MONOGRAPHS IN INEQUALITIES 3

Recent Advances in Hilbert-type Inequalities

A unified treatment of Hilbert-type inequalities

Mario Krnić, Josip Pečarić, Ivan Perić and Predrag Vuković

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Mario Krnić
Faculty of Electrical Engineering and Computing
University of Zagreb
Zagreb, Croatia

Josip Pečarić
Faculty of Textile Technology
University of Zagreb
Zagreb, Croatia

Ivan Perić
Faculty of Food Technology and Biotechnology
University of Zagreb
Zagreb, Croatia

Predrag Vuković
Faculty of Teacher Education
University of Zagreb
Zagreb, Croatia

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Preface

At the beginning of the 20th century, the following inequalities in discrete and integral forms have been established:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin\frac{\pi}{p}}\left[\sum_{m=1}^{\infty} a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty} b_n^q\right]^{\frac{1}{q}}$$

and

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin\frac{\pi}{p}}\left[\int_0^{\infty} f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^{\infty} g^q(y)dy\right]^{\frac{1}{q}}.$$

The first inequality refers to non-negative sequences $(a_m)_{m\in\mathbb{N}} \in l^p$ and $(b_n)_{n\in\mathbb{N}} \in l^q$ which are not zero-sequences, while the second inequality holds for non-negative functions $f \in L^p(\mathbb{R}_+)$ and $g \in L^q(\mathbb{R}_+)$ which are not zero-functions. The common parameters, that is, the exponents p and q appearing in both inequalities are mutually conjugate, that is, they fulfill the condition $\frac{1}{p}+\frac{1}{q}=1$, where $p>1$. The above inequalities were first studied by David Hilbert at the end of the nineteenth century, hence, in his honor, they are referred to as the discrete and the integral Hilbert inequalities.

The Hilbert inequality is one of the most important inequalities in mathematical analysis. Applications of this inequality in diverse fields of mathematics have certainly contributed to its importance.

After its discovery, the Hilbert inequality was studied by numerous authors, who either reproved it using various techniques, or applied and generalized it in many different ways. Such generalizations included inequalities with more general kernels, weight functions and integration sets, extension to a multidimensional case, and so forth. The resulting relations are usually referred to as the Hilbert-type inequalities. On the other hand, Hardy, Littlewood and Pólya [33], noted that to every Hilbert-type inequality one can assign its equivalent form, in the sense that one implies another and vice versa. Such forms are usually called Hardy-Hilbert-type inequalities, since they are closely connected with another famous classical inequality, that is, the Hardy inequality. For a comprehensive inspection of the initial development of the Hilbert inequality, the reader is referred to a classical monograph [33].

Although classical, the Hilbert inequality is still of interest to numerous mathematicians. Nowadays, more than a century after its discovery, this problem area offers diverse possibilities for generalizations and extensions.

The present book is a crown of decennial research of several authors in this area. More precisely, this book is based on some thirty significant papers dealing with Hilbert-type inequalities, published in the course of the last ten years.

We tried to provide a unified approach to Hilbert-type inequalities. More precisely, the original Hilbert inequality can be regarded in a more general setting, with integrals taken over a σ-finite measure space, and with a general kernel and weight functions. In addition, Hilbert-type inequalities can also be considered with exponents which are not mutually conjugate. On the other hand, in recent time a special emphasis has been dedicated to establishing methods for improving original Hilbert-type inequalities. These are the main topics we deal with in this monograph. The book is divided into ten chapters.

In **Chapter 1** a unified treatment of Hilbert-type inequalities with conjugate exponents is established. The most general form of the Hilbert inequality involves integrals taken over a σ-finite measure space, a general kernel and the weight functions. A special emphasis is given to Hilbert-type inequalities with homogeneous kernels. In addition, considerable attention is dedicated to finding the best possible constant factors appearing in some classes of inequalities. Observe here that a majority of results in this and other chapters will be given in two equivalent forms.

In **Chapter 2** we extend Hilbert-type inequalities derived in the previous chapter to the case of non-conjugate exponents. It should be observed here that the problem of finding the best possible constant factors is not considered in Chapter 2 since it seems to be very hard and remains still open. Additionally, we study some operators between certain weighted Lebesgue spaces, arising from the Hardy-Hilbert form of the corresponding Hilbert-type inequality.

In **Chapter 3** we consider Hilbert-type inequalities involving real valued kernel and weight functions defined on $\mathbb{R}^n$. Such results will be derived by virtue of the so-called Selberg integral formula.

In **Chapter 4** we derive two types of refined discrete Hilbert-type inequalities by means of the Euler-Maclaurin summation formula, depending on whether the corresponding kernel is of class C^2 or C^4. In addition, some particular refinements are also established, due to the above summation formula.

In **Chapter 5** a different approach for improving discrete Hilbert-type inequalities is presented. Such improvements are derived by virtue of the Hermite-Hadamard inequality.

In **Chapter 6** we deal with refinements of some particular Hilbert-type inequalities involving the Laplace transform.

In **Chapter 7** a particular class of the so-called Hilbert-Pachpatte-type inequalities is studied. These inequalities are closely connected with Hilbert-type inequalities.

In **Chapter 8** another famous classical inequality, closely connected to the Hilbert inequality, is studied. That is the Hardy inequality. A unified treatment of Hardy-type inequalities with non-conjugate exponents is established.

In **Chapter 9** Hilbert-type inequalities are considered in a more general function space. Namely, all results in previous chapters were related to the weighted Lebesgue spaces. In this chapter Hilbert-type inequalities are established in the weighted Orlicz spaces.

In **Chapter 10** we list another set of recent results of numerous authors, interesting on its own right, which are closely connected with the theory exposed in this book. Namely, some related inequalities and refinements are given without the proof.

Throughout the whole book, presented results are compared with previously known from the literature. Moreover, at the end of a section or a chapter we cite the corresponding references for the results presented there. In addition, we also quote references which are closely connected with presented topics.

Since this monograph is based on numerous papers written by different authors, the terminology in the book is not quite unified. On the other hand, we suppose that the reader is very familiar with mathematical analysis and we mostly use the standard notation. However, to avoid misunderstandings, some extra notation and definitions are presented when it is necessary. Further, in statements of some theorems, conditions concerning convergence of series and integrals are omitted. If nothing else is explicitly stated, they are assumed to be convergent.

Authors

Contents

Chapter 1

Hilbert-type inequalities with conjugate exponents

One of the most important inequalities in modern mathematics is the well-known Hilbert inequality. Applications of this inequality in various branches of mathematics have certainly contributed to its importance. David Hilbert was the first mathematician who started to deal with the Hilbert inequality, by considering its discrete form. He did not even think that he had opened the space for numerous researches whose results will be far-reaching and fruitful.

Shortly after discovering the discrete form, the integral form of the Hilbert inequality was also established, as well as the generalization for the case of conjugate exponents. During subsequent decades, the Hilbert inequality was also generalized in many different ways by some famous authors. Nowadays, more than a century after Hilbert's discovery, this problem area is still of interest and provides some possibilities for further generalizations.

In this chapter we present some basic generalizations of the Hilbert inequality. After the short historical overview, we expose a recent important generalization, which provides a unified treatment to this inequality with conjugate exponents. In particular, in that result the integrals are taken with σ-finite measures, which includes both integral and discrete case.

The above mentioned main result is then applied to homogeneous functions, which yields numerous interesting examples. Also, the consideration of such examples in particular settings yields numerous results, previously known from the literature. Moreover, all results presented in two-dimensional case can naturally be extended to a multidimensional case.

Finally, numerous inequalities in this chapter include the corresponding constant factor on their right-hand sides. By the classical Hilbert inequality such constant factor was the best possible in the sense that it cannot be replaced with the smaller constant so that the resulting inequality still remains valid. We shall also present here some recent results which include such best possible constant factors.

1.1 Historical overview

We begin this overview with a bilinear form

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n},$$

associated to sequences of real numbers $(a_m)_{m\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, which was first studied by D. Hilbert at the end of the nineteenth century. Hilbert discovered a natural upper bound of this double series and laid the foundations for the theory that will follow. Thus, we present here some basic theorems which arose immediately from Hilbert's considerations.

Theorem 1.1 *Let p and q be mutually conjugate exponents, that is, $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $(a_m)_{m\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be any two sequences of non-negative real numbers such that $0<\sum_{m=1}^{\infty}a_m^p<\infty$ and $0<\sum_{n=1}^{\infty}b_n^q<\infty$. Then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{\sin\frac{\pi}{p}}\left[\sum_{m=1}^{\infty}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}b_n^q\right]^{\frac{1}{q}}. \tag{1.1}$$

The integral form of the previous theorem reads as follows:

Theorem 1.2 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $f,g:\mathbb{R}_+\to\mathbb{R}$ be any two non-negative Lebesgue measurable functions such that $0<\int_0^\infty f^p(x)dx<\infty$ and $0<\int_0^\infty g^q(y)dy<\infty$. Then*

$$\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{x+y}dxdy<\frac{\pi}{\sin\frac{\pi}{p}}\left[\int_0^\infty f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty g^q(y)dy\right]^{\frac{1}{q}}. \tag{1.2}$$

Remark 1.1 Suppose that p and q are mutually conjugate parameters, i.e. $\frac{1}{p}+\frac{1}{q}=1$, and let $p>1$. Then it follows that $q>1$. On the other hand, if $0<p<1$, then $q<0$, and analogously, if $0<q<1$, then $p<0$.

As we see, in the above theorems the same constant factor appears on the right-hand sides of both inequalities. It was proved by Hardy and Riesz that this constant factor was the best possible.

Theorem 1.3 *The constant $\pi/\sin(\pi/p)$ appearing in (1.1) and (1.2) is the best possible.*

The previous three results are taken from the classical monograph [33], in a slightly altered form. The case of $p = q = 2$ in Theorem 1.1 was first proved by Hilbert in his lectures about integral equations. The lack of that old proof consisted in the fact that Hilbert didn't know to determine the optimal constant factor π. That drawback was removed by Shur in 1911, who also proved the integral version of the inequality. The extensions to arbitrary pair of positive mutually conjugate exponents are due to G.H. Hardy and M. Riesz.

Some other proofs, as well as various generalizations are due to the following mathematicians: L. Fejér, E. Francis, G. H. Hardy, J. Littlewood, H. Mulholland, P. Owen, G. Pólya, F. Riesz, M. Riesz, I. Schur, G. Szegö. Nevertheless, the inequalities (1.1) and (1.2) remained known as the discrete and the integral Hilbert inequalities. For more details about the initial development of the Hilbert inequality the reader is referred to [33, Chapter 9]. It should be noticed here that generalizations of inequalities (1.1) and (1.2) will be referred to as the Hilbert-type inequalities.

However, we provide two more results from the 1920s, which will also play an important role in further investigations. Namely, Hardy, Littlewood and Pólya noted that to every Hilbert-type inequality one can assign its equivalent form, in the sense that one implies another and vice versa. For example, the equivalent form assigned to inequality (1.1) is contained in the following theorem.

Theorem 1.4 *Let $p > 1$ and let $(a_m)_{m\in\mathbb{N}}$ be the sequence of non-negative real numbers such that $0 < \sum_{m=1}^{\infty} a_m^p < \infty$. Then*

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right)^p < \left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^p \sum_{m=1}^{\infty} a_m^p. \tag{1.3}$$

Obviously, the integral analogue of inequality (1.3) is analogous, with the sum replaced with the integral, and a sequence with a non-negative real function. Such inequalities, derived from the Hilbert-type inequalities will be referred to as the Hardy-Hilbert-type inequalities. Moreover, the Hilbert-type and the Hardy-Hilbert-type inequalities will sometimes simply be referred to as the Hilbert-type inequalities.

Already at that time, the sharper version of inequality (1.1) was also known. That result is presented in the following theorem.

Theorem 1.5 *Under the same assumptions as in Theorem 1.1, we have*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{m+n+1} < \frac{\pi}{\sin\frac{\pi}{p}}\left[\sum_{m=0}^{\infty} a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=0}^{\infty} b_n^q\right]^{\frac{1}{q}}. \tag{1.4}$$

Inequalities (1.1) and (1.4) are known in the literature as "the Hilbert double series theorems".

These theorems were inspiration to numerous mathematicians. During the 20th century numerous proofs, generalizations and applications of the Hilbert inequality were discovered and it would be impossible to count them here.

Nowadays, more than a century after the discovery of the Hilbert inequality, this research area is still interesting to numerous authors. As an illustration, we indicate here some generalizations obtained in the last ten years. One of the possible extensions arises from studying various kernels. Namely, in presented results such kernel was the function $K(x,y)=(x+y)^{-1}$. In 1998, considering the kernel $K(x,y)=(x+y)^{-s}$, $s>0$, B. Yang was the first one to have included the well-known Beta function into the study of Hilbert-type inequalities (see [138]). Recall that the Beta function is an integral function defined by

$$B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}dt, \qquad a,b>0. \tag{1.5}$$

For example, in [152] one can find the following result in the integral form.

Theorem 1.6 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $s>2-\min\{p,q\}$. If $f,g:\mathbb{R}_+\to\mathbb{R}$ are non-negative measurable functions such that $0<\int_0^\infty x^{1-s}f^p(x)dx<\infty$ and $0<\int_0^\infty y^{1-s}g^q(y)dy<\infty$, then*

$$\begin{aligned}&\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^s}dxdy\\ &< B\left(\frac{p+s-2}{p},\frac{q+s-2}{q}\right)\left[\int_0^\infty x^{1-s}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{1-s}g^q(y)dy\right]^{\frac{1}{q}}\end{aligned} \tag{1.6}$$

and

$$\begin{aligned}&\int_0^\infty y^{(s-1)(p-1)}\left[\int_0^\infty \frac{f(x)}{(x+y)^s}dx\right]^p dy\\ &< B^p\left(\frac{p+s-2}{p},\frac{q+s-2}{q}\right)\int_0^\infty x^{1-s}f^p(x)dx.\end{aligned} \tag{1.7}$$

Moreover, these two inequalities are equivalent and include the best possible constant factors on their right-hand sides.

The multidimensional extension of inequality (1.6), involving the usual Gamma function, has also been derived in the above mentioned paper [152]. Recall that the Gamma function is an integral

$$\Gamma(a)=\int_0^\infty t^{a-1}e^{-t}dt, \qquad a>0. \tag{1.8}$$

Theorem 1.7 *Suppose that p_i are mutually conjugate exponents, i.e. $\sum_{i=1}^n \frac{1}{p_i}=1$, $p_i>1$, $i=1,2,\ldots,n$, and let $s>n-\min_{1\le i\le n}\{p_i\}$. If $f_i:\mathbb{R}_+\to\mathbb{R}$, $i=1,2,\ldots,n$, are non-negative measurable functions satisfying $0<\int_0^\infty {x_i}^{n-1-s}f_i^{p_i}(x_i)dx_i<\infty$, then*

$$\begin{aligned}&\int_0^\infty\cdots\int_0^\infty \frac{\prod_{i=1}^n f_i(x_i)}{(\sum_{i=1}^n x_i)^s}dx_1dx_2\ldots dx_n\\ &< \frac{1}{\Gamma(s)}\prod_{i=1}^n\Gamma\left(\frac{p_i+s-n}{p_i}\right)\left[\int_0^\infty {x_i}^{n-1-s}f_i^{p_i}(x_i)dx_i\right]^{\frac{1}{p_i}}.\end{aligned} \tag{1.9}$$

Remark 1.2 The above notation for the Beta and the Gamma function will be used throughout the book. The basic relationship between the Beta and the Gamma functions is given by

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},\ a,b>0, \tag{1.10}$$

and this formula will often be exploited. For more details about the Beta and the Gama functions, as well as about their meromorphic extensions to the set of complex numbers, the reader is referred to [1].

It is interesting that the n-dimensional inequality (1.9) also posses its equivalent form, which will be discussed in this chapter.

On the other hand, another possible generalization of the presented results is the investigation of the inequalities of the same type, but where the integrals are taken over a bounded interval in $\mathbb{R}_+$. Guided by that idea, K. Jichang and T. Rassias [42], obtained the following result.

Theorem 1.8 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ be a non-negative homogeneous symmetric function of degree $-s$, where $\max\{\frac{1}{p},\frac{1}{q}\}<s$. If $K(1,y)$ is strictly decreasing in y and $f,g:\mathbb{R}_+\to\mathbb{R}$ are non-negative measurable functions, then*

$$\begin{aligned}
&\int_a^b\int_a^b K(x,y)f(x)g(y)dxdy \\
&\leq \left[\int_a^b \Big(I(q)-\varphi(q,x)\Big)x^{1-s}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_a^b \Big(I(p)-\varphi(p,y)\Big)y^{1-s}g^q(y)dy\right]^{\frac{1}{q}},
\end{aligned} \tag{1.11}$$

where

$$\varphi(r,x) = \left(\frac{a}{x}\right)^{1-\frac{1}{r}}\int_0^1 K(1,u)u^{-\frac{1}{r}}du + \left(\frac{x}{b}\right)^{s+\frac{1}{r}-1}\int_0^1 K(1,u)u^{s+\frac{1}{r}-2}du,$$

$I(r)=\int_0^\infty K(1,u)u^{-\frac{1}{r}}du$, $r\in\{p,q\}$, and $0\leq a<b\leq\infty$.

In the next section, the integrals will be taken over more general sets.

1.2 A unified treatment of Hilbert-type inequalities with conjugate exponents

In the previous historical overview we have seen the classical Hilbert inequality in both discrete and integral case. Moreover, throughout years numerous extensions of these inequalities were derived. However, all these results were given in either integral form, with respect to the Lebesgue measure, or in the discrete form.

The main objective of this section is to present a general result which unifies the integral and discrete cases. This can be done by observing a more general integral. Namely, the classical Hilbert inequality is a consequence of the Hölder inequality and the Fubini theorem. In general, the Fubini theorem holds for the integrals with σ-finite measures, therefore, such measures will be considered.

The most important examples of σ-finite measures are the Lebesgue measure and the counting measure. The Lebesgue measure yields the classical integral case, while the counting measure provides the discrete case.

Further, it is well-known that if one of the mutually conjugate exponents in the Hölder inequality is negative, then the sign of the inequality is reversed (see [103]). Hence, we shall also be concerned with the Hilbert-type inequalities with the reversed sign of inequality. Such inequalities will be referred to as the reverse inequalities.

Now we present the most general form of the Hilbert inequality in the setting described above. It should be noticed here that we suppose that all integrals converge, and such types of conditions will often be omitted. Moreover, integrals will be taken over a general measure space. Results that follow are provided in two equivalent forms: the Hilbert and the Hardy-Hilbert forms.

Theorem 1.9 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let Ω be a measure space with positive σ-finite measures μ_1 and μ_2. Let $K:\Omega\times\Omega\to\mathbb{R}$ and $\varphi,\psi:\Omega\to\mathbb{R}$ be non-negative measurable functions. If the functions F and G are defined by $F(x)=\int_\Omega K(x,y)\psi^{-p}(y)d\mu_2(y)$ and $G(y)=\int_\Omega K(x,y)\varphi^{-q}(x)d\mu_1(x)$, then for all non-negative measurable functions f and g on Ω the inequalities*

$$\int_\Omega\int_\Omega K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y)$$
$$\leq\left[\int_\Omega \varphi^p(x)F(x)f^p(x)d\mu_1(x)\right]^{\frac{1}{p}}\left[\int_\Omega \psi^q(y)G(y)g^q(y)d\mu_2(y)\right]^{\frac{1}{q}} \tag{1.12}$$

and

$$\int_\Omega G^{1-p}(y)\psi^{-p}(y)\left[\int_\Omega K(x,y)f(x)d\mu_1(x)\right]^p d\mu_2(y)$$
$$\leq\int_\Omega \varphi^p(x)F(x)f^p(x)d\mu_1(x) \tag{1.13}$$

hold and are equivalent.

If $0<p<1$, then the reverse inequalities in (1.12) and (1.13) are valid, as well as the inequality

$$\int_\Omega F^{1-q}(x)\varphi^{-q}(x)\left[\int_\Omega K(x,y)g(y)d\mu_2(y)\right]^q d\mu_1(x)$$
$$\leq\int_\Omega \psi^q(y)G(y)g^q(y)d\mu_2(y). \tag{1.14}$$

Proof. The left-hand side of inequality (1.12) can be rewritten in the following form:

$$\int_\Omega\int_\Omega K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y)=\int_\Omega\int_\Omega K(x,y)f(x)\frac{\varphi(x)}{\psi(y)}g(y)\frac{\psi(y)}{\varphi(x)}d\mu_1(x)d\mu_2(y).$$

Now, applying the Hölder inequality to the above relation yields

$$\begin{aligned}
&\int_\Omega\int_\Omega K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y)\\
&\leq \left[\int_\Omega\int_\Omega K(x,y)f^p(x)\frac{\varphi^p(x)}{\psi^p(y)}d\mu_1(x)d\mu_2(y)\right]^{\frac{1}{p}}\\
&\quad\times\left[\int_\Omega\int_\Omega K(x,y)g^q(y)\frac{\psi^q(y)}{\varphi^q(x)}d\mu_1(x)d\mu_2(y)\right]^{\frac{1}{q}}.
\end{aligned}$$

Finally, using the Fubini theorem and definitions of functions F and G we obtain (1.12).

Now, we are going to show the equivalence of inequalities (1.12) and (1.13). For that sake, suppose that inequality (1.12) holds. Defining the function g by

$$g(y)=G^{1-p}(y)\psi^{-p}(y)\left[\int_\Omega K(x,y)f(x)d\mu_1(x)\right]^{p-1},$$

taking into account that $\frac{1}{p}+\frac{1}{q}=1$, and using (1.12), we have

$$\begin{aligned}
&\int_\Omega G^{1-p}(y)\psi^{-p}(y)\left[\int_\Omega K(x,y)f(x)d\mu_1(x)\right]^p d\mu_2(y)\\
&=\int_\Omega\int_\Omega K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y)\\
&\leq\left[\int_\Omega \varphi^p(x)F(x)f^p(x)d\mu_1(x)\right]^{\frac{1}{p}}\left[\int_\Omega \psi^q(y)G(y)g^q(y)d\mu_2(y)\right]^{\frac{1}{q}}\\
&=\left[\int_\Omega \varphi^p(x)F(x)f^p(x)d\mu_1(x)\right]^{\frac{1}{p}}\\
&\quad\times\left[\int_\Omega G^{1-p}(y)\psi^{-p}(y)\left[\int_\Omega K(x,y)f(x)d\mu_1(x)\right]^p d\mu_2(y)\right]^{\frac{1}{q}},
\end{aligned}$$

that is, we get (1.13).

On the other hand, suppose that inequality (1.13) holds. In that case, another use of the Hölder inequality yields

$$\begin{aligned}
&\int_\Omega\int_\Omega K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y)\\
&=\int_\Omega\left[\psi^{-1}(y)G^{-\frac{1}{q}}(y)\int_\Omega K(x,y)f(x)d\mu_1(x)\right]\psi(y)G^{\frac{1}{q}}(y)g(y)d\mu_2(y)\\
&\leq\left[\int_\Omega G^{1-p}(y)\psi^{-p}(y)\left(\int_\Omega K(x,y)f(x)d\mu_1(x)\right)^p d\mu_2(y)\right]^{\frac{1}{p}}\\
&\quad\times\left[\int_\Omega \psi^q(y)G(y)g^q(y)d\mu_2(y)\right]^{\frac{1}{q}}\\
&\leq\left[\int_\Omega \varphi^p(x)F(x)f^p(x)d\mu_1(x)\right]^{\frac{1}{p}}\left[\int_\Omega \psi^q(y)G(y)g^q(y)d\mu_2(y)\right]^{\frac{1}{q}},
\end{aligned}$$

which implies (1.12). Therefore, inequalities (1.12) and (1.13) are equivalent.

The reverse inequalities, as well as their equivalence, are derived in the same way by virtue of the reverse Hölder inequality. □

Remark 1.3 The equality in the previous theorem is possible if and only if it holds in the Hölder inequality, that is, if

$$\left[f(x)\frac{\varphi(x)}{\psi(y)}\right]^p = C\left[g(y)\frac{\psi(y)}{\varphi(x)}\right]^q, \quad \text{a. e. on } \Omega,$$

where C is a positive constant. In that case we have

$$f(x) = C_1\varphi^{-q}(x) \quad \text{and} \quad g(y) = C_2\psi^{-p}(y) \quad \text{a. e. on } \Omega, \tag{1.15}$$

for some constants C_1 and C_2, which is possible if and only if

$$\int_\Omega F(x)\varphi^{-q}(x)d\mu_1(x) < \infty \quad \text{and} \quad \int_\Omega G(y)\psi^{-p}(y)d\mu_2(y) < \infty. \tag{1.16}$$

Otherwise, the inequalities in Theorem 1.9 are strict.

In some applications of the previous theorem it will be more convenient to bound the functions $F(x)$ and $G(y)$. Of course, such result follows immediately from Theorem 1.9.

Theorem 1.10 *Suppose that the assumptions as in Theorem* 1.9 *are fulfilled and let* $F_1, G_1 : \Omega \to \mathbb{R}$ *be non-negative measurable functions such that* $F(x) \le F_1(x)$ *and* $G(y) \le G_1(y)$*, a. e. on* Ω*. Then the inequalities*

$$\int_\Omega\int_\Omega K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y)$$
$$\le \left[\int_\Omega \varphi^p(x)F_1(x)f^p(x)d\mu_1(x)\right]^{\frac{1}{p}}\left[\int_\Omega \psi^q(y)G_1(y)g^q(y)d\mu_2(y)\right]^{\frac{1}{q}} \tag{1.17}$$

and

$$\int_\Omega {G_1}^{1-p}(y)\psi^{-p}(y)\left[\int_\Omega K(x,y)f(x)d\mu_1(x)\right]^p d\mu_2(y)$$
$$\le \int_\Omega \varphi^p(x)F_1(x)f^p(x)d\mu_1(x) \tag{1.18}$$

hold and are equivalent.

If $0 < p < 1$, $F(x) \ge F_1(x)$, *and* $G(y) \le G_1(y)$, *then the reverse inequalities in* (1.17) *and* (1.18) *hold, as well as the inequality*

$$\int_\Omega {F_1}^{1-q}(x)\varphi^{-q}(x)\left[\int_\Omega K(x,y)g(y)d\mu_2(y)\right]^q d\mu_1(x)$$
$$\le \int_\Omega \psi^q(y)G_1(y)g^q(y)d\mu_2(y). \tag{1.19}$$

The reverse inequalities are also equivalent.

Remark 1.4 The general Hilbert-type inequalities presented in this section have been obtained by M. Krnić and J. Pečarić in [53].

1.3 Applications to homogeneous kernels

Theorem 1.9 from the previous section has unified the classical integral and discrete cases of the Hilbert inequality. In order to approach to some well-known results from the literature, we study here some particular choices of kernels and weight functions.

In this section we consider homogeneous kernels of negative degree of homogeneity, equipped with some additional properties. Recall that a function $K:\Omega\times\Omega\to\mathbb{R}$ is said to be homogeneous of degree $-s$, $s>0$, if $K(tx,ty)=t^{-s}K(x,y)$ for every $x,y\in\Omega$ and $t\in\mathbb{R}$ such that tx, $ty\in\Omega$. In addition, for such homogeneous function we define $k(\alpha)$ as

$$k(\alpha)=\int_0^\infty K(1,u)u^{-\alpha}du, \tag{1.20}$$

provided that the above integral converges for $1-s<\alpha<1$.

We study here the integral case, that is, the Lebesgue integral. The integrals are taken over an arbitrary interval of non-negative real numbers, i.e. $(a,b)\subseteq\mathbb{R}_+$, $0\le a<b\le\infty$, and the weight functions are chosen to be power functions.

Theorem 1.11 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $K:(a,b)\times(a,b)\to\mathbb{R}$ be a non-negative homogeneous function of degree $-s$, $s>0$, strictly decreasing in both variables. If A_1 and A_2 are real parameters such that $A_1\in(\frac{1-s}{q},\frac{1}{q})$, $A_2\in(\frac{1-s}{p},\frac{1}{p})$, then for all non-negative measurable functions $f,g:(a,b)\to\mathbb{R}$ the inequalities*

$$\begin{aligned}&\int_a^b\int_a^b K(x,y)f(x)g(y)dxdy\\ &\le\left[\int_a^b\big(k(pA_2)-\varphi_1(pA_2,x)\big)x^{1-s+p(A_1-A_2)}f^p(x)dx\right]^{\frac{1}{p}}\\ &\quad\times\left[\int_a^b\big(k(2-s-qA_1)-\varphi_2(2-s-qA_1,y)\big)y^{1-s+q(A_2-A_1)}g^q(y)dy\right]^{\frac{1}{q}}\end{aligned} \tag{1.21}$$

and

$$\begin{aligned}&\int_a^b\big(k(2-s-qA_1)-\varphi_2(2-s-qA_1,y)\big)^{1-p}y^{(p-1)(s-1)+p(A_1-A_2)}\\ &\quad\times\left[\int_a^b K(x,y)f(x)dx\right]^p dy\\ &\le\int_a^b\big(k(pA_2)-\varphi_1(pA_2,x)\big)x^{1-s+p(A_1-A_2)}f^p(x)dx\end{aligned} \tag{1.22}$$

hold and are equivalent, where

$$\varphi_1(\alpha,x)=\left(\frac{a}{x}\right)^{1-\alpha}\int_0^1 K(1,u)u^{-\alpha}du+\left(\frac{x}{b}\right)^{s+\alpha-1}\int_0^1 K(u,1)u^{s+\alpha-2}du,$$

$$\varphi_2(\alpha,y)=\left(\frac{a}{y}\right)^{s+\alpha-1}\int_0^1 K(u,1)u^{s+\alpha-2}du+\left(\frac{y}{b}\right)^{1-\alpha}\int_0^1 K(1,u)u^{-\alpha}du.$$

If $0<p<1$, $b=\infty$, and $K(x,y)$ is strictly decreasing in x and strictly increasing in y, then the reverse inequalities in (1.21) and (1.22) are valid for every $A_1\in(\frac{1}{q},\frac{1-s}{q})$ and $A_2\in(\frac{1-s}{p},\frac{1}{p})$, as well as the inequality

$$\int_a^\infty \left(k(pA_2)-\varphi_1(pA_2,x)\right)^{1-q}x^{(q-1)(s-1)+q(A_2-A_1)}\left[\int_a^\infty K(x,y)g(y)dy\right]^q dx$$
$$\le \int_a^\infty \left(k(2-s-qA_1)-\varphi_2(2-s-qA_1,y)\right)y^{1-s+q(A_2-A_1)}g(y)^q dy.$$

Moreover, if $0<p<1$, $a=0$, and $K(x,y)$ is strictly increasing in x and strictly decreasing in y, then the reverse inequalities in (1.21) and (1.22) hold for every $A_1\in(\frac{1}{q},\frac{1-s}{q})$ and $A_2\in(\frac{1-s}{p},\frac{1}{p})$, as well as the inequality

$$\int_0^b \left(k(pA_2)-\varphi_1(pA_2,x)\right)^{1-q}x^{(q-1)(s-1)+q(A_2-A_1)}\left[\int_0^b K(x,y)g(y)dy\right]^q dx$$
$$\le \int_0^b \left(k(2-s-qA_1)-\varphi_2(2-s-qA_1,y)\right)y^{1-s+q(A_2-A_1)}g(y)^q dy.$$

Proof. We only prove inequality (1.21). After substituting the power functions $\varphi(x)=x^{A_1}$ and $\psi(y)=y^{A_2}$ in (1.12), the homogeneity of the kernel K and the substitution $u=\frac{y}{x}$ yield the following relation:

$$\int_a^b\int_a^b K(x,y)f(x)g(y)dxdy$$
$$\le\left[\int_a^b x^{1-s+p(A_1-A_2)}\left(\int_{\frac{a}{x}}^{\frac{b}{x}}K(1,u)u^{-pA_2}du\right)f^p(x)dx\right]^{\frac{1}{p}}$$
$$\times\left[\int_a^b y^{1-s+q(A_2-A_1)}\left(\int_{\frac{y}{b}}^{\frac{y}{a}}K(1,u)u^{qA_1+s-2}du\right)g^q(y)dy\right]^{\frac{1}{q}}.$$

In addition, considering the function $l(y)=y^{\alpha-1}\int_0^y K(1,u)u^{-\alpha}du$, $\alpha<1$, the integration by parts yields equality

$$l'(y)=y^{\alpha-2}\int_0^y u^{1-\alpha}\frac{\partial K(1,u)}{\partial u}du.$$

Since the kernel K is strictly decreasing in both variables, it follows that $l'(y)<0$, $y\in\mathbb{R}_+$, that is, l is strictly decreasing on $\mathbb{R}_+$.

On the other hand, since

$$\int_{\frac{a}{x}}^{\frac{b}{x}}K(1,u)u^{-pA_2}du=\int_0^\infty K(1,u)u^{-pA_2}du-\int_0^{\frac{a}{x}}K(1,u)u^{-pA_2}du$$
$$-\int_0^{\frac{x}{b}}K(u,1)u^{pA_2+s-2}du,$$

and due to the fact that l is strictly decreasing on $\mathbb{R}_+$, we obtain the estimate

$$\int_{\frac{a}{x}}^{\frac{b}{x}} K(1,u)u^{-pA_2}du \le k(pA_2) - \varphi_1(pA_2,x)$$

and similarly,

$$\int_{\frac{y}{b}}^{\frac{y}{a}} K(1,u)u^{qA_1+s-2}du \le k(2-s-qA_1) - \varphi_2(2-s-qA_1,y),$$

so the result follows from Theorem 1.9. Note also that the intervals defining the parameters A_1 and A_2 arise from the assumption on the convergence of integral (1.20). □

Remark 1.5 If the kernel K in the previous theorem is a symmetric function, then $k(2-s-qA_1) = k(qA_1)$. Then, setting $A_1 = A_2 = \frac{1}{pq}$ in Theorem 1.11, provided that $\max\{\frac{1}{p},\frac{1}{q}\} < s$, we get Theorem 1.8 (see also [42]).

Remark 1.6 In order to justify the convergence interval $(1-s,1)$ for the integral $k(\alpha)$ defined by (1.20), observe that the homogeneity of the kernel K implies the following sequence of identities:

$$k(\alpha) = \int_0^\infty K\left(\frac{1}{u},1\right)u^{-s-\alpha}du = \int_0^\infty K(u,1)u^{s+\alpha-2}du.$$

On the other hand, assuming that K is strictly decreasing in each argument, K is strictly positive on $\mathbb{R}_+\times\mathbb{R}_+$. In particular, for $\alpha \ge 1$, monotonicity of K in the second argument and the fact that $K(1,1) > 0$ yield

$$k(\alpha) = \int_0^\infty K(1,u)u^{-\alpha}du \ge \int_0^1 K(1,u)u^{-\alpha}du \ge K(1,1)\int_0^1 u^{-\alpha}du = \infty.$$

Analogous result holds also for $\alpha \le 1-s$, since

$$\begin{aligned} k(\alpha) &= \int_0^\infty K(u,1)u^{s+\alpha-2}du \ge \int_0^1 K(u,1)u^{s+\alpha-2}du \\ &\ge K(1,1)\int_0^1 u^{s+\alpha-2}du = \infty. \end{aligned}$$

Therefore, the interval $(1-s,1)$, considered in definition (1.20), covers all arguments α for which $k(\alpha)$ may converge. The same conclusion on convergence of $k(\alpha)$ can be drawn if we consider a function K increasing in each argument and such that $K(1,1) > 0$.

It is interesting to consider a particular case of the previous theorem, that is, when the integrals are taken over the whole set $\mathbb{R}_+$. Then, $a = 0$, $b = \infty$, and we obtain the corresponding inequalities for an arbitrary non-negative homogeneous function of degree $-s$.

Corollary 1.1 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ be a non-negative homogeneous function of degree $-s$, $s>0$. If A_1 and A_2 are real parameters such that $A_1\in(\frac{1-s}{q},\frac{1}{q})$, $A_2\in(\frac{1-s}{p},\frac{1}{p})$, then for all non-negative measurable functions $f,g:\mathbb{R}_+\to\mathbb{R}$ the inequalities*

$$\begin{aligned}&\int_0^\infty\int_0^\infty K(x,y)f(x)g(y)dxdy\\&\le L\left[\int_0^\infty x^{1-s+p(A_1-A_2)}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{1-s+q(A_2-A_1)}g^q(y)dy\right]^{\frac{1}{q}}\end{aligned}\tag{1.23}$$

and

$$\begin{aligned}&\int_0^\infty y^{(p-1)(s-1)+p(A_1-A_2)}\left[\int_0^\infty K(x,y)f(x)dx\right]^p dy\\&\le L^p\int_0^\infty x^{1-s+p(A_1-A_2)}f^p(x)dx\end{aligned}\tag{1.24}$$

hold and are equivalent, where $L=k^{\frac{1}{p}}(pA_2)k^{\frac{1}{q}}(2-s-qA_1)$.

If $0<p<1$, then the reverse inequalities in (1.23) and (1.24) are valid for every $A_1\in(\frac{1}{q},\frac{1-s}{q})$ and $A_2\in(\frac{1-s}{p},\frac{1}{p})$, as well as the inequality

$$\begin{aligned}&\int_0^\infty x^{(q-1)(s-1)+q(A_2-A_1)}\left[\int_0^\infty K(x,y)g(y)dy\right]^q dx\\&\le L^q\int_0^\infty y^{1-s+q(A_2-A_1)}g^q(y)dy.\end{aligned}\tag{1.25}$$

Inequalities (1.23) and (1.24), as well as their reverse inequalities are equivalent. Moreover, equality in the above relations holds if and only if $f=0$ or $g=0$ a.e. on $\mathbb{R}_+$.

Proof. The proof follows immediately from Theorem 1.11 by substituting $a=0$ and $b=\infty$. Moreover, condition (1.15) gives the nontrivial case of equality in (1.23), while condition (1.16) leads to the divergent integrals. Hence, the observed inequalities are strict, unless $f=0$ or $g=0$ a. e. on $\mathbb{R}_+$. □

In the sequel we consider some generalizations of Theorem 1.11. For example, utilizing the substitution $u=x+\mu$ and $v=y+\mu$, $\mu\ge0$, we have:

Theorem 1.12 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $K:(a+\mu,b+\mu)\times(a+\mu,b+\mu)\to\mathbb{R}$ be a non-negative homogeneous function of degree $-s$, $s>0$, strictly decreasing in both variables. If A_1 and A_2 are real parameters such that $A_1\in(\frac{1-s}{q},\frac{1}{q})$, $A_2\in(\frac{1-s}{p},\frac{1}{p})$, then* *for all non-negative measurable functions $f,g:(a,b)\to\mathbb{R}$ the inequalities*

$$\begin{aligned}&\int_a^b\int_a^b K(x+\mu,y+\mu)f(x)g(y)dxdy\\&\le\left[\int_a^b\big(k(pA_2)-\psi_1(pA_2,x,\mu)\big)(x+\mu)^{1-s+p(A_1-A_2)}f^p(x)dx\right]^{\frac{1}{p}}\\&\quad\times\left[\int_a^b\big(k(2-s-qA_1)-\psi_2(2-s-qA_1,y,\mu)\big)(y+\mu)^{1-s+q(A_2-A_1)}g^q(y)dy\right]^{\frac{1}{q}}\end{aligned}\tag{1.26}$$

and

$$\begin{aligned}&\int_a^b(k(2-s-qA_1)-\psi_2(2-s-qA_1,y,\mu))^{1-p}(y+\mu)^{(p-1)(s-1)+p(A_1-A_2)}\\&\quad\times\left[\int_a^b K(x+\mu,y+\mu)f(x)dx\right]^p dy\\&\le\int_a^b\big(k(pA_2)-\psi_1(pA_2,x,\mu)\big)(x+\mu)^{1-s+p(A_1-A_2)}f^p(x)dx\end{aligned}\tag{1.27}$$

hold and are equivalent, where

$$\psi_1(\alpha,x,\lambda)=\left(\frac{a+\lambda}{x+\lambda}\right)^{1-\alpha}\int_0^1 K(1,u)u^{-\alpha}du+\left(\frac{x+\lambda}{b+\lambda}\right)^{s+\alpha-1}\int_0^1 K(u,1)u^{s+\alpha-2}du,$$

$$\psi_2(\alpha,y,\lambda)=\left(\frac{a+\lambda}{y+\lambda}\right)^{s+\alpha-1}\int_0^1 K(u,1)u^{s+\alpha-2}du+\left(\frac{y+\lambda}{b+\lambda}\right)^{1-\alpha}\int_0^1 K(1,u)u^{-\alpha}du.$$

If $0<p<1$, $b=\infty$, and $K(x,y)$ is strictly decreasing in x and strictly increasing in y, then the reverse inequalities in (1.26) *and* (1.27) *hold for all $A_1\in(\frac{1}{q},\frac{1-s}{q})$ and $A_2\in(\frac{1-s}{p},\frac{1}{p})$, as well as the inequality*

$$\begin{aligned}&\int_a^\infty(k(pA_2)-\psi_1(pA_2,x,\mu))^{1-q}(x+\mu)^{(q-1)(s-1)+q(A_2-A_1)}\\&\quad\times\left[\int_a^\infty K(x+\mu,y+\mu)g(y)dy\right]^q dx\\&\le\int_a^\infty(k(2-s-qA_1)-\psi_2(2-s-qA_1,y,\mu))(y+\mu)^{1-s+q(A_2-A_1)}g^q(y)dy.\end{aligned}\tag{1.28}$$

Moreover, inequalities (1.26) *and* (1.27)*, as well as their reverses, are equivalent.*

Remark 1.7 Considering Theorem 1.12 with a symmetric kernel and parameters $A_1=A_2=\frac{2\lambda}{pq}$, provided that $0<1-\frac{2\lambda}{p}<s$, $0<1-\frac{2\lambda}{q}<s$, we obtain the corresponding result from [42].

Remark 1.8 Some other ways of generalizing Theorem 1.11 arise from various substitutions. For example, in [53] the authors also use the substitution $u = Ax^{\alpha}$ and $v = By^{\beta}$, where $A, B, \alpha, \beta > 0$. Such results are here omitted. It should be noticed here that the results in this section are taken from the above mentioned paper [53]. In addition, for some more specific Hilbert-type inequalities with a homogeneous kernel the reader is referred to [164] and [181].

1.4 Examples. The best possible constants

This section is dedicated to Hilbert-type inequalities with some particular homogeneous kernels and weight functions. Numerous interesting examples will be given here. Moreover, the best possible constant factors will be derived in some particular settings.

1.4.1 Integral case

We start with the classical integral case. We are concerned here with Corollary 1.1 from the previous section. It is not hard to see that this corollary covers Theorems 1.2 and 1.6, presented in the historical overview at the beginning of this chapter.

Namely, if $K(x,y) = (x+y)^{-s}$, $s > 0$, then the integral (1.20) is expressed in terms of the Beta function, that is, $k(\alpha) = B(1-\alpha, s+\alpha-1)$. Hence, in this setting the constant factor L on the right-hand sides of inequalities (1.23) and (1.24) takes the form

$$L = B^{\frac{1}{p}}(1-pA_2, s+pA_2-1)B^{\frac{1}{q}}(1-qA_1, s+qA_1-1).$$

Moreover, if $A_1 = A_2 = \frac{2-s}{pq}$, then the above constant coincides with the constant factor on the right-hand side of inequality (1.6). Hence, Corollary 1.1 can be regarded as an extension of both Theorems 1.2 and 1.6.

Further, if $K(x,y) = \max\{x,y\}^{-s}$, $s > 0$, then the above constant L, included in (1.23) and (1.24), reads

$$L = \frac{s}{(1-pA_2)^{\frac{1}{p}}(1-qA_1)^{\frac{1}{q}}(s+pA_2-1)^{\frac{1}{p}}(s+qA_1-1)^{\frac{1}{q}}}.$$

Similarly, for $A_1 = A_2 = \frac{2-s}{pq}$ the above constant factor reduces to

$$L = \frac{pqs}{(p+s-2)(q+s-2)},$$

and the resulting Hilbert-type inequality coincides with the one from [42].

On the other hand, Hilbert-type inequalities in Theorems 1.2 and 1.6, as well as the above mentioned result from [42], include the best possible constant factor.

Our main task is to determine conditions under which the constant factor $L = k^{\frac{1}{p}}(pA_2)k^{\frac{1}{q}}(2-s-qA_1)$ is the best possible in inequalities (1.23) and (1.24). Observe that inequalities

(1.2) and (1.6) include the best possible constant factors without any exponent. Guided by that fact we are going to simplify the constant factor L from Corollary 1.1. More precisely, if we set the condition

$$pA_2 + qA_1 = 2 - s, \tag{1.29}$$

then the constant factor L in Corollary 1.1 reduces to $L = k(pA_2)$.

In the sequel, we are going to show that, under the above condition (1.29) and assuming some weak conditions on the kernel, inequalities in Corollary 1.1 include the best possible constant factors. In order to prove that result we need the following lemma.

Lemma 1.1 *Let p and q be conjugate parameters with $p > 1$. If $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a non-negative measurable function such that $K(1,t)$ is bounded on $(0,1)$, then*

$$\int_1^\infty x^{-\varepsilon-1}\left(\int_0^{1/x} K(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dt\right)dx = O(1),\ \varepsilon \to 0^+, \tag{1.30}$$

where $A_2 < \frac{1}{p}$.

Proof. Using the assumptions, we have $K(1,t) \le C$ for some $C > 0$ and every $t \in (0,1)$. Let $\varepsilon > 0$ be such that $\varepsilon < pq\left(\frac{1}{p} - A_2\right)$. We have

$$\begin{aligned}\int_1^\infty x^{-1-\varepsilon}\left(\int_0^{1/x} K(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dt\right)dx &\le C\int_1^\infty x^{-1-\varepsilon}\left(\int_0^{1/x} t^{-pA_2-\frac{\varepsilon}{q}}dt\right)dx \\ = \frac{C}{1-pA_2-\frac{\varepsilon}{q}}\int_1^\infty x^{pA_2+\frac{\varepsilon}{q}-\varepsilon-2}dx &= \frac{C}{\left(1-pA_2-\frac{\varepsilon}{q}\right)\left(1-pA_2+\frac{\varepsilon}{p}\right)},\end{aligned}$$

wherefrom (1.30) follows. □

Theorem 1.13 *Suppose that the assumptions of Corollary 1.1 are fulfilled. Additionally, if the kernel $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is such that $K(1,t)$ is bounded on $(0,1)$, and if the parameters A_1 and A_2 fulfill condition (1.29), then the constants $L = k(pA_2)$ and $L^p = k^p(pA_2)$ are the best possible in both inequalities (1.23) and (1.24).*

Proof. For this purpose, with $0 < \varepsilon < pq\left(\frac{1}{p} - A_2\right)$, set $f(x) = x^{-qA_1-\frac{\varepsilon}{p}}\chi_{[1,\infty)}(x)$ and $g(y) = y^{-pA_2-\frac{\varepsilon}{q}}\chi_{[1,\infty)}(y)$, where χ_A is the characteristic function of a set A.
Now, suppose that there exists a smaller constant $0 < M < L$ such that inequality (1.23) holds. Let I denote the right-hand side of (1.23). Then,

$$I = M\left(\int_1^\infty x^{-\varepsilon-1}dx\right)^{\frac{1}{p}}\left(\int_1^\infty y^{-\varepsilon-1}dy\right)^{\frac{1}{q}} = \frac{M}{\varepsilon}. \tag{1.31}$$

Applying respectively the Fubini theorem, substitution $t = \frac{y}{x}$, and Lemma 1.1, we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty K(x,y)f(x)g(y)dxdy \\
&= \int_1^\infty x^{-qA_1-\frac{\varepsilon}{p}} \left(\int_1^\infty K(x,y)y^{-pA_2-\frac{\varepsilon}{q}}dy \right) dx \\
&= \int_1^\infty x^{-\varepsilon-1} \left(\int_0^\infty K(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dt - \int_0^{x^{-1}} K(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dt \right) dx \\
&= \frac{1}{\varepsilon}\left[k\left(pA_2 + \frac{\varepsilon}{q} \right) + o(1) \right]. \qquad (1.32)
\end{aligned}$$

From (1.23), (1.31), and (1.32) we get

$$k\left(pA_2 + \frac{\varepsilon}{q} \right) + o(1) < M. \qquad (1.33)$$

Now, letting $\varepsilon \to 0^+$, relation (1.33) yields a contradiction with the assumption $M < L = k(pA_2)$.

Finally, equivalence of inequalities (1.23) and (1.24) means that the constant $L^p = k^p(pA_2)$ is also the best possible in (1.24). The proof is now completed. □

As we see, the previous theorem covers the problem of finding the best possible constant factors for a quite weak condition on homogeneous kernel and parameters A_1, A_2 satisfying (1.29). We have already considered the kernel $K(x,y) = (x+y)^{-s}$, $s > 0$. This kernel fulfills the above mentioned condition from Theorem 1.13, hence, the best possible constant in that case takes the form $B(1-pA_2, 1-qA_1)$. Hilbert-type inequalities with this kernel and parameters A_1, A_2 were extensively studied in recent papers [10], [11], [52], [53], and [54].

Some other examples of the best possible constant factors arise from various choices of kernels. For example, considering the kernel $K(x,y) = (x+y+\max\{x,y\})^{-s}$, $s > 0$, the best possible constant factor $k(pA_2)$ from Theorem 1.13 becomes

$$\frac{2^{-s}}{pA_2+s-1} F\left(s, s+pA_2-1; s+pA_2; -1/2\right) + \frac{2^{-s}}{1-pA_2} F\left(s, 1-pA_2; 2-pA_2; -1/2\right),$$

where $F(\alpha,\beta;\gamma;z)$ denotes the Gaussian hypergeometric function, that is,

$$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 \frac{t^{\beta-1}(1-t)^{\gamma-\beta-1}}{(1-zt)^{\alpha}}dt, \ \gamma > \beta > 0, z < 1. \qquad (1.34)$$

The above kernel with degree of homogeneity equal to -1 was also discussed in [81].

We conclude this subsection with some particular Hilbert-type inequalities equipped with homogeneous kernels of degree -1, involving the best possible constant factors.

Remark 1.9 Setting $s=1$, $A_1=A_2=\frac{1}{pq}$ in Corollary 1.1, inequalities (1.23) and (1.24) become respectively

$$\int_0^\infty\int_0^\infty K(x,y)f(x)g(y)dxdy \le k(1/q)\left[\int_0^\infty f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty g^q(y)dy\right]^{\frac{1}{q}} \tag{1.35}$$

and

$$\int_0^\infty\left[\int_0^\infty K(x,y)f(x)dx\right]^p dy \le k^p(1/q)\int_0^\infty f^p(x)dx. \tag{1.36}$$

The following kernels K are homogenous with bounded $K(1,t)$ on $(0,1)$. For each of these functions we compute constants $L=k(1/q)$ and $L_2=k(1/2)$, that is, when $p=q=2$:

$$K(x,y)=\frac{1}{x+y+\max\{x,y\}},$$
$$L=\frac{1}{2}qF\left(1,\frac{1}{q};1+\frac{1}{q};-\frac{1}{2}\right)+\frac{1}{2}pF\left(1,\frac{1}{p};1+\frac{1}{p};-\frac{1}{2}\right),$$
$$L_2=\sqrt{2}(\pi-2\arctan\sqrt{2});$$
$$K(x,y)=\frac{1}{x+y+\min\{x,y\}},$$
$$L=qF\left(1,\frac{1}{q};1+\frac{1}{q};-2\right)+pF\left(1,\frac{1}{p};1+\frac{1}{p};-2\right), L_2=2\sqrt{2}\arctan\sqrt{2};$$
$$K(x,y)=\frac{1}{|x-y|+\max\{x,y\}},$$
$$L=\frac{1}{2}qF\left(1,\frac{1}{q};1+\frac{1}{q};\frac{1}{2}\right)+\frac{1}{2}pF\left(1,\frac{1}{p};1+\frac{1}{p};\frac{1}{2}\right), L_2=2\operatorname{artanh}\frac{1}{\sqrt{2}};$$
$$K(x,y)=\frac{1}{x+y-\min\{x,y\}}=\frac{1}{\max\{x,y\}}, L=pq, L_2=4;$$
$$K(x,y)=\frac{1}{x+y+\frac{2}{\frac{1}{x}+\frac{1}{y}}}, L_2=\sqrt{\frac{2}{3}}\pi;$$
$$K(x,y)=\frac{1}{x+y-\frac{2}{\frac{1}{x}+\frac{1}{y}}}, L=\frac{\pi}{2}\left(\frac{1}{\cos\frac{\pi}{2p}}+\frac{1}{\cos\frac{\pi}{2q}}\right), L_2=\pi\sqrt{2};$$
$$K(x,y)=\frac{1}{x+y+\sqrt{xy}}, L_2=\frac{4\pi}{3\sqrt{3}};$$
$$K(x,y)=\frac{1}{x+y-\sqrt{xy}}, L_2=\frac{8\pi}{3\sqrt{3}};$$
$$K(x,y)=\frac{x^{\lambda-1}+y^{\lambda-1}}{x^\lambda+y^\lambda}, L=\frac{\pi}{\lambda}\left(\frac{1}{\sin\frac{\pi}{\lambda p}}+\frac{1}{\sin\frac{\pi}{\lambda q}}\right), \lambda\ge 1;$$
$$K(x,y)=\frac{x^{\lambda-1}-y^{\lambda-1}}{x^\lambda-y^\lambda}, L=\frac{\pi}{\lambda}\left(\cot\frac{\pi}{\lambda p}+\cot\frac{\pi}{\lambda q}\right), \lambda>1;$$

$$K(x,y)=\frac{\log y-\log x}{y-x}, L=\frac{\pi^2}{\left(\sin\frac{\pi}{p}\right)^2}, L_2=\pi^2.$$

Since parameters $s=1, A_1=A_2=\frac{1}{pq}$ fulfill condition $pA_2+qA_1=2-s$, all these constant factors are the best possible in both inequalities (1.35) and (1.36).

1.4.2 Discrete case

Discrete case of the Hilbert inequality is more complicated than the integral one. Namely, in order to obtain discrete forms of the corresponding integral inequalities, it is necessary to do some further estimates, which requires some additional conditions.

In Section 1.1 we encountered the Hilbert double series theorems, those were inequalities (1.1) and (1.4). Moreover, the corresponding equivalent form assigned to (1.1) is inequality (1.3), while the equivalent form assigned to (1.4) was derived in [142].

Recently, M. Krnić and J. Pečarić (see [52]), obtained the following discrete version of the Hilbert inequality with conjugate parameters $p,q>1$ and real parameters $A,B,\alpha,\beta,s>0$,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(Am^{\alpha}+Bn^{\beta})^s}\le M\left[\sum_{m=1}^{\infty}m^{\alpha(1-s)+\alpha p(A_1-A_2)+(p-1)(1-\alpha)}a_m^p\right]^{\frac{1}{p}}$$
$$\times\left[\sum_{n=1}^{\infty}n^{\beta(1-s)+\beta q(A_2-A_1)+(q-1)(1-\beta)}b_n^q\right]^{\frac{1}{q}}, \tag{1.37}$$

where $(a_m)_{m\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ are non-negative real sequences, $A_1\in\left(\max\{\frac{1-s}{q},\frac{\alpha-1}{\alpha q}\},\frac{1}{q}\right)$, $A_2\in\left(\max\{\frac{1-s}{p},\frac{\beta-1}{\beta p}\},\frac{1}{p}\right)$ and

$$M=\alpha^{-\frac{1}{q}}\beta^{-\frac{1}{p}}A^{\frac{2-s}{p}+A_1-A_2-1}B^{\frac{2-s}{q}+A_2-A_1-1}$$
$$\times B^{\frac{1}{p}}(1-pA_2,s-1+pA_2)B^{\frac{1}{q}}(1-qA_1,s-1+qA_1).$$

The equivalent form that corresponds to (1.37) is also derived in [52].

Similarly, considering parameters ϕ, ψ, and λ, such that $0<\phi,\psi\le 1$ and $\phi+\psi=\lambda$, B. Yang [162], obtained the following pair of equivalent inequalities

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\log\left(\frac{m}{n}\right)a_m b_n}{m^{\lambda}-n^{\lambda}}<\left[\frac{\pi}{\lambda\sin\left(\frac{\pi\phi}{\lambda}\right)}\right]^2\left[\sum_{n=1}^{\infty}n^{p(1-\phi)-1}a_n^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\psi)-1}b_n^q\right]^{\frac{1}{q}} \tag{1.38}$$

and

$$\sum_{n=1}^{\infty}n^{p\psi-1}\left[\sum_{m=1}^{\infty}\frac{\log\left(\frac{m}{n}\right)a_m b_n}{m^{\lambda}-n^{\lambda}}\right]^p<\left[\frac{\pi}{\lambda\sin\left(\frac{\pi\phi}{\lambda}\right)}\right]^{2p}\sum_{n=1}^{\infty}n^{p(1-\phi)-1}a_n^p, \tag{1.39}$$

which hold for all non-negative conjugate exponents and non-negative sequences fulfilling $0 < \sum_{n=1}^{\infty} n^{p(1-\phi)-1} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{q(1-\psi)-1} b_n^q < \infty$. Moreover, the constant factors included in the right-hand sides of inequalities are the best possible.

On the other hand, B. Yang and T. M. Rassias [152] (see also [149]), studied the kernel expressed in terms of the logarithm function. They obtained the following pair of equivalent inequalities,

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{a_m b_n}{\log mn} < \frac{\pi}{\sin \pi/p} \left[\sum_{n=2}^{\infty} n^{p-1} a_n^p\right]^{\frac{1}{p}} \left[\sum_{n=2}^{\infty} n^{q-1} b_n^q\right]^{\frac{1}{q}} \tag{1.40}$$

and

$$\sum_{n=2}^{\infty} \frac{1}{n} \left(\frac{a_m}{\log mn}\right)^p < \left[\frac{\pi}{\sin \pi/p}\right]^p \sum_{n=2}^{\infty} n^{p-1} a_n^p, \tag{1.41}$$

which hold for non-negative conjugate exponents and non-negative sequences such that $0 < \sum_{n=2}^{\infty} n^{p-1} a_n^p < \infty$ and $0 < \sum_{n=2}^{\infty} n^{q-1} b_n^q < \infty$. Moreover, the constant factors $\pi/\sin(\pi/p)$ and $[\pi/\sin(\pi/p)]^p$, on the right-hand sides of inequalities (1.40) and (1.41), are the best possible. Observe that the above inequality (1.40) for $p = q = 2$ is also known as the Mulholland inequality.

Clearly, the kernel involved in the previous two inequalities, as well as in (1.37) is non-homogeneous, while the kernel in (1.38) and (1.39) is homogeneous.

However, utilizing suitable substitutions, these non-homogeneous kernels can also be interpreted as the homogeneous ones. Thus, in the sequel we provide discrete forms of Hilbert-type inequalities with a general homogeneous kernel. The same conditions as in the integral case are assumed on the convergence of the integral $k(\alpha)$, defined by (1.20).

The following result contains discrete Hilbert-type inequalities for a homogeneous kernel in both equivalent forms. Discrete weight functions involve here differentiable real functions. In addition, for the reader's convenience, we introduce here the following notation: $H(r)$, $r > 0$, denotes the set of all non-negative differentiable functions $u : \mathbb{R}_+ \to \mathbb{R}$ satisfying the following conditions:

(i) u is strictly increasing on $\mathbb{R}_+$ and there exists $x_0 \in \mathbb{R}_+$ such that $u(x_0) = 1$,

(ii) $\lim_{x\to\infty} u(x) = \infty$, $\frac{u'(x)}{[u(x)]^r}$ is decreasing on $\mathbb{R}_+$.

Theorem 1.14 *Let* $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, *and let* $s > 0$. *Further, suppose that* $A_1 \in \left(\max\{\frac{1-s}{q}, 0\}, \frac{1}{q}\right)$, $A_2 \in \left(\max\{\frac{1-s}{p}, 0\}, \frac{1}{p}\right)$, $u \in H(qA_1)$ *and* $v \in H(pA_2)$. *If* $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ *is a non-negative homogeneous function of degree* $-s$, *strictly decreasing* *in each argument, then the inequalities*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K(u(m),v(n))a_m b_n \leq L\left[\sum_{m=1}^{\infty}[u(m)]^{1-s+p(A_1-A_2)}[u'(m)]^{1-p}a_m^p\right]^{\frac{1}{p}} \times \left[\sum_{n=1}^{\infty}[v(n)]^{1-s+q(A_2-A_1)}[v'(n)]^{1-q}b_n^q\right]^{\frac{1}{q}} \tag{1.42}$$

and

$$\sum_{n=1}^{\infty}[v(n)]^{(s-1)(p-1)+p(A_1-A_2)}v'(n)\left[\sum_{m=1}^{\infty}K(u(m),v(n))a_m\right]^p \leq L^p\sum_{m=1}^{\infty}[u(m)]^{(1-s)+p(A_1-A_2)}[u'(m)]^{1-p}a_m^p \tag{1.43}$$

hold for all non-negative sequences $(a_m)_{m\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, *where*

$$L = k^{\frac{1}{p}}(pA_2)k^{\frac{1}{q}}(2-s-qA_1). \tag{1.44}$$

Moreover, inequalities (1.42) *and* (1.43) *are equivalent.*

Proof. Rewrite inequality (1.12) for the counting measure on $\mathbb{N}$, $(\varphi\circ u)(m) = [u(m)]^{A_1}[u'(m)]^{-\frac{1}{q}}$, $(\psi\circ v)(n) = [v(n)]^{A_2}[v'(n)]^{-\frac{1}{p}}$, and the sequences $(a_m)_{m\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$. Clearly, these substitutions are well-defined, since u and v are injective functions. Thus, in this setting we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K(u(m),v(n))a_m b_n \leq \left[\sum_{m=1}^{\infty}[u(m)]^{pA_1}[u'(m)]^{1-p}(F\circ u)(m)a_m^p\right]^{\frac{1}{p}} \times \left[\sum_{n=1}^{\infty}[v(n)]^{qA_2}[v'(n)]^{1-q}(G\circ v)(n)b_n^q\right]^{\frac{1}{q}}, \tag{1.45}$$

where

$$(F\circ u)(m) = \sum_{n=1}^{\infty}\frac{K(u(m),v(n))v'(n)}{[v(n)]^{pA_2}}$$

and

$$(G\circ v)(n) = \sum_{m=1}^{\infty}\frac{K(u(m),v(n))u'(m)}{[u(m)]^{qA_1}}.$$

Now, since the kernel K is strictly decreasing in each argument and $u \in H(qA_1)$, $v \in H(pA_2)$, it follows that $F \circ u$ and $G \circ v$ are strictly decreasing. Hence, we have

$$(F \circ u)(m) < \int_0^\infty \frac{K(u(m), v(y))}{[v(y)]^{pA_2}} v'(y)dy, \tag{1.46}$$

since the left-hand side of this inequality is obviously the lower Darboux sum for the integral on the right-hand side of inequality. Further, utilizing substitution $v(y) = tu(m)$ and homogeneity of the kernel K, we have

$$\int_0^\infty \frac{K(u(m), v(y))}{[v(y)]^{pA_2}} v'(y)dy = [u(m)]^{1-s-pA_2} \int_0^\infty K(1,t)t^{-pA_2}dt,$$

so by virtue of (1.20) and (1.46) we get

$$(F \circ u)(m) < [u(m)]^{1-s-pA_2} k(pA_2). \tag{1.47}$$

By the similar arguments as for the function $F \circ u$, we obtain

$$\begin{aligned}
(G \circ v)(m) &< \int_0^\infty \frac{K(u(x), v(n))}{[u(x)]^{qA_1}} u'(x)dx \\
&= [v(n)]^{1-s-qA_1} \int_0^\infty K(t,1)t^{-qA_1}dt \\
&= [v(n)]^{1-s-qA_1} \int_0^\infty K(1,t)t^{-2+s+qA_1}dt \\
&= [v(n)]^{1-s-qA_1} k(2-s-qA_1).
\end{aligned} \tag{1.48}$$

Finally, relations (1.45), (1.47), and (1.48) imply inequality (1.42).

On the other hand, if we rewrite inequality (1.13) with the counting measure on $\mathbb{N}$ and the same functions as in the proof of inequality (1.42), after using estimates (1.47) and (1.48), we also obtain (1.43). □

Clearly, Theorem 1.14 covers discrete Hilbert and Hardy-Hilbert-type inequalities with homogeneous kernels, decreasing in both arguments.

Remark 1.10 Suppose $(a_m)_{m\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are non-negative real sequences, not identically equal to trivial zero sequence. Then, according to estimates (1.47) and (1.48), it follows that inequalities (1.42) and (1.43) are sharp. In other words, equalities in (1.42) and (1.43) hold if and only if $a_m \equiv 0$ or $b_n \equiv 0$.

Remark 1.11 If the homogeneous kernel $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a symmetric function, that is, $K(x,y) = K(y,x)$, for all $x, y \in \mathbb{R}_+$, then the constant L, defined by (1.44), simplifies to $L = k^{\frac{1}{p}}(pA_2)k^{\frac{1}{q}}(qA_1)$.

As emphasized above, inequalities (1.42) and (1.43) are sharp if the sequences $(a_m)_{m\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are not identically equal to the zero sequence. Therefore, it is interesting to consider the problem of finding the best possible constant factors for inequalities (1.42) and (1.43).

The main idea in obtaining the best possible constant factor is a reduction of constant defined by (1.44) to the form without exponents, which was already considered in the integral case. Thus, the parameters A_1 and A_2 fulfill (1.29), that is, $pA_2 + qA_1 = 2 - s$, which implies that $k(pA_2) = k(2 - s - qA_1)$. In such a way, the constant factor L from Theorem 1.14 becomes

$$L^* = k(pA_2). \tag{1.49}$$

Moreover, under assumption (1.29), inequalities (1.42) and (1.43) respectively read

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K(u(m),v(n))a_m b_n \le L^* \left[\sum_{m=1}^{\infty} [u(m)]^{-1+pqA_1}[u'(m)]^{1-p}a_m^p\right]^{\frac{1}{p}} \times \left[\sum_{n=1}^{\infty} [v(n)]^{-1+pqA_2}[v'(n)]^{1-q}b_n^q\right]^{\frac{1}{q}} \tag{1.50}$$

and

$$\sum_{n=1}^{\infty} [v(n)]^{(p-1)(1-pqA_2)}v'(n)\left[\sum_{m=1}^{\infty} K(u(m),v(n))a_m\right]^p \le (L^*)^p \sum_{m=1}^{\infty} [u(m)]^{-1+pqA_1}[u'(m)]^{1-p}a_m^p. \tag{1.51}$$

The following theorem shows that the constants on the right-hand sides of inequalities (1.50) and (1.51) are the best possible.

Theorem 1.15 *Suppose that parameters p, q, s, A_1, A_2, and the functions $u,v:\mathbb{R}_+ \to \mathbb{R}$, $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ are defined as in the statement of Theorem* 1.14. *If parameters A_1 and A_2 fulfill condition $pA_2 + qA_1 = 2 - s$, then the constant factors L^* and $(L^*)^p$ are the best possible in inequalities* (1.50) *and* (1.51).

Proof. It is enough to show that L^* is the best possible constant factor in inequality (1.50), since (1.50) and (1.51) are equivalent. For this purpose, we consider sequences $\widetilde{a}_m = [u(m)]^{-qA_1-\frac{\varepsilon}{p}}u'(m)$ and $\widetilde{b}_n = [v(n)]^{-pA_2-\frac{\varepsilon}{q}}v'(n)$, where $\varepsilon > 0$ is sufficiently small number. Since $u \in H(qA_1)$ we may assume that u is strictly increasing in $\mathbb{R}_+$, and there exists $x_0 \in \mathbb{R}_+$ such that $u(x_0) = 1$. Therefore, considering integral sums, we have

$$\begin{aligned}
\frac{1}{\varepsilon} &= \int_1^{\infty} [u(x)]^{-1-\varepsilon}d[u(x)] < \sum_{m=1}^{\infty} [u(m)]^{-1-\varepsilon}u'(m) \\
&= \sum_{m=1}^{\infty} [u(m)]^{-1+pqA_1}[u'(m)]^{1-p}\widetilde{a}_m^p \\
&< \zeta(1) + \int_1^{\infty} [u(x)]^{-1-\varepsilon}d[u(x)] = \zeta(1) + \frac{1}{\varepsilon},
\end{aligned}$$

where the function ζ is defined by $\zeta(x) = [u(x)]^{-1-\varepsilon}u'(x)$. Hence, we have

$$\sum_{m=1}^{\infty}[u(m)]^{-1+pqA_1}[u'(m)]^{1-p}\widetilde{a}_m^p = \frac{1}{\varepsilon} + O(1), \tag{1.52}$$

and similarly,

$$\sum_{n=1}^{\infty}[v(n)]^{-1+pqA_2}[v'(n)]^{1-q}\widetilde{b}_n^q = \frac{1}{\varepsilon} + O(1). \tag{1.53}$$

Now, suppose that there exists a positive constant M, smaller than L^*, such that (1.50) holds after replacing L^* with M. Then, combining relations (1.52) and (1.53) with inequality (1.50), we get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}K(u(m),v(n))\widetilde{a}_m\widetilde{b}_n < \frac{1}{\varepsilon}(M+o(1)). \tag{1.54}$$

On the other hand, we can also estimate the left-hand side of inequality (1.50). Namely, inserting the above defined sequences $(\widetilde{a}_m)_{m\in\mathbb{N}}$ and $(\widetilde{b}_n)_{n\in\mathbb{N}}$ in the left-hand side of (1.50), we easily obtain

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}K(u(m),v(n))\widetilde{a}_m\widetilde{b}_n \\
&\quad > \int_1^{\infty}[u(x)]^{-qA_1-\frac{\varepsilon}{p}}\left[\int_1^{\infty}K(u(x),v(y))[v(y)]^{-pA_2-\frac{\varepsilon}{q}}d(v(y))\right]d(u(x)) \\
&\quad = \int_1^{\infty}[u(x)]^{-1-\varepsilon}\left[\int_{\frac{1}{u(x)}}^{\infty}K(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dt\right]d(u(x)).
\end{aligned} \tag{1.55}$$

Moreover, since the kernel K is strictly decreasing in both arguments, it follows that $K(1,0) > K(1,t)$, for $t > 0$, so we have

$$\begin{aligned}
\int_{\frac{1}{u(x)}}^{\infty}K(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dt &> \int_0^{\infty}K(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dt - K(1,0)\int_0^{\frac{1}{u(x)}}t^{-pA_2-\frac{\varepsilon}{q}}dt \\
&= k\left(pA_2+\frac{\varepsilon}{q}\right) - \frac{K(1,0)}{1-pA_2-\frac{\varepsilon}{q}}[u(x)]^{-1+pA_2+\frac{\varepsilon}{q}},
\end{aligned}$$

and consequently

$$\begin{aligned}
&\int_1^{\infty}[u(x)]^{-1-\varepsilon}\left[\int_{\frac{1}{u(x)}}^{\infty}K(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dt\right]d(u(x)) \\
&\quad \geq \frac{1}{\varepsilon}k\left(pA_2+\frac{\varepsilon}{q}\right) - \frac{K(1,0)}{\left(1-pA_2-\frac{\varepsilon}{q}\right)\left(1-pA_2+\frac{\varepsilon}{p}\right)}.
\end{aligned} \tag{1.56}$$

In other words, relations (1.55) and (1.56) yield the lower bound for the left-hand side of inequality (1.50):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}K(u(m),v(n))\widetilde{a}_m\widetilde{b}_n > \frac{1}{\varepsilon}(L^*+o(1)). \tag{1.57}$$

Finally, comparing relations (1.54) and (1.57), and letting $\varepsilon \to 0^+$, it follows that $L^* \leq M$, which contradicts the assumption that the constant M is smaller than L^*. This means that L^* is the best possible constant in inequality (1.50). □

We conclude this discussion with a few remarks which connect Theorems 1.14 and 1.15 with particular results presented at the beginning of this subsection.

Remark 1.12 Observe that Theorem 1.14 is a generalization of inequality (1.37) (see also [52]). Moreover, Theorem 1.15 yields the form of inequality (1.37) with the best possible constant factor. Namely, putting the kernel $K(x,y) = (x+y)^{-s}$, $s > 0$, and power functions $u(x) = Ax^{\alpha}$ and $v(y) = By^{\beta}$, A, B, α, $\beta > 0$, in (1.50), we obtain the corresponding form of inequality (1.37), with the best possible constant factor

$$\alpha^{-\frac{1}{q}}\beta^{-\frac{1}{p}}A^{-1+qA_1}B^{-1+pA_2}B(1-pA_2, 1-qA_1).$$

Remark 1.13 If $s = 1$, then parameters $A_1 = A_2 = \frac{1}{pq}$ fulfill condition (1.29). Thus, putting these parameters in (1.50) and (1.51), together with kernel $K(x,y) = (x+y)^{-1}$ and differentiable functions $u(x) = v(x) = \log(x+1)$, we obtain inequalities (1.40) and (1.41) with the best possible constants (see also [152]).

Remark 1.14 Since parameters $A_1 = A_2 = \frac{2-s}{pq}$, where $2 - \min\{p,q\} < s < 2$, fulfill condition (1.29), they can be substituted in inequalities (1.50) and (1.51). In addition, considering the kernel $K(x,y) = (x+y)^{-s}$, $s > 0$, and differentiable functions $u(x) = v(x) = x + \nu$, $0 \leq \nu < 1$, inequalities (1.50) and (1.51) reduce to

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n+2\nu)^s} \leq S_1\left[\sum_{m=1}^{\infty}(m+\nu)^{1-s}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}(n+\nu)^{1-s}b_n^q\right]^{\frac{1}{q}} \tag{1.58}$$

and

$$\sum_{n=1}^{\infty}(n+\nu)^{(p-1)(s-1)}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m+n+2\nu)^s}\right]^p \leq S_1^p\sum_{m=1}^{\infty}(m+\nu)^{1-s}a_m^p, \tag{1.59}$$

where the constant factors $S_1 = B(\frac{1}{p}+\frac{s-1}{q}, \frac{1}{q}+\frac{s-1}{p})$ and S_1^p are the best possible. If $s = 1$, then S_1 becomes $\pi/\sin(\pi/p)$. Thus, setting $\nu = \frac{1}{2}$ and $s = 1$ in (1.58) and (1.59), we obtain the sharper version of the Hilbert double series theorem, as well as its equivalent form (see also [142]).

Remark 1.15 Some particular discrete Hilbert-type inequalities regarding homogeneous kernels are also obtained in [54]. They can be derived as the consequences of Theorems 1.14 and 1.15. For example, setting $K(x,y) = (x+y)^{-s}$, $s > 0$, $u(x) = xa^x$, $v(y) = ya^y$, $a > 1$, inequalities (1.50) and (1.51) respectively read

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(ma^m+na^n)^s} &\leq L^*\left[\sum_{m=1}^{\infty}(ma^m)^{-1+pqA_1}(a^m+ma^m\log a)^{1-p}a_m^p\right]^{\frac{1}{p}}\\ &\quad\times\left[\sum_{n=1}^{\infty}(na^n)^{-1+pqA_2}(a^n+na^n\log a)^{1-p}b_n^q\right]^{\frac{1}{q}}\end{aligned}$$

and

$$\sum_{n=1}^{\infty}(na^n)^{(p-1)(1-pqA_2)}(a^n+na^n\log a)\left[\sum_{m=1}^{\infty}\frac{a_m}{(ma^m+na^n)^s}\right]^p$$
$$\leq (L^*)^p\sum_{m=1}^{\infty}(ma^m)^{-1+pqA_1}(a^m+ma^m\log a)^{1-p}a_m^p,$$

where $L^*=B(1-pA_2,1-qA_1)$.

Similarly, if $K(x,y)=(x+y)^{-s}$, $s>0$, $u(x)=x\arctan x$, $v(y)=y\arctan y$, then inequalities (1.50) and (1.51) become

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_mb_n}{(m\arctan m+n\arctan n)^s}\leq L^*\left[\sum_{m=1}^{\infty}\omega_p(m)a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}\omega_q(n)b_n^q\right]^{\frac{1}{q}}$$

and

$$\sum_{n=1}^{\infty}(n\arctan n)^{(p-1)(1-pqA_2)}\left(\arctan n+\frac{n}{1+n^2}\right)$$
$$\times\left[\sum_{m=1}^{\infty}\frac{a_m}{(m\arctan m+n\arctan n)^s}\right]^p\leq (L^*)^p\sum_{m=1}^{\infty}\omega_p(m){a_m}^p,$$

where $L^*=B(1-pA_2,1-qA_1)$ and

$$\omega_r(x)=(x\arctan x)^{1-s-r(A_2-A_1)}\left(\arctan x+\frac{x}{1+x^2}\right)^{1-r},\ r\in\{p,q\}.$$

Of course, the above inequalities include the best possible constants. For some other examples arising from various choices of weight functions, the reader is referred to [54].

1.4.3 Some further estimates

In this subsection we study a few particular Hilbert-type inequalities involving the homogeneous kernel $K(x,y)=(x+y)^{-s}$, $s>0$. In addition to the Hilbert inequality, the following results will be derived with a help of some additional estimates that arise from this particular setting. More precisely, we shall use Theorems 1.9 and 1.10, as well as various methods for estimating the integral of type

$$\int_{\frac{a}{x}}^{\frac{b}{x}}K(1,u)u^{-\alpha}du.$$

The first in the series of results is the following pair of inequalities.

Corollary 1.2 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $s>0$. If $(a,b)\subseteq\mathbb{R}_+$, then the inequalities*

$$\begin{aligned}
&\int_a^b\int_a^b \frac{f(x)g(y)}{(x+y)^s}dxdy\\
&\le B\left(\frac{s}{2},\frac{s}{2}\right)\left\{\int_a^b\left[1-\frac{1}{2}\left(\frac{a}{x}\right)^{\frac{s}{2}}-\frac{1}{2}\left(\frac{x}{b}\right)^{\frac{s}{2}}\right]x^{-\frac{sp}{2}+p-1}f^p(x)dx\right\}^{\frac{1}{p}}\\
&\qquad\times\left\{\int_a^b\left[1-\frac{1}{2}\left(\frac{a}{y}\right)^{\frac{s}{2}}-\frac{1}{2}\left(\frac{y}{b}\right)^{\frac{s}{2}}\right]y^{-\frac{sq}{2}+q-1}g^q(y)dy\right\}^{\frac{1}{q}}
\end{aligned}\tag{1.60}$$

and

$$\begin{aligned}
&\int_a^b\left[1-\frac{1}{2}\left(\frac{a}{y}\right)^{\frac{s}{2}}-\frac{1}{2}\left(\frac{y}{b}\right)^{\frac{s}{2}}\right]^{1-p}y^{\frac{sp}{2}-1}\left[\int_a^b\frac{f(x)}{(x+y)^s}dx\right]^p dy\\
&\le B^p\left(\frac{s}{2},\frac{s}{2}\right)\int_a^b\left[1-\frac{1}{2}\left(\frac{a}{x}\right)^{\frac{s}{2}}-\frac{1}{2}\left(\frac{x}{b}\right)^{\frac{s}{2}}\right]x^{-\frac{sp}{2}+p-1}f^p(x)dx
\end{aligned}\tag{1.61}$$

hold for all non-negative measurable functions $f,g:(a,b)\to\mathbb{R}$. In addition, inequalities (1.60) and (1.61) are equivalent.

Proof. Considering Theorem 1.9 with the kernel $K(x,y)=(x+y)^{-s}$ and weight functions $\varphi(x)=x^{\frac{2-s}{2q}}$, $\psi(y)=y^{\frac{2-s}{2p}}$, we have

$$\begin{aligned}
&\int_a^b\int_a^b\frac{f(x)g(y)}{(x+y)^s}dxdy\\
&\le\left[\int_a^b x^{-\frac{sp}{2}+p-1}\left(\int_{\frac{a}{x}}^{\frac{b}{x}}\frac{u^{\frac{s}{2}-1}}{(1+u)^s}du\right)f^p(x)dx\right]^{\frac{1}{p}}\\
&\quad\times\left[\int_a^b y^{-\frac{sq}{2}+q-1}\left(\int_{\frac{y}{b}}^{\frac{y}{a}}\frac{u^{\frac{s}{2}-1}}{(1+u)^s}du\right)g^q(y)dy\right]^{\frac{1}{q}}.
\end{aligned}$$

Now, we are going to estimate integrals in the above inequality, dependent on variables x and y. Taking into account an obvious relation

$$B\left(\frac{s}{2},\frac{s}{2}\right)=2\int_1^\infty\frac{u^{\frac{s}{2}-1}}{(1+u)^s}du=2\alpha^{\frac{s}{2}}\int_\alpha^\infty\frac{u^{\frac{s}{2}-1}}{(\alpha+u)^s}du,$$

and inequality

$$\int_\alpha^\infty\frac{u^{\frac{s}{2}-1}}{(\alpha+u)^s}du<\int_\alpha^\infty\frac{u^{\frac{s}{2}-1}}{(1+u)^s}du,$$

where $\alpha>1$, we have

$$\int_\alpha^\infty\frac{u^{\frac{s}{2}-1}}{(1+u)^s}du>\frac{1}{2}\alpha^{-\frac{s}{2}}B\left(\frac{s}{2},\frac{s}{2}\right),\qquad\alpha>1.\tag{1.62}$$

Finally, considering the relation

$$\int_{\frac{a}{x}}^{\frac{b}{x}} \frac{u^{\frac{s}{2}-1}}{(1+u)^s}du = B\left(\frac{s}{2},\frac{s}{2}\right) - \int_{\frac{b}{x}}^{\infty} \frac{u^{\frac{s}{2}-1}}{(1+u)^s}du - \int_{\frac{x}{a}}^{\infty} \frac{u^{\frac{s}{2}-1}}{(1+u)^s}du,$$

and (1.62), we obtain the estimate

$$\int_{\frac{a}{x}}^{\frac{b}{x}} \frac{u^{\frac{s}{2}-1}}{(1+u)^s}du < B\left(\frac{s}{2},\frac{s}{2}\right)\left[1-\frac{1}{2}\left(\frac{a}{x}\right)^{\frac{s}{2}}-\frac{1}{2}\left(\frac{x}{b}\right)^{\frac{s}{2}}\right],$$

which yields inequality (1.60). Equivalent form (1.61) follows in a similar way. □

Remark 1.16 Combining the well-known arithmetic-geometric mean inequality

$$\frac{1}{2}\left(\frac{a}{x}\right)^{\frac{s}{2}}+\frac{1}{2}\left(\frac{x}{b}\right)^{\frac{s}{2}} \geq \left(\frac{a}{b}\right)^{\frac{s}{4}},$$

with inequalities (1.60) and (1.61), we have

$$\begin{aligned}&\int_a^b\int_a^b \frac{f(x)g(y)}{(x+y)^s}dxdy\\ &\leq B\left(\frac{s}{2},\frac{s}{2}\right)\left[1-\left(\frac{a}{b}\right)^{\frac{s}{4}}\right]\left[\int_a^b x^{-\frac{sp}{2}+p-1}f(x)^p dx\right]^{\frac{1}{p}}\left[\int_a^b y^{-\frac{sq}{2}+q-1}g(y)^q dy\right]^{\frac{1}{q}}\end{aligned}$$

and

$$\int_a^b y^{\frac{sp}{2}-1}\left[\int_a^b \frac{f(x)}{(x+y)^s}dx\right]^p dy \leq \left[B\left(\frac{s}{2},\frac{s}{2}\right)\left(1-\left(\frac{a}{b}\right)^{\frac{s}{4}}\right)\right]^p \int_a^b x^{-\frac{sp}{2}+p-1}f(x)^p dx.$$

Putting $p=q=2$ in these inequalities, we obtain a pair of inequalities derived in [152]. Moreover, if $a=0$ and $b=\infty$, the above inequalities reduce to corresponding relations obtained in [11].

We finish this section with another specific Hilbert-type inequality referring to kernel $K(x,y)=(x+y)^{-s}$, $s>0$.

Corollary 1.3 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, $s>0$, and let A_1 and A_2 be real parameters such that $A_1 \in (\frac{1-s}{q},\frac{1}{q})$ and $A_2 \in (\frac{1-s}{p},\frac{1}{p})$. If $Q=k_{l_1}^{\frac{1}{p}}(pA_2)k_{l_2}^{\frac{1}{q}}(qA_1)$, $l_1=\frac{1-pA_2}{s}$, $l_2=\frac{1-qA_1}{s}$, and*

$$k_l(\alpha)=\int_{\frac{a^l-b^l}{b(b^{l-1}-a^{l-1})}}^{\frac{a^l-b^l}{a(b^{l-1}-a^{l-1})}} \frac{u^{-\alpha}}{(1+u)^s}du,$$

then the inequalities

$$\begin{aligned}&\int_a^b\int_a^b \frac{f(x)g(y)}{(x+y)^s}dxdy\\ &\leq Q\left[\int_a^b x^{1-s+p(A_1-A_2)}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_a^b y^{1-s+q(A_2-A_1)}g^q(y)dy\right]^{\frac{1}{q}} \qquad (1.63)\end{aligned}$$

and

$$\int_a^b y^{(p-1)(s-1)+p(A_1-A_2)}\left[\int_a^b \frac{f(x)}{(x+y)^s}dx\right]^p dy \le Q^p \int_a^b x^{1-s+p(A_1-A_2)} f^p(x)dx \tag{1.64}$$

hold for all non-negative measurable functions $f,g:(a,b)\to\mathbb{R}$ *and are equivalent.*

Proof. We start as in the proof of Corollary 1.2, but for the estimate of the integral

$$\int_{\frac{a}{x}}^{\frac{b}{x}} \frac{u^{-\alpha}}{(1+u)^s}du,$$

we use the fact that the function $f(x)=\int_{a/x}^{b/x}u^{-\alpha}(1+u)^{-s}du$, $x\in\mathbb{R}_+$, attains its maximum value at $x=\frac{ab^l-ba^l}{a^l-b^l}$, $l=\frac{1-\alpha}{s}$. □

Remark 1.17 Setting $A_1=A_2=\frac{2-s}{pq}$, provided that $s>2-\min\{p,q\}$, inequalities (1.63) and (1.64) reduce respectively to

$$\int_a^b\int_a^b \frac{f(x)g(y)}{(x+y)^s}dxdy \le Q_1\left[\int_a^b x^{1-s}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_a^b y^{1-s}g^q(y)dy\right]^{\frac{1}{q}}$$

and

$$\int_a^b y^{(s-1)(p-1)}\left[\int_a^b \frac{f(x)}{(x+y)^s}dx\right]^p dy \le {Q_1}^p\int_a^b x^{1-s}f^p(x)dx,$$

where

$$Q_1 = k^{\frac{1}{p}}_{\frac{q+s-2}{qs}}\left(\frac{2-s}{q}\right)k^{\frac{1}{q}}_{\frac{p+s-2}{ps}}\left(\frac{2-s}{p}\right).$$

Similarly, if $A_1=\frac{2-s}{2q}$ and $A_2=\frac{2-s}{2p}$, inequalities (1.63) and (1.64) respectively read

$$\int_a^b\int_a^b \frac{f(x)g(y)}{(x+y)^s}dxdy$$
$$\le k_{\frac{1}{2}}\left(\frac{2-s}{2}\right)\left[\int_a^b x^{-\frac{sp}{2}+p-1}f(x)^pdx\right]^{\frac{1}{p}}\left[\int_a^b y^{-\frac{sq}{2}+q-1}g(y)^qdy\right]^{\frac{1}{q}}$$

and

$$\int_a^b y^{\frac{sp}{2}-1}\left[\int_a^b \frac{f(x)}{(x+y)^s}dx\right]^p dy \le k^p_{\frac{1}{2}}\left(\frac{2-s}{2}\right)\int_a^b x^{-\frac{sp}{2}+p-1}f(x)^pdx.$$

Remark 1.18 General results from this section, covering the best possible constant factors for Hilbert-type inequalities with a homogeneous kernel in integral and discrete case are established in papers [63] and [111]. Particular inequalities in subsection 1.4.3 are taken from paper [53]. For the similar problem area, the reader is referred to the following references: [13], [36], [38], [80], [85], [101], [122], [126], [132], [135], [146], [149], [155], [162], [164], [176], [181], [182], and [183].

1.5 Refined Hilbert-type inequalities via the refined Hölder inequality

In this section we provide an interesting improvement of the general Hilbert-type inequality based on the improvement of the Hölder inequality, obtained by H. Leping et. al. in [75].

For the reader's convenience, we first introduce some definitions. Let f and g be elements of the inner product space of measurable functions, where the inner product is denoted by $\langle f,g\rangle$. Further, let $S_r(f,u)$ be defined as

$$S_r(f,u)=\langle f^{\frac{r}{2}},u\rangle ||f||_r^{-\frac{r}{2}},$$

where u is the unit vector and $||f||_r=\sqrt[r]{\langle f^{\frac{r}{2}},f^{\frac{r}{2}}\rangle}$. Clearly, $S_r(f,u)=0$ when the vector u selected is orthogonal to $f^{\frac{r}{2}}$.

By virtue of the positive definiteness of the Gramm matrix, G. Mingzhe and L. Debnath [98], derived an important inequality

$$\langle f,g\rangle^2 \leq ||f||^2||g||^2-(||f||x-||g||y)^2=||f||^2||g||^2\left[1-r(h)\right], \tag{1.65}$$

where $r(h)=\left(\frac{y}{||f||}-\frac{x}{||g||}\right)^2$, $x=\langle g,h\rangle$, $y=\langle f,h\rangle$, $||h||=1$, and $xy\geq 0$. Here, $||\cdot||$ denotes the usual 2-norm in L^2 space. In addition, equality in (1.65) holds if and only if vectors f and g are linearly dependent or vector h is a linear combination of f and g, provided $xy=0$, $x\neq y$. It should be noticed here that inequality (1.65) is a consequence of an earlier result of Mitrović (see paper [104]).

Now, regarding the previous definitions, the above mentioned refinement of the Hölder inequality from [75] asserts that

$$\langle f,g\rangle \leq ||f||_p||g||_q[1-R(h)]^m, \tag{1.66}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, $R(h)=[S_p(f,h)-S_q(g,h)]^2\neq 0$, $||h||=1$, $m=\min\{\frac{1}{p},\frac{1}{q}\}$, provided $f^{\frac{p}{2}}$, $g^{\frac{q}{2}}$, h are linearly independent.

In order to explain the idea from paper [75], we derive here improvement of the reverse Hölder inequality. Moreover, the integrals will be taken with $\sigma-$finite measures, as in Section 1.2, and the corresponding inner product will be defined as
$\langle f,g\rangle=\int_\Omega K(x)f(x)g(x)d\mu(x)$.

Lemma 1.2 *Let* $\frac{1}{p}+\frac{1}{q}=1$, $0<p<1$, *and let* $K:\Omega\to\mathbb{R}$, $f:\Omega\to\mathbb{R}$, $g:\Omega\to\mathbb{R}$, $h:\Omega\to\mathbb{R}$ *be non-negative measurable functions such that* $f^{\frac{p}{2}}$, $g^{\frac{q}{2}}$ *and* h *are linearly independent. Then,*

$$\langle f,g\rangle \geq ||f||_p||g||_q\left[1-R(h)\right]^{\frac{1}{q}}, \tag{1.67}$$

where $R(h)=[S_p(f,h)-S_q(g,h)]^2\neq 0$ *and* $||h||=1$.

Proof. The inner product $\langle f,g\rangle$ can be rewritten in the following form:

$$\langle f,g\rangle = \int_\Omega K(x)\left(f^{\frac{p}{q}}(x)g(x)\right)f^{1-\frac{p}{q}}(x)d\mu(x).$$

Now, since parameters $A=\frac{q}{2}$ and $B=\frac{q}{q-2}$ are conjugate, that is, $\frac{1}{A}+\frac{1}{B}=1$, applying the reverse Hölder inequality to the above expression, we have

$$\begin{aligned}\langle f,g\rangle &\geq \left[\int_\Omega K(x)\left(f^{\frac{p}{q}}(x)g(x)\right)^A d\mu(x)\right]^{\frac{1}{A}}\left[\int_\Omega K(x)\left(f^{1-\frac{p}{q}}(x)\right)^B d\mu(x)\right]^{\frac{1}{B}}\\ &= \langle f^{\frac{p}{2}},g^{\frac{q}{2}}\rangle ||f||_p{}^{p(1-\frac{2}{q})}. \end{aligned} \tag{1.68}$$

On the other hand, replacing f and g with $f^{\frac{p}{2}}$ and $g^{\frac{q}{2}}$ in (1.65), we obtain

$$\langle f^{\frac{p}{2}},g^{\frac{q}{2}}\rangle^2 \leq ||f||_p{}^p||g||_q{}^q\left[1-R(h)\right], \tag{1.69}$$

that is, (1.67). □

In the sequel we provide an extension of Theorem 1.9 via the above described improvement of the Hölder inequality. In the following two theorems, the exponent m indicates $m=\min\{\frac{1}{p},\frac{1}{q}\}$, where p and q are conjugate exponents. Moreover, regarding the above definitions, we denote $R(\overline{f},\overline{g},\overline{h})=(S_p(\overline{f},\overline{h})-S_q(\overline{g},\overline{h}))^2$, where $S_p(\overline{f},\overline{h})=\langle\overline{f}^{\frac{p}{2}},\overline{h}\rangle||\overline{f}||_p{}^{-\frac{p}{2}}$. Obviously, $S_p(\overline{f},\overline{h})$ depends on the inner product. In order to obtain improved Hilbert-type inequalities, the inner product will be defined as

$$\langle\overline{f},\overline{g}\rangle=\int_\Omega\int_\Omega K(x,y)\overline{f}(x,y)\overline{g}(x,y)d\mu_1(x)d\mu_2(y). \tag{1.70}$$

Theorem 1.16 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let Ω be a measure space with positive σ-finite measures μ_1 and μ_2. Let $K:\Omega\times\Omega\to\mathbb{R}$ and $\varphi,\psi:\Omega\to\mathbb{R}$ be non-negative measurable functions. If the functions F and G are defined by $F(x)=\int_\Omega K(x,y)\psi^{-p}(y)d\mu_2(y)$ and $G(y)=\int_\Omega K(x,y)\varphi^{-q}(x)d\mu_1(x)$, then for all non-negative measurable functions f and g on Ω the inequality*

$$\begin{aligned}&\int_\Omega\int_\Omega K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y)\\ &\leq(1-R(\overline{f},\overline{g},\overline{h}))^m\left[\int_\Omega\varphi^p(x)F(x)f^p(x)d\mu_1(x)\right]^{\frac{1}{p}}\\ &\quad\times\left[\int_\Omega\psi^q(y)G(y)g^q(y)d\mu_2(y)\right]^{\frac{1}{q}}\end{aligned} \tag{1.71}$$

holds, where $\overline{f}(x,y)=f(x)\frac{\varphi(x)}{\psi(y)}$, $\overline{g}(x,y)=g(y)\frac{\psi(y)}{\varphi(x)}$, and $\overline{h}(x,y)$ is such that

$$\int_\Omega\int_\Omega K(x,y)\overline{h}^2(x,y)d\mu_1(x)d\mu_2(y)=1.$$

If $0<p<1$, then the reverse inequality in (1.71) holds.

Proof. Inequality (1.71) is an immediate consequence of the relation

$$\int_\Omega \int_\Omega K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y) = \int_\Omega \int_\Omega K(x,y)\overline{f}(x,y)\overline{g}(x,y)d\mu_1(x)d\mu_2(y)$$

and inequality (1.66). On the other hand, the reverse inequality is a consequence of Lemma 1.2 and the above relation. □

In addition, replacing g in (1.71) with the function

$$\widetilde{g}(y) = G^{1-p}(y)\psi^{-p}(y)\left(\int_\Omega K(x,y)f(x)d\mu_1(x)\right)^{p-1},$$

we also obtain an improvement of the Hardy-Hilbert-type inequality (1.13).

Theorem 1.17 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let functions K, φ, ψ, F, G be defined as in the statement of Theorem* 1.16. *Then the inequality*

$$\begin{aligned}\int_\Omega G^{1-p}(y)\psi^{-p}(y)\left[\int_\Omega K(x,y)f(x)d\mu_1(x)\right]^p d\mu_2(y)\\ \leq (1-R(\overline{f},\overline{\widetilde{g}},\overline{h}))^{mp}\int_\Omega \varphi^p(x)F(x)f^p(x)d\mu_1(x)\end{aligned} \tag{1.72}$$

holds for all non-negative measurable functions $f:\Omega\to\mathbb{R}$, provided the functions $\overline{f}$, $\overline{\widetilde{g}}$, $\overline{h}$ are defined as in Theorem 1.16. *If $0<p<1$, then the reverse inequality in* (1.72) *is valid.*

Proof. Since $\frac{1}{p}+\frac{1}{q}=1$, utilizing (1.71) we have

$$\begin{aligned}&\int_\Omega G^{1-p}(y)\psi^{-p}(y)\left(\int_\Omega K(x,y)f(x)d\mu_1(x)\right)^p d\mu_2(y)\\ &= \int_\Omega\int_\Omega K(x,y)f(x)\widetilde{g}(y)d\mu_1(x)d\mu_2(y)\\ &\leq (1-R(\overline{f},\overline{\widetilde{g}},\overline{h}))^m\left[\int_\Omega \varphi^p(x)F(x)f^p(x)d\mu_1(x)\right]^{\frac{1}{p}}\left[\int_\Omega \psi^q(y)G(y)\widetilde{g}^q(y)d\mu_2(y)\right]^{\frac{1}{q}}\\ &= \left[\int_\Omega \varphi^p(x)F(x)f^p(x)d\mu_1(x)\right]^{\frac{1}{p}}\\ &\quad\times\left[\int_\Omega G^{1-p}(y)\psi^{-p}(y)\left(\int_\Omega K(x,y)f(x)d\mu_1(x)\right)^p d\mu_2(y)\right]^{\frac{1}{q}},\end{aligned}$$

that is, we get (1.72). The reverse inequality is derived in the same way, by virtue of Lemma 1.2. □

Remark 1.19 Note that inequalities (1.71) and (1.72) present refinements of inequalities (1.12) and (1.13) from Theorem 1.9.

