MONOGRAPHS IN INEQUALITIES 10

Inequalities of Opial and Jensen

Improvements of Opial-type inequalities with applications to fractional calculus

Maja Andrić, Josip Pečarić and Ivan Perić

ELEMENT

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Preface

In 1960, the Polish mathematician Zdzisław Opial proved the next integral inequality ([64]), which now bear his name:

> Let $x(t) \in C^1[0,h]$ be such that $x(0) = x(h) = 0$ and $x(t) > 0$ for $t \in (0,h)$. Then
> $$\int_0^h \left|x(t)x'(t)\right| dt \leq \frac{h}{4} \int_0^h \left(x'(t)\right)^2 dt, \tag{1}$$
> where the constant $h/4$ is the best possible.

This integral inequality, containing the derivative of the function, is recognized as a fundamental result in the analysis of qualitative properties of a solution of differential equations (see [5, 61] and the references cited therein). Over the last five decades, an enormous amount of work has been done on the Opial inequality: several simplifications of the original proof, various extensions, generalizations and discrete analogues. More details can be found in the monograph by Agarwal and Pang [5] which is dedicated to the theory of Opial-type inequalities and its applications in theory of differential and difference equations.

Motivated with Opial-type inequalities, together with Jensen's inequality, we improve some known results and obtain new, interesting inequalities. For such inequalities we construct functionals and give its mean value theorems. These Cauchy type mean value theorems are used for Stolarsky type means, all defined by the observed inequalities, and also, they are used to prove the n-exponential convexity for the functionals.

We study Opial-type inequalities not only for ordinary derivatives, but also for fractional derivatives which leads us to the fractional calculus. It is a theory of differential and integral operators of non-integer order that has become very useful due to its many applications in almost all the applied sciences. We study the Riemann-Liouville fractional integrals and three types of fractional derivatives (the Riemann-Liouville, the Caputo and the Canavati type), in the real domain. Obtaining improvements of composition identities for the above mentioned fractional derivatives, we apply them on the fractional differentiation inequalities that have applications in the fractional differential equations; the most important ones are in establishing the uniqueness of the solution of initial problems and giving upper bounds to their solutions. We give refinements, generate new extensions and generalizations of some known Opial-type inequalities, investigate the possibility of obtaining the best possible constant, compare results obtained by different methods and present some new inequalities involving fractional integrals and fractional derivatives. Each Opial-type inequality is observed for the left-sided and the right-sided fractional derivatives, emphasizing special cases when order of derivatives belongs to $\mathbb{N}_0$, reducing to classical Opial-type inequality for ordinary derivatives. The book is divided in nine chapters.

In the first chapter, notation, terms and overview of some important results are listed for integrable functions, continuous functions, absolutely continuous functions and convex functions with the Jensen inequality. Also, an overview of method of producing n-exponentially convex and exponentially convex functions is given. Finally, Opial-type inequalities due to Beesack, Wirtinger, Willett, Godunova, Levin, Rozanova, Fink, Agarwal, Pang, Alzer are listed.

In Chapter 2 we give definitions and basic properties of Riemann-Liouville fractional integral and three types of fractional derivatives: the Riemann-Liouville, the Caputo and the Canavati type. Fractional integrals and fractional derivatives are observed in the real domain. Also, improvements of the known composition identities for fractional derivatives are presented. Each of three types of fractional derivatives is specially treated, first for the left-sided and then for the right-sided fractional derivatives. An overview of conditions under which composition identities are valid is given. An attention was paid on the role of initial conditions for a function involved in composition identities and the mutual respect among the Riemann-Liouville fractional integrals and above mentioned fractional derivatives.

In Chapter 3, extensions and generalizations of Opial's inequalities due to Willett, Godunova, Levin and Rozanova are obtained using Jensen's inequality. Cauchy type mean value theorems are proved and used in studying Stolarsky type means defined by the obtained inequalities. An elegant method of producing n-exponentially convex and exponentially convex functions is applied. Also, Willett's and Rozanova's generalizations of Opial's inequality are extended to multidimensional inequalities.

In Chapter 4, extensions of Opial-type integral inequalities are used to obtain generalizations of inequalities due to Mitrinović and Pečarić for convex and for relative convex functions. Again, Cauchy type mean value theorems are given, as well as Stolarsky type means defined by the observed integral inequalities. Further, n-exponentially convex and exponentially convex functions are produced. Also, some new Opial-type inequalities are given for different types of fractional integrals and fractional derivatives as applications.

Obtained results from previous chapters are applied to the fractional differentiation inequalities in Chapter 5 and Chapter 6. Opial-type inequalities are studied and inequalities involving the Riemann-Liouville, the Caputo and the Canavati fractional derivatives are presented. Some generalizations, extensions and refinements of Opial-type inequalities are given and some new fractional differentiation inequalities are obtained. Possibilities to obtain the best possible constants are investigated and a comparison of results obtained by different methods is given. Special cases of order of fractional derivatives are emphasized, in which inequalities are reduced to the classical Opial's, Beesack's, Wirtinger's, Fink's, Agarwal-Pang's or Alzer's inequality for ordinary derivatives.

A new general inequalities for integral operators with a kernel and applications to a Green function are presented in Chapter 7. Inequalities are observed on a measure space (Ω, Σ, μ) for two functions, convex and concave. Results are applied to numerous symmetric functions and new results involving a Green function, Lidstone's and Hermite's interpolating polynomials are obtained.

The last chapter starts with improvements of some Opial-type inequalities in one variable, following with multidimensional integral inequalities and theirs discrete versions. These inequalities are similar to those of Nirenberg, Opial, Poincaré, Serrin, Sobolev and Wirtinger. In this chapter, some elementary techniques such as appropriate integral representations of functions, appropriate summation representations of discrete functions and inequalities involving means are used to establish multidimensional integral (and discrete) inequalities.

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Contents

Chapter 1

Preliminaries

1.1 Spaces of integrable, continuous and absolutely continuous functions

In this section we listed definitions and properties of integrable functions, continuous functions, absolutely continuous functions and basic properties of the Laplace transform. Also we give required notation, terms and overview of some important results (more details could be found in monographs [57, 59, 70, 74]).

L_p spaces

Let $[a,b]$ be a finite interval in $\mathbb{R}$, where $-\infty \leq a < b \leq \infty$. We denote by $L_p[a,b]$, $1 \leq p < \infty$, the space of all Lebesgue measurable functions f for which $\int_a^b |f(t)|^p\, dt < \infty$, where

$$||f||_p = \left(\int_a^b |f(t)|^p \; dt \right)^{\frac{1}{p}},$$

and by $L_\infty[a,b]$ the set of all functions measurable and essentially bounded on $[a,b]$ with

$$||f||_\infty = \operatorname{ess\,sup} \{ |f(x)| \colon x \in [a,b] \}.$$

Theorem 1.1 (INTEGRAL HÖLDER'S INEQUALITY) *Let $p,q \in \mathbb{R}$ such that $1 \leq p,q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $f,g \colon [a,b] \to \mathbb{R}$ be integrable functions such that $f \in L_p[a,b]$ and*

$g \in L_q[a,b]$. Then

$$\int_a^b |f(t)\,g(t)|\,dt \le ||f||_p\,||g||_q\,. \tag{1.1}$$

Equality in (1.1) holds if and only if $A\,|f(t)|^p = B\,|g(t)|^q$ almost everywhere, where A and B are constants.

Spaces of continuous and absolutely continuous functions

We denote by $C^n[a,b]$, $n \in \mathbb{N}_0$, the space of functions which are n times continuously differentiable on $[a,b]$, that is

$$C^n[a,b] = \left\{ f : [a,b] \to \mathbb{R} \colon f^{(k)} \in C[a,b]\,, k = 0,1,\ldots,n \right\}.$$

In particular, $C^0[a,b] = C[a,b]$ is the space of continuous functions on $[a,b]$ with the norm

$$||f||_{C^n} = \sum_{k=0}^{n} ||f^{(k)}||_C = \sum_{k=0}^{n} \max_{x\in[a,b]} |f^{(k)}(x)|\,,$$

and for $C[a,b]$

$$||f||_C = \max_{x\in[a,b]} |f(x)|\,.$$

Lemma 1.1 *The space $C^n[a,b]$ consists of those and only those functions f which are represented in the form*

$$f(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} \varphi(t)\,dt + \sum_{k=0}^{n-1} c_k (x-a)^k\,, \tag{1.2}$$

where $\varphi \in C[a,b]$ and c_k are arbitrary constants $(k = 0,1,\ldots,n-1)$.
Moreover,

$$\varphi(t) = f^{(n)}(t)\,, \quad c_k = \frac{f^{(k)}(a)}{k!}\ (k = 0,1,\ldots,n-1)\,. \tag{1.3}$$

By $C_a^n[a,b]$ we denote the subspace of the space $C^n[a,b]$ defined by

$$C_a^n[a,b] = \left\{ f \in C^n[a,b] \colon f^{(k)}(a) = 0\,, k = 0,1,\ldots,n-1 \right\}.$$

For $f \in C^n[a,b]$ and $0 \le \mu < 1$ we define

$$|f|_{n,\mu} = \sup \left\{ \frac{\left| f^{(n)}(x) - f^{(n)}(y) \right|}{|x-y|^{\mu}} : x,y \in [a,b], x \ne y \right\}.$$

Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, n the integral part of α (notation $n = [\alpha]$) and let $\mu = \alpha - n$. By $\mathscr{D}^{\alpha}[a,b]$ we denote the space

$$\mathscr{D}^{\alpha}[a,b] = \left\{ f \in C^n[a,b] \colon\ |f|_{n,\mu} < \infty \right\},$$

and by $\mathscr{D}_a^\alpha[a,b]$ the subspace of the space $\mathscr{D}^\alpha[a,b]$

$$\mathscr{D}_a^\alpha[a,b] = \left\{ f \in \mathscr{D}^\alpha[a,b] \colon f^{(k)}(a) = 0\,, k = 0,1,\ldots,n \right\}.$$

Specially, for $\alpha = n \in \mathbb{N}$ we have $\mathscr{D}^n[a,b] = C^n[a,b]$ and $\mathscr{D}_a^n[a,b] = C_a^n[a,b]$.

The space of absolutely continuous functions on a finite interval $[a,b]$ is denoted by $AC[a,b]$. It is known that $AC[a,b]$ coincides with the space of primitives of Lebesgue integrable functions $L_1[a,b]$ (see Kolmogorov and Fomin [53, Chapter 33.2]):

$$f \in AC[a,b] \quad \Leftrightarrow \quad f(x) = f(a) + \int_a^x \varphi(t)\,dt\,, \quad \varphi \in L_1[a,b]\,,$$

and therefore an absolutely continuous function f has an integrable derivative $f'(x) = \varphi(x)$ almost everywhere na $[a,b]$. We denote by $AC^n[a,b]$, $n \in \mathbb{N}$, the space

$$AC^n[a,b] = \left\{ f \in C^{n-1}[a,b] \colon f^{(n-1)} \in AC[a,b] \right\}.$$

In particular, $AC^1[a,b] = AC[a,b]$.

Lemma 1.2 *The space $AC^n[a,b]$ consists of those and only those functions which can be represented in the form* (1.2), *where $\varphi \in L_1[a,b]$ and c_k are arbitrary constants* ($k = 0,1,\ldots,n-1$).
Moreover, (1.3) *holds.*

The next theorem has numerous applications involving multiple integrals.

Theorem 1.2 (FUBINI'S THEOREM) *Let $(X,\mathscr{M},\mu)$ and $(Y,\mathscr{N},\nu)$ be σ-finite measure spaces and f $\mu\times\nu$-measurable function on $X\times Y$. If $f \geq 0$, then next integrals are equal*

$$\int\limits_{X\times Y} f(x,y)\,d(\mu\times\nu)(x,y)\,, \int\limits_X \left(\int\limits_Y f(x,y)\,d\nu(y) \right) d\mu(x) \;\; \text{and} \;\; \int\limits_Y \left(\int\limits_X f(x,y)\,d\mu(x) \right) d\nu(y).$$

If f is a complex function, then above equalities hold with additional requirement

$$\int\limits_{X\times Y} |f(x,y)|\,d(\mu\times\nu)(x,y) < \infty.$$

Next equalities are consequences of this theorem:

$$\int_a^b dx \int_c^d f(x,y)\,dy = \int_c^d dy \int_a^b f(x,y)\,dx;$$

$$\int_a^b dx \int_a^x f(x,y)\,dy = \int_a^b dy \int_y^b f(x,y)\,dx. \tag{1.4}$$

The gamma and beta functions

The *gamma function* Γ is the function of complex variable defined by Euler's integral of second kind

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt\,, \quad \Re(z) > 0\,. \tag{1.5}$$

This integral is convergent for each $z \in \mathbb{C}$ such that $\Re(z) > 0$. It has next property

$$\Gamma(z+1) = z\Gamma(z)\,, \quad \Re(z) > 0\,,$$

from which follows

$$\Gamma(n+1) = n!\,, \quad n \in \mathbb{N}_0\,.$$

For domain $\Re(z) \leq 0$ we have

$$\Gamma(z) = \frac{\Gamma(z+n)}{(z)_n}\,, \quad \Re(z) > -n;\ n \in \mathbb{N};\ z \notin \mathbb{Z}_0^- = \{0,-1,-2,\dots\}\,, \tag{1.6}$$

where $(z)_n$ is the *Pochhammer's symbol* defined for $z \in \mathbb{C}$ and $n \in \mathbb{N}_0$ by

$$(z)_0 = 1; \quad (z)_n = z(z+1)\cdots(z+n-1),\ n \in \mathbb{N}\,.$$

The gamma function is analytic in complex plane except in $0,-1,-2,\dots$ which are simple poles.

The *beta function* is the function of two complex variables defined by Euler's integral of the first kind

$$B(z,w) = \int_0^1 t^{z-1}\,(1-t)^{w-1}\, dt\,, \quad \Re(z),\Re(w) > 0\,. \tag{1.7}$$

It is related to the gamma function with

$$B(z,w) = \frac{\Gamma(z)\,\Gamma(w)}{\Gamma(z+w)}\,, \quad z,w \notin \mathbb{Z}_0^-\,,$$

which gives

$$B(z+1,w) = \frac{z}{z+w} B(z,w)\,.$$

Next we proceed with examples of integrals often used in proofs and calculations in this book.

Example 1.1 Let $\alpha,\beta > 0$ and $x \in [a,b]$. Then by substitution $t = x - s(x-a)$ we have

$$\begin{aligned}\int_a^x (x-t)^{\alpha-1}(t-a)^{\beta-1}\, dt &= \int_0^1 (x-a)^{\alpha+\beta-1} s^{\alpha-1}(1-s)^{\beta-1}\, ds \\ &= B(\alpha,\beta)(x-a)^{\alpha+\beta-1}\,.\end{aligned}$$

Analogously, by substitution $t = x + s(b-x)$, it follows

$$\int_x^b (t-x)^{\alpha-1}(b-t)^{\beta-1}\, dt = B(\alpha,\beta)\,(b-x)^{\alpha+\beta-1}\,.$$

Example 1.2 Let $\alpha,\beta>0$, $f\in L_1[a,b]$ and $x\in[a,b]$. Then interchanging the order of integration and evaluating the inner integral we obtain

$$\begin{aligned}\int_a^x (x-t)^{\alpha-1}\int_a^t (t-s)^{\beta-1} f(s)\,ds\,dt &= \int_{s=a}^x f(s)\int_{t=s}^x (x-t)^{\alpha-1}(t-s)^{\beta-1}\,dt\,ds\\ &= B(\alpha,\beta)\int_a^x (x-s)^{\alpha+\beta-1} f(s)\,ds\,.\end{aligned}$$

Analogously,

$$\int_x^b (t-x)^{\alpha-1}\int_t^b (s-t)^{\beta-1} f(s)\,ds\,dt = B(\alpha,\beta)\int_x^b (s-x)^{\alpha+\beta-1} f(s)\,ds\,.$$

The Laplace transform

Let $f\colon [0,\infty)\to\mathbb{R}$ be a function such that mapping $t\mapsto e^{-\sigma t}\,|f(t)|$, $\sigma>0$, is integrable on $[0,\infty)$. Then for each $p\geq\sigma$ the Lebesgue integral

$$F(p)=\int_0^\infty e^{-pt} f(t)\,dt \tag{1.8}$$

exists. The mapping $f\mapsto F$ is called the *Laplace transform* and noted with $\mathscr{L}$, that is

$$\mathscr{L}[f](p)=F(p)\,.$$

Sufficient conditions for the Laplace transform existence are that function f is locally integrable and exponentially bounded in ∞, that is $|f(t)|\leq Me^{\sigma t}$ for $t>\varepsilon$, where M, σ and ε are constant. The *abscissa of convergence* σ_0 is the smallest value of σ for which $|f(t)|\leq Me^{\sigma t}$.

Example 1.3 Let $f\colon [0,\infty)\to\mathbb{R}$, $f(t)=t^\alpha$, where $\alpha>-1$. Obviously $|f(t)|=t^\alpha<e^{\alpha t}$ for $t>0$ and $\alpha\geq 0$. For $-1<\alpha<0$, the function f is locally integrable and $t^\alpha\leq 1$ for $t\geq 1$. Therefore, by substitution $pt=x$, the Laplace transform has the form

$$\mathscr{L}[f](p)=\int_0^\infty e^{-pt}\,t^\alpha\,dt=\frac{1}{p^{\alpha+1}}\int_0^\infty e^{-x}x^\alpha\,dx=\frac{\Gamma(\alpha+1)}{p^{\alpha+1}}\,.$$

We give some properties and rules of the Laplace transform, and important uniqueness theorem ([74, Teorem 6.3]):

$$\textit{convolution:}\quad \mathscr{L}\left[\int_0^t f(t-\tau)\,g(\tau)\,d\tau\right](p)=\mathscr{L}[f](p)\,\mathscr{L}[g](p)$$

$$\textit{differentiation:}\quad \mathscr{L}\left[f^{(n)}\right](p)=p^n\mathscr{L}[f](p)-\sum_{k=1}^n p^{n-k}f^{(k-1)}(0)$$

Theorem 1.3 (UNIQUENESS THEOREM) *Let $f,g\colon [0,\infty)\to\mathbb{R}$ be two functions for which the Laplaceova transform exists. If*

$$\int_0^\infty e^{-pt} f(t)\,dt=\int_0^\infty e^{-pt} g(t)\,dt$$

for each p on common area of convergence, then $f(t)=g(t)$ for almost every $t\in[0,\infty)$.

1.2 Convex functions and Jensen's inequalities

Definitions and properties of convex functions and Jensen's inequality, with more details, could be found in monographs [61, 62, 67].

Let I be an interval in $\mathbb{R}$.

Definition 1.1 *A function $f : I \to \mathbb{R}$ is called convex if*

$$f((1-\lambda)x+\lambda y) \leq (1-\lambda)f(x)+\lambda f(y) \tag{1.9}$$

for all points x and y in I and all $\lambda \in [0,1]$. It is called strictly convex if the inequality (1.9) *holds strictly whenever x and y are distinct points and $\lambda \in (0,1)$. If $-f$ is convex (respectively, strictly convex) then we say that f is concave (respectively, strictly concave). If f is both convex and concave, then f is said to be affine.*

Lemma 1.3 (THE DISCRETE CASE OF JENSEN'S INEQUALITY) *A real-valued function f defined on an interval I is convex if and only if for all $x_1,\dots,x_n$ in I and all scalars $\lambda_1,\dots,\lambda_n$ in $[0,1]$ with $\sum_{k=1}^{n}\lambda_k = 1$ we have*

$$f\left(\sum_{k=1}^{n}\lambda_k x_k\right) \leq \sum_{k=1}^{n}\lambda_k f(x_k). \tag{1.10}$$

The above inequality is strict if f is strictly convex, all the points x_k are distinct and all scalars λ_k are positive.

Theorem 1.4 (JENSEN) *Let $f : I \to \mathbb{R}$ be a continuous function. Then f is convex if and only if f is midpoint convex, that is,*

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \tag{1.11}$$

for all $x,y \in I$.

Corollary 1.1 *Let $f : I \to \mathbb{R}$ be a continuous function. Then f is convex if and only if*

$$f(x+h)+f(x-h)-2f(x) \geq 0 \tag{1.12}$$

for all $x \in I$ and all $h > 0$ such that both $x+h$ and $x-h$ are in I.

Proposition 1.1 (THE OPERATIONS WITH CONVEX FUNCTIONS) **(i)** *The addition of two convex functions (defined on the same interval) is a convex function; if one of them is strictly convex, then the sum is also strictly convex.*

(ii) *The multiplication of a (strictly) convex function with a positive scalar is also a (strictly) convex function.*

(iii) *The restriction of every (strictly) convex function to a subinterval of its domain is also a (strictly) convex function.*

(iv) *If $f : I \to \mathbb{R}$ is a convex (respectively a strictly convex) function and $g : \mathbb{R} \to \mathbb{R}$ is a nondecreasing (respectively an increasing) convex function, then $g \circ f$ is convex (respectively strictly convex)*

(v) *Suppose that f is a bijection between two intervals I and J. If f is increasing, then f is (strictly) convex if and only if f^{-1} is (strictly) concave. If f is a decreasing bijection, then f and f^{-1} are of the same type of convexity.*

Definition 1.2 *If g is strictly monotonic, then f is said to be (strictly) convex with respect to g if $f \circ g^{-1}$ is (strictly) convex.*

Proposition 1.2 *If $x_1, x_2, x_3 \in I$ are such that $x_1 < x_2 < x_3$, then the function $f : I \to \mathbb{R}$ is convex if and only if the inequality*

$$(x_3 - x_2)\, f(x_1) + (x_1 - x_3)\, f(x_2) + (x_2 - x_1)\, f(x_3) \;\geq\; 0$$

holds.

Proposition 1.3 *If f is a convex function on an interval I and if $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \;\leq\; \frac{f(y_2) - f(y_1)}{y_2 - y_1}\,.$$

If the function f is concave, then the inequality reverses.

The following theorems concern derivatives of convex functions.

Theorem 1.5 *Let $f : I \to \mathbb{R}$ be convex. Then*

(i) *f is Lipschitz on any closed interval in I;*

(ii) *f'_+ and f'_- exist and are increasing in I, and $f'_- \leq f'_+$ (if f is strictly convex, then these derivatives are strictly increasing);*

(iii) *f' exists, except possibly on a countable set, and on the complement of which it is continuous.*

Proposition 1.4 *Suppose that $f : I \to \mathbb{R}$ is a twice differentiable function. Then*

(i) *f is convex if and only if $f'' \geq 0$;*

(ii) *f is strictly convex if and only if $f'' \geq 0$ and the set of points where f'' vanishes does not include intervals of positive length.*

Next we need divided differences, commonly used when dealing with functions that have different degree of smoothness.

Definition 1.3 *Let $f\colon I \to \mathbb{R}$, $n \in \mathbb{N}_0$ and let $x_0, x_1, \ldots, x_n \in I$ be mutually different points. The n-th order divided difference of a function at $x_0, \ldots, x_n$ is defined recursively by*

$$\begin{aligned}
[x_i; f] &= f(x_i), \quad i = 0, 1, \ldots, n\,,\\
[x_0, x_1; f] &= \frac{[x_0; f] - [x_1; f]}{x_0 - x_1} = \frac{f(x_0) - f(x_1)}{x_0 - x_1}\,,\\
[x_0, x_1, x_2; f] &= \frac{[x_0, x_1; f] - [x_1, x_2; f]}{x_0 - x_2}\,,\\
&\vdots\\
[x_0, \ldots, x_n; f] &= \frac{[x_0, \ldots, x_{n-1}; f] - [x_1, \ldots, x_n; f]}{x_0 - x_n}\,.
\end{aligned} \tag{1.13}$$

Remark 1.1 The value $[x_0, x_1, x_2; f]$ is independent of the order of the points x_0, x_1 and x_2. This definition may be extended to include the case in which some or all the points coincide. Namely, taking the limit $x_1 \to x_0$ in (1.13), we get

$$\lim_{x_1 \to x_0} [x_0, x_1, x_2; f] = [x_0, x_0, x_2; f] = \frac{f(x_0) - f(x_2) - f'(x_0)(x_0 - x_2)}{(x_0 - x_2)^2}\,,\ x_2 \neq x_0$$

provided that f' exists, and furthermore, taking the limits $x_i \to x_0$, $i = 1, 2$ in (1.13), we get

$$\lim_{x_2 \to x_0} \lim_{x_1 \to x_0} [x_0, x_1, x_2; f] = [x_0, x_0, x_0; f] = \frac{f''(x_0)}{2}$$

provided that f'' exists.

Definition 1.4 *A function $f: I \to \mathbb{R}$ is said to be n-convex ($n \in \mathbb{N}_0$) if for all choices of $n+1$ distinct points $x_0, \ldots, x_n \in I$, the n-th order divided difference of f satisfies*

$$[x_0, \ldots, x_n; f] \geq 0\,. \tag{1.14}$$

Thus the 1-convex functions are the nondecreasing functions, while the 2-convex functions are precisely the classical convex functions.

Definition 1.5 *A function $f: I \to (0, \infty)$ is called log-convex if*

$$f\left((1-\lambda)x + \lambda y\right) \leq f(x)^{1-\lambda}\, f(y)^{\lambda} \tag{1.15}$$

for all points x and y in I and all $\lambda \in [0, 1]$.

If a function $f: I \to \mathbb{R}$ is log-convex, then it is also convex, which is a consequence of the weighted AG-inequality.

We end this section with the integral form of Jensen's inequality.

Theorem 1.6 (INTEGRAL JENSEN'S INEQUALITY) *Let $(\Omega, \mathscr{A}, \mu)$ be a finite measure space, $0 < \mu(\Omega) < \infty$ and let $f\colon \Omega \to I$ be a μ-integrable function. If $\varphi\colon I \to \mathbb{R}$ is convex function, then next inequality holds*

$$\varphi\left(\frac{1}{\mu(\Omega)} \int_{\Omega} f\, d\mu\right) \leq \frac{1}{\mu(\Omega)} \int_{\Omega} (\varphi \circ f)\, d\mu\,. \tag{1.16}$$

If φ is strictly convex, then in (1.16) *we have equality if and only f is constant μ-almost everywhere on Ω.*

1.3 Exponential convexity

Following definitions and properties of exponentially convex functions comes from [28], also [66]. Let I be an interval in $\mathbb{R}$.

Definition 1.6 *A function $\psi\colon I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if*

$$\sum_{i,j=1}^{n} \xi_i \xi_j \psi\left(\frac{x_i + x_j}{2}\right) \geq 0$$

holds for all choices $\xi_i \in \mathbb{R}$ and $x_i \in I$, $i = 1, \ldots, n$.

A function $\psi\colon I \to \mathbb{R}$ is n-exponentially convex if it is n-exponentially convex in the Jensen sense and continuous on I.

Remark 1.2 It is clear from the definition that $1-$exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also, n-exponentially convex functions in the Jensen sense are $k-$exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $k \leq n$.

By definition of positive semi-definite matrices and some basic linear algebra we have the following proposition.

Proposition 1.5 *If ψ is an n-exponentially convex in the Jensen sense, then the matrix $\left[\psi\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{k}$ is a positive semi-definite matrix for all $k \in \mathbb{N}, k \leq n$. Particularly,* $\det\left[\psi\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{k} \geq 0$ *for all $k \in \mathbb{N}$, $k \leq n$.*

Definition 1.7 *A function $\psi\colon I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.*

A function $\psi\colon I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Remark 1.3 It is known (and easy to show) that $\psi : I \to (0, \infty)$ is log-convex in the Jensen sense if and only if

$$\alpha^2 \psi(x) + 2\alpha\beta\psi\left(\frac{x+y}{2}\right) + \beta^2 \psi(y) \geq 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a function is log-convex in the Jensen sense if and only if it is $2-$exponentially convex in the Jensen sense.

Also, using basic convexity theory it follows that a function is log-convex if and only if it is $2-$exponentially convex.

One of the main features of exponentially convex functions is its integral representation given by Bernstein ([32]) in the following theorem.

Theorem 1.7 *The function $\psi\colon I \to \mathbb{R}$ is exponentially convex on I if and only if*

$$\psi(x) = \int_{-\infty}^{\infty} e^{tx} d\sigma(t), \quad x \in I$$

for some non-decreasing function $\sigma : \mathbb{R} \to \mathbb{R}$.

1.4 Opial-type inequalities

In 1960. Opial published an inequality involving integrals of a function and its derivative, which now bear his name ([64]). Over the last five decades, an enormous amount of work has been done on Opial's inequality: several simplifications of the original proof, various extensions, generalizations and discrete analogues. More details can be found in the monograph by Agarwal and Pang [5] which is dedicated to the theory of Opial-type inequalities and its applications in theory of differential and difference equations. We observe Beesack's, Wirtinger's, Willett's, Godunova-Levin's, Rozanova's, Fink's, Agarwal-Pang's and Alzer's versions of Opial's inequality.

Theorem 1.8 (OPIAL'S INEQUALITY) *Let $f \in C^1[0,h]$ be such that $f(0) = f(h) = 0$ and $f(x) > 0$ for $x \in (0,h)$. then*

$$\int_0^h \left|f(x)f'(x)\right| dx \leq \frac{h}{4}\int_0^h \left[f'(x)\right]^2 dx, \tag{1.17}$$

where constant $h/4$ is the best possible.

The novelty of Opial's result is thus in establishing the best possible constant $h/4$.

Example 1.4 It is easy to construct the function which satisfy equality in (1.17). For instance, let f be defined by

$$f(x) = \begin{cases} cx, & 0 \leq x \leq \frac{h}{2} \\ c(h-x), & \frac{h}{2} \leq x \leq h \end{cases}$$

where $c > 0$ is arbitrary constant. Although this function is not derivable in $t = h/2$, it could be approximated by the function belonging to $C^1[0,h]$ that satisfy (1.17). Then constant $h/4$ is the best possible.

Opial's inequality (1.17) holds even if function f' has discontinuity at $t = h/2$, provided that f is absolutely continuous on both of the subintervals $[0, \frac{h}{2}]$ and $[\frac{h}{2}, h]$, with $f(0) = f(h) = 0$. Also, the positivity requirement of f on $(0,h)$ is unnecessary, that is, next Beesack's inequality holds ([31]).

Theorem 1.9 (BEESACK'S INEQUALITY) *Let $f \in AC[0,h]$ be such that $f(0)=0$. Then*

$$\int_0^h |f(x)f'(x)|\,dx \le \frac{h}{2}\int_0^h [f'(x)]^2\,dx. \tag{1.18}$$

Equality in (1.18) *holds if and only if $f(x)=cx$, where c is a constant.*

Theorem 1.10 (WIRTINGER'S INEQUALITY) *Let $f\colon [0,h]\to\mathbb{R}$ be such that $f'\in L_2[0,h]$. If $f(0)=f(h)=0$, then*

$$\int_0^h [f(x)]^2\,dx \le \left(\frac{h}{\pi}\right)^2\int_0^h [f'(x)]^2\,dx. \tag{1.19}$$

Equality in (1.19) *holds if and only if $f(x)=c\sin\frac{\pi x}{h}$, where c is a constant.*

Remark 1.4 A weaker form of Opial's inequality can be obtained by combining Cauchy-Schwarz-Buniakowski's inequality and Wirtinger's inequality:

$$\int_0^h |f(x)f'(x)|\,dx \le \left(\int_0^h |f(x)|^2\,dx\right)^{\frac{1}{2}}\left(\int_0^h |f'(x)|^2\,dx\right)^{\frac{1}{2}} \le \frac{h}{\pi}\int_0^h [f'(x)]^2\,dx.$$

Next inequality involving $x^{(n)}$, $n\ge 1$, is given by Willett [75] (see also [5, p. 128]).

Theorem 1.11 (WILLETT'S INEQUALITY) *Let $x\in C^n[0,h]$ be such that $x^{(i)}(0)=0$, $i=0,\dots,n-1$, $n\ge 1$. Then*

$$\int_0^h \left|x(t)x^{(n)}(t)\right|dt \;\le\; \frac{h^n}{2}\int_0^h \left|x^{(n)}(t)\right|^2 dt. \tag{1.20}$$

More generalizations and extensions of Willett's inequality are done by Boyd in [33].

Following generalization of Opial's inequality is due to Godunova and Levin [46] (see also [5, p. 74]).

Theorem 1.12 (GODUNOVA-LEVIN'S INEQUALITY) *Let f be a convex and increasing function on $[0,\infty)$ with $f(0)=0$. Further, let x be absolutely continuous on $[a,\tau]$ and $x(a)=0$. Then, the following inequality holds*

$$\int_a^\tau f'(|x(t)|)\,|x'(t)|\,dt \le f\left(\int_a^\tau |x'(t)|\,dt\right). \tag{1.21}$$

An extension of the inequality (1.21) is embodied in the following inequality by Rozanova [69] (see also [5, p. 82]).

Theorem 1.13 (ROZANOVA'S INEQUALITY) *Let f, g be convex and increasing functions on $[0,\infty)$ with $f(0)=0$, and let $p(t)\ge 0$, $p'(t)>0$, $t\in[a,\tau]$ with $p(a)=0$. Further, let x be absolutely continuous on $[a,\tau]$ and $x(a)=0$. Then, the following inequality holds*

$$\int_a^\tau p'(t)\,g\left(\frac{|x'(t)|}{p'(t)}\right)f'\left(p(t)\,g\left(\frac{|x(t)|}{p(t)}\right)\right)dt \le f\left(\int_a^\tau p'(t)\,g\left(\frac{|x'(t)|}{p'(t)}\right)dt\right). \tag{1.22}$$

Moreover, equality holds in (1.22) *for the function $x(t)=c\,p(t)$.*

Remark 1.5 The condition in the two previous theorems that function f is to be increasing is actually unneeded, and also, the condition $g \geq 0$ is missing in Theorem 1.13 (it can be easily seen from proofs of the theorems).

Among inequalities of Opial-type, there is a class of inequality involving higher order derivatives. First we have Fink's inequality ([45]).

Theorem 1.14 (FINK'S INEQUALITY) *Let $q \geq 1$, $\frac{1}{p}+\frac{1}{q}=1$, $n \geq 2$ and $0 \leq i \leq j \leq n-1$. Let $f \in AC^n[0,h]$ be such that $f(0)=f'(0)=\cdots=f^{(n-1)}(0)=0$ and $f^{(n)} \in L_q[0,h]$. Then*

$$\int_0^h \left|f^{(i)}(x)\, f^{(j)}(x)\right| dx \leq C h^{2n-i-j+1-\frac{2}{q}} \left(\int_0^h \left|f^{(n)}(x)\right|^q dx\right)^{\frac{2}{q}}, \tag{1.23}$$

where $C=C(n,i,j,q)$ is given by

$$C=\left[2^{\frac{1}{q}}\,(n-i-1)!\,(n-j)!\,[p\,(n-j)+1]^{\frac{1}{p}}\,[p\,(2n-i-j-1)+2]^{\frac{1}{p}}\right]^{-1}. \tag{1.24}$$

Inequality (1.23) is sharp for $j=i+1$, where equality in this case is achieved for $q>1$ and function f such that

$$f(x)=\frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}\,(h-t)^{\frac{p}{q}(n-i-1)}\,dt\,.$$

Remark 1.6 Agarwal and Pang proved in [65] that Fink's inequality does not hold for $i=j$, and that is not necessary to assume that $f^{(k)}(0)=0$ for $k<i$.

Next inequality is due to Agarwal and Pang ([65]).

Theorem 1.15 (AGARWAL-PANG'S INEQUALITY) *Let $n \in \mathbb{N}$ and $f \in AC^n[0,h]$ be such that $f(0)=f'(0)=\cdots=f^{(n-1)}(0)=0$. Let w_1 and w_2 be positive, measurable functions on $[0,h]$. Let $r_i>0$, $i=0,\ldots,n-1$, and let $r=\sum_{i=0}^{n-1} r_i$. Let $s_k>1$ and $\frac{1}{s_k}+\frac{1}{s_k'}=1$ for $k=1,2$, and $q \in \mathbb{R}$ such that $q>s_2$. Further, let*

$$P=\left(\int_0^h [w_2(x)]^{-\frac{s_2'}{q}}\,dx\right)^{\frac{r}{s_2'}}<\infty,$$

$$Q=\left(\int_0^h [w_1(x)]^{s_1'}\,dx\right)^{\frac{1}{s_1'}}<\infty.$$

Then

$$\int_0^h w_1(x)\prod_{i=0}^{n-1}\left|f^{(i)}(x)\right|^{r_i} dx \leq C h^{\rho+\frac{1}{s_1}} \left(\int_0^h w_2(x)\left|f^{(n)}(x)\right|^q dx\right)^{\frac{r}{q}}, \tag{1.25}$$

where $\rho=\sum\limits_{i=0}^{n-1} I r_i+\sigma r$, $I=n-i-1$, $\sigma=\frac{1}{s_2}-\frac{1}{q}$, and $C=C(n,\{r_i\},w_1,w_2,s_1,s_2,q)$ is given by

$$C \leq QP\prod_{i=0}^{n-1}(I!)^{-r_i}\left[\frac{I}{\sigma}+1\right]^{-r_i\sigma}\left[\sum_{i=0}^{n-1} I r_i s_1+\sigma r s_1+1\right]^{-\frac{1}{s_1}},$$

provided that integral on the right side in (1.25) exists.

Alzer's inequalities are given in [10, 11], where second one includes higher order derivatives of two functions.

Theorem 1.16 (ALZER'S INEQUALITY 1) *Let $n \in \mathbb{N}$ and $f \in C^n[a,b]$ be such that $f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0$. Let w be continuous, positive, decreasing function on $[a,b]$. Let $r_i \geq 0$, $i = 0, \ldots, n-1$, and $\sum_{i=0}^{n-1} r_i = 1$. Let $p \geq 1$, $q > 0$ and $\sigma = 1/(p+q)$. Then*

$$\int_a^b w(x) \left(\prod_{i=0}^{n-1} \left| f^{(i)}(x) \right|^{r_i} \right)^p \left| f^{(n)}(x) \right|^q dx \leq A_1 \int_a^b w(x) \left| f^{(n)}(x) \right|^{p+q} dx, \tag{1.26}$$

where

$$A_1 = \sigma q^{\sigma q} \left[n - \sum_{i=1}^{n-1} i r_i \right]^{-\sigma p} (b-a)^{(n - \sum\limits_{i=1}^{n-1} i r_i) p} \prod_{i=0}^{n-1} \left[\left(\frac{1-\sigma}{n-i-\sigma} \right)^{1-\sigma} \frac{1}{(n-i-1)!} \right]^{r_i p}.$$

Theorem 1.17 (ALZER'S INEQUALITY 2) *Let $p \geq 0$, $q > 0$, $r > 1$ and $r > q$. Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$, $0 \leq k \leq n-1$. Let $w_1 \geq 0$ and $w_2 > 0$ be measurable functions on $[a,b]$. Further, let $f, g \in AC^n[a,b]$ be such that $f^{(i)}(a) = g^{(i)}(a) = 0$ for $i = 0, \ldots, n-1$ and let integrals $\int_a^b w_2(x) |f^{(n)}(x)|^r dx$ and $\int_a^b w_2(x) |g^{(n)}(x)|^r dx$ exist. Then*

$$\begin{aligned} &\int_a^b w_1(x) \left[\left| g^{(k)}(x) \right|^p \left| f^{(n)}(x) \right|^q + \left| f^{(k)}(x) \right|^p \left| g^{(n)}(x) \right|^q \right] dx \\ &\leq A_2 \left(\int_a^b w_2(x) \left[\left| f^{(n)}(x) \right|^r + \left| g^{(n)}(x) \right|^r \right] dx \right)^{\frac{p+q}{r}}, \end{aligned} \tag{1.27}$$

where

$$A_2 = \frac{2M}{[(n-k-1)!]^p} \left[\frac{q}{2(p+q)} \right]^{\frac{q}{r}} \left[\int_a^b [w_1(x)]^{\frac{r}{r-q}} [w_2(x)]^{\frac{q}{q-r}} [s(x)]^{\frac{p(r-1)}{r-q}} dx \right]^{\frac{r-q}{r}},$$

$$s(x) = \int_a^x (x-u)^{\frac{r(n-k-1)}{r-1}} [w_2(u)]^{\frac{1}{1-r}} du,$$

$$M = \begin{cases} \left(1 - 2^{-\frac{p}{q}}\right)^{\frac{q}{r}}, & p \geq q, \\ 2^{-\frac{p}{r}}, & p \leq q. \end{cases}$$

Chapter 2

Fractional integrals and fractional derivatives

Fractional calculus is a theory of differential and integral operators of non-integer order. This chapter contains definitions and basic properties of the Riemann-Liouville fractional integral and three main types of fractional derivatives (more detailed information may be found in [38, 51, 68, 72]). The last part of the chapter is based on our results involving composition identities for fractional derivatives: Andrić, Pečarić and Perić [23, 25, 26]. At the same time we investigate the role of the initial conditions on functions included in composition identities, and also relations between the order of the Riemann-Liouville fractional integrals and mentioned fractional derivatives.

Fractional integrals and fractional derivatives will be observed in the real domain. Let $[a,b] \subset \mathbb{R}$ be a finite interval, that is $-\infty < a < b < \infty$. For the integral part of a real number α we use notation $[\alpha]$. Also, Γ is the gamma function defined by (1.5) on $\mathbb{R}^+$, and by (1.6) on $\mathbb{R}_0^- \setminus \mathbb{Z}_0^-$. Throughout this chapter let $x \in [a,b]$.

2.1 The Riemann-Liouville fractional integrals

In [48] G. H. Hardy showed that the Riemann-Liouville fractional integrals are defined for a function $f \in L_1[a,b]$, existing almost everywhere on $[a,b]$. Also, which is in accordance with the classical theorem of Vallée-Poussin and the Young convolution theorem, he proved $J_{a+}^{\alpha} f, J_{b-}^{\alpha} f \in L_1[a,b]$.

Definition 2.1 *Let $\alpha > 0$ and $f \in L_1[a,b]$. The left-sided and the right-sided Riemann-Liouville fractional integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order α are defined by*

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt\,, \quad x \in [a,b]\,, \tag{2.1}$$

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\,dt\,, \quad x \in [a,b]\,. \tag{2.2}$$

For $\alpha = n \in \mathbb{N}$ fractional integrals are actually n-fold integrals, that is

$$\begin{aligned} J^{n}_{a+}f(x) &= \int_a^x dt_1 \int_a^{t_1} dt_2 \cdots \int_a^{t_{n-1}} f(t_n)\,dt_n \\ &= \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)\,dt\,, \end{aligned} \tag{2.3}$$

$$\begin{aligned} J^{n}_{b-}f(x) &= \int_x^b dt_1 \int_{t_1}^{b} dt_2 \cdots \int_{t_{n-1}}^{b} f(t_n)\,dt_n \\ &= \frac{1}{(n-1)!}\int_x^b (t-x)^{n-1} f(t)\,dt\,. \end{aligned} \tag{2.4}$$

Example 2.1 Let $\alpha, \beta > 0$, $f(x) = (x-a)^{\beta-1}$ and $g(x) = (b-x)^{\beta-1}$. By Example 1.1, for the left-sided Riemann-Liouville fractional integral of a function f we have

$$\begin{aligned} J^{\alpha}_{a+}(x-a)^{\beta-1} &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}(t-a)^{\beta-1}\,dt \\ &= \frac{(x-a)^{\alpha+\beta-1}}{\Gamma(\alpha)} B(\alpha,\beta) \\ &= \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1}\,. \end{aligned}$$

Analogously, the right-sided Riemann-Liouville fractional integral of a function g is

$$J^{\alpha}_{b-}(b-x)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(b-x)^{\alpha+\beta-1}\,.$$

Example 2.2 Let $\alpha > 0$ and $\lambda \in \mathbb{R}$. By using Taylor series for the exponential function we have

$$\begin{aligned} J^{\alpha}_{a+}e^{\lambda x} &= J^{\alpha}_{a+}\left(e^{\lambda a}e^{\lambda(x-a)}\right) \\ &= J^{\alpha}_{a+}\left[e^{\lambda a}\sum_{n=0}^{\infty}\frac{\lambda^n (x-a)^n}{n!}\right] \\ &= e^{\lambda a}\sum_{n=0}^{\infty}\frac{\lambda^n}{\Gamma(n+1)} J^{\alpha}_{a+}(x-a)^n \end{aligned}$$

$$= e^{\lambda a}(x-a)^{\alpha}\sum_{n=0}^{\infty}\frac{\lambda^n(x-a)^n}{\Gamma(\alpha+n+1)},$$

$$J_{b-}^{\alpha}e^{\lambda x} = e^{\lambda b}(b-x)^{\alpha}\sum_{n=0}^{\infty}\frac{(-\lambda)^n(b-x)^n}{\Gamma(\alpha+n+1)}.$$

Next we give some properties of the Riemann-Liouville fractional integral, basically presented by Samko et al. in [72] and by Canavati in [38]. Those result we will unify and give complete proofs. We start with a following lemma by Canavati ([38]): the Riemann-Liouville fractional integral of a continuous function is also continuous function.

Lemma 2.1 *Let $\alpha > 0$ and $f \in C[a,b]$. Then for each $x,y \in [a,b]$ we have*

$$\left|J_{a+}^{\alpha}f(x) - J_{a+}^{\alpha}f(y)\right| \le \frac{||f||_C}{\Gamma(\alpha+1)}\left(2\,|x-y|^{\alpha} + |(x-a)^{\alpha} - (y-a)^{\alpha}|\right). \tag{2.5}$$

In particular, if $0 < \alpha < 1$, then

$$\left|J_{a+}^{\alpha}f(x) - J_{a+}^{\alpha}f(y)\right| \le \frac{3\,||f||_C}{\Gamma(\alpha+1)}\,|x-y|^{\alpha}. \tag{2.6}$$

Proof. Let $x < y$. Then

$$\begin{aligned}
&J_{a+}^{\alpha}f(x) - J_{a+}^{\alpha}f(y)\\
&= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\,dt \;-\; \frac{1}{\Gamma(\alpha)}\int_a^y (y-t)^{\alpha-1}f(t)\,dt\\
&= \frac{1}{\Gamma(\alpha)}\int_a^x \left[(x-t)^{\alpha-1} - (y-t)^{\alpha-1}\right]f(t)\,dt \;-\; \frac{1}{\Gamma(\alpha)}\int_x^y (y-t)^{\alpha-1}f(t)\,dt.
\end{aligned}$$

$$\begin{aligned}
&\left|J_{a+}^{\alpha}f(x) - J_{a+}^{\alpha}f(y)\right|\\
&\le \frac{1}{\Gamma(\alpha)}\int_a^x \left|(x-t)^{\alpha-1} - (y-t)^{\alpha-1}\right||f(t)|\,dt \;+\; \frac{1}{\Gamma(\alpha)}\left|\int_x^y (y-t)^{\alpha-1}f(t)\,dt\right|\\
&\le \frac{||f||_C}{\Gamma(\alpha)}\int_a^x \left[(x-t)^{\alpha-1} - (y-t)^{\alpha-1}\right]dt \;+\; \frac{||f||_C}{\Gamma(\alpha)}\int_x^y (y-t)^{\alpha-1}\,dt\\
&= \frac{||f||_C}{\Gamma(\alpha+1)}\left(2(y-x)^{\alpha} \;+\; (x-a)^{\alpha} - (y-a)^{\alpha}\right)\\
&\le \frac{||f||_C}{\Gamma(\alpha+1)}\left(2\,|x-y|^{\alpha} \;+\; |(x-a)^{\alpha} - (y-a)^{\alpha}|\right).
\end{aligned}$$

The same inequality follows for $x > y$, that is (2.5) holds. If $0 < \alpha < 1$, then $\left||a|^{\alpha} - |b|^{\alpha}\right| \le |a-b|^{\alpha}$, and

$$\begin{aligned}
\left|J_{a+}^{\alpha}f(x) - J_{a+}^{\alpha}f(y)\right| &\le \frac{||f||_C}{\Gamma(\alpha+1)}\left(2\,|x-y|^{\alpha} + |(x-a)^{\alpha} - (y-a)^{\alpha}|\right)\\
&\le \frac{3\,||f||_C}{\Gamma(\alpha+1)}\,|x-y|^{\alpha}.
\end{aligned}$$

□

We give lemma for the right-sided Riemann-Liouville fractional integrals. The proof is analogous to the previous one, and is omitted.

Lemma 2.2 *Let $\alpha > 0$ and $f \in C[a,b]$. Then for each $x, y \in [a,b]$ we have*

$$\left|J^{\alpha}_{b-}f(x) - J^{\alpha}_{b-}f(y)\right| \le \frac{||f||_C}{\Gamma(\alpha+1)} \left(2|x-y|^{\alpha} + |(b-x)^{\alpha} - (b-y)^{\alpha}|\right). \tag{2.7}$$

In particular, if $0 < \alpha < 1$, then

$$\left|J^{\alpha}_{b-}f(x) - J^{\alpha}_{b-}f(y)\right| \le \frac{3||f||_C}{\Gamma(\alpha+1)} |x-y|^{\alpha}. \tag{2.8}$$

Corollary 2.1 *Let $\alpha > 0$ and $f \in C[a,b]$. Then $J^{\alpha}_{a+}f, J^{\alpha}_{b-}f \in C[a,b]$.*

Next we observe the composition of fractional integrals (see Samko et al. [72], Section 2).

Lemma 2.3 *Let $\alpha, \beta > 0$ and $f \in L_p[a,b]$, $1 \le p \le \infty$. Then for almost every $x \in [a,b]$ we have*

$$J^{\alpha}_{a+}J^{\beta}_{a+}f(x) = J^{\alpha+\beta}_{a+}f(x), \qquad J^{\alpha}_{b-}J^{\beta}_{b-}f(x) = J^{\alpha+\beta}_{b-}f(x). \tag{2.9}$$

If $f \in C[a,b]$ or $\alpha + \beta > 1$, then equalities (2.9) hold for each x in $[a,b]$.

Proof. Straightforward calculations with Example 1.2 gives us

$$\begin{aligned}
J^{\alpha}_{a+}J^{\beta}_{a+}f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} J^{\beta}_{a+}f(t)\,dt \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x (x-t)^{\alpha-1} \int_a^t (t-s)^{\beta-1} f(s)\,ds\,dt \\
&= \frac{1}{\Gamma(\alpha+\beta)} \int_a^x (x-s)^{\alpha+\beta-1} f(s)\,ds \\
&= J^{\alpha+\beta}_{a+}f(x).
\end{aligned}$$

Analogously for the right-sided fractional integrals follows

$$J^{\alpha}_{b-}J^{\beta}_{b-}f(x) = \frac{1}{\Gamma(\alpha+\beta)} \int_x^b (s-x)^{\alpha+\beta-1} f(s)\,ds = J^{\alpha+\beta}_{b-}f(x).$$

If $f \in C[a,b]$, then $J^{\beta}_{a+}f \in C[a,b]$ by Lemma 2.1, and also $J^{\alpha}_{a+}J^{\beta}_{a+}f \in C[a,b]$, $J^{\alpha+\beta}_{a+}f \in C[a,b]$. Hence, two function $J^{\alpha}_{a+}J^{\beta}_{a+}f$ and $J^{\alpha+\beta}_{a+}f$ coincide almost everywhere on $[a,b]$, and by continuity follows that they coincide on whole interval $[a,b]$. If $f \in L_p[a,b]$ and $\alpha + \beta > 1$, then

$$J^{\alpha}_{a+}J^{\beta}_{a+}f = J^{\alpha+\beta}_{a+}f = J^{\alpha+\beta-1}_{a+}J^{1}_{a+}f$$

almost everywhere on $[a,b]$. Since $J^{1}_{a+}f$ is continuous function, then $J^{\alpha+\beta}_{a+} = J^{\alpha+\beta-1}_{a+}J^{1}_{a+}f \in C[a,b]$, that is once again they coincide on whole interval $[a,b]$ due to continuity.

The same goes for the right-sided Riemann-Liouville fractional integral, so we conclude that equalities (2.9) hold for each x in $[a,b]$. $\square$

The homogeneous Abel integral equation has only trivial solution (see Samko et al. [72], Section 2.4).

Lemma 2.4 *Let $\alpha > 0$ and $f \in L_1[a,b]$. Then integral equations $J^{\alpha}_{a+}f = 0$ and $J^{\alpha}_{b-}f = 0$ have only trivial solution $f = 0$ (almost everywhere).*

Proof. Let $J^{\alpha}_{a+}f = 0$. If $0 < \alpha < 1$, then by Lemma 2.3 follows $J^{1}_{a+}f = J^{1-\alpha}_{a+}J^{\alpha}_{a+}f = 0$. Now we have

$$f = \frac{d}{dx}\int_a^x f(t)\,dt = \frac{d}{dx}J^{1}_{a+}f = 0\,.$$

Let $\alpha \geq 1$, $m = [\alpha]$, $\alpha = m+\beta, 0 \leq \beta < 1$. If $\beta = 0$, then $\alpha = m \in \mathbb{N}$, and by (2.3) follows $f = \frac{d^m}{dx^m}J^{m}_{a+}f = 0$. Let $\beta > 0$. Again by Lemma 2.3 follows

$$J^{\beta}_{a+}J^{m}_{a+}f = J^{\alpha}_{a+}f = 0,$$

and by just proven, for $0 < \beta < 1$ we have $J^{m}_{a+}f = 0$ and also $f = 0$. The proof is analogous for the right-sided Riemann-Liouville fractional integral. $\square$

Lemma 2.1 and Lemma 2.2 showed that the Riemann-Liouville fractional integral of continuous function is also continuous function. Moreover, for the image of the Riemann-Liouville fractional integral of continuous function we have next result by Canavati ([38]).

Lemma 2.5 *Let $\alpha > 0$ and $f \in C[a,b]$. Then $J^{\alpha}_{a+}f \in \mathscr{D}^{\alpha}_{a}[a,b]$ and $J^{\alpha}_{b-}f \in \mathscr{D}^{\alpha}_{b}[a,b]$.*

Proof. Let $m = [\alpha]$ and $\mu = \alpha - m$. For $\mu = 0$, that is $\alpha = m \in \mathbb{N}$ $(m \geq 1)$, we use (2.3) and Lemma 2.3

$$\frac{d^k}{dx^k}J^{m}_{a+}f(x) = \frac{d^k}{dx^k}J^{k}_{a+}J^{m-k}_{a+}f(x) = J^{m-k}_{a+}f(x)\,, \quad k = 0,1,\ldots,m-1\,,$$

that is $\frac{d^k}{dx^k}J^{m}_{a+}f(a) = 0$, for $k = 0,1,\ldots,m-1$ (since f is continuous at a), and then $J^{m}_{a+}f \in C^{m}_{a}[a,b] = \mathscr{D}^{m}_{a}[a,b]$.
Let $0 < \alpha < 1$. Then by Lemma 2.1 we have (2.6),

$$\left|J^{\alpha}_{a+}f(x) - J^{\alpha}_{a+}f(y)\right| \leq \frac{3\,||f||_C}{\Gamma(\alpha+1)}\,|x-y|^{\alpha}\,,$$

that is $J^{\alpha}_{a+}f \in C[a,b]$. Since $0 < \alpha < 1$, then $m = 0$ and

$$\left|J^{\alpha}_{a+}f\right|_{m,\mu} = \sup\left\{\frac{\left|J^{\alpha}_{a+}f(x) - J^{\alpha}_{a+}f(y)\right|}{|x-y|^{\mu}}\right\} \leq \frac{3\,||f||_C}{\Gamma(\alpha+1)} < \infty,$$

that is $J^{\alpha}_{a+}f \in \mathscr{D}^{\alpha}[a,b]$. Further, $J^{\alpha}_{a+}f(a) = 0$, $m = 0$, and then $J^{\alpha}_{a+}f \in \mathscr{D}^{\alpha}_{a}[a,b]$.
Let $\alpha > 1$ $(m \geq 1)$ and $0 < \mu < 1$. Then

$$\left|\frac{d^m}{dx^m}J^{\alpha}_{a+}f(x) - \frac{d^m}{dx^m}J^{\alpha}_{a+}f(y)\right| = \left|J^{\mu}_{a+}f(x) - J^{\mu}_{a+}f(y)\right| \leq \frac{3\,||f||_C}{\Gamma(\alpha+1)}\,|x-y|^{\mu}\,,$$

that is $\left|J^{\alpha}_{a+}f\right|_{m,\mu} \leq \frac{3||f||_C}{\Gamma(\alpha+1)} < \infty$ and $J^{\alpha}_{a+}f \in \mathscr{D}^{\alpha}[a,b]$. Again, using Lemma 2.3, (2.3) and continuity of f at a, we have $\frac{d^k}{dx^k}J^{\alpha}_{a+}f(a) = J^{\alpha-k}_{a+}f(a) = 0$ za $k = 0,1,\dots,m$, that is $J^{\alpha}_{a+}f \in \mathscr{D}^{\alpha}_a[a,b]$.
The proof is analogous for the right-sided Riemann-Liouville fractional integral. □

Since $\mathscr{D}^{\alpha}_a[a,b], \mathscr{D}^{\alpha}_b[a,b] \subseteq \mathscr{D}^{\alpha}[a,b] \subseteq C^{[\alpha]}[a,b]$, next corollary is valid.

Corollary 2.2 *Let $\alpha > 0$, $m = [\alpha]$ and $f \in C[a,b]$. Then $J^{\alpha}_{a+}f, J^{\alpha}_{b-}f \in C^m[a,b]$.*

Next result by Samko et al. ([72]) shows that the Riemann-Liouville fractional integral is bounded operator on $L_p[a,b]$.

Lemma 2.6 *Let $\alpha > 0$ and $1 \leq p \leq \infty$. Then the Riemann-Liouville fractional integrals are bounded on $L_p[a,b]$, that is*

$$||J^{\alpha}_{a+}f||_p \leq K||f||_p\,, \quad ||J^{\alpha}_{b-}f||_p \leq K||f||_p\,, \tag{2.10}$$

where

$$K = \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}.$$

For $C[a,b]$ we have

$$||J^{\alpha}_{a+}f||_C \leq K||f||_C\,, \quad ||J^{\alpha}_{b-}f||_C \leq K||f||_C\,. \tag{2.11}$$

Proof. We give a proof for the left-sided fractional integrals in spaces $L_p[a,b]$ and $C[a,b]$. The proof for the right-sided fractional integrals is analogous. By Jensen's inequality (1.16) and Fubini's theorem follows

$$\int_a^b (x-a)^{\alpha}\left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}\left|J^{\alpha}_{a+}f(x)\right|\right)^p dx \tag{2.12}$$

$$\begin{aligned}
&= \int_a^b (x-a)^{\alpha}\left(\frac{\alpha}{(x-a)^{\alpha}}\int_a^x (x-t)^{\alpha-1}|f(t)|\,dt\right)^p dx \\
&= \int_a^b (x-a)^{\alpha}\left(\int_a^x (x-t)^{\alpha-1}|f(t)|\,dt \Big/ \int_a^x (x-t)^{\alpha-1}\,dt\right)^p dx \\
&\leq \int_a^b (x-a)^{\alpha}\left(\int_a^x (x-t)^{\alpha-1}|f(t)|^p\,dt \Big/ \int_a^x (x-t)^{\alpha-1}\,dt\right) dx \\
&= \int_a^b \alpha\int_a^x (x-t)^{\alpha-1}\,|f(t)|^p\,dt\,dx \\
&= \int_a^b \alpha|f(t)|^p\int_t^b (x-t)^{\alpha-1}\,dx\,dt \\
&= \int_a^b |f(t)|^p\,(b-t)^{\alpha}\,dt\,.
\end{aligned} \tag{2.13}$$

Since $x \in [a,b]$ and $\alpha(1-p) < 0$, for (2.12) we have

$$\int_a^b (x-a)^{\alpha}\left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}\left|J^{\alpha}_{a+}f(x)\right|\right)^p dx$$

$$
\begin{aligned}
&= \int_a^b (x-a)^{\alpha(1-p)} [\Gamma(\alpha+1)]^p \left|J_{a+}^{\alpha} f(x)\right|^p dx \\
&\geq (b-a)^{\alpha(1-p)} [\Gamma(\alpha+1)]^p \int_a^b \left|J_{a+}^{\alpha} f(x)\right|^p dx, \qquad (2.14)
\end{aligned}
$$

and for (2.13)

$$
\int_a^b (b-t)^{\alpha} |f(t)|^p dt \leq (b-a)^{\alpha} \int_a^b |f(t)|^p dt. \qquad (2.15)
$$

Now from (2.14) and (2.15) follows

$$
(b-a)^{\alpha(1-p)} [\Gamma(\alpha+1)]^p \int_a^b \left|J_{a+}^{\alpha} f(x)\right|^p dx \leq (b-a)^{\alpha} \int_a^b |f(t)|^p dt,
$$

that is

$$
\int_a^b \left|J_{a+}^{\alpha} f(x)\right|^p dx \leq \left[\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right]^p \int_a^b |f(t)|^p dt. \qquad (2.16)
$$

If we use exponent $1/p$ for both sides of inequality (2.16), we get that the left-sided Riemann-Liouville fractional integrals are bounded on $L_p[a,b]$.
For $C[a,b]$ we have inequality

$$
||J_{a+}^{\alpha} f||_C = \max_{x\in[a,b]} |J_{a+}^{\alpha} f(x)| \leq \max_{x\in[a,b]} |J_{a+}^{\alpha} 1| \cdot ||f||_C
$$

and by Example 2.1 for $\beta=1$ we have $J_{a+}^{\alpha} 1 = (x-a)^{\alpha}/\Gamma(\alpha+1)$, that is $\max\limits_{x\in[a,b]} |J_{a+}^{\alpha} 1| = K$. $\square$

At the end of this section, we give our result showing that for $\alpha \in (0,1]$ the Riemann-Liouville fractional integral of an absolutely continuous function is also absolutely continuous.

Proposition 2.1 *Let $n \in \mathbb{N}$, $0 < \alpha \leq 1$ and $f \in AC^n[a,b]$. Then $J_{a+}^{\alpha} f \in AC^n[a,b]$.*

Proof. Let $f \in AC^n[a,b]$, that is $f \in C^{n-1}[a,b]$ and $f^{(n-1)} \in AC[a,b]$. Let

$$
g(x) = f(x) - \sum_{and=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k.
$$

The statement $J_{a+}^{\alpha} f \in AC^n[a,b]$ will follow if we prove that $J_{a+}^{\alpha} g \in AC^n[a,b]$.

First we prove that $J_{a+}^{\alpha} g \in C^{n-1}[a,b]$, that is $\frac{d^k}{dx^k} J_{a+}^{\alpha} g \in C[a,b]$ for $k = 0, \dots, n-1$. Notice that $g(a) = g'(a) = \cdots = g^{(n-1)}(a) = 0$. Using integration by parts we get

$$
\begin{aligned}
J_{a+}^{\alpha} g(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} g(t)\, dt \\
&= \frac{1}{\Gamma(\alpha)} \left[-\frac{(x-t)^{\alpha}}{\alpha} g(t)\Big|_{t=a}^{x} + \int_a^x \frac{(x-t)^{\alpha}}{\alpha} g'(t)\, dt \right] \\
&= \frac{1}{\Gamma(\alpha+1)} \int_a^x (x-t)^{\alpha} g'(t)\, dt.
\end{aligned}
$$

If we repeat this procedure k times ($k \le n$), then

$$J_{a+}^{\alpha} g(x) = \frac{1}{\Gamma(\alpha+k)} \int_a^x (x-t)^{\alpha+k-1} g^{(k)}(t)\,dt = J_{a+}^{\alpha+k} g^{(k)}(x). \tag{2.17}$$

Since $f^{(k)}$ is continuous for $k = 0, \dots, n-1$, then $g^{(k)}$ is also continuous, and since

$$\frac{d^k}{dx^k} J_{a+}^{\alpha+k} g^{(k)}(x) = \frac{d^k}{dx^k} J_{a+}^{k} J_{a+}^{\alpha} g^{(k)}(x) = J_{a+}^{\alpha} g^{(k)}(x), \tag{2.18}$$

then by Lemma 2.1, and Corollary 2.1, follows that $J_{a+}^{\alpha} g^{(k)}$ is continuous too. Hence, $\frac{d^k}{dx^k} J_{a+}^{\alpha} g \in C[a,b]$ for $k = 0, \dots, n-1$, that is $J_{a+}^{\alpha} g \in C^{n-1}[a,b]$.

It remains to show that $\frac{d^{n-1}}{dx^{n-1}} J_{a+}^{\alpha} g \in AC[a,b]$, that is $\frac{d^n}{dx^n} J_{a+}^{\alpha} g \in L_1[a,b]$. If we put $k = n$ in (2.17), then as in (2.18) we get

$$\frac{d^n}{dx^n} J_{a+}^{\alpha} g(x) = \frac{d^n}{dx^n} J_{a+}^{\alpha+n} g^{(n)}(x) = J_{a+}^{\alpha} g^{(n)}(x).$$

Since $g^{(n)} = f^{(n)}$, and $f^{(n)}$ is integrable, that also $J_{a+}^{\alpha} g^{(n)} \in L_1[a,b]$. $\square$

We give analogous lemma for the right-sided Riemann-Liouville fractional integrals. The proof is omitted.

Proposition 2.2 *Let $n \in \mathbb{N}$, $0 < \alpha \le 1$ and $f \in AC^n[a,b]$. Then $J_{b-}^{\alpha} f \in AC^n[a,b]$.*

2.2 The Riemann-Liouville fractional derivatives

Definition 2.2 *Let $\alpha > 0$, $n = [\alpha] + 1$ and $f : [a,b] \to \mathbb{R}$. The left-sided and the right-sided Riemann-Liouville fractional derivatives $D_{a+}^{\alpha} f$ and $D_{b-}^{\alpha} f$ of order α are defined by*

$$\begin{aligned} D_{a+}^{\alpha} f(x) &= \frac{d^n}{dx^n} J_{a+}^{n-\alpha} f(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} f(t)\,dt, \end{aligned} \tag{2.19}$$

$$\begin{aligned} D_{b-}^{\alpha} f(x) &= (-1)^n \frac{d^n}{dx^n} J_{b-}^{n-\alpha} f(x) \\ &= \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b (t-x)^{n-\alpha-1} f(t)\,dt. \end{aligned} \tag{2.20}$$

In particular, if $0 < \alpha < 1$, then

$$D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} f(t)\,dt, \tag{2.21}$$

$$D_{b-}^{\alpha} f(x) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (t-x)^{-\alpha} f(t)\, dt\,. \tag{2.22}$$

For $\alpha = n \in \mathbb{N}$ we have

$$D_{a+}^{n} f(x) = f^{(n)}(x)\,, \quad D_{b-}^{n} f(x) = (-1)^n f^{(n)}(x)\,, \tag{2.23}$$

and for $\alpha = 0$

$$D_{a+}^{0} f(x) = D_{b-}^{0} f(x) = f(x)\,. \tag{2.24}$$

We will also use the notations

$$J_{a+}^{-\alpha} f := D_{a+}^{\alpha} f,\; J_{b-}^{-\alpha} f := D_{b-}^{\alpha} f, \quad \alpha > 0\,. \tag{2.25}$$

Example 2.3 Let $\alpha \geq 0$, $\beta > 0$, $f(x) = (x-a)^{\beta-1}$ and $g(x) = (b-x)^{\beta-1}$. By Example 2.1, for the left-sided Riemann-Liouville fractional derivative of the function f we have

$$\begin{aligned} D_{a+}^{\alpha}(x-a)^{\beta-1} &= \frac{d^n}{dx^n} J_{a+}^{n-\alpha}(x-a)^{\beta-1} \\ &= \frac{d^n}{dx^n} \frac{\Gamma(\beta)}{\Gamma(n-\alpha+\beta)} (x-a)^{n-\alpha+\beta-1} \\ &= \frac{\Gamma(\beta)}{\Gamma(-\alpha+\beta)} (x-a)^{-\alpha+\beta-1}\,. \end{aligned} \tag{2.26}$$

Analogously, the right-sided Riemann-Liouville fractional derivative of the function g is

$$\begin{aligned} D_{b-}^{\alpha}(b-x)^{\beta-1} &= (-1)^n \frac{d^n}{dx^n} J_{b-}^{n-\alpha}(b-x)^{\beta-1} \\ &= \frac{\Gamma(\beta)}{\Gamma(-\alpha+\beta)} (b-x)^{-\alpha+\beta-1}\,. \end{aligned} \tag{2.27}$$

In particular, if $\beta = 1$, then $f(x) = g(x) = 1$ and we have

$$D_{a+}^{\alpha}(1) = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}, \quad D_{b-}^{\alpha}(1) = \frac{(b-x)^{-\alpha}}{\Gamma(1-\alpha)}\,,$$

that is, the Riemann-Liouville fractional derivatives of a constant function are, in general, not equal to zero.
On the other hand, for $j = 1, 2, \ldots, n$,

$$D_{a+}^{\alpha}(x-a)^{\alpha-j} = \frac{\Gamma(\alpha-j+1)}{\Gamma(1-j)} (x-a)^{-j} = 0\,, \qquad D_{b-}^{\alpha}(b-x)^{\alpha-j} = 0\,, \tag{2.28}$$

since the gamma function has simple poles in $0, -1, -2, \ldots$

From (2.28) we derive the following result indicating that the functions $(x-a)^{\alpha-j}$, and $(b-x)^{\alpha-j}$, play the same role for fractional derivatives as the constants do in usual differentiation.

Corollary 2.3 *Let $\alpha > 0$ and $n = [\alpha] + 1$.*

(*i*) *The equality $D^\alpha_{a+} f(x) = 0$ is valid if and only if $f(x) = \sum_{j=1}^{n} c_j (x-a)^{\alpha-j}$, where $c_j \in \mathbb{R}$ $(j = 1, \ldots, n)$ are arbitrary constants.*
In particular, when $0 < \alpha \le 1$, the relation $D^\alpha_{a+} f(x) = 0$ holds if and only if $f(x) = c(x-a)^{\alpha-1}$ with every $c \in \mathbb{R}$.

(*ii*) *The equality $D^\alpha_{b-} f(x) = 0$ is valid if and only if $f(x) = \sum_{j=1}^{n} d_j (b-x)^{\alpha-j}$, where $d_j \in \mathbb{R}$ $(j = 1, \ldots, n)$ are arbitrary constants.*
In particular, when $0 < \alpha \le 1$, the relation $D^\alpha_{b-} f(x) = 0$ holds if and only if $f(x) = d(b-x)^{\alpha-1}$ with every $d \in \mathbb{R}$.

Example 2.4 Let $\alpha > 0$ and $\lambda \in \mathbb{R}$. For the exponential function we have

$$\begin{aligned}
D^\alpha_{a+} e^{\lambda x} &= D^\alpha_{a+} \left(e^{\lambda a} e^{\lambda (x-a)} \right) \\
&= D^\alpha_{a+} \left[e^{\lambda a} \sum_{k=0}^{\infty} \frac{\lambda^k (x-a)^k}{k!} \right] \\
&= e^{\lambda a} \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(k+1)} D^\alpha_{a+} (x-a)^k \\
&= \frac{e^{\lambda a}}{(x-a)^\alpha} \sum_{k=n}^{\infty} \frac{\lambda^k (x-a)^k}{\Gamma(-\alpha+k+1)}, \\
D^\alpha_{b-} e^{\lambda x} &= \frac{e^{\lambda b}}{(b-x)^\alpha} \sum_{k=n}^{\infty} \frac{(-\lambda)^k (b-x)^k}{\Gamma(-\alpha+k+1)}.
\end{aligned}$$

We proceed with conditions for the existence of fractional derivatives in the space $AC^n[a,b]$.

Theorem 2.1 *Let $\alpha \ge 0$ and $n = [\alpha] + 1$. If $f \in AC^n[a,b]$, then the Riemann-Liouville fractional derivatives $D^\alpha_{a+} f$ and $D^\alpha_{b-} f$ exist almost everywhere on $[a,b]$ and can be represented in the forms*

$$D^\alpha_{a+} f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)} (x-a)^{k-\alpha} + \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)\, dt, \quad (2.29)$$

$$D^\alpha_{b-} f(x) = \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(b)}{\Gamma(k-\alpha+1)} (b-x)^{k-\alpha} + \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (t-x)^{n-\alpha-1} f^{(n)}(t)\, dt. \quad (2.30)$$

Proof. Since $f \in AC^n[a,b]$ then by Lemma 1.2, Example 1.2 and Example 2.3 follows

$$\begin{aligned}
&D^\alpha_{a+} f(x) \\
&= D^\alpha_{a+} \left(\sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k + \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f^{(n)}(t)\, dt \right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} D_{a+}^{\alpha} (x-a)^k + D_{a+}^{\alpha} J_{a+}^{n} f^{(n)}(x) \\
&= \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} (x-a)^{k-\alpha} \\
&\quad + \frac{1}{\Gamma(n-\alpha)\Gamma(n)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} \int_a^t (t-s)^{n-1} f^{(n)}(s)\, ds\, dt \\
&= \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k+1-\alpha)} (x-a)^{k-\alpha} \\
&\quad + \frac{1}{\Gamma(n-\alpha)\Gamma(n)} B(n-\alpha, n) \frac{d^n}{dx^n} \int_a^x (x-s)^{2n-\alpha-1} f^{(n)}(s)\, ds \\
&= \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k+1-\alpha)} (x-a)^{k-\alpha} + \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-s)^{n-\alpha-1} f^{(n)}(s)\, ds\,.
\end{aligned}
$$

In last step we use (2.3) and the fact that for the ordinary derivatives and n-fold integrals $\frac{d^n}{dx^n} J_{a+}^n \varphi = \varphi$ is valid, that is

$$
\frac{d^n}{dx^n} J_{a+}^{2n-\alpha} f^{(n)}(x) = \frac{d^n}{dx^n} J_{a+}^{n} J_{a+}^{n-\alpha} f^{(n)}(x) = J_{a+}^{n-\alpha} f^{(n)}(x)\,.
$$

The equality (2.30) follows similarly by using next representation of the function $g \in AC^n[a,b]$:

$$
g(x) = \frac{(-1)^n}{(n-1)!} \int_x^b (t-x)^{n-1} \phi(t)\, dt + \sum_{k=0}^{n-1} (-1)^k d_k (b-x)^k\,, \tag{2.31}
$$

where

$$
\phi = g^{(n)} \in L_1[a,b], \quad d_k = \frac{g^{(k)}(b)}{k!} \; (k = 0, 1, \ldots, n-1)\,. \tag{2.32}
$$

□

Corollary 2.4 *Let* $0 \le \alpha < 1$ *and* $f \in AC[a,b]$. *Then*

$$
D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \left[(x-a)^{-\alpha} f(a) + \int_a^x (x-t)^{-\alpha} f'(t)\, dt \right], \tag{2.33}
$$

$$
D_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \left[(b-x)^{-\alpha} f(b) - \int_x^b (t-x)^{-\alpha} f'(t)\, dt \right]. \tag{2.34}
$$

Let us now consider a relation between fractional differentiation and fractional integration. It is well known that differentiation and integration are reciprocal operation if integration is applied first, that is $(d/dx) \int_a^x f(t)\, dt = f(x)$ while $\int_a^x f'(t)\, dt = f(x) - f(a)$. In the same way for $n \in \mathbb{N}$ we have $(d^n/dx^n) J_{a+}^n f \equiv f$, but $J_{a+}^n f^{(n)} \neq f$, differing from f by a polynomial of the order $n-1$. Similarly, for $\alpha > 0$, we will always have $D_{a+}^{\alpha} J_{a+}^{\alpha} f \equiv f$, but $J_{a+}^{\alpha} D_{a+}^{\alpha} f \neq f$, since it may arise the linear combinations of functions $(x-a)^{\alpha-k}$,

$k = 1,2,\dots,[\alpha]+1$. We will observe this in detail in Theorem 2.4. First, let us define the following spaces of functions:

Let $\alpha > 0$ and $1 \le p \le \infty$. By $J^{\alpha}_{a+}(L_p)$ and $J^{\alpha}_{b-}(L_p)$ denote next two spaces of functions

$$J^{\alpha}_{a+}(L_p) = \left\{ f\colon\, f = J^{\alpha}_{a+}\varphi,\ \varphi \in L_p[a,b] \right\}, \tag{2.35}$$

$$J^{\alpha}_{b-}(L_p) = \left\{ f\colon\, f = J^{\alpha}_{b-}\psi,\ \psi \in L_p[a,b] \right\}. \tag{2.36}$$

The characterization of the space $J^{\alpha}_{a+}(L_1)$ is given by the following theorem.

Theorem 2.2 *Let $\alpha > 0$ and $n = [\alpha]+1$. Then $f \in J^{\alpha}_{a+}(L_1)$ if and only if*

$$J^{n-\alpha}_{a+} f \in AC^n[a,b], \tag{2.37}$$

$$\frac{d^k}{dx^k} J^{n-\alpha}_{a+} f(a) = 0, \quad k = 0,1,\dots,n-1. \tag{2.38}$$

Proof. Necessity. Let $f \in J^{\alpha}_{a+}(L_1)$, i.e. $f = J^{\alpha}_{a+}\varphi$, where $\varphi \in L_1[a,b]$. By Lemma 2.3 follows

$$J^{n-\alpha}_{a+} f = J^{n-\alpha}_{a+} J^{\alpha}_{a+}\varphi = J^{n}_{a+}\varphi.$$

Using Lemma 1.2 we know that functions from $AC^n[a,b]$ have the form

$$J^{n}_{a+}\varphi(x) + \sum_{k=0}^{n-1} c_k (x-a)^k,$$

where $\varphi \in L_1[a,b]$, and c_k are constants. Hence, $J^{n-\alpha}_{a+} f \in AC^n[a,b]$ and

$$c_k = \frac{1}{k!}\frac{d^k}{dx^k} J^{n-\alpha}_{a+} f(a) = 0,$$

showing that conditions (2.37) and (2.38) are valid.
Sufficiency. Let conditions (2.37) and (2.38) hold. Then $J^{n-\alpha}_{a+} f = J^{n}_{a+}\varphi$ follows from Lemma 1.2 for $\varphi \in L_1[a,b]$. Again we use Lemma 2.3 which gives us

$$J^{n-\alpha}_{a+} f = J^{n}_{a+}\varphi = J^{n-\alpha}_{a+} J^{\alpha}_{a+}\varphi,$$

i.e.

$$J^{n-\alpha}_{a+}(f - J^{\alpha}_{a+}\varphi) = 0.$$

Since $n-\alpha > 0$, by Lemma 2.4 follows $f - J^{\alpha}_{a+}\varphi = 0$, that is $f \in J^{\alpha}_{a+}(L_1)$. □

Analogously, for $J^{\alpha}_{b-}(L_1)$ we have next theorem in which for function $g \in AC^n[a,b]$ we use (2.31) and (2.32).

Theorem 2.3 *Let $\alpha > 0$ and $n = [\alpha]+1$. Then $g \in J^{\alpha}_{b-}(L_1)$ if and only if*

$$J^{n-\alpha}_{b-} g \in AC^n[a,b], \tag{2.39}$$

$$\frac{d^k}{dx^k} J^{n-\alpha}_{b-} g(b) = 0, \quad k = 0,1,\dots,n-1. \tag{2.40}$$

Definition 2.3 *Let $\alpha > 0$ and $n = [\alpha] + 1$. A function $f \in L_1[a,b]$ is said to have a left-sided integrable fractional derivative $D^\alpha_{a+} f$ if $J^{n-\alpha}_{a+} f \in AC^n[a,b]$, and to have a right-sided integrable fractional derivative $D^\alpha_{b-} f$ if $J^{n-\alpha}_{b-} f \in AC^n[a,b]$.*

If $D^\alpha_{a+} f = (d^n/dx^n) J^{n-\alpha}_{a+} f$ exists in the usual sense, i.e. $J^{n-\alpha}_{a+} f$ is differentiable n times at every point, then, f has a derivative in the sense of Definition 2.3. This holds also for the right-sided fractional derivative.

Lemma 2.7 *Let $\alpha > 0$ and $n = [\alpha] + 1$. Then $f \in L_1[a,b]$ has the left-sided integrable fractional derivative $D^\alpha_{a+} f$ if and only if*

$$D^{\alpha-k}_{a+} f \in C[a,b], \quad k = 1, \ldots, n, \tag{2.41}$$

$$D^{\alpha-1}_{a+} f \in AC[a,b]. \tag{2.42}$$

Also, $f \in J^\alpha_{a+}(L_1)$ if and only if f has the left-sided integrable fractional derivative $D^\alpha_{a+} f$ such that

$$D^{\alpha-k}_{a+} f(a) = 0, \quad k = 1, \ldots, n. \tag{2.43}$$

Proof. Let f has the left-sided integrable fractional derivative $D^\alpha_{a+} f$, i.e. $J^{n-\alpha}_{a+} f \in AC^n[a,b]$. Then $J^{n-\alpha}_{a+} f \in C^{n-1}[a,b]$, that is

$$\frac{d^k}{dx^k} J^{n-\alpha}_{a+} f \in C[a,b], \quad k = 0, \ldots, n-1$$

and $\dfrac{d^{n-1}}{dx^{n-1}} J^{n-\alpha}_{a+} f \in AC[a,b]$. From $[\alpha - n + k] + 1 = k$ follows

$$\frac{d^k}{dx^k} J^{n-\alpha}_{a+} f(x) = \frac{d^k}{dx^k} J^{k-(\alpha-n+k)}_{a+} f(x) = D^{\alpha-n+k}_{a+} f(x), \quad k = 0, 1, \ldots, n-1,$$

which shows that conditions (2.41) and (2.42) are equivalent with (2.37). Also, a condition (2.43) is equivalent with (2.38). In a case $k = n$ we have $n - \alpha \le 1$, and by (2.25) follows $D^{\alpha-n}_{a+} f = J^{n-\alpha}_{a+} f$. □

Lemma 2.8 *Let $\alpha > 0$ and $n = [\alpha] + 1$. Then $f \in L_1[a,b]$ has the right-side integrable fractional derivative $D^\alpha_{b-} f$ if and only if*

$$D^{\alpha-k}_{b-} f \in C[a,b], \quad k = 1, \ldots, n, \tag{2.44}$$

$$D^{\alpha-1}_{b-} f \in AC[a,b]. \tag{2.45}$$

Also, $f \in J^\alpha_{b-}(L_1)$ if and only if f has the right-sided integrable fractional derivative $D^\alpha_{b-} f$ such that

$$D^{\alpha-k}_{b-} f(b) = 0, \quad k = 1, \ldots, n. \tag{2.46}$$

Now we can present theorem about the composition of the fractional integration operator with the fractional differentiation operator.

Theorem 2.4 *Let $\alpha > 0$, $n = [\alpha]+1$ and $1 \le p \le \infty$.*

(i) *If $\varphi \in L_p[a,b]$, then relations*

$$D^{\alpha}_{a+}J^{\alpha}_{a+}\varphi = \varphi\,, \quad D^{\alpha}_{b-}J^{\alpha}_{b-}\varphi = \varphi \tag{2.47}$$

hold almost everywhere on $[a,b]$.

(ii) *If $f \in J^{\alpha}_{a+}(L_p)$ and $g \in J^{\alpha}_{b-}(L_p)$, then*

$$J^{\alpha}_{a+}D^{\alpha}_{a+}f = f\,, \quad J^{\alpha}_{b-}D^{\alpha}_{b-}g = g\,. \tag{2.48}$$

(iii) *If $f \in L_1[a,b]$ has the left-sided integrable fractional derivative $D^{\alpha}_{a+}f$, then the equality*

$$J^{\alpha}_{a+}D^{\alpha}_{a+}f(x) = f(x) - \sum_{k=1}^{n} \frac{D^{\alpha-k}_{a+}f(a)}{\Gamma(\alpha-k+1)}\,(x-a)^{\alpha-k} \tag{2.49}$$

holds almost everywhere on $[a,b]$.
If $g \in L_1[a,b]$ has the right-sided integrable fractional derivative $D^{\alpha}_{b-}g$, then the equality

$$J^{\alpha}_{b-}D^{\alpha}_{b-}g(x) = g(x) - \sum_{k=1}^{n} \frac{(-1)^{n-k}D^{\alpha-k}_{b-}g(b)}{\Gamma(\alpha-k+1)}\,(b-x)^{\alpha-k} \tag{2.50}$$

holds almost everywhere on $[a,b]$.
In particular, if $0 < \alpha < 1$, then

$$J^{\alpha}_{a+}D^{\alpha}_{a+}f(x) = f(x) - \frac{J^{1-\alpha}_{a+}f(a)}{\Gamma(\alpha)}\,(x-a)^{\alpha-1}\,, \tag{2.51}$$

$$J^{\alpha}_{b-}D^{\alpha}_{b-}g(x) = g(x) - \frac{J^{1-\alpha}_{b-}g(b)}{\Gamma(\alpha)}\,(b-x)^{\alpha-1}\,, \tag{2.52}$$

while for $\alpha = n \in \mathbb{N}$ we have

$$J^{n}_{a+}D^{n}_{a+}f(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^{k}\,, \tag{2.53}$$

$$J^{n}_{b-}D^{n}_{b-}g(x) = g(x) - \sum_{k=0}^{n-1} \frac{(-1)^{k}g^{(k)}(b)}{k!}(b-x)^{k}\,. \tag{2.54}$$

Proof. We give a proof for the left-sided fractional integrals and derivatives, since for the right-sided it follows analogously.

(i) Let $\varphi \in L_p[a,b]$. Then

$$D^{\alpha}_{a+}J^{\alpha}_{a+}\varphi(x) = \frac{1}{\Gamma(\alpha)\,\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-t)^{n-\alpha-1}\int_a^t (t-s)^{\alpha-1}\varphi(s)\,ds\,dt\,.$$

Interchanging the order of integration, as in Example 1.2, we get

$$D^{\alpha}_{a+}J^{\alpha}_{a+}\varphi(x) = \frac{1}{\Gamma(n)}\frac{d^n}{dx^n}\int_a^x (x-s)^{n-1}\,\varphi(s)\,ds\,. \tag{2.55}$$

Since $\dfrac{d^n}{dx^n}J^n_{a+}\varphi = \varphi$, then relations in (2.47) follows from (2.3) and (2.55).

(ii) With the assumption $f \in J^{\alpha}_{a+}(L_p)$, the relation (2.48) follows immediately from (2.47), that is

$$J^{\alpha}_{a+}D^{\alpha}_{a+}f = J^{\alpha}_{a+}D^{\alpha}_{a+}J^{\alpha}_{a+}\varphi = J^{\alpha}_{a+}\varphi = f\,.$$

(iii) It remains for us to prove (2.49). Let $f \in L_1[a,b]$ has the left-sided integrable fractional derivative $D^{\alpha}_{a+}f$, that is $J^{n-\alpha}_{a+}f \in AC^n[a,b]$. Then by Lemma 1.2 we have

$$J^{n-\alpha}_{a+}f(x) = J^n_{a+}\frac{d^n}{dx^n}J^{n-\alpha}_{a+}f(x) \;+\; \sum_{k=0}^{n-1}\frac{(x-a)^k}{\Gamma(k+1)}\frac{d^k}{dx^k}J^{n-\alpha}_{a+}f(a)\,,$$

and since $[\alpha-n+k]+1 = k$, also follows

$$J^{n-\alpha}_{a+}f(x) = J^n_{a+}D^{\alpha}_{a+}f(x) \;+\; \sum_{k=0}^{n-1}\frac{(x-a)^k}{\Gamma(k+1)}D^{\alpha-n+k}_{a+}f(a)\,.$$

Since $n-\alpha > 0$ and $k+1 > 0$ for $k = 0,\dots,n-1$, we can use Example 2.3

$$D^{n-\alpha}_{a+}(x-a)^k = \frac{\Gamma(k+1)}{\Gamma(\alpha-n+k+1)}\,(x-a)^{\alpha-n+k}\,,$$

and since $\alpha-n+k+1 > 0$, by Example 2.1 we have

$$J^{n-\alpha}_{a+}(x-a)^{\alpha-n+k} = \frac{\Gamma(\alpha-n+k+1)}{\Gamma(k+1)}\,(x-a)^k\,,$$

that is

$$J^{n-\alpha}_{a+}D^{n-\alpha}_{a+}(x-a)^k = (x-a)^k\,.$$

Using Lemma 2.3 follows

$$\begin{aligned}
J^{n-\alpha}_{a+}f(x) &= J^{n-\alpha}_{a+}J^{\alpha}_{a+}D^{\alpha}_{a+}f(x) + J^{n-\alpha}_{a+}D^{n-\alpha}_{a+}\sum_{k=0}^{n-1}\frac{(x-a)^k}{\Gamma(k+1)}D^{\alpha-n+k}_{a+}f(a)\\
&= J^{n-\alpha}_{a+}\left(J^{\alpha}_{a+}D^{\alpha}_{a+}f(x) + \sum_{k=0}^{n-1}\frac{D^{n-\alpha}_{a+}(x-a)^k}{\Gamma(k+1)}D^{\alpha-n+k}_{a+}f(a)\right)\\
&= J^{n-\alpha}_{a+}\left(J^{\alpha}_{a+}D^{\alpha}_{a+}f(x) + \sum_{k=0}^{n-1}\frac{(x-a)^{\alpha-n+k}}{\Gamma(\alpha-n+k+1)}D^{\alpha-n+k}_{a+}f(a)\right)\\
&= J^{n-\alpha}_{a+}\left(J^{\alpha}_{a+}D^{\alpha}_{a+}f(x) + \sum_{k=1}^{n}\frac{(x-a)^{\alpha-k}}{\Gamma(\alpha-k+1)}D^{\alpha-k}_{a+}f(a)\right)\,,
\end{aligned}$$

where the last equality follows from changing the order of summation. Finally, since $n-\alpha > 0$ we have (2.49) which follows by Lemma 2.4. □

Corollary 2.5 *Let $\alpha > \beta > 0$, $1 \le p \le \infty$ and $\varphi \in L_p[a,b]$. Then relations*

$$D^{\beta}_{a+} J^{\alpha}_{a+} \varphi = J^{\alpha-\beta}_{a+} \varphi , \quad D^{\beta}_{b-} J^{\alpha}_{b-} \varphi = J^{\alpha-\beta}_{b-} \varphi \tag{2.56}$$

hold almost everywhere on $[a,b]$.
In particular, if $\beta = k \in \mathbb{N}$ and $\alpha > k$, then

$$\frac{d^k}{dx^k} J^{\alpha}_{a+} \varphi(x) = J^{\alpha-k}_{a+} \varphi(x) , \quad \frac{d^k}{dx^k} J^{\alpha}_{b-} \varphi(x) = (-1)^k J^{\alpha-k}_{b-} \varphi(x) . \tag{2.57}$$

Corollary 2.6 *Let $\alpha > 0$ and $m \in \mathbb{N}$.*

(i) *If the left-sided fractional derivatives $D^{\alpha}_{a+} f$ and $D^{\alpha+m}_{a+} f$ exist, then*

$$\frac{d^m}{dx^m} D^{\alpha}_{a+} f(x) = D^{\alpha+m}_{a+} f(x) . \tag{2.58}$$

(ii) *If the right-sided fractional derivatives $D^{\alpha}_{b-} f$ and $D^{\alpha+m}_{b-} f$, then*

$$\frac{d^m}{dx^m} D^{\alpha}_{b-} f(x) = (-1)^m D^{\alpha+m}_{b-} f(x) . \tag{2.59}$$

Lemma 2.9 *Let $\alpha > \beta > 0$, $n = [\alpha] + 1$ and $m = [\beta] + 1$. Let $f \in L_1[a,b]$ has the left-sided integrable fractional derivative $D^{\beta}_{a+} f$ and let $g \in L_1[a,b]$ has the right-sided integrable fractional derivative $D^{\beta}_{b-} g$. Then relations*

$$J^{\alpha}_{a+} D^{\beta}_{a+} f(x) = J^{\alpha-\beta}_{a+} f(x) - \sum_{k=1}^{m} \frac{(x-a)^{\alpha-k}}{\Gamma(\alpha-k+1)} D^{\beta-k}_{a+} f(a) , \tag{2.60}$$

$$J^{\alpha}_{b-} D^{\beta}_{-b} g(x) = J^{\alpha-\beta}_{b-} g(x) - \sum_{k=1}^{m} \frac{(-1)^{m-k}(b-x)^{\alpha-k}}{\Gamma(\alpha-k+1)} D^{\beta-k}_{b-} g(b) , \tag{2.61}$$

hold almost everywhere on $[a,b]$.

