$ in (7.3). Then $\frac{2(1+r)}{p+2r+1} = \frac{1+s}{p+s}$ since $2(1+r) = 1+s$, so that (7.3) can be rewritten as follows;

$$A^{1+s} \geq \left(A^{\frac{s}{2}} B^p A^{\frac{s}{2}}\right)^{\frac{1+s}{p+s}} \qquad \text{for } p \geq 1, \text{ and } s \in [1,3]. \tag{7.4}$$

Consequently (7.2) and (7.4) ensure that (7.2) holds for *any* $r \in [0,3]$ since $r \in [0,1]$ and $s = 2r+1 \in [1,3]$. Repeating this process, we should obtain that (7.1) holds for *any* $r \geq 0$ and so (ii) is shown.

If $A \geq B > 0$, then $B^{-1} \geq A^{-1} > 0$. Then by (ii), for each $r \geq 0$, $B^{\frac{-(p+r)}{q}} \geq \left(B^{\frac{-r}{2}} A^{-p} B^{\frac{-r}{2}}\right)^{\frac{1}{q}}$ holds for each p and q such that $p \geq 0$, $q \geq 1$ and $(1+r)q \geq p+r$. Taking inverses of both sides, we have (i) and so the proof of Theorem F is complete. $\square$

Theorem 7.2 *If $A \geq B \geq 0$, then the following inequalities hold.*

$$\text{(i)} \qquad \left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}} \geq B^{1+r}$$

$$\text{(ii)} \qquad A^{1+r} \geq \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}}$$

for $p \geq 1$ and $r \geq 0$.

Proof. We have only to put $q = \frac{p+r}{1+r} \geq 1$ if $p \geq 1$ and $r \geq 0$ in Theorem F. $\square$

Remark 7.1 *Theorem 7.2 is the essential part of Theorem F since Theorem F in case $p \in [0,1]$ is trivial by the Löwner-Heinz inequality, and we shall state several applications of Theorem 7.2 in the forthcoming sections.*

We show that Theorem F is equivalent to the following Theorem 7.3.

Theorem 7.3 *If $A \geq C \geq B \geq 0$, then for each $r \geq 0$*

$$(\star) \qquad \left(C^{\frac{r}{2}} A^p C^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq \left(C^{\frac{r}{2}} C^p C^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq \left(C^{\frac{r}{2}} B^p C^{\frac{r}{2}}\right)^{\frac{1}{q}}$$

for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

Proof of equivalence between Theorem F and Theorem 7.3.

Theorem F $\Longrightarrow$ Theorem 7.3.

The first inequality of $(\star)$ follows by (i) of Theorem F and also the second one of $(\star)$ follows by (ii) of Theorem F.

Theorem 7.3 $\Longrightarrow$ Theorem F.

Put $B = C$ in $(\star)$ of Theorem 7.3, then we have (i) of Theorem F. Also put $A = C$ in $(\star)$ of Theorem 7.3, then we have (ii) of Theorem F.

Whence a proof of equivalence relation between Theorem F and Theorem 7.3 is complete. $\square$

Theorem 7.3 implies the following equivalence relation;

Theorem 7.4 (CHARACTERIZATION OF C IN THEOREM 7.3) *$A \geq C \geq B \geq 0$ holds if and only if*

$$(\clubsuit) \qquad \left(C^{\frac{r}{2}}A^pC^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq \left(C^{\frac{r}{2}}C^pC^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq \left(C^{\frac{r}{2}}B^pC^{\frac{r}{2}}\right)^{\frac{1}{q}}$$

holds for all $r \geq 0$, $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

Proof. A proof of "only if" part follows by Theorem 7.3 and also a proof of "if" part follows by putting $r = 0$ and $p = q = 1$ in (♣). □

We remark that Theorem 7.4 is a characterization of C satisfying $A \geq C \geq B \geq 0$ by using the operator inequality (♣).

We state the best possibility of Theorem F as follows. We omit the proof.

Theorem 7.5 (TANAHASHI) *Let $p > 0$, $q > 0$ and $r > 0$. If $(1+r)q < p+r$ or $0 < q < 1$, then there exist positive invertible operators A and B with $A \geq B \geq 0$ which do not satisfy the inequality*

$$A^{\frac{p+r}{q}} \geq \left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{1}{q}}.$$

Theorem 7.5 asserts that the domain drawn for p, q and r in the Figure 7.1 of Theorem F is the best possible domain.

Notice that Theorem 7.5 easily ensures the following result.

Theorem 7.6 *Let $p > 1$ and $r > 0$. If $\alpha > 1$, there exist positive invertible operators A and B such that $A \geq B > 0$ and*

$$A^{(1+r)\alpha} \not\geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{(1+r)\alpha}{p+r}}.$$

Theorem G (GENERALIZED FURUTA INEQUALITY). *If $A \geq B \geq 0$ with $A > 0$, then for $t \in [0,1]$ and $p \geq 1$*

$$\text{(G-1)} \qquad A^{1-t+r} \geq \left\{A^{\frac{r}{2}}\left(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}}\right)^s A^{\frac{r}{2}}\right\}^{\frac{1-t+r}{(p-t)s+r}} \qquad \textit{for } s \geq 1 \textit{ and } r \geq t.$$

Theorem G can be regarded as an extension of Theorem 7.2.

Proof of Theorem G. We may assume that B is invertible. First of all, we prove that if $A \geq B \geq 0$ with $A > 0$, then

$$A \geq \left\{A^{\frac{t}{2}}\left(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}}\right)^s A^{\frac{t}{2}}\right\}^{\frac{1}{(p-t)s+t}} \quad \text{for } t \in [0,1],\ p \geq 1 \text{ and } s \geq 1. \tag{7.5}$$

In the case of $2 \geq s \geq 1$, as $s-1, \frac{1}{(p-t)s+t} \in [0,1]$ and $A^t \geq B^t$ by the Löwner-Heinz inequality, so by Lemma 7.1 and the Löwner-Heinz inequality we have

$$\begin{aligned} B_1 &= \left\{A^{\frac{t}{2}}\left(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}}\right)^sA^{\frac{t}{2}}\right\}^{\frac{1}{(p-t)s+t}} = \left\{B^{\frac{p}{2}}\left(B^{\frac{p}{2}}A^{-t}B^{\frac{p}{2}}\right)^{s-1}B^{\frac{p}{2}}\right\}^{\frac{1}{(p-t)s+t}} \\ &\leq \left\{B^{\frac{p}{2}}\left(B^{\frac{p}{2}}B^{-t}B^{\frac{p}{2}}\right)^{s-1}B^{\frac{p}{2}}\right\}^{\frac{1}{(p-t)s+t}} = B \leq A = A_1 \end{aligned} \tag{7.6}$$

for $t \in [0,1]$, $p \geq 1$ and $2 \geq s \geq 1$. Repeating (7.6) for $A_1 \geq B_1 \geq 0$, then we have

$$A_1 \geq \left\{A_1^{\frac{t_1}{2}}\left(A_1^{\frac{-t_1}{2}}B_1^{p_1}A_1^{\frac{-t_1}{2}}\right)^{s_1}A_1^{\frac{t_1}{2}}\right\}^{\frac{1}{(p_1-t_1)s_1+t_1}} \tag{7.7}$$

for $t_1 \in [0,1]$, $p_1 \geq 1$ and $2 \geq s_1 \geq 1$.

Put $t_1 = t \in [0,1]$ and $p_1 = (p-t)s+t \geq 1$ in (7.7). Then we obtain

$$\begin{aligned} A &\geq \left\{A^{\frac{t}{2}}\left[A^{\frac{-t}{2}}A^{\frac{t}{2}}\left(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}}\right)^sA^{\frac{t}{2}}A^{\frac{-t}{2}}\right]^{s_1}A^{\frac{t}{2}}\right\}^{\frac{1}{(p-t)ss_1+t}} \\ &= \left\{A^{\frac{t}{2}}\left(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}}\right)^{ss_1}A^{\frac{t}{2}}\right\}^{\frac{1}{(p-t)ss_1+t}} \end{aligned} \tag{7.8}$$

for $t \in [0,1]$, $p \geq 1$ and $4 \geq ss_1 \geq 1$.

Repeating this process from (7.6) to (7.8), we obtain (7.5) *for* $t \in [0,1]$, $p \geq 1$ *and any* $s \geq 1$. Put $A_2 = A$ and $B_2 = \left\{A^{\frac{t}{2}}\left(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}}\right)^sA^{\frac{t}{2}}\right\}^{\frac{1}{(p-t)s+t}}$ in (7.5). Applying (ii) of Theorem F for $A_2 \geq B_2 \geq 0$ by (7.5) for $t \in [0,1]$, $p \geq 1$ and $s \geq 1$, so we have

$$A_2^{1+r_2} \geq \left(A_2^{\frac{r_2}{2}}B_2^{p_2}A_2^{\frac{r_2}{2}}\right)^{\frac{1+r_2}{p_2+r_2}} \quad \text{for } p_2 \geq 1 \text{ and } r_2 \geq 0. \tag{7.9}$$

We have only to put $r_2 = r-t \geq 0$ and $p_2 = (p-t)s+t \geq 1$ in (7.9) to obtain the desired inequality (G-1) in Theorem G, so the proof of Theorem G is complete. □

Recall that for positive invertible operators A and B, the order $\log A \geq \log B$ is said to be the chaotic order (denoted by $A \gg B$) and this order is weaker than the usual order $A \geq B > 0$ as seen in Example 1.7, that is, $\log t$ is operator monotone.

Theorem 7.7 *Let* A *and* B *be positive invertible operators. Then the following* (i), (ii) *and* (iii) *are mutually equivalent:*

(i) $A \gg B$ *(i.e.,* $\log A \geq \log B$*).*

(ii) $A^r \geq \left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{r}{p+r}}$ *for all* $p \geq 0$ *and* $r \geq 0$.

(iii) $\left(B^{\frac{r}{2}}A^pB^{\frac{r}{2}}\right)^{\frac{r}{p+r}} \geq B^r$ *for all* $p \geq 0$ *and* $r \geq 0$.

Proof. (i) $\Longrightarrow$ (ii). We recall the following obvious and crucial formula

$$(\star\star) \qquad \delta\lim_{n\to\infty}\left(1_H+\tfrac{1}{n}\log X\right)^n = X \text{ for any } X>0.$$

The hypothesis $\log A \geq \log B$ ensures

$$\delta A_1 = 1_H + \frac{\log A}{n} \geq 1_H + \frac{\log B}{n} = B_1$$

for sufficiently large natural number n. Applying (ii) of Theorem F to A_1 and B_1, we have

$$A_1^{nr} \geq \left(A_1^{\frac{nr}{2}} B_1^{np} A_1^{\frac{nr}{2}}\right)^{\frac{nr}{np+nr}} \qquad \text{for all } p \geq 0 \text{ and } r \geq 0 \tag{7.10}$$

since $q = \frac{np+nr}{nr}$ satisfies the required condition of Theorem F. When $n \to \infty$, (7.10) ensures (ii) by $(\star\star)$.

(ii) $\Longrightarrow$ (i). Taking logarithm of both sides of (ii) and refining, we have

$$r(p+r)\log A \geq r\log\left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right) \qquad \text{for all } p \geq 0 \text{ and } r \geq 0$$

by the operator monotonicity of the logarithm, and tending $r \to +0$, we obtain $\log A \geq \log B$.

The implication $(i) \Longleftrightarrow (iii)$ is shown by the same way of the proof of $(i) \Longleftrightarrow (ii)$. □

In order to prove the best possibility of Theorem G, we prepare the following result which is nothing but a slight modification of Theorem 7.5.

Theorem 7.8 *Let $p>0$, $q>0$, $r>0$ and $\delta>0$. If $0<q<1$ or $(\delta+r)q<p+r$, then there exist positive invertible operators A and B such that $A^{\delta} \geq B^{\delta}$ and*

$$A^{\frac{p+r}{q}} \ngeq \left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{1}{q}}. \tag{7.11}$$

Proof. Assume $0<q<1$ or $(\delta+r)q<p+r$. Put $p_1=\frac{p}{\delta}>0$ and $r_1=\frac{r}{\delta}>0$, then $(\delta+r)q<p+r$ is equivalent to $(1+r_1)q<p_1+r_1$. By Theorem 7.5, there exist positive invertible operators A_1 and B_1 such that $A_1 \geq B_1 > 0$ and

$$A_1^{\frac{p_1+r_1}{q}} \ngeq \left(A_1^{\frac{r_1}{2}}B_1^{p_1}A_1^{\frac{r_1}{2}}\right)^{\frac{1}{q}}. \tag{7.12}$$

Here we put $A=A_1^{\frac{1}{\delta}}>0$ and $B=B_1^{\frac{1}{\delta}}>0$, then $A_1=A^{\delta}$ and $B_1=B^{\delta}$, so that $A_1 \geq B_1$ is equivalent to $A^{\delta} \geq B^{\delta}$ and (7.12) is equivalent to (7.11). Therefore A and B satisfy both $A^{\delta} \geq B^{\delta}$ and (7.11). Hence the proof is complete. □

Theorem 7.9 *Let $p>0$, $q>0$ and $r>0$. If $rq<p+r$, then there exist positive invertible operators A and B such that $\log A \geq \log B$ and*

$$A^{\frac{p+r}{q}} \ngeq \left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{1}{q}}. \tag{7.13}$$

Proof. Assume $rq < p+r$. Since $0 < \frac{p+r}{q} - r$, there exists a $\delta > 0$ such that $0 < \delta < \frac{p+r}{q} - r$, that is, $(\delta + r)q < p+r$. By Theorem 7.8, there exist positive invertible operators A and B such that $A^{\delta} \geq B^{\delta}$ and (7.13).

$A^{\delta} \geq A^{\delta}$ ensures $\log A \geq \log B$ by the operator monotonicity of the logarithm function and $\delta > 0$, so that A and B satisfy both $\log A \geq \log B$ and (7.13). Hence the proof is complete. □

Theorem 7.9 can be easily rewritten in the following form.

Theorem 7.10 *Let $p > 0$ and $r > 0$. If $\alpha > 1$, then there exist positive invertible operators A and B such that $\log A \geq \log B$ and*

$$A^{r\alpha} \not\geq \left(A^{\frac{r}{2}} B^{p} A^{\frac{r}{2}}\right)^{\frac{r\alpha}{p+r}}. \tag{7.14}$$

Next we prove the best possibility of Theorem G as follows.

Theorem 7.11 *Let $p \geq 1$, $t \in [0,1]$, $r \geq t$ and $s \geq 1$. If*

$$\delta \frac{1-t+r}{(p-t)s+r} < \alpha, \tag{7.15}$$

then there exist positive invertible operators A and B such that $A \geq B > 0$ and

$$A^{\{(p-t)s+r\}\alpha} \not\geq \left\{A^{\frac{r}{2}} \left(A^{\frac{-t}{2}} B^{p} A^{\frac{-t}{2}}\right)^{s} A^{\frac{r}{2}}\right\}^{\alpha}. \tag{7.16}$$

Proof. (a) In the case of $t \in [0,1)$. Assume that

$$S \geq T > 0 \quad \text{ensures} \quad S^{(1-t+r)\alpha} \geq \left\{S^{\frac{r}{2}} \left(S^{\frac{-t}{2}} T^{p} S^{\frac{-t}{2}}\right)^{s} S^{\frac{r}{2}}\right\}^{\frac{1-t+r}{(p-t)s+r}\alpha} \tag{7.17}$$

for $p \geq 1$, $t \in [0,1)$, $r \geq t$, $s \geq 1$ and $\alpha > 1$.

On the other hand, $A \geq B > 0$ ensures the following (7.18) by (ii) of Theorem F:

$$A^{1+r_1} \geq \left(A^{\frac{r_1}{2}} B^{p_1} A^{\frac{r_1}{2}}\right)^{\frac{1+r_1}{p_1+r_1}} \quad \text{for } p_1 \geq 1 \text{ and } r_1 \geq 0. \tag{7.18}$$

Put $p_1 = \frac{p-t}{1-t} \geq 1$ and $r_1 = \frac{t}{1-t} \geq 0$ in (7.18). Then (7.18) implies

$$A^{\frac{1}{1-t}} \geq \left(A^{\frac{t}{2(1-t)}} B^{\frac{p-t}{1-t}} A^{\frac{t}{2(1-t)}}\right)^{\frac{1}{p}}. \tag{7.19}$$

Put $S = A^{\frac{1}{1-t}}$ and $T = \left(A^{\frac{t}{2(1-t)}} B^{\frac{p-t}{1-t}} A^{\frac{t}{2(1-t)}}\right)^{\frac{1}{p}}$. Then $S \geq T > 0$ by (7.19) and applying (7.17), we have

$$S^{(1-t+r)\alpha} \geq \left\{S^{\frac{r}{2}} \left(S^{\frac{-t}{2}} T^{p} S^{\frac{-t}{2}}\right)^{s} S^{\frac{r}{2}}\right\}^{\frac{1-t+r}{(p-t)s+r}\alpha}. \tag{7.20}$$

(7.20) is equivalent to the following

$$
\begin{aligned}
A^{(1+\frac{r}{1-t})\alpha} &\geq \left[A^{\frac{r}{2(1-t)}}\left\{A^{\frac{-t}{2(1-t)}}\left(A^{\frac{t}{2(1-t)}}B^{\frac{p-t}{1-t}}A^{\frac{t}{2(1-t)}}\right)^{\frac{p}{p}}A^{\frac{-t}{2(1-t)}}\right\}^{s}A^{\frac{r}{2(1-t)}}\right]^{\frac{1-t+r}{(p-t)s+r}\alpha} \\
&= \left(A^{\frac{r}{2(1-t)}}B^{\frac{p-t}{1-t}s}A^{\frac{r}{2(1-t)}}\right)^{\frac{1+\frac{r}{1-t}}{\frac{p-t}{1-t}s+\frac{r}{1-t}}\alpha}. \qquad (7.21)
\end{aligned}
$$

Put $r_2 = \frac{r}{1-t} \geq 0$ and $p_2 = \frac{p-t}{(1-t)}s \geq 1$ in (7.21). Then (7.21) is equivalent to

$$A^{(1+r_2)\alpha} \geq \left(A^{\frac{r_2}{2}}B^{p_2}A^{\frac{r_2}{2}}\right)^{\frac{1+r_2}{p_2+r_2}\alpha} \quad \text{for } p_2 \geq 1,\ r_2 \geq 0 \text{ and } \alpha > 1. \qquad (7.22)$$

This contradiction proves the result in the case of $t \in [0,1)$ by Theorem 7.6.

(b) In the case of $t = 1$. Assume that

$$S \geq T > 0 \quad \text{ensures} \quad S^{r\alpha} \geq \left\{S^{\frac{r}{2}}\left(S^{\frac{-1}{2}}T^{p}S^{\frac{-1}{2}}\right)^{s}S^{\frac{r}{2}}\right\}^{\frac{r}{(p-1)s+r}\alpha}. \qquad (7.23)$$

for $p \geq 1$, $r \geq 1$, $s \geq 1$ and $\alpha > 1$.

