MONOGRAPHS IN INEQUALITIES 6

Inequalities of Hardy and Jensen

New Hardy type inequalities with general kernels

Kristina Krulić Himmelreich, Josip Pečarić and Dora Pokaz

ELEMENT

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Preface

The main motivation for writing this book was to present some general aspects of generalizations, refinements, and variants of famous Hardy's inequality (see [48], [49], [50])

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)\,dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(x)\,dx,\ p > 1 \tag{0.1}$$

where f is a non-negative function, such that $f \in L^p(\mathbb{R}_+)$. Rewriting (0.1) with the function f replaced with $f^{1/p}$ and then letting $p \to \infty$ we obtain the limiting case of Hardy's inequality:

$$\int_0^\infty \exp\left(\frac{1}{x} \int_0^x \log f(t)\,dt \right) dx < e \int_0^\infty f(x)\,dx, \tag{0.2}$$

which holds for all positive functions $f \in L^1(\mathbb{R}_+)$. This inequality is referred to as Pólya–Knopp's inequality. It was first published by K. Knopp [69] in 1928, but it was certainly known before since Hardy himself (see [49, p. 156]) claimed that it was G. Pólya who pointed it out to him earlier. Note that the discrete version of (0.2) is surely due to T. Carleman [17].

In this book an integral operator with general non-negative kernel on measure spaces with positive σ-finite measure is considered and some new weighted Hardy type inequalities for convex functions and refinements of weighted Hardy type inequalities for superquadratic functions are obtained. Moreover, some refinements of weighted Hardy type inequalities for convex functions and some new refinements of discrete Hardy type inequalities are given. Furthermore, improvements and reverses of new weighted Hardy type inequalities with integral operators are stated and proved. New Cauchy type mean is introduced and monotonicity property of this mean is proved. By using the concept of the subdifferential of a convex function, we refine the general Boas-type inequality. Furthermore, we get some new inequalities for superquadratic and subquadratic functions as well as for functions which can be bounded by non-negative convex or superquadratic function. The Boas functional and related inequality allow us to adjust Lagrange and Cauchy mean value theorems to the context and in that way define a new class of two-parametric means of the Cauchy-type. In the context of the maximal operator, we obtain similar results. We also give some interesting, one-dimensional and multidimensional, examples related to balls and cones in $\mathbb{R}^n$.

Conventions. All measures are assumed to be positive, all functions are assumed to be measurable, and expressions of the form $0 \cdot \infty$, $\frac{0}{0}$, $\frac{a}{\infty}$ $(a \in \mathbb{R})$, and $\frac{\infty}{\infty}$ are taken to be equal to zero. Further, we set $\mathbb{N}_k = \{1, 2, \ldots, k\}$ for $k \in \mathbb{N}$. For a real parameter $0 \neq p \neq 1$, we denote by p' its conjugate exponent $p' = \frac{p}{p-1}$, that is, $\frac{1}{p} + \frac{1}{p'} = 1$. We denote by $|\Omega|_\mu$ the measure of a measurable set Ω with respect to the measure μ. In particular, we use the symbol $|\ |_1$ as an abbreviation for $\|\ \|_{L^1(\Omega_1,\mu_1)}$. Also, by a weight function (shortly: a weight) we mean a non-negative measurable function on the actual set. An interval I in $\mathbb{R}$ is any convex subset of $\mathbb{R}$, while by $\operatorname{Int} I$ we denote its interior. By $\mathbb{R}_+$ we denote the set of all positive real numbers i.e. $\mathbb{R}_+ = (0,\infty)$. $B(\,\cdot\,;\cdot,\cdot)$ denotes the incomplete Beta function, defined by

$$B(x;a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt,\ x \in [0,1],\ a,b > 0.$$

As usual, $B(a,b) = B(1;a,b)$ stands for the standard Beta function. For $R > 0$ we denote by $B(R)$ a ball in $\mathbb{R}^n$ centred at the origin and of radius R, that is, $B(R) = \{\mathbf{x} \in \mathbb{R}^n\colon |\mathbf{x}| \leq R\}$, where $|\mathbf{x}|$ denotes the Euclidean norm of $\mathbf{x} \in \mathbb{R}^n$. By its dual set we mean the set $\mathbb{R}^n \setminus B(R) = \{\mathbf{x} \in \mathbb{R}^n\colon |\mathbf{x}| > R\}$. By S^{n-1} we denote the surface of the unit ball $B(1)$, namely $S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n\colon |\mathbf{x}| = 1\}$, and by $|S^{n-1}|$ its area.

We thank our colleagues Marjan Praljak, Ph. D. and Sajid Iqbal for their valuable comments and suggestions that improved the final version of this book.

Contents

Chapter 1

Definitions and basic results

1.1 Convex functions

Convex functions are very important in the theory of inequalities. The third chapter of the classical book of Hardy, Littlewood and Pólya [51] is devoted to the theory of convex functions (see also [82]). In this section we give some of the results concerning convex functions.

Definition 1.1 *Let I be an interval in $\mathbb{R}$. A function $\Phi\colon I \to \mathbb{R}$ is called* convex *if*

$$\Phi(\lambda x + (1-\lambda)y) \leq \lambda\Phi(x) + (1-\lambda)\Phi(y) \tag{1.1}$$

for all points $x, y \in I$ and all $\lambda \in [0,1]$. It is called strictly convex *if the inequality* (1.1) *holds strictly whenever x and y are distinct points and $\lambda \in (0,1)$.*

If $-\Phi$ is convex (respectively, strictly convex) then we say that Φ is concave *(respectively,* strictly concave*). If Φ is both convex and concave, Φ is said to be* affine.

Remark 1.1 (a) For $x, y \in I$, $p, q \geq 0$, $p+q > 0$, (1.1) is equivalent to

$$\Phi\left(\frac{px+qy}{p+q}\right) \leq \frac{p}{p+q}\Phi(x) + \frac{q}{p+q}\Phi(y).$$

(b) The simple geometrical interpretation of (1.1) is that the graph of Φ lies below its chords.

(c) If x_1, x_2, x_3 are three points in I such that $x_1 < x_2 < x_3$, then (1.1) is equivalent to

$$\begin{vmatrix} x_1 & \Phi(x_1) & 1 \\ x_2 & \Phi(x_2) & 1 \\ x_3 & \Phi(x_3) & 1 \end{vmatrix} = (x_3 - x_2)\Phi(x_1) + (x_1 - x_3)\Phi(x_2) + (x_2 - x_1)\Phi(x_3) \geq 0$$

which is equivalent to

$$\Phi(x_2) \leq \frac{x_2 - x_3}{x_1 - x_3}\Phi(x_1) + \frac{x_1 - x_2}{x_1 - x_3}\Phi(x_3),$$

or, more symmetrically and without the condition of monotonicity on x_1, x_2, x_3

$$\frac{\Phi(x_1)}{(x_1 - x_2)(x_1 - x_3)} + \frac{\Phi(x_2)}{(x_2 - x_3)(x_2 - x_1)} + \frac{\Phi(x_3)}{(x_3 - x_1)(x_3 - x_2)} \geq 0$$

Definition 1.2 *Let I be an interval in $\mathbb{R}$. A function $\Phi : I \to \mathbb{R}$ is called* convex in the Jensen sense*, or* J-convex on I (midconvex, midpoint convex) *if for all points $x, y \in I$ the inequality*

$$\Phi\left(\frac{x+y}{2}\right) \leq \frac{\Phi(x) + \Phi(y)}{2} \tag{1.2}$$

holds. A J-convex function is said to be strictly J-convex *if for all pairs of points $(x, y), x \neq y$, strict inequality holds in (1.2).*

In the context of continuity the following criteria of equivalence of (1.1) and (1.2) is valid.

Theorem 1.1 *Let $\Phi : I \to \mathbb{R}$ be a continuous function. Then Φ is a convex function if and only if Φ is a J-convex function.*

Inequality (1.1) can be extended to the convex combinations of finitely many points in I by mathematical induction. These extensions are known as discrete and integral Jensen's inequality.

Theorem 1.2 (THE DISCRETE CASE OF JENSEN'S INEQUALITY) *A function $\Phi : I \to \mathbb{R}$ is convex if and only if for all $x_1, ..., x_n \in I$ and all scalars $p_1, ..., p_n \in [0, 1]$ with $P_n = \sum_1^n p_i$ we have*

$$\Phi\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i \Phi(x_i). \tag{1.3}$$

Inequality (1.3) is strict if Φ is a strictly convex function, all points x_i, $i = 1, ..., n, n \in \mathbb{N}$ are disjoint and all scalars p_i are positive.

Now, we introduce some necessary notation and recall some basic facts about convex functions, log-convex functions (see e.g. [65], [82], [92]) as well as exponentially convex functions (see e.g [15], [79], [81]).

In 1929, S. N. Bernstein introduced the notion of exponentially convex function in [15]. Later on D.V. Widder in [100] introduced these functions as a sub-class of convex function in a given interval (a, b) (for details see [100], [101]).

Definition 1.3 *A positive function Φ is said to be* logarithmically convex *on an interval $I \subseteq \mathbb{R}$ if $\log \Phi$ is a convex function on I, or equivalently if for all $x, y \in I$ and all $\alpha \in [0,1]$*

$$\Phi(\alpha x + (1-\alpha)y) \leq \Phi^{\alpha}(x)\Phi^{1-\alpha}\Phi(y).$$

For such function Φ, we shortly say Φ is log-convex. A positive function Φ is log-convex in the Jensen sense *if for each $x, y \in I$*

$$\Phi^2\left(\frac{x+y}{2}\right) \leq \Phi(x)\Phi(y)$$

holds, i.e., if $\log \Phi$ is convex in the Jensen sense.

Remark 1.2 A function Φ is log-convex on an interval I, if and only if for all $x_1, x_2, x_3 \in I$, $x_1 < x_2 < x_3$, it holds

$$[\Phi(x_2)]^{x_3-x_1} \leq [\Phi(x_1)]^{x_3-x_2}[\Phi(x_3)]^{x_2-x_1}. \tag{1.4}$$

Furthermore, if $x_1, x_2, y_1, y_2 \in I$ are such that $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then

$$\left(\frac{\Phi(x_2)}{\Phi(x_1)}\right)^{\frac{1}{x_2-x_1}} \leq \left(\frac{\Phi(y_2)}{\Phi(y_1)}\right)^{\frac{1}{y_2-y_1}}. \tag{1.5}$$

Inequality (1.5) is known as Galvani's theorem for log-convex functions $\Phi \colon I \to \mathbb{R}$.

We continue with the definition of exponentially convex function as originally given in [15] by Berstein (see also [9], [79], [81]).

Definition 1.4 *A function $\Phi \colon (a,b) \to \mathbb{R}$ is* exponentially convex *if it is continuous and*

$$\sum_{i,j=1}^{n} t_i t_j \Phi(x_i + x_j) \geq 0, \tag{1.6}$$

holds for every $n \in \mathbb{N}$ and all sequences $(t_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ of real numbers, such that $x_i + x_j \in (a,b)$, $1 \leq i, j \leq n$.

Moreover, the condition (1.6) can be replaced with a more suitable condition

$$\sum_{i,j=1}^{n} t_i t_j \Phi\left(\frac{x_i + x_j}{2}\right) \geq 0, \tag{1.7}$$

which has to hold for all $n \in \mathbb{N}$, all sequences $(t_n)_{n\in\mathbb{N}}$ of real numbers, and all sequences $(x_n)_{n\in\mathbb{N}}$ in (a,b). More precisely, a function $\Phi \colon (a,b) \to \mathbb{R}$ is exponentially convex if and only if it is continuous and fulfils (1.7). Condition (1.7) means that the matrix $\left[\Phi(\frac{x_i+x_j}{2})\right]_{i,j=1}^{n}$ is positive semi-definite matrix. Hence, its determinant must be non-negative. For $n = 2$ this means that it holds

$$\Phi(x_1)\Phi(x_2) - \Phi^2\left(\frac{x_1+x_2}{2}\right) \geq 0,$$

hence, exponentially convex function is log-convex in the Jensen sense, and, being continuous, it is also log-convex function.

We continue with the definition of n-exponentially convex function.

Definition 1.5 *A function $\Phi : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if*

$$\sum_{i,j=1}^{n} t_i t_j \Phi\left(\frac{x_i + x_j}{2}\right) \geq 0$$

holds for all choices of $t_i \in \mathbb{R}$, $x_i \in I$, $i = 1, \ldots, n$.
A function $\Phi : I \to \mathbb{R}$ is n-exponentially convex on I if it is n-exponentially convex in the Jensen sense and continuous on I.

Remark 1.3 *It is clear from the definition that 1-exponentially convex functions in the Jensen sense are in fact non-negative functions. Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $k \leq n$.*

Proposition 1.1 *Let I be an open interval in $\mathbb{R}$. If Φ is n-exponentially convex in the Jensen sense on J then the matrix $\left[\Phi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k}$ is positive semi-definite matrix for all $k \in \mathbb{N}$, $k \leq n$. Particularly*

$$\det\left[\Phi\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{k} \geq 0, \; for\ all\ k \in \mathbb{N},\ k \leq n.$$

Definition 1.6 *Let I be an open interval in $\mathbb{R}$. A function $\Phi : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense on I for all $n \in \mathbb{N}$.*

Remark 1.4 *It follows that a function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense.*
Also, using basic convexity theory it follows that a function is log-convex if and only if it is 2-exponentially convex.

It is easily seen that a convex function is continuous on the interior of its domain, but it may not be continuous at the boundary points of the domain.

Theorem 1.3 *If $\Phi : I \to \mathbb{R}$ is a convex function, then Φ satisfies the Lipschitz condition on any closed interval $[a,b]$ contained in the interior of I, that is, there exists a constant K so that for any two points $x, y \in [a,b]$,*

$$|\Phi(x) - \Phi(y)| \leq K|x - y|.$$

Now, we continue with derivative of a convex function. The derivative of a convex function is best studied in terms of the left and right derivatives defined by

$$\Phi'_-(x) = \lim_{y \nearrow x} \frac{\Phi(y) - \Phi(x)}{y - x}, \; \Phi'_+(x) = \lim_{y \searrow x} \frac{\Phi(y) - \Phi(x)}{y - x}.$$

The following result concerning the left and the right derivative of a convex function can be seen e.g. in [92].

Theorem 1.4 *Let I be an interval in $\mathbb{R}$ and $\Phi\colon I \to \mathbb{R}$ be convex. Then*

(i) *Φ'_- and Φ'_+ exist and are increasing on I, and $\Phi'_- \le \Phi'_+$ (if Φ is strictly convex, then these derivatives are strictly increasing);*

(ii) *Φ' exists, except possibly on a countable set, and on the complement of which it is continuous.*

Theorem 1.5 (a) *$\Phi\colon [a,b] \to \mathbb{R}$ is (strictly) convex if there exists an (strictly) increasing function $f\colon [a,b] \to \mathbb{R}$ and a real number c $(a < c < b)$ such that for all x and $a < x < b$,*

$$\Phi(x) = \Phi(c) + \int_c^x f(t)dt.$$

(b) *If Φ is differentiable, then Φ is (strictly) convex if Φ' is (strictly) increasing.*

(c) *If Φ'' exists on (a,b), then Φ is convex if $\Phi''(x) \ge 0$. If $\Phi''(x) > 0$, then Φ is strictly convex.*

Example 1.1 (a) The exponential function $\Phi : \mathbb{R} \to \mathbb{R}$, $\Phi(x) = e^x$ is a strictly convex function.

(b) Let $\Phi : \mathbb{R}_+ \to \mathbb{R}$ be defined by $\Phi(x) = x^p$, $p \in \mathbb{R} \setminus \{0\}$. Obviously, $\Phi'(x) = px^{p-1}$ and the function Φ is convex for $p \in \mathbb{R} \setminus [0,1)$, concave for $p \in (0,1]$, and affine for $p = 1$.

Remark 1.5 Let I be an open interval and let $h \in C^2(I)$ be such that h'' is bounded, that is, $m \le h'' \le M$. Then the functions Φ_1, Φ_2 defined by

$$\Phi_1(t) = \frac{M}{2}t^2 - h(t), \qquad \Phi_2(t) = h(t) - \frac{m}{2}t^2$$

are convex.

The geometric characterization depends upon the idea of a support line. The following result can be seen e.g. in [92].

Theorem 1.6 (a) *$\Phi\colon (a,b) \to \mathbb{R}$ is convex if there is at least one line of support for Φ at each $x_0 \in (a,b)$, i.e.,*

$$\Phi(x) \ge \Phi(x_0) + \lambda(x - x_0), \forall x \in (a,b),$$

where λ depends on x_0 and is given by $\lambda = \Phi'(x_0)$ when Φ' exists, and $\lambda \in [\Phi'_-(x_0), \Phi'_+(x_0)]$ when $\Phi'_-(x_0) \ne \Phi'_+(x_0)$.

(b) *$\Phi\colon (a,b) \to \mathbb{R}$ is convex if the function $\Phi(x) - \Phi(x_0) - \lambda(x - x_0)$ (the difference between the function and its support) is decreasing for $x < x_0$ and increasing for $x > x_0$.*

Definition 1.7 *Let $\Phi : I \longrightarrow \mathbb{R}$ be a convex function, then the sub-differential of Φ at x, denoted by $\partial\Phi(x)$, is defined as*

$$\partial\Phi(x) = \{\alpha \in \mathbb{R} : \Phi(y) - \Phi(x) - \alpha(y-x) \geq 0,\, y \in I\}.$$

There is a connection between a convex function and its sub-differential. It is well-known that $\partial\Phi(x) \neq \emptyset$ for all $x \in \mathrm{Int}I$. More precisely, at each point $x \in \mathrm{Int}I$ we have $-\infty < \Phi'_-(x) \leq \Phi'_+(x) < \infty$ and

$$\partial\Phi(x) = [\Phi'_-(x), \Phi'_+(x)],$$

while the set on which Φ is not differentiable is at most countable. Moreover, each function $\varphi : I \longrightarrow \mathbb{R}$ such that $\varphi(x) \in \partial\Phi(x)$, whenever $x \in \mathrm{Int}I$, is increasing on $\mathrm{Int}I$. For any such function φ and arbitrary $x \in \mathrm{Int}I$, $y \in I$ we have

$$\Phi(y) - \Phi(x) - \varphi(x)(y-x) \geq 0$$

and further

$$\begin{aligned} \Phi(y) - \Phi(x) - \varphi(x)(y-x) &= |\Phi(y) - \Phi(x) - \varphi(x)(y-x)| \\ &\geq \left||\Phi(y) - \Phi(x)| - |\varphi(x)| \cdot |y-x|\right|. \end{aligned} \tag{1.8}$$

On the other hand, if $\Phi : I \to \mathbb{R}$ is a concave function, that is, $-\Phi$ is convex, then $\partial\Phi(x) = \{\alpha \in \mathbb{R} : \Phi(x) - \Phi(y) - \alpha(x-y) \geq 0,\, y \in I\}$ denotes the superdifferential of Φ at the point $x \in I$. For all $x \in \mathrm{Int}I$, in this setting we have $-\infty < \Phi'_+(x) \leq \Phi'_-(x) < \infty$ and $\partial\Phi(x) = [\Phi'_+(x), \Phi'_-(x)] \neq \emptyset$. Hence, the inequality

$$\Phi(x) - \Phi(y) - \varphi(x)(x-y) \geq 0$$

holds for all $x \in \mathrm{Int}I$, $y \in I$, and all real functions φ on I, such that $\varphi(z) \in \partial\Phi(z)$, $z \in \mathrm{Int}I$. Finally, we get

$$\begin{aligned} \Phi(x) - \Phi(y) - \varphi(x)(x-y) &= |\Phi(x) - \Phi(y) - \varphi(x)(x-y)| \\ &\geq \left||\Phi(y) - \Phi(x)| - |\varphi(x)| \cdot |y-x|\right|. \end{aligned} \tag{1.9}$$

Note that, although the symbol $\partial\Phi(x)$ has two different notions, it will be clear from the context whether it applies to a convex or to a concave function Φ. Many further information on convex and concave functions can be found e.g. in the monographs [82] and [92] and in references cited therein.

1.2 Superquadratic and subquadratic functions

The concept of superquadratic and subquadratic functions is introduced by Abramovich, Jameson and Sinnamon in [4] (see also [3]).

Definition 1.8 *(See* [4, Definition 2.1].*) A function $\varphi : [0,\infty) \to \mathbb{R}$ is superquadratic provided that for all $x \geq 0$ there exists a constant $C_x \in \mathbb{R}$ such that*

$$\varphi(y) - \varphi(x) - \varphi(|y-x|) \geq C_x (y-x), \tag{1.10}$$

for all $y \geq 0$. We say that φ is subquadratic if $-\varphi$ is superquadratic. We say that φ is a strictly superquadratic function if for $x \neq y, x, y \neq 0$ there is strict inequality in (1.10). *We say that φ is a strictly subquadratic function if $-\varphi$ is a strictly superquadratic function.*

Lemma 1.1 *(See* [4, Theorem 2.3].) Let (Ω,μ) be a probability measure space. The inequality

$$\varphi\left(\int_\Omega f(s)d\mu(s)\right) \leq \int_\Omega \varphi(f(s))d\mu(s) - \int_\Omega \varphi\left(\left|f(s) - \int_\Omega f(s)d\mu(s)\right|\right) d\mu(s) \tag{1.11}$$

holds for all probability measures μ and all non-negative $\mu-$integrable functions f if and only if φ is superquadratic. Moreover, (1.11) holds in the reversed direction if and only if φ is subquadratic.

Proof. See [4] and [3] for the details. □

Definition 1.9 *A function $f : [0,\infty) \to \mathbb{R}$ is superadditive provided $f(x+y) \geq f(x) + f(y)$ for all $x, y \geq 0$. If the reverse inequality holds, then f is said to be subadditive.*

Lemma 1.2 *(See* [4, Lemma 3.1].) Suppose $\varphi : [0,\infty) \to \mathbb{R}$ is continuously differentiable and $\varphi(0) \leq 0$. If φ' is superadditive or $\frac{\varphi'(x)}{x}$ is nondecreasing, then φ is superquadratic.

Proof. See [4] for details. □

Remark 1.6 By Lemma 1.2, the function $\varphi(x) = x^p$ is superquadratic for $p \geq 2$ and subquadratic for $1 < p \leq 2$. Therefore, by Lemma 1.1, for $p \geq 2$ the inequality

$$\left(\int_\Omega f(s)d\mu(s)\right)^p \leq \int_\Omega f^p(s)d\mu(s) - \int_\Omega \left|f(s) - \int_\Omega f(s)d\mu(s)\right|^p d\mu(s)$$

holds and the reversed inequality holds when $1 < p \leq 2$ (see also [2, Example 1, p. 1448]).

1.3 Operator convex functions

We shall first recall the definition of an operator convex function.

Definition 1.10 *Let I be a real interval of any type. A continuous function $f : I \to \mathbb{R}$ is said to be operator convex if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

holds for each $\lambda \in [0,1]$ and every pair of self-adjoint x and y (acting) on an infinite dimensional Hilbert space H with spectra in I (the ordering is defined by setting $x \leq y$ if $y - x$ is positive semi-definite).

Let f be an operator convex function defined on an interval I. Ch. Davis [34] proved a Schwartz type inequality

$$f(\Phi(x)) \leq \Phi(f(x)),$$

where $\Phi : A \to B(H)$ is a unital completely positive linear map from a C^*-algebra A to linear operators on a Hilbert space H and x is a self-adjoint element in A with spectrum in I.

Let us recall the definition of a unital field. Assume that there is a field $(\Phi_t)_{t\in T}$ of positive linear mappings $\Phi_t : A \to B$ from A to another C^*-algebra B. We say that such a field is continuous if the function $t \to \Phi_t(x)$ is continuous for every $x \in A$. If the C^*-algebras are unital and the field $t \to \Phi_t(\mathbf{1})$ is integrable with integral $\mathbf{1}$, that is $\int_T \Phi_t(\mathbf{1}) d\mu(t) = \mathbf{1}$, we say that $(\Phi_t)_{t\in T}$ is unital.

In particular, F. Hansen et al. [46] proved the following result:

Theorem 1.7 *Let $f : I \to \mathbb{R}$ be an operator convex function defined on an interval I, and let A and B be unital C^*-algebras. If $(\Phi_t)_{t\in T}$ is a unital field of positive linear mappings $\Phi_t : A \to B$ defined on a locally compact space T with a bounded positive Radon measure μ, then the Jensen type inequality*

$$f\left(\int_T \Phi_t(x_t)) d\mu(t)\right) \leq \int_T \Phi_t(f(x_t)) d\mu(t)$$

holds for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in A with spectra contained in I.

1.4 Fractional integrals and fractional derivatives

First, let us recall some facts about fractional derivatives needed in the sequel, for more details see e.g. [9], [43].

Let $0 < a < b \leq \infty$. By $C^m([a,b])$ we denote the space of all functions on $[a,b]$ which have continuous derivatives up to order m, and $AC([a,b])$ is the space of all absolutely continuous functions on $[a,b]$. By $AC^m([a,b])$ we denote the space of all functions $g \in C^{m-1}([a,b])$ with $g^{(m-1)} \in AC([a,b])$. For any $\alpha \in \mathbb{R}$ we denote by $[\alpha]$ the integral part of α (the integer k satisfying $k \leq \alpha < k+1$) and $\lceil \alpha \rceil$ is the ceiling of α ($\min\{n \in \mathbb{N}, n \geq \alpha\}$). By $L_1(a,b)$ we denote the space of all functions integrable on the interval (a,b), and by $L_\infty(a,b)$ the set of all functions measurable and essentially bounded on (a,b). Clearly, $L_\infty(a,b) \subset L_1(a,b)$.

Now, we give well known definitions of the Riemann-Liouville fractional integrals, see [67]. Let $[a,b]$ be a finite interval on real axis $\mathbb{R}$. The Riemann-Liouville fractional integrals $I^\alpha_{a+}f$ and $I^\alpha_{b-}f$ of order $\alpha > 0$ are defined by

$$I^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(y)(x-y)^{\alpha-1}dy, \;\; (x > a)$$

and

$$I^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b f(y)(y-x)^{\alpha-1}dy, \;\; (x < b)$$

respectively. Here $\Gamma(\alpha)$ is the Gamma function. These integrals are called the right-sided and left-sided fractional integrals. Some recent results involving Riemann-Liouville fractional integrals can be seen in e.g [10], [11], [61] and [63]. We denote some properties of the operators $I^\alpha_{a_+}f$ and $I^\alpha_{b_-}f$ of order $\alpha > 0$, see also [96]. The first result yields that the fractional integral operators $I^\alpha_{a_+}f$ and $I^\alpha_{b_-}f$ are bounded in $L_p(a,b)$, $1 \leq p \leq \infty$, that is

$$\|I^\alpha_{a_+}f\|_p \leq K\|f\|_p, \quad \|I^\alpha_{b_-}f\|_p \leq K\|f\|_p, \tag{1.12}$$

where

$$K = \frac{(b-a)^\alpha}{\alpha\Gamma(\alpha)}.$$

Inequality (1.12), that is the result involving the left-sided fractional integral, was proved by G. H. Hardy in one of his first papers, see [49]. He did not write down the constant, but the calculation of the constant was hidden inside his proof. Inequality (1.12) is referred to as inequality of G. H. Hardy.

Next we give result with respect to the generalized Riemann-Liouville fractional derivative. Let us recall the definition.

Let $\alpha > 0$ and $n = [\alpha] + 1$ where $[\cdot]$ is the integral part. We define the generalized Riemann-Liouville fractional derivative of f of order α by

$$(D_a^\alpha f)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-y)^{n-\alpha-1} f(y)\, dy.$$

In addition, we stipulate

$$D_a^0 f := f =: I_a^0 f, \quad I_a^{-\alpha} f := D_a^\alpha f \text{ if } \alpha > 0.$$

If $\alpha \in \mathbb{N}$ then $D_a^\alpha f = \frac{d^\alpha f}{dx^\alpha}$, the ordinary α-order derivative.

The space $I_a^\alpha(L(a,b))$ is defined as the set of all functions f on $[a,b]$ of the form $f = I_a^\alpha \varphi$ for some $\varphi \in L(a,b)$, [96, Chapter 1, Definition 2.3]. According to Theorem 2.3 in [96, p. 43], the latter characterization is equivalent to the condition

$$I_a^{n-\alpha} f \in AC^n[a,b], \tag{1.13}$$

$$\frac{d^j}{dx^j} I_a^{n-\alpha} f(a) = 0, \quad j = 0, 1, \ldots, n-1.$$

A function $f \in L(a,b)$ satisfying (1.13) is said to have an integrable fractional derivative $D_a^\alpha f$, [96, Chapter1, Definition 2.4].

The following lemma summarizes conditions in identity for generalized Riemann-Liouville fractional derivative.

Lemma 1.3 *Let $\beta > \alpha \geq 0$, $n = [\beta] + 1$, $m = [\alpha] + 1$. Identity*

$$D_a^\alpha f(x) = \frac{1}{\Gamma(\beta-\alpha)} \int_a^x (x-y)^{\beta-\alpha-1} D_a^\beta f(y)\, dy, \quad x \in [a,b].$$

is valid if one of the following conditions holds:

(i) $f \in I_a^\beta(L(a,b))$.

(ii) $I_a^{n-\beta} f \in AC^n[a,b]$ *and* $D_a^{\beta-k} f(a) = 0$ *for* $k = 1, \ldots n$.

(iii) $D_a^{\beta-k} f \in C[a,b]$ *for* $k = 1, \ldots, n$, $D_a^{\beta-1} f \in AC[a,b]$ *and* $D_a^{\beta-k} f(a) = 0$ *for* $k = 1, \ldots n$.

(iv) $f \in AC^n[a,b]$, $D_a^\beta f \in L(a,b)$, $D_a^\alpha f \in L(a,b)$, $\beta - \alpha \notin \mathbb{N}$, $D_a^{\beta-k} f(a) = 0$ *for* $k = 1, \ldots, n$ *and* $D_a^{\alpha-k} f(a) = 0$ *for* $k = 1, \ldots, m$.

(v) $f \in AC^n[a,b]$, $D_a^\beta f \in L(a,b)$, $D_a^\alpha f \in L(a,b)$, $\beta - \alpha = l \in \mathbb{N}$, $D_a^{\beta-k} f(a) = 0$ *for* $k = 1, \ldots, l$.

(vi) $f \in AC^n[a,b]$, $D_a^\beta f \in L(a,b)$, $D_a^\alpha f \in L(a,b)$ *and* $f(a) = f'(a) = \cdots = f^{(n-2)}(a) = 0$.

(vii) $f \in AC^n[a,b]$, $D_a^\beta f \in L(a,b)$, $D_a^\alpha f \in L(a,b)$, $\beta \notin \mathbb{N}$ *and* $D_a^{\beta-1} f$ *is bounded in a neighborhood of* $t = a$.

The definition of Canavati-type fractional derivative is given in [9] but we will use the Canavati-type fractional derivative given in [13] with some new conditions. Now we define Canavati-type fractional derivative ($\nu-$fractional derivative of f). We consider

$$C^{\nu}([0,1]) = \{f \in C^{n}([0,1]) : I_{1-\bar{\nu}} f^{(n)} \in C^{1}([0,1])\},$$

$\nu > 0$, $n = [\nu]$, $[.]$ is the integral part, and $\bar{\nu} = \nu - n, 0 \leq \bar{\nu} < 1$.
For $f \in C^{\nu}([0,1])$, the Canavati-ν fractional derivative of f is defined by

$$D^{\nu} f = D I_{1-\bar{\nu}} f^{(n)},$$

where $D = d/dx$.

Lemma 1.4 *Let $\nu > \gamma \geq 0$, $n = [\nu]$, $m = [\gamma]$. Let $f \in C^{\nu}([0,1])$, be such that $f^{(i)}(0) = 0$, $i = m, m+1, ..., n-1$. Then*

(i) $f \in C^{\gamma}([0,1])$

(ii) $(D^{\gamma} f)(x) = \frac{1}{\Gamma(\nu-\gamma)} \int\limits_{0}^{x} (x-t)^{\nu-\gamma-1} (D^{\nu} f)(t) dt$,

for every $x \in [a,b]$.

Next, we define Caputo fractional derivative, for details see [9, p. 449]. Let $\nu \geq 0$, $n = \lceil \nu \rceil$, $g \in AC^{n}([a,b])$. The Caputo fractional derivative is given by

$$D_{*a}^{\nu} g(t) = \frac{1}{\Gamma(n-\nu)} \int\limits_{a}^{x} \frac{g^{(n)}(y)}{(x-y)^{\nu-n+1}} dy,$$

for all $x \in [a,b]$. The above function exists almost everywhere for $x \in [a,b]$.

We continue with the following lemma that is given in [12].

Lemma 1.5 *Let $\nu > \gamma \geq 0$, $n = [\nu] + 1$, $m = [\gamma] + 1$ and $f \in AC^{n}([a,b])$. Suppose that one of the following conditions hold:*

(a) $\nu, \gamma \notin \mathbb{N}_0$ *and* $f^{(i)}(a) = 0$ *for* $i = m, ..., n-1$

(b) $\nu \in \mathbb{N}_0, \gamma \notin \mathbb{N}_0$ *and* $f^{(i)}(a) = 0$ *for* $i = m, ..., n-2$

(c) $\nu \notin \mathbb{N}_0, \gamma \in \mathbb{N}_0$ *and* $f^{(i)}(a) = 0$ *for* $i = m-1, ..., n-1$

(d) $\nu \in \mathbb{N}_0, \gamma \in \mathbb{N}_0$ *and* $f^{(i)}(a) = 0$ *for* $i = m-1, ..., n-2$.

Then

$$D_{*a}^{\gamma} f(x) = \frac{1}{\Gamma(\nu-\gamma)} \int\limits_{a}^{x} (x-y)^{\nu-\gamma-1} D_{*a}^{\nu} f(y) dy$$

for all $a \leq x \leq b$.

Now, we define Hadamard-type fractional integrals. Let (a,b) be finite or infinite interval of $\mathbb{R}_+$ and $\alpha > 0$. The left- and right-sided Hadamard-type fractional integrals of order $\alpha > 0$ are given by

$$(J^{\alpha}_{a_+} f)(x) = \frac{1}{\Gamma(\alpha)} \int\limits_a^x \left(\log \frac{x}{y}\right)^{\alpha-1} \frac{f(y)dy}{y}, \quad x > a$$

and

$$(J^{\alpha}_{b_-} f)(x) = \frac{1}{\Gamma(\alpha)} \int\limits_x^b \left(\log \frac{y}{x}\right)^{\alpha-1} \frac{f(y)dy}{y}, \quad x < b$$

respectively.

We continue with definitions and some properties of the fractional integrals of a function f with respect to a given function g. For details see e.g. [67, p. 99].

Let (a,b), $-\infty \le a < b \le \infty$ be a finitive or infinitive interval of the real line $\mathbb{R}$ and $\alpha > 0$. Also let g be an increasing function on $(a,b]$ such that g' is continuous on (a,b). The left- and right-sided fractional integrals of a function f with respect to another function g in (a,b) are given by

$$(I^{\alpha}_{a+;g} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t)f(t)dt}{[g(x)-g(t)]^{1-\alpha}}, x > a \tag{1.14}$$

and

$$(I^{\alpha}_{b-;g} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t)f(t)dt}{[g(t)-g(x)]^{1-\alpha}}, x < b, \tag{1.15}$$

respectively.

Remark 1.7 If $g(x) = x$, then $I^{\alpha}_{a+;x} f$ reduces to $I^{\alpha}_{a_+} f$ and $I^{\alpha}_{b-;x} f$ reduces to $I^{\alpha}_{b_-} f$, that is to Riemann-Liouville fractional integrals. Notice also that Hadamard fractional integrals of order α are special case of the left- and right-sided fractional integrals of a function f with respect to another function $g(x) = \log(x)$ in $[a,b]$ where $0 \le a < b \le \infty$.

We also recall the definition of the Erdelyi-Kóber type fractional integrals. For details see [96] (also see [35, p, 154]).

Let $(a,b), (0 \le a < b \le \infty)$ be finite or infinite interval of $\mathbb{R}_+$ Let $\alpha > 0, \sigma > 0$, and $\eta \in \mathbb{R}$. The left- and right-sided Erdelyi-Kóber type fractional integral of order $\alpha > 0$ are defined by

$$(I^{\alpha}_{a+;\sigma;\eta} f)(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int\limits_a^x \frac{t^{\sigma\eta+\sigma-1} f(t)dt}{(x^{\sigma}-t^{\sigma})^{1-\alpha}}, \quad (x > a)$$

and

$$(I^{\alpha}_{b-;\sigma;\eta} f)(x) = \frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)} \int_x^b \frac{t^{\sigma(1-\eta-\alpha)-1} f(t)dt}{(t^{\sigma}-x^{\sigma})^{1-\alpha}}, \quad (x < b)$$

respectively.

We conclude this section with multidimensional fractional integrals. Such type of fractional integrals are usually generalization of the corresponding one-dimensional fractional integral and fractional derivative.

For $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, ..., \alpha_n)$, we use the following notations:

$$\Gamma(\alpha) = (\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)),\ [\mathbf{a}, \mathbf{b}] = [a_1, b_1] \times \cdots \times [a_n, b_n],$$

and by $\mathbf{x} > \mathbf{a}$ we mean $x_1 > a_1, ..., x_n > a_n$.

We define the mixed Riemann-Liouville fractional integrals of order $\alpha > 0$ as

$$(I^{\alpha}_{\mathbf{a}_+} f)(\mathbf{x}) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^{x_1} \cdots \int_{a_n}^{x_n} f(\mathbf{t})(\mathbf{x} - \mathbf{t})^{\alpha-1} d\mathbf{t},\ (\mathbf{x} > \mathbf{a})$$

and

$$(I^{\alpha}_{\mathbf{b}_-} f)(\mathbf{x}) = \frac{1}{\Gamma(\alpha)} \int_{x_1}^{b_1} \cdots \int_{x_n}^{b_n} f(\mathbf{t})(\mathbf{t} - \mathbf{x})^{\alpha-1} d\mathbf{t},\ (\mathbf{x} < \mathbf{b}).$$

Chapter 2

Some new Hardy-type inequalities with general kernels

First, we present some previous recent results and some other preliminaries.

2.1 Preliminaries

Hardy's and Pólya-Knopp's inequality were already given in Preface, see (0.1) and (0.2) respectively. We recall other important inequalities: if $p > 1$ and f is a non-negative function such that $f \in L^p(\mathbb{R}_+)$, then

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}\, dx \right)^p dy \leq \left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \right)^p \int_0^\infty f^p(y)\, dy, \tag{2.1}$$

and if in addition $g \in L^q(\mathbb{R}_+)$ where $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\, dx\, dy \leq \frac{\pi}{\sin \frac{\pi}{p}} \left(\int_0^\infty f^p(x)\, dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)\, dy \right)^{\frac{1}{q}}. \tag{2.2}$$

Moreover, (2.2) is sometimes called Hilbert's or Hardy-Hilbert's inequality even if Hilbert himself only considered the case $p = 2$ (L^p -spaces were defined much later). Note that the constants $\left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^p$ and $\frac{\pi}{\sin\frac{\pi}{p}}$, respectively appearing on the right-hand sides of (2.1) - (2.2), are the best possible, that is, neither of them can be replaced with any smaller constant. Also the constants $\left(\frac{p}{p-1}\right)^p$ and e respectively appearing on the right-hand sides of (0.1) - (0.2), are the best possible, We also note that Hardy's inequality (0.1) shows that the Hardy operator H, defined by setting

$$Hf(x) = \frac{1}{x}\int_0^x f(t)\,dt, \tag{2.3}$$

maps L^p into itself with operator norm $p/(p-1)$. Similarly, (2.1) shows that the operator A, defined by setting

$$Af(x) = \int_0^\infty f(t)(t+x)^{-1}\,dt,$$

maps L^p into itself with operator norm $\pi/(\sin\pi/p)$.

It is now natural to generalize the operators above to the following ones:

$$H_k f(x) = \frac{1}{K(x)}\int_0^x f(t)k(x,t)\,dt, \tag{2.4}$$

where

$$K(x) = \int_0^x k(x,t)\,dt < \infty$$

and (more generally)

$$A_k f(x) = \frac{1}{K(x)}\int_0^\infty f(t)k(x,t)\,dt, \tag{2.5}$$

where now

$$K(x) = \int_0^\infty k(x,t)\,dt < \infty.$$

Here $k(x,y)$ is a general measurable and non-negative function, a so called kernel.

Since Hardy, Hilbert and Pólya established inequalities (0.1), (0.2), (2.1) and (2.2), they have been investigated and generalized in several directions. Further information and remarks concerning the rich history of the integral inequalities mentioned above can be found e.g. in the monographs [51, 68, 74, 75, 82, 92], expository papers [26, 64, 73], and the references cited therein. Besides, here we also emphasize the papers [15, 20, 25, 27, 28, 30, 65, 66, 77, 93, 103, 105], all of which to some extent have guided us in the research we present here.

Recently it was pointed out by S. Kaijser et al. [66] that both (0.1) and (0.2) are just special cases of much more general (Hardy-Knopp's type) inequality

$$\int_0^\infty \Phi\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\frac{dx}{x} \le \int_0^\infty \Phi(f(x))\,\frac{dx}{x}, \tag{2.6}$$

where Φ is a convex function on $\mathbb{R}_+$ and $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ a positive function. Inequality (2.6) follows by using a standard application of Jensen's inequality and the Fubini theorem. By taking $\Phi(x) = x^p$ and $\Phi(x) = e^x$ they obtained an elegant new proof of inequalities (0.1) and (0.2) and showed that both Hardy's and Pólya-Knopp's inequality can be derived by using only a convexity argument.

S. Kaijser et al. [65] proved a more general inequality of Hardy-Knopp's type with a kernel

$$\int_0^\infty u(x)\Phi(A_k f(x))\,\frac{dx}{x} \le \int_0^\infty v(x)\Phi(f(x))\,\frac{dx}{x}, \tag{2.7}$$

where $0 < b \le \infty$, $k : (0,b)\times(0,b) \to \mathbb{R}$ and $u : (0,b) \to \mathbb{R}$ are non-negative functions, such that

$$K(x) = \int_0^x k(x,y)\,dy > 0,\ x \in (0,b), \tag{2.8}$$

and

$$v(y) = y\int_y^b u(x)\,\frac{k(x,y)}{K(x)}\,\frac{dx}{x} < \infty,\ y \in (0,b),$$

Φ is a convex function on an interval $I \subseteq \mathbb{R}$, $f : (0,b) \to \mathbb{R}$ is a function with values in I, and

$$A_k f(x) = \frac{1}{K(x)}\int_0^x k(x,y) f(y)\,dy,\ x \in (0,b). \tag{2.9}$$

On the other hand, Godunova [38] (see also [92, Chapter VIII,p. 233], [39], [41]) proved that the inequality

$$\int_{\mathbb{R}_+^n} \Phi\left(\frac{1}{x_1\cdots x_n}\int_{\mathbb{R}_+^n} l\left(\frac{y_1}{x_1},\dots,\frac{y_n}{x_n}\right) f(y_1,\dots,y_n)d\mathbf{y}\right)\frac{d\mathbf{x}}{x_1\cdots x_n}$$
$$\le \int_{\mathbb{R}_+^n} \frac{\Phi(f(\mathbf{x}))}{x_1\cdots x_n}d\mathbf{x}, \tag{2.10}$$

holds for a non-negative function $l : \mathbb{R}_+^n \to \mathbb{R}_+$, such that

$$\int_{\mathbb{R}_+^n} l(\mathbf{x})d\mathbf{x} = 1,$$

a convex function $\Phi : [0,\infty) \to [0,\infty)$, and a non-negative function f on $\mathbb{R}_+^n$, such that the function $x \mapsto \Phi(f(\mathbf{x}))/(x_1\cdots x_n)$ is integrable on $\mathbb{R}_+^n$.

Remark 2.1 By using the result given in (2.10) Godunova obtained many general inequalities which include, Hardy's (0.1), Pólya-Knopp's (0.2) and Hardy-Hilbert's inequality (2.2). For more details see [92, Chapter VIII,p. 234]).

The following result was recently proved by Kaijser et al. [65]:

Theorem 2.1 *Let u be a weight function on $(0,b)$, $0<b\le\infty$, and let $k(x,y)\ge 0$ on $(0,b)\times(0,b)$. Assume that $\frac{k(x,y)u(x)}{xK(x)}$ is locally integrable on $(0,b)$ for each fixed $y\in(0,b)$ and define v by*

$$v(y)=y\int_y^b \frac{k(x,y)}{K(x)}u(x)\frac{dx}{x}<\infty,\ y\in(0,b).$$

If Φ is a positive and convex function on (a,c), $-\infty\le a<c\le\infty$, then

$$\int_0^b u(x)\Phi(H_k f(x))\frac{dx}{x}\le\int_0^b v(x)\Phi(f(x))\frac{dx}{x}, \tag{2.11}$$

for all f with $a<f(x)<c$, $0\le x\le b$, where H_k is defined by (2.4).

In the same paper the dual operator $H_{\overline{k}}$, defined by

$$H_{\overline{k}}f(x):=\frac{1}{\overline{K}(x)}\int_x^\infty k(x,y)f(y)dy, \tag{2.12}$$

where $\overline{K}(x)=\int_x^\infty k(x,y)dy<\infty$, was studied and the following result was proved:

Theorem 2.2 *For $0\le b<\infty$, let u be a weight function such that $\frac{k(x,y)u(x)}{x\overline{K}(x)}$ is locally integrable on (b,∞) for every fixed $y\in(b,\infty)$. Let the function v be defined by*

$$v(y)=y\int_b^y \frac{k(x,y)}{\overline{K}(x)}u(x)\frac{dx}{x}<\infty,\ y\in(b,\infty).$$

If Φ is a positive and convex function on (a,c), $-\infty\le a<c\le\infty$, then

$$\int_b^\infty u(x)\Phi(H_{\overline{k}}f(x))\frac{dx}{x}\le\int_b^\infty v(x)\Phi(f(x))\frac{dx}{x}, \tag{2.13}$$

for all f with $a<f(x)<c$, $0\le x\le b$, where $H_{\overline{k}}$ is defined by (2.12).

The most general result so far for the operator H_k is the following by Kaijser et. al [65, Theorem 4.4]:

Theorem 2.3 *Let $1 < p \leq q < \infty, 0 < b \leq \infty, s \in (1, p)$, let Φ be a convex and strictly monotone function on $I = (a, c), -\infty \leq a < c \leq \infty$, let H_k be defined by (2.4) and let $u(x)$ and $v(x)$ be weight functions on $[0, b]$. Then the inequality*

$$\left(\int_0^b u(x) \left[\Phi(H_k f(x))\right]^q \frac{dx}{x} \right)^{\frac{1}{q}} \leq C \left(\int_0^b v(x) \Phi^p(f(x)) \frac{dx}{x} \right)^{\frac{1}{p}} \tag{2.14}$$

holds for some finite constant C and all functions f such that $Im f \subseteq I$ if

$$A(s) := \sup_{0<t\leq b} \left(\int_t^b u(x) \left(\frac{k(x,t)}{K(x)} \right)^q V^{\frac{q(p-s)}{p}}(x) \frac{dx}{x} \right)^{\frac{1}{q}} V^{\frac{s-1}{p}}(t) < \infty$$

holds, where $V(t) := \int_0^t v^{1-p'}(x) x^{p'-1} dx$. Moreover, if C is the best constant in (2.14), then

$$C \leq \inf_{1<s<p} \left(\frac{p-1}{p-s} \right)^{\frac{1}{p'}} A(s).$$

For our further discussions we also mention the following recent result by Oguntuase et. al [83]:

Theorem 2.4 *Let $\mathbf{b} \in (0, \infty], -\infty \leq a < c \leq \infty$ and let Φ be a positive function on $[a, c]$. Suppose that the weight function u defined on $(\mathbf{0}, \mathbf{b})$ is non-negative such that $\frac{u(x_1,\ldots,x_n)}{x_1^2 \cdots x_n^2}$ is locally integrable on $(\mathbf{0}, \mathbf{b})$ and the weight function v is defined by*

$$v(t_1, \ldots, t_n) = t_1 \cdots t_n \int_{t_1}^{b_1} \cdots \int_{t_n}^{b_n} \frac{u(x_1, \ldots, x_n)}{x_1^2 \cdots x_n^2} dx_1 \cdots dx_n, \, \mathbf{t} \in (\mathbf{0}, \mathbf{b}).$$

(i) If Φ is convex, then

$$\int_0^{b_1} \cdots \int_0^{b_n} u(x_1, \ldots, x_n) \Phi \left(\frac{1}{x_1 \cdots x_n} \int_0^{x_1} \cdots \int_0^{x_n} f(t_1, \ldots, t_n) dt_1 \cdots dt_n \right) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}$$
$$\leq \int_0^{b_1} \cdots \int_0^{b_n} v(x_1, \ldots, x_n) \Phi\left(f(x_1, \ldots, x_n)\right) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}$$

holds for every function f on $(\mathbf{0}, \mathbf{b})$ such that $a < f(x_1, \ldots, x_n) < c$.

(ii) If Φ is concave, then

$$\int_0^{b_1} \cdots \int_0^{b_n} u(x_1, \ldots, x_n) \Phi \left(\frac{1}{x_1 \cdots x_n} \int_0^{x_1} \cdots \int_0^{x_n} f(t_1, \ldots, t_n) dt_1 \cdots dt_n \right) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}$$
$$\geq \int_0^{b_1} \cdots \int_0^{b_n} v(x_1, \ldots, x_n) \Phi\left(f(x_1, \ldots, x_n)\right) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}$$

holds for every function f on $(\mathbf{0},\mathbf{b})$ such that $a < f(x_1,\ldots,x_n) < c$.

Remark 2.2 Also the obvious dual result was formulated and proved in [83]. For further developments in this directions even with a general kernel see [84] and [87].

2.2 The main results

In the sequel let $(\Omega_1,\Sigma_1,\mu_1)$, $(\Omega_2,\Sigma_2,\mu_2)$ be measure spaces and let A_k from (2.5) be generalized as follows:

$$A_k f(x) := \frac{1}{K(x)} \int_{\Omega_2} k(x,y) f(y)\, d\mu_2(y), \tag{2.15}$$

where $f : \Omega_2 \to \mathbb{R}$ is a measurable function, $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is a measurable and non-negative kernel and

$$K(x) := \int_{\Omega_2} k(x,y)\, d\mu_2(y) < \infty,\, x \in \Omega_1. \tag{2.16}$$

Our first result reads (see also [70]):

Theorem 2.5 *Let $(\Omega_1,\Sigma_1,\mu_1)$ and $(\Omega_2,\Sigma_2,\mu_2)$ be measure spaces with positive σ-finite measures, u be a weight function on Ω_1, k a non-negative measurable function on $\Omega_1 \times \Omega_2$, and K be defined on Ω_1 by* (2.16).
Suppose that $K(x) > 0$ for all $x \in \Omega_1$, that the function $x \mapsto u(x)\frac{k(x,y)}{K(x)}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$, and that v is defined on Ω_2 by

$$v(y) := \int_{\Omega_1} u(x) \frac{k(x,y)}{K(x)}\, d\mu_1(x) < \infty. \tag{2.17}$$

If Φ is a convex function on an interval $I \subseteq \mathbb{R}$, then the inequality

$$\int_{\Omega_1} u(x)\Phi(A_k f(x))\, d\mu_1(x) \le \int_{\Omega_2} v(y)\Phi(f(y))\, d\mu_2(y) \tag{2.18}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$, such that $Im f \subseteq I$, where A_k is defined by (2.15).

Proof. For an arbitrary $x \in \Omega_1$ let the function $h_x : \Omega_2 \to \mathbb{R}$ be defined by $h_x(y) = f(y) - A_k f(x)$. Then we have

$$\begin{aligned}\int_{\Omega_2} k(x,y)h_x(y)\, d\mu_2(y) &= \int_{\Omega_2} k(x,y)f(y)\, d\mu_2(y) - \int_{\Omega_2} k(x,y)A_k f(x)\, d\mu_2(y)\\ &= K(x)A_k f(x) - A_k f(x)\int_{\Omega_2} k(x,y)\, d\mu_2(y)\\ &= 0,\ x \in \Omega_1. \end{aligned} \tag{2.19}$$

First we must prove that $A_k f(x) \in I$, for all $x \in \Omega_1$. The motivation for this is that $A_k f(x)$ is simply a generalized mean and since $f(y) \in I$ for all $y \in \Omega_2$ (by assumption) also the mean $A_k f(x) \in I$. We also include a more formal proof of this fact:
Suppose that there exists $x_0 \in \Omega_1$ such that $A_k f(x_0) \notin I$. Since I is an interval in $\mathbb{R}$ and $f(\Omega_2) \subseteq I$, it follows that either $A_k f(x_0) > f(y)$ for all $y \in \Omega_2$, or $A_k f(x_0) < f(y)$ for all $y \in \Omega_2$. Hence, the function h_{x_0} is is either strictly positive or strictly negative on Ω_2. Moreover, $k(x_0,y)h_{x_0}(y) \geq 0$ for all $y \in \Omega_2$ or $k(x_0,y)h_{x_0}(y) \leq 0$ for all $y \in \Omega_2$. On the other hand, $K(x_0) > 0$ implies that there is a set $\widetilde{\Omega_2} \in \Sigma_2$ such that $\mu_2(\widetilde{\Omega_2}) > 0$ and $k(x_0,y) > 0$, $y \in \widetilde{\Omega_2}$. Therefore, the function $y \mapsto k(x_0,y)h_{x_0}(y)$ does not change the sign on Ω_2 and is strictly positive or strictly negative on $\widetilde{\Omega_2}$, so $\int_{\Omega_2} k(x_0,y)h_{x_0}(y)\,d\mu_2(y) \neq 0$, which contradicts (2.19). Thus $A_k f(x) \in I$, for all $x \in \Omega_1$. Note that if $A_k f(x)$ is an end-point of I for some $x \in \Omega_1$ (in cases when I is not an open interval), then h_x (or $-h_x$) will be a non-negative function whose integral over Ω_2, with respect to the measure μ_2, is equal to 0. Therefore, $h_x \equiv 0$, that is, $f(y) = A_k f(x)$ holds for μ_2-a.e. $y \in \Omega_2$.
Now, let us prove the inequality (2.18). By using Jensen's inequality and the Fubini theorem we find that

$$
\begin{aligned}
\int_{\Omega_1} u(x)\Phi\left(A_k f(x)\right) d\mu_1(x) &= \int_{\Omega_1} u(x)\Phi\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)f(y)d\mu_2(y)\right) d\mu_1(x) \\
&\leq \int_{\Omega_1} \frac{u(x)}{K(x)}\left(\int_{\Omega_2} k(x,y)\Phi(f(y))d\mu_2(y)\right) d\mu_1(x) \\
&= \int_{\Omega_2} \Phi(f(y))\left(\int_{\Omega_1} u(x)\frac{k(x,y)}{K(x)}d\mu_1(x)\right) d\mu_2(y) \\
&= \int_{\Omega_2} v(y)\Phi(f(y))d\mu_2(y)
\end{aligned}
$$

and the proof is complete. □

Example 2.1 By applying Theorem 2.5 with $\Omega_1 = \Omega_2 = (0,\infty)$ and $k(x,y) = 1$, $0 \leq y \leq x$, $k(x,y) = 0$, $y > x$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$ and $u(x) = \frac{1}{x}$ (so that $v(y) = \frac{1}{y}$), we obtain (2.6) which, in its turn, is equivalent to the original Hardy inequality (0.1) when $\Phi(u) = u^p, p > 1$. □

Example 2.2 Let $\Omega_1 = \Omega_2 = (0,\infty)$, replace $d\mu_1(x)$ and $d\mu_2(y)$ by the Lebesgue measures dx and dy, respectively, let $k(x,y) = \frac{(\frac{y}{x})^{-1/p}}{x+y}$, $p > 1$ and $u(x) = \frac{1}{x}$. Then $K(x) = K = \frac{\pi}{\sin(\pi/p)}$ and $v(y) = \frac{1}{y}$. Let $\Phi(u) = u^p$ then inequality (2.18) reads:

$$
\begin{aligned}
K^{-p}\int_0^\infty \left(\int_0^\infty \left(\frac{y}{x}\right)^{-1/p} \frac{f(y)}{x+y}dy\right)^p \frac{dx}{x} &= K^{-p}\int_0^\infty \left(\int_0^\infty \frac{f(y)}{x+y}y^{-1/p}dy\right)^p dx \\
&\leq \int_0^\infty f^p(y)\frac{dy}{y}
\end{aligned}
$$

Replace $f(t)t^{-1/p}$ with $f(t)$ and we get Hilbert's inequality (2.1). □

Example 2.3 Let $\Omega_1 = \Omega_2 = (0,b)$, $0 < b \leq \infty$, replace $d\mu_1(x)$ and $d\mu_2(y)$ by the Lebesque measures dx and dy, respectively, and let $k(x,y) = 0$ for $x < y \leq b$. Then A_k coincides with the operator H_k defined by (2.4) and if also $u(x)$ is replaced by $u(x)/x$ and $v(x)$ by $v(x)/x$, then (2.18) coincides with (2.11) and we see that Theorem 2.1 is a special case of Theorem 2.5. □

Example 2.4 By arguing as in Example 2.3 but $\Omega_1 = \Omega_2 = (b,\infty)$, $0 \leq b < \infty$ and with kernels such that $k(x,y) = 0$ for $b \leq y < x$ we find that now (2.18) coincides with (2.13) so that also Theorem 2.2 is a special case of Theorem 2.5. □

In the previous examples we derived only inequalities over some subsets of $\mathbb{R}_+$. However, Theorem 2.5 covers much more general situations. We can apply that result to n-dimensional cells in $\mathbb{R}_+^n$ and thus, in particular, obtain a generalization of the Godunova inequality (2.10).

Before presenting our results, it is necessary to introduce some further notation. For $\mathbf{u},\mathbf{v} \in \mathbb{R}_+^n$, $\mathbf{u} = (u_1,u_2,\dots,u_n)$, $\mathbf{v} = (v_1,v_2,\dots,v_n)$, let

$$\frac{\mathbf{u}}{\mathbf{v}} = \left(\frac{u_1}{v_1},\frac{u_2}{v_2},\dots,\frac{u_n}{v_n}\right) \quad \text{and} \quad \mathbf{u}^{\mathbf{v}} = u_1^{v_1}u_2^{v_2}\cdots u_n^{v_n}.$$

In particular, $\mathbf{u}^{\mathbf{1}} = \prod_{i=1}^n u_i$, $\mathbf{u}^{\mathbf{2}} = \left(\prod_{i=1}^n u_i\right)^2$, and $\mathbf{u}^{-\mathbf{1}} = \left(\prod_{i=1}^n u_i\right)^{-1}$, where $\mathbf{n} = (n,n,\dots,n)$. We write $\mathbf{u} < \mathbf{v}$ if componentwise $u_i < v_i$, $i = 1,\dots,n$. Relations $\leq$, $>$, and $\geq$ are defined analogously. Finally, we denote $(\mathbf{0},\mathbf{b}) = \{\mathbf{x} \in \mathbb{R}_+^n : \mathbf{0} < \mathbf{x} < \mathbf{b}\}$ and $(\mathbf{b},\infty) = \{\mathbf{x} \in \mathbb{R}_+^n : \mathbf{b} < \mathbf{x} < \infty\}$.

Applying Theorem 2.5 with $\Omega_1 = \Omega_2 = \mathbb{R}_+^n$, the Lebesgue measure $d\mu_1(\mathbf{x}) = d\mathbf{x}$ and $d\mu_2(\mathbf{y}) = d\mathbf{y}$, and the kernel $k : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}$ of the form $k(\mathbf{x},\mathbf{y}) = l\left(\frac{\mathbf{y}}{\mathbf{x}}\right)$, where $l : \mathbb{R}_+^n \to \mathbb{R}$ is a non-negative measurable function, we obtain the following corollary.

Corollary 2.1 *Let l and u be non-negative measurable functions on $\mathbb{R}_+^n$, such that $0 < L(\mathbf{x}) = \mathbf{x}^{\mathbf{1}} \int_{\mathbb{R}_+^n} l(\mathbf{y})\,d\mathbf{y} < \infty$ for all $\mathbf{x} \in \mathbb{R}_+^n$, and that the function $\mathbf{x} \mapsto u(\mathbf{x})\frac{l\left(\frac{\mathbf{y}}{\mathbf{x}}\right)}{L(\mathbf{x})}$ is integrable on $\mathbb{R}_+^n$ for each fixed $\mathbf{y} \in \mathbb{R}_+^n$. Let the function v be defined on $\mathbb{R}_+^n$ by*

$$v(\mathbf{y}) = \int_{\mathbb{R}_+^n} u(\mathbf{x})\frac{l\left(\frac{\mathbf{y}}{\mathbf{x}}\right)}{L(\mathbf{x})}\,d\mathbf{x}.$$

If Φ is a convex function on an interval $I \subseteq \mathbb{R}$, then the inequality

$$\int_{\mathbb{R}_+^n} u(\mathbf{x})\Phi\left(\frac{1}{L(\mathbf{x})}\int_{\mathbb{R}_+^n} l\left(\frac{\mathbf{y}}{\mathbf{x}}\right)f(\mathbf{y})d\mathbf{y}\right)d\mathbf{x} \leq \int_{\mathbb{R}_+^n} v(\mathbf{y}))\Phi(f(\mathbf{y}))d\mathbf{y}$$

holds for all measurable functions $f : \mathbb{R}_+^n \to \mathbb{R}$ such that $Im f \subseteq I$.

Example 2.5 Especially, for $\int_{\mathbb{R}_+^n} l(\mathbf{t})\,d\mathbf{t} = 1$ and $u(\mathbf{x}) = \mathbf{x}^{-1}$, Corollary 2.1 reduces to Godunova's inequality (2.10). This shows that Corollary 2.1 is a genuine generalization of the Godunova inequality (2.10).

We shall continue by stating a somewhat more general theorem, which is of a type described in Theorem 2.3 but for general measures. More exactly, we state the following generalization of Theorem 2.5:

Theorem 2.6 *Let $0 < p \le q < \infty$ and let the assumptions in Theorem 2.5 be satisfied but now with*

$$v(y) := \left(\int_{\Omega_1} u(x) \left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}} d\mu_1(x)\right)^{\frac{p}{q}} < \infty. \tag{2.20}$$

If Φ is a positive convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\left(\int_{\Omega_1} u(x)\,[\Phi(A_k f(x))]^{\frac{q}{p}}\, d\mu_1(x)\right)^{\frac{1}{q}} \le \left(\int_{\Omega_2} v(y)\Phi(f(y))\,d\mu_2(y)\right)^{\frac{1}{p}} \tag{2.21}$$

holds for all measurable functions $f : \Omega_2 \to R$, such that $Im f \subseteq I$.

Proof. As in the proof of Theorem 2.5 we first note that $A_k f(x) \in I$, for all $x \in \Omega_1$. Moreover, by using Jensen's inequality and then Minkowski's general integral inequality we find that

$$\begin{aligned}
&\left(\int_{\Omega_1} u(x)\,[\Phi\,(A_k f(x))]^{\frac{q}{p}}\, d\mu_1(x)\right)^{\frac{1}{q}} \\
&= \left(\int_{\Omega_1} u(x) \left[\Phi\left(\frac{1}{K(x)} \int_{\Omega_2} k(x,y) f(y)\, d\mu_2(y)\right)\right]^{\frac{q}{p}} d\mu_1(x)\right)^{\frac{1}{q}} \\
&\le \left(\int_{\Omega_1} u(x) \left[\frac{1}{K(x)} \int_{\Omega_2} k(x,y)\Phi(f(y))\, d\mu_2(y)\right]^{\frac{q}{p}} d\mu_1(x)\right)^{\frac{1}{q}} \\
&\le \left(\int_{\Omega_2} \Phi(f(y)) \left(\int_{\Omega_1} u(x) \left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}} d\mu_1(x)\right)^{\frac{p}{q}} d\mu_2(y)\right)^{\frac{1}{p}} \\
&= \left(\int_{\Omega_2} v(y)\Phi(f(y))\, d\mu_2(y))\right)^{\frac{1}{p}}
\end{aligned}$$

and the proof is complete. □

For the case $p = q$ we obtain Theorem 2.5 and as expected by applying Theorem 2.6 we obtain the following further generalization of the Godunova result:

Corollary 2.2 *Let $0 < p \leq q < \infty$ and let the assumptions in Corollary 2.1 be satisfied with v defined by*

$$v(\mathbf{y}) = \left(\int_{\mathbb{R}^n_+} u(\mathbf{x}) \left(\frac{l\left(\frac{\mathbf{y}}{\mathbf{x}}\right)}{L(\mathbf{x})} \right)^{\frac{q}{p}} d\mathbf{x} \right)^{\frac{p}{q}}.$$

If Φ is a positive convex function on an interval $I \subseteq \mathbb{R}$, then the inequality

$$\left(\int_{\mathbb{R}^n_+} u(\mathbf{x}) \left[\Phi\left(\frac{1}{L(\mathbf{x})} \int_{\mathbb{R}^n_+} l\left(\frac{\mathbf{y}}{\mathbf{x}}\right) f(\mathbf{y}) d\mathbf{y} \right) \right]^{\frac{q}{p}} d\mathbf{x} \right)^{\frac{1}{q}} \leq \left(\int_{\mathbb{R}^n_+} v(\mathbf{y}) \Phi(f(\mathbf{y})) d\mathbf{y} \right)^{\frac{1}{p}}$$

holds for all measurable functions $f : \mathbb{R}^n_+ \to \mathbb{R}$ such that $Imf \subseteq I$.

Proof. The proof only consists of obvious modifications in the proof of Corollary 2.1 so we omit the details. □

Example 2.6 By using Theorem 2.6 with $\Omega_1 = \Omega_2 = (0,b)$, $0 < b \leq \infty$, $k(x,y) = 0$ for $x < y < b$, $u(x)$ replaced by $u(x)/x$ and $v(y)$ replaced by $v(y)/y$ we obtain the inequality

$$\left(\int_0^b u(x) \left[\Phi(H_k f(x))\right]^{\frac{q}{p}} \frac{d\mu_1(x)}{x} \right)^{\frac{1}{q}} \leq \left(\int_0^b v(y) \Phi(f(y)) \frac{d\mu_2(y)}{y} \right)^{\frac{1}{p}},$$

where $v(y)$ is defined by (2.20). For Φ replaced by Φ^p, $1 < p \leq q < \infty$ (Φ^p is convex function) this inequality is similar to (2.14). However, these results are not comparable but we conjecture that Theorem 2.3 can be generalized also to the case with general measures even to a multidimensional setting. □

We finish this Section by stating the following useful fact:

Remark 2.3 Let the assumptions of Theorem 2.6 be satisfied. By applying Theorem 2.6 with $\Phi(x) = x$ we get the following inequality:

$$\left(\int_{\Omega_1} u(x) \left[A_k f(x)\right]^{\frac{q}{p}} d\mu_1(x) \right)^{\frac{1}{q}} \leq \left(\int_{\Omega_2} v(y) f(y) \, d\mu_2(y) \right)^{\frac{1}{p}}. \tag{2.22}$$

Now replace $f(x)$ with $\Phi(f(x))$ and we get that

$$\left(\int_{\Omega_1} u(x) \left[A_k \Phi(f(x))\right]^{\frac{q}{p}} d\mu_1(x) \right)^{\frac{1}{q}} \leq \left(\int_{\Omega_2} v(y) \Phi(f(y)) \, d\mu_2(y) \right)^{\frac{1}{p}}. \tag{2.23}$$

On the other hand, applying Jensen's inequality to the left side of inequality (2.23) we obtain that

$$
\begin{aligned}
&\left(\int_{\Omega_1} u(x)\,[A_k\Phi(f(x))]^{\frac{q}{p}}\,d\mu_1(x)\right)^{\frac{1}{q}} \\
&= \left(\int_{\Omega_1} u(x)\left[\frac{1}{K(x)}\int_{\Omega_2} k(x,y)\Phi(f(y))\,d\mu_2(y)\right]^{\frac{q}{p}} d\mu_1(x)\right)^{\frac{1}{q}} \\
&\geq \left(\int_{\Omega_1} u(x)\left[\Phi\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)f(y)\,d\mu_2(y)\right)\right]^{\frac{q}{p}} d\mu_1(x)\right)^{\frac{1}{q}} \\
&= \left(\int_{\Omega_1} u(x)\,[\Phi(A_k f(x))]^{\frac{q}{p}}\,d\mu_1(x)\right)^{\frac{1}{q}},
\end{aligned}
$$

i.e., by (2.23) that (2.21) holds. We conclude that if the assumptions of Theorem 2.6 hold, then each of (2.21), (2.22) and (2.23) holds and are equivalent.

2.3 Remarks and examples

Remark 2.4 By applying Theorem 2.5 for special cases e.g. for kernels with additional homogeneity properties, $\Phi(u) = u^p$, $p > 1$, and making some obvious variable transformations we obtain what in the literature is usually called Hilbert type or Hardy-Hilbert type inequalities, see e.g. Example 2.2 for the original case.

However, by keeping our convex functions we obtain further generalizations of Hilbert type inequalities. Here we only give two simple examples.

Example 2.7 Let $\Omega_1 = \Omega_2 = (0,\infty), d\mu_1(x) = dx$, $d\mu_2(y) = dy$. For $k(x,y) = (x+y)^{-s}$, $s > 1$, we have $K(x) = \frac{x^{1-s}}{s-1}$ and

$$
v(y) = (s-1)\int_0^\infty (x+y)^{-s}x^{s-1}u(x)dx.
$$

Let $u(x) = x^{1-t-s}$, $t \in (1-s,1)$.
Then we have

$$
v(y) = (s-1)\int_0^\infty (x+y)^{-s}x^{s-1}x^{1-t-s}dx = (s-1)y^{1-t-s}B(1-t,s+t-1),
$$

where $B(.,.)$ is the usual Beta function.
By applying Theorem 2.5 we get the following inequality:

$$\int_0^\infty x^{1-t-s}\Phi(A_k f(x))dx \le (s-1)B(1-t,s+t-1)\int_0^\infty y^{1-t-s}\Phi(f(y))dy,$$

where Φ is a convex function and $A_k f(x)$ is defined by (2.15). □

Example 2.8 Let $\Omega_1 = \Omega_2 = (0,\infty), d\mu_1(x) = dx, d\mu_2(y) = dy$,

$$u(x) = x^{-2\alpha} \text{ and } k(x,y) = \frac{\log y - \log x}{y-x}\left(\frac{y}{x}\right)^{-\alpha}, \alpha \in (0,1).$$

Evidently, it is homogeneous of degree -1, $K(x)$ converges for all $\alpha \in (0,1)$, and we have

$$\begin{aligned} K(x) &= \int_0^\infty \frac{\log y - \log x}{y-x}\left(\frac{y}{x}\right)^{-\alpha} dy = \int_0^\infty \frac{\log u}{u-1}u^{-\alpha}du \\ &= \int_{-\infty}^{\infty} \frac{te^{(1-\alpha)t}}{e^t-1}dt = \Psi'(\alpha) + \Psi'(1-\alpha) = \frac{\pi^2}{sin^2 \pi\alpha}, \end{aligned}$$

where $\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, $x > 0$, is the Digamma function and we used the identity $\Psi(1-x) = \Psi(x) + \pi \cot \pi x$, $x \in (0,1)$ (for details on Ψ see [1]). Then we have

$$\begin{aligned} v(y) &= \frac{sin^2 \pi\alpha}{\pi^2}\int_0^\infty \frac{\log x - \log y}{x-y}\left(\frac{y}{x}\right)^{\alpha} y^{-2\alpha}dx \\ &= \frac{sin^2 \pi\alpha}{\pi^2} y^{-2\alpha}\int_0^\infty \frac{\log u}{u-1}u^{-\alpha}du = y^{-2\alpha}, \ (x = yu) \end{aligned}$$

Therefore, by applying (2.18) we get the following inequality:

$$\int_0^\infty \Phi\left(\frac{sin^2 \pi\alpha}{\pi^2}\int_0^\infty \frac{\log y - \log x}{y-x}\left(\frac{y}{x}\right)^{-\alpha} f(y)dy\right) x^{-2\alpha}dx \le \int_0^\infty y^{-2\alpha}\Phi(f(y))dy,$$

where Φ is a convex function. □

Moreover, by applying our result with the convex function $\Phi(x) = e^x$ and making some suitable variable transformations we obtain what in the literature is called Pólya-Knopp type inequalities. We give the following example:

Example 2.9 Let the assumptions in Theorem 2.5 be satisfied. Then, by applying (2.18) with $\Phi(x) = e^x$, and f replaced by $\log f^p$, $p > 0$ we obtain that

$$\int_{\Omega_1} u(x) \left[\exp\left(\frac{1}{K(x)} \int_{\Omega_2} k(x,y) \log f(y)\, d\mu_2(y)\right)\right]^p d\mu_1(x) \\ \leq \int_{\Omega_2} v(y) f^p(y)\, d\mu_2(y), \tag{2.24}$$

where $k(x,y), K(x), u(x)$ and $v(y)$ are defined as in Theorem 2.5 In particular, if $p = 1, \Omega_1 = \Omega_2 = (0,\infty), k(x,y) = 1, 0 < y < x, k(x,y) = 0, y \geq x$. (so that $K(x) = x$), $d\mu_1(x) = dx$, $d\mu_2(y) = dy$, $u(x) = 1/x$ (so that $v(x) = 1/x$) replacing $f(x)/x$ by $f(x)$ and making a simple calculation we find that (2.24) is equal to

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \log f(y)\, dy\right) dx \leq e \int_0^\infty f(y)\, dy,$$

which is the classical form of Pólya-Knopp's inequality. □

We continue with the following special cases of Theorem 2.5 (see [63]).

Let $(\Omega_1,\Sigma_1,\mu_1)$ and $(\Omega_2,\Sigma_2,\mu_2)$ be measure spaces with positive $\sigma-$finite measures, $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a non-negative function, and K be defined by (2.16). Let $U(k)$ denote the class of measurable functions $g : \Omega_1 \to \mathbb{R}$ with the representation

$$g(x) = \int_{\Omega_2} k(x,y) f(y) d\mu_2(y), \tag{2.25}$$

where $f : \Omega_2 \to \mathbb{R}$ is a measurable function.

If we substitute $k(x,y)$ by $k(x,y)f_2(y)$ and f by $\frac{f_1}{f_2}$, where $f_i : \Omega_2 \to \mathbb{R}, (i = 1,2)$ are measurable functions in Theorem 2.5 we obtain the following result.

Theorem 2.7 *Let $(\Omega_1,\Sigma_1,\mu_1)$ and $(\Omega_2,\Sigma_2,\mu_2)$ be measure spaces with σ-finite measures, u be a weight function on Ω_1, k be a non-negative measurable function on $\Omega_1 \times \Omega_2$. Assume that the function $x \mapsto u(x)\frac{k(x,y)}{g_2(x)}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$. Define v on Ω_2 by*

$$v(y) := f_2(y) \int_{\Omega_1} u(x) \frac{k(x,y)}{g_2(x)} d\mu_1(x) < \infty. \tag{2.26}$$

If $\Phi : I \to \mathbb{R}$ is a convex function and $\frac{g_1(x)}{g_2(x)}, \frac{f_1(y)}{f_2(y)} \in I$, then the inequality

$$\int_{\Omega_1} u(x) \Phi\left(\frac{g_1(x)}{g_2(x)}\right) d\mu_1(x) \leq \int_{\Omega_2} v(y) \Phi\left(\frac{f_1(y)}{f_2(y)}\right) d\mu_2(y), \tag{2.27}$$

holds for all $g_i \in U(k), (i = 1,2)$ and for all measurable functions $f_i : \Omega_2 \to \mathbb{R}$, $(i = 1,2)$.

Remark 2.5 If Φ is strictly convex on I and $\frac{f_1(x)}{f_2(x)}$ is non-constant, then the inequality given in (2.27) is strict.

Remark 2.6 If we take $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$ and $d\mu_2(y) = dy$ in Theorem 2.7, we obtain the result given in Theorem 2.1 in [63].

Here we give Hardy's inequality in quotient.

Theorem 2.8 *Let u be a weight function defined on $(0,\infty)$. Define v on $(0,\infty)$ by*

$$v(y) = f_2(y) \int_y^\infty \left(\int_0^x f_2(y)dy \right)^{-1} u(x)dx < \infty. \tag{2.28}$$

If Φ is a convex function on the interval $I \subseteq \mathbb{R}$, then the following inequality

$$\int_0^\infty u(x)\Phi\left(\frac{\int_0^x f_1(y)dy}{\int_0^x f_2(y)dy} \right) dx \leq \int_0^\infty v(y)\Phi\left(\frac{f_1(y)}{f_2(y)} \right) dy \tag{2.29}$$

holds for all measurable functions $f_i : (0,\infty) \to \mathbb{R}, (i = 1,2)$, such that $\frac{f_1(y)}{f_2(y)} \in I$.

Proof. Rewrite the inequality (2.27) with $\Omega_1 = \Omega_2 = \mathbb{R}_+$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$. Let us define the kernel $k : \mathbb{R}_+^2 \to \mathbb{R}$ by

$$k(x,y) = \begin{cases} 1, & 0 < y \leq x; \\ 0, & y > x, \end{cases} \tag{2.30}$$

then g_i defined in (2.25) takes the form

$$g_i(x) = \int_0^x f_i(y)dy. \tag{2.31}$$

Substitute $g_i(x),\ (i = 1,2)$ in (2.27), so we get (2.29). □

Example 2.10 If we take $\Phi(x) = x^p$, $p \geq 1$ and particular weight function $u(x) = \frac{1}{x^2} \int_0^x f_2(y)dy$, $x \in (0,\infty)$ in (2.28), we obtain $v(y) = \frac{f_2(y)}{y}$ and the inequality (2.29) becomes

$$\int_0^\infty \left(\int_0^x f_1(y)dy \right)^p \left(\int_0^x f_2(y)dy \right)^{1-p} \frac{dx}{x^2} \leq \int_0^\infty f_1^p(y) f_2^{1-p}(y) \frac{dy}{y}. \tag{2.32}$$

If we put $f_2(y) = 1$, in (2.32), we obtain the following inequality (for details see [70] and [66]):

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f_1(y)dy \right)^p \frac{dx}{x} \leq \int_0^\infty f_1^p(y) \frac{dy}{y}. \tag{2.33}$$

Remark 2.7 As a special case of Theorem 2.7 the well-known Pólya-Knopp, Hardy-Hilbert and other inequalities in quotients can be given, but her we omit the details.

Chapter 3

On an inequality of G. H. Hardy

In this chapter we give result involving the inequality of G. H. Hardy (1.12). To start with let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive $\sigma-$finite measures, $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a non-negative function, and K be defined by (2.16). Let $U(k)$ denote the class of functions $g : \Omega_1 \to \mathbb{R}$ with the representation

$$g(x) = \int_{\Omega_2} k(x,y) f(y) d\mu_2(y),$$

where $f : \Omega_2 \to \mathbb{R}$ is a measurable function.

Our first result is given in the following theorem (see [61]).

Theorem 3.1 *Let u be a weight function on Ω_1, k a non-negative measurable function on $\Omega_1 \times \Omega_2$, and K be defined on Ω_1 by (2.16). Assume that the function $x \mapsto u(x)\frac{k(x,y)}{K(x)}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$ and v is defined by (2.17). If $\phi : (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality*

$$\int_{\Omega_1} u(x)\phi\left(\left|\frac{g(x)}{K(x)}\right|\right) d\mu_1(x) \leq \int_{\Omega_2} v(y)\phi(|f(y)|) d\mu_2(y) \tag{3.1}$$

holds for all $g \in U(k)$ such that $g(x) = \int_{\Omega_2} k(x,y) f(y) d\mu_2(y)$.

Proof. By using Jensen's inequality and the Fubini theorem, since ϕ is increasing function, we find that

$$\begin{aligned}\int_{\Omega_1} u(x)\phi\left(\left|\frac{g(x)}{K(x)}\right|\right)d\mu_1(x) &= \int_{\Omega_1} u(x)\phi\left(\left|\frac{1}{K(x)}\int_{\Omega_2} k(x,y)f(y)d\mu_2(y)\right|\right)d\mu_1(x)\\ &\le \int_{\Omega_1}\frac{u(x)}{K(x)}\left(\int_{\Omega_2} k(x,y)\phi(|f(y)|)d\mu_2(y)\right)d\mu_1(x)\\ &= \int_{\Omega_2}\phi(|f(y)|)\left(\int_{\Omega_1} u(x)\frac{k(x,y)}{K(x)}d\mu_1(x)\right)d\mu_2(y)\\ &= \int_{\Omega_2} v(y)\phi(|f(y)|)d\mu_2(y)\end{aligned}$$

and the proof is complete. □

As a special case of Theorem 3.1 we get the following result.

Corollary 3.1 *Let u be a weight function on (a,b) and $\alpha > 0$. $I^{\alpha}_{a+}f$ denotes the Riemann-Liouville fractional integral of f. Define v on (a,b) by*

$$v(y) := \alpha\int_y^b u(x)\frac{(x-y)^{\alpha-1}}{(x-a)^{\alpha}}dx < \infty.$$

If $\phi : (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_a^b u(x)\phi\left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}|I^{\alpha}_{a+}f(x)|\right)dx \le \int_a^b v(y)\phi(|f(y)|)\,dy \tag{3.2}$$

holds.

Proof.
Applying Theorem 3.1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \frac{(x-y)^{\alpha-1}}{\Gamma(\alpha)}, & a \le y \le x;\\ 0, & x < y \le b\end{cases}$$

we get that $K(x) = \frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)}$ and $g(x) = I^{\alpha}_{a+}f(x)$, so (3.2) follows. □

Remark 3.1 In particular for the weight function $u(x) = (x-a)^{\alpha}$, $x \in (a,b)$ in Corollary 3.1 we obtain the inequality

$$\int_a^b (x-a)^{\alpha}\phi\left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}|I^{\alpha}_{a+}f(x)|\right)dx \le \int_a^b (b-y)^{\alpha}\phi(|f(y)|)\,dy. \tag{3.3}$$

Although (3.1) holds for all convex and increasing functions, some choices of ϕ are of particular interest. Namely, we shall consider power function. Let $q > 1$ and the function $\phi : \mathbb{R}_+ \to \mathbb{R}$ be defined by $\phi(x) = x^q$, then (3.3) reduces to

$$\int_a^b (x-a)^{\alpha}\left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}|I^{\alpha}_{a+}f(x)|\right)^q dx \le \int_a^b (b-y)^{\alpha}|f(y)|^q\,dy. \tag{3.4}$$

Since $x \in (a,b)$ and $\alpha(1-q) < 0$, then we obtain that the left-hand side of (3.4) satisfies

$$\int_a^b (x-a)^\alpha \left(\frac{\Gamma(\alpha+1)}{(x-a)^\alpha} |I_{a+}^\alpha f(x)| \right)^q dx$$
$$\geq (b-a)^{\alpha(1-q)} (\Gamma(\alpha+1))^q \int_a^b |I_{a+}^\alpha f(x)|^q \, dx \tag{3.5}$$

and the right-hand side of (3.4) satisfies

$$\int_a^b (b-y)^\alpha |f(y)|^q \, dy \leq (b-a)^\alpha \int_a^b |f(y)|^q \, dy. \tag{3.6}$$

Combining (3.5) and (3.6) we get

$$\int_a^b |I_{a+}^\alpha f(x)|^q \, dx \leq \left(\frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \right)^q \int_a^b |f(y)|^q \, dy. \tag{3.7}$$

Taking power $\frac{1}{q}$ on both sides we obtain (1.12), that is the inequality of G. H. Hardy.

Corollary 3.2 *Let u be a weight function on (a,b) and $\alpha > 0$. $I_{b-}^\alpha f$ denotes the Riemann-Liouville fractional integral of f. Define v on (a,b) by*

$$v(y) := \alpha \int_a^y u(x) \frac{(y-x)^{\alpha-1}}{(b-x)^\alpha} dx < \infty.$$

If $\phi : (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_a^b u(x) \phi \left(\frac{\Gamma(\alpha+1)}{(b-x)^\alpha} |I_{b-}^\alpha f(x)| \right) dx \leq \int_a^b v(y) \phi(|f(y)|) \, dy \tag{3.8}$$

holds.

Proof.
Applying Theorem 3.1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \frac{(y-x)^{\alpha-1}}{\Gamma(\alpha)}, & x < y \leq b; \\ 0, & a \leq y \leq x \end{cases}$$

we get that $K(x) = \frac{(b-x)^\alpha}{\Gamma(\alpha+1)}$ and $g(x) = I_{b-}^\alpha f(x)$, so (3.8) follows. $\square$

Remark 3.2 In particular for the weight function $u(x) = (b-x)^\alpha$, $x \in (a,b)$ in Corollary 3.2 we obtain the inequality

$$\int_a^b (b-x)^\alpha \phi \left(\frac{\Gamma(\alpha+1)}{(b-x)^\alpha} |I_{b-}^\alpha f(x)| \right) dx \leq \int_a^b (y-a)^\alpha \phi(|f(y)|) \, dy. \tag{3.9}$$

Let $q > 1$ and the function $\phi : \mathbb{R}_+ \to \mathbb{R}$ be defined by $\phi(x) = x^q$. Then (3.9) reduces to

$$\int_a^b (b-x)^\alpha \left(\frac{\Gamma(\alpha+1)}{(b-x)^\alpha} |I_{b-}^\alpha f(x)| \right)^q dx \le \int_a^b (y-a)^\alpha |f(y)|^q \, dy. \tag{3.10}$$

Since $x \in (a,b)$ and $\alpha(1-q) < 0$, then we obtain that the left-hand side of (3.10) satisfies

$$\int_a^b (b-x)^\alpha \left(\frac{\Gamma(\alpha+1)}{(b-x)^\alpha} |I_{b-}^\alpha f(x)| \right)^q dx \ge (b-a)^{\alpha(1-q)} (\Gamma(\alpha+1))^q \int_a^b |I_{b-}^\alpha f(x)|^q \, dx \tag{3.11}$$

and the right-hand side of (3.10) satisfies

$$\int_a^b (y-a)^\alpha |f(y)|^q \, dy \le (b-a)^\alpha \int_a^b |f(y)|^q \, dy. \tag{3.12}$$

Combining (3.11) and (3.12) we get

$$\int_a^b |I_{b-}^\alpha f(x)|^q \, dx \le \left(\frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \right)^q \int_a^b |f(y)|^q \, dy. \tag{3.13}$$

Taking power $\frac{1}{q}$ on both sides we obtain (1.12), that is the inequality of G. H. Hardy.

Theorem 3.2 *Let $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$, $I_{a+}^\alpha f$ and $I_{b-}^\alpha f$ denote the Riemann-Liouville fractional integral of f. Then the following inequalities*

$$\int_a^b |I_{a+}^\alpha f(x)|^q \, dx \le C \int_a^b |f(y)|^q \, dy \tag{3.14}$$

and

$$\int_a^b |I_{b-}^\alpha f(x)|^q \, dx \le C \int_a^b |f(y)|^q \, dy \tag{3.15}$$

hold, where $C = \frac{(b-a)^{q\alpha}}{(\Gamma(\alpha))^q q\alpha (p(\alpha-1)+1)^{q-1}}$.

Proof.
We will prove only inequality (3.14), since the proof of (3.15) is analogous. We have

$$|(I_{a+}^\alpha f)(x)| \le \frac{1}{\Gamma(\alpha)} \int_a^x |f(t)| (x-t)^{\alpha-1} dt.$$

Then by the Hölder inequality the right-hand of the above inequality is

$$\begin{aligned}
&\le \frac{1}{\Gamma(\alpha)} \left(\int_a^x (x-t)^{p(\alpha-1)} dt \right)^{\frac{1}{p}} \left(\int_a^x |f(t)|^q dt \right)^{\frac{1}{q}} \\
&= \frac{1}{\Gamma(\alpha)} \frac{(x-a)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left(\int_a^x |f(t)|^q dt \right)^{\frac{1}{q}} \\
&\le \frac{1}{\Gamma(\alpha)} \frac{(x-a)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left(\int_a^b |f(t)|^q dt \right)^{\frac{1}{q}}.
\end{aligned}$$

Thus, we have

$$|(I^{\alpha}_{a+}f)(x)| \leq \frac{1}{\Gamma(\alpha)} \frac{(x-a)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left(\int\limits_a^b |f(t)|^q dt \right)^{\frac{1}{q}}, \text{ for every } x \in [a,b].$$

Consequently we find

$$|(I^{\alpha}_{a+}f)(x)|^q \leq \frac{1}{(\Gamma(\alpha))^q} \frac{(x-a)^{q(\alpha-1)+\frac{q}{p}}}{(p(\alpha-1)+1)^{\frac{q}{p}}} \left(\int\limits_a^b |f(t)|^q dt \right)$$

and we obtain

$$\int_a^b |I^{\alpha}_{a+}f(x)|^q\, dx \leq \frac{(b-a)^{q(\alpha-1)+\frac{q}{p}+1}}{(\Gamma(\alpha))^q (q(\alpha-1)+\frac{q}{p}+1)(p(\alpha-1)+1)^{\frac{q}{p}}} \int_a^b |f(t)|^q\, dt.$$

□

Remark 3.3 For $\alpha \geq 1$ inequalities (3.14) and (3.15) are refinements of (1.12) since

$$q\alpha(p(\alpha-1)+1)^{q-1} \geq q\alpha^q > \alpha^q, \text{ so } C < \left(\frac{(b-a)^{\alpha}}{\alpha\Gamma(\alpha)} \right)^q.$$

We proved that Theorem 3.2 is a refinement of (1.12) and Corollary 3.1 and 3.2 are generalizations of (1.12).

Next we give results with respect to the generalized Riemann-Liouville fractional derivative.

Corollary 3.3 *Let u be a weight function on (a,b) and let the assumptions in Lemma 1.3 be satisfied. Define v on (a,b) by*

$$v(y) := (\beta-\alpha) \int_y^b u(x) \frac{(x-y)^{\beta-\alpha-1}}{(x-a)^{\beta-\alpha}} dx < \infty.$$

If $\phi : (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_a^b u(x)\phi \left(\frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}} |D^{\alpha}_a f(x)| \right) dx \leq \int_a^b v(y)\phi(|D^{\beta}_a f(y)|)\, dy \tag{3.16}$$

holds.

Proof. Applying Theorem 3.1 with $\Omega_1 = \Omega_2 = (a,b)$,

$$k(x,y) = \begin{cases} \frac{(x-y)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}, & a \leq y \leq x; \\ 0, & x < y \leq b \end{cases}$$

we get that $K(x) = \frac{(x-a)^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}$. Replace f by $D^{\beta}_a f$. Then, by Lemma 1.3 $g(x) = (D^{\alpha}_a f)(x)$ and we get (3.16). □

Remark 3.4 In particular for the weight function $u(x) = (x-a)^{\beta-\alpha}$, $x \in (a,b)$ in Corollary 3.3 we obtain the inequality

$$\int_a^b (x-a)^{\beta-\alpha}\phi\left(\frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}}|D_a^\alpha f(x)|\right)dx \le \int_a^b (b-y)^{\beta-\alpha}\phi(|D_a^\beta f(y)|)\,dy.$$

Let $q>1$ and the function $\phi:\mathbb{R}_+\to\mathbb{R}$ be defined by $\phi(x)=x^q$. Then, after some calculation, we obtain

$$\int_a^b |D_a^\alpha f(x)|^q\,dx \le \left(\frac{(b-a)^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}\right)^q \int_a^b |D_a^\beta f(y)|^q\,dy.$$

In the next Corollary results involving Canavati-type fractional derivative ($\nu-$fractional derivative of f) are presented.

Corollary 3.4 *Let u be a weight function on (a,b) and let the assumptions in Lemma 1.4 be satisfied. Define v on (a,b) by*

$$v(y) := (\nu-\gamma)\int_y^b u(x)\frac{(x-y)^{\nu-\gamma-1}}{(x-x_0)^{\nu-\gamma}}dx < \infty.$$

If $\phi:(0,\infty)\to\mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_a^b u(x)\phi\left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}|D_a^\gamma f(x)|\right)dx \le \int_a^b v(y)\phi(|D_a^\nu f(y)|)\,dy \tag{3.17}$$

holds.

Proof. Applying Theorem 3.1 with $\Omega_1=\Omega_2=(a,b)$,

$$k(x,y) = \begin{cases} \frac{(x-y)^{\nu-\gamma-1}}{\Gamma(\nu-\gamma)}, & a\le y\le x;\\ 0, & x<y\le b\end{cases}$$

we get that $K(x)=\frac{(x-a)^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}$. Replace f by $D_a^\nu f$. Then, by Lemma 1.4 $g(x)=(D_a^\gamma f)(x)$ and we get (3.17). □

Remark 3.5 In particular for the weight function $u(x)=(x-a)^{\nu-\gamma}$, $x\in(a,b)$, in Corollary 3.4 we obtain the inequality

$$\int_a^b (x-a)^{\nu-\gamma}\phi\left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}|D_a^\gamma f(x)|\right)dx \le \int_a^b (b-y)^{\nu-\gamma}\phi(|D_a^\nu f(y)|)\,dy. \tag{3.18}$$

Let $q>1$ and the function $\phi:\mathbb{R}_+\to\mathbb{R}$ be defined by $\phi(x)=x^q$. Then (3.18) reduces to

$$(\Gamma(\nu-\gamma+1))^q\int_a^b (x-a)^{(\nu-\gamma)(1-q)}|D_a^\gamma f(x)|^q\,dx \le \int_a^b (b-y)^{\nu-\gamma}|D_a^\nu f(y)|^q\,dy.$$

Since $x \in [a,b]$ and $(\nu-\gamma)(1-q) \le 0$, we further obtain

$$\int_a^b |D_a^\gamma f(x)|^q\,dx \le \left(\frac{(b-a)^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}\right)^q \int_a^b |D_a^\nu f(y)|^q\,dy. \tag{3.19}$$

Taking power $\frac{1}{q}$ on both sides of (3.19) we obtain

$$\|(D_a^\gamma f(x)\|_q \le \frac{(b-a)^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}\|D_a^\nu f(y)\|_q.$$

When $\gamma = 0$ we find

$$(\Gamma(\nu+1))^q \int_a^b (x-a)^{\nu(1-q)}|f(x)|^q\,dx \le \int_a^b (b-y)^\nu |D_a^\nu f(y)|^q\,dy,$$

that is

$$\|f\|_q \le \frac{(b-a)^\nu}{\Gamma(\nu+1)}\|D_a^\nu f(y)\|_q.$$

In the next Corollary we give results with respect to the Caputo fractional derivative.

Corollary 3.5 *Let u be a weight function on (a,b) and $\nu > 0$. $D_{*a}^\nu g$ denotes the Caputo fractional derivative of g. Define v on (a,b) by*

$$v(y) := (n-\nu)\int_y^b u(x)\frac{(x-y)^{n-\nu-1}}{(x-a)^{n-\nu}}dx < \infty.$$

If $\phi : (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_a^b u(x)\phi\left(\frac{\Gamma(n-\nu+1)}{(x-a)^{n-\nu}}|D_{*a}^\nu g(x)|\right)dx \le \int_a^b v(y)\phi(|g^{(n)}(y)|)\,dy \tag{3.20}$$

holds.

Proof.
Applying Theorem 3.1 with $\Omega_1 = \Omega_2 = (a,b)$,

$$k(x,y) = \begin{cases} \frac{(x-y)^{n-\nu-1}}{\Gamma(n-\nu)}, & a \le y \le x; \\ 0, & x < y \le b \end{cases}$$

we get that $K(x) = \frac{(x-a)^{n-\nu}}{\Gamma(n-\nu+1)}$. Replace f by $g^{(n)}$, so g becomes $D_{*a}^\nu g$ and (3.20) follows. □

Remark 3.6 In particular for the weight function $u(x) = (x-a)^{n-\nu}$, $x \in (a,b)$ in Corollary 3.5 we obtain the inequality

$$\int_a^b (x-a)^{n-\nu}\phi\left(\frac{\Gamma(n-\nu+1)}{(x-a)^{n-\nu}}|D_{*a}^\nu g(x)|\right)dx \le \int_a^b (b-y)^{n-\nu}\phi(|g^{(n)}(y)|)\,dy.$$

Let $q > 1$ and the function $\phi : \mathbb{R}_+ \to \mathbb{R}$ be defined by $\phi(x) = x^q$. Then, after some calculation, we obtain

$$\int_a^b |D_{*a}^\nu g(x)|^q \, dx \le \left(\frac{(b-a)^{n-\nu}}{\Gamma(n-\nu+1)} \right)^q \int_a^b |g^{(n)}(y)|^q \, dy.$$

Taking power $\frac{1}{q}$ on both sides we obtain

$$\|D_{*a}^\nu g(x)\|_q \le \frac{(b-a)^{n-\nu}}{\Gamma(n-\nu+1)} \|g^{(n)}(y)\|_q.$$

Theorem 3.3 *Let $p,q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $n - \nu > \frac{1}{q}$, $D_{*a}^\nu f(x)$ denotes the Caputo fractional derivative of f. Then the following inequality*

$$\int_a^b |D_{*a}^\nu f(x)|^q \, dx \le \frac{(b-a)^{q(n-\nu)}}{(\Gamma(n-\nu))^q (p(n-\nu-1)+1)^{\frac{q}{p}} q(n-\nu)} \int_a^b |f^{(n)}(y)|^q \, dy$$

holds.

Proof. Similar to the proof of Theorem 3.2. □

Corollary 3.6 *Let u be a weight function on (a,b) and $\nu > 0$. $D_{*a}^\nu f$ denotes the Caputo fractional derivative of f and the assumptions in Lemma 1.5 are satisfied. Define v on (a,b) by*

$$v(y) := (\nu - \gamma) \int_y^b u(x) \frac{(x-y)^{\nu-\gamma-1}}{(x-a)^{\nu-\gamma}} dx < \infty.$$

If $\phi : (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_a^b u(x) \phi \left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} |D_{*a}^\gamma f(x)| \right) dx \le \int_a^b v(y) \phi(|D_{*a}^\nu f(y)|) \, dy \tag{3.21}$$

holds.

Proof.
Applying Theorem 3.1 with $\Omega_1 = \Omega_2 = (a,b)$,

$$k(x,y) = \begin{cases} \frac{(x-y)^{\nu-\gamma-1}}{\Gamma(\nu-\gamma)}, & a \le y \le x; \\ 0, & x < y \le b \end{cases}$$

we get that $K(x) = \frac{(x-a)^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}$. Replace f by $D_{*a}^\nu f$, so g becomes $D_{*a}^\gamma f$ and (3.21) follows. □

Remark 3.7 In particular for the weight function $u(x) = (x-a)^{\nu-\gamma}$, $x \in (a,b)$, in Corollary 3.6 we obtain the inequality

$$\int_a^b (x-a)^{\nu-\gamma} \phi\left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} |D^{\gamma}_{*a} f(x)|\right) dx \leq \int_a^b (b-y)^{\nu-\gamma} \phi(|D^{\nu}_{*a} f(y)|)\, dy.$$

Let $q > 1$ and the function $\phi : \mathbb{R}_+ \to \mathbb{R}$ be defined by $\phi(x) = x^q$. Then, after some calculation we obtain

$$\int_a^b \left|D^{\gamma}_{*a} f(x)\right|^q dx \leq \left(\frac{(b-a)^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}\right)^q \int_a^b |D^{\nu}_{*a} f(y)|^q\, dy.$$

For $\gamma = 0$, we obtain

$$\int_a^b |f(x)|^q\, dx \leq \left(\frac{(b-a)^{\nu}}{\Gamma(\nu+1)}\right)^q \int_a^b |D^{\nu}_{*a} f(y)|^q\, dy.$$

We continue with results involving fractional integrals of a function f with respect to a given function g.

Corollary 3.7 *Let u be a weight function on (a,b), g be an increasing function on $(a,b]$ such that g' is a continuous function on (a,b) and $\alpha > 0$. $I^{\alpha}_{a+;g} f$ denotes the left-sided fractional integral of a function f with respect to the function g in $[a,b]$. Define v on (a,b) by*

$$v(y) := \alpha g'(y) \int_y^b u(x) \frac{(g(x)-g(y))^{\alpha-1}}{(g(x)-g(a))^{\alpha}} dx < \infty.$$

If $\phi : (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_a^b u(x) \phi\left(\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} |I^{\alpha}_{a+;g} f(x)|\right) dx \leq \int_a^b v(y) \phi(|f(y)|)\, dy \tag{3.22}$$

holds.

Proof.
Applying Theorem 3.1 with $\Omega_1 = \Omega_2 = (a,b)$,

$$k(x,y) = \begin{cases} \frac{1}{\Gamma(\alpha+1)} \frac{g'(y)}{(g(x)-g(y))^{1-\alpha}}, & a \leq y \leq x; \\ 0, & x < y \leq b \end{cases}$$

we get that $K(x) = \frac{1}{\Gamma(\alpha+1)} (g(x)-g(a))^{\alpha}$, so (3.22) follows. □

Remark 3.8 In particular for the weight function $u(x) = g'(x)(g(x)-g(a))^{\alpha}$, $x \in (a,b)$ in Corollary 3.7 we obtain the inequality

$$\int_a^b g'(x)(g(x)-g(a))^{\alpha} \phi\left(\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} |I^{\alpha}_{a+;g} f(x)|\right) dx$$
$$\leq \int_a^b g'(y)(g(b)-g(y))^{\alpha} \phi(|f(y)|)\, dy. \tag{3.23}$$

Let $q>1$ and the function $\phi:\mathbb{R}_+\to\mathbb{R}$ be defined by $\phi(x)=x^q$. Then (3.23) reduces to

$$(\Gamma(\alpha+1))^q\int_a^b g'(x)(g(x)-g(a))^{\alpha(1-q)}|I^{\alpha}_{a+;g}f(x)|^q\,dx$$
$$\leq\int_a^b g'(y)(g(b)-g(y))^{\alpha}|f(y)|^q\,dy.$$

Since $x\in(a,b)$, $\alpha(1-q)<0$ and g is increasing we have $(g(x)-g(a))^{\alpha(1-q)}>(g(b)-g(a))^{\alpha(1-q)}$ and $(g(b)-g(y))^{\alpha}<(g(b)-g(a))^{\alpha}$ so we obtain

$$\int_a^b g'(x)|I^{\alpha}_{a+;g}f(x)|^q\,dx\leq\left(\frac{(g(b)-g(a))^{\alpha}}{\Gamma(\alpha+1)}\right)^q\int_a^b g'(y)|f(y)|^q\,dy. \tag{3.24}$$

Remark 3.9 If $g(x)=x$, then $I^{\alpha}_{a+;x}f(x)$ reduces to $I^{\alpha}_{a+}f(x)$ the Riemann-Liouville fractional integral, and (3.24) becomes (3.7).