Proof. Using (2.49) and Example 2.1 we have

$$\begin{aligned} J^{\alpha}_{a+} D^{\beta}_{a+} f(x) &= J^{\alpha-\beta}_{a+} J^{\beta}_{a+} D^{\beta}_{a+} f(x) \\ &= J^{\alpha-\beta}_{a+} \left(f(x) - \sum_{k=1}^{m} \frac{(x-a)^{\beta-k}}{\Gamma(\beta-k+1)} D^{\beta-k}_{a+} f(a) \right) \\ &= J^{\alpha-\beta}_{a+} f(x) - \sum_{k=1}^{m} \frac{J^{\alpha-\beta}_{a+} (x-a)^{\beta-k}}{\Gamma(\beta-k+1)} D^{\beta-k}_{a+} f(a) \\ &= J^{\alpha-\beta}_{a+} f(x) - \sum_{k=1}^{m} \frac{(x-a)^{\alpha-k}}{\Gamma(\alpha-k+1)} D^{\beta-k}_{a+} f(a) . \end{aligned}$$

Analogously follows equality (2.61) for the right-sided fractional integrals and derivatives. □

The following lemma contains some assertions from Theorem 2.4 and its corollaries, applied on functions $f \in C^n[a,b]$.

Lemma 2.10 *Let $\alpha > \beta > 0$ and $n = [\alpha] + 1$.*

(i) *If $\varphi \in C[a,b]$, then (2.47) holds at every point $x \in [a,b]$.*

(ii) *If $\varphi \in C[a,b]$, then (2.56) and (2.57) hold at every point $x \in [a,b]$.*

(iii) *If $f \in C[a,b]$ and $J_{a+}^{n-\alpha} f \in C^n[a,b]$, then (2.49) holds at every point $x \in [a,b]$. In particular, if $f \in C^n[a,b]$, then (2.53) holds at every point $x \in [a,b]$.*

(iv) *If $g \in C[a,b]$ and $J_{b-}^{n-\alpha} g \in C^n[a,b]$, then (2.50) holds at every point $x \in [a,b]$. In particular, if $g \in C^n[a,b]$, then (2.54) holds at every point $x \in [a,b]$.*

Relations (2.49) and (2.50) represent Taylor's formula for the Riemann-Liouville fractional derivatives.

Corollary 2.7 (TAYLOR'S FORMULA FOR THE RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVES) *Let $\alpha > 0$ and $n = [\alpha] + 1$. Let $f \in L_1[a,b]$ has the left-sided integrable fractional derivative $D_{a+}^{\beta} f$ and let $g \in L_1[a,b]$ has the right-sided integrable fractional derivative $D_{b-}^{\beta} g$. Then*

$$f(x) = \sum_{k=1}^{n} \frac{D_{a+}^{\alpha-k} f(a)}{\Gamma(\alpha-k+1)} (x-a)^{\alpha-k} + J_{a+}^{\alpha} D_{a+}^{\alpha} f(x) , \tag{2.62}$$

$$g(x) = \sum_{k=1}^{n} \frac{(-1)^{n-k} D_{b-}^{\alpha-k} g(b)}{\Gamma(\alpha-k+1)} (b-x)^{\alpha-k} + J_{b-}^{\alpha} D_{b-}^{\alpha} g(x) . \tag{2.63}$$

Proposition 2.3 *Let $\alpha > 0$, $n = [\alpha] + 1$ and $f \in AC^n[a,b]$.*

(i) *If $f(a) = f'(a) = \cdots = f^{(n-2)}(a) = 0$, then $D_{a+}^{\alpha-1} f(a) = D_{a+}^{\alpha-2} f(a) = \cdots = D_{a+}^{\alpha-n} f(a) = 0$.*

(ii) *If $\alpha \notin \mathbb{N}$ and $D_{a+}^{\alpha-1} f$ is bounded in a neighborhood of $x = a$, then $f(a) = f'(a) = \cdots = f^{(n-2)}(a) = 0$.*

Proof.
(i) Let $f^{(i)}(a) = 0$ for $i = 0, \ldots, n-2$. Then by Theorem 2.1 for $k = 1, \ldots, n-1$ we have

$$\begin{aligned} & D_{a+}^{\alpha-k} f(x) \\ &= \sum_{i=0}^{n-k-1} \frac{f^{(i)}(a)}{\Gamma(i-\alpha+k+1)} (x-a)^{i-\alpha+k} + \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n-k)}(t)\, dt \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n-k)}(t)\, dt , \end{aligned}$$

that is $D_{a+}^{\alpha-k} f(a) = 0$, since $f^{(n-k)} \in C^{n-k}[a,b]$, $k = 1, \ldots, n-1$. For $k = n$ we have $\alpha - n < 0$, that is $D_{a+}^{\alpha-n} f(x) = J_{a+}^{n-\alpha} f(x)$ which gives us $D_{a+}^{\alpha-n} f(a) = 0$ (we use the fact $f \in C[a,b]$).

(ii) Let $\alpha \notin \mathbb{N}$ and let $D_{a+}^{\alpha-1} f$ be bounded in a neighborhood of $x = a$. For $k = 1$ follows

$$D_{a+}^{\alpha-1} f(x) = \sum_{i=0}^{n-2} \frac{f^{(i)}(a)}{\Gamma(i-\alpha+2)} (x-a)^{i-\alpha+1} + \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n-1)}(t)\, dt\,. \tag{2.64}$$

For $0 < \alpha < 1$ there is nothing to prove. Suppose $\alpha > 1$. Multiplying (2.64) with $(x-a)^{\alpha-1}$ we get

$$\begin{aligned}
&(x-a)^{\alpha-1} D_{a+}^{\alpha-1} f(x) \\
&= \sum_{i=0}^{n-2} \frac{f^{(i)}(a)}{\Gamma(i-\alpha+2)} (x-a)^i + \frac{(x-a)^{\alpha-1}}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n-1)}(t)\, dt \\
&= \frac{f(a)}{\Gamma(-\alpha+2)} + \sum_{i=1}^{n-2} \frac{f^{(i)}(a)}{\Gamma(i-\alpha+2)} (x-a)^i + \frac{(x-a)^{\alpha-1}}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n-1)}(t)\, dt\,.
\end{aligned} \tag{2.65}$$

Taking $\lim\limits_{x\to a}$ of the both sides of (2.65) it follows $f(a) = 0$. For $1 < \alpha < 2$ the proof is complete. For $\alpha > 2$ the proof proceeds analogously by induction using successive multiplications with $(x-a)^{\alpha-i}$, $i = 1,\dots,n-1$. □

The analogous proposition holds for the right-sided Riemann-Liouville fractional derivatives.

Proposition 2.4 *Let $\alpha > 0$, $n = [\alpha]+1$ and $f \in AC^n[a,b]$.*

(i) *If $f(b) = f'(b) = \cdots = f^{(n-2)}(b) = 0$, then $D_{b-}^{\alpha-1} f(b) = D_{b-}^{\alpha-2} f(b) = \cdots = D_{b-}^{\alpha-n} f(b) = 0$.*

(ii) *If $\alpha \notin \mathbb{N}$ and $D_{b-}^{\alpha-1} f$ is bounded in a neighborhood of $x = b$, then $f(b) = f'(b) = \cdots = f^{(n-2)}(b) = 0$.*

If we apply Proposition 2.3 and Proposition 2.4 on assertion (iii) of Theorem 2.4, then next corollary holds.

Corollary 2.8 *Let $\alpha > 0$ and $n = [\alpha]+1$. Let $f \in L_1[a,b]$ has the left-sided integrable fractional derivative $D_{a+}^{\beta} f$ and let $g \in L_1[a,b]$ has the right-sided integrable fractional derivative $D_{b-}^{\beta} g$.*

(i) *If $f(a) = f'(a) = \cdots = f^{(n-2)}(a) = 0$, then*

$$f(x) = J_{a+}^{\alpha} D_{a+}^{\alpha} f(x)\,. \tag{2.66}$$

(ii) *If $g(b) = g'(b) = \cdots = g^{(n-2)}(b) = 0$, then*

$$g(x) = J_{b-}^{\alpha} D_{b-}^{\alpha} g(x)\,. \tag{2.67}$$

The next theorem is an extension of Lemma 2.3, and for its proof we use Theorem 2.4.

Theorem 2.5 *Let $1 \le p \le \infty$. The relation*

$$J_{a+}^{\alpha} J_{a+}^{\beta} \varphi = J_{a+}^{\alpha+\beta} \varphi \tag{2.68}$$

is valid in any of the following cases

(i) $\beta > 0,\ \alpha + \beta > 0\ i\ \varphi \in L_p[a,b]$;

(ii) $\beta < 0,\ \alpha > 0\ i\ \varphi \in J_{a+}^{-\beta}(L_p)$;

(iii) $\alpha < 0,\ \alpha + \beta < 0\ i\ \varphi \in J_{a+}^{-\alpha-\beta}(L_p)$.

Proof.

(i) Let $\beta > 0$ and $\varphi \in L_p[a,b]$. The case when $\alpha > 0$ (and then also $\alpha + \beta > 0$) is proven in Lemma 2.3. Suppose $\alpha < 0$ and $\alpha + \beta > 0$. Then

$$J_{a+}^{\alpha} J_{a+}^{\beta} \varphi = D_{a+}^{-\alpha} J_{a+}^{-\alpha+\alpha+\beta} \varphi = D_{a+}^{-\alpha} J_{a+}^{-\alpha} J_{a+}^{\alpha+\beta} \varphi .$$

Since $-\alpha > 0$, we can apply Theorem 2.4 (i) from which we obtain (2.68).

(ii) Let $\beta < 0$, $\alpha > 0$ and $\varphi \in J_{a+}^{-\beta}(L_p)$. Then $\varphi = J_{a+}^{-\beta} \psi$, where $\psi \in L_p[a,b]$ and

$$J_{a+}^{\alpha+\beta} \varphi = J_{a+}^{\alpha+\beta} J_{a+}^{-\beta} \psi .$$

Since $\alpha + \beta + (-\beta) > 0$, then according to case (i) we have

$$J_{a+}^{\alpha+\beta} J_{a+}^{-\beta} \psi = J_{a+}^{\alpha} \psi ,$$

and then by Theorem 2.4 (i), for $-\beta > 0$ and $\psi \in L_p[a,b]$ follows

$$\psi = D_{a+}^{-\beta} J_{a+}^{-\beta} \psi = D_{a+}^{-\beta} \varphi .$$

Hence,

$$J_{a+}^{\alpha+\beta} \varphi = J_{a+}^{\alpha} \psi = J_{a+}^{\alpha} D_{a+}^{-\beta} \varphi = J_{a+}^{\alpha} J_{a+}^{\beta} \varphi .$$

(iii) Let $\alpha < 0$, $\alpha + \beta < 0$ and $\varphi \in J_{a+}^{-\alpha-\beta}(L_p)$. Then $\varphi = J_{a+}^{-\alpha-\beta} \psi$, where $\psi \in L_p[a,b]$. We have

$$J_{a+}^{\alpha} J_{a+}^{\beta} \varphi = J_{a+}^{\alpha} J_{a+}^{\beta} J_{a+}^{-\alpha-\beta} \psi = J_{a+}^{\alpha} J_{a+}^{-\alpha} \psi$$

where the last step follows according to case (i). Again, by Theorem 2.4 (i), for $-\alpha - \beta > 0$ and $\psi \in L_p[a,b]$ we have

$$\psi = D_{a+}^{-\alpha-\beta} J_{a+}^{-\alpha-\beta} \psi = D_{a+}^{-\alpha-\beta} \varphi ,$$

which leads to

$$J_{a+}^{\alpha} J_{a+}^{\beta} \varphi = D_{a+}^{-\alpha} J_{a+}^{-\alpha} \psi = \psi = J_{a+}^{\alpha+\beta} \varphi .$$

□

The analogous theorem holds for the composition of the right-sided fractional integrals.

Theorem 2.6 *Let $1 \le p \le \infty$. The relation*

$$J_{b-}^{\alpha} J_{b-}^{\beta} \varphi = J_{b-}^{\alpha+\beta} \varphi \tag{2.69}$$

is valid in any of the following cases

(*i*) $\beta > 0$, $\alpha + \beta > 0$ *i* $\varphi \in L_p[a,b]$;

(*ii*) $\beta < 0$, $\alpha > 0$ *i* $\varphi \in J_{b-}^{-\beta}(L_p)$;

(*iii*) $\alpha < 0$, $\alpha + \beta < 0$ *i* $\varphi \in J_{b-}^{-\alpha-\beta}(L_p)$.

2.3 The Caputo fractional derivatives

The second type of fractional derivatives that we observe is the Caputo-type, which we define using the Riemann-Liouville fractional derivatives. Again, let $x \in [a,b]$. For $\alpha \ge 0$ we define n in the following way:

$$n = [\alpha] + 1\,, \text{ for } \alpha \notin \mathbb{N}_0; \quad n = \alpha\,, \text{ for } \alpha \in \mathbb{N}_0\,. \tag{2.70}$$

Definition 2.4 *Let $\alpha \ge 0$, n given by (2.70) and $f : [a,b] \to \mathbb{R}$. The left-sided Caputo fractional derivative ${}^{C}D_{a+}^{\alpha} f$ is defined by*

$${}^{C}D_{a+}^{\alpha} f(x) = D_{a+}^{\alpha} \left[f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k+1)} (x-a)^k \right], \tag{2.71}$$

while the right-sided Caputo fractional derivative ${}^{C}D_{b-}^{\alpha} f$ is defined by

$${}^{C}D_{b-}^{\alpha} f(x) = D_{b-}^{\alpha} \left[f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{\Gamma(k+1)} (b-x)^k \right]. \tag{2.72}$$

If $\alpha \notin \mathbb{N}_0$ and f is a function for which the Caputo fractional derivatives of order α exist together with the Riemann-Liouville fractional derivatives, then, according to Example 2.3, we have

$${}^{C}D_{a+}^{\alpha} f(x) = D_{a+}^{\alpha} f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)} (x-a)^{k-\alpha}, \tag{2.73}$$

$${}^{C}D_{b-}^{\alpha} f(x) = D_{b-}^{\alpha} f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{\Gamma(k-\alpha+1)} (b-x)^{k-\alpha}. \tag{2.74}$$

In particular, for $0 < \alpha < 1$ hold

$${}^{C}D_{a+}^{\alpha} f(x) = D_{a+}^{\alpha} f(x) - \frac{f(a)}{\Gamma(1-\alpha)} (x-a)^{-\alpha}, \tag{2.75}$$

$$ {}^{C}D^{\alpha}_{b-}f(x) = D^{\alpha}_{b-}f(x) - \frac{f(b)}{\Gamma(1-\alpha)}(b-x)^{-\alpha}, \tag{2.76} $$

and for $\alpha = n \in \mathbb{N}_0$

$$ {}^{C}D^{n}_{a+}f(x) = f^{(n)}(x), \quad {}^{C}D^{n}_{b-}f(x) = (-1)^n f^{(n)}(x). \tag{2.77} $$

From the definition of the Caputo fractional derivatives it is clear that for $\alpha \notin \mathbb{N}_0$ the Caputo coincide with the Riemann-Liouville fractional derivatives if derivatives at the boundary point vanishes, i.e.

$$ {}^{C}D^{\alpha}_{a+}f(x) = D^{\alpha}_{a+}f(x), \quad f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0, \tag{2.78} $$

$$ {}^{C}D^{\alpha}_{b-}f(x) = D^{\alpha}_{b-}f(x), \quad f(b) = f'(b) = \cdots = f^{(n-1)}(b) = 0. \tag{2.79} $$

Theorem 2.7 *Let $\alpha \geq 0$ and n given by (2.70). If $f \in AC^n[a,b]$, then the left-sided and the right-sided Caputo fractional derivatives ${}^{C}D^{\alpha}_{a+}f$ and ${}^{C}D^{\alpha}_{b-}f$ exist almost everywhere on $[a,b]$.*

(*i*) *If $\alpha \notin \mathbb{N}_0$, then for ${}^{C}D^{\alpha}_{a+}f$ and ${}^{C}D^{\alpha}_{b-}f$ we have*

$$ {}^{C}D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)\,dt = J^{n-\alpha}_{a+}f^{(n)}(x), \tag{2.80} $$

$$ {}^{C}D^{\alpha}_{b-}f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b (t-x)^{n-\alpha-1} f^{(n)}(t)\,dt = (-1)^n J^{n-\alpha}_{b-}f^{(n)}(x). \tag{2.81} $$

In particular, for $0 < \alpha < 1$ and $f \in AC[a,b]$ hold

$$ {}^{C}D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(1-\alpha)}\int_a^x (x-t)^{-\alpha} f'(t)\,dt = J^{1-\alpha}_{a+}f'(x), \tag{2.82} $$

$$ {}^{C}D^{\alpha}_{b-}f(x) = \frac{-1}{\Gamma(1-\alpha)}\int_x^b (t-x)^{-\alpha} f'(t)\,dt = -J^{1-\alpha}_{b-}f'(x). \tag{2.83} $$

(*ii*) *If $\alpha = n \in \mathbb{N}_0$, then for ${}^{C}D^{n}_{a+}f$ and ${}^{C}D^{n}_{b-}f$ the relation (2.77) holds. In particular,*

$$ {}^{C}D^{0}_{a+}f(x) = {}^{C}D^{0}_{b-}f(x) = f(x). \tag{2.84} $$

Proof.
(*i*) Let $\alpha \notin \mathbb{N}_0$. Then

$$ \begin{aligned} {}^{C}D^{\alpha}_{a+}f(x) &= D^{\alpha}_{a+}\left[f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(k+1)}(x-a)^k\right] \\ &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-t)^{n-\alpha-1}\left[f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(k+1)}(t-a)^k\right]dt. \end{aligned} $$

Integrating by parts the inner integral and differentiating leads to

$$
\begin{aligned}
{}^{C}D^{\alpha}_{a+}f(x) &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\left\{-\frac{(x-t)^{n-\alpha}}{n-\alpha}\left[f(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(k+1)}(t-a)^k\right]\Big|_{t=a}^{x}\right.\\
&\quad\left.+\int_a^x\frac{(x-t)^{n-\alpha}}{n-\alpha}\left[f'(t)-\sum_{k=1}^{n-1}\frac{f^{(k)}(a)}{\Gamma(k)}(t-a)^{k-1}\right]dt\right\}\\
&= \frac{1}{\Gamma(n-\alpha+1)}\frac{d^n}{dx^n}\int_a^x(x-t)^{n-\alpha}\left[f'(t)-\sum_{k=1}^{n-1}\frac{f^{(k)}(a)}{\Gamma(k)}(t-a)^{k-1}\right]dt\,.
\end{aligned}
$$

Repeating the process we arrive to

$$
{}^{C}D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(n-\alpha+2)}\frac{d^n}{dx^n}\int_a^x(x-t)^{n-\alpha+1}\left[f''(t)-\sum_{k=2}^{n-1}\frac{f^{(k)}(a)}{\Gamma(k-1)}(t-a)^{k-2}\right]dt\,,
$$

and if we repeat in n times, then

$$
{}^{C}D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(n-\alpha+n)}\frac{d^n}{dx^n}\int_a^x(x-t)^{n-\alpha+n-1}f^{(n)}(t)\,dt = \frac{d^n}{dx^n}J^{2n-\alpha}_{a+}f^{(n)}(x)\,.
$$

Hence

$$
CD^{\alpha}_{a+}f(x) = \frac{d^n}{dx^n}J^{n}_{a+}J^{n-\alpha}_{a+}f^{(n)}(x) = J^{n-\alpha}_{a+}f^{(n)}(x)\,.
$$

Analogously follows (2.81) for the right-sided Caputo fractional derivatives.

(ii) We prove (2.77) for the left-sided Caputo fractional derivatives (analogously follows for the right-sided). Let $\alpha = n \in \mathbb{N}_0$. Since $f \in AC^n[a,b]$, by Lemma 1.2 and (2.47) we get

$$
{}^{C}D^{n}_{a+}f(x) = D^{n}_{a+}\left[f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(k+1)}(x-a)^k\right] = D^{n}_{a+}J^{n}_{a+}f^{(n)}(x) = f^{(n)}(x)\,. \tag{2.85}
$$

□

The following theorem is analogous to the previous one for functions $f \in C^n[a,b]$.

Theorem 2.8 *Let $\alpha \geq 0$, n given by (2.70) and $f \in C^n[a,b]$. Then the Caputo fractional derivatives ${}^{C}D^{\alpha}_{a+}f$ and ${}^{C}D^{\alpha}_{b-}f$ are continuous on $[a,b]$.*

(i) *If $\alpha \notin \mathbb{N}_0$, then ${}^{C}D^{\alpha}_{a+}f$ and ${}^{C}D^{\alpha}_{b-}f$ are represented by (2.80) and (2.81) respectively. Moreover*

$$
{}^{C}D^{\alpha}_{a+}f(a) = {}^{C}D^{\alpha}_{b-}f(b) = 0\,. \tag{2.86}
$$

In particular, fractional derivatives have forms (2.82) and (2.83) for $0 < \alpha < 1$, respectively.

(ii) *If $\alpha = n \in \mathbb{N}_0$, then for ${}^{C}D^{n}_{a+}f$ and ${}^{C}D^{n}_{b-}f$ holds (2.77). In particular, for $\alpha = 0$ holds (2.84).*

Proof.

(i) Let $f \in C^n[a,b]$. Relations (2.80) and (2.81) are proved as in Theorem 2.7. The continuity of functions ${}^C D^{\alpha}_{a+} f$ and ${}^C D^{\alpha}_{b-} f$ follows by Corollary 2.2, since $n-\alpha > 0$ and $f^{(n)} \in C[a,b]$ gives us

$$J^{n-\alpha}_{a+} f^{(n)}, J^{n-\alpha}_{b-} f^{(n)} \in C^{[n-\alpha]}[a,b] = C^1[a,b].$$

The relations in (2.86) follow from the inequality

$$|J^{n-\alpha}_{a+} f^{(n)}(x)| \le |J^{n-\alpha}_{a+} 1| \cdot ||f^{(n)}||_C$$

and Example 2.1 when $\beta = 1$

$$J^{n-\alpha}_{a+} 1 = \frac{(x-a)^{n-\alpha}}{\Gamma(n-\alpha+1)},$$

hence, $\lim\limits_{x\to a+} {}^C D^{\alpha}_{a+} f(x) = \lim\limits_{x\to a+} J^{n-\alpha}_{a+} f^{(n)}(x) = 0$. Analogously,

$$|J^{n-\alpha}_{b-} f^{(n)}(x)| \le \frac{(b-x)^{n-\alpha}}{\Gamma(n-\alpha+1)} ||f^{(n)}||_C,$$

that is $\lim\limits_{x\to b-} {}^C D^{\alpha}_{b-} f(x) = 0$.

(ii) Let $\alpha = n \in \mathbb{N}_0$. Then ${}^C D^{n}_{a+} f(x) = f^{(n)}(x)$ follows as in (2.85), using Lemma 1.1 and Lemma 2.10 (i). Analogously follows for the right-sided Caputo fractional derivatives. □

Example 2.5 Let $\alpha > 0$, n given by (2.70), $\beta > n$ and let $f(x) = (x-a)^{\beta-1}$ and $g(x) = (b-x)^{\beta-1}$. Using Example 2.1, the left-sided Caputo fractional derivative of the f is

$$\begin{aligned} {}^C D^{\alpha}_{a+} (x-a)^{\beta-1} &= J^{n-\alpha}_{a+} \frac{d^n}{dx^n} (x-a)^{\beta-1} \\ &= (\beta-1)(\beta-2)\cdots(\beta-n)\, J^{n-\alpha}_{a+} (x-a)^{\beta-n-1} \\ &= (\beta-1)(\beta-2)\cdots(\beta-n)\, \frac{\Gamma(\beta-n)}{\Gamma(-\alpha+\beta)} (x-a)^{-\alpha+\beta-1} \\ &= \frac{\Gamma(\beta)}{\Gamma(-\alpha+\beta)} (x-a)^{-\alpha+\beta-1}. \end{aligned} \tag{2.87}$$

Analogously, the right-sided Caputo fractional derivative of the function g is

$$\begin{aligned} {}^C D^{\alpha}_{b-} (b-x)^{\beta-1} &= (-1)^n J^{n-\alpha}_{b-} \frac{d^n}{dx^n} (b-x)^{\beta-1} \\ &= \frac{\Gamma(\beta)}{\Gamma(-\alpha+\beta)} (b-x)^{-\alpha+\beta-1}. \end{aligned} \tag{2.88}$$

Hence, for $\beta > n$ relations (2.87) and (2.26) coincide, as well as (2.88) and (2.27), which is true since we have (2.78) and (2.79).

But, if $\beta = 1$, then $f(x) = g(x) = 1$ and $J^{n-\alpha}_{a+} f^{(n)}(x) = J^{n-\alpha}_{b-} g^{(n)}(x) = 0$, that is

$$^{C}D^{\alpha}_{a+}(1) = {}^{C}D^{\alpha}_{b-}(1) = 0\,.$$

Therefore, unlike with the Riemann-Liouville fractional derivatives, we have that the Caputo fractional derivative of a constant function is equal zero.
Moreover, for $k = 0, 1, \dots, n-1$ we have

$$^{C}D^{\alpha}_{a+}(x-a)^k = {}^{C}D^{\alpha}_{b-}(b-x)^k = 0\,. \tag{2.89}$$

Example 2.6 Let $\alpha > 0$ and $\lambda \in \mathbb{R}$. For the exponential function we have

$$\begin{aligned}
^{C}D^{\alpha}_{a+} e^{\lambda x} &= {}^{C}D^{\alpha}_{a+}\left(e^{\lambda a} e^{\lambda (x-a)}\right) \\
&= {}^{C}D^{\alpha}_{a+}\left[e^{\lambda a} \sum_{k=0}^{\infty} \frac{\lambda^k (x-a)^k}{k!}\right] \\
&= e^{\lambda a} \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(k+1)} {}^{C}D^{\alpha}_{a+}(x-a)^k \\
&= \frac{e^{\lambda a}}{(x-a)^{\alpha}} \sum_{k=n}^{\infty} \frac{\lambda^k (x-a)^k}{\Gamma(-\alpha+k+1)}\,, \\
^{C}D^{\alpha}_{b-} e^{\lambda x} &= \frac{e^{\lambda b}}{(b-x)^{\alpha}} \sum_{k=n}^{\infty} \frac{(-\lambda)^k (b-x)^k}{\Gamma(-\alpha+k+1)}\,.
\end{aligned}$$

Theorem 2.9 *Let $\alpha > 0$ and n given by* (2.70).

(i) *If $f \in L_{\infty}[a,b]$, then*

$$^{C}D^{\alpha}_{a+} J^{\alpha}_{a+} f = f\,, \quad {}^{C}D^{\alpha}_{b-} J^{\alpha}_{b-} f = f\,. \tag{2.90}$$

(ii) *If $f \in AC^n[a,b]$, then*

$$J^{\alpha}_{a+}\, {}^{C}D^{\alpha}_{a+} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k\,, \tag{2.91}$$

$$J^{\alpha}_{b-}\, {}^{C}D^{\alpha}_{b-} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(b)}{k!}(b-x)^k\,. \tag{2.92}$$

In particular, for $0 < \alpha \le 1$ and $f \in AC[a,b]$, hold

$$J^{\alpha}_{a+}\, {}^{C}D^{\alpha}_{a+} f(x) = f(x) - f(a)\,, \tag{2.93}$$

$$J^{\alpha}_{b-}\, {}^{C}D^{\alpha}_{b-} f(x) = f(x) - f(b)\,. \tag{2.94}$$

Proof.

(i) By Corollary 2.5, for $k=0,1,\ldots,n-1$ we have (2.57), that is $\frac{d^k}{dx^k}J^\alpha_{a+}f(x) = J^{\alpha-k}_{a+}f(x)$. Further,

$$\begin{aligned} |J^{\alpha-k}_{a+}f(x)| &\le \frac{1}{\Gamma(\alpha-k)}\int_a^x \left|(x-t)^{\alpha-k-1}f(t)\right| dt \\ &\le \frac{1}{\Gamma(\alpha-k)}\left(\int_a^x (x-t)^{\alpha-k-1}\,dt\right)||f||_\infty \\ &= \frac{(x-a)^{\alpha-k}}{\Gamma(\alpha-k+1)}\,||f||_\infty\,, \end{aligned}$$

and since f is bounded in a follows

$$\lim_{x\to a+} J^{\alpha-k}_{a+}f(x) = \lim_{x\to a+}\frac{d^k}{dx^k}J^\alpha_{a+}f(x) = 0\,, \quad k=0,1,\ldots,n-1\,.$$

Now from the definition of the Caputo fractional derivatives and Theorem 2.4 (i) follows

$${}^CD^\alpha_{a+}J^\alpha_{a+}f(x) = D^\alpha_{a+}\left[J^\alpha_{a+}f(x) - \sum_{k=0}^{n-1}\frac{(x-a)^k}{\Gamma(k+1)}\frac{d^k}{dx^k}J^\alpha_{a+}f(a)\right] = f(x)\,.$$

In the case of the right-sided fractional integrals and derivatives, proof follows analogously.

(ii) Let $\alpha\notin\mathbb{N}$ and $f\in AC^n[a,b]$. Then by the case (i) of Theorem 2.7 for ${}^CD^\alpha_{a+}f$ and ${}^CD^\alpha_{b-}f$ hold (2.80) and (2.81), respectively. Using Lemma 2.3 we have

$$J^\alpha_{a+}\,{}^CD^\alpha_{a+}f = J^\alpha_{a+}J^{n-\alpha}_{a+}f^{(n)} = J^n_{a+}D^n_{a+}f\,,$$

$$J^\alpha_{b-}\,{}^CD^\alpha_{b-}f = (-1)^n J^\alpha_{b-}J^{n-\alpha}_{b-}f^{(n)} = J^n_{b-}D^n_{b-}f\,.$$

If we apply (2.53) and (2.54) on relations above, we get (2.91) and (2.92).
If $\alpha\in\mathbb{N}$ then we apply (ii) of Theorem 2.7 and then once more (2.53), (2.54). □

The analogous theorem holds for the functions in the space $C^n[a,b]$.

Theorem 2.10 *Let $\alpha>0$ and n given by* (2.70).

(i) *If $f\in C[a,b]$, then* (2.90) *hold for every $x\in[a,b]$.*

(ii) *If $f\in C^n[a,b]$, then* (2.91) *and* (2.92) *holds for every $x\in[a,b]$. In particular, for $0<\alpha\le 1$ and $f\in C[a,b]$ relations* (2.93) *and* (2.94) *hold for every $x\in[a,b]$.*

Corollary 2.9 (TAYLOR'S FORMULA FOR THE CAPUTO FRACTIONAL DERIVATIVES) *Let $\alpha\ge 0$, n given by* (2.70) *and $f,g\in AC^n[a,b]$. Then*

$$f(x) = \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k + J^\alpha_{a+}\,{}^CD^\alpha_{a+}f(x)\,, \tag{2.95}$$

$$g(x) = \sum_{k=0}^{n-1}\frac{(-1)^k g^{(k)}(b)}{k!}(b-x)^k + J^\alpha_{b-}\,{}^CD^\alpha_{b-}g(x)\,. \tag{2.96}$$

Corollary 2.10 *Let $\alpha \geq 0$, n given by (2.70) and $f, g \in AC^n[a,b]$.*

(i) *If $f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0$, then*

$$f(x) = J^{\alpha}_{a+}\, {}^{C}D^{\alpha}_{a+} f(x)\,. \tag{2.97}$$

(ii) *If $g(b) = g'(b) = \cdots = g^{(n-1)}(b) = 0$, then*

$$g(x) = J^{\alpha}_{b-}\, {}^{C}D^{\alpha}_{b-} g(x)\,. \tag{2.98}$$

2.4 The Canavati fractional derivatives

Let $\alpha > 0$ and $n = [\alpha] + 1$. With $C^{\alpha}_{a+}[a,b]$ and $C^{\alpha}_{b-}[a,b]$ we denote subspaces of $C^{n-1}[a,b]$, defined by

$$C^{\alpha}_{a+}[a,b] = \left\{ f \in C^{n-1}[a,b] : J^{n-\alpha}_{a+} f^{(n-1)} \in C^1[a,b] \right\}, \tag{2.99}$$

$$C^{\alpha}_{b-}[a,b] = \left\{ f \in C^{n-1}[a,b] : J^{n-\alpha}_{b-} f^{(n-1)} \in C^1[a,b] \right\}. \tag{2.100}$$

Definition 2.5 *Let $\alpha > 0$, $n = [\alpha] + 1$, $f \in C^{\alpha}_{a+}[a,b]$ and $g \in C^{\alpha}_{b-}[a,b]$. The left-sided Canavati fractional derivative ${}^{C_1}D^{\alpha}_{a+} f$ is defined by*

$${}^{C_1}D^{\alpha}_{a+} f(x) = \frac{d}{dx} J^{n-\alpha}_{a+} f^{(n-1)}(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{n-\alpha-1} f^{(n-1)}(t)\, dt\,, \tag{2.101}$$

while the right-sided Canavati fractional derivative ${}^{C_1}D^{\alpha}_{b-} g$ is defined by

$${}^{C_1}D^{\alpha}_{b-} g(x) = (-1)^n \frac{d}{dx} J^{n-\alpha}_{b-} g^{(n-1)}(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d}{dx} \int_x^b (t-x)^{n-\alpha-1} g^{(n-1)}(t)\, dt\,. \tag{2.102}$$

For $0 < \alpha < 1$ the Canavati fractional derivatives coincide with the Riemann-Liouville, that is

$${}^{C_1}D^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} f(t)\, dt = D^{\alpha}_{a+} f(x)\,, \tag{2.103}$$

$${}^{C_1}D^{\alpha}_{b-} g(x) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (t-x)^{-\alpha} g(t)\, dt = D^{\alpha}_{b-} g(x)\,. \tag{2.104}$$

For $\alpha = n \in \mathbb{N}$ hold

$${}^{C_1}D^{n}_{a+} f(x) = f^{(n)}(x)\,, \quad {}^{C_1}D^{n}_{b-} f(x) = (-1)^n f^{(n)}(x)\,, \tag{2.105}$$

while for $\alpha = 0$ we stipulate

$${}^{C_1}D^{0}_{a+} f(x) = {}^{C_1}D^{0}_{b-} f(x) = f(x)\,. \tag{2.106}$$

Example 2.7 Let $\alpha > 0$, $n = [\alpha]+1$, $\beta > n-1$ and let $f(x) = (x-a)^{\beta-1}$, $g(x) = (b-x)^{\beta-1}$. By Example 2.1 for the left-sided Canavati fractional derivative of the function f we have

$$\begin{aligned}
&{}^{C_1}D^{\alpha}_{a+}(x-a)^{\beta-1}\\
&= \frac{d}{dx} J^{n-\alpha}_{a+} \frac{d^{n-1}}{dx^{n-1}}(x-a)^{\beta-1}\\
&= (\beta-1)(\beta-2)\cdots(\beta-n+1)\frac{d}{dx} J^{n-\alpha}_{a+}(x-a)^{\beta-n}\\
&= (\beta-1)(\beta-2)\cdots(\beta-n+1)\frac{\Gamma(\beta-n+1)}{\Gamma(-\alpha+\beta+1)}\frac{d}{dx}(x-a)^{-\alpha+\beta}\\
&= \frac{\Gamma(\beta)}{\Gamma(-\alpha+\beta)}(x-a)^{-\alpha+\beta-1}. \qquad (2.107)
\end{aligned}$$

Analogously, the right-sided Canavati fractional derivative of the function g is

$$\begin{aligned}
{}^{C_1}D^{\alpha}_{b-}(b-x)^{\beta-1} &= (-1)^n \frac{d}{dx} J^{n-\alpha}_{b-} \frac{d^{n-1}}{dx^{n-1}}(b-x)^{\beta-1}\\
&= \frac{\Gamma(\beta)}{\Gamma(-\alpha+\beta)}(b-x)^{-\alpha+\beta-1}. \qquad (2.108)
\end{aligned}$$

Hence, for $\beta > n-1$ relations (2.107) and (2.26) coincide, as well as (2.108) and (2.27).
If $\beta = 1$, then $f(x) = g(x) = 1$ and $\frac{d}{dx}J^{n-\alpha}_{a+}f^{(n-1)}(x) = \frac{d}{dx}J^{n-\alpha}_{b-}g^{(n-1)}(x) = 0$, that is

$$ {}^{C_1}D^{\alpha}_{a+}(1) = {}^{C_1}D^{\alpha}_{b-}(1) = 0. $$

Therefore, the Canavati fractional derivative (as well as the Caputo) of a constant function is equal zero.
Moreover, for $k = 0,1,\ldots,n-2$ we have

$$ {}^{C_1}D^{\alpha}_{a+}(x-a)^k = {}^{C_1}D^{\alpha}_{b-}(b-x)^k = 0. \qquad (2.109) $$

Example 2.8 Let $\alpha > 0$ and $\lambda \in \mathbb{R}$. For the exponential function we have

$$\begin{aligned}
{}^{C_1}D^{\alpha}_{a+}e^{\lambda x} &= {}^{C_1}D^{\alpha}_{a+}\left(e^{\lambda a}e^{\lambda(x-a)}\right)\\
&= {}^{C_1}D^{\alpha}_{a+}\left[e^{\lambda a}\sum_{k=0}^{\infty}\frac{\lambda^k(x-a)^k}{k!}\right]\\
&= e^{\lambda a}\sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(k+1)}\,{}^{C_1}D^{\alpha}_{a+}(x-a)^k\\
&= \frac{e^{\lambda a}}{(x-a)^{\alpha}}\sum_{k=n-1}^{\infty}\frac{\lambda^k(x-a)^k}{\Gamma(-\alpha+k+1)},\\
{}^{C_1}D^{\alpha}_{b-}e^{\lambda x} &= \frac{e^{\lambda b}}{(b-x)^{\alpha}}\sum_{k=n-1}^{\infty}\frac{(-\lambda)^k(b-x)^k}{\Gamma(-\alpha+k+1)}.
\end{aligned}$$

Theorem 2.11 *Let $\alpha > 0$, $n = [\alpha]+1$, $f \in C^{\alpha}_{a+}[a,b]$ and $g \in C^{\alpha}_{b-}[a,b]$.*

(i)

$$^{C_1}D^{\alpha}_{a+} J^{\alpha}_{a+} f = f\,, \quad ^{C_1}D^{\alpha}_{b-} J^{\alpha}_{b-} g = g\,. \tag{2.110}$$

(ii) *If $\alpha \geq 1$, then*

$$J^{\alpha}_{a+}\, {}^{C_1}D^{\alpha}_{a+} f(x) = f(x) - \sum_{k=0}^{n-2} \frac{f^{(k)}(a)}{k!}(x-a)^k\,, \tag{2.111}$$

$$J^{\alpha}_{b-}\, {}^{C_1}D^{\alpha}_{b-} g(x) = g(x) - \sum_{k=0}^{n-2} \frac{(-1)^k g^{(k)}(b)}{k!}(b-x)^k\,. \tag{2.112}$$

For $0 < \alpha < 1$ hold

$$J^{\alpha}_{a+}\, {}^{C_1}D^{\alpha}_{a+} f(x) = f(x)\,, \tag{2.113}$$

$$J^{\alpha}_{b-}\, {}^{C_1}D^{\alpha}_{b-} g(x) = g(x)\,. \tag{2.114}$$

Proof.

(i) Let $f \in C^{\alpha}_{a+}[a,b]$. Then $f \in L_1[a,b]$, so we use Lemma 2.3 and the relation (2.57) from Corollary 2.5 (since $n-1 < \alpha$), which lead to

$$\begin{aligned} ^{C_1}D^{\alpha}_{a+} J^{\alpha}_{a+} f(x) &= \frac{d}{dx} J^{n-\alpha}_{a+} \frac{d^{n-1}}{dt^{n-1}} J^{\alpha}_{a+} f(t) = \frac{d}{dx} J^{n-\alpha}_{a+} J^{\alpha+1-n}_{a+} f(t) \\ &= \frac{d}{dx} J^{1}_{a+} f(x) = f(x)\,, \end{aligned}$$

$$\begin{aligned} ^{C_1}D^{\alpha}_{b-} J^{\alpha}_{b-} g(x) &= (-1)^n \frac{d}{dx} J^{n-\alpha}_{b-} \frac{d^{n-1}}{dt^{n-1}} J^{\alpha}_{b-} g(t) = -\frac{d}{dx} J^{n-\alpha}_{b-} J^{\alpha+1-n}_{b-} g(t) \\ &= -\frac{d}{dx} J^{1}_{b-} g(x) = g(x)\,. \end{aligned}$$

(ii) Let $\alpha \notin \mathbb{N}$. If $0 < \alpha < 1$, then the Canavati fractional derivative coincides with the Riemann-Liouville (for which hold (2.33) and (2.34)). Therefore

$$\begin{aligned} J^{\alpha}_{a+}\, {}^{C_1}D^{\alpha}_{a+} f(x) &= \frac{1}{\Gamma(1-\alpha)} J^{\alpha}_{a+} \left[(x-a)^{-\alpha} f(a) + \int_a^x (x-t)^{-\alpha} f'(t)\,dt \right] \\ &= \frac{f(a)}{\Gamma(1-\alpha)} J^{\alpha}_{a+} (x-a)^{-\alpha} + J^{\alpha}_{a+} J^{1-\alpha}_{a+} f'(x) \\ &= f(a) + J^{1}_{a+} \frac{d}{dx} f(x) = f(x)\,, \end{aligned}$$

$$\begin{aligned} J^{\alpha}_{b-}\, {}^{C_1}D^{\alpha}_{b-} g(x) &= \frac{1}{\Gamma(1-\alpha)} J^{\alpha}_{b-} \left[(b-x)^{-\alpha} g(b) - \int_x^b (t-x)^{-\alpha} g'(t)\,dt \right] \\ &= \frac{g(b)}{\Gamma(1-\alpha)} J^{\alpha}_{b-} (b-x)^{-\alpha} - J^{\alpha}_{b-} J^{1-\alpha}_{b-} g'(x) \end{aligned}$$

$$= g(b) - J^1_{b-} \frac{d}{dx} g(x) \;=\; g(x)\,.$$

Let $\alpha > 1$. Since $J^{n-\alpha}_{a+} f^{(n-1)}, J^{n-\alpha}_{b-} g^{(n-1)} \in C^1[a,b]$, by Theorem 2.5, Theorem 2.6, (2.101) and (2.102), we arrive to

$$J^{\alpha}_{a+}\, {}^{C_1}D^{\alpha}_{a+} f(x) = J^{\alpha}_{a+} \frac{d}{dx} J^{n-\alpha}_{a+} f^{(n-1)}(x) = J^{\alpha-1}_{a+} J^{n-\alpha}_{a+} f^{(n-1)}(x) = J^{n-1}_{a+} D^{n-1}_{a+} f(x)\,,$$

$$\begin{aligned} J^{\alpha}_{b-}\, {}^{C_1}D^{\alpha}_{b-} g(x) \;&=\; (-1)^n J^{\alpha}_{b-} \frac{d}{dx} J^{n-\alpha}_{b-} g^{(n-1)}(x) \\ &=\; (-1)^{n-1} J^{\alpha-1}_{b-} J^{n-\alpha}_{b-} \left[(-1)^{n-1} D^{n-1}_{b-} g(x)\right] \\ &=\; J^{n-1}_{b-} D^{n-1}_{b-} g(x)\,. \end{aligned}$$

If we apply (2.53) and (2.54) on relations above, we get (2.111) and (2.112).
If $\alpha \in \mathbb{N}$, then we apply (2.105) and once more (2.53), (2.54). □

Corollary 2.11 (TAYLOR'S FORMULA FOR THE CANAVATI FRACTIONAL DERIVATIVES) *Let $f \in C^{\alpha}_{a+}[a,b]$ and $g \in C^{\alpha}_{b-}[a,b]$.*

(*i*) *If $\alpha > 1$, $\alpha \notin \mathbb{N}$ and $n = [\alpha]+1$, then*

$$f(x) = \sum_{k=0}^{n-2} \frac{f^{(k)}(a)}{k!}(x-a)^k \;+\; J^{\alpha}_{a+}\, {}^{C_1}D^{\alpha}_{a+} f(x)\,, \tag{2.115}$$

$$g(x) = \sum_{k=0}^{n-2} \frac{(-1)^k g^{(k)}(b)}{k!}(b-x)^k \;+\; J^{\alpha}_{b-}\, {}^{C_1}D^{\alpha}_{b-} g(x)\,. \tag{2.116}$$

(*ii*) *If $0 \le \alpha < 1$, then*

$$f(x) = J^{\alpha}_{a+}\, {}^{C_1}D^{\alpha}_{a+} f(x)\,, \tag{2.117}$$

$$g(x) = J^{\alpha}_{b-}\, {}^{C_1}D^{\alpha}_{b-} g(x)\,. \tag{2.118}$$

Corollary 2.12 *Let $\alpha \ge 1$ and $n = [\alpha]+1$.*

(*i*) *If $f \in C^{\alpha}_{a+}[a,b]$ is such that $f(a) = f'(a) = \cdots = f^{(n-2)}(a) = 0$, then* (2.117) *holds.*

(*ii*) *If $g \in C^{\alpha}_{b-}[a,b]$ is such that $g(b) = g'(b) = \cdots = g^{(n-2)}(b) = 0$, then* (2.118) *holds.*

2.5 Relations between different types of fractional derivatives

Since all of the observe types of fractional derivatives are connected to each other, in this section we will examine their relations.

First we observe the special case when $\alpha = n \in \mathbb{N}_0$.

Corollary 2.13 *If $\alpha = n \in \mathbb{N}_0$, then*

$$D^n_{a+} f(x) = {}^C D^n_{a+} f(x) = {}^{C_1} D^n_{a+} f(x) = f^{(n)}(x)\,,$$

$$D^n_{b-} f(x) = {}^C D^n_{b-} f(x) = {}^{C_1} D^n_{b-} f(x) = (-1)^n f^{(n)}(x)\,.$$

In particular,

$$D^0_{a+} f(x) = {}^C D^0_{a+} f(x) = {}^{C_1} D^0_{a+} f(x) = f(x)\,,$$
$$D^0_{b-} f(x) = {}^C D^0_{b-} f(x) = {}^{C_1} D^0_{b-} f(x) = f(x)\,.$$

From now on let $\alpha \notin \mathbb{N}_0$.

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Relations between these two types of fractional derivatives follows from Theorem 2.1 and Theorem 2.7.

Corollary 2.14 *Let $\alpha > 0$, $n = [\alpha] + 1$ and $f \in AC^n[a,b]$. Then*

$$D^\alpha_{a+} f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)} (x-a)^{k-\alpha} + {}^C D^\alpha_{a+} f(x)\,, \tag{2.119}$$

$$D^\alpha_{b-} f(x) = \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(b)}{\Gamma(k-\alpha+1)} (b-x)^{k-\alpha} + {}^C D^\alpha_{b-} f(x)\,. \tag{2.120}$$

Corollary 2.15 *Let $\alpha > 0$, $n = [\alpha] + 1$ and $f \in AC^n[a,b]$.*

(i) If $f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0$, then $D^\alpha_{a+} f(x) = {}^C D^\alpha_{a+} f(x)$.

(ii) If $f(b) = f'(b) = \cdots = f^{(n-1)}(b) = 0$, then $D^\alpha_{b-} f(x) = {}^C D^\alpha_{b-} f(x)$.

Remark 2.1 Let g be the function defined by

$$g(x) = \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t)\,dt = J^n_{a+} f(x)\,.$$

Then $g^{(n)}(x) = f(x)$ and holds

$$D^\alpha_{a+} f(x) = \frac{d^n}{dx^n} J^{n-\alpha}_{a+} f(x) = \frac{d^n}{dx^n} J^{n-\alpha}_{a+} g^{(n)}(x)\,,$$

that is

$$D^\alpha_{a+} f(x) = \frac{d^n}{dx^n} {}^C D^\alpha_{a+} g(x)\,. \tag{2.121}$$

Analogously,

$$D^\alpha_{b-} f(x) = \frac{d^n}{dx^n} {}^C D^\alpha_{b-} g(x)\,. \tag{2.122}$$

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Using integration by parts we obtain

$$\begin{aligned}
{}^{C_1}D^{\alpha}_{a+}f(x) &= \frac{1}{\Gamma(n-\alpha)}\,\frac{d}{dx}\int_a^x (x-t)^{n-\alpha-1} f^{(n-1)}(t)\,dt \\
&= \frac{1}{\Gamma(n-\alpha)}\,\frac{d}{dx}\left[\frac{(x-a)^{n-\alpha}}{n-\alpha} f^{(n-1)}(a) + \int_a^x \frac{(x-t)^{n-\alpha}}{n-\alpha}\, f^{(n)}(t)\,dt\right] \\
&= \frac{f^{(n-1)}(a)}{\Gamma(n-\alpha)}(x-a)^{n-\alpha-1} \;+\; \frac{d}{dx} J^{n-\alpha+1}_{a+} f^{(n)}(x)\,.
\end{aligned}$$

By (2.3) and Lemma 2.3 we have

$$\frac{d}{dx} J^{n-\alpha+1}_{a+} f^{(n)}(x) = \frac{d}{dx} J^{1}_{a+} J^{n-\alpha}_{a+} f^{(n)}(x) = {}^{C}D^{\alpha}_{a+}f(x)\,,$$

that is

$${}^{C_1}D^{\alpha}_{a+}f(x) = \frac{f^{(n-1)}(a)}{\Gamma(n-\alpha)}(x-a)^{n-\alpha-1} \;+\; {}^{C}D^{\alpha}_{a+}f(x)\,.$$

The analogous equality holds for the right-sided fractional derivatives, contained in the following corollaries.

Corollary 2.16 *Let $\alpha > 0$ and $n = [\alpha]+1$.*

(i) *If $f \in AC^n[a,b] \cap C^{\alpha}_{a+}[a,b]$, then*

$${}^{C_1}D^{\alpha}_{a+}f(x) = \frac{f^{(n-1)}(a)}{\Gamma(n-\alpha)}\,(x-a)^{n-\alpha-1} \;+\; {}^{C}D^{\alpha}_{a+}f(x)\,. \tag{2.123}$$

(ii) *If $f \in AC^n[a,b] \cap C^{\alpha}_{b-}[a,b]$, then*

$${}^{C_1}D^{\alpha}_{b-}f(x) = (-1)^{n-1}\,\frac{f^{(n-1)}(b)}{\Gamma(n-\alpha)}\,(b-x)^{n-\alpha-1} \;+\; {}^{C}D^{\alpha}_{b-}f(x)\,. \tag{2.124}$$

Corollary 2.17 *Let $\alpha > 0$ and $n = [\alpha]+1$.*

(i) *If $f \in AC^n[a,b] \cap C^{\alpha}_{a+}[a,b]$ and $f^{(n-1)}(a) = 0$, then ${}^{C_1}D^{\alpha}_{a+}f(x) = {}^{C}D^{\alpha}_{a+}f(x)$.*

(ii) *If $f \in AC^n[a,b] \cap C^{\alpha}_{b-}[a,b]$ and $f^{(n-1)}(b) = 0$, then ${}^{C_1}D^{\alpha}_{b-}f(x) = {}^{C}D^{\alpha}_{b-}f(x)$.*

Remark 2.2 We also have

$${}^{C_1}D^{\alpha}_{a+}f'(x) = \frac{d}{dx} J^{n-\alpha}_{a+} f^{(n)}(x)\,,$$

that is

$${}^{C_1}D^{\alpha}_{a+}f'(x) = \frac{d}{dx}\,{}^{C}D^{\alpha}_{a+}f(x)\,, \tag{2.125}$$

$${}^{C_1}D^{\alpha}_{b-}f'(x) = \frac{d}{dx}\,{}^{C}D^{\alpha}_{b-}f(x)\,. \tag{2.126}$$

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The following corollaries are obtained by combinations od the two previous relations between different types of fractional derivatives.

Corollary 2.18 *Let* $\alpha > 1$ *and* $n = [\alpha] + 1$.

(i) *If* $f \in AC^n[a,b] \cap C^{\alpha}_{a+}[a,b]$, *then*

$$D^{\alpha}_{a+} f(x) = \sum_{k=0}^{n-2} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)} (x-a)^{k-\alpha} + {}^{C_1}D^{\alpha}_{a+} f(x) . \tag{2.127}$$

(ii) *If* $f \in AC^n[a,b] \cap C^{\alpha}_{b-}[a,b]$, *then*

$$D^{\alpha}_{b-} f(x) = \sum_{k=0}^{n-2} \frac{(-1)^k f^{(k)}(b)}{\Gamma(k-\alpha+1)} (b-x)^{k-\alpha} + {}^{C_1}D^{\alpha}_{b-} f(x) . \tag{2.128}$$

Corollary 2.19 *Let* $\alpha > 1$ *and* $n = [\alpha] + 1$.

(i) *If* $f \in AC^n[a,b] \cap C^{\alpha}_{a+}[a,b]$ *and* $f(a) = f'(a) = \cdots = f^{(n-2)}(a) = 0$, *then*

$$D^{\alpha}_{a+} f(x) = {}^{C_1}D^{\alpha}_{a+} f(x) .$$

(ii) *If* $f \in AC^n[a,b] \cap C^{\alpha}_{b-}[a,b]$ *and* $f(b) = f'(b) = \cdots = f^{(n-2)}(b) = 0$, *then*

$$D^{\alpha}_{b-} f(x) = {}^{C_1}D^{\alpha}_{b-} f(x) .$$

Corollary 2.20 *Let* $0 < \alpha < 1$.

(i) *If* $f \in AC^n[a,b] \cap C^{\alpha}_{a+}[a,b]$, *then* $D^{\alpha}_{a+} f(x) = {}^{C_1}D^{\alpha}_{a+} f(x)$.

(ii) *If* $f \in AC^n[a,b] \cap C^{\alpha}_{b-}[a,b]$, *then* $D^{\alpha}_{b-} f(x) = {}^{C_1}D^{\alpha}_{b-} f(x)$.

Remark 2.3 Let h be the function defined by

$$h(x) = \frac{1}{\Gamma(n-1)} \int_a^x (x-t)^{n-2} f(t)\, dt = J^{n-1}_{a+} f(x) .$$

Then $h^{(n-1)}(x) = f(x)$ and holds

$$D^{\alpha}_{a+} f(x) = \frac{d^n}{dx^n} J^{n-\alpha}_{a+} f(x) = \frac{d^{n-1}}{dx^{n-1}} \frac{d}{dx} J^{n-\alpha}_{a+} h^{(n-1)}(x) ,$$

that is

$$D^{\alpha}_{a+} f(x) = \frac{d^{n-1}}{dx^{n-1}} {}^{C_1}D^{\alpha}_{a+} h(x) . \tag{2.129}$$

Analogously,

$$D^{\alpha}_{b-} f(x) = \frac{d^{n-1}}{dx^{n-1}} {}^{C_1}D^{\alpha}_{b-} h(x) . \tag{2.130}$$

Finally, we give relations between the left-sided and the right-sided Riemann-Liouville fractional integrals, and also for the observed types of fractional derivatives. We use the *reflection operator* Q:

$$(Q\varphi)(x) = \varphi(a+b-x).$$

Proposition 2.5 *Let* $\alpha > 0$. *Then*

$$QJ^{\alpha}_{a+} = J^{\alpha}_{b-}Q, \quad QJ^{\alpha}_{b-} = J^{\alpha}_{a+}Q, \tag{2.131}$$

$$QD^{\alpha}_{a+} = D^{\alpha}_{b-}Q, \quad QD^{\alpha}_{b-} = D^{\alpha}_{a+}Q, \tag{2.132}$$

$$Q\,{}^{C}D^{\alpha}_{a+} = {}^{C}D^{\alpha}_{b-}Q, \quad Q\,{}^{C}D^{\alpha}_{b-} = {}^{C}D^{\alpha}_{a+}Q, \tag{2.133}$$

$$Q\,{}^{C_1}D^{\alpha}_{a+} = {}^{C_1}D^{\alpha}_{b-}Q, \quad Q\,{}^{C_1}D^{\alpha}_{b-} = {}^{C_1}D^{\alpha}_{a+}Q. \tag{2.134}$$

Proof. Using the substitution $a+b-t = s \in [b,x]$ we have

$$\begin{aligned} QJ^{\alpha}_{a+}f(x) &= J^{\alpha}_{a+}f(a+b-x) \\ &= \frac{1}{\Gamma(\alpha)}\int_a^{a+b-x}(a+b-x-t)^{\alpha-1}f(t)\,dt \\ &= \frac{1}{\Gamma(\alpha)}\int_x^b (s-x)^{\alpha-1}f(a+b-s)\,ds \\ &= J^{\alpha}_{b-}Qf(x). \end{aligned}$$

This gives us

$$QD^{\alpha}_{a+}f = (-1)^n\left(QJ^{n-\alpha}_{a+}f\right)^{(n)} = (-1)^n\left(J^{n-\alpha}_{b-}Qf\right)^{(n)} = D^{\alpha}_{b-}Qf.$$

Further,

$$Q\,{}^{C}D^{\alpha}_{a+}f = Q\left(J^{n-\alpha}_{a+}f^{(n)}\right) = (-1)^n J^{n-\alpha}_{b-}(Qf)^{(n)} = {}^{C}D^{\alpha}_{b-}Qf,$$

$$Q\,{}^{C_1}D^{\alpha}_{a+}f = -\left(QJ^{n-\alpha}_{a+}f^{(n-1)}\right)' = (-1)^n\left(J^{n-\alpha}_{b-}(Qf)^{(n-1)}\right)' = {}^{C_1}D^{\alpha}_{b-}Qf.$$

The proofs for the QJ^{α}_{b-}, QD^{α}_{b-}, $Q\,{}^{C}D^{\alpha}_{b-}$ and $Q\,{}^{C_1}D^{\alpha}_{b-}$ are analogous. □

2.6 Composition identities for fractional derivatives

In general, for $\alpha > \beta \geq 0$ and $x \in [a,b]$ the composition identity has a form

$$\mathbf{D}^{\beta}f(x) = \frac{1}{\Gamma(\alpha-\beta)}\int_a^x (x-t)^{\alpha-\beta-1}\mathbf{D}^{\alpha}f(t)\,dt, \tag{2.135}$$

where equality (2.135) holds for all observed types, that is, $\mathbf{D}$ can be the left-sided Riemann-Liouville D, the Caputo ${}^{C}D$ or the Canavati ${}^{C_1}D$ fractional derivative (an analogous identity holds for the right-sided fractional derivatives).

2.6.1 Composition identities for the Riemann-Liouville fractional derivatives

The theorem about the composition identity for the left-sided Riemann-Liouville fractional derivatives was firstly given by Handley-Koliha-Pečarić in [47]:

Theorem 2.12 *Let $\alpha > \beta \geq 0$ and $n = [\alpha] + 1$. Let $f \in L_1[a,b]$ has integrable fractional derivative $D^{\alpha}_{a+} f \in L_\infty[a,b]$ such that $D^{\alpha-k}_{a+} f(a) = 0$, $k = 1,\ldots,n$. Then*

$$D^{\beta}_{a+} f(x) = \frac{1}{\Gamma(\alpha-\beta)} \int_a^x (x-t)^{\alpha-\beta-1} D^{\alpha}_{a+} f(t)\,dt\,, \quad x \in [a,b]\,.$$

In the following theorem, Andrić-Pečarić-Perić give another approach to the composition identity, and use the Laplace transform as an elegant technique of proof ([25]).

Theorem 2.13 *Let $\alpha > \beta \geq 0$, $n = [\alpha] + 1$, $m = [\beta] + 1$ and let $f \in AC^n[a,b]$ be such that $D^{\alpha}_{a+} f, D^{\beta}_{a+} f \in L_1[a,b]$.*

(i) If $\alpha - \beta \notin \mathbb{N}$ and f is such that $D^{\alpha-k}_{a+} f(a) = 0$ for $k = 1,\ldots,n$ and $D^{\beta-k}_{a+} f(a) = 0$ for $k = 1,\ldots,m$, then

$$D^{\beta}_{a+} f(x) = \frac{1}{\Gamma(\alpha-\beta)} \int_a^x (x-t)^{\alpha-\beta-1} D^{\alpha}_{a+} f(t)\,dt\,, \quad x \in [a,b]\,. \tag{2.136}$$

(ii) If $\alpha - \beta = l \in \mathbb{N}$ and f is such that $D^{\alpha-k}_{a+} f(a) = 0$ for $k = 1,\ldots,l$, then (2.136) holds.

Proof. (i) Define auxiliary function $h\colon [0,\infty) \to \mathbb{R}$ with

$$h(x) = \begin{cases} f(x+a)\,,\ x \in [0, b-a] \\ \sum\limits_{k=0}^{n} \frac{f^{(k)}(b)}{k!} (x-b+a)^k\,,\ x \geq b-a \end{cases}. \tag{2.137}$$

Obviously $h \in AC^n[0,\infty)$, $D^{\alpha-k}_{0+} h(0) = 0$, $k = 1,\ldots,n$ and $D^{\beta-k}_{0+} h(0) = 0$, $k = 1,\ldots,m$. Also h has polynomial growth at ∞, so the Laplace transform of h exists. Notice that both sides of (2.136) are integrable functions. The composition identity (2.136) will follow if we prove that for every $x \geq 0$ holds

$$\begin{aligned} &\frac{1}{\Gamma(m-\beta)} \frac{d^m}{dx^m} \int_0^x (x-t)^{m-\beta-1} h(t)\,dt \\ &= \frac{1}{\Gamma(\alpha-\beta)\Gamma(n-\alpha)} \int_0^x (x-t)^{\alpha-\beta-1} \frac{d^n}{dt^n} \int_0^t (t-y)^{n-\alpha-1} h(y)\,dy\,dt\,. \end{aligned} \tag{2.138}$$

Using convolution for the Laplace transform, for the right side of the equality (2.138) we have

$$\mathscr{L}\left[\frac{1}{\Gamma(\alpha-\beta)\Gamma(n-\alpha)} \int_0^x (x-t)^{\alpha-\beta-1} \frac{d^n}{dt^n} \int_0^t (t-y)^{n-\alpha-1} h(y)\,dy\,dt\right](p)$$

$$= \frac{1}{\Gamma(\alpha-\beta)\,\Gamma(n-\alpha)} \mathscr{L}\left[x^{\alpha-\beta-1}\right](p)\,\mathscr{L}\left[\frac{d^n}{dt^n}\int_0^t (t-y)^{n-\alpha-1}\,h(y)\,dy\right](p)\,.$$

By substitution $px = s$, where p is a variable from the definition of the Laplace transform (1.8), we obtain

$$\mathscr{L}\left[x^{\alpha-\beta-1}\right](p) = \int_0^\infty e^{-px}\,x^{\alpha-\beta-1}\,dx = \frac{1}{p^{\alpha-\beta}}\int_0^\infty e^{-s}\,s^{\alpha-\beta-1}\,ds = \frac{\Gamma(\alpha-\beta)}{p^{\alpha-\beta}}\,,$$

and by the rule of differentiation of the Laplace transform follows

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha-\beta)\,\Gamma(n-\alpha)} \mathscr{L}\left[x^{\alpha-\beta-1}\right](p)\,\mathscr{L}\left[\left(\frac{d}{dt}\right)^n \int_0^t (t-y)^{n-\alpha-1}\,h(y)\,dy\right](p) \\
&= \frac{1}{p^{\alpha-\beta}\,\Gamma(n-\alpha)}\left\{p^n\,\mathscr{L}\left[\int_0^t (t-y)^{n-\alpha-1}\,h(y)\,dy\right](p)\right. \\
&\qquad \left. - \sum_{k=0}^{n-1} p^k\,\frac{d^{n-k-1}}{dt^{n-k-1}}\left(\int_0^t (t-y)^{n-\alpha-1}\,h(y)\,dy\right)(0)\right\} \\
&= \frac{p^{n-\alpha+\beta}}{\Gamma(n-\alpha)} \mathscr{L}\left[t^{n-\alpha-1}\right](p)\mathscr{L}[h](p) - \frac{1}{p^{\alpha-\beta}}\sum_{k=1}^{n} p^{k-1} D_{0+}^{\alpha-k} h(0) \qquad (2.139) \\
&= p^\beta\,\mathscr{L}[h](p)\,. \qquad (2.140)
\end{aligned}$$

For the left side of the equality (2.138) we have

$$\begin{aligned}
&\mathscr{L}\left[\frac{1}{\Gamma(m-\beta)}\,\frac{d^m}{dx^m}\int_0^x (x-t)^{m-\beta-1}\,h(t)\,dt\right](p) \\
&= \frac{1}{\Gamma(m-\beta)}\left\{p^m\,\mathscr{L}\left[\int_0^x (x-t)^{m-\beta-1}\,h(t)\,dt\right](p)\right. \\
&\qquad \left. - \sum_{k=0}^{m-1} p^k\,\frac{d^{m-k-1}}{dx^{m-k-1}}\left(\int_0^x (x-t)^{m-\beta-1}\,h(t)\,dt\right)(0)\right\} \\
&= \frac{p^m}{\Gamma(m-\beta)} \mathscr{L}\left[x^{m-\beta-1}\right](p)\,\mathscr{L}[h](p) - \sum_{k=1}^{m} p^{k-1} D_{0+}^{\beta-k} h(0) \qquad (2.141) \\
&= p^\beta\,\mathscr{L}[h](p)\,. \qquad (2.142)
\end{aligned}$$

Using (2.140) and (2.142) it follows that both sides of (2.138) have the same Laplace transform, so by Theorem 1.3 we conclude that equality in (2.138) holds for every $x \geq 0$.

(ii) Notice that from $\alpha = \beta + l$, $l \in \mathbb{N}$, follows $n = m + l$, $n - \alpha = m - \beta$. Again we use function $h\colon [0,\infty) \to \mathbb{R}$ defined with (2.137) and conclude

$$D_{0+}^{\beta+l-k} h(0) = 0, \quad k = 1,\ldots,l\,, \qquad (2.143)$$

and also h has polynomial growth at ∞, that is, the Laplace transform of h exists. The composition identity (2.136) will follow if we prove that for every $x \geq 0$ holds

$$\frac{1}{\Gamma(m-\beta)}\,\frac{d^m}{dx^m}\int_0^x (x-t)^{m-\beta-1}\,h(t)\,dt$$

$$= \frac{1}{\Gamma(l)\,\Gamma(m-\beta)} \int_0^x (x-t)^{l-1} \frac{d^{m+l}}{dt^{m+l}} \int_0^t (t-y)^{m-\beta-1} h(y)\,dy\,dt\,. \tag{2.144}$$

As in the proof of the previous claim, using standard properties of the Laplace transform, relations (2.139) and (2.140), for the right side of the equality (2.144) we have

$$\mathscr{L}\left[\frac{1}{\Gamma(l)\,\Gamma(m-\beta)} \int_0^x (x-t)^{l-1} \frac{d^{m+l}}{dt^{m+l}} \int_0^t (t-y)^{m-\beta-1} h(y)\,dy\,dt\right](p)$$
$$= p^{\beta}\mathscr{L}[h](p) - \frac{1}{p^l}\sum_{k=1}^{m+l} p^{k-1} D_{0+}^{\beta+l-k} h(0)\,.$$

From (2.143) follows

$$\begin{aligned}\frac{1}{p^l}\sum_{k=1}^{m+l} p^{k-1} D_{0+}^{\beta+l-k} h(0) &= \sum_{k=l+1}^{m+l} p^{k-l-1} D_{0+}^{\beta+l-k} h(0)\\ &= \sum_{k=1}^{m} p^{(k+l)-l-1} D_{0+}^{\beta+l-(k+l)} h(0) \;=\; \sum_{k=1}^{m} p^{k-1} D_{0+}^{\beta-k} h(0)\,,\end{aligned}$$

that is

$$\mathscr{L}\left[\frac{1}{\Gamma(l)\,\Gamma(m-\beta)} \int_0^x (x-t)^{l-1} \frac{d^{m+l}}{dt^{m+l}} \int_0^t (t-y)^{m-\beta-1} h(y)\,dy\,dt\right](p)$$
$$= p^{\beta}\mathscr{L}[h](p) - \sum_{k=1}^{m} p^{k-1} D_{0+}^{\beta-k} h(0)\,. \tag{2.145}$$

As in (2.141) and (2.142), for the left side we get

$$\mathscr{L}\left[\frac{1}{\Gamma(m-\beta)} \frac{d^m}{dx^m} \int_0^x (x-t)^{m-\beta-1} h(t)\,dt\right](p)$$
$$= p^{\beta}\,\mathscr{L}[h](p) - \sum_{k=1}^{m} p^{k-1} D_{0+}^{\beta-k} h(0)\,. \tag{2.146}$$

Applying Theorem 1.3, from the equality of the Laplace transforms (2.145) and (2.146), follows that for every $x \geq 0$ equality (2.144) holds. □

According to Proposition 2.3, we summarize conditions for the identity (2.136).

Corollary 2.21 *Let $\alpha > \beta \geq 0$, $n = [\alpha]+1$ and $m = [\beta]+1$. The composition identity* (2.136) *is valid in one of the following conditions holds:*

(*i*) $f \in J_{a+}^{\alpha}(L_1)$.

(*ii*) $J_{a+}^{n-\alpha} f \in AC^n[a,b]$ *and* $D_{a+}^{\alpha-k} f(a) = 0$ *for* $k = 1,\dots n$.

(*iii*) $D_{a+}^{\alpha-1} f \in AC[a,b]$, $D_{a+}^{\alpha-k} f \in C[a,b]$ *and* $D_{a+}^{\alpha-k} f(a) = 0$ *for* $k = 1,\dots n$.

(iv) $f \in AC^n[a,b]$, $D^\alpha_{a+}f$, $D^\beta_{a+}f \in L_1[a,b]$, $\alpha - \beta \notin \mathbb{N}$, $D^{\alpha-k}_{a+}f(a) = 0$ *for* $k = 1,\dots,n$ *and* $D^{\beta-k}_{a+}f(a) = 0$ *for* $k = 1,\dots,m$.

(v) $f \in AC^n[a,b]$, $D^\alpha_{a+}f, D^\beta_{a+}f \in L_1[a,b]$, $\alpha - \beta = l \in \mathbb{N}$ *and* $D^{\alpha-k}_{a+}f(a) = 0$ *for* $k = 1,\dots,l$.

(vi) $f \in AC^n[a,b]$, $D^\alpha_{a+}f, D^\beta_{a+}f \in L_1[a,b]$ *and* $f^{(k)}(a) = 0$ *for* $k = 0,\dots,n-2$.

(vii) $f \in AC^n[a,b]$, $D^\alpha_{a+}f, D^\beta_{a+}f \in L_1[a,b]$, $\alpha \notin \mathbb{N}$ *and* $D^{\alpha-1}_{a+}f$ *is bounded in a neighborhood of* $t = a$.

Next theorem gives us the composition identity for the right-sided Riemann-Liouville fractional derivatives.

Theorem 2.14 *Let* $\alpha > \beta \geq 0$, $n = [\alpha]+1$, $m = [\beta]+1$ *and let* $f \in AC^n[a,b]$ *be such that* $D^\alpha_{b-}f, D^\beta_{b-}f \in L_1[a,b]$.

(i) If $\alpha - \beta \notin \mathbb{N}$ *and* f *is such that* $D^{\alpha-k}_{b-}f(b) = 0$ *for* $k = 1,\dots,n$ *and* $D^{\beta-k}_{b-}f(b) = 0$ *for* $k = 1,\dots,m$, *then*

$$D^\beta_{b-}f(x) = \frac{1}{\Gamma(\alpha-\beta)} \int_x^b (t-x)^{\alpha-\beta-1} D^\alpha_{b-}f(t)\,dt\,, \quad x \in [a,b]\,. \tag{2.147}$$

(ii) If $\alpha - \beta = l \in \mathbb{N}$ *and* f *is such that* $D^{\alpha-k}_{b-}f(b) = 0$ *for* $k = 1,\dots,l$, *then* (2.147) *holds.*

Proof. According to relations (2.131) and (2.132) which use the reflection operator Q, the proof of this theorem follows from Theorem 2.13, mentioned relations and

$$\begin{aligned} D^\beta_{b-}f &= Q\left(QD^\beta_{b-}f\right) = Q\left(D^\beta_{a+}Qf\right) \\ &= Q\left(J^{\alpha-\beta}_{a+}D^\alpha_{a+}Qf\right) = J^{\alpha-\beta}_{b-}Q\left(D^\alpha_{a+}Qf\right) \\ &= J^{\alpha-\beta}_{b-}D^\alpha_{b-}Q(Qf) = J^{\alpha-\beta}_{b-}D^\alpha_{b-}f\,. \end{aligned}$$

□

Corollary 2.22 *Let* $\alpha > \beta \geq 0$, $n = [\alpha]+1$ *and* $m = [\beta]+1$. *The composition identity* (2.147) *is valid if one of the following conditions holds:*

(i) $f \in J^\alpha_{b-}(L_1)$.

(ii) $J^{n-\alpha}_{b-}f \in AC^n[a,b]$ *and* $D^{\alpha-k}_{b-}f(b) = 0$ *for* $k = 1,\dots n$.

(iii) $D^{\alpha-1}_{b-}f \in AC[a,b]$, $D^{\alpha-k}_{b-}f \in C[a,b]$ *and* $D^{\alpha-k}_{b-}f(b) = 0$ *for* $k = 1,\dots n$.

(iv) $f \in AC^n[a,b]$, $D^\alpha_{b-}f, D^\beta_{b-}f \in L_1[a,b]$, $\alpha - \beta \notin \mathbb{N}$, $D^{\alpha-k}_{b-}f(b) = 0$ *for* $k = 1,\dots,n$ *and* $D^{\beta-k}_{b-}f(b) = 0$ *for* $k = 1,\dots,m$.

(v) $f \in AC^n[a,b]$, $D^{\alpha}_{b-}f, D^{\beta}_{b-}f \in L_1[a,b]$, $\alpha - \beta = l \in \mathbb{N}$ *and* $D^{\alpha-k}_{b-}f(b) = 0$ *for* $k = 1,\dots,l$.

(vi) $f \in AC^n[a,b]$, $D^{\alpha}_{b-}f, D^{\beta}_{b-}f \in L_1[a,b]$ *and* $f^{(k)}(b) = 0$ *for* $k = 0,\dots,n-2$.

(vii) $f \in AC^n[a,b]$, $D^{\alpha}_{b-}f, D^{\beta}_{b-}f \in L_1[a,b]$, $\alpha \notin \mathbb{N}$ *and* $D^{\alpha-1}_{b-}f$ *is bounded in a neighborhood of* $t = b$.

2.6.2 Composition identities for the Caputo fractional derivatives

The theorem on the composition identity for the left-sided Caputo fractional derivatives was proven by Anastassiou in [12].

Theorem 2.15 *Let* $\alpha \geq \beta + 1$, $\beta \geq 0$, $n = [\alpha] + 1$ *and* $m = [\beta] + 1$. *Assume* $f \in AC^n[a,b]$ *is such that* $f^{(i)}(a) = 0$ *for* $i = 0,1,\dots,n-1$, *and* ${}^C D^{\alpha}_{a+}f \in L_\infty[a,b]$. *Then*

$$ {}^C D^{\beta}_{a+}f \in C[a,b]\,, \quad {}^C D^{\beta}_{a+}f(x) = J^{m-\beta}_{a+} f^{(m)}(x)\,, $$

$$ {}^C D^{\beta}_{a+}f(x) = \frac{1}{\Gamma(\alpha-\beta)} \int_a^x (x-t)^{\alpha-\beta-1}\, {}^C D^{\alpha}_{a+}f(t)\,dt\,, \quad x \in [a,b]\,. $$

In [26], Andrić-Pečarić-Perić proved that one doesn't need all vanishing left-sided (right-sided) derivatives of the function f at point a (at point b) and that condition $\alpha \geq \beta + 1$ can be relaxed. First we have theorem involving the left-sided Caputo fractional derivatives.

Theorem 2.16 *Let* $\alpha > \beta \geq 0$, n *and* m *given by* (2.70). *Let* $f \in AC^n[a,b]$ *be such that* ${}^C D^{\alpha}_{a+}f, {}^C D^{\beta}_{a+}f \in L_1[a,b]$ *and* $f^{(i)}(a) = 0$ *for* $i = m,\dots,n-1$. *Then*

$$ {}^C D^{\beta}_{a+}f(x) = \frac{1}{\Gamma(\alpha-\beta)} \int_a^x (x-t)^{\alpha-\beta-1}\, {}^C D^{\alpha}_{a+}f(t)\,dt\,, \quad x \in [a,b]\,. \tag{2.148} $$

Proof. Let $\alpha > \beta \geq 0$. We give a proof for $\alpha, \beta \notin \mathbb{N}$ when $n = [\alpha] + 1$ and $m = [\beta] + 1$. The proofs for $\alpha \in \mathbb{N}$ or $\beta \in \mathbb{N}$ or $\alpha, \beta \in \mathbb{N}$ are analogous.
Using the Fubini theorem and change of variables (1.4), also Example 1.2, we have

$$ \begin{aligned} & \frac{1}{\Gamma(\alpha-\beta)} \int_a^x (x-y)^{\alpha-\beta-1}\, {}^C D^{\alpha}_{a+}f(y)\,dy \\ &= \frac{1}{\Gamma(\alpha-\beta)\Gamma(n-\alpha)} \int_{y=a}^x \int_{t=a}^y (x-y)^{\alpha-\beta-1}(y-t)^{n-\alpha-1} f^{(n)}(t)\,dt\,dy \\ &= \frac{1}{\Gamma(\alpha-\beta)\Gamma(n-\alpha)} \int_{t=a}^x f^{(n)}(t) \int_{y=t}^x (x-y)^{\alpha-\beta-1}(y-t)^{n-\alpha-1}\,dy\,dt \\ &= \frac{B(\alpha-\beta, n-\alpha)}{\Gamma(\alpha-\beta)\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\beta-1} f^{(n)}(t)\,dt \\ &= \frac{1}{\Gamma(n-\beta)} \int_a^x (x-t)^{n-\beta-1} f^{(n)}(t)\,dt\,. \end{aligned} $$

Hence, for $\alpha > \beta \geq 0$ we have

$$\frac{1}{\Gamma(\alpha-\beta)} \int_a^x (x-y)^{\alpha-\beta-1}\, {}^C D_{a+}^{\alpha} f(y)\, dy = \frac{1}{\Gamma(n-\beta)} \int_a^x (x-t)^{n-\beta-1} f^{(n)}(t)\, dt\,.$$

Case 1. Let $[\alpha] = [\beta]$, that is $n = m$. Then (2.148) follows with no boundary conditions.

Case 2. Let $[\alpha] > [\beta]$. Then $[\alpha] > \beta$ also, and therefore $n - \beta - 1 > 0$. Using integration by parts, it follows

$$\begin{aligned}
&\frac{1}{\Gamma(n-\beta)} \int_a^x (x-t)^{n-\beta-1} f^{(n)}(t)\, dt \\
&= \frac{1}{\Gamma(n-\beta)} \left[(x-t)^{n-\beta-1} f^{(n-1)}(t) \Big|_a^x + (n-\beta-1) \int_a^x (x-t)^{n-\beta-2} f^{(n-1)}(t)\, dt \right] \\
&= \left| \; f^{(n-1)}(a) = 0 \; \right| \\
&= \frac{1}{\Gamma(n-\beta-1)} \int_a^x (x-t)^{n-\beta-2} f^{(n-1)}(t)\, dt\,.
\end{aligned}$$

Case 2a. Let $[\alpha] = [\beta] + 1$, that is $m = n - 1$. Then with boundary condition $f^{(n-1)}(a) = 0$ follows (2.148).

Case 2b. Let $[\alpha] > [\beta] + 1$. Then $n - \beta - 2 > 0$ and

$$\begin{aligned}
&\frac{1}{\Gamma(n-\beta-1)} \int_a^x (x-t)^{n-\beta-2} f^{(n-1)}(t)\, dt \\
&= \frac{1}{\Gamma(n-\beta-1)} \left[(x-t)^{n-\beta-2} f^{(n-2)}(t) \Big|_a^x + (n-\beta-2) \int_a^x (x-t)^{n-\beta-3} f^{(n-2)}(t)\, dt \right] \\
&= \left| \; f^{(n-2)}(a) = 0 \; \right| \\
&= \frac{1}{\Gamma(n-\beta-2)} \int_a^x (x-t)^{n-\beta-3} f^{(n-2)}(t)\, dt\,.
\end{aligned}$$

Case 2b.1. Let $[\alpha] = [\beta] + 2$, that is $m = n - 2$. Then with boundary conditions $f^{(n-1)}(a) = f^{(n-2)}(a) = 0$ follows (2.148).

Case 2b.2. Continuing in this way, in the last step, when $m = n - (n - m)$, we have that (2.148) is valid with boundary conditions $f^{(n-1)}(a) = \cdots = f^{(m)}(a) = 0$. $\square$

Remark 2.4 We also give an alternative proof of Theorem 2.16, using the Laplace transform. *Proof.* Let $\alpha > \beta \geq 0$, $\alpha, \beta \notin \mathbb{N}$, that is $n = [\alpha] + 1$ and $m = [\beta] + 1$. Let $f^{(i)}(a) = 0$ for $i = m, \ldots, n-1$. Then by Lemma 2.3 we have

$$J_{a+}^{\alpha-\beta}\, {}^C D_{a+}^{\alpha} f = J_{a+}^{\alpha-\beta} J_{a+}^{n-\alpha} f^{(n)} = J_{a+}^{n-\beta} f^{(n)}\,.$$

Set $g = f^{(m)}$. Now (2.148) can be written as

$$J_{a+}^{m-\beta} g(x) = J_{a+}^{n-\beta} g^{(n-m)}(x), \tag{2.149}$$

where $x \in [a,b]$ and $g(a) = g'(a) = \cdots = g^{(n-m-1)}(a) = 0$. Define auxiliary function $h : [0,\infty) \to \mathbb{R}$ with

$$h(x) = \begin{cases} g(x+a)\,, \; x \in [0, b-a] \\ \sum_{k=0}^{n-m} \frac{g^{(k)}(b)}{k!}(x-b+a)^k\,, \; x \geq b-a \end{cases}. \tag{2.150}$$

Obviously $h(0) = h'(0) = \cdots = h^{(n-m-1)}(0) = 0$. Also h has polynomial growth at ∞, so the Laplace transform exists. The composition identity (2.149) will follow if we prove that for every $x \geq 0$ holds

$$\frac{1}{\Gamma(m-\beta)} \int_0^x (x-t)^{m-\beta-1} h(t)\,dt = \frac{1}{\Gamma(n-\beta)} \int_0^x (x-t)^{n-\beta-1} h^{(n-m)}(t)\,dt\,. \tag{2.151}$$

Using standard properties of the Laplace transform, for the left side of the equality (2.151) we have

$$\begin{aligned} &\mathscr{L}\left[\frac{1}{\Gamma(m-\beta)} \int_0^x (x-t)^{m-\beta-1} h(t)\,dt\right](p) \\ &= \frac{1}{\Gamma(m-\beta)} \mathscr{L}\left[x^{m-\beta-1}\right](p)\mathscr{L}[h](p) \,=\, p^{\beta-m}\mathscr{L}[h](p)\,. \end{aligned} \tag{2.152}$$

For the right side of the equality (2.151) we have

$$\begin{aligned} &\mathscr{L}\left[\frac{1}{\Gamma(n-\beta)} \int_0^x (x-t)^{n-\beta-1} h^{(n-m)}(t)\,dt\right](p) \\ &= \frac{1}{\Gamma(n-\beta)} \mathscr{L}\left[x^{n-\beta-1}\right](p)\mathscr{L}\left[h^{(n-m)}\right](p) \\ &= p^{\beta-n} \cdot p^{n-m}\mathscr{L}[h](p) = p^{\beta-m}\mathscr{L}[h](p)\,. \end{aligned} \tag{2.153}$$

By Theorem 1.3, since both sides have the same Laplace transform (2.152) and (2.153), it follows that equality holds in (2.151) for every $x \geq 0$. □

Next theorem gives us the composition identity for the right-sided Caputo fractional derivatives.

Theorem 2.17 *Let $\alpha > \beta \geq 0$, n and m given by (2.70). Let $f \in AC^n[a,b]$ be such that ${}^C D^\alpha_{b-} f, {}^C D^\beta_{b-} f \in L_1[a,b]$ and $f^{(i)}(b) = 0$ for $i = m, \ldots, n-1$. Then*

$${}^C D^\beta_{b-} f(x) = \frac{1}{\Gamma(\alpha-\beta)} \int_x^b (t-x)^{\alpha-\beta-1}\, {}^C D^\alpha_{b-} f(t)\,dt\,, \quad x \in [a,b]\,. \tag{2.154}$$

Proof. Let $\alpha > \beta \geq 0$. We give proof for $\alpha, \beta \notin \mathbb{N}$ when $n = [\alpha]+1$ and $m = [\beta]+1$. The proofs for $\alpha \in \mathbb{N}$ or $\beta \in \mathbb{N}$ or $\alpha, \beta \in \mathbb{N}$ are analogous.

Using the Fubini theorem and change of variables (1.4), also Example 1.2, we have

$$\frac{1}{\Gamma(\alpha-\beta)} \int_x^b (y-x)^{\alpha-\beta-1}\, {}^C D^\alpha_{b-} f(y)\,dy$$

$$
\begin{aligned}
&= \frac{(-1)^n}{\Gamma(\alpha-\beta)\,\Gamma(n-\alpha)} \int_{y=x}^{b}\int_{t=y}^{b} (y-x)^{\alpha-\beta-1}(t-y)^{n-\alpha-1} f^{(n)}(t)\,dt\,dy \\
&= \frac{(-1)^n}{\Gamma(\alpha-\beta)\,\Gamma(n-\alpha)} \int_{t=x}^{b} f^{(n)}(t) \int_{y=x}^{t} (y-x)^{\alpha-\beta-1}(t-y)^{n-\alpha-1}\,dy\,dt \\
&= \frac{(-1)^n B(\alpha-\beta, n-\alpha)}{\Gamma(\alpha-\beta)\,\Gamma(n-\alpha)} \int_x^b (t-x)^{n-\beta-1} f^{(n)}(t)\,dt \\
&= \frac{(-1)^n}{\Gamma(n-\beta)} \int_x^b (t-x)^{n-\beta-1} f^{(n)}(t)\,dt\,.
\end{aligned}
$$

Hence, for $\alpha > \beta \geq 0$ we have

$$
\frac{1}{\Gamma(\alpha-\beta)} \int_x^b (y-x)^{\alpha-\beta-1}\, {}^{C}D_{b-}^{\alpha} f(y)\,dy = \frac{(-1)^n}{\Gamma(n-\beta)} \int_x^b (t-x)^{n-\beta-1} f^{(n)}(t)\,dt\,.
$$

Case 1. Let $[\alpha] = [\beta]$, that is $n = m$. Then (2.154) follows with no boundary conditions.

Case 2. Let $[\alpha] > [\beta]$. Then $[\alpha] > \beta$ also, and therefore $n-\beta-1 > 0$. Using integration by parts, it follows

$$
\begin{aligned}
&\frac{(-1)^n}{\Gamma(n-\beta)} \int_x^b (t-x)^{n-\beta-1} f^{(n)}(t)\,dt \\
&= \frac{(-1)^n}{\Gamma(n-\beta)} \left[(t-x)^{n-\beta-1} f^{(n-1)}(t) \Big|_x^b - (n-\beta-1) \int_x^b (t-x)^{n-\beta-2} f^{(n-1)}(t)\,dt \right] \\
&= \left|\ f^{(n-1)}(b) = 0\ \right| \\
&= \frac{(-1)^{n-1}}{\Gamma(n-\beta-1)} \int_x^b (t-x)^{n-\beta-2} f^{(n-1)}(t)\,dt\,.
\end{aligned}
$$

Case 2a. Let $[\alpha] = [\beta]+1$, that is $m = n-1$. Then with boundary condition $f^{(n-1)}(b) = 0$ follows (2.154).

Case 2b. Let $[\alpha] > [\beta]+1$. Then $n-\beta-2 > 0$ and

$$
\begin{aligned}
&\frac{(-1)^{n-1}}{\Gamma(n-\beta-1)} \int_x^b (t-x)^{n-\beta-2} f^{(n-1)}(t)\,dt \\
&= \frac{(-1)^{n-1}}{\Gamma(n-\beta-1)} \left[(t-x)^{n-\beta-2} f^{(n-2)}(t) \Big|_x^b - (n-\beta-2) \int_x^b (t-x)^{n-\beta-3} f^{(n-2)}(t)\,dt \right] \\
&= \left|\ f^{(n-2)}(b) = 0\ \right| \\
&= \frac{(-1)^{n-2}}{\Gamma(n-\beta-2)} \int_x^b (t-x)^{n-\beta-3} f^{(n-2)}(t)\,dt\,.
\end{aligned}
$$

Case 2b.1. Let $[\alpha] = [\beta]+2$, that is $m = n-2$. Then with boundary conditions $f^{(n-1)}(b) = f^{(n-2)}(b) = 0$ follows (2.154).