For positive invertible operators A and B, $\log A \geq \log B$ ensures the following (7.24) by Theorem 7.7

$$A \geq \left(A^{\frac{1}{2}}B^{p-1}A^{\frac{1}{2}}\right)^{\frac{1}{p}}. \qquad (7.24)$$

Put $S = A$ and $T = \left(A^{\frac{1}{2}}B^{p-1}A^{\frac{1}{2}}\right)^{\frac{1}{p}}$. Then $S \geq T > 0$ by (7.24) and applying (7.23), we have

$$S^{r\alpha} \geq \left\{S^{\frac{r}{2}}\left(S^{\frac{-1}{2}}T^{p}S^{\frac{-1}{2}}\right)^{s}S^{\frac{r}{2}}\right\}^{\frac{r}{(p-1)s+r}\alpha}. \qquad (7.25)$$

(7.25) is equivalent to the following

$$
\begin{aligned}
A^{r\alpha} &\geq \left[A^{\frac{r}{2}}\left\{A^{\frac{-1}{2}}\left(A^{\frac{1}{2}}B^{p-1}A^{\frac{1}{2}}\right)^{\frac{p}{p}}A^{\frac{-1}{2}}\right\}^{s}A^{\frac{r}{2}}\right]^{\frac{r}{(p-1)s+r}\alpha} \\
&= \left(A^{\frac{r}{2}}B^{(p-1)s}A^{\frac{r}{2}}\right)^{\frac{r}{(p-1)s+r}\alpha} \qquad (7.26)
\end{aligned}
$$

Put $p_3 = (p-1)s > 0$ in (7.26). Then we have

$$A^{r\alpha} \geq \left(A^{\frac{r}{2}}B^{p_3}A^{\frac{r}{2}}\right)^{\frac{r}{p_3+r}\alpha} \quad \text{for } p_3 > 0,\ r \geq 1 \text{ and } \alpha > 1.$$

This contradiction proves the result in the case of $t = 1$ by Theorem 7.10.

Hence the proof is complete by (a) and (b). $\square$

7.2 Operator functions associated with Theorem G

We show the following equivalence relation between Theorem G and related operator functions.

Theorem 7.12 *The following* (i),(ii),(iii) *and* (iv) *hold and follow from each other.*
(i) If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geq 1$,

$$A^{1-t+r} \geq \left\{A^{\frac{r}{2}}\left(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}}\right)^sA^{\frac{r}{2}}\right\}^{\frac{1-t+r}{(p-t)s+r}} \qquad \text{for } r \geq t \text{ and } s \geq 1.$$

(ii) If $A \geq B \geq 0$ with $A > 0$, then for each $1 \geq q \geq t \geq 0$ and $p \geq q$,

$$A^{q-t+r} \geq \left\{A^{\frac{r}{2}}\left(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}}\right)^sA^{\frac{r}{2}}\right\}^{\frac{q-t+r}{(p-t)s+r}} \qquad \text{for } r \geq t \text{ and } s \geq 1.$$

(iii) If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geq 1$,

$$F_{p,t}(A,B,r,s) = A^{\frac{-r}{2}}\left\{A^{\frac{r}{2}}\left(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}}\right)^sA^{\frac{r}{2}}\right\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is a decreasing function for $r \geq t$ and $s \geq 1$.
(iv) If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$, $q \geq 0$ and $p \geq t$,

$$G_{p,q,t}(A,B,r,s) = A^{\frac{-r}{2}}\left\{A^{\frac{r}{2}}\left(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}}\right)^sA^{\frac{r}{2}}\right\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is a decreasing function for $r \geq t$ and $s \geq 1$ such that $(p-t)s \geq q-t$.

Proof. We may assume that both A and B are invertible.

(iv) $\Longrightarrow$ (iii). We have only to put $q = 1$ in (iv).

(iii) $\Longrightarrow$ (i). $A \geq B \geq 0$ and the monotonicity of $F_{p,t}(A,B,r,s)$ ensure

$$A^{1-t} \geq A^{\frac{-t}{2}}BA^{\frac{-t}{2}} = F_{p,t}(A,B,t,1) \geq F_{p,t}(A,B,r,s)$$

so that we have (i).

(i) $\Longrightarrow$ (ii). Put $A_1 = A^q$ and $B_1 = B^q$ for $q \in [0,1]$. Then $A_1 \geq B_1 \geq 0$ holds by the Löwner-Heinz theorem (Theorem 1.8) in Section 1.2. Put $p_1 = \frac{p}{q} \geq 1$, $t_1 = \frac{t}{q}$ and $r_1 = \frac{r}{q}$. Then we have only to apply (i) on $A_1 \geq B_1$.

(ii) $\Longrightarrow$ (iv). Put $q = t$ in (ii). Then if $A \geq B \geq 0$, then for each $t \in [0,1]$ and $p \geq t$

$$A^r \geq \left\{A^{\frac{r}{2}}\left(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}}\right)^sA^{\frac{r}{2}}\right\}^{\frac{r}{(p-t)s+r}} \quad \text{for } r \geq t \text{ and } s \geq 1. \tag{7.27}$$

(a) **Decreasing of $G_{p,q,t}(A,B,r,s)$ for s.** Put $D = A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}}$. Applying Lemma 7.1 to (7.27) and the Löwner-Heinz theorem (Theorem 1.8) in Section 1.2, we obtain for each $t \in [0,1]$, $p \geq t$, $s \geq 1$ and $r \geq t$

$$\left(D^{\frac{s}{2}}A^rD^{\frac{s}{2}}\right)^{\frac{(p-t)w}{(p-t)s+r}} \geq D^w \quad \text{for } s \geq w \geq 0. \tag{7.28}$$

Then we have

$$\begin{aligned}
f(s) &= \left\{A^{\frac{r}{2}}\left(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}}\right)^sA^{\frac{r}{2}}\right\}^{\frac{q-t+r}{(p-t)s+r}} \\
&= \left(A^{\frac{r}{2}}D^sA^{\frac{r}{2}}\right)^{\frac{q-t+r}{(p-t)s+r}} \\
&= \left\{\left(A^{\frac{r}{2}}D^sA^{\frac{r}{2}}\right)^{\frac{(p-t)(s+w)+r}{(p-t)s+r}}\right\}^{\frac{q-t+r}{(p-t)(s+w)+r}} \\
&= \left\{A^{\frac{r}{2}}D^{\frac{s}{2}}\left(D^{\frac{s}{2}}A^rD^{\frac{s}{2}}\right)^{\frac{(p-t)w}{(p-t)s+r}}D^{\frac{s}{2}}A^{\frac{r}{2}}\right\}^{\frac{q-t+r}{(p-t)(s+w)+r}} && \text{by Lemma 7.1} \\
&\geq \left(A^{\frac{r}{2}}D^{s+w}A^{\frac{r}{2}}\right)^{\frac{q-t+r}{(p-t)(s+w)+r}} \\
&= f(s+w)
\end{aligned}$$

and the last inequality holds by (7.28) and the Löwner-Heinz theorem since $\frac{q-t+r}{(p-t)(s+w)+r} \in [0,1]$ holds, so the proof of (a) is complete since $G_{p,q,t}(A,B,r,s) = A^{\frac{-r}{2}}f(s)A^{\frac{-r}{2}}$.

(b) **Decreasing of $F_{p,q,t}(A,B,r,s)$ for r.** Applying the Löwner-Heinz theorem to (7.27), if $A \geq B \geq 0$, then for each $t \in [0,1]$, $p \geq t$, $s \geq 1$ and $r \geq t$

$$A^u \geq \left(A^{\frac{u}{2}}D^sA^{\frac{u}{2}}\right)^{\frac{u}{(p-t)s+r}} \quad \text{for } r \geq u \geq 0. \tag{7.29}$$

Then we have

$$
\begin{aligned}
G_{p,q,t}(A,B,r,s) &= A^{\frac{-r}{2}}\left\{A^{\frac{r}{2}}\left(A^{\frac{-t}{2}}B^{p}A^{\frac{-t}{2}}\right)^{s}A^{\frac{r}{2}}\right\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}\\
&= D^{\frac{s}{2}}\left(D^{\frac{s}{2}}A^{r}D^{\frac{s}{2}}\right)^{\frac{q-t-(p-t)s}{(p-t)s+r}}D^{\frac{s}{2}} \quad \text{by Lemma 7.1}\\
&= D^{\frac{s}{2}}\left\{\left(D^{\frac{s}{2}}A^{r}D^{\frac{s}{2}}\right)^{\frac{(p-t)s+r+u}{(p-t)s+r}}\right\}^{\frac{q-t-(p-t)s}{(p-t)s+r+u}}D^{\frac{s}{2}}\\
&= D^{\frac{s}{2}}\left\{D^{\frac{s}{2}}A^{\frac{r}{2}}\left(A^{\frac{r}{2}}D^{s}A^{\frac{r}{2}}\right)^{\frac{u}{(p-t)s+r}}A^{\frac{r}{2}}D^{\frac{s}{2}}\right\}^{\frac{q-t-(p-t)s}{(p-t)s+r+u}}D^{\frac{s}{2}} \quad \text{by Lemma 7.1}\\
&\geq D^{\frac{s}{2}}\left(D^{\frac{s}{2}}A^{r+u}D^{\frac{s}{2}}\right)^{\frac{q-t-(p-t)s}{(p-t)s+r+u}}D^{\frac{s}{2}}\\
&= G_{p,q,t}(A,B,r+u,s)
\end{aligned}
$$

and the last inequality holds by (7.29) and the Löwner-Heinz theorem since $\frac{q-t-(p-t)s}{(p-t)s+r+u} \in [-1,0]$. Consequently we obtain (iv) by (a) and (b), so the proof is complete. □

Corollary 7.13 *If $A \geq B > 0$, then the following inequalities* (i) *and* (ii) *hold*

(i) $$\delta\left\{B^{\frac{t}{2}}\left(B^{\frac{-t}{2}}A^{p}B^{\frac{-t}{2}}\right)^{s}B^{\frac{t}{2}}\right\}^{\frac{1}{(p-t)s+r}} \geq A \geq B \geq \left\{A^{\frac{t}{2}}\left(A^{\frac{-t}{2}}B^{p}A^{\frac{-t}{2}}\right)^{s}A^{\frac{t}{2}}\right\}^{\frac{1}{(p-t)s+r}}$$

(ii) $$\delta B^{\frac{-(r-t)}{2}}\left(B^{\frac{r-t}{2}}A^{p}B^{\frac{r-t}{2}}\right)^{\frac{1-t+r}{p-t+r}}B^{\frac{-(r-t)}{2}} \geq A \geq \delta B \geq A^{\frac{-(r-t)}{2}}\left(A^{\frac{r-t}{2}}B^{p}A^{\frac{r-t}{2}}\right)^{\frac{1-t+r}{p-t+r}}A^{\frac{-(r-t)}{2}}$$

for each $t \in [0,1]$, $p \geq 1$, $r \geq t$ and $s \geq 1$.

Proof. (i) Theorem 7.12 yields

$$F_{p,t}(A,B,t,1) \geq F_{p,t}(A,B,t,s) \geq F_{p,t}(A,B,r,s)$$

for $t \in [0,1]$, $p \geq 1$, $r \geq t$ and $s \geq 1$, so that we have the latter half inequality, and the former one follows by the letter one by taking inverses of both sides as seen in the proof of (i) via (ii) of Theorem F.

(ii) Theorem 7.12 yields

$$F_{p,t}(A,B,t,1) \geq F_{p,t}(A,B,r,1) \geq F_{p,t}(A,B,r,s)$$

for $t \in [0,1]$, $p \geq 1$, $r \geq t$ and $s \geq 1$, so that we have the latter half inequality, and the former is easily shown as the same way as in (i). □

Corollary 7.14 *If $A \geq B > 0$, then the following inequality holds*

$$\delta B^{\frac{-r}{2}}\left(B^{\frac{r}{2}}A^pB^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}}B^{\frac{-r}{2}} \geq A \geq B \geq A^{\frac{-r}{2}}\left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}}A^{\frac{-r}{2}}$$

for $p \geq 1$ and $r \geq 0$.

Proof. We have only to put $t = 0$ in (ii) of Corollary 7.13. □

Remark 7.2 *Corollary 7.14 easily yields Theorem 7.2.*

Corollary 7.15 *If $A \geq B > 0$, then the following* (i) *and* (ii) *hold:*

(i) $\delta f(p,r) = B^{\frac{-r}{2}}\left(B^{\frac{r}{2}}A^pB^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}}B^{\frac{-r}{2}}$ *is an increasing function of both $p \geq 1$ and $r \geq 0$.*

(ii) $\delta g(p,r) = A^{\frac{-r}{2}}\left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}}A^{\frac{-r}{2}}$ *is a decreasing function of both $p \geq 1$ and $r \geq 0$.*

Proof. (ii) Put $t = 0$ and $p = 1$ in (iii) of Theorem 7.12, and then replace s by p.

(i) Since $B^{-1} \geq A^{-1}$ holds, (ii) yields that

$$\delta B^{\frac{r}{2}}(B^{\frac{-r}{2}}A^{-p}B^{\frac{-r}{2}})^{\frac{1+r}{p+r}}B^{\frac{r}{2}}$$

is a decreasing function of both $p \geq 1$ and $r \geq 0$, so that we have (i) by taking inverse. □

Remark 7.3 *Corollary 7.15 easily implies Corollary 7.14.*

Corollary 7.16 *If $A \geq B > 0$, then the following* (i) *and* (ii) *hold:*

(i) For any fixed $t \geq 0$,

$$\delta f(p,r) = B^{\frac{-r}{2}}\left(B^{\frac{r}{2}}A^pB^{\frac{r}{2}}\right)^{\frac{t+r}{p+r}}B^{\frac{-r}{2}} \text{ is an increasing function of both } p \geq t \text{ and } r \geq 0.$$

(ii) For any fixed $t \geq 0$,

$$\delta g(p,r) = A^{\frac{-r}{2}}\left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{t+r}{p+r}}A^{\frac{-r}{2}} \text{ is a decreasing function of both } p \geq t \text{ and } r \geq 0.$$

Proof. (i) (i) of Corollary 7.15 ensures that if $A \geq B > 0$, then

$$\delta f(p',r') = B^{\frac{-r'}{2}}(B^{\frac{r'}{2}}A^{p'}B^{\frac{r'}{2}})^{\frac{1+r'}{p'+r'}}B^{\frac{-r'}{2}}$$

is an increasing function of both $p' \geq 1$ and $r' \geq 0$. We have only to put $p' = \frac{p}{t} \geq 1$ and $r' = \frac{r}{t} \geq 0$

(ii) It follow from the same way as one in (i) by using (ii) of Corollary 7.15. □

7.3 Chaotic order and the relative operator entropy

In this section, as applications of Furuta inequality, we pick up several inequalities for the relative operator entropy.

We recall that the relative operator entropy $S(A|B)$ is defined by

$$S(A|B) = A^{\frac{1}{2}}\left(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}$$

for invertible positive operators A and B on a Hilbert space H.

We begin with characterizations of the chaotic order.

Theorem 7.17 *Let A and B be positive invertible operators. Then the following assertions are mutually equivalent.*

(I) $A \gg B$ (i.e., $\log A \geq \log B$).

(II$_1$) $A^u \geq \left(A^{\frac{u}{2}}B^pA^{\frac{u}{2}}\right)^{\frac{u}{p+u}}$ *for all* $p \geq 0$ *and all* $u \geq 0$,

(II$_2$) $A^u \geq \left(A^{\frac{u}{2}}B^{p_0}A^{\frac{u}{2}}\right)^{\frac{u}{p_0+u}}$ *for a fixed positive number* p_0 *and for all* u *such that* $u \in [0,u_0]$, *where* u_0 *is a fixed positive number.*

(III$_1$) $\log A^{p+u} \geq \log\left(A^{\frac{u}{2}}B^pA^{\frac{u}{2}}\right)$ *for all* $p \geq 0$ *and all* $u \geq 0$.

(III$_2$) $\log A^{u+p_0} \geq \log\left(A^{\frac{u}{2}}B^{p_0}A^{\frac{u}{2}}\right)$ *for a fixed positive number* p_0 *and for all* u *such that* $u \in [0,u_0]$, *where* u_0 *is a fixed positive number.*

Proof. (I)$\Longleftrightarrow$(II$_1$) is shown in Theorem 7.7.

(III$_2$)$\Longrightarrow$(I). We have only to put $u = 0$ in (III$_2$).

(II$_1$)$\Longrightarrow$(II$_2$)$\Longrightarrow$(III$_2$) and (II$_1$)$\Longrightarrow$(III$_1$)$\Longrightarrow$(III$_2$) are obviously since $\log t$ is operator monotone. Hence the proof is complete. □

Theorem 7.18 *Let A, B and C be positive invertible operators. Then the following assertions are mutually equivalent.*

(I) $C \gg A \gg B$ (i.e., $\log C \geq \log A \geq \log B$).

(II$_1$) $\left(A^{\frac{u}{2}}C^pA^{\frac{u}{2}}\right)^{\frac{u}{p+u}} \geq A^u \geq \left(A^{\frac{u}{2}}B^pA^{\frac{u}{2}}\right)^{\frac{u}{p+u}}$ *for all* $p \geq 0$ *and all* $u \geq 0$.

(II$_2$) $\left(A^{\frac{u}{2}}C^{p_0}A^{\frac{u}{2}}\right)^{\frac{u}{p_0+u}} \geq A^u \geq \left(A^{\frac{u}{2}}B^{p_0}A^{\frac{u}{2}}\right)^{\frac{u}{p_0+u}}$ *for a fixed positive number* p_0 *and for all* u *such that* $u \in [0,u_0]$, *where* u_0 *is a fixed positive number.*

(III$_1$) $\log\left(A^{\frac{u}{2}}C^pA^{\frac{u}{2}}\right) \geq \log A^{p+u} \geq \log\left(A^{\frac{u}{2}}B^pA^{\frac{u}{2}}\right)$ *for all* $p \geq 0$ *and all* $u \geq 0$,

(III$_2$) $\log\left(A^{\frac{u}{2}}C^{p_0}A^{\frac{u}{2}}\right) \geq \log A^{p_0+u} \geq \log\left(A^{\frac{u}{2}}B^{p_0}A^{\frac{u}{2}}\right)$ *for a fixed positive number* p_0 *and for all u such that* $u \in [0,u_0]$, *where* u_0 *is a fixed positive number.*

(IV$_1$) $S(A^{-u}|C^p) \geq S(A^{-u}|A^p) \geq S(A^{-u}|B^p)$ *for all* $p \geq 0$ *and all* $u \geq 0$.

(IV$_2$) $S(A^{-u}|C^{p_0}) \geq S(A^{-u}|A^{p_0}) \geq S(A^{-u}|B^{p_0})$ *for a fixed positive number* p_0 *and for all u such that* $u \in [0,u_0]$ *,where* u_0 *is a fixed positive number.*

Proof. (I)$\Longleftrightarrow$(II$_1$)$\Longleftrightarrow$(II$_2$)$\Longleftrightarrow$(III$_1$)$\Longleftrightarrow$(III$_2$) is easy by Theorem 7.17.