Remark 1.20 Clearly, the method developed in this section can be applied to results from Sections 1.3 and 1.4. For more details about this problem area the reader is referred to [58]. Some other methods of improving Hilbert-type inequalities will be studied in Chapters 4, 5, and 6.

1.6 Multidimensional Hilbert-type inequalities

1.6.1 General form

Regarding Theorem 1.9, in this section we derive multidimensional forms of the general Hilbert-type and Hardy-Hilbert-type inequality. In such setting we deal with conjugate parameters $p_1, p_2, \ldots, p_n$, $n \geq 2$. Recall that these parameters fulfill condition $\sum_{i=1}^n \frac{1}{p_i} = 1$. If all parameters are non-negative, then $p_i > 1$, $i = 1, 2, \ldots, n$.

The following result includes integrals taken over general subsets of $\mathbb{R}_+$, with respect to σ-finite measures.

Theorem 1.18 *Let $\sum_{i=1}^n \frac{1}{p_i} = 1$, $p_i > 1$, $i = 1, 2, \ldots, n$, and let Ω be measure space with σ-finite measures μ_i, $i = 1, 2, \ldots, n$. Further, suppose that $K : \Omega^n \to \mathbb{R}$ and $\phi_{ij} : \Omega \to \mathbb{R}$, $i, j = 1, \ldots, n$, are non-negative measurable functions such that $\prod_{i,j=1}^n \phi_{ij}(x_j) = 1$. If $\frac{1}{q} = \sum_{i=1}^{n-1} \frac{1}{p_i}$ and functions F_i, h are defined by*

$$F_i(x_i) = \int_{\Omega^{n-1}} K(x_1, \ldots, x_n) \prod_{j=1, j\neq i}^n \phi_{ij}^{p_i}(x_j) \times d\mu_1(x_1) \ldots d\mu_{i-1}(x_{i-1}) d\mu_{i+1}(x_{i+1}) \ldots d\mu_n(x_n),\ i = 1, 2, \ldots, n,$$

$h = \phi_{nn}^{-q} F_n^{1-q}$, then for all non-negative measurable functions $f_i : \Omega \to \mathbb{R}$, $i = 1, 2, \ldots, n$, the inequalities

$$\int_{\Omega^n} K(x_1, \ldots, x_n) \prod_{i=1}^n f_i(x_i) d\mu_1(x_1) \cdots d\mu_n(x_n) \leq \prod_{i=1}^n \left[\int_\Omega F_i(x_i) f_i^{p_i}(x_i) \phi_{ii}^{p_i}(x_i) d\mu_i(x_i) \right]^{\frac{1}{p_i}} \tag{1.73}$$

and

$$\int_\Omega h(x_n) \left[\int_{\Omega^{n-1}} K(x_1, \ldots, x_n) \prod_{i=1}^{n-1} f_i(x_i) d\mu_1(x_1) \cdots d\mu_{n-1}(x_{n-1}) \right]^q d\mu_n(x_n) \leq \prod_{i=1}^{n-1} \left[\int_\Omega F_i(x_i) f_i^{p_i}(x_i) \phi_{ii}^{p_i}(x_i) d\mu_i(x_i) \right]^{\frac{q}{p_i}} \tag{1.74}$$

hold and are equivalent.

If $p_i > 0$, $i \in \{1,2,\dots,n\}$, and $p_k < 0$, $k \neq i$, then the reverse inequality in (1.73) *holds. Moreover, if $p_i > 0$, $i \in \{1,2,\dots,n-1\}$, and $p_k < 0$, $k \neq i$, then the reverse inequality in* (1.74) *holds. In addition, inequality* (1.74) *holds also when $p_n > 0$ and $p_k < 0$, $k \neq n$.*

Proof. We first prove inequality (1.73). Applying the Hölder inequality we have

$$\begin{aligned}
&\int_{\Omega^n} K(x_1,\dots,x_n)\prod_{i=1}^{n} f_i(x_i)d\mu_1(x_1)\dots d\mu_n(x_n)\\
&= \int_{\Omega^n} K(x_1,\dots,x_n)\prod_{i=1}^{n}\Big(f_i(x_i)\prod_{j=1}^{n}\phi_{ij}(x_j)\Big)d\mu_1(x_1)\dots d\mu_n(x_n)\\
&\leq \prod_{i=1}^{n}\left[\int_{\Omega^n} K(x_1,\dots,x_n)f_i^{p_i}(x_i)\prod_{j=1}^{n}\phi_{ij}^{p_i}(x_j)d\mu_1(x_1)\dots d\mu_n(x_n)\right]^{\frac{1}{p_i}},
\end{aligned}$$

so the Fubini theorem and definitions of functions F_i, $i = 1,2,\dots,n$, yield (1.73).

In order to prove inequality (1.74), we define $I(x_n)$ as

$$I(x_n) = \int_{\Omega^{n-1}} K(x_1,\dots,x_n)\prod_{i=1}^{n-1} f_i(x_i)d\mu_1(x_1)\dots d\mu_{n-1}(x_{n-1}).$$

Now, putting function

$$f_n(x_n) = h(x_n)\cdot(I(x_n))^{q-1}$$

in inequality (1.73), we get

$$\begin{aligned}
I = \int_{\Omega} h(x_n)(I(x_n))^q d\mu_n(x_n) &\leq \prod_{i=1}^{n-1}\left[\int_{\Omega}\phi_{ii}^{p_i}(x_i)F_i(x_i)f_i^{p_i}(x_i)d\mu_i(x_i)\right]^{\frac{1}{p_i}}\\
&\times\left[\int_{\Omega} h^{p_n}(x_n)(I(x_n))^{p_n(q-1)}F_n(x_n)\phi_{nn}^{p_n}(x_n)d\mu_n(x_n)\right]^{1-\frac{1}{q}}.
\end{aligned}$$

Since $h(x_n) = \phi_{nn}^{-q}(x_n)F_n^{1-q}(x_n)$ and $\frac{1}{q} = \sum_{i=1}^{n-1}\frac{1}{p_i}$, the above inequality can be rewritten in the form,

$$I \leq \prod_{i=1}^{n-1}\left[\int_{\Omega} F_i(x_i)f_i^{p_i}(x_i)\phi_{ii}^{p_i}(x_i)d\mu_i(x_i)\right]^{\frac{1}{p_i}}\cdot I^{1-\frac{1}{q}},$$

that is, we obtain (1.74).

Reverse inequalities are derived in the same way, by virtue of the reverse Hölder inequality.

It remains to prove that inequalities (1.73) and (1.74) are equivalent. It is enough to check that inequality (1.73) follows from (1.74). For this purpose, suppose that inequality

(1.74) holds. Then we have

$$\begin{aligned}
&\int_{\Omega^n} K(x_1,\dots,x_n)\prod_{i=1}^{n} f_i(x_i)d\mu_1(x_1)\dots d\mu_n(x_n)\\
&=\int_{\Omega} \phi_{nn}{}^{-1}(x_n)F_n{}^{-\frac{1}{p_n}}(x_n)\left[\int_{\Omega^{n-1}} K(x_1,\dots,x_n)\prod_{i=1}^{n-1} f_i(x_i)d\mu_1(x_1)\dots d\mu_{n-1}(x_{n-1})\right]\\
&\quad\times F_n{}^{\frac{1}{p_n}}(x_n)f_n(x_n)\phi_{nn}(x_n)d\mu_n(x_n).
\end{aligned}$$

In addition, the Hölder inequality with conjugate exponents q and p_n yields

$$\begin{aligned}
&\int_{\Omega^n} K(x_1,\dots,x_n)\prod_{i=1}^{n} f_i(x_i)d\mu_1(x_1)\dots d\mu_n(x_n)\\
&\le\left\{\int_{\Omega} h(x_n)\left[\int_{\Omega^{n-1}} K(x_1,\dots,x_n)\prod_{i=1}^{n-1} f_i(x_i)d\mu_1(x_1)\dots d\mu_{n-1}(x_{n-1})\right]^q d\mu_n(x_n)\right\}^{\frac{1}{q}}\\
&\quad\times\left[\int_{\Omega}\phi_{nn}^{p_n}(x_n)F_n(x_n)f_n^{p_n}(x_n)d\mu_n(x_n)\right]^{\frac{1}{p_n}},
\end{aligned}$$

and the result follows from (1.74). □

Remark 1.21 Considering the proof of the previous theorem, it follows that the equality in (1.73) and (1.74) is possible if and only if it holds in the Hölder inequality. More precisely, this means that functions $f_i^{p_i}(x_i)\prod_{j=1}^{n}\phi_{ij}^{p_i}(x_j)$ are effectively proportional. Hence, equality in (1.73) and (1.74) holds if and only if

$$f_i(x_i)=C_i\phi_{ii}(x_i)^{\frac{p_i}{1-p_i}},\ i=1,\dots,n, \tag{1.75}$$

for some constants $C_i\ge 0$. That is possible only if the functions

$$\frac{\prod_{j=1,j\ne i}^{n}\phi_{jj}{}^{\frac{p_j}{1-p_j}}(x_j)}{\prod_{j=1,j\ne i}^{n}\phi_{ij}{}^{p_j}(x_j)},\quad i=1,2,\dots,n,$$

are appropriate constants, and

$$\int_{\Omega} F_i(x_i)\phi_{ii}^{\frac{p_i}{1-p_i}}(x_i)d\mu_i(x_i)<\infty,\quad i=1,2,\dots n.$$

Otherwise, the inequalities in Theorem 1.18 are strict.

Remark 1.22 It should be emphasized here that Theorem 1.18 is a multidimensional extension of Theorem 1.9 from Section 1.2.

1.6.2 Application to homogeneous kernels

In the sequel we consider multidimensional versions of Hilbert inequality (i. e. Theorem 1.18) equipped with $\Omega = \mathbb{R}_+$, Lebesgue measures μ_i, $i = 1, \ldots, n$, a non-negative homogeneous kernel $K : \mathbb{R}_+^n \to \mathbb{R}$, and the weight functions $\phi_{ij}(x_j) = x_j^{A_{ij}}$, where $A_{ij} \in \mathbb{R}$, $i, j = 1, \ldots, n$.

We define multidimensional version of integral $k(\alpha)$ (see definition (1.20)), but in this section it will be more convenient to slightly change the definition:

$$k(\alpha_1, \ldots, \alpha_{n-1}) = \int_{\mathbb{R}_+^{n-1}} K(1, t_1 \ldots, t_{n-1}) t_1^{\alpha_1} \cdots t_{n-1}^{\alpha_{n-1}} dt_1 \cdots dt_{n-1}, \tag{1.76}$$

where we assume that $k(\alpha_1, \ldots, \alpha_{n-1}) < \infty$ for $\alpha_1, \ldots, \alpha_{n-1} > -1$ and $\alpha_1 + \cdots + \alpha_{n-1} + n < s + 1$.

Theorem 1.19 *Let $\sum_{i=1}^n \frac{1}{p_i} = 1$, $p_i > 1$, $i = 1, 2, \ldots, n$, and let $K : \mathbb{R}_+^n \to \mathbb{R}$ be a non-negative measurable homogeneous function of degree $-s$, $s > 0$. Further, let A_{ij}, $i, j = 1, \ldots, n$, be real parameters such that $\sum_{i=1}^n A_{ij} = 0$ for $j = 1, \ldots, n$. If $f_i : \mathbb{R}_+ \to \mathbb{R}$, $f_i \neq 0$, $i = 1, \ldots, n$, are non-negative measurable functions, then the inequalities*

$$\int_{\mathbb{R}_+^n} K(x_1, \ldots, x_n) \prod_{i=1}^n f_i(x_i) dx_1 \cdots dx_n < L \prod_{i=1}^n \left[\int_0^\infty x_i^{n-s-1+p_i\alpha_i} f_i^{p_i}(x_i) dx_i \right]^{\frac{1}{p_i}} \tag{1.77}$$

and

$$\begin{aligned} \int_0^\infty x_n^{(1-q)(n-1-s)-q\alpha_n} \left[\int_{\mathbb{R}_+^{n-1}} K(x_1, \ldots, x_n) \prod_{i=1}^{n-1} f_i(x_i) dx_1 \cdots dx_{n-1} \right]^q dx_n \\ < L^q \prod_{i=1}^{n-1} \left[\int_0^\infty x_i^{n-1-s+p_i\alpha_i} f_i^{p_i}(x_i) dx_i \right]^{\frac{q}{p_i}} \end{aligned} \tag{1.78}$$

hold and are equivalent, where

$$\begin{aligned} L = k^{\frac{1}{p_1}}(p_1 A_{12}, \ldots, p_1 A_{1n}) \cdot k^{\frac{1}{p_2}}(s - n - p_2(\alpha_2 - A_{22}), p_2 A_{23}, \ldots, p_2 A_{2n}) \\ \cdots k^{\frac{1}{p_n}}(p_n A_{n2}, \ldots, p_n A_{n,n-1}, s - n - p_n(\alpha_n - A_{nn})), \end{aligned} \tag{1.79}$$

$\frac{1}{q} = \sum_{i=1}^{n-1} \frac{1}{p_i}$, $\alpha_i = \sum_{j=1}^n A_{ij}$, $p_i A_{ij} > -1$, $i \neq j$, $p_i(A_{ii} - \alpha_i) > n - s - 1$.

Proof. Set $\phi_{ij}(x_j) = x_j^{A_{ij}}$ in Theorem 1.18, where $\sum_{i=1}^n A_{ij} = 0$ for every $j = 1, \ldots, n$. It is enough to calculate the functions $F_i(x_i)$, $i = 1, \ldots, n$. Using homogeneity of the kernel K and obvious change of variables, we have

$$\begin{aligned} F_1(x_1) &= \int_{\mathbb{R}_+^{n-1}} K(x_1, x_2, \ldots, x_n) \prod_{j=2}^n x_j^{p_1 A_{1j}} dx_2 \cdots dx_n \\ &= \int_{\mathbb{R}_+^{n-1}} x_1^{-s} K(1, x_2/x_1, \ldots, x_n/x_1) \prod_{j=2}^n x_j^{p_1 A_{1j}} dx_2 \cdots dx_n \\ &= x_1^{n-1-s+p_1(\alpha_1 - A_{11})} k(p_1 A_{12}, \ldots, p_1 A_{1n}). \end{aligned}$$

On the other hand, using homogeneity of K and the change of variables

$$x_1 = x_2 \cdot \frac{1}{t_2},\ x_i = x_2 \cdot \frac{t_i}{t_2},\ i = 3,\dots,n, \text{ so that } \frac{\partial(x_1,x_3,\dots,x_n)}{\partial(t_2,t_3,\dots,t_n)} = x_2^{n-1} t_2^{-n},$$

where $\frac{\partial(x_1,x_3,\dots,x_n)}{\partial(t_2,t_3,\dots,t_n)}$ denotes the Jacobian of the transformation, we have

$$\begin{aligned} F_2(x_2) &= \int_{\mathbb{R}_+^{n-1}} K(x_1,x_2,\dots,x_n) \prod_{j=1, j\neq 2}^{n} x_j^{p_2 A_{2j}} dx_1 dx_3 \cdots dx_n \\ &= \int_{\mathbb{R}_+^{n-1}} x_1^{-s} K(1, x_2/x_1,\dots,x_n/x_1) \prod_{j=1, j\neq 2}^{n} x_j^{p_2 A_{2j}} dx_1 dx_3 \cdots dx_n \\ &= x_2^{n-1-s+p_2(\alpha_2 - A_{22})} k(s-n-p_2(\alpha_2 - A_{22}), p_2 A_{23},\dots,p_2 A_{2n}). \end{aligned}$$

In a similar manner we obtain

$$\begin{aligned} F_i(x_i) &= x_i^{n-s-1+p_i(\alpha_i - A_{ii})} \\ &\quad \times k(p_i A_{i2},\dots,p_i A_{i,i-1}, s-n-p_i(\alpha_i - A_{ii}), p_i A_{i,i+1},\dots,p_i A_{in}), \end{aligned}$$

for $i = 3,\dots,n$. This gives inequalities (1.77) and (1.78) with inequality sign $\leq$.

Finally, condition (1.75) immediately gives that nontrivial case of equality in (1.77) and (1.78) leads to the divergent integrals. This completes the proof. □

Motivated by the ideas from Section 1.4, we can also establish conditions under which the constant factors L and L^q are the best possible in inequalities (1.77) and (1.78). In order to obtain such factors, it is natural to impose the following conditions on the parameters A_{ij}:

$$\begin{aligned} p_j A_{ji} &= s-n-p_i(\alpha_i - A_{ii}),\ i = 2,\dots,n,\ j \neq i, \\ p_i A_{ik} &= p_j A_{jk},\ k \neq i,j,\ k \neq 1. \end{aligned} \tag{1.80}$$

The missing cases $i = 1$ and $k = 1$ can be deduced from (1.80) as follows:

$$p_1(\alpha_1 - A_{11}) = p_1 A_{1j} + p_1 \sum_{i\neq 1,j} A_{1i} = s-n-p_j(\alpha_j - A_{jj}) + p_j \sum_{i \neq j,1} A_{ji} = s-n-p_j A_{j1},$$

where $j \neq 1$. Thus, the complete set of conditions is

$$\begin{aligned} p_j A_{ji} &= s-n-p_i(\alpha_i - A_{ii}),\ i,j = 1,2,\dots,n,\ i \neq j, \\ p_i A_{ik} &= p_j A_{jk},\ k \neq i,j. \end{aligned} \tag{1.81}$$

Theorem 1.20 *Suppose that real parameters A_{ij}, $i,j = 1,\dots,n$, fulfill conditions from Theorem* 1.19 *and conditions given in* (1.81). *If the kernel $K : \mathbb{R}_+^n \to \mathbb{R}$ is as in Theorem* 1.19 *and for every $i = 2,\dots,n$*

$$K(1,t_2,\dots,t_i,\dots,t_n) \leq CK(1,t_2,\dots,0,\dots,t_n),\ 0 \leq t_i \leq 1,\ t_j \geq 0,\ j \neq i, \tag{1.82}$$

for some $C > 0$, then the constants L and L^q are the best possible in inequalities (1.77) *and* (1.78). *In this case $L = k(p_1 A_{12}, p_1 A_{13},\dots,p_1 A_{1n})$.*

Proof. It is easy to see that $n-s+p_i\alpha_i = -p_i\widetilde{A}_i$, where $\widetilde{A}_i = p_1A_{1i}$ for $i \neq 1$, and $\widetilde{A}_1 = p_nA_{n1}$. Hence, inequality (1.77) can be rewritten as

$$\int_{\mathbb{R}_+^n} K(x_1,\dots,x_n)\prod_{i=1}^n f_i(x_i)dx_1\cdots dx_n < L\prod_{i=1}^n\left[\int_0^\infty x_i^{-1-p_i\widetilde{A}_i} f_i^{p_i}(x_i)dx_i\right]^{\frac{1}{p_i}}, \tag{1.83}$$

where $L = k(\widetilde{A}_2,\dots,\widetilde{A}_n)$.

Now, suppose that the above constant factor L is not the best possible. Then, there exists a positive constant M, smaller than L, such that inequality (1.83) still holds after replacing L by M. For this purpose, set

$$\widetilde{f}_i(x_i) = \begin{cases} 0, & x \in (0,1) \\ x_i^{\widetilde{A}_i - \frac{\varepsilon}{p_i}}, & x \in [1,\infty) \end{cases}, \quad i = 1,\dots,n,$$

where $0 < \varepsilon < \min_{1\le i\le n}\{p_i + p_i\widetilde{A}_i\}$. If we substitute these functions in (1.83), then the right-hand side of the inequality becomes $\frac{M}{\varepsilon}$, since

$$\prod_{i=1}^n\left[\int_0^\infty x_i^{-1-p_i\widetilde{A}_i}\widetilde{f}_i^{p_i}(x_i)dx_i\right]^{\frac{1}{p_i}} = \frac{1}{\varepsilon}. \tag{1.84}$$

Further, let J denotes the left-hand side of inequality (1.83), for the above choice of functions $\widetilde{f}_i$. Utilizing substitutions $u_i = \frac{x_i}{x_1}$, $i = 2,\dots,n$, in J, we find that

$$J = \int_1^\infty x_1^{-1-\varepsilon}\left[\int_{\frac{1}{x_1}}^\infty \cdots \int_{\frac{1}{x_1}}^\infty K(1,u_2,\dots,u_n)\prod_{i=2}^n u_i^{\widetilde{A}_i - \frac{\varepsilon}{p_i}} du_2\dots du_n\right]dx_1. \tag{1.85}$$

Moreover, J can be estimated as

$$\begin{aligned} J &\geq \int_1^\infty x_1^{-1-\varepsilon}\left[\int_{\mathbb{R}_+^{n-1}} K(1,u_2,\dots,u_n)\prod_{i=2}^n u_i^{\widetilde{A}_i-\frac{\varepsilon}{p_i}}du_2\dots du_n\right]dx_1 \\ &\quad - \int_1^\infty x_1^{-1-\varepsilon}\sum_{j=2}^n I_j(x_1)dx_1 \\ &= \frac{1}{\varepsilon}k\left(\widetilde{A}_2 - \frac{\varepsilon}{p_2},\dots,\widetilde{A}_n - \frac{\varepsilon}{p_n}\right) - \int_1^\infty x_1^{-1-\varepsilon}\sum_{j=2}^n I_j(x_1)dx_1, \end{aligned} \tag{1.86}$$

where for $j = 2,\dots,n$, $I_j(x_1)$ is defined by

$$I_j(x_1) = \int_{D_j} K(1,u_2,\dots,u_n)\prod_{i=2}^n u_i^{\widetilde{A}_i - \frac{\varepsilon}{p_i}}du_2\dots du_n,$$

with $D_j = \{(u_2,u_3,\dots,u_n); 0 < u_j \le \frac{1}{x_1},\ 0 < u_k < \infty, k \neq j\}$.

Without losing generality, it is enough to estimate the integral $I_2(x_1)$. The case of $n=2$ was proved in Theorem 1.13 (see Section 1.4). For $n\geq 3$ we have

$$\begin{aligned}
I_2(x_1) &\leq C\left[\int_{\mathbb{R}_+^{n-2}} K(1,0,u_3,\ldots,u_n)\prod_{i=3}^{n} u_i^{\widetilde{A}_i-\frac{\varepsilon}{p_i}}du_3\ldots du_n\right]\\
&\quad\times\int_0^{\frac{1}{x_1}} u_2^{\widetilde{A}_2-\frac{\varepsilon}{p_2}}du_2\\
&= C\left(1-\frac{\varepsilon}{p_2}+\widetilde{A}_2\right)^{-1} x_1^{\frac{\varepsilon}{p_2}-\widetilde{A}_2-1}k\left(\widetilde{A}_3-\frac{\varepsilon}{p_3},\ldots,\widetilde{A}_n-\frac{\varepsilon}{p_n}\right),
\end{aligned}$$

where $k\big(\widetilde{A}_3-\frac{\varepsilon}{p_3},\ldots,\widetilde{A}_n-\frac{\varepsilon}{p_n}\big)$ is well-defined since obviously $\widetilde{A}_3+\cdots+\widetilde{A}_n<s-n+2$. Hence, we have $I_j(x_1)=x_1^{\varepsilon/p_j-\widetilde{A}_j-1}O_j(1)$ for $\varepsilon\to 0^+$, $j=2,\ldots,n$, and consequently

$$\int_1^{\infty} x_1^{-1-\varepsilon}\sum_{j=2}^{n} I_j(x_1)dx_1 = O(1). \tag{1.87}$$

Therefore, taking into account (1.84), (1.86), and (1.87), it follows that $L\leq M$ when $\varepsilon\to 0^+$, which is an obvious contradiction. This means that the constant L is the best possible in (1.83). Moreover, since the equivalence preserves the best possible constant, the proof is completed. □

Remark 1.23 If the parameters A_{ij} are defined by $A_{ii}=\frac{(n-s)(p_i-1)}{p_i^2}$ and $A_{ij}=\frac{s-n}{p_ip_j}$, $i\neq j$, $i,j=1,\ldots,n$, then we have

$$\sum_{i=1}^{n} A_{ij} = (s-n)\sum_{i\neq j}\frac{1}{p_ip_j}+(n-s)\frac{p_j-1}{p_j^2} = \frac{s-n}{p_j}\left(\sum_{i=1}^{n}\frac{1}{p_i}-1\right)=0,$$

for $j=1,2,\ldots,n$. Obviously, due to the symmetry, it follows that $\alpha_i=\sum_{j=1}^{n}A_{ij}=0$, for $i=1,\ldots,n$. Moreover, parameters A_{ij}, $i,j=1,\ldots,n$, fulfill the set of conditions (1.81), hence, in this case Theorem 1.20 yields the following inequality

$$\int_{\mathbb{R}_+^n} K(x_1,\ldots,n)\prod_{i=1}^{n} f_i(x_i)dx_1\cdots dx_n < L\prod_{i=1}^{n}\left[\int_0^{\infty} x_i^{n-s-1}f_i^{p_i}(x_i)dx_i\right]^{\frac{1}{p_i}},$$

where $L=k\big(\frac{s-n}{p_2},\ldots,\frac{s-n}{p_n}\big)$ is the best possible constant. For $s=n-1$ we obtain the non-weighted case with the best possible constant $L=k\big(-\frac{1}{p_2},\ldots,-\frac{1}{p_n}\big)$ (compare with Remark 1.9).

1.6.3 Examples

We proceed with various examples of multidimensional Hilbert-type inequalities. In order to establish some particular results, we first indicate here a few lemmas. For the proof of the following lemma the reader is referred to [152].

Lemma 1.3 *If $n \in \mathbb{N}$, $r_i > 0$, $i = 1, \ldots, n$, then*

$$\int_{\mathbb{R}_+^{n-1}} \frac{\prod_{i=1}^{n-1} u_i^{r_i-1}}{\left(1+\sum_{i=1}^{n-1} u_i\right)^{\sum_{i=1}^{n} r_i}} du_1 \ldots du_{n-1} = \frac{\prod_{i=1}^{n} \Gamma(r_i)}{\Gamma\left(\sum_{i=1}^{n} r_i\right)}, \tag{1.88}$$

where Γ is the usual Gama function.

Moreover, the trivial substitution $u_i = t_i^{\lambda}$, $i = 1, \ldots, n-1$, applied to relation (1.88), yields another integral formula:

Lemma 1.4 *If $n \in \mathbb{N}$, $s, \lambda > 0$, $\beta_i > -1$, $i = 1, \ldots, n-1$, and $\sum_{i=1}^{n-1} \beta_i < \lambda s - n + 1$, then*

$$\begin{aligned} &\int_{\mathbb{R}_+^{n-1}} \frac{\prod_{i=1}^{n-1} t_i^{\beta_i}}{\left(1+\sum_{i=1}^{n-1} t_i^{\lambda}\right)^{s}} dt_1 \ldots dt_{n-1} \\ &= \frac{1}{\Gamma(s)\lambda^{n-1}} \left[\prod_{i=1}^{n-1} \Gamma\left(\frac{\beta_i+1}{\lambda}\right)\right] \Gamma\left(s - \frac{1}{\lambda}\sum_{i=1}^{n-1}(\beta_i+1)\right). \end{aligned} \tag{1.89}$$

The following lemma will be needed when considering a particular 3-dimensional case.

Lemma 1.5 *Let $s > 0$, $a \geq 0$, and $b > 0$. Further, let α_1, $\alpha_2 > -1$, $\alpha_1 + \alpha_2 < s - 2$, and*

$$k(\alpha_1, \alpha_2) = \int_{\mathbb{R}_+^2} \frac{t_1^{\alpha_1} t_2^{\alpha_2}}{(a \min\{1, t_1, t_2\} + b \max\{1, t_1, t_2\})^s} dt_1 dt_2.$$

Then

$$\begin{aligned} k(\alpha_1, \alpha_2) = &\frac{b^{-s}}{(\alpha_1+1)(\alpha_2+1)} \sum_{i=1}^{2} F\left(s, \alpha_i+1; \alpha_i+2; -\frac{a}{b}\right) \\ &- \frac{b^{-s}}{(\alpha_1+1)(\alpha_2+1)} F\left(s, \alpha_1+\alpha_2+2; \alpha_1+\alpha_2+3; -\frac{a}{b}\right) \\ &+ b^{-s} \sum_{i=1}^{2} \int_0^1 t_i^{\alpha_i} F\left(s, s-\alpha_{i+1}-1; s-\alpha_{i+1}; -\frac{a}{b} t_i\right) dt_i \\ &+ \frac{b^{-s}(\alpha_1+\alpha_2+2)}{(\alpha_1+1)(\alpha_2+1)(s-\alpha_1-\alpha_2-1)} \\ &\quad \times F\left(s, s-\alpha_1-\alpha_2-2; s-\alpha_1-\alpha_2-1; -\frac{a}{b}\right) \\ &- b^{-s} \sum_{i=1}^{2} \frac{1}{(\alpha_i+1)(s-\alpha_{i+1}-1)} F\left(s, s-\alpha_{i+1}-1; s-\alpha_{i+1}; -\frac{a}{b}\right), \end{aligned} \tag{1.90}$$

where F denotes hypergeometric function (1.34) and the indices are taken modulo 2.

Proof. By virtue of the Fubini theorem we have $k(\alpha_1, \alpha_2) = I_1 + I_2$, where

$$I_1 = \int_0^1 t_1^{\alpha_1} \left(\int_0^{\infty} \frac{t_2^{\alpha_2}}{(a \min\{t_1, t_2\} + b \max\{1, t_2\})^s} dt_2\right) dt_1$$

and

$$I_2=\int_1^\infty t_1^{\alpha_1}\left(\int_0^\infty \frac{t_2^{\alpha_2}}{(a\min\{1,t_2\}+b\max\{t_1,t_2\})^s}dt_2\right)dt_1.$$

In what follows, we shall express integral I_1 by a hypergeometric function. It is easy to see that $I_1=I_{11}+I_{12}$, where

$$I_{11}=\int_0^1 t_1^{\alpha_1}\left(\int_0^1 \frac{t_2^{\alpha_2}}{(a\min\{t_1,t_2\}+b)^s}dt_2\right)dt_1$$

and

$$I_{12}=\int_0^1 t_1^{\alpha_1}\left(\int_1^\infty \frac{t_2^{\alpha_2}}{(at_1+bt_2)^s}dt_2\right)dt_1.$$

The integral I_{11} can be transformed in the following way:

$$I_{11}=\int_0^1 t_1^{\alpha_1}\left(\int_0^{t_1} \frac{t_2^{\alpha_2}}{(at_2+b)^s}dt_2\right)dt_1+\int_0^1 t_1^{\alpha_1}\left(\int_{t_1}^1 \frac{t_2^{\alpha_2}}{(at_1+b)^s}dt_2\right)dt_1. \tag{1.91}$$

Applying the classical calculus, we have

$$\begin{aligned}
&\int_0^1 t_1^{\alpha_1}\left(\int_0^{t_1} \frac{t_2^{\alpha_2}}{(at_2+b)^s}dt_2\right)dt_1\\
&=\int_0^1 t_2^{\alpha_2}\left(\int_{t_2}^1 \frac{t_1^{\alpha_1}}{(at_2+b)^s}dt_1\right)dt_2=\int_0^1 t_2^{\alpha_2}(at_2+b)^{-s}\left(\int_{t_2}^1 t_1^{\alpha_1}dt_1\right)dt_2\\
&=\frac{b^{-s}}{\alpha_1+1}\int_0^1 t_2^{\alpha_2}(1-t_2^{\alpha_1+1})\left(1+\frac{a}{b}t_2\right)^{-s}dt_2\\
&=\frac{b^{-s}}{\alpha_1+1}\left[\frac{1}{\alpha_2+1}F\left(s,\alpha_2+1;\alpha_2+2;-\frac{a}{b}\right)\right.\\
&\left.-\frac{1}{\alpha_1+\alpha_2+2}F\left(s,\alpha_1+\alpha_2+2;\alpha_1+\alpha_2+3;-\frac{a}{b}\right)\right],
\end{aligned}\tag{1.92}$$

and similarly,

$$\begin{aligned}
&\int_0^1 t_1^{\alpha_1}\left(\int_{t_1}^1 \frac{t_2^{\alpha_2}}{(at_1+b)^s}dt_2\right)dt_1\\
&=\frac{b^{-s}}{\alpha_2+1}\left[\frac{1}{\alpha_1+1}F\left(s,\alpha_1+1;\alpha_1+2;-\frac{a}{b}\right)\right.\\
&\left.-\frac{1}{\alpha_1+\alpha_2+2}F\left(s,\alpha_1+\alpha_2+2;\alpha_1+\alpha_2+3;-\frac{a}{b}\right)\right].
\end{aligned}\tag{1.93}$$

Now, setting (1.92) and (1.93) in (1.91), we obtain

$$\begin{aligned}
I_{11}&=\frac{b^{-s}}{(\alpha_1+1)(\alpha_2+1)}\sum_{i=1}^2 F\left(s,\alpha_i+1;\alpha_i+2;-\frac{a}{b}\right)\\
&-\frac{b^{-s}}{(\alpha_1+1)(\alpha_2+1)}F\left(s,\alpha_1+\alpha_2+2;\alpha_1+\alpha_2+3;-\frac{a}{b}\right),
\end{aligned}\tag{1.94}$$

while the substitution $u = \frac{1}{t_2}$ yields

$$
\begin{aligned}
I_{12} &= \int_0^1 t_1^{\alpha_1} \left(\int_1^\infty \frac{t_2^{\alpha_2}}{(at_1+bt_2)^s} dt_2 \right) dt_1 \\
&= b^{-s} \int_0^1 t_1^{\alpha_1} \left(\int_0^1 u^{s-\alpha_2-2} \left(1+\frac{a}{b}t_1 u\right)^{-s} du \right) dt_1 \\
&= b^{-s} \int_0^1 t_1^{\alpha_1} F\left(s, s-\alpha_2-1; s-\alpha_2; -\frac{a}{b}t_1\right) dt_1. \qquad (1.95)
\end{aligned}
$$

Finally, from (1.94) and (1.95) we have

$$
\begin{aligned}
I_1 &= I_{11} + I_{12} = \\
&= \frac{b^{-s}}{(\alpha_1+1)(\alpha_2+1)} \sum_{i=1}^{2} F\left(s, \alpha_i+1; \alpha_i+2; -\frac{a}{b}\right) \\
&- \frac{b^{-s}}{(\alpha_1+1)(\alpha_2+1)} F\left(s, \alpha_1+\alpha_2+2; \alpha_1+\alpha_2+3; -\frac{a}{b}\right) \\
&+ b^{-s} \int_0^1 t_1^{\alpha_1} F\left(s, s-\alpha_2-1; s-\alpha_2; -\frac{a}{b}t_1\right) dt_1. \qquad (1.96)
\end{aligned}
$$

Repeating the above procedure for the integrals

$$
I_{21} = \int_1^\infty t_1^{\alpha_1} \left(\int_0^1 \frac{t_2^{\alpha_2}}{(at_2+bt_1)^s} dt_2 \right) dt_1
$$

and

$$
I_{22} = \int_1^\infty t_1^{\alpha_1} \left(\int_1^\infty \frac{t_2^{\alpha_2}}{(a+b\max\{t_1,t_2\})^s} dt_2 \right) dt_1,
$$

since $I_{21} + I_{22} = I_2$, we obtain (1.90), as claimed. □

Multidimensional Hilbert-type inequalities that follow will be based on Theorems 1.19 and 1.20. More precisely, under the assumptions of Theorem 1.20, inequalities (1.77) and (1.78) can be rewritten as

$$
\int_{\mathbb{R}_+^n} K(x_1,\ldots,x_n) \prod_{i=1}^{n} f_i(x_i) dx_1 \cdots dx_n < L \prod_{i=1}^{n} \left[\int_0^\infty x_i^{-1-p_i\widetilde{A}_i} f_i^{p_i}(x_i) dx_i \right]^{\frac{1}{p_i}} \qquad (1.97)
$$

and

$$
\begin{aligned}
&\int_0^\infty x_n^{(1-q)(-1-p_n\widetilde{A}_n)} \left[\int_{\mathbb{R}_+^{n-1}} K(x_1,\ldots,x_n) \prod_{i=1}^{n-1} f_i(x_i) dx_1 \cdots dx_{n-1} \right]^q dx_n \\
&\qquad < L^q \prod_{i=1}^{n-1} \left[\int_0^\infty x_i^{-1-p_i\widetilde{A}_i} f_i^{p_i}(x_i) dx_i \right]^{\frac{q}{p_i}}, \qquad (1.98)
\end{aligned}
$$

where $\widetilde{A}_i = p_1 A_{1i}$ for $i \neq 1$, $\widetilde{A}_1 = p_n A_{n1}$, $\frac{1}{q} = \sum_{i=1}^{n-1} \frac{1}{p_i}$, $p_1 A_{1j} > -1$, $i \neq j$, and $p_1(A_{11} - \alpha_1) > n - s - 1$. Moreover, constant factors $L = k(\widetilde{A}_2, \ldots, \widetilde{A}_n)$ and L^q are the best possible in inequalities (1.97) and (1.98).

We first consider a particular case of parameters A_{ij}, $i, j = 1, \ldots, n$, which fulfill the set of conditions (1.81), necessary in establishing the inequalities with the best possible constant factors. These are the parameters

$$A_{ij} = \frac{s - p_j}{p_i p_j}, \quad i \neq j, \quad \text{and} \quad A_{ii} = \frac{(s - p_i)(1 - p_i)}{p_i^2}. \tag{1.99}$$

Hence, considering inequalities (1.97) and (1.98) with parameters A_{ij}, $i, j = 1, \ldots, n$, defined by (1.99), we have the following consequence:

Corollary 1.4 *Let $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, $p_i > 1$, $i = 1, 2, \ldots, n$, and let $\frac{1}{q} = \sum_{i=1}^{n-1} \frac{1}{p_i}$. Further, suppose that $K : \mathbb{R}_+^n \to \mathbb{R}$ is a non-negative measurable homogeneous function of degree $-s$, $s > 0$, fulfilling condition (1.82). If $f_i : \mathbb{R}_+ \to \mathbb{R}$, $f_i \neq 0$, $i = 1, \ldots, n$, are non-negative measurable functions, then inequalities*

$$\int_{\mathbb{R}_+^n} K(x_1, \ldots, x_n) \prod_{i=1}^{n} f_i(x_i) dx_1 \cdots dx_n < L \prod_{i=1}^{n} \left[\int_0^\infty x_i^{p_i - s - 1} f_i^{p_i}(x_i) dx_i \right]^{\frac{1}{p_i}} \tag{1.100}$$

and

$$\int_0^\infty x_n^{\frac{s}{p_n - 1} - 1} \left[\int_{\mathbb{R}_+^{n-1}} K(x_1, \ldots, x_n) \prod_{i=1}^{n-1} f_i(x_i) dx_1 \cdots dx_{n-1} \right]^q dx_n$$
$$< L^q \prod_{i=1}^{n-1} \left[\int_0^\infty x_i^{p_i - s - 1} f_i^{p_i}(x_i) dx_i \right]^{\frac{q}{p_i}} \tag{1.101}$$

hold and are equivalent, where the constant factors $L = k\left(\frac{s-p_2}{p_2}, \ldots, \frac{s-p_n}{p_n}\right)$ and L^q are the best possible in both inequalities.

Remark 1.24 Observe that the kernel $K(x_1, \ldots, x_n) = (x_1 + \ldots + x_n)^{-s}$, $s > 0$, fulfills condition (1.82) from Theorem 1.20 i.e. Corollary 1.4. In this case, having in mind integral formula (1.88), the above constant factor $L = k\left(\frac{s-p_2}{p_2}, \ldots, \frac{s-p_n}{p_n}\right)$ can be expressed in terms of the Gamma function i.e. $L = \frac{1}{\Gamma(s)} \prod_{i=1}^{n} \Gamma\left(\frac{s}{p_i}\right)$. This particular case was studied by B. Yang in paper [154]. Moreover, some other particular cases regarding this kernel were extensively studied in [156].

Remark 1.25 Setting $n = 2$, $K(x, y) = (a \min\{x, y\} + b \max\{x, y\})^{-1}$, $p = q = 2$ in Corollary 1.4, we obtain the result from [82] with the best possible constant factor $D(a, b) = k\left(-\frac{1}{2}\right) = \frac{4}{b} F\left(1, \frac{1}{2}; \frac{3}{2}; -\frac{a}{b}\right)$. For each choice of parameters a, b we compute the best possible constant $D(a, b)$:

(i) $a = b = 1$, $D(1, 1) = \pi$, as in [82],

(ii) $a = 0$, $b = 1$, $D(0, 1) = 4$, as in [82],

(iii) $a = 1$, $b = 2$, $D(1,2) = 2\sqrt{2}\arctan\left(\frac{1}{\sqrt{2}}\right)$,

(iv) $a = 1$, $b = 3$, $D(1,3) = \frac{2\pi}{3\sqrt{3}}$,

(v) $a = 2$, $b = 1$, $D(2,1) = 2\sqrt{2}\arctan\sqrt{2}$.

In order to exploit Lemma 1.4, in the sequel we consider the homogeneous kernel $K(x_1, \dots, x_n) = \frac{\sum_{i=1}^n x_i^{s(\lambda-1)}}{(\sum_{i=1}^n x_i^\lambda)^s}$, $s, \lambda > 0$. Clearly, this kernel fulfills the assumptions of Corollary 1.4, hence, as a consequence, we have the following pair of inequalities with the constant factors expressed in terms of the Gamma function:

Corollary 1.5 *Let $\sum_{i=1}^n \frac{1}{p_i} = 1$, $p_i > 1$, $i = 1, 2, \dots, n$, and let $\frac{1}{q} = \sum_{i=1}^{n-1} \frac{1}{p_i}$. If $f_i : \mathbb{R}_+ \to \mathbb{R}$, $f_i \neq 0$, $i = 1, \dots, n$, are non-negative measurable functions, then inequalities*

$$\int_{\mathbb{R}_+^n} \frac{\sum_{i=1}^n x_i^{s(\lambda-1)}}{\left(\sum_{i=1}^n x_i^\lambda\right)^s} \prod_{i=1}^n f_i(x_i) dx_1 \dots dx_n < L_1 \prod_{i=1}^n \left[\int_0^\infty x_i^{p_i - s - 1} f_i^{p_i}(x_i) dx_i\right]^{\frac{1}{p_i}} \tag{1.102}$$

and

$$\left[\int_0^\infty x_n^{\frac{s}{p_n - 1} - 1} \left(\int_{\mathbb{R}_+^{n-1}} \frac{\sum_{i=1}^n x_i^{s(\lambda-1)}}{\left(\sum_{i=1}^n x_i^\lambda\right)^s} \prod_{i=1}^{n-1} f_i(x_i) dx_1 \cdots dx_{n-1}\right)^q dx_n\right]^{\frac{1}{q}}$$
$$< L_1 \prod_{i=1}^{n-1} \left[\int_0^\infty x_i^{p_i - s - 1} f_i^{p_i}(x_i) dx_i\right]^{\frac{1}{p_i}} \tag{1.103}$$

hold and are equivalent, where the constant factor

$$L_1 = \frac{1}{\Gamma(s)\lambda^{n-1}} \sum_{j=1}^n \left[\left(\prod_{i=1, i\neq j}^n \Gamma\left(\frac{s}{p_i \lambda}\right)\right) \cdot \Gamma\left(\frac{s p_j(\lambda - 1) + s}{p_j \lambda}\right)\right] \tag{1.104}$$

is the best possible in both inequalities.

Proof. It is enough to calculate the constant $L_1 = k\left(\frac{s-p_2}{p_2}, \dots, \frac{s-p_n}{p_n}\right)$. Using definition (1.76) of the integral $k(\alpha_1, \dots, \alpha_{n-1})$, L_1 can be represented in the form

$$L_1 = \int_{\mathbb{R}_+^{n-1}} \frac{1 + t_1^{s(\lambda-1)} + \dots + t_{n-1}^{s(\lambda-1)}}{\left(1 + \sum_{i=1}^n t_i^\lambda\right)^s} t_1^{\frac{s}{p_2} - 1} \dots t_{n-1}^{\frac{s}{p_n} - 1} dt_1 \dots dt_{n-1} = \sum_{k=0}^{n-1} I_k,$$

where

$$I_0 = \int_{\mathbb{R}_+^{n-1}} \frac{t_1^{\frac{s}{p_2} - 1} \dots t_{n-1}^{\frac{s}{p_n} - 1}}{\left(1 + \sum_{i=1}^n t_i^\lambda\right)^s} dt_1 \dots dt_{n-1}$$

and

$$I_k = \int_{\mathbb{R}_+^{n-1}} \frac{t_1^{\frac{s}{p_2} - 1} \dots t_k^{s(\lambda-1) + \frac{s}{p_{k+1}} - 1} \dots t_{n-1}^{\frac{s}{p_n} - 1}}{\left(1 + \sum_{i=1}^n t_i^\lambda\right)^s} dt_1 \dots dt_{n-1}, \quad \text{for } k = 1, \dots, n-1.$$

Now, taking into account Lemma 1.4, it follows that

$$I_0 = \frac{1}{\Gamma(s)\lambda^{n-1}} \left[\prod_{i=2}^{n} \Gamma\left(\frac{s}{p_i\lambda}\right)\right] \cdot \Gamma\left(\frac{sp_1(\lambda-1)+s}{p_1\lambda}\right)$$

and

$$I_k = \frac{1}{\Gamma(s)\lambda^{n-1}} \left[\prod_{i=1, i\neq k+1}^{n} \Gamma\left(\frac{s}{p_i\lambda}\right)\right] \cdot \Gamma\left(\frac{sp_{k+1}(\lambda-1)+s}{p_{k+1}\lambda}\right), \ k = 1,\dots,n-1,$$

that is, we have (1.104). □

We conclude this subsection with some particular 3-dimensional Hilbert-type inequalities with constant factors expressed in terms of a hypergeometric function. The following result arises from Corollary 1.4 for the case of kernel $K(x_1,x_2,x_3) = (a\min\{x_1, x_2,x_3\} + b\max\{x_1,x_2,x_3\})^{-s}$, $a \geq 0$, $b > 0$. Clearly, K has degree of homogeneity $-s$, $s > 0$, and fulfills condition (1.82).

Corollary 1.6 *Let $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, $p_i > 1$, $i = 1,2,3$, and let $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$. Then inequalities*

$$\int_{\mathbb{R}_+^3} \frac{f_1(x_1)f_2(x_2)f_3(x_3)}{(a\min\{x_1,x_2,x_3\} + b\max\{x_1,x_2,x_3\})^s} dx_1dx_2dx_3 < L_2 \prod_{i=1}^{3} \left[\int_0^\infty x_i^{p_i-s-1} f_i^{p_i}(x_i)dx_i\right]^{\frac{1}{p_i}} \tag{1.105}$$

and

$$\left[\int_0^\infty x_3^{\frac{s}{p_3-1}-1} \left(\int_{\mathbb{R}_+^2} \frac{f_1(x_1)f_2(x_2)}{(a\min\{x_1,x_2,x_3\} + b\max\{x_1,x_2,x_3\})^s} dx_1dx_2\right)^q dx_3\right]^{\frac{1}{q}} < L_2 \prod_{i=1}^{2} \left[\int_0^\infty x_i^{p_i-s-1} f_i^{p_i}(x_i)dx_i\right]^{\frac{1}{p_i}} \tag{1.106}$$

hold for all non-negative measurable functions $f_i : \mathbb{R}_+ \to \mathbb{R}$, $f_i \neq 0$, $i = 1,2,3$, where

$$\begin{aligned} L_2 = {} & \frac{p_2p_3}{b^s s^2} \sum_{i=2}^{3} F\left(s, \frac{s}{p_i}; \frac{s}{p_i} + 1; -\frac{a}{b}\right) - \frac{p_2p_3}{b^s s^2} F\left(s, \frac{s}{p_2} + \frac{s}{p_3}; \frac{s}{p_2} + \frac{s}{p_3} + 1; -\frac{a}{b}\right) \\ & + b^{-s} \sum_{i=2}^{3} \int_0^1 t^{\frac{s}{p_i}-1} F\left(s, \frac{s}{p_1} + \frac{s}{p_i}; \frac{s}{p_1} + \frac{s}{p_i} + 1; -\frac{a}{b}t\right) dt \\ & + \frac{p_1(p_2+p_3)}{b^s s(s+p_1)} F\left(s, \frac{s}{p_1}; \frac{s}{p_1} + 1; -\frac{a}{b}\right) \\ & - \frac{p_2p_3}{b^s s^2} \sum_{i=2}^{3} \frac{1}{p_i - 1} F\left(s, s - \frac{s}{p_i}; s - \frac{s}{p_i} + 1; -\frac{a}{b}\right). \end{aligned}$$

Moreover, the constant factor L_2 is the best possible in both inequalities.

In particular, for $s = 1$, $p_1 = p_2 = p_3 = 3$ and $a = b = 1$, we have $L_2 = \frac{55\log 2}{2} - \frac{3\pi\sqrt{3}}{2}$.

Proof. Making use of Corollary 1.4 in the case of homogeneous kernel $K(x_1,x_2,x_3) = (a\min\{x_1, x_2,x_3\} + b\max\{x_1,x_2,x_3\})^{-s}$, it is enough to calculate the constant $L_2 = k(\frac{s-p_2}{p_2}, \frac{s-p_3}{p_3})$. Now the result follows from Lemma 1.5. □

Remark 1.26 Setting the kernel $K(x_1,x_2,x_3) = (x_1+x_2+x_3-\min\{x_1,x_2,x_3\})^{-s}$, $s > 0$, in Corollary 1.4, we obtain the corresponding Hilbert-type inequalities with the best possible constant factor

$$\begin{aligned}L_3 = k\left(\frac{s-p_2}{p_2}, \frac{s-p_3}{p_3}\right) &= \frac{p_2+p_3}{s}F\left(s, \frac{s}{p_2}+\frac{s}{p_3}; \frac{s}{p_2}+\frac{s}{p_3}+1; -1\right)\\ &+\frac{1}{s}\sum_{i=2}^{3} p_i F\left(s, \frac{s}{p_1}+\frac{s}{p_i}; \frac{s}{p_1}+\frac{s}{p_i}+1; -1\right)\\ &+\frac{p_1(p_2+p_3)}{s^2}F\left(s, \frac{s}{p_1}; \frac{s}{p_1}+1; -1\right)\\ &-\frac{p_2p_3}{s^2}\sum_{i=2}^{3}\frac{1}{p_i-1}F\left(s, s-\frac{s}{p_i}; s-\frac{s}{p_i}+1; -1\right).\end{aligned}$$

In particular, if $s = 1$, $p_1 = p_2 = p_3 = 3$, we have $L_3 = \frac{10\pi\sqrt{3}}{3} + \log 4$.

1.6.4 Inequalities with product-type homogeneous kernels and Schur polynomials

In this subsection we study some particular product-type homogeneous kernels and investigate associated Hilbert-type inequalities. In some cases, the corresponding constant factors can be expressed in terms of Shur polynomials.

At the beginning, we shall be concerned with the homogeneous function $K : \mathbb{R}_+^n \to \mathbb{R}$, defined by $K(x_1, \ldots, x_n) = \prod_{i=1}^{n-1}(x_1 + a_i^2 x_{i+1})$, where $a_i > 0$, $i = 1,\ldots,n-1$. Clearly, kernel K has degree of homogeneity equal to $-(n-1)$, and, in addition, fulfills condition (1.82) from Theorem 1.20. In such a way, we obtain the corresponding Hilbert-type inequalities with the best possible constant factors.

Corollary 1.7 *Let $\sum_{i=1}^{n}\frac{1}{p_i} = 1$, $p_i > 1$, $i = 1,2,\ldots,n$, and let $\frac{1}{q} = \sum_{i=1}^{n-1}\frac{1}{p_i}$. If $a_i > 0$, $i = 1,\ldots,n-1$, and $f_i : \mathbb{R}_+ \to \mathbb{R}$, $f_i \neq 0$, $i = 1,\ldots,n$, are non-negative measurable functions, then inequalities*

$$\int_{\mathbb{R}_+^n}\frac{\prod_{i=1}^{n} f_i(x_i)}{\prod_{i=1}^{n-1}(x_1+a_i^2x_{i+1})}dx_1\ldots dx_n < M_1\prod_{i=1}^{n}\left[\int_0^\infty f_i^{p_i}(x_i)dx_i\right]^{\frac{1}{p_i}} \tag{1.107}$$

and

$$\begin{aligned}&\int_0^\infty\left[\int_{\mathbb{R}_+^{n-1}}\frac{\prod_{i=1}^{n-1} f_i(x_i)}{\prod_{i=1}^{n-1}(x_1+a_i^2x_{i+1})}dx_1\ldots dx_{n-1}\right]^q dx_n\\ &< M_1^q\prod_{i=1}^{n-1}\left[\int_0^\infty f_i^{p_i}(x_i)dx_i\right]^{\frac{q}{p_i}} \qquad (1.108)\end{aligned}$$

hold and are equivalent, where

$$M_1 = \pi^{n-1} \prod_{i=2}^{n} \frac{a_{i-1}^{2(1-p_i)/p_i}}{\sin(\pi/p_i)}. \tag{1.109}$$

Moreover, constant factors M_1 *and* M_1^q *are the best possible in inequalities* (1.107) *and* (1.108).

Proof. Considering parameters $A_{ii} = \frac{p_i-1}{p_i{}^2}$ and $A_{ij} = -\frac{1}{p_i p_j}$, $i \neq j$, $i,j = 1,\ldots,n$, we have

$$\sum_{i=1}^{n} A_{ij} = -\sum_{i \neq j} \frac{1}{p_i p_j} + \frac{p_j - 1}{p_j{}^2} = \frac{1}{p_j}\left(-\sum_{i=1}^{n} \frac{1}{p_i} + 1\right) = 0,$$

for $j = 1,2,\ldots,n$, that is, $\alpha_i = \sum_{j=1}^{n} A_{ij} = 0$, $i = 1,\ldots,n$, due to the symmetry. In addition, these parameters satisfy the set of conditions (1.81), which is necessary in obtaining the best possible constant factors.

Further, utilizing inequalities (1.97), (1.98), and the above parameters, we have $-1 - p_i\widetilde{A}_i = 0$, $i = 1,\ldots,n$, and $(1-q)(-1-p_n\widetilde{A}_n) = 0$. Hence, it is enough to calculate the constant $M_1 = k\big(-\frac{1}{p_2},\ldots,-\frac{1}{p_n}\big)$ in the case when $K(x_1,\ldots,x_n) = \prod_{i=1}^{n-1}(x_1 + a_i^2 x_{i+1})$.

Exploiting (1.76), we have

$$\begin{aligned} M_1 &= k\left(-\frac{1}{p_2},\ldots,-\frac{1}{p_n}\right) = \int_{\mathbb{R}_+^{n-1}} \frac{t_1^{-\frac{1}{p_2}} \cdots t_{n-1}^{-\frac{1}{p_n}}}{\prod_{i=1}^{n-1}(1+a_i^2 t_i)} dt_1 \ldots dt_{n-1} \\ &= \prod_{i=1}^{n-1}\left(\int_0^\infty \frac{t_i^{-\frac{1}{p_{i+1}}}}{1+a_i^2 t_i} dt_i\right) = \pi^{n-1} \prod_{i=2}^{n} \frac{a_{i-1}^{2(1-p_i)/p_i}}{\sin(\pi/p_i)}, \end{aligned}$$

and the proof is completed. □

The rest of this subsection is dedicated to the two-dimensional kernel $K(x,y) = \prod_{i=1}^{m}(x + a_i^2 y)^{-1}$, where m is an integer and $a_i > 0$, $i = 1,\ldots,m$, are real parameters. Observe that the function $K(x,y)$ fulfills condition (1.82) from Theorem 1.20. In addition, we shall derive here two-dimensional Hilbert-type inequalities whose constant factors can be expressed in terms of Shur polynomials.

In order to establish the corresponding inequalities, we need the following auxiliary result. For the reader's convenience, let $f[x_1,\ldots,x_n]$ denote the well-known divided difference defined by

$$f[x_1,\ldots,x_n] = \sum_{i=1}^{n} \frac{f(x_i)}{\prod_{j \neq i}(x_i - x_j)}.$$

Lemma 1.6 *Let* m *be an integer. If* $-1 < \alpha < m-1$, $\alpha \neq 0,1,\ldots,m-2$, *and* $a_i > 0$, $i = 1,\ldots,m$, *then*

$$\int_0^\infty \frac{x^\alpha dx}{\prod_{i=1}^{m}(x+a_i^2)} = (-1)^m \frac{\pi}{\sin(\alpha\pi)} f[a_1^2,\ldots,a_m^2], \quad f(x) = x^\alpha. \tag{1.110}$$

For $k = 0, 1, \dots, m-2$, we have

$$\int_0^\infty \frac{x^k dx}{\prod_{i=1}^m (x+a_i^2)} = (-1)^{m+k} f_k[a_1^2, \dots, a_m^2], \quad f_k(x) = x^k \log x. \tag{1.111}$$

Proof. The proof is based on mathematical induction. First, for $m = 1$, using the substitution $t = \frac{x}{a_1^2}$, we obtain

$$\int_0^\infty \frac{x^\alpha dx}{x+a_1^2} = a_1^{2\alpha} \int_0^\infty t^\alpha (1+t)^{-1} dt, \tag{1.112}$$

where $-1 < \alpha < 0$. Further, by using the substitution $x = \frac{1}{1+t}$, relation (1.112) and the definition of the usual Beta function, we have

$$\begin{aligned}\int_0^\infty \frac{x^\alpha dx}{x+a_1^2} &= a_1^{2\alpha} \int_0^1 (1-x)^\alpha x^{-\alpha-1} dx \\ &= a_1^{2\alpha} B(1+\alpha, -\alpha) = a_1^{2\alpha} \frac{\pi}{\sin(-\alpha\pi)}.\end{aligned} \tag{1.113}$$

Starting from the following induction hypothesis

$$\int_0^\infty \frac{x^\alpha dx}{\prod_{i=1}^{m-1} (x+a_i^2)} = (-1)^{m-1} \frac{\pi}{\sin(\alpha\pi)} f[a_1^2, \dots, a_{m-1}^2], \quad f(x) = x^\alpha, \tag{1.114}$$

where $-1 < \alpha < m-2$, $\alpha \neq 0, 1, \dots, m-3$, we shall calculate the integral

$$I_m = \int_0^\infty \frac{x^\alpha dx}{\prod_{i=1}^m (x+a_i^2)}, \quad -1 < \alpha < m-1, \quad \alpha \neq 0, 1, \dots, m-2.$$

We treat three cases. If $-1 < \alpha < 0$, we use (1.113) and find that

$$\begin{aligned} I_m &= \sum_{i=1}^m \frac{1}{\prod_{j\neq i}(a_j^2 - a_i^2)} \int_0^\infty \frac{x^\alpha dx}{x+a_1^2} \\ &= \sum_{i=1}^m \frac{1}{\prod_{j\neq i}(a_j^2 - a_i^2)} a_i^{2\alpha} \frac{\pi}{\sin(-\alpha\pi)} \\ &= (-1)^m \frac{\pi}{\sin(\alpha\pi)} f[a_1^2, \dots, a_m^2], \; f(x) = x^\alpha. \end{aligned} \tag{1.115}$$

If $m-2 < \alpha < m-1$, we use the substitution $x = \frac{1}{t}$ and (1.115). More precisely, we have

$$\begin{aligned} I_m &= \int_0^\infty \frac{t^{m-\alpha-2}}{\prod_{i=1}^m (\frac{1}{a_i^2} + t)} dt \\ &= \frac{1}{\prod_{i=1}^m a_i^2} (-1)^m \frac{\pi}{\sin[(m-\alpha-2)\pi]} g[1/a_1^2, \dots, 1/a_m^2], \end{aligned} \tag{1.116}$$

where $g(x) = x^{m-\alpha-2}$. It is easy to check that the following relation is valid:

$$g[1/a_1^2, \dots, 1/a_m^2] = \sum_{i=1}^m \frac{\left(\prod_{j=1}^m a_j^2\right) a_i^{2\alpha}}{\prod_{j\neq i}(a_j^2 - a_i^2)}. \tag{1.117}$$

Now, relations (1.116), (1.117), and the formula $\sin[(m-\alpha-2)\pi]=(-1)^{m-1}\sin(\alpha\pi)$ imply (1.110).

On the other hand, if $0<\alpha<m-2$, we again use the induction hypothesis and obtain

$$\begin{aligned}
I_m &= \frac{1}{a_m^2-a_1^2}\left[\int_0^\infty \frac{x^\alpha dx}{\prod_{i=1}^{m-1}(x+a_i^2)}-\int_0^\infty \frac{x^\alpha dx}{\prod_{i=2}^{m}(x+a_i^2)}\right]\\
&= (-1)^m\frac{\pi}{\sin(\alpha\pi)}\frac{f[a_2^2,\ldots,a_m^2]-f[a_1^2,\ldots,a_{m-1}^2]}{a_m^2-a_1^2}\\
&= (-1)^m\frac{\pi}{\sin(\alpha\pi)}f[a_1^2,\ldots,a_m^2],\quad f(x)=x^\alpha.
\end{aligned}$$

At the end of the proof, it is necessary to consider the critical cases of the integrals

$$\int_0^\infty \frac{x^k dx}{\prod_{i=1}^{m}(x+a_i^2)},\quad k=0,1,\ldots,m-2.$$

Applying the classical calculus, we have

$$\begin{aligned}
\int_0^\infty \frac{x^k dx}{\prod_{i=1}^{m}(x+a_i^2)} &= \lim_{\alpha\to k}\int_0^\infty \frac{x^\alpha dx}{\prod_{i=1}^{m}(x+a_i^2)}\\
&= (-1)^m\pi\lim_{\alpha\to k}\frac{1}{\sin(\alpha\pi)}\sum_{i=1}^m\frac{a_i^{2\alpha}}{\prod_{j\neq i}(a_i^2-a_j^2)}\\
&= \frac{(-1)^m}{\cos(k\pi)}\sum_{i=1}^m\frac{a_i^{2k}\log a_i^2}{\prod_{j\neq i}(a_i^2-a_j^2)}=(-1)^{m+k}f_k[a_1^2,\ldots,a_m^2],
\end{aligned}$$

where $f_k(x)=x^k\log x$. That completes the proof. □

The previous lemma enables us to derive the following two-dimensional Hilbert-type inequalities with the kernel $K(x,y)=\prod_{i=1}^m(x+a_i^2y)^{-1}$, $a_i>0$, $i=1,\ldots,m$.

Corollary 1.8 *Let m be an integer, let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let α be real parameter such that $-1<\alpha<m-1$. Further, let*

$$L_2=(-1)^m\frac{\pi}{\sin(\alpha\pi)}f[a_1^2,\ldots,a_m^2],\quad f(x)=x^\alpha,$$

for $\alpha\neq 0,1,\ldots,m-2$, and

$$L_2=(-1)^{m+\alpha}f_\alpha[a_1^2,\ldots,a_m^2],\quad f_\alpha(x)=x^\alpha\log x,$$

for $\alpha=0,1,\ldots,m-2$. If $a_i>0$, $i=1,\ldots,m$, and $f_1,f_2:\mathbb{R}_+\to\mathbb{R}$, $f_1,f_2\neq 0$, are non-negative measurable functions, then the inequalities

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{f_1(x)f_2(y)}{(x+a_1^2y)\ldots(x+a_m^2y)}dxdy\\
&< L_2\left[\int_0^\infty x^{-1-\alpha p}f_1^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{-1-(m-\alpha-2)q}f_2^q(y)dy\right]^{\frac{1}{q}} \qquad (1.118)
\end{aligned}$$

and

$$\begin{aligned}&\int_0^\infty y^{(1-p)(-1-(m-\alpha-2)q)}\left[\int_0^\infty \frac{f_1(x)}{(x+a_1^2y)\dots(x+a_m^2y)}dx\right]^p dy\\&< L_2^p\int_0^\infty x^{-1-\alpha p}f_1^p(x)dx \qquad (1.119)\end{aligned}$$

hold and are equivalent. Moreover, the constant factors L_2 and L_2^p are the best possible in (1.118) *and* (1.119).

Proof. Exploiting inequalities (1.97) and (1.98), for $n=2$, with the kernel $K(x,y)=\prod_{i=1}^m(x+a_i^2y)^{-1}$ and parameters $p_1=p$, $p_2=q$, $\widetilde{A}_1=\alpha$, $\widetilde{A}_2=m-\alpha-2$, we see that it is enough to calculate the constant $L_2=k(\widetilde{A}_2)$. Utilizing substitution $u=\frac{1}{x}$, we have

$$L_2=k(m-\alpha-2)=\int_0^\infty \frac{u^{m-\alpha-2}du}{\prod_{i=1}^m(1+a_i^2u)}=\int_0^\infty \frac{x^\alpha dx}{\prod_{i=1}^m(x+a_i^2)},$$

and the result follows from Lemma 1.6. □

A class of Hilbert-type inequalities derived in the previous corollary involves the best possible constant factor

$$L_2=\int_0^\infty \frac{x^\alpha\,dx}{\prod_{i=1}^m\left(x+a_i^2\right)}=(-1)^m\frac{\pi}{\sin(\alpha\pi)}f\left[a_1^2,\dots,a_m^2\right], \qquad (1.120)$$

where $f(x)=x^\alpha$, $-1<\alpha<m-1$. In the sequel we study some other forms of expression (1.120), which will bring us to Shur polynomials.

It is well-known that the divided difference can be expressed via determinants:

$$f\left[a_1^2,\dots,a_m^2\right]=\frac{\det\begin{bmatrix} f\left(a_1^2\right) & f\left(a_2^2\right) & \cdots & f\left(a_m^2\right)\\ a_1^{2(m-2)} & a_2^{2(m-2)} & \cdots & a_m^{2(m-2)}\\ \vdots & \vdots & \vdots & \vdots\\ a_1^2 & a_2^2 & \cdots & a_m^2\\ 1 & 1 & \cdots & 1\end{bmatrix}}{\det\begin{bmatrix} a_1^{2(m-1)} & a_2^{2(m-1)} & \cdots & a_m^{2(m-1)}\\ a_1^{2(m-2)} & a_2^{2(m-2)} & \cdots & a_m^{2(m-2)}\\ \vdots & \vdots & \vdots & \vdots\\ a_1^2 & a_2^2 & \cdots & a_m^2\\ 1 & 1 & \cdots & 1\end{bmatrix}}. \qquad (1.121)$$

Calculating the Vandermonde determinant, we have

$$\det\begin{bmatrix} a_1^{2(m-1)} & a_2^{2(m-1)} & \cdots & a_m^{2(m-1)} \\ a_1^{2(m-2)} & a_2^{2(m-2)} & \cdots & a_m^{2(m-2)} \\ \vdots & \vdots & \vdots & \vdots \\ a_1^2 & a_2^2 & \cdots & a_m^2 \\ 1 & 1 & \cdots & 1 \end{bmatrix} = \det\begin{bmatrix} a_1^{m-1} & a_2^{m-1} & \cdots & a_m^{m-1} \\ a_1^{m-2} & a_2^{m-2} & \cdots & a_m^{m-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_1 & a_2 & \cdots & a_m \\ 1 & 1 & \cdots & 1 \end{bmatrix} \cdot \prod_{1\le i<j\le m} (a_i+a_j). \tag{1.122}$$

Setting $m=3$ and $\alpha=\frac{1}{2}$, we have (see also [133]):

$$\int_0^\infty \frac{x^{\frac{1}{2}}\,dx}{\left(x+a_1^2\right)\left(x+a_2^2\right)\left(x+a_3^2\right)} = \frac{\pi}{(a_1+a_2)(a_1+a_3)(a_2+a_3)}.$$

Similarly, for $m=4$ and $\alpha=\frac{1}{2}$, by calculating the integral in (1.120) or resolving the above determinants, we have

$$\int_0^\infty \frac{x^{\frac{1}{2}}\,dx}{\prod_{i=1}^4\left(x+a_i^2\right)} = \frac{\sum_{i=1}^4 a_i}{\prod_{1\le i<j\le 4}(a_i+a_j)}\pi.$$

Suppose that $\alpha=\frac{2l-3}{2}$, $l=2,\ldots,m$ (for $l=1$ see below). The numerator of (1.121) is for $l<m$ equal to

$$\det\begin{bmatrix} a_1^{2l-3} & a_2^{2l-3} & \cdots & a_m^{2l-3} \\ a_1^{2(m-2)} & a_2^{2(m-2)} & \cdots & a_m^{2(m-2)} \\ \vdots & \vdots & \vdots & \vdots \\ a_1^2 & a_2^2 & \cdots & a_m^2 \\ 1 & 1 & \cdots & 1 \end{bmatrix} = (-1)^{m-l}\det\begin{bmatrix} a_1^{2(m-2)} & a_2^{2(m-2)} & \cdots & a_m^{2(m-2)} \\ \vdots & \vdots & \vdots & \vdots \\ a_1^{2(l-1)} & a_2^{2(l-1)} & \cdots & a_m^{2(l-1)} \\ a_1^{2l-3} & a_2^{2l-3} & \cdots & a_m^{2l-3)} \\ a_1^{2(l-2)} & a_2^{2(l-2)} & \cdots & a_m^{2(l-2)} \\ \vdots & \vdots & \vdots & \vdots \\ a_1^2 & a_2^2 & \cdots & a_m^2 \\ 1 & 1 & \cdots & 1 \end{bmatrix}. \tag{1.123}$$

Using $\sin\frac{2l-3}{2}\pi=(-1)^l$, we have

$$L_2=\int_0^\infty \frac{x^{\frac{2l-3}{2}}\,dx}{\prod_{i=1}^m\left(x+a_i^2\right)} = \frac{\det\begin{bmatrix} a_1^{2(m-2)} & a_2^{2(m-2)} & \cdots & a_m^{2(m-2)} \\ \vdots & \vdots & \vdots & \vdots \\ a_1^{2(l-1)} & a_2^{2(l-1)} & \cdots & a_m^{2(l-1)} \\ a_1^{2l-3} & a_2^{2l-3} & \cdots & a_m^{2l-3)} \\ a_1^{2(l-2)} & a_2^{2(l-2)} & \cdots & a_m^{2(l-2)} \\ \vdots & \vdots & \vdots & \vdots \\ a_1^2 & a_2^2 & \cdots & a_m^2 \\ 1 & 1 & \cdots & 1 \end{bmatrix}}{\mathrm{V}(a_1,a_2,\ldots,a_m)\cdot\prod_{1\le i<j\le m}(a_i+a_j)}\pi, \tag{1.124}$$

where $\mathrm{V}(a_1,a_2,\ldots,a_m)$ denotes the Vandermonde determinant.

Recall that the Schur polynomial of a given integer partition $d = d_1 + d_2 + \cdots + d_m$, $d_1 \geq d_2 \geq \cdots \geq d_m \geq 0$, is defined by

$$S_{(d_1,d_2,\ldots,d_m)}(x_1,x_2,\ldots,x_m) = \frac{\det\begin{bmatrix} x_1^{d_1+m-1} & x_2^{d_1+m-1} & \cdots & x_m^{d_1+m-1} \\ x_1^{d_2+m-2} & x_2^{d_2+m-2} & \cdots & x_m^{d_2+m-2} \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{d_m} & x_2^{d_m} & \cdots & x_m^{d_m} \end{bmatrix}}{\det\begin{bmatrix} x_1^{m-1} & x_2^{m-1} & \cdots & x_m^{m-1} \\ x_1^{m-2} & x_2^{m-2} & \cdots & x_m^{m-2} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}}. \tag{1.125}$$

The Schur polynomial can be expanded as a sum of monomials

$$S_{(d_1,d_2,\ldots,d_m)}(x_1,x_2,\ldots,x_m) = \sum_T x_1^{t_1}\cdots x_m^{t_m}, \tag{1.126}$$

where the summation is over all semi-standard Young tableaux T of shape $(d_1,d_2,\ldots,d_m)$. The exponents $t_1,\ldots,t_m$ give the weight of T, in which each t_i counts the occurrences of the number i in T (see [3] and [89]).

It is obvious from (1.121) (the case $l = 2m-3$), (1.124) and (1.125) that

$$\int_0^\infty \frac{x^{\frac{2m-3}{2}}\,dx}{\prod_{i=1}^m (x+a_i^2)} = \frac{S_{(m-2,m-2,m-3,\ldots,1,0)}(a_1,a_2,\ldots,a_m)}{\prod_{i<j}(a_i+a_j)}\pi,$$

$$\int_0^\infty \frac{x^{\frac{2l-3}{2}}\,dx}{\prod_{i=1}^m (x+a_i^2)} = \frac{S_{(m-3,m-4\ldots,l-2,l-2,l-2,\ldots 1,0)}(a_1,a_2,\ldots,a_m)}{\prod_{i<j}(a_i+a_j)}\pi,\ l = 3,\ldots,m-1,$$

$$\int_0^\infty \frac{x^{\frac{1}{2}}\,dx}{\prod_{i=1}^m (x+a_i^2)} = \frac{S_{(m-3,m-4,\ldots 1,0,0,0)}(a_1,a_2,\ldots,a_m)}{\prod_{i<j}(a_i+a_j)}\pi,$$

where $d = \frac{(m-3)(m-2)}{2} + 2(l-2)$, $l = 2,\ldots,m$, is degree of the obtained Schur polynomials.

To illustrate the above results we provide the following examples:

(i) For $m = 2$ the only admissible case is $\alpha = \frac{1}{2}$. It follows:

$$\int_0^\infty \frac{x^{\frac{1}{2}}\,dx}{\prod_{i=1}^2 (x+a_i^2)} = \frac{S_{(0,0)}(a_1,a_2)}{a_1+a_2}\pi = \frac{\pi}{a_1+a_2}.$$

(ii) For $m = 3$, the admissible cases are $\alpha = \frac{1}{2}$ and $\alpha = \frac{3}{2}$, that is, we have

$$\int_0^\infty \frac{x^{\frac{1}{2}}\,dx}{\prod_{i=1}^3 (x+a_i^2)} = \frac{S_{(0,0,0)}(a_1,a_2,a_3)}{\prod_{i<j}(a_i+a_j)}\pi = \frac{\pi}{\prod_{i<j}(a_i+a_j)},$$

$$\int_0^\infty \frac{x^{\frac{3}{2}}\,dx}{\prod_{i=1}^3 (x+a_i^2)} = \frac{S_{(1,1,0)}(a_1,a_2,a_3)}{\prod_{i<j}(a_i+a_j)}\pi = \frac{\sum_{i<j} a_i a_j}{\prod_{i<j}(a_i+a_j)}\pi.$$

(iii) For $m = 4$, the admissible cases are $\alpha = \frac{1}{2}$, $\alpha = \frac{3}{2}$, and $\alpha = \frac{5}{2}$, that is,

$$\int_0^\infty \frac{x^{\frac{1}{2}}\,dx}{\prod_{i=1}^4 \left(x+a_i^2\right)} = \frac{S_{(1,0,0,0)}\left(a_1,a_2,a_3,a_4\right)}{\prod_{i<j}\left(a_i+a_j\right)}\pi = \frac{\sum_{i=1}^4 a_i}{\prod_{i<j}\left(a_i+a_j\right)}\pi,$$

$$\int_0^\infty \frac{x^{\frac{3}{2}}\,dx}{\prod_{i=1}^4 \left(x+a_i^2\right)} = \frac{S_{(1,1,1,0)}\left(a_1,a_2,a_3,a_4\right)}{\prod_{i<j}\left(a_i+a_j\right)}\pi = \frac{\sum_{i<j<k} a_i a_j a_k}{\prod_{i<j}\left(a_i+a_j\right)}\pi,$$

$$\begin{aligned}\int_0^\infty \frac{x^{\frac{5}{2}}\,dx}{\prod_{i=1}^4 \left(x+a_i^2\right)} &= \frac{S_{(2,2,1,0)}\left(a_1,a_2,a_3,a_4\right)}{\prod_{i<j}\left(a_i+a_j\right)}\pi \\ &= \frac{\sum_{i<j<k} a_i^2 a_j^2 a_k + a_i^2 a_j a_k^2 + a_i a_j^2 a_k^2 + 2\sum_{i<j<k<l} a_i^2 a_j a_k a_l}{\prod_{i<j}\left(a_i+a_j\right)}\pi \\ &\quad + \frac{a_i a_j^2 a_k a_l + a_i a_j a_k^2 a_l + a_i a_j a_k a_l^2}{\prod_{i<j}\left(a_i+a_j\right)}\pi.\end{aligned}$$

The above method doesn't work in the case $\alpha = -\frac{1}{2}$. We proceed as follows. By virtue of the substitution $x = \frac{1}{t}$, using the case $l = 2m-3$ and the basic properties of determinants, we have:

$$\begin{aligned}&\int_0^\infty \frac{x^{-\frac{1}{2}}\,dx}{\prod_{i=1}^m \left(x+a_i^2\right)} \\ &= \frac{1}{\prod_{i=1}^m a_i^2}\int_0^\infty \frac{t^{\frac{2m-3}{2}}\,dt}{\prod_{i=1}^m \left(t+\frac{1}{a_i^2}\right)} = \frac{1}{\prod_{i=1}^m a_i^2}\,\frac{S_{(m-2,m-2,m-3,\dots,1,0)}\left(\frac{1}{a_1},\dots,\frac{1}{a_m}\right)}{\prod_{i<j}\left(\frac{1}{a_i}+\frac{1}{a_j}\right)}\pi \\ &= \frac{1}{\prod_{i=1}^m a_i^2}\,\frac{\prod_{i=1}^m \left(\frac{1}{a_i}\right)^{2m-3}}{\prod_{i=1}^m \left(\frac{1}{a_i}\right)^{m-1}}\,\frac{S_{(m-2,m-3,\dots,1,0,0)}\left(a_1,\dots,a_m\right)}{\frac{1}{\prod_{i=1}^m a_i^{m-1}}\prod_{i<j}\left(a_i+a_j\right)}\pi \\ &= \frac{1}{\prod_{i=1}^m a_i}\,\frac{S_{(m-2,m-3,\dots,1,0,0)}\left(a_1,\dots,a_m\right)}{\prod_{i<j}\left(a_i+a_j\right)}\pi.\end{aligned}$$

Remark 1.27 A general form of multidimensional Hilbert-type inequality (Subsection 1.6.1) is established in [156], while the application to homogeneous kernels and the existence of the best possible constant factors (Subsection 1.6.2) is derived in [111]. Most of the examples in Subsection 1.6.3 are taken from [113], while inequalities including Shur polynomials can be found in [112]. For some related results, the reader can also consult the following papers: [19], [23], [37], [52] [128], [152], and [156].

Chapter 2

Hilbert-type inequalities with non-conjugate exponents

The previous chapter was dedicated to a unified treatment of Hilbert-type inequalities and numerous applications of general results. All these results included conjugate exponents.

The question is whether it is possible to establish the corresponding inequalities related to Hilbert-type inequalities where the exponents are not conjugate. The answer to that question appeared to be true. This problem was dealt by some famous mathematicians, such as F. F. Bonsall, G. H. Hardy, V. Levin, J. Littlewood, G. Pólya, in the first half of the twentieth century, an later, by E. K. Godunova. This bring us to the concept of non-conjugate parameters.

Suppose that p and q are real parameters, such that

$$p > 1,\ q > 1,\ \frac{1}{p} + \frac{1}{q} \geq 1, \tag{2.1}$$

and let $p' = \frac{p}{p-1}$ and $q' = \frac{q}{q-1}$ respectively be their conjugate exponents, that is, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Further, define

$$\lambda = \frac{1}{p'} + \frac{1}{q'} \tag{2.2}$$

and observe that $0 < \lambda \leq 1$ holds for all p and q as in (2.1). In particular, equality $\lambda = 1$ holds in (2.2) if and only if $q = p'$, that is, only if p and q are mutually conjugate. Otherwise, we have $0 < \lambda < 1$, and such parameters p and q will be referred to as non-conjugate exponents.

Considering p, q, and λ as in (2.1) and (2.2), Hardy, Littlewood, and Pólya [33], proved that there exists a constant $C_{p,q}$, dependent only on the parameters p and q, such that the following Hilbert-type inequality holds for all non-negative functions $f \in L^p(\mathbb{R}_+)$ and $g \in L^q(\mathbb{R}_+)$:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\,dxdy \leq C_{p,q} \|f\|_{L^p(\mathbb{R}_+)} \|g\|_{L^q(\mathbb{R}_+)}. \tag{2.3}$$

However, the original proof did not bring any information about the value of the best possible constant $C_{p,q}$. That drawback was improved by Levin [79], who obtained an explicit upper bound for $C_{p,q}$,

$$C_{p,q} \leq \left(\pi \operatorname{cosec} \frac{\pi}{\lambda p'}\right)^\lambda. \tag{2.4}$$

This was an interesting result, since the right-hand side of (2.4) reduces to the previously known sharp constant $\pi \operatorname{cosec}(\pi/p')$ when the exponents p and q are conjugate (see Theorem 1.2, Chapter 1). A simpler proof of (2.4), based on a single application of the Hölder inequality, was given later by F. F. Bonsall [9].

In spite of its trivial appearance, Bonsall's idea was useful for investigating the inequalities for multiple integrals involving non-conjugate parameters. So, he obtained the following inequality for $n = 3$: Let $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \geq 1$ with $p_i > 1$, $\frac{1}{p_i} + \frac{1}{p_i'} = 1$, $i = 1,2,3$, and $\lambda = \frac{1}{2}\left(\frac{1}{p_1'} + \frac{1}{p_2'} + \frac{1}{p_3'}\right)$. Then,

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{f(x)g(y)h(z)}{(x+y+z)^{2\lambda}}\,dxdydz \leq k \|f\|_{L^{p_1}(\mathbb{R}_+)} \|g\|_{L^{p_2}(\mathbb{R}_+)} \|h\|_{L^{p_3}(\mathbb{R}_+)}, \tag{2.5}$$

with an explicit upper bound expressed in terms of the usual Gamma function:

$$k \leq \left[\Gamma\left(\frac{1}{\lambda p_1'}\right) \Gamma\left(\frac{1}{\lambda p_2'}\right) \Gamma\left(\frac{1}{\lambda p_3'}\right)\right]^\lambda.$$

Although Bonsall established the concept of n non-conjugate parameters, there were no results in that direction.

Moreover, in the same paper, with p, q and λ as in (2.1) and (2.2), Bonsall proved another interesting Hilbert-type inequality,

$$\begin{aligned} \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\,dxdy &\leq B^\lambda\left(\frac{1}{p'}, \frac{1}{q'}\right) \| f \|_{L^p(\mathbb{R}_+)}^{\frac{p}{q'}} \| g \|_{L^q(\mathbb{R}_+)}^{\frac{q}{p'}} \\ &\times \left[\int_0^\infty \int_0^\infty \frac{x^{\frac{1}{p'}} y^{\frac{1}{q'}}}{(x+y)^\lambda} f^p(x) g^q(y)\,dxdy\right]^{1-\lambda}, \end{aligned} \tag{2.6}$$

with the best possible constant factor $B^\lambda\left(\frac{1}{p'}, \frac{1}{q'}\right)$, expressed in terms of the usual Beta function.

Remark 2.1 In the above relations, $L^p(\mathbb{R}_+)$ denotes the Lebesgue space consisting of all measurable functions $f:\mathbb{R}_+\to\mathbb{R}$ with a finite norm $\|f\|_{L^p(\mathbb{R}_+)}=[\int_0^\infty |f(x)|^p dx]^{\frac{1}{p}}$. More generally, when considering a measure space Ω with a positive σ-finite measure μ, the corresponding Lebesgue space will be denoted by $L^p(\mu)$, or simply L^p.

Although inequality (2.6) involves the best possible constant factor, there is still no evidence that the constant factors in inequalities (2.3) and (2.5) are the best possible. This problem seems to be very hard and remains still open. Hence, the problem of the best possible constant factors will not be considered in this chapter.

The main objective of this chapter is to extend Hilbert-type inequalities with conjugate exponents to the setting of non-conjugate exponents. First we derive the general forms of Hilbert-type inequalities with non-conjugate exponents in two-dimensional, and later on, in multidimensional case. Accordingly, all results from the previous chapter can be extended to this non-conjugate setting. In addition, we are going to consider some particular non-homogeneous kernels, yielding the constant factors expressed in terms of generalized hypergeometric functions. At the end of this chapter, we study some particular operators between the weighted Lebesgue spaces, that naturally arise from the established Hardy-Hilbert-type inequalities.

2.1 General form

To provide a basis for main results, in this section we first discuss general inequalities of the Hilbert-type and Hardy-Hilbert-type with non-conjugate exponents. These equivalent relations are stated and proved in the following theorem.

Theorem 2.1 *Let p, q, and λ be real parameters as in* (2.1) *and* (2.2)*, and let Ω_1 and Ω_2 be measure spaces with positive σ-finite measures μ_1 and μ_2 respectively. Let K be a non-negative measurable function on $\Omega_1\times\Omega_2$, φ a measurable, a.e. positive function on Ω_1, and ψ a measurable, a.e. positive function on Ω_2. If the functions F on Ω_1 and G on Ω_2 are defined by*

$$F(x)=\left[\int_{\Omega_2} K(x,y)\psi^{-q'}(y)\,d\mu_2(y)\right]^{\frac{1}{q'}},\quad x\in\Omega_1, \tag{2.7}$$

and

$$G(y)=\left[\int_{\Omega_1} K(x,y)\varphi^{-p'}(x)\,d\mu_1(x)\right]^{\frac{1}{p'}},\quad y\in\Omega_2, \tag{2.8}$$

then for all non-negative measurable functions f on Ω_1 and g on Ω_2 the inequalities

$$\int_{\Omega_1}\int_{\Omega_2} K^{\lambda}(x,y)f(x)g(y)\,d\mu_1(x)d\mu_2(y)\leq \|\varphi F f\|_{L^p(\mu_1)}\|\psi G g\|_{L^q(\mu_2)} \tag{2.9}$$

and

$$\left\{\int_{\Omega_2}\left[(\psi G)^{-1}(y)\int_{\Omega_1}K^{\lambda}(x,y)f(x)\,d\mu_1(x)\right]^{q'}d\mu_2(y)\right\}^{\frac{1}{q'}} \le \|\varphi F f\|_{L^p(\mu_1)} \tag{2.10}$$

hold and are equivalent.

Proof. We prove inequality (2.9) first. Let K, φ, and ψ be as in the statement of Theorem 2.1 and let f and g be arbitrary non-negative measurable functions on Ω_1 and Ω_2 respectively. Since $\frac{1}{q'}+\frac{1}{p'}+1-\lambda=1$, the left-hand side of relation (2.9) can be written as

$$\begin{aligned}
&\int_{\Omega_1}\int_{\Omega_2}K^{\lambda}(x,y)f(x)g(y)\,d\mu_1(x)d\mu_2(y)\\
&=\int_{\Omega_1}\int_{\Omega_2}\left[K(x,y)\psi^{-q'}(y)(\varphi^p F^{p-q'}f^p)(x)\right]^{\frac{1}{q'}}\left[K(x,y)\varphi^{-p'}(x)(\psi^q G^{q-p'}g^q)(y)\right]^{\frac{1}{p'}}\\
&\quad\times\left[(\varphi F f)^p(x)\,(\psi G g)^q(y)\right]^{1-\lambda}d\mu_1(x)d\mu_2(y).
\end{aligned}\tag{2.11}$$

Now, utilizing the Hölder inequality, either with the parameters $q', p', \frac{1}{1-\lambda} > 1$ in the case of non-conjugate exponents p and q, or with the parameters p and p' when $q' = p$ (that is, when $\lambda = 1$), and then applying the Fubini theorem, we obtain that the right-hand side of (2.11) is not greater than

$$\begin{aligned}
&\left\{\int_{\Omega_1}\left[\int_{\Omega_2}K(x,y)\psi^{-q'}(y)\,d\mu_2(y)\right](\varphi^p F^{p-q'}f^p)(x)\,d\mu_1(x)\right\}^{\frac{1}{q'}}\\
&\qquad\times\left\{\int_{\Omega_2}\left[\int_{\Omega_1}K(x,y)\varphi^{-p'}(x)\,d\mu_1(x)\right](\psi^q G^{q-p'}g^q)(y)\,d\mu_2(y)\right\}^{\frac{1}{p'}}\\
&\qquad\times\left[\int_{\Omega_1}(\varphi F f)^p(x)\,d\mu_1(x)\right]^{1-\lambda}\left[\int_{\Omega_2}(\psi G g)^q(y)\,d\mu_2(y)\right]^{1-\lambda}\\
&=\left[\int_{\Omega_1}(\varphi F f)^p(x)\,d\mu_1(x)\right]^{\frac{1}{q'}+1-\lambda}\left[\int_{\Omega_2}(\psi G g)^q(y)\,d\mu_2(y)\right]^{\frac{1}{p'}+1-\lambda}\\
&=\|\varphi F f\|_{L^p(\mu_1)}\|\psi G g\|_{L^q(\mu_2)},
\end{aligned}$$

so (2.9) is proved. The further step is to prove that (2.9) implies (2.10) to hold for all non-negative measurable functions f on Ω_1. In particular, for any such f and the function g defined by

$$g(y) = (\psi G)^{-q'}(y)\left[\int_{\Omega_1}K^{\lambda}(x,y)f(x)\,d\mu_1(x)\right]^{\frac{q'}{q}},\ y \in \Omega_2,$$

applying the Fubini theorem, the left-hand side of (2.9) becomes

$$\begin{aligned}
L &= \int_{\Omega_1}\int_{\Omega_2}K^{\lambda}(x,y)f(x)(\psi G)^{-q'}(y)\left[\int_{\Omega_1}K^{\lambda}(x,y)f(x)\,d\mu_1(x)\right]^{\frac{q'}{q}}d\mu_1(x)d\mu_2(y)\\
&= \int_{\Omega_2}\left[(\psi G)^{-1}(y)\int_{\Omega_1}K^{\lambda}(x,y)f(x)\,d\mu_1(x)\right]^{q'}d\mu_2(y),
\end{aligned}$$

that is, the integral on the left-hand side of (2.10), while on the right-hand side of (2.9) we have

$$
\begin{aligned}
R &= \|\varphi F f\|_{L^p(\mu_1)} \left\{ \int_{\Omega_2} (\psi G)^{q(1-q')}(y) \left[\int_{\Omega_1} K^{\lambda}(x,y) f(x)\, d\mu_1(x) \right]^{q'} d\mu_2(y) \right\}^{\frac{1}{q}} \\
&= \|\varphi F f\|_{L^p(\mu_1)} L^{\frac{1}{q}}.
\end{aligned}
$$

Hence,

$$
L \leq \|\varphi F f\|_{L^p(\mu_1)} L^{\frac{1}{q}},
$$

which directly yields (2.10), so the implication (2.9) $\Rightarrow$ (2.10) is proved. Conversely, by using the Hölder inequality for the conjugate exponents q and q', together with the relation (2.10), for arbitrary $f, g \geq 0$ we have

$$
\begin{aligned}
&\int_{\Omega_1} \int_{\Omega_2} K^{\lambda}(x,y) f(x) g(y)\, d\mu_1(x) d\mu_2(y) \\
&= \int_{\Omega_2} (\psi G g)(y) \left[(\psi G)^{-1}(y) \int_{\Omega_1} K^{\lambda}(x,y) f(x)\, d\mu_1(x) \right] d\mu_2(y) \\
&\leq \|\psi G g\|_{L^q(\mu_2)} \left\{ \int_{\Omega_2} \left[(\psi G)^{-1}(y) \int_{\Omega_1} K^{\lambda}(x,y) f(x)\, d\mu_1(x) \right]^{q'} d\mu_2(y) \right\}^{\frac{1}{q'}} \\
&\leq \|\varphi F f\|_{L^p(\mu_1)} \|\psi G g\|_{L^q(\mu_2)}.
\end{aligned}
$$

Thus, (2.10) implies (2.9), so these inequalities are equivalent. The proof is now completed. □

Remark 2.2 The sign of inequality in (2.9) depends only on the parameters p', q', and λ, since the crucial step in proving this relation was in applying the Hölder inequality. Therefore, we can consider exponents which provide the reverse inequality in (2.9). In particular, if the parameters p and q from Theorem 2.1 are such that

$$
p < 0,\ 0 < q < 1,\ \frac{1}{p} + \frac{1}{q} \leq 1, \tag{2.12}
$$

and λ is defined by (2.2), we have $0 < p' < 1$, $q' < 0$, and $1 - \lambda \leq 0$, so the inequality in (2.9) is reversed as a direct consequence of the reversed Hölder inequality. The same result is achieved also with the parameters p and q satisfying

$$
0 < p < 1,\ q < 0,\ \frac{1}{p} + \frac{1}{q} \leq 1, \tag{2.13}
$$

since from (2.13) we obtain $p' < 0$, $0 < q' < 1$, and $1 - \lambda \leq 0$. Moreover, by using the same arguments, parameters $p, q \in (0,1)$ give another sufficient condition for the reverse inequality in (2.9). In that case we have $p', q' < 0$, and $1 - \lambda > 0$.

Remark 2.3 Equality in (2.9) holds if and only if it holds in the Hölder inequality, that is, if the functions $K\psi^{-q'}\varphi^p F^{p-q'} f^p$, $K\varphi^{-p'}\psi^q G^{q-p'} g^q$, and $(\varphi F f)^p(\psi G g)^q$ are effectively proportional on $\Omega_1\times\Omega_2$. Of course, this trivially happens if at least one of the functions involved in the left-hand side of (2.9) is the zero-function. To discuss other non-trivial cases of equality in (2.9), we can without loss of generality assume that the functions K, f, and g are positive. Under such assumptions, equality in (2.9) occurs if and only if there exist positive real constants α_1, β_1, and γ_1, such that the relations

$$\begin{aligned}\alpha_1 K(x,y)\psi^{-q'}(y)(\varphi^p F^{p-q'} f^p)(x) &= \beta_1 K(x,y)\varphi^{-p'}(x)(\psi^q G^{q-p'} g^q)(y)\\ &= \gamma_1\,(\varphi F f)^p\,(x)\,(\psi G g)^q\,(y)\end{aligned}$$

hold for a. e. $(x,y)\in\Omega_1\times\Omega_2$. These equalities can be written in a more suitable form, as

$$\alpha_1(\varphi^{p+p'}F^{p-q'}f^p)(x) = \beta_1(\psi^{q+q'}G^{q-p'}g^q)(y), \quad \text{for a.e. } (x,y)\in\Omega_1\times\Omega_2, \tag{2.14}$$

and

$$\alpha_1 K(x,y) = \gamma_1 F^{q'}(x)(\psi^{q+q'}G^q g^q)(y), \quad \text{for a.e. } (x,y)\in\Omega_1\times\Omega_2. \tag{2.15}$$

Since the left-hand side of (2.14) depends only on $x\in\Omega_1$, while the right-hand side is a single-variable function of $y\in\Omega_2$, (2.14) holds only if

$$\varphi^{p+p'}F^{p-q'}f^p = \alpha^p = const. \quad \text{a.e. on } \Omega_1$$

and

$$\psi^{q+q'}G^{q-p'}g^q = \beta^p = const. \quad \text{a.e. on } \Omega_2,$$

for some positive real constants α and β. Taking into account $1+\frac{p'}{p}=p'$ and $1+\frac{q'}{q}=q'$, these identities can be finally transformed to

$$f = \alpha\varphi^{-p'}F^{\frac{q'}{p}-1} \quad \text{a.e. on } \Omega_1 \ \text{ and } \ g = \beta\psi^{-q'}G^{\frac{p'}{q}-1} \quad \text{a.e. on } \Omega_2. \tag{2.16}$$

Moreover, combining (2.16) with (2.15), we obtain

$$K = \gamma F^{q'}G^{p'} \quad \text{a.e. on } \Omega_1\times\Omega_2, \tag{2.17}$$

for some positive constant γ. Therefore, we proved that the conditions (2.16) and (2.17) are necessary and sufficient for equality in (2.9). Moreover, it is clear that the equality in (2.10) holds only if it holds in (2.9).