Analogous to Corollary 3.7, we obtain the following result.

Corollary 3.8 *Let u be a weight function on (a,b), g be an increasing function on $(a,b]$ such that g' is a continuous function on (a,b) and $\alpha>0$. $I^{\alpha}_{b-;g}f$ denotes the right-sided fractional integral of a function f with respect to the function g in $[a,b]$. Define v on (a,b) by*

$$v(y):=\alpha g'(y)\int_a^y u(x)\frac{(g(y)-g(x))^{\alpha-1}}{(g(b)-g(x))^{\alpha}}dx<\infty.$$

If $\phi:(0,\infty)\to\mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_a^b u(x)\phi\left(\frac{\Gamma(\alpha+1)}{(g(b)-g(x))^{\alpha}}|I^{\alpha}_{b-;g}f(x)|\right)dx\leq\int_a^b v(y)\phi(|f(y)|)\,dy$$

holds.

Remark 3.10 In particular for the weight function $u(x)=g'(x)(g(b)-g(x))^{\alpha}$, $x\in(a,b)$, and for the function $\phi(x)=x^q$, $q>1$ after some calculation we obtain

$$\int_a^b g'(x)|I^{\alpha}_{b-;g}f(x)|^q\,dx\leq\left(\frac{(g(b)-g(a))^{\alpha}}{\Gamma(\alpha+1)}\right)^q\int_a^b g'(y)|f(y)|^q\,dy. \tag{3.25}$$

Remark 3.11 If $g(x)=x$, then $I^{\alpha}_{b-;x}f(x)$ reduces to $I^{\alpha}_{b-}f(x)$ the Riemann-Liouville fractional integral and (3.25) becomes (3.13).

The refinements of (3.24) and (3.25) for $\alpha>\frac{1}{q}$ are given in the following theorem.

Theorem 3.4 *Let $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\alpha>\frac{1}{q}$, $I^{\alpha}_{a+;g}f$ and $I^{\alpha}_{b-;g}f$ denotes the left-sided and right-sided fractional integral of a function f with respect to another function g in $[a,b]$. Then the following inequalities*

$$\int_a^b |I^{\alpha}_{a+;g}f(x)|^q g'(x)\,dx\leq\frac{(g(b)-g(a))^{\alpha q}}{\alpha q(\Gamma(\alpha))^q(p(\alpha-1)+1)^{\frac{q}{p}}}\int_a^b |f(y)|^q g'(y)\,dy$$

and

$$\int_a^b |I_{b-;g}^{\alpha} f(x)|^q g'(x)\,dx \le \frac{(g(b)-g(a))^{\alpha q}}{\alpha q(\Gamma(\alpha))^q (p(\alpha-1)+1)^{\frac{q}{p}}} \int_a^b |f(y)|^q g'(y)\,dy$$

hold.

We continue by giving results for Hadamard type fractional integrals.

The Hadamard fractional integrals of order α are special case of the left- and right-sided fractional integrals of a function f with respect to the function $g(x) = \log(x)$ in (a,b) where $0 \le a < b \le \infty$, so (3.24) reduces to

$$\int_a^b |(J_{a+}^{\alpha} f)(x)|^q \frac{dx}{x} \le \left(\frac{(\log \frac{b}{a})^{\alpha}}{\Gamma(\alpha+1)} \right)^q \int_a^b |f(y)|^q \frac{dy}{y} \tag{3.26}$$

and (3.25) becomes

$$\int_a^b |(J_{b-}^{\alpha} f)(x)|^q \frac{dx}{x} \le \left(\frac{(\log \frac{b}{a})^{\alpha}}{\Gamma(\alpha+1)} \right)^q \int_a^b |f(y)|^q \frac{dy}{y}. \tag{3.27}$$

Also, from Theorem 3.4 we obtain refinements of (3.26) and (3.27) for $\alpha > \frac{1}{q}$

$$\int_a^b |(J_{a+}^{\alpha} f)(x)|^q \frac{dx}{x} \le \frac{(\log \frac{b}{a})^{q\alpha}}{q\alpha(\Gamma(\alpha))^q (p(\alpha-1)+1)^{\frac{q}{p}}} \int_a^b |f(y)|^q \frac{dy}{y}$$

and

$$\int_a^b |(J_{b-}^{\alpha} f)(x)|^q \frac{dx}{x} \le \frac{(\log \frac{b}{a})^{q\alpha}}{q\alpha(\Gamma(\alpha))^q (p(\alpha-1)+1)^{\frac{q}{p}}} \int_a^b |f(y)|^q \frac{dy}{y}.$$

Some results involving Hadamard type fractional integrals are given in [67, p. 110]. Here we mention the following result that can not be compared with our result.

Let $\alpha > 0$, $1 \le p \le \infty$ and $0 \le a < b \le \infty$. Then the operators $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ are bounded in $L_p(a,b)$ as follows:

$$\|J_{a+}^{\alpha} f\|_p \le K_1 \|f\|_p \text{ and } \|J_{b-}^{\alpha} f\|_p \le K_2 \|f\|_p,$$

where

$$K_1 = \frac{1}{\Gamma(\alpha)} \int_0^{\log(b/a)} t^{\alpha-1} e^{\frac{t}{p}}\,dt$$

$$K_2 = \frac{1}{\Gamma(\alpha)} \int_0^{\log(b/a)} t^{\alpha-1} e^{-\frac{t}{p}}\,dt.$$

Now we give results involving Erdélyi-Kober type fractional integral.

Corollary 3.9 *Let u be a weight function on (a,b), ${}_2F_1(a,b;c;z)$ denotes the hypergeometric function and $I^{\alpha}_{a+;\sigma;\eta}f$ denotes the Erdélyi-Kober type fractional left-sided integral. Define v by*

$$v(y) = \alpha\sigma y^{\sigma\eta+\sigma-1}\int_y^b u(x)\frac{x^{-\sigma\eta}(x^{\sigma}-y^{\sigma})^{\alpha-1}}{(x^{\sigma}-a^{\sigma})^{\alpha}_{2}F_1(\alpha,-\eta;\alpha+1;1-\left(\frac{a}{x}\right)^{\sigma})}dx < \infty.$$

If $\phi : (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_a^b u(x)\phi\left(\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^{\sigma}\right)^{\alpha}{}_2F_1(\alpha,-\eta;\alpha+1;1-\left(\frac{a}{x}\right)^{\sigma})}|I^{\alpha}_{a+;\sigma;\eta}f(x)|\right)dx$$
$$\leq \int_a^b v(y)\phi(|f(y)|)\,dy \tag{3.28}$$

holds.

Proof. Applying Theorem 3.1 with $\Omega_1 = \Omega_2 = (a,b)$,

$$k(x,y) = \begin{cases} \frac{1}{\Gamma(\alpha)}\frac{\sigma x^{-\sigma(\alpha+\eta)}}{(x^{\sigma}-y^{\sigma})^{1-\alpha}}y^{\sigma\eta+\sigma-1}, & a \leq y \leq x; \\ 0, & x < y \leq b \end{cases}$$

we get that $K(x) = \frac{1}{\Gamma(\alpha+1)}\left(1-\left(\frac{a}{x}\right)^{\sigma}\right)^{\alpha}{}_2F_1(\alpha,-\eta;\alpha+1;1-\left(\frac{a}{x}\right)^{\sigma})$, so (3.28) follows. □

Remark 3.12 In particular, for the weight function $u(x) = x^{\sigma-1}(x^{\sigma}-a^{\sigma})^{\alpha}{}_2F_1(x)$ $({}_2F_1(x)$ $={}_2F_1(\alpha,-\eta;\alpha+1;1-\left(\frac{a}{x}\right)^{\sigma})$ and ${}_2F_1(y) ={}_2F_1(\alpha,\eta;\alpha+1;1-\left(\frac{b}{y}\right)^{\sigma}))$ in Corollary 3.9 we obtain the inequality

$$\int_a^b x^{\sigma-1}(x^{\sigma}-a^{\sigma})^{\alpha}{}_2F_1(x)\phi\left(\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^{\sigma}\right)^{\alpha}{}_2F_1(x)}|I^{\alpha}_{a+;\sigma;\eta}f(x)|\right)dx$$
$$\leq \int_a^b y^{\sigma-1}(b^{\sigma}-y^{\sigma})^{\alpha}{}_2F_1(y)\phi(|f(y)|)\,dy.$$

Remark 3.13 Similar results can be given for Erdélyi-Kober type fractional right-sided integral, for details see [61].

In the previous corollaries we derived only inequalities over some subsets of $\mathbb{R}$. However, Theorem 3.1 covers much more general situations. We conclude this section with multidimensional fractional integrals.

Corollary 3.10 *Let u be a weight function on $(\mathbf{a},\mathbf{b})$ and $\alpha > 0$. $I^{\alpha}_{\mathbf{a}+}f$ denotes the mixed Riemann-Liouville fractional integral of f. Define v on $(\mathbf{a},\mathbf{b})$ by*

$$v(\mathbf{y}) := \alpha\int_{y_1}^{b_1}\cdots\int_{y_n}^{b_n} u(\mathbf{x})\frac{(\mathbf{x}-\mathbf{y})^{\alpha-1}}{(\mathbf{x}-\mathbf{a})^{\alpha}}d\mathbf{x} < \infty.$$

If $\phi : (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} u(\mathbf{x})\phi\left(\frac{\Gamma(\alpha+1)}{(\mathbf{x}-\mathbf{a})^{\alpha}}|I^{\alpha}_{\mathbf{a}+} f(\mathbf{x})|\right) d\mathbf{x} \le \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} v(\mathbf{y})\phi(|f(\mathbf{y})|)\, d\mathbf{y} \tag{3.29}$$

holds for all measurable functions $f : (\mathbf{a},\mathbf{b}) \to \mathbb{R}$.

Proof.
Applying Theorem 3.1 with $\Omega_1 = \Omega_2 = (\mathbf{a},\mathbf{b})$,

$$k(x,y) = \begin{cases} \frac{(\mathbf{x}-\mathbf{y})^{\alpha-1}}{\Gamma(\alpha)}, & \mathbf{a} \le \mathbf{y} \le \mathbf{x}; \\ 0, & \mathbf{x} < \mathbf{y} \le \mathbf{b} \end{cases}$$

we get that $K(\mathbf{x}) = \frac{(\mathbf{x}-\mathbf{a})^{\alpha}}{\Gamma(\alpha+1)}$ and $g(\mathbf{x}) = I^{\alpha}_{\mathbf{a}+} f(\mathbf{x})$, so (3.29) follows. □

Remark 3.14 Analogous to Remark 3.1 and 3.2 we obtain multidimensional version of inequality (1.12) for $q > 1$:

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} |I^{\alpha}_{\mathbf{a}+} f(\mathbf{x})|^g\, d\mathbf{x} \le \left(\frac{(\mathbf{b}-\mathbf{a})^{\alpha}}{\Gamma(\alpha+1)}\right)^q \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} |f(\mathbf{y})|^q\, d\mathbf{y}$$

and

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} |I^{\alpha}_{\mathbf{b}-} f(\mathbf{x})|^g\, d\mathbf{x} \le \left(\frac{(\mathbf{b}-\mathbf{a})^{\alpha}}{\Gamma(\alpha+1)}\right)^q \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} |f(\mathbf{y})|^q\, d\mathbf{y}.$$

3.1 New inequalities involving fractional integrals and derivatives

If we substitute $k(x,y)$ by $k(x,y)f_2(y)$ and f by $\frac{f_1}{f_2}$, where $f_i : \Omega_2 \to \mathbb{R}, (i = 1,2)$ are measurable functions, in Theorem 3.1 we obtain the following result (see [60]).

Theorem 3.5 *Let $f_i : \Omega_2 \to \mathbb{R}$ be measurable functions, $g_i \in U(f_i)$, $(i = 1,2)$, where $g_2(x) > 0$ for every $x \in \Omega_1$. Let u be a weight function on Ω_1, k be a non-negative measurable function on $\Omega_1 \times \Omega_2$. Assume that the function $x \mapsto u(x)\frac{f_2(y)k(x,y)}{g_2(x)}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$. Define v on Ω_2 by*

$$v(y) := f_2(y)\int_{\Omega_1} \frac{u(x)k(x,y)}{g_2(x)}\, d\mu_1(x) < \infty. \tag{3.30}$$

If $\phi : (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_{\Omega_1} u(x)\phi\left(\left|\frac{g_1(x)}{g_2(x)}\right|\right) d\mu_1(x) \le \int_{\Omega_2} v(y)\phi\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right) d\mu_2(y),$$

holds.

Remark 3.15 If ϕ is strictly convex and $\frac{f_1(x)}{f_2(x)}$ is non-constant, then the inequality in Theorem 3.5 is strict.

Remark 3.16 As a special case of Theorem 3.5 for $\Omega_1 = \Omega_2 = [a,b]$ and $d\mu_1(x) = dx$, $d\mu_1(y) = dy$ we obtain the result in [81] (see also [92, p. 236]).

As a special case of Theorem 3.5 we obtain the following results involving Riemann-Liouville fractional integrals, the Canavati-type fractional derivative, the Caputo fractional derivative, Hadamard-type fractional integrals, Erdélyi-Kober type fractional integrals (for details see [60]).

Corollary 3.11 *Let u be a weight function on (a,b) and $\alpha > 0$. $I^{\alpha}_{b-}g$ denotes the right-sided Riemann-Liouville fractional integral of g. Define v on (a,b) by*

$$v(y) = \frac{f_2(y)}{\Gamma(\alpha)} \int_a^y \frac{u(x)(y-x)^{\alpha-1}}{I^{\alpha}_{b-} f_2(x)} dx < \infty.$$

If $\phi : (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_a^b u(x)\phi\left(\left|\frac{I^{\alpha}_{b-} f_1(x)}{I^{\alpha}_{b-} f_2(x)}\right|\right) dx \leq \int_a^b v(y)\phi\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right) dy.$$

holds.

Remark 3.17 The result involving the left-sided Riemann-Liouville fractional integral is given in Corollary 2.4 in [63].

Next we give results with respect to the generalized Riemann-Liouville fractional derivative.

Corollary 3.12 *Let u be a weight function on (a,b) and let the assumptions in Lemma* 1.3 *be satisfied. Define v on (a,b) by*

$$v(y) = \frac{D^{\beta}_a f_2(y)}{\Gamma(\beta-\alpha)} \int_y^b \frac{u(x)(x-y)^{\beta-\alpha-1}}{D^{\alpha}_a f_2(x)} dx < \infty.$$

If $\phi : (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_a^b u(x)\phi\left(\left|\frac{D^{\alpha}_a f_1(x)}{D^{\alpha}_a f_2(x)}\right|\right) dx \leq \int_a^b v(y)\phi\left(\left|\frac{D^{\beta}_a f_1(y)}{D^{\beta}_a f_2(y)}\right|\right) dy$$

holds.

Now we give results involving the Canavati-type fractional derivative.

Corollary 3.13 *Let u be a weight function on (a,b) and let the assumptions in Lemma 1.4 be satisfied. Define $v(y)$ on (a,b) by*

$$v(y) = \frac{D_a^\nu f_2(y)}{\Gamma(\nu-\gamma)} \int_y^b \frac{u(x)(x-y)^{\nu-\gamma-1}}{D_a^\gamma f_2(x)} dx < \infty.$$

If $\phi : (0,\infty) \to \mathbb{R}$ is convex and increasing, then the inequality

$$\int_a^b u(x)\phi\left(\left|\frac{D_a^\gamma f_1(x)}{D_a^\gamma f_2(x)}\right|\right) dx \le \int_a^b v(y)\phi\left(\left|\frac{D_a^\nu f_1(y)}{D_a^\nu f_2(y)}\right|\right) dy$$

holds.

We continue with results involving the Caputo fractional derivative.

Corollary 3.14 *Let u be a weight function on (a,b) and $\nu \ge 0$. $D_{*a}^\nu f$ denotes the Caputo fractional derivative of f. Define $v(y)$ on (a,b) by*

$$v(y) = \frac{f_2^{(n)}(y)}{\Gamma(n-\nu)} \int_y^b \frac{u(x)(x-y)^{n-\nu-1}}{D_{*a}^\nu f_2(x)} dx < \infty.$$

If $\phi : (0,\infty) \to \mathbb{R}$ is convex and increasing, then the inequality

$$\int_a^b u(x)\phi\left(\left|\frac{D_{*a}^\nu f_1(x)}{D_{*a}^\nu f_2(x)}\right|\right) dx \le \int_a^b v(y)\phi\left(\left|\frac{f_1^{(n)}(y)}{f_2^{(n)}(y)}\right|\right) dy$$

holds.

Corollary 3.15 *Let u be a weight function on (a,b) and let the assumptions in Lemma 1.5 be satisfied. Define $v(y)$ on (a,b) by*

$$v(y) = \frac{D_{*a}^\nu f_2(y)}{\Gamma(\nu-\gamma)} \int_y^b \frac{u(x)(x-y)^{\nu-\gamma-1}}{D_{*a}^\gamma f_2(x)} dx < \infty.$$

If $\phi : (0,\infty) \to \mathbb{R}$ is convex and increasing, then the inequality

$$\int_a^b u(x)\phi\left(\left|\frac{D_{*a}^\gamma f_1(x)}{D_{*a}^\gamma f_2(x)}\right|\right) dx \le \int_a^b v(y)\phi\left(\left|\frac{D_{*a}^\nu f_1(y)}{D_{*a}^\nu f_2(y)}\right|\right) dy$$

holds.

Now we continue with results involving the Hadamard-type fractional integrals.

Corollary 3.16 *Let u be a weight function and $\alpha > 0$. $J^{\alpha}_{a+}f$ denotes the left-sided Hadamard-type fractional integral. Define*

$$v(y) = \frac{f_2(y)}{y\Gamma(\alpha)} \int_y^b u(x) \left(\log\frac{x}{y}\right)^{\alpha-1} \frac{1}{(J^{\alpha}_{a+}f_2)(x)} dx < \infty.$$

If $\phi : (0,\infty) \to \mathbb{R}$ is convex and increasing, then the inequality

$$\int_a^b u(x)\phi\left(\left|\frac{J^{\alpha}_{a+}f_1(x)}{J^{\alpha}_{a+}f_2(x)}\right|\right) dx \leq \int_a^b v(y)\phi\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right) dy$$

holds.

Similarly we obtain the following Corollary.

Corollary 3.17 *Let u be a weight function and $\alpha > 0$. $J^{\alpha}_{b-}f$ denotes the right-sided Hadamard-type fractional integral. Define*

$$v(y) = \frac{f_2(y)}{y\Gamma(\alpha)} \int_y^b u(x) \left(\log\frac{y}{x}\right)^{\alpha-1} \frac{1}{(J^{\alpha}_{b-}f_2)(x)} dx < \infty.$$

If $\phi : (0,\infty) \to \mathbb{R}$ is convex and increasing, then the inequality

$$\int_a^b u(x)\phi\left(\left|\frac{J^{\alpha}_{b-}f_1(x)}{J^{\alpha}_{b-}f_2(x)}\right|\right) dx \leq \int_a^b v(y)\phi\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right) dy$$

holds.

Corollary 3.18 *Let u be a weight function, $I^{\alpha}_{a+;\sigma;\eta}f$ denotes the left-sided Erdelyi-Kóber type fractional integral of function f of order $\alpha > 0$. Define v on (a,b) by*

$$v(y) = \frac{f_2(y)}{\Gamma(\alpha)} \int_y^b \frac{u(x)\sigma x^{-\sigma(\alpha+\eta)} y^{\sigma\eta+\sigma-1}}{(x^{\sigma}-y^{\sigma})^{1-\alpha}(I^{\alpha}_{a+;\sigma;\eta}f_2)(x)} dx < \infty.$$

If $\phi : (0,\infty) \to \mathbb{R}$ is convex and increasing, then the inequality

$$\int_a^b u(x)\phi\left(\left|\frac{I^{\alpha}_{a+;\sigma;\eta}f_1(x)}{I^{\alpha}_{a+;\sigma;\eta}f_2(x)}\right|\right) dx \leq \int_a^b v(y)\phi\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right) dy$$

holds.

Similarly we obtain the following Corollary.

Corollary 3.19 *Let u be a weight function, $I^{\alpha}_{b_-;\sigma;\eta}f$ denotes the right-sided Erdelyi-Kóber type fractional integral of a function f. Define v on (a,b) by*

$$v(y) = \frac{f_2(y)}{\Gamma(\alpha)} \int_a^y \frac{u(x)\sigma x^{\sigma\eta} y^{\sigma(1-\alpha-\eta)-1}}{(y^{\sigma}-x^{\sigma})^{1-\alpha}(I^{\alpha}_{b_-;\sigma;\eta}f_2)(x)} dx < \infty.$$

If $\phi : (0,\infty) \to \mathbb{R}$ is convex and increasing, then the inequality

$$\int_a^b u(x)\phi\left(\left|\frac{I^{\alpha}_{b_-;\sigma;\eta}f_1(x)}{I^{\alpha}_{b_-;\sigma;\eta}f_2(x)}\right|\right) dx \le \int_a^b v(y)\phi\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right) dy$$

holds.

As a special case of Theorem 3.5 we obtain results involving the generalized Riemann-Liouville fractional derivative.

Corollary 3.20 *Let u be a weight function on (a,b) and let the assumptions in Lemma* 1.3 *be satisfied. Define v on (a,b) by*

$$v(y) = \frac{D^{\beta}_a f_2(y)}{\Gamma(\beta-\alpha)} \int_y^b \frac{u(x)(x-y)^{\beta-\alpha-1}}{D^{\alpha}_a f_2(x)} dx < \infty.$$

If $\phi : (0,\infty) \to \mathbb{R}$ is convex and increasing function, then the inequality

$$\int_a^b u(x)\phi\left(\left|\frac{D^{\alpha}_a f_1(x)}{D^{\alpha}_a f_2(x)}\right|\right) dx \le \int_a^b v(y)\phi\left(\left|\frac{D^{\beta}_a f_1(y)}{D^{\beta}_a f_2(y)}\right|\right) dy$$

holds.

We continue this chapter with results involving the mixed Riemann-Liouville fractional integral of f.

Corollary 3.21 *Let u be a weight function on $(\mathbf{a},\mathbf{b})$ and $\alpha > 0$. $I^{\alpha}_{\mathbf{a}_+}f$ denotes the mixed Riemann-Liouville fractional integral of f. Define v on $(\mathbf{a},\mathbf{b})$ by*

$$v(\mathbf{y}) := \frac{f_2(\mathbf{y})}{\Gamma(\alpha)} \int_{y_1}^{b_1} \cdots \int_{y_n}^{b_n} u(\mathbf{x}) \frac{(\mathbf{x}-\mathbf{y})^{\alpha-1}}{(I^{\alpha}_{\mathbf{a}_+}f_2)(\mathbf{x})} d\mathbf{x} < \infty.$$

If $\phi : (0,\infty) \to \mathbb{R}$ is convex and increasing, then the inequality

$$\int_{a_1}^{b_1} \dots \int_{a_1}^{b_1} u(\mathbf{x})\phi\left(\left|\frac{I^{\alpha}_{\mathbf{a}_+}f_1(\mathbf{x})}{I^{\alpha}_{\mathbf{a}_+}f_2(\mathbf{x})}\right|\right) d\mathbf{x} \le \int_{a_1}^{b_1} \dots \int_{a_1}^{b_1} v(\mathbf{y})\phi\left(\left|\frac{f_1(\mathbf{y})}{f_2(\mathbf{y})}\right|\right) d\mathbf{y}$$

holds.

Corollary 3.22 *Let u be a weight function on $(\mathbf{a},\mathbf{b})$ and $\alpha > 0$. $I^{\alpha}_{\mathbf{b}-}f$ denotes the mixed Riemann-Liouville fractional integral of f. Define v on $(\mathbf{a},\mathbf{b})$ by*

$$v(\mathbf{y}) := \frac{f_2(\mathbf{y})}{\Gamma(\alpha)} \int_{a_1}^{y_1} \cdots \int_{a_n}^{y_n} u(\mathbf{x}) \frac{(\mathbf{y}-\mathbf{x})^{\alpha-1}}{(I^{\alpha}_{\mathbf{b}-} f_2)(\mathbf{x})} d\mathbf{x} < \infty.$$

If $\phi : (0,\infty) \to \mathbb{R}$ is convex and increasing, then the inequality

$$\int_{a_1}^{b_1} \dots \int_{a_1}^{b_1} u(\mathbf{x}) \phi \left(\left| \frac{I^{\alpha}_{\mathbf{b}-} f_1(\mathbf{x})}{I^{\alpha}_{\mathbf{b}-} f_2(\mathbf{x})} \right| \right) d\mathbf{x} \le \int_{a_1}^{b_1} \dots \int_{a_1}^{b_1} v(\mathbf{y}) \phi \left(\left| \frac{f_1(\mathbf{y})}{f_2(\mathbf{y})} \right| \right) d\mathbf{y}$$

holds.

Note that Theorem 3.5 can be generalized for convex functions of several variables.

Theorem 3.6 *Let $g_i \in U(f_i)$, $(i = 1,2,3)$, where $g_2(x) > 0$ for every $x \in \Omega_1$. Let u be a weight function on Ω_1, k be a non-negative measurable function on $\Omega_1 \times \Omega_2$. Let v be defined by* (3.30)*. If $\phi : (0,\infty) \times (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality*

$$\int_{\Omega_1} u(x) \phi \left(\left| \frac{g_1(x)}{g_2(x)} \right|, \left| \frac{g_3(x)}{g_2(x)} \right| \right) d\mu_1(x) \le \int_{\Omega_2} v(y) \phi \left(\left| \frac{f_1(y)}{f_2(y)} \right|, \left| \frac{f_3(y)}{f_2(y)} \right| \right) d\mu_2(y) \tag{3.31}$$

holds.

Remark 3.18 Apply Theorem 3.6 with $\Omega_1 = \Omega_2 = [a,b]$ and $d\mu_1(x) = dx$, $d\mu_2(y) = dy$. Then

$$v(y) = f_2(y) \int_a^b \frac{u(x)k(x,y)}{g_2(x)} dx$$

and (3.31) reduces to

$$\int_a^b u(x) \phi \left(\left| \frac{g_1(x)}{g_2(x)} \right|, \left| \frac{g_3(x)}{g_2(x)} \right| \right) dx \le \int_a^b v(y) \phi \left(\left| \frac{f_1(y)}{f_2(y)} \right|, \left| \frac{f_3(y)}{f_2(y)} \right| \right) dy$$

This result is given in [81] (see also [92, p. 236]).

3.2 Improvements of an inequality of G. H. Hardy

Using Theorem 2.6, we will give some special cases for different fractional integrals and fractional derivatives to establish new Hardy-type inequalities (see [56]).

Our first result involving fractional integral of f with respect to another increasing function g is given in the following theorem and from this we obtain the case of Riemann-Liouville fractional integrals and Hadamard fractional integrals.

Theorem 3.7 *Let $0 < p \le q < \infty$, u be a weight function on (a,b), g be an increasing function on $(a,b]$ such that g' is continues on (a,b), $I^{\alpha}_{a+;g}f$ denotes the left-sided fractional integral of f with respect to the increasing function g. Let v be defined on (a,b) by*

$$v(y) := \alpha g'(y)\left(\int_y^b u(x)\left(\frac{(g(x)-g(y))^{\alpha-1}}{(g(x)-g(a))^{\alpha}}\right)^{\frac{q}{p}}dx\right)^{\frac{p}{q}} < \infty. \tag{3.32}$$

If Φ is a non-negative convex function on an interval $I \subseteq \mathbb{R}$, then the inequality

$$\left(\int_a^b u(x)\left[\Phi\left(\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}}I^{\alpha}_{a+;g}f(x)\right)\right]^{\frac{q}{p}}dx\right)^{\frac{1}{q}} \le \left(\int_a^b v(y)\Phi(f(y))\,dy\right)^{\frac{1}{p}} \tag{3.33}$$

holds for all measurable functions $f : (a,b) \to \mathbb{R}$, such that $Im f \subseteq I$.

Proof. Applying Theorem 2.6 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \frac{g'(y)}{\Gamma(\alpha)(g(x)-g(y))^{1-\alpha}}, & a \le y \le x; \\ 0, & x < y \le b, \end{cases}$$

we get that $K(x) = \frac{1}{\Gamma(\alpha+1)}(g(x)-g(a))^{\alpha}$, $A_k f(x) = \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}}I^{\alpha}_{a+;g}f(x)$ and the inequality in (2.21) reduces to (3.33) with v defined by (3.32). □

Corollary 3.23 *Let $0 < p \le q < \infty$, $s \ge 1, \alpha > 1 - \frac{p}{q}$, g be an increasing function on $(a,b]$ such that g' is continues on (a,b), $I^{\alpha}_{a+;g}f$ denotes the left-sided fractional integral of f with respect to the increasing function g. Then the inequality*

$$\left(\int_a^b g'(x)(I^{\alpha}_{a+;g}f(x))^{\frac{sq}{p}}dx\right)^{\frac{1}{q}} \le \frac{\alpha^{\frac{1}{p}}(g(b)-g(a))^{\frac{q(\alpha s-1)+p}{pq}}}{((\alpha-1)\frac{q}{p}+1)^{\frac{1}{q}}(\Gamma(\alpha+1))^{\frac{s}{p}}}\left(\int_a^b g'(y)f^s(y)dy\right)^{\frac{1}{p}} \tag{3.34}$$

holds.

Proof. For particular convex function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$, $\Phi(x) = x^s, s \ge 1$ and weight function $u(x) = g'(x)(g(x)-g(a))^{\frac{\alpha q}{p}}, x \in (a,b)$ in (3.33) we get $v(y) = (\alpha g'(y)(g(b)-g(y))^{\alpha-1+\frac{p}{q}})/(((\alpha-1)\frac{q}{p}+1)^{\frac{p}{q}})$ and (3.33) becomes

$$\left(\int_a^b g'(x)(g(x)-g(a))^{\frac{\alpha q}{p}(1-s)}(I^{\alpha}_{a+;g}f(x))^{\frac{sq}{p}}dx\right)^{\frac{1}{q}}$$

$$\le \frac{\alpha^{\frac{1}{p}}}{((\alpha-1)\frac{q}{p}+1)^{\frac{1}{q}}(\Gamma(\alpha+1))^{\frac{s}{p}}}\left(\int_a^b g'(y)(g(b)-g(y))^{\alpha-1+\frac{p}{q}}f^s(y)dy\right)^{\frac{1}{p}}.$$

Since $(g(x)-g(a))^{\frac{\alpha q}{p}(1-s)} \geq (g(b)-g(a))^{\frac{\alpha q}{p}(1-s)}$ and $(g(b)-g(y))^{\alpha-1+\frac{p}{q}} \leq (g(b)-g(a))^{\alpha-1+\frac{p}{q}}$ due to $\alpha > 1-\frac{p}{q}$, we obtain (3.34). □

Remark 3.19 Similar result can be obtained for the right-sided fractional integral of f with respect to another increasing function g, but here we omit the details.

We continue with results involving the Riemman-Liouville and Hadamard-type fractional integrals. If $g(x)=x$, then $I^{\alpha}_{a+;x}f(x)$ reduces to $I^{\alpha}_{a+}f(x)$, the left-sided Riemann-Liouville fractional integral, and if $g(x)=\log(x)$ in $[a,b]$ where $0 \leq a < b \leq \infty$, then $I^{\alpha}_{a+;x}f(x)$ reduces to $J^{\alpha}_{a+}f(x)$, the left-sided Hadamrd-type fractional integral.

Corollary 3.24 *Let $0 < p \leq q < \infty$, u be a weight function on (a,b), $I^{\alpha}_{a+}f$ denotes the left-sided Riemann-Liouville fractional integral of f. Let v be defined on (a,b) by*

$$v(y) := \alpha \left(\int_y^b u(x) \left(\frac{(x-y)^{\alpha-1}}{(x-a)^{\alpha}} \right)^{\frac{q}{p}} dx \right)^{\frac{p}{q}} < \infty.$$

If Φ is a non-negative convex function on an interval $I \subseteq \mathbb{R}$, then the inequality

$$\left(\int_a^b u(x) \left[\Phi \left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} I^{\alpha}_{a+}f(x) \right) \right]^{\frac{q}{p}} dx \right)^{\frac{1}{q}} \leq \left(\int_a^b v(y)\Phi(f(y))\,dy \right)^{\frac{1}{p}}$$

holds for all measurable functions $f:(a,b)\to\mathbb{R}$, such that $Im f \subseteq I$.

Corollary 3.25 *Let $0<p\leq q<\infty$, $s\geq 1$, $\alpha > 1-\frac{p}{q}$, $I^{\alpha}_{a+}f$ denotes the left-sided Riemann-Liouville fractional integral of f. Then the inequality*

$$\left(\int_a^b (I^{\alpha}_{a+}f(x))^{\frac{sq}{p}} dx \right)^{\frac{1}{q}} \leq \frac{\alpha^{\frac{1}{p}}(b-a)^{\frac{q(\alpha s-1)+p}{pq}}}{((\alpha-1)\frac{q}{p}+1)^{\frac{1}{q}}(\Gamma(\alpha+1))^{\frac{s}{p}}} \left(\int_a^b f^s(y)dy \right)^{\frac{1}{p}}$$

holds.

Corollary 3.26 *Let $0 < p \leq q < \infty$, $s \geq 1$, $\alpha > 1-\frac{p}{q}$, $J^{\alpha}_{a+}f$ denotes the Hadamard-type fractional integrals of f. Then the following inequality holds*

$$\left(\int_a^b (J^{\alpha}_{a+}f(x))^{\frac{sq}{p}} \frac{dx}{x} \right)^{\frac{1}{q}} \leq \frac{\alpha^{\frac{1}{p}}(\log b-\log a)^{\frac{q(\alpha s-1)+p}{pq}}}{((\alpha-1)\frac{q}{p}+1)^{\frac{1}{q}}(\Gamma(\alpha+1))^{\frac{s}{p}}} \left(\int_a^b f^s(y)\frac{dy}{y} \right)^{\frac{1}{p}}.$$

Next we give result with respect to the generalized Riemann-Liouville fractional derivative.

Theorem 3.8 *Let* $0 < p \le q < \infty$, *u be a weight function on* (a,b), $\beta > \alpha \ge 0$ *and let the assumptions of Lemma* 1.3 *be satisfied. Let v be defined on* (a,b) *by*

$$v(y) := (\beta-\alpha)\left(\int_y^b u(x)\left(\frac{(x-y)^{\beta-\alpha-1}}{(x-a)^{\beta-\alpha}}\right)^{\frac{q}{p}} dx\right)^{\frac{p}{q}} < \infty.$$

If Φ *is a non-negative convex function on an interval* $I \subseteq \mathbb{R}$, *then the inequality*

$$\left(\int_a^b u(x)\left[\Phi\left(\frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}} D_a^{\alpha} f(x)\right)\right]^{\frac{q}{p}} dx\right)^{\frac{1}{q}} \le \left(\int_a^b v(y)\Phi\left(D_a^{\beta} f(y)\right) dy\right)^{\frac{1}{p}} \tag{3.35}$$

holds for all measurable functions $f : (a,b) \to \mathbb{R}$, *such that* $Im f \subseteq I$.

Proof. Applying Theorem 2.6 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \frac{(x-y)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}, & a \le y \le x; \\ 0, & x < y \le b, \end{cases}$$

we get that $K(x) = \frac{(x-a)^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}$. Replace f by $D_a^{\beta} f$. Then $A_k f(x) = \frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}} D_a^{\alpha} f(x)$ and the inequality given in (2.21) reduces to (3.35). □

If we take $\Phi(x) = x^s, s \ge 1$ and $u(x) = (x-a)^{\frac{(\beta-\alpha)q}{p}}, x \in (a,b)$, similarly to the proof of Corollary 3.23 we obtain the following result.

Corollary 3.27 *Let* $0 < p \le q < \infty$, $s > 1$, $\beta - \alpha > 1 - \frac{p}{q}$ *and let the assumption of Lemma* 1.3 *be satisfied. Then the following inequality holds*

$$\left(\int_a^b (D_a^{\alpha} f(x))^{\frac{sq}{p}} dx\right)^{\frac{1}{q}} \le \frac{(\beta-\alpha)^{\frac{1}{p}}(b-a)^{\frac{q((\beta-\alpha)s-1)+p}{pq}}}{((\beta-\alpha-1)\frac{q}{p}+1)^{\frac{1}{q}}(\Gamma(\beta-\alpha+1))^{\frac{s}{p}}}\left(\int_a^b (D_a^{\beta} f(y))^s dy\right)^{\frac{1}{p}}.$$

In the following Theorem, we will construct a new inequality for the Canavati-type fractional derivative.

Theorem 3.9 *Let* $0 < p \le q < \infty$, $\nu > \gamma > 0$, *u be a weight function on* (a,b) *and the assumptions in Lemma* 1.4 *be satisfied,* $D_a^{\gamma} f$ *denotes the Canavati-type fractional derivative of* f. *Let v be defined on* (a,b) *by*

$$v(y) := (\nu-\gamma)\left(\int_y^b u(x)\left(\frac{(x-y)^{\nu-\gamma-1}}{(x-a)^{\nu-\gamma}}\right)^{\frac{q}{p}} dx\right)^{\frac{p}{q}} < \infty.$$

If Φ is a non-negative convex function on an interval $I \subseteq \mathbb{R}$, then the inequality

$$\left(\int_a^b u(x) \left[\Phi \left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_a^\gamma f(x) \right) \right]^{\frac{q}{p}} dx \right)^{\frac{1}{q}} \leq \left(\int_a^b v(y) \Phi \left(D_a^\nu f(y) \right) dy \right)^{\frac{1}{p}} \tag{3.36}$$

holds for all measurable functions $f : (a,b) \to \mathbb{R}$, such that $Im f \subseteq I$.

Proof. Applying Theorem 2.6 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \frac{(x-y)^{\nu-\gamma-1}}{\Gamma(\nu-\gamma)}, & a \leq y \leq x; \\ 0, & x < y \leq b \end{cases}$$

we get that $K(x) = \frac{(x-a)^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}$. Replace f by $D_a^\nu f$. Then the inequality given in (2.21) reduces to (3.36). □

Example 3.1 If we take $\Phi(x) = x^s, s \geq 1$, $\nu - \gamma > 1 - \frac{p}{q}$ and weight function $u(x) = (x-a)^{\frac{(\nu-\gamma)q}{p}}$, $x \in (a,b)$ in (3.36), after some calculations we obtain

$$\left(\int_a^b \left(D_a^\gamma f(x) \right)^{\frac{sq}{p}} dx \right)^{\frac{1}{q}} \leq \frac{(\nu-\gamma)^{\frac{1}{p}} (b-a)^{\frac{q((\nu-\gamma)s-1)+p}{pq}}}{((\nu-\gamma-1)\frac{q}{p}+1)^{\frac{1}{q}} (\Gamma(\nu-\gamma+1))^{\frac{s}{p}}} \left(\int_a^b \left(D_a^\nu f(y) \right)^s dy \right)^{\frac{1}{p}}.$$

Next, we give the result for the Caputo fractional derivative.

Theorem 3.10 *Let $0 < p \leq q < \infty$, u be a weight function on (a,b) and $D_{*a}^\nu f$ denotes the Caputo fractional derivative of f. Let v be defined on (a,b) by*

$$v(y) := (n-\nu) \left(\int_y^b u(x) \left(\frac{(x-y)^{n-\nu-1}}{(x-a)^{n-\nu}} \right)^{\frac{q}{p}} dx \right)^{\frac{p}{q}} < \infty.$$

If Φ is a non-negative convex function on an interval $I \subseteq \mathbb{R}$, then the inequality

$$\left(\int_a^b u(x) \left[\Phi \left(\frac{\Gamma(n-\nu+1)}{(x-a)^{n-\nu}} D_{*a}^\nu f(x) \right) \right]^{\frac{q}{p}} dx \right)^{\frac{1}{q}} \leq \left(\int_a^b v(y) \Phi \left(f^{(n)}(y) \right) dy \right)^{\frac{1}{p}} \tag{3.37}$$

holds for all measurable functions $f : (a,b) \to \mathbb{R}$, such that $Im f \subseteq I$.

Proof. Applying Theorem 2.6 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \frac{(x-y)^{n-\nu-1}}{\Gamma(n-\nu)}, & a \leq y \leq x; \\ 0, & x < y \leq b \end{cases}$$

we get that $K(x) = \frac{(x-a)^{n-\nu}}{\Gamma(n-\nu+1)}$. Replace f by $f^{(n)}$. Then the inequality given in (2.21) reduces to (3.37). □

Example 3.2 If we take $\Phi(x) = x^s, s \geq 1$, $n-\nu > 1-\frac{p}{q}$ and weight function $u(x) = (x-a)^{\frac{(n-\nu)q}{p}}$, $x \in (a,b)$, in (3.37) we obtain

$$\left(\int_a^b (D^{\nu}_{*a}f(x))^{\frac{sq}{p}}dx\right)^{\frac{1}{q}} \leq \frac{(n-\nu)^{\frac{1}{p}}(b-a)^{\frac{q((n-\nu)s-1)+p}{pq}}}{((n-\nu-1)\frac{q}{p}+1)^{\frac{1}{q}}(\Gamma(n-\nu+1))^{\frac{s}{p}}}\left(\int_a^b (f^{(n)}(y))^s dy\right)^{\frac{1}{p}}.$$

Theorem 3.11 *Let $0 < p \leq q < \infty$, u be a weight function on (a,b) and the assumptions in Lemma 1.5 be satisfied. $D^{\nu}_{*a}f$ denotes the Caputo fractional derivative of f. Let v be defined on (a,b) by*

$$v(y) := (\nu-\gamma)\left(\int_y^b u(x)\left(\frac{(x-y)^{\nu-\gamma-1}}{(x-a)^{\nu-\gamma}}\right)^{\frac{q}{p}}dx\right)^{\frac{p}{q}} < \infty.$$

If Φ is a non-negative convex function on an interval $I \subseteq \mathbb{R}$, then the inequality

$$\left(\int_a^b u(x)\left[\Phi\left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}D^{\gamma}_{*a}f(x)\right)\right]^{\frac{q}{p}}dx\right)^{\frac{1}{q}} \leq \left(\int_a^b v(y)\Phi\left(D^{\nu}_{*a}f(y)\right)dy\right)^{\frac{1}{p}} \tag{3.38}$$

holds for all measurable functions $f:(a,b)\to\mathbb{R}$, such that $Imf \subseteq I$.

Proof. Applying Theorem 2.6 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \frac{(x-y)^{\nu-\gamma-1}}{\Gamma(\nu-\gamma)}, & a \leq y \leq x; \\ 0, & x < y \leq b \end{cases}$$

we get that $K(x) = \frac{(x-a)^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}$. Replace f by $D^{\nu}_{*a}f$. Then the inequality given in (2.21) reduces to (3.38). □

Now, we give the following result that involves the Erdélyi-Kober type fractional integrals.

Theorem 3.12 *Let $0 < p \leq q < \infty$, u be a weight function on (a,b), $I^{\alpha}_{a+;\sigma;\eta}f$ denotes the Erdélyi-Kober type fractional integrals of f, ${}_2F_1(a,b;c;z)$ denotes the hypergeometric function. Let v be defined on (a,b) by*

$$v(y) := \alpha\left(\int_y^b u(x)\left(\frac{\sigma x^{-\sigma\eta}y^{\sigma\eta+\sigma-1}}{(x^{\sigma}-y^{\sigma})^{1-\alpha}(x^{\sigma}-a^{\sigma})^{\alpha}}\right)^{\frac{q}{p}}dx\right)^{\frac{p}{q}} < \infty.$$

If Φ is a non-negative convex function on an interval $I \subseteq \mathbb{R}$, then the inequality

$$\left(\int_a^b u(x)\left[\Phi\left(\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^{\sigma}\right)^{\alpha}{}_2F_1(x)}I^{\alpha}_{a+;\sigma;\eta}f(x)\right)\right]^{\frac{q}{p}}dx\right)^{\frac{1}{q}} \leq \left(\int_a^b v(y)\Phi(f(y))dy\right)^{\frac{1}{p}} \tag{3.39}$$

holds for all measurable functions $f:(a,b)\to\mathbb{R}$*, such that* $Imf\subseteq I$ *where* ${}_2F_1(x) = {}_2F_1\left(-\eta,\alpha;\alpha+1;1-\left(\frac{a}{x}\right)^{\sigma}\right)$.

Proof. Applying Theorem 2.6 with $\Omega_1=\Omega_2=(a,b)$, $d\mu_1(x)=dx, d\mu_2(y)=dy$,

$$k(x,y)=\begin{cases}\frac{1}{\Gamma(\alpha)}\frac{\sigma x^{-\sigma(\alpha+\eta)}}{(x^{\sigma}-y^{\sigma})^{1-\alpha}}y^{\sigma\eta+\sigma-1}, & a\le y\le x;\\ 0, & x<y\le b,\end{cases}$$

we get that $K(x)=\frac{1}{\Gamma(\alpha+1)}\left(1-\left(\frac{a}{x}\right)^{\sigma}\right)^{\alpha}{}_2F_1(-\eta,\alpha;\alpha+1;1-\left(\frac{a}{x}\right)^{\sigma})$.

Then the inequality (2.21) becomes (3.39). □

Example 3.3 If we take $\Phi(x)=x^s, s\ge 1$ and weight function $u(x)$ $=x^{\sigma-1}\left((x^{\sigma}-a^{\sigma})^{\alpha}{}_2F_1(x)\right)^{\frac{q}{p}}, x\in(a,b)$ in (3.39), after some calculations we obtain

$$\left(\int_a^b ({}_2F_1(x))^{\frac{q}{p}(1-s)}\left(I^{\alpha}_{a+;\sigma;\eta}f(x)\right)^{\frac{sq}{p}}dx\right)^{\frac{1}{q}}\le C\left(\int_a^b ({}_2F_1(y))f^s(y)dy\right)^{\frac{1}{p}}$$

where

$$C=\frac{\alpha^{\frac{1}{p}}\sigma^{\frac{q-p}{pq}}b^{\frac{\sigma-1}{p}}(b^{\sigma}-a^{\sigma})^{\frac{q(\alpha s-1)+p}{pq}}}{a^{\frac{p\sigma-p+qs\sigma\alpha}{pq}}((\alpha-1)\frac{q}{p}+1)^{\frac{1}{q}}(\Gamma(\alpha+1))^{\frac{s}{p}}},$$

$${}_2F_1(x)={}_2F_1\left(-\eta,\alpha;\alpha+1;1-\left(\frac{a}{x}\right)^{\sigma}\right) \text{ and } {}_2F_1(y)={}_2F_1\left(\eta,\alpha;\alpha+1;1-\left(\frac{b}{y}\right)^{\sigma}\right).$$

Remark 3.20 Similar result can be obtained for the right-sided Erdélyi-Kober type fractional integrals, but we omit the details here.

Chapter 4

Some new refined Hardy-type inequalities with kernels

In this chapter we state and prove a new general refined weighted Hardy-type inequality for convex functions and integral operator and also for monotone convex functions. We point out that the obtained result generalizes and refines the classical one-dimensional Hardy, Pólya-Knopp, Hardy-Hilbert inequalities and related dual inequalities. We show that our results may be seen as generalizations of some recent results related to Riemann-Liouville and Weyl's operator, as well as a generalization and a refinement of the so-called Godunova's inequality.

4.1 New general refined Hardy-type inequalities with kernels

Now, we are ready to state and prove the central result of this chapter, that is, a new refined general weighted Hardy-type inequality with a non-negative kernel and related to an arbitrary convex function. It is given in the following theorem (see [24]).

Theorem 4.1 *Let* $(\Omega_1, \Sigma_1, \mu_1)$ *and* $(\Omega_2, \Sigma_2, \mu_2)$ *be measure spaces with positive* σ*-finite measures, u be a weight function on* Ω_1*, k a non-negative measurable function on* $\Omega_1 \times \Omega_2$*, and K be defined on* Ω_1 *by* (2.16)*. Suppose that* $K(x) > 0$ *for all* $x \in \Omega_1$*, that the function*

$x \mapsto u(x)\frac{k(x,y)}{K(x)}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$, and that v is defined on Ω_1 by (2.17). *If Φ is a convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \mathrm{Int}I$, then the inequality*

$$\begin{aligned}\int_{\Omega_2} v(y)\Phi(f(y))\,d\mu_2(y) - \int_{\Omega_1} u(x)\Phi(A_k f(x))\,d\mu_1(x) \\ \geq \int_{\Omega_1} \frac{u(x)}{K(x)} \int_{\Omega_2} k(x,y)\,\big|\,|\Phi(f(y)) - \Phi(A_k f(x))| \\ - |\varphi(A_k f(x))| \cdot |f(y) - A_k f(x)|\,\big|\,d\mu_2(y)\,d\mu_1(x)\end{aligned} \tag{4.1}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$ such that $f(y) \in I$ for all $y \in \Omega_2$, where $A_k f$ is defined on Ω_1 by (2.15).
If Φ is a monotone convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \mathrm{Int}\,I$, then the inequality

$$\begin{aligned}\int_{\Omega_2} v(y)\Phi(f(y))\,d\mu_2(y) - \int_{\Omega_1} u(x)\Phi(A_k f(x))\,d\mu_1(x) \\ \geq \left| \int_{\Omega_1} \frac{u(x)}{K(x)} \int_{\Omega_2} sgn(f(y) - A_k f(x))k(x,y)\Big[\Phi(f(y)) - \Phi(A_k f(x)) \right. \\ \left. -|\varphi(A_k f(x))| \cdot (f(y) - A_k f(x))\Big]\,d\mu_2(y)\,d\mu_1(x) \right|\end{aligned} \tag{4.2}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$ such that $f(y) \in I$ for all $y \in \Omega_2$, where $A_k f$ is defined by (2.15).

Proof. First, $A_k f(x) \in I$, for all $x \in \Omega_1$ (see the proof of Theorem 2.5). To prove inequality (4.1), observe that for all $r \in \mathrm{Int}I$ and $s \in I$ we have

$$\Phi(s) - \Phi(r) - \varphi(r)(s - r) \geq 0,$$

where $\varphi : I \longrightarrow \mathbb{R}$ is any function such that $\varphi(x) \in \partial\Phi(x)$ for $x \in \mathrm{Int}I$, and hence

$$\begin{aligned}\Phi(s) - \Phi(r) - \varphi(r)(s - r) &= |\Phi(s) - \Phi(r) - \varphi(r)(s - r)| \\ &\geq \big|\,|\Phi(s) - \Phi(r)| - |\varphi(r)|\,|s - r|\,\big|.\end{aligned} \tag{4.3}$$

Especially, in the case when $A_k f(x) \in \mathrm{Int}I$, by substituting $r = A_k f(x)$ and $s = f(y)$ in (4.3) we get

$$\begin{aligned}\Phi(f(y)) - \Phi(A_k f(x)) - \varphi(A_k f(x))\,[f(y) - A_k f(x)] \\ \geq \big|\,|\Phi(f(y)) - \Phi(A_k f(x))| - |\varphi(A_k f(x))|\,|f(y) - A_k f(x)|\,\big|.\end{aligned} \tag{4.4}$$

Observe that (4.4) holds even if $A_k f(x)$ is an endpoint of I, since in that case both sides of inequality (4.4) are equal to 0 for μ_2-a.e. $y \in \Omega_2$. Multiplying (4.4) by $u(x)\frac{k(x,y)}{K(x)} \geq 0$ for a fixed $x \in \Omega_1$, and then integrating it over Ω_2 and Ω_1 respectively, we obtain

$$\begin{aligned}
&\int_{\Omega_1}\int_{\Omega_2} u(x)\frac{k(x,y)}{K(x)}\Phi(f(y))\,d\mu_2(y)\,d\mu_1(x) \\
&\quad - \int_{\Omega_1}\int_{\Omega_2} u(x)\frac{k(x,y)}{K(x)}\Phi(A_k f(x))\,d\mu_2(y)\,d\mu_1(x) \\
&\quad - \int_{\Omega_1}\int_{\Omega_2} u(x)\frac{k(x,y)}{K(x)}\varphi(A_k f(x))\,[f(y) - A_k f(x)]\,d\mu_2(y)\,d\mu_1(x) \\
&\geq \int_{\Omega_1}\int_{\Omega_2} u(x)\frac{k(x,y)}{K(x)}\,||\Phi(f(y)) - \Phi(A_k f(x))| \\
&\quad - |\varphi(A_k f(x))| \cdot |f(y) - A_k f(x)||\,d\mu_2(y)\,d\mu_1(x).
\end{aligned} \tag{4.5}$$

By using Fubini's theorem and the definition (2.17) of the weight v, the first integral on the left-hand side od (4.5) becomes

$$\begin{aligned}
&\int_{\Omega_1}\int_{\Omega_2} u(x)\frac{k(x,y)}{K(x)}\Phi(f(y))\,d\mu_2(y)\,d\mu_1(x) \\
&= \int_{\Omega_2}\Phi(f(y))\left(\int_{\Omega_1} u(x)\frac{k(x,y)}{K(x)}\,d\mu_1(x)\right)d\mu_2(y) \\
&= \int_{\Omega_2} v(y)\Phi(f(y))\,d\mu_2(y),
\end{aligned} \tag{4.6}$$

while for the second integral on the left-hand side of (4.5) we have

$$\begin{aligned}
&\int_{\Omega_1}\int_{\Omega_2} u(x)\frac{k(x,y)}{K(x)}\Phi(A_k f(x))\,d\mu_2(y)\,d\mu_1(x) \\
&= \int_{\Omega_1} u(x)\Phi(A_k f(x))\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)\,d\mu_2(y)\right)d\mu_1(x) \\
&= \int_{\Omega_1} u(x)\Phi(A_k f(x))\,d\mu_1(x).
\end{aligned} \tag{4.7}$$

Finally, applying (4.5) and (2.19) and we similarly get

$$\begin{aligned}
&\int_{\Omega_1}\int_{\Omega_2} u(x)\frac{k(x,y)}{K(x)}\varphi(A_k f(x))\,[f(y) - A_k f(x)]\,d\mu_2(y)\,d\mu_1(x) \\
&= \int_{\Omega_1}\frac{u(x)}{K(x)}\varphi(A_k f(x))\left(\int_{\Omega_2} k(x,y)h_x(y)\,d\mu_2(y)\right)d\mu_1(x) = 0,
\end{aligned} \tag{4.8}$$

so (4.1) holds by combining (4.5), (4.6), (4.7), and (4.8). Now, we prove inequality (4.2). Consider the case, when Φ is non-decreasing on the interval I. For a fixed $x \in \Omega_1$, let

$\Omega_2' = \{y \in \Omega_2 : f(y) > A_k f(x)\}$. Then

$$\begin{aligned}
&\int_{\Omega_2} k(x,y)|\Phi(f(y)) - \Phi(A_k f(x))| d\mu_2(y) \\
&= \int_{\Omega_2'} k(x,y)[\Phi(f(y)) - \Phi(A_k f(x)))] d\mu_2(y) \\
&\quad + \int_{\Omega_2 \setminus \Omega_2'} k(x,y)[\Phi(A_k f(x)) - \Phi(f(y))] d\mu_2(y) \\
&= \int_{\Omega_2'} k(x,y)\Phi(f(y)) d\mu_2(y) - \int_{\Omega_2 \setminus \Omega_2'} k(x,y)\Phi(f(y)) d\mu_2(y) \\
&\quad - \Phi(A_k f(x)) \int_{\Omega_2'} k(x,y) d\mu_2(y) + \Phi(A_k f(x)) \int_{\Omega_2 \setminus \Omega_2'} k(x,y) d\mu_2(y) \\
&= \int_{\Omega_2} sgn(f(y) - A_k f(x)) k(x,y)[\Phi(f(y)) - \Phi(A_k f(x)] d\mu_2(y). \qquad (4.9)
\end{aligned}$$

Similarly, we can write

$$\begin{aligned}
&\int_{\Omega_2} k(x,y)|f(y) - (A_k f(x))| d\mu_2(y) \\
&= \int_{\Omega_2} sgn(f(y) - A_k f(x)) k(x,y)(f(y) - A_k f(x)) d\mu_2(y). \qquad (4.10)
\end{aligned}$$

From (4.1), (4.9) and (4.10), we get (4.2).

The case, when Φ is non-increasing can be discussed in a similar way. □

Remark 4.1 Let Φ be a concave function (that is, $-\Phi$ is convex). Then for all $r \in \mathrm{Int}I$ and $s \in I$ we have

$$\Phi(r) - \Phi(s) - \varphi(r)(r-s) \geq 0,$$

and (4.3) reads

$$\begin{aligned}
\Phi(r) - \Phi(s) - \varphi(r)(r-s) &= |\Phi(r) - \Phi(s) - \varphi(r)(r-s)| \\
&\geq \big| |\Phi(s) - \Phi(r)| - |\varphi(r)| \cdot |s-r| \big|,
\end{aligned}$$

where φ is an arbitrary real function on I such that $\varphi(x) \in \partial\Phi(x) = [\Phi'_+(x), \Phi'_-(x)]$, for all $x \in \mathrm{Int}I$. Hence, in this setting, (4.1) holds with its left-hand side replaced with

$$\int_{\Omega_1} u(x)\Phi(A_k f(x))\, d\mu_1(x) - \int_{\Omega_2} v(y)\Phi(f(y))\, d\mu_2(y).$$

If Φ is monotone concave, then the order of terms on the left-hand side of (4.2) is reversed.

Remark 4.2 Since the right-hand side of (4.1) is non-negative, we get (2.18) as an immediate consequence of Theorem 4.1 and Remark 4.1 Consequently, our new result can be regarded as a refinement of the general weighted Hardy-type inequality (2.18). The same holds also for a concave function Φ.

Although (4.1) holds for all convex (or concave) functions, some choices of Φ are of particular interest. Namely, we shall consider power and exponential functions. To start with, let $p \in \mathbb{R} \setminus \{0\}$ and the function $\Phi : \mathbb{R}_+ \to \mathbb{R}$ be defined by $\Phi(x) = x^p$. Obviously, Φ is monotone, $\varphi(x) = \Phi'(x) = px^{p-1}$, $x \in \mathbb{R}_+$, so Φ is convex for $p \in \mathbb{R} \setminus [0,1)$, concave for $p \in (0,1]$, and affine, that is, both convex and concave for $p = 1$. In this setting, we get the following consequence of Theorem 4.1.

Corollary 4.1 *Let $\Omega_1, \Omega_2, \mu_1, \mu_2, u, k, K$, and v be as in Theorem 4.1 Let $p \in \mathbb{R}$ be such that $p \neq 0$, $f : \Omega_2 \to \mathbb{R}$ be a non-negative measurable function (positive for $p < 0$), $A_k f$ be defined by (2.15),*

$$R_{p,k}f(x,y) = \left| \left| f^p(y) - A_k^p f(x) \right| - |p| \cdot |A_k f(x)|^{p-1} \left| f(y) - A_k f(x) \right| \right|, \tag{4.11}$$

and

$$M_{p,k}f(x,y) = f^p(y) - A_k^p f(x) - |p| \cdot |A_k f(x)|^{p-1} (f(y) - A_k f(x)) \tag{4.12}$$

for $x \in \Omega_1$, $y \in \Omega_2$. If $p \geq 1$ or $p < 0$, then the inequalities

$$\begin{aligned} &\int_{\Omega_2} v(y) f^p(y)\, d\mu_2(y) - \int_{\Omega_1} u(x) A_k^p f(x)\, d\mu_1(x) \\ &\geq \int_{\Omega_1} \frac{u(x)}{K(x)} \int_{\Omega_2} k(x,y) R_{p,k} f(x,y)\, d\mu_2(y)\, d\mu_1(x), \end{aligned} \tag{4.13}$$

$$\begin{aligned} &\int\limits_{\Omega_2} v(y) f^p(y) d\mu_2(y) - \int\limits_{\Omega_1} u(x) A_k^p f(x) d\mu_1(x) \\ &\geq \left| \int\limits_{\Omega_1} \frac{u(x)}{K(x)} \int\limits_{\Omega_2} sgn(f(y) - A_k f(x)) k(x,y) M_{p,k} f(x,y)\, d\mu_2(y)\, d\mu_1(x) \right| \end{aligned} \tag{4.14}$$

hold, while for $p \in (0,1)$ relations (4.13) and (4.14) hold with

$$\int_{\Omega_1} u(x) A_k^p f(x)\, d\mu_1(x) - \int_{\Omega_2} v(y) f^p(y)\, d\mu_2(y)$$

on its left-hand side.

Remark 4.3 Note that relations (4.13) and (4.14) are trivial for $p = 1$, since both of its sides are equal to 0.

On the other hand, for the convex function $\Phi : \mathbb{R} \to \mathbb{R}$, $\Phi(x) = e^x$, we have $\varphi(x) = \Phi'(x) = e^x$ and the following new general refined weighted Pólya-Knopp-type inequality with a kernel, which is a generalization and refinement of the classical Pólya-Knopp's inequality (0.2).

Corollary 4.2 *Let $\Omega_1, \Omega_2, \mu_1, \mu_2, u, k, K$, and v be as in Theorem* 4.1 *and let $p \in \mathbb{R}$, $p \neq 0$. Then the inequality*

$$\begin{aligned}\int_{\Omega_2} v(y) f^p(y)\, d\mu_2(y) - \int_{\Omega_1} u(x) G_k^p f(x)\, d\mu_1(x) \\ \geq \int_{\Omega_1} \frac{u(x)}{K(x)} \int_{\Omega_2} k(x,y) S_{p,k} f(x,y)\, d\mu_2(y)\, d\mu_1(x)\end{aligned} \tag{4.15}$$

holds for all positive measurable functions f on Ω_2, where $G_k f(x)$ and $S_{p,k} f(x,y)$ are defined for $x \in \Omega_1$ and $y \in \Omega_2$ by

$$G_k f(x) = \exp\left(\frac{1}{K(x)} \int_{\Omega_2} k(x,y) \log f(y)\, d\mu_2(y)\right) \tag{4.16}$$

and

$$S_{p,k} f(x,y) = \left| \left| f^p(y) - G_k^p f(x) \right| - |p|\, G_k^p(x) \left| \log \frac{f(y)}{G_k f(x)} \right| \right|. \tag{4.17}$$

In particular, for $p = 1$ we get

$$\begin{aligned}\int_{\Omega_2} v(y) f(y)\, d\mu_2(y) - \int_{\Omega_1} u(x) G_k f(x)\, d\mu_1(x) \geq \int_{\Omega_1} \frac{u(x)}{K(x)} \int_{\Omega_2} k(x,y) \times \\ \times \left| |f(y) - G_k f(x)| - G_k(x) \left| \log \frac{f(y)}{G_k f(x)} \right| \right| d\mu_2(y)\, d\mu_1(x).\end{aligned} \tag{4.18}$$

Moreover, relations (4.15) *and* (4.18) *are equivalent.*
Let $p > 0$, G_k be defined by (4.16),

$$P_{p,k} f(x,y) = f^p(y) - G_k^p f(x) - p\, |G_k^p(x)| \log \frac{f(y)}{G_k f(x)} \tag{4.19}$$

and $f : \Omega_2 \to \mathbb{R}$ be a positive measurable function. Then the following inequality holds

$$\begin{aligned}\int_{\Omega_2} v(y) f^p(y) d\mu_2(y) - \int_{\Omega_1} u(x) (G_k^p f(x)) d\mu_1(x) \\ \geq \left| \int_{\Omega_1} \frac{u(x)}{K(x)} \int_{\Omega_2} sgn(f(y) - G_k f(x)) k(x,y) P_{p,k} f(x,y) d\mu_2(y)\, d\mu_1(x) \right|.\end{aligned}$$

Proof. Apply (4.1) with $\Phi : \mathbb{R} \to \mathbb{R}$, $\Phi(x) = e^x$, and replace the function f with $p \log f$. Note that $G_k f = \exp(A_k(\log f))$ and $G_k f^p = G_k^p f$, so equivalence of (4.15) and (4.18) is evident. □

Now, we consider the simplest kernels k, that is, those with separate variables. As a corollary of Theorem 4.1 in this setting, we get a refined general Jensen's inequality.

Corollary 4.3 *Suppose Ω is a measure space with a positive σ-finite measure μ, $m \in L^1(\Omega,\mu)$ is a non-negative function such that $|m|_1 > 0$, a real function Φ is convex on an interval $I \subseteq \mathbb{R}$, and $\varphi : I \to \mathbb{R}$ is any function such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \mathrm{Int}I$. Then the inequality*

$$\begin{aligned}\int_\Omega m(y)\Phi(f(y))\,d\mu(y) - |m|_1\Phi(A_m f)\\ \geq \int_\Omega m(y)\,\big|\,|\Phi(f(y)) - \Phi(A_m f)| - |\varphi(A_m f)|\cdot|f(y) - A_m f|\,\big|\,d\mu(y)\end{aligned} \tag{4.20}$$

holds for all measurable functions $f : \Omega \to \mathbb{R}$ with values in I, where

$$A_m f = \frac{1}{|m|_1}\int_\Omega m(y)f(y)\,d\mu(y). \tag{4.21}$$

If the function Φ is concave, the order of integrals on the left-hand side of (4.20) *is reversed.*

Proof. Suppose that in Theorem 4.1 we have $\Omega_2 = \Omega$, $\mu_2 = \mu$, $u \in L^1(\Omega_1,\mu_1)$ such that $|u|_1 > 0$, and k of the form $k(x,y) = l(x)m(y)$, for some positive measurable function $l : \Omega_1 \to \mathbb{R}$. Then $K(x) = |m|_1 l(x)$ and $A_k f(x) = A_m f \in I$, $x \in \Omega_1$, while $v(y) = \frac{|u|_1}{|m|_1}m(y)$, $y \in \Omega$. Thus, (4.1) reduces to (4.20) and it does not depend on Ω_1, l, and u. □

Remark 4.4 Observe that, for $0 < |\Omega|_\mu < \infty$ and $m(y) \equiv 1$ on Ω, we have $|m|_1 = |\Omega|_\mu$, so (4.20) becomes the classical refined Jensen's inequality

$$\begin{aligned}\frac{1}{|\Omega|_\mu}\int_\Omega \Phi(f(y))\,d\mu(y) - \Phi(Af)\\ \geq \frac{1}{|\Omega|_\mu}\int_\Omega \big|\,|\Phi(f(y)) - \Phi(Af)| - |\varphi(Af)|\cdot|f(y) - Af|\,\big|\,d\mu(y),\end{aligned}$$

where

$$Af = \frac{1}{|\Omega|_\mu}\int_\Omega f(y)\,d\mu(y). \tag{4.22}$$

In the sequel, the general results are applied to particular measure spaces, convex functions, weights, and kernels. This enables us to refine and even generalize some important inequalities previously known from the literature.

First, we consider an one-dimensional setting, with intervals in $\mathbb{R}$ and the Lebesgue measure, to get refined Hardy and Pólya-Knopp-type inequalities, as well as related dual relations. In the following theorem, we state and prove a refinement of a Hardy-type inequality obtained by S. Kaijser et al. in [65].

Theorem 4.2 *Let $0 < b \leq \infty$ and $k : (0,b)\times(0,b) \to \mathbb{R}$ be a non-negative measurable function, such that*

$$K(x) = \int_0^x k(x,y)\,dy > 0,\ x \in (0,b). \tag{4.23}$$

Let a weight $u:(0,b)\to\mathbb{R}$ *be such that the function* $x\mapsto \frac{u(x)}{x}\cdot\frac{k(x,y)}{K(x)}$ *is integrable on* (y,b) *for each fixed* $y\in(0,b)$*, and let the function* $w:(0,b)\to\mathbb{R}$ *be defined by*

$$w(y)=y\int_y^b \frac{k(x,y)}{K(x)}u(x)\,\frac{dx}{x}.$$

If Φ *is a convex function on an interval* $I\subseteq\mathbb{R}$ *and* $\varphi:I\to\mathbb{R}$ *is such that* $\varphi(x)\in\partial\Phi(x)$ *for all* $x\in\operatorname{Int}I$*, then the inequality*

$$\begin{aligned}\int_0^b w(y)\Phi(f(y))\,\frac{dy}{y}-\int_0^b u(x)\Phi(A_kf(x))\,\frac{dx}{x}\geq\int_0^b\frac{u(x)}{K(x)}\int_0^x k(x,y)\times\\ \times\big|\,|\Phi(f(y))-\Phi(A_kf(x))|-|\varphi(A_kf(x))|\cdot|f(y)-A_kf(x)|\,\big|\,dy\,\frac{dx}{x}\end{aligned}\tag{4.24}$$

holds for all measurable functions $f:(0,b)\to\mathbb{R}$ *with values in* I *and for* A_kf *defined by*

$$A_kf(x)=\frac{1}{K(x)}\int_0^x k(x,y)f(y)\,dy,\ x\in(0,b).\tag{4.25}$$

If the function Φ *is concave, the order of integrals on the left-hand side of* (4.24) *is reversed. If* Φ *is monotone convex on the interval* $I\subseteq\mathbb{R}$ *and* $\varphi:I\to\mathbb{R}$ *is such that* $\varphi(x)\in\partial\Phi(x)$ *for all* $x\in IntI$*, then the following inequality*

$$\begin{aligned}&\int_0^b w(y)\Phi(f(y))\,\frac{dy}{y}-\int_0^b u(x)\Phi(A_kf(x))\,\frac{dx}{x}\\ &\geq\Bigg|\int_0^b\frac{u(x)}{K(x)}\int_0^x sgn(f(y)-A_kf(x))k(x,y)\Big[\Phi(f(y)-\Phi(A_kf(x))\\ &\quad-|\varphi(A_kf(x))|.(f(y)-A_kf(x))\Big]dy\,\frac{dx}{x}\Bigg|\end{aligned}\tag{4.26}$$

holds for all measurable functions $f:(0,b)\to\mathbb{R}$ *with values in* I*.*

Proof. Denote $T_1=\{(x,y)\in\mathbb{R}^2_+:0<y\leq x<b\}$ and set $\Omega_1=\Omega_2=(0,b)$ in Theorem 4.1. Relation (4.24) follows from (4.1) by replacing $d\mu_1(x)$, $d\mu_2(y)$, $u(x)$, and $k(x,y)$ respectively with dx, dy, $\frac{u(x)}{x}$, and $k(x,y)\chi_{T_1}(x,y)$. In this case, (2.15) reduces to (4.25), while (2.16) becomes (4.23). Moreover, $w(y)=yv(y)$, $y\in(0,b)$. Similarly we obtain (4.26). □

Remark 4.5 Since the right-hand side of inequality (4.24) is non-negative, Theorem 4.2 can be seen as a refinement of Theorem 1.5 in [65]. In particular, for $k(x,y)\equiv1$, $x,y\in$

$(0,b)$, and the classical Hardy's operator

$$Hf(x) = \frac{1}{x}\int_0^x f(y)\,dy,\ x \in (0,b),$$

we get a refinement of Theorem 1 in [30], that is, the refined Hardy-type inequality for convex functions,

$$\begin{aligned}
&\int_0^b w(y)\Phi(f(y))\,\frac{dy}{y} - \int_0^b u(x)\Phi(Hf(x))\,\frac{dx}{x} \\
&\geq \int_0^b \frac{u(x)}{x^2}\int_0^x \big|\,|\Phi(f(y)) - \Phi(Hf(x))| - |\varphi(Hf(x))|\cdot|f(y) - Hf(x)|\,\big|\,dy\,dx,
\end{aligned}$$

where

$$w(y) = y\int_y^b \frac{u(x)}{x^2}\,dx,\ y \in (0,b).$$

Observing that the right-hand side of the above inequality is greater than

$$\begin{aligned}
&\left|\int_0^b u(x)\int_0^x |\Phi(f(y)) - \Phi(Hf(x))|\,dy\,\frac{dx}{x^2}\right. \\
&\qquad \left. - \int_0^b u(x)\,|\varphi(Hf(x))|\int_0^x |f(y) - Hf(x)|\,dy\,\frac{dx}{x^2}\right|,
\end{aligned}$$

we obtain Theorem 2.2 in [21]. Therefore, Theorem 4.2 generalizes the result mentioned.

Applying Theorem 4.2 to power and exponential functions, we get the following two corollaries.

Corollary 4.4 *Let $0 < b \leq \infty$ and k, K, u, and w be as in Theorem 4.2. Let $p \in \mathbb{R}$ be such that $p \neq 0$, f be a non-negative measurable function on $(0,b)$ (f positive for $p < 0$), and let $A_k f, R_{p,k} f$ and $M_{p,k}$ be defined by (4.25), (4.11) and (4.12) respectively. If $p > 1$ or $p < 0$, then*

$$\int_0^b w(y)f^p(y)\,\frac{dy}{y} - \int_0^b u(x)A_k^p f(x)\,\frac{dx}{x} \geq \int_0^b \frac{u(x)}{K(x)}\int_0^x k(x,y)R_{p,k}f(x,y)\,dy\,\frac{dx}{x}, \tag{4.27}$$

while for $p \in (0,1)$ the order of integrals on the left-hand side of (4.27) is reversed. If $p = 1$, then both-hand sides of (4.27) are equal to 0.

Let $p > 1$ and f be a positive measurable function on $(0,b)$. Then the following inequality *holds*

$$\int_0^b w(y)f^p(y)\frac{dy}{y} - \int_0^b u(x)\,(A_k f(x))^p\,\frac{dx}{x}$$
$$\geq \left|\int_0^b \frac{u(x)}{K(x)}\int_0^x sgn(f(y)-A_k f)k(x,y)M_{p,k}f(x,y)\,dy\,\frac{dx}{x}\right|. \tag{4.28}$$

If $p \in (0,1)$, then the order of terms on the left-hand side of relation (4.28) is reversed.

Corollary 4.5 *Let $0 < b \leq \infty$, k, K, u, and w be as in Theorem 4.2, and let $p \in \mathbb{R}$ be such that $p \neq 0$. If f is a positive measurable function on $(0,b)$,*

$$G_k f(x) = \exp\left(\frac{1}{K(x)}\int_0^x k(x,y)\log f(y)\,dy\right),\ x \in (0,b),$$

and $S_{p,k}f$ is defined by (4.17), then

$$\int_0^b w(y)f^p(y)\,\frac{dy}{y} - \int_0^b u(x)G_k^p f(x)\,\frac{dx}{x} \geq \int_0^b \frac{u(x)}{K(x)}\int_0^x k(x,y)S_{p,k}f(x,y)\,dy\,\frac{dx}{x}. \tag{4.29}$$

Moreover, for $p = 1$ we have

$$\int_0^b w(y)f(y)\,\frac{dy}{y} - \int_0^b u(x)G_k f(x)\,\frac{dx}{x} \geq \int_0^b \frac{u(x)}{K(x)}\int_0^x k(x,y)\times$$
$$\times\left|\,|f(y)-G_k f(x)| - G_k(x)\left|\log\frac{f(y)}{G_k f(x)}\right|\right|dy\,\frac{dx}{x} \tag{4.30}$$

and relations (4.29) and (4.30) are equivalent.
Let $p > 1$ and f be a positive measurable function on $(0,b)$. Then the following inequality holds

$$\int_0^b w(y)f^p(y)\frac{dy}{y} - \int_0^b u(x)(G_k^p f(x))\frac{dx}{x}$$
$$\geq \left|\int_0^b \frac{u(x)}{K(x)}\int_0^x sgn(f(y)-G_k f(x))k(x,y)P_{p,k}f(x,y)dy\,\frac{dx}{x}\right|,$$

where $P_{p,k}$ is given by (4.19).

The above results can be applied to some important particular kernels. Namely, in the following example we discuss refined Hardy and Pólya-Knopp-type inequalities related to the Riemann-Liouville operator

$$R_\gamma f(x) = \frac{\gamma}{x^\gamma} \int_0^x (x-y)^{\gamma-1} f(y)\, dy, \tag{4.31}$$

where $\gamma \in \mathbb{R}_+$. Of course, for $\gamma = 1$ we have $R_1 = H$, that is, the classical Hardy's integral operator.

Example 4.1 Suppose $0 < b \le \infty$, $\gamma \in \mathbb{R}_+$, and T_1 is as in the proof of Theorem 4.2. If $u(x) \equiv 1$, $k(x,y) = \frac{\gamma}{x^\gamma}(x-y)^{\gamma-1}\chi_{T_1}(x,y)$, and $R_\gamma f(x)$ is as in (4.31), then inequality (4.24) reads

$$\begin{aligned}\int_0^b \left(1-\frac{y}{b}\right)^\gamma \Phi(f(y))\,\frac{dy}{y} - \int_0^b \Phi(R_\gamma f(x))\,\frac{dx}{x} \ge \gamma \int_0^b \int_0^x (x-y)^{\gamma-1} \times \\ \times \Big| \left|\Phi(f(y)) - \Phi(R_\gamma f(x))\right| - \left|\varphi(R_\gamma f(x))\right| \cdot \left|f(y) - R_\gamma f(x)\right| \Big|\, dy\,\frac{dx}{x^{\gamma+1}},\end{aligned} \tag{4.32}$$

so we obtained a refinement of Example 4.2 in [65]. We also obtain the following result, since (4.26) becomes

$$\begin{aligned}&\int_0^b \left(1-\frac{y}{b}\right)^\gamma \Phi(f(y))\,\frac{dy}{y} - \int_0^b \Phi\left(R_\gamma f(x)\right)\frac{dx}{x} \\ &\ge \Bigg| \gamma \int_0^b \int_0^x sgn(f(y) - A_k f)(x-y)^{\gamma-1}\Big[\Phi(f(y) - \Phi(R_\gamma f(x)) \\ &\quad - |\varphi(R_\gamma f(x))| \cdot (f(y) - R_\gamma f(x))\Big]\, dy\,\frac{dx}{x^{\gamma+1}}\Bigg|.\end{aligned}$$

As in Corollaries 4.4 and 4.5, relation (4.32) can be considered with Φ being a power or exponential function. In particular, let $p,k \in \mathbb{R}$ be such that $\frac{k-1}{p} > 0$, f be a non-negative function on $(0,b)$ (positive for $p < 0$), and

$$Rf(x) = \int_0^x \left[1 - \left(\frac{y}{x}\right)^{\frac{k-1}{p}}\right]^{\gamma-1} f(y)\, dy,\ x \in (0,b).$$

Rewrite (4.32) for $\Phi(x) = x^p$ and substitute $b^{\frac{k-1}{p}}$ and $f\left(y^{\frac{p}{k-1}}\right) y^{\frac{p}{k-1}-1}$ instead of b and $f(y)$

respectively. After suitable variable changes, for $p \geq 1$ and $p < 0$ we get

$$\begin{aligned}
&\left(\frac{p}{\gamma(k-1)}\right)^p \int_0^b \left[1-\left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right]^\gamma x^{p-k} f^p(x)\,dx - \int_0^b x^{-k} R^p f(x)\,dx \\
&\geq \Bigg|\left(\frac{p}{\gamma(k-1)}\right)^{p-1} \int_0^b x^{\frac{1-k}{p}-1} \int_0^x \left[1-\left(\frac{y}{x}\right)^{\frac{k-1}{p}}\right]^{\gamma-1} y^{\frac{k-1}{p}-1} \times \\
&\quad \times \left| y^{p-k+1} f^p(y) - \left(\frac{\gamma(k-1)}{p}\right)^p x^{1-k} R^p f(x)\right| dy\,dx \\
&\quad - |p| \int_0^b x^{-k} R^{p-1} f(x) \int_0^x \left[1-\left(\frac{y}{x}\right)^{\frac{k-1}{p}}\right]^{\gamma-1} \times \\
&\quad \times \left| f(y) - \frac{k-1}{p}\cdot\frac{\gamma}{y}\left(\frac{y}{x}\right)^{\frac{k-1}{p}} R f(x)\right| dy\,dx\Bigg|,
\end{aligned} \tag{4.33}$$

while for $p \in (0,1)$ the order of integrals on the left-hand side of (4.33) is reversed. Note that for $\gamma = 1$ inequality (4.33) reduces to the refined strengthened Hardy's inequality from Corollary 3.1 in [21]. Moreover, for $b = \infty$ and $p = k$ we obtain a refinement of the classical Hardy's inequality (0.1).

On the other hand, for $\gamma = 1$, $\Phi(x) = e^x$, a positive function f on $(0,b)$, $f(y)$ replaced with $\log(yf(y))$, and

$$Gf(x) = \exp\left(\frac{1}{x}\int_0^x \log f(y)\,dy\right),\ x \in (0,b), \tag{4.34}$$

relation (4.32) becomes

$$\begin{aligned}
e\int_0^b \left(1-\frac{y}{b}\right) f(y)\,dy - \int_0^b Gf(x)\,dx &\geq \Bigg| \int_0^b \int_0^x |eyf(y) - xGf(x)|\,dy\,\frac{dx}{x^2} \\
&\quad - \int_0^b Gf(x) \int_0^x \left|\log\left(\frac{eyf(y)}{xGf(x)}\right)\right| dy\,\frac{dx}{x}\Bigg|,
\end{aligned}$$

so we obtained the refined strengthened Pólya-Knopp's inequality from Corollary 3.3 in [21]. In the case when $b = \infty$, we get a refinement of the classical Pólya-Knopp's inequality (0.2). □

We continue by formulating results dual to Theorem 4.2 and its corollaries. They are derived from Theorem 4.1 by similar arguments. The following theorem is dual to Theorem 4.2.

Theorem 4.3 *For $0 \leq b < \infty$, let $k : (b,\infty)\times(b,\infty) \to \mathbb{R}$ be a non-negative measurable function, such that*

$$\tilde{K}(x) = \int_x^\infty k(x,y)\, dy > 0,\ x \in (b,\infty), \tag{4.35}$$

and a weight $u : (b,\infty) \to \mathbb{R}$ be such that the function $x \mapsto \frac{u(x)}{x}\cdot\frac{k(x,y)}{\tilde{K}(x)}$ is integrable on (b,y) for each fixed $y \in (b,\infty)$. Let the function $\tilde{w} : (b,\infty) \to \mathbb{R}$ be defined by

$$\tilde{w}(y) = y\int_b^y \frac{k(x,y)}{\tilde{K}(x)}\, u(x)\,\frac{dx}{x}.$$

If Φ is a convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \mathrm{Int}I$, then the inequality

$$\begin{aligned}&\int_b^\infty \tilde{w}(y)\Phi(f(y))\,\frac{dy}{y} - \int_b^\infty u(x)\Phi(\tilde{A}_k f(x))\frac{dx}{x} \geq \int_b^\infty \frac{u(x)}{\tilde{K}(x)}\int_x^\infty k(x,y)\times\\ &\times\Big|\big|\Phi(f(y)) - \Phi(\tilde{A}_k f(x))\big| - \big|\varphi(\tilde{A}_k f(x))\big|\cdot\big|f(y)-\tilde{A}_k f(x)\big|\Big|\, dy\,\frac{dx}{x}\end{aligned} \tag{4.36}$$

holds for all measurable functions $f : (b,\infty) \to \mathbb{R}$ with values in I and for $\tilde{A}_k f$ defined by

$$\tilde{A}_k f(x) = \frac{1}{\tilde{K}(x)}\int_x^\infty k(x,y)f(y)\, dy,\ x \in (b,\infty). \tag{4.37}$$

If the function Φ is concave, the order of integrals on the left-hand side of (4.37) is reversed.
If Φ is a monotone convex on an interval $I \subseteq \mathbb{R}$, and $\varphi : I \to \mathbb{R}$ such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in IntI$, the the following inequality

$$\begin{aligned}&\int_b^\infty \widetilde{w}(y)\Phi\left(f(y)\right)\frac{dy}{y} - \int_b^\infty u(x)\Phi\left(\widetilde{A}_k f(x)\right)\frac{dx}{x}\\ &\geq \Bigg|\int_b^\infty \frac{u(x)}{\widetilde{K}(x)}\int_x^\infty sgn(f(y)-\widetilde{A}_k f)k(x,y)\Big[\Phi(f(y)) - \Phi(\widetilde{A}_k f(x))\\ &-|\varphi(\widetilde{A}_k f(x))|.(f(y)-\widetilde{A}_k f(x))\Big]\,dy\,\frac{dx}{x}\Bigg|\end{aligned} \tag{4.38}$$

holds for all measurable functions $f : (b,\infty) \to \mathbb{R}$ with values in I.

Proof. Let $T_2 = \{(x,y) \in \mathbb{R}^2_+ : b < x \leq y < \infty\}$. Inequality (4.36) follows directly from Theorem 4.1, rewritten with $\Omega_1 = \Omega_2 = (b,\infty)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$, and with $\frac{u(x)}{x}$ and $k(x,y)\chi_{T_2}(x,y)$ instead of $u(x)$ and $k(x,y)$. Note that (2.15) and (2.16) respectively become (4.37) and (4.35), while $\tilde{w}(y) = yv(y)$, $y \in (b,\infty)$. Similarly we obtain (4.38). □

Remark 4.6 Note that Theorem 4.3 provides a refinement of Theorem 4.3 in [65]. Furthermore, by setting $k(x,y) = \frac{1}{y^2}$, $x,y \in (b,\infty)$, and denoting

$$\tilde{H}f(x) = x\int_x^\infty f(y)\,\frac{dy}{y^2},\; x \in (b,\infty),$$

inequality (4.36) reduces to the following refined dual Hardy-type inequality for convex functions:

$$\begin{aligned}&\int_b^\infty \tilde{w}(y)\Phi(f(y))\,\frac{dy}{y} - \int_b^\infty u(x)\Phi(\tilde{H}f(x))\,\frac{dx}{x}\\ &\geq \int_b^\infty u(x)\int_x^\infty \left|\left|\Phi(f(y)) - \Phi(\tilde{H}f(x))\right| - \left|\varphi(\tilde{H}f(x))\right|\cdot\left|f(y) - \tilde{H}f(x)\right|\right|\frac{dy}{y^2}\,dx,\end{aligned}$$

where

$$\tilde{w}(y) = \frac{1}{y}\int_b^y u(x)\,dx,\; y \in (b,\infty).$$

Since the right-hand side of this inequality is not less than

$$\begin{aligned}&\left|\int_b^\infty u(x)\int_x^\infty \left|\Phi(f(y)) - \Phi(\tilde{H}f(x))\right|\frac{dy}{y^2}\,dx\right.\\ &\left.\quad - \int_b^\infty u(x)\left|\varphi(\tilde{H}f(x))\right|\int_x^\infty \left|f(y) - \tilde{H}f(x)\right|\frac{dy}{y^2}\,dx\right|,\end{aligned}$$

as a consequence of our result we get Theorem 2.3 in [21]. Similarly, we obtain as a special case of (4.38) the following result

$$\begin{aligned}&\int_b^\infty \left(1 - \frac{b}{y}\right)\Phi\left(f(y)\right)\frac{dy}{y} - \int_b^\infty \Phi\left(\widetilde{H}f(x)\right)\frac{dx}{x}\\ &\geq \left|\int_b^\infty\int_x^\infty sgn(f(y) - \widetilde{H}f(x))\left[\Phi(f(y)) - \Phi(\widetilde{H}f(x))\right.\right.\\ &\qquad \left.\left. - |\varphi(\widetilde{H}f(x))|\cdot(f(y) - \widetilde{H}f(x))\right]\frac{dy}{y^2}\,dx\right|.\end{aligned}$$

The next two corollaries are dual to Corollary 4.4 and Corollary 4.5.

Corollary 4.6 *Let $0 \leq b < \infty$ and let k, $\tilde{K}$, u, and $\tilde{w}$ be as in Theorem* 4.3. *For $p \in \mathbb{R}$, $p \neq 0$, and a non-negative measurable function f on (b,∞) (f positive for $p < 0$), let $\tilde{A}_k f$ be defined by* (4.37) *and*

$$\tilde{R}_{p,k}f(x,y) = \left|\left|f^p(y) - \tilde{A}_k^p f(x)\right| - |p|\cdot\left|\tilde{A}_k f(x)\right|^{p-1}\left|f(y) - \tilde{A}_k f(x)\right|\right|,$$

for $x,y \in (b,\infty)$. Then the inequality

$$\int_b^\infty \tilde{w}(y)f^p(y)\,\frac{dy}{y} - \int_b^\infty u(x)\tilde{A}_k^p f(x)\,\frac{dx}{x} \geq \int_b^\infty \frac{u(x)}{\tilde{K}(x)}\int_x^\infty k(x,y)\tilde{R}_{p,k}f(x,y)\,dy\,\frac{dx}{x} \tag{4.39}$$

holds for $p > 1$ *and* $p < 0$. *For* $p \in (0,1)$ *the order of integrals on the left-hand side of* (4.39) *is reversed, while for* $p = 1$ *its both-hand sides are equal to* 0.
Let $p > 1$, f *be a non-negative measurable function on* (b,∞). *Then the following inequality holds*

$$\int_b^\infty \widetilde{w}(y) f^p(y) \frac{dy}{y} - \int_b^\infty u(x) \left(\widetilde{A}_k f(x)\right)^p \frac{dx}{x}$$

$$\geq \left| \int_b^\infty \frac{u(x)}{\widetilde{K}(x)} \int_x^\infty sgn(f(y) - \widetilde{A}_k f) k(x,y) \left[f^p(y) - (\widetilde{A}_k f(x))^p \right.\right.$$

$$\left.\left. -p|\widetilde{A}_k f(x)|^{p-1}.(f(y) - \widetilde{A}_k f(x)) \right] dy \frac{dx}{x} \right|.$$

Corollary 4.7 *Suppose that* $p \in \mathbb{R} \setminus \{0\}$, $0 \leq b < \infty$, *and that* k, $\tilde{K}$, u, *and* $\tilde{w}$ *are as in Theorem* 4.3. *If* f *is a positive measurable function on* (b,∞),

$$\tilde{G}_k f(x) = \exp\left(\frac{1}{\tilde{K}(x)} \int_x^\infty k(x,y) \log f(y)\, dy \right), \; x \in (b,\infty),$$

and

$$\tilde{S}_{p,k} f(x,y) = \left| \left| f^p(y) - \tilde{G}_k^p f(x) \right| - |p|\, \tilde{G}_k^p(x) \left| \log \frac{f(y)}{\tilde{G}_k f(x)} \right| \right|, \; x,y \in (b,\infty),$$

then the inequality

$$\int_b^\infty \tilde{w}(y) f^p(y) \frac{dy}{y} - \int_b^\infty u(x) \tilde{G}_k^p f(x) \frac{dx}{x} \geq \int_b^\infty \frac{u(x)}{\tilde{K}(x)} \int_x^\infty k(x,y) \tilde{S}_{p,k} f(x,y)\, dy \frac{dx}{x} \tag{4.40}$$

holds. In particular, for $p = 1$ *we have*

$$\int_b^\infty \tilde{w}(y) f(y) \frac{dy}{y} - \int_b^\infty u(x) \tilde{G}_k f(x) \frac{dx}{x} \geq \int_b^\infty \frac{u(x)}{\tilde{K}(x)} \int_x^\infty k(x,y) \times$$

$$\times \left| \left| f(y) - \tilde{G}_k f(x) \right| - \tilde{G}_k(x) \left| \log \frac{f(y)}{\tilde{G}_k f(x)} \right| \right| dy \frac{dx}{x} \tag{4.41}$$

and relations (4.40) *and* (4.41) *are equivalent.*

We conclude this section by giving results dual to those from Example 4.1, that is, by explicating refined Hardy and Pólya-Knopp-type inequalities related to Weyl's fractional integral operator

$$W_\gamma f(x) = \gamma x \int_x^\infty (y-x)^{\gamma-1} f(y) \frac{dy}{y^{\gamma+1}}, \tag{4.42}$$

where $\gamma \in \mathbb{R}_+$. Note that $W_1 = \tilde{H}$, that is, for $\gamma = 1$ we get the classical dual Hardy's integral operator and related inequalities.

Example 4.2 Let $0 \le b < \infty$, $\gamma \in \mathbb{R}_+$, and T_2 be as in the proof of Theorem 4.3. For $u(x) \equiv 1$, $k(x,y) = \gamma \frac{x}{y^{\gamma+1}}(y-x)^{\gamma-1}\chi_{T_2}(x,y)$, and $W_\gamma f(x)$ as in (4.42), inequality (4.36) becomes

$$\int_b^\infty \left(1-\frac{b}{y}\right)^\gamma \Phi(f(y))\,\frac{dy}{y} - \int_b^\infty \Phi(W_\gamma f(x))\,\frac{dx}{x} \ge \gamma \int_b^\infty \int_x^\infty (y-x)^{\gamma-1} \times$$
$$\times \left| \left|\Phi(f(y)) - \Phi(W_\gamma f(x))\right| - \left|\varphi(W_\gamma f(x))\right| \cdot \left|f(y) - W_\gamma f(x)\right| \right| \frac{dy}{y^{\gamma+1}}\,dx. \tag{4.43}$$

Similarly, as s special case of (4.38) we obtain the following result

$$\int_b^\infty \left(1-\frac{b}{y}\right)^\gamma \Phi(f(y))\,\frac{dy}{y} - \int_b^\infty \Phi\left(W_\gamma f(x)\right)\frac{dx}{x}$$
$$\ge \left|\gamma \int_b^\infty \int_x^\infty sgn(f(y) - W_\gamma f)(y-x)^{\gamma-1} \left(\Phi(f(y) - \Phi(W_\gamma f(x))\right.\right.$$
$$\left.\left. - |\varphi(W_\gamma f)| \cdot (f(y) - W_\gamma f(x))\,\frac{dy}{y^{\gamma+1}}\,dx\right)\right|.$$

Now, we apply (4.43) to power and exponential functions. Namely, let $p,k \in \mathbb{R}$ be such that $\frac{p}{1-k} > 0$, f be a non-negative measurable function on (b,∞) (f positive for $p<0$),

$$Wf(x) = \int_x^\infty \left[1-\left(\frac{x}{y}\right)^{\frac{1-k}{p}}\right]^{\gamma-1} f(y)\,dy,\ x \in (b,\infty),$$

and $\Phi(x) = x^p$. Rewrite (4.43) with $b^{\frac{1-k}{p}}$ and $f\left(y^{\frac{p}{1-k}}\right)y^{\frac{p}{1-k}+1}$ instead of b and $f(y)$ respectively. After some variable substitutions, for $p \ge 1$ and $p < 0$ we obtain the inequality

$$\left(\frac{p}{\gamma(1-k)}\right)^p \int_b^\infty \left[1-\left(\frac{b}{x}\right)^{\frac{1-k}{p}}\right]^\gamma x^{p-k} f^p(x)\,dx - \int_b^\infty x^{-k} W^p f(x)\,dx$$
$$\ge \left| \left(\frac{p}{\gamma(1-k)}\right)^{p-1} \int_b^\infty x^{\frac{1-k}{p}-1} \int_x^\infty \left[1-\left(\frac{x}{y}\right)^{\frac{1-k}{p}}\right]^{\gamma-1} y^{\frac{k-1}{p}-1} \times \right.$$
$$\times \left| y^{p-k+1} f^p(y) - \left(\frac{\gamma(1-k)}{p}\right)^p x^{1-k} W^p f(x)\right| dy\,dx$$
$$- |p| \int_b^\infty x^{-k} W^{p-1} f(x) \int_x^\infty \left[1-\left(\frac{x}{y}\right)^{\frac{1-k}{p}}\right]^{\gamma-1} \times$$
$$\left. \times \left| f(y) - \frac{(1-k)}{p}\cdot\frac{\gamma}{y}\left(\frac{x}{y}\right)^{\frac{1-k}{p}} Wf(x)\right| dy\,dx \right|. \tag{4.44}$$

For $p \in (0,1)$, relation (4.44) holds with integrals on its left-hand side written in the reverse order. Moreover, if $\gamma = 1$, then (4.44) becomes the refined strengthened dual Hardy's inequality from Corollary 3.2 in [21].

In the case when $\gamma = 1$, $\Phi(x) = e^x$, f is a positive function on (b,∞), and

$$\tilde{G}f(x) = \exp\left(x \int_x^\infty \log f(y)\,\frac{dy}{y^2}\right),\ x \in (b,\infty),$$

after substituting $\log(yf(y))$ instead of $f(y)$, relation (4.43) reads

$$\begin{aligned}\frac{1}{e}\int_b^\infty \left(1-\frac{b}{x}\right) f(x)\,dx - \int_b^\infty \tilde{G}f(x)\,dx \geq \Bigg| &\int_b^\infty \int_x^\infty \left|\frac{1}{e}yf(y) - x\tilde{G}f(x)\right| \frac{dy}{y^2}\,dx \\ &- \int_b^\infty x\tilde{G}f(x) \int_x^\infty \left|\log\left(\frac{yf(y)}{ex\tilde{G}f(x)}\right)\right| \frac{dy}{y^2}\,dx\Bigg|,\end{aligned}$$

that is, it is reduced to the refined strengthened dual Pólya-Knopp's inequality given in Corollary 3.4 in [21]. □

4.2 One-dimensional refined Hardy-Hilbert-type inequalities

We continue the above analysis by considering some important kernels related to $\Omega_1 = \Omega_2 = \mathbb{R}_+$ and by assuming that $d\mu_1(x) = dx$, $d\mu_2(y) = dy$, and that $\Phi : \mathbb{R}_+ \to \mathbb{R}$ is given by $\Phi(x) = x^p$, where $p \in \mathbb{R}$, $p \neq 0$. In this setting, Corollary 4.1 provides new refinements of some well-known one-dimensional Hardy-Hilbert-type inequalities.

First, we obtain a generalization and a refinement of the classical Hardy-Hilbert's inequality (2.2). It is given in the following example.

Example 4.3 For $p \in \mathbb{R}\setminus\{0\}$, let $s \in \mathbb{R}$ be such that $\frac{s-2}{p}, \frac{s-2}{p'} > -1$ and the kernel $k : \mathbb{R}_+^2 \to \mathbb{R}$ be defined by $k(x,y) = \left(\frac{y}{x}\right)^{\frac{s-2}{p}} (x+y)^{-s}$. Let $\alpha \in \left(-\frac{s-2}{p'} - 1, \frac{s-2}{p} + 1\right)$ be arbitrary and the weight $u : \mathbb{R}_+ \to \mathbb{R}$ be given by $u(x) = x^{\alpha-1}$. Set

$$C_1 = B\left(\frac{s-2}{p} - \alpha + 1, \frac{s-2}{p'} + \alpha + 1\right) \quad\text{and}\quad C_2 = B\left(\frac{s-2}{p} + 1, \frac{s-2}{p'} + 1\right),$$

where $B(\cdot,\cdot)$ denotes the usual Beta function. Let f be a non-negative function on $\mathbb{R}_+$ (positive, if $p < 0$) and Sf its generalized Stieltjes transform,

$$Sf(x) = \int_0^\infty \frac{f(y)}{(x+y)^s}\,dy,\ x \in \mathbb{R}_+$$

(see [8] and [97] for further information). Corollary 4.1, rewritten with $f(y)y^{\frac{2-s}{p}}$ instead of $f(y)$, implies that the inequality

$$
\begin{aligned}
&C_1C_2^{p-1}\int_0^\infty y^{\alpha-s+1}f^p(y)\,dy-\int_0^\infty x^{\alpha+(s-1)(p-1)}S^pf(x)\,dx\\
&\geq\Bigg|C_2^{p-1}\int_0^\infty x^{\alpha+\frac{s-2}{p'}}\int_0^\infty \frac{y^{\frac{s-2}{p}}}{(x+y)^s}\left|f^p(y)y^{2-s}-\frac{x^{(s-1)(p-1)+1}}{C_2^p}S^pf(x)\right|dy\,dx\\
&\quad-|p|\int_0^\infty x^{\alpha+(s-1)(p-1)}\int_0^\infty \frac{S^{p-1}f(x)}{(x+y)^s}\left|f(y)-\frac{1}{C_2}x^{\frac{s-2}{p'}+1}y^{\frac{s-2}{p}}Sf(x)\right|dy\,dx\Bigg|
\end{aligned}
\tag{4.45}
$$

holds for $p\geq 1$ and $p<0$, while for $p\in(0,1)$ it holds with the reverse order of the integrals on its left-hand side. In particular, for $\alpha=0$ we get a refinement of the general Hardy-Hilbert-type inequality from [104], with the best possible constant $C=C_2^p=B^p\left(\frac{s-2}{p}+1,\frac{s-2}{p'}+1\right)$. Moreover, for $p>1$, $\alpha=0$, and $s=1$, we have $C_1=C_2=B\left(\frac{1}{p},\frac{1}{p'}\right)=\frac{\pi}{\sin\frac{\pi}{p}}$, so relation (4.45) provides a new refinement of the classical Hardy-Hilbert's inequality (2.2). Analogously, from (4.14) we can also obtain refinement of the classical Hardy-Hilbert's inequality (2.2), but we omit the details here. □

Similarly, in the next example we generalize and refine the classical Hardy-Littlewood-Pólya's inequality

$$
\int_0^\infty\left(\int_0^\infty\frac{f(y)}{\max\{x,y\}}\right)^p dy\leq (pp')^p\int_0^\infty f^p(y)\,dy, \tag{4.46}
$$

which holds for $1<p<\infty$ and non-negative functions $f\in L^p(\mathbb{R}_+)$.

Example 4.4 Let the real parameters p, s, α, and the weight function u be as in Example 4.3. Define the kernel $k:\mathbb{R}_+^2\to\mathbb{R}$ by $k(x,y)=\left(\frac{y}{x}\right)^{\frac{s-2}{p}}\max\{x,y\}^{-s}$ and for a non-negative function f on $\mathbb{R}_+$ (positive for $p<0$) set

$$
Lf(x)=\int_0^\infty\frac{f(y)}{\max\{x,y\}^s}\,dy,\ x\in\mathbb{R}_+.
$$

Finally, denote

$$
D_1=\frac{pp's}{(p-p\alpha+s-2)(\alpha p'+p'+s-2)}\quad\text{and}\quad D_2=\frac{pp's}{(p+s-2)(p'+s-2)}.
$$

Applying the same procedure as in Example 4.3, we obtain that the inequality

$$
\begin{aligned}
& D_1 D_2^{p-1} \int_0^\infty y^{\alpha-s+1} f^p(y)\,dy - \int_0^\infty x^{\alpha+(s-1)(p-1)} L^p f(x)\,dx \\
\geq & \Bigg| D_2^{p-1} \int_0^\infty x^{\alpha+\frac{s-2}{p'}} \int_0^\infty \frac{y^{\frac{s-2}{p}}}{\max\{x,y\}^s} \left| f^p(y) y^{2-s} - \frac{x^{(s-1)(p-1)+1}}{D_2^p} L^p f(x) \right| dy\,dx \\
& - |p| \int_0^\infty x^{\alpha+(s-1)(p-1)} \int_0^\infty \frac{L^{p-1} f(x)}{\max\{x,y\}^s} \left| f(y) - \frac{1}{D_2} x^{\frac{s-2}{p'}+1} y^{\frac{s-2}{p}} L f(x) \right| dy\,dx \Bigg|
\end{aligned}
\tag{4.47}
$$

holds for $p \geq 1$ and $p < 0$, while for $p \in (0,1)$ it holds with the integrals on its left-hand side given in the reverse order. Note that the constant $C = D_2^p = \left[\frac{pp's}{(p+s-2)(p'+s-2)}\right]^p$ is the best possible for the Hardy-Littlewood-Pólya-type inequalities with $\alpha = 0$. As a special case, for $p > 1$, $\alpha = 0$, and $s = 1$, we get $D_1 = D_2 = pp'$, that is, relation (4.47) is a new refinement of the classical Hardy-Littlewood-Pólya's inequality (see [51] for further details). Analogously, from (4.14) we can also obtain a refinement of the classical Hardy-Littlewood-Pólya's inequality, but we omit the details here. □

To calculate the integrals in our last example of the refined Hardy-Hilbert-type inequalities, we used the well-known reflection formula for the Digamma function ψ,

$$
\int_0^\infty \frac{\log x}{x-1} x^{-\alpha}\,dx = \psi'(1-\alpha) + \psi'(\alpha) = \frac{\pi^2}{\sin^2 \pi\alpha},
$$

where $\alpha \in (0,1)$ (for details on ψ see [1]).