(III$_1$)$\Longleftrightarrow$(IV$_1$) and (III$_2$)$\Longleftrightarrow$(IV$_2$) are obtained by the definition of the relative operator entropy. □

Corollary 7.19 *Let* A,B *and* C *be positive invertible operators. If* $C \gg A^{-1} \gg B$, *then*

$$S(A|C) \geq -2A\log A \geq S(A|B).$$

Proof. Put $p = u = 1$ and replace A by A^{-1} in (IV$_1$) of Theorem 7.18. Then

$$S(A|C) \geq S(A|A^{-1}) \geq S(A|B)$$

and the proof is complete since $S(A|A^{-1}) = -2A\log A$. □

The following theorem is an improvement of Theorem 5.12.

Theorem 7.20 *Let A and B be positive invertible operators. For any positive number* x_0, *the following inequality holds;*

$$(\log x_0 - 1)A + \delta\frac{1}{x_0}B \geq S(A|B) \geq (1-\log x_0)A - \delta\frac{1}{x_0}AB^{-1}A.$$

In particular, $S(A|B) = 0$ *holds if and only if* $A = B$.

Proof. First of all, we cite the following obvious inequality for any positive numbers x and x_0

$$\log x_0 - 1 + \frac{x}{x_0} \geq \log x \geq 1 - \log x_0 - \frac{1}{x_0 x}. \tag{7.30}$$

We can interchange x with positive operator $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in (7.30), then

$$\begin{aligned} A^{\frac{1}{2}}\left(\log x_0 - 1 + \frac{1}{x_0}A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}} &\geq A^{\frac{1}{2}}\left(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}} \\ &\geq A^{\frac{1}{2}}\left(1-\log x_0 - \frac{1}{x_0}A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}\right)A^{\frac{1}{2}}, \end{aligned}$$

that is,

$$(\log x_0 - 1)A + \delta\frac{1}{x_0}B \geq S(A|B) \geq (1-\log x_0)A - \delta\frac{1}{x_0}AB^{-1}A.$$

For the proof of the latter part, put $x_0 = 1$ and $S(A|B) = 0$, then

$$-A + B \geq 0 \geq A - AB^{-1}A,$$

that is, $B \geq A$ and $AB^{-1}A \geq A$. The latter inequality is equivalent to $A \geq B$, so that $A = B$ holds. That is, $S(A|B) = 0$ ensures $A = B$, and the reverse implication is trivial by the definition of $S(A|B)$. Hence the proof is complete. □

Next, we study operator functions associated with the chaotic order.

Theorem 7.21 *Let A and B be positive invertible operators. Then the following assertions are mutually equivalent.*

(I) $A \gg B$ (i.e., $\log A \geq \log B$).

(II) *For any fixed $t \geq 0$,*

$$F(p,r) = B^{\frac{-r}{2}}\left(B^{\frac{r}{2}}A^pB^{\frac{r}{2}}\right)^{\frac{t+r}{p+r}}B^{\frac{-r}{2}} \text{ is an increasing function of both } p \geq t \text{ and } r \geq 0.$$

(III) *For any fixed $t \geq 0$,*

$$G(p,r) = A^{\frac{-r}{2}}\left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{t+r}{p+r}}A^{\frac{-r}{2}} \text{ is a decreasing function of both } p \geq t \text{ and } r \geq 0.$$

Proof. (I) $\Longrightarrow$ (III) $\log A \geq \log B$ is equivalent to the following (1) by Theorem 7.18

$$A^r \geq \left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{r}{p+r}} \quad \text{for all } r \geq 0 \text{ and } p \geq 0 \tag{1}$$

and (1) is also equivalent to the following (2) by Lemma 7.1

$$\left(B^{\frac{p}{2}}A^rB^{\frac{p}{2}}\right)^{\frac{p}{r+p}} \geq B^p \quad \text{for all } p \geq 0 \text{ and } r \geq 0. \tag{2}$$

Applying the Löwner-Heinz theorem to (1) and (2), we have the following (3) and (4) respectively

$$A^u \geq \left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{u}{p+r}} \quad \text{for all } r \geq u \geq 0 \text{ and } p \geq 0 \tag{3}$$

$$\left(B^{\frac{p}{2}}A^rB^{\frac{p}{2}}\right)^{\frac{w}{r+p}} \geq B^w \quad \text{for all } p \geq w \geq 0 \text{ and } r \geq 0. \tag{4}$$

(a) $G(p,r)$ **is a decreasing function of** p**.**

$$g(p,r) = \left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{t+r}{p+r}}$$

$$= \left\{ \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}} \right)^{\frac{p+w+r}{p+r}} \right\}^{\frac{t+r}{p+w+r}}$$

$$= \left\{ A^{\frac{r}{2}} B^{\frac{p}{2}} \left(B^{\frac{p}{2}} A^r B^{\frac{p}{2}} \right)^{\frac{w}{p+r}} B^{\frac{p}{2}} A^{\frac{r}{2}} \right\}^{\frac{t+r}{p+w+r}} \quad \text{by Lemma 7.1}$$

$$\geq \left(A^{\frac{r}{2}} B^{p+w} A^{\frac{r}{2}} \right)^{\frac{t+r}{p+w+r}}$$

$$= g(p+w,r)$$

and the last inequality holds by (4) and the Löwner-Heinz theorem since $\frac{t+r}{p+w+r} \in [0,1]$, so that $= g(p,r)$ is a decreasing of p, and $G(p,r) = A^{\frac{-r}{2}} g(p,r) A^{\frac{-r}{2}}$ is also a decreasing function of p.

(b) $G(p,r)$ **is a decreasing function of** r**.**

$$G(p,r) = A^{\frac{-r}{2}} \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}} \right)^{\frac{t+r}{p+r}} A^{\frac{-r}{2}}$$

$$= B^{\frac{p}{2}} \left(B^{\frac{p}{2}} A^r B^{\frac{p}{2}} \right)^{\frac{t-p}{p+r}} B^{\frac{p}{2}} \quad \text{by Lemma 7.1}$$

$$= B^{\frac{p}{2}} \left\{ \left(B^{\frac{p}{2}} A^r B^{\frac{p}{2}} \right)^{\frac{r+u+p}{p+r}} \right\}^{\frac{t-p}{r+u+p}} B^{\frac{p}{2}}$$

$$= B^{\frac{p}{2}} \left\{ B^{\frac{p}{2}} A^{\frac{r}{2}} \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}} \right)^{\frac{u}{p+r}} A^{\frac{r}{2}} B^{\frac{p}{2}} \right\}^{\frac{t-p}{r+u+p}} B^{\frac{p}{2}} \quad \text{by Lemma 7.1}$$

$$\geq B^{\frac{p}{2}} \left(B^{\frac{p}{2}} A^{r+u} B^{\frac{p}{2}} \right)^{\frac{t-p}{r+u+p}} B^{\frac{p}{2}}$$

$$= G(p,r+u)$$

and the last inequality holds by (3) and the Löwner-Heinz theorem since $\frac{t-p}{r+u+p} \in [-1,0]$, so that $G(p,r)$ is a decreasing function of r. Whence the proof of (I) $\Longrightarrow$ (III) is complete by (a) and (b).

(III) $\Longrightarrow$ (I) Assume (III). Then $G(p,0) \geq G(p,r)$ with $t=0$, that is,

$$1_H \geq A^{\frac{-r}{2}} \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}} \right)^{\frac{r}{p+r}} A^{\frac{-r}{2}} \quad \text{for all } p \geq 0 \text{ and } r \geq 0,$$

that is,

$$A^r \geq \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}} \right)^{\frac{r}{p+r}} \quad \text{for all } p \geq 0 \text{ and } r \geq 0,$$

so that $\log A \geq \log B$ by Theorem 7.7. Therefore (I) $\Longleftrightarrow$ (III) is proved.

(I) $\Longleftrightarrow$ (II) Since $\log A \geq \log B$ is equivalent to $\log B^{-1} \geq \log A^{-1}$, so that by applying this latter condition to (I) $\Longleftrightarrow$ (III), (I) is equivalent to the following (5).

(5) For any fixed $t \geq 0$

$$B^{\frac{r}{2}}\left(B^{\frac{-r}{2}}A^{-p}B^{\frac{-r}{2}}\right)^{\frac{t+r}{p+r}}B^{\frac{r}{2}} \text{ is a decreasing function of } p \geq t \text{ and } r \geq 0,$$

(5) is equivalent to the following (6)

(6) For any fixed $t \geq 0$

$$F(p,r) = B^{\frac{-r}{2}}\left(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}}\right)^{\frac{t+r}{p+r}}B^{\frac{-r}{2}} \text{ is an increasing function of } p \geq t \text{ and } r \geq 0,$$

so that (I) $\Longleftrightarrow$ (II). Whence the proof of Theorem 7.21 is complete. $\square$

7.4 Notes

A proof of the Furuta inequality is due to [70] and we also refer to [54], [111] and [73]. Theorem 7.3 and Theorem 7.4 are due to Cho, Furuta, J.I.Lee and W.Y.Lee [23]. An excellent and tough proof of the best possibility of the Furuta inequality is obtained in Tanahashi [183].

A proof of the generalized Furuta inequality is due to [76] and we refer to [62], [94] and [81]. The best possibility of the generalized Furuta inequality is contained in Tanahashi [184] and alternative proofs are in [196] and [65].

The spacial case of Theorem 7.7 appeared in Ando [5]. Theorem 7.12 is due to Furuta, Hashimoto and Ito [90]. Corollary 7.16 is due to Furuta [74] and M.Fujii, Furuta and Kamei [55].

For our exposition we have used a work of Furuta [84] and [85].

Chapter 8

Mond-Pečarić ideas in operator order

In this chapter, we observe the operator order $\geq$ and the chaotic order $\gg$ in the algebra $\mathscr{B}(H)$ according to Definition 1.1 and Definition 4.1. We study some characterizations of the operator order and the chaotic one by virtue of the Kantorovich inequality. We call them Kantorovich type inequalities of the operator order and the chaotic one.

8.1 Fundamental results

When we observe the inequalities which preserve the operator order, we recall the Hansen-Pedersen theorem (Theorem 1.11). For a function $f \in \mathscr{C}([0,\infty))$, if $f(t) \geq 0$ for all $t \in [0,\infty)$, then f is operator monotone if and only if it is operator concave. So if f is a convex function, then it can not be an operator monotone function. For example, the Löwner-Heinz Theorem asserts that $A,B \in \mathscr{B}^+(H)$, $A \geq B \geq 0$ imply $A^p \geq B^p$ for $1 \geq p \geq 0$. However, $A \geq B \geq 0$ does not imply $A^2 \geq B^2$ in general. As an application of the Kantorovich inequality we show that a function $f(t) = t^2$ preserves the operator order in the following sense:

Theorem 8.1 *Let $A,B \in \mathscr{B}^+(H)$ be positive operators on a Hilbert space H with* $\mathsf{Sp}(B) \subseteq$

$[m,M]$ for some scalars $M > m > 0$. Then

$$A \geq B > 0 \quad \textit{imply} \quad \frac{(M+m)^2}{4Mm}A^2 \geq B^2.$$

Proof. By the Kantorovich inequality (Theorem 1.29), we have

$$\begin{aligned}(B^2x,x) &\leq \frac{(M+m)^2}{4Mm}(Bx,x)^2 \quad \text{by} \quad M1_H \geq B \geq m1_H > 0 \\ &\leq \frac{(M+m)^2}{4Mm}(Ax,x)^2 \quad \text{by} \quad A \geq B \geq 0 \\ &\leq \frac{(M+m)^2}{4Mm}(A^2x,x) \quad \text{by the Hölder-McCarthy inequality}\end{aligned}$$

for every unit vector $x \in H$. Therefore it follows that $\frac{(M+m)^2}{4Mm}A^2 \geq B^2$. □

The number $\frac{(M+m)^2}{4Mm}$ is called the Kantorovich constant.
Moreover, we have the following complementary result to Theorem 8.1.

Theorem 8.2 *Let $A,B \in \mathscr{B}^{++}(H)$ be positive invertible operators on H with $\mathsf{Sp}(A) \subseteq [n,N]$ for some scalars $N > n > 0$. Then*

$$A \geq B > 0 \quad \textit{imply} \quad \frac{(N+n)^2}{4Nn}A^2 \geq B^2.$$

Proof. Since $B^{-1} \geq A^{-1}$ and $\frac{1}{n}1_H \geq A^{-1} \geq \frac{1}{N}1_H > 0$, it follows from Theorem 8.1 that

$$\frac{(N+n)^2}{4Nn}B^{-2} = \frac{(\frac{1}{N}+\frac{1}{n})^2}{4\frac{1}{N}\frac{1}{n}}B^{-2} \geq A^{-2}$$

and hence $\frac{(N+n)^2}{4Nn}A^2 \geq B^2$. □

Generally, though the power function $f(t) = t^p$ $(0 \leq p \leq 1)$ is operator monotone, it follows that $A \geq B \geq 0$ does not always ensure $A^p \geq B^p$ for any $p > 1$. Related to this result, we have an extension of Theorem 8.1 and Theorem 8.2, which plays a fundamental role in this chapter.

Theorem 8.3 *Let $A,B \in \mathscr{B}^{+}(H)$ be positive operators on H with $\mathsf{Sp}(A) \subseteq [m_1,M_1]$ and $\mathsf{Sp}(B) \subseteq [m_2,M_2]$ for some scalars $M_j > m_j > 0\,(j=1,2)$. If $A \geq B > 0$, then the following inequalities hold:*

$$\left(\frac{M_1}{m_1}\right)^{p-1}A^p \geq K(m_1,M_1,p)A^p \geq B^p, \tag{8.1}$$

$$\left(\frac{M_2}{m_2}\right)^{p-1}A^p \geq K(m_2,M_2,p)A^p \geq B^p \tag{8.2}$$

for all $p \geq 1$, where $K(m,M,p)$ is defined by (2.78) in § 2.7:

$$K(m,M,p) = \frac{(mM^p - Mm^p)}{(p-1)(M-m)} \left(\frac{p-1}{p} \frac{M^p - m^p}{mM^p - Mm^p} \right)^p . \tag{8.3}$$

Proof. Put $h = \frac{M_1}{m_1} > 1$. By (vi) of Theorem 2.54, we have the first inequality in (8.1) and (8.2). Next, since $M_2 1_H \geq B \geq m_2 1_H > 0$, it follows from Theorem 2.53 that

$$K(m_2, M_2, p)(Bx,x)^p \geq (B^p x, x)$$

for every unit vector $x \in H$. Therefore, we have for all $p > 1$

$$\begin{aligned} (B^p x,x) &\leq K(m_2,M_2,p)(Bx,x)^p \\ &\leq K(m_2,M_2,p)(Ax,x)^p \quad \text{by} \quad A \geq B \geq 0 \\ &\leq K(m_2,M_2,p)(A^p x,x) \quad \text{by the Hölder-McCarthy inequality} \end{aligned}$$

for every unit vector $x \in H$. Therefore it follows that $K(m_2,M_2,p)A^p \geq B^p$.

On the other hand, since $B^{-1} \geq A^{-1}$ and $\frac{1}{m_1} 1_H \geq A^{-1} \geq \frac{1}{M_1} 1_H > 0$, it follows that

$$K\left(\frac{1}{M_1}, \frac{1}{m_1}, p\right) B^{-p} \geq A^{-p} \quad \text{for } p > 1.$$

Since $K\left(\frac{1}{M_1}, \frac{1}{m_1}, p\right) = K(m_1,M_1,p)$ by Theorem 2.54, taking inverse of both sides we have $K(m_1,M_1,p)^{-1} B^p \leq A^p$ and hence we have the desired inequality. □

For positive invertible operators A and B, we denote the chaotic order by $A \gg B$ if $\log A \geq \log B$. We give some characterizations of the chaotic order by applying Theorem 8.3 and Theorem 7.7.

Firstly, we show Kantorovich type operator inequalities of the chaotic order which are parallel to Theorem 8.3.

Theorem 8.4 *Let $A,B \in \mathscr{B}^{++}(H)$ be positive invertible operators with $\mathrm{Sp}(B) \subseteq [m,M]$ for some scalars $M > m > 0$. If $\log A \geq \log B$, then*

$$\left(\frac{M}{m}\right)^p A^p \geq K(m,M,p+1)A^p \geq B^p \qquad \text{for all } p \geq 0,$$

where the Kantorovich constant $K(m,M,p)$ is defined by (8.3).

Proof. Put $r = 1$ in (iii) of Theorem 7.7, then $\log A \geq \log B$ ensures the following inequality

$$\left(B^{\frac{1}{2}} A^p B^{\frac{1}{2}}\right)^{\frac{1}{p+1}} \geq B \qquad \text{for all } p \geq 0.$$

Put $A_1 = \left(B^{\frac{1}{2}} A^p B^{\frac{1}{2}}\right)^{\frac{1}{p+1}}$ and $B_1 = B$, then A_1 and B_1 satisfy $A_1 \geq B_1 > 0$ and $M 1_H \geq B_1 \geq m 1_H > 0$. Applying Theorem 8.3 to A_1 and B_1, we have

$$\left(\frac{M}{m}\right)^{p_1 - 1} \left(B^{\frac{1}{2}} A^p B^{\frac{1}{2}}\right)^{\frac{p_1}{p+1}} \geq K(m,M,p_1) \left(B^{\frac{1}{2}} A^p B^{\frac{1}{2}}\right)^{\frac{p_1}{p+1}} \geq B^{p_1} \tag{8.4}$$

for $p \geq 0$ and $p_1 \geq 1$.

Put $p_1 = p+1 \geq 1$ in (8.4) and multiply $B^{-\frac{1}{2}}$ on both sides, then we have

$$\left(\frac{M}{m}\right)^p A^p \geq K(m,M,p+1)A^p \geq B^p \qquad \text{for all } p \geq 0.$$

Hence the proof of Theorem 8.4 is complete. □

Theorem 8.5 *Let $A,B \in \mathscr{B}^{++}(H)$ be positive invertible operators with $\mathrm{Sp}(B) \subseteq [m,M]$ for some scalars $M > m > 0$. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) $\frac{(M^p+m^p)^2}{4M^pm^p}A^p \geq B^p$ *for all* $p \geq 0$.

Proof. $(i) \Longrightarrow (ii)$. Put $r = p$ in (iii) of Theorem 7.7, then $\log A \geq \log B$ ensures the following inequality

$$\left(B^{\frac{p}{2}}A^pB^{\frac{p}{2}}\right)^{\frac{1}{2}} \geq B^p \qquad \text{for all } p \geq 0.$$

Put $A_1 = \left(B^{\frac{p}{2}}A^pB^{\frac{p}{2}}\right)^{\frac{1}{2}}$ and $B_1 = B^p$, then A_1 and B_1 satisfy $A_1 \geq B_1 > 0$ and $M^p 1_H \geq B_1 \geq m^p 1_H > 0$. Applying Theorem 8.3 to A_1 and B_1, we have

$$K(m^p,M^p,p_1)\left(B^{\frac{p}{2}}A^pB^{\frac{p}{2}}\right)^{\frac{p_1}{2}} \geq (B^p)^{p_1} \tag{8.5}$$

for $p \geq 0$ and $p_1 \geq 1$.