As an example of the function K which fulfills (2.17), here we mention

$$K(x,y) = \frac{\varphi^{p'}(x)\psi^{q'}(y)}{\mu_1(\Omega_1)\mu_2(\Omega_2)}, \quad (x,y)\in\Omega_1\times\Omega_2,$$

where the sets Ω_1 and Ω_2 are such that $\mu_1(\Omega_1),\mu_2(\Omega_2)<\infty$ and the functions φ and ψ are arbitrary, as in Theorem 2.1. In particular, in this setting we have

$$F = \mu_1(\Omega_1)^{-\frac{1}{q'}}\varphi^{\frac{p'}{q'}} \quad \text{and} \quad G = \mu_2(\Omega_2)^{-\frac{1}{p'}}\psi^{\frac{q'}{p'}},$$

so K fulfills (2.17) with $\gamma=1$. Equality in (2.9) is attained for $f=\alpha\varphi^{-1-\frac{p'}{q'}}$ and $g=\beta\psi^{-1-\frac{q'}{p'}}$, where α and β are positive constants.

Remark 2.4 Considering the case of conjugate exponents, that is, when $q = p'$ and $\lambda = 1$, Theorem 2.1 reduces to Theorem 1.9 from Section 1.2. In other words, Theorem 2.1 is the non-conjugate extension of Theorem 1.9. This extension was established recently in [16].

2.2 The case of a homogeneous kernel

Similarly as in Section 1.3, we apply here general results from Theorem 2.1 to non-negative homogeneous functions $K : \Omega \subseteq \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ with a negative degree of homogeneity. Recall that $K : \Omega \subseteq \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is assumed to be homogeneous of degree $-s$, $s > 0$, such that $k(\alpha) = \int_0^\infty K(1,u)u^{-\alpha}du < \infty$ for $1 - s < \alpha < 1$ (see (1.20), Section 1.3).

In this way all results from Section 1.3 can be extended to the case of non-conjugate exponents. For example, non-conjugate version of Theorem 1.11 reads as follows.

Theorem 2.2 *Let p, q, and λ be as in* (2.1) *and* (2.2)*, and let $K : (a,b) \times (a,b) \to \mathbb{R}$ be a non-negative homogeneous function of degree $-s$, $s > 0$, strictly decreasing in both arguments. Further, suppose that A_1 and A_2 are real parameters such that $A_1 \in \left(\frac{1-s}{p'}, \frac{1}{p'}\right)$, $A_2 \in \left(\frac{1-s}{q'}, \frac{1}{q'}\right)$. If the functions φ_1 and φ_2 are defined as in the statement of Theorem* 1.11*, then for all non-negative measurable functions f and g on (a,b) the inequalities*

$$\begin{aligned}
&\int_a^b \int_a^b K^\lambda(x,y) f(x) g(y) dx dy \\
&\leq \left[\int_a^b \left(k(q'A_2) - \varphi_1(q'A_2, x)\right)^{\frac{p}{q'}} x^{\frac{p}{q'}(1-s)+p(A_1-A_2)} f^p(x) dx\right]^{\frac{1}{p}} \\
&\quad \times \left[\int_a^b \left(k(2-s-p'A_1) - \varphi_2(2-s-p'A_1, y)\right)^{\frac{q}{p'}} y^{\frac{q}{p'}(1-s)+q(A_2-A_1)} g^q(y) dy\right]^{\frac{1}{q}}
\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}
&\left[\int_a^b y^{\frac{q'}{p'}(s-1)+q'(A_1-A_2)} \left(k(2-s-p'A_1) - \varphi_2(2-s-p'A_1, y)\right)^{-\frac{q'}{p'}} \right. \\
&\qquad \left. \times \left(\int_a^b K^\lambda(x,y) f(x) dx\right)^{q'} dy\right]^{\frac{1}{q'}} \\
&\leq \left[\int_a^b \left(k(q'A_2) - \varphi_1(q'A_2, x)\right)^{\frac{p}{q'}} x^{\frac{p}{q'}(1-s)+p(A_1-A_2)} f^p(x) dx\right]^{\frac{1}{p}}
\end{aligned} \tag{2.19}$$

hold and are equivalent.

Proof. In order to prove (2.18) set $\varphi(x)=x^{A_1}$ and $\psi(y)=y^{A_2}$ in general inequality (2.9). Taking into account the substitution $u=\frac{y}{x}$, we obtain

$$\begin{aligned}&\int_a^b\int_a^b K^\lambda(x,y)f(x)g(y)dxdy\\&\le\left[\int_a^b x^{\frac{p}{q'}(1-s)+p(A_1-A_2)}\left(\int_{\frac{a}{x}}^{\frac{b}{x}}K(1,u)u^{-q'A_2}du\right)^{\frac{p}{q'}}f^p(x)dx\right]^{\frac{1}{p}}\\&\times\left[\int_a^b y^{\frac{q}{p'}(1-s)+q(A_2-A_1)}\left(\int_{\frac{y}{b}}^{\frac{y}{a}}K(1,u)u^{p'A_1+s-2}du\right)^{\frac{q}{p'}}g^q(y)dy\right]^{\frac{1}{q}}.\end{aligned}$$

Now, the rest of the proof follows the same lines as the proof of Theorem 1.11. □

Remark 2.5 According to Remark 2.2, we discuss here conditions under which the reverse inequalities in Theorem 2.2 are fulfilled. Firstly, if $0<p<1$, $0<q<1$ and K is as in the statement of Theorem 2.2, then the reverse inequalities in (2.9) and (2.10) hold. On the other hand, if the conditions (2.12) are satisfied, then the reverse inequalities in Theorem 2.2 are valid if $a=0$ and K is strictly increasing in first argument and strictly decreasing in second argument, or if $b=\infty$ and K is strictly decreasing in first argument and strictly increasing in second argument. The remaining case (2.13) which also provides reverse inequalities is analyzed similarly. Observe that in the case of reversed inequalities we have to adjust the intervals for parameters A_1 and A_2.

Of course, the most important case of Theorem 2.2 is with integrals over $\mathbb{R}_+$, that is, when $a=0$ and $b=\infty$. The corresponding equivalent Hilbert-type and Hardy-Hilbert-type inequalities are given in the following corollary.

Corollary 2.1 *Assume that p, q, and λ are as in* (2.1) *and* (2.2)*, and $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ is a non-negative homogeneous function of degree $-s$, $s>0$. Then the inequalities*

$$\begin{aligned}&\int_0^\infty\int_0^\infty K^\lambda(x,y)f(x)g(y)dxdy\\&\le L'\left[\int_0^\infty x^{\frac{p}{q'}(1-s)+p(A_1-A_2)}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{\frac{q}{p'}(1-s)+q(A_2-A_1)}g^q(y)dy\right]^{\frac{1}{q}}\end{aligned}\tag{2.20}$$

and

$$\begin{aligned}&\left[\int_0^\infty y^{\frac{q'}{p'}(s-1)+q'(A_1-A_2)}\left(\int_0^\infty K^\lambda(x,y)f(x)dx\right)^{q'}dy\right]^{\frac{1}{q'}}\\&\le L'\left[\int_0^\infty x^{\frac{p}{q'}(1-s)+p(A_1-A_2)}f^p(x)dx\right]^{\frac{1}{p}}\end{aligned}\tag{2.21}$$

hold for all parameters $A_1\in\left(\frac{1-s}{p'},\frac{1}{p'}\right)$, $A_2\in\left(\frac{1-s}{q'},\frac{1}{q'}\right)$, and for all non-negative measurable functions f and g on $\mathbb{R}_+$, where $L'=k^{\frac{1}{q'}}(q'A_2)k^{\frac{1}{p'}}(2-s-p'A_1)$. Moreover, these inequalities are equivalent. In addition, reverse inequalities are valid under conditions appearing in Remark 2.2.

Remark 2.6 Considering the above Corollary 2.1, equipped with the kernel $K(x,y) = (x+y)^{-s}$, $s>0$, the constant on the right-hand sides of (2.20) and (2.21) is expressed in terms of the Beta function, i.e.

$$L' = B^{\frac{1}{q'}}(1-q'A_2, s+q'A_2-1)B^{\frac{1}{p'}}(1-p'A_1, s+p'A_1-1).$$

For the kernel $K(x,y) = \max\{x,y\}^{-s}$, $s>0$, this constant reads

$$L' = \frac{s^{\lambda}}{(1-q'A_2)^{\frac{1}{q'}}(1-p'A_1)^{\frac{1}{p'}}(s+q'A_2-1)^{\frac{1}{q'}}(s+p'A_1-1)^{\frac{1}{p'}}},$$

while for the homogeneous kernel of degree -1, given by $K(x,y) = \frac{\log y - \log x}{y-x}$, we have

$$L' = \pi^{2\lambda}\sin^{-\frac{2}{p'}}(A_1p')\sin^{-\frac{2}{q'}}(A_2q').$$

Note that in the conjugate case, these three constants coincide with the constants obtained in Subsection 1.4.1 (see Chapter 1).

Remark 2.7 As we have already mentioned, all results from Section 1.3 can be extended to non-conjugate case. Here they are omitted, and for more details the reader is referred to [16]. Moreover, diverse methods presented in Section 1.4, except the theorems with the best possible constant factors, can be combined with a general method for non-conjugate Hilbert-type inequalities. Namely, the problem of determining the best possible constants in non-conjugate case is not resolved yet and still remains open.

As an illustration, we only provide non-conjugate extension of Corollary 1.2. Namely, if p, q, and λ are as in (2.1) and (2.2), and $s>0$, then the non-conjugate versions of inequalities (1.60) and (1.61) respectively read as

$$\begin{aligned}
&\int_a^b\int_a^b \frac{f(x)g(y)}{(x+y)^{s\lambda}}\,dxdy \\
&\leq B^{\lambda}\left(\frac{s}{2},\frac{s}{2}\right)\left\{\int_a^b\left[1-\frac{1}{2}\left(\frac{a}{x}\right)^{\frac{s}{2}}-\frac{1}{2}\left(\frac{x}{b}\right)^{\frac{s}{2}}\right]^{\frac{p}{q'}} x^{-\frac{s}{2}p\lambda+p-1}f^p(x)\,dx\right\}^{\frac{1}{p}} \\
&\quad\times\left\{\int_a^b\left[1-\frac{1}{2}\left(\frac{a}{y}\right)^{\frac{s}{2}}-\frac{1}{2}\left(\frac{y}{b}\right)^{\frac{s}{2}}\right]^{\frac{q}{p'}} y^{-\frac{s}{2}q\lambda+q-1}g^q(y)\,dy\right\}^{\frac{1}{q}}
\end{aligned}$$

and

$$\begin{aligned}
&\left\{\int_a^b\left[1-\frac{1}{2}\left(\frac{a}{y}\right)^{\frac{s}{2}}-\frac{1}{2}\left(\frac{y}{b}\right)^{\frac{s}{2}}\right]^{-\frac{q'}{p'}} y^{\frac{s}{2}q'\lambda-1}\left[\int_a^b\frac{f(x)}{(x+y)^{s\lambda}}\,dx\right]^{q'}dy\right\}^{\frac{1}{q'}} \\
&\leq B^{\lambda}\left(\frac{s}{2},\frac{s}{2}\right)\left\{\int_a^b\left[1-\frac{1}{2}\left(\frac{a}{x}\right)^{\frac{s}{2}}-\frac{1}{2}\left(\frac{x}{b}\right)^{\frac{s}{2}}\right]^{\frac{p}{q'}} x^{-\frac{s}{2}p\lambda+p-1}f^p(x)\,dx\right\}^{\frac{1}{p}}.
\end{aligned}$$

We conclude this section with some discrete examples. Namely, Theorem 2.1, rewritten with the counting measure on $\mathbb{N}$, leads to some interesting Hilbert-type inequalities related to sequences of non-negative real numbers. The following results include homogeneous kernel strictly decreasing in each argument.

Theorem 2.3 *Let p, q, and λ be as in* (2.1) *and* (2.2), $A,B,\alpha,\beta>0$, *and let $n,n'\in\mathbb{N}$, $n<n'$. Further, suppose that $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ is a non-negative homogeneous function of degree $-s$, $s>0$, strictly decreasing in both arguments. If the functions ζ_1, ζ_2 are defined by*

$$\zeta_1(\gamma,x)=\left(\frac{a}{x}\right)^{\alpha(1-\gamma)}\int_0^{\frac{B}{A}a^{\beta-\alpha}}K(1,u)u^{-\gamma}du+\left(\frac{x}{b}\right)^{\alpha(s+\gamma-1)}\int_0^{\frac{A}{B}b^{\alpha-\beta}}K(u,1)u^{s+\gamma-2}du,$$

$$\zeta_2(\gamma,y)=\left(\frac{a}{y}\right)^{\beta(s+\gamma-1)}\int_0^{\frac{A}{B}a^{\alpha-\beta}}K(u,1)u^{s+\gamma-2}du+\left(\frac{y}{b}\right)^{\beta(1-\gamma)}\int_0^{\frac{B}{A}b^{\beta-\alpha}}K(1,u)u^{-\gamma}du,$$

and $M'=\alpha^{-\frac{1}{p'}}\beta^{-\frac{1}{q'}}A^{\frac{1-s}{q'}+A_1-A_2-\frac{1}{p'}}B^{\frac{1-s}{p'}+A_2-A_1-\frac{1}{q'}}$, *then the inequalities*

$$\begin{aligned}&\sum_{i=n+1}^{n'}\sum_{j=n+1}^{n'}K^{\lambda}(Ai^{\alpha},Bj^{\beta})a_ib_j\\ &\leq M'\left[\sum_{i=n+1}^{n'}\left(k(q'A_2)-\zeta_1(q'A_2,i)\right)^{\frac{p}{q'}}i^{\frac{\alpha p}{q'}(1-s)+\alpha p(A_1-A_2)-(p-1)(\alpha-1)}a_i^p\right]^{\frac{1}{p}}\\ &\quad\times\left[\sum_{j=n+1}^{n'}\left(k(2-s-p'A_1)-\zeta_2(2-s-p'A_1,j)\right)^{\frac{q}{p'}}j^{\frac{\beta q}{p'}(1-s)+\beta q(A_2-A_1)-(q-1)(\beta-1)}b_j^q\right]^{\frac{1}{q}}\end{aligned}\tag{2.22}$$

and

$$\begin{aligned}&\left[\sum_{j=n+1}^{n'}j^{\beta q'(A_1-A_2)+\frac{\beta q'}{p'}(s-1)+\beta-1}\left(k(2-s-p'A_1)-\zeta_2(2-s-p'A_1,j)\right)^{-\frac{q'}{p'}}\right.\\ &\quad\left.\times\left(\sum_{i=n+1}^{n'}K^{\lambda}(Ai^{\alpha},Bj^{\beta})a_i\right)^{q'}\right]^{\frac{1}{q'}}\\ &\leq M'\left[\sum_{i=n+1}^{n'}\left(k(q'A_2)-\zeta_1(q'A_2,i)\right)^{\frac{p}{q'}}i^{\frac{\alpha p}{q'}(1-s)+\alpha p(A_1-A_2)-(p-1)(\alpha-1)}a_i^p\right]^{\frac{1}{p}}\end{aligned}\tag{2.23}$$

hold for all real parameters A_1, A_2 such that $A_1\in\left(\max\{\frac{1-s}{p'},\frac{\alpha-1}{\alpha p'}\},\frac{1}{p'}\right)$, $A_2\in\left(\max\{\frac{1-s}{q'},\frac{\beta-1}{\beta q'}\},\frac{1}{q'}\right)$, and for all non-negative sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$. In addition, these inequalities are equivalent.

Proof. In order to prove inequality (2.22), rewrite Theorem 2.1 for the counting measure on $\mathbb{N}$, $K_{ij} = K(Ai^\alpha, Bj^\beta)$, $\varphi_i = (Ai^\alpha)^{A_1+\frac{1}{p'\alpha}-\frac{1}{p'}}$, $\psi_j = (Bj^\beta)^{A_2+\frac{1}{q'\beta}-\frac{1}{q'}}$, and the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$. In this setting, inequality (2.9) becomes

$$\sum_{i=n+1}^{n'}\sum_{j=n+1}^{n'} K^\lambda(Ai^\alpha, Bj^\beta)\, a_i b_j$$
$$\leq \left[\sum_{i=n+1}^{n'} (Ai^\alpha)^{pA_1+(p-1)\frac{1-\alpha}{\alpha}} F_i^p a_i^p\right]^{\frac{1}{p}} \left[\sum_{j=n+1}^{n'} (Bj^\beta)^{qA_2+(q-1)\frac{1-\beta}{\beta}} G_j^q b_j^q\right]^{\frac{1}{q}},$$

where

$$F_i = \left[\sum_{j=n+1}^{n'} \frac{K(Ai^\alpha, Bj^\beta)}{(Bj^\beta)^{q'A_2+\frac{1}{\beta}-1}}\right]^{\frac{1}{q'}} \quad \text{and} \quad G_j = \left[\sum_{i=n+1}^{n'} \frac{K(Ai^\alpha, Bj^\beta)}{(Ai^\alpha)^{p'A_1+\frac{1}{\alpha}-1}}\right]^{\frac{1}{p'}}.$$

Now, since the kernel K is strictly decreasing in both variables and $p'A_1 + \frac{1}{\alpha} - 1 \geq 0$, $q'A_2 + \frac{1}{\beta} - 1 \geq 0$, we have that

$$F_i \leq \int_n^{n'} \frac{K(Ai^\alpha, By^\beta)}{(By^\beta)^{q'A_2+\frac{1}{\beta}-1}} dy \quad \text{and} \quad G_j \leq \int_n^{n'} \frac{K(Ax^\alpha, Bj^\beta)}{(Ax^\alpha)^{p'A_1+\frac{1}{\alpha}-1}} dx,$$

so the result follows due to the homogeneity of kernel K. □

An important consequence of Theorem 2.3 is the following corollary for infinite series.

Corollary 2.2 *Let p, q, and λ be as in (2.1) and (2.2), $A, B, \alpha, \beta > 0$, and let $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be a non-negative homogeneous function of degree $-s$, $s > 0$, strictly decreasing in both variables. Then the inequalities*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} K^\lambda(Ai^\alpha, Bj^\beta) a_i b_j$$
$$\leq L'M' \left[\sum_{i=1}^{\infty} i^{\frac{\alpha p}{q'}(1-s)+\alpha p(A_1-A_2)-(p-1)(\alpha-1)} a_i^p\right]^{\frac{1}{p}}$$
$$\times \left[\sum_{j=1}^{\infty} j^{\frac{\beta q}{p'}(1-s)+\beta q(A_2-A_1)-(q-1)(\beta-1)} b_j^q\right]^{\frac{1}{q}} \tag{2.24}$$

and

$$\left[\sum_{j=1}^{\infty} j^{\beta q'(A_1-A_2)+\frac{\beta q'}{p'}(s-1)+\beta-1} \left(\sum_{i=1}^{\infty} K^\lambda(Ai^\alpha, Bj^\beta) a_i\right)^{q'}\right]^{\frac{1}{q'}}$$
$$\leq L'M' \left[\sum_{i=1}^{\infty} i^{\frac{\alpha p}{q'}(1-s)+\alpha p(A_1-A_2)-(p-1)(\alpha-1)} a_i^p\right]^{\frac{1}{p}} \tag{2.25}$$

hold for all real parameters A_1, A_2 *such that* $A_1 \in \left(\max\{\frac{1-s}{p'}, \frac{\alpha-1}{\alpha p'}\}, \frac{1}{p'}\right)$, $A_2 \in \left(\max\{\frac{1-s}{q'}, \frac{\beta-1}{\beta q'}\}, \frac{1}{q'}\right)$, *and for all non-negative sequences* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$, *where* L' *and* M' *are defined respectively in Corollary* 2.1 *and Theorem* 2.3. *Moreover, these inequalities are equivalent.*

Remark 2.8 The Hilbert-type inequalities in Corollary 2.2 represent non-conjugate extensions of the corresponding inequalities from [52] and [53]. The results presented in this section are a part of the recent paper [16] by Čižmešija et. al.

2.3 Godunova-type inequalities

So far, we have considered integrals taken over certain subsets of $\mathbb{R}_+$, that is, one-dimensional case. Since Theorem 2.1 covers more general settings, we apply that result to n-dimensional space $\mathbb{R}^n_+$. Before presenting such results, it is necessary to introduce some notation for this section. For $\mathbf{x},\mathbf{y} \in \mathbb{R}^n_+$, $\mathbf{x} = (x_1,x_2,\ldots,x_n)$, $\mathbf{y} = (y_1,y_2,\ldots,y_n)$, we define

$$\frac{\mathbf{y}}{\mathbf{x}} = \left(\frac{y_1}{x_1},\frac{y_2}{x_2},\ldots,\frac{y_n}{x_n}\right) \quad \text{and} \quad \mathbf{x}^{\mathbf{y}} = {x_1}^{y_1}{x_2}^{y_2}\cdots{x_n}^{y_n}.$$

Moreover, $\mathbf{1}$ denotes the n-tuple $(1,1,\ldots,1)$, and the vector is multiplied by a scalar in the usual way.

Motivated by the one-dimensional case (see relation (1.20), Chapter 1), we also define

$$k(\mathbf{a}) = \int_{\mathbb{R}^n_+} K(\mathbf{t})\mathbf{t}^{-\mathbf{a}}d\mathbf{t},$$

where the function $K : \mathbb{R}^n_+ \to \mathbb{R}$ and the parameter $\mathbf{a} \in \mathbb{R}^n_+$ are such that the above integral converges.

Now, considering the weight functions $\varphi(\mathbf{x}) = \mathbf{x}^{\mathbf{A_1}}$, $\psi(\mathbf{y}) = \mathbf{y}^{\mathbf{A_2}}$, where $\mathbf{A_1} = (A_{11},A_{12},\ldots,A_{1n})$, $\mathbf{A_2} = (A_{21},A_{22},\ldots,A_{2n})$, and replacing $K(\mathbf{x},\mathbf{y})$ with $\mathbf{x}^{-\mathbf{s}}K\left(\frac{\mathbf{y}}{\mathbf{x}}\right)$ in Theorem 2.1, we get the following result.

Theorem 2.4 *Suppose that p, q, and* λ *are as in* (2.1) *and* (2.2). *If* $\mathbf{s} \in \mathbb{R}^n_+$, $K : \mathbb{R}^n_+ \to \mathbb{R}$ *is a non-negative measurable function, and the parameters* $\mathbf{A_1},\mathbf{A_2} \in \mathbb{R}^n$ *are such that* $L' = k^{\frac{1}{q'}}(q'\mathbf{A_2})k^{\frac{1}{p'}}(2\cdot\mathbf{1}-\mathbf{s}-p'\mathbf{A_1}) < \infty$, *then the inequalities*

$$\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} \mathbf{x}^{-\lambda\mathbf{s}}K^{\lambda}\left(\frac{\mathbf{y}}{\mathbf{x}}\right) f(\mathbf{x})g(\mathbf{y})d\mathbf{x}d\mathbf{y}$$
$$\leq L'\left[\int_{\mathbb{R}^n_+} \mathbf{x}^{\frac{p}{q'}(\mathbf{1}-\mathbf{s})+p(\mathbf{A_1}-\mathbf{A_2})} f^p(\mathbf{x})d\mathbf{x}\right]^{\frac{1}{p}} \left[\int_{\mathbb{R}^n_+} \mathbf{y}^{\frac{q}{p'}(\mathbf{1}-\mathbf{s})+q(\mathbf{A_2}-\mathbf{A_1})} g^q(\mathbf{y})d\mathbf{y}\right]^{\frac{1}{q}}$$

and

$$\left[\int_{\mathbb{R}_+^n} \mathbf{y}^{\frac{q'}{p'}(\mathbf{s}-\mathbf{1})+q'(\mathbf{A_1}-\mathbf{A_2})} \left(\int_{\mathbb{R}_+^n} \mathbf{x}^{-\lambda\mathbf{s}} K^{\lambda}\left(\frac{\mathbf{y}}{\mathbf{x}}\right) f(\mathbf{x})d\mathbf{x}\right)^{q'} d\mathbf{y}\right]^{\frac{1}{q'}}$$
$$\leq L' \left[\int_{\mathbb{R}_+^n} \mathbf{x}^{\frac{p}{q'}(\mathbf{1}-\mathbf{s})+p(\mathbf{A_1}-\mathbf{A_2})} f^p(\mathbf{x})d\mathbf{x}\right]^{\frac{1}{p}}$$

hold for all non-negative measurable functions $f,g:\mathbb{R}_+^n\to\mathbb{R}$ and are equivalent.

Remark 2.9 For $n=1$ we have that $x^{-s}K\left(\frac{y}{x}\right)$ is a homogeneous function of degree $-s$, hence Theorem 2.4 may be regarded as an n-dimensional generalization of Corollary 2.1.

We explicitly state two particular cases of Theorem 2.4, obtained for some special choices of parameters. The first one considers $\mathbf{s}=s\mathbf{1}$, $\mathbf{A_1}=A_1\mathbf{1}$, and $\mathbf{A_2}=A_2\mathbf{1}$, where s,A_1,A_2 are real numbers.

Corollary 2.3 *Suppose that p, q, and λ are as in* (2.1) *and* (2.2). *If $\mathbf{s}\in\mathbb{R}_+$, $K:\mathbb{R}_+^n\to\mathbb{R}$ is a non-negative measurable function, and the parameters $A_1,A_2\in\mathbb{R}$ are such that $L'=k^{\frac{1}{q'}}(q'A_2\mathbf{1})k^{\frac{1}{p'}}((2-s-p'A_1)\mathbf{1})<\infty$, then the inequalities*

$$\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n} \mathbf{x}^{-\lambda s\mathbf{1}} K^{\lambda}\left(\frac{\mathbf{y}}{\mathbf{x}}\right) f(\mathbf{x})g(\mathbf{y})d\mathbf{x}d\mathbf{y}$$
$$\leq L'\left[\int_{\mathbb{R}_+^n} \mathbf{x}^{\left[\frac{p}{q'}(1-s)+p(A_1-A_2)\right]\mathbf{1}} f^p(\mathbf{x})d\mathbf{x}\right]^{\frac{1}{p}} \left[\int_{\mathbb{R}_+^n} \mathbf{y}^{\left[\frac{q}{p'}(1-s)+q(A_2-A_1)\right]\mathbf{1}} g^q(\mathbf{y})d\mathbf{y}\right]^{\frac{1}{q}}$$

and

$$\left[\int_{\mathbb{R}_+^n} \mathbf{y}^{\left[\frac{q'}{p'}(s-1)+q'(A_1-A_2)\right]\mathbf{1}} \left(\int_{\mathbb{R}_+^n} \mathbf{x}^{-\lambda\mathbf{s}} K^{\lambda}\left(\frac{\mathbf{y}}{\mathbf{x}}\right) f(\mathbf{x})d\mathbf{x}\right)^{q'} d\mathbf{y}\right]^{\frac{1}{q'}}$$
$$\leq L'\left[\int_{\mathbb{R}_+^n} \mathbf{x}^{\left[\frac{p}{q'}(1-s)+p(A_1-A_2)\right]\mathbf{1}} f^p(\mathbf{x})d\mathbf{x}\right]^{\frac{1}{p}}$$

hold for all non-negative measurable functions $f,g:\mathbb{R}_+^n\to\mathbb{R}$ and are equivalent.

The second special case of Theorem 2.4, and also the concluding result in this section, presents an inequality of E. K. Godunova from [24]. Namely, if we put $A_1=\frac{2-s}{p'}$, $A_2=0$, $K=u^{\frac{1}{\lambda}}$ in Corollary 2.3, and consider the functions $\widetilde{f},\widetilde{g}:\mathbb{R}_+^n\to\mathbb{R}$, defined by $\widetilde{f}(\mathbf{x})=\mathbf{x}^{\left[-\frac{1}{p'}+(s-1)\lambda\right]\mathbf{1}}f(\mathbf{x})$, $\widetilde{g}(\mathbf{y})=\mathbf{y}^{\frac{1}{p'}\cdot\mathbf{1}}g(\mathbf{y})$, we get the following result.

Corollary 2.4 *Let p, q, and λ be as in* (2.1) *and* (2.2). *If $\mathbf{s}\in\mathbb{R}_+$ and $u:\mathbb{R}_+^n\to\mathbb{R}$ is a non-negative measurable function, then the inequalities*

$$\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n} \mathbf{x}^{-\left(\frac{1}{p'}+\lambda\right)\mathbf{1}} \mathbf{y}^{\frac{1}{p'}\mathbf{1}} u\left(\frac{\mathbf{y}}{\mathbf{x}}\right) f(\mathbf{x})g(\mathbf{y})d\mathbf{x}d\mathbf{y} \leq ||u||_{L^{\frac{1}{\lambda}}(\mathbb{R}_+^n)} ||f||_{L^{p'}(\mathbb{R}_+^n)} ||g||_{L^{q'}(\mathbb{R}_+^n)},$$

and

$$\left[\int_{\mathbb{R}_+^n} \mathbf{y}^{\frac{q'}{p'}\mathbf{1}} \left(\int_{\mathbb{R}_+^n} \mathbf{x}^{-\left(\frac{1}{p'}+\lambda\right)\mathbf{1}} u\left(\frac{\mathbf{y}}{\mathbf{x}}\right) f(\mathbf{x}) d\mathbf{x}\right)^{q'} d\mathbf{y}\right]^{\frac{1}{q'}} \le ||u||_{L^{\frac{1}{\lambda}}(\mathbb{R}_+^n)} ||f||_{L^{p'}(\mathbb{R}_+^n)}$$

hold for all non-negative measurable functions $f, g : \mathbb{R}_+^n \to \mathbb{R}$ *and are equivalent.*

Remark 2.10 Since the first inequality in Corollary 2.4 was proved by E. K. Godunova in [24], all inequalities presented in this section will be referred to as the Godunova-type inequalities. These inequalities are established in [16]. In addition, Hilbert-type inequalities with vector variables will be extensively studied in Chapter 3.

2.4 Multidimensional case

The main objective of this section is to extend Theorem 2.1 to a multidimensional case. The three-dimensional version of the Hilbert-type inequality, that is, the relation (2.5), was given by F.F. Bonsall [9], in 1950s. Although Bonsall also established conditions for the set of n non-conjugate parameters, there were no results in that direction.

In order to obtain our general results we introduce here n-dimensional extension of non-conjugate exponents, defined in [9]. Let $n \in \mathbb{N}$, $n \ge 2$, and let real parameters $p_1, \dots, p_n$ be such that

$$p_1, \dots, p_n > 1, \quad \sum_{i=1}^{n} \frac{1}{p_i} \ge 1. \tag{2.26}$$

Define

$$\lambda = \frac{1}{n-1} \sum_{i=1}^{n} \frac{1}{p_i'} \quad \text{and} \quad \frac{1}{q_i} = \lambda - \frac{1}{p_i'}, \quad i = 1, \dots, n, \tag{2.27}$$

where $\frac{1}{p_i} + \frac{1}{p_i'} = 1$, $i = 1, \dots, n$. On the other hand, for any choice of parameters as in (2.26), it follows from (2.27) that

$$\frac{1}{q_i} + (1-\lambda) = \frac{1}{p_i}, \quad i = 1, \dots, n, \tag{2.28}$$

and

$$\sum_{i=1}^{n} \frac{1}{q_i} + (1-\lambda) = 1. \tag{2.29}$$

Hence, in order to apply the Hölder inequality with exponents $q_1, \dots, q_n, \frac{1}{1-\lambda}$, we need to require

$$\frac{1}{q_i} > 0, \quad i = 1, \dots, n. \tag{2.30}$$

Note that for $n \geq 3$ conditions (2.26) and (2.27) do not automatically imply (2.30). More precisely, since (2.26) and (2.27) give only

$$\frac{1}{q_i} > \frac{2-n}{n-1}\frac{1}{p_i'}, \quad i = 1, \ldots, n,$$

some of q_i may be negative. For example, for $p_1 = 2$ and $p_2 = p_3 = \frac{20}{19}$ we have $\frac{1}{q_1} = -\frac{1}{5} < 0$. Therefore, the condition (2.30) is not redundant.

Observe that for $\lambda = 1$ the above parameters reduce to the conjugate case, that is, $\sum_{i=1}^{n} \frac{1}{p_i} = 1$ and $p_i = q_i$, $i = 1, 2, \ldots, n$.

Now, we are ready to state and prove general forms of multidimensional Hilbert-type inequalities with non-conjugate exponents. These inequalities will include integrals taken over general subsets of $\mathbb{R}_+$, equipped with σ-finite measures. Of course, the following extension may also be regarded as a non-conjugate version of Theorem 1.18.

Theorem 2.5 *Let λ, p_i, p_i', q_i, $i = 1, 2, \ldots, n$, $n \geq 2$, fulfill relations* (2.26), (2.27) *and* (2.30), *and let Ω be a measure space with σ-finite measures μ_i, $i = 1, 2, \ldots, n$. Further, suppose that $K : \Omega^n \to \mathbb{R}$ and $\phi_{ij} : \Omega \to \mathbb{R}$, $i, j = 1, \ldots, n$, are non-negative measurable functions such that $\prod_{i,j=1}^{n} \phi_{ij}(x_j) = 1$ a.e. on Ω^n. If the functions F_i, $i = 1, 2, \ldots, n$, are defined by*

$$F_i(x_i) = \left[\int_{\Omega^{n-1}} K(x_1, \ldots, x_n) \prod_{j=1, j \neq i}^{n} \phi_{ij}^{q_i}(x_j) d\mu_1(x_1) \ldots d\mu_{j-1}(x_{j-1}) \right.$$
$$\left. \times d\mu_{j+1}(x_{j+1}) \ldots d\mu_n(x_n) \right]^{\frac{1}{q_i}},$$

then for all non-negative measurable functions $f_i : \Omega \to \mathbb{R}$, $i = 1, 2, \ldots, n$, the inequalities

$$\int_{\Omega^n} K^{\lambda}(x_1, \ldots, x_n) \prod_{i=1}^{n} f_i(x_i) d\mu_1(x_1) \ldots d\mu_n(x_n) \leq \prod_{i=1}^{n} \|\phi_{ii} F_i f_i\|_{L^{p_i}(\mu_i)} \tag{2.31}$$

and

$$\left\{ \int_{\Omega} \left[\frac{1}{(\phi_{nn} F_n)(x_n)} \int_{\Omega^{n-1}} K^{\lambda}(x_1, \ldots, x_n) \prod_{i=1}^{n-1} f_i(x_i) d\mu_1(x_1) \ldots d\mu_{n-1}(x_{n-1}) \right]^{p_n'} d\mu_n(x_n) \right\}^{\frac{1}{p_n'}}$$
$$\leq \prod_{i=1}^{n-1} \|\phi_{ii} F_i f_i\|_{L^{p_i}(\mu_i)} \tag{2.32}$$

hold and are equivalent.

Proof. The left-hand side of inequality (2.31) can be rewritten as

$$\begin{aligned}
&\int_{\Omega^n} K^\lambda(x_1,\dots,x_n)\prod_{i=1}^n f_i(x_i)d\mu_1(x_1)\dots d\mu_n(x_n)\\
&= \int_{\Omega^n}\prod_{i=1}^n\left[K(x_1,\dots,x_n)\phi_{ii}{}^{p_i}(x_i)\prod_{j=1,j\neq i}^n \phi_{ij}^{q_i}(x_j)F_i{}^{p_i-q_i}(x_i)f_i{}^{p_i}(x_i)\right]^{\frac{1}{q_i}}\\
&\quad\times\left[\prod_{i=1}^n(\phi_{ii}F_if_i)^{p_i}(x_i)\right]^{1-\lambda} d\mu_1(x_1)\dots d\mu_n(x_n).
\end{aligned}$$

In addition, since $\sum_{i=1}^n \frac{1}{q_i}+1-\lambda=1$, $q_i>1$, $0<\lambda\le 1$, applying the Hölder inequality to the above relation yields

$$\begin{aligned}
&\int_{\Omega^n} K^\lambda(x_1,\dots,x_n)\prod_{i=1}^n f_i(x_i)d\mu_1(x_1)\dots d\mu_n(x_n)\\
&\le \prod_{i=1}^n\left[\int_\Omega(\phi_{ii}F_if_i)^{p_i}(x_i)d\mu_i(x_i)\right]^{\frac{1}{q_i}}\prod_{i=1}^n\left[\int_\Omega(\phi_{ii}F_if_i)^{p_i}(x_i)d\mu_i(x_i)\right]^{1-\lambda}.
\end{aligned}$$

Finally, since $\frac{1}{q_i}+1-\lambda=\frac{1}{p_i}$, we obtain inequality (2.31).

Now, we show that inequalities (2.31) and (2.32) are equivalent. Suppose that inequality (2.31) is valid. Setting the function $f_n:\Omega\to\mathbb{R}$, defined by

$$f_n(x_n)=\left[\frac{1}{(\phi_{nn}F_n)^{p_n}(x_n)}\int_{\Omega^{n-1}}K^\lambda(x_1,\dots,x_n)\prod_{i=1}^{n-1}f_i(x_i)d\mu_1(x_1)\dots d\mu_{n-1}(x_{n-1})\right]^{\frac{p_n'}{p_n}},$$

in inequality (2.31), we have

$$I(x_n)^{p_n'}\le\prod_{i=1}^{n-1}\left[\int_\Omega(\phi_{ii}F_if_i)^{p_i}(x_i)d\mu_i(x_i)\right]^{\frac{1}{p_i}} I(x_n)^{\frac{p_n'}{p_n}},$$

where $I(x_n)$ denotes the left-hand side of inequality (2.32). Clearly, this relation represents inequality (2.32).

It remains to prove that inequality (2.31) is a consequence of (2.32). Assume, therefore, that inequality (2.32) holds. The left-hand side of (2.31) can be rewritten as

$$\begin{aligned}
&\int_{\Omega^n} K^\lambda(x_1,\dots,x_n)\prod_{i=1}^n f_i(x_i)d\mu_1(x_1)\dots d\mu_n(x_n)\\
&= \int_\Omega(\phi_{nn}F_nf_n)(x_n)\left[\frac{1}{(\phi_{nn}F_n)(x_n)}\int_{\Omega^{n-1}}K^\lambda(x_1,\dots,x_n)\right.\\
&\quad\left.\times\prod_{i=1}^{n-1}f_i(x_i)d\mu_1(x_1)\dots d\mu_{n-1}(x_{n-1})\right]d\mu_n(x_n),
\end{aligned}$$

hence, applying the Hölder inequality with conjugate exponents p_n and p_n', we have

$$\int_{\Omega^n} K^{\lambda}(x_1,...,x_n)\prod_{i=1}^{n} f_i(x_i)d\mu_1(x_1)...d\mu_n(x_n) \le \|\phi_{nn}F_nf_n\|_{L^{p_n}(\mu_n)}I(x_n),$$

and the result follows from (2.32). $\square$

Remark 2.11 Since the crucial step in proving the previous theorem was in applying the Hölder inequality, equality in (2.31) holds if and only if the functions

$$K(x_1,...,x_n)\phi_{ii}{}^{p_i}(x_i)\prod_{j=1,j\neq i}^{n}\phi_{ij}^{q_i}(x_j)F_i{}^{p_i-q_i}(x_i)f_i{}^{p_i}(x_i), \quad i=1,2,\dots n,$$

and $\prod_{i=1}^{n}(\phi_{ii}F_if_i)^{p_i}(x_i)$ are effectively proportional. Clearly, this trivially happens if at least one of the functions f_i, $i=1,2,\dots,n$, is a zero-function. Otherwise, these conditions can be rewritten in a more suitable form, yielding the explicit expressions for the functions and kernel, that is, $f_i(x_i)=C_i\phi_{ii}(x_i)^{\frac{q_i}{1-\lambda q_i}}F_i(x_i)^{(1-\lambda)q_i}$, $i=1,2,\dots n$, and $K(x_1,x_2,\dots,x_n)=C\prod_{i=1}^{n}F_i^{q_i}(x_i)$, where C and C_i are positive constants. It is possible only if the functions

$$\frac{\prod_{j=1,j\neq i}^{n}\phi_{jj}^{\frac{\lambda q_j}{1-\lambda q_j}}(x_j)}{\prod_{j=1,j\neq i}^{n}\phi_{ij}{}^{\lambda q_j}(x_j)}, \quad i=1,2,\dots,n,$$

are appropriate constants, and

$$\int_{\Omega} F_i^{q_i}(x_i)\phi_{ii}^{\frac{q_i}{1-\lambda q_i}}(x_i)d\mu_i(x_i)<\infty, \quad i=1,2,\dots n.$$

Otherwise, the inequalities in Theorem 2.5 are strict.

Remark 2.12 If the parameters p_i, $i=1,2,\dots,n$, are chosen in such a way that

$$q_j>0, j\in\{1,2,\dots n\}, \quad q_i<0, i\neq j, \quad \text{and} \quad \lambda<1, \tag{2.33}$$

or

$$q_i<0, \quad i=1,2,\dots,n, \tag{2.34}$$

then the reverse inequalities in (2.31) and (2.32) hold, due to the reverse Hölder inequality.

As an application of Theorem 2.5, we consider the case of homogeneous kernel $K:\mathbb{R}_+^n\to\mathbb{R}$, defined by $K(x_1,...,x_n)=(x_1+\cdots+x_n)^{-s}$, $s>0$, and the power weight functions $\phi_{ij}:\mathbb{R}_+^n\to\mathbb{R}$, $\phi_{ij}(x_j)=x_j^{A_{ij}}$, $A_{ij}\in\mathbb{R}$, with respect to Lebesgue measures dx_i, $i=1,2,\dots,n$ on $\mathbb{R}_+$. The parameters A_{ij} fulfill relations $\sum_{i=1}^{n}A_{ij}=0$, $j=1,\dots,n$, so that the condition $\prod_{i,j=1}^{n}\phi_{ij}(x_j)=1$ is fulfilled. Regarding the form of the kernel, we obtain inequalities with the constant factor expressed in terms of the Gamma function.

Theorem 2.6 *Let λ, p_i, p_i', q_i, $i = 1,2,\ldots,n$, $n \geq 2$, be as in* (2.26), (2.27), *and* (2.30), *and let A_{ij}, $i,j = 1,\ldots,n$, be real parameters such that $\sum_{i=1}^n A_{ij} = 0$ for $j = 1,\ldots,n$. If $s > 0$, $\alpha_i = \sum_{j=1}^n A_{ij}$, $A_{ij} > -\frac{1}{q_i}$, $i \neq j$, $A_{ii} - \alpha_i > \frac{n-s-1}{q_i}$, and*

$$K = \frac{1}{\Gamma^\lambda(s)} \prod_{i=1}^n \Gamma^{\frac{1}{q_i}}(s-n+1-q_i\alpha_i+q_iA_{ii}) \prod_{i,j=1, i\neq j}^n \Gamma^{\frac{1}{q_i}}(q_iA_{ij}+1),$$

then the inequalities

$$\int_{\mathbb{R}_+^n} \frac{\prod_{i=1}^n f_i(x_i)}{(\sum_{j=1}^n x_j)^{\lambda s}} dx_1 \ldots dx_n < K \prod_{i=1}^n \left[\int_0^\infty x_i^{\frac{p_i}{q_i}(n-1-s)+p_i\alpha_i} f_i^{p_i}(x_i) dx_i\right]^{\frac{1}{p_i}} \tag{2.35}$$

and

$$\left[\int_0^\infty x_n^{(1-\lambda p_n')(n-1-s)-p_n'\alpha_n} \left(\int_{\mathbb{R}_+^{n-1}} \frac{\prod_{i=1}^{n-1} f_i(x_i)}{(\sum_{j=1}^n x_j)^{\lambda s}} dx_1 \ldots dx_{n-1}\right)^{p_n'} dx_n\right]^{\frac{1}{p_n'}}$$
$$< K \prod_{i=1}^{n-1} \left[\int_0^\infty x_i^{\frac{p_i}{q_i}(n-1-s)+p_i\alpha_i} f_i^{p_i}(x_i) dx_i\right]^{\frac{1}{p_i}} \tag{2.36}$$

hold for all non-negative measurable functions $f_i : \mathbb{R}_+ \to \mathbb{R}$, $f_i \neq 0$, $i = 1,\ldots,n$. Moreover, these inequalities are equivalent.

Another way of extending Hilbert-type inequalities with non-conjugate exponents to the multidimensional setting arises from inequality (2.6), presented at the beginning of this chapter.

Theorem 2.7 *Let $n \in \mathbb{N}$, $n \geq 2$, and let parameters λ, p_i, q_i, $i = 1,2,\ldots,n$, be as in* (2.26), (2.27), *and* (2.30). *Let $\mu_1,\ldots,\mu_n$ be positive σ-finite measures on Ω. If $K : \Omega^n \to \mathbb{R}$, $F_i : \Omega^n \to \mathbb{R}$, $\phi_{ij} : \Omega \to \mathbb{R}$, $i,j = 1,\ldots,n$, are non-negative measurable functions such that*

$$\prod_{i,j=1}^n \phi_{ij}(x_j) = 1, \quad \text{a.e. on } \Omega^n, \tag{2.37}$$

then the inequality

$$\int_{\Omega^n} K(x_1,\ldots,x_n) \prod_{i=1}^n f_i(x_i)\, d\mu_1(x_1)\ldots d\mu_n(x_n)$$
$$\leq \prod_{i=1}^n \left[\int_{\Omega^n} \left(KF_i^{p_i-q_i}\right)(x_1,\ldots,x_n)(\phi_{ii}f_i)^{p_i}(x_i) \prod_{j\neq i} \phi_{ij}^{q_i}(x_j)\, d\mu_1(x_1)\ldots d\mu_n(x_n)\right]^{\frac{1}{q_i}}$$
$$\times \left[\int_{\Omega^n} K(x_1,\ldots,x_n) \prod_{i=1}^n F_i^{p_i}(x_1,\ldots,x_n)(\phi_{ii}f_i)^{p_i}(x_i)\, d\mu_1(x_1)\ldots d\mu_n(x_n)\right]^{1-\lambda} \tag{2.38}$$

holds for all non-negative measurable functions $f_i : \Omega \to \mathbb{R}$, $i = 1,\ldots,n$.

Proof. Note that from (2.28) we have $\frac{p_i}{q_i}+p_i(1-\lambda)=1$, $i=1,\ldots,n$. Using this and (2.29), the left-hand side of (2.38) can be written as

$$\begin{aligned}
&\int_{\Omega^n} K(x_1,\ldots,x_n)\prod_{i=1}^n f_i(x_i)\,d\mu_1(x_1)\ldots d\mu_n(x_n)\\
&=\int_{\Omega^n} K^{\sum_{i=1}^n \frac{1}{q_i}+1-\lambda}(x_1,\ldots,x_n)\prod_{i=1}^n f_i^{\frac{p_i}{q_i}+p_i(1-\lambda)}(x_i)\\
&\quad\times\prod_{i=1}^n F_i^{\frac{p_i}{q_i}-1+p_i(1-\lambda)}(x_1,\ldots,x_n)\prod_{i=1}^n \phi_{ii}^{\frac{p_i}{q_i}+p_i(1-\lambda)}(x_i)\\
&\quad\times\prod_{j\neq i}\phi_{ij}(x_j)\,d\mu_1(x_1)\ldots d\mu_n(x_n)\\
&=\int_{\Omega^n}\prod_{i=1}^n\left[\left(KF_i^{p_i-q_i}\right)(x_1,\ldots,x_n)(\phi_{ii}f_i)^{p_i}(x_i)\prod_{j\neq i}\phi_{ij}^{q_i}(x_j)\right]^{\frac{1}{q_i}}\\
&\quad\times\left[K(x_1,\ldots,x_n)\prod_{i=1}^n F_i^{p_i}(x_1,\ldots,x_n)(\phi_{ii}f_i)^{p_i}(x_i)\right]^{1-\lambda} d\mu_1(x_1)\ldots d\mu_n(x_n).
\end{aligned}$$

The inequality (2.38) now follows by using the Hölder inequality with the exponents $q_1,\ldots,q_n,\frac{1}{1-\lambda}$. □

Remark 2.13 Observe that without loss of generality the condition (2.37) from the statement of Theorem 2.7 can be replaced by $\prod_{i=1}^n\phi_{ij}(x_j)=1$ a.e. on Ω, for $j=1,\ldots,n$, since (2.37) implies that

$$\prod_{i=1}^n\phi_{ij}(x_j)=c_j=const,\quad j=1,\ldots,n, \tag{2.39}$$

where $c_1\cdots c_n=1$.

Remark 2.14 Obviously, (2.38) becomes equality if at least one of the functions involved in its left-hand side is a zero-function. Otherwise, equality holds if and only if it holds in the Hölder inequality, that is, only if the functions $KF_i^{p_i-q_i}(\phi_{ii}f_i)^{p_i}\prod_{j\neq i}\phi_{ij}^{q_i}$, $i=1,\ldots,n$, and $K\prod_{i=1}^n(F_i\phi_{ii}f_i)^{p_i}$ are effectively proportional. Therefore, equality in (2.38) occurs if and only if there exist positive constants α_i, β_{ij}, $i,j=1,\ldots,n$, $j\neq i$, such that

$$KF_i^{p_i-q_i}(\phi_{ii}f_i)^{p_i}\prod_{l\neq i}\phi_{il}^{q_i}=\alpha_i K\prod_{l=1}^n(F_l\phi_{ll}f_l)^{p_l},\quad i=1,\ldots,n, \tag{2.40}$$

and

$$KF_i^{p_i-q_i}(\phi_{ii}f_i)^{p_i}\prod_{l\neq i}\phi_{il}^{q_i}=\beta_{ij}KF_j^{p_j-q_j}(\phi_{jj}f_j)^{p_j}\prod_{l\neq j}\phi_{jl}^{q_j},\quad i\neq j. \tag{2.41}$$

Moreover, the relation (2.40) is equivalent to

$$F_i^{-q_i}=\alpha_i\prod_{l\neq i}\phi_{il}^{-q_i}(F_l\phi_{ll}f_l)^{p_l},\quad i=1,\ldots,n. \tag{2.42}$$

In the special case when $F_i \equiv F_i(x_i)$, $i = 1,\ldots,n$, the functions f_i and ϕ_{ij} from (2.41) and (2.42) can be expressed explicitly in terms of ϕ_{ii}. More precisely, from (2.42) we have

$$F_i \equiv const, \quad i = 1,\ldots,n, \tag{2.43}$$

directly, since the right-hand side of this relation depends on x_l, $l \neq i$, while its left-hand side in this setting is a function of x_i. Considering this, (2.41) becomes

$$(\phi_{ii} f_i)^{p_i}\, \phi_{ji}^{-q_j} = \gamma_{ij}\,(\phi_{jj} f_j)^{p_j} \prod_{l\neq i,j} \phi_{jl}^{q_j} \prod_{l\neq i} \phi_{il}^{-q_i}, \quad i \neq j, \tag{2.44}$$

for some positive constants γ_{ij}. Thus,

$$(\phi_{ii} f_i)^{p_i}\, \phi_{ji}^{-q_j} \equiv const, \quad i = 1,\ldots,n, \; j \neq i, \tag{2.45}$$

where again we exploited the fact that the left-hand side of (2.44) depends only on x_i, while its right-hand side is a function of x_l, $l = 1,\ldots,n$, $l \neq i$. The relation (2.45) further implies that $\phi_{ji}^{q_j}\phi_{li}^{-q_l} = const$, $i = 1,\ldots,n$, $j,l \neq i$, which combined with (2.39) gives

$$\phi_{ii}\phi_{ji}^{q_j \sum_{l\neq i} \frac{1}{q_l}} \equiv const, \quad i = 1,\ldots,n, \; j \neq i. \tag{2.46}$$

Since by (2.27) and (2.29) we have $q_j \sum_{l\neq i} \frac{1}{q_l} = \frac{q_j}{p_i'}$, the relation (2.46) can be transformed into

$$\phi_{ii}^{p_i'}\phi_{ji}^{q_j} \equiv const, \quad i = 1,\ldots,n, \; j \neq i, \tag{2.47}$$

while (2.45) becomes

$$f_i^{p_i} \equiv C_i \phi_{ii}^{-(p_i+p_i')}, \quad i = 1,\ldots,n, \tag{2.48}$$

for some positive constants C_i, $i = 1,\ldots,n$. Hence, if $F_i \equiv F_i(x_i)$, the conditions (2.43), (2.47), and (2.48) are necessary and sufficient for equality in (2.38).

Remark 2.15 If the parameters p_i, $i = 1,2,\ldots,n$, in Theorem 2.7 are such that

$$0 < p_i < 1, \qquad \frac{n-1}{p_i} + 1 < \sum_{j=1}^{n} \frac{1}{p_j}, \quad i = 1,\ldots,n, \tag{2.49}$$

and λ, and q_i, $i = 1,2,\ldots,n$, are defined by (2.27), then the sign of inequality in (2.38) is reversed. To justify this assertion, observe that the first inequality in (2.49) gives $\frac{1}{p_i'} < 0$, $i = 1,\ldots,n$, so we have $\lambda < 0$. Similarly, from the second relation in (2.49) it follows that

$$\frac{1}{q_i} = \lambda - \frac{1}{p_i'} = \frac{1}{n-1}\left(\frac{n-1}{p_i} + 1 - \sum_{j=1}^{n} \frac{1}{p_j}\right) < 0, \quad i = 1,\ldots,n.$$

Therefore, $q_i < 0$, $i = 1,\ldots,n$, and $0 < \frac{1}{1-\lambda} < 1$, so (2.38) holds with the reverse inequality as a direct consequence of the reverse Hölder inequality. The same result is also achieved with the parameters p_i, $i = 1,2,\ldots,n$, satisfying

$$\sum_{i=1}^{n} \frac{1}{p_i} < 1 \quad \text{and} \quad 0 < p_l < 1, \quad \frac{n-1}{p_i} + 1 < \sum_{j=1}^{n} \frac{1}{p_j}, \quad i \neq l, \tag{2.50}$$

for some $l \in \{1,\ldots,n\}$, since from (2.50) we obtain $\frac{1}{1-\lambda} < 0$, $q_l > 0$, and $q_i < 0$, $i \neq l$.

To conclude this section, we restate Theorem 2.7 in the case of $n = 2$. This result is interesting in its own right, since it will be applied in the next chapter, where we shall consider Hilbert-type inequalities in some particular cases.

Theorem 2.8 *Let p, q, and λ be as in* (2.1) *and* (2.2). *Let μ_1 and μ_2 be positive σ-finite measures on Ω. If K, F, and G are non-negative measurable functions on Ω^2 and φ and ψ are non-negative measurable functions on Ω, then the inequality*

$$\begin{aligned}
&\int_{\Omega^2} K(x,y)f(x)g(y)\,d\mu_1(x)d\mu_2(y) \\
&\leq \left[\int_{\Omega^2} (KF^{p-q'})(x,y)\psi^{-q'}(y)(\varphi f)^p(x)\,d\mu_1(x)d\mu_2(y)\right]^{\frac{1}{q'}} \\
&\quad \times \left[\int_{\Omega^2} (KG^{q-p'})(x,y)\varphi^{-p'}(x)(\psi g)^q(y)\,d\mu_1(x)d\mu_2(y)\right]^{\frac{1}{p'}} \\
&\quad \times \left[\int_{\Omega^2} (KF^pG^q)(x,y)(\varphi f)^p(x)(\psi g)^q(y)\,d\mu_1(x)d\mu_2(y)\right]^{1-\lambda}
\end{aligned} \tag{2.51}$$

holds for all non-negative measurable functions f and g on Ω.

Proof. The proof follows directly from Theorem 2.7 using substitutions $p_1 = p$, $p_2 = q$, $q_1 = q'$, $q_2 = p'$, $\phi_{11} = \varphi$, and $\phi_{22} = \psi$. Observe that from $\phi_{11}\phi_{21} = 1$ and $\phi_{12}\phi_{22} = 1$ we have $\phi_{21} = \frac{1}{\varphi}$ and $\phi_{12} = \frac{1}{\psi}$. □

Remark 2.16 If we rewrite Theorem 2.8 with $\Omega = \mathbb{R}_+$, Lebesgue measures, the kernel $K(x,y) = (x+y)^{-\lambda}$, and with functions $F(x,y) = G(x,y) \equiv 1$, $\varphi(x) = x^{\frac{1}{pp'}}$, and $\psi(y) = y^{\frac{1}{qq'}}$, we obtain (2.6). Hence, Theorem 2.8 can be viewed as a generalization of the mentioned Bonsall result (2.6) from [9].

Remark 2.17 Theorem 2.5 and its consequences are taken from [12], while Theorem 2.7 is obtained in [17].

2.5 Examples with hypergeometric functions

We have already discussed that the general method with non-conjugate exponents, developed in this chapter, can be combined with particular settings and diverse methods presented in Section 1.4. In addition, we have seen some Hilbert-type inequalities involving hypergeometric functions.

Therefore, we consider here some particular settings in which hypergeometric functions occur in a more general manner. We start with a classic example.

2.5.1 Hilbert-type inequalities and Gaussian hypergeometric function

Gaussian hypergeometric function is the formal power series in $z \in \mathbb{C}$ with three parameters, defined in terms of rising factorial powers:

$$F(a,b;c;z) = \sum_{k \geq 0} \frac{a^{\overline{k}} b^{\overline{k}}}{c^{\overline{k}}} \cdot \frac{z^k}{k!}, \ a,b,c,z \in \mathbb{R}, \ |z| < 1. \tag{2.52}$$

Here, the rising factorial power is $a^{\overline{k}} = a(a+1)(a+2)\cdots(a+k-1)$, $k \in \mathbb{N}$, and $a^{\overline{0}} = 1$, $a \neq 0$.

To avoid division by zero, c is neither zero nor negative integer. The series (2.52) is often called the Gaussian hypergeometric function, because many of its interesting properties were first proved by Gauss. In fact, it was the only hypergeometric series until the second half of nineteenth century, when everything was generalized to arbitrary number of parameters. For more details, the reader is referred to [26].

Relation (1.34) from Chapter 1 represents the integral representation of the above power series. Moreover, knowing the relation between the Gamma and the Beta function, (1.34) can also be rewritten as

$$F(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt, \tag{2.53}$$

where $c > b > 0$ and $|z| < 1$.

In order to obtain Hilbert-type inequality with constant factor expressed in terms of $F(a,b;c;z)$, we are going to rewrite relation (2.53) in a more suitable form.

Lemma 2.1 *Suppose $a,b,c,\alpha,\gamma \in \mathbb{R}$ are such that $a+c > b > 0$ and $0 < \alpha < 2\gamma$. Then,*

$$\int_0^\infty \frac{x^{b-1}}{(1+\alpha x)^a (1+\gamma x)^c} dx = \gamma^{-b} B(b,a+c-b) F\Big(a,b;a+c;1-\frac{\alpha}{\gamma}\Big). \tag{2.54}$$

Proof. Consider the integral $I = \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt$. Using the substitutions $1-t = \frac{1}{1+u}$, $u = \gamma x$, $\gamma > 0$, and the abbreviation $\alpha = (1-z)\gamma$, we obtain

$$I = \gamma^b \int_0^\infty \frac{x^{b-1}}{(1+\alpha x)^a (1+\gamma x)^{c-a}} dx.$$

Now, utilizing (2.53) we have

$$\int_0^\infty \frac{x^{b-1}}{(1+\alpha x)^a (1+\gamma x)^{c-a}} dx = \gamma^{-b} B(b,c-b) F\Big(a,b;c;1-\frac{\alpha}{\gamma}\Big).$$

Finally, replacing $c-a$ with c in the previous formula, we get (2.54). □

Now, considering Corollary 2.1 with the homogeneous kernel

$$K(x,y) = (x+\alpha_1 y)^{-s_1}(x+\alpha_2 y)^{-s_2}, \tag{2.55}$$

where $\alpha_1, \alpha_2 > 0$, $\frac{1}{2} < \frac{\alpha_1}{\alpha_2} < 2$, and $s_1 + s_2 > 0$, it follows that the corresponding constant factor is a product of two integrals of the form (2.54). Moreover, the degree of homogeneity of the above kernel is $-(s_1+s_2)$, so Corollary 2.1 yields the following consequence.

Corollary 2.5 *Let p, q, and λ be as in* (2.1) *and* (2.2)*, and let $\alpha_1, \alpha_2 > 0$, $\frac{1}{2} < \frac{\alpha_1}{\alpha_2} < 2$, $s_1 + s_2 > 0$. If $f, g : \mathbb{R}_+ \to \mathbb{R}$ are non-negative measurable functions, then the inequalities*

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+\alpha_1 y)^{\lambda s_1}(x+\alpha_2 y)^{\lambda s_2}} dxdy \\
&\le K' \left[\int_0^\infty x^{\frac{p}{q'}(1-s_1-s_2)+p(A_1-A_2)} f^p(x)dx \right]^{\frac{1}{p}} \\
&\times \left[\int_0^\infty y^{\frac{q}{p'}(1-s_1-s_2)+q(A_2-A_1)} g^q(y)dy \right]^{\frac{1}{q}}
\end{aligned} \tag{2.56}$$

and

$$\begin{aligned}
&\left[\int_0^\infty y^{\frac{q'}{p'}(s_1+s_2-1)+q'(A_1-A_2)} \left(\int_0^\infty \frac{f(x)}{(x+\alpha_1 y)^{\lambda s_1}(x+\alpha_2 y)^{\lambda s_2}} dx \right)^{q'} dy \right]^{\frac{1}{q'}} \\
&\le K' \left[\int_0^\infty x^{\frac{p}{q'}(1-s_1-s_2)+p(A_1-A_2)} f^p(x)dx \right]^{\frac{1}{p}}
\end{aligned} \tag{2.57}$$

hold for all $A_1 \in (\frac{1-s_1-s_2}{p'}, \frac{1}{p'})$ and $A_2 \in (\frac{1-s_1-s_2}{q'}, \frac{1}{q'})$, where

$$\begin{aligned}
K' &= \alpha_1^{\frac{1-s_1}{p'}-A_1} \alpha_2^{A_2-\frac{1}{q'}-\frac{s_2}{p'}} B^{\frac{1}{q'}}(1-q'A_2, s_1+s_2+q'A_2-1) \\
&\times B^{\frac{1}{p'}}(1-p'A_1, s_1+s_2+p'A_1-1) F^{\frac{1}{q'}}\left(s_1, 1-q'A_2; s_1+s_2; 1-\frac{\alpha_1}{\alpha_2}\right) \\
&\times F^{\frac{1}{p'}}\left(s_2, 1-p'A_1; s_1+s_2; 1-\frac{\alpha_1}{\alpha_2}\right).
\end{aligned} \tag{2.58}$$

Moreover, inequalities (2.56) *and* (2.57) *are equivalent.*

Remark 2.18 Every hypergeometric series always has the value 1 when $z = 0$. Hence, if $\alpha_1 = \alpha_2$, then the hypergeometric part of the above constant (2.58) takes the value of 1 and inequalities (2.56) and (2.57) reduce to already known cases, considered in Section 1.4.

On the other hand, hypergeometric series (2.52) converges also for $z = 1$ when b is a non-positive integer or $c > a+b$. In addition, since

$$F(a,b;c;1) = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}, \tag{2.59}$$

(see [26]), relation (2.54) also holds for $\alpha = 0$ and reduces to the well-known formula for the Beta function:

$$\int_0^\infty \frac{x^{b-1}}{(1+\gamma x)^c} dx = \gamma^{-b} B(b, c-b), \quad \text{where } c > b > 0. \tag{2.60}$$

Therefore, if $\alpha_1 = 0$ or $\alpha_2 = 0$, the previous corollary also yields inequalities without hypergeometric part. Such cases are omitted here.

We conclude the previous discussion with a remark about the best possible constant factors appearing in inequalities (2.56) and (2.57), which can be achieved in the conjugate case.

Remark 2.19 If p and q are conjugate exponents, then the kernel (2.55) fulfills conditions as in Theorem 1.13 (Chapter 1). Moreover, since K is homogeneous of degree $-(s_1+s_2)$, the parameters A_1 and A_2, that provide the best possible constant factor, must fulfill the relation $pA_2+qA_1 = 2-s_1-s_2$. Under the above assumptions and utilizing the so-called Euler identity $F(a,b;c;z) = (1-z)^{c-a-b}F(c-a,c-b;c;z)$ (see [26]), the constant factor K' from Corollary 2.5 reduces to

$$\alpha_2{}^{pA_2-1}B(1-pA_2, 1-qA_1)F\left(s_1, 1-pA_2; s_1+s_2; 1-\frac{\alpha_1}{\alpha_2}\right)$$

and is the best possible.

2.5.2 Hilbert-type inequalities and generalized hypergeometric functions ${}_mF_n$

Gaussian hypergeometric function is naturally extended to an arbitrary number of parameters, which gives generalized hypergeometric function. Such generalized series also have integral representations. More precisely, we shall use the so-called Poisson-type integral representations, in order to obtain multidimensional Hilbert-type inequalities in such settings.

Hence, before obtaining such multidimensional inequalities, we introduce the notion of a generalized hypergeometric function ${}_mF_n$, as well as its integral representations.

By a generalized hypergeometric function ${}_mF_n$ we mean the sum of the series

$${}_mF_n(a_1,\ldots,a_m;b_1,\ldots,b_n;z) = \sum_{k=0}^{\infty} \frac{a_1^{\overline{k}}a_2^{\overline{k}}\ldots a_m^{\overline{k}}}{b_1^{\overline{k}}b_2^{\overline{k}}\ldots b_n^{\overline{k}}}\cdot\frac{z^k}{k!},$$

where $a_i^{\overline{k}}$, $b_i^{\overline{k}}$ are the rising factorial powers and $z\in\mathbb{C}$, in domain of its convergence: $\Omega = \{|z|<\infty\}$ for $m\le n$ and $\Omega = \{|z|<1\}$ for $m=n+1$, or its analytical continuation in $\{|z|>1, |\arg(1-z)|<\pi\}$, in the latter case. One may also consider z as a real variable $z\in[0,\infty)$.

The paper [50] provides a unified treatment of generalized hypergeometric functions by means of a generalized fractional calculus. More precisely, hypergeometric functions ${}_mF_n$ are separated into three classes depending on whether $m<n$, $m=n$ or $m=n+1$. Further, hypergeometric functions of each class are represented as generalized fractional integrals or derivatives of three basic elementary functions:

$$\cos_{n-m+1}(z)\ (m<n), \qquad z^{\alpha}\exp z\ (m=n), \qquad z^{\alpha}(1-z)^{\beta}\ (m=n+1).$$

Here, $\cos_t(z)$ is the so-called generalized cosine function of order $t \geq 2$, defined as

$$\cos_t(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{kt}}{(kt)!}, \quad t \geq 2,$$

where $\cos z = \cos_2(z)$. The above mentioned representations lead to several new integral formulas for ${}_mF_n$ functions and allow their study in a unified way. Moreover, the generalized fractional calculus is developed in [49].

Now, we introduce the Poisson-type integral representations of the above classes of hypergeometric functions ${}_mF_n$, established in [50].

1° **First case**: $m < n$. If the conditions

$$b_k > \frac{k}{n-m+1}, \; k = 1,2,\dots,n-m, \qquad b_{n-m+k} > a_k > 0, \; k = 1,2,\dots,m, \tag{2.61}$$

are fulfilled, then the following Poisson-type integral representation is valid:

$$\begin{aligned}
&{}_mF_n(a_1,\dots,a_m;b_1,\dots,b_n;-z) \\
&= C\int_0^1 \cdots \int_0^1 \prod_{k=1}^{n-m} \left[\frac{(1-t_k)^{b_k-(k/(n-m+1))-1}}{\Gamma(b_k-(k/(n-m+1)))} t_k{}^{(k/(n-m+1))-1} \right] \\
&\times \prod_{k=n-m+1}^{n} \left[\frac{(1-t_k)^{b_k-a_{k-n+m}-1}}{\Gamma(b_k-a_{k-n+m})} t_k{}^{a_{k-n+m}-1} \right] \\
&\times \cos_{n-m+1}\left[(n-m+1)(zt_1\dots t_n)^{1/(n-m+1)} \right] dt_1\dots dt_n.
\end{aligned} \tag{2.62}$$

The constant C is defined by $C = \sqrt{\frac{n-m+1}{(2\pi)^{n-m}}} \frac{\prod_{j=1}^{n}\Gamma(b_j)}{\prod_{j=1}^{m}\Gamma(a_j)}$.

2° **Second case:** $m = n$. Assuming that

$$b_k > a_k > 0, \; k = 1,2,\dots,n, \tag{2.63}$$

we have

$$\begin{aligned}
&{}_nF_n(a_1,\dots,a_n;b_1,\dots,b_n;z) \\
&= E\int_0^1 \cdots \int_0^1 \prod_{k=1}^{n} \left[\frac{(1-t_k)^{b_k-a_k-1} t_k{}^{a_k-1}}{\Gamma(b_k-a_k)} \right] \exp(zt_1\dots t_n) dt_1\dots dt_n,
\end{aligned} \tag{2.64}$$

where the constant E is defined by $E = \prod_{j=1}^{n} \frac{\Gamma(b_j)}{\Gamma(a_j)}$.

3° **Third case:** $m = n+1$. If the conditions

$$b_k > a_{k+1} > 0, \;\; k = 1,2,\dots,n, \tag{2.65}$$

are satisfied, then the following Poisson-type integral representation is valid:

$$\begin{aligned}
&{}_{n+1}F_n(a_1,\dots,a_{n+1};b_1,\dots,b_n;\pm z) \\
&= M\int_0^1 \cdots \int_0^1 \prod_{k=1}^{n} (1-t_k)^{b_k-a_{k+1}-1} t_k{}^{a_{k+1}-1} (1 \mp zt_1\dots t_n)^{-a_1} dt_1\dots dt_n.
\end{aligned} \tag{2.66}$$

The constant M is defined by $M = \prod_{j=1}^{n} \frac{\Gamma(b_j)}{\Gamma(a_{j+1})\Gamma(b_j - a_{j+1})}$.

Clearly, the above integral representations are valid for complex numbers z in the corresponding domains of convergence. These representations will be crucial in obtaining examples with multidimensional Hilbert-type inequalities. More precisely, our next step is to find appropriate kernels such that the formulas for the functions F_i, $i = 1,2,\ldots,n$, from Theorem 2.5, reduce to the above stated Poisson-type integral representations.

The kernel involving exponential function

We are going to find a more suitable form of integral representation (2.64). Namely, utilizing substitutions $1 - t_i = \frac{1}{1+x_i}$, $i = 1,2,\ldots,n$, and the well-known relationship between the Beta and the Gamma function, i.e. $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, $x, y > 0$, we have

$$\begin{aligned} &\int_{\mathbb{R}_+^n} \prod_{i=1}^{n} \frac{x_i^{a_i - 1}}{(1+x_i)^{b_i}} \exp\left(x \prod_{i=1}^{n} \frac{x_i}{1+x_i} \right) dx_1 dx_2 \ldots dx_n \\ &= {}_nF_n(\mathbf{a}; \mathbf{b}; x) \prod_{i=1}^{n} B(a_i, b_i - a_i), \end{aligned} \tag{2.67}$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$. However, the coordinates a_i and b_i fulfill conditions as in (2.63).

Considering Theorem 2.5 with the kernel $K : \mathbb{R}_+^n \to \mathbb{R}$, defined by

$$K(x_1, x_2, \ldots, x_n) = \frac{\exp\left(\prod_{i=1}^{n} \frac{x_i}{1+x_i} \right)}{\prod_{i=1}^{n} (1+x_i)^{b_i}}, \tag{2.68}$$

and the power weight functions $\varphi_{ij} : \mathbb{R}_+ \to \mathbb{R}$, $i, j = 1,2,\ldots,n$, the above integral representation appears when calculating the constant factors involved in the corresponding inequality.

Theorem 2.9 *Let λ, p_i, p_i', q_i, $i = 1,2,\ldots,n$, $n \geq 2$, be real parameters fulfilling* (2.26), (2.27), *and* (2.30), *and let A_{ij}, $i, j = 1,\ldots,n$, be real parameters such that $\sum_{i=1}^{n} A_{ij} = 0$ for $j = 1,\ldots,n$. If $\beta' = \prod_{i,j=1, i\neq j}^{n} B^{\frac{1}{q_i}}(1 + q_i A_{ij}, b_j - 1 - q_i A_{ij})$ and $f_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1,\ldots,n$, are non-negative measurable functions, then the inequalities*

$$\begin{aligned} &\int_{\mathbb{R}_+^n} \frac{\prod_{i=1}^{n} f_i(x_i)}{\prod_{i=1}^{n} (1+x_i)^{\lambda b_i}} \exp^{\lambda}\left(\prod_{i=1}^{n} \frac{x_i}{1+x_i} \right) dx_1 dx_2 \ldots dx_n \\ &\leq \beta' \prod_{i=1}^{n} \left[\int_0^\infty x_i^{p_i A_{ii}} (1+x_i)^{(1-\lambda) p_i b_i - b_i} \right. \\ &\quad \left. \times {}_{n-1}F_{n-1}{}^{1-(1-\lambda)p_i}\left(\mathbf{1} + q_i \mathbf{A_i}; \mathbf{b_i}; \frac{x_i}{1+x_i} \right) f_i^{p_i}(x_i) dx_i \right]^{\frac{1}{p_i}} \end{aligned} \tag{2.69}$$

and

$$\left\{ \int_0^\infty x_n{}^{-p_n' A_{nn}} (1+x_n)^{b_n(\lambda p_n'-1)} {}_{n-1}F_{n-1}{}^{1-\lambda p_n'} \left(\mathbf{1}+q_n\mathbf{A_n};\mathbf{b_n};\frac{x_n}{1+x_n} \right) \right.$$
$$\left. \times \left[\int_{\mathbb{R}_+^{n-1}} \frac{\prod_{i=1}^{n-1} f_i(x_i)}{\prod_{i=1}^{n}(1+x_i)^{\lambda b_i}} \exp^\lambda \left(\prod_{i=1}^{n} \frac{x_i}{1+x_i} \right) dx_1 dx_2 \ldots dx_{n-1} \right]^{p_n'} dx_n \right\}^{\frac{1}{p_n'}}$$
$$\leq \beta' \prod_{i=1}^{n-1} \left[\int_0^\infty x_i{}^{p_i A_{ii}} (1+x_i)^{(1-\lambda)p_i b_i - b_i} \right.$$
$$\left. \times {}_{n-1}F_{n-1}{}^{1-(1-\lambda)p_i} \left(\mathbf{1}+q_i\mathbf{A_i};\mathbf{b_i};\frac{x_i}{1+x_i} \right) f_i{}^{p_i}(x_i)dx_i \right]^{\frac{1}{p_i}} \tag{2.70}$$

hold for all parameters A_{ij}, $i \neq j$, *such that* $q_iA_{ij} \in (-1, b_j - 1)$, *where* $\mathbf{1} + q_i\mathbf{A_i} = (1+q_iA_{i1}, 1+q_iA_{i2}, \ldots, 1+q_iA_{i,i-1}, 1+q_iA_{i,i+1}, \ldots, 1+q_iA_{in})$ *and* $\mathbf{b_i} = (b_1, b_2, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$. *Moreover, these inequalities are equivalent.*

Proof. The proof is based on a simple use of Theorem 2.5. Taking into account notation from Theorem 2.5, as well as considering the kernel defined by (2.68) and the power weight functions $\phi_{ij}(x_j) = x_j^{A_{ij}}$, $i, j = 1, 2, \ldots, n$, we have

$$F_i(x_i) = (1+x_i)^{-\frac{b_i}{q_i}} \left[\int_{\mathbb{R}_+^{n-1}} \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{x_j{}^{(1+q_iA_{ij})-1}}{(1+x_j)^{b_j}} \exp\left(\frac{x_i}{1+x_i} \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{x_j}{1+x_j} \right) \right.$$
$$\left. \times dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n \right]^{\frac{1}{q_i}}.$$

Clearly, the above power functions are well-defined, that is, $\prod_{i,j=1}^n \phi_{ij}(x_j) = 1$, since $\sum_{i=1}^n A_{ij} = 0$ for $j = 1, \ldots, n$. Now, exploiting integral representation (2.67), we have

$$F_i(x_i) = (1+x_i)^{-\frac{b_i}{q_i}} \beta_{n-1}F_{n-1}{}^{\frac{1}{q_i}} \left(\mathbf{1}+q_i\mathbf{A_i};\mathbf{b_i};\frac{x_i}{1+x_i} \right),$$

that is, after substituting the expressions for $F_i(x_i)$, $i = 1, 2, \ldots, n$, in (2.31) and (2.32) (see Theorem 2.5), we obtain desired inequalities. It should be noticed here that parameters A_{ij}, $i \neq j$, fulfill conditions $q_iA_{ij} \in (-1, b_j - 1)$, $i, j = 1, 2, \ldots, n$, since the arguments of the Beta function are positive. □

Remark 2.20 Considering some particular values of parameters A_{ij}, $i, j = 1, 2, \ldots, n$, we can simplify the constant β' from Theorem 2.9. For example, taking the arithmetic mean of the borders of intervals defining parameters A_{ij}, $i \neq j$, we have $A_{ij} = \frac{b_j - 2}{2q_i}$, hence the constant factor becomes $\beta' = \prod_{i=1}^n B^{\frac{1}{p_i'}}\left(\frac{b_i}{2}, \frac{b_i}{2}\right)$. In that case, the parameters A_{ii}, $i = 1, 2, \ldots, n$, are defined by $A_{ii} = -\frac{b_j - 2}{2p_i'}$.

Since $0 < \frac{t}{1+t} < 1$, $t \in \mathbb{R}_+$, we have that ${}_{n-1}F_{n-1}\left(\mathbf{1}+q_i\mathbf{A_i};\mathbf{b_i};\frac{x_i}{1+x_i}\right) < {}_{n-1}F_{n-1}$ $(\mathbf{1}+q_i\mathbf{A_i};\mathbf{b_i};1)$. Applying this estimate to inequalities (2.69) and (2.70), we obtain the constant factor expressed in terms of the hypergeometric function ${}_{n-1}F_{n-1}$.

Corollary 2.6 *Under the assumptions of Theorem* 2.9*, the inequalities*

$$\begin{aligned}&\int_{\mathbb{R}_+^n} \frac{\prod_{i=1}^n f_i(x_i)}{\prod_{i=1}^n (1+x_i)^{\lambda b_i}} \exp^{\lambda}\left(\prod_{i=1}^n \frac{x_i}{1+x_i}\right) dx_1 dx_2 \dots dx_n \\ &\leq \beta_H' \prod_{i=1}^n \left[\int_0^\infty x_i^{p_iA_{ii}}(1+x_i)^{(1-\lambda)p_ib_i-b_i} f_i^{p_i}(x_i)dx_i\right]^{\frac{1}{p_i}} \end{aligned} \tag{2.71}$$

and

$$\begin{aligned}&\left\{\int_0^\infty x_n^{-p_n'A_{nn}}(1+x_n)^{b_n(\lambda p_n'-1)}\right. \\ &\left.\times\left[\int_{\mathbb{R}_+^{n-1}} \frac{\prod_{i=1}^{n-1} f_i(x_i)}{\prod_{i=1}^n (1+x_i)^{\lambda b_i}} \exp^{\lambda}\left(\prod_{i=1}^n \frac{x_i}{1+x_i}\right) dx_1 dx_2 \dots dx_{n-1}\right]^{p_n'} dx_n\right\}^{\frac{1}{p_n'}} \\ &\leq \beta_H' \prod_{i=1}^{n-1} \left[\int_0^\infty x_i^{p_iA_{ii}}(1+x_i)^{(1-\lambda)p_ib_i-b_i} f_i^{p_i}(x_i)dx_i\right]^{\frac{1}{p_i}} \end{aligned} \tag{2.72}$$

hold and are equivalent, where

$$\beta_H' = \beta' \prod_{i=1}^n {}_{n-1}F_{n-1}^{\frac{1}{q_i}}(\mathbf{1}+q_i\mathbf{A_i};\mathbf{b_i};1) \tag{2.73}$$

and β' is defined in Theorem 2.9*.*

It remains to investigate the other two integral representations (2.62) and (2.66). We use the same procedure as for the kernel involving the exponential function.

The kernel involving cosine function

We consider here integral representation (2.62) for $m = n-1$. Applying the substitutions $1 - t_i = \frac{1}{1+x_i}$, $i = 1,2,\dots,n$, this representation can be expressed in a more suitable form

$$\begin{aligned}&\int_{\mathbb{R}_+^n} \prod_{i=1}^n \frac{x_i^{a_{i-1}-1}}{(1+x_i)^{b_i}} \cos\left[2\left(x\prod_{i=1}^n \frac{x_i}{1+x_i}\right)^{\frac{1}{2}}\right] dx_1 dx_2 \dots dx_n \\ &= \left(\pi+\frac{1}{2}\right)^{-\frac{1}{2}} {}_{n-1}F_n(\mathbf{a};\mathbf{b};-x)\, B\left(b_1-\frac{1}{2},\frac{1}{2}\right)\prod_{i=2}^n B(a_{i-1},b_i-a_{i-1}), \end{aligned} \tag{2.74}$$

where $\mathbf{a} = (a_1,a_2,\dots,a_{n-1})$, $\mathbf{b} = (b_1,b_2,\dots,b_n)$ and $a_0 = \frac{1}{2}$. Of course, the coordinates of vectors $\mathbf{a}$ and $\mathbf{b}$ fulfill conditions as in (2.61).

The previous integral formula will be needed when applying Theorem 2.5 to the kernel $K:\mathbb{R}_+^n \to \mathbb{R}$ defined by

$$K(x_1,x_2,\dots,x_n) = \frac{\cos\left(\prod_{i=1}^n \frac{x_i}{1+x_i}\right)^{\frac{1}{2}}}{\prod_{i=1}^n (1+x_i)^{b_i}}, \tag{2.75}$$

and the power weight functions $\varphi_{ij}:\mathbb{R}_+ \to \mathbb{R}$, $i,j=1,2,\dots,n$.

Theorem 2.10 *Let λ, p_i, p_i', q_i, $i=1,2,\dots,n$, $n \geq 2$, be as in* (2.26), (2.27) *and* (2.30), *and let A_{ij}, $i,j=1,\dots,n$, be real parameters such that $\sum_{i=1}^n A_{ij}=0$ for $j=1,\dots,n$. If $\gamma' = \left(\pi+\frac{1}{2}\right)^{-\frac{\lambda}{2}} \beta' \prod_{i=1}^n B^{\frac{1}{q_i}}\left(b_{i+1}-\frac{1}{2},\frac{1}{2}\right)$ (β' is defined in Theorem* 2.9*), and $f_i:\mathbb{R}_+\to\mathbb{R}$, $i=1,\dots,n$, are non-negative measurable functions, then the inequalities*

$$\begin{aligned}
&\int_{\mathbb{R}_+^n} \frac{\prod_{i=1}^n f_i(x_i)}{\prod_{i=1}^n (1+x_i)^{\lambda b_i}} \cos^\lambda \left(\prod_{i=1}^n \frac{x_i}{1+x_i}\right)^{\frac{1}{2}} dx_1 dx_2 \dots dx_n \\
&\leq \gamma' \prod_{i=1}^n \left[\int_0^\infty x_i^{p_i A_{ii}} (1+x_i)^{(1-\lambda)p_i b_i - b_i} \right. \\
&\qquad \left. \times {}_{n-2}F_{n-1}{}^{1-(1-\lambda)p_i}\left(\mathbf{1}+q_i\mathbf{A_{i,i+1}};\mathbf{b_i};\frac{x_i}{4(1+x_i)}\right) f_i^{p_i}(x_i)dx_i\right]^{\frac{1}{p_i}}
\end{aligned} \tag{2.76}$$

and

$$\begin{aligned}
&\left\{\int_0^\infty x_n^{-p_n' A_{nn}} (1+x_n)^{b_n(\lambda p_n'-1)} {}_{n-2}F_{n-1}{}^{1-\lambda p_n'}\left(\mathbf{1}+q_n\mathbf{A_{n1}};\mathbf{b_n};\frac{x_n}{4(1+x_n)}\right)\right. \\
&\quad \left. \times \left[\int_{\mathbb{R}_+^{n-1}} \frac{\prod_{i=1}^{n-1} f_i(x_i)}{\prod_{i=1}^n (1+x_i)^{\lambda b_i}} \cos^\lambda \left(\prod_{i=1}^n \frac{x_i}{1+x_i}\right)^{\frac{1}{2}} dx_1 dx_2 \dots dx_{n-1}\right]^{p_n'} dx_n\right\}^{\frac{1}{p_n'}} \\
&\leq \gamma' \prod_{i=1}^{n-1} \left[\int_0^\infty x_i^{p_i A_{ii}} (1+x_i)^{(1-\lambda)p_i b_i - b_i} \right. \\
&\qquad \left. \times {}_{n-2}F_{n-1}{}^{1-(1-\lambda)p_i}\left(\mathbf{1}+q_i\mathbf{A_{i,i+1}};\mathbf{b_i};\frac{x_i}{4(1+x_i)}\right) f_i^{p_i}(x_i)dx_i\right]^{\frac{1}{p_i}}
\end{aligned} \tag{2.77}$$

hold for all parameters A_{ij}, $i,j=1,2,\dots,n$, and b_i, $i=1,2,\dots,n$, such that $q_iA_{ij}\in(-1,b_j-1)$, $j-i\notin\{0,1,1-n\}$, $q_iA_{i,i+1}=-\frac{1}{2}$, $b_i>\frac{1}{2}$, $i=1,2,\dots,n$, and where $\mathbf{1}+q_i\mathbf{A_{i,i+1}} = (1+q_iA_{i1},\dots,1+q_iA_{i,i-1},1+q_iA_{i,i+2},\dots,1+q_iA_{in})$, $\mathbf{b_i}=(b_1,b_2,\dots,b_{i-1},b_{i+1},\dots,b_n)$. Moreover, these inequalities are equivalent.

Proof. Considering Theorem 2.5 with the kernel defined by (2.75) and the functions

$\phi_{ij}(x_j) = x_j^{A_{ij}}$, $i, j = 1, 2, \ldots, n$, we have

$$F_i(x_i) = (1+x_i)^{-\frac{b_i}{q_i}} \left[\int_{\mathbb{R}_+^{n-1}} \frac{x_{i+1}^{(1+q_iA_{i,i+1})-1}}{(1+x_{i+1})^{b_{i+1}}} \prod_{\substack{j=1\\ j\neq i,i+1}}^{n} \frac{x_j{}^{(1+q_iA_{ij})-1}}{(1+x_j)^{b_j}} \right.$$
$$\left. \times \cos\left[2\left(\frac{x_i}{4(1+x_i)} \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{x_j}{1+x_j} \right)^{\frac{1}{2}} \right] dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n \right]^{\frac{1}{q_i}}.$$

Hence, the above integral representation (2.74) yields

$$F_i(x_i) = (1+x_i)^{-\frac{b_i}{q_i}} \beta'\left(\pi + \frac{1}{2}\right)^{-\frac{1}{2q_i}} B^{\frac{1}{q_i}}\left(b_{i+1} - \frac{1}{2}, \frac{1}{2}\right)$$
$$\times {}_{n-2}F_{n-1}{}^{\frac{1}{q_i}}\left(\mathbf{1} + q_i\mathbf{A_{i,i+1}}; \mathbf{b_i}; \frac{x_i}{4(1+x_i)}\right),$$

where β' is defined in Theorem 2.9, and the result follows. Note also that we assume congruence modulo n for the parameters A_{ij}, i.e. $A_{n,n+1} = A_{n1}$. □

Remark 2.21 Taking into account the obvious estimate

$${}_{n-2}F_{n-1}\left(\mathbf{1} + q_i\mathbf{A_{i,i+1}}; \mathbf{b_i}; \frac{x_i}{4(1+x_i)}\right) < {}_{n-2}F_{n-1}\left(\mathbf{1} + q_i\mathbf{A_{i,i+1}}; \mathbf{b_i}; 1\right),$$

which holds for all $x_i \in \mathbb{R}_+$, we also obtain the inequalities as those from Corollary 2.6, with the kernel (2.75) and the corresponding constant factor

$$\gamma' \prod_{i=1}^{n} {}_{n-2}F_{n-1}{}^{\frac{1}{q_i}}\left(\mathbf{1} + q_i\mathbf{A_{i,i+1}}; \mathbf{b_i}; 1\right),$$

where γ' is defined in Theorem 2.10.

Fractional kernel

It remains to consider the remaining case, that is, the integral representation (2.66). It can be rewritten in a more convenient form, that is,

$$\int_{\mathbb{R}_+^n} \prod_{i=1}^{n} \frac{x_i{}^{a_{i+1}-1}}{(1+x_i)^{b_i}} \left(1 - x\prod_{i=1}^{n} \frac{x_i}{1+x_i}\right)^{-a_1} dx_1 dx_2 \ldots dx_n$$
$$= {}_{n+1}F_n(\mathbf{a}; \mathbf{b}; x) \prod_{i=1}^{n} B(a_{i+1}, b_i - a_{i+1}), \tag{2.78}$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_{n+1})$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ are such that $b_i > a_{i+1} > 0$, $i = 1, 2, \ldots, n$.

The previous integral representation is essential when considering Theorem 2.5 with the fractional kernel $K:\mathbb{R}_+^n \to \mathbb{R}$, defined by

$$K(x_1,x_2,\ldots,x_n) = \frac{\left(1-\prod_{i=1}^n \frac{x_i}{1+x_i}\right)^{-s}}{\prod_{i=1}^n (1+x_i)^{b_i}}, \quad s>0, \tag{2.79}$$

and the power weight functions.

Theorem 2.11 *Let λ, p_i, p_i', q_i, $i=1,2,\ldots,n$, $n\geq 2$, be as in* (2.26), (2.27) *and* (2.30), *and let A_{ij}, $i,j=1,\ldots,n$, be real parameters such that $\sum_{i=1}^n A_{ij}=0$ for $j=1,\ldots,n$. If the constant β' is as in Theorem* 2.9 *and $f_i:\mathbb{R}_+\to\mathbb{R}$, $i=1,\ldots,n$, are non-negative measurable functions, then the inequalities*

$$\begin{aligned}
&\int_{\mathbb{R}_+^n} \frac{\prod_{i=1}^n f_i(x_i)}{\prod_{i=1}^n (1+x_i)^{\lambda b_i}} \left(1-\prod_{i=1}^n \frac{x_i}{1+x_i}\right)^{-\lambda s} dx_1 dx_2 \ldots dx_n \\
&\leq \beta' \prod_{i=1}^n \left[\int_0^\infty x_i^{p_i A_{ii}} (1+x_i)^{(1-\lambda)p_i b_i - b_i}\right. \\
&\qquad \left.\times {}_nF_{n-1}{}^{1-(1-\lambda)p_i}\left(s, \mathbf{1}+q_i\mathbf{A_i}; \mathbf{b_i}; \frac{x_i}{1+x_i}\right) dx_i\right]^{\frac{1}{p_i}}
\end{aligned} \tag{2.80}$$

and

$$\begin{aligned}
&\left\{\int_0^\infty x_n^{-p_n' A_{nn}} (1+x_n)^{b_n(\lambda p_n'-1)} {}_nF_{n-1}{}^{1-\lambda p_n'}\left(s, \mathbf{1}+q_n\mathbf{A_n}; \mathbf{b_n}; \frac{x_n}{1+x_n}\right)\right. \\
&\quad \left.\times \left[\int_{\mathbb{R}_+^{n-1}} \frac{\prod_{i=1}^{n-1} f_i(x_i)}{\prod_{i=1}^n (1+x_i)^{\lambda b_i}} \left(1-\prod_{i=1}^n \frac{x_i}{1+x_i}\right)^{-\lambda s} dx_1 dx_2 \ldots dx_{n-1}\right]^{p_n'} dx_n\right\}^{\frac{1}{p_n'}} \\
&\leq \beta' \prod_{i=1}^{n-1} \left[\int_0^\infty x_i^{p_i A_{ii}} (1+x_i)^{(1-\lambda)p_i b_i - b_i}\right. \\
&\qquad \left.\times {}_nF_{n-1}{}^{1-(1-\lambda)p_i}\left(s, \mathbf{1}+q_i\mathbf{A_i}; \mathbf{b_i}; \frac{x_i}{1+x_i}\right) dx_i\right]^{\frac{1}{p_i}}
\end{aligned} \tag{2.81}$$

hold for all parameters A_{ij}, $i\neq j$, such that $q_iA_{ij}\in(-1,b_j-1)$, $i,j=1,2,\ldots,n$, where $\mathbf{1}+q_i\mathbf{A_i}=(1+q_iA_{i1},1+q_iA_{i2},\ldots,1+q_iA_{i,i-1},1+q_iA_{i,i+1},\ldots,1+q_iA_{in})$ and $\mathbf{b_i}=(b_1,b_2,\ldots,b_{i-1},b_{i+1},\ldots,b_n)$. Moreover, these inequalities are equivalent.

Proof. Applying Theorem 2.5 with the fractional kernel (2.79) and the power weight functions $\phi_{ij}(x_j)=x_j^{A_{ij}}$, $i,j=1,2,\ldots,n$, and taking into account the integral representation (2.78), we have that

$$F_i(x_i) = (1+x_i)^{-\frac{b_i}{q_i}} \beta_n F_{n-1}{}^{\frac{1}{q_i}}\left(s, \mathbf{1}+q_i\mathbf{A_i}; \mathbf{b_i}; \frac{x_i}{1+x_i}\right),$$

so the result follows. □

Remark 2.22 It is obvious that the estimate

$$ {}_nF_{n-1}\left(s,\mathbf{1}+q_i\mathbf{A_i};\mathbf{b_i};\frac{x_i}{1+x_i}\right) < {}_nF_{n-1}\left(s,\mathbf{1}+q_i\mathbf{A_i};\mathbf{b_i};1\right) $$

holds for all $x_i \in \mathbb{R}_+$. Therefore, we also obtain the inequalities as those from Corollary 2.6, with the fractional kernel (2.79) and the corresponding constant factor $\beta' \prod_{i=1}^{n} {}_nF_{n-1}^{\frac{1}{q_i}}\left(s,\mathbf{1}+q_i\mathbf{A_i};\mathbf{b_i};1\right)$, where β' is defined in Theorem 2.9.

Remark 2.23 Regarding Remark 2.11, it follows that the equality in any of inequalities from this subsection is possible if and only if at least one of the functions $f_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1,\dots,n$, is the zero function. Namely, the equality is possible only for the kernels with separated variables, which is not the case here.

Remark 2.24 Hilbert-type inequalities from this section related to Gaussian hypergeometric function are derived in [62], while the multidimensional extension via the Poisson-type integral representations is developed in [60].

2.6 Hilbert-type inequalities and related operators

So far, we have considered Hilbert-type inequalities with homogeneous kernels of negative degree of homogeneity. The reason for this was in the fact that we required the kernel to be decreasing in both arguments (see Remark 1.6, Chapter 1). Such requirement was essential in order to obtain some estimates when considering inequalities with integrals taken over bounded intervals in $\mathbb{R}_+$. However, when considering the integrals taken over $\mathbb{R}_+$, such requirement was redundant.

On the other hand, assuming the convergence, Hilbert-type inequalities with integrals taken over the set $\mathbb{R}_+$ can also be considered for homogeneous kernels of zero-degree.

In this section we shall be more concerned with an equivalent form of the Hilbert-type inequality, that is, with the Hardy-Hilbert-type inequality. Namely, the Hardy-Hilbert form of inequality provides the possibility of defining certain integral operators between the weighted Lebesgue spaces and determining their norms in some particular cases.

The previous program will be carried out for the Hardy-Hilbert-type inequalities including a homogeneous kernel with zero-degree of homogeneity. First, we are going to derive appropriate inequalities, and then, to consider the related operators between the weighted Lebesgue spaces, which naturally arise from these inequalities.

2.6.1 Hilbert-type inequalities involving a homogeneous function of zero-degree

In this subsection we give a unified treatment of Hilbert-type inequalities with homogeneous kernels of zero-degree.

The results that follow are considered in the setting with non-conjugate exponents, in a slightly generalized form. More precisely, the kernel includes two differentiable functions with some additional properties. We start with some definitions and notation that will be valid throughout this section.

Let (a,b) be an interval on the real line and let $u,v:(a,b)\to\mathbb{R}$ be non-negative measurable functions satisfying the following conditions:

(i) u and v are differentiable on (a,b);

(ii) u and v are strictly increasing on (a,b);

(iii) $\lim_{x\to a^+} u(t)=v(t)=0$ and $\lim_{x\to b^-} u(t)=v(t)=\infty$.

In this section, by $k_0:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ we denote a non-negative measurable homogeneous function of zero-degree. Also, we deal with the integral

$$c_0(\alpha)=\int_0^\infty k_0(1,t)t^{-\alpha}dt. \tag{2.82}$$

We consider only the parameters α such that (2.82) converges.

The above functions u, v and k_0 will be essential in defining the corresponding kernel. More precisely, $K_0:(a,b)\times(a,b)\to\mathbb{R}$ denotes a non-negative measurable function defined by

$$K_0(x,y)=k_0\left(u(x),v(y)\right), \tag{2.83}$$

where u and v are assumed to fulfill conditions (i)-(iii).

First we provide Hilbert-type and Hardy-Hilbert-type inequalities involving the above kernel K_0.

