MONOGRAPHS IN INEQUALITIES 7

Steffensen's and Related Inequalities

A Comprehensive Survey and Recent Advances

Josip Pečarić, Ksenija Smoljak and Sanja Varošanec

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Preface

The aim of this book is to present a comprehensive overview of results related to the famous Steffensen's inequality

$$\int_{b-\lambda}^{b} f(t)dt \leq \int_{a}^{b} f(t)g(t)dt \leq \int_{a}^{a+\lambda} f(t)dt,$$

where f and g are integrable functions defined on (a,b), f is nonincreasing, $0 \leq g \leq 1$ and $\lambda = \int_a^b g(t)dt$.

The aforementioned Steffensen's inequality has a corresponding version for sums instead of integrals, i.e. we have the discrete Steffensen's inequality

$$\sum_{n=y-s+1}^{y} f(n) \leq \sum_{n=x}^{y} f(n)\phi(n) \leq \sum_{n=x}^{x+s-1} f(n)$$

where $0 \leq \phi \leq 1$, f is nonincreasing and $s = \sum_{n=x}^{y} \phi(n)$.

Since its appearance in 1918 Steffensen's inequality has been a subject of investigation by many mathematicians. The book is devoted to generalizations and refinements of Steffensen's inequality and its connection to other inequalities, such as Gauss', Jensen-Steffensen's, Hölder's and Iyengar's inequality.

We start with different proofs, simple modifications and variants of Steffensen's inequality from the beginning of its investigation. Furthermore, we give a survey of weaker conditions on functions f and g and conditions for the inverse Steffensen's inequality. The book also contains L^p generalizations, generalizations for convex functions, refinements and sharpened versions, multidimensional generalizations and measure spaces generalizations of Steffensen's inequality. Further, an integration over two intervals with overleaping and with two different weights also give new results of a Steffensen-type.

Estimating the difference between two weighted integral means, obtained by using weighted Montgomery identity, Taylor's formula and interpolating polynomials, we give different generalizations of Steffensen's inequality. Using fractional integrals, such as Riemann-Liouville's, Hadamard's and Erdély-Kober's, and fractional derivatives, such as Riemann-Liouville's, Caputo's and Canavati's, we obtain various Steffensen-type inequalities. We use Lagrange-type and Cauchy-type mean value theorems related to some generalizations and in this way we define new classes of two-parametric means of a Cauchy-type.

We conclude the book with applications of Steffensen's inequality related to Hölder's and Iyengar's inequality, with a section on $a(x)-$monotonic functions and a short survey of other applications in statistics, functional equations, time scales and special functions.

Contents

Chapter 1

Basic results and definitions

1.1 Convex functions

In this section we give definitions and some properties of convex functions. Convex functions are very important in the theory of inequalities. The third chapter of the classical book by Hardy, Littlewood and Pólya [60] is devoted to the theory of convex functions (see also [97]).

Definition 1.1 *Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is called* convex *if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \tag{1.1}$$

for all points $x, y \in I$ and all $\lambda \in [0,1]$. It is called strictly convex *if the inequality in (1.1) holds strictly whenever x and y are distinct points and $\lambda \in (0,1)$.*

If the inequality in (1.1) is reversed, then f is said to be concave*. It is called* strictly concave *if the reversed inequality in (1.1) holds strictly whenever x and y are distinct points and $\lambda \in (0,1)$.*

If f is both convex and concave, f is said to be affine.

Remark 1.1 (a) For $x, y \in I$, $p, q \geq 0$, $p+q > 0$, (1.1) is equivalent to

$$f\left(\frac{px+qy}{p+q}\right) \leq \frac{pf(x)+qf(y)}{p+q}.$$

(b) A simple geometrical interpretation of (1.1) is that the graph of f lies below its chords.

(c) If x_1, x_2, x_3 are three points in I such that $x_1 < x_2 < x_3$, then (1.1) is equivalent to

$$\begin{vmatrix} x_1 & f(x_1) & 1 \\ x_2 & f(x_2) & 1 \\ x_3 & f(x_3) & 1 \end{vmatrix} = (x_3 - x_2)f(x_1) + (x_1 - x_3)f(x_2) + (x_2 - x_1)f(x_3) \geq 0$$

which is equivalent to

$$f(x_2) \leq \frac{x_2 - x_3}{x_1 - x_3} f(x_1) + \frac{x_1 - x_2}{x_1 - x_3} f(x_3),$$

or, more symmetrically and without the condition of monotonicity on x_1, x_2, x_3, to

$$\frac{f(x_1)}{(x_1 - x_2)(x_1 - x_3)} + \frac{f(x_2)}{(x_2 - x_3)(x_2 - x_1)} + \frac{f(x_3)}{(x_3 - x_1)(x_3 - x_2)} \geq 0.$$

Proposition 1.1 *If f is a convex function on I and if $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(y_2) - f(y_1)}{y_2 - y_1}.$$

If the function f is concave, the inequality is reversed.

Definition 1.2 *Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is called* convex in the Jensen sense*, or* J-convex on I (midconvex, midpoint convex) *if for all points $x, y \in I$ the inequality*

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2} \tag{1.2}$$

holds. A J-convex function is said to be strictly J-convex *if for all pairs of points $(x,y), x \neq y$, strict inequality holds in (1.2).*

In the context of continuity the following criterion of equivalence of (1.1) and (1.2) is valid.

Theorem 1.1 *Let $f : I \to \mathbb{R}$ be a continuous function. Then f is a convex function if and only if f is a J-convex function.*

Definition 1.3 *Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is called* Wright convex *function if for each $x \leq y$, $z \geq 0$, $x, y + z \in I$, the inequality*

$$f(x+z) - f(x) \leq f(y+z) - f(y)$$

holds.

Next, we want to define convex functions of higher order, but first we need to define divided differences.

Definition 1.4 *Let f be a function defined on $[a,b]$. The n-th* order divided difference *of f at distinct points $x_0, x_1, \ldots, x_n$ in $[a,b]$ is defined recursively by*

$$[x_j; f] = f(x_j), \quad j = 0, \ldots, n$$

and

$$[x_0, x_1, \ldots, x_n; f] = \frac{[x_1, \ldots, x_n; f] - [x_0, \ldots, x_{n-1}; f]}{x_n - x_0}. \tag{1.3}$$

Remark 1.2 The value $[x_0, x_1, \ldots, x_n; f]$ is independent of the order of the points $x_0, \ldots, x_n$. The previous definition can be extended to include the case in which some or all of the points coincide by assuming that $x_0 \leq \cdots \leq x_k$ and letting

$$\underbrace{[x, \ldots, x}_{j+1 \text{ times}}; f] = \frac{f^{(j)}(x)}{j!},$$

provided that $f^{(j)}(x)$ exists. Note that (1.3) is equivalent to

$$[x_0, \ldots, x_n; f] = \sum_{k=0}^{n} \frac{f(x_k)}{\omega'(x_k)}, \text{ where } \omega'(x_k) = \prod_{\substack{j=0 \\ j \neq k}}^{n} (x_k - x_j).$$

Definition 1.5 *Let $n \in \mathbb{N}_0$. A function $f : [a,b] \to \mathbb{R}$ is said to be n-*convex *on $[a,b]$ if and only if for every choice of $n+1$ distinct points $x_0, x_1, \ldots, x_n$ in $[a,b]$*

$$[x_0, x_1, \ldots, x_n; f] \geq 0. \tag{1.4}$$

*If the inequality in (1.4) is reversed, the function f is said to be n-*concave *on $[a,b]$. If the inequality is strict, f is said to be a* strictly $n-$convex ($n-$concave) *function.*

Remark 1.3 Particularly, $0-$convex functions are nonnegative functions, $1-$convex functions are nondecreasing functions, $2-$convex functions are convex functions.

Theorem 1.2 *If $f^{(n)}$ exists, then f is $n-$convex if and only if $f^{(n)} \geq 0$.*

Definition 1.6 *A positive function f is said to be* logarithmically convex *on an interval $I \subseteq \mathbb{R}$ if $\log f$ is a convex function on I, or equivalently if for all $x, y \in I$ and all $\alpha \in [0,1]$*

$$f(\alpha x + (1-\alpha)y) \leq f^{\alpha}(x) f^{1-\alpha}(y). \tag{1.5}$$

For such a function f, we shortly say that f is log-convex.
It is said to be log-concave *if the inequality in (1.5) is reversed.*

Definition 1.7 *A positive function f is said to be* log-convex in the Jensen sense *if for all $x, y \in I$*

$$f^2\left(\frac{x+y}{2}\right) \leq f(x) f(y)$$

holds, i.e. if $\log f$ is convex in the Jensen sense.

As a consequence of results from Remark 1.1 (c) and Proposition 1.1 we get the following inequality for a log-convex function f and $a < b < c$:

$$[f(b)]^{c-a} \leq [f(a)]^{c-b}\,[f(c)]^{b-a}. \tag{1.6}$$

Corollary 1.1 *For a log-convex function f on an interval I and $p,q,r,s \in I$ such that $p \leq r$, $q \leq s$, $p \neq q$, $r \neq s$, it holds*

$$\left(\frac{f(p)}{f(q)}\right)^{\frac{1}{p-q}} \leq \left(\frac{f(r)}{f(s)}\right)^{\frac{1}{r-s}}. \tag{1.7}$$

Inequality (1.7) is known as Galvani's theorem for log-convex functions $f : I \to \mathbb{R}$.

1.2 Exponentially convex functions

In this section we introduce the definition of exponential convexity as given by Bernstein in [27] (see also [13], [93], [94]). Throughout this section I is an open interval in $\mathbb{R}$.

Definition 1.8 *A function $h : I \to \mathbb{R}$ is said to be* exponentially convex *on I if it is continuous and*

$$\sum_{i,j=1}^{n} \xi_i \xi_j h(x_i + x_j) \geq 0$$

for every $n \in \mathbb{N}$ and all sequences $(\xi_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ of real numbers, such that $x_i + x_j \in I$, $1 \leq i,j \leq n$.

The following Proposition follows directly from the previous definition.

Proposition 1.2 *For a function $h : I \to \mathbb{R}$ the following statements are equivalent:*

(i) h is exponentially convex

(ii) h is continuous and

$$\sum_{i,j=1}^{n} \xi_i \xi_j h\left(\frac{x_i + x_j}{2}\right) \geq 0, \tag{1.8}$$

for all $n \in \mathbb{N}$, all sequences $(\xi_n)_{n\in\mathbb{N}}$ of real numbers, and all sequences $(x_n)_{n\in\mathbb{N}}$ in I.

Note that for $n = 1$, it follows from (1.8) that an exponentially convex function is nonnegative.

Directly from the definition of positive semi-definite matrix and inequality (1.8) we get the following result.

Corollary 1.2 *If h is exponentially convex on I, then the matrix*

$$\left[h\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{n}$$

is a positive semi-definite matrix. In particular,

$$\det\left[h\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{n} \geq 0, \tag{1.9}$$

for every $n \in \mathbb{N}$ and every choice of $x_i \in I,\ i = 1,\ldots,n$.

Remark 1.4 Note that for $n = 2$ from (1.9) we obtain

$$h(x_1)h(x_2) - h^2\left(\frac{x_1+x_2}{2}\right) \geq 0.$$

Hence, an exponentially convex function is log-convex in the Jensen sense, and, being continuous, it is also a log-convex function.

We continue with the definition of n-exponentially convex functions.

Definition 1.9 *A function $h : I \to \mathbb{R}$ is n-*exponentially convex in the Jensen sense *on I if*

$$\sum_{i,j=1}^{n} \xi_i \xi_j h\left(\frac{x_i+x_j}{2}\right) \geq 0$$

for all choices of $\xi_i \in \mathbb{R}$ and $x_i \in I,\ i = 1,\ldots,n$.

A function $h : I \to \mathbb{R}$ is *n-exponentially convex* on I if it is n-exponentially convex in the Jensen sense and continuous on I.

It is clear from the definition that 1-exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \in \mathbb{N},\ k \leq n$.

A function $h : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is $n-$exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

Remark 1.5 It is known that $h : I \to \mathbb{R}$ is log-convex in the Jensen sense if and only if for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$

$$\alpha^2 h(x) + 2\alpha\beta h\left(\frac{x+y}{2}\right) + \beta^2 h(y) \geq 0.$$

It follows that a positive function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense. Similarly, a positive function is log-convex if and only if it is 2-exponentially convex.

1.3 The Gamma function and the Gauss hypergeometric function

In this section we give definitions and basic properties of the Gamma function and the Gauss hypergeometric function. More details about these functions can be found e.g. in [73].

The Gamma function $\Gamma(z)$ is a function of complex variable defined by the Euler integral of the second kind

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re(z) > 0.$$

This integral is convergent for every $z \in \mathbb{C}$ with $\Re(z) > 0$. The Gamma function has a property

$$\Gamma(z+1) = z\Gamma(z), \quad \Re(z) > 0,$$

and a simple consequence of it is the following identity

$$\Gamma(n+1) = n!, \quad n \in \mathbb{N}_0.$$

Extension of the Gamma function to $\Re(z) \leq 0$ is given by

$$\Gamma(z) = \frac{\Gamma(z+n)}{(z)_n}, \quad \Re(z) > -n; \quad n \in \mathbb{N}; \quad z \notin \mathbb{Z}_0^- = \{0,-1,-2,\ldots\},$$

where $(z)_n$ is the Pochhammer symbol defined for $z \in \mathbb{C}$ and $n \in \mathbb{N}_0$ by

$$(z)_0 = 1; \quad (z)_n = z(z+1)\cdots(z+n-1), n \in \mathbb{N}. \tag{1.10}$$

The Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ is defined as the sum of the hypergeometric series

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \tag{1.11}$$

where $|z| < 1$; $a,b \in \mathbb{C}$, $c \in \mathbb{C} \setminus \mathbb{Z}_0^-$. The series in (1.11) is absolutely convergent for $|z| < 1$ and for $|z| = 1$, when $\Re(c-a-b) > 0$.

The Euler integral representation of the hypergeometric function is given by

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt$$

where $0 < \Re(b) < \Re(c)$; and $|\arg(1-z)| < \pi$.

Basic properties of the Gauss hypergeometric function are:

$${}_2F_1(b,a;c;z) = {}_2F_1(a,b;c;z),$$

$$ {}_2F_1(a,b;c;0) = {}_2F_1(0,b;c;z) = 1, $$

$$ {}_2F_1(a,b;b;z) = (1-z)^{-a}, $$

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \Re(c-a-b) > 0. $$

For the Gauss hypergeometric function, the following Euler transformation formula holds

$$ {}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\, {}_2F_1(c-a,c-b;c;z). $$

1.4 Fractional integrals and fractional derivatives

In this section we give definitions and properties of fractional integrals and fractional derivatives. More details can be found in [16], [59], [73] and [139].

First, we recall definitions and properties of integrable, continuous and absolutely continuous functions.

By $C^m[a,b]$, $m \in \mathbb{N}_0$, we denote the space of all functions which are m times continuously differentiable on $[a,b]$, i.e.

$$ C^m[a,b] = \{f : [a,b] \to \mathbb{R} : f^{(k)} \in C[a,b],\ k = 0,1,\ldots,m\}. $$

By $AC[a,b]$ we denote the space of all absolutely continuous functions on the finite interval $[a,b]$, i.e. $-\infty < a < b < \infty$. By $AC^m[a,b]$, $m \in \mathbb{N}$, we denote the space

$$ AC^m[a,b] = \{f \in C^{m-1}[a,b] : f^{(m-1)} \in AC[a,b]\}. $$

Obviously, $AC^1[a,b] = AC[a,b]$.

Let $[a,b]$ be an interval in $\mathbb{R}$, where $-\infty \le a < b \le \infty$. We denote by $L^p[a,b]$, $1 \le p < \infty$, the space of Lebesgue measurable functions f such that $\int_a^b |f(t)|^p dt < \infty$ with the norm

$$ \|f\|_p = \left(\int_a^b |f(t)|^p dt \right)^{\frac{1}{p}}, $$

and by $L^\infty[a,b]$ the space of all measurable and almost everywhere bounded functions on $[a,b]$, with the norm

$$ \|f\|_\infty = \operatorname{ess\,sup}\{|f(x)| : x \in [a,b]\}. $$

For any $\alpha \in \mathbb{R}$ we denote by $[\alpha]$ the integral part of α i.e. $[\alpha]$ is the integer k satisfying $k \le \alpha < k+1$.

The Riemann-Liouville fractional integral

Let $[a,b]$ be a finite interval in $\mathbb{R}$, i.e. $-\infty < a < b < \infty$. *The left-sided Riemann-Liouville fractional integral* $I^{\alpha}_{a+}f$ of order $\alpha > 0$ is defined by

$$I^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(y)(x-y)^{\alpha-1}dy, \quad x \in [a,b].$$

For $\alpha = n \in \mathbb{N}$ the definition of the left-sided Riemann-Liouville fractional integral coincides with the $n-$th integral of the form

$$I^{n}_{a+}f(x) = \int_a^x dy_1 \int_a^{y_1} dy_2 \cdots \int_a^{y_{n-1}} f(y_n)dy_n = \frac{1}{(n-1)!} \int_a^x (x-y)^{n-1}f(y)dy.$$

The generalized Riemann-Liouville fractional derivative

The left-sided generalized Riemann-Liouville fractional derivative $D^{\alpha}_{a+}f$ of order $\alpha > 0$ is defined by

$$D^{\alpha}_{a+}f(x) := \frac{d^n}{dx^n} I^{n-\alpha}_{a+}f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-y)^{n-\alpha-1}f(y)dy, x \in [a,b],$$

where $n = [\alpha]+1$.

For $\alpha = n \in \mathbb{N}$ we have $D^{n}_{a+}f(x) = f^{(n)}(x)$, while for $\alpha = 0$ we put $D^{0}_{a+}f(x) = f(x)$. Also, we use

$$I^{-\alpha}_{a+}f := D^{\alpha}_{a+}f \text{ if } \alpha > 0.$$

Definition 1.10 *Let $\alpha > 0$ and $1 \leq p \leq \infty$. By $I^{\alpha}_{a+}(L^p)$ we denote the following space of functions*

$$I^{\alpha}_{a+}(L^p) = \{f : f = I^{\alpha}_{a+}\varphi, \, \varphi \in L^p[a,b]\}.$$

A characterization of the space $I^{\alpha}_{a+}(L^1)$ is given in the following theorem.

Theorem 1.3 *Let $\alpha > 0$ and $n = [\alpha]+1$. A function f belongs to $I^{\alpha}_{a+}(L^1)$ if and only if*

$$I^{n-\alpha}_{a+}f \in AC^n[a,b],$$

$$\frac{d^j}{dx^j} I^{n-\alpha}_{a+}f(a) = 0, \quad j = 0,1,\dots,n-1.$$

Composition identity for the left-sided generalized Riemann-Liouville fractional derivative is given by Handley, Koliha and Pečarić in [59]. We use the following lemma which summarizes conditions in composition identity for the left-sided generalized Riemann-Liouville fractional derivatives given in [20].

Lemma 1.1 *Let $\beta > \alpha \geq 0$, $n = [\beta]+1$, $m = [\alpha]+1$. Composition identity*

$$D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\beta-\alpha)} \int_a^x (x-y)^{\beta-\alpha-1} D^{\beta}_{a+}f(y)\, dy, \quad x \in [a,b]$$

is valid if one of the following conditions holds:

(*i*) $f \in I_{a+}^{\beta}\left(L^1\right)$.

(*ii*) $I_{a+}^{n-\beta} f \in AC^n[a,b]$ *and* $D_{a+}^{\beta-k} f(a) = 0$ *for* $k = 1, \dots n$.

(*iii*) $D_{a+}^{\beta-1} f \in AC[a,b]$, $D_{a+}^{\beta-k} f \in C[a,b]$ *and* $D_{a+}^{\beta-k} f(a) = 0$ *for* $k = 1, \dots n$.

(*iv*) $f \in AC^n[a,b]$, $D_{a+}^{\beta} f, D_{a+}^{\alpha} f \in L^1[a,b]$, $\beta - \alpha \notin \mathbb{N}$, $D_{a+}^{\beta-k} f(a) = 0$ *for* $k = 1, \dots, n$ *and* $D_{a+}^{\alpha-k} f(a) = 0$ *for* $k = 1, \dots, m$.

(*v*) $f \in AC^n[a,b]$, $D_{a+}^{\beta} f, D_{a+}^{\alpha} f \in L^1[a,b]$, $\beta - \alpha = l \in \mathbb{N}$, $D_{a+}^{\beta-k} f(a) = 0$ *for* $k = 1, \dots, l$.

(*vi*) $f \in AC^n[a,b]$, $D_{a+}^{\beta} f, D_{a+}^{\alpha} f \in L^1[a,b]$ *and* $f^{(k)}(a) = 0$ *for* $k = 0, \dots, n-2$.

(*vii*) $f \in AC^n[a,b]$, $D_{a+}^{\beta} f, D_{a+}^{\alpha} f \in L^1[a,b]$, $\beta \notin \mathbb{N}$ *and* $D_{a+}^{\beta-1} f$ *is bounded in a neighborhood of* $t = a$.

The Caputo fractional derivative

The following type of fractional derivative which we use is the Caputo fractional derivative. We give the definition from [16].

Definition 1.11 *Let* $\alpha > 0$, $n = [\alpha] + 1$, $f \in AC^n[a,b]$. The Caputo fractional derivative $D_{*a}^{\alpha} f$ *is defined by*

$$D_{*a}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

for every $t \in [a,b]$.

For $\alpha = n \in \mathbb{N}$ we have $D_{*a}^{n} f(x) = f^{(n)}(x)$, while for $\alpha = 0$ we put $D_{*a}^{0} f(x) = f(x)$.

The Canavati fractional derivative

The definition of the Canavati fractional derivative is given in [16], but we use it with some new conditions given in [19].

Let $\alpha > 0$ and $n = [\alpha] + 1$. By $C_{a+}^{\alpha}[a,b]$ we denote a space defined by

$$C_{a+}^{\alpha}[a,b] = \{f \in C^{n-1}[a,b] : I_{a+}^{n-\alpha} f^{(n-1)} \in C^1[a,b]\}.$$

Definition 1.12 *Let* $\alpha > 0$, $n = [\alpha] + 1$. The left-sided Canavati fractional derivative *of* $f \in C_{a+}^{\alpha}[a,b]$, *denoted by* ${}^{C_1}D_{a+}^{\alpha} f$, *is defined by*

$${}^{C_1}D_{a+}^{\alpha} f(x) = \frac{d}{dx} I_{a+}^{n-\alpha} f^{(n-1)}(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{n-\alpha-1} f^{(n-1)}(t) dt.$$

For $\alpha = n \in \mathbb{N}$ we have ${}^{C_1}D_{a+}^{n} f(x) = f^{(n)}(x)$, while for $\alpha = 0$ we put ${}^{C_1}D_{a+}^{0} f(x) = f(x)$.

A theorem on composition identity for the left-sided Canavati fractional derivative is proved by Anastassiou in [16]. We use an improvement of that theorem with weaker conditions given in [19].

Lemma 1.2 *Let $\beta > \alpha \geq 0$, $n = [\beta]+1$, $m = [\alpha]+1$. Let $f \in C^{\beta}_{a+}[a,b]$ such that $f^{(i)}(a) = 0$, $i = m-1,\dots,n-2$. Then $f \in C^{\alpha}_{a+}[a,b]$ and*

$$^{C_1}D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\beta-\alpha)}\int_a^x (x-t)^{\beta-\alpha-1}\, {}^{C_1}D^{\beta}_{a+}f(t)dt, \quad x \in [a,b].$$

The fractional integral of a function f with respect to a given function g

Let $(a,b)\,(-\infty \leq a < b \leq \infty)$ be a finite or infinite interval in $\mathbb{R}$ and let $\alpha > 0$. Let g be an increasing function on (a,b) such that g' is continuous on (a,b). *The left-sided fractional integral of a function f with respect to a given function g* on $[a,b]$ is defined by

$$I^{\alpha}_{a+;g}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{g'(y)f(y)dy}{[g(x)-g(y)]^{1-\alpha}}, \quad x > a.$$

Remark 1.6 If $g(x) = x$, then $I^{\alpha}_{a+;x}f$ coincides with the left-sided Riemann-Liouville fractional integral $I^{\alpha}_{a+}f$.

The Hadamard fractional integral

Let (a,b) $(0 \leq a < b \leq \infty)$ be a finite or infinite interval in $\mathbb{R}^+$ and let $\alpha > 0$. *The left-sided Hadamard fractional integral* of order $\alpha > 0$ is defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\log\frac{x}{y}\right)^{\alpha-1}\frac{f(y)dy}{y}, \quad a < x < b.$$

Note that the left-sided Hadamard fractional integral of order α is a special case of the left-sided fractional integral of a function f with respect to the given function g, where $g(x) = \log x$ on $[a,b]$ where $0 < a < b \leq \infty$.

The Erdélyi-Kober fractional integral

Let $(a,b)\,(0 \leq a < b \leq \infty)$ be a finite or infinite interval in $\mathbb{R}^+$. Let $\alpha > 0, \sigma > 0$ and $\eta \in \mathbb{R}$. *The left-sided Erdélyi-Kober fractional integral* of order $\alpha > 0$ is defined by

$$I^{\alpha}_{a+;\sigma;\eta}f(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\int_a^x \frac{y^{\sigma\eta+\sigma-1}f(y)dy}{(x^{\sigma}-y^{\sigma})^{1-\alpha}}, \quad a < x < b.$$

The mixed Riemann-Liouville fractional integral

Multidimensional fractional integrals are natural generalizations of corresponding one-dimensional fractional integrals.
For $\mathbf{x} = (x_1,...,x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1,...,\alpha_n) \in \mathbb{R}^n$, we use the following notation:

$$\Gamma(\alpha) = \Gamma(\alpha_1)\cdot\ldots\cdot\Gamma(\alpha_n); \quad \mathbf{x}^{\alpha} = x_1^{\alpha_1}\ldots x_n^{\alpha_n}; \quad \frac{\partial}{\partial\mathbf{x}} = \frac{\partial}{\partial x_1}\cdots\frac{\partial}{\partial x_n};$$

$$[\mathbf{a},\mathbf{b}] = [a_1,b_1]\times\cdots\times[a_n,b_n]; \mathbf{a} = (a_1,\dots,a_n) \in \mathbb{R}^n, \mathbf{b} = (b_1,\dots,b_n) \in \mathbb{R}^n$$

and by $\mathbf{x} > \mathbf{a}$ we mean $x_1 > a_1, \ldots, x_n > a_n$.

The left-sided mixed Riemann-Liouville fractional integral of order $\alpha > 0$ is defined by

$$I^{\alpha}_{\mathbf{a}+} f(\mathbf{x}) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^{x_1} \cdots \int_{a_n}^{x_n} f(\mathbf{t})(\mathbf{x}-\mathbf{t})^{\alpha-\mathbf{1}} d\mathbf{t}, \quad (\mathbf{x} > \mathbf{a}).$$

Chapter 2

Steffensen's inequality

2.1 Introduction

There are many results related to Steffensen's inequality and it is still the subject of investigation by many mathematicians. This inequality was firstly given and proved by J. F. Steffensen in 1918 in paper [142]. However, Steffensen's inequality did not appear in the work Inequalities by Hardy, Littlewood and Pólya from 1934 (see [60]), which assembled almost all known inequalities of that time. Also, Steffensen's paper [142] was not reviewed in Jahrbuch über die Fortschritte der Mathematik.

The original version has the following form.

Theorem 2.1 *Suppose that f and g are integrable functions defined on (a,b), f is nonincreasing and for each $t \in (a,b)$ $0 \leq g(t) \leq 1$. Then*

$$\int_{b-\lambda}^{b} f(t)dt \leq \int_{a}^{b} f(t)g(t)dt \leq \int_{a}^{a+\lambda} f(t)dt, \tag{2.1}$$

where

$$\lambda = \int_{a}^{b} g(t)dt.$$

Proof. The proof of the second inequality in (2.1) goes as follows:

$$\begin{aligned}
\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt &= \int_a^{a+\lambda} [1-g(t)]f(t)dt - \int_{a+\lambda}^b f(t)g(t)dt \\
&\geq f(a+\lambda)\int_a^{a+\lambda} [1-g(t)]dt - \int_{a+\lambda}^b f(t)g(t)dt \\
&= f(a+\lambda)\left[\lambda - \int_a^{a+\lambda} g(t)dt\right] - \int_{a+\lambda}^b f(t)g(t)dt \\
&= f(a+\lambda)\int_{a+\lambda}^b g(t)dt - \int_{a+\lambda}^b f(t)g(t)dt \\
&= \int_{a+\lambda}^b g(t)[f(a+\lambda)-f(t)]dt \geq 0.
\end{aligned}$$

The first inequality in (2.1) is proved in a similar way, but it also follows from the second one. One merely sets $G(t) = 1-g(t)$ and $\Lambda = \int_a^b G(t)dt$. Since $0 \leq g(t) \leq 1$ on (a,b) it implies $0 \leq G(t) \leq 1$ on (a,b) and $b-a = \lambda + \Lambda$. Using the second inequality in (2.1) we obtain

$$\begin{aligned}
\int_a^b f(t)G(t)dt &\leq \int_a^{a+\Lambda} f(t)dt, \\
\int_a^b f(t)[1-g(t)]dt &\leq \int_a^{b-\lambda} f(t)dt, \\
\int_a^b f(t)dt - \int_a^{b-\lambda} f(t)dt &\leq \int_a^b f(t)g(t)dt.
\end{aligned}$$

Hence,

$$\int_{b-\lambda}^b f(t)dt \leq \int_a^b f(t)g(t)dt,$$

which is the first inequality in (2.1). □

A simple modification of the original Steffensen's inequality was given by Hayashi in [61]. In fact, using the substitution $g(t)/A$ for $g(t)$ in (2.1) we get the following statement which is the starting point for investigation of Iyengar inequalities which will be described in Chapter 9.

Theorem 2.2 *Let f and g be integrable functions defined on $[a,b]$ such that f is nonincreasing and for each $t \in [a,b]$ $0 \leq g(t) \leq A$ (A is a constant > 0). Then*

$$A\int_{b-\lambda}^b f(t)dt \leq \int_a^b f(t)g(t)dt \leq A\int_a^{a+\lambda} f(t)dt, \tag{2.2}$$

where

$$\lambda = \frac{1}{A}\int_a^b g(t)dt.$$

The following variant of Steffensen's inequality was proved by Apéry in the paper [21]. In the proof Apéry used an identity which gave a new approach to the proof and a further generalization of the original inequality.

Theorem 2.3 *Let f be nonincreasing on $(0,\infty)$ and let g be a measurable function on $[0,\infty)$ such that $0 \leq g \leq A$, (A is a constant $\neq 0$). Then*

$$\int_0^\infty f(t)g(t)dt \leq A\int_0^\lambda f(t)dt,$$

where

$$\lambda = \frac{1}{A}\int_0^\infty g(t)dt.$$

In the proof Apéry used the following identity

$$\int_0^\infty f(t)g(t)dt = A\int_0^\lambda f(t)dt - \int_0^\lambda [A-g(t)][f(t)-f(\lambda)]dt - \int_\lambda^\infty g(t)[f(\lambda)-f(t)]dt,$$

from which the statement of the theorem holds immediately.

Using the idea of Apéry's proof, Mitrinović stated in [91] that inequalities in (2.1) follow from the identities

$$\begin{aligned} &\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt \\ &= \int_a^{a+\lambda} [f(t)-f(a+\lambda)][1-g(t)]dt + \int_{a+\lambda}^b [f(a+\lambda)-f(t)]g(t)dt, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt \\ &= \int_a^{b-\lambda} [f(t)-f(b-\lambda)]g(t)dt + \int_{b-\lambda}^b [f(b-\lambda)-f(t)][1-g(t)]dt. \end{aligned}$$

In the same paper [91] Mitrinović gave Davies' proof of the second inequality in (2.1) which is based on the consideration of the function H defined by

$$H(x) = \int_a^{a+\lambda(x)} f(t)dt - \int_a^x f(t)g(t)dt,$$

where

$$\lambda(x) = \int_a^x g(t)dt.$$

The derivative of H is

$$H'(x) = f(a+\lambda(x))g(x) - f(x)g(x)$$

and it is nonnegative because from $0 \leq g \leq 1$ we obtain that $a+\lambda(x) \leq x$ and under the assumption that f is nonincreasing we have that $f(a+\lambda(x)) \geq f(x)$. Since function H has a zero for $x=a$ and its derivative is nonnegative we have that $H(x) \geq 0$ for all $x \in [a,b]$ and especially, $H(b) \geq 0$ which is the right-hand side of Steffensen's inequality.
This proof is valid for smooth functions, but it can be extended to other functions by an appropriate approximation.

By applying Steffensen's inequality to appropriate functions, in [85] Masjed-Jamei, Qi and Srivastava obtained the following Steffensen's type inequalities:

Theorem 2.4 *If f and g are integrable functions such that f is nonincreasing and*

$$-\frac{\sigma}{b-a}\left(1-\frac{1}{q}\right) \leq g(x) \leq 1-\frac{\sigma}{b-a}\left(1-\frac{1}{q}\right)$$

on (a,b), where $q \neq 0$ and

$$\sigma = q\int_a^b g(x)dx,$$

then

$$\begin{aligned} \int_{b-\sigma}^{b} f(x)dx - \frac{\sigma}{b-a}\left(1-\frac{1}{q}\right)\int_a^b f(x)dx &\leq \int_a^b f(x)g(x)dx \\ &\leq \int_a^{a+\sigma} f(x)dx - \frac{\sigma}{b-a}\left(1-\frac{1}{q}\right)\int_a^b f(x)dx. \end{aligned} \tag{2.4}$$

Proof. Let $p,q \in \mathbb{R}$ and let us consider the functions

$$F(x) = f(x) + p\int_a^b f(x)dx \quad \text{and} \quad G(x) = g(x) + \frac{q-1}{b-a}\int_a^b g(x)dx.$$

Since the function F is nonincreasing and $G(x) \in [0,1]$, Steffensen's inequalities for the functions F and G have the form (2.4). □

In this place we give another proof of inequality (2.1) requiring f to be nonnegative. This proof was given by Bellman in [25].

Suppose there exists no interval on which f vanishes and define a function u by

$$\int_a^{u(s)} f(t)dt = \int_a^s f(t)g(t)dt. \tag{2.5}$$

Then for $s, s+h \in [a,b], h > 0$

$$\begin{aligned} \int_{u(s)}^{u(s+h)} f(t)dt &= \int_a^{u(s+h)} f(t)dt - \int_a^{u(s)} f(t)dt \\ &= \int_a^{s+h} f(t)g(t)dt - \int_a^s f(t)g(t)dt = \int_s^{s+h} f(t)g(t)dt \geq 0, \end{aligned}$$

which means that $u(s+h) \geq u(s)$, i.e. function u is a nondecreasing function. Since $0 \leq g(t) \leq 1$ and $f(t) \geq 0$, $t \in [a,b]$ we have

$$0 \leq \int_a^s f(t)g(t)dt \leq \int_a^s f(t)dt, \quad a < s < b.$$

So,

$$\int_a^{u(s)} f(t)dt \leq \int_a^s f(t)dt$$

which means that $u(s) \leq s$. Now

$$|u(s+h)-u(s)|\cdot v = \left|\int_a^{u(s+h)} f(t)dt - \int_a^{u(s)} f(t)dt\right|$$
$$= \left|\int_{u(s)}^{u(s+h)} f(t)dt\right| = \left|\int_s^{s+h} f(t)g(t)dt\right| \leq |h| f(s),$$

where $\inf f(t) \leq v \leq \sup f(t)$ on $[s,s+h]$ if $h>0$ or $[s+h,s]$ if $h<0$. So, the function u is continuous. Differentiating equality (2.5) we obtain

$$f(u)\frac{du}{ds} = f(s)g(s) \quad (a.e.).$$

Using the assumption that f is nonincreasing and $u(s) \leq s$ we have that $\frac{du}{ds} = \frac{f(s)}{f(u)}g(s) \leq g(s)$. Therefore

$$\int_a^s du \leq \int_a^s g(t)dt \quad \text{i.e.} \quad u(s) \leq a + \int_a^s g(t)dt.$$

From this and from equality (2.5) there follows the second inequality in (2.1).

Marjanović gave a short proof of Steffensen's inequality using the following theorem given by Steffensen in his paper [144] from 1925.

Theorem 2.5 *Let g_1 and g_2 be functions defined on $[a,b]$ such that*

$$\int_a^x g_1(t)dt \geq \int_a^x g_2(t)dt$$

for all $x \in [a,b]$ and

$$\int_a^b g_1(t)dt = \int_a^b g_2(t)dt.$$

If f is a nondecreasing function on $[a,b]$, then

$$\int_a^b f(x)g_1(x)dx \leq \int_a^b f(x)g_2(x)dx. \tag{2.6}$$

If f is a nonincreasing function, the inequality in (2.6) is reversed.

Proof. Put $g(x) = g_1(x) - g_2(x)$ and $G(x) = \int_a^x g(t)dt$. Then, under the above hypothesis,

$$G(x) \geq 0 \ \ (a \leq x \leq b) \quad \text{and} \quad G(a) = G(b) = 0.$$

Applying integration by parts, we get

$$\int_a^b f(t)g(t)dt = \int_a^b f(t)dG(t) = f(t)G(t)|_a^b - \int_a^b G(t)df(t) = -\int_a^b G(t)df(t).$$

If f is nondecreasing, then $\int_a^b G(t)df(t) \geq 0$, so $\int_a^b f(t)g(t)dt \leq 0$ which has to be proved. □

Marjanović considered inequality (2.6) to give the following short proof of Steffensen's inequality (see [84]).

Let us define the functions g_1 and g_2 as

$$g_1(x) = \begin{cases} 1, & \text{for } x \in [a, a+\lambda) \\ 0, & \text{for } x \in [a+\lambda, b] \end{cases}$$

and $g_2(x) = g(x)$, where $\lambda = \int_a^b g(x)dx$. Using Theorem 2.5 for a nonincreasing function f we have

$$\int_a^{a+\lambda} f(x)dx = \int_a^b f(x)g_1(x)dx \geq \int_a^b f(x)g(x)dx,$$

which proves the second inequality in (2.1). The first inequality in (2.1) is derived in a similar way.

For the sake of completeness, let us mention that Rakić in [137] proved Steffensen's inequality using a proof which is directly connected to the definition of the integral.

2.2 Weaker conditions

As we already mentioned, identity (2.3) is a starting point for studying the conditions of Steffensen's inequality and eventually changing them. Namely, Milovanović and Pečarić in their paper [90], using integration by parts in identity (2.3), obtained weaker conditions on the function g. Vasić and Pečarić in paper [149] showed that these weaker conditions are necessary and sufficient. Hence, we have the following theorem.

Theorem 2.6 *Let f and g be integrable functions on $[a,b]$ and let $\lambda = \int_a^b g(t)dt$.*

a) The second inequality in (2.1) holds for every nonincreasing function f if and only if

$$\int_a^x g(t)dt \leq x-a \quad \text{and} \quad \int_x^b g(t)dt \geq 0 \quad \text{for every } x \in [a,b]. \tag{2.7}$$

b) The first inequality in (2.1) holds for every nonincreasing function f if and only if

$$\int_x^b g(t)dt \leq b-x \quad \text{and} \quad \int_a^x g(t)dt \geq 0 \quad \text{for every } x \in [a,b].$$

Proof.

a) Applying integration by parts, identity (2.3) becomes

$$\begin{aligned}\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt = &-\int_a^{a+\lambda} \left(\int_a^x (1-g(t))dt\right) df(x) \\ &- \int_{a+\lambda}^b \left(\int_x^b g(t)dt\right) df(x),\end{aligned} \tag{2.8}$$

from where we conclude that the condition $0 \le g(t) \le 1$ can be replaced by the weaker conditions

$$\int_a^x g(t)dt \le x - a \quad \text{for every } x \in [a, a+\lambda] \text{ and}$$
$$\int_x^b g(t)dt \ge 0 \quad \text{for every } x \in [a+\lambda, b]. \tag{2.9}$$

The previous conditions are also necessary. In fact, if x is any element of $[a,b]$, then let us define the function f as

$$f(t) = \begin{cases} 1, & t \le x \\ 0, & t > x. \end{cases}$$

Using the second inequality in (2.1) we obtain

$$\int_a^x g(t)dt = \int_a^b f(t)g(t)dt \le \int_a^{a+\lambda} f(t)dt = \begin{cases} x-a, & \text{for } x \in [a, a+\lambda] \\ \lambda, & \text{for } x \in (a+\lambda, b]. \end{cases} \tag{2.10}$$

It is obvious that for $x \in (a+\lambda, b]$ the inequality $x - a > \lambda$ holds. So, we conclude that

$$\int_a^x g(t)dt \le x - a \qquad \text{for every } x \in [a,b].$$

From the same result (2.10) we have that if $x \in (a+\lambda, b]$, then $\int_a^x g(t)dt \le \lambda$. On the other hand $\lambda = \int_a^b g(t)dt$, so we conclude that

$$\int_x^b g(t)dt \ge 0 \qquad \text{for } x \in (a+\lambda, b].$$

If $x \in [a, a+\lambda]$, then we have

$$\int_x^b g(t)dt = \int_a^b g(t)dt - \int_a^x g(t)dt = \lambda - \int_a^x g(t)dt \ge \lambda - (x-a) \ge 0.$$

So, $\int_x^b g(t)dt \ge 0$ for every $x \in [a,b]$. We get that if the second inequality in (2.1) holds for every nonincreasing function, then conditions in (2.7) hold.

b) This is proved similarly as in a). □

Remark 2.1 In his paper [31] Cao repeated these weaker conditions for Steffensen's inequality.

Previous results involve weakening of assumptions on function g, while the next results will point out that the assumption on the function f to be nonincreasing is a very strong condition. Pečarić and Varošanec in paper [129] proved the following result.

Theorem 2.7 *Let $f,g:[a,b]\to\mathbb{R}$ be integrable functions and $\lambda=\int_a^b g(t)dt$. If f satisfies*

$$(QD)\quad \begin{cases} f(t)\geq f(c) & \text{for } t\in[a,c] \\ f(d)\leq f(t)\leq f(c) & \text{for } t\in[c,d] \\ f(t)\leq f(d) & \text{for } t\in[d,b], \end{cases}$$

where $c=\min\{a+\lambda,b-\lambda\}$, $d=\max\{a+\lambda,b-\lambda\}$, $c,d\in[a,b]$, and if g satisfies

$$(QB)\quad \begin{cases} 0\leq g(t)\leq 1 & \text{for } t\in[a,c]\cup[d,b] \\ g(t)\geq 0 & \text{for } t\in[c,d] \text{ when } c=a+\lambda \\ g(t)\leq 1 & \text{for } t\in[c,d] \text{ when } c=b-\lambda, \end{cases}$$

then (2.1) is true.

It is obvious that if f is a nonincreasing function, then f satisfies condition (QD) and if $g(t)\in[0,1]$ for $t\in[a,b]$, then g has property (QB). Hence, the classical Steffensen inequality becomes a special case of Theorem 2.7. The proof of that theorem is also based on the Apéry identity.

In [90] Milovanović and Pečarić generalized Theorem 2.6 in the case when function f is convex of order n. They used the following result from [89]:

Theorem 2.8 *Let $x\mapsto f(x)$ be a convex function of order n $(n\geq 1)$ on $[a,b]$. Then, for every $c\in[a,b]$, the function $x\mapsto \frac{G(x)}{(x-c)^n}$ is nondecreasing on $[a,b]$, where*

$$G(x)=f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(c)}{k!}(x-c)^k,$$

with $f^{(k)}(c)$ being the right derivative for $c=a$ and the left derivative for $c=b$.

Generalizations for a convex function of order n are given in the following theorems.

Theorem 2.9 *Let functions f and g satisfy conditions:*

(1) *f is convex of order n $(n\in\mathbb{N})$;*

(2) $f^{(k)}(a)=0, \quad k=0,1,\ldots,n-1$;

(3) *for all $x\in[a,b]$*

$$\int_a^x (x-a)^n g(x)dx\leq\frac{(x-a)^{n+1}}{n+1} \text{ and } \int_x^b (x-a)^n g(x)dx\geq 0.$$

Then

$$\int_a^{a+\lambda_1} f(x)dx \leq \int_a^b f(x)g(x)dx,$$

where

$$\lambda_1 = \left[(n+1)\int_a^b (x-a)^n g(x)dx\right]^{\frac{1}{n+1}}. \tag{2.11}$$

Proof. Using Theorem 2.8 for $c = a$ and the assumption for a function f, we get that the function $x \mapsto \frac{f(x)}{(x-a)^n}$ is nondecreasing. Let us define the functions g_1 and g_2 as

$$g_1(x) = \begin{cases} 1, & x \in [a, a+\lambda_1] \\ 0, & x \in (a+\lambda_1, b] \end{cases}$$

and $g_2(x) = g(x)$, where λ_1 is given by (2.11). Then we get

$$\int_a^x (t-a)^n g_1(t)dt \geq \int_a^x (t-a)^n g_2(t)dt \quad (\forall x \in [a,b])$$

and

$$\int_a^b (t-a)^n g_1(t)dt = \int_a^b (t-a)^n g_2(t)dt.$$

Setting in Theorem 2.5:

$$g_1(x) \to (x-a)^n g_1(x),\ g_2(x) \to (x-a)^n g_2(x),\ f(x) \to \frac{f(x)}{(x-a)^n}$$

we obtain

$$\int_a^b f(x)g_1(x)dx \leq \int_a^b f(x)g_2(x)dx$$

and then, using the definition of function g_1, we get

$$\int_a^{a+\lambda_1} f(x)dx = \int_a^b f(x)g_1(x)dx \leq \int_a^b f(x)g(x)dx,$$

which proves the theorem. □

Theorem 2.10 *Let functions f and g satisfy conditions:*

(1) *f is convex of order n $(n \in \mathbb{N})$;*

(2) $f^{(k)}(b) = 0, \quad k = 0, 1, \ldots, n-1$;

(3) *for all $x \in [a,b]$*

$$\int_x^b (b-x)^n g(x)dx \leq \frac{(b-x)^{n+1}}{n+1} \quad \text{and} \quad \int_a^x (b-x)^n g(x)dx \geq 0.$$

If n is an even number, the inequality

$$\int_a^b f(x)g(x)dx \leq \int_{b-\lambda_2}^b f(x)dx$$

holds, where

$$\lambda_2 = \left[(n+1)\int_a^b (b-x)^n g(x)dx\right]^{\frac{1}{n+1}}.$$

If n is an odd number, the reverse inequality holds.

Proof. Similar to the proof of Theorem 2.9. □

Remark 2.2 If $0 \leq g \leq 1$, the condition (3) in Theorem 2.9 (and Theorem 2.10) is fulfilled.

2.3 Gauss inequality

In [55] Gauss mentioned the following inequality:

If f is a nonnegative nonincreasing function and $a > 0$, then provided the integrals involved exist,

$$\int_a^\infty f(x)dx \leq \frac{4}{9a^2}\int_0^\infty x^2 f(x)dx. \tag{2.12}$$

That result was generalized by Volkov (see [150], [151]).

Theorem 2.11 *Let f be a nonincreasing nonnegative function and g be a differentiable increasing function such that $g(x) \geq x$, $x \in (0,\infty)$. If the integral on the right-hand side of inequality (2.13) exists, then the integral on the left-hand side of inequality (2.13) exists, too, and the following inequality is valid:*

$$\int_{g(0)}^\infty f(x)dx \leq \int_0^\infty f(x)g'(x)dx. \tag{2.13}$$

Putting $g(x) = \frac{4x^3}{27a^2} + a$, $a > 0$, inequality (2.13) is reduced to Gauss' inequality (2.12).

Volkov also gave a multidimensional version in [151].

Theorem 2.12 *Let D be a starlike region and $(r,\varphi_1,\varphi_2,\ldots,\varphi_{n-1})$ a point from D. If*

(a) $f(r,\varphi_1,\varphi_2,\ldots,\varphi_{n-1}) \leq 0$, $(r, \operatorname{grad} f) \geq 0$,

(b) $g(r,\varphi_1,\varphi_2,\ldots,\varphi_{n-1}) \geq r$, $(r, \operatorname{grad} g) \geq 0$

then

$$\int_{\omega}^{g}\int_{g(0)}^{h} f(r,e)drd\omega \geq \int_{\omega}\int_{0}^{h} f(r,e)g_r'(r,e)drd\omega,$$

where ω *is an* $(n-1)$*-dimensional sphere in* $\mathbb{R}^n$, $e=(\varphi_1,\varphi_2,\ldots,\varphi_{n-1})\in\omega$ *and* $r=h(\varphi_1,\varphi_2,\ldots,\varphi_{n-1})$ *is the equation of the border of region D.*

In [130] Petschke proved the following result which includes a finite segment of integration.

Theorem 2.13 *Let* $f:[0,1]\to\mathbb{R}^+$ *be a nonincreasing function and* $\mu\in(0,1)$, *and* $\alpha,\beta\in[0,\infty)$, $\alpha\neq\beta$.

1) If $\beta>\alpha$ *and* $t_0=\left(\frac{\beta+1}{\beta-\alpha}\right)^{\frac{1}{\alpha+1}}\mu<1$, *then*

$$\mu^{\beta-\alpha}\int_{\mu}^{1} f(x)x^{\alpha}dx \leq \left(\frac{\beta-\alpha}{\beta+1}\right)^{\frac{\beta-\alpha}{\alpha+1}}\int_{0}^{1} f(x)x^{\beta}dx. \tag{2.14}$$

2) If $\beta\leq\alpha$ *and* $t_0\geq 1$, *then*

$$\frac{1}{1-\mu^{\alpha+1}}\int_{\mu}^{1} f(x)x^{\alpha}dx \leq \frac{\beta+1}{\alpha+1}\int_{0}^{1} f(x)x^{\beta}dx.$$

As a consequence of inequality (2.14) Petschke obtained the following theorem in [130].

Theorem 2.14 *Let* $f:[0,\infty)\to\mathbb{R}^+$ *be a nonincreasing function. Then for* $\lambda>0$ *and* $\beta>\alpha\geq 0$ *we have*

$$\lambda^{\beta-\alpha}\int_{\lambda}^{\infty} f(x)x^{\alpha}dx \leq \left(\frac{\beta-\alpha}{\beta+1}\right)^{\frac{\beta-\alpha}{\alpha+1}}\int_{0}^{\infty} f(x)x^{\beta}dx.$$

In fact, this inequality is a special case of Volkov's result (2.13) when

$$g(x)=\frac{1}{\lambda^{\beta-\alpha}(\beta-\alpha+1)}\left(\frac{\beta-\alpha}{\beta+1}\right)^{\frac{\beta-\alpha}{\alpha+1}}x^{\beta-\alpha+1}+\lambda.$$

In [113] Pečarić proved the following result.

Theorem 2.15 *Let* $G:[a,b]\to\mathbb{R}$ *be an increasing function and let* $f:I\to\mathbb{R}$ *be a nonincreasing function (I is an interval from* $\mathbb{R}$ *such that* $a,b,G(a),G(b)\in I$*). If* $G(x)\geq x$ *then*

$$\int_{G(a)}^{G(b)} f(x)dx \leq \int_{a}^{b} f(x)G'(x)dx. \tag{2.15}$$

If $G(x)\leq x$, *the reverse inequality in (2.15) is valid.*

Proof. Using the substitution $G(x) = z$ we get

$$\int_a^b f(x)G'(x)dx = \int_a^b f(x)dG(x) = \int_{G(a)}^{G(b)} f(G^{-1}(z))dz.$$

If $G(z) \geq z$, then $G^{-1}(z) \leq z$ and $f(G^{-1}(z)) \geq f(z)$. So we have

$$\int_{G(a)}^{G(b)} f(G^{-1}(z))dz \geq \int_{G(a)}^{G(b)} f(z)dz,$$

and (2.15) is valid. Of course, if $G(z) \leq z$, we get the reverse inequality. □

It is interesting that this inequality includes as special cases three famous inequalities which were obtained in independent ways: Volkov's, Steffensen's and Ostrowski's inequality. Volkov's inequality (2.13) was already mentioned as a generalization of Gauss' inequality (2.12).

Ostrowski's result has the following form (see [99]).

Let f be a nonincreasing function on $[0,a]$ and g be a nondecreasing continuous function with continuous derivative and $g(t) \leq t$ for $0 \leq t \leq a$ with $g(0) = 0$. Then

$$\int_0^a f(t)g'(t)dt \leq \int_0^{g(a)} f(t)dt. \tag{2.16}$$

It is obvious that (2.15) is a generalization of (2.13) and (2.16).

Now, we show that Steffensen's inequality follows from (2.15).

If we let $G(x) = a + \int_a^x g(t)dt$ in Theorem 2.15, where g is a nonnegative function, then in the case $G(x) \leq x$, i.e.,

$$\int_a^x g(t)dt \leq x - a,$$

we get the second inequality in (2.1). For the first inequality we let $G(x) = b - \int_x^b g(t)dt$ in Theorem 2.15, for $G(x) \geq x$, i.e.,

$$\int_x^b g(t)dt \leq b - x.$$

Hence, we get the first inequality in (2.1).

Note that here we used the weaker conditions for Steffensen's inequality given in Theorem 2.6.

In [15] Alzer gave a lower bound for Gauss' inequality (2.12). In fact, he proved the following theorem.

Theorem 2.16 *Let $g : [a,b] \to \mathbb{R}$ be increasing, convex and differentiable, and let $f : I \to \mathbb{R}$ be a onincreasing function. Then*

$$\int_a^b f(s(x))g'(x)dx \leq \int_{g(a)}^{g(b)} f(x)dx \leq \int_a^b f(t(x))g'(x)dx, \tag{2.17}$$

where

$$s(x) = \frac{g(b) - g(a)}{b - a}(x - a) + g(a) \tag{2.18}$$

and

$$t(x) = g'(x_0)(x - x_0) + g(x_0), \quad x_0 \in [a,b]. \tag{2.19}$$

(I is an interval containing $a, b, g(a), g(b), t(a)$ and $t(b)$.)

If either g is concave or f is nondecreasing, then the reversed inequalities hold.

Proof. Let g be convex and f be nonincreasing. Denote $h(x) = f(g(x))$. Then h is also nonincreasing. Since g is convex for all $x \in [a,b]$ we have

$$t(x) \le g(x) \le s(x).$$

This implies

$$g^{-1}(t(x)) \le x \le g^{-1}(s(x)) \quad \text{and} \quad h(g^{-1}(t(x))) \ge h(x) \ge h(g^{-1}(s(x))).$$

Since g is increasing we have

$$h(g^{-1}(t(x))) \cdot g'(x) \ge h(x) \cdot g'(x) \ge h(g^{-1}(s(x))) \cdot g'(x).$$

Hence,

$$\int_a^b h(g^{-1}(t(x))) \cdot g'(x)dx \ge \int_a^b h(x) \cdot g'(x)dx \ge \int_a^b h(g^{-1}(s(x))) \cdot g'(x)dx. \tag{2.20}$$

Now, from (2.20) and

$$\int_a^b h(x)g'(x)dx = \int_{g(a)}^{g(b)} h(g^{-1}(y))dy \quad \text{(with substitution } y = g(x))$$

we get (2.17). □

Setting in (2.17) $a = 0$, $b \ge x_0 = \frac{k}{\sqrt[3]{2}}$ and $g(x) = \frac{1}{k^2}x^3 + k$, and then letting b tend to ∞, we get Gauss' inequality for a nonincreasing f.
Under the additional assumption that f is nonnegative and $b = k$, Alzer obtained

$$3\int_0^k x^2 f(x+k)dx \le k^2 \int_k^\infty f(x)dx,$$

where the constant 3 cannot be replaced by a larger number.

2.4 Inverse Steffensen's inequality

In this section we give conditions for inverse inequalities in (2.1). Results given in this section were given by Pečarić in [109].

Theorem 2.17 *Let $f : I \to \mathbb{R}$, $g : [a,b] \to \mathbb{R}$ ($[a,b] \subset I$, I is an interval in $\mathbb{R}$) be integrable functions, $a+\lambda \in I$ where $\lambda = \int_a^b g(t)dt$. Then*

$$\int_a^{a+\lambda} f(t)dt \le \int_a^b f(t)g(t)dt \tag{2.21}$$

holds for every nonincreasing function f if and only if

$$\int_a^x g(t)dt \ge x-a \text{ for } x \in [a,a+\lambda] \text{ and } \int_x^b g(t)dt \le 0 \text{ for } x \in (a+\lambda,b] \tag{2.22}$$

and $0 \le \lambda \le b-a$; or

$$\int_a^x g(t)dt \ge x-a \quad \text{for } x \in [a,b]; \tag{2.23}$$

or

$$\int_x^b g(t)dt \le 0 \quad \text{for } x \in [a,b]. \tag{2.24}$$

Proof. For

$$f(t) = \begin{cases} 1, & t \le x \\ 0, & t > x \end{cases}$$

for all $x \in I$, we get from (2.21) that (2.22) or (2.23) or (2.24) must be satisfied.
Now, we show the other direction.
Let $0 \le \lambda \le b-a$. Then

$$\begin{aligned} &\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt = \int_a^{a+\lambda} (f(t)-f(a+\lambda))(1-g(t))dt \\ &+ \int_{a+\lambda}^b (f(a+\lambda)-f(t))g(t)dt = -\int_a^{a+\lambda} \left(\int_a^x (1-g(t))dt \right) df(x) \\ &- \int_{a+\lambda}^b \left(\int_x^b g(t)dt \right) df(x) \le 0. \end{aligned}$$

If $\lambda > b-a$, then

$$\begin{aligned} &\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt = \int_a^b f(t)(1-g(t))dt + \int_b^{a+\lambda} f(t)dt \\ &= \int_a^b (f(t)-f(b))(1-g(t))dt + \int_b^{a+\lambda} (a+\lambda-x)df(x) \\ &= -\int_a^b \left(\int_a^x (1-g(t))dt \right) df(x) + \int_b^{a+\lambda} (a+\lambda-x)df(x) \le 0. \end{aligned}$$

Now, let $\lambda < 0$. Then

$$\begin{aligned}\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt &= -\int_{a+\lambda}^a f(t)dt - \int_a^b f(t)g(t)dt \\ &= \int_a^b g(t)(f(a)-f(t))dt + \int_{a+\lambda}^b (x-a-\lambda)df(x) \\ &= -\int_a^b \left(\int_x^b g(t)dt\right) df(x) + \int_{a+\lambda}^a (x-a-\lambda)df(x) \le 0.\end{aligned}$$

□

Analoguously, in the same paper, the following theorem is given.

Theorem 2.18 *Let $f: I \to \mathbb{R}$, $g: [a,b] \to \mathbb{R}$ ($[a,b] \subset I$) be integrable functions, $b-\lambda \in I$ where $\lambda = \int_a^b g(t)dt$. Then*

$$\int_a^b f(t)g(t)dt \le \int_{b-\lambda}^b f(t)dt \tag{2.25}$$

holds for every nonincreasing function f if and only if

$$\int_a^x g(t)dt \le 0 \text{ for } x \in [a, b-\lambda] \text{ and } \int_x^b g(t)dt \ge b-x \text{ for } x \in (b-\lambda, b]$$

and $0 \le \lambda \le b-a$; or

$$\int_x^b g(t)dt \ge b-x \quad \text{for } x \in [a,b];$$

or

$$\int_a^x g(t)dt \le 0 \quad \text{for } x \in [a,b].$$

The following theorems are also proved in [109].

Theorem 2.19 *Let $g: [a,b] \to \mathbb{R}$ be an integrable function for which there exists $c \in [a,b]$ such that $g(x) \ge 1$ for $x \in [a,c]$ and $g(x) \le 0$ for $x \in (c,b]$. Then (2.21) holds for every nonincreasing function $f: I \to \mathbb{R}$ provided that $[a,b] \subset I$ and $a+\lambda \in I$.*

Proof. Let $0 \le \lambda \le b-a$. Suppose that $c \le a+\lambda$. Then it is obvious

$$\int_a^x g(t)dt \ge x-a \text{ for } x \in [a,c] \text{ and } \int_x^b g(t)dt \le 0 \text{ for } x \in [a+\lambda, b].$$

Suppose that for some $x_1 \in (c, a+\lambda)$ we have $\int_a^{x_1} g(t)dt < x_1 - a$. Since $\int_{x_1}^b g(t)dt \le 0$, we have $\int_a^b g(t)dt < x_1 - a$, i.e. $a+\lambda < x_1$, what is, evidentily, a contradiction. Analoguously, in the case $c > a+\lambda$ we can prove that (2.22) also holds.
Now let $\lambda > b-a$. Then $\int_a^x g(t)dt \ge x-a$ for $x \in [a,c]$ is obvious. For some $x \in (c,b]$ we have

$$\int_a^x g(t)dt = \int_a^b g(t)dt - \int_x^b g(t)dt \ge \int_a^b g(t)dt \ge b-a \ge x-a,$$

i.e., the condition (2.23) holds. Similarly, in the case $\lambda < 0$, we can prove that (2.24) holds. So, from Theorem 2.17 we obtain Theorem 2.19. □

Theorem 2.20 *Let $g : [a,b] \to \mathbb{R}$ be an integrable function for which there exists $c \in [a,b]$ such that $g(x) \leq 0$ for $x \in [a,c]$ and $g(x) \geq 1$ for $x \in (c,b]$. Then (2.25) holds for every nonincreasing function $f : I \to \mathbb{R}$ provided that $[a,b] \subset I$ and $b - \lambda \in I$.*

Theorem 2.21 *Let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $g(x) \geq 1$ (or $g(x) \leq 0$) for every $x \in [a,b]$. Then the reverse inequalities in (2.1) hold for every nonincreasing function $f : I \to \mathbb{R}$ provided that $a + \lambda, b - \lambda \in I$.*

Proof. This is a consequence of Theorems 2.19 and 2.20. □

2.5 Jensen-Steffensen's inequality

Jensen's inequality for convex functions is one of the most important inequalities in mathematics and statistics. Some well known inequalities can be obtained from it. For more details see e.g. [94] and [122].

Theorem 2.22 (JENSEN'S INEQUALITY) *If I is an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ is convex, $\mathbf{x} = (x_1, \dots, x_n) \in I^n$ $(n \geq 2)$, $\mathbf{p} = (p_1, \dots, p_n)$ is a positive $n-$tuple (i.e. $p_i > 0$) and $P_n = \sum_{i=1}^n p_i$, then*

$$f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^n p_i f(x_i). \tag{2.26}$$

If f is strictly convex, then (2.26) is strict unless $x_1 = \cdots = x_n$.

The integral version of Jensen's inequality is

$$\varphi\left(\frac{\int_a^b f(x)d\sigma(x)}{\int_a^b d\sigma(x)}\right) \leq \frac{\int_a^b \varphi(f(x))d\sigma(x)}{\int_a^b d\sigma(x)}$$

which holds for all convex φ and $f \in L^1(a,b)$ and a nonnegative measure σ.

Reasonable question is whether positivity of numbers p_i in Jensen's inequality can be relaxed at the expense of restricting $\mathbf{x} = (x_1, \dots, x_n)$. In 1919 Steffensen answered on this question with the following theorem (see [143]).

Theorem 2.23 (JENSEN-STEFFENSEN'S INEQUALITY) *If $f : I \to \mathbb{R}$ is a convex function, $\mathbf{x}$ is a real monotone $n-$tuple from I^n and $\mathbf{p}$ is a real $n-$tuple such that*

$$0 \leq P_k = \sum_{i=1}^k p_i \leq P_n, \quad (1 \leq k \leq n-1), \quad P_n > 0, \tag{2.27}$$

then (2.26) holds. If f is a strictly convex function, then inequality (2.26) is strict except when $x_1 = \cdots = x_n$.

This inequality is evidently more general than Jensen's inequality since numbers p_i need not necessarily be positive.

An integral analogue of Jensen-Steffensen's inequality is given in the following theorem.

Theorem 2.24 *If f is a convex function, g is a monotone function and p satisfies*

$$0 \le \int_a^x p(x)dx \le \int_a^b p(x)dx \quad (\forall x \in [a,b]), \quad \int_a^b p(x)dx > 0,$$

then

$$f\left(\frac{\int_a^b p(x)g(x)dx}{\int_a^b p(x)dx}\right) \le \frac{\int_a^b p(x)f(g(x))dx}{\int_a^b p(x)dx}. \tag{2.28}$$

In 1919 Steffensen derived Jensen-Steffensen's inequality using the second inequality in (2.1) (see [104] and [143]). In 1970 Bullen derived Steffensen's inequality using Jensen-Steffensen's inequality (see [30]). Therefore, Steffensen's and Jensen-Steffensen's inequality are equivalent.

In [104] Pečarić derived Jensen-Steffensen's inequality from Steffensen's inequality using the idea of the proof of Olkin's inequality (given in [95, p. 113] and [98]). Olkin's inequality is given in the following theorem.

Theorem 2.25 *Let $1 \ge h_1 \ge \cdots \ge h_n \ge 0$ and $a_1 \ge \cdots \ge a_n \ge 0$. Let f be a convex function on $[0,a_1]$. Then*

$$\left(1-\sum_{i=1}^n (-1)^{i-1}h_i\right)f(0)+\sum_{i=1}^n (-1)^{i-1}h_i f(a_i) \ge f\left(\sum_{i=1}^n (-1)^{i-1}h_i a_i\right).$$

The proof of Jensen-Steffensen's inequality using Steffensen's inequality given in [104] is the following.

Let $a_1 \ge \cdots \ge a_n$ and let f be a continuous convex function on $[a_n,a_1]$. Let

$$g(t) = g_k \quad \text{for} \quad a_{k+1} < t \le a_k \quad (k=1,\ldots,n-1)$$

where

$$g_k = \frac{\sum_{i=1}^k p_i}{\sum_{i=1}^n p_i} \quad (k=1,\ldots,n-1), \quad \sum_{i=1}^n p_i > 0.$$

Then

$$\lambda = \int_{a_n}^{a_1} g(t)dt = (a_1-a_2)g_1+\cdots+(a_{n-1}-a_n)g_{n-1} = \frac{\sum_{i=1}^n p_i a_i}{\sum_{i=1}^n p_i} - a_n.$$

From the convexity of f it follows that $x \mapsto -f'(x)$ is nonincreasing function. Applying the right-hand side of Steffensen's inequality to the function $-f'(x)$ we obtain

$$\int_{a_n}^{a_1} f'(t)g(t)dt \ge \int_{a_n}^{a_n+\lambda} f'(t)dt$$

thus we get

$$\sum_{k=1}^{n-1}(f(a_k)-f(a_{k+1}))g_k \geq f(a_n+\lambda)-f(a_n)$$

from which we obtain (2.26). From the condition $0 \leq g \leq 1$ we get (2.27) directly. Thus we proved that in this case (2.26) holds for a nonincreasing sequence. Having in mind that

$$0 \leq \sum_{i=k}^{n} p_i \leq \sum_{i=1}^{n} p_i \quad (k=1,\dots,n) \Leftrightarrow 0 \leq \sum_{i=1}^{k} p_i \leq \sum_{i=1}^{n} p_i \quad (k=1,\dots,n)$$

it can easily be shown that (2.26) also holds for a nondecreasing sequence.
Hence, Theorem 2.23 is proved for every continuous function convex on $[a_n,a_1]$. However, if a function is convex on $[a_n,a_1]$ it is continuous on (a_n,a_1) (see [95, p. 17]). Therefore, this proof is valid for the function

$$F(x)=\begin{cases} \lim\limits_{x\to a_n^+} f(x), & \text{for } x=a_n \\ f(x), & \text{for } x\in(a_n,a_1) \\ \lim\limits_{x\to a_1^-} f(x), & \text{for } x=a_1 \end{cases}$$

if the function f is convex. Then, it is obvious that

$$f(a_n) \geq F(a_n) \quad \text{and} \quad f(a_1) \geq F(a_1). \tag{2.29}$$

In [104] Pečarić also discussed the effect of the end-points on inequality (2.26). Discussion is made for the following cases

$$1)\, 0<\lambda<a_1-a_n, \quad 2)\, \lambda=0 \text{ or } \lambda=a_1-a_n.$$

Let us observe the first case, i.e. $0<\lambda<a_1-a_n$. Summing for the same points, without loss of generality, we can suppose $a_1>a_2>\cdots>a_n$. From (2.29) and conditions $p_1 \geq 0$, $p_n \geq 0$ we have

$$f\left(\frac{\sum_{i=1}^{n} p_i a_i}{\sum_{i=1}^{n} p_i}\right)=F\left(\frac{\sum_{i=1}^{n} p_i a_i}{\sum_{i=1}^{n} p_i}\right) \leq \frac{\sum_{i=1}^{n} p_i F(a_i)}{\sum_{i=1}^{n} p_i} \leq \frac{\sum_{i=1}^{n} p_i f(a_i)}{\sum_{i=1}^{n} p_i}.$$

Now, let us observe the second case. Firstly, let $\lambda=a_1-a_n$. From the proof it is obvious that it is valid when

$$\frac{\sum_{i=1}^{k} p_i}{\sum_{i=1}^{n} p_i}=1 \quad (k=1,\dots,n-1) \Rightarrow \sum_{i=k}^{n} p_i=0 \quad (k=2,\dots,n),$$

i.e. $p_1 \neq 0$, $p_2=\cdots=p_n=0$. Analogously, when $\lambda=0$ we get $p_n \neq 0$, $p_1=\cdots=p_{n-1}=0$. It can easily be shown that in this case the equality in (2.26) holds. Hence, the proof of Theorem 2.23 is complete.

Weaker conditions for Steffensen's inequality given by (2.7) can be used to extend the conditions for validing inequality (2.26). Hence, similar to the previous proof the following theorem can be proved (see [104]).

Theorem 2.26 *Let p_i $(i = 1,\dots,n)$ and $a_1 \geq \cdots \geq a_n$ be real numbers which satisfy*

$$\sum_{i=1}^{k} p_i(x-a_i) \geq 0, \qquad \sum_{i=k+1}^{n} p_i(x-a_i) \leq 0, \qquad \text{for all} \quad x \in (a_{k+1}, a_k] \tag{2.30}$$

for $k = 1,\dots,n-1$ and $\sum_{i=1}^{n} p_i > 0$. Then (2.26) holds for every convex function f.

Remark 2.3 It is easy to prove that (2.26) is valid if $a_1 \leq \cdots \leq a_n$ and the reverse inequalities in (2.30) hold.

The corresponding integral analogue is given in the following theorem (see [104]).

Theorem 2.27 *If the function p and monotone function g satisfy conditions*

$$0 \leq \int_a^x p(t)|g(x)-g(t)|dt \leq \int_a^b p(t)|g(x)-g(t)|dt, \quad \forall x \in [a,b],$$
$$\int_a^b p(t)dt > 0,$$

then for every convex function f inequality (2.28) holds.

Now we consider Bullen's proof of Steffensen's inequality using Jensen-Steffensen's inequality (see [30]). First we recall that if f is nonincreasing function and $F(x) = \int_a^x f(t)dt$, then F is concave. The following theorem needed in Bullen's proof is proved in [30].

Theorem 2.28 *If F is a continuous concave function and $x_1 \leq \cdots \leq x_n$ and if $p_1,\dots,p_n$ are real numbers satisfying (2.27), then the reverse inequality in (2.26) holds.*

The idea of Bullen's proof is to obtain Steffensen's inequality (2.1) for a function g in a certain class of step functions. Then it is done for a Riemann integrable function g and at the end for integrable function g.

Let f be nonincreasing and let $a = a_0 < a_1 < \cdots < a_n = b$ be a partition of $[a,b]$. Suppose that g is the step function

$$g(x) = c_k, \quad a_k \leq x < a_{k+1}, \quad k = 0,1,\dots,n-1$$

such that $0 \leq g \leq 1$. This implies that $0 \leq c_k \leq 1$, $k = 0,1,\dots,n-1$. Now the right-hand side inequality in (2.1) reduces to

$$F(a_0) + \sum_{k=0}^{n-1} c_k(F(a_{k+1}) - F(a_k)) \leq F\left(a_0 + \sum_{k=0}^{n-1} c_k(a_{k+1}-a_k)\right), \tag{2.31}$$

where $F(x) = \int_a^x f(t)dt$. Since f is nonincreasing and $F(x) = \int_a^x f(t)dt$, F is concave. Hence, (2.31) follows from Theorem 2.28. Similar arguments can be used for the left-hand side Steffensen's inequality.

To complete the proof Bullen considered a class of Riemann integrable functions. He supposed that $g = \lim g_k$, where $g_k, (k = 1, 2, \ldots)$, are step functions. Furthermore, Bullen extended this result to all integrable functions g to complete the proof.

In [30] Bullen noted that this procedure can be used to extend other results from sums to integrals. He mentioned that integral Rado and Popoviciu inequalities can be obtained in this manner.

In [29] Boas considered integral Jensen-Steffensen inequality. Boas gave proof of Jensen-Steffensen's inequality which begins with reproducing Zygmund's proof of Jensen's inequality and uses the second mean value theorem for Stieltjes integral (for details see [29]).

The following generalization of Jensen- Steffensen's inequality is given in [110]. It is a consequence of generalization of Steffensen's inequality which will be described in Section 3.2.

Theorem 2.29 *Let $f : [a,b] \to \mathbb{R}$ and $H : [0, b-a] \to \mathbb{R}$ be differentiable functions such that $x \mapsto f'(x)/H'(x-a)$ is a nondecreasing function, H is an increasing function and $H(0) = 0$. If $\mathbf{a}$ is a monotonous $n-$tuple and $\mathbf{p}$ is a real $n-$tuple such that*

$$0 \leq P_k \leq P_n, \quad P_n > 0 \quad \left(P_k = \sum_{i=1}^{k} p_i, \, k = 1, \ldots, n \right),$$

then

$$f\left(a + H^{-1}\left(\frac{1}{P_n} \sum_{i=1}^{n} p_i H(a_i - a) \right) \right) \leq \frac{1}{P_n} \sum_{i=1}^{n} p_i f(a_i). \tag{2.32}$$

If $x \mapsto f'(x)/H'(x-a)$ is a nonincreasing function, then the reverse inequality in (2.32) holds.

Proof. Let $\mathbf{a}$ be a nondecreasing $n-$tuple. By substitutions $f(t) \to f'(t)$, $h(t) \to H'(t-a)$ and $g(t) = g_i$ $(g_i = P_i/P_n)$ for $a_{i-1} < t \leq a_i$ $(a_0 = a)$, from Theorem 3.15 we obtain Theorem 2.29. $\square$

The following remarks are given in [110].

Remark 2.4 If f and H are twice differentiable functions, then the condition that $x \mapsto f'(x)/H'(x-a)$ is a nondecreasing function can be replaced by the condition

$$f''(x)H'(x-a) - f'(x)H''(x-a) \geq 0.$$

This result for $a = 0$ is given in [72].

Remark 2.5 Let f be a $(k+1)-$convex function such that $f^{(m)}(a) = 0$ $(m = 1, \ldots, k-1)$. Then $x \mapsto f'(x)/(x-a)^{k-1}$ is a nondecreasing function and (2.32) becomes

$$f\left(a + \left(\frac{1}{P_n} \sum_{i=1}^{n} p_i (a_i - a)^k \right)^{\frac{1}{k}} \right) \leq \frac{1}{P_n} \sum_{i=1}^{n} p_i f(a_i).$$

If f is a $(k+1)-$concave function, then the reverse inequality holds. This is also given in [106] and [148]. Proof given in [106] based on the generalization of Steffensen's inequality for convex functions given in Section 3.2 (Theorem 3.17).

Remark 2.6 The condition for f in Theorem 2.29 can be weakened i.e. we can only suppose that the function $x \mapsto f(a+H^{-1}(t))$ is convex (concave) on $[0, H(b-a)]$. This result, for $a=0$ and $H(x)=x^k$, is given in [72]. (See also [106]).

In [108] Pečarić gave necessary and sufficient conditions for inverse of Jensen-Steffensen's inequality (see also [109]).

Theorem 2.30 *Let* $\mathbf{x}$ *and* $\mathbf{p}$ *be two* $n-$*tuples of real numbers such that* $x_i \in I$ $(1 \le i \le n)$ *and* $P_n > 0$*. The reverse inequality in (2.26) holds for every convex function* $f : I \to \mathbb{R}$ *and for every monotonic* $n-$*tuple* $\mathbf{x}$ *if and only if there exists* $m \in \{1, \ldots, n\}$ *such that*

$$P_k \le 0\,(k < m), \quad \overline{P_k} \le 0\,(k > m),$$

where $\overline{P_k} = P_n - P_{k-1}$.

Proof. We recall the proof from [109] which shows that this theorem can be obtained from Theorem 2.19. Let $x_1 \ge \cdots \ge x_n$ and

$$g(t) = g_k,\, x_{k+1} < t \le x_k\,(1 \le k \le n-1),\, g_k = P_k/P_n.$$

Then

$$\lambda = \int_{x_n}^{x_1} g(t)dt = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i - x_n.$$

Since f can be approximated uniformly in $[a,b]$ by polynomials with a nonnegative second derivative there is no loss of generality in assuming that $f'(x)$ exists and it is nondecreasing, i.e. we have that $x \mapsto -f'(x)$ is a nonincreasing function, and then, from (2.21), we obtain (2.26) with the reverse inequality. □

2.6 Discrete Steffensen's inequality

In his paper [142] Steffensen gave a corresponding theorem for sums instead of integrals. The discrete Steffensen's inequality was also mentioned in Hayashi's paper [62].

Theorem 2.31 *Let* $0 \le \phi \le 1$ *and let* f *be a nonincreasing function. Then*

$$\sum_{n=y-s+1}^{y} f(n) \le \sum_{n=x}^{y} f(n)\phi(n) \le \sum_{n=x}^{x+s-1} f(n)$$

where $s = \sum_{n=x}^{y} \phi(n)$.

We have the following convention about sum when limits of summation are not integers. If $x = \alpha - \theta_1$, $y = \beta + \theta_2$, where α and β are integers, $0 \leq \theta_i < 1$, $i = 1,2$ and $y \geq x-1$, we use the convenient notation

$$\sum_{n=x}^{y} u(n) = \theta_1 u(\alpha-1) + \sum_{n=\alpha}^{\beta} u(n) + \theta_2 u(\beta+1), \tag{2.33}$$

with the understanding that

$$\sum_{n=\alpha}^{\alpha-1} u(n) = 0.$$

Since the sum depends only on the integer values of the argument n, we may put

$$u(n+\theta) = u(n), \quad 0 \leq \theta < 1.$$

With this convention, (2.33) can be written as

$$\sum_{n=x}^{y} u(n) = \int_x^{y+1} u(t)dt, \quad y \geq x-1.$$

The following is a discrete analogue of Steffensen's inequality given by Evard and Gauchman in [41]. Evard and Gauchman obtained the following discrete case applying Corollary 3.4, which is a consequence of generalized Steffensen's inequality over a general measure space given in Theorem 3.65.

Theorem 2.32 *Let c be a positive real number. Let $(x_i)_{i=1}^n$ be a nonincreasing finite sequence of nonnegative real numbers, and let $(y_i)_{i=1}^n$ be a finite sequence of real numbers in $[0,c]$. Let $k_1, k_2 \in \{1,\dots,n\}$ be such that*

$$k_2 \leq \frac{\sum_{i=1}^n y_i}{c} \leq k_1.$$

Then

$$\sum_{i=n-k_2+1}^{n} x_i \leq \frac{1}{c}\sum_{i=1}^{n} x_i y_i \leq \sum_{i=1}^{k_1} x_i.$$

For $c = 1$ we obtain the following corollary and give a simple proof which was obtained by Liu in [81].

Corollary 2.1 *Let $(x_i)_{i=1}^n$ be a nonincreasing finite sequence of nonnegative real numbers, and let $(y_i)_{i=1}^n$ be a finite sequence of real numbers such that $0 \leq y_i \leq 1$ for every i. Let $k_1, k_2 \in \{1,\dots,n\}$ be such that*

$$k_2 \leq \sum_{i=1}^{n} y_i \leq k_1.$$

Then

$$\sum_{i=n-k_2+1}^{n} x_i \leq \sum_{i=1}^{n} x_i y_i \leq \sum_{i=1}^{k_1} x_i. \tag{2.34}$$

Proof. Let us consider the difference $\sum_{i=1}^{k_1} x_i - \sum_{i=1}^{n} x_i y_i$. By elementary transformations we have

$$\begin{aligned}
\sum_{i=1}^{k_1} x_i - \sum_{i=1}^{n} x_i y_i &= \sum_{i=1}^{k_1} (1-y_i)x_i - \sum_{i=k_1+1}^{n} x_i y_i \\
&\geq x_{k_1} \sum_{i=1}^{k_1} (1-y_i) - \sum_{i=k_1+1}^{n} x_i y_i = x_{k_1} \left(k_1 - \sum_{i=1}^{k_1} y_i \right) - \sum_{i=k_1+1}^{n} x_i y_i \\
&\geq x_{k_1} \left(\sum_{i=1}^{n} y_i - \sum_{i=1}^{k_1} y_i \right) - \sum_{i=k_1+1}^{n} x_i y_i = x_{k_1} \sum_{i=k_1+1}^{n} y_i - \sum_{i=k_1+1}^{n} x_i y_i \\
&= \sum_{i=k_1+1}^{n} (x_{k_1} - x_i) y_i \geq 0
\end{aligned}$$

where the first inequality holds because $(x_i)_{i=1}^n$ is nonincreasing, while the second inequality holds since $\sum_{i=1}^{n} y_i \leq k_1$ by assumption of Corollary. So the second inequality in (2.34) is proved.

The first inequality in (2.34) is proved similarly. □

Inequality (2.34) is called the discrete Steffensen's inequality.

The following corollary was obtained by Evard and Gauchman in [41]. They applied integral inequalities on the composition of functions to the function $f = \sum_{i=1}^{n} x_i \chi_{(i-1,i]}$.

Corollary 2.2 *Let $\alpha, \beta, \gamma \in \mathbb{R}$. Let $x_1, \ldots, x_n$ be nonnegative real numbers. Let A and B be positive real numbers such that*

$$\max\{x_1^\alpha, \ldots, x_n^\alpha\} \leq B^\alpha, \quad \sum_{i=1}^{n} x_i^\alpha = A^\alpha, \quad \sum_{i=1}^{n} x_i^\beta = B^\beta.$$

Let $k_1, k_2 \in \{1, \ldots, n\}$ be such that $k_1 \leq \left(\frac{A}{B}\right)^\alpha \leq k_2$. Then:

(i) If $\frac{\beta-\alpha-\gamma}{\alpha} \leq 0$ and $\alpha \neq 0$, then there are k_1 numbers among the numbers $x_1^\gamma, \ldots, x_n^\gamma$ whose sum is at most B^γ.

(ii) If $\frac{\beta-\alpha-\gamma}{\alpha} \geq 0$ and $\alpha \neq 0$, then there are k_2 numbers among the numbers $x_1^\gamma, \ldots, x_n^\gamma$ whose sum is at least B^γ.

Applying Corollary 2.2 with $\alpha = 1$, $\beta = 2$ and $\gamma = 1$ Evard and Gauchman obtained the following corollary.

Corollary 2.3 *Let $(x_i)_{i=1}^n$ be a finite sequence of nonnegative real numbers. Let A, B be positive real numbers such that*

$$\sum_{i=1}^{n} x_i \leq A, \quad \sum_{i=1}^{n} x_i^2 \geq B^2.$$

Let $k \in \{1, \ldots, n\}$ be such that $k \geq \frac{A}{B}$. Then there are k numbers among the numbers $x_1, \ldots, x_n$ whose sum is at least B.

To give an application of Corollary 2.3 Evard and Gauchman showed that this corollary gives an immediate solution to the problem proposed in the Moscow Mathematical Olympiad in 1954. Problem was the following:
A hundered positive numbers $x_1,\ldots,x_{100}$ satisfy conditions

$$x_1+\cdots+x_{100}<300, \qquad x_1^2+\cdots+x_{100}^2>10000.$$

Show that among them, there are three numbers whose sum is greater than 100.
Solution of that problem given in [41] is to apply Corollary 2.3 with $n=100$, $A=300$, $B=100$ and $k=3$.

The following result was proved by Gauchman in [51].

Theorem 2.33 *Let $l\geq 0$ be a real number, $(x_i)_{i=1}^n$ be a nonincreasing finite sequence of real numbers in $[l,\infty)$ and $(y_i)_{i=1}^n$ be a finite sequence of nonnegative real numbers. Let $\Phi:[l,\infty)\to[0,\infty)$ be increasing, convex and such that $\Phi(xy)\geq\Phi(x)\Phi(y)$ for all $x,y,xy\geq l$. Let $k\in\{1,\ldots,n\}$ be such that $k\geq l$ and $\Phi(k)\geq\sum_{i=1}^n y_i$. Then either*

$$\sum_{i=1}^n\Phi(x_i)y_i\leq\Phi\left(\sum_{i=1}^k x_i\right) \quad \text{or} \quad \sum_{i=1}^k y_i\geq 1.$$

Proof. Since $(x_i)_{i=1}^n$ is nonincreasing and Φ is increasing we have

$$\Phi\left(\sum_{i=1}^k x_i\right)\geq\Phi\left(kx_k\right) \text{ and } \Phi\left(\sum_{i=1}^k x_i\right)\geq\Phi\left(x_i+(k-1)x_k\right). \tag{2.35}$$

If $x_i>x_k$ for some $i=1,2,\ldots,k-1$, then by Proposition 1.1 for convex function Φ we have

$$\frac{\Phi(x_i)-\Phi(x_k)}{x_i-x_k}\leq\frac{\Phi(x_i+(k-1)x_k)-\Phi(x_k+(k-1)x_k)}{x_i-x_k}$$

i.e.

$$\Phi(x_i)-\Phi(x_k)\leq\Phi(x_i+(k-1)x_k)-\Phi(kx_k).$$

If $x_i=x_k$ for some $i=1,2,\ldots,k-1$, then the above inequality also holds. Using (2.35) we get

$$\Phi(x_i)-\Phi(x_k)\leq\Phi\left(\sum_{i=1}^k x_i\right)-\Phi(kx_k). \tag{2.36}$$

for $i = 1,2,\dots,k$. Now simple transformations on sums give us

$$\begin{aligned}
\sum_{i=1}^{n}\Phi(x_i)y_i &= \sum_{i=1}^{k}\Phi(x_i)y_i + \sum_{i=k+1}^{n}\Phi(x_i)y_i \le \sum_{i=1}^{k}\Phi(x_i)y_i + \Phi(x_k)\sum_{i=k+1}^{n} y_i \\
&= \sum_{i=1}^{k}\Phi(x_i)y_i + \Phi(x_k)\left(\sum_{i=1}^{n} y_i - \sum_{i=1}^{k} y_i\right) = \sum_{i=1}^{k}(\Phi(x_i)-\Phi(x_k))y_i + \Phi(x_k)\sum_{i=1}^{n} y_i \\
&\le \sum_{i=1}^{k}\left[\Phi\left(\sum_{i=1}^{k} x_i\right) - \Phi(kx_k)\right]y_i + \Phi(x_k)\sum_{i=1}^{n} y_i \\
&\le \left[\Phi\left(\sum_{i=1}^{k} x_i\right) - \Phi(kx_k)\right]\sum_{i=1}^{k} y_i + \Phi(x_k)\Phi(k) \le \left[\Phi\left(\sum_{i=1}^{k} x_i\right) - \Phi(kx_k)\right]\sum_{i=1}^{k} y_i + \Phi(kx_k).
\end{aligned}$$

Then

$$\begin{aligned}
&\left[\Phi\left(\sum_{i=1}^{k} x_i\right) - \Phi(kx_k)\right]\left(\sum_{i=1}^{k} y_i - 1\right) \\
&= \left[\Phi\left(\sum_{i=1}^{k} x_i\right) - \Phi(kx_k)\right]\sum_{i=1}^{k} y_i - \Phi\left(\sum_{i=1}^{k} x_i\right) + \Phi(kx_k) \ge \sum_{i=1}^{n}\Phi(x_i)y_i - \Phi\left(\sum_{i=1}^{k} x_i\right).
\end{aligned}$$

If $\sum_{i=1}^{k} y_i - 1 \ge 0$, then the statement of theorem is valid. If $\sum_{i=1}^{k} y_i - 1 \le 0$, then using (2.35) we get

$$\sum_{i=1}^{n}\Phi(x_i)y_i - \Phi\left(\sum_{i=1}^{k} x_i\right) \le 0$$

and the proof has been established. □

As mentioned by Gauchman, Theorem 2.33 has simple form if $\Phi(x)=x^{\alpha}$, where $\alpha\ge 1$.

Theorem 2.34 *Let $(x_i)_{i=1}^{n}$ be a nonincreasing finite sequence of nonnegative real numbers and let $(y_i)_{i=1}^{n}$ be a finite sequence of nonnegative real numbers. Assume that $\alpha \ge 1$. Let $k \in \{1,\dots,n\}$ be such that*

$$k \ge \left(\sum_{i=1}^{n} y_i\right)^{\frac{1}{\alpha}}.$$

Then either

$$\sum_{i=1}^{n} x_i^{\alpha} y_i \le \left(\sum_{i=1}^{k} x_i\right)^{\alpha} \quad \text{or} \quad \sum_{i=1}^{k} y_i \ge 1.$$

As an application of Theorem 2.34 Gauchman obtained the following theorem in [51].

Theorem 2.35 *Let α and β be real numbers such that $\alpha \geq 1+\beta$, $0 \leq \beta \leq 1$. Let $(x_i)_{i=1}^n$ be a nonincreasing sequence of nonnegative real numbers. Assume that*

$$\sum_{i=1}^n x_i \leq A, \quad \sum_{i=1}^n x_i^\alpha \geq B^\alpha,$$

where A and B are positive real numbers. Let $k \in \{1,2,\ldots,n\}$ be such that $k \geq \left(\frac{A}{B}\right)^{\frac{\beta}{\alpha-1}}$. Then

$$\sum_{i=1}^k x_i^\beta \geq B^\beta. \tag{2.37}$$

Proof. Let us define the sequence $(y_i)_{i=1}^n$ as following: $y_i = \frac{x_i}{B}$, $i=1,2,\ldots,n$. Then we have

$$\sum_{i=1}^n y_i = \frac{1}{B}\sum_{i=1}^n x_i \leq \frac{A}{B},$$

and since k is a number such that $k \geq \left(\frac{A}{B}\right)^{\frac{\beta}{\alpha-1}}$ we get that $k \geq \left(\sum_{i=1}^n y_i\right)^{\frac{\beta}{\alpha-1}}$. Sequences $(x_i^\beta)_{i=1}^n$ (instead of $(x_i)_{i=1}^n$), $(y_i)_{i=1}^n$ and number $\frac{\alpha-1}{\beta} > 1$ satisfy the assumptions of Theorem 2.34. Then either

$$\sum_{i=1}^n (x_i^\beta)^{\frac{\alpha-1}{\beta}} y_i \leq \left(\sum_{i=1}^k x_i^\beta\right)^{\frac{\alpha-1}{\beta}} \quad \text{or} \quad \sum_{i=1}^k y_i \geq 1,$$

i.e.

$$\sum_{i=1}^n \frac{x_i^\alpha}{B} \leq \left(\sum_{i=1}^k x_i^\beta\right)^{\frac{\alpha-1}{\beta}} \quad \text{or} \quad \sum_{i=1}^k x_i \geq B. \tag{2.38}$$

If the first inequality holds, then we have

$$\sum_{i=1}^k x_i^\beta \geq \left(\frac{1}{B}\sum_{i=1}^n x_i^\alpha\right)^{\frac{\beta}{\alpha-1}} \geq \left(\frac{1}{B}B^\alpha\right)^{\frac{\beta}{\alpha-1}} = B^\beta.$$

If the second inequality holds, then by the well-known inequality for sums of order p, ([122, p.165]), we have

$$\left(\sum_{i=1}^k x_i^\beta\right)^{1/\beta} \geq \sum_{i=1}^k x_i \text{ for } 0 \leq \beta \leq 1,$$

i.e. together with (2.38) we obtain $\sum_{i=1}^k x_i^\beta \geq B^\beta$. Therefore, in both cases we have (2.37). □

Remark 2.7 For $\beta = 1$, Theorem 2.35 was given in [41] by Evard and Gauchman.