Example 4.5 As in the previous examples, let $p \in \mathbb{R}$, $p \neq 0$. For $\alpha \in (0,1)$, let the kernel k be defined on $\mathbb{R}_+^2$ by $k(x,y) = \frac{\log y - \log x}{y-x} \left(\frac{x}{y}\right)^\alpha$ and the weight $u : \mathbb{R}_+ \to \mathbb{R}$ by $u(x) = x^\beta$, where $\beta \in (-\alpha-1,-1)$. For a non-negative function f on $\mathbb{R}_+$ (positive for $p < 0$), let

$$
Mf(x) = \int_0^\infty \frac{\log y - \log x}{y-x} f(y)\,dy,\ x \in \mathbb{R}_+.
$$

Corollary 4.1, applied with the function $y \mapsto f(y)y^\alpha$ instead of f, then implies the inequality

$$
\begin{aligned}
&\frac{\pi^{2p}}{\sin^{2(p-1)}\pi\alpha\cdot\sin^2\pi(\alpha+\beta)}\int_0^\infty y^{p\alpha+\beta}f^p(y)\,dy-\int_0^\infty x^{p\alpha+\beta}M^p f(x)\,dx\\
&\geq\Bigg|\left(\frac{\pi}{\sin\pi\alpha}\right)^{2(p-1)}\int_0^\infty x^{\alpha+\beta}\int_0^\infty\frac{\log y-\log x}{y-x}y^{-\alpha}\times\\
&\qquad\times\left|f^p(y)y^{p\alpha}-\left(\frac{\sin\pi\alpha}{\pi}\right)^{2p}x^{p\alpha}M^p f(x)\right|dy\,dx\\
&\quad-|p|\int_0^\infty x^{p\alpha+\beta}M^{p-1}f(x)\int_0^\infty\frac{\log y-\log x}{y-x}\times\\
&\qquad\times\left|f(y)-\frac{\sin^2\pi\alpha}{\pi^2}\left(\frac{x}{y}\right)^{\alpha}Mf(x)\right|dy\,dx\Bigg| \qquad (4.48)
\end{aligned}
$$

for $p\geq 1$ and $p<0$, while for $p\in(0,1)$ the order of the integrals on the left-hand side of (4.48) is reversed. Especially, for $p>1$, $\alpha=\frac{1}{p}$, and $\beta=-1$, the left-hand side of (4.48) becomes

$$
\left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^{2p}\int_0^\infty f^p(y)\,dy-\int_0^\infty M^p f(x)\,dx.
$$

Since the above expression is positive (unless $f\equiv 0$) and bounded from below by a positive constant, relation (4.48) provides a generalization and a refinement of another classical Hardy-Hilbert-type inequality. Analogously, from (4.14) we can also obtain a refinement of the Hardy-Hilbert-type inequality, but we omit the details here. □

4.3 Refined Godunova-type inequalities

We can apply Theorem 4.1 to n-dimensional cells in $\mathbb{R}^n_+$. As a consequence, a generalization and a refinement of Godunova's inequality (2.10) is derived. Applying Theorem 4.1 with $\Omega_1=\Omega_2=\mathbb{R}^n_+$, the Lebesgue measure $d\mu_1(\mathbf{x})=d\mathbf{x}$ and $d\mu_2(\mathbf{y})=d\mathbf{y}$, and the kernel $k:\mathbb{R}^n_+\times\mathbb{R}^n_+\to\mathbb{R}$ of the form $k(\mathbf{x},\mathbf{y})=l\left(\frac{\mathbf{y}}{\mathbf{x}}\right)$, where $l:\mathbb{R}^n_+\to\mathbb{R}$ is a non-negative measurable function, we obtain the following theorem. We omit the results involving monotone convex function since they are obtained analogously.

Theorem 4.4 *Let l and u be non-negative measurable functions on $\mathbb{R}^n_+$, such that $0<L(\mathbf{x})=\mathbf{x}^{\mathbf{1}}\int_{\mathbb{R}^n_+}l(\mathbf{y})\,d\mathbf{y}<\infty$ for all $\mathbf{x}\in\mathbb{R}^n_+$, and that the function $\mathbf{x}\mapsto u(\mathbf{x})\frac{l\left(\frac{\mathbf{y}}{\mathbf{x}}\right)}{L(\mathbf{x})}$ is integrable* *on $\mathbb{R}^n_+$ for each fixed $\mathbf{y} \in \mathbb{R}^n_+$. Let the function v be defined on $\mathbb{R}^n_+$ by*

$$v(\mathbf{y}) = \int_{\mathbb{R}^n_+} u(\mathbf{x}) \frac{l\left(\frac{\mathbf{y}}{\mathbf{x}}\right)}{L(\mathbf{x})}\, d\mathbf{x}.$$

If Φ is a convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is any function such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$, then the inequality

$$\begin{aligned}\int_{\mathbb{R}^n_+} v(\mathbf{y})\Phi(f(\mathbf{y}))\, d\mathbf{y} &- \int_{\mathbb{R}^n_+} u(\mathbf{x})\Phi(A_l f(\mathbf{x}))\, d\mathbf{x} \\ &\geq \int_{\mathbb{R}^n_+} \frac{u(\mathbf{x})}{L(\mathbf{x})} \int_{\mathbb{R}^n_+} l\left(\frac{\mathbf{y}}{\mathbf{x}}\right) R_{\Phi,l} f(\mathbf{x},\mathbf{y})\, d\mathbf{y}\, d\mathbf{x} \qquad (4.49)\end{aligned}$$

holds for all measurable functions $f : \mathbb{R}^n_+ \to \mathbb{R}$ with values in I, where $A_l f(\mathbf{x})$ and $R_{\Phi,l} f(\mathbf{x},\mathbf{y})$ are defined for $\mathbf{x},\mathbf{y} \in \mathbb{R}^n_+$ by

$$A_l f(\mathbf{x}) = \frac{1}{L(\mathbf{x})} \int_{\mathbb{R}^n_+} l\left(\frac{\mathbf{y}}{\mathbf{x}}\right) f(\mathbf{y})\, d\mathbf{y}$$

and

$$R_{\Phi,l} f(\mathbf{x},\mathbf{y}) = \big|\, |\Phi(f(\mathbf{y})) - \Phi(A_l f(\mathbf{x}))| - |\varphi(A_l f(\mathbf{x}))| \cdot |f(\mathbf{y}) - A_l f(\mathbf{x})|\, \big|. \qquad (4.50)$$

If the function Φ is concave, the order of integrals on the left-hand side of (4.49) *is reversed.*

Especially, for $\int_{\mathbb{R}^n_+} l(\mathbf{t})\, d\mathbf{t} = 1$ and $u(\mathbf{x}) = \mathbf{x}^{-\mathbf{1}}$, Theorem 4.4 becomes the following refinement of Godunova's inequality (2.10).

Corollary 4.8 *Let $l : \mathbb{R}^n_+ \to \mathbb{R}$ be a non-negative measurable function and $\int_{\mathbb{R}^n_+} l(\mathbf{t})\, d\mathbf{t} = 1$. If Φ is a convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is any function such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$, then the inequality*

$$\int_{\mathbb{R}^n_+} \Phi(f(\mathbf{y}))\, \frac{d\mathbf{y}}{\mathbf{y}} - \int_{\mathbb{R}^n_+} \Phi(A_l f(\mathbf{x}))\, \frac{d\mathbf{x}}{\mathbf{x}} \geq \int_{\mathbb{R}^n_+} \mathbf{x}^{-\mathbf{2}} \int_{\mathbb{R}^n_+} l\left(\frac{\mathbf{y}}{\mathbf{x}}\right) R_{\Phi,l} f(\mathbf{x},\mathbf{y})\, d\mathbf{y}\, d\mathbf{x} \qquad (4.51)$$

holds for all measurable functions $f : \mathbb{R}^n_+ \to \mathbb{R}$ with values in I, where

$$A_l f(\mathbf{x}) = \mathbf{x}^{-\mathbf{1}} \int_{\mathbb{R}^n_+} l\left(\frac{\mathbf{y}}{\mathbf{x}}\right) f(\mathbf{y})\, d\mathbf{y},\ \mathbf{x} \in \mathbb{R}^n_+,$$

and $R_{\Phi,l} f$ is defined by (4.50). *If Φ is concave, the integrals on the left-hand side of* (4.51) *are given in the reverse order.*

To conclude this section, we give n-dimensional analogues of some previous results, that is, some new multidimensional refined general Hardy-type inequalities. These results can be regarded as refinements of those obtained in [86]. Namely, the following theorem is a refinement of Lemma 2.1 in [86].

Theorem 4.5 *Suppose that* $\mathbf{0} < \mathbf{b} \leq \infty$*, that* u *is a weight on* $(\mathbf{0}, \mathbf{b})$ *such that the function* $\mathbf{x} \mapsto \frac{u(\mathbf{x})}{\mathbf{x}^2}$ *is locally integrable in* $(\mathbf{0}, \mathbf{b})$*, and that the weight* w *is defined by*

$$w(\mathbf{y}) = \mathbf{y}^{\mathbf{1}} \int_{(\mathbf{y}, \mathbf{b})} u(\mathbf{x}) \frac{d\mathbf{x}}{\mathbf{x}^2}, \ \mathbf{y} \in (\mathbf{0}, \mathbf{b}).$$

Let $\Phi : I \to \mathbb{R}$ *be a convex function and* $\varphi : I \to \mathbb{R}$ *be any function such that* $\varphi(x) \in \partial\Phi(x)$ *for all* $x \in \operatorname{Int} I$*. If* $f : (\mathbf{0}, \mathbf{b}) \to \mathbb{R}$ *is a measurable function such that* $f(\mathbf{y}) \in I$ *for all* $y \in (\mathbf{0}, \mathbf{b})$*, and* $Hf(\mathbf{x})$ *and* $R_\Phi f(\mathbf{x}, \mathbf{y})$ *are defined for* $\mathbf{x}, \mathbf{y} \in (\mathbf{0}, \mathbf{b})$ *by*

$$Hf(\mathbf{x}) = \mathbf{x}^{-\mathbf{1}} \int_{(\mathbf{0}, \mathbf{x})} f(\mathbf{y})\, d\mathbf{y}$$

and

$$R_\Phi f(\mathbf{x}, \mathbf{y}) = \left| |\Phi(f(\mathbf{y})) - \Phi(Hf(\mathbf{x}))| - |\varphi(Hf(\mathbf{x}))| \cdot |f(\mathbf{y}) - Hf(\mathbf{x})| \right|,$$

then

$$\begin{aligned} \int_{(\mathbf{0}, \mathbf{b})} w(\mathbf{y}) \Phi(f(\mathbf{y})) \frac{d\mathbf{y}}{\mathbf{y}^{\mathbf{1}}} &- \int_{(\mathbf{0}, \mathbf{b})} u(\mathbf{x}) \Phi(Hf(\mathbf{x})) \frac{d\mathbf{x}}{\mathbf{x}^{\mathbf{1}}} \\ &\geq \int_{(\mathbf{0}, \mathbf{b})} u(\mathbf{x}) \int_{(\mathbf{0}, \mathbf{b})} R_\Phi f(\mathbf{x}, \mathbf{y})\, d\mathbf{y} \frac{d\mathbf{x}}{\mathbf{x}^2}. \end{aligned} \tag{4.52}$$

If Φ *is concave, the order of integrals on the left-hand side of* (4.52) *is reversed.*

Proof. Let $S_1 = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : \mathbf{0} < \mathbf{y} \leq \mathbf{x} < \mathbf{b}\}$ and $\Omega_1 = \Omega_2 = (\mathbf{0}, \mathbf{b})$. The proof follows directly from Theorem 4.1, applied with $d\mu_1(\mathbf{x}) = d\mathbf{x}$, $d\mu_2(\mathbf{y}) = d\mathbf{y}$, $k(\mathbf{x}, \mathbf{y}) = \chi_{S_1}$, and with $u(\mathbf{x})$ replaced with $\frac{u(\mathbf{x})}{\mathbf{x}^{\mathbf{1}}}$. Note that $w(\mathbf{y}) = \mathbf{y}^{\mathbf{1}} v(\mathbf{y})$. $\square$

Remark 4.7 Observe that for $u(\mathbf{x}) \equiv 1$ we have $w(\mathbf{y}) = \left(\mathbf{1} - \frac{\mathbf{y}}{\mathbf{b}}\right)^{\mathbf{1}}$.

Our last result is dual to Theorem 4.5 and provides a refinement of Lemma 2.3 in [86].

Theorem 4.6 *For* $\mathbf{0} \leq \mathbf{b} < \infty$*, let* $u : (\mathbf{b}, \infty) \to \mathbb{R}$ *be a locally integrable weight in* $(\mathbf{b}, \infty)$*, and the weight* w *be given by*

$$w(\mathbf{y}) = \mathbf{y}^{-\mathbf{1}} \int_{(\mathbf{b}, \mathbf{y})} u(\mathbf{x})\, d\mathbf{x}, \ \mathbf{y} \in (\mathbf{b}, \infty).$$

Suppose $\Phi : I \to \mathbb{R}$ *is a convex function and* $\varphi : I \to \mathbb{R}$ *is any function such that* $\varphi(x) \in \partial\Phi(x)$ *for all* $x \in \operatorname{Int} I$*. If* $f : (\mathbf{b}, \infty) \to \mathbb{R}$ *is a measurable function such that* $f(\mathbf{y}) \in I$ *for all* $y \in (\mathbf{b}, \infty)$*, and* $\tilde{H}f(\mathbf{x})$ *and* $\tilde{R}_\Phi f(\mathbf{x}, \mathbf{y})$ *are defined for* $\mathbf{x}, \mathbf{y} \in (\mathbf{b}, \infty)$ *by*

$$\tilde{H}f(\mathbf{x}) = \mathbf{x}^{\mathbf{1}} \int_{(\mathbf{x}, \infty)} f(\mathbf{y}) \frac{d\mathbf{y}}{\mathbf{y}^2}$$

and

$$\tilde{R}_\Phi f(\mathbf{x}, \mathbf{y}) = \left| \left|\Phi(f(\mathbf{y})) - \Phi(\tilde{H}f(\mathbf{x}))\right| - \left|\varphi(\tilde{H}f(\mathbf{x}))\right| \cdot \left|f(\mathbf{y}) - \tilde{H}f(\mathbf{x})\right| \right|,$$

then the inequality

$$\int_{(\mathbf{b},\infty)} w(\mathbf{y})\Phi(f(\mathbf{y}))\,\frac{d\mathbf{y}}{\mathbf{y}^{\mathbf{1}}} - \int_{(\mathbf{b},\infty)} u(\mathbf{x})\Phi(\tilde{H}f(\mathbf{x}))\,\frac{d\mathbf{x}}{\mathbf{x}^{\mathbf{1}}}$$
$$\geq \int_{(\mathbf{b},\infty)} u(\mathbf{x}) \int_{(\mathbf{x},\infty)} \tilde{R}_{\Phi} f(\mathbf{x},\mathbf{y})\,\frac{d\mathbf{y}}{\mathbf{y}^{\mathbf{2}}}\,d\mathbf{x} \tag{4.53}$$

holds. If the function Φ is concave, the order of integrals on the left-hand side of (4.53) *is reversed.*

Proof. Let $S_2 = \{(\mathbf{x},\mathbf{y}) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : \mathbf{b} < \mathbf{x} \leq \mathbf{y} < \infty\}$ and $\Omega_1 = \Omega_2 = (\mathbf{b},\infty)$. The proof follows directly from Theorem 4.1, rewritten with the Lebesgue measures $d\mu_1(\mathbf{x}) = d\mathbf{x}$, $d\mu_2(\mathbf{y}) = d\mathbf{y}$, the kernel $k(\mathbf{x},\mathbf{y}) = \mathbf{y}^{-\mathbf{2}}\chi_{S_2}(\mathbf{x},\mathbf{y})$, and with the weight $\frac{u(\mathbf{x})}{\mathbf{x}^{\mathbf{1}}}$ instead of $u(\mathbf{x})$. Note that $w(\mathbf{y}) = \mathbf{y}^{\mathbf{1}}v(\mathbf{y})$. □

Remark 4.8 Observe that for $u(\mathbf{x}) \equiv 1$ we get $w(\mathbf{y}) = \left(\mathbf{1} - \frac{\mathbf{b}}{\mathbf{y}}\right)^{\mathbf{1}}$.

4.4 Refinements of an inequality of G. H. Hardy

Let us continue by taking a non-negative difference between the right-hand side and the left-hand side of the refined Hardy-type inequality given in (4.1) (see [59]).

$$\begin{aligned}\psi(\Phi) \;=\; & \int_{\Omega_2} v(y)\Phi(f(y))\,d\mu_2(y) - \int_{\Omega_1} u(x)\Phi(A_k f(x))\,d\mu_1(x) \\ & - \int_{\Omega_1} \frac{u(x)}{K(x)} \int_{\Omega_2} k(x,y)\,\big|\,|\Phi(f(y)) - \Phi(A_k f(x))| \\ & \qquad -|\varphi(A_k f(x))| \cdot |f(y) - A_k f(x)|\,\big|\,d\mu_2(y)\,d\mu_1(x).\end{aligned} \tag{4.54}$$

We can also take the non-negative difference of the left-hand side and the right-hand side of the inequality given in (4.2) by taking $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$, $\Phi(x) = x^s, s \geq 1$ as (see [59]):

$$\begin{aligned}\Upsilon(s) \;=\; & \int_{\Omega_2} v(y) f^s(y)\,d\mu_2(y) - \int_{\Omega_1} u(x)(A_k f(x))^s\,d\mu_1(x) \\ & - \left| \int_{\Omega_1} \frac{u(x)}{K(x)} \int_{\Omega_2} sgn(f(y) - A_k f)k(x,y) \left[f^s(y) - (A_k f(x))^s \right.\right. \\ & \left.\left. \qquad -|\varphi(A_k f(x))| \cdot (f(y) - A_k f(x)) \right] d\mu_2(y)\,d\mu_1(x) \right|.\end{aligned} \tag{4.55}$$

We will give some special cases for different fractional integrals and fractional derivatives to establish new inequalities for non-negative differences given in (4.54) and (4.55).

Our first result involving fractional integral of f with respect to another increasing function g is given in the following Theorem.

Theorem 4.7 *Let $s \geq 1$, $\alpha > 0$, $f \geq 0$, g be increasing function on (a,b) such that g' is continuous on (a,b), $I^{\alpha}_{a+;g}f$ denotes the left-sided fractional integral of f with respect to another increasing function g and $\psi_1 : \mathbb{R} \to [0,\infty)$. Then the following inequalities hold:*

$$0 \leq \psi_1(s) \leq H_1(s) - U_1(s) \leq H_1(s) \tag{4.56}$$

and

$$0 \leq \Upsilon_1(s) \leq H_1(s) - F_1(s) \leq H_1(s), \tag{4.57}$$

where

$$\begin{aligned}\psi_1(s) = & \left[\int_a^b g'(y)(g(b)-g(y))^{\alpha} f^s(y)dy \right. \\ & \left. - \int_a^b g'(x)(g(x)-g(a))^{\alpha} \left(\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g}f(x) \right)^s dx \right] - U_1(s)\end{aligned} \tag{4.58}$$

with

$$\begin{aligned}U_1(s) = & \alpha \int_a^b \int_a^x g'(x)g'(y)(g(x)-g(y))^{\alpha-1} \left| \left| f^s(y) - \left(\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g}f(x) \right)^s \right| \right. \\ & \left. -s \left| \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g}f(x) \right|^{s-1} \cdot \left| f(y) - \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g}f(x) \right| \right| dydx,\end{aligned}$$

$$\begin{aligned}\Upsilon_1(s) = & \int_a^b g'(y)(g(b)-g(y))^{\alpha} f^s(y)dy \\ & -(\Gamma(\alpha+1))^s \int_a^b g'(x)(g(x)-g(a))^{\alpha(1-s)} \left(I^{\alpha}_{a+;g}f(x) \right)^s dx - F_1(s)\end{aligned}$$

with

$$
\begin{aligned}
F_1(s) \;=\; \alpha \Bigg| & \int_a^b \int_a^x sgn\left(f(y) - \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x)\right) g'(x) \\
& \times g'(y)(g(x)-g(y))^{\alpha-1} \left[f^s(y) - \left(\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x)\right)^s \right. \\
& \left. - s\left|\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x)\right|^{s-1} \cdot \left(f(y) - \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x)\right)\right] dydx \Bigg|
\end{aligned}
$$

and

$$
\begin{aligned}
H_1(s) \;=\; (g(b)-g(a))^{\alpha(1-s)} & \left[(g(b)-g(a))^{\alpha s} \int_a^b f^s(y) g'(y) dy \right. \\
& \left. - (\Gamma(\alpha+1))^s \int_a^b (I^{\alpha}_{a+;g} f(x))^s g'(x) dx \right].
\end{aligned}
$$

Proof. We will prove only (4.56), since the proof of (4.57) is analogous. Rewriting equation (4.54) with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$ing

$$
k(x,y) = \begin{cases} \frac{g'(y)}{\Gamma(\alpha)(g(x)-g(y))^{1-\alpha}}, & a \le y \le x; \\ 0, & x < y \le b, \end{cases}
$$

we get that $K(x) = \frac{1}{\Gamma(\alpha+1)}(g(x)-g(a))^{\alpha}$ and $A_k f(x) = \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x)$. For particular weight function $u(x) = g'(x)(g(x)-g(a))^{\alpha}$, we obtain $v(y) = g'(y)(g(b)-g(y))^{\alpha}$. If we take $\Phi(x) = x^s, s \ge 1$, $x \in \mathbb{R}_+$, then we obtain (4.58). Since

$$
\begin{aligned}
U_1(s) = \alpha \int_a^b \int_a^x & g'(x) g'(y)(g(x)-g(y))^{\alpha-1} \Bigg| \left| f^s(y) - \left(\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x)\right)^s \right| \\
& - s\left|\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x)\right|^{s-1} \cdot \left| f(y) - \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x)\right| \Bigg| dy dx \ge 0.
\end{aligned}
$$

Then

$$\begin{aligned}
\psi_1(s) &\leq \int_a^b g'(y)(g(b)-g(y))^\alpha f^s(y)dy \\
&\quad - \int_a^b g'(x)(g(x)-g(a))^\alpha \left(\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^\alpha} I^\alpha_{a+;g} f(x)\right)^s dx \\
&\leq (g(b)-g(a))^{\alpha(1-s)} \left[(g(b)-g(a))^{\alpha s} \int_a^b f^s(y) g'(y)dy \right. \\
&\quad \left. - (\Gamma(\alpha+1))^s \int_a^b (I^\alpha_{a+;g} f(x))^s g'(x)dx\right] \\
&= H_1(s)
\end{aligned}$$

□

Here, we give a first special case for the Riemman-Liouville fractional integral. If $g(x) = x$, then $I^\alpha_{a+;x} f(x)$ reduces to the $I^\alpha_{a+} f(x)$ left-sided Riemann-Liouville fractional integral, and the following result follows.

Corollary 4.9 *Let $s \geq 1$, $\alpha > 0$, $f \geq 0$, $I^\alpha_{a+} f$ denotes the left-sided Riemann-Liouville fractional integral of f and $\psi_2 : \mathbb{R} \to [0,\infty)$. Then the following inequalities hold:*

$$0 \leq \psi_2(s) \leq H_2(s) - U_2(s) \leq H_2(s)$$

and

$$0 \leq \Upsilon_2(s) \leq H_2(s) - F_2(s) \leq H_2(s),$$

where

$$\psi_2(s) = \left[\int_a^b (b-y)^\alpha f^s(y)dy - \int_a^b (x-a)^\alpha \left(\frac{\Gamma(\alpha+1)}{(x-a)^\alpha} I^\alpha_{a+} f(x)\right)^s dx\right] - U_2(s)$$

with

$$\begin{aligned}
U_2(s) &= \alpha \int_a^b \int_a^x (x-y)^{\alpha-1} \left| \left| f^s(y) - \left(\frac{\Gamma(\alpha+1)}{(x-a)^\alpha} I^\alpha_{a+} f(x)\right)^s \right| \right. \\
&\quad \left. - s \left|\frac{\Gamma(\alpha+1)}{(x-a)^\alpha} I^\alpha_{a+} f(x)\right|^{s-1} \cdot \left| f(y) - \frac{\Gamma(\alpha+1)}{(x-a)^\alpha} I^\alpha_{a+} f(x)\right| \right| dydx,
\end{aligned}$$

$$\Upsilon_2(s) = \int_a^b (b-y)^\alpha f^s(y)dy - (\Gamma(\alpha+1))^s \int_a^b (x-a)^{\alpha(1-s)} \left(I^\alpha_{a+} f(x)\right)^s dx - F_2(s),$$

with

$$F_2(s)=\alpha\left|\int_a^b\int_a^x sgn\left(f(y)-\frac{\Gamma(\alpha+1)}{(x-a)^\alpha}I^\alpha_{a+}f(x)\right)(x-y)^{\alpha-1}\left[f^s(y)-\left(\frac{\Gamma(\alpha+1)}{(x-a)^\alpha}I^\alpha_{a+}f(x)\right)^s\right.\right.$$
$$\left.\left.-s\left|\frac{\Gamma(\alpha+1)}{(x-a)^\alpha}I^\alpha_{a+}f(x)\right|^{s-1}\cdot\left(f(y)-\frac{\Gamma(\alpha+1)}{(x-a)^\alpha}I^\alpha_{a+}f(x)\right)\right]dydx\right|$$

and

$$H_2(s)=(b-a)^{\alpha(1-s)}\left[(b-a)^{\alpha s}\int_a^b f^s(y)dy-(\Gamma(\alpha+1))^s\int_a^b(I^\alpha_{a+}f(x))^s dx\right].$$

We continue with the result for the Hadamard-type fractional integral. If we take $g(x)=\log x$ in (4.56) the following result is obtained.

Corollary 4.10 *Let $s\geq 1$, $\alpha>0$, $f\geq 0$, $J^\alpha_{a_+}f$ denotes the Hadamard-type fractional integrals of f and $\psi_3:\mathbb{R}\to[0,\infty)$. Then the following inequality holds*

$$0\leq\psi_3(s)\leq H_3(s)-U_3(s)\leq H_3(s)$$

and

$$0\leq\Upsilon_3(s)\leq H_3(s)-F_3(s)\leq H_3(s),$$

where

$$\begin{aligned}\psi_3(s)&=\int_a^b\frac{(\log b-\log y)^\alpha}{y}f^s(y)dy\\&-\int_a^b\frac{(\log x-\log a)^\alpha}{x}\left(\frac{\Gamma(\alpha+1)}{(\log x-\log a)^\alpha}J^\alpha_{a_+}f(x)\right)^s dx-U_3(s)\end{aligned}$$

with

$$\begin{aligned}U_3(s)&=\alpha\int_a^b\int_a^x\frac{(\log x-\log y)^{\alpha-1}}{xy}\left|\left|f^s(y)-\left(\frac{\Gamma(\alpha+1)}{(\log x-\log a)^\alpha}J^\alpha_{a_+}f(x)\right)^s\right|\right.\\&\left.-s\left|\frac{\Gamma(\alpha+1)}{(\log x-\log a)^\alpha}J^\alpha_{a_+}f(x)\right|^{s-1}\cdot\left|f(y)-\frac{\Gamma(\alpha+1)}{(\log x-\log a)^\alpha}J^\alpha_{a_+}f(x)\right|\right|dydx,\end{aligned}$$

$$\Upsilon_3(s)=\int_a^b(\log b-\log y)^\alpha f^s(y)\frac{dy}{y}$$
$$-(\Gamma(\alpha+1))^s\int_a^b(\log x-\log a)^{\alpha(1-s)}\left(J^\alpha_{a_+}f(x)\right)^s\frac{dx}{x}-F_3(s)$$

with

$$
\begin{aligned}
F_3(s) = \alpha \Bigg| \int_a^b \int_a^x & sgn \left(f(y) - \frac{\Gamma(\alpha+1)}{(\log x - \log a)^\alpha} J_{a_+}^\alpha f(x) \right) (\log x - \log y)^{\alpha-1} \\
& \times \left[f^s(y) - \left(\frac{\Gamma(\alpha+1)}{(\log x - \log a)^\alpha} J_{a_+}^\alpha f(x) \right)^s - s \left| \frac{\Gamma(\alpha+1)}{(\log x - \log a)^\alpha} J_{a_+}^\alpha f(x) \right|^{s-1} \right. \\
& \left. \times \left(f(y) - \frac{\Gamma(\alpha+1)}{(\log x - \log a)^\alpha} J_{a_+}^\alpha f(x) \right) \right] \frac{dy}{y} \frac{dx}{x} \Bigg|
\end{aligned}
$$

and

$$
\begin{aligned}
H_3(s) \;=\; (\log b - \log a)^{\alpha(1-s)} & \left[(\log b - \log a)^{\alpha s} \int_a^b f^s(y) \frac{dy}{y} \right. \\
& \left. - (\Gamma(\alpha+1))^s \int_a^b (J_{a_+}^\alpha f(x))^s \frac{dx}{x} \right].
\end{aligned}
$$

Next we give results with respect to the generalized Riemann-Liouville fractional derivative.

Theorem 4.8 *Let $s \geq 1$, and let the assumptions in Lemma 1.3 be satisfied. Let $\psi_4 : \mathbb{R} \to [0,\infty)$. Then for non-negative functions f, $D_a^\beta f$ and $D_a^\alpha f$ the following inequalities hold:*

$$0 \leq \psi_4(s) \leq H_4(s) - U_4(s) \leq H_4(s)$$

$$0 \leq \Upsilon_4(s) \leq H_4(s) - F_4(s) \leq H_4(s),$$

where

$$
\begin{aligned}
\psi_4(s) \;=\; & \left[\int_a^b (b-y)^{\beta-\alpha} (D_a^\beta f(y))^s dy \right. \\
& \left. - \int_a^b (x-a)^{\beta-\alpha} \left(\frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}} D_a^\alpha f(x) \right)^s dx \right] - U_4(s)
\end{aligned}
$$

with

$$
\begin{aligned}
U_4(s) = (\beta-\alpha) \int_a^b \int_a^x (x-y)^{\beta-\alpha-1} & \Bigg| \left| (D_a^\beta f(y))^s - \left(\frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}} D_a^\alpha f(x) \right)^s \right| \\
& - s \left| \frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}} D_a^\alpha f(x) \right|^{s-1} \cdot \left| D_a^\beta f(y) - \frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}} D_a^\alpha f(x) \right| \Bigg| dy dx,
\end{aligned}
$$

$$\begin{aligned}\Upsilon_4(s) &= \int_a^b (b-y)^{\beta-\alpha}(D_a^\beta f(y))^s dy \\ &\quad -(\Gamma(\beta-\alpha+1))^s \int_a^b (x-a)^{(\beta-\alpha)(1-s)}(D_a^\alpha f(x))^s dx - F_4(s),\end{aligned}$$

with

$$\begin{aligned}F_4(s) = (\beta-\alpha)\Bigg|\int_a^b \int_a^x & sgn\left(D_a^\beta f(y) - \frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}} D_a^\alpha f(x)\right)(x-y)^{\beta-\alpha-1} \\ \times & \left[(D_a^\beta f(y))^s - \left(\frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}} D_a^\alpha f(x)\right)^s - s\left|\frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}} D_a^\alpha f(x)\right|^{s-1}\right. \\ \times & \left.\left(D_a^\beta f(y) - \frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}} D_a^\alpha f(x)\right)\right] dydx\Bigg|\end{aligned}$$

and

$$\begin{aligned}H_4(s) &= (b-a)^{(\beta-\alpha)(1-s)}\left[(b-a)^{(\beta-\alpha)s}\int_a^b (D_a^\beta f(y))^s dy\right. \\ &\quad \left. - (\Gamma(\beta-\alpha+1))^s \int_a^b (D_a^\alpha f(x))^s dx\right].\end{aligned}$$

Proof. Similar to the proof of Theorems 3.8 and 4.7. □

In the following Theorem, we will construct new inequality for the Canavati-type fractional derivative.

Theorem 4.9 *Let $s \geq 1$ and let the assumptions in Lemma 1.4 be satisfied. Let $D_a^\gamma f$ denotes the Canavati-type fractional derivative of f and $\psi_5 : \mathbb{R} \to [0,\infty)$. Then for non-negative functions f, $D_a^\nu f$ and $D_a^\gamma f$ the following inequalities hold:*

$$\begin{aligned}0 \leq \psi_5(s) \leq H_5(s) - U_5(s) \leq H_5(s), \\ 0 \leq \Upsilon_5(s) \leq H_5(s) - F_5(s) \leq H_5(s),\end{aligned}$$

where

$$\begin{aligned}\psi_5(s) &= \left[\int_a^b (b-y)^{\nu-\gamma}(D_a^\nu f(y))^s dy\right. \\ &\quad \left. - \int_a^b (x-a)^{\nu-\gamma}\left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_a^\gamma f(x)\right)^s dx\right] - U_5(s),\end{aligned}$$

with

$$U_5(s) = (\nu-\gamma)\int_a^b\int_a^x (x-y)^{\nu-\gamma-1}\left|\left|(D_a^\nu f(y))^s - \left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}D_a^\gamma f(x)\right)^s\right|\right.$$
$$\left. -s\left|\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}D_a^\gamma f(x)\right|^{s-1}\cdot\left|D_a^\nu f(y) - \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}D_a^\gamma f(x)\right|\right| dy\,dx,$$

$$\Upsilon_5(s) = \int_a^b (b-y)^{\nu-\gamma}(D_a^\nu f(y))^s dy$$
$$-(\Gamma(\nu-\gamma+1))^s\int_a^b (x-a)^{(\nu-\gamma)(1-s)}(D_a^\gamma f(x))^s\,dx - F_5(s)$$

with

$$F_5(s) = (\nu-\gamma)\left|\int_a^b\int_a^x sgn\left(D_a^\nu f(y) - \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}D_a^\gamma f(x)\right)(x-y)^{\nu-\gamma-1}\right.$$
$$\times\left[(D_a^\nu f(y))^s - \left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}D_a^\gamma f(x)\right)^s - s\left|\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}D_a^\gamma f(x)\right|^{s-1}\right.$$
$$\left.\left.\times\left(D_a^\nu f(y) - \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}D_a^\gamma f(x)\right)\right] dydx\right|$$

and

$$H_5(s) = (b-a)^{(\nu-\gamma)(1-s)}\left[(b-a)^{(\nu-\gamma)s}\int_a^b (D_a^\nu f(y))^s dy\right.$$
$$\left. - (\Gamma(\nu-\gamma+1))^s\int_a^b (D_a^\gamma f(x))^s dx\right].$$

Proof. Similar to the proof of Theorems 3.9 and 4.7. □

Next we give following results that involve a new inequality for the Caputo fractional derivative.

Theorem 4.10 *Let $s \geq 1$ and let the assumptions in Lemma 1.5 be satisfied. Let $D_{*a}^\gamma f$ denotes the Caputo fractional derivative of f and $\psi_6 : \mathbb{R} \to [0,\infty)$. Then for non-negative functions f, $D_{*a}^\nu f$ and $D_{*a}^\gamma f$ the following inequalities hold:*

$$0 \leq \psi_6(s) \leq H_6(s) - U_6(s) \leq H_6(s),$$

$$0 \leq \Upsilon_6(s) \leq H_6(s) - F_6(s) \leq H_6(s),$$

where

$$\begin{aligned}\psi_6(s) = &\left[\int_a^b (b-y)^{\nu-\gamma}(D_{*a}^{\nu} f(y))^s dy \right.\\ &\left. - \int_a^b (x-a)^{\nu-\gamma}\left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_{*a}^{\gamma} f(x)\right)^s dx\right] - U_6(s)\end{aligned}$$

with

$$\begin{aligned}U_6(s) = &(\nu-\gamma)\int_a^b \int_a^x (x-y)^{\nu-\gamma-1}\left|\left|(D_{*a}^{\nu} f(y))^s - \left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_{*a}^{\gamma} f(x)\right)^s\right|\right.\\ &\left. - s\left|\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_{*a}^{\gamma} f(x)\right|^{s-1} \cdot \left|D_{*a}^{\nu} f(y) - \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_{*a}^{\gamma} f(x)\right|\right| dy\, dx,\end{aligned}$$

$$\begin{aligned}\Upsilon_6(s) = &\int_a^b (b-y)^{\nu-\gamma}(D_{*a}^{\nu} f(y))^s dy\\ &- (\Gamma(\nu-\gamma+1))^s \int_a^b (x-a)^{(\nu-\gamma)(1-s)}\left(D_{*a}^{\gamma} f(x)\right)^s dx - F_6(s)\end{aligned}$$

with

$$\begin{aligned}F_6(s) = &(\nu-\gamma)\left|\int_a^b \int_a^x sgn\left(D_{*a}^{\nu} f(y) - \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_{*a}^{\gamma} f(x)\right)(x-y)^{\nu-\gamma-1}\right.\\ &\times\left[(D_{*a}^{\nu} f(y))^s - \left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_{*a}^{\gamma} f(x)\right)^s\right.\\ &\left.\left. - s\left|\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_{*a}^{\gamma} f(x)\right|^{s-1} \cdot \left(D_{*a}^{\nu} f(y) - \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_{*a}^{\gamma} f(x)\right)\right] dy dx\right|\end{aligned}$$

and

$$\begin{aligned}H_6(s) = (b-a)^{(\nu-\gamma)(1-s)}&\left[(b-a)^{(\nu-\gamma)s}\int_a^b (D_{*a}^{\nu} f(y))^s dy\right.\\ &\left. - (\Gamma(\nu-\gamma+1))^s \int_a^b (D_{*a}^{\gamma} f(x))^s dx\right].\end{aligned}$$

Proof. Similar to the proof of Theorems 3.11 and 4.7. □

Theorem 4.11 *Let $s \geq 1$, $\alpha > 0$, $f \geq 0$, $I^{\alpha}_{a+;\sigma;\eta} f$ denotes the Erdélyi-Kober type fractional integrals of f, ${}_2F_1(a,b;c;z)$ denotes the hypergeometric function and $\psi_7 : \mathbb{R} \to [0,\infty)$. Then the following inequalities hold*

$$0 \leq \psi_7(s) \leq H_7(s) - U_7(s) \leq H_7(s),$$

$$0 \leq \Upsilon_7(s) \leq H_7(s) - F_7(s) \leq H_7(s),$$

where

$$\begin{aligned}\psi_7(s) = &\left[\int_a^b y^{\sigma-1}(b^\sigma - y^\sigma)^\alpha {}_2F_1(y) f^s(y)dy \right.\\ &\left. -\int_a^b x^{\sigma-1}(x^\sigma - a^\sigma)^\alpha {}_2F_1(x)\left(\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha {}_2F_1(x)} I^{\alpha}_{a+;\sigma;\eta} f(x)\right)^s dx\right] - U_7(s)\end{aligned}$$

with

$$\begin{aligned}U_7(s) = &\alpha\sigma \int_a^b\int_a^x \left(\frac{y}{x}\right)^{\sigma\eta} \frac{(xy)^{\sigma-1}}{(x^\sigma - y^\sigma)^{1-\alpha}} \left\| f^s(y) - \left(\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha {}_2F_1(x)} I^{\alpha}_{a+;\sigma;\eta} f(x)\right)^s \right| \\ &-s\left|\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha {}_2F_1(x)} I^{\alpha}_{a+;\sigma;\eta} f(x)\right|^{s-1} \cdot \\ &\left| f(y) - \frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha {}_2F_1(x)} I^{\alpha}_{a+;\sigma;\eta} f(x)\right\| dy dx,\end{aligned}$$

$$\begin{aligned}\Upsilon_7(s) = &\int_a^b y^{\sigma-1}(b^\sigma - y^\sigma)^\alpha {}_2F_1(y) f^s(y)dy \\ &-\int_a^b x^{\alpha\sigma s+\sigma-1}((x^\sigma - a^\sigma)^\alpha {}_2F_1(x))^{1-s}\left(I^{\alpha}_{a+;\sigma;\eta} f(x)\right)^s dx - F_7(s)\end{aligned}$$

with

$$F_7(s)=\alpha\sigma\left|\int_a^b\int_a^x sgn\left(f(y)-\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha {}_2F_1(x)}I^\alpha_{a+;\sigma;\eta}f(x)\right)\frac{x^{\sigma-\sigma\eta-1}y^{\sigma\eta+\sigma-1}}{(x^\sigma-y^\sigma)^{1-\alpha}}\right.$$
$$\times\left[\left(f^s(y)-\left(\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha {}_2F_1(x)}I^\alpha_{a+;\sigma;\eta}f(x)\right)^s\right.\right.$$
$$-s\left|\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha {}_2F_1(x)}I^\alpha_{a+;\sigma;\eta}f(x)\right|^{s-1}$$
$$\left.\left.\left.\times\left(f(y)-\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha {}_2F_1(x)}I^\alpha_{a+;\sigma;\eta}f(x)\right)\right]dydx\right|$$

and

$$H_7(s) = (b^\sigma-a^\sigma)^{\alpha(1-s)}\left[(b^\sigma-a^\sigma)^{\alpha s}b^{\sigma-1}\int_a^b {}_2F_1(y)f^s(y)dy\right.$$
$$\left.-\,a^{\sigma-1+\alpha\sigma s}(\Gamma(\alpha+1))^s\int_a^b(({}_2F_1(x))^{1-s}I^\alpha_{a+;\sigma;\eta}f(x))^s dx\right],$$

$${}_2F_1(x) = {}_2F_1\left(-\eta,\alpha;\alpha+1;1-\left(\frac{a}{x}\right)^\sigma\right) \text{ and } {}_2F_1(y) = {}_2F_1\left(\eta,\alpha;\alpha+1;1-\left(\frac{b}{y}\right)^\sigma\right).$$

Proof. Similar to the proof of Theorems 3.12 and 4.7. □

Remark 4.9 Similar result can be obtained for the right-sided fractional integral of f with respect to another increasing function g, the right-sided Riemann-Liouville fractional integral, the right-sided Hadamard-type fractional integrals and for the right-sided Erdélyi-Kober type fractional integrals but we omit the details here.

Chapter 5

Refinements of Hardy-type inequalities for the case $0 < p \leq q < \infty$

We state and prove a new class of refined general Hardy-type inequalities related to the weighted Lebesgue spaces L^p and L^q, where $0 < p \leq q < \infty$, convex functions and the integral operators A_k.

5.1 A new class of general Hardy-type inequalities with kernels

To begin with, in this section we provide a new class of sufficient conditions on weight functions u and w, and on a kernel k, for a modular inequality involving the Hardy-type operator A_k, defined by (2.15), to hold. The first result in that direction is given in the following theorem (see [22]).

Theorem 5.1 *Let* $0 < p \leq q < \infty$. *Let* $(\Omega_1, \Sigma_1, \mu_1)$ *and* $(\Omega_2, \Sigma_2, \mu_2)$ *be measure spaces with positive* σ*-finite measures, u be a weight function on* Ω_1*, w be a* μ_2*-a.e. positive function on* Ω_2*, k be a non-negative measurable function on* $\Omega_1 \times \Omega_2$*, and K be defined on*

Ω_1 *by* (2.16). *Suppose that* $K(x) > 0$ *for all* $x \in \Omega_1$ *and that the function* $x \mapsto u(x)\left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}}$ *is integrable on* Ω_1 *for each fixed* $y \in \Omega_2$. *Let* Φ *be a non-negative convex function on an interval* $I \subseteq \mathbb{R}$. *If*

$$A = \sup_{y\in\Omega_2} w^{-\frac{1}{p}}(y)\left(\int_{\Omega_1} u(x)\left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}} d\mu_1(x)\right)^{\frac{1}{q}} < \infty,$$

then there exists a positive real constant C, *such that the inequality*

$$\left(\int_{\Omega_1} u(x)\Phi^{\frac{q}{p}}(A_k f(x))\, d\mu_1(x)\right)^{\frac{1}{q}} \le C\left(\int_{\Omega_2} w(y)\Phi(f(y))\, d\mu_2(y)\right)^{\frac{1}{p}} \tag{5.1}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$ *with values in* I *and* $A_k f$ *defined on* Ω_1 *by* (2.15). *Moreover, if* C *is the smallest constant for* (5.1) *to hold, then* $C \le A$.

Proof. By using Jensen's inequality, monotonicity of the power functions $\alpha \mapsto \alpha^t$ for a positive exponent t, and then Minkowski's inequality, we find that

$$\begin{aligned}
&\left(\int_{\Omega_1} u(x)\Phi^{\frac{q}{p}}(A_k f(x))\, d\mu_1(x)\right)^{\frac{1}{q}} \\
&= \left(\int_{\Omega_1} u(x)\left[\Phi\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)f(y)\, d\mu_2(y)\right)\right]^{\frac{q}{p}} d\mu_1(x)\right)^{\frac{1}{q}} \\
&\le \left(\int_{\Omega_1} u(x)\left[\frac{1}{K(x)}\int_{\Omega_2} k(x,y)\Phi(f(y))\, d\mu_2(y)\right]^{\frac{q}{p}} d\mu_1(x)\right)^{\frac{1}{q}} \\
&\le \left(\int_{\Omega_2}\left(w^{-\frac{q}{p}}(y)\int_{\Omega_1} u(x)\left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}} d\mu_1(x)\right)^{\frac{p}{q}} w(y)\Phi(f(y))\, d\mu_2(y)\right)^{\frac{1}{p}} \\
&\le A\left(\int_{\Omega_2} w(y)\Phi(f(y))\, d\mu_2(y)\right)^{\frac{1}{p}}.
\end{aligned}$$

Hence, (5.1) holds with $C = A$, so the proof is complete. □

Following the same lines as in the proof of Theorem 5.1, we get the next corollary.

Corollary 5.1 *Let* $-\infty < q \le p < 0$ *and let the assumptions of Theorem* 5.1 *be satisfied with a positive convex function* Φ. *If*

$$B = \inf_{y\in\Omega_2} w^{-\frac{1}{p}}(y)\left(\int_{\Omega_1} u(x)\left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}} d\mu_1(x)\right)^{\frac{1}{q}} < \infty,$$

then there exists a positive real constant C, such that the inequality

$$\left(\int_{\Omega_1} u(x)\Phi^{\frac{q}{p}}(A_k f(x))\,d\mu_1(x)\right)^{\frac{1}{q}} \geq C\left(\int_{\Omega_2} w(y)\Phi(f(y))\,d\mu_2(y)\right)^{\frac{1}{p}} \tag{5.2}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$ with values in Ω_2. Moreover, if C is the largest constant for (5.2) *to hold, then $C \geq B$.*

Now, we apply Theorem 5.1 to n-dimensional cells in $\mathbb{R}^n_+$ and in this setting, Theorem 5.1 reads as follows.

Corollary 5.2 *Let $0 < p \leq q < \infty$ and $\mathbf{0} < \mathbf{b} \leq \infty$. Let u be a non-negative and v be a positive function on $(\mathbf{0}, \mathbf{b})$ and let Φ be a non-negative convex function on an interval $I \subseteq \mathbb{R}$. If*

$$A = \sup_{\mathbf{y}\in(\mathbf{0},\mathbf{b})} \left(\frac{\mathbf{y}}{v(\mathbf{y})}\right)^{\frac{1}{p}} \left(\int_{(\mathbf{y},\mathbf{b})} u(\mathbf{x})\,\mathbf{x}^{-\frac{q}{p}-1} d\mathbf{x}\right)^{\frac{1}{q}} < \infty,$$

then there exists a positive real constant C, such that the inequality

$$\left(\int_{(\mathbf{0},\mathbf{b})} u(\mathbf{x})\Phi^{\frac{q}{p}}(Hf(\mathbf{x}))\,\frac{d\mathbf{x}}{\mathbf{x}^{\mathbf{1}}}\right)^{\frac{1}{q}} \leq C\left(\int_{(\mathbf{0},\mathbf{b})} v(\mathbf{y})\Phi(f(\mathbf{y}))\,\frac{d\mathbf{y}}{\mathbf{y}^{\mathbf{1}}}\right)^{\frac{1}{p}} \tag{5.3}$$

holds for all measurable functions $f : (\mathbf{0}, \mathbf{b}) \to \mathbb{R}$ with values in I and

$$Hf(\mathbf{x}) = \mathbf{x}^{-\mathbf{1}} \int_{(\mathbf{0},\mathbf{x})} f(\mathbf{y})\,d\mathbf{y},\ \mathbf{x} \in (\mathbf{0}, \mathbf{b}).$$

Moreover, if A is the smallest constant for (5.3) *to hold, then $C \leq A$.*

Proof. Let $S_n = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^n : \mathbf{0} < \mathbf{y} \leq \mathbf{x} < \mathbf{b}\}$ and $\Omega_1 = \Omega_2 = (\mathbf{0}, \mathbf{b})$. The proof follows directly from Theorem 5.1, applied with $d\mu_1(\mathbf{x}) = d\mathbf{x}$, $d\mu_2(\mathbf{y}) = d\mathbf{y}$, $k = \chi_{S_n}$, and with $\frac{u(\mathbf{x})}{\mathbf{x}^{\mathbf{1}}}$ instead of $u(\mathbf{x})$, $\mathbf{x} \in (\mathbf{0}, \mathbf{b})$. Observe that $w(\mathbf{y}) = \mathbf{y}^{-\mathbf{1}}v(\mathbf{y})$, $\mathbf{y} \in (\mathbf{0}, \mathbf{b})$. □

Remark 5.1 The result given in Corollary 5.2 was published in [65, Theorem 3.1], so we see that Theorem 3.1 from [65] is just a special case of our Theorem 5.1. □

5.1.1 Further results involving fractional integrals and derivatives

Our first result deals with the fractional integral of f with respect to an increasing function g (see [55]).

Theorem 5.2 *Let $0 < p \leq q < \infty$, $\alpha > 0$, u be a weight function on (a, b), ω be an a.e. positive function on (a, b), g be an increasing function on (a, b) such that g' is continuous on (a, b), $I^{\alpha}_{a+;g} f$ denotes the left-sided fractional integral of f with respect to another* *increasing function g and let Φ be a non-negative convex function on an interval $I \subseteq \mathbb{R}$. If*

$$A = \sup_{y \in (a,b)} \omega^{\frac{-1}{p}}(y) \left(\int_y^b u(x) \left(\frac{\alpha g'(y)(g(x)-g(y))^{\alpha-1}}{(g(x)-g(a))^{\alpha}} \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C such that the inequality

$$\left(\int_a^b u(x) \left[\Phi\left(\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x) \right) \right]^{\frac{q}{p}} dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b \omega(y) \Phi(f(y)) dy \right)^{\frac{1}{p}} \tag{5.4}$$

holds. Moreover, if C is the smallest constant for (5.4) to hold, then $C \leq A$.

Proof. Similar to the proof of Theorems 3.7 and 5.1. □

Here, we give a first special case for the Riemman-Liouville fractional integral. If $g(x) = x$, then $I^{\alpha}_{a+;x} f(x)$ reduces to the $I^{\alpha}_{a+} f(x)$ left-sided Riemann-Liouville fractional integral, so the following result follows.

Corollary 5.3 *Let $0 < p \leq q < \infty$, $\alpha > 0$, u be a weight function on (a,b), ω be an a.e. positive function on (a,b), $I^{\alpha}_{a+} f$ denotes the left-sided Riemann-Liouville fractional integral of f and let Φ be a non-negative convex function on an interval $I \subseteq \mathbb{R}$. If*

$$A = \sup_{y \in (a,b)} \omega^{\frac{-1}{p}}(y) \left(\int_y^b u(x) \left(\frac{\alpha (x-y)^{\alpha-1}}{(x-a)^{\alpha}} \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C such that the inequality

$$\left(\int_a^b u(x) \left[\Phi\left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} I^{\alpha}_{a+} f(x) \right) \right]^{\frac{q}{p}} dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b \omega(y) \Phi(f(y)) dy \right)^{\frac{1}{p}} \tag{5.5}$$

holds. Moreover, if C is the smallest constant for (5.5) to hold, then $C \leq A$.

Since the Hadamard fractional integrals of order α are special cases of the left- and right-sided fractional integrals of a function f with respect to the function $g(x) = \log(x)$ on (a,b), where $0 \leq a < b \leq \infty$, the following result follows.

Corollary 5.4 *Let $0 < p \leq q < \infty, \alpha > 0$, u be a weight function on (a,b), ω be an a.e. positive function on (a,b), $J^{\alpha}_{a+} f$ denotes the Hadamard-type fractional integrals of f and let Φ be a non-negative convex function on an interval $I \subseteq \mathbb{R}$. If*

$$A = \sup_{y \in (a,b)} \omega^{\frac{-1}{p}}(y) \left(\int_y^b u(x) \left(\frac{\alpha (\log x - \log y)^{\alpha-1}}{y(\log x - \log a)^{\alpha}} \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C such that the inequality

$$\left(\int_a^b u(x)\left[\Phi\left(\frac{\Gamma(\alpha+1)}{(\log x-\log a)^{\alpha}}J^{\alpha}_{a+}f(x)\right)\right]^{\frac{q}{p}}dx\right)^{\frac{1}{q}} \le C\left(\int_a^b \omega(y)\Phi(f(y))dy\right)^{\frac{1}{p}} \tag{5.6}$$

holds. Moreover, if C is the smallest constant for (5.6) *to hold, then* $C \le A$.

Next we give the result with respect to the generalized Riemann-Liouville fractional derivative.

Corollary 5.5 *Let* $0 < p \le q < \infty$, $\beta > \alpha \ge 0$, *u be a weight function on* (a,b), ω *be an a.e. positive function on* (a,b), $D^{\alpha}_{a}f$ *denotes the generalized Riemann-Liouville fractional derivative of f and let the assumption of Lemma* 1.3 *be satisfied and let* Φ *be a non-negative convex function on an interval* $I \subseteq \mathbb{R}$. *If*

$$A = \sup_{y\in(a,b)} \omega^{\frac{-1}{p}}(y)\left(\int_y^b u(x)\left(\frac{(\beta-\alpha)(x-y)^{\beta-\alpha-1}}{(x-a)^{\beta-\alpha}}\right)^{\frac{q}{p}}dx\right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C such that the inequality

$$\left(\int_a^b u(x)\left[\Phi\left(\frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}}D^{\alpha}_{a}f(x)\right)\right]^{\frac{q}{p}}dx\right)^{\frac{1}{q}} \le C\left(\int_a^b \omega(y)\Phi(D^{\beta}_{a}(f(y))dy\right)^{\frac{1}{p}} \tag{5.7}$$

holds. Moreover, if C is the smallest constant for (5.7) *to hold, then* $C \le A$.

Proof. Similar to the proof of Theorems 3.8 and 5.1. □

In the following Corollary, we construct a new inequality for the Canavati-type fractional derivative.

Corollary 5.6 *Let* $0 < p \le q < \infty$, *u be a weight function on* (a,b), ω *be an a.e. positive function on* (a,b), *and let the assumptions in Lemma* 1.4 *be satisfied.* $D^{\gamma}_{a}f$ *denotes the Canavati-type fractional derivative of f and let* Φ *be a non-negative convex function on an interval* $I \subseteq \mathbb{R}$. *If*

$$A = \sup_{y\in(a,b)} \omega^{\frac{-1}{p}}(y)\left(\int_y^b u(x)\left(\frac{(\nu-\gamma)(x-y)^{\nu-\gamma-1}}{(x-a)^{\nu-\gamma}}\right)^{\frac{q}{p}}dx\right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C such that the inequality

$$\left(\int_a^b u(x)\left[\Phi\left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}D^{\gamma}_{a}f(x)\right)\right]^{\frac{q}{p}}dx\right)^{\frac{1}{q}} \le C\left(\int_a^b \omega(y)\Phi(D^{\nu}_{a}(f(y))dy\right)^{\frac{1}{p}} \tag{5.8}$$

holds. Moreover, if C is the smallest constant for (5.8) *to hold, then* $C \le A$.

Proof. Similar to the proof of Theorems 3.9 and 5.1. □

We prove the following result as a special case of Theorem 5.1 to construct a new inequality for the Caputo fractional derivative.

Corollary 5.7 *Let $0 < p \leq q < \infty$, u be a weight function, ω be an a.e. positive function on (a,b), and $D^{\alpha}_{*a}f$ denotes the Caputo fractional derivative of f, $f \in AC^n([a,b])$ and let Φ be a non-negative convex function on an interval $I \subseteq \mathbb{R}$. If*

$$A = \sup_{y\in(a,b)} \omega^{\frac{-1}{p}}(y) \left(\int_y^b u(x) \left(\frac{(n-\alpha)(x-y)^{n-\alpha-1}}{(x-a)^{n-\alpha}} \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C such that the inequality

$$\left(\int_a^b u(x) \left[\Phi\left(\frac{\Gamma(n-\alpha+1)}{(x-a)^{n-\alpha}} D^{\alpha}_{*a}f(x) \right) \right]^{\frac{q}{p}} dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b \omega(y)\Phi(f^{(n)}(y))dy \right)^{\frac{1}{p}} \tag{5.9}$$

holds. Moreover, if C is the smallest constant for (5.9) to hold, then $C \leq A$.

Proof. Similar to the proof of Theorems 3.10 and 5.1. □

Corollary 5.8 *Let $0 < p \leq q < \infty$, u be a weight function, ω be an a.e. positive function on (a,b), and let the assumptions in Lemma 1.5 be satisfied. $D^{\gamma}_{*a}f$ denotes the Caputo fractional derivative of f, $f \in AC^n([a,b])$ and let Φ be a non-negative convex function on an interval $I \subseteq \mathbb{R}$. If*

$$A = \sup_{y\in(a,b)} \omega^{\frac{-1}{p}}(y) \left(\int_y^b u(x) \left(\frac{(\nu-\gamma)(x-y)^{\nu-\gamma-1}}{(x-a)^{\nu-\gamma}} \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C such that the inequality

$$\left(\int_a^b u(x) \left[\Phi\left(\frac{\Gamma(n-\alpha+1)}{(x-a)^{\nu-\gamma}} D^{\gamma}_{*a}f(x) \right) \right]^{\frac{q}{p}} dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b \omega(y)\Phi(D^{\nu}_{*a}(f(y))dy \right)^{\frac{1}{p}} \tag{5.10}$$

holds. Moreover, if C is the smallest constant for (5.10) to hold, then $C \leq A$.

Proof. Similar to the proof of Theorems 3.11 and 5.1. □

Now, we give the following result.

Corollary 5.9 *Let $0<p\le q<\infty$, u be a weight function, ω be an a.e. positive function on (a,b), $I^{\alpha}_{a+;\sigma;\eta}f$ denotes the Erdélyi-Kober type fractional integrals of f, and ${}_2F_1(a,b;c;z)$ denotes the hypergeometric function and let Φ be a non-negative convex function on an interval $I\subseteq\mathbb{R}$. If*

$$A=\sup_{y\in(a,b)}\omega^{\frac{-1}{p}}(y)\left(\int_y^b u(x)\left(\frac{\alpha\,\sigma\,x^{-\sigma\eta}y^{\sigma\eta+\sigma-1}(x^{\sigma}-y^{\sigma})^{\alpha-1}}{(x^{\sigma}-a^{\sigma})^{\alpha}\;{}_2F_1(x)}\right)^{\frac{q}{p}}dx\right)^{\frac{1}{q}}<\infty,$$

$${}_2F_1(x)={}_2F_1\left(-\eta,\alpha;\alpha+1;1-\left(\frac{a}{x}\right)^{\sigma}\right),$$

then there exists a positive constant C such that the inequality

$$\left(\int_a^b u(x)\left[\Phi\left(\left(\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^{\sigma}\right)^{\alpha}\;{}_2F_1(x)}I^{\alpha}_{a+;\sigma;\eta}f(x)\right)^{\frac{q}{p}}dx\right)\right]^{\frac{1}{q}}\right.\le C\left(\int_a^b\omega(y)\Phi(f(y))dy\right)^{\frac{1}{p}} \tag{5.11}$$

holds. Moreover, if C is the smallest constant for (5.11) to hold, then $C\le A$.

Proof. Similar to the proof of Theorems 3.12 and 5.1. □

Remark 5.2 Similar result can be obtained for the right-sided Erdélyi-Kober type fractional integrals, but we omit the details here.

Our analysis continues by providing a new two-parametric class of sufficient conditions for a weighted modular inequality involving the operator A_k to hold. The conditions obtained depend on a real parameter s and a positive function V on Ω_2. That result is given in the following theorem.

Theorem 5.3 *Let $1<p\le q<\infty$. Let $(\Omega_1,\Sigma_1,\mu_1)$ and $(\Omega_2,\Sigma_2,\mu_2)$ be measure spaces with positive σ-finite measures, u be a weight function on Ω_1, v be a measurable μ_2-a.e. positive function on Ω_2, k be a non-negative measurable function on $\Omega_1\times\Omega_2$, and K be defined on Ω_1 by (2.16). Let $K(x)>0$ for all $x\in\Omega_1$ and let the function $x\mapsto u(x)\left(\frac{k(x,y)}{K(x)}\right)^q$ be integrable on Ω_1 for each fixed $y\in\Omega_2$. Suppose that $\Phi:I\to[0,\infty)$ is a bijective convex function on an interval $I\subseteq\mathbb{R}$. If there exist a real parameter $s\in(1,p)$ and a positive measurable function $V:\Omega_2\to\mathbb{R}$ such that*

$$A(s,V)=F(V,v)\sup_{y\in\Omega_2}V^{\frac{s-1}{p}}(y)\left[\int_{\Omega_1}u(x)\left(\frac{k(x,y)}{K(x)}\right)^q d\mu_1(x)\right]^{\frac{1}{q}}<\infty,$$

where

$$F(V,v)=\left(\int_{\Omega_2}V^{\frac{-p'(s-1)}{p}}(y)v^{1-p'}(y)\,d\mu_2(y)\right)^{\frac{1}{p'}},$$

then there is a positive real constant C such that the inequality

$$\left(\int_{\Omega_1} u(x)\Phi^q(A_k f(x))\,d\mu_1(x)\right)^{\frac{1}{q}} \le C\left(\int_{\Omega_2} v(y)\Phi^p(f(y))\,d\mu_2(y)\right)^{\frac{1}{p}} \tag{5.12}$$

holds for all measurable functions $f:\Omega_2 \to \mathbb{R}$ *with values in* I *and* $A_k f$ *defined on* Ω_1 *by* (2.15). *Moreover, if* C *is the best possible constant in* (5.12), *then*

$$C \le \inf_{\substack{1<s<p \\ V>0}} A(s,V). \tag{5.13}$$

Proof. Let $f:\Omega_2 \to \mathbb{R}$ be an arbitrary measurable function with values in I. Applying Jensen's inequality to the left-hand side of (5.12) we get

$$\begin{aligned}&\left(\int_{\Omega_1} u(x)\Phi^q(A_k f(x))\,d\mu_1(x)\right)^{\frac{1}{q}}\\ &\quad\le \left[\int_{\Omega_1} u(x)\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)\Phi(f(y))\,d\mu_2(y)\right)^q d\mu_1(x)\right]^{\frac{1}{q}}.\end{aligned}$$

Hence, to prove inequality (5.12) it suffices to prove that there is a real constant $C > 0$, independent on f, such that

$$\begin{aligned}&\left[\int_{\Omega_1} u(x)\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)\Phi(f(y))\,d\mu_2(y)\right)^q d\mu_1(x)\right]^{\frac{1}{q}}\\ &\qquad\le C\left(\int_{\Omega_2} v(y)\Phi^p(f(y))\,d\mu_2(y)\right)^{\frac{1}{p}}.\end{aligned} \tag{5.14}$$

Taking into account properties of the function Φ, let $g:\Omega_2 \to \mathbb{R}$ be defined by $\Phi(g(y)) = v(y)\Phi^p(f(y))$. Then $g(\Omega_2) \subseteq I$ holds and (5.14) is equivalent to

$$\begin{aligned}&\left[\int_{\Omega_1} u(x)\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)\Phi^{\frac{1}{p}}(g(y))v^{-\frac{1}{p}}(y)\,d\mu_2(y)\right)^q d\mu_1(x)\right]^{\frac{1}{q}}\\ &\qquad\le C\left(\int_{\Omega_2} \Phi(g(y))\,d\mu_2(y)\right)^{\frac{1}{p}}.\end{aligned} \tag{5.15}$$

Therefore, instead of proving (5.14), we prove that (5.15) holds for all measurable functions $g:\Omega_2 \to \mathbb{R}$ with values in I. Applying Hölder's inequality, monotonicity of the power functions $\alpha \mapsto \alpha^t$ for positive exponents t, Minkowski's inequality, and the definitions of $F(V,v)$ and $A(s,V)$, we get the following sequence of inequalities involving an arbitrary positive measurable function $V:\Omega_2\to\mathbb{R}$:

$$
\begin{aligned}
&\left\{\int_{\Omega_1} u(x)\left[\frac{1}{K(x)}\int_{\Omega_2} k(x,y)\Phi^{\frac{1}{p}}(g(y))v^{-\frac{1}{p}}(y)\,d\mu_2(y)\right]^q d\mu_1(x)\right\}^{\frac{1}{q}}\\
&=\left\{\int_{\Omega_1}\frac{u(x)}{K^q(x)}\left[\int_{\Omega_2}\left(k(x,y)\Phi^{\frac{1}{p}}(g(y))V^{\frac{s-1}{p}}(y)\right)\left(V^{\frac{1-s}{p}}(y)v^{-\frac{1}{p}}(y)\right)d\mu_2(y)\right]^q d\mu_1(x)\right\}^{\frac{1}{q}}\\
&\le\left\{\int_{\Omega_1}\frac{u(x)}{K^q(x)}\left(\int_{\Omega_2} k^p(x,y)\Phi(g(y))V^{s-1}(y)\,d\mu_2(y)\right)^{\frac{q}{p}}\times\right.\\
&\qquad\left.\times\left(\int_{\Omega_2} V^{-\frac{p'(s-1)}{p}}(y)v^{1-p'}(y)\,d\mu_2(y)\right)^{\frac{q}{p'}} d\mu_1(x)\right\}^{\frac{1}{q}}\\
&=F(V,v)\left\{\int_{\Omega_1}\frac{u(x)}{K^q(x)}\left(\int_{\Omega_2} k^p(x,y)\Phi(g(y))V^{s-1}(y)\,d\mu_2(y)\right)^{\frac{q}{p}} d\mu_1(x)\right\}^{\frac{1}{q}}\\
&\le F(V,v)\left\{\int_{\Omega_2}\Phi(g(y))V^{s-1}(y)\left[\int_{\Omega_1} u(x)\left(\frac{k(x,y)}{K(x)}\right)^q d\mu_1(x)\right]^{\frac{p}{q}} d\mu_2(y)\right\}^{\frac{1}{p}}\\
&\le A(s,V)\left(\int_{\Omega_2}\Phi(g(y))\,d\mu_2(y)\right)^{\frac{1}{p}}.
\end{aligned}
\tag{5.16}
$$

Thus, inequalities (5.15) and (5.14) hold. Relation (5.12) follows by considering (5.13), so the proof is complete. □

By modifying Theorem 5.3 for the setting from relations (2.8) and (2.9), we obtain the following result.

Theorem 5.4 *Let $1<p\le q<\infty$, $1<s<p$, and $0<b\le\infty$. Let u be a weight function on $(0,b)$, w be an a.e. positive measurable function on $(0,b)$, and k be a non-negative measurable function on $(0,b)\times(0,b)$ satisfying (2.8). Let I be an interval in $\mathbb{R}$ and $\Phi: I\to[0,\infty)$ be a bijective convex function. If*

$$V(y)=\int_0^y w^{1-p'}(x)x^{p'-1}\,dx<\infty \tag{5.17}$$

holds almost everywhere in $(0,b)$ and

$$A(s)=\sup_{0<y<b}\left(\int_y^b u(x)\left(\frac{k(x,y)}{K(x)}\right)^q V^{\frac{q(p-s)}{p}}(x)\,\frac{dx}{x}\right)^{\frac{1}{q}} V^{\frac{s-1}{p}}(y)<\infty, \tag{5.18}$$

then there exists a positive real constant C such that

$$\left(\int_0^b u(x)\Phi^q(A_kf(x))\,\frac{dx}{x}\right)^{\frac{1}{q}}\le C\left(\int_0^b w(x)\Phi^p(f(x))\,\frac{dx}{x}\right)^{\frac{1}{p}} \tag{5.19}$$

holds for all measurable functions $f : (0,b) \to \mathbb{R}$ *with values in* I *and the Hardy-type operator* A_k *defined by* (2.15)*. Moreover, if* C *is the best possible constant in* (5.19)*, then*

$$C \le \inf_{1<s<p} \left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}} A(s).$$

Proof. Denote $S_1 = \{(x,y) \in \mathbb{R}^2 : 0 < y \le x < b\}$ and set $\Omega_1 = \Omega_2 = (0,b)$. In Theorem 5.3, replace $d\mu_1(x)$, $d\mu_2(y)$, $u(x)$, $v(y)$, and k respectively with dx, dy, $\frac{u(x)}{x}$, $\frac{w(y)}{y}$, and $k\chi_{S_1}$. In this setting, inequality (5.12) reduces to (5.19). Moreover, following the lines of the proof of Theorem 5.3, the first inequality in (5.16) becomes

$$\begin{aligned}
&\left\{\int_0^b u(x)\left[\frac{1}{K(x)}\int_0^x k(x,y)\Phi^{\frac{1}{p}}(g(y))\left(\frac{y}{w(y)}\right)^{\frac{1}{p}}(y)\,dy\right]^q \frac{dx}{x}\right\}^{\frac{1}{q}} \\
&\le \left\{\int_0^b \frac{u(x)}{K^q(x)}\left(\int_0^x k^p(x,y)\Phi(g(y))V^{s-1}(y)\,dy\right)^{\frac{q}{p}} \times \right. \\
&\qquad \left. \times \left(\int_0^x V^{-\frac{p'(s-1)}{p}}(y)w^{1-p'}(y)y^{p'-1}\,dy\right)^{\frac{q}{p'}} \frac{dx}{x}\right\}^{\frac{1}{q}}. \qquad (5.20)
\end{aligned}$$

Since definition (5.17) yields

$$\int_0^x V^{-\frac{p'(s-1)}{p}}(y)w^{1-p'}(y)y^{p'-1}\,dy = \frac{p-1}{p-s}V^{\frac{p-s}{p-1}}(x),\; x \in (0,b),$$

the right-hand side of (5.20) is further equal to

$$\left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}}\left\{\int_0^b \frac{u(x)}{K^q(x)}V^{\frac{q(p-s)}{p}}(x)\left(\int_0^x k^p(x,y)\Phi(g(y))V^{s-1}(y)\,dy\right)^{\frac{q}{p}}\frac{dx}{x}\right\}^{\frac{1}{q}}.$$

As in (5.16), the rest of the proof follows by applying Minkowski's inequality and definition (5.18) of $A(s)$. □

Remark 5.3 The result of Theorem 5.4 is given in [65, Theorem 4.4]. Hence, Theorem 4.4 in [65] can be seen as a special case of Theorem 5.3. □

We also give results involving fractional integrals and fractional derivatives.

Our first result deals with the fractional integral of f with respect to an increasing function g.

Theorem 5.5 *Let $1 < p \le q < \infty$, $\alpha > 0$, u be a weight function on (a,b), v be an a.e. positive function on (a,b), g be an increasing function on (a,b) such that g' is continuous on (a,b), $I^{\alpha}_{a+;g}f$ denotes the left-sided fractional integral of f with respect to another increasing function g. Let I be an interval in $\mathbb{R}$ and $\Phi : I \to [0,\infty)$ be a bijective convex function. If there exist a real parameter $s \in (1,p)$ and a positive measurable function $V : (a,b) \to \mathbb{R}$ such that*

$$A(s,V) = \left(\int_a^b V^{\frac{-p'(s-1)}{p}}(y) v^{1-p'}(y) dy \right)^{\frac{1}{p'}}$$

$$\times \sup_{y\in(a,b)} V^{\frac{s-1}{p}}(y) \left(\int_y^b u(x) \left(\frac{\alpha g'(y)(g(x)-g(y))^{\alpha-1}}{(g(x)-g(a))^{\alpha}} \right)^q dx \right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C such that the inequality

$$\left(\int_a^b u(x) \left[\Phi \left(\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x) \right) \right]^q dx \right)^{\frac{1}{q}} \le C \left(\int_a^b v(y) \Phi^p(f(y)) dy \right)^{\frac{1}{p}} \tag{5.21}$$

holds. Moreover, if C is the smallest constant for (5.21) to hold, then

$$C \le \inf_{\substack{1<s<p \\ V>0}} A(s,V).$$

Proof. Similar to the proof of Theorems 3.7 and 5.3. □

As in previous examples we can give special cases of Theorem 5.5 for the Riemman-Liouville fractional integrals and the Hadamard-type fractional integral, but we omit the details here.

Next we give the result with respect to the generalized Riemann-Liouville fractional derivative.

Corollary 5.10 *Let $1 < p \le q < \infty$, $\beta > \alpha \ge 0$, u be a weight function on (a,b), v be an a.e. positive function on (a,b), $D^{\alpha}_a f$ denotes the generalized Riemann-Liouville fractional derivative of f, let the assumptions of Lemma 1.3 be satisfied. Let I be an interval in $\mathbb{R}$ and $\Phi : I \to [0,\infty)$ be a bijective convex function. If there exist a real parameter $s \in (1,p)$ and a positive measurable function $V : (a,b) \to \mathbb{R}$ such that*

$$A(s,V) = \left(\int_a^b V^{\frac{-p'(s-1)}{p}}(y) v^{1-p'}(y) dy \right)^{\frac{1}{p'}}$$

$$\times \sup_{y\in(a,b)} V^{\frac{s-1}{p}}(y) \left(\int_y^b u(x) \left(\frac{(\beta-\alpha)(x-y)^{\beta-\alpha-1}}{(x-a)^{\beta-\alpha}} \right)^q dx \right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C such that the inequality

$$\left(\int_a^b u(x) \left[\Phi\left(\frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}} D_a^{\alpha} f(x) \right) \right]^q dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b v(y) \Phi^p \left(D_a^{\beta} f(y) \right) dy \right)^{\frac{1}{p}} \tag{5.22}$$

holds. Moreover, if C is the smallest constant for (5.22) *to hold, then*

$$C \leq \inf_{\substack{1<s<p \\ V>0}} A(s,V).$$

Proof. Similar to the proof of Theorems 3.8 and 5.3. □

In the following Corollary we obtain a new inequality for the Canavati-type fractional derivative.

Corollary 5.11 *Let $1 < p \leq q < \infty$, u be a weight function on (a,b), v be an a.e. positive function on (a,b), and let the assumptions in Lemma 1.4 be satisfied. Let I be an interval in $\mathbb{R}$ and $\Phi : I \to [0,\infty)$ be a bijective convex function. If there exist a real parameter $s \in (1,p)$ and a positive measurable $V : (a,b) \to \mathbb{R}$ function such that*

$$\begin{aligned} A(s,V) = & \left(\int_a^b V^{\frac{-p'(s-1)}{p}}(y) v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \\ & \times \sup_{y \in (a,b)} V^{\frac{s-1}{p}}(y) \left(\int_y^b u(x) \left(\frac{(\nu-\gamma)(x-y)^{\nu-\gamma-1}}{(x-a)^{\nu-\gamma}} \right)^q dx \right)^{\frac{1}{q}} < \infty, \end{aligned}$$

then there exists a positive constant C, such that the inequality

$$\left(\int_a^b u(x) \left[\Phi\left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_a^{\gamma} f(x) \right) \right]^q dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b v(y) \Phi^p \left(D_a^{\nu} f(y) \right) dy \right)^{\frac{1}{p}} \tag{5.23}$$

holds. Moreover, if C is the smallest constant for (5.23) *to hold, then*

$$C \leq \inf_{\substack{1<s<p \\ V>0}} A(s,V).$$

Proof. Similar to the proof of Theorems 3.9 and 5.3. □

Next, we give the result for the Caputo fractional derivative.

Corollary 5.12 *Let $1 < p \leq q < \infty$, u be a weight function on (a,b), v be an a.e. positive function on (a,b), and $D_{*a}^{\nu} f$ denotes the Caputo fractional derivative of f, $f \in AC^n([a,b])$.*

Let I be an interval in $\mathbb{R}$ and $\Phi : I \to [0,\infty)$ be a bijective convex function. If there exist a real parameter $s \in (1,p)$ and a positive measurable function $V : (a,b) \to \mathbb{R}$ such that

$$A(s,V) = \left(\int_a^b V^{\frac{-p'(s-1)}{p}}(y) v^{1-p'}(y) dy \right)^{\frac{1}{p'}}$$

$$\times \sup_{y \in (a,b)} V^{\frac{s-1}{p}}(y) \left(\int_y^b u(x) \left(\frac{(n-\nu)(x-y)^{n-\nu-1}}{(x-a)^{n-\nu}} \right)^q dx \right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C such that the inequality

$$\left(\int_a^b u(x) \left[\Phi\left(\frac{\Gamma(n-\nu+1)}{(x-a)^{n-\nu}} D^{\nu}_{*a} f(x) \right) \right]^q dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b v(y) \Phi^p \left(f^{(n)}(y) \right) dy \right)^{\frac{1}{p}} \tag{5.24}$$

holds. Moreover, if C is the smallest constant for (5.24) to hold, then

$$C \leq \inf_{\substack{1<s<p \\ V>0}} A(s,V).$$

Proof. Similar to the proof of Theorems 3.10 and 5.3. □

Corollary 5.13 *Let $1 < p \leq q < \infty$, u be a weight function on (a,b), v be an a.e. positive function on (a,b), and let the assumptions in Lemma 1.5 be satisfied. Let $D^{\gamma}_{*a} f$ denotes the Caputo fractional derivative of f, $f \in AC^n([a,b])$. Let I be an interval in $\mathbb{R}$ and $\Phi : I \to [0,\infty)$ be a bijective convex function. If there exist a real parameter $s \in (1,p)$ and a positive measurable function $V : (a,b) \to \mathbb{R}$ such that*

$$A(s,V) = \left(\int_a^b V^{\frac{-p'(s-1)}{p}}(y) v^{1-p'}(y) dy \right)^{\frac{1}{p'}}$$

$$\times \sup_{y \in (a,b)} V^{\frac{s-1}{p}}(y) \left(\int_y^b u(x) \left(\frac{(\nu-\gamma)(x-y)^{\nu-\gamma-1}}{(x-a)^{\nu-\gamma}} \right)^q dx \right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C such that the inequality

$$\left(\int_a^b u(x) \left[\Phi\left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D^{\gamma}_{*a} f(x) \right) \right]^q dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b v(y) \Phi^p \left(D^{\nu}_{*a} f(y) \right) dy \right)^{\frac{1}{p}} \tag{5.25}$$

holds. Moreover, if C is the smallest constant for (5.25) to hold, then

$$C \leq \inf_{\substack{1<s<p \\ V>0}} A(s,V).$$

Proof. Similar to the proof of Theorems 3.11 and 5.3. □

Now, we give the following result for the Erdélyi-Kober type fractional integrals.

Corollary 5.14 *Let $1 < p \le q < \infty, \alpha > 0$, u be a weight function on (a,b), v be an a.e. positive function on (a,b), $I^{\alpha}_{a+;\sigma;\eta} f$ denotes the Erdélyi-Kober type fractional integrals of f, and ${}_2F_1(a,b;c;z)$ denotes the hypergeometric function. Let I be an interval in $\mathbb{R}$ and $\Phi : I \to [0,\infty)$ be a bijective convex function. If there exist a real parameter $s \in (1,p)$ and a positive measurable function $V : (a,b) \to \mathbb{R}$ such that*

$$A(s,V) = \left(\int_a^b V^{\frac{-p'(s-1)}{p}}(y) v^{1-p'}(y) dy \right)^{\frac{1}{p'}}$$

$$\times \sup_{y\in(a,b)} V^{\frac{s-1}{p}}(y) \left(\int_y^b u(x) \left(\frac{\alpha\sigma x^{-\sigma\eta} y^{\sigma\eta+\sigma-1} (x^{\sigma}-y^{\sigma})^{\alpha-1}}{(x^{\sigma}-a^{\sigma})^{\alpha}\, {}_2F_1(x)} \right)^q dx \right)^{\frac{1}{q}} < \infty,$$

where ${}_2F_1(x) = {}_2F_1\left(-\eta, \alpha; \alpha+1; 1-\left(\frac{a}{x}\right)^{\sigma}\right)$, then there exists a positive constant C such that the inequality

$$\left(\int_a^b u(x) \left[\Phi\left(\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^{\sigma}\right)^{\alpha} {}_2F_1(x)} I^{\alpha}_{a+;\sigma;\eta} f(x) \right) \right]^q dx \right)^{\frac{1}{q}}$$

$$\le C \left(\int_a^b v(y) \Phi^p(f(y))\, dy \right)^{\frac{1}{p}} \tag{5.26}$$

holds. Moreover, if C is the smallest constant for (5.26) to hold, then

$$C \le \inf_{\substack{1<s<p \\ V>0}} A(s,V).$$

Proof. Similar to the proof of Theorem 3.12 and 5.3. □

Remark 5.4 Similar result can be obtained for the right-sided fractional integral of f with respect to an increasing function g, the right-sided Riemann-Liouville fractional integral, the right-sided Hadamard-type fractional integrals, the right-sided Erdélyi-Kober type fractional integrals, but we omit the details here.

5.2 Refined Hardy-type inequalities with kernels

The rest of this chapter is dedicated to new refined inequalities related to the general Hardy-type operator A_k with a non-negative kernel, defined by (2.15). We state and prove the central result of this section, that is, a new general refined weighted Hardy-type inequality with a non-negative kernel, related to an arbitrary non-negative convex function. It is given in the following theorem.

Theorem 5.6 *Let $t \in \mathbb{R}_+$, $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive σ-finite measures, u be a weight function on Ω_1, k a non-negative measurable function on $\Omega_1 \times \Omega_2$, and K be defined on Ω_1 by* (2.16). *Suppose that $K(x) > 0$ for all $x \in \Omega_1$, that the function $x \mapsto u(x)\left(\frac{k(x,y)}{K(x)}\right)^t$ is integrable on Ω_1 for each fixed $y \in \Omega_2$, and that v is defined on Ω_2 by*

$$v(y) = \left(\int_{\Omega_1} u(x)\left(\frac{k(x,y)}{K(x)}\right)^t d\mu_1(x)\right)^{\frac{1}{t}}.$$

If Φ is a non-negative convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$, then the inequality

$$\begin{aligned}\left(\int_{\Omega_2} v(y)\Phi(f(y))\,d\mu_2(y)\right)^t &- \int_{\Omega_1} u(x)\Phi^t(A_k f(x))\,d\mu_1(x)\\ &\geq t\int_{\Omega_1}\frac{u(x)}{K(x)}\Phi^{t-1}(A_k f(x))\int_{\Omega_2} k(x,y)r(x,y)\,d\mu_2(y)\,d\mu_1(x)\end{aligned} \tag{5.27}$$

holds for all $t \geq 1$ and all measurable functions $f : \Omega_2 \to \mathbb{R}$ with values in I, where $A_k f$ is defined on Ω_1 by (2.15) *and the function $r : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is defined by*

$$r(x,y) = \big|\,|\Phi(f(y)) - \Phi(A_k f(x))| - |\varphi(A_k f(x))| \cdot |f(y) - A_k f(x)|\,\big|. \tag{5.28}$$

If $t \in (0,1]$ and the function $\Phi : I \to \mathbb{R}$ is positive and concave, then the order of the terms on the left-hand side of (5.27) *is reversed, that is, the inequality*

$$\begin{aligned}\int_{\Omega_1} u(x)\Phi^t(A_k f(x))\,d\mu_1(x) &- \left(\int_{\Omega_2} v(y)\Phi(f(y))\,d\mu_2(y)\right)^t\\ &\geq t\int_{\Omega_1}\frac{u(x)}{K(x)}\Phi^{t-1}(A_k f(x))\int_{\Omega_2} k(x,y)r(x,y)\,d\mu_2(y)\,d\mu_1(x)\end{aligned} \tag{5.29}$$

holds.
Let the function $r_1 : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be defined by

$$r_1(x,y) = \Big[\Phi(f(y)) - \Phi(A_k f(x)) - |\varphi(A_k f(x))| \cdot (f(y) - A_k f(x))\Big]. \tag{5.30}$$

If Φ is non-negative monotone convex on the interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \mathrm{Int}\, I$, then the inequality

$$\begin{aligned}
&\left(\int_{\Omega_2} v(y)\Phi\left(f(y)\right) d\mu_2(y)\right)^t - \int_{\Omega_1} u(x)\Phi^t\left(A_k f(x)\right) d\mu_1(x) \\
&\ge t\left|\int_{\Omega_1} \frac{u(x)}{K(x)}\Phi^{t-1}(A_k f(x))\int_{\Omega_2} sgn(f(y)-A_k f(x))k(x,y)r_1(x,y)d\mu_2(y)d\mu_1(x)\right|
\end{aligned} \tag{5.31}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$ such that $f(y) \in I$, for all $y \in \Omega_2$ where $A_k f$ is defined by (2.15).
If Φ is non-negative monotone concave, then the order of the terms on the left-hand side of (5.31) *is reversed.*

Proof. First, fix an arbitrary $x \in \Omega_1$. It is not hard to see that $A_k f(x) \in I$. Moreover, for the function $h_x : \Omega_2 \to \mathbb{R}$ defined by $h_x(y) = f(y) - A_k f(x)$ we have

$$\int_{\Omega_2} k(x,y)h_x(y)\, d\mu_2(y) = 0,\ x \in \Omega_1. \tag{5.32}$$

Now, suppose that Φ is a convex function. If $A_k f(x) \in \mathrm{Int} I$, then for all $y \in \Omega_2$ by substituting $r = A_k f(x)$, $s = f(y)$ in (1.8) and multiplying the inequality obtained by $\frac{k(x,y)}{K(x)} \ge 0$, we get

$$\frac{k(x,y)}{K(x)}\left[\Phi(f(y)) - \Phi(A_k f(x)) - \varphi(A_k f(x))h_x(y)\right] \ge \frac{k(x,y)}{K(x)}\, r(x,y). \tag{5.33}$$

Relation (5.33) holds even if $A_k f(x)$ is an endpoint of I. In that case, the function h_x is either non-negative or non-positive on Ω_2, so (5.32) and non-negativity of the kernel k imply that $k(x,y)h_x(y) = 0$ for μ_2-a.e. $y \in \Omega_2$. Therefore, the identity $h_x(y) = 0$, that is, $f(y) = A_k f(x)$ holds whenever $k(x,y) > 0$ and we conclude that the both sides of inequality (5.33) are equal to 0 for μ_2-a.e. $y \in \Omega_2$. Since $K(x) > 0$, notice that the set of all $y \in \Omega_2$ such that $k(x,y) > 0$ is of a positive μ_2 measure.

Integrating (5.33) over Ω_2 we obtain

$$\begin{aligned}
\frac{1}{K(x)}\int_{\Omega_2} k(x,y)\Phi(f(y))\, d\mu_2(y) &- \frac{1}{K(x)}\int_{\Omega_2} k(x,y)\Phi(A_k f(x))\, d\mu_2(y) \\
&- \frac{1}{K(x)}\int_{\Omega_2} k(x,y)\varphi(A_k f(x))h_x(y)\, d\mu_2(y) \\
&\ge \frac{1}{K(x)}\int_{\Omega_2} k(x,y)r(x,y)\, d\mu_2(y).
\end{aligned} \tag{5.34}$$

Observe that the second integral on the left-hand side of (5.34) is equal to

$$\frac{1}{K(x)}\int_{\Omega_2} k(x,y)\Phi(A_k f(x))\, d\mu_2(y) = \Phi(A_k f(x)),$$

while applying (5.32) we get

$$\frac{1}{K(x)} \int_{\Omega_2} k(x,y)\varphi(A_k f(x))h_x(y)\, d\mu_2(y) = 0.$$

Hence, (5.34) reduces to

$$\Phi(A_k f(x)) + \frac{1}{K(x)} \int_{\Omega_2} k(x,y)r(x,y)\, d\mu_2(y) \le \frac{1}{K(x)} \int_{\Omega_2} k(x,y)\Phi(f(y))\, d\mu_2(y).$$

Let $t \ge 1$. Since the functions Φ, k, and r are non-negative and the power functions with positive exponents are strictly increasing on $[0,\infty)$, we further have

$$\begin{aligned}
&\Phi^t(A_k f(x)) + t\,\frac{\Phi^{t-1}(A_k f(x))}{K(x)} \int_{\Omega_2} k(x,y)r(x,y)\, d\mu_2(y) \\
&\le \left(\Phi(A_k f(x)) + \frac{1}{K(x)} \int_{\Omega_2} k(x,y)r(x,y)\, d\mu_2(y)\right)^t \\
&\le \left(\frac{1}{K(x)} \int_{\Omega_2} k(x,y)\Phi(f(y))\, d\mu_2(y)\right)^t, \qquad (5.35)
\end{aligned}$$

where the first inequality in (5.35) is a consequence of Bernoulli's inequality. Multiplying (5.35) by $u(x)$, integrating the inequalities obtained over Ω_1 and then applying Minkowski's inequality to the right-hand side of the second inequality, we get the following sequence of inequalities:

$$\begin{aligned}
&\int_{\Omega_1} u(x)\Phi^t(A_k f(x))\, d\mu_1(x) \\
&\quad + t\int_{\Omega_1} \frac{u(x)}{K(x)}\Phi^{t-1}(A_k f(x)) \int_{\Omega_2} k(x,y)r(x,y)\, d\mu_2(y)\, d\mu_1(x) \\
&\le \int_{\Omega_1} u(x)\left(\Phi(A_k f(x)) + \frac{1}{K(x)} \int_{\Omega_2} k(x,y)r(x,y)\, d\mu_2(y)\right)^t d\mu_1(x) \\
&\le \int_{\Omega_1} u(x)\left(\frac{1}{K(x)} \int_{\Omega_2} k(x,y)\Phi(f(y))\, d\mu_2(y)\right)^t d\mu_1(x) \\
&= \left\{\left[\int_{\Omega_1} u(x)\left(\frac{1}{K(x)} \int_{\Omega_2} k(x,y)\Phi(f(y))\, d\mu_2(y)\right)^t d\mu_1(x)\right]^{\frac{1}{t}}\right\}^t \\
&\le \left\{\int_{\Omega_2} \Phi(f(y))\left[\int_{\Omega_1} u(x)\left(\frac{k(x,y)}{K(x)}\right)^t d\mu_1(x)\right]^{\frac{1}{t}} d\mu_2(y)\right\}^t \\
&= \left(\int_{\Omega_2} \Phi(f(y))v(y)\, d\mu_2(y)\right)^t,
\end{aligned}$$

so (5.27) holds. The proof for a concave function Φ and $t \in (0,1]$ is similar. Namely, by the same arguments as for convex functions, from (1.9) we first obtain

$$\frac{k(x,y)}{K(x)}\left[\Phi(A_k f(x)) - \Phi(f(y)) + \varphi(A_k f(x))h_x(y)\right] \ge \frac{k(x,y)}{K(x)}\, r(x,y),$$

$x \in \Omega_1, y \in \Omega_2$, then

$$\begin{aligned}
&\Phi^t(A_k f(x)) - t\,\frac{\Phi^{t-1}(A_k f(x))}{K(x)} \int_{\Omega_2} k(x,y) r(x,y)\, d\mu_2(y) \\
&\geq \left(\Phi(A_k f(x)) - \frac{1}{K(x)} \int_{\Omega_2} k(x,y) r(x,y)\, d\mu_2(y)\right)^t \\
&\geq \left(\frac{1}{K(x)} \int_{\Omega_2} k(x,y) \Phi(f(y))\, d\mu_2(y)\right)^t,
\end{aligned}$$

and finally

$$\begin{aligned}
&\int_{\Omega_1} u(x) \Phi^t(A_k f(x))\, d\mu_1(x) \\
&\quad - t \int_{\Omega_1} \frac{u(x)}{K(x)} \Phi^{t-1}(A_k f(x)) \int_{\Omega_2} k(x,y) r(x,y)\, d\mu_2(y)\, d\mu_1(x) \\
&\quad \geq \int_{\Omega_1} u(x) \left(\Phi(A_k f(x)) - \frac{1}{K(x)} \int_{\Omega_2} k(x,y) r(x,y)\, d\mu_2(y)\right)^t d\mu_1(x) \\
&\quad \geq \left(\int_{\Omega_2} \Phi(f(y)) v(y)\, d\mu_2(y)\right)^t
\end{aligned}$$

that is, we get (5.29). The proof of (5.31) is analogous to the proof of (4.2). □

Remark 5.5 In particular, for $t = 1$ inequality (5.27) reduces to

$$\begin{aligned}
&\int_{\Omega_2} v(y) \Phi(f(y))\, d\mu_2(y) - \int_{\Omega_1} u(x) \Phi(A_k f(x))\, d\mu_1(x) \\
&\quad \geq \int_{\Omega_1} \frac{u(x)}{K(x)} \int_{\Omega_2} k(x,y) r(x,y)\, d\mu_2(y)\, d\mu_1(x)
\end{aligned} \tag{5.36}$$

where in this setting v is defined as in (2.17). Moreover, by analyzing the proof of Theorem 5.6, we see that (5.36) holds for all convex functions $\Phi : I \to \mathbb{R}$, that is, Φ does not need to be non-negative. Similarly, if Φ is any real concave function on I (not necessarily positive), then (5.36) holds with the reversed order of the terms on its left-hand side. This result was already proved in Theorem 4.1. □

Remark 5.6 Rewriting (5.27) with $t = \frac{q}{p} \geq 1$, that is, with $0 < p \leq q < \infty$ or $-\infty < q \leq p < 0$, and with an arbitrary non-negative convex function Φ, we obtain

$$\begin{aligned}
&\left(\int_{\Omega_2} v(y) \Phi(f(y))\, d\mu_2(y)\right)^{\frac{q}{p}} - \int_{\Omega_1} u(x) \Phi^{\frac{q}{p}}(A_k f(x))\, d\mu_1(x) \\
&\quad \geq \frac{q}{p} \int_{\Omega_1} \frac{u(x)}{K(x)} \Phi^{\frac{q}{p}-1}(A_k f(x)) \int_{\Omega_2} k(x,y) r(x,y)\, d\mu_2(y)\, d\mu_1(x) \geq 0,
\end{aligned} \tag{5.37}$$

where v is defined by (2.20). Also (5.31) becomes

$$\left(\int_{\Omega_2} v(y)\Phi(f(y))\,d\mu_2(y)\right)^{\frac{q}{p}} - \int_{\Omega_1} u(x)\Phi^{\frac{q}{p}}(A_k f(x))\,d\mu_1(x)$$
$$\geq \frac{q}{p}\left|\int_{\Omega_1}\frac{u(x)}{K(x)}\Phi^{\frac{q}{p}-1}(A_k f(x))\int_{\Omega_2} sgn(f(y)-A_k f(x))k(x,y)r_1(x,y)d\mu_2(y)d\mu_1(x)\right| \tag{5.38}$$

Therefore, we get (2.21) as an immediate consequence of Theorem 5.6 and our inequality (5.27) is a refinement of (2.21). Especially, if $p \geq 1$ or $p < 0$ (in that case, Φ should be positive), then the function Φ^p is convex as well, so by replacing Φ with Φ^p relation (5.37) becomes

$$\|\Phi f\|^q_{L^p_v(\Omega_2,\mu_2)} - \|\Phi(A_k f)\|^q_{L^q_u(\Omega_1,\mu_1)}$$
$$\geq \frac{q}{p}\int_{\Omega_1}\frac{u(x)}{K(x)}\Phi^{q-p}(A_k f(x))\int_{\Omega_2} k(x,y)r_p(x,y)\,d\mu_2(y)\,d\mu_1(x), \tag{5.39}$$

where for $x \in \Omega_1$, $y \in \Omega_2$ we set

$$r_p(x,y) = \big|\,|\Phi^p(f(y)) - \Phi^p(A_k f(x))|$$
$$-|p|\,\Phi^{p-1}(A_k f(x))\,|\varphi(A_k f(x))|\cdot|f(y)-A_k f(x)|\,\big|.$$

On the other hand, if Φ is a positive concave function and $t = \frac{q}{p} \in (0,1]$, that is, $0 < q \leq p < \infty$ or $-\infty < p \leq q < 0$, then (5.37) holds with the reversed order of the terms on its left-hand side. Moreover, if $p \in (0,1]$, then the function Φ^p is concave, so the order of the terms on the left-hand side of (5.39) is reversed. □

Now, we consider some particularly interesting convex (or concave) functions in (5.27), namely, power and exponential functions. We start with the function $\Phi : \mathbb{R}_+ \to \mathbb{R}$, $\Phi(x) = x^p$, where $p \in \mathbb{R}$, $p \neq 0$. For $p \geq 1$ and $p < 0$, this function is convex, while it is concave for $p \in (0,1]$. In both cases we have $\varphi(x) = px^{p-1}$, $x \in \mathbb{R}_+$. In this setting, we obtain the following direct consequence of Theorem 5.6 and Remark 5.6.

Corollary 5.15 *Suppose that $p,q \in \mathbb{R}$, $\frac{q}{p} > 0$, that Ω_1, Ω_2, μ_1, μ_2, u, k, and K are as in Theorem 5.6, that the function $x \mapsto u(x)\left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$, and that the function v is defined on Ω_2 by (2.20). Further, suppose that $f : \Omega_2 \to \mathbb{R}$ is a non-negative measurable function (positive in the case when $p < 0$), that $A_k f$ is defined on Ω_1 by (2.15), $R_{p,k}f(x,y)$ is defined by (4.11) and $M_{p,k}f(x,y)$ is defined by (4.12). If $1 \leq p \leq q < \infty$ or $-\infty < q \leq p < 0$, then the inequalities*

$$\|f\|^q_{L^p_v(\Omega_2,\mu_2)} - \|A_k f\|^q_{L^q_u(\Omega_1,\mu_1)}$$
$$\geq \frac{q}{p}\int_{\Omega_1}\frac{u(x)}{K(x)}(A_k f(x))^{q-p}\int_{\Omega_2} k(x,y)R_{p,k}f(x,y)d\mu_2(y)d\mu_1(x) \tag{5.40}$$

and

$$\|f\|^q_{L^p_v(\Omega_2,\mu_2)} - \|A_k f\|^q_{L^q_u(\Omega_1,\mu_1)} \ge$$
$$\frac{q}{p}\left|\int_{\Omega_1}\frac{u(x)}{K(x)}(A_k f(x))^{q-p}\int_{\Omega_2} sgn(f(y)-A_k f(x))k(x,y)M_{p,k}f(x,y)d\mu_2(y)d\mu_1(x)\right| \tag{5.41}$$

hold, while for $0 < q \le p < 1$ *relations* (5.40) *and* (5.41) *hold with the reversed order of terms on its left-hand side.*

Remark 5.7 For $p = q$ in Corollary 5.15, we obtain Corollary 4.1. Moreover, for $p = q = 1$, relations (5.40) and (5.41) are trivial since its both sides are equal to 0. □

Our analysis continues by considering the convex function $\Phi : \mathbb{R} \to \mathbb{R}$, $\Phi(x) = e^x$. Then $\varphi = \Phi' = \Phi$ and we obtain the following new general refined weighted Pólya-Knopp-type inequality with a kernel, which is a generalization of a result from Corollary 4.2.

Corollary 5.16 *Let* $p,q \in \mathbb{R}$ *be such that* $0 < p \le q < \infty$ *or* $-\infty < q \le p < 0$. *Let* Ω_1, Ω_2, μ_1, μ_2, u, k, *and* K *be as in Theorem* 5.6, *the function* $x \mapsto u(x)\left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}}$ *be integrable on* Ω_1 *for each fixed* $y \in \Omega_2$, *and the function* v *be defined on* Ω_1 *by* (2.20). *Then the inequalities*

$$\|f\|^q_{L^p_v(\Omega_2,\mu_2)} - \|G_k f\|^q_{L^q_u(\Omega_1,\mu_1)}$$
$$\ge \frac{q}{p}\int_{\Omega_1}\frac{u(x)}{K(x)}(G_k f(x))^{q-p}\int_{\Omega_2} k(x,y)S_{p,k}f(x,y)\,d\mu_2(y)\,d\mu_1(x)$$

and

$$\|f\|^q_{L^p_v(\Omega_2,\mu_2)} - \|G_k f\|^q_{L^q_u(\Omega_1,\mu_1)}$$
$$\ge \frac{q}{p}\left|\int_{\Omega_1}\frac{u(x)}{K(x)}G_k^{q-p}f(x)\int_{\Omega_2} sgn(f(y)-G_k f(x))k(x,y)P_{p,k}f(x,y)d\mu_2(y)\,d\mu_1(x)\right|.$$

hold for all positive measurable functions f *on* Ω_2, *where* $G_k f(x)$, $S_{p,k}f(x,y)$ *and* $P_{p,k}f(x,y)$ *for* $x \in \Omega_1$ *and* $y \in \Omega_2$ *are defined by* (4.16), (4.17) *and* (4.19).

Proof. See the proof of Corollary 4.2. □

We conclude this section by considering the simplest kernels k, that is, those with separate variables.