Theorem 2.12 *Let p, q, and λ be as in* (2.1) *and* (2.2)*, and let $u,v:(a,b)\to\mathbb{R}$ be non-negative measurable functions fulfilling conditions* (i)-(iii). *If $K_0:(a,b)\times(a,b)\to\mathbb{R}$ is a non-negative measurable function defined by* (2.83)*, and A_1, A_2 are real parameters such that $c_0(2-p'A_1)<\infty$, $c_0(q'A_2)<\infty$, then the inequalities*

$$\begin{aligned}\int_a^b\int_a^b K_0^\lambda(x,y)f(x)g(y)\,dxdy &\le c_0^{\frac{1}{p'}}(2-p'A_1)c_0^{\frac{1}{q'}}(q'A_2)\\ &\times\left[\int_a^b \frac{u^{(A_1-A_2)p+\frac{p}{q'}}(x)}{(u')^{p-1}(x)}f^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_a^b \frac{v^{(A_2-A_1)q+\frac{q}{p'}}(y)}{(v')^{q-1}(y)}g^q(y)\,dy\right]^{\frac{1}{q}}\end{aligned} \tag{2.84}$$

and

$$\begin{aligned}&\left\{\int_a^b \frac{v'(y)}{v^{(A_2-A_1)q'+\frac{q'}{p'}}(y)}\left[\int_a^b K_0^\lambda(x,y)f(x)\,dx\right]^{q'}dy\right\}^{\frac{1}{q'}}\\ &\le c_0^{\frac{1}{p'}}(2-p'A_1)c_0^{\frac{1}{q'}}(q'A_2)\left[\int_a^b \frac{u^{(A_1-A_2)p+\frac{p}{q'}}(x)}{(u')^{p-1}(x)}f^p(x)\,dx\right]^{\frac{1}{p}}\end{aligned} \tag{2.85}$$

hold for all non-negative measurable functions $f,g:(a,b)\to\mathbb{R}$ and are equivalent. Equalities in (2.84) *and* (2.85) *hold if and only if $f=0$ or $g=0$ a.e. on (a,b).*

Proof. Similarly as for homogeneous kernels of negative degree, this theorem is also a consequence of Theorem 2.1. Namely, using notation from Theorem 2.1, we put the kernel K_0, defined by (2.83), and the weight functions

$$\varphi(x)=\frac{u^{A_1}(x)}{(u')^{\frac{1}{p'}}(x)},\qquad \psi(y)=\frac{v^{A_2}(y)}{(v')^{\frac{1}{q'}}(y)},\qquad x\in(a,b),$$

in inequalities (2.9) and (2.10). Then, defining $v(y)=tu(x)$ and using the change of variable, we get

$$\begin{aligned} F(x) &= \left[\int_a^b k_0(u(x),v(y))v^{-q'A_2}(y)v'(y)dy\right]^{\frac{1}{q'}} \\ &= u^{\frac{1}{q'}-A_2}(x)\left[\int_0^\infty k_0(1,t)t^{-q'A_2}dt\right]^{\frac{1}{q'}} = c_0^{\frac{1}{q'}}(q'A_2)u^{\frac{1}{q'}-A_2}(x),\quad x\in(a,b). \end{aligned}$$

Using the same argument, we also have

$$G(y)=c_0^{\frac{1}{p'}}(2-p'A_1)v^{\frac{1}{p'}-A_1}(x),\quad y\in(a,b),$$

which proves relations (2.84) and (2.85).

The case of equality follows immediately from Remark 2.3. □

We continue with some basic examples of functions u and v, fulfilling conditions (i)-(iii). For example, functions $u,v:\mathbb{R}_+\to\mathbb{R}$, defined by $u(x)=Ax^\mu$, $v(y)=By^\nu$, where $A,B,\mu,\nu>0$, fulfill the above mentioned conditions, hence the following result is a direct consequence of Theorem 2.12.

Corollary 2.7 *Suppose p, q, and λ are as in* (2.1) *and* (2.2)*, and $k_0:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ is a non-negative measurable homogeneous function of zero-degree. Let A_1, A_2 be real parameters such that $c_0(2-p'A_1)<\infty$ and $c_0(q'A_2)<\infty$, and let $A,B,\mu,\nu>0$. Then the inequalities*

$$\begin{aligned} &\int_0^\infty\int_0^\infty k_0^\lambda(Ax^\mu,By^\nu)f(x)g(y)\,dxdy \\ &\quad\le c_0^{\frac{1}{p'}}(2-p'A_1)c_0^{\frac{1}{q'}}(q'A_2)C_0\left[\int_0^\infty x^{\left(A_1-A_2+\frac{1}{q'}\right)p\mu+(p-1)(1-\mu)}f^p(x)\,dx\right]^{\frac{1}{p}} \\ &\quad\quad\times\left[\int_0^\infty y^{\left(A_2-A_1+\frac{1}{p'}\right)q\nu+(q-1)(1-\nu)}g^q(y)\,dy\right]^{\frac{1}{q}} \end{aligned} \tag{2.86}$$

and

$$\left\{\int_0^\infty y^{\left(A_1-A_2-\frac{1}{p'}\right)q'\nu+\nu-1}\left[\int_0^\infty k_0^\lambda(Ax^\mu,By^\nu)f(x)\,dx\right]^{q'}dy\right\}^{\frac{1}{q'}}$$
$$\leq c_0^{\frac{1}{p'}}(2-p'A_1)c_0^{\frac{1}{q'}}(q'A_2)C_0\left[\int_0^\infty x^{\left(A_1-A_2+\frac{1}{q'}\right)p\mu+(p-1)(1-\mu)}f^p(x)\,dx\right]^{\frac{1}{p}} \quad (2.87)$$

hold for all non-negative measurable functions $f,g:\mathbb{R}_+\to\mathbb{R}$*, where*

$$C_0=\mu^{-\frac{1}{p'}}\nu^{-\frac{1}{q'}}A^{A_1-A_2+\frac{1}{q'}-\frac{1}{p'}}B^{A_2-A_1+\frac{1}{p'}-\frac{1}{q'}}.$$

Moreover, these inequalities are equivalent. In addition, equalities in (2.86) *and* (2.87) *hold if and only if* $f=0$ *or* $g=0$ *a.e. on* $\mathbb{R}_+$.

In the sequel we consider another interesting kernel dependent on homogeneous function $k_0:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ of zero-degree. Namely, let $\widetilde{K}_0:(a,b)\times(a,b)\to\mathbb{R}$ be a non-negative measurable function defined by

$$\widetilde{K}_0(x,y)=k_0\left(1,u(x)v(y)\right), \quad (2.88)$$

where the functions u and v fulfill conditions (i)-(iii). The following result is an analogue of Theorem 2.12.

Theorem 2.13 *Let* p*,* q*, and* λ *be as in* (2.1) *and* (2.2)*, and let* $u,v:(a,b)\to\mathbb{R}$ *be non-negative measurable functions fulfilling conditions* (i)-(iii)*. Further, suppose* $\widetilde{K}_0:(a,b)\times(a,b)\to\mathbb{R}$ *is a non-negative measurable function defined by* (2.88)*, and* A_1*,* A_2 *are real parameters such that* $c_0(p'A_1)<\infty$*,* $c_0(q'A_2)<\infty$*. Then the inequalities*

$$\int_a^b\int_a^b\widetilde{K}_0^\lambda(x,y)f(x)g(y)\,dxdy\leq c_0^{\frac{1}{p'}}(p'A_1)c_0^{\frac{1}{q'}}(q'A_2)$$
$$\times\left[\int_a^b\frac{u^{(A_1+A_2)p-\frac{p}{q'}}(x)}{(u')^{p-1}(x)}f^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_a^b\frac{v^{(A_1+A_2)q-\frac{q}{p'}}(y)}{(v')^{q-1}(y)}g^q(y)\,dy\right]^{\frac{1}{q}} \quad (2.89)$$

and

$$\left\{\int_a^b\frac{v'(y)}{v^{(A_1+A_2)q'-\frac{q'}{p'}}(y)}\left[\int_a^b\widetilde{K}_0^\lambda(x,y)f(x)\,dx\right]^{q'}dy\right\}^{\frac{1}{q'}}$$
$$\leq c_0^{\frac{1}{p'}}(p'A_1)c_0^{\frac{1}{q'}}(q'A_2)\left[\int_a^b\frac{u^{(A_1+A_2)p-\frac{p}{q'}}(x)}{(u')^{p-1}(x)}f^p(x)\,dx\right]^{\frac{1}{p}} \quad (2.90)$$

hold for all non-negative measurable functions $f,g:(a,b)\to\mathbb{R}$ *and are equivalent. Equalities in* (2.89) *and* (2.90) *hold if and only if* $f=0$ *or* $g=0$ *a.e. on* (a,b).

Proof. It follows directly from Theorem 2.1. □

As an application of Theorem 2.13, we consider exponential functions $u, v : \mathbb{R} \to \mathbb{R}$, that is, $u(x) = \exp x$ and $v(y) = \exp y$.

Corollary 2.8 *Suppose p, q, and λ are as in* (2.1) *and* (2.2)*, and $k_0 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a non-negative measurable homogeneous function of zero-degree. If A_1 and A_2 are real parameters such that $c_0(p'A_1) < \infty$ and $c_0(q'A_2) < \infty$, then the inequalities*

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k_0^{\lambda}(1,\exp(x+y)) f(x) g(y)\,dxdy &\le c_0^{\frac{1}{p'}}(p'A_1) c_0^{\frac{1}{q'}}(q'A_2)\\ &\times \left[\int_{-\infty}^{\infty} \exp\left[\left((A_1+A_2)p+1-2p+\frac{p}{q}\right)x\right] f^p(x)\,dx\right]^{\frac{1}{p}}\\ &\times \left[\int_{-\infty}^{\infty} \exp\left[\left((A_1+A_2)q+1-2q+\frac{q}{p}\right)y\right] g^q(y)\,dy\right]^{\frac{1}{q}}\end{aligned} \tag{2.91}$$

and

$$\begin{aligned}&\left\{\int_{-\infty}^{\infty} \exp\left[\left(1-(A_1+A_2)q'+\frac{q'}{p'}\right)y\right]\left[\int_{-\infty}^{\infty} k_0^{\lambda}(1,\exp(x+y)) f(x)\,dx\right]^{q'} dy\right\}^{\frac{1}{q'}}\\ &\le c_0^{\frac{1}{p'}}(p'A_1) c_0^{\frac{1}{q'}}(q'A_2)\\ &\times \left[\int_{-\infty}^{\infty} \exp\left[\left((A_1+A_2)p+1-2p+\frac{p}{q}\right)x\right] f^p(x)\,dx\right]^{\frac{1}{p}}\end{aligned} \tag{2.92}$$

hold for all non-negative measurable functions $f, g : \mathbb{R} \to \mathbb{R}$ and are equivalent. Equalities in (2.91) *and* (2.92) *hold if and only if $f = 0$ or $g = 0$ a.e on $\mathbb{R}$.*

2.6.2 On some related Hilbert-type operators

The Hardy-Hilbert-type inequalities (2.85) and (2.90) allow us precise definition of Hilbert-type operators and some conclusions about their norms as well. We consider here the weighted Lebesgue space $L_w^r(a,b)$ consisting of all measurable functions $f : (a,b) \to \mathbb{R}$ with a finite norm $\|f\|_{L_w^r(a,b)} = \left[\int_a^b w(x)|f^r(x)|\,dx\right]^{\frac{1}{r}}$. Here, $r > 1$ and $w : (a,b) \to \mathbb{R}$ is a non-negative measurable weight function.

Hence, in the setting with p, q and λ as in (2.1) and (2.2), the kernel K_0 defined by (2.83), and the weight functions

$$\Phi(x) = \frac{u^{(A_1-A_2)p+\frac{p}{q'}}(x)}{(u')^{p-1}(x)} \quad \text{and} \quad \Psi(y) = \frac{v^{(A_2-A_1)q+\frac{q}{p'}}(y)}{(v')^{q-1}(y)}, \quad x, y \in (a,b),$$

we can define the operator $\mathscr{T}_0 : L_{\Phi}^p(a,b) \to L_{\Psi^{1-q'}}^{q'}(a,b)$ as

$$(\mathscr{T}_0 f)(y) = \int_a^b K_0^{\lambda}(x,y) f(x)\,dx, \quad y \in (a,b). \tag{2.93}$$

Clearly, the operator $\mathscr{T}_0$ is well-defined since inequality (2.85) implies that $\mathscr{T}_0 f \in L^{q'}_{\Psi^{1-q'}}(a,b)$. Moreover, considering the norm of operator $\mathscr{T}_0$, that is,

$$\|\mathscr{T}_0\| = \sup_{f \in L^p_\Phi(a,b), f \neq 0} \frac{\|\mathscr{T}_0 f\|_{L^{q'}_{\Psi^{1-q'}}(a,b)}}{\|f\|_{L^p_\Phi(a,b)}}, \tag{2.94}$$

it follows that the operator $\mathscr{T}_0$ is bounded. In other words, inequality (2.85) yields the upper bound for the norm of this operator, i.e.

$$\|\mathscr{T}_0\| \leq c_0^{\frac{1}{p'}}(2 - p'A_1) c_0^{\frac{1}{q'}}(q'A_2).$$

The same type of discussion can also be applied when considering the Hilbert-type operator related to kernel $\widetilde{K}_0$, defined by (2.88). More precisely, if we denote

$$\widetilde{\Phi}(x) = \frac{u^{(A_1+A_2)p - \frac{p}{q'}}(x)}{(u')^{p-1}(x)} \text{ and } \widetilde{\Psi}(y) = \frac{v^{(A_1+A_2)q - \frac{q}{p'}}(y)}{(v')^{q-1}(y)}, \qquad x, y \in (a,b),$$

we can define the operator $\widetilde{\mathscr{T}_0} : L^p_{\widetilde{\Phi}}(a,b) \to L^{q'}_{\widetilde{\Psi}^{1-q'}}(a,b)$ as

$$\big(\widetilde{\mathscr{T}_0} f\big)(y) = \int_a^b \widetilde{K}_0^\lambda(x,y) f(x)\, dx, \quad y \in (a,b). \tag{2.95}$$

The operator $\widetilde{\mathscr{T}_0}$ is well-defined since inequality (2.90) implies $\widetilde{\mathscr{T}_0} f \in L^{q'}_{\widetilde{\Psi}^{1-q'}}(a,b)$. In addition, since the norm of operator $\widetilde{\mathscr{T}_0}$ is defined by

$$\|\widetilde{\mathscr{T}_0}\| = \sup_{f \in L^p_{\widetilde{\Phi}}(a,b), f \neq 0} \frac{\|\widetilde{\mathscr{T}_0} f\|_{L^{q'}_{\widetilde{\Psi}^{1-q'}}(a,b)}}{\|f\|_{L^p_{\widetilde{\Phi}}(a,b)}}, \tag{2.96}$$

inequality (2.90) yields the upper bound for the norm, that is,

$$\|\widetilde{\mathscr{T}_0}\| \leq c_0^{\frac{1}{p'}}(p'A_1) c_0^{\frac{1}{q'}}(q'A_2).$$

We consider now some particular cases in which we are able to find the norm of operators $\widetilde{T}$ and $\widetilde{T}_0$. Obviously, this problem is equivalent to the problem of finding the best possible constants in inequalities (2.85) and (2.90).

The problem of finding the best possible constants in Hilbert-type inequalities with non-conjugate exponents is still open. We solved the mentioned problem in the case of conjugate exponents, as in Section 1.4 (Chapter 1).

Hence, in order to obtain the best possible constants in Theorems 2.12 and 2.13, in the case of conjugate exponents, we exploit Theorem 1.13. The parameters A_1 and A_2 should fulfill condition (1.29) for $s = 0$, that is, $pA_2 + qA_1 = 2$, and the homogeneous kernel of zero-degree should be such that $k_0(1,t)$ is bounded on $(0,1)$. Under these assumptions, the constant factor on the right-hand sides of inequalities (2.84) and (2.85) reduces to $c_0(pA_2)$.

Theorem 2.14 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, let A_1 and A_2 be real parameters such that $qA_1+pA_2=2$, and let $c_0(pA_2)<\infty$. If the function $k_0(1,t)$ is bounded on $(0,1)$, then the constant $c_0(pA_2)$ is the best possible in both inequalities* (2.84) *and* (2.85).

According to the above discussion, the previous result also provides the norm of the operator $\mathscr{T}_0$, defined by (2.93), in the case of conjugate exponents.

Corollary 2.9 *Suppose that the assumptions of Theorem* 2.14 *are fulfilled. Then the norm of operator $\mathscr{T}_0 : L^p_{\Phi}(a,b) \to L^p_{\Psi^{1-p}}(a,b)$, defined by the formula $(\mathscr{T}_0 f)(y) = \int_a^b K_0(x,y)f(x)dx$, $y \in (a,b)$, is $\|\mathscr{T}_0\| = c_0(pA_2)$.*

Utilizing a suitable change of variables, Theorem 1.13 can also be adjusted in establishing the best possible constant in Theorem 2.13. It turns out that the parameters A_1 and A_2 satisfy $pA_2 = qA_1$, providing the same constant factor $c_0(pA_2)$ as in the previous case.

Theorem 2.15 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, let A_1 and A_2 be real parameters such that $qA_1 = pA_2$, and let $c_0(pA_2)<\infty$. If the function $k_0(1,t)$ is bounded on $(0,1)$, then the constant factor $c_0(pA_2)$ is the best possible in both inequalities* (2.89) *and* (2.90).

We conclude this subsection with the operator analogue of Theorem 2.15.

Corollary 2.10 *Suppose that the assumptions as in Theorem* 2.15 *are fulfilled. Then, the norm of the operator $\widetilde{\mathscr{T}_0} : L^p_{\widetilde{\Phi}}(a,b) \to L^p_{\widetilde{\Psi}^{1-p}}(a,b)$, defined by $\big(\widetilde{\mathscr{T}_0} f\big)(y) = \int_a^b \widetilde{K}_0(x,y)f(x)dx$, $y \in (a,b)$, is $\|\widetilde{\mathscr{T}_0}\| = c_0(pA_2)$.*

2.6.3 On some related Hardy-type operators

In the previous subsection we have considered two examples of Hilbert-type operators. We can also generate some other operators related to inequalities (2.85) and (2.90). More precisely, if $k_0 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a homogeneous function of zero-degree, then the function $\overline{k}_0 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, defined by

$$\overline{k}_0(x,y) = k_0(x,y)\chi_{x\geq y}(x,y) = \begin{cases} 0, & x<y, \\ k_0(x,y), & x \geq y, \end{cases} \tag{2.97}$$

is also homogeneous of zero-degree. Now, we use the same procedure as in the previous subsection, but with this function $\overline{k}_0$. The kernel $\overline{k}_0$ looks like the classical Hardy kernel (see Chapter 8), so the corresponding operators will be referred to as the Hardy-type operators. Moreover, since

$$c_0(\alpha) = \int_0^\infty \overline{k}_0(1,t)t^{-\alpha}dt = \int_0^1 k_0(1,t)t^{-\alpha}dt,$$

we define

$$\overline{c}_0(\alpha) = \int_0^1 k_0(1,t)t^{-\alpha}dt. \tag{2.98}$$

Bearing in mind the notation from the previous subsection, we define the operator $\overline{\mathscr{T}}_0 : L^p_{\Phi}(a,b) \to L^{q'}_{\Psi^{1-q'}}(a,b)$ by

$$\left(\overline{\mathscr{T}}_0 f\right)(y) = \int_{u^{(-1)}(v(y))}^{b} K_0^{\lambda}(x,y) f(x)\, dx, \quad y \in (a,b), \tag{2.99}$$

and $\overline{\overline{\mathscr{T}}}_0 : L^p_{\widetilde{\Phi}}(a,b) \to L^{q'}_{\widetilde{\Psi}^{1-q'}}(a,b)$ by

$$\left(\overline{\overline{\mathscr{T}}}_0 f\right)(y) = \int_{a}^{u^{(-1)}\left(\frac{1}{v(y)}\right)} \widetilde{K}_0^{\lambda}(x,y) f(x)\, dx, \quad y \in (a,b), \tag{2.100}$$

where $u^{(-1)}$ denotes the inverse of the function u.

The following two corollaries show that these definitions are meaningful. In other words, we provide the corresponding analogues of Theorems 2.12 and 2.13 for the above Hardy-type kernel of zero-degree.

Corollary 2.11 *Let p, q, and λ be as in* (2.1) *and* (2.2)*, and let $u,v : (a,b) \to \mathbb{R}$ be non-negative measurable functions fulfilling conditions* (i)-(iii)*. Further, suppose $K_0 : (a,b) \times (a,b) \to \mathbb{R}$ is a non-negative measurable function defined by* (2.83)*. If A_1 and A_2 are real parameters such that $\overline{c}_0(2-p'A_1) < \infty$ and $\overline{c}_0(q'A_2) < \infty$, then the inequalities*

$$\int_a^b \int_{u^{(-1)}(v(y))}^{b} K_0^{\lambda}(x,y) f(x) g(y)\, dx dy \le \overline{c}_0^{\frac{1}{p'}}(2-p'A_1)\overline{c}_0^{\frac{1}{q'}}(q'A_2)$$
$$\times \left[\int_a^b \frac{u^{(A_1-A_2)p+\frac{p}{q'}}(x)}{(u')^{p-1}(x)} f^p(x)\, dx\right]^{\frac{1}{p}} \left[\int_a^b \frac{v^{(A_2-A_1)q+\frac{q}{p'}}(y)}{(v')^{q-1}(y)} g^q(y)\, dy\right]^{\frac{1}{q}} \tag{2.101}$$

and

$$\left\{\int_a^b \frac{v'(y)}{v^{(A_2-A_1)q'+\frac{q'}{p'}}(y)} \left[\int_{u^{(-1)}(v(y))}^{b} K_0^{\lambda}(x,y) f(x)\, dx\right]^{q'} dy\right\}^{\frac{1}{q'}}$$
$$\le \overline{c}_0^{\frac{1}{p'}}(2-p'A_1)\overline{c}_0^{\frac{1}{q'}}(q'A_2) \left[\int_a^b \frac{u^{(A_1-A_2)p+\frac{p}{q'}}(x)}{(u')^{p-1}(x)} f^p(x)\, dx\right]^{\frac{1}{p}} \tag{2.102}$$

hold for all non-negative measurable functions $f,g : (a,b) \to \mathbb{R}$ and are equivalent. Equalities in (2.101) *and* (2.102) *hold if and only if $f = 0$ or $g = 0$ a.e. on (a,b).*

Corollary 2.12 *Let p, q, and λ be as in* (2.1) *and* (2.2)*, and let $u,v : (a,b) \to \mathbb{R}$ be non-negative measurable functions fulfilling conditions* (i)-(iii)*. Further, suppose $\widetilde{K}_0 : (a,b) \times (a,b) \to \mathbb{R}$ is a non-negative measurable function defined by* (2.88) *and A_1, A_2 are real* *parameters such that* $\overline{c}_0(p'A_1) < \infty$ *and* $\overline{c}_0(q'A_2) < \infty$*. Then the inequalities*

$$\int_a^b \int_a^{u^{(-1)}\left(\frac{1}{v(y)}\right)} \widetilde{K}_0^\lambda(x,y)f(x)g(y)\,dxdy \le \overline{c}_0^{\frac{1}{p'}}(p'A_1)\overline{c}_0^{\frac{1}{q'}}(q'A_2)$$
$$\times \left[\int_a^b \frac{u^{(A_1+A_2)p-\frac{p}{q'}}(x)}{(u')^{p-1}(x)} f^p(x)\,dx\right]^{\frac{1}{p}} \left[\int_a^b \frac{v^{(A_1+A_2)q-\frac{q}{p'}}(y)}{(v')^{q-1}(y)} g^q(y)\,dy\right]^{\frac{1}{q}} \tag{2.103}$$

and

$$\left\{\int_a^b \frac{v'(y)}{v^{(A_1+A_2)q'-\frac{q'}{p'}}(y)} \left[\int_a^{u^{(-1)}\left(\frac{1}{v(y)}\right)} \widetilde{K}_0^\lambda(x,y)f(x)\,dx\right]^{q'} dy\right\}^{\frac{1}{q'}}$$
$$\le \overline{c}_0^{\frac{1}{p'}}(p'A_1)\overline{c}_0^{\frac{1}{q'}}(q'A_2)\left[\int_a^b \frac{u^{(A_1+A_2)p-\frac{p}{q'}}(x)}{(u')^{p-1}(x)} f^p(x)\,dx\right]^{\frac{1}{p}} \tag{2.104}$$

hold for all non-negative measurable functions $f,g:(a,b)\to\mathbb{R}$ *and are equivalent. Equalities in* (2.103) *and* (2.104) *hold if and only if* $f=0$ *or* $g=0$ *a.e. on* (a,b)*.*

Remark 2.25 Regarding Corollaries 2.11 and 2.12, one can easily obtain analogues of Corollaries 2.7 and 2.8. It suffices to replace the constant factor $c_0(\alpha)$ with $\overline{c}_0(\alpha)$ and change the integration intervals according to definitions (2.99) and (2.100). For example, if $u(x)=\exp x$ and $v(y)=\exp y$, then $u^{(-1)}(v(y))=y$ and $u^{(-1)}\left(\frac{1}{v(y)}\right)=-y$.

Remark 2.26 The discussion about the best constants, carried out in the previous subsection, holds for the Hardy-type operators (2.99) and (2.100), as well. More precisely, if $p,q>1$ are conjugate exponents and A_1,A_2 are such that $pA_2+qA_1=2$, then $\overline{c}_0(pA_2)$ is the best possible constant in inequalities (2.101) and (2.102). Moreover, the same constant is also the best possible in (2.103) and (2.104), provided that $qA_1=pA_2$ in the conjugate setting. Of course, under the above assumptions, the norms of operators $\overline{\mathscr{T}}_0$ and $\overline{\overline{\mathscr{T}}}_0$ are both equal to $\overline{c}_0(pA_2)$.

2.6.4 Applications

We conclude this section with some consequences of Theorem 2.12, Theorem 2.13, Corollary 2.11 and Corollary 2.12, obtained by a suitable choice of parameters A_1 and A_2. Namely, if $A_1=\frac{1}{p'}$ and $A_2=\frac{1}{q'}$, then the inequalities (2.84) and (2.85) respectively read as

$$\int_a^b \int_a^b k_0^\lambda(u(x),v(y))f(x)g(y)\,dxdy$$
$$\le c_0^\lambda(1)\left[\int_a^b \left(\frac{u}{u'}\right)^{p-1}(x)f^p(x)\,dx\right]^{\frac{1}{p}} \left[\int_a^b \left(\frac{v}{v'}\right)^{p-1}(y)g^q(y)\,dy\right]^{\frac{1}{q}} \tag{2.105}$$

and

$$\left\{\int_a^b \left(\frac{v'}{v}\right)(y)\left[\int_a^b k_0^\lambda(u(x),v(y))f(x)\,dx\right]^{q'} dy\right\}^{\frac{1}{q'}}$$

$$\leq c_0^\lambda(1)\left[\int_a^b \left(\frac{u}{u'}\right)^{p-1}(x)f^p(x)\,dx\right]^{\frac{1}{p}}. \tag{2.106}$$

Remark 2.27 For the same choice of parameters A_1 and A_2, i.e. $A_1 = \frac{1}{p'}$ and $A_2 = \frac{1}{q'}$, Theorem 2.13 yields the same inequalities as (2.105) and (2.106), with the kernel $k_0(1,u(x)v(y))$ instead of $k_0(u(x),v(y))$. In other words, inequalities (2.105) and (2.106) also hold after replacing the kernel $k_0(u(x),v(y))$ with $k_0(1,u(x)v(y))$.

The same setting also yields the corresponding result for the Hardy-type kernel (2.97). Namely, for $A_1 = \frac{1}{p'}$ and $A_2 = \frac{1}{q'}$, Corollary 2.11 yields inequalities

$$\int_a^b \int_{u^{(-1)}(v(y))}^b k_0^\lambda(u(x),v(y))f(x)g(y)\,dxdy$$

$$\leq \overline{c}_0^\lambda(1)\left[\int_a^b \left(\frac{u}{u'}\right)^{p-1}(x)f^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_a^b \left(\frac{v}{v'}\right)^{p-1}(y)g^q(y)\,dy\right]^{\frac{1}{q}} \tag{2.107}$$

and

$$\left\{\int_a^b \left(\frac{v'}{v}\right)(y)\left[\int_{u^{(-1)}(v(y))}^b k_0^\lambda(u(x),v(y))f(x)\,dx\right]^{q'} dy\right\}^{\frac{1}{q'}}$$

$$\leq \overline{c}_0^\lambda(1)\left[\int_a^b \left(\frac{u}{u'}\right)^{p-1}(x)f^p(x)\,dx\right]^{\frac{1}{p}}. \tag{2.108}$$

Remark 2.28 According to Corollary 2.12, inequalities (2.107) and (2.108) also hold after replacing the kernel $k_0(u(x),v(y))$ with $k_0(1,u(x)v(y))$ and replacing integration interval $(u^{(-1)}(v(y)),b)$ with $\left(a,u^{(-1)}\left(\frac{1}{v(y)}\right)\right)$.

Remark 2.29 Considering the conjugate case, parameters that generate inequalities (2.105), (2.106), (2.107), and (2.108) become $A_1 = \frac{1}{q}$ and $A_2 = \frac{1}{p}$. These parameters fulfill condition $qA_1 + pA_2 = 2$, providing the best possible constant factors $c_0(1)$ and $\overline{c}_0(1)$.

In order to complete the previous discussion, we provide some examples of homogeneous kernels with zero-degree of homogeneity, which generate the corresponding constant factors expressed in terms of the Beta and the Gamma function.

Example 2.1 *Let $\alpha > 0$, $\beta > -1$ and*

$$k_0(x,y) = \left(\frac{\min\{x,y\}}{\max\{x,y\}}\right)^\alpha \left|\log\left(\frac{y}{x}\right)\right|^\beta.$$

Then,

$$\int_0^\infty \left(\frac{\min\{1,t\}}{\max\{1,t\}}\right)^\alpha |\log t|^\beta t^{-1}dt = \int_0^1 t^{\alpha-1}(-\log t)^\beta dt + \int_1^\infty t^{-\alpha-1}(\log t)^\beta dt.$$

Since,

$$\int_0^1 t^{\alpha-1}(-\log t)^\beta dt = \int_1^\infty t^{-\alpha-1}(\log t)^\beta dt = \int_0^\infty e^{-\alpha v} v^\beta dv = \frac{\Gamma(\beta+1)}{\alpha^{\beta+1}},$$

the above constant factors become

$$c_0(1) = \frac{2\Gamma(\beta+1)}{\alpha^{\beta+1}} \qquad \textit{and} \qquad \overline{c}_0(1) = \frac{\Gamma(\beta+1)}{\alpha^{\beta+1}}.$$

Example 2.2 *For the homogeneous function defined by*

$$k_0(x,y) = \frac{\min\{x,y\}}{\max\{x,y\}} \arctan\left(\frac{y}{x}\right),$$

we have

$$\int_0^\infty \frac{\min\{1,t\}}{\max\{1,t\}} t^{-1} \arctan t\, dt = \int_0^1 \arctan t\, dt + \int_1^\infty t^{-2} \arctan t\, dt.$$

Clearly, the above two integrals can be resolved by using integration by parts, yielding the constant factors

$$c_0(1) = \frac{\pi}{2} \qquad \textit{and} \qquad \overline{c}_0(1) = \frac{\pi}{4} - \frac{\log 2}{2}.$$

Example 2.3 *Let* $0 < \alpha < 1$ *and*

$$k_0(x,y) = \left(\frac{\min\{x,y\}}{|x-y|}\right)^\alpha.$$

Then,

$$\int_0^\infty \left(\frac{\min\{1,t\}}{|1-t|}\right)^\alpha t^{-1}dt = \int_0^1 t^{\alpha-1}(1-t)^{-\alpha}dt + \int_1^\infty t^{-1}(t-1)^{-\alpha}dt.$$

Since

$$\int_0^1 t^{\alpha-1}(1-t)^{-\alpha}dt = \int_1^\infty t^{-1}(t-1)^{-\alpha}dt = B(1-\alpha,\alpha),$$

we obtain the constant factors expressed in terms of the Beta function:

$$c_0(1) = 2B(1-\alpha,\alpha) \qquad \textit{and} \qquad \overline{c}_0(1) = B(1-\alpha,\alpha).$$

Remark 2.30 Results in this section are taken from [166]. However, similar Hilbert-type and Hardy-type operators can also be derived for homogeneous kernels of arbitrary degree of homogeneity. For more details about similar results, the reader is referred to [37], [158], [160], [163], [165], [167].

Chapter 3

Hilbert-type inequalities with vector variables

This chapter deals with Hilbert-type inequalities involving real valued functions with vector arguments. We start this overview with the so-called doubly weighted Hardy-Littlewood-Sobolev inequality of Stein and Weiss, [118],

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)g(y)}{|x|^{\alpha}|x-y|^{s}|y|^{\beta}}\,dxdy \le C_{\alpha,\beta,p,q,n}\parallel f\parallel_{L^p(\mathbb{R}^n)}\parallel g\parallel_{L^q(\mathbb{R}^n)}, \tag{3.1}$$

which holds for $n\in\mathbb{N}$, $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}>1$, λ as in (2.2), $0\le\alpha<\frac{n}{p'}$, $0\le\beta<\frac{n}{q'}$, $s=n\lambda-\alpha-\beta$, and all non-negative functions $f\in L^p(\mathbb{R}^n)$ and $g\in L^q(\mathbb{R}^n)$. Here, and throughout this chapter, $|x|$ denotes the Euclidean norm of the vector $x\in\mathbb{R}^n$.

In [83], Lieb proved the existence of optimizers for (3.1), that is, functions f and g which, when inserted into (3.1), yield equality with the smallest possible constant $C_{\alpha,\beta,p,q,n}$. Moreover, for $p=q$ and $\alpha=\beta=0$, the constant and maximizing functions were explicitly computed in [83]. In particular, Lieb proved that

$$C_{0,0,p,p,n}=\pi^{\frac{n}{p'}}\frac{\Gamma\left(\frac{n}{2}-\frac{n}{p'}\right)}{\Gamma\left(\frac{n}{p}\right)}\left[\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(n)}\right]^{\frac{2}{p'}-1},$$

where Γ is the Gamma function. Unfortunately, neither $C_{\alpha,\beta,p,q,n}$ nor the optimizers are known for any other choice of the parameters appearing in (3.1). It was only shown (see e. g. [84]) that for the classical Hardy-Littlewood-Sobolev inequality, that is, for (3.1) with

$\alpha=\beta=0$ and $s=n\lambda$, the estimate

$$C_{0,0,p,q,n} \leq \frac{(p')^{\lambda}+(q')^{\lambda}}{(1-\lambda)pq}\left(\frac{\lambda}{n}|\mathbb{S}^{n-1}|\right)^{\lambda} \tag{3.2}$$

holds, where

$$|\mathbb{S}^{n-1}| = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \tag{3.3}$$

is the area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. For more details about the Hardy-Littlewood-Sobolev inequality, the reader is referred to [83] and [84].

One of the most important tasks in this chapter is to derive the explicit upper bounds for the general case of the doubly weighted Hardy-Littlewood-Sobolev inequality (3.1). This can be done by virtue of Theorem 2.8 from the previous chapter and the Selberg integral formula (see e. g. [119]). Moreover, the Selberg integral formula and the appropriate results from the previous chapter will also be utilized in deriving numerous particular Hilbert-type inequalities with vector variables.

On the other hand, some vector extensions of the usual Beta function will also be utilized in obtaining some particular Hilbert-type inequalities. Finally, at the end of this chapter we shall present some multidimensional Hilbert-type inequalities including a general homogeneous kernel and power weight functions whose arguments are norms of the corresponding vectors.

3.1 Explicit upper bounds for the doubly weighted Hardy-Littlewood-Sobolev inequality

The main goal of this section is to derive a form of the doubly weighted Hardy-Littlewood-Sobolev inequality (3.1) with an explicit constant factor on its right-hand side. In fact, we derive explicit upper bounds for the sharp constant $C_{\alpha,\beta,p,q,n}$ in (3.1).

Main results in this section will be based on Theorem 2.8 (see Section 2.4), and the well-known Selberg integral formula

$$\begin{aligned}\int_{\mathbb{R}^{kn}} |x_k|^{\alpha_k-n}\left(\prod_{i=1}^{k-1}|x_{i+1}-x_i|^{\alpha_i-n}\right)|x_1-y|^{\alpha_0-n}\,dx_1\ldots dx_k\\ = \frac{\Gamma_n(\alpha_0)\cdots\Gamma_n(\alpha_k)}{\Gamma_n(\alpha_0+\ldots+\alpha_k)}\,|y|^{\alpha_0+\ldots+\alpha_k-n},\end{aligned} \tag{3.4}$$

for arbitrary $k,n\in\mathbb{N}$, $y\in\mathbb{R}^n$, and $0<\alpha_0,\ldots,\alpha_k<n$ such that $0<\sum_{i=0}^{k}\alpha_i<n$, where $\Gamma_n(\alpha)=\frac{\pi^{\frac{n}{2}}2^{\alpha}\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n}{2}-\frac{\alpha}{2}\right)}$ is the n-dimensional Gamma function. For $k=1$, the product appearing in (3.4) is defined to be equal to 1. In [119], Stein derived the above Selberg integral

formula with two parameters using the Riesz potential (see also [25]). It should be noticed here that

$$\Gamma_n(n-\alpha) = \frac{(2\pi)^n}{\Gamma_n(\alpha)}, \quad 0 < \alpha < n. \tag{3.5}$$

The Selberg integral formula is very useful in numerous parts of mathematics, especially in representation theory and in mathematical physics.

In order to establish our main results, we first reformulate (3.4) in a form which will be more suitable for our computations.

Lemma 3.1 *Suppose $k, n \in \mathbb{N}$, $0 < \beta_1, \ldots, \beta_k, s < n$ are such that $\sum_{i=1}^{k} \beta_i + s > kn$, and $y \in \mathbb{R}^n$. Then*

$$\begin{aligned}\int_{\mathbb{R}^{kn}} &\frac{|x_1|^{-\beta_1} \ldots |x_k|^{-\beta_k}}{|x_1 + \ldots + x_k + y|^s} \, dx_1 \ldots dx_k \\ &= \frac{\Gamma_n(n-\beta_1) \cdots \Gamma_n(n-\beta_k) \Gamma_n(n-s)}{\Gamma_n((k+1)n - \beta_1 - \ldots - \beta_k - s)} \, |y|^{kn - \beta_1 - \ldots - \beta_k - s}.\end{aligned} \tag{3.6}$$

Proof. Set $\alpha_i = n - \beta_{i+1}$, $i = 0, \ldots, k-1$, and $\alpha_k = n - s$. Substituting first $t_1 = x_1 + y$ and then $t_2 = t_1 + x_2$, the left-hand side of (3.6) becomes

$$\begin{aligned}&\int_{\mathbb{R}^{kn}} \frac{|x_1|^{-\beta_1} \ldots |x_k|^{-\beta_k}}{|x_1 + \ldots + x_k + y|^s} \, dx_1 \ldots dx_k \\ &= \int_{\mathbb{R}^{kn}} \frac{|t_1 - y|^{\alpha_0 - n} |x_2|^{\alpha_1 - n} \ldots |x_k|^{\alpha_{k-1} - n}}{|t_1 + x_2 + \ldots + x_k|^{n - \alpha_k}} \, dt_1 dx_2 \ldots dx_k \\ &= \int_{\mathbb{R}^{kn}} \frac{|t_1 - y|^{\alpha_0 - n} |t_2 - t_1|^{\alpha_1 - n} |x_3|^{\alpha_2 - n} \ldots |x_k|^{\alpha_{k-1} - n}}{|t_2 + x_3 + \ldots + x_k|^{n - \alpha_k}} \, dt_1 dt_2 dx_3 \ldots dx_k.\end{aligned} \tag{3.7}$$

After the sequence of similar substitutions $t_i = t_{i-1} + x_i$, $i = 2, \ldots, k$, the last line of (3.7) is finally equal to

$$\begin{aligned}&\int_{\mathbb{R}^{kn}} |t_k|^{\alpha_k - n} \left(\prod_{i=1}^{k-1} |t_{i+1} - t_i|^{\alpha_i - n} \right) |t_1 - y|^{\alpha_0 - n} \, dt_1 \ldots dt_k \\ &= \frac{\Gamma_n(\alpha_0) \cdots \Gamma_n(\alpha_k)}{\Gamma_n(\alpha_0 + \ldots + \alpha_k)} \, |y|^{\alpha_0 + \ldots + \alpha_k - n} \\ &= \frac{\Gamma_n(n-\beta_1) \cdots \Gamma_n(n-\beta_k) \Gamma_n(n-s)}{\Gamma_n((k+1)n - \beta_1 - \ldots - \beta_k - s)} \, |y|^{kn - \beta_1 - \ldots - \beta_k - s},\end{aligned}$$

where the last two equalities are obtained by the Selberg integral formula (3.4) and by replacing α_i by the corresponding expressions including β_i. □

Since the case $k = 1$ of Lemma 3.1 will be of our special interest, we state it as a separate result.

Lemma 3.2 *Let $n \in \mathbb{N}$ and $y \in \mathbb{R}^n$. If $0 < \beta, s < n$ are such that $\beta + s > n$, then*

$$\int_{\mathbb{R}^n} \frac{|x|^{-\beta}}{|x + y|^s} \, dx = \frac{\Gamma_n(n-\beta) \Gamma_n(n-s)}{\Gamma_n(2n - \beta - s)} \, |y|^{n - \beta - s}.$$

We can now obtain the doubly weighted Hardy-Littlewood-Sobolev inequality (3.1) mentioned above. More precisely, we utilize Theorem 2.8, that is, inequality (2.51) with $\Omega = \mathbb{R}^n$ and the kernel $K(x,y) = |x|^{-\alpha}|x-y|^{-s}|y|^{-\beta}$.

The first step is to consider the case when the function $g \in L^q(\mathbb{R}^n)$ on the left-hand side of (2.51) is symmetric-decreasing, that is, $g(x) \geq g(y)$ whenever $|x| \leq |y|$. For such function and $y \in \mathbb{R}^n$, $y \neq 0$, we have

$$\begin{aligned} g^q(y) &\leq \frac{1}{|B(|y|)|}\int_{B(|y|)} g^q(x)\,dx \\ &\leq \frac{1}{|B(|y|)|}\int_{\mathbb{R}^n} g^q(x)\,dx = \frac{n}{|\mathbb{S}^{n-1}|}|y|^{-n} \parallel g \parallel^q_{L^q(\mathbb{R}^n)}, \end{aligned} \tag{3.8}$$

where $B(|y|)$ denotes the ball of radius $|y|$ in $\mathbb{R}^n$, centered at the origin, and $|B(|y|)| = |y|^n \frac{|\mathbb{S}^{n-1}|}{n}$ is its volume.

Theorem 3.1 *Let $n \in \mathbb{N}$, $p > 1$ and $q > 1$ be such that $\frac{1}{p} + \frac{1}{q} > 1$, and let λ be defined by* (2.2)*. Let $0 \leq \alpha < \frac{n}{p'}$, $0 \leq \beta < \frac{n}{q'}$, and $s = n\lambda - \alpha - \beta$. Then the inequality*

$$\begin{aligned} &\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^{\alpha}|x-y|^s|y|^{\beta}}\,dxdy \\ &\quad\leq \frac{(2\pi)^{2n}\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\lambda-1}}{\Gamma_n\left(\frac{n}{p}+\alpha\right)\Gamma_n\left(\frac{n}{q}+\beta\right)\Gamma_n(s)} \parallel f \parallel_{L^p(\mathbb{R}^n)} \parallel g \parallel_{L^q(\mathbb{R}^n)} \end{aligned} \tag{3.9}$$

holds for all non-negative functions $f \in L^p(\mathbb{R}^n)$ and symmetric-decreasing functions $g \in L^q(\mathbb{R}^n)$.

Proof. Suppose that in Theorem 2.8 (see Section 2.4) we have $\Omega = \mathbb{R}^n$, $K(x,y) = |x|^{-\alpha}|x-y|^{-s}|y|^{-\beta}$, $F(x,y) = G(x,y) \equiv 1$, $\varphi(x) = |x|^{\frac{n}{pp'}}$, $\psi(y) = |y|^{\frac{n}{qq'}}$, and the Lebesgue measure dx. In this case, the left-hand side of inequality (2.51) reads

$$L = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^{\alpha}|x-y|^s|y|^{\beta}}\,dxdy, \tag{3.10}$$

while its right-hand side is a product $I_1^{\frac{1}{q'}} I_2^{\frac{1}{p'}} I_3^{1-\lambda}$, where

$$I_1 = \int_{\mathbb{R}^{2n}} \frac{|x|^{\frac{n}{p'}}|y|^{-\frac{n}{q}}}{|x|^{\alpha}|x-y|^s|y|^{\beta}} f^p(x)\,dxdy, \quad I_2 = \int_{\mathbb{R}^{2n}} \frac{|x|^{-\frac{n}{p}}|y|^{\frac{n}{q'}}}{|x|^{\alpha}|x-y|^s|y|^{\beta}} g^q(y)\,dxdy,$$

$$I_3 = \int_{\mathbb{R}^{2n}} \frac{|x|^{\frac{n}{p'}}|y|^{\frac{n}{q'}}}{|x|^{\alpha}|x-y|^s|y|^{\beta}} f^p(x)g^q(y)\,dxdy.$$

Therefore, applying the Fubini theorem, Lemma 3.2, identity (3.5), and the fact that $\alpha+\beta+s=n\lambda$, we have

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^n} |x|^{\frac{n}{p'}-\alpha} f^p(x) \int_{\mathbb{R}^n} \frac{|y|^{-\left(\frac{n}{q}+\beta\right)}}{|x-y|^s}\, dydx \\
&= \int_{\mathbb{R}^n} |x|^{\frac{n}{p'}-\alpha} f^p(x) \int_{\mathbb{R}^n} \frac{|z|^{-\left(\frac{n}{q}+\beta\right)}}{|z+x|^s}\, dzdx \\
&= \frac{\Gamma_n\left(n-\frac{n}{q}-\beta\right)\Gamma_n(n-s)}{\Gamma_n\left(2n-\frac{n}{q}-\beta-s\right)} \int_{\mathbb{R}^n} |x|^{\frac{n}{p'}-\alpha+n-\frac{n}{q}-\beta-s} f^p(x)\, dx \\
&= \frac{(2\pi)^{2n}}{\Gamma_n\left(\frac{n}{p}+\alpha\right)\Gamma_n\left(\frac{n}{q}+\beta\right)\Gamma_n(s)} \parallel f \parallel^p_{L^p(\mathbb{R}^n)} .
\end{aligned}$$

Analogously,

$$I_2 = \frac{(2\pi)^{2n}}{\Gamma_n\left(\frac{n}{p}+\alpha\right)\Gamma_n\left(\frac{n}{q}+\beta\right)\Gamma_n(s)} \parallel g \parallel^q_{L^q(\mathbb{R}^n)},$$

and, by (3.8),

$$\begin{aligned}
I_3 &\leq \frac{n}{|\mathbb{S}^{n-1}|} \parallel g \parallel^q_{L^q(\mathbb{R}^n)} \int_{\mathbb{R}^n} |x|^{\frac{n}{p'}-\alpha} f^p(x) \int_{\mathbb{R}^n} \frac{|y|^{-\left(\frac{n}{q}+\beta\right)}}{|x-y|^s}\, dydx \\
&= \frac{n}{|\mathbb{S}^{n-1}|} \frac{(2\pi)^{2n}}{\Gamma_n\left(\frac{n}{p}+\alpha\right)\Gamma_n\left(\frac{n}{q}+\beta\right)\Gamma_n(s)} \parallel f \parallel^p_{L^p(\mathbb{R}^n)} \parallel g \parallel^q_{L^q(\mathbb{R}^n)} .
\end{aligned}$$

Finally, (3.9) follows by combining (3.10) and the expressions we have obtained for the integrals I_1, I_2, and I_3. □

To obtain an analogous result for arbitrary non-negative functions $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, we utilize the general rearrangement inequality. Recall that for a Borel set $A \subset \mathbb{R}^n$ of finite Lebesgue measure, we define A^*, the symmetric rearrangement of A, to be the open ball centered at origin whose volume is that of A. The symmetric-decreasing rearrangement of a characteristic function of a set A is $\chi_A^* = \chi_{A^*}$, so if $f : \mathbb{R}^n \to \mathbb{C}$ is a Borel measurable function vanishing at infinity, the symmetric-decreasing rearrangement f^* of a function f is defined by

$$f^*(x) = \int_0^\infty \chi^*_{\{|f|>t\}}(x)dt.$$

Now, the general rearrangement inequality asserts that

$$I(f_1, f_2, \ldots, f_m) \leq I(f_1^*, f_2^*, \ldots, f_m^*), \tag{3.11}$$

where $f_1, f_2, \ldots, f_m$ are non-negative functions on $\mathbb{R}^n$, vanishing at infinity, and

$$I(f_1, f_2, \ldots, f_m) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j \left(\sum_{i=1}^{k} b_{ij} x_i \right) dx_1 dx_2 \ldots dx_k,$$

$k \leq m$, $b_{ij} \in \mathbb{R}$, $1 \leq i \leq k$, $1 \leq j \leq m$. In the following result we also use the fact that for parameter $\gamma > 0$ the function $h : \mathbb{R}^n \to \mathbb{R}$, $h(x) = |x|^{-\gamma}$, is symmetric-decreasing and vanishes at infinity, which implies that $h^* = h$. For more details about rearrangements of sets and functions, the reader is referred to [84].

Theorem 3.2 *Let $n \in \mathbb{N}$, $p > 1$ and $q > 1$ be such that $\frac{1}{p} + \frac{1}{q} > 1$, and set λ as in (2.2). Suppose $0 \leq \alpha < \frac{n}{p'}$, $0 \leq \beta < \frac{n}{q'}$, and $s = n\lambda - \alpha - \beta$. Then the inequality (3.9) holds for all non-negative functions $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$.*

Proof. Since $x \to |x|^{-\alpha}$, $x \to |x|^{-s}$, and $x \to |x|^{-\beta}$ are symmetric-decreasing functions vanishing at infinity, the general rearrangement inequality (3.11) implies that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^{\alpha}|x-y|^{s}|y|^{\beta}}\, dxdy \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f^*(x)g^*(y)}{|x|^{\alpha}|x-y|^{s}|y|^{\beta}}\, dxdy. \tag{3.12}$$

Clearly, by Theorem 3.1, the right-hand side of (3.12) is not greater than

$$K_{\alpha,\beta,p,q,n} \parallel f^* \parallel_{L^p(\mathbb{R}^n)} \parallel g^* \parallel_{L^q(\mathbb{R}^n)} = K_{\alpha,\beta,p,q,n} \parallel f \parallel_{L^p(\mathbb{R}^n)} \parallel g \parallel_{L^q(\mathbb{R}^n)}, \tag{3.13}$$

where $K_{\alpha,\beta,p,q,n}$ is the constant from the right-hand side of (3.9). To obtain equality in (3.13), we have used the fact that the symmetric-decreasing rearrangement is norm preserving (see e.g. [84]). $\square$

Remark 3.1 Note that $C_{\alpha,\beta,p,q,n} \leq K_{\alpha,\beta,p,q,n}$, where $C_{\alpha,\beta,p,q,n}$ is the sharp constant for (3.1) and $K_{\alpha,\beta,p,q,n}$ is the constant factor involved in the right-hand side of (3.9). Hence, Theorem 3.2 provides new explicit upper bounds for the doubly weighted Hardy-Littlewood-Sobolev inequality. In particular, for $\alpha = \beta = 0$ we have

$$K_{0,0,p,q,n} = \frac{(2\pi)^{2n} \left(\frac{|\mathbb{S}^{n-1}|}{n} \right)^{\lambda - 1}}{\Gamma_n\left(\frac{n}{p}\right) \Gamma_n\left(\frac{n}{q}\right) \Gamma_n(n\lambda)}, \tag{3.14}$$

while for $p = q$ the above constant reduces to

$$K_{0,0,p,p,n} = \frac{(2\pi)^{2n} \left(\frac{|\mathbb{S}^{n-1}|}{n} \right)^{\frac{2}{p'} - 1}}{\Gamma_n^2\left(\frac{n}{p}\right) \Gamma_n\left(\frac{2n}{p'}\right)} = \pi^{\frac{n}{p'}} \frac{\Gamma\left(\frac{n}{2} - \frac{n}{p'}\right)}{\Gamma\left(\frac{n}{p'}\right)} \left[\frac{\Gamma\left(\frac{n}{2p'}\right)}{\Gamma\left(\frac{n}{2p}\right)} \right]^2 \left[\Gamma\left(\frac{n}{2} + 1\right) \right]^{1 - \frac{2}{p'}}.$$

Although (3.2) provides a better estimate for $C_{0,0,p,q,n}$ than (3.14), it is important to note that Theorem 3.2 covers all admissible choices of the parameters p, q, α, and β in (3.1), so the main contribution of this section is in extending the mentioned Lieb result, presented at the beginning of this chapter.

3.2 Trilinear version of standard Beta integral

Considering Lemma 3.2, we may regard the Selberg integral formula as the $k-$fold generalization of the standard Beta integral on $\mathbb{R}^n$. On the other hand, by virtue of the Fourier transform (see [25]), the so-called trilinear version of the standard Beta integral is obtained, that is,

$$\int_{\mathbb{R}^n} \frac{|t|^{\alpha+\beta-2n}}{|x-t|^{\alpha}|y-t|^{\beta}} dt = B(\alpha,\beta,n)\frac{|x-y|^{n-\alpha-\beta}}{|x|^{n-\beta}|y|^{n-\alpha}}, \tag{3.15}$$

where $x,y \in \mathbb{R}^n$, $x \neq y \neq 0$, $0<\alpha,\beta<n$, $\alpha+\beta>n$, and

$$B(\alpha,\beta,n) = \pi^{\frac{n}{2}} \frac{\Gamma\left(\frac{n-\alpha}{2}\right)\Gamma\left(\frac{n-\beta}{2}\right)\Gamma\left(\frac{\alpha+\beta-n}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta}{2}\right)\Gamma\left(n-\frac{\alpha+\beta}{2}\right)}.$$

Taking into account definition (3.5) of n-dimensional Gamma function, the previous formula can be rewritten as

$$B(\alpha,\beta,n) = \frac{\Gamma_n(n-\alpha)\Gamma_n(n-\beta)}{\Gamma_n(2n-\alpha-\beta)}. \tag{3.16}$$

It is still unclear whether or not there is a corresponding $k-$fold analogue of (3.15). In spite of that, we shall use the trilinear formula (3.15) to obtain a $2-$fold inequality of the Hilbert-type for the kernel $K(x,y) = |x-y|^{\alpha-n}|x+y|^{\beta-n}$, where $0<\alpha,\beta<n$, $\alpha+\beta<n$. Of course, we shall rewrite formula (3.15) in a more suitable form. Namely, replacing y with $-x$ in (3.15), we obtain

$$\int_{\mathbb{R}^n} \frac{|t|^{-\alpha-\beta}}{|x-t|^{n-\alpha}|x+t|^{n-\beta}} dt = 2^{\alpha+\beta-n}B^*(\alpha,\beta,n)|x|^{-n}, \tag{3.17}$$

where

$$B^*(\alpha,\beta,n) = \frac{\Gamma_n(\alpha)\Gamma_n(\beta)}{\Gamma_n(\alpha+\beta)}. \tag{3.18}$$

Now, the corresponding Hilbert-type inequality and its equivalent form are the immediate consequences of Theorem 2.1 and the above relation (3.17).

Theorem 3.3 *Let p, q, and λ be as in* (2.1) *and* (2.2)*, and let α and β be real parameters satisfying $0<\alpha,\beta<n$ and $\alpha+\beta<n$. Then, the inequalities*

$$\int_{\mathbb{R}^{2n}} \frac{f(x)g(y)}{|x-y|^{\lambda(n-\alpha)}|x+y|^{\lambda(n-\beta)}} dxdy \tag{3.19}$$

$$\leq N\left[\int_{\mathbb{R}^n} |x|^{(p-1)(\alpha+\beta+n)-pn\lambda} f^p(x)dx\right]^{\frac{1}{p}} \left[\int_{\mathbb{R}^n} |y|^{(q-1)(\alpha+\beta+n)-qn\lambda} g^q(y)dy\right]^{\frac{1}{q}}$$

and

$$\left\{\int_{\mathbb{R}^n} |y|^{n(\lambda q'-1)-\alpha-\beta}\left[\int_{\mathbb{R}^n} \frac{f(x)dx}{|x-y|^{\lambda(n-\alpha)}|x+y|^{\lambda(n-\beta)}}\right]^{q'} dy\right\}^{\frac{1}{q'}} \tag{3.20}$$
$$\leq N\left[\int_{\mathbb{R}^n} |x|^{(p-1)(\alpha+\beta+n)-pn\lambda} f^p(x)dx\right]^{\frac{1}{p}}$$

hold and are equivalent, where $N = 2^{\lambda(\alpha+\beta-n)}B^*(\alpha,\beta,n)^{\lambda}$.

Proof. Putting $K(x,y) = |x-y|^{\alpha-n}|x+y|^{\beta-n}$, $\varphi(x) = |x|^{\frac{\alpha+\beta}{p'}}$, $\psi(y) = |y|^{\frac{\alpha+\beta}{q'}}$ in (2.9) and (2.10) (see Theorem 2.1), and utilizing the formula (3.17), we obtain (3.19) and (3.20) respectively, as claimed. □

Real parameters α and β from the previous theorem fulfill the condition $\alpha+\beta<n$. In what follows we shall obtain similar inequalities which are, in some way, complementary to inequalities (3.19) and (3.20). Such inequalities will be derived by virtue of Theorem 2.8 (see Section 2.4).

Theorem 3.4 *Let p, q, and* λ *be as in* (2.1) *and* (2.2)*, and let* α *and* β *be real parameters satisfying* $0<\alpha<n$, $0<\beta<n$, $\alpha+\beta = n\left(\frac{1}{p}+\frac{1}{q}\right) > n$. *If f and g are non-negative functions such that* $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, *then the following inequalities hold and are equivalent:*

$$\int_{\mathbb{R}^{2n}} \frac{f(x)g(y)}{|x-y|^{n-\alpha}|x+y|^{n-\beta}}dxdy \leq \left(\frac{n}{|\mathbb{S}^{n-1}|}\right)^{1-\lambda} C(p,q;\alpha,\beta;n)\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}, \tag{3.21}$$

$$\left\{\int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} \frac{f(x)dx}{|x-y|^{n-\alpha}|x+y|^{n-\beta}}\right]^{q'} dy\right\}^{\frac{1}{q'}} \leq \left(\frac{n}{|\mathbb{S}^{n-1}|}\right)^{1-\lambda} C(p,q;\alpha,\beta;n)\|f\|_{L^p(\mathbb{R}^n)}. \tag{3.22}$$

Here,

$$C(p,q;\alpha,\beta;n) = \int_{\mathbb{R}^n} \frac{|x|^{-\frac{n}{q}}dx}{|e_1-x|^{n-\alpha}|e_1+x|^{n-\beta}}, \tag{3.23}$$

where $e_1 = (1,0,\dots,0) \in \mathbb{R}^n$ *and* $|\mathbb{S}^{n-1}|$ *is the Lebesgue measure of the unit sphere in* $\mathbb{R}^n$.

Proof. Since we shall use the general rearrangement inequality (3.11), it is enough to prove the inequalities for symmetric-decreasing functions f and g. First, applying Theorem 2.8, we have

$$\int_{\mathbb{R}^{2n}} \frac{f(x)g(y)}{|x-y|^{n-\alpha}|x+y|^{n-\beta}}dxdy \leq I_1^{\frac{1}{q'}} I_2^{\frac{1}{p'}} I_3^{1-\lambda}, \tag{3.24}$$

where

$$I_1 = \int_{\mathbb{R}^{2n}} \frac{|x|^{\frac{n}{p'}} |y|^{-\frac{n}{q}}}{|x-y|^{n-\alpha}|x+y|^{n-\beta}} f^p(x)\, dxdy,$$

$$I_2 = \int_{\mathbb{R}^{2n}} \frac{|x|^{-\frac{n}{p}} |y|^{\frac{n}{q'}}}{|x-y|^{n-\alpha}|x+y|^{n-\beta}} g^q(y)\, dxdy,$$

$$I_3 = \int_{\mathbb{R}^{2n}} \frac{|x|^{\frac{n}{p'}} |y|^{\frac{n}{q'}}}{|x-y|^{n-\alpha}|x+y|^{n-\beta}} f^p(x) g^q(y)\, dxdy.$$

Further, using the change of variables $y = |x|u$ (so $dy = |x|^n du$) and rotational invariance of the Lebesgue integral in $\mathbb{R}^n$, we easily get:

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^n} |x|^{\frac{n}{p'}} f^p(x) \int_{\mathbb{R}^n} \frac{|y|^{-\frac{n}{q}}}{|x-y|^{n-\alpha}|x+y|^{n-\beta}}\, dydx \\
&= \int_{\mathbb{R}^n} |x|^{\frac{n}{p'}-\frac{n}{q}+\alpha+\beta-n} f^p(x) \int_{\mathbb{R}^n} \frac{|u|^{-\frac{n}{q}}}{\left|\frac{x}{|x|}-u\right|^{n-\alpha}\left|\frac{x}{|x|}+u\right|^{n-\beta}}\, dudx \\
&= \int_{\mathbb{R}^n} \frac{|u|^{-\frac{n}{q}} du}{|e_1-u|^{n-\alpha}\,|e_1+u|^{n-\beta}}\, \|f\|^p_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

Analogously,

$$I_2 = \int_{\mathbb{R}^n} \frac{|u|^{-\frac{n}{p}} du}{|e_1-u|^{n-\alpha}\,|e_1+u|^{n-\beta}}\, \|g\|^q_{L^q(\mathbb{R}^n)},$$

and by (3.8),

$$I_3 \le \frac{n}{|\mathbb{S}^{n-1}|} \int_{\mathbb{R}^n} \frac{|u|^{-\frac{n}{q}} du}{|e_1-u|^{n-\alpha}\,|e_1+u|^{n-\beta}}\, \|f\|^p_{L^p(\mathbb{R}^n)}\, \|g\|^q_{L^q(\mathbb{R}^n)}.$$

It remains to prove that

$$\int_{\mathbb{R}^n} \frac{|x|^{-\frac{n}{p}} dx}{|e_1-x|^{n-\alpha}|e_1+x|^{n-\beta}} = \int_{\mathbb{R}^n} \frac{|x|^{-\frac{n}{q}} dx}{|e_1-x|^{n-\alpha}|e_1+x|^{n-\beta}}.$$

We transform the left integral in polar coordinates using $x = t\theta$, $t \ge 0$, $\theta \in |\mathbb{S}^{n-1}|$, and $t = \frac{1}{u}$ to obtain:

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{|x|^{-\frac{n}{p}} dx}{|e_1-x|^{n-\alpha}|e_1+x|^{n-\beta}} &= \int_{\mathbb{S}^{n-1}} d\theta \int_0^\infty \frac{t^{-\frac{n}{p}} t^{n-1} dt}{|e_1-t\theta|^{n-\alpha}|e_1+t\theta|^{n-\beta}} \\
&= \int_{\mathbb{S}^{n-1}} d\theta \int_0^\infty \frac{t^{-\frac{n}{p}} t^{n-1} dt}{(1+t^2-2t(e_1,\theta))^{\frac{n-\alpha}{2}}(1+t^2+2t(e_1,\theta))^{\frac{n-\beta}{2}}} \\
&= \int_{\mathbb{S}^{n-1}} d\theta \int_0^\infty \frac{u^{\frac{n}{p}-\alpha-\beta} u^{n-1} du}{(1+u^2-2u(e_1,\theta))^{\frac{n-\alpha}{2}}(1+u^2+2u(e_1,\theta))^{\frac{n-\beta}{2}}} \\
&= \int_{\mathbb{R}^n} \frac{|x|^{-\frac{n}{q}} dx}{|e_1-x|^{n-\alpha}|e_1+x|^{n-\beta}}.
\end{aligned}$$

To complete the proof, we need to consider the general case, that is, arbitrary non-negative functions f and g. Since $x \longrightarrow |x|^{n-\alpha}$, $x \longrightarrow |x|^{n-\beta}$ are symmetric-decreasing functions vanishing at infinity, the general rearrangement inequality implies that

$$\int_{\mathbb{R}^{2n}} \frac{f(x)g(y)}{|x-y|^{n-\alpha}|x+y|^{n-\beta}}\, dxdy \leq \int_{\mathbb{R}^{2n}} \frac{f^*(x)g^*(y)}{|x-y|^{n-\alpha}|x+y|^{n-\beta}}\, dxdy. \tag{3.25}$$

Clearly, by (3.24), the right-hand side of (3.25) is not greater than

$$\begin{aligned}&\left(\frac{n}{|\mathbb{S}^{n-1}|}\right)^{1-\lambda} C(p,q;\alpha,\beta;n)\, \|f^*\|_{L^p(\mathbb{R}^n)}\|g^*\|_{L^q(\mathbb{R}^n)}\\ &\qquad = \left(\frac{n}{|\mathbb{S}^{n-1}|}\right)^{1-\lambda} C(p,q;\alpha,\beta;n)\, \|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)},\end{aligned} \tag{3.26}$$

where $C(p,q;\alpha,\beta;n)$ is the constant from the right-hand side of (3.21). To get equality in (3.26), we used the fact that the symmetric-decreasing rearrangement is norm preserving.

On the other hand, substituting the function

$$g(y) = \left[\int_{\mathbb{R}^n} \frac{f(x)}{|x-y|^{n-\alpha}|x+y|^{n-\beta}} dx\right]^{\frac{q'}{q}}$$

in (3.21), we obtain inequality (3.22). To show that inequality (3.22) implies (3.21), we proceed in the same way as in the proof of Theorem 2.1. □

It is very interesting to consider the case $n = 1$ in the previous theorem. Namely, in that case the constant $C(p,q;\alpha,\beta;n)$ can be expressed in terms of the usual Beta function and the Gaussian hypergeometric function (see formulas (1.5) and (1.34), Chapter 1). More precisely, utilizing the above mentioned definitions, it follows easily that the following identity holds for $0 < d_1, d_2, d_3 < 1$ and $d_1 + d_2 + d_3 > 1$:

$$\begin{aligned}&\int_{\mathbb{R}} |t|^{-d_2}|1-t|^{-d_3}|1+t|^{-d_1} dt =\\ &\quad B(1-d_2,1-d_3)F(d_1,1-d_2;2-d_2-d_3;-1)\\ &\quad +B(1-d_2,1-d_1)F(d_3,1-d_2;2-d_2-d_1;-1)\\ &\quad +B(d_1+d_2+d_3-1,1-d_3)F(d_1,d_1+d_2+d_3-1;d_1+d_2;-1)\\ &\quad +B(d_1+d_2+d_3-1,1-d_1)F(d_3,d_1+d_2+d_3-1;d_3+d_2;-1).\end{aligned}$$

Hence, for $n = 1$ we have:

Corollary 3.1 *Let p, q, and λ be as in* (2.1) *and* (2.2)*, and let α and β be real parameters satisfying* $0 < \alpha < 1$, $0 < \beta < 1$, *and* $\alpha + \beta = \frac{1}{p} + \frac{1}{q} > 1$. *If $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ are non-negative functions, then the inequalities*

$$\int_{\mathbb{R}^2} \frac{f(x)g(y)}{|x-y|^{1-\alpha}|x+y|^{1-\beta}} dxdy \leq 2^{\lambda-1} C(p,q;\alpha,\beta)\|f\|_{L^p(\mathbb{R})}\|g\|_{L^q(\mathbb{R})} \tag{3.27}$$

and

$$\left\{\int_{\mathbb{R}}\left[\int_{\mathbb{R}}\frac{f(x)dx}{|x-y|^{1-\alpha}|x+y|^{1-\beta}}\right]^{q'}dy\right\}^{\frac{1}{q'}} \leq 2^{\lambda-1}C(p,q;\alpha,\beta)\|f\|_{L^p(\mathbb{R})} \tag{3.28}$$

hold and are equivalent, where

$$\begin{aligned} C(p,q;\alpha,\beta) &= B\big(\frac{1}{q'},\alpha\big)F\big(1-\beta,\frac{1}{q'};\frac{1}{q'}+\alpha;-1\big) \\ &+B\big(\frac{1}{q'},\beta\big)F\big(1-\alpha,\frac{1}{q'};\frac{1}{q'}+\beta;-1\big)+B\big(\frac{1}{p'},\alpha\big)F\big(1-\beta,\frac{1}{p'};\frac{1}{p'}+\alpha;-1\big) \\ &+B\big(\frac{1}{p'},\beta\big)F\big(1-\alpha,\frac{1}{p'};\frac{1}{p'}+\beta;-1\big). \end{aligned} \tag{3.29}$$

The following corollary should be compared with Theorem 3.3.

Corollary 3.2 *If p, q, and λ are as in* (2.1) *and* (2.2)*, and $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$ are non-negative functions, then the following inequalities hold and are equivalent:*

$$\int_{\mathbb{R}^2}\frac{f(x)g(y)dxdy}{|x^2-y^2|^{\frac{\lambda}{2}}} \leq 2^{\lambda-1}\left(B\big(1-\frac{\lambda}{2},\frac{1}{2p'}\big)+B\big(1-\frac{\lambda}{2},\frac{1}{2q'}\big)\right)\|f\|_{L^p(\mathbb{R})}\|g\|_{L^q(\mathbb{R})}, \tag{3.30}$$

$$\left[\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\frac{f(x)dx}{|x^2-y^2|^{\frac{\lambda}{2}}}\right)^{q'}dy\right]^{\frac{1}{q'}} \leq 2^{\lambda-1}\left(B\big(1-\frac{\lambda}{2},\frac{1}{2p'}\big)+B\big(1-\frac{\lambda}{2},\frac{1}{2q'}\big)\right)\|f\|_{L^p(\mathbb{R})}. \tag{3.31}$$

Proof. Set $\alpha=\beta=1-\frac{\lambda}{2}$ in the previous corollary. □

It should be noticed here that inequalities (3.30) and (3.31) could not be obtained from Theorem 3.3. In other words, there are no α and β for which the kernel appearing in inequalities (3.19) and (3.20) would reduce to $|x^2-y^2|^{-\frac{\lambda}{2}}$.

3.3 Multiple Hilbert-type inequalities via the Selberg integral formula

In this section we derive some multidimensional versions of Hilbert-type inequalities with the help of the Selberg integral formula. The following result can be regarded as the corresponding analogue of Theorem 2.6.

Theorem 3.5 *Let λ, p_i, p_i', q_i, $i = 1,2,\ldots,k$, $k \geq 2$, be real numbers satisfying* (2.26), (2.27), *and* (2.30), *and let A_{ij}, $i, j = 1,\ldots,k$, be real parameters such that $\sum_{i=1}^{k} A_{ij} = 0$ for $j = 1,\ldots,k$. If $n \in \mathbb{N}$, $0 < s < n$, $A_{ij} \in (-\frac{n}{q_i}, 0)$, $\alpha_i - A_{ii} < \frac{s-(k-1)n}{q_i}$, and*

$$K = \frac{1}{\Gamma_n^{\lambda}(s)} \prod_{i,j=1, i\neq j}^{k} \Gamma_n^{\frac{1}{q_i}}(n + q_i A_{ij}) \prod_{i=1}^{k} \Gamma_n^{\frac{1}{q_i}}(s - (k-1)n - q_i\alpha_i + q_i A_{ii}),$$

then the inequalities

$$\int_{\mathbb{R}^{kn}} \frac{\prod_{i=1}^{k} f_i(x_i)}{|\sum_{i=1}^{k} x_i|^{\lambda s}} dx_1 dx_2 \cdots dx_k \leq K \prod_{i=1}^{k} \left[\int_{\mathbb{R}^n} |x_i|^{\frac{p_i(k-1)n - p_i s}{q_i} + p_i\alpha_i} f_i^{p_i}(x_i) dx_i\right]^{\frac{1}{p_i}} \tag{3.32}$$

and

$$\left\{\int_{\mathbb{R}^n} |x_k|^{-\frac{p_k'}{q_k}[(k-1)n-s] - p_k'\alpha_k} \left[\int_{\mathbb{R}^{(k-1)n}} \frac{\prod_{i=1}^{k-1} f_i(x_i)}{|\sum_{i=1}^{k} x_i|^{\lambda s}} dx_1 dx_2 \cdots dx_{k-1}\right]^{p_k'} dx_k\right\}^{\frac{1}{p_k'}}$$

$$\leq K \prod_{i=1}^{k-1} \left[\int_{\mathbb{R}^n} |x_i|^{\frac{p_i(k-1)n - p_i s}{q_i} + p_i\alpha_i} f_i^{p_i}(x_i) dx_i\right]^{\frac{1}{p_i}} \tag{3.33}$$

hold for all non-negative measurable functions $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1,\ldots,k$. Here $\alpha_i = \sum_{j=1}^{k} A_{ij}$, $i = 1,2,\ldots,k$. Moreover, these inequalities are equivalent.

Proof. We use the general result in the non-conjugate multidimensional setting, that is, Theorem 2.5. Namely, we consider inequalities (2.31) and (2.32) with $\Omega = \mathbb{R}^n$, $\phi_{ij}(x_j) = |x_j|^{A_{ij}}$, $i, j = 1,\ldots,k$, $K(x_1,\ldots,x_k) = |x_1 + \cdots + x_k|^{-s}$, and the Lebesgue measure.

Now, using the notation from Theorem 2.5, the Selberg integral formula (3.6) yields

$$F_i(x_i) = \left[\frac{\prod_{j=1, j\neq i}^{k} \Gamma_n(n + q_i A_{ij}) \Gamma_n(n-s)}{\Gamma_n(kn + q_i\alpha_i - q_i A_{ii} - s)}\right]^{\frac{1}{q_i}} |x_i|^{\frac{(k-1)n-s}{q_i} + \alpha_i - A_{ii}},$$

so inequalities (2.31) and (2.32) reduce to (3.32) and (3.33) respectively. $\square$

Remark 3.2 Observe that, according to Remark 2.11, equalities in (3.32) and (3.33) hold if and only if at least one of the functions f_i is equal to zero a.e. on $\mathbb{R}^n$. Otherwise, inequalities (3.32) and (3.33) are strict.

As an application of Theorem 3.5, we consider some particular choices of parameters A_{ij}, $i, j = 1,2,\ldots,k$. For example, if $A_{ii} = (nk - s)\frac{\lambda q_i - 1}{q_i^2}$ and $A_{ij} = (s - nk)\frac{1}{q_i q_j}$, $i \neq j$, $i, j = 1,2,\ldots,k$, then

$$\sum_{i=1}^{k} A_{ij} = \sum_{i\neq j} \frac{s - nk}{q_i q_j} + (nk - s)\left(\frac{\lambda q_j - 1}{q_j^2}\right) = \frac{s - nk}{q_j}\left(\sum_{i=1}^{k} \frac{1}{q_i} - \lambda\right) = 0,$$

for $j = 1,2,\ldots,k$, that is, these parameters fulfill conditions of Theorem 3.5. Moreover, due to the symmetry, we also have $\alpha_i = \sum_{j=1}^{k} A_{ij} = 0$ for $i = 1,2,\ldots,k$, and hence, as a consequence we obtain the following result.

Corollary 3.3 *Let λ, p_i, p_i', q_i, $i = 1,2,\dots,k$, $k \geq 2$, be real numbers satisfying* (2.26), (2.27), *and* (2.30). *If $n \in \mathbb{N}$, $0 < nk - s < n\min\{p_i, q_j, i, j = 1,2,\dots k\}$, and*

$$L = \frac{1}{\Gamma_n^{\lambda}(s)} \prod_{i=1}^{k} \Gamma_n^{\frac{1}{p_i'}} \left(n - \frac{nk-s}{q_i}\right) \prod_{i=1}^{k} \Gamma_n^{\frac{1}{q_i}} \left(n - \frac{nk-s}{p_i}\right),$$

then the inequalities

$$\int_{\mathbb{R}^{kn}} \frac{\prod_{i=1}^{k} f_i(x_i)}{|\sum_{i=1}^{k} x_i|^{\lambda s}} dx_1 dx_2 \cdots dx_k \leq L \prod_{i=1}^{k} \left[\int_{\mathbb{R}^n} |x_i|^{\frac{p_i(k-1)n - p_i s}{q_i}} f_i^{p_i}(x_i) dx_i\right]^{\frac{1}{p_i}} \tag{3.34}$$

and

$$\begin{aligned}
&\left\{\int_{\mathbb{R}^n} |x_k|^{-\frac{p_k'}{q_k}[(k-1)n-s]} \left[\int_{\mathbb{R}^{(k-1)n}} \frac{\prod_{i=1}^{k-1} f_i(x_i)}{|\sum_{i=1}^{k} x_i|^{\lambda s}} dx_1 dx_2 \cdots dx_{k-1}\right]^{p_k'} dx_k\right\}^{\frac{1}{p_k'}} \\
&\leq L \prod_{i=1}^{k-1} \left[\int_{\mathbb{R}^n} |x_i|^{\frac{p_i(k-1)n - p_i s}{q_i}} f_i^{p_i}(x_i) dx_i\right]^{\frac{1}{p_i}}
\end{aligned} \tag{3.35}$$

hold for all non-negative measurable functions $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1,\dots,k$. Moreover, these inequalities are equivalent. The equality in both inequalities holds if and only if at least one of the functions f_i, $i = 1,2,\dots,k$, is equal to zero a.e. on $\mathbb{R}^n$.

Remark 3.3 Similarly as in the previous corollary, defining $A_{ii} = \frac{n(\lambda q_i - 1)}{\lambda q_i^2}$ and $A_{ij} = -\frac{n}{\lambda q_i q_j}$, $i \neq j$, $i, j = 1,2,\dots k$, we have

$$\sum_{i=1}^{k} A_{ij} = \sum_{j=1}^{k} A_{ij} = \sum_{i \neq j} -\frac{n}{\lambda q_i q_j} + \frac{n(\lambda q_j - 1)}{\lambda q_j{}^2} = -\frac{n}{\lambda q_j}\left(\sum_{i=1}^{k} \frac{1}{q_i} - \lambda\right) = 0,$$

for $j = 1,2,\dots,k$. Hence, substituting these parameters in Theorem 3.5, assuming that $(k-1)n - s < \frac{n}{\lambda p_i'} < n$, $i = 1,2,\dots,k$, we obtain the same inequalities as in Corollary 3.3, with the constant

$$L' = \frac{1}{\Gamma_n^{\lambda}(s)} \prod_{i=1}^{k} \Gamma_n^{\lambda - \frac{1}{q_i}} \left(\frac{n}{\lambda p_i'}\right) \prod_{i=1}^{k} \Gamma_n^{\frac{1}{q_i}} \left(s + \frac{n}{\lambda p_i'} - (k-1)n\right),$$

instead of L.

Inequalities in Corollary 3.3 and Remark 3.3 are interesting since they provide the best possible constant factors in the conjugate case. This will be explained in the next section. It should also be noticed here that these inequalities can be regarded as the n-fold extensions of the corresponding results from [12] and [156].

We proceed with multidimensional Hilbert-type inequalities related to those in Section 3.1. Namely, in Section 3.1, by virtue of Theorem 2.8 and Lemma 3.2, we have obtained Hardy-Littlewood-Sobolev inequality which includes a pair of non-conjugate parameters. However, the method used in deriving Hardy-Littlewood-Sobolev inequality can also be extended to a multidimensional non-conjugate setting. More precisely, we shall utilize Theorem 2.7 applied to the kernel $K(x_1,\ldots,x_k)=|x_1+\ldots+x_k|^{-(k-1)n\lambda}$ on $\mathbb{R}^{kn}$, and Lemma 3.1. The corresponding Hilbert-type inequality is given in the following theorem.

Theorem 3.6 *Let $n\in\mathbb{N}$, $k\in\mathbb{N}$, $k\geq 2$, and assume that λ, p_i, q_i, $i=1,2,\ldots,k$, are as in* (2.26), (2.27), *and* (2.30). *If $0<\lambda<\frac{1}{k-1}$, then the inequality*

$$\int_{\mathbb{R}^{kn}} \frac{f_1(x_1)\cdots f_k(x_k)}{|x_1+\ldots+x_k|^{(k-1)n\lambda}}\,dx_1\ldots dx_k \leq \frac{(2\pi)^{kn}\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{(k-1)(\lambda-1)}}{\Gamma_n((k-1)n\lambda)\prod_{i=1}^{k}\Gamma_n\left(\frac{n}{p_i}\right)}\parallel f_1\parallel_{L^{p_1}(\mathbb{R}^n)}\cdots\parallel f_k\parallel_{L^{p_k}(\mathbb{R}^n)} \tag{3.36}$$

holds for all non-negative functions $f_i\in L^{p_i}(\mathbb{R}^n)$, $i=1,\ldots,k$.

Proof. First, we consider a simpler special case of the functions appearing in (3.36). Namely, suppose that $f_2,\ldots,f_k$ are symmetric-decreasing functions. To prove the above assertion, we rewrite Theorem 2.7 with $\Omega=\mathbb{R}^n$, $K(x_1,\ldots,x_k)=|x_1+\ldots+x_k|^{-(k-1)n\lambda}$, $F_i(x_1,\ldots,x_k)\equiv 1$, $\phi_{ij}(x_j)=|x_j|^{A_{ij}}$, where

$$A_{ij}=\begin{cases} \dfrac{n}{p_ip_i'}, & i=j \\[2ex] -\dfrac{n}{q_ip_j}, & i\neq j, \end{cases} \tag{3.37}$$

and with Lebesgue measures dx_i, for $i,j=1,\ldots,k$. Then the left-hand side of (2.38) in Theorem 2.7 becomes

$$L=\int_{\mathbb{R}^{kn}} \frac{f_1(x_1)\cdots f_k(x_k)}{|x_1+\ldots+x_k|^{(k-1)n\lambda}}\,dx_1\ldots dx_k, \tag{3.38}$$

while the right-hand side of this inequality is the product of $k+1$ factors,

$$R=I_1^{\frac{1}{q_1}}\cdots I_k^{\frac{1}{q_k}}I_{k+1}^{1-\lambda}, \tag{3.39}$$

where

$$I_i=\int_{\mathbb{R}^{kn}} \frac{|x_i|^{\frac{n}{p_i'}}\prod_{j\neq i}|x_j|^{-\frac{n}{p_j}}}{|x_1+\ldots+x_k|^{(k-1)n\lambda}}\,f_i^{p_i}(x_i)\,dx_1\ldots dx_k,\ \ i=1,\ldots,k,$$

and

$$I_{k+1}=\int_{\mathbb{R}^{kn}} \frac{\prod_{i=1}^{k}|x_i|^{\frac{n}{p_i'}}}{|x_1+\ldots+x_k|^{(k-1)n\lambda}}\prod_{i=1}^{k}f_i^{p_i}(x_i)\,dx_1\ldots dx_k.$$

Before calculating these integrals, observe that from (2.26), (2.27), and (2.30) (see Section 2.4), we obtain that $0 < \frac{n}{p_i} < n$ and

$$\begin{aligned}\sum_{j\neq i}\frac{n}{p_j}+(k-1)n\lambda &= n\sum_{j\neq i}\left(\frac{1}{p_j}+\lambda\right) = n\sum_{j\neq i}\left(\frac{1}{q_j}+1\right)\\ &= (k-1)n+n\sum_{j\neq i}\frac{1}{q_j} > (k-1)n,\end{aligned}$$

for all $i\in\{1,\ldots,k\}$. Moreover, the conditions of Theorem 3.6 imply also that $0<(k-1)n\lambda<n$. Therefore, applying the Fubini theorem, Lemma 3.1, and (3.5), for $i=1,\ldots,k$, we get

$$\begin{aligned}I_i &= \int_{\mathbb{R}^n}|x_i|^{\frac{n}{p_i'}}f_i^{p_i}(x_i)\int_{\mathbb{R}^{(k-1)n}}\frac{\prod_{j\neq i}|x_j|^{-\frac{n}{p_j}}}{|x_1+\ldots+x_k|^{(k-1)n\lambda}}dx_1\ldots dx_{i-1}dx_{i+1}\ldots dx_k dx_i\\ &= \frac{\Gamma_n(n-(k-1)n\lambda)\prod_{j\neq i}\Gamma_n\left(n-\frac{n}{p_j}\right)}{\Gamma_n\left(kn-\sum_{j\neq i}\frac{n}{p_j}-(k-1)n\lambda\right)}\int_{\mathbb{R}^n}|x_i|^{\frac{n}{p_i'}+(k-1)n-\sum_{j\neq i}\frac{n}{p_j}-(k-1)n\lambda}f_i^{p_i}(x_i)\,dx_i\\ &= \frac{(2\pi)^{kn}}{\Gamma_n((k-1)n\lambda)\prod_{j=1}^k\Gamma_n\left(\frac{n}{p_j}\right)}\parallel f_i\parallel_{L^{p_i}(\mathbb{R}^n)}^{p_i}. \qquad (3.40)\end{aligned}$$

To estimate the last integral I_{k+1} in (3.39), we use the assumption that the functions $f_2,\ldots,f_k$ are symmetric-decreasing. Hence, we can use relation (3.8) to obtain that

$$f_i^{p_i}(x_i)\le\frac{n}{|\mathbb{S}^{n-1}|}|x_i|^{-n}\parallel f_i\parallel_{L^{p_i}(\mathbb{R}^n)}^{p_i}$$

holds for all $x_i\in\mathbb{R}^n$, $x_i\neq 0$. Again, according to the Fubini theorem, Lemma 3.1, and identity (3.5), similarly to the procedure used in (3.40), we obtain

$$\begin{aligned}I_{k+1} &\le \left(\frac{n}{|\mathbb{S}^{n-1}|}\right)^{k-1}\prod_{i=2}^k\parallel f_i\parallel_{L^{p_i}(\mathbb{R}^n)}^{p_i}\\ &\quad\times\int_{\mathbb{R}^{kn}}\frac{|x_1|^{\frac{n}{p_1'}}\prod_{i=2}^k|x_i|^{\frac{n}{p_i'}-n}}{|x_1+\ldots+x_k|^{(k-1)n\lambda}}f_1^{p_1}(x_1)\,dx_1\ldots dx_k\\ &= \left(\frac{n}{|\mathbb{S}^{n-1}|}\right)^{k-1}\prod_{i=2}^k\parallel f_i\parallel_{L^{p_i}(\mathbb{R}^n)}^{p_i}\\ &\quad\times\int_{\mathbb{R}^n}|x_1|^{\frac{n}{p_1'}}f_1^{p_1}(x_1)\int_{\mathbb{R}^{(k-1)n}}\frac{\prod_{i=2}^k|x_i|^{-\frac{n}{p_i}}}{|x_1+\ldots+x_k|^{(k-1)n\lambda}}dx_2\ldots dx_k\,dx_1\\ &= \left(\frac{n}{|\mathbb{S}^{n-1}|}\right)^{k-1}\frac{\Gamma_n(n-(k-1)n\lambda)\prod_{i=2}^k\Gamma_n\left(n-\frac{n}{p_i}\right)}{\Gamma_n\left(kn-\sum_{i=2}^k\frac{n}{p_i}-(k-1)n\lambda\right)}\prod_{i=1}^k\parallel f_i\parallel_{L^{p_i}(\mathbb{R}^n)}^{p_i}\\ &= \left(\frac{n}{|\mathbb{S}^{n-1}|}\right)^{k-1}\frac{(2\pi)^{kn}}{\Gamma_n((k-1)n\lambda)\prod_{i=1}^k\Gamma_n\left(\frac{n}{p_i}\right)}\prod_{i=1}^k\parallel f_i\parallel_{L^{p_i}(\mathbb{R}^n)}^{p_i}. \qquad (3.41)\end{aligned}$$

Now, we arrive at inequality (3.36) for this case by combining (3.38), (3.39), (3.40), and (3.41). To complete the proof, we need to consider the general case, that is, arbitrary non-negative functions $f_2, \dots, f_k$. Since the function $x \to |x|^{-(k-1)n\lambda}$ is symmetric-decreasing and vanishes at infinity, by the general rearrangement inequality we have

$$\begin{aligned}
&\int_{\mathbb{R}^{kn}} \frac{f_1(x_1)\cdots f_k(x_k)}{|x_1+\ldots+x_k|^{(k-1)n\lambda}}\, dx_1\ldots dx_k \le \int_{\mathbb{R}^{kn}} \frac{f_1^*(x_1)\cdots f_k^*(x_k)}{|x_1+\ldots+x_k|^{(k-1)n\lambda}}\, dx_1\ldots dx_k \\
&\le \frac{(2\pi)^{kn}\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{(k-1)(\lambda-1)}}{\Gamma_n((k-1)n\lambda)\prod_{i=1}^k \Gamma_n\left(\frac{n}{p_i}\right)} \| f_1^* \|_{L^{p_1}(\mathbb{R}^n)} \cdots \| f_k^* \|_{L^{p_k}(\mathbb{R}^n)} \\
&= \frac{(2\pi)^{kn}\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{(k-1)(\lambda-1)}}{\Gamma_n((k-1)n\lambda)\prod_{i=1}^k \Gamma_n\left(\frac{n}{p_i}\right)} \| f_1 \|_{L^{p_1}(\mathbb{R}^n)} \cdots \| f_k \|_{L^{p_k}(\mathbb{R}^n)} .
\end{aligned}$$

As in the proof of Theorem 3.2, here we used the fact that $f_2^*, \dots, f_k^*$ are symmetric-decreasing functions and that the mapping $f \to f^*$ is norm preserving. □

Remark 3.4 Note that the proof of Theorem 3.6 is, in fact, based on the Selberg integral formula (3.4). Some further applications of this formula can be found in [25].

Setting $k = 2$ and $k = 3$ in Theorem 3.6, we get the following consequences.

Corollary 3.4 *Let $n \in \mathbb{N}$, $p > 1$ and $q > 1$ be such that $\frac{1}{p} + \frac{1}{q} > 1$, and let λ be defined by* (2.2). *Then*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x+y|^{n\lambda}}\, dx\, dy \le \frac{(2\pi)^{2n}\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\lambda-1}}{\Gamma_n(n\lambda)\Gamma_n\left(\frac{n}{p}\right)\Gamma_n\left(\frac{n}{q}\right)} \| f \|_{L^p(\mathbb{R}^n)} \| g \|_{L^q(\mathbb{R}^n)} \tag{3.42}$$

holds for all non-negative functions $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$. In particular, if $n = 1$, then (3.42) *becomes*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(x)g(y)}{|x+y|^{\lambda}}\, dx\, dy \le 2^{\lambda-1}\sqrt{\pi}\, \frac{B\left(\frac{1}{2p'}, \frac{1}{2q'}\right)}{B\left(\frac{1}{2p}, \frac{1}{2q}\right)} \frac{\Gamma\left(\frac{1}{2}-\frac{\lambda}{2}\right)}{\Gamma\left(1-\frac{\lambda}{2}\right)} \| f \|_{L^p(\mathbb{R})} \| g \|_{L^q(\mathbb{R})} .$$

Corollary 3.5 *Let $n \in \mathbb{N}$ and assume that parameters $p_1, p_2, p_3, \lambda, q_1, q_2, q_3$ satisfy* (2.26), (2.27), *and* (2.30). *If $0 < \lambda < \frac{1}{2}$, then*

$$\begin{aligned}
&\int_{\mathbb{R}^{3n}} \frac{f(x)g(y)h(z)}{|x+y+z|^{2n\lambda}}\, dx\, dy\, dz \\
&\le \frac{(2\pi)^{3n}\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{2(\lambda-1)}}{\Gamma_n(2n\lambda)\Gamma_n\left(\frac{n}{p_1}\right)\Gamma_n\left(\frac{n}{p_2}\right)\Gamma_n\left(\frac{n}{p_3}\right)} \| f \|_{L^{p_1}(\mathbb{R}^n)} \| g \|_{L^{p_2}(\mathbb{R}^n)} \| h \|_{L^{p_3}(\mathbb{R}^n)}
\end{aligned} \tag{3.43}$$

holds for all non-negative functions $f \in L^{p_1}(\mathbb{R}^n)$*,* $g \in L^{p_2}(\mathbb{R}^n)$*, and* $h \in L^{p_3}(\mathbb{R}^n)$*. In particular, if* $n = 1$*, then* (3.43) *reads*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(x)g(y)h(z)}{|x+y+z|^{2\lambda}}\,dx\,dy\,dz$$

$$\leq 2^{2(\lambda-1)}\pi \frac{\Gamma\left(\frac{1}{2}-\lambda\right)}{\Gamma(\lambda)} \prod_{i=1}^{3} \frac{\Gamma\left(\frac{1}{2p_i'}\right)}{\Gamma\left(\frac{1}{2p_i}\right)} \parallel f \parallel_{L^{p_1}(\mathbb{R})} \parallel g \parallel_{L^{p_2}(\mathbb{R})} \parallel h \parallel_{L^{p_3}(\mathbb{R})}.$$

Remark 3.5 Note that in all presented applications of Theorem 2.7 we considered $F_i \equiv 1$, $i = 1, \ldots, k$, while Theorem 2.8 was applied with $F = G \equiv 1$ (see Section 3.1). Obviously, according to the conditions from the statements of these theorems, we can use any other non-negative functions F_i and, consequently, take the infimum of the right-hand sides of the obtained inequalities over all such functions. Therefore, to conclude this section, we mention the following open problem: Can this approach give sharp Hilbert-type inequalities, that is, do there exist such functions F_i that the related inequalities are obtained with the best possible constants on their right-hand sides?

3.4 The best constants

In this section we are focused on Theorem 3.5 in the conjugate setting, that is, when $\sum_{i=1}^{k} \frac{1}{p_i} = 1$ and $\lambda = 1$. The main task is to determine the conditions on parameters A_{ij}, $i, j = 1, \ldots, k$, under which the constant factor on the right-hand sides of inequalities (3.32) and (3.33) is the best possible. In the conjugate case, the constant K, involved in the above mentioned inequalities, reduces to

$$K = \frac{1}{\Gamma_n(s)} \prod_{i,j=1, i\neq j}^{k} \Gamma_n^{\frac{1}{p_i}}(n + p_i A_{ij}) \prod_{i=1}^{k} \Gamma_n^{\frac{1}{p_i}}(s - (k-1)n - p_i\alpha_i + p_i A_{ii}).$$

Taking into account the method as in Section 1.6 (Chapter 1), we are going to simplify the above constant K, to obtain the expression without exponents. Hence, we impose the following conditions on the parameters A_{ij}:

$$s - (k-1)n + p_i A_{ii} - p_i\alpha_i = n + p_j A_{ji}, \quad j \neq i, \quad i, j = 1, 2, \ldots, k. \tag{3.44}$$

In this case the above constant K reads

$$K^* = \frac{1}{\Gamma_n(s)} \prod_{i=1}^{k} \Gamma_n(n + \widetilde{A}_i), \tag{3.45}$$

where

$$\widetilde{A}_i = p_j A_{ji},\ j \neq i, \text{ and } -n < \widetilde{A}_i < 0. \tag{3.46}$$

It follows easily that then

$$\sum_{i=1}^{k}\widetilde{A}_i = s - kn, \tag{3.47}$$

so that inequalities (3.32) and (3.33) with the parameters A_{ij} fulfilling (3.44), reduce to

$$\int_{\mathbb{R}^{kn}} \frac{\prod_{i=1}^{k} f_i(x_i)}{|\sum_{i=1}^{k} x_i|^s} dx_1 dx_2 \cdots dx_k \leq K^* \prod_{i=1}^{k}\left[\int_{\mathbb{R}^n} |x_i|^{-n-p_i\widetilde{A}_i} f_i^{p_i}(x_i)dx_i\right]^{\frac{1}{p_i}} \tag{3.48}$$

and

$$\left\{\int_{\mathbb{R}^n} |x_k|^{(1-p_k')(-n-p_k\widetilde{A}_k)}\left[\int_{\mathbb{R}^{(k-1)n}} \frac{\prod_{i=1}^{k-1} f_i(x_i)}{|\sum_{i=1}^{k} x_i|^s} dx_1 dx_2 \cdots dx_{k-1}\right]^{p_k'} dx_k\right\}^{\frac{1}{p_k'}}$$
$$\leq K^* \prod_{i=1}^{k-1}\left[\int_{\mathbb{R}^n} |x_i|^{-n-p_i\widetilde{A}_i} f_i^{p_i}(x_i)dx_i\right]^{\frac{1}{p_i}}. \tag{3.49}$$

Inequalities (3.48) and (3.49) involve the best possible constant factor K^* on their right-hand sides. To show our assertion, we first need to establish some auxiliary results.