Case 2b.2. By inductions, in the last step, when $m = n-(n-m)$, we have that (2.154) is valid with boundary conditions $f^{(n-1)}(b) = \cdots = f^{(m)}(b) = 0$. $\square$

Remark 2.5 Previous theorem can be proven using the reflection operator Q from Proposition 2.5 and by Theorem 2.16:

$$\begin{aligned} {}^{C}D^{\beta}_{b-}f &= Q\left(Q\,{}^{C}D^{\beta}_{b-}f\right) = Q\left({}^{C}D^{\beta}_{a+}Qf\right) \\ &= Q\left(J^{\alpha-\beta}_{a+}\,{}^{C}D^{\alpha}_{a+}Qf\right) = J^{\alpha-\beta}_{b-}\,Q\left({}^{C}D^{\alpha}_{a+}Qf\right) \\ &= J^{\alpha-\beta}_{b-}\,{}^{C}D^{\alpha}_{b-}Q(Qf) = J^{\alpha-\beta}_{b-}\,{}^{C}D^{\alpha}_{b-}f\,. \end{aligned}$$

2.6.3 Composition identities for the Canavati fractional derivatives

For the left-sided Canavati fractional derivatives, the composition identity was proven by Anastassiou in [12].

Theorem 2.18 *Let $\alpha \ge \beta+1$, $\beta \ge 0$, $n = [\alpha]+1$ and $m = [\beta]+1$. Let $f \in C^{\alpha}_{a+}[a,b]$ be such that $f^{(i)}(a) = 0$, $i = 0,1,\dots,n-2$. Then ${}^{C_1}D^{\beta}_{a+}f \in C[a,b]$ and*

$${}^{C_1}D^{\beta}_{a+}f(x) = \frac{1}{\Gamma(\alpha-\beta)}\int_a^x (x-t)^{\alpha-\beta-1}\,{}^{C_1}D^{\alpha}_{a+}f(t)\,dt\,, \quad x \in [a,b]\,.$$

In [23], Andrić-Pečarić-Perić improved this with relaxed restrictions on orders of fractional derivatives in the composition identity, and vanishing derivatives of the function f at point a.

Theorem 2.19 *Let $\alpha > \beta \ge 0$, $n = [\alpha]+1$, $m = [\beta]+1$. Let $f \in C^{\alpha}_{a+}[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m-1,\dots,n-2$. Then $f \in C^{\beta}_{a+}[a,b]$ and*

$${}^{C_1}D^{\beta}_{a+}f(x) = \frac{1}{\Gamma(\alpha-\beta)}\int_a^x (x-t)^{\alpha-\beta-1}\,{}^{C_1}D^{\alpha}_{a+}f(t)\,dt\,, \quad x \in [a,b]\,. \tag{2.155}$$

Proof. Let $m = n$, that is $\alpha - n > \beta - m$. By Lemma 2.3 we have

$$J^{\alpha-\beta}_{a+}J^{n-\alpha}_{a+}f^{(n-1)} = J^{n-\beta}_{a+}f^{(n-1)} = J^{m-\beta}_{a+}f^{(m-1)}\,.$$

Since $J^{n-\alpha}_{a+}f^{(n-1)} \in C^1[a,b]$, then $J^{n-\alpha}_{a+}f^{(n-1)}(a) = 0$, and using integrations by parts we have

$$\begin{aligned} &J^{\alpha-\beta+1}_{a+}\,{}^{C_1}D^{\alpha}_{a+}f(x) \\ &= \frac{1}{\Gamma(\alpha-\beta+1)}\int_a^x (x-t)^{\alpha-\beta}\left(\frac{d}{dt}J^{n-\alpha}_{a+}f^{(n-1)}(t)\right)dt \\ &= \frac{1}{\Gamma(\alpha-\beta+1)}\Bigg[(x-t)^{\alpha-\beta}J^{n-\alpha}_{a+}f^{(n-1)}(t)\Big|_a^x \end{aligned}$$

$$+ (\alpha-\beta)\int_a^x (x-t)^{\alpha-\beta-1} J_{a+}^{n-\alpha} f^{(n-1)}(t)\,dt\Bigg]$$
$$= \frac{1}{\Gamma(\alpha-\beta)}\int_a^x (x-t)^{\alpha-\beta-1} J_{a+}^{n-\alpha} f^{(n-1)}(t)\,dt$$
$$= J_{a+}^{\alpha-\beta} J_{a+}^{n-\alpha} f^{(n-1)}(x)\,,$$

that is,

$$J_{a+}^{m-\beta} f^{(m-1)} = J_{a+}^{\alpha-\beta+1}\, {}^{C_1}D_{a+}^{\alpha} f\,. \tag{2.156}$$

Since ${}^{C_1}D_{a+}^{\alpha} f(x) = \frac{d}{dx} J_{a+}^{n-\alpha} f^{(n-1)}(x) \in C[a,b]$ and $\alpha-\beta+1>1$, by Corollary 2.2 follows $J_{a+}^{m-\beta} f^{(m-1)} \in C^1[a,b]$.

If $n>m$, then using $f^{(m-1)}(a)=\cdots=f^{(n-2)}(a)=0$ and integration by parts, we have

$$\begin{aligned}
&J_{a+}^{n-m} f^{(n-1)}(x)\\
&= \frac{1}{\Gamma(n-m)}\int_a^x (x-t)^{n-m-1} f^{(n-1)}(t)\,dt\\
&= \frac{1}{\Gamma(n-m)}\left[(x-t)^{n-m-1} f^{(n-2)}(t)\Big|_a^x + (n-m-1)\int_a^x (x-t)^{n-m-2} f^{(n-2)}(t)\,dt\right]\\
&= \frac{1}{\Gamma(n-m-1)}\int_a^x (x-t)^{n-m-2} f^{(n-2)}(t)\,dt \;=\; \cdots\\
&= \int_a^x f^{(m)}(t)\,dt \;=\; f^{(m-1)}(x)\,.
\end{aligned}$$

Therefore

$$J_{a+}^{m-\beta} f^{(m-1)} = J_{a+}^{m-\beta} J_{a+}^{n-m} f^{(n-1)} = J_{a+}^{n-\beta} f^{(n-1)}.$$

The result again follows from Corollary 2.2 since $n-\beta>1$ and $f^{(n-1)}\in C[a,b]$. This proves that $f\in C_{a+}^{\beta}[a,b]$.

For the proof of the composition identity (2.155) we use the Laplace transform. Set $g=f^{(m-1)}$. Now (2.155) can be written as

$$\frac{d}{dx} J_{a+}^{m-\beta} g(x) = J_{a+}^{\alpha-\beta} \frac{d}{dx} J_{a+}^{n-\alpha} g^{(n-m)}(x)\,,$$

where $x\in[a,b]$ and $g(a)=g'(a)=\cdots=g^{(n-m-1)}(a)=0$. Define auxiliary function $h:[0,\infty)\to\mathbb{R}$ with

$$h(x) = \begin{cases} g(x+a)\,,\ x\in[0,b-a] \\ \sum\limits_{k=0}^{n-m-1} \frac{g^{(k)}(b)}{k!}(x-b+a)^k\,,\ x\geq b-a \end{cases}. \tag{2.157}$$

Obviously $h\in C^{n-m}[0,\infty)$ and $h(0)=h'(0)=\cdots=h^{(n-m-1)}(0)=0$. Also h has polynomial growth at ∞, so the Laplace transform of h exists. The composition identity (2.155) will follow if we prove that for $x\geq 0$ holds

$$\frac{1}{\Gamma(m-\beta)}\frac{d}{dx}\int_0^x (x-t)^{m-\beta-1} h(t)\,dt$$

$$= \frac{1}{\Gamma(\alpha-\beta)\,\Gamma(n-\alpha)} \int_0^x (x-t)^{\alpha-\beta-1} \frac{d}{dt} \int_0^t (t-y)^{n-\alpha-1} h^{(n-m)}(y)\,dy\,dt\,. \tag{2.158}$$

Using standard properties of the Laplace transform, for the left side of the equality (2.158) we have

$$\begin{aligned}
&\mathscr{L}\left[\frac{1}{\Gamma(m-\beta)}\frac{d}{dx}\int_0^x (x-t)^{m-\beta-1} h(t)\,dt\right](p)\\
&= \frac{p}{\Gamma(m-\beta)}\mathscr{L}\left[\int_0^x (x-t)^{m-\beta-1} h(t)\,dt\right](p)\\
&= \frac{p}{\Gamma(m-\beta)}\mathscr{L}\left[x^{m-\beta-1}\right](p)\mathscr{L}[h](p) = p^{-m+\beta+1}\mathscr{L}[h](p)\,.
\end{aligned} \tag{2.159}$$

On the other hand we have

$$\begin{aligned}
&\mathscr{L}\left[\frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-\beta)}\int_0^x (x-t)^{\alpha-\beta-1}\frac{d}{dt}\int_0^t (t-y)^{n-\alpha-1} h^{(n-m)}(y)\,dy\,dt\right](p)\\
&= \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-\beta)}\mathscr{L}\left[x^{\alpha-\beta-1}\right](p)\mathscr{L}\left[\frac{d}{dt}\int_0^t (t-y)^{n-\alpha-1} h^{(n-m)}(y)\,dy\right](p)\\
&= \frac{p^{\beta-\alpha}}{\Gamma(n-\alpha)}\,p\,\mathscr{L}\left[t^{n-\alpha-1}\right](p)\mathscr{L}\left[h^{(n-m)}\right](p)\\
&= p^{\beta-\alpha}\frac{p}{p^{n-\alpha}}\,p^{n-m}\mathscr{L}[h](p) = p^{-m+\beta+1}\mathscr{L}[h](p)\,.
\end{aligned} \tag{2.160}$$

By Theorem 1.3, since both sides have the same Laplace transform (2.159) and (2.160), it follows that the equality in (2.158) holds for every $x \geq 0$. □

In the following theorem we give a proof of the composition identity for the right-sided Canavati fractional derivatives.

Theorem 2.20 *Let $\alpha > \beta \geq 0$, $n = [\alpha]+1$ and $m = [\beta]+1$. Let $f \in C^{\alpha}_{b-}[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m-1,\ldots,n-2$. Then $f \in C^{\beta}_{b-}[a,b]$ and*

$$^{C_1}D^{\beta}_{b-}f(x) = \frac{1}{\Gamma(\alpha-\beta)}\int_x^b (t-x)^{\alpha-\beta-1}\,{}^{C_1}D^{\alpha}_{b-}f(t)\,dt\,, \quad x \in [a,b]\,. \tag{2.161}$$

Proof. As in the previous theorem, analogously we can prove that $f \in C^{\beta}_{b-}[a,b]$. Let write the identity (2.161) as

$$(-1)^m \frac{d}{dx} J^{m-\beta}_{b-} f^{(m-1)}(x) = (-1)^n J^{\alpha-\beta}_{b-}\frac{d}{dx} J^{n-\alpha}_{b-} f^{(n-1)}(x)\,.$$

Let $n = m$. Since $J^{n-\alpha}_{b-} f^{(n-1)} \in C^1[a,b]$, then $J^{n-\alpha}_{b-} f^{(n-1)}(b) = 0$. Using integration by part, analogously to relation (2.156), we get

$$(-1)^{m-1} J^{m-\beta}_{b-} f^{(m-1)} = J^{\alpha-\beta+1}_{b-}\,{}^{C_1}D^{\alpha}_{b-} f\,. \tag{2.162}$$

With this and relation (2.57) from Corollary 2.5, we have

$$\begin{aligned} {}^{C_1}D^{\beta}_{b-}f(x) &= (-1)^m \frac{d}{dx} J^{m-\beta}_{b-} f^{(m-1)} \\ &= (-1)\frac{d}{dx} J^{\alpha-\beta+1}_{b-}\, {}^{C_1}D^{\alpha}_{b-}f(x) \;=\; J^{\alpha-\beta}_{b-}\, {}^{C_1}D^{\alpha}_{b-}f(x)\,. \end{aligned}$$

Let $n > m$. Since $f^{(m-1)}(b) = f^{(m)}(b) = \cdots = f^{(n-2)}(b) = 0$, then

$$\begin{aligned} &J^{n-m}_{b-} f^{(n-1)}(x) \\ &= \frac{1}{\Gamma(n-m)} \int_x^b (t-x)^{n-m-1} f^{(n-1)}(t)\,dt \\ &= \frac{1}{\Gamma(n-m)} \left[(t-x)^{n-m-1} f^{(n-2)}(t)\Big|_x^b - (n-m-1)\int_x^b (t-x)^{n-m-2} f^{(n-2)}(t)\,dt \right] \\ &= \frac{-1}{\Gamma(n-m-1)} \int_x^b (t-x)^{n-m-2} f^{(n-2)}(t)\,dt \;=\; \cdots \\ &= (-1)^{n-m-1} \int_x^b f^{(m)}(t)\,dt \;=\; (-1)^{n-m} f^{(m-1)}(x)\,. \end{aligned}$$

Now, by (i) from Theorem 2.6 follows

$$ {}^{C_1}D^{\beta}_{b-}f(x) = (-1)^m \frac{d}{dx} J^{m-\beta}_{b-} f^{(m-1)}(x) = (-1)^n \frac{d}{dx} J^{n-\beta}_{b-} f^{(n-1)}(x)\,. $$

If we apply Corollary 2.5 and relation (2.57) with $n-\beta > 1$ (since $n > m$), and then again case (i) from Theorem 2.6, we get

$$ {}^{C_1}D^{\beta}_{b-}f(x) = (-1)^{n+1} J^{n-\beta-1}_{b-} f^{(n-1)}(x) = (-1)^{n+1} J^{\alpha-\beta-1}_{b-} J^{n-\alpha}_{b-} f^{(n-1)}(x)\,. $$

If $\alpha-\beta-1 > 0$, then identity (2.161) holds, that is

$$ {}^{C_1}D^{\beta}_{b-}f(x) = (-1)^{n+1} J^{\alpha-\beta-1}_{b-} J^{n-\alpha}_{b-} f^{(n-1)}(x) = (-1)^n J^{\alpha-\beta}_{b-} \frac{d}{dx} J^{n-\alpha}_{b-} f^{(n-1)}(x)\,. $$

If $\alpha-\beta-1 < 0$ then $J^{\alpha-\beta-1}_{b-}\varphi = D^{-\alpha+\beta+1}_{b-}\varphi$ and we use (2.34) from Corollary 2.4

$$ D^{-\alpha+\beta+1}_{b-}\varphi(x) = -J^{\alpha-\beta}_{b-} \frac{d}{dx}\varphi(x), $$

where $\varphi(x) = J^{n-\alpha}_{b-} f^{(n-1)}(x)$ and $\varphi(b) = 0$ since $J^{n-\alpha}_{b-} f^{(n-1)} \in C^1[a,b]$. $\square$

Remark 2.6 We give a simple proof of the previous theorem using the reflection operator Q from Proposition 2.5, and Theorem 2.19:

$$\begin{aligned} {}^{C_1}D^{\beta}_{b-}f &= Q\left(Q\,{}^{C_1}D^{\beta}_{b-}f\right) \;=\; Q\left({}^{C_1}D^{\beta}_{a+}Qf\right) \\ &= Q\left(J^{\alpha-\beta}_{a+}\,{}^{C_1}D^{\alpha}_{a+}Qf\right) \;=\; J^{\alpha-\beta}_{b-}\, Q\left({}^{C_1}D^{\alpha}_{a+}Qf\right) \\ &= J^{\alpha-\beta}_{b-}\,{}^{C_1}D^{\alpha}_{b-}Q(Qf) \;=\; J^{\alpha-\beta}_{b-}\,{}^{C_1}D^{\alpha}_{b-}f\,. \end{aligned}$$

Chapter 3

Jensen-Opial type inequalities

In this chapter we consider extensions and generalizations of Opial's inequalities due to Willett, Godunova, Levin and Rozanova, all obtained using Jensen's inequality. Cauchy type mean value theorems are proved and used in studying Stolarsky type means defined by the obtained inequalities. Also, a method of producing n-exponentially convex and exponentially convex functions is applied. This chapter is based on our results: Andrić, Barbir and Pečarić [20, 21].

3.1 The Godunova-Levin and related inequalities

First inequality is motivated by Willett's inequality (1.20) and Godunova-Levin's inequality (1.21) together with its application given in [5, Theorem 2.17.2]. In a special case, it is an improvement of Willett's inequality (see Remark 3.1). We will use Jensen's inequality for integrals (see for example [67])

$$\phi\left(\frac{\int_\alpha^\beta \psi(t)\,r(t)\,dt}{\int_\alpha^\beta r(t)\,dt}\right) \leq \frac{\int_\alpha^\beta \phi\left(\psi(t)\right) r(t)\,dt}{\int_\alpha^\beta r(t)\,dt}, \tag{3.1}$$

where $\phi(t)$ is convex on $[\gamma,\delta]$, $\gamma \leq \psi(t) \leq \delta$ for all $t \in [\alpha,\beta]$, $r(t) > 0$ and all integrals in (3.1) exist.

Theorem 3.1 *Let f be a convex function on $[0,\infty)$ with $f(0)=0$. Further, let $x \in AC^n[a,b]$ be such that $x^{(i)}(a) = x^{(i)}(b) = 0$, $i = 0,\dots,n-1$, $n \geq 1$. If f is a differentiable function, then the following inequality holds*

$$\int_a^b f'\left(|x(t)|\right)|x^{(n)}(t)|\,dt \;\leq\; \frac{2(n-1)!}{(b-a)^n}\int_a^b f\left(\frac{(b-a)^n|x^{(n)}(t)|}{2(n-1)!}\right)dt\,. \tag{3.2}$$

Proof. We start a proof as in Godunova and Levin's generalizations of Opial's inequality given in [5, Theorem 2.17.1, 2.17.2].

Let $a < \tau < b$. Let

$$\begin{aligned} y(t) &= \int_a^t \int_a^{t_{n-1}} \cdots \int_a^{t_1} |x^{(n)}(s)|\,ds\,dt_1 \cdots dt_{n-1} \\ &= \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}|x^{(n)}(s)|\,ds\,, \end{aligned} \tag{3.3}$$

$t \in [a,\tau]$, so that $y^{(n)}(t) = |x^{(n)}(t)|$ and $y(t) \geq |x(t)|$. It is clear that for each $0 \leq i \leq n-1$, $y^{(i)}(t) \geq 0$ and nondecreasing on $[a,\tau]$. Therefore, in view of $y^{(i)}(a) = 0$, $i = 0,\dots,n-1$, it follows

$$y(t) \;\leq\; \frac{(\tau-a)^{n-1}}{(n-1)!}\,y^{(n-1)}(t)\,, \quad t \in [a,\tau]\,, \tag{3.4}$$

that is

$$y(t) \;\leq\; \frac{(b-a)^{n-1}}{(n-1)!}\,y^{(n-1)}(t)\,, \quad t \in [a,\tau]\,. \tag{3.5}$$

Since f' is an increasing function, it follows

$$\begin{aligned} &\int_a^\tau f'\left(|x(t)|\right)|x^{(n)}(t)|\,dt \\ &\leq \int_a^\tau f'\left(y(t)\right)y^{(n)}(t)\,dt \\ &\leq \int_a^\tau f'\left(\frac{(b-a)^{n-1}}{(n-1)!}\,y^{(n-1)}(t)\right)y^{(n)}(t)\,dt \\ &= \frac{(n-1)!}{(b-a)^{n-1}}\int_a^\tau \frac{d}{dt}\left[f\left(\frac{(b-a)^{n-1}}{(n-1)!}\,y^{(n-1)}(t)\right)\right]dt \\ &= \frac{(n-1)!}{(b-a)^{n-1}}f\left(\frac{(b-a)^{n-1}}{(n-1)!}\,y^{(n-1)}(\tau)\right) \\ &= \frac{(n-1)!}{(b-a)^{n-1}}f\left(\frac{(b-a)^{n-1}}{(n-1)!}\int_a^\tau |x^{(n)}(t)|\,dt\right). \end{aligned} \tag{3.6}$$

Next we consider the interval $[\tau,b]$. Thus, by defining

$$y(t) \;=\; \int_t^b \int_{t_{n-1}}^b \cdots \int_{t_1}^b |x^{(n)}(s)|\,ds\,dt_1 \cdots dt_{n-1}$$

$$= \frac{1}{(n-1)!} \int_t^b (s-t)^{n-1} |x^{(n)}(s)| \, ds, \tag{3.7}$$

$t \in [\tau, b]$, analogously we obtain

$$\int_\tau^b f'(|x(t)|) \, |x^{(n)}(t)| \, dt \le \frac{(n-1)!}{(b-a)^{n-1}} f\left(\frac{(b-a)^{n-1}}{(n-1)!} \int_\tau^b |x^{(n)}(t)| \, dt \right). \tag{3.8}$$

Let τ be such that

$$\int_a^\tau |x^{(n)}(t)| \, dt = \int_\tau^b |x^{(n)}(t)| \, dt = \frac{1}{2} \int_a^b |x^{(n)}(t)| \, dt. \tag{3.9}$$

From (3.6), (3.8), (3.9) and Jensen's inequality (3.1), we have

$$\begin{aligned}
\int_a^b f'(|x(t)|) \, |x^{(n)}(t)| \, dt &\le \frac{2(n-1)!}{(b-a)^{n-1}} f\left(\frac{(b-a)^{n-1}}{2(n-1)!} \int_a^b |x^{(n)}(t)| \, dt \right) \\
&= \frac{2(n-1)!}{(b-a)^{n-1}} f\left(\frac{1}{b-a} \int_a^b \frac{(b-a)^n |x^{(n)}(t)|}{2(n-1)!} \, dt \right) \\
&\le \frac{2(n-1)!}{(b-a)^n} \int_a^b f\left(\frac{(b-a)^n |x^{(n)}(t)|}{2(n-1)!} \right) dt.
\end{aligned}$$

□

Remark 3.1 For $f(x) = x^r$, $r \ge 1$, the inequality (3.2) becomes

$$\int_a^b |x(t)|^{r-1} |x^{(n)}(t)| \, dt \le \frac{(b-a)^{n(r-1)}}{2^{r-1} r \, [(n-1)!]^{r-1}} \int_a^b |x^{(n)}(t)|^r \, dt.$$

Notice, if we use (3.4) instead of (3.5) and apply Jensen's inequality on (3.6), then for $r = 2$ we get

$$\int_a^\tau |x(t)| \, |x^{(n)}(t)| \, dt \le \frac{(\tau-a)^n}{2\,(n-1)!} \int_a^\tau |x^{(n)}(t)|^2 \, dt,$$

which is substantial improvement of Willett's inequality (1.20) on $[a, \tau]$. Specially, when $r = 2$ and $n = 1$, for (3.2) we get

$$\int_a^b |x(t)| \, |x'(t)| \, dt \le \frac{b-a}{4} \int_a^b |x'(t)|^2 \, dt,$$

which is Opial's inequality (1.17) on $[a, b]$.

A special case of the previous theorem for $n = 1$ is given in the next corollary.

Corollary 3.1 *Let f be a convex function on $[0, \infty)$ with $f(0) = 0$. Further, let x be absolutely continuous on $[a, b]$, satisfying $x(a) = x(b) = 0$. If f is a differentiable function, then the following inequality holds*

$$\int_a^b f'(|x(t)|) \, |x'(t)| \, dt \le \frac{2}{b-a} \int_a^b f\left(\frac{(b-a)\,|x'(t)|}{2} \right) dt. \tag{3.10}$$

The same method is used on Theorem 1.13, which produces next generalized inequality.

Theorem 3.2 *Let f be a convex function on $[0,\infty)$ with $f(0)=0$. Let g be convex, nonnegative and increasing on $[0,\infty)$. Let $w'(t)>0$, $t\in[a,\tau]$ with $w(a)=0$. Further, let $x\in AC^n[a,\tau]$ be such that $x^{(i)}(a)=0$, $i=0,\dots,n-1$, $n\geq 1$. If f is a differentiable function, then the following inequality holds*

$$\begin{aligned}&\int_a^\tau w'(t)\,g\left(\frac{(\tau-a)^{n-1}}{(n-1)!}\frac{|x^{(n)}(t)|}{w'(t)}\right)f'\left(w(t)\,g\left(\frac{|x(t)|}{w(t)}\right)\right)dt\\&\leq\frac{1}{w(\tau)}\int_a^\tau f\left(w(\tau)\,g\left(\frac{(\tau-a)^{n-1}}{(n-1)!}\frac{|x^{(n)}(t)|}{w'(t)}\right)\right)w'(t)\,dt\,.\end{aligned}\tag{3.11}$$

Proof.

As in the proof of the previous theorem for $t\in[a,\tau]$ we have $y^{(n)}(t)=|x^{(n)}(t)|$, $y(t)\geq|x(t)|$ and

$$y(t)\;\leq\;\frac{(\tau-a)^{n-1}}{(n-1)!}\,y^{(n-1)}(t)\,,\quad t\in[a,\tau]\,.$$

Since g is increasing, then from Jensen's inequality (3.1) follows

$$\begin{aligned}g\left(\frac{|x(t)|}{w(t)}\right)\;&\leq\;g\left(\frac{y(t)}{w(t)}\right)\;\leq\;g\left(\frac{(\tau-a)^{n-1}}{(n-1)!}\frac{y^{(n-1)}(t)}{w(t)}\right)\\&=g\left(\frac{\frac{(\tau-a)^{n-1}}{(n-1)!}\int_a^t w'(s)\frac{|x^{(n)}(s)|}{w'(s)}\,ds}{\int_a^t w'(s)\,ds}\right)\\&\leq\frac{1}{w(t)}\int_a^t w'(s)\,g\left(\frac{(\tau-a)^{n-1}}{(n-1)!}\frac{y^{(n)}(s)}{w'(s)}\right)ds\,.\end{aligned}$$

Using the above inequality and increasing property of f', we obtain

$$\begin{aligned}&\int_a^\tau w'(t)\,g\left(\frac{(\tau-a)^{n-1}}{(n-1)!}\frac{|x^{(n)}(t)|}{w'(t)}\right)f'\left(w(t)\,g\left(\frac{|x(t)|}{w(t)}\right)\right)dt\\&\leq\int_a^\tau w'(t)\,g\left(\frac{(\tau-a)^{n-1}}{(n-1)!}\frac{y^{(n)}(t)}{w'(t)}\right)f'\left(\int_a^t w'(s)\,g\left(\frac{(\tau-a)^{n-1}}{(n-1)!}\frac{y^{(n)}(s)}{w'(s)}\right)ds\right)dt\\&=\int_a^\tau\frac{d}{dt}\left[f\left(\int_a^t w'(s)\,g\left(\frac{(\tau-a)^{n-1}}{(n-1)!}\frac{y^{(n)}(s)}{w'(s)}\right)ds\right)\right]dt\\&=f\left(\int_a^\tau w'(t)\,g\left(\frac{(\tau-a)^{n-1}}{(n-1)!}\frac{y^{(n)}(t)}{w'(t)}\right)dt\right)\\&=f\left(\int_a^\tau w'(t)\,g\left(\frac{(\tau-a)^{n-1}}{(n-1)!}\frac{|x^{(n)}(t)|}{w'(t)}\right)dt\right).\end{aligned}\tag{3.12}$$

Finally, by Jensen's inequality (3.1), we have

$$\begin{aligned}
&\int_a^\tau w'(t)\,g\left(\frac{(\tau-a)^{n-1}}{(n-1)!}\frac{|x^{(n)}(t)|}{w'(t)}\right) f'\left(w(t)\,g\left(\frac{|x(t)|}{w(t)}\right)\right) dt \\
&\leq f\left(\int_a^\tau w'(t)\,g\left(\frac{(\tau-a)^{n-1}}{(n-1)!}\frac{|x^{(n)}(t)|}{w'(t)}\right) dt\right) \\
&= f\left(\frac{1}{w(\tau)}\int_\alpha^\tau w(\tau)\,w'(t)\,g\left(\frac{(\tau-\alpha)^{n-1}}{(n-1)!}\frac{|x^{(n)}(t)|}{w'(t)}\right) dt\right) \qquad (3.13)\\
&= f\left(\frac{\int_\alpha^\tau w(\tau)\,g\left(\frac{(\tau-\alpha)^{n-1}}{(n-1)!}\frac{|x^{(n)}(t)|}{w'(t)}\right) w'(t)\,dt}{\int_\alpha^\tau w'(t)\,dt}\right) \\
&\leq \frac{1}{w(\tau)}\int_\alpha^\tau f\left(w(\tau)\,g\left(\frac{(\tau-\alpha)^{n-1}}{(n-1)!}\frac{|x^{(n)}(t)|}{w'(t)}\right)\right) w'(t)\,dt\,. \qquad (3.14)
\end{aligned}$$

□

For $n = 1$ we have the next inequality.

Corollary 3.2 *Let f be a convex function on $[0,\infty)$ with $f(0) = 0$. Let g be convex, nonnegative and increasing on $[0,\infty)$. Let $w'(t) > 0$, $t \in [a,\tau]$ with $w(a) = 0$. Further, let x be absolutely continuous on $[a,\tau]$, satisfying $x(a) = 0$. If f is a differentiable function, then the following inequality holds*

$$\begin{aligned}
&\int_a^\tau w'(t)\,g\left(\frac{|x'(t)|}{w'(t)}\right) f'\left(w(t)\,g\left(\frac{|x(t)|}{w(t)}\right)\right) dt \\
&\leq \frac{1}{w(\tau)}\int_a^\tau f\left(w(\tau)\,g\left(\frac{|x'(t)|}{w'(t)}\right)\right) w'(t)\,dt\,. \qquad (3.15)
\end{aligned}$$

We finish this section with a similar inequality (and its special case for $n = 1$) obtain by using $(\tau - a)$ instead of $w(\tau)$ in (3.13). The proof is omitted.

Theorem 3.3 *Let f be a convex function on $[0,\infty)$ with $f(0) = 0$. Let g be convex, nonnegative and increasing on $[0,\infty)$. Let $w'(t) > 0$, $t \in [a,\tau]$ with $w(a) = 0$. Further, let $x \in AC^n[a,\tau]$ be such that $x^{(i)}(a) = 0$, $i = 0,\ldots,n-1$, $n \geq 1$. If f is a differentiable function, then the following inequality holds*

$$\begin{aligned}
&\int_a^\tau w'(t)\,g\left(\frac{(\tau-a)^{n-1}}{(n-1)!}\frac{|x^{(n)}(t)|}{w'(t)}\right) f'\left(w(t)\,g\left(\frac{|x(t)|}{w(t)}\right)\right) dt \\
&\leq \frac{1}{\tau-a}\int_a^\tau f\left((\tau-a)\,w'(t)\,g\left(\frac{(\tau-a)^{n-1}}{(n-1)!}\frac{|x^{(n)}(t)|}{w'(t)}\right)\right) dt\,. \qquad (3.16)
\end{aligned}$$

Corollary 3.3 *Let f be a convex function on $[0,\infty)$ with $f(0)=0$. Let g be convex, nonnegative and increasing on $[0,\infty)$. Let $w'(t)>0$, $t\in[a,\tau]$ with $w(a)=0$. Further, let x be absolutely continuous on $[a,\tau]$, satisfying $x(a)=0$. If f is a differentiable function, then the following inequality holds*

$$\begin{aligned}
&\int_a^\tau w'(t)\,g\left(\frac{|x'(t)|}{w'(t)}\right) f'\left(w(t)\,g\left(\frac{|x(t)|}{w(t)}\right)\right)dt \\
&\le \frac{1}{\tau-a}\int_a^\tau f\left((\tau-a)w'(t)\,g\left(\frac{|x'(t)|}{w'(t)}\right)\right)dt\,. \qquad (3.17)
\end{aligned}$$

3.2 Mean value theorems and exponential convexity

Motivated by the inequalities (1.21), (3.2), (3.10) and (1.22), (3.11), (3.15), we define next functionals:

$$\Phi_1(f)=f\left(\int_a^\tau |x'(t)|\,dt\right)-\int_a^\tau f'(|x(t)|)\,|x'(t)|\,dt\,, \qquad (3.18)$$

where function x is as in Theorem 1.12;

$$\Phi_2(f)=\frac{2}{b-a}\int_a^b f\left(\frac{(b-a)\,|x'(t)|}{2}\right)dt-\int_a^b f'(|x(t)|)\,|x'(t)|\,dt\,, \qquad (3.19)$$

where function x is as in Corollary 3.1;

$$\begin{aligned}
\Phi_3(f) &= \frac{2(n-1)!}{(b-a)^n}\int_a^b f\left(\frac{(b-a)^n\,|x^{(n)}(t)|}{2(n-1)!}\right)dt \\
&\quad -\int_a^b f'(|x(t)|)\,|x^{(n)}(t)|\,dt\,, \qquad (3.20)
\end{aligned}$$

where function x is as in Theorem 3.1;

$$\begin{aligned}
\Phi_4(f) &= f\left(\int_a^\tau w'(t)\,g\left(\frac{|x'(t)|}{w'(t)}\right)dt\right) \\
&\quad -\int_a^\tau w'(t)\,g\left(\frac{|x'(t)|}{w'(t)}\right) f'\left(w(t)\,g\left(\frac{|x(t)|}{w(t)}\right)\right)dt\,, \qquad (3.21)
\end{aligned}$$

where functions g, w and x are as in Theorem 1.13;

$$\begin{aligned}
\Phi_5(f) &= \frac{1}{w(\tau)}\int_a^\tau f\left(w(\tau)\,g\left(\frac{|x'(t)|}{w'(t)}\right)\right)w'(t)\,dt \\
&\quad -\int_a^\tau w'(t)\,g\left(\frac{|x'(t)|}{w'(t)}\right) f'\left(w(t)\,g\left(\frac{|x(t)|}{w(t)}\right)\right)dt\,, \qquad (3.22)
\end{aligned}$$

where functions g, w and x are as in Corollary 3.2;

$$\begin{aligned}\Phi_6(f) &= \frac{1}{w(\tau)}\int_a^\tau f\left(w(\tau)\,g\left(\frac{(\tau-a)^{n-1}}{(n-1)!}\frac{|x^{(n)}(t)|}{w'(t)}\right)\right)w'(t)\,dt \\ &\quad - \int_a^\tau w'(t)\,g\left(\frac{(\tau-a)^{n-1}}{(n-1)!}\frac{|x^{(n)}(t)|}{w'(t)}\right)f'\left(w(t)\,g\left(\frac{|x(t)|}{w(t)}\right)\right)dt\,, \end{aligned} \tag{3.23}$$

where functions g, w and x are as in Theorem 3.2 (and, in all functionals, f is a differentiable function with $f(0)=0$).
If f is a convex function, Theorems 1.12, 1.13, 3.2, 3.2 and Corollaries 3.1, 3.2, imply that $\Phi_i(f)\geq 0$, $i=1,\ldots,6$.

Next we give mean value theorems for the functionals Φ_i. First consider the functional Φ_1. Let $0\leq |x'|\leq M$. It follows

$$0\leq \int_a^\tau |x'(t)|\,dt \leq M(\tau-a)$$

and

$$0\leq |x(t)|\leq \int_a^t |x'(s)|\,ds\leq M(t-a)\leq M(\tau-a)\,.$$

Hence, for the functional Φ_1 let $f: I_1\to\mathbb{R}$ where

$$I_1=[0,M(\tau-a)]\,. \tag{3.24}$$

Analogously, with $|x(t)|\leq M(\tau-a)$ and $|x(t)|\leq M(b-\tau)$, for the functional Φ_2 follows

$$I_2=\left[0,\frac{M(b-a)}{2}\right]. \tag{3.25}$$

For Φ_3 and $0\leq \left|x^{(n)}\right|\leq M$ we have

$$\frac{(b-a)^n\,|x^{(n)}(t)|}{2(n-1)!}\leq \frac{M(b-a)^n}{2(n-1)!}$$

and

$$|x(t)|\leq \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}|x^{(n)}(s)|\,ds\leq \frac{M(t-a)^n}{n!}\leq \frac{M(\tau-a)^n}{n!}$$

from which we obtain

$$I_3=\left[0,\frac{M(b-a)^n}{2n!}\right]. \tag{3.26}$$

Next, for the functional Φ_4 let $0<m\leq w'\leq M_1$, $0\leq |x'|\leq M$ and $g\geq 0$. Then $0\leq \frac{|x'|}{w'}\leq \frac{M}{m}$.
It follows

$$m(\tau-a)\min_{\left[0,\frac{M}{m}\right]} g\leq \int_a^\tau w'(t)\,g\left(\frac{|x'(t)|}{w'(t)}\right)dt\leq M_1(\tau-a)\max_{\left[0,\frac{M}{m}\right]} g\,.$$

Also

$$0 \le \frac{|x(t)|}{w(t)} \le \frac{\int_a^t |x'(s)|\,ds}{\int_a^t w'(s)\,ds} \le \frac{M}{m}.$$

Since obviously $w(t) \le M_1(\tau - a)$, we get

$$0 \le w(t)\,g\left(\frac{|x(t)|}{w(t)}\right) \le M_1(\tau - a) \max_{\left[0,\frac{M}{m}\right]} g.$$

Hence, for the functional Φ_4 we have

$$I_4 = \left[0, M_1(\tau - a) \max_{\left[0,\frac{M}{m}\right]} g\right]. \tag{3.27}$$

For the functional Φ_5 we have

$$0 \le w(\tau)\,g\left(\frac{|x'(t)|}{w'(t)}\right) \le M_1(\tau - a) \max_{\left[0,\frac{M}{m}\right]} g$$

which gives us the same interval, that is

$$I_5 = I_4. \tag{3.28}$$

Finally, for Φ_6 and $0 \le \left|x^{(n)}\right| \le M$ we have

$$w(\tau)\,g\left(\frac{(\tau - a)^{n-1}\,|x^{(n)}(t)|}{(n-1)!\,w'(t)}\right) \le M_1(\tau - a) \max_{\left[0,\frac{M(\tau-a)^{n-1}}{m(n-1)!}\right]} g.$$

Also

$$0 \le \frac{|x(t)|}{w(t)} \le \frac{M(\tau - a)^{n-1}}{mn!}$$

and

$$0 \le w(t)\,g\left(\frac{|x(t)|}{w(t)}\right) \le M_1(\tau - a) \max_{\left[0,\frac{M(\tau-a)^{n-1}}{mn!}\right]} g,$$

from which we obtain

$$I_6 = \left[0, M_1(\tau - a) \max_{\left[0,\frac{M(\tau-a)^{n-1}}{mn!}\right]} g\right]. \tag{3.29}$$

Define next conditions:

$(A1)$ Let x be absolutely continuous on $[a,\tau]$, $x(a) = 0$ and $0 \le |x'| \le M$.

$(A2)$ Let x be absolutely continuous on $[a,b]$, $x(a) = x(b) = 0$ and $0 \le |x'| \le M$.

$(A3)$ Let $x \in AC^n[a,b]$, $x^{(i)}(a) = x^{(i)}(b) = 0$ for $i = 0,\dots,n-1$ $(n \geq 1)$ and $0 \leq \left|x^{(n)}\right| \leq M$.

$(A4)$ Let g be convex, nonnegative and increasing function on $[0,\infty)$. Let $w(t) \geq 0$, $w'(t) > 0$, $t \in [a,\tau]$, $w(a) = 0$ and $0 < m \leq w' \leq M_1$. Further, let x be absolutely continuous on $[a,\tau]$, $x(a) = 0$ and $0 \leq |x'| \leq M$.

$(A5)$ Let g be convex, nonnegative and increasing function on $[0,\infty)$. Let $w(t) \geq 0$, $w'(t) > 0$, $t \in [a,\tau]$, $w(a) = 0$ and $0 < m \leq w' \leq M_1$. Further, let x be absolutely continuous on $[a,\tau]$, $x(a) = 0$ and $0 \leq |x'| \leq M$.

$(A6)$ Let g be convex, nonnegative and increasing function on $[0,\infty)$. Let $w(t) \geq 0$, $w'(t) > 0$, $t \in [a,\tau]$, $w(a) = 0$ and $0 < m \leq w' \leq M_1$. Further, let $x \in AC^n[a,\tau]$, $x^{(i)}(a) = 0$ for $i = 0,\dots,n-1$ $(n \geq 1)$ and $0 \leq \left|x^{(n)}\right| \leq M$.

Theorem 3.4 *Let conditions (Ai) hold for the functionals Φ_i $(i = 1,\dots,6)$, respectively. Let $f \in C^2(I_i)$ and $f(0) = 0$. Then there exists $\xi \in I_i$ such that*

$$\Phi_i(f) = \frac{f''(\xi)}{2}\Phi_i(f_0), \quad (i = 1,\dots,6), \tag{3.30}$$

where $f_0(t) = t^2$.

Proof. We give a proof for the functional Φ_1. Since $f \in C^2(I_1)$, there exist real numbers $m = \min_{t \in I_1} f''(t)$ and $M = \max_{t \in I_1} f''(t)$. It is easy to show that the functions f_1 and f_2 defined by

$$\begin{aligned} f_1(t) &= \frac{M}{2}t^2 - f(t), \\ f_2(t) &= f(t) - \frac{m}{2}t^2 \end{aligned}$$

are convex. Therefore $\Phi_1(f_1) \geq 0$, $\Phi_1(f_2) \geq 0$, and we get

$$\frac{m}{2}\Phi_1(f_0) \leq \Phi_1(f) \leq \frac{M}{2}\Phi_1(f_0).$$

If $\Phi_1(f_0) = 0$ there is nothing to prove. Suppose $\Phi_1(f_0) > 0$. We have

$$m \leq \frac{2\Phi_1(f)}{\Phi_1(f_0)} \leq M.$$

Hence, there exists $\xi \in I_1$ such that

$$\Phi_1(f) = \frac{f''(\xi)}{2}\Phi_1(f_0).$$

□

Theorem 3.5 *Let conditions (Ai) hold for the functionals Φ_i $(i=1,\ldots,6)$, respectively. Let $f,u \in C^2(I_i)$ with $f(0)=u(0)=0$. Then there exists $\xi \in I_i$ such that*

$$\frac{\Phi_i(f)}{\Phi_i(u)} = \frac{f''(\xi)}{u''(\xi)}, \quad (i=1,\ldots,6), \tag{3.31}$$

provided that the denominators are non-zero.

Proof. We give a proof for the functional Φ_1. Define $h \in C^2(I_1)$ by $h = c_1 f - c_2 u$, where

$$c_1 = \Phi_1(u), \;\; c_2 = \Phi_1(f).$$

Now using Theorem 3.4 there exists $\xi \in I_1$ such that

$$\left(c_1 \frac{f''(\xi)}{2} - c_2 \frac{u''(\xi)}{2} \right) \Phi_1(f_0) = 0.$$

Since $\Phi_1(f_0) \neq 0$ (otherwise we have a contradiction with $\Phi_1(u) \neq 0$ by Theorem 3.4), we get

$$\frac{\Phi_1(f)}{\Phi_1(u)} = \frac{f''(\xi)}{u''(\xi)}.$$

□

An elegant method of producing n-exponentially convex and exponentially convex functions is given in [50]. We use this to prove the n-exponential convexity for the functionals Φ_i, $i=1,\ldots,6$. Next theorem is analogous to the one given in [66, Theorem 3.9] and we give a proof for the reader's convenience.

Note here that for functionals Φ_i $(i=1,\ldots,6)$, intervals I are defined by (3.24) – (3.29), respectively.

Theorem 3.6 *Let $\Upsilon = \{f_s : s \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $s \mapsto [y_0,y_1,y_2;f_s]$ is n-exponentially convex in the Jensen sense on J for every three mutually different points $y_0,y_1,y_2 \in I$. Let Φ_i $(i=1,\ldots,6)$ be linear functionals defined as in (3.18) – (3.23). Then $s \mapsto \Phi_i(f_s)$ is n-exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Phi_i(f_s)$ is also continuous on J, then it is n-exponentially convex on J.*

Proof. For $\xi_i \in \mathbb{R}$, $s_i \in J$, $i=1,\ldots,n$, we define the function

$$h(y) = \sum_{i,j=1}^{n} \xi_i \xi_j f_{\frac{s_i+s_j}{2}}(y).$$

Using the assumption that the function $s \mapsto [y_0,y_1,y_2;f_s]$ is n-exponentially convex in the Jensen sense, we have

$$[y_0,y_1,y_2;h] = \sum_{i,j=1}^{n} \xi_i \xi_j [y_0,y_1,y_2;f_{\frac{s_i+s_j}{2}}] \geq 0,$$

which in turn implies that h is a convex function on I. Therefore we have $\Phi_i(h) \geq 0$, $i = 1,\ldots,6$. Hence

$$\sum_{i,j=1}^{n} \xi_i \xi_j \Phi_i(f_{\frac{s_i+s_j}{2}}) \geq 0.$$

We conclude that the function $s \mapsto \Phi_i(f_s)$ is n-exponentially convex on J in the Jensen sense. If the function $s \mapsto \Phi_i(f_s)$ is also continuous on J, then it is n-exponentially convex by definition. □

Corollary 3.4 *Let $\Upsilon = \{f_s : s \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Let Φ_i $(i = 1,\ldots,6)$ be linear functionals defined as in $(3.18) - (3.23)$. Then $s \mapsto \Phi_i(f_s)$ is exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Phi_i(f_s)$ is continuous on J, then it is exponentially convex on J.*

Let us denote means for $f_s, f_q \in \Omega$ by

$$\mu_{s,q}(\Phi_i, \Omega) = \begin{cases} \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)} \right)^{\frac{1}{s-q}} & , s \neq q, \\ \exp\left(\frac{\frac{d}{ds}\Phi_i(f_s)}{\Phi_i(f_s)} \right) & , s = q. \end{cases} \tag{3.32}$$

Theorem 3.7 *Let $\Omega = \{f_s : s \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is $2-$exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Let Φ_i $(i = 1,\ldots,6)$ be linear functionals defined as in $(3.18) - (3.23)$. Then the following statements hold:*

(i) If the function $s \mapsto \Phi_i(f_s)$ is continuous on J, then it is $2-$exponentially convex function on J. If the function $s \mapsto \Phi_i(f_s)$ is additionally positive, then it is also log-convex on J, and for $r, s, t \in J$ such that $r < s < t$, we have

$$(\Phi_i(f_s))^{t-r} \leq (\Phi_i(f_r))^{t-s} (\Phi_i(f_t))^{s-r}, \; i = 1,\ldots,6. \tag{3.33}$$

(ii) If the function $s \mapsto \Phi_i(f_s)$ is positive and differentiable on J, then for every $s, q, r, t \in J$, such that $s \leq r$ and $q \leq t$, we have

$$\mu_{s,q}(\Phi_i, \Omega) \leq \mu_{r,t}(\Phi_i, \Omega), \; i = 1,\ldots,6. \tag{3.34}$$

Proof. (i) The first part is an immediate consequence of Theorem 3.6 and in second part log-convexity on J follows from Remark 1.3. Since $s \mapsto \Phi_i(f_s)$ is positive, for $r, s, t \in J$ such that $r < s < t$, with $f(s) = \log \Phi_i(f_s)$ in Proposition 1.2, we have

$$(t-s)\log \Phi_i(f_r) + (r-t)\log \Phi_i(f_s) + (s-r)\log \Phi_i(f_t) \geq 0.$$

This is equivalent to inequality (3.33).
(ii) The function $s \mapsto \Phi_i(f_s)$ is log-convex on J by (i), that is, the function $s \mapsto \log \Phi_i(f_s)$ is convex on J. Applying Proposition 1.3 we get

$$\frac{\log \Phi_i(f_s) - \log \Phi_i(f_q)}{s-q} \le \frac{\log \Phi_i(f_r) - \log \Phi_i(f_t)}{r-t} \tag{3.35}$$

for $s \le r$, $q \le t$, $s \ne q$, $r \ne t$, and therefore we have

$$\mu_{s,q}(\Phi_i, \Omega) \le \mu_{r,t}(\Phi_i, \Omega).$$

Cases $s = q$ and $r = t$ follows from (3.35) as limit cases. □

Remark 3.2 Results from Theorem 3.6, Corollary 3.4 and Theorem 3.7 still hold when two of the points $y_0, y_1, y_2 \in I$ coincide, for a family of differentiable functions f_s such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense). Furthermore, they still hold when all three points coincide for a family of twice differentiable functions with the same property. The proofs can be obtained by recalling Remark 1.1 and suitable characterization of convexity.

3.3 Applications to Stolarsky type means

We use Cauchy type mean value Theorems 3.4 and 3.5 for Stolarsky type means, defined by the functionals Φ_i, $i = 1, \ldots, 6$. Several families of functions which fulfil conditions of Theorem 3.6, Corollary 3.4 and Theorem 3.7 (and Remark 3.2) that we present here, enable us to construct large families of functions which are exponentially convex.

Example 3.1 Consider a family of functions

$$\Omega_1 = \{f_s : \mathbb{R} \to \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$f_s(x) = \begin{cases} \frac{e^{sx}-1}{s^2}, & s \ne 0, \\ \frac{x^2}{2}, & s = 0. \end{cases}$$

Since $\frac{d^2 f_s}{dx^2}(x) = e^{sx} > 0$, then f_s is convex on $\mathbb{R}$ for every $s \in \mathbb{R}$, and $s \mapsto \frac{d^2 f_s}{dx^2}(x)$ is exponentially convex by definition.
Analogously as in the proof of Theorem 3.6 we conclude that $s \mapsto [y_0, y_1, y_2; f_s]$ is exponentially convex (and so exponentially convex in the Jensen sense).

Notice that $f_s(0)=0$. By Corollary 3.4 we have that $s\mapsto\Phi_i(f_s)$ $(i=1,\ldots,6)$ is exponentially convex in the Jensen sense. It is easy to verify that this mappings are continuous (although mapping $s\mapsto f_s$ is not continuous for $s=0$), so they are exponentially convex.

For this family of functions, $\mu_{s,q}(\Phi_i,\Omega_1)$ $(i=1,\ldots,6)$ from (3.32) become

$$\mu_{s,q}(\Phi_i,\Omega_1)=\begin{cases}\left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}}, & s\neq q,\\ \exp\left(\frac{\Phi_i(id\cdot f_s)}{\Phi_i(f_s)}-\frac{2}{s}\right), & s=q\neq 0,\\ \exp\left(\frac{\Phi_i(id\cdot f_0)}{3\Phi_i(f_0)}\right), & s=q=0,\end{cases}$$

and using (3.34), they are monotonous functions in parameters s and q.

Consider for example the functional Φ_1. If Φ_1 is positive, then Theorem 3.5 applied for $f=f_s\in\Omega_1$ and $u=f_q\in\Omega_1$ yields that there exists $\xi\in I_1=[0,M(\tau-a)]$ such that

$$e^{(s-q)\xi}=\frac{\Phi_1(f_s)}{\Phi_1(f_q)}.$$

It follows that

$$M_{s,q}(\Phi_1,\Omega_1)\;=\;\log\mu_{s,q}(\Phi_1,\Omega_1)$$

satisfy $0\le M_{s,q}(\Phi_1,\Omega_1)\le M(\tau-a)$, which shows that $M_{s,q}(\Phi_1,\Omega_1)$ is a mean, and by (3.34) it is a monotonous mean. Analogously follows

$$0\;\le\;M_{s,q}(\Phi_2,\Omega_1)\;\le\;\frac{M(b-a)}{2},\tag{3.36}$$

$$0\;\le\;M_{s,q}(\Phi_3,\Omega_1)\;\le\;\frac{M(b-a)^n}{2n!},\tag{3.37}$$

$$0\;\le\;M_{s,q}(\Phi_4,\Omega_1)\;\le M_1(\tau-a)\max_{[0,\frac{M}{m}]}g,\tag{3.38}$$

$$0\;\le\;M_{s,q}(\Phi_5,\Omega_1)\;\le\;M_1(\tau-a)\max_{[0,\frac{M}{m}]}g,\tag{3.39}$$

$$0\;\le\;M_{s,q}(\Phi_6,\Omega_1)\;\le\;M_1(\tau-a)\max_{\left[0,\frac{M(\tau-a)^{n-1}}{mn!}\right]}g,\tag{3.40}$$

hence, $M_{s,q}(\Phi_i,\Omega_1)$ are also monotonous means, $i=2,\ldots,6$.

Example 3.2 Consider a family of functions

$$\Omega_2=\{g_s:[0,\infty)\to\mathbb{R}:s\in\mathbb{R}\}$$

defined by

$$g_s(x)=\begin{cases}\frac{(x+1)^s-1}{s(s-1)}, & s\neq 0,1,\\ -\log(x+1), & s=0,\\ (x+1)\log(x+1), & s=1.\end{cases}$$

Here, $\frac{d^2 g_s}{dx^2}(x) = (x+1)^{s-2} = e^{(s-2)\log(x+1)} > 0$ which shows that g_s is convex for $x > 0$ and $s \mapsto \frac{d^2 g_s}{dx^2}(x)$ is exponentially convex by definition. Also, $g_s(0) = 0$. Arguing as in Example 3.1 we get that the mapping $s \mapsto \Phi_i(g_s)$ is exponentially convex and also log-convex. Hence, for $r,s,t \in J$ such that $r < s < t$, we have

$$(\Phi_i(g_s))^{t-r} \leq (\Phi_i(g_r))^{t-s} (\Phi_i(g_t))^{s-r}, \ i = 1,\ldots,6. \tag{3.41}$$

Particularly we observe the functional

$$\Phi_2(f) = \frac{2}{b-a}\int_a^b f\left(\frac{(b-a)|x'(t)|}{2}\right) dt - \int_a^b f'(|x(t)|)|x'(t)|\,dt$$

and obtain

$$\Phi_2(g_s) = \begin{cases} -\frac{2}{s(s-1)} + \frac{2}{(b-a)s(s-1)} \int\limits_a^b \left(\frac{(b-a)|x'(t)|}{2} + 1\right)^s dt \\ -\frac{1}{s-1}\int\limits_a^b (|x(t)|+1)^{s-1}|x'(t)|\,dt, & s \neq 0,1, \\ -\frac{2}{b-a}\int\limits_a^b \log\left(\frac{(b-a)|x'(t)|}{2} + 1\right) dt + \int\limits_a^b \frac{|x'(t)|}{|x(t)|+1}\,dt, & s = 0, \\ \frac{2}{b-a}\int\limits_a^b \left(\frac{(b-a)|x'(t)|}{2} + 1\right) \log\left(\frac{(b-a)|x'(t)|}{2} + 1\right) dt \\ -\int\limits_a^b |x'(t)|[\log(|x(t)|+1)+1]\,dt, & s = 1. \end{cases}$$

Further, for this family of functions, $\mu_{s,q}(\Phi_i,\Omega_2)$ from (3.32) is equal to

$$\mu_{s,q}(\Phi_i,\Omega_2) = \begin{cases} \left(\frac{\Phi_i(g_s)}{\Phi_i(g_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{1-2s}{s(s-1)} - \frac{\Phi_i(g_0 g_s)}{\Phi_i(g_s)} - \frac{1}{s(s-1)}\frac{\Phi_i(g_0)}{\Phi_i(g_s)}\right), & s = q \neq 0,1, \\ \exp\left(1 - \frac{\Phi_i(g_0^2)}{2\Phi_i(g_0)}\right), & s = q = 0, \\ \exp\left(-1 - \frac{\Phi_i(g_0 g_1)}{2\Phi_i(g_1)}\right), & s = q = 1, \end{cases}$$

and by (3.34) it is monotonous in parameters s and q.

Hence, for the functional Φ_2 we get

$$
\mu_{s,q}(\Phi_2,\Omega_2) = \begin{cases}
\left(\dfrac{q(q-1)\left[-2+\frac{2}{b-a}\int_a^b\left(\frac{(b-a)|x'(t)|}{2}+1\right)^s dt - s\int_a^b(|x(t)|+1)^{s-1}|x'(t)|dt\right]}{s(s-1)\left(-2+\frac{2}{b-a}\int_a^b\left(\frac{(b-a)|x'(t)|}{2}+1\right)^q dt - q\int_a^b(|x(t)|+1)^{q-1}|x'(t)|dt\right]} \right)^{\frac{1}{s-q}}, \\
s \neq q, \\
\exp\left(\dfrac{1-2s}{s(s-1)} + \dfrac{\frac{2}{b-a}\int_a^b \log\left(\frac{(b-a)|x'(t)|}{2}+1\right)\left(\frac{(b-a)|x'(t)|}{2}+1\right)^s dt}{-2+\int_a^b\left[\frac{2}{a-b}\left(\frac{(b-a)|x'(t)|}{2}+1\right)^s - s(|x(t)|+1)^{s-1}|x'(t)|\right]dt} \right. \\
\left. - \dfrac{\int_a^b(|x(t)|+1)^{s-1}|x'(t)|(1+s\log(|x(t)|+1))dt}{-2+\int_a^b\left[\frac{2}{a-b}\left(\frac{(b-a)|x'(t)|}{2}+1\right)^s - s(|x(t)|+1)^{s-1}|x'(t)|\right]dt} \right), \\
s = q \neq 0,1, \\
\exp\left(1 - \dfrac{\int_a^b\left[\log^2\left(\frac{(b-a)|x'(t)|}{2}+1\right) - (b-a)\log(|x(t)|+1)\frac{|x'(t)|}{|x(t)|+1}\right]dt}{\int_a^b\left[\frac{(b-a)|x'(t)|}{|x(t)|+1} - 2\log\left(\frac{(b-a)|x'(t)|}{2}+1\right)\right]dt} \right), \\
s = q = 0, \\
\exp\left(-1 + \dfrac{\frac{2}{b-a}\int_a^b\left(\frac{(b-a)|x'(t)|}{2}+1\right)\log^2\left(\frac{(b-a)|x'(t)|}{2}+1\right)dt}{2\int_a^b\left[\frac{2}{b-a}\left(\frac{(b-a)|x'(t)|}{2}+1\right)\log\left(\frac{(b-a)|x'(t)|}{2}+1\right) - |x'(t)|(\log(|x(t)|+1)+1)\right]dt} \right. \\
\left. - \dfrac{\int_a^b |x'(t)|\left(\log^2(|x(t)|+1)+2\log(|x(t)|+1)\right)dt}{2\int_a^b\left[\frac{2}{b-a}\left(\frac{(b-a)|x'(t)|}{2}+1\right)\log\left(\frac{(b-a)|x'(t)|}{2}+1\right) - |x'(t)|(\log(|x(t)|+1)+1)\right]dt} \right), \\
s = q = 1.
\end{cases}
$$

Applying Theorem 3.5 it follows that there exists $\xi \in I_2$ such that

$$(\xi+1)^{s-q} = \frac{\Phi_2(g_s)}{\Phi_2(g_q)}.$$

Since the function $\xi \mapsto (\xi+1)^{s-q}$ is invertible for $s \neq q$, we have

$$0 \leq \left(\frac{\Phi_2(g_s)}{\Phi_2(g_q)}\right)^{\frac{1}{s-q}} \leq \frac{M(b-a)}{2}$$

which together with the fact that $\mu_{s,q}(\Phi_2,\Omega_2)$ is continuous, symmetric and monotonous, shows that $\mu_{s,q}(\Phi_2,\Omega_2)$ is a mean (analogously follows for $m_{s,q}(\Phi_i,\Omega_2)$, $i = 1,3,\ldots,6$).

Example 3.3 Consider a family of functions

$$\Omega_3 = \{h_s : [0,\infty) \to \mathbb{R} : s > 0\}$$

defined by

$$h_s(x) = \begin{cases} \frac{s^{-x}-1}{\log^2 s}, & s \neq 1, \\ \frac{x^2}{2}, & s = 1. \end{cases}$$

Since $s \mapsto \frac{d^2h_s}{dx^2}(x) = s^{-x}$ is the Laplace transform of a nonnegative function ([74]), that is $s^{-x} = \dfrac{1}{\Gamma(x)}\displaystyle\int_0^\infty e^{-st}t^{x-1}\,dt$, it is exponentially convex on $(0,\infty)$. Obviously h_s are convex functions for every $s > 0$ and $h_s(0) = 0$.
For this family of functions, $\mu_{s,q}(\Phi_i,\Omega_3)$ from (3.32) is equal to

$$\mu_{s,q}(\Phi_i,\Omega_3) = \begin{cases} \left(\frac{\Phi_i(h_s)}{\Phi_i(h_q)}\right)^{\frac{1}{s-q}}, & s \neq q\,, \\ \exp\left(-\frac{\Phi_i(id\cdot h_s)}{s\Phi_i(h_s)} - \frac{2}{s\log s}\right), & s = q \neq 1\,, \\ \exp\left(-\frac{\Phi_i(id\cdot h_1)}{3\Phi_i(h_1)}\right), & s = q = 1\,, \end{cases}$$

and it is monotonous in parameters s and q by (3.34).

Applying Theorem 3.5 it follows that there exists $\xi \in I_1$ such that

$$\left(\frac{s}{q}\right)^{-\xi} = \frac{\Phi_1(h_s)}{\Phi_1(h_q)}.$$

Hence,

$$M_{s,q}(\Phi_1,\Omega_3) = -L(s,q)\log\mu_{s,q}(\Phi_1,\Omega_3),$$

satisfies $0 \leq M_{s,q}(\Phi_1,\Omega_3) \leq M(\tau - a)$, which shows that $M_{s,q}(\Phi_1,\Omega_3)$ is a mean.
$L(s,q)$ is the logarithmic mean defined by

$$L(s,q) = \begin{cases} \frac{s-q}{\log s - \log q}\,, & s \neq q\,, \\ s\,, & s = q\,. \end{cases}$$

Analogously, for the family Ω_3, we get as in (3.36) – (3.40), that is, $M_{s,q}(\Phi_i,\Omega_3)$ are also monotonous means, $i = 2,\ldots,6$.

Example 3.4 Consider a family of functions

$$\Omega_4 = \{k_s : [0,\infty) \to \mathbb{R} : s > 0\}$$

defined by

$$k_s(x) = \frac{e^{-x\sqrt{s}} - 1}{s}.$$

Again we conclude, since $s \mapsto \frac{d^2k_s}{dx^2}(x) = e^{-x\sqrt{s}}$ is the Laplace transform of a nonnegative function ([74]), that is $e^{-x\sqrt{s}} = \dfrac{s}{2\sqrt{\pi}}\displaystyle\int_0^\infty \frac{e^{-st}e^{-x^2/4t}}{t\sqrt{t}}\,dt$, it is exponentially convex on $(0,\infty)$. For every $s > 0$, k_s are convex functions and $k_s(0) = 0$.
For this family of functions, $\mu_{s,q}(\Phi_i,\Omega_4)$ from (3.32) is equal to

$$\mu_{s,q}(\Phi_i,\Omega_4) = \begin{cases} \left(\frac{\Phi_i(k_s)}{\Phi_i(k_q)}\right)^{\frac{1}{s-q}}, & s \neq q\,, \\ \exp\left(-\frac{\Phi_i(id\cdot k_s)}{2\sqrt{s}\Phi_i(k_s)} - \frac{1}{s}\right), & s = q\,, \end{cases}$$

and by (3.34) it is monotonous in parameters s and q.

Applying Theorem 3.5 it follows that there exists $\xi \in I_1$ such that

$$e^{-\xi(\sqrt{s}-\sqrt{q})} = \frac{\Phi_i(k_s)}{\Phi_i(k_q)}.$$

Hence,

$$M_{s,q}(\Phi_1,\Omega_4) = -\left(\sqrt{s}+\sqrt{q}\right)\log \mu_{s,q}(\Phi_1,\Omega_4)$$

satisfies $0 \le M_{s,q}(\Phi_1,\Omega_4) \le M(\tau - a)$, which shows that $M_{s,q}(\Phi_1,\Omega_4)$ is a mean. Analogously, for the family Ω_4, follows (3.36) – (3.40), that is, $M_{s,q}(\Phi_i,\Omega_4)$ are also monotonous means, $i = 2,\dots,6$.

3.4 Multidimensional generalizations of Opial-type inequalities

In this section Willett's (Theorem 3.1) and Rozanova's (Theorem 3.3) generalizations of Opial's inequality are extended to multidimensional inequalities.

For this we use following notation:

Let $\Omega = \prod_{j=1}^{m}[a_j,b_j]$ and $|\Omega| = \prod_{j=1}^{m}(b_j - a_j)$. Let $t = (t_1,\dots,t_m)$ be a general point in Ω, $\Omega_t = \prod_{j=1}^{m}[a_j,t_j]$ and $dt = dt_1\dots dt_m$. Further, let $Du(x) = \frac{d}{dx}u(x)$, $D_k u(t_1,\dots,t_m) = \frac{\partial}{\partial t_k}u(t_1,\dots,t_m)$ and $D^k u(t_1,\dots,t_m) = D_1\cdots D_k u(t_1,\dots,t_m)$, $1 \le k \le m$. Let $\Omega' = \prod_{j=2}^{m}[a_j,b_j]$ and $dt' = dt_2\dots,dt_m$. Let $D^{j,l}u(t_1,\dots,t_m) = \frac{\partial^{jl}}{\partial t_j^l \dots \partial t_1^l}u(t_1,\dots,t_m)$, $1 \le j \le m$, $1 \le l \le n$. Also, by $C^{mn}(\Omega)$ we denote the space of all functions u on Ω which have continuous derivatives $D^{j,l}u$ for $j = 1,\dots,m$ and $l = 1,\dots,n$.

In order to generalized Theorem 3.1 and Theorem 3.2, we follow idea of the next theorem by Brnetić-Pečarić ([35]) which presents multidimensional inequality.

Theorem 3.8 *Let $m \ge 2$ and let x_i, $D^j x_i$, $i = 1,\dots,p$, $j = 1,\dots,m$, be real-valued continuous functions on Ω with*

$$x_i(t)|_{t_j=a_j} = 0, \quad i = 1,\dots,p,\ j = 1,\dots,m$$

or

$$x_i(t)|_{t_1=a_1} = D^1 x_i(t)|_{t_2=a_2} = \dots = D^{m-1}x_i(t)|_{t_m=a_m} = 0, \quad i = 1,\dots,p.$$

Let f be a nonnegative and differentiable function on $[0,\infty)^p$ with $f(0,\dots,0) = 0$ such that $D_i f$, $i = 1,\dots,p$, are nonnegative, continuous and nondecreasing on $[0,\infty)^p$. Then the integral inequality holds

$$\int_{\Omega} \left(\sum_{i=1}^{p} D_i f\left(|x_1(t)|,\dots,|x_p(t)|\right) |D^m x_i(t)| \right) dt$$

$$\leq f\left(\int_\Omega |D^m x_1(t)|\, dt, \ldots, \int_\Omega \left|D^m x_p(t)\right| dt\right) \tag{3.42}$$

holds.

For proof, we will use next lemma about convex function of several variables ([67, page 11]).

Lemma 3.1 *Suppose f is defined on the open convex set $U \subset \mathbb{R}^n$. If f is convex (strictly) on U and the gradient vector $f'(x)$ exists throughout U, then f' is (strictly) increasing on U.*

First theorem is a generalization of Theorem 3.8.

Theorem 3.9 *Let $m,n,p \in \mathbb{N}$. Let f be a nonnegative and differentiable function on $[0,\infty)^p$, with $f(0,\ldots,0) = 0$. Further, for $i = 1,\ldots,p$ let $x_i \in C^{mn}(\Omega)$ be such that $D^{j,l}x_i(t)|_{t_j=a_j} = 0$, where $j = 1,\ldots,m$ and $l = 0,\ldots,n-1$. Also, let $D_i f$, $i = 1,\ldots,p$, be nonnegative, continuous and nondecreasing on $[0,\infty)^p$. Then the following inequality holds*

$$\begin{aligned}
&\int_\Omega \left(\sum_{i=1}^p D_i f\left(|x_1(t)|,\ldots,\left|x_p(t)\right|\right) |D^{m,n} x_i(t)|\right) dt \\
&\leq \frac{(n-1)!^m}{|\Omega|^{n-1}} f\left(\frac{|\Omega|^{n-1}}{(n-1)!^m} \int_\Omega |D^{m,n} x_1(t)|\, dt, \ldots,\right. \\
&\qquad \left.\frac{|\Omega|^{n-1}}{(n-1)!^m} \int_\Omega \left|D^{m,n} x_p(t)\right| dt\right).
\end{aligned} \tag{3.43}$$

Proof. We extend technique used in Theorem 3.1 and Theorem 3.2 on multidimensional case. For continuous function $g : \Omega \to \mathbb{R}$ we should define $y : \Omega \to \mathbb{R}$ such that

$$D^{m,n} y(x_1,\ldots,x_m) = \frac{\partial^{mn} y}{\partial x_m^n \cdots \partial x_1^n} = g(x_1,\ldots,x_m) \tag{3.44}$$

and

$$y(x_1,\ldots,x_m) = \frac{1}{(n-1)!^m} \int_{\Omega_x} \prod_{j=1}^m (x_j - t_j)^{n-1} g(t_1,\ldots,t_m)\, dt_1 \cdots dt_m, \tag{3.45}$$

where $\Omega_x = \prod_{j=1}^m [a_j, x_j]$.
Define

$$y(x) = \int_a^x dt^1 \int_a^{t^1} dt^2 \cdots \int_a^{t^{n-2}} dt^{n-1} \int_a^{t^{n-1}} g(t^n)\, dt^n \tag{3.46}$$

or, in different notations

$$y(x) = \int_{\Omega_x} dt^1 \int_{\Omega_{t^1}} dt^2 \cdots \int_{\Omega_{t^{n-2}}} dt^{n-1} \int_{\Omega_{t^{n-1}}} g(t^n)\, dt^n, \tag{3.47}$$

where $a=(a_1,\ldots,a_m)$, $x=(x_1,\ldots,x_m)$, $t^i=(t_1^i,\ldots,t_m^i)$, $dt^i=dt_1^i\cdots dt_m^i$, $i=1,\ldots,n$ and $\Omega_{t^i}=\prod_{j=1}^m[a_j,t_j^i]$, $\Omega_{t^i}\subseteq\Omega_{t^{i-1}}$, $i=1,\ldots,n-1$.
Since g is a continuous function, (3.44) obviously follows.
Obviously, integrals on the right-hand side of (3.46) or (3.47), can be written as iterations of the integrals of the form

$$\int_{a_j}^{x_j} dt_j^1 \int_{a_j}^{t_j^1} dt_j^2 \cdots \int_{a_j}^{t_j^{n-2}} dt_j^{n-1} \int_{a_j}^{t_j^{n-1}} \widetilde{g}(t_j^n)\,dt_j^n\,,$$

which are known (and easy to deduce by interchanging the order of integration) to be equal to

$$\frac{1}{(n-1)!}\int_{a_j}^{x_j}(x_j-t_j^n)^{n-1}\widetilde{g}(t_j^n)\,dt_j^n\,,$$

$j=1,\ldots,m$, from which (3.45) easily follows.

Let

$$y_i(t)=\frac{1}{(n-1)!^m}\int\limits_{\Omega_t}\prod_{j=1}^m(t_j-s_j)^{n-1}|D^{m,n}x_i(s)|\,ds\,, \tag{3.48}$$

for $t\in\Omega$, $i=1,\ldots,p$. Hence $D^{m,n}y_i(t)=|D^{m,n}x_i(t)|$ and $y_i(t)\ge|x_i(t)|$. It is easy to conclude that for each $l=0,\ldots,n-1$ we have $D^{j,l}y_i(t)\ge 0$ and nondecreasing on Ω ($i=1,\ldots,p$ and $j=1,\ldots,m$). From $D^{j,l}y_i(t)|_{t_j=a_j}=0$ follows

$$y_i(t)\le\frac{|\Omega|^{n-1}}{(n-1)!^m}D^{m,n-1}y_i(t)\,,\quad t\in\Omega\,.$$

Define

$$u_i(t)=\frac{|\Omega|^{n-1}}{(n-1)!^m}D^{m,n-1}y_i(t)$$

for $t\in\Omega$ and $i=1,\ldots,p$. Since D_if are nonnegative, continuous and nondecreasing on $[0,\infty)^p$, it follows

$$\begin{aligned}
&\int\limits_{\Omega}\left[\sum_{i=1}^p D_if\left(|x_1(t)|,\ldots,|x_p(t)|\right)|D^{m,n}x_i(t)|\right]dt\\
&\le\int\limits_{\Omega}\left[\sum_{i=1}^p D_if\left(y_1(t),\ldots,y_p(t)\right)D^{m,n}y_i(t)\right]dt\\
&\le\int\limits_{\Omega}\left[\sum_{i=1}^p D_if\left(\frac{|\Omega|^{n-1}}{(n-1)!^m}D^{m,n-1}y_1(t),\ldots,\frac{|\Omega|^{n-1}}{(n-1)!^m}D^{m,n-1}y_p(t)\right)\right.\\
&\qquad\left.D^{m,n}y_i(t)\right]dt\\
&\le\int_{a_1}^{b_1}\left[\sum_{i=1}^p D_if\left(u_1(t_1,b_2,\ldots,b_m),\ldots,u_p(t_1,b_2,\ldots,b_m)\right)\times\int\limits_{\Omega'}D^{m,n}y_i(t)\,dt'\right]dt_1
\end{aligned}\tag{3.49}$$

$$\leq \int_{a_1}^{b_1} \left[\sum_{i=1}^{p} D_i f\left(u_1(t_1,b_2,\dots,b_m),\dots,u_p(t_1,b_2,\dots,b_m)\right) \right.$$
$$\left. \frac{(n-1)!^m}{|\Omega|^{n-1}} D_1 u_i(t_1,b_2\dots,b_m) \right] dt_1$$
$$= \frac{(n-1)!^m}{|\Omega|^{n-1}} \int_{a_1}^{b_1} \frac{d}{dt_1} \left[f\left(u_1(t_1,b_2,\dots,b_m),\dots,u_p(t_1,b_2,\dots,b_m)\right)\right] dt_1$$
$$= \frac{(n-1)!^m}{|\Omega|^{n-1}} f\left(u_1(b_1,b_2,\dots,b_m),\dots,u_p(b_1,b_2,\dots,b_m)\right)$$
$$= \frac{(n-1)!^m}{|\Omega|^{n-1}} f\left(\frac{|\Omega|^{n-1}}{(n-1)!^m} \int_\Omega |D^{m,n}x_1(t)|\,dt,\dots, \right.$$
$$\left. \frac{|\Omega|^{n-1}}{(n-1)!^m} \int_\Omega \left|D^{m,n}x_p(t)\right| dt \right). \tag{3.50}$$

□

Remark 3.3 For $n = 1$ the inequality (3.43) becomes the inequality (3.42).

Next we proceed with inequality for convex function f.

Theorem 3.10 *Let $m,n,p \in \mathbb{N}$. Let f be a convex and differentiable function on $[0,\infty)^p$ with $f(0,\dots,0) = 0$. Further, for $i = 1,\dots,p$ let $x_i \in C^{mn}(\Omega)$ be such that $D^{j,l}x_i(t)|_{t_j=a_j} =$ 0, where $j = 1,\dots,m$ and $l = 0,\dots,n-1$. Then the following inequality holds*

$$\int_\Omega \left(\sum_{i=1}^{p} D_i f\left(|x_1(t)|,\dots,|x_p(t)|\right) |D^{m,n}x_i(t)| \right) dt$$
$$\leq \frac{(n-1)!^m}{|\Omega|^n} \int_\Omega f\left(\frac{|\Omega|^n}{(n-1)!^m} |D^{m,n}x_1(t)|,\dots, \right.$$
$$\left. \frac{|\Omega|^n}{(n-1)!^m} \left|D^{m,n}x_p(t)\right| \right) dt. \tag{3.51}$$

Proof. As in the proof of the previous theorem we obtain (3.43) with the difference of applying Lemma 3.1 in (3.49) since f is a convex function. Then, from Jensen's inequality (3.1), we have

$$\int_\Omega \left[\sum_{i=1}^{p} D_i f\left(|x_1(t)|,\dots,|x_p(t)|\right) |D^{m,n}x_i(t)| \right] dt$$
$$\leq \frac{(n-1)!^m}{|\Omega|^{n-1}} f\left(\frac{|\Omega|^{n-1}}{(n-1)!^m} \int_\Omega |D^{m,n}x_1(t)|\,dt,\dots, \right.$$

$$\left.\frac{|\Omega|^{n-1}}{(n-1)!^m}\int_\Omega \left|D^{m,n}x_p(t)\right| dt\right)$$

$$= \frac{(n-1)!^m}{|\Omega|^{n-1}} f\left(\frac{1}{|\Omega|}\int_\Omega \frac{|\Omega|^n}{(n-1)!^m}\left|D^{m,n}x_1(t)\right| dt, \ldots,\right.$$

$$\left.\frac{1}{|\Omega|}\int_\Omega \frac{|\Omega|^n}{(n-1)!^m}\left|D^{m,n}x_p(t)\right| dt\right)$$

$$\le \frac{(n-1)!^m}{|\Omega|^n}\int_\Omega f\left(\frac{|\Omega|^n}{(n-1)!^m}\left|D^{m,n}x_1(t)\right|, \ldots,\right.$$

$$\left.\frac{|\Omega|^n}{(n-1)!^m}\left|D^{m,n}x_p(t)\right|\right) dt\,.$$

□

Remark 3.4 As a special case for $p = 1$ and $m = 1$, Theorem 3.1 is reobtained.

Next theorem is a multidimensional generalization of Theorem 3.3.

Theorem 3.11 *Let $m,n,p \in \mathbb{N}$. Let f be a convex and differentiable function on $[0,\infty)^p$ with $f(0,\ldots,0) = 0$. Let g_i be convex, nonnegative and increasing on $[0,\infty)$ for $i = 1,\ldots,p$. For $i = 1,\ldots,p$ let $h_i : \Omega \to [0,\infty)$ be such that $D^m h_i$ is nonnegative with $D^{j-1}h_i(t)|_{t_j=a_j} = 0$, $j = 1,\ldots,m$. Further, for $i = 1,\ldots,p$ let $x_i \in C^{mn}(\Omega)$ be such that $D^{j,l}x_i(t)|_{t_j=a_j} = 0$, where $j = 1,\ldots,m$ and $l = 0,\ldots,n-1$. Then the following inequality holds*

$$\int_\Omega \left(\sum_{i=1}^p D_i f\left(h_1(t)g_1\left(\frac{|x_1(t)|}{h_1(t)}\right), \ldots, h_p(t)g_p\left(\frac{|x_p(t)|}{h_p(t)}\right)\right)\right.$$

$$\left.\times D^m h_i(t) g_i\left(\frac{|\Omega|^{n-1}}{(n-1)!^m}\frac{|D^{m,n}x_i(t)|}{D^m h_i(t)}\right)\right) dt$$

$$\le \frac{1}{|\Omega|}\int_\Omega f\left(|\Omega|\, D^m h_1(t)\, g_1\left(\frac{|\Omega|^{n-1}}{(n-1)!^m}\frac{|D^{m,n}x_1(t)|}{D^m h_1(t)}\right), \ldots,\right.$$

$$\left.|\Omega|\, D^m h_p(t)\, g_p\left(\frac{|\Omega|^{n-1}}{(n-1)!^m}\frac{|D^{m,n}x_p(t)|}{D^m h_p(t)}\right)\right) dt. \tag{3.52}$$

Proof. As in the proof of Theorem 3.9, for $i = 1,\ldots,p$, $t \in \Omega$ we have $D^{m,n}y_i(t) = |D^{m,n}x_i(t)|$, $y_i(t) \ge |x_i(t)|$ and

$$y_i(t) \le \frac{|\Omega|^{n-1}}{(n-1)!^m} D^{m,n-1} y_i(t).$$

From Jensen's inequality, monotonicity and convexity of each g_i $(i=1,\ldots,p)$, we have

$$\begin{aligned}
g_i\left(\frac{|x_i(t)|}{h_i(t)}\right) &\le g_i\left(\frac{y_i(t)}{h_i(t)}\right) \le g_i\left(\frac{|\Omega|^{n-1}}{(n-1)!^m}\frac{D^{m,n-1}y_i(t)}{h_i(t)}\right)\\
&= g_i\left(\frac{\frac{|\Omega|^{n-1}}{(n-1)!^m}\int_{\Omega_t} D^m h_i(s)\frac{|D^{m,n}x_i(s)|}{D^m h_i(s)}ds}{\int_{\Omega_t} D^m h_i(s)ds}\right)\\
&\le \frac{1}{h_i(t)}\int_{\Omega_t} D^m h_i(s) g_i\left(\frac{|\Omega|^{n-1}}{(n-1)!^m}\frac{D^{m,n}y_i(s)}{D^m h_i(s)}\right)ds\,.
\end{aligned}$$

Define

$$U_i(s) = D^m h_i(s) g_i\left(\frac{|\Omega|^{n-1}}{(n-1)!^m}\frac{D^{m,n}y_i(s)}{D^m h_i(s)}\right)$$

for $t\in\Omega$ and $i=1,\ldots,p$. Hence,

$$\begin{aligned}
&\int_\Omega \left[\sum_{i=1}^p D_i f\left(h_1(t)g_1\left(\frac{|x_1(t)|}{h_1(t)}\right),\ldots,h_p(t)g_p\left(\frac{|x_p(t)|}{h_p(t)}\right)\right)\right.\\
&\qquad\left.\times D^m h_i(t) g_i\left(\frac{|\Omega|^{n-1}}{(n-1)!^m}\frac{D^{m,n}|x_i(t)|}{D^m h_i(t)}\right)\right]dt\\
&\le \int_\Omega \left[\sum_{i=1}^p D_i f\left(\int_{\Omega_t} D^m h_1(s) g_1\left(\frac{|\Omega|^{n-1}}{(n-1)!^m}\frac{D^{m,n}y_1(s)}{D^m h_1(s)}\right)ds,\ldots,\right.\right.\\
&\qquad\left.\int_{\Omega_t} D^m h_p(s) g_p\left(\frac{|\Omega|^{n-1}}{(n-1)!^m}\frac{D^{m,n}y_p(s)}{D^m h_p(s)}\right)ds\right)\\
&\qquad\left.\times D^m h_i(t) g_i\left(\frac{|\Omega|^{n-1}}{(n-1)!^m}\frac{D^{m,n}y_i(t)}{D^m h_i(t)}\right)\right]dt\\
&= \int_\Omega \left[\sum_{i=1}^p D_i f\left(\int_{\Omega_t} U_1(s)\,ds,\ldots,\int_{\Omega_t} U_p(s)\,ds\right)U_i(t)\right]dt\\
&= f\left(\int_\Omega U_1(t)\,dt,\ldots,\int_\Omega U_p(t)\,dt\right)\\
&= f\left(\int_\Omega D^m h_1(t) g_1\left(\frac{|\Omega|^{n-1}}{(n-1)!^m}\frac{D^{m,n}y_1(t)}{D^m h_1(t)}\right)dt,\ldots,\right.\\
&\qquad\left.\int_\Omega D^m h_p(t) g_p\left(\frac{|\Omega|^{n-1}}{(n-1)!^m}\frac{D^{m,n}y_p(t)}{D^m h_p(t)}\right)dt\right).\\
&= f\left(\int_\Omega D^m h_1(t) g_1\left(\frac{|\Omega|^{n-1}}{(n-1)!^m}\frac{|D^{m,n}x_1(t)|}{D^m h_1(t)}\right)dt,\ldots,\right.
\end{aligned}$$

$$\int_\Omega D^m h_p(t) g_p\left(\frac{|\Omega|^{n-1}}{(n-1)!^m}\frac{\left|D^{m,n}x_p(t)\right|}{D^m h_p(t)}\right)dt\Bigg). \tag{3.53}$$

Finally, by Jensen's inequality we obtain

$$\begin{aligned}
&\int_\Omega \Bigg[\sum_{i=1}^{p} D_i f\left(h_1(t)g_1\left(\frac{|x_1(t)|}{h_1(t)}\right),\ldots,h_p(t)g_p\left(\frac{|x_p(t)|}{h_p(t)}\right)\right)\\
&\qquad \times D^m h_i(t) g_i\left(\frac{|\Omega|^{n-1}}{(n-1)!^m}\frac{D^{m,n}|x_i(t)|}{D^m h_i(t)}\right)\Bigg]dt\\
&\le f\Bigg(\int_\Omega D^m h_1(t) g_1\left(\frac{|\Omega|^{n-1}}{(n-1)!^m}\frac{\left|D^{m,n}x_1(t)\right|}{D^m h_1(t)}\right)dt,\ldots,\\
&\qquad \int_\Omega D^m h_p(t) g_p\left(\frac{|\Omega|^{n-1}}{(n-1)!^m}\frac{\left|D^{m,n}x_p(t)\right|}{D^m h_p(t)}\right)dt\Bigg)\\
&= f\Bigg(\frac{1}{|\Omega|}\int_\Omega |\Omega|\, D^m h_1(t)\, g_1\left(\frac{|\Omega|^{n-1}}{(n-1)!^m}\frac{\left|D^{m,n}x_1(t)\right|}{D^m h_1(t)}\right)dt,\ldots,\\
&\qquad \frac{1}{|\Omega|}\int_\Omega |\Omega|\, D^m h_p(t)\, g_p\left(\frac{|\Omega|^{n-1}}{(n-1)!^m}\frac{\left|D^{m,n}x_p(t)\right|}{D^m h_p(t)}\right)dt\Bigg)\\
&\le \frac{1}{|\Omega|}\int_\Omega f\Bigg(|\Omega|\, D^m h_1(t)\, g_1\left(\frac{|\Omega|^{n-1}}{(n-1)!^m}\frac{\left|D^{m,n}x_1(t)\right|}{D^m h_1(t)}\right),\ldots,\\
&\qquad |\Omega|\, D^m h_p(t)\, g_p\left(\frac{|\Omega|^{n-1}}{(n-1)!^m}\frac{\left|D^{m,n}x_p(t)\right|}{D^m h_p(t)}\right)\Bigg)dt\,.
\end{aligned}$$

□

Remark 3.5 Theorem 3.2 follows for $p = 1$ and $m = 1$. Also, the inequality (3.53) is an extension of the inequality given in [34, Theorem 1].

Chapter 4

Generalizations of the Mitrinović-Pečarić inequalities

In this chapter a certain class of convex functions in Opial-type integral inequality is considered. We give extensions of Opial-type integral inequalities and use them to obtain generalizations of inequalities due to Mitrinović and Pečarić for convex and for relative convex functions. Cauchy type mean value theorems are proved and used in studying Stolarsky type means defined by the observed integral inequalities. Also, a method of producing n-exponentially convex and exponentially convex functions is applied. Applications with respect to fractional derivatives and fractional integrals are also given. Some new Opial-type inequalities are given for different types of fractional integrals and fractional derivatives. This chapter is based on our results: Andrić, Barbir, Farid, Iqbal and Pečarić [17, 18, 19].

4.1 The Mitrinović-Pečarić inequality for convex functions

We say that a function $u : [a,b] \to \mathbb{R}$ belongs to the class $U(v,K)$ if it admits the representation

$$u(x) = \int_a^x K(x,t)\, v(t)\, dt\,, \tag{4.1}$$

where v is a continuous function and K is an arbitrary nonnegative kernel such that $v(x) > 0$ implies $u(x) > 0$ for every $x \in [a,b]$. We also assume that all integrals under consideration exist and are finite.

The following inequality is given by Mitrinović and Pečarić in [60] (also see [5, p. 89] and [67, p. 236]).

Theorem 4.1 *Let $u_1 \in U(v_1,K)$, $u_2 \in U(v_2,K)$ and $v_2(x) > 0$ for every $x \in [a,b]$. Further, let $\phi(u)$ and $f(u)$ be convex and increasing for $u \geq 0$ and $f(0) = 0$. If f is a differentiable function and $M = \max K(x,t)$, then*

$$M \int_a^b v_2(t)\,\phi\left(\left|\frac{v_1(t)}{v_2(t)}\right|\right) f'\left(u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right)\right) dt$$

$$\leq f\left(M \int_a^b v_2(t)\,\phi\left(\left|\frac{v_1(t)}{v_2(t)}\right|\right) dt\right). \tag{4.2}$$

In the following theorem we give the generalization of the inequality (4.2).

Theorem 4.2 *Let $u_1 \in U(v_1,K)$, $u_2 \in U(v_2,K)$ and $v_2(x) > 0$ for every $x \in [a,b]$. Further, let $\phi(u)$ be convex, nonnegative and increasing for $u \geq 0$, $f(u)$ be convex for $u \geq 0$, and $f(0) = 0$. If f is a differentiable function and $M = \max K(x,t)$, then these inequalities are valid:*

$$M \int_a^b v_2(t)\,\phi\left(\left|\frac{v_1(t)}{v_2(t)}\right|\right) f'\left(u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right)\right) dt$$

$$\leq f\left(M \int_a^b v_2(t)\,\phi\left(\left|\frac{v_1(t)}{v_2(t)}\right|\right) dt\right) \tag{4.3}$$

$$\leq \frac{1}{b-a} \int_a^b f\left(M(b-a)v_2(t)\,\phi\left(\left|\frac{v_1(t)}{v_2(t)}\right|\right)\right) dt. \tag{4.4}$$

Proof. On the right side of the inequality (4.3), if we multiply and divide by the factor $(b-a)$ inside and outside the integral and use Jensen's inequality for the function f, then we obtain the inequality (4.4). □

The condition of Theorem 4.1 that the function f is increasing is actually unneeded. From the proof of the theorem [67, p. 236] one can see that this property is never used, therefore we omit it here. Also, a condition that is missing in Theorem 4.1 is that ϕ has to be nonnegative, which we add.

4.1.1 Mean value theorems and exponential convexity

Motivated by the inequalities given in Theorem 4.2, we define two functionals as:

$$\Phi_1(f) = f\left(M \int_a^b v_2(t)\,\phi\left(\left|\frac{v_1(t)}{v_2(t)}\right|\right) dt\right)$$

$$-M\int_a^b v_2(t)\,\phi\left(\left|\frac{v_1(t)}{v_2(t)}\right|\right) f'\left(u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right)\right)dt \tag{4.5}$$

$$\begin{aligned}\Phi_2(f) &= \frac{1}{b-a}\int_a^b f\left(M(b-a)v_2(t)\,\phi\left(\left|\frac{v_1(t)}{v_2(t)}\right|\right)\right)dt\\ &\quad -f\left(M\int_a^b v_2(t)\,\phi\left(\left|\frac{v_1(t)}{v_2(t)}\right|\right)dt\right),\end{aligned} \tag{4.6}$$

where f is a differentiable function with $f(0)=0$, and M, ϕ, u_i, v_i $(i=1,2)$ are as in Theorem 4.1.

If f is a convex function, then Theorem 4.2 implies that $\Phi_i(f)\geq 0$ $(i=1,2)$.