Corollary 5.17 *Let* $p,q \in \mathbb{R}$, $\frac{q}{p} > 0$. *Let* (Ω,Σ,μ) *be a measure space with a positive* σ*-finite measure* μ, *let* $m \in L^1(\Omega,\mu)$ *be a non-negative function such that* $|m|_1 > 0$, Φ *be a non-negative convex function on an interval* $I \subseteq \mathbb{R}$, *and* $\varphi : I \to \mathbb{R}$ *be any function such* *that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$. Let $f : \Omega \to \mathbb{R}$ be a measurable function with values in I and let $A_m f$ be defined by* (4.21). *If $0 < p \leq q < \infty$ or $-\infty < q \leq p < 0$, then the inequality*

$$[A_m(\Phi \circ f)]^{\frac{q}{p}} - \Phi^{\frac{q}{p}}(A_m f) \geq \frac{q}{p} \Phi^{\frac{q}{p}-1}(A_m f) \cdot A_m r \tag{5.42}$$

holds, where $r(y) = \left| |\Phi(f(y)) - \Phi(A_m f)| - |\varphi(A_m f)| \cdot |f(y) - A_m f| \right|$, $y \in \Omega$. If Φ is a positive concave function and $0 < q \leq p < \infty$ or $-\infty < p \leq q < 0$, then (5.42) *holds with the reversed order of the terms on its left-hand side.*

Proof. Suppose that in Theorem 5.6 and in relation (5.37) we have $\Omega_2 = \Omega$, $\mu_2 = \mu$, $u \in L^1(\Omega_1, \mu_1)$ such that $|u|_1 > 0$, and k of the form $k(x,y) = l(x)m(y)$, for some positive measurable function $l : \Omega_1 \to \mathbb{R}$. Then $K(x) = |m|_1 l(x)$ and $A_k f(x) = A_m f \in I$, $x \in \Omega_1$, while $v(y) = \frac{|u|_1^{\frac{p}{q}}}{|m|_1} m(y)$, $y \in \Omega$. Thus, (5.37) reduces to (5.42) and it does not depend on Ω_1, l, and u. □

Remark 5.8 Observe that for $0 < |\Omega|_\mu < \infty$ and $m(y) \equiv 1$ on Ω we have $|m|_1 = |\Omega|_\mu$, so (5.42) becomes the generalized refined Jensen's inequality

$$[A(\Phi \circ f)]^{\frac{q}{p}} - \Phi^{\frac{q}{p}}(Af) \geq \frac{q}{p} \Phi^{\frac{q}{p}-1}(Af) \cdot Ar$$

where Af is defined by (4.22) and

$$r(y) = \left| |\Phi(f(y)) - \Phi(Af)| - |\varphi(Af)| \cdot |f(y) - Af| \right|, y \in \Omega.$$

Notice that, for $p = q$ we obtain the classical refined Jensen's inequality that was obtained in Corollary 4.3. □

5.3 Generalized one-dimensional Hardy's and Pólya-Knopp's inequality

In the following three sections, general results from Section 5.2 are applied to some usual measure spaces, convex functions, weights and kernels and new refinements and generalizations of the inequalities mentioned in the Introduction are derived. We start with the standard one-dimensional setting, that is, by considering intervals in $\mathbb{R}$ and the Lebesgue measure, and obtain generalized refined Hardy and Pólya-Knopp-type inequalities, as well as related dual inequalities. In the following theorem we generalize and refine inequality (2.7).

Theorem 5.7 *Let $0 < b \le \infty$ and $k : (0,b) \times (0,b) \to \mathbb{R}$, $u : (0,b) \to \mathbb{R}$ be non-negative measurable functions satisfying* (2.8) *and*

$$w(y) = y \left(\int_y^b u(x) \left(\frac{k(x,y)}{K(x)} \right)^{\frac{q}{p}} \frac{dx}{x} \right)^{\frac{p}{q}} < \infty, \; y \in (0,b).$$

If $0 < p \le q < \infty$ or $-\infty < q \le p < 0$, Φ is a non-negative convex function on an interval $I \subseteq \mathbb{R}$, and $\varphi : I \to \mathbb{R}$ is such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \mathrm{Int} I$, then the inequality

$$\left(\int_0^b w(y)\Phi(f(y)) \frac{dy}{y} \right)^{\frac{q}{p}} - \int_0^b u(x)\Phi^{\frac{q}{p}}(A_k f(x)) \frac{dx}{x}$$
$$\ge \frac{q}{p} \int_0^b \frac{u(x)}{K(x)} \Phi^{\frac{q}{p}-1}(A_k f(x)) \int_0^x k(x,y) r(x,y)\, dy \frac{dx}{x} \tag{5.43}$$

holds for all measurable functions $f : (0,b) \to \mathbb{R}$ with values in I, where $A_k f$ and r are respectively defined by (4.25) *and* (5.28)*. If Φ is non-negative monotone convex on the interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in IntI$, then the following inequality*

$$\left(\int_0^b w(y)\Phi(f(y)) \frac{dy}{y} \right)^{\frac{q}{p}} - \int_0^b u(x)\Phi^{\frac{q}{p}}(A_k f(x)) \frac{dx}{x}$$
$$\ge \frac{q}{p} \left| \int_0^b \frac{u(x)}{K(x)} \Phi^{\frac{q}{p}-1}(A_k f(x)) \int_0^x sgn(f(y) - A_k f(x)) k(x,y) r_1(x,y) dy \frac{dx}{x} \right| \tag{5.44}$$

holds for all measurable functions $f : (0,b) \to \mathbb{R}$ such that $f(y) \in I$, for all $y \in (0,b)$ where $A_k f$ and r_1 are respectively defined by (4.25) *and* (5.30)*.*
If $0 < q \le p < \infty$ or $-\infty < p \le q < 0$, and Φ is a non-negative (monotone) concave function, then (5.43) *and* (5.44) *hold with the reversed order of the integrals on its left-hand side.*

Proof. Let S_1, Ω_1, and Ω_2 be as in the proof of Theorem 5.4. Relations (5.43) and (5.44) follow from (5.37) by replacing $d\mu_1(x)$, $d\mu_2(y)$, $u(x)$, $v(y)$, and k respectively with dx, dy, $\frac{u(x)}{x}$, $\frac{w(y)}{y}$, and $k\chi_{S_1}$. □

In the following theorem we formulate a result dual to Theorem 5.7.

Theorem 5.8 *For $0 \le b < \infty$, let $k : (b,\infty) \times (b,\infty) \to \mathbb{R}$ and $u : (b,\infty) \to \mathbb{R}$ be non-negative measurable functions satisfying* (4.35) *and*

$$\tilde{w}(y) = y \left(\int_b^y u(x) \left(\frac{k(x,y)}{\tilde{K}(x)} \right)^{\frac{q}{p}} \frac{dx}{x} \right)^{\frac{p}{q}} < \infty, \; y \in (b,\infty).$$

If $0 < p \leq q < \infty$ or $-\infty < q \leq p < 0$, Φ is a non-negative convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \mathrm{Int}I$, then the inequality

$$\left(\int_b^\infty \tilde{w}(y)\Phi(f(y))\,\frac{dy}{y}\right)^{\frac{q}{p}} - \int_b^\infty u(x)\Phi^{\frac{q}{p}}(\tilde{A}_k f(x))\,\frac{dx}{x}$$
$$\geq \frac{q}{p}\int_b^\infty \frac{u(x)}{\tilde{K}(x)}\Phi^{\frac{q}{p}-1}(\tilde{A}_k f(x))\int_x^\infty k(x,y)\tilde{r}(x,y)\,dy\,\frac{dx}{x} \tag{5.45}$$

holds for all measurable functions $f : (b,\infty) \to \mathbb{R}$ with values in I and for $\tilde{A}_k f(x)$ defined by (4.37) and $\tilde{r}(x,y)$ defined by

$$\tilde{r}(x,y) = \left|\,|\Phi(f(y)) - \Phi(\tilde{A}_k f(x))| - |\varphi(\tilde{A}_k f(x))| \cdot |f(y) - \tilde{A}_k f(x)|\,\right|,$$

where $x,y \in (b,\infty)$. If Φ is non-negative monotone convex on the interval $I \subseteq \mathbb{R}$, and $\varphi : I \to \mathbb{R}$ is such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in IntI$, then the following inequality

$$\left(\int_b^\infty \widetilde{w}(y)\Phi(f(y))\,\frac{dy}{y}\right)^{\frac{q}{p}} - \int_b^\infty u(x)\Phi^{\frac{q}{p}}\left(\widetilde{A}_k f(x)\right)\frac{dx}{x}$$
$$\geq \frac{q}{p}\left|\int_b^\infty \frac{u(x)}{\widetilde{K}(x)}\Phi^{\frac{q}{p}-1}\left(\widetilde{A}_k f(x)\right)\int_x^\infty sgn(f(y)-\widetilde{A}_k f)k(x,y)\tilde{r}_1(x,y)dy\frac{dx}{x}\right| \tag{5.46}$$

holds for all measurable functions $f : (b,\infty) \to \mathbb{R}$ such that $f(y) \in I$, for all $y \in (b,\infty)$, where $\widetilde{A}_k f$ is defined by (4.37) and $\tilde{r}_1(x,y)$ is defined by

$$\tilde{r}_1(x,y) = \left[\Phi(f(y)) - \Phi(\widetilde{A}_k f(x)) - |\varphi(\widetilde{A}_k f(x))| \cdot (f(y) - \widetilde{A}_k f(x))\right].$$

If $0 < q \leq p < \infty$ or $-\infty < p \leq q < 0$, and Φ is a non-negative (monotone) concave function, the order of the integrals on the left-hand side of (5.45) and (5.46) is reversed.

Proof. Let $S_2 = \{(x,y) \in \mathbb{R}^2 : b < x \leq y < \infty\}$. Inequality (5.45) follows directly from (5.37), rewritten with $\Omega_1 = \Omega_2 = (b,\infty)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$, and with $\frac{u(x)}{x}$, $\frac{w(y)}{y}$, and $k\chi_{S_2}$ instead of $u(x)$, $v(y)$, and k. □

Remark 5.9 For $p = q$ Theorem 5.7 and Theorem 5.8 respectively reduce to [65, Theorem 3.1] and [65, Theorem 4.3]. In particular, (5.43) refines (2.7). Of course, in that case, the function Φ does not need to be non-negative. □

The rest of this section is dedicated to generalizations and refinements of the well-known Hardy's and Pólya-Knopp's inequality (0.1) and (0.2) and of their dual inequalities. Since they direct consequences of the above results, we state them as examples. We don't emphasise results with non-negative monotone and convex functions since they can be obtained in a similar way.

Example 5.1 Let $0<b\le\infty$, $\gamma\in\mathbb{R}_+$, $p,q\in\mathbb{R}$ be such that $\frac{q}{p}>0$, and let S_1 be as in the proofs of Theorem 5.4 and Theorem 5.7. Let the kernel $k:(0,b)\times(0,b)\to\mathbb{R}$ be defined by $k(x,y)=\frac{\gamma}{x^\gamma}(x-y)^{\gamma-1}\chi_{S_1}$ and $u(x)\equiv 1$. If $\frac{q}{p}\ge 1$, $\gamma>1-\frac{p}{q}$, Φ is a non-negative convex function on an interval $I\subseteq\mathbb{R}$ and $f:(0,b)\to\mathbb{R}$ is a function with values in I, then (5.43) reads

$$\left(\int_0^b w_\gamma(y)\Phi(f(y))\,\frac{dy}{y}\right)^{\frac{q}{p}}-\int_0^b\Phi^{\frac{q}{p}}(R_\gamma f(x))\,\frac{dx}{x}$$
$$\ge\gamma\frac{q}{p}\int_0^b\Phi^{\frac{q}{p}-1}(R_\gamma f(x))\int_0^x(x-y)^{\gamma-1}r_\gamma(x,y)\,dy\,\frac{dx}{x^{\gamma+1}},\tag{5.47}$$

where R_γ is the Riemann-Liouville operator given by (4.31), while for $x,y\in(0,b)$ we set

$$w_\gamma(y)=\gamma\left(\int_0^{1-\frac{y}{b}}t^{(\gamma-1)\frac{q}{p}}(1-t)^{\frac{q}{p}-1}dt\right)^{\frac{p}{q}}=\gamma B^{\frac{p}{q}}\left(1-\frac{y}{b};(\gamma-1)\frac{q}{p}+1,\frac{q}{p}\right)$$

and

$$r_\gamma(x,y)=\big|\left|\Phi(f(y))-\Phi(R_\gamma f(x))\right|-\left|\varphi(R_\gamma f(x))\right|\cdot\left|f(y)-R_\gamma f(x)\right|\big|.$$

Observe that $B(\cdot;\cdot,\cdot)$ denotes the incomplete Beta function defined in Introduction. In the case when $\frac{q}{p}\in(0,1]$ and Φ is non-negative and concave, the order of the terms on the left-hand side of (5.47) is reversed and the inequality obtained holds for any $\gamma>0$.

Rewriting (5.47) with some suitable parameters and with Φ being a power function, we get a new refined Hardy's inequality. Namely, let $\Phi(x)=x^p$, $k\in\mathbb{R}$ be such that $\frac{k-1}{p}>0$,

$$w_{\gamma,k}(y)=B^{\frac{p}{q}}\left(1-\left(\frac{y}{b}\right)^{\frac{k-1}{p}};(\gamma-1)\frac{q}{p}+1,\frac{q}{p}\right)y^{p-k},\ y\in(0,b),$$

f be a non-negative function on $(0,b)$ (positive, if $p<0$) and

$$Rf(x)=\int_0^x\left[1-\left(\frac{y}{x}\right)^{\frac{k-1}{p}}\right]^{\gamma-1}f(y)\,dy,\ x\in(0,b).$$

For $1\le p\le q<\infty$ or $-\infty<q\le p<0$, replace b and $f(y)$ in (5.47) respectively with $b^{\frac{k-1}{p}}$

and $f\left(y^{\frac{p}{k-1}}\right)y^{\frac{p}{k-1}-1}$. After a sequence of suitable variable changes, we get the inequality

$$\begin{aligned}
&\gamma\left(\frac{p}{\gamma(k-1)}\right)^{q+1-\frac{q}{p}}\left(\int_0^b w_{\gamma,k}(y)f^p(y)\,dy\right)^{\frac{q}{p}}-\int_0^b x^{\frac{q}{p}(1-k)-1}(Rf(x))^q\,dx\\
&\geq\frac{q}{p}\left|\left(\frac{p}{\gamma(k-1)}\right)^{p-1}\int_0^b x^{\frac{k-1}{p}(p-q-1)-1}(Rf(x))^{q-p}\int_0^x\left[1-\left(\frac{y}{x}\right)^{\frac{k-1}{p}}\right]^{\gamma-1}\times\right.\\
&\qquad\times y^{\frac{k-1}{p}-1}\left|y^{p-k+1}f^p(y)-\left(\frac{\gamma(k-1)}{p}\right)^p x^{1-k}(Rf(x))^p\right|dy\,dx\\
&\quad-|p|\int_0^b x^{\frac{1-k}{p}q-1}(Rf(x))^{q-1}\int_0^x\left[1-\left(\frac{y}{x}\right)^{\frac{k-1}{p}}\right]^{\gamma-1}\times\\
&\qquad\left.\times\left|f(y)-\frac{\gamma(k-1)}{py}\left(\frac{y}{x}\right)^{\frac{k-1}{p}}Rf(x)\right|dy\,dx\right|.
\end{aligned}\tag{5.48}$$

For $0<q\leq p<1$, the order of the terms on the left-hand side of relation (5.48) is reversed. Notice that for $b=\infty$, $p=q=k>1$ and $\gamma=1$ inequality (5.48) reduces to a refinement of the classical Hardy's inequality (0.1). It can be seen that our result generalizes refined and strengthened Hardy-type inequalities from [21].

On the other hand, rewriting (5.47) with $\Phi(x)=e^x$ and $\gamma=1$, as well as with the function $y\mapsto\log(yf(y))$ instead of a positive function $f:(0,b)\to\mathbb{R}$, we derive the following new refined strengthened Pólya-Knopp-type inequality:

$$\begin{aligned}
&\frac{p}{q}e^{\frac{q}{p}}\left(\int_0^b\left[1-\left(\frac{y}{b}\right)^{\frac{q}{p}}\right]^{\frac{p}{q}}f(y)\,dy\right)^{\frac{q}{p}}-\int_0^b x^{\frac{q}{p}-1}(Gf(x))^{\frac{q}{p}}\,dx\\
&\geq\frac{q}{p}\left|\int_0^b x^{\frac{q}{p}-3}(Gf(x))^{\frac{q}{p}-1}\int_0^x|eyf(y)-xGf(x)|\,dy\,dx\right.\\
&\qquad\left.-\int_0^b x^{\frac{q}{p}-2}(Gf(x))^{\frac{q}{p}}\int_0^x\left|\log\left(\frac{eyf(y)}{xGf(x)}\right)\right|dy\,dx\right|,
\end{aligned}\tag{5.49}$$

where $\frac{q}{p}\geq 1$ and Gf is defined by (4.34). For $p=q$ relation (5.49) reduces to a refined strengthened Pólya-Knopp's inequality from [21]. Moreover, for $b=\infty$ we obtain a refinement of the classical Pólya-Knopp's inequality (0.2). □

The following example provides results dual to those from Example 5.1.

Example 5.2 Suppose $0\leq b<\infty$, $\gamma\in\mathbb{R}_+$, $p,q\in\mathbb{R}$ are such that $\frac{q}{p}>0$, and S_2 is as in the proof of Theorem 5.8. Define the kernel $k:(b,\infty)\times(b,\infty)\to\mathbb{R}$ and the weight function

$u:(b,\infty)\to\mathbb{R}$ as $k(x,y)=\gamma\frac{x}{y^{\gamma+1}}(y-x)^{\gamma-1}\chi_{S_2}(x,y)$ and $u(x)\equiv 1$. For $\frac{q}{p}\ge 1$, $\gamma>1-\frac{p}{q}$, a non-negative convex function Φ on an interval $I\subseteq\mathbb{R}$ and a function $f:(b,\infty)\to\mathbb{R}$ with values in I, inequality (5.45) becomes

$$\left(\int_b^\infty \tilde{w}_\gamma(y)\Phi(f(y))\,\frac{dy}{y}\right)^{\frac{q}{p}}-\int_b^\infty \Phi^{\frac{q}{p}}(W_\gamma f(x))\,\frac{dx}{x}$$
$$\ge \gamma\frac{q}{p}\int_b^\infty \Phi^{\frac{q}{p}-1}(W_\gamma f(x))\int_x^\infty (y-x)^{\gamma-1}\tilde{r}_\gamma(x,y)\,\frac{dy}{y^{\gamma+1}}\,dx, \tag{5.50}$$

where W_γ denotes Weyl's operator defined by (4.42), and for $x,y\in(b,\infty)$ we define $\tilde{w}_\gamma(y)=\gamma B^{\frac{p}{q}}\left(1-\frac{b}{y};(\gamma-1)\frac{q}{p}+1,\frac{q}{p}\right)$ and $\tilde{r}_\gamma(x,y)=\left|\left|\Phi(f(y))-\Phi(W_\gamma f(x))\right|-\left|\varphi(W_\gamma f(x))\right|\cdot\left|f(y)-W_\gamma f(x)\right|\right|$. If $\frac{q}{p}\in(0,1]$ and Φ is non-negative and concave, (5.50) holds for all $\gamma>0$ with the reversed order of the terms on its left-hand side.

As in Example 5.1, to get a new refined dual Hardy's inequality, we rewrite (5.50) with $\Phi(x)=x^p$. More precisely, let $k\in\mathbb{R}$ be such that $\frac{p}{1-k}>0$,

$$\tilde{w}_{\gamma,k}(y)=B^{\frac{p}{q}}\left(1-\left(\frac{b}{y}\right)^{\frac{1-k}{p}};(\gamma-1)\frac{q}{p}+1,\frac{q}{p}\right)y^{p-k},\ y\in(b,\infty),$$

f be a non-negative function on (b,∞) (positive, if $p<0$) and

$$Wf(x)=\int_x^\infty\left[1-\left(\frac{x}{y}\right)^{\frac{1-k}{p}}\right]^{\gamma-1} f(y)\,dy,\ x\in(b,\infty).$$

For $1\le p\le q<\infty$ or $-\infty<q\le p<0$, substitute $b^{\frac{1-k}{p}}$ and $f\left(y^{\frac{p}{1-k}}\right)y^{\frac{p}{1-k}+1}$ in (5.50) respectively for b and $f(y)$. After some computations, we obtain the inequality

$$\gamma\left(\frac{p}{\gamma(1-k)}\right)^{q+1-\frac{q}{p}}\left(\int_b^\infty \tilde{w}_{\gamma,k}(y)f^p(y)\,dy\right)^{\frac{q}{p}}-\int_b^\infty x^{\frac{q}{p}(1-k)-1}(Wf(x))^q\,dx$$
$$\ge\frac{q}{p}\left|\left(\frac{p}{\gamma(1-k)}\right)^{p-1}\int_b^\infty x^{\frac{1-k}{p}(q-p+1)-1}(Wf(x))^{q-p}\int_x^\infty\left[1-\left(\frac{x}{y}\right)^{\frac{1-k}{p}}\right]^{\gamma-1}\times\right.$$
$$\times\, y^{\frac{k-1}{p}-1}\left|y^{p-k+1}f^p(y)-\left(\frac{\gamma(1-k)}{p}\right)^p x^{1-k}(Wf(x))^p\right|dy\,dx$$
$$-|p|\int_b^\infty x^{\frac{1-k}{p}q-1}(Wf(x))^{q-1}\int_x^\infty\left[1-\left(\frac{x}{y}\right)^{\frac{1-k}{p}}\right]^{\gamma-1}\times$$
$$\left.\times\left|f(y)-\frac{\gamma(1-k)}{py}\left(\frac{x}{y}\right)^{\frac{1-k}{p}}Wf(x)\right|dy\,dx\right|. \tag{5.51}$$

For $0 < q \leq p < 1$, relation (5.51) holds with the reversed order of the terms on its left-hand side. When $p = q$ (5.51) becomes a refined and strengthened dual Hardy's inequality from Example 4.2.

Finally, for $\frac{q}{p} \geq 1$, $\gamma = 1$, $\Phi(x) = e^x$ and $y \mapsto \log(yf(y))$ instead of a positive function $f : (b, \infty) \to \mathbb{R}$, inequality (5.50) becomes

$$\frac{p}{q} e^{-\frac{q}{p}} \left(\int_b^\infty \left[1 - \left(\frac{b}{y} \right)^{\frac{q}{p}} \right]^{\frac{p}{q}} f(y)\, dy \right)^{\frac{q}{p}} - \int_b^\infty x^{\frac{q}{p}-1} (\tilde{G}f(x))^{\frac{q}{p}}\, dx$$
$$\geq \frac{q}{p} \left| \int_b^\infty x^{\frac{q}{p}-1} (\tilde{G}f(x))^{\frac{q}{p}-1} \int_x^\infty \left| e^{-1} y f(y) - x\tilde{G}f(x) \right| \frac{dy}{y^2}\, dx \right.$$
$$\left. - \int_b^\infty x^{\frac{q}{p}} (\tilde{G}f(x))^{\frac{q}{p}} \int_x^\infty \left| \log \frac{yf(y)}{ex\tilde{G}f(x)} \right| \frac{dy}{y^2}\, dx \right|,$$

where

$$\tilde{G}f(x) = \exp\left(x \int_x^\infty \log f(y) \frac{dy}{y^2} \right), \; y \in (b, \infty).$$

Thus, we proved a new refined strengthened dual Pólya-Knopp's inequality. Its special case $p = q$ was already considered in [21] and in Example 4.2. □

5.4 Generalized one-dimensional Hardy-Hilbert's inequality

In this section, we consider Theorem 5.6, that is, inequalities (5.37) and (5.38) , with some important kernels related to $\Omega_1 = \Omega_2 = \mathbb{R}_+$ and $\Phi : \mathbb{R}_+ \to \mathbb{R}$, $\Phi(x) = x^p$, where $p \in \mathbb{R}$, $p \neq 0$. We also assume that $d\mu_1(x) = dx$ and $d\mu_2(y) = dy$.

In the first example, we generalize and refine the classical Hardy-Hilbert's inequality (2.2).

Example 5.3 Let $p, q, s \in \mathbb{R}$ be such that $\frac{q}{p} > 0$ and $\frac{s-2}{p}, \frac{s-2}{p'} > -1$, and let $\alpha \in \left(-\frac{q}{p}\left(\frac{s-2}{p'} + 1\right), \frac{q}{p}\left(\frac{s-2}{p} + 1\right) \right)$. Denote

$$C_1 = B\left(\frac{q}{p}\left(\frac{s-2}{p} + 1 \right) - \alpha, \frac{q}{p}\left(\frac{s-2}{p'} + 1 \right) + \alpha \right)$$

and

$$C_2 = B\left(\frac{s-2}{p} + 1, \frac{s-2}{p'} + 1 \right),$$

where $B(\cdot,\cdot)$ is the usual Beta function, and define $k:\mathbb{R}_+^2\to\mathbb{R}$ and $u:\mathbb{R}_+\to\mathbb{R}$ respectively by $k(x,y)=\left(\frac{y}{x}\right)^{\frac{s-2}{p}}(x+y)^{-s}$ and $u(x)=x^{\alpha-1}$. Finally, let f be a non-negative function on $\mathbb{R}_+$ (positive, if $p<0$) and Sf its generalized Stieltjes transform,

$$Sf(x)=\int_0^\infty \frac{f(y)}{(x+y)^s}\,dy,\ x\in\mathbb{R}_+$$

(see [8] and [97] for further information). Rewriting (5.37) and (5.38) with the above parameters and with $f(y)y^{\frac{2-s}{p}}$ instead of $f(y)$, for $1\le p\le q<\infty$ or $-\infty<q\le p<0$ we obtain the inequalities

$$\begin{aligned}
&C_1C_2^{\frac{q}{p'}}\left(\int_0^\infty y^{\alpha\frac{p}{q}-s+1}f^p(y)\,dy\right)^{\frac{q}{p}}-\int_0^\infty x^{\alpha-1+\frac{q}{p'}(s-1)+\frac{q}{p}}(Sf(x))^q\,dx\\
&\ge\frac{q}{p}\left|C_2^{p-1}\int_0^\infty x^{\alpha+q-p+\frac{s-2}{p'}(q-p+1)}(Sf(x))^{q-p}\times\right.\\
&\qquad\times\int_0^\infty \frac{y^{\frac{s-2}{p}}}{(x+y)^s}\left|f^p(y)y^{2-s}-C_2^{-p}x^{(p-1)(s-1)+1}(Sf(x))^p\right|dy\,dx\\
&\qquad-|p|\int_0^\infty x^{\alpha+q+\frac{s-2}{p'}q-1}(Sf(x))^{q-1}\times\\
&\qquad\left.\times\int_0^\infty (x+y)^{-s}\left|f(y)-C_2^{-1}x^{\frac{s-2}{p'}+1}y^{\frac{s-2}{p}}Sf(x)\right|dy\,dx\right| \qquad (5.52)
\end{aligned}$$

and

$$\begin{aligned}
&C_1C_2^{\frac{q}{p'}}\left(\int_0^\infty y^{\alpha\frac{p}{q}-s+1}f^p(y)\,dy\right)^{\frac{q}{p}}-\int_0^\infty x^{\alpha-1+\frac{q}{p'}(s-1)+\frac{q}{p}}(Sf(x))^q\,dx\\
&\ge\frac{q}{p}\left|C_2^{p-1}\int_0^\infty x^{\alpha+q-p+\left(\frac{s-2}{p'}\right)(q-p+1)}(Sf(x))^{q-p}\right.\\
&\quad\times\int_0^\infty sgn\left(y^{\frac{s-2}{p}}f(y)-x^{\frac{s-2}{p'}+1}B_2^{-1}Sf(x)\right)\\
&\quad\times\frac{y^{\frac{s-2}{p}}}{(x+y)^s}\left[y^{2-s}f^p(y)-\left(x^{\frac{s-2}{p'}+1}C_2^{-1}Sf(x)\right)^p-p\left|x^{\frac{s-2}{p'}+1}C_2^{-1}Sf(x)\right|^{p-1}\right.\\
&\quad\left.\left.\times\left(y^{\frac{2-s}{p}}f(y)-x^{\frac{s-2}{p'}+1}C_2^{-1}Sf(x)\right)\right]dy\,dx\right|, \qquad (5.53)
\end{aligned}$$

while for $0 < q \leq p < 1$ the order of the terms on the left-hand side of (5.52) and (5.53) is reversed. The case $p = q$ was already studied in Example 4.3. In particular, for $p = q > 1$, $\alpha = 0$ and $s = 1$ we have $C_1 = C_2 = B\left(\frac{1}{p}, \frac{1}{p'}\right) = \frac{\pi}{\sin\frac{\pi}{p}}$, so (5.52) provides a new generalization and refinement of the classical Hardy-Hilbert's inequality (2.2). □

Similarly, in the next example we generalize and refine the classical Hardy-Littlewood-Pólya's inequality (4.46).

Example 5.4 Let the parameters p, q, s, α and the functions u and f be as in Example 5.3. Define $k : \mathbb{R}_+^2 \to \mathbb{R}$ by $k(x,y) = \left(\frac{y}{x}\right)^{\frac{s-2}{p}} \max\{x,y\}^{-s}$ and the transform Lf as

$$Lf(x) = \int_0^\infty \frac{f(y)}{\max\{x,y\}^s}\, dy,\ x \in \mathbb{R}_+.$$

Finally, set

$$D_1 = \frac{p^2 p' qs}{(\alpha pp' + p'q + qs - 2q)(pq + qs - \alpha p^2 - 2q)}$$

and

$$D_2 = \frac{pp's}{(p+s-2)(p'+s-2)}.$$

Considering $1 \leq p \leq q < \infty$, or $-\infty < q \leq p < 0$, and $f(y)y^{\frac{2-s}{p}}$ instead of $f(y)$, relation (5.37) and (5.38) become

$$\begin{aligned}
&D_1 D_2^{\frac{q}{p'}} \left(\int_0^\infty y^{\alpha\frac{p}{q}-s+1} f^p(y)\, dy \right)^{\frac{q}{p}} - \int_0^\infty x^{\alpha-1+\frac{q}{p'}(s-1)+\frac{q}{p}} (Lf(x))^q\, dx \\
&\geq \frac{q}{p} \Bigg| D_2^{p-1} \int_0^\infty x^{\alpha+q-p+\frac{s-2}{p'}(q-p+1)} (Lf(x))^{q-p} \times \\
&\qquad \times \int_0^\infty \frac{y^{\frac{s-2}{p}}}{\max\{x,y\}^s} \left| f^p(y) y^{2-s} - D_2^{-p} x^{(p-1)(s-1)+1} (Lf(x))^p \right| dy\, dx \\
&\qquad - |p| \int_0^\infty x^{\alpha+q+\frac{s-2}{p'}q-1} (Lf(x))^{q-1} \times \\
&\qquad \times \int_0^\infty \max\{x,y\}^{-s} \left| f(y) - D_2^{-1} x^{\frac{s-2}{p'}+1} y^{\frac{s-2}{p}} Lf(x) \right| dy\, dx \Bigg| \qquad (5.54)
\end{aligned}$$

and

$$
\begin{aligned}
&D_1 D_2^{\frac{q}{p'}} \left(\int_0^\infty y^{\alpha\frac{p}{q}-s+1} f^p(y)\,dy \right)^{\frac{q}{p}} - \int_0^\infty x^{\alpha-1+\frac{q}{p'}(s-1)+\frac{q}{p}} (Lf(x))^q\,dx \\
&\ge \frac{q}{p} \left| D_2^{p-1} \int_0^\infty x^{\alpha+q-p+\left(\frac{s-2}{p'}\right)(q-p+1)} (Lf(x))^{q-p} \right. \\
&\times \int_0^\infty sgn\left(y^{\frac{2-s}{p}} f(y) - x^{\frac{s-2}{p'}+1} D_2^{-1} Lf(x) \right) \\
&\times \frac{y^{\frac{s-2}{p}}}{\max\{x,y\}^s} \left[y^{2-s} f^p(y) - \left(x^{\frac{s-2}{p'}+1} D_2^{-1} Lf(x) \right)^p - p \left| x^{\frac{s-2}{p'}+1} D_2^{-1} Lf(x) \right|^{p-1} \right. \\
&\left. \left. \times \left(y^{\frac{2-s}{p}} f(y) - x^{\frac{s-2}{p'}+1} D_2^{-1} Lf(x) \right) \right] dy\,dx \right|. \qquad (5.55)
\end{aligned}
$$

If $0 < q \le p < 1$, the order of the terms on the left-hand side of (5.54) and (5.55) is reversed. For $p = q$, (5.54) reduces to Example 4.4. Moreover, since for $p = q > 1$, $\alpha = 0$ and $s = 1$ we have $D_1 = D_2 = pp'$, our result generalizes and refines (4.46). □

We complete this section with another refined Hardy-Hilbert-type inequality, making use of the well-known reflection formula for the Digamma function ψ and of the fact that

$$
Z(a,b) = \int_0^\infty t^b e^{-at} \left(1 - e^{-t}\right)^b dt < \infty,\ a \in \mathbb{R}_+,\ b \ge 1.
$$

More precisely, $Z(a,b) = \Gamma(b+1)\phi_b^*(1, b+1, a)$, where ϕ_μ^* is the so-called unified Riemann-Zeta function,

$$
\phi_\mu^*(z,s,a) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at} \left(1 - ze^{-t}\right)^{-\mu} dt,
$$

where $\mu \ge 1$, $\mathrm{Re}\,a > 0$ and either $|z| \le 1$, $z \ne 1$ and $\mathrm{Re}\,s > 0$, or $z = 1$ and $\mathrm{Re}\,s > \mu$ (for more information regarding the unified Riemann-Zeta function, see e.g. [42]).

Example 5.5 Suppose that $\alpha \in (0,1)$ and $p, q, \beta \in \mathbb{R}$ are such that $\frac{q}{p} \ge 1$ and $\alpha\frac{q}{p} + \beta \in \left(-1, \frac{q}{p} - 1\right)$. Define the kernel $k : \mathbb{R}_+^2 \to \mathbb{R}$ by $k(x,y) = \frac{\log y - \log x}{y - x} \left(\frac{x}{y}\right)^\alpha$ and the weight function $u : \mathbb{R}_+ \to \mathbb{R}$ by $u(x) = x^\beta$. Finally, denote

$$
Mf(x) = \int_0^\infty \frac{\log y - \log x}{y - x} f(y)\,dy,\ x \in \mathbb{R}_+,
$$

where f is a non-negative function on $\mathbb{R}_+$ (positive, if $p < 0$),

$$E_1 = \int_0^\infty \left(\frac{\log t}{t-1}\right)^{\frac{q}{p}} t^{\alpha\frac{q}{p}+\beta}\,dt = Z\left(\alpha\frac{q}{p}+\beta+1,\frac{q}{p}\right)+Z\left(\frac{q}{p}-\alpha\frac{q}{p}-\beta-1,\frac{q}{p}\right)$$

and

$$E_2 = \int_0^\infty \frac{\log t}{t-1}\,t^{-\alpha}\,dt = \frac{\pi^2}{\sin^2\pi\alpha}.$$

Applying (5.37) and (5.38) to the above parameters and to $f(y)$ replaced with $f(y)y^{\alpha}$, we get the inequalities

$$\begin{aligned}
&E_1 E_2^{\frac{q}{p'}}\left(\int_0^\infty y^{\alpha p+(\beta+1)\frac{p}{q}-1} f^p(y)\,dy\right)^{\frac{q}{p}} - \int_0^\infty x^{\alpha q+\beta}(Mf(x))^q\,dx \\
&\geq \frac{q}{p}\left| E_2^{p-1}\int_0^\infty x^{\alpha(q-p+1)+\beta}(Mf(x))^{q-p}\times\right. \\
&\qquad\times\int_0^\infty y^{-\alpha}\,\frac{\log y-\log x}{y-x}\left|f^p(y)y^{\alpha p}-E_2^{-p}x^{\alpha p}(Mf(x))^p\right|dy\,dx \\
&\quad -|p|\int_0^\infty x^{\alpha q+\beta}(Mf(x))^{q-1}\times \\
&\qquad\left.\times\int_0^\infty \frac{\log y-\log x}{y-x}\left|f(y)-E_2^{-1}\left(\frac{x}{y}\right)^{\alpha}Mf(x)\right|dy\,dx\right|
\end{aligned} \tag{5.56}$$

and

$$\begin{aligned}
&E_1 E_2^{\frac{q}{p'}}\left(\int_0^\infty y^{\alpha p+(\beta+1)\frac{p}{q}-1} f^p(y)\,dy\right)^{\frac{q}{p}} - \int_0^\infty x^{\alpha q+\beta}(Mf(x))^q\,dx \\
&\geq \frac{q}{p}\left| E_2^{p-1}\int_0^\infty x^{\alpha(q-p+1)+\beta}(Mf(x))^{q-p}\int_0^\infty sgn\left(y^{\alpha}f(y)-x^{\alpha}E_2^{-1}Mf(x)\right)\right. \\
&\times y^{-\alpha}\frac{\ln y-\ln x}{y-x}\left[y^{\alpha p}f^p(y)-\left(x^{\alpha}E_2^{-1}Mf(x)\right)^p-p\left|x^{\alpha}E_2^{-1}Mf(x)\right|^{p-1}\right. \\
&\left.\left.\times\left(y^{\alpha}f(y)-x^{\alpha}E_2^{-1}Mf(x)\right)\right]dy\,dx\right|.
\end{aligned}$$

Notice that for $p = q$ we have

$$E_1 = \int_0^\infty \frac{\log t}{t-1}\,t^{\alpha+\beta}\,dt = \frac{\pi^2}{\sin^2\pi(\alpha+\beta)}$$

and (5.56) reduces to the Hardy-Hilbert-type inequality obtained in Example 4.5. Therefore our result can be seen as its generalization. □

5.5 General Godunova-type inequalities

We conclude the chapter with a multidimensional result related to Godunova's inequality (2.10). Namely, let $\Omega_1 = \Omega_2 = \mathbb{R}_+^n$, $d\mu_1(\mathbf{x}) = d\mathbf{x}$, $d\mu_2(\mathbf{y}) = d\mathbf{y}$, let $\frac{\mathbf{y}}{\mathbf{x}}$ and $\mathbf{x}^{\mathbf{y}}$ be as in Section 5.1, and let the kernel $k : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}$ be of the form $k(\mathbf{x},\mathbf{y}) = l\left(\frac{\mathbf{y}}{\mathbf{x}}\right)$, where $l : \mathbb{R}_+^n \to \mathbb{R}$ is a non-negative measurable function.

Applying Theorem 5.6 to this setting, we get the following generalization and refinement of Godunova's inequality (2.10).

Theorem 5.9 *Let $0 < p \le q < \infty$ or $-\infty < q \le p < 0$. Let l and u be non-negative measurable functions on $\mathbb{R}_+^n$, such that $0 < L(\mathbf{x}) = \mathbf{x}^{\mathbf{1}} \int_{\mathbb{R}_+^n} l(\mathbf{y})\, d\mathbf{y} < \infty$ for all $\mathbf{x} \in \mathbb{R}_+^n$, and that the function $\mathbf{x} \mapsto u(\mathbf{x}) \left(\frac{l\left(\frac{\mathbf{y}}{\mathbf{x}}\right)}{L(\mathbf{x})}\right)^{\frac{q}{p}}$ is integrable on $\mathbb{R}_+^n$ for each fixed $\mathbf{y} \in \mathbb{R}_+^n$. Let the function v be defined on $\mathbb{R}_+^n$ by*

$$v(\mathbf{y}) = \left(\int_{\mathbb{R}_+^n} u(\mathbf{x}) \left(\frac{l\left(\frac{\mathbf{y}}{\mathbf{x}}\right)}{L(\mathbf{x})} \right)^{\frac{q}{p}} d\mathbf{x} \right)^{\frac{p}{q}}.$$

If Φ is a non-negative convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$, then the inequality

$$\begin{aligned} &\left(\int_{\mathbb{R}_+^n} v(\mathbf{y}) \Phi(f(\mathbf{y}))\, d\mathbf{y} \right)^{\frac{q}{p}} - \int_{\mathbb{R}_+^n} u(\mathbf{x}) \Phi^{\frac{q}{p}}(A_l f(\mathbf{x}))\, d\mathbf{x} \\ &\ge \frac{q}{p} \int_{\mathbb{R}_+^n} \frac{u(\mathbf{x})}{L(\mathbf{x})} \Phi^{\frac{q}{p}-1}(A_l f(\mathbf{x})) \int_{\mathbb{R}_+^n} l\left(\frac{\mathbf{y}}{\mathbf{x}}\right) r(\mathbf{x},\mathbf{y})\, d\mathbf{y}\, d\mathbf{x} \end{aligned} \tag{5.57}$$

holds for all measurable functions $f : \mathbb{R}_+^n \to \mathbb{R}$ with values in I, where $A_l f(\mathbf{x})$ and $r(\mathbf{x},\mathbf{y})$, for $\mathbf{x},\mathbf{y} \in \mathbb{R}_+^n$ are defined by

$$A_l f(\mathbf{x}) = \frac{1}{L(\mathbf{x})} \int_{\mathbb{R}_+^n} l\left(\frac{\mathbf{y}}{\mathbf{x}}\right) f(\mathbf{y})\, d\mathbf{y} \tag{5.58}$$

and

$$r(\mathbf{x},\mathbf{y}) = \big|\, |\Phi(f(\mathbf{y})) - \Phi(A_l f(\mathbf{x}))| - |\varphi(A_l f(\mathbf{x}))| \cdot |f(\mathbf{y}) - A_l f(\mathbf{x})| \,\big|.$$

If Φ is a positive concave function and $0 < q \le p < \infty$ or $-\infty < p \le q < 0$, then (5.57) holds with the reversed order of the terms on its left-hand side.

If Φ is non-negative monotone convex on the interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is any function such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in Int\, I$, then the inequality

$$\begin{aligned}&\left(\int_{\mathbb{R}^n_+} v(\mathbf{y})\Phi(f(\mathbf{y}))\,d\mathbf{y}\right)^{\frac{q}{p}} - \int_{\mathbb{R}^n_+} u(\mathbf{x})\Phi^{\frac{q}{p}}(A_l f(\mathbf{x}))\,d\mathbf{x} \\ &\geq \frac{q}{p}\left|\int\limits_{\mathbb{R}^n_+} \frac{u(\mathbf{x})}{L(\mathbf{x})}\Phi^{\frac{q}{p}-1}(A_l f(\mathbf{x}))\int\limits_{\Omega_2} sgn(f(\mathbf{y}) - A_l f(\mathbf{x}))l\left(\frac{\mathbf{y}}{\mathbf{x}}\right)\times \right. \\ &\left.\Big[\Phi(f(\mathbf{y})) - \Phi(A_l f(\mathbf{x})) - |\varphi(A_l f(\mathbf{x}))| \cdot (f(\mathbf{y}) - A_l f(\mathbf{x}))\Big] d\mathbf{y}d\mathbf{x}\right| \end{aligned} \tag{5.59}$$

holds for all measurable functions $f : \mathbb{R}^n_+ \to \mathbb{R}$ such that $f(\mathbf{y}) \in I$ for all $\mathbf{y} \in \mathbb{R}^n_+$ where $A_l f$ is defined by (5.58).

If Φ is a positive monotone concave function, then the order of the terms on the left-hand side of (5.59) *is reversed.*

Remark 5.10 Observe that for $p = q$ inequality (5.57) reduces to Theorem 4.4. If, additionally, $\int_{\mathbb{R}^n_+} l(\mathbf{y})\,d\mathbf{y} = 1$ and $u(\mathbf{x}) = \mathbf{x}^{-\mathbf{1}}$, we get a refinement of (2.10). □

The above results can be rewritten with particular convex (or concave) functions, for example, with power and exponential functions. This leads to multidimensional analogues of corollaries and examples from Sections 5.3 and 5.4. Due to the lack of space, we omit them here.

5.6 Generalized G. H. Hardy-type inequality

Let us continue by taking the non-negative difference of the left-hand side and the right-hand side of the inequality given in Theorem 5.6 with $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$, $\Phi(x) = x^s, s \geq 1$ as:

$$\begin{aligned}\rho(s) = &\left(\int\limits_{\Omega_2} v(y)f^s(y)d\mu_2(y)\right)^{\frac{q}{p}} - \int\limits_{\Omega_1} u(x)(A_k f(x))^{\frac{sq}{p}}d\mu_1(x) \\ &- \frac{q}{p}\int\limits_{\Omega_1} \frac{u(x)}{K(x)}(A_k f(x))^{s(\frac{q}{p}-1)}\int\limits_{\Omega_2} k(x,y)r(x,y)d\mu_2(y)d\mu_1(x),\end{aligned} \tag{5.60}$$

where $r(x,y)$ is defined by (5.28).

We can also take the non-negative difference of the left-hand side and the right-hand side of the inequality given in (5.31) with $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$, $\Phi(x) = x^s, s \geq 1$ as:

$$\begin{aligned}
\pi(s) = &\left(\int_{\Omega_2} v(y) f^s(y) d\mu_2(y) \right)^{\frac{q}{p}} - \int_{\Omega_1} u(x) \left(A_k f(x)\right)^{\frac{sq}{p}} d\mu_1(x) \\
&- \frac{q}{p} \left| \int_{\Omega_1} \frac{u(x)}{K(x)} \Phi^{\frac{q}{p}-1} \left(A_k f(x)\right) \int_{\Omega_2} sgn(f(y) - A_k f(x)) k(x,y) \right. \\
&\left. \times \left[f^s(y) - (A_k f(x))^s - s|A_k f(x)|^{s-1} \cdot (f(y) - A_k f(x)) \right] d\mu_2(y) d\mu_1(x) \right|.
\end{aligned} \tag{5.61}$$

In the following sections we will give results related to $\rho(s)$ defined by (5.60). Results involving $\pi(s)$ defined by (5.61) can be obtained in a similar way. For more details see [62].

5.6.1 G. H. Hardy-type inequalities for fractional integrals

In the following theorem, our first result involving the fractional integral of f with respect to an increasing function g is given. We give results for the Riemann-Liouville fractional integrals and Hadamard-type fractional integrals as an applications of this theorem.

Theorem 5.10 *Let $0 < p \leq q < \infty$, $s \geq 1$, $\alpha > 1 - \frac{p}{q}$, $f \geq 0$, g be increasing function on (a,b) such that g' be continuous on (a,b). Then the following inequality holds:*

$$0 \leq \rho_1(s) \leq \overline{H}_1(s) - M_1(s) \leq \overline{H}_1(s),$$

where

$$\begin{aligned}
\rho_1(s) \;=\; &\frac{\alpha^{\frac{q}{p}}}{(\alpha-1)\frac{q}{p}+1} \left(\int_a^b g'(y)(g(b)-g(y))^{\alpha-1+\frac{p}{q}} f^s(y) dy \right)^{\frac{q}{p}} \\
&- (\Gamma(\alpha+1))^{\frac{sq}{p}} \int_a^b g'(x)(g(x)-g(a))^{\frac{\alpha q}{p}(1-s)} \left(I^{\alpha}_{a+;g} f(x)\right)^{\frac{sq}{p}} dx - M_1(s),
\end{aligned}$$

$$\begin{aligned}
M_1(s) = &\frac{\alpha q (\Gamma(\alpha+1))^{s(\frac{q}{p}-1)}}{p} \int_a^b g'(x)(g(x)-g(a))^{\frac{\alpha(q-p)(1-s)}{p}} \left(I^{\alpha}_{a+;g} f(x)\right)^{s(\frac{q}{p}-1)} \\
&\times \int_a^x \frac{g'(y) r_1(x,y)}{(g(x)-g(y))^{1-\alpha}} dy dx,
\end{aligned}$$

$$
\begin{aligned}
r_1(x,y) = \Bigg| & \left| f^s(y) - \left(\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^\alpha} I^\alpha_{a+;g} f(x) \right)^s \right| \\
& - s \left| \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^\alpha} I^\alpha_{a+;g} f(x) \right|^{s-1} \cdot \left| f(y) - \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^\alpha} I^\alpha_{a+;g} f(x) \right| \Bigg|,
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{H}_1(s) = (g(b)-g(a))^{\frac{\alpha q}{p}(1-s)} & \left[\frac{\alpha^{\frac{q}{p}} (g(b)-g(a))^{\frac{q(\alpha s-1)+p}{p}}}{(\alpha-1)\frac{q}{p}+1} \left(\int_a^b g'(y) f^s(y) dy \right)^{\frac{q}{p}} \right. \\
& \left. - (\Gamma(\alpha+1))^{\frac{sq}{p}} \int_a^b g'(x) (I^\alpha_{a+;g} f(x))^{\frac{sq}{p}} dx \right].
\end{aligned}
$$

Proof. Applying Theorem 5.6 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(y) = dy$,

$$
k(x,y) = \begin{cases} \frac{g'(y)}{\Gamma(\alpha)(g(x)-g(y))^{1-\alpha}}, & a < y \le x; \\ 0, & x < y \le b, \end{cases}
$$

we get that $K(x) = \frac{1}{\Gamma(\alpha+1)}(g(x)-g(a))^\alpha$ and $A_k f(x) = \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^\alpha} I^\alpha_{a+;g} f(x)$. For particular weight function $u(x) = g'(x)(g(x)-g(a))^{\frac{\alpha q}{p}}$, $x \in (a,b)$, we get $v(y) = (\alpha g'(y)(g(b)-g(y))^{\alpha-1+\frac{p}{q}})/(((\alpha-1)\frac{q}{p}+1)^{\frac{p}{q}})$, so (5.60) takes the form

$$
\begin{aligned}
\rho_1(s) = & \frac{\alpha^{\frac{q}{p}}}{(\alpha-1)\frac{q}{p}+1} \left(\int_a^b g'(y)(g(b)-g(y))^{\alpha-1+\frac{p}{q}} f^s(y) dy \right)^{\frac{q}{p}} \\
& - (\Gamma(\alpha+1))^{\frac{sq}{p}} \int_a^b g'(x)(g(x)-g(a))^{\frac{\alpha q(1-s)}{p}} \left(I^\alpha_{a+;g} f(x) \right)^{\frac{sq}{p}} dx - M_1(s).
\end{aligned}
$$

Since $\frac{\alpha q}{p}(1-s) \le 0$, g is increasing and $M_1(s) \ge 0$, we obtain that

$$
\begin{aligned}
0 \le \rho_1(s) \le & \frac{\alpha^{\frac{q}{p}} (g(b)-g(a))^{(\alpha-1)\frac{q}{p}+1}}{(\alpha-1)\frac{q}{p}+1} \left(\int_a^b g'(y) f^s(y) dy \right)^{\frac{q}{p}} \\
& - (g(b)-g(a))^{\frac{\alpha q}{p}(1-s)} (\Gamma(\alpha+1))^{\frac{sq}{p}} \int_a^b g'(x) (I^\alpha_{a+;g} f(x))^{\frac{sq}{p}} dx - M_1(s) \\
= & \overline{H}_1(s) - M_1(s) \\
\le & \overline{H}_1(s).
\end{aligned}
$$

This completes the proof.

Remark 5.11 Similar result can be obtained for the right-sided fractional integral of f with respect to an increasing function g, but we omit the details here.

Here, we give a first special case for the Riemman-Liouville fractional integral. If $g(x) = x$, then $I^{\alpha}_{a+;x}f(x)$ reduces to $I^{\alpha}_{a+}f(x)$, the left-sided Riemann-Liouville fractional integra,l and the following result follows.

Corollary 5.18 *Let* $0 < p \leq q < \infty$, $\alpha > 1 - \frac{p}{q}$, $s \geq 1$,$f \geq 0$. *Then the following inequality holds*

$$0 \leq \rho_2(s) \leq \overline{H}_2(s) - M_2(s) \leq \overline{H}_2(s),$$

where

$$\begin{aligned}
\rho_2(s) = {} & \frac{\alpha^{\frac{q}{p}}}{(\alpha-1)\frac{q}{p}+1}\left(\int_a^b (b-y)^{\alpha-1+\frac{p}{q}} f^s(y)dy\right)^{\frac{q}{p}} \\
& -(\Gamma(\alpha+1))^{\frac{sq}{p}} \int_a^b (x-a)^{\frac{\alpha q}{p}(1-s)} \left(I^{\alpha}_{a+}f(x)\right)^{\frac{sq}{p}} dx - M_2(s),
\end{aligned}$$

$$\begin{aligned}
M_2(s) = {} & \frac{\alpha q(\Gamma(\alpha+1))^{s(\frac{q}{p}-1)}}{p} \int_a^b (x-a)^{\frac{\alpha(q-p)(1-s)}{p}} \left(I^{\alpha}_{a+}f(x)\right)^{s(\frac{q}{p}-1)} \\
& \times \int_a^x \frac{r_2(x,y)}{(x-y)^{1-\alpha}} dy dx,
\end{aligned}$$

$$\begin{aligned}
r_2(x,y) = {} & \left| \left| f^s(y) - \left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} I^{\alpha}_{a+}f(x)\right)^s \right| \right. \\
& \left. -s\left|\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} I^{\alpha}_{a+}f(x)\right|^{s-1} \cdot \left| f(y) - \frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} I^{\alpha}_{a+}f(x)\right| \right|,
\end{aligned}$$

and

$$\begin{aligned}
\overline{H}_2(s) = {} & (b-a)^{\frac{\alpha q}{p}(1-s)} \left[\frac{\alpha^{\frac{q}{p}}(b-a)^{\frac{q(\alpha s-1)+p}{p}}}{(\alpha-1)\frac{q}{p}+1} \left(\int_a^b f^s(y)dy\right)^{\frac{q}{p}} \right. \\
& \left. -(\Gamma(\alpha+1))^{\frac{sq}{p}} \int_a^b \left(I^{\alpha}_{a+}f(x)\right)^{\frac{sq}{p}} dx \right].
\end{aligned}$$

Now we continue with a result involving the Hadamard-type fractional integrals.

Corollary 5.19 *Let* $0 < p \le q < \infty$, $s \ge 1$, $\alpha > 1 - \frac{p}{q}$, $f \ge 0$. *Then the following inequality holds*

$$0 \le \rho_3(s) \le \overline{H}_3(s) - M_3(s) \le \overline{H}_3(s),$$

where

$$\begin{aligned}
\rho_3(s) &= \frac{\alpha^{\frac{q}{p}}}{(\alpha-1)\frac{q}{p}+1}\left(\int_a^b (\log b - \log y)^{\alpha-1+\frac{p}{q}} f^s(y)\frac{dy}{y}\right)^{\frac{q}{p}} \\
&\quad -(\Gamma(\alpha+1))^{\frac{sq}{p}}\int_a^b (\log x - \log a)^{\frac{\alpha q}{p}(1-s)}\left(J^{\alpha}_{a+}f(x)\right)^{\frac{sq}{p}}\frac{dx}{x} - M_3(s),
\end{aligned}$$

$$\begin{aligned}
M_3(s) &= \frac{\alpha q(\Gamma(\alpha+1))^{s(\frac{q}{p}-1)}}{p}\int_a^b (\log x - \log a)^{\frac{\alpha(q-p)(1-s)}{p}}\left(J^{\alpha}_{a+}f(x)\right)^{s(\frac{q}{p}-1)} \\
&\quad \times \int_a^x \frac{r_3(x,y)}{(\log x - \log y)^{1-\alpha}}\frac{dy}{y}\frac{dx}{x},
\end{aligned}$$

$$\begin{aligned}
r_3(x,y) &= \left|\left|f^s(y) - \left(\frac{\Gamma(\alpha+1)}{(\log x - \log a)^{\alpha}}J^{\alpha}_{a+}f(x)\right)^s\right|\right. \\
&\quad \left. -s\left|\frac{\Gamma(\alpha+1)}{(\log x - \log a)^{\alpha}}J^{\alpha}_{a+}f(x)\right|^{s-1}\cdot\left|f(y) - \frac{\Gamma(\alpha+1)}{(\log x - \log a)^{\alpha}}J^{\alpha}_{a+}f(x)\right|\right|,
\end{aligned}$$

and

$$\begin{aligned}
\overline{H}_3(s) &= (\log b - \log a)^{\frac{\alpha q}{p}(1-s)}\left[\frac{\alpha^{\frac{q}{p}}(\log b - \log a)^{\frac{q(\alpha s-1)+p}{p}}}{(\alpha-1)\frac{q}{p}+1}\left(\int_a^b f^s(y)\frac{dy}{y}\right)^{\frac{q}{p}}\right. \\
&\quad \left. -(\Gamma(\alpha+1))^{\frac{sq}{p}}\int_a^b \left(J^{\alpha}_{a+}f(x)\right)^{\frac{sq}{p}}\frac{dx}{x}\right].
\end{aligned}$$

Now, we give the following result involving the Erdélyi-Kober type fractional integrals.

Theorem 5.11 *Let* $0 < p \le q < \infty$, $s \ge 1$, $\alpha > 1 - \frac{p}{q}$, $f \ge 0$ *and* ${}_2F_1(a,b;c;z)$ *denotes the hypergeometric function. Then the following inequality holds*

$$0 \le \rho_4(s) \le \overline{H}_4(s) - M_4(s) \le \overline{H}_4(s),$$

where

$$\begin{aligned}
\rho_4(s) &= \frac{\alpha^{\frac{q}{p}}\sigma^{\frac{q}{p}-1}}{(\alpha-1)\frac{q}{p}+1}\left(\int_a^b y^{\sigma-1}\,{}_2F_1(y)\,(b^\sigma-y^\sigma)^{\alpha-1+\frac{p}{q}} f^s(y)dy\right)^{\frac{q}{p}} \\
&\quad -(\Gamma(\alpha+1))^{\frac{sq}{p}}\int_a^b x^{\frac{\sigma\alpha sq}{p}+\sigma-1}\left((x^\sigma-a^\sigma)^\alpha\,{}_2F_1(x)\right)^{\frac{q(1-s)}{p}}\left(I^\alpha_{a+;\sigma;\eta}f(x)\right)^{\frac{sq}{p}}dx \\
&\quad -M_4(s)
\end{aligned}$$

$$\begin{aligned}
M_4(s) &= \frac{\alpha q\sigma(\Gamma(\alpha+1))^{\frac{sq}{p}}}{p}\int_a^b x^{\sigma\alpha s(\frac{q}{p}-1)+\sigma-\sigma\eta-1}\left((x^\sigma-a^\sigma)^\alpha\,{}_2F_1(x)\right)^{\frac{(q-p)(1-s)}{p}} \\
&\quad \times\left(I^\alpha_{a+;\sigma;\eta}f(x)\right)^{s(\frac{q}{p}-1)}\int_a^x \frac{r_4(x,y)y^{\sigma\eta+\sigma-1}}{(x^\sigma-y^\sigma)^{1-\alpha}}dydx,
\end{aligned}$$

$$\begin{aligned}
r_4(x,y) &= \left\|f^s(y)-\left(\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha\,{}_2F_1(x)}I^\alpha_{a+;\sigma;\eta}f(x)\right)^s\right| \\
&\quad -s\left|\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha\,{}_2F_1(x)}I^\alpha_{a+;\sigma;\eta}f(x)\right|^{s-1}\cdot \\
&\quad \left|f(y)-\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha\,{}_2F_1(x)}I^\alpha_{a+;\sigma;\eta}f(x)\right\|,
\end{aligned}$$

$$\begin{aligned}
\overline{H}_4(s) &= (b^\sigma-a^\sigma)^{\alpha\frac{q}{p}(1-s)}\left[\frac{\alpha^{\frac{q}{p}}\sigma^{\frac{q}{p}-1}b^{(\sigma-1)\frac{q}{p}}(b^\sigma-a^\sigma)^{\frac{q(\alpha s-1)+p}{p}}}{(\alpha-1)\frac{q}{p}+1}\left(\int_a^b {}_2F_1(y)f^s(y)dy\right)^{\frac{q}{p}}\right. \\
&\quad \left.-a^{\sigma\alpha s\frac{q}{p}+\sigma-1}(\Gamma(\alpha+1))^{\frac{sq}{p}}\int_a^b \left({}_2F_1(x)\right)^{\frac{q}{p}(1-s)}\left(I^\alpha_{a+;\sigma;\eta}f(x)\right)^{\frac{sq}{p}}dx\right],
\end{aligned}$$

$${}_2F_1(x) = {}_2F_1\left(-\eta,\alpha;\alpha+1;1-\left(\frac{a}{x}\right)^\sigma\right) \text{ and } {}_2F_1(y) = {}_2F_1\left(\eta,\alpha;\alpha+1;1-\left(\frac{b}{y}\right)^\sigma\right).$$

Proof. Similar to the proof of Theorems 3.12 and 5.10. □

Remark 5.12 Similar result can be obtained for the right sided Erdélyi-Kober type fractional integrals, but we omit the details here.

5.6.2 G. H. Hardy-type inequalities for fractional derivatives

In the following Theorem, we will construct a new inequality for the Canavati-type fractional derivative.

Theorem 5.12 *Let $0 < p \leq q < \infty$, $s \geq 1$, $\nu - \gamma > 1 - \frac{p}{q}$ and the assumptions in Lemma 1.4 be satisfied. Then for non-negative functions $D_a^\nu f$ and $D_a^\gamma f$, the following inequality holds*

$$0 \leq \rho_5(s) \leq \overline{H}_5(s) - M_5(s) \leq \overline{H}_5(s),$$

where

$$\begin{aligned}\rho_5(s) \;=\; & \frac{(\nu-\gamma)^{\frac{q}{p}}}{(\nu-\gamma-1)\frac{q}{p}+1}\left(\int_a^b (b-y)^{\nu-\gamma-1+\frac{p}{q}}(D_a^\nu f(y))^s dy\right)^{\frac{q}{p}} \\ & -(\Gamma(\nu-\gamma+1))^{\frac{sq}{p}}\int_a^b (x-a)^{\frac{(\nu-\gamma)q(1-s)}{p}}\,(D_a^\gamma f(x))^{\frac{sq}{p}}\,dx - M_5(s),\end{aligned}$$

$$\begin{aligned}M_5(s) \;=\; & \frac{q(\nu-\gamma)(\Gamma(\nu-\gamma+1))^{s(\frac{q}{p}-1)}}{p}\int_a^b (x-a)^{\frac{(\nu-\gamma)(q-p)(1-s)}{p}}\,(D_a^\gamma f(x))^{s(\frac{q}{p}-1)} \\ & \times \int_a^x r_5(x,y)(x-y)^{\nu-\gamma-1} dy dx,\end{aligned}$$

$$\begin{aligned}r_5(x,y) \;=\; & \left|\left|(D_a^\nu f(y))^s - \left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^\gamma}D_a^\gamma f(x)\right)^s\right|\right. \\ & \left. -s\left|\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}D_a^\gamma f(x)\right|^{s-1}\left|D_a^\nu f(y) - \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}D_a^\gamma f(x)\right|\right|,\end{aligned}$$

and

$$\begin{aligned}\overline{H}_5(s) \;=\; & (b-a)^{(\nu-\gamma)\frac{q}{p}(1-s)}\left[\frac{(\nu-\gamma)^{\frac{q}{p}}(b-a)^{\frac{q((\nu-\gamma)s-1)+p}{p}}}{(\nu-\gamma-1)\frac{q}{p}+1}\left(\int_a^b (D_a^\nu f(y))^s dy\right)^{\frac{q}{p}}\right. \\ & \left. -\,(\Gamma(\nu-\gamma+1))^{\frac{sq}{p}}\int_a^b (D_a^\gamma f(x))^{\frac{sq}{p}}dx\right].\end{aligned}$$

Proof. Similar to the proof of Theorems 3.9 and 5.10. □

Next, we give the result for the Caputo fractional derivative.

Theorem 5.13 *Let $0 < p \le q < \infty$, $s \ge 1$, $\nu - \gamma > 1 - \frac{p}{q}$ and the assumptions in Lemma 1.5 be satisfied. Then for non-negative functions $D^{\nu}_{*a}f$ and $D^{\gamma}_{*a}f$, the following inequality holds*

$$0 \le \rho_6(s) \le \overline{H}_6(s) - M_6(s) \le \overline{H}_6(s),$$

where

$$\begin{aligned}\rho_6(s) &= \frac{(\nu-\gamma)^{\frac{q}{p}}}{(\nu-\gamma-1)\frac{q}{p}+1}\left(\int_a^b (b-y)^{\nu-\gamma-1+\frac{p}{q}}(D^{\nu}_{*a}f(y))^s dy\right)^{\frac{q}{p}} \\ &\quad -(\Gamma(\nu-\gamma+1))^{\frac{sq}{p}}\int_a^b (x-a)^{\frac{(\nu-\gamma)q(1-s)}{p}}\left(D^{\gamma}_{*a}f(x)\right)^{\frac{sq}{p}}dx,\end{aligned}$$

$$\begin{aligned}M_6(s) &= \frac{q(\nu-\gamma)(\Gamma(\nu-\gamma+1))^{s(\frac{q}{p}-1)}}{p}\int_a^b (x-a)^{\frac{(\nu-\gamma)(q-p)(1-s)}{p}}\left(D^{\gamma}_{*a}f(x)\right)^{s(\frac{q}{p}-1)} \\ &\quad \times \int_a^x r_6(x,y)(x-y)^{\nu-\gamma-1}dy\,dx,\end{aligned}$$

$$\begin{aligned}r_6(x,y) &= \Bigg|\left|(D^{\nu}_{*a}f(y))^s - \left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\gamma}}D^{\gamma}_{*a}f(x)\right)^s\right| \\ &\quad - s\left|\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}D^{\gamma}_{*a}f(x)\right|^{s-1}\cdot\left|D^{\nu}_{*a}f(y) - \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}D^{\gamma}_{*a}f(x)\right|\Bigg|,\end{aligned}$$

and

$$\begin{aligned}\overline{H}_6(s) &= (b-a)^{(\nu-\gamma)\frac{q}{p}(1-s)}\left[\frac{(\nu-\gamma)^{\frac{q}{p}}(b-a)^{\frac{q((\nu-\gamma)s-1)+p}{p}}}{(\nu-\gamma-1)\frac{q}{p}+1}\left(\int_a^b (D^{\nu}_{*a}f(y))^s dy\right)^{\frac{q}{p}}\right. \\ &\quad \left. - (\Gamma(\nu-\gamma+1))^{\frac{sq}{p}}\int_a^b (D^{\gamma}_{*a}f(x))^{\frac{sq}{p}}dx\right].\end{aligned}$$

Proof. Similar to the proof of Theorem 3.11 and 5.10. □

Chapter 6

Bounds for Hardy-type differences

In this chapter we prove and discuss improvements and reverses of new weighted Hardy type inequalities with integral operators. We introduce a new Cauchy type mean and prove a monotonicity property of this mean.

6.1 The main results with applications

Lemma 6.1 *For $s \in \mathbb{R}$, let function the $\varphi_s : (0,\infty) \to \mathbb{R}$ be defined by*

$$\varphi_s(x) = \begin{cases} \frac{x^s}{s(s-1)}, & s \neq 0,1 \\ -\log x, & s = 0 \\ x \log x, & s = 1 \end{cases}. \tag{6.1}$$

Then $\varphi_s''(x) = x^{s-2}$, that is, φ_s is a convex function.

Lemma 6.2 *For $s \in \mathbb{R}$, let the function $\psi_s : \mathbb{R} \to [0,\infty)$ be defined by*

$$\psi_s(x) = \begin{cases} \frac{1}{s^2} e^{sx}, & s \neq 0 \\ \frac{1}{2} x^2, & s = 0 \end{cases}. \tag{6.2}$$

Then $\psi_s''(x) = e^{sx}$*, that is,* ψ_s *is a convex function.*

Our first result reads (see [36]):

Theorem 6.1 *Let* $(\Omega_1, \Sigma_1, \mu_1)$*,* $(\Omega_2, \Sigma_2, \mu_2)$ *be measure spaces with* σ*-finite measures and* $u : \Omega_1 \to \mathbb{R}$ *be a weight function. Let* I *be a compact interval of* $\mathbb{R}$*,* $h \in C^2(I)$*, and* $f : \Omega_2 \to \mathbb{R}$ *a measurable function such that* $Imf \subseteq I$*. Then there exists* $\eta \in I$ *such that*

$$\begin{aligned}
&\int_{\Omega_2} v(y)h(f(y))\,d\mu_2(y) - \int_{\Omega_1} u(x)h(A_k f(x))\,d\mu_1(x) \\
&= \frac{h''(\eta)}{2}\left[\int_{\Omega_2} v(y)f^2(y)\,d\mu_2(y) - \int_{\Omega_1} u(x)(A_k f(x))^2\,d\mu_1(x)\right], \qquad (6.3)
\end{aligned}$$

where $A_k f$ *and* v *are defined by* (2.15) *and* (2.17)*.*

Proof. Since h'' is continuous on the segment $I \subseteq \mathbb{R}$, there exist $m = \min_{x\in I} h''(x)$ and $M = \max_{x\in I} h''(x)$. Then by applying Theorem 2.5 on functions Φ_1, Φ_2 from Remark 1.5, the following two inequalities hold:

$$\int_{\Omega_2} v(y)\Phi_1(f(y))\,d\mu_2(y) \geq \int_{\Omega_1} u(x)\Phi_1(A_k f(x))\,d\mu_1(x),$$

$$\int_{\Omega_2} v(y)\Phi_2(f(y))\,d\mu_2(y) \geq \int_{\Omega_1} u(x)\Phi_2(A_k f(x))\,d\mu_1(x).$$

It follows,

$$\begin{aligned}
&\frac{m}{2}\left\{\int_{\Omega_2} v(y)f^2(y)\,d\mu_2(y) - \int_{\Omega_1} u(x)(A_k f(x))^2\,d\mu_1(x)\right\} \\
&\leq \int_{\Omega_2} v(y)h(f(y))\,d\mu_2(y) - \int_{\Omega_1} u(x)h(A_k f(x))\,d\mu_1(x) \\
&\leq \frac{M}{2}\left\{\int_{\Omega_2} v(y)f^2(y)\,d\mu_2(y) - \int_{\Omega_1} u(x)(A_k f(x))^2\,d\mu_1(x)\right\}.
\end{aligned}$$

The function h'' is continuous, so $Imh'' = [m, M]$. Therefore, there exists $\eta \in I$ such that (6.3) holds. □

Theorem 6.2 *Let the conditions of Theorem* 2.5 *be satisfied and* φ_s *be defined by* (6.1)*. Let* f *be a positive function. Then the function* $\xi : \mathbb{R} \to [0, \infty)$ *defined by*

$$\xi(s) = \int_{\Omega_2} v(y)\varphi_s(f(y))\,d\mu_2(y) - \int_{\Omega_1} u(x)\varphi_s(A_k f(x))\,d\mu_1(x), \qquad (6.4)$$

is exponentially convex.

Proof. Let us prove first that ξ is continuous on $\mathbb{R}$. Obviously, it is continuous on $\mathbb{R}\setminus\{0,1\}$, which easily follows from the Lebesgue monotone convergence theorem. Suppose $s\to 0$:

$$\begin{aligned}\lim_{s\to 0}\xi(s) &= \lim_{s\to 0}\int_{\Omega_2} v(y)\frac{f^s(y)}{s(s-1)}\,d\mu_2(y)-\int_{\Omega_1} u(x)\frac{(A_kf(x))^s}{s(s-1)}\,d\mu_1(x)\\ &= \lim_{s\to 0}\frac{\int_{\Omega_2} v(y)f^s(y)\,d\mu_2(y)-\int_{\Omega_1} u(x)(A_kf(x))^s\,d\mu_1(x)}{s(s-1)}\end{aligned} \tag{6.5}$$

Since

$$\lim_{s\to 0}\int_{\Omega_2} v(y)f^s(y)\,d\mu_2(y)-\int_{\Omega_1} u(x)(A_kf(x))^s\,d\mu_1(x)=0,$$

by L'Hospital's rule, the limit in (6.5) is equal to

$$\begin{aligned}\lim_{s\to 0}\xi(s)&=\lim_{s\to 0}\frac{\int_{\Omega_2} v(y)f^s(y)\log f(y)\,d\mu_2(y)-\int_{\Omega_1} u(x)(A_kf(x))^s\log(A_kf(x))\,d\mu_1(x)}{2s-1}\\ &=-\int_{\Omega_2} v(y)\log f(y)\,d\mu_2(y)+\int_{\Omega_1} u(x)\log(A_kf(x))\,d\mu_1(x)\\ &=\int_{\Omega_2} v(y)\varphi_0(f(y))\,d\mu_2(y)-\int_{\Omega_1} u(x)\varphi_0(A_kf(x))\,d\mu_1(x)=\xi(0)\end{aligned}$$

In the same way, for $s=1$ we get

$$\lim_{s\to 1}\xi(s)=\int_{\Omega_2} v(y)f(y)\log f(y)\,d\mu_2(y)-\int_{\Omega_1} u(x)A_kf(x)\log(A_kf(x))\,d\mu_1(x)=\xi(1)$$

Hence, ξ is continuous on $\mathbb{R}$. Let $n\in\mathbb{N}$, $t_i\in\mathbb{R}$, and $p_i\in\mathbb{R}$, $i=1,2,\ldots,n$ be arbitrary. Denote

$$p_{ij}=\frac{p_i+p_j}{2},$$

and define the function $\Phi\colon\mathbb{R}_+\to\mathbb{R}$ by

$$\Phi(x)=\sum_{i,j}^{n} t_it_j\varphi_{p_{ij}}(x)$$

Then

$$\Phi''(x)=\sum_{i,j}^{n} t_it_jx^{p_{ij}-2}=\left(\sum_{i}^{n} t_ix^{\frac{p_i}{2}-1}\right)^2\geq 0,$$

so Φ is a convex function on $\mathbb{R}_+$.

Now, we can apply the result from Theorem 2.5 to the function Φ defined above, and obtain

$$\sum_{i,j}^{n} t_it_j\xi(p_{ij})\geq 0$$

concluding positive semi-definiteness. Since ξ is continuous, it is exponentially convex function. □

Remark 6.1 The function ξ being exponentially convex is also a log-convex function. Then, by Remark 1.2, the following inequality holds

$$[\xi(r)]^{q-p} \leq [\xi(p)]^{q-r}[\xi(q)]^{r-p} \tag{6.6}$$

for every choice of $p,q,r \in \mathbb{R}$ such that $p<r<q$.

As a consequence of Theorem 6.2, we prove an improvement and reverse of strengthened Hardy's inequality and its dual.

Theorem 6.3 *Let $k,b,\gamma \in \mathbb{R}$ be such that $k \neq 1$, $b>0$ and $\gamma>0$, let f be a non-negative function, and let $p \in \mathbb{R}\backslash\{0,1\}$.*

(i) If $\frac{p}{k-1}>0$ and $r<p<t$, then

$$\begin{aligned}
&\frac{1}{p(p-1)}\left\{\left(\frac{p}{k-1}\right)^p \int_0^b \left[1-\left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right]^\gamma x^{p-k} f^p(x)dx\right. \\
&\qquad \left. -\gamma^p \int_0^b x^{-k}\left(\int_0^x \left[1-\left(\frac{y}{x}\right)^{\frac{k-1}{p}}\right]^{\gamma-1} f(y)dy\right)^p dx\right\} \\
&\leq \left(\frac{p}{k-1}\right)^p [R(r)]^{\frac{t-p}{t-r}}[R(t)]^{\frac{p-r}{t-r}}.
\end{aligned} \tag{6.7}$$

If $p<t<r$ or $t<r<p$, then (6.7) holds with reversed sign of inequality, where

$$\begin{aligned}
R(r) \;=\; &\int_0^b \left[1-\left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right]^\gamma \varphi_r\left(x^{\frac{p-k+1}{p}} f(x)\right)\frac{dx}{x} \\
&-\int_0^b \varphi_r\left(\frac{\gamma(k-1)}{p} x^{\frac{1-k}{p}} \int_0^x \left[1-\left(\frac{y}{x}\right)^{\frac{k-1}{p}}\right]^{\gamma-1} f(y)dy\right)\frac{dx}{x}.
\end{aligned}$$

(ii) If $\frac{p}{1-k}>0$ and $r<p<t$, then

$$\begin{aligned}
&\frac{1}{p(p-1)}\left\{\left(\frac{p}{1-k}\right)^p \int_b^\infty \left[1-\left(\frac{b}{x}\right)^{\frac{1-k}{p}}\right]^\gamma x^{p-k} f^p(x)dx\right. \\
&\qquad \left. -\gamma^p \int_b^\infty x^{-k}\left(\int_x^\infty \left[1-\left(\frac{x}{y}\right)^{\frac{k-1}{p}}\right]^{\gamma-1} f(y)dy\right)^p dx\right\} \\
&\leq \left(\frac{p}{1-k}\right)^p [W(t)]^{\frac{p-r}{t-r}}[W(r)]^{\frac{t-p}{t-r}}.
\end{aligned} \tag{6.8}$$

If $p < t < r$ or $t < r < p$, then (6.8) *holds with reversed sign of inequality, where*

$$\begin{aligned} W(r) \;=\; & \int_b^\infty \left[1-\left(\frac{b}{x}\right)^{\frac{1-k}{p}}\right]^\gamma \varphi_r\left(x^{\frac{p-k+1}{p}} f(x)\right)\frac{dx}{x} \\ & -\int_b^\infty \varphi_r\left(\frac{\gamma(1-k)}{p}x^{\frac{1-k}{p}}\int_x^\infty \left[1-\left(\frac{x}{y}\right)^{\frac{1-k}{p}}\right]^{\gamma-1} f(y)dy\right)\frac{dx}{x}. \end{aligned}$$

Proof. The proof follows from Theorem 6.2 and Remark 6.1 by choosing $\Omega_1 = \Omega_2 = (0,b)$ and replacing μ_1 and μ_2 by the Lebesgue measure. Choosing

$$k(x,y) = \begin{cases} \frac{\gamma}{x^\gamma}(x-y)^{\gamma-1}, & 0<y\le x<b, \gamma>0, \\ 0, & x<y \end{cases}$$

and $u(x) = \frac{1}{x}$, we obtain $K(x) = 1$, $v(y) = \frac{1}{y}\left(1-\frac{y}{b}\right)^\gamma$ and Riemann-Liouville operator

$$R_\gamma f(x) = A_k f(x) = \frac{\gamma}{x^\gamma}\int_0^x (x-y)^{\gamma-1} f(y)dy.$$

Then (6.4) becomes

$$F(p) = \int_0^b \left(1-\frac{x}{b}\right)^\gamma \varphi_p(f(x))\frac{dx}{x} - \int_0^b \varphi_p(R_\gamma f(x))\frac{dx}{x} \tag{6.9}$$

and (6.6) becomes

$$[F(p)]^{t-r} \le [F(r)]^{t-p}\,[F(t)]^{p-r} \tag{6.10}$$

for every choice $r,p,t \in \mathbb{R}$, such that $r<p<t$. We know that $F(p)$ is a log-convex function. To obtain (6.7) replace the parameter b in (6.9) by $b^{(k-1)/p}$ and choose for f the function $x \mapsto f(x^{p/(k-1)})x^{p/(k-1)-1}$.

Then, after suitable variable changes it follows

$$\begin{aligned} F(p) \;=\; & \frac{k-1}{p}\left\{\int_0^b \left[1-\left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right]^\gamma \varphi_p\left(x^{\frac{p-k+1}{p}} f(x)\right)\frac{dx}{x}\right. \\ & \left.-\int_0^b \varphi_p\left(\frac{\gamma(k-1)}{p}x^{\frac{1-k}{p}}\int_0^x \left[1-\left(\frac{y}{x}\right)^{\frac{k-1}{p}}\right]^{\gamma-1} f(y)dy\right)\frac{dx}{x}\right\}. \end{aligned}$$

Now (6.10) reduces to

$$\int_0^b \left[1-\left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right]^\gamma \varphi_p\left(x^{\frac{p-k+1}{p}} f(x)\right)\frac{dx}{x}$$
$$-\int_0^b \varphi_p\left(\frac{\gamma(k-1)}{p} x^{\frac{1-k}{p}} \int_0^x \left[1-\left(\frac{y}{x}\right)^{\frac{k-1}{p}}\right]^{\gamma-1} f(y)dy\right)\frac{dx}{x}$$
$$\leq [R(r)]^{\frac{t-p}{t-r}} [R(t)]^{\frac{p-r}{t-r}}.$$

For $p \in \mathbb{R}\backslash\{0,1\}$ we get (6.7).

By taking substitutions $r \to t$, $p \to r$, $t \to p$ or $r \to p$, $p \to t$, $t \to r$ in (6.10), we get reversed sign of inequality in (6.7).

To prove (6.8), let us take $\Omega_1 = \Omega_2 = (b,\infty)$ and let μ_1, μ_2 be the Lebesgue measure. Choosing

$$k(x,y) = \begin{cases} \frac{\gamma(y-x)^{\gamma-1}x}{y^{\gamma+1}}, & b < x \leq y < \infty,\ \gamma > 0, \\ 0, & y < x \end{cases}$$

and $u(x) = \frac{1}{x}$ we obtain $K(x) = 1$, $v(y) = \frac{1}{y}\left(1-\frac{b}{y}\right)^\gamma$ and Weyl's operator

$$W_\gamma f(x) = A_k f(x) = \gamma x \int_x^\infty (y-x)^{\gamma-1} f(y)\frac{dy}{y^{\gamma+1}}.$$

Then (6.4) becomes

$$\tilde{F}(p) = \int_b^\infty \left(1-\frac{b}{x}\right)^\gamma \varphi_p(f(x))\frac{dx}{x} - \int_b^\infty \varphi_p(W_\gamma f(x))\frac{dx}{x} \tag{6.11}$$

and (6.6) becomes

$$[\tilde{F}(p)]^{t-r} \leq \left[\tilde{F}(r)\right]^{t-p} \left[\tilde{F}(t)\right]^{p-r} \tag{6.12}$$

for every choice $r,p,t \in \mathbb{R}$, such that $r < p < t$. We know that $\tilde{F}(p)$ is log-convex. To obtain (6.8) it is sufficient to replace the parameter b in (6.11) by $b^{(1-k)/p}$ and replace function f by $x \mapsto f(x^{p/(1-k)})x^{p/(1-k)+1}$. Then it follows

$$\begin{aligned}\tilde{F}(p) &= \frac{1-k}{p}\Bigg\{\int_b^\infty \left[1-\left(\frac{b}{x}\right)^{\frac{1-k}{p}}\right]^\gamma \varphi_p\left(x^{\frac{p-k+1}{p}} f(x)\right)\frac{dx}{x} \\ &\quad - \int_b^\infty \varphi_p\left(\frac{\gamma(1-k)}{p} x^{\frac{1-k}{p}} \int_x^\infty \left[1-\left(\frac{x}{y}\right)^{\frac{1-k}{p}}\right]^{\gamma-1} f(y)dy\right)\frac{dx}{x}\Bigg\}.\end{aligned}$$

From here (6.12) reduces to

$$\int_b^\infty \left[1-\left(\frac{b}{x}\right)^{\frac{1-k}{p}}\right]^\gamma \varphi_p\left(x^{\frac{p-k+1}{p}} f(x)\right)\frac{dx}{x}$$
$$-\int_b^\infty \varphi_p\left(\frac{\gamma(1-k)}{p} x^{\frac{1-k}{p}} \int_x^\infty \left[1-\left(\frac{x}{y}\right)^{\frac{1-k}{p}}\right]^{\gamma-1} f(y)dy\right)\frac{dx}{x}$$
$$\le [W(r)]^{\frac{t-p}{t-r}}\,[W(t)]^{\frac{p-r}{t-r}}.$$

For $p \in \mathbb{R}\backslash\{0,1\}$ we get (6.8).

By taking substitutions $r \to t$, $p \to r$, $t \to p$ and $r \to p$, $p \to t$, $t \to r$ in (6.12), we get reversed sign of inequality in (6.8). □

Remark 6.2 For $\gamma = 1$ in (6.7) and (6.8), we obtain the result from Theorem 4.1 in [53].

Let us discuss an improvement and reverse of the classical Hardy-Hilbert's inequality (2.1).

Theorem 6.4 *Let the assumptions of Theorem* 6.2 *be satisfied and* $H : \mathbb{R} \to [0,\infty)$ *be a function defined by*

$$H(s) = \frac{1}{s(s-1)}\left[\int_0^\infty f^s(y)\,dy - \left(\frac{\sin\frac{\pi}{s}}{\pi}\right)^s \int_0^\infty\left(\int_0^\infty \frac{f(y)}{x+y}dy\right)^s dx\right]. \tag{6.13}$$

Then

$$H(s) \le \left\{\frac{1}{r(r-1)}\left[\int_0^\infty f^r(y)\,dy - \left(\frac{\sin\frac{\pi}{s}}{\pi}\right)^r \int_0^\infty\left(\int_0^\infty \frac{f(y)}{x+y}dy\right)^r x^{\frac{r}{s}-1}dx\right]\right\}^{\frac{t-s}{t-r}} \times$$
$$\times\left\{\frac{1}{t(t-1)}\left[\int_0^\infty f^t(y)\,dy - \left(\frac{\sin\frac{\pi}{s}}{\pi}\right)^t \int_0^\infty\left(\int_0^\infty \frac{f(y)}{x+y}dy\right)^t x^{\frac{t}{s}-1}dx\right]\right\}^{\frac{s-r}{t-r}} \tag{6.14}$$

for $1<r<s<t$, *and*

$$H(s) \ge \left\{\frac{1}{r(r-1)}\left[\int_0^\infty f^r(y)\,dy - \left(\frac{\sin\frac{\pi}{s}}{\pi}\right)^r \int_0^\infty\left(\int_0^\infty \frac{f(y)}{x+y}dy\right)^r x^{\frac{r}{s}-1}dx\right]\right\}^{\frac{t-s}{t-r}} \times$$
$$\times\left\{\frac{1}{t(t-1)}\left[\int_0^\infty f^t(y)\,dy - \left(\frac{\sin\frac{\pi}{s}}{\pi}\right)^t \int_0^\infty\left(\int_0^\infty \frac{f(y)}{x+y}dy\right)^t x^{\frac{t}{s}-1}dx\right]\right\}^{\frac{s-r}{t-r}} \tag{6.15}$$

for $1<s<t<r$ *or* $1<t<r<s$.

Proof. Let us take $\Omega_1 = \Omega_2 = (0,\infty)$ and the Lebesgue measure for μ_1 and μ_2 in Theorem 6.2 and Remark 6.1. Choosing $k(x,y) = \frac{(\frac{y}{x})^{-1/s}}{x+y}$, $s > 1$ and $u(x) = \frac{1}{x}$, we obtain $K(x) = K = \frac{\pi}{\sin(\pi/s)}$, $v(y) = \frac{1}{y}$. Replace $f(t)t^{-1/s}$ with $f(t)$, then we get (6.13). By Theorem 6.2 (6.13) is a log-convex function. Now, for $1 < r < s < t$ apply Remark 1.2 on (6.13) and we obtain (6.14).

If in (6.14) we take substitions $r \to t$, $s \to r$, $t \to s$ or $r \to s$, $s \to t$, $t \to r$, (6.15) follows. □

Using the function ψ_s instead of φ_s, the following result follows.

Theorem 6.5 *Let the conditions of Theorem* 2.5 *be satisfied,* ψ_s *be defined by* (6.2) *and let* f *be a positive function. Then the function* $\zeta : \mathbb{R} \to [0,\infty)$ *defined by*

$$\zeta(s) = \int_{\Omega_2} v(y)\psi_s(f(y))\,d\mu_2(y) - \int_{\Omega_1} u(x)\psi_s(A_k f(x))\,d\mu_1(x), \tag{6.16}$$

is exponentially convex.

Proof. The proof is analogous to the proof of Theorem 6.2. We just have to show that ζ is a continuous function on $\mathbb{R}$. Obviously, it is continuous on $\mathbb{R}\setminus\{0\}$, which easily follows from the Lebesgue monotone convergence theorem. Suppose $s \to 0$:

$$\begin{aligned}\lim_{s\to 0}\zeta(s) &= \lim_{s\to 0}\int_{\Omega_2} v(y)\frac{1}{s^2}e^{sf(y)}\,d\mu_2(y) - \int_{\Omega_1} u(x)\frac{1}{s^2}e^{sA_k f(x)}\,d\mu_1(x)\\ &= \lim_{s\to 0}\frac{\int_{\Omega_2} v(y)e^{sf(y)}\,d\mu_2(y) - \int_{\Omega_1} u(x)e^{sA_k f(x)}\,d\mu_1(x)}{s^2}. \end{aligned}\tag{6.17}$$

Since

$$\lim_{s\to 0}\int_{\Omega_2} v(y)e^{sf(y)}\,d\mu_2(y) - \int_{\Omega_1} u(x)e^{sA_k f(x)}\,d\mu_1(x) = 0,$$

we can use L'Hospital rule and obtain that (6.17) is equal to

$$\lim_{s\to 0}\zeta(s) = \lim_{s\to 0}\frac{\int_{\Omega_2} v(y)e^{sf(y)}f(y)\,d\mu_2(y) - \int_{\Omega_1} u(x)e^{sA_k f(x)}A_k f(x)\,d\mu_1(x)}{2s}.$$

We know that

$$\lim_{s\to 0}\int_{\Omega_2} v(y)e^{sf(y)}f(y)\,d\mu_2(y) - \int_{\Omega_1} u(x)e^{sA_k f(x)}A_k f(x)\,d\mu_1(x) = 0,$$

so we use L'Hospital rule once again and obtain the following

$$\begin{aligned}\lim_{s\to 0}\zeta(s) &= \lim_{s\to 0}\frac{\int_{\Omega_2} v(y)e^{sf(y)}f^2(y)\,d\mu_2(y) - \int_{\Omega_1} u(x)e^{sA_k f(x)}(A_k f(x))^2\,d\mu_1(x)}{2}\\ &= \frac{1}{2}\int_{\Omega_2} v(y)f^2(y)\,d\mu_2(y) + \frac{1}{2}\int_{\Omega_1} u(x)(A_k f(x))^2\,d\mu_1(x)\\ &= \int_{\Omega_2} v(y)\psi_0(f(y))\,d\mu_2(y) - \int_{\Omega_1} u(x)\psi_0(A_k f(x))\,d\mu_1(x) = \zeta(0).\end{aligned}$$

The proof is completed. □

Remark 6.3 It is proved that the function ζ is exponentially convex, hence it is also a log-convex function. Then, by Remark 1.2 the following inequality holds

$$[\zeta(r)]^{q-p} \le [\zeta(p)]^{q-r}[\zeta(q)]^{r-p} \tag{6.18}$$

for every choice of $p,q,r \in \mathbb{R}$, such that $p<r<q$.

Now we discuss an improvement and reverse of Polya-Knopp's inequality (0.2) and of its dual.

Theorem 6.6 *Let f be a positive function, and ψ_s be defined by* (6.2).
(*i*) *If $r<1<t$, then*

$$e\int_0^b \left(1-\frac{x}{b}\right) f(x)dx - \int_0^b \exp\left(\frac{1}{x}\int_0^x \log f(y)\,dy\right) dx \le e\,[P(r)]^{\frac{t-1}{t-r}}\,[P(t)]^{\frac{1-r}{t-r}}. \tag{6.19}$$

If $1<t<r$ or $t<r<1$, then (6.19) *holds with reversed sign of inequality, where*

$$P(s) = \int_0^b \left(1-\frac{x}{b}\right)\psi_s(\log(xf(x)))\frac{dx}{x} - \int_0^b \psi_s\left(\frac{1}{x}\int_0^x \log(yf(y))dy\right)\frac{dx}{x}.$$

(*ii*) *If $r<1<t$, then*

$$\int_b^\infty \left(1-\frac{b}{x}\right) f(x)dx - e\int_b^\infty \exp\left(x\int_x^\infty \log f(y)\,\frac{dy}{y^2}\right) dx \le [\tilde{P}(r)]^{\frac{t-1}{t-r}}\,[\tilde{P}(t)]^{\frac{1-r}{t-r}}. \tag{6.20}$$

If $1<t<r$ or $t<r<1$, then (6.20) *holds with reversed sign of inequality, where*

$$\tilde{P}(s) = \int_b^\infty \left(1-\frac{b}{x}\right)\psi_s(\log(xf(x)))\frac{dx}{x} - \int_b^\infty \psi_s\left(\int_x^\infty x\log(yf(y))\frac{dy}{y^2}\right)\frac{dx}{x}.$$

Proof. The proof follows from Theorem 6.5 and Remark 6.3 by choosing $\Omega_1=\Omega_2=(0,b)$ and replacing μ_1, μ_2 by the Lebesgue measure. Let

$$k(x,y) = \begin{cases} \frac{1}{x},\ 0<y\le x<b, \\ 0,\ x<y \end{cases}$$

and $u(x)=\frac{1}{x}$, then we obtain $K(x)=1$, $v(y)=\frac{1}{y}\left(1-\frac{y}{b}\right)$ and Hardy's operator (the Riemann-Liouville operator for $\gamma=1$)

$$R_1 f(x) = A_k f(x) = \frac{1}{x}\int_0^x f(y)dy.$$

Then (6.16) becomes

$$P(s)=\int_0^b \left(1-\frac{x}{b}\right)\psi_s(f(x))\frac{dx}{x}-\int_0^b \psi_s(R_1 f(x))\frac{dx}{x}$$

and (6.18) becomes

$$[P(s)]^{t-r}\le [P(r)]^{t-s}\,[P(t)]^{s-r} \tag{6.21}$$

for every choice $r,s,t\in\mathbb{R}$, such that $r<s<t$. We know that $P(s)$ is a log-convex function. To obtain (6.19) choose for f the function $x\mapsto \log(xf(x))$. Then we obtain

$$P(s)=\int_0^b \left(1-\frac{x}{b}\right)\psi_s(\log(xf(x))\frac{dx}{x}-\int_0^b \psi_s\left(\frac{1}{x}\int_0^x \log(yf(y))dy\right)\frac{dx}{x}.$$

From here for $s=1$ (6.21) reduces to (6.19).

By substitution $r\to t$, $1\to r$, $t\to 1$ or $r\to 1$, $1\to t$, $t\to r$ in (6.21), we get reversed sign of inequality in (6.19).

To prove (6.20), choose $\Omega_1=\Omega_2=(b,\infty)$ and replace μ_1, μ_2 by the Lebesgue measure. Let

$$k(x,y)=\begin{cases}\frac{x}{y^2}, & b<x\le y<\infty,\\ 0, & y<x\end{cases}$$

and $u(x)=\frac{1}{x}$, then we obtain $K(x)=1$, $v(y)=\frac{1}{y}\left(1-\frac{b}{y}\right)$ and dual Hardy's operator (Weyl's operator for $\gamma=1$)

$$W_1 f(x)=A_k f(x)=x\int_x^\infty f(y)\frac{dy}{y^2}.$$

Then (6.16) becomes

$$\tilde{P}(s)=\int_b^\infty \left(1-\frac{b}{x}\right)\psi_s(f(x))\frac{dx}{x}-\int_b^\infty \psi_s(W_1 f(x))\frac{dx}{x}$$

and (6.18) becomes

$$[\tilde{P}(s)]^{t-r}\le \left[\tilde{P}(r)\right]^{t-s}\left[\tilde{P}(t)\right]^{s-r} \tag{6.22}$$

for every choice $r,s,t\in\mathbb{R}$, such that $r<s<t$. We know that $\tilde{P}(s)$ is a log-convex function. To obtain (6.20) choose for f the function $x\mapsto \log(xf(x))$. Then we obtain

$$\tilde{P}(s)=\int_b^\infty \left(1-\frac{x}{b}\right)\psi_s(\log(xf(x))\frac{dx}{x}-\int_b^\infty \psi_s\left(x\int_x^\infty \log(yf(y))\frac{dy}{y^2}\right)\frac{dx}{x}.$$

From here, for $s=1$ (6.22) reduces to (6.20).

By substitution $r\to t$, $s\to r$, $t\to s$ or $r\to s$, $s\to t$, $t\to r$ in (6.22) we get reversed sign of inequality in (6.20), so the proof is completed. □

Corollary 6.1 *Let φ_s be defined by* (6.1) *and l and u be non-negative measurable functions on $\mathbb{R}^n_+$ such that $0 < L(\mathbf{x}) = \mathbf{x}^{\mathbf{1}} \int_{\mathbb{R}^n_+} l(\mathbf{y})\,d\mathbf{y} < \infty$ for all $\mathbf{x} \in \mathbb{R}^n_+$, and that the function $\mathbf{x} \mapsto u(\mathbf{x})\frac{l\left(\frac{\mathbf{y}}{\mathbf{x}}\right)}{L(\mathbf{x})}$ is integrable on $\mathbb{R}^n_+$ for each fixed $\mathbf{y} \in \mathbb{R}^n_+$. Let the function v be defined on $\mathbb{R}^n_+$ by*

$$v(\mathbf{y}) = \int_{\mathbb{R}^n_+} u(\mathbf{x}) \frac{l\left(\frac{\mathbf{y}}{\mathbf{x}}\right)}{L(\mathbf{x})}\,d\mathbf{x}$$

and

$$A_l f(\mathbf{x}) = \frac{1}{L(\mathbf{x})} \int_{\mathbb{R}^n_+} l\left(\frac{\mathbf{y}}{\mathbf{x}}\right) f(\mathbf{y})\,d\mathbf{y}.$$

Then

$$G(s) = \int_{\mathbb{R}^n_+} v(\mathbf{y})\varphi_s(f(\mathbf{y}))d\mathbf{y} - \int_{\mathbb{R}^n_+} u(\mathbf{x})\varphi_s(A_l f(\mathbf{x}))d\mathbf{x}$$

is a log-convex function, that is

$$G(s) \leq [G(r)]^{\frac{t-s}{t-r}} [G(t)]^{\frac{s-r}{t-r}}$$

for $r < s < t$, and

$$G(s) \geq [G(r)]^{\frac{t-s}{t-r}} [G(t)]^{\frac{s-r}{t-r}}$$

for $s < t < r$ or $t < r < s$.

Corollary 6.2 *Let ψ_s be defined by* (6.2)*, and the other assumptions of Corollary* 6.1 *be satisfied. Then the following inequality holds:*

$$\tilde{G}(s) \leq \left[\tilde{G}(r)\right]^{\frac{t-s}{t-r}} \left[\tilde{G}(t)\right]^{\frac{s-r}{t-r}}$$

for $r < s < t$, and

$$\tilde{G}(s) \geq \left[\tilde{G}(r)\right]^{\frac{t-s}{t-r}} \left[\tilde{G}(t)\right]^{\frac{s-r}{t-r}}$$

for $s < t < r$ or $t < r < s$, where

$$\tilde{G}(s) = \int_{\mathbb{R}^n_+} v(\mathbf{y})\psi_s(f(\mathbf{y}))d\mathbf{y} - \int_{\mathbb{R}^n_+} u(\mathbf{x})\psi_s(A_l f(\mathbf{x}))d\mathbf{x}.$$

6.2 Cauchy means

Theorem 6.7 *Assume that all conditions of Theorem* 6.2 *are satisfied. Let I be a compact interval in $\mathbb{R}$ and $g,h \in C^2(I)$ such that $h''(x) \neq 0$ for every $x \in I$. Let $f : \Omega_2 \to \mathbb{R}$ be a measurable function such that $Im f \subseteq I$ and*

$$\int_{\Omega_2} v(y)h(f(y))\,d\mu_2(y) - \int_{\Omega_1} u(x)h(A_k f(x))\,d\mu_1(x) \neq 0. \tag{6.23}$$

Then there exists $\eta \in I$ such that

$$\frac{g''(\eta)}{h''(\eta)} = \frac{\int_{\Omega_2} v(y)g(f(y))\,d\mu_2(y) - \int_{\Omega_1} u(x)g(A_k f(x))\,d\mu_1(x)}{\int_{\Omega_2} v(y)h(f(y))\,d\mu_2(y) - \int_{\Omega_1} u(x)h(A_k f(x))\,d\mu_1(x)}.$$

Proof. Let us denote

$$\begin{aligned} c_1 &= \int_{\Omega_2} v(y)h(f(y))\,d\mu_2(y) - \int_{\Omega_1} u(x)h(A_k f(x))\,d\mu_1(x), \\ c_2 &= \int_{\Omega_2} v(y)g(f(y))\,d\mu_2(y) - \int_{\Omega_1} u(x)g(A_k f(x))\,d\mu_1(x). \end{aligned}$$

Now, apply (6.3) to the function $c_1 g - c_2 h$. The following equality follows:

$$\begin{aligned} c_1 &\left[\int_{\Omega_2} v(y)g(f(y))\,d\mu_2(y) - \int_{\Omega_1} u(x)g(A_k f(x))\,d\mu_1(x)\right] \\ &\quad -c_2\left[\int_{\Omega_2} v(y)h(f(y))\,d\mu_2(y) - \int_{\Omega_1} u(x)h(A_k f(x))\,d\mu_1(x)\right] \\ &= \frac{c_1 g''(\xi) - c_2 h''(\xi)}{2}\left[\int_{\Omega_2} v(y)f^2(y)\,d\mu_2(y) - \int_{\Omega_1} u(x)(A_k f(x))^2\,d\mu_1(x)\right] \end{aligned} \tag{6.24}$$

After a short calculation, it is easy to see that the left-hand side of (6.24) is equal to 0 and, thus, the right-hand side as well.
If we apply Theorem 6.1 to the function h we get the following

$$\begin{aligned} &\int_{\Omega_2} v(y)h(f(y))\,d\mu_2(y) - \int_{\Omega_1} u(x)h(A_k f(x))\,d\mu_1(x) \\ &= \frac{h''(\eta)}{2}\left[\int_{\Omega_2} v(y)f^2(y)\,d\mu_2(y) - \int_{\Omega_1} u(x)(A_k f(x))^2\,d\mu_1(x)\right]. \end{aligned}$$

From (6.23) we conclude that the term in the square brackets on the right-hand side of (6.24) is not equal to 0. It follows that $c_1 g''(\xi) - c_2 h''(\xi) = 0$, so the proof is completed. □

Corollary 6.3 *Assume that all conditions of Theorem* 6.7 *are satisfied. If $f : \Omega_2 \to I$ is a measurable function, then for $p \neq s$, $p, s \neq 0, 1$ there exists $\eta \in I$ such that*

$$\eta^{p-s} = \frac{s(s-1)[\int_{\Omega_2} v(y)f^p(y)\,d\mu_2(y) - \int_{\Omega_1} u(x)(A_k f(x))^p\,d\mu_1(x)]}{p(p-1)[\int_{\Omega_2} v(y)f^s(y)\,d\mu_2(y) - \int_{\Omega_1} u(x)(A_k f(x))^s\,d\mu_1(x)]}. \tag{6.25}$$

Proof. Apply Theorem 6.7 to $g(x) = \frac{x^p}{p(p-1)}$, $h(x) = \frac{x^s}{s(s-1)}$, $p \neq s$, $p, s \neq 0, 1$ and (6.25) follows.