Lemma 3.3 *Let $k \geq 2$ be an integer, $x_k \in \mathbb{R}^n$, and $x_k \neq 0$. We define*

$$I_1^{\varepsilon}(x_k) = \int_{K^n(\varepsilon)} |x_1|^{\widetilde{A}_1}\left[\int_{\mathbb{R}^{(k-2)n}} \frac{\prod_{i=2}^{k-1} |x_i|^{\widetilde{A}_i}}{|\sum_{i=1}^{k} x_i|^s} dx_2 \ldots dx_{k-1}\right] dx_1,$$

where $\varepsilon > 0$, $K^n(\varepsilon)$ is the closed n-dimensional ball of radius ε, and parameters $\widetilde{A}_i$, $i = 1,2,\ldots,k$, are defined by (3.46). *Then there exists a positive constant C_k such that*

$$I_1^{\varepsilon}(x_k) \leq C_k \varepsilon^{n+\widetilde{A}_1} |x_k|^{-2n-\widetilde{A}_1-\widetilde{A}_k}, \quad \text{when} \quad \varepsilon \to 0. \tag{3.50}$$

Proof. We distinguish two cases. If $k = 2$ we have

$$I_1^{\varepsilon}(x_2) = \int_{K^n(\varepsilon)} \frac{|x_1|^{\widetilde{A}_1}}{|x_1 + x_2|^s} dx_1.$$

Letting $\varepsilon \to 0$, it follows that there exist a positive constant c_2 such that $I_1^{\varepsilon}(x_2) \leq c_2 |x_2|^{-s}$ $\int_{K^n(\varepsilon)} |x_1|^{\widetilde{A}_1} dx_1$. The previous integral can be calculated by using n-dimensional spherical coordinates. More precisely, we have

$$\int_{K^n(\varepsilon)} |x_1|^{\widetilde{A}_1} dx_1$$
$$= \int_0^{\pi} \cdots \int_0^{\pi}\int_0^{2\pi}\int_0^{\varepsilon} r^{n+\widetilde{A}_1-1} \sin^{n-2}\theta_{n-1} \sin^{n-3}\theta_{n-2} \cdots \sin\theta_2 dr d\theta_1 \ldots d\theta_{n-1}$$
$$= \int_0^{\varepsilon} r^{n+\widetilde{A}_1-1} dr \int_{|\mathbb{S}^{n-1}|} dS = \frac{|\mathbb{S}^{n-1}|\varepsilon^{n+\widetilde{A}_1}}{n+\widetilde{A}_1}. \tag{3.51}$$

Consequently,

$$I_1^{\varepsilon}(x_2) \le \frac{c_2|\mathbb{S}^{n-1}|\varepsilon^{n+\widetilde{A}_1}}{n+\widetilde{A}_1}|x_2|^{-s},$$

so the inequality holds when $\varepsilon \to 0$, since $-2n-\widetilde{A}_1-\widetilde{A}_2=-s$ holds for $k=2$. Further, if $k>2$, then by letting $\varepsilon \to 0$, since $|x_1| \to 0$, we easily conclude that there exists a positive constant c_k such that

$$I_1^{\varepsilon}(x_k) \le c_k \left[\int_{K^n(\varepsilon)} |x_1|^{\widetilde{A}_1} dx_1\right] \left[\int_{\mathbb{R}^{(k-2)n}} \frac{\prod_{i=2}^{k-1}|x_i|^{\widetilde{A}_i}}{|\sum_{i=2}^{k} x_i|^s} dx_2 \dots dx_{k-1}\right]. \tag{3.52}$$

We have already calculated the first integral in inequality (3.52), and the second one is the Selberg integral. Namely, utilizing formulas (3.5), (3.6) (see Section 3.1), and (3.47) we have

$$\begin{aligned} &\int_{\mathbb{R}^{(k-2)n}} \frac{\prod_{i=2}^{k-1}|x_i|^{\widetilde{A}_i}}{|\sum_{i=2}^{k} x_i|^s} dx_2 \dots dx_{k-1} \\ &= \frac{\Gamma_n(2n+\widetilde{A}_1+\widetilde{A}_k)\prod_{i=2}^{k-1}\Gamma_n(n+\widetilde{A}_i)}{\Gamma_n(s)}|x_k|^{-2n-\widetilde{A}_1-\widetilde{A}_k}. \end{aligned} \tag{3.53}$$

Finally, combining (3.51), (3.52), and (3.53) we obtain inequality (3.50) and the proof is completed. □

Lemma 3.4 *Let $k \ge 2$ be an integer and $x_k \in \mathbb{R}^n$. We define*

$$I_1^{\varepsilon^{-1}}(x_k) = \int_{\mathbb{R}^n \setminus K^n(\varepsilon^{-1})} |x_1|^{\widetilde{A}_1} \left[\int_{\mathbb{R}^{(k-2)n}} \frac{\prod_{i=2}^{k-1}|x_i|^{\widetilde{A}_i}}{|\sum_{i=1}^{k} x_i|^s} dx_2 \dots dx_{k-1}\right] dx_1,$$

where $\varepsilon > 0$, $K^n(\varepsilon^{-1})$ is the closed n-dimensional ball of radius ε^{-1}, and parameters $\widetilde{A}_i$, $i=1,2,\dots,k$, are defined by (3.46). Then there exists a positive constant D_k such that

$$I_1^{\varepsilon^{-1}}(x_k) \le D_k \varepsilon^{n+\widetilde{A}_k}, \quad when \quad \varepsilon \to 0. \tag{3.54}$$

Proof. We again consider two cases. If $k=2$ we have

$$I_1^{\varepsilon^{-1}}(x_2) = \int_{\mathbb{R}^n \setminus K^n(\varepsilon^{-1})} \frac{|x_1|^{\widetilde{A}_1}}{|x_1+x_2|^s} dx_1.$$

If $\varepsilon \to 0$, then $|x_1| \to \infty$, hence there exist a positive constant d_2 such that

$$I_1^{\varepsilon^{-1}}(x_2) \le d_2 \int_{\mathbb{R}^n \setminus K^n(\varepsilon^{-1})} |x_1|^{\widetilde{A}_1 - s} dx_1.$$

Now, utilizing spherical coordinates for calculating the integral on the right-hand side of the above inequality, we obtain

$$I_1^{\varepsilon^{-1}}(x_2) \le \frac{d_2|\mathbb{S}^{n-1}|}{n+\widetilde{A}_2}\varepsilon^{n+\widetilde{A}_2}.$$

Moreover, for $k > 2$, using (3.5), (3.6), and (3.47) we have

$$\begin{aligned}&\int_{\mathbb{R}^{(k-2)n}} \frac{\prod_{i=2}^{k-1}|x_i|^{\widetilde{A}_i}}{|\sum_{i=1}^{k} x_i|^s} dx_2 \ldots dx_{k-1} \\ &= \frac{\Gamma_n(2n+\widetilde{A}_1+\widetilde{A}_k)\prod_{i=2}^{k-1}\Gamma_n(n+\widetilde{A}_i)}{\Gamma_n(s)}|x_1+x_k|^{-2n-\widetilde{A}_1-\widetilde{A}_k}. \qquad (3.55)\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}I_1^{\varepsilon^{-1}}(x_k) &= \frac{\Gamma_n(2n+\widetilde{A}_1+\widetilde{A}_k)\prod_{i=2}^{k-1}\Gamma_n(n+\widetilde{A}_i)}{\Gamma_n(s)} \\ &\times \int_{\mathbb{R}^n\setminus K^n(\varepsilon^{-1})} |x_1|^{\widetilde{A}_1}|x_1+x_k|^{-2n-\widetilde{A}_1-\widetilde{A}_k}dx_1. \qquad (3.56)\end{aligned}$$

Letting $\varepsilon \to 0$, it follows that $|x_1| \to \infty$, so there exist a positive constant d_k such that

$$I_1^{\varepsilon^{-1}}(x_k) \le d_k \int_{\mathbb{R}^n\setminus K^n(\varepsilon^{-1})} |x_1|^{-2n-\widetilde{A}_k}dx_1.$$

Finally, since

$$\int_{\mathbb{R}^n\setminus K^n(\varepsilon^{-1})} |x_1|^{-2n-\widetilde{A}_k}dx_1 = \frac{|\mathbb{S}^{n-1}|\varepsilon^{n+\widetilde{A}_k}}{n+\widetilde{A}_k},$$

inequality (3.54) holds. □

Now, we are able to obtain the best possible constant factors in (3.48) and (3.49). Clearly, inequalities (3.48) and (3.49) do not contain parameters A_{ij}, $i,j = 1,2,\ldots,k$, so we can regard these inequalities with $\widetilde{A}_i$, $i = 1,2,\ldots,k$, as primitive parameters.

Theorem 3.7 *The constant K^* is the best possible in both inequalities* (3.48) *and* (3.49).

Proof. Let $K^n(\varepsilon)$ be the closed n-dimensional ball of radius ε, centered at the origin, and let $0 < \varepsilon < 1$. We define the functions $\widetilde{f}_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1,2,\ldots,k$, in the following way:

$$\widetilde{f}_i(x_i) = \begin{cases} |x_i|^{\widetilde{A}_i}, & x_i \in K^n(\varepsilon^{-1}) \setminus K^n(\varepsilon) \\ 0, & \text{otherwise.}\end{cases}$$

Inserting the above functions in (3.48), the right-hand side of (3.48) becomes

$$K^*\prod_{i=1}^{k}\left(\int_{K^n(\varepsilon^{-1})\setminus K^n(\varepsilon)} |x_i|^{-n}dx_i\right)^{\frac{1}{p_i}} = K^*\int_{K^n(\varepsilon^{-1})\setminus K^n(\varepsilon)} |x_i|^{-n}dx_i.$$

In addition, utilizing n-dimensional spherical coordinates we have

$$\int_{K^n(\varepsilon^{-1})\setminus K^n(\varepsilon)} |x_i|^{-n}dx_i = \int_{\varepsilon}^{\varepsilon^{-1}} r^{-1}dr \int_{|\mathbb{S}^{n-1}|} dS = |\mathbb{S}^{n-1}|\log\frac{1}{\varepsilon^2},$$

so that the right-hand side of inequality (3.48) reads

$$K^*|\mathbb{S}^{n-1}|\log\frac{1}{\varepsilon^2}. \tag{3.57}$$

Now, let J denotes the left-hand side of inequality (3.48). Using the Fubini theorem, for the above choice of functions $\widetilde{f_i}$, we have

$$\begin{aligned}
J &= \int_{\left(K^n(\varepsilon^{-1})\setminus K^n(\varepsilon)\right)^k} \frac{\prod_{i=1}^k |x_i|^{\widetilde{A}_i}}{|\sum_{i=1}^k x_i|^s} dx_1 dx_2 \dots dx_k \\
&= \int_{K^n(\varepsilon^{-1})\setminus K^n(\varepsilon)} |x_k|^{\widetilde{A}_k} \left[\int_{\left(K^n(\varepsilon^{-1})\setminus K^n(\varepsilon)\right)^{k-1}} \frac{\prod_{i=1}^{k-1} |x_i|^{\widetilde{A}_i}}{|\sum_{i=1}^k x_i|^s} dx_1 dx_2 \dots dx_{k-1}\right] dx_k.
\end{aligned}$$

Note that the integral J can be transformed in the following way: $J = J_1 - J_2 - J_3$, where

$$\begin{aligned}
J_1 &= \int_{K^n(\varepsilon^{-1})\setminus K^n(\varepsilon)} |x_k|^{\widetilde{A}_k} \left[\int_{\mathbb{R}^{(k-1)n}} \frac{\prod_{i=1}^{k-1} |x_i|^{\widetilde{A}_i}}{|\sum_{i=1}^k x_i|^s} dx_1 dx_2 \dots dx_{k-1}\right] dx_k, \\
J_2 &= \int_{K^n(\varepsilon^{-1})\setminus K^n(\varepsilon)} |x_k|^{\widetilde{A}_k} \sum_{j=1}^{k-1} I_j^{\varepsilon}(x_k) dx_k, \\
J_3 &= \int_{K^n(\varepsilon^{-1})\setminus K^n(\varepsilon)} |x_k|^{\widetilde{A}_k} \sum_{j=1}^{k-1} I_j^{\varepsilon^{-1}}(x_k) dx_k.
\end{aligned}$$

Here, for $j = 1, 2, \dots, k-1$, the integrals $I_j^{\varepsilon}(x_k)$ and $I_j^{\varepsilon^{-1}}(x_k)$ are defined by

$$I_j^{\varepsilon}(x_k) = \int_{\mathbb{P}_j} \frac{\prod_{i=1}^{k-1} |x_i|^{\widetilde{A}_i}}{|\sum_{i=1}^k x_i|^s} dx_1 dx_2 \dots dx_{k-1},$$

where $\mathbb{P}_j = \{(U_1, U_2, \dots, U_{k-1}); U_j = K^n(\varepsilon), U_l = \mathbb{R}^n, l \neq j\}$, and

$$I_j^{\varepsilon^{-1}}(x_k) = \int_{\mathbb{Q}_j} \frac{\prod_{i=1}^{k-1} |x_i|^{\widetilde{A}_i}}{|\sum_{i=1}^k x_i|^s} dx_1 dx_2 \dots dx_{k-1},$$

where $\mathbb{Q}_j = \{(U_1, U_2, \dots, U_{k-1}); U_j = \mathbb{R}^n \setminus K^n(\varepsilon^{-1}), U_l = \mathbb{R}^n, l \neq j\}$.

Now, our aim is to find the lower bound for J. The first integral J_1 can easily be computed by virtue of the Selberg integral formula (3.6). More precisely, since parameters $\widetilde{A}_i$ fulfill relation (3.47), it follows that

$$\int_{\mathbb{R}^{(k-1)n}} \frac{\prod_{i=1}^{k-1} |x_i|^{\widetilde{A}_i}}{|\sum_{i=1}^k x_i|^s} dx_1 dx_2 \dots dx_{k-1} = K^*|x_k|^{-\widetilde{A}_k - n},$$

and consequently,

$$J_1 = K^*|\mathbb{S}^{n-1}|\log\frac{1}{\varepsilon^2}. \tag{3.58}$$

In the sequel, we show that the parts J_2 and J_3 converge when $\varepsilon \to 0$. For that sake, without loss of generality, it is enough to estimate the integrals

$$\int_{K^n(\varepsilon^{-1})\setminus K^n(\varepsilon)} |x_k|^{\widetilde{A}_k} I_1^{\varepsilon}(x_k)dx_k \quad \text{and} \quad \int_{K^n(\varepsilon^{-1})\setminus K^n(\varepsilon)} |x_k|^{\widetilde{A}_k} I_1^{\varepsilon^{-1}}(x_k)dx_k.$$

Utilizing Lemma 3.4 and n-dimensional spherical coordinates we have

$$\begin{aligned}\int_{K^n(\varepsilon^{-1})\setminus K^n(\varepsilon)} |x_k|^{\widetilde{A}_k} I_1^{\varepsilon}(x_k)dx_k &\le C_k \varepsilon^{n+\widetilde{A}_1} \int_{K^n(\varepsilon^{-1})\setminus K^n(\varepsilon)} |x_k|^{-2n-\widetilde{A}_1} dx_k \\ &= C_k |\mathbb{S}^{n-1}| \varepsilon^{n+\widetilde{A}_1} \int_{\varepsilon}^{\varepsilon^{-1}} r^{-n-\widetilde{A}_1-1} dr \\ &= \frac{C_k |\mathbb{S}^{n-1}|}{n+\widetilde{A}_1} \left(1-\varepsilon^{2(n+\widetilde{A}_1)}\right).\end{aligned}$$

Similarly, we also have

$$\begin{aligned}\int_{K^n(\varepsilon^{-1})\setminus K^n(\varepsilon)} |x_k|^{\widetilde{A}_k} I_1^{\varepsilon^{-1}}(x_k)dx_k &\le D_k \varepsilon^{n+\widetilde{A}_k} \int_{K^n(\varepsilon^{-1})\setminus K^n(\varepsilon)} |x_k|^{\widetilde{A}_k} dx_k \\ &= \frac{D_k |\mathbb{S}^{n-1}|}{n+\widetilde{A}_k} \left(1-\varepsilon^{2(n+\widetilde{A}_k)}\right).\end{aligned}$$

Clearly, since $n+\widetilde{A}_i > 0$, $i = 1,2,\dots,k$, the above computation shows that $J_2 + J_3 \le O(1)$, when $\varepsilon \to 0$. Hence, taking into account relation (3.58), it follows that J is bounded from below by

$$K^* |\mathbb{S}^{n-1}| \log \frac{1}{\varepsilon^2} - O(1), \quad \text{when} \quad \varepsilon \to 0. \tag{3.59}$$

Now, suppose that there exists a positive constant L^*, $0 < L^* < K^*$, such that inequality (3.48) holds with the constant L^* instead of K^*. In that case, for the above choice of functions $\widetilde{f}_i$, the right-hand side of inequality (3.48) becomes $L^* |\mathbb{S}^{n-1}| \log \frac{1}{\varepsilon^2}$. Since $L^* |\mathbb{S}^{n-1}| \log \frac{1}{\varepsilon^2} \ge J$, relation (3.59) implies that

$$(K^* - L^*)\, |\mathbb{S}^{n-1}| \log \frac{1}{\varepsilon^2} \le O(1), \quad \text{when} \quad \varepsilon \to 0. \tag{3.60}$$

Finally, letting $\varepsilon \to 0$, we obtain a contradiction, since the left-hand side of inequality (3.60) goes to infinity. This shows that the constant K^* is the best possible in (3.48). Due to the equivalence, K^* is also the best possible in inequality (3.49). □

Remark 3.6 A straightforward computation shows that the parameters A_{ij}, providing the inequalities from Corollary 3.3, fulfill the set of conditions (3.44) in the conjugate case. Then, the constant L from Corollary 3.3 becomes

$$L = \frac{1}{\Gamma_n(s)} \prod_{i=1}^{k} \Gamma_n\left(n - \frac{nk-s}{p_i}\right),$$

and that is the best possible constant in the corresponding inequalities.

Remark 3.7 Hilbert-type inequalities from this chapter, obtained by virtue of the Selberg integral formula, are taken from [17] and [61].

3.5 Some related inequalities with norms

So far, in this chapter we have studied Hilbert-type inequalities established by virtue of the Selberg integral formula. As distinguished from the previous sections, we consider here multidimensional Hilbert-type inequalities with a homogeneous kernel and the power weight functions whose arguments are α-norms of the corresponding vectors. Recall, if $t=(t_1,t_2,\dots,t_n)\in\mathbb{R}^n$, then α-norm ($\alpha\geq 1$) of vector t is $|t|_\alpha=({t_1}^\alpha+{t_2}^\alpha+\cdots+{t_n}^\alpha)^{\frac{1}{\alpha}}$.

We shall be concerned here with a general homogeneous kernel $K_\alpha:\mathbb{R}_+^k\to\mathbb{R}$ of degree $-s$, $s>0$. Hence, the constant factors in the corresponding inequalities will include the integral

$$k_\alpha(\beta_1,\dots,\beta_{k-1})=\int_{\mathbb{R}_+^{k-1}}K_\alpha(1,t_1\dots,t_{k-1})t_1^{\beta_1}\cdots t_{k-1}^{\beta_{k-1}}dt_1\cdots dt_{k-1}.$$

The above integral is assumed to converge for $\beta_1,\dots,\beta_{k-1}>-1$ and $\beta_1+\cdots+\beta_{k-1}+k<s+1$ (see relation (1.76), Section 1.6, Chapter 1).

On the other hand, since the arguments of the kernel and the weight functions will be expressed by means of α-norm, constant factors in the corresponding Hilbert-type inequalities will also include the area of the unit sphere in $\mathbb{R}^n$. The area of the unit sphere in view of the α-norm, denoted here by $|\mathbb{S}^{n-1}|_\alpha$, is

$$|\mathbb{S}^{n-1}|_\alpha=\frac{2^n\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}. \tag{3.61}$$

It should be noticed here that the above formula (3.61) coincides with (3.3) when $\alpha=2$.

The following Theorem can be regarded as a vector extension of Theorem 1.19 (see Section 1.6, Chapter 1), in view of α-norm.

Theorem 3.8 *Let λ,p_i,p_i',q_i, $i=1,2,\dots,k$, $k\geq 2$, be parameters satisfying conditions* (2.26)*,* (2.27)*, and* (2.30)*, and let A_{ij}, $i,j=1,\dots,k$, be such that $\sum_{i=1}^k A_{ij}=0$ for $j=1,\dots,k$, $q_iA_{ij}>-n$, $i\neq j$, and $q_i(A_{ii}-\alpha_i)>(k-1)n-s$, where $n\in\mathbb{N}$, $s>0$, and $\alpha_i=\sum_{j=1}^k A_{ij}$, $i=1,2,\dots,k$. If $K_\alpha:\mathbb{R}_+^k\to\mathbb{R}$ is a non-negative measurable homogeneous function of degree $-s$ and*

$$\begin{aligned}L=\;&\frac{|\mathbb{S}^{n-1}|_\alpha^{(k-1)\lambda}}{2^{(k-1)n\lambda}}k_\alpha^{\frac{1}{q_1}}(n-1+q_1A_{12},\dots,n-1+q_1A_{1k})\\&\times k_\alpha^{\frac{1}{q_2}}(s-(k-1)n-1-q_2(\alpha_2-A_{22}),n-1+q_2A_{23},\dots,n-1+q_2A_{2k})\\&\cdots k_\alpha^{\frac{1}{q_k}}(n-1+q_kA_{k2},\dots,n-1+q_kA_{k,k-1},s-(k-1)n-1-q_k(\alpha_k-A_{kk})),\end{aligned} \tag{3.62}$$

then the inequalities

$$\int_{\mathbb{R}_+^{nk}} K_\alpha^\lambda(|x_1|_\alpha,\dots,|x_k|_\alpha)\prod_{i=1}^k f_i(x_i)dx_1\dots dx_k$$
$$< L\prod_{i=1}^k \left[\int_{\mathbb{R}_+^n} |x_i|_\alpha^{\frac{p_i}{q_i}[(k-1)n-s]+p_i\alpha_i} f_i^{p_i}(x_i)dx_i\right]^{\frac{1}{p_i}} \tag{3.63}$$

and

$$\left[\int_{\mathbb{R}_+^n} |x_k|_\alpha^{-\frac{p_k'}{q_k}[(k-1)n-s]-p_k'\alpha_k}\left(\int_{\mathbb{R}_+^{n(k-1)}} K_\alpha^\lambda(|x_1|_\alpha,\dots,|x_k|_\alpha)\right.\right.$$
$$\left.\left.\times\prod_{i=1}^{k-1} f_i(x_i)dx_1\dots dx_{k-1}\right)^{p_k'}dx_k\right]^{\frac{1}{p_k'}}$$
$$< L\prod_{i=1}^{k-1}\left[\int_{\mathbb{R}_+^n} |x_i|_\alpha^{\frac{p_i}{q_i}[(k-1)n-s]+p_i\alpha_i} f_i^{p_i}(x_i)dx_i\right]^{\frac{1}{p_i}} \tag{3.64}$$

hold for all non-negative measurable functions $f_i:\mathbb{R}_+^n\to\mathbb{R}$, $f_i\neq 0$, $i=1,\dots,k$. *Moreover, these inequalities are equivalent.*

Proof. We utilize Theorem 1.18 (see Section 1.6) with the homogeneous kernel $K_\alpha(|x_1|_\alpha,\dots,|x_k|_\alpha)$, the weight functions $\phi_{ij}(x_j)=|x_j|^{A_{ij}}$, and Lebesgue measures on $\mathbb{R}_+^n$. Using notation from Theorem 1.18, it is enough to calculate the functions $F_i(x_i)$, $i=1,\dots,k$. Utilizing n-dimensional spherical coordinates we find that

$$\begin{aligned} F_1^{q_1}(x_1) &= \int_{\mathbb{R}_+^{n(k-1)}} K_\alpha(|x_1|_\alpha,\dots,|x_k|_\alpha)\prod_{j=2}^k |x_j|^{q_1A_{1j}}dx_2\cdots dx_k \\ &= \frac{|\mathbb{S}^{n-1}|_\alpha^{k-1}}{2^{(k-1)n}}\int_{\mathbb{R}_+^{k-1}} K_\alpha(|x_1|_\alpha,t_2,\dots,t_k)\prod_{j=2}^k t_j^{n-1+q_1A_{1j}}dt_2\cdots dt_k. \end{aligned}$$

Moreover, due to homogeneity of the kernel K_α and using the change of variables $u_i=\frac{t_i}{|x_1|_\alpha}$, $i=2,\dots,k$, we obtain

$$\begin{aligned} F_1^{q_1}(x_1) &= \frac{|\mathbb{S}^{n-1}|_\alpha^{k-1}}{2^{(k-1)n}}\int_{\mathbb{R}_+^{k-1}} |x_1|_\alpha^{-s}K_\alpha(1,u_2,\dots,u_k) \\ &\quad\times\prod_{j=2}^k (|x_1|_\alpha u_j)^{n-1+q_1A_{1j}}|x_1|_\alpha^{k-1}du_2\dots du_k \\ &= \frac{|\mathbb{S}^{n-1}|_\alpha^{k-1}}{2^{(k-1)n}}|x_1|_\alpha^{(k-1)n-s+q_1(\alpha_1-A_{11})}k_\alpha(n-1+q_1A_{12},\dots,n-1+q_1A_{1k}). \end{aligned}$$

Similarly,

$$\begin{aligned} F_2^{q_2}(x_2) &= \int_{\mathbb{R}_+^{n(k-1)}} K_\alpha(|x_1|_\alpha,\dots,|x_k|_\alpha) \prod_{j=1,j\neq 2}^{k} |x_j|^{q_2A_{2j}} dx_1 dx_3 \dots dx_k \\ &= \frac{|\mathbb{S}^{n-1}|_\alpha^{k-1}}{2^{(k-1)n}} \int_{\mathbb{R}_+^{k-1}} t_1^{-s} K_\alpha(1,|x_2|_\alpha/t_1, t_3/t_1,\dots,t_k/t_1) \\ &\quad\times \prod_{j=1,j\neq 2}^{k} t_j^{n-1+q_2A_{2j}} dt_1 dt_3 \dots dt_k. \end{aligned}$$

Hence, the change of variables $t_1 = |x_2|_\alpha u_2^{-1}$, $t_i = |x_2|_\alpha u_2^{-1} u_i$, $i = 3,\dots,k$, with the Jacobian

$$\frac{\partial(t_1,t_3,\dots,t_k)}{\partial(u_2,u_3,\dots,u_k)} = |x_2|_\alpha^{k-1} u_2^{-k},$$

yields

$$\begin{aligned} F_2^{q_2}(x_2) &= \frac{|\mathbb{S}^{n-1}|_\alpha^{k-1}}{2^{(k-1)n}} |x_2|_\alpha^{(k-1)n-s+q_2(\alpha_2-A_{22})} \\ &\quad\times \int_{\mathbb{R}_+^{k-1}} K_\alpha(1,u_2,\dots,u_k) u_2^{s-(k-1)n-q_2(\alpha_2-A_{22})} \prod_{j=3}^{k} u_j^{n-1+q_2A_{2j}} du_2 \dots du_k \\ &= \frac{|\mathbb{S}^{n-1}|_\alpha^{k-1}}{2^{(k-1)n}} |x_2|_\alpha^{(k-1)n-s-q_2(\alpha_2-A_{22})} \\ &\quad\times k_\alpha(s-(k-1)n-1-q_2(\alpha_2-A_{22}), n-1+q_2A_{23},\dots,n-1+q_2A_{2k}). \end{aligned}$$

In a similar manner we obtain

$$\begin{aligned} F_i^{q_i}(x_i) &= \frac{|\mathbb{S}^{n-1}|_\alpha^{k-1}}{2^{(k-1)n}} |x_i|_\alpha^{(k-1)n-s+q_i(\alpha_i-A_{ii})} \\ &\quad\times k_\alpha(n-1+q_iA_{i2},\dots,n-1+q_iA_{i,i-1}, s-(k-1)n-1-q_i(\alpha_i-A_{ii}), \\ &\quad n-1+q_iA_{i,i+1},\dots,n-1+q_iA_{ik}), \quad i=3,\dots,k, \end{aligned}$$

which yields inequalities (3.63) and (3.64) with the sign $\leq$. Finally, Remark 1.21 (see Section 1.6) provides the sharpness of the obtained inequalities. □

In order to obtain the best constants in (3.63) and (3.64), we consider now these inequalities in the conjugate setting. Similarly to the previous section, we impose certain conditions on parameters A_{ij}, $i,j = 1,2,\dots,k$, to obtain a simpler form of the constant L defined by (3.62). More precisely, if the parameters A_{ij} fulfill conditions

$$n + p_jA_{ji} = s-(k-1)n - p_i(\alpha_i - A_{ii}), \quad j \neq i, \quad i,j = 1,2,\dots,k, \tag{3.65}$$

then, in the conjugate case, constant L reduces to

$$L^* = \frac{|\mathbb{S}^{n-1}|_\alpha^{(k-1)}}{2^{(k-1)n}} k_\alpha(n-1+\widetilde{A}_2,\dots,n-1+\widetilde{A}_k), \tag{3.66}$$

where $\widetilde{A}_i = p_1 A_{1i}$, $i \neq 1$, and $\widetilde{A}_1 = p_k A_{k1}$. Moreover, inequalities (3.63) and (3.64) reduce to

$$\int_{\mathbb{R}_+^{nk}} K_\alpha(|x_1|_\alpha, \ldots, |x_k|_\alpha) \prod_{i=1}^k f_i(x_i) dx_1 \ldots dx_k$$
$$< L^* \prod_{i=1}^k \left[\int_{\mathbb{R}_+^n} |x_i|_\alpha^{-n-p_i\widetilde{A}_i} f_i^{p_i}(x_i) dx_i \right]^{\frac{1}{p_i}} \tag{3.67}$$

and

$$\left[\int_{\mathbb{R}_+^n} |x_k|_\alpha^{(1-p_k')(-n-p_k\widetilde{A}_k)} \left(\int_{\mathbb{R}_+^{n(k-1)}} K_\alpha(|x_1|_\alpha, \ldots, |x_k|_\alpha) \right.\right.$$
$$\left.\left. \times \prod_{i=1}^{k-1} f_i(x_i) dx_1 \ldots dx_{k-1} \right)^{p_k'} dx_k \right]^{\frac{1}{p_k'}}$$
$$< L^* \prod_{i=1}^{k-1} \left[\int_{\mathbb{R}_+^n} |x_i|_\alpha^{-n-p_i\widetilde{A}_i} f_i^{p_i}(x_i) dx_i \right]^{\frac{1}{p_i}}. \tag{3.68}$$

The following result asserts that L^* is the best possible constant in inequalities (3.67) and (3.68), under some weak conditions on the kernel K_α.

Theorem 3.9 *Suppose that real parameters A_{ij}, $i, j = 1, \ldots, k$, fulfill conditions of Theorem* 3.8 *(conjugate case) and conditions given in* (3.65). *If the kernel $K_\alpha : \mathbb{R}_+^k \to \mathbb{R}$ satisfies conditions of Theorem* 3.8 *and for every $i = 2, \ldots, k$,*

$$K_\alpha(1, t_2, \ldots, t_i, \ldots, t_k) \leq C K_\alpha(1, t_2, \ldots, 0, \ldots, t_k),\ 0 \leq t_i \leq 1,\ t_j \geq 0,\ j \neq i,$$

for some $C > 0$, then L^* is the best possible constant in inequalities (3.67) *and* (3.68).

Proof. Suppose that inequality (3.67) holds for all non-negative measurable functions f_i, $i = 1, 2, \ldots, k$, when L^* is replaced with a smaller positive constant L_1. To prove our assertion, we consider inequality (3.67) with the constant L_1 and the functions $\widetilde{f}_{i,\varepsilon} : \mathbb{R}_+^n \to \mathbb{R}$, defined by

$$\widetilde{f}_{i,\varepsilon}(x_i) = \begin{cases} 0, & |x_i|_\alpha < 1 \\ |x_i|_\alpha^{\widetilde{A}_i - \frac{\varepsilon}{p_i}}, & |x_i|_\alpha \geq 1 \end{cases},\quad i = 1, \ldots, k,$$

where $0 < \varepsilon < \min_{1 \leq i \leq k}\{p_i + p_i\widetilde{A}_i\}$. Utilizing n-dimensional spherical coordinates, the right-hand side of (3.67) becomes

$$L_1 \prod_{i=1}^k \left[\int_{|x_i|_\alpha \geq 1} |x_i|_\alpha^{-n-\varepsilon} dx_i \right]^{\frac{1}{p_i}} = \frac{L_1 |\mathbb{S}^{n-1}|_\alpha}{2^n} \int_1^\infty t^{-1-\varepsilon} dt = \frac{L_1 |\mathbb{S}^{n-1}|_\alpha}{2^n \varepsilon}. \tag{3.69}$$

Further, let J denote the left-hand side of inequality (3.67), for the above choice of functions $\widetilde{f}_{i,\varepsilon}$. Using the change of variables $u_i = \frac{t_i}{t_1}$, $i \neq 2$, we have

$$\begin{aligned}
J &= \int_{|x_1|_\alpha \geq 1} \cdots \int_{|x_k|_\alpha \geq 1} K_\alpha(|x_1|_\alpha, \ldots, |x_k|_\alpha) \prod_{i=1}^{k} |x_i|_\alpha^{\widetilde{A}_i - \frac{\varepsilon}{p_i}} dx_1 \ldots dx_k \\
&= \frac{|\mathbb{S}^{n-1}|_\alpha^k}{2^{kn}} \int_1^\infty \cdots \int_1^\infty K_\alpha(t_1, \ldots, t_k) \prod_{i=1}^{k} t_i^{n-1+\widetilde{A}_i - \frac{\varepsilon}{p_i}} dt_1 \ldots dt_k \\
&= \frac{|\mathbb{S}^{n-1}|_\alpha^k}{2^{kn}} \int_1^\infty t_1^{-1-\frac{\varepsilon}{\beta}} \left(\int_{\frac{1}{t_1}}^\infty \cdots \int_{\frac{1}{t_1}}^\infty K_\alpha(1, u_2, \ldots, u_k) \prod_{i=2}^{k} u_i^{n-1+\widetilde{A}_i - \frac{\varepsilon}{p_i}} du_2 \ldots du_k \right) dt_1,
\end{aligned}$$

so, J can be estimated from below as follows:

$$\begin{aligned}
J \geq \frac{|\mathbb{S}^{n-1}|_\alpha^k}{2^{kn}} \int_1^\infty t_1^{-1-\varepsilon} \left(\int_0^\infty \cdots \int_0^\infty K_\alpha(1, u_2, \ldots, u_k) \prod_{i=2}^{k} u_i^{n-1+\widetilde{A}_i - \frac{\varepsilon}{p_i}} du_2 \ldots du_k \right) dt_1 \\
- \frac{|\mathbb{S}^{n-1}|_\alpha^k}{2^{kn}} \int_1^\infty t_1^{-1-\varepsilon} \sum_{j=2}^{k} I_j(t_1) dt_1, \qquad (3.70)
\end{aligned}$$

where

$$I_j(t_1) = \int_{\mathbb{D}_j} K_\alpha(1, u_2, \ldots, u_k) \prod_{i=2}^{k} u_i^{n-1+\widetilde{A}_i - \frac{\varepsilon}{p_i}} du_2 \ldots du_k, \; j = 2, \ldots, k,$$

and $\mathbb{D}_j = \{(u_2, \ldots, u_k); 0 < u_j < \frac{1}{t_1}, 0 < u_l < \infty, l \neq j\}$.

Now, the rest of the proof follows the same lines as the proof of Theorem 1.20 (see Section 1.6, Chapter 1). Namely, utilizing the estimate $\int_1^\infty t_1^{-1-\varepsilon} \sum_{j=2}^{k} I_j(t_1) dt_1 \leq O(1)$, (3.69), and (3.70), it follows that $L^* \leq L_1$, which contradicts assumption that L_1 is smaller than L^*. □

Observe that Theorem 3.9 may be regarded as an extension of Theorem 1.20 from Section 1.6.

Remark 3.8 Let $k_s : \mathbb{R}_+^2 \to \mathbb{R}$ be a non-negative homogeneous kernel of degree $-s$, $s > 0$, and let parameters A_1 and A_2 fulfill relation $qA_1 + pA_2 = m + n - s$, where p and q are conjugate parameters, and $m, n \in \mathbb{N}$. Assuming that there exists $\delta > 0$ such that $c_s(\eta) = \int_0^\infty k_s(1,t) t^{-\eta} dt < \infty$ for $\eta \in [pA_2 + 1 - n - \delta, pA_2 + 1 - n]$, Yang and Krnić [167], obtained a pair of equivalent inequalities

$$\begin{aligned}
&\int_{\mathbb{R}_+^m} \int_{\mathbb{R}_+^n} k_s\left(|x|_\alpha, |y|_\beta\right) f(x) g(y) dx dy \\
&\leq C_s \left[\int_{\mathbb{R}_+^m} |x|_\alpha^{pqA_1 - m} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_{\mathbb{R}_+^n} |y|_\beta^{pqA_2 - n} g^q(y) dy \right]^{\frac{1}{q}}
\end{aligned}$$

and

$$\left\{\int_{\mathbb{R}_+^n} |y|_\beta{}^{pqA_1+p(s-m)-n}\left[\int_{\mathbb{R}_+^m} k_s\left(|x|_\alpha,|y|_\beta\right) f(x)dx\right]^p dy\right\}^{\frac{1}{p}}$$
$$\leq C_s\left[\int_{\mathbb{R}_+^m} |x|_\alpha^{pqA_1-m} f^p(x)dx\right]^{\frac{1}{p}},$$

with the best possible constant expressed in terms of the usual Gamma function:

$$C_s = \left[\frac{\Gamma^m\left(\frac{1}{\alpha}\right)}{\alpha^{m-1}\Gamma\left(\frac{m}{\alpha}\right)}\right]^{\frac{1}{q}}\left[\frac{\Gamma^n\left(\frac{1}{\beta}\right)}{\beta^{n-1}\Gamma\left(\frac{n}{\beta}\right)}\right]^{\frac{1}{p}} c_s(pA_2+1-n).$$

It should be noticed here that these inequalities include two different norms $|\cdot|_\alpha$ and $|\cdot|_\beta$. Obviously, the above inequalities may also be extended to a multidimensional setting, as in Theorem 3.8.

To conclude this section we consider Theorem 3.8 for some particular choices of the kernel K_α and parameters A_{ij}, $i,j=1,2,\dots k$. More precisely, we are concerned here with the homogeneous kernel $K_\alpha(t_1,t_2,\dots,t_k)=\left(\sum_{i=1}^k t_i^\beta\right)^{-s}$ of degree $-\beta s$. In this case we have

$$k_\alpha(\beta_1,\dots,\beta_{k-1}) = \frac{1}{\beta^{k-1}\Gamma(s)}\Gamma\left(s-\sum_{i=1}^{k-1}\frac{\beta_i+1}{\beta}\right)\prod_{i=1}^{k-1}\Gamma\left(\frac{\beta_i+1}{\beta}\right)$$

(see Lemma 1.4, Section 1.6, Chapter 1), so Theorem 3.8 yields the following result:

Corollary 3.6 *Let λ, p_i, p_i', q_i, $i=1,2,\dots,k$, $k\geq 2$, be parameters satisfying* (2.26), (2.27), *and* (2.30), *and let A_{ij}, $i,j=1,\dots,k$, be real parameters such that $\sum_{i=1}^k A_{ij}=0$ for $j=1,\dots,k$, $q_iA_{ij}>-n$, $i\neq j$, and $q_i(A_{ii}-\alpha_i)>(k-1)n-\beta s$, where $\alpha_i=\sum_{j=1}^k A_{ij}$, $i=1,2,\dots,k$, $n\in\mathbb{N}$, and $s>0$. If*

$$K = \frac{\Gamma^{\lambda(k-1)n}\left(\frac{1}{\alpha}\right)}{\beta^{\lambda(k-1)}\alpha^{\lambda(n-1)(k-1)}\Gamma^{\lambda(k-1)}\left(\frac{n}{\alpha}\right)\Gamma^\lambda(s)}$$
$$\times\prod_{i=1}^k\Gamma^{\frac{1}{q_i}}\left(s-\frac{(k-1)n+q_i\alpha_i-q_iA_{ii}}{\beta}\right)\prod_{i,j=1,i\neq j}^k\Gamma^{\frac{1}{q_i}}\left(\frac{n+q_iA_{ij}}{\beta}\right),$$

then the inequalities

$$\int_{\mathbb{R}_+^{nk}}\frac{\prod_{i=1}^k f_i(x_i)}{\left(\sum_{i=1}^k |x_i|_\alpha{}^\beta\right)^{\lambda s}}dx_1dx_2\cdots dx_k$$
$$\leq K\prod_{i=1}^k\left[\int_{\mathbb{R}_+^n}|x_i|_\alpha{}^{\frac{p_i(k-1)n-p_i\beta s}{q_i}+p_i\alpha_i} f_i^{p_i}(x_i)dx_i\right]^{\frac{1}{p_i}} \tag{3.71}$$

and

$$\begin{aligned}
&\left\{ \int_{\mathbb{R}_+^n} |x_k|_\alpha^{-\frac{p_k'}{q_k}[(k-1)n-\beta s]-p_k'\alpha_k} \right.\\
&\quad\times \left.\left[\int_{\mathbb{R}_+^{n(k-1)}} \frac{\prod_{i=1}^{k-1} f_i(x_i)}{\left(\sum_{i=1}^{k} |x_i|_\alpha{}^\beta\right)^{\lambda s}} dx_1 dx_2 \cdots dx_{k-1} \right]^{p_k'} dx_k \right\}^{\frac{1}{p_k'}}\\
&\leq K \prod_{i=1}^{k-1} \left[\int_{\mathbb{R}_+^n} |x_i|_\alpha^{\frac{p_i(k-1)n-p_i\beta s}{q_i}+p_i\alpha_i} f_i^{p_i}(x_i) dx_i \right]^{\frac{1}{p_i}} \qquad (3.72)
\end{aligned}$$

hold for all non-negative measurable functions $f_i : \mathbb{R}_+^n \to \mathbb{R}$, $i = 1,\ldots,k$, and are equivalent.

Remark 3.9 Introducing the parameters $A_{ii} = (nk-s)\frac{\lambda q_i - 1}{q_i^2}$ and $A_{ij} = (s-nk)\frac{1}{q_i q_j}$, $i \neq j$, it follows that $\sum_{i=1}^k A_{ij} = \sum_{j=1}^k A_{ij} = 0$ for $i, j = 1,2,\ldots,k$. Hence, in this setting inequalities (3.71) and (3.72) become

$$\int_{\mathbb{R}_+^{nk}} \frac{\prod_{i=1}^{k} f_i(x_i)}{\left(\sum_{i=1}^{k} |x_i|_\alpha{}^\beta\right)^{\lambda s}} dx_1 dx_2 \cdots dx_k \leq L \prod_{i=1}^{k} \left[\int_{\mathbb{R}_+^n} |x_i|_\alpha^{\frac{p_i(k-1)n-p_i\beta s}{q_i}} f_i^{p_i}(x_i) dx_i \right]^{\frac{1}{p_i}}$$

and

$$\begin{aligned}
&\left\{ \int_{\mathbb{R}_+^n} |x_k|_\alpha^{-\frac{p_k'}{q_k}[(k-1)n-\beta s]} \left[\int_{(\mathbb{R}_+^n)^{k-1}} \frac{\prod_{i=1}^{k-1} f_i(x_i)}{\left(\sum_{i=1}^{k} |x_i|_\alpha{}^\beta\right)^{\lambda s}} dx_1 dx_2 \cdots dx_{k-1} \right]^{p_k'} dx_k \right\}^{\frac{1}{p_k'}}\\
&\leq L \prod_{i=1}^{k-1} \left[\int_{\mathbb{R}_+^n} |x_i|_\alpha^{\frac{p_i(k-1)n-p_i\beta s}{q_i}} f_i^{p_i}(x_i) dx_i \right]^{\frac{1}{p_i}},
\end{aligned}$$

with the constant factor

$$\begin{aligned}
L &= \frac{\Gamma^{\lambda(k-1)n}\left(\frac{1}{\alpha}\right)}{\beta^{\lambda(k-1)}\alpha^{\lambda(n-1)(k-1)}\Gamma^{\lambda(k-1)}\left(\frac{n}{\alpha}\right)\Gamma^{\lambda}(s)} \prod_{i=1}^{k} \Gamma^{\frac{1}{q_i}}\left(\frac{\beta s + n(p_i - k)}{\beta p_i}\right)\\
&\quad\times \prod_{i=1}^{k} \Gamma^{\lambda - \frac{1}{q_i}}\left(\frac{\beta s + n(q_i - k)}{\beta q_i}\right),
\end{aligned}$$

where we assume that $nk - \beta s < n \min\{p_i, q_j, i, j = 1,2,\ldots k\}$.

Observe that the above parameters A_{ij} fulfill the set of conditions (3.65) in the conjugate case, that is, when $\lambda = 1$ and $p_i = q_i$, $i = 1,2,\ldots,n$. Therefore, in the conjugate case, the above constant L is the best possible in the corresponding inequalities.

Remark 3.10 Let $A_i \in \mathbb{R}$, $i = 1,2,\ldots,k$. Defining the parameters $A_{ii} = A_i$, $A_{ii+1} = -A_{i+1}$, $A_{ij} = 0$ for $|i-j| > 1$, $i, j = 1,2,\ldots k$, where indices are taken modulo k, we have $\sum_{i=1}^{n} A_{ij} = A_{j-1j} + A_{jj} = -A_j + A_j = 0$. Hence, Corollary 3.6 yields

$$\int_{\mathbb{R}_+^{nk}} \frac{\prod_{i=1}^{k} f_i(x_i)}{\left(\sum_{i=1}^{k} |x_i|_\alpha{}^\beta\right)^{\lambda s}} dx_1 dx_2 \cdots dx_k$$
$$\leq M \prod_{i=1}^{k} \left[\int_{\mathbb{R}_+^{n}} |x_i|_\alpha{}^{\frac{p_i(k-1)n - p_i\beta s}{q_i} + p_i(A_i - A_{i+1})} f_i^{p_i}(x_i) dx_i\right]^{\frac{1}{p_i}} \tag{3.73}$$

and

$$\left\{\int_{\mathbb{R}_+^{n}} |x_k|_\alpha{}^{-\frac{p_k'}{q_k}[(k-1)n-\beta s] - p_k'(A_k - A_1)}\right.$$
$$\left.\times \left[\int_{\mathbb{R}_+^{n(k-1)}} \frac{\prod_{i=1}^{k-1} f_i(x_i)}{\left(\sum_{i=1}^{k} |x_i|_\alpha{}^\beta\right)^{\lambda s}} dx_1 dx_2 \cdots dx_{k-1}\right]^{p_k'} dx_k\right\}^{\frac{1}{p_k'}}$$
$$\leq M \prod_{i=1}^{k-1} \left[\int_{\mathbb{R}_+^{n}} |x_i|_\alpha{}^{\frac{p_i(k-1)n - p_i\beta s}{q_i} + p_i(A_i - A_{i+1})} f_i^{p_i}(x_i) dx_i\right]^{\frac{1}{p_i}}, \tag{3.74}$$

with the constant factor

$$M = \frac{\Gamma^{\lambda(k-1)n}\left(\frac{1}{\alpha}\right)\Gamma^{\lambda(k-2)}\left(\frac{n}{\beta}\right)}{\beta^{\lambda(k-1)}\alpha^{\lambda(n-1)(k-1)}\Gamma^{\lambda(k-1)}\left(\frac{n}{\alpha}\right)\Gamma^{\lambda}(s)} \prod_{i=1}^{k} \Gamma^{\frac{1}{q_i}}\left(\frac{\beta s + q_i A_{i+1} - (k-1)n}{\beta}\right)$$
$$\times \prod_{i=1}^{k} \Gamma^{\frac{1}{q_i}}\left(\frac{n - q_i A_{i+1}}{\beta}\right),$$

provided that $A_i \in \left(\frac{(k-1)n-\beta s}{q_{i-1}}, \frac{n}{q_{i-1}}\right)$, $i = 1,2,\ldots,k$.

Remark 3.11 Hilbert-type inequalities in this section are established in [59] and [127]. For some related particular results, the reader is referred to [125] and [174]. In addition, results from this section for $n = 1$ and $\beta = 1$ were also studied in [12], [111], and [156].

Chapter 4

Applying the Euler-Maclaurin summation formula

The starting point in this chapter is the well-known Euler-Maclaurin summation formula which asserts that

$$\begin{aligned}\int_a^b f(x)dx &= \frac{(b-a)}{2}[f(a)+f(b)] \\ &\quad -\sum_{k=1}^{n-1}\frac{(b-a)^{2k}}{(2k)!}B_{2k}\left[f^{(2k-1)}(b)-f^{(2k-1)}(a)\right]+\rho_{2n}(f),\end{aligned} \tag{4.1}$$

where B_k is the corresponding Bernoulli number and the remainder $\rho_{2n}(f)$ is

$$\rho_{2n}(f)=\frac{(b-a)^{2n-1}}{(2n-1)!}\int_a^b f^{(2n-1)}(t)\gamma^*_{2n-1}\left(\frac{b-t}{b-a}\right)dt$$

or

$$\rho_{2n}(f)=\frac{(b-a)^{2n}}{(2n)!}\int_a^b f^{(2n)}(t)\gamma^*_{2n}\left(\frac{b-t}{b-a}\right)dt,$$

depending on whether the function $f:(a,b)\to\mathbb{R}$ has a continuous derivative of order $2n-1$ or $2n$, that is, f is of class C^{2n-1} or C^{2n}. Here, γ^*_n denotes the periodic function $\gamma^*_n(x)=B^*_n(x)-B_n$, where $B^*_n(x)$ and B_n are the corresponding Bernoulli polynomial and the Bernoulli number. For a comprehensive inspection on Bernoulli polynomials and numbers, as well as on the above summation formula, the reader is referred to [1] and [67].

The Euler-Maclaurin summation formula is a very useful tool in obtaining refinements of discrete Hilbert-type inequalities. Recently, Mingzhe and Xuemei [100], obtained the following refinement of the Hilbert-type inequality by means of the above summation formula.

Theorem 4.1 *Suppose $p \geq q > 1$ are conjugate exponents. If $1-\frac{q}{p} < s \leq 2$, then the inequality*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{(m+n+1)^s} < \left[\sum_{m=0}^{\infty}\omega_q(s,m)a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=0}^{\infty}\omega_p(s,n)b_n^q\right]^{\frac{1}{q}} \tag{4.2}$$

holds for all non-negative sequences $(a_m)_{m\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0} \neq 0$, where

$$\omega_r(s,n) = \left(n+\frac{1}{2}\right)^{1-s} B\left(\frac{p-2+s}{p},\frac{q-2+s}{q}\right) - \frac{(2-s)(r+2-s)}{4r(r+s-2)(2n+1)^{s-\frac{2-s}{r}}}.$$

Since $B\left(\frac{p-2+s}{p},\frac{q-2+s}{q}\right) = \pi/\sin(\pi/p)$ for $s=1$, the above inequality (4.2) yields a refinement of the Hilbert double series theorem (1.4) (see Chapter 1).

On the other hand, Jichang and Debnath obtained in [41] the following refinement dealing with a symmetric homogeneous kernel of class C^4.

Theorem 4.2 *Suppose $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and $\frac{1}{2} \leq \mu < \frac{1}{2}\min\{p,q\}$. Further, let $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be a non-negative homogeneous symmetric function of degree $-s$, $s>0$, and of class C^4. If $(-1)^n\frac{\partial^n K}{\partial y^n}(1,y) \geq 0$, $n=0,1,2,3,4$, and $\frac{\partial^m K}{\partial y^m}(1,y)y^{-\frac{2\mu}{r}} \to 0$ when $y\to\infty$, $m=0,1$, then the inequality*

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}K(m+\mu,n+\mu)a_m b_n \\
&< \left\{\sum_{m=0}^{\infty}\left[I(q,\mu)-\phi_q(m,s,\mu)\right](m+\mu)^{1-s}a_m^p\right\}^{\frac{1}{p}} \\
&\quad\times\left\{\sum_{n=0}^{\infty}\left[I(p,\mu)-\phi_p(n,s,\mu)\right](n+\mu)^{1-s}b_n^q\right\}^{\frac{1}{q}}
\end{aligned} \tag{4.3}$$

holds for all non-negative sequences $(a_m)_{m\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0} \neq 0$, where

$$\begin{aligned}
\phi_r(m,s,\mu) &= \left(\frac{\mu}{m+\mu}\right)^{1-\frac{2\mu}{r}}\left\{K\left(1,\frac{\mu}{m+\mu}\right)\left[\frac{1}{1-\frac{2\mu}{r}}-\frac{1}{2\mu}\left(1+\frac{1}{3r}\right)\right]\right. \\
&\quad\left.-\frac{1}{24\mu(m+\mu)}\frac{\partial K}{\partial y}\left(1,\frac{\mu}{m+\mu}\right)\right\} > 0
\end{aligned}$$

and $I(r,\mu)=\int_0^{\infty}K(1,u)u^{-\frac{2\mu}{r}}du < \infty$.

Motivated by Theorems 4.1 and 4.2, in this chapter we develop some general methods for improving discrete Hilbert-type inequalities via the Euler-Maclaurin summation formula. More precisely, the method used in the proof of Theorem 4.1 can be utilized in obtaining refinements of Hilbert-type inequalities with a general homogeneous kernel of class C^2, fulfilling some additional properties. Similarly, Theorem 4.2 can also be extended to include homogeneous kernels of class C^4 which are generally not symmetric. Such extensions will be given in both equivalent forms, as in the previous chapters, and in the setting of non-conjugate parameters.

Finally, the last section of this chapter is dedicated to some particular refinements of Hilbert-type inequalities with the kernel $K(x,y) = (x+y)^{-s}$, $s > 0$. Such extensions will also be established by virtue of the Euler-Maclaurin summation formula.

4.1 Inequalities for kernels of class C^2

4.1.1 Auxiliary results

In order to obtain refinements of discrete Hilbert-type inequalities with a homogeneous kernel of class C^2, we first provide some auxiliary results, derived by virtue of the Euler-Maclaurin summation formula.

Lemma 4.1 *Let $f : [M,N] \to \mathbb{R}$, $M,N \in \mathbb{N}$, be a continuously differentiable function. Then the following equality holds*

$$\sum_{k=M}^{N} f(k) = \int_M^N f(x)dx + \frac{1}{2}[f(M)+f(N)] + \int_M^N \gamma_1^*(x)f'(x)dx, \tag{4.4}$$

where $\gamma_1^*(x) = x - \lfloor x \rfloor - \frac{1}{2}$.

Proof. Making use of the Euler-Maclaurin summation formula (4.1) with $a = K$ and $b = K+1$, where $k = M, M+1, \ldots, N-1$, we obtain

$$\int_K^{K+1} f(x)dx = \frac{f(K)+f(K+1)}{2} + \int_K^{K+1} \gamma_1^*(x)f'(x)dx,$$

so the result follows by summing the above equalities. □

Remark 4.1 In particular, if the function f from the previous lemma is defined on $[0,\infty)$, and $f(x) \to 0^+$ when $x \to \infty$, then relation (4.4) yields (see also [100]):

$$\sum_{k=0}^{\infty} f(k) = \int_0^{\infty} f(x)dx + \frac{1}{2}f(0) + \int_0^{\infty} \gamma_1^*(x)f'(x)dx.$$

Utilizing the fact that $\gamma_1^*(x) = x - \lfloor x \rfloor - \frac{1}{2}$ is a periodic function with the period equal to 1, we also obtain the following estimate:

Lemma 4.2 *Let $\varphi : [M,N] \to \mathbb{R}$, $M,N \in \mathbb{N}$, be strictly decreasing function. Then,*

$$-\int_M^N \gamma_1^*(x)\varphi(x)dx < \frac{1}{8}\left[\varphi(M) - \varphi(N)\right], \tag{4.5}$$

where $\gamma_1^*(x) = x - \lfloor x \rfloor - \frac{1}{2}$.

Proof. Since $\int_K^{K+1} \gamma_1^*(x)dx = 0$, $K \in \mathbb{N}$, we have

$$\begin{aligned}
-\int_M^N \gamma_1^*(x)\varphi(x)dx &= \sum_{k=M}^{N-1} \int_k^{k+1} -\gamma_1^*(x)\left(\varphi(x) - \varphi(k+\frac{1}{2})\right)dx \\
&= \sum_{k=M}^{N-1} \int_k^{k+\frac{1}{2}} -\gamma_1^*(x)\left(\varphi(k) - \varphi(k+\frac{1}{2})\right)dx \\
&\quad + \sum_{k=M}^{N-1} \int_{k+\frac{1}{2}}^{k+1} \gamma_1^*(x)\left(\varphi(k+\frac{1}{2}) - \varphi(k+1)\right)dx + \sum_{k=M}^{N-1} \alpha_k \\
&= \frac{1}{8}\left(\varphi(M) - \varphi(N)\right) + \sum_{k=M}^{N-1} \alpha_k,
\end{aligned}$$

where

$$\alpha_k = \int_k^{k+\frac{1}{2}} -\gamma_1^*(x)\left(\varphi(x) - \varphi(k)\right)dx + \int_{k+\frac{1}{2}}^{k+1} \gamma_1^*(x)\left(\varphi(k+1) - \varphi(x)\right)dx.$$

Finally, since φ is strictly decreasing, it follows that $\alpha_k < 0$ and (4.5) is proved. □

Remark 4.2 Assuming that φ is defined on $[0,\infty)$ and $\varphi(x) \to 0^+$ when $x \to \infty$, inequality (4.5) reads:

$$-\int_0^\infty \gamma_1^*(x)\varphi(x)dx < \frac{1}{8}\varphi(0).$$

Now, by virtue of the above lemmas, we derive the following estimate referring to a homogeneous kernel of class C^2.

Lemma 4.3 *Let $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be a non-negative homogeneous function of degree $-s$, $s > 0$, and of class C^2, such that*

$$\frac{\partial K}{\partial y}(x,y) < 0,\ \frac{\partial^2 K}{\partial y^2}(x,y) > 0\ \lim_{y\to\infty} K(x,y) = \lim_{y\to\infty} \frac{\partial K}{\partial y}(x,y) = 0,\ x \in \mathbb{R}_+. \tag{4.6}$$

Further, let f and $F_{q'}$ be defined by

$$f(y) = K(m+\mu, y+\mu)\frac{(m+\mu)^{q'A_1}}{(y+\mu)^{q'A_2}} \tag{4.7}$$

and

$$F_{q'}(m,s,\mu) = (m+\mu)^{1-s+q'(A_1-A_2)} \int_0^{\frac{\mu}{m+\mu}} K(1,y)y^{-q'A_2}dy, \tag{4.8}$$

where A_1, A_2, μ, and q' are real parameters such that $0 \le A_2 < \frac{1}{q'}$ and $\mu \ge \frac{1}{2}$. Then,

$$
\begin{aligned}
F_{q'}(m,s,\mu) &- \frac{1}{2}f(0) - \int_0^\infty \gamma_1^*(y)f'(y)dy \\
&> \frac{(m+\mu)^{q'A_1}}{\mu^{q'A_2}}\left[K(m+\mu,m+\mu)\left(\frac{\mu}{1-q'A_2} - \frac{q'A_2}{8\mu} - \frac{1}{2}\right)\right. \\
&\quad \left. - \frac{\partial K}{\partial y}(m+\mu,m+\mu)\left(\frac{\mu^2}{2} - \frac{1}{8}\right)\right],
\end{aligned}
\tag{4.9}
$$

where $\gamma_1^*(y) = y - \lfloor y \rfloor - \frac{1}{2}$.

Proof. Utilizing the integration by parts, we obtain the following identity:

$$
\begin{aligned}
&\int_0^{\frac{\mu}{m+\mu}} K(1,y)y^{-q'A_2}dy \\
&= K\left(1, \frac{\mu}{m+\mu}\right)\frac{\mu^{1-q'A_2}}{(m+\mu)^{1-q'A_2}(1-q'A_2)} - \frac{1}{1-q'A_2}\int_0^{\frac{\mu}{m+\mu}} \frac{\partial K}{\partial y}(1,y)y^{1-q'A_2}dy.
\end{aligned}
$$

In addition, since $\frac{\partial^2 K}{\partial y^2}(x,y) > 0$ for all $x \in \mathbb{R}_+$, it follows that the function $\frac{\partial K}{\partial y}(1,y)$ is strictly increasing, so that

$$
\begin{aligned}
&\int_0^{\frac{\mu}{m+\mu}} K(1,y)y^{-q'A_2}dy \\
&> K\left(1, \frac{\mu}{m+\mu}\right)\frac{\mu^{1-q'A_2}}{(m+\mu)^{1-q'A_2}(1-q'A_2)} - \frac{\partial K}{\partial y}\left(1, \frac{\mu}{m+\mu}\right)\int_0^{\frac{\mu}{m+\mu}} y^{1-q'A_2}dy \\
&\ge K\left(1, \frac{\mu}{m+\mu}\right)\frac{\mu^{1-q'A_2}}{(m+\mu)^{1-q'A_2}(1-q'A_2)} - \frac{1}{2}\frac{\partial K}{\partial y}\left(1, \frac{\mu}{m+\mu}\right)\frac{\mu^{2-q'A_2}}{(m+\mu)^{2-q'A_2}}.
\end{aligned}
$$

Now, taking into account the homogeneity of K, the above estimate yields:

$$
F_{q'}(m,s,\mu) > \frac{(m+\mu)^{q'A_1}}{\mu^{q'A_2-1}}\left[\frac{1}{1-q'A_2}K(m+\mu,\mu) - \frac{\mu}{2}\frac{\partial K}{\partial y}(m+\mu,\mu)\right]. \tag{4.10}
$$

On the other hand, it is obvious that the function $-f'$ fulfills conditions of Lemma 4.2. Moreover, since $-f'(x) \to 0^+$, when $x \to \infty$, relation (4.5) provides the estimate

$$
\int_0^\infty \gamma_1^*(y)f'(y)dy < \frac{(m+\mu)^{q'A_1}}{\mu^{q'A_2}}\left[\frac{q'A_2}{8\mu}K(m+\mu,\mu) - \frac{1}{8}\frac{\partial K}{\partial y}(m+\mu,\mu)\right]. \tag{4.11}
$$

Clearly, the above estimates (4.10) and (4.11) imply the inequality

$$
\begin{aligned}
&F_{q'}(m,s,\mu)-\frac{1}{2}f(0)-\int_0^\infty \gamma_1^*(y)f'(y)dy\\
&\quad>\frac{(m+\mu)^{q'A_1}}{\mu^{q'A_2}}\left[K(m+\mu,\mu)\left(\frac{\mu}{1-q'A_2}-\frac{q'A_2}{8\mu}-\frac{1}{2}\right)\right.\\
&\qquad\left.-\frac{\partial K}{\partial y}(m+\mu,\mu)\left(\frac{\mu^2}{2}-\frac{1}{8}\right)\right].
\end{aligned}
$$

Finally, since $\frac{\partial K}{\partial y}(x,y)<0$ and $\frac{\partial^2 K}{\partial y^2}(x,y)>0$, we have $K(m+\mu,\mu)>K(m+\mu,m+\mu)$ and $\frac{\partial K}{\partial y}(m+\mu,\mu)<\frac{\partial K}{\partial y}(m+\mu,m+\mu)$, so (4.9) holds. □

Remark 4.3 It should be noticed here that the right-hand side of inequality (4.9) is non-negative since $\frac{\mu}{1-q'A_2}-\frac{q'A_2}{8\mu}-\frac{1}{2}\geq 0$ and $\mu\geq\frac{1}{2}$.

Since we deal with homogeneous kernels which are in general not symmetric, we shall also use the following estimate complementary to (4.9).

Lemma 4.4 *Let $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ be a non-negative homogeneous function of degree $-s$, $s>0$, and of class C^2, such that*

$$
\frac{\partial K}{\partial x}(x,y)<0,\ \frac{\partial^2 K}{\partial x^2}(x,y)>0,\ \lim_{x\to\infty}K(x,y)=\lim_{x\to\infty}\frac{\partial K}{\partial x}(x,y)=0,\ y\in\mathbb{R}_+. \tag{4.12}
$$

Further, let f and $F_{p'}$ be defined by

$$
f(x)=K(x+\mu,n+\mu)\frac{(n+\mu)^{p'A_2}}{(x+\mu)^{p'A_1}} \tag{4.13}
$$

and

$$
F_{p'}(n,s,\mu)=(n+\mu)^{1-s+p'(A_2-A_1)}\int_0^{\frac{\mu}{n+\mu}}K(x,1)x^{-p'A_1}dx, \tag{4.14}
$$

where A_1, A_2, μ, and p' are real parameters such that $0\leq A_1<\frac{1}{p'}$ and $\mu\geq\frac{1}{2}$. Then,

$$
\begin{aligned}
&F_{p'}(n,s,\mu)-\frac{1}{2}f(0)-\int_0^\infty \gamma_1^*(x)f'(x)dx\\
&\quad>\frac{(n+\mu)^{p'A_2}}{\mu^{p'A_1}}\left[K(m+\mu,m+\mu)\left(\frac{\mu}{1-p'A_1}-\frac{p'A_1}{8\mu}-\frac{1}{2}\right)\right.\\
&\qquad\left.-\frac{\partial K}{\partial x}(m+\mu,m+\mu)\left(\frac{\mu^2}{2}-\frac{1}{8}\right)\right],
\end{aligned} \tag{4.15}
$$

where $\gamma_1^*(x)=x-\lfloor x\rfloor-\frac{1}{2}$.

Remark 4.4 Similarly to (4.9), the right-hand side of inequality (4.15) is also non-negative since $\frac{\mu}{1-p'A_1}-\frac{p'A_1}{8\mu}-\frac{1}{2}\geq 0$ and $\mu\geq\frac{1}{2}$.

4.1.2 Refined discrete Hilbert-type inequalities

Estimates from the previous subsection enable us to derive refinements of discrete Hilbert-type inequalities with homogeneous kernels. Our aim here is to extend Theorem 4.1 to include homogeneous kernels of class C^2. Similarly to Section 1.3 (Chapter 1) we use the notation $k(\alpha) = \int_0^\infty K(1,u)u^{-\alpha}du$, where $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a non-negative homogeneous function of degree $-s$, $s > 0$, provided that $k(\alpha) < \infty$ for $\min\{1-s,0\} < \alpha < \max\{1,2-s\}$. The main result is presented in the setting of non-conjugate exponents.

Theorem 4.3 *Let p, q, and λ be as in* (2.1) *and* (2.2)*, and let $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be a non-negative homogeneous function of degree $-s$, $s > 0$, and of class C^2, fulfilling conditions* (4.6) *and* (4.12)*. If $A_1 \in \left(0, \frac{1}{p'}\right)$, $A_2 \in \left(0, \frac{1}{q'}\right)$, and $\mu \geq \frac{1}{2}$, then the inequalities*

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} K^{\lambda}(m+\mu, n+\mu)a_m b_n \\
&< \left[\sum_{m=0}^{\infty} \Omega_1^{\frac{p}{q'}}(m,q')a_m^p\right]^{\frac{1}{p}} \left[\sum_{n=0}^{\infty} \Omega_2^{\frac{q}{p'}}(n,p')b_n^q\right]^{\frac{1}{q}}
\end{aligned} \tag{4.16}$$

and

$$\left[\sum_{n=0}^{\infty} \Omega_2^{-\frac{q'}{p'}}(n,p')\left(\sum_{m=0}^{\infty} K^{\lambda}(m+\mu,n+\mu)a_m\right)^{q'}\right]^{\frac{1}{q'}} < \left[\sum_{m=0}^{\infty} \Omega_1^{\frac{p}{q'}}(m,q')a_m^p\right]^{\frac{1}{p}} \tag{4.17}$$

hold for all non-negative sequences $(a_m)_{m\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0} \neq 0$, where $\Omega_i(h,r) = (h+\mu)^{1-s+r(A_i-A_{i+1})}k(\alpha_i) - \Gamma_i(h,r)$, $i = 1,2$, $\alpha_1 = q'A_2$, $\alpha_2 = 2-s-p'A_1$, and

$$\begin{aligned}
\Gamma_i(h,r) &= \frac{(h+\mu)^{rA_i}}{\mu^{rA_{i+1}}} \cdot \left[K(h+\mu,h+\mu)\left(\frac{\mu}{1-rA_{i+1}} - \frac{rA_{i+1}}{8\mu} - \frac{1}{2}\right)\right. \\
&\quad \left. - \frac{\partial K}{\partial x_{i+1}}(h+\mu,h+\mu)\left(\frac{\mu^2}{2}-\frac{1}{8}\right)\right], \quad x_1 = x, x_2 = y.
\end{aligned}$$

Moreover, these inequalities are equivalent.

Proof. It is enough to show inequality (4.16) since the equivalent form (4.17) follows from Theorem 2.1 (see also Theorem 1.10, Chapter 1). If we rewrite (2.9) (Theorem 2.1) with the weight functions $\varphi(x) = (x+\mu)^{A_1}$, $\psi(y) = (y+\mu)^{A_2}$, and the counting measure, we obtain the inequality

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} K^{\lambda}(m+\mu,n+\mu)a_m b_n \leq \left[\sum_{m=0}^{\infty} \Phi^{\frac{p}{q'}}(q')a_m^p\right]^{\frac{1}{p}} \left[\sum_{n=0}^{\infty} \Phi^{\frac{q}{p'}}(p')b_n^q\right]^{\frac{1}{q}},$$

where the weight functions are defined by $\Phi(q') = \sum_{n=0}^{\infty} K(m+\mu,n+\mu)\frac{(m+\mu)^{q'A_1}}{(n+\mu)^{q'A_2}}$ and $\Phi(p') = \sum_{m=0}^{\infty} K(m+\mu,n+\mu)\frac{(n+\mu)^{p'A_2}}{(m+\mu)^{p'A_1}}$.

On the other hand, the function f defined by (4.7) is continuously differentiable on $[0,\infty)$ and $f(y)\to 0^+$, when $y\to\infty$. Hence, in view of Lemma 4.1 we have

$$\Phi(q') = \int_0^\infty f(y)dy + \frac{1}{2}f(0) + \int_0^\infty \gamma_1^*(y)f'(y)dy.$$

Moreover, the homogeneity of the kernel K implies that

$$\int_0^\infty f(y)dy = (m+\mu)^{1-s+q'(A_1-A_2)}k(q'A_2) - F_{q'}(m,s,\mu),$$

where $F_{q'}(m,s,\mu)$ is defined by (4.8). Clearly, the above two relations yield

$$\Phi(q') = (m+\mu)^{1-s+q'(A_1-A_2)}k(q'A_2) - \left(F_{q'}(m,s,\mu) - \frac{1}{2}f(0) - \int_0^\infty \gamma_1^*(y)f'(y)dy\right),$$

that is, $\Phi(q') < \Omega_1(m,q')$, by virtue of inequality (4.9). Analogously, by virtue of (4.15) we obtain $\Phi(p') < \Omega_2(n,p')$, so (4.16) is proved. □

The kernel $K(x,y) = (x+y)^{-s}$, $s>0$, fulfills conditions of Theorem 4.3. Now, considering the above theorem with this kernel and the parameters $A_1 = A_2 = \frac{2-s}{p'q'}$, $\mu = \frac{1}{2}$, we obtain an extension of Theorem 4.1 to non-conjugate case.

Corollary 4.1 *Let p, q, and λ satisfy* (2.1) *and* (2.2)*, and let*

$$\max\left\{1-\min\left\{\frac{1}{p'-1},\frac{1}{q'-1}\right\},2-\min\{p',q'\}\right\} < s \le 2.$$

Then the inequalities

$$\sum_{m=0}^\infty\sum_{n=0}^\infty \frac{a_m b_n}{(m+n+1)^{\lambda s}} < \left[\sum_{m=0}^\infty \Omega_{p'}^{\frac{p}{q'}}(m)a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=0}^\infty \Omega_{q'}^{\frac{q}{p'}}(n)b_n^q\right]^{\frac{1}{q}} \tag{4.18}$$

and

$$\left[\sum_{n=0}^\infty \Omega_{q'}^{-\frac{q'}{p'}}(n)\left(\sum_{m=0}^\infty \frac{a_m}{(m+n+1)^{\lambda s}}\right)^{q'}\right]^{\frac{1}{q'}} < \left[\sum_{m=0}^\infty \Omega_{p'}^{\frac{p}{q'}}(m)a_m^p\right]^{\frac{1}{p}} \tag{4.19}$$

hold for all non-negative sequences $(a_m)_{m\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}\neq 0$, where

$$\Omega_r(h) = \left(h+\frac{1}{2}\right)^{1-s}B\left(\frac{r+s-2}{r},\frac{rs-r-s+2}{r}\right) - \frac{(2-s)(r+2-s)}{4r(r+s-2)(2h+1)^{s-\frac{2-s}{r}}}.$$

Moreover, these inequalities are equivalent.

In the sequel we derive a version of Theorem 4.3 dealing with finite sums. The corresponding results will include the weight functions

$$\begin{aligned}\Phi_i(h,r) &= (h+\mu)^{1-s+r(A_i-A_{i+1})}k(\alpha_i) - \frac{(h+\mu)^{rA_i}}{(M+\mu)^{rA_{i+1}}}\Lambda_i(h,r)\\ &\quad - \frac{(h+\mu)^{rA_i}}{(N+\mu)^{rA_{i+1}}}\Delta_i(h,r), \quad i=1,2,\end{aligned} \tag{4.20}$$

where

$$\Lambda_i(h,r) = K_i(h,M)\left[\frac{M+\mu}{1-rA_{i+1}} - \frac{rA_{i+1}}{8(M+\mu)} - \frac{1}{2}\right] - \frac{\partial K_i}{\partial x_{i+1}}(h,M)\left[\frac{(M+\mu)^2}{2} - \frac{1}{8}\right]$$

$$\Delta_i(h,r) = K_i(h,N)\left[\frac{N+\mu}{s+rA_{i+1}-1} + \frac{rA_{i+1}}{8(N+\mu)} - \frac{1}{2}\right] - \frac{(h+\mu)(N+\mu)}{s(s+1)}\frac{\partial K_i}{\partial x_i}(h,N)$$
$$-\frac{1}{8}\frac{\partial K_i}{\partial x_{i+1}}(h,N),$$

$\alpha_1 = q'A_2$, $\alpha_2 = 2-s-p'A_1$, $x_1 = x$, $x_2 = y$, $K_1(h,H) = K(h+\mu,H+\mu)$, and $K_2(h,H) = K(H+\mu,h+\mu)$.

Theorem 4.4 *Let p, q, and λ satisfy* (2.1) *and* (2.2)*, and let $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ be a non-negative homogeneous function of degree $-s$, $s>0$, and of class C^2, fulfilling conditions*

$$\frac{\partial K}{\partial x}(x,y) < 0,\ \frac{\partial^2 K}{\partial x^2}(x,y) > 0,\ \frac{\partial K}{\partial y}(x,y) < 0,\ \frac{\partial^2 K}{\partial y^2}(x,y) > 0.$$

If $M,N\in\mathbb{N}$, and $A_1\in\left(\max\{\frac{1-s}{p'},0\},\frac{1}{p'}\right)$, $A_2\in\left(\max\{\frac{1-s}{q'},0\},\frac{1}{q'}\right)$, $\mu\ge 0$, then the inequalities

$$\sum_{m=M}^{N}\sum_{n=M}^{N} K^{\lambda}(m+\mu,n+\mu)a_m b_n$$
$$< \left[\sum_{m=M}^{N}\Phi_1^{\frac{p}{q'}}(m,q')a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=M}^{N}\Phi_2^{\frac{q}{p'}}(n,p')b_n^q\right]^{\frac{1}{q}} \tag{4.21}$$

and

$$\left[\sum_{n=M}^{N}\Phi_2^{-\frac{q'}{p'}}(n,p')\left(\sum_{m=M}^{N}K^{\lambda}(m+\mu,n+\mu)a_m\right)^{q'}\right]^{\frac{1}{q'}} < \left[\sum_{m=M}^{N}\Phi_1^{\frac{p}{q'}}(m,q')a_m^p\right]^{\frac{1}{p}} \tag{4.22}$$

hold for all non-negative sequences $(a_m)_{m\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}\neq 0$, where the weight functions are defined by (4.20)*. Moreover, these inequalities are equivalent.*

Proof. The proof follows the same lines as the proof of Theorem 4.3, except that we consider the bounded interval $[M,N]$ instead of $[0,\infty)$. Applying Theorem 2.1 (Chapter 2), we obtain inequality

$$\sum_{m=M}^{N}\sum_{n=M}^{N} K^{\lambda}(m+\mu,n+\mu)a_m b_n \le \left(\sum_{m=M}^{N}\Phi^{\frac{p}{q'}}(q')a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=M}^{N}\Phi^{\frac{q}{p'}}(p')b_n^q\right)^{\frac{1}{q}},$$

with the weight functions $\Phi(q') = \sum_{n=M}^{N}K(m+\mu,n+\mu)\frac{(m+\mu)^{q'A_1}}{(n+\mu)^{q'A_2}}$ and $\Phi(p') = \sum_{m=M}^{N}K(m+\mu,n+\mu)\frac{(n+\mu)^{p'A_2}}{(m+\mu)^{p'A_1}}$.

On the other hand, applying Lemma 4.1 to the function f defined by (4.7), we have

$$\Phi(q') = \int_M^N f(y)dy + \frac{1}{2}(f(M)+f(N)) + \int_M^N \gamma_1^*(y) f'(y)dy, \tag{4.23}$$

while the homogeneity of the kernel K implies

$$\begin{aligned}\int_M^N f(y)dy &= (m+\mu)^{1-s+q'(A_1-A_2)}k(q'A_2) - (m+\mu)^{1-s+q'(A_1-A_2)} \\ &\quad \times \left[\int_0^{\frac{M+\mu}{m+\mu}} K(1,t)t^{-q'A_2}dt + \int_0^{\frac{m+\mu}{N+\mu}} K(t,1)t^{s+q'A_2-1}dt\right].\end{aligned}$$

Finally, using (4.23), Lemma 4.2 and the same estimates as in Lemma 4.3 we have $\Phi(q') < \Phi_1(m,q')$, and similarly, $\Phi(p') < \Phi_2(n,p')$, which completes the proof. □

Remark 4.5 It should be noticed here that Theorem 4.4 also covers Theorem 4.3. Namely, let $M=0$ and $N \to \infty$. Then, taking into account the conditions

$$\lim_{x\to 0} K(x,y) = \lim_{y\to 0} K(x,y) = \lim_{x\to 0} \frac{\partial K}{\partial x}(x,y) = \lim_{y\to 0} \frac{\partial K}{\partial y}(x,y) = 0$$

and $\mu \geq \frac{1}{2}$, Theorem 4.4 reduces to Theorem 4.3.

We conclude this section with the finite sum version of Corollary 4.1.

Corollary 4.2 *Let p, q, and λ satisfy* (2.1) *and* (2.2)*, and let*

$$\max\left\{1-\min\left\{\frac{1}{p'-1}, \frac{1}{q'-1}\right\}, 2-\min\{p',q'\}\right\} < s \leq 2.$$

Then,

$$\sum_{m=M}^N \sum_{n=M}^N \frac{a_m b_n}{(m+n+1)^{\lambda s}} < \left[\sum_{m=M}^N \Phi_{p'}^{\frac{p}{q'}}(m) a_m^p\right]^{\frac{1}{p}} \left[\sum_{n=M}^N \Phi_{q'}^{\frac{q}{p'}}(n) b_n^q\right]^{\frac{1}{q}} \tag{4.24}$$

and

$$\left[\sum_{n=M}^N \Phi_{q'}^{-\frac{q'}{p'}}(n) \left(\sum_{m=M}^N \frac{a_m}{(m+n+1)^{\lambda s}}\right)^{q'}\right]^{\frac{1}{q'}} < \left[\sum_{m=M}^N \Phi_{p'}^{\frac{p}{q'}}(m) a_m^p\right]^{\frac{1}{p}}, \tag{4.25}$$

where the weight functions are defined by

$$\begin{aligned}\Phi_r(h) &= \left(h+\frac{1}{2}\right)^{1-s} B\left(\frac{r+s-2}{r}, \frac{rs-r-s+2}{r}\right) \\ &\quad - \left(\frac{2h+1}{2M+1}\right)^{\frac{2-s}{r}} \phi_{M,r}(h) - \left(\frac{2h+1}{2N+1}\right)^{\frac{2-s}{r}} \phi_{N,r}(h), \quad r = p', q',\end{aligned}$$

$$
\begin{aligned}
\phi_{M,r}(h) &= \frac{1}{(M+h+1)^s}\left[\frac{r(2M+1)}{2(r+s-2)} - \frac{2-s}{4r(2M+1)} + \frac{sM(M+1)}{2(M+h+1)} - \frac{1}{2}\right], \\
\phi_{N,r}(h) &= \frac{1}{(N+h+1)^s}\left[\frac{r(2N+1)}{2(rs-r-s+2)} + \frac{2-s}{4r(2N+1)}\right. \\
&\quad \left. + \frac{2(2N+1)(2h+1)+s(s+1)}{8(s+1)(N+h+1)} - \frac{1}{2}\right].
\end{aligned}
$$

Moreover, these inequalities are equivalent.

4.2 Inequalities for kernels of class C^4

4.2.1 Auxiliary results

Motivated by Theorem 4.2, in this section we deal with Hilbert-type inequalities with a homogeneous kernel of class C^4. Similarly to the previous section, we first prove a few auxiliary results, derived by means of the Euler-Maclaurin summation formula.

Lemma 4.5 *Suppose $f : [M,N] \to \mathbb{R}$, $M,N \in \mathbb{N}$, is of class C^4 and let $f^{(4)}(x) \geq 0$, $x \in [M,N]$. Then the following inequality holds:*

$$\sum_{k=M}^{N} f(k) < \int_M^N f(x)dx + \frac{1}{2}\left[f(M)+f(N)\right] + \frac{1}{12}\left[f'(N)-f'(M)\right]. \tag{4.26}$$

Proof. Applying the Euler-Maclaurin summation formula to the function $f : [M,N] \to \mathbb{R}$, we have

$$
\begin{aligned}
\sum_{k=M}^{N} f(k) &= \int_M^N f(x)dx + \frac{1}{2}\left[f(M)+f(N)\right] + \frac{1}{12}\left[f'(N)-f'(M)\right] \\
&\quad + \frac{1}{24}\int_M^N f^{(4)}(t)(B_4 - B_4(t-\lfloor t \rfloor))dt,
\end{aligned}
$$

where B_4 and $B_4(t)$ respectively denote the corresponding Bernoulli number and the Bernoulli polynomial. Since $B_4 = -\frac{1}{30}$ and the sign of $B_4 - B_4(t-\lfloor t \rfloor)$ is the same as the sign of B_4, we obtain (4.26). □

Remark 4.6 In particular, if the function f from the previous lemma is defined on $[0,\infty)$, and $f(x), f'(x) \to 0$ when $x \to \infty$, then relation (4.26) yields (see also [41]):

$$\sum_{k=0}^{\infty} f(k) < \int_0^{\infty} f(t)dt + \frac{1}{2}f(0) - \frac{1}{12}f'(0).$$

The following two estimates refer to a homogeneous kernel $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ of degree $-s$, $s>0$, and of class C^4. Like in the previous section, the integral $k(\alpha)=\int_0^\infty K(1,u)u^{-\alpha}du$ is assumed to converge for $\min\{1-s,0\}<\alpha<\max\{1,2-s\}$.

Lemma 4.6 *Let $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ be a non-negative homogeneous function of degree $-s$, $s>0$, and of class C^4, fulfilling*

$$(-1)^n\frac{\partial^n K}{\partial y^n}(x,y)>0,\ n=1,2,3,4,\ \text{and}\ \lim_{y\to\infty}K(x,y)=\lim_{y\to\infty}\frac{\partial K}{\partial y}(x,y)=0, \tag{4.27}$$

for all $x,y\in\mathbb{R}_+$. Further, let $\Theta_{q'}$ be defined by

$$\Theta_{q'}(m,s,\mu)=(m+\mu)^{q'A_1}\sum_{n=0}^{\infty}K(m+\mu,n+\mu)(n+\mu)^{-q'A_2}, \tag{4.28}$$

where A_1, A_2, μ, and q' are real parameters such that $0\le A_2<\frac{1}{q'}$ and $\mu\ge\frac{1}{2}$. Then,

$$\Theta_{q'}(m,s,\mu)<(m+\mu)^{1-s+q'(A_1-A_2)}\left[k(q'A_2)-\theta_{q'}(m,s,\mu)\right], \tag{4.29}$$

where

$$\begin{aligned}\theta_{q'}(m,s,\mu) &= \left(\frac{\mu}{m+\mu}\right)^{1-q'A_2}\left\{K\left(1,\frac{\mu}{m+\mu}\right)\left[\frac{1}{1-q'A_2}-\frac{1}{2\mu}\left(1+\frac{q'A_2}{6\mu}\right)\right]\right.\\ &\left.\quad-\frac{1}{24\mu(m+\mu)}\frac{\partial K}{\partial y}\left(1,\frac{\mu}{m+\mu}\right)\right\}.\end{aligned}$$

Proof. Setting $f(y)=K(m+\mu,y+\mu)(m+\mu)^{q'A_1}(y+\mu)^{-q'A_2}$, we have $\Theta_{q'}(m,s,\mu)=\sum_{n=0}^\infty f(n)$, so Lemma 4.5 yields the inequality

$$\Theta_{q'}(m,s,\mu)<(m+\mu)^{1-s+q'(A_1-A_2)}\left[k(q'A_2)-\omega_{q'}(m,s,\mu)\right],$$

where

$$\begin{aligned}\omega_{q'}(m,s,\mu) &= \int_0^{\frac{\mu}{m+\mu}}K(1,t)t^{-q'A_2}dt-\left(\frac{1}{2\mu}+\frac{q'A_2}{12\mu^2}\right)\left(\frac{\mu}{m+\mu}\right)^{1-q'A_2}K\left(1,\frac{\mu}{m+\mu}\right)\\ &\quad+\frac{1}{12\mu^2}\left(\frac{\mu}{m+\mu}\right)^{2-q'A_2}\frac{\partial K}{\partial y}\left(1,\frac{\mu}{m+\mu}\right).\end{aligned}$$

Moreover, applying the integration by parts twice, we have

$$\begin{aligned}\int_0^{\frac{\mu}{m+\mu}}K(1,t)t^{-q'A_2}dt &= \frac{1}{1-q'A_2}\left(\frac{\mu}{m+\mu}\right)^{1-q'A_2}K\left(1,\frac{\mu}{m+\mu}\right)\\ &\quad-\frac{1}{(1-q'A_2)(2-q'A_2)}\left(\frac{\mu}{m+\mu}\right)^{2-q'A_2}\frac{\partial K}{\partial y}\left(1,\frac{\mu}{m+\mu}\right)\\ &\quad+\frac{1}{(1-q'A_2)(2-q'A_2)}\int_0^{\frac{\mu}{m+\mu}}\frac{\partial^2 K}{\partial y^2}(1,t)t^{2-q'A_2}dt.\end{aligned}$$

Finally, since the last term in the above relation is non-negative and since the inequality

$$\frac{1}{(1-q'A_2)(2-q'A_2)}-\frac{1}{12\mu^2}>\frac{1}{24\mu^2}$$

holds for $0<q'A_2<1$ and $\mu\geq\frac{1}{2}$, it follows that $\omega_{q'}(m,s,\mu)>\theta_{q'}(m,s,\mu)$, which completes the proof. $\square$

Since we are concerned with homogeneous kernels which are in general not symmetric, we shall also utilize the result which is, in some way, complementary to Lemma 4.6.

Lemma 4.7 *Let $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ be a non-negative homogeneous function of degree $-s$, $s>0$, and of class C^4, fulfilling*

$$(-1)^n\frac{\partial^n K}{\partial x^n}(x,y)>0,\ n=1,2,3,4,\ \textit{and}\ \lim_{x\to\infty}K(x,y)=\lim_{x\to\infty}\frac{\partial K}{\partial x}(x,y)=0, \tag{4.30}$$

for all $x,y\in\mathbb{R}_+$. Further, let $\Theta_{p'}$ be defined by

$$\Theta_{p'}(n,s,\mu)=(n+\mu)^{p'A_2}\sum_{m=0}^{\infty}K(m+\mu,n+\mu)(m+\mu)^{-p'A_1}, \tag{4.31}$$

where A_1, A_2, μ, and p' are real parameters such that $0\leq A_1<\frac{1}{p'}$ and $\mu\geq\frac{1}{2}$. Then,

$$\Theta_{p'}(n,s,\mu)<(n+\mu)^{1-s+p'(A_2-A_1)}\left[k(2-s-p'A_1)-\theta_{p'}(n,s,\mu)\right], \tag{4.32}$$

where

$$\begin{aligned}\theta_{p'}(n,s,\mu)\ =\ &\left(\frac{\mu}{n+\mu}\right)^{1-p'A_1}\left\{K\left(\frac{\mu}{n+\mu},1\right)\left[\frac{1}{1-p'A_1}-\frac{1}{2\mu}\left(1+\frac{p'A_1}{6\mu}\right)\right]\right.\\ &\left.-\frac{1}{24\mu(n+\mu)}\frac{\partial K}{\partial x}\left(\frac{\mu}{n+\mu},1\right)\right\}.\end{aligned}$$

Remark 4.7 It should be noticed here that the terms $\theta_{q'}(m,s,\mu)$ and $\theta_{p'}(n,s,\mu)$ in (4.29) and (4.32) are non-negative.

4.2.2 Refined discrete Hilbert-type inequalities

In the sequel we extend Theorem 4.2 to hold for a homogeneous kernel of class C^4, in the non-conjugate case. Such extension is a simple consequence of the general Hilbert-type inequality in non-conjugate setting (Theorem 2.1, Chapter 2) and Lemmas 4.6 and 4.7.

Theorem 4.5 *Let p, q, and λ satisfy* (2.1) *and* (2.2)*, and let $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ be a non-negative homogeneous function of degree $-s$, $s>0$, and of class C^4, fulfilling conditions*

(4.27) *and* (4.30). *If* $A_1 \in \left(0, \frac{1}{p'}\right)$, $A_2 \in \left(0, \frac{1}{q'}\right)$, *and* $\mu \geq \frac{1}{2}$, *then the inequalities*

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} K^{\lambda}(m+\mu,n+\mu)a_m b_n \\
&< \left\{\sum_{m=0}^{\infty}(m+\mu)^{\frac{p}{q'}(1-s)+p(A_1-A_2)}\left[k(q'A_2)-\theta_{q'}(m,s,\mu)\right]^{\frac{p}{q'}} a_m^p\right\}^{\frac{1}{p}} \\
&\quad \times \left\{\sum_{n=0}^{\infty}(n+\mu)^{\frac{q}{p'}(1-s)+q(A_2-A_1)}\left[k(2-s-p'A_1)-\theta_{p'}(n,s,\mu)\right]^{\frac{q}{p'}} b_n^q\right\}^{\frac{1}{q}} \qquad (4.33)
\end{aligned}$$

and

$$\begin{aligned}
&\left\{\sum_{n=0}^{\infty}(n+\mu)^{q'(A_1-A_2)+\frac{q'}{p'}(s-1)}\left[k(2-s-p'A_1)-\theta_{p'}(n,s,\mu)\right]^{-\frac{q'}{p'}}\right. \\
&\quad \left. \times \left[\sum_{m=0}^{\infty} K^{\lambda}(m+\mu,n+\mu)a_m\right]^{q'}\right\}^{\frac{1}{q'}} \\
&< \left\{\sum_{m=0}^{\infty}(m+\mu)^{\frac{p}{q'}(1-s)+p(A_1-A_2)}\left[k(q'A_2)-\theta_{q'}(m,s,\mu)\right]^{\frac{p}{q'}} a_m^p\right\}^{\frac{1}{p}} \qquad (4.34)
\end{aligned}$$

hold for all non-negative sequences $(a_m)_{m\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0} \neq 0$, *where the functions* $\theta_{q'}(m,s,\mu)$ *and* $\theta_{p'}(n,s,\mu)$ *are defined in Lemmas* 4.6 *and* 4.7. *Moreover, these inequalities are equivalent.*

Proof. Applying Theorem 2.1 (Chapter 2) to discrete setting with a suitable power weight functions, we have

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} K^{\lambda}(m+\mu,n+\mu)a_m b_n < \left[\sum_{m=0}^{\infty}\Theta_{q'}^{\frac{p}{q'}}(m,s,\mu)a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=0}^{\infty}\Theta_{p'}^{\frac{q}{p'}}(n,s,\mu)b_n^q\right]^{\frac{1}{q}},$$

where $\Theta_{q'}(m,s,\mu)$ and $\Theta_{p'}(n,s,\mu)$ are defined by (4.28) and (4.31) respectively. Now, inequality (4.33) follows from estimates (4.29) and (4.32). $\square$

The previous theorem with the parameters $A_1 = A_2 = \frac{2\mu}{p'q'}$, implies the following:

Corollary 4.3 *Let p, q, and* λ *satisfy* (2.1) *and* (2.2), *and let* $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ *be a non-negative homogeneous function of degree* $-s$, $s > 0$, *and of class* C^4, *fulfilling conditions*

(4.27) *and* (4.30). *If* $\frac{1}{2} \leq \mu < \frac{1}{2}\min\{p', q'\}$, *then the inequalities*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} K^{\lambda}(m+\mu, n+\mu)a_m b_n$$
$$< \left\{\sum_{m=0}^{\infty}(m+\mu)^{\frac{p}{q'}(1-s)}\left[k\left(\frac{2\mu}{p'}\right) - \phi_{p'}(m,s,\mu)\right]^{\frac{p}{q'}} a_m^p\right\}^{\frac{1}{p}}$$
$$\times \left\{\sum_{n=0}^{\infty}(n+\mu)^{\frac{q}{p'}(1-s)}\left[k\left(2-s-\frac{2\mu}{q'}\right) - \phi_{q'}(m,s,\mu)\right]^{\frac{q}{p'}} b_n^q\right\}^{\frac{1}{q}} \qquad (4.35)$$

and

$$\left\{\sum_{n=0}^{\infty}(n+\mu)^{\frac{q'}{p'}(s-1)}\left[k\left(2-s-\frac{2\mu}{q'}\right) - \phi_{q'}(n,s,\mu)\right]^{-\frac{q'}{p'}}\right.$$
$$\left.\times \left[\sum_{m=0}^{\infty} K^{\lambda}(m+\mu, n+\mu)a_m\right]^{q'}\right\}^{\frac{1}{q'}}$$
$$< \left\{\sum_{m=0}^{\infty}(m+\mu)^{\frac{p}{q'}(1-s)}\left[k\left(\frac{2\mu}{p'}\right) - \theta_{p'}(m,s,\mu)\right]^{\frac{p}{q'}} a_m^p\right\}^{\frac{1}{p}} \qquad (4.36)$$

hold for all non-negative sequences $(a_m)_{m\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0} \neq 0$, *where the functions* $\theta_{q'}(m,s,\mu)$ *and* $\theta_{p'}(n,s,\mu)$ *are defined in Lemmas* 4.6 *and* 4.7. *Moreover, these inequalities are equivalent.*

Remark 4.8 Assuming that the kernel $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ from Corollary 4.3 is symmetric, we have that $k(2-s-\alpha) = k(\alpha)$. Then, in the conjugate case, that is, when $\lambda = 1$, inequality (4.35) reduces to (4.3) from Theorem 4.2.

The rest of this section is dedicated to some particular homogeneous kernels fulfilling conditions (4.27) and (4.30). We start with the well-known example $K(x,y) = (x+y)^{-s}$, $s > 0$. In this case, the corresponding Hilbert-type inequalities involve the constant factors expressed in terms of the Beta function. Hence, denoting $b(\alpha) = B(1-\alpha, s+\alpha-1)$, we have the following result:

Corollary 4.4 *Suppose* p, q, *and* λ *satisfy conditions* (2.1) *and* (2.2), *and let* $A_1 \in \left(\max\{\frac{1-s}{p'}, 0\}, \frac{1}{p'}\right)$, $A_2 \in \left(\max\{\frac{1-s}{q'}, 0\}, \frac{1}{q'}\right)$, *and* $\mu \geq \frac{1}{2}$. *Then the inequalities*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_m b_n}{(m+n+2\mu)^{\lambda s}}$$
$$< \left\{\sum_{m=0}^{\infty}(m+\mu)^{\frac{p}{q'}(1-s)+p(A_1-A_2)}\left[b(q'A_2) - \psi_{q'}(m,s,\mu)\right]^{\frac{p}{q'}} a_m^p\right\}^{\frac{1}{p}}$$
$$\times \left\{\sum_{n=0}^{\infty}(n+\mu)^{\frac{q}{p'}(1-s)+q(A_2-A_1)}\left[b(p'A_1) - \psi_{p'}(n,s,\mu)\right]^{\frac{q}{p'}} b_n^q\right\}^{\frac{1}{q}} \qquad (4.37)$$

and

$$\Bigg\{\sum_{n=0}^{\infty}(n+\mu)^{q'(A_1-A_2)+\frac{q'}{p'}(s-1)}\big[b(p'A_1)-\psi_{p'}(n,s,\mu)\big]^{-\frac{q'}{p'}} \times\left[\sum_{m=0}^{\infty}\frac{a_m}{(m+n+2\mu)^{\lambda s}}\right]^{q'}\Bigg\}^{\frac{1}{q'}} < \left\{\sum_{m=0}^{\infty}(m+\mu)^{\frac{p}{q'}(1-s)+p(A_1-A_2)}\big[b(q'A_2)-\psi_{q'}(m,s,\mu)\big]^{\frac{p}{q'}}a_m^p\right\}^{\frac{1}{p}} \tag{4.38}$$

hold for all non-negative sequences $(a_m)_{m\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}\neq 0$, *where*

$$\psi_{q'}(m,s,\mu)=\left(\frac{\mu}{m+\mu}\right)^{1-q'A_2}\frac{1}{2^s}\left[\frac{1}{1-q'A_2}-\frac{1}{2\mu}\left(1+\frac{q'A_2}{6\mu}\right)\right]$$

and

$$\psi_{p'}(n,s,\mu)=\left(\frac{\mu}{n+\mu}\right)^{1-p'A_1}\frac{1}{2^s}\left[\frac{1}{1-p'A_1}-\frac{1}{2\mu}\left(1+\frac{p'A_1}{6\mu}\right)\right].$$

In addition, these inequalities are equivalent.