Now, we give mean value theorems for the functionals Φ_i $(i=1,2)$.
Let $0<m_2\leq v_2\leq M_2$, $0\leq |v_1|\leq M_1$ and $\phi\geq 0$. Then $0\leq\left|\frac{v_1}{v_2}\right|\leq\frac{M_1}{m_2}$. It follows

$$m_2 M(b-a)\min_{\left[0,\frac{M_1}{m_2}\right]}\phi\leq M\int_a^b v_2(t)\,\phi\left(\left|\frac{v_1(t)}{v_2(t)}\right|\right)dt\leq M_2 M(b-a)\max_{\left[0,\frac{M_1}{m_2}\right]}\phi\,.$$

Also

$$0\leq\left|\frac{u_1(t)}{u_2(t)}\right|\leq\frac{M_1\int_a^t K(x,\tau)\,d\tau}{m_2\int_a^t K(x,\tau)\,d\tau}=\frac{M_1}{m_2}\,.$$

Since obviously $|u_2(t)|\leq MM_2(b-a)$, we have

$$0\leq u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right)\leq MM_2(b-a)\max_{\left[0,\frac{M_1}{m_2}\right]}\phi.$$

Hence, from now on let $f:I\to\mathbb{R}$ where

$$I=\left[0,MM_2(b-a)\max_{\left[0,\frac{M_1}{m_2}\right]}\phi\right]. \tag{4.7}$$

Theorem 4.3 *Let $u_1\in U(v_1,K)$, $u_2\in U(v_2,K)$ and $v_2(x)>0$ for every $x\in[a,b]$. Further, let $\phi(u)$ be convex, nonnegative and increasing for $u\geq 0$. Let $f\in C^2(I)$ and $f(0)=0$. Then there exists $\xi\in I$ such that*

$$\Phi_i(f)=\frac{f''(\xi)}{2}\Phi_i(f_0),\quad (i=1,2), \tag{4.8}$$

where $f_0(x)=x^2$.

Proof. Since $f \in C^2(I)$, there exist real numbers $m = \min_{x\in I} f''(x)$ and $M = \max_{x\in I} f''(x)$. Hence, the functions f_1 and f_2 defined by

$$\begin{aligned} f_1(x) &= \frac{M}{2}x^2 - f(x), \\ f_2(x) &= f(x) - \frac{m}{2}x^2 \end{aligned}$$

are convex. Therefore $\Phi_i(f_1) \geq 0$, $\Phi_i(f_2) \geq 0$ $(i = 1,2)$, and we get

$$\frac{m}{2}\Phi_i(f_0) \;\leq\; \Phi_i(f) \;\leq\; \frac{M}{2}\Phi_i(f_0).$$

If $\Phi_i(f_0) = 0$, then there is nothing to prove. Suppose $\Phi_i(f_0) > 0$. We have

$$m \leq \frac{2\Phi_i(f)}{\Phi_i(f_0)} \leq M.$$

Hence, there exists $\xi \in I$ such that

$$\Phi_i(f) = \frac{f''(\xi)}{2}\Phi_i(f_0), \quad (i = 1,2).$$

This completes the proof. □

Theorem 4.4 *Let $u_1 \in U(v_1,K)$, $u_2 \in U(v_2,K)$ and $v_2(x) > 0$ for every $x \in [a,b]$. Further, let $\phi(u)$ be convex, nonnegative and increasing for $u \geq 0$. Let $f,g \in C^2(I)$ and $f(0) = g(0) = 0$. Then there exists $\xi \in I$ such that*

$$\frac{\Phi_i(f)}{\Phi_i(g)} = \frac{f''(\xi)}{g''(\xi)}, \quad (i = 1,2), \tag{4.9}$$

provided that the denominators are non-zero.

Proof. Define $h \in C^2(I)$ by $h = c_1 f - c_2 g$, where

$$c_1 = \Phi_i(g), \;\; c_2 = \Phi_i(f), \quad (i = 1,2).$$

Now using Theorem 4.3 with $f = h$ there exists $\xi \in I$ such that

$$\left(c_1\frac{f''(\xi)}{2} - c_2\frac{g''(\xi)}{2}\right)\Phi_i(f_0) = 0, \quad (i = 1,2).$$

Since $\Phi_i(f_0) \neq 0$ (otherwise we have a contradiction with $\Phi_i(g) \neq 0$ by Theorem 4.3), we get

$$\frac{\Phi_i(f)}{\Phi_i(g)} = \frac{f''(\xi)}{g''(\xi)}, \quad (i = 1,2).$$

This completes the proof. □

We continue with the method of exponential convexity given in [50]. We use this to prove the n-exponential convexity for the functionals Φ_i $(i = 1,2)$. The next theorem is analogous to the one given in [66, Theorem 3.9] and we give a proof for the reader's convenience.
Note here that for the functionals Φ_i $(i = 1,2)$ interval I is defined by (4.7).

Theorem 4.5 *Let $\Upsilon = \{f_s : s \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is n-exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Let Φ_i $(i = 1, 2)$ be linear functionals defined as in (4.5) and (4.6). Then $s \mapsto \Phi_i(f_s)$ is n-exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Phi_i(f_s)$ is also continuous on J, then it is n-exponentially convex on J.*

Proof. For $\xi_i \in \mathbb{R}$, $s_i \in J$, $i = 1, \ldots, n$, we define the function

$$g(y) = \sum_{i,j=1}^{n} \xi_i \xi_j f_{\frac{s_i+s_j}{2}}(y).$$

Using the assumption that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is n-exponentially convex in the Jensen sense, we have

$$[y_0, y_1, y_2; g] = \sum_{i,j=1}^{n} \xi_i \xi_j [y_0, y_1, y_2; f_{\frac{s_i+s_j}{2}}] \geq 0,$$

which in turn implies that g is a convex function on I. Therefore we have $\Phi_i(g) \geq 0$ $(i = 1, 2)$. Hence

$$\sum_{i,j=1}^{n} \xi_i \xi_j \Phi_i(f_{\frac{s_i+s_j}{2}}) \geq 0.$$

We conclude that the function $s \mapsto \Phi_i(f_s)$ is n-exponentially convex on J in the Jensen sense. If the function $s \mapsto \Phi_i(f_s)$ is also continuous on J, then $s \mapsto \Phi_i(f_s)$ is n-exponentially convex by definition. $\square$

Corollary 4.1 *Let $\Upsilon = \{f_s : s \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Let Φ_i $(i = 1, 2)$ be linear functionals defined as in (4.5) and (4.6). Then $s \mapsto \Phi_i(f_s)$ is exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Phi_i(f_s)$ is continuous on J, then it is exponentially convex on J.*

Let us denote a mean for $f_s, f_q \in \Omega$ by

$$\mu_{s,q}(\Phi_i, \Omega) = \begin{cases} \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)} \right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{\frac{d}{ds}\Phi_i(f_s)}{\Phi_i(f_s)} \right), & s = q. \end{cases} \tag{4.10}$$

Theorem 4.6 *Let $\Omega = \{f_s : s \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is 2-exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Let Φ_i $(i = 1, 2)$ be linear functionals defined as in (4.5) and (4.6). Then the following statements hold:*

(i) *If the function $s \mapsto \Phi_i(f_s)$ is continuous on J, then it is 2-exponentially convex function on J. If the function $s \mapsto \Phi_i(f_s)$ is additionally positive, then it is also log-convex on J, and for $r,s,t \in J$ such that $r < s < t$, we have*

$$(\Phi_i(f_s))^{t-r} \leq (\Phi_i(f_r))^{t-s}\,(\Phi_i(f_t))^{s-r}\,,\ i = 1,2. \tag{4.11}$$

(ii) *If the function $s \mapsto \Phi_i(f_s)$ is strictly positive and differentiable on J, then for every $s,q,r,t \in J$, such that $s \leq r$ and $q \leq t$, we have*

$$\mu_{s,q}(\Phi_i,\Omega)\ \leq\ \mu_{r,t}(\Phi_i,\Omega)\,,\ i = 1,2. \tag{4.12}$$

Proof. (i) The first part is an immediate consequence of Theorem 4.5 and in second part log-convexity on J follows from Remark 1.3. Since $s \mapsto \Phi_i(f_s)$ is positive, for $r,s,t \in J$ such that $r < s < t$, with $f(s) = \log \Phi_i(f_s)$ in Proposition 1.2, we have

$$(t-s)\log \Phi_i(f_r) + (r-t)\log \Phi_i(f_s) + (s-r)\log \Phi_i(f_t)\ \geq\ 0\,.$$

This is equivalent to inequality (4.11).
(ii) The function $s \mapsto \Phi_i(f_s)$ is log-convex on J by (i), that is, the function $s \mapsto \log \Phi_i(f_s)$ is convex on J. Applying Proposition 1.3 we get

$$\frac{\log \Phi_i(f_s) - \log \Phi_i(f_q)}{s-q}\ \leq\ \frac{\log \Phi_i(f_r) - \log \Phi_i(f_t)}{r-t} \tag{4.13}$$

for $s \leq r$, $q \leq t$, $s \neq q$, $r \neq t$, and therefore we have

$$\mu_{s,q}(\Phi_i,\Omega)\ \leq\ \mu_{r,t}(\Phi_i,\Omega)\,.$$

Cases $s = q$ and $r = t$ follows from (4.13) as limit cases. □

Remark 4.1 Results from Theorem 4.5, Corollary 4.1 and Theorem 4.6 still hold when two of the points $y_0, y_1, y_2 \in I$ coincide, for a family of differentiable functions f_s such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense). Furthermore, they still hold when all three points coincide for a family of twice differentiable functions with the same property. The proofs can be obtained by recalling Remark 1.1 and suitable characterization of convexity.

Remark 4.2 As we prove the n-exponential convexity of the functionals Φ_1 and Φ_2 obtained from the inequalities given in (4.3) and (4.4), similarly we can define the functionals from the inequalities given in (4.14), (4.17), (4.19), (4.21), (4.24) and (4.26) and prove the n-exponential convexity of our defined functionals.

4.1.2 Applications to Stolarsky type means

We use Cauchy type mean value Theorem 4.3 and Theorem 4.4 for Stolarsky type means, defined by the functional Φ_i $(i=1,2)$. Several families of functions which fulfil conditions of Theorem 4.5, Corollary 4.1 and Theorem 4.6 (and Remark 4.1) that we present here, enable us to construct large families of functions which are exponentially convex.

Example 4.1 Consider a family of functions

$$\Omega_1 = \{f_s : \mathbb{R} \to \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$f_s(x) = \begin{cases} \frac{e^{sx}-1}{s^2}, & s \neq 0, \\ \frac{x^2}{2}, & s = 0. \end{cases}$$

Since $\frac{d^2 f_s}{dx^2}(x) = e^{sx} > 0$, then f_s is convex on $\mathbb{R}$ for every $s \in \mathbb{R}$, and $s \mapsto \frac{d^2 f_s}{dx^2}(x)$ is exponentially convex by definition.
Analogously as in the proof of Theorem 4.5 we conclude that $s \mapsto [y_0, y_1, y_2; f_s]$ is exponentially convex (and so exponentially convex in the Jensen sense).
Notice that $f_s(0) = 0$. By Corollary 4.1 we have that $s \mapsto \Phi_i(f_s)$ $(i=1,2)$ is exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous (although mapping $s \mapsto f_s$ is not continuous for $s=0$), so it is exponentially convex.
For this family of functions, $\mu_{s,q}(\Phi_i, \Omega_1)$ $(i=1,2)$ from (4.10) is equal to

$$\mu_{s,q}(\Phi_i, \Omega_1) = \begin{cases} \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{\Phi_i(id \cdot f_s)}{\Phi_i(f_s)} - \frac{2}{s}\right), & s = q \neq 0, \\ \exp\left(\frac{\Phi_i(id \cdot f_0)}{3\Phi_i(f_0)}\right), & s = q = 0, \end{cases}$$

and using (4.12) it is a monotonous in parameters s and q.

If Φ_i is positive, $(i=1,2)$, then Theorem 4.4 applied for $f = f_s \in \Omega_1$ and $g = f_q \in \Omega_1$ yields that there exists $\xi \in I = \left[0, MM_2\,(b-a)\max_{\left[0,\frac{M_1}{m_2}\right]} \phi\right]$ such that

$$e^{(s-q)\xi} = \frac{\Phi_i(f_s)}{\Phi_i(f_q)}.$$

It follows that

$$M_{s,q}(\Phi_i, \Omega_1) = \log \mu_{s,q}(\Phi_i, \Omega_1)$$

satisfy $0 \leq M_{s,q}(\Phi_i, \Omega_1) \leq MM_2\,(b-a)\max_{\left[0,\frac{M_1}{m_2}\right]} \phi$, which shows that $M_{s,q}(\Phi_i, \Omega_1)$ is a mean, and by (4.12) it is a monotonous mean, $i=1,2$.

Example 4.2 Consider a family of functions

$$\Omega_2 = \{g_s : [0, \infty) \to \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$g_s(x) = \begin{cases} \frac{(x+1)^s-1}{s(s-1)}, & s \neq 0,1, \\ -\log(x+1), & s=0, \\ (x+1)\log(x+1), & s=1. \end{cases}$$

Here, $\frac{d^2 g_s}{dx^2}(x) = (x+1)^{s-2} = e^{(s-2)\log(x+1)} > 0$ which shows that g_s is convex for $x>0$ and $s \mapsto \frac{d^2 g_s}{dx^2}(x)$ is exponentially convex by definition. Also, $g_s(0)=0$. Arguing as in Example 4.1 we get that the mapping $s \mapsto \Phi_i(g_s)$ is exponentially convex and also log-convex. For this family of functions, $\mu_{s,q}(\Phi_i,\Omega_2)$ from (4.10) is equal to

$$\mu_{s,q}(\Phi_i,\Omega_2) = \begin{cases} \left(\frac{\Phi_i(g_s)}{\Phi_i(g_q)}\right)^{\frac{1}{s-q}}, & s\neq q, \\ \exp\left(\frac{1-2s}{s(s-1)} - \frac{\Phi_i(g_0 g_s)}{\Phi_i(g_s)} - \frac{1}{s(s-1)}\frac{\Phi_i(g_0)}{\Phi_i(g_s)}\right), & s=q\neq 0,1, \\ \exp\left(1-\frac{\Phi_i(g_0^2)}{2\Phi_i(g_0)}\right), & s=q=0, \\ \exp\left(-1-\frac{\Phi_i(g_0 g_1)}{2\Phi_i(g_1)}\right), & s=q=1, \end{cases}$$

and by (4.12) it is monotonous in parameters s and q.

Using Theorem 4.4 it follows that there exists $\xi \in I$ such that

$$(\xi+1)^{s-q} = \frac{\Phi_i(g_s)}{\Phi_i(g_q)}.$$

Since the function $\xi \mapsto (\xi+1)^{s-q}$ is invertible for $s \neq q$, we have

$$0 \leq \left(\frac{\Phi_i(g_s)}{\Phi_i(g_q)}\right)^{\frac{1}{s-q}} \leq MM_2\,(b-a) \max_{\left[0,\frac{M_1}{m_2}\right]} \phi$$

which together with the fact that $\mu_{s,q}(\Phi_i,\Omega_2)$ is continuous, symmetric and monotonous, shows that $\mu_{s,q}(\Phi_i,\Omega_2)$ is a mean, $i=1,2$.

Example 4.3 Consider a family of functions

$$\Omega_3 = \{h_s : [0,\infty) \to \mathbb{R} : s>0\}$$

defined by

$$h_s(x) = \begin{cases} \frac{s^{-x}-1}{\log^2 s}, & s\neq 1, \\ \frac{x^2}{2}, & s=1. \end{cases}$$

Since $s \mapsto \frac{d^2 h_s}{dx^2}(x) = s^{-x}$ is the Laplace transform of a nonnegative function ([74]), that is $s^{-x} = \dfrac{1}{\Gamma(x)}\displaystyle\int_0^\infty e^{-st}t^{x-1}\,dt$, it is exponentially convex on $(0,\infty)$. Obviously h_s are convex

functions for every $s>0$ and $h_s(0)=0$.
For this family of functions, $\mu_{s,q}(\Phi_i,\Omega_3)$ from (4.10) is equal to

$$\mu_{s,q}(\Phi_i,\Omega_3) = \begin{cases} \left(\frac{\Phi_i(h_s)}{\Phi_i(h_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(-\frac{\Phi_i(id\cdot h_s)}{s\Phi_i(h_s)} - \frac{2}{s\log s}\right), & s=q\neq 1, \\ \exp\left(-\frac{\Phi_i(id\cdot h_1)}{3\Phi_i(h_1)}\right), & s=q=1, \end{cases}$$

and it is monotonous in parameters s and q by (4.12).

Using Theorem 4.4 it follows that there exists $\xi \in I$ such that

$$\left(\frac{s}{q}\right)^{-\xi} = \frac{\Phi_i(h_s)}{\Phi_i(h_q)}.$$

Hence,

$$M_{s,q}(\Phi_i,\Omega_3) = -L(s,q)\log \mu_{s,q}(\Phi_i,\Omega_3),$$

satisfies $0 \leq M_{s,q}(\Phi_i,\Omega_3) \leq MM_2\,(b-a)\max_{\left[0,\frac{M_1}{m_2}\right]}\phi$, which shows that $M_{s,q}(\Phi_i,\Omega_3)$ is a mean, $i=1,2$.
$L(s,q)$ is the logarithmic mean defined by

$$L(s,q) = \begin{cases} \frac{s-q}{\log s - \log q}, & s\neq q, \\ s, & s=q. \end{cases}$$

Example 4.4 Consider a family of functions

$$\Omega_4 = \{k_s : [0,\infty) \to \mathbb{R} : s>0\}$$

defined by

$$k_s(x) = \frac{e^{-x\sqrt{s}}-1}{s}.$$

Again we conclude, since $s \mapsto \frac{d^2k_s}{dx^2}(x) = e^{-x\sqrt{s}}$ is the Laplace transform of a nonnegative function ([74]), that is $e^{-x\sqrt{s}} = \frac{s}{2\sqrt{\pi}}\int\limits_0^\infty \frac{e^{-st}e^{-x^2/4t}}{t\sqrt{t}}\,dt$ it is exponentially convex on $(0,\infty)$.
For every $s>0$, k_s are convex functions and $k_s(0)=0$.
For this family of functions, $\mu_{s,q}(\Phi_i,\Omega_4)$ from (4.10) is equal to

$$\mu_{s,q}(\Phi_i,\Omega_4) = \begin{cases} \left(\frac{\Phi_i(k_s)}{\Phi_i(k_q)}\right)^{\frac{1}{s-q}}, & s\neq q, \\ \exp\left(-\frac{\Phi_i(id\cdot k_s)}{2\sqrt{s}\Phi_i(k_s)} - \frac{1}{s}\right), & s=q, \end{cases}$$

and by (4.12) it is monotonous in parameters s and q.

Using Theorem 4.4 it follows that there exists $\xi \in I$ such that

$$e^{-\xi\left(\sqrt{s}-\sqrt{q}\right)} = \frac{\Phi_i(k_s)}{\Phi_i(k_q)}.$$

Hence,

$$M_{s,q}(\Phi_i,\Omega_4) = -\left(\sqrt{s}+\sqrt{q}\right)\log \mu_{s,q}(\Phi_i,\Omega_4)$$

satisfies $0 \leq M_{s,q}(\Phi_i,\Omega_4) \leq MM_2\,(b-a)\max_{\left[0,\frac{M_1}{m_2}\right]}\phi$, which shows that $M_{s,q}(\Phi_i,\Omega_4)$ is a mean, $i=1,2$.

4.1.3 Opial-type inequalities for fractional integrals and fractional derivatives

In this section we present some new Opial-type inequalities involving fractional integrals and fractional derivatives, based on inequalities given in Theorem 4.2. For this we need fractional integrals of a function with respect to another function, the Riemann-Liouville and the Hadamard fractional integrals ([51, Section 2.1, 2.5 and 2.7]).

Let (a,b), $-\infty \leq a < b \leq \infty$, be a finite or infinite interval of the real line $\mathbb{R}$ and $\alpha > 0$. Also let g be an increasing function on (a,b) and g' be a continuous function on (a,b). The *left-sided and right-sided fractional integrals of a function f with respect to another function g* on $[a,b]$ are given by

$$J^{\alpha}_{a+;g}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{g'(t)f(t)dt}{[g(x)-g(t)]^{1-\alpha}}, \quad x>a,$$

$$J^{\alpha}_{b-;g}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \frac{g'(t)f(t)dt}{[g(t)-g(x)]^{1-\alpha}}, \quad x<b,$$

respectively.

Theorem 4.7 *Let $\alpha \geq 1$, $\phi(u)$ be convex, nonnegative and increasing for $u \geq 0$ and let $f(u)$ be convex for $u \geq 0$ with $f(0)=0$. If f is a differentiable function, then these inequalities are valid:*

$$\begin{aligned}
&\frac{(g(b)-g(a))^{\alpha-1}\max\limits_{x\in[a,b]} g'(x)}{\Gamma(\alpha)}\int_a^b u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right) f'\left(J^{\alpha}_{a+;g}u_2(t)\,\phi\left(\left|\frac{J^{\alpha}_{a+;g}u_1(t)}{J^{\alpha}_{a+;g}u_2(t)}\right|\right)\right)dt \\
&\quad\leq f\left(\frac{(g(b)-g(a))^{\alpha-1}\max\limits_{x\in[a,b]} g'(x)}{\Gamma(\alpha)}\int_a^b u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right)dt\right) \\
&\quad\leq \frac{1}{b-a}\int_a^b f\left(\frac{(b-a)(g(b)-g(a))^{\alpha-1}\max\limits_{x\in[a,b]} g'(x)}{\Gamma(\alpha)}u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right)\right)dt. \qquad (4.14)
\end{aligned}$$

Proof. We use Theorem 4.2 with the kernel

$$K(x,t) = \begin{cases} \frac{g'(t)}{\Gamma(\alpha)(g(x)-g(t))^{1-\alpha}}, & a < t \le x \\ 0, & x < t \le b \end{cases}. \tag{4.15}$$

For $\alpha \ge 1$, we get

$$M = \max K(x,t) = \frac{(g(b)-g(a))^{\alpha-1} \max\limits_{x\in[a,b]} g'(x)}{\Gamma(\alpha)}.$$

If we replace u_i by $J^{\alpha}_{a+;g}u_i$ and v_i by u_i $(i=1,2)$ in inequalities given in (4.3) an (4.4), then the inequality (4.14) follows. $\square$

A similar result follows for the right-sided fractional integrals of a function f with respect to another function g.

Theorem 4.8 *Let $\alpha \ge 1$, $\phi(u)$ be convex, nonnegative and increasing for $u \ge 0$ and let $f(u)$ be convex for $u \ge 0$ with $f(0) = 0$. If f is a differentiable function, then these inequalities are valid:*

$$\begin{aligned}
&\frac{(g(b)-g(a))^{\alpha-1} \max\limits_{x\in[a,b]} g'(x)}{\Gamma(\alpha)} \int_a^b u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right) f'\left(J^{\alpha}_{b-;g}u_2(t)\,\phi\left(\left|\frac{J^{\alpha}_{b-;g}u_1(t)}{J^{\alpha}_{b-;g}u_2(t)}\right|\right)\right) dt \\
&\le f\left(\frac{(g(b)-g(a))^{\alpha-1} \max\limits_{x\in[a,b]} g'(x)}{\Gamma(\alpha)} \int_a^b u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right) dt\right) \\
&\le \frac{1}{b-a}\int_a^b f\left(\frac{(b-a)(g(b)-g(a))^{\alpha-1} \max\limits_{x\in[a,b]} g'(x)}{\Gamma(\alpha)} u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right)\right) dt.
\end{aligned} \tag{4.16}$$

If $g(x) = x$, then $J^{\alpha}_{a+;x}f(x)$ reduces to $J^{\alpha}_{a+}f(x)$, i.e. the left-sided Riemann-Liouville fractional integral. Same follows for the right-sided fractional integral. This gives us the next results.

Corollary 4.2 *Let $\alpha \ge 1$, $\phi(u)$ be convex, nonnegative and increasing for $u \ge 0$ and let $f(u)$ be convex for $u \ge 0$ with $f(0) = 0$. If f is a differentiable function, then these inequalities are valid:*

$$\begin{aligned}
&\frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} \int_a^b u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right) f'\left(J^{\alpha}_{a+}u_2(t)\,\phi\left(\left|\frac{J^{\alpha}_{a+}u_1(t)}{J^{\alpha}_{a+}u_2(t)}\right|\right)\right) dt \\
&\le f\left(\frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} \int_a^b u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right) dt\right) \\
&\le \frac{1}{b-a}\int_a^b f\left(\frac{(b-a)^{\alpha}}{\Gamma(\alpha)} u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right)\right) dt.
\end{aligned} \tag{4.17}$$

Corollary 4.3 *Let $\alpha \geq 1$, $\phi(u)$ be convex, nonnegative and increasing for $u \geq 0$ and let $f(u)$ be convex for $u \geq 0$ with $f(0) = 0$. If f is a differentiable function, then these inequalities are valid:*

$$\begin{aligned}
&\frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} \int_a^b u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right) f'\left(J_{b-}^{\alpha} u_2(t)\,\phi\left(\left|\frac{J_{b-}^{\alpha} u_1(t)}{J_{b-}^{\alpha} u_2(t)}\right|\right)\right) dt \\
&\quad \leq f\left(\frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} \int_a^b u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right) dt\right) \\
&\quad \leq \frac{1}{b-a} \int_a^b f\left(\frac{(b-a)^{\alpha}}{\Gamma(\alpha)} u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right)\right) dt.
\end{aligned} \tag{4.18}$$

Let (a,b) be a finite or infinite interval of $\mathbb{R}^+$ and $\alpha > 0$. The *left-sided and right-sided Hadamard fractional integrals* of order $\alpha > 0$ are given by

$${}^{H}J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log\frac{x}{t}\right)^{\alpha-1} \frac{f(t)dt}{t}, \quad x > a,$$

$${}^{H}J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\log\frac{t}{x}\right)^{\alpha-1} \frac{f(t)dt}{t}, \quad x < b.$$

Notice that the Hadamard fractional integrals of order α are special cases of the left-sided and right-sided fractional integrals of a function f with respect to a function $g(x) = \log x$ on (a,b), where $0 \leq a < b \leq \infty$.

Corollary 4.4 *Let $\alpha \geq 1$, $\phi(u)$ be convex, nonnegative and increasing for $u \geq 0$ and let $f(u)$ be convex for $u \geq 0$ with $f(0) = 0$. If f is a differentiable function, then these inequalities are valid:*

$$\begin{aligned}
&\frac{(\log b - \log a)^{\alpha-1}}{a\Gamma(\alpha)} \int_a^b u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right) f'\left({}^{H}J_{a+}^{\alpha} u_2(t)\,\phi\left(\left|\frac{{}^{H}J_{a+}^{\alpha} u_1(t)}{{}^{H}J_{a+}^{\alpha} u_2(t)}\right|\right)\right) dt \\
&\quad \leq f\left(\frac{1}{a\Gamma(\alpha)} (\log b - \log a)^{\alpha-1} \int_a^b u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right) dt\right) \\
&\quad \leq \frac{1}{b-a} \int_a^b f\left(\frac{(b-a)(\log b - \log a)^{\alpha-1}}{a\Gamma(\alpha)} u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right)\right) dt.
\end{aligned} \tag{4.19}$$

Corollary 4.5 *Let $\alpha \geq 1$, $\phi(u)$ be convex, nonnegative and increasing for $u \geq 0$ and let $f(u)$ be convex for $u \geq 0$ with $f(0) = 0$. If f is a differentiable function, then these* *inequalities are valid:*

$$
\begin{aligned}
&\frac{(\log b-\log a)^{\alpha-1}}{a\Gamma(\alpha)}\int_a^b u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right) f'\left({}^{H}J_{b-}^{\alpha}u_2(t)\,\phi\left(\left|\frac{{}^{H}J_{b-}^{\alpha}u_1(t)}{{}^{H}J_{b-}^{\alpha}u_2(t)}\right|\right)\right)dt \\
&\quad\le f\left(\frac{1}{a\Gamma(\alpha)}(\log b-\log a)^{\alpha-1}\int_a^b u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right)dt\right) \\
&\quad\le \frac{1}{b-a}\int_a^b f\left(\frac{(b-a)(\log b-\log a)^{\alpha-1}}{a\Gamma(\alpha)}u_2(t)\,\phi\left(\left|\frac{u_1(t)}{u_2(t)}\right|\right)\right)dt. \qquad (4.20)
\end{aligned}
$$

Next inequalities include the Riemann-Liouville fractional derivatives. The composition identity for the left-sided fractional derivatives is valid if one of conditions $(i)-(vii)$ in Corollary 2.21 holds. For the right-sided Riemann-Liouville fractional derivatives we use Corollary 2.22.

Theorem 4.9 *Let $\beta\ge 0$ and $\alpha>\beta+1$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.21 holds for $\{\alpha,\beta,u_i\}$, $i=1,2$. Let $D_{a+}^{\alpha}u_2(x)>0$ on $[a,b]$. Further, let $\phi(u)$ be convex, nonnegative and increasing for $u\ge 0$ and let $f(u)$ be convex for $u\ge 0$ with $f(0)=0$. Then for a.e. $x\in[a,b]$ hold*

$$
\begin{aligned}
&\frac{(b-a)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\int_a^b D_{a+}^{\alpha}u_2(t)\,\phi\left(\left|\frac{D_{a+}^{\alpha}u_1(t)}{D_{a+}^{\alpha}u_2(t)}\right|\right) f'\left(D_{a+}^{\beta}u_2(t)\,\phi\left(\left|\frac{D_{a+}^{\beta}u_1(t)}{D_{a+}^{\beta}u_2(t)}\right|\right)\right)dt \\
&\quad\le f\left(\frac{(b-a)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\int_a^b D_{a+}^{\alpha}u_2(t)\,\phi\left(\left|\frac{D_{a+}^{\alpha}u_1(t)}{D_{a+}^{\alpha}u_2(t)}\right|\right)dt\right) \\
&\quad\le \frac{1}{b-a}\int_a^b f\left(\frac{(b-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta)}D_{a+}^{\alpha}u_2(t)\,\phi\left(\left|\frac{D_{a+}^{\alpha}u_1(t)}{D_{a+}^{\alpha}u_2(t)}\right|\right)\right)dt. \qquad (4.21)
\end{aligned}
$$

Proof. We use Theorem 4.2 with the kernel

$$
K(x,t)=\begin{cases}\frac{(x-t)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}, & a<t\le x\\ 0, & x<t\le b\end{cases}. \qquad (4.22)
$$

For $\alpha>\beta+1$, we get

$$
M=\max K(x,t)=\frac{(b-a)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}.
$$

If we replace u_i by $D_{a+}^{\beta}u_i$ and v_i by $D_{a+}^{\alpha}u_i$ $(i=1,2)$ in inequalities given in (4.3) and (4.4), then the inequality (4.21) follows. □

Theorem 4.10 *Let $\beta\ge 0$ and $\alpha>\beta+1$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.22 holds for $\{\alpha,\beta,u_i\}$, $i=1,2$. Let $D_{b-}^{\alpha}u_2(x)>0$ on $[a,b]$. Further, let $\phi(u)$*

be convex, nonnegative and increasing for $u \geq 0$ and let $f(u)$ be convex for $u \geq 0$ with $f(0) = 0$. Then for a.e. $x \in [a,b]$ hold

$$\begin{aligned}
&\frac{(b-a)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \int_a^b D_{b-}^{\alpha} u_2(t)\, \phi\left(\left|\frac{D_{b-}^{\alpha} u_1(t)}{D_{b-}^{\alpha} u_2(t)}\right|\right) f'\left(D_{b-}^{\beta} u_2(t)\, \phi\left(\left|\frac{D_{b-}^{\beta} u_1(t)}{D_{b-}^{\beta} u_2(t)}\right|\right)\right) dt \\
&\leq f\left(\frac{(b-a)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \int_a^b D_{b-}^{\alpha} u_2(t)\, \phi\left(\left|\frac{D_{b-}^{\alpha} u_1(t)}{D_{b-}^{\alpha} u_2(t)}\right|\right) dt\right) \\
&\leq \frac{1}{b-a} \int_a^b f\left(\frac{(b-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta)} D_{b-}^{\alpha} u_2(t)\, \phi\left(\left|\frac{D_{b-}^{\alpha} u_1(t)}{D_{b-}^{\alpha} u_2(t)}\right|\right)\right) dt. \qquad (4.23)
\end{aligned}$$

Next inequalities include the Caputo fractional derivatives. The composition identity for the left-sided fractional derivatives is given in Theorem 2.16. For the right-sided Caputo fractional derivatives we use Theorem 2.17. The proofs of these theorems are similar to the proof of Theorem 4.9.

Theorem 4.11 *Let $\beta \geq 0$, $\alpha > \beta + 1$ and n, m given by* (2.70). *Let $u_i \in AC^n[a,b]$ be such that $u_i^{(k)}(a) = 0$ for $k = m, \ldots, n-1$, $i = 1,2$. Let ${}^C D_{a+}^{\alpha} u_2(x) > 0$ on $[a,b]$. Further, let $\phi(u)$ be convex, nonnegative and increasing for $u \geq 0$ and let $f(u)$ be convex for $u \geq 0$ with $f(0) = 0$. Then for a.e. $x \in [a,b]$ hold*

$$\begin{aligned}
&\frac{(b-a)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \int_a^b {}^C D_{a+}^{\alpha} u_2(t)\, \phi\left(\left|\frac{{}^C D_{a+}^{\alpha} u_1(t)}{{}^C D_{a+}^{\alpha} u_2(t)}\right|\right) f'\left({}^C D_{a+}^{\beta} u_2(t)\, \phi\left(\left|\frac{{}^C D_{a+}^{\beta} u_1(t)}{{}^C D_{a+}^{\beta} u_2(t)}\right|\right)\right) dt \\
&\leq f\left(\frac{(b-a)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \int_a^b {}^C D_{a+}^{\alpha} u_2(t)\, \phi\left(\left|\frac{{}^C D_{a+}^{\alpha} u_1(t)}{{}^C D_{a+}^{\alpha} u_2(t)}\right|\right) dt\right) \\
&\leq \frac{1}{b-a} \int_a^b f\left(\frac{(b-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta)} {}^C D_{a+}^{\alpha} u_2(t)\, \phi\left(\left|\frac{{}^C D_{a+}^{\alpha} u_1(t)}{{}^C D_{a+}^{\alpha} u_2(t)}\right|\right)\right) dt. \qquad (4.24)
\end{aligned}$$

Theorem 4.12 *Let $\beta \geq 0$, $\alpha > \beta + 1$ and n, m given by* (2.70). *Let $u_i \in AC^n[a,b]$ be such that $u_i^{(k)}(b) = 0$ for $k = m, \ldots, n-1$, $i = 1,2$. Let ${}^C D_{b-}^{\alpha} u_2(x) > 0$ on $[a,b]$. Further, let $\phi(u)$ be convex, nonnegative and increasing for $u \geq 0$ and let $f(u)$ be convex for $u \geq 0$ with $f(0) = 0$. Then for a.e. $x \in [a,b]$ hold*

$$\begin{aligned}
&\frac{(b-a)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \int_a^b {}^C D_{b-}^{\alpha} u_2(t)\, \phi\left(\left|\frac{{}^C D_{b-}^{\alpha} u_1(t)}{{}^C D_{b-}^{\alpha} u_2(t)}\right|\right) f'\left({}^C D_{b-}^{\beta} u_2(t)\, \phi\left(\left|\frac{{}^C D_{b-}^{\beta} u_1(t)}{{}^C D_{b-}^{\beta} u_2(t)}\right|\right)\right) dt \\
&\leq f\left(\frac{(b-a)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \int_a^b {}^C D_{b-}^{\alpha} u_2(t)\, \phi\left(\left|\frac{{}^C D_{b-}^{\alpha} u_1(t)}{{}^C D_{b-}^{\alpha} u_2(t)}\right|\right) dt\right) \\
&\leq \frac{1}{b-a} \int_a^b f\left(\frac{(b-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta)} {}^C D_{b-}^{\alpha} u_2(t)\, \phi\left(\left|\frac{{}^C D_{b-}^{\alpha} u_1(t)}{{}^C D_{b-}^{\alpha} u_2(t)}\right|\right)\right) dt. \qquad (4.25)
\end{aligned}$$

Next inequalities include the Canavati fractional derivatives. The composition identity for the left-sided fractional derivatives is given in Theorem 2.19. For the right-sided Canavati fractional derivatives we use Theorem 2.20. The proofs of these theorems are similar to the proof of Theorem 4.9.

Theorem 4.13 *Let $\beta \geq 0$, $\alpha > \beta+1$, $n = [\alpha]+1$, $m = [\beta]+1$. Let $u_i \in C^{\alpha}_{a+}[a,b]$ be such that $u_i^{(k)}(a) = 0$ for $k = m-1,\dots,n-2$, $i = 1,2$. Let ${}^{C_1}D^{\alpha}_{a+}u_2(x) > 0$ on $[a,b]$. Further, let $\phi(u)$ be convex, nonnegative and increasing for $u \geq 0$ and let $f(u)$ be convex for $u \geq 0$ with $f(0) = 0$. Then for a.e. $x \in [a,b]$ hold*

$$\begin{aligned}
&\frac{(b-a)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \int_a^b {}^{C_1}D^{\alpha}_{a+}u_2(t)\,\phi\left(\left|\frac{{}^{C_1}D^{\alpha}_{a+}u_1(t)}{{}^{C_1}D^{\alpha}_{a+}u_2(t)}\right|\right) f'\left({}^{C_1}D^{\beta}_{a+}u_2(t)\,\phi\left(\left|\frac{{}^{C_1}D^{\beta}_{a+}u_1(t)}{{}^{C_1}D^{\beta}_{a+}u_2(t)}\right|\right)\right) dt \\
&\quad \leq f\left(\frac{(b-a)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \int_a^b {}^{C_1}D^{\alpha}_{a+}u_2(t)\,\phi\left(\left|\frac{{}^{C_1}D^{\alpha}_{a+}u_1(t)}{{}^{C_1}D^{\alpha}_{a+}u_2(t)}\right|\right) dt\right) \\
&\quad \leq \frac{1}{b-a} \int_a^b f\left(\frac{(b-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta)} {}^{C_1}D^{\alpha}_{a+}u_2(t)\,\phi\left(\left|\frac{{}^{C_1}D^{\alpha}_{a+}u_1(t)}{{}^{C_1}D^{\alpha}_{a+}u_2(t)}\right|\right)\right) dt. \qquad (4.26)
\end{aligned}$$

Theorem 4.14 *Let $\beta \geq 0$, $\alpha > \beta+1$, $n = [\alpha]+1$, $m = [\beta]+1$. Let $u_i \in C^{\alpha}_{b-}[a,b]$ be such that $u_i^{(k)}(b) = 0$ for $k = m-1,\dots,n-2$, $i = 1,2$. Let ${}^{C_1}D^{\alpha}_{b-}u_2(x) > 0$ on $[a,b]$. Further, let $\phi(u)$ be convex, nonnegative and increasing for $u \geq 0$ and let $f(u)$ be convex for $u \geq 0$ with $f(0) = 0$. Then for a.e. $x \in [a,b]$ hold*

$$\begin{aligned}
&\frac{(b-a)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \int_a^b {}^{C_1}D^{\alpha}_{b-}u_2(t)\,\phi\left(\left|\frac{{}^{C_1}D^{\alpha}_{b-}u_1(t)}{{}^{C_1}D^{\alpha}_{b-}u_2(t)}\right|\right) f'\left({}^{C_1}D^{\beta}_{b-}u_2(t)\,\phi\left(\left|\frac{{}^{C_1}D^{\beta}_{b-}u_1(t)}{{}^{C_1}D^{\beta}_{b-}u_2(t)}\right|\right)\right) dt \\
&\quad \leq f\left(\frac{(b-a)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \int_a^b {}^{C_1}D^{\alpha}_{b-}u_2(t)\,\phi\left(\left|\frac{{}^{C_1}D^{\alpha}_{b-}u_1(t)}{{}^{C_1}D^{\alpha}_{b-}u_2(t)}\right|\right) dt\right) \\
&\quad \leq \frac{1}{b-a} \int_a^b f\left(\frac{(b-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta)} {}^{C_1}D^{\alpha}_{b-}u_2(t)\,\phi\left(\left|\frac{{}^{C_1}D^{\alpha}_{b-}u_1(t)}{{}^{C_1}D^{\alpha}_{b-}u_2(t)}\right|\right)\right) dt. \qquad (4.27)
\end{aligned}$$

4.2 The Mitrinović-Pečarić inequality for relative convex functions

We observe following Opial-type inequality due to Agarwal and Pang ([5, p. 180]).

Theorem 4.15 *Let $u \in C^{n-1}[a,b]$ be such that $u^{(i)}(a) = 0$, $0 \le i \le n-1$ where $n \ge 1$. Let $u^{(n-1)}$ be absolutely continuous and $\int_a^b |u^{(n)}(t)|^q\,dt < \infty$ with $\frac{1}{p}+\frac{1}{q} = 1$. Then*

$$\begin{aligned}
&\int_a^b |u(t)|^p |u^{(n)}(t)|^q\,dt \\
&\le \frac{q}{p+q}\left[\frac{(b-a)^{n-\frac{1}{q}}}{(n-1)!}\left(\frac{q-1}{nq-1}\right)^{\frac{q-1}{q}}\right]^p \left(\int_a^b |u^{(n)}(t)|^q\,dt\right)^{\frac{p+q}{q}} \\
&\le \frac{q}{p+q}\left[\frac{1}{(n-1)!}\left(\frac{q-1}{nq-1}\right)^{\frac{q-1}{q}}\right]^p (b-a)^{np}\int_a^b |u^{(n)}(t)|^{p+q}\,dt\,.
\end{aligned} \tag{4.28}$$

Our object is to give a generalization of their result by extending some known Opial-type integral inequalities, which will in a special case give inequality (4.28) (see Remark 4.3). Inequalities that we extend are given in the following two theorems. Those are inequalities by Mitrinović and Pečarić for relative convex functions ([46]; see also [67, p. 237], [5, p. 90]), and for them we need characterization as in previous chapter:

We say that a function $u : [a,b] \longrightarrow \mathbb{R}$ belongs to the class $U_1(v,K)$ if it admits the representation

$$u(x) = \int_a^x K(x,t)v(t)\,dt\,, \tag{4.29}$$

where v is a continuous function and K is an arbitrary nonnegative kernel such that $v(x) > 0$ implies $u(x) > 0$ for every $x \in [a,b]$. We also assume that all integrals under consideration exist and are finite.

Theorem 4.16 *Let $\phi : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Let $u \in U_1(v,K)$ where $(\int_a^x (K(x,t))^p\,dt)^{\frac{1}{p}} \le M$ and $\frac{1}{p}+\frac{1}{q} = 1$. Then*

$$\int_a^b |u(x)|^{1-q}\phi'(|u(x)|)|v(x)|^q\,dx \le \frac{q}{M^q}\,\phi\left(M\left(\int_a^b |v(x)|^q\,dx\right)^{\frac{1}{q}}\right). \tag{4.30}$$

If the function $\phi(x^{\frac{1}{q}})$ is concave, then the reverse inequality holds.

A similar result follows by using another class $U_2(v,K)$ of functions $u : [a,b] \longrightarrow \mathbb{R}$ which admits representation

$$u(x) = \int_x^b K(x,t)v(t)\,dt. \tag{4.31}$$

Theorem 4.17 *Let $\phi : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Let $u \in U_2(v,K)$ where $\left(\int_x^b (K(x,t))^p dt\right)^{\frac{1}{p}} \leq N$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\int_a^b |u(x)|^{1-q} \phi'(|u(x)|)|v(x)|^q\, dx \;\leq\; \frac{q}{N^q}\, \phi\Big(N\Big(\int_a^b |v(x)|^q\, dx\Big)^{\frac{1}{q}}\Big). \tag{4.32}$$

If the function $\phi(x^{\frac{1}{q}})$ is concave, then the reverse inequality holds.

Now we extend inequalities (4.30) and (4.32), and use them to obtain a generalization of the inequality (4.28).

Theorem 4.18 *Let $\phi : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Let $u \in U_1(v,K)$ where $(\int_a^x (K(x,t))^p\, dt)^{\frac{1}{p}} \leq M$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\begin{aligned}
&\int_a^b |u(x)|^{1-q}\, \phi'(|u(x)|)|v(x)|^q\, dx \\
&\leq \frac{q}{M^q}\, \phi\Big(M\Big(\int_a^b |v(x)|^q\, dx\Big)^{\frac{1}{q}}\Big) &(4.33)\\
&\leq \frac{q}{M^q\,(b-a)} \int_a^b \phi\left((b-a)^{\frac{1}{q}} M|v(x)|\right) dx. &(4.34)
\end{aligned}$$

If the function $\phi(x^{\frac{1}{q}})$ is concave, then reverse inequalities hold.

Proof. Inequality (4.33) holds by Theorem 4.16. Since $\phi(x^{\frac{1}{q}})$ is convex, the following Jensen's inequality holds

$$\phi\Big(\Big(\frac{1}{b-a}\int_a^b g(t)\, dt\Big)^{\frac{1}{q}}\Big) \leq \frac{1}{b-a}\int_a^b \phi\left(g^{\frac{1}{q}}(t)\right) dt\,. \tag{4.35}$$

Applying (4.35) on (4.33) we get (4.34). □

Next we have a special case when $\phi(x) = x^{p+q}$.

Corollary 4.6 *Let $u \in U_1(v,K)$ where $(\int_a^x (K(x,t))^p\, dt)^{\frac{1}{p}} \leq M$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\begin{aligned}
\int_a^b |u(x)|^p |v(x)|^q\, dx &\leq \frac{qM^p}{p+q}\left(\int_a^b |v(x)|^q\, dx\right)^{\frac{p+q}{q}} \\
&\leq \frac{qM^p(b-a)^{\frac{p}{q}}}{p+q} \int_a^b |v(x)|^{p+q}\, dx\,. &(4.36)
\end{aligned}$$

Remark 4.3 If we put $v(x) = u^{(n)}(x)$ in (4.36), then we get a generalization of Agarwal-Pang's inequality (4.28) (the inequality (4.28) follows for additionally $M = \frac{(b-a)^{n-\frac{1}{q}}}{(n-1)!} \cdot \left(\frac{q-1}{nq-1}\right)^{\frac{q-1}{q}}$). The same applies for the Corollary 4.7.

The following results are obtained by extending the inequality (4.32).

Theorem 4.19 *Let $\phi : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Let $u \in U_2(v,K)$ where $\left(\int_x^b (K(x,t))^p\, dt\right)^{\frac{1}{p}} \le N$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\begin{aligned} &\int_a^b |u(x)|^{1-q}\phi'(|u(x)|)|v(x)|^q dx \\ &\le \frac{q}{N^q}\,\phi\left(N\left(\int_a^b |v(x)|^q\, dx\right)^{\frac{1}{q}}\right) \\ &\le \frac{q}{N^q(b-a)}\int_a^b \phi\left((b-a)^{\frac{1}{q}}N|v(x)|\right) dx. \end{aligned} \tag{4.37}$$

If the function $\phi(x^{\frac{1}{q}})$ is concave, then reverse inequalities hold.

Proof. As in the proof of the previous theorem, inequalities follow from Theorem 4.17 and Jensen's inequality (4.35). □

Corollary 4.7 *Let $u \in U_2(v,K)$ where $\left(\int_x^b (K(x,t))^p\, dt\right)^{\frac{1}{p}} \le N$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\begin{aligned} \int_a^b |u(x)|^p|v(x)|^q\, dx &\le \frac{qN^p}{p+q}\left(\int_a^b |v(x)|^q\, dx\right)^{\frac{p+q}{q}} \\ &\le \frac{qN^p(b-a)^{\frac{p}{q}}}{p+q}\int_a^b |v(x)|^{p+q}\, dx. \end{aligned} \tag{4.38}$$

4.2.1 Mean value theorems and exponential convexity

Motivated by the inequality (4.34), we define next functional:

$$\begin{aligned} \Psi_\phi(u,v) \;=\; &\frac{q}{M^q(b-a)}\int_a^b \phi\left((b-a)^{\frac{1}{q}}M|v(x)|\right) dx \\ &- \int_a^b |u(x)|^{1-q}\,\phi'(|u(x)|)|v(x)|^q\, dx, \end{aligned} \tag{4.39}$$

where functions ϕ, u and v are as in Theorem 4.18.
If $\phi(x^{\frac{1}{q}})$ is a convex function ($q > 1$), then by Theorem 4.18 $\Psi_\phi(u,v) \ge 0$.

For our results we need Definition 1.2 and next lemma from [43].

Lemma 4.1 *Let $I \subseteq (0,\infty)$, $\phi \in C^2(I)$, $g(x) = x^q$, $q > 1$ and let*

$$m_1 \le \frac{\xi\phi''(\xi) - (q-1)\phi'(\xi)}{q^2\xi^{2q-1}} \le M_1\,,$$

for all $\xi \in I$. Then the functions ϕ_1, ϕ_2 defined as

$$\phi_1(x) = \frac{M_1 x^{2q}}{2} - \phi(x) \tag{4.40}$$

$$\phi_2(x) = \phi(x) - \frac{m_1 x^{2q}}{2} \tag{4.41}$$

are convex functions with respect to $g(x) = x^q$, that is $\phi_i(x^{\frac{1}{q}})$ $(i = 1,2)$ are convex.

Next two theorems are our main results, and they follow methods used in [43, 44].

Theorem 4.20 *Let $\phi : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Let $u \in U_1(v,K)$ where $(\int_a^x (K(x,t))^p\,dt)^{\frac{1}{p}} \leq M$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $\phi \in C^2(I)$, where $I \subseteq (0,\infty)$ is closed interval, then there exists $\xi \in I$ such that the following equality holds*

$$\Psi_\phi(u,v) = \frac{\xi\,\phi''(\xi) - (q-1)\,\phi'(\xi)}{2\,q\,\xi^{2q-1}} \cdot \left((b-a)M^q \int_a^b |v(x)|^{2q}\,dx - 2\int_a^b |u(x)|^q\,|v(x)|^q\,dx \right). \tag{4.42}$$

Proof. Suppose that $\psi(y)$ is bounded and $\min(\psi(y)) = m_1$, $\max(\psi(y)) = M_1$ where

$$\psi(y) = \frac{y\,\phi''(y) - (q-1)\phi'(y)}{q^2 y^{2q-1}}.$$

If we apply Theorem 4.18 for ϕ_1 defined by (4.40), then inequality (4.34) becomes

$$\frac{q}{M^q\,(b-a)} \int_a^b \phi\left((b-a)^{\frac{1}{q}} M|v(x)|\right) dx - \int_a^b |u(x)|^{1-q}\,\phi'(|u(x)|)|v(x)|^q\,dx \leq \frac{q M_1}{2}\left((b-a)M^q \int_a^b |v(x)|^{2q}\,dx - 2\int_a^b |u(x)|^q\,|v(x)|^q\,dx\right). \tag{4.43}$$

Similarly, if we apply Theorem 4.18 for ϕ_2 defined by (4.41), then inequality (4.34) becomes

$$\frac{q}{M^q\,(b-a)} \int_a^b \phi\left((b-a)^{\frac{1}{q}} M|v(x)|\right) dx - \int_a^b |u(x)|^{1-q}\,\phi'(|u(x)|)|v(x)|^q\,dx \geq \frac{q m_1}{2}\left((b-a)M^q \int_a^b |v(x)|^{2q}\,dx - 2\int_a^b |u(x)|^q\,|v(x)|^q\,dx\right). \tag{4.44}$$

By combining the above two inequalities with the fact

$$m_1 \leq \frac{y\,\phi''(y) - (q-1)\phi'(y)}{q^2 y^{2q-1}} \leq M_1,$$

there exists $\xi \in I$ such that (4.42) follows. □

Theorem 4.21 *Let $\phi_1, \phi_2 : [0,\infty) \longrightarrow \mathbb{R}$ be differentiable functions such that for $q > 1$ the function $\phi_i(x^{\frac{1}{q}})$ is convex and $\phi_i(0) = 0$, $i = 1,2$. Let $u \in U_1(v,K)$ where $(\int_a^x (K(x,t))^p\,dt)^{\frac{1}{p}} \leq M$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $\phi_1, \phi_2 \in C^2(I)$, where $I \subseteq (0,\infty)$ is closed interval and*

$$(b-a)M^q \int_a^b |v(x)|^{2q}\,dx - 2\int_a^b |u(x)|^q |v(x)|^q dx \neq 0\,,$$

then there exists an $\xi \in I$ such that we have

$$\frac{\Psi_{\phi_1}(u,v)}{\Psi_{\phi_2}(u,v)} = \frac{\xi\,\phi_1''(\xi) - (q-1)\,\phi_1'(\xi)}{\xi\,\phi_2''(\xi) - (q-1)\,\phi_2'(\xi)}\,, \tag{4.45}$$

provided the denominators are not equal to zero.

Proof. Let us consider $h \in C^2(I)$ defined by

$$h = \Psi_{\phi_2}(u,v)\,\phi_1 - \Psi_{\phi_1}(u,v)\,\phi_2\,.$$

For this function, (4.39) gives us $\Psi_h(u,v) = 0$. By Theorem 4.20 used on h follows that there exists $\xi \in I$ such that

$$\Psi_{\phi_2}(u,v)\frac{\xi\phi_1''(\xi) - (q-1)\phi_1'(\xi)}{2q\,\xi^{2q-1}} - \Psi_{\phi_1}(u,v)\frac{\xi\phi_2''(\xi) - (q-1)\phi_2'(\xi)}{2q\,\xi^{2q-1}}$$
$$\cdot\left((b-a)M^q \int_a^b |v(x)|^{2q}\,dx - 2\int_a^b |u(x)|^q|v(x)|^q dx\right) = 0.$$

From this we get (4.45). □

Remark 4.4 By considering nonnegative difference of inequality given in Theorem 4.19, similar results can be done analogously (for details see [44]).

We continue with the method of producing n-exponentially convex and exponentially convex functions given in [50], to prove the n-exponential convexity for the functional $\Psi_\phi(u,v)$ defined by (4.39).

Theorem 4.22 *Let J be an interval in $\mathbb{R}$ and $\Upsilon = \{\phi_s : s \in J\}$ be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $s \mapsto [y_0, y_1, y_2; F_{\phi_s}]$ is n-exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$, where $F_{\phi_s}(y) = \phi_s(y^{\frac{1}{q}})$. Let $\Psi_{\phi_s}(u,v)$ be a linear functional defined by* (4.39). *Then $s \mapsto \Psi_{\phi_s}(u,v)$ is n-exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Psi_{\phi_s}(u,v)$ is also continuous on J, then it is n-exponentially convex on J.*

Proof. For $\xi_i \in \mathbb{R}$, $s_i \in J$, $i = 1,\ldots,n$, we define the function

$$h(y) = \sum_{i,j=1}^{n} \xi_i\xi_j\phi_{\frac{s_i+s_j}{2}}(y)\,.$$

Set

$$H(y) = h(y^{\frac{1}{q}}).$$

Using the assumption that the function $s \mapsto [y_0, y_1, y_2; F_{\phi_s}]$ is n-exponentially convex in the Jensen sense, we have

$$[y_0, y_1, y_2; H] = \sum_{i,j=1}^{n} \xi_i \xi_j [y_0, y_1, y_2; F_{\phi_{\frac{s_i+s_j}{2}}}] \geq 0,$$

which in turn implies that H is a convex function on I. Therefore we have $\Psi_h(u,v) \geq 0$. Hence

$$\sum_{i,j=1}^{n} \xi_i \xi_j \Psi_{\phi_{\frac{s_i+s_j}{2}}}(u,v) \geq 0.$$

We conclude that the function $s \mapsto \Psi_{\phi_s}(u,v)$ is n-exponentially convex on J in the Jensen sense. If the function $s \mapsto \Psi_{\phi_s}(u,v)$ is also continuous on J, then it is n-exponentially convex by definition. □

Corollary 4.8 *Let J be an interval in $\mathbb{R}$ and $\Upsilon = \{\phi_s : s \in J\}$ be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $s \mapsto [y_0, y_1, y_2; F_{\phi_s}]$ is exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$, where $F_{\phi_s}(y) = \phi_s(y^{\frac{1}{q}})$. Let $\Psi_{\phi_s}(u,v)$ be a linear functional defined by* (4.39). *Then $s \mapsto \Psi_{\phi_s}(u,v)$ is exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Psi_{\phi_s}(u,v)$ is continuous on J, then it is exponentially convex on J.*

Let us denote means for $\phi_s, \phi_p \in \Omega$ by

$$\mu_{s,p}(\Psi,\Omega) = \begin{cases} \left(\dfrac{\Psi_{\phi_s}(u,v)}{\Phi_{\phi_p}(u,v)}\right)^{\frac{1}{s-p}}, & s \neq p, \\ \exp\left(\dfrac{\frac{d}{ds}\Psi_{\phi_s}(u,v)}{\Psi_{\phi_s}(u,v)}\right), & s = p. \end{cases} \tag{4.46}$$

Theorem 4.23 *Let J be an interval in $\mathbb{R}$ and $\Omega = \{\phi_s : s \in J\}$ be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $s \mapsto [y_0, y_1, y_2; F_{\phi_s}]$ is 2-exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$, where $F_{\phi_s}(y) = \phi_s(y^{\frac{1}{q}})$. Let $\Psi_{\phi_s}(u,v)$ be a linear functional defined by* (4.39). *Then the following statements hold:*

(i) If the function $s \mapsto \Psi_{\phi_s}(u,v)$ is continuous on J, then it is 2-exponentially convex function on J. If the function $s \mapsto \Psi_{\phi_s}(u,v)$ is additionally positive, then it is also log-convex on J, and for $r,s,t \in J$ such that $r < s < t$, we have

$$\left(\Psi_{\phi_s}(u,v)\right)^{t-r} \leq \left(\Psi_{\phi_r}(u,v)\right)^{t-s} \left(\Psi_{\phi_t}(u,v)\right)^{s-r}. \tag{4.47}$$

(ii) If the function $s \mapsto \Psi_{\phi_s}(u,v)$ is positive and differentiable on J, then for every $s,p,r,t \in J$, such that $s \leq r$ and $p \leq t$, we have

$$\mu_{s,p}(\Psi,\Omega) \leq \mu_{r,t}(\Psi,\Omega). \tag{4.48}$$

Proof. (i) The first part is an immediate consequence of Theorem 4.22 and in second part log-convexity on J follows from Remark 1.3. Since $s \mapsto \Psi_{\phi_s}(u,v)$ is positive, for $r,s,t \in J$ such that $r < s < t$, with $f(s) = \log \Psi_{\phi_s}(u,v)$ in Proposition 1.2, we have

$$(t-s)\log \Psi_{\phi_r}(u,v) + (r-t)\log \Psi_{\phi_s}(u,v) + (s-r)\log \Psi_{\phi_t}(u,v) \;\geq\; 0\,.$$

This is equivalent to inequality (4.47).
(ii) The function $s \mapsto \Psi_{\phi_s}(u,v)$ is log-convex on J by (i), that is, the function $s \mapsto \log \Psi_{\phi_s}(u,v)$ is convex on J. Applying Proposition 1.3 we get

$$\frac{\log \Psi_{\phi_s}(u,v) - \log \Psi_{\phi_p}(u,v)}{s-p} \;\leq\; \frac{\log \Psi_{\phi_r}(u,v) - \log \Psi_{\phi_t}(u,v)}{r-t} \tag{4.49}$$

for $s \leq r$, $p \leq t$, $s \neq p$, $r \neq t$, and therefore we have

$$\mu_{s,p}(\Psi,\Omega) \;\leq\; \mu_{r,t}(\Psi,\Omega)\,.$$

Cases $s = p$ and $r = t$ follows from (4.49) as limit cases. □

Remark 4.5 Results from Theorem 4.22, Corollary 4.8 and Theorem 4.23 still hold when two of the points $y_0, y_1, y_2 \in I$ coincide, for a family of differentiable functions ϕ_s such that the function $s \mapsto [y_0, y_1, y_2; F_{\phi_s}]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense). Furthermore, they still hold when all three points coincide for a family of twice differentiable functions with the same property. The proofs can be obtained by recalling Remark 1.1 and suitable characterization of convexity.

4.2.2 Applications to Stolarsky type means

We use Cauchy type mean value Theorem 4.20 and Theorem 4.21 for Stolarsky type means and functional $\Psi_\phi(u,v)$. Several families of functions which fulfil conditions of Theorem 4.22, Corollary 4.8 and Theorem 4.23 (and Remark 4.5) that we present here, enable us to construct large families of functions which are exponentially convex.

Example 4.5 Consider a family of functions

$$\Omega_1 = \{\phi_s : [0,\infty) \to \mathbb{R} : s > 0\}$$

defined for $q > 1$ by

$$\phi_s(x) = \begin{cases} \frac{q^2}{s(s-q)}\, x^s, & s \neq 0, q, \\ q x^q \log x, & s = q. \end{cases}$$

Then $\left[\phi_s(x^{\frac{1}{q}})\right]'' = x^{\frac{s-2q}{q}} = e^{\frac{s-2q}{q}\ln x} > 0$ which show that ϕ_s is convex function with respect to $g(x) = x^q$ for $x > 0$, and $s \mapsto \left[\phi_s(x^{\frac{1}{q}})\right]''$ is exponentially convex by definition. Notice $\phi_s(0) = 0$, with the convention $0 \log 0 = 0$.

Analogously as in the proof of Theorem 4.22 we conclude that $s \mapsto [y_0, y_1, y_2; F_{\phi_s}]$ is exponentially convex (and so exponentially convex in the Jensen sense), where $F_{\phi_s}(y) = \phi_s(y^{\frac{1}{q}})$. By Corollary 4.8 we have that $s \mapsto \Psi_{\phi_s}(u,v)$ is exponentially convex in the Jensen sense. It is easy to verify that this mappings are continuous, so they are exponentially convex.

Hence, we have

$$\Psi_{\phi_s}(u,v) = \begin{cases} \frac{q^3(b-a)^{\frac{s}{q}-1}M^{s-q}}{s(s-q)} \int\limits_a^b |v(x)|^s \, dx - \frac{q^2}{s-q} \int\limits_a^b |u(x)|^{s-q} |v(x)|^q \, dx, \\ s \neq 0, q, \\ q^2 \int\limits_a^b |v(x)|^q \log\left[(b-a)^{\frac{1}{q}} M|v(x)|\right] dx - q \int\limits_a^b |v(x)|^q \left[q \log|u(x)| + 1\right] dx, \\ s = q. \end{cases}$$

For this family of functions, $\mu_{s,t}(\Psi, \Omega_1)$ from (4.46) becomes

$$\mu_{s,t}(\Psi, \Omega_1) = \begin{cases} \left(\frac{\Psi_{\phi_s}(u,v)}{\Psi_{\phi_t}(u,v)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(\frac{q-2s}{s(s-q)} + \frac{\Psi_{\phi_s \cdot \log}(u,v)}{\Psi_{\phi_s}(u,v)}\right), & s = t \neq q, \\ \exp\left(-\frac{1}{q} + \frac{\Psi_{\phi_q \cdot \log}(u,v)}{2\Psi_{\phi_q}(u,v)}\right), & s = t = q, \end{cases}$$

and by (4.48) it is monotonous in parameters s and t.
For the functional $\Psi_\phi(u,v)$ we get

$$\mu_{s,t}(\Psi, \Omega_1) = \begin{cases} \left(\frac{qt(t-q)(b-a)^{\frac{s}{q}-1}M^{s-q}\int\limits_a^b |v(x)|^s \, dx - s\int\limits_a^b |u(x)|^{s-q}|v(x)|^q \, dx}{qs(s-q)(b-a)^{\frac{t}{q}-1}M^{t-q}\int\limits_a^b |v(x)|^t \, dx - t\int\limits_a^b |u(x)|^{t-q}|v(x)|^q \, dx}\right)^{\frac{1}{s-t}}, s \neq t, \\ \exp\left(\frac{q-2s}{s(s-q)}\right. \\ \left. + \frac{q(b-a)^{\frac{s}{q}-1}M^{s-q}\int\limits_a^b |v(x)|^s \log\left[(b-a)^{\frac{1}{q}} M|v(x)|\right] dx - \int\limits_a^b [s\log|u(x)|+1]|u(x)|^{s-q}|v(x)|^q \, dx}{q(b-a)^{\frac{s}{q}-1}M^{s-q}\int\limits_a^b |v(x)|^s \, dx - s\int\limits_a^b |u(x)|^{s-q}|v(x)|^q \, dx}\right), \\ s = t \neq q, \\ \exp\left(-\frac{1}{q} + \frac{q\int\limits_a^b |v(x)|^q \log^2\left[(b-a)^{\frac{1}{q}} M|v(x)|\right] dx - \int\limits_a^b [q\log|u(x)|+2]|v(x)|^q \log|u(x)| dx}{2q\int\limits_a^b |v(x)|^q \log\left[(b-a)^{\frac{1}{q}} M|v(x)|\right] dx - 2\int\limits_a^b [q\log|u(x)|+1]|v(x)|^q \, dx}\right), \\ s = t = q. \end{cases}$$

Example 4.6 Consider a family of functions

$$\Omega_2 = \{\varphi_s : [0, \infty) \to \mathbb{R} : s \in \mathbb{R}\}$$

defined for $q > 1$ by

$$\varphi_s(x) = \begin{cases} \frac{e^{sx^q}-1}{s^2}, & s \neq 0, \\ \frac{x^{2q}}{2}, & s = 0. \end{cases}$$

Since $\left[\varphi(x^{\frac{1}{q}})\right]'' = e^{sx} > 0$, then φ_s is convex function with respect to $g(x) = x^q$ for $x > 0$, and $s \mapsto \left[\varphi_s(x^{\frac{1}{q}})\right]''$ is exponentially convex by definition. Notice that $\varphi_s(0) = 0$. Arguing as in the previous example, we get that the mapping $s \mapsto \Phi_{\varphi_s}(u,v)$ is exponentially convex.

We have

$$\Psi_{\varphi_s}(u,v) = \begin{cases} \frac{q}{s^2 M^q (b-a)} \int_a^b \{\exp[s(b-a)M^q |v(x)|^q] - 1\}\, dx \\ -\frac{q}{s}\int_a^b |v(x)|^q \exp[s|u(x)|^q] dx, & s \neq 0, \\ \frac{q(b-a)M^q}{2} \int_a^b |v(x)|^{2q} dx - q\int_a^b |u(x)|^q |v(x)|^q\, dx, & s = 0. \end{cases}$$

For this family of functions, $\mu_{s,t}(\Phi, \Omega_2)$ from (4.46) becomes

$$\mu_{s,t}(\Psi,\Omega_2) = \begin{cases} \left(\frac{\Psi_{\varphi_s}(u,v)}{\Psi_{\varphi_t}(u,v)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(-\frac{2}{s} + \frac{\Psi_{x^q \cdot \varphi_s}(u,v)}{\Psi_{\varphi_s}(u,v)}\right), & s = t \neq 0, \\ \exp\left(\frac{\Psi_{x^q \cdot \varphi_0}(u,v)}{3\Psi_{\varphi_0}(u,v)}\right), & s = t = 0, \end{cases}$$

and by (4.48) it is monotonous in parameters s and t.
For the functional $\Psi_\varphi(u,v)$ we get

$$\mu_{s,t}(\Psi,\Omega_2) = \begin{cases} \left(\frac{s^{-2}\int_a^b \{\exp[s(b-a)M^q|v(x)|^q]-1\}dx - s^{-1}M^q(b-a)\int_a^b \exp(s|u(x)|^q)|v(x)|^q\, dx}{t^{-2}\int_a^b \{\exp[t(b-a)M^q|v(x)|^q]-1\}dx - t^{-1}M^q(b-a)\int_a^b \exp(t|u(x)|^q)|v(x)|^q\, dx}\right)^{\frac{1}{s-t}}, \\ s \neq t, \\ \exp\left(-\frac{2}{s} + \frac{\int_a^b \exp[s(b-a)M^q|v(x)|^q]|v(x)|^q\, dx - \int_a^b (1+s|u(x)|^q)\exp(s|u(x)|^q)|v(x)|^q\, dx}{\frac{1}{M^q(b-a)}\int_a^b \{\exp[s(b-a)M^q|v(x)|^q]-1\}dx - s\int_a^b \exp(s|u(x)|^q)|v(x)|^q dx}\right), \\ s = t \neq 0, \\ \exp\left(\frac{(b-a)^2 M^{2q}\int_a^b |v(x)|^{3q}dx - 3\int_a^b |u(x)|^{2q}|v(x)|^q dx}{3(b-a)M^q\int_a^b |v(x)|^{2q}dx - 6\int_a^b |u(x)|^q|v(x)|^q dx}\right), \\ s = t = 0. \end{cases}$$

4.2.3 Opial-type inequalities for fractional integrals and fractional derivatives

First result is based on Theorem 4.18 and the left-sided Riemann-Liouville fractional integrals.

Theorem 4.24 *Let $\phi : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Further, let $\alpha > \frac{1}{q}$, $\frac{1}{p} + \frac{1}{q} = 1$ and $v \in L_1[a,b]$. Then following inequalities hold*

$$\int_a^b \left|J_{a+}^\alpha v(x)\right|^{1-q} \phi'\left(\left|J_{a+}^\alpha v(x)\right|\right) |v(x)|^q\, dx$$

$$\leq \frac{q\Gamma^q(\alpha)\, p^{\frac{q}{p}}\left(\alpha - \frac{1}{q}\right)^{\frac{q}{p}}}{(b-a)^{q\alpha-1}}\, \phi\left(\frac{(b-a)^{\alpha-\frac{1}{q}}}{\Gamma(\alpha)\, p^{\frac{1}{p}}\left(\alpha-\frac{1}{q}\right)^{\frac{1}{p}}}\left(\int_a^b |v(x)|^q\, dx\right)^{\frac{1}{q}}\right)$$

$$\leq \frac{q\Gamma^q(\alpha)\, p^{\frac{q}{p}} \left(\alpha-\frac{1}{q}\right)^{\frac{q}{p}}}{(b-a)^{q\alpha}} \int_a^b \phi\left(\frac{(b-a)^{\alpha}\,|v(x)|}{\Gamma(\alpha)\, p^{\frac{1}{p}}\left(\alpha-\frac{1}{q}\right)^{\frac{1}{p}}}\right) dx. \tag{4.50}$$

If the function $\phi(x^{\frac{1}{q}})$ is concave, then reverse inequalities hold.

Proof. For $x \in [a,b]$ let

$$K(x,t) = \begin{cases} \frac{1}{\Gamma(\alpha)}(x-t)^{\alpha-1}, & a \leq t \leq x; \\ 0, & x < t \leq b, \end{cases}$$

$$u(x) = J^{\alpha}_{a+} v(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} v(t)\, dt, \tag{4.51}$$

$$P(x) = \left(\int_a^x (K(x,t))^p\, dt\right)^{\frac{1}{p}} = \frac{(x-a)^{\alpha-\frac{1}{q}}}{\Gamma(\alpha)\left[p\left(\alpha-\frac{1}{q}\right)\right]^{\frac{1}{p}}}.$$

It is easy to see that for $\alpha > \frac{1}{q}$ the function P is increasing on $[a,b]$, thus

$$\max_{x\in[a,b]} P(x) = \frac{(b-a)^{\alpha-\frac{1}{q}}}{\Gamma(\alpha)\, p^{\frac{1}{p}}\left(\alpha-\frac{1}{q}\right)^{\frac{1}{p}}} = M.$$

Hence $(\int_a^x K(x,t)^p dt)^{\frac{1}{p}} \leq M$, which with the function u defined by (4.51) and Theorem 4.18 gives us (4.50). □

Corollary 4.9 *Let $\alpha > \frac{1}{q}$, $\frac{1}{p}+\frac{1}{q} = 1$ and $v \in L_1[a,b]$. Then following inequalities hold*

$$\int_a^b \left|J^{\alpha}_{a+} v(x)\right|^p |v(x)|^q\, dx \leq \frac{q\,(b-a)^{p\left(\alpha-\frac{1}{q}\right)}}{(p+q)\,\Gamma^p(\alpha)\, p\left(\alpha-\frac{1}{q}\right)} \left(\int_a^b |v(x)|^q dx\right)^{\frac{p+q}{q}}$$
$$\leq \frac{q\,(b-a)^{p\alpha}}{(p+q)\,\Gamma^p(\alpha)\, p\left(\alpha-\frac{1}{q}\right)} \int_a^b |v(x)|^{p+q} dx.$$

A similar results follows for the right-sided Riemann-Liouville fractional integrals.

Theorem 4.25 *Let $\phi : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Further, let $\alpha > \frac{1}{q}$, $\frac{1}{p}+\frac{1}{q} = 1$ and $v \in L_1[a,b]$. Then following inequalities hold*

$$\int_a^b \left|J^{\alpha}_{b-} v(x)\right|^{1-q} \phi'\left(\left|J^{\alpha}_{b-} v(x)\right|\right) |v(x)|^q\, dx$$

$$\leq \frac{q\Gamma^q(\alpha)\,p^{\frac{q}{p}}\left(\alpha-\frac{1}{q}\right)^{\frac{q}{p}}}{(b-a)^{q\alpha-1}}\,\phi\left(\frac{(b-a)^{\alpha-\frac{1}{q}}}{\Gamma(\alpha)\,p^{\frac{1}{p}}\left(\alpha-\frac{1}{q}\right)^{\frac{1}{p}}}\left(\int_a^b |v(x)|^q dx\right)^{\frac{1}{q}}\right)$$

$$\leq \frac{q\Gamma^q(\alpha)\,p^{\frac{q}{p}}\left(\alpha-\frac{1}{q}\right)^{\frac{q}{p}}}{(b-a)^{q\alpha}}\int_a^b \phi\left(\frac{(b-a)^{\alpha}|v(x)|}{\Gamma(\alpha)\,p^{\frac{1}{p}}\left(\alpha-\frac{1}{q}\right)^{\frac{1}{p}}}\right)dx. \tag{4.52}$$

If the function $\phi(x^{\frac{1}{q}})$ is concave, then reverse inequalities hold.

Proof. The proof is similar to the proof of Theorem 4.24. □

Corollary 4.10 *Let $\alpha > \frac{1}{q}$, $\frac{1}{p}+\frac{1}{q}=1$ and $v \in L_1[a,b]$. Then following inequalities hold*

$$\int_a^b \left|J_{b-}^{\alpha} v(x)\right|^p |v(x)|^q\, dx \leq \frac{q\,(b-a)^{p\left(\alpha-\frac{1}{q}\right)}}{(p+q)\,\Gamma^p(\alpha)\,p\left(\alpha-\frac{1}{q}\right)}\left(\int_a^b |v(x)|^q dx\right)^{\frac{p+q}{q}}$$
$$\leq \frac{q\,(b-a)^{p\alpha}}{(p+q)\,\Gamma^p(\alpha)\,p\left(\alpha-\frac{1}{q}\right)}\int_a^b |v(x)|^{p+q} dx.$$

Next, we observe the Caputo fractional derivatives (left-sided and then right-sided).

Theorem 4.26 *Let $\phi : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Further, let $\alpha \geq 0$, n given by (2.70) and $v \in AC^n[a,b]$. If $n-\alpha > \frac{1}{q}$ and $\frac{1}{p}+\frac{1}{q}=1$, then following inequalities hold*

$$\int_a^b \left|{}^C D_{a+}^{\alpha} v(x)\right|^{1-q} \phi'\left(\left|{}^C D_{a+}^{\alpha} v(x)\right|\right)\left|v^{(n)}(x)\right|^q dx$$
$$\leq \frac{q\Gamma^q(n-\alpha)\,p^{\frac{q}{p}}\left(n-\alpha-\frac{1}{q}\right)^{\frac{q}{p}}}{(b-a)^{q(n-\alpha)-1}}$$
$$\cdot\,\phi\left(\frac{(b-a)^{n-\alpha-\frac{1}{q}}}{\Gamma(n-\alpha)\,p^{\frac{1}{p}}\left(n-\alpha-\frac{1}{q}\right)^{\frac{1}{p}}}\left(\int_a^b \left|v^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}\right)$$
$$\leq \frac{q\Gamma^q(n-\alpha)\,p^{\frac{q}{p}}\left(n-\alpha-\frac{1}{q}\right)^{\frac{q}{p}}}{(b-a)^{q(n-\alpha)}}$$
$$\cdot\int_a^b \phi\left(\frac{(b-a)^{n-\alpha}\left|v^{(n)}(x)\right|}{\Gamma(n-\alpha)\,p^{\frac{1}{p}}\left(n-\alpha-\frac{1}{q}\right)^{\frac{1}{p}}}\right)dx. \tag{4.53}$$

If the function $\phi(x^{\frac{1}{q}})$ is concave, then reverse inequalities hold.

Proof. For $x \in [a,b]$ let

$$K(x,t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}(x-t)^{n-\alpha-1}, & a \le t \le x; \\ 0, & x < t \le b. \end{cases}$$

$$u(x) = {}^C D^{\alpha}_{a+} v(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} v^{(n)}(t)\, dt\,, \tag{4.54}$$

$$Q(x) = \left(\int_a^x (K(x,t))^p dt \right)^{\frac{1}{p}} = \frac{(x-a)^{n-\alpha-\frac{1}{q}}}{\Gamma(n-\alpha)\left[p\left(n-\alpha-\frac{1}{q}\right)\right]^{\frac{1}{p}}}\,.$$

For $n-\alpha > \frac{1}{q}$ the function Q is increasing on $[a,b]$, thus

$$\max_{x\in[a,b]} Q(x) = \frac{(b-a)^{n-\alpha-\frac{1}{q}}}{\Gamma(n-\alpha)\, p^{\frac{1}{p}} \left(n-\alpha-\frac{1}{q}\right)^{\frac{1}{p}}} = M\,.$$

Hence $(\int_a^x K(x,t)^p dt)^{\frac{1}{p}} \le M$, which with $v = v^{(n)}$, u as in (4.54) and Theorem 4.18 gives us (4.53). □

Corollary 4.11 *Let $\alpha \ge 0$, n given by* (2.70) *and $v \in AC^n[a,b]$. If $n-\alpha > \frac{1}{q}$ and $\frac{1}{p}+\frac{1}{q} = 1$, then following inequalities hold*

$$\begin{aligned} &\int_a^b \left|{}^C D^{\alpha}_{a+} v(x)\right|^p \left|v^{(n)}(x)\right|^q dx \\ &\le \frac{q\,(b-a)^{p\left(n-\alpha-\frac{1}{q}\right)}}{(p+q)\,\Gamma^p(n-\alpha)\, p\left(n-\alpha-\frac{1}{q}\right)} \left(\int_a^b \left|v^{(n)}(x)\right|^q dx \right)^{\frac{p+q}{q}} \\ &\le \frac{q\,(b-a)^{p(n-\alpha)}}{(p+q)\,\Gamma^p(n-\alpha)\, p\left(n-\alpha-\frac{1}{q}\right)} \int_a^b \left|v^{(n)}(x)\right|^{p+q} dx\,. \end{aligned}$$

Theorem 4.27 *Let $\phi : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Further, let $\alpha \ge 0$, n given by* (2.70) *and $v \in AC^n[a,b]$. For $\frac{1}{p}+\frac{1}{q} = 1$ and even n such that $n-\alpha > \frac{1}{q}$ following inequalities hold*

$$\begin{aligned} &\int_a^b \left|{}^C D^{\alpha}_{b-} v(x)\right|^{1-q} \phi'\left(\left|{}^C D^{\alpha}_{b-} v(x)\right|\right) \left|v^{(n)}(x)\right|^q dx \\ &\le \frac{q\,\Gamma^q(n-\alpha)\, p^{\frac{q}{p}} \left(n-\alpha-\frac{1}{q}\right)^{\frac{q}{p}}}{(b-a)^{q(n-\alpha)-1}} \\ &\quad \cdot \phi\left(\frac{(b-a)^{n-\alpha-\frac{1}{q}}}{\Gamma(n-\alpha)\, p^{\frac{1}{p}} \left(n-\alpha-\frac{1}{q}\right)^{\frac{1}{p}}} \left(\int_a^b \left|v^{(n)}(x)\right|^q dx \right)^{\frac{1}{q}} \right) \end{aligned}$$

$$\leq \frac{q\,\Gamma^q(n-\alpha)\,p^{\frac{q}{p}}\left(n-\alpha-\frac{1}{q}\right)^{\frac{q}{p}}}{(b-a)^{q(n-\alpha)}}$$

$$\cdot \int_a^b \phi\left(\frac{(b-a)^{n-\alpha}\left|v^{(n)}(x)\right|}{\Gamma(n-\alpha)\,p^{\frac{1}{p}}\left(n-\alpha-\frac{1}{q}\right)^{\frac{1}{p}}}\right)dx. \tag{4.55}$$

If the function $\phi(x^{\frac{1}{q}})$ is concave, then reverse inequalities hold.

Proof. The proof is similar to the proof of Theorem 4.26. □

Corollary 4.12 *Let $\alpha \geq 0$, n given by (2.70) and $v \in AC^n[a,b]$. For $\frac{1}{p}+\frac{1}{q}=1$ and even n such that $n-\alpha > \frac{1}{q}$ following inequalities hold*

$$\int_a^b \left|{}^C D^{\alpha}_{b-}v(x)\right|^p \left|v^{(n)}(x)\right|^q dx$$

$$\leq \frac{q\,(b-a)^{p\left(n-\alpha-\frac{1}{q}\right)}}{(p+q)\,\Gamma^p(n-\alpha)\,p\left(n-\alpha-\frac{1}{q}\right)}\left(\int_a^b \left|v^{(n)}(x)\right|^q dx\right)^{\frac{p+q}{q}}$$

$$\leq \frac{q\,(b-a)^{p(n-\alpha)}}{(p+q)\,\Gamma^p(n-\alpha)\,p\left(n-\alpha-\frac{1}{q}\right)}\int_a^b \left|v^{(n)}(x)\right|^{p+q} dx.$$

For the following inequality we use the composition identity for the left-sided Caputo fractional derivatives given in Theorem 2.16.

Theorem 4.28 *Let $\phi : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q>1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0)=0$. Further, let $\frac{1}{p}+\frac{1}{q}=1$, $\alpha-\beta>\frac{1}{q}$, $\beta \geq 0$, n and m given by (2.70) for α and β respectively. Let $v \in AC^n[a,b]$ be such that $v^{(i)}(a)=0$ for $i=m,\dots,n-1$. Let ${}^C D^{\alpha}_{a+}v \in L_q[a,b]$ and ${}^C D^{\beta}_{a+}v \in L_1[a,b]$. Then following inequalities hold*

$$\int_a^b \left|{}^C D^{\beta}_{a+}v(x)\right|^{1-q} \phi'\left(\left|{}^C D^{\beta}_{a+}v(x)\right|\right)\left|{}^C D^{\alpha}_{a+}v(x)\right|^q dx$$

$$\leq \frac{q\,\Gamma^q(\alpha-\beta)\,p^{\frac{q}{p}}\left(\alpha-\beta-\frac{1}{q}\right)^{\frac{q}{p}}}{(b-a)^{q(\alpha-\beta)-1}}$$

$$\cdot\,\phi\left(\frac{(b-a)^{\alpha-\beta-\frac{1}{q}}}{\Gamma(\alpha-\beta)\,p^{\frac{1}{p}}\left(\alpha-\beta-\frac{1}{q}\right)^{\frac{1}{p}}}\left(\int_a^b \left|{}^C D^{\alpha}_{a+}v(x)\right|^q dx\right)^{\frac{1}{q}}\right)$$

$$\leq \frac{q\,\Gamma^q(\alpha-\beta)\,p^{\frac{q}{p}}\left(\alpha-\beta-\frac{1}{q}\right)^{\frac{q}{p}}}{(b-a)^{q(\alpha-\beta)}}$$

$$\cdot \int_a^b \phi \left(\frac{(b-a)^{\alpha-\beta} \left| {}^C D_{a+}^{\alpha} v(x) \right|}{\Gamma(\alpha-\beta)\, p^{\frac{1}{p}} \left(\alpha-\beta-\frac{1}{q}\right)^{\frac{1}{p}}} \right) dx. \tag{4.56}$$

If the function $\phi(x^{\frac{1}{q}})$ is concave, then reverse inequalities hold.

Proof. For $x \in [a,b]$ let

$$K(x,t) = \begin{cases} \frac{1}{\Gamma(\alpha-\beta)}(x-t)^{\alpha-\beta-1}, & a \le t \le x; \\ 0, & x < t \le b. \end{cases}$$

$$u(x) = {}^C D_{a+}^{\beta} v(x) = \frac{1}{\Gamma(\alpha-\beta)} \int_a^x (x-t)^{\alpha-\beta-1}\, {}^C D_{a+}^{\alpha} v(t)\, dt, \tag{4.57}$$

$$R(x) = \left(\int_a^x (K(x,t))^p dt \right)^{\frac{1}{p}} = \frac{(x-a)^{\alpha-\beta-\frac{1}{q}}}{\Gamma(\alpha-\beta) \left[p \left(\alpha-\beta-\frac{1}{q}\right) \right]^{\frac{1}{p}}}.$$

For $\alpha-\beta > \frac{1}{q}$ the function R is increasing on $[a,b]$, thus

$$\max_{x\in[a,b]} R(x) = \frac{(b-a)^{\alpha-\beta-\frac{1}{q}}}{\Gamma(\alpha-\beta)\, p^{\frac{1}{p}} \left(\alpha-\beta-\frac{1}{q}\right)^{\frac{1}{p}}} = M.$$

Hence $(\int_a^x K(x,t)^p dt)^{\frac{1}{p}} \le M$, which with $v = {}^C D_{a+}^{\alpha} v$, u as in (4.57) and Theorem 4.18 gives us (4.56). □

Corollary 4.13 *Let $\frac{1}{p}+\frac{1}{q}=1$, $\alpha-\beta > \frac{1}{q}$, $\beta \ge 0$, n and m given by* (2.70) *for α and β respectively. Let $v \in AC^n[a,b]$ be such that $v^{(i)}(a)=0$ for $i=m,\dots,n-1$. Let ${}^C D_{a+}^{\alpha} v \in L_{p+q}[a,b]$ and ${}^C D_{a+}^{\beta} v \in L_1[a,b]$. Then following inequalities hold*

$$\begin{aligned} &\int_a^b \left| {}^C D_{a+}^{\beta} v(x) \right|^p \left| {}^C D_{a+}^{\alpha} v(x) \right|^q dx \\ &\le \frac{q\,(b-a)^{p\left(\alpha-\beta-\frac{1}{q}\right)}}{(p+q)\,\Gamma^p(\alpha-\beta)\, p \left(\alpha-\beta-\frac{1}{q}\right)} \left(\int_a^b \left| {}^C D_{a+}^{\alpha} v(x) \right|^q dx \right)^{\frac{p+q}{q}} \\ &\le \frac{q\,(b-a)^{p(\alpha-\beta)}}{(p+q)\,\Gamma^p(\alpha-\beta)\, p \left(\alpha-\beta-\frac{1}{q}\right)} \int_a^b \left| {}^C D_{a+}^{\alpha} v(x) \right|^{p+q} dx. \end{aligned}$$

The composition identity for the right-sided Caputo fractional derivatives is given in Theorem 2.17.

Theorem 4.29 *Let $\phi : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Further, let $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha - \beta > \frac{1}{q}$, $\beta \geq 0$, n and m given by (2.70) for α and β respectively. Let $v \in AC^n[a,b]$ be such that $v^{(i)}(b) = 0$ for $i = m, \ldots, n-1$. Let ${}^C D^{\alpha}_{b-} v \in L_q[a,b]$ and ${}^C D^{\beta}_{b-} v \in L_1[a,b]$. Then for even m and n following inequalities hold*

$$\begin{aligned}
&\int_a^b \left|{}^C D^{\beta}_{b-} v(x)\right|^{1-q} \phi'\left(\left|{}^C D^{\beta}_{b-} v(x)\right|\right) \left|{}^C D^{\alpha}_{b-} v(x)\right|^q dx \\
&\leq \frac{q \Gamma^q(\alpha-\beta)\, p^{\frac{q}{p}} \left(\alpha - \beta - \frac{1}{q}\right)^{\frac{1}{p}}}{(b-a)^{q(\alpha-\beta)-1}} \\
&\quad \cdot \phi\left(\frac{(b-a)^{\alpha-\beta-\frac{1}{q}}}{\Gamma(\alpha-\beta)\, p^{\frac{1}{p}} \left(\alpha-\beta-\frac{1}{q}\right)^{\frac{1}{p}}} \left(\int_a^b \left|{}^C D^{\alpha}_{b-} v(x)\right|^q dx \right)^{\frac{1}{q}} \right) \\
&\leq \frac{q \Gamma^q(\alpha-\beta)\, p^{\frac{q}{p}} \left(\alpha - \beta - \frac{1}{q}\right)^{\frac{1}{p}}}{(b-a)^{q(\alpha-\beta)}} \\
&\quad \cdot \int_a^b \phi\left(\frac{(b-a)^{\alpha-\beta} \left|{}^C D^{\alpha}_{b-} v(x)\right|}{\Gamma(\alpha-\beta)\, p^{\frac{1}{p}} \left(\alpha-\beta-\frac{1}{q}\right)^{\frac{1}{p}}} \right) dx. \qquad (4.58)
\end{aligned}$$

If the function $\phi(x^{\frac{1}{q}})$ is concave, then reverse inequalities hold.

Proof. The proof is similar to the proof of Theorem 4.28. □

Corollary 4.14 *Let $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha - \beta > \frac{1}{q}$, $\beta \geq 0$, n and m given by (2.70) for α and β respectively. Let $v \in AC^n[a,b]$ be such that $v^{(i)}(b) = 0$ for $i = m, \ldots, n-1$. Let ${}^C D^{\alpha}_{b-} v \in L_{p+q}[a,b]$ and ${}^C D^{\beta}_{b-} v \in L_1[a,b]$. Then for even m and n following inequalities hold*

$$\begin{aligned}
&\int_a^b \left|{}^C D^{\beta}_{b-} v(x)\right|^p \left|{}^C D^{\alpha}_{b-} v(x)\right|^q dx \\
&\leq \frac{q\,(b-a)^{p\left(\alpha-\beta-\frac{1}{q}\right)}}{(p+q)\,\Gamma^p(\alpha-\beta)\, p\left(\alpha-\beta-\frac{1}{q}\right)} \left(\int_a^b \left|{}^C D^{\alpha}_{b-} v(x)\right|^q dx \right)^{\frac{p+q}{q}} \\
&\leq \frac{q\,(b-a)^{p(\alpha-\beta)}}{(p+q)\,\Gamma^p(\alpha-\beta)\, p\left(\alpha-\beta-\frac{1}{q}\right)} \int_a^b \left|{}^C D^{\alpha}_{b-} v(x)\right|^{p+q} dx.
\end{aligned}$$

Results given in Theorems and Corollaries 4.26–4.14 can be analogously done for two other types of fractional derivatives that we observe: the Canavati type and the Riemann-Liouville type. Here, as an example inequality for each type of fractional derivatives, we give inequality analogous to the (4.56) obtain with the composition identity, together with a counterpart, a special case when $\phi(x) = x^{p+q}$, for the left-sided fractional derivatives. Proofs are omitted.

Following two results include the left-sided Canavati fractional derivatives using the composition identity given in Theorem 2.19.