Put $p_1 = 2 \geq 1$ in (8.5) and multiply $B^{-\frac{p}{2}}$ on both sides, then we have

$$\frac{(M^p+m^p)^2}{4M^pm^p}A^p = K(m^p,M^p,2)A^p \geq B^p \qquad \text{for all } p \geq 0.$$

Hence the proof of $(i) \Longrightarrow (ii)$ is complete.

$(ii) \Longrightarrow (i)$. Taking logarithm of both sides of (ii) since $\log t$ is operator monotone, we have

$$\log\left\{\left(\frac{(M^p+m^p)^2}{4M^pm^p}\right)^{\frac{1}{p}} A\right\} \geq \log B \qquad \text{for all } p \geq 0. \tag{8.6}$$

Noting that

$$\lim_{p\to+0}\left(\frac{M^p+m^p}{2}\right)^{\frac{1}{p}} = \sqrt{Mm},$$

we have

$$\lim_{p\to 0}\left(\frac{(M^p+m^p)^2}{4M^pm^p}\right)^{\frac{1}{p}} = \lim_{p\to 0}\frac{1}{Mm}\left(\frac{M^p+m^p}{2}\right)^{\frac{2}{p}} = \frac{1}{Mm}\left(\sqrt{Mm}\right)^2 = 1.$$

Therefore, letting $p \to 0$ in (8.6), we have $\log A \geq \log B$. □

Theorem 8.6 *Let $A,B \in \mathscr{B}^{++}(H)$ be positive invertible operators with $\mathsf{Sp}(B) \subseteq [m,M]$ for some scalars $M > m > 0$. If $\log A \geq \log B$, then*

$$K\left(m^r, M^r, \frac{p+r}{r}\right) A^p \geq B^p \qquad \text{for all } p \geq 0 \text{ and } r \geq 0,$$

where $K(m,M,p)$ is defined by (8.3).

Proof. It follows from Theorem 7.7 that $\log A \geq \log B$ is equivalent to the following inequality

$$\left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}}\right)^{\frac{r}{p+r}} \geq B^r \qquad \text{for all } p > 0 \text{ and } r > 0.$$

Put $A_1 = \left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}}\right)^{\frac{r}{p+r}}$ and $B_1 = B^r$, then A_1 and B_1 satisfy $A_1 \geq B_1 > 0$ and $M^r 1_H \geq B_1 \geq m^r 1_H > 0$. Applying Theorem 8.3 to A_1 and B_1, we have

$$K(m^r, M^r, p_1) A_1^{p_1} \geq B_1^{p_1} \qquad \text{for all } p_1 \geq 1. \tag{8.7}$$

Put $p_1 = \frac{p+r}{r} \geq 1$ in (8.7), then we have

$$K\left(m^r, M^r, \frac{p+r}{r}\right) B^{\frac{r}{2}} A^p B^{\frac{r}{2}} \geq B^{p+r}. \tag{8.8}$$

By multiplying $B^{-\frac{r}{2}}$ on both sides of (8.8), we have

$$K\left(m^r, M^r, 1 + \frac{p}{r}\right) A^p \geq B^p \qquad \text{for all } p \geq 0 \text{ and } r > 0.$$

Hence the proof of Theorem 8.6 is complete. □

Theorem 8.7 *Let $A,B \in \mathscr{B}^{++}(H)$ be positive invertible operators with $\mathsf{Sp}(B) \subseteq [m,M]$ for some scalars $M > m > 0$. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) $S(h,p) A^p \geq B^p$ *for all* $p \geq 0$,

where $h = M/m$ and the Specht ratio $S(h,p)$ is defined as (2.73).

Proof. $(i) \Longrightarrow (ii)$. By Theorem 8.6, the chaotic order $\log A \geq \log B$ implies

$$K\left(m^r, M^r, \frac{p+r}{r}\right) A^p \geq B^p \qquad \text{for all } p \geq 0 \text{ and } r \geq 0.$$

Letting $r \to 0$, we have $S(h,p) A^p \geq B^p$ for all $p > 0$ since $K\left(m^r, M^r, \frac{p+r}{r}\right) \to S(h,p)$ as $r \to 0$ by Theorem 2.56.

$(ii) \Longrightarrow (i)$. By taking logarithm of both sides of (ii), we have

$$\log\left(S(h,p)^{\frac{1}{p}} A\right) \geq \log B \qquad \text{for all } p > 0.$$

Then letting $p \to 0$, we have $\log A \geq \log B$ since $S(h,p)^{\frac{1}{p}} \to 1$ as $p \to 0$ by (v) of Lemma 2.47.

Hence the proof of Theorem 8.7 is complete. □

Remark 8.1 *Theorem 8.7 gives a more precise sufficient condition for the chaotic order than Theorem 8.5 and Theorem 8.6 since*

$$\frac{(M^p+m^p)^2}{4M^pm^p} = K(m^p,M^p,2) = K\left(m^p,M^p,\frac{p+p}{p}\right) \geq S(h,p) \quad \textit{for all } p>0$$

and

$$h^p = \left(\frac{M}{m}\right)^p \geq K\left(m^r,M^r,\frac{p+r}{r}\right) \geq S(h,p) \quad \textit{for all } p>0 \textit{ and } r>0.$$

Using a generalized Furuta inequality, we have the following results on the chaotic order and the operator one.

Theorem 8.8 *Let $A,B \in \mathscr{B}^{++}(H)$ be positive invertible operators with $\mathsf{Sp}(A) \subseteq [m,M]$ for some scalars $M>m>0$. Then the following assertions are mutually equivalent.*

(i) $\log A \geq \log B$.

(ii) For each $\alpha \in [0,1]$, $p \geq 0$ and $u \geq 0$,

$$\frac{(M^{(p+\alpha u)s}+m^{(p+\alpha u)s})^2}{4M^{(p+\alpha u)s}m^{(p+\alpha u)s}}A^{(p+\alpha u)s} \geq \left(A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}}\right)^s$$

for all $s \geq 1$ and $(p+\alpha u)s \geq (1-\alpha)u$.

(iii) For each $\alpha \in [0,1]$, $p \geq u \geq 0$,

$$\frac{(M^{(p+\alpha u)s}+m^{(p+\alpha u)s})^2}{4M^{(p+\alpha u)s}m^{(p+\alpha u)s}}A^{(p+\alpha u)s} \geq \left(A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}}\right)^s$$

for all $s \geq 1$.

(iv)

$$\frac{(M^p+m^p)^2}{4M^pm^p}A^p \geq B^p \quad \textit{for all } p \geq 0.$$

Proof. $(i) \Longrightarrow (ii)$. For each $p \geq 0$ and $u \geq 0$, put $A_1 = A^u$ and $B_1 = \left(A^{\frac{u}{2}}B^pA^{\frac{u}{2}}\right)^{\frac{u}{p+u}}$ in (ii) of Theorem 7.7. Then we have $A_1 \geq B_1 \geq 0$. By the generalized Furuta inequality, it follows that for each $t \in [0,1]$

$$A_1^{\frac{(p_1-t)s+r}{q}} \geq \left\{A_1^{\frac{r}{2}}\left(A_1^{-\frac{t}{2}}B_1^{p_1}A_1^{-\frac{t}{2}}\right)^sA_1^{-\frac{r}{2}}\right\}^{\frac{1}{q}}$$

for all $s \geq 1$, $p_1 \geq 1$, $q \geq 1$ and the following conditions

$$r \geq t, \tag{8.9}$$

$$(1-t+r)q \geq (p_1-t)s+r. \tag{8.10}$$

Put $p_1 = \frac{p+u}{u} \geq 1$ in the case $u > 0$, $q = 2$, $r = (p_1 - t)s$ and also put $\alpha = 1 - t$ in (8.9) and (8.10). Then (8.10) is satisfied, so the only required condition (8.9) is equivalent to the following

$$(p+\alpha u)s \geq (1-\alpha)u. \tag{8.11}$$

Therefore, we have for each $\alpha \in [0,1]$, $p \geq 0$ and $u \geq 0$

$$A^{(p+\alpha u)s} \geq \left\{ A^{\frac{(p+\alpha u)s}{2}} \left(A^{\frac{\alpha u}{2}} B^p A^{\frac{\alpha u}{2}} \right)^s A^{\frac{(p+\alpha u)s}{2}} \right\}^{\frac{1}{2}}$$

for $s \geq 1$ and the condition (8.11). Since $M^{(p+\alpha u)s} \geq A^{(p+\alpha u)s} \geq m^{(p+\alpha u)s} > 0$, it follows from Theorem 8.2 that

$$\frac{(M^{(p+\alpha u)s} + m^{(p+\alpha u)s})^2}{4M^{(p+\alpha u)s} m^{(p+\alpha u)s}} A^{(p+\alpha u)s} \geq \left(A^{\frac{\alpha u}{2}} B^p A^{\frac{\alpha u}{2}} \right)^s$$

for all $s \geq 1$ and $(p+\alpha u)s \geq (1-\alpha)u$.

$(ii) \Longrightarrow (iii)$. Put $p \geq u \geq 0$ in (ii). Then the required condition $(p+\alpha u)s \geq (1-\alpha)u$ is satisfied, so we have (iii).

$(iii) \Longrightarrow (iv)$. We have only to put $u = 0$ or $\alpha = 0$ and $s = 1$ in (iii).

$(iv) \Longrightarrow (i)$ is shown by Theorem 8.5. □

Theorem 8.9 *Let $A, B \in \mathscr{B}^+(H)$ be positive operators with $\mathsf{Sp}(A) \subseteq [m, M]$ for some scalars $M > m > 0$. Then the following assertions are mutually equivalent.*

(i) $A \geq B$.

(ii) For each $t \in [0,1]$,

$$\frac{(M^{(p-t)s} + m^{(p-t)s})^2}{4M^{(p-t)s} m^{(p-t)s}} A^{(p-t)s} \geq \left(A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}} \right)^s$$

for all $p \geq 1$ and $s \geq 1$ such that $(p-t)s \geq t$.

(iii)

$$\left(\frac{(M^{(p-1)s} + m^{(p-1)s})^2}{4M^{(p-1)s} m^{(p-1)s}} \right)^{\frac{1}{s}} A^p \geq B^p$$

for all $s \geq 1$ and $p \geq \frac{1}{s} + 1$.

(iv)

$$\left(\frac{M}{m} \right)^{p-1} A^p \geq B^p \qquad \text{for all } p \geq 1.$$

To prove Theorem 8.9, we need the following lemma.

Lemma 8.10 *If $M > m > 0$, then*

$$\lim_{s\to+\infty}\left(\frac{(M^s+m^s)^2}{4M^sm^s}\right)^{\frac{1}{s}} = \frac{M}{m}.$$

Proof. Put $x = \frac{M}{m} > 1$, then it follows from L'Hospital's theorem that

$$\lim_{s\to+\infty}\frac{\log(1+x^s)^2}{s} = \lim_{s\to+\infty}\frac{2x^s\log x}{1+x^s} = \log x^2.$$

Therefore we have

$$\begin{aligned}\lim_{s\to+\infty}\left(\frac{(M^s+m^s)^2}{4M^sm^s}\right)^{\frac{1}{s}} &= \lim_{s\to+\infty}\left(\frac{(1+x^s)^2}{4x^s}\right)^{\frac{1}{s}}\\ &= \lim_{s\to+\infty}\frac{(1+x^s)^{\frac{2}{s}}}{4^{1/s}x} = \frac{M}{m}\end{aligned}$$

□

Proof of Theorem 8.9.

$(i) \Longrightarrow (ii)$. Since $A \ge B \ge 0$ and $A > 0$, if we put $q = 2$ in a generalized Furuta inequality, then for $p \ge 1$, $s \ge 1$ and $t \in [0,1]$

$$A^{\frac{(p-t)s+r}{2}} \ge \left\{A^{\frac{r}{2}}\left(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}}\right)^sA^{\frac{r}{2}}\right\}^{\frac{1}{2}}$$

holds under the following conditions (8.12) and (8.13)

$$r \ge t \tag{8.12}$$

$$2(1-t+r) \ge (p-t)s+r. \tag{8.13}$$

If we moreover put $r = (p-t)s$, then (8.13) is satisfied and (8.12) is equivalent to the following

$$(p-t)s \ge t. \tag{8.14}$$

Therefore we have for $t \in (0,1]$, $p \ge 1$ and $s \ge 1$

$$A^{(p-t)s} \ge \left\{A^{\frac{(p-t)s}{2}}\left(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}}\right)^sA^{\frac{(p-t)s}{2}}\right\}^{\frac{1}{2}}$$

for the condition (8.14). Since $M^{(p-t)s} \ge A^{(p-t)s} \ge m^{(p-t)s} > 0$, the proof is complete by Theorem 8.2.

$(ii) \Longrightarrow (iii)$. If we put $t = 1$ in (ii), then we have (iii) by the Löwner-Heinz theorem.

$(iii) \Longrightarrow (iv)$. If we put $s \to +\infty$, then we have (iv) by Lemma 8.10.

$(iv) \Longrightarrow (i)$. If we put $p = 1$, then we have (i). □

By Theorem 8.9, we have the following corollary which is a parallel result with Theorem 8.5.

Corollary 8.11 *Let $A, B \in \mathscr{B}^+(H)$ be positive operators with $\mathrm{Sp}(A) \subseteq [m, M]$ for some scalars $M > m > 0$. If $A \geq B \geq 0$, then*

$$\frac{(M^{p-1}+m^{p-1})^2}{4M^{p-1}m^{p-1}}A^p \geq B^p \qquad \textit{for all } p \geq 2.$$

Proof. Put $s = 1$ in (iii) of Theorem 8.9. □

Let A and B be positive invertible operators on H. We consider an order $A^\delta \geq B^\delta$ for $\delta \in (0,1]$ which interpolates the operator order $A \geq B$ and the chaotic order $A \gg B$ continuously. The following theorem is easily obtained by Theorem 8.9.

Theorem 8.12 *Let $A, B \in \mathscr{B}^{++}(H)$ be positive invertible operators with $\mathrm{Sp}(A) \subseteq [m, M]$ for some scalars $M > m > 0$. If $A^\delta \geq B^\delta$ for $\delta \in (0,1]$, then*

$$\left(\frac{(M^{(p-\delta)s}+m^{(p-\delta)s})^2}{4M^{(p-\delta)s}m^{(p-\delta)s}}\right)^{\frac{1}{s}} A^p \geq B^p \qquad \textit{for all } s \geq 1 \textit{ and } p \geq (\tfrac{1}{s}+1)\delta.$$

Remark 8.2 *Theorem 8.12 interpolates Theorem 8.3 and Theorem 8.4 by means of the Kantorovich constant. Let A and B be positive invertible operators with $\mathrm{Sp}(A) \subseteq [m, M]$ for some scalars $M > m > 0$. Then the following assertions holds.*

(i) $A \geq B$ *implies* $\left(\frac{M}{m}\right)^{p-1} A^p \geq B^p$ *for all* $p \geq 1$.

(ii) $A^\delta \geq B^\delta$ *implies* $\left(\frac{(M^{(p-\delta)s}+m^{(p-\delta)s})^2}{4M^{(p-\delta)s}m^{(p-\delta)s}}\right)^{\frac{1}{s}} A^p \geq B^p$

for all $s \geq 1$ *and* $p \geq (\frac{1}{s}+1)\delta$.

(iii) $A \gg B$ *implies* $\left(\frac{M}{m}\right)^{p} A^p \geq B^p$ *for all* $p \geq 0$.

It follows that the Kantorovich constant of (ii) interpolates the scalar of (i) and (iii) continuously. In fact, if we put $\delta = 1$ and $s \to \infty$ in (ii), then we have (i), also if we put $\delta \to 0$ and $s \to \infty$ in (ii), then we have (iii).

Moreover, Theorem 8.12 interpolates Theorem 8.5 and Corollary 8.11 by means of the Kantorovich constant.

(i) $A \geq B$ *implies* $\frac{(M^{p-1}+m^{p-1})^2}{4M^{p-1}m^{p-1}}A^p \geq B^p$ *for all* $p \geq 2$.

(ii) $A^\delta \geq B^\delta$ *implies* $\left(\frac{(M^{(p-\delta)s}+m^{(p-\delta)s})^2}{4M^{(p-\delta)s}m^{(p-\delta)s}}\right)^{\frac{1}{s}} A^p \geq B^p$ *for all* $s \geq 1$ *and* $p \geq (\frac{1}{s}+1)\delta$.

(iii) $A \gg B$ *implies* $\frac{(M^p+m^p)^2}{4M^p m^p}A^p \geq B^p$ *for all* $p > 0$.

The Kantorovich constant of (ii) interpolates the scalar of (i) and (iii). In fact, if we put $\delta = 1$ and $s = 1$ in (ii), then we have (i), also if we put $s = 1$ and $\delta \to 0$ in (ii), then we have (iii).

We have the following result on preserving the operator order, which is parallel to Theorem 8.3.

Theorem 8.13 *If $A,B \in \mathscr{B}^+(H)$, $\mathsf{Sp}(B) \subseteq [m,M]$, $M > m > 0$ and $A \geq B > 0$, then*

$$A^p + C(m,M,p)1_H \geq B^p, \qquad \text{for all } p > 1,$$

where

$$C(m,M,p) = \frac{mM^p - Mm^p}{M-m}\left\{K(m,M,p)^{\frac{1}{p-1}} - 1\right\} \geq 0.$$

Moreover, these extensions are discussed by many authors and a distinction between the usual order and the chaotic one is clarified in the framework of Kantorovich type inequalities.

8.2 General form preserving the operator order

We show the order preserving operator inequalities under a more general setting, based on Kantorovich type inequalities for convex functions due to Mond-Pečarić in § 2.4. In this section, we assume that $M > m > 0$.

Theorem 8.14 *Let $A,B \in \mathscr{B}_h(H)$ with $\mathsf{Sp}(B) \subseteq [m,M]$, $f \in \mathscr{C}([m,M])$ be a convex function and $g \in \mathscr{C}(U)$, where $U \supseteq [m,M] \cup \mathsf{Sp}(A)$. Suppose that either of the following conditions holds: (i) g is increasing convex on U or (ii) g is decreasing concave on U. If $A \geq B$, then for a given $\alpha \in \mathbb{R}_+$ in the case (i) or $\alpha \in \mathbb{R}_-$ in the case (ii)*

$$\alpha g(A) + \beta 1_H \geq f(B) \tag{8.15}$$

holds for $\beta = \max_{m \leq t \leq M}\left\{\mu_f t + \nu_f - \alpha g(t)\right\}$, where

$$\mu_f = \frac{f(M)-f(m)}{M-m} \qquad \text{and} \qquad \nu_f = \frac{Mf(m)-mf(M)}{M-m}.$$

Proof. Let $x \in H$ be such that $(x,x) = 1$. Since αg is convex, then it follows from Theorem 1.2 that

$$\alpha(g(A)x,x) \geq \alpha g((Ax,x)).$$

On the other hand, since f is convex, then it follows from Theorem 2.25 that

$$\alpha g((Bx,x)) + \beta \geq (f(B)x,x)$$

for $\beta = \max_{m \leq t \leq M}\{\mu_f t + \nu_f - \alpha g(t)\}$. By the increase of αg we have

$$\alpha g((Ax,x)) \geq \alpha g((Bx,x)).$$

Therefore, combining three inequalities above we have

$$\alpha(g(A)x,x) + \beta \geq \alpha g((Ax,x)) + \beta \geq \alpha g((Bx,x)) + \beta \geq (f(B)x,x).$$

□

Remark 8.3 Let the hypothesis of Teorema 8.14 be satisfied. If αg is a strictly convex differentiable function on $[m,M]$, then by Theorem 2.25 the constant β may be defined as the unique solution of $f'(t) = \mu_f/\alpha$ when $f'(m) < \mu_f/\alpha < f'(M)$, otherwise t_o is defined as M or m according to $\mu_f/\alpha \geq f'(M)$ or $f'(m) \geq \mu_f/\alpha$.