In [53] Gauchman obtained the following theorem using the generalization of Steffensen's inequality given in Theorem 3.67.

Theorem 2.36 *Let α and β be real numbers such that $\alpha > 1$, $\beta - \alpha + 1 \geq 0$ and let $(x_i)_{i=1}^n$ be a nonincreasing sequence of nonnegative real numbers such that $\sum_{i=1}^n x_i \leq A$, $\sum_{i=1}^n x_i^\alpha \geq B^\alpha$, where A and B are positive real numbers. Let $k \in \{1, \ldots, n\}$ be such that*

$$k \geq \left(\frac{A}{B}\right)^{\frac{\alpha\beta}{(\alpha-1)(\beta+1)}}.$$

Then

$$\sum_{i=1}^{k} x_i^\beta \geq \left(\frac{B^{\alpha\beta}}{A^{\beta-\alpha+1}}\right)^{\frac{\beta}{(\alpha-1)(\beta+1)}}.$$

Taking $\alpha = 2$, $\beta = 1$ in Theorem 2.36 we obtain Corollary 2.3. Taking $\beta = 1$ in Theorem 2.36 Gauchman obtained the following.

Corollary 2.4 *Let α be real number such that $1 < \alpha \leq 2$ and let $(x_i)_{i=1}^n$ be a nonincreasing sequence of nonnegative real numbers such that $\sum_{i=1}^n x_i \leq A$, $\sum_{i=1}^n x_i^\alpha \geq B^\alpha$, where A and B are positive real numbers. Let $k \in \{1, \ldots, n\}$ be such that*

$$k \geq \left(\frac{A}{B}\right)^{\frac{\alpha}{2(\alpha-1)}}.$$

Then

$$\sum_{i=1}^{k} x_i \geq \left(\frac{B^\alpha}{A^{2-\alpha}}\right)^{\frac{1}{2(\alpha-1)}}.$$

Remark 2.8 Corollary 2.4 complements the result given in Theorem 2.35 for $\beta = 1$.

In [101] Pachpatte proved the following two theorems.

Theorem 2.37 *Let $(u_n)_{n\in\mathbb{N}}$ be a nonincreasing sequence of nonnegative real numbers. If $(c_n)_{n\in\mathbb{N}}$ is a sequence of real numbers such that $0 \leq c_n \leq A$ (A is a constant different from zero), then*

$$\sum_{n=1}^{\infty} c_n u_n \leq A \sum_{n=1}^{\lambda} u_n,$$

where

$$\lambda = \frac{1}{A} \sum_{n=1}^{\infty} c_n. \tag{2.39}$$

Remark 2.9 As noted in [129], Pachpatte omitted that λ must be an integer.

Theorem 2.38 *Let $(u_n)_{n\in\mathbb{N}}$ be a nondecreasing sequence of nonnegative real numbers. Let $(c_n)_{n\in\mathbb{N}}$ be a sequence as in Theorem 2.37. Then*

$$A\sum_{n=1}^{\lambda} u_n \le \sum_{n=1}^{\infty} c_n u_n,$$

where λ is defined by (2.39).

Considering one dimensional case with discrete measure μ in Theorem 3.55, Pečarić and Varošanec obtained the following theorem (see [129]).

Theorem 2.39 *Let $(f_n)_n$ and $(g_n)_n$ be real $N-$tuples, A a real number and L an integer such that $AL = \sum_{n=1}^{N} g_n$ and one of the following cases is satisfied:*

(1) $g_n \le A$ and $f_n \ge f_L$ for $n = 1,\dots,L$;
$g_n \ge 0$ and $f_n \le f_L$ for $n = L+1,\dots,N$;

(2) $g_n \ge A$ and $f_n \le f_L$ for $n = 1,\dots,L$;
$g_n \le 0$ and $f_n \ge f_L$ for $n = L+1,\dots,N$.

Then

$$\sum_{n=1}^{N} f_n g_n \le A\sum_{n=1}^{L} f_n.$$

Remark 2.10 The case where f is nonincreasing and $0 \le g(x_n) \le A$ is discussed by Pachpatte in Theorem 2.37.

The discrete version of weaker conditions in the right-hand Steffensen's inequality is based on the following identity (see [129]):

$$\begin{aligned}\sum_{n=1}^{N} f_n g_n - A\sum_{n=1}^{L} f_n - f_{L+1}\left(\sum_{n=1}^{N} g_n - AL\right) &= \sum_{n=1}^{L}(f_{L+1}-f_n)(A-g_n) - \sum_{n=L+1}^{N}(f_{L+1}-f_n)g_n \\ &= \sum_{n=1}^{L}\Delta f_{n+1}\left(nA - \sum_{k=1}^{n} g_k\right) + \sum_{n=L+1}^{N-1}\Delta f_{n+1}\sum_{k=n+1}^{N} g_k,\end{aligned}$$

where $\Delta f_{n+1} = f_{n+1} - f_n$.

Using the above-mentioned identity we have the following discrete version of the right-hand Steffensen's inequality.

Theorem 2.40 *If $(f_n)_n$ is nonincreasing sequence, $A \in \mathbb{R}$, $L \in \mathbb{N}$ and $(g_n)_n$ is such that*

$$\sum_{k=1}^{n} g_k \le An \text{ and } \sum_{k=n+1}^{N} g_k \ge 0 \text{ for all } n = 1,2,\dots,N-1$$

and

$$f_{L+1}\left(\sum_{n=1}^{N} g_n - AL\right) \le 0,$$

then we have

$$\sum_{n=1}^{N} f_n g_n \leq A \sum_{n=1}^{L} f_n.$$

Remark 2.11 The discrete version of weaker conditions in the left-hand Steffensen's inequality can be obtained similarly.

In [141] Shi and Wu proved the following theorem using the theory of majorization.

Theorem 2.41 *Let $(x_i)_{i=1}^n$ be a nonincreasing finite sequence of real numbers, and let $(y_i)_{i=1}^n$ be a finite sequence of real numbers such that for every i, $0 \leq y_i \leq 1$. Let $k_1, k_2 \in \{1, \dots, n\}$ be such that $k_2 \leq \sum_{i=1}^n y_i \leq k_1$. Then*

$$\sum_{i=n-k_2+1}^{n} x_i + \left(\sum_{i=1}^{n} y_i - k_2 \right) x_n \leq \sum_{i=1}^{n} x_i y_i \leq \sum_{i=1}^{k_1} x_i - \left(k_1 - \sum_{i=1}^{n} y_i \right) x_n.$$

As a consequence of Theorem 2.41 Shi and Wu obtained the following refinement of Steffensen's inequality.

Corollary 2.5 *Let $(x_i)_{i=1}^n$ be a nonincreasing finite sequence of nonnegative real numbers and let $(y_i)_{i=1}^n$ be a finite sequence of real numbers such that for every i, $0 \leq y_i \leq 1$. Let $k_1, k_2 \in \{1, \dots, n\}$ be such that $k_2 \leq \sum_{i=1}^n y_i \leq k_1$. Then*

$$\begin{aligned}\sum_{i=n-k_2+1}^{n} x_i &\leq \sum_{i=n-k_2+1}^{n} x_i + \left(\sum_{i=1}^{n} y_i - k_2 \right) x_n \leq \sum_{i=1}^{n} x_i y_i \\ &\leq \sum_{i=1}^{k_1} x_i - \left(k_1 - \sum_{i=1}^{n} y_i \right) x_n \leq \sum_{i=1}^{k_1} x_i.\end{aligned}$$

2.7 Steffensen pairs

Using Corollary 2.3 Gauchman introduced Steffensen pairs in [52].

Definition 2.1 *Let $\varphi : [c,\infty) \to [0,\infty)$, $c \geq 0$ and $\tau : (0,\infty) \to (0,\infty)$ be two increasing functions. We say that (φ,τ) is a* Steffensen pair *on $[c,\infty)$ if the following is satisfied:*
If $x_1,\ldots,x_n$ are real numbers such that $x_i \geq c$ for all i, A and B are positive real numbers, and

$$\sum_{i=1}^{n} x_i \leq A, \quad \sum_{i=1}^{n} \varphi(x_i) \geq \varphi(B),$$

then for any $k \in \{1,\ldots,n\}$ such that $k \geq \tau\left(\frac{A}{B}\right)$, there are k numbers among $x_1,\ldots,x_n$ whose sum is larger than or equal to B.

As noted in [52] some results given in Section 2.6 can be reformulated using the definition of Steffensen pairs. Firstly, Corollary 2.3 can be reformulated as follows.

Proposition 2.1 *(x^2,x) is a Steffensen pair on $[0,\infty)$.*

Next, for $\beta = 1$, Theorem 2.35 can be reformulated in the following way.

Proposition 2.2 *If $\alpha \geq 2$, then $(x^{\alpha}, x^{\frac{1}{\alpha-1}})$ is a Steffensen pair on $[0,\infty)$.*

Now we give some Gauchman's examples of Steffensen pairs (see [52]).

Theorem 2.42 *Let $\psi : [c,\infty) \to [0,\infty)$ where $c \geq 0$ be nonndecreasing and convex. Assume that ψ satisfies the following condition:*

$$\psi(xy) \geq \psi(x)g(y) \quad \text{for all } x \geq c, y \geq 1,$$

where $g : [1,\infty) \to [0,\infty)$ is increasing. Set $\varphi(x) = x\psi(x)$, $\tau(x) = g^{-1}(x)$, where g^{-1} is the inverse function of g. Then (φ,τ) is a Steffensen pair on $[c,\infty)$.

Proof. Let $x_1,\ldots,x_n$ be real numbers such that $x_1 \geq x_2 \geq \ldots \geq x_n \geq c$ and let A and B be positive real numbers such that

$$\sum_{i=1}^{n} x_i \leq A \text{ and } \sum_{i=1}^{n} \varphi(x_i) \geq \varphi(B).$$

Let us suppose that $k \in \{1,2,\ldots,n\}$ is a number such that $k \geq g^{-1}(\frac{A}{B})$, i.e. $A \leq Bg(k)$. Then

$$A\psi(x_k) \leq B\psi(x_k)g(k) \leq B\psi(kx_k).$$

By Proposition 1.1 for convex function ψ and $i \leq k-1$ we have the following

$$\psi(x_i) - \psi(x_k) \leq \psi(x_i + (k-1)x_k) - \psi(x_k + (k-1)x_k).$$

Multiplying by x_i and adding all inequalities for $1,2,\ldots,k$ we obtain

$$\sum_{i=1}^{k}\varphi(x_i)-\psi(x_k)\sum_{i=1}^{k}x_i\le\sum_{i=1}^{k}x_i\psi(x_i+(k-1)x_k)-\psi(kx_k)\sum_{i=1}^{k}x_i$$
$$\le\psi(\sum_{i=1}^{k}x_i)\sum_{i=1}^{k}x_i-\psi(kx_k)\sum_{i=1}^{k}x_i=\varphi(\sum_{i=1}^{k}x_i)-\psi(kx_k)\sum_{i=1}^{k}x_i$$

since ψ is nondecreasing and $\sum_{i=1}^{k}x_i\ge x_i+(k-1)x_k$ for any $i\in\{1,2,\ldots,k\}$. Let us consider the difference $B-\sum_{i=1}^{k}x_i$ and multiply it by a positive number $\psi(kx_k)$. Using the inequality just proved and assumptions of Theorem we get

$$\psi(kx_k)(B-\sum_{i=1}^{k}x_i)=B\psi(kx_k)-\psi(kx_k)\sum_{i=1}^{k}x_i$$
$$\ge A\psi(x_k)-\psi(x_k)\sum_{i=1}^{k}x_i+\varphi(B)-\varphi(\sum_{i=1}^{k}x_i)$$
$$=\psi(x_k)(A-\sum_{i=1}^{k}x_i)+(\varphi(B)-\varphi(\sum_{i=1}^{k}x_i))\ge\varphi(B)-\varphi(\sum_{i=1}^{k}x_i).$$

Let us assume that the conclusion is wrong, i.e. $B-\sum_{i=1}^{k}x_i>0$. Then from $\psi(\sum_{i=1}^{k}x_i)\ge\psi(kx_k)$ we have

$$(B-\sum_{i=1}^{k}x_i)\psi(\sum_{i=1}^{k}x_i)\ge(B-\sum_{i=1}^{k}x_i)\psi(kx_k)\ge\varphi(B)-\varphi(\sum_{i=1}^{k}x_i)$$

i.e. $B\psi(B)=\varphi(B)\le B\psi(\sum_{i=1}^{k}x_i)$. Since ψ is nondecreasing, it follows that $B\le\sum_{i=1}^{k}x_i$ which is a contradiction with the above assumption. $\square$

In the following remark we pointed out some examples of Steffensen pairs.

Remark 2.12 (i) Let $\alpha\ge 2$, $\psi(x)=x^{\alpha-1}$. Then $\psi(xy)=\psi(x)g(y)$. Hence $\varphi(x)=x^{\alpha}$, $\tau(x)=x^{\frac{1}{\alpha-1}}$, and we obtain Proposition 2.2.

(ii) Let $f:[0,\infty)\to\mathbb{R}$ be a twice differentiable function on $[0,\infty)$ such that $f'(x)\ge 1$ and $f''(x)\ge 0$ for all $x\ge 0$. Assume that $f(0)=0$. Then the functions ψ and g from $[1,\infty)$ into $[0,\infty)$ given by

$$\psi=g=\exp\circ f\circ\log$$

satisfy the conditions of Theorem 2.42.

In [52] an example of the function f is given: $f(x)=\sum_{i=1}^{\infty}a_ix^i$ is a sum of series converging on $[0,\infty)$ with $a_1\ge 1$, $a_i\ge 0$ for $i=2,3,\ldots$.

(iii) If $\alpha\ge 1$, then $\left(x\exp(x^{\alpha}-1),(1+\log x)^{\frac{1}{\alpha}}\right)$ is a Steffensen pair on $[1,\infty)$.

(iv) Let a and b be real numbers satisfying conditions $b > a > 1$ and $\sqrt{ab} \geq e$. Set

$$\varphi(x) = \begin{cases} \frac{x^{1+\log b} - x^{1+\log a}}{\log x}, & \text{if } x > 1, \\ \log b - \log a, & \text{if } x = 1. \end{cases}$$

$$\tau(x) = x^{1/\log\sqrt{ab}}.$$

Then (φ, τ) is a Steffensen pair on $[1, \infty)$.

In [133] Qi and Cheng established some new Steffensen pairs.

Theorem 2.43 *If a and b are real numbers satisfying $b > a > 1$ or $b > a^{-1} > 1$, and $\sqrt{ab} \geq e$, then*

$$\left(x \int_a^b t^{\log x - 1} dt, x^{\frac{1}{\log\sqrt{ab}}} \right)$$

is a Steffensen pair on $[1, \infty)$.
If a and b are real numbers satisfying $b > a > 1$ and $\sqrt{ab} \geq e$, then

$$\left(x \int_a^b (\log t)^n t^{\log x - 1} dt, x^{\frac{n+2}{n+1} \frac{(\log b)^{n+1} - (\log a)^{n+1}}{(\log b)^{n+2} - (\log a)^{n+2}}} \right)$$

are Steffensen pairs on $[1, \infty)$ for any positive integer n.

Proof is similar to the following theorem, so here we omit details. (See [133]).

Remark 2.13 As noted in [133], Theorem 2.43 generalizes Remark 2.12 (iv).

In [134] more Steffensen pairs are established by Qi and Guo. They also proved the following generalization of Theorem 2.43.

Theorem 2.44 *Let $a, b \in \mathbb{R}$, let $p \neq 0$ be a nonnegative and integrable function and f a positive and integrable function on the interval $[a, b]$.*

(i) If the inequality

$$\int_a^b p(u) du \leq \int_a^b p(u) \log f(u) du \tag{2.40}$$

holds, then

$$\left(x \int_a^b p(u)[f(u)]^{\log x} du, x^{\frac{\int_a^b p(u) du}{\int_a^b p(u) \log f(u) du}} \right)$$

is a Steffensen pair on $[1, \infty)$.

(ii) If $f(u) \geq 1$ and inequality (2.40) holds, then

$$\left(x \int_a^b p(u)[f(u)]^{\log x} [\log f(u)]^n du, x^{\frac{\int_a^b p(u)[\log f(u)]^n du}{\int_a^b p(u)[\log f(u)]^{n+1} du}} \right)$$

are Steffensen pairs on $[1, \infty)$ for any positive integer n.

Proof. We define

$$h(t)=\int_a^b p(u)f^t(u)du, \quad t\in\mathbb{R}.$$

Since $f(u)\geq 1$ on $[a,b]$, it is clear that

$$h^{(n)}(t)=\int_a^b p(u)f^t(u)[\log f(u)]^n du \geq 0.$$

Furthermore, for $n\geq 0$ and $x\geq 0$, if (2.40) holds, then $h^{(n+1)}(x)\geq h^{(n)}(x)$. Let us define functions ψ and g as following: $\psi(x)=h^{(n)}(\log x)$ for $x\geq 1$, $n\geq 0$ and

$$g(x)=x^{\frac{\int_a^b p(u)[\log f(u)]^{n+1}du}{\int_a^b p(u)[\log f(u)]^n du}} \quad \text{for } x\geq 1.$$

It is easy to see that ψ is increasing and convex. Since $f(u)\geq 1$, for $n\geq 1$, we have

$$\frac{h^{(n)}(x+y)}{h^{(n)}(x)}=\frac{\int_a^b p(u)[f(u)]^{x+y}[\log f(u)]^n du}{\int_a^b p(u)[f(u)]^x[\log f(u)]^n du}\geq \exp\left(y\cdot\frac{\int_a^b p(u)[\log f(u)]^{n+1}du}{\int_a^b p(u)[\log f(u)]^n du}\right),$$

where we use the monotonicity property $M(x+y,x)\geq M(0,0)$ for the generalized weighted mean M defined by

$$M(r,s)=\begin{cases}\left(\frac{\int p(u)[\log f(u)]^n f^s(u)du}{\int p(u)[\log f(u)]^n f^r(u)du}\right)^{\frac{1}{s-r}}, & r\neq s\\ \exp\left(\frac{\int p(u)[\log f(u)]^{n+1} f^s(u)du}{\int p(u)[\log f(u)]^n f^s(u)du}\right), & r=s.\end{cases}$$

Therefore, for $x,y\geq 1$,

$$\frac{\psi(xy)}{\psi(x)}=\frac{h^{(n)}(\log(xy))}{h^{(n)}(\log x)}=\frac{h^{(n)}(\log x+\log y)}{h^{(n)}(\log x)}\geq y^{\frac{\int_a^b p(u)[\log f(u)]^{n+1}du}{\int_a^b p(u)[\log f(u)]^n du}}=g(y).$$

So, all assumptions of Theorem 2.42 are satisfied and by that theorem (φ,τ), where $\varphi(x)=x\psi(x)$, $\tau(x)=g^{-1}(x)$ for $x\geq 1$ and $n\geq 1$ are Steffensen pairs on $[1,\infty)$. $\square$

Chapter 3

Generalizations of Steffensen's inequality

3.1 L^p generalizations

In 1959 Bellman gave the following generalization of Steffensen's inequality (see [25]).

Theorem 3.1 *Let f be a nonnegative and nonincreasing function on $[a,b]$ and $f \in L^p[a,b]$, $p > 1$. Let $g(t) \geq 0$ on $[a,b]$ and $\int_a^b g^q(t)dt \leq 1$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left(\int_a^b f(t)g(t)dt\right)^p \leq \int_a^{a+\lambda} f^p(t)dt \quad \left(\lambda = \left(\int_a^b g(t)dt\right)^p\right). \tag{3.1}$$

As noted by Godunova, Levin and Čebaevskaja in [57] Bellman's result is incorrect as stated. This was also noted by Godunova and Levin in [56], where they gave generalization for $0 < p \leq 1$. Their generalization is a consequence of a more general result given in [56] and will be described in Section 3.4.

Another corrected version of Bellman's inequality, for $p \geq 1$, is given by Bergh in [26].

Theorem 3.2 *Let f and g be positive functions on $(0,\infty)$, f nonincreasing and g measurable. Assume that, for some $p \geq 1$, $f \in L^p + L^\infty$ and $g \in L^q \cap L^1$, with*

$$\|f\|_{L^q} = 1, \quad \|g\|_{L^1} = t \quad \left(\frac{1}{p} + \frac{1}{q} = 1\right).$$

Then

$$\int_0^\infty f(x)g(x)dx \le 2^{\frac{1}{q}} \left(\int_0^{t^p} f^p(x)dx\right)^{\frac{1}{p}}$$

holds, where $2^{1/q}$ cannot be replaced by a smaller constant.

Proof of this theorem is based on an estimation of the $K-$functional in the theory of interpolation spaces. For more details see [26].

In [111] Pečarić showed that the Bellman generalization of Steffensen's inequality, with very simple modifications of conditions, is true.

Theorem 3.3 *Let $f:[0,1]\to\mathbb{R}$ be a nonnegative and nonincreasing function and let $g:[0,1]\to\mathbb{R}$ be an integrable function such that $0\le g\le 1$. If $p\ge 1$, then*

$$\left(\int_0^1 g(t)f(t)dt\right)^p \le \int_0^\lambda f^p(t)dt \tag{3.2}$$

where

$$\lambda = \left(\int_0^1 g(t)dt\right)^p. \tag{3.3}$$

Proof. Using the Jensen inequality for the convex function $\Phi(x)=x^p$ $(p\ge 1)$, we have

$$\left(\int_0^1 g(t)f(t)dt\right)^p \le \left(\int_0^1 g(t)dt\right)^{p-1} \int_0^1 g(t)f^p(t)dt.$$

To complete the proof we must prove

$$\left(\int_0^1 g(t)dt\right)^{p-1} \int_0^1 g(t)f^p(t)dt \le \int_0^\lambda f^p(t)dt.$$

Since f is nonincreasing we have

$$\begin{aligned}
\int_0^\lambda f^p(t)\left(1-g(t)\left(\int_0^1 g(s)ds\right)^{p-1}\right)dt &\ge f^p(\lambda)\int_0^\lambda \left(1-g(t)\left(\int_0^1 g(s)ds\right)^{p-1}\right)dt \\
&= f^p(\lambda)\left(\lambda - \left(\int_0^1 g(s)ds\right)^{p-1}\int_0^\lambda g(t)dt\right) \\
&= f^p(\lambda)\left(\left(\int_0^1 g(s)ds\right)^p - \left(\int_0^1 g(s)ds\right)^{p-1}\int_0^\lambda g(t)dt\right) \\
&= f^p(\lambda)\left(\int_0^1 g(s)ds\right)^{p-1}\int_\lambda^1 g(t)dt.
\end{aligned}$$

It follows

$$\begin{aligned}
&\int_0^\lambda f^p(t)dt - \left(\int_0^1 g(t)dt\right)^{p-1} \int_0^1 g(t)f^p(t)dt \\
&= \int_0^\lambda f^p(t)\left(1 - g(t)\left(\int_0^1 g(s)ds\right)^{p-1}\right)dt - \left(\int_0^1 g(s)ds\right)^{p-1} \int_\lambda^1 f^p(t)g(t)dt \\
&\geq \left(\int_0^1 g(s)ds\right)^{p-1}\left(f^p(\lambda)\int_\lambda^1 g(t)dt - \int_\lambda^1 g(t)f^p(t)dt\right) \\
&= \left(\int_0^1 g(s)ds\right)^{p-1} \int_\lambda^1 g(t)\left(f^p(\lambda) - f^p(t)\right)dt \geq 0.
\end{aligned}$$

$\square$

Remark 3.1 If the functions f and g are defined on $[a,b]$, using the substitution $x = (b-a)t + a$, the corresponding result for Bellman's generalization can be obtained.

Cao gave another correction of Bellman's result in [32].

Theorem 3.4 *Let f be a nonnegative and nonincreasing function on $[a,b]$ and $f \in L^p[a,b]$, $p > 1$. Let function g satisfy relations $g \geq 0$ on $[a,b]$ and $\int_a^b g^q(t)dt \leq 1$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left(\int_a^b f(t)g(t)dt\right)^p \leq \int_a^{a+\lambda} f^p(t)dt$$

where

$$\lambda = \begin{cases} \left(\frac{f(a+0)}{f(b-0)}\right)^{p-1}\left(\int_a^b g(t)dt\right)^p, & f(b-0) > 0 \\ b-a, & f(b-0) = 0. \end{cases}$$

In [111] Pečarić gave the following result.

Theorem 3.5 *Let $f : [0,1] \to \mathbb{R}$ be a nonincreasing function and let $g : [0,1] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq 1$. If $p \geq 1$, then*

$$\frac{\int_0^1 g(t)f(t)dt}{\int_0^1 g(t)dt} \leq \frac{1}{\lambda}\int_0^\lambda f(t)dt, \tag{3.4}$$

where λ is given by (3.3).

Remark 3.2 For $p = 1$ we have Steffensen's inequality.

Using substitution

$$g(t) = \frac{\lambda G(t)}{\int_a^b G(t)dt},$$

where $\lambda > 0$ and $\int_a^b G(t)dt > 0$, Pečarić obtained the following modification of Steffensen's inequality (see [112]). This result is an extension of Theorem 3.5.

Theorem 3.6 *Assume that two integrable functions f and G are defined on the interval $[a,b]$, f is nonincreasing, and*

$$0 \leq \lambda G(t) \leq \int_a^b G(t)dt \quad (\forall t \in [a,b]), \tag{3.5}$$

where λ is a positive number. Then

$$\frac{1}{\lambda}\int_{b-\lambda}^b f(t)dt \leq \frac{\int_a^b f(t)G(t)dt}{\int_a^b G(t)dt} \leq \frac{1}{\lambda}\int_a^{a+\lambda} f(t)dt. \tag{3.6}$$

In [92] Mitrinović and Pečarić gave necessary and sufficient conditions for inequality (3.6). Inequality (3.6) is true for each nonincreasing function f if and only if for every $x \in [a,b]$

$$0 \leq \lambda \int_x^b G(t)dt \leq (b-x)\int_a^b G(t)dt$$

and

$$0 \leq \lambda \int_a^x G(t)dt \leq (x-a)\int_a^b G(t)dt.$$

The second inequality in (3.6) is valid if and only if for every $x \in [a,b]$

$$\lambda \int_a^x G(t)dt \leq (x-a)\int_a^b G(t)dt \quad \text{and} \quad \int_x^b G(t)dt \geq 0. \tag{3.7}$$

In [112] Pečarić gave the following generalization of Theorem 3.3.

Theorem 3.7 *Let $f : [a,b] \to \mathbb{R}$ be a nonnegative nonincreasing function and let $G : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leq G \leq 1$. If $p \geq 1$, then*

$$\frac{1}{(b-a)^{p-1}}\left(\int_a^b G(x)f(x)dx\right)^p \leq \int_a^{a+\lambda} f^p(x)dx \tag{3.8}$$

holds, where

$$\lambda = \frac{1}{(b-a)^{p-1}}\left(\int_a^b G(x)dx\right)^p.$$

Proof. Under the assumptions of this theorem (3.5) is valid. So, from the second inequality in (3.6), for the nonincreasing function $x \mapsto f^p(x)$, we have

$$\left(\int_a^b G(t)dt\right)^{p-1}\int_a^b f^p(t)G(t)dt \leq (b-a)^{p-1}\int_a^{a+\lambda} f^p(x)dx.$$

On the other hand, using Jensen's inequality for convex function $u(x) = x^p$ $(p \geq 1)$, we have

$$\left(\int_a^b G(t)f(t)dt\right)^p \leq \left(\int_a^b G(t)dt\right)^{p-1}\int_a^b G(t)f^p(t)dt.$$

So we obtain (3.8). □

Analogously, Pečarić obtained the following theorem in [112].

Theorem 3.8 *Let $f : [a,b] \to \mathbb{R}$ be a nonnegative nonincreasing function and let $G : [a,b] \to \mathbb{R}$ be an integrable function such that*

$$0 \leq G(x) \left(\int_a^b G(t)dt \right)^{p-1} \leq 1 \quad (\forall x \in [a,b]).$$

If $p \geq 1$, then

$$\left(\int_a^b G(x)f(x)dx \right)^p \leq \int_a^{a+\lambda} f^p(t)dt,$$

where

$$\lambda = \left(\int_a^b G(t)dt \right)^p.$$

Remark 3.3 For $a = 0$, $b = 1$, from Theorem 3.8, we obtain Theorem 3.3. For $p \in \mathbb{N}$ we have Corollary 9 from [105].

In [71] Jiang obtained the following result.

Theorem 3.9 *Let $f : [a,b] \to \mathbb{R}$ be a nonnegative nonincreasing function, $g : [a,b] \to \mathbb{R}$ be an integrable function, $0 \leq \frac{g(x)}{(\int_a^b g(t)dt)^{p-1}} \leq M$, $x \in [a,b]$ and M be a positive constant. Then*

$$\left(\int_a^b f(t)g(t)dt \right)^p \leq M \int_a^{a+\lambda} f^p(t)dt \text{ if } p \geq 1,$$

and

$$M \int_{b-\lambda}^b f^p(t)dt \leq \left(\int_a^b f(t)g(t)dt \right)^p \text{ if } p \leq 1,$$

where $\lambda = \frac{1}{M} \left(\int_a^b g(t)dx \right)^p$.

In [57] Godunova, Levin and Čebaevskaja gave the following two results. In [92] Mitrinović and Pečarić showed that these results are consequences of their necessary and sufficient conditions for inequality (3.6). We recall the proof from [92].

Theorem 3.10 *Let f be a nonnegative nonincreasing function on $[a,b]$, and let ϕ be an increasing convex function on $[0,\infty\rangle$ with $\phi(0) = 0$. If g is a nonnegative nondecreasing function on $[a,b]$ such that there exists nonnegative function g_1, defined by the equation*

$$g_1(x)\,\phi\left(\frac{g(x)}{g_1(x)} \right) = 1 \tag{3.9}$$

and that $\int_a^b g_1(t)dt \leq 1$, then the following inequality is valid

$$\phi\left(\frac{\int_a^b f(t)g(t)dt}{\int_a^b g(t)dt} \right) \leq \frac{1}{\lambda} \int_a^{a+\lambda} \phi(f(t))dt,$$

where

$$\lambda = \phi\left(\int_a^b g(t)dt \right).$$

Proof. Applying the second inequality in (3.6) and Jensen's inequality for convex functions, we have that

$$\phi\left(\frac{\int_a^b f(t)g(t)dt}{\int_a^b g(t)dt}\right) \leq \frac{\int_a^b g(t)\phi(f(t))dt}{\int_a^b g(t)dt} \leq \frac{1}{\lambda}\int_a^{a+\lambda}\phi(f(t))dt. \tag{3.10}$$

Now, from the necessary and sufficient conditions for inequality (3.6) given by (3.7), we obtain that inequality (3.10) is valid if and only if

$$\phi\left(\int_a^b g(t)dt\right)\int_a^x g(t)dt \leq (x-a)\int_a^b g(t)dt \quad \text{and} \quad \int_x^b g(t)dt \geq 0 \tag{3.11}$$

holds for every $x \in [a,b]$. Since g is nonnegative, the second condition in (3.11) is obviously satisfied. On the other hand, the increasing convex function ϕ with $\phi(0) = 0$ is starshaped, that is $\phi(cx) \leq c\phi(x)$, $(0 < c \leq 1)$. Therefore, by (3.9), (3.10) and Jensen's inequality, we have

$$\begin{aligned}\phi\left(\int_a^b g(t)dt\right) &= \phi\left(\int_a^b g_1(t)dt\,\frac{\int_a^b g(t)dt}{\int_a^b g_1(t)dt}\right) \leq \left(\int_a^b g_1(t)dt\right)\,\phi\left(\frac{\int_a^b g_1(t)\frac{g(t)}{g_1(t)}dt}{\int_a^b g_1(t)dt}\right)\\ &\leq \int_a^b g_1(t)\,\phi\left(\frac{g(t)}{g_1(t)}\right)dt = \int_a^b dt = (b-a).\end{aligned}$$

Since g is a nondecreasing function, we have

$$\frac{1}{b-a}\int_a^b g(t)dt \geq \frac{1}{x-a}\int_a^x g(t)dt,$$

i.e.

$$(b-a)\int_a^x g(t)dt \leq (x-a)\int_a^b g(t)dt.$$

Hence, the first condition in (3.11) is also satisfied. □

In [82] Liu gave a generalization of the previous theorem for Stieltjes integral. We omit the proof because it is very similar to the previous one.

Theorem 3.11 *Let f and h be nonnegative decreasing functions defined on $[a,b]$, and let Φ be an increasing convex function on $[0,\infty)$ with $\Phi(0) = 0$. If g is a nonnegative increasing function defined on $[a,b]$ such that there exists the nonnegative function g_1 which satisfies*

$$\int_a^b g_1(t)\Phi\left(\frac{g(t)}{g_1(t)}\right)d\mu(t) \leq \int_a^b h(t)d\mu(t)$$

and $\int_a^b g_1(t)d\mu(t) \leq 1$, then

$$\Phi\left(\frac{\int_a^b f(t)g(t)d\mu(t)}{\int_a^b g(t)d\mu(t)}\right) \leq \frac{\int_a^{\mu^{-1}(\mu(a)+\lambda)} h(t)\Phi(f(t))d\mu(t)}{\int_a^{\mu^{-1}(\mu(a)+\lambda)} h(t)d\mu(t)}$$

holds, where λ is given by

$$\int_a^{\mu^{-1}(\mu(a)+\lambda)} h(t)d\mu(t) = \Phi\left(\int_a^b g(t)d\mu(t)\right).$$

For $\phi(u) = u^p$ $(p > 1)$, Theorem 3.10 becomes the following:

Theorem 3.12 *Let f be a nonnegative nonincreasing function on $[a,b]$, $f \in L^p(a,b)$, and let g be nonnegative and nondecreasing on $[a,b]$ such that $\int_a^b g^q(t)dt \leq 1$, where $p > 1$ and $q = \frac{p}{p-1}$. Then (3.1) holds.*

Remark 3.4 From the above proof it follows that condition (3.9) can be replaced by

$$g_1(x)\,\phi\left(\frac{g(x)}{g_1(x)}\right) \leq 1$$

or, more generally, by

$$\int_a^b g_1(x)\,\phi\left(\frac{g(x)}{g_1(x)}\right) dx \leq b-a.$$

In 1998 Pachpatte established further generalizations of inequality given in Theorem 3.3 (see [102]).

Theorem 3.13 *Let f,g,h be real-valued integrable functions defined on $[0,1]$ such that $f(t) \geq 0$, $h(t) \geq 0$, $t \in [0,1]$, f/h is nonincreasing on $[0,1]$ and $0 \leq g(t) \leq A$, $t \in [0,1]$, where A is a real positive constant. If $p \geq 1$, then*

$$\left(\int_0^1 g(t)f(t)dt\right)^p \leq A^p \int_0^\lambda f^p(t)dt, \tag{3.12}$$

where λ is the solution of the equation

$$\int_0^\lambda h^p(t)dt = \frac{1}{A^p}\left(\int_0^1 h^p(t)g(t)dt\right)\left(\int_0^1 g(t)dt\right)^{p-1}.$$

Proof. Applying Hölder's inequality on the left-hand side of inequality (3.12) we have

$$\begin{aligned}\left(\int_0^1 g(t)f(t)dt\right)^p &= \left(\int_0^1 g^{\frac{p-1}{p}}(t)g^{\frac{1}{p}}(t)f(t)dt\right)^p \\ &\leq \left(\int_0^1 g(t)dt\right)^{p-1}\left(\int_0^1 g(t)f^p(t)dt\right).\end{aligned} \tag{3.13}$$

In order to prove inequality (3.12) we must prove that

$$\left(\int_0^1 g(t)dt\right)^{p-1}\left(\int_0^1 g(t)f^p(t)dt\right) \leq A^p \int_0^\lambda f^p(t)dt. \tag{3.14}$$

First, we have that

$$
\begin{aligned}
&A^p \int_0^\lambda f^p(t)dt - \left(\int_0^1 g(s)ds\right)^{p-1} \int_0^\lambda g(t)f^p(t)dt \\
&= \int_0^\lambda f^p(t)\left(A^p - g(t)\left(\int_0^1 g(s)ds\right)^{p-1}\right)dt \\
&= \int_0^\lambda \left(\frac{f(t)}{h(t)}\right)^p h^p(t)\left(A^p - g(t)\left(\int_0^1 g(s)ds\right)^{p-1}\right)dt \\
&\geq \left(\frac{f(\lambda)}{h(\lambda)}\right)^p \left(A^p \int_0^\lambda h^p(t)dt - \left(\int_0^1 g(s)ds\right)^{p-1} \int_0^\lambda h^p(t)g(t)dt\right) \\
&= \left(\frac{f(\lambda)}{h(\lambda)}\right)^p \left(\int_0^1 g(s)ds\right)^{p-1} \left(\int_0^1 h^p(t)g(t)dt - \int_0^\lambda h^p(t)g(t)dt\right) \\
&= \left(\frac{f(\lambda)}{h(\lambda)}\right)^p \left(\int_0^1 g(s)ds\right)^{p-1} \int_\lambda^1 h^p(t)g(t)dt.
\end{aligned}
$$

Now, inequality (3.14) can be proved as follows

$$
\begin{aligned}
&A^p \int_0^\lambda f^p(t)dt - \left(\int_0^1 g(t)dt\right)^{p-1} \left(\int_0^1 g(t)f^p(t)dt\right) \\
&= A^p \int_0^\lambda f^p(t)dt - \left(\int_0^1 g(s)ds\right)^{p-1} \int_0^\lambda g(t)f^p(t)dt \\
&+ \left(\int_0^1 g(s)ds\right)^{p-1} \int_0^\lambda g(t)f^p(t)dt - \left(\int_0^1 g(s)ds\right)^{p-1} \int_0^1 g(t)f^p(t)dt \\
&\geq \left(\frac{f(\lambda)}{h(\lambda)}\right)^p \left(\int_0^1 g(s)ds\right)^{p-1} \int_\lambda^1 h^p(t)g(t)dt - \left(\int_0^1 g(s)ds\right)^{p-1} \int_\lambda^1 g(t)f^p(t)dt \\
&= \left(\int_0^1 g(s)ds\right)^{p-1} \int_\lambda^1 h^p(t)g(t)\left(\left(\frac{f(\lambda)}{h(\lambda)}\right)^p - \left(\frac{f(t)}{h(t)}\right)^p\right)dt \geq 0.
\end{aligned}
$$

Inequality (3.12) now follows from (3.13) and (3.14). □

Theorem 3.14 *Let f,g,h,p be as in Theorem 3.13. Then*

$$
\left(\int_0^1 g(t)dt\right)^{p-1} \int_0^1 g(t)f(t)dt \leq A^p \int_0^\lambda f(t)dt, \tag{3.15}
$$

where λ is the solution of the equation

$$
\int_0^\lambda h(t)dt = \frac{1}{A^p}\left(\int_0^1 h(t)g(t)dt\right)\left(\int_0^1 g(t)dt\right)^{p-1}.
$$

Inequality (3.15) is a variant of the inequality given in Theorem 3.3 and when $A = 1$, $p = 1$, $h(t) = 1$, it reduces to the right-hand side of Steffensen's inequality with $a = 0$ and $b = 1$. Similar result with $p = 1$ and $A = 1$ is given by Pečarić in Theorem 3.15.

Remark 3.5 Some L^p generalizations can also be found in [145], [146] and [147]. Furthermore, some results similar to Steffensen's inequality via Hölder's, Minkowski's and Hardy-Hilbert's inequalities are given in [147].

3.2 Pečarić, Mercer and Wu-Srivastava generalizations

In 1982 Pečarić proved the following generalization of Steffensen's inequality (see [110]).

Theorem 3.15 *Let h be a positive integrable function on $[a,b]$ and f be an integrable function such that $x \mapsto f(x)/h(x)$ is nondecreasing on $[a,b]$. If g is a real-valued integrable function such that $0 \le g(x) \le 1$ for every $x \in [a,b]$, then*

$$\int_a^b f(t)g(t)dt \ge \int_a^{a+\lambda} f(t)dt \tag{3.16}$$

holds, where λ is the solution of the equation

$$\int_a^{a+\lambda} h(t)dt = \int_a^b h(t)g(t)dt.$$

If $x \mapsto f(x)/h(x)$ is a nonincreasing function, then the reverse inequality in (3.16) holds.

Proof. Transformation of the difference between the right-hand side and the left-hand side of inequality (3.16) gives

$$\begin{aligned}
&\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt = \int_a^{a+\lambda} (1-g(t))f(t)dt - \int_{a+\lambda}^b f(t)g(t)dt \\
&\le \frac{f(a+\lambda)}{h(a+\lambda)} \int_a^{a+\lambda} h(t)(1-g(t))dt - \int_{a+\lambda}^b f(t)g(t)dt \\
&= \frac{f(a+\lambda)}{h(a+\lambda)} \left(\int_a^b h(t)g(t)dt - \int_a^{a+\lambda} h(t)g(t)dt \right) - \int_{a+\lambda}^b f(t)g(t)dt \\
&= \int_{a+\lambda}^b g(t)h(t) \left(\frac{f(a+\lambda)}{h(a+\lambda)} - \frac{f(t)}{h(t)} \right) dt \le 0.
\end{aligned}$$

□

By substitutions $g(x) \to 1 - g(x)$ and $\lambda \to b - a - \lambda$, Theorem 3.15 becomes:

Theorem 3.16 *Let the conditions of Theorem 3.15 be fulfilled. Then*

$$\int_a^b f(t)g(t)dt \leq \int_{b-\lambda}^b f(t)dt$$

where λ is the solution of the equation

$$\int_{b-\lambda}^b h(t)dt = \int_a^b h(t)g(t)dt. \tag{3.17}$$

For $h(x) = 1$ we have Steffensen's inequality.

The following generalization of Steffensen's inequality for a convex function of order n is a version of Theorem 2.9 with a stronger condition on the function g (see [122, p. 193]).

Theorem 3.17 *Let g be an integrable function such that $0 \leq g(x) \leq 1$ for every $x \in [a,b]$.*

(a) If the function $f : [a,b] \to \mathbb{R}$ is convex of order n with $f^{(k)}(a) = 0$, $k = 0,\ldots,n-2$, then (3.16) holds with

$$\lambda = \left(n \int_a^b (t-a)^{n-1} g(t)dt \right)^{\frac{1}{n}}. \tag{3.18}$$

(b) If f is a nonnegative and concave function of order n with $f^{(k)}(a) = 0$, $k = 0,\ldots,n-2$, then the reverse of the inequality in (3.16) holds.

Proof. Let f be an $n-$convex function such that $f^{(k)}(a) = 0$, $(k = 0,1,\ldots,n-2)$. Then $x \mapsto \frac{f(x)}{(x-a)^{n-1}}$ is a nondecreasing function. Applying Theorem 3.15 on function $\frac{f(x)}{(x-a)^{n-1}}$ we have that (3.16) is valid where λ is defined by (3.18). □

In [87] Mercer proved following generalization of Steffensen's inequality.

Theorem 3.18 *Let f,g and h be integrable functions on (a,b) with f nonincreasing and $0 \leq g \leq h$. Then*

$$\int_{b-\lambda}^b f(t)h(t)dt \leq \int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)h(t)dt, \tag{3.19}$$

where λ is given by

$$\int_a^{a+\lambda} h(t)dt = \int_a^b g(t)dt. \tag{3.20}$$

As noted by Wu and Srivastava in [155] and by Liu in [83] the generalization due to Mercer is incorrect as stated. They have proved that it is true if we add the condition:

$$\int_{b-\lambda}^b h(t)dt = \int_a^b g(t)dt. \tag{3.21}$$

As proved by Pečarić, Perušić and Smoljak in [117], the corrected version of Mercer's results follows from Theorems 3.15 and 3.16, and it is stated as following.

Theorem 3.19 *Let h be a positive integrable function on $[a,b]$ and f,g be integrable functions on $[a,b]$ such that f is nonincreasing on $[a,b]$ and $0 \leq g \leq h$ for every $x \in [a,b]$.*

a) Then

$$\int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)h(t)dt,$$

where λ is given by (3.20).

b) Then

$$\int_{b-\lambda}^b f(t)h(t)dt \leq \int_a^b f(t)g(t)dt,$$

where λ is given by (3.21).

Proof. Putting substitutions $g(t) \mapsto g(t)/h(t)$ and $f(t) \mapsto f(t)h(t)$ in Theorems 3.15 and 3.16 we obtain the statements of this theorem. □

Mercer also gave the following theorem in [87].

Theorem 3.20 *Let f,g,h and k be integrable functions on (a,b) with $k > 0$, f/k nonincreasing and $0 \leq g \leq h$. Then*

$$\int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)h(t)dt, \tag{3.22}$$

where λ is the solution of the equation

$$\int_a^{a+\lambda} h(t)k(t)dt = \int_a^b g(t)k(t)dt. \tag{3.23}$$

If f/k is a nondecreasing function, then the reverse of the inequality in (3.22) holds.

Let us show that it is equivalent to Theorem 3.15. Let us suppose that the assumptions of Theorem 3.20 hold. Then for $h \equiv 1$ we obtain Theorem 3.15.
Oppositely, taking $h(t) \mapsto k(t)h(t)$, $g(t) \mapsto g(t)/h(t)$ and $f(t) \mapsto f(t)h(t)$ in Theorem 3.15 we obtain Theorem 3.20. Hence, Theorems 3.15 and 3.20 are equivalent.

Motivated by Theorem 3.20 the following theorem, which is equivalent to Theorem 3.16, was obtained in [117].

Theorem 3.21 *Let f,g,h and k be integrable functions on (a,b) with $k > 0$, f/k nonincreasing and $0 \leq g \leq h$. Then*

$$\int_a^b f(t)g(t)dt \geq \int_{b-\lambda}^b f(t)h(t)dt, \tag{3.24}$$

where λ satisfies

$$\int_{b-\lambda}^b h(t)k(t)dt = \int_a^b g(t)k(t)dt. \tag{3.25}$$

If f/k is a nondecreasing function, then the reverse of the inequality in (3.24) holds.

Proof. Take $h(t) \mapsto k(t)h(t)$, $g(t) \mapsto g(t)/h(t)$ and $f(t) \mapsto f(t)h(t)$ in Theorem 3.16. □

Remark 3.6 From Theorems 3.20 and 3.21 taking $k \equiv 1$ we can obtain the corrected Mercer's results given in Theorem 3.19.

Next, we give the corrected version of Mercer's results given by Wu and Srivastava in [155]. Note that this is not only corrected but also a refined version of Mercer's result.

Theorem 3.22 *Let f,g and h be integrable functions on $[a,b]$ with f nonincreasing and let $0 \le g \le h$. Then the following integral inequalities hold true*

$$\begin{aligned}\int_{b-\lambda}^{b} f(t)h(t)dt &\le \int_{b-\lambda}^{b} (f(t)h(t) - [f(t)-f(b-\lambda)][h(t)-g(t)])dt \\ &\le \int_{a}^{b} f(t)g(t)dt \\ &\le \int_{a}^{a+\lambda} (f(t)h(t) - [f(t)-f(a+\lambda)][h(t)-g(t)])dt \\ &\le \int_{a}^{a+\lambda} f(t)h(t)dt,\end{aligned} \tag{3.26}$$

where λ satisfies

$$\int_{a}^{a+\lambda} h(t)dt = \int_{a}^{b} g(t)dt = \int_{b-\lambda}^{b} h(t)dt.$$

Proof. The proof is based on the following identities:

$$\begin{aligned}\int_{a}^{b} f(t)g(t)dt = \int_{a}^{a+\lambda} &(f(t)h(t) - [f(t)-f(a+\lambda)][h(t)-g(t)])dt \\ &+ \int_{a+\lambda}^{b} [f(t)-f(a+\lambda)]g(t)dt,\end{aligned} \tag{3.27}$$

and

$$\begin{aligned}\int_{a}^{b} f(t)g(t)dt = \int_{b-\lambda}^{b} &(f(t)h(t) - [f(t)-f(b-\lambda)][h(t)-g(t)])dt \\ &+ \int_{a}^{b-\lambda} [f(t)-f(b-\lambda)]g(t)dt.\end{aligned} \tag{3.28}$$

Let us prove the first one. Transformation of the right-hand side of the identity gives the following

$$
\begin{aligned}
&\int_a^{a+\lambda}(f(t)h(t)-[f(t)-f(a+\lambda)][h(t)-g(t)])dt+\int_{a+\lambda}^b[f(t)-f(a+\lambda)]g(t)dt\\
&=\int_a^{a+\lambda}[f(t)g(t)+f(a+\lambda)(h(t)-g(t))]dt+\int_{a+\lambda}^b f(t)g(t)dt-f(a+\lambda)\int_{a+\lambda}^b g(t)dt\\
&=\int_a^b f(t)g(t)dt+f(a+\lambda)\left[\int_a^{a+\lambda}(h(t)-g(t))dt-\int_{a+\lambda}^b g(t)dt\right]\\
&=\int_a^b f(t)g(t)dt+f(a+\lambda)\left[\int_a^{a+\lambda}h(t)dt-\int_a^b g(t)dt\right]\\
&=\int_a^b f(t)g(t)dt
\end{aligned}
$$

where in the last equality we use the property of λ, i.e. $\int_a^{a+\lambda}h(t)dt=\int_a^b g(t)dt$.

The second identity can be proved in a similar manner.

Since f is nonincreasing on $[a,b]$ we get $f(t)\geq f(b-\lambda)$ for all $t\in[a,b-\lambda]$ and $f(t)\leq f(b-\lambda)$ for all $t\in[b-\lambda,b]$. Then

$$\int_a^{b-\lambda}[f(t)-f(b-\lambda)]g(t)dt\geq 0$$

and

$$\int_{b-\lambda}^b[f(t)-f(b-\lambda)][h(t)-g(t)]dt\leq 0.$$

Using (3.27) and the above inequalities we obtain

$$\int_a^b f(t)g(t)dt\geq\int_{b-\lambda}^b(f(t)h(t)-[f(t)-f(b-\lambda)][h(t)-g(t)])dt\geq\int_{b-\lambda}^b f(t)h(t)dt.$$

Similarly, we obtain

$$\int_a^b f(t)g(t)dt\leq\int_a^{a+\lambda}(f(t)h(t)-[f(t)-f(a+\lambda)][h(t)-g(t)])dt\leq\int_a^{a+\lambda}f(t)h(t)dt.$$

□

As noted by Wu and Srivastava in [155], for $h(t)\equiv 1$, Theorem 3.22 gives a refinement of Steffensen's inequality. Separating inequalities given in (3.26) into two parts we can obtain weaker conditions on λ. Those results are given in the following theorems (see [117]).

Theorem 3.23 *Let f,g and h be integrable functions on $[a,b]$ with f nonincreasing and let $0\leq g\leq h$. Then*

$$
\begin{aligned}
\int_a^b f(t)g(t)dt&\leq\int_a^{a+\lambda}(f(t)h(t)-[f(t)-f(a+\lambda)][h(t)-g(t)])dt\\
&\leq\int_a^{a+\lambda}f(t)h(t)dt,
\end{aligned}\tag{3.29}
$$

where λ is given by (3.20).
If f is a nondecreasing function, then the reverse inequalities in (3.29) hold.

Proof. Similar to the proof of the right-hand side inequalities in Theorem 3.22. □

Theorem 3.24 *Let f,g and h be integrable functions on $[a,b]$ with f nonincreasing and let $0 \leq g \leq h$. Then*

$$\begin{aligned}\int_{b-\lambda}^{b} f(t)h(t)dt &\leq \int_{b-\lambda}^{b} (f(t)h(t) - [f(t) - f(b-\lambda)][h(t) - g(t)])dt \\ &\leq \int_{a}^{b} f(t)g(t)dt\end{aligned} \tag{3.30}$$

where λ is given by (3.21).
If f is a nondecreasing function, then the reverse inequalities in (3.30) hold.

Proof. Similar to the proof of the left-hand side inequalities in Theorem 3.22. □

In the following theorems Pečarić, Perušić and Smoljak obtained a refined version of the results given in Theorems 3.20 and 3.21 (see [117]).

Theorem 3.25 *Let k be a positive integrable function on $[a,b]$ and f,g,h be integrable functions on $[a,b]$ such that f/k is nonincreasing and $0 \leq g \leq h$. Then*

$$\begin{aligned}\int_{a}^{b} f(t)g(t)dt &\leq \int_{a}^{a+\lambda} \left(f(t)h(t) - \left[\frac{f(t)}{k(t)} - \frac{f(a+\lambda)}{k(a+\lambda)}\right] k(t)[h(t) - g(t)]\right) dt \\ &\leq \int_{a}^{a+\lambda} f(t)h(t)dt,\end{aligned} \tag{3.31}$$

where λ is given by (3.23).
If f/k is a nondecreasing function, then the reverse inequalities in (3.31) hold.

Proof. Take $g(t) \mapsto k(t)g(t)$, $f(t) \mapsto f(t)/k(t)$ and $h(t) \mapsto k(t)h(t)$ in Theorem 3.23. □

Theorem 3.26 *Let k be a positive integrable function on $[a,b]$ and f,g,h be integrable functions on $[a,b]$ such that f/k is nonincreasing and $0 \leq g \leq h$. Then*

$$\begin{aligned}\int_{b-\lambda}^{b} f(t)h(t)dt &\leq \int_{b-\lambda}^{b} \left(f(t)h(t) - \left[\frac{f(t)}{k(t)} - \frac{f(b-\lambda)}{k(b-\lambda)}\right] k(t)[h(t) - g(t)]\right) dt \\ &\leq \int_{a}^{b} f(t)g(t)dt\end{aligned} \tag{3.32}$$

where λ is given by (3.25).
If f/k is a nondecreasing function, then the reverse inequalities in (3.32) hold.

Proof. Take $g(t) \mapsto k(t)g(t)$, $f(t) \mapsto f(t)/k(t)$ and $h(t) \mapsto k(t)h(t)$ in Theorem 3.24. □

Remark 3.7 From Theorems 3.25 and 3.26 taking $k \equiv 1$ we obtain a refinement of the corrected Mercer's results given in Theorem 3.19. Putting $h \equiv 1$ in Theorems 3.25 and 3.26 we get a refinement of Pečarić's results given in Theorems 3.15 and 3.16.

Furthermore, Wu and Srivastava proved a new sharpened and generalized version of inequality (3.19). We separate this result into two theorems to obtain weaker conditions on λ. The original result can be found in [155].

Theorem 3.27 *Let f,g,h and ψ be integrable functions on $[a,b]$ with f nonincreasing and let $0 \le \psi \le g \le h-\psi$. Then*

$$\int_a^b f(t)g(t)dt \le \int_a^{a+\lambda} f(t)h(t)dt - \int_a^b |f(t)-f(a+\lambda)|\,\psi(t)dt$$

where λ is given by (3.20).

Proof. Using identity (3.27) we get

$$\begin{aligned}
&\int_a^{a+\lambda} f(t)h(t)dt - \int_a^b f(t)(t)dt \\
&\quad = \int_a^{a+\lambda} [f(t)-f(a+\lambda)][h(t)-g(t)]dt - \int_{a+\lambda}^b [f(t)-f(a+\lambda)]g(t)dt \\
&\quad \ge \int_a^{a+\lambda} |f(t)-f(a+\lambda)|\psi(t)dt + \int_{a+\lambda}^b |f(a+\lambda)-f(t)|\psi(t)dt \\
&\quad = \int_a^b |f(t)-f(a+\lambda)|\psi(t)dt
\end{aligned}$$

and the proof is established. □

Similarly, the following theorem holds.

Theorem 3.28 *Let f,g,h and ψ be integrable functions on $[a,b]$ with f nonincreasing and let $0 \le \psi \le g \le h-\psi$. Then*

$$\int_{b-\lambda}^b f(t)h(t)dt + \int_a^b |f(t)-f(b-\lambda)|\,\psi(t)dt \le \int_a^b f(t)g(t)dt$$

where λ is given by (3.21).

In [117] the following sharpenings of Theorems 3.20 and 3.21 are given based on results from Theorems 3.27 and 3.28.

Theorem 3.29 *Let k be a positive integrable function on $[a,b]$ and f,g,h,ψ be integrable functions on $[a,b]$ with f/k nonincreasing and $0 \le \psi \le g \le h-\psi$.*

a) Then

$$\int_a^b f(t)g(t)dt \le \int_a^{a+\lambda} f(t)h(t)dt - \int_a^b \left|\left(\frac{f(t)}{k(t)} - \frac{f(a+\lambda)}{k(a+\lambda)}\right)\right| k(t)\psi(t)dt$$

where λ is given by (3.23).

b) Then

$$\int_{b-\lambda}^{b} f(t)h(t)dt + \int_a^b \left| \left(\frac{f(t)}{k(t)} - \frac{f(b-\lambda)}{k(b-\lambda)} \right) k(t)\psi(t) \right| dt \leq \int_a^b f(t)g(t)dt$$

where λ is given by (3.25).

Proof. Putting substitutions $g(t) \mapsto k(t)g(t)$, $f(t) \mapsto f(t)/k(t)$, $h(t) \mapsto k(t)h(t)$ and $\psi(t) \mapsto k(t)\psi(t)$ in Theorems 3.27 and 3.28 we obtain statements of this theorem. □

Remark 3.8 Taking $k \equiv 1$ in Theorem 3.29 we obtain a sharpened and generalized versions of Theorem 3.19.

If $h \equiv 1$, then inequalities from Theorem 3.29 become sharpenings of Steffensen's inequalities.

Motivated by the weaker conditions for the function g in Steffensen's inequality given by Milovanović and Pečarić in [90], weaker conditions for Theorems 3.20 and 3.21 are obtained in [117].

Theorem 3.30 *Let k be a positive integrable function on $[a,b]$, let f,g,h be integrable functions on $[a,b]$ such that h is nonnegative and let f/k be a nonincreasing integrable function on $[a,b]$. Then*

$$\int_a^x k(t)g(t)dt \leq \int_a^x k(t)h(t)dt \quad \textit{and} \quad \int_x^b k(t)g(t)dt \geq 0, \quad \forall x \in [a,b] \tag{3.33}$$

if and only if

$$\int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)h(t)dt \tag{3.34}$$

where λ is defined by (3.23).
If f/k is a nondecreasing function, then (3.33) holds if and only if the reverse of the inequality in (3.34) holds.

Proof. Using the identity

$$\begin{aligned}
&\int_a^{a+\lambda} f(t)h(t)dt - \int_a^b f(t)g(t)dt \\
&= \int_a^{a+\lambda} \left[\frac{f(t)}{k(t)} - \frac{f(a+\lambda)}{k(a+\lambda)} \right] k(t)[h(t)-g(t)]dt + \int_{a+\lambda}^b \left[\frac{f(a+\lambda)}{k(a+\lambda)} - \frac{f(t)}{k(t)} \right] k(t)g(t)dt
\end{aligned} \tag{3.35}$$

and applying integration by parts we obtain

$$\begin{aligned}
\int_a^{a+\lambda} f(t)h(t)dt - \int_a^b f(t)g(t)dt = &- \int_a^{a+\lambda} \left(\int_a^x k(t)[h(t)-g(t)]dt \right) d\left(\frac{f(x)}{k(x)} \right) \\
&- \int_{a+\lambda}^b \left(\int_x^b k(t)g(t)dt \right) d\left(\frac{f(x)}{k(x)} \right).
\end{aligned}$$

From here we conclude that, for nonincreasing function f/k, (3.34) holds when

$$\int_a^x k(t)h(t)dt \geq \int_a^x k(t)g(t)dt, \quad \text{for } a \leq x \leq a+\lambda \tag{3.36}$$

and

$$\int_x^b k(t)g(t)dt \geq 0, \quad \text{for } a+\lambda \leq x \leq b. \tag{3.37}$$

For $a+\lambda \leq x \leq b$, since h is nonnegative, k is positive and (3.37) holds, we have

$$\begin{aligned}\int_a^x k(t)g(t)dt &= \int_a^b k(t)g(t)dt - \int_x^b k(t)g(t)dt \\ &= \int_a^{a+\lambda} k(t)h(t)dt - \int_x^b k(t)g(t)dt \leq \int_a^{a+\lambda} k(t)h(t)dt \leq \int_a^x k(t)h(t)dt.\end{aligned}$$

On the other hand, for $a \leq x \leq a+\lambda$, since h is nonnegative, k is positive and (3.36) holds, we have

$$\begin{aligned}\int_x^b k(t)g(t)dt &= \int_a^b k(t)g(t) - \int_a^x k(t)g(t)dt \\ &= \int_a^{a+\lambda} k(t)h(t)dt - \int_a^x k(t)g(t)dt \\ &\geq \int_a^{a+\lambda} k(t)h(t)dt - \int_a^x k(t)h(t)dt = \int_x^{a+\lambda} k(t)h(t)dt \geq 0.\end{aligned}$$

Hence, (3.36) and (3.37) are equivalent to (3.33).
Now, we prove that conditions (3.33) are also necessary. In fact, for f defined by

$$f(t) = \begin{cases} k(t), & t \leq x \\ 0, & t > x, \end{cases} \qquad \forall x \in [a,b]$$

we have that f/k is a nonincreasing function. Now, from (3.34) we can obtain

$$\int_a^x k(t)g(t)dt \leq \int_a^x k(t)h(t)dt \quad \text{and} \quad \int_x^b k(t)g(t)dt \geq 0, \quad \forall x \in [a,b].$$

In a similar way we can obtain the reverse inequality in (3.34) for f/k nondecreasing. Thus the proof is completed. □

Theorem 3.31 *Let k be a positive integrable function on $[a,b]$, let f,g,h be integrable functions on $[a,b]$ such that h is nonnegative and let f/k be a nonincreasing integrable function on $[a,b]$. Then*

$$\int_x^b k(t)g(t)dt \leq \int_x^b k(t)h(t)dt \quad \textit{and} \quad \int_a^x k(t)g(t)dt \geq 0, \quad \forall x \in [a,b] \tag{3.38}$$

if and only if

$$\int_a^b f(t)g(t)dt \geq \int_{b-\lambda}^b f(t)h(t)dt \tag{3.39}$$

where λ is defined by (3.25).
If f/k is a nondecreasing function, then (3.38) holds if and only if the reverse of the inequality in (3.39) holds.

Proof. Using the identity

$$\begin{aligned}
&\int_{b-\lambda}^{b} f(t)h(t)dt - \int_{a}^{b} f(t)g(t)dt \\
&= \int_{b-\lambda}^{b} \left[\frac{f(t)}{k(t)} - \frac{f(b-\lambda)}{k(b-\lambda)}\right] k(t)[h(t)-g(t)]dt + \int_{a}^{b-\lambda} \left[\frac{f(b-\lambda)}{k(b-\lambda)} - \frac{f(t)}{k(t)}\right] k(t)g(t)dt
\end{aligned} \tag{3.40}$$

and applying integration by parts we obtain

$$\begin{aligned}
&\int_{b-\lambda}^{b} f(t)h(t)dt - \int_{a}^{b} f(t)g(t)dt \\
&= \int_{b-\lambda}^{b} \left(\int_{x}^{b} k(t)[h(t)-g(t)]dt\right) d\left(\frac{f(x)}{k(x)}\right) + \int_{a}^{b-\lambda} \left(\int_{a}^{x} k(t)g(t)dt\right) d\left(\frac{f(x)}{k(x)}\right).
\end{aligned}$$

Now, similar conclusions as in the proof of Theorem 3.30 completes the proof. □

Remark 3.9 For $k \equiv 1$, from Theorems 3.30 and 3.31, we obtain results similar to the results with weaker conditions given by Liu in [83] and Mercer in [87].

In the following theorems we give weaker conditions for refinements given in Theorems 3.25 and 3.26.

Theorem 3.32 *Let k be a positive integrable function on $[a,b]$, let f,g,h be integrable functions on $[a,b]$ such that h is nonnegative and let f/k be a nonincreasing integrable function on $[a,b]$. Let λ be defined by (3.23). If (3.33) holds, then (3.31) is valid.*
If f/k is a nondecreasing function, the reverse inequalities in (3.31) hold.

Proof. Using identity (3.35) and applying integration by parts we obtain

$$\begin{aligned}
&\int_{a}^{a+\lambda} f(t)h(t)dt - \int_{a}^{b} f(t)g(t)dt - \int_{a}^{a+\lambda} \left[\frac{f(t)}{k(t)} - \frac{f(a+\lambda)}{k(a+\lambda)}\right] k(t)[h(t)-g(t)]dt \\
&= \int_{a+\lambda}^{b} \left[\frac{f(a+\lambda)}{k(a+\lambda)} - \frac{f(t)}{k(t)}\right] k(t)g(t)dt = -\int_{a+\lambda}^{b} \left(\int_{x}^{b} k(t)g(t)dt\right) d\left(\frac{f(x)}{k(x)}\right).
\end{aligned}$$

From here we conclude that the left-hand side inequality in (3.31) holds when (3.37) holds. Furthermore, we have

$$\begin{aligned}
&\int_{a}^{a+\lambda} \left[\frac{f(t)}{k(t)} - \frac{f(a+\lambda)}{k(a+\lambda)}\right] k(t)[h(t)-g(t)]dt \\
&= -\int_{a}^{a+\lambda} \left(\int_{a}^{x} k(t)[h(t)-g(t)]dt\right) d\left(\frac{f(x)}{k(x)}\right).
\end{aligned}$$

So, if (3.36) holds, then

$$\int_a^{a+\lambda}\left[\frac{f(t)}{k(t)}-\frac{f(a+\lambda)}{k(a+\lambda)}\right]k(t)[h(t)-g(t)]dt \geq 0.$$

Hence, the right-hand side inequality in (3.31) holds.
As showed in the proof of Theorem 3.30, (3.36) and (3.37) are equivalent to (3.33). So the proof is completed. □

Theorem 3.33 *Let k be a positive integrable function on $[a,b]$, let f,g,h be integrable functions on $[a,b]$ and let f/k be a nonincreasing integrable function on $[a,b]$. Let λ be defined by (3.23). If*

$$\int_x^b k(t)g(t)dt \geq 0, \quad \text{for } a+\lambda \leq x \leq b,$$

then

$$\int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)h(t)dt - \int_a^{a+\lambda}\left[\frac{f(t)}{k(t)}-\frac{f(a+\lambda)}{k(a+\lambda)}\right]k(t)[h(t)-g(t)]dt. \quad (3.41)$$

If we additionally have

$$\int_a^x k(t)h(t)dt \geq \int_a^x k(t)g(t)dt, \quad \text{for } a \leq x \leq a+\lambda$$

then (3.31) holds.
If f/k is a nondecreasing function, the reverse inequalities in (3.41) and (3.31) hold.

Proof. Similar to the proof of Theorem 3.32. □

Theorem 3.34 *Let k be a positive integrable function on $[a,b]$, let f,g,h be integrable functions on $[a,b]$ such that h is nonnegative and let f/k be a nonincreasing integrable function on $[a,b]$. Let λ be defined by (3.25). If (3.38) holds, then (3.32) is valid.*
If f/k is a nondecreasing function, the reverse inequalities in (3.32) hold.

Proof. Similar to the proof of Theorem 3.32 using identity (3.40). □

Theorem 3.35 *Let k be a positive integrable function on $[a,b]$, let f,g,h be integrable functions on $[a,b]$ and let f/k be a nonincreasing integrable function on $[a,b]$. Let λ be defined by (3.25). If*

$$\int_a^x k(t)g(t)dt \geq 0, \quad \text{for } a \leq x \leq b-\lambda,$$

then

$$\int_{b-\lambda}^b f(t)h(t)dt - \int_{b-\lambda}^b\left[\frac{f(t)}{k(t)}-\frac{f(b-\lambda)}{k(b-\lambda)}\right]k(t)[h(t)-g(t)]dt \leq \int_a^b f(t)g(t)dt. \quad (3.42)$$

If we additionally have

$$\int_x^b k(t)h(t)dt \geq \int_x^b k(t)g(t)dt, \quad \text{for } b-\lambda \leq x \leq b$$

then (3.32) holds.
If f/k is a nondecreasing function, the reverse inequalities in (3.42) and (3.32) hold.

Proof. Similar to the proof of Theorem 3.34. □

Wu and Srivastava also gave a general result on an improved version of Steffensen's inequality by introducing additional parameters λ_1 and λ_2. This result is given in the following theorem.

Theorem 3.36 *Let f and g be integrable functions defined on $[a,b]$ with f nonincreasing. Also let*

$$0 \leq \lambda_1 \leq \int_a^b g(t)dt \leq \lambda_2 \leq b-a$$

and $0 \leq M \leq g \leq 1-M$. Then

$$\begin{aligned}
&\int_{b-\lambda_1}^b f(t)dt + f(b)\left(\int_a^b g(t)dt - \lambda_1\right) + M\int_a^b \left|f(t) - f\left(b - \int_a^b g(t)dt\right)\right| dt \\
&\leq \int_a^b f(t)g(t)dt \\
&\leq \int_a^{a+\lambda_2} f(t)dt - f(b)\left(\lambda_2 - \int_a^b g(t)dt\right) - M\int_a^b \left|f(t) - f\left(a + \int_a^b g(t)dt\right)\right| dt.
\end{aligned} \tag{3.43}$$

Proof. Let ψ be a constant function, i.e. $\psi(t) = M$, $t \in [a,b]$. Then f and g satisfy the assumptions of Theorem 3.27 with $h \equiv 1$ and $\psi(t) = M$. So, we get

$$\int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)dt - M\int_a^b |f(t) - f(a+\lambda)|dt$$

where $\lambda = \int_a^b g(t)dt$.

Let us prove the second inequality in (3.43). Using notation $\lambda = \int_a^b g(t)dt$, the fact that $\lambda_2 - \lambda = \int_{a+\lambda}^{a+\lambda_2} dt$ and the above inequality we get

$$\begin{aligned}
&\int_a^{a+\lambda_2} f(t)dt - f(b)\left(\lambda_2 - \int_a^b g(t)dt\right) - M\int_a^b \left|f(t) - f\left(a + \int_a^b g(t)dt\right)\right| dt \\
&= \int_a^{a+\lambda_2} f(t)dt - f(b)(\lambda_2 - \lambda) - M\int_a^b |f(t) - f(a+\lambda)|dt \\
&\geq \int_a^{a+\lambda_2} f(t)dt - f(b)\int_{a+\lambda}^{a+\lambda_2} dt - \int_a^{a+\lambda} f(t)dt + \int_a^b f(t)g(t)dt \\
&= \int_{a+\lambda}^{a+\lambda_2} f(t)dt - \int_{a+\lambda}^{a+\lambda_2} f(b)dt + \int_a^b f(t)g(t)dt \\
&= \int_{a+\lambda}^{a+\lambda_2} (f(t) - f(b))dt + \int_a^b f(t)g(t)dt \geq \int_a^b f(t)g(t)dt
\end{aligned}$$

where in the last inequality we use $f(t) \geq f(b)$ for any $t \in [a,b]$.
In the same way we prove the first inequality in (3.43). □

It is clear that Steffensen's inequality follows as a special case of inequality (3.43) when $M = 0$ and $\lambda_1 = \lambda_2$.

And finally, let us mention here one generalization involving Stieltjes integral due to Liu. In [82] he proved

Theorem 3.37 *Let f, g and h be $\mu-$integrable functions defined on $[a,b]$ with f nonincreasing and $0 \leq g \leq h$. Then*

$$\int_a^b f(t)g(t)d\mu(t) \leq \int_a^{\mu^{-1}(\mu(a)+\lambda)} f(t)h(t)d\mu(t),$$

where λ satisfies

$$\int_a^{\mu^{-1}(\mu(a)+\lambda)} h(t)d\mu(t) = \int_a^b g(t)d\mu(t). \tag{3.44}$$

Proof. After direct computation we get

$$\begin{aligned}
&\int_a^{\mu^{-1}(\mu(a)+\lambda)} f(t)h(t)d\mu(t) - \int_a^b f(t)g(t)d\mu(t) \\
&= \int_a^{\mu^{-1}(\mu(a)+\lambda)} [h(t)-g(t)]f(t)d\mu(t) - \int_{\mu^{-1}(\mu(a)+\lambda)}^b f(t)g(t)d\mu(t) \\
&\geq f(\mu^{-1}(\mu(a)+\lambda)) \int_a^{\mu^{-1}(\mu(a)+\lambda)} [h(t)-g(t)]d\mu(t) - \int_{\mu^{-1}(\mu(a)+\lambda)}^b f(t)g(t)d\mu(t) \\
&= f(\mu^{-1}(\mu(a)+\lambda)) \left[\int_a^b g(t)d\mu(t) - \int_a^{\mu^{-1}(\mu(a)+\lambda)} g(t)d\mu(t) \right] \\
&- \int_{\mu^{-1}(\mu(a)+\lambda)}^b f(t)g(t)d\mu(t) \\
&= \int_{\mu^{-1}(\mu(a)+\lambda)}^b [f(\mu^{-1}(\mu(a)+\lambda)) - f(t)]g(t)d\mu(t) \geq 0.
\end{aligned}$$

□

Of course, in the same paper, Liu gave a generalization of the second Steffensen inequality.

Theorem 3.38 *Let f, g and h be $\mu-$integrable functions defined on $[a,b]$ with f nonincreasing and $0 \leq g \leq h$. Then*

$$\int_{\mu^{-1}(\mu(b)-\lambda)}^{b} f(t)h(t)d\mu(t) \leq \int_{a}^{b} f(t)g(t)d\mu(t),$$

where λ satisfies

$$\int_{\mu^{-1}(\mu(b)-\lambda)}^{b} h(t)d\mu(t) = \int_{a}^{b} g(t)d\mu(t). \tag{3.45}$$

He also obtained results which are consequences of substitution $f \mapsto f/k$, $g \mapsto gk$, and results with weaker conditions.

Theorem 3.39 *Let g and h be nonnegative $\mu-$integrable functions defined on $[a,b]$. Then*

$$\int_{a}^{x} g(t)d\mu(t) \leq \int_{a}^{x} h(t)d\mu(t) \quad \text{and} \quad \int_{x}^{b} g(t)d\mu(t) \geq 0, \quad \forall x \in [a,b]$$

is a necessary and sufficient condition for

$$\int_{a}^{b} f(t)g(t)d\mu(t) \leq \int_{a}^{\mu^{-1}(\mu(a)+\lambda)} f(t)h(t)d\mu(t)$$

to hold for all nonincreasing functions f defined on $[a,b]$, where λ is given by (3.44).

Theorem 3.40 *Let g and h be nonnegative $\mu-$integrable functions defined on $[a,b]$. Then*

$$\int_{a}^{x} g(t)d\mu(t) \geq 0 \quad \text{and} \quad \int_{x}^{b} g(t)d\mu(t) \leq \int_{x}^{b} h(t)d\mu(t), \quad \forall x \in [a,b]$$

is a necessary and sufficient condition for

$$\int_{\mu^{-1}(\mu(b)-\lambda)}^{b} f(t)h(t)d\mu(t) \leq \int_{a}^{b} f(t)g(t)d\mu(t)$$

to hold for all nonincreasing functions f defined on $[a,b]$, where λ is given by (3.45).

3.3 Cerone's generalizations

We begin this section with generalizations of Steffensen's inequality given by Cerone in [33]. As we see Cerone's generalization of Steffensen's inequality allows bounds involving any two subintervals instead of restricting them to include the end points.

Theorem 3.41 *Let $f, g : [a,b] \to \mathbb{R}$ be integrable functions on $[a,b]$ and let f be nonincreasing. Further, let $0 \le g \le 1$ and*

$$\lambda = \int_a^b g(t)dt = d_i - c_i, \tag{3.46}$$

where $[c_i, d_i] \subseteq [a,b]$ for $i = 1,2$ and $d_1 \le d_2$. Then

$$\int_{c_2}^{d_2} f(t)dt - r(c_2,d_2) \le \int_a^b f(t)g(t)dt \le \int_{c_1}^{d_1} f(t)dt + R(c_1,d_1) \tag{3.47}$$

holds, where

$$r(c_2,d_2) = \int_{d_2}^b (f(c_2) - f(t))g(t)dt \ge 0$$

and

$$R(c_1,d_1) = \int_a^{c_1} (f(t) - f(d_1))g(t)dt \ge 0.$$

Proof. Let us prove the second inequality. Let us consider the corresponding difference:

$$\begin{aligned}
&\int_{c_1}^{d_1} f(t)dt + R(c_1,d_1) - \int_a^b f(t)g(t)dt \\
&= \int_{c_1}^{d_1} f(t)dt + \int_a^{c_1} (f(t) - f(d_1))g(t)dt - \int_a^b f(t)g(t)dt \\
&= \int_{c_1}^{d_1} (f(t) - f(d_1))(1 - g(t))dt + \int_{d_1}^b (f(d_1) - f(t))g(t)dt
\end{aligned}$$

where we use $\int_{c_1}^{d_1} dt = \int_a^b g(t)dt$.

Since $0 \le g \le 1$ and f is nonincreasing, the terms under the integral sign are nonnegative, hence the first sum in this chain is nonnegative, i.e.

$$\int_{c_1}^{d_1} f(t)dt + R(c_1,d_1) \ge \int_a^b f(t)g(t)dt.$$

The first inequality in (3.47) follows from the identity

$$\begin{aligned}
&\int_a^b f(t)g(t)dt - \int_{c_2}^{d_2} f(t)dt + r(c_2,d_2) \\
&\qquad = \int_{c_2}^{d_2} (f(c_2) - f(t))(1 - g(t))dt + \int_a^{c_2} (f(t) - f(c_2))g(t)dt.
\end{aligned}$$

□

If in Theorem 3.41 we take $c_1 = a$ and $d_1 = a + \lambda$, then $R(a, a+\lambda) = 0$. Further, taking $d_2 = b$ and $c_2 = b - \lambda$, then $r(b - \lambda, b) = 0$. Thus we obtain Steffensen's inequality. Since $\lambda = \int_a^b g(t)dt$ and $0 \le g \le 1$, then $c_2 = b - \lambda \ge a$ and $d_1 = a + \lambda \le b$ giving $[c_i, d_i] \subseteq [a,b]$. Hence, Theorem 3.41 can be viewed as a generalization of Steffensen's inequality for two subintervals which have equal lengths and which boundaries are not necessarily at the bounds of $[a,b]$.

Remark 3.10 In [10] Aglić Aljinović, Pečarić and Perušić showed that identity (3.47) can also be proved differently than in [33] using Steffensen's inequality. Indeed, in order to prove the right-hand side inequality in (3.47) we observe

$$\begin{aligned}
&\int_a^b f(t)g(t)\,dt - \int_{c_1}^{d_1} f(t)\,dt \\
&= \int_a^b (f(t)-f(d_1))\,g(t)\,dt + f(d_1)\int_a^b g(t)\,dt - \int_{c_1}^{d_1} f(t)\,dt \\
&= \int_a^b (f(t)-f(d_1))\,g(t)\,dt + f(d_1)\lambda - \int_{c_1}^{c_1+\lambda} f(t)\,dt \\
&= \int_a^b (f(t)-f(d_1))\,g(t)\,dt - \int_{c_1}^{c_1+\lambda} (f(t)-f(d_1))\,dt
\end{aligned}$$

and apply the right-hand Steffensen's inequality for nonincreasing function $f(t)-f(d_1)$ on the interval $[c_1,b]$

$$\int_{c_1}^b (f(t)-f(d_1))\,g(t)\,dt \le \int_{c_1}^{c_1+\lambda_1} (f(t)-f(d_1))\,dt.$$

Here we have $\lambda_1 = \int_{c_1}^b g(t)\,dt$ and thus obviously $\lambda_1 \le \lambda$ which leads us to

$$\int_{c_1}^{c_1+\lambda_1} (f(t)-f(d_1))\,dt \le \int_{c_1}^{c_1+\lambda} (f(t)-f(d_1))\,dt.$$

Finally

$$\begin{aligned}
&\int_a^b (f(t)-f(d_1))\,g(t)\,dt - \int_{c_1}^{c_1+\lambda} (f(t)-f(d_1))\,dt \\
&\le \int_a^b (f(t)-f(d_1))\,g(t)\,dt - \int_{c_1}^b (f(t)-f(d_1))\,g(t)\,dt = R(c_1,d_1)
\end{aligned}$$

and the proof is complete. In a similar manner, the left-hand side inequality in (3.47) can be proved.

The following corollary from [33] gives bounds that can be more easily evaluated.

Corollary 3.1 *Let the conditions of Theorem 3.41 hold. Then*

$$\int_{c_2}^b f(t)dt - (b-d_2)f(c_2) \le \int_a^b f(t)g(t)dt \le \int_a^{d_1} f(t)dt - (c_1-a)f(d_1). \tag{3.48}$$

Proof. From Theorem 3.41, using that $0 \le g \le 1$, we have

$$0 \le r(c_2,d_2) = \int_{d_2}^b (f(c_2)-f(t))g(t)dt \le \int_{d_2}^b (f(c_2)-f(t))dt = (b-d_2)f(c_2) - \int_{d_2}^b f(t)dt$$

and so

$$\int_{c_2}^{d_2} f(t)dt - r(c_2,d_2) \ge \int_{c_2}^{d_2} f(t)dt - (b-d_2) + \int_{d_2}^{b} f(t)dt.$$

So we obtain the left-hand side inequality in (3.48). Similarly we obtain the right-hand side inequality in (3.48). □

Theorem 3.42 *Let $f,g:[a,b]\to\mathbb{R}$ be integrable functions on $[a,b]$ and let f be nonincreasing. Further, let $g\ge 0$ and $G(x)=\int_a^x g(t)dt$ with $\lambda = G(b) = d_i - c_i$ where $[c_i,d_i]\subset [a,b]$ for $i=1,2$ and $d_1<d_2$. Then*

$$\begin{aligned}\int_{c_2}^{d_2} f(y)dy - \lambda[\mathscr{M}(f;c_2,d_2)-f(b)] &\le \int_a^b f(x)g(x)dx \\ &\le \int_{c_1}^{d_1} f(y)dy + \lambda[f(a)-\mathscr{M}(f;c_1,d_1)]\end{aligned} \tag{3.49}$$

where

$$\mathscr{M}(f;a,b) = \frac{1}{b-a}\int_a^b f(x)dx. \tag{3.50}$$

Proof. Let us transform the difference $\int_a^b f(x)g(x)dx - \int_{c_2}^{d_2} f(y)dy$ using the identity $1 = \frac{\int_a^b g(x)dx}{d_2-c_2}$ and integration by parts:

$$\begin{aligned}\int_a^b f(x)g(x)dx - \int_{c_2}^{d_2} f(y)dy &= \int_a^b f(x)g(x)dx - \frac{1}{d_2-c_2}\int_{c_2}^{d_2} f(y)dy \int_a^b g(x)dx \\ &= \int_a^b f(x)g(x)dx - \mathscr{M}(f;c_2,d_2)\int_a^b g(x)dx = \int_a^b g(x)[f(x)-\mathscr{M}(f;c_2,d_2)]dx \\ &= G(x)[f(x)-\mathscr{M}(f;c_2,d_2)]|_a^b - \int_a^b G(x)df(x) = \lambda[f(b)-\mathscr{M}(f;c_2,d_2)] - \int_a^b G(x)df(x)\end{aligned}$$

since $G(b)=\lambda$ and $G(a)=0$. Now we have

$$\int_a^b f(x)g(x)dx - \int_{c_2}^{d_2} f(y)dy + \lambda[\mathscr{M}(f;c_2,d_2)-f(b)] = -\int_a^b G(x)df(x) \ge 0$$

since f is nonincreasing and $g\ge 0$. So, the first inequality is proved.
The second inequality follows from the identity

$$\int_a^b f(x)g(x)dx - \int_{c_1}^{d_1} f(y)dy = \lambda[f(a)-\mathscr{M}(f;c_1,d_1)] + \int_a^b [\lambda - G(x)]df(x).$$

□

As noted by Cerone, the left-hand and right-hand inequalities in (3.49) can be simplified to $\lambda f(b)$ and $\lambda f(a)$ respectively since

$$\int_c^d f(y)dy = \lambda \mathscr{M}(f;c,d).$$

Hence, (3.49) becomes

$$\lambda f(b) \leq \int_a^b f(x)g(x)dx \leq \lambda f(a).$$

This result can also be obtained directly since

$$\inf_{x\in[a,b]} f(x)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \sup_{x\in[a,b]} f(x)\int_a^b g(x)dx.$$

Cerone's result for function f/k is generalized by Pečarić, Perušić and Smoljak in [118]. First we give the following lemma in which some useful identities are given.

Lemma 3.1 *Let k be a positive integrable function on $[a,b]$ and $f,g,h : [a,b] \to \mathbb{R}$ be integrable functions on $[a,b]$. Further, let $[c,d] \subseteq [a,b]$ with $\int_c^d h(t)k(t)dt = \int_a^b g(t)k(t)dt$. Then the following identities hold:*

$$\begin{aligned}\int_c^d f(t)h(t)dt - \int_a^b f(t)g(t)dt &= \int_a^c \left(\frac{f(d)}{k(d)} - \frac{f(t)}{k(t)}\right) g(t)k(t)dt \\ &+ \int_c^d \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) k(t)[h(t)-g(t)]dt + \int_d^b \left(\frac{f(d)}{k(d)} - \frac{f(t)}{k(t)}\right) g(t)k(t)dt\end{aligned} \tag{3.51}$$

and

$$\begin{aligned}\int_a^b f(t)g(t)dt - \int_c^d f(t)h(t)dt &= \int_a^c \left(\frac{f(t)}{k(t)} - \frac{f(c)}{k(c)}\right) g(t)k(t)dt \\ &+ \int_c^d \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right) k(t)[h(t)-g(t)]dt + \int_d^b \left(\frac{f(t)}{k(t)} - \frac{f(c)}{k(c)}\right) g(t)k(t)dt.\end{aligned} \tag{3.52}$$

Proof. We have

$$\begin{aligned}&\int_c^d f(t)h(t)dt - \int_a^b f(t)g(t)dt = \int_c^d k(t)[h(t)-g(t)]\frac{f(t)}{k(t)}dt \\ &- \left[\int_a^c \frac{f(t)}{k(t)}g(t)k(t)dt + \int_d^b \frac{f(t)}{k(t)}g(t)k(t)dt\right] = \int_a^c \left(\frac{f(d)}{k(d)} - \frac{f(t)}{k(t)}\right) g(t)k(t)dt \\ &+ \int_c^d \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) k(t)[h(t)-g(t)]dt + \int_d^b \left(\frac{f(d)}{k(d)} - \frac{f(t)}{k(t)}\right) g(t)k(t)dt \\ &+ \frac{f(d)}{k(d)}\left[\int_c^d k(t)h(t)dt - \int_a^c g(t)k(t)dt - \int_c^d k(t)g(t)dt - \int_d^b g(t)k(t)dt\right].\end{aligned} \tag{3.53}$$

Since

$$\int_c^d k(t)h(t)dt = \int_a^b k(t)g(t)dt,$$

we have

$$\frac{f(d)}{k(d)}\left[\int_c^d k(t)h(t)dt - \int_a^c g(t)k(t)dt - \int_c^d k(t)g(t)dt - \int_d^b g(t)k(t)dt\right] = 0.$$

Hence, (3.51) follows from (3.53).
Furthermore, (3.52) can be obtained in a similar way so the proof is completed. □

Now we proceed to the generalization of Cerone's result, [118].

Theorem 3.43 *Let k be a positive integrable function on $[a,b]$ and $f,g,h:[a,b]\to\mathbb{R}$ be integrable functions on $[a,b]$ such that f/k is nonincreasing. Further, let $0\le g\le h$ and $\int_c^d h(t)k(t)dt = \int_a^b g(t)k(t)dt$, where $[c,d]\subseteq[a,b]$. Then*

$$\int_a^b f(t)g(t)dt \le \int_c^d f(t)h(t)dt + R_g(c,d) \tag{3.54}$$

holds, where

$$R_g(c,d) = \int_a^c \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) g(t)k(t)dt \ge 0. \tag{3.55}$$

If f/k is a nondecreasing function, then the inequalities in (3.54) and (3.55) are reversed.

Proof. Since f/k is nonincreasing, k is positive and $0\le g\le h$ we have

$$\int_c^d \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) k(t)[h(t)-g(t)]dt \ge 0, \tag{3.56}$$

$$\int_d^b \left(\frac{f(d)}{k(d)} - \frac{f(t)}{k(t)}\right) g(t)k(t)dt \ge 0 \tag{3.57}$$

and $R_g(c,d)\ge 0$. Now, from (3.51), (3.56) and (3.57) we have

$$\begin{aligned}
&\int_c^d f(t)h(t)dt - \int_a^b f(t)g(t)dt + \int_a^c \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) g(t)k(t)dt \\
&= \int_c^d \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) k(t)[h(t)-g(t)]dt + \int_d^b \left(\frac{f(d)}{k(d)} - \frac{f(t)}{k(t)}\right) g(t)k(t)dt \\
&\ge 0.
\end{aligned} \tag{3.58}$$

Hence, inequality (3.54) holds. □

In a similar way we obtain a lower bound for $\int_a^b f(t)g(t)dt$.

Theorem 3.44 *Let k be a positive integrable function on $[a,b]$ and $f,g,h:[a,b]\to\mathbb{R}$ be integrable functions on $[a,b]$ such that f/k is nonincreasing. Further, let $0\leq g\leq h$ and $\int_c^d h(t)k(t)dt=\int_a^b g(t)k(t)dt$, where $[c,d]\subseteq[a,b]$. Then*

$$\int_c^d f(t)h(t)dt-r_g(c,d)\leq\int_a^b f(t)g(t)dt \tag{3.59}$$

holds, where

$$r_g(c,d)=\int_d^b\left(\frac{f(c)}{k(c)}-\frac{f(t)}{k(t)}\right)g(t)k(t)dt\geq 0. \tag{3.60}$$

If f/k is a nondecreasing function, then the inequalities in (3.59) and (3.60) are reversed.

Proof. Since f/k is nonincreasing, k is positive and $0\leq g\leq h$ we have

$$\int_a^c\left(\frac{f(t)}{k(t)}-\frac{f(c)}{k(c)}\right)k(t)g(t)dt\geq 0, \tag{3.61}$$

$$\int_c^d\left(\frac{f(c)}{k(c)}-\frac{f(t)}{k(t)}\right)k(t)[h(t)-g(t)]dt\geq 0 \tag{3.62}$$

and $r_g(c,d)\geq 0$. Now, from (3.52), (3.61) and (3.62) we have

$$\begin{aligned}&\int_a^b f(t)g(t)dt-\int_c^d f(t)h(t)dt+\int_d^b\left(\frac{f(c)}{k(c)}-\frac{f(t)}{k(t)}\right)g(t)k(t)dt\\&=\int_a^c\left(\frac{f(t)}{k(t)}-\frac{f(c)}{k(c)}\right)g(t)k(t)dt+\int_c^d\left(\frac{f(c)}{k(c)}-\frac{f(t)}{k(t)}\right)k(t)[h(t)-g(t)]dt\geq 0.\end{aligned} \tag{3.63}$$

Hence, inequality (3.59) holds. □

Remark 3.11 If we take $c=a$ and $d=a+\lambda$ in Theorem 3.43 we obtain Mercer's generalization of the right-hand side Steffensen's inequality given in Theorem 3.20. If we take $c=b-\lambda$ and $d=b$ in Theorem 3.44, then we obtain a generalization of the left-hand side Steffensen's inequality which is given in Theorem 3.21.

In the previous section we proved generalization of the Wu and Srivastava refinement of Steffensen's inequality for nonincreasing function f/k. In the following theorems we generalize these results to obtain bounds which involve any two subintervals (see [118]).