Proof. It follows from Theorem 4.5, setting $K(x,y)=(x+y)^{-s}$ and observing that $\psi_{q'}(m,s,\mu)<\theta_{q'}(m,s,\mu)$ and $\psi_{p'}(n,s,\mu)<\theta_{p'}(n,s,\mu)$. □

Remark 4.9 It should be noticed here that Corollary 4.4 provides refinements of inequalities (1.58) and (1.59) (see Remark 1.14, Chapter 1). This can be seen by considering inequalities (4.37) and (4.38) in the conjugate case, with the parameters $\mu=\frac{1}{2}$ and $A_1=A_2=\frac{2-s}{pq}$, provided that $\max\left\{1-\min\left\{\frac{1}{p-1},\frac{1}{q-1}\right\},2-\min\{p,q\}\right\}<s\leq 2$. In such a way we also obtain refinement of the Hilbert double series theorem (1.4) (see Chapter 1).

Our next application of Theorem 4.5 refers to the homogeneous kernel $K(x,y)=(x^s+y^s)^{-1}$, $s>0$. Namely, this kernel fulfills conditions (4.27) and (4.30), hence, denoting $b_s(\alpha)=\frac{1}{s}B(\frac{1-\alpha}{s},\frac{s+\alpha-1}{s})$, we obtain the following consequence:

Corollary 4.5 *Let* p, q, *and* λ *satisfy* (2.1) *and* (2.2), *and let* $A_1\in\left(\max\{\frac{1-s}{p'},0\},\frac{1}{p'}\right)$, $A_2\in\left(\max\{\frac{1-s}{q'},0\},\frac{1}{q'}\right)$, *and* $\mu\geq\frac{1}{2}$. *Then the inequalities*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_mb_n}{[(m+\mu)^s+(n+\mu)^s]^{\lambda}} < \left\{\sum_{m=0}^{\infty}(m+\mu)^{\frac{p}{q'}(1-s)+p(A_1-A_2)}\big[b_s(q'A_2)-2^{s-1}\psi_{q'}(m,s,\mu)\big]^{\frac{p}{q'}}a_m^p\right\}^{\frac{1}{p}} \times\left\{\sum_{n=0}^{\infty}(n+\mu)^{\frac{q}{p'}(1-s)+q(A_2-A_1)}\big[b_s(p'A_1)-2^{s-1}\psi_{p'}(n,s,\mu)\big]^{\frac{q}{p'}}b_n^q\right\}^{\frac{1}{q}} \tag{4.39}$$

and

$$\left\{\sum_{n=0}^{\infty}(n+\mu)^{q'(A_1-A_2)+\frac{q'}{p'}(s-1)}\left[b_s(p'A_1)-2^{s-1}\psi_{p'}(n,s,\mu)\right]^{-\frac{q'}{p'}}\right.$$
$$\left.\times\left[\sum_{m=0}^{\infty}\frac{a_m}{[(m+\mu)^s+(n+\mu)^s]^\lambda}\right]^{q'}\right\}^{\frac{1}{q'}}$$
$$<\left\{\sum_{m=0}^{\infty}(m+\mu)^{\frac{p}{q'}(1-s)+p(A_1-A_2)}\left[b_s(q'A_2)-2^{s-1}\psi_{q'}(m,s,\mu)\right]^{\frac{p}{q'}}a_m^p\right\}^{\frac{1}{p}} \tag{4.40}$$

hold for all non-negative sequences $(a_m)_{m\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}\neq 0$, where the functions $\psi_{q'}(m,s,\mu)$ and $\psi_{p'}(n,s,\mu)$ are defined in the previous corollary. In addition, these inequalities are equivalent.

Remark 4.10 Observe that special cases of Corollaries 4.4 and 4.5, involving the conjugate setting and the parameters $A_1=A_2=\frac{2\mu}{pq}$, were studied in [41].

To conclude this section, we consider Corollary 4.5 with $A_1=\frac{1}{p'}\left(1-\frac{s}{r}\right)$ and $A_2=\frac{1}{q'}\left(1-\frac{s}{t}\right)$, where $\frac{1}{r}+\frac{1}{t}=1$, $r>1$, and $0<s\leq 1$. A weaker version of the following result (without weight functions $\psi_{q'}$ and $\psi_{p'}$) was obtained in [156].

Corollary 4.6 *Suppose p, q, and λ are as in (2.1) and (2.2). If $\frac{1}{r}+\frac{1}{t}=1$, $r>1$, and $0<s\leq 1$, then the inequalities*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_mb_n}{[(m+\mu)^s+(n+\mu)^s]^\lambda}$$
$$<\left\{\sum_{m=0}^{\infty}(m+\mu)^{p\left(\frac{1}{p'}-\frac{s\lambda}{r}\right)}\left[\frac{\pi}{s\sin\left(\frac{\pi}{r}\right)}-2^{s-1}\psi_{q'}(m,s,\mu)\right]^{\frac{p}{q'}}a_m^p\right\}^{\frac{1}{p}}$$
$$\times\left\{\sum_{n=0}^{\infty}(n+\mu)^{q\left(\frac{1}{q'}-\frac{s\lambda}{t}\right)}\left[\frac{\pi}{s\sin\left(\frac{\pi}{t}\right)}-2^{s-1}\psi_{p'}(n,s,\mu)\right]^{\frac{q}{p'}}b_n^q\right\}^{\frac{1}{q}} \tag{4.41}$$

and

$$\left\{\sum_{n=0}^{\infty}(n+\mu)^{\frac{sq'\lambda}{t}-1}\left[\frac{\pi}{s\sin\left(\frac{\pi}{t}\right)}-2^{s-1}\psi_{p'}(n,s,\mu)\right]^{-\frac{q'}{p'}}\right.$$
$$\left.\times\left[\sum_{m=0}^{\infty}\frac{a_m}{[(m+\mu)^s+(n+\mu)^s]^\lambda}\right]^{q'}\right\}^{\frac{1}{q'}}$$
$$<\left\{\sum_{m=0}^{\infty}(m+\mu)^{p\left(\frac{1}{p'}-\frac{s\lambda}{r}\right)}\left[\frac{\pi}{s\sin\left(\frac{\pi}{r}\right)}-2^{s-1}\psi_{q'}(m,s,\mu)\right]^{\frac{p}{q'}}a_m^p\right\}^{\frac{1}{p}} \tag{4.42}$$

hold for all non-negative sequences $(a_m)_{m\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0} \neq 0$, *where* $\psi_{q'}(m,s,\mu)$ *and* $\psi_{p'}(n,s,\mu)$ *are defined in Corollary* 4.4. *In addition, these inequalities are equivalent.*

4.3 Some particular refinements

Discrete Hilbert-type inequalities are more complicated than the corresponding integral inequalities, since we use some additional estimates in order to obtain a suitable form of inequality. Of course, this causes some extra conditions on functions and parameters involved in the corresponding discrete inequality. For example, the general Theorem 1.14 (Chapter 1) dealing with homogeneous kernels, includes kernels which are strictly decreasing in both arguments, while the corresponding integral result, that is, Corollary 1.1, holds for an arbitrary non-negative homogeneous kernel.

The starting point in this section is the following consequence of the above mentioned Theorem 1.14, that is, a pair of equivalent inequalities

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^s} \leq L\left[\sum_{m=1}^{\infty} m^{1-s+p(A_1-A_2)}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty} n^{1-s+q(A_2-A_1)}b_n^q\right]^{\frac{1}{q}} \tag{4.43}$$

and

$$\sum_{n=1}^{\infty} n^{(s-1)(p-1)+p(A_1-A_2)}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m+n)^s}\right]^p \leq L^p\sum_{m=1}^{\infty} m^{1-s+p(A_1-A_2)}a_m^p, \tag{4.44}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, $s>0$, $A_1\in(\max\left\{0,\frac{1-s}{q}\right\},\frac{1}{q})$, $A_2\in(\max\left\{0,\frac{1-s}{p},\right\},\frac{1}{p})$, and the constant L is expressed in terms of the Beta function, i.e. $L=B^{\frac{1}{p}}(1-A_2p,s-1+A_2p)B^{\frac{1}{q}}(1-A_1q,s-1+A_1q)$.

Note that parameters A_1 and A_2 in (4.43) and (4.44) are non-negative, while the corresponding result in the integral case (Corollary 1.1, Chapter 1) refers to parameters which can be negative, that is, $A_1\in(\frac{1-s}{q},\frac{1}{q})$, $A_2\in(\frac{1-s}{p},\frac{1}{p})$. In other words, the intervals for parameters A_1 and A_2 in the discrete case are smaller than the corresponding intervals in the integral case.

On the other hand, the above intervals $(\max\left\{0,\frac{1-s}{q}\right\},\frac{1}{q})$ and $(\max\left\{0,\frac{1-s}{p},\right\},\frac{1}{p})$ for parameters A_1 and A_2 can also be extended so that inequalities (4.43) and (4.44) still hold. This can be done by virtue of the Euler-Maclaurin summation formula, utilizing similar methods as in the previous two sections. Hence, in order to establish extensions of inequalities (4.43) and (4.44), we first mention some preliminary estimates.

Let $f:[1,\infty)\to\mathbb{R}$ be a non-negative continuously differentiable function such that $\sum_{k=1}^{\infty} f(k)<\infty$ and $\int_1^{\infty} f(t)dt<\infty$. It follows from Lemma 4.1 that

$$\sum_{k=1}^{\infty} f(k)=\int_1^{\infty} f(t)dt+\frac{1}{2}f(1)+\int_1^{\infty}\gamma_1^*(t)f'(t)dt, \tag{4.45}$$

where $\gamma_1^*(t)=t-\lfloor t\rfloor-\frac{1}{2}$.

Moreover, if $f:[1,\infty)\to\mathbb{R}$ is of class C^4 such that $f^{(r)}(\infty)=0$, $r=0,1,2,3,4$, $f^{(2r)}(x)>0$, and $f^{(2r-1)}(x)<0$, $r=1,2$, then the following sequence of inequalities hold:

$$-\frac{1}{12}f(1)<\int_1^{\infty}\gamma_1^*(t)f(t)dt<-\frac{1}{12}f(1)+\frac{1}{720}f''(1)<0. \tag{4.46}$$

The above sequence of inequalities was established in [147], by virtue of the Euler-Maclaurin summation formula. Relations (4.45) and (4.46) will be utilized in extending inequalities (4.43) and (4.44).

First, we define the functions $f_{s,\alpha,n}(t)=t^{-\alpha}(t+n)^{-s}$, where $-1\le\alpha<1$, $0<s\le 14$, $n\in\mathbb{N}$, and

$$Q_{s,\alpha}(n)=\int_0^1 f_{s,\alpha,n}(t)dt-\frac{1}{2}f_{s,\alpha,n}(1)-\int_1^{\infty} f'_{s,\alpha,n}(t)\gamma_1^*(t)dt,$$

where $\gamma_1^*(t)=t-\lfloor t\rfloor-\frac{1}{2}$. The following lemma asserts that under the above assumptions $Q_{s,\alpha}(n)$ is a non-negative function.

Lemma 4.8 *If* $-1\le\alpha<1$ *and* $0<s\le 14$*, then* $Q_{s,\alpha}(n)>0$.

Proof. Utilizing the integration by parts three times, we have

$$\begin{aligned}
\int_0^1 f_{s,\alpha,n}(t)dt &= \frac{1}{(1-\alpha)(n+1)^s}+\frac{s}{1-\alpha}\int_0^1\frac{t^{1-\alpha}}{(t+n)^{s+1}}dt\\
&= \frac{1}{(1-\alpha)(n+1)^s}+\frac{s}{(1-\alpha)(2-\alpha)(n+1)^{s+1}}\\
&\quad+\frac{s(s+1)}{(1-\alpha)(2-\alpha)}\int_0^1\frac{t^{2-\alpha}}{(t+n)^{s+2}}dt\\
&= \frac{1}{(1-\alpha)(n+1)^s}+\frac{s}{(1-\alpha)(2-\alpha)(n+1)^{s+1}}\\
&\quad+\frac{s(s+1)}{(1-\alpha)(2-\alpha)(3-\alpha)(n+1)^{s+2}}\\
&\quad+\frac{s(s+1)(s+2)}{(1-\alpha)(2-\alpha)(3-\alpha)}\int_0^1\frac{t^{3-\alpha}}{(t+n)^{s+3}}dt.
\end{aligned}$$

Further, since the function $f_{s,\alpha,n}:\mathbb{R}_+\to\mathbb{R}$ is non-negative, we obtain the inequality

$$\begin{aligned}
\int_0^1 f_{s,\alpha,n}(t)dt &> \frac{1}{(1-\alpha)}\left[\frac{1}{(n+1)^s}+\frac{s}{(2-\alpha)(n+1)^{s+1}}\right.\\
&\quad\left.+\frac{s(s+1)}{(2-\alpha)(3-\alpha)(n+1)^{s+2}}\right].
\end{aligned}\tag{4.47}$$

In addition,

$$-\frac{1}{2}f_{s,\alpha,n}(1) = -\frac{1}{2(n+1)^s}. \tag{4.48}$$

It remains to estimate the last term in the expression for $Q_{s,\alpha}(n)$. The first derivative of $f_{s,\alpha,n}$ is

$$f'_{s,\alpha,n}(t) = \frac{nst^{-\alpha-1}}{(t+n)^{s+1}} - \frac{(s+\alpha)t^{-\alpha-1}}{(t+n)^s},$$

that is,

$$-\int_1^\infty f'_{s,\alpha,n}(t)\gamma_1^*(t)dt = \int_1^\infty g_1(t)\gamma_1^*(t)dt - \int_1^\infty g_2(t)\gamma_1^*(t)dt,$$

where

$$g_1(t) = \frac{(s+\alpha)t^{-\alpha-1}}{(t+n)^s} \qquad \text{and} \qquad g_2(t) = \frac{nst^{-\alpha-1}}{(t+n)^{s+1}}.$$

Now, using (4.46), we obtain

$$\int_1^\infty g_1(t)\gamma_1^*(t)dt > -\frac{1}{12}g_1(1) = -\frac{s+\alpha}{12(n+1)^s}$$

and

$$\begin{aligned}
-\int_1^\infty & g_2(t)\gamma_1^*(t)dt \\
&> \frac{1}{12}g_2(1) - \frac{1}{720}g_2''(1) \\
&= \frac{ns}{12(n+1)^{s+1}} - \frac{ns}{720}\left[\frac{(s+1)(s+2)}{(n+1)^{s+3}} + \frac{2(s+1)(\alpha+1)}{(n+1)^{s+2}} + \frac{(\alpha+1)(\alpha+2)}{(n+1)^{s+1}}\right] \\
&> \frac{(n+1)s-s}{12(n+1)^{s+1}} - \frac{s}{720}\left[\frac{(s+1)(s+2)}{(n+1)^{s+2}} + \frac{2(s+1)(\alpha+1)}{(n+1)^{s+1}} + \frac{(\alpha+1)(\alpha+2)}{(n+1)^{s}}\right],
\end{aligned}$$

which implies

$$\begin{aligned}
-\int_1^\infty & f'_{s,\alpha,n}(t)\gamma_1^*(t)dt \\
&> -\frac{\alpha}{12(n+1)^s} - \frac{s}{12(n+1)^{s+1}} \\
&\quad - \frac{s}{720}\left[\frac{(s+1)(s+2)}{(n+1)^{s+2}} + \frac{2(s+1)(\alpha+1)}{(n+1)^{s+1}} + \frac{(\alpha+1)(\alpha+2)}{(n+1)^{s}}\right].
\end{aligned} \tag{4.49}$$

Finally, taking into account (4.47), (4.48), and (4.49), we obtain

$$Q_{s,\alpha}(n) > \frac{1}{(n+1)^s}Q_0(s,\alpha) + \frac{1}{(n+1)^{s+1}}Q_1(s,\alpha) + \frac{1}{(n+1)^{s+2}}Q_2(s,\alpha),$$

where

$$
\begin{aligned}
Q_0(s,\alpha) &= \frac{1}{1-\alpha} - \frac{1}{2} - \frac{\alpha}{12} - \frac{s(\alpha+1)(\alpha+2)}{720}, \\
Q_1(s,\alpha) &= \frac{s}{(1-\alpha)(2-\alpha)} - \frac{s}{12} - \frac{s(s+1)(\alpha+1)}{360}, \\
Q_2(s,\alpha) &= \frac{s(s+1)}{(1-\alpha)(2-\alpha)(3-\alpha)} - \frac{s(s+1)(s+2)}{720}.
\end{aligned}
$$

It is enough to show that the expressions $Q_0(s,\alpha)$, $Q_1(s,\alpha)$, and $Q_2(s,\alpha)$ are non-negative. Obviously, we have $Q_2(s,\alpha) > s(s+1)\left(\frac{1}{24} - \frac{1}{45}\right) > 0$ and $Q_1(s,\alpha) > s\left(\frac{1}{6} - \frac{1}{12} - \frac{1}{12}\right) = 0$. In addition, if $s \le 14$ then $\frac{s}{720} \le \frac{1}{24}$, so that

$$
Q_0(s,\alpha) \ge \frac{1}{1-\alpha} - \frac{1}{2} - \frac{\alpha}{12} - \frac{(\alpha+1)(\alpha+2)}{24} = \frac{\alpha^3 + 4\alpha^2 + 9\alpha + 10}{24(1-\alpha)}.
$$

It is easy to check that the function $f(\alpha) = \alpha^3 + 4\alpha^2 + 9\alpha + 10$ is strictly increasing. Since $f(-1) = 4$, it follows that $Q_0(s,\alpha) > 0$. The proof is now completed. □

In order to extend inequalities (4.43) and (4.44), it is necessary to establish a suitable upper bound on the weight function

$$
\omega_{s,\alpha_1,\alpha_2}(n) = \sum_{m=1}^{\infty} \frac{1}{(m+n)^s} \frac{n^{\alpha_1}}{m^{\alpha_2}}, \tag{4.50}
$$

where $0 < s \le 14$, and either $\alpha_2 \in (1-s, 1)$ for $s \le 2$, or $\alpha_2 \in [-1, 1)$ for $s > 2$. Of course, due to the form of the weight function, such a bound will involve the Beta function.

Lemma 4.9 *If $0 < s \le 14$, and either $\alpha_2 \in (1-s,1)$ for $s \le 2$, or $\alpha_2 \in [-1,1)$ for $s > 2$, then*

$$
\omega_{s,\alpha_1,\alpha_2}(n) < n^{1-s+\alpha_1-\alpha_2} B(1-\alpha_2, s+\alpha_2-1). \tag{4.51}
$$

Proof. Since $f_{s,\alpha_2,n}(t) = t^{-\alpha_2}(t+n)^{-s}$, we have $\omega_{s,\alpha_1,\alpha_2}(n) = n^{\alpha_1} \sum_{m=1}^{\infty} f_{s,\alpha_2,n}(m)$. Further, by virtue of Lemma 4.8, it follows that

$$
\omega_{s,\alpha_1,\alpha_2}(n) = n^{\alpha_1} \left[\int_0^{\infty} f_{s,\alpha_2,n}(t)dt - Q_{s,\alpha_2}(n) \right] < n^{\alpha_1} \int_0^{\infty} f_{s,\alpha_2,n}(t)dt.
$$

Finally, utilizing the change of variables $x = nt$, we have

$$
\int_0^{\infty} f_{s,\alpha_2,n}(t)dt = n^{1-s-\alpha_2} \int_0^{\infty} \frac{x^{-\alpha_2}}{(1+x)^s} dx,
$$

so that (4.51) holds due to the well-known formula

$$
\int_0^{\infty} \frac{x^{-\alpha_2}}{(1+x)^s} dx = B(1-\alpha_2, s+\alpha_2-1).
$$

□

The following result extends inequalities (4.43) and (4.44) in the sense that the parameters A_1 and A_2 can be chosen from a larger interval.

Theorem 4.6 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and $0<s\leq 14$. If either $A_1\in\left(\frac{1-s}{q},\frac{1}{q}\right)$, $A_2\in\left(\frac{1-s}{p},\frac{1}{p}\right)$ for $s\leq 2$, or $A_1\in\left[-\frac{1}{q},\frac{1}{q}\right)$, $A_2\in\left[-\frac{1}{p},\frac{1}{p}\right)$ for $s>2$, then inequalities (4.43) and (4.44) hold for all non-negative sequences $(a_m)_{m\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}\neq 0$. In addition, these inequalities are equivalent.*

Proof. Making use of Theorem 1.9 (Chapter 1), it is enough to show that (4.43) holds under conditions of this theorem.

Clearly, considering the discrete form of inequality (1.12) (Theorem 1.9) with the kernel $K(m,n)=(m+n)^{-s}$, and the weight functions $\varphi(m)=m^{A_1}$, $\varphi(n)=n^{A_2}$, it follows that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^s}\leq\left[\sum_{m=1}^{\infty}\omega_{s,pA_1,pA_2}(m)a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}\omega_{s,qA_2,qA_1}(n)b_n^q\right]^{\frac{1}{q}},$$

where $\omega_{s,\alpha_1,\alpha_2}$ is defined by (4.50). Now, the result follows from Lemma 4.9. □

Remark 4.11 Obviously, Theorem 4.6 can also be extended to the case of non-conjugate exponents. For more details, the reader is referred to [55].

We know from Section 1.4 (Chapter 1) that the constant L appearing in (4.43) and (4.44) is the best possible if the parameters $A_1\in\left(\max\left\{0,\frac{1-s}{q}\right\},\frac{1}{q}\right)$ and $A_2\in\left(\max\left\{0,\frac{1-s}{p},\right\},\frac{1}{p}\right)$ fulfill condition $pA_2+qA_1=2-s$. The following theorem asserts that L is also the best constant in the context of Theorem 4.6.

Theorem 4.7 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let A_1, A_2, s be real parameters fulfilling conditions of Theorem 4.6. If $pA_2+qA_1=2-s$, then L is the best constant in inequalities (4.43) and (4.44).*

Proof. Due to the equivalence, it is enough to show that L is the best constant in (4.43), under conditions of the theorem. Clearly, if $pA_2+qA_1=2-s$, this constant reduces to $L=B(1-pA_2,pA_2+s-1)$.

Applying (4.43) to the sequences $a_m=m^{-qA_1-\frac{\varepsilon}{p}}$ and $b_n=n^{-pA_2-\frac{\varepsilon}{q}}$, $\varepsilon>0$, its right-hand side becomes $L\sum_{n=1}^{\infty}n^{-1-\varepsilon}$.

On the other hand, for the above choice of sequences $(a_m)_{m\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^s}=\sum_{n=1}^{\infty}n^{-pA_2-\frac{\varepsilon}{q}}\left(\sum_{m=1}^{\infty}h(m)\right),\tag{4.52}$$

where $h(t) = \frac{t^{-qA_1-\frac{\varepsilon}{p}}}{(n+t)^s}$. Further, it follows that

$$\int_0^\infty h(t)dt = n^{1-s-qA_1-\frac{\varepsilon}{p}} B\left(1-qA_1-\frac{\varepsilon}{p}, qA_1+\frac{\varepsilon}{p}+s-1\right) \tag{4.53}$$

and

$$\begin{aligned}\int_0^1 h(t)dt &= n^{1-s-qA_1-\frac{\varepsilon}{p}} \int_0^{\frac{1}{n}} \frac{x^{-qA_1-\frac{\varepsilon}{p}}}{(1+x)^s}dx \\ &< n^{1-s-qA_1-\frac{\varepsilon}{p}} \int_0^{\frac{1}{n}} x^{-qA_1-\frac{\varepsilon}{p}}dx = n^{-s}.\end{aligned} \tag{4.54}$$

Moreover, taking into account that

$$h'(t) = \frac{ns}{(n+t)^{s+1}t^{qA_1+\frac{\varepsilon}{p}+1}} - \frac{s+qA_1+\frac{\varepsilon}{p}}{(n+t)^s t^{qA_1+\frac{\varepsilon}{p}+1}},$$

we have

$$\int_1^\infty h'(t)\gamma_1^*(t)dt > -\frac{ns}{12(n+1)^{s+1}} > -\frac{s}{12n^s}, \tag{4.55}$$

by virtue of (4.46). Now, utilizing the summation formula

$$\sum_{m=1}^\infty h(m) = \int_0^\infty h(t)dt - \int_0^1 h(t)dt + \frac{1}{2}h(1) + \int_1^\infty h'(t)\gamma_1^*(t)dt$$

and relations (4.53)–(4.55), we obtain

$$\sum_{m=1}^\infty h(m) > B(\varepsilon)n^{1-s-qA_1-\frac{\varepsilon}{p}} - \frac{1}{n^s} - \frac{s}{12n^s}, \tag{4.56}$$

where $B(\varepsilon) = B\left(1-qA_1-\frac{\varepsilon}{p}, qA_1+\frac{\varepsilon}{p}+s-1\right)$. Moreover, relations (4.52) and (4.56) yield the estimate

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{(m+n)^s} > B(\varepsilon)\sum_{n=1}^\infty \frac{1}{n^{1+\varepsilon}} - \left(1+\frac{s}{12}\right)\sum_{n=1}^\infty \frac{1}{n^{s+pA_2+\frac{\varepsilon}{q}}}. \tag{4.57}$$

Now, assuming that there exists a constant C, $0 < C < L$, such that (4.43) holds after replacing L by C, the above discussion implies that

$$(L-C)\sum_{n=1}^\infty \frac{1}{n^{1+\varepsilon}} < \left(1+\frac{s}{12}\right)\sum_{n=1}^\infty \frac{1}{n^{s+pA_2+\frac{\varepsilon}{q}}}.$$

Finally, letting $\varepsilon \to 0$, it follows that $L \le C$, which contradicts the assumption $0 < C < L$. $\square$

Our first application of Theorem 4.6 refers to parameters $A_1 = A_2 = \frac{2-s}{pq}$. These parameters satisfy $pA_2 + qA_1 = 2-s$, and hence, yield the inequalities involving the best constants.

Corollary 4.7 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $2-\min\{p,q\}<s\leq 2+\max\{p,q\}$. If $L_1=B\left(\frac{p+s-2}{p},\frac{q+s-2}{q}\right)$, then the inequalities*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^s}<L_1\left[\sum_{m=1}^{\infty}m^{1-s}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{1-s}b_n^q\right]^{\frac{1}{q}} \tag{4.58}$$

and

$$\sum_{n=1}^{\infty}n^{(s-1)(p-1)}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m+n)^s}\right]^p<L_1^p\sum_{m=1}^{\infty}m^{1-s}a_m^p \tag{4.59}$$

hold for all non-negative sequences $(a_m)_{m\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}\neq 0$. In addition, these inequalities are equivalent and L_1 is the best constant in both of them.

Remark 4.12 It should be noticed here that inequalities (4.58) and (4.59) were also obtained by Yang (see [148], [152], and [153]), but only for $2-\min\{p,q\}<s\leq 2$. On the other hand, considering the special case when $p=q=2$ and utilizing (4.46), Yang (see [147] and [152]) showed that inequalities (4.58) and (4.59) hold for $0<s\leq 4$, which obviously coincides with the above result.

Another application of Theorems 4.6 and 4.7 refers to parameters $A_1=\frac{2-s}{2q}$, $A_2=\frac{2-s}{2p}$, and is closely connected with the above mentioned papers [147] and [152].

Corollary 4.8 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $0<s\leq 4$. If $L_2=B\left(\frac{s}{2},\frac{s}{2}\right)$, then the inequalities*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^s}<L_2\left[\sum_{m=1}^{\infty}m^{-\frac{ps}{2}+p-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{-\frac{qs}{2}+q-1}b_n^q\right]^{\frac{1}{q}} \tag{4.60}$$

and

$$\sum_{n=1}^{\infty}n^{\frac{ps}{2}-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m+n)^s}\right]^p<L_2^p\sum_{m=1}^{\infty}m^{-\frac{ps}{2}+p-1}a_m^p \tag{4.61}$$

hold for all non-negative sequences $(a_m)_{m\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}\neq 0$. In addition, these inequalities are equivalent and L_2 is the best constant in both of them.

Remark 4.13 The main results presented in this section are derived in [55] by Krnić and Pečarić. In addition, general refinements of Hilbert-type inequalities dealing with homogeneous kernels of class C^2 and C^4 (Sections 4.1 and 4.2) are established in [56] and [57] by the same authors.

Chapter 5

Applying the Hermite-Hadamard inequality

This chapter provides a different approach for improving discrete Hilbert-type inequalities. The method we develop here, is based on the famous Hermite-Hadamard inequality regarding convex functions.

Recall that $f:[a,b]\to\mathbb{R}$ is a convex function if

$$f(tx+(1-t)y)\le tf(x)+(1-t)f(y), \tag{5.1}$$

for all $x,y\in[a,b]$ and $t\in[0,1]$. The Hermite-Hadamard inequality asserts that

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(t)dt\le\frac{f(a)+f(b)}{2}, \tag{5.2}$$

where $f:[a,b]\to\mathbb{R}$ is a convex function.

The main objective of this chapter is to establish a general method for refining discrete Hilbert-type inequalities via the above inequality (5.2).

By virtue of the Hermite-Hadamard inequality, in Section 5.1 we establish the basic theorem regarding discrete Hilbert-type inequalities with a general kernel and weight functions. Further, in Section 5.2 the general result is then applied to discrete inequalities with homogeneous kernels. Moreover, in Section 5.3 some particular homogeneous kernels are considered and compared with actual results, known from the literature. Finally, in Section 5.4 some cases of non-homogeneous kernels are also considered.

Observe that refined Hilbert-type inequalities that we derive in this chapter, are considered in the case of non-conjugate parameters.

5.1 Basic theorem and some remarks

When dealing with discrete Hilbert-type inequalities, some integral bounds are used for certain sums. In Sections 1.4 and 2.2 such sums were recognized as the lower Darboux sums for the corresponding integrals. As we have already seen, this fact required monotonic decrease of the function that defines the integral sum. For example, in the statement of Theorem 1.14 (Chapter 1) the kernel was strictly decreasing in each argument.

As distinguished from the above discussion, in this section we are going to adjust the Hermite-Hadamard inequality, in order to bound the integral sum with the integral. Of course, this requires some extra assumptions concerning convexity, but as a consequence, we shall obtain better results than in the previously discussed case.

Now, we state and prove the main result that will be the basis for the results in this chapter.

Theorem 5.1 *Let p, q, and λ satisfy* (2.1) *and* (2.2)*, and let $m,M,n,N \in \mathbb{N}$ be such that $m < M$ and $n < N$. Suppose that $K : [m-\frac{1}{2}, M+\frac{1}{2}] \times [n-\frac{1}{2}, N+\frac{1}{2}] \to \mathbb{R}$, $\varphi : [m-\frac{1}{2}, M+\frac{1}{2}] \to \mathbb{R}$, $\psi : [n-\frac{1}{2}, N+\frac{1}{2}] \to \mathbb{R}$ are non-negative measurable functions fulfilling the following conditions:*

(i) the functions $K(i,t)\psi^{-q'}(t)$ are convex on interval $[n-\frac{1}{2}, N+\frac{1}{2}]$ for every $i = m, m+1, \ldots, M$;

(ii) the functions $K(t,j)\varphi^{-p'}(t)$ are convex on interval $[m-\frac{1}{2}, M+\frac{1}{2}]$ for every $j = n, n+1, \ldots, N$.

Then the inequality

$$\sum_{i=m}^{M}\sum_{j=n}^{N} K^{\lambda}(i,j)a_i b_j \leq \left[\sum_{i=m}^{M} \varphi^p(i)\left[\int_{n-\frac{1}{2}}^{N+\frac{1}{2}} \frac{K(i,t)}{\psi^{q'}(t)}dt\right]^{\frac{p}{q'}} a_i^p\right]^{\frac{1}{p}} \left[\sum_{j=n}^{N} \psi^q(j)\left[\int_{m-\frac{1}{2}}^{M+\frac{1}{2}} \frac{K(t,j)}{\varphi^{p'}(t)}dt\right]^{\frac{q}{p'}} b_j^q\right]^{\frac{1}{q}} \tag{5.3}$$

holds for all sequences $(a_i)_{i\in\mathbb{N}}$ and $(b_j)_{j\in\mathbb{N}}$ of non-negative real numbers.

Proof. We utilize the fundamental Hilbert-type inequality in the non-conjugate case, that is, Theorem 2.1. More precisely, using inequality (2.9) in a suitable discrete setting, it follows that

$$\sum_{i=m}^{M}\sum_{j=n}^{N} K^{\lambda}(i,j)a_i b_j \leq \left[\sum_{i=m}^{M} \varphi^p(i)F_i^p a_i^p\right]^{\frac{1}{p}} \left[\sum_{j=n}^{N} \psi^q(j)G_j^q b_j^q\right]^{\frac{1}{q}}, \tag{5.4}$$

where

$$F_i = \left[\sum_{j=n}^{N} \frac{K(i,j)}{\psi^{q'}(j)} \right]^{\frac{1}{q'}}, \quad i = m, m+1, \ldots, M,$$

and

$$G_j = \left[\sum_{i=m}^{M} \frac{K(i,j)}{\varphi^{p'}(i)} \right]^{\frac{1}{p'}}, \quad j = n, n+1, \ldots, N.$$

The further step is to estimate the terms of the sequences F_i, $i = m, m+1, \ldots, M$, and G_j, $j = n, n+1, \ldots, N$, via the Hermite-Hadamard inequality.

Since the functions $K(i,t)\psi^{-q'}(t)$ are convex on interval $[n-\frac{1}{2}, N+\frac{1}{2}]$ for every $i = m, m+1, \ldots, M$, applying the Hermite-Hadamard inequality to intervals $[j-\frac{1}{2}, j+\frac{1}{2}]$ yields the following inequalities:

$$\frac{K(i,j)}{\psi^{q'}(j)} \le \int_{j-\frac{1}{2}}^{j+\frac{1}{2}} \frac{K(i,t)}{\psi^{q'}(t)} dt, \quad j = n, n+1, \ldots, N.$$

Now, summing these inequalities we have

$$\sum_{j=n}^{N} \frac{K(i,j)}{\psi^{q'}(j)} \le \int_{n-\frac{1}{2}}^{N+\frac{1}{2}} \frac{K(i,t)}{\psi^{q'}(t)} dt, \quad i = m, m+1, \ldots, M,$$

that is,

$$F_i \le \left[\int_{n-\frac{1}{2}}^{N+\frac{1}{2}} \frac{K(i,t)}{\psi^{q'}(t)} dt \right]^{\frac{1}{q'}}, \quad i = m, m+1, \ldots, M. \tag{5.5}$$

Finally, the same conclusion can be drawn by exploiting the convexity of functions $K(t,j)$ $\varphi^{-p'}(t)$, $j = n, n+1, \ldots, N$, on interval $[m-\frac{1}{2}, M+\frac{1}{2}]$. In that case we have the estimates

$$G_j \le \left[\int_{m-\frac{1}{2}}^{M+\frac{1}{2}} \frac{K(t,j)}{\varphi^{p'}(t)} dt \right]^{\frac{1}{p'}}, \quad j = n, n+1, \ldots, N, \tag{5.6}$$

and the proof is now completed. □

Remark 5.1 Assuming the convergence of the series and integrals, the proof of Theorem 5.1 covers inequality (5.3) also for $M = N = \infty$. In that case, inequality (5.3) reads

$$\begin{aligned} &\sum_{i=m}^{\infty} \sum_{j=n}^{\infty} K^{\lambda}(i,j) a_i b_j \\ &\le \left[\sum_{i=m}^{\infty} \varphi^p(i) \left[\int_{n-\frac{1}{2}}^{\infty} \frac{K(i,t)}{\psi^{q'}(t)} dt \right]^{\frac{p}{q'}} a_i^p \right]^{\frac{1}{p}} \left[\sum_{j=n}^{\infty} \psi^q(j) \left[\int_{m-\frac{1}{2}}^{\infty} \frac{K(t,j)}{\varphi^{p'}(t)} dt \right]^{\frac{q}{p'}} b_j^q \right]^{\frac{1}{q}}. \end{aligned} \tag{5.7}$$

Remark 5.2 Let us explain here why the method presented in Theorem 5.1 yields a better result than the method used in Sections 1.4 and 2.2, regarding discrete case. Namely, using the notation of Theorem 5.1, the method exploited in the above mentioned sections is based on the assumption that the functions $K(i,t)\psi^{-q'}(t)$ and $K(t,j)\varphi^{-p'}(t)$, $i=m,m+1,\dots,M$, $j=n,n+1,\dots,N$, are strictly decreasing on their domains. Hence, $\sum_{i=m}^{M} K(i,t)\psi^{-q'}(t)$ and $\sum_{j=n}^{N} K(t,j)\varphi^{-p'}(t)$ are the lower Darboux sums for the corresponding integrals, that is,

$$\sum_{j=n}^{N} \frac{K(i,j)}{\psi^{q'}(j)} \le \int_{n-1}^{N} \frac{K(i,t)}{\psi^{q'}(t)} dt, \quad \sum_{i=m}^{M} \frac{K(i,j)}{\varphi^{p'}(i)} \le \int_{m-1}^{M} \frac{K(t,j)}{\varphi^{p'}(t)} dt,$$

$i=m,m+1,\dots,M$, $j=n,n+1,\dots,N$, provided that all functions are defined on the corresponding intervals. Clearly, due to the described monotonicity, these estimates are less precise than estimates (5.5) and (5.6).

Remark 5.3 The inequality sign in (5.4) depends only on the parameters p', q', and λ, since the crucial step in proving this relation was in applying the Hölder inequality (see Theorem 2.1). Therefore, the reverse inequality in (5.4) holds under conditions of Remark 2.2. On the other hand, in order to obtain the reverse inequality of (5.3), the assumptions of Remark 2.2 should be consistent with the estimates for F_i, $i=m,m+1,\dots,M$, and G_j, $j=n,n+1,\dots,N$ (see the proof of Theorem 5.1). In other words, estimates (5.5) and (5.6) should also hold with the reverse inequality, and that is possible only if $p',q'<0$, which implies that $0<p,q<1$. Comparing this with Remark 2.2, it follows that the reverse inequality in (5.3) holds if and only if $0<p,q<1$.

Remark 5.4 The equivalent form assigned to (5.3), that is, the Hardy-Hilbert-type inequality, reads

$$\left\{\sum_{j=n}^{N} \psi^{-q'}(j)\left[\int_{m-\frac{1}{2}}^{M+\frac{1}{2}} \frac{K(t,j)}{\varphi^{p'}(t)} dt\right]^{-\frac{q'}{p'}} \left[\sum_{i=m}^{M} K^{\lambda}(i,j)a_i\right]^{q'}\right\}^{\frac{1}{q'}}$$
$$\le \left[\sum_{i=m}^{M} \varphi^{p}(i)\left[\int_{n-\frac{1}{2}}^{N+\frac{1}{2}} \frac{K(i,t)}{\psi^{q'}(t)} dt\right]^{\frac{p}{q'}} a_i^{p}\right]^{\frac{1}{p}}, \tag{5.8}$$

and is established by substituting the sequence

$$b_j = \psi^{-q'}(j)\left[\int_{m-\frac{1}{2}}^{M+\frac{1}{2}} \frac{K(t,j)}{\varphi^{p'}(t)} dt\right]^{-\frac{q'}{p'}} \left[\sum_{i=m}^{M} K^{\lambda}(i,j)a_i\right]^{\frac{q'}{q}}, \quad j=n,n+1,\dots,N,$$

in (5.3). Such inequalities will not be considered in this chapter.

5.2 A unified approach to inequalities with a homogeneous kernel

In this section we are concerned with Hilbert-type inequalities involving homogeneous kernels with negative degree of homogeneity, which are defined for all positive arguments. Moreover, the weight functions will be chosen to be the power functions.

Remark 5.5 When dealing with a homogeneous kernel and the power weight functions, conditions (i) and (ii) from Theorem 5.1, referring to convexity, can be rewritten in a more suitable form. Namely, suppose that the function $f(t) = K(1,t)t^{-a}$ is convex on interval $[\frac{2n-1}{2M}, \frac{2N+1}{2m}]$, where K is homogeneous function of degree $-s$, $s > 0$ and $n < N$, $m < M$ are positive integers. Then, introducing the functions $f_i(t) = K(i,t)t^{-a}$, $i = m, m+1, \ldots, M$, we have

$$\begin{aligned} f_i(\lambda t_1 + (1-\lambda)t_2) &= i^{-a-s} f\left(\lambda \frac{t_1}{i} + (1-\lambda)\frac{t_2}{i}\right) \\ &\leq i^{-a-s}\lambda f\left(\frac{t_1}{i}\right) + i^{-a-s}(1-\lambda) f\left(\frac{t_2}{i}\right) = \lambda f_i(t_1) + (1-\lambda) f_i(t_2), \end{aligned}$$

for $t_1, t_2 \in [n - \frac{1}{2}, N + \frac{1}{2}]$ and $0 \leq \lambda \leq 1$. This means that the functions f_i, $i = m, m+1, \ldots, M$, are also convex on interval $[n - \frac{1}{2}, N + \frac{1}{2}]$. By the same arguments, convexity of the function $g(t) = K(t,1)t^{-a}$ on the interval $[\frac{2m-1}{2N}, \frac{2M+1}{2n}]$ implies convexity of the functions $g_j(t) = K(t,j)t^{-a}$, $j = n, n+1, \ldots, N$, on interval $[m - \frac{1}{2}, M + \frac{1}{2}]$.

Therefore, if $\varphi(t) = t^{A_1}$ and $\psi(t) = t^{A_2}$, then, using the notation of Theorem 5.1, the conditions

(i') function $K(1,t)\psi^{-q'}(t)$ is convex on interval $[\frac{2n-1}{2M}, \frac{2N+1}{2m}]$;

(i") function $K(t,1)\varphi^{-p'}(t)$ is convex on interval $[\frac{2m-1}{2N}, \frac{2M+1}{2n}]$;

imply conditions (i) and (ii) from Theorem 5.1. In particular, if $m = n = 1$ and $M = N = \infty$, then convexity of the functions $K(1,t)t^{-A_2 q'}$ and $K(t,1)t^{-A_1 p'}$ on $\mathbb{R}_+$ implies convexity of functions $K(i,t)t^{-A_2 q'}$ and $K(t,j)t^{-A_1 p'}$ on $\mathbb{R}_+$ for all $i, j \in \mathbb{N}$. This fact will frequently be used in order to make checking of the convexity conditions easier.

Now, in order to present the main result concerning homogeneous kernels, we define the integral

$$k(\alpha; r_1, r_2) = \int_{r_1}^{r_2} K(1,t)t^{-\alpha} dt, \quad 0 \leq r_1 < r_2 \leq \infty, \tag{5.9}$$

where the arguments α, r_1 and r_2 are such that (5.9) converges. In addition, if $r_1 = 0$ and $r_2 = \infty$, then the integral $k(\alpha; 0, \infty)$ will be denoted by $k(\alpha)$, for short, as in the previous chapters.

Theorem 5.2 *Let p, q, and λ satisfy* (2.1) *and* (2.2), *let $m, M, n, N \in \mathbb{N}$ be such that $m < M$, $n < N$ and let $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be a non-negative homogeneous function of degree $-s$, $s > 0$. If A_1 and A_2 are real parameters such that the functions $K(1,t)t^{-A_2 q'}$ and $K(t,1)t^{-A_1 p'}$ are convex on intervals $[\frac{2n-1}{2M}, \frac{2N+1}{2m}]$ and $[\frac{2m-1}{2N}, \frac{2M+1}{2n}]$ respectively, then the inequality*

$$
\begin{aligned}
&\sum_{i=m}^{M}\sum_{j=n}^{N} K^{\lambda}(i,j)a_i b_j \\
&\leq \left[\sum_{i=m}^{M} i^{\frac{p}{q'}(1-s)+p(A_1-A_2)} k^{\frac{p}{q'}}\left(A_2 q'; \tfrac{2n-1}{2i}, \tfrac{2N+1}{2i}\right) a_i^p\right]^{\frac{1}{p}} \\
&\quad \times \left[\sum_{j=n}^{N} j^{\frac{q}{p'}(1-s)+q(A_2-A_1)} k^{\frac{q}{p'}}\left(2-A_1 p'-s; \tfrac{2j}{2M+1}, \tfrac{2j}{2m-1}\right) b_j^q\right]^{\frac{1}{q}}
\end{aligned}
\tag{5.10}
$$

holds for all sequences $(a_i)_{i\in\mathbb{N}}$ and $(b_j)_{j\in\mathbb{N}}$ of non-negative real numbers.

Proof. Exploiting Theorem 5.1 with the power weight functions $\varphi(i) = i^{A_1}$ and $\psi(j) = j^{A_2}$, and taking into account the homogeneity of kernel $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, we have

$$
\begin{aligned}
\int_{n-\frac{1}{2}}^{N+\frac{1}{2}} K(i,t)t^{-A_2 q'}dt &= i^{1-s-A_2 q'} \int_{\frac{2n-1}{2i}}^{\frac{2N+1}{2i}} K(1,t)t^{-A_2 q'}dt \\
&= i^{1-s-A_2 q'} k\left(A_2 q'; \tfrac{2n-1}{2i}, \tfrac{2N+1}{2i}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{m-\frac{1}{2}}^{M+\frac{1}{2}} K(t,j)t^{-A_1 p'}dt &= j^{1-s-A_1 p'} \int_{\frac{2j}{2M+1}}^{\frac{2j}{2m-1}} K(1,t)t^{s+A_1 p'-2}dt \\
&= j^{1-s-A_1 p'} k\left(2-A_1 p'-s; \tfrac{2j}{2M+1}, \tfrac{2j}{2m-1}\right).
\end{aligned}
$$

Hence, the result follows from (5.3). □

An important consequence of Theorem 5.2 is the corresponding result for infinite series, that is, when $m = n = 1$ and $M = N = \infty$.

Corollary 5.1 *Let p, q, and λ satisfy* (2.1) *and* (2.2), *and let $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be a non-negative homogeneous function of degree $-s$, $s > 0$. If A_1 and A_2 are real parameters such that the functions $K(1,t)t^{-A_2 q'}$ and $K(t,1)t^{-A_1 p'}$ are convex on $\mathbb{R}_+$, then the inequality*

$$
\begin{aligned}
&\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} K^{\lambda}(i,j)a_i b_j \\
&\leq \left[\sum_{i=1}^{\infty} i^{\frac{p}{q'}(1-s)+p(A_1-A_2)} k^{\frac{p}{q'}}\left(A_2 q'; \tfrac{1}{2i}, \infty\right) a_i^p\right]^{\frac{1}{p}} \\
&\quad \times \left[\sum_{j=1}^{\infty} j^{\frac{q}{p'}(1-s)+q(A_2-A_1)} k^{\frac{q}{p'}}\left(2-A_1 p'-s; 0, 2j\right) b_j^q\right]^{\frac{1}{q}}
\end{aligned}
\tag{5.11}
$$

holds for all sequences $(a_i)_{i\in\mathbb{N}}$ and $(b_j)_{j\in\mathbb{N}}$ of non-negative real numbers.

Remark 5.6 According to the obvious estimates

$$k\big(A_2q';\tfrac{1}{2i},\infty\big) \le k\left(A_2q'\right) \quad \text{and} \quad k(2-A_1p'-s;0,2j) \le k\left(2-A_1p'-s\right),$$

which are valid for all $i,j\in\mathbb{N}$, it follows that the right-hand side of inequality (5.11) is not greater than the right-hand side of inequality (2.24) for $A=B=\alpha=\beta=1$ (see Section 2.2). In such a way we get the interpolating sequence of inequalities, that is, inequality (5.11) refines inequality (2.24). As we have already discussed, the convexity assumptions in Corollary 5.1 yield a better result than the monotonicity assumptions of the kernel in each of its arguments.

5.3 Examples with homogeneous kernels

In the sequel we consider Corollary 5.1 in some particular cases of homogeneous kernels. Our first example refers to the well-known homogeneous kernel $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$, defined by $K(x,y)=(x+y)^{-s}$, $s>0$. In such a way we shall obtain the weight functions expressed in terms of the incomplete Beta function. Recall that the incomplete Beta function is defined by

$$B_r(a,b)=\int_0^r t^{a-1}(1-t)^{b-1}\,dt, \quad a,b>0. \tag{5.12}$$

Clearly, if $r=1$ the incomplete Beta function reduces to the usual Beta function (1.5) and obviously, $B_r(a,b)\le B(a,b)$, $a,b>0$, $0\le r\le 1$. Hence, in the above setting, Corollary 5.1 reduces to the following form:

Corollary 5.2 *Let p, q, and λ satisfy* (2.1) *and* (2.2)*, and let $s>0$. If A_1 and A_2 are real parameters such that $A_1\in\big(\max\{\frac{1-s}{p'},0\},\frac{1}{p'}\big)$ and $A_2\in\big(\max\{\frac{1-s}{q'},0\},\frac{1}{q'}\big)$, then the inequality*

$$\begin{aligned}
&\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{a_ib_j}{(i+j)^{\lambda s}}\\
&\le\left[\sum_{i=1}^{\infty} i^{\frac{p}{q'}(1-s)+p(A_1-A_2)}B_{\frac{2i}{2i+1}}^{\frac{p}{q'}}(s+A_2q'-1,1-A_2q')a_i^p\right]^{\frac{1}{p}}\\
&\quad\times\left[\sum_{j=1}^{\infty} j^{\frac{q}{p'}(1-s)+q(A_2-A_1)}B_{\frac{2j}{2j+1}}^{\frac{q}{p'}}(s+A_1p'-1,1-A_1p')b_j^q\right]^{\frac{1}{q}}
\end{aligned}\tag{5.13}$$

holds for all sequences $(a_i)_{i\in\mathbb{N}}$ and $(b_j)_{j\in\mathbb{N}}$ of non-negative real numbers.

Proof. In order to apply Corollary 5.1, we have to check that the functions $K(1,t)t^{-A_2q'}$ and $K(t,1)t^{-A_1p'}$ are convex on $\mathbb{R}_+$, where $K(x,y) = (x+y)^{-s}$. Due to the symmetry of K, it suffices to show that the function $f(t) = (1+t)^{-s}t^{-a}$ is convex on $\mathbb{R}_+$ for $a > 0$. Its second derivative is equal to

$$f''(t) = \frac{(s+a)(s+a+1)t^2+2a(s+a+1)t+a(a+1)}{t^{a+2}(1+t)^{s+2}}, \tag{5.14}$$

that is, $f''(t) > 0$ for $t \in \mathbb{R}_+$, since $a > 0$ and $s > 0$.

Since the assumptions of Corollary 5.1 are fulfilled, we are ready to apply inequality (5.11) in the case of homogeneous kernel $K(x,y) = (x+y)^{-s}$.

From the definition of the incomplete Beta function and passing to the new variable $t = \frac{1}{u} - 1$, we have

$$\begin{aligned} k\big(A_2q'; \tfrac{1}{2i}, \infty\big) &= \int_{\frac{1}{2i}}^{\infty} \frac{t^{-A_2q'}}{(1+t)^s} dt = \int_0^{\frac{2i}{2i+1}} u^{s+A_2q'-2}(1-u)^{-A_2q'} du \\ &= B_{\frac{2i}{2i+1}}(s+A_2q'-1, 1-A_2q'). \end{aligned}$$

Similarly, the change of variable $t = \frac{u}{1-u}$ yields

$$\begin{aligned} k(2-A_1p'-s; 0, 2j) &= \int_0^{2j} \frac{t^{s+A_1p'-2}}{(1+t)^s} dt = \int_0^{\frac{2j}{2j+1}} u^{s+A_1p'-2}(1-u)^{-A_1p'} du \\ &= B_{\frac{2j}{2j+1}}(s+A_1p'-1, 1-A_1p'), \end{aligned}$$

that is, the result follows from inequality (5.11).

Note also that the intervals defining the parameters A_1 and A_2 are established due to the domain of the incomplete Beta function and the convexity of the functions $(1+t)^s t^{-A_1p'}$ and $(1+t)^s t^{-A_2q'}$. □

Remark 5.7 Since the incomplete Beta function is bounded from above by the usual Beta function with the same arguments, the estimates

$$B_{\frac{2i}{2i+1}}(s+A_2q'-1, 1-A_2q') \le B(s+A_2q'-1, 1-A_2q')$$

and

$$B_{\frac{2j}{2j+1}}(s+A_1p'-1, 1-A_1p') \le B(s+A_1p'-1, 1-A_1p'),$$

hold for all $i, j \in \mathbb{N}$. Moreover, since the right-hand sides of the above estimates do not depend on i and j, it follows that the right-hand side of (5.13) does not exceed

$$L \left[\sum_{i=1}^{\infty} i^{\frac{p}{q'}(1-s)+p(A_1-A_2)} a_i^p\right]^{\frac{1}{p}} \left[\sum_{j=1}^{\infty} j^{\frac{q}{p'}(1-s)+q(A_2-A_1)} b_j^q\right]^{\frac{1}{q}},$$

where $L = B^{\frac{1}{q'}}(s+A_2q'-1, 1-A_2q')B^{\frac{1}{p'}}(s+A_1p'-1, 1-A_1p')$. Of course, we again obtain the interpolating sequence of inequalities and inequality (5.13) refines some known results related to the Beta function (see e.g. paper [16]).

Remark 5.8 According to the previous remark, when taking the values of $A_1 = A_2 = \frac{2-s}{\lambda p'q'}$, where $\frac{2-s}{\lambda p'}, \frac{2-s}{\lambda q'} \in (\max\{1-s,0\},1)$, Corollary 5.2, together with Remark 5.7, provides the following inequalities

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{a_i b_j}{(i+j)^{\lambda s}} &\leq \left[\sum_{i=1}^{\infty} i^{\frac{(1-s)p}{q'}} B^{\frac{p}{q'}}_{\frac{2i}{2i+1}}\left(\tfrac{s+\lambda q'-2}{\lambda q'}, \tfrac{s+\lambda p'-2}{\lambda p'}\right) a_i^p\right]^{\frac{1}{p}} \\ &\quad\times \left[\sum_{j=1}^{\infty} j^{\frac{(1-s)q}{p'}} B^{\frac{q}{p'}}_{\frac{2j}{2j+1}}\left(\tfrac{s+\lambda p'-2}{\lambda p'}, \tfrac{s+\lambda q'-2}{\lambda q'}\right) b_j^q\right]^{\frac{1}{q}} \\ &\leq B^{\lambda}\left(\tfrac{s+\lambda p'-2}{\lambda p'}, \tfrac{s+\lambda q'-2}{\lambda q'}\right)\left[\sum_{i=1}^{\infty} i^{\frac{(1-s)p}{q'}} a_i^p\right]^{\frac{1}{p}}\left[\sum_{j=1}^{\infty} j^{\frac{(1-s)q}{p'}} b_j^q\right]^{\frac{1}{q}},\end{aligned}$$

where we used the fact that the usual Beta function is symmetric in its arguments.

Moreover, considering the kernel of degree -1 in the conjugate case, that is, when $s=1$, $\lambda=1$, $p=q'$, and $q=p'$, the above inequalities reduce to

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{a_i b_j}{i+j} &\leq \left[\sum_{i=1}^{\infty} B_{\frac{2i}{2i+1}}\left(\tfrac{1}{q}, \tfrac{1}{p}\right) a_i^p\right]^{\frac{1}{p}}\left[\sum_{j=1}^{\infty} B_{\frac{2j}{2j+1}}\left(\tfrac{1}{p}, \tfrac{1}{q}\right) b_j^q\right]^{\frac{1}{q}} \\ &\leq \frac{\pi}{\sin\frac{\pi}{p}}\left[\sum_{i=1}^{\infty} a_i^p\right]^{\frac{1}{p}}\left[\sum_{j=1}^{\infty} b_j^q\right]^{\frac{1}{q}},\end{aligned}$$

since $B\left(\frac{1}{p},\frac{1}{q}\right) = \pi/\sin(\pi/p)$. Note that this relation interpolates Hilbert double series theorem (1.1) (see Chapter 1). In addition, if $p=q=2$, then $B_{\frac{2i}{2i+1}}\left(\frac{1}{2},\frac{1}{2}\right) = \arctan\sqrt{2i}$, since $B_r\left(\frac{1}{2},\frac{1}{2}\right) = 2\arctan\sqrt{\frac{r}{1-r}}, 0 \leq r \leq 1$.

To end the previous discussion of some particular choices of parameters A_1 and A_2, we give yet another example in which we are able to find the explicit formulas for the incomplete Beta functions. It is a content of the following remark.

Remark 5.9 If $1 < s < 2$ then the parameters $A_1 = \frac{2-s}{p'}$ and $A_2 = \frac{2-s}{q'}$ are well-defined in the sense of Corollary 5.2. Considering inequality (5.13) in this particular case, we see that all terms with the incomplete Beta function have the form $B_r(1,s-1)$. Now, since

$$B_r(1,s-1) = \frac{1}{s-1}\left[1-(1-r)^{s-1}\right], \quad 0 \leq r \leq 1,$$

inequality (5.13), together with Remark 5.7, implies the following inequalities:

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{a_i b_j}{(i+j)^{\lambda s}} &\leq (s-1)^{-\lambda}\left[\sum_{i=1}^{\infty} i^{(3-s)(p-1)-\lambda p}\left[1-(2i+1)^{1-s}\right]^{\frac{p}{q'}} a_i^p\right]^{\frac{1}{p}} \\ &\quad\times\left[\sum_{j=1}^{\infty} i^{(3-s)(q-1)-\lambda q}\left[1-(2j+1)^{1-s}\right]^{\frac{q}{p'}} b_j^q\right]^{\frac{1}{q}} \\ &\leq (s-1)^{-\lambda}\left[\sum_{i=1}^{\infty} i^{(3-s)(p-1)-\lambda p} a_i^p\right]^{\frac{1}{p}}\left[\sum_{j=1}^{\infty} i^{(3-s)(q-1)-\lambda q} b_j^q\right]^{\frac{1}{q}}.\end{aligned}$$

We continue with some other examples of homogeneous kernels. The next result refers to the homogeneous kernel $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$, defined by $K(x,y)=\min^{-r}\{x,y\}\max^{r-s}\{x,y\}$, $s>0$, $r\in\left(\frac{s}{2},\frac{s}{2}+1\right)$, to which Corollary 5.1 also applies.

Corollary 5.3 *Suppose that p, q, and λ satisfy* (2.1) *and* (2.2)*, and let $s,r>0$ be real parameters such that $r\in\left(\frac{s}{2},\frac{s}{2}+1\right)$. If A_1 and A_2 are real parameters such that $A_1\in\left(\frac{r-s+1}{p'},\frac{1-r}{p'}\right)$ and $A_2\in\left(\frac{r-s+1}{q'},\frac{1-r}{q'}\right)$, then the inequality*

$$\begin{aligned}&\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{a_i b_j}{\min^{\lambda r}\{i,j\}\max^{\lambda(s-r)}\{i,j\}} \\ &\leq\left[\sum_{i=1}^{\infty} i^{\frac{p}{q'}(1-s)+p(A_1-A_2)}\left[k(A_2q')-\frac{(2i)^{A_2q'+r-1}}{1-A_2q'-r}\right]^{\frac{p}{q'}} a_i^p\right]^{\frac{1}{p}} \\ &\times\left[\sum_{j=1}^{\infty} j^{\frac{q}{p'}(1-s)+q(A_2-A_1)}\left[k(A_1p')-\frac{(2j)^{A_1p'+r-1}}{1-A_1p'-r}\right]^{\frac{q}{p'}} b_j^q\right]^{\frac{1}{q}}\end{aligned}\tag{5.15}$$

holds for all sequences $(a_i)_{i\in\mathbb{N}}$ and $(b_j)_{j\in\mathbb{N}}$ of non-negative real numbers, where

$$k(a)=\frac{s-2r}{(1-a-r)(a+s-r-1)}.$$

Proof. We have to check that the kernel $K(x,y)=\min^{-r}\{x,y\}\max^{r-s}\{x,y\}$ fulfills conditions of Corollary 5.1. Due to the symmetry, it suffices to show that the function $f(t)=K(1,t)t^{-a}$ is convex on $\mathbb{R}_+$ for $a\in(r-s+1,1-r)$. Clearly, f is defined by

$$f(t)=\begin{cases} t^{-a-r}, & 0<t\leq 1\\ t^{r-a-s}, & t>1.\end{cases}$$

Obviously, f is convex on intervals $(0,1]$ and $(1,\infty)$. Moreover, we have $f'_-(1)=-a-r<r-a-s=f'_+(1)$, which means that f is convex on $\mathbb{R}_+$.

Now, exploiting Corollary 5.1 we have

$$k\big(A_2q';\tfrac{1}{2i},\infty\big) = \int_{\frac{1}{2i}}^{1} t^{-A_2q'-r}dt + \int_1^\infty t^{r-A_2q'-s}dt = k(A_2q') - \frac{(2i)^{A_2q'+r-1}}{1-A_2q'-r}$$

and

$$\begin{aligned} k(2-A_1p'-s;0,2j) &= \int_0^1 t^{A_1p'+s-r-2}dt + \int_1^{2j} t^{A_1p'+r-2}dt \\ &= k(A_1p') - \frac{(2j)^{A_1p'+r-1}}{1-A_1p'-r}, \end{aligned}$$

that is, we get inequality (5.15) from (5.11). □

So far, we have considered homogeneous kernels with the negative degree of homogeneity. This restriction was adjusted to particular settings including the Beta and the incomplete Beta function. However, assuming the convergence, Theorem 5.2 and Corollary 5.1 are also meaningful for homogeneous kernels with the non-negative degree of homogeneity.

Our next example deals with a particular homogeneous kernel of zero-degree, that is, $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$, defined by $K(x,y)=\frac{x}{y}$. Let us emphasize some significant characteristics of this kernel. In contrast to $K(x,y)=(x+y)^{-s}$, the kernel $K(x,y)=\frac{x}{y}$ is not symmetric, and is not strictly decreasing in both of its arguments. On the other hand, it fulfills convexity conditions of Corollary 5.1, hence we have the following result:

Corollary 5.4 *Suppose that p, q, and λ satisfy conditions* (2.1) *and* (2.2). *If A_1 and A_2 are real parameters such that $A_1>\frac{2}{p'}$ and $A_2>0$, then the inequality*

$$\begin{aligned} &\sum_{i=1}^\infty\sum_{j=1}^\infty \left(\frac{i}{j}\right)^\lambda a_ib_j \\ &\le \frac{2^{A_1+A_2-\frac{2}{p'}}}{(A_1p'-2)^{\frac{1}{p'}}(A_2q')^{\frac{1}{q'}}}\left[\sum_{i=1}^\infty i^{\frac{p}{q'}+pA_1}a_i^p\right]^{\frac{1}{p}}\left[\sum_{j=1}^\infty j^{-\frac{q}{p'}+qA_2}b_j^q\right]^{\frac{1}{q}} \end{aligned} \tag{5.16}$$

holds for all sequences $(a_i)_{i\in\mathbb{N}}$ and $(b_j)_{j\in\mathbb{N}}$ of non-negative real numbers.

Proof. In order to exploit Corollary 5.1 for the homogeneous kernel $K(x,y)=\frac{x}{y}$, we easily check that the functions $K(1,t)t^{-A_2q'}=t^{-1-A_2q'}$ and $K(t,1)t^{-A_1p'}=t^{1-A_1p'}$ are convex on $\mathbb{R}_+$ for the parameters A_1 and A_2 as in the statement of this corollary. Therefore we have

$$k\big(A_2q';\tfrac{1}{2i},\infty\big) = \int_{\frac{1}{2i}}^{\infty} t^{-1-A_2q'}dt = \frac{(2i)^{A_2q'}}{A_2q'}$$

and similarly,

$$k(2-A_1p';0,2j) = \int_0^{2j} t^{A_1p'-3}dt = \frac{(2j)^{A_1p'-2}}{A_1p'-2},$$

so the result follows from (5.11). □

5.4 A non-conjugate example

The method developed in this chapter is often useful when dealing with some non-homogeneous kernels. Here we are going to consider the kernel $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$, defined by $K(x,y)=(1+xy)^{-s}$, $s>0$. Clearly, we start here with a general result, that is, with Theorem 5.1, since the kernel is non-homogeneous. As a consequence, we obtain the following:

Corollary 5.5 *Let p, q, and λ satisfy conditions* (2.1) *and* (2.2)*, and let $s>0$. If A_1 and A_2 are real parameters such that $A_1\in\left(\max\{\frac{1-s}{p'},0\},\frac{1}{p'}\right)$ and $A_2\in\left(\max\{\frac{1-s}{q'},0\},\frac{1}{q'}\right)$, then the inequality*

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}&\frac{a_ib_j}{(1+ij)^{\lambda s}}\\ &\le\left[\sum_{i=1}^{\infty} i^{-\frac{p}{q'}+p(A_1+A_2)}B^{\frac{p}{q'}}_{\frac{2}{i+2}}(s+A_2q'-1,1-A_2q')a_i^p\right]^{\frac{1}{p}}\\ &\quad\times\left[\sum_{j=1}^{\infty} j^{-\frac{q}{p'}+q(A_1+A_2)}B^{\frac{q}{p'}}_{\frac{2}{j+2}}(s+A_1p'-1,1-A_1p')b_j^q\right]^{\frac{1}{q}}\end{aligned}\tag{5.17}$$

holds for all sequences $(a_i)_{i\in\mathbb{N}}$ and $(b_j)_{j\in\mathbb{N}}$ of non-negative real numbers.

Proof. In order to be able to apply (5.7), we have to check convexity conditions (i) and (ii) from Theorem 5.1, for the kernel $K(x,y)=(1+xy)^{-s}$ and the weight functions $\varphi(x)=x^{A_1}$ and $\psi(y)=y^{A_2}$.

Due to the symmetry, it suffices to show that the functions $g_i(t)=(1+it)^{-s}t^{-a}$, $i\in\mathbb{N}$, are convex on $\mathbb{R}_+$ for $a>0$. These functions can be rewritten as $g_i(t)=i^af(it)$, where $f(t)=(1+t)^{-s}t^{-a}$. The second derivative is $g_i''(t)=i^{a+2}f''(it)$, $i\in\mathbb{N}$. In addition, the second derivative of f is given by (5.14), which proves the convexity of functions g_i, $i\in\mathbb{N}$, on $\mathbb{R}_+$.

Therefore, the assumptions of Theorem 5.1 are fulfilled, so we use (5.7) with $m=n=1$. Using the change of variable $u=\frac{1}{it+1}$, $i\in\mathbb{N}$, we conclude that

$$\begin{aligned}\int_{\frac{1}{2}}^{\infty}\frac{K(i,t)}{\psi^{q'}(t)}dt &= \int_{\frac{1}{2}}^{\infty}(1+it)^{-s}t^{-A_2q'}dt = i^{q'A_2-1}\int_0^{\frac{2}{i+2}}u^{s+A_2q'-2}(1-u)^{-A_2q'}du\\ &= i^{q'A_2-1}B_{\frac{2}{i+2}}(s+A_2q'-1,1-A_2q').\end{aligned}$$

Due to the symmetry, we also have

$$\int_{\frac{1}{2}}^{\infty}\frac{K(t,j)}{\varphi^{p'}(t)}dt=\int_{\frac{1}{2}}^{\infty}(1+jt)^{-s}t^{-A_1p'}dt=j^{p'A_1-1}B_{\frac{2}{j+2}}(s+A_1p'-1,1-A_1p'),$$

where $j\in\mathbb{N}$, so the result follows. □

Remark 5.10 Since the incomplete Beta function is bounded from above by the usual Beta function with the same parameters, the right-hand side of inequality (5.17) does not exceed

$$L\left[\sum_{i=1}^{\infty} i^{-\frac{p}{q'}+p(A_1+A_2)} a_i^p\right]^{\frac{1}{p}} \left[\sum_{j=1}^{\infty} j^{-\frac{q}{p'}+q(A_1+A_2)} b_j^q\right]^{\frac{1}{q}}, \tag{5.18}$$

where $L = B^{\frac{1}{q'}}(s+A_2q'-1, 1-A_2q')B^{\frac{1}{p'}}(s+A_1p'-1, 1-A_1p')$. This also yields interpolating sequence of inequalities that we have already discussed in the previous section. As an illustration, we consider a particular choice of parameters A_1 and A_2 as in Remark 5.8, that is, $A_1 = A_2 = \frac{2-s}{\lambda p'q'}$, where $\frac{2-s}{\lambda p'}, \frac{2-s}{\lambda q'} \in (\max\{1-s, 0\}, 1)$. In this case, the above expression (5.18) reads

$$B^{\lambda}\left(\frac{s+\lambda p'-2}{\lambda p'}, \frac{s+\lambda q'-2}{\lambda q'}\right) \left[\sum_{i=1}^{\infty} i^{\frac{p}{q'}\left[\frac{2(2-s)}{\lambda p'}-1\right]} a_i^p\right]^{\frac{1}{p}} \left[\sum_{j=1}^{\infty} j^{\frac{q}{p'}\left[\frac{2(2-s)}{\lambda q'}-1\right]} b_j^q\right]^{\frac{1}{q}}. \tag{5.19}$$

In the conjugate case, expression (5.19) represents the right-hand side of the appropriate inequality from [159]. Hence, relation (5.17) may be regarded as both a refinement and a generalization of the above mentioned result from [159].

Remark 5.11 The method of improving Hilbert-type inequalities via the Hermite-Hadamard inequality, presented in this chapter, has been recently developed in [66].

Chapter 6

Hilbert-type inequalities and the Laplace transform

In this short chapter we study refinements of some particular Hilbert-type inequalities involving the Laplace transform. Let us recall that if $f : \mathbb{R}_+ \to \mathbb{R}$ is a Lebesgue measurable function, then the Laplace transform $\mathscr{L} f$ of f is defined by $(\mathscr{L} f)(x) = \int_0^\infty \exp(-xt) f(t)dt$, for each x such that the above integral converges.

Considering the Hilbert-type inequality with the kernel $(x+y)^{-s}$, $s > 0$, Peachey [108], obtained the following interpolating sequence of inequalities in the case of conjugate parameters.

Theorem 6.1 *Let $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, and let $b > -\frac{1}{p}$, $c > -\frac{1}{q}$. If $f, g : \mathbb{R}_+ \to \mathbb{R}$, $f, g \neq 0$, are non-negative measurable functions with the respective Laplace transforms $\mathscr{L} f$ and $\mathscr{L} g$, then*

$$\begin{aligned}\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^{b+c+1}} dxdy &\leq \frac{1}{\Gamma(b+c+1)} \|t^b (\mathscr{L} f)(t)\|_{L^p(\mathbb{R}_+)} \|t^c (\mathscr{L} g)(t)\|_{L^q(\mathbb{R}_+)} \\ &< B\left(b + \frac{1}{p}, c + \frac{1}{q}\right) \|x^{1-b-\frac{2}{p}} f(x)\|_{L^p(\mathbb{R}_+)} \|y^{1-c-\frac{2}{q}} g(y)\|_{L^q(\mathbb{R}_+)}. \qquad (6.1)\end{aligned}$$

Note that the middle term in (6.1) that includes norms with the Laplace transforms of functions f and g, interpolates between the left-hand side and the right-hand side of the corresponding Hilbert-type inequality. Moreover, taking into account the form of the kernel, the above inequalities include the constant factors expressed in terms of the Beta and Gamma functions.

In the same paper Peachey also derived an analogue of the above interpolating sequence of inequalities, in the setting with non-conjugate parameters.

Theorem 6.2 *Suppose that* $p>1$, $q>1$, $\frac{1}{p}+\frac{1}{q}\geq 1$, $q\leq r\leq p'$, $b+\frac{1}{r'}>0$, *and* $c+\frac{1}{r}>0$. *If* $f,g:\mathbb{R}_+\to\mathbb{R}$, $f,g\neq 0$, *are non-negative measurable functions with the respective Laplace transforms* $\mathscr{L}f$ *and* $\mathscr{L}g$, *then*

$$\begin{aligned}\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^{b+c+1}}dxdy &\leq \frac{1}{\Gamma(b+c+1)}\|t^b(\mathscr{L}f)(t)\|_{L^{r'}(\mathbb{R}_+)}\,\|t^c(\mathscr{L}g)(t)\|_{L^r(\mathbb{R}_+)}\\ &< C\|x^{\frac{1}{p'}-\frac{1}{r'}-b}f(x)\|_{L^p(\mathbb{R}_+)}\,\|y^{\frac{1}{q'}-\frac{1}{r}-c}g(y)\|_{L^q(\mathbb{R}_+)},\end{aligned}\tag{6.2}$$

where $\beta=\frac{1}{p'}+\frac{1}{r'}$, $\gamma=\frac{1}{q'}+\frac{1}{r}$, *and*

$$C=\beta^{b+\frac{1}{r'}}\gamma^{c+\frac{1}{r}}\Gamma^{\beta}\left(\frac{b}{\beta}+\frac{1}{r'\beta}\right)\Gamma^{\gamma}\left(\frac{c}{\gamma}+\frac{1}{r\gamma}\right)\Gamma^{-1}(b+c+1).$$

Our aim in this chapter is to derive multidimensional versions of Theorems 6.1 and 6.2 with a more general parameters. In other words, we shall extend inequalities (6.1) and (6.2) regarding multidimensional conjugate and non-conjugate parameters. In such a way we shall obtain refinements of some particular Hilbert-type inequalities from Chapters 1 and 2.

Basic results will be established by virtue of the general Hardy-Hilbert-type inequality in both conjugate and non-conjugate setting.

6.1 The case of conjugate parameters

To prove the main result we first establish the following two lemmas.

Lemma 6.1 *Let* $a>-1$. *If* $f_i:\mathbb{R}_+\to\mathbb{R}$, $i=1,\dots,n$, *are non-negative measurable functions with the respective Laplace transforms* $\mathscr{L}f_i$, $i=1,\dots,n$, *then*

$$\int_{\mathbb{R}_+^n}\frac{\prod_{i=1}^n f_i(x_i)}{(\sum_{j=1}^n x_j)^{a+1}}dx_1\dots dx_n=\frac{1}{\Gamma(a+1)}\int_0^\infty t^a\prod_{i=1}^n(\mathscr{L}f_i)(t)dt.\tag{6.3}$$

Proof. The proof is obtained using a simple application of the Fubini theorem:

$$\begin{aligned}\int_0^\infty t^a\prod_{i=1}^n(\mathscr{L}f_i)(t)dt &= \int_0^\infty t^a\left(\prod_{i=1}^n\int_0^\infty \exp(-tx_i)f_i(x_i)dx_i\right)dt\\ &=\int_{\mathbb{R}_+^n}\prod_{i=1}^n f_i(x_i)\left(\int_0^\infty \exp(-t(x_1+\ldots+x_n))t^a dt\right)dx_1\ldots dx_n\\ &=\int_{\mathbb{R}_+^n}\frac{\Gamma(a+1)\prod_{i=1}^n f_i(x_i)}{(\sum_{j=1}^n x_j)^{a+1}}dx_1\ldots dx_n.\end{aligned}$$

□

The following lemma is a consequence of the general Hardy-Hilbert-type inequality (1.13) (see Chapter 1), observed in a suitable setting.

Lemma 6.2 *Suppose* $\frac{1}{p}+\frac{1}{q}=1$, $p>1$. *If* $f:\mathbb{R}_+\to\mathbb{R}$, $f\neq 0$, *is a non-negative measurable function with the Laplace transform* $\mathscr{L}f$, *then*

$$\begin{aligned}&\int_0^\infty y^{p-1-p(B+C)}\left(\mathscr{L}f\right)^p(y)dy\\&\quad<\Gamma(1-pC)\Gamma^{p-1}(1-qB)\int_0^\infty x^{p(B+C)-1}f^p(x)dx,\end{aligned}\tag{6.4}$$

where $B<\frac{1}{q}$ *and* $C<\frac{1}{p}$.

Proof. Inequality (6.4) is an immediate consequence of (1.13) (see Theorem 1.9, Chapter 1). Namely, using the notation from the mentioned theorem and setting $K(x,y)=\exp(-yx)$, $\varphi(x)=x^B$, and $\psi(y)=y^C$, we have

$$F(x)=\int_0^\infty \psi^{-p}(y)K(x,y)dy=\int_0^\infty y^{-pC}\exp(-xy)dy=x^{pC-1}\Gamma(1-pC)$$

and similarly, $G(y)=\int_0^\infty \varphi^{-q}(x)K(x,y)dx=y^{qB-1}\Gamma(1-qB)$, so that inequality (1.13) yields

$$\begin{aligned}&\Gamma^{1-p}(1-qB)\int_0^\infty y^{(qB-1)(1-p)-pC}\left(\int_0^\infty \exp(-xy)f(x)dx\right)^p dy\\&\quad\leq\Gamma(1-pC)\int_0^\infty x^{p(B+C)-1}f^p(x)dx.\end{aligned}\tag{6.5}$$

Observe that equality in (6.5) holds if and only if $f(x)=K_1x^{-qB}$ for arbitrary non-negative constant K_1 (see Remark 1.3, Chapter 1). Clearly, this condition immediately gives that nontrivial case of equality in (6.5) leads to a divergent integral.

Finally, since $(qB-1)(1-p)-pC=p-1-p(B+C)$, the above relation (6.5) yields (6.4). □

Now, we are ready to state and prove the basic result of this section.

Theorem 6.3 *Let* $s>0$, $\sum_{i=1}^n\frac{1}{p_i}=1$, $p_i>1$, $\frac{1}{p_i}+\frac{1}{q_i}=1$, $i=1,\dots,n$, *and* $\sum_{i=1}^n\alpha_i=0$. *If* $f_i:\mathbb{R}_+\to\mathbb{R}$, $f_i\neq 0$, $i=1,\dots,n$, *are non-negative measurable functions with the respective Laplace transforms* $\mathscr{L}f_i$, *then*

$$\begin{aligned}\int_{\mathbb{R}_+^n}\frac{\prod_{i=1}^n f_i(x_i)}{(\sum_{j=1}^n x_j)^s}dx_1\dots dx_n&\leq\frac{1}{\Gamma(s)}\prod_{i=1}^n\|t^{\frac{1}{q_i}+\frac{s-n}{p_i}-\alpha_i}\left(\mathscr{L}f_i\right)(t)\|_{L^{p_i}(\mathbb{R}_+)}\\&<\frac{1}{\Gamma(s)}\prod_{i=1}^n\left[\Gamma^{\frac{1}{q_i}}(1-q_iB_i)\Gamma^{\frac{1}{p_i}}(1-p_iC_i)\right]\\&\quad\times\prod_{i=1}^n\left[\int_0^\infty x_i^{n-1-s+p_i\alpha_i}f_i^{p_i}(x_i)dx_i\right]^{\frac{1}{p_i}},\end{aligned}\tag{6.6}$$

where $B_i+C_i=\frac{n-s}{p_i}+\alpha_i$, $B_i<\frac{1}{q_i}$, *and* $C_i<\frac{1}{p_i}$, $i=1,\dots,n$.

Proof. Utilizing Lemma 6.1 and setting exponents β_i, $i=1,\dots,n$, such that $\sum_{i=1}^n \beta_i = s-1$, we have

$$\begin{aligned}\int_{\mathbb{R}_+^n} \frac{\prod_{i=1}^n f_i(x_i)}{(\sum_{j=1}^n x_j)^s} dx_1 \dots dx_n &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \prod_{i=1}^n (\mathscr{L} f_i)(t) dt \\ &= \frac{1}{\Gamma(s)} \int_0^\infty \prod_{i=1}^n t^{\beta_i} (\mathscr{L} f_i)(t) dt.\end{aligned}$$

Moreover, applying the Hölder inequality, we have that

$$\int_{\mathbb{R}_+^n} \frac{\prod_{i=1}^n f_i(x_i)}{(\sum_{j=1}^n x_j)^s} dx_1 \dots dx_n \le \frac{1}{\Gamma(s)} \prod_{i=1}^n \left[\int_0^\infty t^{p_i\beta_i} (\mathscr{L} f_i)^{p_i}(t) dt\right]^{\frac{1}{p_i}}. \tag{6.7}$$

On the other hand, setting

$$\beta_i = \frac{1}{q_i} - (B_i + C_i), \qquad i=1,\dots,n, \tag{6.8}$$

where $B_i < \frac{1}{q_i}$ and $C_i < \frac{1}{p_i}$, and taking into account Lemma 6.2, we have

$$\begin{aligned}\left[\int_0^\infty t^{p_i\beta_i} (\mathscr{L} f_i)^{p_i}(t) dt\right]^{\frac{1}{p_i}} &= \left[\int_0^\infty t^{p_i-1-p_i(B_i+C_i)} (\mathscr{L} f_i)^{p_i}(t) dt\right]^{\frac{1}{p_i}} \\ &< \Gamma^{\frac{1}{q_i}}(1-q_iB_i)\Gamma^{\frac{1}{p_i}}(1-p_iC_i) \left[\int_0^\infty x_i^{p_i(B_i+C_i)-1} f_i^{p_i}(x_i) dx_i\right]^{\frac{1}{p_i}}.\end{aligned} \tag{6.9}$$

Now, putting $B_i + C_i = \frac{n-s}{p_i} + \alpha_i$ in (6.8), it follows that $\sum_{i=1}^n \beta_i = s-1$, that is, inequality (6.9) reads

$$\begin{aligned}\left[\int_0^\infty t^{p_i\beta_i} (\mathscr{L} f_i)^{p_i}(t) dt\right]^{\frac{1}{p_i}} &= \left[\int_0^\infty t^{\frac{p_i}{q_i}-n+s-p_i\alpha_i} (\mathscr{L} f_i)^{p_i}(t) dt\right]^{\frac{1}{p_i}} \\ &< \Gamma^{\frac{1}{q_i}}(1-q_iB_i)\Gamma^{\frac{1}{p_i}}(1-p_iC_i) \left[\int_0^\infty x_i^{n-1-s+p_i\alpha_i} f_i^{p_i}(x_i) dx_i\right]^{\frac{1}{p_i}}.\end{aligned} \tag{6.10}$$

Finally, utilizing (6.7) and (6.10), we obtain (6.6). □

Remark 6.1 Applying Lemma 6.2 to parameters $B = \frac{1}{q}\left(\frac{1}{q} - \beta\right)$ and $C = \frac{1}{p}\left(\frac{1}{q} - \beta\right)$, where $\frac{1}{p} + \beta > 0$, we have $\Gamma(1-pC)\Gamma^{p-1}(1-qB) = \Gamma^p(\frac{1}{p} + \beta)$, that is,

$$\|t^\beta (\mathscr{L} f)(t)\|_{L^p(\mathbb{R}_+)} < \Gamma\left(\beta + \frac{1}{p}\right) \|x^{1-\beta-\frac{2}{p}} f(x)\|_{L^p(\mathbb{R}_+)}. \tag{6.11}$$

The above inequality was also established in [108]. Moreover, utilizing (6.11) in the proof of Theorem 6.3, the constant $\frac{1}{\Gamma(s)} \prod_{i=1}^n [\Gamma^{\frac{1}{q_i}}(1-q_iB_i)\Gamma^{\frac{1}{p_i}}(1-p_iC_i)]$ becomes $\frac{1}{\Gamma(s)} \prod_{i=1}^n \Gamma(1 - \frac{n-s}{p_i} - \alpha_i)$.

We restate Theorem 6.3 in the case of $n = 2$. This result is interesting in its own right, since it may be regarded as a generalization of the mentioned result from [108], as well as a refinement of inequality (1.23) with the kernel $K(x,y) = (x+y)^{-s}$ (see Corollary 1.1, Chapter 1).

Theorem 6.4 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and $s>0$. If $f,g:\mathbb{R}_+\to\mathbb{R}$, $f,g\neq 0$, are non-negative measurable functions with the respective Laplace transforms $\mathscr{L}f$ and $\mathscr{L}g$, then*

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^s}dxdy\\
&\quad\leq \frac{1}{\Gamma(s)}\|t^{\frac{1}{q}+\frac{s-2}{p}+A_2-A_1}\left(\mathscr{L}f\right)(t)\|_{L^p(\mathbb{R}_+)}\ \|t^{\frac{1}{p}+\frac{s-2}{q}+A_1-A_2}\left(\mathscr{L}g\right)(t)\|_{L^q(\mathbb{R}_+)}\\
&\quad< L\left[\int_0^\infty x^{1-s+p(A_1-A_2)}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty x^{1-s+q(A_2-A_1)}g^q(x)dx\right]^{\frac{1}{q}}, \qquad (6.12)
\end{aligned}$$

where $L = B^{\frac{1}{p}}(1-pA_2, s-1+pA_2)B^{\frac{1}{q}}(1-qA_1, s-1+qA_1)$, and $A_1 \in (\frac{1-s}{q},\frac{1}{q})$, $A_2 \in (\frac{1-s}{p},\frac{1}{p})$.

Proof. The proof follows directly from Theorem 6.3 defining $p_1 = q_2 = p$, $p_2 = q_1 = q$, $B_i = A_i$, $C_i = \frac{2-s}{q_i} - A_{i+1}$, $\alpha_i = A_i - A_{i+1}$ for $i = 1,2$ (the indices are taken modulo 2). □

In order to obtain inequality (6.1), we consider (6.12) with the parameters $A_1 = \frac{1-b}{q} - \frac{1}{pq}$ and $A_2 = \frac{1-c}{p} - \frac{1}{pq}$. Obviously, since $A_1 \in \left(\frac{1-s}{q},\frac{1}{q}\right)$ and $A_2 \in \left(\frac{1-s}{p},\frac{1}{p}\right)$, it follows that $0 < b+\frac{1}{p} < s$ and $0 < c+\frac{1}{q} < s$.

Corollary 6.1 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and $s>0$. If $f,g:\mathbb{R}_+\to\mathbb{R}$, $f,g\neq 0$, are non-negative measurable functions with the respective Laplace transforms $\mathscr{L}f$ and $\mathscr{L}g$, then*

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^s}dxdy\\
&\quad\leq \frac{1}{\Gamma(s)}\|t^{\frac{1}{p}(s-c-1)+\frac{b}{q}}\left(\mathscr{L}f\right)(t)\|_{L^p(\mathbb{R}_+)}\ \|t^{\frac{1}{q}(s-b-1)+\frac{c}{p}}\left(\mathscr{L}g\right)(t)\|_{L^q(\mathbb{R}_+)}\\
&\quad< C_1\left[\int_0^\infty x^{(p-1)(1-b)+c-s}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty x^{(q-1)(1-c)+b-s}g^q(x)dx\right]^{\frac{1}{q}}, \qquad (6.13)
\end{aligned}$$

where $C_1 = B^{\frac{1}{p}}\left(c+\frac{1}{q}, s-c-\frac{1}{q}\right)B^{\frac{1}{q}}\left(b+\frac{1}{p}, s-b-\frac{1}{p}\right)$.

Remark 6.2 If $s = b+c+1$, the above inequality (6.13) reduces to (6.1), that is, Theorem 6.3 can be seen as a generalization of the Peachey result from [108].

6.2 The case of non-conjugate parameters

The following result is an extension of Lemma 6.2 to the setting of non-conjugate exponents.

Lemma 6.3 *Let p, p', q, q', and λ satisfy (2.1) and (2.2). If $f:\mathbb{R}_+\to\mathbb{R}$, $f\neq 0$, is a non-negative measurable function with the Laplace transform $\mathscr{L}f$, then*

$$\left[\int_0^\infty y^{\frac{q'}{p'}-q'(B+C)}(\mathscr{L}f)^{q'}(y)dy\right]^{\frac{1}{q'}}$$
$$<\lambda^{\lambda-(B+C)}\Gamma^{\frac{1}{p'}}(1-p'B)\Gamma^{\frac{1}{q'}}(1-q'C)\left[\int_0^\infty x^{p(B+C)-\frac{p}{q'}}f^p(x)dx\right]^{\frac{1}{p}}, \tag{6.14}$$

where $B<\frac{1}{p'}$ and $C<\frac{1}{q'}$.

Proof. We utilize general Hardy-Hilbert-type inequality (2.10) with non-conjugate exponents (Theorem 2.1, Chapter 2). Similarly to the procedure used in the proof of Lemma 6.2, setting $K(x,y)=\exp(-\frac{xy}{\lambda})$, $\varphi(x)=x^B$, and $\psi(y)=y^C$ in (2.10), we obtain inequality (6.14). It follows also that the equality in (6.14) is possible only in the trivial case (see Remark 2.3, Chapter 2). □

In order to obtain a non-conjugate version of Theorem 6.3, we introduce real parameters r_i' such that $p_i\le r_i'$, $i=1,\dots,n$, and $\sum_{i=1}^n\frac{1}{r_i'}=1$. For example, we can define $\frac{1}{r_i'}=\frac{1}{q_i}+\frac{1-\lambda}{n}$ or $r_i'=(n-n\lambda+\lambda)p_i$, $i=1,\dots,n$. Now, utilizing Lemma 6.3 we obtain the following general result:

Theorem 6.5 *Let $s>0$ and $p_1,\dots,p_n,\lambda$ be as in (2.26) and (2.27). Let $r_1',\dots,r_n'$ and $\alpha_1,\dots,\alpha_n$ be such that $p_i\le r_i'$, $i=1,\dots,n$, $\sum_{i=1}^n\frac{1}{r_i'}=1$, and $\sum_{i=1}^n\alpha_i=0$. If $f_i:\mathbb{R}_+\to\mathbb{R}$, $f_i\neq 0$, $i=1,\dots,n$, are non-negative measurable functions with the respective Laplace transforms $\mathscr{L}f_i$, then*

$$\int_{\mathbb{R}_+^n}\frac{\prod_{i=1}^n f_i(x_i)}{(\sum_{j=1}^n x_j)^s}dx_1\dots dx_n\le\frac{1}{\Gamma(s)}\prod_{i=1}^n\|t^{\frac{1}{p_i'}-(B_i+C_i)}(\mathscr{L}f_i)(t)\|_{L^{r_i'}(\mathbb{R}_+)}$$
$$<\frac{1}{\Gamma(s)}\prod_{i=1}^n\left[\gamma_i^{\gamma_i-(B_i+C_i)}\Gamma^{\frac{1}{p_i'}}(1-p_i'B_i)\Gamma^{\frac{1}{r_i'}}(1-r_i'C_i)\right]$$
$$\times\prod_{i=1}^n\left[\int_0^\infty x_i^{p_i(B_i+C_i)-\frac{p_i}{r_i'}}f_i^{p_i}(x_i)dx_i\right]^{\frac{1}{p_i}}, \tag{6.15}$$

where $\gamma_i=\frac{1}{p_i'}+\frac{1}{r_i'}$, $B_i<\frac{1}{p_i'}$, and $C_i<\frac{1}{r_i'}$, $i=1,\dots,n$.

Proof. Let β_i, $i=1,\dots,n$, be parameters such that $\sum_{i=1}^n \beta_i = s-1$. Similarly to the proof of Theorem 6.3, applying Lemma 6.1 and the Hölder inequality, we have

$$\int_{\mathbb{R}_+^n} \frac{\prod_{i=1}^n f_i(x_i)}{(\sum_{j=1}^n x_j)^s} dx_1 \dots dx_n \le \frac{1}{\Gamma(s)} \prod_{i=1}^n \left[\int_0^\infty t^{r_i'\beta_i} (\mathscr{L} f_i)^{r_i'}(t) dt \right]^{\frac{1}{r_i'}}, \tag{6.16}$$

where $\sum_{i=1}^n \frac{1}{r_i'} = 1$ and $p_i \le r_i'$, $i=1,\dots,n$.

On the other hand, setting

$$\beta_i = \frac{1}{p_i'} - (B_i + C_i), \qquad i=1,\dots,n, \tag{6.17}$$

where $B_i < \frac{1}{p_i'}$ and $C_i < \frac{1}{r_i'}$, and taking into account Lemma 6.3, we obtain

$$\begin{aligned} \left[\int_0^\infty t^{r_i'\beta_i} (\mathscr{L} f_i)^{r_i'}(t) dt \right]^{\frac{1}{r_i'}} &= \left[\int_0^\infty t^{\frac{r_i'}{p_i'} - r_i'(B_i+C_i)} (\mathscr{L} f_i)^{r_i'}(t) dt \right]^{\frac{1}{r_i'}} \\ &< \gamma_i^{\gamma_i - (B_i+C_i)} \Gamma^{\frac{1}{p_i'}}(1-p_i'B_i) \Gamma^{\frac{1}{r_i'}}(1-r_i'C_i) \left[\int_0^\infty x_i^{p_i(B_i+C_i) - \frac{p_i}{r_i'}} f_i^{p_i}(x_i) dx_i \right]^{\frac{1}{p_i}}, \end{aligned} \tag{6.18}$$

where $\gamma_i = \frac{1}{p_i'} + \frac{1}{r_i'}$. Finally, relations (6.16) and (6.18) yield (6.15). □

Setting $B_i + C_i = \frac{n-s+1}{r_i'} - \frac{1}{p_i} + \alpha_i$ in (6.17), we have $\sum_{i=1}^n \beta_i = s-1$, that is, inequality (6.18) reduces to

$$\begin{aligned} \| t^{\frac{s-n-1}{r_i'} + 1 - \alpha_i} (\mathscr{L} f_i)(t) \|_{L^{r_i'}(\mathbb{R}_+)} &< \gamma_i^{\frac{s-n}{r_i'} + 1 - \alpha_i} \Gamma^{\frac{1}{p_i'}}(1-p_i'B_i) \Gamma^{\frac{1}{r_i'}}(1-r_i'C_i) \\ &\times \left[\int_0^\infty x_i^{\frac{p_i}{r_i'}(n-s) - 1 + p_i\alpha_i} f_i^{p_i}(x_i) dx_i \right]^{\frac{1}{p_i}}. \end{aligned} \tag{6.19}$$

Now, exploiting (6.19) we obtain the following result:

Corollary 6.2 *Let $s > 0$ and $p_1,\dots,p_n,\lambda$ be as in (2.26) and (2.27). Let $r_1',\dots,r_n'$, $\alpha_1,\dots,\alpha_n$ satisfy conditions of Theorem 6.5. If $f_i : \mathbb{R}_+ \to \mathbb{R}$, $f_i \ne 0$, $i=1,\dots,n$, are non-negative measurable functions with the respective Laplace transforms $\mathscr{L} f_i$, then*

$$\begin{aligned} \int_{\mathbb{R}_+^n} \frac{\prod_{i=1}^n f_i(x_i)}{(\sum_{j=1}^n x_j)^s} dx_1 \dots dx_n &\le \frac{1}{\Gamma(s)} \prod_{i=1}^n \| t^{\frac{s-n-1}{r_i'} + 1 - \alpha_i} (\mathscr{L} f_i)(t) \|_{L^{r_i'}(\mathbb{R}_+)} \\ &< \frac{1}{\Gamma(s)} \prod_{i=1}^n \left[\gamma_i^{\frac{s-n}{r_i'} + 1 - \alpha_i} \Gamma^{\frac{1}{p_i'}}(1-p_i'B_i) \Gamma^{\frac{1}{r_i'}}(1-r_i'C_i) \right] \\ &\times \prod_{i=1}^n \left[\int_0^\infty x_i^{\frac{p_i}{r_i'}(n-s) - 1 + p_i\alpha_i} f_i^{p_i}(x_i) dx_i \right]^{\frac{1}{p_i}}, \end{aligned} \tag{6.20}$$

where $\gamma_i = \frac{1}{p_i'} + \frac{1}{r_i'}$, $B_i + C_i = \frac{n-s+1}{r_i'} - \frac{1}{p_i} + \alpha_i$, $B_i < \frac{1}{p_i'}$, and $C_i < \frac{1}{r_i'}$, $i=1,\dots,n$.

Remark 6.3 Setting $\frac{1}{r_i'}=\frac{1}{q_i}+\frac{1-\lambda}{n}$, $\alpha_i=\frac{1}{p_i}-\frac{1}{r_i'}(n-s)$, $B_i=\frac{1}{r_i'+p_i'}$, and $C_i=\frac{p_i'}{r_i'(r_i'+p_i')}$ in (6.20), we get the following inequalities:

$$\begin{aligned}
&\int_{\mathbb{R}_+^n}\frac{\prod_{i=1}^n f_i(x_i)}{(\sum_{j=1}^n x_j)^{(n-1)\lambda}}dx_1\dots dx_n\\
&\leq\frac{1}{\Gamma((n-1)\lambda)}\prod_{i=1}^n\|t^{\frac{2}{p_i'}-\frac{(n-1)\lambda+1}{n}}\left(\mathscr{L}f_i\right)(t)\|_{L^{\frac{nq_i}{n+q_i(1-\lambda)}}(\mathbb{R}_+)}\\
&<\frac{1}{\Gamma((n-1)\lambda)}\left[\lambda+\frac{1-\lambda}{n}\right]^{(n-1)\lambda}\left[\prod_{i=1}^n\Gamma\left(\frac{n}{(n-1)\lambda+1}\cdot\frac{1}{p_i'}\right)\right]^{\lambda+\frac{1-\lambda}{n}}\prod_{i=1}^n\|f_i\|_{L^{p_i}(\mathbb{R}_+)}.
\end{aligned}$$

Obviously, according to the conditions from the statement of Corollary 6.2, we can use other choices of parameters r_i', and consequently, take the infimum of the right-hand sides of the obtained inequalities over all such parameters r_i'.