The following theorem is a complementary result to Theorem 8.14:

Theorem 8.15 *Let $A,B \in \mathscr{B}_h(H)$ with $\mathsf{Sp}(A) \subseteq [m,M]$, $f \in \mathscr{C}([m,M])$ be a concave function and $g \in \mathscr{C}(U)$, where $U \supseteq [m,M] \cup \mathsf{Sp}(B)$. Suppose that either of the following conditions holds: (i) g is increasing concave on U or (ii) g is decreasing convex on U. If $A \geq B$, then for a given $\alpha \in \mathbb{R}_+$ in the case (i) or $\alpha \in \mathbb{R}_-$ in the case (ii)*

$$f(A) \geq \alpha g(B) + \beta 1_H, \tag{8.16}$$

holds for $\beta = \min_{m \leq t \leq M} \{\mu_f t + \nu_f - \alpha g(t)\}$.

Remark 8.4 Let the hypothesis of Teorema 8.15 be satisfied. If αg is a strictly concave differentiable function on $[m,M]$, then the constant β may be defined as the unique solution of $f'(t) = \mu_f/\alpha$ when $f'(M) < \mu_f/\alpha < f'(m)$, otherwise t_o is defined as M or m according to $\mu_f/\alpha \leq f'(M)$ or $f'(m) \leq \mu_f/\alpha$.

If we put $\alpha = 1$ in Theorems 8.14 and 8.15 we have the following corollary.

Corollary 8.16 *Let $A,B \in \mathscr{B}_h(H)$, $\mathsf{Sp}(B) \subseteq [m,M]$ (resp. $\mathsf{Sp}(A) \subseteq [m,M]$). Let $f \in \mathscr{C}([m,M])$ be a convex (resp. concave) function and $g \in \mathscr{C}(U)$ be an increasing convex (resp. increasing concave) function, where $U \supseteq [m,M] \cup \mathsf{Sp}(A) \cup \mathsf{Sp}(B)$. If $A \geq B$, then*

$$g(A) + \beta 1_H \geq f(B) \qquad (\text{resp. } f(A) \geq g(B) + \beta 1_H),$$

holds for

$$\beta = \max_{m \leq t \leq M} \{\mu_f t + \nu_f - g(t)\} \quad (\text{resp. } \beta = \min_{m \leq t \leq M} \{\mu_f t + \nu_f - g(t)\}).$$

If we choose α such that $\beta = 0$ in Theorems 8.14 and 8.15, then we have the following corollary:

Corollary 8.17 *Let $A,B \in \mathscr{B}_h(H)$, $\mathsf{Sp}(B) \subseteq [m,M]$ (resp. $\mathsf{Sp}(A) \subseteq [m,M]$). Let $f \in \mathscr{C}([m,M])$ be a convex (resp. concave) function and $g \in \mathscr{C}(U)$, where $U \supseteq [m,M] \cup \mathsf{Sp}(A) \cup \mathsf{Sp}(B)$. Suppose that either of the following conditions holds:*
(I) *g is increasing convex (resp. concave) on U, $g > 0$ on $[m,M]$ and $f(m) > 0$, $f(M) > 0$,*
(II) *g is increasing convex (resp. concave) on U, $g < 0$ on $[m,M]$ and $f(m) < 0$, $f(M) < 0$,*
(III) *g is decreasing concave (resp. convex) on U, $g > 0$ on $[m,M]$ and $f(m) < 0$, $f(M) < 0$,*
(IV) *g is decreasing concave (resp. convex) on U, $g < 0$ on $[m,M]$ and $f(m) > 0$, $f(M) > 0$.*
If $A \geq B$, then

$$\alpha_1 g(A) \geq f(B) \qquad (\text{resp. } f(A) \geq \alpha_2 g(B))$$

holds for

$$\alpha_1 = \max_{m \le t \le M} \left\{ \frac{\mu_f t + \nu_f}{g(t)} \right\} \quad \left(\textit{resp.} \quad \alpha_2 = \min_{m \le t \le M} \left\{ \frac{\mu_f t + \nu_f}{g(t)} \right\} \right)$$

in case (I) *and* (III), *or*

$$\alpha_1 = \min_{m \le t \le M} \left\{ \frac{\mu_f t + \nu_f}{g(t)} \right\} \quad \left(\textit{resp.} \quad \alpha_2 = \max_{m \le t \le M} \left\{ \frac{\mu_f t + \nu_f}{g(t)} \right\} \right)$$

in case (II) *and* (IV).

Remark 8.5 Let the hypothesis of Corollary 8.17 be satisfied. Suppose that additionally either of the following conditions holds: (a) g is a strictly convex twice differentiable function on $[m,M]$ in case (I) or (II) or (b) g is strictly concave two differentiable on $[m,M]$ in case (III) or (IV), then by Corollary 2.26 α_1 and α_2 may be defined more precisely as follows: $\alpha_1 = \alpha_2 = \left(\mu_f t_o + \nu_f\right)/g(t_o)$, where $t_o \in [m,M]$ is defined as the unique solution of $\mu_f g(t) = g'(t)(\mu_f t + \nu_f)$ if $f(M)g'(M)/g(M) > \mu_f > f(m)g'(m)/g(m)$, otherwise t_o is defined as M or m according to $\mu_f \ge f(M)g'(M)/g(M)$ or $f(m)g'(m)/g(m) \ge \mu_f$.

8.3 Form preserving the operator order for convex function

Applying results in § 8.2, in this section we show function order preserving operator inequalities.

We recall that if f is a convex function, then it can not be operator monotone. We show that a convex function preserves the operator order in the following sense:

Theorem 8.18 *Let $A,B \in \mathscr{B}_h(H)$, $\mathsf{Sp}(B) \subseteq [m,M]$, $f \in \mathscr{C}(U)$ be a strictly convex increasing differentiable function, where $U \supseteq [m,M] \cup \mathsf{Sp}(A)$. If $A \ge B$, then for a given $\alpha \in \mathbb{R}_+$*

$$\alpha f(A) + \beta 1_H \ge f(B),$$

hold for $\beta = \mu_f t_o + \nu_f - \alpha f(t_o)$ and $t_o \in [m,M]$ is defined as the unique solution of $f'(t) = \mu_f/\alpha$ when $f'(m) < \mu_f/\alpha < f'(M)$, otherwise t_0 is defined as M or m according to $f'(M) \le \mu_f/\alpha$ or $\mu_f/\alpha \le f'(m)$.

Proof. If we put $g = f$ in Theorem 8.14 and Remark 8.3, then we have this theorem. □

Though a concave increasing function is not always operator monotone, we have the following theorem which is a complementary result to Theorem 8.18.

Theorem 8.19 *Let $A,B \in \mathscr{B}_h(H)$, $\mathsf{Sp}(A) \subseteq [m,M]$, $f \in \mathscr{C}(U)$ be a strictly concave increasing differentiable function, where $U \supseteq [m,M] \cup \mathsf{Sp}(B)$. If $A \geq B$, then for a given $\alpha \in \mathbb{R}_+$*

$$f(A) \geq \alpha f(B) + \beta 1_H$$

holds for $\beta = \mu_f t_o + \nu_f - \alpha f(t_o)$ and $t_o \in [m,M]$ is defined as the unique solution of $f'(t) = \mu_f/\alpha$ when $f'(M) < \mu_f/\alpha < f'(m)$, otherwise t_0 is defined as M or m according to $f'(M) \geq \mu_f/\alpha$ or $\mu_f/\alpha \geq f'(m)$.

If we put $g \equiv f$ in Corollaries 8.16 and 8.17 then we have the following two results on functions preserving the operator order.

Corollary 8.20 *Let $A,B \in \mathscr{B}_h(H)$, $\mathsf{Sp}(B) \subseteq [m,M]$ (resp. $\mathsf{Sp}(A) \subseteq [m,M]$). Let $f \in \mathscr{C}(U)$ be a strictly convex (resp. strictly concave) increasing differentiable function, where $U \supseteq [m,M] \cup \mathsf{Sp}(A) \cup \mathsf{Sp}(B)$. If $A \geq B$, then*

$$f(A) + \beta\, 1_H \geq f(B) \qquad (\textit{resp. } f(A) \geq f(B) + \beta\, 1_H),$$

where $\beta = \mu_f t_o + \nu_f - f(t_o)$ and $t_o \in (m,M)$ is the unique solution of $f'(t) = \mu_f$.

Corollary 8.21 *Let $A,B \in \mathscr{B}_h(H)$, $\mathsf{Sp}(B) \subseteq [m,M]$ (resp. $\mathsf{Sp}(A) \subseteq [m,M]$). Let $f \in \mathscr{C}(U)$ be a convex (resp. concave) increasing function, where $U \supseteq [m,M] \cup \mathsf{Sp}(A) \cup \mathsf{Sp}(B)$. Suppose that either of the following conditions holds:* (i) *$f > 0$ on $[m,M]$ or* (ii) *$f < 0$ on $[m,M]$. If $A \geq B$, then*

$$\alpha_1 f(A) \geq f(B) \qquad (\textit{resp. } f(A) \geq \alpha_2 f(B)),$$

where

$$\alpha_1 = \max_{m \leq t \leq M} \left\{ \frac{\mu_f t + \nu_f}{f(t)} \right\} \quad \left(\textit{resp. } \alpha_2 = \min_{m \leq t \leq M} \left\{ \frac{\mu_f t + \nu_f}{f(t)} \right\} \right), \quad \textit{in case (i)}$$

or

$$\alpha_1 = \min_{m \leq t \leq M} \left\{ \frac{\mu_f t + \nu_f}{f(t)} \right\} \quad \left(\textit{resp. } \alpha_2 = \max_{m \leq t \leq M} \left\{ \frac{\mu_f t + \nu_f}{f(t)} \right\} \right), \quad \textit{in case (ii)}.$$

Moreover, if f is a twice differentiable function on U, then a value of $\alpha_{1,2}$ may be determined more precisely as follows: $\alpha_{1,2} = \frac{\mu_f t_0 + \nu_f}{f(t_0)}$, where $t_0 \in [m,M]$ is the unique solution of $\mu_f f(t) = f'(t)(\mu_f t + \nu_f)$.

We show a function order version of Theorem 8.3:

Corollary 8.22 *Let $A,B \in \mathscr{B}_h(H)$, $\mathsf{Sp}(B) \subseteq [m,M]$, $f \in \mathscr{C}(U)$ be a strictly convex increasing twice differentiable function, where $U \supseteq [m,M] \cup \mathsf{Sp}(A) \cup \mathsf{Sp}(B)$. Let $f > 0$ on U. If $A \geq B$, then:*

$$\frac{f'(M)}{f'(m)} f(A) \geq \alpha f(A) \geq f(B),$$

where $\alpha = (\mu_f t_o + \nu_f)/f(t_o)$ and $t_o \in (m,M)$ is the unique solution of $\mu_f f(t) = f'(t)(\mu_f t + \nu_f)$.

Proof. By the assumption of f, we have $\mu \leq f'(M)$ and $0 < f'(m) \leq f'(t_o)$, where $t_o \in (m,M)$ is such that $\mu f(t_o) = f'(t_o)(f(m) + \mu(t_o - m))$. Then we have

$$0 < \alpha = \frac{f(m) + \mu(t_o - m)}{f(t_o)} = \frac{\mu}{f'(t_o)} \leq \frac{f'(M)}{f'(m)}.$$

Therefore Corollary 8.21 implies $\frac{f'(M)}{f'(m)} f(A) \geq \alpha f(A) \geq f(B)$. □

Remark 8.6 *If we put $f(t) = t^p$ in Corollary 8.22, then we have*

$$\frac{f'(M)}{f'(m)} = \left(\frac{M}{m}\right)^{p-1}.$$

If we put $f(t) = t^p$ for $p > 1$ in Theorem 8.18, then we have the following corollary.

Corollary 8.23 *Let $A, B \in \mathscr{B}_h(H)$ with $\mathsf{Sp}(B) \subseteq [m,M]$. If $A \geq B$, then for a given $\alpha > 0$*

$$\alpha A^p + \beta 1_H \geq B^p \qquad \textit{for all } \; p > 1,$$

where

$$\beta = \begin{cases} \alpha(p-1)\left(\frac{1}{\alpha p}\frac{M^p - m^p}{M-m}\right)^{\frac{p}{p-1}} + \frac{Mm^p - mM^p}{M-m} & \textit{if} \quad pm^{p-1} \leq \frac{M^p - m^p}{\alpha(M-m)} \leq pM^{p-1}, \\ \max\{M^p - \alpha M^p, m^p - \alpha m^p\} & \textit{otherwise.} \end{cases}$$

Remark 8.7 We have Theorem 8.13 if we put $\alpha = 1$ in Corollary 8.23 and Theorem 8.3 if we choose α such that $\beta = 0$ in Corollary 8.23. As a matter of fact, if we put $\beta = \alpha(p-1)\left(\frac{1}{\alpha p}\frac{M^p - m^p}{M-m}\right)^{\frac{p}{p-1}} + \frac{Mm^p - mM^p}{M-m} = 0$, then we obtain that the constant α coincides with $K(m,M,p)$ defined as (8.3). Also, since $m \leq \frac{p}{p-1}\frac{mM^p - Mm^p}{M-m} \leq M$ for $M > m > 0$, we have that α satisfies the condition $pm^{p-1} \leq \frac{M^p - m^p}{\alpha(M-m)} \leq pM^{p-1}$. Therefore we have Theorem 8.3.

If we put $f(t) = e^t$ in Theorem 8.18, we have the following corollary.

Corollary 8.24 *Let $A, B \in \mathscr{B}_h(H)$ with $\mathsf{Sp}(B) \subseteq [m,M]$. If $A \geq B$, then for a given $\alpha \in \mathbb{R}_+$*

$$\alpha e^A + \beta 1_H \geq e^B,$$

holds for

$$\beta = \begin{cases} \frac{e^M - e^m}{M-m} \log \frac{e^M - e^m}{\alpha(M-m)} + \frac{(M+1)e^m - (m+1)e^M}{M-m} & \textit{if} \quad m \leq \log \frac{e^M - e^m}{\alpha(M-m)} \leq M, \\ (1-\alpha)e^M & \textit{if} \quad M < \log \frac{e^M - e^m}{\alpha(M-m)}, \\ (1-\alpha)e^m & \textit{if} \quad \log \frac{e^M - e^m}{\alpha(M-m)} < m. \end{cases}$$

In particular,

$$S(e^{M-m},1)e^A \geq e^B$$

and

$$e^A + L(e^m,e^M)\log S(e^{M-m},1)1_H \geq e^B,$$

where the Specht ratio and the Mond-Shisha difference are as follows:

$$S(e^{M-m},1) = \frac{e^M - e^m}{M-m}\exp\left(\frac{(M+1)e^m-(m+1)e^M}{e^M-e^m}\right)$$

and

$$L(e^m,e^M)\log S(e^{M-m},1) = \frac{(M+1)e^m-(m+1)e^M}{M-m} + \frac{e^M-e^m}{M-m}\log\left(\frac{e^M-e^m}{M-m}\right).$$

8.4 Kantorovich type inequalities under the operator order

In this section, as applications of our results in § 8.2 on power functions, we show a generalization of Theorem 8.3, Theorem 8.13 and two variable versions of Kantorovich type operator inequalities under the operator order.

We start with the following corollary, which follows from Theorem 8.14 if we put $f(t) = t^p$, $p \in \mathbb{R}\backslash[0,1)$ and $g(t) = t^q$, $q > 1$.

Corollary 8.25 *Let $A,B \in \mathscr{B}^+(H)$ with $\mathsf{Sp}(B) \subseteq [m,M]$. If $A \geq B > 0$, then for a given $\alpha \in \mathbb{R}_+$*

$$\alpha A^q + \beta 1_H \geq B^p \qquad \text{for all } p \in \mathbb{R}\backslash[0,1),\ q>1,$$

where

$$\beta = \begin{cases} \alpha(q-1)\left(\frac{1}{\alpha q}\frac{M^p-m^p}{M-m}\right)^{\frac{q}{q-1}} + \frac{Mm^p-mM^p}{M-m} & \text{if} \quad m < \left(\frac{1}{\alpha q}\mu_{t^p}\right)^{\frac{1}{q-1}} < M, \\ \max\{m^p-\alpha m^q, M^p-\alpha M^q\} & \text{otherwise.} \end{cases}$$

The following theorem is a two variable version of Theorem 8.3.

Theorem 8.26 *Let $A,B \in \mathscr{B}^+(H)$ with $\mathsf{Sp}(B) \subseteq [m,M]$. If $A \geq B > 0$, then*

$$\frac{M^{p-1}}{m^{q-1}}A^q \geq K(m,M,p,q)A^q \geq B^p \qquad \text{for all } p>1,\ q>1, \tag{8.17}$$

where

$K(m,M,p,q)$

$$= \begin{cases} \dfrac{(q-1)^{q-1}}{q^q} \dfrac{(M^p - m^p)^q}{(M-m)(mM^p - Mm^p)^{q-1}} & \text{if} \quad qm^{p-1} < \frac{M^p - m^p}{M-m} < qM^{p-1}, \\ m^{p-q} & \text{if} \quad \frac{M^p - m^p}{M-m} \le qm^{p-1}, \\ M^{p-q} & \text{if} \quad qM^{p-1} \le \frac{M^p - m^p}{M-m}. \end{cases} \tag{8.18}$$

In particular,

$$\frac{(M^p - m^p)^2}{4mM(M-m)(M^{p-1} - m^{p-1})} A^2 \ge B^p \qquad \text{for all } p > 1. \tag{8.19}$$

Remark 8.8 *We recall that the constant $K(m,M,p,q)$ is defined as*

$$K(m,M,t^p,q) = \frac{mM^p - Mm^p}{(q-1)(M-m)} \left(\frac{q-1}{q} \frac{M^p - m^p}{mM^p - Mm^p} \right)^q$$

in (2.20) of § 2.2. However, for the sake of convenience, we define $K(m,M,p,q)$ by (8.18) above. Because expression (8.18) has arisen no confusion and is simplified.

To prove Theorem 8.26, we need the following lemma.