Theorem 3.45 *Let k be a positive integrable function on $[a,b]$ and $f,g,h:[a,b]\to\mathbb{R}$ be integrable functions on $[a,b]$ such that f/k is nonincreasing. Further, let $0\leq g\leq h$ and $\int_c^d h(t)k(t)dt=\int_a^b g(t)k(t)dt$, where $[c,d]\subseteq[a,b]$.*

a) Then

$$\begin{aligned}\int_a^b f(t)g(t)dt&\leq\int_c^d f(t)h(t)dt-\int_c^d\left(\frac{f(t)}{k(t)}-\frac{f(d)}{k(d)}\right)k(t)[h(t)-g(t)]dt\\&+R_g(c,d)\leq\int_c^d f(t)h(t)dt+R_g(c,d)\end{aligned} \tag{3.64}$$

holds, where $R_g(c,d)$ is defined by (3.55).

b) Then

$$\begin{aligned}\int_c^d f(t)h(t)dt - r_g(c,d) \le \int_c^d f(t)h(t)dt \\ + \int_c^d \left(\frac{f(c)}{k(c)} - \frac{f(t)}{k(t)}\right) k(t)[h(t)-g(t)]dt - r_g(c,d) \le \int_a^b f(t)g(t)dt\end{aligned} \tag{3.65}$$

holds, where $r_g(c,d)$ is defined by (3.60).

If f/k is a nondecreasing function, then the inequalities in (3.64) and (3.65) are reversed.

Proof. Similar to the proof of Theorem 3.43 and Theorem 3.44. □

Remark 3.12 If we take $c=a$ and $d=a+\lambda$ in Theorem 3.45 a), or $c=b-\lambda$ and $d=b$ in Theorem 3.45 b), we obtain refinements given in Theorem 3.25 and Theorem 3.26.

Let us see what we can say about weaker conditions on the function g.

Using identity (3.58) and applying integration by parts in it we obtain that under assumptions of Theorem 3.43 and with assumptions

$$\int_c^x k(t)g(t)dt \le \int_c^x k(t)h(t)dt,\ c \le x \le d \text{ and } \int_x^b k(t)g(t)dt \ge 0,\ d \le x \le b, \tag{3.66}$$

inequality (3.54) is valid, [118].

Putting $c=a$ and $d=a+\lambda$ in (3.66) we obtain the following conditions

$$\begin{aligned}\int_a^x k(t)g(t)dt \le \int_a^x k(t)h(t)dt,\ a \le x \le a+\lambda \\ \text{and } \int_x^b k(t)g(t)dt \ge 0,\ a+\lambda \le x \le b.\end{aligned} \tag{3.67}$$

In Theorem 3.30 it is proved that for nonnegative function h, conditions (3.67) are equivalent to

$$\int_a^x k(t)g(t)dt \le \int_a^x k(t)h(t)dt \quad \text{and} \quad \int_x^b k(t)g(t)dt \ge 0, \quad \forall x \in [a,b].$$

Hence, we again obtain the sufficient conditions given in Theorem 3.30.

In a similar manner, we conclude that the condition $0 \le g \le h$ in Theorem 3.44 can be substituted by the assumptions

$$\begin{aligned}\int_x^d k(t)g(t)dt \le \int_x^d k(t)h(t)dt, c \le x \le d \\ \text{and } \int_a^x k(t)g(t)dt \ge 0, a \le x \le c.\end{aligned} \tag{3.68}$$

Putting $c=b-\lambda$ and $d=b$ in (3.68) we obtain the conditions

$$\begin{aligned}\int_x^b k(t)g(t)dt \le \int_x^b k(t)h(t)dt,\ b-\lambda \le x \le b \\ \text{and } \int_a^x k(t)g(t)dt \ge 0,\ a \le x \le b-\lambda.\end{aligned} \tag{3.69}$$

In Theorem 3.31 it is proved that for a nonnegative function h, conditions (3.69) are equivalent to

$$\int_x^b k(t)g(t)dt \leq \int_x^b k(t)h(t)dt \quad \text{and} \quad \int_a^x k(t)g(t)dt \geq 0, \quad \forall x \in [a,b].$$

Hence, we again obtain the sufficient conditions given in Theorem 3.31.

Taking $k \equiv 1$ and $h \equiv 1$ in (3.66) and (3.68) we obtain weaker conditions for the function g in Cerone's generalization of Steffensen's inequality.

Theorem 3.46 *Let $f,g:[a,b] \to \mathbb{R}$ be integrable functions on $[a,b]$ such that f is nonincreasing. Let $\lambda = d-c = \int_a^b g(t)dt$, where $[c,d] \subseteq [a,b]$.*

a) If

$$\int_c^x g(t)dt \leq x-c, \quad c \leq x \leq d \quad \text{and} \quad \int_x^b g(t)dt \geq 0, \quad d \leq x \leq b,$$

then

$$\int_a^b f(t)g(t)dt \leq \int_c^d f(t)dt + \int_a^c (f(t)-f(d))\,g(t)dt.$$

b) If

$$\int_x^d g(t)dt \leq d-x, \quad c \leq x \leq d \quad \text{and} \quad \int_a^x g(t)dt \geq 0, \quad a \leq x \leq c,$$

then

$$\int_c^d f(t)dt - \int_d^b (f(c)-f(t))\,g(t)dt \leq \int_a^b f(t)g(t)dt.$$

From the above discussion, it is obvious that if the condition $0 \leq g \leq h$ in Theorem 3.45 is substituted with (3.66), then (3.64) holds, while if that condition is substituted with (3.68), then (3.65) is valid. Namely, Theorem 3.45 has the following form.

Theorem 3.47 *Let k be a positive integrable function on $[a,b]$ and $f,g,h:[a,b] \to \mathbb{R}$ be integrable functions on $[a,b]$ such that f/k is nonincreasing. Let $\int_c^d h(t)k(t)dt = \int_a^b g(t)k(t)dt$, where $[c,d] \subseteq [a,b]$. If*

$$\int_c^x k(t)g(t)dt \leq \int_c^x k(t)h(t)dt,\ c \leq x \leq d \text{ and } \int_x^b k(t)g(t)dt \geq 0,\ d \leq x \leq b,$$

holds, then

$$\begin{aligned}\int_a^b f(t)g(t)dt &\leq \int_c^d f(t)h(t)dt + R_g(c,d) - \int_c^d \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) k(t)[h(t)-g(t)]dt \\ &\leq \int_c^d f(t)h(t)dt + R_g(c,d),\end{aligned} \tag{3.70}$$

where $R_g(c,d) = \int_a^c \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) g(t)k(t)dt$.

Proof. Using identity (3.51) and applying integration by parts we obtain

$$\begin{aligned}
&\int_c^d f(t)h(t)dt - \int_a^b f(t)g(t)dt + \int_a^c \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) g(t)k(t)dt \\
&- \int_c^d \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) k(t)[h(t)-g(t)]dt = \int_d^b \left(\frac{f(d)}{k(d)} - \frac{f(t)}{k(t)}\right) g(t)k(t)dt \\
&= -\int_d^b \left(\int_x^b g(t)k(t)dt\right) d\left(\frac{f(x)}{k(x)}\right) \geq 0
\end{aligned}$$

when

$$\int_x^b k(t)g(t)dt \geq 0, \quad d \leq x \leq b.$$

Furthermore,

$$\int_c^d \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) k(t)[h(t)-g(t)]dt = -\int_c^d \left(\int_c^x k(t)[h(t)-g(t)]dt\right) d\left(\frac{f(x)}{k(x)}\right) \geq 0$$

when

$$\int_c^x k(t)g(t)dt \leq \int_c^x k(t)h(t)dt, \quad c \leq x \leq d.$$

Hence (3.70) holds. □

The following theorem states what happens if only the second condition in (3.66) is valid.

Theorem 3.48 *Let k be a positive integrable function on $[a,b]$ and $f,g,h:[a,b]\to\mathbb{R}$ be integrable functions on $[a,b]$ such that f/k is nonincreasing. Let $\int_c^d h(t)k(t)dt = \int_a^b g(t)k(t)dt$, where $[c,d]\subseteq[a,b]$. If*

$$\int_x^b k(t)g(t)dt \geq 0 \quad \text{for } d \leq x \leq b,$$

then

$$\begin{aligned}
\int_a^b f(t)g(t)dt \leq \int_c^d f(t)h(t)dt &+ \int_a^c \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) g(t)k(t)dt \\
&- \int_c^d \left(\frac{f(t)}{k(t)} - \frac{f(d)}{k(d)}\right) k(t)[h(t)-g(t)]dt.
\end{aligned}$$

If we additionally have

$$\int_c^x k(t)g(t)dt \leq \int_c^x k(t)h(t)dt \quad \text{for } c \leq x \leq d,$$

then (3.70) holds.

Proof. Similar to the proof of Theorem 3.47. □

Of course, as with Theorem 3.47, we state a weaker version of Theorem 3.45 b) and an analogue of Theorem 3.48 as following:

Theorem 3.49 *Let k be a positive integrable function on $[a,b]$ and $f,g,h:[a,b]\to\mathbb{R}$ be integrable functions on $[a,b]$ such that f/k is nonincreasing. Let $\int_c^d h(t)k(t)dt=\int_a^b g(t)k(t)dt$, where $[c,d]\subseteq[a,b]$. If*

$$\int_a^x k(t)g(t)dt \geq 0 \quad \text{for } a\leq x\leq c,$$

then

$$\int_c^d f(t)h(t)dt - \int_d^b \left(\frac{f(c)}{k(c)}-\frac{f(t)}{k(t)}\right) g(t)k(t)dt \\ + \int_c^d \left(\frac{f(c)}{k(c)}-\frac{f(t)}{k(t)}\right) k(t)[h(t)-g(t)]dt \leq \int_a^b f(t)g(t)dt.$$

If we additionally have

$$\int_x^d k(t)g(t)dt \leq \int_x^d k(t)h(t)dt \quad \text{for } c\leq x\leq d,$$

then

$$\int_c^d f(t)h(t)dt - r_g(c,d) \leq \int_c^d f(t)h(t)dt - r_g(c,d) \\ + \int_c^d \left(\frac{f(c)}{k(c)}-\frac{f(t)}{k(t)}\right) k(t)[h(t)-g(t)]dt \leq \int_a^b f(t)g(t)dt, \tag{3.71}$$

where $r_g(c,d)=\int_d^b \left(\frac{f(c)}{k(c)}-\frac{f(t)}{k(t)}\right) g(t)k(t)dt$.

Proof. Similar to the proof of Theorem 3.47 using identity (3.52). □

3.4 Multidimensional and measure spaces generalizations

In [105] Pečarić gave the multidimensional Steffensen's inequality. First, we give the following theorem which is used in the proof of the multidimensional version.

Theorem 3.50 *Let $p:[a,b]^m\to\mathbb{R}$ and $r:[a,b]\to\mathbb{R}$ be integrable functions. For all nonnegative nonincreasing functions $f_j:[a,b]\to\mathbb{R}$ $(1\leq j\leq m)$*

$$\int_a^b \cdots \int_a^b p(x_1,\ldots,x_m)f_1(x_1)\cdots f_m(x_m)dx_1\cdots dx_m \leq \int_a^b r(x)f_1(x)\cdots f_m(x)dx \tag{3.72}$$

if and only if

$$P(y_1,\dots,y_m) \le R(\min\{y_1,\dots,y_m\}) \quad (\forall y_1,\dots,y_m \in [a,b]) \tag{3.73}$$

where

$$P(y_1\dots,y_m) = \int_a^{y_1}\cdots\int_a^{y_m} p(x_1,\dots,x_m)dx_1\cdots dx_m$$

and $R(y) = \int_a^y r(y)dy$.
If in (3.73) the reverse inequality holds, then the reverse inequality in (3.72) holds.

The multidimensional Steffensen's inequality is given in the following theorem.

Theorem 3.51 *Let* $f_j : [a,b] \to \mathbb{R}$ *(*$1 \le j \le m$*) and* $p : [a,b]^m \to \mathbb{R}$ *be nonnegative integrable functions such that* f_j *(*$1 \le j \le m$*) are nonincreasing functions and*

$$0 \le p(x_1,\dots,x_m) \le 1 \quad (\forall x_j \in [a,b],\ 1 \le j \le m).$$

Then

$$\begin{aligned}\frac{1}{(b-a)^{m-1}}\int_a^b\cdots\int_a^b p(x_1,\dots,x_m)f_1(x_1)\cdots f_m(x_m)dx_1\cdots dx_m& \\ \le \int_a^{a+\lambda} f_1(x)\cdots f_m(x)dx&\end{aligned} \tag{3.74}$$

where

$$\lambda = \frac{1}{(b-a)^{m-1}}\int_a^b\cdots\int_a^b p(x_1,\dots,x_m)dx_1\cdots dx_m.$$

Proof. Let $r(x) = (b-a)^{m-1}$ for $x \in [a,a+\lambda]$ and $r(x) = 0$ for $x \in (a+\lambda,b]$. Then, (3.74) could be written in the form (3.72). Now, we show that condition (3.73) is satisfied.
If $\min\{y_1,\dots,y_m\} \le a+\lambda$, then we have

$$\begin{aligned}P(y_1,\dots,y_m) &\le \int_a^{y_1}\cdots\int_a^{y_m} dx_1\cdots dx_m = (y_1-a)(y_2-a)\dots(y_m-a)\\ &\le (b-a)^{m-1}(\min\{y_1,\dots,y_m\}-a) = R(\min\{y_1,\dots y_m\}).\end{aligned}$$

If $\min\{y_1,\dots,y_m\} \ge a+\lambda$, then

$$P(y_1,\dots,y_m) \le P(b,\dots,b) = (b-a)^{m-1}\lambda = R(\min\{y_1,\dots,y_m\}).$$

□

Pečarić and Varošanec in [129] proved Steffensen's inequality for functions in several variables given in the following theorem. Let $\Lambda = (\lambda_1,\dots,\lambda_n)$, $\mathbf{t} = (t_1,\dots,t_n)$ and let $I = [\mathbf{a},\mathbf{b}] = \{\mathbf{t} : a_i \le t_i \le b_i, 1 \le i \le n\} = \prod_{i=1}^n [a_i,b_i]$.

Theorem 3.52 *Let $f : I = [\mathbf{a},\mathbf{b}] \to \mathbb{R}$ be an integrable function such that each function $f_k : t \mapsto f(x_1,x_2,\ldots,x_{k-1},t,x_{k+1},\ldots,x_n)$, $k = 1,\ldots,n$, has property (QD) described in Theorem 2.7. If $g(\mathbf{x}) = g_1(x_1)g_2(x_2)\cdots g_n(x_n)$ where each g_i is a nonnegative integrable function satisfying condition (QB), then*

$$\int_{[\mathbf{b}-\Lambda,\mathbf{b}]} f(\mathbf{x})d\mathbf{x} \le \int_{[\mathbf{a},\mathbf{b}]} f(\mathbf{x})g(\mathbf{x})d\mathbf{x} \le \int_{[\mathbf{a},\mathbf{a}+\Lambda]} f(\mathbf{x})d\mathbf{x} \tag{3.75}$$

holds, where $d\mathbf{x} = dx_1dx_2\cdots dx_n$ and $\lambda_i = \int_{a_i}^{b_i} g_i(t)dt$.

Proof. If the function f satisfies the assumptions, then the functions

$$F_{k+1} : t \mapsto \int_{a_k}^{a_k+\lambda_k} \cdots \int_{a_1}^{a_1+\lambda_1} f(x_1,\ldots,x_k,t,x_{k+2},\ldots,x_n)dx_1\cdots dx_k,$$

and $F_1 : t \mapsto f(t,x_2,\ldots,x_n)$ also satisfy property (QD) for all $k = 1,\ldots,n-1$. So using Steffensen's inequality for the functions F_k and g_k for $k = 1,\ldots,n$ we have

$$\begin{aligned}
&\int_I f(\mathbf{x})g_1(x_1)\cdots g_n(x_n)d\mathbf{x} \\
&= \int_{a_n}^{b_n} \cdots \int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} f(\mathbf{x})g_1(x_1)dx_1 \right) g_2(x_2)\cdots g_n(x_n)dx_2\cdots dx_n \\
&\le \int_{a_n}^{b_n} \cdots \int_{a_2}^{b_2} \left(\int_{a_1}^{a_1+\lambda_1} f(\mathbf{x})dx_1 \right) g_2(x_2)\cdots g_n(x_n)dx_2\cdots dx_n \\
&= \int_{a_n}^{b_n} \cdots \int_{a_3}^{b_3} \left(\int_{a_2}^{b_2} \left(\int_{a_1}^{a_1+\lambda_1} f(\mathbf{x})dx_1 \right) g_2(x_2)dx_2 \right) g_3(x_3)\cdots g_n(x_n)dx_3\cdots dx_n \\
&= \int_{a_n}^{b_n} \cdots \int_{a_3}^{b_3} \left(\int_{a_2}^{b_2} F_2(x_2)g_2(x_2)dx_2 \right) g_3(x_3)\cdots g_n(x_n)dx_3\cdots dx_n \\
&\le \cdots \le \int_{[\mathbf{a},\mathbf{a}+\Lambda]} f(\mathbf{x})d\mathbf{x}.
\end{aligned}$$

Similarly, we can prove the second inequality. □

Using the same idea, Pečarić and Varošanec obtained the multidimensional generalization given in the following theorem (see [129]).

Theorem 3.53 *Let f be a nondecreasing function in each variable and g_i, $i = 1,\ldots,n$ nonnegative integrable functions such that*

$$0 \le \int_{x_i}^{b_i} g_i(t)dt \le b_i - x_i \quad \text{and} \quad 0 \le \int_{a_i}^{x_i} g_i(t)dt \le x_i - a_i$$

for all $x_i \in [a_i,b_i]$, $i = 1,\ldots,n$. Then (3.75) holds.

Another Steffensen-type inequality for functions of several variables is given in the following theorem (see [129]).

Theorem 3.54 *Let λ_i $(i = 1,\dots,n)$ be nonnegative real numbers less than or equal to $b_i - a_i$.*

(a) Let $f : I = [\mathbf{a},\mathbf{b}] \to \mathbb{R}$ be an integrable function with property (QD) in each variable. If $g_i : [a_i,b_i] \to \mathbb{R}$ are integrable functions such that

$$0 \le \lambda_i g_i(x) \le \int_{a_i}^{b_i} g_i(t)dt \quad \text{for all } x \in [a_i,b_i],\ i = 1,\dots,n,$$

then

$$\frac{\int_{[\mathbf{b}-\Lambda,\mathbf{b}]} f(\mathbf{x})d\mathbf{x}}{\lambda_1\cdots\lambda_n} \le \frac{\int_{[\mathbf{a},\mathbf{b}]} f(\mathbf{x})g_1(x_1)\cdots g_n(x_n)d\mathbf{x}}{\int_{[\mathbf{a},\mathbf{b}]} g_1(x_1)\cdots g_n(x_n)d\mathbf{x}} \le \frac{\int_{[\mathbf{a},\mathbf{a}+\Lambda]} f(\mathbf{x})d\mathbf{x}}{\lambda_1\cdots\lambda_n}. \tag{3.76}$$

(b) Let f be a nondecreasing function in each variable and g_i positive integrable functions such that

$$0 \le \lambda_i \int_{x_i}^{b_i} g_i(t)dt \le (b_i - x_i)\int_{a_i}^{b_i} g_i(t)dt$$

and

$$0 \le \lambda_i \int_{a_i}^{x_i} g_i(t)dt \le (x_i - a_i)\int_{a_i}^{b_i} g_i(t)dt$$

for all $x_i \in [a_i,b_i], i = 1,\dots,n$. Then (3.76) holds.

Proof. The functions

$$G_i(t) = \frac{\lambda_i g_i(t)}{\int_{a_i}^{b_i} g_i(x)dx}$$

satisfy the assumptions of Theorem 3.52 and, under these substitutions, inequality (3.75) becomes inequality (3.76). □

We use the following notation:

$$t = (t_1,\dots,t_n), I = [\mathbf{a},\mathbf{b}] = \{\mathbf{t} : a_i \le t_i \le b_i, 1 \le i \le n\} = \prod_{i=1}^{n}[a_i,b_i].$$

If $\mathbf{t}$ and $\mathbf{x}$ are two $n-$tuples, then by $\mathbf{t}+\mathbf{x}$ we mean an $n-$tuple $(t_1+x_1,\dots,t_n+x_n)$.

The multidimensional generalization of the right-hand side of the Steffensen inequality is given in the following theorem (see [129]).

Theorem 3.55 *Let μ be a measure such that I is a finite $\mu-$measurable set, and f, g and fg be $\mu-$integrable functions on I. Let $\Lambda = (\lambda_1,\dots,\lambda_n)$ be a positive $n-$tuple, $I_1 := [\mathbf{a},\mathbf{a}+\Lambda] \subset I$ and let A be a real number such that*

$$A\mu(I_1) = \int_I g d\mu,$$

and let one of the following cases holds $\mu-$almost everywhere:

(1) $g(\mathbf{t}) \leq A$ *and* $f(\mathbf{t}) \geq f(\mathbf{a}+\Lambda)$ *for* $\mathbf{t} \in I_1$;
$g(\mathbf{t}) \geq 0$ *and* $f(\mathbf{t}) \leq f(\mathbf{a}+\Lambda)$ *for* $\mathbf{t} \in I \setminus I_1$;

(2) $g(\mathbf{t}) \geq A$ *and* $f(\mathbf{t}) \leq f(\mathbf{a}+\Lambda)$ *for* $\mathbf{t} \in I_1$;
$g(\mathbf{t}) \leq 0$ *and* $f(\mathbf{t}) \geq f(\mathbf{a}+\Lambda)$ *for* $\mathbf{t} \in I \setminus I_1$.

Then we have

$$A \int_{I_1} f d\mu \geq \int_I f g d\mu. \tag{3.77}$$

Proof. It is easy to verify that the following identity holds:

$$\begin{aligned} A \int_{I_1} f(\mathbf{t}) d\mu - \int_I f(\mathbf{t}) g(\mathbf{t}) d\mu &= \int_{I_1} (f(\mathbf{t}) - f(\mathbf{a}+\Lambda))(A - g(\mathbf{t})) d\mu \\ + \int_{I \setminus I_1} (f(\mathbf{a}+\Lambda) - f(\mathbf{t})) g(\mathbf{t}) d\mu &+ f(\mathbf{a}+\Lambda) \left(A\mu(I_1) - \int_I g(\mathbf{t}) d\mu \right). \end{aligned}$$

The last term is equal to zero, and f and g satisfy either case (1) or (2), so inequality (3.77) is valid. □

A generalization of the left-hand side of Steffensen's inequality is given in the following theorem (see [129]).

Theorem 3.56 *Let μ be a measure such that $I := [\mathbf{a}, \mathbf{b}]$ is a finite $\mu-$measurable set, and f, g and fg be $\mu-$integrable functions on I. Let $\Lambda = (\lambda_1, \ldots, \lambda_n)$ be a positive $n-$tuple, $I_2 := [\mathbf{b} - \Lambda, \mathbf{b}] \subset I$ and let A be a real number such that*

$$A\mu(I_2) = \int_I g d\mu,$$

and let one of the following cases holds $\mu-$almost everywhere:

(1) $g(\mathbf{t}) \leq A$ *and* $f(\mathbf{t}) \leq f(\mathbf{b}-\Lambda)$ *for* $\mathbf{t} \in I_2$;
$g(\mathbf{t}) \geq 0$ *and* $f(\mathbf{t}) \geq f(\mathbf{b}-\Lambda)$ *for* $\mathbf{t} \in I \setminus I_2$;

(2) $g(\mathbf{t}) \geq A$ *and* $f(\mathbf{t}) \geq f(\mathbf{b}-\Lambda)$ *for* $\mathbf{t} \in I_2$;
$g(\mathbf{t}) \leq 0$ *and* $f(\mathbf{t}) \leq f(\mathbf{b}-\Lambda)$ *for* $\mathbf{t} \in I \setminus I_2$.

Then

$$A \int_{I_2} f d\mu \leq \int_I f g d\mu.$$

Proof. The proof is similar to the proof of Theorem 3.55 based on the following identity:

$$\begin{aligned} \int_I f g d\mu - A \int_{I_2} f d\mu &= \int_{I \setminus I_2} (f(\mathbf{t}) - f(\mathbf{b}-\Lambda)) g(\mathbf{t}) d\mu \\ + \int_{I_2} (f(\mathbf{b}-\Lambda) - f(\mathbf{t}))(A - g(\mathbf{t})) d\mu &+ f(\mathbf{b}-\Lambda) \left(\int_I g d\mu - A\mu(I_2) \right). \end{aligned}$$

□

Remark 3.13 For $n = 1$, $A = 1$ and $\mu = \mu_F$, where μ_F is the Stieltjes measure with $F(x) = x$, from Theorems 3.55 and 3.56 we obtain the generalized one-dimensional Steffensen's inequality given in Theorem 2.7.

Let $f \in M_0$ where M_0 is the class of nonnegative nondecreasing functions defined on interval $[a,b]$. Then $f(x) = \int_a^x d\nu(t)$ for some nonnegative Borel measure ν. We introduce the notation $x_+ = \max(x,0)$. Also x_+^n denotes $(x_+)^n$ except that 0^0 is interpreted as 0. Thus the characteristic function of $[t,\infty)$ is $(x-t)_+^0$. Now the above formula for $f \in M_0$ may be written as

$$f(x) = \int_a^b (x-t)_+^0 d\nu(t).$$

The class of functions which we consider generalizes this formula. Let M_k denote the class of functions f defined on the interval $[a,b]$ with the representation

$$f(x) = \int_a^b (x-t)_+^k d\nu(t), x \in [a,b], \tag{3.78}$$

for some nonnegative regular Borel measure ν. Note that k need not be an integer, although the case of great interest is when it is an integer. In particular, M_1 is the class of increasing convex functions with a value zero at a. More generally, if $f \in C^{(n+1)}[a,b]$ with $f^{(i)}(a) = 0$, $i = 0,\dots,n-1$, and $f^{(n)} \geq 0, f^{(n+1)} \geq 0$ on $[a,b]$, then $f \in M_n$.

In [42] Fink proved the following theorem.

Theorem 3.57 *Let μ be a (signed) regular Borel measure such that $\int_a^b |d\mu| < \infty$. Then*

$$\int_a^b f(x) d\mu(x) \geq 0 \quad \textit{for all } f \in M_k \tag{3.79}$$

if and only if

$$\int_a^b (x-t)_+^k d\mu(x) \geq 0 \quad \textit{for } t \in [a,b]. \tag{3.80}$$

Proof. Using representation (3.78) in (3.79) and Fubini's theorem, (3.79) is equivalent to

$$\int_a^b d\nu(t) \int_a^b (x-t)_+^k d\mu(x) \geq 0$$

for all nonnegative Borel measures ν. This holds if and only if (3.80) is valid. □

In [42] Fink proved generalizations of Steffensen's inequality for the class M_k. We give Fink's results on the interval $[a,b] \subset \mathbb{R}_0^+$.

Theorem 3.58 *Let σ be a (signed) regular Borel measure such that $\int_a^b |d\sigma| < \infty$. Then*

$$\int_a^b f(x) d\sigma(x) \geq \int_a^{a+\lambda} f(x) dx \tag{3.81}$$

for all $f \in M_k$ if and only if

$$\int_a^b (x-t)_+^k d\sigma(x) \geq 0, \quad t \in [a,b] \tag{3.82}$$

and

$$\lambda \leq \min_{a\leq t\leq b} \left\{ (t-a) + \left((k+1) \int_a^b (x-t)_+^k d\sigma(x) \right)^{\frac{1}{k+1}} \right\}. \tag{3.83}$$

Therefore the best possible choice is for equality in (3.83).

Proof. We apply Theorem 3.57 to the measure $d\mu = d\sigma - (a+\lambda-x)_+^0 dx$. Then (3.81) is equivalent to (3.79). Thus condition (3.80) is equivalent to

$$\int_a^b (x-t)_+^k d\sigma(x) \geq \int_a^b (x-t)_+^k (a+\lambda-x)_+^0 dx. \tag{3.84}$$

Since the right-hand side in (3.84) is nonnegative, (3.82) is necessary. Now taking $a \leq t \leq b$, (3.84) becomes

$$\int_a^b (x-t)_+^k d\sigma(x) \geq \frac{(a+\lambda-t)^{k+1}}{k+1}$$

and in turn

$$\lambda \leq (t-a) + \left((k+1) \int_a^b (x-t)_+^k d\sigma(x) \right)^{\frac{1}{k+1}}, \quad \text{for } t \in [a, a+\lambda]. \tag{3.85}$$

But since (3.82) holds, inequality (3.85) is true if $t \geq a+\lambda$. Thus (3.83) is necessary and sufficient since we may reverse all of the above steps. □

Remark 3.14 Let $f \in C^n[a,b]$ be an $n-$convex function with $f^{(k)}(a) = 0$, $k = 0, \ldots, n-2$ and $f^{(n-1)} \geq 0$. Then $f \in M_{n-1}$. Hence, we can apply Theorem 3.17 on function $f \in M_{n-1}$. Furthermore, by replacing gdx by $d\sigma$ we obtain (3.81) with

$$\lambda = \left(n \int_a^b (x-a)^{n-1} d\sigma(x) \right)^{1/n}.$$

Theorem 3.59 *If $\int_a^b |d\sigma| < \infty$, then the inequality*

$$\int_a^b f(x) d\sigma(x) \leq \int_{b-\lambda}^b f(x) dx \tag{3.86}$$

holds for all $f \in M_k$ if and only if

$$\int_a^b (x-t)_+^k d\sigma(x) \leq \frac{(b-t)^{k+1}}{k+1}, \quad t \in [a,b] \tag{3.87}$$

and

$$b-\lambda \leq \min_{a\leq t\leq b} \left\{ t + \left[(b-t)^{k+1} - (k+1) \int_a^b (x-t)_+^k d\sigma(x) \right]^{\frac{1}{k+1}} \right\}. \tag{3.88}$$

In particular, the best choice for $b-\lambda$ is equality in (3.88).

Proof. Similarly as in the proof of Theorem 3.58, we apply Theorem 3.57 to the measure $d\mu = (x-(b-\lambda))_+^0 dx - d\sigma$. □

If M_k^* is the class of functions $f \in C^{(k+1)}(a,b)$ with $f^{(k+1)}(x) \geq 0$ on $[a,b]$, then $f(x) - \sum_{j=0}^{k} f^{(j)}(a)\frac{x^j}{j!} \in M_k$. Hence, a particular case of Theorem 3.58 is given in the following corollary (see [42]).

Corollary 3.2 *If*

$$\int_a^x t^k d\sigma(t) \leq \frac{x^{k+1}}{k+1}, \quad \int_x^b t^k d\sigma(t) \geq 0, \quad a \leq x \leq b$$

and

$$\lambda = \left((k+1)\int_a^b s^k d\sigma(s)\right)^{\frac{1}{k+1}},$$

then (3.81) holds for all $f \in M_k$.

In several important instances, as in Corollary 3.2, the formula for λ is given for a specific choice of t, in that case the minimum is attained at $t=a$. Fink and Jodeit showed in [44] that if $f \in M_k$, then $f(x)x^{-k} \in M_0$. In general, the converse is not true. Version of Steffensen's inequality for $f(x)x^{-k}$ is given in the following theorem (see [42]).

Theorem 3.60 *If $f(x)/x^k \in M_0$, then*

(i)

$$\int_a^{a+\lambda} f(x)dx \leq \int_a^b f(x)d\sigma(x)$$

holds when

$$\int_t^b x^k d\sigma(x) \geq 0, \quad t \in [a,b] \tag{3.89}$$

and

$$a+\lambda = \min_{a \leq t \leq b}\left\{t^{k+1} + (k+1)\int_t^b x^k d\sigma(x)\right\}^{\frac{1}{k+1}};$$

(ii)

$$\int_a^b f(x)d\sigma(x) \leq \int_{b-\lambda}^b f(x)dx$$

holds when

$$\int_t^b x^k d\sigma(x) \leq \frac{b^{k+1}-t^{k+1}}{k+1}, \quad t \in [a,b] \tag{3.90}$$

and

$$b-\lambda = \min_{a \leq t \leq b}\left\{b^{k+1} - (k+1)\int_t^b x^k d\sigma(x)\right\}^{\frac{1}{k+1}}.$$

In particular, if

$$\int_a^t x^k d\sigma(x) \le \frac{t^{k+1}}{k+1},$$

then

$$a+\lambda = \left[(k+1)\int_a^b x^k d\sigma(x)\right]^{\frac{1}{k+1}}.$$

If (3.89) holds as well as (3.90), then

$$b-\lambda = \left[b-(k+1)\int_a^b x^k d\sigma(x)\right]^{\frac{1}{k+1}}.$$

Proof. We apply Theorem 3.57 with $k=0$ and f replaced by $f(x)/x^k$. To prove (i) we take $d\mu = x^k d\sigma - x^k(a+\lambda-x)_+^0 dx$ and to prove (ii) we take $d\mu = x^k(x-b+\lambda)_+^0 dx - x^k d\sigma$. □

Also, in Fink's paper [42] the multidimensional version of Steffensen's inequality is given. If $\mathbf{x} \in \mathbb{R}^n$ with nonnegative components, then $\int_{\mathbf{0}}^{\mathbf{x}} d\nu(\mathbf{t})$ means the multiple integral

$$\int\int_{0\le t_i\le x_i} \cdots \int d\nu(t_1,\ldots,t_n).$$

Let $\overline{M_0}$ be the class of functions with representation

$$f(\mathbf{x}) = \int_{\mathbf{0}}^{\mathbf{x}} d\nu(\mathbf{t})$$

for some nonnegative regular Borel measure ν.

Theorem 3.61 *Let $f \in \overline{M_0}$ and σ be a regular Borel measure such that $\int_{\mathbf{0}}^{\mathbf{1}} |d\sigma| < \infty$ and for every union of cubes E, $\int_E d\sigma \le \text{volume}(E)$. If $\int_{\mathbf{t}}^{\mathbf{1}} d\sigma \ge 0$ for all $\mathbf{t} \in [0,1]^n$, then*

$$\int_{\mathbf{0}}^{\lambda} f dx \le \int_{\mathbf{0}}^{\mathbf{1}} f d\sigma$$

where λ is the vector $(c,c,\ldots,c)$ with $c = 1-(1-\int_{\mathbf{0}}^{\mathbf{1}} d\sigma)^{1/n}$.

Theorem 3.62 *Assume $\int_{\mathbf{t}}^{\mathbf{1}} d\sigma \le \prod_{i=1}^n (1-t_i)$ and $f \in \overline{M_0}$. If*

$$\sup_{0\le t_i\le 1} \prod_{i=1}^n (1-t_i)^{-1} \int_{\mathbf{t}}^{\mathbf{1}} d\sigma \le (1-c)^n, \quad \lambda = (c,\ldots,c),$$

then

$$\int_{\mathbf{0}}^{\mathbf{1}} f d\sigma \le \int_{\lambda}^{\mathbf{1}} f dx.$$

In [56] Godunova and Levin gave a generalization of Steffensen's inequality using nonnegative kernel. Let F_K be the class of functions $f : [0,a] \to \mathbb{R}$ with the representation

$$f(t) = \int_X K(x,t)d\sigma(x), \tag{3.91}$$

where $K(x,t) \geq 0$ when $x \in X$, $t \in [0,a]$ and $\int_X d\sigma(x) = 1$ where σ is nondecreasing.

Theorem 3.63 *Let φ be a positive, increasing and concave function such that $\frac{\varphi'}{\varphi''}$ is a convex function and let*

$$h(x) = \int_0^x dh(t),\, g(x) = \int_0^x dg(t),\, h(0) = g(0) = 0,\, h(a) = 1,$$

where g and h are nondecreasing on $[0,a]$. The inequality

$$\int_0^a f(t)dg(t) \leq \varphi^{-1}\left(\int_0^a \varphi(f(t))dh(t)\right),$$

holds for each $f \in F_K$ if and only if for $x \in X$

$$\int_0^a K(x,t)dg(t) \leq \varphi^{-1}\left(\int_0^a \varphi(K(x,t))dh(t)\right).$$

Corollary 3.3 *If a function f is decreasing on $[0,a]$, h, g and φ satisfy the conditions of Theorem 3.63 and the function h has an inverse h^{-1}, then*

$$g(x) \leq \frac{\varphi^{-1}(\varphi(f(0)-f(a))h(x))}{f(0)-f(a)} = r(x), \tag{3.92}$$

holds for all $x \in [0,a]$ if and only if

$$\int_0^a f(t)dg(t) \leq \varphi^{-1}\left(\int_0^c \varphi(f(t))dh(t)\right), \tag{3.93}$$

where

$$c = h^{-1}\left(\frac{\varphi(g(a)[f(0)-f(a)])}{\varphi(f(0)-f(a))}\right) \leq a.$$

If in (3.92) and (3.93) we put $\varphi(u) = u$, $g(t) = \int_0^t G(\tau)d\tau$ and $h(t) = t$, we obtain Steffensen's inequality. The corrected version of Bellman's generalization follows from (3.93). For $\varphi(u) = u^p$, $0 < p \leq 1$, $g(t) = \int_0^t G(\tau)d\tau$, $h(t) = t$ we obtain

$$\int_0^a f(t)G(t)dt \leq \left(\int_0^{(\int_0^a G(\tau)d\tau)^p} f^p(t)dt\right)^{\frac{1}{p}}. \tag{3.94}$$

It is also proved that inequality (3.94) holds for any decreasing function f and $0 < p \leq 1$ if and only if

$$\int_0^x G(\tau)d\tau \leq x^{\frac{1}{p}}, \quad x \in [0,a].$$

In [138] Sadikova proved some further results for function $f \in F_K$ which covered the results of Godunova and Levin. First, she gave the following lemma.

Lemma 3.2 *Let $K(a,b)$ be integrable on $A \times B$, $\int_A d\mu(a) = 1$, $\int_B d\rho(b) = 1$, and let φ and ψ be continuous monotone real functions on $(0,\infty)$ such that*

1) $1/\varphi'$ is nondecreasing for increasing φ or $1/\varphi'$ is nonincreasing for decreasing φ

and

2) ψ' is nondecreasing for increasing ψ or ψ' is nonincreasing for decreasing ψ.

Then

$$\int_B \varphi^{-1}\left(\int_A \varphi(K(a,b))d\mu(a)\right)d\rho(b) \leq \psi^{-1}\left(\int_A \psi\left(\int_B K(a,b)d\rho(b)\right)d\mu(a)\right) \tag{3.95}$$

and

$$\int_B \psi^{-1}\left(\int_A \psi(K(a,b))d\mu(a)\right)d\rho(b) \geq \varphi^{-1}\left(\int_A \varphi\left(\int_B K(a,b)d\rho(b)\right)d\mu(a)\right). \tag{3.96}$$

Proof. The proof is based on the relationship between quasiarithmetic means M_{id} and M_φ where M_φ is defined by

$$M_\varphi = \varphi^{-1}\left(\int_X p(x)\varphi(f(x))dx\right), \quad \int_X p(x)dx = 1.$$

Namely, the inequality

$$M_\varphi \leq M_\psi \tag{3.97}$$

holds if the quotient $\frac{\psi'}{\varphi'}$ is:

(a) nondecreasing if continuous functions φ and ψ have the same monotonicity;

or

(b) nonincreasing if functions φ and ψ have the opposite monotonicity.

If φ satisfies the assumptions of Lemma, applying (3.97) on φ and id, we get

$$\varphi^{-1}\left(\int_A \varphi(K(a,b))d\mu(a)\right) \leq \int_A K(a,b)d\mu(a).$$

Integrating over B we have

$$\begin{aligned}\int_B \varphi^{-1}\left(\int_A \varphi(K(a,b))d\mu(a)\right)d\rho(b) &\leq \int_B\int_A K(a,b)d\mu(a)d\rho(b) \\ &= \int_A\int_B K(a,b)d\rho(b)d\mu(a)\end{aligned} \tag{3.98}$$

where we use Fubini's theorem. If we apply (3.97) on the functions id and ψ we get

$$\int_B K(a,b)d\rho(b) \leq \psi^{-1}\left(\int_B \psi(K(a,b))d\rho(b)\right)$$

and after integrating over A we get

$$\int_A \int_B K(a,b)d\rho(b)d\mu(a) \le \int_A \psi^{-1}\left(\int_B \psi(K(a,b))d\rho(b)\right)d\mu(a). \tag{3.99}$$

Form (3.98) and (3.99) we obtain inequality (3.95).
The similar proof holds for inequality (3.96). □

Theorem 3.64 *Let $f \in F_K$. Let the functions φ, ψ satisfy the conditions of Lemma 3.2, $h(x) = \int_0^x dh(t)$, $g(x) = \int_0^x dg(t)$, $x \in X$, $h(0) = g(0) = 0$, $h(a) = 1$ where g, h are nondecreasing functions. If*

$$\psi^{-1}\left(\int_0^a \psi(K(x,t))dh(t)\right) \le \int_0^a K(x,t)dg(t) \le \varphi^{-1}\left(\int_0^a \varphi(K(x,t))dh(t)\right), \tag{3.100}$$

then

$$\varphi^{-1}\left(\int_0^a \varphi(f(t))dh(t)\right) \le \int_0^a f(t)dg(t) \le \psi^{-1}\left(\int_0^a \psi(f(t))dh(t)\right).$$

Proof. Integrating all terms in (3.100) over X we get ($T = [0,a]$)

$$\begin{aligned}\int_X \psi^{-1}\left(\int_T \psi(K(x,t))dh(t)\right)d\sigma(x) &\le \int_X \int_T K(x,t)dg(t)d\sigma(x)\\ &\le \int_X \varphi^{-1}\left(\int_T \varphi(K(x,t))dh(t)\right)d\sigma(x).\end{aligned} \tag{3.101}$$

Using inequality (3.96) with $A = T$, $B = X$, $d\mu(a) = dh(t)$, $d\rho(b) = d\sigma(x)$ we get

$$\varphi^{-1}\left(\int_T \varphi\left(\int_X K(x,t)d\sigma(x)\right)dh(t)\right) \le \int_X \psi^{-1}\left(\int_T \psi(K(x,t))dh(t)\right)d\sigma(x). \tag{3.102}$$

Using inequality (3.95) with the same substitutions we get

$$\int_X \varphi^{-1}\left(\int_T \varphi(K(x,t))dh(t)\right)d\sigma(x) \le \psi^{-1}\left(\int_T \psi\left(\int_X K(x,t)d\sigma(x)\right)dh(t)\right). \tag{3.103}$$

The middle term in (3.101) is equal to $\int_T f(t)dg(t)$ by Fubini's theorem and by representation of $f \in F_K$. Combining (3.101), (3.102) and (3.103) we get the statement of Theorem. □

Gauchman and Evard extended Steffensen's inequality to integrals over general measure space (see [41], [51], [53]).

Before we give Evard and Gauchman's generalized Steffensen's inequality over general measure spaces which are not necessairly of finite measure, given in paper [41], we define upper-separating and lower-separating subsets for some function. Let $(X, \mathscr{A}, \mu)$ denote a measure space such that $0 < \mu(X) \le \infty$.

Definition 3.1 *Let $f \in L^0(X)$. Let $(U,c) \in \mathscr{A} \times \mathbb{R}$. We say that the pair (U,c) is* upper-separating *for f if and only if*

$$\operatorname*{ess\,sup}_{x \in X \setminus U} f(x) \leq c \leq \operatorname*{ess\,inf}_{x \in U} f(x).$$

In this case, we also say that the subset U of X is upper-separating for f. We say that the pair (U,c) is lower-separating *for f if and only if the pair $(X \setminus U, c)$ is upper-separating for f. In this case, we also say that the subset U of X is lower-separating for f.*

Theorem 3.65 *Let $f, g \in L^1(X)$ be such that $g \geq 0$ almost everywhere. Let $U, V \in \mathscr{A}$ be respectively upper-, lower-separating subsets for f. Let $h \in L^1(U \cup V)$ be such that $g \leq h$ almost everywhere on $U \cup V$ and*

$$\int_U h d\mu = \int_V h d\mu = \int_X g d\mu. \tag{3.104}$$

Then

$$\int_V f h d\mu \leq \int_X f g d\mu \leq \int_U f h d\mu. \tag{3.105}$$

Proof. The set U is an upper-separating, so there exists $c \in \mathbb{R}$ such that $\operatorname{ess\,sup}_{x \in X \setminus U} f(x) \leq c \leq \operatorname{ess\,inf}_{x \in U} f(x)$. Using hypothesis (3.104), we get

$$\begin{aligned} \int_X f g d\mu &= \int_{X \setminus U} f g d\mu + \int_U f g d\mu \leq c\left(\int_X g d\mu - \int_U g d\mu\right) + \int_U f g d\mu \\ &= c \int_U (h - g) d\mu + \int_U f g d\mu \leq \int_U f(h - g) d\mu + \int_U f g d\mu = \int_U f h d\mu. \end{aligned}$$

So the second inequality in (3.105) is proved. We obtain the first inequality of (3.105) by applying the second inequality to $\tilde{f} = (-f)$ and $\tilde{U} = V$, and using the fact that a pair $(U,c) \in \mathscr{A} \times \mathbb{R}$ is lower-separating for f if and only if the pair $(U, -c)$ is upper-separating for $(-f)$. □

Remark 3.15 In Theorem 3.65, the introduction of function h is equivalent to the change of measure $d\nu = h d\mu$. So it allowed Gauchman and Evard to include the case where $\mu(U) = \mu(V) = \infty$.

In the following corollary Gauchman and Evard proved the case where g is bounded and the separating subsets have finite measure.

Corollary 3.4 *Let $f, g \in L^1(X)$ be such that $g \geq 0$ almost everywhere and $\|g\|_\infty < \infty$. Let c be a positive real number such that $c \geq \|g\|_\infty$. Let $U, V \in \mathscr{A}$ be respectively upper-, lower-, separating for f and have the same measure*

$$\mu(U) = \mu(V) = \frac{1}{c}\|g\|_1. \tag{3.106}$$

Then

$$\int_V f d\mu \leq \frac{1}{c} \int_X f g d\mu \leq \int_U f d\mu.$$

Proof. Let $h(x) = c$ for all $x \in U \cup V$. Since $\mu(U \cup V) \leq \mu(U) + \mu(V) < \infty$, we have $h \in L^1(U \cup V)$. Moreover, (3.106) implies that g and h satisfy condition (3.104). Besides, the hypothesis $c \geq \|g\|_\infty$ implies that $h \geq g$ almost everywhere on $U \cup V$. Therefore, the conclusion follows by Theorem 3.65. □

Remark 3.16 For $c = 1$ and X an interval in $\mathbb{R}$, Corollary 3.4 gives the original Steffensen's inequality.

Evard and Gauchman applied their generalization of Steffensen's inequality given in Theorem 3.65 to obtain integral inequalities on composed functions. For more details see [41].

In[51] Gauchman proved the following theorem.

Theorem 3.66 *Let $l \geq 0$ be a real number. Let $\Phi : [l, \infty) \to \mathbb{R}$ be convex increasing and such that $\Phi(xy) \geq \Phi(x)\Phi(y)$ for all $x, y, xy \geq l$. Let $f, g \in L^1(X)$ be such that $f \geq l$ and $g \geq 0$ almost everywhere. Let λ be a real number such that $\Phi(\lambda) = \int_X g d\mu$. Assume that $0 \leq \lambda \leq \mu(X)$ and let (U, c) be an upper-separating pair for f such that $\mu(U) = \lambda$. Assume that*

$$f - c \leq \int_U (f - c) d\mu \tag{3.107}$$

almost everywhere on U. Then either

$$\int_X (\Phi \circ f) g d\mu \leq \Phi\left(\int_U f d\mu\right) \quad \text{or} \quad \int_U g d\mu \geq 1. \tag{3.108}$$

Proof. Without loss of generality let us suppose that $\mu(\{x \in X : f(x) > c\}) > 0$. Then from (3.107) we get $\int_U (f - c) d\mu > 0$. By integrating assumption (3.107) we get

$$\int_U (f - c) d\mu \leq \int_U (f - c) d\mu \cdot \mu(U)$$

which together with $\int_U (f - c) d\mu > 0$ gives $\mu(U) \geq 1$.

The assumption of convexity of Φ implies that Φ is a Wright-convex function, i.e., for any $x < y$, $z > 0$

$$\Phi(x) - \Phi(y) \leq \Phi(x + z) - \Phi(y + z).$$

Putting in the above-mentioned inequality $x \to f$, $y \to c$, $z \to c(\mu(U) - 1)$ we obtain

$$\begin{aligned} \Phi \circ f - \Phi(c) &\leq \Phi(f - c + c\mu(U)) - \Phi(c\mu(U)) \quad \text{a.e.} \\ &\leq \Phi\left(\int_U (f - c) d\mu + \int_U c d\mu\right) - \Phi(c\mu(U)) \\ &= \Phi\left(\int_U f d\mu\right) - \Phi(c\lambda) \end{aligned} \tag{3.109}$$

where in the last inequality we use that Φ is increasing. Multiplying by g and integrating over U we get

$$\int_U (\Phi \circ f) g d\mu - \int_U g \Phi(c) d\mu \leq \int_U \Phi\left(\int_U f d\mu\right) g d\mu - \int_U \Phi(c\lambda) g d\mu. \tag{3.110}$$

On the other hand, on $X\setminus U$ we get $f \le c$, i.e. $\Phi\circ f \le \Phi(c)$ since Φ is increasing, and finally

$$\int_{X\setminus U} (\Phi\circ f) g\,d\mu \le \Phi(c)\int_{X\setminus U} g\,d\mu$$

from which the following inequality follows:

$$\int_X ((\Phi\circ f)g - \Phi(c)g)\,d\mu \le \int_U ((\Phi\circ f)g - \Phi(c)g)\,d\mu. \tag{3.111}$$

Using (3.110) and (3.111) we obtain

$$\int_X (\Phi\circ f) g\,d\mu \le \int_U (\Phi\circ f) g\,d\mu - \Phi(c)\int_U g\,d\mu + \Phi(c)\Phi(\lambda).$$

Subtracting $\Phi(\int_U f\,d\mu)$ from the both sides of the above inequality and using submultiplicativity of Φ we get

$$\int_X (\Phi\circ f) g\,d\mu - \Phi\left(\int_U f\,d\mu\right) \le \left[\Phi\left(\int_U f\,d\mu\right) - \Phi(c\lambda)\right]\left(\int_U g\,d\mu - 1\right). \tag{3.112}$$

Since (U,c) is upper-separating for f, $f \ge c$ on U. Hence,

$$\int_U f\,d\mu \ge c\lambda \quad \text{and therefore} \quad \Phi\left(\int_U f\,d\mu\right) - \Phi(c\lambda) \ge 0$$

because Φ is increasing.

Assume first that

$$\Phi\left(\int_U f\,d\mu\right) - \Phi(c\lambda) = 0.$$

In that case

$$\Phi\left(\int_U f\,d\mu\right) = \Phi\left(\int_U c\,d\mu\right)$$

and, since Φ is increasing,

$$\int_U f\,d\mu = \int_U c\,d\mu, \quad \text{i.e.} \quad \int_U (f-c)\,d\mu = 0.$$

Since $f \ge c$ on U, we obtain that $f = c$ almost everywhere on U. Then

$$\Phi\left(\int_U f\,d\mu\right) = \Phi\left(\int_U c\,d\mu\right) = \Phi(c\lambda) \ge \Phi(c)\Phi(\lambda).$$

Since (U,c) is upper-separating for f, we obtain that $f = c$ almost everywhere on U and $f \le c$ almost everywhere on $X\setminus U$. Hence $f \le c$ almost everywhere on X. It follows that

$$\begin{aligned}\Phi\left(\int_U f\,d\mu\right) - \int_X (\Phi\circ f) g\,d\mu &\ge \Phi(c)\Phi(\lambda) - \int_X (\Phi\circ f) g\,d\mu \\ &\ge \Phi(c)\Phi(\lambda) - \int_X \Phi(c) g\,d\mu \\ &= \Phi(c)\left[\Phi(\lambda) - \int_X g\,d\mu\right] = 0.\end{aligned}$$

Hence, in the first case the theorem is proved. If we assume that $\Phi\left(\int_U f d\mu\right) - \Phi(c\lambda) > 0$, then (3.112) implies (3.108). $\square$

Remark 3.17 If we take the measure space to be a closed interval of $\mathbb{R}$, we obtain a more simplier case of Theorem 3.66. That result and a similar result for discrete case with appropriate consequences are given in [51].

In [53] Gauchman extended Pečarić's results given in Theorems 3.7 and 3.8 to the case of integrals over a general measure spaces. Those extensions are given in the following theorems. First we define continuous measure space.

Definition 3.2 *We say that a measure space $(X, \mathscr{A}, \mu)$ is* continuous *if and only if for all $A, B \in \mathscr{A}$ such that $A \subseteq B$ and $\mu(A) < \mu(B)$, there exists an increasing mapping ϕ : $[\mu(A), \mu(B)] \to \mathscr{A}$ such that $\phi(\mu(A)) = A$, $\phi(\mu(B)) = B$ and $\mu(\phi(u)) = u$ for all $u \in [\mu(A), \mu(B)]$.*

Theorem 3.67 *Let $(X, \mathscr{A}, \mu)$ be a continuous measure space with $\mu(X) < \infty$. Let $\alpha \geq 1$ be a real number and let f and g be functions on X such that f, $f^\alpha \cdot g \in L^1(X)$ and $f, g \geq 0$ almost everywhere on X. Set $\lambda = \left(\int_X g d\mu\right)^\alpha$. Let U be an upper-separating subset for f such that $\mu(U) = \lambda$. Assume that $g \cdot \left(\int_X g d\mu\right)^{\alpha-1} \leq 1$ almost everywhere on U. Then*

$$\left(\int_X gf d\mu\right)^\alpha \leq \int_U f^\alpha d\mu.$$

Proof. Let us denote $G = \int_X g d\mu$. By Jensen's inequality for the convex function x^α, $\alpha \geq 1$,

$$\left(\int_X f \cdot g d\mu\right)^\alpha \leq G^{\alpha-1} \int_X f^\alpha \cdot g d\mu.$$

Hence, it is enough to prove that

$$G^{\alpha-1} \int_X f^\alpha \cdot g d\mu \leq \int_U f^\alpha d\mu.$$

We proceed as follows:

$$\begin{aligned}
\int_U f^\alpha d\mu - G^{\alpha-1} \int_X f^\alpha \cdot g d\mu &= \int_U f^\alpha d\mu - G^{\alpha-1} \left(\int_U f^\alpha \cdot g d\mu + \int_{X \setminus U} f^\alpha \cdot g d\mu\right) \\
&= \int_U f^\alpha \left(1 - g \cdot G^{\alpha-1}\right) d\mu - G^{\alpha-1} \int_{X \setminus U} f^\alpha \cdot g d\mu.
\end{aligned}$$

By assumption, U is an upper-separating subset for f, so, there exists a real number c such that $f \geq c$ almost everywhere on U and $f \leq c$ almost everywhere on $X \setminus U$. Since

$1-g\cdot G^{\alpha-1}\geq 0$ almost everywhere on U, we obtain

$$\begin{aligned}\int_U f^\alpha d\mu - G^{\alpha-1}\int_X f^\alpha\cdot g d\mu &\geq c^\alpha\int_U\left(1-g\cdot G^{\alpha-1}\right)d\mu - G^{\alpha-1}\int_{X\setminus U} f^\alpha\cdot g d\mu\\ &= c^\alpha\left(\mu(U)-G^{\alpha-1}\int_U g d\mu\right)-G^{\alpha-1}\int_{X\setminus U} f^\alpha\cdot g d\mu\\ &= c^\alpha\left(G^\alpha-G^{\alpha-1}\int_U g d\mu\right)-G^{\alpha-1}\int_{X\setminus U} f^\alpha g d\mu\\ &= G^{\alpha-1}\int_{X\setminus U} g\cdot(c^\alpha-f^\alpha)d\mu\geq 0,\end{aligned}$$

where the last inequality holds since $f\leq c$ almost everywhere on $X\setminus U$. □

In the similar way we can prove the following theorem.

Theorem 3.68 *Let $(X,\mathscr{A},\mu)$ be a continuous measure space with $\mu(X)<\infty$. Let $\alpha\geq 1$ be a real number and let $f,g\in L^1(X)$ be such that $f\geq 0$ and $0\leq g\leq 1$ almost everywhere on X. Set $\lambda=\frac{1}{(\mu(X))^{\alpha-1}}\left(\int_X g d\mu\right)^\alpha$. Let U be an upper-separating subset for f such that $\mu(U)=\lambda$. Then*

$$\frac{1}{(\mu(X))^{\alpha-1}}\left(\int_X g\cdot f d\mu\right)^\alpha\leq\int_U f^\alpha d\mu.$$

To obtain Theorems 3.7 and 3.8 we must take $X=[a,b]$ and assume that f is nonincreasing in Theorems 3.67 and 3.68.

In [63] Imoru extended Theorem 3.5. We assume that μ is a nonnegative function on $[a,b]$, $-\infty<a<b<\infty$ and that f and g are nonnegative real-valued functions on $[a,b]$ which are Lebesgue-Stieltjes integrable with respect to μ on $[a,b]$ with $\int_a^b d\mu(x)=1$, i.e. $\int_{[a,b]} d\mu=1$.

Theorem 3.69 *Let $f:\mathbb{R}^+\to\mathbb{R}^+$ be a nonnegative and nonincreasing function which is Lebesgue-Stieltjes integrable with respect to μ. Let $\int_{[a,b]} g d\mu>0$ and*

$$\Psi\left(\int_{[a,b]} g d\mu\right)=K\int_{[a,b]} g d\mu,$$

where K is some positive constant and $\Psi:\mathbb{R}^+\to\mathbb{R}^+$ is a nonnegative continuous function which satisfies

$$\Psi\left(\int_{[a,b]} g d\mu\right)=\Psi(b-a)\int_{[a,\lambda]} d\mu=\Psi(b-a)\int_{[\xi,b]} d\mu.$$

If $0\leq g(x)\leq K^{-1}\Psi(b-a)$, $a\leq\xi\leq x\leq b$, $a\leq x\leq\lambda\leq b$, then

$$\Psi(b-a)\int_{[\xi,b]} f d\mu\leq K\int_{[a,b]} f g d\mu\leq\Psi(b-a)\int_{[a,\lambda]} f d\mu. \tag{3.113}$$

Proof. We have

$$\begin{aligned}
&\Psi(b-a)\int_{[a,\lambda]} f d\mu - K\int_{[a,b]} fg d\mu \\
&= \int_{[a,\lambda]} f\cdot(\Psi(b-a)-Kg)d\mu - K\int_{[\lambda,b]} fg d\mu \\
&\geq f(\lambda)\int_{[a,\lambda]}(\Psi(b-a)-Kg)d\mu - K\int_{[\lambda,b]} fg d\mu \\
&= f(\lambda)\left(\Psi(b-a)\int_{[a,\lambda]} d\mu - K\int_{[a,\lambda]} g d\mu\right) - K\int_{[\lambda,b]} fg d\mu \\
&= f(\lambda)\left(\Psi\left(\int_{[a,b]} g d\mu\right) - K\int_{[a,\lambda]} g d\mu\right) - K\int_{[\lambda,b]} fg d\mu \\
&= f(\lambda)\left(K\int_{[a,b]} g d\mu - K\int_{[a,\lambda]} g d\mu\right) - K\int_{[\lambda,b]} fg d\mu \\
&= Kf(\lambda)\int_{[\lambda,b]} g d\mu - K\int_{[\lambda,b]} fg d\mu \\
&= K\int_{[\lambda,b]}(f(\lambda)-f)g d\mu \geq 0.
\end{aligned}$$

Hence we obtain the right-hand side inequality in (3.113). The left-hand side inequality can be obtained in a similar way. □

Remark 3.18 If $\Psi(u) = u^p$, $p \geq 1$ and f is nonincreasing on (a,b), inequality (3.113) reduces to

$$\frac{\int_{[\xi,b]} f d\mu}{\left(\int_{[a,b]} g d\mu\right)^p} \leq (b-a)^{-p}\frac{\int_{[a,b]} fg d\mu}{\int_{[a,b]} g d\mu} \leq \frac{\int_{[a,\lambda]} f d\mu}{\left(\int_{[a,b]} g d\mu\right)^p},$$

where

$$(b-a)^p\int_{[a,\lambda]} d\mu = \left(\int_{[a,b]} g d\mu\right)^p = (b-a)^p\int_{[\xi,b]} d\mu.$$

When $a = 0$, $b = 1$ and $d\mu = dx$ we obtain (3.4) from Theorem 3.5.

Remark 3.19 If $\Psi(u) = \exp(u)$ inequality (3.113) gives

$$\frac{\int_{[\xi,b]} f d\mu}{\exp\left(\int_{[a,b]} g d\mu\right)} \leq e^{a-b}\frac{\int_{[a,b]} fg d\mu}{\int_{[a,b]} g d\mu} \leq \frac{\int_{[a,\lambda]} f d\mu}{\exp\left(\int_{[a,b]} g d\mu\right)},$$

where

$$e^{b-a}\int_{[a,\lambda]} d\mu = \exp\left(\int_{[a,b]} g d\mu\right) = e^{b-a}\int_{[\xi,b]} d\mu,$$

with $a \leq \lambda$, $\xi \leq b$.

Imoru also proved extensions of Steffensen's inequality for $P-$admissible functions (see [63]).

Definition 3.3 *Let P and Q be the classes of all nonnegative continuous convex and concave functions on $\mathbb{R}^+$ respectively and let Ψ be a nonnegative continuous function on $\mathbb{R}^+$. We say that Ψ is $P-$*admissible *if for every $u,v \in \mathbb{R}^+$, $v \neq 0$, there exists $\Phi \in P$ such that*

$$\Psi(u) \leq \Phi\left(\frac{u}{v}\right)\Psi(v). \tag{3.114}$$

*The function Ψ is said to be $Q-$*admissible *if for every $u,v \in \mathbb{R}^+$, $v \neq 0$, there exists $\Phi \in Q$ such that*

$$\Psi(u) \geq \Phi\left(\frac{u}{v}\right)\Psi(v).$$

Remark 3.20 The power function $\Psi(u) = u^p$ is $P-$admissible or $Q-$ admissible according as $p \geq 1$ or $0 < p \leq 1$. The function $\Psi(u) = \exp(u)$, $u \in \mathbb{R}^+$ is $P-$admissible.

Theorem 3.70 *Let the real-valued function Ψ be $P-$admissible and let Ψ and g satisfy the hypothesis of Theorem 3.69. If f is a nonnegative function on (a,b), which is Lebesgue-Stieltjes integrable with respect to μ and $\Phi \circ f$ is nonincreasing on (a,b), $\Phi \in P$, then*

$$\Psi\left(\int_{[a,b]} fgd\mu\right) \leq \Psi(b-a)\int_{[a,\lambda]}(\Phi \circ f)d\mu. \tag{3.115}$$

Proof. Since Ψ is $P-$admissible, Ψ satisfies inequality (3.114). Consequently, using Jensen's inequality

$$\begin{aligned}
\Psi\left(\int_{[a,b]} fgd\mu\right) &\leq \Phi\left(\frac{\int_{[a,b]} fgd\mu}{\int_{[a,b]} gd\mu}\right)\Psi\left(\int_{[a,b]} gd\mu\right) \\
&\leq \frac{\int_{[a,b]}(\Phi \circ f)gd\mu}{\int_{[a,b]} gd\mu}\Psi\left(\int_{[a,b]} gd\mu\right) \\
&= K\int_{[a,b]}(\Phi \circ f)gd\mu \leq \Psi(b-a)\int_{[a,b]}(\Phi \circ f)d\mu,
\end{aligned}$$

where the last inequality follows from an application of Theorem 3.69. □

Remark 3.21 If $\Psi(u) = \Phi(u) = u^p$, $p \geq 1$ and f is nonincreasing, then inequality (3.115) yields

$$\left(\int_{[a,b]} fgd\mu\right)^p \leq (b-a)^p\int_{[a,\lambda]} f^p d\mu,$$

where

$$(b-a)^p\int_{[a,\lambda]} d\mu = \left(\int_{[a,b]} gd\mu\right)^p.$$

When $a = 0$, $b = 1$ and $d\mu = dx$, we obtain (3.2).

Remark 3.22 If $\Psi(u) = \Phi(u) = \exp(u)$, then inequality (3.115) becomes

$$\exp\left(\int_{[a,b]} fgd\mu\right) \leq e^{b-a}\int_{[a,\lambda]}\exp(f)d\mu,$$

where

$$e^{b-a}\int_{[a,\lambda]} d\mu = \exp\left(\int_{[a,b]} g d\mu\right).$$

An analoguous result holds for $Q-$admissible functions.

Theorem 3.71 *Let the function Ψ be $Q-$admissible and let Ψ and g satisfy the hypothesis of Theorem 3.69. If f is a nonnegative function on (a,b), which is Lebesgue-Stieltjes integrable with respect to μ and $\Phi\circ f$ is nonincreasing on (a,b), $\Phi\in Q$, then*

$$\Psi(b-a)\int_{[\xi,b]}(\Phi\circ f)d\mu \le \Psi\left(\int_{[a,b]} fgd\mu\right).$$

3.5 Steffensen inequality and $L^1 - L^\infty$ estimates of weighted integrals

First, we recall some properties which will be used in this section.

If $\Phi : [0,\infty)\to\mathbb{R}$ is convex and continuous, then Φ' (defined at all but countably many points of $(0,\infty)$) is in $L^1(0,a)$ for every $a>0$ and $\int_0^a \Phi'(r)dr = \Phi(a)-\Phi(0)$. Also, the right derivative Φ'_+ is defined and finite at every point of $(0,\infty)$ and nondecreasing. Since it coincides with Φ' whenever Φ' is defined, it follows that $\Phi'_+\in L^1(0,a)$.

In [135] Rabier gave the following modern formulation of Steffensen's inequality.

Lemma 3.3 *Let $\Phi : [0,\infty)\to\mathbb{R}$ be convex and continuous with $\Phi(0)=0$. If $\alpha>0$ and $f\in L^\infty(0,\alpha)$, $f\ge 0$ and $\|f\|_\infty\le 1$, then $f\Phi'\in L^1(0,\alpha)$ and*

$$\Phi\left(\int_0^\alpha f(r)dr\right) \le \int_0^\alpha f(r)\Phi'(r)dr. \tag{3.116}$$

Proof. $f\Phi'\in L^1(0,\alpha)$ follows from $f\in L^\infty(0,\alpha)$ and $\Phi'\in L^1(0,\alpha)$.
Let us assume that $f>0$ almost everywhere, so that $F(r):=\int_0^r f(t)dt$ is well defined and increasing. Next,

$$\Phi\left(\int_0^\alpha f(r)dr\right) = \Phi(F(\alpha)) = \int_0^{F(\alpha)}\Phi'(s)ds = \int_0^{F(\alpha)}\Phi'_+(s)ds.$$

The change of variable $s:=F(r)$ yields $\Phi\left(\int_0^\alpha f(r)dr\right) = \int_0^\alpha \Phi'_+(F(r))f(r)dr$. Now, $F(r)\le r$ from the assumption $\|f\|_\infty\le 1$, so that $\Phi'_+(F(r))\le\Phi'_+(r)$ if $r>0$ by the monotonicity of Φ'_+. Thus, $\Phi\left(\int_0^\alpha f(r)dr\right)\le\int_0^\alpha \Phi'(r)f(r)dr$ since $f>0$ and $\Phi'_+=\Phi'$ almost everywhere.

If $f\ge 0$, let $\varepsilon>0$ be given and set $f_\varepsilon(r):=\frac{f(r)+\varepsilon\alpha}{1+\varepsilon\alpha}$, so that $0<f_\varepsilon\le 1$ almost everywhere on $(0,\alpha)$. From the previous part of the proof

$$\Phi\left(\int_0^\alpha f_\varepsilon(r)dr\right) = \Phi\left(\frac{\left(\int_0^\alpha f(r)dr\right)+\varepsilon\alpha^2}{1+\varepsilon\alpha}\right) \le \frac{1}{1+\varepsilon\alpha}\left(\int_0^\alpha \Phi'(r)f(r)dr+\varepsilon\alpha\Phi(\alpha)\right).$$

When $\varepsilon \to 0$, the left-hand side tends to $\Phi\left(\int_0^\alpha f(r)dr\right)$ by continuity of Φ and the right-hand side tends to $\int_0^\alpha f(r)\Phi'(r)dr$. This completes the proof. □

As showed in [135] (3.116) is equivalent to the right-hand side of Steffensen's inequality (2.1). Choosing $a = 0$, $b = \alpha$ and $g = -\Phi'$ in the right-hand side of (2.1) we obtain (3.116). Conversely, choosing $\alpha = b-a$, $\Phi(r) = -\int_a^{a+r} g(s)ds$ (so that $\Phi'(r) = -g(a+r)$ and extending g by $g(s) := g(b)$ for $s > b$) and $f(r)$ changed into $f(a+r)$ in (3.116) we obtain the right-hand side of (2.1).

In Theorem 2.3 we gave the variant of Steffensen's inequality for $b = \infty$ obtained by Apéry. In the following theorem we give natural extension of (3.116) to the infinite interval $(0,\infty)$ obtained by Rabier in [135]. We recall Rabier's theorem withouth the proof.

Theorem 3.72 *Let $\Phi : [0,\infty) \to \mathbb{R}$ be convex and continuous with $\Phi(0) = 0$. If $f \in L^\infty(0,\infty)$, $f \geq 0$ and $\|f\|_\infty \leq 1$, then $\int_0^\infty f(r)\Phi'(r)dr$ is well defined in $\mathbb{R}\cup\{\pm\infty\}$ and*

$$\Phi\left(\int_0^\infty f(r)dr\right) \leq \int_0^\infty f(r)\Phi'(r)dr.$$

Changing f into $\|f\|_\infty^{-1} f$ in Theorem 3.72, Rabier obtained the following corollary.

Corollary 3.5 *Let $\Phi : [0,\infty) \to \mathbb{R}$ be convex and continuous with $\Phi(0) = 0$. If $f \in L^\infty(0,\infty)$, $f \geq 0$ and $f \neq 0$, then $\int_0^\infty f(r)\Phi'(r)dr$ is well defined in $\mathbb{R}\cup\{\pm\infty\}$ and*

$$\|f\|_\infty \Phi\left(\frac{1}{\|f\|_\infty}\int_0^\infty f(r)dr\right) \leq \int_0^\infty f(r)\Phi'(r)dr.$$

In the following theorem we recall the extension of Corollary 3.5 to functions f on $\mathbb{R}^N$ obtained by Rabier in [135]. First, let us denote by ω_N the volume of the unit ball in $\mathbb{R}^N$.

Theorem 3.73 *Let $\Phi : [0,\infty) \to \mathbb{R}$ be convex with $\Phi(0) = 0$. If $f \in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, $f \neq 0$, then $\int_{\mathbb{R}^N} |f(x)|\Phi'(|x|^N)dx$ is well defined in $\mathbb{R}\cup\{\infty\}$ and*

$$\omega_N\|f\|_\infty \Phi\left(\frac{\|f\|_1}{\omega_N\|f\|_\infty}\right) \leq \int_{\mathbb{R}^N} |f(x)|\Phi'(|x|^N)dx. \tag{3.117}$$

The proof of the previous theorem involves changing variables through diffeomorphisms of $\mathbb{R}^N \setminus \{0\}$ in integrals that may be $\pm\infty$. For details see [135].

As noted by Rabier a variety of other inequalities can be derived from (3.117) changing the function or variable.

In [136] Rabier proved the following.

Theorem 3.74 *If $\Phi : [0,\infty) \to \mathbb{R}$ is convex and continuous with $\Phi(0) = 0$ and if $q \in (1,\infty)$, $q' := \frac{q}{q-1}$, then*

$$\Phi\left(\int_0^\infty f(r)dr\right) \leq C\int_0^\infty f(r)\Phi'(r^{1/q'})dr$$

holds for every $f \in L^q(0,\infty)$, $f \geq 0$ with $||f||_q \leq 1$ when $C = 1$.

In general, both sides may be $\pm\infty$. Rabier also derived related inequalities for $f \in L^1(\mathbb{R}^N)\cap L^q(\mathbb{R}^N)$, $f \neq 0$.

Furthermore, in [136] Rabier identified the range of the admissible constants C and, in particular, characterized the optimal constant when $\Phi \geq 0$ or $\Phi \leq 0$.

3.6 Steffensen-type inequalities involving convex functions

Let us begin with the definition of a new class of functions introduced by Pečarić and Smoljak in [127] that extends the class of convex functions.

Definition 3.4 *Let $f : [a,b] \to \mathbb{R}$ be a function and $c \in (a,b)$. We say that f belongs to class $\mathcal{M}_1^c[a,b]$ (f belongs to class $\mathcal{M}_2^c[a,b]$) if there exists a constant A such that the function $F(x) = f(x) - Ax$ is nonincreasing (nondecreasing) on $[a,c]$ and nondecreasing (nonincreasing) on $[c,b]$.*

Let us show that, if $f \in \mathcal{M}_1^c[a,b]$ or $f \in \mathcal{M}_2^c[a,b]$ and $f'(c)$ exists, then $f'(c) = A$. We show this for $f \in \mathcal{M}_1^c[a,b]$. Since F is nonincreasing on $[a,c]$ and nondecreasing on $[c,b]$ for every distinct points $x_1,x_2 \in [a,c]$ and $y_1,y_2 \in [c,b]$ we have

$$[x_1,x_2;F] = [x_1,x_2;f] - A \leq 0 \leq [y_1,y_2;f] - A = [y_1,y_2;F].$$

Therefore, if $f'_-(c)$ and $f'_+(c)$ exist, letting $x_i \nearrow c$ and $y_i \searrow c$, $i = 1,2$ we get

$$f'_-(c) \leq A \leq f'_+(c). \tag{3.118}$$

In the following lemma and theorem we give connection between the class of functions $\mathcal{M}_1^c[a,b]$ and the class of convex functions which was obtained in [127].

Lemma 3.4 *If $f : [a,b] \to \mathbb{R}$ is convex (concave), then $f \in \mathcal{M}_1^c[a,b]$ ($f \in \mathcal{M}_2^c[a,b]$) for every $c \in (a,b)$.*

Proof. If f is convex, then f'_- and f'_+ exist (see [122]). Hence, for every $x_1,x_2 \in [a,c]$ and $y_1,y_2 \in [c,b]$ it holds

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \leq f'_-(c) \leq f'_+(c) \leq \frac{f(y_2)-f(y_1)}{y_2-y_1}.$$

Therefore, for every $A \in [f'_-(c), f'_+(c)]$ the function $F(x) = f(x) - Ax$ satisfies

$$\frac{F(x_2)-F(x_1)}{x_2-x_1} \leq 0 \leq \frac{F(y_2)-F(y_1)}{y_2-y_1},$$

so F is nonincreasing on $[a,c]$ and nondecreasing on $[c,b]$. □

Theorem 3.75 *If $f \in \mathcal{M}_1^c[a,b]$ ($f \in \mathcal{M}_2^c[a,b]$) for every $c \in (a,b)$, then f is convex (concave).*

Proof. We give the proof for $f \in \mathcal{M}_1^c[a,b]$. First, let us recall the characterization of convexity given in [122]: the function g is convex if and only if the function

$$(x,y) \mapsto [x,y;g] = \frac{g(x)-g(y)}{x-y}$$

is nondecreasing in both variables.

For every $c \in (a,b)$ there exists constant A_c such that the function $F_c(x) = f(x) - A_c x$ is nonincreasing on $[a,c]$ and nondecreasing on $[c,b]$. So for every $x_1 \neq x_2 \leq c \leq y_1 \neq y_2$ we have

$$\frac{F_c(x_2)-F_c(x_1)}{x_2-x_1} = \frac{f(x_2)-f(x_1)}{x_2-x_1} - A_c \leq 0 \leq \frac{f(y_2)-f(y_1)}{y_2-y_1} - A_c = \frac{F_c(y_2)-F_c(y_1)}{y_2-y_1}.$$

Particularly, for $u < v < w$ we have

$$\frac{f(v)-f(u)}{v-u} \leq A_v \leq \frac{f(w)-f(v)}{w-v}. \tag{3.119}$$

Now, let $x_1, x_2, y \in [a,b]$ be arbitrary. If $y < x_1 < x_2$, applying (3.119) we get

$$\frac{f(x_1)-f(y)}{x_1-y} \leq A_{x_1} \leq \frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{f(x_2)-f(y)}{x_2-x_1} - \frac{f(x_1)-f(y)}{x_2-x_1}.$$

By multiplying the above inequality with $\frac{x_2-x_1}{x_2-y} > 0$ and simplifying we get

$$\frac{f(x_1)-f(y)}{x_1-y} \leq \frac{f(x_2)-f(y)}{x_2-y}.$$

Similarly for the cases $x_1 < y < x_2$ and $x_1 < x_2 < y$. So we can conclude that the function $(x,y) \mapsto [x,y;f]$ is nondecreasing in variable x. By symmetry, the same thing holds for variable y, so the proof is completed. $\square$

Taking into account Lemma 3.4 and Theorem 3.75, we can describe the property from Definition 3.4 as "convexity at point c". Therefore, function f is convex on $[a,b]$ if and only if it is convex at every $c \in (a,b)$.

In the following theorems we give Steffensen-type inequalities for the class of functions that are convex at point c obtained in [127].

Theorem 3.76 *Let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq 1$. Let $c \in (a,b)$, $\lambda_1 = \int_a^c g(t)dt$ and $\lambda_2 = \int_c^b g(t)dt$. If $f \in \mathcal{M}_1^c[a,b]$ and*

$$\int_a^b tg(t)dt = a\lambda_1 + b\lambda_2 + \frac{\lambda_1^2 - \lambda_2^2}{2}, \tag{3.120}$$

then

$$\int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda_1} f(t)dt + \int_{b-\lambda_2}^b f(t)dt \tag{3.121}$$

holds.
If $f \in \mathcal{M}_2^c[a,b]$ and (3.120) holds, the inequality in (3.121) is reversed.

Proof. We give the proof for $f \in \mathscr{M}_1^c[a,b]$. Let $F(x) = f(x) - Ax$, where A is the constant from Definition 3.4. Since $F : [a,c] \to \mathbb{R}$ is nonincreasing we can apply the right-hand side of Steffensen's inequality on function F, so

$$\int_a^c F(t)g(t)dt \le \int_a^{a+\lambda_1} F(t)dt.$$

Hence, we obtain

$$\begin{aligned} 0 &\le \int_a^{a+\lambda_1} F(t)dt - \int_a^c F(t)g(t)dt \\ &= \int_a^{a+\lambda_1} f(t)dt - \int_a^c f(t)g(t)dt - A\left(a\lambda_1 + \frac{\lambda_1^2}{2} - \int_a^c tg(t)dt\right). \end{aligned} \tag{3.122}$$

Further, since $F : [c,b] \to \mathbb{R}$ is nondecreasing we can apply the left-hand side of Steffensen's inequality on function F, so

$$\int_c^b F(t)g(t)dt \le \int_{b-\lambda_2}^b F(t)dt.$$

Hence, we obtain

$$\begin{aligned} 0 &\ge \int_c^b F(t)g(t)dt - \int_{b-\lambda_2}^b F(t)dt \\ &= \int_c^b f(t)g(t)dt - \int_{b-\lambda_2}^b f(t)dt - A\left(\int_c^b tg(t)dt - b\lambda_2 + \frac{\lambda_2^2}{2}\right). \end{aligned} \tag{3.123}$$

Now from (3.122) and (3.123) we obtain

$$\int_a^{a+\lambda_1} f(t)dt + \int_{b-\lambda_2}^b f(t)dt - \int_a^b f(t)g(t)dt \ge A\left(a\lambda_1 + b\lambda_2 + \frac{\lambda_1^2 - \lambda_2^2}{2} - \int_a^b tg(t)dt\right).$$

Hence, if $\int_a^b tg(t)dt = a\lambda_1 + b\lambda_2 + \frac{\lambda_1^2-\lambda_2^2}{2}$, then (3.121) holds.
Proof for $f \in \mathscr{M}_2^c[a,b]$ is similar so we omit the details. □

Remark 3.23 It is obvious from the proof that condition (3.120) can be weakened. That is, for $f \in \mathscr{M}_1^c[a,b]$ inequality (3.121) still holds if (3.120) is replaced by the weaker condition

$$A\left(a\lambda_1 + b\lambda_2 + \frac{\lambda_1^2 - \lambda_2^2}{2} - \int_a^b tg(t)dt\right) \ge 0, \tag{3.124}$$

where A is a constant from Definition 3.4. Also, for $f \in \mathscr{M}_2^c[a,b]$ the reverse inequality in (3.121) holds if (3.120) is replaced by (3.124) with the reverse inequality.

Additionaly, condition (3.120) can be further weakened if the function f is monotonic. Since (3.118) holds, for nondecreasing function $f \in \mathscr{M}_1^c[a,b]$ or for nonincreasing function $f \in \mathscr{M}_2^c[a,b]$, from (3.124) we obtain that (3.120) can be weakened to

$$\int_a^b tg(t)dt \le a\lambda_1 + b\lambda_2 + \frac{\lambda_1^2 - \lambda_2^2}{2}. \tag{3.125}$$

Further, if $f \in \mathscr{M}_1^c[a,b]$ is nonincreasing or $f \in \mathscr{M}_2^c[a,b]$ is nondecreasing, (3.120) can be weakened to (3.125) with the reverse inequality.

Theorem 3.77 *Let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq 1$. Let $c \in (a,b)$, $\lambda_1 = \int_a^c g(t)dt$ and $\lambda_2 = \int_c^b g(t)dt$. If $f \in \mathscr{M}_1^c[a,b]$ and*

$$\int_a^b tg(t)dt = c(\lambda_1 + \lambda_2) + \frac{\lambda_2^2 - \lambda_1^2}{2}, \tag{3.126}$$

then

$$\int_a^b f(t)g(t)dt \geq \int_{c-\lambda_1}^{c+\lambda_2} f(t)dt \tag{3.127}$$

holds.
If $f \in \mathscr{M}_2^c[a,b]$ and (3.126) holds, the inequality in (3.127) is reversed.

Proof. We give the proof for $f \in \mathscr{M}_1^c[a,b]$. Let $F(x) = f(x) - Ax$. Since $F : [a,c] \to \mathbb{R}$ is nonincreasing applying the left-hand side of Steffensen's inequality on function F we obtain

$$0 \leq \int_a^c f(t)g(t)dt - \int_{c-\lambda_1}^c f(t)dt - A\left(\int_a^c tg(t)dt - c\lambda_1 + \frac{\lambda_1^2}{2}\right). \tag{3.128}$$

Further, since $F : [c,b] \to \mathbb{R}$ is nondecreasing applying the right-hand side of Steffensen's inequality on function F we obtain

$$0 \geq \int_c^{c+\lambda_2} f(t)dt - \int_c^b f(t)g(t)dt - A\left(c\lambda_2 + \frac{\lambda_2^2}{2} - \int_c^b tg(t)dt\right). \tag{3.129}$$

Now from (3.128) and (3.129) we obtain

$$\int_a^b f(t)g(t)dt - \int_{c-\lambda_1}^{c+\lambda_2} f(t)dt \geq A\left(\int_a^b tg(t)dt - c(\lambda_1 + \lambda_2) + \frac{\lambda_1^2 - \lambda_2^2}{2}\right).$$

Hence, if $\int_a^b tg(t)dt = c(\lambda_1 + \lambda_2) + \frac{\lambda_2^2 - \lambda_1^2}{2}$, then (3.127) holds.
Proof for $f \in \mathscr{M}_2^c[a,b]$ is similar so we omit the details. □

Remark 3.24 For $f \in \mathscr{M}_1^c[a,b]$ inequality (3.127) still holds if condition (3.126) is replaced by the weaker condition

$$A\left(\int_a^b tg(t)dt - c(\lambda_1 + \lambda_2) + \frac{\lambda_1^2 - \lambda_2^2}{2}\right) \geq 0, \tag{3.130}$$

where A is a constant from Definition 3.4. Also, for $f \in \mathscr{M}_2^c[a,b]$ the reverse inequality in (3.127) holds if (3.126) is replaced by (3.130) with the reverse inequality.

Additionaly, condition (3.126) can be further weakened if the function f is monotonic. Since (3.118) holds, for nondecreasing function $f \in \mathscr{M}_1^c[a,b]$ or for nonincreasing function $f \in \mathscr{M}_2^c[a,b]$, from (3.130) we obtain that (3.126) can be weakened to

$$\int_a^b tg(t)dt \geq c(\lambda_1 + \lambda_2) + \frac{\lambda_2^2 - \lambda_1^2}{2}. \tag{3.131}$$

Further, if $f \in \mathscr{M}_1^c[a,b]$ is nonincreasing or $f \in \mathscr{M}_2^c[a,b]$ is nondecreasing, (3.126) can be weakened to (3.131) with the reverse inequality.

As a consequence of Theorems 3.76 and 3.77 we obtain Steffensen type inequalities that involve convex functions.

Corollary 3.6 *Let $g : [a,b] \to \mathbb{R}$ be integrable function such that $0 \leq g \leq 1$. Let $c \in (a,b)$, $\lambda_1 = \int_a^c g(t)dt$ and $\lambda_2 = \int_c^b g(t)dt$. If $f : [a,b] \to \mathbb{R}$ is a convex function and (3.120) holds, then (3.121) holds.*
If $f : [a,b] \to \mathbb{R}$ is a concave function and (3.120) holds, the inequality in (3.121) is reversed.

Proof. Since f is convex, from Lemma 3.4 we have that $f \in \mathscr{M}_1^c[a,b]$ for every $c \in (a,b)$. So we can apply Theorem 3.76. □

Corollary 3.7 *Let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq 1$. Let $c \in (a,b)$, $\lambda_1 = \int_a^c g(t)dt$ and $\lambda_2 = \int_c^b g(t)dt$. If $f : [a,b] \to \mathbb{R}$ is a convex function and (3.126) holds, then (3.127) holds.*
If $f : [a,b] \to \mathbb{R}$ is a concave function and (3.126) holds, the inequality in (3.127) is reversed.

Proof. Similarly as in the proof of Corollary 3.6 we have that $f \in \mathscr{M}_1^c[a,b]$ for every $c \in (a,b)$. So we can apply Theorem 3.77. □

As noted in [127], the condition $0 \leq g \leq 1$ in Theorem 3.76 and Corollary 3.6 can be replaced by the weaker conditions

$$\int_a^x g(t)dt \leq x-a \quad \text{and} \quad \int_x^c g(t)dt \geq 0 \quad \text{for every } x \in [a,c] \tag{3.132}$$

and

$$\int_x^b g(t)dt \leq b-x \quad \text{and} \quad \int_c^x g(t)dt \geq 0 \quad \text{for every } x \in [c,b]. \tag{3.133}$$

Further, the condition $0 \leq g \leq 1$ in Theorem 3.77 and Corollary 3.7 can be replaced by weaker conditions

$$\int_x^c g(t)dt \leq c-x \quad \text{and} \quad \int_a^x g(t)dt \geq 0 \quad \text{for every } x \in [a,c] \tag{3.134}$$

and

$$\int_c^x g(t)dt \leq x-c \quad \text{and} \quad \int_x^b g(t)dt \geq 0 \quad \text{for every } x \in [c,b]. \tag{3.135}$$

The mentioned weaker conditions follow from Theorem 2.6.

3.7 Steffensen-type inequalities for $s-$convex functions

First, let us define $s-$convex functions in the second sense.

Definition 3.5 *A function $f:\mathbb{R}^+\to\mathbb{R}$ is said to be $s-$*convex in the second sense *if*

$$f(\alpha x+\beta y)\le \alpha^s f(x)+\beta^s f(y)$$

for all $x,y\in[0,\infty)$, $\alpha,\beta\ge 0$ with $\alpha+\beta=1$ and for some fixed $s\in(0,1]$.

For $s=1$, $s-$convexity reduces to the ordinary convexity of functions defined on $[0,\infty)$. In [14] Alomari proved the following inequalities involving $s-$convexity.

Theorem 3.78 *Let $f,g:[a,b]\subset\mathbb{R}^+\to\mathbb{R}$ be integrable and $0\le g\le 1$ such that $\int_a^b g(t)f'(t)dt$ exists. If f is absolutely continuous on $[a,b]$ such that $|f'|$ is $s-$convex in the second sense on $[a,b]$, then*

$$\left|\int_a^{a+\lambda} f(t)dt-\int_a^b f(t)g(t)dt\right|\le\frac{1}{(s+1)(s+2)}\left[\lambda^2|f'(a)|+(b-a-\lambda)^2|f'(b)|\right]$$
$$+\frac{1}{s+2}\left[\lambda^2+(b-a-\lambda)^2\right]|f'(a+\lambda)| \tag{3.136}$$

and

$$\left|\int_a^{b} f(t)g(t)dt-\int_{b-\lambda}^b f(t)dt\right|\le\frac{1}{(s+1)(s+2)}\left[\lambda^2|f'(b)|+(b-a-\lambda)^2|f'(a)|\right]$$
$$+\frac{1}{s+2}\left[\lambda^2+(b-a-\lambda)^2\right]|f'(b-\lambda)| \tag{3.137}$$

where $\lambda=\int_a^b g(t)dt$.

Proof. Utilizing the triangle inequality on identity (2.8) and using $s-$convexity of $|f'|$ we obtain (3.136). For details see [14]. □

As noted by Alomari, for $s=1$ inequality (3.136) becomes

$$\left|\int_a^{a+\lambda} f(t)dt-\int_a^b f(t)g(t)dt\right|\le\frac{1}{6}\lambda^2|f'(a)|+\frac{1}{3}[\lambda^2+(b-a-\lambda)^2]|f'(a+\lambda)|$$
$$+\frac{1}{6}(b-a-\lambda)^2|f'(b)|$$

and inequality (3.137) becomes

$$\left|\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt\right| \leq \frac{1}{6}\lambda^2|f'(b)| + \frac{1}{3}[\lambda^2 + (b-a-\lambda)^2]|f'(b-\lambda)| + \frac{1}{6}(b-a-\lambda)^2|f'(a)|.$$

Using identity (2.8) Alomari obtained the following.

Theorem 3.79 *Let $f,g : [a,b] \subset \mathbb{R}^+ \to \mathbb{R}$ be integrable and $0 \leq g \leq 1$ such that $\int_a^b g(t)f'(t)dt$ exists. If f is absolutely continuous on $[a,b]$ with $|f'|$ $s-$convex in the second sense on $[a,b]$ for some fixed $s \in (0,1]$, then we have*

$$\begin{aligned}
&\left|\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt\right| \\
&\leq \frac{1}{(s+1)}\left[\int_{a+\lambda}^b g(t)dt\right]\left[\lambda|f'(a)| + (b-a)|f'(a+\lambda)| + (b-a-\lambda)|f'(b)|\right] \\
&\leq \frac{b-a-\lambda}{s+1}\left[\lambda|f'(a)| + (b-a)|f'(a+\lambda)| + (b-a-\lambda)|f'(b)|\right]
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt\right| \\
&\leq \frac{1}{(s+1)}\left[\int_{b-\lambda}^b g(t)dt\right]\left[(b-a-\lambda)|f'(a)| + (b-a)|f'(b-\lambda)| + \lambda|f'(b)|\right] \\
&\leq \frac{\lambda}{s+1}\left[(b-a-\lambda)|f'(a)| + (b-a)|f'(b-\lambda)| + \lambda|f'(b)|\right]
\end{aligned}$$

where $\lambda = \int_a^b g(t)dt$.

Furthermore, Alomari also obtained Steffensen-type inequalities involving $s-$concavity.

Theorem 3.80 *Let $f,g : [a,b] \subset \mathbb{R}^+ \to \mathbb{R}$ be integrable and $0 \leq g \leq 1$ such that $\int_a^b g(t)f'(t)dt$ exists. If f is absolutely continuous on $[a,b]$ with $|f'|$ $s-$concave in the second sense on $[a,b]$ for some fixed $s \in (0,1]$, then we have*

$$\begin{aligned}
&\left|\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt\right| \\
&\leq 2^{s-1}\left[\int_{a+\lambda}^b g(t)dt\right]\left[\lambda\left|f'\left(a+\frac{\lambda}{2}\right)\right| + (b-a-\lambda)\left|f'\left(\frac{a+b+\lambda}{2}\right)\right|\right] \\
&\leq 2^{s-1}(b-a-\lambda)\left[\lambda\left|f'\left(a+\frac{\lambda}{2}\right)\right| + (b-a-\lambda)\left|f'\left(\frac{a+b+\lambda}{2}\right)\right|\right]
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt\right| \\
&\leq 2^{s-1}\left[\int_{b-\lambda}^b g(t)dt\right]\left[(b-a-\lambda)\left|f'\left(\frac{a+b-\lambda}{2}\right)\right| + \lambda\left|f'\left(b-\frac{\lambda}{2}\right)\right|\right] \\
&\leq \lambda 2^{s-1}\left[(b-a-\lambda)\left|f'\left(\frac{a+b-\lambda}{2}\right)\right| + \lambda\left|f'\left(b-\frac{\lambda}{2}\right)\right|\right]
\end{aligned}$$

where $\lambda = \int_a^b g(t)dt$.