Theorem 4.30 *Let $\phi : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Further, let $\frac{1}{p}+\frac{1}{q} = 1$, $\alpha-\beta > \frac{1}{q}$, $\beta \geq 0$, $n = [\alpha]+1$ and $m = [\beta]+1$. Let $v \in C^{\alpha}_{a+}[a,b]$ be such that $v^{(i)}(a) = 0$ for $i = m-1,\dots,n-2$. Let ${}^{C_1}D^{\alpha}_{a+}v \in L_q[a,b]$. Then following inequalities hold*

$$
\begin{aligned}
&\int_a^b \left|{}^{C_1}D^{\beta}_{a+}v(x)\right|^{1-q} \phi'\left(\left|{}^{C_1}D^{\beta}_{a+}v(x)\right|\right) \left|{}^{C_1}D^{\alpha}_{a+}v(x)\right|^q dx \\
&\leq \frac{q\,\Gamma^q(\alpha-\beta)\,p^{\frac{q}{p}}\left(\alpha-\beta-\frac{1}{q}\right)^{\frac{1}{p}}}{(b-a)^{q(\alpha-\beta)-1}} \\
&\quad \cdot \phi\left(\frac{(b-a)^{\alpha-\beta-\frac{1}{q}}}{\Gamma(\alpha-\beta)\,p^{\frac{1}{p}}\left(\alpha-\beta-\frac{1}{q}\right)^{\frac{1}{p}}}\left(\int_a^b \left|{}^{C_1}D^{\alpha}_{a+}v(x)\right|^q dx\right)^{\frac{1}{q}}\right) \\
&\leq \frac{q\,\Gamma^q(\alpha-\beta)\,p^{\frac{q}{p}}\left(\alpha-\beta-\frac{1}{q}\right)^{\frac{1}{p}}}{(b-a)^{q(\alpha-\beta)}} \\
&\quad \cdot \int_a^b \phi\left(\frac{(b-a)^{\alpha-\beta}\left|{}^{C_1}D^{\alpha}_{a+}v(x)\right|}{\Gamma(\alpha-\beta)\,p^{\frac{1}{p}}\left(\alpha-\beta-\frac{1}{q}\right)^{\frac{1}{p}}}\right) dx. \qquad (4.59)
\end{aligned}
$$

If the function $\phi(x^{\frac{1}{q}})$ is concave, then reverse inequalities hold.

Corollary 4.15 *Let $\frac{1}{p}+\frac{1}{q} = 1$, $\alpha-\beta > \frac{1}{q}$, $\beta \geq 0$, $n = [\alpha]+1$ and $m = [\beta]+1$. Let $v \in C^{\alpha}_{a+}[a,b]$ be such that $v^{(i)}(a) = 0$ for $i = m-1,\dots,n-2$. Let ${}^{C_1}D^{\alpha}_{a+}v \in L_{p+q}[a,b]$. Then following inequalities hold*

$$
\begin{aligned}
&\int_a^b \left|{}^{C_1}D^{\beta}_{a+}v(x)\right|^{p} \left|{}^{C_1}D^{\alpha}_{a+}v(x)\right|^q dx \\
&\leq \frac{q\,(b-a)^{p\left(\alpha-\beta-\frac{1}{q}\right)}}{(p+q)\,\Gamma^p(\alpha-\beta)\,p\left(\alpha-\beta-\frac{1}{q}\right)}\left(\int_a^b \left|{}^{C_1}D^{\alpha}_{a+}v(x)\right|^q dx\right)^{\frac{p+q}{q}} \\
&\leq \frac{q\,(b-a)^{p(\alpha-\beta)}}{(p+q)\,\Gamma^p(\alpha-\beta)\,p\left(\alpha-\beta-\frac{1}{q}\right)}\int_a^b \left|{}^{C_1}D^{\alpha}_{a+}v(x)\right|^{p+q} dx.
\end{aligned}
$$

We end with results for the left-sided Riemann-Liouville fractional derivatives using its composition identity which is valid if one of conditions $(i)-(vii)$ in Corollary 2.21 holds.

Theorem 4.31 *Let $\phi : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q>1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0)=0$. Further, let $\frac{1}{p}+\frac{1}{q}=1$, $\alpha-\beta>\frac{1}{q}$, $\beta\geq 0$. Suppose that one of conditions $(i)-(vii)$ in the Corollary 2.21 holds for $\{\alpha,\beta,v\}$ and let $D^{\alpha}_{a+}v\in L_q[a,b]$. Then following inequalities hold*

$$\begin{aligned}
&\int_a^b \left|D^{\beta}_{a+}v(x)\right|^{1-q}\phi'\left(\left|D^{\beta}_{a+}v(x)\right|\right)\left|D^{\alpha}_{a+}v(x)\right|^q dx \\
&\leq \frac{q\Gamma^q(\alpha-\beta)\,p^{\frac{q}{p}}\left(\alpha-\beta-\frac{1}{q}\right)^{\frac{1}{p}}}{(b-a)^{q(\alpha-\beta)-1}} \\
&\quad\cdot\phi\left(\frac{(b-a)^{\alpha-\beta-\frac{1}{q}}}{\Gamma(\alpha-\beta)\,p^{\frac{1}{p}}\left(\alpha-\beta-\frac{1}{q}\right)^{\frac{1}{p}}}\left(\int_a^b\left|D^{\alpha}_{a+}v(x)\right|^q dx\right)^{\frac{1}{q}}\right) \\
&\leq \frac{q\Gamma^q(\alpha-\beta)\,p^{\frac{q}{p}}\left(\alpha-\beta-\frac{1}{q}\right)^{\frac{1}{p}}}{(b-a)^{q(\alpha-\beta)}} \\
&\quad\cdot\int_a^b\phi\left(\frac{(b-a)^{\alpha-\beta}\left|D^{\alpha}_{a+}v(x)\right|}{\Gamma(\alpha-\beta)\,p^{\frac{1}{p}}\left(\alpha-\beta-\frac{1}{q}\right)^{\frac{1}{p}}}\right)dx.
\end{aligned}\tag{4.60}$$

If the function $\phi(x^{\frac{1}{q}})$ is concave, then reverse inequalities hold.

Corollary 4.16 *Let $\frac{1}{p}+\frac{1}{q}=1$, $\alpha-\beta>\frac{1}{q}$, $\beta\geq 0$. Suppose that one of conditions $(i)-(vii)$ in the Corollary 2.21 holds for $\{\alpha,\beta,v\}$ and let $D^{\alpha}_{a+}v\in L_{p+q}[a,b]$. Then following inequalities hold*

$$\begin{aligned}
&\int_a^b\left|D^{\beta}_{a+}v(x)\right|^p\left|D^{\alpha}_{a+}v(x)\right|^q dx \\
&\leq \frac{q\,(b-a)^{p\left(\alpha-\beta-\frac{1}{q}\right)}}{(p+q)\,\Gamma^p(\alpha-\beta)\,p\left(\alpha-\beta-\frac{1}{q}\right)}\left(\int_a^b\left|D^{\alpha}_{a+}v(x)\right|^q dx\right)^{\frac{p+q}{q}} \\
&\leq \frac{q\,(b-a)^{p(\alpha-\beta)}}{(p+q)\,\Gamma^p(\alpha-\beta)\,p\left(\alpha-\beta-\frac{1}{q}\right)}\int_a^b\left|D^{\alpha}_{a+}v(x)\right|^{p+q}dx.
\end{aligned}$$

4.2.4 Opial-type equalities for fractional integrals and fractional derivatives

Now we give some Opial-type equalities for fractional integrals and fractional derivatives as an application of Theorem 4.20 and Theorem 4.21. First we observe the left-sided

Riemann-Liouville fractional integrals.

Theorem 4.32 *Let $\phi : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Further, let $\phi \in C^2(I)$, where $I \subseteq \mathbb{R}_+$ is a closed interval. Let $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$ and $v \in L_1[a,b]$. Then there exists $\xi \in I$ such that the following equality holds*

$$\begin{aligned}\Psi_\phi(J^\alpha_{a+}v,v) &= \frac{\xi\phi''(\xi) - (q-1)\phi'(\xi)}{2q\xi^{2q-1}} \\ &\cdot \left[\frac{(b-a)^{q\alpha}}{\Gamma^q(\alpha)\left[p(\alpha - \frac{1}{q})\right]^{\frac{q}{p}}} \int_a^b |v(x)|^{2q}dx - 2\int_a^b |J^\alpha_{a+}v(x)|^q |v(x)|^q dx \right]. \qquad (4.61)\end{aligned}$$

Proof. We follow the same idea as in [43, Theorem 6] and [17, Theorem 3.1]. For $x \in [a,b]$ let

$$K(x,t) = \begin{cases} \frac{1}{\Gamma(\alpha)}(x-t)^{\alpha-1}, & a \le t \le x; \\ 0, & x < t \le b, \end{cases}$$

$$u(x) = J^\alpha_{a+}v(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}v(t)\,dt\,, \qquad (4.62)$$

$$P(x) = \left(\int_a^x (K(x,t))^p\,dt\right)^{\frac{1}{p}} = \frac{(x-a)^{\alpha-\frac{1}{q}}}{\Gamma(\alpha)\left[p\left(\alpha-\frac{1}{q}\right)\right]^{\frac{1}{p}}}.$$

It is easy to see that for $\alpha > \frac{1}{q}$ the function P is increasing on $[a,b]$, thus

$$\max_{x\in[a,b]} P(x) = \frac{(b-a)^{\alpha-\frac{1}{q}}}{\Gamma(\alpha)\,p^{\frac{1}{p}}\left(\alpha-\frac{1}{q}\right)^{\frac{1}{p}}} = M\,.$$

Hence $(\int_a^x K(x,t)^p dt)^{\frac{1}{p}} \le M$, which with the function u defined by (4.62) and Theorem 4.20 gives us (4.61). □

Theorem 4.33 *Let $\phi_1, \phi_2 : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi_i(x^{\frac{1}{q}})$ is convex and $\phi_i(0) = 0, i = 1,2$. Further, let $\phi_1, \phi_2 \in C^2(I)$, where $I \subseteq \mathbb{R}_+$ is a closed interval. Let $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$ and $v \in L_1[a,b]$. Then there exists $\xi \in I$ such that the following equality holds*

$$\frac{\Psi_{\phi_1}(J^\alpha_{a+}v,v)}{\Psi_{\phi_2}(J^\alpha_{a+}v,v)} = \frac{\xi\,\phi_1''(\xi) - (q-1)\phi_1'(\xi)}{\xi\,\phi_2''(\xi) - (q-1)\phi_2'(\xi)},$$

provided that denominators are not equal to zero.

Proof. It follows directly for the function u defined by (4.62) and Theorem 4.21. □

Using Theorems 4.19, 4.20 and 4.21, analogous results follows for the right-sided Riemann-Liouville fractional integrals. The proofs are similar and omitted.

Theorem 4.34 *Let $\phi : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Further, let $\phi \in C^2(I)$, where $I \subseteq \mathbb{R}_+$ is a closed interval. Let $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$ and $v \in L_1[a,b]$. Then there exists $\xi \in I$ such that the following equality holds*

$$\begin{aligned}\Psi_\phi(J^\alpha_{b-}v, v) &= \frac{\xi\phi''(\xi) - (q-1)\phi'(\xi)}{2q\xi^{2q-1}} \\ &\cdot \left[\frac{(b-a)^{q\alpha}}{\Gamma^q(\alpha)\left[p(\alpha - \frac{1}{q})\right]^{\frac{q}{p}}} \int_a^b |v(x)|^{2q}dx - 2\int_a^b |J^\alpha_{b-}v(x)|^q |v(x)|^q dx \right]. \end{aligned} \tag{4.63}$$

Theorem 4.35 *Let $\phi_1, \phi_2 : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi_i(x^{\frac{1}{q}})$ is convex and $\phi_i(0) = 0, i = 1,2$. Further, let $\phi_1, \phi_2 \in C^2(I)$, where $I \subseteq \mathbb{R}_+$ is a closed interval. Let $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$ and $v \in L_1[a,b]$. Then there exists $\xi \in I$ such that the following equality holds*

$$\frac{\Psi_{\phi_1}(J^\alpha_{b-}v, v)}{\Psi_{\phi_2}(J^\alpha_{b-}v, v)} = \frac{\xi\phi_1''(\xi) - (q-1)\phi_1'(\xi)}{\xi\phi_2''(\xi) - (q-1)\phi_2'(\xi)},$$

provided that denominators are not equal to zero.

Next, we observe the Caputo fractional derivatives. The proofs for the equalities involving the right-sided Caputo fractional derivatives are omitted.

Theorem 4.36 *Let $\phi : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Further, let $\phi \in C^2(I)$, where $I \subseteq \mathbb{R}_+$ is a closed interval. Let $\alpha \geq 0$, n given by (2.70) and $v \in AC^n[a,b]$. If $n - \alpha > \frac{1}{q}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then there exists $\xi \in I$ such that the following equality holds*

$$\begin{aligned}\Psi_\phi({}^C D^\alpha_{a+}v, v^{(n)}) &= \frac{\xi\phi''(\xi) - (q-1)\phi'(\xi)}{2q\xi^{2q-1}} \\ &\cdot \left[\frac{(b-a)^{q(n-\alpha)}}{\Gamma^q(n-\alpha)\left[p\left(n-\alpha-\frac{1}{q}\right)\right]^{\frac{q}{p}}} \int_a^b |v^{(n)}(x)|^{2q}dx - 2\int_a^b |{}^C D^\alpha_{a+}v(x)|^q |v^{(n)}(x)|^q dx \right]. \end{aligned} \tag{4.64}$$

Proof. For $x \in [a,b]$ let

$$K(x,t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}(x-t)^{n-\alpha-1}, & a \leq t \leq x; \\ 0, & x < t \leq b. \end{cases}$$

$$u(x) = {}^C D^\alpha_{a+}v(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} v^{(n)}(t)\, dt, \tag{4.65}$$

$$Q(x) = \left(\int_a^x (K(x,t))^p dt \right)^{\frac{1}{p}} = \frac{(x-a)^{n-\alpha-\frac{1}{q}}}{\Gamma(n-\alpha)\left[p\left(n-\alpha-\frac{1}{q}\right)\right]^{\frac{1}{p}}}.$$

For $n-\alpha>\frac{1}{q}$ the function Q is increasing on $[a,b]$, thus

$$\max_{x\in[a,b]} Q(x)=\frac{(b-a)^{n-\alpha-\frac{1}{q}}}{\Gamma(n-\alpha)\,p^{\frac{1}{p}}\left(n-\alpha-\frac{1}{q}\right)^{\frac{1}{p}}}=M\,.$$

Hence $(\int_a^x K(x,t)^p dt)^{\frac{1}{p}}\leq M$, which with $v=v^{(n)}$, u as in (4.65) and Theorem 4.20 gives us (4.64). □

Theorem 4.37 *Let $\phi_1,\phi_2:[0,\infty)\longrightarrow\mathbb{R}$ be a differentiable function such that for $q>1$ the function $\phi_i(x^{\frac{1}{q}})$ is convex and $\phi_i(0)=0, i=1,2$. Further, let $\phi_1,\phi_2\in C^2(I)$, where $I\subseteq\mathbb{R}_+$ is a closed interval. Let $\alpha\geq 0$, n given by (2.70) and $v\in AC^n[a,b]$. If $n-\alpha>\frac{1}{q}$ and $\frac{1}{p}+\frac{1}{q}=1$, then there exists $\xi\in I$ such that the following equality holds*

$$\frac{\Psi_{\phi_1}({}^CD^{\alpha}_{a+}v,v^{(n)})}{\Psi_{\phi_2}({}^CD^{\alpha}_{a+}v,v^{(n)})}=\frac{\xi\phi_1''(\xi)-(q-1)\phi_1'(\xi)}{\xi\phi_2''(\xi)-(q-1)\phi_2'(\xi)},$$

provided that denominators are not equal to zero.

Proof. It follows directly for the function u defined by (4.65) and Theorem 4.21. □

Theorem 4.38 *Let $\phi:[0,\infty)\longrightarrow\mathbb{R}$ be a differentiable function such that for $q>1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0)=0$. Further, let $\phi\in C^2(I)$, where $I\subseteq\mathbb{R}_+$ is a closed interval. Let $\alpha\geq 0$, n given by (2.70) and $v\in AC^n[a,b]$. If $n-\alpha>\frac{1}{q}$ and $\frac{1}{p}+\frac{1}{q}=1$, then there exists $\xi\in I$ such that the following equality holds*

$$\begin{aligned}\Psi_{\phi}({}^CD^{\alpha}_{b-}v,v^{(n)}) &= \frac{\xi\phi''(\xi)-(q-1)\phi'(\xi)}{2q\xi^{2q-1}}\\ &\cdot\left[\frac{(b-a)^{q(n-\alpha)}}{\Gamma^q(n-\alpha)\left[p\left(n-\alpha-\frac{1}{q}\right)\right]^{\frac{q}{p}}}\int_a^b|v^{(n)}(x)|^{2q}dx-2\int_a^b|{}^CD^{\alpha}_{b-}v(x)|^q|v^{(n)}(x)|^q dx\right].\end{aligned}\tag{4.66}$$

Theorem 4.39 *Let $\phi_1,\phi_2:[0,\infty)\longrightarrow\mathbb{R}$ be a differentiable function such that for $q>1$ the function $\phi_i(x^{\frac{1}{q}})$ is convex and $\phi_i(0)=0, i=1,2$. Further, let $\phi_1,\phi_2\in C^2(I)$, where $I\subseteq\mathbb{R}_+$ is a closed interval. Let $\alpha\geq 0$, n given by (2.70) and $v\in AC^n[a,b]$. If $n-\alpha>\frac{1}{q}$ and $\frac{1}{p}+\frac{1}{q}=1$, then there exists $\xi\in I$ such that the following equality holds*

$$\frac{\Psi_{\phi_1}({}^CD^{\alpha}_{b-}v,v^{(n)})}{\Psi_{\phi_2}({}^CD^{\alpha}_{b-}v,v^{(n)})}=\frac{\xi\phi_1''(\xi)-(q-1)\phi_1'(\xi)}{\xi\phi_2''(\xi)-(q-1)\phi_2'(\xi)},$$

provided that denominators are not equal to zero.

We continue with extensions that require the composition identity for the left-sided Caputo fractional derivatives, given in Theorem 2.16.

Theorem 4.40 *Let $\phi : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Further, let $\phi \in C^2(I)$, where $I \subseteq \mathbb{R}_+$ is a closed interval. Let $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha - \beta > \frac{1}{q}$, $\beta \geq 0$, n and m given by (2.70) for α and β respectively. Let $v \in AC^n[a,b]$ be such that $v^{(i)}(a) = 0$ for $i = m,...,n-1$. Let ${}^CD^{\alpha}_{a+}v \in L_q[a,b]$ and ${}^CD^{\beta}_{a+}v \in L_1[a,b]$. Then there exists $\xi \in I$ such that the following equality holds*

$$\Psi_{\phi}({}^CD^{\alpha}_{a+}v, {}^CD^{\beta}_{a+}v) = \frac{\xi\phi''(\xi) - (q-1)\phi'(\xi)}{2q\xi^{2q-1}}$$
$$\cdot \left[\frac{(b-a)^{q(\alpha-\beta)}}{\Gamma^q(\alpha-\beta)\left[p\left(\alpha-\beta-\frac{1}{q}\right)\right]^{\frac{q}{p}}} \int_a^b |{}^CD^{\beta}_{a+}v(x)|^{2q}dx - 2\int_a^b |{}^CD^{\alpha}_{a+}v(x)|^q |{}^CD^{\beta}_{a+}v(x)|^q dx \right]. \tag{4.67}$$

Proof. For $x \in [a,b]$ let

$$K(x,t) = \begin{cases} \frac{1}{\Gamma(\alpha-\beta)}(x-t)^{\alpha-\beta-1}, & a \leq t \leq x; \\ 0, & x < t \leq b. \end{cases}$$

$$u(x) = {}^CD^{\beta}_{a+}v(x) = \frac{1}{\Gamma(\alpha-\beta)} \int_a^x (x-t)^{\alpha-\beta-1}\, {}^CD^{\alpha}_{a+}v(t)\, dt\,, \tag{4.68}$$

$$R(x) = \left(\int_a^x (K(x,t))^p dt \right)^{\frac{1}{p}} = \frac{(x-a)^{\alpha-\beta-\frac{1}{q}}}{\Gamma(\alpha-\beta)\left[p\left(\alpha-\beta-\frac{1}{q}\right)\right]^{\frac{1}{p}}}.$$

For $\alpha - \beta > \frac{1}{q}$ the function R is increasing on $[a,b]$, thus

$$\max_{x\in[a,b]} R(x) = \frac{(b-a)^{\alpha-\beta-\frac{1}{q}}}{\Gamma(\alpha-\beta)\, p^{\frac{1}{p}} \left(\alpha-\beta-\frac{1}{q}\right)^{\frac{1}{p}}} = M.$$

Hence $(\int_a^x K(x,t)^p dt)^{\frac{1}{p}} \leq M$, which with $v = {}^CD^{\alpha}_{a+}v$, u as in (4.68) and Theorem 4.20 gives us (4.67). □

Theorem 4.41 *Let $\phi_1, \phi_2 : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi_i(x^{\frac{1}{q}})$ is convex and $\phi_i(0) = 0, i = 1,2$. Further, let $\phi_1, \phi_2 \in C^2(I)$, where $I \subseteq \mathbb{R}_+$ is a closed interval. Let $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha - \beta > \frac{1}{q}$, $\beta \geq 0$, n and m given by (2.70) for α and β respectively. Let $v \in AC^n[a,b]$ be such that $v^{(i)}(a) = 0$ for $i = m,...,n-1$. Let ${}^CD^{\alpha}_{a+}v \in L_q[a,b]$ and ${}^CD^{\beta}_{a+}v \in L_1[a,b]$. Then there exists $\xi \in I$ such that the following equality holds*

$$\frac{\Psi_{\phi_1}({}^CD^{\alpha}_{a+}v, {}^CD^{\beta}_{a+}v)}{\Psi_{\phi_2}({}^CD^{\alpha}_{a+}v, {}^CD^{\beta}_{a+}v)} = \frac{\xi\phi_1''(\xi) - (q-1)\phi_1'(\xi)}{\xi\phi_2''(\xi) - (q-1)\phi_2'(\xi)},$$

provided that denominators are not equal to zero.

Proof. It follows directly for $v = {}^{C}D^{\alpha}_{a+}v$, u defined by (4.68) and Theorem 4.21. □

The composition identity for the right-sided Caputo fractional derivatives is given in Theorem 2.17.

Theorem 4.42 *Let $\phi : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Further, let $\phi \in C^2(I)$, where $I \subseteq \mathbb{R}_+$ is a closed interval. Let $\frac{1}{p}+\frac{1}{q} = 1$, $\alpha - \beta > \frac{1}{q}$, $\beta \geq 0$, n and m given by (2.70) for α and β respectively. Let $v \in AC^n[a,b]$ be such that $v^{(i)}(b) = 0$ for $i = m,\ldots,n-1$. Let ${}^{C}D^{\alpha}_{b-}v \in L_q[a,b]$ and ${}^{C}D^{\beta}_{b-}v \in L_1[a,b]$. Then there exists $\xi \in I$ such that the following equality holds*

$$\begin{aligned}\Psi_{\phi}({}^{C}D^{\alpha}_{b-}v, {}^{C}D^{\beta}_{b-}v) &= \frac{\xi\phi''(\xi) - (q-1)\phi'(\xi)}{2q\xi^{2q-1}} \\ &\cdot \left[\frac{(b-a)^{q(\alpha-\beta)}}{\Gamma^q(\alpha-\beta)\left[p\left(\alpha-\beta-\frac{1}{q}\right)\right]^{\frac{q}{p}}} \int_a^b |{}^{C}D^{\beta}_{b-}v(x)|^{2q}dx - 2\int_a^b |{}^{C}D^{\alpha}_{b-}v(x)|^q |{}^{C}D^{\beta}_{b-}v(x)|^q dx \right].\end{aligned} \tag{4.69}$$

Theorem 4.43 *Let $\phi_1, \phi_2 : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi_i(x^{\frac{1}{q}})$ is convex and $\phi_i(0) = 0, i = 1,2$. Further, let $\phi_1, \phi_2 \in C^2(I)$, where $I \subseteq \mathbb{R}_+$ is a closed interval. Let $\frac{1}{p}+\frac{1}{q} = 1$, $\alpha-\beta > \frac{1}{q}$, $\beta \geq 0$, n and m given by (2.70) for α and β respectively. Let $v \in AC^n[a,b]$ be such that $v^{(i)}(b) = 0$ for $i = m,\ldots,n-1$. Let ${}^{C}D^{\alpha}_{b-}v \in L_q[a,b]$ and ${}^{C}D^{\beta}_{b-}v \in L_1[a,b]$. Then there exists $\xi \in I$ such that the following equality holds*

$$\frac{\Psi_{\phi_1}({}^{C}D^{\alpha}_{b-}v, {}^{C}D^{\beta}_{b-}v)}{\Psi_{\phi_2}({}^{C}D^{\alpha}_{b-}v, {}^{C}D^{\beta}_{b-}v)} = \frac{\xi\phi_1''(\xi) - (q-1)\phi_1'(\xi)}{\xi\phi_2''(\xi) - (q-1)\phi_2'(\xi)},$$

provided that denominators are not equal to zero.

Results given for the Caputo fractional derivatives can be analogously done for two other types of fractional derivatives that we observe: the Canavati type and the Riemann-Liouville type. Here, as an example equality for each type of fractional derivatives, we give equality analogous to the (4.67) obtain with the composition identity, for the left-sided fractional derivatives. Proofs are omitted.

Following result include the left-sided Canavati fractional derivatives using the composition identity given in Theorem 2.19.

Theorem 4.44 *Let $\phi : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Further, let $\phi \in C^2(I)$, where $I \subseteq \mathbb{R}_+$ is a closed interval. Let $\frac{1}{p}+\frac{1}{q} = 1$, $\alpha-\beta > \frac{1}{q}$, $\beta \geq 0$, $n = [\alpha]+1$ and $m = [\beta]+1$. Let $v \in C^{\alpha}_{\beta+}[a,b]$ be such that $v^{(i)}(a) = 0$ for $i = m-1,\ldots,n-2$. Let ${}^{C}D^{\alpha}_{a+}v \in L_q[a,b]$. Then there exists $\xi \in I$ such that the following equality holds*

$$\Psi_{\phi}({}^{C_1}D^{\alpha}_{a+}v, {}^{C_1}D^{\beta}_{a+}v) = \frac{\xi\phi''(\xi) - (q-1)\phi'(\xi)}{2q\xi^{2q-1}}$$

$$\cdot \left[\frac{(b-a)^{q(\alpha-\beta)}}{\Gamma^q(\alpha-\beta)\left[p\left(\alpha-\beta-\frac{1}{q}\right)\right]^{\frac{q}{p}}} \int_a^b |^{C_1}D^{\alpha}_{a+}v(x)|^{2q}dx - 2\int_a^b |^{C_1}D^{\beta}_{a+}v(x)|^q|^{C_1}D^{\alpha}_{a+}v(x)|^q dx \right].$$

We end with the result for the left-sided Riemann-Liouville fractional derivatives using its composition identity which is valid if one of conditions $(i)-(vii)$ in Corollary 2.21 holds.

Theorem 4.45 *Let $\phi : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Further, let $\phi \in C^2(I)$, where $I \subseteq \mathbb{R}_+$ is a closed interval. Let $\frac{1}{p}+\frac{1}{q} = 1$, $\alpha-\beta > \frac{1}{q}$, $\beta \geq 0$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.21 holds for $\{\alpha,\beta,v\}$ and let ${}^CD^{\alpha}_{a+}v \in L_q[a,b]$. Then there exists $\xi \in I$ such that the following equality holds*

$$\Psi_\phi(D^{\alpha}_{a+}v, D^{\beta}_{a+}v) = \frac{\xi\phi''(\xi)-(q-1)\phi'(\xi)}{2q\xi^{2q-1}}$$

$$\cdot \left[\frac{(b-a)^{q(\alpha-\beta)}}{\Gamma^q(\alpha-\beta)\left[p\left(\alpha-\beta-\frac{1}{q}\right)\right]^{\frac{q}{p}}} \int_a^b |D^{\alpha}_{a+}v(x)|^{2q}dx - 2\int_a^b |D^{\beta}_{a+}v(x)|^q|D^{\alpha}_{a+}v(x)|^q dx \right].$$

Chapter 5

Generalizations of Opial-type inequalities for fractional derivatives

The monograph by Agarwal and Pang [5] gives an overview of Opial-type differential inequalities (including ordinary derivatives) and its applications. Here we present Opial-type inequalities involving the Riemann-Liouville, the Caputo and the Canavati fractional derivatives and obtain theirs generalizations, extensions, improvements and refinements. We observe inequalities with two fractional derivatives of a function on the left side of an inequality, and emphasize special cases when order of derivatives belongs to $\mathbb{N}_0$, reducing to classical Opial's, Beesack's, Wirtinger's or Fink's inequality for ordinary derivatives, given in Section 1.4. Known generalizations and extensions of Opial-type inequalities involving fractional derivatives ([12, 14, 15]) we give under new conditions using results from Section 2.6, and also we give some new fractional differentiation inequalities. Further, we investigate the possibility of obtaining the best possible constant and compare results obtained by different methods. Each inequality is given for the Riemann-Liouville, the Caputo and the Canavati fractional derivatives, the left-sided and the right-sided. This chapter is based on our results: Andrić, Pečarić and Perić [16, 23, 26].

5.1 Inequalities with fractional derivatives of order α and β

Let $\mathbf{D}$ be a fractional derivative (the Riemann-Liouville, the Caputo or the Canavati type). The first inequality that we observe has a form

$$\int_a^b |\mathbf{D}^\beta f(t)|\,|\mathbf{D}^\alpha f(t)|\,dt \le K\left(\int_a^b |\mathbf{D}^\alpha f(t)|^q\,dt\right)^{\frac{2}{q}},$$

where $\alpha > \beta \ge 0$, $K > 0$ is a constant and $q \in \mathbb{R}$.

This inequality for the left-sided Riemann-Liouville fractional derivatives is given in [14], and for the Caputo and the Canavati fractional derivatives in [12]. In the following theorems we use results from Chapter 2.6, and give new conditions under which inequalities hold. Some of the improved results for the Caputo fractional derivatives, Andrić-Pečarić-Perić give in [26].

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Depending on parameters p and q, we differ next inequalities, first including the left-sided Riemann-Liouville fractional derivatives, and then the right-sided. Thereby we use composition identities for the Riemann-Liouville fractional derivatives given in Theorem 2.13 and Theorem 2.14. The composition identity for the left-sided fractional derivatives is valid if one of conditions $(i)-(vii)$ in Corollary 2.21 holds. For the right-sided we use Corollary 2.22.

Theorem 5.1 *Let $p,q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $\beta \ge 0$, $\alpha > \beta + \frac{1}{q}$ and $D^\alpha_{a+}f \in L_q[a,b]$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.21 holds. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x |D^\beta_{a+}f(t)|\,|D^\alpha_{a+}f(t)|\,dt \le K_1\,(x-a)^{\alpha-\beta-1+\frac{2}{p}}\left(\int_a^x \left|D^\alpha_{a+}f(t)\right|^q dt\right)^{\frac{2}{q}}, \tag{5.1}$$

where

$$K_1 = \left[\Gamma(\alpha-\beta)\,(p(\alpha-\beta-1)+1)^{\frac{1}{p}}\,(p(\alpha-\beta-1)+2)^{\frac{1}{p}}\,2^{\frac{1}{q}}\right]^{-1}. \tag{5.2}$$

Inequality (5.1) *is sharp for $\alpha = \beta+1$, where equality is attained for a function f such that $D^\alpha_{a+}f(t) = 1$, $t \in [a,x]$.*

Proof. Using Theorem 2.13, the triangle inequality and Hölder's inequality we have

$$|D^\beta_{a+}f(t)| \le \frac{1}{\Gamma(\alpha-\beta)}\int_a^t (t-\tau)^{\alpha-\beta-1}\left|D^\alpha_{a+}f(\tau)\right|d\tau \tag{5.3}$$

$$\le \frac{1}{\Gamma(\alpha-\beta)}\left(\int_a^t (t-\tau)^{p(\alpha-\beta-1)}\,d\tau\right)^{\frac{1}{p}}\left(\int_a^t |D^\alpha_{a+}f(\tau)|^q\,d\tau\right)^{\frac{1}{q}} \tag{5.4}$$

$$= \frac{1}{\Gamma(\alpha-\beta)} \frac{(t-a)^{\alpha-\beta-1+\frac{1}{p}}}{[p(\alpha-\beta-1)+1]^{\frac{1}{p}}} \left(\int_a^t |D_{a+}^{\alpha} f(\tau)|^q \, d\tau \right)^{\frac{1}{q}} .$$

Let $z(t) = \int_a^t |D_{a+}^{\alpha} f(\tau)|^q \, d\tau$. Then $z'(t) = |D_{a+}^{\alpha} f(t)|^q$, that is $|D_{a+}^{\alpha} f(t)| = (z'(t))^{\frac{1}{q}}$, hence

$$|D_{a+}^{\beta} f(t)| \, |D_{a+}^{\alpha} f(t)| \leq \frac{1}{\Gamma(\alpha-\beta)} \frac{(t-a)^{\alpha-\beta-1+\frac{1}{p}}}{[p(\alpha-\beta-1)+1]^{\frac{1}{p}}} \left[z(t) \, z'(t) \right]^{\frac{1}{q}} . \tag{5.5}$$

Again we use Hölder's inequality to obtain

$$\begin{aligned} &\int_a^x (t-a)^{\alpha-\beta-1+\frac{1}{p}} \left[z(t) \, z'(t) \right]^{\frac{1}{q}} \, dt \\ &\leq \left(\int_a^x (t-a)^{p(\alpha-\beta-1)+1} dt \right)^{\frac{1}{p}} \left(\int_a^x z(t) \, z'(t) \, dt \right)^{\frac{1}{q}} \\ &= \frac{(x-a)^{\alpha-\beta-1+\frac{2}{p}}}{[p(\alpha-\beta-1)+2]^{\frac{1}{p}}} \left[\frac{1}{2} \left(\int_a^x |D_{a+}^{\alpha} f(t)|^q \, dt \right)^2 \right]^{\frac{1}{q}} . \end{aligned} \tag{5.6}$$

From (5.5) and (5.6) follows

$$\begin{aligned} &\int_a^x |D_{a+}^{\beta} f(t)| \, |D_{a+}^{\alpha} f(t)| \, dt \\ &\leq \frac{1}{\Gamma(\alpha-\beta) \, [p(\alpha-\beta-1)+1]^{\frac{1}{p}}} \int_a^x (t-a)^{\alpha-\beta-1+\frac{1}{p}} \left[z(t) \, z'(t) \right]^{\frac{1}{q}} \, dt \\ &\leq \frac{1}{\Gamma(\alpha-\beta) \, [p(\alpha-\beta-1)+1]^{\frac{1}{p}}} \frac{(x-a)^{\alpha-\beta-1+\frac{2}{p}}}{2^{\frac{1}{q}} \, [p(\alpha-\beta-1)+2]^{\frac{1}{p}}} \left(\int_a^x |D_{a+}^{\alpha} f(t)|^q \, dt \right)^{\frac{2}{q}} , \end{aligned}$$

which gives inequality (5.1).

Using the equality condition in Hölder's inequality we have equality in (5.4) if and only if $|D_{a+}^{\alpha} f(\tau)|^q = K(t-\tau)^{p(\alpha-\beta-1)}$ for some constant $K \geq 0$ and every $\tau \in [a,t]$. Since $D_{a+}^{\alpha} f(\tau)$ depends only on τ, this implies that $\alpha-\beta-1=0$. Due to homogeneous property of inequality (5.1) we can take that $D_{a+}^{\alpha} f(\tau) = 1$, which, by Theorem 2.13, gives us

$$D_{a+}^{\beta} f(t) = \frac{1}{\Gamma(\alpha-\beta)} \int_a^t (t-\tau)^{\alpha-\beta-1} D_{a+}^{\alpha} f(\tau) \, d\tau = \frac{1}{\Gamma(1)} \int_a^t d\tau = t-a ,$$

that is

$$\int_a^x |D_{a+}^{\beta} f(t)| \, |D_{a+}^{\alpha} f(t)| \, dt = \int_a^x |D_{a+}^{\beta} f(t)| \, dt = \int_a^x (t-a) \, dt = \frac{1}{2}(x-a)^2 . \tag{5.7}$$

On the other hand $K_1 \, (x-a)^{\alpha-\beta-1+\frac{2}{p}} = \frac{1}{2}(x-a)^{\frac{2}{p}}$, and we have

$$K_1 \, (x-a)^{\alpha-\beta-1+\frac{2}{p}} \left(\int_a^x \left| D_{a+}^{\alpha} f(t) \right|^q \, dt \right)^{\frac{2}{q}}$$

$$= \frac{1}{2}(x-a)^{\frac{2}{p}} \left(\int_a^x dt \right)^{\frac{2}{q}} = \frac{1}{2}(x-a)^{\frac{2}{p}} (x-a)^{\frac{2}{q}} = \frac{1}{2}(x-a)^2 . \tag{5.8}$$

According to (5.7) and (5.8) we conclude that equality in (5.1) holds for a function f such that $D^{\alpha}_{a+}f(t) = 1$ for every $t \in [a,x]$. □

Remark 5.1 Let $\alpha = 1$, $\beta = 0$, $q = 2$ and $x = b$. Then $K_1 = \frac{1}{2}$ and inequality (5.1) becomes Beesack's inequality (1.18) on interval $[a,b]$

$$\int_a^b \left| f(t) f'(t) \right| dt \le \frac{b-a}{2} \int_a^b \left[f'(t) \right]^2 dt . \tag{5.9}$$

Boundary condition $D^{\alpha-1}_{a+} f(a) = f(a) = 0$ follows from conditions in Theorem 2.13.

The following result deals with the extreme case of the preceding theorem when $p = 1$ and $q = \infty$.

Proposition 5.1 *Let $\alpha > \beta \ge 0$ and $D^{\alpha}_{a+}f \in L_\infty[a,b]$. Suppose that one of conditions $(i) - (vii)$ in Corollary 2.21 holds. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x |D^{\beta}_{a+}f(t)|\,|D^{\alpha}_{a+}f(t)|\, dt \le \frac{(x-a)^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} \left\| D^{\alpha}_{a+}f \right\|^2_\infty . \tag{5.10}$$

Proof. Using Theorem 2.13, the triangle inequality and Hölder's inequality we have

$$\begin{aligned} |D^{\beta}_{a+}f(t)| &\le \frac{1}{\Gamma(\alpha-\beta)} \int_a^t (t-\tau)^{\alpha-\beta-1} \left| D^{\alpha}_{a+}f(\tau) \right| d\tau \\ &\le \frac{1}{\Gamma(\alpha-\beta)} \left(\int_a^t (t-\tau)^{\alpha-\beta-1} d\tau \right) \|D^{\alpha}_{a+}f\|_\infty \\ &= \frac{(t-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \left\| D^{\alpha}_{a+}f \right\|_\infty . \end{aligned}$$

Hence,

$$\begin{aligned} &\int_a^x |D^{\beta}_{a+}f(t)|\,|D^{\alpha}_{a+}f(t)|\, dt \\ &\le \frac{1}{\Gamma(\alpha-\beta+1)} \left(\int_a^x (t-a)^{\alpha-\beta} |D^{\alpha}_{a+}f(\tau)|\, d\tau \right) \|D^{\alpha}_{a+}f\|_\infty \\ &\le \frac{1}{\Gamma(\alpha-\beta+1)} \left(\int_a^x (t-a)^{\alpha-\beta} d\tau \right) \|D^{\alpha}_{a+}f\|^2_\infty \\ &= \frac{1}{\Gamma(\alpha-\beta+2)} (x-a)^{\alpha-\beta+1} \left\| D^{\alpha}_{a+}f \right\|^2_\infty . \end{aligned}$$

□

Now we present a counterpart of Theorem 5.1, a case when $p \in (0,1)$ and $q < 0$. Since $q < 0$, a sufficient condition is $\alpha > \beta$, but we need $1/D^{\alpha}_{a+}f \in L_q[a,b]$. The proof is similar to the proof of Theorem 5.1, apart that we have equality in (5.3), since $D^{\alpha}_{a+}f$ has a fixed sign on $[a,b]$, and then we use reverse Hölder's inequality.

Theorem 5.2 *Let $p \in (0,1)$, $q < 0$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $\alpha > \beta \geq 0$ and let $D^{\alpha}_{a+}f \in L_q[a,b]$ be of fixed sign on $[a,b]$, with $1/D^{\alpha}_{a+}f \in L_q[a,b]$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.21 holds. Then reverse inequality in (5.1) holds.*

Following inequalities include the right-sided Riemann-Liouville fractional derivatives.

Theorem 5.3 *Let $p,q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $\beta \geq 0$, $\alpha > \beta + \frac{1}{q}$ and $D^{\alpha}_{b-}f \in L_q[a,b]$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.22 holds. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b |D^{\beta}_{b-}f(t)|\,|D^{\alpha}_{b-}f(t)|\,dt \leq K_1\,(b-x)^{\alpha-\beta-1+\frac{2}{p}} \left(\int_x^b \left|D^{\alpha}_{b-}f(t)\right|^q dt\right)^{\frac{2}{q}}, \tag{5.11}$$

where K_1 is given by (5.2).

Inequality (5.11) is sharp for $\alpha = \beta+1$, where equality is attained for a function f such that $D^{\alpha}_{b-}f(t) = 1$, $t \in [x,b]$.

We continue with cases for $p=1$ and $p \in (0,1)$.

Proposition 5.2 *Let $\alpha > \beta \geq 0$ and $D^{\alpha}_{b-}f \in L_{\infty}[a,b]$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.22 holds. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b |D^{\beta}_{b-}f(t)|\,|D^{\alpha}_{b-}f(t)|\,dt \leq \frac{(b-x)^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} \left\|D^{\alpha}_{b-}f\right\|^2_{\infty}. \tag{5.12}$$

Theorem 5.4 *Let $p \in (0,1)$, $q < 0$ and $\frac{1}{p}+\frac{1}{q}=1$. Let $\alpha > \beta \geq 0$ and let $D^{\alpha}_{b-}f \in L_q[a,b]$ be of fixed sign on $[a,b]$, with $1/D^{\alpha}_{b-}f \in L_q[a,b]$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.22 holds. Then reverse inequality in (5.11) holds.*

Next corollary is a simple consequence of inequalities (5.1), (5.11) and elementary inequality $(x+y)^{\mu} \geq x^{\mu}+y^{\mu}$ which holds for $\mu \geq 1$.

Corollary 5.1 *Suppose that assumptions of Theorem 5.1 and Theorem 5.3 hold. If $1 < q \leq 2$, then*

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}} |D^{\beta}_{a+}f(t)|\,|D^{\alpha}_{a+}f(t)|\,dt + \int_{\frac{a+b}{2}}^b |D^{\beta}_{b-}f(t)|\,|D^{\alpha}_{b-}f(t)|\,dt \\
&\leq K_1 \left(\frac{b-a}{2}\right)^{\alpha-\beta-1+\frac{2}{p}} \left(\int_a^{\frac{a+b}{2}} \left|D^{\alpha}_{a+}f(t)\right|^q dt + \int_{\frac{a+b}{2}}^b \left|D^{\alpha}_{b-}f(t)\right|^q dt\right)^{\frac{2}{q}}.
\end{aligned} \tag{5.13}$$

Inequality (5.13) is sharp for $\alpha = \beta+1$ and $q=2$.

Remark 5.2 Let $\alpha = 1$, $\beta = 0$ and $q = 2$. Then $K_1 = \frac{1}{2}$ and inequality (5.13) implies classical Opial's inequality (1.17) on interval $[a,b]$

$$\int_a^b \left|f(x)f'(x)\right| dx \leq \frac{b-a}{4} \int_a^b \left[f'(x)\right]^2 dx. \tag{5.14}$$

Boundary conditions $D_{a+}^{\alpha-1}f(a) = f(a) = 0$ and $D_{b-}^{\alpha-1}f(b) = f(b) = 0$ follow from conditions in Theorem 2.13 and Theorem 2.14.

THE CAPUTO FRACTIONAL DERIVATIVES

The compositions identities for the Caputo fractional derivatives are given in Theorem 2.16 and Theorem 2.17. Inequalities in Theorem 5.5, Theorem 5.7 and Corollary 5.2 are given by Andrić-Pečarić-Perić in [26].

Theorem 5.5 *Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $\beta \geq 0$, $\alpha > \beta + \frac{1}{q}$ and n, m given by* (2.70). *Let $f \in AC^n[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m,\dots,n-1$ and let ${}^C D_{a+}^{\alpha} f \in L_q[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x |{}^C D_{a+}^{\beta} f(t)| |{}^C D_{a+}^{\alpha} f(t)| \, dt \leq K_1 \, (x-a)^{\alpha-\beta-1+\frac{2}{p}} \left(\int_a^x |{}^C D_{a+}^{\alpha} f(t)|^q \, dt \right)^{\frac{2}{q}}, \tag{5.15}$$

where K_1 is given by (5.2).

Inequality (5.15) *is sharp for $\alpha = \beta + 1$, where equality is attained for a function f such that ${}^C D_{a+}^{\alpha} f(t) = 1$, $t \in [a,b]$.*

Remark 5.3 Let $\alpha = 1$, $\beta = 0$, $q = 2$ and $x = b$. Then $K_1 = \frac{1}{2}$ and inequality (5.15) implies Beesack's inequality (1.18), that is (5.9). We have $n = 1$, $m = 0$ and boundary condition $f(a) = 0$ follows from conditions in Theorem 2.16.

Proposition 5.3 *Let $\alpha > \beta \geq 0$ and n, m given by* (2.70). *Let $f \in AC^n[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m,\dots,n-1$ and let ${}^C D_{a+}^{\alpha} f \in L_\infty[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x |{}^C D_{a+}^{\beta} f(t)| |{}^C D_{a+}^{\alpha} f(t)| \, dt \leq \frac{(x-a)^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} \left\| {}^C D_{a+}^{\alpha} f \right\|_\infty^2. \tag{5.16}$$

Theorem 5.6 *Let $p \in (0,1)$, $q < 0$ and $\frac{1}{p}+\frac{1}{q}=1$. Let $\alpha > \beta \geq 0$ and n, m given by* (2.70). *Let $f \in AC^n[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m,\dots,n-1$ and let ${}^C D_{a+}^{\alpha} f \in L_q[a,b]$ be of fixed sign on $[a,b]$, with $1/{}^C D_{a+}^{\alpha} f \in L_q[a,b]$. Then reverse inequality in* (5.15) *holds.*

Following inequalities include the right-sided Caputo fractional derivatives.

Theorem 5.7 *Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $\beta \geq 0$, $\alpha > \beta + \frac{1}{q}$ and n, m given by* (2.70). *Let $f \in AC^n[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m,\dots,n-1$ and let ${}^C D_{b-}^{\alpha} f \in L_q[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b |{}^C D_{b-}^{\beta} f(t)| |{}^C D_{b-}^{\alpha} f(t)| \, dt \leq K_1 \, (b-x)^{\alpha-\beta-1+\frac{2}{p}} \left(\int_x^b |{}^C D_{b-}^{\alpha} f(t)|^q \, dt \right)^{\frac{2}{q}}, \tag{5.17}$$

where K_1 is given by (5.2).

Inequality (5.17) *is sharp for $\alpha = \beta + 1$, where equality is attained for a function f such that ${}^C D_{b-}^{\alpha} f(t) = 1$, $t \in [a,b]$.*

Proposition 5.4 *Let $\alpha > \beta \geq 0$ and n,m given by* (2.70). *Let $f \in AC^n[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m, \ldots, n-1$ and let ${}^C D^{\alpha}_{b-} f \in L_\infty[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b |{}^C D^{\beta}_{b-} f(t)|\,|{}^C D^{\alpha}_{b-} f(t)|\,dt \leq \frac{(b-x)^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} \left\|{}^C D^{\alpha}_{b-} f\right\|^2_\infty . \tag{5.18}$$

Theorem 5.8 *Let $p \in (0,1)$, $q < 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $\alpha > \beta \geq 0$ and n, m given by* (2.70). *Let $f \in AC^n[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m, \ldots, n-1$ and let ${}^C D^{\alpha}_{b-} f \in L_q[a,b]$ be of fixed sign on $[a,b]$, with $1/{}^C D^{\alpha}_{b-} f \in L_q[a,b]$. Then reverse inequality in* (5.17) *holds.*

Corollary 5.2 *Suppose that assumptions of Theorem 5.5 and Theorem 5.7 holds. If $1 < q \leq 2$, then*

$$\int_a^{\frac{a+b}{2}} |{}^C D^{\beta}_{a+} f(t)|\,|{}^C D^{\alpha}_{a+} f(t)|\,dt + \int_{\frac{a+b}{2}}^b |{}^C D^{\beta}_{b-} f(t)|\,|{}^C D^{\alpha}_{b-} f(t)|\,dt$$
$$\leq K_1 \left(\frac{b-a}{2}\right)^{\alpha-\beta-1+\frac{2}{p}} \left(\int_a^{\frac{a+b}{2}} \left|{}^C D^{\alpha}_{a+} f(t)\right|^q dt + \int_{\frac{a+b}{2}}^b \left|{}^C D^{\alpha}_{b-} f(t)\right|^q dt\right)^{\frac{2}{q}} . \tag{5.19}$$

Inequality (5.19) *is sharp for $\alpha = \beta + 1$ and $q = 2$.*

Remark 5.4 Let $\alpha = 1$, $\beta = 0$ and $q = 2$. Then $K_1 = \frac{1}{2}$ and inequality (5.19) implies classical Opial's inequality (1.17), that is (5.14). We have $n = 1$, $m = 0$ and boundary conditions $f(a) = f(b) = 0$ follow from conditions in Theorem 2.16 and Theorem 2.17.

THE CANAVATI FRACTIONAL DERIVATIVES

Composition identities for the Canavati fractional derivatives are given in Theorem 2.19 and Theorem 2.20. First we present inequalities involving the left-sided Canavati fractional derivatives, and then the right-sided.

Theorem 5.9 *Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $\beta \geq 0$, $\alpha > \beta + \frac{1}{q}$, $n = [\alpha] + 1$ and $m = [\beta] + 1$. Let $f \in C^{\alpha}_{a+}[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m-1, \ldots, n-2$ and let ${}^{C_1} D^{\alpha}_{a+} f \in L_q[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x |{}^{C_1} D^{\beta}_{a+} f(t)|\,|{}^{C_1} D^{\alpha}_{a+} f(t)|\,dt \leq K_1 (x-a)^{\alpha-\beta-1+\frac{2}{p}} \left(\int_a^x \left|{}^{C_1} D^{\alpha}_{a+} f(t)\right|^q dt\right)^{\frac{2}{q}} , \tag{5.20}$$

where K_1 is given by (5.2).

Inequality (5.20) *is sharp for $\alpha = \beta + 1$, where equality is attained for a function f such that ${}^{C_1} D^{\alpha}_{a+} f(t) = 1$, $t \in [a,b]$.*

Remark 5.5 Let $\alpha = 1$, $\beta = 0$, $q = 2$ and $x = b$. Then $K_1 = \frac{1}{2}$ and inequality (5.20) implies Beesack's inequality (1.18), that is (5.9). Boundary condition $f(a) = 0$ follows from conditions in Theorem 2.19 since $n = 2$ and $m = 1$.

Proposition 5.5 *Let $\alpha > \beta \geq 0$, $n = [\alpha]+1$ and $m = [\beta]+1$. Let $f \in C^{\alpha}_{a+}[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m-1,\dots,n-2$ and let ${}^{C_1}D^{\alpha}_{a+}f \in L_\infty[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x |{}^{C_1}D^{\beta}_{a+}f(t)|\,|{}^{C_1}D^{\alpha}_{a+}f(t)|\,dt \leq \frac{(x-a)^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} \left\|{}^{C_1}D^{\alpha}_{a+}f\right\|^2_\infty . \tag{5.21}$$

Theorem 5.10 *Let $p \in (0,1)$, $q < 0$ and $\frac{1}{p}+\frac{1}{q} = 1$. Let $\alpha > \beta \geq 0$, $n = [\alpha]+1$ and $m = [\beta]+1$. Let $f \in C^{\alpha}_{a+}[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m-1,\dots,n-2$ and let ${}^{C_1}D^{\alpha}_{a+}f \in L_q[a,b]$ be of fixed sign on $[a,b]$, with $1/{}^{C_1}D^{\alpha}_{a+}f \in L_q[a,b]$. Then reverse inequality in (5.20) holds.*

Following inequalities include the right-sided Canavati fractional derivatives.

Theorem 5.11 *Let $p,q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$. Let $\beta \geq 0$, $\alpha > \beta + \frac{1}{q}$, $n = [\alpha]+1$ and $m = [\beta]+1$. Let $f \in C^{\alpha}_{b-}[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m-1,\dots,n-2$ and let ${}^{C_1}D^{\alpha}_{b-}f \in L_q[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b |{}^{C_1}D^{\beta}_{b-}f(t)|\,|{}^{C_1}D^{\alpha}_{b-}f(t)|\,dt \leq K_1\,(b-x)^{\alpha-\beta-1+\frac{2}{p}} \left(\int_x^b \left|{}^{C_1}D^{\alpha}_{b-}f(t)\right|^q dt\right)^{\frac{2}{q}}, \tag{5.22}$$

where K_1 is given by (5.2).

Inequality (5.22) is sharp for $\alpha = \beta+1$, where equality is attained for a function f such that ${}^{C_1}D^{\alpha}_{b-}f(t) = 1$, $t \in [a,b]$.

Proposition 5.6 *Let $\alpha > \beta \geq 0$, $n = [\alpha]+1$ and $m = [\beta]+1$. Let $f \in C^{\alpha}_{b-}[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m-1,\dots,n-2$ and let ${}^{C_1}D^{\alpha}_{b-}f \in L_\infty[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b |{}^{C_1}D^{\beta}_{b-}f(t)|\,|{}^{C_1}D^{\alpha}_{b-}f(t)|\,dt \leq \frac{(b-x)^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} \left\|{}^{C_1}D^{\alpha}_{b-}f\right\|^2_\infty . \tag{5.23}$$

Theorem 5.12 *Let $p \in (0,1)$, $q < 0$ and $\frac{1}{p}+\frac{1}{q} = 1$. Let $\alpha > \beta \geq 0$, $n = [\alpha]+1$ and $m = [\beta]+1$. Let $f \in C^{\alpha}_{b-}[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m-1,\dots,n-2$ and let ${}^{C_1}D^{\alpha}_{b-}f \in L_q[a,b]$ be of fixed sign on $[a,b]$, with $1/{}^{C_1}D^{\alpha}_{b-}f \in L_q[a,b]$. Then reverse inequality in (5.22) holds.*

Corollary 5.3 *Suppose that assumptions of Theorem 5.9 and Theorem 5.11 hold. If t $1 < q \leq 2$, then*

$$\int_a^{\frac{a+b}{2}} |{}^{C_1}D^{\beta}_{a+}f(t)|\,|{}^{C_1}D^{\alpha}_{a+}f(t)|\,dt + \int_{\frac{a+b}{2}}^b |{}^{C_1}D^{\beta}_{b-}f(t)|\,|{}^{C_1}D^{\alpha}_{b-}f(t)|\,dt$$

$$\leq K_1\left(\frac{b-a}{2}\right)^{\alpha-\beta-1+\frac{2}{p}}\left(\int_a^{\frac{a+b}{2}}\left|{}^{C_1}D^{\alpha}_{a+}f(t)\right|^q dt+\int_{\frac{a+b}{2}}^{b}\left|{}^{C_1}D^{\alpha}_{b-}f(t)\right|^q dt\right)^{\frac{2}{q}}. \tag{5.24}$$

Inequality (5.24) *is sharp for $\alpha=\beta+1$ and $q=2$.*

Remark 5.6 Let $\alpha=1$, $\beta=0$ and $q=2$. Then $K_1=\frac{1}{2}$ and inequality (5.24) implies classical Opial's inequality (1.17), that is (5.14). Boundary conditions $f(a)=f(b)=0$ follow from conditions of Theorem 2.19 and Theorem 2.20 since $n=2$ and $m=1$.

5.2 Inequalities with fractional derivatives of order α, β and $\beta+1$

Next we observe an inequality with a form

$$\int_a^b |\mathbf{D}^{\beta}f(t)|\,|\mathbf{D}^{\beta+1}f(t)|\,dt\leq K\left(\int_a^b |\mathbf{D}^{\alpha}f(t)|^q\,dt\right)^{\frac{2}{q}},$$

where $\alpha>\beta+1$, $\beta\geq 0$, $K>0$ is a constant and $q\in\mathbb{R}$.

This inequality for the left-sided Riemann-Liouville fractional derivatives is given in [14], and for the Caputo and the Canavati fractional derivatives in [12]. Now we give them under improved conditions. Some of the improved results for the Caputo fractional derivatives, Andrić-Pečarić-Perić give in [26].

THE RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVES

Theorem 5.13 *Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $\beta\geq 0$, $\alpha>\beta+1$ and $D^{\alpha}_{a+}f\in L_q[a,b]$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.21 holds. Then for a.e. $x\in[a,b]$ holds*

$$\int_a^x |D^{\beta}_{a+}f(t)|\,|D^{\beta+1}_{a+}f(t)|\,dt\leq K_2\,(x-a)^{2(\alpha-\beta-\frac{1}{q})}\left(\int_a^x |D^{\alpha}_{a+}f(t)|^q\,dt\right)^{\frac{2}{q}}, \tag{5.25}$$

where

$$K_2=\left[2\Gamma^2(\alpha-\beta)\,(p(\alpha-\beta-1)+1)^{\frac{2}{p}}\right]^{-1}. \tag{5.26}$$

Inequality (5.25) *is sharp and equality in* (5.25) *is attained for a function f such that $D^{\alpha}_{a+}f(t)=(x-t)^{\frac{p(\alpha-\beta-1)}{q}}$, $t\in[a,x]$.*

Proof. Using Theorem 2.13 and triangle inequality we have

$$|D^{\beta}_{a+}f(t)| \le \frac{1}{\Gamma(\alpha-\beta)}\int_a^t (t-\tau)^{\alpha-\beta-1}|D^{\alpha}_{a+}f(\tau)|\,d\tau := U(t).$$

Since $\alpha > \beta+1$, then by (i) in Theorem 2.5 follows

$$\begin{aligned}|D^{\beta+1}_{a+}f(t)| &\le \frac{1}{\Gamma(\alpha-\beta-1)}\int_a^t (t-\tau)^{\alpha-\beta-2}|D^{\alpha}_{a+}f(\tau)|\,d\tau \\ &= J^{\alpha-\beta-1}_{a+}|D^{\alpha}_{a+}f(t)| = J^{-1}_{a+}J^{\alpha-\beta}_{a+}|D^{\alpha}_{a+}f(t)| \\ &= D^{1}_{a+}J^{\alpha-\beta}_{a+}|D^{\alpha}_{a+}f(t)| = U'(t).\end{aligned}$$

Now we use Hölder's inequality

$$\begin{aligned}&\int_a^x |D^{\beta}_{a+}f(t)|\,|D^{\beta+1}_{a+}f)(t)|\,dt \\ &\le \int_a^x U(t)U'(t)\,dt = \frac{1}{2}\left[U^2(x)-U^2(a)\right] = \frac{1}{2}U^2(x) \\ &= \frac{1}{2\left[\Gamma(\alpha-\beta)\right]^2}\left(\int_a^x (x-t)^{\alpha-\beta-1}\,|D^{\alpha}_{a+}f(t)|\,dt\right)^2 \\ &\le \frac{1}{2\left[\Gamma(\alpha-\beta)\right]^2}\left(\int_a^x (x-t)^{p(\alpha-\beta-1)}\,dt\right)^{\frac{2}{p}}\left(\int_a^x |D^{\alpha}_{a+}f(t)|^q\,dt\right)^{\frac{2}{q}} \qquad (5.27) \\ &= \frac{1}{2\left[\Gamma(\alpha-\beta)\right]^2}\,\frac{(x-a)^{\frac{2p(\alpha-\beta-1)+2}{p}}}{\left[p(\alpha-\beta-1)+1\right]^{\frac{2}{p}}}\left(\int_a^x |D^{\alpha}_{a+}f(t)|^q\,dt\right)^{\frac{2}{q}},\end{aligned}$$

which gives inequality (5.25).

Using the equality condition in Hölder's inequality, we have equality in (5.27) if and only if $|D^{\alpha}_{a+}f(t)|^q = K(x-t)^{p(\alpha-\beta-1)}$ for some constant $K \ge 0$ and every $t \in [a,x]$. □

We continue with cases for $p=1$ and $p\in(0,1)$.

Proposition 5.7 *Let $\beta \ge 0$, $\alpha > \beta+1$ and $D^{\alpha}_{a+}f \in L_{\infty}[a,b]$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.21 holds. Then for a.e. $x\in[a,b]$ holds*

$$\int_a^x |D^{\beta}_{a+}f(t)|\,|D^{\beta+1}_{a+}f(t)|\,dt \le \frac{(x-a)^{2(\alpha-\beta)}}{2\Gamma^2(\alpha-\beta+1)}\left\|D^{\alpha}_{a+}f\right\|^2_{\infty}. \qquad (5.28)$$

Proof. Using Theorem 2.13, the triangle inequality and Hölder's inequality we have

$$\begin{aligned}|D^{\beta}_{a+}f(t)| &\le \frac{1}{\Gamma(\alpha-\beta)}\int_a^t (t-\tau)^{\alpha-\beta-1}|D^{\alpha}_{a+}f(\tau)|\,d\tau \\ &\le \frac{1}{\Gamma(\alpha-\beta)}\left(\int_a^t (t-\tau)^{\alpha-\beta-1}\,d\tau\right)\|D^{\alpha}_{a+}f\|_{\infty} \\ &= \frac{(t-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\|D^{\alpha}_{a+}f\|_{\infty}.\end{aligned}$$

Also,

$$|D_{a+}^{\beta+1} f(t)| \leq \frac{(t-a)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \|D_{a+}^{\alpha} f\|_{\infty}.$$

Hence,

$$\begin{aligned}
&\int_a^x |D_{a+}^{\beta} f(t)|\,|D_{a+}^{\beta+1} f(t)|\,dt \\
&\leq \frac{1}{\Gamma(\alpha-\beta)\,\Gamma(\alpha-\beta+1)} \left(\int_a^x (t-a)^{2(\alpha-\beta)-1}\,dt\right) \|D_{a+}^{\alpha} f\|_{\infty}^2 \\
&= \frac{(x-a)^{2(\alpha-\beta)}}{\Gamma(\alpha-\beta)\,\Gamma(\alpha-\beta+1)\,2(\alpha-\beta)} \|D_{a+}^{\alpha} f\|_{\infty}^2.
\end{aligned}$$

□

Theorem 5.14 *Let $p \in (0,1)$, $q < 0$ and $\frac{1}{p}+\frac{1}{q}=1$. Let $\beta \geq 0$, $\alpha > \beta+1$ and let $D_{a+}^{\alpha} f \in L_q[a,b]$ be of fixed sign on $[a,b]$, with $1/D_{a+}^{\alpha} f \in L_q[a,b]$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.21 holds. Then reverse inequality in (5.25) holds.*

Following inequalities include the right-sided Riemann-Liouville fractional derivatives.

Theorem 5.15 *Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $\beta \geq 0$, $\alpha > \beta+1$ and $D_{b-}^{\alpha} f \in L_q[a,b]$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.22 holds. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b |D_{b-}^{\beta} f(t)|\,|D_{b-}^{\beta+1} f(t)|\,dt \leq K_2\,(b-x)^{2(\alpha-\beta-\frac{1}{q})} \left(\int_x^b \left|D_{b-}^{\alpha} f(t)\right|^q dt\right)^{\frac{2}{q}}, \tag{5.29}$$

where K_2 is given by (5.26).

Inequality (5.29) *is sharp and equality in* (5.29) *is attained for a function f such that $D_{b-}^{\alpha} f(t) = (t-x)^{\frac{p(\alpha-\beta-1)}{q}}$, $t \in [x,b]$.*

Next we observe cases for $p=1$ and $p \in (0,1)$.

Proposition 5.8 *Let $\beta \geq 0$, $\alpha > \beta+1$ and $D_{b-}^{\alpha} f \in L_{\infty}[a,b]$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.22 holds. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b |D_{b-}^{\beta} f(t)|\,|D_{b-}^{\beta+1} f(t)|\,dt \leq \frac{(b-x)^{2(\alpha-\beta)}}{2\Gamma^2(\alpha-\beta+1)} \left\|D_{b-}^{\alpha} f\right\|_{\infty}^2. \tag{5.30}$$

Theorem 5.16 *Let $p \in (0,1)$, $q < 0$ and $\frac{1}{p}+\frac{1}{q}=1$. Let $\beta \geq 0$, $\alpha > \beta+1$ and let $D_{b-}^{\alpha} f \in L_q[a,b]$ be of fixed sign on $[a,b]$, with $1/D_{b-}^{\alpha} f \in L_q[a,b]$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.22 holds. Then reverse inequality in (5.29) holds.*

THE CAPUTO FRACTIONAL DERIVATIVES

Inequalities in Theorem 5.17 and Theorem 5.19 are given by Andrić-Pečarić-Perić in [26].

Theorem 5.17 *Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $\beta\geq 0$, $\alpha>\beta+1$ and n, m given by* (2.70)*. Let $f\in AC^n[a,b]$ be such that $f^{(i)}(a)=0$ for $i=m,\dots,n-1$ and let ${}^CD^{\alpha}_{a+}f\in L_q[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$\int_a^x |{}^CD^{\beta}_{a+}f(t)||{}^CD^{\beta+1}_{a+}f(t)|\,dt\leq K_2\,(x-a)^{2(\alpha-\beta-\frac{1}{q})}\left(\int_a^x \left|{}^CD^{\alpha}_{a+}f(t)\right|^q dt\right)^{\frac{2}{q}}, \tag{5.31}$$

where K_2 is given by (5.26)*.*

Inequality (5.31) *is sharp and equality in* (5.31) *is attained for a function f such that ${}^CD^{\alpha}_{a+}f(t)=(x-t)^{\frac{p(\alpha-\beta-1)}{q}}$, $t\in[a,x]$.*

Proposition 5.9 *Let $\beta\geq 0$, $\alpha>\beta+1$ and n, m given by* (2.70)*. Let $f\in AC^n[a,b]$ be such that $f^{(i)}(a)=0$ for $i=m,\dots,n-1$ and let ${}^CD^{\alpha}_{a+}f\in L_\infty[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$\int_a^x |{}^CD^{\beta}_{a+}f(t)||{}^CD^{\beta+1}_{a+}f(t)|\,dt\leq \frac{(x-a)^{2(\alpha-\beta)}}{2\Gamma^2(\alpha-\beta+1)}\left\|{}^CD^{\alpha}_{a+}f\right\|_\infty^2. \tag{5.32}$$

Theorem 5.18 *Let $p\in(0,1)$, $q<0$ and $\frac{1}{p}+\frac{1}{q}=1$. Let $\beta\geq 0$, $\alpha>\beta+1$ and n, m given by* (2.70)*. Let $f\in AC^n[a,b]$ be such that $f^{(i)}(a)=0$ for $i=m,\dots,n-1$ and let ${}^CD^{\alpha}_{a+}f\in L_q[a,b]$ be of fixed sign on $[a,b]$, with $1/{}^CD^{\alpha}_{a+}f\in L_q[a,b]$. Then reverse inequality in* (5.31) *holds.*

Following inequalities include the right-sided Caputo fractional derivatives.

Theorem 5.19 *Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $\beta\geq 0$, $\alpha>\beta+1$ and n, m given by* (2.70)*. Let $f\in AC^n[a,b]$ be such that $f^{(i)}(b)=0$ for $i=m,\dots,n-1$ and let ${}^CD^{\alpha}_{b-}f\in L_q[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$\int_x^b |{}^CD^{\beta}_{b-}f(t)||{}^CD^{\beta+1}_{b-}f(t)|\,dt\leq K_2\,(b-x)^{2(\alpha-\beta-\frac{1}{q})}\left(\int_x^b \left|{}^CD^{\alpha}_{b-}f(t)\right|^q dt\right)^{\frac{2}{q}}, \tag{5.33}$$

where K_2 is given by (5.26)*.*

Inequality (5.33) *is sharp and equality in* (5.33) *is attained for a function f such that ${}^CD^{\alpha}_{b-}f(t)=(t-x)^{\frac{p(\alpha-\beta-1)}{q}}$, $t\in[x,b]$.*

Proposition 5.10 *Let $\beta\geq 0$, $\alpha>\beta+1$ and n, m given by* (2.70)*. Let $f\in AC^n[a,b]$ be such that $f^{(i)}(b)=0$ for $i=m,\dots,n-1$ and let ${}^CD^{\alpha}_{b-}f\in L_\infty[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$\int_x^b |{}^CD^{\beta}_{b-}f(t)||{}^CD^{\beta+1}_{b-}f(t)|\,dt\leq \frac{(b-x)^{2(\alpha-\beta)}}{2\Gamma^2(\alpha-\beta+1)}\left\|{}^CD^{\alpha}_{b-}f\right\|_\infty^2. \tag{5.34}$$

Theorem 5.20 *Let $p\in(0,1)$, $q<0$ and $\frac{1}{p}+\frac{1}{q}=1$. Let $\beta\geq 0$, $\alpha>\beta+1$ and n, m given by* (2.70)*. Let $f\in AC^n[a,b]$ be such that $f^{(i)}(b)=0$ for $i=m,\dots,n-1$ and let ${}^CD^{\alpha}_{b-}f\in L_q[a,b]$ be of fixed sign on $[a,b]$, with $1/{}^CD^{\alpha}_{b-}f\in L_q[a,b]$. Then reverse inequality in* (5.33) *holds.*

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Theorem 5.21 *Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $\beta\geq 0$, $\alpha>\beta+1$, $n=[\alpha]+1$ and $m=[\beta]+1$. Let $f\in C^{\alpha}_{a+}[a,b]$ be such that $f^{(i)}(a)=0$ for $i=m-1,\dots,n-2$ and let ${}^{C_1}D^{\alpha}_{a+}f\in L_q[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$\int_a^x |{}^{C_1}D^{\beta}_{a+}f(t)||{}^{C_1}D^{\beta+1}_{a+}f(t)|\,dt \leq K_2\,(x-a)^{2(\alpha-\beta-\frac{1}{q})}\left(\int_a^x \left|{}^{C_1}D^{\alpha}_{a+}f(t)\right|^q dt\right)^{\frac{2}{q}}, \quad (5.35)$$

where K_2 is given by (5.26).

Inequality (5.35) *is sharp and equality in* (5.35) *is attained for a function f such that ${}^{C_1}D^{\alpha}_{a+}f(t)=(x-t)^{\frac{p(\alpha-\beta-1)}{q}}$, $t\in[a,x]$.*

Proposition 5.11 *Let $\beta\geq 0$, $\alpha>\beta+1$, $n=[\alpha]+1$ and $m=[\beta]+1$. Let $f\in C^{\alpha}_{a+}[a,b]$ be such that $f^{(i)}(a)=0$ for $i=m-1,\dots,n-2$ and let ${}^{C_1}D^{\alpha}_{a+}f\in L_\infty[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$\int_a^x |{}^{C_1}D^{\beta}_{a+}f(t)||{}^{C_1}D^{\beta+1}_{a+}f(t)|\,dt \leq \frac{(x-a)^{2(\alpha-\beta)}}{2\Gamma^2(\alpha-\beta+1)}\left\|{}^{C_1}D^{\alpha}_{a+}f\right\|^2_\infty. \quad (5.36)$$

Theorem 5.22 *Let $p\in(0,1)$, $q<0$ and $\frac{1}{p}+\frac{1}{q}=1$. Let $\beta\geq 0$, $\alpha>\beta+1$, $n=[\alpha]+1$ and $m=[\beta]+1$. Let $f\in C^{\alpha}_{a+}[a,b]$ be such that $f^{(i)}(a)=0$ for $i=m-1,\dots,n-2$ and let ${}^{C_1}D^{\alpha}_{a+}f\in L_q[a,b]$ be of fixed sign on $[a,b]$, with $1/{}^{C_1}D^{\alpha}_{a+}f\in L_q[a,b]$. Then reverse inequality in* (5.35) *holds.*

We continue with the right-sided Canavati fractional derivatives.

Theorem 5.23 *Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $\beta\geq 0$, $\alpha>\beta+1$, $n=[\alpha]+1$ and $m=[\beta]+1$. Let $f\in C^{\alpha}_{b-}[a,b]$ be such that $f^{(i)}(b)=0$ for $i=m-1,\dots,n-2$ and let ${}^{C_1}D^{\alpha}_{b-}f\in L_q[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$\int_x^b |{}^{C_1}D^{\beta}_{b-}f(t)||{}^{C_1}D^{\beta+1}_{b-}f(t)|\,dt \leq K_2\,(b-x)^{2(\alpha-\beta-\frac{1}{q})}\left(\int_x^b \left|{}^{C_1}D^{\alpha}_{b-}f(t)\right|^q dt\right)^{\frac{2}{q}}, \quad (5.37)$$

where K_2 is given by (5.26).

Inequality (5.37) *is sharp and equality in* (5.37) *is attained for a function f such that ${}^{C_1}D^{\alpha}_{b-}f(t)=(t-x)^{\frac{p(\alpha-\beta-1)}{q}}$, $t\in[x,b]$.*

Proposition 5.12 *Let $\beta\geq 0$, $\alpha>\beta+1$, $n=[\alpha]+1$ and $m=[\beta]+1$. Let $f\in C^{\alpha}_{b-}[a,b]$ be such that $f^{(i)}(b)=0$ for $i=m-1,\dots,n-2$ and let ${}^{C_1}D^{\alpha}_{b-}f\in L_\infty[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$\int_x^b |{}^{C_1}D^{\beta}_{b-}f(t)||{}^{C_1}D^{\beta+1}_{b-}f(t)|\,dt \leq \frac{(b-x)^{2(\alpha-\beta)}}{2\Gamma^2(\alpha-\beta+1)}\left\|{}^{C_1}D^{\alpha}_{b-}f\right\|^2_\infty. \quad (5.38)$$

Theorem 5.24 *Let $p\in(0,1)$, $q<0$ and $\frac{1}{p}+\frac{1}{q}=1$. Let $\beta\geq 0$, $\alpha>\beta+1$, $n=[\alpha]+1$ and $m=[\beta]+1$. Let $f\in C^{\alpha}_{b-}[a,b]$ such that $f^{(i)}(b)=0$ for $i=m-1,\dots,n-2$ and let ${}^{C_1}D^{\alpha}_{b-}f\in L_q[a,b]$ be of fixed sign on $[a,b]$, with $1/{}^{C_1}D^{\alpha}_{b-}f\in L_q[a,b]$. Then reverse inequality in* (5.37) *holds.*

5.3 Inequalities with fractional derivatives of order α, β_1 and β_2

Following inequality is motivated by Fink's inequality (1.23) for ordinary derivatives whose form is

$$\int_a^b |\mathbf{D}^{\beta_1} f(t)|\,|\mathbf{D}^{\beta_2} f(t)|\,dt \le K\left(\int_a^b |\mathbf{D}^{\alpha} f(t)|^q\,dt\right)^{\frac{2}{q}},$$

where $\alpha > \beta_i \ge 0$, $K > 0$ is a constant and $q \in \mathbb{R}$.

This inequality is given in [15] for the left-sided Riemann-Liouville fractional derivatives and for the Caputo in [12] (obtained by Fink's method from [45]). Here we will give them under improved conditions using results from Section 2.6 Inequality for the left-sided Canavati fractional derivatives (here Theorem 5.38), Andrić-Pečarić-Perić give in [23] where improved composition identity for the left-sided Canavati fractional derivatives is used. Also, another estimation for the same inequality using method different from Fink's is obtained in [23] with a comparison of results (here Theorem 5.37 and Remark 5.9). Finally, here we will give analogous results for inequalities involving the Riemann-Liouville and the Caputo fractional derivatives. Some of the improved results for the Caputo fractional derivatives, Andrić-Pečarić-Perić give in [26].

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Theorem 5.25 *Let $p,q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $\beta_i \ge 0$, $\alpha > \beta_i + \frac{1}{q}$ for $i=1,2$ and let $D^{\alpha}_{a+} f \in L_q[a,b]$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.21 holds for $\{\alpha,\beta_1\}$ and $\{\alpha,\beta_2\}$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x |D^{\beta_1}_{a+} f(t)|\,|D^{\beta_2}_{a+} f(t)|\,dt \le K_3\,(x-a)^{2\alpha-\beta_1-\beta_2-1+\frac{2}{p}}\left(\int_a^x \left|D^{\alpha}_{a+} f(t)\right|^q dt\right)^{\frac{2}{q}}, \tag{5.39}$$

where

$$K_3 = \left[\left(2\alpha-\beta_1-\beta_2-1+\frac{2}{p}\right)\prod_{i=1}^{2}\Gamma(\alpha-\beta_i)\,(p(\alpha-\beta_i-1)+1)^{\frac{1}{p}}\right]^{-1}. \tag{5.40}$$

Proof. Using Theorem 2.13 and the triangle inequality, for $i=1,2$ follows

$$\begin{aligned}
|D^{\beta_i}_{a+} f(t)| &\le \frac{1}{\Gamma(\alpha-\beta_i)}\int_a^t (t-\tau)^{\alpha-\beta_i-1}\left|D^{\alpha}_{a+} f(\tau)\right|d\tau \\
&\le \frac{1}{\Gamma(\alpha-\beta_i)}\left(\int_a^t (t-\tau)^{p(\alpha-\beta_i-1)}\,d\tau\right)^{\frac{1}{p}}\left(\int_a^t |D^{\alpha}_{a+} f(\tau)|^q\,d\tau\right)^{\frac{1}{q}} \\
&= \frac{1}{\Gamma(\alpha-\beta_i)}\frac{(t-a)^{\alpha-\beta_i-1+\frac{1}{p}}}{[p(\alpha-\beta_i-1)+1]^{\frac{1}{p}}}\left(\int_a^t |D^{\alpha}_{a+} f(\tau)|^q\,d\tau\right)^{\frac{1}{q}}
\end{aligned}$$

$$\leq \frac{1}{\Gamma(\alpha-\beta_i)} \frac{(t-a)^{\alpha-\beta_i-1+\frac{1}{p}}}{[p(\alpha-\beta_i-1)+1]^{\frac{1}{p}}} \left(\int_a^x |D_{a+}^{\alpha} f(\tau)|^q \, d\tau\right)^{\frac{1}{q}}.$$

Hence,

$$\begin{aligned}
&\int_a^x |D_{a+}^{\beta_1} f(t)| \, |D_{a+}^{\beta_2} f(t)| \, dt \\
&\leq \frac{1}{\prod_{i=1}^2 \Gamma(\alpha-\beta_i) \, [p(\alpha-\beta_i-1)+1]^{\frac{1}{p}}} \left(\int_a^x |D_{a+}^{\alpha} f(\tau)|^q \, d\tau\right)^{\frac{2}{q}} \\
&\qquad \cdot \int_a^x (t-a)^{2\alpha-\beta_1-\beta_2-2+\frac{2}{p}} \, dt \\
&= \frac{1}{\prod_{i=1}^2 \Gamma(\alpha-\beta_i) \, [p(\alpha-\beta_i-1)+1]^{\frac{1}{p}}} \left(\int_a^x |D_{a+}^{\alpha} f(\tau)|^q \, d\tau\right)^{\frac{2}{q}} \\
&\qquad \cdot \frac{(x-a)^{2\alpha-\beta_1-\beta_2-1+\frac{2}{p}}}{2\alpha-\beta_1-\beta_2-1+\frac{2}{p}}.
\end{aligned}$$

□

Using a different technique due to Fink ([45]), we obtain yet another estimation for the inequality (5.39).

Theorem 5.26 *Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $\beta_1 \geq 0$, $\alpha > \beta_2 \geq \beta_1+1$ and $D_{a+}^{\alpha} f \in L_q[a,b]$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.21 holds for $\{\alpha,\beta_1\}$ and $\{\alpha,\beta_2\}$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x |D_{a+}^{\beta_1} f(t)| \, |D_{a+}^{\beta_2} f(t)| \, dt \leq F \, (x-a)^{2\alpha-\beta_1-\beta_2-1+\frac{2}{p}} \left(\int_a^x \left|D_{a+}^{\alpha} f(t)\right|^q \, dt\right)^{\frac{2}{q}}, \qquad (5.41)$$

where

$$F = \left[\Gamma(\alpha-\beta_1)\Gamma(\alpha-\beta_2+1)[p(\alpha-\beta_2)+1]^{\frac{1}{p}} [p(2\alpha-\beta_1-\beta_2-1)+2]^{\frac{1}{p}} \, 2^{\frac{1}{q}}\right]^{-1}. \qquad (5.42)$$

Inequality (5.41) is sharp for $\beta_2=\beta_1+1$, where equality is attained for a function f such that $D_{a+}^{\alpha} f(t) = (x-t)^{\frac{p(\alpha-\beta_2)}{q}}$, $t \in [a,x]$.

Proof. Set $\delta_1 = \alpha-\beta_1-1$ and $\delta_2 = \alpha-\beta_2-1$. Then $\delta_1-\delta_2-1 \geq 0$ since $\beta_2 \geq \beta_1+1$. Let $a \leq \tau \leq \sigma \leq t \leq x$. Then

$$\begin{aligned}
&(t-\tau)^{\delta_1}(t-\sigma)^{\delta_2} + (t-\sigma)^{\delta_1}(t-\tau)^{\delta_2} \\
&= (t-\tau)^{\delta_1-\delta_2-1}(t-\tau)^{\delta_2+1}(t-\sigma)^{\delta_2} + (t-\sigma)^{\delta_1-\delta_2-1}(t-\sigma)^{\delta_2+1}(t-\tau)^{\delta_2} \\
&\leq (x-\tau)^{\delta_1-\delta_2-1}[(t-\tau)^{\delta_2+1}(t-\sigma)^{\delta_2} + (t-\sigma)^{\delta_2+1}(t-\tau)^{\delta_2}]. \qquad (5.43)
\end{aligned}$$

In the last step (5.43) we use $\delta_1 - \delta_2 - 1 \geq 0$, and it is obvious that for $\delta_1 - \delta_2 - 1 = 0$ equality holds in (5.43). Let

$$(t-\tau)_+^r = \begin{cases} (t-\tau)^r, & a \leq \tau < t \leq x, \\ 0 \quad , & a \leq t \leq \tau \leq x. \end{cases} \tag{5.44}$$

Using integration by parts we get

$$\begin{aligned} &\int_a^x \left[(t-\tau)_+^{\delta_2+1}(t-\sigma)_+^{\delta_2}\right] dt \\ &= (t-\tau)_+^{\delta_2+1} \frac{(t-\sigma)_+^{\delta_2+1}}{\delta_2+1}\Big|_a^x - \int_a^x \left[(t-\tau)_+^{\delta_2}(t-\sigma)_+^{\delta_2+1}\right] dt \\ &= \frac{(x-\tau)^{\delta_2+1}(x-\sigma)^{\delta_2+1}}{\delta_2+1} - \int_a^x \left[(t-\tau)_+^{\delta_2}(t-\sigma)_+^{\delta_2+1}\right] dt, \end{aligned}$$

that is

$$\int_a^x [(t-\tau)_+^{\delta_2+1}(t-\sigma)_+^{\delta_2} + (t-\sigma)_+^{\delta_2+1}(t-\tau)_+^{\delta_2}]\, dt = \frac{1}{\delta_2+1}[(x-\tau)(x-\sigma)]^{\delta_2+1}. \tag{5.45}$$

Now from (5.43) and (5.45) we have

$$\int_a^x [(t-\tau)_+^{\delta_1}(t-\sigma)_+^{\delta_2} + (t-\sigma)_+^{\delta_1}(t-\tau)_+^{\delta_2}]\, dt \leq \frac{1}{(\alpha-\beta_2)}(x-\tau)^{\delta_1}(x-\sigma)^{\delta_2+1}. \tag{5.46}$$

Next we abbreviate

$$c_1 := [\Gamma(\alpha-\beta_2)\Gamma(\alpha-\beta_1)]^{-1}, \quad c_2 := [\Gamma(\alpha-\beta_2+1)\Gamma(\alpha-\beta_1)]^{-1},$$

$$c_3 := p(\alpha-\beta_2)+1, \quad \varepsilon := 2\alpha-\beta_1-\beta_2-1+\frac{1}{p}.$$

Let $a \leq t \leq x$ and $i = 1,2$. Using Theorem 2.13 we have

$$D_{a+}^{\beta_i} f(t) = \frac{1}{\Gamma(\alpha-\beta_i)} \int_a^x (t-\tau)_+^{\delta_i} D_{a+}^{\alpha} f(\tau)\, d\tau.$$

With this representation, the triangle inequality, inequality (5.46) and Hölder's inequality, we obtain

$$\begin{aligned} &\int_a^x |D_{a+}^{\beta_1} f(t)|\,|D_{a+}^{\beta_2} f(t)|\, dt \\ &\leq c_1 \int_a^x \left(\int_a^x |D_{a+}^{\alpha} f(\tau)|\,(t-\tau)_+^{\delta_1}\, d\tau\right)\left(\int_a^x |D_{a+}^{\alpha} f(\sigma)|\,(t-\sigma)_+^{\delta_2}\, d\sigma\right) dt \qquad (5.47) \\ &= c_1 \int_a^x |D_{a+}^{\alpha} f(\tau)| \left\{\int_\tau^x |D_{a+}^{\alpha} f(\sigma)| \right. \\ &\qquad \left. \cdot \left(\int_a^x \left[(t-\tau)_+^{\delta_1}(t-\sigma)_+^{\delta_2} + (t-\sigma)_+^{\delta_1}(t-\tau)_+^{\delta_2}\right] dt\right) d\sigma \right\} d\tau \end{aligned}$$

$$\leq c_2 \int_a^x |D_{a+}^{\alpha} f(\tau)| \left(\int_\tau^x |D_{a+}^{\alpha} f(\sigma)| (x-\tau)^{\delta_1} (x-\sigma)^{\delta_2+1} d\sigma \right) d\tau$$

$$\leq c_2 \int_a^x |D_{a+}^{\alpha} f(\tau)| (x-\tau)^{\delta_1} \left(\int_\tau^x |D_{a+}^{\alpha} f(\sigma)|^q d\sigma \right)^{\frac{1}{q}} \left(\int_\tau^x (x-\sigma)^{p(\delta_2+1)} d\sigma \right)^{\frac{1}{p}} d\tau \tag{5.48}$$

$$= c_2 c_3^{-\frac{1}{p}} \int_a^x |D_{a+}^{\alpha} f(\tau)| (x-\tau)^{\varepsilon} \left(\int_\tau^x |D_{a+}^{\alpha} f(\sigma)|^q d\sigma \right)^{\frac{1}{q}} d\tau$$

$$\leq c_2 c_3^{-\frac{1}{p}} \left\{ \int_a^x |D_{a+}^{\alpha} f(\tau)|^q \left(\int_\tau^x |D_{a+}^{\alpha} f(\sigma)|^q d\sigma \right) d\tau \right\}^{\frac{1}{q}} \left(\int_a^x (x-\tau)^{\varepsilon p} d\tau \right)^{\frac{1}{p}} \tag{5.49}$$

$$= c_2 c_3^{-\frac{1}{p}} (\varepsilon p+1)^{-\frac{1}{p}} (x-a)^{\varepsilon+\frac{1}{p}} \left\{ \frac{1}{2} \left(\int_a^x |D_{a+}^{\alpha} f(\tau)|^q d\tau \right)^2 \right\}^{\frac{1}{q}}.$$

In the last step we use: let $G(\tau) = \int\limits_\tau^x |D_{a+}^{\alpha} f(\sigma)|^q d\sigma$, then $G'(\tau) = -|D_{a+}^{\alpha} f(\tau)|^q$ and

$$\int_a^x -G'(\tau)G(\tau)\, d\tau = -\frac{1}{2}\int_a^x dG^2(\tau) = \frac{1}{2}\left(\int_a^x |D_{a+}^{\alpha} f(\tau)|^q d\tau \right)^2.$$

It is obvious that in the case $\beta_2 = \beta_1 + 1$ we have equality in (5.46). Using the equality condition for Hölder's inequality, we have equality in (5.48) if and only if $|D_{a+}^{\alpha} f(\sigma)|^q = K(x-\sigma)^{p(\delta_2+1)}$ for some constant $K \geq 0$ and every $\sigma \in [a,x]$. Let's prove that equality holds in (5.49) (equality in (5.47) is obvious):

$$\int_a^x |D_{a+}^{\alpha} f(\tau)| (x-\tau)^{\varepsilon} \left(\int_\tau^x |D_{a+}^{\alpha} f(\sigma)|^q d\sigma \right)^{\frac{1}{q}} d\tau$$

$$= \int_a^x K^{\frac{1}{q}} (x-\tau)^{\frac{p(\delta_2+1)}{q}} (x-\tau)^{\varepsilon} \left(\int_\tau^x K(x-\sigma)^{p(\delta_2+1)} d\sigma \right)^{\frac{1}{q}} d\tau$$

$$= \int_a^x K^{\frac{1}{q}} (x-\tau)^{\frac{p(\delta_2+1)}{q}+\varepsilon} \left(\frac{K(x-\tau)^{p(\delta_2+1)+1}}{p(\delta_2+1)+1} \right)^{\frac{1}{q}} d\tau$$

$$= \frac{K^{\frac{2}{q}}}{[p(\delta_2+1)+1]^{\frac{1}{q}}} \int_a^x (x-\tau)^{\frac{2p(\delta_2+1)}{q}+\varepsilon+\frac{1}{q}} d\tau$$

$$= \frac{K^{\frac{2}{q}}}{[p(\delta_2+1)+1]^{\frac{1}{q}}} (x-a)^{\frac{2p(\delta_2+1)}{q}+\varepsilon+\frac{1}{q}+1} \frac{1}{\frac{2p}{q}(\delta_2+1)+\varepsilon+1+\frac{1}{q}}.$$

Since $\frac{p}{q} = p-1$ and from $\beta_2 = \beta_1 + 1$ follows $\varepsilon = 2(\delta_2+1)+\frac{1}{p}$, we have

$$\int_a^x |D_{a+}^{\alpha} f(\tau)| (x-\tau)^{\varepsilon} \left(\int_\tau^x |D_{a+}^{\alpha} f(\sigma)|^q d\sigma \right)^{\frac{1}{q}} d\tau$$

$$= \frac{K^{\frac{2}{q}}}{[p(\delta_2+1)+1]^{\frac{1}{q}}} \frac{(x-a)^{2p(\delta_2+1)+2}}{2p(\delta_2+1)+2}. \tag{5.50}$$

On the other hand

$$\begin{aligned}
&\left\{\int_a^x |D_{a+}^{\alpha} f(\tau)|^q \left(\int_\tau^x |D_{a+}^{\alpha} f(\sigma)|^q \, d\sigma\right) d\tau\right\}^{\frac{1}{q}} \left(\int_a^x (x-\tau)^{\varepsilon p} \, d\tau\right)^{\frac{1}{p}} \\
&= \frac{(x-a)^{\varepsilon+\frac{1}{p}}}{(\varepsilon p+1)^{\frac{1}{p}}} \left\{\int_a^x K(x-\tau)^{p(\delta_2+1)} \left(\int_\tau^x K(x-\sigma)^{p(\delta_2+1)} \, d\sigma\right) d\tau\right\}^{\frac{1}{q}} \\
&= \frac{(x-a)^{\varepsilon+\frac{1}{p}}}{[2p(\delta_2+1)+2]^{\frac{1}{p}}} \frac{K^{\frac{2}{q}}}{[p(\delta_2+1)+1]^{\frac{1}{q}}} \left\{\int_a^x (x-\tau)^{p(\delta_2+1)} (x-\tau)^{p(\delta_2+1)+1} \, d\tau\right\}^{\frac{1}{q}} \\
&= \frac{(x-a)^{\varepsilon+\frac{1}{p}}}{[2p(\delta_2+1)+2]^{\frac{1}{p}}} \frac{K^{\frac{2}{q}}}{[p(\delta_2+1)+1]^{\frac{1}{q}}} \frac{(x-a)^{\frac{2p(\delta_2+1)+2}{q}}}{[2p(\delta_2+1)+2]^{\frac{1}{q}}} \\
&= \frac{K^{\frac{2}{q}}}{[p(\delta_2+1)+1]^{\frac{1}{q}}} \frac{(x-a)^{2p(\delta_2+1)+2}}{2p(\delta_2+1)+2}.
\end{aligned} \tag{5.51}$$

According to (5.50) and (5.51) we conclude that for $|D_{a+}^{\alpha} f(\sigma)|^q = K(x-\sigma)^{p(\delta_2+1)}$ equality holds in (5.49). □

Remark 5.7 Let $\alpha = n \in \mathbb{N}$, $\beta_1 = i \in \mathbb{N}$, $\beta_2 = j \in \mathbb{N}$, $i < j \leq n-1$ and $x = b$. Then inequality (5.41) implies Fink's inequality (1.23) on interval $[a,b]$

$$\int_a^b \left| f^{(i)}(t) f^{(j)}(t) \right| dt \leq F\,(a-b)^{2n-i-j+1-\frac{2}{q}} \left(\int_a^b \left| f^{(n)}(t) \right|^q dt\right)^{\frac{2}{q}} \tag{5.52}$$

with $F = C(n,i,j,q)$ from relation (1.24). By Notation 1.6 we know that sufficient condition for the vanishing derivatives is $f^{(k)}(a) = 0$ for $k = i, \ldots, n-1$. Since $\alpha - \beta_i \in \mathbb{N}$, we use (ii) in Theorem 2.13, which gives us boundary conditions $D_{a+}^{n-k} f(a) = 0$, $k = 1, \ldots, n-i$, that is $D_{a+}^{k} f(a) = f^{(k)}(a) = 0$ for $= i, \ldots, n-1$.