Lemma 8.27 *Let $p > 1, q > 1$ and $h > 1$. If $q \le \frac{h^p - 1}{h-1} \le qh^{p-1}$, then*

$$h^{p-1} \ge \frac{(q-1)^{q-1}}{q^q} \frac{(h^p - 1)^q}{(h-1)(h^p - h)^{q-1}}. \tag{8.20}$$

Proof. Let $l(t) = \mu t + \nu$, $t_o = \frac{q}{q-1} \frac{-\nu}{\mu}$ and $g(t) = t^q$, where $\mu = \frac{h^p - 1}{h-1}$, $\nu = \frac{h - h^p}{h-1}$. Since $p,q,h > 1$, then $\mu \ge 0$ and $\nu \le 0$. We see that the condition $q \le \frac{h^p - 1}{h-1} \le qh^{p-1}$ is equivalent to the condition $1 \le t_o \le h$ and we have

$$\max_{1 \le t_o \le h} \left\{ \frac{l(t)}{g(t)} \right\} = \frac{\mu t_o + \nu}{t_o^q} = \frac{(q-1)^{q-1}}{q^q} \frac{(h^p - 1)^q}{(h-1)(h^p - h)^{q-1}}.$$

Put $l_1(t) = \frac{\mu t + \nu}{t}$ and $g_1(t) = t^{q-1}$. Then $l_1(t)$ and $g_1(t)$ are increasing and we have $l_1(h) \ge l_1(t_o) > 0$ and $g_1(t_o) \ge g_1(1) > 0$. Hence we have

$$h^{p-1} = \frac{l_1(h)}{g_1(1)} \ge \frac{l_1(t_o)}{g_1(t_o)} = \frac{\mu t_o + \nu}{t_o} \frac{1}{t_o^{q-1}},$$

as desired inequality (8.20). □

Proof of Theorem 8.26. We prove (8.17). Put $h = \frac{M}{m} > 1$. If $qm^{p-1} \le \frac{M^p - m^p}{M-m} \le qM^{p-1}$, then it follows from (8.20) that

$$\frac{M^{p-1}}{m^{q-1}} = m^{q-p} h^{p-1} \ge m^{q-p} \frac{(q-1)^{q-1}}{q^q} \frac{(h^p - 1)^q}{(h-1)(h^p - h)^{q-1}} = K(m,M,p,q).$$

Otherwise, we see that $\frac{M^{p-1}}{m^{q-1}} \geq \frac{m^{p-1}}{m^{q-1}}$ and $\frac{M^{p-1}}{m^{q-1}} \geq \frac{M^{p-1}}{M^{q-1}}$. Then we have the left hand inequality of (8.17). We have the right hand inequality of (8.17) if we choose α such that $\beta = 0$ in Corollary 8.25. The inequality (8.19) follows from (8.17) if we put $p = 2$ and if we take into account that

$$\frac{(M^p - m^p)^2}{4mM(M-m)(M^{p-1} - m^{p-1})} \geq K(m,M,p,2)$$

holds for all $p > 1$. □

Remark 8.9 (i) If we put $q = p$ in Theorem 8.26, then the assumption $pm^{p-1} \leq \frac{M^p - m^p}{M-m} \leq pM^{p-1}$ is automatically satisfied by the convexity of $f(t) = t^p$ and then the constant $K(m,M,p,p)$ coincides with $K(m,M,p)$. Therefore we have Theorem 8.3.

(ii) If we put $p = 2$ in (8.19) then the constant in it coincides with the Kantorovich constant $\frac{(M+m)^2}{4Mm}$.

(iii) We remark that the following inequality

$$\frac{(q-1)^{q-1}}{q^q} \frac{(M^p - m^p)^q}{(M-m)(mM^p - Mm^p)^{q-1}} \geq K(m,M,p,q)$$

generally holds for all $p > 1$ and $q > 1$.

Constants in the next theorem are considered as two variable versions of the Kantorovich constant. The heart of the extension is exactly the Furuta inequality (Theorem F).

Theorem 8.28 *If $A \geq B > 0$ with $\mathrm{Sp}(B) \subseteq [m,M]$, then*

$$\frac{(M^{p+q-2} - m^{p+q-2})^2}{4m^{q-1}M^{q-1}(M^{q-1} - m^{q-1})(M^{p-1} - m^{p-1})} A^q \geq B^p \quad \text{for all } p > 1 \text{ and } q > 2.$$

Proof. It follows from the Furuta inequality that for each $r > 0$

$$\left(B^{\frac{r}{2}} A^q B^{\frac{r}{2}}\right)^{\frac{1}{2}} \geq B^{r+1}$$

holds for $q > 2$ such that $q = r+2$. If we put $A_1 = \left(B^{\frac{r}{2}} A^q B^{\frac{r}{2}}\right)^{\frac{1}{2}}$ and $B_1 = B^{r+1}$, then $A_1 \geq B_1 > 0$ and $M^{r+1}1_H \geq B_1 \geq m^{r+1}1_H > 0$. Applying (8.19) to A_1 and B_1 gives

$$\frac{\left((M^{1+r})^{\frac{p+r}{1+r}} - (m^{1+r})^{\frac{p+r}{1+r}}\right)^2}{4m^{1+r}M^{1+r}(M^{1+r} - m^{1+r})\left((M^{1+r})^{\frac{p-1}{1+r}} - (m^{1+r})^{\frac{p-1}{1+r}}\right)} A_1^2 \geq B_1^{\frac{p+r}{1+r}}$$

for all $p > 1$. Therefore, we have

$$\frac{(M^{p+r} - m^{p+r})^2}{4m^{1+r}M^{1+r}(M^{1+r} - m^{1+r})(M^{p-1} - m^{p-1})} B^{\frac{r}{2}} A^q B^{\frac{r}{2}} \geq B^{p+r}$$

for all $p > 1$ and $q > 2$. Multiplying by $B^{\frac{-r}{2}}$ on both sides gives

$$\frac{(M^{p+q-2} - m^{p+q-2})^2}{4m^{q-1}M^{q-1}(M^{q-1} - m^{q-1})(M^{p-1} - m^{p-1})}A^q \geq B^p \quad \text{for all } p > 1 \text{ and } q > 2.$$

□

Remark 8.10 *If we put $p = 2$ and $q = 2$ in Theorem 8.28, then it follows that the constant in Theorem 8.28 coincides with the Kantorovich constant $\frac{(M+m)^2}{4Mm}$.*

We show the next theorem as a generalization of Theorem 8.26.

Theorem 8.29 *If $A \geq B > 0$ and $\mathrm{Sp}(B) \subseteq [m,M]$, then*

$$K(m^r, M^r, \frac{p-1+r}{r}, \frac{q-1+r}{r})A^q \geq B^p \quad \text{for all } p > 1,\ q > 1 \text{ and } r > 1,$$

where $K(m,M,p,q)$ is defined as (8.18) in Theorem 8.26.

Proof. By the Furuta inequality, we have that $A \geq B$ ensures

$$\left(B^{\frac{r}{2}}A^qB^{\frac{r}{2}}\right)^{\frac{1+r}{q+r}} \geq B^{1+r} \quad \text{for all } q > 1 \text{ and } r > 0.$$

If we put $A_1 = \left(B^{\frac{r}{2}}A^qB^{\frac{r}{2}}\right)^{\frac{1+r}{q+r}}$ and $B_1 = B^{1+r}$, then $A_1 \geq B_1 > 0$ and $M^{1+r}\ 1_H \geq B_1 \geq m^{1+r}1_H > 0$. Applying Theorem 8.26 to A_1 and B_1, we obtain

$$K(m^{1+r}, M^{1+r}, p_1, q_1)A_1^{q_1} \geq B_1^{p_1} \qquad \text{for all } p_1 > 1 \text{ and } q_1 > 1.$$

We put $p_1 = \frac{p+r}{1+r} > 1$ and $q_1 = \frac{q+r}{1+r} > 1$ and have

$$K\left(m^{1+r}, M^{1+r}, \frac{p+r}{1+r}, \frac{q+r}{1+r}\right)B^{\frac{r}{2}}A^qB^{\frac{r}{2}} \geq B^{p+r} \quad \text{for all } p > 1 \text{ and } q > 1.$$

Multiplying by $B^{\frac{-r}{2}}$ on both sides and replacing r by $r-1$ give

$$K\left(m^r, M^r, \frac{p-1+r}{r}, \frac{q-1+r}{r}\right)A^q \geq B^p \quad \text{for all } p > 1,\ q > 1 \text{ and } r > 1.$$

□

8.5 Chaotic order version

In this section, we show Kantorovich type order preserving operator inequalities associated with the chaotic order, which are parallel to the operator order versions in § 8.4.

We first show a chaotic order version of Theorem 8.26.

Theorem 8.30 *Let $A,B \in \mathscr{B}^{++}(H)$ and $\mathsf{Sp}(B) \subseteq [m,M]$. If $\log A \geq \log B$, then*

$$\frac{M^p}{m^q}A^q \geq K(m,M,p+1,q+1)A^q \geq B^p \qquad \text{for all } p>0 \text{ and } q>0,$$

where $K(m,M,p,q)$ is defined as (8.18) in Theorem 8.26.

Proof. If we put $r=1$ in Theorem 7.7, then $\log A \geq \log B$ ensures

$$\left(B^{\frac{1}{2}}A^qB^{\frac{1}{2}}\right)^{\frac{1}{q+1}} \geq B \quad \text{for all } q>0.$$

If we put $A_1 = \left(B^{\frac{1}{2}}A^qB^{\frac{1}{2}}\right)^{\frac{1}{q+1}}$ and $B_1 = B$, then $A_1 \geq B_1 > 0$ and $M1_H \geq B_1 \geq m1_H > 0$. Applying Theorem 8.26 to A_1 and B_1, we obtain

$$\frac{M^{(p+1)-1}}{m^{(q+1)-1}}A_1^{q+1} \geq K(m,M,p+1,q+1)A_1^{q+1} \geq B_1^{p+1} \quad \text{for all } p>0 \text{ and } q>0.$$

Multiplying by $B^{\frac{-1}{2}}$ on both sides, it follows that

$$\frac{M^p}{m^q}A^q \geq K(m,M,p+1,q+1)A^q \geq B^p \quad \text{for all } p>0 \text{ and } q>0.$$

□

Next, we shall show a chaotic order version of Theorem 8.28.

Theorem 8.31 *Let $A,B \in \mathscr{B}^{++}(H)$ with $\mathsf{Sp}(B) \subseteq [m,M]$. If $\log A \geq \log B$, then*

$$\frac{(M^{p+q}-m^{p+q})^2}{4m^qM^q(M^q-m^q)(M^p-m^p)}A^q \geq B^p \qquad \text{for all } p>0 \text{ and } q>0.$$

Proof. If we put $r=p$ in Theorem 7.7, then $\log A \geq \log B$ ensures

$$(B^{\frac{q}{2}}A^qB^{\frac{q}{2}})^{\frac{1}{2}} \geq B^q \quad \text{for all } q>0.$$

If we put $A_1 = (B^{\frac{q}{2}}A^qB^{\frac{q}{2}})^{\frac{1}{2}}$ and $B_1 = B^q$, then $A_1 \geq B_1 > 0$ and $M^q1_H \geq B_1 \geq m^q1_H > 0$. Applying (8.19) to A_1 and B_1, we obtain

$$\frac{\left((M^q)^{\frac{p+q}{q}} - (m^q)^{\frac{p+q}{q}}\right)^2}{4m^qM^q(M^q-m^q)\left((M^q)^{\frac{p}{q}} - (m^q)^{\frac{p}{q}}\right)}A_1^2 \geq B_1^{\frac{p+q}{q}} \quad \text{for all } p>0 \text{ and } q>0.$$

By rearranging this inequality, we have

$$\frac{(M^{p+q}-m^{p+q})^2}{4m^qM^q(M^q-m^q)(M^p-m^p)}B^{\frac{q}{2}}A^qB^{\frac{q}{2}} \geq B^{p+q} \quad \text{for all } p>0 \text{ and } q>0.$$

Multiplying by $B^{\frac{-q}{2}}$ on both sides, it follows that

$$\frac{(M^{p+q}-m^{p+q})^2}{4m^qM^q(M^q-m^q)(M^p-m^p)}A^q \geq B^p \quad \text{for all } p>0 \text{ and } q>0.$$

□

We show the following result as a generalization of Theorem 8.30.

Theorem 8.32 *Let $A,B \in \mathscr{B}^{++}(H)$ and $\mathsf{Sp}(B) \subseteq [m,M]$. If $\log A \geq \log B$, then*

$$K\left(m^r, M^r, \frac{p+r}{r}, \frac{q+r}{r}\right)A^q \geq B^p \qquad \textit{for all } p>0,\ q>0 \textit{ and } r>0,$$

where $K(m,M,p,q)$ is defined as (8.18) in Theorem 8.26.

Proof. By Theorem 7.7, $\log A \geq \log B$ ensures

$$\left(B^{\frac{r}{2}}A^qB^{\frac{r}{2}}\right)^{\frac{r}{q+r}} \geq B^r \quad \text{for all } q>0 \text{ and } r>0.$$

If we put $A_1 = \left(B^{\frac{r}{2}}A^qB^{\frac{r}{2}}\right)^{\frac{r}{q+r}}$ and $B_1 = B^r$, then $A_1 \geq B_1 > 0$ and $M^r 1_H \geq B_1 \geq m^r 1_H > 0$. Applying Theorem 8.26 to A_1 and B_1, we obtain

$$K(m^r, M^r, p_1, q_1)A_1^{q_1} \geq B_1^{p_1} \quad \text{for all } p_1>1 \text{ and } q_1>1.$$

If we put $p_1 = \frac{p+r}{r} > 1$ and $q_1 = \frac{q+r}{r} > 1$, then we have

$$K\left(m^r, M^r, \frac{p+r}{r}, \frac{q+r}{r}\right)B^{\frac{r}{2}}A^qB^{\frac{r}{2}} \geq B^{p+r}.$$

Multiplying by $B^{\frac{-r}{2}}$ on both sides, it follows that

$$K\left(m^r, M^r, \frac{p+r}{r}, \frac{q+r}{r}\right)A^q \geq B^p \quad \text{for all } p>0,\ q>0 \text{ and } r>0.$$

□

The following result is a two variable version of a characterization of the chaotic order via the Specht ratio by Theorem 8.7.

Theorem 8.33 *Let $A,B \in \mathscr{B}^{++}(H)$ with $\mathsf{Sp}(B) \subseteq [m,M]$. Then $\log A \geq \log B$ is equivalent to*

$$S(h,p,q)A^q \geq B^p \qquad \textit{for all } p>0 \textit{ and } q>0$$

where $h = \frac{M}{m} > 1$ and

$$S(h,p,q) = \begin{cases} m^{p-q}\frac{(h^p-1)h^{\frac{q}{h^p-1}}}{eq\log h} & \textit{if } \ q < \frac{h^p-1}{\log h} < qh^p, \\ m^{p-q} & \textit{if } \ \frac{h^p-1}{\log h} \leq q, \\ M^{p-q} & \textit{if } \ qh^p \leq \frac{h^p-1}{\log h}. \end{cases} \tag{8.21}$$

Proof. For given $p > 0$ and $q > 0$, suppose that $q < \frac{h^p-1}{\log h} < qh^p$. Then $\frac{q+r}{r} < \frac{h^{p+r}-1}{h^p-1} < \frac{q+r}{r}h^p$ holds for sufficient small $r > 0$. It follows from Theorem 8.32 that $\log A \geq \log B$ implies $K(m^r, M^r, \frac{p+r}{r}, \frac{q+r}{r})A^q \geq B^p$. Since $\left(\frac{q}{q+r}\right)^{\frac{1}{r}} \to \frac{1}{e^{1/q}}$ and $\left(\frac{h^{p+r}-1}{h^p-1}\right)^{1/r} \to h^{\frac{h^p}{h^p-1}}$ as $r \to +0$, we have

$$\begin{aligned}
K\left(m^r, M^r, \frac{p+r}{r}, \frac{q+r}{r}\right) &= \frac{\left(\frac{q}{r}\right)^{\frac{q}{r}}}{\left(\frac{q+r}{r}\right)^{\frac{q+r}{r}}} \frac{(M^{p+r}-m^{p+r})^{\frac{q+r}{r}}}{(M^r-m^r)(m^r M^{p+r}-M^r m^{p+r})^{\frac{q}{r}}} \\
&= \frac{\left(\frac{q}{r}\right)^{\frac{q}{r}}}{\left(\frac{q+r}{r}\right)^{\frac{q+r}{r}}} m^{p-q} \frac{(h^{p+r}-1)^{\frac{q+r}{r}}}{(h^r-1)(h^{p+r}-h^r)^{\frac{q}{r}}} \\
&= m^{p-q} \frac{\left(\frac{q}{r}\right)^{\frac{q}{r}}}{\left(\frac{q+r}{r}\right)^{\frac{q}{r}}\left(\frac{q+r}{r}\right)} \frac{(h^{p+r}-1)(h^{p+r}-1)^{\frac{q}{r}}}{(h^r-1)h^q(h^p-1)^{\frac{q}{r}}} \\
&= \frac{m^{p-q}}{h^q}\left(\left(\frac{r}{q+r}\frac{h^{p+r}-1}{h^r-1}\right)^{\frac{1}{q}}\left(\frac{q}{q+r}\frac{h^{p+r}-1}{h^p-1}\right)^{\frac{1}{r}}\right)^q \\
&\to \frac{m^{p-q}}{h^q}\left(\left(\frac{1}{\log h}\frac{h^p-1}{q}\right)^{\frac{1}{q}}\left(\frac{1}{e^{1/q}}h^{\frac{h^p}{h^p-1}}\right)\right)^q \\
&= m^{p-q}\frac{(h^p-1)h^{\frac{q}{h^p-1}}}{eq\log h},
\end{aligned}$$

as $r \to +0$. Therefore we have

$$m^{p-q}\frac{(h^p-1)h^{\frac{q}{h^p-1}}}{eq\log h}A^q \geq B^p.$$

Suppose that $\frac{h^p-1}{\log h} \leq q$. Then $\frac{h^{p+r}-1}{h^r-1} \leq \frac{q+r}{r}$ holds for sufficient small $r > 0$ and we have $K\left(m^r, M^r, \frac{p+r}{r}, \frac{q+r}{r}\right) = m^{p-q}$. Similarly we have $K(m^r, M^r, \frac{p+r}{r}, \frac{q+r}{r}) = M^{p-q}$ in the case $qh^p \leq \frac{h^p-1}{\log h}$. Therefore we have the desired inequalities by Theorem 8.32.

Conversely, suppose that $S(h,p,q)A^q \geq B^p$ for all $p > 0$ and $q > 0$. If we put $q = p$, then we have that $p \leq \frac{h^p-1}{\log h} \leq ph^p$ holds for all $p > 0$. Therefore the constant $S(h,p,p)$ coincides with the Specht ratio $S(h,p)$ defined by (2.73). Then it follows from Theorem 8.7 that $S(h,p)A^p \geq B^p$ for all $p > 0$ implies $\log A \geq \log B$. $\square$

8.6 Kantorovich type inequalities via generalized Furuta inequality

In this section, as an application of both Furuta inequality and the generalized Furuta inequality we show a generalization of Kantorovich type order preserving operator inequalities by means of a generalized Kantorovich constant.

Let $A,B \in \mathscr{B}^{++}(H)$. We consider the class of orders $A^{\delta} \geq B^{\delta}$ for $\delta \in (0,1]$, which interpolates the usual order $A \geq B$ and the chaotic order $A \gg B$ continuously.