Proof. Utilizing the triangle inequality on identity (2.8) and using $s-$concavity of $|f'|$ we obtain (3.136). For details see [14]. □

Chapter 4

Generalizations of Steffensen's inequality via weighted Montgomery identity

4.1 Generalizations via weighted Montgomery identity

Let $w : [a,b] \to \mathbb{R}$ be a weight function, i.e. an integrable function such that $\int_a^b w(t)\,dt \neq 0$ and $W(x) = \int_a^x w(t)\,dt$, $x \in [a,b]$. Let also $f : [a,b] \to \mathbb{R}$ be a continuous function of bounded variation. Then the *weighted Montgomery identity* given by Pečarić in [107], states

$$f(x) - \frac{1}{\int_a^b w(t)\,dt} \int_a^b f(t)\,w(t)\,dt = \int_a^b P_w(x,t)\,df(t) \tag{4.1}$$

where $P_w(x,t)$ is the weighted Peano kernel, defined by

$$P_w(x,t) = \begin{cases} \frac{W(t)}{W(b)}, & a \leq t \leq x \\ \frac{W(t)}{W(b)} - 1, & x < t \leq b. \end{cases}$$

Assumptions $W(t) = 0$ for $t \leq a$ and $W(t) = \int_a^b w(t)\,dt$ for $t \geq b$ allow us to subtract two weighted Montgomery identities, one for the interval $[a,b]$ and the other for $[c,d]$. In such a way the next result is obtained in [5].

Theorem 4.1 *Let $f:[a,b]\cup[c,d]\to\mathbb{R}$ be a continuous function of bounded variation on $[a,b]\cup[c,d]$, $w:[a,b]\to\mathbb{R}$ and $u:[c,d]\to\mathbb{R}$ some weight functions such that $\int_a^b w(t)\,dt\neq 0$, $\int_c^d u(t)\,dt\neq 0$ and*

$$W(x)=\begin{cases}0, & x<a\\ \int_a^x w(t)\,dt, & a\le x\le b\\ \int_a^b w(t)\,dt, & x>b,\end{cases}\qquad U(x)=\begin{cases}0, & x<c\\ \int_c^x u(t)\,dt & c\le x\le d\\ \int_c^d u(t)\,dt, & x>d,\end{cases}$$

and $[a,b]\cap[c,d]\neq\emptyset$. Then, for both cases $[c,d]\subseteq[a,b]$ and $[a,b]\cap[c,d]=[c,b]$ (and also for $[a,b]\subseteq[c,d]$ and $[a,b]\cap[c,d]=[a,d]$) the next formula is valid

$$\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)f(t)\,dt-\frac{1}{\int_c^d u(t)\,dt}\int_c^d u(t)f(t)\,dt=\int_{\min\{a,c\}}^{\max\{b,d\}}K(t)\,df(t)$$

where

$$K(t)=P_u(x,t)-P_w(x,t),\ t\in[\min\{a,c\},\max\{b,d\}]$$

and $P_u(x,t)$, $P_w(x,t)$ are given by

$$P_w(x,t)=\begin{cases}\frac{W(t)}{W(b)}, & a\le t\le x\\[2ex] \frac{W(t)}{W(b)}-1, & x<t\le b,\end{cases}$$

$$P_u(x,t)=\begin{cases}\frac{U(t)}{U(d)}, & c\le t\le x\\[2ex] \frac{U(t)}{U(d)}-1, & x<t\le d,\end{cases}$$

and thus

$$K(t)=\begin{cases}-\frac{W(t)}{W(b)}, & t\in[a,c)\\[2ex] -\frac{W(t)}{W(b)}+\frac{U(t)}{U(d)}, & t\in[c,d] \quad \text{if}\ \ [c,d]\subseteq[a,b],\\[2ex] 1-\frac{W(t)}{W(b)}, & t\in(d,b]\end{cases}\tag{4.2}$$

$$K(t)=\begin{cases}-\frac{W(t)}{W(b)}, & t\in[a,c)\\[2ex] -\frac{W(t)}{W(b)}+\frac{U(t)}{U(d)}, & t\in[c,b) \quad \text{if}\ \ [a,b]\cap[c,d]=[c,b].\\[2ex] \frac{U(t)}{U(d)}-1, & t\in[b,d]\end{cases}\tag{4.3}$$

This identity enables us to estimate the difference between two weighted integral means, each having its own weight, on two different intersecting intervals $[a,b]$ and $[c,d]$ for four possible cases when $[a,b]\cap[c,d]\neq\emptyset$. First two cases are when one interval is a subset of the other $[c,d]\subseteq[a,b]$ and overlapping intervals $[a,b]\cap[c,d]=[c,b]$. The other two cases are obtained by replacement $a\leftrightarrow c$, $b\leftrightarrow d$.

The special case of this identity for normalized weight function was obtained in [9].

Now we give a generalization of Steffensen's inequality via estimate of the difference of two weighted integral means obtained by Aglić Aljinović, Pečarić and Perušić in [10].

Theorem 4.2 *Let $f:[a,b]\cup[c,d]\to\mathbb{R}$ be a continuous function of bounded variation on $[a,b]\cup[c,d]$, $w:[a,b]\to\mathbb{R}$ and $u:[c,d]\to\mathbb{R}$ some weight functions, such that $\int_a^b w(t)\,dt\neq 0$, $\int_c^d u(t)\,dt\neq 0$ and $[a,b]\cap[c,d]\neq\emptyset$. Then*

$$\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)f(t)\,dt\leq\frac{1}{\int_c^d u(t)\,dt}\int_c^d u(t)f(t)\,dt \tag{4.4}$$

holds for every nonincreasing function f if and only if $[c,d]\subseteq[a,b]$ and

$$\frac{W(x)}{W(b)}\leq 0\text{ for } x\in[a,c),\qquad \frac{W(x)}{W(b)}\leq\frac{U(x)}{U(d)}\text{ for } x\in[c,d],\ \frac{W(x)}{W(b)}\leq 1\text{ for } x\in(d,b], \tag{4.5}$$

or $[a,b]\cap[c,d]=[c,b]$ and

$$\frac{W(x)}{W(b)}\leq 0\text{ for } x\in[a,c),\qquad \frac{W(x)}{W(b)}\leq\frac{U(x)}{U(d)}\text{ for } x\in[c,b),\ 1\leq\frac{U(x)}{U(d)}\text{ for } x\in[b,d]. \tag{4.6}$$

Proof. If $[c,d]\subseteq[a,b]$, we apply (4.4) for

$$f(t)=\begin{cases}1, & t\leq x\\ 0, & t>x,\end{cases} \tag{4.7}$$

with $x\in[a,c)$, $x\in[c,d]$, $x\in(d,b]$, respectively, and inequalities in (4.5) follow. Similarly, if $[a,b]\cap[c,d]=[c,b]$, we apply (4.4) for f with $x\in[a,c)$, $x\in[c,b)$, $x\in[b,d]$, respectively, and inequalities in (4.6) follow.

Conversely, utilizing (4.1) for every nonincreasing function f, in both cases $[c,d]\subseteq[a,b]$ and $[a,b]\cap[c,d]=[c,b]$ we have $K(t)\geq 0$, $t\in[\min\{a,c\},\max\{b,d\}]$ and thus $\int_{\min\{a,c\}}^{\max\{b,d\}}K(t)\,df(t)\leq 0$. □

Remark 4.1 If f is a nondecreasing function, inequality (4.4) is reversed.

Theorem 4.3 *Let $f:[a,b]\cup[a,a+\lambda]\to\mathbb{R}$ be a continuous function of bounded variation on $[a,b]\cup[a,a+\lambda]$ and let $w:[a,b]\to\mathbb{R}$ and $u:[a,a+\lambda]\to\mathbb{R}$ be some weight functions such that $\int_a^b w(t)\,dt=\int_a^{a+\lambda}u(t)\,dt$. Then*

$$\int_a^{a+\lambda}u(t)f(t)\,dt\leq\int_a^b w(t)f(t)\,dt \tag{4.8}$$

holds for every nonincreasing function f *if and only if* $0 < \lambda \leq b-a$ *and*

$$\begin{gathered}\int_a^x u(t)\,dt \leq \int_a^x w(t)\,dt \text{ for } x \in [a,a+\lambda] \\ \text{and } \int_x^b w(t)\,dt \leq 0 \text{ for } x \in (a+\lambda,b];\end{gathered} \tag{4.9}$$

or $\lambda > b-a$ *and*

$$\begin{gathered}\int_a^x u(t)\,dt \leq \int_a^x w(t)\,dt \text{ for } x \in [a,b] \\ \text{and } \int_x^{a+\lambda} u(t)\,dt \geq 0 \text{ for } x \in (b,a+\lambda].\end{gathered} \tag{4.10}$$

Proof. If $0 < \lambda \leq b-a$ and if inequality (4.8) holds we apply it for f defined by (4.7) with $x \in [a,a+\lambda]$, $x \in (a+\lambda,b]$, respectively, and inequalities in (4.9) follow. Similarly, if $\lambda > b-a$, we apply (4.8) for f with $x \in [a,b]$, $x \in (b,a+\lambda]$, and inequalities in (4.10) follow.

Conversely, from Theorem 4.1 applied with $[c,d] = [a,a+\lambda]$ we have

$$\int_a^b w(t)f(t)\,dt - \int_a^{a+\lambda} u(t)f(t)\,dt = \alpha \int_a^{\max\{b,a+\lambda\}} K(t)\,df(t)$$

where $\alpha = \int_a^b w(t)\,dt = \int_a^{a+\lambda} u(t)\,dt$, that is, $\alpha = W(b) = U(a+\lambda)$. First, we consider the case $0 < \lambda \leq b-a$. We have $\max\{b,a+\lambda\} = b$. By utilizing (4.2) we obtain

$$\alpha K(t) = \begin{cases} U(t) - W(t), & t \in [a,a+\lambda] \\ \alpha - W(t), & t \in (a+\lambda,b]. \end{cases}$$

Since f is nonincreasing, if $\int_a^x u(t)\,dt \leq \int_a^x w(t)\,dt$ for $x \in [a,a+\lambda]$ and $\alpha - \int_a^x w(t)\,dt = \int_x^b w(t)\,dt \leq 0$ for $x \in (a+\lambda,b]$, we have $\alpha K(t) \leq 0$ and therefore $\int_a^b \alpha K(t)\,df(t) \geq 0$.

In case $\lambda > b-a$, we have $\max\{b,a+\lambda\} = a+\lambda$, and by utilizing (4.3)

$$\alpha K(t) = \begin{cases} U(t) - W(t), & t \in [a,b] \\ U(t) - \alpha, & t \in (b,a+\lambda]. \end{cases}$$

Again, since f is nonincreasing, if $\int_a^x u(t)\,dt \leq \int_a^x w(t)\,dt$ for $x \in [a,b]$ and $\int_a^x u(t)\,dt - \alpha = -\int_x^{a+\lambda} u(t)\,dt \leq 0$ for $x \in (b,a+\lambda]$, we have $\alpha K(t) \leq 0$ and therefore $\int_a^b \alpha K(t)\,df(t) \geq 0$. □

Corollary 4.1 *Let* $f : [a,b] \cup [a,a+\lambda] \to \mathbb{R}$ *and* $g : [a,b] \to \mathbb{R}$ *be integrable functions, where* $\lambda = \int_a^b g(t)\,dt$. *Then*

$$\int_a^{a+\lambda} f(t)\,dt \leq \int_a^b f(t)g(t)\,dt \tag{4.11}$$

holds for every nonincreasing function f if and only if $0 < \lambda \le b-a$ and

$$x-a \le \int_a^x g(t)\,dt \text{ for } x \in [a,a+\lambda] \text{ and } \int_x^b g(t)\,dt \le 0 \text{ for } x \in (a+\lambda,b];$$

or $\lambda > b-a$ and

$$x-a \le \int_a^x g(t)\,dt \quad \text{for } x \in [a,b].$$

Proof. Apply Theorem 4.3 with weight functions $w(t) = g(t)$ for $t \in [a,b]$ and $u(t) = 1$ for $t \in [a,a+\lambda]$. □

Theorem 4.4 *Let $f : [a,b] \cup [b-\lambda,b] \to \mathbb{R}$ be a continuous function of bounded variation on $[a,b] \cup [b-\lambda,b]$ and let $w : [a,b] \to \mathbb{R}$ and $u : [b-\lambda,b] \to \mathbb{R}$ be some weight functions such that $\int_a^b w(t)\,dt = \int_{b-\lambda}^b u(t)\,dt$. Then*

$$\int_a^b w(t) f(t)\,dt \le \int_{b-\lambda}^b u(t) f(t)\,dt \tag{4.12}$$

holds for every nonincreasing function f if and only if $0 < \lambda \le b-a$ and

$$\begin{aligned} &\int_a^x w(t)\,dt \le 0 \text{ for } x \in [a,b-\lambda] \\ \text{and } &\int_{b-\lambda}^x u(t)\,dt \ge \int_a^x w(t)\,dt \quad \text{for } x \in (b-\lambda,b]; \end{aligned} \tag{4.13}$$

or $\lambda > b-a$ and

$$\begin{aligned} &\int_{b-\lambda}^x u(t)\,dt \ge 0 \text{ for } x \in [b-\lambda,a] \\ \text{and } &\int_{b-\lambda}^x u(t)\,dt \ge \int_a^x w(t)\,dt \text{ for } x \in (a,b]. \end{aligned} \tag{4.14}$$

Proof. If $0 < \lambda \le b-a$ we apply (4.12) for f defined by (4.7) with $x \in [a,b-\lambda]$, $x \in (b-\lambda,b]$, and inequalities in (4.13) follow. Similarly, if $\lambda > b-a$, we apply (4.12) for f with $x \in [b-\lambda,a]$, $x \in (a,b]$ and (4.14) follows.

Conversely from Theorem 4.1 applied with $[c,d] = [b-\lambda,b]$ we have

$$\int_a^b w(t) f(t)\,dt - \int_{b-\lambda}^b u(t) f(t)\,dt = \alpha \int_{\min\{a,b-\lambda\}}^b K(t)\,df(t)$$

where $\alpha = \int_a^b w(t)\,dt = \int_{b-\lambda}^b u(t)\,dt$ and in case $0 < \lambda \le b-a$

$$\alpha K(t) = \begin{cases} -W(t), & t \in [a,b-\lambda] \\ U(t) - W(t), & t \in (b-\lambda,b], \end{cases}$$

while in case $\lambda > b-a$

$$\alpha K(t) = \begin{cases} U(t), & t \in [b-\lambda, a] \\ U(t) - W(t), & t \in (a,b]. \end{cases}$$

The rest of the proof can be obtained by proceeding in a similar way as in the proof of Theorem 4.3. □

Corollary 4.2 *Let $f : [a,b] \cup [b-\lambda, b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ be integrable functions, where $\lambda = \int_a^b g(t)\,dt$. Then*

$$\int_a^b f(t)g(t)\,dt \le \int_{b-\lambda}^b f(t)\,dt \tag{4.15}$$

holds for every nonincreasing function f if and only if $0 < \lambda \le b-a$ and

$$\int_a^x g(t)\,dt \le 0 \text{ for } x \in [a, b-\lambda] \text{ and } b - x \le \int_x^b g(t)\,dt \text{ for } x \in (b-\lambda, b];$$

or $\lambda > b-a$ and

$$b - x \le \int_x^b g(t)\,dt \text{ for } x \in [a,b].$$

Proof. Apply Theorem 4.4 with weight functions $w(t) = g(t)$ for $t \in [a,b]$ and $u(t) = 1$ for $t \in [b-\lambda, b]$. □

Corollaries 4.1 and 4.2 were obtained by Pečarić in [109] (see also Theorem 2.6). Here we showed that they can also be obtained from the generalization of Steffensen's inequality given here which was obtained by Aglić Aljinović, Pečarić and Perušić in [10].

Remark 4.2 If f is a nondecreasing function, inequalities (4.8), (4.11), (4.12) and (4.15) are reversed.

Finally, we give a generalization of Cerone's result given in Theorem 3.41.

Theorem 4.5 *Let $f : [a,b] \to \mathbb{R}$ be nonincreasing. Also, let $w : [a,b] \to [0,\infty)$ and $u_i : [c_i, d_i] \to [0,\infty)$, $i = 1,2$, be some weight functions, such that $\int_a^b w(t)\,dt = \int_{c_i}^{d_i} u_i(t)\,dt \neq 0$ and $0 \le w(t) \le u_i(t)$, $t \in [c_i, d_i]$, where $[c_i, d_i] \subset [a,b]$ for $i = 1,2$ and $c_1 \le c_2$. Then*

$$\int_{c_2}^{d_2} u_2(t) f(t)\,dt - r(c_2, d_2) \le \int_a^b w(t) f(t)\,dt \le \int_{c_1}^{d_1} u_1(t) f(t)\,dt + R(c_1, d_1) \tag{4.16}$$

holds, where

$$r(c_2, d_2) = \int_{d_2}^b (f(c_2) - f(t))\, w(t)\,dt \ge 0$$

and

$$R(c_1, d_1) = \int_a^{c_1} (f(t) - f(d_1))\, w(t)\,dt \ge 0.$$

Proof. First we prove the right-hand side of inequality (4.16). We denote $\lambda = \int_a^b w(t)\,dt = \int_{c_i}^{d_i} u_i(t)\,dt \neq 0$, $i = 1,2$. Multiplying (4.1) with λ and utilizing (4.2) we have

$$\begin{aligned}
&\int_a^b w(t) f(t)\,dt - \int_{c_1}^{d_1} u_1(t) f(t)\,dt \\
&= -\int_a^{c_1} W(t)\,df(t) + \int_{c_1}^{d_1} (-W(t)+U_1(t))\,df(t) + \int_{d_1}^{b} (\lambda - W(t))\,df(t).
\end{aligned}$$

Since $\lambda - W(t) = \int_t^b w(t) \geq 0$ and f is nonincreasing, we have $\int_{d_1}^b (\lambda - W(t))\,df(t) \leq 0$. Thus, changing the order of integration leads us to

$$\begin{aligned}
&\int_a^b w(t) f(t)\,dt - \int_{c_1}^{d_1} u_1(t) f(t)\,dt \\
&\leq -\int_a^{c_1} W(t)\,df(t) + \int_{c_1}^{d_1} (-W(t)+U_1(t))\,df(t) \\
&= -\int_a^{d_1} W(t)\,df(t) + \int_{c_1}^{d_1} U_1(t)\,df(t) \\
&= -\int_a^{d_1} \left(\int_a^t w(s)\,ds\right) df(t) + \int_{c_1}^{d_1} \left(\int_{c_1}^t u_1(s)\,ds\right) df(t) \\
&= -\int_a^{d_1} \left(\int_s^{d_1} df(t)\right) w(s)\,ds + \int_{c_1}^{d_1} \left(\int_s^{d_1} df(t)\right) u(s)\,ds \\
&= -\int_a^{d_1} (f(d_1) - f(s))\,w(s)\,ds + \int_{c_1}^{d_1} (f(d_1) - f(s))\,u(s)\,ds \\
&= \int_a^{c_1} (f(s) - f(d_1))\,w(s)\,ds + \int_{c_1}^{d_1} (f(d_1) - f(s))\,(u(s) - w(s))\,ds \\
&\leq \int_a^{c_1} (f(s) - f(d_1))\,w(s)\,ds = R(c_1,d_1).
\end{aligned}$$

The last inequality holds since $f(d_1) \leq f(s)$ and $u(s) \geq w(s)$ for $s \in [c_1,d_1]$. The left-hand side inequality in (4.16) can be proved in a similar manner. □

Remark 4.3 If we take $u_i(x) = 1$, $x \in [c_i,d_i]$, for $i = 1,2$ the previous theorem reduces to Cerone's Theorem 3.41.

Now we give estimates of the left-hand and the right-hand side of generalizations of Steffensen's inequality given in this section. Those estimates were also obtained in [10].

Theorem 4.6 *Let $f : [a,b] \cup [a,a+\lambda] \to \mathbb{R}$ be a continuous function of bounded variation on $[a,b] \cup [a,a+\lambda]$ and let $w : [a,b] \to \mathbb{R}$ and $u : [a,a+\lambda] \to \mathbb{R}$ be some weight functions, such that $\int_a^b w(t)\,dt \neq 0$, $\int_a^{a+\lambda} u(t)\,dt \neq 0$. Let also $W(x) = \int_a^x w(t)\,dt$, $x \in [a,b]$ and*

$U(x)=\int_a^x u(t)\,dt$, $x\in[a,a+\lambda]$. Then, if $a+\lambda\le b$, it holds that

$$\left|\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)f(t)\,dt-\frac{1}{\int_a^{a+\lambda}u(t)\,dt}\int_a^{a+\lambda}u(t)f(t)\,dt\right| \le \bigvee_a^b(f)\cdot\max\left\{\max_{t\in[a,a+\lambda]}\left|\frac{U(t)}{U(a+\lambda)}-\frac{W(t)}{W(b)}\right|,\max_{t\in[a+\lambda,b]}\left|1-\frac{W(t)}{W(b)}\right|\right\} \tag{4.17}$$

and if $\lambda\ge b-a$

$$\left|\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)f(t)\,dt-\frac{1}{\int_a^{a+\lambda}u(t)\,dt}\int_a^{a+\lambda}u(t)f(t)\,dt\right| \le \bigvee_a^{a+\lambda}(f)\cdot\max\left\{\max_{t\in[a,b]}\left|\frac{U(t)}{U(a+\lambda)}-\frac{W(t)}{W(b)}\right|,\max_{t\in[b,a+\lambda]}\left|\frac{U(t)}{U(a+\lambda)}-1\right|\right\}, \tag{4.18}$$

where $\bigvee_a^b(f)$ is the total variation of function f. Both inequalities are sharp.

Proof. Applying Theorem 4.1 with $[c,d]=[a,a+\lambda]$ we obtain

$$\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)f(t)\,dt-\frac{1}{\int_a^{a+\lambda}u(t)\,dt}\int_a^{a+\lambda}u(t)f(t)\,dt=\int_a^{\max\{b,a+\lambda\}}K(t)\,df(t).$$

Since $K(t)$ is continuous on $[a,b]$ and f is a function of bounded variation on $[a,b]\cup[a,a+\lambda]$, in the case $a+\lambda\le b$ we have

$$\left|\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)f(t)\,dt-\frac{1}{\int_a^{a+\lambda}u(t)\,dt}\int_a^{a+\lambda}u(t)f(t)\,dt\right| =\left|\int_a^b K(t)\,df(t)\right|\le\bigvee_a^b(f)\cdot\sup_{t\in[a,b]}|K(t)|$$

where $K(t)$ is given by

$$K(t)=\begin{cases}\frac{U(t)}{U(a+\lambda)}-\frac{W(t)}{W(b)}, & t\in[a,a+\lambda]\\[2ex] 1-\frac{W(t)}{W(b)}, & t\in(a+\lambda,b].\end{cases}$$

Thus (4.17) follows. In order to prove the sharpness of (4.17) consider the nonincreasing function f defined by

$$f(t)=\begin{cases}1, & t\in[a,a+\lambda]\\ 0, & t\in(a+\lambda,b],\end{cases}$$

and weight functions $w(t)=1$, $t\in[a,b]$, $u(t)=1$, $t\in[a,a+\lambda]$. It is easy to check that then equality in (4.17) holds. In case $a+\lambda\ge b$ we have

$$K(t)=\begin{cases}\frac{U(t)}{U(a+\lambda)}-\frac{W(t)}{W(b)}, & t\in[a,b]\\[2ex] \frac{U(t)}{U(a+\lambda)}-1, & t\in(b,a+\lambda],\end{cases}$$

and the proof of (4.18) can be obtained in the similar manner. In this case, to prove the sharpness of (4.18), we consider

$$f(t) = \begin{cases} 1, & t \in [a,b] \\ 0, & t \in (b, a+\lambda], \end{cases}$$

and weight functions $w(t) = 1$, $t \in [a,b]$, $u(t) = 1$, $t \in [a, a+\lambda]$. This completes the proof. □

Corollary 4.3 *Suppose that all the assumptions of the previous theorem hold. Additionally, assume that $g : [a,b] \to \mathbb{R}$ is an integrable function. Let $G(x) = \int_a^x g(t)\,dt$, $x \in [a,b]$ and $\lambda = G(b)$. Then, if $a + \lambda \le b$, it holds that*

$$\left| \int_a^b g(t) f(t)\,dt - \int_a^{a+\lambda} f(t)\,dt \right| \le \bigvee_a^b (f) \cdot \max \left\{ \max_{t \in [a,a+\lambda]} |t - a - G(t)|, \max_{t \in [a+\lambda, b]} |\lambda - G(t)| \right\}$$

and if $\lambda \ge b - a$

$$\left| \int_a^b g(t) f(t)\,dt - \int_a^{a+\lambda} f(t)\,dt \right| \le \bigvee_a^{a+\lambda} (f) \cdot \max \left\{ \max_{t \in [a,b]} |t - a - G(t)|, \max_{t \in [b,a+\lambda]} |t - a - \lambda| \right\}.$$

Both inequalities are sharp.

Proof. Apply Theorem 4.6 with weight functions $w(t) = g(t)$ for $t \in [a,b]$ and $u(t) = 1$ for $t \in [a, a+\lambda]$. □

Theorem 4.7 *Let $f : [a,b] \cup [b-\lambda, b] \to \mathbb{R}$ be a continuous function of bounded variation on $[a,b] \cup [b-\lambda, b]$ and let $w : [a,b] \to \mathbb{R}$ and $u : [b-\lambda, b] \to \mathbb{R}$ be some weight functions, such that $\int_a^b w(t)\,dt \neq 0$, $\int_{b-\lambda}^b u(t)\,dt \neq 0$. Let also $W(x) = \int_a^x w(t)\,dt$, $x \in [a,b]$ and $U(x) = \int_{b-\lambda}^x u(t)\,dt$, $x \in [b-\lambda, b]$. Then, if $a + \lambda \le b$, it holds that*

$$\begin{aligned} &\left| \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) f(t)\,dt - \frac{1}{\int_{b-\lambda}^b u(t)\,dt} \int_{b-\lambda}^b u(t) f(t)\,dt \right| \\ &\le \bigvee_a^b (f) \cdot \max \left\{ \max_{t \in [a, b-\lambda]} \left| \frac{W(t)}{W(b)} \right|, \max_{t \in [b-\lambda, b]} \left| \frac{U(t)}{U(b)} - \frac{W(t)}{W(b)} \right| \right\} \end{aligned} \tag{4.19}$$

and if $\lambda \ge b - a$

$$\begin{aligned} &\left| \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) f(t)\,dt - \frac{1}{\int_{b-\lambda}^b u(t)\,dt} \int_{b-\lambda}^b u(t) f(t)\,dt \right| \\ &\le \bigvee_{b-\lambda}^b (f) \cdot \max \left\{ \max_{t \in [b-\lambda, a]} \left| \frac{U(t)}{U(b)} \right|, \max_{t \in [a,b]} \left| \frac{U(t)}{U(b)} - \frac{W(t)}{W(b)} \right| \right\}. \end{aligned} \tag{4.20}$$

Both inequalities are sharp.

Proof. Applying Theorem 4.1 with $[c,d]=[b-\lambda,b]$ we obtain

$$\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)f(t)\,dt-\frac{1}{\int_{b-\lambda}^b u(t)\,dt}\int_{b-\lambda}^b u(t)f(t)\,dt=\int_{\min\{a,b-\lambda\}}^b K(t)\,df(t)$$

where if $a+\lambda\le b$

$$K(t)=\begin{cases}-\frac{W(t)}{W(b)}, & t\in[a,b-\lambda]\\ \frac{U(t)}{U(b)}-\frac{W(t)}{W(b)}, & t\in(b-\lambda,b],\end{cases}$$

and if $a+\lambda\ge b$

$$K(t)=\begin{cases}\frac{U(t)}{U(b)}, & t\in[b-\lambda,a]\\ \frac{U(t)}{U(b)}-\frac{W(t)}{W(b)}, & t\in(a,b].\end{cases}$$

The rest of the proof of (4.19) and (4.20) is similar to the proof of Theorem 4.6. In order to prove the sharpness of (4.19) consider the nonincreasing function f defined by

$$f(t)=\begin{cases}1, & t\in[a,b-\lambda]\\ 0, & t\in(b-\lambda,b],\end{cases}$$

weight functions $w(t)=1, t\in[a,b]$, $u(t)=1, t\in[b-\lambda,b]$, and to prove the sharpness of (4.20) consider

$$f(t)=\begin{cases}1, & t\in[b-\lambda,a]\\ 0, & t\in(a,b],\end{cases}$$

weight functions $w(t)=1, t\in[a,b]$, $u(t)=1, t\in[b-\lambda,b]$. This completes the proof. □

Corollary 4.4 *Suppose that all the assumptions of the previous theorem hold. Additionally, assume that $g:[a,b]\to\mathbb{R}$ is an integrable function. Let $G(x)=\int_a^x g(t)\,dt$, $x\in[a,b]$ and $\lambda=G(b)$. Then, if $a+\lambda\le b$, it holds that*

$$\left|\int_a^b g(t)f(t)\,dt-\int_{b-\lambda}^b f(t)\,dt\right|\le\bigvee_a^b(f)\cdot\max\left\{\max_{t\in[a,b-\lambda]}|-G(t)|,\ \max_{t\in[b-\lambda,b]}|t-b+\lambda-G(t)|\right\}$$

and if $\lambda\ge b-a$

$$\left|\int_a^b g(t)f(t)\,dt-\int_{b-\lambda}^b f(t)\,dt\right|\le\bigvee_{b-\lambda}^b(f)\cdot\max\left\{\max_{t\in[b-\lambda,a]}|t-b+\lambda|,\ \max_{t\in[a,b]}|t-b+\lambda-G(t)|\right\}.$$

Both inequalities are sharp.

Proof. Apply Theorem 4.7 with weight functions $w(t)=g(t)$ for $t\in[a,b]$ and $u(t)=1$ for $t\in[b-\lambda,b]$. □

Applying Theorem 4.1 for $[c,d]=[a,a+\lambda]$ and for $[c,d]=[b-\lambda,b]$, with an additional assumption that f is differentiable and that $|f'|^p$ is an integrable function, analogous inequalities for L^p spaces as in [5] could be obtained.

4.2 Generalizations via n weight functions

We begin this section with the following weighted Euler identity given in [8].

Theorem 4.8 *Let $f : [a,b] \to \mathbb{R}$ be n-times differentiable on $[a,b], n \in \mathbb{N}$ with $f^{(n)} : [a,b] \to \mathbb{R}$ integrable on $[a,b]$. Let $w_i : [a,b] \to [0,\infty)$, $i = 1,..,n$ be a sequence of n integrable functions satisfying $\int_a^b w_i(t)\,dt = 1$ and $W_i(t) = \int_a^t w_i(x)\,dx$ for $t \in [a,b]$, $W_i(t) = 0$ for $t < a$ and $W_i(t) = 1$ for $t > b$, for all $i = 1,..,n$. For any $x \in [a,b]$ define weighted Peano kernel:*

$$P_{w_i}(x,t) = \begin{cases} W_i(t), & a \le t \le x \\ W_i(t) - 1, & x < t \le b. \end{cases}$$

Then

$$\begin{aligned} & f(x) - \int_a^b w_1(t) f(t)\,dt - \sum_{k=0}^{n-2} \left(\int_a^b w_{k+2}(t) f^{(k+1)}(t)\,dt \right) \times \\ & \times \left(\int_a^b \cdots \int_a^b P_{w_1}(x,t_1) \prod_{i=1}^{k} P_{w_{i+1}}(t_i,t_{i+1})\,dt_1 \cdots dt_{k+1} \right) \\ & = \int_a^b \cdots \int_a^b P_{w_1}(x,t_1) \prod_{i=1}^{n-1} P_{w_{i+1}}(t_i,t_{i+1}) f^{(n)}(t_n)\,dt_1 \cdots dt_n. \end{aligned} \tag{4.21}$$

For $n = 1$, identity (4.21) reduces to the weighted Montgomery identity given by (4.1), for $\int_a^b w(t)dt = 1$, i.e. it reduces to

$$f(x) - \int_a^b w_1(t) f(t)\,dt = \int_a^b P_{w_1}(x,t_1) f'(t_1)\,dt_1.$$

Next, we subtract two generalized weighted Montgomery identities (4.21) to obtain identity for the difference between two weighted integral means, each having its own weight, on two different intersecting intervals $[a,b]$ and $[c,d]$. As mentioned in the previous section we have four possible cases when $[a,b] \cap [c,d] \neq \emptyset$.

For that purpose we denote

$$\begin{aligned} T_{w_1,..,w_n}^{[a,b]}(x) = & \sum_{k=0}^{n-2} \left(\frac{1}{\int_a^b w_{k+2}(t)\,dt} \int_a^b w_{k+2}(t) f^{(k+1)}(t)\,dt \right) \times \\ & \times \left(\int_a^b \cdots \int_a^b P_{w_1}(x,t_1) \prod_{i=1}^{k} P_{w_{i+1}}(t_i,t_{i+1})\,dt_1 \cdots dt_{k+1} \right). \end{aligned}$$

The following results were obtained by Aglić Aljinović, Pečarić and Perušić in [11].

Theorem 4.9 *Let $f:[a,b]\cup[c,d]\to\mathbb{R}$ be n-times differentiable on $[a,b]\cup[c,d]$, $n\in\mathbb{N}$ with $f^{(n)}:[a,b]\to\mathbb{R}$ integrable on $[a,b]\cup[c,d]$. Let $w_i:[a,b]\to[0,\infty)$, $i=1,..,n$ be a sequence of n integrable functions, $W_i(t)=\int_a^t w_i(x)\,dx$ for $t\in[a,b]$, $W_i(t)=0$ for $t<a$ and $W_i(t)=\int_a^b w_i(x)\,dx$ for $t>b$, for all $i=1,..,n$. Also, let $u_i:[c,d]\to[0,\infty)$, $i=1,..,n$ be a sequence of n integrable functions, $U_i(t)=\int_c^t u_i(x)\,dx$ for $t\in[c,d]$, $U_i(t)=0$ for $t<c$ and $U_i(t)=\int_c^d u_i(x)\,dx$ for $t>d$, for all $i=1,..,n$. For any $x\in[a,b]\cup[c,d]$ define weighted Peano kernels:*

$$P_{w_i}(x,t)=\begin{cases}\frac{1}{W_i(b)}W_i(t), & a\le t\le x\\ \frac{1}{W_i(b)}W_i(t)-1, & x<t\le b\\ 0, & t\notin[a,b],\end{cases}$$

$$P_{u_i}(x,t)=\begin{cases}\frac{1}{U_i(d)}U_i(t), & c\le t\le x\\ \frac{1}{U_i(d)}U_i(t)-1, & x<t\le d\\ 0, & t\notin[c,d].\end{cases}$$

If $W_i(b)\neq 0$ and $U_i(d)\neq 0$, $i=1,..,n$, then for any $x\in[a,b]\cap[c,d]$ it holds

$$\frac{1}{\int_c^d u_1(t)\,dt}\int_c^d u_1(t)f(t)\,dt-\frac{1}{\int_a^b w_1(t)\,dt}\int_a^b w_1(t)f(t)\,dt-T^{[a,b]}_{w_1,..,w_n}(x)$$
$$+T^{[c,d]}_{u_1,..,u_n}(x)=\int_{\min\{a,c\}}^{\max\{b,d\}}K(x,t_1,\dots,t_n)f^{(n)}(t_n)\,dt_n \tag{4.22}$$

where

$$K(x,t_1,\dots,t_n)=\int_{\min\{a,c\}}^{\max\{b,d\}}\cdots\int_{\min\{a,c\}}^{\max\{b,d\}}\left[P_{w_1}(x,t_1)\prod_{i=1}^{n-1}P_{w_{i+1}}(t_i,t_{i+1})\right. \tag{4.23}$$
$$\left.-P_{u_1}(x,t_1)\prod_{i=1}^{n-1}P_{u_{i+1}}(t_i,t_{i+1})\right]dt_1\cdots dt_{n-1}.$$

Proof. We apply (4.21) with n normalized weight functions $W_i(b)/w_i(t)$, $t\in[a,b]$, $i=1,..,n$, and then once again with n normalized weight functions $U_i(d)/u_i(t)$, $t\in[c,d]$, $i=1,..,n$. Subtracting these two identities we obtain

$$\begin{aligned}&\frac{\int_c^d u_1(t)f(t)\,dt}{\int_c^d u_1(t)\,dt}-\frac{\int_a^b w_1(t)f(t)\,dt}{\int_a^b w_1(t)\,dt}-T^{[a,b]}_{w_1,..,w_n}(x)+T^{[c,d]}_{u_1,..,u_n}(x)\\ &=\int_a^b\cdots\int_a^b P_{w_1}(x,t_1)\prod_{i=1}^{n-1}P_{w_{i+1}}(t_i,t_{i+1})f^{(n)}(t_n)\,dt_1\cdots dt_n\\ &-\int_c^d\cdots\int_c^d P_{u_1}(x,t_1)\prod_{i=1}^{n-1}P_{u_{i+1}}(t_i,t_{i+1})f^{(n)}(t_n)\,dt_1\cdots dt_n\\ &=\int_{\min\{a,c\}}^{\max\{b,d\}}K(x,t_1,\dots,t_n)f^{(n)}(t_n)\,dt_n\end{aligned}$$

and (4.22) is proved. □

Consider the sequence $(B_k(t), k \geq 0)$ of Bernoulli polynomials which is uniquely determined by the following identities:

$$B_k'(t) = kB_{k-1}(t), \quad k \geq 1, \ B_0(t) = 1$$

and

$$B_k(t+1) - B_k(t) = kt^{k-1}, \quad k \geq 0.$$

The values $B_k = B_k(0)$, $k \geq 0$ are known as the Bernoulli numbers. For our purposes, the first five Bernoulli polynomials are

$$\begin{aligned} &B_0(t) = 1, \ B_1(t) = t - \frac{1}{2}, \ B_2(t) = t^2 - t + \frac{1}{6}, \\ &B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t, \ B_4(t) = t^4 - 2t^3 + t^2 - \frac{1}{30}. \end{aligned} \tag{4.24}$$

Let $\left(B_k^*(t), k \geq 0\right)$ be the sequence of periodic functions with period 1, related to Bernoulli polynomials as

$$B_k^*(t) = B_k(t), \quad 0 \leq t < 1, \qquad B_k^*(t+1) = B_k^*(t), \quad t \in \mathbb{R}.$$

From the properties of Bernoulli polynomials it easily follows that $B_0^*(t) = 1, B_1^*$ is discontinuous function with a jump of -1 at each integer, while B_k^* , $k \geq 2$, are continuous functions.

Corollary 4.5 *Let $f : [a,b] \cup [c,d] \to \mathbb{R}$ be n-times differentiable on $[a,b] \cup [c,d], n \in \mathbb{N}$ with $f^{(n)} : [a,b] \to \mathbb{R}$ integrable on $[a,b] \cup [c,d]$. Let $w : [a,b] \to [0,\infty)$ and $u : [c,d] \to [0,\infty)$ be integrable weight functions, $W(t) = \int_a^t w(x)\,dx$ for $t \in [a,b]$, $W(t) = 0$ for $t < a$ and $W(t) = \int_a^b w(x)\,dx$ for $t > b$, $U(t) = \int_c^t u(x)\,dx$ for $t \in [c,d]$, $U(t) = 0$ for $t < c$ and $U(t) = \int_c^d u(x)\,dx$ for $t > d$. If $W(b) \neq 0$ and $U(d) \neq 0$, then for any $x \in [a,b] \cap [c,d]$ it holds*

$$\begin{aligned} &\frac{1}{\int_c^d u(t)\,dt} \int_c^d u(t) f(t)\,dt - \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) f(t)\,dt - T_w^{[a,b]}(x) + T_u^{[c,d]}(x) \\ &= \frac{(b-a)^{n-2}}{(n-1)!} \int_a^b \left(\int_a^b P_w(x,s) \left[B_{n-1}\left(\frac{s-a}{b-a}\right) - B_{n-1}^*\left(\frac{s-t}{b-a}\right) \right] ds \right) f^{(n)}(t)\,dt \\ &- \frac{(d-c)^{n-2}}{(n-1)!} \int_c^d \left(\int_c^d P_u(x,s) \left[B_{n-1}\left(\frac{s-c}{d-c}\right) - B_{n-1}^*\left(\frac{s-t}{d-c}\right) \right] ds \right) f^{(n)}(t)\,dt \end{aligned} \tag{4.25}$$

where

$$T_w^{[a,b]}(x) = \sum_{k=0}^{n-2} \frac{(b-a)^{k-1}}{k!} \left(\int_a^b P_w(x,t) B_k\left(\frac{t-a}{b-a}\right) dt \right) \left(f^{(k)}(b) - f^{(k)}(a) \right).$$

Proof. We apply identity (4.22) with $w_1 \equiv w$, $w_i \equiv \frac{1}{b-a}$, $i = 2,..,n$ and $u_1 \equiv u$, $u_i \equiv \frac{1}{d-c}$, $i = 2,..,n$. Then $P_{w_i}(x,t)$ and $P_{u_i}(x,t)$ for $i = 2,..,n$ reduce to

$$P_{a,b}(x,t) = \begin{cases} \frac{t-a}{b-a}, & a \le t \le x \\ \frac{t-b}{b-a}, & x < t \le b \\ 0, & t \notin [a,b] \end{cases} \quad \text{and} \quad P_{c,d}(x,t) = \begin{cases} \frac{t-c}{d-c}, & c \le t \le x \\ \frac{t-d}{d-c}, & x < t \le d \\ 0, & t \notin [c,d]. \end{cases}$$

Since the the following two identities hold (see [4])

$$\int_a^b \cdots \int_a^b P_{a,b}(x,s_1) \left(\prod_{i=1}^{k-1} P_{a,b}(s_i,s_{i+1}) \right) ds_1 \cdots ds_k = \frac{(b-a)^k}{k!} B_k \left(\frac{x-a}{b-a} \right)$$

and

$$\int_a^b \cdots \int_a^b P_{a,b}(x,s_1) \left(\prod_{i=1}^{n-2} P_{a,b}(s_i,s_{i+1}) \right) ds_1 \cdots ds_{n-2} = \frac{(b-a)^{n-2}}{(n-1)!} \left[B_{n-1} \left(\frac{x-a}{b-a} \right) - B_{n-1}^* \left(\frac{x-s_n}{b-a} \right) \right]$$

it follows that

$$\frac{1}{b-a} \int_a^b \cdots \int_a^b P_w(x,t_1) \prod_{i=1}^{k} P_{a,b}(t_i,t_{i+1})\, dt_1 \cdots dt_{k+1} = \frac{(b-a)^{k-1}}{k!} \left(\int_a^b P_w(x,t) B_k \left(\frac{t-a}{b-a} \right) dt \right)$$

and

$$\begin{aligned} &\int_a^b \cdots \int_a^b P_w(x,t_1) \prod_{i=1}^{n-1} P_{a,b}(t_i,t_{i+1}) f^{(n)}(t_n)\, dt_1 \cdots dt_n \\ &= \frac{(b-a)^{n-2}}{(n-1)!} \int_a^b \left(\int_a^b P_w(x,s) \left[B_{n-1} \left(\frac{s-a}{b-a} \right) - B_{n-1}^* \left(\frac{s-t}{b-a} \right) \right] ds \right) f^{(n)}(t)\, dt. \end{aligned}$$

Consequently $T_{w_1,..,w_n}^{[a,b]}(x)$ reduces to

$$\begin{aligned} T_w^{[a,b]}(x) &= \frac{1}{b-a} \sum_{k=0}^{n-2} \left(\int_a^b \cdots \int_a^b P_w(x,t_1) \prod_{i=1}^{k} P_{a,b}(t_i,t_{i+1})\, dt_1 \cdots dt_{k+1} \right) \times \\ &\times \left(f^{(k)}(b) - f^{(k)}(a) \right) = \sum_{k=0}^{n-2} \frac{(b-a)^{k-1}}{k!} \times \\ &\times \left(\int_a^b P_w(x,t) B_k \left(\frac{t-a}{b-a} \right) dt \right) \left(f^{(k)}(b) - f^{(k)}(a) \right) \end{aligned}$$

and similarly $T^{[c,d]}_{u_1,..,u_n}(x)$ to $T^{[c,d]}_{u}(x)$. Finally

$$\begin{aligned}
&\int_{\min\{a,c\}}^{\max\{b,d\}} K(x,t_1,\ldots,t_n) f^{(n)}(t_n)\,dt_n \\
&= \frac{(b-a)^{n-2}}{(n-1)!}\int_a^b \left(\int_a^b P_w(x,s)\left[B_{n-1}\left(\frac{s-a}{b-a}\right) - B^*_{n-1}\left(\frac{s-t}{b-a}\right)\right] ds\right)\times \\
&\times f^{(n)}(t)\,dt - \frac{(d-c)^{n-2}}{(n-1)!}\int_c^d \left(\int_c^d P_u(x,s)\left[B_{n-1}\left(\frac{s-c}{d-c}\right)\right.\right. \\
&\left.\left.-B^*_{n-1}\left(\frac{s-t}{d-c}\right)\right] ds\right) f^{(n)}(t)\,dt
\end{aligned}$$

and identity (4.22) reduces to identity (4.25). □

Corollary 4.6 *Let $f:[a,b]\cup[c,d]\to\mathbb{R}$ be n-times differentiable on $[a,b]\cup[c,d]$, $n\in\mathbb{N}$ with $f^{(n)}:[a,b]\to\mathbb{R}$ integrable on $[a,b]\cup[c,d]$. Let $w:[a,b]\to[0,\infty)$ and $u:[c,d]\to[0,\infty)$ be integrable weight functions, $W(t)=\int_a^t w(x)\,dx$ for $t\in[a,b]$, $W(t)=0$ for $t<a$ and $W(t)=\int_a^b w(x)\,dx$ for $t>b$, $U(t)=\int_c^t u(x)\,dx$ for $t\in[c,d]$, $U(t)=0$ for $t<c$ and $U(t)=\int_c^d u(x)\,dx$ for $t>d$. If $W(b)\neq 0$ and $U(d)\neq 0$, then for any $x\in[a,b]\cap[c,d]$ it holds*

$$\begin{aligned}
&\frac{1}{\int_c^d u(t)\,dt}\int_c^d u(t) f(t)\,dt - \frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t) f(t)\,dt - T^{[a,b]}_{w,n}(x) + T^{[c,d]}_{u,n}(x) \\
&= \int_{\min\{a,c\}}^{\max\{b,d\}} \widehat{K}(x,t_1,\ldots,t_n) f^{(n)}(t_n)\,dt_n \qquad (4.26)
\end{aligned}$$

where

$$\begin{aligned}
T^{[a,b]}_{w,n}(x) = \sum_{k=0}^{n-2}&\left(\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t) f^{(k+1)}(t)\,dt\right) \\
&\times\left(\int_a^b\cdots\int_a^b P_w(x,t_1)\prod_{i=1}^{k} P_w(t_i,t_{i+1})\,dt_1\cdots dt_{k+1}\right)
\end{aligned}$$

and

$$\begin{aligned}
\widehat{K}(x,t_1,\ldots,t_n) = \int_{\min\{a,c\}}^{\max\{b,d\}}\cdots\int_{\min\{a,c\}}^{\max\{b,d\}}&\left[P_w(x,t_1)\prod_{i=1}^{n-1} P_w(t_i,t_{i+1})\right. \\
&\left.-P_u(x,t_1)\prod_{i=1}^{n-1} P_u(t_i,t_{i+1})\right] dt_1\cdots dt_{n-1}.
\end{aligned}$$

Proof. We apply identity (4.22) with $w_i\equiv w$, $i=1,..,n$. Then $T^{[a,b]}_{w_1,..,w_n}(x)$, $T^{[c,d]}_{u_1,..,u_n}(x)$ and $K(x,t_1,\ldots,t_n)$ reduce to $T^{[a,b]}_{w,n}(x)$, $T^{[c,d]}_{u,n}(x)$ and $\widehat{K}(x,t_1,\ldots,t_n)$ respectively. □

Identity (4.25) was obtained in [7]. Special case for uniform normalized weight function w for the case $[c,d] \subseteq [a,b]$ was obtained in [115] and for the case $[a,b] \cap [c,d] = [c,b]$ in [9].

Identity (4.26) for uniform normalized weight function w for $c = d$ as a limit case and $n = 2$ was obtained in [17] and for $n = 3$ in [4].

Theorem 4.10 *Let $f : [a,b] \cup [c,d] \to \mathbb{R}$ be n-convex function on $[a,b] \cup [c,d]$, $n \in \mathbb{N}$. Let $w_i : [a,b] \to [0,\infty)$, $i = 1,..,n$ be a sequence of n integrable functions, $W_i(t) = \int_a^t w_i(x)\,dx$ for $t \in [a,b]$, $W_i(t) = 0$ for $t < a$ and $W_i(t) = \int_a^b w_i(x)\,dx$ for $t > b$, for all $i = 1,..,n$. Also, let $u_i : [c,d] \to [0,\infty)$, $i = 1,..,n$ be a sequence of n integrable functions, $U_i(t) = \int_c^t u_i(x)\,dx$ for $t \in [c,d]$, $U_i(t) = 0$ for $t < c$ and $U_i(t) = \int_c^d u_i(x)\,dx$ for $t > d$, for all $i = 1,..,n$. If $K(x,t_1,\ldots,t_n) \geq 0$, where $K(x,t_1,\ldots,t_n)$ is the function defined by (4.23), then for any $x \in [a,b] \cap [a,a+\lambda]$ it holds*

$$\frac{1}{\int_a^b w_1(t)\,dt} \int_a^b w_1(t) f(t)\,dt + T_{w_1,..,w_n}^{[a,b]}(x) \leq \frac{1}{\int_c^d u_1(t)\,dt} \int_c^d u_1(t) f(t)\,dt + T_{u_1,..,u_n}^{[c,d]}(x). \tag{4.27}$$

Proof. Since f is an n-convex function, without loss of generality we can assume (see [122, p. 293]) that $f^{(n)}$ exists and is continuous. Using (4.22), inequality (4.27) follows. □

Inequality (4.27) also holds if f is n-concave and $K(x,t_1,\ldots,t_n) \leq 0$. If f is n-concave and $K(x,t_1,\ldots,t_n) \geq 0$ or f is n-convex and $K(x,t_1,\ldots,t_n) \leq 0$, inequality (4.27) is reversed.

In the following corollaries we give generalizations of Steffensen's inequality using previous general results. These results were obtained in [11].

Corollary 4.7 *Let $f : [a,b] \cup [a,a+\lambda] \to \mathbb{R}$ be an n-convex function on $[a,b] \cup [a,a+\lambda]$, $n \in \mathbb{N}$. Let $w_i : [a,b] \to [0,\infty)$, $i = 1,..,n$ and $u_i : [a,a+\lambda] \to [0,\infty)$, $i = 1,..,n$ be a sequence of weight functions as in Theorem 4.9. If $K(x,t_1,\ldots,t_n) \geq 0$ where $K(x,t_1,\ldots,t_n)$ is the function defined by (4.23), then for any $x \in [a,b] \cap [a,a+\lambda]$ it holds:*

$$\begin{aligned} \frac{1}{\int_a^b w_1(t)\,dt} \int_a^b w_1(t) f(t)\,dt + T_{w_1,..,w_n}^{[a,b]}(x) \leq \frac{1}{\int_a^{a+\lambda} u_1(t)\,dt} &\int_a^{a+\lambda} u_1(t) f(t)\,dt \\ &+ T_{u_1,..,u_n}^{[a,a+\lambda]}(x). \end{aligned} \tag{4.28}$$

If f is an n-concave function and $K(x,t_1,\ldots,t_n) \leq 0$, inequality (4.28) holds.

Proof. Apply Theorem 4.10 with $[c,d] = [a,a+\lambda]$. □

For every differentiable, nonincreasing function $f : [a,b] \cup [a,a+\lambda] \to \mathbb{R}$ and some weight functions $w : [a,b] \to [0,\infty)$ and $u : [a,a+\lambda] \to [0,\infty)$ such that $\int_a^b w(t)\,dt = \int_a^{a+\lambda} u(t)\,dt$ inequality (4.28) for $n = 1$ reduces to

$$\int_a^b w(t) f(t)\,dt \leq \int_a^{a+\lambda} u(t) f(t)\,dt$$

while condition $K(x,t_1,\ldots,t_n) \le 0$ reduces to

$$\begin{aligned} \int_a^x u(t)\,dt \ge \int_a^x w(t)\,dt \text{ for } x \in [a,a+\lambda] \\ \text{and } \int_x^b w(t)\,dt \ge 0 \text{ for } x \in (a+\lambda,b] \end{aligned} \tag{4.29}$$

in case $0 < \lambda \le b-a$ and to

$$\begin{aligned} \int_a^x u(t)\,dt \ge \int_a^x w(t)\,dt \text{for } x \in [a,b] \\ \text{and } \int_x^{a+\lambda} u(t)\,dt \le 0 \text{ for } x \in (b,a+\lambda] \end{aligned}$$

in case $\lambda > b-a$.

Further for $u \equiv 1$ we have $\int_a^b w(t)\,dt = \int_a^{a+\lambda} u(t)\,dt = \lambda$. Thus if $0 \le w(t) \le 1$ for $t \in [a,b]$, then $\lambda \le b-a$ and it's easy to see that (4.29) is fulfilled. In such a way the right-hand side of Steffensen's inequality (2.1) is recaptured.

Corollary 4.8 *Let $f : [a,b] \cup [b-\lambda,b] \to \mathbb{R}$ be an n-convex function on $[a,b] \cup [b-\lambda,b]$, $n \in \mathbb{N}$. Let $w_i : [a,b] \to [0,\infty)$, $i = 1,..,n$ and $u_i : [b-\lambda,b] \to [0,\infty)$, $i = 1,..,n$ be a sequence of weight functions as in Theorem 4.9. If $K(x,t_1,\ldots,t_n) \le 0$ where $K(x,t_1,\ldots,t_n)$ is the function defined by (4.23), then for any $x \in [a,b] \cap [b-\lambda,b]$ it holds:*

$$\begin{aligned} \frac{1}{\int_a^b w_1(t)\,dt} \int_a^b w_1(t) f(t)\,dt + T_{w_1,..,w_n}^{[a,b]}(x) \ge \frac{1}{\int_{b-\lambda}^b u_1(t)\,dt} \int_{b-\lambda}^b u_1(t) f(t)\,dt \\ + T_{u_1,..,u_n}^{[b-\lambda,b]}(x). \end{aligned} \tag{4.30}$$

If f is an n-concave function and $K(x,t_1,\ldots,t_n) \ge 0$, inequality (4.30) holds.

Proof. Apply Theorem 4.10 with $[c,d] = [b-\lambda,b]$. □

For every differentiable, nonincreasing function $f : [a,b] \cup [b-\lambda,b] \to \mathbb{R}$ and some weight functions $w : [a,b] \to [0,\infty)$ and $u : [b-\lambda,b] \to [0,\infty)$ such that $\int_a^b w(t)\,dt = \int_{b-\lambda}^b u(t)\,dt$ inequality (4.28) for $n = 1$ reduces to

$$\int_a^b w(t) f(t)\,dt \ge \int_{b-\lambda}^b u(t) f(t)\,dt$$

while condition $K(x,t_1,\ldots,t_n) \ge 0$ reduces to

$$\begin{aligned} \int_a^x w(t)\,dt \ge 0 \text{ for } x \in [a,b-\lambda] \\ \text{and } \int_{b-\lambda}^x u(t)\,dt \le \int_a^x w(t)\,dt \ \text{ for } x \in (b-\lambda,b] \end{aligned} \tag{4.31}$$

in case $0 < \lambda \leq b-a$ and to

$$\int_{b-\lambda}^{x} u(t)\,dt \leq 0 \text{ for } x \in [b-\lambda, a]$$
$$\text{and } \int_{b-\lambda}^{x} u(t)\,dt \leq \int_{a}^{x} w(t)\,dt \text{ for } x \in (a,b]$$

in case $\lambda > b-a$.

Further, for $u \equiv 1$ we have $\int_a^b w(t)\,dt = \int_{b-\lambda}^b u(t)\,dt = \lambda$. Thus if $0 \leq w(t) \leq 1$ for $t \in [a,b]$, then $\lambda \leq b-a$ and it's easy to see that (4.31) is fulfilled since

$$x-b+\lambda = \int_{b-\lambda}^{x} u(t)\,dt \leq \int_{a}^{x} w(t)\,dt = \lambda - \int_{x}^{b} w(t)\,dt.$$

In such a way the left-hand side of Steffensen's inequality (2.1) is recaptured.

Now we give L^p inequalities obtained by Aglić Aljinović, Pečarić and Perušić in [11].

Theorem 4.11 *Suppose that all the assumptions of Theorem 4.9 hold. Additionally assume (p,q) is a pair of conjugate exponents, and $f^{(n)} \in L^p_{[a,b]\cup[c,d]}$. Then the following inequality holds*

$$\begin{aligned}
&\left| \frac{1}{\int_c^d u_1(t)\,dt} \int_c^d u_1(t) f(t)\,dt - T^{[a,b]}_{w_1,..,w_n}(x) - \frac{1}{\int_a^b w_1(t)\,dt} \int_a^b w_1(t) f(t)\,dt + T^{[c,d]}_{u_1,..,u_n}(x) \right| \\
&\leq \|K(x,t_1,\ldots,t_{n-1},\cdot)\|_{q,[\min\{a,c\},\max\{b,d\}]} \left\| f^{(n)} \right\|_{p,[\min\{a,c\},\max\{b,d\}]}
\end{aligned} \tag{4.32}$$

Inequality (4.32) is sharp for $1 < p \leq \infty$ and for $p=1$ the constant $\|K(x,t_1,\ldots,t_{n-1},\cdot)\|_{q,[\min\{a,c\},\max\{b,d\}]}$ is the best possible.

Proof. By taking the modulus on (4.22) and applying the Hölder inequality we obtain

$$\begin{aligned}
&\left| \frac{1}{\int_c^d u_1(t)\,dt} \int_c^d u_1(t) f(t)\,dt - T^{[a,b]}_{w_1,..,w_n}(x) - \frac{1}{\int_a^b w_1(t)\,dt} \int_a^b w_1(t) f(t)\,dt + T^{[c,d]}_{u_1,..,u_n}(x) \right| \\
&= \left| \int_{\min\{a,c\}}^{\max\{b,d\}} K(x,t_1,\ldots,t_n) f^{(n)}(t_n)\,dt_n \right| \leq \|K(x,t_1,\ldots,t_{n-1},\cdot)\|_{q,[\min\{a,c\},\max\{b,d\}]} \left\| f^{(n)} \right\|_{p,[\min\{a,c\},\max\{b,d\}]}
\end{aligned}$$

Let us denote $C(t) = K(x,t_1,\ldots,t_{n-1},t)$. For the proof of the sharpness we find a function f for which the equality in (4.32) is obtained.

For $1 < p < \infty$ take f to be such that

$$f^{(n)}(t) = sgn\, C(t) \cdot |C(t)|^{\frac{1}{p-1}}.$$

For $p = \infty$ take

$$f^{(n)}(t) = sgn\, C(t).$$

For $p = 1$ we shall prove that

$$\left| \int_{\min\{a,c\}}^{\max\{b,d\}} C(t) f^{(n)}(t)\,dt \right| \leq \max_{t\in[\min\{a,c\},\max\{b,d\}]} |C(t)| \left(\int_{\min\{a,c\}}^{\max\{b,d\}} \left| f^{(n)}(t) \right| dt \right) \tag{4.33}$$

is the best possible inequality.

If $n \geq 2$ the function $C(t)$ is continuous except in points $\max\{a,c\}$ and $\min\{b,d\}$ where it has a finite jump. If $n = 1$ it is continuous. Thus we have four possibilities:

1. $|C(t)|$ attains its maximum at $t_0 \in [\min\{a,c\}, \max\{b,d\}]$ and $C(t_0) > 0$. Then for $\varepsilon > 0$ small enough define $f_\varepsilon(t)$ by

$$f_\varepsilon(t) = \begin{cases} 0, & \min\{a,c\} \leq t \leq t_0 - \varepsilon \\ \frac{1}{\varepsilon n!}(t - t_0 + \varepsilon)^n, & t_0 - \varepsilon \leq t \leq t_0 \\ \frac{1}{n!}(t - t_0 + \varepsilon)^{n-1}, & t_0 \leq t \leq \max\{b,d\}. \end{cases}$$

Thus

$$\left| \int_{\min\{a,c\}}^{\max\{b,d\}} C(t) f_\varepsilon^{(n)}(t) dt \right| = \left| \int_{t_0-\varepsilon}^{t_0} C(t) \frac{1}{\varepsilon} dt \right| = \frac{1}{\varepsilon} \int_{t_0-\varepsilon}^{t_0} C(t) dt.$$

Now, from inequality (4.33) we have

$$\frac{1}{\varepsilon} \int_{t_0-\varepsilon}^{t_0} C(t) dt \leq \frac{1}{\varepsilon} C(t_0) \int_{t_0-\varepsilon}^{t_0} dt = C(t_0).$$

Since

$$\lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \frac{1}{\varepsilon} \int_{t_0-\varepsilon}^{t_0} C(t) dt = C(t_0)$$

the statement follows.

2. $|C(t)|$ attains its maximum at $t_0 \in [\min\{a,c\}, \max\{b,d\}]$ and $C(t_0) < 0$. Then for $\varepsilon > 0$ small enough define $f_\varepsilon(t)$ by

$$f_\varepsilon(t) = \begin{cases} \frac{1}{n!}(t_0 - t)^{n-1}, & \min\{a,c\} \leq t \leq t_0 - \varepsilon \\ -\frac{1}{\varepsilon n!}(t_0 - t)^n, & t_0 - \varepsilon \leq t \leq t_0 \\ 0, & t_0 \leq t \leq \max\{b,d\}, \end{cases}$$

and the rest of the proof is similar as above.

3. $|C(t)|$ does not attain a maximum on $[\min\{a,c\}, \max\{b,d\}]$ and let $t_0 \in [\min\{a,c\}, \max\{b,d\}]$ be such that

$$\sup_{t \in [\min\{a,c\}, \max\{b,d\}]} |C(t)| = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} |f(t_0 + \varepsilon)|$$

If $\lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} f(t_0 + \varepsilon) > 0$, we take

$$f_\varepsilon(t) = \begin{cases} 0, & \min\{a,c\} \leq t \leq t_0 \\ \frac{1}{\varepsilon n!}(t - t_0)^n, & t_0 \leq t \leq t_0 + \varepsilon \\ \frac{1}{n!}(t - t_0)^{n-1}, & t_0 + \varepsilon \leq t \leq \max\{b,d\}, \end{cases}$$

and, similarly as before, we have

$$\left|\int_{\min\{a,c\}}^{\max\{b,d\}} C(t) f_\varepsilon^{(n)}(t)dt\right| = \left|\int_{t_0}^{t_0+\varepsilon} C(t)\frac{1}{\varepsilon}dt\right| = \frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} C(t)dt,$$

$$\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} C(t)dt \le \frac{1}{\varepsilon}C(t_0)\int_{t_0}^{t_0+\varepsilon} dt = C(t_0),$$

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} C(t)dt = C(t_0)$$

and the statement follows.

4. $|C(t)|$ does not attain a maximum on $[\min\{a,c\},\max\{b,d\}]$ and let $t_0 \in [\min\{a,c\},\max\{b,d\}]$ be such that

$$\sup_{t\in[\min\{a,c\},\max\{b,d\}]} |C(t)| = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} |f(t_0+\varepsilon)|.$$

If $\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} f(t_0+\varepsilon) < 0$, we take

$$f_\varepsilon(t) = \begin{cases} \frac{1}{n!}(t-t_0-\varepsilon)^{n-1}, & \min\{a,c\} \le t \le t_0 \\ -\frac{1}{\varepsilon n!}(t-t_0-\varepsilon)^n, & t_0 \le t \le t_0+\varepsilon \\ 0, & t_0+\varepsilon \le t \le \max\{b,d\}, \end{cases}$$

and the rest of the proof is similar as above. □

Corollary 4.9 *Let $f : [a,b]\cup[a,a+\lambda] \to \mathbb{R}$ be such that $f' \in L^p_{[a,b]\cup[a,a+\lambda]}$ and $g : [a,b] \to \mathbb{R}$ be an integrable function such that $\lambda = \int_a^b g(t)\,dt$. Let also $G(x) = \int_a^x g(t)\,dt$, $x \in [a,b]$. Then the following two sharp inequalities hold for $1 < p \le \infty$ and for $0 \le \lambda \le b-a$*

$$\left|\int_a^b f(t)g(t)\,dt - \int_a^{a+\lambda} f(t)\,dt\right|$$
$$\le \left(\int_a^{a+\lambda} |t-a-G(t)|^q\,dt + \int_{a+\lambda}^b |\lambda - G(t)|^q\,dt\right)^{\frac{1}{q}} \left\|f'\right\|_{p,[a,\max\{b,a+\lambda\}]}$$

while for $\lambda > b-a$

$$\left|\int_a^b f(t)g(t)\,dt - \int_a^{a+\lambda} f(t)\,dt\right|$$
$$\le \left(\int_a^b |t-a-G(t)|^q\,dt + \int_b^{a+\lambda} |t-a-\lambda|^q\,dt\right)^{\frac{1}{q}} \left\|f'\right\|_{p,[a,\max\{b,a+\lambda\}]}.$$

In case $p = 1$ and $0 \le \lambda \le b-a$ we have two following two best possible inequalities

$$\left|\int_a^b f(t)g(t)\,dt - \int_a^{a+\lambda} f(t)\,dt\right|$$
$$\le \max\left\{\max_{t\in[a,a+\lambda]} |t-a-G(t)|, \max_{t\in[a+\lambda,b]} |\lambda - G(t)|\right\} \left\|f'\right\|_{1,[a,\max\{b,a+\lambda\}]}$$

while for $\lambda > b-a$

$$\left| \int_a^b f(t)\, g(t)\, dt - \int_a^{a+\lambda} f(t)\, dt \right|$$
$$\leq \max\left\{ \max_{t\in[a,b]} |t-a-G(t)|,\ \max_{t\in[b,a+\lambda]} |t-a-\lambda| \right\} \left\| f' \right\|_{1,[a,\max\{b,a+\lambda\}]}.$$

Proof. Apply Theorem 4.11 with $n=1$ and weight functions $w_1(t) = g(t)$ for $t\in[a,b]$ and $u_1(t) = 1$ for $t\in[a,a+\lambda]$. We have $\int_a^b g(t)\, dt = \int_a^{a+\lambda} dt = \lambda$ and, consequently,

$$\left| \int_a^b f(t)\, g(t)\, dt - \int_a^{a+\lambda} f(t)\, dt \right| = \left| \lambda \int_a^{\max\{b,a+\lambda\}} K(t)\, f'(t)\, dt \right|$$

where

$$\lambda K(t) = \begin{cases} t-a-\int_a^t g(s)\, ds, & t\in[a,a+\lambda] \\ \int_t^b g(s)\, ds, & t\in(a+\lambda,b] \end{cases} \quad \text{if } a+\lambda \leq b,$$

$$\lambda K(t) = \begin{cases} t-a-\int_a^t g(s)\, ds, & t\in[a,b] \\ t-a-\lambda, & t\in(b,a+\lambda] \end{cases} \quad \text{if } a+\lambda \geq b,$$

and the proof follows. □

Corollary 4.10 *Let* $f:[a,b]\cup[b-\lambda,b]\to\mathbb{R}$ *be such that* $f'\in L^p_{[a,b]\cup[b-\lambda,b]}$ *and* $g:[a,b]\to\mathbb{R}$ *integrable function such that* $\lambda = \int_a^b g(t)\, dt$. *Let also* $G(x) = \int_a^x g(t)\, dt$, $x\in[a,b]$. *Then the following two sharp inequalities hold for* $1<p\leq\infty$ *and for* $0\leq\lambda\leq b-a$

$$\left| \int_a^b f(t)\, g(t)\, dt - \int_{b-\lambda}^b f(t)\, dt \right|$$
$$\leq \left(\int_a^{b-\lambda} |-G(t)|^q\, dt + \int_{b-\lambda}^b |t-b+\lambda-G(t)|^q\, dt \right)^{\frac{1}{q}} \left\| f' \right\|_{p,[a,\max\{b,a+\lambda\}]}$$

while for $\lambda > b-a$

$$\left| \int_a^b f(t)\, g(t)\, dt - \int_{b-\lambda}^b f(t)\, dt \right|$$
$$\leq \left(\int_{b-\lambda}^a |t-b+\lambda|^q\, dt + \int_a^b |t-b+\lambda-G(t)|^q\, dt \right)^{\frac{1}{q}} \left\| f' \right\|_{p,[a,\max\{b,a+\lambda\}]}.$$

In case $p=1$ *and* $0\leq\lambda\leq b-a$ *we have two following two best possible inequalities*

$$\left| \int_a^b f(t)\, g(t)\, dt - \int_{b-\lambda}^b f(t)\, dt \right|$$
$$\leq \max\left\{ \max_{t\in[a,b-\lambda]} |-G(t)|,\ \max_{t\in[b-\lambda,b]} |t-b+\lambda-G(t)| \right\} \left\| f' \right\|_{1,[a,\max\{b,a+\lambda\}]}$$

while for $\lambda > b-a$

$$\begin{aligned}&\left|\int_a^b f(t)\,g(t)\,dt - \int_{b-\lambda}^b f(t)\,dt\right| \\ &\le \max\left\{\max_{t\in[b-\lambda,a]}|t-b+\lambda|\,,\ \max_{t\in[a,b]}|t-b+\lambda-G(t)|\right\}\left\|f'\right\|_{1,[a,\max\{b,a+\lambda\}]}.\end{aligned}$$

Proof. Apply Theorem 4.11 with $n=1$ and weight functions $w_1(t)=g(t)$ for $t\in[a,b]$ and $u_1(t)=1$ for $t\in[b-\lambda,b]$. We have $\int_a^b g(t)\,dt=\int_{b-\lambda}^b dt=\lambda$ and consequently

$$\left|\int_a^b f(t)\,g(t)\,dt - \int_{b-\lambda}^b f(t)\,dt\right| = \left|\lambda\int_{\min\{a,b-\lambda\}}^b K(t)\,f'(t)\,dt\right|$$

where

$$\lambda K(t) = \begin{cases} -G(t), & t\in[a,b-\lambda] \\ t-b+\lambda-G(t), & t\in(b-\lambda,b] \end{cases} \quad \text{if } a+\lambda\le b,$$

$$\lambda K(t) = \begin{cases} t-b+\lambda, & t\in[b-\lambda,a] \\ t-b+\lambda-G(t), & t\in(a,b] \end{cases} \quad \text{if } a+\lambda\ge b,$$

and the proof follows. □

4.3 Generalizations via Fink identity and related results

In [43] Fink obtained the following identity:

$$\begin{aligned}\frac{1}{n}\left(f(x)+\sum_{k=1}^{n-1}F_k(x)\right) &- \frac{1}{b-a}\int_a^b f(t)dt \\ &= \frac{1}{n!(b-a)}\int_a^b (x-t)^{n-1}k(t,x)f^{(n)}(t)dt,\end{aligned} \tag{4.34}$$

where

$$F_k(x)=\frac{n-k}{k!}\cdot\frac{f^{(k-1)}(a)(x-a)^k-f^{(k-1)}(b)(x-b)^k}{b-a},$$

$$k(t,x)=\begin{cases} t-a, & a\le t\le x\le b\\ t-b, & a\le x<t\le b.\end{cases}$$

In [12] Aglić Aljinović, Pečarić and Vukelić gave the extension of weighted Montgomery identity (4.1) using identity (4.34). Further they obtained some new generalizations of the estimations of the difference of two weighted integral means. First is when $[c,d]\subseteq[a,b]$ and the second when $[a,b]\cap[c,d]=[c,b]$. Other two possible cases, when $[a,b]\cap[c,d]\neq\emptyset$ we simply get by substitutions $a\leftrightarrow c$, $b\leftrightarrow d$.

Theorem 4.12 *Let $f:[a,b]\cup[c,d]\to\mathbb{R}$ be such that $f^{(n-1)}$ is an absolutely continuous function on $[a,b]$ for some $n>1$, and let $w:[a,b]\to[0,\infty)$ and $u:[c,d]\to[0,\infty)$. Then if $[a,b]\cap[c,d]\neq\emptyset$ and $x\in[a,b]\cap[c,d]$, we have*

$$\begin{aligned}\frac{1}{\int_a^b w(t)dt}\int_a^b w(t)f(t)\,dt &- \frac{1}{\int_c^d u(t)dt}\int_c^d u(t)f(t)\,dt - T_{w,n}^{[a,b]}(x)\\ &+T_{u,n}^{[c,d]}(x)=\int_{\min\{a,c\}}^{\max\{b,d\}}K_n(x,y)\,f^{(n)}(y)\,dy,\end{aligned}$$

where

$$T_{w,n}^{[a,b]}(x)=\sum_{k=1}^{n-1}F_k^{[a,b]}(x)-\frac{1}{\int_a^b w(t)dt}\sum_{k=1}^{n-1}\int_a^b w(t)\,F_k^{[a,b]}(t)dt,$$

and, in case $[c,d]\subseteq[a,b]$,

$$K_n(x,y)=\begin{cases}\frac{-1}{(n-2)!(b-a)}\left[\int_a^b P_w(x,t)\,(t-y)^{n-2}k^{[a,b]}(y,t)dt\right], & y\in[a,c]\\[2ex] \frac{-1}{(n-2)!(b-a)}\left[\int_a^b P_w(x,t)\,(t-y)^{n-2}k^{[a,b]}(y,t)dt\right] \\ +\frac{1}{(n-2)!(d-c)}\left[\int_c^d P_u(x,t)\,(t-y)^{n-2}k^{[c,d]}(y,t)dt\right], & y\in(c,d]\\[2ex] \frac{-1}{(n-2)!(b-a)}\left[\int_a^b P_w(x,t)\,(t-y)^{n-2}k^{[a,b]}(y,t)dt\right], & y\in(d,b],\end{cases}$$

and in case $[a,b]\cap[c,d]=[c,b]$,

$$K_n(x,y)=\begin{cases}\frac{-1}{(n-2)!(b-a)}\left[\int_a^b P_w(x,t)(t-y)^{n-2}k^{[a,b]}(y,t)dt\right], & y\in[a,c]\\[2ex] \frac{-1}{(n-2)!(b-a)}\left[\int_a^b P_w(x,t)(t-y)^{n-2}k^{[a,b]}(y,t)dt\right] \\ +\frac{1}{(n-2)!(d-c)}\left[\int_c^d P_u(x,t)(t-y)^{n-2}k^{[c,d]}(y,t)dt\right], & y\in(c,b]\\[2ex] \frac{1}{(n-2)!(d-c)}\left[\int_c^d P_u(x,t)(t-y)^{n-2}k^{[c,d]}(y,t)dt\right], & y\in(b,d].\end{cases}$$

The following Ostrowski type inequality was obtained in [12].

Theorem 4.13 *Assume (p,q) is a pair of conjugate exponents. Let $\left|f^{(n)}\right|^p:[a,b]\to\mathbb{R}$ be an integrable function for some $n>1$. Then for $x\in[a,b]\cap[c,d]$ we have*

$$\begin{aligned}&\left|\frac{1}{\int_a^b w(t)dt}\int_a^b w(t)f(t)dt-\frac{1}{\int_c^d u(t)dt}\int_c^d u(t)f(t)dt-T_{w,n}^{[a,b]}(x)\right.\\&\left.+T_{u,n}^{[c,d]}(x)\right|\leq\left(\int_{\min\{a,c\}}^{\max\{b,d\}}|K_n(x,y)|^q dy\right)^{\frac{1}{q}}\left\|f^{(n)}\right\|_p.\end{aligned}\tag{4.35}$$

The constant $\left(\int_{\min\{a,c\}}^{\max\{b,d\}}|K_n(x,y)|^q dy\right)^{1/q}$ in inequality (4.35) is sharp for $1<p\leq\infty$ and the best possible for $p=1$.

In [121] Pečarić, Perušić and Vukelić obtained the following results. Directly from Theorem 4.12 we get the following:

Theorem 4.14 *Let $[a,b]\cap[c,d]\neq\emptyset$. Let $f:[a,b]\cup[c,d]\to\mathbb{R}$ be n-covex on $[a,b]$ for some $n>1$ and let $w:[a,b]\to[0,\infty)$ and $u:[c,d]\to[0,\infty)$. Then for $x\in[a,b]\cap[c,d]$ and*

$$K_n(x,y)\geq 0,\tag{4.36}$$

we have

$$\frac{1}{\int_a^b w(t)dt}\int_a^b w(t)f(t)dt-T_{w,n}^{[a,b]}(x)\geq\frac{1}{\int_c^d u(t)dt}\int_c^d u(t)f(t)dt-T_{u,n}^{[c,d]}(x).\tag{4.37}$$

If the reversed inequality in (4.36) is valid, then the reversed inequality in (4.37) is also valid.

For $u(t)=1$ and $\lambda=\int_a^b w(t)dt=d-c$ in inequality (4.37) we get an inequality related to the left-hand side of inequality (3.47).

For $a\leftrightarrow c,\ b\leftrightarrow d, w\leftrightarrow u,\ u(t)=1$ and $\lambda=\int_a^b w(t)dt=d-c$ in inequality (4.37) we get an inequality related to the right-hand side of inequality (3.47).

Corollary 4.11 *Let $\lambda > 0$ and let $f : [a,b] \cup [a,a+\lambda] \to \mathbb{R}$ be n-covex on $[a,b] \cup [a,a+\lambda]$ for some $n > 1$ and $w : [a,b] \to [0,\infty)$. Then for $x \in [a,b] \cap [a,a+\lambda]$ and*

$$K_n(x,y) \geq 0, \tag{4.38}$$

we have

$$\frac{1}{\int_a^b w(t)dt} \int_a^b w(t) f(t) dt - T_{w,n}^{[a,b]}(x) \geq \frac{1}{\lambda} \int_a^{a+\lambda} f(t) dt - T_{1,n}^{[a,a+\lambda]}(x), \tag{4.39}$$

where, in case $a+\lambda \leq b$,

$$K_n(x,y) = \begin{cases} \frac{-1}{(n-2)!(b-a)} \left[\int_a^b P_w(x,t)(t-y)^{n-2} k^{[a,b]}(y,t) dt\right] \\ + \frac{1}{\lambda(n-2)!} \left[\int_a^{a+\lambda} P_1(x,t)(t-y)^{n-2} k^{[a,a+\lambda]}(y,t) dt\right], & y \in [a,a+\lambda] \\ \\ \frac{-1}{(n-2)!(b-a)} \left[\int_a^b P_w(x,t)(t-y)^{n-2} k^{[a,b]}(y,t) dt\right], & y \in (a+\lambda,b], \end{cases}$$

and in case $a+\lambda \geq b$,

$$K_n(x,y) = \begin{cases} \frac{-1}{(n-2)!(b-a)} \left[\int_a^b P_w(x,t)(t-y)^{n-2} k^{[a,b]}(y,t) dt\right] \\ + \frac{1}{\lambda(n-2)!} \left[\int_a^{a+\lambda} P_1(x,t)(t-y)^{n-2} k^{[a,a+\lambda]}(y,t) dt\right], & y \in [a,b] \\ \\ \frac{1}{\lambda(n-2)!} \left[\int_a^{a+\lambda} P_1(x,t)(t-y)^{n-2} k^{[a,a+\lambda]}(y,t) dt\right], & y \in (b,a+\lambda]. \end{cases}$$

If the reversed inequality in (4.38) is valid, then the reversed inequality in (4.39) is also valid.

Proof. We put $c = a,\ d = a+\lambda$ and $u(t) = 1$ in inequality (4.37) to get inequality (4.39). □

Corollary 4.12 *Assume (p,q) is a pair of conjugate exponents. Let $\left|f^{(n)}\right|^p : [a,b] \to \mathbb{R}$ be an R-integrable function for some $n > 1$. Then for any $x \in [a,b] \cap [a,a+\lambda]$ we have*

$$\begin{aligned} &\left|\frac{1}{\int_a^b w(t)dt} \int_a^b w(t) f(t) dt - \frac{1}{\lambda} \int_a^{a+\lambda} f(t) dt - T_{w,n}^{[a,b]}(x) + T_{1,n}^{[a,a+\lambda]}(x)\right| \\ &\leq \left(\int_a^{\max\{b,a+\lambda\}} |K_n(x,y)|^q dy\right)^{\frac{1}{q}} \left\|f^{(n)}\right\|_p . \end{aligned} \tag{4.40}$$

The constant $\left(\int_a^{\max\{b,a+\lambda\}} |K_n(x,y)|^q dy\right)^{1/q}$ in inequality (4.40) is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof. We put $c = a,\ d = a+\lambda$ and $u(t) = 1$ in inequality (4.35) to get inequality (4.40). □

Remark 4.4 For $n=1$ and $\lambda \le b-a$, $K_1(x,y)$ becomes:

$$K_1(x,y) = \begin{cases} \frac{y-a}{\lambda} - \frac{1}{\int_a^b w(t)dt} \int_a^y w(t)dt, & y \in [a, a+\lambda] \\ \frac{1}{\int_a^b w(t)dt} \int_y^b w(t)dt, & y \in (a+\lambda, b]. \end{cases}$$

So, if $\lambda \int_a^y w(t)dt \le (y-a)\int_a^b w(t)dt$ and $f'(x) \ge 0$, inequality (4.39) becomes

$$\frac{1}{\int_a^b w(t)dt} \int_a^b w(t) f(t)\, dt \ge \frac{1}{\lambda} \int_a^{a+\lambda} f(t)\, dt,$$

which is the right-hand side of the reversed generalized Steffensen's inequality given by (3.6).

Remark 4.5 For $n=1$ and $\int_a^b w(t)dt = \lambda$, $K_1(x,y)$ becomes:

$$K_1(x,y) = \begin{cases} \frac{1}{\lambda} \int_a^y (1-w(t))dt, & y \in [a, a+\lambda] \\ \frac{1}{\lambda} \int_y^b w(t)dt, & y \in (a+\lambda, b]. \end{cases}$$

So, if $w(t) \le 1$ and $f'(x) \ge 0$, inequality (4.39) becomes

$$\int_a^b w(t) f(t)\, dt \ge \int_a^{a+\lambda} f(t)\, dt,$$

which is the right-hand side of the reversed Steffensen's inequality.

Corollary 4.13 *Let $f : [a,b] \cup [b-\lambda, b] \to \mathbb{R}$ be n-convex on $[a,b]$ for some $n > 1$ and $w : [a,b] \to [0,\infty)$. Then if $\lambda > 0$, $x \in [a,b] \cap [b-\lambda, b]$ and*

$$K_n(x,y) \ge 0, \tag{4.41}$$

we have

$$\frac{1}{\lambda} \int_{b-\lambda}^b f(t)\, dt - T_{1,n}^{[b-\lambda,b]}(x) \ge \frac{1}{\int_a^b w(t)dt} \int_a^b w(t) f(t)\, dt - T_{w,n}^{[a,b]}(x), \tag{4.42}$$

where, in case $b-\lambda \le a$,

$$K_n(x,y) = \begin{cases} \frac{-1}{\lambda(n-2)!} \left[\int_{b-\lambda}^b P_1(x,t)(t-y)^{n-2} k^{[b-\lambda,b]}(y,t)dt \right], & y \in [b-\lambda, a] \\ \frac{-1}{\lambda(n-2)!} \left[\int_{b-\lambda}^b P_1(x,t)(t-y)^{n-2} k^{[b-\lambda,b]}(y,t)dt \right] & \\ + \frac{1}{(n-2)!(b-a)} \left[\int_a^b P_w(x,t)(t-y)^{n-2} k^{[a,b]}(y,t)dt \right], & y \in (a,b], \end{cases}$$

and in case $a \le b-\lambda$,

$$K_n(x,y) = \begin{cases} \frac{1}{(n-2)!(b-a)} \left[\int_a^b P_w(x,t)(t-y)^{n-2} k^{[a,b]}(y,t)dt \right] & y \in [a, b-\lambda] \\ \frac{-1}{\lambda(n-2)!} \left[\int_{b-\lambda}^b P_1(x,t)(t-y)^{n-2} k^{[b-\lambda,b]}(y,t)dt \right] & \\ + \frac{1}{(n-2)!(b-a)} \left[\int_a^b P_w(x,t)(t-y)^{n-2} k^{[a,b]}(y,t)dt \right], & y \in (b-\lambda, b]. \end{cases}$$

If the reversed inequality in (4.41) is valid, then the reversed inequality in (4.42) is also valid.

Proof. We substitute $a \leftrightarrow c,\ b \leftrightarrow d, w \leftrightarrow u$, and put $c = b-\lambda,\ d = b,\ u(t) = 1$ in inequality (4.37) to get inequality (4.42). □

Corollary 4.14 *Assume (p,q) is a pair of conjugate exponents. Let $\left|f^{(n)}\right|^p : [a,b] \to \mathbb{R}$ be an R-integrable function for some $n > 1$. Then we have*

$$\left|\frac{1}{\lambda}\int_{b-\lambda}^{b} f(t)\,dt - \frac{1}{\int_a^b w(t)dt}\int_a^b w(t)f(t)\,dt - T_{1,n}^{[b-\lambda,b]}(x) + T_{w,n}^{[a,b]}(x)\right|$$
$$\leq \left(\int_{\min\{a,b-\lambda\}}^{b} |K_n(x,y)|^q\,dy\right)^{\frac{1}{q}} \left\|f^{(n)}\right\|_p \tag{4.43}$$

for every $x \in [a,b] \cap [b-\lambda,b]$. The constant $\left(\int_{\min\{a,b-\lambda\}}^{b} |K_n(x,y)|^q\,dy\right)^{1/q}$ in inequality (4.43) is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof. We substitute $a \leftrightarrow c,\ b \leftrightarrow d, w \leftrightarrow u$, and put $c = b-\lambda,\ d = b,\ u(t) = 1$ in inequality (4.35) to get inequality (4.43). □

Remark 4.6 For $n = 1$ and $\lambda \leq b-a$, $K_1(x,y)$ becomes:

$$K_1(x,y) = \begin{cases} \frac{1}{\int_a^b w(t)dt}\int_a^y w(t)dt, & y \in [a,b-\lambda] \\ \frac{b-y}{\lambda} - \frac{1}{\int_a^b w(t)dt}\int_y^b w(t)dt, & y \in (b-\lambda,b]. \end{cases}$$

So, if $\lambda\int_y^b w(t)dt \leq (b-y)\int_a^b w(t)dt$ and $f'(x) \geq 0$, inequality (4.42) becomes

$$\frac{1}{\lambda}\int_{b-\lambda}^{b} f(t)\,dt \geq \frac{1}{\int_a^b w(t)dt}\int_a^b w(t)\,f(t)\,dt,$$

which is the left-hand side of the reversed generalized Steffensen's inequality given by (3.6).

Remark 4.7 For $n = 1$ and $\int_a^b w(t)dt = \lambda$, $K_1(x,y)$ becomes:

$$K_1(x,y) = \begin{cases} \frac{1}{\lambda}\int_a^y w(t)dt, & y \in [a,b-\lambda] \\ \frac{1}{\lambda}\int_y^b (1-w(t))dt, & y \in (b-\lambda,b]. \end{cases}$$

So, if $w(t) \leq 1$ and $f'(x) \geq 0$, inequality (4.42) becomes

$$\int_{b-\lambda}^{b} f(t)\,dt \geq \int_a^b w(t)\,f(t)\,dt,$$

which is the left-hand side of the reversed Steffensen's inequality.

Chapter 5

Generalizations of Steffensen's inequality via Taylor's formula

5.1 Generalizations via Taylor's formula

Results given in this section were obtained by Jakšetić, Pečarić and Perušić in [69]. The following simple lemma will be useful for results that follow.

Lemma 5.1 *Let $f : [a,b] \cup [c,d] \to \mathbb{R}$, where $c \in [a,b]$, be a $n-$times differentiable function and let $u : [a,b] \to \mathbb{R}$ and $w : [c,d] \to \mathbb{R}$ be integrable functions. Then*

$$\int_a^b u(x)f(x)dx - \int_c^d w(x)f(x)dx - T_a^{f,u} + T_a^{f,w} = \frac{1}{(n-1)!}\int_a^{\max\{b,d\}} f^{(n)}(t)g(t)dt \quad (5.1)$$

where

$$T_a^{f,u} = \sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{i!} \int_a^b u(x)(x-a)^i dx$$

and in case $a \le c < b < d$,

$$g(t) = \begin{cases} \int_t^c u(x)(x-t)^{n-1}dx + \int_c^b (u(x)-w(x))(x-t)^{n-1}dx \\ -\int_b^d w(x)(x-t)^{n-1}dx, & t \in [a,c] \\ \int_t^b (u(x)-w(x))(x-t)^{n-1}dx - \int_b^d w(x)(x-t)^{n-1}dx, & t \in [c,b] \\ -\int_t^d w(x)(x-t)^{n-1}dx, & t \in [b,d] \end{cases} \tag{5.2}$$

and in case $a \le c < d \le b$,

$$g(t) = \begin{cases} \int_t^c u(x)(x-t)^{n-1}dx + \int_c^d (u(x)-w(x))(x-t)^{n-1}dx \\ +\int_d^b u(x)(x-t)^{n-1}dx, & t \in [a,c] \\ \int_t^d (u(x)-w(x))(x-t)^{n-1}dx + \int_d^b u(x)(x-t)^{n-1}dx, & t \in [c,d] \\ \int_t^b u(x)(x-t)^{n-1}dx, & t \in [d,b]. \end{cases} \tag{5.3}$$

Proof. This follows from Taylor's formula

$$f(x) - \sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{i!}(x-a)^i = \int_a^x f^{(n)}(t)\frac{(x-t)^{n-1}}{(n-1)!}dt$$

and Fubini's theorem. □

Theorem 5.1 *Let* $f : [a,b] \cup [c,d] \to \mathbb{R}$, *where* $c \in [a,b]$ *be a* $n-$*convex function and let* $u : [a,b] \to \mathbb{R}$ *and* $w : [c,d] \to \mathbb{R}$ *be integrable functions. If the function* $g : [a,b] \cup [c,d] \to \mathbb{R}$, *defined with (5.2) and (5.3), is nonnegative on* $[a,b] \cup [c,d]$. *Then*

$$\int_a^b u(x)f(x)dx - \int_c^d w(x)f(x)dx \ge T_a^{f,u} - T_a^{f,w}. \tag{5.4}$$

If g is nonpositive on $[a,b] \cup [c,d]$, *then inequality (5.4) is reversed.*

Proof. Without loss of generality we can assume that f is $n-$times differentiable and $f^{(n)} \ge 0$. The result now follows from Lemma 5.1. □

Using previous results for some special weight functions and integral limits we obtain generalization of Steffensen's inequality.

Theorem 5.2 *Suppose that* $f : [a,b] \to \mathbb{R}$ *is* $n-$*convex function and* $u : [a,b] \to \mathbb{R}$ *is integrable on* $[a,b]$ *such that* $0 \le u \le 1$, *on* $[a,b]$.