Remark 5.8 Let f be such that $D_{a+}^{\alpha} f \in C[a,b]$. Then

$$\lim_{\beta_2 \to \alpha - 0} D_{a+}^{\beta_2} f(t) = \lim_{\beta_2 \to \alpha - 0} \frac{1}{\Gamma(\alpha-\beta_2)} \int_a^t (t-\tau)^{\alpha-\beta_2-1} D_{a+}^{\alpha} f(\tau) \, d\tau,$$

so we can formally compare estimations from Theorem 5.1 and Theorem 5.26, obtained in a different way. Setting $\alpha = \beta_2$ and $\beta = \beta_1$, we have

$$F = \frac{1}{2^{\frac{1}{q}} \Gamma(\alpha-\beta) \, [p(\alpha-\beta-1)+2]^{\frac{1}{p}}}$$

so obviously $K_1 < F$ for $\alpha > \beta + 1$, hence, estimation in Theorem 5.1 is better then estimation in Theorem 5.26.

Remark 5.9 Constants K_3 and F from the two previous theorems are in general not comparable, but there are cases when we can do that. Notice

$$\frac{F}{K_3} = \frac{(\alpha-\beta_1-1+\frac{1}{p})^{\frac{1}{p}}\,(\alpha-\beta_2-1+\frac{1}{p})^{\frac{1}{p}}\,(2\alpha-\beta_1-\beta_2-1+\frac{2}{p})^{\frac{1}{q}}}{2^{\frac{1}{q}}\,(\alpha-\beta_2)\,(\alpha-\beta_2+\frac{1}{p})^{\frac{1}{p}}}.$$

We want to find cases when $K_3 < F$. Set $\alpha-\beta_2 = d_2$, $\beta_2-\beta_1 = d_1 \geq 1$. Then $\alpha-\beta_1 = d_1+d_2$ and inequality $K_3 < F$ is equivalent to

$$\frac{1}{(d_1+d_2-1+\frac{1}{p})^{\frac{1}{p}}\,(d_1+2d_2-1+\frac{2}{p})^{1-\frac{1}{p}}} < \frac{(d_2-1+\frac{1}{p})^{\frac{1}{p}}}{2^{1-\frac{1}{p}}\,d_2\,(d_2+\frac{1}{p})^{\frac{1}{p}}}. \tag{5.53}$$

If d_1 is big enough, then the left side of (5.53) tends to zero, while the right side is independent of d_1. Therefore, in this case $K_3 < F$.

Let $d_1 = 1$, that is $\beta_2 = \beta_1+1$ (see the discussion on sharpness in Theorem 5.26). Then the reverse inequality in (5.53) is equivalent to

$$\frac{1}{(d_2+\frac{1}{p})^{\frac{1}{p}}\,(2d_2+\frac{2}{p})^{1-\frac{1}{p}}} > \frac{(d_2-1+\frac{1}{p})^{\frac{1}{p}}}{2^{1-\frac{1}{p}}\,d_2\,(d_2+\frac{1}{p})^{\frac{1}{p}}},$$

that is

$$\frac{pd_2+1}{pd_2-p+1} > \left(1+\frac{1}{pd_2}\right)^p,$$

which is equivalent to inequality

$$\left(\frac{pd_2+1-p}{pd_2+1}\right)^{\frac{1}{p}} < \frac{pd_2}{1+pd_2}. \tag{5.54}$$

This (5.54) is a simple consequence of Bernoulli's inequality, since

$$\left(\frac{pd_2+1-p}{pd_2+1}\right)^{\frac{1}{p}} = \left(1+\frac{-p}{pd_2+1}\right)^{\frac{1}{p}} < 1+\frac{1}{p}\cdot\frac{-p}{1+pd_2} = \frac{pd_2}{1+pd_2}.$$

Therefore, if $\beta_2 = \beta_1+1$, then $F < K_3$, and this is in accordance with Theorem 5.26.

An illustrative case is $p = q = 2$. Then (5.53) is equivalent to

$$12(d_1-1)\,d_2^2+2(2\,d_1^2-4\,d_1+1)\,d_2-2\,d_1^2+d_1 > 0.$$

That is, $K_3 < F$ is equivalent to

$$d_2 > \tilde{d}_2 = \frac{-2d_1^2+4d_1-1+\sqrt{4d_1^4+8d_1^3-16d_1^2+4d_1+1}}{12(d_1-1)}.$$

Notice that $\lim_{d_1\to 1}\tilde{d}_2 = \infty$ and $\lim_{d_1\to\infty}\tilde{d}_2 = \frac{1}{2}$. Roughly speaking, for $d_1 \approx 1$ or $d_2 \approx 0.5$ estimation in Theorem 5.26 is better that estimation in Theorem 5.25, otherwise the opposite conclusion holds. For example, for $d_1 = 2$ and $d_2 > \frac{-1+\sqrt{73}}{12} \approx 0.62867$ or for $d_2 = 1$ and $d_1 > \frac{-5+\sqrt{105}}{4} \approx 1.31174$ estimation in Theorem 5.25 is better than estimation in Theorem 5.26.

We continue with cases for $p = 1$, and then for $p \in (0,1)$, in which we use method from Theorem 5.25.

Proposition 5.13 *Let $\alpha > \beta_i \geq 0$ for $i = 1,2$ and $D^{\alpha}_{a+}f \in L_\infty[a,b]$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.21 holds for $\{\alpha,\beta_1\}$ and $\{\alpha,\beta_2\}$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x |D^{\beta_1}_{a+}f(t)|\,|D^{\beta_2}_{a+}f(t)|\,dt \leq \frac{(x-a)^{2\alpha-\beta_1-\beta_2+1}}{(2\alpha-\beta_1-\beta_2+1)\prod_{i=1}^{2}\Gamma(\alpha-\beta_i+1)}\left\|D^{\alpha}_{a+}f\right\|_\infty^2. \tag{5.55}$$

Proof. Using Theorem 2.13 and the triangle inequality, for $i = 1,2$ we have

$$\begin{aligned}
|D^{\beta_i}_{a+}f(t)| &\leq \frac{1}{\Gamma(\alpha-\beta_i)}\int_a^t (t-\tau)^{\alpha-\beta_i-1}\left|D^{\alpha}_{a+}f(\tau)\right|d\tau \\
&\leq \frac{1}{\Gamma(\alpha-\beta_i)}\left(\int_a^t (t-\tau)^{\alpha-\beta_i-1}\,d\tau\right)\|D^{\alpha}_{a+}f\|_\infty = \frac{(t-a)^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)}\|D^{\alpha}_{a+}f\|_\infty.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_a^x |D^{\beta_1}_{a+}f(t)|\,|D^{\beta_2}_{a+}f(t)|\,dt \\
&\leq \frac{1}{\prod_{i=1}^{2}\Gamma(\alpha-\beta_i+1)}\|D^{\alpha}_{a+}f\|_\infty^2\int_a^x (t-a)^{2\alpha-\beta_1-\beta_2}\,dt \\
&= \frac{1}{\prod_{i=1}^{2}\Gamma(\alpha-\beta_i+1)}\|D^{\alpha}_{a+}f\|_\infty^2\,\frac{(x-a)^{2\alpha-\beta_1-\beta_2+1}}{2\alpha-\beta_1-\beta_2+1}.
\end{aligned}$$

□

Theorem 5.27 *Let $p \in (0,1)$, $q < 0$ and $\frac{1}{p}+\frac{1}{q} = 1$. Let $\alpha > \beta_i \geq 0$ for $i = 1,2$. Let $D^{\alpha}_{a+}f \in L_q[a,b]$ be of fixed sign on $[a,b]$, with $1/D^{\alpha}_{a+}f \in L_q[a,b]$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.21 holds for $\{\alpha,\beta_1\}$ and $\{\alpha,\beta_2\}$. Then reverse inequality in (5.39) holds.*

Next inequalities include the right-sided Riemann-Liouville fractional derivatives.

Theorem 5.28 *Let $p,q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $\beta_i \geq 0$, $\alpha > \beta_i + \frac{1}{q}$ for $i=1,2$ and $D^{\alpha}_{b-}f \in L_q[a,b]$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.22 holds for $\{\alpha,\beta_1\}$ and $\{\alpha,\beta_2\}$. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b |D^{\beta_1}_{b-}f(t)|\,|D^{\beta_2}_{b-}f(t)|\,dt \leq K_3\,(b-x)^{2\alpha-\beta_1-\beta_2-1+\frac{2}{p}} \left(\int_x^b \left|D^{\alpha}_{b-}f(t)\right|^q dt\right)^{\frac{2}{q}}, \tag{5.56}$$

where K_3 is given by (5.40).

In the following theorem again we use Fink's method and obtain new estimation for the inequality (5.56). The same comparison of constants holds as with the left-sided Riemann-Liouville fractional derivatives, that is for Theorem 5.28 and Theorem 5.29 Remark 5.9 is valid also.

Theorem 5.29 *Let $p,q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $\beta_1 \geq 0$, $\alpha > \beta_2 \geq \beta_1 + 1$ and $D^{\alpha}_{b-}f \in L_q[a,b]$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.22 holds for $\{\alpha,\beta_1\}$ and $\{\alpha,\beta_2\}$. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b |D^{\beta_1}_{b-}f(t)|\,|D^{\beta_2}_{b-}f(t)|\,dt \leq F\,(b-x)^{2\alpha-\beta_1-\beta_2-1+\frac{2}{p}} \left(\int_x^b \left|D^{\alpha}_{b-}f(t)\right|^q dt\right)^{\frac{2}{q}}, \tag{5.57}$$

where F is given by (5.42).

Inequality (5.57) *is sharp for $\beta_2 = \beta_1 + 1$, where equality is attained for a function f such that $D^{\alpha}_{b-}f(t) = (x-t)^{\frac{p(\alpha-\beta_2)}{q}}$, $t \in [x,b]$.*

A technique from Theorem 5.28 is used in the following cases for $p=1$ and $p \in (0,1)$.

Proposition 5.14 *Let $\alpha > \beta_i \geq 0$ for $i=1,2$ and $D^{\alpha}_{b-}f \in L_\infty[a,b]$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.22 holds for $\{\alpha,\beta_1\}$ and $\{\alpha,\beta_2\}$. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b |D^{\beta_1}_{b-}f(t)|\,|D^{\beta_2}_{b-}f(t)|\,dt \leq \frac{(b-x)^{2\alpha-\beta_1-\beta_2+1}}{(2\alpha-\beta_1-\beta_2+1)\prod_{i=1}^{2}\Gamma(\alpha-\beta_i+1)} \left\|D^{\alpha}_{b-}f\right\|^2_\infty . \tag{5.58}$$

Theorem 5.30 *Let $p \in (0,1)$, $q < 0$ and $\frac{1}{p}+\frac{1}{q}=1$. Let $\alpha > \beta_i \geq 0$ for $i=1,2$. Let $D^{\alpha}_{b-}f \in L_q[a,b]$ be of fixed sign on $[a,b]$, with $1/D^{\alpha}_{b-}f \in L_q[a,b]$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.22 holds for $\{\alpha,\beta_1\}$ and $\{\alpha,\beta_2\}$. Then reverse inequality in* (5.56) *holds.*

Next two corollaries represent generalization of the classical Opial's inequality. In the first one, its inequality is a consequence of inequalities (5.39) and (5.56), and in the second corollary a consequence of inequalities (5.41) and (5.57).

Corollary 5.4 *Suppose that assumptions of Theorem 5.25 and Theorem 5.28 hold and let* $1 < q \leq 2$. *Then*

$$\int_a^{\frac{a+b}{2}} |D_{a+}^{\beta_1} f(t)||D_{a+}^{\beta_2} f(t)|\,dt + \int_{\frac{a+b}{2}}^b |D_{b-}^{\beta_1} f(t)||D_{b-}^{\beta_2} f(t)|\,dt$$
$$\leq K_3 \left(\frac{b-a}{2}\right)^{2\alpha-\beta_1-\beta_2-1+\frac{2}{p}} \left(\int_a^{\frac{a+b}{2}} \left|D_{a+}^{\alpha} f(t)\right|^q dt + \int_{\frac{a+b}{2}}^b \left|D_{b-}^{\alpha} f(t)\right|^q dt\right)^{\frac{2}{q}}. \tag{5.59}$$

Remark 5.10 Let $\alpha = 1$, $\beta_1 = \beta_2 = 0$ and $q = 2$. Then $K_3 = \frac{1}{2}$ and inequality (5.59) implies Wirtinger-type inequality (1.19) on interval $[a,b]$

$$\int_a^b [f(x)]^2\,dx \leq \frac{(b-a)^2}{8} \int_a^b \left[f'(x)\right]^2 dx. \tag{5.60}$$

Boundary conditions $D_{a+}^{\alpha-1} f(a) = f(a) = 0$ and $D_{b-}^{\alpha-1} f(b) = f(b) = 0$ follow from conditions of Theorem 2.13 and Theorem 2.14. Notice that the best possible estimation of the Wirtinger inequality $\left(\frac{b-a}{\pi}\right)^2$ is not obtained here.

Corollary 5.5 *Suppose that assumptions of Theorem 5.26 and Theorem 5.29 hold and let* $1 < q \leq 2$. *Then*

$$\int_a^{\frac{a+b}{2}} |D_{a+}^{\beta_1} f(t)||D_{a+}^{\beta_2} f(t)|\,dt + \int_{\frac{a+b}{2}}^b |D_{b-}^{\beta_1} f(t)||D_{b-}^{\beta_2} f(t)|\,dt$$
$$\leq F \left(\frac{b-a}{2}\right)^{2\alpha-\beta_1-\beta_2-1+\frac{2}{p}} \left(\int_a^{\frac{a+b}{2}} \left|D_{a+}^{\alpha} f(t)\right|^q dt + \int_{\frac{a+b}{2}}^b \left|D_{b-}^{\alpha} f(t)\right|^q dt\right)^{\frac{2}{q}}. \tag{5.61}$$

Inequality (5.61) *is sharp for* $\beta_2 = \beta_1 + 1$ *and* $q = 2$.

Wirtinger-type inequality does not follow from Corollary 5.5 due to conditions on α, β_1 and β_2.

THE CAPUTO FRACTIONAL DERIVATIVES

Inequalities in Theorem 5.32, Theorem 5.35 and Corollary 5.6 are given by Andrić-Pečarić-Perić in [26].

Theorem 5.31 *Let* $p,q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *For* $i = 1,2$ *let* $\beta_i \geq 0$, $\alpha > \beta_i + \frac{1}{q}$ *and* m_i, n *given by* (2.70). *Let* $m = \min\{m_1, m_2\}$. *Let* $f \in AC^n[a,b]$ *be such that* $f^{(i)}(a) = 0$ *for* $i = m, \dots, n-1$ *and let* ${}^C D_{a+}^{\alpha} f \in L_q[a,b]$. *Then for a.e.* $x \in [a,b]$ *holds*

$$\int_a^x |{}^C D_{a+}^{\beta_1} f(t)||{}^C D_{a+}^{\beta_2} f(t)|\,dt \leq K_3\,(x-a)^{2\alpha-\beta_1-\beta_2-1+\frac{2}{p}} \left(\int_a^x \left|{}^C D_{a+}^{\alpha} f(t)\right|^q dt\right)^{\frac{2}{q}}, \tag{5.62}$$

where K_3 *is given by* (5.40).

Next theorem gives a second estimation of the inequality (5.62).

Theorem 5.32 *Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $\alpha>\beta_2\geq\beta_1+1$, $\beta_1\geq 0$ and n, m_1 given by* (2.70). *Let $f\in AC^n[a,b]$ be such that $f^{(i)}(a)=0$ for $i=m_1,\ldots,n-1$ and let ${}^CD^{\alpha}_{a+}f\in L_q[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$\int_a^x |{}^CD^{\beta_1}_{a+}f(t)|\,|{}^CD^{\beta_2}_{a+}f(t)|\,dt\leq F\,(x-a)^{2\alpha-\beta_1-\beta_2-1+\frac{2}{p}}\left(\int_a^x |{}^CD^{\alpha}_{a+}f(t)|^q\,dt\right)^{\frac{2}{q}}, \quad (5.63)$$

where F is given by (5.42).

Inequality (5.63) *is sharp for $\beta_2=\beta_1+1$, where equality is attained for a function f such that ${}^CD^{\alpha}_{a+}f(t)=(x-t)^{\frac{p(\alpha-\beta_2)}{q}}$, $t\in[a,x]$.*

Remark 5.11 Let $\alpha=n\in\mathbb{N}$, $\beta_1=i\in\mathbb{N}$, $\beta_2=j\in\mathbb{N}$, $i<j\leq n-1$ and $x=b$. Then inequality (5.63) implies Fink's inequality (1.23), that is (5.52). Also, $F=C(n,i,j,q)$ from (1.24). By Notation 1.6 and conditions from Theorem 2.16, we have boundary conditions for Fink's inequality: $f^{(k)}(a)=0$ for $k=[\beta_1],\ldots,[\alpha]-1$, that is $f^{(k)}(a)=0$ for $k=i,\ldots,n-1$.

Next cases are for $p=1$ and $p\in(0,1)$, obtained by technique from Theorem 5.31.

Proposition 5.15 *For $i=1,2$ let $\alpha>\beta_i\geq 0$ and m_i, n given by* (2.70). *Let $m=\min\{m_1,m_2\}$. Let $f\in AC^n[a,b]$ be such that $f^{(i)}(a)=0$ for $i=m,\ldots,n-1$ and let ${}^CD^{\alpha}_{a+}f\in L_\infty[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$\int_a^x |{}^CD^{\beta_1}_{a+}f(t)|\,|{}^CD^{\beta_2}_{a+}f(t)|\,dt\leq\frac{(x-a)^{2\alpha-\beta_1-\beta_2+1}}{(2\alpha-\beta_1-\beta_2+1)\prod\limits_{i=1}^{2}\Gamma(\alpha-\beta_i+1)}\left\|{}^CD^{\alpha}_{a+}f\right\|_\infty^2. \quad (5.64)$$

Theorem 5.33 *Let $p\in(0,1)$, $q<0$ and $\frac{1}{p}+\frac{1}{q}=1$. For $i=1,2$ let $\alpha>\beta_i\geq 0$ and m_i, n given by* (2.70). *Let $m=\min\{m_1,m_2\}$. Let $f\in AC^n[a,b]$ be such that $f^{(i)}(a)=0$ for $i=m,\ldots,n-1$ and let ${}^CD^{\alpha}_{a+}f\in L_q[a,b]$ be of fixed sign on $[a,b]$, with $1/{}^CD^{\alpha}_{a+}f\in L_q[a,b]$. Then reverse inequality in* (5.62) *holds.*

We proceed with inequalities for the right-sided Caputo fractional derivatives.

Theorem 5.34 *Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. For $i=1,2$ let $\beta_i\geq 0$, $\alpha>\beta_i+\frac{1}{q}$ and m_i, n given by* (2.70). *Let $m=\min\{m_1,m_2\}$. Let $f\in AC^n[a,b]$ be such that $f^{(i)}(b)=0$ for $i=m,\ldots,n-1$ and let ${}^CD^{\alpha}_{b-}f\in L_q[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$\int_x^b |{}^CD^{\beta_1}_{b-}f(t)|\,|{}^CD^{\beta_2}_{b-}f(t)|\,dt\leq K_3\,(b-x)^{2\alpha-\beta_1-\beta_2-1+\frac{2}{p}}\left(\int_x^b |{}^CD^{\alpha}_{b-}f(t)|^q\,dt\right)^{\frac{2}{q}}, \quad (5.65)$$

where K_3 is given by (5.40).

Theorem 5.35 *Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $\alpha>\beta_2\geq\beta_1+1$, $\beta_1\geq 0$ and n, m_1 given by* (2.70). *Let $f\in AC^n[a,b]$ be such that $f^{(i)}(b)=0$ for $i=m_1,\dots,n-1$ and let ${}^CD^{\alpha}_{b-}f\in L_q[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$\int_x^b |{}^CD^{\beta_1}_{b-}f(t)||{}^CD^{\beta_2}_{b-}f(t)|\,dt \leq F\,(b-x)^{2\alpha-\beta_1-\beta_2-1+\frac{2}{p}}\left(\int_x^b \left|{}^CD^{\alpha}_{b-}f(t)\right|^q dt\right)^{\frac{2}{q}}, \quad (5.66)$$

where F is given by (5.42).

Inequality (5.66) *is sharp for $\beta_2=\beta_1+1$, where equality is attained for a function f such that ${}^CD^{\alpha}_{b-}f(t)=(x-t)^{\frac{p(\alpha-\beta_2)}{q}}$, $t\in[x,b]$.*

Next we have cases for $p=1$ and $p\in(0,1)$, obtained with method from Theorem 5.34.

Proposition 5.16 *For $i=1,2$ let $\alpha>\beta_i\geq 0$ and m_i, n given by* (2.70). *Let $m=\min\{m_1,m_2\}$. Let $f\in AC^n[a,b]$ be such that $f^{(i)}(b)=0$ for $i=m,\dots,n-1$ and let ${}^CD^{\alpha}_{b-}f\in L_\infty[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$\int_x^b |{}^CD^{\beta_1}_{b-}f(t)||{}^CD^{\beta_2}_{b-}f(t)|\,dt \leq \frac{(b-x)^{2\alpha-\beta_1-\beta_2+1}}{(2\alpha-\beta_1-\beta_2+1)\prod_{i=1}^{2}\Gamma(\alpha-\beta_i+1)}\left\|{}^CD^{\alpha}_{b-}f\right\|_\infty^2. \quad (5.67)$$

Theorem 5.36 *Let $p\in(0,1)$, $q<0$ and $\frac{1}{p}+\frac{1}{q}=1$. For $i=1,2$ let $\alpha>\beta_i\geq 0$, and m_i, n given by* (2.70). *Let $m=\min\{m_1,m_2\}$. Let $f\in AC^n[a,b]$ be such that $f^{(i)}(b)=0$ for $i=m,\dots,n-1$ and let ${}^CD^{\alpha}_{b-}f\in L_q[a,b]$ be of fixed sign on $[a,b]$, with $1/{}^CD^{\alpha}_{b-}f\in L_q[a,b]$. Then reverse inequality in* (5.65) *holds.*

A comparison of estimations in Remark 5.9 is also valid for Theorems 5.31, 5.32 and Theorems 5.34, 5.35.

Corollary 5.6 *Suppose that assumptions of Theorem 5.31 and Theorem 5.34 hold and let $1<q\leq 2$. Then*

$$\int_a^{\frac{a+b}{2}} |{}^CD^{\beta_1}_{a+}f(t)||{}^CD^{\beta_2}_{a+}f(t)|\,dt+\int_{\frac{a+b}{2}}^b |{}^CD^{\beta_1}_{b-}f(t)||{}^CD^{\beta_2}_{b-}f(t)|\,dt$$
$$\leq K_3\left(\frac{b-a}{2}\right)^{2\alpha-\beta_1-\beta_2-1+\frac{2}{p}}\left(\int_a^{\frac{a+b}{2}}\left|{}^CD^{\alpha}_{a+}f(t)\right|^q dt+\int_{\frac{a+b}{2}}^b\left|{}^CD^{\alpha}_{b-}f(t)\right|^q dt\right)^{\frac{2}{q}}. \quad (5.68)$$

Remark 5.12 Let $\alpha=1$, $\beta_1=\beta_2=0$ and $q=2$. Then $K_3=\frac{1}{2}$ and inequality (5.68) implies Wirtinger-type inequality (1.19), that is (5.60). Boundary conditions $f(a)=f(b)=0$ follow from conditions of Theorem 2.16 and Theorem 2.17 since $n=1$ and $m=0$.

Corollary 5.7 *Suppose that assumptions of Theorem 5.32 and Theorem 5.35 hold and let $1 < q \leq 2$. Then*

$$\int_a^{\frac{a+b}{2}} |^C D_{a+}^{\beta_1} f(t)||^C D_{a+}^{\beta_2} f(t)|\,dt + \int_{\frac{a+b}{2}}^b |^C D_{b-}^{\beta_1} f(t)||^C D_{b-}^{\beta_2} f(t)|\,dt$$
$$\leq F\left(\frac{b-a}{2}\right)^{2\alpha-\beta_1-\beta_2-1+\frac{2}{p}} \left(\int_a^{\frac{a+b}{2}} \left|^C D_{a+}^{\alpha} f(t)\right|^q dt + \int_{\frac{a+b}{2}}^b \left|^C D_{b-}^{\alpha} f(t)\right|^q dt\right)^{\frac{2}{q}}. \tag{5.69}$$

Inequality (5.69) *is sharp for $\beta_2 = \beta_1 + 1$ and $q = 2$.*

THE CANAVATI FRACTIONAL DERIVATIVES

Theorem 5.37 *Let $p,q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $\beta_i \geq 0$, $\alpha > \beta_i + \frac{1}{q}$, $m_i = [\beta_i] + 1$ for $i = 1,2$ and $n = [\alpha] + 1$, $m = \min\{m_1, m_2\}$. Let $f \in C_{a+}^{\alpha}[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m-1, \dots, n-2$ and let ${}^{C_1}D_{a+}^{\alpha} f \in L_q[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x |^{C_1} D_{a+}^{\beta_1} f(t)||^{C_1} D_{a+}^{\beta_2} f(t)|\,dt \leq K_3\,(x-a)^{2\alpha-\beta_1-\beta_2-1+\frac{2}{p}} \left(\int_a^x \left|^{C_1} D_{a+}^{\alpha} f(t)\right|^q dt\right)^{\frac{2}{q}}, \tag{5.70}$$

where K_3 is given by (5.40).

Theorem 5.38 *Let $p,q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $\alpha > \beta_2 \geq \beta_1 + 1$, $\beta_1 \geq 0$ and $n = [\alpha] + 1$, $m_1 = [\beta_1] + 1$. Let $f \in C_{a+}^{\alpha}[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m_1 - 1, \dots, n-2$ and let ${}^{C_1}D_{a+}^{\alpha} f \in L_q[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x |^{C_1} D_{a+}^{\beta_1} f(t)||^{C_1} D_{a+}^{\beta_2} f(t)|\,dt \leq F\,(x-a)^{2\alpha-\beta_1-\beta_2-1+\frac{2}{p}} \left(\int_a^x \left|^{C_1} D_{a+}^{\alpha} f(t)\right|^q dt\right)^{\frac{2}{q}}, \tag{5.71}$$

where F is given by (5.42).

Inequality (5.71) *is sharp for $\beta_2 = \beta_1 + 1$, where equality is attained for a function f such that ${}^{C_1}D_{a+}^{\alpha} f(t) = (x-t)^{\frac{p(\alpha-\beta_2)}{q}}$, $t \in [a,x]$.*

Remark 5.13 Let $\alpha = n \in \mathbb{N}$, $\beta_1 = i \in \mathbb{N}$, $\beta_2 = j \in \mathbb{N}$, $i < j \leq n-1$ and $x = b$. Then inequality (5.71) implies Fink's inequality (1.23), that is (5.52). Also, $F = C(n,i,j,q)$ from relation (1.24). By Notation 1.6 and conditions in Theorem 2.19, we have boundary conditions for Fink's inequality: $f^{(k)}(a) = 0$ for $k = [\beta_1] + 1 - 1, \dots, [\alpha] + 1 - 2$, that is $f^{(k)}(a) = 0$ for $k = i, \dots, n-1$.

A technique from Theorem 5.37 is used for cases when $p = 1$ and $p \in (0,1)$.

Proposition 5.17 *Let $\alpha > \beta_i \geq 0$, $m_i = [\beta_i] + 1$ for $i = 1,2$ and $n = [\alpha] + 1$, $m = \min\{m_1, m_2\}$. Let $f \in C^{\alpha}_{a+}[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m-1, \dots, n-2$ and let ${}^{C_1}D^{\alpha}_{a+} f \in L_\infty[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x |{}^{C_1}D^{\beta_1}_{a+} f(t)| \, |{}^{C_1}D^{\beta_2}_{a+} f(t)| \, dt \leq \frac{(x-a)^{2\alpha-\beta_1-\beta_2+1}}{(2\alpha-\beta_1-\beta_2+1) \prod_{i=1}^{2} \Gamma(\alpha-\beta_i+1)} \left\| {}^{C_1}D^{\alpha}_{a+} f \right\|_\infty^2 . \tag{5.72}$$

Theorem 5.39 *Let $p \in (0,1)$, $q < 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $\alpha > \beta_i \geq 0$, $m_i = [\beta_i] + 1$ for $i = 1,2$ and $n = [\alpha] + 1$, $m = \min\{m_1, m_2\}$. Let $f \in C^{\alpha}_{a+}[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m-1, \dots, n-2$ and let ${}^{C_1}D^{\alpha}_{a+} f \in L_q[a,b]$ be of fixed sign on $[a,b]$, with $1/{}^{C_1}D^{\alpha}_{a+} f \in L_q[a,b]$. Then reverse inequality in (5.70) holds.*

We continue with inequalities for the right-sided Canavati fractional derivatives.

Theorem 5.40 *Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $\beta_i \geq 0$, $\alpha > \beta_i + \frac{1}{q}$, $m_i = [\beta_i] + 1$ for $i = 1,2$ and $n = [\alpha] + 1$, $m = \min\{m_1, m_2\}$. Let $f \in C^{\alpha}_{b-}[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m-1, \dots, n-2$ and let ${}^{C_1}D^{\alpha}_{b-} f \in L_q[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b |{}^{C_1}D^{\beta_1}_{b-} f(t)| \, |{}^{C_1}D^{\beta_2}_{b-} f(t)| \, dt \leq K_3 \, (b-x)^{2\alpha-\beta_1-\beta_2-1+\frac{2}{p}} \left(\int_x^b \left| {}^{C_1}D^{\alpha}_{b-} f(t) \right|^q dt \right)^{\frac{2}{q}} , \tag{5.73}$$

where K_3 is given by (5.40).

Theorem 5.41 *Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $\alpha > \beta_2 \geq \beta_1 + 1$, $\beta_1 \geq 0$ and $n = [\alpha] + 1$, $m_1 = [\beta_1] + 1$. Let $f \in C^{\alpha}_{b-}[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m_1 - 1, \dots, n-2$ and let ${}^{C_1}D^{\alpha}_{b-} f \in L_q[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b |{}^{C_1}D^{\beta_1}_{b-} f(t)| \, |{}^{C_1}D^{\beta_2}_{b-} f(t)| \, dt \leq F \, (b-x)^{2\alpha-\beta_1-\beta_2-1+\frac{2}{p}} \left(\int_x^b \left| {}^{C_1}D^{\alpha}_{b-} f(t) \right|^q dt \right)^{\frac{2}{q}} , \tag{5.74}$$

where F is given by (5.42).

Inequality (5.74) *is sharp for $\beta_2 = \beta_1 + 1$, where equality is attained for a function f such that ${}^{C_1}D^{\alpha}_{b-} f(t) = (x-t)^{\frac{p(\alpha-\beta_2)}{q}}$, $t \in [x,b]$.*

For cases $p = 1$ and $p \in (0,1)$ we use method from Theorem 5.40.

Proposition 5.18 *Let $\alpha > \beta_i \geq 0$, $m_i = [\beta_i] + 1$ for $i = 1,2$ and $n = [\alpha] + 1$, $m = \min\{m_1, m_2\}$. Let $f \in C^{\alpha}_{b-}[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m-1, \dots, n-2$ and let ${}^{C_1}D^{\alpha}_{b-} f \in L_\infty[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b |{}^{C_1}D^{\beta_1}_{b-} f(t)| \, |{}^{C_1}D^{\beta_2}_{b-} f(t)| \, dt \leq \frac{(b-x)^{2\alpha-\beta_1-\beta_2+1}}{(2\alpha-\beta_1-\beta_2+1) \prod_{i=1}^{2} \Gamma(\alpha-\beta_i+1)} \left\| {}^{C_1}D^{\alpha}_{b-} f \right\|_\infty^2 . \tag{5.75}$$

Theorem 5.42 *Let $p \in (0,1)$, $q < 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $\alpha > \beta_i \geq 0$, $m_i = [\beta_i] + 1$ for $i = 1,2$ and $n = [\alpha] + 1$, $m = \min\{m_1, m_2\}$. Let $f \in C^{\alpha}_{b-}[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m-1, \ldots, n-2$ and let ${}^{C_1}D^{\alpha}_{b-}f \in L_q[a,b]$ be of fixed sign on $[a,b]$, with $1/{}^{C_1}D^{\alpha}_{b-}f \in L_q[a,b]$. Then reverse inequality in (5.73) holds.*

A comparison of estimations in Remark 5.9 is also valid for Theorems 5.37, 5.38 and Theorems 5.40, 5.41.

Corollary 5.8 *Suppose that assumptions of Theorem 5.37 and Theorem 5.40 hold and let $1 < q \leq 2$. Then*

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}} |{}^{C_1}D^{\beta_1}_{a+}f(t)|\,|{}^{C_1}D^{\beta_2}_{a+}f(t)|\,dt + \int_{\frac{a+b}{2}}^{b} |{}^{C_1}D^{\beta_1}_{b-}f(t)|\,|{}^{C_1}D^{\beta_2}_{b-}f(t)|\,dt \\
&\leq K_3 \left(\frac{b-a}{2}\right)^{2\alpha-\beta_1-\beta_2-1+\frac{2}{p}} \left(\int_a^{\frac{a+b}{2}} \left|{}^{C_1}D^{\alpha}_{a+}f(t)\right|^q dt + \int_{\frac{a+b}{2}}^{b} \left|{}^{C_1}D^{\alpha}_{b-}f(t)\right|^q dt\right)^{\frac{2}{q}}.
\end{aligned} \tag{5.76}$$

Remark 5.14 Let $\alpha = 1$, $\beta_1 = \beta_2 = 0$ and $q = 2$. Then $K_3 = \frac{1}{2}$ and inequality (5.76) implies Wirtinger-type inequality (1.19), that is (5.60). Boundary conditions $f(a) = f(b) = 0$ follow from conditions of Theorem 2.19 and Theorem 2.20 since $n = 2$ and $m = 1$.

Corollary 5.9 *Suppose that assumptions of Theorem 5.38 and Theorem 5.41 hold and let $1 < q \leq 2$. Then*

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}} |{}^{C_1}D^{\beta_1}_{a+}f(t)|\,|{}^{C_1}D^{\beta_2}_{a+}f(t)|\,dt + \int_{\frac{a+b}{2}}^{b} |{}^{C_1}D^{\beta_1}_{b-}f(t)|\,|{}^{C_1}D^{\beta_2}_{b-}f(t)|\,dt \\
&\leq F \left(\frac{b-a}{2}\right)^{2\alpha-\beta_1-\beta_2-1+\frac{2}{p}} \left(\int_a^{\frac{a+b}{2}} \left|{}^{C_1}D^{\alpha}_{a+}f(t)\right|^q dt + \int_{\frac{a+b}{2}}^{b} \left|{}^{C_1}D^{\alpha}_{b-}f(t)\right|^q dt\right)^{\frac{2}{q}}.
\end{aligned} \tag{5.77}$$

Inequality (5.77) is sharp for $\beta_2 = \beta_1 + 1$ and $q = 2$.

Next we observe a weighted version of the previous inequality for $q = 2$, with a form

$$\int_a^b w(t)\,|\mathbf{D}^{\beta_1}f(t)|\,|\mathbf{D}^{\beta_2}f(t)|\,dt \leq K \left(\int_a^b [w(t)]^2\,dt\right)^{\frac{1}{2}} \int_a^b |\mathbf{D}^{\alpha}f(t)|^2\,dt,$$

where $w \in C[a,b]$ is a nonnegative weight function, $\alpha > \beta_i \geq 0$ and $K > 0$ is a constant.

This inequality is given in [12] for the left-sided Canavati fractional derivatives, with estimation K_4 (here Theorem 5.55, given under new conditions). In [23] Andrić-Pečarić-Perić give two more estimations of this inequality obtained with several applications of Hölder's inequality on different factors with different indices (here Theorem 5.56 and Theorem 5.57) involving the Canavati fractional derivatives. We give analogous results for the inequalities involving the Riemann-Liouville and the Caputo fractional derivatives.

THE RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVES

Theorem 5.43 *Let $\beta_i \geq 0$, $\alpha > \beta_i + \frac{5}{6}$ for $i = 1,2$ and $D^{\alpha}_{a+}f \in L_2[a,b]$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.21 holds for $\{\alpha,\beta_1\}$ and $\{\alpha,\beta_2\}$. Let w be continuous nonnegative weight function on $[a,x]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x w(t)\,|D^{\beta_1}_{a+}f(t)|\,|D^{\beta_2}_{a+}f(t)|\,dt \leq K_4\,\Omega(x)\,(x-a)^{2\alpha-\beta_1-\beta_2-\frac{1}{2}} \int_a^x \left|D^{\alpha}_{a+}f(t)\right|^2 dt\,, \quad (5.78)$$

where

$$\Omega(x) = \left(\int_a^x [w(t)]^2\,dt\right)^{\frac{1}{2}}, \quad (5.79)$$

$$K_4 = \left[\left(4\alpha - 2\beta_1 - 2\beta_2 - \frac{7}{3}\right)^{\frac{1}{2}} \prod_{i=1}^{2} \Gamma(\alpha-\beta_i)\,(6\alpha-6\beta_i-5)^{\frac{1}{6}}\right]^{-1}. \quad (5.80)$$

Proof. Set $\delta_i = \alpha - \beta_i - 1$, $i = 1,2$. Using Theorem 2.13, the triangle inequality and Hölder's inequality, for $t \in [a,x]$ we have

$$\begin{aligned}
&\int_a^x w(t)\,|D^{\beta_1}_{a+}f(t)|\,|D^{\beta_2}_{a+}f(t)|\,dt \\
&\leq \left(\int_a^x [w(t)]^2\,dt\right)^{\frac{1}{2}} \left(\int_a^x \prod_{i=1}^{2} |D^{\beta_i}_{a+}f(t)|^2\,dt\right)^{\frac{1}{2}} \\
&\leq \Omega(x)\,\frac{1}{\prod_{i=1}^{2}\Gamma(\delta_i+1)} \left\{\int_a^x \prod_{i=1}^{2}\left(\int_a^t (t-\tau)^{\delta_i}|D^{\alpha}_{a+}f(\tau)|\,d\tau\right)^2 dt\right\}^{\frac{1}{2}}.
\end{aligned}$$

Again we use Hölder's inequality for $\{p=3, q=\frac{3}{2}\}$

$$\int_a^t (t-\tau)^{\delta_i}|D^{\alpha}_{a+}f(\tau)|\,d\tau \leq \left(\int_a^t d\tau\right)^{\frac{1}{3}} \left(\int_a^t (t-\tau)^{\frac{3}{2}\delta_i}|D^{\alpha}_{a+}f(\tau)|^{\frac{3}{2}}\,d\tau\right)^{\frac{2}{3}},$$

and $\{p=4, q=\frac{4}{3}\}$

$$\int_a^t (t-\tau)^{\frac{3}{2}\delta_i}|D^{\alpha}_{a+}f(\tau)|^{\frac{3}{2}}\,d\tau \leq \left(\int_a^t (t-\tau)^{6\delta_i}\,d\tau\right)^{\frac{1}{4}} \left(\int_a^t |D^{\alpha}_{a+}f(\tau)|^2\,d\tau\right)^{\frac{3}{4}}.$$

Hence,

$$\begin{aligned}
&\int_a^x w(t)\,|D^{\beta_1}_{a+}f(t)|\,|D^{\beta_2}_{a+}f(t)|\,dt \\
&\leq \frac{\Omega(x)}{\prod_{i=1}^{2}\Gamma(\delta_i+1)} \left\{\int_a^x (t-a)^{\frac{4}{3}} \prod_{i=1}^{2}\left(\int_a^t (t-\tau)^{\frac{3}{2}\delta_i}|D^{\alpha}_{a+}f(\tau)|^{\frac{3}{2}}\,d\tau\right)^{\frac{4}{3}} dt\right\}^{\frac{1}{2}}
\end{aligned}$$

$$\le \frac{\Omega(x)}{\prod_{i=1}^{2}\Gamma(\delta_i+1)}\left\{\int_a^x (x-a)^{\frac{4}{3}}\left(\int_a^t |D_{a+}^{\alpha}f(\tau)|^2\,d\tau\right)^2 \prod_{i=1}^{2}\left(\frac{(t-a)^{6\delta_i+1}}{6\delta_i+1}\right)^{\frac{1}{3}} dt\right\}^{\frac{1}{2}}$$

$$= \frac{\Omega(x)\,(x-a)^{\frac{2}{3}}}{\prod_{i=1}^{2}\Gamma(\delta_i+1)\,(6\delta_i+1)^{\frac{1}{6}}}\left\{\int_a^x\left(\int_a^t |D_{a+}^{\alpha}f(\tau)|^2\,d\tau\right)^2 (t-a)^{2\delta_1+2\delta_2+\frac{2}{3}}\,dt\right\}^{\frac{1}{2}}$$

$$\le \frac{\Omega(x)\,(x-a)^{\frac{2}{3}}}{\prod_{i=1}^{2}\Gamma(\delta_i+1)\,(6\delta_i+1)^{\frac{1}{6}}}\int_a^x |D_{a+}^{\alpha}f(\tau)|^2\,d\tau\left(\int_a^x (t-a)^{2\delta_1+2\delta_2+\frac{2}{3}}\,dt\right)^{\frac{1}{2}}$$

$$= \frac{\Omega(x)\,(x-a)^{\frac{2}{3}}}{\prod_{i=1}^{2}\Gamma(\delta_i+1)\,(6\delta_i+1)^{\frac{1}{6}}}\int_a^x |D_{a+}^{\alpha}f(\tau)|^2\,d\tau\,\frac{(x-a)^{\delta_1+\delta_2+\frac{5}{6}}}{(2\delta_1+2\delta_2+\frac{5}{3})^{\frac{1}{2}}}$$

$$= \frac{\Omega(x)}{\prod_{i=1}^{2}\Gamma(\alpha-\beta_i)\,(6\alpha-6\beta_i-5)^{\frac{1}{6}}}\int_a^x |D_{a+}^{\alpha}f(\tau)|^2\,d\tau\frac{(x-a)^{2\alpha-\beta_1-\beta_2-\frac{1}{2}}}{(4\alpha-2\beta_1-2\beta_1-\frac{7}{3})^{\frac{1}{2}}}.$$

□

Theorem 5.44 *Let $\beta_i \ge 0$, $\alpha > \beta_i + \frac{5}{6}$ for $i = 1,2$ and $D_{a+}^{\alpha}f \in L_2[a,b]$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.21 holds for $\{\alpha,\beta_1\}$ and $\{\alpha,\beta_2\}$. Let w be continuous nonnegative weight function on $[a,x]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x w(t)\,|D_{a+}^{\beta_1}f(t)|\,|D_{a+}^{\beta_2}f(t)|\,dt \le K_5\,\Omega(x)\,(x-a)^{2\alpha-\beta_1-\beta_2-\frac{1}{2}}\int_a^x \left|D_{a+}^{\alpha}f(t)\right|^2 dt\,, \quad (5.81)$$

where Ω is given by (5.79) *and*

$$K_5 = \left[(4\alpha-2\beta_1-2\beta_2-1)^{\frac{1}{2}}\prod_{i=1}^{2}\Gamma(\alpha-\beta_i)\,(6\alpha-6\beta_i-5)^{\frac{1}{6}}\right]^{-1}. \quad (5.82)$$

Proof. Set $\delta_i = \alpha-\beta_i-1$, $i = 1,2$. Using Theorem 2.13, the triangle inequality and Hölder's inequality, for $t \in [a,x]$ we have

$$\int_a^x w(t)\,|D_{a+}^{\beta_1}f(t)|\,|D_{a+}^{\beta_2}f(t)|\,dt$$

$$\le \left(\int_a^x [w(t)]^2\,dt\right)^{\frac{1}{2}}\left(\int_a^x \prod_{i=1}^{2}|D_{a+}^{\beta_i}f(t)|^2\,dt\right)^{\frac{1}{2}}$$

$$\le \Omega(x)\,\frac{1}{\prod_{i=1}^{2}\Gamma(\delta_i+1)}\left\{\int_a^x \prod_{i=1}^{2}\left(\int_a^t (t-\tau)^{\delta_i}|D_{a+}^{\alpha}f(\tau)|\,d\tau\right)^2 dt\right\}^{\frac{1}{2}}.$$

Again by Hölder's inequality for $\{p=3, q=\frac{3}{2}\}$ we have

$$\int_a^t (t-\tau)^{\delta_i} |D_{a+}^{\alpha} f(\tau)| \, d\tau \le \left(\int_a^t d\tau \right)^{\frac{1}{3}} \left(\int_a^t (t-\tau)^{\frac{3}{2}\delta_i} |D_{a+}^{\alpha} f(\tau)|^{\frac{3}{2}} \, d\tau \right)^{\frac{2}{3}},$$

and for $\{p=4, q=\frac{4}{3}\}$

$$\int_a^t (t-\tau)^{\frac{3}{2}\delta_i} |D_{a+}^{\alpha} f(\tau)|^{\frac{3}{2}} \, d\tau \le \left(\int_a^t (t-\tau)^{6\delta_i} \, d\tau \right)^{\frac{1}{4}} \left(\int_a^t |D_{a+}^{\alpha} f(\tau)|^2 \, d\tau \right)^{\frac{3}{4}}.$$

Hence,

$$\begin{aligned}
&\int_a^x w(t) \, |D_{a+}^{\beta_1} f(t)| \, |D_{a+}^{\beta_2} f(t)| \, dt \\
&\le \frac{\Omega(x)}{\prod_{i=1}^{2} \Gamma(\delta_i+1)} \left\{ \int_a^x (t-a)^{\frac{4}{3}} \prod_{i=1}^{2} \left(\int_a^t (t-\tau)^{\frac{3}{2}\delta_i} |D_{a+}^{\alpha} f(\tau)|^{\frac{3}{2}} \, d\tau \right)^{\frac{4}{3}} dt \right\}^{\frac{1}{2}} \\
&\le \frac{\Omega(x)}{\prod_{i=1}^{2} \Gamma(\delta_i+1)} \left\{ \int_a^x (t-a)^{\frac{4}{3}} \left(\int_a^t |D_{a+}^{\alpha} f(\tau)|^2 \, d\tau \right)^2 \prod_{i=1}^{2} \left(\frac{(t-a)^{6\delta_i+1}}{6\delta_i+1} \right)^{\frac{1}{3}} dt \right\}^{\frac{1}{2}} \\
&\le \frac{\Omega(x)}{\prod_{i=1}^{2} \Gamma(\delta_i+1) \, (6\delta_i+1)^{\frac{1}{6}}} \int_a^x |D_{a+}^{\alpha} f(\tau)|^2 \, d\tau \left(\int_a^x (t-a)^{2\delta_1+2\delta_2+2} \, dt \right)^{\frac{1}{2}} \\
&= \frac{\Omega(x)}{\prod_{i=1}^{2} \Gamma(\delta_i+1) \, (6\delta_i+1)^{\frac{1}{6}}} \int_a^x |D_{a+}^{\alpha} f(\tau)|^2 \, d\tau \, \frac{(x-a)^{\delta_1+\delta_2+\frac{3}{2}}}{(2\delta_1+2\delta_2+3)^{\frac{1}{2}}} \\
&= \frac{\Omega(x)}{\prod_{i=1}^{2} \Gamma(\alpha-\beta_i) \, (6\alpha-6\beta_i-5)^{\frac{1}{6}}} \int_a^x |D_{a+}^{\alpha} f(\tau)|^2 \, d\tau \frac{(x-a)^{2\alpha-\beta_1-\beta_2-\frac{1}{2}}}{(4\alpha-2\beta_1-2\beta_1-1)^{\frac{1}{2}}}.
\end{aligned}$$

□

Theorem 5.45 *Let $\beta_i \ge 0$, $\alpha > \beta_i + \frac{1}{2}$ for $i = 1,2$ and $D_{a+}^{\alpha} f \in L_2[a,b]$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.21 holds for $\{\alpha, \beta_1\}$ and $\{\alpha, \beta_2\}$. Let w be continuous nonnegative weight function on $[a,x]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x w(t) \, |D_{a+}^{\beta_1} f(t)| \, |D_{a+}^{\beta_2} f(t)| \, dt \le K_6 \, \Omega(x) \, (x-a)^{2\alpha-\beta_1-\beta_2-\frac{1}{2}} \int_a^x \left| D_{a+}^{\alpha} f(t) \right|^2 dt, \quad (5.83)$$

where Ω is given by (5.79) and

$$K_6 = \left[(4\alpha-2\beta_1-2\beta_2-1)^{\frac{1}{2}} \prod_{i=1}^{2} \Gamma(\alpha-\beta_i) \, (2\alpha-2\beta_i-1)^{\frac{1}{2}} \right]^{-1}. \quad (5.84)$$

Proof. Set $\delta_i = \alpha - \beta_i - 1$, $i = 1,2$. Using Theorem 2.13, the triangle inequality and Hölder's inequality, for $t \in [a,x]$ we have

$$\begin{aligned}
&\int_a^x w(t)\,|D_{a+}^{\beta_1} f(t)|\,|D_{a+}^{\beta_2} f(t)|\,dt \\
&\leq \int_a^x w(t) \prod_{i=1}^{2} \left(\frac{1}{\Gamma(\delta_i+1)} \int_a^t (t-\tau)^{\delta_i} \,|D_{a+}^{\alpha} f(\tau)|\,d\tau \right) dt \\
&\leq \frac{1}{\prod_{i=1}^{2} \Gamma(\delta_i+1)} \int_a^x w(t) \left(\int_a^t |D_{a+}^{\alpha} f(\tau)|^2\,d\tau \right) \prod_{i=1}^{2} \left(\int_a^t (t-\tau)^{2\delta_i}\,d\tau \right)^{\frac{1}{2}} dt \\
&\leq \frac{1}{\prod_{i=1}^{2} \Gamma(\delta_i+1)\,(2\delta_i+1)^{\frac{1}{2}}} \left(\int_a^x |D_{a+}^{\alpha} f(\tau)|^2\,d\tau \right) \int_a^x w(t)\,(t-a)^{\delta_1+\delta_2+1}\,dt \\
&\leq \frac{1}{\prod_{i=1}^{2} \Gamma(\delta_i+1)\,(2\delta_i+1)^{\frac{1}{2}}} \left(\int_a^x |D_{a+}^{\alpha} f(\tau)|^2\,d\tau \right) \\
&\qquad \cdot \left(\int_a^x [w(t)]^2\,dt \right)^{\frac{1}{2}} \left(\int_a^x (t-a)^{2\delta_1+2\delta_2+2}\,dt \right)^{\frac{1}{2}} \\
&= \frac{\Omega(x)}{\prod_{i=1}^{2} \Gamma(\delta_i+1)\,(2\delta_i+1)^{\frac{1}{2}}} \left(\int_a^x |D_{a+}^{\alpha} f(\tau)|^2\,d\tau \right) \frac{(x-a)^{\delta_1+\delta_2+\frac{3}{2}}}{(2\delta_1+2\delta_2+3)^{\frac{1}{2}}} \\
&= \frac{\Omega(x)}{\prod_{i=1}^{2} \Gamma(\alpha-\beta_i)\,(2\alpha-2\beta_i-1)^{\frac{1}{2}}} \left(\int_a^x |D_{a+}^{\alpha} f(\tau)|^2\,d\tau \right) \frac{(x-a)^{2\alpha-\beta_1-\beta_2-\frac{1}{2}}}{(4\alpha-2\beta_1-2\beta_2-1)^{\frac{1}{2}}}.
\end{aligned}$$

□

Remark 5.15 Comparing three constants K_i from Theorem 5.43, Theorem 5.44 and Theorem 5.45, we conclude

$$K_6 \leq K_5 < K_4. \tag{5.85}$$

The second inequality $K_5 < K_4$ is obvious, and inequality $K_6 \leq K_5$ is equivalent to

$$\frac{\sqrt[3]{6\alpha-6\beta_1-5}}{2\alpha-2\beta_1-1}\,\frac{\sqrt[3]{6\alpha-6\beta_2-5}}{2\alpha-2\beta_2-1} \leq 1, \tag{5.86}$$

for $\alpha-\beta_i > \frac{5}{6}$, $i=1,2$. Equality in (5.86) holds for $\alpha-\beta_i = 1$ when $K_6 = K_5 = 3^{-\frac{1}{2}}$. In all other cases, since $\frac{\sqrt[3]{3x-5}}{x-1} < 1$ for $x > \frac{5}{3}$ and $x \neq 2$, holds $K_6 < K_5$.

Following inequalities include the right-sided Riemann-Liouville fractional derivatives.

Theorem 5.46 *Let $\beta_i \geq 0$, $\alpha > \beta_i + \frac{5}{6}$ for $i = 1,2$ and $D^{\alpha}_{b-}f \in L_2[a,b]$. Suppose that one of conditions $(i) - (vii)$ in Corollary 2.22 holds for $\{\alpha,\beta_1\}$ and $\{\alpha,\beta_2\}$. Let w be continuous nonnegative weight function on $[x,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b w(t)\,|D^{\beta_1}_{b-}f(t)|\,|D^{\beta_2}_{b-}f(t)|\,dt \leq K_4\,\widetilde{\Omega}(x)\,(b-x)^{2\alpha-\beta_1-\beta_2-\frac{1}{2}} \int_x^b \left|D^{\alpha}_{b-}f(t)\right|^2 dt\,, \quad (5.87)$$

where

$$\widetilde{\Omega}(x) = \left(\int_x^b [w(t)]^2\,dt\right)^{\frac{1}{2}} \quad (5.88)$$

and K_4 is given by (5.80).

Theorem 5.47 *Let $\beta_i \geq 0$, $\alpha > \beta_i + \frac{5}{6}$ for $i = 1,2$ and $D^{\alpha}_{b-}f \in L_2[a,b]$. Suppose that one of conditions $(i) - (vii)$ in Corollary 2.22 holds for $\{\alpha,\beta_1\}$ and $\{\alpha,\beta_2\}$. Let w be continuous nonnegative weight function on $[x,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x w(t)\,|D^{\beta_1}_{b-}f(t)|\,|D^{\beta_2}_{b-}f(t)|\,dt \leq K_5\,\widetilde{\Omega}(x)\,(b-x)^{2\alpha-\beta_1-\beta_2-\frac{1}{2}} \int_a^x \left|D^{\alpha}_{b-}f(t)\right|^2 dt\,, \quad (5.89)$$

where $\widetilde{\Omega}$ and K_5 are given by (5.88) *and* (5.82), *respectively.*

Theorem 5.48 *Let $\beta_i \geq 0$, $\alpha > \beta_i + \frac{1}{2}$ for $i = 1,2$ and $D^{\alpha}_{b-}f \in L_2[a,b]$. Suppose that one of conditions $(i) - (vii)$ in Corollary 2.22 holds for $\{\alpha,\beta_1\}$ and $\{\alpha,\beta_2\}$. Let w be continuous nonnegative weight function on $[x,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x w(t)\,|D^{\beta_1}_{b-}f(t)|\,|D^{\beta_2}_{b-}f(t)|\,dt \leq K_6\,\widetilde{\Omega}(x)\,(x-a)^{2\alpha-\beta_1-\beta_2-\frac{1}{2}} \int_a^x \left|D^{\alpha}_{b-}f(t)\right|^2 dt\,, \quad (5.90)$$

where $\widetilde{\Omega}$ and K_6 are given by (5.88) *and* (5.84), *respectively.*

For Theorems 5.46, 5.47 and 5.48, a comparison of estimations in Remark 5.15 is also valid.

THE CAPUTO FRACTIONAL DERIVATIVES

Theorem 5.49 *Let $i = 1,2$, $\beta_i \geq 0$, $\alpha > \beta_i + \frac{5}{6}$ and m_i, n given by* (2.70). *Let $m = \min\{m_1,m_2\}$. Let $f \in AC^n[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m,\ldots,n-1$ and let ${}^CD^{\alpha}_{a+}f \in L_2[a,b]$. Let w be continuous nonnegative weight function on $[a,x]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x w(t)\,|{}^CD^{\beta_1}_{a+}f(t)|\,|{}^CD^{\beta_2}_{a+}f(t)|\,dt \leq K_4\,\Omega(x)\,(x-a)^{2\alpha-\beta_1-\beta_2-\frac{1}{2}} \int_a^x \left|{}^CD^{\alpha}_{a+}f(t)\right|^2 dt\,, \quad (5.91)$$

where Ω and K_4 are given by (5.79) *and* (5.80), *respectively.*

Theorem 5.50 *Let $i = 1,2$, $\beta_i \geq 0$, $\alpha > \beta_i + \frac{5}{6}$ and m_i, n given by* (2.70). *Let $m = \min\{m_1,m_2\}$. Let $f \in AC^n[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m,\ldots,n-1$ and let ${}^CD^{\alpha}_{a+}f \in$*

$L_2[a,b]$. *Let w be continuous nonnegative weight function on $[a,x]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x w(t)\,|^C D_{a+}^{\beta_1} f(t)|\,|^C D_{a+}^{\beta_2} f(t)|\,dt \le K_5\,\Omega(x)\,(x-a)^{2\alpha-\beta_1-\beta_2-\frac{1}{2}} \int_a^x \left|^C D_{a+}^{\alpha} f(t)\right|^2 dt\,, \tag{5.92}$$

where Ω and K_5 are given by (5.79) *and* (5.82)*, respectively.*

Theorem 5.51 *Let $i=1,2$, $\beta_i \ge 0$, $\alpha > \beta_i + \frac{1}{2}$ and m_i, n given by* (2.70)*. Let $m = \min\{m_1,m_2\}$. Let $f \in AC^n[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m,\dots,n-1$ and let ${}^C D_{a+}^{\alpha} f \in L_2[a,b]$. Let w be continuous nonnegative weight function on $[a,x]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x w(t)\,|^C D_{a+}^{\beta_1} f(t)|\,|^C D_{a+}^{\beta_2} f(t)|\,dt \le K_6\,\Omega(x)\,(x-a)^{2\alpha-\beta_1-\beta_2-\frac{1}{2}} \int_a^x \left|^C D_{a+}^{\alpha} f(t)\right|^2 dt\,, \tag{5.93}$$

where Ω and K_6 are given by (5.79) *and* (5.84)*, respectively.*

We continue with the inequalities for the right-sided Caputo fractional derivatives.

Theorem 5.52 *Let $i=1,2$, $\beta_i \ge 0$, $\alpha > \beta_i + \frac{5}{6}$ and m_i, n given by* (2.70)*. Let $m = \min\{m_1,m_2\}$. Let $f \in AC^n[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m,\dots,n-1$ and let ${}^C D_{b-}^{\alpha} f \in L_2[a,b]$. Let w be continuous nonnegative weight function on $[x,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b w(t)\,|^C D_{b-}^{\beta_1} f(t)|\,|^C D_{b-}^{\beta_2} f(t)|\,dt \le K_4\,\widetilde{\Omega}(x)\,(b-x)^{2\alpha-\beta_1-\beta_2-\frac{1}{2}} \int_x^b \left|^C D_{b-}^{\alpha} f(t)\right|^2 dt\,, \tag{5.94}$$

where $\widetilde{\Omega}$ and K_4 are given by (5.88) *and* (5.80)*, respectively.*

Theorem 5.53 *Let $i=1,2$, $\beta_i \ge 0$, $\alpha > \beta_i + \frac{5}{6}$ and m_i, n given by* (2.70)*. Let $m = \min\{m_1,m_2\}$. Let $f \in AC^n[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m,\dots,n-1$ and let ${}^C D_{b-}^{\alpha} f \in L_2[a,b]$. Let w be continuous nonnegative weight function on $[x,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x w(t)\,|^C D_{b-}^{\beta_1} f(t)|\,|^C D_{b-}^{\beta_2} f(t)|\,dt \le K_5\,\widetilde{\Omega}(x)\,(b-x)^{2\alpha-\beta_1-\beta_2-\frac{1}{2}} \int_a^x \left|^C D_{b-}^{\alpha} f(t)\right|^2 dt\,, \tag{5.95}$$

where $\widetilde{\Omega}$ and K_5 are given by (5.88) *and* (5.82)*, respectively.*

Theorem 5.54 *Let $i=1,2$, $\beta_i \ge 0$, $\alpha > \beta_i + \frac{1}{2}$ and m_i, n given by* (2.70)*. Let $m = \min\{m_1,m_2\}$. Let $f \in AC^n[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m,\dots,n-1$ and let ${}^C D_{b-}^{\alpha} f \in L_2[a,b]$. Let w be continuous nonnegative weight function on $[x,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x w(t)\,|^C D_{b-}^{\beta_1} f(t)|\,|^C D_{b-}^{\beta_2} f(t)|\,dt \le K_6\,\widetilde{\Omega}(x)\,(x-a)^{2\alpha-\beta_1-\beta_2-\frac{1}{2}} \int_a^x \left|^C D_{b-}^{\alpha} f(t)\right|^2 dt\,, \tag{5.96}$$

where $\widetilde{\Omega}$ and K_6 are given by (5.88) *and* (5.84)*, respectively.*

Since these inequalities for the Caputo fractional derivatives have same constants K_4, K_5 and K_6, as inequalities for the Riemann-Liouville fractional derivatives, then Remark 5.15 is also valid here.

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Theorem 5.55 *Let $\beta_i \geq 0$, $\alpha > \beta_i + \frac{5}{6}$, $m_i = [\beta_i] + 1$ for $i = 1,2$ and $n = [\alpha] + 1$, $m = \min\{m_1, m_2\}$. Let $f \in C^{\alpha}_{a+}[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m-1, \ldots, n-2$ and let ${}^{C_1}D^{\alpha}_{a+} f \in L_2[a,b]$. Let w be continuous nonnegative weight function on $[a,x]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x w(t)\,|{}^{C_1}D^{\beta_1}_{a+} f(t)|\,|{}^{C_1}D^{\beta_2}_{a+} f(t)|\,dt \leq K_4\,\Omega(x)\,(x-a)^{2\alpha-\beta_1-\beta_2-\frac{1}{2}} \int_a^x \left|{}^{C_1}D^{\alpha}_{a+} f(t)\right|^2 dt\,, \tag{5.97}$$

where Ω and K_4 are given by (5.79) and (5.80), respectively.

Theorem 5.56 *Let $\beta_i \geq 0$, $\alpha > \beta_i + \frac{5}{6}$, $m_i = [\beta_i] + 1$ for $i = 1,2$ and $n = [\alpha] + 1$, $m = \min\{m_1, m_2\}$. Let $f \in C^{\alpha}_{a+}[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m-1, \ldots, n-2$ and let ${}^{C_1}D^{\alpha}_{a+} f \in L_2[a,b]$. Let w be continuous nonnegative weight function on $[a,x]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x w(t)\,|{}^{C_1}D^{\beta_1}_{a+} f(t)|\,|{}^{C_1}D^{\beta_2}_{a+} f(t)|\,dt \leq K_5\,\Omega(x)\,(x-a)^{2\alpha-\beta_1-\beta_2-\frac{1}{2}} \int_a^x \left|{}^{C_1}D^{\alpha}_{a+} f(t)\right|^2 dt\,, \tag{5.98}$$

where Ω and K_5 are given by (5.79) and (5.82), respectively.

Theorem 5.57 *Let $\beta_i \geq 0$, $\alpha > \beta_i + \frac{1}{2}$, $m_i = [\beta_i] + 1$ for $i = 1,2$ and $n = [\alpha] + 1$, $m = \min\{m_1, m_2\}$. Let $f \in C^{\alpha}_{a+}[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m-1, \ldots, n-2$ and let ${}^{C_1}D^{\alpha}_{a+} f \in L_2[a,b]$. Let w be continuous nonnegative weight function on $[a,x]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x w(t)\,|{}^{C_1}D^{\beta_1}_{a+} f(t)|\,|{}^{C_1}D^{\beta_2}_{a+} f(t)|\,dt \leq K_6\,\Omega(x)\,(x-a)^{2\alpha-\beta_1-\beta_2-\frac{1}{2}} \int_a^x \left|{}^{C_1}D^{\alpha}_{a+} f(t)\right|^2 dt\,, \tag{5.99}$$

where Ω and K_6 are given by (5.79) and (5.84), respectively.

Next inequalities include the right-sided Canavati fractional derivatives.

Theorem 5.58 *Let $\beta_i \geq 0$, $\alpha > \beta_i + \frac{5}{6}$, $m_i = [\beta_i] + 1$ for $i = 1,2$ and $n = [\alpha] + 1$, $m = \min\{m_1, m_2\}$. Let $f \in C^{\alpha}_{b-}[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m-1, \ldots, n-2$ and let ${}^{C_1}D^{\alpha}_{b-} f \in L_2[a,b]$. Let w be continuous nonnegative weight function on $[x,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b w(t)\,|{}^{C_1}D^{\beta_1}_{b-} f(t)|\,|{}^{C_1}D^{\beta_2}_{b-} f(t)|\,dt \leq K_4\,\widetilde{\Omega}(x)\,(b-x)^{2\alpha-\beta_1-\beta_2-\frac{1}{2}} \int_x^b \left|{}^{C_1}D^{\alpha}_{b-} f(t)\right|^2 dt\,, \tag{5.100}$$

where $\widetilde{\Omega}$ and K_4 are given by (5.88) and (5.80), respectively.

Theorem 5.59 *Let $\beta_i \geq 0$, $\alpha > \beta_i + \frac{5}{6}$, $m_i = [\beta_i] + 1$ for $i = 1,2$ and $n = [\alpha] + 1$, $m = \min\{m_1, m_2\}$. Let $f \in C^{\alpha}_{b-}[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m-1, \ldots, n-2$ and let ${}^{C_1}D^{\alpha}_{b-}f \in L_2[a,b]$. Let w be continuous nonnegative weight function on $[x,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x w(t)\, |{}^{C_1}D^{\beta_1}_{b-}f(t)|\, |{}^{C_1}D^{\beta_2}_{b-}f(t)|\, dt \leq K_5\, \widetilde{\Omega}(x)\, (b-x)^{2\alpha-\beta_1-\beta_2-\frac{1}{2}} \int_a^x \left|{}^{C_1}D^{\alpha}_{b-}f(t)\right|^2 dt\,, \tag{5.101}$$

where $\widetilde{\Omega}$ and K_5 are given by (5.88) and (5.82), respectively.

Theorem 5.60 *Let $\beta_i \geq 0$, $\alpha > \beta_i + \frac{1}{2}$, $m_i = [\beta_i] + 1$ for $i = 1,2$ and $n = [\alpha] + 1$, $m = \min\{m_1, m_2\}$. Let $f \in C^{\alpha}_{b-}[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m-1, \ldots, n-2$ and let ${}^{C_1}D^{\alpha}_{b-}f \in L_2[a,b]$. Let w be continuous nonnegative weight function on $[x,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x w(t)\, |{}^{C_1}D^{\beta_1}_{b-}f(t)|\, |{}^{C_1}D^{\beta_2}_{b-}f(t)|\, dt \leq K_6\, \widetilde{\Omega}(x)\, (x-a)^{2\alpha-\beta_1-\beta_2-\frac{1}{2}} \int_a^x \left|{}^{C_1}D^{\alpha}_{b-}f(t)\right|^2 dt\,, \tag{5.102}$$

where $\widetilde{\Omega}$ and K_6 are given by (5.88) and (5.84), respectively.

A comparison of constants K_4, K_5 and K_6 from Remark 5.15 is also valid for the Canavati fractional derivatives.

Chapter 6

Extensions of Opial-type inequalities for fractional derivatives

In this chapter we observe Opial-type inequalities involving fractional derivatives of order α and β_i, $i = 1, \ldots, N$, and also obtain theirs generalizations, extensions, improvements and refinements. As in the previous chapter, we investigate the possibility of obtaining the best possible constant and compare results obtained by different methods. This chapter is based on our results: Andrić, Pečarić and Perić [16, 23, 24, 25, 27].

6.1 Extensions of the Fink Opial-type inequality

Our first inequality is the multiple generalization of Fink's inequality (1.23)

$$\int_a^b \prod_{i=1}^N |\mathbf{D}^{\beta_i} f(t)|\, dt \leq K \left(\int_a^b |\mathbf{D}^{\alpha} f(t)|^q\, dt \right)^{\frac{N}{q}},$$

where $\alpha > \beta_i \geq 0$, $K > 0$ is a constant and $q \in \mathbb{R}$.

This inequality is a special case of Opial-type inequality due to Agarwal-Pang, which we observe in section 3.2.2. It is given for the left-sided Riemann-Liouville fractional derivatives in [25] by Andrić-Pečarić-Perić where Agarwal-Pang's method from [2] (here Theorem 6.1) is used as well as generalization of Fink's method from [45] (Theorem 6.2). The comparison of the obtained results is also given in [25] (Remark 6.1).

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Theorem 6.1 *Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $\beta_i \geq 0$, $\alpha > \beta_i + \frac{1}{q}$ for $i=1,\ldots N$, $N \in \mathbb{N}$, $N \geq 2$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.21 holds for all pairs $\{\alpha, \beta_i\}$, $i=1,\ldots,N$. Let $D^{\alpha}_{a+}f \in L_q[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x \prod_{i=1}^N |D^{\beta_i}_{a+}f(t)|\,dt \leq T_1\,(x-a)^{\sum_{i=1}^N(\alpha-\beta_i)+1-\frac{N}{q}} \left(\int_a^x |D^{\alpha}_{a+}f(t)|^q\,dt\right)^{\frac{N}{q}}, \tag{6.1}$$

where

$$T_1 = \left[\left(\prod_{i=1}^N \Gamma(\alpha-\beta_i)\left(\alpha-\beta_i-\frac{1}{q}\right)^{\frac{1}{p}}\right) p^{\frac{N}{p}} \left(\sum_{i=1}^N(\alpha-\beta_i)+1-\frac{N}{q}\right)\right]^{-1}. \tag{6.2}$$

Proof. Set $\delta_i = \alpha-\beta_i-1$, $i=1,\ldots,N$. Using Theorem 2.13, the triangle inequality and Hölder's inequality we have

$$\begin{aligned}
&\int_a^x \prod_{i=1}^N |D^{\beta_i}_{a+}f(t)|\,dt \\
&\leq \frac{1}{\prod\limits_{i=1}^N \Gamma(\delta_i+1)} \int_a^x \prod_{i=1}^N \left(\int_a^t (t-\tau)^{\delta_i}|D^{\alpha}_{a+}f(\tau)|\,d\tau\right) dt \\
&\leq \frac{1}{\prod\limits_{i=1}^N \Gamma(\delta_i+1)} \int_a^x \prod_{i=1}^N \left[\left(\int_a^t (t-\tau)^{p\delta_i}\,d\tau\right)^{\frac{1}{p}} \left(\int_a^t |D^{\alpha}_{a+}f(\tau)|^q\,d\tau\right)^{\frac{1}{q}}\right] dt \\
&= \frac{1}{\prod\limits_{i=1}^N \Gamma(\delta_i+1)\,(p\delta_i+1)^{\frac{1}{p}}} \int_a^x (t-a)^{\sum_{i=1}^N \delta_i+\frac{N}{p}} \left(\int_a^t |D^{\alpha}_{a+}f(\tau)|^q\,d\tau\right)^{\frac{N}{q}} dt \\
&\leq \frac{1}{\prod\limits_{i=1}^N \Gamma(\delta_i+1)\,(p\delta_i+1)^{\frac{1}{p}}} \left(\int_a^x |D^{\alpha}_{a+}f(\tau)|^q\,d\tau\right)^{\frac{N}{q}} \frac{(x-a)^{\sum_{i=1}^N \delta_i+1+\frac{N}{p}}}{\sum\limits_{i=1}^N \delta_i+1+\frac{N}{p}}.
\end{aligned}$$

□

In the next theorem we will obtain new estimation of inequality (6.1) using generalized Fink' method from [45].

Theorem 6.2 *Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $\beta_i \geq 0$, $\alpha > \beta_1 \geq \beta_i + 1$ for $i=2,\dots N$, $N \in \mathbb{N}$, $N \geq 2$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.21 holds for all pairs $\{\alpha,\beta_i\}$, $i=1,\dots,N$. Let $D^{\alpha}_{a+}f \in L_q[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$\int_a^x \prod_{i=1}^N |D^{\beta_i}_{a+}f(t)|\,dt \leq F_N\,(x-a)^{\sum_{i=1}^N(\alpha-\beta_i)+1-\frac{N}{q}} \left(\int_a^x |D^{\alpha}_{a+}f(t)|^q\,dt\right)^{\frac{N}{q}}, \tag{6.3}$$

where

$$F_N = \left[\left(\prod_{i=1}^N \Gamma(\alpha-\beta_i)\right)(\alpha-\beta_1)\,p^{\frac{N}{p}}\left(\alpha-\beta_1+\frac{1}{p}\right)^{\frac{N-1}{p}} N^{\frac{1}{q}}\left(\sum_{i=1}^N(\alpha-\beta_i)+1-\frac{N}{q}\right)^{\frac{1}{p}}\right]^{-1}. \tag{6.4}$$

Inequality (6.3) *is sharp for $\beta_1=\beta_2+1=\cdots=\beta_N+1$, where equality is attained for a function f such that $D^{\alpha}_{a+}f(t)=(x-t)^{\frac{p(\alpha-\beta_1)}{q}}$, $t\in[a,x]$.*

Proof. Set $\delta_i=\alpha-\beta_i-1$, $i=1,\dots,N$. Using Theorem 2.13, the triangle inequality and Hölder's inequality we have

$$\begin{aligned}
&\int_a^x \prod_{i=1}^N |D^{\beta_i}_{a+}f(t)|\,dt \\
&\leq \frac{1}{\prod_{i=1}^N \Gamma(\delta_i+1)} \int_a^x \left(\prod_{i=1}^N \int_a^x |D^{\alpha}_{a+}f(\tau_i)|(t-\tau_i)_+^{\delta_i}\,d\tau_i\right) dt \\
&= \frac{1}{\prod_{i=1}^N \Gamma(\delta_i+1)} \int_{[a,x]^N} \prod_{i=1}^N |D^{\alpha}_{a+}f(\tau_i)| \left(\int_a^x \prod_{i=1}^N (t-\tau_i)_+^{\delta_i}\,dt\right) d\tau_1\cdots d\tau_N \\
&= \frac{1}{\prod_{i=1}^N \Gamma(\delta_i+1)} \int_{[a,x]^N} \prod_{i=1}^N |D^{\alpha}_{a+}f(\tau_i)| \left(\int_{\max\{\tau_1,\dots,\tau_N\}}^x \prod_{i=1}^N (t-\tau_i)^{\delta_i}\,dt\right) d\tau_1\cdots d\tau_N \\
&= \frac{1}{\prod_{i=1}^N \Gamma(\delta_i+1)} \int_{\Delta_N} \prod_{i=1}^N |D^{\alpha}_{a+}f(\tau_i)| \left(\int_{\tau_N}^x \sum_{\pi\in S_N} \prod_{i=1}^N (t-\tau_i)^{\delta_{\pi(i)}}\,dt\right) d\tau_N\cdots d\tau_1,
\end{aligned} \tag{6.5}$$

where $\Delta_N=\{(\tau_1,\dots,\tau_N): a\leq\tau_1\leq\cdots\leq\tau_N\leq x\}$ and S_N is the group of all permutations of the set $\{1,2,\dots,N\}$. The last equality in (6.5) follows by dividing the cube $[a,x]^N$ into parts where $a\leq\tau_{\pi(1)}\leq\tau_{\pi(2)}\leq\cdots\leq\tau_{\pi(N)}\leq x$ for some $\pi\in S_N$, and symmetry of the involved expressions. Suppose that $\pi\in S_N$ is given and suppose that $j\in\{1,\dots,N\}$ is such that $\pi(j)=1$. Then

$$\prod_{i=1}^N (t-\tau_i)^{\delta_{\pi(i)}} = \prod_{i\neq j}(t-\tau_i)^{\delta_{\pi(i)}-\delta_1-1}(t-\tau_j)^{\delta_1}\prod_{i\neq j}(t-\tau_i)^{\delta_1+1}$$

$$\leq (t-\tau_1)^{\sum_{i=2}^N \delta_i-(N-1)(\delta_1+1)}(t-\tau_j)^{\delta_1}\prod_{i\neq j}(t-\tau_i)^{\delta_1+1}. \tag{6.6}$$

Since the group S_N has $N!$ permutations, then $\pi(j)=1$ is on a certain position in permutation exactly $(N-1)!$ times. Therefore using (6.6) we have

$$\begin{aligned}
&\int_{\tau_N}^x \sum_{\pi\in S_N}\prod_{i=1}^N (t-\tau_i)^{\delta_{\pi(i)}}\,dt \\
&\leq \int_{\tau_N}^x \sum_{\pi\in S_N}\left[(t-\tau_1)^{\sum_{i=2}^N \delta_i-(N-1)(\delta_1+1)}(t-\tau_j)^{\delta_1}\prod_{i\neq j}(t-\tau_i)^{\delta_1+1}\right]dt \\
&= \int_{\tau_N}^x (N-1)!\sum_{j=1}^N (t-\tau_1)^{\sum_{i=2}^N \delta_i-(N-1)(\delta_1+1)}(t-\tau_j)^{\delta_1}\prod_{i\neq j}(t-\tau_i)^{\delta_1+1}\,dt \\
&= (N-1)!\int_{\tau_N}^x (t-\tau_1)^{\sum_{i=2}^N \delta_i-(N-1)(\delta_1+1)}\sum_{j=1}^N (t-\tau_j)^{\delta_1}\prod_{i\neq j}(t-\tau_i)^{\delta_1+1}\,dt \\
&= \frac{(N-1)!}{\delta_1+1}\int_{\tau_N}^x (t-\tau_1)^{\sum_{i=2}^N \delta_i-(N-1)(\delta_1+1)}\frac{d}{dt}\left(\prod_{i=1}^N (t-\tau_i)^{\delta_1+1}\right)dt\,.
\end{aligned}$$

Using integration by parts we get

$$\begin{aligned}
&\int_{\tau_N}^x (t-\tau_1)^{\sum_{i=2}^N \delta_i-(N-1)(\delta_1+1)}\frac{d}{dt}\left(\prod_{i=1}^N (t-\tau_i)^{\delta_1+1}\right)dt \\
&\leq (x-\tau_1)^{\sum_{i=2}^N \delta_i-(N-1)(\delta_1+1)}\prod_{i=1}^N (x-\tau_i)^{\delta_1+1} \\
&= (x-\tau_1)^{\sum_{i=2}^N \delta_i-(N-2)(\delta_1+1)}\prod_{i=2}^N (x-\tau_i)^{\delta_1+1},
\end{aligned}$$

which leads to

$$\int_{\tau_N}^x \sum_{\pi\in S_N}\prod_{i=1}^N (t-\tau_i)^{\delta_{\pi(i)}}\,dt \leq \frac{(N-1)!}{\delta_1+1}(x-\tau_1)^{\sum_{i=2}^N \delta_i-(N-2)(\delta_1+1)}\prod_{i=2}^N (x-\tau_i)^{\delta_1+1}.$$

Set $A=\frac{(N-1)!}{(\delta_1+1)\prod_{i=1}^N\Gamma(\delta_i+1)}$. Then

$$\begin{aligned}
&\int_a^x \prod_{i=1}^N |D_{a+}^{\beta_i}f(t)|\,dt \\
&\leq \frac{1}{\prod_{i=1}^N\Gamma(\delta_i+1)}\int_{\Delta_N}\prod_{i=1}^N |D_{a+}^{\alpha}f(\tau_i)|\left(\int_{\tau_N}^x \sum_{\pi\in S_N}\prod_{i=1}^N (t-\tau_i)^{\delta_{\pi(i)}}\,dt\right)d\tau_N\cdots d\tau_1 \\
&\leq A\int_{\Delta_N}\prod_{i=1}^N |D_{a+}^{\alpha}f(\tau_i)|(x-\tau_1)^{\sum_{i=2}^N \delta_i-(N-2)(\delta_1+1)}\prod_{i=2}^N (x-\tau_i)^{\delta_1+1}\,d\tau_N\cdots d\tau_1
\end{aligned}$$

$$= A \int_a^x (x-\tau_1)^{\sum_{i=2}^N \delta_i - (N-2)(\delta_1+1)} |D_{a+}^\alpha f(\tau_1)|\, d\tau_1 \int_{\tau_1}^x (x-\tau_2)^{\delta_1+1} |D_{a+}^\alpha f(\tau_2)|\, d\tau_2$$

$$\cdots \int_{\tau_{N-1}}^x (x-\tau_N)^{\delta_1+1} |D_{a+}^\alpha f(\tau_N)|\, d\tau_N\,. \tag{6.7}$$

Applying Hölder's inequality on the last integral in (6.7) we get

$$\begin{aligned}
&\int_{\tau_{N-1}}^x (x-\tau_N)^{\delta_1+1} |D_{a+}^\alpha f(\tau_N)|\, d\tau_N \\
&\leq \left(\int_{\tau_{N-1}}^x (x-\tau_N)^{p(\delta_1+1)}\, d\tau_N \right)^{\frac{1}{p}} \left(\int_{\tau_{N-1}}^x |D_{a+}^\alpha f(\tau_N)|^q\, d\tau_N \right)^{\frac{1}{q}} \qquad (6.8)\\
&= \frac{(x-\tau_{N-1})^{\delta_1+1+\frac{1}{p}}}{[p(\delta_1+1)+1]^{\frac{1}{p}}} \left(\int_{\tau_{N-1}}^x |D_{a+}^\alpha f(\tau_N)|^q\, d\tau_N \right)^{\frac{1}{q}}.
\end{aligned}$$

Again we use Hölder's inequality, that is

$$\begin{aligned}
&\int_{\tau_{N-2}}^x (x-\tau_{N-1})^{2(\delta_1+1)+\frac{1}{p}} |D_{a+}^\alpha f(\tau_{N-1})| \left(\int_{\tau_{N-1}}^x |D_{a+}^\alpha f(\tau_N)|^q\, d\tau_N \right)^{\frac{1}{q}} d\tau_{N-1} \\
&\leq \left(\int_{\tau_{N-2}}^x (x-\tau_{N-1})^{2p(\delta_1+1)+1}\, d\tau_{N-1} \right)^{\frac{1}{p}} \\
&\quad \cdot \left[\int_{\tau_{N-2}}^x |D_{a+}^\alpha f(\tau_{N-1})|^q \left(\int_{\tau_{N-1}}^x |D_{a+}^\alpha f(\tau_N)|^q\, d\tau_N \right) d\tau_{N-1} \right]^{\frac{1}{q}} \\
&= \frac{(x-\tau_{N-2})^{2(\delta_1+1)+\frac{2}{p}}}{[2p(\delta_1+1)+2]^{\frac{1}{p}}} \frac{1}{2^{\frac{1}{q}}} \left(\int_{\tau_{N-2}}^x |D_{a+}^\alpha f(\tau_N)|^q\, d\tau_N \right)^{\frac{2}{q}}.
\end{aligned}$$

Notice

$$[2p(\delta_1+1)+2]^{\frac{1}{p}} 2^{\frac{1}{q}} = 2^{\frac{1}{p}+\frac{1}{q}} [p(\delta_1+1)+1]^{\frac{1}{p}} = 2\,[p(\delta_1+1)+1]^{\frac{1}{p}}\,.$$

Next step gives us

$$\begin{aligned}
&\int_{\tau_{N-3}}^x (x-\tau_{N-2})^{3(\delta_1+1)+\frac{2}{p}} |D_{a+}^\alpha f(\tau_{N-2})| \left(\int_{\tau_{N-2}}^x |D_{a+}^\alpha f(\tau_N)|^q\, d\tau_N \right)^{\frac{2}{q}} d\tau_{N-2} \\
&\leq \left(\int_{\tau_{N-3}}^x (x-\tau_{N-2})^{3p(\delta_1+1)+2} d\tau_{N-2} \right)^{\frac{1}{p}} \\
&\quad \cdot \left[\int_{\tau_{N-3}}^x |D_{a+}^\alpha f(\tau_{N-2})|^q \left(\int_{\tau_{N-2}}^x |D_{a+}^\alpha f(\tau_N)|^q\, d\tau_N \right)^2 d\tau_{N-2} \right]^{\frac{1}{q}} \\
&= \frac{(x-\tau_{N-3})^{3(\delta_1+1)+\frac{3}{p}}}{[3p(\delta_1+1)+3]^{\frac{1}{p}}} \frac{1}{3^{\frac{1}{q}}} \left(\int_{\tau_{N-3}}^x |D_{a+}^\alpha f(\tau_N)|^q\, d\tau_N \right)^{\frac{3}{q}}.
\end{aligned}$$

Finally, we have

$$
\begin{aligned}
&\int_a^x \prod_{i=1}^N |D_{a+}^{\beta_i} f(t)|\,dt \\
&\le \frac{A}{(N-2)!\,[p(\delta_1+1)+1]^{\frac{N-2}{p}}} \int_a^x (x-\tau_1)^{\sum_{i=2}^N \delta_i-(N-2)(\delta_1+1)} |D_{a+}^{\alpha} f(\tau_1)|\,d\tau_1 \\
&\qquad \cdot \frac{(x-\tau_1)^{(N-1)(\delta_1+1)+\frac{N-1}{p}}}{[(N-1)p(\delta_1+1)+N-1]^{\frac{1}{p}}\,(N-1)^{\frac{1}{q}}} \left(\int_{\tau_1}^x |D_{a+}^{\alpha} f(\tau_N)|^q\,d\tau_N\right)^{\frac{N-1}{q}} d\tau_1 \\
&= \frac{A}{(N-1)!\,[p(\delta_1+1)+1]^{\frac{N-1}{p}}} \int_a^x (x-\tau_1)^{\sum_{i=1}^N \delta_i+1+\frac{N-1}{p}} |D_{a+}^{\alpha} f(\tau_1)|\,d\tau_1 \\
&\qquad \cdot \left(\int_{\tau_1}^x |D_{a+}^{\alpha} f(\tau_N)|^q\,d\tau_N\right)^{\frac{N-1}{q}} d\tau_1 \\
&\le \frac{1}{\prod_{i=1}^N \Gamma(\delta_i+1)\,(\delta_1+1)\,[p(\delta_1+1)+1]^{\frac{N-1}{p}}} \left(\int_a^x (x-\tau_1)^{p\sum_{i=1}^N \delta_i+p+N-1}\,d\tau_1\right)^{\frac{1}{p}} \\
&\qquad \cdot \left[\int_a^x |D_{a+}^{\alpha} f(\tau_1)|^q \left(\int_{\tau_1}^x |D_{a+}^{\alpha} f(\tau_N)|^q\,d\tau_N\right)^{N-1} d\tau_1\right]^{\frac{1}{q}} \\
&= \frac{1}{\prod_{i=1}^N \Gamma(\delta_i+1)\,(\delta_1+1)\,[p(\delta_1+1)+1]^{\frac{N-1}{p}}} \cdot \frac{(x-a)^{\sum_{i=1}^N \delta_i+1+\frac{N}{p}}}{[p(\sum_{i=1}^N \delta_i+1)+N]^{\frac{1}{p}}} \\
&\qquad \cdot \frac{1}{N^{\frac{1}{q}}} \left(\int_a^x |D_{a+}^{\alpha} f(\tau_N)|^q\,d\tau_N\right)^{\frac{N}{q}}.
\end{aligned}
$$

This proves the inequality (6.3).