Theorem 8.34 *Let $A,B \in \mathscr{B}^{++}(H)$ with $\mathsf{Sp}(A) \subseteq [m,M]$ for some scalars $M > m > 0$. Then the following statements are mutually equivalent for each $\delta \in (0,1]$:*

(i) $A^{\delta} \geq B^{\delta}$.

(ii) For each $n > 0$ and $\alpha \in [0,1]$

$$\overline{K}\left(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, n\frac{(q-\delta+\alpha u)s-\alpha u}{(p-\delta+\alpha u)s-\alpha u}+1, n+1\right) A^{(q-\delta+\alpha u)s} \geq \left(A^{\frac{\alpha u-\delta}{2}} B^p A^{\frac{\alpha u-\delta}{2}}\right)^s$$

holds for $s \geq 1$, $p \geq \delta$, $q \geq \delta$ and $u \geq \delta$ with $(p-\delta+\alpha u)s \geq (\alpha+n)u$.

(iii) For each $n > 0$

$$\overline{K}\left(m^{\frac{(p-\delta)s}{n}}, M^{\frac{(p-\delta)s}{n}}, n\frac{q-\delta}{p-\delta}+1, n+1\right)^{\frac{1}{s}} A^q \geq B^p$$

holds for $s \geq 1$, $p \geq \delta$ and $q \geq \delta$ with $(p-\delta)s \geq n\delta$.

(iv) $\frac{M^{p-\delta}}{m^{q-\delta}} A^q \geq B^p$ *holds for $p \geq \delta$ and $q \geq \delta$.*

Here the constant $\overline{K}(m,M,p,q) = M^{q-p}m^{q-p}K(m,M,p,q)$ where $K(m,M,p,q)$ is defined in (8.18), i.e.,
$\overline{K}(m,M,p,q)$

$$= \begin{cases} \dfrac{(q-1)^{q-1}}{q^q} \dfrac{(M^p-m^p)^q M^{q-p} m^{q-p}}{(M-m)(mM^p-Mm^p)^{q-1}} & \text{if } qm^{p-1} < \frac{M^p-m^p}{M-m} < qM^{p-1}, \\ M^{q-p} & \text{if } \frac{M^p-m^p}{M-m} \leq qm^{p-1}, \\ m^{q-p} & \text{if } qM^{p-1} \leq \frac{M^p-m^p}{M-m}. \end{cases} \tag{8.22}$$

In order to prove Theorem 8.34, we need the following lemma and theorem. The following lemma shows that Furuta inequality interpolates the operator order and the chaotic one.

Lemma 8.35 *Let $A,B \in \mathscr{B}^{++}(H)$. Then the following statements are mutually equivalent for each $\delta \in [0,1]$:*

(i) $A^{\delta} \geq B^{\delta}$*, where the case* $\delta = 0$ *means* $A \gg B$.

(ii) $A^{p+\delta} \geq \left(A^{\frac{p}{2}} B^{p+\delta} A^{\frac{p}{2}}\right)^{\frac{p+\delta}{2p+\delta}}$ *for all* $p \geq 0$.

(iii) $A^{u+\delta} \geq \left(A^{\frac{u}{2}} B^{p+\delta} A^{\frac{u}{2}}\right)^{\frac{u+\delta}{p+u+\delta}}$ *for all* $p \geq 0$ *and* $u \geq 0$.

Proof. In the case of $0 < \delta \leq 1$, Furuta inequality ensures Lemma 8.35. In the case of $\delta = 0$, Theorem 7.7 implis Lemma 8.35. □

By Theorem 8.26 we obtain the following theorem.

Theorem 8.36 *Let* $A, B \in \mathscr{B}^{++}(H)$ *with* $\mathsf{Sp}(A) \subseteq [m, M]$ *for some scalars* $M > m > 0$*. If* $A \geq B > 0$*, then*

$$\frac{M^{q-1}}{m^{p-1}} A^p \geq \overline{K}(m,M,p,q) A^p \geq B^q \qquad \text{for all } p > 1 \text{ and } q > 1, \tag{8.23}$$

where $\overline{K}(m,M,p,q)$ *is defined in (8.22).*

Proof. As $0 < A^{-1} \leq B^{-1}$ and $M^{-1} 1_H \leq A^{-1} \leq m^{-1} 1_H$, by applying Theorem 8.26 we have

$$A^{-p} \leq \frac{(q-1)^{q-1}}{q^q} \frac{(m^{-p} - M^{-p})^q}{(m^{-1} - M^{-1})(M^{-1} m^{-p} - m^{-1} M^{-p})^{q-1}} B^{-q}$$

if $qM^{-(p-1)} \leq \frac{m^{-p} - m^{-p}}{m^{-1} - M^{-1}} \leq qm^{-(p-1)}$,

$$A^{-p} \leq M^{-(p-q)} B^{-q} \qquad \text{if } \frac{m^{-p} - M^{-p}}{m^{-1} - M^{-1}} < qM^{-(p-1)}$$

and

$$A^{-p} \leq m^{-(p-q)} B^{-q} \qquad \text{if } qm^{-(p-1)} < \frac{m^{-p} - M^{-p}}{m^{-1} - M^{-1}}.$$

Then a simple calculation implies

$$A^{-p} \leq \frac{(q-1)^{q-1}}{q^q} \frac{(M^p - m^p)^q M^{q-p} m^{q-p}}{(M-m)(mM^p - Mm^p)^{q-1}} B^{-q} = M^{q-p} m^{q-p} K(m,M,p,q) B^{-q}$$

if $qm^{p-1} \leq \frac{M^p - m^p}{M - m} \leq qM^{p-1}$,

$$A^{-p} \leq M^{q-p} B^{-q} = M^{q-p} m^{q-p} m^{p-q} B^{-q} = M^{q-p} m^{q-p} K(m,M,p,q) B^{-q}$$

if $\frac{M^p - m^p}{M - m} < qm^{p-1}$ and

$$A^{-p} \leq m^{q-p} B^{-q} = M^{q-p} m^{q-p} M^{p-q} B^{-q} = M^{q-p} m^{q-p} K(m,M,p,q) B^{-q}$$

if $qM^{p-1} < \frac{M^p - m^p}{M - m}$.

We obtain the right hand inequality in (8.23) by taking inverses in both sides of inequalities above. As we have from Theorem 8.26 that $K(m,M,p,q) \leq \frac{M^{p-1}}{m^{q-1}}$, we obtain

$$B^q \leq M^{q-p} m^{q-p} K(m,M,p,q) A^p \leq \frac{M^{q-1}}{m^{p-1}} A^p \qquad \text{for all } p > 1 \text{ and } q > 1,$$

so the proof of theorem is complete. □

Proof of Theorem 8.34.

(i)$\Longrightarrow$(ii). For given $p \geq \delta$ and $u \geq \delta$, put $A_1 = A^u$ and $B_1 = \left(A^{\frac{u-\delta}{2}} B^p A^{\frac{u-\delta}{2}}\right)^{\frac{u}{p+u-\delta}}$ in (iii) of Lemma 8.35. Then we have $A_1 \geq B_1 \geq 0$. By the generalized Furuta inequality, it follows that for each $\delta \in [0,1]$,

$$A_1^{\frac{(p_1-t)s+r}{q_1}} \geq \left\{ A_1^{\frac{r}{2}} \left(A_1^{-\frac{t}{2}} B_1^{p_1} A_1^{-\frac{t}{2}} \right)^s A_1^{\frac{r}{2}} \right\}^{\frac{1}{q_1}} \tag{8.24}$$

holds for all $s \geq 1$, $p_1 \geq 1$, $q_1 \geq 1$ satisfying the following two conditions

$$r \geq t, \tag{8.25}$$

$$(1-t+r)q_1 \geq (p_1-t)s+r. \tag{8.26}$$

For given $n > 0$, $\alpha \in [0,1]$ and $s \geq 1$, we put $p_1 = \frac{p+u-\delta}{u}$, $q_1 = n+1 \geq 1$, $\alpha = 1-t$ and $r = \frac{(p-\delta+\alpha u)s}{nu} - \frac{n+1}{n}\alpha$. Then (8.25) is equivalent to the assumption in (ii)

$$(p-\delta+\alpha u)s \geq (n+\alpha)u. \tag{8.27}$$

and (8.26) is satisfied as the equality holds.

Therefore (8.24) implies that

$$A^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}} \geq \left\{ A^{\frac{(p-\delta+\alpha u)s-(n+1)\alpha u}{2n}} \left(A^{\frac{\alpha u-\delta}{2}} B^p A^{\frac{\alpha u-\delta}{2}} \right)^s A^{\frac{(p-\delta+\alpha u)s-(n+1)\alpha u}{2n}} \right\}^{\frac{1}{n+1}} \tag{8.28}$$

holds for $n > 0$, $p \geq \delta$, $\alpha \in [0,1]$ and $s \geq 1$ with the condition (8.27). By raising the left hand side to power $n\frac{(q-\delta+\alpha u)s-\alpha u}{(p-\delta+\alpha u)s-\alpha u} + 1$ for some $q \geq \delta$ and the right hand side to power $n+1$, it follows from Theorem 8.36 that

$$\begin{aligned} &\overline{K}\left(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, n\frac{(q-\delta+\alpha u)s-\alpha u}{(p-\delta+\alpha u)s-\alpha u}+1, n+1\right) \\ &\qquad \cdot A^{(q-\delta+\alpha u)s-\alpha u+\frac{(p-\delta+\alpha u)s-\alpha u}{n}} \\ &\geq A^{\frac{(p-\delta+\alpha u)s-(n+1)\alpha u}{2n}} \left(A^{\frac{\alpha u-\delta}{2}} B^p A^{\frac{\alpha u-\delta}{2}} \right)^s A^{\frac{(p-\delta+\alpha u)s-(n+1)\alpha u}{2n}}. \end{aligned} \tag{8.29}$$

By rearranging (8.29), we have the desired inequality

$$\begin{aligned} &\overline{K}\left(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, n\frac{(q-\delta+\alpha u)s-\alpha u}{(p-\delta+\alpha u)s-\alpha u}+1, n+1\right) A^{(q-\delta+\alpha u)s} \\ &\qquad \geq \left(A^{\frac{\alpha u-\delta}{2}} B^p A^{\frac{\alpha u-\delta}{2}} \right)^s, \end{aligned}$$

so that (i) $\Longrightarrow$ (ii) is proved.

(ii) $\Longrightarrow$ (iii). If we put $\alpha = 0$ and $u = \delta$ in (ii), then we obtain (iii).

(iii) $\Longrightarrow$ (iv). If we put $x = \frac{M}{m}$ in (iii), then we have

$$\begin{aligned} &\overline{K}\left(m^{\frac{(p-\delta)s}{n}}, M^{\frac{(p-\delta)s}{n}}, n\frac{q-\delta}{p-\delta}+1, n+1\right)^{\frac{1}{s}} \\ &= \left(\frac{n^n}{(n+1)^{n+1}} M^{(p-q)s} \frac{(x^{(q-\delta)s+\frac{(p-\delta)s}{n}}-1)^{n+1}}{(x^{\frac{(p-\delta)s}{n}}-1)(x^{(q-\delta)s+\frac{(p-\delta)s}{n}}-x^{\frac{(p-\delta)s}{n}})^n} \right)^{\frac{1}{s}} \\ &\to 1 \cdot M^{p-q} \frac{x^{(n+1)(q-\delta+\frac{p-\delta}{n})}}{x^{\frac{p-\delta}{n}} x^{p-\delta} x^{n(q-\delta)}} = \frac{M^{p-\delta}}{m^{q-\delta}} \quad \text{as } s \to \infty \end{aligned}$$

if $(n+1)m^{(q-\delta)s} \leq \frac{M^{(q-\delta)s+(p-\delta)s/n}-m^{(q-\delta)s+(p-\delta)s/n}}{M^{(p-\delta)s/n}-m^{(p-\delta)s/n}} \leq (n+1)M^{(q-\delta)s}$.
But, if $\frac{M^{(q-\delta)s+(p-\delta)s/n}-m^{(q-\delta)s+(p-\delta)s/n}}{M^{(p-\delta)s/n}-m^{(p-\delta)s/n}} < (n+1)m^{(q-\delta)s}$, then

$$\begin{aligned}
&\overline{K}\left(m^{\frac{(p-\delta)s}{n}}, M^{\frac{(p-\delta)s}{n}}, n\frac{q-\delta}{p-\delta}+1, n+1\right)^{\frac{1}{s}} \\
&= \left(\left(M^{\frac{(p-\delta)s}{n}}\right)^{n-n\frac{q-\delta}{p-\delta}}\right)^{\frac{1}{s}} = M^{p-q} \leq \frac{M^{p-\delta}}{m^{q-\delta}}.
\end{aligned}$$

Similarly, if $(n+1)M^{(q-\delta)s} < \frac{M^{(q-\delta)s+(p-\delta)s/n}-m^{(q-\delta)s+(p-\delta)s/n}}{M^{(p-\delta)s/n}-m^{(p-\delta)s/n}}$, then

$$\overline{K}\left(m^{\frac{(p-\delta)s}{n}}, M^{\frac{(p-\delta)s}{n}}, n\frac{q-\delta}{p-\delta}+1, n+1\right)^{\frac{1}{s}} = m^{p-q} \leq \frac{M^{p-\delta}}{m^{q-\delta}}.$$

Hence it follows from (iii) that

$$\frac{M^{p-\delta}}{m^{q-\delta}}A^q \geq B^p \qquad \text{holds for all } p \geq \delta \text{ and } q \geq \delta.$$

(iv) $\Longrightarrow$ (i). If we put $p = q = \delta$ in (iv), then we obtain (i). □

By Theorem 8.34, we obtain the following Kantorovich type order preserving operator inequalities under the operator order and the chaotic one.

Corollary 8.37 *Let $A, B \in \mathscr{B}^{++}(H)$ with $\mathrm{Sp}(A) \subseteq [m, M]$ for some scalars $M > m > 0$. Then the following statements are mutually equivalent:*

(i) $A \geq B$.

(ii) For each $n > 0$ and $\alpha \in [0,1]$

$$\begin{aligned}
&\overline{K}\left(m^{\frac{(p-1+\alpha u)s-\alpha u}{n}}, M^{\frac{(p-1+\alpha u)s-\alpha u}{n}}, n\frac{(q-1+\alpha u)s-\alpha u}{(p-1+\alpha u)s-\alpha u}+1, n+1\right)A^{(q-1+\alpha u)s} \\
&\qquad \geq \left(A^{\frac{\alpha u-1}{2}}B^pA^{\frac{\alpha u-1}{2}}\right)^s
\end{aligned}$$

holds for $s \geq 1$, $p \geq 1$, $q \geq 1$ and $u \geq 1$ with $(p-1+\alpha u)s \geq (\alpha+n)u$.

(iii) For each $n > 0$

$$\overline{K}\left(m^{\frac{(p-1)s}{n}}, M^{\frac{(p-1)s}{n}}, n\frac{q-1}{p-1}+1, n+1\right)^{\frac{1}{s}} A^q \geq B^p$$

holds for $s \geq 1$, $p \geq 1$ and $q \geq 1$ with $(p-1)s \geq n$.

(iv) $\frac{M^{p-1}}{m^{q-1}}A^q \geq B^p$ holds for $p \geq 1$ and $q \geq 1$,

where $\overline{K}(m, M, p, q)$ is defined in (8.22).

Proof. Put $\delta = 1$ in Theorem 8.34. □

Corollary 8.38 *Let $A,B \in \mathscr{B}^{++}(H)$ with $\mathsf{Sp}(A) \subseteq [m,M]$ for some scalars $M > m > 0$. Then the following statements are mutually equivalent:*

(i) $A \gg B$.

(ii) For each $n > 0$ and $\alpha \in [0,1]$

$$\overline{K}\left(m^{\frac{(p+\alpha u)s-\alpha u}{n}}, M^{\frac{(p+\alpha u)s-\alpha u}{n}}, n\frac{(q-1+\alpha u)s-\alpha u}{(p-1+\alpha u)s-\alpha u}+1, n+1\right) A^{(q+\alpha u)s} \geq \left(A^{\frac{\alpha u-1}{2}} B^p A^{\frac{\alpha u-1}{2}}\right)^s$$

holds for $s \geq 1$, $p \geq 0$, $q \geq 0$ and $u \geq 0$ with $(p+\alpha u)s \geq (\alpha+n)u$.

(iii) For each $n > 0$

$$\overline{K}\left(m^{\frac{ps}{n}}, M^{\frac{ps}{n}}, n\frac{q}{p}+1, n+1\right)^{\frac{1}{s}} A^q \geq B^p \quad \textit{holds for } s \geq 1,\ p \geq 0 \textit{ and } q \geq 0,$$

where $\overline{K}(m,M,p,q)$ is defined in (8.22).

Proof. By virtue of Theorem 8.34, if we put $\delta = 0$ in (i), (ii) and (iii) in Theorem 8.34, then we have (i)$\Rightarrow$ (ii) $\Rightarrow$ (iii) of Corollary 8.38. If we put $p = q$ and $s = 1$ in (iii), then

$$K\left(m^{\frac{p}{n}}, M^{\frac{p}{n}}, n+1\right) A^p \geq B^p \qquad \text{for all } p \geq 1 \text{ and } n > 0.$$

Since

$$K\left(m^{\frac{p}{n}}, M^{\frac{p}{n}}, n+1\right) = K\left(m^{\frac{p}{n}}, M^{\frac{p}{n}}, \frac{\frac{p}{n}+p}{\frac{p}{n}}\right) \mapsto S(h,p) \quad \text{as } n \to \infty,$$

we have (i) by Theorem 8.7. □

By Theorem 8.34, we have the following parameterized result.

Corollary 8.39 *Let $A,B \in \mathscr{B}^{++}(H)$ with $\mathsf{Sp}(A) \subseteq [m,M]$ for some scalars $M > m > 0$. Then the following statements are mutually equivalent for each $\delta \in (0,1]$:*

(i) $A^\delta \geq B^\delta$.

(ii) $\overline{K}\left(m^r, M^r, 1+\frac{q-\delta}{r}, 1+\frac{p-\delta}{r}\right) A^q \geq B^p$ for all $p \geq \delta$, $q \geq \delta$ and $r \geq \delta$.

Proof. (i) $\Longrightarrow$ (ii). If we put $n = \frac{p-\delta}{n}$ and $s = 1$ in (iii) of Theorem 8.34, then we obtain (ii) of Corollary 8.39.

(ii) $\Longrightarrow$ (i). It follows from Theorem 8.36 that

$$\overline{K}\left(m^r, M^r, 1+\frac{q-\delta}{r}, 1+\frac{p-\delta}{r}\right) \leq \frac{(M^r)^{\frac{p-\delta}{r}}}{(m^r)^{\frac{q-\delta}{r}}} = \frac{M^{p-\delta}}{m^{q-\delta}}.$$

Hence we have (i) of Corollary 8.39 by (iv) $\Longrightarrow$ (iii) of Theorem 8.34. □

We obtain the following theorem if we put $p = q$ in Theorem 8.34.

Theorem 8.40 *Let $A,B \in \mathscr{B}^{++}(H)$ with $\mathsf{Sp}(A) \subseteq [m,M]$ for some scalars $M > m > 0$. Then the following statements are mutually equivalent for each $\delta \in (0,1]$:*

(i) $A^{\delta} \geq B^{\delta}$.