We restate Corollary 6.2 for the case $n=2$. This result is interesting in its own right, and it will be applied to obtain Theorem 6.2.

Corollary 6.3 *Suppose p, p', q, q', and λ satisfy* (2.1) *and* (2.2)*, and let $s>0$. Further, let $p\leq r'\leq q'$, $\frac{1}{r'}+\frac{1}{r}=1$, and $\alpha_1+\alpha_2=0$. If $f,g:\mathbb{R}_+\to\mathbb{R}$, $f,g\neq 0$, are non-negative measurable functions with the respective Laplace transforms $\mathscr{L}f$ and $\mathscr{L}g$, then*

$$\begin{aligned}
&\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{(x+y)^s}dxdy\\
&\leq\frac{1}{\Gamma(s)}\|t^{\frac{s-3}{r'}+1-\alpha_1}\left(\mathscr{L}f\right)(t)\|_{L^{r'}(\mathbb{R}_+)}\|t^{\frac{s-3}{r}+1-\alpha_2}\left(\mathscr{L}g\right)(t)\|_{L^{r}(\mathbb{R}_+)}\\
&<\frac{M}{\Gamma(s)}\|x^{\frac{2-s}{r'}-\frac{1}{p}+\alpha_1}f(x)\|_{L^p(\mathbb{R}_+)}\|y^{\frac{2-s}{r}-\frac{1}{q}+\alpha_2}g(y)\|_{L^q(\mathbb{R}_+)},
\end{aligned}\tag{6.21}$$

where

$$M=\gamma_1^{\frac{s-2}{r'}+1-\alpha_1}\gamma_2^{\frac{s-2}{r}+1-\alpha_2}\Gamma^{\frac{1}{p'}}(1-p'B_1)\Gamma^{\frac{1}{r'}}(1-r'C_1)\Gamma^{\frac{1}{q'}}(1-q'B_2)\Gamma^{\frac{1}{r}}(1-rC_2),$$

$\gamma_1=\frac{1}{p'}+\frac{1}{r'}$, $\gamma_2=\frac{1}{q'}+\frac{1}{r}$, $B_1+C_1=\frac{3-s}{r'}-\frac{1}{p}+\alpha_1$, $B_2+C_2=\frac{3-s}{r}-\frac{1}{q}+\alpha_2$, $B_1<\frac{1}{p'}$, $C_1<\frac{1}{r'}$, $B_2<\frac{1}{q'}$, *and* $C_2<\frac{1}{r}$.

Proof. The proof follows directly from Corollary 6.2 using substitutions $p_1=p$, $p_2=q$, $p_1'=q_2=p'$, $p_2'=q_1=q'$, $r_1'=r'$, and $r_2'=r$. □

Remark 6.4 Setting $s=b+c+1$, $\alpha_1=\frac{c-2}{r'}+1-\frac{b}{r}$, $\alpha_2=-\alpha_1$, $B_1=\frac{1}{p'}-\frac{1}{p'\gamma_1}(b+\frac{1}{r'})$, $C_1=\frac{1}{r'}-\frac{1}{r'\gamma_1}(b+\frac{1}{r'})$, $B_2=\frac{1}{q'}-\frac{1}{q'\gamma_2}(c+\frac{1}{r})$, and $C_2=\frac{1}{r}-\frac{1}{r\gamma_2}(c+\frac{1}{r})$ in Corollary 6.3, we obtain Theorem 6.2.

Remark 6.5 Set $s=\lambda$, $\alpha_1=\frac{\lambda-2}{r'}+\frac{1}{p}$, $\alpha_2=-\alpha_1$, $B_1=\frac{1}{r'p'}$, $C_1=\frac{1}{r'p}$, $B_2=\frac{1}{rq'}$, and $C_2=\frac{1}{rq}$, $p\le r'\le q'$, in Corollary 6.3. Then inequality (6.21) becomes

$$\begin{aligned}\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy &\le \frac{1}{\Gamma(\lambda)}\|t^{\frac{1}{p'}-\frac{1}{r'}}(\mathscr{L}f)(t)\|_{L^{r'}(\mathbb{R}_+)}\|t^{\frac{1}{q'}-\frac{1}{r}}(\mathscr{L}g)(t)\|_{L^{r}(\mathbb{R}_+)}\\ &< \frac{M}{\Gamma(\lambda)}\|f\|_{L^p(\mathbb{R}_+)}\|g\|_{L^q(\mathbb{R}_+)},\end{aligned} \tag{6.22}$$

where $\gamma_1=\frac{1}{p'}+\frac{1}{r'}$, $\gamma_2=\frac{1}{q'}+\frac{1}{r}$, and $M=\gamma_1^{\frac{1}{p'}}\gamma_2^{\frac{1}{q'}}\Gamma^{\frac{1}{p'}}\left(\frac{1}{r}\right)\Gamma^{\frac{1}{r'}}\left(\frac{1}{p'}\right)\Gamma^{\frac{1}{q'}}\left(\frac{1}{r'}\right)\Gamma^{\frac{1}{r}}\left(\frac{1}{q'}\right)$. It should be noticed here that in conjugate case, the constant $\frac{M}{\Gamma(\lambda)}$ reduces to the best constant $\pi/\sin(\pi/p)$.

Remark 6.6 Similarly to Corollary 6.2, if we put $B_i+C_i=\frac{1}{r_i'}+\frac{1}{q_i}(n-1-\frac{s}{\lambda})+\alpha_i$, $i=1,\dots,n$, in (6.17), then $\sum_{i=1}^n\beta_i=s-1$. Thus, inequality (6.15) from Theorem 6.5 becomes

$$\begin{aligned}\int_{\mathbb{R}_+^n}\frac{\prod_{i=1}^n f_i(x_i)}{(\sum_{j=1}^n x_j)^s}dx_1\dots dx_n &\le \frac{1}{\Gamma(s)}\prod_{i=1}^n\|t^{\lambda-\frac{1}{r_i'}-\frac{1}{q_i}(n-\frac{s}{\lambda})-\alpha_i}(\mathscr{L}f_i)(t)\|_{L^{r'}(\mathbb{R}_+)}\\ &< \frac{1}{\Gamma(s)}\prod_{i=1}^n\left[\gamma_i^{\lambda-\frac{1}{q_i}(n-\frac{s}{\lambda})-\alpha_i}\Gamma^{\frac{1}{p_i'}}(1-p_i'B_i)\Gamma^{\frac{1}{r_i'}}(1-r_i'C_i)\right]\\ &\quad\times\prod_{i=1}^n\left[\int_0^\infty x_i^{\frac{p_i}{q_i}(n-1-\frac{s}{\lambda})+p_i\alpha_i}f_i^{p_i}(x_i)dx_i\right]^{\frac{1}{p_i}},\end{aligned} \tag{6.23}$$

where γ_i, r_i', α_i are as in Corollary 6.2. Observe that the integrals on the right-hand side of the second inequality in (6.23) coincide with integrals on the right-hand side of inequality (2.35) (see Chapter 2), when s is replaced by $\frac{s}{\lambda}$.

Remark 6.7 Multidimensional refinements of Hilbert-type inequalities via the Laplace transform, presented in this chapter, are derived in [110] by Pečarić et.al.

Chapter 7

A class of Hilbert-Pachpatte-type inequalities

In this chapter we investigate a particular class of the so-called Hilbert-Pachpatte-type inequalities which are closely connected to Hilbert-type inequalities. An interesting feature of this class is that it controls the size (in the sense of L^p or l^p spaces) of the modified Hilbert transform of a function or of a series with the size of its derivative or its backward differences, respectively.

We start this overview with the results of Lü [86], in a slightly altered form, in both continuous and discrete case. For a sequence $(a_n)_{n\in\mathbb{N}_0}$, the sequence $(\nabla a_n)_{n\in\mathbb{N}}$ is defined by $\nabla a_n = a_n - a_{n-1}$, while for a function $u:\mathbb{R}_+ \to \mathbb{R}$, u' denotes its usual derivative.

Theorem 7.1 *Let* $\frac{1}{p}+\frac{1}{q}=1,\ p>1,$ *and let* $s>2-\min\{p,q\}$. *If* $f,g:\mathbb{R}_+\to\mathbb{R}$ *are absolutely continuous functions such that* $f(0)=g(0)=0$, *then*

$$\begin{aligned}&\int_0^\infty\int_0^\infty \frac{|f(x)||g(y)|}{(qx^{p-1}+py^{q-1})(x+y)^s}dxdy\\ &\leq \frac{B\left(\frac{q+s-2}{q},\frac{p+s-2}{p}\right)}{pq}\left[\int_0^\infty\int_0^x x^{1-s}|f'(\tau)|^p d\tau dx\right]^{\frac{1}{p}}\left[\int_0^\infty\int_0^y y^{1-s}|g'(\delta)|^q d\delta dy\right]^{\frac{1}{q}}.\end{aligned} \tag{7.1}$$

Theorem 7.2 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $2-\min\{p,q\}<s\leq 2$. If $(a_m)_{m\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$ are sequences of real numbers such that $a_0=b_0=0$, then the following inequality hold:*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{|a_m||b_n|}{(qm^{p-1}+pn^{q-1})(m+n)^s}$$
$$\leq \frac{B\left(\frac{q+s-2}{q},\frac{p+s-2}{p}\right)}{pq}\left[\sum_{m=1}^{\infty}\sum_{\tau=1}^{m}m^{1-s}|\nabla a_\tau|^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}\sum_{\delta=1}^{n}n^{1-s}|\nabla b_\delta|^q\right]^{\frac{1}{q}}. \tag{7.2}$$

Due to the form of the kernel, the above Hilbert-Pachpatte-type inequalities include the constant factors expressed in terms of the usual Beta function. Our main task here is to obtain generalizations of the above inequalities which include arbitrary kernels and weight functions, with a special emphasis on homogeneous kernels. This can be done in a simpler way than in [86], by virtue of the general Hilbert-type and Hardy-Hilbert-type inequalities from Chapters 1 and 2. Hence, generalizations that follow will be given in both Hilbert and Hardy-Hilbert form. Moreover, the established results will also be considered in the setting of non-conjugate exponents.

7.1 Integral case

We start with the following general result.

Theorem 7.3 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$, $\varphi,\psi:\mathbb{R}_+\to\mathbb{R}$ be non-negative functions. If $f,g:\mathbb{R}_+\to\mathbb{R}$ are absolutely continuous functions such that $f(0)=g(0)=0$, and $F(x)=\int_0^\infty K(x,y)\psi^{-p}(y)dy$, $G(y)=\int_0^\infty K(x,y)\varphi^{-q}(x)dx$, then the following inequalities hold:*

$$\int_0^\infty\int_0^\infty\frac{K(x,y)|f(x)||g(y)|}{qx^{p-1}+py^{q-1}}dxdy$$
$$\leq \int_0^\infty\int_0^\infty K(x,y)|f(x)||g(y)|d(x^{\frac{1}{p}})d(y^{\frac{1}{q}})$$
$$\leq \frac{1}{pq}\left[\int_0^\infty\int_0^x\varphi^p(x)F(x)|f'(\tau)|^p d\tau dx\right]^{\frac{1}{p}}\left[\int_0^\infty\int_0^y\psi^q(y)G(y)|g'(\delta)|^q d\delta dy\right]^{\frac{1}{q}} \tag{7.3}$$

and

$$\int_0^\infty G^{1-p}(y)\psi^{-p}(y)\left[\int_0^\infty K(x,y)|f(x)|d(x^{\frac{1}{p}})\right]^p dy$$
$$\leq \frac{1}{p^p}\int_0^\infty\int_0^x\varphi^p(x)F(x)|f'(\tau)|^p d\tau dx. \tag{7.4}$$

Proof. Utilizing the Hölder inequality (see also [86]), we have

$$|f(x)||g(y)| \le x^{\frac{1}{q}} y^{\frac{1}{p}} \left[\int_0^x |f'(\tau)|^p d\tau\right]^{\frac{1}{p}} \left[\int_0^y |g'(\delta)|^q d\delta\right]^{\frac{1}{q}}. \tag{7.5}$$

Taking into account (7.5) and the well-known Young inequality

$$xy \le \frac{x^p}{p} + \frac{y^q}{q}, \quad x \ge 0, \quad y \ge 0, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad p > 1,$$

we observe that

$$\frac{pq|f(x)||g(y)|}{qx^{p-1} + py^{q-1}} \le \frac{|f(x)||g(y)|}{x^{\frac{1}{q}} y^{\frac{1}{p}}} \le \left[\int_0^x |f'(\tau)|^p d\tau\right]^{\frac{1}{p}} \left[\int_0^y |g'(\delta)|^q d\delta\right]^{\frac{1}{q}}$$

and therefore,

$$\begin{aligned} & pq \int_0^\infty \int_0^\infty \frac{K(x,y)|f(x)||g(y)|}{qx^{p-1} + py^{q-1}} dxdy \\ & \le \int_0^\infty \int_0^\infty \frac{K(x,y)}{x^{\frac{1}{q}} y^{\frac{1}{p}}} |f(x)||g(y)| dxdy \\ & \le \int_0^\infty \int_0^\infty K(x,y) \left[\int_0^x |f'(\tau)|^p d\tau\right]^{\frac{1}{p}} \left[\int_0^y |g'(\delta)|^q d\delta\right]^{\frac{1}{q}} dxdy. \end{aligned} \tag{7.6}$$

Now, putting

$$f_1(x) = \left[\int_0^x |f'(\tau)|^p d\tau\right]^{\frac{1}{p}}, \qquad g_1(y) = \left[\int_0^y |g'(\delta)|^q d\delta\right]^{\frac{1}{q}}$$

in the general Hilbert-type inequality (1.12) in the conjugate case (Theorem 1.9, Chapter 1), we obtain

$$\begin{aligned} & \int_0^\infty \int_0^\infty K(x,y) f_1(x) g_1(y) dxdy \\ \le & \left[\int_0^\infty \varphi^p(x) F(x) f_1^p(x) dx\right]^{\frac{1}{p}} \left[\int_0^\infty \psi^q(y) G(y) g_1^q(y) dy\right]^{\frac{1}{q}} \\ = & \left[\int_0^\infty \int_0^x \varphi^p(x) F(x) |f'(\tau)|^p d\tau dx\right]^{\frac{1}{p}} \left[\int_0^\infty \int_0^y \psi^q(y) G(y) |g'(\delta)|^q d\delta dy\right]^{\frac{1}{q}}. \end{aligned} \tag{7.7}$$

Finally, using (7.6) and (7.7) we obtain (7.3).

The second inequality (7.4) follows from the general Hardy-Hilbert-type inequality (1.13) in the conjugate case (Theorem 1.9, Chapter 1), and the inequality $|f(x)| \le x^{\frac{1}{q}} \left(\int_0^x |f'(t)|^p dt\right)^{\frac{1}{p}}$. □

In the sequel we apply the above Theorem 7.3 to a homogeneous kernel. Similarly to the previous chapters we use the notation $k(\alpha) = \int_0^\infty K(1,u) u^{-\alpha} du$, where $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a non-negative homogeneous function of degree $-s$, $s > 0$, provided $k(\alpha) < \infty$ for $1 - s < \alpha < 1$.

Corollary 7.1 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ be a non-negative symmetric homogeneous function of degree $-s$, $s>0$. If $f,g:\mathbb{R}_+\to\mathbb{R}$ are absolutely continuous functions such that $f(0)=g(0)=0$, then*

$$\begin{aligned}
\int_0^\infty\int_0^\infty \frac{K(x,y)|f(x)||g(y)|}{qx^{p-1}+py^{q-1}}dxdy &\le \int_0^\infty\int_0^\infty K(x,y)|f(x)||g(y)|d(x^{\frac{1}{p}})d(y^{\frac{1}{q}})\\
&\le \frac{L}{pq}\left[\int_0^\infty\int_0^x x^{1-s+p(A_1-A_2)}|f'(\tau)|^p d\tau dx\right]^{\frac{1}{p}}\\
&\quad\times\left[\int_0^\infty\int_0^y y^{1-s+q(A_2-A_1)}|g'(\delta)|^q d\delta dy\right]^{\frac{1}{q}}
\end{aligned}\tag{7.8}$$

and

$$\begin{aligned}
&\int_0^\infty y^{(p-1)(s-1)+p(A_1-A_2)}\left[\int_0^\infty K(x,y)|f(x)|d(x^{\frac{1}{p}})\right]^p dy\\
&\quad\le\left(\frac{L}{p}\right)^p\int_0^\infty\int_0^x x^{1-s+p(A_1-A_2)}|f'(\tau)|^p d\tau dx,
\end{aligned}\tag{7.9}$$

where $A_1\in(\frac{1-s}{q},\frac{1}{q})$, $A_2\in(\frac{1-s}{p},\frac{1}{p})$, and $L=k^{\frac{1}{p}}(pA_2)k^{\frac{1}{q}}(qA_1)$.

Proof. We prove (7.8) only. Let $F(x)$ and $G(y)$ be as in the statement of Theorem 7.3. Setting $\varphi(x)=x^{A_1}$ and $\psi(y)=y^{A_2}$, we obtain

$$\begin{aligned}
&\int_0^\infty\int_0^x \varphi^p(x)F(x)|f'(\tau)|^p d\tau dx\\
&=\int_0^\infty\int_0^x |f'(\tau)|^p\left[\int_0^\infty K(x,y)\left(\frac{x}{y}\right)^{pA_2}dy\right]x^{p(A_1-A_2)}d\tau dx\\
&=k(pA_2)\int_0^\infty\int_0^x x^{1-s+p(A_1-A_2)}|f'(\tau)|^p d\tau dx,
\end{aligned}\tag{7.10}$$

and similarly,

$$\int_0^\infty\int_0^y \psi^q(y)G(y)|g'(\delta)|^q d\delta dy = k(qA_1)\int_0^\infty\int_0^y y^{1-s+q(A_2-A_1)}|g'(\delta)|^q d\delta dy.\tag{7.11}$$

Now, relations (7.3), (7.10), and (7.11) yield (7.8). □

Our first application of Corollary 7.1 refers to the homogeneous kernel $K(x,y)=(x+y)^{-s}$, $s>0$.

Corollary 7.2 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $s>0$. If $f,g:\mathbb{R}_+\to\mathbb{R}$ are absolutely* *continuous functions such that* $f(0) = g(0) = 0$, *then*

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{|f(x)||g(y)|}{(qx^{p-1}+py^{q-1})(x+y)^s}dxdy \le \int_0^\infty\int_0^\infty \frac{|f(x)||g(y)|}{(x+y)^s}d(x^{\frac{1}{p}})d(y^{\frac{1}{q}}) \\
&\le \frac{L_1}{pq}\left[\int_0^\infty\int_0^x x^{1-s+p(A_1-A_2)}|f'(\tau)|^p d\tau dx\right]^{\frac{1}{p}} \\
&\quad\times\left[\int_0^\infty\int_0^y y^{1-s+q(A_2-A_1)}|g'(\delta)|^q d\delta dy\right]^{\frac{1}{q}}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\infty y^{(p-1)(s-1)+p(A_1-A_2)}\left[\int_0^\infty \frac{|f(x)|}{(x+y)^s}d(x^{\frac{1}{p}})\right]^p dy \\
&\le \left(\frac{L_1}{p}\right)^p \int_0^\infty\int_0^x x^{1-s+p(A_1-A_2)}|f'(\tau)|^p d\tau dx,
\end{aligned}$$

where $A_1 \in (\frac{1-s}{q}, \frac{1}{q})$, $A_2 \in (\frac{1-s}{p}, \frac{1}{p})$, *and* $L_1 = B^{\frac{1}{p}}(1-pA_2, s-1+pA_2)B^{\frac{1}{q}}(1-qA_1, s-1+qA_1)$.

Remark 7.1 Setting $A_1 = A_2 = \frac{2-s}{pq}$ in Corollary 7.2, provided $s > 2-\min\{p,q\}$, we obtain Theorem 7.1.

Considering the homogeneous kernel $K(x,y) = \frac{\log\frac{y}{x}}{y-x}$ of degree -1, we obtain the following consequence of Corollary 7.1:

Corollary 7.3 *Let* $\frac{1}{p}+\frac{1}{q} = 1$ *and* $p > 1$. *If* $f,g : \mathbb{R}_+ \to \mathbb{R}$ *are absolutely continuous functions such that* $f(0) = g(0) = 0$, *then*

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{\log\frac{y}{x}|f(x)||g(y)|}{(qx^{p-1}+py^{q-1})(y-x)}dxdy \le \int_0^\infty\int_0^\infty \frac{\log\frac{y}{x}|f(x)||g(y)|}{y-x}d(x^{\frac{1}{p}})d(y^{\frac{1}{q}}) \\
&\le \frac{L_2}{pq}\left[\int_0^\infty\int_0^x x^{p(A_1-A_2)}|f'(\tau)|^p d\tau dx\right]^{\frac{1}{p}}\left[\int_0^\infty\int_0^y y^{q(A_2-A_1)}|g'(\delta)|^q d\delta dy\right]^{\frac{1}{q}}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\infty y^{p(A_1-A_2)}\left[\int_0^\infty \frac{|f(x)|\log\frac{y}{x}}{y-x}d(x^{\frac{1}{p}})\right]^p dy \\
&\le \left(\frac{L_2}{p}\right)^p \int_0^\infty\int_0^x x^{p(A_1-A_2)}|f'(\tau)|^p d\tau dx,
\end{aligned}$$

where $A_1 \in (0, \frac{1}{q})$, $A_2 \in (0, \frac{1}{p})$, *and* $L_2 = \pi^2(\sin pA_2\pi)^{-\frac{2}{p}}(\sin qA_1\pi)^{-\frac{2}{q}}$.

We conclude the above discussion regarding symmetric homogeneous kernels with the function $K(x,y) = \max\{x,y\}^{-s}$, $s > 0$.

Corollary 7.4 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $s>0$. If $f,g:\mathbb{R}_+\to\mathbb{R}$ are absolutely continuous functions such that $f(0)=g(0)=0$, then*

$$\int_0^\infty\int_0^\infty \frac{|f(x)||g(y)|}{(qx^{p-1}+py^{q-1})\max\{x,y\}^s}dxdy \le \int_0^\infty\int_0^\infty \frac{|f(x)||g(y)|}{\max\{x,y\}^s}d(x^{\frac{1}{p}})d(y^{\frac{1}{q}})$$
$$\le \frac{L_3}{pq}\left[\int_0^\infty\int_0^x x^{1-s+p(A_1-A_2)}|f'(\tau)|^p d\tau dx\right]^{\frac{1}{p}}$$
$$\times\left[\int_0^\infty\int_0^y y^{1-s+q(A_2-A_1)}|g'(\delta)|^q d\delta dy\right]^{\frac{1}{q}}$$

and

$$\int_0^\infty y^{(p-1)(s-1)+p(A_1-A_2)}\left[\int_0^\infty \frac{|f(x)|}{\max\{x,y\}^s}d(x^{\frac{1}{p}})\right]^p dy$$
$$\le \left(\frac{L_3}{p}\right)^p\int_0^\infty\int_0^x x^{1-s+p(A_1-A_2)}|f'(\tau)|^p d\tau dx,$$

where $A_1\in(\frac{1-s}{q},\frac{1}{q})$, $A_2\in(\frac{1-s}{p},\frac{1}{p})$, and

$$L_3=\frac{s}{(1-pA_2)^{\frac{1}{p}}(1-qA_1)^{\frac{1}{q}}(s+pA_2-1)^{\frac{1}{p}}(s+qA_1-1)^{\frac{1}{q}}}.$$

By virtue of general multidimensional Hilbert-type inequalities in conjugate setting (Theorem 1.18, Chapter 1), Theorem 7.3 can also be extended to the case of n conjugate exponents. We provide here the multidimensional version of inequality (7.3).

Theorem 7.4 *Let $\sum_{i=1}^n \frac{1}{p_i}=1$, $p_i>1$, and let $\alpha_i=\prod_{j=1,j\neq i}^n p_j$, $i=1,2,\dots,n$. If $K:\mathbb{R}_+^n\to\mathbb{R}$, $\phi_{ij}:\mathbb{R}_+\to\mathbb{R}$, $i,j=1,\dots,n$, are non-negative functions such that $\prod_{i,j=1}^n\phi_{ij}(x_j)=1$, and $f_i:\mathbb{R}_+\to\mathbb{R}$, $i=1,\dots,n$, are absolutely continuous functions such that $f_i(0)=0$, $i=1,\dots,n$, then*

$$\int_{\mathbb{R}_+^n}\frac{K(x_1,\dots,x_n)\prod_{i=1}^n|f_i(x_i)|}{\sum_{i=1}^n\alpha_i x_i^{p_i-1}}dx_1\dots dx_n$$
$$\le\int_{\mathbb{R}_+^n}K(x_1,\dots,x_n)\prod_{i=1}^n|f_i(x_i)|d(x_1^{\frac{1}{p_1}})\dots d(x_n^{\frac{1}{p_n}})$$
$$\le\frac{1}{p_1\dots p_n}\prod_{i=1}^n\left[\int_0^\infty\int_0^{x_i}\phi_{ii}^{p_i}(x_i)F_i(x_i)|f_i'(\tau_i)|^{p_i}d\tau_i dx_i\right]^{\frac{1}{p_i}},$$

where $F_i(x_i)=\int_{\mathbb{R}_+^{n-1}}K(x_1,\dots,x_n)\prod_{j=1,j\neq i}^n\phi_{ij}^{p_i}(x_j)dx_1\dots dx_{i-1}dx_{i+1}\dots dx_n$ for $i=1,\dots,n$.

7.2 Discrete case

As we have already seen, Hilbert-Pachpatte-type inequalities can also be considered in the discrete setting. We first obtain a general result covering homogeneous kernels. It is established by virtue of discrete Hilbert-type inequalities for homogeneous kernels, that is, Theorem 1.14 (Chapter 1).

Theorem 7.5 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $A, B, \alpha, \beta > 0$. If $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ is a non-negative homogeneous function of degree $-s$, $s>0$, strictly decreasing in both arguments and $(a_m)_{m\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$ are sequences of real numbers such that $a_0=b_0=0$, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{K(Am^{\alpha},Bn^{\beta})|a_m||b_n|}{qm^{p-1}+pn^{q-1}} \le \frac{1}{pq}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{K(Am^{\alpha},Bn^{\beta})|a_m||b_n|}{m^{\frac{1}{q}}n^{\frac{1}{p}}}$$
$$\le \frac{N}{pq}\left[\sum_{m=1}^{\infty}\sum_{\tau=1}^{m} m^{\alpha(1-s)+\alpha p(A_1-A_2)+(p-1)(1-\alpha)}|\nabla a_\tau|^p\right]^{\frac{1}{p}}$$
$$\times\left[\sum_{n=1}^{\infty}\sum_{\delta=1}^{n} n^{\beta(1-s)+\beta q(A_2-A_1)+(q-1)(1-\beta)}|\nabla b_\delta|^q\right]^{\frac{1}{q}} \tag{7.12}$$

and

$$\sum_{n=1}^{\infty} n^{\beta(s-1)(p-1)+p\beta(A_1-A_2)+\beta-1}\left[\sum_{m=1}^{\infty}K(Am^{\alpha},Bn^{\beta})\frac{|a_m|}{m^{\frac{1}{q}}}\right]^p$$
$$\le N^p\sum_{m=1}^{\infty}\sum_{\tau=1}^{m} m^{\alpha(1-s)+\alpha p(A_1-A_2)+(p-1)(1-\alpha)}|\nabla a_\tau|^p, \tag{7.13}$$

where $A_1\in(\max\{\frac{1-s}{q},\frac{\alpha-1}{\alpha q}\},\frac{1}{q})$, $A_2\in(\max\{\frac{1-s}{p},\frac{\beta-1}{\beta p}\},\frac{1}{p})$, and

$$N=\alpha^{-\frac{1}{q}}\beta^{-\frac{1}{p}}A^{\frac{2-s}{p}+A_1-A_2-1}B^{\frac{2-s}{q}+A_2-A_1-1}k^{\frac{1}{p}}(pA_2)k^{\frac{1}{q}}(2-s-qA_1).$$

Proof. Similarly to the proof of Theorem 7.3, utilizing the Hölder and the Young inequalities, we have

$$\frac{pq|a_m||b_n|}{qm^{p-1}+pn^{q-1}} \le \frac{|a_m||b_n|}{m^{\frac{1}{q}}n^{\frac{1}{p}}} \le \left[\sum_{\tau=1}^{m}|\nabla a_\tau|^p\right]^{\frac{1}{p}}\left[\sum_{\delta=1}^{n}|\nabla b_\delta|^q\right]^{\frac{1}{q}},$$

and therefore,

$$pq\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{K(Am^{\alpha},Bn^{\beta})|a_m||b_n|}{qm^{p-1}+pn^{q-1}} \le \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{K(Am^{\alpha},Bn^{\beta})|a_m||b_n|}{m^{\frac{1}{q}}n^{\frac{1}{p}}}$$
$$\le \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}K(Am^{\alpha},Bn^{\beta})\left[\sum_{\tau=1}^{m}|\nabla a_\tau|^p\right]^{\frac{1}{p}}\left[\sum_{\delta=1}^{n}|\nabla b_\delta|^q\right]^{\frac{1}{q}}. \tag{7.14}$$

Now, setting $\widetilde{a}_m = (\sum_{\tau=1}^{m} |\nabla a_\tau|^p)^{\frac{1}{p}}$, $\widetilde{b}_n = \left(\sum_{\delta=1}^{n} |\nabla b_\delta|^q\right)^{\frac{1}{q}}$, $u(m) = Am^\alpha$, and $v(n) = Bn^\beta$ in (1.42) (Theorem 1.14, Chapter 1), we have

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K(Am^\alpha, Bn^\beta)\widetilde{a}_m\widetilde{b}_n \\
&\le N\left[\sum_{m=1}^{\infty} m^{\alpha(1-s)+\alpha p(A_1-A_2)+(p-1)(1-\alpha)}\widetilde{a}_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty} n^{\beta(1-s)+\beta q(A_2-A_1)+(q-1)(1-\beta)}\widetilde{b}_n^q\right]^{\frac{1}{q}} \\
&= N\left[\sum_{m=1}^{\infty}\sum_{\tau=1}^{m} m^{\alpha(1-s)+\alpha p(A_1-A_2)+(p-1)(1-\alpha)}|\nabla a(\tau)|^p\right]^{\frac{1}{p}} \\
&\quad\times\left[\sum_{n=1}^{\infty}\sum_{\delta=1}^{n} n^{\beta(1-s)+\beta q(A_2-A_1)+(q-1)(1-\beta)}|\nabla b(\delta)|^q\right]^{\frac{1}{q}}.
\end{aligned}\tag{7.15}$$

Finally, using (7.14) and (7.15), we obtain (7.12).

The second inequality (7.13) follows from (1.43) (Theorem 1.14, Chapter 1) and using $|a_m| \le m^{\frac{1}{q}} (\sum_{\tau=1}^{m} |\nabla a_\tau|^p)^{\frac{1}{p}}$. □

Remark 7.2 The above Theorem 7.5 can be regarded as a generalization of Theorem 7.2. Namely, applying Theorem 7.5 to $K(x,y) = (x+y)^{-s}$, $s > 0$, $A = B = \alpha = \beta = 1$, and $A_1 = A_2 = \frac{2-s}{pq}$, assuming $2 - \min\{p,q\} < s \le 2$, we obtain Theorem 7.2.

We conclude this section with an interesting extension of Theorem 7.2. Namely, by virtue of Corollary 4.7 (Chapter 4), we can obtain a larger interval of admissible values of the parameter s. Recall that this corollary was established by means of the Euler-Maclaurin summation formula.

Now, following the same lines as in the proof of Theorem 7.5 and utilizing the above mentioned Corollary 4.7 instead of Theorem 1.14, we obtain the following extension of Theorem 7.2.

Corollary 7.5 *Let $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, and let $2 - \min\{p,q\} < s \le 2 + \min\{p,q\}$. If $(a_m)_{m\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$ are sequences of real numbers such that $a_0 = b_0 = 0$, then*

$$\begin{aligned}
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{|a_m||b_n|}{(qm^{p-1}+pn^{q-1})(m+n)^s} &\le \frac{1}{pq}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{|a_m||b_n|}{m^{\frac{1}{q}}n^{\frac{1}{p}}(m+n)^s} \\
&\le \frac{N_1}{pq}\left[\sum_{m=1}^{\infty}\sum_{\tau=1}^{m} m^{1-s}|\nabla a_\tau|^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}\sum_{\delta=1}^{n} n^{1-s}|\nabla b_\delta|^q\right]^{\frac{1}{q}}
\end{aligned}\tag{7.16}$$

and

$$\sum_{n=1}^{\infty} n^{(s-1)(p-1)}\left[\sum_{m=1}^{\infty}\frac{|a_m|}{m^{\frac{1}{q}}(m+n)^s}\right]^p \le N_1^p\sum_{m=1}^{\infty}\sum_{\tau=1}^{m} m^{1-s}|\nabla a_\tau|^p, \tag{7.17}$$

where $N_1 = B(\frac{s+q-2}{q}, \frac{s+p-2}{p})$.

7.3 Non-conjugate exponents

Having in mind Theorem 2.1 (Chapter 2), Hilbert-Pachpatte-type inequalities considered in this chapter can also be extended to the case of non-conjugate exponents. Let p, p', q, q', and λ be as in (2.1) and (2.2). To obtain analogous results in the case of non-conjugate exponents, we introduce real parameters r', r such that $p \le r' \le q'$ and $\frac{1}{r'} + \frac{1}{r} = 1$. For example, we can define $\frac{1}{r'} = \frac{1}{q'} + \frac{1-\lambda}{2}$ or $r' = (2-\lambda)p$.

It is easy to see that inequalities

$$x^{\frac{1}{p'}} y^{\frac{1}{q'}} \le \frac{1}{rr'}\left(rx^{\frac{r'}{p'}} + r'y^{\frac{r}{q'}}\right),\ x \ge 0,\ y \ge 0 \tag{7.18}$$

and

$$|f(x)||g(y)| \le x^{\frac{1}{p'}} y^{\frac{1}{q'}} \left[\int_0^x |f'(\tau)|^p d\tau\right]^{\frac{1}{p}} \left[\int_0^y |g'(\delta)|^q d\delta\right]^{\frac{1}{q}} \tag{7.19}$$

hold, provided $f, g : \mathbb{R}_+ \to \mathbb{R}$ are absolutely continuous functions.

Now, utilizing Theorem 2.1 and inequalities (7.18) and (7.19), we obtain the following general result for non-conjugate exponents, in the same way as in the proof of Theorem 7.3.

Theorem 7.6 *Let p, p', q, q', and λ satisfy (2.1) and (2.2), and let r', r be real parameters such that $p \le r' \le q'$ and $\frac{1}{r'} + \frac{1}{r} = 1$. If $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}$ are non-negative functions and $f, g : \mathbb{R}_+ \to \mathbb{R}$ are absolutely continuous functions such that $f(0) = g(0) = 0$, then*

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{K^\lambda(x,y)|f(x)||g(y)|}{rx^{\frac{r'}{p'}} + r'y^{\frac{r}{q'}}} dxdy \\
&\le \frac{pq}{rr'} \int_0^\infty \int_0^\infty K^\lambda(x,y)|f(x)||g(y)|d(x^{\frac{1}{p}})d(y^{\frac{1}{q}}) \\
&\le \frac{1}{rr'} \left[\int_0^\infty \int_0^x (\varphi F)^p(x)|f'(\tau)|^p d\tau dx\right]^{\frac{1}{p}} \left[\int_0^\infty \int_0^y (\psi G)^q(y)|g'(\delta)|^q d\delta dy\right]^{\frac{1}{q}}
\end{aligned} \tag{7.20}$$

and

$$\begin{aligned}
&\left[\int_0^\infty \left[\frac{1}{\psi G(y)} \int_0^\infty K^\lambda(x,y)|f(x)|d(x^{\frac{1}{p}})\right]^{q'} dy\right]^{\frac{1}{q'}} \\
&\le \frac{1}{p}\left[\int_0^\infty \int_0^x (\varphi F)^p(x)|f'(\tau)|^p d\tau dx\right]^{\frac{1}{p}},
\end{aligned} \tag{7.21}$$

where $F(x) = \left(\int_0^\infty K(x,y)\psi^{-q'}(y)dy\right)^{\frac{1}{q'}}$ and $G(y) = \left(\int_0^\infty K(x,y)\varphi^{-p'}(x)dx\right)^{\frac{1}{p'}}$.

Obviously, Theorem 7.6 is a generalization of Theorem 7.3. Namely, setting $\lambda = 1$, $r' = p$, and $r = q$ in Theorem 7.6, inequalities (7.20) and (7.21) reduce to (7.3) and (7.4) respectively.

The following consequence of Theorem 7.6 may be regarded as a non-conjugate extension of Corollary 7.1.

Corollary 7.6 *Let p, p', q, q', and λ satisfy* (2.1) *and* (2.2)*, and let $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be a non-negative symmetric homogeneous function of degree $-s$, $s > 0$. If $f, g : \mathbb{R}_+ \to \mathbb{R}$ are absolutely continuous functions such that $f(0) = g(0) = 0$, then*

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{K^\lambda(x,y)|f(x)||g(y)|}{qx^{(p-1)(2-\lambda)} + py^{(q-1)(2-\lambda)}} dxdy \\
&\leq \frac{1}{2-\lambda} \int_0^\infty \int_0^\infty K^\lambda(x,y)|f(x)||g(y)| d(x^{\frac{1}{p}}) d(y^{\frac{1}{q}}) \\
&\leq \frac{M}{pq(2-\lambda)} \left[\int_0^\infty \int_0^x x^{\frac{p}{q'}(1-s)+p(A_1-A_2)} |f'(\tau)|^p d\tau dx \right]^{\frac{1}{p}} \\
&\quad \times \left[\int_0^\infty \int_0^y y^{\frac{q}{p'}(1-s)+q(A_2-A_1)} |g'(\delta)|^q d\delta dy \right]^{\frac{1}{q}}
\end{aligned} \tag{7.22}$$

and

$$\begin{aligned}
&\left[\int_0^\infty y^{\frac{q'}{p'}(s-1)+q'(A_1-A_2)} \left[\int_0^\infty K^\lambda(x,y)|f(x)| d(x^{\frac{1}{p}}) \right]^{q'} dy \right]^{\frac{1}{q'}} \\
&\leq \frac{M}{p} \left[\int_0^\infty \int_0^x x^{\frac{p}{q'}(1-s)+p(A_1-A_2)} |f'(\tau)|^p d\tau dx \right]^{\frac{1}{p}},
\end{aligned} \tag{7.23}$$

where $A_1 \in (\frac{1-s}{p'}, \frac{1}{p'})$, $A_2 \in (\frac{1-s}{q'}, \frac{1}{q'})$, and $M = k^{\frac{1}{p'}}(p'A_1) k^{\frac{1}{q'}}(q'A_2)$.

Proof. The proof follows directly from Theorem 7.6, setting $r' = (2-\lambda)p$, $r = (2-\lambda)q$, $\varphi(x) = x^{A_1}$, and $\psi(y) = y^{A_2}$ in inequalities (7.20) and (7.21). Namely, if $F(x)$ and $G(y)$ are defined as in the statement of Theorem 7.6, it follows that

$$\begin{aligned}
(\varphi F)^p(x) &= x^{pA_1} \left(\int_0^\infty K(x,y) y^{-q'A_2} dy \right)^{\frac{p}{q'}} \\
&= x^{pA_1 - pA_2} \left(\int_0^\infty K(x,y) \left(\frac{x}{y} \right)^{q'A_2} dy \right)^{\frac{p}{q'}} \\
&= x^{\frac{p}{q'}(1-s)+p(A_1-A_2)} k^{\frac{p}{q'}}(q'A_2),
\end{aligned} \tag{7.24}$$

and similarly,

$$(\psi G)^q(y) = y^{\frac{q}{p'}(1-s)+q(A_2-A_1)} k^{\frac{q}{p'}}(p'A_1). \tag{7.25}$$

Now, utilizing (7.20), (7.21), (7.24), and (7.25) we obtain (7.22) and (7.23). □

Of course, remaining results from previous two sections can also be extended to include non-conjugate parameters. Here, they are omitted.

Remark 7.3 The general Hilbert-Pachpatte-type inequalities in this chapter, as well as their consequences, are taken from [109]. For related results and some other forms of Hilbert-Pachpatte-type inequalities, the reader is referred to [28], [29], [35], [47], [105], [106], [107], and [171].

Chapter 8

General Hardy-type inequalities with non-conjugate exponents

In 1925, G. H. Hardy stated and proved in [33] the following integral inequality:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \|f\|_{L^p(\mathbb{R}_+)}, \tag{8.1}$$

where $p > 1$, and $f \in L^p(\mathbb{R}_+)$ is a non-negative function. This is the original form of the Hardy integral inequality, which later on has been extensively studied and used as a model example for investigations of more general integral inequalities.

During subsequent decades, the Hardy inequality was generalized in many different ways. Roughly speaking, the Hardy inequality was extended to what we call nowadays the general Hardy inequality, or the Hardy-type inequality,

$$\left[\int_a^b \left(\int_a^x f(y)\,dy\right)^{q'} u(x)\,dx\right]^{\frac{1}{q'}} \leq C_{p,q'}\left(\int_a^b f^p(x)v(x)\,dx\right)^{\frac{1}{p}},\ f \geq 0, \tag{8.2}$$

with parameters a,b,p,q', such that $-\infty \leq a < b \leq \infty$, $0 < q' \leq \infty$, $1 \leq p \leq \infty$, and with u,v given weight functions. The main problem in connection with the Hardy inequality is to determine conditions on the parameters p,q' and on the weight functions u,v under which the inequality holds for some classes of functions.

The Hardy inequality plays important role in various parts of mathematics, especially in functional and spectral analysis, where one investigates properties of the Hardy operator, like continuity and compactness, and also its behavior in more general function spaces. For a more details about the Hardy inequality, its history and related results, the reader is referred to [33], [69], [71], and [103].

Although classical, the Hardy inequality is still a field of interest to numerous authors. In [14] and [15], A. Čizmešija and J. Pečarić investigated finite sections of the Hardy inequality, i.e. inequalities of the same type, where the integrals are taken over certain subsets of $\mathbb{R}_+$. In such a way they obtained some generalizations and refinements of (8.1). For example, in [15], they proved that

$$\int_0^b x^{-k}\left(\int_0^x f(y)\,dy\right)^p dx \le \left(\frac{p}{k-1}\right)^p \int_0^b \left[1-\left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right] x^{p-k} f^p(x)\,dx, \tag{8.3}$$

where $0<b<\infty$, $1<p,k<\infty$, $f\ge 0$, and $x^{1-\frac{k}{p}}f\in L^p(0,b)$.

It is well known that the Hardy inequality is closely connected to the Hilbert inequality. That connection may be explained in a more general setting. Namely, Theorem 1.9 (Chapter 1) provides a unified treatment of Hilbert-type inequalities with conjugate exponents. In addition, as a consequence of the above mentioned theorem, M. Krnić and J. Pečarić [53], extended the Hardy integral inequality to cover the case when p and p' are conjugate exponents. More precisely, they obtained the following pair of equivalent inequalities:

$$\begin{aligned}
&\int_a^b\int_a^y (hg)(y)f(x)\,d\mu_1(x)d\mu_2(y)\\
&\qquad\le \left[\int_a^b \varphi^p(x)\left(\int_x^b H(y)\,d\mu_2(y)\right) f^p(x)\,d\mu_1(x)\right]^{\frac{1}{p}}\\
&\qquad\quad\times\left[\int_a^b (\psi^{p'}h)(y)\left(\int_a^y \varphi^{-p'}(x)\,d\mu_1(x)\right) g^{p'}(y)\,d\mu_2(y)\right]^{\frac{1}{p'}}
\end{aligned}\tag{8.4}$$

and

$$\begin{aligned}
&\int_a^b H(y)\left(\int_a^y \varphi^{-p'}(x)\,d\mu_1(x)\right)^{1-p}\left(\int_a^y f(x)\,d\mu_1(x)\right)^p d\mu_2(y)\\
&\qquad\le \int_a^b (\psi^{p'}h)(y)\left(\int_a^y \varphi^{-p'}(x)\,d\mu_1(x)\right) g^{p'}(y)\,d\mu_2(y),
\end{aligned}\tag{8.5}$$

where $p>1$, μ_1,μ_2 are positive σ-finite measures, h,f,g,φ,ψ are measurable, positive functions a.e. on (a,b), and $H=h\psi^{-p}$. Inequality (8.5) extends (8.1) and (8.3), as well as numerous results known from the literature. Therefore, the inequalities deduced from (8.5) will be referred to as the Hardy-type inequalities.

On the other hand, Theorem 2.1 (Chapter 2) covers the Hilbert-type inequalities with non-conjugate exponents. The main objective of this chapter is to extend the general Hardy-type inequality to the case of non-conjugate exponents. This will be done with the help of the above mentioned result regarding non-conjugate exponents.

This chapter is organized in the following way: In Section 8.1 we state and prove a pair of equivalent Hilbert and Hardy-type inequalities with non-conjugate exponents p and q, in the context of general measure spaces with positive σ-finite measures, and to the Hardy-type kernels. In Section 8.2 we discuss duality in Hardy-type inequalities, that is, we obtain dual analogues of the results in Section 8.1. Further, in Section 8.3 general results are applied to special Hardy-type kernels and power weight functions with integrals taken over intervals in $\mathbb{R}_+$. In such a way, a numerous new inequalities with explicit constant factors on their right-hand sides are obtained. In Section 8.4 we estimate some factors included in the inequalities from the previous section, depending on non-conjugate parameters and the exponents of power weight functions. Section 8.5 is dedicated to some uniform bounds for constant factors in Hardy-type inequalities. We perform a detailed analysis of optimal constants, depending on non-conjugate parameters p and q. Finally, in the last section, we synthesize the methods developed in Sections 8.3, 8.4, and 8.5.

8.1 General inequalities of the Hardy-type

In this section we prove the main result that extends relations (8.4) and (8.5) to the case of non-conjugate exponents.

Let (a,b) be an interval in $\mathbb{R}$, $T=\{(x,y)\in\mathbb{R}^2 : a<x\leq y<b\}$, and let μ_1 and μ_2 be positive $\sigma-$finite measures on (a,b). We define the Hardy-type kernel $K:(a,b)\times(a,b)\to\mathbb{R}$ by

$$K(x,y)=h(y)\chi_T(x,y)=\begin{cases} h(y) & , x\leq y, \\ 0 & , x>y, \end{cases} \tag{8.6}$$

where h is a measurable, a.e. positive function on (a,b). Further, we define the functions $F:(a,b)\to\mathbb{R}$ and $G:(a,b)\to\mathbb{R}$ by

$$\begin{aligned} F(x) &= \left[\int_x^b h(y)\psi^{-q'}(y)\,d\mu_2(y)\right]^{\frac{1}{q'}}, \quad x\in(a,b), \\ G(y) &= \left[h(y)\int_a^y \varphi^{-p'}(x)\,d\mu_1(x)\right]^{\frac{1}{p'}}, \quad y\in(a,b), \end{aligned} \tag{8.7}$$

where ψ and φ are measurable, a.e. positive functions on (a,b).

We also introduce the related Hardy-type operator by the formula

$$(Hf)(y)=\int_a^y f(x)\,d\mu_1(x), \qquad y\in(a,b). \tag{8.8}$$

Now, we are ready to state and prove the main result.

Theorem 8.1 *Let p, q, and λ be real parameters satisfying* (2.1) *and* (2.2)*, and let μ_1 and μ_2 be σ-finite measures on (a,b), $-\infty\leq a<b\leq\infty$. Let h, φ, ψ be measurable,*

a.e. positive functions on (a,b)*, and let* H *be the operator defined by* (8.8)*. If the functions* F *and* G *are defined by* (8.7)*, then the inequalities*

$$\int_a^b (h^\lambda g)(y)\,(Hf)(y)\,d\mu_2(y) \le \|\varphi F f\|_{L^p(\mu_1)} \|\psi G g\|_{L^q(\mu_2)} \tag{8.9}$$

and

$$\left\{\int_a^b (h\psi^{-q'})(y)\left[\int_a^y \varphi^{-p'}(x)\,d\mu_1(x)\right]^{-\frac{q'}{p'}} (Hf)^{q'}(y)\,d\mu_2(y)\right\}^{\frac{1}{q'}} \le \|\varphi F f\|_{L^p(\mu_1)} \tag{8.10}$$

hold for all non-negative functions f *and* g *on* (a,b)*, such that* $\varphi F f \in L^p(\mu_1)$ *and* $\psi G g \in L^q(\mu_2)$*, and they are equivalent.*

Proof. We follow the same lines as in the proof of Theorem 2.1 (see Chapter 2). Namely, the left-hand side of (8.9) can be rewritten as

$$\begin{aligned} L &= \int_a^b \int_a^y (h^\lambda g)(y) f(x)\,d\mu_1(x) d\mu_2(y) \\ &= \int_a^b \int_a^y \left[(h\psi^{-q'})(y)(\varphi^p F^{p-q'} f^p)(x)\right]^{\frac{1}{q'}} \times \left[\varphi^{-p'}(x)(h\psi^q G^{q-p'} g^q)(y)\right]^{\frac{1}{p'}} \\ &\qquad \times \left[(\varphi F f)^p(x)\,(\psi G g)^q(y)\right]^{1-\lambda} d\mu_1(x) d\mu_2(y), \end{aligned} \tag{8.11}$$

since $\frac{1}{q'}+\frac{1}{p'}+(1-\lambda)=1$. Further, utilizing the Hölder inequality, either with the parameters $q', p', \frac{1}{1-\lambda} > 1$ in the case of non-conjugate exponents p and q, or with the parameters p and p' when $q' = p$, and then applying the Fubini theorem, we obtain that L does not exceed

$$\begin{aligned} R = &\left\{\int_a^b \left[\int_x^b (h\psi^{-q'})(y)\,d\mu_2(y)\right] (\varphi^p F^{p-q'} f^p)(x)\,d\mu_1(x)\right\}^{\frac{1}{q'}} \\ &\times \left\{\int_a^b \left[h(y)\int_a^y \varphi^{-p'}(x)\,d\mu_1(x)\right] (\psi^q G^{q-p'} g^q)(y)\,d\mu_2(y)\right\}^{\frac{1}{p'}} \\ &\times \left\{\int_a^b (\psi G g)^q(y) \int_a^y (\varphi F f)^p(x)\,d\mu_1(x) d\mu_2(y)\right\}^{1-\lambda}. \end{aligned}$$

Now, exploiting definitions (8.7), the above expression can be rewritten as

$$\begin{aligned} R = &\|\varphi F f\|_{L^p(\mu_1)}^{\frac{p}{q'}} \|\psi G g\|_{L^q(\mu_2)}^{\frac{q}{p'}} \\ &\times \left\{\|\varphi F f\|_{L^p(\mu_1)}^p \|\psi G g\|_{L^q(\mu_2)}^q - \int_a^b \int_y^b (\varphi F f)^p(x)\,(\psi G g)^q(y)\,d\mu_1(x) d\mu_2(y)\right\}^{1-\lambda}. \end{aligned} \tag{8.12}$$

Of course, relations (8.11) and (8.12) yield

$$L \leq R. \tag{8.13}$$

Moreover, using (8.12), we easily obtain

$$R \leq \|\varphi F f\|_{L^p(\mu_1)}^{\frac{p}{q'}+p(1-\lambda)} \|\psi G g\|_{L^q(\mu_2)}^{\frac{q}{p'}+q(1-\lambda)} = \|\varphi F f\|_{L^p(\mu_1)} \|\psi G g\|_{L^q(\mu_2)},$$

so that (8.9) is proved.

The further step is to prove that (8.9) implies that (8.10) holds for all non-negative measurable functions f on (a,b). In particular, for any such f and the function

$$g(y) = (\psi^{-q'} h^{1-\lambda})(y) \left[\int_a^y \varphi^{-p'}(x)\, d\mu_1(x)\right]^{-\frac{q'}{p'}} (Hf)^{q'-1}(y), \quad y \in (a,b), \tag{8.14}$$

applying the Fubini theorem, the left-hand side of (8.9) becomes

$$L_f = \int_a^b (h\psi^{-q'})(y) \left[\int_a^y \varphi^{-p'}(x)\, d\mu_1(x)\right]^{-\frac{q'}{p'}} (Hf)^{q'}(y)\, d\mu_2(y),$$

that is, we get the integral on the left-hand side of (8.10), while on the right-hand side of (8.9) we have

$$\begin{aligned} R_f = \|\varphi F f\|_{L^p(\mu_1)} \Bigg\{ &\int_a^b (h\psi^{-q'})(y) \left[\int_a^y \varphi^{-p'}(x)\, d\mu_1(x)\right]^{-\frac{q'}{p'}} \\ &\times (Hf)^{q'}(y)\, d\mu_2(y) \Bigg\}^{\frac{1}{q}} = \|\varphi F f\|_{L^p(\mu_1)} L_f^{\frac{1}{q}}. \end{aligned}$$

Hence, $L_f \leq \|\varphi F f\|_{L^p(\mu_1)} L_f^{\frac{1}{q}}$, which directly yields (8.10). Conversely, utilizing the Hölder inequality for conjugate exponents q and q', together with relation (8.10) and definitions (8.7), we have

$$\begin{aligned} &\int_a^b (h^\lambda g)(y)(Hf)(y)\, d\mu_2(y) \\ &= \int_a^b (\psi G g)(y) \left[(\psi G)^{-1}(y) h^\lambda(y)(Hf)(y)\right] d\mu_2(y) \\ &\leq \|\psi G g\|_{L^q(\mu_2)} \left\{ \int_a^b (h\psi^{-q'})(y) \left[\int_a^y \varphi^{-p'}(x)\, d\mu_1(x)\right]^{-\frac{q'}{p'}} (Hf)^{q'}(y)\, d\mu_2(y) \right\}^{\frac{1}{q'}} \\ &\leq \|\varphi F f\|_{L^p(\mu_1)} \|\psi G g\|_{L^q(\mu_2)}. \end{aligned}$$

Thus, (8.10) implies (8.9), so these inequalities are equivalent. □

Remark 8.1 At the first sight, the proof of Theorem 8.1 is redundant since inequalities (8.9) and (8.10) follow from inequalities (2.9) and (2.10) (Theorem 2.1, Chapter 2) equipped with the Hardy-type kernel. On the other hand, in the proof of Theorem 8.1, we have obtained inequality (8.13) which is a refinement of inequality (8.9). Let us write that inequality once again, i.e.

$$\int_a^b (h^\lambda g)(y)\,(Hf)\,(y)\,d\mu_2(y) \le \|\varphi F f\|_{L^p(\mu_1)}^{\frac{p}{q'}} \|\psi G g\|_{L^q(\mu_2)}^{\frac{q}{p'}}$$
$$\times \left\{ \int_a^b \int_a^y (\varphi F f)^p\,(x)\,(\psi G g)^q\,(y)\,d\mu_1(x) d\mu_2(y) \right\}^{1-\lambda}. \tag{8.15}$$

Clearly, substituting the function g, defined by (8.14), in the above inequality, we obtain its equivalent Hardy-type form

$$\left\{ \int_a^b (h\psi^{-q'})(y) \left[\int_a^y \varphi^{-p'}(x)\,d\mu_1(x) \right]^{-\frac{q'}{p'}} (Hf)^{q'}\,(y)\,d\mu_2(y) \right\}^{\frac{1}{p}}$$
$$\le \|\varphi F f\|_{L^p(\mu_1)}^{\frac{p}{q'}} \cdot \left\{ \int_a^b (h\psi^{-q'})(y) \left[\int_a^y \varphi^{-p'}(x)\,d\mu_1(x) \right]^{-\frac{q'}{p'}} \right.$$
$$\left. \times (Hf)^{q'}\,(y) \int_a^y (\varphi F f)^p\,d\mu_1(x) d\mu_2(y) \right\}^{1-\lambda}. \tag{8.16}$$

Inequality (8.16) is also a slight refinement of (8.10). It should be noticed here that these refinements hold only in non-conjugate case.

Remark 8.2 Taking into account the form of Hardy-type kernel, it follows from Remark 2.3 (Chapter 2) that the equality in (8.9) and (8.10) holds only in the trivial case, that is, when $f = 0$ or $g = 0$ a.e. on (a,b). In addition, the reverse inequalities in (8.9) and (8.10) hold if the conditions (2.12) or (2.13) are fulfilled (see Remark 2.2, Chapter 2).

Remark 8.3 In the case of conjugate exponents, that is, when $q = p'$ and $\lambda = 1$, inequalities (8.9) and (8.10) reduce to relations (8.4) and (8.5). Thus, Theorem 8.1 may be regarded as an extension of the corresponding results from [53] to the case of non-conjugate exponents. Clearly, reverse inequalities in (8.4) and (8.5) hold if $0 \ne p < 1$.

8.2 General inequalities with dual Hardy-type kernel

One of important properties of the Hardy inequality is duality. Here we obtain dual analogues of the relations from the previous section.

Let $(a,b) \subseteq \mathbb{R}$, let $\widetilde{T} = \{(x,y) \in \mathbb{R}^2 : a < y \leq x < b\}$, and let μ_1 and μ_2 be positive $\sigma-$finite measures on (a,b). We define the dual Hardy-type kernel $\widetilde{K} : (a,b) \times (a,b) \to \mathbb{R}$ by

$$\widetilde{K}(x,y) = h(y)\chi_{\widetilde{T}}(x,y) = \begin{cases} h(y) & , \ x \geq y, \\ 0 & , \ x < y, \end{cases} \tag{8.17}$$

where h is a measurable function which is a.e. positive on (a,b). Moreover, we define the functions $\widetilde{F} : (a,b) \to \mathbb{R}$ and $\widetilde{G} : (a,b) \to \mathbb{R}$ by

$$\begin{aligned} \widetilde{F}(x) &= \left[\int_a^x (h\psi^{-q'})(y)\, d\mu_2(y)\right]^{\frac{1}{q'}}, \ \ x \in (a,b), \\ \widetilde{G}(y) &= \left[h(y)\int_y^b \varphi^{-p'}(x)\, d\mu_1(x)\right]^{\frac{1}{p'}}, \ \ y \in (a,b), \end{aligned} \tag{8.18}$$

where ψ and φ are measurable functions that are a. e. positive on (a,b) with respect to the corresponding σ-finite measures.

Further, the dual Hardy-type operator with respect to the operator H in (8.8) is defined in the following way:

$$(\widetilde{H}f)(y) = \int_y^b f(x)\, d\mu_1(x), \qquad y \in (a,b). \tag{8.19}$$

In this setting, we obtain a dual analogue of Theorem 8.1.

Theorem 8.2 *Let p, q, and λ be real parameters satisfying* (2.1) *and* (2.2)*, and let μ_1 and μ_2 be σ-finite measures on (a,b), $-\infty \leq a < b \leq \infty$. Let h, φ, ψ be measurable, a.e. positive functions on (a,b), and let $\widetilde{H}$ be the operator defined by* (8.19)*. If the functions $\widetilde{F}$ and $\widetilde{G}$ are defined by* (8.18)*, then the inequalities*

$$\int_a^b (h^\lambda g)(y)(\widetilde{H}f)(y)\, d\mu_2(y) \leq \|\varphi\widetilde{F}f\|_{L^p(\mu_1)}\|\psi\widetilde{G}g\|_{L^q(\mu_2)} \tag{8.20}$$

and

$$\left\{\int_a^b (h\psi^{-q'})(y)\left[\int_y^b \varphi^{-p'}(x)\, d\mu_1(x)\right]^{-\frac{q'}{p'}} (\widetilde{H}f)^{q'}(y)\, d\mu_2(y)\right\}^{\frac{1}{q'}} \leq \|\varphi\widetilde{F}f\|_{L^p(\mu_1)} \tag{8.21}$$

hold for all non-negative functions f and g on (a,b), such that $\varphi\widetilde{F}f \in L^p(\mu_1)$ and $\psi\widetilde{G}g \in L^q(\mu_2)$, and are equivalent.

Proof. It follows the same lines as the proof of Theorem 8.1. □

Remark 8.4 The equality in dual inequalities (8.20) and (8.21) holds only in the trivial case, that is, when $f = 0$ or $g = 0$ a.e. on (a,b). Moreover, the discussion about the reverse inequalities in (8.20) and (8.21) is the same as in Remark 8.2.

Remark 8.5 Similarly to Remark 8.1, one easily obtains a refinement of (8.20) in the non-conjugate case, that is,

$$\int_a^b (h^\lambda g)(y)(\widetilde{H}f)(y)\,d\mu_2(y) \le \|\varphi\widetilde{F}f\|_{L^p(\mu_1)}^{\frac{p}{q'}} \|\psi\widetilde{G}g\|_{L^q(\mu_2)}^{\frac{q}{p'}} \times \left\{ \int_a^b \int_y^b (\varphi\widetilde{F}f)^p(x)(\psi\widetilde{G}g)^q(y)\,d\mu_1(x)d\mu_2(y) \right\}^{1-\lambda}, \tag{8.22}$$

with sharp inequality for $f,g \neq 0$ a.e. on (a,b). Furthermore, inserting the function g defined by

$$g(y) = (\psi^{-q'}h^{1-\lambda})(y)\left[\int_y^b \varphi^{-p'}(x)\,d\mu_1(x)\right]^{-\frac{q'}{p'}} (\widetilde{H}f)^{q'-1}(y), \quad y \in (a,b),$$

in (8.22), we obtain the inequality

$$\begin{aligned}
&\left\{ \int_a^b (h\psi^{-q'})(y)\left[\int_y^b \varphi^{-p'}(x)\,d\mu_1(x)\right]^{-\frac{q'}{p'}} (\widetilde{H}f)^{q'}(y)\,d\mu_2(y) \right\}^{\frac{1}{p}} \\
&\le \|\varphi\widetilde{F}f\|_{L^p(\mu_1)}^{\frac{p}{q'}} \left\{ \int_a^b (h\psi^{-q'})(y)\left[\int_y^b \varphi^{-p'}(x)\,d\mu_1(x)\right]^{-\frac{q'}{p'}} \right. \\
&\quad \left. \times (\widetilde{H}f)^{q'}(y) \int_y^b (\varphi\widetilde{F}f)^p\,d\mu_1(x)d\mu_2(y) \right\}^{1-\lambda},
\end{aligned} \tag{8.23}$$

which can be regarded as a refinement of (8.21).

The most interesting case in connection with dual inequalities appears when $(a,b) \subseteq \mathbb{R}_+$. Namely, we show that Theorems 8.1 and 8.2 are equivalent in the case of Lebesgue measures.

Theorem 8.3 *Let $0 \le a < b \le \infty$ and let $d\mu_1(x) = dx$, $d\mu_2(y) = dy$. Then inequalities* (8.9) *and* (8.20) *are equivalent. Moreover, inequalities* (8.10) *and* (8.21) *are equivalent as well.*

Proof. Suppose that inequality (8.9) holds for an arbitrary interval $(a,b) \subseteq \mathbb{R}_+$ and arbitrary non-negative measurable functions φ, ψ, h, f, g on (a,b). We define $\widetilde{a} = \frac{1}{b}$ and $\widetilde{b} = \frac{1}{a}$, with conventions $\widetilde{a} = 0$ for $b = \infty$ and $\widetilde{b} = \infty$ for $a = 0$. We also define the functions $\widetilde{h}, \widetilde{\varphi}, \widetilde{\psi}, \widetilde{f}$, and $\widetilde{g}$ on $(\widetilde{a},\widetilde{b})$ by $\widetilde{h}(t) = h\left(\frac{1}{t}\right)$, $\widetilde{\varphi}(t) = t^{\frac{2}{p'}}\varphi\left(\frac{1}{t}\right)$, $\widetilde{\psi}(t) = t^{\frac{2}{q'}}\psi\left(\frac{1}{t}\right)$, $\widetilde{f}(t) = t^{-2}f\left(\frac{1}{t}\right)$, and $\widetilde{g}(t) = t^{-2}g\left(\frac{1}{t}\right)$.

Rewrite (8.9) with these new parameters. More precisely, using the change of variables $x = \frac{1}{u}$ and $y = \frac{1}{v}$, the left-hand side of (8.9) becomes

$$\int_{\widetilde{a}}^{\widetilde{b}} (\widetilde{h}^{\lambda}\widetilde{g})(y)t(H\widetilde{f})(y)\,dy = \int_a^b (h^{\lambda}g)(v)\int_v^b f(u)\,du\,dv,$$

that is, the left-hand side of inequality (8.20). Analogously, for the first factor on the right-hand side of (8.9) we have

$$\|\widetilde{\varphi}F\widetilde{f}\|^p_{L^p(\widetilde{a},\widetilde{b})} = \int_a^b (\varphi f)^p(u)\left[\int_a^u (h\psi^{-q'})(v)dv\right]^{\frac{p}{q'}} du,$$

which obviously represents the first factor on the right-hand side of inequality (8.20). The same argument holds for the second factor on the right-hand side of (8.9). Thus, inequality (8.9) implies (8.20). In the same manner, one obtains the reverse implication, so inequalities (8.9) and (8.20) are equivalent.

Finally, pairs of inequalities (8.9) and (8.10) as well as (8.20) and (8.21) are equivalent (see Theorems 8.1 and 8.2), which implies the equivalence of (8.10) and (8.21). □

8.3 Some special Hardy-type kernels and weight functions

In this section, we consider the case of Lebesgue measure for some particular Hardy-type kernels and weight functions. Namely, let $0 \le a < b \le \infty$ and let $h,\varphi,\psi : (a,b) \to \mathbb{R}$ be defined by $h(y) = \frac{1}{y}$, $\varphi(x) = x^{A_1}$, $\psi(y) = y^{A_2}$, $A_1, A_2 \in \mathbb{R}$, respectively. As it was shown in the previous section, it is sufficient to consider only the Hardy-type inequalities in Theorem 8.1, since their dual inequalities are equivalent with them.

In particular, we have to distinguish the following cases:

$$0 < a < b < \infty, \tag{8.24}$$
$$0 = a < b < \infty, \tag{8.25}$$
$$0 < a < b = \infty, \tag{8.26}$$
$$0 = a < b = \infty, \tag{8.27}$$

since one obtains different integration formulas for the functions F and G, defined by (8.7).

In particular, if $0 < a < b < \infty$, then

$$F(x) = \begin{cases} |q'A_2|^{-\frac{1}{q'}} x^{-A_2} \left|1 - \left(\frac{x}{b}\right)^{q'A_2}\right|^{\frac{1}{q'}}, & A_2 \neq 0, \\ \left(\log \frac{b}{x}\right)^{\frac{1}{q'}}, & A_2 = 0, \end{cases} \tag{8.28}$$

and

$$G(y)=\begin{cases}|1-p'A_1|^{-\frac{1}{p'}}y^{-A_1}\left|1-\left(\frac{a}{y}\right)^{1-p'A_1}\right|^{\frac{1}{p'}}, & A_1\neq\frac{1}{p'},\\ y^{-\frac{1}{p'}}\left(\log\frac{y}{a}\right)^{\frac{1}{p'}}, & A_1=\frac{1}{p'}.\end{cases}\tag{8.29}$$

It should be noticed here that we have included the cases $A_1>\frac{1}{p'}$ and $A_2<0$, by means of the modulus function. In this setting, we obtain four corollaries arising from Theorem 8.1.

If $A_1\neq\frac{1}{p'}$ and $A_2\neq 0$, then we have the following result:

Corollary 8.1 *Let $-\infty<a<b<\infty$, assume that real parameters p, q, and λ satisfy* (2.1) *and* (2.2), *let A_1, A_2 be real parameters such that $A_1\neq\frac{1}{p'}$, $A_2\neq 0$, and let H be the operator defined by* (8.8). *Then the inequalities*

$$\begin{aligned}&\int_a^b y^{-\lambda}g(y)(Hf)(y)\,dy\\ &\quad\leq\frac{|1-p'A_1|^{-\frac{1}{p'}}}{|q'A_2|^{\frac{1}{q'}}}\left[\int_a^b x^{(A_1-A_2)p}\left|1-\left(\frac{x}{b}\right)^{q'A_2}\right|^{\frac{p}{q'}}f^p(x)\,dx\right]^{\frac{1}{p}}\\ &\quad\quad\times\left[\int_a^b y^{(A_2-A_1)q}\left|1-\left(\frac{a}{y}\right)^{1-p'A_1}\right|^{\frac{q}{p'}}g^q(y)\,dy\right]^{\frac{1}{q}}\end{aligned}\tag{8.30}$$

and

$$\begin{aligned}&\left[\int_a^b y^{(A_1-A_2-\lambda)q'}\left|1-\left(\frac{a}{y}\right)^{1-p'A_1}\right|^{-\frac{q'}{p'}}(Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}}\\ &\quad\leq\frac{|1-p'A_1|^{-\frac{1}{p'}}}{|q'A_2|^{\frac{1}{q'}}}\left[\int_a^b x^{(A_1-A_2)p}\left|1-\left(\frac{x}{b}\right)^{q'A_2}\right|^{\frac{p}{q'}}f^p(x)\,dx\right]^{\frac{1}{p}}\end{aligned}\tag{8.31}$$

hold for all non-negative measurable functions f and g on (a,b), and are equivalent.

The case of $A_1\neq\frac{1}{p'}$, $A_2=0$ is described in the following result.

Corollary 8.2 *Let $-\infty<a<b<\infty$, let p, q, and λ be as in* (2.1) *and* (2.2), *assume that A_1 is a real parameter such that $A_1\neq\frac{1}{p'}$, and let H be the operator defined by* (8.8). *Then the inequalities*

$$\begin{aligned}\int_a^b y^{-\lambda}g(y)(Hf)(y)\,dy&\leq|1-p'A_1|^{-\frac{1}{p'}}\left[\int_a^b x^{A_1p}\left(\log\frac{b}{x}\right)^{\frac{p}{q'}}f^p(x)\,dx\right]^{\frac{1}{p}}\\ &\quad\times\left[\int_a^b y^{-A_1q}\left|1-\left(\frac{a}{y}\right)^{1-p'A_1}\right|^{\frac{q}{p'}}g^q(y)\,dy\right]^{\frac{1}{q}}\end{aligned}\tag{8.32}$$

and

$$\left[\int_a^b y^{(A_1-\lambda)q'}\left|1-\left(\frac{a}{y}\right)^{1-p'A_1}\right|^{-\frac{q'}{p'}}(Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}}$$
$$\leq |1-p'A_1|^{-\frac{1}{p'}}\left[\int_a^b x^{A_1p}\left(\log\frac{b}{x}\right)^{\frac{p}{q'}} f^p(x)\,dx\right]^{\frac{1}{p}} \tag{8.33}$$

hold for all non-negative measurable functions f and g on (a,b), and are equivalent.

If $A_1 = \frac{1}{p'}$ and $A_2 \neq 0$, we obtain the following corollary.

Corollary 8.3 *Let $-\infty < a < b < \infty$, let p, q, and λ be as in* (2.1) *and* (2.2)*, assume that A_2 is a real parameter such that $A_2 \neq 0$, and let H be the operator defined by* (8.8)*. Then the inequalities*

$$\int_a^b y^{-\lambda} g(y)(Hf)(y)\,dy$$
$$\leq |q'A_2|^{-\frac{1}{q'}}\left[\int_a^b x^{(1-A_2)p-1}\left|1-\left(\frac{x}{b}\right)^{q'A_2}\right|^{\frac{p}{q'}} f^p(x)\,dx\right]^{\frac{1}{p}}$$
$$\times\left[\int_a^b y^{(A_2-\frac{1}{p'})q}\left(\log\frac{y}{a}\right)^{\frac{q}{p'}} g^q(y)\,dy\right]^{\frac{1}{q}} \tag{8.34}$$

and

$$\left[\int_a^b y^{-q'A_2-1}\left(\log\frac{y}{a}\right)^{-\frac{q'}{p'}}(Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}}$$
$$\leq |q'A_2|^{-\frac{1}{q'}}\left[\int_a^b x^{(1-A_2)p-1}\left|1-\left(\frac{x}{b}\right)^{q'A_2}\right|^{\frac{p}{q'}} f^p(x)\,dx\right]^{\frac{1}{p}} \tag{8.35}$$

hold for all non-negative measurable functions f and g on (a,b), and are equivalent.

Finally, if $A_1 = \frac{1}{p'}$ and $A_2 = 0$ we have:

Corollary 8.4 *Let $-\infty < a < b < \infty$, let p, q, and λ be as in* (2.1) *and* (2.2)*, and let H be the operator defined by* (8.8)*. Then the inequalities*

$$\int_a^b y^{-\lambda} g(y)(Hf)(y)\,dy \leq \left[\int_a^b x^{p-1}\left(\log\frac{b}{x}\right)^{\frac{p}{q'}} f^p(x)\,dx\right]^{\frac{1}{p}}$$
$$\times\left[\int_a^b y^{-\frac{q}{p'}}\left(\log\frac{y}{a}\right)^{\frac{q}{p'}} g^q(y)\,dy\right]^{\frac{1}{q}} \tag{8.36}$$

and

$$\left[\int_a^b y^{-1}\left(\log\frac{y}{a}\right)^{-\frac{q'}{p'}}(Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}} \le \left[\int_a^b x^{p-1}\left(\log\frac{b}{x}\right)^{\frac{p}{q'}} f^p(x)\,dx\right]^{\frac{1}{p}} \tag{8.37}$$

hold for all non-negative measurable functions f and g on (a,b), and are equivalent.

Now, we consider the second case, that is, (8.25), where $a=0$. Then F is defined by (8.28), and

$$G(y) = (1-p'A_1)^{-\frac{1}{p'}} y^{-A_1}, \qquad y\in(0,b), \tag{8.38}$$

where $1-p'A_1>0$. In this case, we obtain the following two results, dependent on the value of the parameter A_2 ($A_2\neq 0$ or $A_2=0$).

Corollary 8.5 *Let p, q, and λ be as in* (2.1) *and* (2.2)*, let $0<b<\infty$, assume that A_1, A_2 are real parameters such that $p'A_1<1$, $A_2\neq 0$, and let H be the operator defined by* (8.8)*. Then the inequalities*

$$\begin{aligned}\int_0^b y^{-\lambda} g(y)(Hf)(y)\,dy &\le \frac{(1-p'A_1)^{-\frac{1}{p'}}}{|q'A_2|^{\frac{1}{q'}}}\left[\int_0^b x^{(A_1-A_2)p}\left|1-\left(\frac{x}{b}\right)^{q'A_2}\right|^{\frac{p}{q'}} f^p(x)\,dx\right]^{\frac{1}{p}} \\ &\quad\times\left[\int_0^b y^{(A_2-A_1)q} g^q(y)\,dy\right]^{\frac{1}{q}}\end{aligned} \tag{8.39}$$

and

$$\begin{aligned}&\left[\int_0^b y^{(A_1-A_2-\lambda)q'}(Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}} \\ &\quad\le \frac{(1-p'A_1)^{-\frac{1}{p'}}}{|q'A_2|^{\frac{1}{q'}}}\left[\int_0^b x^{(A_1-A_2)p}\left|1-\left(\frac{x}{b}\right)^{q'A_2}\right|^{\frac{p}{q'}} f^p(x)\,dx\right]^{\frac{1}{p}}\end{aligned} \tag{8.40}$$

hold for all non-negative measurable functions f and g on $(0,b)$, and are equivalent.

Corollary 8.6 *Let p, q, and λ be as in* (2.1) *and* (2.2)*, let $0<b<\infty$, assume that A_1 is a real parameter such that $p'A_1<1$, and let H be the operator defined by* (8.8)*. Then the inequalities*

$$\begin{aligned}\int_0^b y^{-\lambda} g(y)(Hf)(y)\,dy &\le (1-p'A_1)^{-\frac{1}{p'}}\left[\int_0^b x^{A_1 p}\left(\log\frac{b}{x}\right)^{\frac{p}{q'}} f^p(x)\,dx\right]^{\frac{1}{p}} \\ &\quad\times\left[\int_0^b y^{-A_1 q} g^q(y)\,dy\right]^{\frac{1}{q}}\end{aligned} \tag{8.41}$$

and

$$\left[\int_0^b y^{(A_1-\lambda)q'}(Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}}$$
$$\leq (1-p'A_1)^{-\frac{1}{p'}}\left[\int_0^b x^{A_1 p}\left(\log\frac{b}{x}\right)^{\frac{p}{q'}} f^p(x)\,dx\right]^{\frac{1}{p}} \tag{8.42}$$

hold for all non-negative measurable functions f and g on $(0,b)$, and are equivalent.

The next case (8.26) includes $b=\infty$. Then G is defined by (8.29) and

$$F(x) = (q'A_2)^{-\frac{1}{q'}}x^{-A_2}, \qquad x\in(a,\infty), \tag{8.43}$$

where $q'A_2>0$. In this setting, we obtain the following two results, depending on value of the parameter A_1 ($A_1\neq\frac{1}{p'}$ or $A_1=\frac{1}{p'}$).

Corollary 8.7 *Let $0<a<\infty$, let p, q, and λ be as in* (2.1) *and* (2.2)*, assume that A_1, A_2 are two real parameters such that $A_1\neq\frac{1}{p'}$, $q'A_2>0$, and let H be the operator defined by* (8.8)*. Then the inequalities*

$$\int_a^\infty y^{-\lambda}g(y)(Hf)(y)\,dy$$
$$\leq \frac{|1-p'A_1|^{-\frac{1}{p'}}}{(q'A_2)^{\frac{1}{q'}}}\left[\int_a^\infty x^{(A_1-A_2)p}f^p(x)\,dx\right]^{\frac{1}{p}}$$
$$\times\left[\int_a^\infty y^{(A_2-A_1)q}\left|1-\left(\frac{a}{y}\right)^{1-p'A_1}\right|^{\frac{q}{p'}} g^q(y)\,dy\right]^{\frac{1}{q}} \tag{8.44}$$

and

$$\left[\int_a^\infty y^{(A_1-A_2-\lambda)q'}\left|1-\left(\frac{a}{y}\right)^{1-p'A_1}\right|^{-\frac{q'}{p'}}(Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}}$$
$$\leq \frac{|1-p'A_1|^{-\frac{1}{p'}}}{(q'A_2)^{\frac{1}{q'}}}\left[\int_a^\infty x^{(A_1-A_2)p}f^p(x)\,dx\right]^{\frac{1}{p}} \tag{8.45}$$

hold for all non-negative measurable functions f and g on (a,∞), and are equivalent.

Corollary 8.8 *Let p, q, and λ be as in* (2.1) *and* (2.2)*, let $0<a<\infty$, assume that A_2 is a real parameter such that $q'A_2>0$, and let H be the operator defined by* (8.8)*. Then the inequalities*

$$\int_a^\infty y^{-\lambda}g(y)(Hf)(y)\,dy \leq (q'A_2)^{-\frac{1}{q'}}\left[\int_a^\infty x^{(1-A_2)p-1}f^p(x)\,dx\right]^{\frac{1}{p}}$$
$$\times\left[\int_a^\infty y^{q(A_2-\frac{1}{p'})}\left(\log\frac{y}{a}\right)^{\frac{q}{p'}} g^q(y)\,dy\right]^{\frac{1}{q}} \tag{8.46}$$

and

$$\left[\int_a^\infty y^{-q'A_2-1}\left(\log\frac{y}{a}\right)^{-\frac{q'}{p'}}(Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}} \\ \leq (q'A_2)^{-\frac{1}{q'}}\left[\int_a^\infty x^{(1-A_2)p-1}f^p(x)\,dx\right]^{\frac{1}{p}} \tag{8.47}$$

hold for all non-negative measurable functions f and g on (a,∞), and are equivalent.

Finally, we consider the case of (8.27), that is, $a=0$ and $b=\infty$. In that case, the functions F and G are defined by (8.43) and (8.38) respectively, where $1-p'A_1>0$ and $q'A_2>0$. Hence, we have only one possibility described by the following corollary.

Corollary 8.9 *Suppose p, q, and λ are as in* (2.1) *and* (2.2)*, A_1 and A_2 are real parameters such that $p'A_1<1$, $q'A_2>0$, and H is the operator defined by* (8.8)*. Then the inequalities*

$$\int_0^\infty y^{-\lambda}g(y)(Hf)(y)\,dy \leq \frac{(1-p'A_1)^{-\frac{1}{p'}}}{(q'A_2)^{\frac{1}{q'}}}\left[\int_0^\infty x^{(A_1-A_2)p}f^p(x)\,dx\right]^{\frac{1}{p}} \\ \times\left[\int_0^\infty y^{(A_2-A_1)q}g^q(y)\,dy\right]^{\frac{1}{q}} \tag{8.48}$$

and

$$\left[\int_0^\infty y^{(A_1-A_2-\lambda)q'}(Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}} \\ \leq \frac{(1-p'A_1)^{-\frac{1}{p'}}}{(q'A_2)^{\frac{1}{q'}}}\left[\int_0^\infty x^{(A_1-A_2)p}f^p(x)\,dx\right]^{\frac{1}{p}} \tag{8.49}$$

hold for all non-negative measurable functions f and g on $\mathbb{R}_+$, and are equivalent.

Remark 8.6 Some results from this section, considered in conjugate setting, can be found in [53]. Hence, the above relations may be regarded as an extension to non-conjugate setting.

8.4 Further analysis of parameters

We proceed with estimates for some constant factors included in Hardy-type inequalities from the previous section, depending on non-conjugate exponents and on the real parameters A_1 and A_2. Applying these estimates, we shall get closer to the classical Hardy inequality. More precisely, retaining the notation from the previous section, the estimates

$$\left|1-\left(\frac{a}{y}\right)^{1-p'A_1}\right| \le \left|1-\left(\frac{a}{b}\right)^{1-p'A_1}\right|, \ y\in(a,b),\ A_1\neq\frac{1}{p'}, \tag{8.50}$$

and

$$\left|1-\left(\frac{x}{b}\right)^{q'A_2}\right| \le \left|1-\left(\frac{a}{b}\right)^{q'A_2}\right|, \ x\in(a,b),\ A_2\neq 0, \tag{8.51}$$

where $0<a<b<\infty$, hold. In addition, assuming that $0<a<b<\infty$, the estimates

$$\log\frac{b}{x}\le\log\frac{b}{a}, \ x\in(a,b), \tag{8.52}$$

and

$$\log\frac{y}{a}\le\log\frac{b}{a}, \ y\in(a,b), \tag{8.53}$$

are obviously valid for the logarithm function.

Our aim here is to apply the above estimates to the results obtained in Section 8.3. In such a way, we shall simplify these inequalities by obtaining the corresponding constant factors included in the right-hand sides of inequalities. These constant factors will be expressed in terms of the function $l:\mathbb{R}\to\mathbb{R}$, defined by

$$l(\alpha)=\begin{cases}\frac{1-\left(\frac{a}{b}\right)^{\alpha}}{\alpha}, & \alpha\neq 0,\\ \log\frac{b}{a}, & \alpha=0,\end{cases} \tag{8.54}$$

where $0<a<b<\infty$. Obviously, l is a continuous function since $\lim_{\alpha\to 0} l(\alpha)=l(0)$.

Combining Corollary 8.1 and estimates (8.50), (8.51), we obtain the following pair of inequalities.

Corollary 8.10 *Suppose p, q, and λ are as in* (2.1) *and* (2.2), $-\infty<a<b<\infty$, *A_1 and A_2 are real parameters such that $A_1\neq\frac{1}{p'}$, $A_2\neq 0$, and H is the operator defined by* (8.8). *Then the inequalities*

$$\int_a^b y^{-\lambda} g(y)(Hf)(y)\,dy \le l^{\frac{1}{p'}}(1-p'A_1)\,l^{\frac{1}{q'}}(q'A_2)$$
$$\times\left[\int_a^b x^{(A_1-A_2)p} f^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_a^b y^{(A_2-A_1)q} g^q(y)\,dy\right]^{\frac{1}{q}} \tag{8.55}$$

and

$$\left[\int_a^b y^{(A_1-A_2-\lambda)q'}(Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}}$$
$$\leq l^{\frac{1}{p'}}(1-p'A_1)l^{\frac{1}{q'}}(q'A_2)\left[\int_a^b x^{(A_1-A_2)p}f^p(x)\,dx\right]^{\frac{1}{p}} \tag{8.56}$$

hold for all non-negative measurable functions f and g on (a,b), and are equivalent.

Note that inequality (8.56) has a form of the classical Hardy inequality (8.1). Now, let us compare inequalities (8.31) and (8.56). The left-hand side of inequality (8.31) is not less than the corresponding side of (8.56), while the right-hand side of (8.31) is not greater than the corresponding side of (8.56). Thus, we can regard (8.31) as both generalization and refinement of the classical Hardy inequality. The same reasoning will be valid for the remaining results of the Hardy-type in Section 8.3.

Of course, in a similar way, we obtain the results that correspond to Corollaries 8.2, 8.3 and 8.4.

Corollary 8.11 *Let $-\infty < a < b < \infty$, let p, q, and λ be as in* (2.1) *and* (2.2), *assume that A_1 is a real parameter such that $A_1 \neq \frac{1}{p'}$, and let H be the operator defined by* (8.8). *Then the inequalities*

$$\int_a^b y^{-\lambda}g(y)(Hf)(y)\,dy \leq l^{\frac{1}{p'}}(1-p'A_1)l^{\frac{1}{q'}}(0)\left[\int_a^b x^{A_1p}f^p(x)\,dx\right]^{\frac{1}{p}}$$
$$\times\left[\int_a^b y^{-A_1q}g^q(y)\,dy\right]^{\frac{1}{q}} \tag{8.57}$$

and

$$\left[\int_a^b y^{(A_1-\lambda)q'}(Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}}$$
$$\leq l^{\frac{1}{p'}}(1-p'A_1)l^{\frac{1}{q'}}(0)\left[\int_a^b x^{A_1p}f^p(x)\,dx\right]^{\frac{1}{p}} \tag{8.58}$$

hold for all non-negative measurable functions f and g on (a,b), and are equivalent.

Corollary 8.12 *Suppose p, q, and λ are as in* (2.1) *and* (2.2), *$0 < a < b < \infty$, A_2 is a real parameter such that $A_2 \neq 0$, and H is the operator defined by* (8.8). *Then the inequalities*

$$\int_a^b y^{-\lambda}g(y)(Hf)(y)\,dy \leq l^{\frac{1}{p'}}(0)l^{\frac{1}{q'}}(q'A_2)\left[\int_a^b x^{(1-A_2)p-1}f^p(x)\,dx\right]^{\frac{1}{p}}$$
$$\times\left[\int_a^b y^{(A_2-\frac{1}{p'})q}g^q(y)\,dy\right]^{\frac{1}{q}} \tag{8.59}$$

and

$$\left[\int_a^b y^{-q'A_2-1}(Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}} \leq l^{\frac{1}{p'}}(0)\,l^{\frac{1}{q'}}(q'A_2)\left[\int_a^b x^{(1-A_2)p-1}f^p(x)\,dx\right]^{\frac{1}{p}} \tag{8.60}$$

hold for all non-negative measurable functions f and g on (a,b), and are equivalent.

Corollary 8.13 *Let $0<a<b<\infty$. Suppose p, q, and λ are as in* (2.1) *and* (2.2) *and H is the operator defined by* (8.8)*. Then the inequalities*

$$\int_a^b y^{-\lambda}g(y)(Hf)(y)\,dy \leq l^{\lambda}(0)\left[\int_a^b x^{p-1}f^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_a^b y^{-\frac{q}{p'}}g^q(y)\,dy\right]^{\frac{1}{q}} \tag{8.61}$$

and

$$\left[\int_a^b y^{-1}(Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}} \leq l^{\lambda}(0)\left[\int_a^b x^{p-1}f^p(x)\,dx\right]^{\frac{1}{p}} \tag{8.62}$$

hold for all non-negative measurable functions f and g on (a,b), and are equivalent.

Remark 8.7 It is easy to see that reverse inequalities are not valid in Corollaries 8.10–8.13.

Finally, utilizing the established estimates, we can also obtain results that correspond to Corollaries 8.5 and 8.7. Since $a=0$ or $b=\infty$, we do not need to express the constant factors in terms of the function l.

Corollary 8.14 *Let $0<b<\infty$, let p, q, and λ be as in* (2.1) *and* (2.2)*, assume that A_1, A_2 are two real parameters such that $p'A_1<1$, $q'A_2>0$, and let H be the operator defined by* (8.8)*. Then the inequalities*

$$\int_0^b y^{-\lambda}g(y)(Hf)(y)\,dy \leq \frac{(1-p'A_1)^{-\frac{1}{p'}}}{(q'A_2)^{\frac{1}{q'}}}\left[\int_0^b x^{(A_1-A_2)p}f^p(x)\,dx\right]^{\frac{1}{p}} \times\left[\int_0^b y^{(A_2-A_1)q}g^q(y)\,dy\right]^{\frac{1}{q}} \tag{8.63}$$

and

$$\left[\int_0^b y^{(A_1-A_2-\lambda)q'}(Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}} \leq \frac{(1-p'A_1)^{-\frac{1}{p'}}}{(q'A_2)^{\frac{1}{q'}}}\left[\int_0^b x^{(A_1-A_2)p}f^p(x)\,dx\right]^{\frac{1}{p}} \tag{8.64}$$

hold for all non-negative measurable functions f and g on $(0,b)$, and are equivalent.

Remark 8.8 If the parameters p and q satisfy conditions (2.12) (see Remark 2.2, Chapter 2), then the inequalities (8.63) and (8.64) are reversed.

Corollary 8.15 *Suppose p, q, and λ are as in* (2.1) *and* (2.2), $0 < a < \infty$, *A_1 and A_2 are real parameters such that $p'A_1 < 1$, $q'A_2 > 0$, and H is the operator defined by* (8.8). *Then the inequalities*

$$\int_a^\infty y^{-\lambda} g(y)(Hf)(y)\,dy \leq \frac{(1-p'A_1)^{-\frac{1}{p'}}}{(q'A_2)^{\frac{1}{q'}}} \left[\int_a^\infty x^{(A_1-A_2)p} f^p(x)\,dx\right]^{\frac{1}{p}} \times \left[\int_a^\infty y^{(A_2-A_1)q} g^q(y)\,dy\right]^{\frac{1}{q}} \tag{8.65}$$

and

$$\left[\int_a^\infty y^{(A_1-A_2-\lambda)q'} (Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}} \leq \frac{(1-p'A_1)^{-\frac{1}{p'}}}{(q'A_2)^{\frac{1}{q'}}} \left[\int_a^\infty x^{(A_1-A_2)p} f^p(x)\,dx\right]^{\frac{1}{p}} \tag{8.66}$$

hold for all non-negative measurable functions f and g on (a,∞), and are equivalent.

Remark 8.9 If p and q are non-conjugate exponents which fulfill conditions (2.13) (see Remark 2.2, Chapter 2), then the inequalities (8.65) and (8.66) are reversed.

8.5 Uniform bounds of constant factors

We investigate here some further estimates for Hardy-type inequalities. First, recall that Corollary 8.10 was obtained from Corollary 8.1 by means of estimates (8.50) and (8.51). On the other hand, we may apply uniform upper bound $1 - t^x \leq 1$, $t \in (0,1)$, $x \geq 0$, to Corollary 8.1. The corresponding result, under some stronger conditions, is contained in the following corollary.

Corollary 8.16 *Let $0 < a < b < \infty$, let p, q, and λ be as in* (2.1) *and* (2.2), *assume that A_1, A_2 are two real parameters such that $p'A_1 < 1$, $q'A_2 > 0$, and let H be the operator defined by* (8.8). *Then the inequalities*

$$\int_a^b y^{-\lambda} g(y)(Hf)(y)\,dy \leq \frac{(1-p'A_1)^{-\frac{1}{p'}}}{(q'A_2)^{\frac{1}{q'}}} \left[\int_a^b x^{(A_1-A_2)p} f^p(x)\,dx\right]^{\frac{1}{p}} \times \left[\int_a^b y^{(A_2-A_1)q} g^q(y)\,dy\right]^{\frac{1}{q}} \tag{8.67}$$

and

$$\left[\int_a^b y^{(A_1-A_2-\lambda)q'}(Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}}$$
$$\leq \frac{(1-p'A_1)^{-\frac{1}{p'}}}{(q'A_2)^{\frac{1}{q'}}}\left[\int_a^b x^{(A_1-A_2)p} f^p(x)\,dx\right]^{\frac{1}{p}} \tag{8.68}$$

hold for all non-negative measurable functions f and g on (a,b), and are equivalent.

Remark 8.10 Comparing Corollaries 8.9, 8.14, and 8.15 with Corollary 8.16, we easily conclude that Corollary 8.16 holds also if $a = 0$ or $b = \infty$.

The constant factor on the right-hand sides of inequalities (8.67) and (8.68) depends on the parameters A_1 and A_2, while the corresponding integrals are dependent only on the parameter $A = A_1 - A_2$. Hence, it is interesting to consider such constant factor for a fixed value of A. Then $A_2 = A_1 - A$ and the constant factor can be regarded as a function

$$C(A_1) = (1-p'A_1)^{-\frac{1}{p'}}(q'A_1 - q'A)^{-\frac{1}{q'}}. \tag{8.69}$$

It is interesting to find the optimal value for the constant factor (8.69). More precisely, depending on the inequality sign, we find maximal or minimal values for this factor. Having in mind Remark 8.2, we have to consider three cases:

1. $p,q > 1$, $\lambda \geq 1$
   In this case we have $A < \frac{1}{p'}$ and $A_1 \in \left(A, \frac{1}{p'}\right)$, so, we have to find

   $$\inf_{A<x<\frac{1}{p'}} C(x) = \inf_{A<x<\frac{1}{p'}} (1-p'x)^{-\frac{1}{p'}}(q'x - q'A)^{-\frac{1}{q'}}.$$

   One easily obtains that $C'(x) = 0$ if and only if

   $$x_0 = \frac{1+q'A}{p'+q'}. \tag{8.70}$$

   Further, since $x_0 \in \left(A, \frac{1}{p'}\right)$ and $C''(x_0) > 0$, it follows that $C(x)$ attains its minimum value on the interval $\left(A, \frac{1}{p'}\right)$ at the point x_0. Hence, a straightforward computation gives the following value of the minimal constant factor:

   $$\inf_{A<x<\frac{1}{p'}} C(x) = C(x_0) = \left(\frac{p'\lambda}{1-p'A}\right)^{\lambda}.$$

2. $p < 0$, $q \in (0,1)$, $\lambda \geq 1$
   It is easy to see that $A_1 \in \left(-\infty, \min\left\{\frac{1}{p'}, A\right\}\right)$, so we distinguish two cases, depending on the relationship between the parameters A and $\frac{1}{p'}$.

If $A < \frac{1}{p'}$, we conclude, by a similar reasoning as in the previous case, that the function $C(x)$ attains its maximal value on the interval $(-\infty, A)$ at the point x_0 defined by (8.70). Finally, since $C(A) = 0$, we have

$$\sup_{x \le A} C(x) = C(x_0) = \left(\frac{p'\lambda}{1-p'A}\right)^{\lambda}.$$

If $A \ge \frac{1}{p'}$, then the stationary point (8.70) does not belong to the interval $\left(-\infty, \frac{1}{p'}\right)$ and $C(x)$ is strictly increasing on that interval. Further, since $\lim_{x \to \frac{1}{p'}-} C(x) = \infty$, there is no upper bound for the constant factor $C(x)$ in this case.

3. $p \in (0,1)$, $q < 0$, $\lambda \ge 1$
   Here, we have to find the optimal value of $C(x)$ on the interval $\left(\max\left\{\frac{1}{p'}, A\right\}, \infty\right)$. Similarly as above, we have to consider two cases. For $A < \frac{1}{p'}$, it follows that the function $C(x)$ attains its maximal value on the interval $\left(\frac{1}{p'}, \infty\right)$ at the point defined by (8.70). Moreover, since $C\left(\frac{1}{p'}\right) = 0$, we have

$$\sup_{x \ge \frac{1}{p'}} C(x) = C(x_0) = \left(\frac{p'\lambda}{1-p'A}\right)^{\lambda}.$$

   If $A \ge \frac{1}{p'}$, then the stationary point (8.70) is not contained in the interval (A, ∞) and $C(x)$ is strictly decreasing on that interval. Since $\lim_{x \to A+} C(x) = \infty$, there is no upper bound for the constant factor $C(x)$ in this case.

According to the previous analysis, we have just proved the following result.