Consider the case $\beta_1 = \beta_2 + 1 = \cdots = \beta_N + 1$. Using the equality condition in Hölder's inequality we have equality in (6.8) if and only if $|D_{a+}^{\alpha} f(\tau_N)|^q = K(x-\tau_N)^{p(\delta_1+1)}$ for some constant $K \ge 0$ and every $\tau_N \in [\tau_{N-1}, x]$. Straightforward calculation will show that for a function $D_{a+}^{\alpha} f(t) = (x-t)^{\frac{p(\alpha-\beta_1)}{q}}$ inequality (6.3) is sharp.

$$
F_N\,(x-a)^{\sum\limits_{i=1}^{N}(\alpha-\beta_i)+1-\frac{N}{q}} = \frac{(x-a)^{N(\alpha-\beta_1)+\frac{N}{p}}}{N\,[p(\alpha-\beta_1)+1]^{\frac{N}{p}}\,[\Gamma(\alpha-\beta_1)]^N\,(\alpha-\beta_1)^N},
$$

$$
\left(\int_a^x |D_{a+}^{\alpha} f(t)|^q\,dt\right)^{\frac{N}{q}} = \left(\int_a^x K^q (x-t)^{p(\alpha-\beta_1)}\,dt\right)^{\frac{N}{q}} = \frac{K^N\,(x-a)^{\frac{Np}{q}(\alpha-\beta_1)+\frac{N}{q}}}{[p(\alpha-\beta_1)+1]^{\frac{N}{q}}},
$$

and for the right side of the inequality (6.3) we have

$$
\frac{K^N\,(x-a)^{Np(\alpha-\beta_1)+N}}{N\,[p(\alpha-\beta_1)+1]^N\,[\Gamma(\alpha-\beta_1)]^N\,(\alpha-\beta_1)^N}. \tag{6.9}
$$

For the left side of the inequality (6.3) we get

$$
\begin{aligned}
\int_a^x \prod_{i=1}^N |D_{a+}^{\beta_i} f(t)|\, dt &= \int_a^x |D_{a+}^{\beta_1} f(t)||D_{a+}^{\beta_1-1} f(t)|^{N-1}\, dt = \frac{|D_{a+}^{\beta_1-1} f(x)|^N}{N} \\
&= \frac{1}{N[\Gamma(\alpha-\beta_1+1)]^N} \left(\int_a^x (x-\tau)^{\alpha-\beta_1} |D_{a+}^{\alpha} f(\tau)|\, d\tau \right)^N \\
&= \frac{K^N}{N(\alpha-\beta_1)^N [\Gamma(\alpha-\beta_1)]^N} \left(\int_a^x (x-\tau)^{p(\alpha-\beta_1)}\, d\tau \right)^N \\
&= \frac{K^N (x-a)^{Np(\alpha-\beta_1)+N}}{N(\alpha-\beta_1)^N [\Gamma(\alpha-\beta_1)]^N [p(\alpha-\beta_1)+1]^N}, \qquad (6.10)
\end{aligned}
$$

so (6.9) and (6.10) are equal. This proves theorem. □

Remark 6.1 The constants T_1 and F_N from the two previous theorems are in general not comparable, but there are cases when we can do that. Although the constat F_N gives the best possible estimation in the case $\beta_1 = \beta_2 + 1 = \cdots = \beta_N + 1$, it seems that constant T_1 gives more uniform estimation which is partially justified by the following discussion (in Remark 5.9 we give comparison for $N = 2$). Notice

$$
\frac{F_N}{T_1} = \frac{\left[\sum_{i=1}^N (\alpha-\beta_i) + 1 - N + \frac{N}{p} \right]^{\frac{1}{q}} \prod_{i=1}^N \left(\alpha - \beta_i - 1 + \frac{1}{p} \right)^{\frac{1}{p}}}{N^{\frac{1}{q}} (\alpha-\beta_1) \left(\alpha - \beta_1 + \frac{1}{p} \right)^{\frac{N-1}{p}}}.
$$

We want to find cases when $T_1 < F_N$. Set $\alpha - \beta_1 = d$, $\beta_1 - \beta_i = \delta_i \geq 1$ for $i = 2, \ldots, N$. Then inequality $T_1 < F_N$ is equivalent to

$$
\frac{1}{\left(Nd + \sum_{i=2}^N \delta_i + 1 - N + \frac{N}{p} \right)^{1-\frac{1}{p}} \prod_{i=2}^N \left(d + \delta_i - 1 + \frac{1}{p} \right)^{\frac{1}{p}}} < \frac{\left(d - 1 + \frac{1}{p} \right)^{\frac{1}{p}}}{N^{1-\frac{1}{p}} d \left(d + \frac{1}{p} \right)^{\frac{N-1}{p}}}. \qquad (6.11)
$$

If δ_i are big enough, then the left side of (6.11) tends to zero, while the right side depends only of d. Therefore, in this case $T_1 < F_N$.

Let $\delta_i = 1$, that is $\beta_1 = \beta_i + 1$, $i = 2, \ldots, N$ (see the discussion of sharpness in Theorem 6.2). Then the reverse inequality (6.11) is equivalent to

$$
\frac{1}{\left(Nd + \frac{N}{p} \right)^{1-\frac{1}{p}} \left(d + \frac{1}{p} \right)^{\frac{N-1}{p}}} > \frac{\left(d - 1 + \frac{1}{p} \right)^{\frac{1}{p}}}{N^{1-\frac{1}{p}} d \left(d + \frac{1}{p} \right)^{\frac{N-1}{p}}},
$$

that is

$$
\frac{pd+1}{pd-p+1} > \left(1 + \frac{1}{pd} \right)^p.
$$

This is equivalent to inequality

$$\left(\frac{pd+1-p}{pd+1}\right)^{\frac{1}{p}} < \frac{pd}{1+pd}$$

which is a simple consequence of Bernoulli's inequality. This is in accordance with Theorem 6.2. Numerical calculations indicate that there is a very narrow area around the best possible case $\beta_1 = \beta_i + 1$, $i = 2,\dots,N$, where F_N gives better estimation than T_1.

In the following cases, when $p = 1$ and $p \in (0,1)$, we use a method from 6.1.

Proposition 6.1 *Let $\alpha > \beta_i \geq 0$ for $i = 1,\dots N$, $N \in \mathbb{N}$, $N \geq 2$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.21 holds for all pairs $\{\alpha,\beta_i\}$, $i = 1,\dots,N$. Let $D^{\alpha}_{a+}f \in L_\infty[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x \prod_{i=1}^N |D^{\beta_i}_{a+}f(t)|\,dt \leq \frac{(x-a)^{\sum\limits_{i=1}^N(\alpha-\beta_i)+1}}{\left[\sum\limits_{i=1}^N(\alpha-\beta_i)+1\right]\prod\limits_{i=1}^N \Gamma(\alpha-\beta_i+1)} \left\|D^{\alpha}_{a+}f(t)\right\|^N_\infty. \tag{6.12}$$

Proof. Set $\delta_i = \alpha - \beta_i - 1$, $i = 1,\dots,N$. Using Theorem 2.13, the triangle inequality and Hölder's inequality we have

$$\begin{aligned}
\int_a^x \prod_{i=1}^N |D^{\beta_i}_{a+}f(t)|\,dt &\leq \frac{1}{\prod_{i=1}^N\Gamma(\delta_i+1)}\int_a^x\prod_{i=1}^N\left(\int_a^t(t-\tau)^{\delta_i}|D^{\alpha}_{a+}f(\tau)|\,d\tau\right)dt\\
&\leq \frac{1}{\prod_{i=1}^N\Gamma(\delta_i+1)}\int_a^x\prod_{i=1}^N\left[\left(\int_a^t(t-\tau)^{\delta_i}\,d\tau\right)\left\|D^{\alpha}_{a+}f\right\|_\infty\right]dt\\
&= \frac{\left\|D^{\alpha}_{a+}f\right\|^N_\infty}{\prod_{i=1}^N\Gamma(\delta_i+2)}\int_a^x(t-a)^{\sum_{i=1}^N\delta_i+N}\,dt\\
&= \frac{\left\|D^{\alpha}_{a+}f\right\|^N_\infty}{\prod_{i=1}^N\Gamma(\delta_i+2)}\,\frac{(x-a)^{\sum_{i=1}^N\delta_i+N+1}}{\sum_{i=1}^N\delta_i+N+1}.
\end{aligned}$$

□

Theorem 6.3 *Let $p \in (0,1)$, $q < 0$ and $\frac{1}{p}+\frac{1}{q} = 1$. Let $\alpha > \beta_i \geq 0$ for $i = 1,\dots N$, $N \in \mathbb{N}$, $N \geq 2$. Let $D^{\alpha}_{a+}f \in L_q[a,b]$ be of fixed sign on $[a,b]$, with $1/D^{\alpha}_{a+}f \in L_q[a,b]$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.21 holds for all pairs $\{\alpha,\beta_i\}$, $i = 1,\dots,N$. Then reverse inequality in* (6.1) *holds.*

Next inequalities include the right-sided Riemann-Liouville fractional derivatives.

Theorem 6.4 *Let $p,q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$. Let $\beta_i \geq 0$, $\alpha > \beta_i + \frac{1}{q}$ for $i = 1,\dots N$, $N \in \mathbb{N}$, $N \geq 2$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.22 holds for all pairs $\{\alpha,\beta_i\}$, $i = 1,\dots,N$. Let $D^{\alpha}_{b-}f \in L_q[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b \prod_{i=1}^N |D^{\beta_i}_{b-}f(t)|\,dt \leq T_1\,(b-x)^{\sum\limits_{i=1}^N(\alpha-\beta_i)+1-\frac{N}{q}}\left(\int_x^b |D^{\alpha}_{b-}f(t)|^q\,dt\right)^{\frac{N}{q}}, \tag{6.13}$$

where T_1 is given by (6.2).

In the following theorem again we use Fink's method to obtain a new estimation for the inequality (6.13). The same comparison holds as with the left-sided Riemann-Liouville fractional derivatives, i.e., for Theorem 6.4 and Theorem 6.5, Remark 6.1 is valid also.

Theorem 6.5 *Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $\beta_i \geq 0$, $\alpha > \beta_1 \geq \beta_i + 1$ for $i=2,\dots N$, $N \in \mathbb{N}$, $N \geq 2$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.22 holds for all pairs $\{\alpha,\beta_i\}$, $i=1,\dots,N$. Let $D_{b-}^{\alpha} f \in L_q[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b \prod_{i=1}^N |D_{b-}^{\beta_i} f(t)|\,dt \leq F_N\,(b-x)^{\sum\limits_{i=1}^N(\alpha-\beta_i)+1-\frac{N}{q}} \left(\int_x^b |D_{b-}^{\alpha} f(t)|^q\,dt\right)^{\frac{N}{q}}, \tag{6.14}$$

where F_N is given by (6.4).

Inequality (6.14) *is sharp for $\beta_1 = \beta_2 + 1 = \cdots = \beta_N + 1$, where equality is attained for a function f such that $D_{b-}^{\alpha} f(t) = (t-x)^{\frac{p(\alpha-\beta_1)}{q}}$, $t \in [x,b]$.*

In the following cases, when $p=1$ and $p \in (0,1)$, we use a method from Theorem 6.4.

Proposition 6.2 *Let $\alpha > \beta_i \geq 0$ for $i=1,\dots N$, $N \in \mathbb{N}$, $N \geq 2$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.22 holds for all pairs $\{\alpha,\beta_i\}$, $i=1,\dots,N$. Let $D_{b-}^{\alpha} f \in L_\infty[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b \prod_{i=1}^N |D_{b-}^{\beta_i} f(t)|\,dt \leq \frac{(b-x)^{\sum\limits_{i=1}^N(\alpha-\beta_i)+1}}{\left[\sum\limits_{i=1}^N(\alpha-\beta_i)+1\right]\prod\limits_{i=1}^N \Gamma(\alpha-\beta_i+1)} \left\|D_{b-}^{\alpha} f(t)\right\|_\infty^N. \tag{6.15}$$

Theorem 6.6 *Let $p \in (0,1)$, $q<0$ and $\frac{1}{p}+\frac{1}{q}=1$. Let $\alpha > \beta_i \geq 0$ for $i=1,\dots N$, $N \in \mathbb{N}$, $N \geq 2$. Let $D_{b-}^{\alpha} f \in L_q[a,b]$ be of fixed sign on $[a,b]$, with $1/D_{b-}^{\alpha} f \in L_q[a,b]$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.22 holds for all pairs $\{\alpha,\beta_i\}$, $i=1,\dots,N$. Then reverse inequality in* (6.1) *holds.*

THE CAPUTO FRACTIONAL DERIVATIVES

Theorem 6.7 *Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $N\in\mathbb{N}$, $N\geq 2$, $i=1,\dots,N$, $\beta_i\geq 0$, $\alpha>\beta_i+\frac{1}{q}$, and m_i, n given by* (2.70) *with $m=\min\{m_i\}$. Let $f\in AC^n[a,b]$ be such that $f^{(i)}(a)=0$ for $i=m,\dots,n-1$ and let ${}^C D^{\alpha}_{a+}f\in L_q[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$\int_a^x \prod_{i=1}^N |{}^C D^{\beta_i}_{a+}f(t)|\,dt \leq T_1\,(x-a)^{\sum\limits_{i=1}^N(\alpha-\beta_i)+1-\frac{N}{q}}\left(\int_a^x |{}^C D^{\alpha}_{a+}f(t)|^q\,dt\right)^{\frac{N}{q}}, \tag{6.16}$$

where T_1 is given by (6.2).

Next theorem gives a new estimation of the inequality (6.16).

Theorem 6.8 *Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $N\in\mathbb{N}$, $N\geq 2$, $i=1,\dots,N$, $\alpha>\beta_1\geq\beta_i+1$, $\beta_i\geq 0$, and m_i, n given by* (2.70) *with $m=\min\{m_i\}$. Let $f\in AC^n[a,b]$ be such that $f^{(i)}(a)=0$ for $i=m,\dots,n-1$ and let ${}^C D^{\alpha}_{a+}f\in L_q[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$\int_a^x \prod_{i=1}^N |{}^C D^{\beta_i}_{a+}f(t)|\,dt \leq F_N\,(x-a)^{\sum\limits_{i=1}^N(\alpha-\beta_i)+1-\frac{N}{q}}\left(\int_a^x |{}^C D^{\alpha}_{a+}f(t)|^q\,dt\right)^{\frac{N}{q}}, \tag{6.17}$$

where F_N is given by (2.70).

Inequality (6.17) *is sharp for $\beta_1=\beta_2+1=\cdots=\beta_N+1$, where equality is attained for a function f such that ${}^C D^{\alpha}_{a+}f(t)=(x-t)^{\frac{p(\alpha-\beta_1)}{q}}$, $t\in[a,x]$.*

In the following cases, when $p=1$ and $p\in(0,1)$, we use a method from Theorem 6.7.

Proposition 6.3 *Let $N\in\mathbb{N}$, $N\geq 2$, $i=1,\dots,N$, $\alpha>\beta_i\geq 0$, and m_i, n given by* (2.70) *with $m=\min\{m_i\}$. Let $f\in AC^n[a,b]$ be such that $f^{(i)}(a)=0$ for $i=m,\dots,n-1$ and let ${}^C D^{\alpha}_{a+}f\in L_q[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$\int_a^x \prod_{i=1}^N |{}^C D^{\beta_i}_{a+}f(t)|\,dt \leq \frac{(x-a)^{\sum\limits_{i=1}^N(\alpha-\beta_i)+1}}{\left[\sum\limits_{i=1}^N(\alpha-\beta_i)+1\right]\prod\limits_{i=1}^N\Gamma(\alpha-\beta_i+1)}\left\|{}^C D^{\alpha}_{a+}f(t)\right\|_{\infty}^N. \tag{6.18}$$

Theorem 6.9 *Let $p\in(0,1)$, $q<0$ and $\frac{1}{p}+\frac{1}{q}=1$. Let $N\in\mathbb{N}$, $N\geq 2$, $i=1,\dots,N$, $\alpha>\beta_i\geq 0$, and m_i, n given by* (2.70) *with $m=\min\{m_i\}$. Let $f\in AC^n[a,b]$ be such that $f^{(i)}(a)=0$ for $i=m,\dots,n-1$. Let ${}^C D^{\alpha}_{a+}f\in L_q[a,b]$ be of fixed sign on $[a,b]$, with $1/{}^C D^{\alpha}_{a+}f\in L_q[a,b]$. Then reverse inequality in* (6.16) *holds.*

Following inequalities include the right-sided Caputo fractional derivatives.

Theorem 6.10 *Let $p,q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $N \in \mathbb{N}$, $N \geq 2$, $i=1,\ldots,N$, $\beta_i \geq 0$, $\alpha > \beta_i + \frac{1}{q}$, and m_i, n given by (2.70) with $m=\min\{m_i\}$. Let $f \in AC^n[a,b]$ be such that $f^{(i)}(b)=0$ for $i=m,\ldots,n-1$ and let ${}^CD^{\alpha}_{b-}f \in L_q[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b \prod_{i=1}^N |{}^CD^{\beta_i}_{b-}f(t)|\,dt \leq T_1\,(b-x)^{\sum_{i=1}^N(\alpha-\beta_i)+1-\frac{N}{q}} \left(\int_x^b |{}^CD^{\alpha}_{b-}f(t)|^q\,dt\right)^{\frac{N}{q}}, \tag{6.19}$$

where T_1 is given by (6.2).

Theorem 6.11 *Let $p,q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $N \in \mathbb{N}$, $N \geq 2$, $i=1,\ldots,N$, $\alpha > \beta_1 \geq \beta_i + 1$, $\beta_i \geq 0$, and m_i, n given by (2.70) with $m=\min\{m_i\}$. Let $f \in AC^n[a,b]$ be such that $f^{(i)}(b)=0$ for $i=m,\ldots,n-1$ and let ${}^CD^{\alpha}_{b-}f \in L_q[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b \prod_{i=1}^N |{}^CD^{\beta_i}_{b-}f(t)|\,dt \leq F_N\,(b-x)^{\sum_{i=1}^N(\alpha-\beta_i)+1-\frac{N}{q}} \left(\int_x^b |{}^CD^{\alpha}_{b-}f(t)|^q\,dt\right)^{\frac{N}{q}}, \tag{6.20}$$

where F_N is given by (6.4).

Inequality (6.20) is sharp for $\beta_1=\beta_2+1=\cdots=\beta_N+1$, where equality is attained for a function f such that ${}^CD^{\alpha}_{b-}f(t)=(t-x)^{\frac{p(\alpha-\beta_1)}{q}}$, $t \in [x,b]$.

In the following cases, when $p=1$ and $p \in (0,1)$, we use a method from Theorem 6.10.

Proposition 6.4 *Let $N \in \mathbb{N}$, $N \geq 2$, $i=1,\ldots,N$, $\alpha > \beta_i \geq 0$, and m_i, n given by (2.70) with $m=\min\{m_i\}$. Let $f \in AC^n[a,b]$ be such that $f^{(i)}(b)=0$ for $i=m,\ldots,n-1$ and let ${}^CD^{\alpha}_{b-}f \in L_\infty[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b \prod_{i=1}^N |{}^CD^{\beta_i}_{b-}f(t)|\,dt \leq \frac{(b-x)^{\sum_{i=1}^N(\alpha-\beta_i)+1}}{\left[\sum_{i=1}^N(\alpha-\beta_i)+1\right]\prod_{i=1}^N\Gamma(\alpha-\beta_i+1)} \left\|{}^CD^{\alpha}_{b-}f(t)\right\|_\infty^N. \tag{6.21}$$

Theorem 6.12 *Let $p \in (0,1)$, $q < 0$ and $\frac{1}{p}+\frac{1}{q}=1$. Let $N \in \mathbb{N}$, $N \geq 2$, $i=1,\ldots,N$, $\alpha > \beta_i \geq 0$, and m_i, n given by (2.70) with $m=\min\{m_i\}$. Let $f \in AC^n[a,b]$ be such that $f^{(i)}(b)=0$ for $i=m,\ldots,n-1$. Let ${}^CD^{\alpha}_{b-}f \in L_q[a,b]$ be of fixed sign on $[a,b]$, with $1/{}^CD^{\alpha}_{b-}f \in L_q[a,b]$. Then reverse inequality in (6.16) holds.*

The same comparison as in Remark 6.1 holds also for Theorems 6.7, 6.8 and Theorems 6.10, 6.11.

THE CANAVATI FRACTIONAL DERIVATIVES

Theorem 6.13 *Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $N\in\mathbb{N}$, $N\geq 2$, $i=1,\ldots,N$, $\beta_i\geq 0$, $\alpha>\beta_i+\frac{1}{q}$, $n=[\alpha]+1$, $m_i=[\beta_i]+1$ with $m=\min\{m_i\}$. Let $f\in C^{\alpha}_{a+}[a,b]$ be such that $f^{(i)}(a)=0$ for $i=m-1,\ldots,n-2$ and let ${}^{C_1}D^{\alpha}_{a+}f\in L_q[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$\int_a^x\prod_{i=1}^N|{}^{C_1}D^{\beta_i}_{a+}f(t)|\,dt\leq T_1\,(x-a)^{\sum\limits_{i=1}^N(\alpha-\beta_i)+1-\frac{N}{q}}\left(\int_a^x|{}^{C_1}D^{\alpha}_{a+}f(t)|^q\,dt\right)^{\frac{N}{q}}, \tag{6.22}$$

where T_1 is given by (6.2).

Next theorem gives a new estimation of the inequality (6.22).

Theorem 6.14 *Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $N\in\mathbb{N}$, $N\geq 2$, $i=1,\ldots,N$, $\alpha>\beta_1\geq\beta_i+1$, $\beta_i\geq 0$, $n=[\alpha]+1$, $m_i=[\beta_i]+1$ with $m=\min\{m_i\}$. Let $f\in C^{\alpha}_{a+}[a,b]$ be such that $f^{(i)}(a)=0$ for $i=m-1,\ldots,n-2$ and let ${}^{C_1}D^{\alpha}_{a+}f\in L_q[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$\int_a^x\prod_{i=1}^N|{}^{C_1}D^{\beta_i}_{a+}f(t)|\,dt\leq F_N\,(x-a)^{\sum\limits_{i=1}^N(\alpha-\beta_i)+1-\frac{N}{q}}\left(\int_a^x|{}^{C_1}D^{\alpha}_{a+}f(t)|^q\,dt\right)^{\frac{N}{q}}, \tag{6.23}$$

where F_N is given by (2.70).

Inequality (6.23) *is sharp for $\beta_1=\beta_2+1=\cdots=\beta_N+1$, where equality is attained for a function f such that ${}^{C_1}D^{\alpha}_{a+}f(t)=(x-t)^{\frac{p(\alpha-\beta_1)}{q}}$, $t\in[a,x]$.*

In the following cases, when $p=1$ and $p\in(0,1)$, we use a method from Theorem 6.13.

Proposition 6.5 *Let $N\in\mathbb{N}$, $N\geq 2$, $i=1,\ldots,N$, $\alpha>\beta_i\geq 0$, $n=[\alpha]+1$, $m_i=[\beta_i]+1$ with $m=\min\{m_i\}$. Let $f\in C^{\alpha}_{a+}[a,b]$ be such that $f^{(i)}(a)=0$ for $i=m-1,\ldots,n-2$ and let ${}^{C_1}D^{\alpha}_{a+}f\in L_q[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$\int_a^x\prod_{i=1}^N|{}^{C_1}D^{\beta_i}_{a+}f(t)|\,dt\leq\frac{(x-a)^{\sum\limits_{i=1}^N(\alpha-\beta_i)+1}}{\left[\sum\limits_{i=1}^N(\alpha-\beta_i)+1\right]\prod\limits_{i=1}^N\Gamma(\alpha-\beta_i+1)}\left\|{}^{C_1}D^{\alpha}_{a+}f(t)\right\|_\infty^N. \tag{6.24}$$

Theorem 6.15 *Let $p\in(0,1)$, $q<0$ and $\frac{1}{p}+\frac{1}{q}=1$. Let $N\in\mathbb{N}$, $N\geq 2$, $i=1,\ldots,N$, $\alpha>\beta_i\geq 0$, $n=[\alpha]+1$, $m_i=[\beta_i]+1$ with $m=\min\{m_i\}$. Let $f\in C^{\alpha}_{a+}[a,b]$ be such that $f^{(i)}(a)=0$ for $i=m-1,\ldots,n-2$. Let ${}^{C_1}D^{\alpha}_{a+}f\in L_q[a,b]$ be of fixed sign on $[a,b]$, with $1/{}^{C_1}D^{\alpha}_{a+}f\in L_q[a,b]$. Then reverse inequality in* (6.22) *holds.*

Following inequalities include the right-sided Canavati fractional derivatives.

Theorem 6.16 *Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $N\in\mathbb{N}$, $N\geq 2$, $i=1,\ldots,N$, $\beta_i\geq 0$, $\alpha>\beta_i+\frac{1}{q}$, $n=[\alpha]+1$, $m_i=[\beta_i]+1$ with $m=\min\{m_i\}$. Let $f\in C_{b-}^{\alpha}[a,b]$ be such that $f^{(i)}(b)=0$ for $i=m-1,\ldots,n-2$ and let ${}^{C_1}D_{b-}^{\alpha}f\in L_q[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$\int_x^b\prod_{i=1}^N|{}^{C_1}D_{b-}^{\beta_i}f(t)|\,dt\leq T_1\,(b-x)^{\sum\limits_{i=1}^N(\alpha-\beta_i)+1-\frac{N}{q}}\left(\int_x^b|{}^{C_1}D_{b-}^{\alpha}f(t)|^q\,dt\right)^{\frac{N}{q}}, \tag{6.25}$$

where T_1 is given by (6.2).

Theorem 6.17 *Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $N\in\mathbb{N}$, $N\geq 2$, $i=1,\ldots,N$, $\alpha>\beta_1\geq\beta_i+1$, $\beta_i\geq 0$, $n=[\alpha]+1$, $m_i=[\beta_i]+1$ with $m=\min\{m_i\}$. Let $f\in C_{b-}^{\alpha}[a,b]$ be such that $f^{(i)}(b)=0$ for $i=m-1,\ldots,n-2$ and let ${}^{C_1}D_{b-}^{\alpha}f\in L_q[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$\int_x^b\prod_{i=1}^N|{}^{C_1}D_{b-}^{\beta_i}f(t)|\,dt\leq F_N\,(b-x)^{\sum\limits_{i=1}^N(\alpha-\beta_i)+1-\frac{N}{q}}\left(\int_x^b|{}^{C_1}D_{b-}^{\alpha}f(t)|^q\,dt\right)^{\frac{N}{q}}, \tag{6.26}$$

where F_N is given by (6.4).

Inequality (6.26) *is sharp for $\beta_1=\beta_2+1=\cdots=\beta_N+1$, where equality is attained for a function f such that ${}^{C_1}D_{b-}^{\alpha}f(t)=(t-x)^{\frac{p(\alpha-\beta_1)}{q}}$, $t\in[x,b]$.*

In the following cases, when $p=1$ and $p\in(0,1)$, we use a method from Theorem 6.16.

Proposition 6.6 *Let $N\in\mathbb{N}$, $N\geq 2$, $i=1,\ldots,N$, $\alpha>\beta_i\geq 0$, $n=[\alpha]+1$, $m_i=[\beta_i]+1$ with $m=\min\{m_i\}$. Let $f\in C_{b-}^{\alpha}[a,b]$ be such that $f^{(i)}(b)=0$ for $i=m-1,\ldots,n-2$ and let ${}^{C_1}D_{b-}^{\alpha}f\in L_\infty[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$\int_x^b\prod_{i=1}^N|{}^{C_1}D_{b-}^{\beta_i}f(t)|\,dt\leq\frac{(b-x)^{\sum\limits_{i=1}^N(\alpha-\beta_i)+1}}{\left[\sum\limits_{i=1}^N(\alpha-\beta_i)+1\right]\prod\limits_{i=1}^N\Gamma(\alpha-\beta_i+1)}\left\|{}^{C_1}D_{b-}^{\alpha}f(t)\right\|_\infty^N. \tag{6.27}$$

Theorem 6.18 *Let $p\in(0,1)$, $q<0$ and $\frac{1}{p}+\frac{1}{q}=1$. Let $N\in\mathbb{N}$, $N\geq 2$, $i=1,\ldots,N$, $\alpha>\beta_i\geq 0$, $n=[\alpha]+1$, $m_i=[\beta_i]+1$ with $m=\min\{m_i\}$. Let $f\in C_{b-}^{\alpha}[a,b]$ be such that $f^{(i)}(b)=0$ for $i=m-1,\ldots,n-2$. Let ${}^{C_1}D_{b-}^{\alpha}f\in L_q[a,b]$ be of fixed sign on $[a,b]$, with $1/{}^{C_1}D_{b-}^{\alpha}f\in L_q[a,b]$. Then reverse inequality in* (6.22) *holds.*

The same comparison as in Remark 6.1 holds also for Theorems 6.13, 6.14 and Theorems 6.16, Theorem 6.17.

6.2 Extensions of the Agarwal-Pang Opial-type inequality

Next inequality is motivated by the extension of Agarwal-Pang given in (1.25), which for fractional derivatives has a form

$$\begin{aligned}&\int_a^b w_1(t)\prod_{i=1}^N |\mathbf{D}^{\beta_i} f(t)|^{r_i}\,dt\\ &\leq K\left(\int_a^b [w_1(t)]^{\gamma}\,dt\right)^{\frac{1}{\gamma}}\left(\int_a^b [w_2(t)]^{\delta}\,dt\right)^{\varepsilon}\left(\int_a^b w_2(t)\,|\mathbf{D}^{\alpha} f(t)|^q\,dt\right)^{\frac{r}{q}},\end{aligned}$$

where $w_1, w_2 \in C[a,b]$ are positive weight functions, $\alpha > \beta_i \geq 0$, $K > 0$ is a constant, $r = \sum r_i$ and $q, \gamma, \delta, \varepsilon \in \mathbb{R}$.

This inequality is given for the left-sided Riemann-Liouville fractional derivatives in [15], and for the Caputo fractional derivatives in [12]. In the following theorems we use results from Chapter 2.6, and give new conditions under which inequalities hold. An inequality for the left-sided Canavati fractional derivatives (here Theorem 6.27), Andrić-Pečarić-Perić give in [23] where improved the composition identity for the left-sided Canavati fractional derivatives is used.

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Theorem 6.19 *Let $N \in \mathbb{N}$, $N \geq 2$, $\alpha > \beta_i \geq 0$ for $i = 1, \ldots, N$. Suppose that one of conditions $(i) - (vii)$ in Corollary 2.21 holds for all pairs $\{\alpha, \beta_i\}$, $i = 1, \ldots, N$. Let w_1, w_2 be continuous positive weight functions on $[a,x]$. Let $r_i > 0$, $r = \sum_{i=1}^N r_i$, $s_k > 1$ and $\frac{1}{s_k} + \frac{1}{s_k'} = 1$ for $k = 1, 2$. Let $q > s_2$, $\sigma = \frac{1}{s_2} - \frac{1}{q}$, $\rho = \sum_{i=1}^N r_i(\alpha - \beta_i - 1) + r\sigma$ and $\alpha > \beta_i + 1 - \sigma$ for $i = 1, \ldots, N$. Let $D_{a+}^{\alpha} f \in L_q[a,b]$. Then for a.e. $x \in [a,b]$ hold*

$$\begin{aligned}&\int_a^x w_1(t)\prod_{i=1}^N |D_{a+}^{\beta_i} f(t)|^{r_i}\,dt\\ &\leq T_2\,P(x)\int_a^x w_1(t)\,(t-a)^{\rho}\,dt\left(\int_a^x w_2(t)\,|D_{a+}^{\alpha} f(t)|^q\,dt\right)^{\frac{r}{q}} \qquad (6.28)\\ &\leq T_3\,P(x)\,Q(x)\,(x-a)^{\rho+\frac{1}{s_1}}\left(\int_a^x w_2(t)\,|D_{a+}^{\alpha} f(t)|^q\,dt\right)^{\frac{r}{q}}, \qquad (6.29)\end{aligned}$$

where

$$P(x) = \left(\int_a^x [w_2(t)]^{-\frac{s_2'}{q}}\,dt\right)^{\frac{r}{s_2'}}, \qquad (6.30)$$

$$Q(x) = \left(\int_a^x [w_1(t)]^{s_1'}\,dt\right)^{\frac{1}{s_1'}}, \qquad (6.31)$$

$$T_2 = \frac{\sigma^{r\sigma}}{\prod_{i=1}^{N} \left[\Gamma(\alpha-\beta_i)\,(\alpha-\beta_i-1+\sigma)^{\sigma}\right]^{r_i}}, \tag{6.32}$$

$$T_3 = \frac{T_2}{(\rho s_1+1)^{\frac{1}{s_1}}}. \tag{6.33}$$

Proof. We notice that conditions on s_2 and q ensure $\sigma = \frac{1}{s_2} - \frac{1}{q} > 0$. Set $\delta_i = \alpha - \beta_i - 1$, $i = 1, \ldots, N$. Using Theorem 2.13 and the triangle inequality, for $t \in [a,x]$ we have

$$|D^{\beta_i}_{a+} f(t)| \le \frac{1}{\Gamma(\delta_i+1)} \int_a^t (t-\tau)^{\delta_i}\,|D^{\alpha}_{a+} f(\tau)|\,d\tau\,.$$

Since

$$\int_a^t (t-\tau)^{\delta_i}|D^{\alpha}_{a+} f(\tau)|\,d\tau = \int_a^t [w_2(\tau)]^{-\frac{1}{q}}\,[w_2(\tau)]^{\frac{1}{q}}\,(t-\tau)^{\delta_i}|D^{\alpha}_{a+} f(\tau)|\,d\tau\,,$$

applying Hölder's inequality for $\{\frac{q}{q-1}, q\}$ we obtain

$$\le \left(\int_a^t [w_2(\tau)]^{-\frac{1}{q-1}}\,(t-\tau)^{\frac{q}{q-1}\delta_i}\,d\tau\right)^{\frac{q-1}{q}} \left(\int_a^x w_2(\tau)\,|D^{\alpha}_{a+} f(\tau)|^q\,d\tau\right)^{\frac{1}{q}}.$$

Now follows

$$\begin{aligned}
&\int_a^x w_1(t)\prod_{i=1}^{N} |D^{\beta_i}_{a+} f(t)|^{r_i}\,dt \\
&\le \frac{1}{\prod_{i=1}^{N}[\Gamma(\delta_i+1)]^{r_i}} \int_a^x w_1(t)\prod_{i=1}^{N}\left(\int_a^t (t-\tau)^{\delta_i}\,|D^{\alpha}_{a+} f(\tau)|\,d\tau\right)^{r_i} dt \\
&\le \frac{1}{\prod_{i=1}^{N}[\Gamma(\delta_i+1)]^{r_i}} \left(\int_a^x w_2(\tau)\,|D^{\alpha}_{a+} f(\tau)|^q\,d\tau\right)^{\frac{r}{q}} \\
&\qquad \cdot \int_a^x w_1(t)\prod_{i=1}^{N}\left(\int_a^t [w_2(\tau)]^{-\frac{1}{q-1}}\,(t-\tau)^{\frac{q}{q-1}\delta_i}\,d\tau\right)^{\frac{q-1}{q}r_i} dt\,.
\end{aligned}$$

Again by Hölder's inequality for $\frac{s_2'(q-1)}{q}$ and $\frac{s_2'(q-1)}{s_2'(q-1)-q} = \frac{q-1}{q\sigma}$ follow

$$\begin{aligned}
&\int_a^x w_1(t)\prod_{i=1}^{N} |D^{\beta_i}_{a+} f(t)|^{r_i}\,dt \\
&\le \frac{1}{\prod_{i=1}^{N}[\Gamma(\delta_i+1)]^{r_i}} \left(\int_a^x w_2(\tau)\,|D^{\alpha}_{a+} f(\tau)|^q\,d\tau\right)^{\frac{r}{q}} \\
&\qquad \cdot \int_a^x w_1(t)\prod_{i=1}^{N}\left(\int_a^x [w_2(t)]^{-\frac{s_2'}{q}}\,dt\right)^{\frac{r_i}{s_2'}} \left(\int_a^t (t-\tau)^{\frac{\delta_i}{\sigma}}\,d\tau\right)^{\sigma r_i} dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{P(x)\,\sigma^{r\sigma}}{\prod\limits_{i=1}^{N}[\Gamma(\delta_i+1)]^{r_i}\,(\delta_i+\sigma)^{r_i\sigma}} \left(\int_a^x w_2(\tau)\,|D_{a+}^{\alpha}f(\tau)|^q\,d\tau\right)^{\frac{r}{q}} \int_a^x w_1(t)\,(t-a)^{\rho}\,dt \\
&\le T_2\,P(x)\left(\int_a^x w_2(\tau)\,|D_{a+}^{\alpha}f(\tau)|^q\,d\tau\right)^{\frac{r}{q}} \left(\int_a^x [w_1(t)]^{s_1'}\,dt\right)^{\frac{1}{s_1'}} \left(\int_a^x (t-a)^{\rho s_1}\,dt\right)^{\frac{1}{s_1}} \\
&= \frac{T_2\,Q(x)\,P(x)\,(x-a)^{\rho+\frac{1}{s_1}}}{(\rho s_1+1)^{\frac{1}{s_1}}} \left(\int_a^x w_2(\tau)\,|D_{a+}^{\alpha}f(\tau)|^q\,d\tau\right)^{\frac{r}{q}}.
\end{aligned}
$$

□

Next is the case for $q=\infty$.

Proposition 6.7 *Let $N\in\mathbb{N}$, $N\ge 2$, $\alpha>\beta_i\ge 0$ for $i=1,\dots,N$ and let $D_{a+}^{\alpha}f\in L_\infty[a,b]$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.21 holds for all pairs $\{\alpha,\beta_i\}$, $i=1,\dots,N$. Let w be continuous nonnegative weight function on $[a,x]$. Let $r_i>0$, $r=\sum_{i=1}^N r_i$. Then for a.e. $x\in[a,b]$ holds*

$$
\begin{aligned}
&\int_a^x w(t)\prod_{i=1}^{N}|D_{a+}^{\beta_i}f(t)|^{r_i}\,dt \\
&\le \frac{(x-a)^{\sum\limits_{i=1}^{N} r_i(\alpha-\beta_i)+1}}{\left[\sum\limits_{i=1}^{N} r_i(\alpha-\beta_i)+1\right]\prod\limits_{i=1}^{N}[\Gamma(\alpha-\beta_i+1)]^{r_i}}\,\|w\|_\infty\,\|D_{a+}^{\alpha}f\|_\infty^{r}. \qquad (6.34)
\end{aligned}
$$

Proof. Set $\delta_i=\alpha-\beta_i-1$, $i=1,\dots,N$. Using Theorem 2.13, the triangle inequality and Hölder's inequality, for $t\in[a,x]$ we have

$$
\begin{aligned}
|D_{a+}^{\beta_i}f(t)| &\le \frac{1}{\Gamma(\delta_i+1)}\int_a^t (t-\tau)^{\delta_i}|D_{a+}^{\alpha}f(\tau)|\,d\tau \\
&\le \frac{1}{\Gamma(\delta_i+1)}\left(\int_a^t (t-\tau)^{\delta_i}\,d\tau\right)\|D_{a+}^{\alpha}f\|_\infty \\
&= \frac{(t-a)^{\delta_i+1}}{\Gamma(\delta_i+2)}\|D_{a+}^{\alpha}f\|_\infty.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\int_a^x w(t)\prod_{i=1}^{N}|D_{a+}^{\beta_i}f(t)|^{r_i}\,dt \\
&\le \frac{1}{\prod\limits_{i=1}^{N}[\Gamma(\delta_i+2)]^{r_i}}\left(\int_a^x w(t)\,(t-a)^{\sum_{i=1}^N r_i(\delta_i+1)}\,dt\right)\|D_{a+}^{\alpha}f\|_\infty^{r} \\
&\le \frac{1}{\prod\limits_{i=1}^{N}[\Gamma(\delta_i+2)]^{r_i}}\,\|w\|_\infty\left(\int_a^x (t-a)^{\rho-1}\,dt\right)\|D_{a+}^{\alpha}f\|_\infty^{r}
\end{aligned}
$$

$$= \frac{(x-a)^{\rho}}{\rho \prod_{i=1}^{N} [\Gamma(\delta_i+2)]^{r_i}} \|w\|_\infty \|D_{a+}^{\alpha} f\|_\infty^{r}.$$

□

In the following theorem we have negative q and $s_1, s_2 \in (0,1)$ (which again ensure $\sigma = \frac{1}{s_2} - \frac{1}{q} > 0$). Since $q < 0$, then we use reverse Hölder's inequality and we need a condition $1/D_{a+}^{\alpha} f \in L_q[a,b]$.

Theorem 6.20 *Let $N \in \mathbb{N}$, $N \geq 2$, $\alpha > \beta_i \geq 0$ for $i = 1, \ldots, N$. Suppose that one of conditions $(i) - (vii)$ in Corollary 2.21 holds for all pairs $\{\alpha, \beta_i\}$, $i = 1, \ldots, N$. Let w_1, w_2 be continuous positive weight functions on $[a,x]$. Let $r_i > 0$, $r = \sum_{i=1}^{N} r_i$, $s_k \in (0,1)$ and $\frac{1}{s_k} + \frac{1}{s_k'} = 1$ for $k = 1,2$. Let $q < 0$, $\sigma = \frac{1}{s_2} - \frac{1}{q}$, $\rho = \sum_{i=1}^{N} r_i(\alpha - \beta_i - 1) + r\sigma$ and $\alpha > \beta_i + 1 - \sigma$, $i = 1, \ldots, N$. Let $D_{a+}^{\alpha} f \in L_q[a,b]$ be of fixed sign on $[a,b]$, with $1/D_{a+}^{\alpha} f \in L_q[a,b]$. Then reverse inequalities in* (6.28) *and* (6.29) *hold.*

Next inequalities include the right-sided Riemann-Liouville fractional derivatives.

Theorem 6.21 *Let $N \in \mathbb{N}$, $N \geq 2$, $\alpha > \beta_i \geq 0$ for $i = 1, \ldots, N$. Suppose that one of conditions $(i) - (vii)$ in Corollary 2.22 holds for all pairs $\{\alpha, \beta_i\}$, $i = 1, \ldots, N$. Let w_1, w_2 be continuous positive weight functions on $[a,x]$. Let $r_i > 0$, $r = \sum_{i=1}^{N} r_i$, $s_k > 1$ and $\frac{1}{s_k} + \frac{1}{s_k'} = 1$ for $k = 1,2$. Let $q > s_2$, $\sigma = \frac{1}{s_2} - \frac{1}{q}$, $\rho = \sum_{i=1}^{N} r_i(\alpha - \beta_i - 1) + r\sigma$ and $\alpha > \beta_i + 1 - \sigma$ for $i = 1, \ldots, N$. Let $D_{b-}^{\alpha} f \in L_q[a,b]$. Then for a.e. $x \in [a,b]$ hold*

$$\begin{aligned} &\int_x^b w_1(t) \prod_{i=1}^{N} |D_{b-}^{\beta_i} f(t)|^{r_i}\, dt \\ &\leq T_2 \widetilde{P}(x) \int_x^b w_1(t)\, (b-t)^{\rho}\, dt \left(\int_x^b w_2(t)\, |D_{b-}^{\alpha} f(t)|^q\, dt \right)^{\frac{r}{q}} \qquad (6.35) \\ &\leq T_3 \widetilde{P}(x)\, \widetilde{Q}(x)\, (b-x)^{\rho + \frac{1}{s_1}} \left(\int_x^b w_2(t)\, |D_{b-}^{\alpha} f(t)|^q\, dt \right)^{\frac{r}{q}}, \qquad (6.36) \end{aligned}$$

where T_2 and T_3 are given by (6.32) *and* (6.33)*, and*

$$\widetilde{P}(x) = \left(\int_x^b [w_2(t)]^{-\frac{s_2'}{q}}\, dt \right)^{\frac{r}{s_2'}}, \qquad (6.37)$$

$$\widetilde{Q}(x) = \left(\int_x^b [w_1(t)]^{s_1'}\, dt \right)^{\frac{1}{s_1'}}. \qquad (6.38)$$

Following cases are for $q = \infty$ and $q < 0$.

Proposition 6.8 *Let $N \in \mathbb{N}$, $N \geq 2$, $\alpha > \beta_i \geq 0$ for $i = 1, \ldots, N$ and let $D_{b-}^{\alpha} f \in L_\infty[a,b]$. Suppose that one of conditions $(i) - (vii)$ in Corollary 2.22 holds for all pairs $\{\alpha, \beta_i\}$, $i =$*

$1,\dots,N$. *Let w be continuous nonnegative weight function on $[a,x]$. Let $r_i>0$, $r=\sum_{i=1}^N r_i$. Then for a.e. $x\in[a,b]$ holds*

$$\int_x^b w(t)\prod_{i=1}^N |D_{b-}^{\beta_i}f(t)|^{r_i}\,dt$$

$$\leq \frac{(b-x)^{\sum\limits_{i=1}^N r_i(\alpha-\beta_i)+1}}{\left[\sum\limits_{i=1}^N r_i(\alpha-\beta_i)+1\right]\prod\limits_{i=1}^N[\Gamma(\alpha-\beta_i+1)]^{r_i}}\|w\|_\infty\|D_{b-}^\alpha f\|_\infty^r. \tag{6.39}$$

Theorem 6.22 *Let $N\in\mathbb{N}$, $N\geq 2$, $\alpha>\beta_i\geq 0$ for $i=1,\dots,N$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.22 holds for all pairs $\{\alpha,\beta_i\}$, $i=1,\dots,N$. Let w_1, w_2 be continuous positive weight functions on $[a,x]$. Let $r_i>0$, $r=\sum_{i=1}^N r_i$, $s_k\in(0,1)$ and $\frac{1}{s_k}+\frac{1}{s_k'}=1$ for $k=1,2$. Let $q<0$, $\sigma=\frac{1}{s_2}-\frac{1}{q}$, $\rho=\sum_{i=1}^N r_i(\alpha-\beta_i-1)+r\sigma$ and $\alpha>\beta_i+1-\sigma$, $i=1,\dots,N$. Let $D_{b-}^\alpha f\in L_q[a,b]$ be of fixed sign on $[a,b]$, with $1/D_{b-}^\alpha f\in L_q[a,b]$. Then reverse inequalities in (6.35) and (6.36) hold.*

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Theorem 6.23 *Let $N\in\mathbb{N}$, $N\geq 2$, $i=1,\dots,N$, $\alpha>\beta_i\geq 0$, and m_i, n given by (2.70) with $m=\min\{m_i\}$. Let $f\in AC^n[a,b]$ be such that $f^{(i)}(a)=0$ for $i=m,\dots,n-1$. Let w_1, w_2 be continuous positive weight functions on $[a,x]$. Let $r_i>0$, $r=\sum_{i=1}^N r_i$, $s_k>1$ and $\frac{1}{s_k}+\frac{1}{s_k'}=1$ for $k=1,2$. Let $q>s_2$, $\sigma=\frac{1}{s_2}-\frac{1}{q}$, $\rho=\sum_{i=1}^N r_i(\alpha-\beta_i-1)+r\sigma$ and $\alpha>\beta_i+1-\sigma$ for $i=1,\dots,N$. Let ${}^CD_{a+}^\alpha f\in L_q[a,b]$. Then for a.e. $x\in[a,b]$ hold*

$$\int_a^x w_1(t)\prod_{i=1}^N |{}^CD_{a+}^{\beta_i}f(t)|^{r_i}\,dt$$

$$\leq T_2\,P(x)\int_a^x w_1(t)\,(t-a)^\rho\,dt\left(\int_a^x w_2(t)\,|{}^CD_{a+}^\alpha f(t)|^q\,dt\right)^{\frac{r}{q}} \tag{6.40}$$

$$\leq T_3\,P(x)\,Q(x)\,(x-a)^{\rho+\frac{1}{s_1}}\left(\int_a^x w_2(t)\,|{}^CD_{a+}^\alpha f(t)|^q\,dt\right)^{\frac{r}{q}}, \tag{6.41}$$

where P, Q, T_2 and T_3 are given by (6.30), (6.31), (6.32) and (6.33), respectively.

Proposition 6.9 *Let $N\in\mathbb{N}$, $N\geq 2$, $i=1,\dots,N$, $\alpha>\beta_i\geq 0$, and m_i, n given by (2.70) with $m=\min\{m_i\}$. Let $f\in AC^n[a,b]$ be such that $f^{(i)}(a)=0$ for $i=m,\dots,n-1$ and let ${}^CD_{a+}^\alpha f\in L_\infty[a,b]$. Let w be continuous nonnegative weight function on $[a,x]$. Let $r_i>0$, $r=\sum_{i=1}^N r_i$. Then for a.e. $x\in[a,b]$ holds*

$$\int_a^x w(t)\prod_{i=1}^N |{}^CD_{a+}^{\beta_i}f(t)|^{r_i}\,dt$$

$$\leq \frac{(x-a)^{\sum\limits_{i=1}^{N} r_i(\alpha-\beta_i)+1}}{\left[\sum\limits_{i=1}^{N} r_i(\alpha-\beta_i)+1\right] \prod\limits_{i=1}^{N} [\Gamma(\alpha-\beta_i+1)]^{r_i}} \|w\|_\infty \, \|{}^C D^\alpha_{a+} f\|^r_\infty. \tag{6.42}$$

Theorem 6.24 *Let $N \in \mathbb{N}$, $N \geq 2$, $i = 1, \ldots, N$, $\alpha > \beta_i \geq 0$, and m_i, n given by (2.70) with $m = \min\{m_i\}$. Let $f \in AC^n[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m, \ldots, n-1$. Let w_1, w_2 be continuous positive weight functions on $[a,x]$. Let $r_i > 0$, $r = \sum_{i=1}^N r_i$, $s_k \in (0,1)$ and $\frac{1}{s_k} + \frac{1}{s'_k} = 1$ for $k = 1,2$. Let $q < 0$, $\sigma = \frac{1}{s_2} - \frac{1}{q}$, $\rho = \sum_{i=1}^N r_i(\alpha - \beta_i - 1) + r\sigma$ and $\alpha > \beta_i + 1 - \sigma$, $i = 1, \ldots, N$. Let ${}^C D^\alpha_{a+} f \in L_q[a,b]$ be of fixed sign on $[a,b]$, with $1/{}^C D^\alpha_{a+} f \in L_q[a,b]$. Then reverse inequalities in (6.40) and (6.41) hold.*

Following inequalities include the right-sided Caputo fractional derivatives.

Theorem 6.25 *Let $N \in \mathbb{N}$, $N \geq 2$, $i = 1, \ldots, N$, $\alpha > \beta_i \geq 0$, and m_i, n given by (2.70) with $m = \min\{m_i\}$. Let $f \in AC^n[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m, \ldots, n-1$. Let w_1, w_2 be continuous positive weight functions on $[a,x]$. Let $r_i > 0$, $r = \sum_{i=1}^N r_i$, $s_k > 1$ and $\frac{1}{s_k} + \frac{1}{s'_k} = 1$ for $k = 1,2$. Let $q > s_2$, $\sigma = \frac{1}{s_2} - \frac{1}{q}$, $\rho = \sum_{i=1}^N r_i(\alpha - \beta_i - 1) + r\sigma$ and $\alpha > \beta_i + 1 - \sigma$ for $i = 1, \ldots, N$. Let ${}^C D^\alpha_{b-} f \in L_q[a,b]$. Then for a.e. $x \in [a,b]$ hold*

$$\begin{aligned}
&\int_x^b w_1(t) \prod_{i=1}^N |{}^C D^{\beta_i}_{b-} f(t)|^{r_i}\, dt \\
&\leq T_2 \widetilde{P}(x) \int_x^b w_1(t)\,(b-t)^\rho\, dt \left(\int_x^b w_2(t)\, |{}^C D^\alpha_{b-} f(t)|^q\, dt \right)^{\frac{r}{q}} \qquad (6.43)\\
&\leq T_3 \widetilde{P}(x)\, \widetilde{Q}(x)\, (b-x)^{\rho+\frac{1}{s_1}} \left(\int_x^b w_2(t)\, |{}^C D^\alpha_{b-} f(t)|^q\, dt \right)^{\frac{r}{q}}, \qquad (6.44)
\end{aligned}$$

where $\widetilde{P}$, $\widetilde{Q}$, T_2 and T_3 are given by (6.37), (6.38), (6.32) and (6.33), respectively.

Proposition 6.10 *Let $N \in \mathbb{N}$, $N \geq 2$, $i = 1, \ldots, N$, $\alpha > \beta_i \geq 0$, and m_i, n given by (2.70) with $m = \min\{m_i\}$. Let $f \in AC^n[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m, \ldots, n-1$ and let ${}^C D^\alpha_{b-} f \in L_\infty[a,b]$. Let w be continuous nonnegative weight function on $[a,x]$. Let $r_i > 0$, $r = \sum_{i=1}^N r_i$. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b w(t) \prod_{i=1}^N |{}^C D^{\beta_i}_{b-} f(t)|^{r_i}\, dt$$

$$\leq \frac{(b-x)^{\sum\limits_{i=1}^{N} r_i(\alpha-\beta_i)+1}}{\left[\sum\limits_{i=1}^{N} r_i(\alpha-\beta_i)+1\right] \prod\limits_{i=1}^{N} [\Gamma(\alpha-\beta_i+1)]^{r_i}} \|w\|_\infty \, \|{}^C D^\alpha_{b-} f\|^r_\infty. \tag{6.45}$$

Theorem 6.26 *Let $N \in \mathbb{N}$, $N \geq 2$, $i = 1, \ldots, N$, $\alpha > \beta_i \geq 0$, and m_i, n given by (2.70) with $m = \min\{m_i\}$. Let $f \in AC^n[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m, \ldots, n-1$. Let*

w_1, w_2 be continuous positive weight functions on $[a,x]$. Let $r_i > 0$, $r = \sum_{i=1}^N r_i$, $s_k \in (0,1)$ and $\frac{1}{s_k} + \frac{1}{s_k'} = 1$ for $k = 1,2$. Let $q < 0$, $\sigma = \frac{1}{s_2} - \frac{1}{q}$, $\rho = \sum_{i=1}^N r_i(\alpha - \beta_i - 1) + r\sigma$ and $\alpha > \beta_i + 1 - \sigma$, $i = 1,\dots,N$. Let ${}^C D_{b-}^{\alpha} f \in L_q[a,b]$ be of fixed sign on $[a,b]$, with $1/{}^C D_{b-}^{\alpha} f \in L_q[a,b]$. Then reverse inequalities in (6.43) and (6.44) hold.

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Theorem 6.27 *Let $N \in \mathbb{N}$, $N \geq 2$, $\alpha > \beta_i \geq 0$, $m_i = [\beta_i] + 1$, $m = \min\{m_i\}$ for $i = 1,\dots,N$ and $n = [\alpha] + 1$. Let $f \in C_{a+}^{\alpha}[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m-1,\dots,n-2$. Let w_1, w_2 be continuous positive weight functions on $[a,x]$. Let $r_i > 0$, $r = \sum_{i=1}^N r_i$, $s_k > 1$ and $\frac{1}{s_k} + \frac{1}{s_k'} = 1$ for $k = 1,2$. Let $q > s_2$, $\sigma = \frac{1}{s_2} - \frac{1}{q}$, $\rho = \sum_{i=1}^N r_i(\alpha - \beta_i - 1) + r\sigma$ and $\alpha > \beta_i + 1 - \sigma$ for $i = 1,\dots,N$. Let ${}^{C_1}D_{a+}^{\alpha} f \in L_q[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$
\begin{aligned}
&\int_a^x w_1(t) \prod_{i=1}^N |{}^{C_1}D_{a+}^{\beta_i} f(t)|^{r_i}\, dt \\
&\leq T_2 P(x) \int_a^x w_1(t)\, (t-a)^{\rho}\, dt \left(\int_a^x w_2(t)\, |{}^{C_1}D_{a+}^{\alpha} f(t)|^q\, dt \right)^{\frac{r}{q}} \qquad (6.46) \\
&\leq T_3 P(x)\, Q(x)\, (x-a)^{\rho + \frac{1}{s_1}} \left(\int_a^x w_2(t)\, |{}^{C_1}D_{a+}^{\alpha} f(t)|^q\, dt \right)^{\frac{r}{q}}, \qquad (6.47)
\end{aligned}
$$

where P, Q, T_2 and T_3 are given by (6.30), (6.31), (6.32) and (6.33), respectively.

Proposition 6.11 *Let $N \in \mathbb{N}$, $N \geq 2$, $\alpha > \beta_i \geq 0$, $m_i = [\beta_i] + 1$, $m = \min\{m_i\}$ for $i = 1,\dots,N$ and $n = [\alpha] + 1$. Let $f \in C_{a+}^{\alpha}[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m-1,\dots,n-2$ and let ${}^{C_1}D_{a+}^{\alpha} f \in L_\infty[a,b]$. Let w be continuous nonnegative weight function on $[a,x]$. Let $r_i > 0$, $r = \sum_{i=1}^N r_i$. Then for a.e. $x \in [a,b]$ holds*

$$
\begin{aligned}
&\int_a^x w(t) \prod_{i=1}^N |{}^{C_1}D_{a+}^{\beta_i} f(t)|^{r_i}\, dt \\
&\leq \frac{(x-a)^{\sum\limits_{i=1}^N r_i(\alpha-\beta_i)+1}}{\left[\sum\limits_{i=1}^N r_i(\alpha - \beta_i) + 1\right] \prod\limits_{i=1}^N [\Gamma(\alpha - \beta_i + 1)]^{r_i}} \|w\|_\infty \|{}^{C_1}D_{a+}^{\alpha} f\|_\infty^r. \qquad (6.48)
\end{aligned}
$$

Theorem 6.28 *Let $N \in \mathbb{N}$, $N \geq 2$, $\alpha > \beta_i \geq 0$, $m_i = [\beta_i] + 1$, $m = \min\{m_i\}$ for $i = 1,\dots,N$ and $n = [\alpha] + 1$. Let $f \in C_{a+}^{\alpha}[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m-1,\dots,n-2$. Let w_1, w_2 be continuous positive weight functions on $[a,x]$. Let $r_i > 0$, $r = \sum_{i=1}^N r_i$, $s_k \in (0,1)$ and $\frac{1}{s_k} + \frac{1}{s_k'} = 1$ for $k = 1,2$. Let $q < 0$, $\sigma = \frac{1}{s_2} - \frac{1}{q}$, $\rho = \sum_{i=1}^N r_i(\alpha - \beta_i - 1) + r\sigma$ and $\alpha > \beta_i + 1 - \sigma$, $i = 1,\dots,N$. Let ${}^{C_1}D_{a+}^{\alpha} f \in L_q[a,b]$ be of fixed sign on $[a,b]$, with $1/{}^{C_1}D_{a+}^{\alpha} f \in L_q[a,b]$. Then reverse inequalities in (6.46) and (6.47) hold.*

Following inequalities include the right-sided Canavati fractional derivatives.

Theorem 6.29 *Let $N \in \mathbb{N}$, $N \geq 2$, $\alpha > \beta_i \geq 0$, $m_i = [\beta_i] + 1$, $m = \min\{m_i\}$ for $i = 1, \dots, N$ and $n = [\alpha] + 1$. Let $f \in C^{\alpha}_{b-}[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m-1, \dots, n-2$. Let w_1, w_2 be continuous positive weight functions on $[a,x]$. Let $r_i > 0$, $r = \sum_{i=1}^N r_i$, $s_k > 1$ and $\frac{1}{s_k} + \frac{1}{s'_k} = 1$ for $k = 1,2$. Let $q > s_2$, $\sigma = \frac{1}{s_2} - \frac{1}{q}$, $\rho = \sum_{i=1}^N r_i(\alpha - \beta_i - 1) + r\sigma$ and $\alpha > \beta_i + 1 - \sigma$ for $i = 1, \dots, N$. Let ${}^{C_1}D^{\alpha}_{b-} f \in L_q[a,b]$. Then for a.e. $x \in [a,b]$ hold*

$$
\begin{aligned}
&\int_x^b w_1(t) \prod_{i=1}^N |{}^{C_1}D^{\beta_i}_{b-} f(t)|^{r_i}\, dt \\
&\leq T_2\, \widetilde{P}(x) \int_x^b w_1(t)\, (b-t)^{\rho}\, dt \left(\int_x^b w_2(t)\, |{}^{C_1}D^{\alpha}_{b-} f(t)|^q\, dt \right)^{\frac{r}{q}} \qquad (6.49)\\
&\leq T_3\, \widetilde{P}(x)\, \widetilde{Q}(x)\, (b-x)^{\rho + \frac{1}{s_1}} \left(\int_x^b w_2(t)\, |{}^{C_1}D^{\alpha}_{b-} f(t)|^q\, dt \right)^{\frac{r}{q}}, \qquad (6.50)
\end{aligned}
$$

where $\widetilde{P}$, $\widetilde{Q}$, T_2 and T_3 are given by (6.37), (6.38), (6.32) *and* (6.33), *respectively.*

Proposition 6.12 *Let $N \in \mathbb{N}$, $N \geq 2$, $\alpha > \beta_i \geq 0$, $m_i = [\beta_i] + 1$, $m = \min\{m_i\}$ for $i = 1, \dots, N$ and $n = [\alpha] + 1$. Let $f \in C^{\alpha}_{b-}[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m-1, \dots, n-2$ and let ${}^{C_1}D^{\alpha}_{b-} f \in L_{\infty}[a,b]$. Let w be continuous nonnegative weight function on $[a,x]$. Let $r_i > 0$, $r = \sum_{i=1}^N r_i$. Then for a.e. $x \in [a,b]$ holds*

$$
\begin{aligned}
&\int_x^b w(t) \prod_{i=1}^N |{}^{C_1}D^{\beta_i}_{b-} f(t)|^{r_i}\, dt \\
&\leq \frac{(x-a)^{\sum\limits_{i=1}^N r_i(\alpha-\beta_i)+1}}{\left[\sum\limits_{i=1}^N r_i(\alpha-\beta_i)+1\right] \prod\limits_{i=1}^N [\Gamma(\alpha-\beta_i+1)]^{r_i}} \|w\|_{\infty}\, \|{}^{C_1}D^{\alpha}_{b-} f\|^r_{\infty}. \qquad (6.51)
\end{aligned}
$$

Theorem 6.30 *Let $N \in \mathbb{N}$, $N \geq 2$, $\alpha > \beta_i \geq 0$, $m_i = [\beta_i] + 1$, $m = \min\{m_i\}$ for $i = 1, \dots, N$ and $n = [\alpha] + 1$. Let $f \in C^{\alpha}_{b-}[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m-1, \dots, n-2$. Let w_1, w_2 be continuous positive weight functions on $[a,x]$. Let $r_i > 0$, $r = \sum_{i=1}^N r_i$, $s_k \in (0,1)$ and $\frac{1}{s_k} + \frac{1}{s'_k} = 1$ for $k = 1,2$. Let $q < 0$, $\sigma = \frac{1}{s_2} - \frac{1}{q}$, $\rho = \sum_{i=1}^N r_i(\alpha - \beta_i - 1) + r\sigma$ and $\alpha > \beta_i + 1 - \sigma$, $i = 1, \dots, N$. Let ${}^{C_1}D^{\alpha}_{b-} f \in L_q[a,b]$ be of fixed sign on $[a,b]$, with $1/{}^{C_1}D^{\alpha}_{b-} f \in L_q[a,b]$. Then reverse inequalities in* (6.49) *and* (6.50) *hold.*

6.3 Extensions of the Alzer Opial-type inequalities

We observe two Alzer's inequalities given in Theorem 1.16 and Theorem 1.17. First inequality (1.26) applied on fractional derivatives has a form

$$\int_a^b w(t)\left(\prod_{i=1}^N |\mathbf{D}^{\beta_i} f(t)|^{r_i}\right)^p |\mathbf{D}^{\alpha} f(t)|^q\,dt \le K\left(\int_a^b w(t)\,|\mathbf{D}^{\alpha} f(t)|^{p+q}\,dt\right)^{\frac{pr+q}{p+q}},$$

where $w \in C[a,b]$ is positive weight function, $\alpha > \beta_i \ge 0$, $K > 0$ is a constant, $r = \sum r_i$, and $p,q \in \mathbb{R}$.

This inequality is given only for the ordinary derivatives, so in [24] Andrić-Pečarić-Perić give its fractional version involving the left-sided Canavati fractional derivatives and monotonous weight function (here Theorem 6.47) under new and relaxed condition (using improved composition identity for the Canavati fractional derivatives), as well as non-weighted version (Theorem 6.50), two-weighted version (Theorem 6.67) and version including bounded weight function (Theorem 6.48). We give analogous results for the Riemann-Liouville and the Caputo fractional derivatives.

We start with one-weighted inequalities.

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Theorem 6.31 *Let $N \in \mathbb{N}$, $\alpha > \beta_i \ge 0$ for $i = 1,\dots,N$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.21 holds for all pairs $\{\alpha,\beta_i\}$, $i = 1,\dots,N$. Let w be continuous positive decreasing weight function on $[a,x]$. Let $r_i \ge 0$, $r = \sum_{i=1}^N r_i \ge 1$. Let $p > 0$, $q \ge 0$, $\sigma = \frac{1}{p+q} < 1$, $\rho = \sum_{i=1}^N r_i(\alpha-\beta_i) - r\sigma$ and $\alpha > \beta_i + \sigma$ for $i = 1,\dots,N$. Let $D_{a+}^{\alpha} f \in L_{p+q}[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\begin{aligned}&\int_a^x w(t)\prod_{i=1}^N |D_{a+}^{\beta_i} f(t)|^{r_i p}\,|D_{a+}^{\alpha} f(t)|^q\,dt\\ &\le T_4\,(x-a)^{(\rho+\sigma)p}\,[w(x)]^{\frac{p(1-r)}{p+q}}\left(\int_a^x w(t)\,|D_{a+}^{\alpha} f(t)|^{p+q}dt\right)^{\frac{pr+q}{p+q}},\end{aligned} \tag{6.52}$$

where

$$T_4 = \frac{\sigma^{\sigma p}\,q^{\sigma q}\,(1-\sigma)^{(1-\sigma)rp}}{(\rho+\sigma)^{\sigma p}\,(rp+q)^{\sigma q}\prod_{i=1}^N [\Gamma(\alpha-\beta_i)\,(\alpha-\beta_i-\sigma)^{1-\sigma}]^{r_i p}}. \tag{6.53}$$

Proof. Let $q \ne 0$, $\delta_i = \alpha-\beta_i-1$, $i = 1,\dots,N$. Since w is decreasing, then for $\tau \le t$ holds

$$1 \le \left[\frac{w(\tau)}{w(t)}\right]^{\sigma}. \tag{6.54}$$

Using Theorem 2.13, the triangle inequality and (6.54), for $t \in [a,x]$ we have

$$\begin{aligned}
&\prod_{i=1}^{N} |D_{a+}^{\beta_i} f(t)|^{r_i p} \\
&\le \frac{1}{\prod_{i=1}^{N} [\Gamma(\delta_i+1)]^{r_i p}} \prod_{i=1}^{N} \left[\int_a^t (t-\tau)^{\delta_i} |D_{a+}^{\alpha} f(\tau)|\, d\tau \right]^{r_i p} \\
&\le \frac{1}{\prod_{i=1}^{N} [\Gamma(\delta_i+1)]^{r_i p}} [w(t)]^{-\sigma r p} \prod_{i=1}^{N} \left[\int_a^t (t-\tau)^{\delta_i} [w(\tau)]^{\sigma} |D_{a+}^{\alpha} f(\tau)|\, d\tau \right]^{r_i p}.
\end{aligned}$$

By Hölder's inequality for $\{\frac{1}{1-\sigma}, \frac{1}{\sigma}\}$ follows

$$\begin{aligned}
&\int_a^t (t-\tau)^{\delta_i} [w(\tau)]^{\sigma} |D_{a+}^{\alpha} f(\tau)|\, d\tau \\
&\le \left(\int_a^t (t-\tau)^{\frac{\delta_i}{1-\sigma}}\, d\tau \right)^{1-\sigma} \left(\int_a^t w(\tau) |D_{a+}^{\alpha} f(\tau)|^{\frac{1}{\sigma}}\, d\tau \right)^{\sigma} \\
&= \left(\frac{1-\sigma}{\delta_i+1-\sigma} \right)^{1-\sigma} (t-a)^{\delta_i+1-\sigma} \left(\int_a^t w(\tau) |D_{a+}^{\alpha} f(\tau)|^{\frac{1}{\sigma}}\, d\tau \right)^{\sigma},
\end{aligned}$$

where $\delta_i + 1 - \sigma > 0$, that is $\alpha > \beta_i + \sigma$, $i = 1, \ldots, N$. Therefore

$$\begin{aligned}
&\int_a^x w(t) \prod_{i=1}^{N} |D_{a+}^{\beta_i} f(t)|^{r_i p} |D_{a+}^{\alpha} f(t)|^q\, dt \\
&\le \frac{1}{\prod_{i=1}^{N} [\Gamma(\delta_i+1)]^{r_i p}} \int_a^x [w(t)]^{1-\sigma r p} |D_{a+}^{\alpha} f(t)|^q \\
&\cdot \prod_{i=1}^{N} \left[\left(\frac{1-\sigma}{\delta_i+1-\sigma} \right)^{1-\sigma} (t-a)^{\delta_i+1-\sigma} \left(\int_a^t w(\tau) |D_{a+}^{\alpha} f(\tau)|^{\frac{1}{\sigma}}\, d\tau \right)^{\sigma} \right]^{r_i p} dt \\
&= \prod_{i=1}^{N} \left[\left(\frac{1-\sigma}{\delta_i+1-\sigma} \right)^{1-\sigma} \frac{1}{\Gamma(\delta_i+1)} \right]^{r_i p} \qquad (6.55) \\
&\cdot \int_a^x [w(t)]^{1-\sigma r p} |D_{a+}^{\alpha} f(t)|^q (t-a)^{\sum_{i=1}^{N} (\delta_i+1-\sigma) r_i p} \left(\int_a^t w(\tau) |D_{a+}^{\alpha} f(\tau)|^{\frac{1}{\sigma}}\, d\tau \right)^{\sigma r p} dt. \qquad (6.56)
\end{aligned}$$

Applying Hölder's inequality for $\{\frac{1}{\sigma p}, \frac{1}{\sigma q}\}$ with $\rho p = \sum_{i=1}^{N} (\delta_i + 1 - \sigma) r_i p$, we obtain

$$\begin{aligned}
&\int_a^x [w(t)]^{1-\sigma r p} |D_{a+}^{\alpha} f(t)|^q (t-a)^{\rho p} \left(\int_a^t w(\tau) |D_{a+}^{\alpha} f(\tau)|^{\frac{1}{\sigma}}\, d\tau \right)^{\sigma r p} dt \\
&\le \left(\int_a^x (t-a)^{\frac{\rho}{\sigma}}\, dt \right)^{\sigma p}
\end{aligned}$$

$$
\begin{aligned}
&\cdot\left(\int_a^x [w(t)]^{\frac{1-\sigma rp-\sigma q}{\sigma q}}\, w(t)\,|D^{\alpha}_{a+}f(t)|^{\frac{1}{\sigma}}\left(\int_a^t w(\tau)\,|D^{\alpha}_{a+}f(\tau)|^{\frac{1}{\sigma}}\,d\tau\right)^{\frac{rp}{q}} dt\right)^{\sigma q}\\
&\le (x-a)^{(\rho+\sigma)p}\left(\frac{\sigma}{\rho+\sigma}\right)^{\sigma p}[w(x)]^{\frac{p(1-r)}{p+q}}\left(\frac{q}{rp+q}\right)^{\sigma q}\left(\int_a^x w(t)|D^{\alpha}_{a+}f(t)|^{\frac{1}{\sigma}}\,dt\right)^{\sigma(rp+q)}.
\end{aligned}
\tag{6.57}
$$

In the last step we use

$$
\int_a^x G'(t)G^{\gamma}(t)\,dt = \frac{1}{\gamma+1}\int_a^x d\,G^{\gamma+1}(t) = \frac{1}{\gamma+1}[G^{\gamma+1}(x)-G^{\gamma+1}(a)],
$$

where $G(t)=\int_a^t w(\tau)\,|D^{\alpha}_{a+}f(\tau)|^{\frac{1}{\sigma}}\,d\tau$ and $\gamma=\frac{rp}{q}$, which gives us

$$
\int_a^x G'(t)G^{\gamma}(t)\,dt = \frac{q}{rp+q}\left(\int_a^x w(\tau)\,|D^{\alpha}_{a+}f(\tau)|^{\frac{1}{\sigma}}\,d\tau\right)^{\frac{rp+q}{q}}.
$$

Inequality (6.52) now follows from (6.55) and (6.57).
If $q=0$ (and $\sigma=\frac{1}{p}<1$), then the proof after (6.55) and (6.56) simplifies, that is

$$
\begin{aligned}
&\int_a^x w(t)\prod_{i=1}^N |D^{\beta_i}_{a+}f(t)|^{r_i p}\,dt\\
&\le \prod_{i=1}^N\left[\left(\frac{1-\sigma}{\delta_i+1-\sigma}\right)^{1-\sigma}\frac{1}{\Gamma(\delta_i+1)}\right]^{r_i p}\\
&\quad\cdot\int_a^x [w(t)]^{1-r}\,(t-a)^{\sum_{i=1}^N(\delta_i+1)r_i p-r}\left(\int_a^t w(\tau)\,|D^{\alpha}_{a+}f(\tau)|^p\,d\tau\right)^r dt\\
&\le \frac{(1-\sigma)^{(1-\sigma)rp}}{\prod_{i=1}^N\left[\Gamma(\delta_i+1)\,(\delta_i+1-\sigma)^{1-\sigma}\right]^{r_i p}}\,[w(x)]^{1-r}\\
&\qquad\cdot\left(\int_a^x w(\tau)\,|D^{\alpha}_{a+}f(\tau)|^p\,d\tau\right)^r\int_a^x (t-a)^{\sum_{i=1}^N(\delta_i+1)r_i p-r}\,dt\\
&= \frac{(p-1)^{(p-1)r}}{\prod_{i=1}^N\left[\Gamma(\delta_i+1)\,(p(\delta_i+1)-1)^{\frac{p-1}{p}}\right]^{r_i p}}\,[w(x)]^{1-r}\\
&\qquad\cdot\left(\int_a^x w(\tau)\,|D^{\alpha}_{a+}f(\tau)|^p\,d\tau\right)^r\frac{(x-a)^{\sum_{i=1}^N(\delta_i+1)r_i p-r+1}}{\sum_{i=1}^N(\delta_i+1)r_i p-r+1}.
\end{aligned}
$$

□

If $r=1$, then we have Alzer's inequality (1.26) for the left-sided Riemann-Liouville fractional derivatives.

Corollary 6.1 *Suppose that assumptions of Theorem 6.31 hold and let* $r = 1$. *Then*

$$\begin{aligned} &\int_a^x w(t) \prod_{i=1}^N |D_{a+}^{\beta_i} f(t)|^{r_i p} \, |D_{a+}^{\alpha} f(t)|^q \, dt \\ &\leq \widetilde{T}_4 \, (x-a)^{\sum\limits_{i=1}^N r_i(\alpha-\beta_i)p} \int_a^x w(t) \, |D_{a+}^{\alpha} f(t)|^{p+q} dt \,, \end{aligned} \tag{6.58}$$

where

$$\widetilde{T}_4 = \sigma \, q^{\sigma q} \left[\sum_{i=1}^N r_i(\alpha-\beta_i) \right]^{-\sigma p} \prod_{i=1}^N \left[\left(\frac{1-\sigma}{\alpha-\beta_i-\sigma} \right)^{1-\sigma} \frac{1}{\Gamma(\alpha-\beta_i)} \right]^{r_i p} . \tag{6.59}$$

For the next theorem we suppose that weight function is bounded.

Theorem 6.32 *Let* $N \in \mathbb{N}$, $\alpha > \beta_i \geq 0$ *for* $i = 1, \ldots, N$. *Suppose that one of conditions* $(i) - (vii)$ *in Corollary 2.21 holds for all pairs* $\{\alpha, \beta_i\}$, $i = 1, \ldots, N$. *Let* w *be continuous positive weight function on* $[a,x]$ *such that* $A \leq w(t) \leq B$ *for* $t \in [a,x]$. *Let* $r_i \geq 0$, $r = \sum_{i=1}^N r_i \geq 1$. *Let* $p > 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 1$, $\rho = \sum_{i=1}^N r_i(\alpha-\beta_i) - r\sigma$ *and* $\alpha > \beta_i + \sigma$ *for* $i = 1, \ldots, N$. *Let* $D_{a+}^{\alpha} f \in L_{p+q}[a,b]$. *Then for a.e.* $x \in [a,b]$ *holds*

$$\begin{aligned} &\int_a^x w(t) \prod_{i=1}^N |D_{a+}^{\beta_i} f(t)|^{r_i p} \, |D_{a+}^{\alpha} f(t)|^q \, dt \\ &\leq T_4 \, (x-a)^{(\rho+\sigma)p} \left(\frac{B}{A^r} \right)^{\sigma p} \left(\int_a^x w(t) \, |D_{a+}^{\alpha} f(t)|^{p+q} dt \right)^{\frac{pr+q}{p+q}} , \end{aligned} \tag{6.60}$$

where T_4 *is given by* (6.53).

Proof. The proof of (6.60) is the same as the one for (6.52), except two changes. Instead of inequality (6.54) we use $1 \leq (w(\tau)/A)^{\sigma}$. Moreover, in (6.56) we apply the inequality $w(t) = [w(t)]^{\sigma p} [w(t)]^{\sigma q} \leq B^{\sigma p} [w(t)]^{\sigma q}$. These two changes lead to the inequality (6.60). □

In the following extreme case we don't assume that w is decreasing, and for r we have condition $r > 0$.

Proposition 6.13 *Let* $N \in \mathbb{N}$, $\alpha > \beta_i \geq 0$ *for* $i = 1, \ldots, N$ *and let* $D_{a+}^{\alpha} f \in L_{\infty}[a,b]$. *Suppose that one of conditions* $(i) - (vii)$ *in Corollary 2.21 holds for all pairs* $\{\alpha, \beta_i\}$, $i = 1, \ldots, N$. *Let* w *be continuous nonnegative weight function on* $[a,x]$. *Let* $p > 0$, $q \geq 0$, $r_i \geq 0$, $r = \sum_{i=1}^N r_i > 0$. *Then for a.e.* $x \in [a,b]$ *holds*

$$\begin{aligned} &\int_a^x w(t) \prod_{i=1}^N |D_{a+}^{\beta_i} f(t)|^{r_i p} \, |D_{a+}^{\alpha} f(t)|^q \, dt \\ &\leq \frac{(x-a)^{\sum\limits_{i=1}^N r_i(\alpha-\beta_i)p+1}}{\left[\sum\limits_{i=1}^N r_i(\alpha-\beta_i)p + 1 \right] \prod\limits_{i=1}^N [\Gamma(\alpha-\beta_i+1)]^{r_i p}} \|w\|_{\infty} \, \|D_{a+}^{\alpha} f\|_{\infty}^{rp+q} . \end{aligned} \tag{6.61}$$

Proof. Set $\delta_i = \alpha - \beta_i - 1$, $i = 1,\dots,N$. Using Theorem 2.13 and the triangle inequality, for $t \in [a,x]$ we have

$$\begin{aligned}
\prod_{i=1}^{N} |D_{a+}^{\beta_i} f(t)|^{r_i p} &\leq \frac{1}{\prod_{i=1}^{N} [\Gamma(\delta_i+1)]^{r_i p}} \prod_{i=1}^{N} \left[\int_a^t (t-\tau)^{\delta_i} |D_{a+}^{\alpha} f(\tau)|\, d\tau \right]^{r_i p} \\
&\leq \frac{1}{\prod_{i=1}^{N} [\Gamma(\delta_i+1)]^{r_i p}} \prod_{i=1}^{N} \left[\left(\int_a^t (t-\tau)^{\delta_i}\, d\tau \right) \|D_{a+}^{\alpha} f\|_\infty \right]^{r_i p} \\
&= \frac{(t-a)^{\sum_{i=1}^{N} r_i(\delta_i+1)p}}{\prod_{i=1}^{N} [\Gamma(\delta_i+2)]^{r_i p}} \|D_{a+}^{\alpha} f\|_\infty^{rp}
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_a^x w(t) \prod_{i=1}^{N} |D_{a+}^{\beta_i} f(t)|^{r_i p}\, |D_{a+}^{\alpha} f(t)|^q\, dt \\
&\leq \frac{1}{\prod\limits_{i=1}^{N} [\Gamma(\delta_i+2)]^{r_i p}} \|D_{a+}^{\alpha} f\|_\infty^{rp} \int_a^x w(t)\, (t-a)^{\sum_{i=1}^{N} r_i(\delta_i+1)p}\, |D_{a+}^{\alpha} f(t)|^q\, dt \\
&\leq \frac{1}{\prod\limits_{i=1}^{N} [\Gamma(\delta_i+2)]^{r_i p}} \|D_{a+}^{\alpha} f\|_\infty^{rp+q} \left(\int_a^x w(t)\, (t-a)^{\sum_{i=1}^{N} r_i(\delta_i+1)p}\, dt \right) \\
&\leq \frac{1}{\prod\limits_{i=1}^{N} [\Gamma(\delta_i+2)]^{r_i p}} \|D_{a+}^{\alpha} f\|_\infty^{rp+q} \|w\|_\infty \left(\int_a^x (t-a)^{\sum_{i=1}^{N} r_i(\delta_i+1)p}\, dt \right) \\
&= \frac{1}{\prod\limits_{i=1}^{N} [\Gamma(\delta_i+2)]^{r_i p}} \|D_{a+}^{\alpha} f\|_\infty^{rp+q} \|w\|_\infty \frac{(x-a)^{\sum_{i=1}^{N} r_i(\delta_i+1)p+1}}{\sum\limits_{i=1}^{N} r_i(\delta_i+1)p+1}.
\end{aligned}$$

□

In the next case we have $p+q<0$. Since $\sigma<0$, in (6.54) we have reverse inequality. Further, conditions on p and q allow us to use reverse Hölder's inequalities, for $\{\frac{1}{1-\sigma} \in (0,1), \frac{1}{\sigma} < 0\}$ and $\{\frac{1}{\sigma p} \in (0,1), \frac{1}{\sigma q} < 0\}$. The proof is similar to the proof of Theorem 6.31.

Theorem 6.33 *Let $N \in \mathbb{N}$, $\alpha > \beta_i \geq 0$ for $i = 1,\dots,N$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.21 holds for all pairs $\{\alpha,\beta_i\}$, $i = 1,\dots,N$. Let w be continuous positive decreasing weight function on $[a,x]$. Let $r_i \geq 0$, $r = \sum_{i=1}^{N} r_i \geq 1$. Let $p<0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 0$, $\rho = \sum_{i=1}^{N} r_i(\alpha-\beta_i) - r\sigma$. Let $D_{a+}^{\alpha} f \in L_{p+q}[a,b]$ be of fixed sign on $[a,b]$, with $1/D_{a+}^{\alpha} f \in L_{p+q}[a,b]$. Then reverse inequality in (6.52) holds.*

Next theorem is a non-weighted version of Theorem 6.31. Since we have condition $r>0$ instead of $r \geq 1$, it can't be considered as a corollary of Theorem 6.31.

Theorem 6.34 *Let $N \in \mathbb{N}$, $\alpha > \beta_i \geq 0$ for $i = 1,\dots,N$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.21 holds for all pairs $\{\alpha,\beta_i\}$, $i = 1,\dots,N$. Let $r_i \geq 0$, $r = \sum_{i=1}^N r_i > 0$. Let $p > 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 1$, $\rho = \sum_{i=1}^N r_i(\alpha-\beta_i) - r\sigma$ and $\alpha > \beta_i + \sigma$ for $i = 1,\dots,N$. Let $D^\alpha_{a+}f \in L_{p+q}[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x \prod_{i=1}^N |D^{\beta_i}_{a+}f(t)|^{r_ip}|D^\alpha_{a+}f(t)|^q\,dt \leq T_4\,(x-a)^{(\rho+\sigma)p}\left(\int_a^x |D^\alpha_{a+}f(t)|^{p+q}dt\right)^{\frac{pr+q}{p+q}}, \quad (6.62)$$

where T_4 is given by (6.53).

Inequality (6.62) *is sharp for $\alpha = \beta_i + 1$, $i = 1,\dots,N$ and $q = 1$, where equality is attained for a function f such that $D^\alpha_{a+}f(t) = (x-t)^{\frac{p(\alpha-\beta_1)}{q}}$, $t \in [a,x]$.*

Proof. Let $q \neq 0$, $\delta_i = \alpha - \beta_i - 1$, $i = 1,\dots,N$. Using Theorem 2.13, the triangle inequality and Hölder's inequality for $\{\frac{1}{1-\sigma},\frac{1}{\sigma}\}$, for $t \in [a,x]$ follows

$$\begin{aligned}
&\prod_{i=1}^N |D^{\beta_i}_{a+}f(t)|^{r_ip}\\
&\leq \frac{1}{\prod\limits_{i=1}^N [\Gamma(\delta_i+1)]^{r_ip}} \prod_{i=1}^N \left[\int_a^t (t-\tau)^{\delta_i}|D^\alpha_{a+}f(\tau)|\,d\tau\right]^{r_ip}\\
&\leq \frac{1}{\prod\limits_{i=1}^N [\Gamma(\delta_i+1)]^{r_ip}} \prod_{i=1}^N \left[\left(\int_a^t (t-\tau)^{\frac{\delta_i}{1-\sigma}}\,d\tau\right)^{1-\sigma}\left(\int_a^t |D^\alpha_{a+}f(\tau)|^{\frac{1}{\sigma}}\,d\tau\right)^{\sigma}\right]^{r_ip} \qquad (6.63)\\
&= \frac{1}{\prod\limits_{i=1}^N [\Gamma(\delta_i+1)]^{r_ip}} \prod_{i=1}^N \left[\left(\frac{1-\sigma}{\delta_i+1-\sigma}\right)^{1-\sigma}(t-a)^{\delta_i+1-\sigma}\left(\int_a^t |D^\alpha_{a+}f(\tau)|^{\frac{1}{\sigma}}\,d\tau\right)^{\sigma}\right]^{r_ip}\\
&= \frac{(1-\sigma)^{(1-\sigma)rp}\,(t-a)^{\sum_{i=1}^N(\delta_i+1-\sigma)r_ip}}{\prod\limits_{i=1}^N \left[\Gamma(\delta_i+1)\,(\delta_i+1-\sigma)^{1-\sigma}\right]^{r_ip}}\left(\int_a^t |D^\alpha_{a+}f(\tau)|^{\frac{1}{\sigma}}\,d\tau\right)^{\sigma rp}.
\end{aligned}$$

Now we have

$$\begin{aligned}
\int_a^x \prod_{i=1}^N |D^{\beta_i}_{a+}f(t)|^{r_ip}\,|D^\alpha_{a+}f(t)|^q\,dt &\leq \frac{(1-\sigma)^{(1-\sigma)rp}}{\prod\limits_{i=1}^N \left[\Gamma(\delta_i+1)\,(\delta_i+1-\sigma)^{1-\sigma}\right]^{r_ip}}\\
&\cdot \int_a^x (t-a)^{\sum_{i=1}^N(\delta_i+1-\sigma)r_ip}\left(\int_a^t |D^\alpha_{a+}f(\tau)|^{\frac{1}{\sigma}}\,d\tau\right)^{\sigma rp}|D^\alpha_{a+}f(t)|^q\,dt\,. \qquad (6.64)
\end{aligned}$$

By Hölder's inequality for $\{\frac{1}{\sigma p},\frac{1}{\sigma q}\}$ with $\rho p = \sum\limits_{i=1}^N(\delta_i+1-\sigma)r_ip$, we get

$$\int_a^x \prod_{i=1}^N |D^{\beta_i}_{a+}f(t)|^{r_ip}\,|D^\alpha_{a+}f(t)|^q\,dt$$

$$\leq \frac{(1-\sigma)^{(1-\sigma)rp}}{\prod_{i=1}^{N}\left[\Gamma(\delta_i+1)\,(\delta_i+1-\sigma)^{1-\sigma}\right]^{r_ip}}\left(\int_a^x (t-a)^{\frac{\rho}{\sigma}}\,dt\right)^{\sigma p}$$

$$\cdot\left(\int_a^x |D_{a+}^{\alpha}f(t)|^{\frac{1}{\sigma}}\left(\int_a^t |D_{a+}^{\alpha}f(\tau)|^{\frac{1}{\sigma}}\,d\tau\right)^{\frac{rp}{q}}dt\right)^{\sigma q}$$

$$= \frac{(1-\sigma)^{(1-\sigma)rp}}{\prod_{i=1}^{N}\left[\Gamma(\delta_i+1)\,(\delta_i+1-\sigma)^{1-\sigma}\right]^{r_ip}}\,\frac{\sigma^{\sigma p}\,(x-a)^{(\rho+\sigma)p}}{(\rho+\sigma)^{\sigma p}}$$

$$\cdot\left(\frac{q}{rp+q}\right)^{\sigma q}\left(\int_a^x |D_{a+}^{\alpha}f(t)|^{\frac{1}{\sigma}}\,dt\right)^{\sigma(rp+q)}.$$

If $q=0$ (and $\sigma=\frac{1}{p}<1$) then the proof after (6.64) simplifies, that is

$$\int_a^x \prod_{i=1}^{N}|D_{a+}^{\beta_i}f(t)|^{r_ip}\,dt$$

$$\leq \frac{(1-\sigma)^{(1-\sigma)rp}}{\prod_{i=1}^{N}\left[\Gamma(\delta_i+1)\,(\delta_i+1-\sigma)^{1-\sigma}\right]^{r_ip}}$$

$$\cdot\int_a^x (t-a)^{\sum_{i=1}^{N}(\delta_i+1)r_ip-r}\left(\int_a^t |D_{a+}^{\alpha}f(\tau)|^p\,d\tau\right)^r dt$$

$$\leq \frac{(1-\sigma)^{(1-\sigma)rp}}{\prod_{i=1}^{N}\left[\Gamma(\delta_i+1)\,(\delta_i+1-\sigma)^{1-\sigma}\right]^{r_ip}}$$

$$\cdot\left(\int_a^x |D_{a+}^{\alpha}f(\tau)|^p\,d\tau\right)^r\int_a^x (t-a)^{\sum_{i=1}^{N}(\delta_i+1)r_ip-r}\,dt$$

$$= \frac{(p-1)^{(p-1)r}}{\prod_{i=1}^{N}\left[\Gamma(\delta_i+1)\,(p(\delta_i+1)-1)^{\frac{p-1}{p}}\right]^{r_ip}}$$

$$\cdot\left(\int_a^x |D_{a+}^{\alpha}f(\tau)|^p\,d\tau\right)^r\frac{(x-a)^{\sum_{i=1}^{N}(\delta_i+1)r_ip-r+1}}{\sum_{i=1}^{N}(\delta_i+1)r_ip-r+1}.$$

This proves the inequality (6.62).