Remark 6.4 Notice that $g''(x) = x^{p-2}$ and $h''(x) = x^{s-2}$, so $\frac{g''}{h''}$ is invertible. Then from (6.25) we obtain

$$\inf_{t\in\Omega_2} f(t) \leq \left(\frac{s(s-1)[\int\limits_{\Omega_2} v(y) f^p(y)\, d\mu_2(y) - \int\limits_{\Omega_1} u(x)(A_k f(x))^p\, d\mu_1(x)]}{p(p-1)[\int\limits_{\Omega_2} v(y) f^s(y)\, d\mu_2(y) - \int\limits_{\Omega_1} u(x)(A_k f(x))^s\, d\mu_1(x)]} \right)^{\frac{1}{p-s}} \leq \sup_{t\in\Omega_2} f(t).$$

So,

$$M_{p,s}(f;u) = \left(\frac{s(s-1)[\int\limits_{\Omega_2} v(y) f^p(y)\, d\mu_2(y) - \int\limits_{\Omega_1} u(x)(A_k f(x))^p\, d\mu_1(x)]}{p(p-1)[\int\limits_{\Omega_2} v(y) f^s(y)\, d\mu_2(y) - \int\limits_{\Omega_1} u(x)(A_k f(x))^s\, d\mu_1(x)]} \right)^{\frac{1}{p-s}}$$

for $p \neq s$, $p,s \neq 0,1$ are means. Moreover, we can extend these means to excluded cases. Taking a limit we can define

$$M_{s,s}(f;u) = \exp\left(\frac{1-2s}{s(s-1)} + \frac{\int\limits_{\Omega_2} v(y) f^s(y) \log f(y) d\mu_2(y) - \int\limits_{\Omega_1} u(x)(A_k f(x))^s \log A_k f(x)\, d\mu_1(x)}{\int\limits_{\Omega_2} v(y) f^s(y) d\mu_2(y) - \int\limits_{\Omega_1} u(x)(A_k f(x))^s d\mu_1(x)} \right),$$

$$M_{0,0}(f;u) = \exp\left(\frac{\int\limits_{\Omega_1} u(x) \log A_k f(x)[2+\log(A_k f(x))] d\mu_1(x) - \int\limits_{\Omega_2} v(y) \log f(y)[2+\log f(y)] d\mu_2(y)}{2\int\limits_{\Omega_2} v(y)[1-\log f(y)] d\mu_2(y) - 2\int\limits_{\Omega_1} u(x)[1-\log(A_k f(x)] d\mu_1(x)} \right),$$

$$M_{1,1}(f;u) = \exp\left(\frac{\int\limits_{\Omega_2} v(y) f(y) \log f(y)[\log f(y) - 2] d\mu_2(y)}{2\int\limits_{\Omega_2} v(y) f(y)[1+\log f(y)] d\mu_2(y) - 2\int\limits_{\Omega_1} u(x) A_k f(x)[1+\log(A_k f(x)] d\mu_1(x)} \right) \times$$
$$\times \exp\left(\frac{-\int\limits_{\Omega_1} u(x) A_k f(x) \log A_k f(x)[\log(A_k f(x)) - 2]\, d\mu_1(x)}{2\int\limits_{\Omega_2} v(y) f(y)[1+\log f(y)] d\mu_2(y) - 2\int\limits_{\Omega_1} u(x) A_k f(x)[1+\log(A_k f(x)] d\mu_1(x)} \right).$$

To define the remaining cases we applied Theorem 6.7 with function φ_s and obtained the following

$$M_{0,1}(f;u) = \frac{\int\limits_{\Omega_2} v(y) f(y) \log f(y)\, d\mu_2(y) - \int\limits_{\Omega_1} u(x)(A_k f(x)) \log(A_k f(x))\, d\mu_1(x)}{\int\limits_{\Omega_1} u(x) \log(A_k f(x))\, d\mu_1(x) - \int\limits_{\Omega_2} v(y) \log f(y)\, d\mu_2(y)} = M_{1,0}(f;u).$$

For $s \neq 1$ we get

$$M_{s,1}(f;u) = M_{1,s}(f;u) =$$
$$\left(\frac{\int\limits_{\Omega_2} v(y) f^s(y) d\mu_2(y) - \int\limits_{\Omega_1} u(x)(A_k f(x))^s d\mu_1(x)}{s(s-1)[\int\limits_{\Omega_2} v(y) f(y) \log f(y) d\mu_2(y) - \int\limits_{\Omega_1} u(x)(A_k f(x)) \log(A_k f(x)) \, d\mu_1(x)]} \right)^{\frac{1}{s-1}}$$

and for $s \neq 0$

$$M_{s,0}(f;u) = M_{0,s}(f;u) =$$
$$\left(\frac{\int\limits_{\Omega_2} v(y) f^s(y) d\mu_2(y) - \int\limits_{\Omega_1} u(x)(A_k f(x))^s \, d\mu_1(x)}{s(s-1)[\int\limits_{\Omega_1} u(x) \log(A_k f(x)) d\mu_1(x) - \int\limits_{\Omega_2} v(y) \log f(y) d\mu_2(y)]} \right)^{\frac{1}{s}} .$$

We shall prove that this new mean is monotonic. Note that $M_{p,s}$ is continuous, hence, it is enough to prove monotonicity of mean in case where $r,s,l,p \neq 0,1$ and $r \neq l$, $s \neq p$.

Theorem 6.8 *Let $r \leq s$, $l \leq p$, then the following inequality is valid,*

$$M_{l,r}(f;u) \leq M_{p,s}(f;u) \tag{6.26}$$

that is, the mean $M_{p,s}(f;u)$ is monotonic.

Proof. Since ξ defined in (6.4) is a log-convex function, we can apply Remark 1.2 and get (6.26), so the proof is completed. □

6.3 Further improvements of an inequality of G. H. Hardy

In this section, we obtain some special cases of Theorem 6.2 for different fractional integrals and fractional derivatives to establish new inequalities (see [57]).

Our first result involving the fractional integral of f with respect to an increasing function g is given in the following Theorem.

Theorem 6.9 *Let $s > 1$, $\alpha > 0$, g be an increasing function on $(a,b]$ such that g' is continues on (a,b), $I^{\alpha}_{a+;g}f$ denotes the left sided fractional integral of f with respect to* *another increasing function g. Then the function* $\xi_1 : \mathbb{R} \to [0,\infty)$ *defined by*

$$\begin{aligned}\xi_1(s) &= \frac{1}{s(s-1)}\left[\int_a^b g'(y)(g(b)-g(y))^{\alpha}(f(y))^s dy\right.\\ &\left.-\int_a^b g'(x)(g(x)-g(a))^{\alpha}\left(\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}}I^{\alpha}_{a+;g}f(x)\right)^s dx\right] \end{aligned} \tag{6.27}$$

is exponentially convex and the following inequality holds

$$\xi_1(s) \le H_1(s) \tag{6.28}$$

where

$$H_1(s) = \frac{(g(b)-g(a))^{\alpha(1-s)}}{s(s-1)}\left[(g(b)-g(a))^{\alpha s}\int_a^b f^s(y)dy - (\Gamma(\alpha+1))^s\int_a^b (I^{\alpha}_{a+}f(x))^s dx\right].$$

Proof. Applying Theorem 6.2 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \frac{g'(y)}{\Gamma(\alpha)(g(x)-g(y))^{1-\alpha}}, & a \le y \le x; \\ 0, & x < y \le b, \end{cases}$$

we get that $K(x) = \frac{1}{\Gamma(\alpha+1)}(g(x)-g(a))^{\alpha}$. Then the equation (6.4) becomes

$$\xi_1(s) = \int_a^b v(y)\varphi_s(f(y))dy - \int_a^b u(x)\varphi_s\left(\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}}I^{\alpha}_{a+;g}f(x)\right)dx, \tag{6.29}$$

where φ_s is defined by (6.1). Function ξ_1 is exponentially convex. For particular weight function $u(x) = g'(x)(g(x)-g(a))^{\alpha}$, we obtain $v(y) = g'(y)(g(b)-g(y))^{\alpha}$ and (6.29) reduces to (6.27). □

We continue with results involving the Riemann-Liouville fractional integrals and the Hadamard-type fractional integrals.

Corollary 6.4 *Let* $s > 1$, $\alpha > 0$, $I^{\alpha}_{a+}f$ *denotes the left-sided Riemann-Liouville fractional integral of* f. *Then the function* $\xi_2 : \mathbb{R} \to [0,\infty)$ *defined by*

$$\xi_2(s) = \frac{1}{s(s-1)}\left[\int_a^b (b-y)^{\alpha}f^s(y)dy - \int_a^b (x-a)^{\alpha}\left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}I^{\alpha}_{a+}f(x)\right)^s dx\right]$$

is an exponentially convex function and the following inequality holds

$$\xi_2(s) \le H_2(s),$$

where

$$H_2(s) = \frac{(b-a)^{\alpha(1-s)}}{s(s-1)}\left[(b-a)^{\alpha s}\int_a^b f^s(y)dy - (\Gamma(\alpha+1))^s\int_a^b (I^{\alpha}_{a^+}f(x))^s dx\right].$$

Corollary 6.5 *Let $s > 1$, $\alpha > 0$, $I^{\alpha}_{b-}f$ denotes the right-sided Riemann-Liouville fractional integral of f. Then the function $\xi_3 : \mathbb{R} \to [0,\infty)$ defined by*

$$\xi_3(s) = \frac{1}{s(s-1)}\left[\int_a^b (y-a)^{\alpha} f^s(y)dy - \int_a^b (b-x)^{\alpha}\left(\frac{\Gamma(\alpha+1)}{(b-x)^{\alpha}}I^{\alpha}_{b-}f(x)\right)^s dx\right]$$

is exponentially convex and the following inequality holds

$$\xi_3(s) \le H_3(s),$$

where

$$H_3(s) = \frac{(b-a)^{\alpha(1-s)}}{s(s-1)}\left[(b-a)^{\alpha s}\int_a^b f^s(y)dy - (\Gamma(\alpha+1))^s\int_a^b (I^{\alpha}_{b-}f(x))^s dx\right].$$

The following result is about the Hadamard-type fractional integrals.

Corollary 6.6 *Let $s > 1$, $\alpha > 0$, $J^{\alpha}_{a_+}f$ denotes the Hadamard-type fractional integrals of f. Then the function $\xi_4 : \mathbb{R} \to [0,\infty)$ defined by*

$$\begin{aligned}\xi_4(s) &= \frac{1}{s(s-1)}\left[\int_a^b \frac{(\log b - \log y)^{\alpha}}{y} f^s(y)dy\right.\\ &\quad \left. - \int_a^b \frac{(\log x - \log a)^{\alpha}}{x}\left(\frac{\Gamma(\alpha+1)}{(\log x - \log a)^{\alpha}}(J^{\alpha}_{a_+}f(x))\right)^s dx\right]\end{aligned}$$

is exponentially convex and the following inequality holds

$$\xi_4(s) \le H_4(s)$$

where

$$\begin{aligned}H_4(s) = \frac{1}{s(s-1)}\frac{(\log b - \log a)^{\alpha(1-s)}}{ab}&\left[b(\log b - \log a)^{\alpha s}\int_a^b f^s(y)dy\right.\\ &\quad\left. - a(\Gamma(\alpha+1))^s\int_a^b (J^{\alpha}_{a_+}f(x))^s dx\right].\end{aligned}$$

In the following Theorem, we will construct new inequality for the Canavati-type fractional derivative.

Theorem 6.10 *Let $s>1$ and the assumptions in Lemma 1.4 be satisfied, $D_a^{\gamma}f$ denotes the Canavati-type fractional derivative of f. Then the function $\xi_5:\mathbb{R}\to[0,\infty)$ defined by*

$$\xi_5(s) = \frac{1}{s(s-1)}\left[\int_a^b (b-y)^{\nu-\gamma}(D_a^{\nu}f(y))^s dy - \int_a^b (x-a)^{\nu-\gamma}\left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}D_a^{\gamma}f(x)\right)^s dx\right]$$

is exponentially convex and the following inequality holds

$$\xi_5(s)\le H_5(s),$$

where

$$H_5(s) = \frac{(b-a)^{(\nu-\gamma)(1-s)}}{s(s-1)}\left[(b-a)^{(\nu-\gamma)s}\int_a^b (D_a^{\nu}f(y))^s dy - (\Gamma(\nu-\gamma+1))^s\int_a^b (D_a^{\gamma}f(x))^s dx\right].$$

Proof. Similar to the proof of Theorems 3.9 and 6.9. □

Now, we obtain new inequalities for the Caputo fractional derivative.

Theorem 6.11 *Let $s>1$ and $D_{*a}^{\gamma}f$ denotes the Caputo fractional derivative of f. Then the function $\xi_6:\mathbb{R}\to[0,\infty)$ defined by*

$$\xi_6(s) = \frac{1}{s(s-1)}\left[\int_a^b (b-y)^{n-\alpha}(f^{(n)}(y))^s dy - \int_a^b (x-a)^{n-\alpha}\left(\frac{\Gamma(n-\alpha+1)}{(x-a)^{n-\alpha}}D_{*a}^{\alpha}f(x)\right)^s dx\right]$$

is exponentially convex and the following inequality holds

$$\xi_6(s)\le H_6(s),$$

where

$$H_6(s) = \frac{(b-a)^{(n-\alpha)(1-s)}}{s(s-1)}\left[(b-a)^{(n-\alpha)s}\int_a^b (f^{(n)}(y))^s dy - (\Gamma(n-\alpha+1))^s\int_a^b (D_{*a}^{\alpha}f(x))^s dx\right].$$

Proof. Similar to the proof of Theorems 3.10 and 6.9. □

Theorem 6.12 *Let $s>1$ and the assumptions in Lemma 1.5 be satisfied. $D^{\alpha}_{*a}f$ denotes the Caputo fractional derivative of f. Then the function $\xi_7:\mathbb{R}\to[0,\infty)$ defined by*

$$\xi_7(s) = \frac{1}{s(s-1)}\left[\int_a^b (b-y)^{\alpha-\gamma}(D^{\alpha}_{*a}f(y))^s dy - \int_a^b (x-a)^{\alpha-\gamma}\left(\frac{\Gamma(\alpha-\gamma+1)}{(x-a)^{\alpha-\gamma}}D^{\gamma}_{*a}f(x)\right)^s dx\right]$$

is exponentially convex and the following inequality holds

$$\xi_7(s)\le H_7(s),$$

where

$$H_7(s) = \frac{(b-a)^{(\alpha-\gamma)(1-s)}}{s(s-1)}\left[(b-a)^{(\alpha-\gamma)s}\int_a^b (D^{\alpha}_{*a}f(y))^s dy - (\Gamma(\alpha-\gamma+1))^s\int_a^b (D^{\gamma}_{*a}f(x))^s dx\right].$$

Proof. Similar to the proof of Theorems 3.11 and 6.9. □

Now, we give the following result with the Erdélyi-Kober type fractional integrals.

Theorem 6.13 *Let $s>1$, $I^{\alpha}_{a_+;\sigma;\eta}f$ denotes the Erdélyi-Kober type fractional integrals of f, ${}_2F_1(a,b;c;z)$ denotes the hypergeometric function. Then the function $\xi_8:\mathbb{R}\to[0,\infty)$ defined by*

$$\xi_8(s) = \frac{1}{s(s-1)}\left[\int_a^b y^{\sigma-1}(b^{\sigma}-y^{\sigma})^{\alpha}\,{}_2F_1(y)f^s(y)dy - \int_a^b x^{\sigma-1}(x^{\sigma}-a^{\sigma})^{\alpha}\,{}_2F_1(x)\left(\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^{\sigma}\right)^{\alpha}\,{}_2F_1(x)}I^{\alpha}_{a_+;\sigma;\eta}f(x)\right)^s dx\right]$$

is exponentially convex and the following inequality holds

$$\xi_8(s)\le H_8(s),$$

where

$$H_8(s) = \frac{(b^\sigma - a^\sigma)^{\alpha(1-s)}}{s(s-1)} \left[(b^\sigma - a^\sigma)^{\alpha s} b^{\sigma-1} \int_a^b {}_2F_1(y) f^s(y) dy \right.$$
$$\left. - a^{\sigma-1+\alpha\sigma s} (\Gamma(\alpha+1))^s \int_a^b (({}_2F_1(x))^{1-s} I^\alpha_{a+;\sigma;\eta} f(x))^s dx \right],$$

$${}_2F_1(x) = {}_2F_1(-\eta, \alpha; \alpha+1; 1 - \left(\frac{a}{x}\right)^\sigma) \text{ and } {}_2F_1(y) = {}_2F_1(\eta, \alpha; \alpha+1; 1 - \left(\frac{b}{y}\right)^\sigma).$$

Proof. Similar to the proof of Theorems 3.12 and 6.9. □

In the following theorem we prove some inequalities that follow rom the results above.

Theorem 6.14 *For* $i = 1, \ldots, 8$ *the following inequalities hold*

$$\text{(i).} \qquad [\xi_i(p)]^{\frac{q-r}{q-p}} [\xi_i(q)]^{\frac{r-p}{q-p}} \le H_i(r) \tag{6.30}$$

$$\text{(ii).} \qquad [\xi_i(r)]^{\frac{p-q}{p-r}} [\xi_i(p)]^{\frac{q-r}{p-r}} \le H_i(q) \tag{6.31}$$

$$\text{(iii).} \qquad \xi_i(p) \le [H_i(r)]^{\frac{q-p}{q-r}} [H_i(q)]^{\frac{p-r}{q-r}} \tag{6.32}$$

for every choice of $p, q, r \in \mathbb{R}$ *such that* $1 < r < p < q$.

Proof. We will prove this Theorem just in case $i = 2$, since all other cases are proved analogously.
(i). Since the function ξ_2 is exponentially convex, it is also log-convex. Then for $1 < r < p < q,\ r, p, q \in \mathbb{R}$, (1.4) can be written as

$$[\xi_2(p)]^{q-r} [\xi_2(q)]^{r-p} \le [\xi_2(r)]^{q-p}.$$

This implies that

$$\begin{aligned} &[\xi_2(p)]^{\frac{q-r}{q-p}} [\xi_2(q)]^{\frac{r-p}{q-p}} \\ &\le \frac{(b-a)^{\alpha(1-r)}}{r(r-1)} \left[(b-a)^{\alpha r} \int_a^b f^r(y) dy - (\Gamma(\alpha+1))^r \int_a^b (I^\alpha_{a+} f(x))^r dx \right] \\ &= H_2(r) \end{aligned}$$

so (6.30) follows.

(ii). Now (1.4) can be written as

$$[\xi_2(r)]^{p-q} [\xi_2(p)]^{q-r} \le [\xi_2(q)]^{p-r}.$$

This implies that

$$
\begin{aligned}
&[\xi_2(r)]^{\frac{p-q}{p-r}}[\xi_2(p)]^{\frac{q-r}{p-r}} \\
&\leq \frac{(b-a)^{\alpha(1-q)}}{q(q-1)}\left[(b-a)^{\alpha q}\int_a^b f^q(y)dy-(\Gamma(\alpha+1))^q\int_a^b (I_{a+}^{\alpha}f(x))^q dx\right] \\
&= H_2(q),
\end{aligned}
$$

so (6.31) follows.

(iii). The (1.4) can be written as,

$$[\xi_2(p)]^{\frac{q-r}{p-r}} \leq [\xi_2(r)]^{\frac{q-p}{p-r}}[\xi_2(q)],$$

$$[\xi_2(p)]^{\frac{q-r}{p-r}} \leq [\xi_2(r)]^{\frac{q-p}{p-r}}H_2(q).$$

This implies that

$$\xi_2(p) \leq [H_2(r)]^{\frac{q-p}{q-r}}[H_2(q)]^{\frac{p-r}{q-r}},$$

so (6.32) follows.

6.3.1 Mean value theorems and Cauchy means

Now we will give mean value theorems and means of Cauchy type for different fractional integrals and fractional derivatives. For this purpose we introduce the notation

$$\xi_i(s) := \xi_i(v,\phi_s(f(y));u,\phi_s(A_kf(x))),\ \ for\ \ (i=1,\ldots,8)$$

where A_kf and v are defined by (2.15) and (2.17) respectively.

We will give some special cases of Theorems 6.1 and 6.7 for different fractional integrals and fractional derivatives.

Theorem 6.15 *Let u be a weight function on (a,b), $A_kf(x)$ be defined in (2.15) and v be defined in (2.17). Let I be a compact interval of $\mathbb{R}$, $h \in C^2(I)$ and $\xi_i : \mathbb{R} \to [0,\infty)$. Then there exists $\eta_i \in I$ such that*

$$
\begin{aligned}
&\xi_i(v,h(f(y));u,h(A_kf(x))) \\
&= \frac{h''(\eta_i)}{2}\xi_i(v,(f(y))^2;u,(A_kf(x))^2),\ for\ (i=1,\ldots,8).
\end{aligned}
$$

Theorem 6.16 *Let u be a weight function on (a,b), $A_kf(x)$ be defined in (2.15) and v be defined in (2.17). Let I be a compact interval of $\mathbb{R}$, $g,h \in C^2(I)$ such that $h''(x) \neq 0$ for every $x \in I$, $\xi_i : \mathbb{R} \to [0,\infty)$ and*

$$\xi_i(v,h(f(y));u,h(A_kf(x))) \neq 0.$$

Then there exists $\eta_i \in I$ *such that*

$$\frac{g''(\eta_i)}{h''(\eta_i)} = \frac{\xi_i(v, g(f(y)); u, g(A_k f(x)))}{\xi_i(v, h(f(y)); u, h(A_k f(x)))}, \ for \quad (i = 1, ..., 8).$$

If we apply Theorem 6.16 with $g(x) = \frac{x^p}{p(p-1)}$, $h(x) = \frac{x^s}{s(s-1)}, p \neq s, \ p, s \neq 0, 1$, we get the following result.

Corollary 6.7 *Let u be a weight function on* (a,b), $A_k f(x)$ *be defined in* (2.15) *and v be defined in* (2.17). *Let I be a compact interval of* $\mathbb{R}_+$, $\xi_i : \mathbb{R} \to [0,\infty)$, $(i = 1, ..., 8)$, *then for* $p \neq s, \ p, s \neq 1$, *there exist* $\eta_i \in I$ *such that*

$$\eta_i^{p-s} = \frac{\xi_i(p)}{\xi_i(s)} = \frac{s(s-1)}{p(p-1)} \frac{\xi_i(v, f^p(y); u, (A_k f(x))^p)}{\xi_i(v, f^s(y); u, (A_k f(x))^s)}. \tag{6.33}$$

Remark 6.5 Since $g''(x) = x^{p-2}$ and $h''(x) = x^{s-2}$, $\frac{g''}{h''}$ are invertible. Then from (6.33), we obtain

$$\inf_{t \in [a,b]} f(t) \leq \left(\frac{\xi_i(p)}{\xi_i(s)} \right)^{\frac{1}{p-s}} \leq \sup_{t \in [a,b]} f(t).$$

So,

$$M_i^{p,s}(v, \varphi_s(f(y)); u, \varphi_s(A_k f(x))) = \left(\frac{\xi_i(p)}{\xi_i(s)} \right)^{\frac{1}{p-s}}$$

and

$$M_i^{p,s} := M_i^{p,s}(v, \varphi_s(f(y)); u, \varphi_s(A_k f(x))$$

$p \neq s, \ p, s \neq 0, 1$ are means. Moreover, we can extend these means to the excluded cases. Taking a limit we can define

$$M_i^{p,s} = \begin{cases} \left(\frac{\xi_i(v, \varphi_p(f(y)); u, \varphi_p(A_k f(x)))}{\xi_i(v, \varphi_s(f(y)); u, \varphi_s(A_k f(x)))} \right)^{\frac{1}{p-s}}, & p \neq s, \\ \exp\left(\frac{1-2s}{s(s-1)} - \frac{\xi_i(v, \varphi_s(f(y)) \varphi_0(f(y)); u, \varphi_s(A_k f(x)) \varphi_0(A_k f(x)))}{\xi_i(v, \varphi_s(f(y)); u, \varphi_s(A_k f(x)))} \right), & p = s \neq 0, 1, \\ \exp\left(\frac{-\xi_i(v, \varphi_1(f(y))(\varphi_0(f(y)+2))); u, \varphi_1(A_k f(x))(\varphi_0(A_k f(x)+2))}{2\xi_i(v, f(y) + \varphi_1(f(y)); u, A_k f(x) + \varphi_1(A_k f(x)))} \right), & p = s = 1 \\ \exp\left(\frac{\xi_i(v, (2\varphi_0(f(y)) - \varphi_0^2(f(y))); u, (2\varphi_0(A_k f(x)) - \varphi_0^2(A_k f(x))))}{2\xi_i(v, 1 + \varphi_0 f(y); u, 1 + \varphi_0(A_k f(x)))} \right), & p = s = 0 \end{cases}$$

In the following theorem, we prove monotonicity of the means.

Theorem 6.17 *Let* $r \leq s$, $l \leq p$, *then the following inequality is valid,*

$$M_i^{l,r} \leq M_i^{p,s} \quad for \quad i = 1, ..., 8. \tag{6.34}$$

that is ,the means $M_i^{p,s}$ *are monotonic.*

Proof. Since ξ_i are exponentially convex, we can apply (1.4) and get (6.34). For $r = s$, $l = p$ we get the result by taking limit in (6.34). □

6.4 n-exponential convexity of Hardy-type functionals

In this chapter, we discuss and prove n-exponential convexity of the linear functionals obtained by taking the positive difference of Hardy-type inequalities. Also, we give some examples related to our main results.

Under the assumptions of Theorem 2.5, we define a linear functional by taking the positive difference of the inequality stated in (2.18) as:

$$\Delta_1(\Phi) = \int_{\Omega_2} v(y)\Phi(f(y))\,d\mu_2(y) - \int_{\Omega_1} u(x)\Phi(A_k f(x))\,d\mu_1(x). \tag{6.35}$$

We also define a linear functional by taking the positive difference of the left-hand side and the right-hand side of the inequality given in Theorem 2.7 as:

$$\Delta_2(\Phi) = \int\limits_{\Omega_2} v(y)\Phi\left(\frac{f_1(y)}{f_2(y)}\right) d\mu_2(y) - \int\limits_{\Omega_1} u(x)\Phi\left(\frac{g_1(x)}{g_2(x)}\right) d\mu_1(x) \tag{6.36}$$

6.4.1 The main results

First we give some necessary details about the divided differences. It is important to see that for different degrees of smoothness of a function divided differences are very interesting.
Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a function. Then for distinct points $z_i \in I$, $i = 0,1,2$, the divided differences of first and second order are defined by:

$$[z_i, z_{i+1}; f] = \frac{f(z_{i+1}) - f(z_i)}{z_{i+1} - z_i} \qquad (i = 0,1),$$

$$[z_0, z_1, z_2; f] = \frac{[z_1, z_2; f] - [z_0, z_1; f]}{z_2 - z_0}. \tag{6.37}$$

The values of the divided differences are independent of the order of the points z_0, z_1, z_2 and may be extended to include the cases when some or all of the points are equal, that is

$$[z_0, z_0; f] = \lim_{z_1 \to z_0} [z_0, z_1; f] = f'(z_0), \tag{6.38}$$

provided that f' exists.

Now passing through the limit $z_1 \to z_0$ and replacing z_2 by z in (6.37), we have (see [92, p.16])

$$[z_0, z_0, z; f] = \lim_{z_1 \to z_0} [z_0, z_1, z; f] = \frac{f(z) - f(z_0) - (z - z_0)f'(z_0)}{(z - z_0)^2}, z \neq z_0, \tag{6.39}$$

provided that f' exists. Also passing to the limit $z_i \to z$ $(i = 0,1,2)$ in (6.37), we have

$$[z,z,z;f] = \lim_{z_i \to z}[z_0,z_1,z_2;f] = \frac{f''(z)}{2}, \tag{6.40}$$

provided that f'' exists .
One can observe that if for all $z_0, z_1 \in I$, $[z_0,z_1,f] \geq 0$, then f is increasing on I and if for all $z_0,z_1,z_2 \in I$, $[z_0,z_1,z_2;f] \geq 0$, then f is convex on I.

Now we will produce n-exponentially convex and exponentially convex functions by applying functionals Δ_i, $i = 1,2$ on a given family with the same property. In the sequel J and I will be intervals in $\mathbb{R}$.

Theorem 6.18 *Let $\Gamma = \{\Phi_p : p \in J\}$ be a family of functions defined on I, such that the function $p \mapsto [z_0,z_1,z_2;\Phi_p]$ is n-exponentially convex in the Jensen sense on J for every three distinct points z_0, z_1, $z_2 \in I$. Let Δ_i $(i = 1,2)$ be linear functionals defined by* (6.35), (6.36). *Then the function $p \mapsto \Delta_i(\Phi_p)$ $(i = 1,2)$ is n-exponentially convex in the Jensen sense on J. If the function $p \mapsto \Delta_i(\Phi_p)$ is continuous on J, then it is n-exponentially convex on J.*

Proof. For $a_i \in \mathbb{R}$, $i = 1,...,n$ and $p_i \in J$, $i = 1,...,n$, we define the function

$$\Upsilon(z) = \sum_{i,j=1}^{n} a_i a_j \Phi_{\frac{p_i+p_j}{2}}(z).$$

Using the assumption that the function $p \mapsto [z_0,z_1,z_2;\Phi_p]$ is n-exponentially convex in the Jensen sense, we have

$$[z_0,z_1,z_2;\Upsilon] = \sum_{i,j=1}^{n} a_i a_j [z_0,z_1,z_2;\Phi_{\frac{p_i+p_j}{2}}] \geq 0,$$

which shows that Υ is convex on I and therefore we have $\Delta_i(\Upsilon) \geq 0$ for $(i = 1,2)$. Hence

$$\sum_{i,j=1}^{n} a_i a_j \Delta_i(\Phi_{\frac{p_i+p_j}{2}}) \geq 0.$$

We conclude that the function $p \mapsto \Delta_i(\Phi_p)$ for $(i = 1,2)$ is n-exponentially convex in Jensen sense on J.
If the function $p \mapsto \Delta_i(\Phi_p)$ for $(i = 1,2)$ is also continuous on J, then $p \mapsto \Delta_i(\Phi_p)$ is n-exponentially convex by definition. □

As a direct consequence of the above theorem, we can write the following corollary.

Corollary 6.8 *Let $\Gamma = \{\Phi_p : I \to \mathbb{R}, p \in J \subseteq \mathbb{R}\}$ be a family of functions, such that the function $p \mapsto [z_0,z_1,z_2;\Phi_p]$ is exponentially convex in the Jensen sense on J for every three distinct points z_0, z_1, $z_2 \in I$. Let Δ_i $(i = 1,2)$ be linear functionals defined by* (6.35), (6.36) *respectively. Then $p \mapsto \Delta_i(\Phi_p)$ is exponentially convex in the Jensen sense on J. If the function $p \mapsto \Delta_i(\Phi_p)$ is continuous on J, then it is exponentially convex on J.*

Corollary 6.9 *Let $\Gamma = \{\Phi_p : I \to \mathbb{R}, p \in J \subseteq \mathbb{R}\}$ be a family, such that the function $p \to [z_0, z_1, z_2; \Phi_p]$ is 2-exponentially convex in the Jensen sense on J for every three distinct points $z_0, z_1, z_2 \in I$. Let Δ_i $(i = 1,2)$ be linear functionals defined in* (6.35), (6.36). *Then the following statements hold:*

(i) If the function $p \mapsto \Delta_i(\Phi_p)$ is continuous on J, then it is 2-exponentially convex function on J, thus log-convex on J and for $p,q,r \in I$ such that $p < q < r$, we have

$$\Delta_i(\Phi_q)^{r-p} \leq \Delta_i(\Phi_p)^{r-q} \Delta_i(\Phi_r)^{q-p}, \; i = 1,2.$$

(ii) If the function $p \mapsto \Delta_i(\Phi_p)$ is strictly positive and differentiable on J, then for every $p,q,m,n \in J$ such that $p \leq m$, $q \leq n$, we have

$$\mathscr{B}_{p,q}(f, \Delta_i; \Gamma) \leq \mathscr{B}_{m,n}(f, \Delta_i; \Gamma), \; i = 1,2 \tag{6.41}$$

where

$$\mathscr{B}_{p,q}(f, \Delta_i; \Gamma) = \begin{cases} \left(\frac{\Delta_i(\Phi_p)}{\Delta_i(\Phi_q)} \right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{\frac{d}{dp}\left(\Delta_i(\Phi_p)\right)}{\Delta_i(\Phi_p)} \right), & p = q, \end{cases} \tag{6.42}$$

for $\Phi_p, \Phi_q \in \Gamma$.

Proof. (i) This can be obtained as a direct consequence of Theorem 6.18 and Remark 1.4.
(ii) Since by (i) the function $p \mapsto \Delta_i(\Phi_p)$ for $(i = 1,2)$ is log-convex on J, that is the function $p \mapsto \log \Delta_i(\Phi_p)$ for $(i = 1,2)$ is convex on J. Applying Remark 1.2 we obtain

$$\frac{\log \Delta_i(\Phi_p) - \log \Delta_i(\Phi_q)}{p-q} \leq \frac{\log \Delta_i(\Phi_m) - \log \Delta_i(\Phi_n)}{m-n} \tag{6.43}$$

for $p \leq m,\ q \leq n,\ p \neq q,\ m \neq n$, and we conclude that

$$\mathscr{B}_{p,q}(f, \Delta_i; \Gamma) \leq \mathscr{B}_{m,n}(f, \Delta_i; \Gamma) \text{ for } (i = 1,2).$$

The cases for $p = q$, $m = n$ follow from (6.43) bytaking limit. □

Remark 6.6 Note that the results of Theorem 6.18, Corollary 6.8 and Corollary 6.9 still hold when two of the points $z_0, z_1, z_2 \in I$ coincide for a family of differentiable functions Φ_p such that $p \mapsto [z_0, z_1, z_2; \Phi_p]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense), further, they still hold when all three point coincide for a family of twice differentiable functions with the same property. The proofs are obtained using (6.38), (6.39) and (6.40) respectively and some facts about the exponential convexity.

6.4.2 Examples

We conclude this chapter with the following examples.

Example 6.1 Consider a family of functions

$$\Gamma_1 = \{g_p : (0,\infty) \to (0,\infty) : p \in (0,\infty)\},$$

defined by

$$g_p(t) = \frac{e^{-t\sqrt{p}}}{p}.$$

Since $p \mapsto \frac{d^2 g_p(t)}{dt^2} = e^{-t\sqrt{p}}$, is the Laplace transform of a non-negative function, it is exponentially convex (see [100]). Clearly, g_p is a convex function for each $p > 0$. It is obvious that $\Delta_i(g_p)$ for $(i = 1,2)$ is continuous. It is easy to prove that the function $p \mapsto [z_0, z_1, z_2; g_p]$ is also exponentially convex for arbitrary points $z_0, z_1, z_2 \in I$. For this family of functions, $\mathscr{B}_{p,q}(f, \Delta_i; \Gamma_1)$ becomes

$$\mathscr{B}_{p,q}(f, \Delta_i(g_p); \Gamma_1) = \begin{cases} \left(\frac{\Delta_i(g_p)}{\Delta_i(g_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(-\frac{\Delta_i(id \cdot g_p)}{2\sqrt{p}\Delta_i(g_p)} - \frac{1}{p}\right), & p = q, \end{cases}$$

and from (6.41) it follows that the function $\mathscr{B}_{p,q}(f, \Delta_i; \Gamma_1)$ is monotone in the parameters p and q.

Example 6.2 Let

$$\Gamma_2 = \{h_p : (0,\infty) \to (0,\infty) : p \in (0,\infty)\},$$

be a family of functions defined by

$$h_p(t) = \begin{cases} \frac{p^{-t}}{(\ln p)^2}, & p \in \mathbb{R}_+ \setminus \{1\}, \\ \frac{t^2}{2}, & p = 1. \end{cases}$$

Since $p \mapsto \frac{d^2}{dt^2} h_p(t) = p^{-t}$ is the Laplace transform of a non-negative function (see [100]), it is exponentially convex. Obviously, h_p is a convex function for every $p > 0$. It is easy to prove that the function $p \mapsto [z_0, z_1, z_2; h_p]$ is also exponentially convex for arbitrary points $z_0, z_1, z_2 \in I$. Using Corollary 6.8, it follows that $p \mapsto \Delta_i(h_p)$ for $(i = 1,2)$ is exponentially convex (it is easy to verify that it is continuous) and thus log-convex. From (6.42), we can write

$$\mathscr{B}_{p,q}(f, \Delta_i(h_p); \Gamma_2) = \begin{cases} \left(\frac{\Delta_i(h_p)}{\Delta_i(h_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(-\frac{\Delta_i(id \cdot h_p)}{p\Delta_i(h_p)} - \frac{2}{p \ln p}\right), & p = q \neq 1, \\ \exp\left(-\frac{\Delta_i(id \cdot h_1)}{3\Delta_i(h_1)}\right), & p = q = 1, \end{cases}$$

and from (6.41) we deduce monotonicity of the function $\mathscr{B}_{p,q}(f, \Delta_i(h_p); \Gamma_2)$ in the parameters p and q for $h_p, h_q \in \Gamma_2$.

Example 6.3 Consider a family of functions

$$\Gamma_3 = \{\psi_p : \mathbb{R} \to [0,\infty) : p \in (0,\infty)\},$$

defined with

$$\psi_p(t) = \begin{cases} \frac{1}{p^2}e^{tp}, & p \in \mathbb{R}\setminus\{0\}, \\ \frac{1}{2}t^2, & p = 0. \end{cases}$$

The mapping $p \mapsto \frac{d^2}{dt^2}(\psi_p(t)) = e^{tp}$ is a well known exponentially convex function on $\mathbb{R}$ for every $p \in \mathbb{R}$. Using the analogous arguments as in Theorem 6.18, we also have that $p \mapsto [z_0, z_1, z_2; \psi_p]$ is exponentially convex (also exponentially convex in J-sense). For this family of functions, $\mathscr{B}_{p,q}(f, \Delta_i; \Gamma_3)$ for $(i = 1,2)$, from (6.42) becomes

$$\mathscr{B}_{p,q}(f, \Delta_i(\psi_p); \Gamma_3) = \begin{cases} \left(\frac{\Delta_i(\psi_p)}{\Delta_i(\psi_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{\Delta_i(id\cdot\psi_p)}{\Delta_i(\psi_p)} - \frac{2}{p}\right), & p = q \neq 0, \\ \exp\left(\frac{\Delta_i(id\cdot\psi_0)}{3\Delta_i(\psi_0)}\right), & p = q = 0, \end{cases}$$

and using (6.41) we can see that it is a monotone function in the parameter p and q for $\psi_p, \psi_q \in \Gamma_3$.

Example 6.4 Consider a family of functions

$$\Gamma_4 = \{\phi_p : (0,\infty) \to \mathbb{R} : p \in \mathbb{R}\},$$

defined by

$$\phi_p(t) = \begin{cases} \frac{t^p}{p(p-1)} & p \neq 1, 0, \\ -\ln t & p = 0, \\ t\ln t & p = 1. \end{cases}$$

Since $p \mapsto \frac{d^2}{dt^2}(\phi_p(t)) = t^{p-2} = e^{(p-2)\ln t} > 0$, is the Laplace transform of a non-negative function (see [100]), it is exponentially convex. Obviously ϕ_p is a convex function for every $t > 0$. It is easy to prove that the function $p \mapsto [z_0, z_1, z_2; \phi_p]$ is also exponentially convex for arbitrary points $z_0, z_1, z_2 \in I$. Using Corollary 6.8 it follows that $p \mapsto \Delta_i(\phi_p)$ for $(i = 1,2)$ is exponentially convex (it is easy to verify that it is continuous), and thus log-convex. From (6.42), we see that

$$\mathscr{B}_{p,q}(f, \Delta_i(\phi_p); \Gamma_4) = \begin{cases} \left(\frac{\Delta_i(\phi_p)}{\Delta_i(\phi_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{1-2p}{p(p-1)} - \frac{\Delta_i(\phi_p\phi_0)}{\Delta_i(\phi_p)}\right), & p = q \neq 0, 1, \\ \exp\left(1 - \frac{\Delta_i(\phi_0^2)}{2\Delta_i(\phi_0)}\right), & p = q = 0, \\ \exp\left(-1 - \frac{\Delta_i(\phi_0\phi_1)}{2\Delta_i(\phi_1)}\right), & p = q = 1, \end{cases} \tag{6.44}$$

for $\phi_p, \phi_q \in \Gamma_4$.

Remark 6.7 For the case $i = 1$, the means given in (6.44) were already presented in Remark 6.4 in explicit form.

Chapter 7

Hardy-type inequalities with general kernels and measures via superquadratic functions

7.1 Preliminaries

The following results was recently proved by Oguntuase et al. [89]:

Proposition 7.1 *Let* $\mathbf{b} \in (\mathbf{0}, \infty)$, $u : (\mathbf{0}, \mathbf{b}) \to \mathbb{R}$ *be a weight function which is locally integrable in* $(\mathbf{0}, \mathbf{b})$ *and* v *be defined by*

$$v(\mathbf{t}) = t_1 \cdots t_n \int_{t_1}^{b_1} \cdots \int_{t_n}^{b_n} \frac{u(\mathbf{x})}{x_1^2 \cdots x_n^2} d\mathbf{x}, \mathbf{t} \in (\mathbf{0}, \mathbf{b}). \tag{7.1}$$

Suppose $I = (a, c)$, $0 \le a < c \le \infty$, $\varphi : I \to \mathbb{R}$, *and* $f : (\mathbf{0}, \mathbf{b}) \to \mathbb{R}$ *is an integrable function, such that* $f(\mathbf{x}) \in I$, *for all* $\mathbf{x} \in (\mathbf{0}, \mathbf{b})$.

(i) If φ is superquadratic, then the following inequality holds:

$$
\begin{aligned}
&\int_0^{b_1}\cdots\int_0^{b_n} u(\mathbf{x})\varphi\left(\frac{1}{x_1\cdots x_n}\int_0^{x_1}\cdots\int_0^{x_n} f(\mathbf{t})d\mathbf{t}\right)\frac{d\mathbf{x}}{x_1\cdots x_n}\\
&+\int_0^{b_1}\cdots\int_0^{b_n}\int_{t_1}^{b_1}\cdots\int_{t_n}^{b_n}\varphi\left(\left|f(\mathbf{t})-\frac{1}{x_1\cdots x_n}\int_0^{x_1}\cdots\int_0^{x_n} f(\mathbf{u})d\mathbf{u}\right|\right)\frac{u(\mathbf{x})}{x_1^2\cdots x_n^2}d\mathbf{x}d\mathbf{t}\\
&\le\int_0^{b_1}\cdots\int_0^{b_n} v(\mathbf{x})\varphi(f(\mathbf{x}))\frac{d\mathbf{x}}{x_1\cdots x_n}.
\end{aligned}
\tag{7.2}
$$

(ii) If φ is subquadratic, then (7.2) holds in the reversed direction.

Proposition 7.2 *Let $\mathbf{b}\in(\mathbf{0},\infty)$, $u:(\mathbf{b},\infty)\to\mathbb{R}$ is a weight function which is locally integrable in $(\mathbf{0},\mathbf{b})$ and define v by*

$$
v(\mathbf{t})=\frac{1}{t_1\cdots t_n}\int_{b_1}^{t_1}\ldots\int_{b_n}^{t_n} u(\mathbf{x})d\mathbf{x}<\infty,\ \mathbf{t}\in(\mathbf{b},\infty). \tag{7.3}
$$

Suppose $I=(a,c)$, $0\le a<c\le\infty$, $\varphi:I\to\mathbb{R}$, and $f:(\mathbf{b},\infty)\to\mathbb{R}$ is an integrable function, such that $f(\mathbf{x})\in I$, for all $\mathbf{x}\in(\mathbf{b},\infty)$.

(i) If φ is superquadratic, then the following inequality holds:

$$
\begin{aligned}
&\int_{b_1}^{\infty}\ldots\int_{b_n}^{\infty} u(\mathbf{x})\varphi\left(x_1\cdots x_n\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty} f(\mathbf{t})\frac{d\mathbf{t}}{t_1^2\cdots t_n^2}\right)\frac{d\mathbf{x}}{x_1\cdots x_n}\\
&+\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty}\int_{b_1}^{t_1}\cdots\int_{b_n}^{t_n}\varphi\left(\left|f(\mathbf{t})-x_1\cdots x_n\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty} f(\mathbf{t})\frac{d\mathbf{t}}{t_1^2\cdots t_n^2}\right|\right)\\
&\qquad\times u(\mathbf{x})d\mathbf{x}\frac{d\mathbf{t}}{t_1^2\cdots t_n^2}\\
&\le\int_{b_1}^{\infty}\ldots\int_{b_n}^{\infty} v(\mathbf{x})\varphi(f(vx))\frac{d\mathbf{x}}{x_1\ldots x_n}.
\end{aligned}
\tag{7.4}
$$

(ii) If φ is subquadratic, then the inequality sign in (7.4) is reversed.

7.2 The main results with applications

Probably the first refinement of Hardy's inequality (0.1) is due to D. T. Shum [97] and further developed by C. O. Imoru [54]. These Shum-Imoru results were recently complemented and generalized by L.E. Persson and J.A. Oguntuase in the following way (see [93], Theorem 2.1):

Theorem 7.1 *Let $p,k,b \in \mathbb{R}$ be such that $0<b<\infty$ and one of the following holds:*
(i) $p \geq 1$ and $k>1$,
(ii) $p<0$ and $k<1$.
If $f(x)$ is a non-negative integrable function on $(0,b)$ such that

$$0<\int_0^b x^{p-k} f^p(x)dx<\infty,$$

then the following inequality

$$\int_0^b x^{-k}\left(\int_0^x f(t)dt\right)^p dx+\frac{p}{k-1}b^{1-k}\left(\int_0^b f(t)dt\right)^p \leq\left(\frac{p}{k-1}\right)^p\int_0^b x^{p-k} f^p(x)dx \tag{7.5}$$

holds.
(iii) If $0<p\leq 1$ and $k>1$, then inequality (7.5) holds in the reversed direction.
The constant $\left(\frac{p}{k-1}\right)^p$ on the right hand side of (7.5) is the best possible in all the cases.

Remark 7.1 Note that the statement in Theorem 7.1 in particular means if $p=1, k>1$, then we have equality in (7.5) which can also be seen by performing a simple direct calculation.

Remark 7.2 Also a dual version of Theorem 7.1 (where the integrals $\int_0^b$ are replaced by $\int_b^\infty$) was stated and proved in the same paper, see [93], Theorem 2.2.

The above shows, in particular, that $p=1$ is a natural "breaking point" for Hardy's inequality and also that with the extra term inserted on the left-hand side in (7.5) we even have equality for $p=1$. Another remarkable fact is that by inserting another additional term in (7.5) the natural breaking point in this refined Hardy's inequality is in fact for $p=2$ and equality appears also at $p=2$, see J.A. Oguntuase and al. [88]. In fact, the same authors later on proved these results even in the following multidimensional settings:

Theorem 7.2 *Let $1 < p < \infty$, $\mathbf{k} = (k_1,...,k_n) \in \mathbb{R}^n$ be such that $k_i > 1$ $(i = 1,...,n)$, $0 < \mathbf{b} \leq \infty$, and let the function f be locally integrable on $(\mathbf{0},\mathbf{b})$ such that*

$$0 < \int_0^{b_1} \dots \int_0^{b_n} \prod_{i=1}^n x_i^{p-k_i} f^p(\mathbf{x}) d\mathbf{x} < \infty.$$

(i) *If $p \geq 2$, then*

$$\begin{aligned}
&\int_0^{b_1} \dots \int_0^{b_n} \prod_{i=1}^n x_i^{-k_i} \left(\int_0^{x_1} \dots \int_0^{x_n} f(\mathbf{t}) d\mathbf{t} \right)^p d\mathbf{x} \\
&+ \left(\prod_{i=1}^n \frac{k_i-1}{p} \right) \int_0^{b_1} \dots \int_0^{b_n} \int_{t_1}^{b_1} \dots \int_{t_n}^{b_n} \Bigg| \prod_{i=1}^n \frac{p}{k_i-1} \left(\frac{t_i}{x_i} \right)^{1-\frac{k_i-1}{p}} f(\mathbf{t}) \\
&- \frac{1}{x_1 \dots x_n} \int_0^{x_1} \dots \int_0^{x_n} f(\mathbf{t}) d\mathbf{t} \Bigg|^p \prod_{i=1}^n x_i^{p-k_i-\frac{k_i-1}{p}} d\mathbf{x} \prod_{i=1}^n t_i^{\frac{k_i-1}{p}-1} d\mathbf{t} \\
&\leq \left(\prod_{i=1}^n \frac{p}{k_i-1} \right)^p \int_0^{b_1} \dots \int_0^{b_n} \prod_{i=1}^n \left(1 - \left[\frac{x_i}{b_i} \right]^{\frac{k_i-1}{p}} \right) x_i^{p-k_i} f^p(\mathbf{x}) d\mathbf{x}.
\end{aligned} \tag{7.6}$$

(ii) *If $1 < p \leq 2$, then inequality (7.6) holds in the reversed direction.*

Theorem 7.3 *Let $1 < p < \infty$, $\mathbf{k} = (k_1,...,k_n) \in \mathbb{R}^n$ be such that $k_i < 1$, $i = 1,2,...,n$, $0 \leq \mathbf{b} < \infty$, and let the function f be locally integrable on $(\mathbf{b},\infty)$ and such that*

$$0 < \int_{b_1}^{\infty} \dots \int_{b_n}^{\infty} \prod_{i=1}^n x_i^{p-k_i} f^p(\mathbf{x}) d\mathbf{x} < \infty.$$

(iii) *If $p \geq 2$, then*

$$\begin{aligned}
&\int_{b_1}^{\infty} \dots \int_{b_n}^{\infty} \prod_{i=1}^n x_i^{-k_i} \left(\int_{x_1}^{\infty} \dots \int_{x_n}^{\infty} f(\mathbf{t}) d\mathbf{t} \right)^p d\mathbf{x} \\
&+ \left(\prod_{i=1}^n \frac{1-k_i}{p} \right) \int_{b_1}^{\infty} \dots \int_{b_n}^{\infty} \int_{b_1}^{t_1} \dots \int_{b_n}^{t_n} \Bigg| \prod_{i=1}^n \frac{p}{1-k_i} \left(\frac{t_i}{x_i} \right)^{\frac{1-k_i}{p}+1} f(\mathbf{t}) \\
&- \frac{1}{x_1 \dots x_n} \int_{x_1}^{\infty} \dots \int_{x_n}^{\infty} f(\mathbf{t}) d\mathbf{t} \Bigg|^p \prod_{i=1}^n x_i^{\frac{1-k_i}{p}+p-k_i} d\mathbf{x} \prod_{i=1}^n t_i^{\frac{k_i-1}{p}-1} d\mathbf{t} \\
&\leq \left(\prod_{i=1}^n \frac{p}{1-k_i} \right)^p \int_{b_1}^{\infty} \dots \int_{b_n}^{\infty} \prod_{i=1}^n \left(1 - \left[\frac{b_i}{x_i} \right]^{\frac{1-k_i}{p}} \right) x_i^{p-k_i} f^p(\mathbf{x}) d\mathbf{x}.
\end{aligned} \tag{7.7}$$

(iv) *If $1 < p \leq 2$, then inequality (7.7) holds in the reversed direction.*

Remark 7.3 Note that for the case $p = 2$ both inequalities (7.6) and (7.7) will be equalities so we obtain something like new Parseval type identities with the Hardy and the dual Hardy operators involved.

Remark 7.4 We remark that Propositions 7.1 and 7.2 are crucial for the proofs of Theorems 7.2 and 7.3, respectively. In fact, these proofs are carried out by just applying these Propositions for $\varphi(u) = u^p$ and performing a series of suitable variable transformations (for details, see [89]).

We give generalizations of the above results. Our first result reads (see [5]):

Theorem 7.4 *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive σ-finite measures, u be a weight function on Ω_1, k a non-negative measurable function on $\Omega_1 \times \Omega_2$, and K be defined on Ω_1 by (2.16).*
Suppose that $K(x) > 0$ for all $x \in \Omega_1$, that the function $x \mapsto u(x)\frac{k(x,y)}{K(x)}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$, and that v is defined on Ω_2 by (2.17). Suppose $I = [0,c)$, $c \leq \infty$, $\varphi : I \to \mathbb{R}$. If φ is a superquadratic function, then the inequality

$$\int_{\Omega_1} u(x)\varphi(A_k f(x))d\mu_1(x) + \int_{\Omega_2}\int_{\Omega_1} \varphi\left(|f(y) - A_k f(x)|\right) \frac{u(x)k(x,y)}{K(x)} d\mu_1(x)d\mu_2(y)$$
$$\leq \int_{\Omega_2} v(y)\varphi(f(y))\,d\mu_2(y) \tag{7.8}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$, such that $Imf \subseteq I$, where A_k is defined by (2.15).
If φ is subquadratic, then the inequality sign in (7.8) is reversed.

Proof. We must first prove that $A_k f(x) \in I$, for all $x \in \Omega_1$ (see the proof of Theorem 2.5).
Now, let us prove inequality (7.8). By applying the refined Jensen's inequality (1.11) to the first term on the left hand side of (7.8) and then Fubini theorem, we have that

$$\begin{aligned}
&\int_{\Omega_1} u(x)\varphi(A_k f(x))\,d\mu_1(x) \\
&= \int_{\Omega_1} u(x)\varphi\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)f(y)\,d\mu_2(y)\right) d\mu_1(x) \\
&\leq \int_{\Omega_1} \frac{u(x)}{K(x)}\left(\int_{\Omega_2} k(x,y)\varphi(f(y))\,d\mu_2(y)\right) d\mu_1(x) \\
&\quad - \int_{\Omega_1} \frac{u(x)}{K(x)}\int_{\Omega_2} k(x,y)\varphi\left(|f(y) - A_k f(x)|\right) d\mu_2(y)\,d\mu_1(x) \\
&= \int_{\Omega_2} \varphi(f(y))\left(\int_{\Omega_1} \frac{k(x,y)}{K(x)}u(x)\,d\mu_1(x)\right) d\mu_2(y) \\
&\quad - \int_{\Omega_2}\int_{\Omega_1} \varphi\left(|f(y) - A_k f(x)|\right) \frac{u(x)k(x,y)}{K(x)}\,d\mu_1(x)\,d\mu_2(y) \\
&= \int_{\Omega_2} v(y)\varphi(f(y))\,d\mu_2(y) \\
&\quad - \int_{\Omega_2}\int_{\Omega_1} \varphi\left(|f(y) - A_k f(x)|\right) \frac{u(x)k(x,y)}{K(x)}\,d\mu_1(x)\,d\mu_2(y)
\end{aligned}$$

from which (7.8) follows.
By making the same calculations with a subquadratic φ we see that only the inequality sign in (7.8) will be reversed. The proof is complete. □

Example 7.1 Let $\Omega_1 = \Omega_2 = (\mathbf{0}, \mathbf{b})$, $\mathbf{0} < \mathbf{b} \leq \infty$, replace $d\mu_1(x)$ and $d\mu_2(y)$ by the Lebesque measures $d\mathbf{x}$ and $d\mathbf{y}$, respectively, and $k(\mathbf{x}, \mathbf{y}) = \mathbf{1}$, $\mathbf{0} \leq \mathbf{y} \leq \mathbf{x}$, $k(\mathbf{x}, \mathbf{y}) = \mathbf{0}$, $\mathbf{y} > \mathbf{x}$. Then $K(\mathbf{x}) = x_1 \cdots x_n$ and

$$A_k f(\mathbf{x}) = \frac{1}{x_1 \cdots x_n} \int_0^{x_1} \cdots \int_0^{x_n} f(\mathbf{y}) d\mathbf{y}.$$

Moreover, replace $u(\mathbf{x})$ by $u(\mathbf{x})/x_1 \cdots x_n$ and $v(\mathbf{y})$ by $v(\mathbf{y})/y_1 \cdots y_n$, then, in particular (7.1) coincides with (2.17) and (7.8) coincides with (7.2) and we see that Proposition 7.1 is a special case of Theorem 7.4. □

Remark 7.5 As mentioned before, (see Remark 7.4) Theorem 7.2 follows by using Proposition 7.1 with $\varphi(u) = u^p$ (which is superquadratic for $p \geq 2$ and subquadratic for $1 \leq p \leq 2$) and making some suitable variable transformations (for details see [89]). Hence, Theorem 7.4 implies Theorem 7.2.

Remark 7.6 Note that in the one-dimensional case $(n = 1)$, Example 7.1 reduces to the corresponding Proposition 2.1 in [88]. Moreover, if $u(\mathbf{x}) = \frac{1}{x_1 \cdots x_n}$, then we have

$$v(\mathbf{y}) = \int_{y_1}^{b_1} \cdots \int_{y_n}^{b_n} \frac{d\mathbf{x}}{x_1^2 \cdots x_n^2} = \prod_{i=1}^{n} \frac{1}{y_1 \cdots y_n} \left(1 - \frac{y_i}{b_i}\right), \quad \mathbf{y} \in (\mathbf{0}, \mathbf{b})$$

and we get the inequality given in Remark 2 in [89]. In the one-dimensional case $(n = 1)$, this reduces to Example 4.1 in [88].

Example 7.2 Let $\Omega_1 = \Omega_2 = (\mathbf{b}, \infty)$, $\mathbf{0} \leq \mathbf{b} < \infty$, replace $d\mu_1(x)$ and $d\mu_2(y)$ by the Lebesque measures $d\mathbf{x}$ and $d\mathbf{y}$, respectively and $k(\mathbf{x}, \mathbf{y}) = \frac{1}{y_1^2 \cdots y_n^2}$, $\mathbf{y} \geq \mathbf{x}$, $k(\mathbf{x}, \mathbf{y}) = \mathbf{0}$, $\mathbf{b} \leq \mathbf{y} < \mathbf{x}$. Then $K(\mathbf{x}) = \frac{1}{x_1 \cdots x_n}$ and

$$A_k f(\mathbf{x}) = x_1 \cdots x_n \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \frac{f(\mathbf{y})}{y_1^2 \cdots y_n^2} d\mathbf{y}.$$

Now, by replacing $u(\mathbf{x})$ by $u(\mathbf{x})/x_1 \cdots x_n$ and $v(\mathbf{y})$ by $v(\mathbf{y})/y_1 \cdots y_n$, we see that (7.3) coincides with (2.17) so we conclude that also Proposition 7.2 is a special case of Theorem 7.4. □

Remark 7.7 Analogously as in the discussion in Remark 7.5, we find, according to Example 7.2, that Theorem 7.4 also implies Theorem 7.3.

Remark 7.8 Note that in the one-dimensional case $(n = 1)$, Example 7.2 reduces to the corresponding Propositions 2.2 in [88]. Further, if $u(\mathbf{x}) = \frac{1}{x_1 \cdots x_n}$, then we have

$$v(\mathbf{y}) = \frac{1}{y_1^2 \cdots y_n^2} \int_{b_1}^{y_1} \cdots \int_{b_n}^{y_n} d\mathbf{x} = \prod_{i=1}^{n} \frac{1}{y_1 \cdots y_n} \left(1 - \frac{b_i}{y_i}\right), \quad \mathbf{y} \in (\mathbf{b}, \infty).$$

and we get the inequality given in Remark 3 in [89]. In the one-dimensional case $(n = 1)$, this reduces to Example 4.2 in [88].

As we have seen by applying Theorem 7.4 with $\varphi(u) = u^p$ and special kernels we fairly easily obtain the proofs of Theorems 7.2 and 7.3. More generally we can state the following result:

Corollary 7.1 Let the assumptions in Theorem 7.4 be satisfied.
(i) If $p \geq 2$, then

$$\int_{\Omega_1} u(x) A_k^p f(x)\, d\mu_1(x) + \int_{\Omega_2} \int_{\Omega_1} |f(y) - A_k f(x)|^p \frac{u(x)k(x,y)}{K(x)} d\mu_1(x) d\mu_2(y) \leq \int_{\Omega_2} v(y) f^p(y)\, d\mu_2(y) \tag{7.9}$$

(ii) If $1 < p \leq 2$, then (7.9) holds in the reversed direction.

Proof. Apply Theorem 7.4 with the function $\varphi(x) = x^p$, which is superquadratic for $p \geq 2$ and subquadratic for $1 \leq p \leq 2$. □

Remark 7.9 In particular, by applying Corollary 7.1 with $p = 2$ we obtain the following very general identity (of Parseval type for the generalized Hardy operators involved):

$$\int_{\Omega_1} u(x) A_k^2 f(x)\, d\mu_1(x) + \int_{\Omega_2} \int_{\Omega_1} |f(y) - A_k f(x)|^2 \frac{u(x)k(x,y)}{K(x)} d\mu_1(x)\, d\mu_2(y) = \int_{\Omega_2} v(y) f^2(y)\, d\mu_2(y).$$

By using Corollary 7.1 (and Remark 7.3) with concrete kernels we can obtain refinements of some classical inequalities. Here we give only the following complement and refinement of the Hardy-Hilbert inequality (2.1).

Corollary 7.2 Let $p > 1$ and $f \in L^p(\mathbb{R}_+)$.
If $p \geq 2$, then

$$\begin{aligned} &\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx \right)^p dy \\ &+ \left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \right)^{p-1} \int_0^\infty y^{-\frac{1}{p}} \int_0^\infty \left| f(y) y^{\frac{1}{p}} - \frac{\sin\left(\frac{\pi}{p}\right)}{\pi} x^{\frac{1}{p}} \int_0^\infty \frac{f(y)}{x+y} dy \right|^p \frac{x^{\frac{1}{p}-1}}{x+y} dx\, dy \\ &\leq \left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \right)^p \int_0^\infty f^p(y)\, dy. \end{aligned} \tag{7.10}$$

If $1 < p \leq 2$, then (7.10) holds in the reversed direction.

Proof. Apply Corollary 7.1 with $\Omega_1 = \Omega_2 = (0,\infty)$ and $d\mu_1(x)$ and $d\mu_2(y)$ replaced by the Lebesgue measures dx and dy, respectively, and let $k(x,y) = \frac{(\frac{y}{x})^{-1/p}}{x+y}$, $p > 1$, and $u(x) = \frac{1}{x}$. Then we find that $K(x) = \frac{\pi}{\sin(\pi/p)}$,

$$A_k^p(f(x)) = \left(\frac{\sin(\pi/p)}{\pi}\right)^p x \left(\int_0^\infty \frac{f(y)}{x+y} y^{-1/p} dy\right)^p$$

and

$$\begin{aligned} v(y) &= \frac{\sin(\pi/p)}{\pi} \int_0^\infty \left(\frac{y}{x}\right)^{-1/p} \frac{1}{x(x+y)} dx \\ &= \frac{\sin(\pi/p)}{\pi} \frac{1}{y} \int_0^\infty \left(\frac{y}{x}\right)^{-1/p'} \frac{1}{x+y} dx = \frac{1}{y} \frac{\sin(\pi/p)}{\pi} \frac{\pi}{\sin(\pi/p')} = \frac{1}{y} \end{aligned}$$

(Here, as usual, $\frac{1}{p'} + \frac{1}{p} = 1$).
By replacing, now, $f(x)$ with $f(x)x^{1/p}$ in (7.9) we obtain (7.10).
The proof of the case $1 < p \leq 2$ is the same because then only the inequality sign coming from case (ii) in Corollary 7.2 reverses. The proof is complete. □

Remark 7.10 We note that for $p = 2$ we get equality in (7.10) and this equality is a special case of the general one stated in Remark 7.9.

We finish this section by stating another useful applications of Theorem 7.4.

Corollary 7.3 Let the assumptions in Theorem 7.4 be satisfied and let $|\Omega_1|_{\mu_1}, |\Omega_2|_{\mu_2} < \infty$. Then the inequality

$$\begin{aligned} &\int_{\Omega_1} \varphi\left(\frac{1}{|\Omega_2|_{\mu_2}} \int_{\Omega_2} f(y)\, d\mu_2(y)\right) d\mu_1(x) \\ &+ \frac{1}{|\Omega_2|_{\mu_2}} \int_{\Omega_2} \int_{\Omega_1} \varphi\left(\left| f(y) - \frac{1}{|\Omega_2|_{\mu_2}} \int_{\Omega_2} f(y)\, d\mu_2(y)\right|\right) d\mu_1(x)\, d\mu_2(y) \\ &\leq \frac{|\Omega_1|_{\mu_1}}{|\Omega_2|_{\mu_2}} \int_{\Omega_2} \varphi(f(y))\, d\mu_2(y) \end{aligned} \tag{7.11}$$

holds for all superquadratic functions φ . If the function φ is subquadratic, then (7.11) holds with reverse inequality sign.

Proof. Apply Theorem 7.4 with $k(x,y) \equiv 1$ and $u(x) = 1$. Then

$$K(x) = \int_{\Omega_2} d\mu_2(y) = |\Omega_2|_{\mu_2}$$

and

$$v(y) = \int_{\Omega_1} \frac{1}{|\Omega_2|_{\mu_2}} d\mu_1(x) = \frac{|\Omega_1|_{\mu_1}}{|\Omega_2|_{\mu_2}}$$

and the proof is complete. □

7.3 Remarks

Example 7.3 The inequalities (7.5) and (7.6) for $n = 1$, i.e

$$\begin{aligned}
&\int_0^b x^{-k} \left(\int_0^x f(t)dt \right)^p dx \\
&+ \frac{k-1}{p} \int_0^b \int_t^b \left| \frac{p}{k-1} \left(\frac{t}{x} \right)^{1-\frac{k-1}{p}} f(t) - \frac{1}{x} \int_0^x f(u)du \right|^p x^{p-k-\frac{k-1}{p}} dx t^{\frac{k-1}{p}-1} dt \\
&\le \left(\frac{p}{k-1} \right)^p \int_0^b \left(1 - \left(\frac{x}{b} \right)^{\frac{k-1}{p}} \right) x^{p-k} f^p(x) dx
\end{aligned} \tag{7.12}$$

hold both for $p \geq 2$ and $k > 1$. They can not be easily compared in general. However, (7.12) is better for $p = 2$ because it then reduces to equality. It is also better for the case $b = \infty$ because then the additional term on the left-hand side of (7.5) is equal to zero while the one in (7.12) is not, and still the right-hand side is strictly smaller than the right hand side in (7.12). However, it remains an open question if (7.12) is always better in this case (both inequalities are sharp).
Also for the case $1 \leq p \leq 2$ and $k > 1$ we can not compare these inequalities because then (7.12) holds in the reversed direction. At the endpoint $p = 1$ (7.5) holds with equality, while at the endpoint $p = 2$ (7.12) holds with equality. In particular for $1 < p \leq 2$, $k > 1$ and $b = \infty$ we have that

$$0 \leq -\int_0^\infty x^{-k} \left(\int_0^x f(t)dt \right)^p dx + \left(\frac{p}{k-1} \right)^p \int_0^\infty x^{p-k} f^p(x) dx \leq I_q$$

where

$$I_q = \frac{k-1}{p} \int_0^\infty \int_t^\infty \left| \frac{p}{k-1} \left(\frac{t}{x} \right)^{1-\frac{k-1}{p}} f(t) - \frac{1}{x} \int_0^x f(u)du \right|^p x^{p-k-\frac{k-1}{p}} dx t^{\frac{k-1}{p}-1} dt.$$

Two-sides estimate like this can never be obtained by using only Theorem 7.1.

Remark 7.11 A completely analogous dual example can be obtained by using and comparing Theorem 7.3 with $n = 1$ and the dual version of Theorem 7.1 (see [93], Theorem 2.2).

The function $\Phi(x) = e^x$ is not superquadratic but by working, instead, with the superquadratic function $\Phi(x) = e^x - x - 1$ (see Lemma 1.2) we obtain the following result of Pólya-Knopp type by using Theorem 7.4:

Example 7.4 Assume that the assumptions in Theorem 7.4 are satisfied. Then, by applying Theorem 7.4 with $\Phi(x) = e^x - x - 1$ and $f(x)$ replaced by $\log f(x)$ we obtain the following inequality of (refined) Pólya-Knopp type:

$$\int_{\Omega_1} \exp A_k f(x)\, d\mu_1(x) + I \leq \int_{\Omega_2} f(y)v(y)\, d\mu_2(y),$$

where

$$A_k f(x) = \frac{1}{K(x)} \int_{\Omega_2} k(x,y) \log f(y)\, d\mu_2(y)$$

and

$$\begin{aligned} I = \int_{\Omega_2} \int_{\Omega_1} \exp|\log f(y) - A_k f(x)| - |\log f(y) - A_k f(x)| \frac{u(x)k(x,y)}{K(x)} d\mu_1(x) d\mu_2(y) \\ + \int_{\Omega_2} \log f(y)v(y) d\mu_2(y) - \int_{\Omega_1} (A_k f(x) + 1)\, d\mu_1(x). \end{aligned}$$

□

7.4 Mean value theorems

Let us continue by defining a linear functional as a difference between the right-hand side and the left-hand side of the refined Hardy type inequality (7.8):

$$\begin{aligned} A(\varphi) &= \int_{\Omega_2} \varphi(f(y))v(y) d\mu_2(y) - \int_{\Omega_1} \varphi(A_k f(x))u(x) d\mu_1(x) \\ &\quad - \int_{\Omega_2} \int_{\Omega_1} \varphi\left(|f(y) - A_k f(x)|\right) \frac{u(x)k(x,y)}{K(x)} d\mu_1(x) d\mu_2(y) \end{aligned} \tag{7.13}$$

It is clear that if φ is a superquadratic function, then $A(\varphi) \geq 0$.

Now, we give mean value theorem. First, we state and prove the Lagrange-type mean value theorem (see [37]).

Lemma 7.1 *Suppose that $\varphi \in C^2([0,\infty))$, $-\infty < m \le M < \infty$ be such that*

$$m \le \left(\frac{\varphi'(x)}{x}\right)' = \frac{x\varphi''(x) - \varphi'(x)}{x^2} \le M, \text{ for all } x > 0.$$

Consider the functions $\varphi_1, \varphi_2 : [0,\infty) \to \mathbb{R}$ defined as

$$\varphi_1(x) = \frac{Mx^3}{3} - \varphi(x), \; \varphi_2(x) = \varphi(x) - \frac{mx^3}{3}.$$

Then the functions $x \to \frac{\varphi_1'(x)}{x}$ and $x \to \frac{\varphi_2'(x)}{x}$ are increasing. If $\varphi_i(0) = 0$, $i = 1,2$, then they are superquadratic functions.

Theorem 7.5 *Let $\varphi : [0,\infty) \to \mathbb{R}$, $\varphi(0) = 0$ and the assumptions of Theorem 7.4 be satisfied. Assume that A is a strictly positive functional. If $\frac{\varphi'(x)}{x} \in C^1(0,\infty)$, then there exists $\xi \in (0,\infty)$ such that following equality holds*

$$\begin{aligned} A(\varphi) \;&=\; \frac{1}{3}\frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi^2}\left(\int_{\Omega_2} f^3(y)v(y)d\mu_2(y) - \int_{\Omega_1}(A_k f(x))^3 u(x)d\mu_1(x)\right. \\ &\left. - \int_{\Omega_2}\int_{\Omega_1} |f(y) - A_k f(x)|^3 \frac{u(x)k(x,y)}{K(x)} d\mu_1(x)\,d\mu_2(y)\right), \end{aligned} \tag{7.14}$$

where $A_k f$, K are defined by (2.15) – (2.16), *respectively.*

Proof. 1. Case: Suppose that $\min_{x\in(0,\infty)} (\frac{\varphi'}{x}) = m$ and $\max_{x\in(0,\infty)} (\frac{\varphi'}{x}) = M$ exist. Then by applying Theorem 7.4 on the functions φ_1, φ_2 from Lemma 7.1 the following two inequalities hold:

$$\begin{aligned} A(\varphi) \;&\le\; \frac{M}{3}\left(\int_{\Omega_2} f^3(y)v(y)d\mu_2(y) - \int_{\Omega_1}(A_k f(x))^3 u(x)d\mu_1(x)\right. \\ &\left. - \int_{\Omega_2}\int_{\Omega_1} |f(y) - A_k f(x)|^3 \frac{u(x)k(x,y)}{K(x)} d\mu_1(x)\,d\mu_2(y)\right) \end{aligned}$$

and

$$\begin{aligned} A(\varphi) \;&\ge\; \frac{m}{3}\left(\int_{\Omega_2} f^3(y)v(y)d\mu_2(y) - \int_{\Omega_1}(A_k f(x))^3 u(x)d\mu_1(x)\right. \\ &\left. - \int_{\Omega_2}\int_{\Omega_1} |f(y) - A_k f(x)|^3 \frac{u(x)k(x,y)}{K(x)} d\mu_1(x)\,d\mu_2(y)\right). \end{aligned}$$

Since $\varphi = x^3$ is strictly superquadratic and A is strictly positive

$$\begin{aligned} &\int_{\Omega_2} f^3(y)v(y)d\mu_2(y) - \int_{\Omega_1}(A_k f(x))^3 u(x)d\mu_1(x) \\ &- \int_{\Omega_2}\int_{\Omega_1} |f(y) - A_k f(x)|^3 \frac{u(x)k(x,y)}{K(x)} d\mu_1(x)\,d\mu_2(y) > 0. \end{aligned}$$

By combining the above two inequalities and using the fact

$$m \leq \frac{x\varphi''(x) - \varphi'(x)}{x^2} \leq M$$

we conclude the existance of $\xi \in (0,\infty)$ such that (7.14) holds.
2. Case: Suppose that $\min\limits_{x\in(0,\infty)} (\frac{\varphi'}{x}) = m$ and $\sup\limits_{x\in(0,\infty)} (\frac{\varphi'}{x}) = M$, and assume that M is not a maximum. In this case φ_1 is strictly superquadratic. Then, by applying Theorem 7.4 on the functions φ_1, φ_2 from Lemma 7.1, the following two inequalities hold:

$$\begin{aligned} A(\varphi) < \frac{M}{3} \Bigg(& \int_{\Omega_2} f^3(y)v(y)d\mu_2(y) - \int_{\Omega_1} (A_k f(x))^3 u(x) d\mu_1(x) \\ & - \int_{\Omega_2}\int_{\Omega_1} |f(y) - A_k f(x)|^3 \frac{u(x)k(x,y)}{K(x)} d\mu_1(x)\, d\mu_2(y) \Bigg) \end{aligned}$$

and

$$\begin{aligned} A(\varphi) \geq \frac{m}{3} \Bigg(& \int_{\Omega_2} f^3(y)v(y)d\mu_2(y) - \int_{\Omega_1} (A_k f(x))^3 u(x) d\mu_1(x) \\ & - \int_{\Omega_2}\int_{\Omega_1} |f(y) - A_k f(x)|^3 \frac{u(x)k(x,y)}{K(x)} d\mu_1(x)\, d\mu_2(y) \Bigg). \end{aligned}$$

By combining the above two inequalities and using the fact

$$m \leq \frac{x\varphi''(x) - \varphi'(x)}{x^2} < M$$

we conclude the existance of $\xi \in (0,\infty)$ such that (7.14) holds.
3. Case: Suppose that $\inf\limits_{x\in(0,\infty)} (\frac{\varphi'}{x}) = m$ and $\max\limits_{x\in(0,\infty)} (\frac{\varphi'}{x}) = M$, and assume that m is not a minimum. The proof is analogous to the proof in Case 2.
4. Case: Suppose that $\inf\limits_{x\in(0,\infty)} (\frac{\varphi'}{x}) = m$ and $\sup\limits_{x\in(0,\infty)} (\frac{\varphi'}{x}) = M$, and assume that m is not a minimum and M is not a maximum. The proof is analogous to the proof in Case 2.
In the case where $M = \infty$ and m exists, using just φ_2 we obtain

$$m \leq \frac{x\varphi''(x) - \varphi'(x)}{x^2}$$

when m is the minimum, and strong inequality when m is the infimum. The rest of the proof is as above. □

Theorem 7.6 *Let $\varphi, \psi : [0,\infty) \to \mathbb{R}$, $\varphi(0) = \psi(0) = 0$, the assumptions of Theorem 7.4 be satisfied. Assume that A is a strictly positive functional. If $\frac{\varphi'}{x}, \frac{\psi'}{x} \in C^1(0,\infty)$, then there exists $\xi \in (0,\infty)$ such that*

$$\frac{A(\varphi)}{A(\psi)} = \frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi\psi''(\xi) - \psi'(\xi)},$$

provided the denominators are not equal to zero.

Proof. We consider a function h defined as $h = c_1\varphi - c_2\psi$, where c_1, c_2 are defined by

$$c_1 = A(\psi),\, c_2 = A(\varphi).$$

Then

$$\frac{h'}{x} = c_1\frac{\varphi'}{x} - c_2\frac{\psi'}{x} \in C^1(0,\infty),$$

after a short calculation we obtain that $A(k) = 0$. By Theorem 7.5 there exists ξ such that

$$\begin{aligned}
&(c_1(\xi\varphi''(\xi) - \varphi'(\xi)) - c_2(\xi\psi''(\xi) - \psi'(\xi)))\left(\int_{\Omega_2} f^3(y)v(y)d\mu_2(y)\right. \\
&\left. - \int_{\Omega_1}(A_k f(x))^3 u(x)d\mu_1(x) - \int_{\Omega_2}\int_{\Omega_1}|f(y) - A_k f(x)|^3 \frac{u(x)k(x,y)}{K(x)} d\mu_1(x)d\mu_2(y)\right) \\
&= 0.
\end{aligned} \tag{7.15}$$

Since $\varphi = x^3$ is strictly superquadratic and A is strictly positive

$$\begin{aligned}
&\int_{\Omega_2} f^3(y)v(y)d\mu_2(y) - \int_{\Omega_1}(A_k f(x))^3 u(x)d\mu_1(x) \\
&- \int_{\Omega_2}\int_{\Omega_1}|f(y) - A_k f(x)|^3 \frac{u(x)k(x,y)}{K(x)} d\mu_1(x)\,d\mu_2(y) > 0.
\end{aligned}$$

We conclude that

$$\frac{c_2}{c_1} = \frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi\psi''(\xi) - \psi'(\xi)} = \frac{A(\varphi)}{A(\psi)},$$

provided that the denominator is not zero. This completes the proof. □

As a special case of Theorems 7.5 and 7.6 we obtain the following results:

Example 7.5 Let $\Omega_1 = \Omega_2 = (\mathbf{0},\mathbf{b})$, $\mathbf{0} < \mathbf{b} \le \infty$, replace $d\mu_1(x)$ and $d\mu_2(y)$ by the Lebesgue measures $d\mathbf{x}$ and $d\mathbf{y}$, respectively, and $k(\mathbf{x},\mathbf{y}) = \mathbf{1}$, $\mathbf{0} \le \mathbf{y} \le \mathbf{x}$, $k(\mathbf{x},\mathbf{y}) = \mathbf{0}$, $\mathbf{y} > \mathbf{x}$. Then $K(\mathbf{x}) = x_1 \cdots x_n$ and

$$A_k f(\mathbf{x}) = \frac{1}{x_1 \cdots x_n}\int_0^{x_1} \cdots \int_0^{x_n} f(\mathbf{y})d\mathbf{y}.$$

Moreover, replace $u(\mathbf{x})$ by $u(\mathbf{x})/x_1 \cdots x_n$ and $v(\mathbf{y})$ by $v(\mathbf{y})/y_1 \cdots y_n$, then v coincides with

$$v(\mathbf{t}) = t_1 \ldots t_n \int_{t_1}^{b_1} \ldots \int_{t_n}^{b_n} \frac{u(\mathbf{x})}{x_1^2 \ldots x_n^2} d\mathbf{x}, \mathbf{t} \in (\mathbf{0},\mathbf{b})$$

and A, which we now denote by $\widetilde{A}$, becomes

$$\begin{aligned}
\widetilde{A}(\varphi) &= \int_0^{b_1}\cdots\int_0^{b_n} v(\mathbf{x})\varphi(f(\mathbf{x}))\frac{d\mathbf{x}}{x_1\cdots x_n}\\
&-\int_0^{b_1}\cdots\int_0^{b_n} u(\mathbf{x})\varphi\left(\frac{1}{x_1\cdots x_n}\int_0^{x_1}\cdots\int_0^{x_n} f(\mathbf{t})d\mathbf{t}\right)\frac{d\mathbf{x}}{x_1\cdots x_n}\\
&-\int_0^{b_1}\cdots\int_0^{b_n}\int_{t_1}^{b_1}\cdots\int_{t_n}^{b_n}\varphi\left(\left|f(\mathbf{t})-\frac{1}{x_1\cdots x_n}\int_0^{x_1}\cdots\int_0^{x_n} f(\mathbf{t})d\mathbf{t}\right|\right)\\
&\quad\times\frac{u(\mathbf{x})}{x_1^2\cdots x_n^2}d\mathbf{x}d\mathbf{t}
\end{aligned}$$

and (7.14) takes form

$$\begin{aligned}
\widetilde{A}(\varphi) &= \frac{1}{3}\frac{\xi\varphi''(\xi)-\varphi'(\xi)}{\xi^2}\left(\int_0^{b_1}\cdots\int_0^{b_n} v(\mathbf{x})f^3(\mathbf{x})\frac{d\mathbf{x}}{x_1\cdots x_n}\right.\\
&-\int_0^{b_1}\cdots\int_0^{b_n} u(\mathbf{x})\left(\frac{1}{x_1\cdots x_n}\int_0^{x_1}\cdots\int_0^{x_n} f(\mathbf{t})d\mathbf{t}\right)^3\frac{d\mathbf{x}}{x_1\cdots x_n}\\
&-\int_0^{b_1}\cdots\int_0^{b_n}\int_{t_1}^{b_1}\cdots\int_{t_n}^{b_n}\left|f(\mathbf{t})-\frac{1}{x_1\cdots x_n}\int_0^{x_1}\cdots\int_0^{x_n} f(\mathbf{t})d\mathbf{t}\right|^3\\
&\quad\left.\times\frac{u(\mathbf{x})}{x_1^2\cdots x_n^2}d\mathbf{x}d\mathbf{t}\right).
\end{aligned}$$

Example 7.6 Let $\Omega_1=\Omega_2=(\mathbf{b},\infty)$, $\mathbf{0}\le\mathbf{b}<\infty$, replace $d\mu_1(x)$ and $d\mu_2(y)$ by the Lebesgue measures $d\mathbf{x}$ and $d\mathbf{y}$, respectively and $k(\mathbf{x},\mathbf{y})=\frac{1}{y_1^2\cdots y_n^2}$, $\mathbf{y}\ge\mathbf{x}$, $k(\mathbf{x},\mathbf{y})=\mathbf{0}$, $\mathbf{b}\le\mathbf{y}<\mathbf{x}$. Then $K(\mathbf{x})=\frac{1}{x_1\cdots x_n}$ and

$$A_kf(\mathbf{x})=x_1\cdots x_n\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty}\frac{f(\mathbf{y})}{y_1^2\cdots y_n^2}d\mathbf{y}.$$

Replacing $u(\mathbf{x})$ by $u(\mathbf{x})/x_1...x_n$ and $v(\mathbf{y})$ by $v(\mathbf{y})/y_1...y_n$, we obtain

$$v(\mathbf{t})=\frac{1}{t_1\cdots t_n}\int_{b_1}^{t_1}\cdots\int_{b_n}^{t_n}u(\mathbf{x})d\mathbf{x}\ <\infty,\ \mathbf{t}\in(\mathbf{b},\infty)$$

and A, which we now denote by $\widehat{A}$, becomes

$$\begin{aligned}
\widehat{A}(\varphi) &= \int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty} v(\mathbf{x})\varphi(f(\mathbf{x}))\frac{d\mathbf{x}}{x_1\cdots x_n}\\
&-\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty} u(\mathbf{x})\varphi\left(x_1\cdots x_n\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty} f(\mathbf{t})\frac{d\mathbf{t}}{t_1^2\cdots t_n^2}\right)\frac{d\mathbf{x}}{x_1\cdots x_n}\\
&-\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty}\int_{b_1}^{t_1}\cdots\int_{b_n}^{t_n}\varphi\left(\left|f(\mathbf{t})-x_1\cdots x_n\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty} f(\mathbf{t})\frac{d\mathbf{t}}{t_1^2\cdots t_n^2}\right|\right)\\
&\quad\times u(\mathbf{x})d\mathbf{x}\frac{d\mathbf{t}}{t_1^2\cdots t_n^2}
\end{aligned}$$

and (7.14) takes form

$$
\begin{aligned}
\widehat{A}(\varphi) = {}& \frac{1}{3}\frac{\xi\varphi''(\xi)-\varphi'(\xi)}{\xi^2}\left(\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty} v(\mathbf{x})f^3(\mathbf{x})\frac{d\mathbf{x}}{x_1\cdots x_n}\right.\\
&-\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty} u(\mathbf{x})\left(x_1\cdots x_n\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty} f(\mathbf{t})\frac{d\mathbf{t}}{t_1^2\cdots t_n^2}\right)^3\frac{d\mathbf{x}}{x_1\cdots x_n}\\
&-\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty}\int_{b_1}^{t_1}\cdots\int_{b_n}^{t_n}\left|f(\mathbf{t})-x_1\cdots x_n\int_0^{x_1}\cdots\int_0^{x_n} f(\mathbf{t})\frac{d\mathbf{t}}{t_1^2\cdots t_n^2}\right|^3\\
&\left.\times u(\mathbf{x})d\mathbf{x}\frac{d\mathbf{t}}{t_1^2\cdots t_n^2}\right).
\end{aligned}
$$

We state the following result concerning inequality (2.1) by applying Theorem 7.5 with $\varphi(u)=u^p,\ p\geq 2$.

Example 7.7 Let $\Omega_1=\Omega_2=(0,\infty)$, $\varphi(u)=u^p,\ p\geq 2$ and replace $d\mu_1(x)$ and $d\mu_2(y)$ by the Lebesgue measures dx and dy, respectively, let $k(x,y)=\frac{(\frac{y}{x})^{-1/p}}{x+y}$ and $u(x)=\frac{1}{x}$. Then we find that $K(x)=\frac{\pi}{\sin(\pi/p)}$ and $v(y)=\frac{1}{y}$. Replace $f(x)$ by $f(x)x^{1/p}$, so A, which is now denoted by H_f, becomes

$$
\begin{aligned}
H_f = {}& \int_0^{\infty} f^p(y)\,dy-\left(\frac{\sin\left(\frac{\pi}{p}\right)}{\pi}\right)^p\int_0^{\infty}\left(\int_0^{\infty}\frac{f(x)}{x+y}dx\right)^p dy\\
&-\frac{\sin\left(\frac{\pi}{p}\right)}{\pi}\int_0^{\infty}\int_0^{\infty}\left|f(y)-\frac{\sin\left(\frac{\pi}{p}\right)}{\pi}\left(\frac{x}{y}\right)^{\frac{1}{p}}\int_0^{\infty}\frac{f(y)}{x+y}dy\right|^p\frac{x^{\frac{1}{p}-1}}{x+y}dxdy
\end{aligned}
$$

and (7.14) takes form

$$
\begin{aligned}
H_f = {}& \frac{p(p-2)\xi^{p-3}}{3}\left(\int_0^{\infty} f^3(y)y^{\frac{3}{p}-1}dy-\left(\frac{\sin\left(\frac{\pi}{p}\right)}{\pi}\right)^3\int_0^{\infty}\left(\int_0^{\infty}\frac{f(y)}{x+y}dy\right)^3 x^{\frac{3}{p}-1}dx\right.\\
&\left.-\frac{\sin\left(\frac{\pi}{p}\right)}{\pi}\int_0^{\infty}\int_0^{\infty}\left|f(y)-\frac{\sin\left(\frac{\pi}{p}\right)}{\pi}\left(\frac{x}{y}\right)^{\frac{1}{p}}\int_0^{\infty}\frac{f(y)}{x+y}dy\right|^3\frac{x^{\frac{1}{p}-1}}{x+y}dxdy\right).
\end{aligned}
$$

7.5 Exponential convexity

Lemma 7.2 *Consider the function φ_p for $p > 0$ defined by*

$$\varphi_p(x) = \begin{cases} \frac{x^p}{p(p-2)}, & p \neq 2 \\ \frac{x^2}{2}\log x, & p = 2 \end{cases}. \tag{7.16}$$

Then, with the convention $0\, log\, 0 = 0$, φ_p is superquadratic.

For linear functional A defined by (7.13) we have $A(\varphi_p) \geq 0$ for all $p > 0$.

Lemma 7.3 *Let us define the function*

$$\phi_p(x) = \begin{cases} \frac{pxe^{px} - e^{px} + 1}{p^3}, & p \neq 0 \\ \frac{x^3}{3}, & p = 0. \end{cases} \tag{7.17}$$

Then $\left(\frac{\phi_p'(x)}{x}\right)' = e^{px} > 0$ and $\phi_p(0) = 0$. Therefore ϕ_p is superquadratic.

Properties of the mapping $p \mapsto A(\varphi_p)$ are given in the following theorem:

Theorem 7.7 *For A as in (7.13) and φ_p as in (7.16) we have the following:*

(i) *the mapping $p \mapsto A(\varphi_p)$ is continuous for $p > 0$,*

(ii) *for every $n \in \mathbb{N}$ and $p_i \in \mathbb{R}_+, p_{ij} = \frac{p_i + p_j}{2}$, $i, j = 1, 2, ..., n$, the matrix $[A(\varphi_{p_{ij}})]_{i,j=1}^n$ is positive semi-definite, that is*

$$\det[A(\varphi_{p_{ij}})]_{i,j=1}^n \geq 0,$$

(iii) *the mapping $p \mapsto A(\varphi_p)$ is exponentially convex,*

(iv) *the mapping $p \mapsto A(\varphi_p)$ is log-convex,*

(v) *for $p_i \in \mathbb{R}_+$, $i = 1, 2, 3$, $p_1 < p_2 < p_3$,*

$$[A(\varphi_{p_2})]^{p_3 - p_1} \leq [A(\varphi_{p_1})]^{p_3 - p_2} [A(\varphi_{p_3})]^{p_2 - p_1}.$$

Proof.

(i) Notice that for $p \neq 2$

$$A(\varphi_p) = \frac{1}{p(p-2)} \left[\int_{\Omega_2} f^p(y) v(y) d\mu_2(y) - \int_{\Omega_1} (A_k f(x))^p u(x) d\mu_1(x) \right.$$
$$\left. - \int_{\Omega_2} \int_{\Omega_1} |f(y) - A_k f(x)|^p \frac{u(x)k(x,y)}{K(x)} d\mu_1(x)\, d\mu_2(y) \right]$$

and for $p = 2$

$$A(\varphi_p) = \frac{1}{2}\left[\int_{\Omega_2} f^2(y)\log(f(y))v(y)d\mu_2(y) - \int_{\Omega_1}(A_kf(x))^2\log(A_kf(x))u(x)d\mu_1(x)\right.$$
$$\left. -\int_{\Omega_2}\int_{\Omega_1}|f(y)-A_kf(x)|^2\log|f(y)-A_kf(x)|\frac{u(x)k(x,y)}{K(x)}d\mu_1(x)\,d\mu_2(y)\right]$$

It is, obviously, continuous for $p > 0$, $p \neq 2$. Suppose $p \to 2$:

$$\lim_{p\to 2} A(\varphi_p) = \lim_{p\to 2}\frac{1}{p(p-2)}\left[\int_{\Omega_2} f^p(y)v(y)d\mu_2(y) - \int_{\Omega_1}(A_kf(x))^p u(x)d\mu_1(x)\right.$$
$$\left. -\int_{\Omega_2}\int_{\Omega_1}|f(y)-A_kf(x)|^p\frac{u(x)k(x,y)}{K(x)}d\mu_1(x)\,d\mu_2(y)\right].$$

Since

$$\int_{\Omega_2} f^2(y)v(y)d\mu_2(y) - \int_{\Omega_1}(A_kf(x))^2u(x)d\mu_1(x)$$
$$-\int_{\Omega_2}\int_{\Omega_1}|f(y)-A_kf(x)|^2\frac{u(x)k(x,y)}{K(x)}d\mu_1(x)\,d\mu_2(y) = 0,$$

by applying L'Hospital rule, after a simple calculation, we obtain that

$$\lim_{p\to 2} A(\varphi_p) = A(\varphi_2).$$

Hence, the mapping $p \mapsto A(\varphi_p)$ is continuous for $p > 0$.

(ii) Define the function $F(x) = \sum_{i,j=1}^n u_iu_j\varphi_{p_ij}(x)$, where $p_{ij} = \frac{p_i+p_j}{2}$. Then

$$\left(\frac{F'(x)}{x}\right)' = \sum_{i,j=1}^n u_iu_j\left(\frac{\varphi'_{p_ij}(x)}{x}\right)' = \left(\sum_{i=1}^n u_ix^{\frac{p_i-3}{2}}\right)^2 \geq 0$$

and $F(0) = 0$ implies F is superquadratic, so using this F in place of φ in (7.8) we have

$$A(F) = \sum_{i,j=1}^n u_iu_jA(\varphi_{p_{ij}}) \geq 0$$

from this we have that the matrix $B = [A(\varphi_{\frac{p_i+p_j}{2}})]_{i,j=1}^n$, is positive-semidefinite *i.e.* $\det B \geq 0$.

(iii), (iv) and (v) are trivial consequence of (i), (ii) and definition of exponentially convex and log-convex functions. □

Using the function ϕ_p instead of φ_p, the following result follows.

Theorem 7.8 *For A as in* (7.13) *and* ϕ_p *as in* (7.17) *we have the following:*

(i) *the mapping* $p \mapsto A(\phi_p)$ *is continuous on* $\mathbb{R}$,

(ii) *for every* $n \in \mathbb{N}$ *and* $p_i \in \mathbb{R}, p_{ij} = \frac{p_i+p_j}{2}$, $i,j = 1,2,...,n$, *the matrix* $[A(\phi_{p_{ij}})]_{i,j=1}^n$ *is positive semi-definite, that is*

$$\det[A(\phi_{p_{ij}})]_{i,j=1}^n \geq 0,$$

(iii) *the mapping* $p \mapsto A(\phi_p)$ *is exponentially convex,*

(iv) *the mapping* $p \mapsto A(\phi_p)$ *is log-convex,*

(v) *for* $p_i \in \mathbb{R}$, $i = 1,2,3$, $p_1 < p_2 < p_3$,

$$[A(\phi_{p_2})]^{p_3-p_1} \leq [A(\phi_{p_1})]^{p_3-p_2}[A(\phi_{p_3})]^{p_2-p_1}.$$

7.6 Cauchy means

Theorem 7.6 enables us to define new means, because when the right-hand side, interpreted as a function of ξ and denoted by $K(\xi)$ is invertible, then

$$\xi = K^{-1}\left(\frac{A(\varphi)}{A(\psi)}\right),$$

presents a new Cauchy mean.

Specially, if we choose $\varphi = \varphi_s, \psi = \varphi_r$, where $r,s \in \mathbb{R}_+$, $r \neq s$, $r,s \neq 2$, we obtain

$$\xi^{s-r} = \frac{r(r-2)}{s(s-2)} \times$$
$$\frac{\int_{\Omega_2} f^s(y)v(y)d\mu_2(y) - \int_{\Omega_1}(A_k f(x))^s u(x)d\mu_1(x) - \int_{\Omega_2}\int_{\Omega_1} C^s(x,y)g(x,y)d\mu_1(x)d\mu_2(y)}{\int_{\Omega_2} f^r(y)v(y)d\mu_2(y) - \int_{\Omega_1}(A_k f(x))^r u(x)d\mu_1(x) - \int_{\Omega_2}\int_{\Omega_1} C^r(x,y)g(x,y)d\mu_1(x)d\mu_2(y)},$$

where $C(x,y) = |f(y) - A_k f(x)|$ and $g(x,y) = \frac{u(x)k(x,y)}{K(x)}$.

Now we can define a new family of means.

Definition 7.1 *For* $r,s \in \mathbb{R}_+, r,s \neq 2, r \neq s$ *we define means* $M_{r,s}$ *as follows*

$$M_{s,r} =$$
$$\left(\frac{r(r-2)}{s(s-2)} \frac{\int_{\Omega_2} A_{s,0}(y)d\mu_2(y) - \int_{\Omega_1} B_{s,0}(x)d\mu_1(x) - \int_{\Omega_2}\int_{\Omega_1} C_{s,0}(x,y)d\mu_1(x)d\mu_2(y)}{\int_{\Omega_2} A_{r,0}(y)d\mu_2(y) - \int_{\Omega_1} B_{r,0}(x)d\mu_1(x) - \int_{\Omega_2}\int_{\Omega_1} C_{r,0}(x,y)d\mu_1(x)d\mu_2(y)}\right)^{\frac{1}{s-r}},$$

Taking a limit we can define the excluded cases. For $r \neq 2$ we have

$$M_{r,2} = M_{2,r} =$$

$$\left(\frac{r(r-2)}{2} \frac{\int_{\Omega_2} A_{2,1}(y)d\mu_2(y) - \int_{\Omega_1} B_{2,1}(x)d\mu_1(x) - \int_{\Omega_2}\int_{\Omega_1} C_{2,1}(x,y)d\mu_1(x)d\mu_2(y)}{\int_{\Omega_2} A_{r,0}(y)d\mu_2(y) - \int_{\Omega_1} B_{r,0}(x)d\mu_1(x) - \int_{\Omega_2}\int_{\Omega_1} C_{r,0}(x,y)d\mu_1(x)d\mu_2(y)} \right)^{\frac{1}{2-r}}$$

$$M_{r,r} =$$

$$\exp\left(\frac{\int_{\Omega_2} A_{r,1}(y)d\mu_2(y) - \int_{\Omega_1} B_{r,1}(x)d\mu_1(x) - \int_{\Omega_2}\int_{\Omega_1} C_{r,1}(x,y)d\mu_1(x)d\mu_2(y)}{\int_{\Omega_2} A_{r,0}(y)d\mu_2(y) - \int_{\Omega_1} B_{r,0}(x)d\mu_1(x) - \int_{\Omega_2}\int_{\Omega_1} C_{r,0}(x,y)d\mu_1(x)d\mu_2(y)} - \frac{2r-2}{r(r-2)} \right),$$

and for $r = 2$

$$M_{2,2} =$$

$$\exp\left(\frac{\int_{\Omega_2} A_{2,2}(y)d\mu_2(y) - \int_{\Omega_1} B_{2,2}(x)d\mu_1(x) - \int_{\Omega_2}\int_{\Omega_1} C_{2,2}(x,y)d\mu_1(x)d\mu_2(y)}{\int_{\Omega_2} A_{2,1}(y)d\mu_2(y) - \int_{\Omega_1} B_{2,1}(x)d\mu_1(x) - \int_{\Omega_2}\int_{\Omega_1} C_{2,1}(x,y)d\mu_1(x)d\mu_2(y)} - \frac{1}{2} \right),$$

where

$$A_{p,n}(y) = f^p(y)(\log(f(y))^n v(y),$$

$$B_{p,n}(x) = (A_k f(x))^p (\log(A_k f(x))^n u(x),$$

$$C_{p,n}(x,y) = |f(y) - A_k f(x)|^p \log^n |f(y) - A_k f(x)| g(x,y),\ n = 0,1,2,\ p > 0.$$

Note that these means are symmetric and we can easily check that the special cases in the above definition are limits of the general case. That is,

$$\begin{aligned} M_{r,r} &= \lim_{s\to r} M_{s,r} \\ M_{2,r} &= M_{r,2} = \lim_{s\to 2} M_{s,r}, \\ M_{2,2} &= \lim_{r\to 2} M_{r,r}. \end{aligned}$$

Monotonicity of the means defined above is given in the following theorem.

Theorem 7.9 *Let $s,t,u,v \in \mathbb{R}_+$ be such that $s \le u$, $t \le v$, $s \neq t$, $u \neq v$. Then*

$$M_{t,s} \le M_{v,u}. \tag{7.18}$$

Proof. Since the function $s \mapsto A(\varphi_s)$ is log-convex, then by Remark 1.2 for any $s,t,u,v \in_+$, such that $s \le u$, $t \le v$, $s \neq t$, $u \neq v$, we have

$$\left(\frac{A(\varphi_t)}{A(\varphi_s)}\right)^{\frac{1}{t-s}} \le \left(\frac{A(\varphi_v)}{A(\varphi_u)}\right)^{\frac{1}{v-u}}$$

which is equivalent to (7.18).
When $s = t$ or $u = v$ the inequality follows by taking limits. □

7.7 Inequality of G. H. Hardy and superquadratic functions

We will give some special cases of Theorem 7.7 for different fractional integrals and fractional derivatives to establish new inequalities.

Our first result is for the Riemann-Liouville fractional integral (see [58]).

Theorem 7.10 *Let $s > 2$, $\alpha > 0$, $I^{\alpha}_{a+}f$ denotes the left-sided Riemann-Liouville fractional integral of f. Then the function $A_1 : \mathbb{R} \to [0,\infty)$ defined by*

$$A_1(s) = \frac{1}{s(s-2)}\left[\int_a^b f^s(y)(b-y)^{\alpha}dy - \int_a^b \left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} I^{\alpha}_{a+}f(x)\right)^s (x-a)^{\alpha}dx \right.$$
$$\left. -\alpha \int_a^b\int_y^b \left(\left|f(y) - \frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} I^{\alpha}_{a+}f(x)\right|\right)^s (x-y)^{\alpha-1}dx\,dy\right] \tag{7.19}$$

is exponentially convex and

$$A_1(s) \le H_1(s) \tag{7.20}$$

holds, where

$$H_1(s) = \frac{(b-a)^{\alpha(1-s)}}{s(s-2)}\left[(b-a)^{\alpha s}\int_a^b f^s(y)dy - (\Gamma(\alpha+1))^s \int_a^b (I^{\alpha}_{a+}f(x))^s dx\right].$$

Proof. Applying Theorem 7.7 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \frac{(x-y)^{\alpha-1}}{\Gamma(\alpha)}, & a \le y \le x; \\ 0, & x < y \le b \end{cases}$$

we get that $K(x) = \frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)}$. Then equation (7.13) reduces to

$$A_1(s) = \int_a^b \varphi_s(f(y))v(y)dy - \int_a^b \varphi_s\left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} I^{\alpha}_{a+}f(x)\right) u(x)dx$$
$$-\alpha \int_a^b\int_y^b \varphi_s\left(\left|f(y) - \frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} I^{\alpha}_{a+}f(x)\right|\right) \frac{u(x)(x-y)^{\alpha-1}}{(x-a)^{\alpha}}dx\,dy, \tag{7.21}$$

where φ_s is defined by (7.16). Function A_1 is exponentially convex. Applying (7.21) with particular weight function $u(x) = (x-a)^\alpha$, $x \in (a,b)$, we get (7.19). Notice that

$$\begin{aligned}
A_1(s) = & \frac{1}{s(s-2)} \left[\int_a^b f^s(y)(b-y)^\alpha dy - \int_a^b \left(\frac{\Gamma(\alpha+1)}{(x-a)^\alpha} I_{a+}^\alpha f(x) \right)^s (x-a)^\alpha dx \right. \\
& \left. -\alpha \int_a^b \int_y^b \left(\left| f(y) - \frac{\Gamma(\alpha+1)}{(x-a)^\alpha} I_{a+}^\alpha f(x) \right| \right)^s (x-y)^{\alpha-1} dx dy \right] \\
\le & \frac{1}{s(s-2)} \left[(b-a)^\alpha \int_a^b f^s(y) dy - (b-a)^{\alpha(1-s)} (\Gamma(\alpha+1))^s \int_a^b \left(I_{a+}^\alpha f(x) \right)^s dx \right] \\
= & \frac{(b-a)^{\alpha(1-s)}}{s(s-2)} \left[(b-a)^{\alpha s} \int_a^b f^s(y) dy - (\Gamma(\alpha+1))^s \int_a^b \left(I_{a+}^\alpha f(x) \right)^s dx \right]
\end{aligned}$$

so the inequality (7.20) holds. □

Theorem 7.11 *Let $s > 2$, $\alpha > 0$, $I_{b-}^\alpha f$ denotes the right-sided Riemann-Liouville fractional integral of f. Then the function $A_2 : \mathbb{R} \to [0,\infty)$ defined by*

$$\begin{aligned}
A_2(s) = & \frac{1}{s(s-2)} \left[\int_a^b f^s(y)\,(y-a)^\alpha\, dy - \int_a^b \left(\frac{\Gamma(\alpha+1)}{(b-x)^\alpha} I_{b-}^\alpha f(x) \right)^s (b-x)^\alpha dx \right. \\
& \left. -\alpha \int_a^b \int_y^b \left(\left| f(y) - \frac{\Gamma(\alpha+1)}{(b-x)^\alpha} I_{b-}^\alpha f(x) \right| \right)^s (y-x)^{\alpha-1} dx dy \right]
\end{aligned}$$

is exponentially convex and the following inequality holds

$$A_2(s) \le H_2(s), \tag{7.22}$$

where

$$H_2(s) = \frac{(b-a)^{\alpha(1-s)}}{s(s-2)} \left[(b-a)^{\alpha s} \int_a^b f^s(y) dy - (\Gamma(\alpha+1))^s \int_a^b (I_{b-}^\alpha f(x))^s dx \right].$$

Proof. Similar to the proof of Theorem 7.10. □

Next we give results with respect to the generalized Riemann-Liouville fractional derivative.