(i) *If*

$$\lambda_1 = \left(n\int_a^b u(x)(x-a)^{n-1}dx\right)^{1/n}, \tag{5.5}$$

then we have

$$\int_a^b f(x)u(x)dx - \int_a^{a+\lambda_1} f(x)dx \ge \sum_{i=0}^{n-2} \frac{f^{(i)}(a)}{i!}\left(\int_a^b u(x)(x-a)^i dx - \frac{\lambda_1^{i+1}}{i+1}\right). \tag{5.6}$$

(ii) *If*

$$\lambda_2 = b-a-\left((b-a)^n - n\int_a^b u(x)(x-a)^{n-1}dx\right)^{1/n}, \tag{5.7}$$

then we have

$$\begin{aligned}\int_a^b f(x)u(x)dx - \int_{b-\lambda_2}^b f(x)dx \le \sum_{i=0}^{n-2}\frac{f^{(i)}(a)}{i!}\times\\ \times\left(\int_a^b u(x)(x-a)^i dx - \tfrac{(b-a)^{i+1}-(b-a-\lambda_2)^{i+1}}{i+1}\right).\end{aligned} \tag{5.8}$$

Proof.

(i) We apply Theorem 5.1 for $c=a,\ d=a+\lambda_1$, and functions u and w such that $0\le u\le 1,\ w\equiv 1$. Let us show that $g(t)\ge 0$ on $[a,b]$. From (5.3) we have:

$$g(t)=\begin{cases}\int_t^b u(x)(x-t)^{n-1}dx - \frac{(a+\lambda_1-t)^n}{n}, & t\in[a,a+\lambda_1]\\ \int_t^b u(x)(x-t)^{n-1}dx, & t\in[a+\lambda_1,b].\end{cases} \tag{5.9}$$

Since $g(t)\ge 0$ for $t\in[a+\lambda_1,b]$, we only have to prove

$$\int_t^b u(x)(x-t)^{n-1}dx \ge \frac{(a+\lambda_1-t)^n}{n},\quad t\in[a,a+\lambda_1].$$

For that purpose we modify Fink's proof of Theorem 3.58 as follows:

$$\begin{aligned}&\int_t^b u(x)(x-t)^{n-1}dx = \int_t^b (x-a)^{n-1}\left(\frac{x-t}{x-a}\right)^{n-1}u(x)dx\\ &=(n-1)(t-a)\int_t^b\frac{(x-t)^{n-2}}{(x-a)^n}\left(\int_x^b(s-a)^{n-1}u(s)ds\right)dx\\ &\ge (n-1)(t-a)\int_t^{a+\lambda_1}\frac{(x-t)^{n-2}}{(x-a)^n}\left(\int_x^b(s-a)^{n-1}u(s)ds\right)dx\\ &=(n-1)(t-a)\int_t^{a+\lambda_1}\frac{(x-t)^{n-2}}{(x-a)^n}\left(\frac{\lambda_1^n}{n}-\int_a^x(s-a)^{n-1}u(s)ds\right)dx\\ &\ge (n-1)(t-a)\int_t^{a+\lambda_1}\frac{(x-t)^{n-2}}{(x-a)^n}\left(\frac{\lambda_1^n}{n}-\int_a^x(s-a)^{n-1}ds\right)dx\\ &=\frac{(a+\lambda_1-t)^n}{n}.\end{aligned} \tag{5.10}$$

According to Theorem 5.1

$$\int_a^b f(x)u(x)dx - \int_a^{a+\lambda_1} f(x)dx \ge T_a^{f,u}-T_a^{f,w}$$

and since

$$T_a^{f,u}-T_a^{f,w} = \sum_{i=0}^{n-1}\frac{f^{(i)}(a)}{i!}\left(\int_a^b u(x)(x-a)^i dx - \frac{\lambda_1^{i+1}}{i+1}\right)$$

we have to show that the last summand $\int_a^b u(x)(x-a)^{n-1}dx - \frac{\lambda_1^n}{n}$ is equal to zero. But this follows from the definition of the number λ_1.

(ii) Again, we apply Theorem 5.1 for $c = b-\lambda_2,\ d = b$, and functions u and v such that $0 \le u \le 1,\ w \equiv 1$. We have to show that $g(t) \le 0$ on $[a,b]$. From (5.3) we have:

$$g(t) = \begin{cases} \int_t^b (u(x)-1)(x-t)^{n-1}dx + \frac{(b-\lambda_2-t)^n}{n}, & t \in [a, b-\lambda_2] \\ \int_t^b (u(x)-1)(x-t)^{n-1}dx, & t \in [b-\lambda_2, b] \end{cases} \tag{5.11}$$

Since $g(t) \le 0$ for $t \in [b-\lambda_2, b]$, we only have to prove

$$\int_t^b (u(x)-1)(x-t)^{n-1}dx + \frac{(b-\lambda_2-t)^n}{n} \le 0, \quad t \in [a, b-\lambda_2].$$

But here we again use Fink's argument from (i)-part (with $u(x) \leftrightarrow 1-u(x)$ and $a+\lambda_1 \leftrightarrow b-\lambda_2$). Now (5.8) follows from Theorem 5.1. □

As a special case, from Theorem 5.2 (i) we get Milovanović-Pečarić result given in Theorem 3.17. As a special case of (ii)-part we obtain the following:

Corollary 5.1 *Suppose that $f : [a,b] \to \mathbb{R}$ is n-convex function, $f^{(i)}(a) = 0, i = 0,1,\ldots,n-2$, and $u : [a,b] \to \mathbb{R}$ is integrable on $[a,b]$ such that $0 \le u \le 1$, on $[a,b]$. Let*

$$\lambda_2 = b - a - \left((b-a)^n - n\int_a^b u(x)(x-a)^{n-1}dx\right)^{1/n}.$$

Then

$$\int_{b-\lambda_2}^b f(x)dx \ge \int_a^b f(x)u(x)dx.$$

We now give necessary and sufficient conditions for inequalities (5.6) and (5.8).

Theorem 5.3 *Assume that f is $n-$convex, $0 \le u \le 1$ is integrable on $[a,b]$ and λ_1 and λ_2 are defined by (5.5) and (5.7) respectively.*

(i) Inequality (5.6) holds if and only if the function g defined by (5.9) is nonnegative on $[a,b]$.

(ii) Inequality (5.8) holds if and only if the function g defined by (5.11) is negative on $[a,b]$.

Proof.

(i) Sufficiency is obvious. For necessity we consider functions $f_t(x) = (x-t)_+^{n-1}$, where $t \in [a,b]$. Since f_t is $n-$convex function, for any $t \in [a,b]$, the conclusion follows after we apply inequality (5.6) on family of functions $\{f_t : t \in [a,b]\}$.

(ii) The proof is similar to the proof of the (i)-part. □

Using previous results we can generalize Cerone's result from Theorem 3.41.

Theorem 5.4 *Let $f:[a,b]\to\mathbb{R}$ be a nondecreasing function. Let $u:[a,b]\to\mathbb{R}$ be nonnegative and $w_i:[c_i,d_i]\to\mathbb{R}$, $i=1,2$ integrable functions such that*

$$\int_a^b u(x)dx=\int_{c_i}^{d_i} w_i(x)dx, \tag{5.12}$$

where $[c_i,d_i]\subseteq[a,b]$ for $i=1,2$.

(i) If $u(t)\le w_1(t)$, for $t\in[c_1,d_1]$, then

$$\int_a^b f(t)u(t)dt-\int_{c_1}^{d_1} f(t)w_1(t)dt\ge\int_a^{c_1}(f(t)-f(d_1))u(t)dt.$$

(ii) If $u(t)\le w_2(t)$, for $t\in[c_2,d_2]$, then

$$\int_a^b f(t)u(t)dt-\int_{c_2}^{d_2} f(t)w_2(t)dt\le\int_{d_2}^{b}(f(t)-f(c_2))u(t)dt.$$

Proof. We apply Lemma 5.1 for $n=1$ with $c=c_i,\ d=d_i,\ w=w_i,\ i=1,2$. Then we note

$$T_a^{f,u}-T_a^{f,w_i}=f(a)\left[\int_a^b u(x)dx-\int_{c_i}^{d_i} w_i(x)dx\right]=0,\quad i=1,2.$$

The rest follows directly from integration by parts of the right-hand side of (5.1) in case $n=1$, i.e. integration by parts of $\int_a^b f'(t)g(t)dt$. □

If we put $w_1=w_2=1$ in Theorem 5.4 we get Cerone's result from Theorem 3.41.

We observe that condition (5.12) is more general than Cerone's condition (3.46). This is the stepping stone for $n-$convex case, $n\ge 2$.

Theorem 5.5 *Let $f:[a,b]\to\mathbb{R}$ be $n-$convex function. Let $u:[a,b]\to\mathbb{R}$ be nonnegative and $w_i:[c_i,d_i]\to\mathbb{R}$, $i=1,2$, integrable functions such that*

$$\int_a^b u(x)(x-a)^{n-1}dx=\int_{c_i}^{d_i} w_i(x)(x-a)^{n-1}dx, \tag{5.13}$$

where $[c_i,d_i]\subseteq[a,b]$ for $i=1,2$.

(i) If $u(t)\le w_1(t)$, for $t\in[c_1,d_1]$, then

$$\begin{aligned}\int_a^b f(t)u(t)dt-\int_{c_1}^{d_1} f(t)w_1(t)dt-T_a^{f,u}+T_a^{f,w_1}&\\ \ge\frac{1}{(n-2)!}\int_a^{d_1}(f^{(n-1)}(t)-f^{(n-1)}(d_1))\varphi(t)dt&\end{aligned} \tag{5.14}$$

where

$$\varphi(t)=\begin{cases}\int_t^{c_1} u(x)(x-t)^{n-2}dx+\int_{d_1}^b u(x)(x-t)^{n-2}dx, & t\in[a,c_1]\\ \int_{d_1}^b u(x)(x-t)^{n-2}dx, & t\in[c_1,d_1].\end{cases}$$

(ii) If $u(t) \leq w_2(t)$, for $t \in [c_2, d_2]$, then

$$\begin{aligned} \int_a^b f(t)u(t)dt - \int_{c_2}^{d_2} f(t)w_2(t)dt - T_a^{f,u} + T_a^{f,w_2} \\ \leq \frac{1}{(n-2)!} \int_a^b (f^{(n-1)}(t) - f^{(n-1)}(c_2))\psi(t)dt \end{aligned} \tag{5.15}$$

where

$$\psi(t) = \begin{cases} \int_{c_2}^{d_2} (u(x) - w_2(x))(x-t)^{n-2}dx, & t \in [a, c_2] \\ \int_{d_2}^{b} u(x)(x-t)^{n-2}dx, & t \in [c_2, d_2] \\ \int_t^b u(x)(x-t)^{n-2}dx, & t \in [d_2, b]. \end{cases}$$

Proof. Since f is $n-$convex function it follows that $f^{(n-1)}$ is a nondecreasing function. In order to prove (5.14) we use Lemma 5.1 where the function g is defined by (5.3) ($c = c_1$, $d = d_1$). Condition (5.13) ensures us that $g(a) = g(b) = 0$. Integration by parts gives us

$$\begin{aligned} &\int_a^b f^{(n)}(t)g(t)dt = -\left(\int_a^{c_1} + \int_{c_1}^{d_1} + \int_{d_1}^b\right)\left(f^{(n-1)}(t) - f^{(n-1)}(d_1)\right)g'(t)dt \\ &= (n-1)\int_a^{c_1}\left(f^{(n-1)}(t) - f^{(n-1)}(d_1)\right) \times \\ &\times \left(\int_t^b u(t)(x-t)^{n-2}dx - \int_{c_1}^{d_1} w_1(t)(x-t)^{n-2}dx\right)dt \\ &+ (n-1)\int_{c_1}^{d_1}\left(f^{(n-1)}(t) - f^{(n-1)}(d_1)\right) \times \\ &\times \left(\int_t^b u(x)(x-t)^{n-2}dx - \int_t^{d_1} w_1(t)(x-t)^{n-2}dx\right)dt \\ &+ (n-1)\int_{d_1}^b\left(f^{(n-1)}(t) - f^{(n-1)}(d_1)\right)\left(\int_t^b u(x)(x-t)^{n-2}dx\right)dt \\ &\geq (n-1)\int_a^{c_1}\left(f^{(n-1)}(t) - f^{(n-1)}(d_1)\right) \times \\ &\times \left(\int_t^{c_1} u(x)(x-t)^{n-2}dx + \int_{d_1}^b u(x)(x-t)^{n-2}dx\right)dt \\ &+ (n-1)\int_{c_1}^{d_1}\left(f^{(n-1)}(t) - f^{(n-1)}(d_1)\right)\left(\int_{d_1}^b u(x)(x-t)^{n-2}dx\right)dt. \end{aligned} \tag{5.16}$$

Inequality (5.15) can be deduced in a similar way. □

For $w_1 = 1,\ c_1 = a,\ d_1 = a + \lambda_1$ from (5.13) we get condition (5.5), and for $w_2 = 1,\ c_2 = b - \lambda_2,\ d_2 = b$ from (5.13) we get condition (5.7).

Now, we give the following estimation.

Theorem 5.6 *Assume (p,q) is a pair of conjugate exponents. Let $f : [a,b] \cup [c,d] \to \mathbb{R}$, where $c \in [a,b]$, be a function such that $|f^{(n)}|^p$ is integrable for some $n \geq 2$. Then*

$$\left|\int_a^b u(x)f(x)dx - \int_c^d w(x)f(x)dx - T_a^{f,u} + T_a^{f,w}\right| \leq \frac{1}{(n-1)!}\|g\|_q\|f^{(n)}\|_p.$$

Proof. This follows after we apply Hölder's inequality on (5.1). □

Now we estimate generalization of Steffensen's inequality.

Corollary 5.2 *Assume (p,q) is a pair of conjugate exponents. Let $f:[a,b]\to\mathbb{R}$ be a function such that $|f^{(n)}|^p$ is integrable for some $n\geq 2$ and let λ_1 and λ_2 be defined by (5.5) and (5.7) respectively.*

(i) If the function g is defined by (5.9), then

$$\left|\int_a^b f(x)u(x)dx-\int_a^{a+\lambda_1} f(x)dx-\sum_{i=0}^{n-2}\frac{f^{(i)}(a)}{i!}\times\right.$$
$$\left.\times\left(\int_a^b u(x)(x-a)^i dx-\frac{\lambda_1^{i+1}}{i+1}\right)\right|\leq\frac{1}{(n-1)!}\|g\|_q\|f^{(n)}\|_p.$$

(ii) If the function g is defined by (5.11), then

$$\left|\int_a^b f(x)u(x)dx-\int_{b-\lambda_2}^{b} f(x)dx-\sum_{i=0}^{n-2}\frac{f^{(i)}(a)}{i!}\times\right.$$
$$\left.\times\left(\int_a^b u(x)(x-a)^i dx-\frac{(b-a)^{i+1}-(b-a-\lambda_2)^{i+1}}{i+1}\right)\right|\leq\frac{1}{(n-1)!}\|g\|_q\|f^{(n)}\|_p.$$

5.2 Generalizations by the one-point integral formula

In [74] and [86] the following one-point integral formula is introduced from the general $m-$point integral identity:

$$\int_a^b w(t)f(t)dt=\sum_{j=1}^{n}A_{w,j}(x)f^{(j-1)}(x)+(-1)^n\int_a^b W_{n,w}(t,x)f^{(n)}(t)dt, \tag{5.17}$$

where $f:[a,b]\to\mathbf{R}$ is such that $f^{(n-1)}$ is an absolutely continuous function, $w:[a,b]\to[0,\infty)$ is a weight function and for $x\in[a,b]$

$$A_{w,j}(x)=\frac{(-1)^{j-1}}{(j-1)!}\int_a^b (x-s)^{j-1}w(s)ds,\quad \text{for } j=1,\ldots,n$$

and

$$W_{n,w}(t,x)=\begin{cases} w_{1n}(t)=\frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}w(s)ds & \text{for } t\in[a,x]\\ w_{2n}(t)=\frac{1}{(n-1)!}\int_b^t (t-s)^{n-1}w(s)ds & \text{for } t\in(x,b].\end{cases}$$

Let us define function $W_{n,w}(t,x)$ outside of the interval $[a,b]$ by:

$$W_{n,w}(t,x) = 0, x \notin [a,b].$$

Let us define:

$$T_{w,n}^{[a,b]}(x) := \frac{1}{\int_a^b w(t)dt} \sum_{k=2}^{n} A_{w,k}(x) f^{(k-1)}(x) \qquad \text{for } n \geq 2$$

and $T_{w,1}^{[a,b]}(x) = 0$.

In [6] identity (5.17) is obtained as the extension of the weighted Montgomery identity via Taylor's formula. Also, the difference between two integral means, each having its own weight, w and u defined on two different intervals $[a,b]$ and $[c,d]$ such that $[a,b] \cap [c,d] \neq \emptyset$ is obtained:

$$\begin{aligned} \frac{1}{\int_a^b w(t)dt} \int_a^b w(t)f(t)dt - \frac{1}{\int_c^d u(t)dt} \int_c^d u(t)f(t)dt - T_n^{[a,b]}(x) + T_n^{[c,d]}(x) \\ = \int_{\min\{a,c\}}^{\max\{b,d\}} K_n(t,x) f^{(n)}(t)dt, \end{aligned}$$

where

$$K_n(t,x) \ = \ (-1)^n \left(\frac{W_{n,w}(t,x)}{\int_a^b w(t)dt} - \frac{W_{n,u}(t,x)}{\int_c^d u(t)dt} \right). \tag{5.18}$$

Assume that (p,q) is a pair of conjugate exponents, $1 \leq p,q \leq \infty$. The following inequality is also obtained in [6]: If $f^{(n)} \in L^p[a,d]$, then we have

$$\begin{aligned} \left| \frac{1}{\int_a^b w(t)dt} \int_a^b w(t)f(t)dt - \frac{1}{\int_c^d u(t)dt} \int_c^d u(t)f(t)dt - T_{w,n}^{[a,b]}(x) + T_{u,n}^{[c,d]}(x) \right| \\ \leq ||K_n(\cdot,x)||_q \cdot ||f^{(n)}||_p. \end{aligned} \tag{5.19}$$

The inequality (5.19) is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

The following results were obtained by Kovač, Pečarić and Perušić in [75].

Theorem 5.7 *Let $f : [a,b] \cup [c,d] \to \mathbb{R}$ be an $n-$convex function, $x \in [a,b] \cap [c,d]$ and let $w : [a,b] \to [0,\infty\rangle$ and $u : [c,d] \to [0,\infty\rangle$ be an integrable functions (weights). If $K_n(t,x) \geq 0$ for every $t \in [a,b] \cup [c,d]$, then*

$$\frac{1}{\int_a^b w(t)dt} \int_a^b w(t)f(t)dt - \frac{1}{\int_c^d u(t)dt} \int_c^d u(t)f(t)dt \geq T_{w,n}^{[a,b]}(x) - T_{u,n}^{[c,d]}(x). \tag{5.20}$$

If $K_n(t,x) \leq 0$, for every $t \in [a,b] \cup [c,d]$, then inequality (5.20) is reversed.

Proof. Without loss of generality we can assume that f is $n-$times differentiable and $f^{(n)} \geq 0$. The result now follows from the identity (5.18). □

Theorem 5.8 *Let $\lambda > 0$ and let $f : [a, \max\{b, a+\lambda\}] \to \mathbb{R}$ be $n-$convex function for $n \geq 1$. Let $w : [a,b] \to [0,\infty\rangle$ be integrable on $[a,b]$. If $x \in [a,b] \cap [a, a+\lambda]$ and $K_n(t,x) \geq 0$ for every $t \in [a, \max\{b, a+\lambda\}]$, then we have*

$$\frac{1}{\int_a^b w(t)dt} \int_a^b w(t)f(t)dt - T_{w,n}^{[a,b]}(x) \geq \frac{1}{\lambda} \int_a^{a+\lambda} f(t)dt - T_{1,n}^{[a,a+\lambda]}(x).$$

Proof. We apply Theorem 5.7 with $c = a$, $d = a+\lambda$ and $u \equiv 1$ on $[a, a+\lambda]$. We have two possibilities: $[a, a+\lambda] \subseteq [a,b]$ and $[a,b] \subseteq [a, a+\lambda]$.

In the case $[a, a+\lambda] \subseteq [a,b]$ we have

$$K_n(t,x) =$$
$$\begin{cases} (-1)^n \left[\frac{1}{(n-1)! \int_a^b w(t)dt} \int_a^t (t-s)^{n-1} w(s)ds - \frac{(t-a)^n}{n!\lambda} \right], & a \leq t \leq x \\ (-1)^n \left[\frac{1}{(n-1)! \int_a^b w(t)dt} \int_b^t (t-s)^{n-1} w(s)ds - \frac{(t-a-\lambda)^n}{n!\lambda} \right], & x < t \leq a+\lambda \\ (-1)^n \frac{1}{(n-1)! \int_a^b w(t)dt} \int_b^t (t-s)^{n-1} w(s)ds, & a+\lambda < t \leq b. \end{cases}$$

In the case $[a,b] \subseteq [a, a+\lambda]$ we have

$$K_n(t,x) =$$
$$\begin{cases} (-1)^n \left[\frac{1}{(n-1)! \int_a^b w(t)dt} \int_a^t (t-s)^{n-1} w(s)ds - \frac{(t-a)^n}{n!\lambda} \right], & a \leq t \leq x \\ (-1)^n \left[\frac{1}{(n-1)! \int_a^b w(t)dt} \int_b^t (t-s)^{n-1} w(s)ds - \frac{(t-a-\lambda)^n}{n!\lambda} \right], & x < t \leq b \\ -\frac{(a+\lambda-t)^n}{n!\lambda}, & b < t \leq a+\lambda. \end{cases}$$

□

For $n > 1$ we introduce the following classes of functions:

$$M_n[a,b] := \left\{ w : [a,b] \to [0,1] : \left(\int_a^b w(t)dt \right)^n \leq n \int_a^b (t-a)^{n-1} w(t)dt \right\}$$

and

$$M_n'[a,b] := \left\{ w : [a,b] \to [0,1] : \left(\int_a^b w(t)dt \right)^n \geq n \int_a^b (t-a)^{n-1} w(t)dt \right\}.$$

Let us denote $W := \int_a^b w(t)dt$.

Corollary 5.3 *Let $w : [a,b] \to [0,1]$ be an integrable function on $[a,b]$ and $n > 1$.*

a) If $\lambda = \int_a^b w(t)dt$ and $f : [a,b] \to \mathbb{R}$ is a nondecreasing function, then we have

$$\int_a^b w(t)f(t)dt \geq \int_a^{a+\lambda} f(t)dt.$$

b) If $w \in M_n[a,b]$,

$$\lambda = \left[n \cdot \int_a^b (t-a)^{n-1} w(t)dt \right]^{\frac{1}{n}}$$

and if $f : [a, \max\{b, a+\lambda\}] \to \mathbb{R}$ *is* $n-$*convex function, then we have*

$$\frac{1}{\int_a^b w(t)dt} \int_a^b w(t)f(t)dt - T_{w,n}^{[a,b]}(a) \geq \frac{1}{\lambda} \int_a^{a+\lambda} f(t)dt - T_{1,n}^{[a,a+\lambda]}(a). \tag{5.21}$$

c) If $w \in M_n'[a,b]$,

$$\lambda := \left[\frac{n}{\int_a^b w(t)dt} \cdot \int_a^b (t-a)^{n-1} w(t)dt \right]^{\frac{1}{n-1}}$$

and if $f : [a, \max\{b, a+\lambda\}] \to \mathbb{R}$ *is* $n-$*convex function, then (5.21) holds.*

Proof.

a) Since $\lambda = \int_a^b w(t)dt \leq b-a$, for $x = a$ and $n = 1$ we have

$$K_1(t,a) = \begin{cases} \frac{1}{\lambda} \int_t^b w(s)ds - \frac{a+\lambda-t}{\lambda}, & a \leq t \leq a+\lambda \\ \frac{1}{\lambda} \int_t^b w(s)ds, & a+\lambda < t \leq b. \end{cases}$$

Since $K_1(t,a) \geq 0$ for $t \in [a+\lambda, b]$, we only have to prove

$$\int_t^b w(s)ds \geq a+\lambda-t, \quad t \in [a, a+\lambda].$$

We have

$$\int_t^b w(s)ds = \int_a^b w(s)ds - \int_a^t w(s)ds = \lambda - \int_a^t w(s)ds \geq \lambda - t + a.$$

Hence, $-K_1(t,a) \geq 0$, so the assertion follows from Theorem 5.8.

b) In this case we have

$$\lambda = \left[n \cdot \int_a^b (t-a)^{n-1} w(t)dt \right]^{\frac{1}{n}} \leq \left[n \int_a^b (t-a)^{n-1} dt \right]^{\frac{1}{n}}$$
$$= \left[n \frac{(b-a)^n}{n} \right]^{\frac{1}{n}} = b-a,$$

so for $x = a$ we have

$$K_n(t,a) = \begin{cases} 0, & t = a \\ \frac{1}{(n-1)! \int_a^b w(t)dt} \int_t^b (s-t)^{n-1} w(s)ds - \frac{(a+\lambda-t)^n}{n!\lambda}, & a < t \leq a+\lambda \\ \frac{1}{(n-1)! \int_a^b w(t)dt} \int_t^b (s-t)^{n-1} w(s)ds, & a+\lambda < t \leq b. \end{cases} \tag{5.22}$$

Obviously, $K_n(t,a) \geq 0$, for $t \in \langle a+\lambda, b]$.

In order to prove that $K_n(t,a) \geq 0$, for $t \in [a, a+\lambda]$, we have to prove that

$$\frac{1}{(n-1)!\int_a^b w(t)dt}\int_t^b (s-t)^{n-1}w(s)ds \geq \frac{(a+\lambda-t)^n}{n!\lambda}. \tag{5.23}$$

From (5.10) we have

$$\int_t^b w(s)(s-t)^{n-1}ds \geq \frac{(a+\lambda-t)^n}{n}.$$

From the definition of the class M_n we have that $\lambda \geq \int_a^b w(t)dt$. Hence,

$$\frac{1}{(n-1)!\int_a^b w(t)dt}\int_t^b w(s)(s-t)^{n-1}ds \geq \frac{(a+\lambda-t)^n}{n!\int_a^b w(t)dt} \geq \frac{(a+\lambda-t)^n}{n!\lambda}.$$

So we proved (5.23). Now, the assertion follows from Theorem 5.8.

c) For $w \in M_n'[a,b]$ we have

$$\lambda^{n-1}\int_a^b w(t)dt = n\int_a^b (t-a)^{n-1}w(t)dt \leq \left(\int_a^b w(t)dt\right)^n$$

so $\lambda \leq W \leq b-a$. We have that $K_n(t,a)$ is defined by (5.22). As in (b)-case we get that $K_n(t,a) \geq 0$, $t \in [a,b]$, so the assertion follows from Theorem 5.8. □

Theorem 5.9 *Let $\lambda > 0$, let $f : [\min\{a, b-\lambda\}, b] \to \mathbb{R}$ be an $n-$convex function for $n \geq 1$, and let $w : [a,b] \to [0,\infty\rangle$ be integrable on $[a,b]$. If $x \in [a,b] \cap [b-\lambda, b]$ and $K_n(t,x) \leq 0$ for every $t \in [\min\{a, b-\lambda\}, b]$, then we have*

$$\frac{1}{\lambda}\int_{b-\lambda}^b f(t)dt - T_{1,n}^{[b-\lambda,b]}(x) \geq \frac{1}{\int_a^b w(t)dt}\int_a^b w(t)f(t)dt - T_{w,n}^{[a,b]}(x).$$

Proof. We apply Theorem 5.7 with $c = b-\lambda$, $d = b$ and $u \equiv 1$ on $[b-\lambda, b]$. We have two possibilities: $[b-\lambda, b] \subseteq [a,b]$ and $[a,b] \subseteq [b-\lambda, b]$.

In the case $[b-\lambda, b] \subseteq [a,b]$ we have

$$K_n(t,x) = \begin{cases} \frac{(-1)^n}{(n-1)!\int_a^b w(t)dt}\int_a^t (t-s)^{n-1}w(s)ds, & a \leq t \leq b-\lambda \\ (-1)^n\left[\frac{1}{(n-1)!\int_a^b w(t)dt}\int_b^t (t-s)^{n-1}w(s)ds - \frac{(t-b+\lambda)^n}{n!\lambda}\right], & b-\lambda < t \leq x \\ (-1)^n\left[\frac{1}{(n-1)!\int_a^b w(t)dt}\int_b^t (t-s)^{n-1}w(s)ds - \frac{(t-b)^n}{n!\lambda}\right], & x < t \leq b. \end{cases}$$

In the case $[a,b] \subseteq [b-\lambda,b]$ we have

$$K_n(t,x) = \begin{cases} -\frac{(b-\lambda-t)^n}{n!\lambda}, & b-\lambda \le t \le a \\ (-1)^n \left[\frac{1}{(n-1)!\int_a^b w(t)dt} \int_a^t (t-s)^{n-1} w(s)ds - \frac{(t-b+\lambda)^n}{n!\lambda} \right], & a < t \le x \\ (-1)^n \left[\frac{1}{(n-1)!\int_a^b w(t)dt} \int_a^t (t-s)^{n-1} w(s)ds - \frac{(t-b)^n}{n!\lambda} \right], & x < t \le b. \end{cases}$$

□

If $n = 1$, then a simple consequence of Theorem 5.9 is the right-hand side of the Steffensen inequality.

Chapter 6

Generalizations of Steffensen's inequality by interpolating polynomials

6.1 Generalizations by Lidstone's polynomial

In 1929 G. J. Lidstone [80] introduced a generalization of Taylor's series, today known as a Lidstone series. It approximates a given function in the neighborhood of two points instead of one. Such series have been studied by H. Poritsky (1932), J. M. Wittaker (1934), I. J. Schoenberg (1936), R. P. Boas (1943) and others (see [28], [131], [140], [152]).

Definition 6.1 *Let $f \in C^{\infty}([0,1])$, then* Lidstone series *has the form*

$$\sum_{k=0}^{\infty} \left(f^{(2k)}(0)\Lambda_k(1-x) + f^{(2k)}(1)\Lambda_k(x) \right),$$

where Λ_n is the Lidstone polynomial of degree $2n+1$ defined by the relations

$$\begin{aligned}
&\Lambda_0(t) = t, \\
&\Lambda_n''(t) = \Lambda_{n-1}(t), \\
&\Lambda_n(0) = \Lambda_n(1) = 0, \qquad n \geq 1.
\end{aligned}$$

Explicit representations of Lidstone polynomial are given in [3] and [152]. Some of those representations are given by:

$$\begin{aligned}
\Lambda_n(t) &= (-1)^n \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2n+1}} \sin k\pi t, \quad n \geq 1 \\
\Lambda_n(t) &= \frac{1}{6}\left[\frac{6t^{2n+1}}{(2n+1)!} - \frac{t^{2n-1}}{(2n-1)!}\right] \\
&\quad - \sum_{k=0}^{n-2} \frac{2(2^{2k+3}-1)}{(2k+4)!} B_{2k+4} \frac{t^{2n-2k-3}}{(2n-2k-3)!}, \; n = 1,2,\ldots, \\
\Lambda_n(t) &= \frac{2^{2n+1}}{(2n+1)!} B_{2n+1}\left(\frac{1+t}{2}\right), n = 1,2\ldots,
\end{aligned}$$

where B_{2k+4} is the $(2k+4)$-th Bernoulli number and $B_{2n+1}\left(\frac{1+t}{2}\right)$ is a Bernoulli polynomial. More details about Bernoulli number and Bernoulli polynomial are given in Section 4.2.

In [154] Widder proved the following fundamental lemma:

Lemma 6.1 *If $f \in C^{2n}([0,1])$, then*

$$f(t) = \sum_{k=0}^{n-1} \left[f^{(2k)}(0)\Lambda_k(1-t) + f^{(2k)}(1)\Lambda_k(t)\right] + \int_0^1 G_n(t,s) f^{(2n)}(s)ds,$$

where

$$G_1(t,s) = G(t,s) = \begin{cases} (t-1)s, & \text{if } s < t \\ (s-1)t, & \text{if } t \leq s \end{cases}$$

is the homogeneous Green's function of the differential operator $\frac{d^2}{ds^2}$ on $[0,1]$, and with the successive iterates of $G(t,s)$

$$G_n(t,s) = \int_0^1 G_1(t,p) G_{n-1}(p,s)dp, \qquad n \geq 2.$$

If $[a,b] \cap [c,d] \neq \emptyset$ we have four possible cases for two intervals $[a,b]$ and $[c,d]$. The first case is $[c,d] \subset [a,b]$, the second case is $[a,b] \cap [c,d] = [c,b]$ and other two cases are obtained by changing $a \leftrightarrow c$, $b \leftrightarrow d$. Hence, in the following theorem we only observe first two cases.

In this section by $T_{w,n}^{[a,b]}$ we denote

$$T_{w,n}^{[a,b]} = \sum_{k=0}^{n-1} (b-a)^{2k} \int_a^b w(x) \left[f^{(2k)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right) + f^{(2k)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right] dx.$$

Results which follow in this section were obtained by Pečarić, Perušić and Smoljak in [119]. First we give general results which are used for obtaining generalizations of Steffensen's inequality.

Theorem 6.1 *Let $[a,b]\cap[c,d]\neq\emptyset$ and let $f:[a,b]\cup[c,d]\to\mathbb{R}$ be of class C^{2n} on $[a,b]\cup[c,d]$ for some $n\geq 1$. Let $w:[a,b]\to[0,\infty\rangle$ and $u:[c,d]\to[0,\infty\rangle$. Then we have*

$$\begin{aligned}\int_a^b w(t)f(t)\,dt-\int_c^d u(t)f(t)\,dt-T_{w,n}^{[a,b]}+T_{u,n}^{[c,d]}&\\ =\int_a^{\max\{b,d\}}K_n(s)f^{(2n)}(s)\,ds,&\end{aligned}\tag{6.1}$$

where in case $[c,d]\subseteq[a,b]$,

$$K_n(s)=\begin{cases}(b-a)^{2n-1}\int_a^b w(x)G_n\left(\frac{x-a}{b-a},\frac{s-a}{b-a}\right)dx, & s\in[a,c]\\ \\ (b-a)^{2n-1}\int_a^b w(x)G_n\left(\frac{x-a}{b-a},\frac{s-a}{b-a}\right)dx & \\ -(d-c)^{2n-1}\int_c^d u(x)G_n\left(\frac{x-c}{d-c},\frac{s-c}{d-c}\right)dx, & s\in\langle c,d]\\ \\ (b-a)^{2n-1}\int_a^b w(x)G_n\left(\frac{x-a}{b-a},\frac{s-a}{b-a}\right)dx, & s\in\langle d,b],\end{cases}\tag{6.2}$$

and in case $[a,b]\cap[c,d]=[c,b]$,

$$K_n(s)=\begin{cases}(b-a)^{2n-1}\int_a^b w(x)G_n\left(\frac{x-a}{b-a},\frac{s-a}{b-a}\right)dx, & s\in[a,c]\\ \\ (b-a)^{2n-1}\int_a^b w(x)G_n\left(\frac{x-a}{b-a},\frac{s-a}{b-a}\right)dx & \\ -(d-c)^{2n-1}\int_c^d u(x)G_n\left(\frac{x-c}{d-c},\frac{s-c}{d-c}\right)dx, & s\in\langle c,b]\\ \\ -(d-c)^{2n-1}\int_c^d u(x)G_n\left(\frac{x-c}{d-c},\frac{s-c}{d-c}\right)dx, & s\in\langle b,d].\end{cases}\tag{6.3}$$

Proof. From Lemma 6.1 for $f\in C^{2n}([a,b])$ we have the following identity

$$\begin{aligned}f(x)=\sum_{k=0}^{n-1}(b-a)^{2k}\left[f^{(2k)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right)+f^{(2k)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right]&\\ +(b-a)^{2n-1}\int_a^b G_n\left(\frac{x-a}{b-a},\frac{s-a}{b-a}\right)f^{(2n)}(s)ds.&\end{aligned}\tag{6.4}$$

Multiplying identity (6.4) by $w(x)$, integrating from a to b and using Fubini's theorem we obtain

$$\begin{aligned}\int_a^b w(x)f(x)dx=\sum_{k=0}^{n-1}(b-a)^{2k}\int_a^b w(x)\left[f^{(2k)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right)\right.&\\ \left.+f^{(2k)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right]dx+(b-a)^{2n-1}\int_a^b f^{(2n)}(s)\times&\\ \times\left(\int_a^b w(x)G_n\left(\frac{x-a}{b-a},\frac{s-a}{b-a}\right)dx\right)ds.&\end{aligned}\tag{6.5}$$

A similar identity holds for the weight function u on interval $[c,d]$. Now subtracting those identities for integrals $\int_a^b w(x)f(x)dx$ and $\int_c^d u(x)f(x)dx$ we obtain (6.1). □

Theorem 6.2 *Let $[a,b]\cap[c,d]\neq\emptyset$ and let $f:[a,b]\cup[c,d]\to\mathbb{R}$ be $(2n)-$convex on $[a,b]\cup[c,d]$ and let $w:[a,b]\to[0,\infty\rangle$ and $u:[c,d]\to[0,\infty\rangle$. If*

$$K_n(s)\geq 0, \tag{6.6}$$

then

$$\int_a^b w(t)f(t)\,dt - T_{w,n}^{[a,b]} \geq \int_c^d u(t)f(t)\,dt - T_{u,n}^{[c,d]} \tag{6.7}$$

where in case $[c,d]\subseteq[a,b]$, $K_n(s)$ is defined by (6.2) and in case $[a,b]\cap[c,d]=[c,b]$, $K_n(s)$ is defined by (6.3).

Proof. Since f is $(2n)$-convex, withouth loss of generality we can assume that f is $(2n)-$times differentiable and $f^{(2n)}\geq 0$. Now we can apply Theorem 6.1 to obtain (6.7). □

For a special choice of weights and intervals in previous results we obtain generalization of Steffensen's inequality.

Theorem 6.3 *Let $f:[a,b]\cup[a,a+\lambda]\to\mathbb{R}$ be $(2n)-$convex on $[a,b]\cup[a,a+\lambda]$ and let $w:[a,b]\to[0,\infty\rangle$. Then if*

$$K_n(s)\geq 0, \tag{6.8}$$

we have

$$\int_a^b w(t)f(t)\,dt - T_{w,n}^{[a,b]} \geq \int_a^{a+\lambda} f(t)\,dt - T_{1,n}^{[a,a+\lambda]} \tag{6.9}$$

where in case $a\leq a+\lambda\leq b$,

$$K_n(s)=\begin{cases} (b-a)^{2n-1}\int_a^b w(x)G_n\left(\frac{x-a}{b-a},\frac{s-a}{b-a}\right)dx \\ \quad -\lambda^{2n-1}\int_a^{a+\lambda} G_n\left(\frac{x-a}{\lambda},\frac{s-a}{\lambda}\right)dx, \qquad s\in[a,a+\lambda] \\ \\ (b-a)^{2n-1}\int_a^b w(x)G_n\left(\frac{x-a}{b-a},\frac{s-a}{b-a}\right)dx,\ s\in\langle a+\lambda,b], \end{cases} \tag{6.10}$$

and in case $a\leq b\leq a+\lambda$,

$$K_n(s)=\begin{cases} (b-a)^{2n-1}\int_a^b w(x)G_n\left(\frac{x-a}{b-a},\frac{s-a}{b-a}\right)dx \\ \quad -\lambda^{2n-1}\int_a^{a+\lambda} G_n\left(\frac{x-a}{\lambda},\frac{s-a}{\lambda}\right)dx, \qquad s\in[a,b] \\ \\ \quad -\lambda^{2n-1}\int_a^{a+\lambda} G_n\left(\frac{x-a}{\lambda},\frac{s-a}{\lambda}\right)dx, \qquad s\in\langle b,a+\lambda]. \end{cases} \tag{6.11}$$

Proof. We take $c=a$, $d=a+\lambda$ and $u(t)=1$ in Theorem 6.2. □

Theorem 6.4 *Let $f:[a,b]\cup[b-\lambda,b]\to\mathbb{R}$ be $(2n)-$convex on $[a,b]\cup[b-\lambda,b]$ and let $w:[a,b]\to[0,\infty\rangle$. Then if*

$$K_n(s)\geq 0, \tag{6.12}$$

we have

$$\int_{b-\lambda}^b f(t)\,dt - T_{1,n}^{[b-\lambda,b]}(x) \geq \int_a^b w(t)f(t)\,dt - T_{w,n}^{[a,b]}(x) \tag{6.13}$$

where in case $a \le b-\lambda \le b$,

$$K_n(s) = \begin{cases} -(b-a)^{2n-1}\int_a^b w(x)G_n\left(\frac{x-a}{b-a},\frac{s-a}{b-a}\right)dx, & s\in[a,b-\lambda] \\ \lambda^{2n-1}\int_{b-\lambda}^b G_n\left(\frac{x-b+\lambda}{\lambda},\frac{s-b+\lambda}{\lambda}\right)dx \\ -(b-a)^{2n-1}\int_a^b w(x)G_n\left(\frac{x-a}{b-a},\frac{s-a}{b-a}\right)dx, & s\in\langle b-\lambda,b], \end{cases} \tag{6.14}$$

and in case $b-\lambda \le a \le b$,

$$K_n(s) = \begin{cases} \lambda^{2n-1}\int_{b-\lambda}^b G_n\left(\frac{x-b+\lambda}{\lambda},\frac{s-b+\lambda}{\lambda}\right)dx, & s\in[b-\lambda,a] \\ \lambda^{2n-1}\int_{b-\lambda}^b G_n\left(\frac{x-b+\lambda}{\lambda},\frac{s-b+\lambda}{\lambda}\right)dx \\ -(b-a)^{2n-1}\int_a^b w(x)G_n\left(\frac{x-a}{b-a},\frac{s-a}{b-a}\right)dx, & s\in\langle a,b]. \end{cases} \tag{6.15}$$

Proof. First we change $a\leftrightarrow c$, $b\leftrightarrow d$ and $w\leftrightarrow u$ in Theorem 6.2 and then we take $c=b-\lambda$, $d=b$ and $u(t)=1$. □

Now we will give Ostrowski type inequalities for the previous results.

Theorem 6.5 *Suppose that all assumptions of Theorem 6.1 hold. Assume (p,q) is a pair of conjugate exponents. Let $\left|f^{(2n)}\right|^p : [a,b]\cup[c,d]\to\mathbb{R}$ be an integrable function for some $n\ge 1$. Then we have*

$$\begin{aligned} &\left|\int_a^b w(t)f(t)\,dt - \int_c^d u(t)f(t)\,dt - T_{w,n}^{[a,b]} + T_{u,n}^{[c,d]}\right| \\ &\le \left\|f^{(2n)}\right\|_p \left(\int_a^{\max\{b,d\}} |K_n(s)|^q\,ds\right)^{\frac{1}{q}}. \end{aligned} \tag{6.16}$$

The constant $\left(\int_a^{\max\{b,d\}}|K_n(s)|^q\,ds\right)^{1/q}$ in the inequality (6.16) is sharp for $1<p\le\infty$ and the best possible for $p=1$.

Proof. Using identity (6.1) and applying Hölder's inequality we obtain

$$\begin{aligned} &\left|\int_a^b w(t)f(t)\,dt - \int_c^d u(t)f(t)\,dt - T_{w,n}^{[a,b]} + T_{u,n}^{[c,d]}\right| \\ &= \left|\int_a^{\max\{b,d\}} K_n(s)f^{(2n)}(s)ds\right| \le \left\|f^{(2n)}\right\|_p \left(\int_a^{\max\{b,d\}} |K_n(s)|^q\,ds\right)^{\frac{1}{q}}. \end{aligned}$$

For the proof of the sharpness of the constant $\left(\int_a^{\max\{b,d\}}|K_n(s)|^q\,ds\right)^{\frac{1}{q}}$ we will find a function f for which the equality in (6.16) is obtained.
For $1<p<\infty$ take f to be such that

$$f^{(2n)}(s) = \operatorname{sgn} K_n(s)\,|K_n(s)|^{\frac{1}{p-1}}.$$

For $p=\infty$ take $f^{(2n)}(s)=\operatorname{sgn} K_n(s)$.
For $p=1$ we prove that

$$\left|\int_a^{\max\{b,d\}} K_n(s)\, f^{(2n)}(s)ds\right| \leq \max_{s\in[a,\max\{b,d\}]} |K_n(s)| \left(\int_a^{\max\{b,d\}} \left|f^{(2n)}(s)\right| ds\right) \tag{6.17}$$

is the best possible inequality. Suppose that $|K_n(s)|$ attains its maximum at $s_0 \in [a, \max\{b,d\}]$. First we assume that $K_n(s_0)>0$. For ε small enough we define $f_\varepsilon(s)$ by

$$f_\varepsilon(s)=\begin{cases} 0, & a\leq s\leq s_0 \\ \frac{1}{\varepsilon\,(2n)!}(s-s_0)^{2n}, & s_0\leq s\leq s_0+\varepsilon \\ \frac{1}{(2n)!}(s-s_0)^{2n-1}, & s_0+\varepsilon\leq s\leq \max\{b,d\}. \end{cases}$$

Then for ε small enough

$$\left|\int_a^{\max\{b,d\}} K_n(s) f^{(2n)}(s)ds\right| = \left|\int_{s_0}^{s_0+\varepsilon} K_n(s)\frac{1}{\varepsilon}ds\right| = \frac{1}{\varepsilon}\int_{s_0}^{s_0+\varepsilon} K_n(s)ds.$$

Now from inequality (6.17) we have

$$\frac{1}{\varepsilon}\int_{s_0}^{s_0+\varepsilon} K_n(s)ds \leq K_n(s_0)\int_{s_0}^{s_0+\varepsilon}\frac{1}{\varepsilon}ds = K_n(s_0).$$

Since,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{s_0}^{s_0+\varepsilon} K_n(s)ds = K_n(s_0)$$

the statement follows. In the case $K_n(s_0)<0$ we define

$$f_\varepsilon(s)=\begin{cases} \frac{1}{(2n)!}(s-s_0-\varepsilon)^{2n-1},, & a\leq s\leq s_0 \\ -\frac{1}{\varepsilon\,(2n)!}(s-s_0-\varepsilon)^{2n}, & s_0\leq s\leq s_0+\varepsilon \\ 0, & s_0+\varepsilon\leq s\leq \max\{b,d\}, \end{cases}$$

and the rest of the proof is the same as above. □

Theorem 6.6 *Suppose that all assumptions of Theorem 6.3 hold. Assume (p,q) is a pair of conjugate exponents. Let $\left|f^{(2n)}\right|^p : [a,b]\cup[a,a+\lambda]\to\mathbb{R}$ be an integrable function for some $n\geq 1$. Let $K_n(s)$ be defined by (6.10) in case $a\leq a+\lambda\leq b$ and by (6.11) in case $a\leq b\leq a+\lambda$. Then we have*

$$\begin{aligned} &\left|\int_a^b w(t)f(t)\,dt - \int_a^{a+\lambda} f(t)\,dt - T_{w,n}^{[a,b]} + T_{1,n}^{[a,a+\lambda]}\right| \\ &\leq \left\|f^{(2n)}\right\|_p \left(\int_a^{\max\{b,a+\lambda\}} |K_n(s)|^q ds\right)^{\frac{1}{q}}. \end{aligned} \tag{6.18}$$

The constant $\left(\int_a^{\max\{b,a+\lambda\}} |K_n(s)|^q ds\right)^{1/q}$ in the inequality (6.18) is sharp for $1<p\leq\infty$ and the best possible for $p=1$.

Proof. We take $c = a$, $d = a + \lambda$ and $u(t) = 1$ in Theorem 6.5. □

Theorem 6.7 *Suppose that all assumptions of Theorem 6.4 hold. Assume (p,q) is a pair of conjugate exponents. Let $\left|f^{(2n)}\right|^p : [a,b] \cup [b-\lambda,b] \to \mathbb{R}$ be an integrable function for some $n \geq 1$. Let $K_n(s)$ be defined by (6.14) in case $a \leq b-\lambda \leq b$ and by (6.15) in case $b-\lambda \leq a \leq b$. Then we have*

$$\begin{aligned} &\left| \int_{b-\lambda}^{b} f(t)\,dt - \int_a^b w(t) f(t)\,dt - T_{1,n}^{[b-\lambda,b]} + T_{w,n}^{[a,b]} \right| \\ &\leq \left\| f^{(2n)} \right\|_p \left(\int_{\min\{a,b-\lambda\}}^{b} |K_n(s)|^q\,ds \right)^{\frac{1}{q}}. \end{aligned} \tag{6.19}$$

The constant $\left(\int_{\min\{a,b-\lambda\}}^{b} |K_n(s)|^q\,ds\right)^{1/q}$ in the inequality (6.19) is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof. First we change $a \leftrightarrow c$, $b \leftrightarrow d$ and $w \leftrightarrow u$ in Theorem 6.1 and then we take $c = b-\lambda$, $d = b$ and $u(t) = 1$. The rest of the proof is similar to the proof of Theorem 6.5. □

6.2 Generalizations by Hermite's polynomial

Let $-\infty < a < b < \infty$, and let $a \leq a_1 < a_2 < \ldots < a_r \leq b$, $(r \geq 2)$ be given numbers. For $f \in C^n[a,b]$ a unique polynomial $P_H(t)$ of degree $(n-1)$ exists fulfilling one of the following conditions:

Hermite conditions

$$P_H^{(i)}(a_j) = f^{(i)}(a_j);\ 0 \leq i \leq k_j,\ 1 \leq j \leq r,\ \sum_{j=1}^{r} k_j + r = n,$$

in particular:

Simple Hermite or Osculatory conditions $(n = 2m,\ r = m,\ k_j = 1$ for all $j)$

$$P_O(a_j) = f(a_j),\ P_O'(a_j) = f'(a_j),\ 1 \leq j \leq m,$$

Lagrange conditions $(r = n,\ k_j = 0$ for all $j)$

$$P_L(a_j) = f(a_j),\ 1 \leq j \leq n,$$

Type $(m,n-m)$ **conditions** $(r=2,\ 1\le m\le n-1,\ k_1=m-1,\ k_2=n-m-1)$

$$P_{mn}^{(i)}(a)=f^{(i)}(a),\ 0\le i\le m-1,$$
$$P_{mn}^{(i)}(b)=f^{(i)}(b),\ 0\le i\le n-m-1,$$

One-point Taylor conditions $(r=1,k_1=n-1)$

$$P_T^{(i)}(a)=f^{(i)}(a),\ 0\le i\le n-1.$$

Two-point Taylor conditions $(n=2m,\ r=2,\ k_1=k_2=m-1)$

$$P_{2T}^{(i)}(a)=f^{(i)}(a),\ P_{2T}^{(i)}(b)=f^{(i)}(b),\ 0\le i\le m-1.$$

The associated error $|e_H(t)|$ can be represented in terms of the Green's function $G_H(t,s)$ for the multipoint boundary value problem

$$z^{(n)}(t)=0,\ z^{(i)}(a_j)=0,\ 0\le i\le k_j,\ 1\le j\le r,$$

i.e., the following result holds (see [3]):

Theorem 6.8 *Let $F\in C^n[a,b]$ and let P_H be its Hermite interpolating polynomial. Then*

$$\begin{aligned}F(t)&=P_H(t)+e_H(t)\\&=\sum_{j=1}^{r}\sum_{i=0}^{k_j}H_{ij}(t)F^{(i)}(a_j)+\int_a^b G_H(t,s)F^{(n)}(s)ds,\end{aligned}$$

where H_{ij} are fundamental polynomials of the Hermite basis defined by

$$H_{ij}(t)=\frac{1}{i!}\frac{\omega(t)}{(t-a_j)^{k_j+1-i}}\sum_{k=0}^{k_j-i}\frac{1}{k!}\frac{d^k}{dt^k}\left(\frac{(t-a_j)^{k_j+1}}{\omega(t)}\right)\Big|_{t=a_j}(t-a_j)^k,$$

where

$$\omega(t)=\prod_{j=1}^{r}(t-a_j)^{k_j+1},$$

and G_H is the Green function defined by

$$G_H(t,s)=\begin{cases}\sum\limits_{j=1}^{\ell}\sum\limits_{i=0}^{k_j}\frac{(a_j-s)^{n-i-1}}{(n-i-1)!}H_{ij}(t),\ s\le t\\ -\sum\limits_{j=\ell+1}^{r}\sum\limits_{i=0}^{k_j}\frac{(a_j-s)^{n-i-1}}{(n-i-1)!}H_{ij}(t),\ s\ge t.\end{cases}\tag{6.20}$$

for all $a_\ell\le s\le a_{\ell+1},\ell=0,1,\dots,r$ $(a_0=a,\ a_{r+1}=b)$.

In particular case, for one-point Taylor conditions

$$H_{i1}(t) = \frac{(t-a)^i}{i!}, \qquad i = 0,1,\ldots,n-1,$$

and Green's function G_T is

$$G_T(t,s) = \begin{cases} \frac{(t-s)^{n-1}}{(n-1)!}, & s \le t \\ 0, & s > t, \end{cases}$$

so Theorem 6.8 gives us the classical Taylor theorem with integral reminder:

$$F(t) = \sum_{i=0}^{n-1} (t-a)^i \frac{F^{(i)}(a)}{i!} + \int_a^t (t-s)^{n-1} \frac{F^{(n)}(s)}{n!} ds.$$

For two-point Taylor conditions, $i = 0,1,\ldots,m-1$

$$H_{i1}(t) = \sum_{k=0}^{m-1-i} \binom{m+k-1}{k} \frac{(t-a)^i}{i!} \left(\frac{t-b}{a-b}\right)^m \left(\frac{t-a}{b-a}\right)^k$$

$$H_{i2}(t) = \sum_{k=0}^{m-1-i} \binom{m+k-1}{k} \frac{(t-b)^i}{i!} \left(\frac{t-a}{b-a}\right)^m \left(\frac{x-b}{a-b}\right)^k$$

and Green's function G_{2T} is

$$G_{2T}(t,s) = \begin{cases} \frac{(-1)^m}{(2m-1)!} p^m(t,s) \sum_{j=0}^{m-1} \binom{m-1+j}{j} (t-s)^{m-1-j} q^j(t,s), & s \le t \\ \frac{(-1)^m}{(2m-1)!} q^m(t,s) \sum_{j=0}^{m-1} \binom{m-1+j}{j} (s-t)^{m-1-j} p^j(t,s), & s \ge t \end{cases}$$

and

$$p(t,s) = \frac{(s-a)(b-t)}{b-a}, \; q(t,s) = p(s,t), \; \forall t,s \in [a,b].$$

The following lemma describes properties of the Green function (6.20) (see [24] and [79]).

Lemma 6.2 *The Green function $G_H(t,s)$ has the following properties:*

(i) $\dfrac{G_H(t,s)}{\omega(t)} > 0, \quad a_1 \le t \le a_r, \; a_1 < s < a_r;$

(ii) $G_H(t,s) \le \dfrac{1}{(n-1)!(b-a)} |\omega(t)|;$

(iii) $\displaystyle\int_a^b |G_H(t,s)| ds = \frac{|\omega(t)|}{n!}.$

In this section by $T_{w,n}^{[a,b],H^1}$ and $T_{u,n}^{[c,d],H^2}$ we denote

$$T_{w,n}^{[a,b],H^1} = \sum_{j=1}^{r} \sum_{i=0}^{k_j} f^{(i)}(a_j) \int_a^b w(t) H_{ij}^1(t) dt$$

$$T_{u,n}^{[c,d],H^2} = \sum_{j=1}^{s} \sum_{i=0}^{k_j} f^{(i)}(c_j) \int_c^d u(t) H_{ij}^2(t) dt$$

where H^1 and H^2 concern Hermite basis for knots $-\infty < a \leq a_1 < a_2 < \ldots < a_{r_1} \leq b < \infty$ and $-\infty < c \leq c_1 < c_2 < \ldots < c_{r_2} \leq d < \infty$ respectively.

Results which follow in this section were obtained by Jakšetić, Pečarić and Perušić in [70].

Theorem 6.9 *Let $[a,b] \cap [c,d] \neq \emptyset$ and let $f : [a,b] \cup [c,d] \to \mathbb{R}$ be of class C^n on $[a,b] \cup [c,d]$ for some $n \geq 1$. Let $w : [a,b] \to [0,\infty\rangle$ and $u : [c,d] \to [0,\infty\rangle$. Then we have*

$$\begin{aligned} \int_a^b w(t) f(t) dt - \int_c^d u(t) f(t) dt - T_{w,n}^{[a,b],H^1} + T_{u,n}^{[c,d],H^2} \\ = \int_a^{\max\{b,d\}} K_n(s) f^{(n)}(s) ds, \end{aligned} \tag{6.21}$$

where in case $[c,d] \subseteq [a,b]$,

$$K_n(s) = \begin{cases} \int_a^b w(t) G_{H^1}(t,s) dt, & s \in [a,c] \\ \int_a^b w(t) G_{H^1}(t,s) dt - \int_c^d u(t) G_{H^2}(t,s) dt, & s \in \langle c,d] \\ \int_a^b w(t) G_{H^1}(t,s) dt, & s \in \langle d,b], \end{cases} \tag{6.22}$$

and in case $[a,b] \cap [c,d] = [c,b]$,

$$K_n(s) = \begin{cases} \int_a^b w(t) G_{H^1}(t,s) dt & s \in [a,c] \\ \int_a^b w(t) G_{H^1}(t,s) dt - \int_c^d u(t) G_{H^2}(t,s) dt, & s \in \langle c,b] \\ -\int_c^d u(t) G_{H^2}(t,s) dt, & s \in \langle b,d]. \end{cases} \tag{6.23}$$

Proof. We use Theorem 6.8 to express the function f firstly on knots $-\infty < a \leq a_1 < a_2 < \ldots < a_{r_1} \leq b < \infty$ and then on $-\infty < c \leq c_1 < c_2 < \ldots < c_{r_2} \leq d < \infty$. We multiply both sides with functions w and u respectively, and then integrate both sides. Subtracting this two expressions and using Fubini's theorem we get desired result. □

Using Theorem 6.9 we can get Steffensen's inequality.

Theorem 6.10 *Suppose that f is nondecreasing and w is integrable on $[a,b]$ with $0 \leq w \leq 1$ and*

$$\lambda = \int_a^b w(t)dt.$$

Then we have

$$\int_{b-\lambda}^b f(t)dt \geq \int_a^b w(t)f(t)dt \geq \int_a^{a+\lambda} f(t)dt.$$

Proof. First we prove $\int_a^b f(t)w(t)dt \geq \int_a^{a+\lambda} f(t)dt$. For Hermite polynomials H^1 and H^2 we consider one-point Taylor conditions on $[a,b]$ and $[a,a+\lambda]$ respectively. Then from

$$K_1(s) = \begin{cases} \int_s^b w(t)dt - (a+\lambda) + s, & s \in [a, a+\lambda] \\ \int_s^b w(t)dt, & s \in \langle d, a+\lambda] \end{cases}$$

it follows $K_1(s) \geq 0$. Now (6.21) gives us

$$\int_a^b w(t)f(t)dt - \int_a^{a+\lambda} f(t)dt - f(a)\lambda + f(a)\lambda = \int_a^b K_1(s)f'(s)ds \geq 0,$$

concluding $\int_a^b w(t)f(t)dt - \int_a^{a+\lambda} f(t)dt \geq 0$.
Now we prove $\int_{b-\lambda}^b f(t)dt \geq \int_a^b w(t)f(t)dt$. For Hermite polynomials H^1 and H^2 here we consider one-point Taylor conditions on $[a,b]$ and $[b-\lambda, b]$ respectively. Then

$$K_1(s) = \begin{cases} \int_s^b w(t)dt, & s \in [a, b-\lambda] \\ \int_s^b (w(t)-1)dt, & s \in \langle b-\lambda, b]. \end{cases}$$

Now (6.21) gives us

$$\begin{aligned} \int_a^b w(t)f(t)dt - \int_{b-\lambda}^b f(t)dt - f(a)\lambda + f(b-\lambda)\lambda &= \int_a^b K_1(s)f'(s)ds \\ &\leq \lambda \int_a^{b-\lambda} f'(s)ds, \end{aligned}$$

concluding $\int_a^b w(t)f(t)dt - \int_{b-\lambda}^b f(t)dt \leq 0$. □

Theorem 6.11 *Let $[a,b] \cap [c,d] \neq \emptyset$ and let $f : [a,b] \cup [c,d] \to \mathbb{R}$ be $n-$convex on $[a,b] \cup [c,d]$ and let $w : [a,b] \to [0,\infty\rangle$ and $u : [c,d] \to [0,\infty\rangle$. If*

$$K_n(s) \geq 0, \tag{6.24}$$

then we have

$$\int_a^b w(t) f(t) dt - T_{w,n}^{[a,b],H^1} \geq \int_c^d u(t) f(t) dt - T_{u,n}^{[c,d],H^2} \tag{6.25}$$

where in case $[c,d] \subseteq [a,b]$, $K_n(s)$ is defined by (6.22) and in case $[a,b] \cap [c,d] = [c,b]$, $K_n(s)$ is defined by (6.23).

Proof. Since f is n-convex, without loss of generality we can assume that f is $n-$times differentiable and $f^{(n)} \geq 0$. Now we can apply Theorem 6.9 to obtain (6.25). □

It is easy to find kernels K_n such that (6.24) is fulfilled. For example, if we take $a \leq a_1 < a_2 < \ldots < a_{r_1} \leq b$ and all $k_1, \ldots, k_{r_1}$ are odd and $\sum_{j=1}^{r_1} k_j + r_1 = n$, then $\omega_1(t) = \prod_{j=1}^{r_1}(t-a_j)^{k_j+1} \geq 0$ and according to Lemma 6.2 (i) $G_{H^1}(t,s) \geq 0$. Similarly, if we take $c \leq c_1 < c_2 < \ldots < c_{r_2} = d < \infty$, all $m_1, \ldots, m_{r_2-1}$ are odd and m_{r_2} is even ($\sum_{j=1}^{r_2} m_j + r_2 = n$), then $\omega_2(t) = \prod_{j=1}^{r_2}(t-a_j)^{m_j+1} \leq 0$ and again, according to Lemma 6.2 (i), $G_{H^2}(t,s) \leq 0$. Particularly, in one-point Taylor case this is valid for any $n \in \mathbb{N}$ and in two-point Taylor case this is valid for any even $m \in \mathbb{N}$.

Now we will use Hermite expansion in order to generalize Steffensen's inequality. For special choice of weights and intervals in previous results we obtain generalization of Steffensen's inequality.

Theorem 6.12 *Let $f : [a,b] \cup [a,a+\lambda] \to \mathbb{R}$ be $n-$convex on $[a,b] \cup [a,a+\lambda]$ and let $w : [a,b] \to [0,\infty\rangle$. Then if*

$$K_n(s) \geq 0,$$

we have

$$\int_a^b w(t) f(t)\,dt - T_{w,n}^{[a,b],H^1} \geq \int_a^{a+\lambda} f(t)\,dt - T_{1,n}^{[a,a+\lambda],H^2}$$

where in case $a \leq a+\lambda \leq b$,

$$K_n(s) = \begin{cases} \int_a^b w(t) G_{H^1}(t,s)\,dt - \int_a^{a+\lambda} G_{H^2}(t,s)\,dt, & s \in [a,a+\lambda] \\ \int_a^b w(t) G_{H^1}(t,s)\,dt, & s \in \langle a+\lambda, b], \end{cases}$$

and in case $a \leq b \leq a+\lambda$,

$$K_n(s) = \begin{cases} \int_a^b w(t) G_{H^1}(t,s)\,dt - \int_a^{a+\lambda} G_{H^2}(t,s)\,dt, & s \in [a,b] \\ -\int_a^{a+\lambda} G_{H^2}(t,s)\,dt, & s \in \langle b, a+\lambda]. \end{cases}$$

Proof. We take $c = a$, $d = a+\lambda$ and $u(t) = 1$ in Theorem 6.11. □

Theorem 6.13 *Let $f : [a,b] \cup [b-\lambda,b] \to \mathbb{R}$ be $n-$convex on $[a,b] \cup [b-\lambda,b]$ and let $w : [a,b] \to [0,\infty\rangle$. Then if*

$$K_n(s) \leq 0,$$

we have

$$\int_{b-\lambda}^b f(t)\,dt - T_{1,n}^{[b-\lambda,b],H^2}(x) \geq \int_a^b w(t) f(t)\,dt - T_{w,n}^{[a,b],H^1}(x)$$

where in case $a \leq b-\lambda \leq b$,

$$K_n(s) = \begin{cases} \int_a^b w(t) G_{H^1}(t,s)\,dt, & s \in [a,b-\lambda] \\ \int_a^b w(t) G_{H^1}(t,s)\,dt - \int_{b-\lambda}^b G_{H^2}(t,s)\,dt, & s \in \langle b-\lambda, b], \end{cases}$$

and in case $b-\lambda \le a \le b$,

$$K_n(s) = \begin{cases} -\int_{b-\lambda}^{b} G_{H^2}(t,s)\,dt, & s \in [b-\lambda, a] \\ \int_a^b w(t) G_{H^1}(t,s)\,dt - \int_{b-\lambda}^{b} G_{H^2}(t,s)\,dt, & s \in \langle a, b]. \end{cases}$$

Similar to the proof of Theorem 6.5 we can prove Ostrowski type inequalities for the previous results given in the following theorems.

Theorem 6.14 *Suppose that all assumptions of Theorem 6.11 hold. Additionally assume* (p,q) *is a pair of conjugate exponents. Let* $\left|f^{(n)}\right|^p : [a,b] \cup [c,d] \to \mathbb{R}$ *be an integrable function for some* $n \ge 1$. *Then we have*

$$\begin{aligned} &\left| \int_a^b w(t) f(t)\,dt - \int_c^d u(t) f(t)\,dt - T_{w,n}^{[a,b],H^1} + T_{u,n}^{[c,d],H^2} \right| \\ &\le \left\| f^{(n)} \right\|_p \left(\int_a^{\max\{b,d\}} |K_n(s)|^q\,ds \right)^{\frac{1}{q}}. \end{aligned} \tag{6.26}$$

The constant $\left(\int_a^{\max\{b,d\}} |K_n(s)|^q\,ds \right)^{1/q}$ *in the inequality* (6.26) *is sharp for* $1 < p \le \infty$ *and the best possible for* $p = 1$.

Theorem 6.15 *Suppose that all assumptions of Theorem 6.12 hold. Additionally assume* (p,q) *is a pair of conjugate exponents. Let* $\left|f^{(n)}\right|^p : [a,b] \cup [a, a+\lambda] \to \mathbb{R}$ *be an integrable function for some* $n \ge 1$. *Let* $K_n(s)$ *be defined by* (6.10) *in case* $a \le a+\lambda \le b$ *and by* (6.11) *in case* $a < b \le a+\lambda$. *Then we have*

$$\begin{aligned} &\left| \int_a^b w(t) f(t)\,dt - \int_a^{a+\lambda} f(t)\,dt - T_{w,n}^{[a,b],H^1} + T_{1,n}^{[a,a+\lambda],H^2} \right| \\ &\le \left\| f^{(n)} \right\|_p \left(\int_a^{\max\{b,a+\lambda\}} |K_n(s)|^q\,ds \right)^{\frac{1}{q}}. \end{aligned} \tag{6.27}$$

The constant $\left(\int_a^{\max\{b,a+\lambda\}} |K_n(s)|^q\,ds \right)^{1/q}$ *in the inequality* (6.27) *is sharp for* $1 < p \le \infty$ *and the best possible for* $p = 1$.

Theorem 6.16 *Suppose that all assumptions of Theorem 6.13 hold. Additionally assume* (p,q) *is a pair of conjugate exponents. Let* $\left|f^{(n)}\right|^p : [a,b] \cup [b-\lambda, b] \to \mathbb{R}$ *be an integrable function for some* $n \ge 1$. *Let* $K_n(s)$ *be defined by (6.14) in case* $a \le b-\lambda \le b$ *and by (6.15) in case* $b-\lambda \le a \le b$. *Then we have*

$$\begin{aligned} &\left| \int_a^b w(t) f(t)\,dt - \int_{b-\lambda}^{b} f(t) + T_{1,n}^{[b-\lambda,b],H^2} - T_{w,n}^{[a,b],H^1} \right| \\ &\le \left\| f^{(n)} \right\|_p \left(\int_{\min\{a,b-\lambda\}}^{b} |K_n(s)|^q\,ds \right)^{\frac{1}{q}}. \end{aligned} \tag{6.28}$$

The constant $\left(\int_{\min\{a,b-\lambda\}}^{b} |K_n(s)|^q\, ds\right)^{1/q}$ *in the inequality* (6.28) *is sharp for* $1 < p \le \infty$ *and the best possible for* $p = 1$.

6.3 Generalizations by an Abel-Gontscharoff polynomial

Let $-\infty < a < b < \infty$, and $a \le a_1 < a_2 < \ldots < a_n \le b$ be the given points. For $f \in C^n[a,b]$ the *Abel-Gontscharoff interpolating polynomial* $P_{AG}(t)$ of degree $(n-1)$ satisfying the Abel-Gontscharoff conditions

$$P_{AG}^{(i)}(a_{i+1}) = f^{(i)}(a_{i+1}), \quad 0 \le i \le n-1$$

exists uniquely (see [39], [58]).

This conditions in particular include two-point right focal conditions

$$\begin{aligned}
&P_{AG2}^{(i)}(a_1) = f^{(i)}(a_1),\ 0 \le i \le \alpha,\\
&P_{AG2}^{(i)}(a_2) = f^{(i)}(a_2),\ \alpha + 1 \le i \le n-1,\ a \le a_1 < a_2 \le b.
\end{aligned}$$

First, we give representations of Abel-Gontscharoff interpolating polynomial. For details and proofs see [3].

Theorem 6.17 *The Abel-Gontscharoff interpolating polynomial* $P_{AG}(t)$ *of the function* f *can be expressed as*

$$P_{AG}(t) = \sum_{i=0}^{n-1} T_i(t) f^{(i)}(a_{i+1}),$$

where $T_0(t) = 1$ *and* $T_i(t)$, $1 \le i \le n-1$ *is the unique polynomial of degree* i *satisfying*

$$\begin{aligned}
&T_i^{(k)}(a_{k+1}) = 0,\ 0 \le k \le i-1\\
&T_i^{(i)}(a_{i+1}) = 1
\end{aligned}$$

and it can be written as

$$\begin{aligned}
T_i(t) &= \frac{1}{1!2!\cdots i!}\begin{vmatrix}
1 & a_1 & a_1^2 & \ldots & a_1^{i-1} & a_1^i\\
0 & 1 & 2a_2 & \ldots & (i-1)a_2^{i-2} & ia_2^{i-1}\\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots\\
0 & 0 & 0 & \ldots & (i-1)! & i!a_i\\
1 & t & t^2 & \ldots & t^{i-1} & t^i
\end{vmatrix}\\
&= \int_{a_1}^{t}\int_{a_2}^{t_1}\cdots\int_{a_i}^{t_{i-1}} dt_i dt_{i-1}\cdots dt_1,\ (t_0 = t).
\end{aligned} \tag{6.29}$$

In particular, we have

$$T_0(t) = 1$$
$$T_1(t) = t - a_1$$
$$T_2(t) = \frac{1}{2}\left[t^2 - 2a_2t + a_1(2a_2 - a_1)\right].$$

Corollary 6.1 *The two-point right focal interpolating polynomial $P_{AG2}(t)$ of the function f can be written as*

$$P_{AG2}(t) = \sum_{i=0}^{\alpha} \frac{(t-a_1)^i}{i!} f^{(i)}(a_1) + \sum_{j=0}^{n-\alpha-2}\left[\sum_{i=0}^{j} \frac{(t-a_1)^{\alpha+1+i}(a_1-a_2)^{j-i}}{(\alpha+1+i)!(j-i)!}\right] f^{(\alpha+1+j)}(a_2).$$

The associated error $e_{AG}(t) = f(t) - P_{AG}(t)$ can be represented in terms of the Green function $g_{AG}(t,s)$ of the boundary value problem

$$z^{(n)} = 0,\ z^{(i)}(a_{i+1}) = 0,\ 0 \le i \le n-1$$

and appears as (see [3]):

$$g_{AG}(t,s) = \begin{cases} \sum\limits_{i=0}^{k-1} \frac{T_i(t)}{(n-i-1)!}(a_{i+1}-s)^{n-i-1}, & a_k \le s \le t \\ -\sum\limits_{i=k}^{n-1} \frac{T_i(t)}{(n-i-1)!}(a_{i+1}-s)^{n-i-1}, & t \le s \le a_{k+1} \\ & k = 0,1,\dots,n\ (a_0 = a, a_{n+1} = b) \end{cases} \tag{6.30}$$

Corresponding to the two-point right focal conditions Green's function $g_{AG2}(t,s)$ of the boundary value problem

$$z^{(n)} = 0,\ z^{(i)}(a_1) = 0,\ 0 \le i \le \alpha,\ z^{(i)}(a_2) = 0, \alpha+1 \le i \le n-1$$

is given by (see [3]):

$$g_{AG2}(t,s) = \frac{1}{(n-1)!}\begin{cases} \sum\limits_{i=0}^{\alpha} \binom{n-1}{i}(t-a_1)^i(a_1-s)^{n-i-1}, & a \le s \le t \\ -\sum\limits_{i=\alpha+1}^{n-1} \binom{n-1}{i}(t-a_1)^i(a_1-s)^{n-i-1}, & t \le s \le b. \end{cases} \tag{6.31}$$

Further, for $a_1 \le s, t \le a_2$ the following inequalities hold

$$(-1)^{n-\alpha-1}\frac{\partial^i g_{AG2}(t,s)}{\partial t^i} \ge 0, \quad 0 \le i \le \alpha$$

$$(-1)^{n-i}\frac{\partial^i g_{AG2}(t,s)}{\partial t^i} \ge 0, \quad \alpha+1 \le i \le n-1.$$

Theorem 6.18 *Let $f \in C^n[a,b]$, and let P_{AG} be its Abel-Gontscharoff interpolating polynomial. Then*

$$\begin{aligned} f(t) &= P_{AG}(t) + e_{AG}(t) \\ &= \sum_{i=0}^{n-1} T_i(t) f^{(i)}(a_{i+1}) + \int_a^b g_{AG}(t,s) f^{(n)}(s) ds \end{aligned} \tag{6.32}$$

where T_i is defined by (6.29) and $g_{AG}(t,s)$ is defined by (6.30).

Theorem 6.19 *Let $f \in C^n[a,b]$, and let P_{AG2} be its two-point right focal Abel-Gontscharoff interpolating polynomial. Then*

$$\begin{aligned} f(t) &= P_{AG2}(t) + e_{AG2}(t) \\ &= \sum_{i=0}^{\alpha} \frac{(t-a_1)^i}{i!} f^{(i)}(a_1) + \sum_{j=0}^{n-\alpha-2} \left[\sum_{i=0}^{j} \frac{(t-a_1)^{\alpha+1+i}(a_1-a_2)^{j-i}}{(\alpha+1+i)!(j-i)!} \right] \times \\ &\quad \times f^{(\alpha+1+j)}(a_2) + \int_a^b g_{AG2}(t,s) f^{(n)}(s) ds \end{aligned} \tag{6.33}$$

where $g_{AG2}(t,s)$ is defined by (6.31).

In this section by $T_{w,n}^{[a,b]}$ we will denote

$$T_{w,n}^{[a,b]} = \sum_{i=0}^{n-1} f^{(i)}(a_{i+1}) \int_a^b w(t) T_i(t) dt$$

where $T_i(t)$ is defined by (6.29).

Results given in this section were obtained by Pečarić, Perušić and Smoljak in [120].

Theorem 6.20 *Let $f : [a,b] \cup [c,d] \to \mathbb{R}$ be of class C^n on $[a,b] \cup [c,d]$ for some $n \geq 1$. Let $w : [a,b] \to \mathbb{R}$ and $u : [c,d] \to \mathbb{R}$. Then if $[a,b] \cap [c,d] \neq \emptyset$ we have*

$$\begin{aligned} &\int_a^b w(t) f(t) dt - \int_c^d u(t) f(t) dt - T_{w,n}^{[a,b]} + T_{u,n}^{[c,d]} \\ &\qquad = \int_a^{\max\{b,d\}} K_n(s) f^{(n)}(s) ds, \end{aligned} \tag{6.34}$$

where in case $[c,d] \subseteq [a,b]$,

$$K_n(s) = \begin{cases} \int_a^b w(t) g_{AG}(t,s) dt, & s \in [a,c] \\ \int_a^b w(t) g_{AG}(t,s) dt - \int_c^d u(t) g_{AG}(t,s) dt, & s \in \langle c,d] \\ \int_a^b w(t) g_{AG}(t,s) dt, & s \in \langle d,b], \end{cases} \tag{6.35}$$

and in case $[a,b]\cap[c,d]=[c,b]$,

$$K_n(s)=\begin{cases}\int_a^b w(t)g_{AG}(t,s)\,dt & s\in[a,c]\\ \int_a^b w(t)g_{AG}(t,s)\,dt-\int_c^d u(t)g_{AG}(t,s)\,dt, & s\in\langle c,b]\\ -\int_c^d u(t)g_{AG}(t,s)\,dt, & s\in\langle b,d]\,.\end{cases} \tag{6.36}$$

Proof. Multiplying identity (6.32) by $w(t)$, then integrating from a to b and using Fubini's theorem we obtain

$$\begin{aligned}\int_a^b w(t)f(t)dt&=\sum_{i=0}^{n-1}f^{(i)}(a_{i+1})\int_a^b w(t)T_i(t)dt\\&+\int_a^b f^{(n)}(s)\left(\int_a^b w(t)g_{AG}(t,s)dt\right)ds.\end{aligned} \tag{6.37}$$

Furthermore, multiplying identity (6.32) by $u(t)$, then integrating from c to d and using Fubini's theorem we obtain similar identity to identity (6.37). Now subtracting this two identities we obtain (6.34). □

Remark 6.1 Using the two-point right focal Abel-Gontscharoff polynomial, i.e. using (6.33), inequality (6.34) becomes

$$\begin{aligned}\int_a^b w(t)f(t)\,dt-\int_c^d u(t)f(t)\,dt-Q_{w,n}^{[a,b]}+Q_{u,n}^{[c,d]}\\=\int_a^{\max\{b,d\}}K_n(s)f^{(n)}(s)\,ds,\end{aligned}$$

where $g_{AG}(t,s)$ is replaced by $g_{AG2}(t,s)$ in definition of $K_n(s)$ and by $Q_{w,n}^{[a,b]}$ we denote

$$\begin{aligned}Q_{w,n}^{[a,b]}&=\sum_{i=0}^{\alpha}\frac{f^{(i)}(a_1)}{i!}\int_a^b w(t)(t-a_1)^i dt\\&+\sum_{j=0}^{n-\alpha-2}f^{(\alpha+1+j)}(a_2)\left[\sum_{i=0}^{j}\frac{(a_1-a_2)^{j-i}}{(\alpha+1+j)!(j-i)!}\int_a^b w(t)(t-a_1)^{\alpha+1+i}dt\right].\end{aligned}$$

Theorem 6.21 *Let* $f:[a,b]\cup[c,d]\to\mathbb{R}$ *be* $n-$*convex on* $[a,b]\cup[c,d]$ *and let* $w:[a,b]\to\mathbb{R}$ *and* $u:[c,d]\to\mathbb{R}$. *Then if* $[a,b]\cap[c,d]\neq\emptyset$ *and*

$$K_n(s)\geq 0,$$

we have

$$\int_a^b w(t)f(t)\,dt-T_{w,n}^{[a,b]}\geq\int_c^d u(t)f(t)\,dt-T_{u,n}^{[c,d]} \tag{6.38}$$

where in case $[c,d]\subseteq[a,b]$, $K_n(s)$ *is defined by (6.35) and in case* $[a,b]\cap[c,d]=[c,b]$, $K_n(s)$ *is defined by (6.36).*

Proof. Since f is n-convex, without loss of generality we can assume that f is $n-$times differentiable and $f^{(n)} \geq 0$. Now we can apply Theorem 6.20 to obtain (6.38). □

As in Remark 6.1, using the two-point right focal Abel-Gontscharoff polynomial, inequality (6.38) becomes

$$\int_a^b w(t)\, f(t)\, dt - Q_{w,n}^{[a,b]} \geq \int_c^d u(t) f(t)\, dt - Q_{u,n}^{[c,d]}.$$

For a special choice of weights and intervals in the previous results we obtain generalizations of Steffensen's inequality.

Theorem 6.22 *Let $f : [a,b] \cup [a,a+\lambda] \to \mathbb{R}$ be $n-$convex on $[a,b] \cup [a,a+\lambda]$ and let $w : [a,b] \to \mathbb{R}$. Then if*

$$K_n(s) \geq 0,$$

we have

$$\int_a^b w(t)\, f(t)\, dt - T_{w,n}^{[a,b]} \geq \int_a^{a+\lambda} f(t)\, dt - T_{1,n}^{[a,a+\lambda]}$$

where in case $a \leq a+\lambda \leq b$,

$$K_n(s) = \begin{cases} \int_a^b w(t) g_{AG}(t,s)\, dt - \int_a^{a+\lambda} g_{AG}(t,s)\, dt, & s \in [a,a+\lambda] \\ \int_a^b w(t) g_{AG}(t,s)\, dt, & s \in \langle a+\lambda, b], \end{cases} \tag{6.39}$$

and in case $a \leq b \leq a+\lambda$,

$$K_n(s) = \begin{cases} \int_a^b w(t) g_{AG}(t,s)\, dt - \int_a^{a+\lambda} g_{AG}(t,s)\, dt, & s \in [a,b] \\ -\int_a^{a+\lambda} g_{AG}(t,s)\, dt, & s \in \langle b, a+\lambda]. \end{cases} \tag{6.40}$$

Proof. We take $c = a$, $d = a+\lambda$ and $u(t) = 1$ in Theorem 6.21. □

For $n = 1$ and $\lambda \leq b-a$, $K_1(s)$ becomes

$$K_1(s) = \begin{cases} -\int_a^s w(t)dt + s - a, & s \in [a,a+\lambda] \\ \int_s^b w(t)dt, & s \in \langle a+\lambda, b]. \end{cases}$$

So, if $\int_a^s w(t)dt \leq s-a$ for $a \leq s \leq a+\lambda$ and $\int_s^b w(t)dt \geq 0$ for $a+\lambda < s \leq b$ and f nondecreasing from Theorem 6.22 we have

$$\int_a^b w(t) f(t) dt - f(a+\lambda) \int_a^b w(t) dt \geq \int_a^{a+\lambda} f(t) dt - \lambda f(a+\lambda).$$

Hence, for $\lambda = \int_a^b w(t)dt$ we obtain the right-hand side of Steffensen's inequality for nondecreasing function f.

Theorem 6.23 *Let $f:[a,b]\cup[b-\lambda,b]\to\mathbb{R}$ be $n-$convex on $[a,b]\cup[b-\lambda,b]$ and let $w:[a,b]\to\mathbb{R}$. Then if*

$$K_n(s)\geq 0,$$

we have

$$\int_{b-\lambda}^{b} f(t)\,dt - T_{1,n}^{[b-\lambda,b]} \geq \int_a^b w(t)\,f(t)\,dt - T_{w,n}^{[a,b]}$$

where in case $a\leq b-\lambda\leq b$,

$$K_n(s)=\begin{cases} -\int_a^b w(t)g_{AG}(t,s)\,dt, & s\in[a,b-\lambda] \\ \int_{b-\lambda}^b g_{AG}(t,s)\,dt-\int_a^b w(t)g_{AG}(t,s)\,dt, & s\in\langle b-\lambda,b], \end{cases} \tag{6.41}$$

and in case $b-\lambda\leq a\leq b$,

$$K_n(s)=\begin{cases} \int_{b-\lambda}^b g_{AG}(t,s)\,dt, & s\in[b-\lambda,a] \\ \int_{b-\lambda}^b g_{AG}(t,s)\,dt-\int_a^b w(t)g_{AG}(t,s)\,dt, & s\in\langle a,b]. \end{cases} \tag{6.42}$$

Proof. First we change $a\leftrightarrow c$, $b\leftrightarrow d$ and $w\leftrightarrow u$ in Theorem 6.21. Then we take $c=b-\lambda$, $d=b$ and $u(t)=1$. □

For $n=1$ and $\lambda\leq b-a$, $K_1(s)$ becomes

$$K_1(s)=\begin{cases} \int_a^s w(t)dt, & s\in[a,b-\lambda] \\ b-s-\int_s^b w(t)dt, & s\in\langle b-\lambda,b]. \end{cases}$$

So, if $\int_a^s w(t)dt\geq 0$ for $a\leq s\leq b-\lambda$ and $\int_s^b w(t)dt\leq b-s$ for $b-\lambda<s\leq b$ and f nondecreasing from Theorem 6.23 we have

$$\int_{b-\lambda}^b f(t)dt-\lambda f(b-\lambda)\geq\int_a^b w(t)f(t)dt-f(b-\lambda)\int_a^b w(t)dt.$$

Hence, for $\lambda=\int_a^b w(t)dt$ we obtain the left-hand side of Steffensen's inequality for nondecreasing function f.

Now we will give Ostrowski type inequalities for previous results.

Theorem 6.24 *Suppose that all assumptions of Theorem 6.20 hold. Assume (p,q) is a pair of conjugate exponents. Let $\left|f^{(n)}\right|^p:[a,b]\cup[c,d]\to\mathbb{R}$ be an integrable function for some $n\geq 1$. Then we have*

$$\begin{aligned} &\left|\int_a^b w(t)f(t)\,dt-\int_c^d u(t)\,f(t)\,dt-T_{w,n}^{[a,b]}+T_{u,n}^{[c,d]}\right| \\ &\leq \left\|f^{(n)}\right\|_p\left(\int_a^{\max\{b,d\}}|K_n(s)|^q\,ds\right)^{\frac{1}{q}}. \end{aligned} \tag{6.43}$$

The constant $\left(\int_a^{\max\{b,d\}}|K_n(s)|^q\,ds\right)^{1/q}$ in the inequality (6.43) is sharp for $1<p\leq\infty$ and the best possible for $p=1$.

Proof. Similar to the proof of Theorem 6.5. □

Theorem 6.25 *Suppose that all assumptions of Theorem 6.20 for $c = a$ and $d = a+\lambda$ hold. Assume (p,q) is a pair of conjugate exponents. Let $\left|f^{(n)}\right|^p : [a,b]\cup[a,a+\lambda] \to \mathbb{R}$ be an integrable function for some $n \geq 1$. Let $K_n(s)$ be defined by (6.39) in case $a \leq a+\lambda \leq b$ and by (6.40) in case $a \leq b \leq a+\lambda$. Then we have*

$$\begin{aligned} &\left|\int_a^b w(t)f(t)\,dt - \int_a^{a+\lambda} f(t)\,dt - T_{w,n}^{[a,b]} + T_{1,n}^{[a,a+\lambda]}\right| \\ &\leq \left\|f^{(n)}\right\|_p \left(\int_a^{\max\{b,a+\lambda\}} |K_n(s)|^q\,ds\right)^{\frac{1}{q}}. \end{aligned} \tag{6.44}$$

The constant $\left(\int_a^{\max\{b,a+\lambda\}} |K_n(s)|^q\,ds\right)^{1/q}$ in the inequality (6.44) is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof. We take $c = a$, $d = a+\lambda$ and $u(t) = 1$ in Theorem 6.24. □

Theorem 6.26 *Suppose that all assumptions of Theorem 6.20 for $c = b-\lambda$ and $d = b$ hold. Assume (p,q) is a pair of conjugate exponents. Let $\left|f^{(n)}\right|^p : [a,b]\cup[b-\lambda,b] \to \mathbb{R}$ be an integrable function for some $n \geq 1$. Let $K_n(s)$ be defined by (6.41) in case $a \leq b-\lambda \leq b$ and by (6.42) in case $b-\lambda \leq a \leq b$. Then we have*

$$\begin{aligned} &\left|\int_{b-\lambda}^b f(t)\,dt - \int_a^b w(t)f(t)\,dt - T_{1,n}^{[b-\lambda,b]} + T_{w,n}^{[a,b]}\right| \\ &\leq \left\|f^{(n)}\right\|_p \left(\int_{\min\{a,b-\lambda\}}^b |K_n(s)|^q\,ds\right)^{\frac{1}{q}}. \end{aligned} \tag{6.45}$$

The constant $\left(\int_{\min\{a,b-\lambda\}}^b |K_n(s)|^q\,ds\right)^{1/q}$ in the inequality (6.45) is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof. First we change $a \leftrightarrow c$, $b \leftrightarrow d$ and $w \leftrightarrow u$ in Theorem 6.20 and then we take $c = b-\lambda$, $d = b$ and $u(t) = 1$. The rest of the proof is similar to the proof of Theorem 6.24. □

Chapter 7

Steffensen type inequalities for fractional integrals and derivatives

7.1 Fractional Steffensen type inequalities

Let us recall that M_k denotes the class of functions f defined on interval $[a,b]$ with the representation

$$f(x) = \int_a^b (x-t)_+^k d\nu(t),\, x \in [a,b],$$

for some nonnegative regular Borel measure ν. Results given in this Section are obtained by Krulić, Pečarić and Smoljak in [78].

Remark 7.1 Let $f \in C^n[a,b]$ be $n-$convex function with $f^{(k)}(a) = 0$, $k = 0,\dots,n-2$ and $f^{(n-1)} \geq 0$. Then $f \in M_{n-1}$. Hence, we can apply Theorem 3.17 on function $f \in M_{n-1}$. Furthermore, by replacing gdx by $d\sigma$ we obtain (3.81) where

$$\lambda = \left(n \int_a^b (x-a)^{n-1} d\sigma(x) \right)^{\frac{1}{n}}.$$

Following theorems give the Steffensen type inequality for the Riemann-Liouville fractional integral.

Theorem 7.1 *Let σ be a (signed) regular Borel measure such that $\int_a^b |d\sigma| < \infty$. Then*

$$\int_a^b I_{a+}^{\alpha} f(x) d\sigma(x) \geq I_{a+}^{\alpha+1} f(a+\lambda) \tag{7.1}$$

for all nonnegative functions f if and only if

$$\int_t^b (x-t)^{\alpha-1} d\sigma(x) \geq 0, \quad t \in [a,b] \tag{7.2}$$

and

$$\lambda \leq \min_{a \leq t \leq b} \left\{ t - a + \left(\alpha \int_t^b (x-t)^{\alpha-1} d\sigma(x) \right)^{\frac{1}{\alpha}} \right\}. \tag{7.3}$$

Proof. By the definition of class M_k, a function $g \in M_k$ has representation

$$g(x) = \int_a^b (x-t)_+^k d\nu(t) = \int_a^x (x-t)^k d\nu(t). \tag{7.4}$$

Putting $k = \alpha - 1$ and $d\nu(t) = \frac{f(t)dt}{\Gamma(\alpha)}$, (7.4) becomes

$$g(x) = \int_a^x (x-t)^{\alpha-1} \frac{f(t)dt}{\Gamma(\alpha)} = I_{a+}^{\alpha} f(x).$$

Hence, if $g \in M_k$, then g can be written as the Riemann-Liouville fractional integral of the nonnegative function f. Now we can apply Theorem 3.58 and get

$$\int_a^b I_{a+}^{\alpha} f(x) d\sigma(x) \geq \int_a^{a+\lambda} I_{a+}^{\alpha} f(x) dx \tag{7.5}$$

for all nonnegative functions f if and only if (7.2) and (7.3) hold. Changing the order of integration by Fubini's Theorem, the right-hand side in (7.5) can be written as

$$\begin{aligned}
\int_a^{a+\lambda} \left(\frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} dt \right) dx &= \frac{1}{\Gamma(\alpha)} \int_a^{a+\lambda} \left(f(t) \int_t^{a+\lambda} (x-t)^{\alpha-1} dx \right) dt \\
&= \frac{1}{\Gamma(\alpha+1)} \int_a^{a+\lambda} (a+\lambda-t)^{\alpha} f(t) dt \\
&= I_{a+}^{\alpha+1} f(a+\lambda).
\end{aligned}$$

□

Theorem 7.2 *If $\int_a^b |d\sigma| < \infty$, then the inequality*

$$\int_a^b I_{a+}^{\alpha} f(x) d\sigma(x) \leq I_{a+}^{\alpha+1} f(b) - I_{a+}^{\alpha+1} f(b-\lambda) \tag{7.6}$$

holds for all nonnegative functions f if and only if

$$\int_t^b (x-t)^{\alpha-1} d\sigma(x) \leq \frac{(b-t)^{\alpha}}{\alpha}, \quad t \in [a,b] \tag{7.7}$$

and

$$b-\lambda \leq \min_{a\leq t\leq b}\left\{t+\left[(b-t)^{\alpha}-\alpha\int_t^b (x-t)^{\alpha-1}d\sigma(x)\right]^{\frac{1}{\alpha}}\right\}. \tag{7.8}$$

Proof. Similar to the proof of Theorem 7.1, applying Theorem 3.59 for $k=\alpha-1$ and $d\nu(t)=\frac{f(t)dt}{\Gamma(\alpha)}$. □

Following theorems give the Steffensen type inequality for the Riemann-Liouville fractional derivative.