(ii) For each $n \in \mathbb{N}$ and $\alpha \in [0,1]$

$$K\left(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, n+1\right) A^{(p-\delta+\alpha u)s} \geq \left(A^{\frac{\alpha u-\delta}{2}} B^p A^{\frac{\alpha u-\delta}{2}}\right)^s$$

holds for $s \geq 1$, $p \geq \delta$ and $u \geq \delta$ with $(p-\delta+\alpha u)s \geq (n+\alpha)u$.

(iii) For each $n \in \mathbb{N}$

$$K\left(m^{\frac{(p-\delta)s}{n}}, M^{\frac{(p-\delta)s}{n}}, n+1\right)^{\frac{1}{s}} A^p \geq B^p$$

holds for $s \geq 1$ and $p \geq \delta$ with $(p-\delta)s \geq n\delta$.

We prove that we can not obtain better constant than $1+n$ in $K\left(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, n+1\right)$ in Theorem 8.40, i.e. if we replace $n+1$ with $T+R$ for some $T,R \in \mathbb{R}$, then we obtain that $T > 0$ and $R \geq 1$.

Lemma 8.41 *Let $A,B \in \mathscr{B}^{++}(H)$ with $\mathsf{Sp}(A) \subseteq [m,M]$ for some scalars $M > m > 0$. For each $\alpha \in [0,1]$*

$$K\left(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, T+R\right) A^{(p-\delta+\alpha u)s} \geq \left(A^{\frac{\alpha u-\delta}{2}} B^p A^{\frac{\alpha u-\delta}{2}}\right)^s$$

holds for $T > 0$, $R \geq 1$, $s \geq 1$, $p \geq \delta$ and $u \geq \delta$ with $R(p-\delta+\alpha u)s \geq (\alpha+T)u$ and we have

$$K\left(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, T+R\right) \geq K\left(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, T+1\right).$$

To prove Lemma 8.41 we need the following proposition. We omit the proof.

Proposition 8.42 (YAMAZAKI-YANAGIDA) *Let $K(m,M,p)$ be defined in (8.18). Then*

$$F(p,r,m,M) = K\left(m^r, M^r, \frac{p}{r}+1\right)$$

is an increasing function of p, r and M, and also a decreasing function of m for $p > 0$, $r > 0$ and $M > m > 0$. And the following inequality holds:

$$\left(\frac{M}{m}\right)^p \geq K\left(m^r, M^r, \frac{p}{r}+1\right) \geq 1 \qquad \text{for all } p > 0,\ r > 0 \text{ and } M > m > 0.$$

Proof of Lemma 8.41. We have that the inequality (8.24) holds for $s \geq 1$, $p_1 \geq 1$ and $q_1 \geq 1$ with conditions (8.25) and (8.26). For given $T, R \in \mathbb{R}$, $\alpha \in [0,1]$ and $s \geq 1$, we put $p_1 = \frac{p+u-\delta}{u}$, $q_1 = T + R \geq 1$ and $\alpha = 1 - t$. As we desire that the power of M and m in $K\left(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, T+R\right)$ be $\frac{(p-\delta+\alpha u)s-\alpha u}{T}$, we have $\frac{(p_1-t)s+r}{u(R+T)} = \frac{(p-\delta+\alpha u)s-\alpha u}{T}$. It follows that $r = \frac{R(p-\delta+\alpha u)s}{Tu} - \frac{R+T}{T}\alpha$. The condition (8.25) is equivalent to the assumption in this lemma

$$R(p-\delta+\alpha u)s \geq (\alpha + T)u \tag{8.30}$$

and (8.26) is equivalent to

$$(R-1)\frac{(R+T)}{u}\frac{(p-\delta+\alpha u)s-\alpha u}{T} \geq 0. \tag{8.31}$$

Because $(R-1)\frac{(R+T)}{u}\frac{(p-\delta+\alpha u)s-\alpha u}{T} = (R-1)[(p_1-t)s+r]$ and $(p_1-t)s+r \geq 0$ for $p_1 \geq 1 \geq t \geq 0$, $s \geq 1$ and $r \geq t$, then we obtain $R \geq 1$ from the condition (8.31). Next, because $(p-\delta+\alpha u)s - \alpha u = (p-\delta) + \alpha u(s-1) \geq 0$ for $p \geq \delta$, $u \geq \delta$, $\alpha \in [0,1]$ and $s \geq 1$, then we obtain $T > 0$ from the condition (8.31).

Therefore (8.24) implies that

$$A^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}} \geq \left\{A^{\frac{R(p-\delta+\alpha u)s-(R+T)\alpha u}{2T}}\left(A^{\frac{\alpha u-\delta}{2}}B^pA^{\frac{\alpha u-\delta}{2}}\right)^s A^{\frac{R(p-\delta+\alpha u)s-(R+T)\alpha u}{2T}}\right\}^{\frac{1}{T+R}}$$

holds for $T > 0$, $R \geq 1$, $p \geq \delta$, $\alpha \in [0,1]$ and $s \geq 1$ with the condition (8.30). By raising each sides to power $T+R$, it follows from Theorem 8.40 that

$$\begin{aligned} K&\left(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, T+R\right)A^{(p-\delta+\alpha u)s-\alpha u+R\frac{(p-\delta+\alpha u)s-\alpha u}{T}} \\ &\geq A^{\frac{R(p-\delta+\alpha u)s-(R+T)\alpha u}{2T}}\left(A^{\frac{\alpha u-\delta}{2}}B^pA^{\frac{\alpha u-\delta}{2}}\right)^s A^{\frac{R(p-\delta+\alpha u)s-(R+T)\alpha u}{2T}}. \end{aligned} \tag{8.32}$$

By rearranging (8.32), we have the desired inequality

$$K\left(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, T+R\right)A^{(p-\delta+\alpha u)s} \geq \left(A^{\frac{\alpha u-\delta}{2}}B^pA^{\frac{\alpha u-\delta}{2}}\right)^s.$$

From Proposition 8.42, we have that $F(p,r,m,M) = K(m^r, M^r, \frac{p}{r}+1)$ is an increasing function of p for $p > 0$, $r > 0$ and $M > m > 0$. It follows that $K\left(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, T+R\right)$ is the increasing function of R for $R \geq 1$, $T > 0$, $p \geq \delta$, $u \geq \delta$, $\alpha \in [0,1]$, $\delta \in [0,1]$, $s \geq 1$ and $M > m > 0$. Then

$$K\left(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, T+R\right) \geq K\left(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, T+1\right).$$

□

If we put $n = 1$ in Theorem 8.40, then we obtain the following Kantorovich type inequalities, since $k(m^p, M^p, 2) = \frac{(m^p+M^p)^2}{4m^pM^p}$.

Corollary 8.43 *Let $A,B \in \mathscr{B}^{++}(H)$ with $\mathsf{Sp}(A) \subseteq [m,M]$ for some scalars $M > m > 0$. Then the following statements are mutually equivalent for each $\delta \in (0,1]$:*

(i) $A^{\delta} \geq B^{\delta}$.

(ii) For each $\alpha \in [0,1]$

$$\frac{\left(M^{(p-\delta+\alpha u)s-\alpha u}+m^{(p-\delta+\alpha u)s-\alpha u}\right)^2}{4m^{(p-\delta+\alpha u)s-\alpha u}M^{(p-\delta+\alpha u)s-\alpha u}}A^{(p-\delta+\alpha u)s} \geq \left(A^{\frac{\alpha u-\delta}{2}}B^pA^{\frac{\alpha u-\delta}{2}}\right)^s$$

holds for $s \geq 1$, $p \geq \delta$ *and* $u \geq \delta$ *with* $(p-\delta+\alpha u)s \geq (1+\alpha)u$.

(iii)

$$\left(\frac{(M^{(p-\delta)s}+m^{(p-\delta)s})^2}{4m^{(p-\delta)s}M^{(p-\delta)s}}\right)^{\frac{1}{s}} A^p \geq B^p$$

holds for $s \geq 1$ *and* $p \geq \delta$ *with* $(p-\delta)s \geq \delta$.

(iv) $\left(\frac{M}{m}\right)^{p-\delta} A^p \geq B^p$ *holds for* $p \geq \delta$.

Corollary 8.43 interpolates the following two corollaries by means of the Kantorovich constant.

Corollary 8.44 *Let $A,B \in \mathscr{B}^{++}(H)$ with $\mathsf{Sp}(A) \subseteq [m,M]$ for some scalars $M > m > 0$. Then the following statements are mutually equivalent:*

(i) $A \geq B$.

(ii) For each $\alpha \in [0,1]$

$$\frac{(M^{(p-1+\alpha u)s-\alpha u}+m^{(p-1+\alpha u)s-\alpha u})^2}{4m^{(p-1+\alpha u)s-\alpha u}M^{(p-1+\alpha u)s-\alpha u}}A^{(p-1+\alpha u)s} \geq \left(A^{\frac{\alpha u-1}{2}}B^pA^{\frac{\alpha u-1}{2}}\right)^s$$

holds for $s \geq 1$, $p \geq \delta$ *and* $u \geq \delta$ *with* $(p-1+\alpha u)s \geq (1+\alpha)u$.

(iii)

$$\left(\frac{(M^{(p-1)s}+m^{(p-1)s})^2}{4m^{(p-1)s}M^{(p-1)s}}\right)^{\frac{1}{s}} A^p \geq B^p$$

holds for $s \geq 1$ *and* $p \geq \delta$ *with* $(p-1)s \geq 1$.

(iv) $(\frac{M}{m})^{p-1}A^p \geq B^p$ *holds for* $p \geq 1$.

Proof. Put $\delta = 1$ in Corollary 8.43. □

Corollary 8.45 *Let $A,B \in \mathscr{B}^{++}(H)$ with $\mathsf{Sp}(A) \subseteq [m,M]$ for some scalars $M > m > 0$. Then the following statements are mutually equivalent:*

(i) $A \gg B$.

(ii) *For each* $\alpha \in [0,1]$

$$\frac{(M^{(p+\alpha u)s-\alpha u}+m^{(p+\alpha u)s-\alpha u})^2}{4m^{(p+\alpha u)s-\alpha u}M^{(p+\alpha u)s-\alpha u}}A^{(p+\alpha u)s} \geq \left(A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}}\right)^s$$

holds for $s \geq 1$, $p \geq 0$ *and* $u \geq 0$ *with* $(p+\alpha u)s \geq (1+\alpha)u$.

(iii) $\left(\frac{(M^{ps}+m^{ps})^2}{4m^{ps}M^{ps}}\right)^{\frac{1}{s}} A^p \geq B^p$ *holds for* $s \geq 1$ *and* $p \geq 0$.

Proof. By virtue of Lemma 8.35 and Corollary 8.43, if we put $\delta = 0$ in (i), (ii) and (iii) of Corollary 8.43, then we have (i)$\Rightarrow$ (ii) $\Rightarrow$ (iii) of Corollary 8.45. Also, (iii)$\Rightarrow$(i) is shown in Theorem 8.8 if we put $s = 1$. □

8.7 Form reversing the operator order

The object of this section is to pursue further the study of functions reversing the order of positive operators under a more general setting, which is complementary results to our previous results given in § 8.2–8.4.

The following theorem is similar to Theorem 8.14 but for the reversing order.

Theorem 8.46 *Let* $A,B \in \mathscr{B}_h(H)$ *with* $\mathsf{Sp}(A) \subseteq [m,M]$, $f \in \mathscr{C}([m,M])$ *be a convex function and* $g \in \mathscr{C}(U)$, *where* $U \supseteq [m,M] \cup \mathsf{Sp}(B)$. *Suppose that either of the following conditions holds:* (i) *g is is decreasing convex on U or* (ii) *g is increasing concave on U. If* $A \geq B$, *then for a given* $\alpha \in \mathbb{R}_+$ *in the case* (i) *or* $\alpha \in \mathbb{R}_-$ *in the case* (ii)

$$\alpha g(B) + \beta 1_H \geq f(A) \tag{8.33}$$

holds for $\beta = \max_{m \leq t \leq M}\left\{\mu_f t + \nu_f - \alpha g(t)\right\}$, *where*

$$\mu_f = \frac{f(M)-f(m)}{M-m} \qquad \text{and} \qquad \nu_f = \frac{Mf(m)-mf(M)}{M-m}.$$

Proof. Proof of Theorem 8.46 is quite similar to the proof of Theorem 8.14 gave in Section 8.2. □

The following theorem is a complementary result to Theorem 8.46 and similar to Theorem 8.15, but for the reversing order.

Theorem 8.47 *Let* $A,B \in \mathscr{B}_h(H)$, $\mathsf{Sp}(B) \subseteq [m,M]$, $f \in \mathscr{C}([m,M])$ *be a concave function and* $g \in \mathscr{C}(U)$, *where* $U \supseteq [m,M] \cup \mathsf{Sp}(A)$. *Suppose that either of the following conditions holds:* (i) *g is decreasing concave on U or* (ii) *g is increasing convex on U. If* $A \geq B$, *then for a given* $\alpha \in \mathbb{R}_+$ *in the case* (i) *or* $\alpha \in \mathbb{R}_-$ *in the case* (ii)

$$f(B) \geq \alpha g(A) + \beta 1_H, \tag{8.34}$$

holds for $\beta = \min_{m \leq t \leq M}\left\{\mu_f t + \nu_f - \alpha g(t)\right\}$.

Remark 8.11 *If we put $\alpha = 1$ in Theorems 8.46 and 8.47, then we have the following: Let $A, B \in \mathscr{B}_h(H)$, $\mathrm{Sp}(A) \subseteq [m,M]$ (resp. $\mathrm{Sp}(B) \subseteq [m,M]$). Let $f \in \mathscr{C}([m,M])$ be a convex (resp. concave) function and $g \in \mathscr{C}(U)$ be an decreasing convex (resp. decreasing concave) function, where $U \supseteq [m,M] \cup \mathrm{Sp}(A) \cup \mathrm{Sp}(B)$. If $A \geq B$, then*

$$g(B) + \beta 1_H \geq f(A) \qquad (resp.\ \ f(B) \geq g(A) + \beta 1_H),$$

holds for

$$\beta = \max_{m \leq t \leq M} \{\mu_f t + \nu_f - g(t)\} \quad (resp.\ \ \beta = \min_{m \leq t \leq M} \{\mu_f t + \nu_f - g(t)\}).$$

If we choose α such that $\beta = 0$ in Theorems 8.46 and 8.47, then we have the following corollary:

Corollary 8.48 *Let $A, B \in \mathscr{B}_h(H)$, $\mathrm{Sp}(A) \subseteq [m,M]$ (resp. $\mathrm{Sp}(B) \subseteq [m,M]$). Let $f \in \mathscr{C}([m,M])$ be a convex (resp. concave) function and $g \in \mathscr{C}(U)$, where $U \supseteq [m,M] \cup \mathrm{Sp}(A) \cup \mathrm{Sp}(B)$. Suppose that either of the following conditions holds:*
(I) *g is decreasing convex (resp. concave) on U, $g > 0$ on $[m,M]$ and $f(m) > 0$, $f(M) > 0$,*
(II) *g is decreasing convex (resp. concave)on U, $g < 0$ on $[m,M]$ and $f(m) < 0$, $f(M) < 0$,*
(III) *g is increasing concave (resp. convex) on U, $g > 0$ on $[m,M]$ and $f(m) < 0$, $f(M) < 0$,*
(IV) *g is increasing concave (resp. convex) on U, $g < 0$ on $[m,M]$ and $f(m) > 0$, $f(M) > 0$.*
If $A \geq B$, then

$$\alpha_1 g(B) \geq f(A) \qquad (resp.\ \ f(B) \geq \alpha_2 g(A))$$

holds for

$$\alpha_1 = \max_{m \leq t \leq M} \left\{ \frac{\mu_f t + \nu_f}{g(t)} \right\} \quad \left(resp.\ \ \alpha_2 = \min_{m \leq t \leq M} \left\{ \frac{\mu_f t + \nu_f}{g(t)} \right\} \right)$$

in case (I) *and* (III), *or*

$$\alpha_1 = \min_{m \leq t \leq M} \left\{ \frac{\mu_f t + \nu_f}{g(t)} \right\} \quad \left(resp.\ \ \alpha_2 = \max_{m \leq t \leq M} \left\{ \frac{\mu_f t + \nu_f}{g(t)} \right\} \right)$$

in case (II) *and* (IV).

Remark 8.12 *If we put $f \equiv g$ in Theorems 8.46 and 8.47 and Corollary 8.48 then we can obtain results similar to Theorems 8.18 and 8.19 and Corollary 8.21, but for reversing order.*

In particular, we have the following corollary, which follows from Theorem 8.46 if we put $f(t) = t^p$, $p \in \mathbb{R} \backslash [0,1)$ and $g(t) = t^q$, $q < 0$.

Corollary 8.49 *Let $A, B \in \mathscr{B}^+(H)$, $\mathrm{Sp}(A) \subseteq [m,M]$. If $A \geq B > 0$, then for a given $\alpha \in \mathbb{R}_+$*

$$\alpha B^q + \beta 1_H \geq A^p, \qquad \text{for all } p \in \mathbb{R} \backslash [0,1),\ q < 0,$$

where

$$\beta = \begin{cases} \alpha(q-1)\left(\frac{1}{\alpha q}\mu_{t^p}\right)^{\frac{q}{q-1}} + \nu_{t^p} & \text{if} \quad m < \left(\frac{1}{\alpha q}\mu_{t^p}\right)^{\frac{1}{q-1}} < M, \\ \max\{m^p - \alpha m^q, M^p - \alpha M^q\} & \text{otherwise.} \end{cases}$$

The following two corollaries follow from Corollary 8.49.

Corollary 8.50 *Let $A, B \in \mathscr{B}^+(H)$, $\mathsf{Sp}(A) \subseteq [m, M]$. If $A \geq B > 0$, then*

$$K(m, M, p, q)B^q \geq A^p, \qquad \text{for all } p < 0,\ q < 0,$$

where $K(m, M, p, q)$ is defined in (8.18).

Corollary 8.51 *Let $A, B \in \mathscr{B}^+(H)$ be positive operators on H with $\mathsf{Sp}(A) \subseteq [m_1, M_1]$ and $\mathsf{Sp}(B) \subseteq [m_2, M_2]$ for some scalars $M_j > m_j > 0\,(j = 1, 2)$. If $A \geq B > 0$, then the following inequalities hold*

$$K(m_1, M_1, p)B^p \geq A^p,$$
$$K(m_2, M_2, p)B^p \geq A^p$$

for all $p < -1$, where $K(m, M, p)$ is defined by (8.3).

8.8 Notes

Theorem 8.1 and Theorem 8.2 are due to M.Fujii, Izumino, Nakamoto and Seo [59]. Kantorovich type inequalities of the operator order are firstly considered by Furuta [80] and that of the chaotic order by Yamazaki and Yanagida [199]. Proposition 8.42 is due to [199].

For our exposition we have used [132] and [134].

Further results related to the Kantorovich type inequalities are contained in [92], [175], [51], [52], [58], [63], [89], [101] and [114].

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