Theorem 7.12 *Let $s > 2$, and let the assumptions in Lemma 1.3 be satisfied. Let $D_a^{\alpha} f$ denote the generalized Riemann-Liouville fractional derivative of f of order $\alpha \geq 0$. Then the function $A_3 : \mathbb{R} \to [0,\infty)$ defined by*

$$\begin{aligned} A_3(s) = \frac{1}{s(s-2)} & \left[\int_a^b (b-y)^{\beta-\alpha} (D_a^{\beta} f(y))^s dy \right. \\ & - \int_a^b (x-a)^{\beta-\alpha} \left(\frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}} (D_a^{\alpha} f(x)) \right)^s dx - (\beta-\alpha) \times \\ & \left. \times \int_a^b \int_y^b \left(\left| D^{\alpha} f(y) - \frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}} D_a^{\alpha} f(x) \right| \right)^s (x-y)^{\beta-\alpha-1} dx dy \right] \end{aligned}$$

is exponentially convex and the following inequality holds

$$A_3(s) \leq H_3(s),$$

where

$$\begin{aligned} H_3(s) = \frac{(b-a)^{(\beta-\alpha)(1-s)}}{s(s-2)} & \left[(b-a)^{(\beta-\alpha)s} \int_a^b (D_a^{\beta} f(y))^s dy \right. \\ & \left. - (\Gamma(\beta-\alpha+1))^s \int_a^b (D_a^{\alpha} f(x))^s dx \right]. \end{aligned}$$

Proof. Similar to the proof of Theorems 3.8 and 7.10. □

In the following Theorem, we will construct a new inequality for the Canavati-type fractional derivative.

Theorem 7.13 *Let $s > 2$ and let the assumptions in Lemma 1.4 be satisfied. Let $D_a^{\gamma} f$ denote the Canavati-type fractional derivative of f. Then the function $A_4 : \mathbb{R} \to [0,\infty)$ defined by*

$$\begin{aligned} A_4(s) = \frac{1}{s(s-2)} & \left[\int_a^b (b-y)^{\nu-\gamma} (D_a^{\nu} f(y))^s dy \right. \\ & - \int_a^b (x-a)^{\nu-\gamma} \left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_a^{\gamma} f(x) \right)^s dx - (\nu-\gamma) \times \\ & \left. \times \int_a^b \int_y^b \left(\left| D_a^{\nu} f(y) - \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_a^{\gamma} f(x) \right| \right)^s (x-y)^{\nu-\gamma-1} dx dy \right] \end{aligned}$$

is exponentially convex and the following inequality holds

$$A_4(s) \le H_4(s), \tag{7.23}$$

where

$$\begin{aligned} H_4(s) \;=\; & \frac{(b-a)^{(\nu-\gamma)(1-s)}}{s(s-2)}\left[(b-a)^{(\nu-\gamma)s}\int_a^b (D_a^{\nu} f(y))^s dy\right. \\ & \left. -(\Gamma(\nu-\gamma+1))^s \int_a^b (D_a^{\gamma} f(x))^s dx\right]. \end{aligned}$$

Proof. Similar to the proof of Theorems 3.9 and 7.10. □

Next, new inequalities for the Caputo fractional derivative are given.

Theorem 7.14 *Let $s > 2$ $\nu \ge 0$ and $D_{*a}^{\nu} f$ denote the Caputo fractional derivative of f. Then the function $A_5 : \mathbb{R} \to [0,\infty)$ defined by*

$$\begin{aligned} A_5(s) = & \frac{1}{s(s-2)}\left[\int_a^b (b-y)^{n-\nu}(f^{(n)}(y))^s dy - \int_a^b (x-a)^{n-\nu}\left(\frac{\Gamma(n-\nu+1)}{(x-a)^{n-\nu}} D_{*a}^{\nu} f(x)\right)^s dx\right. \\ & \left. -(n-\nu)\int_a^b\int_y^b \left(\left| f^{(n)}(y) - \frac{\Gamma(n-\nu+1)}{(x-a)^{n-\nu}} D_{*a}^{\nu} f(x)\right|\right)^s (x-y)^{n-\nu-1} dx dy\right] \end{aligned}$$

is exponentially convex and the following inequality holds:

$$A_5(s) \le H_5(s), \tag{7.24}$$

where

$$\begin{aligned} H_5(s) \;=\; & \frac{(b-a)^{(n-\nu)(1-s)}}{s(s-2)}\left[(b-a)^{(n-\nu)s}\int_a^b (f^{(n)}(y))^s dy\right. \\ & \left. -\;(\Gamma(n-\nu+1))^s \int_a^b (D_{*a}^{\nu} f(x))^s dx\right]. \end{aligned}$$

Proof. Similar to the proof of Theorems 3.10 and 7.10. □

Theorem 7.15 *Let $s > 2$ and let the assumptions in Lemma 1.5 be satisfied. Let $D_{*a}^{\gamma} f$ denote the Caputo fractional derivative of f. Then the function $A_6 : \mathbb{R} \to [0,\infty)$ defined by*

$$\begin{aligned} A_6(s) = & \frac{1}{s(s-2)}\left[\int_a^b (b-y)^{\nu-\gamma}(D_{*a}^{\nu} f(y))^s dy - \int_a^b (x-a)^{\nu-\gamma}\left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_{*a}^{\gamma} f(x)\right)^s dx\right. \\ & \left. -(\nu-\gamma)\int_a^b\int_y^b \left(\left| D_{*a}^{\nu} f(y) - \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_{*a}^{\gamma} f(x)\right|\right)^s (x-y)^{\nu-\gamma-1} dx dy\right] \end{aligned}$$

is exponentially convex and the following inequality holds

$$A_6(s) \leq H_6(s), \tag{7.25}$$

where

$$\begin{aligned} H_6(s) &= \frac{(b-a)^{(\nu-\gamma)(1-s)}}{s(s-2)} \left[(b-a)^{(\nu-\gamma)s} \int_a^b (D_{*a}^{\nu} f(y))^s dy \right. \\ &\quad \left. - (\Gamma(\nu-\gamma+1))^s \int_a^b (D_{*a}^{\gamma} f(x))^s dx \right]. \end{aligned}$$

Proof. Similar to the proof of Theorems 3.11 and 7.10. □

Now, we give the following result.

Theorem 7.16 *Let $s>2$, $I^{\alpha}_{a+;\sigma;\eta} f$ denotethe Erdélyi-Kober type fractional integrals of f, ${}_2F_1(a,b;c;z)$ denotes the hypergeometric function. Then the function $A_7 : \mathbb{R} \to [0,\infty)$ defined by*

$$\begin{aligned} A_7(s) = &\frac{1}{s(s-2)} \left[\int_a^b y^{\sigma-1} (b^{\sigma} - y^{\sigma})^{\alpha} {}_2F_1(y) f^s(y) dy \right. \\ &- (\Gamma(\alpha+1))^s \int_a^b x^{\sigma+\sigma\alpha s-1} (x^{\sigma}-a^{\sigma})^{\alpha(s-1)} ({}_2F_1(x))^{1-s} (I^{\alpha}_{a+;\sigma;\eta} f(x))^s dx \\ &- \alpha\sigma \int_a^b \int_y^b \left(\left| f(y) - \frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^{\sigma}\right)^{\alpha} {}_2F_1(x)} I^{\alpha}_{a+;\sigma;\eta} f(x) \right| \right)^s \\ &\left. \times x^{-\sigma\eta+\sigma-1} y^{\sigma\eta+\sigma-1} (x^{\sigma}-y^{\sigma})^{\alpha-1} dx dy \right] \end{aligned}$$

is exponentially convex and the following inequality holds

$$A_7(s) \leq H_7(s), \tag{7.26}$$

where

$$\begin{aligned} H_7(s) &= \frac{(b^{\sigma}-a^{\sigma})^{\alpha(1-s)}}{s(s-2)} \left[(b^{\sigma}-a^{\sigma})^{\alpha s} b^{\sigma-1} \int_a^b {}_2F_1(y) f^s(y) dy \right. \\ &\quad \left. - a^{\sigma-1+\alpha\sigma s} (\Gamma(\alpha+1))^s \int_a^b ({}_2F_1(x))^{1-s} (I^{\alpha}_{a+;\sigma;\eta} f(x))^s dx \right], \end{aligned}$$

$${}_2F_1(x) = {}_2F_1\left(-\eta, \alpha; \alpha+1; 1-\left(\frac{a}{x}\right)^{\sigma}\right) \text{ and } {}_2F_1(y) = {}_2F_1\left(\eta, \alpha; \alpha+1; 1-\left(\frac{b}{y}\right)^{\sigma}\right).$$

Proof. Similar to the proof of Theorems 3.12 and 7.10. □

In the following theorem we will give inequalities that follow from the results given in Theorems 7.10-7.16.

Theorem 7.17 *For* $i = 1, \ldots, 7$ *the following inequalities hold*

$$\text{(i).} \qquad [A_i(p)]^{\frac{q-r}{q-p}} [A_i(q)]^{\frac{r-p}{q-p}} \leq H_i(r)$$

$$\text{(ii).} \qquad [A_i(r)]^{\frac{p-q}{p-r}} [A_i(p)]^{\frac{q-r}{p-r}} \leq H_i(q)$$

$$\text{(iii).} \qquad A_i(p) \leq [H_i(r)]^{\frac{q-p}{q-r}} [H_i(q)]^{\frac{p-r}{q-r}}$$

for every choice of $r, p, q \in \mathbb{R}_+$, *such that* $2 < r < p < q$.

Proof. Similar to the proof of Theorem 6.14. □

Chapter 8

On a new class of refined discrete Hardy-type inequalities

Generalizing certain results of Godunova, [40] (see also [80, Chapter IV, p. 152]), Vasić and Pečarić in [99] proved that the Hardy-type inequality

$$\sum_{m=1}^{\infty} u_m \Phi\left(\sum_{n=1}^{m} k_{mn} a_n\right) \leq \sum_{n=1}^{\infty} v_n \Phi(a_n) \tag{8.1}$$

holds for all non-negative convex functions Φ on an interval $I \subseteq \mathbb{R}$, sequences $(a_n)_{n\in\mathbb{N}}$ in I, sequences $(u_n)_{n\in\mathbb{N}}$ of positive real numbers, and positive real numbers k_{mn}, $m \in \mathbb{N}$, $n = 1,\ldots,m$, such that

$$\sum_{n=1}^{m} k_{mn} = 1,\ m \in \mathbb{N},\ \text{ and } \sum_{m=n}^{\infty} u_m k_{mn} \leq v_n,\ n \in \mathbb{N}. \tag{8.2}$$

Moreover, if the function Φ is concave and the sign of inequality in (8.2) is reversed, then (8.1) holds with the reversed sign of inequality.

As special cases of (8.1) for sequences of positive real numbers $(a_n)_{n\in\mathbb{N}}$, we get the so-called Godunova's inequality

$$\sum_{n=1}^{\infty} \frac{1}{n+1} \left(\frac{1}{n} \sum_{m=1}^{n} a_m\right)^p < \sum_{n=1}^{\infty} \frac{a_n^p}{n}, \tag{8.3}$$

where $p \in \mathbb{R}$, $p > 1$, and Akerberg's inequality

$$\sum_{n=1}^{\infty} \frac{1}{n+1} \left(n! \prod_{m=1}^{n} a_m \right)^{\frac{1}{n}} < \sum_{n=1}^{\infty} a_n, \tag{8.4}$$

obtained by Akerberg in [7]. It can be shown that inequality (8.4) implies the well-known Carleman inequality

$$\sum_{n=1}^{\infty} \left(\prod_{m=1}^{n} a_m \right)^{\frac{1}{n}} < e \sum_{n=1}^{\infty} a_n, \tag{8.5}$$

with the best possible constant e, proved by Carleman in [17].

Motivated by these results, in this chapter we obtain a generalization and a refinement of (8.1) by proving a new refined general weighted discrete Hardy-type inequality with a positive real parameter. As its consequences, obtained by rewriting it for various parameters, kernels, weights, and convex (or concave) functions, we derive new weighted and unweighted generalizations and refinements of inequalities (8.3)-(8.5). Moreover, we show that our result improves and generalizes Carleman's inequality (8.5), that is, we get a new refined weighted strengthened Carleman's inequality. By employing the concepts of exponential and logarithmic convexity, we obtain upper and lower bounds for the left-hand sides of some refined Hardy-type inequalities from the previous section. In particular, we derive upper and lower bounds for the left-hand side of the weighted Godunova's inequality, as well as of the strengthened weighted Carleman's inequality.

8.1 New refined discrete Hardy-type inequalities

Now, we are ready to state and prove a new general refined discrete Hardy-type inequality with a kernel, related to arbitrary non-negative convex functions on real intervals (see [23]).

Theorem 8.1 *Let $t \in \mathbb{R}_+$, $M, N \in \mathbb{N}$, and let non-negative real numbers u_m, v_n, k_{mn}, where $m \in \mathbb{N}_M$, $n \in \mathbb{N}_N$, be such that*

$$K_m = \sum_{n=1}^{N} k_{mn} > 0, \; m \in \mathbb{N}_M, \tag{8.6}$$

and

$$v_n = \left[\sum_{m=1}^{M} u_m \left(\frac{k_{mn}}{K_m} \right)^t \right]^{\frac{1}{t}}, \; n \in \mathbb{N}_N. \tag{8.7}$$

Let Φ be a non-negative convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ be any function such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$. Then the inequality

$$\left(\sum_{n=1}^{N} v_n \Phi(a_n) \right)^t - \sum_{m=1}^{M} u_m \Phi^t(A_m) \geq t \sum_{m=1}^{M} u_m \frac{\Phi^{t-1}(A_m)}{K_m} \sum_{n=1}^{N} k_{mn} r_{mn} \tag{8.8}$$

holds for all $t \geq 1$ *and real numbers* $a_n \in I$, $n \in \mathbb{N}_N$, *where*

$$A_m = \frac{1}{K_m} \sum_{n=1}^{N} k_{mn} a_n \tag{8.9}$$

and

$$r_{mn} = \big|\, |\Phi(a_n) - \Phi(A_m)| - |\varphi(A_m)| \cdot |a_n - A_m| \,\big| . \tag{8.10}$$

If $t \in (0,1]$ *and the function* $\Phi : I \to \mathbb{R}$ *is positive and concave, then the order of the terms on the left-hand side of* (8.9) *is reversed, that is, the inequality*

$$\sum_{m=1}^{M} u_m \Phi^t(A_m) - \left(\sum_{n=1}^{N} v_n \Phi(a_n) \right)^t \geq t \sum_{m=1}^{M} u_m \frac{\Phi^{t-1}(A_m)}{K_m} \sum_{n=1}^{N} k_{mn} r_{mn} \tag{8.11}$$

holds for all $t \in (0,1]$.

Proof. First, note that

$$\sum_{n=1}^{N} k_{mn}(a_n - A_m) = \sum_{n=1}^{N} k_{mn} a_n - A_m \sum_{n=1}^{N} k_{mn} = K_m A_m - A_m K_m = 0 \tag{8.12}$$

holds for all $m \in \mathbb{N}_M$. Further, since $\min_{n \in \mathbb{N}_N} a_n \in I$, $\max_{n \in \mathbb{N}_N} a_n \in I$, and

$$\min_{n \in \mathbb{N}_N} a_n \leq a_n \leq \max_{n \in \mathbb{N}_N} a_n, \; n \in \mathbb{N}_N,$$

we easily get

$$\min_{n \in \mathbb{N}_N} a_n \leq \frac{1}{K_m} \sum_{n=1}^{N} k_{mn} a_n \leq \max_{n \in \mathbb{N}_N} a_n.$$

Therefore, $A_m \in I$ for all $m \in \mathbb{N}_M$. Moreover, if $a_n \in \mathrm{Int} I$, for all $n \in \mathbb{N}_N$, then $A_m \in \mathrm{Int} I$ for all $m \in \mathbb{N}_M$, as well.

Now, we are ready to prove (8.8), so suppose that the function Φ is convex and $t \geq 1$. Fix $m \in \mathbb{N}_M$ and $n \in \mathbb{N}_N$. If $A_m \in \mathrm{Int} I$, then substituting $x = A_m$ and $y = a_n$ in (1.8) yields

$$\Phi(a_n) - \Phi(A_m) - \varphi(A_m)(a_n - A_m) \geq \big|\, |\Phi(a_n) - \Phi(A_m)| - |\varphi(A_m)| \cdot |a_n - A_m| \,\big|$$

and, therefore,

$$\frac{k_{mn}}{K_m} \left[\Phi(a_n) - \Phi(A_m) - \varphi(A_m)(a_n - A_m) \right] \geq \frac{k_{mn}}{K_m} r_{mn}. \tag{8.13}$$

Observe that (8.13) holds trivially also if $k_{mn} = 0$ and A_m is an endpoint of I (if I is not an open interval). Hence, it is only left to analyze the case when A_m is an endpoint of I and $k_{mn} > 0$ (from condition (8.6) we see that such n exists for every $m \in \mathbb{N}_M$). Without loss of generality, assume that A_m is the left endpoint of I, that is, $A_m = \min I$. Then $a_l - A_m \geq 0$ for all $l \in \mathbb{N}_N$, so (8.12) implies that $k_{ml}(a_l - A_m) = 0$ for all $l \in \mathbb{N}_N$. In particular, from $k_{mn} > 0$ we get $a_n = A_m$, so both sides of (8.13) are equal to 0. The case when $A_m = \max I$

is analogous. Thus, (8.13) holds for all $m \in \mathbb{N}_M$ and $n \in \mathbb{N}_N$. Summing it up over $n \in \mathbb{N}_N$ gives

$$\frac{1}{K_m}\sum_{n=1}^{N} k_{mn}\Phi(a_n) - \frac{1}{K_m}\sum_{n=1}^{N} k_{mn}\Phi(A_m) - \frac{\varphi(A_m)}{K_m}\sum_{n=1}^{N} k_{mn}(a_n - A_m) \\ \geq \frac{1}{K_m}\sum_{n=1}^{N} k_{mn}r_{mn}$$

and, by using (8.12), further

$$\Phi(A_m) + \frac{1}{K_m}\sum_{n=1}^{N} k_{mn}r_{mn} \leq \frac{1}{K_m}\sum_{n=1}^{N} k_{mn}\Phi(a_n). \tag{8.14}$$

Since the left-hand side of (8.14) is non-negative and the function $\alpha \mapsto \alpha^t$ is strictly increasing on $[0,\infty)$ for $t \geq 1$, by applying Bernoulli's inequality we obtain

$$\Phi^t(A_m) + t\frac{\Phi^{t-1}(A_m)}{K_m}\sum_{n=1}^{N} k_{mn}r_{mn} \leq \left(\Phi(A_m) + \frac{1}{K_m}\sum_{n=1}^{N} k_{mn}r_{mn}\right)^t \\ \leq \left(\frac{1}{K_m}\sum_{n=1}^{N} k_{mn}\Phi(a_n)\right)^t. \tag{8.15}$$

Multiplying (8.15) by u_m, then summing up over $m \in \mathbb{N}_M$, and applying Minkowski's inequality to the right-hand side, we get

$$\begin{aligned}
&\sum_{m=1}^{M} u_m\Phi^t(A_m) + t\sum_{m=1}^{M} u_m\frac{\Phi^{t-1}(A_m)}{K_m}\sum_{n=1}^{N} k_{mn}r_{mn} \\
&\leq \sum_{m=1}^{M} u_m\left(\Phi(A_m) + \frac{1}{K_m}\sum_{n=1}^{N} k_{mn}r_{mn}\right)^t \leq \sum_{m=1}^{M} u_m\left(\frac{1}{K_m}\sum_{n=1}^{N} k_{mn}\Phi(a_n)\right)^t \\
&= \left\{\left[\sum_{m=1}^{M} u_m\left(\frac{1}{K_m}\sum_{n=1}^{N} k_{mn}\Phi(a_n)\right)^t\right]^{\frac{1}{t}}\right\}^t \\
&\leq \left\{\sum_{n=1}^{N}\Phi(a_n)\left[\sum_{m=1}^{M} u_m\left(\frac{k_{mn}}{K_m}\right)^t\right]^{\frac{1}{t}}\right\}^t = \left(\sum_{n=1}^{N} v_n\Phi(a_n)\right)^t,
\end{aligned}$$

so (8.8) holds. The proof for a concave function Φ and $t \in (0,1]$ is similar. Namely, by the same arguments as for convex functions, from (1.9) we first obtain

$$\frac{k_{mn}}{K_m}\left[\Phi(A_m) - \Phi(a_n) - \varphi(A_m)(A_m - a_n)\right] \geq \frac{k_{mn}}{K_m}r_{mn},\ m \in \mathbb{N}_M,\ n \in \mathbb{N}_N,$$

then

$$\Phi^t(A_m) - t\frac{\Phi^{t-1}(A_m)}{K_m}\sum_{n=1}^{N} k_{mn}r_{mn} \geq \left(\Phi(A_m) - \frac{1}{K_m}\sum_{n=1}^{N} k_{mn}r_{mn}\right)^t$$
$$\geq \left(\frac{1}{K_m}\sum_{n=1}^{N} k_{mn}\Phi(a_n)\right)^t, \; m \in \mathbb{N}_M,$$

and finally

$$\sum_{m=1}^{M} u_m\Phi^t(A_m) - t\sum_{m=1}^{M} u_m\frac{\Phi^{t-1}(A_m)}{K_m}\sum_{n=1}^{N} k_{mn}r_{mn}$$
$$\geq \sum_{m=1}^{M} u_m\left(\Phi(A_m) - \frac{1}{K_m}\sum_{n=1}^{N} k_{mn}r_{mn}\right)^t \geq \left(\sum_{n=1}^{N} v_n\Phi(a_n)\right)^t,$$

that is, we get (8.11). □

Remark 8.1 In particular, for $t = 1$ inequality (8.8) reduces to

$$\sum_{n=1}^{N} v_n\Phi(a_n) - \sum_{m=1}^{M} u_m\Phi(A_m) \geq \sum_{m=1}^{M}\frac{u_m}{K_m}\sum_{n=1}^{N} k_{mn}r_{mn}, \tag{8.16}$$

where in this setting we have

$$v_n = \sum_{m=1}^{M} u_m\frac{k_{mn}}{K_m}, \; m \in \mathbb{N}_M. \tag{8.17}$$

Moreover, by analyzing the proof of Theorem 8.1, we see that (8.16) holds for all convex functions $\Phi : I \to \mathbb{R}$, that is, Φ does not need to be non-negative. Similarly, if Φ is any real concave function on I (not necessarily positive), then (8.16) holds with the reversed order of the terms on its left-hand side. □

Remark 8.2 Rewriting (8.8) with $t = \frac{q}{p} \geq 1$, that is, for $0 < p \leq q < \infty$ or $-\infty < q \leq p < 0$, and with an arbitrary non-negative convex function Φ, we obtain

$$\left(\sum_{n=1}^{N} v_n\Phi(a_n)\right)^{\frac{q}{p}} - \sum_{m=1}^{M} u_m\Phi^{\frac{q}{p}}(A_m) \geq \frac{q}{p}\sum_{m=1}^{M} u_m\frac{\Phi^{\frac{q}{p}-1}(A_m)}{K_m}\sum_{n=1}^{N} k_{mn}r_{mn}, \tag{8.18}$$

where

$$v_n = \left[\sum_{m=1}^{M} u_m\left(\frac{k_{mn}}{K_m}\right)^{\frac{q}{p}}\right]^{\frac{p}{q}}, \; n \in \mathbb{N}_N.$$

Especially, if $p \geq 1$ or $p < 0$ (in that case Φ should be positive), then the function Φ^p is convex as well, so by replacing Φ with Φ^p relation (8.18) becomes

$$\left(\sum_{n=1}^{N} v_n\Phi^p(a_n)\right)^{\frac{q}{p}} - \sum_{m=1}^{M} u_m\Phi^q(A_m) \geq \frac{q}{p}\sum_{m=1}^{M} u_m\frac{\Phi^{q-p}(A_m)}{K_m}\sum_{n=1}^{N} k_{mn}r_{mn}. \tag{8.19}$$

On the other hand, if Φ is a positive concave function and $t = \frac{q}{p} \in (0,1]$, that is, $0 < q \leq p < \infty$ or $-\infty < p \leq q < 0$, then (8.18) holds with the reversed order of the terms on its left-hand side. Moreover, if $p \in (0,1]$, then the function Φ^p is concave, so the order of the terms on the left-hand side of (8.19) is reversed. □

Theorem 8.1 holds even if $M = N = \infty$. More precisely, following a similar procedure as in the proof of Theorem 8.1, we get the following corollary.

Corollary 8.1 *Suppose $t \in \mathbb{R}_+$ and non-negative numbers u_m, v_n, k_{mn}, for $m,n \in \mathbb{N}$, are such that*

$$K_m = \sum_{n=1}^{\infty} k_{mn} \in \mathbb{R}_+,\ m \in \mathbb{N}, \ \text{ and } \ v_n = \left[\sum_{m=1}^{\infty} u_m \left(\frac{k_{mn}}{K_m} \right)^t \right]^{\frac{1}{t}} < \infty,\ n \in \mathbb{N}.$$

If Φ is a non-negative convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is any function such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \mathrm{Int}\, I$, then the inequality

$$\left(\sum_{n=1}^{\infty} v_n \Phi(a_n) \right)^t - \sum_{m=1}^{\infty} u_m \Phi^t(A_m) \geq t \sum_{m=1}^{\infty} u_m \frac{\Phi^{t-1}(A_m)}{K_m} \sum_{n=1}^{\infty} k_{mn} r_{mn} \tag{8.20}$$

holds for all $t \geq 1$ and all real numbers $a_n \in I$, $n \in \mathbb{N}$, such that

$$A_m = \frac{1}{K_m} \sum_{n=1}^{\infty} k_{mn} a_n \in I,\ m \in \mathbb{N}, \tag{8.21}$$

where r_{mn} is defined by (8.10)*. If $t \in (0,1]$ and $\Phi : I \to \mathbb{R}$ is a positive concave function, then the order of the terms on the left-hand side of* (8.20) *is reversed.*

Remark 8.3 If I is a segment, that is, a closed subset of $\mathbb{R}$, condition (8.21) is fulfilled automatically since the series defining K_m converge for all $m \in \mathbb{N}$. Note that this condition can not be omitted in any other general case. Further, according to Remark 8.1, in the case when $t = 1$ the function Φ from Corollary 8.1 needs not to be non-negative (or positive if it is concave). Finally, under conditions of Corollary 8.1, Remark 8.2 holds also with $M = N = \infty$. □

Since the right-hand sides of relations (8.8) and (8.11) are non-negative, the next general discrete Hardy-type inequality follows as a direct consequence of Theorem 8.1

Corollary 8.2 *Let $t \in \mathbb{R}_+$, $M,N \in \mathbb{N}$, and let non-negative real numbers u_m, v_n, k_{mn}, for $m \in \mathbb{N}_M$, $n \in \mathbb{N}_N$, fulfill* (8.6) *and* (8.7)*. If Φ is a non-negative convex function on an interval $I \subseteq \mathbb{R}$, then*

$$\sum_{m=1}^{M} u_m \Phi^t(A_m) \leq \left(\sum_{n=1}^{N} v_n \Phi(a_n) \right)^t \tag{8.22}$$

holds for all $t \geq 1$, real numbers $a_n \in I$, $n \in \mathbb{N}_N$, and A_m defined by (8.9)*. If $t \in (0,1]$ and the function $\Phi : I \to \mathbb{R}$ is positive and concave, then the sign of inequality in* (8.22) *is reversed.*

Remark 8.4 Observing that the right-hand side of (8.19) is non-negative, for $p \geq 1$ and a non-negative convex function Φ we get

$$\left(\sum_{m=1}^{M} u_m \Phi^q(A_m)\right)^{\frac{1}{q}} \leq \left(\sum_{n=1}^{N} v_n \Phi^p(a_n)\right)^{\frac{1}{p}}. \tag{8.23}$$

Obviously, similar arguments can also be applied to other cases analyzed in Remark 8.2 However, we omit their further analysis since here it reflects only to the sign of inequality in (8.23). On the other hand, if non-negative real numbers u_m, v_n, k_{mn}, where $m,n \in \mathbb{N}$, fulfill the conditions of Corollary 8.1, then Corollary 8.2 holds also with $M = N = \infty$. □

8.2 Applications. A new refined Carleman's inequality

In this section we continue earlier analysis by considering some interesting particular cases of Theorem 8.1 and its consequences. Especially, we obtain a refined discrete Jensen's inequality and a refinement and a generalization of the Hardy-type inequality (8.1). As a special case of the Hardy-type inequality obtained, we get a new refined weighted version of Godunova's inequality (8.3). Finally, as our most important result in this section, we state and prove a new refined weighted strengthened Carleman's inequality and show how it refines and generalizes inequality (8.3). More about history, proofs and new developments regarding Carleman's inequality can be found in [31], [64], [94], and in in the references cited in those papers.

First, as a consequence of Theorem 8.1 we obtain a general refined discrete Jensen's inequality.

Theorem 8.2 *Let $\Phi : I \to \mathbb{R}$ be a non-negative convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ be such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \mathrm{Int}I$. Let $t \geq 1$ and $N \in \mathbb{N}$. Then the inequality*

$$\left(\frac{1}{W_N}\sum_{n=1}^{N} w_n \Phi(a_n)\right)^{t} - \Phi^t(A_N) \geq t\,\frac{\Phi^{t-1}(A_N)}{W_N}\sum_{n=1}^{N} w_n r_n \tag{8.24}$$

holds for all real numbers $a_n \in I$ and $w_n \geq 0$, $n \in \mathbb{N}_N$, where

$$W_N = \sum_{n=1}^{N} w_n > 0, \;\; A_N = \frac{1}{W_N}\sum_{n=1}^{N} w_n a_n,$$

and

$$r_n = \big|\,|\Phi(a_n) - \Phi(A_N)| - |\varphi(A_N)| \cdot |a_n - A_N|\,\big|, \; n \in \mathbb{N}_N.$$

If Φ is a positive concave function and $t \in (0,1]$, then the order of the terms on the left-hand side of (8.24) *is reversed.*

Proof. Follows directly from Theorem 8.1, by taking arbitrary $M \in \mathbb{N}$ and positive real numbers u_m and α_m for $m \in \mathbb{N}_M$. Substituting $k_{mn} = \alpha_m w_n$, for all $m \in \mathbb{N}_M$ we get $K_m = \alpha_m W_N$, $A_m = A_N$, and $r_{mn} = r_n$, while $v_n = \dfrac{w_n}{W_N} U_M^{\frac{1}{t}}$ holds for all $n \in \mathbb{N}_N$, where $U_M = \sum_{m=1}^{M} u_m$. Thus, (8.9) reduces to (8.24) and does not depend on M, u_m, and α_m. □

Remark 8.5 For $t = 1$ inequality (8.24) becomes the classical refined discrete Jensen's inequality

$$\frac{1}{W_N}\sum_{n=1}^{N} w_n \Phi(a_n) - \Phi(A_N) \geq \frac{1}{W_N}\sum_{n=1}^{N} w_n r_n \tag{8.25}$$

and the function Φ is not necessarily non-negative. Of course, if the function Φ is concave, relation (8.25) holds with the reversed order of the terms on its left-hand side. □

Observe that Theorem 8.1 and Corollary 8.1 can be easily rewritten with arbitrary $M, N \in \mathbb{N}$ and $K_m = 1$ for all $m \in \mathbb{N}_M$. Here, we emphasize only the case when $M = N = \infty$ since it provides a generalization and a refinement of the Hardy-type inequality (8.1).

Theorem 8.3 *Let I be an interval in $\mathbb{R}$, $\Phi : I \to \mathbb{R}$ be a non-negative convex function, and $\varphi : I \to \mathbb{R}$ be such that $\varphi(x) \in \partial\Phi(x)$, $x \in \operatorname{Int} I$. Let $t \in \mathbb{R}_+$. If real numbers $u_m, v_n, k_{mn} \geq 0$, $m, n \in \mathbb{N}$, are such that*

$$\sum_{n=1}^{\infty} k_{mn} = 1,\ m \in \mathbb{N}, \ \ \textit{and} \ \ v_n = \left(\sum_{m=1}^{\infty} u_m k_{mn}^t\right)^{\frac{1}{t}} < \infty,\ n \in \mathbb{N},$$

if real numbers $a_n \in I$, $n \in \mathbb{N}$, fulfill $A_m = \sum_{n=1}^{\infty} k_{mn} a_n \in I$, $m \in \mathbb{N}$, and if r_{mn} is defined by (8.10)*, then the inequality*

$$\left(\sum_{n=1}^{\infty} v_n \Phi(a_n)\right)^t - \sum_{m=1}^{\infty} u_m \Phi^t(A_m) \geq t \sum_{m=1}^{\infty} u_m \Phi^{t-1}(A_m) \sum_{n=1}^{\infty} k_{mn} r_{mn} \tag{8.26}$$

holds for all $t \geq 1$. If $t \in (0,1]$ and the function Φ is positive and concave, the order of the terms on the left-hand side of (8.26) *is reversed.*

Remark 8.6 Set $k_{mn} = 0$ for $m < n$ in Theorem 8.3. Then

$$\sum_{n=1}^{m} k_{mn} = 1,\ A_m = \sum_{n=1}^{m} k_{mn} a_n,\ m \in \mathbb{N}, \ \ \text{and} \ \ v_n = \left(\sum_{m=n}^{\infty} u_m k_{mn}^t\right)^{\frac{1}{t}},\ n \in \mathbb{N}.$$

Therefore, in this setting (8.26) becomes

$$\begin{aligned}\left(\sum_{n=1}^{\infty} v_n \Phi(a_n)\right)^t &- \sum_{m=1}^{\infty} u_m \Phi^t\left(\sum_{n=1}^{m} k_{mn} a_n\right) \\ &\geq t \sum_{m=1}^{\infty} u_m \Phi^{t-1}\left(\sum_{n=1}^{m} k_{mn} a_n\right) \sum_{n=1}^{m} k_{mn} r_{mn}.\end{aligned} \tag{8.27}$$

In particular, for $t = 1$ we get $v_n = \sum_{m=n}^{\infty} u_m k_{mn}$ and

$$\sum_{n=1}^{\infty} v_n \Phi(a_n) - \sum_{m=1}^{\infty} u_m \Phi\left(\sum_{n=1}^{m} k_{mn} a_n\right) \geq \sum_{m=1}^{\infty} u_m \sum_{n=1}^{m} k_{mn} r_{mn}, \tag{8.28}$$

so (8.26), (8.27), and (8.28) can be, respectively, regarded as two generalizations and a refinement of the Vasić-Pečarić relation (8.1). As in Theorem 8.3, for $t \in (0,1]$ and a positive concave function Φ, inequality (8.27) holds with the reversed order of the terms on its left-hand side. The same goes for (8.28) also, although in this case Φ does not have to be non-negative (or positive, if it is concave). □

Now, we consider some particular functions Φ and non-negative real numbers u_m and k_{mn}. The following result provides a new weighted refinement of Godunova's inequality (8.3). Here we make use of the function $\Phi : \mathbb{R}_+ \to \mathbb{R}$, $\Phi(x) = x^p$, where $p \in \mathbb{R}$, $p \neq 0$. For $p \geq 1$ and $p < 0$ this function is convex, while it is concave for $p \in (0,1]$. In both cases we have $\varphi(x) = px^{p-1}$, $x \in \mathbb{R}_+$.

Theorem 8.4 *Let $N \in \mathbb{N}$, $t \in \mathbb{R}_+$, and $p \in \mathbb{R}$, $p \neq 0$. Let $(w_n)_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers, such that $w_1 > 0$, and let*

$$W_m = \sum_{m=1}^{n} w_n,\ n \in \mathbb{N}. \tag{8.29}$$

If $t \geq 1$ and $p \in \mathbb{R} \setminus [0,1)$, then the inequality

$$\begin{aligned}\left[\sum_{n=1}^{N} w_n \left(\sum_{m=n}^{N} \frac{w_{m+1}}{W_m^t W_{m+1}}\right)^{\frac{1}{t}} a_n^p\right]^t &- \sum_{m=1}^{N} \frac{w_{m+1}}{W_{m+1}} A_m^{pt} \\ &\geq t \sum_{m=1}^{N} \frac{w_{m+1}}{W_m W_{m+1}} A_m^{p(t-1)} \sum_{n=1}^{m} r_{mn} w_n\end{aligned} \tag{8.30}$$

holds for all sequences $(a_n)_{n\in\mathbb{N}}$ of positive real numbers, where

$$A_m = \frac{1}{W_m} \sum_{n=1}^{m} w_n a_n \quad \textit{and} \quad r_{mn} = \left| |a_n^p - A_m^p| - |p| \cdot |A_m|^{p-1} \cdot |a_n - A_m| \right|, \tag{8.31}$$

for $m,n \in \mathbb{N}$. If $t, p \in (0,1]$, then the order of terms on the left-hand side of (8.30) is reversed.

Proof. Note that $w_1 > 0$ implies $W_n > 0$ for all $n \in \mathbb{N}$. In Theorem 8.1, set $\Phi : \mathbb{R}_+ \to \mathbb{R}$, $\Phi(x) = x^p$, $M = N$, $u_m = \dfrac{w_{m+1}}{W_{m+1}}$, and

$$k_{mn} = \begin{cases} \dfrac{w_n}{W_m}, & m \geq n, \\ 0, & \text{otherwise}, \end{cases}$$

for $m,n \in \mathbb{N}_N$. Then we have $K_m = \sum_{n=1}^{m} \frac{w_n}{W_m} = 1$, $m \in \mathbb{N}_N$, and

$$v_n = w_n \left(\sum_{m=n}^{N} \frac{w_{m+1}}{W_m^t W_{m+1}} \right)^{\frac{1}{t}}, \; n \in \mathbb{N}_N,$$

so (8.30) holds. □

According to Theorem 8.3 and Remark 8.6, Theorem 8.4 can be easily extended to $N = \infty$.

Corollary 8.3 *Let $t \in \mathbb{R}_+$ and $p \in \mathbb{R}$, $p \neq 0$. Let $(w_n)_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers and the sequence $(W_n)_{n\in\mathbb{N}}$ be defined by* (8.29). *Let $w_1 > 0$ and $\sum_{m=1}^{\infty} \frac{w_{m+1}}{W_m^t W_{m+1}} < \infty$. If $t \geq 1$ and $p \in \mathbb{R} \setminus [0,1)$, then the inequality*

$$\left[\sum_{n=1}^{\infty} w_n \left(\sum_{m=n}^{\infty} \frac{w_{m+1}}{W_m^t W_{m+1}} \right)^{\frac{1}{t}} a_n^p \right]^t - \sum_{m=1}^{\infty} \frac{w_{m+1}}{W_{m+1}} A_m^{pt} \geq t \sum_{m=1}^{\infty} \frac{w_{m+1}}{W_m W_{m+1}} A_m^{p(t-1)} \sum_{n=1}^{m} r_{mn} w_n \tag{8.32}$$

holds for all sequences $(a_n)_{n\in\mathbb{N}}$ of positive real numbers and A_m, r_{mn} defined by (8.31) *for $m,n \in \mathbb{N}$. If $t,p \in (0,1]$, then* (8.32) *holds with the reversed order of the terms on its left-hand side.*

Remark 8.7 Rewrite Theorem 8.4 with $t = 1$. Then we have

$$v_n = w_n \sum_{m=n}^{N} \frac{w_{m+1}}{W_m W_{m+1}} = \frac{w_n}{W_n} \left(1 - \frac{W_n}{W_{N+1}} \right), \tag{8.33}$$

so for $p \in \mathbb{R} \setminus [0,1]$ we get the inequality

$$\sum_{n=1}^{N} \left(1 - \frac{W_n}{W_{N+1}} \right) \frac{w_n}{W_n} a_n^p - \sum_{m=1}^{N} \frac{w_{m+1}}{W_{m+1}} \left(\frac{1}{W_m} \sum_{n=1}^{m} w_n a_n \right)^p \geq \sum_{m=1}^{N} \frac{w_{m+1}}{W_m W_{m+1}} \sum_{n=1}^{m} r_{mn} w_n, \tag{8.34}$$

while for $p \in (0,1)$ the terms on the left-hand side of (8.34) swap their positions. If $p = 1$, (8.34) holds trivially with both sides equal to 0. On the other hand, denote

$$W_\infty = \sum_{n=1}^{\infty} w_n \tag{8.35}$$

and set $t = 1$ in Corollary 8.3. By using (8.33) and the fact that $0 < W_n \le W_{n+1} \le W_\infty$, that is, $0 \le 1 - \frac{W_n}{W_\infty} \le 1$ for all $n \in \mathbb{N}$, relation (8.34) becomes

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{w_n}{W_n} a_n^p - \sum_{m=1}^{\infty} \frac{w_{m+1}}{W_{m+1}} \left(\frac{1}{W_m} \sum_{n=1}^{m} w_n a_n \right)^p
&\ge \sum_{n=1}^{\infty} \left(1 - \frac{W_n}{W_\infty}\right) \frac{w_n}{W_n} a_n^p - \sum_{m=1}^{\infty} \frac{w_{m+1}}{W_{m+1}} \left(\frac{1}{W_m} \sum_{n=1}^{m} w_n a_n \right)^p \\
&\ge \sum_{m=1}^{\infty} \frac{w_{m+1}}{W_m W_{m+1}} \sum_{n=1}^{m} r_{mn} w_n \ge 0.
\end{aligned}$$

Here we also covered the case when $W_\infty = \infty$. □

Remark 8.8 Theorem 8.4 can be considered in the unweighted case, that is, for $w_n = 1$, $n \in \mathbb{N}$. Then $A_m = \frac{1}{m} \sum_{n=1}^{m} a_n$, $m \in \mathbb{N}$, so relation (8.30) reduces to

$$\left[\sum_{n=1}^{N} \left(\sum_{m=n}^{N} \frac{m^{-t}}{m+1} \right)^{\frac{1}{t}} a_n^p \right]^t - \sum_{m=1}^{N} \frac{1}{m+1} A_m^{pt} \ge t \sum_{m=1}^{N} \frac{1}{m(m+1)} A_m^{p(t-1)} \sum_{n=1}^{m} r_{mn}.$$

Moreover, for $t = 1$ and $p \in \mathbb{R} \setminus [0,1)$ we have

$$\begin{aligned}
\sum_{n=1}^{N} \frac{a_n^p}{n} - \sum_{m=1}^{N} \frac{1}{m+1} \left(\frac{1}{m} \sum_{n=1}^{m} a_n \right)^p
&\ge \sum_{n=1}^{N} \left(1 - \frac{n}{N+1}\right) \frac{a_n^p}{n} - \sum_{m=1}^{N} \frac{1}{m+1} \left(\frac{1}{m} \sum_{n=1}^{m} a_n \right)^p \\
&\ge \sum_{m=1}^{N} \frac{1}{m(m+1)} \sum_{n=1}^{m} r_{mn} \ge 0. \qquad (8.36)
\end{aligned}$$

Finally, for $N = \infty$ inequality (8.30) becomes

$$\sum_{n=1}^{\infty} \frac{a_n^p}{n} - \sum_{m=1}^{\infty} \frac{1}{m+1} \left(\frac{1}{m} \sum_{n=1}^{m} a_n \right)^p \ge \sum_{m=1}^{\infty} \frac{1}{m(m+1)} \sum_{n=1}^{m} r_{mn} \ge 0, \qquad (8.37)$$

so (8.36) and (8.37) respectively provide a finite section and a refinement of Godunova's inequality (8.3). Therefore, Theorem 8.4 can be regarded as a weighted finite section of (8.3), while Corollary 8.3 gives a weighted generalization of Godunova's inequality. □

As the last result in this section, applying Theorem 8.1 to the convex function $\Phi : \mathbb{R} \to \mathbb{R}_+$, $\Phi(x) = e^x$, we obtain a new strengthened weighted Carleman's inequality. Here we have $\varphi = \Phi$. The following theorem provides our first result in that direction.

Theorem 8.5 *Let $N \in \mathbb{N}$ and $t \in [1,\infty)$. If $(w_n)_{n\in\mathbb{N}}$ is a sequence of non-negative real numbers such that $w_1 > 0$ and the sequence $(W_n)_{n\in\mathbb{N}}$ is defined as in* (8.29)*, then the inequality*

$$\left[\sum_{n=1}^{N} w_n W_n \left(\sum_{m=n}^{N} \frac{w_{m+1}}{W_m^t W_{m+1}}\right)^{\frac{1}{t}} a_n\right]^t - \sum_{m=1}^{N} \frac{w_{m+1}}{W_{m+1}} G_m^t$$
$$\geq t \sum_{m=1}^{N} \frac{w_{m+1}}{W_m W_{m+1}} G_m^{t-1} \sum_{n=1}^{m} r_{mn} w_n \tag{8.38}$$

holds for all sequences $(a_n)_{n\in\mathbb{N}}$ of positive real numbers, where

$$G_m = \left[\prod_{n=1}^{m} (W_n a_n)^{w_n}\right]^{\frac{1}{W_m}}, \ m \in \mathbb{N}, \tag{8.39}$$

and

$$r_{mn} = \left| |W_n a_n - G_m| - G_m \left|\log \frac{W_n a_n}{G_m}\right| \right|, \ m,n \in \mathbb{N}. \tag{8.40}$$

In particular, for $t = 1$ relation (8.38) *reduces to*

$$\sum_{n=1}^{N} \left(1 - \frac{W_n}{W_{N+1}}\right) w_n a_n - \sum_{m=1}^{N} \frac{w_{m+1}}{W_{m+1}} \left(\prod_{n=1}^{m} W_n^{w_n}\right)^{\frac{1}{W_m}} \left(\prod_{n=1}^{m} a_n^{w_n}\right)^{\frac{1}{W_m}}$$
$$\geq \sum_{m=1}^{N} \frac{w_{m+1}}{W_m W_{m+1}} \sum_{n=1}^{m} r_{mn} w_n. \tag{8.41}$$

Proof. Follows immediately by rewriting Theorem 8.1 with $M = N$, $\Phi : \mathbb{R} \to \mathbb{R}_+$, $\Phi(x) = e^x$, parameters u_m and k_{mn} as in the proof of Theorem 8.4, and with the sequence $(\log(W_n a_n))_{n\in\mathbb{N}}$ instead of $(a_n)_{n\in\mathbb{N}}$. Then we have $A_m = \log G_m$, $m \in \mathbb{N}$, so (8.38) and (8.41) hold. □

Reformulating Theorem 8.5 for $N = \infty$ as in Theorem 8.3 and Remark 8.6 we get the following corollary.

Corollary 8.4 *Suppose $t \in [1,\infty)$, $(w_n)_{n\in\mathbb{N}}$ is a sequence of non-negative real numbers, and the sequence $(W_n)_{n\in\mathbb{N}}$ is defined by* (8.29)*. If $w_1 > 0$ and $\sum_{m=1}^{\infty} \frac{w_{m+1}}{W_m^t W_{m+1}} < \infty$, then*

$$\left[\sum_{n=1}^{\infty} w_n W_n \left(\sum_{m=n}^{\infty} \frac{w_{m+1}}{W_m^t W_{m+1}}\right)^{\frac{1}{t}} a_n\right]^t - \sum_{m=1}^{\infty} \frac{w_{m+1}}{W_{m+1}} G_m^t$$
$$\geq t \sum_{m=1}^{\infty} \frac{w_{m+1}}{W_m W_{m+1}} G_m^{t-1} \sum_{n=1}^{m} r_{mn} w_n$$

holds for all sequences $(a_n)_{n\in\mathbb{N}}$ *of positive real numbers and* G_m, r_{mn} *respectively defined by* (8.39) *and* (8.40). *In particular, for* $t = 1$ *and* W_∞ *defined by* (8.35), *we get*

$$\sum_{n=1}^{\infty}\left(1-\frac{W_n}{W_\infty}\right)w_n a_n - \sum_{m=1}^{\infty}\frac{w_{m+1}}{W_{m+1}}\left(\prod_{n=1}^{m}W_n^{w_n}\right)^{\frac{1}{W_m}}\left(\prod_{n=1}^{m}a_n^{w_n}\right)^{\frac{1}{W_m}}$$
$$\geq \sum_{m=1}^{\infty}\frac{w_{m+1}}{W_m W_{m+1}}\sum_{n=1}^{m} r_{mn} w_n.$$

Under some additional conditions on weights w_n, the inequalities obtained in Theorem 8.5 and Corollary 8.4 can be seen as finite sections and refinements of the classical weighted Carleman inequality. One such set of conditions is given in the next lemma, interesting in its own right.

Lemma 8.1 *Suppose* $(w_n)_{n\in\mathbb{N}}$ *is a sequence of non-negative real numbers, such that* $w_1 > 0$ *and* $w_1 \geq w_n$, *for* $n = 2, 3, \ldots$. *If the sequence* $(W_n)_{n\in\mathbb{N}}$ *is defined by* (8.29), *then*

$$\frac{1}{W_{m+1}}\left(\prod_{n=1}^{m}W_n^{w_n}\right)^{\frac{1}{W_m}} > \frac{1}{e},\ m\in\mathbb{N}. \tag{8.42}$$

Proof. Since the mapping $x \mapsto \log x$ is strictly increasing on $\mathbb{R}_+$, for arbitrary $0 < a \leq b < \infty$ we have

$$(b-a)\log b \geq \int_a^b \log x\,dx,$$

with the strict inequality if $a < b$. In particular, by substituting $a = W_{n-1}$ and $b = W_n$, we get $b - a = w_n$ and

$$w_n \log W_n \geq \int_{W_{n-1}}^{W_n}\log x\,dx,\ n = 2, 3, \ldots. \tag{8.43}$$

Hence

$$\begin{aligned}\sum_{n=2}^{m+1} w_n \log W_n &\geq \int_{W_1}^{W_{m+1}}\log x\,dx\\ &= W_{m+1}\log W_{m+1} - W_{m+1} - w_1\log W_1 + w_1\end{aligned}$$

holds for an arbitrary $m\in\mathbb{N}$. Therefore,

$$\sum_{n=1}^{m} w_n\log W_n \geq W_m\log W_{m+1} - W_{m+1} + w_1 \geq W_m\log W_{m+1} - W_m, \tag{8.44}$$

where we used the condition $w_1 \geq w_{m+1}$. Observe that at least one of the inequalities in (8.44) is strict. Namely, if there exists $n \in \{2, 3, \ldots, m+1\}$ such that $w_n > 0$, then the sign of inequality in (8.43) is strict and so is the first inequality in (8.44). Otherwise, we have $w_1 > 0 = w_{m+1}$ and the second inequality in (8.44) is strict. Finally,

$$\log\left(\prod_{n=1}^{m}W_n^{w_n}\right) > W_m \log\frac{W_{m+1}}{e},$$

so we get (8.42). □

Remark 8.9 If $w_n = 1$, $n \in \mathbb{N}$, then (8.42) becomes

$$\frac{1}{m+1}\sqrt[m]{m!} > \frac{1}{e},\ m \in \mathbb{N},$$

that is,

$$\frac{m+1}{\sqrt[m]{m!}} < e,\ m \in \mathbb{N}.$$

Thus, Lemma 8.1 provides a class of lower bounds for the constant e. □

Using Lemma 8.1 in Theorem 8.5 and Corollary 8.4 we obtain a new strengthened weighted Carleman's inequality and its finite sections. Here we emphasize only the most important case, that is, the case with $t = 1$. Since the general case can be derived analogously, it is omitted.

Corollary 8.5 *Under the conditions of Theorem* 8.5 *and Lemma* 8.1, *the left-hand side of* (8.41) *is strictly less than*

$$\sum_{n=1}^{N}\left(1-\frac{W_n}{W_{N+1}}\right)w_n a_n - \frac{1}{e}\sum_{m=1}^{N} w_{m+1}\left(\prod_{n=1}^{m} a_n^{w_n}\right)^{\frac{1}{W_m}}.$$

Especially, if $N = \infty$, then the inequalities

$$\begin{aligned}
&\sum_{n=1}^{\infty} w_n a_n - \frac{1}{e}\sum_{m=1}^{\infty} w_{m+1}\left(\prod_{n=1}^{m} a_n^{w_n}\right)^{\frac{1}{W_m}}\\
&\geq \sum_{n=1}^{\infty}\left(1-\frac{W_n}{W_\infty}\right)w_n a_n - \frac{1}{e}\sum_{m=1}^{\infty} w_{m+1}\left(\prod_{n=1}^{m} a_n^{w_n}\right)^{\frac{1}{W_m}}\\
&> \sum_{n=1}^{\infty}\left(1-\frac{W_n}{W_\infty}\right)w_n a_n - \sum_{m=1}^{\infty}\frac{w_{m+1}}{W_{m+1}}\left(\prod_{n=1}^{m} W_n^{w_n}\right)^{\frac{1}{W_m}}\left(\prod_{n=1}^{m} a_n^{w_n}\right)^{\frac{1}{W_m}}\\
&\geq \sum_{m=1}^{\infty}\frac{w_{m+1}}{W_m W_{m+1}}\sum_{n=1}^{m} r_{mn} w_n \geq 0
\end{aligned}$$

hold, where the case when $W_\infty = \infty$ is included as well.

Remark 8.10 For $w_n = 1$, $n \in \mathbb{N}$, relation (8.38) reduces to

$$\left[\sum_{n=1}^{N} n\left(\sum_{m=n}^{N}\frac{m^{-t}}{m+1}\right)^{\frac{1}{t}} a_n\right]^t - \sum_{m=1}^{N}\frac{H_m^t}{m+1} \geq t\sum_{m=1}^{N}\frac{H_m^{t-1}}{m(m+1)}\sum_{n=1}^{m} r_{mn}, \tag{8.45}$$

where

$$H_m = \left(m!\prod_{n=1}^{m} a_n\right)^{\frac{1}{m}} \quad \text{and} \quad r_{mn} = \left|\,|na_n - H_m| - H_m\left|\log\frac{na_n}{H_m}\right|\,\right|,\ m,n \in \mathbb{N}.$$

Since $\sum_{m=1}^{\infty} \frac{m^{-t}}{m+1} < \infty$ for all $t \in [1,\infty)$, note that inequality (8.45) covers also the case when $N = \infty$. On the other hand, Corollary 8.5 and Remark 8.9 imply that

$$\sum_{n=1}^{N}\left(1-\frac{n}{N+1}\right)a_n - \frac{1}{e}\sum_{m=1}^{N}\left(\prod_{n=1}^{m}a_n\right)^{\frac{1}{m}}$$
$$> \sum_{n=1}^{N}\left(1-\frac{n}{N+1}\right)a_n - \sum_{m=1}^{N}\frac{1}{m+1}H_m \geq \sum_{m=1}^{N}\frac{1}{m(m+1)}\sum_{n=1}^{m}r_{mn} \geq 0$$

holds for all $N \in \mathbb{N}$, while for $N = \infty$ we have

$$\begin{aligned}\sum_{n=1}^{\infty}a_n - \frac{1}{e}\sum_{m=1}^{\infty}\left(\prod_{n=1}^{m}a_n\right)^{\frac{1}{m}} &> \sum_{n=1}^{\infty}a_n - \sum_{m=1}^{\infty}\frac{1}{m+1}H_m \\ &\geq \sum_{m=1}^{\infty}\frac{1}{m(m+1)}\sum_{n=1}^{m}r_{mn} \geq 0.\end{aligned}$$

Therefore, our results refine and generalize relation (8.4) and Carleman's inequality (8.5). We take an opportunity to mention that another strengthened weighted Carleman's inequality was obtained by Čižmešija et al. in [31], but that result can be hardly comparable with the inequalities derived in this section. □

8.3 Exponential convexity and Hardy-type inequalities

By employing the concept of logarithmic and exponential convexity, we obtain here upper bounds and some further lower bounds for the left-hand sides of the Hardy-type inequalities from previous two sections, in settings with suitably chosen convex functions Φ and $t = 1$.

According to Lemma 6.1 and Lemma 6.2, all the results can be rewritten with convex functions Φ_s and Ψ_s, $s \in \mathbb{R}$. In particular, observing that the right-hand side of (8.16) is non-negative, from Remark 8.1 we get

$$\sum_{n=1}^{N} v_n\Phi_s(a_n) - \sum_{m=1}^{M} u_m\Phi_s(A_m) \geq 0$$

and

$$\sum_{n=1}^{N} v_n\Psi_s(a_n) - \sum_{m=1}^{M} u_m\Psi_s(A_m) \geq 0,$$

where $M,N \in \mathbb{N}$, and u_m, k_{mn}, K_m, a_n, and A_m are as in Theorem 8.1 ($a_n \in \mathbb{R}_+$ and $a_n \in \mathbb{R}$ in the cases with Φ_s and Ψ_s respectively), while v_n is defined by (8.17). Therefore, under assumptions of Theorem 8.1, the functions $F, G : \mathbb{R} \to \mathbb{R}$,

$$F(s) = \sum_{n=1}^{N} v_n \Phi_s(a_n) - \sum_{m=1}^{M} u_m \Phi_s(A_m) \tag{8.46}$$

and

$$G(s) = \sum_{n=1}^{N} v_n \Psi_s(a_n) - \sum_{m=1}^{M} u_m \Psi_s(A_m), \tag{8.47}$$

are well-defined and non-negative. By proving that they are log-convex, we provide upper bounds and some new lower bounds for the left-hand side of (8.16), in the setting with convex functions Φ_s and Ψ_s. In fact, in the sequel we prove a stronger result, that is, that F and G are exponentially convex functions.

Theorem 8.6 *Let $M,N \in \mathbb{N}$. For $m \in \mathbb{N}_M$ and $n \in \mathbb{N}_N$, let $a_n \in \mathbb{R}_+$, u_m, k_{mn}, K_m, and A_m be as in Theorem* 8.1*, and let v_n be as in* (8.17)*. Then the function $F : \mathbb{R} \to \mathbb{R}$, defined by* (8.46)*, is exponentially convex and the inequality*

$$F(s_2)^{s_3-s_1} \leq F(s_1)^{s_3-s_2} F(s_3)^{s_2-s_1} \tag{8.48}$$

holds for all s_1, s_2, $s_3 \in \mathbb{R}$, such that $s_1 < s_2 < s_3$.

Proof. The first step is to prove that F is continuous on $\mathbb{R}$. Since the mapping $s \mapsto \frac{a^s}{s(s-1)}$ is continuous on $\mathbb{R} \setminus \{0, 1\}$ for all $a \in \mathbb{R}_+$, we only need to prove the continuity of F in $s = 0$ and $s = 1$. Note that

$$\begin{aligned} \sum_{n=1}^{N} v_n - \sum_{m=1}^{M} u_m &= \sum_{n=1}^{N} \sum_{m=1}^{M} u_m \frac{k_{mn}}{K_m} - \sum_{m=1}^{M} u_m \\ &= \sum_{m=1}^{M} u_m \left(\frac{1}{K_m} \sum_{n=1}^{N} k_{mn} \right) - \sum_{m=1}^{M} u_m = 0 \end{aligned} \tag{8.49}$$

and

$$\sum_{n=1}^{N} v_n a_n - \sum_{m=1}^{M} u_m A_m = \sum_{n=1}^{N} a_n \sum_{m=1}^{M} u_m \frac{k_{mn}}{K_m} - \sum_{m=1}^{M} \frac{u_m}{K_m} \sum_{n=1}^{N} k_{mn} a_n = 0. \tag{8.50}$$

Applying the classical L'Hospital's rule, identity (8.49), and the definitions of the functions Φ_s and F, we have

$$\begin{aligned} \lim_{s\to 0} F(s) &= \lim_{s\to 0} \frac{\sum_{n=1}^{N} v_n a_n^s - \sum_{m=1}^{M} u_m A_m^s}{s(s-1)} \\ &= \lim_{s\to 0} \frac{\sum_{n=1}^{N} v_n a_n^s \log a_n - \sum_{m=1}^{M} u_m A_m^s \log A_m}{2s-1} \\ &= \sum_{m=1}^{M} u_m \log A_m - \sum_{n=1}^{N} v_n \log a_n = F(0) \end{aligned}$$

and similarly, by using (8.50),

$$\lim_{s\to 1} F(s) = \sum_{n=1}^{N} v_n a_n \log a_n - \sum_{m=1}^{M} u_m A_m \log A_m = F(1).$$

Hence, F is continuous on $\mathbb{R}$. To prove that it is exponentially convex, it suffices to check condition (1.7). Fix $k \in \mathbb{N}$ and $\alpha_i \in \mathbb{R}$, $s_i \in \mathbb{R}_+$, for $i \in \mathbb{N}_k$. Denote $\Phi = \sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i\alpha_j \Phi_{\frac{s_i+s_j}{2}}$. Lemma 6.1 implies

$$\begin{aligned} \Phi''(x) &= \sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i\alpha_j \Phi''_{\frac{s_i+s_j}{2}}(x) = \sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i\alpha_j x^{\frac{s_i+s_j}{2}-2} \\ &= \left(\sum_{i=1}^{k} \alpha_i x^{\frac{s_i}{2}-1}\right)^2 \geq 0,\ x \in \mathbb{R}_+, \end{aligned} \tag{8.51}$$

so Φ is a convex function on $\mathbb{R}_+$. Thus, applying Corollary 8.2 to Φ and $t = 1$, we get

$$\sum_{n=1}^{N} v_n \Phi(a_n) - \sum_{m=1}^{M} u_m \Phi(A_m) \geq 0$$

and finally

$$\begin{aligned} &\sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i\alpha_j F\left(\frac{s_i+s_j}{2}\right) \\ &= \sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i\alpha_j \left(\sum_{n=1}^{N} v_n \Phi_{\frac{s_i+s_j}{2}}(a_n) - \sum_{m=1}^{M} u_m \Phi_{\frac{s_i+s_j}{2}}(A_m)\right) \\ &= \sum_{n=1}^{N} v_n \sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i\alpha_j \Phi_{\frac{s_i+s_j}{2}}(a_n) - \sum_{m=1}^{M} u_m \sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i\alpha_j \Phi_{\frac{s_i+s_j}{2}}(A_m) \\ &= \sum_{n=1}^{N} v_n \Phi(a_n) - \sum_{m=1}^{M} u_m \Phi(A_m) \geq 0. \end{aligned}$$

Therefore, (1.7) holds and F is exponentially convex. Since every exponentially convex function is log-convex, (8.48) follows directly from Remark 1.2. □

By using similar arguments, we prove exponential convexity of the function G.

Theorem 8.7 *Suppose $M, N \in \mathbb{N}$. For $m \in \mathbb{N}_M$ and $n \in \mathbb{N}_N$, suppose that $a_n \in \mathbb{R}$, u_m, k_{mn}, K_m, and A_m are as in Theorem* 8.1, *and that v_n is as in* (8.17). *Then the function $G : \mathbb{R} \to \mathbb{R}$, given by* (8.47), *is exponentially convex and*

$$G(s_2)^{s_3-s_1} \leq G(s_1)^{s_3-s_2} G(s_3)^{s_2-s_1} \tag{8.52}$$

holds for all s_1, s_2, $s_3 \in \mathbb{R}$, such that $s_1 < s_2 < s_3$.

Proof. Combining (8.49), (8.50), L'Hospital's rule, and the definition of the function G, we obtain

$$\begin{aligned}
\lim_{s\to 0} G(s) &= \lim_{s\to 0} \frac{1}{s^2}\left(\sum_{n=1}^{N} v_n e^{sa_n} - \sum_{m=1}^{M} u_m e^{sA_m}\right)\\
&= \lim_{s\to 0} \frac{1}{2s}\left(\sum_{n=1}^{N} v_n a_n e^{sa_n} - \sum_{m=1}^{M} u_m A_m e^{sA_m}\right)\\
&= \lim_{s\to 0} \frac{1}{2}\left(\sum_{n=1}^{N} v_n a_n^2 e^{sa_n} - \sum_{m=1}^{M} u_m A_m^2 e^{sA_m}\right)\\
&= \frac{1}{2}\left(\sum_{n=1}^{N} v_n a_n^2 - \sum_{m=1}^{M} u_m A_m^2\right) = G(0).
\end{aligned}$$

Since the mapping $s \mapsto \frac{e^{as}}{s^2}$ is continuous on $\mathbb{R}\setminus\{0\}$, we conclude that G is continuous on $\mathbb{R}$. To prove that G is an exponentially convex function, fix $k\in\mathbb{N}$ and $\alpha_i, s_i \in \mathbb{R}$, for $i\in\mathbb{N}_k$. Applying Lemma 6.1 to the function $\Psi = \sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i\alpha_j \Psi_{\frac{s_i+s_j}{2}}$, for all $x\in\mathbb{R}$ we get

$$\Psi''(x) = \sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i\alpha_j \Psi''_{\frac{s_i+s_j}{2}}(x) = \sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i\alpha_j e^{\frac{s_i+s_j}{2}x} = \left(\sum_{i=1}^{k} \alpha_i e^{\frac{s_i}{2}x}\right)^2 \geq 0,$$

so Ψ is convex on $\mathbb{R}$ and

$$\sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i\alpha_j G\left(\frac{s_i+s_j}{2}\right) \geq 0$$

holds as in the proof of Theorem 8.6. Thus, G is exponentially convex and then also log-convex. Relation (8.52) follows directly from Remark 1.2. □

Remark 8.11 Observe that each of inequalities (8.48) and (8.52) implies three further relations suitable for establishing lower and upper bounds for values of F and G. Namely, from (8.48) we obtain that inequalities

$$F(s_2) \leq F(s_1)^{\frac{s_3-s_2}{s_3-s_1}} F(s_3)^{\frac{s_2-s_1}{s_3-s_1}}, \tag{8.53}$$

$$F(s_1) \geq F(s_2)^{\frac{s_3-s_1}{s_3-s_2}} F(s_3)^{\frac{s_1-s_2}{s_3-s_2}} \quad\text{and}\quad F(s_3) \geq F(s_1)^{\frac{s_2-s_3}{s_2-s_1}} F(s_2)^{\frac{s_3-s_1}{s_2-s_1}} \tag{8.54}$$

hold for all $s_1, s_2, s_3 \in \mathbb{R}$, such that $s_1 < s_2 < s_3$, while the same inequalities for G follow from (8.52). □

Remark 8.12 In (8.46) and (8.47), the functions F and G were defined as finite sums of functions, so there were no further conditions on the sequences $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$, $(a_n)_{n\in\mathbb{N}}$, and $(A_n)_{n\in\mathbb{N}}$ needed to apply methods used in the proofs of Theorem 8.6 and Theorem

8.7. Of course, we can also consider the case when $M = N = \infty$, that is, to define F and G respectively by

$$F(s) = \sum_{n=1}^{\infty} v_n \Phi_s(a_n) - \sum_{m=1}^{\infty} u_m \Phi_s(A_m)$$

and

$$G(s) = \sum_{n=1}^{\infty} v_n \Psi_s(a_n) - \sum_{m=1}^{\infty} u_m \Psi_s(A_m).$$

Obviously, then we have to deal with function series and, in order to apply L'Hospital's rule, be able to take limits and differentiate them term by term. Therefore, the sequences of real numbers mentioned above should be such that the function series $\sum_{n=1}^{\infty} v_n a_n^s$ and $\sum_{m=1}^{\infty} u_m A_m^s$ are uniformly convergent in neighbourhoods of $s = 0$ and $s = 1$, and that the function series $\sum_{n=1}^{\infty} v_n e^{a_n s}$ and $\sum_{m=1}^{\infty} u_m e^{A_m s}$ are uniformly convergent in some neighbourhood of $s = 0$. Some such sufficient conditions follow, for example, from the usual Weierstrass's test for uniform convergence. □

Theorem 8.6 and Theorem 8.7, along with Remark 8.11 and Remark 8.12, can be applied to all particular cases of Theorem 8.1 and Corollary 8.1. However, owing to the lack of space, here we mention just the cases related to our improvements of Godunova's and Carleman's inequality.

The following result provides a new lower and upper bound for the left-hand side of the refined weighted Godunova's inequality (8.34).

Corollary 8.6 *Let $N \in \mathbb{N}$ and $p \in \mathbb{R} \setminus \{0, 1\}$. If $(w_n)_{n\in\mathbb{N}}$ is a sequence of non-negative real numbers, such that $w_1 > 0$, and the sequence $(W_n)_{n\in\mathbb{N}}$ is defined by* (8.29)*, then the inequalities*

$$\begin{aligned} &p(p-1) \inf_{(s,t)\in\mathscr{S}_p} F(s)^{\frac{t-p}{t-s}} F(t)^{\frac{p-s}{t-s}} \\ &\geq \sum_{n=1}^{N} \left(1 - \frac{W_n}{W_{N+1}}\right) \frac{w_n}{W_n} a_n^p - \sum_{m=1}^{N} \frac{w_{m+1}}{W_{m+1}} \left(\frac{1}{W_m} \sum_{n=1}^{m} w_n a_n\right)^p \\ &\geq p(p-1) \sup_{(s,t)\in\mathscr{T}_p} F(s)^{\frac{t-p}{t-s}} F(t)^{\frac{p-s}{t-s}} \end{aligned} \tag{8.55}$$

hold for all sequences $(a_n)_{n\in\mathbb{N}}$ of positive real numbers and $F : \mathbb{R} \to \mathbb{R}$ given by

$$F(s) = \sum_{n=1}^{N} \left(1 - \frac{W_n}{W_{N+1}}\right) \frac{w_n}{W_n} \Phi_s(a_n) - \sum_{m=1}^{N} \frac{w_{m+1}}{W_{m+1}} \Phi_s\left(\frac{1}{W_m} \sum_{n=1}^{m} w_n a_n\right),$$

where Φ_s is defined by (6.1) *and*

$$\mathscr{S}_p = \{(s,t) \in \mathbb{R}^2 : s < p < t\}, \ \mathscr{T}_p = \{(s,t) \in \mathbb{R}^2 : p < s < t \ \text{ or } \ s < t < p\}.$$

Proof. Follows directly from Theorem 8.6, applied with $M = N$, u_m and k_{mn} as in the proof of Theorem 8.4, and with v_n defined by (8.33). The first inequality in (8.55) is obtained from (8.53), rewritten with $s_1 = s$, $s_2 = p$, and $s_3 = t$, where $s < p < t$. On the other hand, the second inequality in (8.55) is a consequence of both relations in (8.54). The first of them is rewritten with $s_1 = p$, $s_2 = s$, and $s_3 = t$, where $p < s < t$, and the second with $s_1 = s$, $s_2 = t$, and $s_3 = p$, where $s < t < p$. □

Remark 8.13 In particular, for $w_n = 1$, $n \in \mathbb{N}$, we have

$$F(s) = \sum_{n=1}^{N} \left(1 - \frac{n}{N+1}\right) \frac{1}{n} \Phi_s(a_n) - \sum_{m=1}^{N} \frac{1}{m+1} \Phi_s\left(\frac{1}{m} \sum_{n=1}^{m} a_n\right),$$

so (8.55) becomes

$$\begin{aligned}
&p(p-1) \inf_{(s,t)\in\mathscr{S}_p} F(s)^{\frac{t-p}{t-s}} F(t)^{\frac{p-s}{t-s}} \\
&\geq \sum_{n=1}^{N} \left(1 - \frac{n}{N+1}\right) \frac{a_n^p}{n} - \sum_{m=1}^{N} \frac{1}{m+1} \left(\frac{1}{m} \sum_{n=1}^{m} a_n\right)^p \\
&\geq p(p-1) \sup_{(s,t)\in\mathscr{T}_p} F(s)^{\frac{t-p}{t-s}} F(t)^{\frac{p-s}{t-s}}.
\end{aligned}$$

Under the conditions of Remark 8.12, Corollary 8.6 holds also for $N = \infty$. In that case, W_{N+1} is replaced with W_∞ defined by (8.35), and covers also the case when $W_\infty = \infty$. □

Our final result in this chapter is the following refinement of the weighted Carleman's inequality.

Corollary 8.7 *Suppose $N \in \mathbb{N}$, $(w_n)_{n\in\mathbb{N}}$ is a sequence of non-negative real numbers, such that $w_1 > 0$, and the sequence $(W_n)_{n\in\mathbb{N}}$ is defined by* (8.29). *Then the inequalities*

$$\begin{aligned}
\inf_{(s,t)\in\mathscr{S}_1} G(s)^{\frac{t-1}{t-s}} G(t)^{\frac{1-s}{t-s}} &\geq \sum_{n=1}^{N} \left(1 - \frac{W_n}{W_{N+1}}\right) w_n a_n - \sum_{m=1}^{N} \frac{w_{m+1}}{W_{m+1}} G_m \\
&\geq \sup_{(s,t)\in\mathscr{T}_1} G(s)^{\frac{t-1}{t-s}} G(t)^{\frac{1-s}{t-s}} \qquad (8.56)
\end{aligned}$$

hold for all sequences $(a_n)_{n\in\mathbb{N}}$ of positive real numbers, $(G_n)_{n\in\mathbb{N}}$ defined by (8.39), *and $G : \mathbb{R} \to \mathbb{R}$ given by*

$$G(s) = \sum_{n=1}^{N} \left(1 - \frac{W_n}{W_{N+1}}\right) \frac{w_n}{W_n} \Psi_s\left(\log(W_n a_n)\right) - \sum_{m=1}^{N} \frac{w_{m+1}}{W_{m+1}} \Psi_s\left(\log G_m\right),$$

where Ψ_s is defined by (6.2) *and*

$$\mathscr{S}_1 = \{(s,t) \in \mathbb{R}^2 : s < 1 < t\}, \ \mathscr{T}_1 = \{(s,t) \in \mathbb{R}^2 : 1 < s < t \ \text{ or } \ s < t < 1\}.$$

Proof. A direct consequence of Theorem 8.7, rewritten with $M = N$, u_m and k_{mn} as in the proof of Theorem 8.4, v_n defined by (8.33), and with the sequence $(\log(W_n a_n))_{n\in\mathbb{N}}$ instead of $(a_n)_{n\in\mathbb{N}}$. The first inequality in (8.56) follows from (8.53), rewritten with G, $s_1 = s$, $s_2 = 1$, and $s_3 = t$, for $(s,t) \in \mathscr{S}_1$. The second inequality in (8.56) is obtained by combining both relations in (8.54), rewritten with G. In the first of them we set $s_1 = 1$, $s_2 = s$, and $s_3 = t$, where $1 < s < t$, while in the second relation we substitute $s_1 = s$, $s_2 = t$, and $s_3 = 1$, where $s < t < 1$. □

Remark 8.14 Note that for $w_n = 1$, $n \in \mathbb{N}$, we have

$$G(s) = \sum_{n=1}^{N} \frac{1}{n}\left(1 - \frac{n}{N+1}\right)\Psi_s(\log(na_n)) - \sum_{m=1}^{N} \frac{1}{m+1}\Psi_s(\log H_m),$$

where $H_m = \left(m!\prod_{n=1}^{m} a_n\right)^{\frac{1}{m}}$. Hence, in this setting (8.56) becomes

$$\begin{aligned}\inf_{(s,t)\in\mathscr{S}_1} G(s)^{\frac{t-1}{t-s}} G(t)^{\frac{1-s}{t-s}} &\geq \sum_{n=1}^{N}\left(1 - \frac{n}{N+1}\right)a_n - \sum_{m=1}^{N}\frac{1}{m+1}\left(m!\prod_{n=1}^{m} a_n\right)^{\frac{1}{m}} \\ &\geq \sup_{(s,t)\in\mathscr{T}_1} G(s)^{\frac{t-1}{t-s}} G(t)^{\frac{1-s}{t-s}}.\end{aligned}$$

If the sequence $(a_n)_{n\in\mathbb{N}}$ fulfills the conditions of Remark 8.12, Corollary 8.7 holds also for $N = \infty$ and W_{N+1} replaced with W_∞ defined by (8.35). The case with $W_\infty = \infty$ is included as well. □

Remark 8.15 We can also define a linear functional by taking the positive difference of the inequality stated in (8.22) for $t = 1$ as:

$$\Delta_3(\Phi) = \sum_{n=1}^{N} v_n\Phi(a_n) - \sum_{m=1}^{M} u_m\Phi(A_m) \tag{8.57}$$

All the result in section 6.4 are also valid for (8.57).

Remark 8.16 The well-known Hardy inequality presented in [51] (both in the continuous and discrete settings) has been extensively studied and used as a model for investigation of more general integral inequalities [40, 79, 66, 70, 65]. Recently, several papers have treated the unification and extension of Hardy's continuous and discrete integral inequalities by means of the theory of time scales [95, 90, 91].

Chapter 9

Generalized non-commutative Hardy and Hardy-Hilbert type inequalities

The methods applied in this chapter are based on convexity inequalities, and this is different from the approach taken in the classical literature. It allows us, for $1 < p \leq 2$, to extend Hardy's inequality (0.1) and Hardy-Hilbert's inequality (2.2) from functions whose values are non-negative numbers to functions whose values are positive semi-definite operators.

9.1 The main results

In the sequel let Ω_1 and Ω_2 be locally compact Hausdorff spaces, and let μ_1 and μ_2 denote Radon measures on Ω_1 and Ω_2, respectively. Moreover, let $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a measurable and non-negative kernel and $K(x)$ be defined as in (2.16).

We note that an operator valued function $f\colon (0,\infty) \longrightarrow B(H)$ is said to be weakly measurable if the real functions $x \to (f(x)\xi \mid \eta)$ are measurable for all vectors $\xi, \eta \in H$.

First we present the following generalization of Theorem 2.5, which is crucial for our further investigations but also of independent interest (see [45]):

Theorem 9.1 *Let u be a weight function on Ω_1 and let $k(x,y)$ be a non-negative kernel on $\Omega_1 \times \Omega_2$. Define v as in Theorem 2.5 and assume $v(y) < \infty$ for every $y \in \Omega_2$. Let φ be an operator convex function defined on the positive half-axis and let $f\colon \Omega_2 \longrightarrow B(H)_+$ be a weakly measurable map such that the integral*

$$\int_{\Omega_2} v(y)\varphi(f(y))\,d\mu_2(y)$$

defines a bounded linear operator on a Hilbert space H. Then the operator inequality

$$\int_{\Omega_1} u(x)\varphi\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)f(y)\,d\mu_2(y)\right)d\mu_1(x) \leq \int_{\Omega_2} v(y)\varphi(f(y))\,d\mu_2(y) \tag{9.1}$$

is valid.

Proof. Since the function φ is operator convex we can use Theorem 1.7 and obtain that

$$\varphi\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)f(y)\,d\mu_2(y)\right) \leq \frac{1}{K(x)}\int_{\Omega_2} k(x,y)\varphi(f(y))\,d\mu_2(y).$$

Consequently, by also using the Fubini theorem, we find that

$$\begin{aligned}
&\int_{\Omega_1} u(x)\varphi\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)f(y)\,d\mu_2(y)\right)d\mu_1(x)\\
&\leq \int_{\Omega_1} \frac{u(x)}{K(x)}\left(\int_{\Omega_2} k(x,y)\varphi(f(y))\,d\mu_2(y)\right)d\mu_1(x)\\
&= \int_{\Omega_2} \varphi(f(y))\left(\int_{\Omega_1} u(x)\frac{k(x,y)}{K(x)}\,d\mu_1(x)\right)d\mu_2(y)\\
&= \int_{\Omega_2} v(y)\varphi(f(y))\,d\mu_2(y)
\end{aligned}$$

and the proof is complete. □

In this chapter, for $1 < p \leq 2$, we extend Hardy's inequality (0.1) and Hardy-Hilbert's inequality of the form (2.2) from functions whose values are non-negative numbers to functions whose values are positive semi-definite operators as follows:

Theorem 9.2 *Let $1 < p \leq 2$ be a real number and let $f:(0,\infty) \longrightarrow B(H)_+$ be any weakly measurable map such that the integral*

$$\int_0^\infty f^p(y)dy$$

defines a bounded linear operator on a Hilbert space H. Then we obtain the following inequalities

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)dy\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \tag{9.2}$$

and

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}\,dx \right)^p dy \le \left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \right)^p \int_0^\infty f^p(y)\,dy, \tag{9.3}$$

where the constant $(p/(p-1))^p$ *in* (9.2) *and* $(\pi/\sin(\pi/p))^p$ *in* (9.3) *are the best possible. For* $p>2$ *neither Hardy's inequality* (9.2) *nor Hardy-Hilbert's inequality of the form* (9.3) *hold in general.*

Remark 9.1 Inequality (9.2) was already proved in [44], but we give a new proof in a more general setting and we also prove that it can not be extended to the case $p>2$ in this general form. We also mention that inequality (2.2) can not be extended from functions whose values are non-negative numbers to functions whose values are positive semi-definite operators. By symmetry and the fact that $\sin\pi/p = \sin\pi/q$, $(\frac{1}{p}+\frac{1}{q}=1)$, it is sufficient to formulate (2.2) for $1<p\le 2$.

Proof of (9.3) in Theorem 9.2 We apply the result of Theorem 9.1 with $\Omega_1 = \Omega_2 = (0,\infty)$. Replace $d\mu_1(x)$ and $d\mu_2(y)$ by dx and dy, respectively, and let

$$u(x) = \frac{1}{x} \text{ and } k(x,y) = \frac{(\frac{y}{x})^{-1/p}}{x+y},\ p>1.$$

Then, by making a straightforward calculation with the formula

$$\int_0^\infty \frac{t^{-1/p}}{1+t}\,dt = \frac{\pi}{\sin\frac{\pi}{p}},$$

we find that

$$K(x) = \frac{\pi}{\sin\frac{\pi}{p}} = K \text{ and } v(y) = \frac{1}{y}.$$

We choose for $1<p\le 2$ the operator convex function $\varphi(u)=u^p$ and obtain from (9.1) the inequality

$$\begin{aligned} K^{-p}\int_0^\infty \left(\int_0^\infty \left(\frac{y}{x}\right)^{-1/p} \frac{f(y)}{x+y}\,dy \right)^p \frac{dx}{x} &= K^{-p}\int_0^\infty \left(\int_0^\infty \frac{f(y)}{x+y} y^{-1/p}\,dy \right)^p dx \\ &\le \int_0^\infty f^p(y)\frac{dy}{y}. \end{aligned}$$

Replace $f(t)t^{-1/p}$ with $f(t)$ and we obtain (9.3). The constant $(\pi/\sin(\pi/p))^p$ is of course sharp since it is sharp already in the classical case.

Now, we have to prove that the operator version of Hardy-Hilbert's inequality can not be extended to $p>2$. The function $t\mapsto t^p$ is not convex on (positive semi-definite) two by two matrices [47, Proposition 3.1]. We may, therefore, for a given $p>2$ choose positive semi-definite two by two matrices A and B such that

$$\left(\frac{A+B}{2}\right)^p \not\le \frac{A^p+B^p}{2},\ p>2. \tag{9.4}$$

There is, consequently, a unit vector ξ such that the expectation values

$$\left(\left(\frac{A+B}{2}\right)^p \xi \mid \xi\right) > \left(\frac{A^p+B^p}{2}\xi \mid \xi\right) \tag{9.5}$$

and, therefore, the constant

$$c_p := \left(\frac{A^p+B^p}{2}\xi \mid \xi\right)\left(\left(\frac{A+B}{2}\right)^p \xi \mid \xi\right)^{-1} < 1. \tag{9.6}$$

Let f be an arbitrary non-negative L^p-function in $[0,\infty)$ and set

$$F_n(x) = f(x)\begin{cases} A & \text{if the integer part of } nx \text{ is odd} \\ B & \text{if the integer part of } nx \text{ is even.}\end{cases} \tag{9.7}$$

By Lebesgue's theorem of dominated convergence we obtain that

$$\lim_{n\to\infty}\int_0^\infty\left(\int_0^\infty \frac{F_n(x)}{x+y}dx\right)^p dy = \left(\frac{A+B}{2}\right)^p\int_0^\infty\left(\int_0^\infty \frac{f(x)}{x+y}dx\right)^p dy$$

and

$$\lim_{n\to\infty}\int_0^\infty F_n^p(x)\,dx = \frac{A^p+B^p}{2}\int_0^\infty f^p(x)\,dx.$$

If Hardy-Hilbert's operator inequality were valid for $p>2$ we would obtain, by taking expectation values in the vector ξ,

$$\int_0^\infty\left(\int_0^\infty \frac{f(x)}{x+y}dx\right)^p dy \le c_p\left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right)^p\int_0^\infty f^p(x)dx$$

contradicting that the constant $(\pi/\sin(\pi/p))^p$ is the best possible in Hardy-Hilbert's classical inequality of the form (2.1). □

We now apply the operator convex function $\varphi(x) = x^p$, $1 \le p \le 2$ in Theorem 9.1 and obtain the following result.

Corollary 9.1 *Let u be a weight function on Ω_1 and let $k(x,y)$ be a non-negative kernel on $\Omega_1\times\Omega_2$. We define v as in Theorem 2.5 and assume $v(y)<\infty$ for each $y\in\Omega_2$. Take $1<p\le 2$ and let $f\colon\Omega_2\longrightarrow B(H)_+$ be a weakly measurable map such that the integral*

$$\int_{\Omega_2} v(y)f^p(y)\,d\mu_2(y)$$

defines a bounded linear operator on a Hilbert space H. Then the operator inequality

$$\int_{\Omega_1} u(x)\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)f(y)\,d\mu_2(y)\right)^p d\mu_1(x) \le \int_{\Omega_2} v(y)f^p(y)\,d\mu_2(y)$$

is valid.

9.2 Remarks end examples

Now we consider some special cases of Corollary 9.1 and obtain the following results:

Example 9.1 Let $1 \le p \le 2$ and set $\Omega_1 = \Omega_2 = (0,\infty)$. We replace $d\mu_1(x)$ and $d\mu_2(y)$ by dx and dy, respectively and set

$$k(x,y) = \begin{cases} 1 & 0 \le y < x \\ 0 & y > x \end{cases}$$

and $u(x) = x^{-1}$. Then $v(y) = y^{-1}$ and we obtain that

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(y)dy \right)^p \frac{dx}{x} \le \int_0^\infty f^p(x) \frac{dx}{x}, \tag{9.8}$$

which is the result of Lemma 2.1 in [44]. □

Remark 9.2 By arguing as in Example 9.1 but with $1 < p \le 2$ and the function $f(y) = g(y^{p/(p-1)})t^{1/(p-1)}$ we obtain the following inequality:

$$\int_0^\infty \left(\frac{1}{x} \int_0^x g(y)dy \right)^p dx \le \left(\frac{p}{p-1} \right)^p \int_0^\infty g^p(x)dx, \tag{9.9}$$

which is just (9.2) in Theorem 7.1. This was proved in [44], but we gave a new proof in a more general situation. □

Example 9.2 Let $1 \le p \le 2$, $\Omega_1 = \Omega_2 = (0,b)$, $0 < b \le \infty$, replace $d\mu_1(x)$ and $d\mu_2(y)$ by the Lebesgue measures dx and dy, respectively, $u(x)$ by $u(x)/x$ and $v(x)$ by $v(x)/x$, and let $k(x,y) = 0$ for $x < y \le b$. Then we obtain that

$$\int_0^b u(x) \left(\frac{1}{K(x)} \int_0^x f(y)k(x,y)dy \right)^p \frac{dx}{x} \le \int_0^b v(y) f^p(y) \frac{dy}{y},$$

where

$$v(y) = \int_y^b u(x) \frac{k(x,y)}{K(x)} \frac{dx}{x} < \infty, \, y \in (0,b),$$

and

$$K(x) := \int_0^x k(x,y)\, dy < \infty.$$

□

Remark 9.3 By arguing as in Example 9.2 but with kernels such that

$$k(x,y) = \begin{cases} 1 & 0 \le y \le x, \\ 0 & x < y \le b \end{cases}$$

we find that

$$\int_0^b u(x) \left(\frac{1}{x} \int_0^x f(y)dy \right)^p \frac{dx}{x} \le \int_0^b v(y) f^p(y) \frac{dy}{y}. \tag{9.10}$$

Especially, if the weight function u is chosen to be $u(x) \equiv 1$, then we have that

$$v(y) = \begin{cases} 1 - \frac{y}{b}, \, b < \infty \\ 1, \quad\quad b = \infty, \end{cases}$$

so in the case when $b < \infty$ inequality (9.10) reads

$$\int_0^b \left(\frac{1}{x} \int_0^x f(y)dy \right)^p \frac{dx}{x} \le \int_0^b \left(1 - \frac{x}{b}\right) f^p(x) \frac{dx}{x}, \tag{9.11}$$

while for $b = \infty$ it becomes (9.8). □

Example 9.3 By arguing as in Example 9.2 but with $\Omega_1 = \Omega_2 = (b, \infty)$, $0 \le b < \infty$, and with kernels such that $k(x,y) = 0$ for $b \le y < x$, we find that now the inequality

$$\int_b^\infty u(x) \left(\frac{1}{K(x)} \int_x^\infty f(y)k(x,y)dy \right)^p \frac{dx}{x} \le \int_b^\infty v(y) f^p(y) \frac{dy}{y}$$

is valid, where

$$v(y) = \int_b^y u(x) \frac{k(x,y)}{K(x)} \frac{dx}{x} < \infty, \, y \in (b, \infty),$$

and

$$K(x) = \int_x^\infty k(x,y)dy < \infty.$$

□

Remark 9.4 By arguing as in Example 9.3 but choosing the kernel k such that

$$k(x,y) = \begin{cases} 0 & b \le y < x, \\ \frac{1}{y^2} & x \le y, \end{cases}$$

we obtain the inequality

$$\int_b^\infty u(x) \left(x \int_x^\infty \frac{f(y)}{y^2} dy \right)^p \frac{dx}{x} \le \int_b^\infty v(y) f^p(y) \frac{dy}{y}. \tag{9.12}$$

Especially, if the weight function u is chosen to be $u(x) \equiv 1$, then we find that

$$v(y) = 1 - \frac{b}{y},$$

therefore the relation (9.12) in this setting can be written on the form

$$\int_b^\infty \left(x \int_x^\infty \frac{f(y)}{y^2} dy \right)^p \frac{dx}{x} \leq \int_b^\infty \left(1 - \frac{b}{y}\right) f^p(y) \frac{dy}{y}. \tag{9.13}$$

□

Inspired by these examples we now state the following more general weighted operator inequality:

Theorem 9.3 *Let $k, b, p \in \mathbb{R}$ be such that $k \neq 1$, $b > 0$ and $1 < p \leq 2$ and let $f : (0,\infty) \longrightarrow B(H)_+$ be any weakly measurable map.*

(i) If $k > 1$ and $\int_0^b x^{p-k} f^p(x)dx$ defines a bounded linear operator on H, then

$$\int_0^b x^{-k} \left(\int_0^x f(t)dt \right)^p dx \leq \left(\frac{p}{k-1}\right)^p \int_0^b \left[1 - \left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right] x^{p-k} f^p(x)dx. \tag{9.14}$$

(ii) If $k < 1$ and $\int_b^\infty x^{p-k} f^p(x)dx$ defines a bounded linear operator on H, then

$$\int_b^\infty x^{-k} \left(\int_x^\infty f(t)dt \right)^p dx \leq \left(\frac{p}{1-k}\right)^p \int_b^\infty \left[1 - \left(\frac{b}{x}\right)^{\frac{1-k}{p}}\right] x^{p-k} f^p(x)dx. \tag{9.15}$$

The inequalities (9.14) *and* (9.15) *are sharp in the sense that they can not be improved by inserting a constant less than the one appearing in front of the integrals on the right hand sides.*
For $p > 2$ neither (9.14) *nor* (9.15) *hold in general.*

Proof. Consider first the case when $k > 1$ and replace the parameter b in (9.11) by $a = b^{(k-1)/p}$ and choose for f the function $x \mapsto f(x^{p/(k-1)})x^{p/(k-1)-1}$, c.f. [30, Corollary 2]. Then, with the substitutions $s = y^{p/(k-1)}$ and $t = x^{p/(k-1)}$ respectively, the left hand side of (9.11) becomes

$$\int_0^a \left(\frac{1}{x}\int_0^x f(y^{p/(k-1)})y^{p/(k-1)-1}dy\right)^p \frac{dx}{x}$$
$$=\left(\frac{k-1}{p}\right)^p \int_0^a \left(\frac{1}{x}\int_0^{x^{p/(k-1)}} f(s)ds\right)^p \frac{dx}{x}$$
$$=\left(\frac{k-1}{p}\right)^{p+1} \int_0^b t^{-k}(f(s)ds)^p\, dt.$$

Analogously, by substituting $t = x^{p/(k-1)}$ on the right hand side of (9.11) we obtain

$$\int_0^a \left(1-\frac{x}{a}\right) f^p(x^{p/(k-1)})x^{p(p/(k-1)-1)}\frac{dx}{x}$$
$$=\frac{k-1}{p}\int_0^b \left[1-\left(\frac{t}{b}\right)^{\frac{k-1}{p}}\right] t^{p-k}f^p(t)dt,$$

so relation (9.14) is proved. The sharpness of the constant is obvious since this holds already in the classical situations (see e.g [73] and the references given there). Now, we have to prove that (9.14) can not be extended to the case $p > 2$. Choose positive semi-definite two by two matrices A and B as in (9.4), unit vector ξ as in (9.5) and constant c_p defined by (9.6). Let F_n be defined by (9.7), we then obtain that

$$\lim_{n\to\infty}\int_0^b x^{-k}\left(\int_0^x F_n(t)dt\right)^p dx = \left(\frac{A+B}{2}\right)^p \int_0^b x^{-k}\left(\int_0^x f(t)\,dt\right)^p dx$$

and

$$\lim_{n\to\infty}\int_0^b \left[1-\left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right] x^{p-k}F_n^p(x)\,dx = \frac{A^p+B^p}{2}\int_0^b \left[1-\left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right] x^{p-k}f^p(x)\,dx.$$

If (9.14) were valid for $p > 2$ we would obtain, by taking expectation values in the vector ξ,

$$\int_0^b x^{-k}\left(\int_0^x f(t)dt\right)^p dx \le c_p\left(\frac{p}{k-1}\right)^p \int_0^b \left[1-\left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right] x^{p-k}f^p(x)dx$$

contradicting that the constant $\left(\frac{p}{k-1}\right)^p$ is the best possible in the classical case.
Now, suppose that $k < 1$.
Replacing the function f in (9.13) by $x \mapsto f(x^{p/(1-k)})x^{p/(1-k)+1}$, the parameter b by $a = b^{(1-k)/p}$, and making a similar sequence of substitutions as in the previous case on the left

hand side of (9.13) we obtain

$$\int_a^\infty \left(x \int_x^\infty f(y^{p/(1-k)}) y^{p/(1-k)+1} \frac{dy}{y^2} \right)^p \frac{dx}{x}$$

$$= \left(\frac{1-k}{p} \right)^p \int_a^\infty \left(x \int_{x^{p/(1-k)}}^\infty f(s) ds \right)^p \frac{dx}{x}$$

$$= \left(\frac{1-k}{p} \right)^{p+1} \int_b^\infty t^{-k} \left(\int_t^\infty f(s) ds \right)^p dt,$$

while the right hand side of (9.13) becomes

$$\int_a^\infty \left(1 - \frac{a}{x}\right) f^p(x^{p/(1-k)}) x^{p(p/(1-k)+1)} \frac{dx}{x} = \frac{1-k}{p} \int_b^\infty \left[1 - \left(\frac{b}{t} \right)^{\frac{1-k}{p}} \right] t^{p-k} f^p(t) dt,$$

so relation (9.15) is proved. The sharpness of the constant here is also obvious since it is already so in the classical situation (see e.g. [73] and the references given there). Now, we have to prove that (9.15) can not be extended to the case $p > 2$. We introduce, in analogy with the proof of (9.14), the functions F_n and obtain

$$\lim_{n\to\infty} \int_b^\infty x^{-k} \left(\int_x^\infty F_n(t) dt \right)^p dx = \left(\frac{A+B}{2} \right)^p \int_b^\infty x^{-k} \left(\int_x^\infty f(t)\, dt \right)^p dx$$

and

$$\lim_{n\to\infty} \int_b^\infty \left[1 - \left(\frac{b}{x} \right)^{\frac{1-k}{p}} \right] x^{p-k} F_n^p(x)\, dx = \frac{A^p + B^p}{2} \int_b^\infty \left[1 - \left(\frac{b}{x} \right)^{\frac{1-k}{p}} \right] x^{p-k} f^p(x)\, dx.$$

If (9.15) were valid for $p > 2$ we would obtain, by taking expectation values in the vector ξ, that

$$\int_b^\infty x^{-k} \left(\int_x^\infty f(t) dt \right)^p dx \le c_p \left(\frac{p}{1-k} \right)^p \int_b^\infty \left[1 - \left(\frac{b}{x} \right)^{\frac{1-k}{p}} \right] x^{p-k} f^p(x) dx$$

contradicting that the constant $\left(\frac{p}{1-k}\right)^p$ is the best possible in the classical situation so the proof is complete. □

Remark 9.5 Note that by rewriting (9.14) with $b = \infty$ and $k = p$ we obtain (9.10) and, hence, we have also proved the first part of Theorem 9.2.

We conclude this chapter, by applying the last result to n-dimensional cells in $\mathbb{R}_+^n$ and, thus, obtaining a generalization of the Godunova inequality (2.10). Now, we consider the case $\Omega_1, \Omega_2 = \mathbb{R}_+^n$, $d\mu_1(x) = d\mathbf{x}$, $d\mu_2(y) = d\mathbf{y}$ and the kernel $k : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}$ of the form $k(\mathbf{x}, \mathbf{y}) = l\left(\frac{\mathbf{y}}{\mathbf{x}}\right)$, where $l : \mathbb{R}_+^n \to \mathbb{R}$ is a non-negative measurable function and we obtain the following corollary.

Corollary 9.2 *Let l and u be non-negative measurable functions on $\mathbb{R}_+^n$, such that $0 < L(\mathbf{x}) = \mathbf{x}^{\mathbf{1}} \int_{\mathbb{R}_+^n} l(\mathbf{y})\,d\mathbf{y} < \infty$ for all $\mathbf{x} \in \mathbb{R}_+^n$, and that the function $\mathbf{x} \mapsto u(\mathbf{x})\frac{l\left(\frac{\mathbf{y}}{\mathbf{x}}\right)}{L(\mathbf{x})}$ is integrable on $\mathbb{R}_+^n$ for each fixed $\mathbf{y} \in \mathbb{R}_+^n$. Let the function v be defined on $\mathbb{R}_+^n$ by*

$$v(\mathbf{y}) = \int_{\mathbb{R}_+^n} u(\mathbf{x}) \frac{l\left(\frac{\mathbf{y}}{\mathbf{x}}\right)}{L(\mathbf{x})}\,d\mathbf{x}.$$

If φ is a operator convex function on an interval $I \subseteq \mathbb{R}$, then the inequality

$$\int_{\mathbb{R}_+^n} u(\mathbf{x})\varphi\left(\frac{1}{L(\mathbf{x})}\int_{\mathbb{R}_+^n} l\left(\frac{\mathbf{y}}{\mathbf{x}}\right) f(\mathbf{y})d\mathbf{y}\right) d\mathbf{x} \leq \int_{\mathbb{R}_+^n} v(\mathbf{y})\varphi(f(\mathbf{y}))d\mathbf{y} \tag{9.16}$$

holds for all weekly measurable maps $f : \mathbb{R}_+^n \to B(H)_+$ such that $\int_{\mathbb{R}_+^n} v(\mathbf{y})\varphi(f(\mathbf{y}))d\mathbf{y}$ defines a bounded linear operator on H.

Example 9.4 Now we consider a special case of Corollary 9.2, that is

$$1 = \int_{\mathbb{R}_+^n} l(\mathbf{t})d\mathbf{t}, \quad u(\mathbf{x}) = \frac{1}{x_1 \cdots x_n} = \mathbf{x}^{\mathbf{-1}}.$$

In this case

$$v(\mathbf{y}) = \int_{\mathbb{R}_+^n} \frac{l\left(\frac{\mathbf{y}}{\mathbf{x}}\right)}{x_1^2 \cdots x_n^2} d\mathbf{x} = \int_{\mathbb{R}_+^n} \frac{1}{y_1 \cdots y_n} l(\mathbf{t})d\mathbf{t} = \frac{1}{y_1 \cdots y_n} = \mathbf{y}^{\mathbf{-1}},$$

and we get the Godunova inequality (2.10), which shows that Corollary 9.2 is a genuine generalization of this inequality. □

Chapter 10

Boas-type inequalities

In the previous chapters many results concerning Hardy-type and Pólya-Knopp-type inequalities were given. R. P. Boas gave another direction of generalization of these famous inequalities. In [14], he proved that the inequality

$$\int_0^\infty \Phi\left(\frac{1}{M}\int_0^\infty f(tx)\,dm(t)\right)\frac{dx}{x} \leq \int_0^\infty \Phi(f(x))\,\frac{dx}{x} \tag{10.1}$$

holds for all continuous convex functions $\Phi\colon [0,\infty) \to \mathbb{R}$, measurable non-negative functions $f\colon \mathbb{R}_+ \to \mathbb{R}$, and non-decreasing bounded functions $m\colon [0,\infty) \to \mathbb{R}$, where $M = m(\infty) - m(0) > 0$ and the inner integral on the left-hand side of (10.1) is the Lebesgue-Stieltjes integral with respect to m. After its author, the relation (10.1) was named the Boas inequality. In the case of a concave function Φ, (10.1) holds with the sign of inequality reversed.