Theorem 7.3 *Let σ be a (signed) regular Borel measure such that $\int_a^b |d\sigma|<\infty$ and let assumptions in Lemma 1.1 be satisfied. Then*

$$\int_a^b D^{\alpha}_{a+}f(x)d\sigma(x)\geq \frac{1}{\Gamma(\beta-\alpha+1)}\int_a^{a+\lambda}(a+\lambda-y)^{\beta-\alpha}D^{\beta}_{a+}f(y)dy \tag{7.9}$$

for $f\in L^1(a,b)$ such that $D^{\beta}_{a+}f\geq 0$ if and only if

$$\int_t^b (x-t)^{\beta-\alpha-1}d\sigma(x)\geq 0, t\in[a,b] \tag{7.10}$$

and

$$\lambda \leq \min_{a\leq t\leq b}\left\{t-a+\left((\beta-\alpha)\int_t^b (x-t)^{\beta-\alpha-1}d\sigma(x)\right)^{\frac{1}{\beta-\alpha}}\right\}. \tag{7.11}$$

Proof. Similar to the proof of Theorem 7.1, applying Theorem 3.58 for $k=\beta-\alpha-1$ and $d\nu(t)=\frac{D^{\beta}_{a+}f(t)dt}{\Gamma(\beta-\alpha)}$. □

Theorem 7.4 *Let σ be a (signed) regular Borel measure such that $\int_a^b |d\sigma|<\infty$ and let assumptions in Lemma 1.1 be satisfied. Then*

$$\begin{aligned}\int_a^b D^{\alpha}_{a+}f(x)d\sigma(x)\leq \frac{1}{\Gamma(\beta-\alpha+1)}&\left[\int_a^b D^{\beta}_{a+}f(y)(b-y)^{\beta-\alpha}dy\right.\\ &\left.-\int_a^{b-\lambda} D^{\beta}_{a+}f(y)(b-\lambda-y)^{\beta-\alpha}dy\right]\end{aligned} \tag{7.12}$$

holds for $f\in L_1(a,b)$ such that $D^{\beta}_{a+}f\geq 0$ if and only if

$$\int_t^b (x-t)^{\beta-\alpha-1}d\sigma(x)\leq \frac{(b-t)^{\beta-\alpha}}{\beta-\alpha}, t\in[a,b] \tag{7.13}$$

and

$$b-\lambda \leq \min_{a\leq t\leq b}\left\{t+\left[(b-t)^{\beta-\alpha}-(\beta-\alpha)\int_t^b (x-t)^{\beta-\alpha-1}d\sigma(x)\right]^{\frac{1}{\beta-\alpha}}\right\}. \tag{7.14}$$

Proof. Similar to the proof of Theorem 7.1, applying Theorem 3.59 for $k = \beta - \alpha - 1$ and $d\nu(t) = \frac{D^{\beta}_{a+}f(t)dt}{\Gamma(\beta-\alpha)}$. □

Next we give Steffensen type inequalities for the Caputo fractional derivative.

Theorem 7.5 *Let σ be a (signed) regular Borel measure such that $\int_a^b |d\sigma| < \infty$. Then*

$$\int_a^b D^{\alpha}_{*a}f(x)d\sigma(x) \geq D^{\alpha+1}_{*a}f(a+\lambda) \tag{7.15}$$

for $g \in AC^n[a,b]$ such that $g^{(n)} \geq 0$ and $n = [\alpha]+1$ if and only if

$$\int_t^b (x-t)^{n-\alpha-1}d\sigma(x) \geq 0, t \in [a,b] \tag{7.16}$$

and

$$\lambda \leq \min_{a \leq t \leq b} \left\{ t - a + \left((n-\alpha) \int_t^b (x-t)^{n-\alpha-1} d\sigma(x) \right)^{\frac{1}{n-\alpha}} \right\}. \tag{7.17}$$

Proof. Similar to the proof of Theorem 7.1, applying Theorem 3.58 for $k = n - \alpha - 1$ and $d\nu(t) = \frac{f^{(n)}(t)dt}{\Gamma(n-\alpha)}$. □

Theorem 7.6 *If $\int_a^b |d\sigma| < \infty$, then the inequality*

$$\int_a^b D^{\alpha}_{*a}f(x)d\sigma(x) \leq D^{\alpha+1}_{*a}f(b) - D^{\alpha+1}_{*a}f(b-\lambda) \tag{7.18}$$

holds for $f \in AC^n[a,b]$ such that $f^{(n)} \geq 0$ and $n = [\alpha]+1$ if and only if

$$\int_t^b (x-t)^{n-\alpha-1}d\sigma(x) \leq \frac{(b-t)^{n-\alpha}}{n-\alpha}, t \in [a,b] \tag{7.19}$$

and

$$b - \lambda \leq \min_{a \leq t \leq b} \left\{ t + \left[(b-t)^{n-\alpha} - (n-\alpha) \int_t^b (x-t)^{n-\alpha-1} d\sigma(x) \right]^{\frac{1}{n-\alpha}} \right\}. \tag{7.20}$$

Proof. Similar to the proof of Theorem 7.1, applying Theorem 3.59 for $k = n - \alpha - 1$ and $d\nu(t) = \frac{f^{(n)}(t)dt}{\Gamma(n-\alpha)}$. □

Following theorems give the Steffensen type inequality for the Canavati fractional derivative.

Theorem 7.7 *Let σ be a (signed) regular Borel measure such that $\int_a^b |d\sigma| < \infty$ and let assumptions in Lemma 1.2 be satisfied. Then*

$$\int_a^b {}^{C_1}D_{a+}^{\alpha} f(x)d\sigma(x) \geq \frac{1}{\Gamma(\beta-\alpha+1)} \int_a^{a+\lambda} (a+\lambda-y)^{\beta-\alpha}\, {}^{C_1}D_{a+}^{\beta} f(y)dy \tag{7.21}$$

holds for $f \in C_{a+}^{\beta}[a,b]$ such that ${}^{C_1}D_{a+}^{\beta} f \geq 0$ if and only if (7.10) and (7.11) hold.

Theorem 7.8 *Let σ be a (signed) regular Borel measure such that $\int_a^b |d\sigma| < \infty$ and let assumptions in Lemma 1.2 be satisfied. Then*

$$\begin{aligned}\int_a^b {}^{C_1}D_{a+}^{\alpha} f(x)d\sigma(x) \leq \frac{1}{\Gamma(\beta-\alpha+1)} \Bigg[& \int_a^b {}^{C_1}D_{a+}^{\beta} f(y)(b-y)^{\beta-\alpha} dy \\ & - \int_a^{b-\lambda} {}^{C_1}D_{a+}^{\beta} f(y)(b-\lambda-y)^{\beta-\alpha} dy \Bigg]\end{aligned} \tag{7.22}$$

holds for $f \in C_{a+}^{\beta}[a,b]$ such that ${}^{C_1}D_{a+}^{\beta} f \geq 0$ if and only if (7.13) and (7.14) hold.

The proofs are similar to the proofs of Theorems 7.3 and 7.4.

Now we will define linear functionals, which will be used in the rest of this section, as the difference between the left-hand and the right-hand side of inequalities (7.1), (7.6), (7.9), (7.12), (7.15), (7.18) (7.21) and (7.22).

$$A_1(f) = \int_a^b I_{a+}^{\alpha} f(x)d\sigma(x) - I_{a+}^{\alpha+1} f(a+\lambda), \tag{7.23}$$

$$A_2(f) = I_{a+}^{\alpha+1} f(b) - I_{a+}^{\alpha+1} f(b-\lambda) - \int_a^b I_{a+}^{\alpha} f(x)d\sigma(x), \tag{7.24}$$

$$A_3(f) = \int_a^b D_{a+}^{\alpha} f(x)d\sigma(x) - \frac{1}{\Gamma(\beta-\alpha+1)} \int_a^{a+\lambda} (a+\lambda-y)^{\beta-\alpha} D_{a+}^{\beta} f(y)dy, \tag{7.25}$$

$$\begin{aligned}A_4(f) = & \frac{1}{\Gamma(\beta-\alpha+1)} \Bigg[\int_a^b D_{a+}^{\beta} f(y)(b-y)^{\beta-\alpha} dy \\ & - \int_a^{b-\lambda} D_{a+}^{\beta} f(y)(b-\lambda-y)^{\beta-\alpha} dy \Bigg] - \int_a^b D_{a+}^{\alpha} f(x)d\sigma(x),\end{aligned} \tag{7.26}$$

$$A_5(f) = \int_a^b D_{*a}^{\alpha} f(x)d\sigma(x) - D_{*a}^{\alpha+1} f(a+\lambda), \tag{7.27}$$

$$A_6(f) = D_{*a}^{\alpha+1} f(b) - D_{*a}^{\alpha+1} f(b-\lambda) - \int_a^b D_{*a}^{\alpha} f(x)d\sigma(x). \tag{7.28}$$

$$A_7(f) = \int_a^b {}^{C_1}D_{a+}^{\alpha} f(x)d\sigma(x) - \frac{1}{\Gamma(\beta-\alpha+1)} \int_a^{a+\lambda} (a+\lambda-y)^{\beta-\alpha}\, {}^{C_1}D_{a+}^{\beta} f(y)dy \quad (7.29)$$

$$\begin{aligned} A_8(f) = \frac{1}{\Gamma(\beta-\alpha+1)} &\left[\int_a^b {}^{C_1}D_{a+}^{\beta} f(y)(b-y)^{\beta-\alpha}dy \right. \\ &\left. - \int_a^{b-\lambda} {}^{C_1}D_{a+}^{\beta} f(y)(b-\lambda-y)^{\beta-\alpha}dy \right] - \int_a^b {}^{C_1}D_{a+}^{\alpha} f(x)d\sigma(x) \end{aligned} \quad (7.30)$$

Theorem 7.9 *Let σ be a (signed) regular Borel measure such that $\int_a^b |d\sigma| < \infty$ and $f \in C[a,b]$. Let (7.2) hold and λ be defined as in (7.3). Then there exists $\xi \in [a,b]$ such that*

$$A_1(f) = \frac{f(\xi)}{\Gamma(\alpha+1)} \left(\int_a^b (x-a)^{\alpha} d\sigma(x) - \frac{\lambda^{\alpha+1}}{\alpha+1} \right), \quad (7.31)$$

where A_1 is defined by (7.23).

Proof. Notice that from Theorem 7.1 we have that if $f \geq 0$, then $A_1(f) \geq 0$, so A_1 is a positive linear functional.

Set $m = \min_{x\in[a,b]} f(x)$, $M = \max_{x\in[a,b]} f(x)$. Then $A_1(M-f) \geq 0$ and $A_1(f-m) \geq 0$. Using the definition of Riemann-Liouville fractional integral, linear functional A_1 can be written as

$$A_1(f) = \frac{1}{\Gamma(\alpha)} \int_a^b \int_a^x f(t)(x-t)^{\alpha-1} dt d\sigma(x) - \frac{1}{\Gamma(\alpha+1)} \int_a^{a+\lambda} f(t)(a+\lambda-t)^{\alpha} dt.$$

Therefore

$$\begin{aligned} & m \left[\frac{1}{\Gamma(\alpha+1)} \left(\int_a^b (x-a)^{\alpha} d\sigma(x) - \frac{\lambda^{\alpha+1}}{\alpha+1} \right) \right] \\ & \leq \int_a^b I_{a+}^{\alpha} f(x) d\sigma(x) - \frac{1}{\Gamma(\alpha+1)} \int_a^{a+\lambda} (a+\lambda-t)^{\alpha} f(t) dt \\ & \leq M \left[\frac{1}{\Gamma(\alpha+1)} \left(\int_a^b (x-a)^{\alpha} d\sigma(x) - \frac{\lambda^{\alpha+1}}{\alpha+1} \right) \right], \end{aligned}$$

that is,

$$mA_1(1) \leq A_1(f) \leq MA_1(1).$$

If the function $A_1(1) = 0$, then $A_1(f) = 0$, so (7.31) holds for all $\xi \in [a,b]$. Otherwise,

$$\min_{x\in[a,b]} f(x) = m \leq \frac{A_1(f)}{A_1(1)} \leq M = \max_{x\in[a,b]} f(x), \text{ so } \frac{A_1(f)}{A_1(1)} \in f([a,b]).$$

Since f is continuous, by the classical Bolzano-Weierstrass theorem we have that $\frac{A_1(f)}{A_1(1)} = f(\xi)$ for some $\xi \in [a,b]$. □

Theorem 7.10 *Let σ be a (signed) regular Borel measure such that $\int_a^b |d\sigma| < \infty$ and $f \in C[a,b]$. Let (7.7) hold and λ be defined as in (7.8). Then there exists $\xi \in [a,b]$ such that*

$$A_2(f) = \frac{f(\xi)}{\Gamma(\alpha+1)}\left(\frac{(b-a)^{\alpha+1}-(b-\lambda-a)^{\alpha+1}}{\alpha+1} - \int_a^b (x-a)^{\alpha} d\sigma(x)\right),$$

where A_2 is defined by (7.24).

Proof. Similar to the proof of Theorem 7.9. □

Theorem 7.11 *Let σ be a (signed) regular Borel measure such that $\int_a^b |d\sigma| < \infty$, let assumptions in Lemma 1.1 be satisfied, and $D_{a+}^{\beta} f \in C[a,b]$. Let (7.10) hold and λ be defined as in (7.11). Then there exists $\xi \in [a,b]$ such that*

$$A_3(f) = \frac{D_{a+}^{\beta} f(\xi)}{\Gamma(\beta-\alpha+1)}\left(\int_a^b (x-a)^{\beta-\alpha} d\sigma(x) - \frac{\lambda^{\beta-\alpha+1}}{\beta-\alpha+1}\right),$$

where A_3 is defined by (7.25).

Proof. Put $m = \min\limits_{x\in[a,b]} D_{a+}^{\beta} f(x)$, $M = \max\limits_{x\in[a,b]} D_{a+}^{\beta} f(x)$. We define functions F_1 and F_2 by

$$F_1(t) = M\frac{(t-a)^{\beta}}{\Gamma(\beta+1)} - f(t), \quad F_2(t) = f(t) - m\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}.$$

Then $D_{a+}^{\beta} F_1(x) = M - D_{a+}^{\beta} f(x) \geq 0$, $D_{a+}^{\beta} F_2(x) = D_{a+}^{\beta} f(x) - m \geq 0$, so from Theorem 7.3 we have $A_3(F_1) \geq 0$ and $A_3(F_2) \geq 0$. Hence

$$\begin{aligned}
&\frac{m}{\Gamma(\beta-\alpha+1)}\left(\int_a^b (x-a)^{\beta-\alpha} d\sigma(x) - \frac{\lambda^{\beta-\alpha+1}}{\beta-\alpha+1}\right) \\
&\leq \int_a^b D_{a+}^{\alpha} f(x) d\sigma(x) - \frac{1}{\Gamma(\beta-\alpha+1)}\int_a^{a+\lambda} (a+\lambda-y)^{\beta-\alpha} D_{a+}^{\beta} f(y) dy \\
&\leq \frac{M}{\Gamma(\beta-\alpha+1)}\left(\int_a^b (x-a)^{\beta-\alpha} d\sigma(x) - \frac{\lambda^{\beta-\alpha+1}}{\beta-\alpha+1}\right),
\end{aligned}$$

that is,

$$mA_3(1) \leq A_3(g) \leq MA_3(1).$$

Similar reasoning as in proof of Theorem 7.9 completes the proof. □

Theorem 7.12 *Let σ be a (signed) regular Borel measure such that $\int_a^b |d\sigma| < \infty$ and let assumptions in Lemma 1.1 be satisfied and $D_{a+}^{\beta} f \in C[a,b]$. Let (7.13) hold and λ be* *defined as in (7.14). Then there exists $\xi \in [a,b]$ such that*

$$A_4(f) = \frac{D^{\beta}_{a+}f(\xi)}{\Gamma(\beta-\alpha+1)} \left(\frac{(b-a)^{\beta-\alpha+1} - (b-\lambda-a)^{\beta-\alpha+1}}{\beta-\alpha+1} \right. \\ \left. - \int_a^b (x-a)^{\beta-\alpha} d\sigma(x) \right),$$

where A_4 is defined by (7.26).

Proof. Similar to the proof of Theorem 7.11. □

Theorem 7.13 *Let σ be a (signed) regular Borel measure such that $\int_a^b |d\sigma| < \infty$ and $f \in AC^n[a,b]$. Let (7.16) hold and λ be defined as in (7.17). Then there exists $\xi \in [a,b]$ such that*

$$A_5(f) = \frac{f^{(n)}(\xi)}{\Gamma(n-\alpha+1)} \left(\int_a^b (x-a)^{n-\alpha} d\sigma(x) - \frac{\lambda^{n-\alpha+1}}{n-\alpha+1} \right),$$

where A_5 is defined by (7.27).

Proof. Set $m = \min_{x\in[a,b]} f^{(n)}(x)$, $M = \max_{x\in[a,b]} f^{(n)}(x)$. We define functions F_1 and F_2 by

$$F_1(x) = M\frac{x^n}{n!} - f(x), \quad F_2(x) = f(x) - m\frac{x^n}{n!}.$$

Then $F_1^{(n)}(x) = M - f^{(n)}(x) \geq 0$, $F_2(x) = f^{(n)}(x) - m \geq 0$, so from Theorem 7.5 we have that $A_5(F_1) \geq 0$ and $A_5(F_2) \geq 0$. Using definition of Caputo fractional derivative, linear functional A_5 can be written as

$$A_5(f) = \frac{1}{\Gamma(n-\alpha)} \int_a^b \int_a^x f^{(n)}(t)(x-t)^{n-\alpha-1} dt\, d\sigma(x) \\ - \frac{1}{\Gamma(n-\alpha+1)} \int_a^{a+\lambda} f^{(n)}(t)(a+\lambda-t)^{n-\alpha} dt.$$

Therefore

$$m\left[\frac{1}{\Gamma(n-\alpha+1)} \left(\int_a^b (x-a)^{n-\alpha} d\sigma(x) - \frac{\lambda^{n-\alpha+1}}{n-\alpha+1} \right)\right] \\ \leq \int_a^b D^{\alpha}_{*a} f(x) d\sigma(x) - D^{\alpha+1}_{*a} f(a+\lambda) \\ \leq M\left[\frac{1}{\Gamma(n-\alpha+1)} \left(\int_a^b (x-a)^{n-\alpha} d\sigma(x) - \frac{\lambda^{n-\alpha+1}}{n-\alpha+1} \right)\right],$$

that is,

$$mA_5(1) \leq A_5(f) \leq MA_5(1).$$

Similar reasoning as in proof of Theorem 7.9 completes the proof. □

Theorem 7.14 *Let σ be a (signed) regular Borel measure such that $\int_a^b |d\sigma| < \infty$ and $f \in AC^n[a,b]$. Let (7.19) hold and λ be defined as in (7.20). Then there exists $\xi \in [a,b]$ such that*

$$A_6(f) = \frac{f^{(n)}(\xi)}{\Gamma(n-\alpha+1)} \left(\frac{(b-a)^{n-\alpha+1} - (b-\lambda-a)^{n-\alpha+1}}{n-\alpha+1} - \int_a^b (x-a)^{n-\alpha} d\sigma(x) \right),$$

where A_6 is defined by (7.28).

Proof. Similar to the proof of Theorem 7.13. □

Theorem 7.15 *Let σ be a (signed) regular Borel measure such that $\int_a^b |d\sigma| < \infty$, let assumptions in Lemma 1.2 be satisfied, and ${}^{C_1}D^{\beta}_{a+}f \in C[a,b]$. Let (7.10) hold and λ be defined as in (7.11). Then there exists $\xi \in [a,b]$ such that*

$$A_7(f) = \frac{{}^{C_1}D^{\beta}_{a+}f(\xi)}{\Gamma(\beta-\alpha+1)} \left(\int_a^b (x-a)^{\beta-\alpha} d\sigma(x) - \frac{\lambda^{\beta-\alpha+1}}{\beta-\alpha+1} \right),$$

where A_7 is defined by (7.29).

Proof. Similar to the proof of Theorem 7.11. □

Theorem 7.16 *Let σ be a (signed) regular Borel measure such that $\int_a^b |d\sigma| < \infty$, let assumptions in Lemma 1.2 be satisfied, and ${}^{C_1}D^{\beta}_{a+}f \in C[a,b]$. Let (7.13) hold and λ be defined as in (7.14). Then there exists $\xi \in [a,b]$ such that*

$$A_8(f) = \frac{{}^{C_1}D^{\beta}_{a+}f(\xi)}{\Gamma(\beta-\alpha+1)} \left(\frac{(b-a)^{\beta-\alpha+1} - (b-\lambda-a)^{\beta-\alpha+1}}{\beta-\alpha+1} \right.$$
$$\left. - \int_a^b (x-a)^{\beta-\alpha} d\sigma(x) \right),$$

where A_8 is defined by (7.30).

Proof. Similar to the proof of Theorem 7.11. □

Theorem 7.17 *Let σ be a (signed) regular Borel measure such that $\int_a^b |d\sigma| < \infty$. Let A_1 be positive linear functional defined by (7.23), let (7.2) hold and λ be defined as in (7.3) or let A_2 be positive linear functional defined by (7.24), let (7.7) hold and λ be defined as in (7.8). Let $f_1, f_2 \in C[a,b]$ be such that $f_2(x) \neq 0$ for every $x \in [a,b]$. Then there exists $\xi_i \in [a,b]$ such that*

$$\frac{f_1(\xi_i)}{f_2(\xi_i)} = \frac{A_i(f_1)}{A_i(f_2)}, \quad i = 1,2. \tag{7.32}$$

Proof. Set $\Phi(t) = f_1(t)A_i(f_2) - f_2(t)A_i(f_1)$, $i = 1,2$. Obviously, $A_i(\Phi) = 0$. On the other hand, Theorems 7.31 and 7.32 yield that there exist $\xi_i \in [a,b]$ such that

$$A_i(\Phi) = \Phi(\xi_i) \cdot A_i(1), \quad i = 1,2.$$

Since $A_i(1) \neq 0$, we have that

$$\Phi(\xi_i) = f_1(\xi_i)A_i(f_2) - f_2(\xi_i)A_i(f_1) = 0, \quad i = 1,2.$$

By assumption $f_2(\xi_i) \neq 0$, Theorems 7.31 and 7.32 assure that $A_i(f_2) \neq 0$, $i = 1,2$. Thus, (7.32) follows. □

Remark 7.2 Theorem 7.17 enables us to define new types of means, because if f_1/f_2 has an inverse, from (7.32) we conclude

$$\xi_i = \left(\frac{f_1}{f_2}\right)^{-1}\left(\frac{A_i(f_1)}{A_i(f_2)}\right), \quad i = 1,2.$$

The following three theorems have the proofs which are similar to the proof of Theorem 7.17 and, here, we only write its statements.

Theorem 7.18 *Let σ be a (signed) regular Borel measure such that $\int_a^b |d\sigma| < \infty$ and let assumptions in Lemma 1.1 be satisfied. Let A_3 be positive linear functional defined by (7.25), let (7.10) hold and λ be defined as in (7.11) or let A_4 be positive linear functional defined by (7.26), let (7.13) hold and λ be defined as in (7.14). Let $D^\beta_{a+}f_1, D^\beta_{a+}f_2 \in C[a,b]$ be such that $(D^\beta_a f_2)(x) \neq 0$ for every $x \in [a,b]$. Then there exists $\xi_i \in [a,b]$ such that*

$$\frac{D^\beta_{a+}f_1(\xi_i)}{D^\beta_{a+}f_2(\xi_i)} = \frac{A_i(f_1)}{A_i(f_2)}, \quad i = 3,4. \tag{7.33}$$

Remark 7.3 Theorem 7.18 enables us to define new types of means, because if $D^\beta_{a+}f_1/D^\beta_{a+}f_2$ has an inverse, from (7.33) we conclude

$$\xi_i = \left(\frac{D^\beta_{a+}f_1}{D^\beta_{a+}f_2}\right)^{-1}\left(\frac{A_i(f_1)}{A_i(f_2)}\right), \quad i = 3,4.$$

Theorem 7.19 *Let σ be a (signed) regular Borel measure such that $\int_a^b |d\sigma| < \infty$. Let A_5 be positive linear functional defined by (7.27), let (7.16) hold and λ be defined as in (7.17) or let A_6 be positive linear functional defined by (7.28), let (7.19) hold and λ be defined as in (7.20). Let $f_1, f_2 \in AC^n[a,b]$ be such that $f_2^{(n)}(x) \neq 0$ for every $x \in [a,b]$. Then there exists $\xi_i \in [a,b]$ such that*

$$\frac{f_1^{(n)}(\xi_i)}{f_2^{(n)}(\xi_i)} = \frac{A_i(f_1)}{A_i(f_2)}, \quad i = 5,6. \tag{7.34}$$

Remark 7.4 Theorem 7.19 enables us to define new types of means, because if $f_1^{(n)}/f_2^{(n)}$ has an inverse, from (7.34) we conclude

$$\xi_i = \left(\frac{f_1^{(n)}}{f_2^{(n)}}\right)^{-1}\left(\frac{A_i(f_1)}{A_i(f_2)}\right), \quad i = 5,6.$$

Theorem 7.20 *Let σ be a (signed) regular Borel measure such that $\int_a^b |d\sigma| < \infty$ and let assumptions in Lemma 1.2 be satisfied. Let A_7 be positive linear functional defined by (7.29), let (7.10) hold and λ be defined as in (7.11) or let A_8 be positive linear functional defined by (7.30), let (7.13) hold and λ be defined as in (7.14). Let ${}^{C_1}D_{a+}^{\beta}f_1, {}^{C_1}D_{a+}^{\beta}f_2 \in C[a,b]$ be such that ${}^{C_1}D_{a+}^{\beta}f_2(x) \neq 0$ for every $x \in [a,b]$. Then there exists $\xi_i \in [a,b]$ such that*

$$\frac{{}^{C_1}D_{a+}^{\beta}f_1(\xi_i)}{{}^{C_1}D_{a+}^{\beta}f_2(\xi_i)} = \frac{A_i(f_1)}{A_i(f_2)}, \quad i = 7,8. \tag{7.35}$$

Remark 7.5 Theorem 7.20 enables us to define new types of means, because if ${}^{C_1}D_{a+}^{\beta}f_1/{}^{C_1}D_{a+}^{\beta}f_2$ has an inverse, from (7.35) we conclude

$$\xi_i = \left(\frac{{}^{C_1}D_{a+}^{\beta}f_1}{{}^{C_1}D_{a+}^{\beta}f_2}\right)^{-1}\left(\frac{A_i(f_1)}{A_i(f_2)}\right), \quad i = 7,8.$$

Theorems 7.9 and 7.10 enable us to define various types of means, because if f has inverse we have

$$\xi = f^{-1}\left(\frac{A_i(f)}{A_i(1)}\right) \in [a,b], \quad i = 1,2 \tag{7.36}$$

for A_i defined by (7.23) and (7.24). Specially, let $g : [a,b] \to \mathbb{R}^+$ be continuous function and let $B_i : C[a,b] \to \mathbb{R}$, $i = 1,2$ be normalized positive functionals defined by

$$B_i(\Psi(g)) = \frac{A_i(\Psi(g))}{A_i(1)}, \quad i = 1,2. \tag{7.37}$$

After that we apply (7.36) on function $f(x) = x^r$, $r \neq 0$, we get a functional power mean:

$$M^{[r]}(g, B_i) = \begin{cases} (B_i(g^r))^{1/r}, & r \neq 0 \\ \exp\left(B_i(\log g)\right), & r = 0, \end{cases} \qquad i = 1,2.$$

Theorem 7.21 *Let (7.2) hold and λ be defined as in (7.3) (or let (7.7) hold and λ be defined as in (7.8)). Let $g : [a,b] \to \mathbb{R}^+$ be continuous function and let A_i be linear functional on a vector space of all real, nonnegative and continuous functions on $g([a,b])$ defined by (7.23) (or (7.24)).*

(i) The mapping $t \mapsto A_i(g^t)$ is exponentially convex.

(ii) Let $n \in \mathbb{N}$ and let $t_1, \dots, t_n \in \mathbb{R}$ be arbitrary. Then the matrix $\left[A_i\left(g^{\frac{t_j+t_k}{2}}\right)\right]_{j,k=1}^n$ is a positive semi-definite matrix. Particularly

$$\det\left[A_i\left(g^{\frac{t_j+t_k}{2}}\right)\right]_{j,k=1}^n \geq 0.$$

Proof. For fixed $n \in \mathbb{N}, u_1, \dots, u_n \in \mathbb{R}$ let us consider the function

$$\Psi(x) = \sum_{j,k=1}^n u_j u_k x^{\frac{t_j+t_k}{2}}.$$

Since $\Psi(x) = \left(\sum_{j=1}^n u_j x^{\frac{t_j}{2}}\right)^2 \geq 0$, from Theorem 7.1 we have $A_i(\Psi(g)) \geq 0$, that is

$$\sum_{j,k=1}^n u_j u_k A_i\left(g^{\frac{t_j+t_k}{2}}\right) \geq 0.$$

It is obvious that $t \mapsto A_i(g^t)$ is continuous, so (i) and (ii) follow from Proposition 1.2 and Corollary 1.2. $\square$

Corollary 7.1 *Let $g : [a,b] \to \mathbb{R}^+$ be a continuous function and let A_i be linear functional on a vector space of all real, nonnegative and continuous functions on $g([a,b])$ defined by (7.23) or (7.24). Then*

$$A_i(g^s)^{t-r} \leq A_i(g^r)^{t-s} A_i(g^t)^{s-r}, \quad i = 1,2 \quad za \quad r < s < t. \tag{7.38}$$

Proof. From Theorem 7.21 it follows that $t \mapsto A_1(g^t)$ and $t \mapsto A_2(g^t)$ are exponentially convex functions and hence they are log-convex functions. If $A_i(g^s)$ and $A_i(g^r)$ are greater than zero, then from Corollary 1.1 we have

$$\left(\frac{A_i(g^s)}{A_i(g^r)}\right)^{\frac{1}{s-r}} \leq \left(\frac{A_i(g^t)}{A_i(g^s)}\right)^{\frac{1}{t-s}}, \quad i = 1,2$$

which validates (7.38). If $A_i(g^s) = 0$, then (7.38) obviously holds. If $A_i(g^r) = 0$, then using log-convexity of $t \mapsto A_i(g^t)$ we have

$$(A_i(g^s))^2 \leq A_i(g^{2s-r}) A_i(g^r) = 0, \quad i = 1,2$$

concluding $A_i(g^s) = 0$ and again (7.38) is valid. $\square$

Corollary 7.2 *Let g be a positive continuous function on $[a,b]$ and B_i, $i = 1,2$ be positive normalized linear functionals on a vector space of all real, nonnegative and continuous functions on $[a,b]$ defined by (7.37). Then for all $p,q \in \mathbb{R}$, $p < q$*

$$M^{[p]}(g, B_i) \leq M^{[q]}(g, B_i), \quad i = 1,2. \tag{7.39}$$

Proof. The proof is deduced observing three different cases for p and q in (7.39).
(*Case I.*) $0 < p < q$. If we put $r = 0,\ s = p,\ t = q$ in (7.38) we get

$$B_i(g^p)^q \le B_i(g^q)^p$$

and after we raise both sides of this inequality to the power $1/pq$ we get (7.39).
(*Case II.*) $p < 0 < q$. In this case we put $r = p,\ s = 0,\ t = q$ in (7.38) and we get

$$B_i(g^q)^p \le B_i(g^p)^q.$$

Raising both sides of this inequality to the power $1/pq$ we get (7.39).
(*Case III.*) $p < q < 0$. In this case we put $r = p,\ s = q,\ t = 0$ in (7.38) and we get

$$B_i(g^q)^{-p} \le B_i(g^p)^{-q}.$$

Raising both sides of this inequality to the power $-1/pq$ we get (7.39).
Mapping $p \mapsto M^{[p]}(g, B_i)$ is continuous in zero, so (7.39) is valid for all $p, q \in \mathbb{R},\ p < q$. □

7.2 Generalized fractional Steffensen type inequalities

Results given in this Section are obtained by Pečarić, Perić and Smoljak in [114]. Let $(\Omega_1, \Sigma_1, \mu_1)$ be a measure space with $\sigma-$finite (signed) regular Borel measure and Ω_2 be a set. Let $K : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a nonnegative function and let U denote the class of all functions $f : \Omega_1 \to \mathbb{R}$ such that there exists a measure space $(\Omega_2, \Sigma_2, \mu_2)$ such that μ_2 is nonnegative $\sigma-$finite regular Borel measure and

$$f(x) = \int_{\Omega_2} K(x,y)\, d\mu_2(y), \quad x \in \Omega_1. \tag{7.40}$$

Theorem 7.22 *Let $(\Omega_1, \Sigma_1, \mu_1)$ be a measure space with $\sigma-$finite (signed) regular Borel measure. Then for every $f \in U$*

$$\int_{\Omega_1} f(x)\, d\mu_1(x) \ge 0 \tag{7.41}$$

if and only if

$$\int_{\Omega_1} K(x,y)\, d\mu_1(x) \ge 0 \quad \textit{for } y \in \Omega_2. \tag{7.42}$$

Proof. Using the representation (7.40) in (7.41), and then using Fubini's theorem, (7.41) is equivalent to

$$\int_{\Omega_2} \int_{\Omega_1} K(x,y)\, d\mu_1(x)\, d\mu_2(y) \ge 0. \tag{7.43}$$

Since μ_2 is arbitrary nonnegative regular Borel measure, (7.43) holds if and only if (7.42) holds. □

Theorem 7.23 *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_1, \Sigma_3, \mu_3)$ be measure spaces with $\sigma-$finite (signed) regular Borel measures. Then for every $f \in U$*

$$\int_{\Omega_1} f(x)d\mu_1(x) \geq \int_{\Omega_1} f(x)d\mu_3(x) \tag{7.44}$$

if and only if

$$\int_{\Omega_1} K(x,y)d\mu_1(x) \geq \int_{\Omega_1} K(x,y)d\mu_3(x) \quad \text{for } y \in \Omega_2. \tag{7.45}$$

Proof. Apply Theorem 7.22 on measure μ_1 replaced by $\mu_1 - \mu_3$. □

Remark 7.6 Let $\Omega_1 = \Omega_2 = [a,b]$, $K(x,t) = (x-t)_+^k$, $d\mu_3(x) = \chi_{[a,a+\lambda]}dx$ for nonnegative λ such that $a+\lambda \leq b$ and $d\mu_1(x) = d\sigma(x)$ for some finite (signed) regular Borel measure σ. Then the class U reduces to M_k and (7.44) reduces to (3.81). Furthermore, the condition (7.45) reduces to

$$\int_a^b (x-t)_+^k d\sigma(x) \geq \int_a^{a+\lambda} (x-t)_+^k dx. \tag{7.46}$$

Since the right hand side in (7.46) is nonnegative, (3.82) is necessary. Moreover, from (7.46) we have (3.83) for $a \leq t \leq a+\lambda$. Since (3.82) holds, (3.83) is also true for $t \geq a+\lambda$. Hence, considering the class of functions $f \in M_k$ and finite (signed) regular Borel measure σ, Theorem 7.23 reduces to the Steffensen type inequality given in Theorem 3.58.

Remark 7.7 Let $\Omega_1 = \Omega_2 = [a,b]$, $K(x,t) = (x-t)_+^k$, $d\mu_1(x) = \chi_{[b-\lambda,b]}dx$ for λ nonnegative such that $a \leq b-\lambda$ and $d\mu_3(x) = d\sigma(x)$ for some finite (signed) regular Borel measure σ. Then the class U reduces to M_k and (7.44) reduces to (3.86). Furthermore, the condition (7.45) reduces to

$$\int_a^b (x-t)_+^k d\sigma(x) \leq \int_{b-\lambda}^b (x-t)_+^k dx. \tag{7.47}$$

For $t > b-\lambda$, from (7.47) we have

$$\int_a^b (x-t)_+^k d\sigma(x) \leq \frac{(b-t)^{k+1}}{k+1}. \tag{7.48}$$

Obviously, (7.48) also holds for $t \leq b-\lambda$, so (3.87) is necessary. Moreover, from (7.47) we have (3.88) for $a \leq t \leq b-\lambda$. But since (3.87) holds, (3.88) is also true for $t \geq b-\lambda$. Hence, considering the class of functions $f \in M_k$ and finite (signed) regular Borel measure μ, Theorem 7.23 reduces to the Steffensen type inequality given in Theorem 3.59.

Remark 7.8 Applying Theorem 7.23 with $\Omega_1 = \Omega_2 = [a,b]$, $d\mu_2(y) = f(y)dy$, $d\mu_3(x) = \chi_{[a,a+\lambda]}dx$ (or $d\mu_1(x) = \chi_{[b-\lambda,b]}dx$) for λ nonnegative such that $a+\lambda \leq b$ (or $a \leq b-\lambda$), and

$$K(x,y) = \begin{cases} \frac{(x-y)^{\alpha-1}}{\Gamma(\alpha)}, & a \leq y \leq x \\ 0, & x < y \leq b, \end{cases}$$

we obtain Steffensen type inequalities for the left-sided fractional integral $I_{a+}^{\alpha} f$ given in Theorems 7.1 and 7.2.

Remark 7.9 Let assumptions in Lemma 1.1 be satisfied. Then, applying Theorem 7.23 with $\Omega_1 = \Omega_2 = [a,b]$, $d\mu_2(y) = D^{\beta}_{a+}f(y)dy$, $d\mu_3(x) = \chi_{[a,a+\lambda]}dx$ (or $d\mu_1(x) = \chi_{[b-\lambda,b]}dx$) for λ nonnegative such that $a+\lambda \leq b$ (or $a \leq b-\lambda$) and

$$K(x,y) = \begin{cases} \frac{(x-y)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}, & a \leq y \leq x \\ 0, & x < y \leq b, \end{cases} \tag{7.49}$$

we obtain Steffensen type inequalities for the generalized Riemann-Liouville fractional derivative $D^{\alpha}_{a+}f$ given in Theorems 7.3 and 7.4.

Remark 7.10 Applying Theorem 7.23 with $\Omega_1 = \Omega_2 = [a,b]$, $d\mu_2(y) = f^{(n)}(y)dy$, $d\mu_3(x) = \chi_{[a,a+\lambda]}dx$ (or $d\mu_1(x) = \chi_{[b-\lambda,b]}dx$) for λ nonnegative such that $a+\lambda \leq b$ (or $a \leq b-\lambda$) and

$$K(x,y) = \begin{cases} \frac{(x-y)^{n-\alpha-1}}{\Gamma(n-\alpha)}, & a \leq y \leq x \\ 0, & x < y \leq b, \end{cases}$$

we obtain Steffensen type inequalities for the Caputo fractional derivative $D^{\alpha}_{*a}g$ given in Theorems 7.5 and 7.6.

Remark 7.11 Let assumptions in Lemma 1.2 be satisfied. Then, applying Theorem 7.23 with $\Omega_1 = \Omega_2 = [a,b]$, $d\mu_2(y) = {}^{C_1}D^{\beta}_{a+}f(y)dy$, $d\mu_3(x) = \chi_{[a,a+\lambda]}dx$ (or $d\mu_1(x) = \chi_{[b-\lambda,b]}dx$) for λ nonnegative such that $a+\lambda \leq b$ (or $a \leq b-\lambda$) and K defined by (7.49) we obtain Steffensen type inequalities for generalized Canavati fractional derivative ${}^{C_1}D^{\alpha}_{a+}f$ given in Theorems 7.7 and 7.8.

Following theorems give the Steffensen type inequality for the fractional integral of a function f with respect to another function g.

Theorem 7.24 *Let g be an increasing function on (a,b) such that g' is continuous on (a,b), let μ be $\sigma-$finite (signed) regular Borel measure on $[a,b]$. Then for every nonnegative Borel measurable function f_1*

$$\int_a^b I^{\alpha}_{a+;g}f_1(x)d\mu(x) \geq \frac{1}{\Gamma(\alpha)}\int_a^{a+\lambda} g'(y)f_1(y)\int_y^{a+\lambda}(g(x)-g(y))^{\alpha-1}dxdy \tag{7.50}$$

if and only if

$$\int_y^b (g(x)-g(y))^{\alpha-1}d\mu(x) \geq 0, \quad y \in [a,b] \tag{7.51}$$

and

$$\int_y^{a+\lambda}(g(x)-g(y))^{\alpha-1}dx \leq \int_y^b (g(x)-g(y))^{\alpha-1}d\mu(x). \tag{7.52}$$

Proof. Let (7.50) holds. Let $\Omega_1 = \Omega_2 = [a,b]$, λ be nonnegative real number such that $a+\lambda \leq b$,

$$K(x,y) = \begin{cases} \frac{1}{\Gamma(\alpha)}\frac{g'(y)}{(g(x)-g(y))^{1-\alpha}}, & a \leq y \leq x \\ 0, & x < y \leq b, \end{cases} \tag{7.53}$$

$d\mu_1(x) = d\mu(x)$, $d\mu_2(y) = f_1(y)dy$ and $d\mu_3(x) = \chi_{[a,a+\lambda]}dx$. Notice that class U now reduces to class of functions $I^{\alpha}_{a+;g}f_1$ and that (7.44) reduces to (7.50). Now, from Theorem 7.23 it follows that (7.45) holds. Furthermore, (7.45) reduces to

$$\int_a^b K(x,y)d\mu(x) \geq \int_a^{a+\lambda} K(x,y)dx. \tag{7.54}$$

Since the right-hand side in (7.54) is nonnegative, (7.51) is necessary. Now taking $a \leq y \leq b$, (7.54) is

$$\int_y^b (g(x)-g(y))^{\alpha-1}d\mu(x) \geq \int_y^{a+\lambda} (g(x)-g(y))^{\alpha-1}dx, \quad a \leq y \leq a+\lambda. \tag{7.55}$$

But since (7.51) holds, the inequality (7.55) is true for $y > a+\lambda$. Hence, (7.52) follows.

Conversely, let (7.51) hold and λ be such that (7.52) holds. As above, we see that (7.51) and (7.52) are obtained from (7.45). Now applying Theorem 7.23 it follows that (7.44) holds. Furthermore, from (7.44) we have

$$\int_a^b \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(y)f_1(y)}{(g(x)-g(y))^{1-\alpha}}dy\,d\mu(x) \geq \int_a^{a+\lambda} \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(y)f_1(y)}{(g(x)-g(y))^{1-\alpha}}dy\,dx. \tag{7.56}$$

Using Fubini's theorem, the right-hand side in (7.56) can be written as

$$\frac{1}{\Gamma(\alpha)} \int_a^{a+\lambda} g'(y)f_1(y) \int_y^{a+\lambda} (g(x)-g(y))^{\alpha-1}dx\,dy.$$

So we obtain (7.50). Hence, the proof is completed. □

Theorem 7.25 *Let g be an increasing function on (a,b) such that g' is continuous on (a,b), let μ be $\sigma-$finite (signed) regular Borel measure on $[a,b]$. Then for every nonnegative Borel measurable function f_1*

$$\int_a^b I^{\alpha}_{a+;g}f_1(x)d\mu(x) \leq \frac{1}{\Gamma(\alpha)} \left[\int_a^{b-\lambda} g'(y)f_1(y) \int_{b-\lambda}^b (g(x)-g(y))^{\alpha-1}dx\,dy \right.$$
$$\left. + \int_{b-\lambda}^b g'(y)f_1(y) \int_y^b (g(x)-g(y))^{\alpha-1}dx\,dy \right]$$

if and only if

$$\int_y^b (g(x)-g(y))^{\alpha-1}d\mu(x) \leq \int_{b-\lambda}^b \chi_{[y,b]}(g(x)-g(y))^{\alpha-1}dx, \quad y \in [a,b]. \tag{7.57}$$

Proof. Similar to the proof of Theorem 7.24, applying Theorem 7.23 for $\Omega_1 = \Omega_2 = [a,b]$, $d\mu_1(x) = \chi_{[b-\lambda,b]}dx$, $d\mu_2(y) = f_1(y)dy$, $d\mu_3(x) = d\mu(x)$ and K defined by (7.53). □

Following theorems give the Steffensen type inequality for the Hadamard fractional integral.

Corollary 7.3 *Let μ be $\sigma-$finite (signed) regular Borel measure on $[a,b]$. Then for every nonnegative Borel measurable function f_1*

$$\int_a^b J_{a+}^{\alpha} f_1(x)d\mu(x) \geq \frac{1}{\Gamma(\alpha)} \int_a^{a+\lambda} f_1(y) \int_0^{\log\frac{a+\lambda}{y}} x^{\alpha-1}e^x dx dy$$

if and only if

$$\int_y^b \left(\log\frac{x}{y}\right)^{\alpha-1} d\mu(x) \geq 0, \quad y\in[a,b] \tag{7.58}$$

and

$$\int_y^b \left(\log\frac{x}{y}\right)^{\alpha-1} d\mu(x) \geq y\int_0^{\log\frac{a+\lambda}{y}} x^{\alpha-1}e^x dx. \tag{7.59}$$

Proof. Apply Theorem 7.24 for $g(x)=\log x$. □

Corollary 7.4 *Let μ be $\sigma-$finite (signed) regular Borel measure on $[a,b]$. Then for every nonnegative Borel measurable function f_1*

$$\int_a^b J_{a+}^{\alpha} f_1(x)d\mu(x) \leq \frac{1}{\Gamma(\alpha)} \left[\int_a^{b-\lambda} f_1(y) \int_{\log\frac{b-\lambda}{y}}^{\log\frac{b}{y}} x^{\alpha-1}e^x dx dy \right.$$
$$\left. + \int_{b-\lambda}^b f_1(y) \int_0^{\log\frac{b}{y}} x^{\alpha-1}e^x dx dy\right]$$

if and only if

$$\int_y^b \left(\log\frac{x}{y}\right)^{\alpha-1} d\mu(x) \leq y\int_{\log\frac{b-\lambda}{y}}^{\log\frac{b}{y}} \chi_{[0,\log\frac{b}{y}]} x^{\alpha-1}e^x dx, \quad y\in[a,b]. \tag{7.60}$$

Proof. Apply Theorem 7.25 for $g(x)=\log x$. □

Following theorems give the Steffensen type inequality for the Erdély-Kober fractional integral.

Theorem 7.26 *Let μ be $\sigma-$finite (signed) regular Borel measure on $[a,b]$. Then for every nonnegative Borel measurable function f_1*

$$\int_a^b I_{a+;\sigma;\eta}^{\alpha} f_1(x)d\mu(x) \geq \frac{1}{\Gamma(\alpha)} \int_a^{a+\lambda} y^{\sigma\eta+\sigma-1} f_1(y) \int_{y^\sigma}^{(a+\lambda)^\sigma} \frac{x^{-\alpha-\eta-1+\frac{1}{\sigma}}}{(x-y^\sigma)^{1-\alpha}} dx dy \tag{7.61}$$

if and only if

$$\int_y^b \frac{x^{-\sigma(\alpha+\eta)}}{(x^\sigma-y^\sigma)^{1-\alpha}} d\mu(x) \geq 0, \quad y\in[a,b] \tag{7.62}$$

and

$$\int_y^b \frac{x^{-\sigma(\alpha+\eta)}}{(x^\sigma-y^\sigma)^{1-\alpha}} d\mu(x) \geq \frac{1}{\sigma}\int_{y^\sigma}^{(a+\lambda)^\sigma} \frac{x^{-\alpha-\eta-1+\frac{1}{\sigma}}}{(x-y^\sigma)^{1-\alpha}} dx. \tag{7.63}$$

Proof. Let (7.61) hold. Let $\Omega_1 = \Omega_2 = [a,b]$, λ be nonnegative real number such that $a+\lambda \leq b$,

$$K(x,y) = \begin{cases} \frac{1}{\Gamma(\alpha)} \frac{\sigma x^{-\sigma(\alpha+\eta)} y^{\sigma\eta+\sigma-1}}{(x^\sigma - y^\sigma)^{1-\alpha}}, & a \leq y \leq x \\ 0, & x < y \leq b, \end{cases} \tag{7.64}$$

$d\mu_1(x) = d\mu(x)$, $d\mu_2(y) = f_1(y)dy$ and $d\mu_3(x) = \chi_{[a,a+\lambda]}dx$. Notice that class U now reduces to class of functions $I^\alpha_{a+;\sigma;\eta} f_1$ and that (7.44) reduces to (7.61). Now, from Theorem 7.23 it follows that (7.45) holds. Furthermore, (7.45) reduces to

$$\int_a^b K(x,y)d\mu(x) \geq \int_a^{a+\lambda} K(x,y)dx. \tag{7.65}$$

Since the right-hand side in (7.65) is nonnegative, (7.62) is necessary. Now taking $a \leq y \leq b$, (7.65) is

$$\int_y^b \frac{x^{-\sigma(\alpha+\eta)}}{(x^\sigma - y^\sigma)^{1-\alpha}} d\mu(x) \geq \int_y^{a+\lambda} \frac{x^{-\sigma(\alpha+\eta)}}{(x^\sigma - y^\sigma)^{1-\alpha}} dx, \quad a \leq y \leq a+\lambda. \tag{7.66}$$

But since (7.62) holds, the inequality (7.66) is true for $y > a+\lambda$. Calculating integral on the right-hand side in (7.66), we obtain (7.63).

Conversely, let (7.62) hold and λ be such that (7.63) holds. As above, we see that (7.62) and (7.63) are obtained from (7.45). Now applying Theorem 7.23 it follows that (7.44) holds. Furthermore, from (7.44) we have

$$\begin{aligned} &\int_a^b \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\sigma x^{-\sigma(\alpha+\eta)} y^{\sigma\eta+\sigma-1}}{(x^\sigma - y^\sigma)^{1-\alpha}} f_1(y)dy\,d\mu(x) \geq \\ &\quad \int_a^{a+\lambda} \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\sigma x^{-\sigma(\alpha+\eta)} y^{\sigma\eta+\sigma-1}}{(x^\sigma - y^\sigma)^{1-\alpha}} f_1(y)dy\,dx. \end{aligned} \tag{7.67}$$

Using Fubini's theorem, the right-hand side in (7.67) can be written as

$$\frac{1}{\Gamma(\alpha)} \int_a^{a+\lambda} \sigma y^{\sigma\eta+\sigma-1} f_1(y) \int_y^{a+\lambda} \frac{x^{-\sigma(\alpha+\eta)}}{(x^\sigma - y^\sigma)^{1-\alpha}} dx\,dy. \tag{7.68}$$

Now using the definition of Erdelyi-Köber fractional integral and calculating the inner integral in (7.68) we obtain (7.61). □

Theorem 7.27 *Let μ be $\sigma-$finite (signed) regular Borel measure on $[a,b]$. Then for every nonnegative Borel measurable function f_1*

$$\begin{aligned} \int_a^b I^\alpha_{a+;\sigma;\eta} f_1(x)d\mu(x) \leq \frac{1}{\Gamma(\alpha)} \Bigg[&\int_a^{b-\lambda} y^{\sigma\eta+\sigma-1} f_1(y) \int_{(b-\lambda)^\sigma}^{b^\sigma} \frac{x^{-\alpha-\eta-1+\frac{1}{\sigma}}}{(x - y^\sigma)^{1-\alpha}} dx\,dy \\ &+ \int_{b-\lambda}^b y^{\sigma\eta+\sigma-1} f_1(y) \int_{y^\sigma}^{b^\sigma} \frac{x^{-\alpha-\eta-1+\frac{1}{\sigma}}}{(x - y^\sigma)^{1-\alpha}} dx\,dy \Bigg] \end{aligned}$$

if and only if

$$\int_y^b \frac{x^{-\sigma(\alpha+\eta)}}{(x^\sigma - y^\sigma)^{1-\alpha}} d\mu(x) \leq \int_{(b-\lambda)^\sigma}^{b^\sigma} \chi_{[y^\sigma, b^\sigma]} \frac{x^{-\alpha-\eta-1+\frac{1}{\sigma}}}{(x-y^\sigma)^{1-\alpha}} dx, \quad y \in [a,b]. \tag{7.69}$$

Proof. Similar to the proof of Theorem 7.26, applying Theorem 7.23 for $\Omega_1 = \Omega_2 = [a,b]$, $d\mu_1(x) = \chi_{[b-\lambda,b]}dx$, $d\mu_2(y) = f_1(y)dy$, $d\mu_3(x) = d\mu(x)$ and K defined by (7.64). □

In the previous theorems we derived only the Steffensen type inequalities over some subsets of $\mathbb{R}$. Motivated by [42] we will show that Theorem 7.23 covers much more general situations.

Let $\mathbf{a} = (a_1,\dots,a_n)$, $\mathbf{b} = (b_1,\dots,b_n)$, $\mathbf{x} = (x_1,\dots,x_n)$, $\mathbf{t} = (t_1,\dots,t_n)$ and let $\lambda = (\lambda_1,\dots,\lambda_n)$ be nonnegative such that $\lambda \leq \mathbf{b} - \mathbf{a}$. Now we give the Steffensen type inequalities for the mixed Riemann-Liouville fractional integrals.

Theorem 7.28 *Let μ be $\sigma-$finite (signed) regular Borel measure on $[\mathbf{a},\mathbf{b}]$. Then for every nonnegative Borel measurable function f_1*

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} I_{\mathbf{a}+}^{\alpha} f_1(\mathbf{x}) d\mu(\mathbf{x}) \geq I_{\mathbf{a}+}^{\alpha+\mathbf{1}} f_1(\mathbf{a}+\lambda) \tag{7.70}$$

if and only if

$$\int_{y_1}^{b_1} \cdots \int_{y_n}^{b_n} (\mathbf{x}-\mathbf{y})^{\alpha-\mathbf{1}} d\mu(\mathbf{x}) \geq 0, \quad \mathbf{t} \in [\mathbf{a},\mathbf{b}] \tag{7.71}$$

and

$$\prod_{i=1}^n \frac{(a_i+\lambda_i-y_i)_+^{\alpha_i}}{\alpha_i} \leq \int_{y_1}^{b_1} \cdots \int_{y_n}^{b_n} (\mathbf{x}-\mathbf{y})^{\alpha-\mathbf{1}} d\mu(\mathbf{x}). \tag{7.72}$$

Proof. Let $\Omega_1 = \Omega_2 = [\mathbf{a},\mathbf{b}]$,

$$K(\mathbf{x},\mathbf{y}) = \begin{cases} \frac{(\mathbf{x}-\mathbf{y})^{\alpha-\mathbf{1}}}{\Gamma(\alpha)}, & \mathbf{a} \leq \mathbf{y} \leq \mathbf{x} \\ 0, & \text{otherwise}, \end{cases} \tag{7.73}$$

$d\mu_1(x) = d\mu(\mathbf{x})$, $d\mu_2(y) = f(\mathbf{y})d\mathbf{y}$ and $d\mu_3(x) = \chi_{[\mathbf{a},\mathbf{a}+\lambda]}d\mathbf{x}$. Notice that class U now reduces to class of functions $I_{\mathbf{a}+}^{\alpha} f_1$. Applying Theorem 7.23, from (7.44) we obtain

$$\begin{aligned} \int_{\mathbf{a}}^{\mathbf{b}} \frac{1}{\Gamma(\alpha)} \int_{a_1}^{x_1} \cdots \int_{a_n}^{x_n} \frac{f(\mathbf{y})}{(\mathbf{x}-\mathbf{y})^{\mathbf{1}-\alpha}} d\mathbf{y}\, d\mu(\mathbf{x}) \\ \geq \int_{\mathbf{a}}^{\mathbf{a}+\lambda} \frac{1}{\Gamma(\alpha)} \int_{\mathbf{a}}^{\mathbf{x}} (\mathbf{x}-\mathbf{y})^{\alpha-\mathbf{1}} f(\mathbf{y}) d\mathbf{y}\, d\mathbf{x}. \end{aligned} \tag{7.74}$$

Using Fubini's theorem and then calculating the inner integral, the right-hand side in (7.74) can be written as

$$\frac{1}{\Gamma(\alpha+\mathbf{1})} \int_{a_1}^{a_1+\lambda_1} \cdots \int_{a_n}^{a_n+\lambda_n} f(\mathbf{y})(\mathbf{a}+\lambda-\mathbf{y})^{\alpha} d\mathbf{y}.$$

So we obtain (7.70). From (7.45) we obtain

$$\int_{\mathbf{a}}^{\mathbf{b}} K(\mathbf{x},\mathbf{y})d\mu(\mathbf{x}) \geq \int_{\mathbf{a}}^{\mathbf{a}+\lambda} K(\mathbf{x},\mathbf{y})d\mathbf{x}. \tag{7.75}$$

Since the right-hand side in (7.75) is nonnegative, (7.71) is necessary. Now taking $\mathbf{a} \leq \mathbf{y} \leq \mathbf{b}$, (7.75) is

$$\int_{y_1}^{b_1} \cdots \int_{y_n}^{b_n} (\mathbf{x}-\mathbf{y})^{\alpha-\mathbf{1}} d\mu(\mathbf{x}) \geq \int_{y_1}^{a_1+\lambda_1} \cdots \int_{y_n}^{a_n+\lambda_n} (\mathbf{x}-\mathbf{y})^{\alpha-\mathbf{1}} d\mathbf{x},\ \mathbf{a} \leq \mathbf{y} \leq \mathbf{a}+\lambda. \tag{7.76}$$

Calculating integral on the right-hand side in (7.76), we obtain

$$\int_{y_1}^{b_1} \cdots \int_{y_n}^{b_n} (\mathbf{x}-\mathbf{y})^{\alpha-\mathbf{1}} d\mu(\mathbf{x}) \geq \prod_{i=1}^{n} \frac{(a_i+\lambda_i-y_i)^{\alpha_i}}{\alpha_i}, \quad a_i \leq y_i \leq a_i+\lambda_i.$$

Hence, (7.72) follows. □

Theorem 7.29 *Let μ be $\sigma-$finite (signed) regular Borel measure on $[\mathbf{a},\mathbf{b}]$. Then for every nonnegative Borel measurable function f_1*

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} I_{\mathbf{a}+}^{\alpha} f_1(\mathbf{x}) d\mu(\mathbf{x}) \leq I_{\mathbf{a}+}^{\alpha+\mathbf{1}} f_1(\mathbf{b}) - I_{\mathbf{a}+}^{\alpha+\mathbf{1}} f_1(\mathbf{b}-\lambda) \tag{7.77}$$

if and only if

$$\int_{y_1}^{b_1} \cdots \int_{y_n}^{b_n} (\mathbf{x}-\mathbf{y})^{\alpha-\mathbf{1}} d\mu(\mathbf{x}) \leq \int_{b_1-\lambda_1}^{b_1} \cdots \int_{b_n-\lambda_n}^{b_n} \chi_{[\mathbf{y},\mathbf{b}]}(\mathbf{x}-\mathbf{y})^{\alpha-\mathbf{1}} d\mathbf{x}. \tag{7.78}$$

Proof. Let $\Omega_1 = \Omega_2 = [\mathbf{a},\mathbf{b}]$, K defined by (7.73), $d\mu_1(x) = \chi_{[\mathbf{b}-\lambda,\mathbf{b}]} d\mathbf{x}$, $d\mu_2(y) = f_1(\mathbf{y})d\mathbf{y}$ and $d\mu_3(x) = d\mu(\mathbf{x})$ Notice that class U now reduces to class of functions $I_{\mathbf{a}+}^{\alpha} f_1$. Applying Theorem 7.23, from (7.44) we obtain

$$\begin{aligned} &\int_{\mathbf{a}}^{\mathbf{b}} \frac{1}{\Gamma(\alpha)} \int_{a_1}^{x_1} \cdots \int_{a_n}^{x_n} \frac{f_1(\mathbf{y})}{(\mathbf{x}-\mathbf{y})^{\mathbf{1}} - \alpha} d\mathbf{y}\, d\mu(\mathbf{x}) \\ &\leq \int_{\mathbf{b}-\lambda}^{\mathbf{b}} \frac{1}{\Gamma(\alpha)} \int_{\mathbf{a}}^{\mathbf{x}} (\mathbf{x}-\mathbf{y})^{\alpha-\mathbf{1}} f_1(\mathbf{y}) d\mathbf{y}\, d\mathbf{x}. \end{aligned} \tag{7.79}$$

Using Fubini's theorem and then calculating the inner integral, the right-hand side in (7.79) can be written as

$$\begin{aligned} &\frac{1}{\Gamma(\alpha+\mathbf{1})} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f_1(\mathbf{y})(\mathbf{b}-\mathbf{y})^{\alpha} d\mathbf{y} \\ &\quad - \frac{1}{\Gamma(\alpha+\mathbf{1})} \int_{a_1}^{b_1-\lambda_1} \cdots \int_{a_n}^{b_n-\lambda_n} f_1(\mathbf{y})(\mathbf{b}-\lambda-\mathbf{y})^{\alpha} d\mathbf{y}. \end{aligned}$$

So we obtain (7.77). From (7.45) we obtain

$$\int_{\mathbf{a}}^{\mathbf{b}} K(\mathbf{x},\mathbf{y})d\mu(\mathbf{x}) \leq \int_{\mathbf{b}-\lambda}^{\mathbf{b}} K(\mathbf{x},\mathbf{y})d\mathbf{x}.$$

that is

$$\int_{\mathbf{y}}^{\mathbf{b}} (\mathbf{x}-\mathbf{y})^{\alpha-\mathbf{1}} d\mu(\mathbf{x}) \leq \int_{\mathbf{b}-\lambda}^{\mathbf{b}} \chi_{[\mathbf{y},\mathbf{b}]}(\mathbf{x}-\mathbf{y})^{\alpha-\mathbf{1}} d\mathbf{x}.$$

Hence, (7.78) follows. □

Now, we give linear functionals which are used in the following theorems. This results are obtained by Pečarić, Perić and Smoljak in [114].

For $f \in U$ let

$$A(f) = \int_{\Omega_1} f(x) d\mu_1(x) - \int_{\Omega_1} f(x) d\mu_3(x). \tag{7.80}$$

We define linear functionals involving fractional integrals of function f related to given function g.

Let

$$\begin{aligned} L_1(f_1) = & \int_a^b I_{a+;g}^{\alpha} f_1(x) d\mu(x) \\ & - \frac{1}{\Gamma(\alpha)} \int_a^{a+\lambda} g'(y) f_1(y) \int_y^{a+\lambda} (g(x)-g(y))^{\alpha-1} dx dy, \end{aligned} \tag{7.81}$$

and let

$$\begin{aligned} L_2(f_1) = & \frac{1}{\Gamma(\alpha)} \left[\int_a^{b-\lambda} g'(y) f_1(y) \int_{b-\lambda}^{b} (g(x)-g(y))^{\alpha-1} dx dy \right. \\ & \left. + \int_{b-\lambda}^{b} g'(y) f_1(y) \int_y^b (g(x)-g(y))^{\alpha-1} dx dy \right] - \int_a^b I_{a+;g}^{\alpha} f_1(x) d\mu(x), \end{aligned} \tag{7.82}$$

where f_1 is nonnegative Borel measurable function. Next, we define linear functionals related to Hadamard fractional integral. Let

$$L_3(f_1) = \int_a^b J_{a+}^{\alpha} f_1(x) d\mu(x) - \frac{1}{\Gamma(\alpha)} \int_a^{a+\lambda} f_1(y) \int_0^{\log \frac{a+\lambda}{y}} x^{\alpha-1} e^x dx dy, \tag{7.83}$$

and let

$$\begin{aligned} L_4(f_1) = & \frac{1}{\Gamma(\alpha)} \left[\int_a^{b-\lambda} f_1(y) \int_{\log \frac{b-\lambda}{y}}^{\log \frac{b}{y}} x^{\alpha-1} e^x dx dy \right. \\ & \left. + \int_{b-\lambda}^{b} f_1(y) \int_0^{\log \frac{b}{y}} x^{\alpha-1} e^x dx dy \right] - \int_a^b J_{a+}^{\alpha} f_1(x) d\mu(x), \end{aligned} \tag{7.84}$$

where f_1 is nonnegative Borel measurable function.

At the end we define linear functionals related to Erdélyi-Kober fractional integral. Let

$$\begin{aligned} L_5(f_1) = & \int_a^b I_{a+;\sigma;\eta}^{\alpha} f_1(x) d\mu(x) \\ & - \frac{1}{\Gamma(\alpha)} \int_a^{a+\lambda} y^{\sigma\eta+\sigma-1} f_1(y) \int_{y^{\sigma}}^{(a+\lambda)^{\sigma}} \frac{x^{-\alpha-\eta-1+\frac{1}{\sigma}}}{(x-y^{\sigma})^{1-\alpha}} dx dy, \end{aligned} \tag{7.85}$$

and let

$$\begin{aligned} L_6(f_1) = \frac{1}{\Gamma(\alpha)} \Bigg[&\int_a^{b-\lambda} y^{\sigma\eta+\sigma-1} f_1(y) \int_{(b-\lambda)^\sigma}^{b^\sigma} \frac{x^{-\alpha-\eta-1+\frac{1}{\sigma}}}{(x-y^\sigma)^{1-\alpha}} dx\,dy \\ &+ \int_{b-\lambda}^{b} y^{\sigma\eta+\sigma-1} f_1(y) \int_{y^\sigma}^{b^\sigma} \frac{x^{-\alpha-\eta-1+\frac{1}{\sigma}}}{(x-y^\sigma)^{1-\alpha}} dx\,dy \Bigg] - \int_a^b I^\alpha_{a+;\sigma;\eta} f(x)\,d\mu(x), \end{aligned} \tag{7.86}$$

where f_1 is nonnegative Borel measurable function.

Theorem 7.30 *Let Ω_1 be a compact set. Let $(\Omega_1,\Sigma_1,\mu_1)$ and $(\Omega_1,\Sigma_3,\mu_3)$ be measure spaces with $\sigma-$finite regular Borel measures, let (7.45) hold and let $f \in C(\Omega_1)$. Then there exists $\xi \in \Omega_1$ such that*

$$A(f) = f(\xi)\left(\int_{\Omega_1} d\mu_1(x) - \int_{\Omega_1} d\mu_3(x)\right), \tag{7.87}$$

where A is defined by (7.80).

Proof. Notice that from Theorem 7.23 we have that if $f \geq 0$, then $A(f) \geq 0$, so A is positive linear functional.
Since f is continuous on Ω_1, there exists $m = \min_{x\in\Omega_1} f(x)$ and $M = \max_{x\in\Omega_1} f(x)$. Then $A(M - f) \geq 0$ and $A(f-m) \geq 0$. Therefore

$$\begin{aligned} m\left(\int_{\Omega_1} d\mu_1(x) - \int_{\Omega_1} d\mu_3(x)\right) &\leq \int_{\Omega_1} f(x)d\mu_1(x) - \int_{\Omega_1} f(x)d\mu_3(x) \\ &\leq M\left(\int_{\Omega_1} d\mu_1(x) - \int_{\Omega_1} d\mu_3(x)\right) \end{aligned}$$

that is,

$$mA(1) \leq A(f) \leq MA(1).$$

If the function $A(1) = 0$, then $A(f) = 0$, so (7.87) holds for all $\xi \in \Omega_1$. Otherwise,

$$\min_{x\in\Omega_1} f(x) = m \leq \frac{A(f)}{A(1)} \leq M = \max_{x\in\Omega_1} f(x), \text{ so } \frac{A(f)}{A(1)} \in f(\Omega_1).$$

Since f is continuous, we have that $\frac{A(f)}{A(1)} = f(\xi)$ for some $\xi \in \Omega_1$. □

Theorem 7.31 *Let g be an increasing function on (a,b) such that g' is continuous on (a,b) and let f_1 be nonnegative Borel measurable function such that $f_1 \in C[a,b]$. Let μ be $\sigma-$finite (signed) regular Borel measure, let (7.51) and (7.52) hold. Then there exists $\xi \in [a,b]$ such that*

$$L_1(f_1) = \frac{f_1(\xi)}{\Gamma(\alpha)}\left(\int_a^b \frac{(g(x)-g(a))^\alpha}{\alpha} d\mu(x) - \int_a^{a+\lambda} g'(y) \int_y^{a+\lambda} (g(x)-g(y))^{\alpha-1} dx\,dy\right),$$

where L_1 is defined by (7.81).

Proof. Notice that from Theorem 7.24 we have that if $f_1 \geq 0$, then $L_1(f_1) \geq 0$, so L_1 is a positive linear functional.
Set $m = \min_{x\in[a,b]} f_1(x)$, $M = \max_{x\in[a,b]} f_1(x)$. Then $L_1(M - f_1) \geq 0$ and $L_1(f_1 - m) \geq 0$. Using definition of the left-sided fractional integral of the function f_1 with respect to another function g, the linear functional L_1 can be written as

$$L_1(f_1) = \frac{1}{\Gamma(\alpha)} \int_a^b \int_a^x \frac{g'(y) f_1(y)}{(g(x)-g(y))^{1-\alpha}} dy\, d\mu(x) \\ - \frac{1}{\Gamma(\alpha)} \int_a^{a+\lambda} g'(y) f_1(y) \int_y^{a+\lambda} (g(x)-g(y))^{\alpha-1} dx\, dy.$$

Hence

$$\frac{m}{\Gamma(\alpha)} \left(\int_a^b \frac{(g(x)-g(a))^{\alpha}}{\alpha} d\mu(x) - \int_a^{a+\lambda} g'(y) \int_y^{a+\lambda} (g(x)-g(y))^{\alpha-1} dx\, dy \right) \\ \leq \int_a^b I^{\alpha}_{a+;g} f_1(x) d\mu(x) - \frac{1}{\Gamma(\alpha)} \int_a^{a+\lambda} g'(y) f_1(y) \int_y^{a+\lambda} (g(x)-g(y))^{\alpha-1} dx\, dy \\ \leq \frac{M}{\Gamma(\alpha)} \left(\int_a^b \frac{(g(x)-g(a))^{\alpha}}{\alpha} d\mu(x) - \int_a^{a+\lambda} g'(y) \int_y^{a+\lambda} (g(x)-g(y))^{\alpha-1} dx\, dy \right),$$

that is,

$$mL_1(1) \leq L_1(f_1) \leq ML_1(1).$$

Similar reasoning as in proof of Theorem 7.30 completes the proof. □

The following theorems are very similar to the previous one and its proofs are analogu-ous.

Theorem 7.32 *Let g be an increasing function on (a,b) such that g' is continuous on (a,b) and let f_1 be nonnegative Borel measurable function such that $f_1 \in C[a,b]$. Let μ be $\sigma-$finite (signed) regular Borel measure and let (7.57) hold. Then there exists $\xi \in [a,b]$ such that*

$$L_2(f_1) = \frac{f_1(\xi)}{\Gamma(\alpha)} \left(\int_a^{b-\lambda} g'(y) \int_{b-\lambda}^b (g(x)-g(y))^{\alpha-1} dx\, dy \right. \\ \left. + \int_{b-\lambda}^b g'(y) \int_y^b (g(x)-g(y))^{\alpha-1} dx\, dy - \int_a^b \frac{(g(x)-g(a))^{\alpha}}{\alpha} d\mu(x) \right),$$

where L_2 is defined by (7.82).

Denote

$$HG(x) = {}_2F_1(1-\alpha, \eta+1; \eta+2; x^{\sigma}).$$

Theorem 7.33 *Let μ be $\sigma-$finite (signed) regular Borel measure and let f_1 be nonnega-tive Borel measurable function such that $f_1 \in C[a,b]$.*

(i) If (7.58) and (7.59) hold and $a > 0$, then there exists $\xi \in [a,b]$ such that

$$L_3(f_1) = f_1(\xi)\left(\frac{1}{\Gamma(\alpha+1)}\int_a^b \left(\log\frac{x}{a}\right)^{\alpha} d\mu(x) - \frac{1}{\Gamma(\alpha)}\int_a^{a+\lambda}\int_0^{\log\frac{a+\lambda}{y}} x^{\alpha-1}e^x dx\,dy\right),$$

where L_3 is defined by (7.83).

(ii) If (7.60) holds and $a > 0$, then there exists $\xi \in [a,b]$ such that

$$L_4(f_1) = f_1(\xi)\left(\frac{1}{\Gamma(\alpha)}\int_a^{b-\lambda}\int_{\log\frac{b-\lambda}{y}}^{\log\frac{b}{y}} x^{\alpha-1}e^x dx\,dy + \frac{1}{\Gamma(\alpha)}\int_{b-\lambda}^{b}\int_0^{\log\frac{a+\lambda}{y}} x^{\alpha-1}e^x dx\,dy - \frac{1}{\Gamma(\alpha+1)}\int_a^b \left(\log\frac{x}{a}\right)^{\alpha} d\mu(x)\right),$$

where L_4 is defined by (7.84).

(iii) If (7.62) and (7.63) hold, then there exists $\xi \in [a,b]$ such that

$$L_5(f_1) = f_1(\xi)\left(\frac{\Gamma(\eta+1)}{\Gamma(\alpha+\eta+1)}\int_a^b d\mu(x) - \frac{1}{(\eta+1)\Gamma(\alpha)}\times \int_a^b \left(\frac{a}{x}\right)^{\sigma\eta+\sigma} HG\left(\frac{a}{x}\right) d\mu(x) - \frac{1}{\Gamma(\alpha)}\int_a^{a+\lambda} y^{\sigma\eta+\sigma-1}\int_{y^\sigma}^{(a+\lambda)^\sigma} \frac{x^{-\alpha-\eta-1+\frac{1}{\sigma}}}{(x-y^\sigma)^{1-\alpha}}dx\,dy\right),$$

where L_5 is defined by (7.85).

(iv) If (7.69) holds, then there exists $\xi \in [a,b]$ such that

$$L_6(f_1) = f_1(\xi)\left(\frac{1}{\Gamma(\alpha)}\int_a^b y^{\sigma\eta+\sigma-1}f_1(y)\int_{(b-\lambda)^\sigma}^{b^\sigma} \frac{x^{-\alpha-\eta-1+\frac{1}{\sigma}}}{(x-y^\sigma)^{1-\alpha}}dx\,dy + \frac{1}{\Gamma(\alpha)}\int_{b-\lambda}^{b} y^{\sigma\eta+\sigma-1}f_1(y)\int_{y^\sigma}^{b^\sigma} \frac{x^{-\alpha-\eta-1+\frac{1}{\sigma}}}{(x-y^\sigma)^{1-\alpha}}dx\,dy - \frac{\Gamma(\eta+1)}{\Gamma(\alpha+\eta+1)}\int_a^b d\mu(x) + \frac{1}{(\eta+1)\Gamma(\alpha)}\int_a^b \left(\frac{a}{x}\right)^{\sigma\eta+\sigma} HG\left(\frac{a}{x}\right) d\mu(x)\right),$$

where L_6 is defined by (7.86).

Theorem 7.34 *Let conditions of Theorem 7.30 be satisfied and let $f_1, f_2 \in C(\Omega_1)$ be such that $f_2(x) \neq 0$ for every $x \in \Omega_1$. Then there exists $\xi \in \Omega_1$ such that*

$$\frac{f_1(\xi)}{f_2(\xi)} = \frac{A(f_1)}{A(f_2)}.$$

Proof. Similar to the proof of Theorem 7.17. □

Theorem 7.35 *Let conditions of Theorem 7.31 be satisfied and let $f_1, f_2 \in C[a,b]$ be such that $f_2(x) \neq 0$ for every $x \in [a,b]$. Then there exists $\xi_i \in [a,b]$ such that*

$$\frac{f_1(\xi_i)}{f_2(\xi_i)} = \frac{L_i(f_1)}{L_i(f_2)}, \quad i = 1, \ldots, 6, \tag{7.88}$$

where L_i, $i = 1, \ldots, 6$ are linear functionals defined by (7.81)-(7.86).

Proof. Similar to the proof of Theorem 7.17. □

Remark 7.12 *Theorem 7.35 enables us to define new types of means, because if f_1/f_2 has an inverse, from (7.88) we conclude*

$$\xi_i = \left(\frac{f_1}{f_2}\right)^{-1} \left(\frac{L_i(f_1)}{L_i(f_2)}\right), \quad i = 1, \ldots, 6.$$

Remark 7.13 Class of general inequalities for positive measures related to Steffensen inequality given in papers [56] and [138] can also be used for obtaining new Steffensen type inequalities for fractional integrals and derivatives. See also Section 3.4.

Chapter 8

Means related to Steffensen's inequality

8.1 Error terms for Steffensen's inequality

In 2008. Mercer proved an error term for the right-hand Steffensen's inequality as follows (see [88]).

Theorem 8.1 *Let f' be continuous and let g be integrable on $[a,b]$, with $0 \leq g \leq 1$ and $\lambda = \int_a^b g(t)dt$. Then there exists $\xi \in (a,b)$ such that*

$$\int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt = f'(\xi)\left[\int_a^b t\cdot g(t)dt - \lambda\left(a+\frac{\lambda}{2}\right)\right].$$

Proof. We have

$$\begin{aligned}&\int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt \\ &\quad = \int_a^{a+\lambda}[f(a+\lambda)-f(t)][1-g(t)]dt + \int_{a+\lambda}^b [f(t)-f(a+\lambda)]g(t)dt.\end{aligned}$$

By the Mean Value Theorem there exist $p \in (a, a+\lambda)$ and $q \in (a+\lambda, b)$ such that the right-hand side is equal to

$$\int_a^{a+\lambda} f'(p)[a+\lambda-t][1-g(t)]dt + \int_{a+\lambda}^b f'(q)[t-(a+\lambda)]g(t)dt. \tag{8.1}$$

Expressions $[a+\lambda-t]$ on $[a,a+\lambda]$, $[t-(a+\lambda)]$ on $[a+\lambda,b]$ and $1-g(t)$ are nonnegative, so by Mean Value Theorem for integrals there exist $r,s\in(a,b)$ such that (8.1) is equal to

$$f'(r)\int_a^{a+\lambda}[a+\lambda-t][1-g(t)]dt+f'(s)\int_{a+\lambda}^b[t-(a+\lambda)]g(t)dt. \tag{8.2}$$

Each integral is nonnegative so by the Intermediate Value Theorem there exists $\xi\in(a,b)$ such that (8.2) is equal to

$$\begin{aligned}&f'(\xi)\left[\int_a^{a+\lambda}[a+\lambda-t][1-g(t)]dt+\int_{a+\lambda}^b[t-(a+\lambda)]g(t)dt\right]\\&=f'(\xi)\left[\int_a^b t\cdot g(t)dt-\lambda\left(a+\frac{\lambda}{2}\right)\right].\end{aligned}$$

□

Remark 8.1 Similarly, Mercer obtained an error term for the left-hand Steffensen's inequality:

$$\int_a^b f(t)g(t)dt-\int_{b-\lambda}^b f(t)dt=f'(\xi)\left[\int_a^b t\cdot g(t)dt-\lambda\left(b-\frac{\lambda}{2}\right)\right].$$

Corollary 8.1 *For f', h' continuous and g integrable on $[a,b]$, with $0\le g\le 1$ and $\lambda=\int_a^b g(t)dt$, there exist $\xi,\eta\in(a,b)$ such that*

$$\frac{\int_a^b f(t)g(t)dt-\int_a^{a+\lambda}f(t)dt}{\int_a^b h(t)g(t)dt-\int_a^{a+\lambda}h(t)dt}=\frac{f'(\xi)}{h'(\xi)} \tag{8.3}$$

and

$$\frac{\int_a^b f(t)g(t)dt-\int_{b-\lambda}^b f(t)dt}{\int_a^b h(t)g(t)dt-\int_{b-\lambda}^b h(t)dt}=\frac{f'(\eta)}{h'(\eta)}. \tag{8.4}$$

Proof. For fixed g we define linear functional L by

$$L(f)=\int_a^b f(t)g(t)dt-\int_a^{a+\lambda}f(t)dt$$

and we set $\phi(t)=f(t)L(h)-h(t)L(f)$. By Theorem 8.1 we have

$$L(\phi)=\phi'(\xi)\left[\int_a^b t\cdot g(t)dt-\lambda\left(a+\frac{\lambda}{2}\right)\right].$$

But $L(\phi)=0$ and so $f'(\xi)L(h)-h'(\xi)L(f)=0$, thus the first statement is proved. The proof of second statement is similar. □

Mercer applied Theorem 8.1 to f' and various functions g in order to recast inequalities as equalities involving an error term:

$$\text{Error}(f)=\frac{f''(\xi)}{2}\text{Error}(t^2).$$

Let us recall Hermite-Hadamard inequalities. Suppose that f is convex on $[a,b]$. Then

$$f\left(\frac{a+b}{2}\right)(b-a) \leq \int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}(b-a).$$

Suppose that f'' is continuous on $[a,b]$ and let $c = \frac{a+b}{2}$. Applying Theorem 8.1 to f' and

$$g(t) = \begin{cases} \frac{t-a+\frac{b-a}{2}}{b-a}, & t \in [a,c] \\ \frac{t-b+\frac{b-a}{2}}{b-a}, & t \in (c,b], \end{cases} \quad \left(\text{here } \lambda = \int_a^b g(t)dt = \frac{b-a}{2}\right)$$

we obtain the *trapezoid rule*:

$$\int_a^b f(t)dt - \frac{f(a)+f(b)}{2}(b-a) = -\frac{f''(\xi_1)}{2}\frac{(b-a)^3}{6}.$$

Applying Remark 8.1 to f' and $g(t) = \frac{t-a}{b-a}$, (again $\lambda = \int_a^b g(t)dt = \frac{b-a}{2}$) we obtain the *midpoint rule*:

$$\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right)(b-a) = \frac{f''(\xi_2)}{2}\frac{(b-a)^3}{12}.$$

Hence, Mercer noted that Hermite-Hadamard inequalities follow from Steffensen's which has been overlooked in the literature.

In the same paper Mercer gave error terms for Jensen-Steffensen's, Jensen's and integral Jensen-Steffensen's inequality. Furthermore, Mercer noted that many other error terms for inequalities can be similarly obtained.

In [66] Jakšetić and Pečarić generalized Mercer's results. Their generalization of Theorem 8.1 with weaker conditions on function g is given in the following theorem.

Theorem 8.2 *Assume that f' is continuous and g is integrable function on $[a,b]$ such that*

$$0 \leq \int_x^b g(t)dt \leq b-x \quad \text{and} \quad 0 \leq \int_a^x g(t)dt \leq x-a \quad \text{for every } x \in [a,b]. \tag{8.5}$$

Then there exist $\xi, \eta \in (a,b)$ such that

$$\int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt = f'(\xi)\left[\int_a^b t \cdot g(t)dt - \lambda\left(a+\frac{\lambda}{2}\right)\right]$$

and

$$\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt = f'(\eta)\left[\int_a^b t \cdot g(t)dt - \lambda\left(b-\frac{\lambda}{2}\right)\right].$$

Proof. Jakšetić and Pečarić gave two proofs of this theorem. We recall one of them.

$$
\begin{aligned}
&\int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt \\
&= \int_a^{a+\lambda} \left(\int_t^{a+\lambda} f'(x)dx \right) [1-g(t)]dt + \int_{a+\lambda}^b \left(\int_{a+\lambda}^t f'(x)dx \right) g(t)dt \\
&= \int_a^{a+\lambda} f'(x) \left(\int_a^x [1-g(t)]dt \right) dx + \int_{a+\lambda}^b f'(x) \left(\int_x^b g(t)dt \right) dx \\
&= \int_a^b G(x)f'(x)dx,
\end{aligned}
\tag{8.6}
$$

where

$$
G(x) = \begin{cases} \int_a^x (1-g(t))dt, & a \le x \le a+\lambda, \\ \int_x^b g(t)dt, & a+\lambda \le x \le b. \end{cases} \tag{8.7}
$$

Since $G(x) \ge 0$, $x \in [a,b]$ we conclude that there exists $\xi \in (a,b)$ such that

$$
\begin{aligned}
\int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt &= f'(\xi) \int_a^b G(x)dx \\
&= f'(\xi) \left[\int_a^b t \cdot g(t)dt - \lambda \left(a + \frac{\lambda}{2} \right) \right].
\end{aligned}
$$

Similarly, Jakšetić and Pečarić proved:

$$
\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt = -\int_a^b F(x)f'(x)dx, \tag{8.8}
$$

where

$$
F(x) = \begin{cases} \int_a^x g(t)dt, & a \le x \le b-\lambda, \\ \int_x^b (1-g(t))dt, & b-\lambda \le x \le b. \end{cases} \tag{8.9}
$$

Since $F(x) \ge 0$, $x \in [a,b]$ we conclude that there exists $\eta \in (a,b)$ such that

$$
\begin{aligned}
\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt &= f'(\eta) \int_a^b (-F(x))dx \\
&= f'(\eta) \left[\int_a^b t \cdot g(t)dt - \lambda \left(b - \frac{\lambda}{2} \right) \right].
\end{aligned}
$$

□

In the same paper Jakšetić and Pečarić made an estimation of Steffensen's inequality using Hölder's inequality and integral representations (8.6) and (8.8). This estimation is given in the following corollary.

Corollary 8.2 *Assume that f' is continuous and g is integrable function on $[a,b]$ such that (8.5) holds. Then*

$$
\left| \int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt \right| \le \|f'\|_p \|G\|_q
$$

and

$$\left|\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt\right| \leq \|f'\|_p \|F\|_q,$$

where G *and* F *are given by (8.7) and (8.9) and* $1 < p,q < \infty$, $\frac{1}{p}+\frac{1}{q}=1$.

In [66] Corollary 8.1 was restated with more general conditions.

Corollary 8.3 *For* f', h' *continuous and* g *integrable on* $[a,b]$, *with* $\lambda = \int_a^b g(t)dt$ *and (8.5) there exist* $\xi,\eta \in (a,b)$ *such that (8.3) and (8.4) hold.*

Proof. Similar to the proof of Corollary 8.1. □

Using Corollary 8.3 Jakšetić and Pečarić consider the following means

$$M_1(g;x,y;p,q) = \left\{\frac{q-1}{p-1}\,\frac{\int_x^y t^{p-1}g(t)dt - \frac{(x+\lambda)^p - x^p}{p}}{\int_x^y t^{q-1}g(t)dt - \frac{(x+\lambda)^q - x^q}{q}}\right\}^{\frac{1}{p-q}}, \tag{8.10}$$

and

$$M_2(g;x,y;p,q) = \left\{\frac{q-1}{p-1}\,\frac{\int_x^y t^{p-1}g(t)dt - \frac{y^p-(y-\lambda)^p}{p}}{\int_x^y t^{q-1}g(t)dt - \frac{y^q-(y-\lambda)^q}{q}}\right\}^{\frac{1}{p-q}}, \tag{8.11}$$

where $p \neq q$, $y > x > 0$.

Continuous extensions of means (8.10) and (8.11) can be found in [66]. Monotonicity of this means is also proved in [66].

Theorem 8.3 *Let* $p \leq u$, $q \leq v$. *Then*

$$M_1(g;x,y;p,q) \leq M_1(g;x,y;u,v)$$

and

$$M_2(g;x,y;p,q) \leq M_2(g;x,y;u,v).$$

In [67] Jakšetić and Pečarić generalized mean value Theorem 8.2 and obtain the following.

Theorem 8.4 *Let* $f \in C^1[a,b]$ *be nondecreasing and let* g *be integrable function on* $[a,b]$ *such that (8.5) is valid and* $\lambda = \int_a^b g(t)dt$. *If* $h \in C^1[f(a),f(b)]$ *then there exist* $\eta,\xi \in [f(a),f(b)]$ *such that*

$$\int_a^b h(f(t))g(t)dt - \int_a^{a+\lambda} h(f(t))dt = h'(\xi)\left[\int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt\right]$$

and

$$\int_a^b h(f(t))g(t)dt - \int_{b-\lambda}^b h(f(t))dt = h'(\eta)\left[\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt\right].$$

Also, the following generalization of Corollary 8.3 is given.

Corollary 8.4 *Let $f \in C^1[a,b]$ be strictly monotone function and $h_1, h_2 \in C^1[f(a), f(b)]$, g integrable on $[a,b]$, with $\lambda = \int_a^b g(t)dt$ and (8.5) holds. Then there exist $\xi, \eta \in [f(a), f(b)]$ such that*

$$\frac{\int_a^b h_1(f(t))g(t)dt - \int_a^{a+\lambda} h_1(f(t))dt}{\int_a^b h_2(f(t))g(t)dt - \int_a^{a+\lambda} h_2(f(t))dt} = \frac{h_1'(\xi)}{h_2'(\xi)},$$

and

$$\frac{\int_a^b h_1(f(t))g(t)dt - \int_{b-\lambda}^{b} h_1(f(t))dt}{\int_a^b h_2(f(t))g(t)dt - \int_{b-\lambda}^{b} h_2(f(t))dt} = \frac{h_1'(\eta)}{h_2'(\eta)}.$$

In [67] the following Steffensen means are obtained.

$$S_1(f,g;x,y;r,s) = \left\{ \frac{s}{r} \frac{\int_x^y f^r(t)g(t)dt - \int_x^{x+\lambda} f^r(t)dt}{\int_x^y f^s(t)g(t)dt - \int_x^{x+\lambda} f^s(t)dt} \right\}^{\frac{1}{r-s}}, \tag{8.12}$$

and

$$S_2(f,g;x,y;r,s) = \left\{ \frac{s}{r} \frac{\int_x^y f^r(t)g(t)dt - \int_{y-\lambda}^{y} f^r(t)dt}{\int_x^y f^s(t)g(t)dt - \int_{y-\lambda}^{y} f^s(t)dt} \right\}^{\frac{1}{r-s}}, \tag{8.13}$$

where $(r-s)\cdot r \cdot s \neq 0$.
Continuous extensions of Steffensen means are also given in [67]. Furthermore, monotonicity property of this means is also proved.

8.2 New generalized Steffensen means

Results given in this Section are obtained by Krulić, Pečarić and Smoljak in [77]. Using generalization of Steffensen's inequality given in Theorem 3.15 we obtain the following Lagrange type mean value theorem.