Theorem 8.4 *Let $0 < a < b < \infty$, let p, q, and λ be as in* (2.1) *and* (2.2)*, assume that A is a real parameter such that $A < \frac{1}{p'}$, and let H be the operator defined by* (8.8). *Then the inequalities*

$$\int_a^b y^{-\lambda} g(y)(Hf)(y)\,dy \le \left(\frac{p'\lambda}{1-p'A}\right)^{\lambda} \left[\int_a^b x^{pA} f^p(x)\,dx\right]^{\frac{1}{p}} \times \left[\int_a^b y^{-qA} g^q(y)\,dy\right]^{\frac{1}{q}} \tag{8.71}$$

and

$$\left[\int_a^b y^{(A-\lambda)q'} (Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}} \le \left(\frac{p'\lambda}{1-p'A}\right)^{\lambda} \left[\int_a^b x^{pA} f^p(x)\,dx\right]^{\frac{1}{p}} \tag{8.72}$$

hold for all non-negative measurable functions f and g on (a,b), and are equivalent.

Remark 8.11 If $a = 0$ and p, q, λ are as in (2.12) (see Remark 2.2, Chapter 2), then the inequality signs in (8.71) and (8.72) are reversed. On the other hand, if $b = \infty$ and p, q, λ are as in (2.13) (see Remark 2.2, Chapter 2), the inequality signs in (8.71) and (8.72) are reversed as well.

8.6 Applications

Finally, in the last section, we consider some interesting special cases involving the optimal constant factor in the Hardy-type inequality established in the previous section. Namely, we shall synthesize the methods developed in Sections 8.3, 8.4 and 8.5 for such cases. By virtue of established estimates, we obtain the numerous interpolating inequalities which provide both generalizations and refinements of some recent results, known from the literature.

We can gather the previous discussion in the following two sets of inequalities:

$$\begin{aligned}
&\int_a^b y^{-\lambda} g(y)(Hf)(y)\,dy \\
&\quad\le \left(\frac{p'\lambda}{1-p'A}\right)^{\lambda} \left\{\int_a^b x^{pA}\left[1-\left(\frac{x}{b}\right)^{\frac{1-p'A}{\lambda p'}}\right]^{\frac{p}{q'}} f^p(x)\,dx\right\}^{\frac{1}{p}} \\
&\qquad\times \left\{\int_a^b y^{-qA}\left[1-\left(\frac{a}{y}\right)^{\frac{1-p'A}{\lambda p'}}\right]^{\frac{q}{p'}} g^q(y)\,dy\right\}^{\frac{1}{q}} \\
&\quad\le \left(\frac{p'\lambda}{1-p'A}\right)^{\lambda} \left[1-\left(\frac{a}{b}\right)^{\frac{1-p'A}{\lambda p'}}\right]^{\lambda} \\
&\qquad\times \left[\int_a^b x^{pA} f^p(x)\,dx\right]^{\frac{1}{p}} \left[\int_a^b y^{-qA} g^q(y)\,dy\right]^{\frac{1}{q}} \\
&\quad\le \left(\frac{p'\lambda}{1-p'A}\right)^{\lambda} \left[\int_a^b x^{pA} f^p(x)\,dx\right]^{\frac{1}{p}} \left[\int_a^b y^{-qA} g^q(y)\,dy\right]^{\frac{1}{q}}
\end{aligned} \tag{8.73}$$

and

$$\begin{aligned}
&\left[\int_a^b y^{(A-\lambda)q'}(Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}} \\
&\quad\le \left(\frac{p'\lambda}{1-p'A}\right)^{\lambda} \left[1-\left(\frac{a}{b}\right)^{\frac{1-p'A}{\lambda p'}}\right]^{\frac{1}{p'}} \\
&\qquad\times \left\{\int_a^b x^{pA}\left[1-\left(\frac{x}{b}\right)^{\frac{1-p'A}{\lambda p'}}\right]^{\frac{p}{q'}} f^p(x)\,dx\right\}^{\frac{1}{p}} \\
&\quad\le \left(\frac{p'\lambda}{1-p'A}\right)^{\lambda} \left[1-\left(\frac{a}{b}\right)^{\frac{1-p'A}{\lambda p'}}\right]^{\lambda} \left[\int_a^b x^{pA} f^p(x)\,dx\right]^{\frac{1}{p}}
\end{aligned}$$

$$\leq \left(\frac{p'\lambda}{1-p'A}\right)^{\lambda}\left[\int_a^b x^{pA} f^p(x)\,dx\right]^{\frac{1}{p}}, \tag{8.74}$$

which hold under assumptions of Theorem 8.4. Of course, these sets of inequalities are equivalent and reverse sets of inequalities hold as described in Theorem 8.4. For $A=0$, the above sets of inequalities (8.73) and (8.74) reduce respectively to

$$\begin{aligned}
&\int_a^b y^{-\lambda} g(y)(Hf)(y)\,dy \\
&\quad \leq (p'\lambda)^{\lambda}\left\{\int_a^b \left[1-\left(\frac{x}{b}\right)^{\frac{1}{\lambda p'}}\right]^{\frac{p}{q'}} f^p(x)\,dx\right\}^{\frac{1}{p}} \\
&\qquad \times \left\{\int_a^b \left[1-\left(\frac{a}{y}\right)^{\frac{1}{\lambda p'}}\right]^{\frac{q}{p'}} g^q(y)\,dy\right\}^{\frac{1}{q}} \\
&\quad \leq (p'\lambda)^{\lambda}\left[1-\left(\frac{a}{b}\right)^{\frac{1}{\lambda p'}}\right]^{\lambda}\|f\|_{L^p}\|g\|_{L^q} \leq (p'\lambda)^{\lambda}\|f\|_{L^p}\|g\|_{L^q}
\end{aligned} \tag{8.75}$$

and

$$\begin{aligned}
&\left[\int_a^b y^{-\lambda q'}(Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}} \\
&\quad \leq (p'\lambda)^{\lambda}\left[1-\left(\frac{a}{b}\right)^{\frac{1}{\lambda p'}}\right]^{\frac{1}{p'}}\left\{\int_a^b \left[1-\left(\frac{x}{b}\right)^{\frac{1}{\lambda p'}}\right]^{\frac{p}{q'}} f^p(x)\,dx\right\}^{\frac{1}{p}} \\
&\quad \leq (p'\lambda)^{\lambda}\left[1-\left(\frac{a}{b}\right)^{\frac{1}{\lambda p'}}\right]^{\lambda}\|f\|_{L^p} \leq (p'\lambda)^{\lambda}\|f\|_{L^p}.
\end{aligned} \tag{8.76}$$

Remark 8.12 It follows easily that inequalities (8.75) and (8.76), with $A=0$, are equivalent to inequalities (8.73) and (8.74), with condition $A<\frac{1}{p'}$. Namely, setting

$$a^{1-p'A},\quad b^{1-p'A},\quad x^{\frac{1}{1-p'A}-1}f\left(x^{\frac{1}{1-p'A}}\right),\quad y^{(1-\lambda)\frac{p'A}{1-p'A}}g\left(y^{\frac{1}{1-p'A}}\right)$$

in (8.75), respectively instead of $a, b, f(x), g(y)$, and then, applying the variable substitution theorem, the set of inequalities (8.75) become (8.73). So, the case of $A<\frac{1}{p'}$ is equivalent to the case of $A=0$. Thus, it is enough to observe the cases with $A=0$, since all others follow by suitable substitutions.

Finally, to conclude the chapter, we compare the results obtained in this chapter with some previously known from the literature.

Remark 8.13 Setting $a = 0$ and $b = \infty$ in (8.76), and isolating the outer expressions, we obtain the inequality

$$\left[\int_0^\infty y^{-\lambda q'} (Hf)^{q'}(y)\, dy\right]^{\frac{1}{q'}} \leq \left(p'\lambda\right)^{\lambda} \|f\|_{L^p},$$

which coincides with Opic's estimate (see [69]). Clearly, for $\lambda = 1$, we obtain the Hardy inequality (8.1) in its original form. Moreover, the above inequality in conjugate form can also be found in Kufner's paper [68]. So, the interpolating sets of inequalities established in this section may be regarded as both generalizations and refinements of the above mentioned results.

Remark 8.14 Considering the parameter $A = \lambda - \frac{k}{q'}$, $k > 1$, we have $A < \lambda - \frac{1}{q'} = \frac{1}{p'}$. Hence, setting $A = \lambda - \frac{k}{q'}$ in inequalities (8.73) and (8.74), the optimal constant factor established in Theorem 8.4 takes the following value:

$$C = \left(\frac{\lambda q'}{k-1}\right)^{\lambda}. \tag{8.77}$$

In this setting, inequalities (8.73) and (8.74) provide an extension to non-conjugate case of the corresponding results from already mentioned paper [53]. Moreover, relation (8.74) can be seen as both a refinement and an extension of the corresponding results from [14] and [15]. Namely, in the conjugate case ($p = q'$, $\lambda = 1$) with C defined by (8.77), relation (8.74) provides related results from mentioned papers (for example, see [14], Theorem 2, [15], relation (13), and also relation (8.3) at the beginning of this chapter).

Remark 8.15 A unified approach to Hardy-type inequalities with non-conjugate exponents, presented in this chapter, is developed recently in [18].

Chapter 9

Hilbert-type inequalities in the weighted Orlicz spaces

In the previous chapter we studied general Hardy-type inequalities with non-conjugate exponents. However, all derived results were related with the Lebesgue spaces. Nowadays, the Hardy inequality is investigated in more general spaces, for example in Orlicz spaces, Lorenz spaces, rearrangement invariant spaces and their weighted versions, as well as in general Banach ideal spaces. For a comprehensive survey of recent results about the Hardy inequality in Banach function spaces, the reader is referred to [71].

On the other hand, much less attention has been given to Hilbert-type inequalities in such function spaces. Recently, K. Jichang and L. Debnath [43], obtained two-dimensional Hilbert-type inequality in a weighted Orlicz spaces, including a homogeneous kernel. That result will be cited in the next section, after we present basic definitions and properties of weighted Orlicz spaces.

Our main task in this chapter is to establish a multidimensional Hilbert-type inequality in a weighted Orlicz space. In other words, we shall derive the multidimensional inequality in a weighted Orlicz space that corresponds to the classical inequality (1.73) (Theorem 1.18, Chapter 1).

In the next section we present some basic properties of Orlicz spaces, as well as the above mentioned two-dimensional Hilbert-type inequality, obtained by Jichang and Debnath [43]. Further, in Section 9.2 we state and prove the general multidimensional Hilbert-type inequality in weighted Orlicz spaces. A special emphasis is placed on inequalities including a homogeneous kernel with the negative degree of homogeneity. As an application, in Section 9.3 we derive the Hardy-Hilbert-type inequality related to (1.74) (Theorem 1.18, Chapter 1) in some particular cases. Finally, in the last section the general method regarding Orlicz spaces is applied to the case of the weighted Lebesgue spaces.

9.1 Weighted Orlicz spaces and a two-dimensional Hilbert-type inequality

An Orlicz function $\Phi : [0,\infty\rangle \to [0,\infty\rangle$ is a continuous increasing unbounded function with $\Phi(0) = 0$. Convex Orlicz functions are called Young functions. A Young function Φ is called $N-$function if moreover

$$\lim_{x\to 0^+} \frac{\Phi(x)}{x} = 0, \quad \lim_{x\to\infty} \frac{\Phi(x)}{x} = \infty.$$

For an Orlicz function Φ, a σ-finite measure space (Ω,Σ,μ), and a weight w on Ω, the weighted Orlicz space $L_w^{\Phi}(\mu)$ is defined as the space of all classes of $\mu-$measurable functions $f : \Omega \to \mathbb{R}$ such that the modular

$$\rho_{\Phi,w}\left(\frac{f}{\lambda}\right) = \int_{\Omega} \Phi\left(\frac{|f(x)|}{\lambda}\right) w(x)d\mu(x)$$

is finite for some $\lambda > 0$. Moreover, if Φ is a Young function, then the weighted Orlicz space $L_w^{\Phi}(\mu)$ becomes Banach function space with the Luxemburg-Nakano norm

$$\|f\|_{\Phi,w} = \inf\left\{\lambda > 0 \ : \ \rho_{\Phi,w}\left(\frac{f}{\lambda}\right) \le 1\right\}. \tag{9.1}$$

Recall that if $\Phi(x) = x^p$, $p > 1$, the weighted Orlicz space $L_w^{\Phi}(\mu)$ becomes the weighted Lebesgue space, denoted by $L_w^p(\mu)$, with the norm

$$\|f\|_{L_w^p(\mu)} = \left[\int_{\Omega} w(x) f^p(x) d\mu(x)\right]^{\frac{1}{p}}.$$

In addition, if $\Phi(x) = x^p\left[\log(e+x)\right]^{\alpha}$, $p \ge 1$, $\alpha > 0$, then, in non-weighted case we obtain the classical Zygmund space.

On the other hand, to Young function Φ one can associate its convex conjugate function Φ^* defined by

$$\Phi^*(y) = \sup_{x\ge 0}\left\{xy - \Phi(x)\right\}. \tag{9.2}$$

It is easy to see that the convex conjugate function Φ^* is also a Young function, as well as $(\Phi^*)^* = \Phi$. Moreover, definition of convex conjugate function provides the so called Young inequality

$$xy \le \Phi(x) + \Phi^*(y), \ x,y \ge 0. \tag{9.3}$$

Besides, inverse functions Φ^{-1} and Φ^{*-1} fulfill the following inequalities:

$$x \le \Phi^{-1}(x)\Phi^{*-1}(x) \le 2x, \ x \ge 0. \tag{9.4}$$

The right inequality in (9.4) follows immediately from (9.3), and for the other inequality the reader is referred to [8] or [91].

Further, throughout this chapter the Young function Φ is assumed to be submultiplicative (see [90]), that is,

$$\Phi(xy) \leq \Phi(x)\Phi(y), \ x,y \geq 0. \tag{9.5}$$

Condition (9.5) can be regarded as an extension of Orlicz Δ_2 condition for Young function Φ, that is, there exists a positive constant C such that $\Phi(2x) \leq C\Phi(x)$, $x \geq 0$. For more details about standard theory of Orlicz spaces the reader is referred to [114] and [184], while the weighted theory is developed in the monograph [51].

Now, we are ready to state the two-dimensional Hilbert-type inequality in the weighted Orlicz spaces, obtained by Jichang and Debnath [43].

Theorem 9.1 *Suppose $\Phi, \Phi^* : \mathbb{R}_+ \to \mathbb{R}_+$ are submultiplicative conjugate Young functions and $K : \mathbb{R}_+^2 \to \mathbb{R}$ is a non-negative homogeneous function of degree $-s$, $s > 0$. Further, let $f, g : \mathbb{R}_+ \to \mathbb{R}$ be non-negative functions such that $\|f\|_{\Phi,w} > 0$ and $\|g\|_{\Phi^*,w} > 0$, where $w(x) = x^{1-s}$. Then,*

$$\int_{\mathbb{R}_+^2} K(x,y)f(x)g(y)dxdy \leq (c_1 + c_2)\|f\|_{\Phi,w}\|g\|_{\Phi^*,w}, \tag{9.6}$$

where

$$\begin{aligned} c_1 &= \int_{\mathbb{R}_+} K(1,u)\Phi^{*-1}\left(\frac{1}{u}\right) du < \infty, \\ c_2 &= \int_{\mathbb{R}_+} K(u,1)\Phi^*\left(\frac{1}{\Phi^{-1}\left(\Phi^{*-1}(u)\right)}\right) du < \infty. \end{aligned}$$

9.2 Multidimensional Hilbert-type inequality

Concerning the statement of Theorem 9.1, we see that the authors deal with a pair of submultiplicative conjugate functions. Moreover, in the proof of Theorem 9.1 (see [43]), they used properties (9.3) and (9.4), which hold for a pair of conjugate Young functions.

Guided by that idea, we establish here a class of Young functions fulfilling the above mentioned conditions. In such a way we shall naturally extend Hilbert-type inequality (9.6) to a multidimensional case, and even more, formulate the corresponding result for an arbitrary measurable kernel.

More precisely, we assume that $\Phi_i : [0,\infty\rangle \to [0,\infty\rangle$, $i = 1,2,\ldots,n$, are Young functions satisfying the following inequality:

$$\alpha \prod_{i=1}^{n} x_i \leq \sum_{i=1}^{n} \Phi_i(x_i), \quad x_i \geq 0, \ i = 1,2,\ldots,n, \tag{9.7}$$

where $\alpha \geq 1$. In addition, the following inequality will be assumed to hold for inverses of the above Young functions:

$$x \leq \prod_{i=1}^{n} \Phi_i^{-1}(x), \quad x \geq 0. \tag{9.8}$$

Now we are ready to state and prove the general result, that is, the multidimensional Hilbert-type inequality for an arbitrary measurable kernel.

Theorem 9.2 *Suppose $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1,2,\dots,n$, are submultiplicative Young functions satisfying conditions* (9.7) *and* (9.8). *Let $(\Omega_i,\Sigma_i,\mu_i)$, $i = 1,2,\dots,n$, be σ-finite measure spaces, let $K : \prod_{i=1}^n \Omega_i \to \mathbb{R}$ be a non-negative measurable function, and let $\varphi_{ij} : \Omega_i \times \Omega_j \to \mathbb{R}$, $i \neq j$, be non-negative measurable functions satisfying the condition*

$$\prod_{\substack{i,j=1\\ i\neq j}}^{n} \varphi_{ij}(x_i,x_j) = 1. \tag{9.9}$$

Further, suppose $F_i : \Omega_i \to \mathbb{R}$ are defined by

$$F_i(x_i) = \int_{\prod_{\substack{j=1\\ j\neq i}}^{n} \Omega_j} K(x_1,x_2,\dots,x_n) \prod_{\substack{j=1\\ j\neq i}}^{n} \left[\Phi_i\left(\varphi_{ij}(x_i,x_j)\right) d\mu_j(x_j)\right]. \tag{9.10}$$

If $f_i : \Omega_i \to \mathbb{R}$ are non-negative functions such that $f_i \in L^{\Phi_i}_{F_i}(\mu_i)$ and $\|f_i\|_{\Phi_i,F_i} > 0$, then the following inequality holds:

$$\int_{\prod_{i=1}^n \Omega_i} K(x_1,x_2,\dots,x_n) \left[\prod_{i=1}^{n} f_i(x_i) d\mu_i(x_i)\right] \leq \frac{n}{\alpha} \prod_{i=1}^{n} \|f_i\|_{\Phi_i,F_i}. \tag{9.11}$$

Proof. Applying property (9.8) to the kernel $K : \prod_{i=1}^n \Omega_i \to \mathbb{R}$, we obtain that

$$K(x_1,x_2,\dots,x_n) \leq \prod_{i=1}^{n} \Phi_i^{-1}\left[K(x_1,x_2,\dots,x_n)\right].$$

Consequently, considering functions $\varphi_{ij} : \Omega_i \times \Omega_j \to \mathbb{R}$ fulfilling relation (9.9), we obtain the following inequalities

$$\begin{aligned}
&\int_{\prod_{i=1}^n \Omega_i} K(x_1,x_2,\dots,x_n) \left[\prod_{i=1}^{n} f_i(x_i) d\mu_i(x_i)\right] \\
&\leq \int_{\prod_{i=1}^n \Omega_i} \prod_{i=1}^{n} \left[\Phi_i^{-1}\left[K(x_1,x_2,\dots,x_n)\right] f_i(x_i) \prod_{\substack{j=1\\ j\neq i}}^{n} \varphi_{ij}(x_i,x_j)\right] \prod_{j=1}^{n} d\mu_j(x_j) \\
&\leq \frac{1}{\alpha} \sum_{i=1}^{n} \int_{\prod_{i=1}^n \Omega_i} \Phi_i \left[\Phi_i^{-1}\left[K(x_1,x_2,\dots,x_n)\right] f_i(x_i) \prod_{\substack{j=1\\ j\neq i}}^{n} \varphi_{ij}(x_i,x_j)\right] \prod_{j=1}^{n} d\mu_j(x_j),
\end{aligned}$$

where the second inequality sign holds due to relation (9.7). Note that in the previous relation we write $\prod_{j=1}^n d\mu_j(x_j) = d\mu_1(x_1)d\mu_2(x_2)\dots d\mu_n(x_n)$.

Further, since the Young functions $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2, \dots, n$, are submultiplicative, the Fubini theorem and definition (9.10) of functions $F_i : \Omega_i \to \mathbb{R}$ yield another inequality:

$$\begin{aligned}
&\frac{1}{\alpha}\sum_{i=1}^{n}\int_{\prod_{i=1}^{n}\Omega_i}\Phi_i\left[\Phi_i^{-1}\left[K(x_1,x_2,\dots,x_n)\right]f_i(x_i)\prod_{\substack{j=1\\ j\neq i}}^{n}\varphi_{ij}(x_i,x_j)\right]\prod_{j=1}^{n}d\mu_j(x_j)\\
&\leq \frac{1}{\alpha}\sum_{i=1}^{n}\int_{\prod_{i=1}^{n}\Omega_i}K(x_1,x_2,\dots,x_n)\Phi_i\left(f_i(x_i)\right)\left[\prod_{\substack{j=1\\ j\neq i}}^{n}\Phi_i\left(\varphi_{ij}(x_i,x_j)\right)d\mu_j(x_j)\right]d\mu_i(x_i)\\
&= \frac{1}{\alpha}\sum_{i=1}^{n}\int_{\Omega_i}\Phi_i\left(f_i(x_i)\right)\\
&\qquad\times\left[\int_{\prod_{\substack{j=1\\ j\neq i}}^{n}\Omega_j}K(x_1,x_2,\dots,x_n)\prod_{\substack{j=1\\ j\neq i}}^{n}\left[\Phi_i\left(\varphi_{ij}(x_i,x_j)\right)d\mu_j(x_j)\right]\right]d\mu_i(x_i)\\
&= \frac{1}{\alpha}\sum_{i=1}^{n}\int_{\Omega_i}\Phi_i\left(f_i(x_i)\right)F_i(x_i)d\mu_i(x_i).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\int_{\prod_{i=1}^{n}\Omega_i}K(x_1,x_2,\dots,x_n)\left[\prod_{i=1}^{n}f_i(x_i)d\mu_i(x_i)\right]\\
&\qquad\leq \frac{1}{\alpha}\sum_{i=1}^{n}\int_{\Omega_i}\Phi_i\left(f_i(x_i)\right)F_i(x_i)d\mu_i(x_i).
\end{aligned}\tag{9.12}$$

Now, replacing functions f_i in (9.12) respectively with $f_i/\|f_i\|_{\Phi_i,F_i}$, we have

$$\begin{aligned}
&\int_{\prod_{i=1}^{n}\Omega_i}K(x_1,x_2,\dots,x_n)\left[\prod_{i=1}^{n}\frac{f_i(x_i)}{\|f_i\|_{\Phi_i,F_i}}d\mu_i(x_i)\right]\\
&\qquad\leq \frac{1}{\alpha}\sum_{i=1}^{n}\int_{\Omega_i}\Phi_i\left(\frac{f_i(x_i)}{\|f_i\|_{\Phi_i,F_i}}\right)F_i(x_i)d\mu_i(x_i).
\end{aligned}$$

On the other hand, utilizing the definition of the Luxemburg-Nakano norm, it follows that

$$\int_{\Omega_i}\Phi_i\left(\frac{f_i(x_i)}{\|f_i\|_{\Phi_i,F_i}}\right)F_i(x_i)d\mu_i(x_i)\leq 1,\ i=1,2,\dots,n,$$

which yields

$$\int_{\prod_{i=1}^{n}\Omega_i}K(x_1,x_2,\dots,x_n)\left[\prod_{i=1}^{n}\frac{f_i(x_i)}{\|f_i\|_{\Phi_i,F_i}}d\mu_i(x_i)\right]\leq\frac{n}{\alpha},$$

that is, (9.11), as required. □

Our first application of Theorem 9.2 refers to a homogeneous kernel of the negative degree, together with the Lebesgue measures $d\mu_i(x_i) = dx_i$, $i = 1,2,\dots,n$, on $\mathbb{R}_+$. In that case the weighted Orlicz space $L^{\Phi}_{w}(\mu)$ will simply be denoted by L^{Φ}_{w}. Moreover, we shall deal with a more suitable form of the non-negative functions $\varphi_{ij} : \Omega_i \times \Omega_j \to \mathbb{R}$, $i \neq j$, defined in the previous theorem. In described setting, we have the following result.

Corollary 9.1 *Suppose $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1,2,\dots,n$, are submultiplicative Young functions fulfilling conditions* (9.7) *and* (9.8). *Let $K : \mathbb{R}^n_+ \to \mathbb{R}$ be a non-negative homogeneous function of degree $-s$, $s > 0$, and let $h_{ij} : \mathbb{R}_+ \to \mathbb{R}$, $i \neq j$, be non-negative measurable functions satisfying condition*

$$\prod_{\substack{i,j=1\\ i\neq j}}^{n} h_{ij}\left(\frac{x_j}{x_i}\right) = 1. \tag{9.13}$$

Further, assume that C_i, $i = 1,2,\dots,n$, are real constants defined by

$$C_i = \int_{\mathbb{R}^{n-1}_+} K(u_1,\dots u_{i-1},1,u_{i+1},\dots,u_n) \prod_{\substack{j=1\\ j\neq i}}^{n} \left[\Phi_i(h_{ij}(u_j))du_j\right], \tag{9.14}$$

and $w : \mathbb{R}_+ \to \mathbb{R}$ is the weight function, defined by $w(x) = x^{n-s-1}$.

If $f_i : \mathbb{R}_+ \to \mathbb{R}$ are non-negative functions such that $f_i \in L^{\Phi_i}_{C_i w}$ and $\|f_i\|_{\Phi_i,C_i w} > 0$, $i = 1,2,\dots,n$, then the following inequality holds:

$$\int_{\mathbb{R}^n_+} K(x_1,x_2,\dots,x_n) \left[\prod_{i=1}^{n} f_i(x_i)d\mu_i(x_i)\right] \leq \frac{n}{\alpha} \prod_{i=1}^{n} \|f_i\|_{\Phi_i,C_i w}. \tag{9.15}$$

Proof. We utilize Theorem 9.2 with the functions $\varphi_{ij} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ defined by

$$\varphi_{ij}(x_i,x_j) = h_{ij}\left(\frac{x_j}{x_i}\right), \quad i \neq j.$$

In this setting, the functions $F_i : \mathbb{R}_+ \to \mathbb{R}$ can be rewritten in the following form:

$$F_i(x_i) = \int_{\mathbb{R}^{n-1}_+} K(x_1,\dots,x_n) \prod_{\substack{j=1\\ j\neq i}}^{n} \left[\Phi_i\left(h_{ij}\left(\frac{x_j}{x_i}\right)\right)dx_j\right].$$

Now, taking into account substitutions $x_j = x_i u_j$, $j = 1,\dots,i-1,i+1,\dots,n$, and making use of the homogeneity of the kernel $K : \mathbb{R}^n_+ \to \mathbb{R}$, we have

$$\begin{aligned}
F_i(x_i) &= \int_{\mathbb{R}^{n-1}_+} K(x_i u_1,\dots x_i u_{i-1},x_i,x_i u_{i+1},\dots,x_i u_n) \prod_{\substack{j=1\\ j\neq i}}^{n} \left[\Phi_i\left(h_{ij}(u_j)\right) x_i^{n-1} du_j\right] \\
&= \int_{\mathbb{R}^{n-1}_+} x_i^{-s} K(u_1,\dots u_{i-1},1,u_{i+1},\dots,u_n) \prod_{\substack{j=1\\ j\neq i}}^{n} \left[\Phi_i\left(h_{ij}(u_j)\right) x_i^{n-1} du_j\right] \\
&= x_i^{n-s-1} \int_{\mathbb{R}^{n-1}_+} K(u_1,\dots u_{i-1},1,u_{i+1},\dots,u_n) \prod_{\substack{j=1\\ j\neq i}}^{n} \left[\Phi_i\left(h_{ij}(u_j)\right) du_j\right] \\
&= C_i w(x_i),
\end{aligned}$$

which completes the proof. □

Remark 9.1 It is not hard to find non-negative functions $h_{ij} : \mathbb{R}_+ \to \mathbb{R}$, $i \neq j$, $i, j \in \{1, 2, \ldots, n\}$, fulfilling relation (9.13). Namely, if β_{ij}, $i \neq j$, $i, j \in \{1, 2, \ldots, n\}$, are positive real numbers satisfying

$$\sum_{\substack{i=1 \\ i \neq k}}^{n} \beta_{ik} = \sum_{\substack{j=1 \\ j \neq k}}^{n} \beta_{kj}, \quad k = 1, 2, \ldots, n, \tag{9.16}$$

then the functions $h_{ij}(t) = t^{\beta_{ij}}$, $i \neq j$, obviously fulfill (9.13), since

$$\prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\frac{x_j}{x_i} \right)^{\beta_{ij}} = 1.$$

In the previous corollary referring to a homogeneous kernel of degree $-s$, $s > 0$, we obtained the Hilbert-type inequality for weighted Orlicz spaces where the weight functions were multiples of the particular weight $w(x) = x^{n-s-1}$. On the other hand, utilizing the same method as in the proof of Theorem 9.2, one can obtain the inequality which includes the weighted Orlicz spaces with the same weight function. This is the content of the following theorem.

Theorem 9.3 *Under the assumptions of Corollary* 9.1*, inequality*

$$\int_{\mathbb{R}_+^n} K(x_1, x_2, \ldots, x_n) \left[\prod_{i=1}^{n} f_i(x_i) d\mu_i(x_i) \right] \leq \frac{\sum_{i=1}^{n} C_i}{\alpha} \prod_{i=1}^{n} \|f_i\|_{\Phi_i, w} \tag{9.17}$$

holds for all non-negative functions $f_i : \mathbb{R}_+ \to \mathbb{R}$ *such that* $f_i \in L_w^{\Phi_i}$ *and* $\|f_i\|_{\Phi_i, w} > 0$, $i = 1, 2, \ldots, n$.

Proof. We follow the same procedure as in the proof of Theorem 9.2 and take into account the specific form of functions $F_i : \mathbb{R}_+ \to \mathbb{R}$ deduced in Corollary 9.1, that is $F_i(x_i) = C_i w(x_i)$, $i = 1, 2, \ldots, n$. In this setting, inequality (9.12) can be rewritten as

$$\int_{\mathbb{R}_+^n} K(x_1, x_2, \ldots, x_n) \left[\prod_{i=1}^{n} f_i(x_i) d\mu_i(x_i) \right] \leq \frac{1}{\alpha} \sum_{i=1}^{n} C_i \int_{\mathbb{R}_+} \Phi_i \left(f_i(x_i) \right) w(x_i) dx_i.$$

Now, replacing functions f_i in the above inequality respectively with $f_i / \|f_i\|_{\Phi_i, w}$, $i = 1, 2, \ldots, n$, we have

$$\int_{\mathbb{R}_+^n} K(x_1, x_2, \ldots, x_n) \left[\prod_{i=1}^{n} \frac{f_i(x_i)}{\|f_i\|_{\Phi_i, w}} dx_i \right] \leq \frac{1}{\alpha} \sum_{i=1}^{n} C_i \int_{\mathbb{R}_+} \Phi_i \left(\frac{f_i(x_i)}{\|f_i\|_{\Phi_i, w}} \right) w(x_i) dx_i.$$

Finally, since

$$\int_{\mathbb{R}_+} \Phi_i \left(\frac{f_i(x_i)}{\|f_i\|_{\Phi_i, w}} \right) w(x_i) dx_i \leq 1,$$

definition of the Luxemburg-Nakano norm provides the inequality

$$\int_{\mathbb{R}_+^n} K(x_1,x_2,\ldots,x_n)\left[\prod_{i=1}^{n}\frac{f_i(x_i)}{\|f_i\|_{\Phi_i,w}}dx_i\right] \leq \frac{\sum_{i=1}^{n} C_i}{\alpha},$$

and the proof is completed. □

Remark 9.2 If $n=2$, then the constants C_1 and C_2 included in the inequality (9.17) reduce to

$$C_1 = \int_{\mathbb{R}_+} K(1,u_2)\Phi_1(h_{12}(u_2))du_2 \quad \text{and} \quad C_2 = \int_{\mathbb{R}_+} K(u_1,1)\Phi_2(h_{21}(u_1))du_1.$$

Now, considering a pair of conjugate functions, that is, $\Phi_1 = \Phi$ and $\Phi_2 = \Phi^*$, and defining $h_{12}(u_2) = \Phi^{-1}\left(\Phi^{*-1}\left(\frac{1}{u_2}\right)\right)$, condition (9.13) can be rewritten in the form

$$\Phi^{-1}\left(\Phi^{*-1}(u_1)\right)h_{21}(u_1) = 1, \quad u_1 = \frac{x_1}{x_2} > 0,$$

yielding an explicit formula for the function h_{21}:

$$h_{21}(u_1) = \frac{1}{\Phi^{-1}\left(\Phi^{*-1}(u_1)\right)}.$$

Thus, if $n=2$, $\alpha=1$ and $h_{12}(u_2) = \Phi^{-1}\left(\Phi^{*-1}\left(\frac{1}{u_2}\right)\right)$, inequality (9.17) coincides with inequality (9.6). Therefore, inequality (9.17) may be regarded as a multidimensional extension of (9.6).

9.3 A version of Hardy-Hilbert-type inequality

In this section we derive Hardy-Hilbert-type inequalities associated to Hilbert-type inequalities from the previous section, in a particular case. Namely, in the sequel we obtain Hardy-Hilbert versions of inequalities (9.11), (9.15), and (9.17) assuming that one of Young functions provides the weighted Lebesgue space.

A Hardy-Hilbert-type inequality that corresponds to (9.11) in the above described setting is a content of the following result.

Corollary 9.2 *Let $\frac{1}{r}+\frac{1}{r'}=1$, $r>1$, and let $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i=1,2,\ldots,n$, be submultiplicative Young functions satisfying conditions* (9.7) *and* (9.8), *where $\Phi_n(x) = x^{r'}$. Further, suppose $(\Omega_i,\Sigma_i,\mu_i)$, $i=1,2,\ldots,n$, are σ-finite measure spaces, $K : \prod_{i=1}^{n}\Omega_i \to \mathbb{R}$ is a nonnegative measurable function, and $F_i : \Omega_i \to \mathbb{R}$, $i=1,2,\ldots,n$, are defined by* (9.10).

If $f_i : \Omega_i \to \mathbb{R}$, $i = 1,2,\dots,n-1$, are non-negative functions such that $f_i \in L^{\Phi_i}_{F_i}(\mu_i)$ and $\|f_i\|_{\Phi_i,F_i} > 0$, then

$$\left[\int_{\Omega_n} F_n^{1-r}(x_n)\left[\int_{\prod_{i=1}^{n-1}\Omega_i} K(x_1,x_2,\dots,x_n)\prod_{i=1}^{n-1} f_i(x_i)d\mu_i(x_i)\right]^r d\mu_n(x_n)\right]^{\frac{1}{r}} \leq \frac{n}{\alpha}\prod_{i=1}^{n-1}\|f_i\|_{\Phi_i,F_i}, \tag{9.18}$$

provided the integrals on the left-hand side of the inequality converge.

Proof. Let I denote the left-hand side of (9.18). If we define the function $f_n : \Omega_n \to \mathbb{R}$ by

$$f_n(x_n) = F_n^{1-r}(x_n)\left[\int_{\prod_{i=1}^{n-1}\Omega_i} K(x_1,x_2,\dots,x_n)\prod_{i=1}^{n-1} f_i(x_i)d\mu_i(x_i)\right]^{r-1}, \tag{9.19}$$

then, utilizing the Fubini theorem, the r-th power of I can be rewritten as

$$I^r = \int_{\prod_{i=1}^{n}\Omega_i} K(x_1,x_2,\dots,x_n)\prod_{i=1}^{n} f_i(x_i)d\mu_i(x_i).$$

Now, the Hilbert-type inequality (9.11) yields

$$I^r \leq \frac{n}{\alpha}\prod_{i=1}^{n}\|f_i\|_{\Phi_i,F_i}. \tag{9.20}$$

On the other hand, the Young function $\Phi_n(x) = x^{r'}$ provides the corresponding weighted Lebesgue space $L^{r'}_{F_n}(\mu_n)$. Moreover, taking into account (9.19), we obtain

$$\|f_n\|_{\Phi_n,F_n} = \left[\int_{\Omega_n} f_n^{r'}(x_n)F_n(x_n)d\mu_n(x_n)\right]^{\frac{1}{r'}} = I^{\frac{r}{r'}}. \tag{9.21}$$

Finally, relations (9.20) and (9.21) yield inequality

$$I^{r-\frac{r}{r'}} \leq \frac{n}{\alpha}\prod_{i=1}^{n-1}\|f_i\|_{\Phi_i,F_i},$$

that is, (9.18), since $r - \frac{r}{r'} = 1$. □

The following two corollaries refer to homogeneous kernels. Namely, as a special case, we obtain Hardy-Hilbert-type inequalities that correspond to Hilbert-type inequalities (9.15) and (9.17).

Corollary 9.3 *Let $\frac{1}{r} + \frac{1}{r'} = 1$, $r > 1$, and let $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1,2,\dots,n$, be submultiplicative Young functions fulfilling conditions* (9.7) *and* (9.8)*, where $\Phi_n(x) = x^{r'}$. Let*

$K:\mathbb{R}_+^n\to\mathbb{R}$ be a non-negative homogeneous function of degree $-s$, $s>0$, and let C_i, $i=1,2,\dots,n$, be real constants defined by (9.14).

If $f_i:\mathbb{R}_+\to\mathbb{R}$, $i=1,2,\dots,n-1$, are non-negative functions such that $f_i\in L^{\Phi_i}_{C_iw}$ and $\|f_i\|_{\Phi_i,C_iw}>0$, where $w(x)=x^{n-s-1}$, then

$$\left[\int_{\mathbb{R}_+} x_n^{(n-1-s)(1-r)}\left[\int_{\mathbb{R}_+^{n-1}} K(x_1,x_2,\dots,x_n)\prod_{i=1}^{n-1} f_i(x_i)dx_i\right]^r dx_n\right]^{\frac{1}{r}} \leq \frac{nC_n^{1/r'}}{\alpha}\prod_{i=1}^{n-1}\|f_i\|_{\Phi_i,C_iw}, \tag{9.22}$$

provided the integrals on the left-hand side of the inequality converge.

Corollary 9.4 *Under the assumptions of Corollary* 9.3*, the inequality*

$$\left[\int_{\mathbb{R}_+} x_n^{(n-1-s)(1-r)}\left[\int_{\mathbb{R}_+^{n-1}} K(x_1,x_2,\dots,x_n)\prod_{i=1}^{n-1} f_i(x_i)dx_i\right]^r dx_n\right]^{\frac{1}{r}} \leq \frac{\sum_{i=1}^n C_i}{\alpha}\prod_{i=1}^{n-1}\|f_i\|_{\Phi_i,w} \tag{9.23}$$

holds for all non-negative functions $f_i:\mathbb{R}_+\to\mathbb{R}$ such that $f_i\in L^{\Phi_i}_w$ and $\|f_i\|_{\Phi_i,w}>0$, $i=1,2,\dots,n-1$.

9.4 Some examples in the weighted Lebesgue spaces

In order to conclude this chapter, we provide here some remarks about reduction to the case of the weighted Lebesgue spaces. More precisely, we consider the method developed in Section 9.2 in the case of the weighted Lebesgue spaces.

For that sake, we assume that Φ_i are Young functions defined by $\Phi_i(x_i)=x_i^{p_i}$, $i=1,2,\dots,n$, where $p_i>1$, $i=1,2,\dots,n$, are conjugate exponents, that is, $\sum_{i=1}^n \frac{1}{p_i}=1$.

The above power functions define the appropriate weighted Lebesgue spaces. Moreover, since $p_i>1$, $i=1,2,\dots,n$, are conjugate exponents, the classical Young inequality implies

$$\sum_{i=1}^n x_i^{p_i}\geq\prod_{i=1}^n p_i^{\frac{1}{p_i}}\prod_{i=1}^n x_i,$$

and hence, according to condition (9.7), we can take $\alpha=\prod_{i=1}^n p_i^{\frac{1}{p_i}}$. In addition, since the inverses of Young functions $\Phi_i(x_i) = x_i^{p_i}$ are respectively $\Phi_i^{-1}(x_i) = x_i^{\frac{1}{p_i}}$, we have

$$\prod_{i=1}^{n} \Phi_i^{-1}(x) = \prod_{i=1}^{n} x^{\frac{1}{p_i}} = x^{\sum_{i=1}^{n} \frac{1}{p_i}} = x,$$

which means that condition (9.8) is fulfilled as well. Finally, the above power functions are multiplicative, so the assumptions as in Section 9.2 are also fulfilled.

Remark 9.3 Regarding the above setting, inequality (9.15) can be reduced to a form which includes Lebesgue spaces with the same weight function, as inequality (9.17). Namely, using the notation from Corollary 9.1 and taking into account the above power Young functions, we have

$$\|f_i\|_{\Phi_i,C_iw} = \|f_i\|_{L^{p_i}_{C_iw}} = \left[\int_{\mathbb{R}_+} f_i^{p_i}(x_i) C_i w(x_i) dx_i\right]^{\frac{1}{p_i}} = C_i^{\frac{1}{p_i}} \|f_i\|_{L^{p_i}_w},$$

for $i = 1,2,\ldots,n$. In such a way we obtain an inequality related to (9.17), but with a different constant factor. Moreover, in described setting, inequalities (9.15) and (9.17) yield the inequality

$$\int_{\mathbb{R}_+^n} K(x_1,x_2,\ldots,x_n) \prod_{i=1}^{n} f_i(x_i) dx_i$$
$$\leq \min\left\{ \frac{n\prod_{i=1}^{n} C_i^{\frac{1}{p_i}}}{\prod_{i=1}^{n} p_i^{\frac{1}{p_i}}}, \frac{\sum_{i=1}^{n} C_i}{\prod_{i=1}^{n} p_i^{\frac{1}{p_i}}} \right\} \prod_{i=1}^{n} \|f_i\|_{L^{p_i}_w}. \tag{9.24}$$

Note that we cannot decide which constant factor is smaller.

Remark 9.4 We know from previous sections that the constant factors C_i involved in inequality (9.24) can be explicitly computed for some particular choices of the kernel K and the functions $h_{ij} : \mathbb{R}_+ \to \mathbb{R}$, $i \neq j$, $i, j \in \{1,2,\ldots,n\}$, fulfilling relation (9.13).

For example, considering inequality (9.24) with the kernel $K(x_1,x_2,\ldots,x_n) = (\sum_{i=1}^{n} x_i)^{-s}$, $s > 0$, and the power functions $h_{ij}(t) = t^{\beta_{ij}}$, $i \neq j$, where the parameters β_{ij} fulfill relations (9.16), the above constant factors can be rewritten as

$$C_i = \int_{\mathbb{R}_+^{n-1}} \frac{\prod_{j=1, j\neq i}^{n} u_j^{p_i\beta_{ij}} du_j}{\left(1 + \sum_{j=1, j\neq i}^{n} u_j\right)^s}.$$

Now, taking into account Lemma 1.3 (Section 1.6, Chapter 1), the above integral can be expressed in terms of the usual Gamma function, that is,

$$C_i = \frac{\Gamma\left(s - n + 1 - p_i \sum_{j=1, j\neq i}^{n} \beta_{ij}\right) \prod_{j=1, j\neq i}^{n} \Gamma(p_i\beta_{ij} + 1)}{\Gamma(s)},$$

provided that $s - n + 1 - p_i \sum_{j=1, j\neq i}^{n} \beta_{ij} > 0$ and $p_i\beta_{ij} + 1 > 0$, $j \neq i$.

Remark 9.5 Multidimensional Hilbert-type inequalities in weighted Orlicz spaces, introduced in this chapter, are derived recently in [64].

Chapter 10

Some particular inequalities

In this book we have established a unified treatment of Hilbert-type inequalities in both conjugate and non-conjugate case. We have derived numerous inequalities involving diverse choices of function spaces, sets of integration, kernels and weight functions. We have also presented several methods for refinements of Hilbert-type inequalities.

Finally, in this last section we review some particular results, interesting on its own right, which are closely connected with the theory exposed in this book. More precisely, we give here some related Hilbert-type inequalities as well as some other refinements of Hilbert-type inequalities known from the literature. The following recent results are cited without proofs and are listed in the chronological order. For more details, the reader should consult the corresponding literature.

10.1. G. Mingzhe [95], 1997.

If $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are non-negative sequences such that $\sum_{n=0}^{\infty} a_n^2 < \infty$ and $\sum_{n=0}^{\infty} b_n^2 < \infty$, then

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{m+n+1} \le \left[\sum_{n=0}^{\infty}\omega(n)a_n^2\right]^{\frac{1}{2}}\left[\sum_{n=0}^{\infty}\omega(n)b_n^2\right]^{\frac{1}{2}}, \tag{10.1}$$

where the weight coefficient ω is defined by $\omega(n) = \pi - \frac{\theta(n)}{\sqrt{2n+1}}$ and

$$\theta(n) = 2\sqrt{2n+1}\arctan\left(\frac{1}{\sqrt{2n+1}}\right) - \frac{2n+1}{n+1} > 0$$

for $n \geq 0$. The equality sign in (10.1) holds if and only if $(a_n)_{n\in\mathbb{N}}$ or $(b_n)_{n\in\mathbb{N}}$ is a zero-sequence.

10.2. G. Mingzhe, B. Yang [97], 1998.

Let $q \geq p > 1$ be conjugate exponents. If $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \left[\sum_{n=1}^{\infty}\left(\frac{\pi}{\sin\frac{\pi}{p}} - \frac{\lambda}{n^{\frac{1}{p}}}\right)a_n^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}\left(\frac{\pi}{\sin\frac{\pi}{p}} - \frac{\lambda}{n^{\frac{1}{q}}}\right)b_n^q\right]^{\frac{1}{q}}, \tag{10.2}$$

where $\lambda = 1-\gamma$ and $\gamma = 0.57721566\ldots$ is the Euler constant. In addition, λ is the largest constant that keeps (10.2) valid and is independent of p and q.

10.3. B. Yang [145], 2000.

Let $\frac{1}{p}+\frac{1}{q} = 1$, $p > 1$, and let $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ be non-negative sequences such that $0 < \sum_{n=0}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=0}^{\infty} b_n^q < \infty$. Then the following two inequalities hold and are equivalent:

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{m+n+1} &< \left[\sum_{n=0}^{\infty}\left(\frac{\pi}{\sin\frac{\pi}{p}} - \frac{1}{13(n+1)(2n+1)^{\frac{1}{p}}}\right)a_n^p\right]^{\frac{1}{p}}\\ &\quad\times\left[\sum_{n=0}^{\infty}\left(\frac{\pi}{\sin\frac{\pi}{p}} - \frac{1}{13(n+1)(2n+1)^{\frac{1}{q}}}\right)b_n^q\right]^{\frac{1}{q}}\end{aligned} \tag{10.3}$$

and

$$\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty}\frac{a_m b_n}{m+n+1}\right)^p < \left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^{p-1}\sum_{n=0}^{\infty}\left[\frac{\pi}{\sin\frac{\pi}{p}} - \frac{1}{13(n+1)(2n+1)^{\frac{1}{p}}}\right]a_n^p. \tag{10.4}$$

10.4. M. Bencze, C. J. Zhao [6], 2002.

Let $p \geq 1$, $q \geq 1$ be real parameters and let k, r, e be positive integers. If $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ are non-negative sequences and $A_m = \sum_{s=1}^{m} a_s$, $B_n = \sum_{t=1}^{n} b_t$, then

$$\begin{aligned}&\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{A_m^p B_n^q (mn)^{\frac{2}{e}}}{(m\cdot n^{\frac{1}{e}})^2 + (n\cdot m^{\frac{1}{e}})^2}\\ &\leq \frac{1}{2}pq(kr)^{\frac{e-1}{e}}\left[\sum_{m=1}^{k}(k-m+1)(a_m A_m^{p-1})^e\right]^{\frac{1}{e}}\left[\sum_{n=1}^{r}(r-n+1)(b_n B_n^{q-1})^e\right]^{\frac{1}{e}}.\end{aligned} \tag{10.5}$$

10.5. B. He, Y. Li [35], 2006.

Let $m \geq 1$, $n \geq 1$, $p_i > 1$, $\frac{1}{p_i} + \frac{1}{q_i} = 1$ for $i = 0, 1, 2, 3, 4$, $p_0 = p$, $p_3 = k$, $p_4 = r$ $q_0 = q$, $q_3 = l$, $q_4 = \omega$. Further, suppose $f, g : \mathbb{R}_+ \to \mathbb{R}$ are non-negative measurable functions and let $F(s) = \int_0^s f(\sigma)d\sigma$ and $G(t) = \int_0^t g(\tau)d\tau$. If

$$0 < \int_0^\infty s^{p(1-\frac{\lambda}{r})-1} F_f^p(s)ds < \infty, \quad 0 < \int_0^\infty t^{q(1-\frac{\lambda}{\omega})-1} G_g^q(t)dt < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{F^m(s)G^n(t)}{(ls^{\frac{k}{p_1}} + kt^{\frac{l}{p_2}})(s^\lambda + t^\lambda)} dsdt$$
$$\leq E_1(m,n,k,r,\lambda) \left[\int_0^\infty s^{p(1-\frac{\lambda}{r})-1} F_f^p(s)ds\right]^{\frac{1}{p}} \left[\int_0^\infty t^{q(1-\frac{\lambda}{\omega})-1} G_g^q(t)dt\right]^{\frac{1}{q}}, \quad (10.6)$$

where $E_1(m,n,k,r,\lambda) = \frac{\pi mn}{\lambda kl \sin(\pi/r)}$,

$$F_f(s) = \left[\int_0^s (F^{m-1}(\sigma)f(\sigma))^{q_1} d\sigma\right]^{\frac{1}{q_1}}, \quad \text{and} \quad G_g(t) = \left[\int_0^t (G^{m-1}(\tau)g(\tau))^{q_2} d\tau\right]^{\frac{1}{q_2}}.$$

10.6. B. He, Y. Li [34], 2006.

Let $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, and let $f, g : \mathbb{R}_+ \to \mathbb{R}$ be non-negative measurable functions such that

$$0 < \int_0^\infty (x+1)^{p-1} f^p(x)dx < \infty \quad \text{and} \quad 0 < \int_0^\infty (x+1)^{q-1} g^q(x)dx < \infty.$$

Then the inequalities

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\log(x+1) + \log(y+1) + 1} dxdy$$
$$< \frac{\pi}{\sin\frac{\pi}{p}} \left[\int_0^\infty \omega(x,p) f^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty \omega(x,q) g^q(x)dx\right]^{\frac{1}{q}} \quad (10.7)$$

and

$$\int_0^\infty (\omega(y,q))^{1-p} \left[\int_0^\infty \frac{f(x)}{\log(x+1) + \log(y+1) + 1} dx\right]^p dy$$
$$< \left[\frac{\pi}{\sin\frac{\pi}{p}}\right]^p \int_0^\infty \omega(x,p) f^p(x)dx \quad (10.8)$$

hold and are equivalent, where $\omega(x,r) = [1 - \frac{1 - \frac{s\sin(\pi/r)}{\pi}}{(\log(x+1)+1)^{1/r}}](x+1)^{r-1}$, $r = p, q$, $s = \frac{r}{r-1}$. In addition, the constant factors included on the right-hand sides of inequalities (10.7) and (10.8) are the best possible.

10.7. J. Weijian, G. Mingzhe [130], 2006.

Let

$$\omega(r,x) = x^{(1+x)(1-r)}\left(\frac{1}{2}-\varphi(x)\right)^{r-1},$$

where $r > 1$ and $\varphi(x) = \frac{1-x+x\log x}{2(1+x+x\log x)}$, $x \in \mathbb{R}_+$. If $p \geq q > 1$ are conjugate parameters and $f,g : \mathbb{R}_+ \to \mathbb{R}$ are non-negative measurable functions such that

$$0 < \int_0^\infty \omega(p,x) f^p(x)dx < +\infty, \quad 0 < \int_0^\infty \omega(q,x) g^q(x)dx < +\infty,$$

then

$$\begin{aligned}&\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{ax^{1+x}+by^{1+y}}dxdy\\ &\leq \frac{\mu\pi}{\sin\frac{\pi}{p}}\left[\int_0^\infty \omega(p,x)f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty \omega(q,x)g^q(x)dx\right]^{\frac{1}{q}},\end{aligned} \tag{10.9}$$

where $\mu = (\frac{1}{a})^{\frac{1}{q}}(\frac{1}{b})^{\frac{1}{p}}$. Moreover, the constant factor $\frac{\mu\pi}{\sin\frac{\pi}{p}}$ is the best possible.

10.8. W. T. Sulaiman [122], 2006.

Let $\frac{1}{p}+\frac{1}{q} = 1$, $p > 1$, and let $\lambda > \max\{p,q\}$. Further, suppose $f,g : \mathbb{R}_+ \to \mathbb{R}$ are non-negative functions such that $f(0) = g(0) = 0$, $f(\infty) = g(\infty) = \infty$, $f'(s) \geq 0$, $g'(s) \geq 0$, $s \in \{x^p, y^q\}$, and let $\log f$, $\log g$ be convex functions. If

$$0 < \int_0^\infty \frac{t^{-\frac{p^2}{q^2}}[f(t^p)]^{2-\lambda+\frac{p}{q}}}{[f'(t)]^{\frac{p}{q}}}dt < \infty, \quad 0 < \int_0^\infty \frac{t^{-\frac{q^2}{p^2}}[g(t^q)]^{2-\lambda+\frac{q}{p}}}{[g'(t)]^{\frac{q}{p}}}dt < \infty,$$

then

$$\begin{aligned}&\int_0^\infty\int_0^\infty \frac{f(xy)g(xy)}{(f(x^p)+g(y^q))^\lambda}dxdy\\ &\leq \frac{1}{\sqrt[p]{p}\sqrt[q]{q}}B^{\frac{1}{p}}(p,\lambda-p)B^{\frac{1}{q}}(q,\lambda-q)\\ &\quad\times\left[\int_0^\infty \frac{t^{-\frac{p^2}{q^2}}[f(t^p)]^{2-\lambda+\frac{p}{q}}}{[f'(t)]^{\frac{p}{q}}}dt\right]^{\frac{1}{p}}\left[\int_0^\infty \frac{t^{-\frac{q^2}{p^2}}[g(t^q)]^{2-\lambda+\frac{q}{p}}}{[g'(t)]^{\frac{q}{p}}}dt\right]^{\frac{1}{q}},\end{aligned} \tag{10.10}$$

where B is the usual Beta function.

10.9. W. Wang, D. Xin [129], 2006.

Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\alpha \geq e^{\frac{7}{6}}$, and $s,\, t \in \mathbb{R}$. If $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are non-negative real sequences such that $0<\sum_{n=1}^{\infty}(n^{\frac{1}{q}-s}a_n)^p<\infty$ and $0<\sum_{n=1}^{\infty}(n^{\frac{1}{p}-t}b_n)^q<\infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^s n^t \log \alpha mn} < \left[\sum_{n=1}^{\infty}\left(\frac{\pi}{\sin\frac{\pi}{p}}-\frac{3(p-1)}{8(2\log n+1)^{\frac{1}{p}}}\right)(n^{\frac{1}{q}-s}a_n)^p\right]^{\frac{1}{p}} \times \left[\sum_{n=1}^{\infty}\left(\frac{\pi}{\sin\frac{\pi}{p}}-\frac{3(q-1)}{8(2\log n+1)^{\frac{1}{q}}}\right)(n^{\frac{1}{p}-s}b_n)^q\right]^{\frac{1}{q}}. \tag{10.11}$$

10.10. Z. Lü, H. Xie [88], 2007.

Let $-c\leq a<b<\infty$ and let $f,g:[a,b]\to\mathbb{R}$ be non-negative measurable functions such that $\int_a^b (x+c)f^2(x)dx<\infty$ and $\int_a^b (y+c)g^2(y)dy<\infty$. Then the following inequalities hold and are equivalent:

$$\int_a^b\int_a^b \frac{f(x)g(y)}{\log(x+c)(y+c)}dxdy \leq \left(\pi-4\arctan\sqrt[4]{\frac{\log(a+c)}{\log(b+c)}}\right)\left[\int_a^b (x+c)f^2(x)dx\int_a^b (y+c)g^2(y)dy\right]^{\frac{1}{2}} \tag{10.12}$$

and

$$\int_a^b\left[\int_a^b \frac{f(x)}{\log(x+c)(y+c)}dx\right]^2 dy \leq \left(\pi-4\arctan\sqrt[4]{\frac{\log(a+c)}{\log(b+c)}}\right)^2\int_a^b (x+c)f^2(x)dx. \tag{10.13}$$

10.11. Z. Lü, G. Mingzhe, L. Debnath [87], 2007.

Suppose F is defined as

$$F(s,t)=\|\alpha\|^2 s^2-2\langle\alpha,\beta\rangle st+\|\beta\|^2 t^2, \tag{10.14}$$

where α, β, and γ belong to inner product space E, $\langle\alpha,\beta\rangle$ indicates the inner product of vectors α and β, γ is the unit-vector, α and β are not simultaneously orthogonal to γ, and $s=\langle\beta,\gamma\rangle$ and $t=\langle\alpha,\gamma\rangle$.

Further, suppose $f,\, g\in L^2(\mathbb{R}_+)$, $\phi(x)$, $\psi(x)$ are differentiable functions in $\mathbb{R}_+$, and $\phi(0)=\psi(0)\geq 0$, $\phi(\infty)=\psi(\infty)=\infty$, $\phi'(x)$, $\psi'(x)>0$. If the functions $\phi'(x)$ and $\psi'(x)$

have positive infimums, then

$$
\begin{aligned}
&\left[\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\phi(x)+\psi(y)}dxdy\right]^2\\
&\leq \frac{1}{\inf\{\phi'(x)\}\inf\{\psi'(y)\}}\int_0^\infty\left[\pi-\arctan\sqrt{\frac{\psi(0)}{\phi(x)}}\right]f^2(x)dx\\
&\times\int_0^\infty\left[\pi-\arctan\sqrt{\frac{\phi(0)}{\psi(y)}}\right]g^2(y)dy-F(s,t),
\end{aligned}
\tag{10.15}
$$

where $F(s,t)$ is defined by (10.14) and $F(s,t)>0$.

10.12. H. Leping, G. Xuemei, G. Mingzhe [78], 2008.

If $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are non-negative sequences such that $\sum_{n=1}^\infty a_n^2<+\infty$ and $\sum_{n=1}^\infty b_n^2<+\infty$, then

$$
\begin{aligned}
\left[\sum_{m=1}^\infty\sum_{n=1}^\infty\frac{a_mb_n}{m+n}\right]^4 &\leq \pi^4\left[\left(\sum_{n=1}^\infty a_n^2\right)^2-\left(\sum_{n=1}^\infty\omega(n)a_n^2\right)^2\right]\\
&\times\left[\left(\sum_{n=1}^\infty b_n^2\right)^2-\left(\sum_{n=1}^\infty\omega(n)b_n^2\right)^2\right],
\end{aligned}
\tag{10.16}
$$

where the weight function $\omega(n)$ is defined by

$$
\omega(n)=\frac{\sqrt{n}}{n+1}\left(\frac{\sqrt{n}-1}{\sqrt{n}+1}-\frac{\log n}{\pi}\right).
$$

10.13. H. Leping, G. Mingzhe, Z. Yu [77], 2008.

Suppose f and g are non-negative real functions such that $0<\int_\alpha^\infty(x-\alpha)^{1-\lambda}f^2(x)dx<\infty$ and $0<\int_\alpha^\infty(x-\alpha)^{1-\lambda}g^2(x)dx<\infty$ for $\lambda>\frac{1}{2}$. Then,

$$
\begin{aligned}
&\left[\int_\alpha^\infty\int_\alpha^\infty\frac{f(x)g(y)}{(x-\alpha)^\lambda+(y-\alpha)^\lambda}dxdy\right]^4\\
&\leq\left(\frac{\pi}{\lambda\sin\frac{\pi}{2\lambda}}\right)^4\left[\left(\int_\alpha^\infty(x-\alpha)^{1-\lambda}f^2(x)dx\right)^2-\left(\int_\alpha^\infty\omega_\lambda(x)f^2(x)dx\right)^2\right]\\
&\times\left[\left(\int_\alpha^\infty(x-\alpha)^{1-\lambda}g^2(x)dx\right)^2-\left(\int_\alpha^\infty\omega_\lambda(x)g^2(x)dx\right)^2\right],
\end{aligned}
\tag{10.17}
$$

where the weight function $\omega_\lambda(x)$ is defined by

$$
\omega_\lambda(x)=\begin{cases}(x-\alpha)^{1-\lambda}\left[\frac{(x-\alpha)^{\lambda-1/2}-1}{(x-\alpha)^\lambda-1}-\frac{1}{1+(x-\alpha)^\lambda}\right], & x-\alpha\neq 1,\\ \frac{1}{2}-\frac{1}{2\lambda}, & x-\alpha=1.\end{cases}
$$

10.14. W. T. Sulaiman [124], 2008.

Suppose $p,q>1$ are conjugate parameters. A non-negative mapping $f:\mathbb{R}_+\to\mathbb{R}$ is called (p,q)-Hölder type on $\mathbb{R}_+$ if the inequality $f(xy)\le f^{\frac{1}{p}}(x^p)f^{\frac{1}{q}}(y^q)$ holds for all $x,y\in\mathbb{R}_+$.

Let f and F be non-negative functions of (p,q)-Hölder type on $\mathbb{R}_+$ such that $f(0)=0$ and $f(\infty)=\infty$. Further, suppose that f' exist and is strictly positive on $\mathbb{R}_+$. If $\lambda>0$, $\mu>1$, $\max\{\frac{p-1}{q},\frac{q-1}{p}\}<\frac{\mu-1}{\lambda}<\min\{\frac{p}{q},\frac{q}{p}\}$, and

$$K=B^{\frac{1}{p}}(\lambda p-\mu q+q,\lambda-\lambda p+\mu q-q)B^{\frac{1}{q}}(\lambda q-\mu p+p,\lambda-\lambda q+\mu p-p),$$

where B is the Beta function, then

$$\begin{aligned}&\int_0^\infty\int_0^\infty F(xy)\left(\frac{f(xy)}{1+f(xy)}\right)^\lambda dxdy\\&\le K\left[\int_0^\infty\frac{(f(x^p))^{(pq-2)\mu-\lambda-\frac{p}{q}}F(x^p)}{x^{\frac{p(p-1)}{q}}(f'(x^p))^{\frac{p}{q}}}dx\right]^{\frac{1}{p}}\left[\int_0^\infty\frac{(f(y^q))^{(pq-2)\mu-\lambda-\frac{q}{p}}F(y^q)}{y^{\frac{q(q-1)}{p}}(f'(y^q))^{\frac{q}{p}}}dy\right]^{\frac{1}{q}},\end{aligned}\tag{10.18}$$

provided that integrals on the right-hand side of the inequality converge.

10.15. Z. Yu, G. Xuemei, G. Mingzhe [175], 2009.

Let $\lambda>0$ and let f, g be non-negative functions such that $0<\int_0^\infty x^{1-\lambda}f^2(x)dx<\infty$ and $0<\int_0^\infty x^{1-\lambda}g^2(x)dx<\infty$. If m is a positive integer, then

$$\begin{aligned}&\int_0^\infty\int_0^\infty\frac{(\log x-\log y)^{2m-1}f(x)g(y)}{x^\lambda-y^\lambda}dxdy\\&\qquad<C_B\left[\int_0^\infty x^{1-\lambda}f^2(x)dx\right]^{\frac{1}{2}}\left[\int_0^\infty x^{1-\lambda}g^2(x)dx\right]^{\frac{1}{2}},\end{aligned}\tag{10.19}$$

where the constant factor C_B is defined by

$$C_B=\frac{2^{2m-1}(2^{2m}-1)}{m}\left(\frac{\pi}{\lambda}\right)^{2m}B_m.$$

Here, B_m are the Bernoulli numbers, namely $B_1=\frac{1}{6}$, $B_2=\frac{1}{30}$, $B_3=\frac{1}{42}$, $B_4=\frac{1}{30}$, $B_5=\frac{5}{66}$, and so forth. Moreover, the constant factor C_B on the right-hand side of (10.19) is the best possible.

10.16. W. Yang [169], 2009.

Let $p \geq 1$, $q \geq 1$, $\alpha > 1$, $\gamma > 1$ be real parameters and let k, r be positive integers. If $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ are non-negative sequences and $A_m = \sum_{s=1}^{m} a_s$, $B_n = \sum_{t=1}^{n} b_t$, then

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{A_m^p B_n^q}{\gamma m^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}}+\alpha n^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} \leq C(p,q,k,r;\alpha,\gamma)$$
$$\times\left[\sum_{m=1}^{k}(k-m+1)(A_m^{p-1}a_m)^{\alpha}\right]^{\frac{1}{\alpha}}\left[\sum_{n=1}^{r}(r-n+1)(B_n^{q-1}b_n)^{\gamma}\right]^{\frac{1}{\gamma}}, \quad (10.20)$$

where $C(p,q,k,r;\alpha,\gamma) = \frac{pq}{\alpha+\gamma}k^{\frac{\alpha-1}{\alpha}}r^{\frac{\gamma-1}{\gamma}}$.

10.17. W. Yang [170], 2009.

Let $q_i \geq 1$, $p_i > 1$, $p = \sum_{i=1}^{n}\frac{1}{p_i}$, and $\alpha_i = \prod_{j=1,j\neq i}^{n} p_j$, $i = 1,2,\ldots,n$. Further, suppose a_{i,m_i}, $i = 1,2,\ldots,n$, are non-negative sequences defined for $m_i = 1,2,\ldots,k_i$, where k_i are positive integers and let $A_{i,m_i} = \sum_{s_i=1}^{m_i} a_{i,s_i}$, $i = 1,2,\ldots,n$. Then,

$$\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\frac{\prod_{i=1}^{n}A_{i,m_i}^{q_i}}{\sum_{i=1}^{n}\alpha_i m_i^{(p_i-1)p}}$$
$$\leq C(k_1,\ldots,k_n)\prod_{i=1}^{n}\left[\sum_{m_i=1}^{k_i}(k_i-m_i+1)(A_{i,m_i}^{q_i-1}a_{i,m_i})^{p_i}\right]^{\frac{1}{p_i}}, \quad (10.21)$$

where $C(k_1,\ldots,k_n) = \frac{\prod_{i=1}^{n}q_i}{\sum_{i=1}^{n}\alpha_i}\prod_{i=1}^{n}k_i^{\frac{p_i-1}{p_i}}$.

10.18. W. Yang [171], 2009.

Let us define the operator ∇ by $\nabla u(t) = u(t) - u(t-1)$ for a function $u : \mathbb{N}_0 \to \mathbb{R}$. Further, define the operators $\nabla_1 v(s,t) = v(s,t) - v(s-1,t)$, $\nabla_2 v(s,t) = v(s,t) - v(s,t-1)$, and $\nabla_2\nabla_1 v(s,t) = \nabla_2(\nabla_1 v(s,t)) = \nabla_1(\nabla_2 v(s,t))$ for a function $v : \mathbb{N}_0\times\mathbb{N}_0 \to \mathbb{R}$.

Suppose $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, and $a,b : \mathbb{N}_0\times\mathbb{N}_0 \to \mathbb{R}$ are such that $a(0,t) = b(0,t) = a(s,0) = b(s,0) = 0$. If $0 < \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\tau=1}^{m}\sum_{\delta=1}^{n}|\nabla_2\nabla_1 a(\tau,\delta)|^p < \infty$ and $0 < \sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\sum_{k=1}^{s}\sum_{r=1}^{t}|\nabla_2\nabla_1 b(k,r)|^q < \infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\frac{|a(m,n)||b(s,t)|}{(q(mn)^{p-1}+p(st)^{q-1})(m+s)(n+t)}$$
$$< \frac{\pi^2}{pq\sin^2\frac{\pi}{p}}\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\tau=1}^{m}\sum_{\delta=1}^{n}|\nabla_2\nabla_1 a(\tau,\delta)|^p\right]^{\frac{1}{p}}$$
$$\times\left[\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\sum_{k=1}^{s}\sum_{r=1}^{t}|\nabla_2\nabla_1 b(k,r)|^q\right]^{\frac{1}{q}}. \quad (10.22)$$

10.19. C. J. Zhao, W. S. Cheung [179], 2009.

Let $p \geq 1$, $q \geq 1$, $t > 0$, and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, $\alpha > 1$. Further, suppose $a_{m_1,\dots,m_n}$ and $b_{n_1,\dots,n_n}$ are non-negative sequences defined for $m_i = 1,2,\dots,k_i$, and $n_i = 1,2,\dots,r_i$, where k_i and r_i, $i = 1,\dots,n$, are positive integers. If $A_{m_1,\dots,m_n} = \sum_{s_1=1}^{m_1} \cdots \sum_{s_n=1}^{m_n} a_{s_1,\dots,s_n}$ and $B_{n_1,\dots,n_n} = \sum_{t_1=1}^{n_1} \cdots \sum_{t_n=1}^{n_n} b_{t_1,\dots,t_n}$, then

$$\begin{aligned}
&\sum_{m_1=1}^{k_1} \cdots \sum_{m_n=1}^{k_n} \sum_{n_1=1}^{r_1} \cdots \sum_{n_n=1}^{r_n} \frac{\alpha\beta t^{\frac{1}{\beta}} A_{m_1,\dots,m_n}^p B_{n_1,\dots,n_n}^q}{m_1 \cdots m_n \beta + n_1 \cdots n_n \alpha t} \\
&\leq L(k_1,\dots,k_n,r_1,\dots,r_n,p,q,\alpha,\beta) \\
&\times \left[\sum_{m_1=1}^{k_1} \cdots \sum_{m_n=1}^{k_n} \prod_{j=1}^{n} (k_j - m_j + 1) \left(a_{m_1,\dots,m_n} A_{m_1,\dots,m_n}^{p-1} \right)^{\beta} \right]^{\frac{1}{\beta}} \\
&\times \left[\sum_{n_1=1}^{r_1} \cdots \sum_{n_n=1}^{r_n} \prod_{j=1}^{n} (r_j - n_j + 1) \left(b_{n_1,\dots,n_n} B_{n_1,\dots,n_n}^{q-1} \right)^{\alpha} \right]^{\frac{1}{\alpha}}, \qquad (10.23)
\end{aligned}$$

where

$$L(k_1,\dots,k_n,r_1,\dots,r_n,p,q,\alpha,\beta) = pq(k_1 \cdots k_n)^{\frac{1}{\alpha}} (r_1 \cdots r_n)^{\frac{1}{\beta}}.$$

10.20. Q. Huang [38], 2010.

Suppose $p_i, r_i > 1$, $i = 1,\dots,n$, are real parameters such that $\sum_{i=1}^{n} \frac{1}{p_i} = \sum_{i=1}^{n} \frac{1}{r_i} = 1$ and $\frac{1}{q_n} = 1 - \frac{1}{p_n}$. Further, let α, β, and λ be real parameters such that $\lambda > 0$, $0 < \alpha < 2$, $\beta \geq -\frac{1}{2}$, and $\lambda\alpha \max\{\frac{1}{2-\alpha}, 1\} \leq \min_{1 \leq i \leq n}\{r_i\}$. Then the following two equivalent inequalities

$$\begin{aligned}
&\sum_{m_n=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} \frac{1}{[\sum_{i=1}^{n} (m_i + \beta)^{\alpha}]^{\lambda}} \prod_{i=1}^{n} a_{m_i}^{(i)} \\
&\qquad < \frac{\alpha^{1-n}}{\Gamma(\lambda)} \prod_{i=1}^{n} \Gamma\left(\frac{\lambda}{r_i}\right) \left[\sum_{m_i=1}^{\infty} (m_i + \beta)^{p_i(1-\frac{\lambda\alpha}{r_i})-1} \left(a_{m_i}^{(i)}\right)^{p_i} \right]^{\frac{1}{p_i}}, \qquad (10.24)
\end{aligned}$$

$$\begin{aligned}
&\left[\sum_{m_n=1}^{\infty} (m_n + \beta)^{\frac{\lambda\alpha q_n}{r_n}-1} \left(\sum_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} \frac{\prod_{i=1}^{n-1} a_{m_i}^{(i)}}{[\sum_{i=1}^{n} (m_i + \beta)^{\alpha}]^{\lambda}} \right)^{q_n} \right]^{\frac{1}{q_n}} \\
&\qquad < \frac{\Gamma\left(\frac{\lambda}{r_n}\right)}{\alpha^{n-1}\Gamma(\lambda)} \prod_{i=1}^{n-1} \Gamma\left(\frac{\lambda}{r_i}\right) \left[\sum_{m_i=1}^{\infty} (m_i + \beta)^{p_i(1-\frac{\lambda\alpha}{r_i})-1} \left(a_{m_i}^{(i)}\right)^{p_i} \right]^{\frac{1}{p_i}} \qquad (10.25)
\end{aligned}$$

hold for all non-negative sequences $(a_{m_i}^{(i)})_{m_i \in \mathbb{N}}$, provided that

$$0 < \sum_{m_i=1}^{\infty} (m_i + \beta)^{p_i(1-\frac{\lambda\alpha}{r_i})-1} \left(a_{m_i}^{(i)}\right)^{p_i} < \infty \quad i = 1,\dots,n.$$

Here, Γ is the Gamma function.

10.21. N. Das, S. Sahoo [20], 2010.

Suppose $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let λ, r, s be real parameters such that $0<\lambda\leq 4$, $0<r,s<\min\{2,\lambda\}$, and $r+s=\lambda$. If $A_n=\sum_{k=1}^{n} a_k$ and $B_n=\sum_{k=1}^{n} b_k$, where $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ are non-negative sequences fulfilling $0<\sum_{n=1}^{\infty} a_n^p<\infty$ and $0<\sum_{n=1}^{\infty} b_n^q<\infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^{r-\frac{1}{q}-1}n^{s-\frac{1}{p}-1}}{(m+n)^{\lambda}}A_mB_n<pqB(r,s)\left[\sum_{n=1}^{\infty}a_n^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}b_n^q\right]^{\frac{1}{q}}, \tag{10.26}$$

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{m^{r-\frac{1}{q}-1}n^{s-\frac{1}{p}}}{(m+n)^{\lambda}}A_m\right)^p<(qB(r,s))^p\sum_{n=1}^{\infty}a_n^p, \tag{10.27}$$

where B is the Beta function. In addition, the constant factors $pqB(r,s)$ and $(qB(r,s))^p$ are the best possible in the above equivalent inequalities.

10.22. X. Liu, B. Yang [85], 2010.

Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $\lambda,\lambda_1,\lambda_2$ be real parameters such that $\lambda_1+\lambda_2=\lambda<2$. Further, suppose $k_\lambda:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ is a non-negative homogeneous function of degree $-\lambda$ so that $0<k(\lambda_1)=\int_0^\infty k(u,1)u^{\lambda_1-1}du<\infty$ holds for all $\lambda_1\in(\lambda-1,1)$. If $f,g:\mathbb{R}_+\to\mathbb{R}$ are non-negative functions and $\widetilde{\varphi}(x)=x^{p(2-\lambda-\lambda_1)-1}$, $\psi(y)=y^{q(1-\lambda_2)-1}$,

$$\widetilde{F}_\lambda(x)=\int_x^\infty\frac{1}{t^\lambda}f(t)dt,\quad \widetilde{G}_\lambda(y)=\int_y^\infty\frac{1}{t^\lambda}g(t)dt,$$

so that $0<\|f\|_{L^p_{\widetilde{\varphi}}}=[\int_0^\infty\widetilde{\varphi}(x)|f(x)|^pdx]^{\frac{1}{p}}<\infty$ and $0<\|\widetilde{G}_\lambda\|_{L^q_\psi}<\infty$, then the inequalities

$$\int_0^\infty\int_0^\infty k_\lambda(x,y)\widetilde{F}_\lambda(x)\widetilde{G}_\lambda(y)dxdy<\frac{k(\lambda_1)}{1-\lambda_1}\|f\|_{L^p_{\widetilde{\varphi}}}\|\widetilde{G}_\lambda\|_{L^q_\psi} \tag{10.28}$$

and

$$\left\{\int_0^\infty\psi^{1-p}(y)\left[\int_0^\infty k_\lambda(x,y)\widetilde{F}_\lambda(x)dx\right]^pdy\right\}^{\frac{1}{p}}<\frac{k(\lambda_1)}{1-\lambda_1}\|f\|_{L^p_{\widetilde{\varphi}}} \tag{10.29}$$

hold and are equivalent.

10.23. S. K. Sunanda, C. Nahak, S. Nanda [126], 2010.

Let $(p_k)_{k\in\mathbb{N}}$ and $(q_k)_{k\in\mathbb{N}}$ be non-negative bounded sequences such that $\frac{1}{p_k}+\frac{1}{q_k}=1$, where $p_k>1$ for all $k\in\mathbb{N}$. Then the inequality

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_mb_n}{\log(mn)}<\alpha\sup_{k\geq 1}\left[\frac{\pi}{\sin\frac{\pi}{p_k}}\left(\sum_{n=2}^{\infty}n^{p_k-1}a_n^{p_k}\right)^{\frac{1}{p_k}}\left(\sum_{n=2}^{\infty}n^{q_k-1}b_n^{q_k}\right)^{\frac{1}{q_k}}\right] \tag{10.30}$$

holds for all non-negative sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$, such that $0<\sum_{n=2}^{\infty} n^{p_k-1}a_n^{p_k}<\infty$, $0<\sum_{n=2}^{\infty} n^{q_k-1}b_n^{q_k}<\infty$. Here,

$$\alpha=\sup_{k\ge 1}\frac{1}{p_k}+\sup_{k\ge 1}\frac{1}{q_k}.$$

10.24. Z. Zeng, Z. Xie [176], 2010.

Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let

$$k=\frac{4\pi\sin\frac{r(\beta-\alpha)}{2}\cos\frac{r(\pi-\alpha-\beta)}{2}}{r\cos\frac{r\pi}{2}},$$

where $-1<r<0$ and $0<\alpha<\beta<\pi$. Then the inequalities

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(y)\left|\log\frac{x^2+2xy\cos\alpha+y^2}{x^2+2xy\cos\beta+y^2}\right|dxdy\\ &<k\left[\int_{-\infty}^{\infty}|x|^{p(1+r)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}|x|^{q(1-r)-1}g^q(x)dx\right]^{\frac{1}{q}}\end{aligned}\tag{10.31}$$

and

$$\begin{aligned}&\int_{-\infty}^{\infty}|y|^{pr-1}\left[\int_{-\infty}^{\infty} f(x)\left|\log\frac{x^2+2xy\cos\alpha+y^2}{x^2+2xy\cos\beta+y^2}\right|dx\right]^p dy\\ &<k^p\int_{-\infty}^{\infty}|x|^{p(1+r)-1}f^p(x)dx\end{aligned}\tag{10.32}$$

hold for all non-negative measurable functions $f,g:\mathbb{R}\to\mathbb{R}$, provided that $0<\int_{-\infty}^{\infty}|x|^{p(1+r)-1}f^p(x)dx<\infty$ and $0<\int_{-\infty}^{\infty}|x|^{q(1-r)-1}g^q(x)dx<\infty$. Moreover, the above inequalities are equivalent and include the best possible constant factors on their right-hand sides.

10.25. J. Jin, L. Debnath [45], 2010.

For $p>0$, $n_0\in\mathbb{Z}$, $w(n)\ge 0$, $n\ge n_0$, $n\in\mathbb{Z}$, we define the set of sequences l_{w,n_0}^p by

$$l_{w,n_0}^p=\left\{(a_n)_{n\ge n_0};\|a\|_{p,w}=\Big(\sum_{n=n_0}^{\infty}w(n)|a_n|^p\Big)^{\frac{1}{p}}<\infty\right\}.$$

For conjugate parameters $r,s>1$ denote by $H(r,s)$ the set of all non-negative functions $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ fulfilling the following conditions:
(i) $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ is continuous and decreasing in each variable,
(ii) K is homogeneous of degree -1,
(iii) there exist a constant $M>0$ such that $\lim_{t\to 0^+}K(1,t)=M$. For sufficiently small $\varepsilon\ge 0$ the integral $K_l(\varepsilon)=\int_0^{\infty}K(1,t)t^{-\frac{1+\varepsilon}{l}}dt$, $l=r,s$, exists, where $K_l(0)=C_r$ is a positive constant and $K_l(\varepsilon)=C_r+o(1)$ as $\varepsilon\to 0^+$.

Further, let $F_{n_0}(r)$, $r>1$, $n_0\in\mathbb{Z}$, denote the set of all real-valued functions $\phi(x)$ such that:
(i) $\phi(x)$ is continuously differentiable and strictly increasing on (n_0-1,∞),
(ii) $\phi((n_0-1)+)=0$, $\phi(\infty)=\infty$, $\phi'(x)[\phi(x)]^{-\frac{1}{p}}$ is decreasing on (n_0-1,∞).

Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $r>1$, $\frac{1}{r}+\frac{1}{s}=1$, $m_0,n_0\in\mathbb{Z}$, $K\in H(r,s)$, $\phi\in F_{m_0}(r)$, $\psi\in F_{n_0}(s)$. Setting $w_1(m)=[\phi(m)]^{\frac{p}{r}-1}[\phi'(m)]^{1-p}$, $w_2(n)=[\psi(n)]^{\frac{q}{s}-1}[\psi'(n)]^{1-q}$, $\widetilde{w}_1(n)=\psi'(n)[\psi(n)]^{\frac{p}{r}-1}$, $\widetilde{w}_2(m)=\phi'(m)[\phi(m)]^{\frac{q}{s}-1}$, we define the sequence operator T as follows: for $a\in l^p_{w_1,m_0}$,

$$(Ta)(n)=\sum_{m=m_0}^{\infty}K(\phi(m),\psi(n))a_m,\ n\geq n_0, n\in\mathbb{Z},$$

or, for $b\in l^q_{w_2,n_0}$,

$$(Tb)(m)=\sum_{n=n_0}^{\infty}K(\phi(m),\psi(n))b_n,\ m\geq m_0, m\in\mathbb{Z}.$$

Then we have

$$\|Ta\|_{p,\widetilde{w}_1}\leq C_r\|a\|_{p,w_1}\ \text{ and }\ \|Tb\|_{q,\widetilde{w}_2}\leq C_r\|b\|_{q,w_2}. \tag{10.33}$$

Moreover,

$$\|T\|_p=\|T\|_q=C_r=\int_0^{\infty}K(1,t)t^{-\frac{1}{r}}dt, \tag{10.34}$$

where

$$\|T\|_p=\sup_{a\in l^p_{w_1,m_0}}\frac{\|Ta\|_{p,\widetilde{w}_1}}{\|a\|_{p,w_1}}\ \text{ and }\ \|T\|_q=\sup_{b\in l^q_{w_2,n_0}}\frac{\|Tb\|_{q,\widetilde{w}_2}}{\|b\|_{q,w_2}}.$$

10.26. L. E. Azar [5], 2011.

Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let a,b,c,s,A_1,A_2 be real parameters such that $a,c>0$, $b^2<ac$, $s>0$, $A_1\in(\frac{1-2s}{q},\frac{1}{q})$, $A_2\in(\frac{1-2s}{p},\frac{1}{p})$, and let

$$L_1=a^{\frac{1-pA_2}{2}-s}c^{\frac{pA_2-1}{2}}B(1-pA_2,2s+pA_2-1)F\Big(\frac{1-pA_2}{2},s-\frac{1-pA_2}{2},s+\frac{1}{2};1-\frac{b^2}{ac}\Big),$$

$$L_2=a^{\frac{qA_1-1}{2}}c^{\frac{1-qA_1}{2}-s}B(1-qA_1,2s+qA_1-1)F\Big(\frac{1-qA_1}{2},s-\frac{1-qA_1}{2},s+\frac{1}{2};1-\frac{b^2}{ac}\Big),$$

where B and F respectively denote the usual Beta and the hypergeometric function (see relation (2.53), Chapter 2).

Further, suppose $u,v:(a,b)\to\mathbb{R}$, $-\infty\leq a<b\leq\infty$, are non-negative differentiable strictly increasing functions fulfilling conditions $\lim_{t\to a^+}u(t)=\lim_{t\to a^+}v(t)=0$ and $\lim_{t\to b^-}u(t)=\lim_{t\to b^-}v(t)=\infty$. Then the inequalities

$$\int_a^b\int_a^b\frac{f(x)g(y)}{(au^2(x)+2bu(x)v(y)+cv^2(y))^s}dxdy$$
$$<L_1^{\frac{1}{p}}L_2^{\frac{1}{q}}\left[\int_a^b\frac{u(x)^{1-2s+p(A_1-A_2)}}{u'(x)^{p-1}}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_a^b\frac{v(y)^{1-2s+q(A_2-A_1)}}{v'(y)^{q-1}}g^q(y)dy\right]^{\frac{1}{q}} \tag{10.35}$$

and

$$\int_a^b v(y)^{(2s-1)(p-1)+p(A_1-A_2)} v'(y) \left[\int_a^b \frac{f(x)}{(au^2(x)+2bu(x)v(y)+cv^2(y))^s}dx\right]^p dy$$
$$< L_1 L_2^{\frac{p}{q}} \int_a^b \frac{u(x)^{1-2s+p(A_1-A_2)}}{u'(x)^{p-1}} f^p(x)dx \tag{10.36}$$

hold for all non-negative measurable functions $f,g:(a,b)\to\mathbb{R}$, provided that $0<\int_a^b \frac{u(x)^{1-2s+p(A_1-A_2)}}{u'(x)^{p-1}} f^p(x)dx<\infty$ and $0<\int_a^b \frac{v(y)^{1-2s+q(A_2-A_1)}}{v'(y)^{q-1}} g^q(y)dy<\infty$. Moreover, inequalities (10.35) and (10.36) are equivalent.

10.27. C. T. Chang, J. W. Lan, K. Z. Wang [13], 2011.

For $p>1$, denote by $\mathscr{H}_p$ the set of all non-negative functions $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ having the following properties:
(i) K is homogeneous of degree -1,
(ii) $K(x,1)x^{-\frac{1}{p}}$ is strictly decreasing function of x and $K(1,y)y^{-\frac{1}{q}}$ is strictly decreasing function of y, where $\frac{1}{p}+\frac{1}{q}=1$,
(iii) $k_p(K)=\int_0^\infty K(1,y)y^{-\frac{1}{q}}dy<\infty$.

Let l_p be the Banach space of all complex sequences $x=(x_n)_{n\in\mathbb{N}}$ with the norm $\|x\|_p=(\sum_{n=1}^\infty |x_n|^p)^{\frac{1}{p}}<\infty$ and let $\mathscr{F}(\alpha)$ be a class of all non-negative differentiable functions $f:\mathbb{R}_+\to\mathbb{R}$ such that $\inf_{x>0} f'(x)\geq\alpha>0$.

If $\phi_i\in\mathscr{F}(\alpha_i)$, $i=1,2$, and $K(x,y)f_1(x)f_2(y)\in\mathscr{H}_p$, then

$$\sum_{n,m\geq 1} K(\phi_1(n),\phi_2(m))|a_n||b_m| < \alpha_1^{-\frac{1}{q}}\alpha_2^{-\frac{1}{p}} k_p \Big\|\frac{a}{f_1(\phi_1)}\Big\|_p \Big\|\frac{b}{f_2(\phi_2)}\Big\|_q, \tag{10.37}$$

$$\left[\sum_{m=1}^\infty\left(\sum_{n=1}^\infty K(\phi_1(n),\phi_2(m))f_2(\phi_2(m))|a_n|\right)^p\right]^{\frac{1}{p}} < \alpha_1^{-\frac{1}{q}}\alpha_2^{-\frac{1}{p}} k_p \Big\|\frac{a}{f_1(\phi_1)}\Big\|_p, \tag{10.38}$$

$$\left[\sum_{n=1}^\infty\left(\sum_{m=1}^\infty K(\phi_1(n),\phi_2(m))f_1(\phi_1(n))|b_m|\right)^q\right]^{\frac{1}{q}} < \alpha_1^{-\frac{1}{q}}\alpha_2^{-\frac{1}{p}} k_p \Big\|\frac{b}{f_2(\phi_2)}\Big\|_q, \tag{10.39}$$

where $k_p=k_p(K(x,y)f_1(x)f_2(y))$ and $\frac{a}{f_1(\phi_1)}=(\frac{a}{f_1(\phi_1(n))})_{n\in\mathbb{N}}$, $\frac{b}{f_2(\phi_2)}=(\frac{b}{f_2(\phi_2(m))})_{m\in\mathbb{N}}$ are complex sequences such that $0<\|\frac{a}{f_1(\phi_1)}\|_p, \|\frac{b}{f_2(\phi_2)}\|_q<\infty$.

In addition, if $\lim_{x\to\infty}\phi_i'(x)=\alpha_i$, $i=1,2$, then the constant $\alpha_1^{-\frac{1}{q}}\alpha_2^{-\frac{1}{p}}k_p$ is the best possible in the above inequalities.

10.28. D. Xin, B. Yang [135], 2011.

Suppose $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let

$$k(\lambda)=\frac{\pi}{\sin\lambda\pi}\left[\frac{\sin\lambda\alpha_1}{\sin\alpha_1}+\frac{\sin\lambda(\pi-\alpha_2)}{\sin\alpha_2}\right],$$

where $0 < |\lambda| < 1$ and $0 < \alpha_1 < \alpha_2 < \pi$. Then the inequalities

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \min_{i\in\{1,2\}}\left\{\frac{1}{x^2+2xy\cos\alpha_i+y^2}\right\} f(x)g(y)dxdy$$
$$< k(\lambda)\left[\int_{-\infty}^{\infty}|x|^{-p\lambda-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}|y|^{q\lambda-1}g^q(y)dy\right]^{\frac{1}{q}} \tag{10.40}$$

and

$$\int_{-\infty}^{\infty}|y|^{p(1-\lambda)-1}\left[\int_{-\infty}^{\infty}\min_{i\in\{1,2\}}\left\{\frac{1}{x^2+2xy\cos\alpha_i+y^2}\right\} f(x)dx\right]^p dy$$
$$< k^p(\lambda)\int_{-\infty}^{\infty}|x|^{-p\lambda-1}f^p(x)dx \tag{10.41}$$

hold for all non-negative functions $f,g:\mathbb{R}\to\mathbb{R}$ such that $0<\int_{-\infty}^{\infty}|x|^{-p\lambda-1}f^p(x)dx<\infty$ and $0<\int_{-\infty}^{\infty}|y|^{q\lambda-1}g^q(y)dy<\infty$. Moreover, inequalities (10.40) and (10.41) are equivalent and include the best possible constant factors on their right-hand sides.

10.29. B. Yang, M. Krnić [168], 2012.

Suppose $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $\alpha\in\mathbb{R}$. Further, let $h:\mathbb{R}_+\to\mathbb{R}$ be a non-negative function such that $0<k(\alpha)=\int_0^{\infty}h(t)t^{\alpha-1}dt<\infty$ and $x^{-\alpha}\sum_{n=1}^{\infty}h\left(\frac{n}{x}\right)n^{\alpha-1}<k(\alpha)$, $x\in\mathbb{R}_+$. Then the inequality

$$\int_0^{\infty}f(x)\sum_{n=1}^{\infty}h\left(\frac{n}{x}\right)a_n dx$$
$$< k(\alpha)\left[\int_0^{\infty}x^{p(1+\alpha)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\alpha)-1}a_n^q\right]^{\frac{1}{q}} \tag{10.42}$$

holds for any non-negative function $f:\mathbb{R}_+\to\mathbb{R}$ and any sequence $(a_n)_{n\in\mathbb{N}}$, provided that $0<\int_0^{\infty}x^{p(1+\alpha)-1}f^p(x)dx<\infty$ and $0<\sum_{n=1}^{\infty}n^{q(1-\alpha)-1}a_n^q<\infty$. In addition, the constant factor $k(\alpha)$ is the best possible in (10.42).

10.30. M. Krnić [65], 2012.

Suppose p_i, p_i', q_i, $i=1,2,\ldots,n$, and λ are real parameters fulfilling conditions (2.26), (2.27) and (2.30) (see Section 2.4, Chapter 2). Further, let $(\Omega_i,\Sigma_i,\mu_i)$ be σ-finite measure spaces, and let $K:\prod_{i=1}^n\Omega_i\to\mathbb{R}$, $\phi_{ij}:\Omega_j\to\mathbb{R}$, $f_i:\Omega_i\to\mathbb{R}$, $i,j=1,2,\ldots,n$, be non-negative measurable functions. If $\prod_{i,j=1}^{n}\phi_{ij}(x_j)=1$, then

$$\begin{aligned}
&\frac{\left\{\int_{\mathbf{\Omega}}\left[K^n(\mathbf{x})\prod_{i=1}^{n}(\phi_{ii}\omega_i f_i)^{2p_i}(x_i)\omega_i^{-q_i}(x_i)\prod_{i,j=1,j\neq i}^{n}\phi_{ij}^{q_i}(x_j)\right]^{1/(n+1)}d\mu(\mathbf{x})\right\}^{(n+1)M}}{\prod_{i=1}^{n}\|\phi_{ii}\omega_i f_i\|_{L^{p_i}(\mu_i)}^{2Mp_i}}\\
&\quad\leq\frac{\int_{\mathbf{\Omega}}K^{\lambda}(\mathbf{x})\prod_{i=1}^{n}f_i(x_i)d\mu(\mathbf{x})}{\prod_{i=1}^{n}\|\phi_{ii}\omega_i f_i\|_{L^{p_i}(\mu_i)}}\\
&\quad\leq\frac{\left\{\int_{\mathbf{\Omega}}\left[K^n(\mathbf{x})\prod_{i=1}^{n}(\phi_{ii}\omega_i f_i)^{2p_i}(x_i)\omega_i^{-q_i}(x_i)\prod_{i,j=1,j\neq i}^{n}\phi_{ij}^{q_i}(x_j)\right]^{1/(n+1)}d\mu(\mathbf{x})\right\}^{(n+1)m}}{\prod_{i=1}^{n}\|\phi_{ii}\omega_i f_i\|_{L^{p_i}(\mu_i)}^{2mp_i}},
\end{aligned}\tag{10.43}$$

and

$$\begin{aligned}
&(n+1)m\prod_{i=1}^{n}\|\phi_{ii}\omega_i f_i\|_{L^{p_i}(\mu_i)}\\
&\times\left[1-\frac{\int_{\mathbf{\Omega}}\left[K^n(\mathbf{x})\prod_{i=1}^{n}(\phi_{ii}\omega_i f_i)^{2p_i}(x_i)\omega_i^{-q_i}(x_i)\prod_{i,j=1,j\neq i}^{n}\phi_{ij}^{q_i}(x_j)\right]^{1/(n+1)}d\mu(\mathbf{x})}{\prod_{i=1}^{n}\|\phi_{ii}\omega_i f_i\|_{L^{p_i}(\mu_i)}^{2p_i/(n+1)}}\right]\\
&\leq\prod_{i=1}^{n}\|\phi_{ii}\omega_i f_i\|_{L^{p_i}(\mu_i)}-\int_{\mathbf{\Omega}}K^{\lambda}(\mathbf{x})\prod_{i=1}^{n}f_i(x_i)d\mu(\mathbf{x})\\
&\leq(n+1)M\prod_{i=1}^{n}\|\phi_{ii}\omega_i f_i\|_{L^{p_i}(\mu_i)}\\
&\times\left[1-\frac{\int_{\mathbf{\Omega}}\left[K^n(\mathbf{x})\prod_{i=1}^{n}(\phi_{ii}\omega_i f_i)^{2p_i}(x_i)\omega_i^{-q_i}(x_i)\prod_{i,j=1,j\neq i}^{n}\phi_{ij}^{q_i}(x_j)\right]^{1/(n+1)}d\mu(\mathbf{x})}{\prod_{i=1}^{n}\|\phi_{ii}\omega_i f_i\|_{L^{p_i}(\mu_i)}^{2p_i/(n+1)}}\right],
\end{aligned}\tag{10.44}$$

where $\mathbf{\Omega}=\prod_{i=1}^{n}\Omega_i$, $\mathbf{x}=(x_1,x_2,\dots,x_n)$, $d\mu(\mathbf{x})=\prod_{i=1}^{n}d\mu_i(x_i)$,

$$m=\min\left\{\frac{1}{q_1},\frac{1}{q_2},\dots,\frac{1}{q_n},1-\lambda\right\},M=\max\left\{\frac{1}{q_1},\frac{1}{q_2},\dots,\frac{1}{q_n},1-\lambda\right\},$$

and $\omega_i:\Omega_i\to\mathbb{R}$ is defined by

$$\omega_i(x_i)=\left[\int_{\hat{\mathbf{\Omega}}^i}K(\mathbf{x})\prod_{j=1,j\neq i}^{n}\phi_{ij}^{q_i}(x_j)d\hat{\mu}^i(\mathbf{x})\right]^{\frac{1}{q_i}},$$

where $\hat{\mathbf{\Omega}}^i=\prod_{j=1,j\neq i}^{n}\Omega_j$ and $d\hat{\mu}^i(\mathbf{x})=\prod_{j=1,j\neq i}^{n}d\mu_j(x_j)$.

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