Independently, S. Kaijser et al. [65] (see also the paper [76] of N. Levinson) established the so-called general Hardy-Knopp-type inequality (2.6) for positive measurable functions $f\colon \mathbb{R}_+ \to \mathbb{R}$, and a real convex function Φ on $\mathbb{R}_+$. Later on, A. Čižmešija et al. [30] generalized relation (2.6) to the so-called strengthened Hardy-Knopp-type inequality by inserting a weight function and integrating over intervals of non-negative real numbers. Further, in [21] A. Čižmešija et al. considered a general Borel measure λ on $\mathbb{R}_+$, such that $L = \lambda(\mathbb{R}_+) = \int_0^\infty d\lambda(t) < \infty$, and proved that for a convex function Φ on an interval $I \subseteq \mathbb{R}$ and a weight function u on $\mathbb{R}_+$ the inequality

$$\int_0^\infty u(x)\Phi(Af(x))\,\frac{dx}{x} \leq \frac{1}{L}\int_0^\infty w(x)\Phi(f(x))\,\frac{dx}{x} \tag{10.2}$$

holds for all measurable functions $f\colon \mathbb{R}_+ \to \mathbb{R}$ such that $f(x) \in I$ for all $x \in \mathbb{R}_+$, where

$Af(x) = \dfrac{1}{L}\displaystyle\int_0^\infty f(tx)\,d\lambda(t)$ and $w(x) = \displaystyle\int_0^\infty u\left(\frac{x}{t}\right)d\lambda(t) < \infty$, $x \in \mathbb{R}_+$. They also gave the following refinement of (10.2):

$$\begin{aligned}
&\frac{1}{L}\int_0^\infty w(x)\Phi(f(x))\,\frac{dx}{x} - \int_0^\infty u(x)\Phi(Af(x))\,\frac{dx}{x} \\
&\geq \frac{1}{L}\left|\int_0^\infty \int_0^\infty u(x)|\Phi(f(tx)) - \Phi(Af(x))|\,d\lambda(t)\,\frac{dx}{x}\right. \\
&\quad \left. - \int_0^\infty \int_0^\infty u(x)|\varphi(Af(x))||f(tx) - Af(x)|\,d\lambda(t)\,\frac{dx}{x}\right|,
\end{aligned}$$

where φ denotes any function with values in the subdifferential of Φ.

Observe that a non-decreasing and bounded function $m\colon [0,\infty) \to \mathbb{R}$ such that $M = m(\infty) - m(0) > 0$ induces a finite Borel measure λ on $\mathbb{R}_+$ and vice versa. For such a function and measure, related Lebesgue and Lebesgue-Stieltjes integrals are equivalent. Thus, all the above results can be stated for $Af(x)$ defined by

$$Af(x) = \frac{1}{M}\int_0^\infty f(tx)\,dm(t),\, x \in \mathbb{R}_+,$$

so they refine and generalize inequality (10.1).

The Boas inequality (10.1) has been generalized in other ways as well. One of them is by using the weighted Hardy-Littlewood average $U_\psi f$ defined by

$$U_\psi f(x) = \int_0^1 f(tx)\psi(t)\,dt,$$

where ψ is a non-negative function on $[0,1]$. J. Xiao [102] characterized functions ψ for which U_ψ is bounded on either $L^p(\mathbb{R}^n), p \in [1,\infty]$, or $BMO(\mathbb{R}^n)$. Recall that the space $BMO(\mathbb{R}^n)$ consists of all measurables functions $f \in L^1_{loc}(\mathbb{R}^n)$ with bounded mean oscillation

$$\|f\|_{BMO(\mathbb{R}^n)} = \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|}\int_Q |f(x) - f_Q|dx < \infty,$$

where the supremum is taken over all cubes $Q \subseteq \mathbb{R}^n$ of sides parallel to the axes, $f_Q = |Q|^{-1}\displaystyle\int_Q f(x)dx$ stands for the average of f over Q, and $|Q|$ denotes the measure of Q.

For a function $\psi\colon [0,1] \to [0,\infty)$ and $p \in [1,\infty)$, J. Xiao proved that an operator $U_\psi\colon L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ exists as a bounded operator if and only if $\displaystyle\int_0^1 t^{-\frac{n}{p}}\psi(t)dt < \infty$, while an operator $U_\psi\colon BMO(\mathbb{R}^n) \to BMO(\mathbb{R}^n)$ exists as a bounded operator if and only if $\displaystyle\int_0^1 \psi(t)dt < \infty$. That heorem sharpens and extends the main result from [18] which asserts that if $t^{1-n}\psi(t)$ is bounded on $[0,1]$ then U_ψ is bounded on $BMO(\mathbb{R}^n)$. Although some time has passed by since then, the result still seems to be of interest as it is related closely to the Hardy-Littlewood maximal operators in harmonic analysis. For example, if $\psi \equiv 1$ and $n = 1$, then U_ψ is just reduced to the classical Hardy-Littlewood average $(Uf)(x) = \dfrac{1}{x}\displaystyle\int_0^x f(x)dy, x \neq 0$, which we recognize in classical Hardy inequality (0.1).

Another generalization of (10.1) was given by D. Luor [78] in a setting with σ-finite Borel measures μ and ν on a topological space X and a Borel probability measure λ on $\mathbb{R}_+$. For a λ-balanced Borel set E in X and the measure μ_t defined for all Borel sets $D \subseteq X$ and $t \in \mathbb{R}_+$ by $\mu_t(D) = \mu(t^{-1}D)$, he proved the inequality

$$\int_E \phi(Hf(x))\,d\mu(x) \leq \int_E \phi(f(x)) \left(\int_0^\infty \frac{d\mu_t}{d\nu}(x)\,d\lambda(t) \right) d\nu(x), \tag{10.3}$$

where ϕ is a non-negative convex function, $\mu_t \ll \nu$, $t \in \operatorname{supp}\lambda$, and Hf is the Hardy-Littlewood average of a non-negative Borel function f on X, defined by

$$Hf(x) = \int_0^\infty f(tx)\,d\lambda(t),\ x \in X.$$

Our first goal in this chapter is to obtain the weighted version of the mentioned Luor's result.

10.1 A new weighted Boas-type inequality

After introducing some necessary notation, in this section we state and prove a new weighted general Boas-type inequality in a setting with a topological space and σ-finite Borel measures.

Let λ be a finite Borel measure on $\mathbb{R}_+$. By supp λ we denote its support, that is, the set of all $t \in \mathbb{R}_+$ such that $\lambda(N_t) > 0$ holds for all open neighbourhoods N_t of t. Hence,

$$L = \int_{\text{supp }\lambda} d\lambda(t) = \int_0^\infty d\lambda(t) = \lambda(\mathbb{R}_+) < \infty. \tag{10.4}$$

On the other hand, let X be a topological space equipped with a continuous scalar multiplication $(a, \mathbf{x}) \mapsto a\mathbf{x} \in X$, for $a \in \mathbb{R}_+$, $\mathbf{x} \in X$, such that

$$1\mathbf{x} = \mathbf{x},\ \ a(b\mathbf{x}) = (ab)\mathbf{x},\ \ \mathbf{x} \in X,\ \ a, b \in \mathbb{R}_+.$$

Further, let the Borel set $\Omega \subseteq X$ be λ-balanced, that is, $t\Omega = \{t\mathbf{x} \colon \mathbf{x} \in \Omega\} \subseteq \Omega$, for all $t \in \text{supp }\lambda$. For a Borel measurable function $f \colon \Omega \to \mathbb{R}$, we define its Hardy-Littlewood average Af as

$$Af(\mathbf{x}) = \frac{1}{L}\int_0^\infty f(t\mathbf{x})\,d\lambda(t),\ \mathbf{x} \in \Omega. \tag{10.5}$$

We recall some facts from the measure theory. If μ is a measure on a ring R, a set E in R is said to have finite measure if $\mu(E) < \infty$. The measure of E is σ-finite if there exists a sequence $\{E_n\}$ of sets in R such that

$$E \subset \bigcup_{n=1}^{\infty} E_n \text{ and } \mu(E_n) < \infty, n = 1, 2, \ldots$$

If the measure of every set E in R is finite (or σ-finite), the measure μ is called finite (or σ-finite) on R. If R is an algebra and $\mu(E)$ is finite or σ-finite, then μ is called totally finite or totally σ-finite, respectively. An algebra is usually denoted by Σ.

Let (X,Σ) be a measurable space and μ and ν measures on Σ. We say that ν is absolutely continuous with respect to μ, in symbols $\nu \ll \mu$, if $\nu(E) = 0$ for every measurable set E for which $\mu(E) = 0$. In a suggestively imprecise phrase, $\nu \ll \mu$ means that ν is small whenever μ is small. A fundamental result, known as Radon-Nikodym theorem, concerning absolute continuity, is the following:

Theorem 10.1 (RADON-NIKODYM) *Suppose (X,Σ,ν) is a totally σ-finite measure space. If a σ-finite measure μ on Σ is absolutely continuous with respect to ν, then there exists a finite valued measurable function f on X such that*

$$\mu(E) = \int_E f(x)\, d\nu(x), \tag{10.6}$$

for every measurable set E.

The function f from (10.6) is unique up to a ν-null set. That is, if $\mu(E) = \int_E g(x) d\nu(x)$, $E \in \Sigma$ also holds, then $f = g$ ν-almost everywhere. The function F is commonly written as $\frac{d\mu}{d\nu}$ and is called the Radon-Nikodym derivative. The choice of the notation and the name of f reflects the fact that the function is analogous to a derivative in calculus in the sense that it describes the rate of change of density of one measure with respect to another.

Theorem 10.1 tells if and how it is possible to change from one measure to another. From all properties of the Radon-Nikodym derivative we emphasize just the following one.

Proposition 10.1 *Suppose that measures μ and ν are totally σ-finite such that $\mu \ll \nu$. If f is a μ-integrable function on X, then*

$$\int_X f(x)\, d\mu(x) = \int_X f(x) \frac{d\mu}{d\nu}(x)\, d\nu(x).$$

Finally, suppose that μ and ν are σ-finite Borel measures on X. For $t > 0$ and a Borel set $S \subseteq X$ we define

$$\mu_t(S) = \mu\left(\frac{1}{t} S\right). \tag{10.7}$$

Obviously, μ_t is a σ-finite Borel measure on X for each $t \in \mathbb{R}_+$. Throughout this paper, we suppose that the measures μ_t are absolutely continuous with respect to the measure ν, that is, $\mu_t \ll \nu$ for each $t \in \operatorname{supp} \lambda$. As usual, by $\dfrac{d\mu_t}{d\nu}$ we denote the related Radon-Nikodym derivative.

We start with a generalization of the main theorem in [78], that is, we state and prove a new weighted general Boas-type inequality.

Theorem 10.2 *Let λ be a finite Borel measure on $\mathbb{R}_+$ and L be defined by* (10.4). *Let μ and ν be σ-finite Borel measures on a topological space X, μ_t be defined by* (10.7) *and such that $\mu_t \ll \nu$ for all $t \in \operatorname{supp} \lambda$. Further, let $\Omega \subseteq X$ be a λ-balanced set and u be a non-negative function on X, such that*

$$v(\mathbf{x}) = \int_0^\infty u\left(\frac{1}{t}\mathbf{x}\right) \frac{d\mu_t}{d\nu}(\mathbf{x})\, d\lambda(t) < \infty, \ \mathbf{x} \in \Omega. \tag{10.8}$$

Suppose $\Phi\colon I \to \mathbb{R}$ is a non-negative convex function on an interval $I \subseteq \mathbb{R}$. If $f\colon \Omega \to \mathbb{R}$ is a Borel measurable function, such that $f(\mathbf{x}) \in I$ for all $\mathbf{x} \in \Omega$, and Af is defined by (10.5), *then $Af(\mathbf{x}) \in I$ for all $\mathbf{x} \in \Omega$ and the inequality*

$$\int_\Omega u(\mathbf{x})\Phi(Af(\mathbf{x}))\, d\mu(\mathbf{x}) \le \frac{1}{L}\int_\Omega v(\mathbf{x})\Phi(f(\mathbf{x}))\, d\nu(\mathbf{x}) \tag{10.9}$$

holds. For a non-positive concave function Φ, the sign of inequality in (10.9) *is reversed.*

Proof. For a fixed $\mathbf{x} \in \Omega$, we define the function $h_\mathbf{x}\colon \mathbb{R}_+ \to \mathbb{R}$ as $h_\mathbf{x}(t) = f(t\mathbf{x}) - Af(\mathbf{x})$. Then (10.4) and (10.5) imply

$$\int_0^\infty h_\mathbf{x}(t)\, d\lambda(t) = \int_0^\infty f(t\mathbf{x})\, d\lambda(t) - Af(\mathbf{x})\int_0^\infty d\lambda(t) = 0. \tag{10.10}$$

Since the set Ω is λ-balanced and $f(\Omega) \subseteq I$, it follows that $f(t\mathbf{x}) \in I$ for all $t \in \operatorname{supp} \lambda$ and each $\mathbf{x} \in \Omega$. Suppose that there exists $\mathbf{x}_0 \in \Omega$ such that $Af(\mathbf{x}_0) \notin I$. Then we have either $Af(\mathbf{x}_0) < f(t\mathbf{x}_0)$ for all $t \in \operatorname{supp} \lambda$, or $Af(\mathbf{x}_0) > f(t\mathbf{x}_0)$ for all $t \in \operatorname{supp} \lambda$, so the function $h_{\mathbf{x}_0}$ is either strictly positive or strictly negative on $\mathbb{R}_+$. This contradicts (10.10), so we proved that $Af(\mathbf{x}) \in I$ for all $\mathbf{x} \in \Omega$.

Finally, we prove (10.9). By using Jensen's inequality, Fubini's theorem, the substitution $\mathbf{y} = t\mathbf{x}$, the fact that Ω is λ-balanced and Φ is non-negative, and the Radon-Nikodym theorem, we obtain

$$\begin{aligned}
\int_\Omega u(\mathbf{x})\Phi(Af(\mathbf{x}))\, d\mu(\mathbf{x}) &\le \frac{1}{L}\int_\Omega u(\mathbf{x})\int_0^\infty \Phi(f(t\mathbf{x}))\, d\lambda(t)\, d\mu(\mathbf{x}) \\
&= \frac{1}{L}\int_0^\infty \int_\Omega u(\mathbf{x})\Phi(f(t\mathbf{x}))\, d\mu(\mathbf{x})\, d\lambda(t) \\
&= \frac{1}{L}\int_0^\infty \int_{t\Omega} u\left(\frac{1}{t}\mathbf{y}\right) \Phi(f(\mathbf{y}))\, d\mu_t(\mathbf{y})\, d\lambda(t) \\
&\le \frac{1}{L}\int_0^\infty \int_\Omega u\left(\frac{1}{t}\mathbf{y}\right) \Phi(f(\mathbf{y}))\, d\mu_t(\mathbf{y})\, d\lambda(t) \\
&= \frac{1}{L}\int_0^\infty \int_\Omega u\left(\frac{1}{t}\mathbf{y}\right) \Phi(f(\mathbf{y}))\frac{d\mu_t}{d\nu}(\mathbf{y})\, d\nu(\mathbf{y})\, d\lambda(t) \\
&= \frac{1}{L}\int_\Omega \left(\int_0^\infty u\left(\frac{1}{t}\mathbf{y}\right) \frac{d\mu_t}{d\nu}(\mathbf{y})\, d\lambda(t)\right) \Phi(f(\mathbf{y}))\, d\nu(\mathbf{y}) \\
&= \frac{1}{L}\int_\Omega v(\mathbf{y})\Phi(f(\mathbf{y}))\, d\nu(\mathbf{y}),
\end{aligned}$$

so the proof is completed. □

Notice that the condition on non-negativity of the convex function Φ in Theorem 10.2 can be omitted only in a particular setting with cones in X. More precisely, the following corollary holds.

Corollary 10.1 *If in Theorem* 10.2 *we have* $t\Omega = \Omega$ *for* λ*-a.e.* $t \in \operatorname{supp} \lambda$*, then* (10.9) *holds for all convex functions* Φ *on an interval* $I \subseteq \mathbb{R}$*. In that case, for all concave functions* Φ *relation* (10.9) *holds with the sign of inequality reversed.*

In Theorem 10.2 we considered general measures μ, ν, and λ, a set Ω, and a function Φ. Now, we give an overview of results obtained by specializing inequality (10.9) to some interesting particular settings. First, we consider the classical one-dimensional cases.

Corollary 10.2 *Let* λ *be a finite Borel measure on* $\mathbb{R}_+$ *and* L *be defined by* (10.4)*. Suppose that* $\Omega \subseteq \mathbb{R}_+$ *is a* λ*-balanced set and that* u *is a non-negative function on* $\mathbb{R}_+$*, such that*

$$w(x) = \int_0^\infty u\left(\frac{x}{t}\right) d\lambda(t) < \infty, x \in \Omega. \tag{10.11}$$

Let $\Phi \colon I \to \mathbb{R}$ *be a non-negative convex function on an interval* $I \subseteq \mathbb{R}$*. If* $f \colon \Omega \to \mathbb{R}$ *is a Borel measurable function, such that* $f(x) \in I$ *for all* $x \in \Omega$*, and* Af *is defined by* (10.5)*, then the inequality*

$$\int_\Omega u(x)\Phi(Af(x))\,\frac{dx}{x} \le \frac{1}{L}\int_\Omega w(x)\Phi(f(x))\,\frac{dx}{x} \tag{10.12}$$

holds. If the function Φ *is non-positive and concave, the sign of inequality in* (10.12) *is reversed.*

Proof. It follows directly from Theorem 10.2 if we set $X = \mathbb{R}_+$, the measures μ and ν to be the Lebesgue measures and replace the weight function u with $x \mapsto \dfrac{u(x)}{x}$. For such measures we get $\dfrac{d\mu_t}{d\nu}(x) = \dfrac{1}{t}$, $t \in \mathbb{R}_+$. In this setting, we have

$$v(x) = \int_0^\infty u\left(\frac{x}{t}\right)\cdot\frac{t}{x}\cdot\frac{1}{t}\,d\lambda(t) = \frac{1}{x}\int_0^\infty u\left(\frac{x}{t}\right) d\lambda(t) = \frac{w(x)}{x}\ ,\ x \in \Omega,$$

where the function v is defined by (10.8). □

Notice that inequality (10.12) obviously generalizes (10.2).

Corollary 10.3 *Let* $0 < b \le \infty$*,* u *be a non-negative function on* $(0,b)$ *such that the function* $t \mapsto \dfrac{u(t)}{t^2}$ *is locally integrable in* $(0,b)$*, and let*

$$w(x) = x\int_x^b u(t)\,\frac{dt}{t^2}\ ,\ x \in (0,b).$$

If Φ is a convex function on an interval $I \subseteq \mathbb{R}$, then the inequality

$$\int_0^b u(x)\Phi(Hf(x))\,\frac{dx}{x} \le \int_0^b w(x)\Phi(f(x))\,\frac{dx}{x} \tag{10.13}$$

holds for all functions f on $(0,b)$ with values in I and for Hf defined on $(0,b)$ by (2.3).

Proof. Rewrite Theorem 10.2 with $d\lambda(t) = \chi_{(0,1)}(t)\,dt$, $X = \Omega = \mathbb{R}_+$, $d\mu(x) = \chi_{(0,b)}(x)\,dx$, and $\nu(x) = dx$, as well as with the function $x \mapsto \dfrac{u(x)}{x}\chi_{(0,b)}(x)$ instead of the weight u. Then supp $\lambda = (0,1]$, $L = 1$, $\dfrac{d\mu_t}{d\nu}(x) = \dfrac{1}{t}\chi_{(0,tb)}(x)$,

$$Af(x) = \int_0^1 f(tx)\,dt = Hf(x),$$

and

$$v(x) = \int_0^1 \frac{u\left(\frac{1}{t}x\right)}{\frac{1}{t}x}\cdot\frac{1}{t}\chi_{(0,tb)}(x)\,dt = \frac{1}{x}\int_{\frac{x}{b}}^1 u\left(\frac{x}{t}\right)\,dt = \int_x^b u(y)\,\frac{dy}{y^2} = \frac{w(x)}{x},$$

for $x \in (0,b)$, so (10.13) holds. Since the conditions of Corollary 10.1 are fulfilled, the function Φ does not have to be non-negative. $\square$

The result of Corollary 10.3 can be found in [21], [30], and [65], so Theorem 10.2 can be regarded as its generalization. On the other hand, considering $d\lambda(t) = \chi_{[1,\infty)}(t)\dfrac{dt}{t^2}$, and $d\mu(x) = \chi_{(b,+\infty)}(x)dx$, as in the proof of Corollary 10.3 we get a dual result to (10.13) (see also [21, 30, 65]).

Corollary 10.4 *For $0 \le b < \infty$, suppose $u\colon (b,\infty) \to \mathbb{R}$ is a non-negative function, locally integrable in (b,∞), and w is defined on (b,∞) by*

$$w(x) = \frac{1}{x}\int_b^x u(t)\,dt. \tag{10.14}$$

If Φ is a convex function on an interval $I \subseteq \mathbb{R}$, then the inequality

$$\int_0^\infty u(x)\Phi(\tilde{H}f(x))\,\frac{dx}{x} \le \int_0^\infty w(x)\Phi(f(x))\,\frac{dx}{x}$$

holds for all functions f on (b,∞) with values in I and for $\tilde{H}f$ defined by

$$\tilde{H}f(x) = x\int_x^\infty f(t)\,\frac{dt}{t^2},\ x \in (b,\infty). \tag{10.15}$$

Further corollaries are related to a multidimensional setting with balls in $\mathbb{R}^n$ centred at the origin.

Corollary 10.5 *Suppose that* $0 < b \le \infty$ *and that a positive function* ψ *on* $[0,1]$ *and a non-negative function* u *on* $\mathbb{R}^n$ *are such that*

$$v(\mathbf{x}) = \int_{\frac{|\mathbf{x}|}{b}}^{1} u\left(\frac{1}{t}\mathbf{x}\right) t^{-n}\psi(t)\,dt < \infty, \quad \mathbf{x} \in B(b) \tag{10.16}$$

and

$$P_1 = \int_0^1 \psi(t)\,dt < \infty. \tag{10.17}$$

Suppose that Φ *is a non-negative convex function on an interval* $I \subseteq \mathbb{R}$*. If* $f: B(b) \to \mathbb{R}$ *is a Borel-measurable function such that* $f(\mathbf{x}) \in I$ *for all* $\mathbf{x} \in B(b)$*, then the inequality*

$$\int_{B(b)} u(\mathbf{x})\Phi\left(\frac{1}{P_1}\int_0^1 \psi(t)f(t\mathbf{x})\,dt\right)d\mathbf{x} \le \frac{1}{P_1}\int_{B(b)} v(\mathbf{x})\Phi(f(\mathbf{x}))\,d\mathbf{x} \tag{10.18}$$

holds.

Proof. Follows from Theorem 10.2 and Corollary 10.1 rewritten with $X = \mathbb{R}^n$, $\Omega = B(b)$, $d\lambda(t) = \psi(t)\chi_{(0,1)}(t)\,dt$, $d\mu(\mathbf{x}) = \chi_{B(b)}(\mathbf{x})\,d\mathbf{x}$, and $d\nu(\mathbf{x}) = d\mathbf{x}$. Here we have $\operatorname{supp}\lambda = (0,1]$, $\frac{d\mu_t}{d\nu}(\mathbf{x}) = t^{-n}\chi_{B(tb)}(\mathbf{x})$, and $Af(\mathbf{x}) = \frac{1}{P_1}\int_0^1 \psi(t)f(t\mathbf{x})\,dt$. It is easy to see that in this setting (10.16) reduces to (10.8), and (10.9) becomes (10.18). □

A similar unweighted n-dimensional result can be found in [78]. Applying Corollary 10.5 to some particular u and Φ we get the following result.

Corollary 10.6 *Let* $0 < b \le \infty$*, let the positive function* ψ *on* $[0,1]$ *be such that*

$$v(\mathbf{x}) = \int_{\frac{|\mathbf{x}|}{b}}^{1} t^{-n}\psi(t)\,dt < \infty, \quad \mathbf{x} \in B(b),$$

and let P_1 *be defined by* (10.17)*. If* $f: B(b) \to \mathbb{R}$ *is a non-negative Borel-measurable function, then the inequality*

$$\begin{aligned}\int_{B(b)}\left(\int_0^1 \psi(t)f(t\mathbf{x})\,dt\right)^p d\mathbf{x} &\le P_1^{p-1}\int_{B(b)} v(\mathbf{x})f^p(\mathbf{x})d\mathbf{x} \\ &\le P_1^{p-1}\left(\int_0^1 t^{-n}\psi(t)\,dt\right)\int_{B(b)} f^p(\mathbf{x})\,d\mathbf{x}\end{aligned} \tag{10.19}$$

holds for all $p \in \mathbb{R}\setminus[0,1)$*. If* $p \in (0,1)$*, then the first inequality in* (10.19) *holds with the reversed sign of inequality.*

Proof. The first inequality in (10.19) is equivalent with inequality (10.18), rewritten with $u(\mathbf{x}) \equiv 1$ and with the convex function $\Phi: \mathbb{R}_+ \to \mathbb{R}$, $\Phi(x) = x^p$, $p \in \mathbb{R}\setminus[0,1)$. For $p \in (0,1)$, the function Φ is concave. □

Analogously, we get the following result.

Corollary 10.7 *Suppose that $0 \le b < \infty$, that the positive function ψ on $[1,\infty)$ and the non-negative function u on $\mathbb{R}^n$ are such that*

$$v(\mathbf{x}) = \int_1^{\frac{|\mathbf{x}|}{b}} u\left(\frac{1}{t}\mathbf{x}\right) t^{-n}\psi(t)\,dt < \infty, \quad \mathbf{x} \in \mathbb{R}^n \setminus B(b). \tag{10.20}$$

and

$$P_\infty = \int_1^\infty \psi(t)\ dt < \infty. \tag{10.21}$$

Suppose that Φ is a non-negative convex function on an interval $I \subseteq \mathbb{R}$. If $f\colon \mathbb{R}^n \setminus B(b) \to \mathbb{R}$ is a Borel-measurable function, such that $f(\mathbf{x}) \in I$ for all $\mathbf{x} \in \mathbb{R}^n \setminus B(b)$, then the inequality

$$\int_{\mathbb{R}^n \setminus B(b)} u(\mathbf{x})\Phi\left(\frac{1}{P_\infty}\int_1^\infty \psi(t)f(t\mathbf{x})dt\right)d\mathbf{x} \le \frac{1}{P_\infty}\int_{\mathbb{R}^n \setminus B(b)} v(\mathbf{x})\Phi(f(\mathbf{x}))d\mathbf{x} \tag{10.22}$$

holds.

Proof. The proof follows from Theorem 10.2 if we set $d\lambda(t) = \psi(t)\chi_{(1,\infty)}(t)\,dt$, $X = \mathbb{R}^n$, $\Omega = \mathbb{R}^n \setminus B(b)$, $d\mu(\mathbf{x}) = \chi_{\mathbb{R}^n \setminus B(b)}(\mathbf{x})\,d\mathbf{x}$, and $d\nu(\mathbf{x}) = d\mathbf{x}$. Then we get $\operatorname{supp}\lambda = [1,\infty)$, $\frac{d\mu_t}{d\nu}(\mathbf{x}) = t^{-n}\chi_{\mathbb{R}^n \setminus B(tb)}(\mathbf{x})$, and $Af(\mathbf{x}) = \frac{1}{P_\infty}\int_1^\infty \psi(t)f(t\mathbf{x})\,dt$, so (10.8) and (10.9) become (10.20) and (10.22), respectively. □

An unweighted form of this result can be found in [78].

Corollary 10.8 *Let $0 \le b < \infty$ and let the function $\psi\colon [1,\infty) \to [0,\infty)$ be such that*

$$v(\mathbf{x}) = \int_1^{\frac{|\mathbf{x}|}{b}} t^{-n}\psi(t)\,dt < \infty, \quad \mathbf{x} \in \mathbb{R}^n \setminus B(b).$$

If $f\colon \mathbb{R}^n \setminus B(b) \to \mathbb{R}$ is a Borel-measurable function and P_∞ is defined by (10.21), then the inequality

$$\int_{\mathbb{R}^n \setminus B(b)} \left(\int_1^\infty \psi(t)f(t\mathbf{x})\,dt\right)^p d\mathbf{x} \le P_\infty^{p-1}\int_{\mathbb{R}^n \setminus B(b)} v(\mathbf{x})f^p(\mathbf{x})\,d\mathbf{x} \tag{10.23}$$

holds for all $p \in \mathbb{R} \setminus [0,1)$. For $p \in (0,1)$, the sign of inequality in (10.23) is reversed.

Proof. Again, like in Corollary 10.6, we take $u(\mathbf{x}) \equiv 1$ and the convex function $\Phi\colon \mathbb{R}_+ \to \mathbb{R}$, $\Phi(\mathbf{x}) = x^p$, $p \in \mathbb{R} \setminus [0,1)$. Notice that Φ is concave for $p \in (0,1)$. □

10.2 A new refined weighted Boas-type inequality

We continue our analisys in the same setting as before. Results from this section can be found in [33]. Now we can state and prove a new refined weighted Boas-type inequality.

Theorem 10.3 *Let the measures λ, μ, ν and μ_t, the number L, the set Ω, and functions u and v be as in Theorem* 10.2. *Suppose $\Phi\colon I \to \mathbb{R}$ is a non-negative convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi\colon I \to \mathbb{R}$ is any function fulfilling $\varphi(x) \in \partial\Phi(x)$, for all $x \in \operatorname{Int} I$. If $f\colon \Omega \to \mathbb{R}$ is a Borel measurable function with values in I and Af is defined by* (10.5), *then $Af(\mathbf{x}) \in I$ for all $\mathbf{x} \in \Omega$ and the inequality*

$$\begin{aligned}&\frac{1}{L}\int_\Omega v(\mathbf{x})\Phi(f(\mathbf{x}))\,d\nu(\mathbf{x}) - \int_\Omega u(\mathbf{x})\Phi(Af(\mathbf{x}))\,d\mu(\mathbf{x})\\ &\geq \frac{1}{L}\left|\int_\Omega u(\mathbf{x})\int_0^\infty |\Phi(f(t\mathbf{x})) - \Phi(Af(\mathbf{x}))|\,d\lambda(t)\,d\mu(\mathbf{x})\right.\\ &\quad\left. - \int_\Omega u(\mathbf{x})\int_0^\infty |\varphi(Af(\mathbf{x}))|\cdot|f(t\mathbf{x}) - Af(\mathbf{x})|\,d\lambda(t)\,d\mu(\mathbf{x})\right| \qquad (10.24)\end{aligned}$$

holds. For a non-positive concave function Φ, relation (10.24) *holds with*

$$\int_\Omega u(\mathbf{x})\Phi(Af(\mathbf{x}))\,d\mu(\mathbf{x}) - \frac{1}{L}\int_\Omega v(\mathbf{x})\Phi(f(\mathbf{x}))\,d\nu(\mathbf{x})$$

on its left-hand side.

Proof. Since $f(\mathbf{x}) \in I$ for all $\mathbf{x} \in \Omega$, it is not hard to see that $Af(\mathbf{x}) \in I$, for all $\mathbf{x} \in \Omega$ (see the proof of Theorem 10.2 for details). Suppose the function Φ is convex and non-negative. To prove inequality (10.24), observe that for arbitrary $r \in \operatorname{Int} I$ and $s \in I$, by (1.8), we have

$$\begin{aligned}\Phi(s) - \Phi(r) - \varphi(r)(s-r) &= |\Phi(s) - \Phi(r) - \varphi(r)(s-r)|\\ &\geq ||\Phi(s) - \Phi(r)| - |\varphi(r)|\cdot|s-r||\,.\end{aligned} \qquad (10.25)$$

In particular, for $\mathbf{x} \in \Omega$, such that $Af(\mathbf{x}) \in \operatorname{Int} I$, and for $t \in \operatorname{supp}\Omega$, from (10.25) we get

$$\begin{aligned}&\Phi(f(t\mathbf{x})) - \Phi(Af(\mathbf{x})) - \varphi(Af(\mathbf{x}))\cdot(f(t\mathbf{x}) - Af(\mathbf{x}))\\ &\geq ||\Phi(f(t\mathbf{x})) - \Phi(Af(\mathbf{x}))| - |\varphi(Af(\mathbf{x}))|\cdot|f(t\mathbf{x}) - Af(\mathbf{x})||\,.\end{aligned} \qquad (10.26)$$

On the other hand, if I is not an open interval and $Af(\mathbf{x})$ is an endpoint of I for some $\mathbf{x} \in \Omega$, then either $f(t\mathbf{x}) - Af(\mathbf{x}) \geq 0$ for all $t \in \operatorname{supp}\lambda$, or $f(t\mathbf{x}) - Af(\mathbf{x}) \leq 0$ for all $t \in \operatorname{supp}\lambda$. Since

$$\int_0^\infty (f(t\mathbf{x}) - Af(\mathbf{x}))\,d\lambda(t) = 0, \qquad (10.27)$$

we conclude that $f(t\mathbf{x}) - Af(\mathbf{x}) = 0$ for λ-a.e. $t \in \operatorname{supp}\lambda$, so in that case both sides of (10.26) are equal to 0. Hence, (10.26) holds for all $\mathbf{x} \in \Omega$ and λ-a.e. $t \in \operatorname{supp}\lambda$. Multiplying it by $u(\mathbf{x})$ and then integrating over $\mathbb{R}_+$ and Ω, we obtain the following sequence of inequalities:

$$\begin{aligned}
&\int_\Omega \int_0^\infty u(\mathbf{x})\Phi(f(t\mathbf{x}))\, d\lambda(t)\, d\mu(\mathbf{x}) - \int_\Omega \int_0^\infty u(\mathbf{x})\Phi(Af(\mathbf{x}))\, d\lambda(t)\, d\mu(\mathbf{x}) \\
&\quad - \int_\Omega \int_0^\infty u(\mathbf{x})\varphi(Af(\mathbf{x}))(f(t\mathbf{x}) - Af(\mathbf{x}))\, d\lambda(t)\, d\mu(\mathbf{x}) \\
&\geq \int_\Omega \int_0^\infty u(\mathbf{x}) \big| |\Phi(f(t\mathbf{x})) - \Phi(Af(\mathbf{x}))| - |\varphi(Af(\mathbf{x}))| \\
&\quad \cdot |f(t\mathbf{x}) - Af(\mathbf{x})| \big|\, d\lambda(t)\, d\mu(\mathbf{x}) \\
&\geq \int_\Omega u(\mathbf{x}) \bigg| \int_0^\infty |\Phi(f(t\mathbf{x})) - \Phi(Af(\mathbf{x}))|\, d\lambda(t) \\
&\quad - |\varphi(Af(\mathbf{x}))| \int_0^\infty |f(t\mathbf{x}) - Af(\mathbf{x})|\, d\lambda(t) \bigg|\, d\mu(\mathbf{x}) \\
&\geq \bigg| \int_\Omega u(\mathbf{x}) \int_0^\infty |\Phi(f(t\mathbf{x})) - \Phi(Af(\mathbf{x}))|\, d\lambda(t)\, d\mu(\mathbf{x}) \\
&\quad - \int_\Omega u(\mathbf{x}) |\varphi(Af(\mathbf{x}))| \int_0^\infty |f(t\mathbf{x}) - Af(\mathbf{x})|\, d\lambda(t)\, d\mu(\mathbf{x}) \bigg|.
\end{aligned} \tag{10.28}$$

By using Fubini's and the Radon-Nikodym theorem, the substitution $\mathbf{y} = t\mathbf{x}$, and the fact that the set Ω is λ-balanced and the function Φ is non-negative, the first integral on the left-hand side of (10.28) becomes

$$\begin{aligned}
&\int_\Omega \int_0^\infty u(\mathbf{x})\Phi(f(t\mathbf{x}))\, d\lambda(t)\, d\mu(\mathbf{x}) \\
&= \int_0^\infty \int_\Omega u(\mathbf{x})\Phi(f(t\mathbf{x}))\, d\mu(\mathbf{x})\, d\lambda(t) \\
&= \int_0^\infty \int_{t\Omega} u\left(\frac{1}{t}\mathbf{y}\right) \Phi(f(\mathbf{y}))\, d\mu_t(\mathbf{y})\, d\lambda(t) \\
&\leq \int_0^\infty \int_\Omega u\left(\frac{1}{t}\mathbf{y}\right) \Phi(f(\mathbf{y}))\, d\mu_t(\mathbf{y})\, d\lambda(t) \\
&= \int_0^\infty \int_\Omega u\left(\frac{1}{t}\mathbf{y}\right) \Phi(f(\mathbf{y})) \frac{d\mu_t}{d\nu}(\mathbf{y})\, d\nu(\mathbf{y})\, d\lambda(t) \\
&= \int_\Omega \left(\int_0^\infty u\left(\frac{1}{t}\mathbf{y}\right) \frac{d\mu_t}{d\nu}(\mathbf{y})\, d\lambda(t) \right) \Phi(f(\mathbf{y}))\, d\nu(\mathbf{y}) \\
&= \int_\Omega v(\mathbf{y})\Phi(f(\mathbf{y}))\, d\nu(\mathbf{y}).
\end{aligned} \tag{10.29}$$

Further, the second integral on the left-hand side in (10.28) reduces to

$$\int_\Omega \int_0^\infty u(\mathbf{x})\Phi(Af(\mathbf{x}))\, d\lambda(t)\, d\mu(\mathbf{x}) = L \int_\Omega u(\mathbf{x})\Phi(Af(\mathbf{x}))\, d\mu(\mathbf{x}), \tag{10.30}$$

while the corresponding third integral is equal to 0 since by (10.27) we have

$$\begin{aligned}
&\int_\Omega \int_0^\infty u(\mathbf{x})\varphi(Af(\mathbf{x}))(f(t\mathbf{x}) - Af(\mathbf{x}))\, d\lambda(t)\, d\mu(\mathbf{x}) \\
&= \int_\Omega u(\mathbf{x})\varphi(Af(\mathbf{x}))\left(\int_0^\infty (f(t\mathbf{x}) - Af(\mathbf{x}))\, d\lambda(t)\right) d\mu(\mathbf{x}) = 0.
\end{aligned} \tag{10.31}$$

Finally, (10.24) holds by combining (10.28), (10.29), (10.30) and (10.31).

It remains to prove the last part of the statement of Theorem 10.3. If Φ is a non-positive concave function, then $-\Phi$ is a non-negative convex function so by using (1.9), relation (10.25) becomes

$$\begin{aligned}
\Phi(r) - \Phi(s) - \varphi(r)(r-s) &= |\Phi(r) - \Phi(s) - \varphi(r)(r-s)| \\
&\geq ||\Phi(s) - \Phi(r)| - |\varphi(r)|\,|s-r||,
\end{aligned}$$

where $\varphi\colon I \to \mathbb{R}$ is any function such that $\varphi(x) \in \partial\Phi(x) = [\Phi'_+(x), \Phi'_-(x)]$ for all $x \in \operatorname{Int} I$. Following the same lines as in the proof for a convex function, we get (10.24) with swapped order of two integrals on its left-hand side. □

Remark 10.1 Observe that a pair of inequalities interpolated between the left-hand side and the right-hand side of (10.28) provides other new refinements of (10.24).

Remark 10.2 We can get analogue result to (10.24), if we consider Φ to be convex and monotone function. That assumption can also be applied to results from the following pages.

It is important to notice that the condition on non-negativity of the convex function Φ in Theorem 10.3 can be omitted only in a particular setting with cones in X. More precisely, the following corollary holds.

Corollary 10.9 *If in Theorem* 10.3 *we have* $t\Omega = \Omega$ *for* λ*-a.e.* $t \in \operatorname{supp} \lambda$*, then* (10.24) *holds for all convex functions* Φ *on an interval* $I \subseteq \mathbb{R}$*. In this setting, relation* (10.24) *holds also for all concave functions* Φ *on* $I \subseteq \mathbb{R}$*, but with swapped order of the integrals on its left-hand side.*

Remark 10.3 Observe that Theorem 10.3 generalizes and refines the Boas-type inequality (10.3) obtained by D. Luor [78].

10.3 A general Boas-type inequality with kernels

Now, we analyse Boas-type inequalities with kernels from [33]. Let the setting be as in Section 10.1, except that λ is a σ-finite Borel measure on $\mathbb{R}_+$. By a kernel we mean a non-negative measurable function $k\colon X \times X \to \mathbb{R}$, such that

$$K(\mathbf{x}) = \int_0^\infty k(\mathbf{x}, t\mathbf{x})\, d\lambda(t) < \infty \tag{10.32}$$

for μ-a.e. $\mathbf{x} \in X$. For a λ-balanced set $\Omega \subseteq X$ and a Borel measurable function $f\colon \Omega \to \mathbb{R}$, we define its Hardy-Littlewood average with the kernel k, denoted by $A_k f$, as

$$A_k f(\mathbf{x}) = \frac{1}{K(\mathbf{x})} \int_0^\infty k(\mathbf{x}, t\mathbf{x}) f(t\mathbf{x})\, d\lambda(t), \quad \mathbf{x} \in \Omega. \tag{10.33}$$

A related Boas-type inequality is given as follows.

Theorem 10.4 *Let λ be a σ-finite Borel measure on $\mathbb{R}_+$, let μ and ν be σ-finite Borel measures on a topological space X, and let μ_t, defined by* (10.7)*, be absolutely continuous with respect to the measure ν for all $t \in \operatorname{supp}\lambda$. Let $\Omega \subseteq X$ be a λ-balanced set and u be a non-negative function on X such that*

$$v(\mathbf{x}) = \int_0^\infty u(\mathbf{x}) \frac{k\left(\frac{1}{t}\mathbf{x}, \mathbf{x}\right)}{K\left(\frac{1}{t}\mathbf{x}\right)} \cdot \frac{d\mu_t}{d\nu}(\mathbf{x})\, d\lambda(t) < \infty, \ \ \mathbf{x} \in \Omega, \tag{10.34}$$

where $k\colon X \times X \to \mathbb{R}$ is a non-negative measurable function satisfying (10.32)*. Further, let Φ be a non-negative convex function on an interval $I \subseteq \mathbb{R}$. If $f\colon \Omega \to \mathbb{R}$ is a measurable function such that $f(\mathbf{x}) \in I$ for all $\mathbf{x} \in \Omega$, and $A_k f$ is defined by* (10.33)*, then $A_k f(\mathbf{x}) \in I$, for all $\mathbf{x} \in \Omega$, and the inequality*

$$\int_\Omega u(\mathbf{x}) \Phi(A_k f(\mathbf{x}))\, d\mu(\mathbf{x}) \le \int_\Omega v(\mathbf{x}) \Phi(f(\mathbf{x}))\, d\nu(\mathbf{x}) \tag{10.35}$$

holds. For a non-positive concave function Φ, relation (10.35) *holds with the sign of inequality reversed.*

Proof. First, we need to prove that $A_k f(\mathbf{x}) \in I$ for all $\mathbf{x} \in \Omega$. Otherwise, there exists $\mathbf{x}_0 \in \Omega$ such that $A_k f(\mathbf{x}_0) \notin I$. In that case, we have either $f(t\mathbf{x}_0) - A_k f(\mathbf{x}_0) < 0$ for all $t \in \operatorname{supp}\lambda$, or $f(t\mathbf{x}_0) - A_k f(\mathbf{x}_0) > 0$ for all $t \in \operatorname{supp}\lambda$. On the other hand, the identity

$$\frac{1}{K(\mathbf{x}_0)} \int_0^\infty k(\mathbf{x}_0, t\mathbf{x}_0) \left(f(t\mathbf{x}_0) - A_k f(\mathbf{x}_0)\right) d\lambda(t) = 0$$

and non-negativity of $k(\mathbf{x}_0, t\mathbf{x}_0)$ for all $t \in \operatorname{supp}\lambda$ yield that

$$k(x_0, t\mathbf{x}_0) \left(f(t\mathbf{x}_0) - A_k f(\mathbf{x}_0)\right) = 0, \ t \in \operatorname{supp}\lambda.$$

Since $K(\mathbf{x}_0) > 0$, there exists a set $J \subseteq \operatorname{supp} \lambda$ such that $\lambda(J) > 0$ and $k(\mathbf{x}_0, t\mathbf{x}_0) > 0$ for all $t \in J$. Hence, $f(t\mathbf{x}_0) - A_k f(\mathbf{x}_0) = 0$, $t \in J$, so we came to a contradiction. Therefore, $A_k f(\mathbf{x}) \in I$ for all $\mathbf{x} \in \Omega$.

By using Jensen's inequality, Fubini's theorem, the Radon-Nikodym theorem, the substitution $\mathbf{y} = t\mathbf{x}$, and the properties of the set Ω and the function Φ, we now obtain

$$\begin{aligned}
&\int_\Omega u(\mathbf{x})\Phi(A_k f(\mathbf{x}))\,d\mu(\mathbf{x})\\
&\le \int_\Omega \frac{u(\mathbf{x})}{K(\mathbf{x})}\int_0^\infty k(\mathbf{x},t\mathbf{x})\Phi(f(t\mathbf{x}))\,d\lambda(t)\,d\mu(\mathbf{x})\\
&= \int_0^\infty \int_\Omega u(\mathbf{x})\,\frac{k(\mathbf{x},t\mathbf{x})}{K(\mathbf{x})}\,\Phi(f(t\mathbf{x}))\,d\mu(\mathbf{x})\,d\lambda(t)\\
&= \int_0^\infty \int_{t\Omega} u\left(\frac{1}{t}\mathbf{y}\right)\frac{k\left(\frac{1}{t}\mathbf{y},\mathbf{y}\right)}{K\left(\frac{1}{t}\mathbf{y}\right)}\,\Phi(f(\mathbf{y}))\,d\mu_t(\mathbf{y})\,d\lambda(t)\\
&\le \int_0^\infty \int_{\Omega} u\left(\frac{1}{t}\mathbf{y}\right)\frac{k\left(\frac{1}{t}\mathbf{y},\mathbf{y}\right)}{K\left(\frac{1}{t}\mathbf{y}\right)}\,\Phi(f(\mathbf{y}))\,d\mu_t(\mathbf{y})\,d\lambda(t)\\
&= \int_0^\infty \int_{\Omega} u\left(\frac{1}{t}\mathbf{y}\right)\frac{k\left(\frac{1}{t}\mathbf{y},\mathbf{y}\right)}{K\left(\frac{1}{t}\mathbf{y}\right)}\,\Phi(f(\mathbf{y}))\frac{d\mu_t}{d\nu}(\mathbf{y})\,d\nu(\mathbf{y})\,d\lambda(t)\\
&= \int_\Omega \left(\int_0^\infty u\left(\frac{1}{t}\mathbf{y}\right)\frac{k\left(\frac{1}{t}\mathbf{y},\mathbf{y}\right)}{K\left(\frac{1}{t}\mathbf{y}\right)}\cdot\frac{d\mu_t}{d\nu}(\mathbf{y})\,d\lambda(t)\right)\Phi(f(\mathbf{y}))\,d\nu(\mathbf{y})\\
&= \int_\Omega v(\mathbf{y})\Phi(f(\mathbf{y}))\,d\nu(\mathbf{y}),
\end{aligned}$$

so the proof is completed. □

Moreover, applying similar reasoning as in Section 10.2, we get a refinement of the Boas-type inequality (10.35).

Theorem 10.5 *Suppose λ is a σ-finite Borel measure on $\mathbb{R}_+$, μ and ν are σ-finite Borel measures on a topological space X, and the measures μ_t, defined by* (10.7)*, are absolutely continuous with respect to the measure ν for all $t \in \operatorname{supp}\lambda$. Further, suppose $\Omega \subseteq X$ is a λ-balanced set, u is a non negative function on X and v is defined on Ω by* (10.34)*, where $k\colon X \times X \to R$ is a non-negative measurable function satisfying* (10.32)*. If Φ is a non-negative convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi\colon I \to \mathbb{R}$ is any function such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$, then the inequality*

$$\begin{aligned}
&\int_\Omega v(\mathbf{x})\Phi(f(\mathbf{x}))\,d\nu(\mathbf{x}) - \int_\Omega u(\mathbf{x})\Phi(A_k f(\mathbf{x}))\,d\mu(\mathbf{x})\\
&\ge \left|\int_\Omega \frac{u(\mathbf{x})}{K(\mathbf{x})}\int_0^\infty k(\mathbf{x},t\mathbf{x})\left|\Phi(f(t\mathbf{x})) - \Phi(A_k f(\mathbf{x}))\right|\,d\lambda(t)\,d\mu(\mathbf{x})\right.\\
&\qquad\left. - \int_\Omega \frac{u(\mathbf{x})}{K(\mathbf{x})}\int_0^\infty k(\mathbf{x},t\mathbf{x})\left|\varphi(A_k f(\mathbf{x}))\right|\cdot\left|f(t\mathbf{x}) - A_k f(\mathbf{x})\right|\,d\lambda(t)\,d\mu(\mathbf{x})\right|
\end{aligned} \tag{10.36}$$

holds for all measurable functions $f\colon \Omega \to \mathbb{R}$*, such that* $f(\mathbf{x}) \in I$ *for all* $\mathbf{x} \in \Omega$*, and* $A_k f$ *defined by* (10.33)*. For a non-positive concave function* Φ*, relation* (10.36) *holds with*

$$\int_\Omega u(\mathbf{x})\Phi(A_k f(\mathbf{x}))\,d\mu(\mathbf{x}) - \int_\Omega v(\mathbf{x})\Phi(f(\mathbf{x}))\,d\nu(\mathbf{x})$$

on its left-hand-side.

Proof. The proof follows the same lines as the proof of Theorem 10.4 so we omit details. □

Remark 10.4 Obviously, $t\mathbf{x}$ and $\mathbf{x}$ have "the same direction" in X. However, this drawback can be avoided if, instead of over $\mathbb{R}_+$ and with respect to λ, the integrals in (10.33), (10.34), (10.35), and (10.36) are taken over some suitable group and the related Haar measure.

10.4 Boas-type inequality for superquadratic functions

Here we prove the Boas-type inequality in a setting with general weighted topological spaces and σ-finite measures using the concept of superquadratic and subquadratic functions. The following results can be found in [71].

Theorem 10.6 *Let* λ*,* μ*,* ν*,* μ_t*,* L*,* Ω*,* u *and* v *be as in Theorem* 10.2*. Suppose that* $I = (0,c), c \leq \infty$ *and let* $f\colon \Omega \to \mathbb{R}$ *be a Borel measurable function such that* $f(\mathbf{x}) \in I$ *for all* $\mathbf{x} \in \Omega$*. If* $\varphi\colon I \to \mathbb{R}$ *is a non-negative superquadratic function and* Af *is defined by* (10.5)*, then the inequality*

$$\begin{aligned} &\int_\Omega u(\mathbf{x})\varphi(Af(\mathbf{x}))\,d\mu(\mathbf{x}) + \frac{1}{L}\int_\Omega\int_0^\infty u(\mathbf{x})\varphi(|f(t\mathbf{x}) - Af(\mathbf{x})|)d\lambda(t)d\mu(\mathbf{x}) \\ &\leq \frac{1}{L}\int_\Omega v(\mathbf{x})\varphi(f(\mathbf{x}))\,d\nu(\mathbf{x}) \end{aligned} \tag{10.37}$$

holds.

Proof. We have already proved that $Af(\mathbf{x}) \in I$ for $\mathbf{x} \in I$. So, we only prove the inequality (10.37). By applying the refined Jensen inequality, that is Lemma 1.1, to the first term on the left hand side of inequality (10.37) we obtain

$$\begin{aligned} \int_\Omega u(\mathbf{x})\varphi(Af(\mathbf{x}))\,d\mu(\mathbf{x}) &= \int_\Omega u(\mathbf{x})\varphi\left(\frac{1}{L}\int_0^\infty f(t\mathbf{x})\,d\lambda(t)\right)\,d\mu(\mathbf{x}) \\ &\leq \frac{1}{L}\int_\Omega u(\mathbf{x})\int_0^\infty \varphi(f(t\mathbf{x}))\,d\lambda(t)\,d\mu(\mathbf{x}) \\ &\quad -\frac{1}{L}\int_\Omega u(\mathbf{x})\int_0^\infty \varphi(|f(t\mathbf{x}) - Af(\mathbf{x})|)d\lambda(t)d\mu(\mathbf{x}) \end{aligned} \tag{10.38}$$

The inequality (10.38) can be written as

$$\begin{aligned}
&\int_\Omega u(\mathbf{x})\varphi\left(\frac{1}{L}\int_0^\infty f(t\mathbf{x})\,d\lambda(t)\right)d\mu(\mathbf{x})\\
&+\frac{1}{L}\int_\Omega u(\mathbf{x})\int_0^\infty \varphi(|f(t\mathbf{x})-Af(\mathbf{x})|)d\lambda(t)d\mu(\mathbf{x})\\
&\leq \frac{1}{L}\int_\Omega u(\mathbf{x})\int_0^\infty \varphi(f(t\mathbf{x}))\,d\lambda(t)\,d\mu(\mathbf{x}).
\end{aligned}\tag{10.39}$$

By applying the Fubini theorem to the right hand side of inequality (10.39), than the substitution $\mathbf{y}=t\mathbf{x}$, the fact that Ω is λ-balanced set, φ is a non-negative function and the Radon-Nikodym theorem, we obtain

$$\begin{aligned}
&\frac{1}{L}\int_0^\infty\int_\Omega u(\mathbf{x})\varphi(f(t\mathbf{x}))\,d\mu(\mathbf{x})\,d\lambda(t)\\
&=\frac{1}{L}\int_0^\infty\int_{t\Omega} u\left(\frac{1}{t}\mathbf{y}\right)\varphi(f(\mathbf{y}))\,d\mu_t(\mathbf{y})\,d\lambda(t)\\
&\leq \frac{1}{L}\int_0^\infty\int_{\Omega} u\left(\frac{1}{t}\mathbf{y}\right)\varphi(f(\mathbf{y}))\,d\mu_t(\mathbf{y})\,d\lambda(t)\\
&=\frac{1}{L}\int_0^\infty\int_{\Omega} u\left(\frac{1}{t}\mathbf{y}\right)\varphi(f(\mathbf{y}))\frac{d\mu_t}{d\nu}(\mathbf{y})\,d\nu(\mathbf{y})\,d\lambda(t)\\
&=\frac{1}{L}\int_\Omega\left(\int_0^\infty u\left(\frac{1}{t}\mathbf{y}\right)\frac{d\mu_t}{d\nu}(\mathbf{y})\,d\lambda(t)\right)\varphi(f(\mathbf{y}))\,d\nu(\mathbf{y})\\
&=\frac{1}{L}\int_\Omega v(\mathbf{y})\varphi(f(\mathbf{y}))\,d\nu(\mathbf{y}).
\end{aligned}\tag{10.40}$$

This completes the proof. □

Theorem 10.7 *Let λ, L, μ, ν, u and v be defined as in Theorem* 10.6 *Further, let $\Omega\subseteq X$ be such that $t\Omega=\Omega$, for all $t\in$ supp λ. Suppose that $I=(0,c), c\leq\infty$, $\varphi\colon I\to\mathbb{R}$. If φ is a superquadratic function on an interval I, then the inequality* (10.37) *holds for all Borel measurable functions $f\colon\Omega\to\mathbb{R}$, such that $f(\mathbf{x})\in I$ for all $\mathbf{x}\in\Omega$, where Af is defined by* (10.5).
If φ is a subquadratic function, then the inequality sign in (10.37) *is reversed, that is*

$$\begin{aligned}
&\int_\Omega u(\mathbf{x})\varphi(Af(\mathbf{x}))\,d\mu(\mathbf{x})+\frac{1}{L}\int_\Omega\int_0^\infty u(\mathbf{x})\varphi(|f(t\mathbf{x})-Af(\mathbf{x})|)d\lambda(t)d\mu(\mathbf{x})\\
&\geq \frac{1}{L}\int_\Omega v(\mathbf{x})\varphi(f(\mathbf{x}))\,d\nu(\mathbf{x})
\end{aligned}\tag{10.41}$$

holds.

Proof. By analyzing (10.40), we see that if $t\Omega=\Omega$, for all $t\in$ supp λ, then

$$\begin{aligned}
&\frac{1}{L}\int_0^\infty\int_{t\Omega} u\left(\frac{1}{t}\mathbf{y}\right)\varphi(f(\mathbf{y}))\,d\mu_t(\mathbf{y})\,d\lambda(t)\\
&\quad=\frac{1}{L}\int_0^\infty\int_{\Omega} u\left(\frac{1}{t}\mathbf{y}\right)\varphi(f(\mathbf{y}))\,d\mu_t(\mathbf{y})\,d\lambda(t)
\end{aligned}$$

and (10.37) holds for all superquadratic functions $\varphi\colon I \to \mathbb{R}$, that is, φ does not need to be non-negative.
By making the same calculations with φ subquadratic function, we see that the inequality sign in (10.37) is reversed, that is (10.41) holds. □

Remark 10.5 Notice, that in case $\Omega = \mathbb{R}_+$ (10.37) and (10.41) hold for all superquadratic and all subquadratic functions, respectively.

Now we consider a particular superquadratic function in (10.37), namely $\varphi(x) = x^p$ which is superquadratic for $p \geq 2$ and subquadratic for $1 \leq p \leq 2$. We obtain the following result.

Corollary 10.10 *Let the assumptions in Theorem* 10.6 *be satisfied and let* $\varphi(x) = x^p$.

(i) If $p \geq 2$, *then*

$$\int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^p \, d\mu(\mathbf{x}) + \frac{1}{L}\int_\Omega \int_0^\infty u(\mathbf{x})|f(t\mathbf{x}) - Af(\mathbf{x})|^p d\lambda(t) d\mu(\mathbf{x}) \\ \leq \frac{1}{L}\int_\Omega v(\mathbf{x}) f^p(\mathbf{x}) \, d\nu(\mathbf{x}). \tag{10.42}$$

(ii) If $t\Omega = \Omega$, *for all* $t \in \operatorname{supp} \lambda$ *and* $1 < p \leq 2$, *then* (10.42) *holds in the reversed direction and for* $p = 2$ *we obtain the following very general identity*

$$\int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^2 \, d\mu(\mathbf{x}) + \frac{1}{L}\int_\Omega \int_0^\infty u(\mathbf{x})|f(t\mathbf{x}) - Af(\mathbf{x})|^2 d\lambda(t) d\mu(\mathbf{x}) \\ = \frac{1}{L}\int_\Omega v(\mathbf{x}) f^2(\mathbf{x}) \, d\nu(\mathbf{x}).$$

Corollary 10.11 *Let* λ *be a finite Borel measure on* $\mathbb{R}_+$ *and* L *be defined by* (10.4). *Let* $X = \mathbb{R}_+$ *and let* μ_t *be defined by* (10.7). *Suppose* $\Omega \subseteq \mathbb{R}_+$ *is a* λ*-balanced set and the function* $x \mapsto \dfrac{u(x)}{x}$ *is non-negative on* $\mathbb{R}_+$, *and the function* $w\colon \Omega \to \mathbb{R}$ *is defined by* (10.11). *Suppose that* $I = (0,c), c \leq \infty$, $\varphi\colon I \to \mathbb{R}$. *If* φ *is a non-negative superquadratic function on an interval I, then the inequality*

$$\int_\Omega u(x)\varphi(Af(x))\frac{dx}{x} + \frac{1}{L}\int_\Omega \int_0^\infty u(x)\varphi(|f(tx) - Af(x)|) d\lambda(t) \frac{dx}{x} \\ \leq \frac{1}{L}\int_\Omega w(x)\varphi(f(x))\frac{dx}{x} \tag{10.43}$$

holds for Borel measurable functions $f\colon \Omega \to \mathbb{R}$ *such that* $f(x) \in I$ *for all* $x \in \Omega$, *where* Af *is defined by* (10.5).
If $t\Omega = \Omega$, *for all* $t \in \operatorname{supp} \lambda$, *then* (10.43) *holds for all superquadratic functions* φ *and the inequality sign in* (10.43) *is reversed if* φ *is a subquadratic function.*

Proof. It follows directly from Theorem 10.2 if we set measures μ and ν to be Lebesgue measures since $X = \mathbb{R}_+$. For such measures $\frac{d\mu_t}{d\nu}(x) = \frac{1}{t}$, $t \in \mathbb{R}_+$. For function w we take

$$w(x) = xv(x) = x\int_0^\infty u\left(\frac{x}{t}\right)\cdot\frac{t}{x}\cdot\frac{1}{t}\,d\lambda(t) = \int_0^\infty u\left(\frac{x}{t}\right)\,d\lambda(t)\;,\; x\in\Omega,$$

where function v is defined by (10.8) with the weight function $x \mapsto \frac{u(x)}{x}$ instead of u. $\square$

Remark 10.6 If we apply Corollary 10.11 with $\Omega = \mathbb{R}_+$ and $u(x) = 1$, then $w(x) \equiv L$ and the following inequality holds:

$$\int_0^\infty \varphi\left(\frac{1}{L}\int_0^\infty f(tx)\,d\lambda(t)\right)\frac{dx}{x} + \frac{1}{L}\int_0^\infty\int_0^\infty \varphi(|f(tx) - Af(x)|)d\lambda(t)\frac{dx}{x}$$
$$\leq \int_0^\infty \varphi(f(x))\frac{dx}{x}. \tag{10.44}$$

In particular, for $\varphi(x) = x^p$, we obtain the following (in)equalities.
(i) If $p \geq 2$, then

$$\int_0^\infty \left(\frac{1}{L}\int_0^\infty f(tx)\,d\lambda(t)\right)^p\frac{dx}{x} + \frac{1}{L}\int_0^\infty\int_0^\infty |f(tx) - Af(x)|^p d\lambda(t)\frac{dx}{x}$$
$$\leq \int_0^\infty f^p(x)\frac{dx}{x}. \tag{10.45}$$

(ii) If $1 < p \leq 2$, then (10.45) holds in the reversed direction.
(iii) If $p = 2$, then the following identity holds

$$\int_0^\infty \left(\frac{1}{L}\int_0^\infty f(tx)\,d\lambda(t)\right)^2\frac{dx}{x} + \frac{1}{L}\int_0^\infty\int_0^\infty |f(tx) - Af(x)|^2 d\lambda(t)\frac{dx}{x}$$
$$= \int_0^\infty f^2(x)\frac{dx}{x}.$$

Remark 10.7 As a special case of inequality (10.45) we obtain the refined Hardy and dual Hardy inequality. Let $\alpha > 0$ and

$$d\lambda(t) = \begin{cases} t^{\alpha-1}, & 0 < t \leq 1; \\ 0, & t \geq 1. \end{cases}$$

Then $L = \alpha^{-1}$ and (10.45) becomes

$$\alpha^p \int_0^\infty x^{-1-\alpha p} \left(\int_0^x f(t) t^{\alpha-1} dt \right)^p dx + \alpha \int_0^\infty \int_0^x |f(t) - Af(x)|^p t^{\alpha-1} x^{-\alpha-1} dt dx$$
$$\leq \int_0^\infty f^p(x) \frac{dx}{x}, \tag{10.46}$$

where

$$Af(x) = \alpha x^{-\alpha} \int_0^x t^{\alpha-1} f(t) dt.$$

If we let $f(t) = g(t) t^{1-\alpha}$ and $\alpha = \frac{k-1}{p}$ $(p \geq 2,\, k > 1)$ in (10.46) we have

$$\int_0^\infty x^{-k} \left(\int_0^x g(t) dt \right)^p dx$$
$$\frac{k-1}{p} \int_0^\infty \int_t^\infty \left| \frac{p}{k-1} \left(\frac{t}{x} \right)^{1-\frac{k-1}{p}} g(t) - \frac{1}{x} \int_0^x g(s) ds \right|^p x^{p-k-\frac{k-1}{p}} dx t^{\frac{k-1}{p}-1} dt$$
$$\leq \left(\frac{p}{k-1} \right)^p \int_0^\infty x^{p-k} g^p(x) dx. \tag{10.47}$$

If $1 < p \leq 2$, then (10.47) holds in the reversed direction.
Now, let $\beta > 0$ and

$$d\lambda(t) = \begin{cases} t^{-\beta-1}, & t \geq 1; \\ 0, & 0 < t \leq 1. \end{cases}$$

Then $L = \beta^{-1}$ and (10.45) becomes

$$\beta^p \int_0^\infty x^{\beta p-1} \left(\int_x^\infty f(t) t^{-\beta-1} dt \right)^p dx + \beta \int_0^\infty \int_x^\infty |f(t) - Af(x)|^p t^{-\beta-1} x^{\beta-1} dt dx$$
$$\leq \int_0^\infty f^p(x) \frac{dx}{x}, \tag{10.48}$$

where

$$Af(x) = \beta x^\beta \int_x^\infty t^{-\beta-1} f(t) dt.$$

If we let $f(t)=g(t)t^{1+\beta}$ and $\beta=\frac{1-k}{p}$ $(p\geq 2, k<1)$ in (10.48) we have

$$\int_0^\infty x^{-k}\left(\int_x^\infty g(t)dt\right)^p dx$$
$$+\frac{1-k}{p}\int_0^\infty\int_0^t\left|\frac{p}{1-k}\left(\frac{t}{x}\right)^{1+\frac{1-k}{p}}g(t)-\frac{1}{x}\int_x^\infty g(s)ds\right|^p x^{p-k+\frac{1-k}{p}}dxt^{\frac{k-1}{p}-1}dt$$
$$\leq\left(\frac{p}{1-k}\right)^p\int_0^\infty x^{p-k}g^p(x)dx. \tag{10.49}$$

If $1<p\leq 2$, then (10.49) holds in the reversed direction.
Note that for the case $p=2$ inequalities (10.47) and (10.49) will both be equalities, Parseval type identities with the Hardy and, respectively, the dual Hardy operators.

Remark 10.8 These results can be found in [88] (see Theorem 3.1 and 3.2). Also, some new results involving Hardy type inequalities using the concept of superquadratic and subquadratic function can be found in [5] and [89]. In [89] J. A. Oguntuase et al. proved these results in multidimensional settings. Theorem 10.6 can be applied in multidimensional settings to obtain these results, but here we omit the details.

We continue with two consequences of Theorem 10.6.

Corollary 10.12 *Let $b\in\mathbb{R}_+$ and let $x\mapsto\dfrac{u(x)}{x}$ be a non-negative function on $(0,b)$, such that the function $t\mapsto\dfrac{u(t)}{t^2}$ is locally integrable in $(0,b)$, and let w be defined as in Corollary* 10.3. *If φ is non-negative superquadratic on an interval $I=(0,c)$, $c\leq\infty$, then the inequality*

$$\int_0^b u(x)\varphi(Hf(x))\frac{dx}{x}+\int_0^b\int_0^x u(x)\varphi(|f(t)-Hf(x)|)dt\frac{dx}{x^2}\leq\int_0^b w(x)\varphi(f(x))\frac{dx}{x} \tag{10.50}$$

holds for all functions f on $(0,b)$ with values in I and for $Hf(x)$ defined on $(0,b)$ by (2.3).

Proof. Rewrite Theorem 10.6 with the measures $d\lambda(t)=\chi_{(0,1)}(t)\,dt$, $\mu(x)=\chi_{(0,b)}(x)\,dx$, $d\nu(x)=dx$ and $x\mapsto\dfrac{u(x)}{x}$ instead of the weight function u. Then we get $L=1$, $\dfrac{d\mu_t}{d\nu}(x)=\dfrac{1}{t}\chi_{(0,tb)}(x)$,

$$Af(x)=\int_0^1 f(tx)\,dt=Hf(x)$$

and

$$v(x)=\int_0^1\frac{u\left(\frac{1}{t}x\right)}{\frac{1}{t}x}\cdot\frac{1}{t}\chi_{(0,tb)}(x)\,dt=\frac{1}{x}\int_{\frac{x}{b}}^1 u\left(\frac{x}{t}\right)\,dt=\int_x^b u(y)\frac{dy}{y^2}=\frac{w(x)}{x},$$

for $x \in (0,b)$, so (10.50) holds. □

We give also a dual result to the previous corollary considering $d\lambda(t) = \chi_{[1,\infty)}(t)\dfrac{dt}{t^2}$.

Corollary 10.13 *For $b \geq 0$, suppose u and w are defined on (b,∞) as in Corollary 10.4. If φ is non-negative superquadratic on an interval $I=(0,c)$, $c \leq \infty$, then the inequality*

$$\begin{aligned}
&\int_b^\infty u(x)\varphi(\tilde{H}f(x))\frac{dx}{x} + \int_b^\infty \int_x^\infty u(x)\varphi(|f(t)-\tilde{H}f(x)|)\frac{dt}{t^2}dx \\
&\leq \int_b^\infty w(x)\varphi(f(x))\frac{dx}{x} \qquad (10.51)
\end{aligned}$$

holds for all functions f on (b,∞) with values in I and for $\tilde{H}f(x)$ defined by (10.15).

Let us continue by some results from [72]. We define a linear functional as a difference between the right-hand side and the left-hand side of the refined Boas type inequality (10.37):

$$\begin{aligned}
G(\varphi) &= \frac{1}{L}\int_\Omega v(\mathbf{x})\varphi(f(\mathbf{x}))\,d\nu(\mathbf{x}) - \int_\Omega u(\mathbf{x})\varphi(Af(\mathbf{x}))\,d\mu(\mathbf{x}) \\
&\quad -\frac{1}{L}\int_\Omega\int_0^\infty u(\mathbf{x})\varphi(|f(t\mathbf{x})-Af(\mathbf{x})|)d\lambda(t)d\mu(\mathbf{x}). \qquad (10.52)
\end{aligned}$$

It is clear, from (10.37), that if φ is a superquadratic function, then $G(\varphi) \geq 0$. If we consider $G(\varphi_p)$ for φ_p as in Lemma 7.2 with G as in (10.52) we have that $G(\varphi_p) \geq 0$ for all $p > 0$.

Properties of the mapping $p \mapsto G(\varphi_p)$ are given in the following theorem:

Theorem 10.8 *For G as in* (10.52)*, φ_p as in* (7.16) *and f a positive function, we have the following:*

(i) the mapping $p \mapsto G(\varphi_p)$ is continuous for $p > 0$,

(ii) for every $n \in \mathbb{N}$ and $p_i \in \mathbb{R}_+, p_{ij} = \frac{p_i+p_j}{2}$, $i,j = 1,2,\ldots,n$, the matrix $[G(\varphi_{p_{ij}})]_{i,j=1}^n$ is positive semi-definite, that is,

$$\det[G(\varphi_{p_{ij}})]_{i,j=1}^n \geq 0,$$

(iii) the mapping $p \mapsto G(\varphi_p)$ is exponentially convex,

(iv) the mapping $p \mapsto G(\varphi_p)$ is log-convex, and for $r < s < t$ where $r,s,t \in \mathbb{R}_+$ we have

$$[G(\varphi_s)]^{t-r} \leq [G(\varphi_r)]^{t-s}[G(\varphi_t)]^{s-r}. \qquad (10.53)$$

Proof.

(i) Notice that

$$G(\varphi_p)=\begin{cases} \frac{1}{p(p-2)}\left[\frac{1}{L}\int_\Omega v(\mathbf{x})f^p(\mathbf{x})\,d\nu(\mathbf{x})-\int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^p\,d\mu(\mathbf{x})\right. \\ \qquad\left.-\frac{1}{L}\int_\Omega\int_0^\infty u(\mathbf{x})|f(t\mathbf{x})-Af(\mathbf{x})|^p d\lambda(t)d\mu(\mathbf{x})\right], \quad p\neq 2; \\ \frac{1}{2}\left[\frac{1}{L}\int_\Omega v(\mathbf{x})f^2(\mathbf{x})\log(f(\mathbf{x}))\,d\nu(\mathbf{x})-\int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^2\log(Af(\mathbf{x}))\,d\mu(\mathbf{x})\right. \\ \quad\left.-\frac{1}{L}\int_\Omega\int_0^\infty u(\mathbf{x})(f(t\mathbf{x})-Af(\mathbf{x}))^2\log|f(t\mathbf{x})-Af(\mathbf{x})|d\lambda(t)d\mu(\mathbf{x})\right], \, p=2 \end{cases}$$

It is obviously continuous for $p>0$, $p\neq 2$. Suppose $p\to 2$:

$$\lim_{p\to 2}G(\varphi_p)=\lim_{p\to 2}\frac{1}{p(p-2)}\left[\frac{1}{L}\int_\Omega v(\mathbf{x})f^p(\mathbf{x})\,d\nu(\mathbf{x})-\int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^p\,d\mu(\mathbf{x})\right.$$
$$\left.-\frac{1}{L}\int_\Omega\int_0^\infty u(\mathbf{x})|f(t\mathbf{x})-Af(\mathbf{x})|^p d\lambda(t)d\mu(\mathbf{x})\right]$$

Since

$$\frac{1}{L}\int_\Omega v(\mathbf{x})f^2(\mathbf{x})\,d\nu(\mathbf{x})-\int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^2\,d\mu(\mathbf{x})$$
$$-\frac{1}{L}\int_\Omega\int_0^\infty u(\mathbf{x})|f(t\mathbf{x})-Af(\mathbf{x})|^2 d\lambda(t)d\mu(\mathbf{x})=0$$

applying L'Hospital rule we obtain, after a simple calculation, that

$$\lim_{p\to 2}G(\varphi_p)=G(\varphi_2).$$

Hence, the mapping $p\mapsto G(\varphi_p)$ is continuous for $p>0$.

(ii) Let $n\in\mathbb{N}$ and $u_i\in\mathbb{R}$, $i=1,2,\ldots,n$, be arbitrary. Define the function $F(x)=\sum\limits_{i,j=1}^{n}u_iu_j\varphi_{p_{ij}}(x)$, where $p_{ij}=\frac{p_i+p_j}{2}$. Then

$$\left(\frac{F'(x)}{x}\right)'=\sum_{i,j=1}^{n}u_iu_j\left(\frac{\varphi'_{p_{ij}}(x)}{x}\right)'=\left(\sum_{i=1}^{n}u_ix^{\frac{p_i-3}{2}}\right)^2\geq 0$$

and $F(0)=0$. Hence F is superquadratic. Using this F in the place of φ in (10.52) we have

$$G(F)=\sum_{i,j=1}^{n}u_iu_jG(\varphi_{p_{ij}})\geq 0. \tag{10.54}$$

So, the matrix $[G(\varphi_{p_{ij}})]_{i,j=1}^{n}$ is positive semi-definite.

Properties (iii) and (iv) are trivial consequence of (i), (ii) and definition of exponentially convex and log-convex functions. $\square$

From the inequality (10.43) we have

$$\begin{aligned} B(\varphi) &= \frac{1}{L}\int_{\Omega} w(x)\varphi(f(x))\frac{dx}{x} - \int_{\Omega} u(x)\varphi(Af(x))\frac{dx}{x} \\ &- \frac{1}{L}\int_{\Omega}\int_0^{\infty} u(x)\varphi(|f(tx)-Af(x)|)d\lambda(t)\frac{dx}{x}. \end{aligned} \tag{10.55}$$

Then the following consequence of Theorem 10.8 is proved:

Corollary 10.14 *For B as in (10.55), φ_p as in (7.16) and f a positive function, the mapping $p \mapsto B(\varphi_p)$ is exponentially convex, and for $r<p<t$ where $r,p,t\in\mathbb{R}_+$, we have*

$$B(\varphi_p) \le [B(\varphi_r)]^{\frac{t-p}{t-r}}[B(\varphi_t)]^{\frac{p-r}{t-r}}. \tag{10.56}$$

As a special case of Corollary 10.14, an improvement and reverse of the strengthened Hardy's inequality and its duals are proved.

Theorem 10.9 *Let $k\in\mathbb{R}$ be such that $k\neq 1$, let f be a positive function, and let $p\in\mathbb{R}_+\backslash\{2\}$.*

(i) If $k>1$ and $r<p<t$, where $r,p,t\in\mathbb{R}_+$, then

$$\begin{aligned} &\frac{1}{p(p-2)}\left\{\left(\tfrac{p}{k-1}\right)^p\int_0^{\infty} x^{p-k}f^p(x)dx - \int_0^{\infty} x^{-k}\left(\int_0^x f(y)dy\right)^p dx\right. \\ &\left. -\frac{k-1}{p}\int_0^{\infty}\int_t^{\infty}\left|\frac{p}{k-1}\left(\tfrac{t}{x}\right)^{1-\frac{k-1}{p}} f(t) - \frac{1}{x}\int_0^x f(s)ds\right|^p x^{p-k-\frac{k-1}{p}}dx\, t^{\frac{k-1}{p}-1}dt\right\} \\ &\le \left(\tfrac{p}{k-1}\right)^p [F(\varphi_r)]^{\frac{t-p}{t-r}}[F(\varphi_t)]^{\frac{p-r}{t-r}}. \end{aligned} \tag{10.57}$$

If $p<t<r$ or $t<r<p$, then (10.57) holds with reversed sign of inequality, where

$$\begin{aligned} F(\varphi_r) &= \int_0^{\infty}\varphi_r\left(f(x)x^{1-\frac{k-1}{p}}\right)\frac{dx}{x} - \int_0^{\infty}\varphi_r\left(\frac{k-1}{p}x^{-\frac{k-1}{p}}\int_0^x f(s)ds\right)\frac{dx}{x} \\ &- \frac{k-1}{p}\int_0^{\infty}\int_0^x \varphi_r\left(\left|f(t)t^{1-\frac{k-1}{p}} - \frac{k-1}{p}x^{1-\frac{k-1}{p}}\int_0^x f(s)ds\right|\right)\times \\ &\times x^{-\frac{k-1}{p}-1}t^{\frac{k-1}{p}-1}dt\,dx \end{aligned} \tag{10.58}$$

(ii) If $k<1$ and $r<p<t$, where $r,p,t\in\mathbb{R}_+$, then

$$\frac{1}{p(p-2)}\Bigg\{\left(\tfrac{p}{1-k}\right)^p\int\limits_0^\infty x^{p-k}f^p(x)dx-\int\limits_0^\infty x^{-k}\left(\int\limits_x^\infty f(y)dy\right)^p dx$$
$$-\tfrac{1-k}{p}\int\limits_0^\infty\int_0^t\left|\tfrac{p}{1-k}\left(\tfrac{t}{x}\right)^{1+\frac{1-k}{p}}f(t)-\tfrac{1}{x}\int\limits_x^\infty f(s)ds\right|^p x^{p-k-\frac{k-1}{p}}t^{\frac{k-1}{p}-1}dx\,dt\Bigg\}$$
$$\leq\left(\tfrac{p}{k-1}\right)^p\,[W(\varphi_r)]^{\frac{t-p}{t-r}}\,[W(\varphi_t)]^{\frac{p-r}{t-r}}. \tag{10.59}$$

If $p<t<r$ or $t<r<p$, then (10.59) *holds with reversed sign of inequality, where*

$$\begin{aligned}W(\varphi_r)\;=\;&\int\limits_0^\infty\varphi_r\left(x^{\frac{p-k+1}{p}}f(x)\right)\frac{dx}{x}-\int\limits_0^\infty\varphi_r\left(\frac{1-k}{p}x^{\frac{1-k}{p}}\int\limits_x^\infty f(t)dt\right)\frac{dx}{x}\\&-\frac{1-k}{p}\int\limits_0^\infty\int\limits_x^\infty\varphi_r\left(\left|f(t)t^{1-\frac{k-1}{p}}-\frac{1-k}{p}x^{\frac{1-k}{p}}\int\limits_x^\infty f(t)dt\right|\right)\times\\&\times t^{\frac{k-1}{p}-1}x^{\frac{1-k}{p}-1}dt\,dx.\end{aligned}\tag{10.60}$$

Proof. The proof follows from Corollary 10.14 by choosing $\Omega=\mathbb{R}_+$ and for a weight function $u(x)=1$. We obtain $w(x)=L$ and

$$Af(x)=\frac{1}{L}\int\limits_0^\infty f(tx)d\lambda(t).$$

Then (10.55) becomes

$$B(\varphi)=\int\limits_0^\infty\varphi(f(x))\frac{dx}{x}-\int\limits_0^\infty\varphi(Af(x))\frac{dx}{x}-\frac{1}{L}\int\limits_0^\infty\int_0^\infty\varphi(|f(tx)-Af(x)|)d\lambda(t)\frac{dx}{x}. \tag{10.61}$$

and (10.56) becomes

$$[B(\varphi_p)]^{t-r}\leq[B(\varphi_r)]^{t-p}\,[B(\varphi_t)]^{p-r} \tag{10.62}$$

for every choice $r,p,t\in\mathbb{R}_+$, such that $r<p<t$. We know that $B(\varphi_p)$ is log-convex. Let $\alpha>0$ and

$$d\lambda(t)=\begin{cases}t^{\alpha-1}, & 0<t\leq 1;\\ 0, & t\geq 1.\end{cases}$$

Then $L=\alpha^{-1}$ and

$$\begin{aligned}B(\varphi_p)\;=\;&\int\limits_0^\infty\varphi_p(f(x))\frac{dx}{x}-\int\limits_0^\infty\varphi_p(Af(x))\frac{dx}{x}\\&-\alpha\int\limits_0^\infty\int\limits_0^x\varphi_p(|f(t)-Af(x)|)t^{\alpha-1}x^{-\alpha-1}dtdx\end{aligned}\tag{10.63}$$

where

$$Af(x) = \alpha x^{-\alpha} \int_0^x t^{\alpha-1} f(t)dt.$$

To obtain (10.57) choose for f the function $x \mapsto f(x)x^{1-\alpha}$, where $\alpha = \frac{k-1}{p}$, $p > 0, k > 1$. Then, after some calculation (10.63) reduces to (10.58) and (10.62) reduces to (10.57).

By taking substitutions $r \to t$, $p \to r$, $t \to p$ or $r \to p$, $p \to t$, $t \to r$ in (10.62), we get reversed sign of inequality in (10.57).

To prove (10.59), let us take $\beta > 0$ and

$$d\lambda(t) = \begin{cases} t^{-\beta-1}, & t \geq 1; \\ 0, & 0 < t \leq 1. \end{cases}$$

Then $L = \beta^{-1}$ and

$$\begin{aligned} B(\varphi_p) \; = \; & \int_0^\infty \varphi_p(f(x))\frac{dx}{x} - \int_0^\infty \varphi_p(Af(x))\frac{dx}{x} \\ & -\beta \int_0^\infty \int_x^\infty \varphi_p(|f(t) - Af(x)|)t^{-\beta-1}x^{\beta-1}dtdx \end{aligned} \tag{10.64}$$

where

$$Af(x) = \beta x^{\beta} \int_x^\infty t^{-\beta-1} f(t)dt.$$

To obtain (10.59) choose for f the function $x \mapsto f(t)x^{1+\beta}$, where $\beta = \frac{1-k}{p}$, $p > 0, k < 1$. Then, after some calculation (10.64) reduces to (10.60) and (10.62) reduces to (10.59).

By taking substitutions $r \to t$, $p \to r$, $t \to p$ and $r \to p$, $p \to t$, $t \to r$ in (10.62), we get reversed sign of inequality in (10.59). □

10.4.1 Mean Value Theorems

Now, we give mean value theorem. First, we state and prove the Lagrange-type mean value theorem.

Theorem 10.10 *Let J be a compact interval and $J \subseteq I$. If $\frac{\varphi'}{x} \in C^1(J)$ and $\varphi(0) = 0$ then there exists $\xi \in J$ such that the following equality holds*

$$\begin{aligned} G(\varphi) \; = \; & \frac{1}{3}\frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi^2}\left(\frac{1}{L}\int_\Omega v(\mathbf{x})f^3(\mathbf{x})\,d\nu(\mathbf{x}) - \int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^3\,d\mu(\mathbf{x})\right. \\ & \left. -\frac{1}{L}\int_\Omega\int_0^\infty u(\mathbf{x})|f(t(\mathbf{x})) - Af((\mathbf{x}))|^3 d\lambda(t)d\mu((\mathbf{x}))\right), \end{aligned} \tag{10.65}$$

where L, Af are defined by (10.4) *and* (10.5), *respectively.*

Proof. Since $\left(\frac{\varphi'}{x}\right)'$ is continuous on the compact set J, there exist $\min\left(\left(\frac{\varphi'}{x}\right)'\right) = m$ and $\max\left(\left(\frac{\varphi'}{x}\right)'\right) = M$. Then by applying Theorem 10.6 on functions φ_1, φ_2 from Lemma 7.1 the following two inequalities hold:

$$\begin{aligned} G(\varphi) \leq \frac{M}{3}\Bigg(&\frac{1}{L}\int_\Omega v(\mathbf{x}) f^3(\mathbf{x})\, d\nu(\mathbf{x}) - \int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^3\, d\mu(\mathbf{x}) \\ &-\frac{1}{L}\int_\Omega\int_0^\infty u(\mathbf{x})|f(t\mathbf{x}) - Af(\mathbf{x})|^3 d\lambda(t) d\mu(\mathbf{x})\Bigg) \end{aligned} \tag{10.66}$$

and

$$\begin{aligned} G(\varphi) \geq \frac{m}{3}\Bigg(&\frac{1}{L}\int_\Omega v(\mathbf{x}) f^3(\mathbf{x})\, d\nu(\mathbf{x}) - \int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^3\, d\mu(\mathbf{x}) \\ &-\frac{1}{L}\int_\Omega\int_0^\infty u(\mathbf{x})|f(t\mathbf{x}) - Af(\mathbf{x})|^3 d\lambda(t) d\mu(\mathbf{x})\Bigg). \end{aligned}$$

By combining above two inequalities we have that there exist $\xi \in J$ such that we get (10.65). $\square$

Theorem 10.11 *Let J be a compact interval and $J \subseteq I$. If $\frac{\varphi'}{x}, \frac{\psi'}{x} \in C^1(J)$ and $\varphi(0) = 0, \psi(0) = 0$, then there exists $\xi \in J$ such that*

$$\frac{G(\varphi)}{G(\psi)} = \frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi\psi''(\xi) - \psi'(\xi)},$$

provided that denominators are not equal to zero.

Proof. We denote $c_1 = G(\psi)$, $c_2 = G(\varphi)$. Now, apply (10.65) to the function $h = c_1\varphi - c_2\psi$. Notice that

$$\frac{h'}{x} = c_1\frac{\varphi'}{x} - c_2\frac{\psi'}{x} \in C^1(J), h(0) = 0.$$

The following equality follows

$$\begin{aligned} G(h) = \frac{1}{3}\frac{\xi h''(\xi) - h'(\xi)}{\xi^2}\Bigg(&\frac{1}{L}\int_\Omega v(\mathbf{x}) f^3(\mathbf{x})\, d\nu(\mathbf{x}) - \int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^3\, d\mu(\mathbf{x}) \\ &-\frac{1}{L}\int_\Omega\int_0^\infty u(\mathbf{x})|f(t\mathbf{x}) - Af(\mathbf{x})|^3 d\lambda(t) d\mu(\mathbf{x})\Bigg). \end{aligned} \tag{10.67}$$

After a short calculation, it is easy to see that the left-hand side of (10.67) is equal to 0, so should be the right-hand side. Since $G(\psi) \neq 0$ we conclude that

$$\begin{aligned} \Bigg(&\frac{1}{L}\int_\Omega v(\mathbf{x}) f^3(\mathbf{x})\, d\nu(\mathbf{x}) - \int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^3\, d\mu(\mathbf{x}) \\ &-\frac{1}{L}\int_\Omega\int_0^\infty u(\mathbf{x})|f(t\mathbf{x}) - Af(\mathbf{x})|^3 d\lambda(t) d\mu(\mathbf{x})\Bigg) \neq 0. \end{aligned}$$

It follows that
$$c_1(\xi\varphi''(\xi)-\varphi'(\xi))-c_2(\xi\psi''(\xi)-\psi'(\xi))=0,$$
so the proof is completed. □

10.4.2 Cauchy Means

Theorem 10.11 enables us to define new means when the right-hand side of the equality, function of ξ and denoted by $K(\xi)$, is invertible. Then, by Theorem 10.11 we have
$$\xi=K^{-1}\left(\frac{G(\varphi)}{G(\psi)}\right),$$
what presents a new Cauchy mean.

Specially, if we choose $\varphi=\varphi_s$, $\psi=\varphi_r$, where $r,s\in\mathbb{R}_+$, $r\neq s$, $r,s\neq 2$, we obtain
$$\xi^{s-r}=\frac{r(r-2)}{s(s-2)}$$
$$\frac{\frac{1}{L}\int_\Omega v(\mathbf{x})f^s(\mathbf{x})\,d\nu(\mathbf{x})-\int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^s\,d\mu(\mathbf{x})-\frac{1}{L}\int_\Omega\int_0^\infty u(\mathbf{x})C^s(\mathbf{x})d\lambda(t)d\mu(\mathbf{x})}{\frac{1}{L}\int_\Omega v(\mathbf{x})f^r(\mathbf{x})\,d\nu(\mathbf{x})-\int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^r\,d\mu(\mathbf{x})-\frac{1}{L}\int_\Omega\int_0^\infty u(\mathbf{x})C^r(\mathbf{x})d\lambda(t)d\mu(\mathbf{x})},$$
where $C(\mathbf{x})=|f(t\mathbf{x})-Af(\mathbf{x})|$. We define new means
$$M_{s,r}=\left(\frac{r(r-2)}{s(s-2)}\right.$$
$$\left.\frac{\frac{1}{L}\int_\Omega A_{s,0}(\mathbf{x})d\nu(\mathbf{x})-\int_\Omega B_{s,0}(\mathbf{x})d\mu(\mathbf{x})-\frac{1}{L}\int_\Omega\int_0^\infty C_{s,0}(\mathbf{x})d\lambda(t)d\mu(\mathbf{x})}{\frac{1}{L}\int_\Omega A_{r,0}(\mathbf{x})\,d\nu(\mathbf{x})-\int_\Omega B_{r,0}(\mathbf{x})d\mu(\mathbf{x})-\frac{1}{L}\int_\Omega\int_0^\infty C_{r,0}(\mathbf{x})d\lambda(t)d\mu(\mathbf{x})}\right)^{\frac{1}{s-r}},$$
for $r,s\in\mathbb{R}_+$, $r\neq s$, $r,s\neq 2$. We can extend these means to excluded cases. Taking a limit we define for $r\neq 2$
$$M_{r,r}=$$
$$\exp\left(\frac{\frac{1}{L}\int_\Omega A_{r,1}(\mathbf{x})d\nu(\mathbf{x})-\int_\Omega B_{r,1}(\mathbf{x})d\mu(\mathbf{x})-\frac{1}{L}\int_\Omega\int_0^\infty C_{r,1}(\mathbf{x})d\lambda(t)d\mu(\mathbf{x})}{\frac{1}{L}\int_\Omega A_{r,0}(\mathbf{x})\,d\nu(\mathbf{x})-\int_\Omega B_{r,0}(\mathbf{x})d\mu(\mathbf{x})-\frac{1}{L}\int_\Omega\int_0^\infty C_{r,0}(\mathbf{x})d\lambda(t)d\mu(\mathbf{x})}-\frac{2r-2}{r(r-2)}\right),$$
$$M_{r,2}=M_{2,r}=$$
$$\left(\frac{r(r-2)}{2}\,\frac{\frac{1}{L}\int_\Omega A_{2,1}(\mathbf{x})d\nu(\mathbf{x})-\int_\Omega B_{2,1}(\mathbf{x})d\mu(\mathbf{x})-\frac{1}{L}\int_\Omega\int_0^\infty C_{2,1}(\mathbf{x})d\lambda(t)d\mu(\mathbf{x})}{\frac{1}{L}\int_\Omega A_{r,0}(\mathbf{x})\,d\nu(\mathbf{x})-\int_\Omega B_{r,0}(\mathbf{x})d\mu(\mathbf{x})-\frac{1}{L}\int_\Omega\int_0^\infty C_{r,0}(\mathbf{x})d\lambda(t)d\mu(\mathbf{x})}\right)^{\frac{1}{2-r}}$$
and for $r=2$
$$M_{2,2}=$$
$$\exp\left(\frac{\frac{1}{L}\int_\Omega A_{2,2}(\mathbf{x})d\nu(\mathbf{x})-\int_\Omega B_{2,2}(\mathbf{x})d\mu(\mathbf{x})-\frac{1}{L}\int_\Omega\int_0^\infty C_{2,2}(\mathbf{x})d\lambda(t)d\mu(\mathbf{x})}{\frac{1}{L}\int_\Omega A_{2,1}(\mathbf{x})\,d\nu(\mathbf{x})-\int_\Omega B_{2,1}(\mathbf{x})d\mu(\mathbf{x})-\frac{1}{L}\int_\Omega\int_0^\infty C_{2,1}(\mathbf{x})d\lambda(t)d\mu(\mathbf{x})}-\frac{1}{2}\right),$$
where

$$A_{p,n}(\mathbf{x}) = f^p(\mathbf{x})(\log(f(\mathbf{x})))^n v(\mathbf{x}),$$

$$B_{p,n}(\mathbf{x}) = (Af(\mathbf{x}))^p(\log(Af(\mathbf{x})))^n u(\mathbf{x}),$$

$$C_{p,n}(\mathbf{x}) = |f(t\mathbf{x}) - Af(\mathbf{x})|^p(\log|f(t\mathbf{x}) - Af(\mathbf{x})|)^n, \quad n = 0,1,2,\ p > 0.$$

We shall prove that this new mean is monotonic. Note that $M_{s,r}$ is continuous, hence, it is enough to prove monotonicity of mean in case where $s,r,l,p \in \mathbb{R}_+$ $s,r,l,p \neq 2$ and $s \neq r$, $l \neq p$.

Theorem 10.12 *Let $l \leq s$, $p \leq r$, then the following inequality is valid,*

$$M_{l,p} \leq M_{s,r} \tag{10.68}$$

that is, the mean $M_{s,r}$ is monotonic.

Proof. Since the function $s \mapsto G(\varphi_s)$ is log-convex, we can apply (1.5) and get (10.68), so the proof is completed. □

10.5 Boas-type inequality with constants

In the proof of Theorem 10.2 we have used Jensen's inequality, where convex function is crucial. Now, we generalize the Boas-type inequality to the class of arbitrary non-negative functions Φ bounded from below and above with a convex function multiplied with positive real constants a_1 and a_2.

Theorem 10.13 *Let X, λ, μ, ν, μ_t, L, Ω, u, v and f be as in Theorem 10.2. If $\Phi\colon I \to \mathbb{R}$ is a non-negative function, integrable on an interval $I \subseteq \mathbb{R}$, such that exist a convex function $\Psi\colon I \to \mathbb{R}$ and real numbers a_1 and a_2, $0 < a_1 \leq a_2 < \infty$ such that*

$$a_1\Psi(y) \leq \Phi(y) \leq a_2\Psi(y),\ y \in I, \tag{10.69}$$

then

$$\int_\Omega u(\mathbf{x})\Phi(Af(\mathbf{x}))\,d\mu(\mathbf{x}) \leq \frac{a_2}{a_1}\frac{1}{L}\int_\Omega v(\mathbf{x})\Phi(f(\mathbf{x}))\,d\nu(\mathbf{x}) \tag{10.70}$$

holds.

Proof. By using the condition (10.69) and Theorem 10.2, we immediately get

$$\begin{aligned}
&\int_\Omega u(\mathbf{x})\Phi(Af(\mathbf{x}))\,d\mu(\mathbf{x}) = \int_\Omega u(\mathbf{x})\Phi\left(\frac{1}{L}\int_0^\infty f(t\mathbf{x})\,d\lambda(t)\right)d\mu(\mathbf{x})\\
&\leq a_2\int_\Omega u(\mathbf{x})\Psi\left(\frac{1}{L}\int_0^\infty f(t\mathbf{x})\,d\lambda(t)\right)d\mu(\mathbf{x})\\
&\leq \frac{a_2}{L}\int_\Omega v(\mathbf{y})\Psi(f(\mathbf{y}))\,d\nu(\mathbf{y}) \leq \frac{a_2}{a_1}\frac{1}{L}\int_\Omega v(\mathbf{x})\Phi(f(\mathbf{x}))d\nu(\mathbf{x}).
\end{aligned}$$

□

Following the same idea here we generalize Theorem 10.6. We consider a function Φ bounded from below and above by a superquadratic function multiplied with positive real constants a_1 and a_2.

Theorem 10.14 *Suppose X, λ, μ, ν, μ_t, L, Ω, u, v and f are as in Theorem* 10.6. *If $\Phi\colon I \to \mathbb{R}$ is non-negative function, integrable on an interval $I \subseteq \mathbb{R}$, such that exist superquadratic function $\varphi\colon I \to \mathbb{R}$ and real numbers a_1 and a_2, $0 < a_1 \le a_2 < \infty$ such that*

$$a_1\varphi(y) \le \Phi(y) \le a_2\varphi(y),\ y \in I, \tag{10.71}$$

then

$$\begin{aligned}&\int_\Omega u(\mathbf{x})\Phi(Af(\mathbf{x}))d\mu(\mathbf{x}) + \frac{1}{L}\int_\Omega\int_0^\infty u(\mathbf{x})\Phi\left(|f(t\mathbf{x}) - Af(\mathbf{x}))|\right)d\lambda(t)\,d\mu(\mathbf{x})\\ &\le \frac{a_2}{a_1}\cdot\frac{1}{L}\int_\Omega v(\mathbf{x})\Phi(f(\mathbf{x}))d\nu(\mathbf{x}) \qquad\qquad \textit{holds.}\end{aligned}$$

Proof. Combining the condition (10.71) and inequality (10.37) we obtain

$$\begin{aligned}&\int_\Omega u(\mathbf{x})\Phi(Af(\mathbf{x}))d\mu(\mathbf{x}) + \frac{1}{L}\int_\Omega u(\mathbf{x})\int_0^\infty \Phi(|f(t\mathbf{x}) - Af(\mathbf{x})|)\,d\lambda(t)d\mu(\mathbf{x})\\ &\le a_2\left(\int_\Omega u(\mathbf{x})\varphi(Af(\mathbf{x}))d\mu(\mathbf{x}) + \frac{1}{L}\int_\Omega u(\mathbf{x})\int_0^\infty \varphi(|f(t\mathbf{x}) - Af(\mathbf{x})|)\,d\lambda(t)d\mu(\mathbf{x})\right)\\ &\le \frac{a_2}{L}\int_\Omega v(\mathbf{x})\varphi(f(\mathbf{x}))d\nu(\mathbf{x}) \le \frac{a_2}{a_1 L}\int_\Omega v(\mathbf{x})\Phi(f(\mathbf{x}))d\nu(\mathbf{x}).\end{aligned}$$

□

Remark 10.9 Theorem 10.13 and Theorem 10.14 improve results from [85].

Chapter 11

Multidimensional Hardy and Pólya-Knopp-type inequalities

Notice that the Boas inequality (10.1) unifies some well-known classical inequalities, such as Hardy's and Pólya-Knopp's inequality. In the sequel, we state their strengthened versions, obtained independently by B. Yang et al. [103, 105] and A. Čižmešija et al. [27, 30].

11.1 Overview of the Hardy and Pólya-Knopp-type inequalities

Let $0 < b \le \infty$ and $p, k \in \mathbb{R}$ be such that $\dfrac{p}{k-1} > 0$. If $p \in \mathbb{R} \setminus [0,1]$, f is a non-negative function, $x^{1-\frac{k}{p}} f \in L^p(0,b)$, and

$$F(x) = \int_0^x f(t)\,dt,\ x \in (0,b), \tag{11.1}$$

then

$$\int_0^b x^{-k} F^p(x)\,dx \le \left(\frac{p}{k-1}\right)^p \int_0^b \left[1 - \left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right] x^{p-k} f^p(x)\,dx \tag{11.2}$$

holds, while for $p \in (0,1)$ the sign of inequality in (11.2) is reversed. Moreover, if $0 \le b < \infty$ and parameters $p \in \mathbb{R} \setminus [0,1]$ and $k \in \mathbb{R}$ are such that $\frac{p}{k-1} < 0$, then the inequality

$$\int_b^\infty x^{-k} \tilde{F}^p(x)\,dx \le \left(\frac{p}{1-k}\right)^p \int_b^\infty \left[1-\left(\frac{b}{x}\right)^{\frac{1-k}{p}}\right] x^{p-k} f^p(x)\,dx \tag{11.3}$$

holds for all non-negative functions f such that $x^{1-\frac{k}{p}} f \in L^p(b,\infty)$, where

$$\tilde{F}(x) = \int_x^\infty f(t)\,dt,\ x \in (b,\infty). \tag{11.4}$$

For $p \in (0,1)$ inequality (11.3) holds with the sign of inequality reversed. The constant $\left|\frac{p}{k-1}\right|^p$ is the best possible for both inequalities, that is, it cannot be replaced with any smaller constant. The classical Hardy's inequality follows by taking $b = \infty$ in (11.2), while for $b = 0$ in (11.3) we get its dual inequality.

Corollary 11.1 *Let $0 < b \le \infty$, f be a non-negative function on $(0,b)$, and $p,k \in \mathbb{R}$ be such that $0 \ne p \ne 1$, $k \ne 1$, and $\frac{p}{k-1} > 0$. If $p \in \mathbb{R}\setminus[0,1]$, then the inequality*

$$\begin{aligned}
&\left(\frac{p}{k-1}\right)^p \int_0^b \left[1-\left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right] x^{p-k} f^p(x)\,dx - \int_0^\infty x^{-k} F^p(x)\,dx \\
&\ge \Bigg| \left(\frac{p}{k-1}\right)^{p-1} \int_0^b x^{\frac{1-k}{p}-1} \int_0^x t^{\frac{k-1}{p}-1} \\
&\quad \cdot \left| t^{p-k+1} f^p(t) - \left(\frac{k-1}{p}\right)^p x^{1-k} F^p(x)\right| dt\,dx \\
&\quad - |p| \int_0^b x^{-k} F^{p-1}(x) \int_0^x \left| f(t) - \frac{k-1}{p}\cdot\frac{1}{t}\left(\frac{t}{x}\right)^{\frac{k-1}{p}} F(x)\right| dt\,dx \Bigg|
\end{aligned} \tag{11.5}$$

holds, where F is defined by (11.1). *In the case when $p \in (0,1)$, the order of integrals on the left-hand side in* (11.5) *is reversed.*

On the other hand, suppose $0 \le b < \infty$, f is a non-negative function on (b,∞) and $p,k \in \mathbb{R}$ are such that $0 \ne p \ne 1$, $k \ne 1$, and $\frac{p}{k-1} < 0$. If $p \in (-\infty,0)\cup(1,\infty)$, then the inequality

$$
\begin{aligned}
&\left(\frac{p}{1-k}\right)^p \int_b^\infty \left[1-\left(\frac{b}{x}\right)^{\frac{1-k}{p}}\right] x^{p-k} f^p(x)\,dx - \int_b^\infty x^{-k}\tilde{F}^p(x)\,dx \\
&\geq \Bigg| \left(\frac{p}{1-k}\right)^{p-1} \int_b^\infty x^{\frac{1-k}{p}-1} \int_x^\infty t^{\frac{k-1}{p}-1} \\
&\quad \times \left| t^{p-k+1} f^p(t) - \left(\frac{1-k}{p}\right)^p x^{1-k}\tilde{F}^p(x)\right| dt dx \\
&\quad -|p| \int_b^\infty x^{-k}\tilde{F}^{p-1}(x) \int_x^\infty \left| f(t) - \frac{1-k}{p}\cdot\frac{1}{t}\left(\frac{x}{t}\right)^{\frac{1-k}{p}} \tilde{F}(x)\right| dt dx \Bigg|
\end{aligned}
\tag{11.6}
$$

holds, where $\tilde{F}$ is defined by (11.4). In the case when $p \in (0,1)$, the order of integrals on the left-hand side in (11.6) is reversed.
Notice that for $b = \infty$ (11.5) becomes refinement of the classical Hardy's inequality, while its dual inequality for $b = 0$ becomes refinement of its dual inequality.