Theorem 8.5 *Let $h(x) > 0$ for all $x \in (a,b]$, $h(x) \in C^1[a,b]$ and let f be such that $f(x)/h(x) \in C^1[a,b]$. If g is a real-valued integrable function such that $0 \leq g(x) \leq 1$ for every $x \in [a,b]$, then there exists $\xi \in (a,b)$ such that*

$$\begin{aligned} \int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt =& \frac{f'(\xi)h(\xi) - f(\xi)h'(\xi)}{h^2(\xi)} \\ & \times \left[\int_a^b th(t)g(t)dt - \int_a^{a+\lambda} th(t)dt \right], \end{aligned} \tag{8.14}$$

*where λ satisfies $\int_a^{a+\lambda} h(t)dt = \int_a^b h(t)g(t)dt$, i.e. λ satisfies (**??**).*

Proof. Since $\left(\frac{f}{h}\right)'$ is continuous on $[a,b]$, there exist

$$m = \min_{x\in[a,b]} \frac{f'(x)h(x) - f(x)h'(x)}{h^2(x)}$$

and

$$M = \max_{x\in[a,b]} \frac{f'(x)h(x) - f(x)h'(x)}{h^2(x)}.$$

Let us consider functions $F_1, F_2 : [a,b] \to \mathbb{R}$ defined by

$$F_1(x) = Mxh(x) - f(x) \quad \text{and} \quad F_2(x) = f(x) - mxh(x).$$

Then F_1/h and F_2/h are nondecreasing functions. From Theorem 3.15, for nondecreasing function F_1/h, we obtain

$$\begin{aligned} 0 \le \int_a^b F_1(t)g(t)dt - \int_a^{a+\lambda} F_1(t)dt = M\int_a^b tg(t)h(t)dt - \int_a^b f(t)g(t)dt \\ - M\int_a^{a+\lambda} th(t)dt + \int_a^{a+\lambda} f(t)dt \end{aligned}$$

that is,

$$\int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt \le M\left[\int_a^b tg(t)h(t)dt - \int_a^{a+\lambda} th(t)dt\right].$$

Similarly, for nondecreasing function F_2/h, from Theorem 3.15 we obtain

$$\int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt \ge m\left[\int_a^b tg(t)h(t)dt - \int_a^{a+\lambda} th(t)dt\right].$$

Hence,

$$\begin{aligned} m\left[\int_a^b tg(t)h(t)dt - \int_a^{a+\lambda} th(t)dt\right] &\le \int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt \\ &\le M\left[\int_a^b tg(t)h(t)dt - \int_a^{a+\lambda} th(t)dt\right]. \end{aligned}$$

If $\int_a^b tg(t)h(t)dt - \int_a^{a+\lambda} th(t)dt = 0$, then $\int_a^b f(t)g(t)dt = \int_a^{a+\lambda} f(t)dt$ and (8.14) holds for all $\xi \in (a,b)$. Otherwise,

$$m \le \frac{\int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt}{\int_a^b tg(t)h(t)dt - \int_a^{a+\lambda} th(t)dt} \le M.$$

Since $(f/h)'$ is continuous there exists $\xi \in (a,b)$ such that (8.14) holds and the proof is complete. □

As a special case of Theorem 8.5 for $h \equiv 1$ we obtain Theorem 8.1.

Applying Theorem 3.17 and Theorem 8.5 we obtain the following result:

Corollary 8.5 *Let g be an integrable function such that $0 \le g(x) \le 1$ for every $x \in [a,b]$. Assume that $\frac{f(x)}{(x-a)^{n-1}} \in C^1[a,b]$, f convex function of order n, $n \ge 2$ with $f^{(k)}(a) = 0$, $k = 0,\ldots,n-2$, then there exists $\xi \in (a,b)$ such that*

$$\int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt = \frac{f'(\xi)(\xi-a) - f(\xi)(n-1)}{(\xi-a)^n} \times \\ \times \left[\int_a^b t(t-a)^{n-1}g(t)dt - \frac{\lambda^n(n(a+\lambda)+a)}{n(n+1)} \right],$$

where $\lambda = \left(n \int_a^b (t-a)^{n-1} g(t)dt \right)^{\frac{1}{n}}$, i.e. λ satisfies (3.18).

Furthermore, we obtain the following Cauchy type mean value theorem.

Theorem 8.6 *Let g be a real-valued integrable function such that $0 \le g(x) \le 1$ for every $x \in [a,b]$. Let h be a positive function on $(a,b]$ and derivable on (a,b), f,k be derivable on (a,b) such that $f(x)/h(x), k(x)/h(x) \in C^1[a,b]$ and such that $k'(x)h(x) - k(x)h'(x) \neq 0$ for every $x \in [a,b]$. Then there exists $\xi \in (a,b)$ such that*

$$\frac{\int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt}{\int_a^b k(t)g(t)dt - \int_a^{a+\lambda} k(t)dt} = \frac{f'(\xi)h(\xi) - f(\xi)h'(\xi)}{k'(\xi)h(\xi) - k(\xi)h'(\xi)}, \tag{8.15}$$

*where λ satisfies (**??**).*

Proof. Let us define linear functional

$$L(f) = \int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt. \tag{8.16}$$

Next, we define $\Phi(t) = f(t)L(k) - k(t)L(f)$. Note that

$$\frac{\Phi(t)}{h(t)} = \frac{f(t)}{h(t)}L(k) - \frac{k(t)}{h(t)}L(f) \in C^1[a,b].$$

By Theorem 8.5, there exists $\xi \in (a,b)$ such that

$$L(\Phi) = \frac{\Phi'(\xi)h(\xi) - \Phi(\xi)h'(\xi)}{h^2(\xi)} \left[\int_a^b th(t)g(t)dt - \int_a^{a+\lambda} th(t)dt \right].$$

From $L(\Phi) = 0$ it follows that $\Phi'(\xi)h(\xi) - \Phi(\xi)h'(\xi) = 0$
i.e.

$$[f'(\xi)h(\xi) - f(\xi)h'(\xi)]L(k) - [k'(\xi)h(\xi) - k(\xi)h'(\xi)]L(f) = 0.$$

So (8.15) follows. □

As a special case of Theorem 8.6 for $h \equiv 1$ we obtain Corollary 8.1.
From Corollary 8.5 we obtain the following result.

Theorem 8.7 *Let g be a real-valued integrable function such that $0 \le g(x) \le 1$ for every $x \in [a,b]$. Assume that $\frac{f_i(x)}{(x-a)^{n-1}} \in C^1[a,b]$, $i = 1,2$, f_1, f_2 are convex functions of order n, $n \ge 2$ with $f_1^{(k)}(a), f_2^{(k)}(a) = 0$, $k = 0,\dots,n-2$ and $f_2'(x)(x-a) - f_2(x)(n-1) \neq 0$ for every $x \in (a,b)$. Then there exists $\xi \in (a,b)$ such that*

$$\frac{\int_a^b f_1(t)g(t)dt - \int_a^{a+\lambda} f_1(t)dt}{\int_a^b f_2(t)g(t)dt - \int_a^{a+\lambda} f_2(t)dt} = \frac{f_1'(\xi)(\xi-a) - f_1(\xi)(n-1)}{f_2'(\xi)(\xi-a) - f_2(\xi)(n-1)}, \tag{8.17}$$

where λ satisfies (3.18).

In the sequel we consider family of nondecreasing functions $\{\varphi_p/h : p \in \mathbb{R}\}$, where h is positive function and

$$\varphi_p(x) = \begin{cases} \frac{x^{p-1}}{p-1}h(x), & p \neq 1 \\ h(x)\log x, & p = 1. \end{cases} \tag{8.18}$$

Observe that the function φ_p/h satisfies conditions of Theorem 3.15, so

$$\int_a^b \varphi_p(t)g(t)dt - \int_a^{a+\lambda} \varphi_p(t)dt \ge 0$$

for $a,b > 0$. Hence, for linear functional L defined by (8.16) we have $L(\varphi_p) \ge 0$ for all $p \in \mathbb{R}$.

Theorem 8.8 *For L as in (8.16) and φ_p as in (8.18) we have the following:*

(i) *the mapping $p \mapsto L(\varphi_p)$ is continuous on $\mathbb{R}$,*

(ii) *for every $n \in \mathbb{N}$ and $p_i \in \mathbb{R}, p_{ij} = \frac{p_i+p_j}{2}$, $i,j = 1,2,\dots,n$, the matrix $[L(\varphi_{p_{ij}})]_{i,j=1}^n$ is a positive semi-definite, that is*

$$\det[L(\varphi_{p_{ij}})]_{i,j=1}^n \ge 0,$$

(iii) *the mapping $p \mapsto L(\varphi_p)$ is exponentially convex,*

(iv) *the mapping $p \mapsto L(\varphi_p)$ is log-convex,*

(v) *for $p_i \in \mathbb{R}$, $i = 1,2,3$, $p_1 < p_2 < p_3$,*

$$[L(\varphi_{p_2})]^{p_3-p_1} \le [L(\varphi_{p_1})]^{p_3-p_2}[L(\varphi_{p_3})]^{p_2-p_1}.$$

Proof.

(i) Notice that

$$L(\varphi_p) = \begin{cases} \frac{1}{p-1}\left[\int_a^b t^{p-1}h(t)g(t)dt - \int_a^{a+\lambda} t^{p-1}h(t)dt\right], & p \neq 1 \\ \int_a^b h(t)g(t)\log t\,dt - \int_a^{a+\lambda} h(t)\log t\,dt, & p = 1. \end{cases}$$

It is obviously continuous on $\mathbb{R}\setminus\{1\}$. Now suppose that $p \to 1$, then

$$\lim_{p\to 1} L(\varphi_p) = \lim_{p\to 1} \frac{1}{p-1}\left[\int_a^b t^{p-1}h(t)g(t)dt - \int_a^{a+\lambda} t^{p-1}h(t)dt\right],$$

and applying L'Hospital rule we obtain

$$\begin{aligned}\lim_{p\to 1} L(\varphi_p) &= \lim_{p\to 1}\left(\int_a^b t^{p-1}\log t\, h(t)g(t)dt - \int_a^{a+\lambda} t^{p-1}\log t\, h(t)dt\right)\\ &= \int_a^b \log t\, h(t)g(t)dt - \int_a^{a+\lambda} \log t\, h(t)dt = L(\varphi_1).\end{aligned}$$

Hence, the mapping $p \mapsto L(\varphi_p)$ is continuous on $\mathbb{R}$.

(ii) Let $n\in\mathbb{N}$, $t_i\in\mathbb{R}$, $i=1,2,\ldots,n$ be arbitrary. Define the function $f:\mathbb{R}^+\to\mathbb{R}$ by

$$f(x) = \sum_{i,j=1}^{n} t_i t_j \varphi_{p_{ij}}(x).$$

Then

$$\frac{f(x)}{h(x)} = \sum_{i,j=1}^{n} t_i t_j \frac{\varphi_{p_{ij}}(x)}{h(x)},$$

so

$$\left(\frac{f(x)}{h(x)}\right)' = \sum_{i,j=1}^{n} t_i t_j x^{p_{ij}-2} = \left(\sum_{i=1}^{n} t_i x^{\frac{p_i-2}{2}}\right)^2 \geq 0,$$

hence f/h is a nondecreasing function on $\mathbb{R}^+$. We can apply (3.16) on function f and obtain

$$\int_a^b f(x)g(x)dx \geq \int_a^{a+\lambda} f(x)dx,$$

that is,

$$\sum_{i,j=1}^{n} t_i t_j L(\varphi_{p_{ij}}) \geq 0.$$

So the matrix $[L(\varphi_{p_{ij}})]_{i,j=1}^n$ is positive semi-definite.
(iii), (iv) and (v) are trivial consequences of (i), (ii) and definition of exponentially convex and log-convex functions. □

Theorem 8.6 enables us to define various types of means, because if $\dfrac{f'h-fh'}{k'h-kh'}$ has an inverse, from (8.15) we have

$$\xi = \left(\frac{f'h-fh'}{k'h-kh'}\right)^{-1}\left(\frac{\int_x^y f(t)g(t)dt - \int_x^{x+\lambda} f(t)dt}{\int_x^y k(t)g(t)dt - \int_x^{x+\lambda} k(t)dt}\right).$$

Especially, if we use family of nondecreasing functions $\{\varphi_p/h : p \in \mathbb{R}\}$ and take $f(t) = \varphi_p(t),\ k(t) = \varphi_q(t)$ in (8.15) we obtain the following mean:

$$S(g,h;x,y;p,q) = \left\{ \frac{q-1}{p-1} \cdot \frac{\int_x^y t^{p-1}h(t)g(t)dt - \int_x^{x+\lambda} t^{p-1}h(t)dt}{\int_x^y t^{q-1}h(t)g(t)dt - \int_x^{x+\lambda} t^{q-1}h(t)dt} \right\}^{\frac{1}{p-q}}, \tag{8.19}$$

where $p \neq q,\ p,q \neq 1,\ y > x > 0.$
Notice that, (8.19) can be written as

$$S(g,h;x,y;p,q) = \left(\frac{L(\varphi_p)}{L(\varphi_q)} \right)^{\frac{1}{p-q}}.$$

Moreover, we can extend these means to excluded cases. Taking a limit we can define

$$\begin{aligned} S(g,h;x,y;p,1) &= \left\{ \frac{\int_x^y t^{p-1}h(t)g(t)dt - \int_x^{x+\lambda} t^{p-1}h(t)dt}{(p-1)\left[\int_x^y h(t)g(t)\log t dt - \int_x^{x+\lambda} h(t)\log t dt\right]} \right\}^{\frac{1}{p-1}} \\ &= S(g,h;x,y;1,p), \end{aligned}$$

$$S(g,h;x,y;p,p) = \exp\left\{ \frac{\int_x^y t^{p-1}\log t\, h(t)g(t)dt - \int_x^{x+\lambda} t^{p-1}\log t\, h(t)dt}{\int_x^y t^{p-1}h(t)g(t)dt - \int_x^{x+\lambda} t^{p-1}h(t)dt} - \frac{1}{p-1} \right\},$$

$$S(g,h;x,y;1,1) = \exp\left\{ \frac{\frac{1}{2}\left[\int_x^y h(t)g(t)\log^2 t\,dt - \int_x^{x+\lambda} h(t)\log^2 t\,dt\right]}{\int_x^y h(t)g(t)\log t\,dt - \int_x^{x+\lambda} h(t)\log t\,dt} \right\}.$$

Theorem 8.9 *Let $r \leq p,\ s \leq q$, then the following inequality is valid*

$$S(g,h;x,y;r,s) \leq S(g,h;x,y;p,q) \tag{8.20}$$

for every $x,y \in \mathbb{R}$, $x < y$, that is, the mean $S(g,h;x,y;p,q)$ is monotonic.

Proof. Since the linear operator L defined by (8.16) is log-convex, we can apply Corollary 1.1 to $L(\varphi_p)$ and get (8.20). □

Remark 8.2 For $h \equiv 1$, where g satisfies conditions of Theorem 8.5, we obtain the Steffensen mean $S_1(g;x,y;p-1,q-1)$ given by (8.12).

Furthermore, Theorem 8.7 enables us to define new types of means, because if $\frac{f_1'(t)(t-a)-f_1(t)(n-1)}{f_2'(t)(t-a)-f_2(t)(n-1)}$ has an inverse, from (8.17) we have

$$\xi = \left(\frac{f_1'(\xi)(\xi-a)-f_1(\xi)(n-1)}{f_2'(\xi)(\xi-a)-f_2(\xi)(n-1)} \right)^{-1} \left(\frac{\int_a^b f_1(t)g(t)dt - \int_a^{a+\lambda} f_1(t)dt}{\int_a^b f_2(t)g(t)dt - \int_a^{a+\lambda} f_2(t)dt} \right),$$

where $f_1, f_2 : [a,b] \to \mathbb{R}$ are convex functions of order n, $n \geq 2$ with $f_1^{(k)}(a), f_2^{(k)}(a) = 0$, $k = 0, \ldots, n-2$ and $f_2'(x)(x-a) - f_2(x)(n-1) \neq 0$ for every $x \in (a,b)$. Specially, if we take substitutions $f_p(t) = \frac{(t-a)^{p-1}}{p-1}$, $f_q(t) = \frac{(t-a)^{q-1}}{q-1}$ in (8.17) for $p, q \geq n+1$ and using continuous extension we consider the following mean

$$S_n(g;a,b;p,q) = \left\{ \frac{q-n}{p-n} \cdot \frac{\int_a^b (t-a)^{p-1} g(t)dt - \frac{\lambda^p}{p}}{\int_a^b (t-a)^{q-1} g(t)dt - \frac{\lambda^q}{q}} \right\}^{\frac{1}{p-q}} + a, \tag{8.21}$$

where $p \neq q$, $p, q \geq n+1$, $b > a > 0$.

Moreover, we can extend these means to excluded cases. Taking a limit we can define

$$S_n(g;a,b;q,q) = \exp \left\{ \frac{\int_a^b (t-a)^{q-1} \log(t-a) g(t)dt - \frac{\lambda^q (q \log \lambda - 1)}{q^2}}{\int_a^b (t-a)^{q-1} g(t)dt - \frac{\lambda^q}{q}} \right\} \times$$
$$\times \exp\left(\frac{-1}{q-n}\right) + a$$

Notice that, (8.21) can be written as

$$S_n(g;a,b;p,q) = \left(\frac{(q-n)(p-1)}{(q-1)(p-n)} \frac{L(f_p)}{L(f_q)} \right)^{\frac{1}{p-q}} + a,$$

where L is defined by (8.16).

Theorem 8.10 *Let $n \geq 2$, $r \leq p$, $s \leq q$, $s, p, q, r \geq n+1$. If*

$$\left(\frac{(s-n)(r-1)}{(r-n)(s-1)} \right)^{\frac{1}{r-s}} \leq \left(\frac{(q-n)(p-1)}{(p-n)(q-1)} \right)^{\frac{1}{p-q}},$$

then the following inequality is valid

$$S_n(g;a,b;r,s) \leq S_n(g;a,b;p,q),$$

that is, the mean $S_n(g;a,b;p,q)$ is monotonic.

Proof. The linear operator L defined by (8.16) is log-convex, so we can apply Corollary 1.1 and obtain the following:

$$\left(\frac{L(f_r)}{L(f_s)} \right)^{\frac{1}{r-s}} \leq \left(\frac{L(f_p)}{L(f_q)} \right)^{\frac{1}{p-q}},$$

that is

$$\left(\frac{(s-n)(r-1)L(f_r)}{(r-n)(s-1)L(f_s)} \right)^{\frac{1}{r-s}} + a \leq \left(\frac{(q-n)(p-1)L(f_p)}{(p-n)(q-1)L(f_q)} \right)^{\frac{1}{p-q}} + a.$$

□

More general results were obtained by Pečarić and Smoljak in [126]. This results are given in the following theorem and its corollaries. In the sequel J and K will be intervals in $\mathbb{R}$.

Theorem 8.11 *Let h be a positive function and $\Upsilon = \{f_p/h : p \in K\}$ be a family of functions defined on J such that the function $p \mapsto [x_0,x_1; f_p/h]$ is $n-$exponentially convex in the Jensen sense on K for mutually different points $x_0,x_1 \in J$. Let L be linear functional defined by (8.16). Then $p \mapsto L(f_p)$ is $n-$exponentially convex function in the Jensen sense on K.*
If the function $p \mapsto L(f_p)$ is continuous on K, then it is $n-$exponentially convex on K.

Proof. For $\xi_j \in \mathbb{R}$, $p_j \in K$, $j = 1,\dots,n$ and $p_{jk} = \frac{p_j+p_k}{2}$ we define the function

$$g(x) = \sum_{j,k=1}^{n} \xi_j \xi_k f_{p_{jk}}(x). \tag{8.22}$$

Since $p \mapsto [x_0,x_1; f_p/h]$ is $n-$exponentially convex in the Jensen sense, we have

$$\left[x_0,x_1;\frac{g}{h}\right] = \sum_{j,k=1}^{n} \xi_j \xi_k \left[x_0,x_1;\frac{f_{p_{jk}}}{h}\right] \geq 0,$$

which implies that g/h is a nondecreasing function on J. Therefore, from Theorem 3.15, we have $L(g) \geq 0$. Hence,

$$\sum_{j,k=1}^{n} \xi_j \xi_k L(f_{p_{jk}}) \geq 0.$$

We conclude that the function $p \mapsto L(f_p)$ is $n-$exponentially convex on K in the Jensen sense.
If the function $p \mapsto L(f_p)$ is also continuous on K, then $p \mapsto L(f_p)$ is $n-$exponentially convex by definition. $\square$

The following corollary is a consequence of Theorem 8.11.

Corollary 8.6 *Let h be a positive function and $\Upsilon = \{f_p/h : p \in K\}$ be a family of functions defined on J such that the function $p \mapsto [x_0,x_1; f_p/h]$ is exponentially convex in the Jensen sense on K for mutually different points $x_0,x_1 \in J$. Let L be linear functional defined by (8.16). Then $p \mapsto L(f_p)$ is exponentially convex function in the Jensen sense on K.*
If the function $p \mapsto L(f_p)$ is continuous on K, then it is exponentially convex on K.

Corollary 8.7 *Let h be a positive function and $\Omega = \{f_p/h : p \in K\}$ be a family of functions defined on J such that the function $p \mapsto [x_0,x_1; f_p/h]$ is $2-$exponentially convex in the Jensen sense on K for mutually different points $x_0,x_1 \in J$. Let L be linear functional defined by (8.16). Then the following statements hols:*

(i) If the function $p \mapsto L(f_p)$ is continuous on K, then it is $2-$exponentially convex on K. If $p \mapsto L(f_p)$ is additionally strictly positive it is also log-convex.

(ii) If the function $p \mapsto L(f_p)$ is strictly positive, continuous and differentiable on K, then for every $p,q,u,v \in K$ such that $p \leq u$ and $q \leq v$, we have

$$M_{p,q}(L,\Omega) \leq M_{u,v}(L,\Omega), \tag{8.23}$$

where

$$M_{p,q}(L,\Omega)=\begin{cases}\left(\frac{L(f_p)}{L(f_q)}\right)^{\frac{1}{p-q}}, & p\neq q\\ \exp\left(\frac{\frac{d}{dp}L(f_p)}{L(f_p)}\right), & p=q\end{cases} \tag{8.24}$$

for $f_p/h, f_q/h\in\Omega$.

Proof.

(i) This statement is a consequence of Theorem 8.11 and Remark 1.5.

(ii) Since $p\mapsto L(f_p)$ is continuous and strictly positive, by (i) we have that $p\mapsto L(f_p)$ is log-convex on K, that is, $p\mapsto \log L(f_p)$ is convex on K. Applying Proposition 1.1 we get

$$\frac{\log L(f_p)-\log L(f_q)}{p-q}\leq\frac{\log L(f_u)-\log L(f_v)}{u-v} \tag{8.25}$$

for $p\leq u$, $q\leq v$, $p\neq q$, $u\neq v$. Hence, we conclude that

$$M_{p,q}(L,\Omega)\leq M_{u,v}(L,\Omega),\quad i=2,3,4.$$

Cases $p=q$ and $u=v$ follow from (8.25) as limit cases. □

Remark 8.3 Results from Theorem 8.11 and Corollaries 8.6 and 8.7 still hold when $x_0=x_1\in J$ for a family of differentiable functions with the same property. This follows from Remark 1.2.

Now we will give some examples of families which satisfy previous general results.

Example 8.1 Let h be a positive function and let

$$\Omega_1=\{f_p/h\colon (0,\infty)\to\mathbb{R} : p\in\mathbb{R}\}$$

be a family of functions wheref_p is defined by

$$f_p(x)=\begin{cases}\frac{x^p}{p}h(x), & p\neq 0\\ \log x h(x), & p=0.\end{cases}$$

Since $\frac{d}{dx}\frac{f_p(x)}{h(x)}=x^{p-1}>0$ for $x>0$, f_p/h is a nondecreasing function for $x>0$ and $p\mapsto\frac{d}{dx}\frac{f_p(x)}{h(x)}$ is exponentially convex by definition. Similar as in proof of Theorem 8.11 we have that $p\mapsto[x_0,x_1;f_p/h]$ is exponentially convex (and so exponentially convex in the Jensen sense). Using Corollary 8.6 we conclude that $p\mapsto L(f_p)$ is exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous, so it is exponentially convex. For this family of functions, from (8.24) we have

$$M_{p,q}(L,\Omega_1)=\begin{cases}\left(\frac{L(f_p)}{L(f_q)}\right)^{\frac{1}{p-q}}, & p\neq q\\ \exp\left(\frac{L(f_pf_0)}{L(f_p)}-\frac{1}{p}\right), & p=q\neq 0\\ \exp\left(\frac{L(f_0^2)}{2L(f_0)}\right), & p=q=0.\end{cases}$$

From (8.23) it follows that the function $M_{p,q}(L,\Omega_1)$ is monotonic in parameters p and q.

Theorem 8.17 applied on functions $f_p/h, f_q/h \in \Omega_1$ and functional L implies that there exists $\xi \in (a,b)$ such that

$$\xi^{p-q} = \frac{L(f_p)}{L(f_q)}.$$

Since the function $\xi \mapsto \xi^{p-q}$ is invertible for $p \neq q$ we have

$$a \leq \left(\frac{L(f_p)}{L(f_q)}\right)^{\frac{1}{p-q}} \leq b$$

which together with the fact that $M_{p,q}(L,\Omega_1)$ is continuous, symetric and monotonic shows that $M_{p,q}(L,\Omega_1)$ is a mean. This means are given in explicit form by (8.19) (for function f_p replaced by φ_p defined by (8.18)).

Example 8.2 Let h be a positive function and let

$$\Omega_2 = \{g_p/h\colon \mathbb{R} \to (0,\infty) : p \in \mathbb{R}\}$$

be a family of functions where g_p is defined by

$$g_p(x) = \begin{cases} \frac{e^{px}}{p} h(x), & p \neq 0 \\ xh(x), & p = 0. \end{cases}$$

Since $\frac{d}{dx}\frac{g_p(x)}{h(x)} = e^{px} > 0$, g_p/h is a nondecreasing function on $\mathbb{R}$ for every $p \in \mathbb{R}$ and $p \mapsto \frac{d}{dx}\frac{g_p(x)}{h(x)}$ is exponentially convex by definition. As in Example 8.1 we conclude that $p \mapsto L(g_p)$ is exponentially convex. For this family of functions, from (8.24) we have

$$M_{p,q}(L_1,\Omega_2) = \left(\frac{q}{p}\frac{\int_a^b e^{pt}h(t)g(t)dt - \int_a^{a+\lambda} e^{pt}h(t)dt}{\int_a^b e^{qt}h(t)g(t)dt - \int_a^{a+\lambda} e^{qt}h(t)dt}\right)^{\frac{1}{p-q}}, \quad \text{for } p \neq q;$$

$$M_{p,p}(L_1,\Omega_2) = \exp\left(\frac{\int_a^b e^{pt}th(t)g(t)dt - \int_a^{a+\lambda} e^{pt}th(t)dt}{\int_a^b e^{pt}h(t)g(t)dt - \int_a^{a+\lambda} e^{pt}h(t)dt} - \frac{1}{p}\right), \text{ for } p \neq 0;$$

$$M_{0,0}(L_1,\Omega_2) = \exp\left(\frac{1}{2}\frac{\int_a^b t^2h(t)g(t)dt - \int_a^{a+\lambda} t^2h(t)dt}{\int_a^b th(t)g(t)dt - \int_a^{a+\lambda} th(t)dt}\right).$$

From Theorem 8.17, applied on functions $g_p/h, g_q/h \in \Omega_2$ and functional L, it follows that

$$S_{p,q}(L,\Omega_2) = \log M_{p,q}(L,\Omega_2)$$

satisfies $a \leq S_{p,q}(L,\Omega_2) \leq b$. So $S_{p,q}(L,\Omega_2)$ is monotonic mean by (8.23).

Example 8.3 Let h be a positive function and let

$$\Omega_3 = \{\phi_p/h\colon (0,\infty) \to (0,\infty) : p \in (0,\infty)\}$$

be a family of functions where ϕ_p is defined by

$$\phi_p(x) = \begin{cases} \frac{-p^{-x}}{\log p} h(x), & p \neq 1 \\ x h(x), & p = 1. \end{cases}$$

Since $\frac{d}{dx}\frac{\phi_p(x)}{h(x)} = p^{-x} > 0$ for $p, x \in (0,\infty)$, ϕ_p/h is a nondecreasing function for $x > 0$. $\frac{d}{dx}\frac{\phi_p(x)}{h(x)} = p^{-x}$ is Laplace transform of nonnegative function, so $p \mapsto \frac{d}{dx}\frac{\phi_p(x)}{h(x)}$ is exponentially convex. As in Example 8.1 we conclude that $p \mapsto L(f_p)$ is exponentially convex. For this family of functions, from (8.24) we have

$$M_{p,q}(L_1,\Omega_3) = \left(\frac{\log q}{\log p} \frac{\int_a^b p^{-t} h(t)g(t)dt - \int_a^{a+\lambda} p^{-t}h(t)dt}{\int_a^b q^{-t}h(t)g(t)dt - \int_a^{a+\lambda} q^{-t}h(t)dt} \right)^{\frac{1}{p-q}}, \text{ for } p \neq q,$$

$$M_{p,p}(L_1,\Omega_3) = \exp\left(\frac{-1}{p} \frac{\int_a^b t p^{-t}h(t)g(t)dt - \int_a^{a+\lambda} t p^{-t}h(t)dt}{\int_a^b p^{-t}h(t)g(t)dt - \int_a^{a+\lambda} p^{-t}h(t)dt} - \frac{1}{p\log p} \right),$$

for $p \neq 1$ and

$$M_{1,1}(L_1,\Omega_3) = \exp\left(\frac{-1}{2} \frac{\int_a^b t^2 h(t)g(t)dt - \int_a^{a+\lambda} t^2 h(t)dt}{\int_a^b t h(t)g(t)dt - \int_a^{a+\lambda} t h(t)dt} \right).$$

From Theorem 8.17, applied on functions $\phi_p/h, \phi_q/h \in \Omega_3$ and functional L, it follows that

$$S_{p,q}(L,\Omega_3) = -L(p,q)\log M_{p,q}(L,\Omega_3)$$

satisfies $a \leq S_{p,q}(L,\Omega_3) \leq b$. $L(p,q)$ is logarithmic mean defined by $L(p,q) = \frac{p-q}{\log p - \log q}$. So $S_{p,q}(L,\Omega_3)$ is mean and by (8.23) it is monotonic.

Example 8.4 Let h be a positive function and let

$$\Omega_4 = \{\psi_p/h \colon (0,\infty) \to (0,\infty) : p \in (0,\infty)\}$$

be a family of functions where ψ_p is defined by

$$\psi_p(x) = \frac{-e^{-x\sqrt{p}}}{\sqrt{p}} h(x).$$

Since $\frac{d}{dx}\frac{\psi_p(x)}{h(x)} = e^{-x\sqrt{p}} > 0$, ψ_p/h is a nondecreasing function for $x > 0$. $\frac{d}{dx}\frac{\psi_p(x)}{h(x)} = e^{-x\sqrt{p}}$ is the Laplace transform of nonnegative function, so $p \mapsto \frac{d}{dx}\frac{\psi_p(x)}{h(x)}$ is exponentially convex. As in Example 8.1 we conclude that $p \mapsto L(f_p)$ is exponentially convex. For this family of functions, from (8.24) we have

$$M_{p,q}(L_i,\Omega_4) = \left(\frac{\sqrt{q}}{\sqrt{p}} \frac{\int_a^b e^{-t\sqrt{p}} h(t)g(t)dt - \int_a^{a+\lambda} e^{-t\sqrt{p}}h(t)dt}{\int_a^b e^{-t\sqrt{q}}h(t)g(t)dt - \int_a^{a+\lambda} e^{-t\sqrt{q}}h(t)dt} \right)^{\frac{1}{p-q}}, \text{ for } p \neq q$$

and

$$M_{p,p}(L_i,\Omega_4) = \exp\left(\frac{-1}{2\sqrt{p}}\frac{\int_a^b te^{-t\sqrt{p}}h(t)g(t)dt - \int_a^{a+\lambda} te^{-t\sqrt{p}}h(t)dt}{\int_a^b e^{-t\sqrt{p}}h(t)g(t)dt - \int_a^{a+\lambda} e^{-t\sqrt{p}}h(t)dt} - \frac{1}{2p}\right).$$

From Theorem 8.17, applied on functions $\psi_p/h, \psi_q/h \in \Omega_4$ and functional L, it follows that

$$S_{p,q}(L,\Omega_4) = -(\sqrt{p}+\sqrt{q})\log M_{p,q}(L,\Omega_4)$$

satisfies $a \le S_{p,q}(L,\Omega_4) \le b$. So $S_{p,q}(L,\Omega_4)$ is mean and by (8.23) it is monotonic.

8.3 Gauss–Steffensen means

Results given in this Section are obtained by Krulić, Pečarić and Smoljak in [76]. Motivated by Theorem 2.15, we define linear functional $L : C^1(I) \to \mathbb{R}$ by

$$L(f) = \begin{cases} \int_{G(a)}^{G(b)} f(t)dt - \int_a^b f(t)G'(t)dt, & G(x) \ge x \\ \int_a^b f(t)G'(t)dt - \int_{G(a)}^{G(b)} f(t)dt, & G(x) \le x \end{cases} \tag{8.26}$$

where $G : [a,b] \to \mathbb{R}$ is an increasing and differentiable function such that $a,b,G(a), G(b) \in I$. $L(f)$ is the difference between the left-hand and the right-hand side of the inequality (2.15). Moreover, $L(f) \ge 0$ for all nondecreasing functions f and $L(f) \le 0$ for all nonincreasing functions f.

Note that $L(id) \ge 0$, that is

$$L(id) = \begin{cases} \frac{G^2(b)-G^2(a)}{2} - \int_a^b tG'(t)dt \ge 0, & G(x) \ge x \\ \int_a^b tG'(t)dt - \frac{G^2(b)-G^2(a)}{2} \ge 0, & G(x) \le x. \end{cases}$$

Lagrange type mean value theorem related to functional L is given in the following theorem.

Theorem 8.12 *Let $G : [a,b] \to \mathbb{R}$ be an increasing and differentiable function, $f \in C^1(I)$ such that f' is bounded $(a,b,G(a), G(b) \in I)$. Then there exists $\xi \in I$ such that the equality*

$$\int_{G(a)}^{G(b)} f(t)dt - \int_a^b f(t)G'(t)dt = f'(\xi)\left[\frac{G^2(b)-G^2(a)}{2} - \int_a^b tG'(t)dt\right], \tag{8.27}$$

holds, that is,

$$L(f) = f'(\xi)L(id),$$

where L is defined by (8.26).

Proof. Since f' is continuous and bounded on I there exists $m = \min_{x\in I} f'(x)$ and $M = \max_{x\in I} f'(x)$ both real numbers. Now we consider functions $\Phi_1, \Phi_2 : I \to R$ defined by

$$\Phi_1(x) = f(x) - Mx \quad \text{and} \quad \Phi_2(x) = mx - f(x).$$

Since $\Phi_1, \Phi_2 \in C^1(I)$ and $\Phi_1'(x) = f'(x) - M \leq 0$ and $\Phi_2'(x) = m - f'(x) \leq 0$, functions Φ_1 and Φ_2 are nonincreasing. From Theorem 2.15 we have for $G(x) \geq x$

$$\int_{G(a)}^{G(b)} f(t)dt - \int_a^b f(t)G'(t)dt \leq M\left[\frac{G^2(b) - G^2(a)}{2} - \int_a^b tG'(t)dt\right],$$

that is, $L(f) \leq ML(id)$. Similarly, if we consider a nonincreasing function Φ_2 we obtain $L(f) \geq mL(id)$. Combining these two results we obtain

$$mL(id) \leq L(f) \leq ML(id).$$

If $L(id) = 0$, then $L(f) = 0$, so (8.27) holds for all $\xi \in I$. Otherwise,

$$\min_{x\in I} f'(x) = m \leq \frac{L(f)}{L(id)} \leq M = \max_{x\in I} f'(x), \text{ so } \frac{L(f)}{L(id)} \in f'(I).$$

Since f' is continuous there exists $\xi \in I$ such that $\frac{L(f)}{L(id)} = f'(\xi)$.
To complete the proof we have to consider the case $G(x) \leq x$, but it is analogous to the case $G(x) \geq x$. □

As a special case of Theorem 8.12 we obtain the following corollary, which is in fact Theorem 8.1 and Remark 8.1.

Corollary 8.8 *Assume that f' is continuous and g is integrable function on $[a,b]$ such that $0 \leq g \leq 1$ and $\lambda = \int_a^b g(t)dt$. Then there exist $\xi,\ \eta \in (a,b)$ such that*

$$\int_a^b f(t)g(t)dt - \int_a^{a+\lambda} f(t)dt = f'(\xi)\left[\int_a^b tg(t)dt - \lambda\left(a + \frac{\lambda}{2}\right)\right] \tag{8.28}$$

and

$$\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt = f'(\eta)\left[\int_a^b tg(t)dt - \lambda\left(b - \frac{\lambda}{2}\right)\right]. \tag{8.29}$$

Proof. To prove (8.28) apply Theorem 8.12 with $G(x) = a + \int_a^x g(t)dt$. To obtain (8.29) we take $G(x) = b - \int_x^b g(t)dt$. □

Theorem 8.13 *For $f, h \in C^1(I)$, $h'(x) \neq 0$ for every $x \in I$ and $G : [a,b] \to \mathbb{R}$ an increasing and differentiable function there exists $\xi \in I$ such that*

$$\frac{\int_{G(a)}^{G(b)} f(t)dt - \int_a^b f(t)G'(t)dt}{\int_{G(a)}^{G(b)} h(t)dt - \int_a^b h(t)G'(t)dt} = \frac{f'(\xi)}{h'(\xi)},$$

that is,

$$\frac{L(f)}{L(h)} = \frac{f'(\xi)}{h'(\xi)}$$

where L is defined by (8.26).

Proof. Similar to the proof of Theorem 8.6. □

As a special case of Theorem 8.13 we obtain Corollary 8.1.

Corollary 8.9 *For $\lambda > 0$, $f,\ h \in C^1(\mathbb{R}^+)$, $h'(x) \neq 0$ for every $x \in \mathbb{R}^+$, there exists $\xi \in \mathbb{R}^+$ such that*

$$\frac{\frac{4}{9}\int_0^\infty t^2 f(t)dt - \lambda^2 \int_\lambda^\infty f(t)dt}{\frac{4}{9}\int_0^\infty t^2 h(t)dt - \lambda^2 \int_\lambda^\infty h(t)dt} = \frac{f'(\xi)}{h'(\xi)}. \tag{8.30}$$

Proof. To prove (8.30) we apply Theorem 8.13 for $G(x) = \frac{4x^3}{27\lambda^2} + \lambda$, $\lambda > 0$, $a = 0$, $b \to \infty$, $G(b) \to \infty$. □

We continue with results given by Pečarić and Smoljak in [126].

Theorem 8.14 *Let $\Upsilon = \{f_p : p \in K\}$ be a family of functions defined on J such that the function $p \mapsto [x_0, x_1; f_p]$ is $n-$exponentially convex in the Jensen sense on K for mutually different points $x_0, x_1 \in J$. Let L be linear functional defined by (8.26). Then $p \mapsto L(f_p)$ is $n-$exponentially convex function in the Jensen sense on K.*
If the function $p \mapsto L(f_p)$ is continuous on K, then it is $n-$exponentially convex on K.

Proof. As in proof of Theorem 8.11 we define the function g by (8.22). Since $p \mapsto [x_0, x_1; f_p]$ is $n-$exponentially convex in the Jensen sense, we have

$$[x_0, x_1; g] = \sum_{j,k=1}^{n} \xi_j \xi_k \left[x_0, x_1; f_{p_{jk}}\right] \geq 0,$$

which implies that g is a nondecreasing function on J. Therefore, from Theorem 2.15, we have $L(g) \geq 0$. Hence,

$$\sum_{j,k=1}^{n} \xi_j \xi_k L(f_{p_{jk}}) \geq 0.$$

Similar reasoning as in the proof of Theorem 8.11 completes the proof. □

The following corollaries are consequences of Theorem 8.14.

Corollary 8.10 *Let $\Upsilon = \{f_p : p \in K\}$ be a family of functions defined on J such that the function $p \mapsto [x_0, x_1; f_p]$ is exponentially convex in the Jensen sense on K for mutually different points $x_0, x_1 \in J$. Let L be linear functional defined by (8.26). Then $p \mapsto L(f_p)$ is exponentially convex function in the Jensen sense on K.*
If the function $p \mapsto L(f_p)$ is continuous on K, then it is exponentially convex on K.

Corollary 8.11 *Let $\Omega = \{f_p : p \in K\}$ be a family of functions defined on J such that the function $p \mapsto [x_0, x_1; f_p]$ is $2-$exponentially convex in the Jensen sense on K for mutually different points $x_0, x_1 \in J$. Let L be linear functional defined by (8.26). Then the following statements hols:*

(i) If the function $p \mapsto L(f_p)$ is continuous on K, then it is $2-$exponentially convex on K. If $p \mapsto L(f_p)$ is additionally strictly positive, it is also log-convex.

(ii) If the function $p \mapsto L(f_p)$ is strictly positive, continuous and differentiable on K, then for every $p,q,u,v \in K$ such that $p \leq u$ and $q \leq v$, we have (8.23) where $M_{p,q}(L,\Omega)$ is defined by (8.24) for $f_p, f_q \in \Omega$.

Proof. Similar to the proof of Corollary 8.7. □

Results from Theorem 8.14 and Corollaries 8.10 and 8.11 still hold when $x_0 = x_1 \in J$ for a family of differentiable functions with the same property. This follows from Remark 1.2.

As in previous section we give some families of functions which satisfy this general results.

Example 8.5 Let

$$\Omega_1 = \{f_p \colon (0,\infty) \to \mathbb{R} : p \in \mathbb{R}\}$$

be a family of functions where f_p is defined by

$$f_p(x) = \begin{cases} \frac{x^p}{p}, & p \neq 0 \\ \log x, & p = 0. \end{cases}$$

Similar as in Example 8.1 we obtain $M_{p,q}(L,\Omega_1)$ defined by

$$M_{p,q}(L,\Omega_1) = \begin{cases} \left(\frac{L(f_p)}{L(f_q)}\right)^{\frac{1}{p-q}}, & p \neq q \\ \exp\left(\frac{L(f_p f_0)}{L(f_p)} - \frac{1}{p}\right), & p = q \neq 0 \\ \exp\left(\frac{L(f_0^2)}{2L(f_0)}\right), & p = q = 0. \end{cases}$$

Theorem 8.13 applied on functions $f_p, f_q \in \Omega_1$ and functional L implies that there exists $\xi \in I$ such that

$$\xi^{p-q} = \frac{L(f_p)}{L(f_q)}.$$

Since the function $\xi \mapsto \xi^{p-q}$ is invertible for $p \neq q$ we have

$$\min I \leq \left(\frac{L(f_p)}{L(f_q)}\right)^{\frac{1}{p-q}} \leq \max I$$

which together with the fact that $M_{p,q}(L,\Omega_1)$ is continuous, symetric and monotonic shows that $M_{p,q}(L,\Omega_1)$ is a mean. This means are given in explicit form in [77] (for $p \mapsto p-1$) by

$$M(G;x,y;p,q) = \left(\frac{q-1}{p-1}\,\frac{\int_x^y t^{p-1}G'(t)dt - \frac{G^p(y)-G^p(x)}{p}}{\int_x^y t^{q-1}G'(t)dt - \frac{G^q(y)-G^q(x)}{q}}\right)^{\frac{1}{p-q}}, \tag{8.31}$$

where $p \neq q$, $p,q \neq 1$, $p,q \neq 0$, $y > x > 0$. Continuous extensions of (8.31) can be found in [77].

Remark 8.4 For $G(x) = a + \int_a^x g(t)dt$, where g satisfies conditions of Theorem 8.12, we obtain Steffensen mean

$$S_1(g;x,y;p,q) = \left\{ \frac{q-1}{p-1} \frac{\int_x^y t^{p-1} g(t)dt - \frac{(x+\lambda)^p - x^p}{p}}{\int_x^y t^{q-1} g(t)dt - \frac{(x+\lambda)^q - x^q}{q}} \right\}^{\frac{1}{p-q}}, \tag{8.32}$$

which is in fact $S_1(f,g;x,y;p-1,q-1)$ defined by (8.12) for $f(t) = t$. For $G(x) = b - \int_x^b g(t)dt$ we obtain Steffensen mean

$$S_2(g;x,y;p,q) = \left\{ \frac{q-1}{p-1} \frac{\int_x^y t^{p-1} g(t)dt - \frac{y^p - (y-\lambda)^p}{p}}{\int_x^y t^{q-1} g(t)dt - \frac{y^q - (y-\lambda)^q}{q}} \right\}^{\frac{1}{p-q}}, \tag{8.33}$$

which is in fact $S_2(f,g;x,y;p-1,q-1)$ defined by (8.13) for $f(t) = t$. In (8.32) and (8.33), we assume $p \neq q$, $p,q \neq 1$, $p,q \neq 0$, $y > x > 0$.

Corollary 8.9 enables us to define new means, because if f'/h' has inverse, from (8.30) we have

$$\xi = \left(\frac{f'}{h'}\right)^{-1} \left(\frac{\frac{4}{9}\int_0^\infty t^2 f(t)dt - \lambda^2 \int_\lambda^\infty f(t)dt}{\frac{4}{9}\int_0^\infty t^2 h(t)dt - \lambda^2 \int_\lambda^\infty h(t)dt} \right).$$

Example 8.6 Let

$$\Omega_2 = \{g_p \colon \mathbb{R} \to (0,\infty) : p \in \mathbb{R}\}$$

be a family of functions where g_p is defined by

$$g_p(x) = \begin{cases} \frac{e^{px}}{p}, & p \neq 0 \\ x, & p = 0. \end{cases}$$

Similar as in Example 8.2 we obtain $M_{p,q}(L,\Omega_2)$ defined by

$$M_{p,q}(L,\Omega_2) = \begin{cases} \left(\frac{L(g_p)}{L(g_q)}\right)^{\frac{1}{p-q}}, & p \neq q \\ \exp\left(\frac{L(g_p \cdot g_0)}{L(g_p)} - \frac{1}{p}\right), & p = q \neq 0 \\ \exp\left(\frac{L(g_0^2)}{2L(g_0)}\right), & p = q = 0. \end{cases}$$

From Theorem 8.13, applied on functions $g_p, g_q \in \Omega_2$ and functional L, it follows that

$$S_{p,q}(L,\Omega_2) = \log M_{p,q}(L,\Omega_2)$$

satisfies $\min I \leq S_{p,q}(L,\Omega_2) \leq \max I$. So $S_{p,q}(L,\Omega_2)$ is monotonic mean by (8.23).

Example 8.7 Let

$$\Omega_3 = \{\phi_p \colon (0,\infty) \to (0,\infty) : p \in (0,\infty)\}$$

be a family of functions where ϕ_p is defined by

$$\phi_p(x) = \begin{cases} \frac{-p^{-x}}{\log p}, & p \neq 1 \\ x, & p = 1. \end{cases}$$

Similar as in Example 8.3 we obtain $M_{p,q}(L,\Omega_3)$ defined by

$$M_{p,q}(L_i,\Omega_3) = \begin{cases} \left(\frac{L(\phi_p)}{L(\phi_q)}\right)^{\frac{1}{p-q}}, & p \neq q \\ \exp\left(\frac{-L(\phi_1 \cdot \phi_p)}{pL(\phi_p)} - \frac{1}{p\log p}\right), & p = q \neq 1 \\ \exp\left(\frac{-L(\phi_1^2)}{2L(\phi_1)}\right), & p = q = 1. \end{cases}$$

From Theorem 8.13, applied on functions $\phi_p, \phi_q \in \Omega_3$ and functional L, it follows that

$$S_{p,q}(L,\Omega_3) = -L(p,q)\log M_{p,q}(L,\Omega_3)$$

satisfies $\min I \leq S_{p,q}(L,\Omega_3) \leq \max I$. $L(p,q)$ is logarithmic mean defined by $L(p,q) = \frac{p-q}{\log p - \log q}$. So $S_{p,q}(L,\Omega_3)$ is mean and by (8.23) it is monotonic.

Example 8.8 Let

$$\Omega_4 = \{\psi_p/h \colon (0,\infty) \to (0,\infty) : p \in (0,\infty)\}$$

be a family of functions where ψ_p is defined by

$$\psi_p(x) = \frac{-e^{-x\sqrt{p}}}{\sqrt{p}}$$

Similar as in Example 8.4 we obtain $M_{p,q}(L,\Omega_4)$ defined by

$$M_{p,q}(L,\Omega_4) = \begin{cases} \left(\frac{L(\psi_p)}{L(\psi_q)}\right)^{\frac{1}{p-q}}, & p \neq q \\ \exp\left(\frac{-L(id \cdot \psi_p)}{2\sqrt{p}L(\psi_p)} - \frac{1}{2p}\right), & p = q. \end{cases}$$

From Theorem 8.13, applied on functions $\psi_p, \psi_q \in \Omega_4$ and functional L, it follows that

$$S_{p,q}(L,\Omega_4) = -(\sqrt{p} + \sqrt{q})\log M_{p,q}(L,\Omega_4)$$

satisfies $\min I \leq S_{p,q}(L,\Omega_4) \leq \max I$. So $S_{p,q}(L,\Omega_4)$ is mean and by (8.23) it is monotonic.

Now we will give further generalizations of Gauss-Steffensen's means obtained by Krulić, Pečarić and Smoljak in [76].

Theorem 8.15 *Let $G:[a,b]\to\mathbb{R}$ be an increasing and differentiable function, $f\in C^1(I)$ nondecreasing and f' bounded ($a,b,G(a)$, $G(b)\in I$). If $h\in C^1(f(I))$, then there exists $\xi\in f(I)$ such that*

$$\int_{G(a)}^{G(b)} h(f(t))dt-\int_a^b h(f(t))G'(t)dt=h'(\xi)\left[\int_{G(a)}^{G(b)} f(t)dt-\int_a^b f(t)G'(t)dt\right].$$

Proof. Similar to the proof of Theorem 8.12 (applying Theorem 2.15 to nondecreasing functions $\Phi_1\circ f$ and $\Phi_2\circ f$, where $\Phi_1(x)=Mx-f(x)$, $\Phi_2(x)=f(x)-mx$). □

Corollary 8.12 *Let $f\in C^1(I)$ be a nondecreasing function, $h_1,h_2\in C^1(f(I))$, $h_2'(x)\neq 0$ for every $x\in f(I)$ and $G:[a,b]\to\mathbb{R}$ an increasing and differentiable function. Then there exists $\xi\in f(I)$ such that*

$$\frac{\int_{G(a)}^{G(b)} h_1(f(t))dt-\int_a^b h_1(f(t))G'(t)dt}{\int_{G(a)}^{G(b)} h_2(f(t))dt-\int_a^b h_2(f(t))G'(t)dt}=\frac{h_1'(\xi)}{h_2'(\xi)}.$$

Proof. Similar to the proof of Theorem 8.6. □

Similar as in Example 8.5, Corollary 8.12 applied on functions $f_p,f_q\in\Omega_1$ enables us to define new means. We obtain

$$M_1(G,f;x,y;p,q)=\left(\frac{q-1}{p-1}\frac{\int_x^y f^{p-1}(t)G'(t)dt-\int_{G(y)}^{G(y)} f^{p-1}(t)dt}{\int_x^y f^{q-1}(t)G'(t)dt-\int_{G(y)}^{G(y)} f^{q-1}(t)dt}\right)^{\frac{1}{p-q}},$$

where $p\neq q$, $p,q\neq 1$ $y>x>0$.

This means are considered in [76].

8.4 Note on the inequality of Gauss

Results given in this Section are obtained by Pečarić and Smoljak in [123]. Motivated by Theorem 2.16, in the sequel we will use linear functionals $L_1,L_2:C^1(I)\to\mathbb{R}$ defined by

$$L_1(f)=\int_a^b f(s(x))g'(x)dx-\int_{g(a)}^{g(b)} f(x)dx, \tag{8.34}$$

$$L_2(f)=\int_{g(a)}^{g(b)} f(x)dx-\int_a^b f(t(x))g'(x)dx, \tag{8.35}$$

where $g:[a,b]\to\mathbb{R}$ is increasing, convex and differentiable function, function s is defined by (2.18), function t is defined by (2.19) and $a,b,g(a),g(b),t(a),t(b)\in I$.

Moreover, $L_1(f)\geq 0$, $L_2(f)\geq 0$ for all nondecreasing functions f and $L_1(f)\leq 0$, $L_2(f)\leq 0$ for all nonincreasing functions f.

Theorem 8.16 *Let $g : [a,b] \to \mathbb{R}$ be increasing, convex and differentiable. Let I be a compact interval such that $a, b, g(a), g(b) \in I$, $h_2 : I \to \mathbb{R}$ be nondecreasing and continuous, $J = h_2(I)$, and $h_1 \in C^1(J)$.*

a) Let s be defined by (2.18). Then there exists $\xi \in J$ such that

$$L_1(h_1 \circ h_2) = h_1'(\xi)L_1(h_2),$$

i.e.

$$\int_a^b h_1(h_2(s(x)))g'(x)dx - \int_{g(a)}^{g(b)} h_1(h_2(x))dx =$$
$$h_1'(\xi)\left[\int_a^b h_2(s(x))g'(x)dx - \int_{g(a)}^{g(b)} h_2(x)dx\right],$$

where L_1 is defined by (8.34).

b) Let t be defined by (2.19), $t(a), t(b) \in I$. Then there exists $\eta \in J$ such that

$$L_2(h_1 \circ h_2) = h_1'(\eta)L_2(h_2),$$

where L_2 is defined by (8.35).

Proof.

a) Applying Theorem 2.16 to nondecreasing functions $\Phi_1 \circ h_2$ and $\Phi_2 \circ h_2$, where $\Phi_1(x) = Mx - h_1(x)$ and $\Phi_2(x) = h_1(x) - mx$ and proceeding as in the proof of Theorem 8.12 we prove this theorem.

b) This part can be proved in a similar way. □

Putting $h_2(x) = x$ in the previous theorem we get the following result.

Corollary 8.13 *Let I, J, g, h_1 satisfy assumptions of Theorem 8.16.*

a) If s is defined by (2.18), then there exists $\xi \in I$ such that

$$\int_a^b h_1(s(x))g'(x)dx - \int_{g(a)}^{g(b)} h_1(x)dx = h_1'(\xi)\left[\int_a^b s(x)g'(x)dx - \frac{g^2(b) - g^2(a)}{2}\right].$$

b) If t is defined by (2.19), $t(a), t(b) \in I$, then there exists $\eta \in I$ such that

$$\int_{g(a)}^{g(b)} h_1(x)dx - \int_a^b h_1(t(x))g'(x)dx = h_1'(\eta)\left[\frac{g^2(b) - g^2(a)}{2} - \int_a^b t(x)g'(x)dx\right].$$

Theorem 8.17 *Let $g : [a,b] \to \mathbb{R}$ be increasing, convex and differentiable function. Let I be a compact interval such that $a, b, g(a), g(b) \in I$, $h_2 : I \to \mathbb{R}$ be nondecreasing and continuous, and $J = h_2(I)$. Let $F, H \in C^1(J)$, $H'(x) \neq 0$ for every $x \in J$.*

a) *Let s be defined by (2.18). Then there exists $\xi \in J$ such that*

$$\frac{L_1(F \circ h_2)}{L_1(H \circ h_2)} = \frac{F'(\xi)}{H'(\xi)},$$

where L_1 is defined by (8.34).

b) *Let t be defined by (2.19), $t(a), t(b) \in I$. Then there exists $\eta \in J$ such that*

$$\frac{L_2(F \circ h_2)}{L_2(H \circ h_2)} = \frac{F'(\eta)}{H'(\eta)},$$

where L_2 is defined by (8.35)

Proof. Similar to the proof of Theorem 8.6. □

Corollary 8.14 *Let $g : [a,b] \to \mathbb{R}$ be increasing, convex and differentiable function. Let I be compact interval such that $a, b, g(a)$, $g(b) \in I$. Let $F, H \in C^1(I)$, $H'(x) \neq 0$ for every $x \in I$.*

a) *If s is defined by (2.18), then there exists $\xi \in I$ such that*

$$\frac{\int_a^b F(s(x))g'(x)dx - \int_{g(a)}^{g(b)} F(x)dx}{\int_a^b H(s(x))g'(x)dx - \int_{g(a)}^{g(b)} H(x)dx} = \frac{F'(\xi)}{H'(\xi)}.$$

b) *If t is defined by (2.19), $t(a), t(b) \in I$, then there exists $\eta \in I$ such that*

$$\frac{\int_{g(a)}^{g(b)} F(x)dx - \int_a^b F(t(x))g'(x)dx}{\int_{g(a)}^{g(b)} H(x)dx - \int_a^b H(t(x))g'(x)dx} = \frac{F'(\eta)}{H'(\eta)}.$$

Proof. Apply Theorem 8.17 for $h_2(x) = x$. □

Corollary 8.15 *Let $k > 0$, $F, H \in C^1(\mathbb{R}^+)$, $H'(x) \neq 0$ for every $x \in \mathbb{R}^+$. Then there exists $\xi \in \mathbb{R}^+$ such that*

$$\frac{3\int_0^k x^2 F(x+k)dx - k^2 \int_k^{2k} F(x)dx}{3\int_0^k x^2 H(x+k)dx - k^2 \int_k^{2k} H(x)dx} = \frac{F'(\xi)}{H'(\xi)}. \tag{8.36}$$

Proof. To prove (8.36) apply Theorem 8.17 for $a = 0$, $b = k$, $g(x) = \frac{1}{k^2}x^3 + k$. □

Corollary 8.9 can also be obtained applying Theorem 8.17 b) for $a = 0$, $x_0 = \frac{k}{\sqrt[3]{2}}$, $g(x) = \frac{1}{k^2}x^3 + k$, $b \to \infty$ and $g(b) \to \infty$.

Theorem 8.14 and Corollaries 8.10 and 8.11 hold for functionals L_1 and L_2 defined by (8.34) and (8.35). As in previous section, using Example 8.5 and Theorem 8.17 a), we obtain new means

$$M(h_2, g, s; a, b; p, q) = \left(\frac{q-1}{p-1} \cdot \frac{\int_a^b h_2^{p-1}(s(x))g'(x)dx - \int_{g(a)}^{g(b)} h_2^{p-1}(x)dx}{\int_a^b h_2^{q-1}(s(x))g'(x)dx - \int_{g(a)}^{g(b)} h_2^{q-1}(x)dx}\right)^{\frac{1}{p-q}},$$

where $(p-q)(p-1)(q-1) \neq 0$. Continuous extensions of these means are given in [123].

Furthermore, using Corollary 8.14 a) we can define new means

$$M(g,s;a,b;p,q) = \left(\frac{q-1}{p-1} \cdot \frac{\int_a^b s^{p-1}(x)g'(x)dx - \frac{g^p(b)-g^p(a)}{p}}{\int_a^b s^{q-1}(x)g'(x)dx - \frac{g^q(b)-g^q(a)}{q}} \right)^{\frac{1}{p-q}},$$

where $(p-q)(p-1)(q-1)pq \neq 0$. Continuous extensions of these means are also given in [123].

Theorem 8.17 b) and Corollary 8.14 b) also enable us to define new means, but here we omit the details.

8.5 $n-$exponential convexity of generalizations of Steffensen's inequality

8.5.1 $n-$convex functions

In this subsection we will generate $n-$exponentially and exponentially convex functions from functionals associated with generalizations of Steffensen's inequality for $n-$convex functions. This generalizations are given in Section 4.3, Chapter 5 and Sections 6.2 and 6.3.

Now we will show how to generate means from the differences of two weighted integrals in Section 5.1, and in particular, from generalized Steffensen inequality. This means were obtained by Jakšetić, Pečarić and Perušić in [69]. Similar results related to Section 5.2 were obtained in [75].

First, we define linear functional $A : C^n[a, \max\{b,d\}] \to \mathbb{R}$, under assumptions of Theorem 5.1 with

$$A(f) = \int_a^b u(x)f(x)dx - \int_c^d w(x)f(x)dx - T_a^{f,u} + T_a^{f,w}.$$

Theorem 8.18 *Assume that $u : [a,b] \to \mathbb{R}$ and $w : [c,d] \to \mathbb{R}$ are weight functions and g is defined by (5.2) and (5.3) such that $g \geq 0$ on $[a, \max\{b,d\}]$. Then for any $f \in C^n[a, \max\{b,d\}]$ there exists $\xi \in [a, \max\{b,d\}]$ such that*

$$A(f) = f^{(n)}(\xi)A(e_n) \tag{8.37}$$

where

$$e_n(x) = \frac{(x-a)^n}{n!}. \tag{8.38}$$

Proof. According to Theorem 5.1, $A(h) \geq 0$ for any $n-$convex function $h : [a, \max\{b,d\}] \to \mathbb{R}$.

Let $m = \min f^{(n)}$ and $M = \max f^{(n)}$. For a given function $f \in C^n[a, \max\{b,d\}]$ we define functions $\varphi,\ \psi : [a, \max\{b,d\}] \to \mathbb{R}$ with

$$\varphi(x) = Me_n(x) - f(x)$$

and

$$\psi(x) = f(x) - me_n(x).$$

Now $\varphi^{(n)}(x) = M - f^{(n)}(x) \geq 0$, for any x, and we conclude $A(\varphi) \geq 0$ and then $A(f) \leq M \cdot A(e_n)$. Similarly, from $\psi^{(n)}(x) = f^{(n)}(x) - m \geq 0$ we conclude $m \cdot A(e_n) \leq A(f)$. From $m \cdot A(e_n) \leq A(f) \leq M \cdot A(e_n)$ and continuity of $f^{(n)}$ we conclude that there exists $\xi \in [a, \max\{b,d\}]$ such that (8.37) holds. □

Corollary 8.16 *Assume that $u : [a,b] \to \mathbb{R}$ and $w : [c,d] \to \mathbb{R}$ are weight functions and g is defined by (5.2) and (5.3) such that $g \geq 0$ on $[a, \max\{b,d\}]$ and $A(e_n) \neq 0$, where e_n is defined by (8.38). Then for any $f, h \in C^n[a, \max\{b,d\}]$ there exists $\xi \in [a, \max\{b,d\}]$ such that*

$$\frac{A(f)}{A(h)} = \frac{f^{(n)}(\xi)}{h^{(n)}(\xi)} \tag{8.39}$$

assuming neither of the denominators is equal to zero.

Proof. Define the function $\phi(x) = f(x)A(h) - h(x)A(f)$. According to Theorem 8.18 there exists $\xi \in [a, \max\{b,d\}]$ such that $A(\phi) = \phi^{(n)}(\xi)A(e_n)$. Since $A(\phi) = 0$ it follows that $f^{(n)}(\xi)A(h) - h^{(n)}(\xi)A(f) = 0$. □

Corollary 8.16 enables us to define various types of means, because if $f^{(n)}/h^{(n)}$ has an inverse, from (8.39) we have

$$\xi = \left(\frac{f^{(n)}}{h^{(n)}}\right)^{-1}\left(\frac{A(f)}{A(h)}\right)$$

which means that ξ is mean of numbers $a,\ b,\ c\ , d$ and functions u and v for given functions f and h.

Using results from Section 5.1, we can now make a list of linear functionals which will give us particular examples of Cauchy means.

Using Theorem 5.2: if λ_1 is defined by (5.5) and λ_2 is defined by (5.7) we define

$$\begin{aligned} A_1(f) = &\int_a^b f(x)u(x)dx - \int_a^{a+\lambda_1} f(x)dx \\ &- \sum_{i=0}^{n-2} \frac{f^{(i)}(a)}{i!}\left(\int_a^b u(x)(x-a)^i dx - \frac{\lambda_1^{i+1}}{i+1}\right); \end{aligned} \tag{8.40}$$

$$\begin{aligned} A_2(f) = &\sum_{i=0}^{n-2} \frac{f^{(i)}(a)}{i!}\left(\int_a^b u(x)(x-a)^i dx\right. \\ &\left. - \frac{(b-a)^{i+1} - (b-a-\lambda_2)^{i+1}}{i+1}\right) - \int_a^b f(x)u(x)dx + \int_{b-\lambda_2}^b f(x)dx. \end{aligned} \tag{8.41}$$

$$A_3(f) = \int_{b-\lambda_2}^{b} f(x)dx - \int_{a}^{a+\lambda_1} f(x)dx$$
$$- \sum_{i=0}^{n-2} \frac{f^{(i)}(a)}{i!} \left(\frac{(b-a)^{i+1} - (b-a-\lambda_2)^{i+1}}{i+1} - \frac{\lambda_1^{i+1}}{i+1} \right). \tag{8.42}$$

In a special case, using Corollary 5.1, we define linear functionals

$$A_4(f) = \int_{a}^{b} f(x)u(x)dx - \int_{a}^{a+\lambda_1} f(x)dx \tag{8.43}$$

$$A_5(f) = \int_{b-\lambda_2}^{b} f(x)dx - \int_{a}^{b} f(x)u(x)dx \tag{8.44}$$

$$A_6(f) = \int_{b-\lambda_2}^{b} f(x)dx - \int_{a}^{a+\lambda_1} f(x)dx \tag{8.45}$$

Now, we use an idea from [68] and [116] to generate n-exponentially and exponentially convex functions applying defined functionals.

Theorem 8.19 *Let $\Upsilon = \{f_s : s \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$ such that the function $s \mapsto [x_0, \dots, x_m; f_s]$ is $n-$exponentially convex in the Jensen sense on J for every $m+1$ mutually different points $x_0, \dots, x_m \in I$. Let A_k, $k = 1, \dots, 6$ be linear functionals defined by (8.40)-(8.45). Then $s \mapsto A_k(f_s)$ is $n-$exponentially convex function in the Jensen sense on J.*
If the function $s \mapsto A_k(f_s)$ is continuous on J, then it is $n-$exponentially convex on J.

Proof. For $\xi_i \in \mathbb{R}$, $i = 1, \dots, n$ and $s_i \in J$, $i = 1, \dots, n$, we define the function

$$g(x) = \sum_{i,j=1}^{n} \xi_i \xi_j f_{\frac{s_i+s_j}{2}}(x).$$

Using the assumption that the function $s \mapsto f_s[x_0, \dots, x_m]$ is n-exponentially convex in the Jensen sense, we have

$$g[x_0, \dots, x_m] = \sum_{i,j=1}^{n} \xi_i \xi_j f_{\frac{s_i+s_j}{2}}[x_0, \dots, x_m] \geq 0,$$

which in turn implies that g is a m-convex function on J, so it is $A_k(g) \geq 0$, $k = 1, \dots, 6$, hence

$$\sum_{i,j=1}^{n} \xi_i \xi_j A_k \left(f_{\frac{s_i+s_j}{2}} \right) \geq 0.$$

We conclude that the function $s \mapsto A_k(f_s)$ is n-exponentially convex on J in the Jensen sense.

If the function $s \mapsto A_k(f_s)$ is also continuous on J, then $s \mapsto A_k(f_s)$ is n-exponentially convex by definition. □

The following corollaries are immediate consequences of the above theorem.

Corollary 8.17 *Let $\Upsilon = \{f_s : s \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $s \mapsto f_s[x_0,\ldots,x_m]$ is exponentially convex in the Jensen sense on J for every $m+1$ mutually different points $x_0,\ldots,x_m \in I$. Let $A_k(f)$, $k = 1,\ldots,6$, be linear functionals defined as in (8.40)-(8.45). Then $s \mapsto A_k(f_s)$ is an exponentially convex function in the Jensen sense on J, $k = 1,\ldots,6$. If the function $s \mapsto A_k(f_s)$ is continuous on J, then it is exponentially convex on J.*

Corollary 8.18 *Let $\Upsilon = \{f_s : s \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $s \mapsto f_s[x_0,\ldots,x_m]$ is 2-exponentially convex in the Jensen sense on J for every $m+1$ mutually different points $x_0,\ldots,x_m \in I$. Let $A_k(f)$, $k = 1,\ldots,6$, be linear functional defined as in (8.40)-(8.45). Then the following statements hold:*

(i) If the function $s \mapsto A_k(f_s)$ is continuous on J, then it is 2-exponentially convex function on J. If $s \mapsto A_k(f_s)$ is additionally strictly positive, then it is also log-convex on J. Furthermore, the following inequality holds true:

$$[A_k(f_s)]^{t-r} \leq [A_k(f_r)]^{t-s} [A_k(f_t)]^{s-r}$$

for every choice $r,s,t \in J$, such that $r < s < t$.

(ii) If the function $s \mapsto A_k(f_s)$ is strictly positive and differentiable on J, then for every $s,q,u,v \in J$, such that $s \leq u$ and $q \leq v$, we have

$$\mu_{s,q}(A_k,\Upsilon) \leq \mu_{u,v}(A_k,\Upsilon), \tag{8.46}$$

where

$$\mu_{s,q}(A_k,\Upsilon) = \begin{cases} \left(\frac{A_k(f_s)}{A_k(f_q)}\right)^{\frac{1}{s-q}}, & s \neq q \\ \exp\left(\frac{\frac{d}{ds}A_k(f_s)}{A_k(f_q)}\right), & s = q, \end{cases} \tag{8.47}$$

for $f_s, f_q \in \Upsilon$.

Proof. Similar to the proof of Corollary 8.7. □

Note that the results from above theorem and corollaries still hold when two of the points $x_0,\ldots,x_m \in I$ coincide, say $x_1 = x_0$, for a family of differentiable functions f_s such that the function $s \mapsto f_s[x_0,\ldots,x_m]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, they still hold when all $m+1$ points coincide for a family of m differentiable functions with the same property. The proofs use Remark 1.2 and suitable characterization of convexity.

Now we will present several families of functions which fulfil the conditions of Theorem 8.19, Corollary 8.17 and Corollary 8.18. This enables us to construct examples of exponentially convex functions. Explicit form of this functions is obtained after we calculate explicit action of functionals on a given family.

Example 8.9 Consider a family of functions

$$\Omega_1 = \{f_s : \mathbb{R} \to \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$f_s(x) = \begin{cases} \frac{e^{sx}}{s^n}, & s \neq 0 \\ \frac{x^n}{n!}, & s = 0. \end{cases}$$

Here, $\frac{d^n f_s}{dx^n}(x) = e^{sx} > 0$ which shows that f_s is n-convex on $\mathbb{R}$ for every $s \in \mathbb{R}$ and $s \mapsto \frac{d^n f_s}{dx^n}(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 8.11 we also have that $s \mapsto f_s[x_0, \ldots, x_m]$ is exponentially convex (and so exponentially convex in the Jensen sense). Using Corollary 8.17 we conclude that $s \mapsto A_k(f_s), i = 1, \ldots, 6$, are exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous (although mapping $s \mapsto f_s$ is not continuous for $s = 0$), so it is exponentially convex. For this family of functions, $\mu_{s,q}(A_k, \Omega_1),\ k = 1, \ldots, 6$, from (8.47), becomes

$$\mu_{s,q}(A_k, \Omega_1) = \begin{cases} \left(\frac{A_k(f_s)}{A_k(f_q)} \right)^{\frac{1}{s-q}}, & s \neq q \\ \exp\left(\frac{A_k(id \cdot f_s)}{A_k(f_s)} - \frac{n}{s} \right), & s = q \neq 0 \\ \exp\left(\frac{1}{n+1} \frac{A_k(id \cdot f_0)}{A_k(f_0)} \right), & s = q = 0, \end{cases}$$

where id is the identity function. Also, by Corollary 8.18 it is monotonic function in parameters s and q.

We observe here that $\left(\frac{\frac{d^n f_s}{dx^n}}{\frac{d^n f_q}{dx^n}} \right)^{\frac{1}{s-q}} (\log x) = x$ so using Corollary 8.15 it follows that:

$$M_{s,q}(A_k, \Omega_1) = \log \mu_{s,q}(A_k, \Omega_1), \quad k = 1, \ldots, 6$$

satisfies

$$a \leq M_{s,q}(A_k, \Omega_1) \leq b, \quad k = 1, \ldots, 6.$$

So, $M_{s,q}(A_k, \Omega_1)$ is monotonic mean.

Example 8.10 Consider a family of functions

$$\Omega_2 = \{f_s : (0, \infty) \to \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$g_s(x) = \begin{cases} \frac{x^s}{s(s-1)\cdots(s-n+1)}, & s \notin \{0, 1, \ldots, n-1\} \\ \frac{x^j \log x}{(-1)^{n-1-j} j!(n-1-j)!}, & s = j \in \{0, 1, \ldots, n-1\}. \end{cases}$$

Here, $\frac{d^n g_s}{dx^n}(x) = x^{s-n} > 0$ which shows that g_s is n-convex for $x > 0$ and $s \mapsto \frac{d^n g_s}{dx^n}(x)$ is exponentially convex by definition. Arguing as in Example 8.9 we get that the mappings $s \mapsto A_k(g_s), k = 1, \ldots, 6$ are exponentially convex. For this family of functions

$\mu_{s,q}(A_k,\Omega_2)$, $k=1,\ldots,6$, from (8.47), is now equal to

$$\mu_{s,q}(A_k,\Omega_2)=$$
$$\begin{cases} \left(\frac{A_k(g_s)}{A_k(g_q)}\right)^{\frac{1}{s-q}}, & s\neq q \\ \exp\left((-1)^{n-1}(n-1)!\frac{A_k(g_0g_s)}{A_k(g_s)}+\sum\limits_{i=0}^{n-1}\frac{1}{i-s}\right), & s=q\notin\{0,1,\ldots,n-1\} \\ \exp\left((-1)^{n-1}(n-1)!\frac{A_k(g_0g_s)}{2A_k(g_s)}+\sum\limits_{\substack{i=0\\ i\neq s}}^{n-1}\frac{1}{i-s}\right), & s=q\in\{0,1,\ldots,n-1\}. \end{cases} \tag{8.48}$$

Again, using Corollary 8.15 we conclude that

$$a\leq\left(\frac{A_k(g_s)}{A_k(g_q)}\right)^{\frac{1}{s-q}}\leq b,\quad k=1,\ldots,6.$$

So, $\mu_{s,q}(A_k,\Omega_2), i=1,\ldots,6$ is a monotonic mean.

Example 8.11 Consider a family of functions

$$\Omega_3=\{\phi_s:(0,\infty)\to\mathbb{R}:s\in(0,\infty)\}$$

defined by

$$\phi_s(x)=\begin{cases} \frac{s^{-x}}{(-\log s)^n}, & s\neq 1 \\ \frac{x^n}{n!}, & s=1. \end{cases}$$

Since $\frac{d^n\phi_s}{dx^n}(x)=s^{-x}$ is the Laplace transform of a nonnegative function (see [153]) it is exponentially convex. Obviously ϕ_s are n-convex functions for every $s>0$. For this family of functions, $\mu_{s,q}(A_k,\Omega_3), k=1,\ldots,6$ from (8.47) is equal to

$$\mu_{s,q}(A_k,\Omega_3)=\begin{cases} \left(\frac{A_k(\phi_s)}{Ak(\phi_q)}\right)^{\frac{1}{s-q}}, & s\neq q \\ \exp\left(-\frac{A_k(id\cdot\phi_s)}{s\,A_k(\phi_s)}-\frac{n}{s\log s}\right), & s=q\neq 1 \\ \exp\left(-\frac{1}{n+1}\frac{A_k(id\cdot\phi_1)}{A_k(\phi_1)}\right), & s=q=1, \end{cases}$$

where id is the identity function. This is a monotone function in parameters s and q by (8.46). Using Corollary 8.15 it follows that

$$M_{s,q}(A_k,\Omega_3)=-L(s,q)\log\mu_{s,q}(A_k,\Omega_3),\quad k=1,\ldots,6$$

satisfies

$$a\leq M_{s,q}(A_k,\Omega_3)\leq b.$$

So $M_{s,q}(A_k,\Omega_3)$ is a monotonic mean. $L(s,q)$ is a logarithmic mean defined by

$$L(s,q)=\begin{cases} \frac{s-q}{\log s-\log q}, & s\neq q \\ s, & s=q. \end{cases}$$

Example 8.12 Consider a family of functions

$$\Omega_4 = \{\psi_s : (0,\infty) \to \mathbb{R} : s \in (0,\infty)\}$$

defined by

$$\psi_s(x) = \frac{e^{-x\sqrt{s}}}{(-\sqrt{s})^n}.$$

Since $\frac{d^n\psi_s}{dx^n}(x) = e^{-x\sqrt{s}}$ is the Laplace transform of a nonnegative function (see [153]) it is exponentially convex. Obviously ψ_s are n-convex functions for every $s > 0$. For this family of functions, $\mu_{s,q}(A_k,\Omega_4), k = 1,\ldots,6$ from (8.47) is equal to

$$\mu_{s,q}(A_k,\Omega_4) = \begin{cases} \left(\frac{A_k(\psi_s)}{A_k(\psi_q)}\right)^{\frac{1}{s-q}}, & s \neq q \\ \exp\left(-\frac{A_k(id\cdot\psi_s)}{2\sqrt{s}A_k(\psi_s)} - \frac{n}{2s}\right), & s = q, \end{cases}$$

where id is the identity function. This is monotone function in parameters s and q by (8.46). Using Corollary 8.15 it follows that

$$M_{s,q}(A_k,\Omega_4) = -(\sqrt{s}+\sqrt{q})\log\mu_{s,q}(A_k,\Omega_4), \quad k = 1,\ldots,6$$

satisfies $a \leq M_{s,q}(A_k,\Omega_4) \leq b$, so $M_{s,q}(A_k,\Omega_4)$ is a monotonic mean.

Similarly, we can generate $n-$exponentially and exponentially convex functions from functionals associated with generalizations of Steffensen's inequality given in Section 4.3. Motivated by inequalities (4.37), (4.39) and (4.42) we can define functionals $\Phi_1(f)$, $\Phi_2(f)$ and $\Phi_3(f)$ by

$$\Phi_1(f) = \frac{1}{\int_a^b w(t)dt}\int_a^b w(t)f(t)\,dt - T_{w,n}^{[a,b]}(x) - \frac{1}{\int_c^d u(t)dt}\int_c^d u(t)f(t)\,dt + T_{u,n}^{[c,d]}(x),$$

$$\Phi_2(f) = \frac{1}{\int_a^b w(t)dt}\int_a^b w(t)f(t)\,dt - T_{w,n}^{[a,b]}(x) - \frac{1}{\lambda}\int_a^{a+\lambda} f(t)\,dt + T_{1,n}^{[a,a+\lambda]}(x)$$

and

$$\Phi_3(f) = \frac{1}{\lambda}\int_{b-\lambda}^b f(t)\,dt - T_{1,n}^{[a,b]}(x) - \frac{1}{\int_a^b w(t)dt}\int_a^b w(t)dtf(t)\,dt + T_{w,n}^{[a,b]}(x).$$

Furthermore, we can generate means from Φ_1,Φ_2,Φ_3. For details see [121].

In a similar way in [70] and [120] means from the differences of two weighted integrals, related to results given in Sections 6.2 and 6.3, were generated.

8.5.2 $(2n)-$convex functions

Similar as in previous subsection we will generate $n-$exponentially and exponentially convex functions from functionals associated with generalizations of Steffensen's inequality given in Section 6.1.

Motivated by inequalities (6.7), (6.9) and (6.13) under assumptions of Theorems 6.2, 6.3 and 6.4, respectively, we define following linear functionals:

$$
\begin{aligned}
L_1(f) &= \int_a^b w(t)f(t)\,dt - \int_c^d u(t)f(t)\,dt \\
&- \sum_{k=0}^{n-1}(b-a)^{2k}\int_a^b w(x)\left[f^{(2k)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right)+f^{(2k)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right]dx \\
&+ \sum_{k=0}^{n-1}(d-c)^{2k}\int_c^d u(x)\left[f^{(2k)}(c)\Lambda_k\left(\frac{d-x}{d-c}\right)+f^{(2k)}(d)\Lambda_k\left(\frac{x-c}{d-c}\right)\right]dx,
\end{aligned} \tag{8.49}
$$

$$
\begin{aligned}
L_2(f) &= \int_a^b w(t)f(t)\,dt - \int_a^{a+\lambda} f(t)\,dt \\
&- \sum_{k=0}^{n-1}(b-a)^{2k}\int_a^b w(x)\left[f^{(2k)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right)+f^{(2k)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right]dx \\
&+ \sum_{k=0}^{n-1}\lambda^{2k}\int_a^{a+\lambda}\left[f^{(2k)}(a)\Lambda_k\left(\frac{a+\lambda-x}{\lambda}\right)+f^{(2k)}(a+\lambda)\Lambda_k\left(\frac{x-a}{\lambda}\right)\right]dx,
\end{aligned} \tag{8.50}
$$

$$
\begin{aligned}
L_3(f) &= \int_{b-\lambda}^b f(t)\,dt - \int_a^b w(t)f(t)\,dt \\
&- \sum_{k=0}^{n-1}\lambda^{2k}\int_{b-\lambda}^b\left[f^{(2k)}(b-\lambda)\Lambda_k\left(\frac{b-x}{\lambda}\right)+f^{(2k)}(b)\Lambda_k\left(\frac{x-b+\lambda}{\lambda}\right)\right]dx \\
&+ \sum_{k=0}^{n-1}(b-a)^{2k}\int_a^b w(x)\left[f^{(2k)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right)+f^{(2k)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right]dx.
\end{aligned} \tag{8.51}
$$

Also, we define $I_1=[a,b]\cup[c,d]$, $I_2=[a,b]\cup[a,a+\lambda]$ and $I_3=[a,b]\cup[b-\lambda,b]$.

Under assumptions of Theorems 6.2, 6.3 and 6.4 respectively, it holds $L_i(f)\geq 0$, $i=1,2,3$ for all $(2n)$-convex functions f.

First we will state and prove mean value theorems for defined functionals.

Theorem 8.20 *Let $f: I_i \to \mathbb{R}$ $(i=1,2,3)$ be such that $f\in C^{2n}(I_i)$. If inequalities in (6.6) $(i=1)$, (6.8) $(i=2)$ and (6.12) $(i=3)$ hold, then there exist $\xi_i \in I_i$ such that*

$$L_i(f) = f^{(2n)}(\xi_i)L_i(\zeta), \quad i=1,2,3$$

where $\zeta(x)=\frac{x^{2n}}{(2n)!}$.

Proof. Defining functions $\varphi(x) = \frac{Mx^{2n}}{(2n)!} - f(x)$ and $\psi(x) = f(x) - \frac{mx^{2n}}{(2n)!}$ and proceeding as in the proof of Theorem 8.18 we prove this theorem. □

Corollary 8.19 *Let $f, g : I_i \to \mathbb{R}$ $(i = 1,2,3)$ be such that $f, g \in C^{2n}(I_i)$ and $g^{(2n)}(x) \neq 0$ for every $x \in I_i$. If inequalities in (6.6) $(i = 1)$, (6.8) $(i = 2)$ and (6.12) $(i = 3)$ hold, then there exist $\xi_i \in I_i$ such that*

$$\frac{L_i(f)}{L_i(g)} = \frac{f^{(2n)}(\xi_i)}{g^{(2n)}(\xi_i)}, \quad i = 1,2,3.$$

Proof. Similar to the proof of Corollary 8.16. □

Theorem 8.21 *Let $\Omega = \{f_p : p \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I_i, $i = 1,2,3$ in $\mathbb{R}$ such that the function $p \mapsto f_p[x_0, \ldots, x_{2m}]$ is $n-$exponentially convex in the Jensen sense on J for every $(2m+1)$ mutually different points $x_0, \ldots, x_{2m} \in I_i$, $i = 1,2,3$. Let L_i, $i = 1,2,3$ be linear functionals defined by (8.49)-(8.51). Then $p \mapsto L_i(f_p)$ is $n-$exponentially convex function in the Jensen sense on J. If the function $p \mapsto L_i(f_p)$ is continuous on J, then it is $n-$exponentially convex on J.*

Proof. Similar to the proof of Theorem 8.19. □

Corollary 8.20 *Let $\Omega = \{f_p : p \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I_i, $i = 1,2,3$ in $\mathbb{R}$, such that the function $p \mapsto f_p[x_0, \ldots, x_{2m}]$ is 2-exponentially convex in the Jensen sense on J for every $(2m+1)$ mutually different points $x_0, \ldots, x_{2m} \in I_i$, $i = 1,2,3$. Let L_i, $i = 1,2,3$ be linear functionals defined by (8.49)-(8.51). Then the following statements hold:*

(i) If the function $p \mapsto L_i(f_p)$ is continuous on J, then it is 2-exponentially convex function on J. If $p \mapsto L_i(f_p)$ is additionally strictly positive, then it is also log-convex on J. Furthermore, the following inequality holds true:

$$[L_i(f_s)]^{t-r} \leq [L_i(f_r)]^{t-s} [L_i(f_t)]^{s-r}$$

for every choice $r, s, t \in J$, such that $r < s < t$.

(ii) If the function $p \mapsto L_i(f_p)$ is strictly positive and differentiable on J, then for every $p, q, u, v \in J$, such that $p \leq u$ and $q \leq v$, we have

$$\mu_{p,q}(L_i, \Omega) \leq \mu_{u,v}(L_i, \Omega),$$

where

$$\mu_{p,q}(L_i, \Omega) = \begin{cases} \left(\frac{L_i(f_p)}{L_i(f_q)}\right)^{\frac{1}{p-q}}, & p \neq q \\ \exp\left(\frac{\frac{d}{dp} L_i(f_p)}{L_i(f_p)}\right), & p = q, \end{cases} \tag{8.52}$$

for $f_p, f_q \in \Omega$.

Proof. Similar to the proof of Corollary 8.18. □

Example 8.13 Consider a family of functions

$$\Omega_1 = \{f_p : \mathbb{R} \to [0,\infty) : p \in \mathbb{R}\}$$

defined by

$$f_p(x) = \begin{cases} \frac{e^{px}}{p^{2n}}, & p \neq 0 \\ \frac{x^{2n}}{(2n)!}, & p = 0. \end{cases}$$

As in Example 8.9 we conclude that $p \mapsto L_i(f_p), i = 1,2,3$, are exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous (although mapping $p \mapsto f_p$ is not continuous for $p = 0$), so it is exponentially convex. For this family of functions, $\mu_{p,q}(L_i,\Omega_1),\ i = 1,2,3$, from (8.52), becomes

$$\mu_{p,q}(L_i,\Omega_1) = \begin{cases} \left(\frac{L_i(f_p)}{L_i(f_q)}\right)^{\frac{1}{p-q}}, & p \neq q \\ \exp\left(\frac{L_i(id\cdot f_p)}{L_i(f_p)} - \frac{2n}{p}\right), & p = q \neq 0 \\ \exp\left(\frac{1}{2n+1}\frac{L_i(id\cdot f_0)}{L_i(f_0)}\right), & p = q = 0, \end{cases}$$

where id is the identity function. Also, by Corollary 8.20 it is monotonic function in parameters p and q.
Corollary 8.19 applied on functions $f_p, f_q \in \Omega_1$ and functionals $L_i,\ i = 1,2,3$ implies that there exist $\xi_i \in I_i$ such that

$$e^{(p-q)\xi_i} = \frac{L_i(f_p)}{L_i(f_q)}$$

so it follows that:

$$M_{p,q}(L_i,\Omega_1) = \log \mu_{p,q}(L_i,\Omega_1), \quad i = 1,2,3$$

satisfies

$$\min\{a, b-\lambda, c\} \leq M_{p,q}(L_i,\Omega_1) \leq \max\{a+\lambda, b, d\}, \quad i = 1,2,3.$$

So, $M_{p,q}(L_i,\Omega_1)$ is a monotonic mean.

Example 8.14 Consider a family of functions

$$\Omega_2 = \{g_p : (0,\infty) \to \mathbb{R} : p \in \mathbb{R}\}$$

defined by

$$g_p(x) = \begin{cases} \frac{x^p}{p(p-1)\cdots(p-2n+1)}, & p \notin \{0,1,\ldots,2n-1\} \\ \frac{x^j \log x}{(-1)^{2n-1-j} j!(2n-1-j)!}, & p = j \in \{0,1,\ldots,2n-1\}. \end{cases}$$

As in Example 8.10 for this family of functions $\mu_{p,j}(L_i,\Omega_2),\ i=1,2,3$, from (8.52), is now equal to

$$\mu_{p,q}(L_i,\Omega_2)=$$
$$\begin{cases} \left(\frac{L_i(g_p)}{L_i(g_q)}\right)^{\frac{1}{p-q}}, & p\neq q \\ \exp\left((-1)^{2n-1}(2n-1)!\frac{L_i(g_0 g_p)}{L_i(g_p)}+\sum\limits_{i=0}^{2n-1}\frac{1}{i-p}\right), & p=q\notin\{0,1,\dots,2n-1\} \\ \exp\left((-1)^{2n-1}(2n-1)!\frac{L_i(g_0 g_p)}{2L_i(g_p)}+\sum\limits_{\substack{i=0\\ i\neq p}}^{2n-1}\frac{1}{i-p}\right), & p=q\in\{0,1,\dots,2n-1\}. \end{cases}$$

Again, using Corollary 8.19 we conclude that

$$\min\{a,b-\lambda,c\}\leq\left(\frac{L_i(g_p)}{L_i(g_q)}\right)^{\frac{1}{p-q}}\leq\max\{a+\lambda,b,d\},\quad i=1,2,3.$$

So, $\mu_{p,q}(L_i,\Omega_2), i=1,2,3$ is a mean.

Example 8.15 Consider a family of functions

$$\Omega_3=\{\phi_p:(0,\infty)\to(0,\infty):p\in(0,\infty)\}$$

defined by

$$\phi_p(x)=\begin{cases}\frac{p^{-x}}{(\log p)^{2n}}, & p\neq 1\\ \frac{x^{2n}}{(2n)!}, & p=1.\end{cases}$$

As in Example 8.11 for this family of functions, $\mu_{p,q}(L_i,\Omega_3), i=1,2,3$ from (8.52) is equal to

$$\mu_{p,q}(L_i,\Omega_3)=\begin{cases}\left(\frac{L_i(\phi_p)}{L_i(\phi_q)}\right)^{\frac{1}{p-q}}, & p\neq q\\ \exp\left(-\frac{L_i(id\cdot\phi_p)}{p\ L_i(\phi_p)}-\frac{2n}{p\log p}\right), & p=q\neq 1\\ \exp\left(-\frac{1}{2n+1}\frac{L_i(id\cdot\phi_1)}{L_i(\phi_1)}\right), & p=q=1,\end{cases}$$

where *id* is the identity function. Using Corollary 8.19 it follows that

$$M_{p,q}(L_i,\Omega_3)=-L(p,q)\log\mu_{p,q}(L_i,\Omega_3),\quad i=1,2,3$$

satisfies

$$\min\{a,b-\lambda,c\}\leq M_{p,q}(L_i,\Omega_3)\leq\max\{a+\lambda,b,d\}.$$

So $M_{p,q}(L_i,\Omega_3)$ is a monotonic mean. $L(p,q)$ is a logarithmic mean defined by

$$L(p,q)=\begin{cases}\frac{p-q}{\log p-\log q}, & p\neq q\\ p, & p=q.\end{cases}$$

Example 8.16 Consider a family of functions

$$\Omega_4 = \{\psi_p : (0,\infty) \to (0,\infty) : p \in (0,\infty)\}$$

defined by

$$\psi_p(x) = \frac{e^{-x\sqrt{p}}}{p^n}.$$

As in Example 8.12 for this family of functions, $\mu_{p,q}(L_i,\Omega_4), i = 1,2,3$ from (8.52) is equal to

$$\mu_{p,q}(L_i,\Omega_4) = \begin{cases} \left(\frac{L_i(\psi_p)}{L_i(\psi_q)}\right)^{\frac{1}{p-q}}, & p \neq q \\ \exp\left(-\frac{L_i(id\cdot\psi_p)}{2\sqrt{p}L_i(\psi_p)} - \frac{n}{p}\right), & p = q, \end{cases}$$

where *id* is the identity function. Using Corollary 8.19 it follows that

$$M_{p,q}(L_i,\Omega_4) = -(\sqrt{p}+\sqrt{q})\log\mu_{p,q}(L_i,\Omega_4), \quad i = 1,2,3$$

satisfies $\min\{a,b-\lambda,c\} \leq M_{p,q}(L_i,\Omega_4) \leq \max\{a+\lambda,b,d\}$, so $M_{p,q}(L_i,\Omega_4)$ is a monotonic mean.

Chapter 9

Applications of Steffensen's inequality

9.1 Extension of Hölder's inequality

Integral version of Hölder's inequality is given in the following theorem (see [94, p.106]).

Theorem 9.1 *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on $[a,b]$ and if $|f|^p$ and $|g|^q$ are integrable functions on $[a,b]$ then*

$$\int_a^b |f(x)g(x)|dx \leq \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q dx\right)^{\frac{1}{q}},$$

with equality holding if and only if $A|f(x)|^p = B|g(x)|^q$ almost everywhere, where A and B are constants.

Mudholkar, Freimer and Subbaiah gave in 1984 (see [96]) an extension of Hölder's inequality for monotonic $n-$tuples. In 1987 Iwamoto, Tomkins and Wang presented this result for infinite sequences and gave a corresponding integral analogue (see [64]).