On the other hand, if $0 < b \leq \infty$, $f \in L^1(0,b)$ is a positive function, and

$$
G(x) = \exp\left(\frac{1}{x}\int_0^x \log f(t)\,dt\right),\ x \in (0,b), \tag{11.7}
$$

then

$$
\int_0^b G(x)\,dx \leq e\int_0^b \left(1-\frac{x}{b}\right) f(x)\,dx \tag{11.8}
$$

holds, while the inequality

$$
\int_b^\infty \tilde{G}(x)\,dx \leq \frac{1}{e}\int_b^\infty \left(1-\frac{b}{x}\right) f(x)\,dx \tag{11.9}
$$

holds for $0 \leq b < \infty$, $0 < f \in L^1(b,\infty)$, and

$$
\tilde{G}(x) = \exp\left(x\int_x^\infty \log f(t)\,\frac{dt}{t^2}\right),\ x \in (b,\infty). \tag{11.10}
$$

For $b = \infty$ in (11.8) and for $b = 0$ in (11.9) we respectively get the classical Pólya-Knopp's inequality and its dual inequality. Notice that the constant factors e and $\frac{1}{e}$, respectively involved in the right-hand sides of (11.8) and (11.9), are the best possible.

Corollary 11.2 *Let $0 < b \leq \infty$, f be a positive function on $(0,b)$, and $G(x)$ be defined by* (11.7). *Then*

$$
\begin{aligned}
&e\int_0^b \left(1-\frac{x}{b}\right) f(x)\,dx - \int_0^b G(x)\,dx \\
&\geq \left| \int_0^b \int_0^x |etf(t) - xG(x)|\,dt\,\frac{dx}{x^2} - \int_0^b G(x)\int_0^x \left|\log\frac{etf(t)}{xG(x)}\right| dt\,\frac{dx}{x} \right|
\end{aligned}
$$

holds. On the other hand, if $0 \le b < \infty$, f *is a positive function on* (b,∞), *and* $\tilde{G}(x)$ *is defined by* (11.10), *then*

$$\begin{aligned}&\frac{1}{e}\int_b^\infty \left(1-\frac{b}{x}\right) f(x)\,dx - \int_b^\infty \tilde{G}(x)\,dx \\ &\ge \left|\int_b^\infty \int_x^\infty \left|\frac{1}{e}tf(t) - x\tilde{G}(x)\right| \frac{dt}{t^2}\,dx - \int_b^\infty x\tilde{G}(x)\int_x^b \left|\log\frac{tf(t)}{e\,x\tilde{G}(x)}\right|\frac{dt}{t^2}\,dx\right|.\end{aligned}$$

holds.

In this chapter, we also make use of the following n-dimensional strengthened Hardy's inequality related to the setting with balls in $\mathbb{R}^n$ centered at the origin (see [29] for details). Let $p,k,R \in \mathbb{R}$ be such that $p>1$, $k \ne 1$, and $R>0$. Suppose that f is a non-negative measurable function and the function F is defined on $\mathbb{R}^n$ by

$$F(\mathbf{x}) = \begin{cases} \displaystyle\int_{B(|\mathbf{x}|)} f(\mathbf{y})\,d\mathbf{y}, & k>1, \\ \displaystyle\int_{\mathbb{R}^n\setminus B(|\mathbf{x}|)} f(\mathbf{y})\,d\mathbf{y}, & k<1, \end{cases}$$

where $B(|\mathbf{x}|)$ is a ball in $\mathbb{R}^n$ centered at the origin and of radius $|\mathbf{x}|$, while $|\mathbf{x}|$ denotes the Euclidean norm of $\mathbf{x} \in \mathbb{R}^n$. If $\dfrac{p}{k-1} > 0$, then the inequality

$$\begin{aligned}&\int_{B(R)} |B(|\mathbf{x}|)|^{-k} F^p(\mathbf{x})\,d\mathbf{x} \\ &\le \left(\frac{p}{k-1}\right)^p \int_{B(R)} \left[1-\left(\frac{|B(|\mathbf{x}|)|}{|B(R)|}\right)^{\frac{k-1}{p}}\right] |B(|\mathbf{x}|)|^{p-k} f^p(\mathbf{x})\,d\mathbf{x}\end{aligned} \tag{11.11}$$

holds, while for $\dfrac{p}{k-1} < 0$ we have

$$\begin{aligned}&\int_{\mathbb{R}^n\setminus B(R)} |B(|\mathbf{x}|)|^{-k} F^p(\mathbf{x})\,d\mathbf{x} \\ &\le \left(\frac{p}{1-k}\right)^p \int_{\mathbb{R}^n\setminus B(R)} \left[1-\left(\frac{|B(R)|}{|B(|\mathbf{x}|)|}\right)^{\frac{1-k}{p}}\right] |B(|\mathbf{x}|)|^{p-k} f^p(\mathbf{x})\,d\mathbf{x}.\end{aligned} \tag{11.12}$$

Terms $|B(|\mathbf{x}|)|$ and $|B(R)|$ respectively denote the volumes of $B(|\mathbf{x}|)$ and $B(R)$. The constant $\left(\dfrac{p}{|k-1|}\right)^p$ is the best possible for both inequalities. Observe that the first natural generalization of the classical Hardy's inequality to balls in $\mathbb{R}^n$ was given by M. Christ and L. Grafakos in [19].

Finally, here we state an n-dimensional Pólya-Knopp's inequality, related to (11.11) and (11.12),

$$\int_{B(R)} G(\mathbf{x})\, d\mathbf{x} < e \int_{B(R)} \left(1 - \frac{|B(|\mathbf{x}|)|}{|B(R)|}\right) f(\mathbf{x})\, d\mathbf{x} \tag{11.13}$$

for a positive function f on $B(R)$ and

$$G(\mathbf{x}) = \exp\left(\frac{1}{|B(|\mathbf{x}|)|} \int_{B(|\mathbf{x}|)} \log f(\mathbf{y})\, d\mathbf{y}\right),\ \mathbf{x} \in B(R),$$

as well as its dual inequality

$$\int_{\mathbb{R}^n \setminus B(R)} \tilde{G}(\mathbf{x})\, d\mathbf{x} < \frac{1}{e} \int_{\mathbb{R}^n \setminus B(R)} \left(1 - \frac{|B(R)|}{|B(|\mathbf{x}|)|}\right) f(\mathbf{x})\, d\mathbf{x} \tag{11.14}$$

for a positive function f on $\mathbb{R}^n \setminus B(R)$ and

$$\tilde{G}(\mathbf{x}) = \exp\left(|B(|\mathbf{x}|)| \int_{\mathbb{R}^n \setminus B(|\mathbf{x}|)} \log f(\mathbf{y})\, \frac{d\mathbf{y}}{|B(|\mathbf{x}|)|^2}\right),\ \mathbf{x} \in \mathbb{R}^n \setminus B(R).$$

Moreover, as a consequence of our new refined Boas-type inequality, we derive a new class of Hardy and Pólya-Knopp-type inequalities related to balls in $\mathbb{R}^n$, along with their respective dual inequalities, and prove that constant factors involved in their right-hand sides are the best possible. Finally, we show that our Hardy's and Pólya-Knopp's inequality differ from (11.11), (11.12), (11.13) and (11.14), although for $n = 1$ both classes coincide.

11.2 Refined Boas inequality with balls in $\mathbb{R}^n$

In this section, we apply Theorem 10.3 to a particular multidimensional setting, namely, to balls in $\mathbb{R}^n$ centered at the origin and to their dual sets. The results obtained represent a new class of n-dimensional Hardy and Pólya-Knopp-type inequalities, different from the existing inequalities (11.11), (11.12), (11.13) and (11.14). Moreover, the constant factors appearing on the right-hand sides of our relations are the best possible.

Using polar coordinates in $\mathbb{R}^n$ we can define the ball $B(R)$ by

$$B(R) = \{rS : 0 \le r \le R, S \in S^{n-1}\}.$$

and the volume of the ball $B(R)$ is then

$$\begin{aligned} |B(R)| &= \int_{B(R)} d\mathbf{x} = \int_{|\mathbf{x}| \le R} d\mathbf{x} = \int_0^R r^{n-1} \left(\int_{S^{n-1}} dS\right) dr \\ &= \int_{S^{n-1}} \left(\int_0^R r^{n-1} dr\right) dS = \frac{R^n |S^{n-1}|}{n}, \end{aligned}$$

where $|S^{n-1}|$ is an area of S^{n-1}.

Our first result in this direction is a refinement of an inequality by D. Luor [78, relation (1.14)], related to cones in $\mathbb{R}^n$.

Theorem 11.1 *Let λ be a finite Borel measure on $\mathbb{R}_+$, L be defined by* (10.4), *Ω be a λ-balanced Borel set in $\mathbb{R}^n$, and C be a Borel subset of the unit sphere S^{n-1}. Let u be a non-negative function on $\mathbb{R}^n$, Φ be a non-negative convex function on an interval $I \subseteq \mathbb{R}$, and $\varphi \colon I \to \mathbb{R}$ be any function fulfilling $\varphi(x) \in \partial\Phi(x)$, for all $x \in \operatorname{Int} I$. Finally, let $f \colon \Omega \to \mathbb{R}$ be a measurable function with values in I.*

(i) *If* $\operatorname{supp}\lambda \subseteq (0,1]$, $0 < R \leq \infty$, *and* $\Omega = \Omega_1 = \{\mathbf{x} = rS \colon S \in C, 0 \leq r < R\}$, *then*

$$\frac{1}{L}\int_{\Omega_1} \Phi(f(\mathbf{x}))\left(\int_{\frac{|\mathbf{x}|}{R}}^{1} u\left(\frac{1}{t}\mathbf{x}\right) t^{1-n}\, d\lambda(t)\right)\frac{d\mathbf{x}}{|\mathbf{x}|} - \int_{\Omega_1} u(\mathbf{x})\Phi(A_1 f(\mathbf{x}))\frac{d\mathbf{x}}{|\mathbf{x}|}$$
$$\geq \frac{1}{L}\left|\int_{\Omega_1} u(\mathbf{x})\int_0^1 |\Phi(f(t\mathbf{x})) - \Phi(A_1 f(\mathbf{x}))|\, d\lambda(t)\frac{d\mathbf{x}}{|\mathbf{x}|}\right.$$
$$\left. - \int_{\Omega_1} u(\mathbf{x})\int_0^1 |\varphi(A_1 f(\mathbf{x}))| \cdot |f(t\mathbf{x}) - A_1 f(\mathbf{x})|\, d\lambda(t)\frac{d\mathbf{x}}{|\mathbf{x}|}\right|, \tag{11.15}$$

where $A_1 f(\mathbf{x}) = \displaystyle\int_0^1 f(t\mathbf{x})\, d\lambda(t)$, $\mathbf{x} \in \Omega_1$.

(ii) *If* $\operatorname{supp}\lambda \subseteq [1,\infty)$, $0 \leq R < \infty$, *and* $\Omega = \Omega_2 = \{\mathbf{x} = rS \colon S \in C, R \leq r < \infty\}$, *then*

$$\frac{1}{L}\int_{\Omega_2} \Phi(f(\mathbf{x}))\left(\int_1^{\frac{|\mathbf{x}|}{R}} u\left(\frac{1}{t}\mathbf{x}\right) t^{1-n}\, d\lambda(t)\right)\frac{d\mathbf{x}}{|\mathbf{x}|} - \int_{\Omega_2} u(\mathbf{x})\Phi(A_2 f(\mathbf{x}))\frac{d\mathbf{x}}{|\mathbf{x}|}$$
$$\geq \frac{1}{L}\left|\int_{\Omega_2} u(\mathbf{x})\int_1^\infty |\Phi(f(t\mathbf{x})) - \Phi(A_2 f(\mathbf{x}))|\, d\lambda(t)\frac{d\mathbf{x}}{|\mathbf{x}|}\right.$$
$$\left. - \int_{\Omega_2} u(\mathbf{x})\int_1^\infty |\varphi(A_2 f(\mathbf{x}))| \cdot |f(t\mathbf{x}) - A_2 f(\mathbf{x})|\, d\lambda(t)\frac{d\mathbf{x}}{|\mathbf{x}|}\right|, \tag{11.16}$$

where $A_2 f(\mathbf{x}) = \displaystyle\int_1^\infty f(t\mathbf{x})\, d\lambda(t)$, $\mathbf{x} \in \Omega_2$.

Proof. Relation (11.15) is a direct consequence of Theorem 10.3, rewritten with $X = \mathbb{R}^n, \Omega = \Omega_1, d\mu(\mathbf{x}) = \chi_{\Omega_1}(\mathbf{x})\, d\mathbf{x}$ and $d\nu(\mathbf{x}) = d\mathbf{x}$, as well as with the function u replaced with $\mathbf{x} \mapsto |\mathbf{x}|^{-1}u(\mathbf{x})$. Then we have $\dfrac{d\mu_t}{d\nu}(\mathbf{x}) = t^{-n}\chi_{t\Omega_1}(\mathbf{x})$, $t \in (0,1]$,

$$v(\mathbf{x}) = \int_{\frac{|\mathbf{x}|}{R}}^{1} u\left(\frac{1}{t}\mathbf{x}\right) t^{1-n}\, d\lambda(t),\ \mathbf{x} \in \Omega_1,$$

and $Af(\mathbf{x}) = A_1 f(\mathbf{x})$, $x \in \Omega_1$, so relation (11.11) reduces to (11.15). The proof of (11.16) follows the same lines, by considering $\Omega = \Omega_2$. In such setting we get

$$v(\mathbf{x}) = \int_1^{\frac{|\mathbf{x}|}{R}} u\left(\frac{1}{t}\mathbf{x}\right) t^{1-n}\, d\lambda(t),\ \mathbf{x} \in \Omega_2,$$

and $Af = A_2 f$. $\square$

11.3 Hardy-type inequalities with balls in $\mathbb{R}^n$

By taking $C = S^{n-1}$, that is, by setting a λ-balanced set Ω to be a ball in $\mathbb{R}^n$ centered at the origin or its corresponding dual set, and by choosing a suitable measure λ and a weight function u, we obtain the following sequence of new refined strengthened inequalities of the Hardy type.

Theorem 11.2 *Let $n \in \mathbb{N}$, $p \in \mathbb{R} \setminus [0,1)$ and $k \in \mathbb{R}$, $k \neq n$.*

(i) *If $0 < R \leq \infty$, $\frac{p}{k-n} > 0$, and f is a non-negative measurable function on $B(R)$, then the inequality*

$$\begin{aligned}
&\left(\frac{p}{k-n}\right)^p \int_{B(R)} |\mathbf{x}|^{p-k} \left(1 - \left(\frac{|\mathbf{x}|}{R}\right)^{\frac{k-n}{p}}\right) f^p(\mathbf{x})\, d\mathbf{x} \\
&\quad - \int_{B(R)} |\mathbf{x}|^{-k} \left(Hf(\mathbf{x})\right)^p\, d\mathbf{x} \\
&\geq \left| \left(\frac{p}{k-n}\right)^{p-1} \int_{B(R)} |\mathbf{x}|^{-k} \int_0^1 t^{\frac{k-n}{p}-1} \Bigg| |\mathbf{x}|^p t^{n-k+p} f^p(t\mathbf{x}) \right. \\
&\quad - \left(\frac{k-n}{p}\right)^p (Hf(\mathbf{x}))^p \Bigg|\, dt\, d\mathbf{x} - |p| \int_{B(R)} |\mathbf{x}|^{-k} (Hf(\mathbf{x}))^{p-1} \\
&\quad \left. \cdot \int_0^1 \left| |\mathbf{x}| f(t\mathbf{x}) - \frac{k-n}{p}\, t^{\frac{k-n}{p}-1} Hf(\mathbf{x}) \right| dt\, d\mathbf{x} \right|, \qquad (11.17)
\end{aligned}$$

holds, where

$$Hf(\mathbf{x}) = |\mathbf{x}| \int_0^1 f(t\mathbf{x})\, dt,\ \mathbf{x} \in B(R). \qquad (11.18)$$

(ii) *If $0 \leq R < \infty$, $\frac{p}{k-n} < 0$, and f is a non-negative measurable function on $\mathbb{R}^n \setminus B(R)$, then*

$$\begin{aligned}
&\left(\frac{p}{n-k}\right)^p \int_{\mathbb{R}^n \setminus B(R)} |\mathbf{x}|^{p-k} \left(1 - \left(\frac{R}{|\mathbf{x}|}\right)^{\frac{n-k}{p}}\right) f^p(\mathbf{x})\, d\mathbf{x} \\
&\quad - \int_{\mathbb{R}^n \setminus B(R)} |\mathbf{x}|^{-k} \left(\tilde{H}f(\mathbf{x})\right)^p\, d\mathbf{x} \\
&\geq \left| \left(\frac{p}{n-k}\right)^{p-1} \int_{\mathbb{R}^n \setminus B(R)} |\mathbf{x}|^{-k} \int_1^\infty t^{\frac{k-n}{p}-1} \Bigg| |\mathbf{x}|^p t^{n-k+p} f^p(t\mathbf{x}) \right. \\
&\quad - \left(\frac{n-k}{p}\right)^p (\tilde{H}f(\mathbf{x}))^p \Bigg|\, dt\, d\mathbf{x} - |p| \int_{\mathbb{R}^n \setminus B(R)} |\mathbf{x}|^{-k} (\tilde{H}f(\mathbf{x}))^{p-1} \\
&\quad \left. \cdot \int_1^\infty \left| |\mathbf{x}| f(t\mathbf{x}) - \frac{n-k}{p}\, t^{\frac{k-n}{p}-1} \tilde{H}f(\mathbf{x}) \right| dt\, d\mathbf{x} \right|, \qquad (11.19)
\end{aligned}$$

where

$$\tilde{H}f(\mathbf{x}) = |\mathbf{x}| \int_1^\infty f(t\mathbf{x})\,dt,\ \mathbf{x} \in \mathbb{R}^n \setminus B(R). \tag{11.20}$$

For $p \in (0,1]$ relations (11.17) *and* (11.19) *hold with swapped order of the inegrals on their respective left-hand sides.*

Proof. Follows from Theorem 10.3 and Theorem 11.1 by rewritting (10.24), that is, (11.15) and (11.16), with some particular parameters. Namely, let $X = \mathbb{R}^n$, $I = [0,\infty)$, $u(\mathbf{x}) = |\mathbf{x}|^{-n}$, $d\nu(\mathbf{x}) = d\mathbf{x}$, and $\Phi(x) = x^p$, $p \neq 0$, that is, $\varphi(x) = px^{p-1}$. In the case (i), let also $\Omega = B\left(R^{\frac{k-n}{p}}\right)$, $d\lambda(t) = \chi_{(0,1)}(t)\,dt$ and $d\mu(\mathbf{x}) = \chi_{B\left(R^{\frac{k-n}{p}}\right)}(\mathbf{x})\,d\mathbf{x}$. Then we have $L = 1$, $\dfrac{d\mu_t}{d\nu}(\mathbf{x}) = t^{-n}\chi_{B\left(tR^{\frac{k-n}{p}}\right)}(\mathbf{x})$ and

$$\begin{aligned} v(\mathbf{x}) &= \int_0^1 \left|\frac{1}{t}\mathbf{x}\right|^{-n} t^{-n}\chi_{B\left(tR^{\frac{k-n}{p}}\right)}(\mathbf{x})\,dt = |\mathbf{x}|^{-n}\int_0^1 \chi_{B\left(tR^{\frac{k-n}{p}}\right)}(\mathbf{x})dt \\ &= |\mathbf{x}|^{-n}\int_{\frac{|\mathbf{x}|}{R^{\frac{k-n}{p}}}}^1 dt = |\mathbf{x}|^{-n}\left(1 - \frac{|\mathbf{x}|}{R^{\frac{k-n}{p}}}\right), \quad \mathbf{x} \in B\left(R^{\frac{k-n}{p}}\right), \end{aligned}$$

where we used $0 \le |\mathbf{x}| \le tR^{\frac{k-n}{p}} < R^{\frac{k-n}{p}}$, i.e. $0 \le \dfrac{|\mathbf{x}|}{R^{\frac{k-n}{p}}} \le t < 1$. Replace the function f in (10.24) with the function $g\colon B\left(R^{\frac{k-n}{p}}\right) \to \mathbb{R}$, $g(\mathbf{x}) = |\mathbf{x}|^{\frac{p}{k-n}-1} f\left(|\mathbf{x}|^{\frac{p}{k-n}-1}\mathbf{x}\right)$. Then

$$\begin{aligned} Ag(\mathbf{x}) &= |\mathbf{x}|^{\frac{p}{k-n}-1}\int_0^1 t^{\frac{p}{k-n}-1} f\left(t^{\frac{p}{k-n}}|\mathbf{x}|^{\frac{p}{k-n}}\frac{\mathbf{x}}{|\mathbf{x}|}\right)dt \\ &= \frac{k-n}{p}\,\frac{1}{|\mathbf{x}|}\int_0^{|\mathbf{x}|^{\frac{p}{k-n}}} f\left(\frac{r}{|\mathbf{x}|}\mathbf{x}\right)dr, \end{aligned}$$

where we applied the substitution $r = (t|\mathbf{x}|)^{\frac{p}{k-n}}$. In this setting, by using polar coordinates, the first integral on the left-hand side of inequality (10.24) becomes

$$\begin{aligned} &\int_{B\left(R^{\frac{k-n}{p}}\right)} |\mathbf{x}|^{p\left(\frac{p}{k-n}-1\right)-n}\left(1 - \frac{|\mathbf{x}|}{R^{\frac{k-n}{p}}}\right) f^p\left(|\mathbf{x}|^{\frac{p}{k-n}-1}\mathbf{x}\right)d\mathbf{x} \\ &= \int_{S^{n-1}} dS \int_0^{R^{\frac{k-n}{p}}} r^{p\left(\frac{p}{k-n}-1\right)-1}\left(1 - \frac{r}{R^{\frac{k-n}{p}}}\right) f^p\left(r^{\frac{p}{k-n}}S\right)dr \\ &= \frac{k-n}{p}\int_{S^{n-1}} dS \int_0^R t^{n-1+p-k}\left(1 - \left(\frac{t}{R}\right)^{\frac{k-n}{p}}\right) f^p(tS)\,dt \\ &= \frac{k-n}{p}\int_{B(R)} |\mathbf{x}|^{p-k}\left(1 - \left(\frac{|\mathbf{x}|}{R}\right)^{\frac{k-n}{p}}\right) f^p(\mathbf{x})\,d\mathbf{x}, \end{aligned} \tag{11.21}$$

while the second integral on the left-hand side of (10.24) reduces to

$$
\begin{aligned}
&\left(\frac{k-n}{p}\right)^p \int_{B\left(R^{\frac{k-n}{p}}\right)} |\mathbf{x}|^{-n-p} \left(\int_0^{|\mathbf{x}|^{\frac{p}{k-n}}} f\left(\frac{r}{|\mathbf{x}|}\mathbf{x}\right) dr\right)^p d\mathbf{x} \\
&= \left(\frac{k-n}{p}\right)^p \int_{S^{n-1}} dS \int_0^{R^{\frac{k-n}{p}}} t^{-p-1} \left(\int_0^{t^{\frac{p}{k-n}}} f(rS)\, dr\right)^p dt \\
&= \left(\frac{k-n}{p}\right)^{p+1} \int_{S^{n-1}} dS \int_0^R s^{n-k-1} \left(\int_0^s f(rS)\, dr\right)^p ds \\
&= \left(\frac{k-n}{p}\right)^{p+1} \int_{B(R)} |\mathbf{x}|^{-k} (Hf(\mathbf{x}))^p\, d\mathbf{x}.
\end{aligned} \tag{11.22}
$$

Analogously, on the right-hand side of (10.24) we get

$$
\begin{aligned}
&\left| \int_{B\left(R^{\frac{k-n}{p}}\right)} |\mathbf{x}|^{-n-p} \int_0^1 \left| t^{p\left(\frac{p}{k-n}-1\right)} |\mathbf{x}|^{p\frac{p}{k-n}} f^p\left(t^{\frac{p}{k-n}} |\mathbf{x}|^{\frac{p}{k-n}} \frac{\mathbf{x}}{|\mathbf{x}|}\right) \right.\right. \\
&\quad \left. - \left(\frac{k-n}{p}\right)^p \left(\int_0^{|\mathbf{x}|^{\frac{p}{k-n}}} f\left(\frac{r}{|\mathbf{x}|}\mathbf{x}\right) dr\right)^p \right| dt\, d\mathbf{x} \\
&\quad - \left(\frac{k-n}{p}\right)^{p-1} |p| \int_{B\left(R^{\frac{k-n}{p}}\right)} |\mathbf{x}|^{-n-p} \left(\int_0^{|\mathbf{x}|^{\frac{p}{k-n}}} f\left(\frac{r}{|\mathbf{x}|}\mathbf{x}\right) dr\right)^{p-1} \\
&\quad \left. \cdot \int_0^1 \left| t^{\frac{p}{k-n}-1} |\mathbf{x}|^{\frac{p}{k-n}} f\left(t^{\frac{p}{k-n}} |\mathbf{x}|^{\frac{p}{k-n}} \frac{\mathbf{x}}{|\mathbf{x}|}\right) - \frac{k-n}{p} \int_0^{|\mathbf{x}|^{\frac{p}{k-n}}} f\left(\frac{r}{|\mathbf{x}|}\mathbf{x}\right) dr \right| dt\, d\mathbf{x} \right| \\
&= \left| \int_{S^{n-1}} dS \int_0^{R^{\frac{k-n}{p}}} s^{-p-1} \int_0^1 \left| t^{p\left(\frac{p}{k-n}-1\right)} s^{p\frac{p}{k-n}} f^p\left(t^{\frac{p}{k-n}} s^{\frac{p}{k-n}} S\right) \right.\right. \\
&\quad \left. - \left(\frac{k-n}{p}\right)^p \left(\int_0^{s^{\frac{p}{k-n}}} f(rS) dr\right)^p \right| dt\, ds \\
&\quad - \left(\frac{k-n}{p}\right)^{p-1} |p| \int_{S^{n-1}} dS \int_0^{R^{\frac{k-n}{p}}} s^{-p-1} \left(\int_0^{s^{\frac{p}{k-n}}} f(rS) dr\right)^{p-1} \\
&\quad \left. \cdot \int_0^1 \left| t^{\frac{p}{k-n}-1} s^{\frac{p}{k-n}} f\left(t^{\frac{p}{k-n}} s^{\frac{p}{k-n}} S\right) - \frac{k-n}{p} \int_0^{s^{\frac{p}{k-n}}} f(rS) dr \right| dt\, ds \right| \\
&= \left| \frac{k-n}{p} \int_{S^{n-1}} dS \int_0^R z^{n-k-1} \int_0^1 \left| z^p t^{p\left(\frac{p}{k-n}-1\right)} f^p\left(t^{\frac{p}{k-n}} zS\right) \right.\right.
\end{aligned}
$$

$$
\begin{aligned}
&- \left(\frac{k-n}{p}\right)^p \left(\int_0^z f(rS)\,dr\right)^p \Bigg|\, dt\, dz \\
&- \left(\frac{k-n}{p}\right)^p |p| \int_{S^{n-1}} dS \int_0^R z^{n-k-1} \left(\int_0^z f(rS)\,dr\right)^{p-1} \\
&\cdot \int_0^1 \left| t^{\frac{p}{k-n}-1} z\, f\left(t^{\frac{p}{k-n}} zS\right) - \frac{k-n}{p} \int_0^z f(rS)\,dr \right| dt\, dz \Bigg| \\
= &\left| \left(\frac{k-n}{p}\right)^2 \int_{S^{n-1}} dS \int_0^R z^{n-k-1} \int_0^1 w^{\frac{k-n}{p}-1} \right| z^p w^{n-k+p} f^p(wzS) \\
&- \left(\frac{k-n}{p}\right)^p \left(\int_0^z f(rS)dr\right)^p \Bigg|\, dw\, dz \\
&- \left(\frac{k-n}{p}\right)^{p+1} |p| \int_{S^{n-1}} dS \int_0^R z^{n-k-1} \left(\int_0^z f(rS)dr\right)^{p-1} \\
&\cdot \int_0^1 w^{\frac{k-n}{p}-1} \left| w^{1-\frac{k-n}{p}} z\, f(wzS) - \frac{k-n}{p} \int_0^z f(rS)dr \right| dw\, dz \Bigg| \\
= &\left| \left(\frac{k-n}{p}\right)^2 \int_{B(R)} |\mathbf{x}|^{-k} \int_0^1 t^{\frac{k-n}{p}-1} \right| |\mathbf{x}|^p t^{n-k+p} f^p(t\mathbf{x}) \\
&- \left(\frac{k-n}{p}\right)^p (Hf(\mathbf{x}))^p \Bigg|\, dt\, d\mathbf{x} - \left(\frac{k-n}{p}\right)^{p+1} |p| \int_{B(R)} |\mathbf{x}|^{-k} (Hf(\mathbf{x}))^{p-1} \\
&\cdot \int_0^1 \left| |\mathbf{x}| f(t\mathbf{x}) - \frac{k-n}{p} t^{\frac{k-n}{p}-1} Hf(\mathbf{x}) \right| dt\, d\mathbf{x} \Bigg| . \qquad (11.23)
\end{aligned}
$$

Finally, (11.17) holds by combining (11.21), (11.22) and (11.23).

To obtain relation (11.19), that is, the case (ii), we consider $\Omega = \mathbb{R}^n \setminus B\left(R^{\frac{n-k}{p}}\right)$, $d\lambda(t) = \chi_{(1,\infty)}(t)\frac{dt}{t^2}$, and $d\mu(\mathbf{x}) = \chi_{\mathbb{R}^n \setminus B\left(R^{\frac{n-k}{p}}\right)}(\mathbf{x})\,d\mathbf{x}$. As in the case (i), here we have $L = 1$, $\dfrac{d\mu_t}{d\nu}(\mathbf{x}) = t^{-n}\chi_{\mathbb{R}^n \setminus B\left(tR^{\frac{n-k}{p}}\right)}(\mathbf{x})$ and

$$
\begin{aligned}
v(\mathbf{x}) &= \int_1^\infty \left|\frac{1}{t}\mathbf{x}\right|^{-n} t^{-n} \chi_{\mathbb{R}^n \setminus B(tR^{\frac{n-k}{p}})}(\mathbf{x}) \frac{dt}{t^2} = |\mathbf{x}|^{-n} \int_1^\infty \chi_{\mathbb{R}^n \setminus B(tR^{\frac{n-k}{p}})}(\mathbf{x}) \frac{dt}{t^2} \\
&= |\mathbf{x}|^{-n} \int_1^{\frac{|\mathbf{x}|}{R^{\frac{n-k}{p}}}} \frac{dt}{t^2} = |\mathbf{x}|^{-n} \left(1 - \frac{R^{\frac{n-k}{p}}}{|\mathbf{x}|}\right),
\end{aligned}
$$

where for $\mathbf{x} \in \mathbb{R}^n \setminus B\left(tR^{\frac{n-k}{p}}\right)$ we know $|\mathbf{x}| > tR^{\frac{n-k}{p}} \geq R^{\frac{n-k}{p}}$, i.e. $\dfrac{|\mathbf{x}|}{R^{\frac{n-k}{p}}} > t \geq 1$. Similary as in the proof of (i), in inequality (10.24) we substitute the function f with the function $g\colon \mathbb{R}^n \setminus B\left(R^{\frac{n-k}{p}}\right) \to \mathbb{R}$, $g(\mathbf{x}) = |\mathbf{x}|^{\frac{p}{n-k}+1} f\left(|\mathbf{x}|^{\frac{p}{n-k}-1}\mathbf{x}\right)$. In that case we have

$$
Ag(\mathbf{x}) = |\mathbf{x}|^{\frac{p}{n-k}+1} \int_1^\infty t^{\frac{p}{n-k}-1} f\left(t^{\frac{p}{n-k}} |\mathbf{x}|^{\frac{p}{n-k}} \frac{\mathbf{x}}{|\mathbf{x}|}\right) dt
$$

$$= \frac{n-k}{p}|\mathbf{x}|\int_{|\mathbf{x}|^{\frac{p}{n-k}}}^{\infty} f\left(\frac{r}{|\mathbf{x}|}\mathbf{x}\right)dr$$

with applied substitution $r=(t|\mathbf{x}|)^{\frac{p}{n-k}}$. In this case, by using polar coordinates, the first integral on the left-hand side in (10.24) becomes

$$\int_{\mathbb{R}^n\setminus B\left(R^{\frac{n-k}{p}}\right)}|\mathbf{x}|^{-n+p\left(\frac{p}{n-k}+1\right)}\left(1-\frac{R^{\frac{n-k}{p}}}{|\mathbf{x}|}\right)f^p\left(|\mathbf{x}|^{\frac{p}{n-k}-1}\mathbf{x}\right)d\mathbf{x}$$

$$= \int_{S^{n-1}}dS\int_{R^{\frac{n-k}{p}}}^{\infty} r^{-1+p\left(\frac{p}{n-k}+1\right)}\left(1-\frac{R^{\frac{n-k}{p}}}{r}\right)f^p(r^{\frac{p}{n-k}}S)\,dr$$

$$= \frac{n-k}{p}\int_{S^{n-1}}dS\int_R^{\infty} t^{n-1+p-k}\left(1-\left(\frac{R}{t}\right)^{\frac{n-k}{p}}\right)f^p(tS)\,dt$$

$$= \frac{n-k}{p}\int_{\mathbb{R}^n\setminus B(R)}|\mathbf{x}|^{p-k}\left(1-\left(\frac{R}{|\mathbf{x}|}\right)^{\frac{n-k}{p}}\right)f^p(\mathbf{x})\,d\mathbf{x}, \tag{11.24}$$

while the second integral on the left-hand side in (10.24) reduces to

$$\left(\frac{n-k}{p}\right)^p\int_{\mathbb{R}^n\setminus B\left(R^{\frac{n-k}{p}}\right)}|\mathbf{x}|^{p-n}\left(\int_{|\mathbf{x}|^{\frac{p}{n-k}}}^{\infty} f\left(\frac{r}{|\mathbf{x}|}\mathbf{x}\right)dr\right)^p d\mathbf{x}$$

$$= \left(\frac{n-k}{p}\right)^p\int_{S^{n-1}}dS\int_{R^{\frac{n-k}{p}}}^{\infty} t^{p-1}\left(\int_{t^{\frac{p}{n-k}}}^{\infty} f(rS)\,dr\right)^p dt$$

$$= \left(\frac{n-k}{p}\right)^{p+1}\int_{S^{n-1}}dS\int_R^{\infty} s^{n-k-1}\left(\int_s^{\infty} f(rS)\,dr\right)^p ds$$

$$= \left(\frac{n-k}{p}\right)^{p+1}\int_{\mathbb{R}^n\setminus B(R)}|\mathbf{x}|^{-k}(\tilde{H}f(\mathbf{x}))^p\,d\mathbf{x}. \tag{11.25}$$

Also, by using analogous techniques as in the proof of (10.24), on the right-hand side we have

$$\Bigg|\int_{\mathbb{R}^n\setminus B\left(R^{\frac{n-k}{p}}\right)}|\mathbf{x}|^{-n+p}\int_1^{\infty}\Bigg|t^{p\left(\frac{p}{n-k}+1\right)}|\mathbf{x}|^{p\frac{p}{n-k}}f^p\left(t^{\frac{p}{n-k}}|\mathbf{x}|^{\frac{p}{n-k}}\frac{\mathbf{x}}{|\mathbf{x}|}\right)$$

$$-\left(\frac{n-k}{p}\right)^p\left(\int_{|\mathbf{x}|^{\frac{p}{n-k}}}^{\infty} f\left(\frac{r}{|\mathbf{x}|}\mathbf{x}\right)dr\right)^p\Bigg|\frac{dt}{t^2}\,d\mathbf{x}$$

$$-\left(\frac{n-k}{p}\right)^{p-1}|p|\int_{\mathbb{R}^n\setminus B\left(R^{\frac{n-k}{p}}\right)}|\mathbf{x}|^{-n+p}\left(\int_{|\mathbf{x}|^{\frac{p}{n-k}}}^{\infty} f\left(\frac{r}{|\mathbf{x}|}\mathbf{x}\right)dr\right)^{p-1}$$

$$\cdot\int_1^{\infty}\left|t^{\frac{p}{n-k}+1}|\mathbf{x}|^{\frac{p}{n-k}}f\left(t^{\frac{p}{n-k}}|\mathbf{x}|^{\frac{p}{n-k}}\frac{\mathbf{x}}{|\mathbf{x}|}\right)-\frac{n-k}{p}\int_{|\mathbf{x}|^{\frac{p}{n-k}}}^{\infty} f\left(\frac{r}{|\mathbf{x}|}\mathbf{x}\right)dr\right|\frac{dt}{t^2}\,d\mathbf{x}\Bigg|$$

$$= \Bigg|\int_{S^{n-1}}dS\int_{R^{\frac{n-k}{p}}}^{\infty} s^{p-1}\int_1^{\infty}\Bigg|t^{p\left(\frac{p}{n-k}+1\right)}s^{p\frac{p}{n-k}}f^p\left(t^{\frac{p}{n-k}}s^{\frac{p}{n-k}}S\right)$$

$$
\begin{aligned}
&-\left(\frac{n-k}{p}\right)^p\left(\int_{s^{\frac{p}{n-k}}}^{\infty} f(rS)dr\right)^p\Bigg|\frac{dt}{t^2}\,ds\\
&-\left(\frac{n-k}{p}\right)^{p-1}|p|\int_{S^{n-1}} dS\int_{R^{\frac{n-k}{p}}}^{\infty} s^{p-1}\left(\int_{s^{\frac{p}{n-k}}}^{\infty} f(rS)dr\right)^{p-1}\\
&\int_1^{\infty}\left|t^{\frac{p}{n-k}+1}s^{\frac{p}{n-k}}f\left(t^{\frac{p}{n-k}}s^{\frac{p}{n-k}}S\right)-\frac{n-k}{p}\int_{s^{\frac{p}{n-k}}}^{\infty} f(rS)dr\right|\frac{dt}{t^2}\,ds\Bigg|\\
=&\Bigg|\frac{n-k}{p}\int_{S^{n-1}} dS\int_R^{\infty} z^{n-k-1}\int_1^{\infty}\Bigg|z^p t^{p\left(\frac{p}{n-k}+1\right)} f^p\left(t^{\frac{p}{n-k}}zS\right)\\
&-\left(\frac{n-k}{p}\right)^p\left(\int_z^{\infty} f(rS)\,dr\right)^p\Bigg|\frac{dt}{t^2}\,dz\\
&-\left(\frac{n-k}{p}\right)^p|p|\int_{S^{n-1}} dS\int_R^{\infty} z^{n-k-1}\left(\int_z^{\infty} f(rS)\,dr\right)^{p-1}\\
&\cdot\int_1^{\infty}\left|t^{\frac{p}{n-k}+1}z\,f\left(t^{\frac{p}{n-k}}zS\right)-\frac{n-k}{p}\int_z^{\infty} f(rS)\,dr\right|\frac{dt}{t^2}\,dz\Bigg|\\
=&\Bigg|\left(\frac{n-k}{p}\right)^2\int_{S^{n-1}} dS\int_R^{\infty} z^{n-k-1}\int_1^{\infty}\Bigg|w^{n-k+p}z^p f^p(wzS)-\left(\frac{n-k}{p}\right)^p\left(\int_z^{\infty} f(rS)dr\right)^p\Bigg|\\
&\cdot w^{\frac{k-n}{p}-1}dw\,dz-\left(\frac{n-k}{p}\right)^{p+1}|p|\int_{S^{n-1}} dS\int_R^{\infty} z^{n-k-1}\left(\int_z^{\infty} f(rS)dr\right)^{p-1}\\
&\cdot\int_1^{\infty}\left|w^{\frac{n-k}{p}+1}z\,f(wzS)-\frac{n-k}{p}\int_z^{\infty} f(rS)dr\right|w^{\frac{k-n}{p}-1}dw\,dz\Bigg|\\
=&\Bigg|\left(\frac{n-k}{p}\right)^2\int_{\mathbb{R}^n\setminus B(R)}|\mathbf{x}|^{-k}\int_1^{\infty} t^{\frac{k-n}{p}-1}\Bigg||\mathbf{x}|^p t^{n-k+p} f^p(t\mathbf{x})\\
&-\left(\frac{n-k}{p}\right)^p(\tilde{H}f(\mathbf{x}))^p\Bigg|dt\,d\mathbf{x}-\left(\frac{n-k}{p}\right)^{p+1}|p|\int_{\mathbb{R}^n\setminus B(R)}|\mathbf{x}|^{-k}(\tilde{H}f(\mathbf{x}))^{p-1}\\
&\cdot\int_1^{\infty}\left||\mathbf{x}|f(t\mathbf{x})-\frac{n-k}{p}\,t^{\frac{k-n}{p}-1}\tilde{H}f(\mathbf{x})\right|dt\,d\mathbf{x}\Bigg|. \qquad (11.26)
\end{aligned}
$$

Inequality (11.19) follows by combining (11.24), (11.25) and (11.26), and by multiplying with $\left(\frac{p}{n-k}\right)^{p+1}$. □

For $n = 1$, inequalities (11.17) and (11.19) respectively reduce to inequalities (11.5) and (11.6), obtained in [21] and [28].

11.4 The best constant for Hardy-type inequality

Observe that the right-hand sides of inequalities (11.17) and (11.19) are non-negative. Moreover, the constants involved in their left-hand sides are shown to be the best possible. That result is given in the following theorem.

Theorem 11.3 *Let $n \in \mathbb{N}$, $p \in \mathbb{R} \setminus [0,1)$, and $k \in \mathbb{R}$, $k \neq n$.*

(i) If $0 < R \leq \infty$, $\dfrac{p}{k-n} > 0$, f is a non-negative function on $B(R)$, and Hf is defined by (11.18)*, then*

$$\int_{B(R)} |\mathbf{x}|^{-k} (Hf(\mathbf{x}))^p \, d\mathbf{x} \\ \leq \left(\frac{p}{k-n}\right)^p \int_{B(R)} |\mathbf{x}|^{p-k} \left[1 - \left(\frac{|\mathbf{x}|}{R}\right)^{\frac{k-n}{p}}\right] f^p(\mathbf{x}) \, d\mathbf{x}. \tag{11.27}$$

(ii) If $0 \leq R < \infty$, $\dfrac{p}{k-n} < 0$, f is a non-negative function on $\mathbb{R}^n \setminus B(R)$, and $\tilde{H}f$ is given by (11.20)*, then*

$$\int_{\mathbb{R}^n \setminus B(R)} |\mathbf{x}|^{-k} \left(\tilde{H}f(\mathbf{x})\right)^p d\mathbf{x} \\ \leq \left(\frac{p}{n-k}\right)^p \int_{\mathbb{R}^n \setminus B(R)} |\mathbf{x}|^{p-k} \left[1 - \left(\frac{R}{|\mathbf{x}|}\right)^{\frac{n-k}{p}}\right] f^p(\mathbf{x}) \, d\mathbf{x}. \tag{11.28}$$

The constant $\left|\dfrac{p}{k-n}\right|^p$ is the best possible for both inequalities. For $p \in (0,1]$, the signs of inequality in (11.27) *and* (11.28) *are reversed.*

Proof. We only need to prove that $\left|\dfrac{p}{k-n}\right|^p$ is the best possible constant for inequalities (11.27) and (11.28). Consider the case (i) first. For a sufficiently small $\varepsilon > 0$, and the function $f_\varepsilon \colon B(R) \to \mathbb{R}$ defined by $f_\varepsilon(\mathbf{x}) = |\mathbf{x}|^{\frac{k-n+\varepsilon}{p}-1}$, the left-hand side of (11.27) is equal to

$$\begin{aligned} L_\varepsilon &= \int_{B(R)} |\mathbf{x}|^{-k} \left(|\mathbf{x}| \int_0^1 t^{\frac{k-n+\varepsilon}{p}-1} |\mathbf{x}|^{\frac{k-n+\varepsilon}{p}-1} dt\right)^p d\mathbf{x} \\ &= \int_{B(R)} |\mathbf{x}|^{-n+\varepsilon} \left(\int_0^1 t^{\frac{k-n+\varepsilon}{p}-1} dt\right)^p d\mathbf{x} = \left(\frac{p}{k-n+\varepsilon}\right)^p \int_{B(R)} |\mathbf{x}|^{-n+\varepsilon} \, d\mathbf{x} \end{aligned}$$

$$= \left(\frac{p}{k-n+\varepsilon}\right)^p \int_{S^{n-1}} dS \int_0^R r^{\varepsilon-1}\,dr = \left(\frac{p}{k-n+\varepsilon}\right)^p |S^{n-1}| \cdot \frac{R^\varepsilon}{\varepsilon},$$

while on the right-hand side of (11.27) we get

$$\begin{aligned} R_\varepsilon &= \left(\frac{p}{k-n}\right)^p \int_{B(R)} |\mathbf{x}|^{-n+\varepsilon} \left(1-\left(\frac{|\mathbf{x}|}{R}\right)^{\frac{k-n}{p}}\right) d\mathbf{x} \\ &\le \left(\frac{p}{k-n}\right)^p \int_{B(R)} |\mathbf{x}|^{-n+\varepsilon}\,d\mathbf{x} = \left(\frac{p}{k-n}\right)^p \int_{S^{n-1}} dS \int_0^R r^{\varepsilon-1}\,dr \\ &= \left(\frac{p}{k-n}\right)^p |S^{n-1}| \cdot \frac{R^\varepsilon}{\varepsilon}. \end{aligned}$$

Therefore, $1 \le \frac{R_\varepsilon}{L_\varepsilon} \le \left(\frac{k-n+\varepsilon}{k-n}\right)^p$. Since $\left(\frac{k-n+\varepsilon}{k-n}\right)^p \searrow 1$, as $\varepsilon \searrow 0$, the constant $\left(\frac{p}{k-n}\right)^p$ is the best possible for (11.27). The proof that the constant $\left(\frac{p}{n-k}\right)^p$ is the best possible for (11.28) follows the same lines, considering the function $f_\varepsilon \colon \mathbb{R}^n \setminus B(R) \to \mathbb{R}$, $f_\varepsilon(\mathbf{x}) = |\mathbf{x}|^{\frac{k-n-\varepsilon}{p}-1}$. For this function, on the left-hand side in (11.28) we obtain

$$\begin{aligned} L_\varepsilon &= \int_{\mathbb{R}^n \setminus B(R)} |\mathbf{x}|^{-k} \left(|\mathbf{x}| \int_1^\infty t^{\frac{k-n-\varepsilon}{p}-1} |\mathbf{x}|^{\frac{k-n-\varepsilon}{p}-1} dt\right)^p d\mathbf{x} \\ &= \int_{\mathbb{R}^n \setminus B(R)} |\mathbf{x}|^{-n-\varepsilon} \left(\int_1^\infty t^{\frac{k-n-\varepsilon}{p}-1} dt\right)^p d\mathbf{x} = \left(\frac{p}{n-k+\varepsilon}\right)^p \int_{\mathbb{R}^n \setminus B(R)} |\mathbf{x}|^{-n-\varepsilon}\,d\mathbf{x} \\ &= \left(\frac{p}{n-k+\varepsilon}\right)^p \int_{S^{n-1}} dS \int_R^\infty r^{-\varepsilon-1}\,dr = \left(\frac{p}{n-k+\varepsilon}\right)^p |S^{n-1}| \cdot \frac{1}{\varepsilon R^\varepsilon}, \end{aligned}$$

while on the right-hand side in (11.28) we have

$$\begin{aligned} R_\varepsilon &= \left(\frac{p}{n-k}\right)^p \int_{\mathbb{R}^n \setminus B(R)} |\mathbf{x}|^{-n-\varepsilon} \left(1-\left(\frac{R}{|\mathbf{x}|}\right)^{\frac{n-k}{p}}\right) d\mathbf{x} \\ &\le \left(\frac{p}{n-k}\right)^p \int_{\mathbb{R}^n \setminus B(R)} |\mathbf{x}|^{-n-\varepsilon}\,d\mathbf{x} = \left(\frac{p}{n-k}\right)^p \int_{S^{n-1}} dS \int_R^\infty r^{-\varepsilon-1}\,dr \\ &= \left(\frac{p}{n-k}\right)^p |S^{n-1}| \cdot \frac{1}{\varepsilon R^\varepsilon}, \end{aligned}$$

and therefrom we easily get $1 \le \frac{R_\varepsilon}{L_\varepsilon} \le \left(\frac{n-k+\varepsilon}{n-k}\right)^p$. Since $\left(\frac{n-k+\varepsilon}{n-k}\right)^p \searrow 1$ as $\varepsilon \searrow 0$, the constant $\left(\frac{p}{n-k}\right)^p$ is the best possible for (11.28). □

If $R = \infty$ in (11.27) and $R = 0$ in (11.28), we immediately get the following new multidimensional Hardy-type inequality.

Corollary 11.3 *Let $n \in \mathbb{N}$, $p \in \mathbb{R} \setminus [0,1)$, and $k \in \mathbb{R}$, $k \neq n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a non-negative measurable function and Hf and $\tilde{H}f$ be defined by* (11.18) *and* (11.20) *respectively. If $\frac{p}{k-n} > 0$, then the inequality*

$$\int_{\mathbb{R}^n} |\mathbf{x}|^{-k} (Hf(\mathbf{x}))^p \, d\mathbf{x} \leq \left(\frac{p}{k-n}\right)^p \int_{\mathbb{R}^n} |\mathbf{x}|^{p-k} f^p(\mathbf{x}) \, d\mathbf{x} \tag{11.29}$$

holds, while for $\frac{p}{k-n} < 0$ we have

$$\int_{\mathbb{R}^n} |\mathbf{x}|^{-k} (\tilde{H}f(\mathbf{x}))^p \, d\mathbf{x} \leq \left(\frac{p}{n-k}\right)^p \int_{\mathbb{R}^n} |\mathbf{x}|^{p-k} f^p(\mathbf{x}) \, d\mathbf{x}. \tag{11.30}$$

The constant $\left|\frac{p}{k-n}\right|^p$ is the best possible for both inequalities. Moreover, for $p \in (0,1]$ the signs of inequality in (11.29) *and* (11.30) *are reversed.*

11.5 Pólya-Knopp-type inequalities with balls in $\mathbb{R}^n$

We continue our analysis by obtaining the refined Pólya-Knopp-type inequality. The following theorem generalises and refines inequalities (11.8) and (11.9).

Theorem 11.4 *Let $n \in \mathbb{N}$.*

(i) *If $0 < R \leq \infty$, f is a positive measurable function on $B(R)$, and*

$$Gf(\mathbf{x}) = \exp\left(\int_0^1 \log f(t\mathbf{x}) \, dt\right), \ \mathbf{x} \in B(R), \tag{11.31}$$

then the inequality

$$\begin{aligned} & e^n \int_{B(R)} \left(1 - \frac{|\mathbf{x}|}{R}\right) f(\mathbf{x}) \, d\mathbf{x} - \int_{B(R)} Gf(\mathbf{x}) \, d\mathbf{x} \\ & \geq \left| \int_{B(R)} \int_0^1 |(et)^n f(t\mathbf{x}) - Gf(\mathbf{x})| \, dt \, d\mathbf{x} \right. \\ & \quad \left. - \int_{B(R)} Gf(\mathbf{x}) \int_0^1 \left|\log \frac{(et)^n f(t\mathbf{x})}{Gf(\mathbf{x})}\right| dt \, d\mathbf{x} \right| \end{aligned} \tag{11.32}$$

holds.

(ii) *If $0 \leq R < \infty$, f is a positive measurable function on $\mathbb{R}^n \setminus B(R)$, and*

$$\tilde{G}f(\mathbf{x}) = \exp\left(\int_1^\infty \log f(t\mathbf{x}) \, \frac{dt}{t^2}\right), \ \mathbf{x} \in \mathbb{R}^n \setminus B(R), \tag{11.33}$$

then

$$
\begin{aligned}
& e^{-n} \int_{\mathbb{R}^n \setminus B(R)} \left(1 - \frac{R}{|\mathbf{x}|}\right) f(\mathbf{x})\, d\mathbf{x} - \int_{\mathbb{R}^n \setminus B(R)} \tilde{G}f(\mathbf{x})\, d\mathbf{x} \\
& \geq \left| \int_{\mathbb{R}^n \setminus B(R)} \int_1^\infty \left| \left(\frac{t}{e}\right)^n f(t\mathbf{x}) - \tilde{G}f(\mathbf{x}) \right| \frac{dt}{t^2}\, d\mathbf{x} \right. \\
& \quad \left. - \int_{\mathbb{R}^n \setminus B(R)} \tilde{G}f(\mathbf{x}) \int_1^\infty \left| \log \frac{(et)^n f(t\mathbf{x})}{\tilde{G}f(\mathbf{x})} \right| \frac{dt}{t^2}\, d\mathbf{x} \right|. \qquad (11.34)
\end{aligned}
$$

Proof. Follows from Theorem 10.3 and Theorem 11.1 by considering $X = \mathbb{R}^n$, $I = \mathbb{R}$, $u(\mathbf{x}) = |B(|\mathbf{x}|)|^{-1}$, $d\nu(\mathbf{x}) = d\mathbf{x}$, and $\Phi(x) = \varphi(x) = e^x$. To get (11.32), we also set $\Omega = B(R)$, $d\lambda(t) = \chi_{(0,1)}(t)\, dt$ and $d\mu(\mathbf{x}) = \chi_{B(R)}(\mathbf{x})\, d\mathbf{x}$. In that case, we have $L = 1$, $\frac{d\mu_t}{d\nu}(\mathbf{x}) = t^{-n}\chi_{B(tR)}(\mathbf{x})$ and

$$
\begin{aligned}
v(\mathbf{x}) &= \int_0^1 \frac{1}{|B(|\frac{1}{t}\mathbf{x}|)|}\, t^{-n}\chi_{B(tR)}(\mathbf{x})\, dt = \frac{1}{|B(|\mathbf{x}|)|} \int_0^1 \chi_{B(tR)}(\mathbf{x})\, dt \\
&= \frac{1}{|B(|\mathbf{x}|)|} \int_{\frac{|\mathbf{x}|}{R}}^1 dt = \frac{1}{|B(|\mathbf{x}|)|} \left(1 - \frac{|\mathbf{x}|}{R}\right), \ \mathbf{x} \in B(R),
\end{aligned}
$$

where we know that for $\mathbf{x} \in B(tR)$, $0 \leq |\mathbf{x}| \leq tR \leq R$ holds, i.e. $0 \leq \frac{|\mathbf{x}|}{R} \leq t \leq 1$. Further, replace the function f in (10.24) with the function $g\colon B(R) \to \mathbb{R}$, $g(\mathbf{x}) = \log(|B(|\mathbf{x}|)|f(\mathbf{x}))$. Then we have

$$
\begin{aligned}
Ag(\mathbf{x}) &= \int_0^1 g(t\mathbf{x})dt = \int_0^1 \log\left(t^n |B(|\mathbf{x}|)| f(t\mathbf{x})\right) dt \\
&= n\int_0^1 \log t\, dt + \log|B(|\mathbf{x}|)| + \int_0^1 \log f(t\mathbf{x})dt \\
&= -n + \log|B(|\mathbf{x}|)| + \int_0^1 \log f(t\mathbf{x})dt.
\end{aligned}
$$

The first intergal on the left-hand side of (10.24) becomes

$$
\int_{B(R)} \frac{1}{|B(|\mathbf{x}|)|} \left(1 - \frac{|\mathbf{x}|}{R}\right) |B(|\mathbf{x}|)| f(\mathbf{x})\, d\mathbf{x} = \int_{B(R)} \left(1 - \frac{|\mathbf{x}|}{R}\right) f(\mathbf{x})\, d\mathbf{x}, \qquad (11.35)
$$

and the corresponding second integral reduces to

$$
\begin{aligned}
& \int_{B(R)} \frac{1}{|B(|\mathbf{x}|)|} \exp\left(-n + \log|B(|\mathbf{x}|)| + \log \int_0^1 \log f(t\mathbf{x})\, dt\right) d\mathbf{x} \\
& = e^{-n} \int_{B(R)} \exp\left(\int_0^1 \log f(t\mathbf{x})\, dt\right) d\mathbf{x} = e^{-n} \int_{B(R)} Gf(\mathbf{x})\, d\mathbf{x}. \qquad (11.36)
\end{aligned}
$$

Since $\Phi(g(\mathbf{x})) = \exp\left(\log(|B(|\mathbf{x}|)| f(\mathbf{x})\right) = |B(|\mathbf{x}|)| f(\mathbf{x})$ and

$$
\Phi(Ag(\mathbf{x})) = \exp\left(-n + \log|B(|\mathbf{x}|)| + \int_0^1 \log f(t\mathbf{x})dt\right)
$$

$$= e^{-n}|B(|\mathbf{x}|)|Gf(\mathbf{x}),$$

on the right-hand side of (10.24) we obtain

$$\begin{aligned}
&\bigg| \int_{B(R)} \frac{1}{|B(|\mathbf{x}|)|} \int_0^1 \left|t^n|B(|\mathbf{x}|)|f(t\mathbf{x}) - e^{-n}|B(|\mathbf{x}|)|Gf(\mathbf{x})\right| dt\, d\mathbf{x} \\
&\quad - \int_{B(R)} \frac{1}{|B(|\mathbf{x}|)|} \int_0^1 e^{-n}|B(|\mathbf{x}|)|Gf(\mathbf{x}) \\
&\quad \cdot \left|\log\left(t^n|B(|\mathbf{x}|)|f(t\mathbf{x})\right) + n - \log|B(|\mathbf{x}|)| - \int_0^1 \log f(t\mathbf{x})\, dt\right| dt\, d\mathbf{x}\bigg| \\
&= \bigg| \int_{B(R)} \int_0^1 |t^n f(t\mathbf{x}) - e^{-n}Gf(\mathbf{x})|\, dt\, d\mathbf{x} - e^{-n} \int_{B(R)} Gf(\mathbf{x}) \int_0^1 |n \log t \\
&\quad + \log|B(|\mathbf{x}|)| + \log f(t\mathbf{x}) + n - \log|B(|\mathbf{x}|)| - \log Gf(\mathbf{x})|\, dt\, d\mathbf{x}\bigg| \\
&= \bigg| \int_{B(R)} \int_0^1 \left|t^n f(t\mathbf{x}) - e^{-n}Gf(\mathbf{x})\right| dt\, d\mathbf{x} \\
&\quad - e^{-n} \int_{B(R)} Gf(\mathbf{x}) \int_0^1 \left|\log \frac{(et)^n f(t\mathbf{x})}{Gf(\mathbf{x})}\right| dt\; d\mathbf{x}\bigg|. \qquad (11.37)
\end{aligned}$$

Finally, (11.32) holds by combining (11.35), (11.36) and (11.37), and then multiplying the whole inequality with e^n.

Inequality (11.34) can be derived by a similar technique, by taking $\Omega = \mathbb{R}^n \setminus B(R)$, $d\lambda(t) = \chi_{[1,\infty)}(t)\, \frac{dt}{t^2}$, and $d\mu(\mathbf{x}) = \chi_{\mathbb{R}^n \setminus B(R)}(\mathbf{x})\, d\mathbf{x}$. Then $L = 1$, $\frac{d\mu_t}{d\nu}(\mathbf{x}) = t^{-n}\chi_{\mathbb{R}^n \setminus B(tR)}(\mathbf{x})$ and

$$\begin{aligned}
v(\mathbf{x}) &= \int_1^\infty \left| B\left(\frac{1}{t}|\mathbf{x}|\right)\right|^{-1} t^{-n}\chi_{\mathbb{R}^n \setminus B(tR)}(\mathbf{x})\, \frac{dt}{t^2} = \frac{1}{|B(|\mathbf{x}|)|} \int_1^\infty \chi_{\mathbb{R}^n \setminus B(tR)}(\mathbf{x}) \frac{dt}{t^2} \\
&= \frac{1}{|B(|\mathbf{x}|)|} \int_1^{\frac{|\mathbf{x}|}{R}} \frac{dt}{t^2} = \frac{1}{|B(|\mathbf{x}|)|} \left(1 - \frac{R}{|\mathbf{x}|}\right),\ \mathbf{x} \in \mathbb{R}^n \setminus B(R),
\end{aligned}$$

because we know that for $\mathbf{x} \in \mathbb{R}^n \setminus B(tR)$ is $|\mathbf{x}| > tR \geq R$, i.e. $\frac{|\mathbf{x}|}{R} > t \geq 1$. If we replace the function f from (10.24) with $g\colon \mathbb{R}^n \setminus B(R) \to \mathbb{R}$, $g(\mathbf{x}) = \log\left(|B(|\mathbf{x}|)|f(\mathbf{x})\right)$, we get

$$\begin{aligned}
Ag(\mathbf{x}) &= \int_1^\infty g(t\mathbf{x}) \frac{dt}{t^2} = \int_1^\infty \log\left(t^n|B(|\mathbf{x}|)|f(t\mathbf{x})\right) \frac{dt}{t^2} \\
&= n \int_1^\infty \log t\, \frac{dt}{t^2} + \log|B(|\mathbf{x}|)| + \int_1^\infty \log f(t\mathbf{x}) \frac{dt}{t^2} \\
&= n + \log|B(|\mathbf{x}|)| + \int_1^\infty \log f(t\mathbf{x}) \frac{dt}{t^2}
\end{aligned}$$

and

$$\begin{aligned}
\Phi(Ag(\mathbf{x})) &= \exp\left(n + \log|B(|\mathbf{x}|)| + \int_1^\infty \log f(t\mathbf{x}) \frac{dt}{t^2}\right) \\
&= e^n|B(|\mathbf{x}|)|\tilde{G}f(\mathbf{x}),
\end{aligned}$$

while $\Phi(g(\mathbf{x})) = |B(|\mathbf{x}|)|f(\mathbf{x})$ remains as in (i). The first integral on the left-hand side in the inequality (10.24) becomes

$$\int_{\mathbb{R}^n\setminus B(R)} \frac{1}{|B(|\mathbf{x}|)|}\left(1-\frac{R}{|\mathbf{x}|}\right)|B(|\mathbf{x}|)|f(\mathbf{x})\,d\mathbf{x} = \int_{\mathbb{R}^n\setminus B(R)}\left(1-\frac{R}{|\mathbf{x}|}\right)f(\mathbf{x})\,d\mathbf{x}, \tag{11.38}$$

for the second integral on the left-hand side in (10.24) we get

$$\int_{\mathbb{R}^n\setminus B(R)} \frac{1}{|B(|\mathbf{x}|)|}e^n\,|B(|\mathbf{x}|)|\tilde{G}f(\mathbf{x})d\mathbf{x} = e^n\int_{\mathbb{R}^n\setminus B(R)}\tilde{G}f(\mathbf{x})d\mathbf{x}, \tag{11.39}$$

and on the right-hand side of the inequality (10.24) we obtain

$$\begin{aligned}
&\Bigg|\int_{\mathbb{R}^n\setminus B(R)} \frac{1}{|B(|\mathbf{x}|)|}\int_1^\infty \left|t^n|B(|\mathbf{x}|)|f(t\mathbf{x}) - e^n|B(|\mathbf{x}|)|\tilde{G}f(\mathbf{x})\right|\frac{dt}{t^2}\,d\mathbf{x} \\
&\quad - \int_{\mathbb{R}^n\setminus B(R)} \frac{1}{|B(|\mathbf{x}|)|}\int_1^\infty e^n|B(|\mathbf{x}|)|\tilde{G}f(\mathbf{x}) \\
&\quad \cdot\left|\log\left(t^n|B(|\mathbf{x}|)|f(t\mathbf{x})\right) - n - \log|B(|\mathbf{x}|)| - \int_1^\infty \log f(t\mathbf{x})\frac{dt}{t^2}\right|\frac{dt}{t^2}\,d\mathbf{x}\Bigg| \\
&= \Bigg|\int_{\mathbb{R}^n\setminus B(R)}\int_1^\infty \left|t^n f(t\mathbf{x}) - e^n\tilde{G}f(\mathbf{x})\right|\frac{dt}{t^2}\,d\mathbf{x} - e^n\int_{\mathbb{R}^n\setminus B(R)}\tilde{G}f(\mathbf{x})\int_1^\infty \Big|n\log t \\
&\quad + \log|B(|\mathbf{x}|)| + \log f(t\mathbf{x}) - n - \log|B(|\mathbf{x}|)| - \int_1^\infty \log f(t\mathbf{x})\frac{dt}{t^2}\Big|\frac{dt}{t^2}\,d\mathbf{x}\Bigg| \\
&= \Bigg|\int_{\mathbb{R}^n\setminus B(R)}\int_1^\infty \left|t^n f(t\mathbf{x}) - e^n\tilde{G}f(\mathbf{x})\right|\frac{dt}{t^2}\,d\mathbf{x} \\
&\quad - e^n\int_{\mathbb{R}^n\setminus B(R)}\tilde{G}f(\mathbf{x})\int_1^\infty \left|\log\frac{(et)^n f(t\mathbf{x})}{\tilde{G}f(\mathbf{x})}\right|\frac{dt}{t^2}\,d\mathbf{x}\Bigg|.
\end{aligned} \tag{11.40}$$

Finally, by combining (11.38), (11.39) and (11.40) and by multiplying with e^{-n} we get (11.34). □

11.6 The best constant for the Pólya-Knopp-type inequality

As we have already discused the best constant for the Hardy-type inequality, here we analyse the best constant for the following strenghthened Pólya-Knopp-type inequality.

Theorem 11.5 *Let $n \in \mathbb{N}$.*

(i) If $0 < R \leq \infty$, f is a positive measurable function on $B(R)$, and Gf is defined by (11.31)*, then the inequality*

$$\int_{B(R)} Gf(\mathbf{x})\, d\mathbf{x} \leq e^n \int_{B(R)} \left(1 - \frac{|\mathbf{x}|}{R}\right) f(\mathbf{x})\, d\mathbf{x} \tag{11.41}$$

holds and the constant e^n is the best possible.

(ii) If $0 \leq R < \infty$, f is a positive measurable function on $\mathbb{R}^n \setminus B(R)$, and $\tilde{G}f$ is defined by (11.33)*, then the inequality*

$$\int_{\mathbb{R}^n \setminus B(R)} \tilde{G}f(\mathbf{x})\, d\mathbf{x} \leq e^{-n} \int_{\mathbb{R}^n \setminus B(R)} \left(1 - \frac{R}{|\mathbf{x}|}\right) f(\mathbf{x})\, d\mathbf{x} \tag{11.42}$$

holds and the constant e^{-n} is the best possible.

Proof. Since the right-hand sides of (11.32) and (11.34) are non-negative, inequalities (11.41) and (11.42) are their respective direct consequences. Now, we discuss the best possible constant for (11.41). For arbitrary $\varepsilon > 0$, let the function $f_\varepsilon : B(R) \to \mathbb{R}$ be defined by $f_\varepsilon(\mathbf{x}) = e^{-n}|B(|\mathbf{x}|)|^{\varepsilon-1}$. Calculating the left-hand side of (11.41) for f_ε, we obtain

$$\begin{aligned}
L_\varepsilon &= \int_{B(R)} \exp\left(\int_0^1 \log\left(e^{-n}|B(|t\mathbf{x}|)|^{\varepsilon-1}\right) dt\right) d\mathbf{x} \\
&= \int_{B(R)} \exp\left(-n + \int_0^1 (\varepsilon-1)(n\log t + \log|B(|\mathbf{x}|)|)\, dt\right) d\mathbf{x} \\
&= e^{-n} \int_{B(R)} \exp\left(n(\varepsilon-1)\int_0^1 \log t\, dt + (\varepsilon-1)\log|B(|\mathbf{x}|)|\right) d\mathbf{x} \\
&= e^{-n\varepsilon} \int_{B(R)} |B(|\mathbf{x}|)|^{\varepsilon-1}\, d\mathbf{x} = e^{-n\varepsilon} \int_{B(R)} \left(|S^{n-1}|\frac{|\mathbf{x}|^n}{n}\right)^{\varepsilon-1} d\mathbf{x} \\
&= e^{-n\varepsilon} \frac{1}{n^{\varepsilon-1}} |S^{n-1}|^{\varepsilon-1} \int_{S^{n-1}} dS \int_0^R r^{n-1+n(\varepsilon-1)}\, dr \\
&= e^{-n\varepsilon} \frac{|S^{n-1}|^{\varepsilon-1}}{n^{\varepsilon-1}} |S^{n-1}| \frac{R^{n\varepsilon}}{n\varepsilon} = e^{-n\varepsilon} \frac{|S^{n-1}|^\varepsilon R^{n\varepsilon}}{n^\varepsilon \varepsilon} = e^{-n\varepsilon} \frac{|B(R)|^\varepsilon}{\varepsilon}.
\end{aligned}$$

On the other hand, the right-hand side of (11.41), rewritten for f_ε, can be estimated as

$$\begin{aligned} R_\varepsilon &= e^n \cdot e^{-n} \int_{B(R)} \left(1 - \frac{|\mathbf{x}|}{R}\right) |B(|\mathbf{x}|)|^{\varepsilon-1}\, d\mathbf{x} \\ &\leq \int_{B(R)} |B(|\mathbf{x}|)|^{\varepsilon-1}\, d\mathbf{x} = \frac{|B(R)|^\varepsilon}{\varepsilon}. \end{aligned}$$

Since $1 \leq \dfrac{R_\varepsilon}{L_\varepsilon} \leq e^{n\varepsilon} \searrow 1$, as $\varepsilon \searrow 0$, the constant e^n is the best possible for the inequality (11.41). The proof that e^{-n} is the best possible constant for (11.42) is similar, if the function $f_\varepsilon \colon \mathbb{R}^n \setminus B(R) \to \mathbb{R}$, $f_\varepsilon = e^n |B(|\mathbf{x}|)|^{-\varepsilon-1}$ is considered. In that case, on the left-hand side we have

$$\begin{aligned} L_\varepsilon &= \int_{\mathbb{R}^n \setminus B(R)} \exp\left(\int_1^\infty \log\left(e^n |B(|t\mathbf{x}|)|^{-\varepsilon-1}\right) \frac{dt}{t^2} \right) d\mathbf{x} \\ &= \int_{\mathbb{R}^n \setminus B(R)} \exp\left(n + \int_1^\infty (-\varepsilon-1) \log |t^n B(|\mathbf{x}|)| \frac{dt}{t^2} \right) d\mathbf{x} \\ &= e^n \int_{\mathbb{R}^n \setminus B(R)} \exp\left(n(-\varepsilon-1) \int_1^\infty \log t \frac{dt}{t^2} + (-\varepsilon-1) \log |B(|\mathbf{x}|)| \right) d\mathbf{x} \\ &= e^{-n\varepsilon} \int_{\mathbb{R}^n \setminus B(R)} |B(|\mathbf{x}|)|^{-\varepsilon-1}\, d\mathbf{x} = e^{-n\varepsilon} \int_{\mathbb{R}^n \setminus B(R)} \left(|S^{n-1}| \frac{|\mathbf{x}|^n}{n} \right)^{-\varepsilon-1} \\ &= e^{-n\varepsilon} \frac{1}{n^{-\varepsilon-1}} |S^{n-1}|^{-\varepsilon-1} \int_{S^{n-1}} dS \int_R^\infty r^{n-1+n(-\varepsilon-1)}\, dr \\ &= e^{-n\varepsilon} \frac{1}{n^{-\varepsilon-1}} |S^{n-1}|^{-\varepsilon} \frac{R^{-n\varepsilon}}{n\varepsilon} = e^{-n\varepsilon} \frac{|B(R)|^{-\varepsilon}}{\varepsilon}, \end{aligned}$$

while the right-hand side can be estimated with

$$\begin{aligned} R_\varepsilon &= e^{-n} \cdot e^n \int_{\mathbb{R}^n \setminus B(R)} \left(1 - \frac{R}{|\mathbf{x}|}\right) |B(|\mathbf{x}|)|^{-\varepsilon-1}\, d\mathbf{x} \\ &\leq \int_{\mathbb{R}^n \setminus B(R)} |B(|\mathbf{x}|)|^{-\varepsilon-1} = \frac{|B(R)|^{-\varepsilon}}{\varepsilon}. \end{aligned}$$

Since $1 \leq \dfrac{R_\varepsilon}{L_\varepsilon} \leq e^{n\varepsilon} \searrow 1$, as $\varepsilon \searrow 0$, the constant e^{-n} is the best possible for (11.42). □

For $R = \infty$ in (11.41) and $R = 0$ in (11.42), we get a new multidimensional Pólya-Knopp-type inequality.

Corollary 11.4 *If f is a positive measurable function on $\mathbb{R}^n$, and Gf, $\tilde{G}f$ are respectively defined by* (11.31) *and* (11.33)*, then the inequalities*

$$\int_{\mathbb{R}^n} Gf(\mathbf{x})\, d\mathbf{x} \leq e^n \int_{\mathbb{R}^n} f(\mathbf{x})\, d\mathbf{x}$$

and

$$\int_{\mathbb{R}^n} \tilde{G}f(\mathbf{x})\, d\mathbf{x} \leq e^{-n} \int_{\mathbb{R}^n} f(\mathbf{x})\, d\mathbf{x}$$

hold. The constants e^n and e^{-n} are the best possible.

Remark 11.1 Notice that (11.41) and (11.42) respectively follow from (11.27) and (11.28) by rewriting those inequalities for $k = p > n$, and f replaced with $f^{\frac{1}{p}}$, and by letting $p \to \infty$. In particular, observe that $\lim\limits_{p\to\infty} \left(\dfrac{p}{p-n} \right)^p = e^n$.

Although being related to the setting with balls in $\mathbb{R}^n$ centered at the origin, inequalities (11.27), (11.28) and (11.41), (11.42) are not equivalent with the previously obtained Hardy and Pólya-Knopp-type inequalities (11.11), (11.12), (11.13), and (11.14). Therefore, our inequalities can be considered as a new class of generalizations of the classical Hardy's and Pólya-Knopp's inequality in a multidimensional setting. However, for $n = 1$ inequalities of both type coincide.

Chapter 12

The Boas functional and its properties

In this chapter we study various Boas-type inequalities. By exploring non-negativity of the difference between the right-hand side and the left-hand side in inequality (10.9), we introduce an isotonic linear functional, the so-called Boas functional, i.e.

$$\xi(f) = \frac{1}{L}\int_{\Omega} v(\mathbf{x})\Phi(f(\mathbf{x}))d\nu(\mathbf{x}) - \int_{\Omega} u(\mathbf{x})\Phi(Af(\mathbf{x}))d\mu(\mathbf{x}). \tag{12.1}$$

In a sequel we examine this functional of the Boas type. We present results from [32]. The aim is to explore its properties and a suitable choice of a convex function allow us to state and prove new mean value theorems of the Lagrange and Cauchy-type as well as define a new class of two-parameter Cauchy-type means. One of its properties is monotonicity, which can be raised on some higher level by using some specific class of convex functions, such as logarithmically and exponentially convex functions.

12.1 The Boas functional and exponential convexity

In the previous chapter we gave the examples based on applications of Theorem 10.2 to specific measures and sets. Here, we continue in that direction. With (6.1) we introduced functions φ_s, $s \in \mathbb{R}$. Since φ_s is a convex function for each $s \in \mathbb{R}$, we can apply Corollary 10.1 on each of them.

Corollary 12.1 *Let the conditions of Corollary* 10.1 *be fulfilled with a positive function* f *and let* φ_s *be defined by* (6.1). *Then*

$$\int_\Omega u(\mathbf{x})\varphi_s(Af(\mathbf{x}))\,d\mu(\mathbf{x}) \le \frac{1}{L}\int_\Omega v(\mathbf{x})\varphi_s(f(\mathbf{x}))\,d\nu(\mathbf{x}) \tag{12.2}$$

holds for all $s \in \mathbb{R}$.

Remark 12.1 *Observe that not the all functions* φ_s *are non-negative. Therefore, Corollary 10.1 does not assure inequality* (12.2) *to hold if there exists a set* $S \subseteq \operatorname{supp}\lambda$, $\lambda(S) > 0$, *such that* $t\Omega \subset \Omega, t \in S$.

Corollary 12.1 enables us to define the Boas difference, that is, the non-negative function $\xi : \mathbb{R} \to [0,\infty)$,

$$\xi(s) = \frac{1}{L}\int_\Omega v(\mathbf{x})\varphi_s(f(\mathbf{x}))\,d\nu(\mathbf{x}) - \int_\Omega u(\mathbf{x})\varphi_s(Af(\mathbf{x}))\,d\mu(\mathbf{x}). \tag{12.3}$$

In particular, under the conditions of Corollary 10.2, Corollary 10.3 and Corollary 10.4, we respectively define the following Boas differences:

$$\xi_1(s) = \frac{1}{L}\int_\Omega w(x)\varphi_s(f(x))\,\frac{dx}{x} - \int_\Omega u(x)\varphi_s(Af(x))\,\frac{dx}{x}, \tag{12.4}$$

$$\xi_2(s) = \int_0^b w(x)\varphi_s(f(x))\,\frac{dx}{x} - \int_0^b u(x)\varphi_s(Hf(x))\,\frac{dx}{x}, \tag{12.5}$$

$$\xi_3(s) = \int_b^\infty w(x)\varphi_s(f(x))\,\frac{dx}{x} - \int_b^\infty u(x)\varphi_s(\tilde{H}f(x))\,\frac{dx}{x}. \tag{12.6}$$

The same can be done also with Corollary 10.5 and Corollary 10.7, so we shall omit it.

Remark 12.2 For $u(x) \equiv 1$, in Corollary 10.3 we have $w(x) = x\int_x^b \frac{dt}{t^2} = 1 - \frac{x}{b}$, so (12.5) becomes

$$\xi_2(s) = \int_0^b \left(1 - \frac{x}{b}\right)\varphi_s(f(x))\,\frac{dx}{x} - \int_0^b \varphi_s(Hf(x))\,\frac{dx}{x}.$$

Inequality $\xi_2(s) \ge 0$ was obtained in [53].

Remark 12.3 For $u(x) \equiv 1$, in Corollary 10.4 we have $w(x) = \frac{1}{x}\int_b^x dt = 1 - \frac{b}{x}$, so (12.6) reduces to

$$\xi_3(s) = \int_b^\infty \left(1 - \frac{b}{x}\right)\varphi_s(f(x))\,\frac{dx}{x} - \int_b^\infty \varphi_s(\tilde{H}f(x))\,\frac{dx}{x}.$$

In that case, inequality $\xi_3(s) \ge 0$ was obtained in [53].

By using the definition and properties of an exponential convex function introduced in the first chapter, we obtain the following result. It is Lyapunov-type inequality related to the Boas differences (12.3).

Theorem 12.1 *Let the conditions of Corollary* 10.1 *be fulfilled with a positive function* f *and let* φ_s *be defined by* (6.1). *Then the function* $\xi : \mathbb{R} \to [0,\infty)$ *defined by* (12.3) *is continuous, exponentially convex and the inequality*

$$[\xi(r)]^{q-p} \leq [\xi(p)]^{q-r} \cdot [\xi(q)]^{r-p} \tag{12.7}$$

holds for all $p,q,r \in \mathbb{R}$*, such that* $p < r < q$.

Proof. First, we prove that ξ is continuous on $\mathbb{R}$. Since the mapping $s \mapsto \dfrac{x^s}{s(s-1)}$ is continuous on $\mathbb{R} \setminus \{0,1\}$ for all $x \in \mathbb{R}_+$, we only need to prove the continuity of ξ in $s = 0$ and $s = 1$. Since under the assumptions of Corollary 10.1 we have

$$\frac{1}{L}\int_\Omega v(\mathbf{x})\,d\nu(\mathbf{x}) - \int_\Omega u(\mathbf{x})\,d\mu(\mathbf{x}) = 0, \tag{12.8}$$

the L'Hospital rule [98] implies

$$\begin{aligned}
\lim_{s\to 0}\xi(s) &= \lim_{s\to 0}\left[\frac{1}{L}\int_\Omega v(\mathbf{x})\frac{f^s(\mathbf{x})}{s(s-1)}\,d\nu(\mathbf{x}) - \int_\Omega u(\mathbf{x})\frac{(Af(\mathbf{x}))^s}{s(s-1)}\,d\mu(\mathbf{x})\right]\\
&= \lim_{s\to 0}\frac{1}{s(s-1)}\left[\frac{1}{L}\int_\Omega v(\mathbf{x})f^s(\mathbf{x})\,d\nu(\mathbf{x}) - \int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^s\,d\mu(\mathbf{x})\right]\\
&= \lim_{s\to 0}\frac{1}{2s-1}\left[\frac{1}{L}\int_\Omega v(\mathbf{x})f^s(\mathbf{x})\log f(\mathbf{x})d\nu(\mathbf{x})\right.\\
&\qquad \left. - \int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^s\log\, Af(\mathbf{x})d\mu(\mathbf{x})\right]\\
&= -\frac{1}{L}\int_\Omega v(\mathbf{x})\log f(\mathbf{x})\,d\nu(\mathbf{x}) + \int_\Omega u(\mathbf{x})\log Af(\mathbf{x})\,d\mu(\mathbf{x})\\
&= \frac{1}{L}\int_\Omega v(\mathbf{x})\varphi_0(f(\mathbf{x}))\,d\nu(\mathbf{x}) - \int_\Omega u(\mathbf{x})\varphi_0(Af(\mathbf{x}))\,d\mu(\mathbf{x}) = \xi(0).
\end{aligned}$$

Similary, for $s = 1$, the identity

$$\frac{1}{L}\int_\Omega v(\mathbf{x})f(\mathbf{x})\,d\nu(\mathbf{x}) - \int_\Omega u(\mathbf{x})Af(\mathbf{x})\,d\mu(\mathbf{x}) = 0 \tag{12.9}$$

yields

$$\begin{aligned}
\lim_{s\to 1}\xi(s) &= \frac{1}{L}\int_\Omega v(\mathbf{x})f(\mathbf{x})\log f(\mathbf{x})\,d\nu(\mathbf{x}) - \int_\Omega u(\mathbf{x})Af(\mathbf{x})\log Af(\mathbf{x})\,d\mu(\mathbf{x})\\
&= \xi(1),
\end{aligned}$$

so ξ is continuous on the entire real line. To prove that it is exponentially convex, it suffices to check condition (1.7). Fix $k \in \mathbb{N}$, $\alpha_i \in \mathbb{R}$, and $s_i \in \mathbb{R}$, for $i \in \{1,\ldots,k\}$. Denote $s_{ij} = \dfrac{s_i + s_j}{2}$ and define the function $\Phi\colon \mathbb{R}_+ \to \mathbb{R}$ by $\Phi(x) = \sum_{i=1}^{k}\sum_{j=1}^{k}\alpha_i\alpha_j\varphi_{s_{ij}}(x)$. By using

Lemma 7.1, we easily get

$$\Phi''(x) = \sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i\alpha_j x^{s_{ij}-2} = \left(\sum_{i=1}^{k} \alpha_i x^{\frac{s_i}{2}-1}\right)^2 \geq 0, x \in \mathbb{R}_+,$$

so the function Φ is convex. Thus, applying Corollary 10.1 to this function Φ, we finally get

$$\begin{aligned}
&\sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i\alpha_j \xi(s_{ij}) \\
&= \sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i\alpha_j \left[\frac{1}{L}\int_\Omega v(\mathbf{x})\varphi_{s_{ij}}(f(\mathbf{x}))\,d\nu(\mathbf{x}) - \int_\Omega u(\mathbf{x})\varphi_{s_{ij}}(Af(\mathbf{x}))\,d\mu(\mathbf{x})\right] \\
&= \frac{1}{L}\int_\Omega v(\mathbf{x}) \sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i\alpha_j \varphi_{s_{ij}}(f(\mathbf{x}))\,d\nu(\mathbf{x}) \\
&\quad - \int_\Omega u(\mathbf{x}) \sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i\alpha_j \varphi_{s_{ij}}(Af(\mathbf{x}))\,d\mu(\mathbf{x}) \\
&= \frac{1}{L}\int_\Omega v(\mathbf{x})\Phi(f(\mathbf{x}))\,d\nu(\mathbf{x}) - \int_\Omega u(\mathbf{x})\Phi(Af(\mathbf{x}))\,d\mu(\mathbf{x}) \geq 0.
\end{aligned}$$

Therefore, (1.7) holds and ξ is exponentially convex. Since every exponentially convex function is log-convex, (12.7) follows directly from (1.4). □

Remark 12.4 Theorem 12.1 does not hold without assuming that $t\Omega = \Omega$ for λ-a.e. $t \in \operatorname{supp} \lambda$. This condition was crucial in proving identities (12.8) and (12.9).

As a direct consequence of Theorem 12.1 we get an upper bound for the Boas difference ξ.

Corollary 12.2 *Let the conditions of Theorem* 12.1 *be fulfilled. Then*

$$\xi(r) \leq [\xi(p)]^{\frac{q-r}{q-p}} \cdot [\xi(q)]^{\frac{r-p}{q-p}} \tag{12.10}$$

holds for all $p,q,r \in \mathbb{R}$, such that $p < r < q$.

Remark 12.5 Relation (12.10) can be written as

$$\xi(r) \leq \inf_{\substack{p,q\in\mathbb{R} \\ p<r<q}} [\xi(p)]^{\frac{q-r}{q-p}} \cdot [\xi(q)]^{\frac{r-p}{q-p}}, \ r \in \mathbb{R}.$$

As a consequence of Theorem 12.1, we get the following modified Galvani's theorem generated by the Boas difference ξ.

Corollary 12.3 *Under the conditions of Theorem* 12.1*, the inequality*

$$\left(\frac{\xi(p)}{\xi(r)}\right)^{\frac{1}{p-r}} \leq \left(\frac{\xi(t)}{\xi(s)}\right)^{\frac{1}{t-s}}. \tag{12.11}$$

holds for all $p,r,s,t \in \mathbb{R}$, such that $r \leq s$, $p \leq t$, $r \neq p$, and $s \neq t$.

Proof. Since the function ξ is exponentially convex, thus log-convex, inequality (12.11) follows from (1.5). $\square$

Remark 12.6 The results obtained in Theorem 12.1, Corollary 12.2 and Corollary 12.3 can be rewritten with ξ_i, $i = 1,2,3$, defined by (12.4), (12.5) and (12.6), respectively.

12.2 Mean value theorems related to the Boas functional

Notice that each side of relation (12.11) has a form of a mean, while (12.11) as a whole looks like an inequality between two means of the same type. Here, we justify this conjecture by proving that the expressions mentioned above are means of the Cauchy type. For more information about means and their inequalities see e.g. [16].

Let the measures λ, μ, ν, the number L, the λ-balanced set Ω, the interval I, and the functions u, v, f, and Af be as in Theorem 10.2 and Corollary 10.1. First, we define the linear functional $F\colon C^2(I) \to \mathbb{R}$ by

$$F(h) = \frac{1}{L}\int_{\Omega} v(\mathbf{x})h(f(\mathbf{x}))\,d\nu(\mathbf{x}) - \int_{\Omega} u(\mathbf{x})h(Af(\mathbf{x}))\,d\mu(\mathbf{x}). \tag{12.12}$$

Its properties will enable us to introduce a new class of the Cauchy-type means related to the Boas difference (12.3).

Observe that $F(\varphi_p) = \xi(p)$, $p \in \mathbb{R}$, where the functions φ_p are defined by (6.1) and ξ denotes the Boas difference introduced by (12.3). Hence, F can be considered as a generalized Boas difference. Moreover, according to Theorem 12.1, the mapping $p \mapsto F(\varphi_p)$ is continuous on $\mathbb{R}$.

Next, we have to adjust some known mean value theorems to our context. The first result in this direction is the following Lagrange-type mean value theorem.

Theorem 12.2 *Under the conditions of Corollary* 10.1*, suppose that I is a compact interval in $\mathbb{R}$. If $h \in C^2(I)$, then there exists $c \in I$ such that the identity*

$$F(h) = h''(c) \cdot F(\varphi_2) \tag{12.13}$$

holds, where F is defined by (12.12) *and $\varphi_2\colon I \to \mathbb{R}$, $\varphi_2(x) = \dfrac{x^2}{2}$.*

Proof. Since h'' is continuous on the compact set I, there exist $m = \min_{x\in I} h''(x)$ and $M = \max_{x\in I} h''(x)$. Define $h_m, h_M : I \to \mathbb{R}$ by

$$h_m(x) = h(x) - \frac{m}{2}x^2 = h(x) - m\varphi_2(x),$$

$$h_M(x) = \frac{M}{2}x^2 - h(x) = M\varphi_2(x) - h(x).$$

Since $h_m, h_M \in C^2(I)$ and $h_m''(x) = h''(x) - m \geq 0$, $h_M''(x) = M - h''(x) \geq 0$, for all $x \in I$, we conclude that h_m and h_M are convex functions on I. Therefore, applying Corollary 10.1 to these functions as Φ, we get $F(h_m) \geq 0$ and $F(h_M) \geq 0$. Obviously, $F(h_m) = F(h) - mF(\varphi_2)$ and $F(h_M) = MF(\varphi_2) - F(h)$, so therefrom we obtain

$$mF(\varphi_2) \leq F(h) \leq MF(\varphi_2). \tag{12.14}$$

Notice that function φ_2 is convex, so $F(\varphi_2) \geq 0$ holds by Corollary 10.1. In particular, if $F(\varphi_2) = 0$, then from (12.14) we get $F(h) = 0$, so (12.13) holds for all $c \in I$. On the other hand, if $F(\varphi_2) > 0$, then (12.14) yields $m \leq \dfrac{F(h)}{F(\varphi_2)} \leq M$. Since h'' takes all values from $[m, M]$, there exists $c \in I$ such that

$$h''(c) = \frac{F(h)}{F(\varphi_2)},$$

so the proof is completed. □

Now, we state and prove a new Cauchy-type mean value theorem.

Theorem 12.3 *Let I be a compact interval in $\mathbb{R}$ and $\varphi_2 : I \to \mathbb{R}$ be defined by $\varphi_2(x) = \dfrac{x^2}{2}$. Under the conditions of Corollary* 10.1*, let F be defined by* (12.12) *and let $F(\varphi_2) > 0$. If the functions $h_1, h_2 \in C^2(I)$ are such that $F(h_1), F(h_2) \neq 0$, and $h_2''(x) \neq 0$, for all $x \in I$, then there exists $c \in I$ such that*

$$\frac{h_1''(c)}{h_2''(c)} = \frac{F(h_1)}{F(h_2)}. \tag{12.15}$$

Proof. Define the function $h_0 = F(h_2)h_1 - F(h_1)h_2$. Then $h_0 \in C^2(I)$ and we have $F(h_0) = F(h_2)F(h_1) - F(h_1)F(h_2) = 0$. On the other hand, from Theorem 12.2 we know that there exists $c \in I$ such that $F(h_0) = h_0''(c)F(\varphi_2)$. Since $F(\varphi_2) \neq 0$, we get $h_0''(c) = 0$, that is, $F(h_2)h_1''(c) = F(h_1)h_2''(c)$, which is equivalent to (12.15). □

A special case of Theorem 12.3 related to power functions defined on a compact interval $I \subseteq \mathbb{R}_+$ will be of our special interest. Namely, let $h_1, h_2 : I \to \mathbb{R}$ be defined by $h_1(x) = x^p$ and $h_2(x) = x^r$, where $p, r \in \mathbb{R} \setminus \{0, 1\}$, $p \neq r$. Then $h_1(x) = p(p-1)\varphi_p(x)$, $h_2(x) = r(r-1)\varphi_r(x)$, $h_1''(x) = p(p-1)x^{p-2}$ and $h_2''(x) = r(r-1)x^{r-2}$, where φ_p and φ_r are given by (6.1). Hence, we obtain the following result.

Corollary 12.4 *Let the conditions of Corollary* 10.1 *be fulfilled with a positive function f with values in a compact interval $I \subseteq \mathbb{R}_+$ and let $F(\varphi_s) > 0$, $s \in \mathbb{R} \setminus \{0,1\}$, where φ_s and F are defined by* (6.1) *and* (12.12)*, respectively. Then*

$$\left(\frac{F(\varphi_p)}{F(\varphi_r)}\right)^{\frac{1}{p-r}} \in I, \tag{12.16}$$

for all $p, r \in \mathbb{R}$, $(p-r)p(p-1)r(r-1) \neq 0$.

Proof. Fix $p, r \in \mathbb{R}$, such that $(p-r)p(p-1)r(r-1) \neq 0$. Observe that the power functions h_1 and h_2 defined before the statement of Corollary 12.4 fulfill the conditions of Theorem 12.3. Hence, there exists $c \in I$ such that

$$\frac{p(p-1)c^{p-2}}{r(r-1)c^{r-2}} = \frac{F(h_1)}{F(h_2)}. \tag{12.17}$$

According to the definition (6.1) of the functions φ_s, $s \in \mathbb{R}$, identity (12.17) reads

$$c^{p-r} = \frac{F(\varphi_p)}{F(\varphi_r)},$$

so we get (12.16). □

12.3 Cauchy-type means generated by the Boas functional

Notice that expression (12.16) can be written in the form

$$\left(\frac{F(\varphi_p)}{F(\varphi_r)}\right)^{\frac{1}{p-r}} = \left(\frac{\xi(p)}{\xi(r)}\right)^{\frac{1}{p-r}}$$

according to the definition (12.3) of the Boas difference ξ. As announced, under the conditions of Corollary 12.4, we introduce a new two-variable function M with values in I, defined by

$$M(p,r) = \left(\frac{\xi(p)}{\xi(r)}\right)^{\frac{1}{p-r}}, \quad p, r \in \mathbb{R} \setminus \{0,1\}, \, p \neq r. \tag{12.18}$$

Evidently, M is symmetric, that is, $M(p,r) = M(r,p)$ holds for all $p, r \in \mathbb{R} \setminus \{0,1\}$, $p \neq r$. Moreover, by Theorem 12.1, M is also continuous in both arguments.

Now, we would like to extend this function to $\mathbb{R}^2$. Fix $r \in \mathbb{R} \setminus \{0,1\}$. Applying continuity of the mapping ξ on $\mathbb{R}$, we obtain

$$\lim_{p \to 0} M(r,p) = \lim_{p \to 0} M(p,r) = \lim_{p \to 0} \exp\left(\frac{(\log \xi(p) - \log \xi(r))}{p-r}\right)$$

$$
\begin{aligned}
&= \exp\left(\lim_{p\to 0}\frac{\log\xi(p)-\log\xi(r)}{p-r}\right) = \exp\left(\frac{\log\xi(r)-\log\xi(0)}{r}\right)\\
&= \left(\frac{\xi(r)}{\xi(0)}\right)^{\frac{1}{r}}
\end{aligned}
$$

and, analogously,

$$
\begin{aligned}
\lim_{p\to 1} M(r,p) &= \lim_{p\to 1} M(p,r) = \exp\left(\lim_{p\to 1}\frac{\log\xi(p)-\log\xi(r)}{p-r}\right)\\
&= \exp\left(\frac{\log\xi(1)-\log\xi(r)}{1-r}\right) = \left(\frac{\xi(1)}{\xi(r)}\right)^{\frac{1}{1-r}}.
\end{aligned}
$$

Thus, in order to keep continuity of M, we define

$$
M(0,r) = M(r,0) = \left(\frac{\xi(r)}{\xi(0)}\right)^{\frac{1}{r}} \text{ and } M(1,r) = M(r,1) = \left(\frac{\xi(r)}{\xi(1)}\right)^{\frac{1}{r-1}}, \tag{12.19}
$$

$r\in\mathbb{R}\setminus\{0,1\}$, as in formula (12.18).

Observe that ξ is derivable for $r\in\mathbb{R}\setminus\{0,1\}$ and

$$
\begin{aligned}
\xi'(r) &= \frac{1}{r(r-1)}\left[(1-2r)\xi(r)+\frac{1}{L}\int_\Omega v(\mathbf{x})f^r(\mathbf{x})\log f(\mathbf{x})\,d\nu(\mathbf{x})\right.\\
&\qquad \left.-\int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^r\log Af(\mathbf{x})\,d\mu(\mathbf{x})\right].
\end{aligned}
$$

Therefore, applying L'Hospital rule [98], for $r\in\mathbb{R}\setminus\{0,1\}$ we have

$$
\begin{aligned}
&\lim_{p\to r} M(p,r) = \lim_{p\to r} M(r,p) = \exp\left(\lim_{p\to r}\frac{\log\xi(p)-\log\xi(r)}{p-r}\right)\\
&= \exp\left\{\frac{1}{r(r-1)}\left[1-2r+\frac{1}{\xi(r)}\left(\frac{1}{L}\int_\Omega v(\mathbf{x})f^r(\mathbf{x})\log f(\mathbf{x})d\nu(\mathbf{x})\right.\right.\right.\\
&\qquad \left.\left.\left.-\int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^r\log Af(\mathbf{x})\,d\mu(\mathbf{x})\right)\right]\right\},
\end{aligned}
\tag{12.20}
$$

which enables us to set $M(r,r)=\lim_{p\to r} M(p,r),\ r\in\mathbb{R}\setminus\{0,1\}$. Finally, to define $M(0,0)$ and $M(1,1)$, notice that for $p\in\{0,1\}$ we get

$$
\begin{aligned}
\lim_{r\to p}\xi'(r) &= \lim_{r\to p}\frac{1}{2r-1}\left[-2\xi(r)+(1-2r)\xi'(r)\right.\\
&\quad +\frac{1}{L}\int_\Omega v(\mathbf{x})f^r(\mathbf{x})\log^2 f(\mathbf{x})\,d\nu(\mathbf{x})\\
&\quad \left.-\int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^r\log^2 Af(\mathbf{x})\,d\mu(\mathbf{x})\right]\\
&= 2(-1)^p\xi(p)-\lim_{r\to p}\xi'(r)+(-1)^p\left[\int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^p\log^2 Af(\mathbf{x})\,d\mu(\mathbf{x})\right.
\end{aligned}
$$

$$-\frac{1}{L}\int_{\Omega} v(\mathbf{x}) f^p(\mathbf{x}) \log^2 f(\mathbf{x})\, d\nu(\mathbf{x})\bigg],$$

so

$$\begin{aligned}
\xi'(p) &= \lim_{r\to p} \xi'(r) \\
&= (-1)^p \xi(p) + \frac{(-1)^p}{2}\left[\int_{\Omega} u(\mathbf{x})(Af(\mathbf{x}))^p \log^2 Af(\mathbf{x})\, d\mu(\mathbf{x})\right. \\
&\quad \left. -\frac{1}{L}\int_{\Omega} v(\mathbf{x}) f^p(\mathbf{x}) \log^2 f(\mathbf{x})\, d\nu(\mathbf{x})\right],\ p \in \{0,1\}.
\end{aligned}$$

Hence, we set

$$\begin{aligned}
M(0,0) &= \lim_{r\to 0} M(r,r) = \lim_{r\to 0} M(r,0) = \exp\left(\lim_{r\to 0} \frac{\log\xi(r) - \log\xi(0)}{r}\right) \\
&= \exp\frac{\xi'(0)}{\xi(0)} = \exp\left\{1 + \frac{1}{2\xi(0)}\left[\int_{\Omega} u(\mathbf{x}) \log^2 Af(\mathbf{x})\, d\mu(\mathbf{x})\right.\right. \\
&\quad \left.\left. -\frac{1}{L}\int_{\Omega} v(\mathbf{x}) \log^2 f(\mathbf{x})\, d\nu(\mathbf{x})\right]\right\}
\end{aligned} \tag{12.21}$$

and

$$\begin{aligned}
M(1,1) &= \lim_{r\to 1} M(r,r) = \lim_{r\to 1} M(r,1) = \exp\left(\lim_{r\to 1} \frac{\log\xi(r) - \log\xi(1)}{r-1}\right) \\
&= \exp\frac{\xi'(1)}{\xi(1)} = \exp\left\{-1 + \frac{1}{2\xi(1)}\left[\frac{1}{L}\int_{\Omega} v(\mathbf{x}) f(\mathbf{x}) \log^2 f(\mathbf{x})\, d\nu(\mathbf{x})\right.\right. \\
&\quad \left.\left. -\int_{\Omega} u(\mathbf{x}) Af(\mathbf{x}) \log^2 Af(\mathbf{x})\, d\mu(\mathbf{x})\right]\right\}.
\end{aligned} \tag{12.22}$$

By the above construction, we have obviously defined a continuous function $M\colon \mathbb{R}^2 \to \mathbb{R}$, with values in the compact interval I. Considering its other properties, in fact, we obtained a new class of two-parametic means of the Cauchy-type. Namely, the following theorem holds.

Theorem 12.4 *Under the conditions of Corollary* 12.4*, let the function* $M\colon \mathbb{R}^2 \to \mathbb{R}$ *be defined by relations* (12.18) - (12.22)*. Then M is a continuous and symmetric function with values in the compact interval I, such that the inequality*

$$M(p,r) \le M(q,s) \tag{12.23}$$

holds for all $p,q,r,s \in \mathbb{R},\ p \le q, r \le s.$

Proof. Taking into account the previous construction and analysis, it is only left to prove the monotonicity property (12.23) of M. However, it follows immediately from Corollary 12.3 and from continuity of M. □

Remark 12.7 Further analysis could go in direction for n-exponentially convex ξ. Since the results and technique are analogues, here we shall omit it.

We could further generalize a Hardy-type inequality to the class of arbitrary non-negative functions bounded from below and above with a convex function multiplied with positive real constants. This would enable us to obtain new generalizations of the classical integral Hardy, Hardy-Hilbert, Hardy- Littlewood-Polya, and Polya-Knopp inequalities as well as of Godunovas and of some recently obtained inequalities in multidimensional settings. Also, we could apply a similar idea to functions bounded from below and above with a superquadratic function. This can be found in [6].

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