First, we recall a key notion from Freimer and Mudholkar (see [47, p. 64]).

Lemma 9.1 *Let b be a continuous, positive, nonincreasing and integrable function on $[0,\infty)$ and M a positive, real number. Then there exists a number K, $0 \leq K < M$, such that*

$$b(K) \leq \frac{1}{M-K} \int_K^\infty b(t)dt. \tag{9.1}$$

Integral result of Iwamoto, Tomkins and Wang is given in the following theorem.

Theorem 9.2 *Let b, M, K be as in Lemma 9.1. Then*

$$\int_0^{\infty} a(t)b(t)dt \le \left(\int_0^M a^p(t)dt\right)^{\frac{1}{p}} \left(\int_0^M \hat{b}^q(t)dt\right)^{\frac{1}{q}} \tag{9.2}$$

for every nonincreasing, differentiable function a on $[0,\infty)$ *and* $p>1$*, where* $\frac{1}{p}+\frac{1}{q}=1$ *and*

$$\hat{b}(t) = \begin{cases} b(t), & 0 \le t < K \\ \frac{1}{M-K}\int_K^{\infty} b(t)dt, & K \le t \le M. \end{cases}$$

The inequality in (9.2) is reversed if $p<1$ *and a is a nondecreasing, differentiable function. In both cases, equality holds if* $a^p(t) = c\hat{b}^q(t)$, $0 \le t \le M$ *(where c is a constant) and* $a(t) = a(K)$, $t \ge K$.

First, we prove Theorem 9.2 directly, but in the further text we prove that Theorem 9.2 is a consequence of the Steffensen inequality.

Proof. Since a and b are nonincreasing functions on $[0,\infty)$, we have

$$\int_0^{\infty} a(t)b(t)dt \le \int_0^K a(t)b(t)dt + \int_K^M a(t)b(t)dt + a(M)\int_M^{\infty} b(t)dt. \tag{9.3}$$

The function b is nonincreasing, i.e. $b(s) \le b(K)$ for any $s \in [K,M]$, and K satisfies (9.1), i.e. $b(K) \le \frac{1}{M-K}\int_K^{\infty} b(t)dt$. Hence

$$\int_K^t b(s)ds \le (t-K)b(K) \le \frac{t-K}{M-K}\int_K^{\infty} b(t)dt. \tag{9.4}$$

Integrating by parts and since $a' \le 0$ we obtain

$$\begin{aligned}
\int_K^M a(t)b(t)dt &= \left(a(t)\int_K^t b(s)ds\right)\Big|_K^M - \int_K^M a'(t)\left(\int_K^t b(s)ds\right)dt \\
&\le a(M)\int_K^M b(s)ds - \int_K^M a'(t)(t-K)\frac{\int_K^{\infty} b(t)dt}{M-K}dt \\
&= a(M)\int_K^M b(s)ds - \frac{\int_K^{\infty} b(t)dt}{M-K}\left[a(t)(t-K)\Big|_K^M - \int_K^M a(t)dt\right] \\
&= a(M)\int_K^M b(s)ds - \frac{\int_K^{\infty} b(t)dt}{M-K}\left[a(M)(M-K) - \int_K^M a(t)dt\right] \\
&= a(M)\int_K^M b(s)ds - a(M)\int_K^{\infty} b(t)dt + \frac{\int_K^{\infty} b(t)dt}{M-K}\int_K^M a(t)dt \\
&= \frac{1}{M-K}\int_K^M a(t)dt\int_K^{\infty} b(t)dt - a(M)\int_M^{\infty} b(t)dt
\end{aligned} \tag{9.5}$$

i.e.

$$\int_K^M a(t)b(t)dt + a(M)\int_M^{\infty} b(t)dt \le \frac{1}{M-K}\int_K^M a(t)dt\int_K^{\infty} b(t)dt.$$

Hence, using (9.3) we get

$$\begin{aligned}\int_0^\infty a(t)b(t)dt &\le \int_0^K a(t)b(t)dt + \frac{1}{M-K}\int_K^\infty b(t)dt \int_K^M a(t)dt \\ &= \int_0^K a(t)\hat{b}(t)dt + \int_K^M a(t)\hat{b}(t)dt = \int_0^M a(t)\hat{b}(t)dt.\end{aligned} \tag{9.6}$$

Finally, application of Hölder's inequality on the right-hand side of (9.6) yields (9.2) for $p > 1$. □

Theorem 9.2 appears to exhibit some connections with Steffensen's inequality for monotonic function. Pearce and Pečarić proved that connection using the following generalization of Steffensen's inequality, which is in fact Theorem 3.6 with conditions according to [92].

Theorem 9.3 *(i) Suppose that f and g are integrable functions on $[a,b]$, f is nonincreasing and $\lambda > 0$. If a positive function g satisfies the condition*

$$\lambda \int_a^x g(t)dt \le (x-a)\int_a^b g(t)dt \tag{9.7}$$

for every $x \in [a,b]$, then

$$\frac{\int_a^b f(t)g(t)dt}{\int_a^b g(t)dt} \le \lambda^{-1}\int_a^{a+\lambda} f(t)dt, \tag{9.8}$$

while if a positive function g satisfies

$$\lambda \int_x^b g(t)dt \le (b-x)\int_a^b g(t)dt \tag{9.9}$$

for every $x \in [a,b]$, then

$$\lambda^{-1}\int_{b-\lambda}^b f(t)dt \le \frac{\int_a^b f(t)g(t)dt}{\int_a^b g(t)dt}. \tag{9.10}$$

In either case equality holds if f is constant.

(ii) If f is nondecreasing, the reverse inequalities hold in (9.8) and (9.10).

The following extension of Hölder's inequality using the above generalization of Steffensen's inequality is given in [103].

Theorem 9.4 *Let f and g be two integrable and positive functions defined on $[a,b]$ and let M, K be real numbers satisfying $a \le K < M \le b$.*

(i) *Suppose that for every $x \in [K,b]$ we have*

$$\frac{1}{x-K}\int_K^x g(t)dt \leq \frac{1}{M-K}\int_K^b g(t)dt, \tag{9.11}$$

that $p > 1$, $p^{-1}+q^{-1} = 1$ and that f is nonincreasing. Then

$$\int_a^b f(t)g(t)dt \leq \left(\int_a^M f^p(t)dt\right)^{1/p}\left(\int_a^M \hat{g}^q(t)dt\right)^{1/q}, \tag{9.12}$$

where

$$\hat{g}(t) = \begin{cases} g(t), & a \leq t < K \\ \frac{1}{M-K}\int_K^b g(t)dt, & K \leq t \leq M. \end{cases} \tag{9.13}$$

The inequality in (9.12) is reversed if $p < 1$ and f is a nondecreasing function. In both cases, equality holds in (9.12) if

$$f^p(t) = c\hat{g}^q(t), \quad a \leq t \leq M$$

(where c is constant) and

$$f(t) = f(K), \quad t \in [K,b].$$

(ii) *Suppose that for every $x \in [a,M]$ we have*

$$\frac{1}{M-x}\int_x^M g(t)dt \leq \frac{1}{M-K}\int_a^M g(t)dt, \tag{9.14}$$

that $p > 1$, $p^{-1}+q^{-1} = 1$ and that f is nondecreasing. Then

$$\int_a^b f(t)g(t)dt \leq \left(\int_K^b f^p(t)dt\right)^{1/p}\left(\int_K^b \hat{g}^q(t)dt\right)^{1/q}, \tag{9.15}$$

where

$$\hat{g}(t) = \begin{cases} \frac{1}{M-K}\int_a^M g(t)dt, & K \leq t \leq M \\ g(t), & M < t \leq b. \end{cases} \tag{9.16}$$

The inequality in (9.15) is reversed if $p < 1$ and f is a nonincreasing function. In both cases, equality holds in (9.15) if

$$f^p(t) = c\hat{g}^q(t), \quad K \leq t \leq b$$

(where c is constant) and

$$f(t) = f(M), \quad t \in [a,M].$$

Proof.

(i) By (9.11), condition (9.7) is satisfied with $\lambda = M - K$ and a replaced by K. Hence, by Theorem 9.3 (i) (9.8) holds, that is

$$\int_K^b f(t)g(t)dt \leq (M-K)^{-1}\int_K^b g(t)dt\int_K^M f(t)dt$$
$$= \int_K^M f(t)\hat{g}(t)dt.$$

Hence,

$$\int_a^b f(t)g(t)dt = \int_a^K f(t)g(t)dt + \int_K^b f(t)g(t)dt$$
$$\leq \int_a^K f(t)\hat{g}(t)dt + \int_K^M f(t)\hat{g}(t)dt = \int_a^M f(t)\hat{g}(t)dt.$$

Relation (9.12) now follows from Hölder's inequality.

(ii) By (9.14), condition (9.9) is satisfied with $\lambda = M - K$ and b replaced by M. As f is nondecreasing by Theorem 9.3 (ii) reversed inequality in (9.10) holds, that is

$$\int_a^M f(t)g(t)dt \leq (M-K)^{-1}\int_a^M g(t)dt\int_K^M f(t)dt$$
$$= \int_K^M f(t)\hat{g}(t)dt,$$

and we derive

$$\int_a^b f(t)g(t)dt \leq \int_K^b f(t)\hat{g}(t)dt.$$

Relation (9.15) now follows from Hölder's inequality.

The other cases follow similarly, while the statement of equality follows from the condition for equality in Steffensen's and Hölder's inequalities. □

Theorem 9.2 is a simple consequence of the following corollary for $b \to \infty$.

Corollary 9.1 *(i) Suppose the assumptions of Theorem 9.4(i) are satisfied and further g is nonincreasing. Then Theorem 9.4(i) is also valid if condition (9.11) is replaced by*

$$g(K) \leq (M-K)^{-1}\int_K^b g(t)dt.$$

(ii) Suppose the assumptions of Theorem 9.4(ii) are satisfied and further g is nondecreasing. Then Theorem 9.4(ii) is also valid if condition (9.14) is replaced by

$$g(M) \leq (M-K)^{-1}\int_a^M g(t)dt.$$

Proof. If g is nonincreasing, then

$$\frac{1}{x-K}\int_K^x g(t)dt \le g(K) \le \frac{1}{M-K}\int_K^b g(t)dt,$$

that is, (9.11) holds. Similarly, if g is nondecreasing, then

$$\frac{1}{M-x}\int_x^M g(t)dt \le g(M) \le \frac{1}{M-K}\int_a^M g(t)dt,$$

that is, (9.14) holds. □

Corollary 9.2 *Let f, g be positive, integrable functions on $[a,b]$ and M, K real numbers satisfying $a \le K < M \le b$.*

(i) Suppose f is nonincreasing and g nondecreasing. If $p > 1$, $p^{-1}+q^{-1} = 1$, then (9.12) holds. The inequality in (9.12) is reversed if $p < 1$ and f is nondecreasing. The equality case is as in Theorem 9.4(i).

(ii) Suppose f is nondecreasing and g nonincreasing. If $p > 1$, $p^{-1}+q^{-1} = 1$, then (9.15) holds. The inequality in (9.15) is reversed if $p < 1$ and f is nonincreasing. The equality case is as in Theorem 9.4(ii).

Proof. For (i) we have,

$$(x-K)^{-1}\int_K^x g(t)dt \le (b-K)^{-1}\int_K^b g(t)dt \le (M-K)^{-1}\int_K^b g(t)dt,$$

for all $x \in [K,b]$.
For (ii), we have

$$(M-x)^{-1}\int_x^M g(t)dt \le (M-a)^{-1}\int_a^M g(t)dt \le (M-K)^{-1}\int_a^M g(t)dt,$$

for all $x \in [a,M]$. □

9.2 Improvement of an extension of Hölder-type inequality

Motivated by Theorem 9.4 Pečarić and Smoljak gave the following results in [125].

We use the family of functions φ_u defined by

$$\varphi_u(x) = \begin{cases} \frac{x^u}{u}, & u \ne 0 \\ \log x, & u = 0. \end{cases} \tag{9.17}$$

Theorem 9.5 *Let the conditions of Theorem 9.4(i) be satisfied, f be a nondecreasing function and φ_u be defined by (9.17). Then the function $\xi : \mathbb{R} \to [0,\infty)$ defined by*

$$\xi(u) = \int_a^b \varphi_u(f(t))g(t)dt - \int_a^M \varphi_u(f(t))\hat{g}(t)dt \tag{9.18}$$

is exponentially convex.

Proof. First, let us prove that ξ is continuous on $\mathbb{R}$. It is obviously continuous on $\mathbb{R}\setminus\{0\}$. Using L'Hospital rule limit it is easy to verify that $\lim_{s\to 0}\xi(s) = \xi(0)$, so ξ is continuous on $\mathbb{R}$. Let $n \in \mathbb{N}$, $t_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$, $i = 1,2,\ldots,n$, be arbitrary. Denote

$$u_{ij} = \frac{u_i + u_j}{2},$$

and define the function $\Phi : \mathbb{R}^+ \to \mathbb{R}$ by

$$\Phi(x) = \sum_{i,j=1}^{n} t_i t_j \varphi_{u_{ij}}(x).$$

Using similar reasoning as in proof of Theorem 8.8, we obtain that Φ is a nondecreasing function on $\mathbb{R}^+$. By (9.11), condition (9.7) is satisfied with $\lambda = M - K$ and a replaced by K. Hence, by Theorem 9.3, the reverse inequality in (9.8) holds, so for nondecreasing function $\Phi \circ f$ we obtain

$$\int_K^b \Phi(f(t))g(t)dt \geq \int_K^M \Phi(f(t))\hat{g}(t)dt.$$

By definition

$$\int_a^K \Phi(f(t))g(t)dt = \int_a^K \Phi(f(t))\hat{g}(t)dt,$$

so we obtain

$$\int_a^b \Phi(f(t))g(t)dt \geq \int_a^M \Phi(f(t))\hat{g}(t)dt,$$

that is,

$$\sum_{i,j=1}^{n} t_i t_j \xi(u_{ij}) \geq 0$$

concluding positive semi-definitness. Since ξ is continuous, it is exponentially convex function. □

Remark 9.1 Notice that $\xi(u) \geq 0$ for all nondecreasing functions f and $\xi(u) \leq 0$ for all nonincreasing functions f.
For nonincreasing function f, similar as in Theorem 9.5, we obtain that $-\xi(u)$ is exponentially convex.

Remark 9.2 The function ξ, for f nondecreasing and $-\xi$, for f nonincreasing, being exponentially convex is also log-convex function. Then, by (1.6), the following inequality holds true:

$$[|\xi(s)|]^{t-r} \leq [|\xi(r)|]^{t-s} [|\xi(t)|]^{s-r} \tag{9.19}$$

for every choice $r, s, t \in \mathbb{R}$, such that $r < s < t$.

Theorem 9.6 *Let the conditions of Theorem 9.4(ii) be satisfied, f be a nondecreasing function and φ_u be defined by (9.17). Then the function $\zeta : \mathbb{R} \to [0, \infty)$ defined by*

$$\zeta(u) = \int_K^b \varphi_u(f(t))\hat{g}(t)dt - \int_a^b \varphi_u(f(t))g(t)dt \tag{9.20}$$

is exponentially convex.

Proof. Similar to the proof of Theorem 9.5. □

Remark 9.3 Notice that $\zeta(u) \geq 0$ for all nondecreasing functions f and $\zeta(u) \leq 0$ for all nonincreasing functions f.
For nonincreasing function f, similar as in Theorem 9.6, we obtain that $-\zeta(u)$ is exponentially convex.

Remark 9.4 Similar as in Remark 9.2, the following inequality holds true:

$$[|\zeta(s)|]^{t-r} \leq [|\zeta(r)|]^{t-s} [|\zeta(t)|]^{s-r}$$

for every choice $r, s, t \in \mathbb{R}$, such that $r < s < t$.

As a consequence of Theorems 9.5 and 9.6 an improvement of an extension of Hölder-type inequality can be proved.

Theorem 9.7 *Let f and g be two integrable and positive functions defined on $[a,b]$, let $\hat{g}$ be defined by (9.13) and let M, K be real numbers satisfying $a \leq K < M \leq b$. Suppose that for every $x \in [K,b]$ we have (9.11).*

(i) Suppose that $p > 1$, $p^{-1} + q^{-1} = 1$, $1 < s < t$ and that f is nonincreasing. Then

$$\left(\int_a^M f^p(t)dt\right)^{1/p} \left(\int_a^M \hat{g}^q(t)dt\right)^{1/q} - \int_a^b f(t)g(t)dt \geq [-\xi(s)]^{\frac{t-1}{t-s}} [-\xi(t)]^{\frac{1-s}{t-s}}. \tag{9.21}$$

If $p < 1$ and f is a nondecreasing function, then

$$\int_a^b f(t)g(t)dt - \left(\int_a^M f^p(t)dt\right)^{1/p} \left(\int_a^M \hat{g}^q(t)dt\right)^{1/q} \geq [\xi(s)]^{\frac{t-1}{t-s}} [\xi(t)]^{\frac{1-s}{t-s}}. \tag{9.22}$$

(ii) *Suppose that* $p > 1$, $p^{-1} + q^{-1} = 1$, $r < s < 1$ *and that* f *is nonincreasing. Then*

$$\left(\int_a^M f^p(t)dt\right)^{1/p}\left(\int_a^M \hat{g}^q(t)dt\right)^{1/q} - \int_a^b f(t)g(t)dt \geq [-\xi(s)]^{\frac{1-r}{s-r}}[-\xi(r)]^{\frac{s-1}{s-r}}.$$

If $p < 1$ *and* f *is a nondecreasing function, then*

$$\int_a^b f(t)g(t)dt - \left(\int_a^M f^p(t)dt\right)^{1/p}\left(\int_a^M \hat{g}^q(t)dt\right)^{1/q} \geq [\xi(s)]^{\frac{1-r}{s-r}}[\xi(r)]^{\frac{s-1}{s-r}}.$$

Proof.

(i) Taking substitution $r \to 1$ in (9.19) and then raising both sides of inequality (9.19) to the power $\frac{1}{t-s}$ we obtain

$$|\xi(1)| \geq [|\xi(s)|]^{\frac{t-1}{t-s}}[|\xi(t)|]^{\frac{1-s}{t-s}}.$$

For nonincreasing function f, we have

$$|\xi(1)| = -\xi(1) = \int_a^M f(t)\hat{g}(t)dt - \int_a^b f(t)g(t)dt \geq 0.$$

Now by Hölder's inequality we have

$$\begin{aligned}\left(\int_a^M f^p(t)dt\right)^{1/p}\left(\int_a^M \hat{g}^q(t)dt\right)^{1/q} &- \int_a^b f(t)g(t)dt \\ &\geq \int_a^M f(t)\hat{g}(t)dt - \int_a^b f(t)g(t)dt \\ &= -\xi(1) \geq [-\xi(s)]^{\frac{t-1}{t-s}}[-\xi(t)]^{\frac{1-s}{t-s}}.\end{aligned}$$

Hence, we obtain (9.21).

For nondecreasing function f, we have

$$|\xi(1)| = \xi(1) = \int_a^b f(t)g(t)dt - \int_a^M f(t)\hat{g}(t)dt \geq 0.$$

Now by Hölder's inequality for $p < 1$ we have

$$\begin{aligned}&\int_a^b f(t)g(t)dt - \left(\int_a^M f^p(t)dt\right)^{1/p}\left(\int_a^M \hat{g}^q(t)dt\right)^{1/q} \\ &\quad\geq \int_a^b f(t)g(t)dt - \int_a^M f(t)\hat{g}(t)dt = \xi(1) \geq [\xi(s)]^{\frac{t-1}{t-s}}[\xi(t)]^{\frac{1-s}{t-s}}.\end{aligned}$$

Hence, we obtain (9.22).

(ii) Similar to the proof of (i). □

Theorem 9.8 *Let f and g be two integrable and positive functions defined on $[a,b]$, let $\hat{g}$ be defined by (9.16) and let M, K be real numbers satisfying $a \le K < M \le b$. Suppose that for every $x \in [a,M]$ we have (9.14).*

(i) Suppose that $p > 1$, $p^{-1} + q^{-1} = 1$, $1 < s < t$ and that f is nondecreasing. Then

$$\left(\int_K^b f^p(t)dt\right)^{1/p}\left(\int_K^b \hat{g}^q(t)dt\right)^{1/q} - \int_a^b f(t)g(t)dt \ge [-\zeta(s)]^{\frac{t-1}{t-s}}[-\zeta(t)]^{\frac{1-s}{t-s}}.$$

If $p < 1$ and f is a nonincreasing function, then

$$\int_a^b f(t)g(t)dt - \left(\int_K^b f^p(t)dt\right)^{1/p}\left(\int_K^b \hat{g}^q(t)dt\right)^{1/q} \ge [\zeta(s)]^{\frac{t-1}{t-s}}[\zeta(t)]^{\frac{1-s}{t-s}}.$$

(ii) Suppose that $p > 1$, $p^{-1} + q^{-1} = 1$, $r < s < 1$ and that f is nondecreasing. Then

$$\left(\int_K^b f^p(t)dt\right)^{1/p}\left(\int_K^b \hat{g}^q(t)dt\right)^{1/q} - \int_a^b f(t)g(t)dt \ge [-\zeta(s)]^{\frac{1-r}{s-r}}[-\zeta(r)]^{\frac{s-1}{s-r}}.$$

If $p < 1$ and f is a nonincreasing function, then

$$\int_a^b f(t)g(t)dt - \left(\int_K^b f^p(t)dt\right)^{1/p}\left(\int_K^b \hat{g}^q(t)dt\right)^{1/q} \ge [\zeta(s)]^{\frac{1-r}{s-r}}[\zeta(t)]^{\frac{s-1}{s-r}}.$$

Proof. Similar to the proof of Theorem 9.7. □

Using the same method as in the proof of Theorem 8.12 we get the following result.

Theorem 9.9 *Let f be a nondecreasing function on $[a,b]$ and $h \in C^1[a,b]$.*

(i) If the conditions of Theorem 9.4(i) are satisfied, then there exists $\xi \in [a,b]$ such that

$$\int_a^b h(f(t))g(t)dt - \int_a^M h(f(t))\hat{g}(t)dt = h'(\xi)\left[\int_a^b f(t)g(t)dt - \int_a^M f(t)\hat{g}(t)dt\right].$$

(ii) If the conditions of Theorem 9.4(ii) are satisfied, then there exists $\eta \in [a,b]$ such that

$$\int_K^b h(f(t))\hat{g}(t)dt - \int_a^b h(f(t))g(t)dt = h'(\eta)\left[\int_K^b f(t)\hat{g}(t)dt - \int_a^b f(t)g(t)dt\right].$$

Theorem 9.10 *Let $h, H \in C^1[a,b]$, $H'(x) \neq 0$ for every $x \in [a,b]$.*

(i) If the conditions of Theorem 9.9 (i) are satisfied, then there exists $\xi \in [a,b]$ such that

$$\frac{h'(\xi)}{H'(\xi)} = \frac{\int_a^b h(f(t))g(t)dt - \int_a^M h(f(t))\hat{g}(t)dt}{\int_a^b H(f(t))g(t)dt - \int_a^M H(f(t))\hat{g}(t)dt}. \tag{9.23}$$

(ii) If the conditions of Theorem 9.9 (ii) are satisfied, then there exists $\eta \in [a,b]$ such that

$$\frac{h'(\eta)}{H'(\eta)} = \frac{\int_K^b h(f(t))\hat{g}(t)dt - \int_a^b h(f(t))g(t)dt}{\int_K^b H(f(t))\hat{g}(t)dt - \int_a^b H(f(t))g(t)dt}. \tag{9.24}$$

Proof. Similar to the proof of Theorem 8.6. □

Theorem 9.10 enables us to define new means, because if h'/H' has an inverse, from (9.23) and (9.24) we have

$$\xi = \left(\frac{h'}{H'}\right)^{-1}\left(\frac{\int_a^b h(f(t))g(t)dt - \int_a^M h(f(t))\hat{g}(t)dt}{\int_a^b H(f(t))g(t)dt - \int_a^M H(f(t))\hat{g}(t)dt}\right)$$

and

$$\eta = \left(\frac{h'}{H'}\right)^{-1}\left(\frac{\int_K^b h(f(t))\hat{g}(t)dt - \int_a^b h(f(t))g(t)dt}{\int_K^b H(f(t))\hat{g}(t)dt - \int_a^b H(f(t))g(t)dt}\right).$$

Specially, if we take substitutions $h(t) = t^r$, $H(t) = t^s$ in (9.23) and (9.24) we consider the following expressions

$$M_1(a,b;r,s) = \left(\frac{s}{r}\cdot\frac{\int_a^b f^r(t)g(t)dt - \int_a^M f^r(t)\hat{g}(t)dt}{\int_a^b f^s(t)g(t)dt - \int_a^M f^s(t)\hat{g}(t)dt}\right)^{\frac{1}{r-s}}$$

and

$$M_2(a,b;r,s) = \left(\frac{s}{r}\cdot\frac{\int_K^b f^r(t)\hat{g}(t)dt - \int_a^b f^r(t)g(t)dt}{\int_K^b f^s(t)\hat{g}(t)dt - \int_a^b f^s(t)g(t)dt}\right)^{\frac{1}{r-s}},$$

where $rs(r-s) \neq 0$.

Notice that

$$M_1(a,b;r,s) = \left(\frac{\xi(r)}{\xi(s)}\right)^{\frac{1}{r-s}}, \quad M_2(a,b;r,s) = \left(\frac{\zeta(r)}{\zeta(s)}\right)^{\frac{1}{r-s}},$$

where ξ is defined by (9.18) and ζ defined by (9.20).

Moreover, we can extend these means to excluded cases. Taking a limit we can define

$$M_1(a,b;r,0) = \left(\frac{1}{r}\cdot\frac{\int_a^b f^r(t)g(t)dt - \int_a^M f^r(t)\hat{g}(t)dt}{\int_a^b \log f(t)g(t)dt - \int_a^M \log f(t)\hat{g}(t)dt}\right)^{\frac{1}{r}} = M_1(a,b;0,r),$$

$$M_1(a,b;r,r) = \exp\left(\frac{\int_a^b f^r(t)g(t)\log f(t)dt - \int_a^M f^r(t)\hat{g}(t)\log f(t)dt}{\int_a^b f^r(t)g(t)dt - \int_a^M f^r(t)\hat{g}(t)dt} - \frac{1}{r}\right),$$

$$M_1(a,b;0,0) = \exp\left(\frac{\int_a^b \log^2 f(t)g(t)dt - \int_a^M \log^2 f(t)\hat{g}(t)dt}{2\left(\int_a^b \log f(t)g(t)dt - \int_a^M \log f(t)\hat{g}(t)dt\right)}\right),$$

$$M_2(a,b;r,0) = \left(\frac{1}{r} \cdot \frac{\int_K^b f^r(t)\hat{g}(t)dt - \int_a^b f^r(t)g(t)dt}{\int_K^b \log f(t)\hat{g}(t)dt - \int_a^b \log f(t)g(t)dt} \right)^{\frac{1}{r}} = M_2(a,b;0,r),$$

$$M_2(a,b;r,r) = \exp\left(\frac{\int_K^b f^r(t)\hat{g}(t)\log f(t)dt - \int_a^b f^r(t)g(t)\log f(t)dt}{\int_K^b f^r(t)\hat{g}(t)dt - \int_a^b f^r(t)g(t)dt} - \frac{1}{r} \right),$$

$$M_2(a,b;0,0) = \exp\left(\frac{\int_K^b \log^2 f(t)\hat{g}(t)dt - \int_a^b \log^2 f(t)g(t)dt}{2\left(\int_K^b \log f(t)\hat{g}(t)dt - \int_a^b \log f(t)g(t)dt\right)} \right).$$

Theorem 9.11 *Let $p \le r$, $q \le t$. Then*

$$M_1(a,b;p,q) \le M_1(a,b;r,t) \quad \text{and} \quad M_2(a,b;p,q) \le M_2(a,b;r,t) \tag{9.25}$$

for every $x,y \in \mathbb{R}$, $x<y$.

Proof. Similar to the proof of Theorem 8.9. □

Theorem 9.11 can be used for further generalizations of results given in this section.

9.3 Generalizations of Iyengar's inequality

In 1938. Iyengar proved the following inequality (see [65]):

Theorem 9.12 *Let f be a differentiable function on $[a,b]$ and $|f'(x)| \le M$. Then*

$$\left| \frac{1}{b-a}\int_a^b f(x)dx - \frac{f(a)+f(b)}{2} \right| \le \frac{M(b-a)}{4} - \frac{(f(b)-f(a))^2}{4M(b-a)}. \tag{9.26}$$

Although, Iyengar's inequality has been generalized in various ways we give attention to generalizations obtained using Hayashi's modification of Steffensen's inequality (see Theorem 2.2).

In [2] Agarwal and Dragomir proved the following theorem.

Theorem 9.13 *Let function F be differentiable on $[a,b]$ and $m \le F'(x) \le M$. Then*

$$\begin{aligned} &\left| \frac{1}{b-a}\int_a^b F(x)dx - \frac{F(a)+F(b)}{2} \right| \\ &\le \frac{[F(b)-F(a)-m(b-a)][M(b-a)-F(b)+F(a)]}{2(M-m)(b-a)}. \end{aligned} \tag{9.27}$$

Proof. Let $f(x) = a - x$, $g(x) = F'(x) - m$. Applying inequality (2.2) we obtain

$$(M-m)\int_{b-\lambda}^{b}(a-x)dx \le Q \le (M-m)\int_{a}^{a+\lambda}(a-x)dx, \tag{9.28}$$

where

$$Q = \int_a^b (a-x)(F'(x)-m)dx$$

and

$$\lambda = \frac{1}{M-m}\int_a^b (F'(x)-m)dx = \frac{F(b)-F(a)-m(b-a)}{M-m}.$$

Since

$$\int_{b-\lambda}^{b}(a-x)dx = \frac{1}{2}((b-a-\lambda)^2-(b-a)^2)$$

and

$$\int_a^{a+\lambda}(a-x)dx = \frac{-\lambda^2}{2},$$

inequality (9.28) becomes

$$\alpha_1 = (M-m)\left(\frac{(b-a-\lambda)^2-(b-a)^2}{2}\right) \le Q \le (M-m)\left(\frac{-\lambda^2}{2}\right) = \alpha_2. \tag{9.29}$$

Since,

$$\frac{\alpha_1+\alpha_2}{2} = \frac{m(b-a)^2}{2} - \frac{(b-a)(F(b)-F(a))}{2}$$

and

$$Q = \int_a^b F(x)dx - (b-a)F(b) + \frac{m(b-a)^2}{2},$$

it follows that

$$\left|Q - \frac{\alpha_1+\alpha_2}{2}\right| = \left|\int_a^b F(x)dx - (b-a)\frac{F(a)+F(b)}{2}\right|. \tag{9.30}$$

Inequality (9.29) implies

$$\begin{aligned}\left|Q - \frac{\alpha_1+\alpha_2}{2}\right| &\le \frac{\alpha_2-\alpha_1}{2} = \frac{M-m}{2}\left(-\lambda^2+(b-a)\lambda\right)\\ &= \frac{(F(b)-F(a)-m(b-a))(M(b-a)-F(b)+F(a))}{2(M-m)}.\end{aligned} \tag{9.31}$$

Now combining (9.30) and (9.31) we obtain (9.27). □

Inequality (9.27) reduces to (9.26) if we take $M = -m = \sup_{a\le x\le b}|f'(x)|$.

In Section 2.2 we gave weaker conditions for Steffensen's inequality. Here we observe generalizations of Iyengar's inequality using Hayashi's form of Steffensen's inequality. So first we give weaker conditions for Hayashi's form.

Inequality (2.2) holds for every nonincreasing function f if and only if

$$0 \leq \int_x^b g(t)dt \leq A(b-x) \quad \text{and} \quad 0 \leq \int_a^x g(t)dt \leq A(x-a) \tag{9.32}$$

for all $x \in [a,b]$.

Conditions (9.32) can be written in the following form:

$$0 \leq \lambda \int_x^b g(t)dt \leq (b-x)\int_a^b g(t)dt \tag{9.33}$$

and

$$0 \leq \lambda \int_a^x g(t)dt \leq (x-a)\int_a^b g(t)dt. \tag{9.34}$$

Using this form Elezović and Pečarić obtained inequality (9.27) under a weaker condition on function f (see [40]).

Theorem 9.14 *Let $F : I \subseteq \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on I° and $[a,b] \subset I^\circ$ (I° being the interior of I). Let real numbers m and M satisfy*

$$m \leq \frac{F(x)-F(a)}{x-a} \leq M, \tag{9.35}$$

and

$$m \leq \frac{F(b)-F(x)}{b-x} \leq M, \tag{9.36}$$

for all $x \in [a,b]$. If F' is integrable on $[a,b]$, then (9.27) holds.

Proof. Let us take $f(x) = a-x$, $g(x) = F'(x) - m$ and

$$\lambda = \frac{1}{A}\int_a^b g(t)dt = \frac{F(b)-F(a)-m(b-a)}{M-m}.$$

From (9.35) it holds $\lambda \geq 0$. We claim that for such a choice, conditions (9.33) and (9.34) are satisfied. Namely, it holds

$$\int_x^b (F'(x)-m)dx = F(b)-F(x)-m(b-x) \geq 0$$

and the right side of (9.33) is equivalent to

$$\begin{aligned}\frac{F(b)-F(a)-m(b-a)}{M-m} &\cdot [F(b)-F(x)-m(b-x)] \\ &\leq (b-x)[F(b)-F(a)-m(b-a)]\end{aligned}$$

which is true since (9.36) holds. Therefore (9.33) is satisfied. In the same way, we conclude that (9.34) holds. Hence, we can apply (2.2), and in the same way as in proof of Theorem 9.13 we obtain (9.27). □

Elezović and Pečarić in [40] also noted that for $m = -M$ Theorem 9.14 gives:

Corollary 9.3 *Let $F : I \subseteq \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on I° and $[a,b] \subset I^\circ$ such that it holds*

$$|F(x) - F(a)| \le M(x-a), \qquad |F(b) - F(x)| \le M(b-x)$$

for all $x \in [a,b]$. If F' is integrable on $[a,b]$ then

$$\left| \frac{1}{b-a} \int_a^b F(x)dx - \frac{F(a)+F(b)}{2} \right| \le \frac{M(b-a)}{4} - \frac{(F(b)-F(a))^2}{4M(b-a)}. \tag{9.37}$$

As noted in [40] Theorem 9.14 and Corollary 9.3 are equivalent. Let (9.35) and (9.36) be valid. It is clear that this conditions can be given in the following form

$$|\tilde{F}(x) - \tilde{F}(a)| \le M_1(x-a), \qquad |\tilde{F}(b) - \tilde{F}(x)| \le M_1(b-x)$$

where $\tilde{F}(x) = F(x) - \frac{M+m}{2}x$ and $M_1 = \frac{M-m}{2}$. So if we apply Corollary 9.3 on $\tilde{F}$, i.e. using (9.37) for $\tilde{F}$, we obtain (9.27).

Similar statements under weaker conditions on function F are also obtained in [40].

9.3.1 Weighted generalizations of Iyengar's inequality

In 2002 Cerone proved the following result for the trapezoidal rule (see [34]):

Theorem 9.15 *Let $F : I \subseteq \mathbb{R} \to \mathbb{R}$ be such that $F^{(n-1)}$ is absolutely continuous on I° (I° being the interior of I) and $[a,b] \subset I^\circ$. Assume $m = \inf_{x\in[a,b]} F^{(n)}(x) > -\infty$ and $M = \sup_{x\in[a,b]} F^{(n)}(x) < \infty$. Then*

$$\left| \int_a^b F(x)dx - \sum_{k=1}^{n} E_k(x;a,b) + R - \frac{M-m}{2(n+1)!}(U+L) \right| \le \frac{M-m}{2(n+1)!}(U-L) \tag{9.38}$$

where

$$E_k(x;a,b) = \frac{1}{k!}[(x-a)^k F^{(k-1)}(a) - (x-b)^k F^{(k-1)}(b)]$$

$$R = \frac{m}{(n+1)!}\left[(x-b)^{n+1} - (x-a)^{n+1}\right]$$

$$L = \begin{cases} (\lambda_n^a)^{n+1} + (\lambda_n^b)^{n+1}, & n \text{ even} \\ (x-b+\lambda_n^0)^{n+1} - (x-b)^{n+1}, & n \text{ odd} \end{cases}$$

$$U = \begin{cases} (x-b+\lambda_n^b)^{n+1} - (x-a-\lambda_n^a)^{n+1} + (x-a)^{n+1} \\ -(x-b)^{n+1}, & n \text{ even} \\ (x-a)^{n+1} - (x-a-\lambda_n^0)^{n+1}, & n \text{ odd} \end{cases}$$

$$\lambda_n^0 = \frac{1}{M-m}\left[F^{(n-1)}(b) - F^{(n-1)}(a) - m(b-a)\right], \tag{9.39}$$

$$\lambda_n^a = \frac{1}{M-m}\left[F^{(n-1)}(x) - F^{(n-1)}(a) - m(x-a)\right], \tag{9.40}$$

$$\lambda_n^b = \frac{1}{M-m}\left[F^{(n-1)}(b) - F^{(n-1)}(x) - m(b-x)\right]. \tag{9.41}$$

Proof. Let $f(t) = \frac{(x-t)^n}{n!}$, $x \in [a,b]$ and $g(t) = F^{(n)}(t) - m$. First we assume that n is odd. Then $f(t)$ is nonincreasing function so from Hayashi's inequality (2.2) it follows that

$$L_o \leq I_n \leq U_o$$

where

$$\begin{aligned} I_n &= \int_a^b f(t)g(t)dt = \int_a^b \frac{(x-t)^n}{n!} \left(F^{(n)}(t) - m\right) dt \\ &= \int_a^b \frac{(x-t)^n}{n!} F^{(n)}(t)dt + \frac{m}{(n+1)!} \left((x-b)^{n+1} - (x-a)^{n+1}\right). \end{aligned}$$

Integration by parts gives us:

$$\int_a^b \frac{(x-t)^n}{n!} F^{(n)}(t)dt = \int_a^b F(t)dt - \sum_{k=1}^{n} E_k(x;a,b)$$

Hence,

$$I_n = \int_a^b F(t)dt - \sum_{k=1}^{n} E_k(x;a,b) + \frac{m}{(n+1)!} \left((x-b)^{n+1} - (x-a)^{n+1}\right)$$

The left side is

$$L_o = \frac{M-m}{n!} \int_{b-\lambda_n^0}^{b} (x-t)^n dt = \frac{M-m}{(n+1)!} \left[(x-(b-\lambda_n^0))^{n+1} - (x-b)^{n+1}\right]$$

and the right side

$$U_o = \frac{M-m}{n!} \int_a^{a+\lambda_n^0} (x-t)^n dt = \frac{M-m}{(n+1)!} \left[(x-a)^{n+1} - (x-(a+\lambda_n^0))^{n+1}\right]$$

where λ_n^0 is as in (9.39).

Now using

$$m \leq x \leq M \Leftrightarrow \left| x - \frac{M+m}{2} \right| \leq \frac{M-m}{2} \tag{9.42}$$

we obtain theorem statement in the case when n is odd.

Now we assume that n is even. In this case function $f(t) = \frac{(x-t)^n}{n!}$ is nonincreasing for $t \in [a,x]$ and nondecreasing for $t \in [x,b]$. Note that for a nondecreasing function $f(t)$ inequalities in (2.2) are reversed.

On an interval $[a,x]$ we have:

$$L^a \leq I_n^a \leq U^a \tag{9.43}$$

where

$$\begin{aligned}
I_n^a &= \int_a^x \frac{(x-t)^n}{n!}\left(F^{(n)}(t)-m\right)dt,\\
L^a &= \frac{M-m}{n!}\int_{x-\lambda_n^a}^{x}(x-t)^n dt = \frac{M-m}{(n+1)!}(\lambda_n^a)^{n+1},\\
U^a &= \frac{M-m}{n!}\int_a^{a+\lambda_n^a}(x-t)^n dt = \frac{M-m}{(n+1)!}\left[(x-a)^{n+1}-(x-(a+\lambda_n^a))^{n+1}\right]
\end{aligned}$$

where λ_n^a is as in (9.40).

Similar, on an interval $[x,b]$ we have:

$$L^b \le I_n^b \le U^b \tag{9.44}$$

where

$$\begin{aligned}
I_n^b &= \int_x^b \frac{(x-t)^n}{n!}\left(F^{(n)}(t)-m\right)dt,\\
L^b &= \frac{M-m}{n!}\int_x^{x+\lambda_n^b}(x-t)^n dt = \frac{M-m}{(n+1)!}(\lambda_n^b)^{n+1},\\
U^b &= \frac{M-m}{n!}\int_{b-\lambda_n^b}^{b}(x-t)^n dt = \frac{M-m}{(n+1)!}\left[(x-(b-\lambda_n^b))^{n+1}-(x-b)^{n+1}\right]
\end{aligned}$$

where λ_n^b is as in (9.41).

Now, combining (9.43) and (9.44), we obtain

$$L_e \le I_n \le U_e \tag{9.45}$$

where

$$\begin{aligned}
I_n &= I_n^a + I_n^b = \int_a^b \frac{(x-t)^n}{n!}\left(F^{(n)}(t)-m\right)dt\\
&= \int_a^b F(t)dt - \sum_{k=1}^{n} E_k(x;a,b) + \frac{m}{(n+1)!}\left((x-b)^{n+1}-(x-a)^{n+1}\right),\\
L_e &= L^a + L^b = \frac{M-m}{(n+1)!}\left[(\lambda_n^a)^{n+1}+(\lambda_n^b)^{n+1}\right],\\
U_e &= U^a + U^b = \frac{M-m}{(n+1)!}\Big[(x-a)^{n+1}-(x-a-\lambda_n^a)^{n+1}\\
&\quad + (x-b+\lambda_n^b)^{n+1}-(x-b)^{n+1}\Big].
\end{aligned}$$

Again, from (9.42) statement follows in the case when n is even. □

Taking $n=1$ and $x=(a+b)/2$ in (9.38), produces (9.27) under Agarwal-Dragomir conditions.

Using the same technique similar inequalities were proved in a number of papers. In [1] Agarwal, Čuljak and Pečarić derived inequality (9.38) for an odd n. For an even n, using a somewhat different technique, they obtained a result which involves only the midpoint. In [40], only the case $n=2$ was considered. In [38] more special cases were considered. The results obtained there follow from (9.38) by taking $x=(a+b)/2$ and assuming that function F satisfies $F^{(k)}(a)=(-1)^{k+1}F^{(k)}(b)$, for $1<k<n$.

In [54], Gauchman proved two inequalities involving Taylor's remainder. Let $R_{n,f}(c,x)$ denote the nth Taylor's remainder of function $f(x)$ with center c:

$$R_{n,f}(c,x)=f(x)-\sum_{k=0}^{n}\frac{f^{(k)}(c)}{k!}(x-c)^k.$$

Theorem 9.16 *Let $f:I\to\mathbb{R}$ and $w:I\to\mathbb{R}$ be two functions, $a,b\in I^\circ$, $a<b$ and let $f\in C^{n+1}([a,b])$ and $w\in C([a,b])$. Assume that $m\le f^{(n+1)}(x)\le M$, $m\ne M$ and $w(x)\ge 0$ for each $x\in[a,b]$. Then*

$$\begin{aligned}(i)\quad &\frac{1}{(n+1)!}\int_{b-\lambda_n^0}^{b}(x-b+\lambda_n^0)^{n+1}w(x)dx \\ &\le\frac{1}{M-m}\int_a^b\left[R_{n,f}(a,x)-m\frac{(x-a)^{n+1}}{(n+1)!}\right]w(x)dx \\ &\le\frac{1}{(n+1)!}\int_a^b[(x-a)^{n+1}-(x-a-\lambda_n^0)^{n+1}]w(x)dx \\ &+\frac{(-1)^{n+1}}{(n+1)!}\int_a^{a+\lambda_n^0}(a+\lambda_n^0-x)^{n+1}w(x)dx;\end{aligned}\tag{9.46}$$

$$\begin{aligned}(ii)\quad &\frac{1}{(n+1)!}\int_a^{a+\lambda_n^0}(a+\lambda_n^0-x)^{n+1}w(x)dx \\ &\le\frac{(-1)^{n+1}}{M-m}\int_a^b\left[R_{n,f}(b,x)-m\frac{(x-b)^{n+1}}{(n+1)!}\right]w(x)dx \\ &\le\frac{1}{(n+1)!}\int_a^b[(b-x)^{n+1}-(b-\lambda_n^0-x)^{n+1}]w(x)dx \\ &+\frac{(-1)^{n+1}}{(n+1)!}\int_{b-\lambda_n^0}^{b}(x-b+\lambda_n^0)^{n+1}w(x)dx;\end{aligned}\tag{9.47}$$

where λ_n^0 is defined by (9.39).

Taking $n=0$ in Theorem 9.16 we obtain results given by Cerone and Dragomir in [36] as special cases.

Addition of (9.46) and (9.47) upon taking $n=0$ and $w(x)=1$ followed by division by 2, produces (9.27) again.

In [45] Franjić, Pečarić and Perić gave a generalization of both Theorem 9.15 and Theorem 9.16 in a sense that an inequality involving both the weight $w(x)$ and the parameter x is given.

We introduce:

$$h_k(s,t) = \frac{1}{k!}\int_s^t (x-s)^k w(x)dx$$

for $s,t \in [a,b]$ and $k \in \mathbb{N}$.

Theorem 9.17 *Let $F : [a,b] \to \mathbb{R}$ be such that $F^{(n-1)}$ is absolutely continuous on $[a,b]$. Assume that $m \le F^{(n)}(x) \le M$ for each $x \in [a,b]$. Let $w : I \to \mathbb{R}$ be integrable and such that $w(x) \ge 0$ for each $x \in [a,b]$. Let $\Theta \in [a,b]$. Then, when n is odd we have*

$$\begin{aligned}
&(M-m)h_n(b-\lambda_n^0,\Theta) - Mh_n(b,\Theta) + mh_n(a,\Theta) \\
&\le \int_a^b F(x)w(x)dx + \sum_{k=0}^{n-1}\Big[F^{(k)}(b)h_k(b,\Theta) - F^{(k)}(a)h_k(a,\Theta)\Big] \\
&\le Mh_n(a,\Theta) - mh_n(b,\Theta) - (M-m)h_n(a+\lambda_n^0,\Theta)
\end{aligned} \tag{9.48}$$

and when n is even we have

$$\begin{aligned}
&(M-m)[h_n(\Theta-\lambda_n^a,\Theta) - h_n(\Theta+\lambda_n^b,\Theta)] + m[h_n(a,\Theta) - h_n(b,\Theta)] \\
&\le \int_a^b F(x)w(x)dx + \sum_{k=0}^{n-1}\Big[F^{(k)}(b)h_k(b,\Theta) - F^{(k)}(a)h_k(a,\Theta)\Big] \\
&\le M[h_n(a,\Theta) - h_n(b,\Theta)] + (M-m)[h_n(b-\lambda_n^b,\Theta) - h_n(a+\lambda_n^a,\Theta)],
\end{aligned} \tag{9.49}$$

where λ_n^0, λ_n^a and λ_n^b are defined by (9.39), (9.40) and (9.41), respectively.

Proof. For $\Theta \in [a,b]$, set

$$\begin{aligned}
g_k(x) &= F^{(k)}(x) - m, && k = 0,1,\ldots n \\
f_k(x) &= \frac{1}{k!}\int_x^\Theta (t-x)^k w(t)dt && k = 0,1,\ldots n-1
\end{aligned}$$

for each $x \in [a,b]$. Now we have: $0 \le g_n(x) \le M-m$, for each $x \in [a,b]$, so $g_n(x)$ satisfies the conditions of Theorem 2.2. It is easy to prove that

$$f_k'(x) = -f_{k-1}(x)$$

and from there we conclude that for $x \le \Theta$, function $f_{n-1}(x)$ is nonincreasing. For $x \ge \Theta$ and odd n, $f_{n-1}(x)$ is again nonincreasing. However, for $x \ge \Theta$ and even n, $f_{n-1}(x)$ is nondecreasing. Therefore, inequality (2.2) is in that case reversed.

Let us assume first that n is odd. From (2.2) we get

$$(M-m)\int_{b-\lambda_n^0}^b f_{n-1}(x)dx \le \int_a^b f_{n-1}(x)g_n(x)dx \le (M-m)\int_a^{a+\lambda_n^0} f_{n-1}(x)dx.$$

where

$$\lambda_n^0 = \frac{1}{M-m}\int_a^b (F^{(n)}(x)-m)dx$$

as defined in (9.39). Using integration by parts and the fact that $f'_{n-1}(x) = -f_{n-2}(x)$, we easily obtain

$$\begin{aligned} I_n &= \int_a^b f_{n-1}(x)g_n(x)dx \qquad (9.50)\\ &= \int_a^b F(x)w(x)dx + \sum_{k=0}^{n-1}[F^{(k)}(b)h_k(b,\Theta) - F^{(k)}(a)h_k(a,\Theta)] \\ &\quad -mh_n(a,\Theta) + mh_n(b,\Theta). \end{aligned}$$

The upper bound is

$$U_o = \frac{M-m}{(n-1)!}\int_a^{a+\lambda_n^0}\left[\int_x^\Theta (t-x)^{n-1}w(t)dt\right]dx.$$

Assume first that $\Theta \le a+\lambda_n^0$. Changing the order of integration, we obtain

$$U_o = (M-m)[h_n(a,\Theta) - h_n(a+\lambda_n^0,\Theta)]. \qquad (9.51)$$

Assuming $\Theta \ge a+\lambda_n^0$, we get the same expression for the upper bound again.

Analogously, after changing the order of integration in the case when $\Theta \ge b-\lambda_n^0$, the lower bound equals

$$\begin{aligned} L_o &= \frac{M-m}{(n-1)!}\int_{b-\lambda_n^0}^{b}\left[\int_x^\Theta (t-x)^{n-1}w(t)dt\right]dx \\ &= (M-m)[h_n(b-\lambda_n^0,\Theta) - h_n(b,\Theta)]. \qquad (9.52) \end{aligned}$$

For $\Theta \le b-\lambda_n^0$, we get the same expression and thus, once again, obtain the same expression in both cases. Inequality (9.48) is produced by combining (9.50), (9.51) and (9.52), so the statement is proved for the case when n is odd.

Assume now that n is even. $f_{n-1}(x)$ is nonincreasing on $[a,\Theta]$ so inequality (2.2) gives us:

$$L_e^a \le \int_a^\Theta f_{n-1}(x)g_n(x)dx \le U_e^a. \qquad (9.53)$$

It is easy to check that $a+\lambda_n^a \le \Theta$. We calculate both lower and upper bound by changing the order of integration:

$$U_e^a = (M-m)\int_a^{a+\lambda_n^a} f_{n-1}(x)dx = (M-m)[h_n(a,\Theta) - h_n(a+\lambda_n^a,\Theta)],$$

$$L_e^a = (M-m)\int_{\Theta-\lambda_n^a}^{\Theta} f_{n-1}(x)dx = (M-m)h_n(\Theta-\lambda_n^a,\Theta),$$

where

$$\lambda_n^a = \frac{1}{M-m}\int_a^\Theta (F^{(n)}(x)-m)dx$$

as defined in (9.40).

On $[\Theta, b]$, $f_{n-1}(x)$ is nondecreasing so inequality (2.2) is reversed. We have:

$$L_e^b \leq \int_{\Theta}^{b} f_{n-1}(x) g_n(x) dx \leq U_e^b. \tag{9.54}$$

This time $b - \lambda_n^b \geq \Theta$, so it follows

$$U_e^b = (M-m) \int_{b-\lambda_n^b}^{b} f_{n-1}(x) dx = (M-m)[h_n(b-\lambda_n^b, \Theta) - h_n(b, \Theta)],$$
$$L_e^b = (M-m) \int_{\Theta}^{\Theta+\lambda_n^b} f_{n-1}(x) dx = -(M-m) h_n(\Theta+\lambda_n^b, \Theta),$$

where

$$\lambda_n^b = \frac{1}{M-m} \int_{\Theta}^{b} (F^{(n)}(x) - m) dx$$

as defined in (9.41).

Addition of (9.53) and (9.54) gives:

$$L_e \leq I_n \leq U_e$$

where

$$U_e = U_e^a + U_e^b \qquad \text{and} \qquad L_e = L_e^a + L_e^b,$$

and thus inequality (9.49) is produced. The proof of this theorem is now complete. □

Taking $w(x) = 1$ in Theorem 9.17 recaptures Theorem 9.15. Taking $\Theta = b$ produces inequality (9.46) and $\Theta = a$ produces inequality (9.47). Of course, for $w(x) = 1$, $n = 1$ and $\Theta = (a+b)/2$, we get inequality (9.27) again.

Next, we give an alternative inequality for an even n which generalizes results from [1]. Taking $\Theta = (a+b)/2$ and $w(x) = 1$ will produce results from there.

Theorem 9.18 *Assume assumptions of Theorem 9.17 are valid. Then, for $\Theta \in [a,b]$ and even n, we have*

$$\begin{aligned} & m(h_n(a,\Theta) - h_n(b,\Theta)) + (M-m)|h_n(b-\lambda_n, \Theta)| \\ & \leq \int_a^b F(x) w(x) dx + \sum_{k=0}^{n-1} \left[F^{(k)}(b) h_k(b,\Theta) - F^{(k)}(a) h_k(a,\Theta) \right] \\ & \leq M(h_n(a,\Theta) - h_n(b,\Theta)) - (M-m)|h_n(a+\lambda_n, \Theta)| \end{aligned}$$

where $\lambda_n = \lambda_n^a - \lambda_n^b + b - \Theta$, $0 \leq \lambda_n \leq b-a$.

Proof. We use Hayashi's modification of Steffensen's inequality. Set

$$f_{n-1}(x) = \begin{cases} \frac{1}{(n-1)!} \int_x^{\Theta} (t-x)^{n-1} w(t) dt, & a \leq x \leq \Theta \\ \frac{1}{(n-1)!} \int_{\Theta}^{x} (t-x)^{n-1} w(t) dt, & \Theta \leq x \leq b. \end{cases}$$

From the proof of Theorem 9.17 it is clear that f_{n-1} is nonincreasing on $[a,b]$. Taking

$$g_n(x) = \begin{cases} F^{(n)}(x) - m, & a \le x \le \Theta \\ M - F^{(n)}(x), & \Theta \le x \le b. \end{cases}$$

produces our statement. □

Results concerning Iyengar's inequality given in this section can also be found in [46]. Furthermore, comparison between generalizations of Iyengar's inequality obtained through different methods, for a function f such that $f \in C^2[a,b]$ and $|f''(x)| \le M$ can also be found in [46].

In [132] Qi gave a survey of Iyengar's inequality and its generalization given in Theorem 9.13.

9.4 $a(x)-$monotonic functions

The following well known result can be found in [23, p. 133–134]:

If the linear differential equation

$$u'(t) = a(t)u(t), \quad u(0) = c, \tag{9.55}$$

and the linear differential inequality

$$v'(t) \ge a(t)v(t), \quad v(0) = c, \tag{9.56}$$

are both valid for $0 \le t \le T$, then

$$v(t) \ge u(t), \quad 0 \le t \le T. \tag{9.57}$$

Differential inequality $y'(x) - a(x)y(x) \ge 0$ is sometimes used as a definition of generalized increasing functions. In this section we will observe an analoguous generalization, given by Pečarić and Smoljak in [124], which we will call $a(x)-$monotonic functions.

Definition 9.1 *Let f, a be real functions defined on interval $I \subseteq \mathbb{R}$ such that af is integrable. Function f is called* $a(x)-$increasing *on interval I if for every $x, y \in I$*

$$(y-x)(f(y)-f(x)) \ge (y-x)\int_x^y a(t)f(t)dt \tag{9.58}$$

holds.
Function f is called $a(x)-$decreasing *if the inequality in (9.58) is reversed.*
Function f is called $a(x)-$monotonic *if it satisfies (9.58) or the reversed inequality.*
If two functions are both $a(x)-$increasing, or both $a(x)-$decreasing, we say that they are $a(x)-$monotonic in the same sense.

Notice that for $a(x)=0$, f is monotonic.
If $x \neq y$, (9.58) is equivallent to

$$\frac{f(y)-f(x)}{y-x} \geq \frac{1}{y-x}\int_x^y a(t)f(t)dt.$$

If f is $a(x)-$increasing, $-f$ is $a(x)-$decreasing. So we will only give properties of $a(x)-$increasing functions, because they are the same for $a(x)-$decreasing functions.

Properties of $a(x)-$increasing functions:

1) Let f and g be $a(x)-$increasing functions. Then $f+g$ is $a(x)-$increasing.
   If f and g are $a(x)-$monotonic functions (withouth further specifications), we can't conclude that $f+g$ is $a(x)-$monotonic.

2) If f is $a(x)-$increasing function and λ is nonnegative real number, then λf is $a(x)-$increasing function.

In applications we often use $a(x)-$monotonicity criteria given in the following theorem.

Theorem 9.19 *If f' is a continuous function and af an integrable function on interval I, f is $a(x)-$monotonic on interval I if and only if the function $f'(x)-a(x)f(x)$ is nonnegative or non-positive on I. More precisely, f is $a(x)-$increasing function if and only if $f'(x)-a(x)f(x)\geq 0$; f is $a(x)-$decreasing function if and only if $f'(x)-a(x)f(x)\leq 0$.*

Proof. Let f be $a(x)-$increasing function, then for $x,y\in I$ such that $x\neq y$ we have

$$\frac{f(y)-f(x)}{y-x} \geq \frac{1}{y-x}\int_x^y a(t)f(t)dt.$$

Taking a limit when $y\rightarrow x$ we get $f'(x)\geq a(x)f(x)$.
Conversely, let $f'(x)\geq a(x)f(x)$. For $x,y\in I$ such that $x<y$ we have that

$$\int_x^y f'(t)dt \geq \int_x^y a(t)f(t)dt. \tag{9.59}$$

Since f' is continuous, we have that for every $[x,y]\subset I$, $\int_x^y f'(t)dt = f(y)-f(x)$. Furthermore, since $x<y$ we can multiply inequality (9.59) by $y-x$ and get

$$(y-x)(f(y)-f(x)) \geq (y-x)\int_x^y a(t)f(t)dt.$$

Hence, f is $a(x)-$increasing function.
Using the same reasoning we get criteria for $a(x)-$decreasing functions. So the proof is completed. □

Note that for f differentiable and $a(x)-$monotonic we have:

(i) for $a(x)=\frac{1}{x}$, $\frac{f(x)}{x}$ is monotonic (this case is studied in [128]);

(ii) for $a(x) = \frac{a}{x}$, where a is some constant, $\frac{f(x)}{x^a}$ is monotonic;

(iii) for $a(x) = \frac{h'(x)}{h(x)}$, $\frac{f}{h}$ is monotonic (this case is studied in [77]).

Proof of this remarks follows from $a(x)-$monotonicity criteria given in Theorem 9.19. For example, for $a(x) = \frac{1}{x}$ and $a(x)-$increasing function f, we have $f'(x) \geq \frac{f(x)}{x}$, so for $x > 0$ we have $\left(\frac{f(x)}{x}\right)' = \frac{xf'(x)-f(x)}{x^2} \geq \frac{f(x)-f(x)}{x^2} = 0$, hence $\frac{f(x)}{x}$ is increasing function for $x > 0$.

Theorem 9.20 *A function f is $a(x)-$increasing if and only if the function F defined by*

$$F(x) = f(x) - \int_{x_0}^{x} a(t)f(t)dt \tag{9.60}$$

is increasing.

Proof. Suppose that $y > x$. Then (9.58) is equivalent to

$$f(y) - f(x) \geq \int_{x}^{y} a(t)f(t)dt$$

i.e.

$$f(y) - f(x) \geq \int_{x_0}^{y} a(t)f(t)dt - \int_{x_0}^{x} a(t)f(t)dt$$

i.e.

$$f(y) - \int_{x_0}^{y} a(t)f(t)dt \geq f(x) - \int_{x_0}^{x} a(t)f(t)dt$$

i.e.

$$F(y) \geq F(x).$$

Since we have equivalence in each step, the proof is completed. □

We can apply the function F defined by (9.60) to inequalities for monotonic functions and get inequalities for $a(x)-$monotonic functions. Here we give Steffensen's inequality for $a(x)-$monotonic functions.

Corollary 9.4 *Suppose that f is $a(x)-$increasing and g is integrable on $[b,c]$ with $0 \leq g \leq 1$ and $\lambda = \int_b^c g(x)dx$. Then we have*

$$\begin{aligned}\int_{b}^{b+\lambda} f(x)dx - \int_{b}^{b+\lambda}\int_{x_0}^{x} a(t)f(t)dt\,dx &\leq \int_{b}^{c} f(x)g(x)dx - \int_{b}^{c}\left(g(x)\int_{x_0}^{x} a(t)f(t)dt\right)dx \\ &\leq \int_{c-\lambda}^{c} f(x)dx - \int_{c-\lambda}^{c}\int_{x_0}^{x} a(t)f(t)dt\,dx.\end{aligned} \tag{9.61}$$

The inequalities are reversed for f $a(x)-$decreasing.

Proof. Let the function F be defined by (9.60). Since, F is increasing we can apply Steffensen's inequality, hence

$$\int_{b}^{b+\lambda} F(x)dx \leq \int_{b}^{c} F(x)g(x)dx \leq \int_{c-\lambda}^{c} F(x)dx.$$

By elementary calculation we get (9.61). □

Theorem 9.21 *Let $u(t) = ce^{\int_0^t a(t)dt}$ for $0 \leq t \leq T$. Let v satisfy (9.56) with $a(x) \geq 0$ for $0 \leq x \leq T$ and let $v(0) = c$. Then $v(x) - u(x)$ is an increasing function for $0 \leq x \leq T$.*

Proof. Notice that u is the solution of the differential equation $u'(t) - a(t)u(t) = 0$ with $u(0) = c$, so u satisfies (9.55). Hence (9.57) is valid. Since v satisfies (9.56), from Theorem 9.19 we have that v is $a(x)-$monotonic function. So

$$(y-x)(v(y)-v(x)) \geq (y-x)\int_{x}^{y} a(t)v(t)dt. \tag{9.62}$$

For $x, y \in [0,T]$ such that $x < y$ we can divide (9.62) by $y-x$ and then apply (9.57). We get

$$v(y)-v(x) \geq \int_{x}^{y} a(t)v(t)dt \geq \int_{x}^{y} a(t)u(t)dt = \int_{x}^{y} u'(t)dt = u(y)-u(x).$$

Hence,

$$v(y)-u(y) \geq v(x)-u(x).$$

So the proof is completed. □

Now we give Steffensen's inequality for function $v(x) - u(x)$.

Corollary 9.5 *Let functions u and v be such that conditions of Theorem 9.21 are satisfied. Let g be an integrable function on $[0,T]$ with $0 \leq g \leq 1$ and $\lambda = \int_0^T g(t)dt$. Then we have*

$$\int_{0}^{\lambda} (v(t)-u(t))dt \leq \int_{0}^{T} (v(t)-u(t))g(t)dt \leq \int_{T-\lambda}^{T} (v(t)-u(t))dt. \tag{9.63}$$

Proof. From Theorem 9.21 we have that $v(x) - u(x)$ is an increasing function, so we can apply Steffensen's inequality and get (9.63). □

9.5 Other applications

In this section we will give a short survey of other applications of Steffensen's inequality in statistics, functional equations, time scales and special functions.

Statistics

In [48] Gajek and Okolewski proved some bounds for order and record statistics using Steffensen's inequality. In [49] the same authors proved Steffensen type bounds for expectations of the record statistics via Moriguti's inequality combined with Steffensen's inequality. Furthermore, in [50] Gajek and Okolewski improved some lower and upper Steffensen type bounds for expectations of the record statistics using relationship between distributions of the kth record statistics for different values of k.

Combining Moriguti and Steffensen inequalities, Balakrishnan and Rychlik obtained sharp upper bounds for the expectations of arbitrary linear combinations of order statistics from independent identically distributed samples. They expressed bounds in terms of expectations of the left truncated parent distribution and constants that depend only on the coefficients of the linear combinations. For details see [22].

Functional equations

At the 9th International Conference on Functional Equations and Inequalities held in Złockie in 2003, Corovei proposed to look for f and g such that the middle term in Steffensen's inequality is the arithmetic mean of the other two. Let x and y vary in $[a,b]$ and let us write the relevant functional equation with the unknown functions f and g:

$$\int_x^{x+\gamma(x,y)} f(t)dt + \int_{y-\gamma(x,y)}^{y} f(t)dt = 2\int_x^y f(t)g(t)dt, \tag{9.64}$$

where $\gamma(x,y) := \int_x^y g(t)dt$ and $(x,y) \in [a,b]^2$.

In [37] Choczewski, Corovei and Matkowska dealed with equation (9.64) for a differentiable f and a continuous g.

Theorem 9.22 *Assume that $g : [a,b] \to [0,1]$ is a continuous function and either:*

(i) $g(x) = K$, $x \in (a,b)$ *and* $K \neq \{0,1,\frac{1}{2}\}$
or

(ii) $0 < g(x) < 1$, $x \in (a,b)$ *and either* $g(a) = 0$, $g(b) = 1$ *or* $g(a) = 1$, $g(b) = 0$.

Then the function $f : [a,b] \to \mathbb{R}$, differentiable in $[a,b]$, satisfies equation (9.64) if and only if it is of the form:
in case (i) $f(x) = \alpha x + \beta, \quad x \in [a,b]$,
in case (ii) $f(x) = A, \quad x \in [a,b]$,
where α, β, A are arbitrary real numbers.

They also considered a functional equation related to (9.64), with three, sufficiently regular, unknown functions: f, g and h, the latter replacing limits of integration in (9.64) which contain $\gamma(x,y)$.

Special functions

In [35] Cerone utilised Steffensen's inequality and bounds for the Chebyshev functional to obtain bounds for some classical special functions. The author demonstrated methodologies through obtaining novel and useful bounds for the Bessel function of the first kind, the Beta function and the Zeta function.

Time scales

In [18] Anderson gave a time scale version of Steffensen's inequality using nabla integral as follows:
Let $a,b \in \mathbb{T}_\kappa^\kappa$ and let $f,g : [a,b]_\mathbb{T} \to \mathbb{R}$ be nabla integrable functions, with f of one sign and decreasing and $0 \leq g \leq 1$ on $[a,b]_\mathbb{T}$. Assume $l,\gamma \in [a,b]_\mathbb{T}$ are such that

$$\begin{aligned} b-l \leq \int_a^b g(t)\nabla t \leq y-a \quad &\text{if } f \geq 0, t \in [a,b]_\mathbb{T}, \\ \gamma - a \leq \int_a^b g(t)\nabla t \leq b-l \quad &\text{if } f \leq 0, t \in [a,b]_\mathbb{T}. \end{aligned}$$

Then

$$\int_l^b f(t)\nabla t \leq \int_a^b f(t)g(t)\nabla t \leq \int_a^\gamma f(t)\nabla t.$$

As noted by Ozkan and Yildirim in [100], in Anderson's result we could replace the nabla integrals with delta integrals under the same hypothesis and get an analogous result.

In [100] Ozkan and Yildirim also extended some generalizations of Steffensen's inequality to an arbitrary time scale. They obtained Steffensen's inequality on time scales via the diamond-α dynamic integral, which is defined as a linear combination of the delta and nabla integrals.

Bibliography

[1] R. P. Agarwal, V. Čuljak, J. Pečarić, *Some integral inequalities involving bounded higher order derivatives*, Math. Comput. Modelling. **28**(3) (1998), 51–57.

[2] R. P. Agarwal, S. S. Dragomir, *An application of Hayashi's inequality for differentiable functions*, Computers Math. Applic. **32**(6) (1996), 95–99.

[3] R. P. Agarwal, P. J. Y. Wong, *Error inequalities in polynomial interpolation and their applications*, Mathematics and its Applications 262, Kluwer Academic Publishers Group, Dordrecht, 1993.

[4] A. Aglić Aljinović, M. Matić, J. Pečarić, *Improvements of some Ostrowski type inequalities*, J. Comput. Anal. Appl. **7**(3) (2005), 289–304.

[5] A. Aglić Aljinović, J. Pečarić, *Error estimates for approximations of the Fourier transform of functions in L_p spaces*, Aust. J. Math. Anal. Appl. **1**(2) (2004), Article 10.

[6] A. Aglić Aljinović, J. Pečarić, *On some Ostrowski type inequalities via Montgomery identity and Taylor's formula,* Tamkang J. Math. **36**(3) (2005), 199–218.

[7] A. Aglić Aljinović, J. Pečarić, *The weighted Euler identity*, Math. Inequal. Appl. **8**(2) (2005), 207–221.

[8] A. Aglić Aljinović, J. Pečarić, *Generalizations of weighted Euler identity and Ostrowski type inequalities*, Adv. Stud. Contemp. Math. (Kyungshang) **14**(1) (2007), 141–151.

[9] A. Aglić Aljinović, J. Pečarić, I. Perić, *Estimates of the difference between two weighted integral means via weighted Montgomery identity*, Math. Inequal. Appl. **7**(3) (2004), 315–336.

[10] A. Aglić Aljinović, J. Pečarić, A. Perušić, *Generalizations of Steffensen's inequality via weighted Montgomery identity*, Math. Inequal. Appl., to appear.

[11] A. Aglić Aljinović, J. Pečarić, A. Perušić, *Generalizations of Steffensen's inequality via n weight functions*, submitted.

[12] A. Aglić Aljinović, J. Pečarić, A. Vukelić, *The extension of Montgomery identity via Fink identity with applications*, J. Inequal. Appl. **2005**(1) (2005), 67–80.

[13] N. I. Akhiezer, *The classical moment problem and some related questions in analysis*, Oliver and Boyd Ltd, Edinburgh and London, 1965.

[14] M. W. Alomari, *New inequalities of Steffensen's type for $s-$convex functions*, Afr. Mat. (2013), 1–10

[15] H. Alzer, *On an inequality of Gauss*, Rev. Mat. Univ. Complut. Madrid **4**(2-3) (1991), 179–183.

[16] G. A. Anastassiou, *Fractional differentiation inequalities*, Springer, Dordrecht, 2009.

[17] G. A. Anastassiou, *Univariate Ostrowski inequalites, revisited*, Monatsh. Math. **135**(3) (2002), 175–189.

[18] D. R. Anderson, *Time-scale integral inequalites*, J. Inequal. Pure Appl. Math. **6**(3) (2005), Article 16, 15 pages.

[19] M. Andrić, J. Pečarić, I. Perić, *Improvements of composition rule for the Canavati fractional derivatives and applications to Opial-type inequalities,* Dynam. Systems Appl. **20**(2-3) (2011), 383–394.

[20] M. Andrić, J. Pečarić, I. Perić, *A multiple Opial type inequality for the Riemann-Liouville fractional derivatives*, J. Math. Inequal. **7**(1) (2013), 139–150.

[21] R. Apéry, *Une inégalité sur les fonctions de variable réelle*, Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina **2** (1951), 3–4 (french).

[22] N. Balakrishnan, T. Rychlik, *Evaluating expectations of $L-$statistics by the Steffensen inequality*, Metrika **63**(3) (2006), 371–384.

[23] E. F. Beckenbach, R. Bellman, *Inequalities*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1961.

[24] P. R. Beesack, *On the Green's function of an N-point boundary value problem*, Pacific J. Math. **12**(3) (1962), 801–812.

[25] R. Bellman, *On inequalities with alternating signs*, Proc. Amer. Math. Soc. **10**(5) (1959), 807–809.

[26] J. Bergh, *A generalization of Steffensen's inequality*, J. Math. Anal. Appl. **41**(1) (1973), 187–191.

[27] S. N. Bernstein, *Sur les fonctions absolument monotones*, Acta Math. **52**(1) (1929), 1–66 (french).

[28] R. P. Boas, *Representation of functions by Lidstone series*, Duke Math. J. **10** (1943), 239–245.

[29] R. P. Boas, *The Jensen-Steffensen inequality*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. **302–319** (1970), 1–8.

[30] P. S. Bullen, *The Steffensen inequality*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz. **320-328** (1970), 59–63.

[31] Y. Cao, *On Steffensen's inequality*, J. Math. Res. Exposition **16**(1) (1996), 20–20.

[32] Y. Cao, *The correction of extension of Steffensen's inequality*, J. Math. Res. Exposition **11**(1) (1991), 151–153.

[33] P. Cerone, *On some generalisations of Steffensen's inequality and related results*, J. Inequal. Pure Appl. Math. **2**(3) (2001), Article 28.

[34] P. Cerone, *Generalised trapezoidal rules with error involving bounds of the nth derivative*, Math. Inequal. Appl. **5**(3) (2002), 451–462.

[35] P. Cerone, *Special functions: approximations and bounds*, Appl. Anal. Discrete Math. **1**(1) (2007), 72–91.

[36] P. Cerone, S. S. Dragomir, *On a weighted generalization of Iyengar type inequalities involving bounded first derivative*, Math. Inequal. Appl. **3**(1) (2000), 35–44.

[37] B. Choczewski, I. Corovei, A. Matkowska, *On some functional equations related to Steffensen's inequality*, Ann. Univ. Paedagog. Crac. Stud. Math. **4** (2004), 31–37.

[38] V. Čuljak, N. Elezović, *A note on Steffensen's and Iyengar's inequality*, Glas. Mat. Ser. III **33**(53) (1998), 167–171.

[39] P. J. Davis, *Interpolation and Approximation*, Blaisdell Publishing Co. Ginn and Co., New York-Toronto-London, 1963.

[40] N. Elezović, J. Pečarić, *Steffensen's inequality and estimates of error in trapezoidal rule*, Appl. Math. Lett. **11**(6) (1998), 63–69.

[41] J. C. Evard, H. Gauchman, *Steffensen type inequalites over general measure spaces*, Analysis **17**(2-3) (1997), 301–322.

[42] A. M. Fink, *Steffensen type inequalities*, Rocky Mountain J. Math. **12**(4) (1982), 785–793.

[43] A. M. Fink, *Bounds on the deviation of a function from its averages*, Czechoslovak Math. J. **42**(117) (1992), 289–310.

[44] A.M. Fink, M. Jodeit Jr., *On Chebyshev's other inequality*, Inequalities in Statistics and Probability, IMS Lecture Notes-Monograph Series 5 (1984), 115–120.

[45] I. Franjić, J. Pečarić, I. Perić, *Iyengar type estimate of error in trapezoidal rule*, Int. J. Pure Appl. Math. **26**(3) (2006), 285–295.

[46] I. Franjić, J. Pečarić, I. Perić, A. Vukelić, *Euler integral identity, quadrature formulae and error estimates (from the point of view of inequality theory)*, Monographs in inequalites 2, Element, Zagreb, 2011.

[47] M. Freimer, G. S. Mudholkar, *A class of generalizations of Hölder's inequality*, Inequalities in Statistics and Probability, IMS Lecture Notes - Monograph Series 5 (1984), 59–67.

[48] L. Gajek, A. Okolewski, *Steffensen-type inequalites for order and record statistics*, Ann. Univ. Mariae Curie-Skłodowska, Sect. A **51**(1) (1997) 41–59.

[49] L. Gajek, A. Okolewski, *Sharp bounds on moments of generalized order statistics*, Metrika **52**(1) (2000), 27–43.

[50] L. Gajek, A. Okolewski, *Improved Steffensen type bounds on expectations of record statistics*, Statist. Probab. Lett. **55**(2) (2001), 205–212.

[51] H. Gauchman, *A Steffensen type inequality*, J. Inequal. Pure Appl. Math. **1**(1) (2000), Article 3.

[52] H. Gauchman, *Steffensen pairs and associated inequalites*, J. Inequal. Appl. **5**(1) (2000), 53–61.

[53] H. Gauchman, *On further generalization of Steffensen's inequality*, J. Inequal. Appl. **5**(5) (2000), 505–513.

[54] H. Gauchman, *Some integral inequalities involving Taylor's remainder I*, J. Inequal. Pure Appl. Math. **3**(2) (2002), Article 26.

[55] C. F. Gauss, Theoria combinationis observationum, 1821., German transl. in *Abhandlungen zur Methode der kleinsten Quadrate.* Neudruck. Würzburg 1964, pp. 9 and 12.

[56] E. K. Godunova, V. I. Levin, *Obšćii klass neravenstv, soderžaščii neravenstvo Steffensena*, Mat. Zametki **3**(3) (1968), 339-344 (russian) (eng. A general class of inequalites containing Steffensen's inequality).

[57] E. K. Godunova, V. I. Levin, I. V. Čebaevskaja, *Novye issledovanija po funkcional'nym neravenstvam*, Materialy šestoi mež. fiz.-mat. nauč. konf. Dal'nego Vostoka. Diff. i Int. Uravenija. Tom 3. Habarovsk (1967), 70–77 (russian) (eng. New results about functional inequalites).

[58] V. L. Gontscharoff, *Theory of Interpolation and Approximation of Functions*, Gostekhizdat, Moscow, 1954.

[59] G. D. Handley, J. J. Koliha, J. Pečarić, *Hilbert-Pachpatte type integral inequalities for fractional derivatives*, Fract. Calc. Appl. Anal. **4**(1) (2001), 37–46.

[60] G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, Cambridge Univ. Press, New York 1973.

[61] T. Hayashi, *On curves with monotonous curvature*, Tohoku Math. J. **15** (1919), 236–239.

[62] T. Hayashi, *On certain inequalities*, Tohoku Math. J. **18** (1920), 75–89.

[63] C. O. Imoru, *Some extensions of Steffensen's inequality*, Facta Univ. Ser. Math. Inform. **6** (1991), 7–12.

[64] S. Iwamoto, R. D. Tomkins, C. L. Wang, *Inequalities and mathematical programming, III*, General inequalities 5, Birchäuser Basel (1987), 419–432.

[65] K. S. K. Iyengar, *Note on an inequality*, Math. Student **6** (1938), 75–76.

[66] J. Jakšetić, J. Pečarić, *Steffensen's means*, J. Math. Inequal. **2**(4) (2008), 487–498.

[67] J. Jakšetić, J. Pečarić, *Generalized Steffensen means*, Bull. Iranian Math. Soc. **36** (1) (2010), 83–91.

[68] J. Jakšetić, J. Pečarić, *Exponential convexity method*, J. Convex Anal. **20**(1) (2013), 181–197.

[69] J. Jakšetić, J. Pečarić, A. Perušić, *Steffensen inequality, higher order convexity and exponential convexity*, Rend. Circ. Mat. Palermo, to appear.

[70] J. Jakšetić, J. Pečarić, A. Perušić, *Generalisations of Steffensen's inequality by Hermite's polynomial*, submitted.

[71] T. Jiang, *A generalization of Steffensen's inequality*, Qufu Shifan Daxue Xuebao Ziran Kexue Ban **21**(1) (1995) 29–33 (chinese).

[72] J. D. Kečkić, I. B. Lacković, *Generalizations of some inequalites of P. M. Vasić*, Mat. Vesnik **7**(22) (1970), 68–72.

[73] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, 204 , Elsevier Science B.V., Amsterdam, 2006.

[74] S. Kovač, J. Pečarić, *Weighted version of general integral formula of Euler type*, Math. Inequal. Appl. **13**(3) (2010), 579–599.

[75] S. Kovač, J. Pečarić, A. Perušić, *Estimations of the difference between two weighted integral means and apllication of the Steffensen's inequality*, submitted.

[76] K. Krulić, J. Pečarić, K. Smoljak, *Gauss-Steffensen's means*, Math. Balkanica (N.S.) **24**(1-2) (2010), 177–190.

[77] K. Krulić, J. Pečarić, K. Smoljak, *New generalized Steffensen means*, Collect. Math. **62**(2) (2011), 139–150.

[78] K. Krulić, J. Pečarić, K. Smoljak, *Fractional Steffensen type inequalities*, Complex Var. Elliptic Equ. **56**(10-11) (2011), 975–990.

[79] A. Y. Levin, *Some problems bearing on the oscillation of solutions of linear differential equations*, Soviet Math. Dokl. **4** (1963), 121–124.

[80] G. J. Lidstone, *Notes on the extension of Aitken's theorem (for polynomial interpolation) to the Everett types*, Proc. Edinb. Math. Soc. **2**(2) (1929), 16–19.

[81] Z. Liu, *A simple proof of the discrete Steffensen's inequality*, Tamkang J. Math. **35**(4) (2004), 281–283.

[82] Z. Liu, *More on Steffensen type inequalities*, Soochow J. Math. **31**(3) (2005), 429–439.

[83] Z. Liu, *On extensions of Steffensen's inequality*, J. Math. Anal. Approx. Theory **2**(2) (2007), 132–139.

[84] M. Marjanović, *Some inequalites with convex funcions*, Publ. Inst. Math. (Beograd) (N.S.) **8**(22) (1968), 66–68.

[85] M. Masjed-Jamei, F. Qi, H. M. Srivastava, *Generalizations of some classical inequalities via a special functional property*, Integral Transforms Spec. Funct. **21**(5) (2010), 327–336.

[86] M. Matić, J. Pečarić, N. Ujević, *Generalizations of weighted version of Ostrowski's inequality and some related results*, J. Inequal. Appl. **5**(6) (2000), 639–666.

[87] P. R. Mercer, *Extensions of Steffensen's inequality*, J. Math. Anal. Appl. **246**(1) (2000), 325–329.

[88] P. R. Mercer, *Error terms for Steffensen's, Young's and Chebychev's inequalities*, J. Math. Inequal. **2**(4) (2008), 479–486.

[89] G. V. Milovanović, *O nekim funkcionalnim nejednakostima*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz. **599** (1977), 1–59.

[90] G. Milovanović, J. Pečarić, *The Steffensen inequality for convex function of order n*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz. **634–677** (1979), 97–100.

[91] D. S. Mitrinović, *The Steffensen inequality*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. **247–273** (1969), 1–14.

[92] D. S. Mitrinović, J. E. Pečarić, *On the Bellman generalization of Steffensen's inequality III*, J. Math. Anal. Appl. **135**(1) (1988), 342–345.

[93] D. S. Mitrinović, J. E. Pečarić, *On some inequalities for monotone functions*, Boll. Un. Mat. Ital. B (7) **5**(2) (1991), 407–416.

[94] D. S. Mitrinović, J. E. Pečarić, A. M. Fink, *Classical and new inequalities in analysis*, Mathematics and its Applications (East European Series) 61, Kluwer Academic Publishers, Dordrecht, 1993.

[95] D. S. Mitrinović (in cooperation with P. M. Vasić), *Analytic inequalites*, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

[96] G. S. Mudholkar, M. Freimer, P. Subbaiah, *An extension of Hölder's inequality*, J. Math. Anal. Appl. **102**(2) (1984), 435–441.

[97] C. Niculescu, L.-E. Persson, *Convex functions and their applications. A contemporary approach*, CMC Books in Mathematics 23, Springer, New York, 2006.

[98] I. Olkin, *On inequalites of Szegö and Bellman*, Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 230–231.

[99] A. Ostrowski, *Aufgabensammlung zur Infinitesimalrechnung*, Erster Band: Funktionen einer Variablen, Birkhäuser Verlag, Basel-Stuttgart, 1964 (german).

[100] U. M. Ozkan, H. Yildrim, *Steffensen's integral inequality on time scales*, J. Inequal. Appl. **2007** (2007), Art. ID 46524.

[101] B. G. Pachpatte, *A note on Steffensen type discrete inequalities*, Octogon Math. Mag. **5**(1) (1997), 25–26.

[102] B. G. Pachpatte, *A note on a certain generalization of Steffensen's inequality*, Octogon Math. Mag. **6**(1) (1998), 46–49.

[103] C. E. M. Pearce, J. Pečarić, *On an extension of Hölder's inequality*, Bull. Austral. Math. Soc. **51**(3) (1995), 453–458.

[104] J. E. Pečarić, *On the Jensen-Steffensen inequality*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz. **634-677** (1979), 101–107.

[105] J. E. Pečarić, *A generalization of an inequality of Ky Fan*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. **678–715** (1980), 75–84.

[106] J. E. Pečarić, *Notes on some inequalities of P. M. Vasić*, Publ. Inst. Math. (Beograd) (N. S.) **27**(41) (1980), 189–194.

[107] J. E. Pečarić, *On the Čebyšev inequality*, Bul. Stiint. Tehn. Inst. Politehn. Timisoara **25**(39) (1980), 10–11.

[108] J. E. Pečarić, *Inverse of Jensen-Steffensen's inequality*, Glas. Mat. Ser. III **16**(36) (1981), 229–233.

[109] J. E. Pečarić, *Inverse of Steffensen's inequality*, Glas. Mat. Ser. III **17**(37) (1982), 265–270.

[110] J. E. Pečarić, *Notes on some general inequalities*, Publ. Inst. Math. (Beograd), (N. S.) **32**(46) (1982), 131–135.

[111] J. E. Pečarić, *On the Bellman generalization of Steffensen's inequality*, J. Math. Anal. Appl. **88**(2) (1982), 505–507.

[112] J. E. Pečarić, *On the Bellman generalization of Steffensen's inequality II*, J. Math. Anal. Appl. **104**(2) (1984), 432–434.

[113] J. E. Pečarić, *Connections among some inequalities of Gauss, Steffensen and Ostrowski*, Southeast Asian Bull. Math. **13**(2) (1989), 89–91.

[114] J. Pečarić, I. Perić, K. Smoljak, *Generalized fractional Steffensen type inequalites*, Eurasian Math. J. **3**(4) (2012), 81–98.

[115] J. Pečarić, I. Perić, A. Vukelić, *Estimations of the difference of two integral means via Euler-type identities*, Math. Inequal. Appl. **7**(3) (2004), 365-378.

[116] J. Pečarić, J. Perić, *Improvements of the Giaccardi and the Petrović inequality and related Stolarsky type means*, An. Univ. Craiova Ser. Mat. Inform. **39**(1) (2012), 65–75.

[117] J. Pečarić, A. Perušić, K. Smoljak, *Mercer and Wu-Srivastava generalisations of Steffensen's inequality*, Appl. Math. Comput. **219**(21) (2013), 10548–10558.

[118] J. Pečarić, A. Perušić, K. Smoljak, *Cerone's generalizations of Steffensen's inequality*, Tatra Mt. Math. Publ. to appear.

[119] J. Pečarić, A. Perušić, K. Smoljak, *Generalisations of Steffensen's inequality by Lidstone polynomial*, submitted.

[120] J. Pečarić, A. Perušić, K. Smoljak, *Generalisations of Steffensen's inequality by Abel-Gontscharoff polynomial*, submitted.

[121] J. Pečarić, A. Perušić, A. Vukelić, *Generalisations of Steffensen's inequality via Fink identity and related results*, Advances in Inequalities and Applications, to appear.

[122] J. E. Pečarić, F. Proschan, Y. L. Tong, *Convex functions, partial orderings and statistical applications*, Mathematics in Science and Engineering, 187, Academic Press Inc., Boston, 1992.

[123] J. Pečarić, K. Smoljak, *Note on an inequality of Gauss*, J. Math. Inequal. **5**(2) (2011), 199–211.

[124] J. Pečarić, K. Smoljak, *$a(x)-$monotonic functions and their inequalities*, J. Math. Inequal. **5**(3) (2011), 435–450.

[125] J. Pečarić, K. Smoljak, *Improvement of an extension of Hölder-type inequality*, Anal. Math. **38**(2) (2012), 135–146.

[126] J. Pečarić, K. Smoljak, *Stolarsky type means related to generalizations of Steffensen's and Gauss' inequality*, Acta Math. Vietnam., to appear.

[127] J. Pečarić, K. Smoljak, *Steffensen type inequalities involving convex functions*, Math. Inequal. Appl. to appear.

[128] J. Pečarić, A. ur Rehman, *On logarithmic convexity for power sums and related results*, J. Inequal. Appl. **2008** (2008), Art. ID 389410.

[129] J. Pečarić, S. Varošanec, *Multidimensional and discrete Steffensen inequality*, Southeast Asian Bull. Math. **23**(2) (1999), 277–284.

[130] M. Petschke, *Extremalstrahlen konvexer Kegel und komplementäre Ungleichungen*, Darmstadt, dissertation, 1989.

[131] H. Poritsky, *On certain polynomial and other approximations to analytic functions*, Trans. Amer. Math. Soc. **34**(2) (1932), 274–331.

[132] F. Qi, *Pólya type integral inequalites: origin, variants, proofs, refinements, generalizations, equivalences, and applications*, Math. Inequal. Appl., to appear.

[133] F. Qi, J.-X. Cheng, *Some new Steffensen pairs*, Anal. Math. **29**(3) (2003), 219–226.

[134] F. Qi, B.-N. Guo, *On Steffensen pairs*, J. Math. Anal. Appl. **271**(2) (2002), 534–541.

[135] P. J. Rabier, *Steffensen's inequality and $L^1 - L^\infty$ estimates of weighted integrals*, Proc. Amer. Math. Soc. **140**(2) (2011), 665–675.

[136] P. J. Rabier, *Generalized Steffensen inequalities and their optimal constants*, J. Convex. Anal. **19**(2) (2012), 301–321.

[137] O. Rakić, *A proof of the Steffensen inequality*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz. **412-460** (1973), 49–52.

[138] R. H. Sadikova, *Obšči klass neravenstv soderžćii neravenstvo Steffensena i ego obobščenija*, Izv. Vyssh. Uchebn. Zaved. Mat. **4** (1980), 79–80 (russian) (eng. A general class of inequalities containing the Steffensen inequality and its generalization).

[139] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach. Sci. Publ., Yverdon, 1993.

[140] I. J. Schoenberg, *On certain two-point expansions of integral functions of exponential type*, Bull. Amer. Math. Soc. **42**(4) (1936), 284–288.

[141] H.-N. Shi, S.-H. Wu, *Majorized proof and improvement of the discrete Steffensen's inequality*, Taiwanese J. Math. **11**(4) (2007), 1203–1208.

[142] J. F. Steffensen, *On certain inequalities between mean values and their application to actuarial problems*, Skand. Aktuarietids. (1918), 82–97.

[143] J. F. Steffensen, *On certain inequalites and methods of approximation*, J. Inst. Actuar. **51** (1919), 274–297.

[144] J. F. Steffensen, *On a generalization of certain inequalites by Tchebychef and Jensen*, Skand. Aktuarietids. (1925), 137–147.

[145] W. T. Sulaiman, *Some generalizations of Steffensen's inequality*, Aust. J. Math. Anal. Appl. **2**(1) (2005), Article 9.

[146] W. T. Sulaiman, *Extensions of Steffensen's inequality*, J. Math. Anal. **3**(2) (2012), 10–17.

[147] W. T. Sulaiman, *Some new generalizations of Steffensen's inequality*, British Journal of Mathematics & Computer Science **2**(3) (2012), 176–186.

[148] P. M. Vasić, *Sur une inégalité de M. Petrović*, Mat. Vesnik **5**(20) (1968), 473–478 (french).

[149] P. M. Vasić, J. Pečarić, *Note on the Steffensen inequality*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. **716-734** (1981), 80–82.

[150] V. N. Volkov, *Inequalities connected with an inequality of Gauss*, Volž. Mat. Sb. Vyp. **7** (1969), 11–13 (russian).

[151] V. N. Volkov, *Utočnenie neravenstva Gel'dera*, Moskov. Gos. Ped. Inst. Učen. Zap. **460** (1972), 110–115 (russian) (eng. A refinement of Hölder's inequality)

[152] J. M. Whittaker, *On Lidstone series and two-point expansions of analytic functions*, Proc. London Math. Soc. **36** (1933-1934), 451–469.

[153] D. V. Widder, *The Laplace transform*, Princeton Mathematical Series **6**, Princeton Univ. Press, New Jersey, 1941.

[154] D. V. Widder, *Completly convex functions and Lidstone series*, Trans. Amer. Math. Soc. **51** (1942), 387–398.

[155] S.-H. Wu, H. M. Srivastava, *Some improvements and generalizations of Steffensen's integral inequality*, Appl. Math. Comput. **192**(2) (2007), 422–428.

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