Using the equality condition in Hölder's inequality we have equality in (6.63) if and only if $|D_{a+}^{\alpha}f(\tau)|^{\frac{1}{\sigma}}=K(t-\tau)^{\frac{\delta_i}{1-\sigma}}$ for some constant $K\geq 0$ and every $\tau\in[a,t]$. Since $D_{a+}^{\alpha}f(\tau)$ depends only on τ, then $\delta_i=0$, that is $\alpha=\beta_i+1$, $i=1,\ldots,N$. Due to homogeneous property of inequality (6.62), we can take $D_{a+}^{\alpha}f(\tau)=1$, which gives us $D_{a+}^{\beta_i}f(\tau)=$

$D_{a+}^{\alpha-1}f(\tau) = \tau - a$, $i = 1,\ldots,N$. Substituting this in equality (6.62), for the left side we get

$$\int_a^x \prod_{i=1}^{N} (t-a)^{r_i p}\, dt = \int_a^x (t-a)^{rp}\, dt = \frac{(x-a)^{rp+1}}{rp+1} \tag{6.65}$$

For the right side, with $\rho = r - r\sigma$, follows

$$\begin{aligned} & T_4\,(x-a)^{(\rho+\sigma)p} \left(\int_a^x dt\right)^{\frac{pr+q}{p+q}} \\ &= \frac{\sigma^{\sigma p}\, q^{\sigma q}}{(r-r\sigma+\sigma)^{\sigma p}\,(rp+q)^{\sigma q}}\,(x-a)^{(r-r\sigma+\sigma)p}\,(x-a)^{(pr+q)\sigma} \\ &= \frac{\sigma^{\sigma p}\, q^{\sigma q}}{(r-r\sigma+\sigma)^{\sigma p}\,(rp+q)^{\sigma q}}\,(x-a)^{rp+1}. \end{aligned} \tag{6.66}$$

Hence

$$\frac{1}{rp+1} = \frac{q^{\frac{q}{p+q}}}{[r(p+q)-r+1]^{\frac{p}{p+q}}\,(rp+q)^{\frac{q}{p+q}}}$$

which is equivalent to

$$[r(p+q)-r+1]^p\,[rp+q]^q = q^q(rp+1)^{p+q}. \tag{6.67}$$

For $q = 1$ equality (6.67) obviously holds. For $q = 0$ equality (6.67) implies $r = 0$, which gives trivial identity in (6.62). By simple rearrangements, equation (6.67) becomes

$$\left[1 + r\frac{q-1}{rp+1}\right]^p \left[1 + r\frac{p}{q}\frac{1-q}{rp+1}\right]^q = 1. \tag{6.68}$$

For $p = q$ the left side of equation (6.68) is equal to $\left[1-\left(r\frac{1-p}{rp+1}\right)^2\right]^p$, which is strictly less then 1, except in trivial cases. For $0 < p < q$, $q \neq 1$, $r > 0$, using the Bernoulli inequality, we have

$$\left[1 + r\frac{q-1}{rp+1}\right]^{\frac{p}{q}} \left[1 + r\frac{p}{q}\frac{1-q}{rp+1}\right] < \left[1 + r\frac{p}{q}\frac{q-1}{rp+1}\right]\left[1 + r\frac{p}{q}\frac{1-q}{rp+1}\right],$$

which is obviously strictly less then 1. For $0 < q < p$, $q \neq 1$, $r > 0$, using the Bernoulli inequality, we have

$$\left[1 + r\frac{q-1}{rp+1}\right]\left[1 + r\frac{p}{q}\frac{1-q}{rp+1}\right]^{\frac{q}{p}} < \left[1 + r\frac{q-1}{rp+1}\right]\left[1 + r\frac{1-q}{rp+1}\right],$$

which is again obviously strictly less then 1. It follows that (6.67) holds if and only if $q = 1$. □

Remark 6.2 Let $N=1$, $\alpha=1$, $\beta_1=0$, $r_1=r=1$, $p=q=1$, $a=0$ and $x=h$. Then inequality (6.62) becomes Beesack's inequality

$$\int_0^h \left|f(t)f'(t)\right|dt \le \frac{h}{2}\int_0^h \left[f'(t)\right]^2 dt\,, \tag{6.69}$$

which is valid for any function f absolutely continuous on $[0,h]$ satisfying single boundary condition $f(0)=0$.

Following inequalities include the right-sided Riemann-Liouville fractional derivatives.

Theorem 6.35 *Let $N\in\mathbb{N}$, $\alpha>\beta_i\ge 0$ for $i=1,\dots,N$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.22 holds for all pairs $\{\alpha,\beta_i\}$, $i=1,\dots,N$. Let w be continuous positive increasing weight function on $[x,b]$. Let $r_i\ge 0$, $r=\sum_{i=1}^N r_i\ge 1$. Let $p>0$, $q\ge 0$, $\sigma=\frac{1}{p+q}<1$, $\rho=\sum_{i=1}^N r_i(\alpha-\beta_i)-r\sigma$ and $\alpha>\beta_i+\sigma$ for $i=1,\dots,N$. Let $D_{b-}^{\alpha}f\in L_{p+q}[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$\begin{aligned}&\int_x^b w(t)\prod_{i=1}^N |D_{b-}^{\beta_i}f(t)|^{r_ip}\,|D_{b-}^{\alpha}f(t)|^q\,dt\\ &\le T_4\,(b-x)^{(\rho+\sigma)p}\,[w(x)]^{\frac{p(1-r)}{p+q}}\left(\int_x^b w(t)\,|D_{b-}^{\alpha}f(t)|^{p+q}dt\right)^{\frac{pr+q}{p+q}},\end{aligned} \tag{6.70}$$

where T_4 is given by (6.53).

Corollary 6.2 *Suppose that assumptions of Theorem 6.35 hold and let $r=1$. Then*

$$\begin{aligned}&\int_x^b w(t)\prod_{i=1}^N |D_{b-}^{\beta_i}f(t)|^{r_ip}\,|D_{b-}^{\alpha}f(t)|^q\,dt\\ &\le \widetilde{T}_4\,(b-x)^{\sum\limits_{i=1}^N r_i(\alpha-\beta_i)p}\int_x^b w(t)\,|D_{b-}^{\alpha}f(t)|^{p+q}dt\,,\end{aligned} \tag{6.71}$$

where $\widetilde{T}_4$ is given by (6.59).

Next we have bounded weight function.

Theorem 6.36 *Let $N\in\mathbb{N}$, $\alpha>\beta_i\ge 0$ for $i=1,\dots,N$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.22 holds for all pairs $\{\alpha,\beta_i\}$, $i=1,\dots,N$. Let w be continuous positive weight function on $[x,b]$ such that $A\le w(t)\le B$ for $t\in[x,b]$. Let $r_i\ge 0$, $r=\sum_{i=1}^N r_i\ge 1$. Let $p>0$, $q\ge 0$, $\sigma=\frac{1}{p+q}<1$, $\rho=\sum_{i=1}^N r_i(\alpha-\beta_i)-r\sigma$ and $\alpha>\beta_i+\sigma$ for $i=1,\dots,N$. Let $D_{b-}^{\alpha}f\in L_{p+q}[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$\begin{aligned}&\int_x^b w(t)\prod_{i=1}^N |D_{b-}^{\beta_i}f(t)|^{r_ip}\,|D_{b-}^{\alpha}f(t)|^q\,dt\\ &\le T_4\,(b-x)^{(\rho+\sigma)p}\left(\frac{B}{A^r}\right)^{\sigma p}\left(\int_x^b w(t)\,|D_{b-}^{\alpha}f(t)|^{p+q}dt\right)^{\frac{pr+q}{p+q}},\end{aligned} \tag{6.72}$$

where T_4 is given by (6.53).

Proposition 6.14 *Let $N \in \mathbb{N}$, $\alpha > \beta_i \geq 0$ for $i = 1, \ldots, N$ and let $D^{\alpha}_{b-}f \in L_{\infty}[a,b]$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.22 holds for all pairs $\{\alpha, \beta_i\}$, $i = 1, \ldots, N$. Let w be continuous nonnegative weight function on $[x,b]$. Let $p > 0$, $q \geq 0$, $r_i \geq 0$, $r = \sum_{i=1}^{N} r_i > 0$. Then for a.e. $x \in [a,b]$ holds*

$$\begin{aligned}
&\int_x^b w(t) \prod_{i=1}^{N} |D^{\beta_i}_{b-}f(t)|^{r_i p} \, |D^{\alpha}_{b-}f(t)|^q \, dt \\
&\leq \frac{(b-x)^{\sum_{i=1}^{N} r_i(\alpha-\beta_i)p+1}}{\left[\sum_{i=1}^{N} r_i(\alpha-\beta_i)p+1\right] \prod_{i=1}^{N} [\Gamma(\alpha-\beta_i+1)]^{r_i p}} \|w\|_{\infty} \, \|D^{\alpha}_{b-}f\|_{\infty}^{rp+q}. \qquad (6.73)
\end{aligned}$$

Theorem 6.37 *Let $N \in \mathbb{N}$, $\alpha > \beta_i \geq 0$ for $i = 1, \ldots, N$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.22 holds for all pairs $\{\alpha, \beta_i\}$, $i = 1, \ldots, N$. Let w be continuous positive increasing weight function on $[x,b]$. Let $r_i \geq 0$, $r = \sum_{i=1}^{N} r_i \geq 1$. Let $p < 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 0$, $\rho = \sum_{i=1}^{N} r_i(\alpha-\beta_i) - r\sigma$. Let $D^{\alpha}_{b-}f \in L_{p+q}[a,b]$ be of fixed sign on $[a,b]$, with $1/D^{\alpha}_{b-}f \in L_{p+q}[a,b]$. Then reverse inequality in (6.70) holds.*

Next theorem is a non-weighted version of Theorem 6.35 with weakened condition on r.

Theorem 6.38 *Let $N \in \mathbb{N}$, $\alpha > \beta_i \geq 0$ for $i = 1, \ldots, N$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.22 holds for all pairs $\{\alpha, \beta_i\}$, $i = 1, \ldots, N$. Let $r_i \geq 0$, $r = \sum_{i=1}^{N} r_i > 0$. Let $p > 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 1$, $\rho = \sum_{i=1}^{N} r_i(\alpha-\beta_i) - r\sigma$ and $\alpha > \beta_i + \sigma$ for $i = 1, \ldots, N$. Let $D^{\alpha}_{b-}f \in L_{p+q}[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b \prod_{i=1}^{N} |D^{\beta_i}_{b-}f(t)|^{r_i p} |D^{\alpha}_{b-}f(t)|^q \, dt \leq T_4 \, (b-x)^{(\rho+\sigma)p} \left(\int_x^b |D^{\alpha}_{b-}f(t)|^{p+q} dt \right)^{\frac{pr+q}{p+q}}, \quad (6.74)$$

where T_4 is given by (6.53).

Inequality (6.74) *is sharp for $\alpha = \beta_i + 1$, $i = 1, \ldots, N$ and $q = 1$, where equality is attained for a function f such that $D^{\alpha}_{b-}f(t) = (x-t)^{\frac{p(\alpha-\beta_1)}{q}}$, $t \in [x,b]$.*

Remark 6.3 In order to get classical Opial's inequality (1.17) we need the inequality (6.74) for $N = 1$, $r_1 = r = 1$ and $p = q = 1$:

$$\int_x^b |D^{\beta_1}_{b-}f(t)| \, |D^{\alpha}_{b-}f(t)| \, dt \leq T_4 \, (b-x) \int_x^b |D^{\alpha}_{b-}f(t)|^2 \, dt \qquad (6.75)$$

satisfying $f(b) = 0$. Observe the inequality

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}} |D^{\beta_1}_{a+}f(t)| \, |D^{\alpha}_{a+}f(t)| \, dt + \int_{\frac{a+b}{2}}^b |D^{\beta_1}_{b-}f(t)| \, |D^{\alpha}_{b-}f(t)| \, dt \\
&\leq T_4 \left(\frac{b-a}{2}\right) \left(\int_a^{\frac{a+b}{2}} \left|D^{\alpha}_{a+}f(t)\right|^2 dt + \int_{\frac{a+b}{2}}^b \left|D^{\alpha}_{b-}f(t)\right|^2 dt \right). \qquad (6.76)
\end{aligned}$$

If we put $\alpha = 1$, $\beta_1 = 0$, $a = 0$ and $b = h$, then inequality (6.76) becomes Opial's inequality

$$\int_0^h \left|f(x)f'(x)\right| dx \leq \frac{h}{4}\int_0^h \left[f'(x)\right]^2 dx,$$

having boundary conditions $f(0) = f(h) = 0$.

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Theorem 6.39 *Let $N \in \mathbb{N}$, $i = 1,\ldots,N$, $\alpha > \beta_i \geq 0$ and m_i, n given by (2.70) with $m = \min\{m_i\}$. Let $f \in AC^n[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m,\ldots,n-1$. Let w be continuous positive decreasing weight function on $[a,x]$. Let $r_i \geq 0$, $r = \sum_{i=1}^N r_i \geq 1$. Let $p > 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 1$, $\rho = \sum_{i=1}^N r_i(\alpha - \beta_i) - r\sigma$ and $\alpha > \beta_i + \sigma$ for $i = 1,\ldots,N$. Let ${}^CD^{\alpha}_{a+}f \in L_{p+q}[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\begin{aligned}&\int_a^x w(t)\prod_{i=1}^N |{}^CD^{\beta_i}_{a+}f(t)|^{r_ip}\,|{}^CD^{\alpha}_{a+}f(t)|^q\,dt\\ &\leq T_4\,(x-a)^{(\rho+\sigma)p}\,[w(x)]^{\frac{p(1-r)}{p+q}}\left(\int_a^x w(t)\,|{}^CD^{\alpha}_{a+}f(t)|^{p+q}dt\right)^{\frac{pr+q}{p+q}},\end{aligned} \tag{6.77}$$

where T_4 is given by (6.53).

If $r = 1$, then we have Alzer's inequality (1.26) for the left-sided Caputo fractional derivatives.

Corollary 6.3 *Suppose that assumptions of Theorem 6.39 hold and let $r = 1$. Then*

$$\begin{aligned}&\int_a^x w(t)\prod_{i=1}^N |{}^CD^{\beta_i}_{a+}f(t)|^{r_ip}\,|{}^CD^{\alpha}_{a+}f(t)|^q\,dt\\ &\leq \widetilde{T}_4\,(x-a)^{\sum\limits_{i=1}^N r_i(\alpha-\beta_i)p}\int_a^x w(t)\,|{}^CD^{\alpha}_{a+}f(t)|^{p+q}dt,\end{aligned} \tag{6.78}$$

where $\widetilde{T}_4$ is given by (6.59).

Next we have bounded weight function.

Theorem 6.40 *Let $N \in \mathbb{N}$, $i = 1,\ldots,N$, $\alpha > \beta_i \geq 0$ and m_i, n given by (2.70) with $m = \min\{m_i\}$. Let $f \in AC^n[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m,\ldots,n-1$. Let w be continuous positive weight function on $[a,x]$ such that $A \leq w(t) \leq B$ for $t \in [a,x]$. Let $r_i \geq 0$, $r = \sum_{i=1}^N r_i \geq 1$. Let $p > 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 1$, $\rho = \sum_{i=1}^N r_i(\alpha - \beta_i) - r\sigma$ and $\alpha > \beta_i + \sigma$ for $i = 1,\ldots,N$. Let ${}^CD^{\alpha}_{a+}f \in L_{p+q}[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\begin{aligned}&\int_a^x w(t)\prod_{i=1}^N |{}^CD^{\beta_i}_{a+}f(t)|^{r_ip}\,|{}^CD^{\alpha}_{a+}f(t)|^q\,dt\\ &\leq T_4\,(x-a)^{(\rho+\sigma)p}\left(\frac{B}{A^r}\right)^{\sigma p}\left(\int_a^x w(t)\,|{}^CD^{\alpha}_{a+}f(t)|^{p+q}dt\right)^{\frac{pr+q}{p+q}},\end{aligned} \tag{6.79}$$

where T_4 is given by (6.53).

Proposition 6.15 *Let $N \in \mathbb{N}$, $i = 1,\ldots,N$, $\alpha > \beta_i \geq 0$ and m_i, n given by* (2.70) *with $m = \min\{m_i\}$. Let $f \in AC^n[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m,\ldots,n-1$ and let ${}^C D^{\alpha}_{a+} f \in L_\infty[a,b]$. Let w be continuous nonnegative weight function on $[a,x]$. Let $p > 0$, $q \geq 0$, $r_i \geq 0$, $r = \sum_{i=1}^N r_i > 0$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x w(t) \prod_{i=1}^N |{}^C D^{\beta_i}_{a+} f(t)|^{r_i p}\, |{}^C D^{\alpha}_{a+} f(t)|^q\, dt$$

$$\leq \frac{(x-a)^{\sum\limits_{i=1}^N r_i(\alpha-\beta_i)p+1}}{\left[\sum\limits_{i=1}^N r_i(\alpha-\beta_i)p+1\right] \prod\limits_{i=1}^N [\Gamma(\alpha-\beta_i+1)]^{r_i p}} \|w\|_\infty \|{}^C D^{\alpha}_{a+} f\|_\infty^{rp+q}. \tag{6.80}$$

Theorem 6.41 *Let $N \in \mathbb{N}$, $i = 1,\ldots,N$, $\alpha > \beta_i \geq 0$ and m_i, n given by* (2.70) *with $m = \min\{m_i\}$. Let $f \in AC^n[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m,\ldots,n-1$. Let w be continuous positive decreasing weight function on $[a,x]$. Let $r_i \geq 0$, $r = \sum_{i=1}^N r_i \geq 1$. Let $p < 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 0$, $\rho = \sum_{i=1}^N r_i(\alpha-\beta_i) - r\sigma$. Let ${}^C D^{\alpha}_{a+} f \in L_{p+q}[a,b]$ be of fixed sign on $[a,b]$, with $1/{}^C D^{\alpha}_{a+} f \in L_{p+q}[a,b]$. Then reverse inequality in* (6.77) *holds.*

Next theorem is a non-weighted version of Theorem 6.39 with weakened condition on r.

Theorem 6.42 *Let $N \in \mathbb{N}$, $i = 1,\ldots,N$, $\alpha > \beta_i \geq 0$ and m_i, n given by* (2.70) *with $m = \min\{m_i\}$. Let $f \in AC^n[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m,\ldots,n-1$. Let $r_i \geq 0$, $r = \sum_{i=1}^N r_i > 0$. Let $p > 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 1$, $\rho = \sum_{i=1}^N r_i(\alpha-\beta_i) - r\sigma$ and $\alpha > \beta_i + \sigma$ for $i = 1,\ldots,N$. Let ${}^C D^{\alpha}_{a+} f \in L_{p+q}[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x \prod_{i=1}^N |{}^C D^{\beta_i}_{a+} f(t)|^{r_i p} |{}^C D^{\alpha}_{a+} f(t)|^q\, dt \leq T_4\, (x-a)^{(\rho+\sigma)p} \left(\int_a^x |{}^C D^{\alpha}_{a+} f(t)|^{p+q} dt \right)^{\frac{pr+q}{p+q}}, \tag{6.81}$$

where T_4 is given by (6.53).

Inequality (6.81) *is sharp for $\alpha = \beta_i + 1$, $i = 1,\ldots,N$ and $q = 1$, where equality is attained for a function f such that ${}^C D^{\alpha}_{a+} f(t) = (x-t)^{\frac{p(\alpha-\beta_1)}{q}}$, $t \in [a,x]$.*

Following inequalities include the right-sided Caputo fractional derivatives.

Theorem 6.43 *Let $N \in \mathbb{N}$, $i = 1,\ldots,N$, $\alpha > \beta_i \geq 0$ and m_i, n given by* (2.70) *with $m = \min\{m_i\}$. Let $f \in AC^n[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m,\ldots,n-1$. Let w be continuous positive increasing weight function on $[x,b]$. Let $r_i \geq 0$, $r = \sum_{i=1}^N r_i \geq 1$. Let $p > 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 1$, $\rho = \sum_{i=1}^N r_i(\alpha-\beta_i) - r\sigma$ and $\alpha > \beta_i + \sigma$ for $i = 1,\ldots,N$. Let ${}^C D^{\alpha}_{b-} f \in L_{p+q}[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b w(t) \prod_{i=1}^N |{}^C D^{\beta_i}_{b-} f(t)|^{r_i p}\, |{}^C D^{\alpha}_{b-} f(t)|^q\, dt$$

$$\leq T_4\,(b-x)^{(\rho+\sigma)p}\,[w(x)]^{\frac{p(1-r)}{p+q}}\left(\int_x^b w(t)\,|{}^CD^{\alpha}_{b-}f(t)|^{p+q}dt\right)^{\frac{pr+q}{p+q}}, \tag{6.82}$$

where T_4 *is given by* (6.53).

Corollary 6.4 *Suppose that assumptions of Theorem 6.43 hold and let* $r=1$. *Then*

$$\int_x^b w(t)\prod_{i=1}^N |{}^CD^{\beta_i}_{b-}f(t)|^{r_ip}\,|{}^CD^{\alpha}_{b-}f(t)|^q\,dt$$
$$\leq \widetilde{T}_4\,(b-x)^{\sum\limits_{i=1}^N r_i(\alpha-\beta_i)p}\int_x^b w(t)\,|{}^CD^{\alpha}_{b-}f(t)|^{p+q}dt\,, \tag{6.83}$$

where $\widetilde{T}_4$ *is given by* (6.59).

Next we have bounded weight function.

Theorem 6.44 *Let* $N\in\mathbb{N}$, $i=1,\ldots,N$, $\alpha>\beta_i\geq 0$ *and* m_i, n *given by* (2.70) *with* $m=\min\{m_i\}$. *Let* $f\in AC^n[a,b]$ *be such that* $f^{(i)}(b)=0$ *for* $i=m,\ldots,n-1$. *Let* w *be continuous positive weight function on* $[x,b]$ *such that* $A\leq w(t)\leq B$ *for* $t\in[x,b]$. *Let* $r_i\geq 0$, $r=\sum_{i=1}^N r_i\geq 1$. *Let* $p>0$, $q\geq 0$, $\sigma=\frac{1}{p+q}<1$, $\rho=\sum_{i=1}^N r_i(\alpha-\beta_i)-r\sigma$ *and* $\alpha>\beta_i+\sigma$ *for* $i=1,\ldots,N$. *Let* ${}^CD^{\alpha}_{b-}f\in L_{p+q}[a,b]$. *Then for a.e.* $x\in[a,b]$ *holds*

$$\int_x^b w(t)\prod_{i=1}^N |{}^CD^{\beta_i}_{b-}f(t)|^{r_ip}\,|{}^CD^{\alpha}_{b-}f(t)|^q\,dt$$
$$\leq T_4\,(b-x)^{(\rho+\sigma)p}\left(\frac{B}{A^r}\right)^{\sigma p}\left(\int_x^b w(t)\,|{}^CD^{\alpha}_{b-}f(t)|^{p+q}dt\right)^{\frac{pr+q}{p+q}}, \tag{6.84}$$

where T_4 *is given by* (6.53).

Proposition 6.16 *Let* $N\in\mathbb{N}$, $i=1,\ldots,N$, $\alpha>\beta_i\geq 0$ *and* m_i, n *given by* (2.70) *with* $m=\min\{m_i\}$. *Let* $f\in AC^n[a,b]$ *be such that* $f^{(i)}(b)=0$ *for* $i=m,\ldots,n-1$ *and let* ${}^CD^{\alpha}_{b-}f\in L_\infty[a,b]$. *Let* w *be continuous nonnegative weight function on* $[x,b]$. *Let* $p>0$, $q\geq 0$, $r_i\geq 0$, $r=\sum_{i=1}^N r_i>0$. *Then for a.e.* $x\in[a,b]$ *holds*

$$\int_x^b w(t)\prod_{i=1}^N |{}^CD^{\beta_i}_{b-}f(t)|^{r_ip}\,|{}^CD^{\alpha}_{b-}f(t)|^q\,dt$$
$$\leq \frac{(b-x)^{\sum\limits_{i=1}^N r_i(\alpha-\beta_i)p+1}}{\left[\sum\limits_{i=1}^N r_i(\alpha-\beta_i)p+1\right]\prod\limits_{i=1}^N[\Gamma(\alpha-\beta_i+1)]^{r_ip}}\,\|w\|_\infty\,\|{}^CD^{\alpha}_{b-}f\|_\infty^{rp+q}\,. \tag{6.85}$$

Theorem 6.45 *Let $N \in \mathbb{N}$, $i = 1,\ldots,N$, $\alpha > \beta_i \geq 0$ and m_i, n given by (2.70) with $m = \min\{m_i\}$. Let $f \in AC^n[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m,\ldots,n-1$. Let w be continuous positive increasing weight function on $[x,b]$. Let $r_i \geq 0$, $r = \sum_{i=1}^N r_i \geq 1$. Let $p < 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 0$, $\rho = \sum_{i=1}^N r_i(\alpha - \beta_i) - r\sigma$. Let ${}^C D^{\alpha}_{b-} f \in L_{p+q}[a,b]$ be of fixed sign on $[a,b]$, with $1/{}^C D^{\alpha}_{b-} f \in L_{p+q}[a,b]$. Then reverse inequality in (6.82) holds.*

Theorem 6.46 *Let $N \in \mathbb{N}$, $i = 1,\ldots,N$, $\alpha > \beta_i \geq 0$ and m_i, n given by (2.70) with $m = \min\{m_i\}$. Let $f \in AC^n[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m,\ldots,n-1$. Let $r_i \geq 0$, $r = \sum_{i=1}^N r_i > 0$. Let $p > 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 1$, $\rho = \sum_{i=1}^N r_i(\alpha - \beta_i) - r\sigma$ and $\alpha > \beta_i + \sigma$ for $i = 1,\ldots,N$. Let ${}^C D^{\alpha}_{b-} f \in L_{p+q}[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b \prod_{i=1}^N |{}^C D^{\beta_i}_{b-} f(t)|^{r_i p} |{}^C D^{\alpha}_{b-} f(t)|^q \, dt \leq T_4 \, (b-x)^{(\rho+\sigma)p} \left(\int_x^b |{}^C D^{\alpha}_{b-} f(t)|^{p+q} dt \right)^{\frac{pr+q}{p+q}}, \tag{6.86}$$

where T_4 is given by (6.53).

Inequality (6.86) is sharp for $\alpha = \beta_i + 1$, $i = 1,\ldots,N$ and $q = 1$, where equality is attained for a function f such that ${}^C D^{\alpha}_{b-} f(t) = (x-t)^{\frac{p(\alpha-\beta_1)}{q}}$, $t \in [x,b]$.

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Theorem 6.47 *Let $N \in \mathbb{N}$, $\alpha > \beta_i \geq 0$, $m_i = [\beta_i] + 1$, $m = \min\{m_i\}$ for $i = 1,\ldots,N$ and $n = [\alpha] + 1$. Let $f \in C^{\alpha}_{a+}[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m-1,\ldots,n-2$. Let w be continuous positive decreasing weight function on $[a,x]$. Let $r_i \geq 0$, $r = \sum_{i=1}^N r_i \geq 1$. Let $p > 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 1$, $\rho = \sum_{i=1}^N r_i(\alpha - \beta_i) - r\sigma$ and $\alpha > \beta_i + \sigma$ for $i = 1,\ldots,N$. Let ${}^{C_1} D^{\alpha}_{a+} f \in L_{p+q}[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\begin{aligned} &\int_a^x w(t) \prod_{i=1}^N |{}^{C_1} D^{\beta_i}_{a+} f(t)|^{r_i p} \, |{}^{C_1} D^{\alpha}_{a+} f(t)|^q \, dt \\ &\leq T_4 \, (x-a)^{(\rho+\sigma)p} \, [w(x)]^{\frac{p(1-r)}{p+q}} \left(\int_a^x w(t) \, |{}^{C_1} D^{\alpha}_{a+} f(t)|^{p+q} dt \right)^{\frac{pr+q}{p+q}}, \end{aligned} \tag{6.87}$$

where T_4 is given by (6.53).

If $r = 1$, then we have Alzer's inequality (1.26) for the left-sided Canavati fractional derivatives.

Corollary 6.5 *Suppose that assumptions of Theorem 6.47 hold and let $r = 1$. Then*

$$\begin{aligned} &\int_a^x w(t) \prod_{i=1}^N |{}^{C_1} D^{\beta_i}_{a+} f(t)|^{r_i p} \, |{}^{C_1} D^{\alpha}_{a+} f(t)|^q \, dt \\ &\leq \widetilde{T}_4 \, (x-a)^{\sum\limits_{i=1}^N r_i(\alpha-\beta_i)p} \int_a^x w(t) \, |{}^{C_1} D^{\alpha}_{a+} f(t)|^{p+q} dt, \end{aligned} \tag{6.88}$$

where $\widetilde{T}_4$ is given by (6.59).

Next we have bounded weight function.

Theorem 6.48 *Let $N \in \mathbb{N}$, $\alpha > \beta_i \geq 0$, $m_i = [\beta_i]+1$, $m = \min\{m_i\}$ for $i = 1,\dots,N$ and $n = [\alpha]+1$. Let $f \in C^{\alpha}_{a+}[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m-1,\dots,n-2$. Let w be continuous positive weight function on $[a,x]$ such that $A \leq w(t) \leq B$ for $t \in [a,x]$. Let $r_i \geq 0$, $r = \sum_{i=1}^{N} r_i \geq 1$. Let $p > 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 1$, $\rho = \sum_{i=1}^{N} r_i(\alpha-\beta_i) - r\sigma$ and $\alpha > \beta_i + \sigma$ for $i = 1,\dots,N$. Let ${}^{C_1}D^{\alpha}_{a+}f \in L_{p+q}[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\begin{aligned}
&\int_a^x w(t) \prod_{i=1}^{N} |{}^{C_1}D^{\beta_i}_{a+}f(t)|^{r_i p}\, |{}^{C_1}D^{\alpha}_{a+}f(t)|^{q}\, dt \\
&\leq T_4\, (x-a)^{(\rho+\sigma)p} \left(\frac{B}{A^r}\right)^{\sigma p} \left(\int_a^x w(t)\, |{}^{C_1}D^{\alpha}_{a+}f(t)|^{p+q} dt\right)^{\frac{pr+q}{p+q}}, \qquad (6.89)
\end{aligned}$$

where T_4 is given by (6.53).

Proposition 6.17 *Let $N \in \mathbb{N}$, $\alpha > \beta_i \geq 0$, $m_i = [\beta_i]+1$, $m = \min\{m_i\}$ for $i = 1,\dots,N$ and $n = [\alpha]+1$. Let $f \in C^{\alpha}_{a+}[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m-1,\dots,n-2$ and let ${}^{C_1}D^{\alpha}_{a+}f \in L_{\infty}[a,b]$. Let w be continuous nonnegative weight function on $[a,x]$. Let $p > 0$, $q \geq 0$, $r_i \geq 0$, $r = \sum_{i=1}^{N} r_i > 0$. Then for a.e. $x \in [a,b]$ holds*

$$\begin{aligned}
&\int_a^x w(t) \prod_{i=1}^{N} |{}^{C_1}D^{\beta_i}_{a+}f(t)|^{r_i p}\, |{}^{C_1}D^{\alpha}_{a+}f(t)|^{q}\, dt \\
&\leq \frac{(x-a)^{\sum_{i=1}^{N} r_i(\alpha-\beta_i)p+1}}{\left[\sum_{i=1}^{N} r_i(\alpha-\beta_i)p+1\right] \prod_{i=1}^{N} [\Gamma(\alpha-\beta_i+1)]^{r_i p}}\, \|w\|_{\infty}\, \|{}^{C_1}D^{\alpha}_{a+}f\|_{\infty}^{rp+q}. \qquad (6.90)
\end{aligned}$$

Theorem 6.49 *Let $N \in \mathbb{N}$, $\alpha > \beta_i \geq 0$, $m_i = [\beta_i]+1$, $m = \min\{m_i\}$ for $i = 1,\dots,N$ and $n = [\alpha]+1$. Let $f \in C^{\alpha}_{a+}[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m-1,\dots,n-2$. Let w be continuous positive decreasing weight function on $[a,x]$. Let $r_i \geq 0$, $r = \sum_{i=1}^{N} r_i \geq 1$. Let $p < 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 0$, $\rho = \sum_{i=1}^{N} r_i(\alpha-\beta_i) - r\sigma$. Let ${}^{C_1}D^{\alpha}_{a+}f \in L_{p+q}[a,b]$ be of fixed sign on $[a,b]$, with $1/{}^{C_1}D^{\alpha}_{a+}f \in L_{p+q}[a,b]$. Then reverse inequality in* (6.87) *holds.*

Next theorem is a non-weighted version of Theorem 6.47 with weakened condition on r.

Theorem 6.50 *Let $N \in \mathbb{N}$, $\alpha > \beta_i \geq 0$, $m_i = [\beta_i]+1$, $m = \min\{m_i\}$ for $i = 1,\dots,N$ and $n = [\alpha]+1$. Let $f \in C^{\alpha}_{a+}[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m-1,\dots,n-2$. Let $r_i \geq 0$, $r = \sum_{i=1}^{N} r_i > 0$. Let $p > 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 1$, $\rho = \sum_{i=1}^{N} r_i(\alpha-\beta_i) - r\sigma$ and $\alpha > \beta_i + \sigma$ for $i = 1,\dots,N$. Let ${}^{C_1}D^{\alpha}_{a+}f \in L_{p+q}[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x \prod_{i=1}^{N} |{}^{C_1}D^{\beta_i}_{a+}f(t)|^{r_i p} |{}^{C_1}D^{\alpha}_{a+}f(t)|^{q}\, dt \leq T_4\, (x-a)^{(\rho+\sigma)p} \left(\int_a^x |{}^{C_1}D^{\alpha}_{a+}f(t)|^{p+q} dt\right)^{\frac{pr+q}{p+q}}, \qquad (6.91)$$

where T_4 is given by (6.53).

Inequality (6.91) *is sharp for $\alpha = \beta_i + 1$, $i = 1,\dots,N$ and $q = 1$, where equality is attained for a function f such that ${}^{C_1}D^{\alpha}_{a+}f(t) = (x-t)^{\frac{p(\alpha-\beta_1)}{q}}$, $t \in [a,x]$.*

Following inequalities include the right-sided Canavati fractional derivatives.

Theorem 6.51 *Let $N \in \mathbb{N}$, $\alpha > \beta_i \geq 0$, $m_i = [\beta_i]+1$, $m = \min\{m_i\}$ for $i = 1,\dots,N$ and $n = [\alpha]+1$. Let $f \in C^{\alpha}_{b-}[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m-1,\dots,n-2$. Let w be continuous positive increasing weight function on $[x,b]$. Let $r_i \geq 0$, $r = \sum_{i=1}^N r_i \geq 1$. Let $p > 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 1$, $\rho = \sum_{i=1}^N r_i(\alpha-\beta_i) - r\sigma$ and $\alpha > \beta_i + \sigma$ for $i = 1,\dots,N$. Let ${}^{C_1}D^{\alpha}_{b-}f \in L_{p+q}[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b w(t) \prod_{i=1}^N |{}^{C_1}D^{\beta_i}_{b-}f(t)|^{r_i p}\, |{}^{C_1}D^{\alpha}_{b-}f(t)|^q\, dt$$

$$\leq T_4\,(b-x)^{(\rho+\sigma)p}\,[w(x)]^{\frac{p(1-r)}{p+q}} \left(\int_x^b w(t)\, |{}^{C_1}D^{\alpha}_{b-}f(t)|^{p+q} dt \right)^{\frac{pr+q}{p+q}}, \tag{6.92}$$

where T_4 is given by (6.53).

Corollary 6.6 *Suppose that assumptions of Theorem 6.51 hold and let $r = 1$. Then*

$$\int_x^b w(t) \prod_{i=1}^N |{}^{C_1}D^{\beta_i}_{b-}f(t)|^{r_i p}\, |{}^{C_1}D^{\alpha}_{b-}f(t)|^q\, dt$$

$$\leq \widetilde{T}_4\,(b-x)^{\sum\limits_{i=1}^N r_i(\alpha-\beta_i)p} \int_x^b w(t)\, |{}^{C_1}D^{\alpha}_{b-}f(t)|^{p+q} dt\,, \tag{6.93}$$

where $\widetilde{T}_4$ is given by (6.59).

Next we have bounded weight function.

Theorem 6.52 *Let $N \in \mathbb{N}$, $\alpha > \beta_i \geq 0$, $m_i = [\beta_i]+1$, $m = \min\{m_i\}$ for $i = 1,\dots,N$ and $n = [\alpha]+1$. Let $f \in C^{\alpha}_{b-}[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m-1,\dots,n-2$. Let w be continuous positive weight function on $[x,b]$ such that $A \leq w(t) \leq B$ for $t \in [x,b]$. Let $r_i \geq 0$, $r = \sum_{i=1}^N r_i \geq 1$. Let $p > 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 1$, $\rho = \sum_{i=1}^N r_i(\alpha-\beta_i) - r\sigma$ and $\alpha > \beta_i + \sigma$ for $i = 1,\dots,N$. Let ${}^{C_1}D^{\alpha}_{b-}f \in L_{p+q}[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b w(t) \prod_{i=1}^N |{}^{C_1}D^{\beta_i}_{b-}f(t)|^{r_i p}\, |{}^{C_1}D^{\alpha}_{b-}f(t)|^q\, dt$$

$$\leq T_4\,(b-x)^{(\rho+\sigma)p} \left(\frac{B}{A^r}\right)^{\sigma p} \left(\int_x^b w(t)\, |{}^{C_1}D^{\alpha}_{b-}f(t)|^{p+q} dt \right)^{\frac{pr+q}{p+q}}, \tag{6.94}$$

where T_4 is given by (6.53).

Proposition 6.18 *Let $N \in \mathbb{N}$, $\alpha > \beta_i \geq 0$, $m_i = [\beta_i] + 1$, $m = \min\{m_i\}$ for $i = 1,\dots,N$ and $n = [\alpha]+1$. Let $f \in C^{\alpha}_{b-}[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m-1,\dots,n-2$ and let ${}^{C_1}D^{\alpha}_{b-}f \in L_\infty[a,b]$. Let w be continuous nonnegative weight function on $[x,b]$. Let $p > 0$, $q \geq 0$, $r_i \geq 0$, $r = \sum_{i=1}^{N} r_i > 0$. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b w(t) \prod_{i=1}^{N} |{}^{C_1}D^{\beta_i}_{b-}f(t)|^{r_i p}\, |{}^{C_1}D^{\alpha}_{b-}f(t)|^q\, dt$$
$$\leq \frac{(b-x)^{\sum\limits_{i=1}^{N} r_i(\alpha-\beta_i)p+1}}{\left[\sum\limits_{i=1}^{N} r_i(\alpha-\beta_i)p+1\right] \prod\limits_{i=1}^{N} [\Gamma(\alpha-\beta_i+1)]^{r_i p}} \|w\|_\infty \|{}^{C_1}D^{\alpha}_{b-}f\|_\infty^{rp+q}. \tag{6.95}$$

Theorem 6.53 *Let $N \in \mathbb{N}$, $\alpha > \beta_i \geq 0$, $m_i = [\beta_i]+1$, $m = \min\{m_i\}$ for $i = 1,\dots,N$ and $n = [\alpha]+1$. Let $f \in C^{\alpha}_{b-}[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m-1,\dots,n-2$. Let w be continuous positive increasing weight function on $[x,b]$. Let $r_i \geq 0$, $r = \sum_{i=1}^{N} r_i \geq 1$. Let $p < 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 0$, $\rho = \sum_{i=1}^{N} r_i(\alpha-\beta_i) - r\sigma$. Let ${}^{C_1}D^{\alpha}_{b-}f \in L_{p+q}[a,b]$ be of fixed sign on $[a,b]$, with $1/{}^{C_1}D^{\alpha}_{b-}f \in L_{p+q}[a,b]$. Then reverse inequality in* (6.92) *holds.*

Theorem 6.54 *Let $N \in \mathbb{N}$, $\alpha > \beta_i \geq 0$, $m_i = [\beta_i]+1$, $m = \min\{m_i\}$ for $i = 1,\dots,N$ and $n = [\alpha]+1$. Let $f \in C^{\alpha}_{b-}[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m-1,\dots,n-2$. Let $r_i \geq 0$, $r = \sum_{i=1}^{N} r_i > 0$. Let $p > 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 1$, $\rho = \sum_{i=1}^{N} r_i(\alpha-\beta_i) - r\sigma$ and $\alpha > \beta_i + \sigma$ for $i = 1,\dots,N$. Let ${}^{C_1}D^{\alpha}_{b-}f \in L_{p+q}[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_x^b \prod_{i=1}^{N} |{}^{C_1}D^{\beta_i}_{b-}f(t)|^{r_i p} |{}^{C_1}D^{\alpha}_{b-}f(t)|^q\, dt \leq T_4\, (b-x)^{(\rho+\sigma)p} \left(\int_x^b |{}^{C_1}D^{\alpha}_{b-}f(t)|^{p+q} dt\right)^{\frac{pr+q}{p+q}}, \tag{6.96}$$

where T_4 is given by (6.53).

Inequality (6.96) *is sharp for $\alpha = \beta_i + 1$, $i = 1,\dots,N$ and $q = 1$, where equality is attained for a function f such that ${}^{C_1}D^{\alpha}_{b-}f(t) = (x-t)^{\frac{p(\alpha-\beta_1)}{q}}$, $t \in [x,b]$.*

Next inequality is a two-weighted extension of Alzer's inequality (1.26)

$$\int_a^b w_1(t) \left(\prod_{i=1}^{N} |\mathbf{D}^{\beta_i} f(t)|^{r_i}\right)^p |\mathbf{D}^{\alpha} f(t)|^q\, dt$$
$$\leq K \left(\int_a^b [w_1(t)]^\gamma dt\right)^{\frac{1}{\gamma}} \left(\int_a^b w_2(t)\, |\mathbf{D}^{\alpha} f(t)|^{p+q} dt\right)^{\frac{pr+q}{p+q}},$$

where $w_1, w_2 \in C[a,b]$ are positive weight functions, $\alpha > \beta_i \geq 0$, $K > 0$ is a constant, $r = \sum r_i$, and $p, q, \gamma \in \mathbb{R}$.

It is given for the left-sided Riemann-Liouville fractional derivatives in [41]. In [24] Andrić-Pečarić-Perić give its version involving the left-sided Canavati fractional derivatives under new and relaxed conditions using improved composition identity for the Canavati fractional derivatives and monotonous weight functions (here Theorem 6.67), and also version including bounded weight functions (Theorem 6.68). We give analogous results for the Riemann-Liouville and the Caputo fractional derivatives.

THE RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVES

Theorem 6.55 *Let $N \in \mathbb{N}$, $\alpha > \beta_i \geq 0$ for $i = 1, \ldots, N$. Suppose that one of conditions $(i) - (vii)$ in Corollary 2.21 holds for all pairs $\{\alpha, \beta_i\}$, $i = 1, \ldots, N$. Let w_1 and w_2 be continuous positive weight functions on $[a,x]$ and let w_2 be decreasing function. Let $r_i \geq 0$, $r = \sum_{i=1}^N r_i > 0$. Let $s, s' > 1$ with $\frac{1}{s} + \frac{1}{s'} = 1$. Let $p > 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 1$, $\rho = \sum_{i=1}^N r_i(\alpha - \beta_i) - r\sigma$ and $\alpha > \beta_i + \sigma$ for $i = 1, \ldots, N$. Let $D_{a+}^{\alpha} f \in L_{p+q}[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\begin{aligned}
&\int_a^x w_1(t) \prod_{i=1}^N |D_{a+}^{\beta_i} f(t)|^{r_i p} \, |D_{a+}^{\alpha} f(t)|^q \, dt \\
&\leq T_5 \, R(x) \, (x-a)^{\rho p + \frac{\sigma p}{s}} \, [w_2(x)]^{-\sigma(rp+q)} \left(\int_a^x w_2(t) \, |D_{a+}^{\alpha} f(t)|^{p+q} \, dt \right)^{\frac{pr+q}{p+q}},
\end{aligned} \tag{6.97}$$

where

$$R(x) = \left(\int_a^x [w_1(t)]^{\frac{s'}{p\sigma}} \, dt \right)^{\frac{\sigma p}{s'}}, \tag{6.98}$$

$$T_5 = \frac{\sigma^{\frac{\sigma p}{s}} \, q^{\sigma q} \, (1-\sigma)^{(1-\sigma)rp}}{(\rho s + \sigma)^{\frac{\sigma p}{s}} \, (rp+q)^{\sigma q} \prod_{i=1}^N [\Gamma(\alpha - \beta_i) \, (\alpha - \beta_i - \sigma)^{1-\sigma}]^{r_i p}}. \tag{6.99}$$

Proof. Let $q \neq 0$, $\delta_i = \alpha - \beta_i - 1$, $i = 1, \ldots, N$. Using Theorem 2.13, the triangle inequality and Hölder's inequality for $\{\frac{1}{1-\sigma}, \frac{1}{\sigma}\}$, for $t \in [a,x]$ follows

$$\begin{aligned}
&|D_{a+}^{\beta_i} f(t)| \\
&\leq \frac{1}{\Gamma(\delta_i + 1)} \int_a^t (t-\tau)^{\delta_i} \, |D_{a+}^{\alpha} f(\tau)| \, d\tau \\
&= \frac{1}{\Gamma(\delta_i + 1)} \int_a^t [w_2(\tau)]^{-\sigma} \, [w_2(\tau)]^{\sigma} \, (t-\tau)^{\delta_i} \, |D_{a+}^{\alpha} f(\tau)| \, d\tau \\
&\leq \frac{1}{\Gamma(\delta_i + 1)} \left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}} \, (t-\tau)^{\frac{\delta_i}{1-\sigma}} \, d\tau \right)^{1-\sigma} \left(\int_a^t w_2(\tau) \, \left| D_{a+}^{\alpha} f(\tau) \right|^{\frac{1}{\sigma}} \, d\tau \right)^{\sigma}.
\end{aligned}$$

Now we have

$$\begin{aligned}
&\int_a^x w_1(t) \prod_{i=1}^N |D_{a+}^{\beta_i} f(t)|^{r_i p} \, |D_{a+}^{\alpha} f(t)|^q \, dt \\
&\leq \frac{1}{\prod_{i=1}^N [\Gamma(\delta_i + 1)]^{r_i p}} \int_a^x w_1(t) \, [w_2(t)]^{-\sigma q} \, [w_2(t)]^{\sigma q} \, |D_{a+}^{\alpha} f(t)|^q
\end{aligned} \tag{6.100}$$

$$\cdot\left(\int_a^t w_2(\tau)\,|D^{\alpha}_{a+}f(\tau)|^{\frac{1}{\sigma}}\,d\tau\right)^{\sigma rp}\prod_{i=1}^{N}\left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}}\,(t-\tau)^{\frac{\delta_i}{1-\sigma}}\,d\tau\right)^{(1-\sigma)r_i p} dt\,. \tag{6.101}$$

By Hölder's inequality for $\{\frac{1}{\sigma p},\frac{1}{\sigma q}\}$, we get

$$\begin{aligned}
&\int_a^x w_1(t)\prod_{i=1}^{N}|D^{\beta_i}_{a+}f(t)|^{r_i p}\,|D^{\alpha}_{a+}f(t)|^{q}\,dt\\
&\le \frac{1}{\prod_{i=1}^{N}[\Gamma(\delta_i+1)]^{r_i p}}\\
&\quad\cdot\left[\int_a^x [w_1(t)]^{\frac{1}{p\sigma}}\,[w_2(t)]^{-\frac{q}{p}}\prod_{i=1}^{N}\left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}}\,(t-\tau)^{\frac{\delta_i}{1-\sigma}}\,d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}} dt\right]^{\sigma p}\\
&\quad\cdot\left[\int_a^x w_2(t)\,|D^{\alpha}_{a+}f(t)|^{\frac{1}{\sigma}}\left(\int_a^t w_2(\tau)\,|D^{\alpha}_{a+}f(\tau)|^{\frac{1}{\sigma}}\,d\tau\right)^{\frac{p}{q}r} dt\right]^{\sigma q}\\
&= \frac{1}{\prod_{i=1}^{N}[\Gamma(\delta_i+1)]^{r_i p}}\left(\frac{q}{rp+q}\right)^{\sigma q}\left(\int_a^x w_2(t)\,|D^{\alpha}_{a+}f(t)|^{\frac{1}{\sigma}}\,dt\right)^{\sigma(rp+q)}\\
&\quad\cdot\left[\int_a^x [w_1(t)]^{\frac{1}{p\sigma}}\,[w_2(t)]^{-\frac{q}{p}}\prod_{i=1}^{N}\left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}}\,(t-\tau)^{\frac{\delta_i}{1-\sigma}}\,d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}} dt\right]^{\sigma p}.
\end{aligned} \tag{6.102}$$

Since w_2 is decreasing, we have

$$\begin{aligned}
&\int_a^x [w_1(t)]^{\frac{1}{p\sigma}}\,[w_2(t)]^{-\frac{q}{p}}\prod_{i=1}^{N}\left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}}\,(t-\tau)^{\frac{\delta_i}{1-\sigma}}\,d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}} dt\\
&\le \int_a^x [w_1(t)]^{\frac{1}{p\sigma}}\,[w_2(t)]^{-\frac{q}{p}}\prod_{i=1}^{N}\left(\int_a^t [w_2(t)]^{-\frac{\sigma}{1-\sigma}}\,(t-\tau)^{\frac{\delta_i}{1-\sigma}}\,d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}} dt\\
&= \int_a^x [w_1(t)]^{\frac{1}{p\sigma}}\,[w_2(t)]^{-\frac{q}{p}-r}\prod_{i=1}^{N}\left(\int_a^t (t-\tau)^{\frac{\delta_i}{1-\sigma}}\,d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}} dt\\
&\le \frac{(1-\sigma)^{\frac{(1-\sigma)r}{\sigma}}}{\prod_{i=1}^{N}(\delta_i+1-\sigma)^{\frac{(1-\sigma)r_i}{\sigma}}}\,[w_2(x)]^{-\frac{q}{p}-r}\int_a^x [w_1(t)]^{\frac{1}{p\sigma}}\,(t-a)^{\frac{\Sigma_{i=1}^{N}(\delta_i+1-\sigma)r_i}{\sigma}}\,dt\,.
\end{aligned}$$

Again we use Hölder's inequality for $\{s,s'\}$ to obtain

$$\int_a^x [w_1(t)]^{\frac{1}{p\sigma}}\,(t-a)^{\frac{\rho}{\sigma}}\,dt \le \left(\int_a^x [w_1(t)]^{\frac{s'}{p\sigma}}\,dt\right)^{\frac{1}{s'}}\left(\int_a^x (t-a)^{\frac{\rho}{\sigma}s}\,dt\right)^{\frac{1}{s}}$$

$$= [R(x)]^{\frac{1}{\sigma p}} (x-a)^{\frac{\rho}{\sigma}+\frac{1}{s}} \frac{\sigma^{\frac{1}{s}}}{(\rho s+\sigma)^{\frac{1}{s}}}.$$

Hence,

$$\begin{aligned}
&\left[\int_a^x [w_1(t)]^{\frac{1}{p\sigma}} [w_2(t)]^{-\frac{q}{p}} \prod_{i=1}^N \left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}} (t-\tau)^{\frac{\delta_i}{1-\sigma}} d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}} dt\right]^{\sigma p} \\
&\leq \frac{(1-\sigma)^{(1-\sigma)rp}}{\prod_{i=1}^N (\delta_i+1-\sigma)^{(1-\sigma)r_i p}} [w_2(x)]^{-\sigma(rp+q)} \left[\int_a^x [w_1(t)]^{\frac{1}{p\sigma}} (t-a)^{\frac{\rho}{\sigma}} dt\right]^{\sigma p} \\
&\leq \frac{(1-\sigma)^{(1-\sigma)rp}}{\prod_{i=1}^N (\delta_i+1-\sigma)^{(1-\sigma)r_i p}} [w_2(x)]^{-\sigma(rp+q)} R(x) \frac{\sigma^{\frac{\sigma p}{s}} (x-a)^{\rho p+\frac{\sigma p}{s}}}{(\rho s+\sigma)^{\frac{\sigma p}{s}}}
\end{aligned} \tag{6.103}$$

Inequality (6.97) now follows from (6.102) and (6.103).
If $q=0$ (and $\sigma = \frac{1}{p} < 1$), then the proof after (6.100) and (6.101) simplifies, that is

$$\begin{aligned}
&\int_a^x w_1(t) \prod_{i=1}^N |D_{a+}^{\beta_i} f(t)|^{r_i p} dt \\
&\leq \frac{1}{\prod_{i=1}^N [\Gamma(\delta_i+1)]^{r_i p}} \int_a^x w_1(t) \left(\int_a^t w_2(\tau) |D_{a+}^{\alpha} f(\tau)|^p d\tau\right)^r \\
&\qquad \cdot \prod_{i=1}^N \left(\int_a^t [w_2(\tau)]^{\frac{1}{1-p}} (t-\tau)^{\frac{\delta_i p}{p-1}} d\tau\right)^{(p-1)r_i} dt \\
&\leq \frac{1}{\prod_{i=1}^N [\Gamma(\delta_i+1)]^{r_i p}} \left(\int_a^x w_2(\tau) |D_{a+}^{\alpha} f(\tau)|^p d\tau\right)^r \\
&\qquad \cdot \int_a^x w_1(t) [w_2(t)]^{-r} \prod_{i=1}^N \left(\int_a^t (t-\tau)^{\frac{\delta_i p}{p-1}} d\tau\right)^{(p-1)r_i} dt \\
&= \frac{1}{\prod_{i=1}^N [\Gamma(\delta_i+1)]^{r_i p}} \left(\int_a^x w_2(\tau) |D_{a+}^{\alpha} f(\tau)|^p d\tau\right)^r \frac{(p-1)^{(p-1)r}}{\prod_{i=1}^N [\delta_i p+p-1]^{(p-1)r_i}} \\
&\qquad \cdot \int_a^x w_1(t) [w_2(t)]^{-r} (t-a)^{\Sigma_{i=1}^N (\delta_i p+p-1)r_i} dt \\
&\leq \frac{(p-1)^{(p-1)r} [w_2(x)]^{-r}}{\prod_{i=1}^N \left[\Gamma(\delta_i+1) (\delta_i p+p-1)^{\frac{p-1}{p}}\right]^{r_i p}} \left(\int_a^x w_2(\tau) |D_{a+}^{\alpha} f(\tau)|^p d\tau\right)^r \\
&\qquad \cdot \left(\int_a^x [w_1(t)]^{s'} dt\right)^{\frac{1}{s'}} \left(\int_a^x (t-a)^{\Sigma_{i=1}^N (\delta_i p+p-1)r_i s} dt\right)^{\frac{1}{s}}
\end{aligned}$$

$$= \frac{(p-1)^{(p-1)r}[w_2(x)]^{-r}}{\prod_{i=1}^{N}\left[\Gamma(\delta_i+1)(\delta_i p+p-1)^{\frac{p-1}{p}}\right]^{r_i p}} \left(\int_a^x w_2(\tau)\,|D_{a+}^{\alpha}f(\tau)|^p\,d\tau\right)^r$$
$$\cdot\left(\int_a^x [w_1(t)]^{s'}\,dt\right)^{\frac{1}{s'}} \frac{(x-a)^{\sum_{i=1}^{N}(\delta_i p+p-1)r_i+\frac{1}{s}}}{\left[\sum_{i=1}^{N}(\delta_i p+p-1)r_i+1\right]^{\frac{1}{s}}}.$$

□

For the next theorem we suppose that weight functions are bounded.

Theorem 6.56 *Let $N\in\mathbb{N}$, $\alpha>\beta_i\geq 0$ for $i=1,\dots,N$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.21 holds for all pairs $\{\alpha,\beta_i\}$, $i=1,\dots,N$. Let w_1 and w_2 be continuous positive weight functions on $[a,x]$ such that $w_1(t)\leq B$ and $A\leq w_2(t)$ for $t\in[a,x]$. Let $r_i\geq 0$, $r=\sum_{i=1}^{N}r_i>0$. Let $p>0$, $q\geq 0$, $\sigma=\frac{1}{p+q}<1$, $\rho=\sum_{i=1}^{N}r_i(\alpha-\beta_i)-r\sigma$ and $\alpha>\beta_i+\sigma$ for $i=1,\dots,N$. Let $D_{a+}^{\alpha}f\in L_{p+q}[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$\int_a^x w_1(t)\prod_{i=1}^{N}|D_{a+}^{\beta_i}f(t)|^{r_i p}\,|D_{a+}^{\alpha}f(t)|^q\,dt$$
$$\leq T_4\,(x-a)^{(\rho+\sigma)p}\,BA^{-\sigma(rp+q)}\left(\int_a^x w_2(t)\,|D_{a+}^{\alpha}f(t)|^{p+q}\,dt\right)^{\frac{pr+q}{p+q}}, \tag{6.104}$$

where T_4 is given by (6.53).

Proof. We start the proof with obtained inequality (6.102)

$$\int_a^x w_1(t)\prod_{i=1}^{N}|D_{a+}^{\beta_i}f(t)|^{r_i p}\,|D_{a+}^{\alpha}f(t)|^q\,dt$$
$$\leq \frac{1}{\prod_{i=1}^{N}[\Gamma(\delta_i+1)]^{r_i p}}\left(\frac{q}{rp+q}\right)^{\sigma q}\left(\int_a^x w_2(t)\,|D_{a+}^{\alpha}f(t)|^{\frac{1}{\sigma}}\,dt\right)^{\sigma(rp+q)}$$
$$\cdot\left[\int_a^x [w_1(t)]^{\frac{1}{p\sigma}}\,[w_2(t)]^{-\frac{q}{p}}\prod_{i=1}^{N}\left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}}\,(t-\tau)^{\frac{\delta_i}{1-\sigma}}\,d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}}dt\right]^{\sigma p}.$$

From conditions $w_1(t)\leq B$ and $A\leq w_2(t)$ for $t\in[a,x]$ we have

$$\left[\int_a^x [w_1(t)]^{\frac{1}{p\sigma}}\,[w_2(t)]^{-\frac{q}{p}}\prod_{i=1}^{N}\left(\int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}}\,(t-\tau)^{\frac{\delta_i}{1-\sigma}}\,d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}}dt\right]^{\sigma p}$$
$$\leq BA]^{-\sigma(rp+q)}\left[\int_a^x\prod_{i=1}^{N}\left(\int_a^t (t-\tau)^{\frac{\delta_i}{1-\sigma}}\,d\tau\right)^{\frac{(1-\sigma)r_i}{\sigma}}dt\right]^{\sigma p}$$

$$= BA^{-\sigma(rp+q)} \frac{(1-\sigma)^{(1-\sigma)rp}}{\prod_{i=1}^{N} (\delta_i + 1 - \sigma)^{(1-\sigma)r_i p}} (x-a)^{(\rho+\sigma)p} \frac{\sigma^{\sigma p}}{(\rho+\sigma)^{\sigma p}}. \tag{6.105}$$

The inequality (6.104) now follows from (6.102) and (6.105). □

Remark 6.4 For the extreme case we have one weight function, only w_1, and this is given in Proposition 6.13.

Next is the case for $p+q<0$. Conditions on p and q allow us to use reverse Hölder's inequalities, for $\{\frac{1}{1-\sigma} \in (0,1), \frac{1}{\sigma} < 0\}$ and $\{\frac{1}{\sigma p} \in (0,1), \frac{1}{\sigma q} < 0\}$. The proof is similar to the proof of Theorem 6.55.

Theorem 6.57 *Let $N \in \mathbb{N}$, $\alpha > \beta_i \geq 0$ for $i = 1,\ldots,N$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.21 holds for all pairs $\{\alpha,\beta_i\}$, $i = 1,\ldots,N$. Let w_1 and w_2 be continuous positive weight functions on $[a,x]$ and let w_2 be decreasing function. Let $r_i \geq 0$, $r = \sum_{i=1}^{N} r_i > 0$. Let $s \in (0,1)$, $s' < 0$ with $\frac{1}{s} + \frac{1}{s'} = 1$. Let $p < 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 0$, $\rho = \sum_{i=1}^{N} r_i(\alpha-\beta_i) - r\sigma$. Let $D^{\alpha}_{a+}f \in L_{p+q}[a,b]$ be of fixed sign on $[a,b]$, with $1/D^{\alpha}_{a+}f \in L_{p+q}[a,b]$. Then reverse inequality in (6.97) holds.*

Following inequalities include the right-sided Riemann-Liouville fractional derivatives.

Theorem 6.58 *Let $N \in \mathbb{N}$, $\alpha > \beta_i \geq 0$ for $i = 1,\ldots,N$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.22 holds for all pairs $\{\alpha,\beta_i\}$, $i = 1,\ldots,N$. Let w_1 and w_2 be continuous positive weight functions on $[x,b]$ and let w_2 be increasing function. Let $r_i \geq 0$, $r = \sum_{i=1}^{N} r_i > 0$. Let $s,s' > 1$ with $\frac{1}{s} + \frac{1}{s'} = 1$. Let $p > 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 1$, $\rho = \sum_{i=1}^{N} r_i(\alpha-\beta_i) - r\sigma$ and $\alpha > \beta_i + \sigma$ for $i = 1,\ldots,N$. Let $D^{\alpha}_{b-}f \in L_{p+q}[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\begin{aligned} &\int_x^b w_1(t) \prod_{i=1}^{N} |D^{\beta_i}_{b-}f(t)|^{r_i p} \, |D^{\alpha}_{b-}f(t)|^q \, dt \\ &\leq T_5 \widetilde{R}(x) \, (b-x)^{\rho p + \frac{\sigma p}{s}} \, [w_2(x)]^{-\sigma(rp+q)} \left(\int_x^b w_2(t) \, |D^{\alpha}_{b-}f(t)|^{p+q} \, dt \right)^{\frac{pr+q}{p+q}}, \end{aligned} \tag{6.106}$$

where T_5 is given by (6.99) and

$$\widetilde{R}(x) = \left(\int_x^b [w_1(t)]^{\frac{s'}{p\sigma}} \, dt \right)^{\frac{\sigma p}{s'}}. \tag{6.107}$$

Theorem 6.59 *Let $N \in \mathbb{N}$, $\alpha > \beta_i \geq 0$ for $i = 1,\ldots,N$. Suppose that one of conditions $(i)-(vii)$ in Corollary 2.22 holds for all pairs $\{\alpha,\beta_i\}$, $i = 1,\ldots,N$. Let w_1 and w_2 be continuous positive weight functions on $[x,b]$ such that $w_1(t) \leq B$ and $A \leq w_2(t)$ for $t \in$*

$[x,b]$. Let $r_i \geq 0$, $r = \sum_{i=1}^{N} r_i > 0$. Let $p > 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 1$, $\rho = \sum_{i=1}^{N} r_i(\alpha - \beta_i) - r\sigma$ and $\alpha > \beta_i + \sigma$ for $i = 1, \ldots, N$. Let $D_{b-}^{\alpha} f \in L_{p+q}[a,b]$. Then for a.e. $x \in [a,b]$ holds

$$\int_x^b w_1(t) \prod_{i=1}^{N} |D_{b-}^{\beta_i} f(t)|^{r_i p} \, |D_{b-}^{\alpha} f(t)|^q \, dt$$
$$\leq T_4 \, (b-x)^{(\rho+\sigma)p} \, BA^{-\sigma(rp+q)} \left(\int_x^b w_2(t) \, |D_{b-}^{\alpha} f(t)|^{p+q} \, dt \right)^{\frac{pr+q}{p+q}}, \tag{6.108}$$

where T_4 is given by (6.53).

Remark 6.5 For the extreme case we have one weight function, only w_1, and this is given in Proposition 6.14.

Theorem 6.60 *Let $N \in \mathbb{N}$, $\alpha > \beta_i \geq 0$ for $i = 1, \ldots, N$. Suppose that one of conditions $(i) - (vii)$ in Corollary 2.22 holds for all pairs $\{\alpha, \beta_i\}$, $i = 1, \ldots, N$. Let w_1 and w_2 be continuous positive weight functions on $[x,b]$ and let w_2 be increasing function. Let $r_i \geq 0$, $r = \sum_{i=1}^{N} r_i > 0$. Let $s \in (0,1)$, $s' < 0$ with $\frac{1}{s} + \frac{1}{s'} = 1$. Let $p < 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 0$, $\rho = \sum_{i=1}^{N} r_i(\alpha - \beta_i) - r\sigma$. Let $D_{b-}^{\alpha} f \in L_{p+q}[a,b]$ be of fixed sign on $[a,b]$, with $1/D_{b-}^{\alpha} f \in L_{p+q}[a,b]$. Then reverse inequality in (6.106) holds.*

THE CAPUTO FRACTIONAL DERIVATIVES

Theorem 6.61 *Let $N \in \mathbb{N}$, $i = 1, \ldots, N$, $\alpha > \beta_i \geq 0$ and m_i, n given by (2.70) with $m = \min\{m_i\}$. Let $f \in AC^n[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m, \ldots, n-1$. Let w_1 and w_2 be continuous positive weight functions on $[a,x]$ and let w_2 be decreasing function. Let $r_i \geq 0$, $r = \sum_{i=1}^{N} r_i > 0$. Let $s, s' > 1$ with $\frac{1}{s} + \frac{1}{s'} = 1$. Let $p > 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 1$, $\rho = \sum_{i=1}^{N} r_i(\alpha - \beta_i) - r\sigma$ and $\alpha > \beta_i + \sigma$ for $i = 1, \ldots, N$. Let ${}^{C}D_{a+}^{\alpha} f \in L_{p+q}[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x w_1(t) \prod_{i=1}^{N} |{}^{C}D_{a+}^{\beta_i} f(t)|^{r_i p} \, |{}^{C}D_{a+}^{\alpha} f(t)|^q \, dt$$
$$\leq T_5 \, R(x) \, (x-a)^{\rho p + \frac{\sigma p}{s}} \, [w_2(x)]^{-\sigma(rp+q)} \left(\int_a^x w_2(t) \, |{}^{C}D_{a+}^{\alpha} f(t)|^{p+q} \, dt \right)^{\frac{pr+q}{p+q}}, \tag{6.109}$$

where R and T_5 are given by (6.98) and (6.99), respectively.

For the next theorem we suppose that weight functions are bounded.

Theorem 6.62 *Let $N \in \mathbb{N}$, $i = 1, \ldots, N$, $\alpha > \beta_i \geq 0$ and m_i, n given by (2.70) with $m = \min\{m_i\}$. Let $f \in AC^n[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m, \ldots, n-1$. Let w_1 and w_2 be continuous positive weight functions on $[a,x]$ such that $w_1(t) \leq B$ and $A \leq w_2(t)$ for $t \in$*

$[a,x]$. *Let* $r_i \geq 0$, $r = \sum_{i=1}^N r_i > 0$. *Let* $p > 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 1$, $\rho = \sum_{i=1}^N r_i(\alpha - \beta_i) - r\sigma$ *and* $\alpha > \beta_i + \sigma$ *for* $i = 1,\dots,N$. *Let* ${}^C D^{\alpha}_{a+} f \in L_{p+q}[a,b]$. *Then for a.e.* $x \in [a,b]$ *holds*

$$\begin{aligned} &\int_a^x w_1(t) \prod_{i=1}^N |{}^C D^{\beta_i}_{a+} f(t)|^{r_i p} \, |{}^C D^{\alpha}_{a+} f(t)|^q \, dt \\ &\leq T_4 \, (x-a)^{(\rho+\sigma)p} \, B A^{-\sigma(rp+q)} \left(\int_a^x w_2(t) \, |{}^C D^{\alpha}_{a+} f(t)|^{p+q} \, dt \right)^{\frac{pr+q}{p+q}}, \end{aligned} \tag{6.110}$$

where T_4 *is given by* (6.53).

Theorem 6.63 *Let* $N \in \mathbb{N}$, $i = 1,\dots,N$, $\alpha > \beta_i \geq 0$ *and* m_i, n *given by* (2.70) *with* $m = \min\{m_i\}$. *Let* $f \in AC^n[a,b]$ *be such that* $f^{(i)}(a) = 0$ *for* $i = m,\dots,n-1$. *Let* w_1 *and* w_2 *be continuous positive weight functions on* $[a,x]$ *and let* w_2 *be decreasing function. Let* $r_i \geq 0$, $r = \sum_{i=1}^N r_i > 0$. *Let* $s \in (0,1)$, $s' < 0$ *with* $\frac{1}{s} + \frac{1}{s'} = 1$. *Let* $p < 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 0$, $\rho = \sum_{i=1}^N r_i(\alpha - \beta_i) - r\sigma$. *Let* ${}^C D^{\alpha}_{a+} f \in L_{p+q}[a,b]$ *be of fixed sign on* $[a,b]$, *with* $1/{}^C D^{\alpha}_{a+} f \in L_{p+q}[a,b]$. *Then reverse inequality in* (6.109) *holds.*

Following inequalities include the right-sided Caputo fractional derivatives.

Theorem 6.64 *Let* $N \in \mathbb{N}$, $i = 1,\dots,N$, $\alpha > \beta_i \geq 0$ *and* m_i, n *given by* (2.70) *with* $m = \min\{m_i\}$. *Let* $f \in AC^n[a,b]$ *be such that* $f^{(i)}(b) = 0$ *for* $i = m,\dots,n-1$. *Let* w_1 *and* w_2 *be continuous positive weight functions on* $[x,b]$ *and let* w_2 *be increasing function. Let* $r_i \geq 0$, $r = \sum_{i=1}^N r_i > 0$. *Let* $s, s' > 1$ *with* $\frac{1}{s} + \frac{1}{s'} = 1$. *Let* $p > 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 1$, $\rho = \sum_{i=1}^N r_i(\alpha - \beta_i) - r\sigma$ *and* $\alpha > \beta_i + \sigma$ *for* $i = 1,\dots,N$. *Let* ${}^C D^{\alpha}_{b-} f \in L_{p+q}[a,b]$. *Then for a.e.* $x \in [a,b]$ *holds*

$$\begin{aligned} &\int_x^b w_1(t) \prod_{i=1}^N |{}^C D^{\beta_i}_{b-} f(t)|^{r_i p} \, |{}^C D^{\alpha}_{b-} f(t)|^q \, dt \\ &\leq T_5 \widetilde{R}(x) \, (b-x)^{\rho p + \frac{\sigma p}{s}} \, [w_2(x)]^{-\sigma(rp+q)} \left(\int_x^b w_2(t) \, |{}^C D^{\alpha}_{b-} f(t)|^{p+q} \, dt \right)^{\frac{pr+q}{p+q}}, \end{aligned} \tag{6.111}$$

where $\widetilde{R}$ *and* T_5 *are given by* (6.107) *and* (6.99), *respectively.*

Theorem 6.65 *Let* $N \in \mathbb{N}$, $i = 1,\dots,N$, $\alpha > \beta_i \geq 0$ *and* m_i, n *given by* (2.70) *with* $m = \min\{m_i\}$. *Let* $f \in AC^n[a,b]$ *be such that* $f^{(i)}(b) = 0$ *for* $i = m,\dots,n-1$. *Let* w_1 *and* w_2 *be continuous positive weight functions on* $[x,b]$ *such that* $w_1(t) \leq B$ *and* $A \leq w_2(t)$ *for* $t \in [x,b]$. *Let* $r_i \geq 0$, $r = \sum_{i=1}^N r_i > 0$. *Let* $p > 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 1$, $\rho = \sum_{i=1}^N r_i(\alpha - \beta_i) - r\sigma$ *and* $\alpha > \beta_i + \sigma$ *for* $i = 1,\dots,N$. *Let* ${}^C D^{\alpha}_{a+} f \in L_{p+q}[a,b]$. *Then for a.e.* $x \in [a,b]$ *holds*

$$\int_x^b w_1(t) \prod_{i=1}^N |{}^C D^{\beta_i}_{b-} f(t)|^{r_i p} \, |{}^C D^{\alpha}_{b-} f(t)|^q \, dt$$

$$\leq T_4\,(b-x)^{(\rho+\sigma)p}\,BA^{-\sigma(rp+q)}\left(\int_x^b w_2(t)\,|^C D^{\alpha}_{b-}f(t)|^{p+q}\,dt\right)^{\frac{pr+q}{p+q}}, \tag{6.112}$$

where T_4 is given by (6.53).

Theorem 6.66 *Let $N \in \mathbb{N}$, $i = 1,\dots,N$, $\alpha > \beta_i \geq 0$ and m_i, n given by* (2.70) *with $m = \min\{m_i\}$. Let $f \in AC^n[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m,\dots,n-1$. Let w_1 and w_2 be continuous positive weight functions on $[x,b]$ and let w_2 be increasing function. Let $r_i \geq 0$, $r = \sum_{i=1}^N r_i > 0$. Let $s \in (0,1)$, $s' < 0$ with $\frac{1}{s}+\frac{1}{s'} = 1$. Let $p < 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 0$, $\rho = \sum_{i=1}^N r_i(\alpha-\beta_i) - r\sigma$. Let ${}^C D^{\alpha}_{b-}f \in L_{p+q}[a,b]$ be of fixed sign on $[a,b]$, with $1/{}^C D^{\alpha}_{b-}f \in L_{p+q}[a,b]$. Then reverse inequality in* (6.111) *holds.*

Extreme cases, with only one weight function w_1, are given in Proposition 6.15 and Proposition 6.16.

THE CANAVATI FRACTIONAL DERIVATIVES

Theorem 6.67 *Let $N \in \mathbb{N}$, $\alpha > \beta_i \geq 0$, $m_i = [\beta_i]+1$, $m = \min\{m_i\}$ for $i = 1,\dots,N$ and $n = [\alpha]+1$. Let $f \in C^{\alpha}_{a+}[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m-1,\dots,n-2$. Let w_1 and w_2 be continuous positive weight functions on $[a,x]$ and let w_2 be decreasing function. Let $r_i \geq 0$, $r = \sum_{i=1}^N r_i > 0$. Let $s,s' > 1$ with $\frac{1}{s}+\frac{1}{s'} = 1$. Let $p > 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 1$, $\rho = \sum_{i=1}^N r_i(\alpha-\beta_i) - r\sigma$ and $\alpha > \beta_i + \sigma$ for $i = 1,\dots,N$. Let ${}^{C_1}D^{\alpha}_{a+}f \in L_{p+q}[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x w_1(t)\prod_{i=1}^N |^{C_1}D^{\beta_i}_{a+}f(t)|^{r_ip}\,|^{C_1}D^{\alpha}_{a+}f(t)|^q\,dt$$

$$\leq T_5\,R(x)\,(x-a)^{\rho p+\frac{\sigma p}{s}}\,[w_2(x)]^{-\sigma(rp+q)}\left(\int_a^x w_2(t)\,|^{C_1}D^{\alpha}_{a+}f(t)|^{p+q}\,dt\right)^{\frac{pr+q}{p+q}}, \tag{6.113}$$

where R and T_5 are given by (6.98) *and* (6.99)*, respectively.*

For the next theorem we suppose that weight functions are bounded.

Theorem 6.68 *Let $N \in \mathbb{N}$, $\alpha > \beta_i \geq 0$, $m_i = [\beta_i]+1$, $m = \min\{m_i\}$ for $i = 1,\dots,N$ and $n = [\alpha]+1$. Let $f \in C^{\alpha}_{a+}[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m-1,\dots,n-2$. Let w_1 and w_2 be continuous positive weight functions on $[a,x]$ such that $w_1(t) \leq B$ and $A \leq w_2(t)$ for $t \in [a,x]$. Let $r_i \geq 0$, $r = \sum_{i=1}^N r_i > 0$. Let $p > 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 1$, $\rho = \sum_{i=1}^N r_i(\alpha-\beta_i) - r\sigma$ and $\alpha > \beta_i + \sigma$ for $i = 1,\dots,N$. Let ${}^{C_1}D^{\alpha}_{a+}f \in L_{p+q}[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\int_a^x w_1(t)\prod_{i=1}^N |^{C_1}D^{\beta_i}_{a+}f(t)|^{r_ip}\,|^{C_1}D^{\alpha}_{a+}f(t)|^q\,dt$$

$$\leq T_4\,(x-a)^{(\rho+\sigma)p}\,BA^{-\sigma(rp+q)}\left(\int_a^x w_2(t)\,|^{C_1}D^{\alpha}_{a+}f(t)|^{p+q}\,dt\right)^{\frac{pr+q}{p+q}},$$

where T_4 is given by (6.53).

Theorem 6.69 *Let $N \in \mathbb{N}$, $\alpha > \beta_i \geq 0$, $m_i = [\beta_i]+1$, $m = \min\{m_i\}$ for $i = 1,\dots,N$ and $n = [\alpha]+1$. Let $f \in C^{\alpha}_{a+}[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = m-1,\dots,n-2$. Let w_1 and w_2 be continuous positive weight functions on $[a,x]$ and let w_2 be decreasing function. Let $r_i \geq 0$, $r = \sum_{i=1}^N r_i > 0$. Let $s \in (0,1)$, $s' < 0$ with $\frac{1}{s}+\frac{1}{s'} = 1$. Let $p < 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 0$, $\rho = \sum_{i=1}^N r_i(\alpha-\beta_i) - r\sigma$. Let ${}^{C_1}D^{\alpha}_{a+}f \in L_{p+q}[a,b]$ be of fixed sign on $[a,b]$, with $1/{}^{C_1}D^{\alpha}_{a+}f \in L_{p+q}[a,b]$. Then reverse inequality in (6.113) holds.*

Following inequalities include the right-sided Canavati fractional derivatives.

Theorem 6.70 *Let $N \in \mathbb{N}$, $\alpha > \beta_i \geq 0$, $m_i = [\beta_i]+1$, $m = \min\{m_i\}$ for $i = 1,\dots,N$ and $n = [\alpha]+1$. Let $f \in C^{\alpha}_{b-}[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m-1,\dots,n-2$. Let w_1 and w_2 be continuous positive weight functions on $[x,b]$ and let w_2 be increasing function. Let $r_i \geq 0$, $r = \sum_{i=1}^N r_i > 0$. Let $s,s' > 1$ with $\frac{1}{s}+\frac{1}{s'} = 1$. Let $p > 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 1$, $\rho = \sum_{i=1}^N r_i(\alpha-\beta_i) - r\sigma$ and $\alpha > \beta_i+\sigma$ for $i = 1,\dots,N$. Let ${}^{C_1}D^{\alpha}_{b-}f \in L_{p+q}[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\begin{aligned} &\int_x^b w_1(t)\prod_{i=1}^N |{}^{C_1}D^{\beta_i}_{b-}f(t)|^{r_ip}\,|{}^{C_1}D^{\alpha}_{b-}f(t)|^q\,dt \\ &\leq T_5\,\widetilde{R}(x)\,(b-x)^{\rho p+\frac{\sigma p}{s}}\,[w_2(x)]^{-\sigma(rp+q)}\left(\int_x^b w_2(t)\,|{}^{C_1}D^{\alpha}_{b-}f(t)|^{p+q}\,dt\right)^{\frac{pr+q}{p+q}}, \end{aligned} \tag{6.114}$$

where $\widetilde{R}$ and T_5 are given by (6.107) and (6.99), respectively.

Theorem 6.71 *Let $N \in \mathbb{N}$, $\alpha > \beta_i \geq 0$, $m_i = [\beta_i]+1$, $m = \min\{m_i\}$ for $i = 1,\dots,N$ and $n = [\alpha]+1$. Let $f \in C^{\alpha}_{b-}[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m-1,\dots,n-2$. Let w_1 and w_2 be continuous positive weight functions on $[x,b]$ such that $w_1(t) \leq B$ and $A \leq w_2(t)$ for $t \in [x,b]$. Let $r_i \geq 0$, $r = \sum_{i=1}^N r_i > 0$. Let $p > 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 1$, $\rho = \sum_{i=1}^N r_i(\alpha-\beta_i) - r\sigma$ and $\alpha > \beta_i+\sigma$ for $i = 1,\dots,N$. Let ${}^{C_1}D^{\alpha}_{b-}f \in L_{p+q}[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\begin{aligned} &\int_x^b w_1(t)\prod_{i=1}^N |{}^{C_1}D^{\beta_i}_{b-}f(t)|^{r_ip}\,|{}^{C_1}D^{\alpha}_{b-}f(t)|^q\,dt \\ &\leq T_4\,(b-x)^{(\rho+\sigma)p}\,BA^{-\sigma(rp+q)}\left(\int_x^b w_2(t)\,|{}^{C_1}D^{\alpha}_{b-}f(t)|^{p+q}\,dt\right)^{\frac{pr+q}{p+q}}, \end{aligned} \tag{6.115}$$

where T_4 is given by (6.53).

Theorem 6.72 *Let $N \in \mathbb{N}$, $\alpha > \beta_i \geq 0$, $m_i = [\beta_i]+1$, $m = \min\{m_i\}$ for $i = 1,\dots,N$ and $n = [\alpha]+1$. Let $f \in C^{\alpha}_{b-}[a,b]$ be such that $f^{(i)}(b) = 0$ for $i = m-1,\dots,n-2$. Let w_1 and w_2 be continuous positive weight functions on $[x,b]$ and let w_2 be increasing function. Let $r_i \geq 0$, $r = \sum_{i=1}^N r_i > 0$. Let $s \in (0,1)$, $s' < 0$ with $\frac{1}{s}+\frac{1}{s'} = 1$. Let $p < 0$, $q \geq 0$, $\sigma = \frac{1}{p+q} < 0$, $\rho = \sum_{i=1}^N r_i(\alpha-\beta_i) - r\sigma$. Let ${}^{C_1}D^{\alpha}_{b-}f \in L_{p+q}[a,b]$ be of fixed sign on $[a,b]$, with $1/{}^{C_1}D^{\alpha}_{b-}f \in L_{p+q}[a,b]$. Then reverse inequality in (6.114) holds.*

Extreme cases, with only one weight function w_1, are given in Proposition 6.17 and Proposition 6.18.

The last inequality that we observe in this section is Alzer's inequality involving two functions (1.27). Applied on fraction derivatives it has a form

$$\begin{aligned}&\int_a^x w_1(t)\left[|\mathbf{D}^\beta g(t)|^p\,|\mathbf{D}^\alpha f(t)|^q+|\mathbf{D}^\beta f(t)|^p\,|\mathbf{D}^\alpha g(t)|^q\right]dt\\&\leq K\left(\int_a^x w_2(t)\left[|\mathbf{D}^\alpha f(t)|^r+|\mathbf{D}^\alpha g(t)|^r\right]dt\right)^{\frac{p+q}{r}},\end{aligned}$$

where $w_1,w_2\in C[a,b]$ are positive weight functions, $\alpha>\beta\geq 0$, $K>0$ is a constant and $p,q,r\in\mathbb{R}$.

This inequality is given in [12] for all three types of fractional derivatives. Its estimation is based on inequality due to Agarwal and Pang ([5]) given for ordinary derivatives. Since Alzer improved Agarwal-Pang's inequality in [11], Andrić-Pečarić-Perić applied Alzer's result on inequalities involving fractional derivatives in [27]. Some new inequalities, for the case $r<0$, is also given in [27].

THE RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVES

Theorem 6.73 *Let $\alpha>\beta\geq 0$ and suppose that one of conditions $(i)-(vii)$ in Corollary 2.21 holds for $\{\alpha,\beta,f\}$ and $\{\alpha,\beta,g\}$. Let $w_1\geq 0$ and $w_2>0$ be continuous weight functions on $[a,x]$. Let $r>1$, $r>q>0$ and $p\geq 0$. Let $D^\alpha_{a+}f, D^\alpha_{a+}g\in L_r[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$\begin{aligned}&\int_a^x w_1(t)\left[|D^\beta_{a+}g(t)|^p\,|D^\alpha_{a+}f(t)|^q+|D^\beta_{a+}f(t)|^p\,|D^\alpha_{a+}g(t)|^q\right]dt\\&\leq T_6\,M\,S(x)\left(\int_a^x w_2(t)\left[|D^\alpha_{a+}f(t)|^r+|D^\alpha_{a+}g(t)|^r\right]dt\right)^{\frac{p+q}{r}},\end{aligned}\tag{6.116}$$

where

$$S(x)=\left[\int_a^x [w_1(t)]^{\frac{r}{r-q}}\,[w_2(t)]^{\frac{q}{q-r}}\,[s(t)]^{\frac{p(r-1)}{r-q}}\,dt\right]^{\frac{r-q}{r}},\tag{6.117}$$

$$s(t)=\int_a^t (t-\tau)^{\frac{r(\alpha-\beta-1)}{r-1}}\,[w_2(\tau)]^{\frac{1}{1-r}}\,d\tau,\tag{6.118}$$

$$M=\begin{cases}\left(1-2^{-\frac{p}{q}}\right)^{\frac{q}{r}}, & p\geq q,\\ 2^{-\frac{p}{r}}, & p\leq q,\end{cases}\tag{6.119}$$

$$T_6=\frac{2}{[\Gamma(\alpha-\beta)]^p}\left[\frac{q}{2(p+q)}\right]^{\frac{q}{r}}.\tag{6.120}$$

Proof. Using Theorem 2.13, the triangle inequality and Hölder's inequality for $\{\frac{r}{r-1},r\}$, for $t\in[a,x]$ we have

$$|D^\beta_{a+}g(t)|\leq\frac{1}{\Gamma(\alpha-\beta)}\int_a^t (t-\tau)^{\alpha-\beta-1}\,|D^\alpha_{a+}g(\tau)|\,d\tau$$

$$= \frac{1}{\Gamma(\alpha-\beta)} \int_a^t (t-\tau)^{\alpha-\beta-1} \, [w_2(\tau)]^{-\frac{1}{r}} \, [w_2(\tau)]^{\frac{1}{r}} \, |D^{\alpha}_{a+} g(\tau)| \, d\tau$$

$$\leq \frac{1}{\Gamma(\alpha-\beta)} \left(\int_a^t (t-\tau)^{\frac{r(\alpha-\beta-1)}{r-1}} \, [w_2(\tau)]^{\frac{1}{1-r}} \, d\tau \right)^{\frac{r-1}{r}} \left(\int_a^t w_2(\tau) \, |D^{\alpha}_{a+} g(\tau)|^r \, d\tau \right)^{\frac{1}{r}}$$

$$= \frac{1}{\Gamma(\alpha-\beta)} \, [s(t)]^{\frac{r-1}{r}} \, [G(t)]^{\frac{1}{r}} \, , \tag{6.121}$$

where

$$G(t) = \int_a^t w_2(\tau) \, |D^{\alpha}_{a+} g(\tau)|^r \, d\tau \, . \tag{6.122}$$

Let

$$F(t) = \int_a^t w_2(\tau) \, |D^{\alpha}_{a+} f(\tau)|^r \, d\tau \, . \tag{6.123}$$

Then $F'(t) = w_2(t) \, |D^{\alpha}_{a+} f(t)|^r$, that is

$$|D^{\alpha}_{a+} f(t)|^q = \left[F'(t)\right]^{\frac{q}{r}} \, [w_2(t)]^{-\frac{q}{r}} \, . \tag{6.124}$$

Now (6.121) and (6.124) imply

$$w_1(t) \, |D^{\beta}_{a+} g(t)|^p \, |D^{\alpha}_{a+} f(t)|^q \leq h(t) \, [G(t)]^{\frac{p}{r}} \left[F'(t)\right]^{\frac{q}{r}} \, , \tag{6.125}$$

where

$$h(t) = \frac{1}{[\Gamma(\alpha-\beta)]^p} \, w_1(t) \, [w_2(t)]^{-\frac{q}{r}} \, [s(t)]^{\frac{p(r-1)}{r}} \, . \tag{6.126}$$

Integrating (6.125) and applying Hölder's inequality for $\{\frac{r}{r-q}, \frac{r}{q}\}$, we obtain

$$\begin{aligned} &\int_a^x w_1(t) \, |D^{\beta}_{a+} g(t)|^p \, |D^{\alpha}_{a+} f(t)|^q \, dt \\ &\leq \int_a^x h(t) \, [G(t)]^{\frac{p}{r}} \left[F'(t)\right]^{\frac{q}{r}} dt \\ &\leq \left(\int_a^x [h(t)]^{\frac{r}{r-q}} \, dt \right)^{\frac{r-q}{r}} \left(\int_a^x [G(t)]^{\frac{p}{q}} \, F'(t) \, dt \right)^{\frac{q}{r}} . \end{aligned} \tag{6.127}$$

Similarly we get

$$\begin{aligned} &\int_a^x w_1(t) \, |D^{\beta}_{a+} f(t)|^p \, |D^{\alpha}_{a+} g(t)|^q \, dt \\ &\leq \left(\int_a^x [h(t)]^{\frac{r}{r-q}} \, dt \right)^{\frac{r-q}{r}} \left(\int_a^x [F(t)]^{\frac{p}{q}} \, G'(t) \, dt \right)^{\frac{q}{r}} . \end{aligned} \tag{6.128}$$

In the next step we need simple inequalities

$$c_{\varepsilon}(A+B)^{\varepsilon} \leq A^{\varepsilon} + B^{\varepsilon} \leq d_{\varepsilon}(A+B)^{\varepsilon} \, , \quad (A, B \geq 0) \, , \tag{6.129}$$

where

$$c_{\varepsilon} = \begin{cases} 1 \, , \; 0 \leq \varepsilon \leq 1 \, , \\ 2^{1-\varepsilon} \, , \; \varepsilon \geq 1 \, , \end{cases} , \qquad d_{\varepsilon} = \begin{cases} 2^{1-\varepsilon} \, , \; 0 \leq \varepsilon \leq 1 \, , \\ 1 \, , \; \varepsilon \geq 1 \, . \end{cases}$$

Therefore, from (6.127), (6.128) and (6.129), with $r > q$, we conclude

$$\begin{aligned}
&\int_a^x w_1(t)\left[|D^{\beta}_{a+}g(t)|^p\,|D^{\alpha}_{a+}f(t)|^q + |D^{\beta}_{a+}f(t)|^p\,|D^{\alpha}_{a+}g(t)|^q\right]dt \\
&\le \left(\int_a^x [h(t)]^{\frac{r}{r-q}}\,dt\right)^{\frac{r-q}{r}}\left[\left(\int_a^x [G(t)]^{\frac{p}{q}}F'(t)\,dt\right)^{\frac{q}{r}} + \left(\int_a^x [F(t)]^{\frac{p}{q}}G'(t)\,dt\right)^{\frac{q}{r}}\right] \\
&\le \left(\int_a^x [h(t)]^{\frac{r}{r-q}}\,dt\right)^{\frac{r-q}{r}} 2^{1-\frac{q}{r}}\left(\int_a^x \left[[G(t)]^{\frac{p}{q}}F'(t) + [F(t)]^{\frac{p}{q}}G'(t)\right]dt\right)^{\frac{q}{r}}. \qquad (6.130)
\end{aligned}$$

Since $G(a) = F(a) = 0$, then with (6.129) follows

$$\begin{aligned}
&\int_a^x \left[[G(t)]^{\frac{p}{q}}F'(t) + [F(t)]^{\frac{p}{q}}G'(t)\right]dt \\
&= \int_a^x \left[[G(t)]^{\frac{p}{q}} + [F(t)]^{\frac{p}{q}}\right]\left[G'(t) + F'(t)\right]dt - \int_a^x \left[[G(t)]^{\frac{p}{q}}G'(t) + [F(t)]^{\frac{p}{q}}F'(t)\right]dt \\
&\le d_{\frac{p}{q}}\int_a^x [G(t)+F(t)]^{\frac{p}{q}}\,[G(t)+F(t)]'\,dt - \frac{q}{p+q}\left[G(x)^{\frac{p}{q}+1} + F(x)^{\frac{p}{q}+1}\right] \\
&= \frac{q}{p+q}d_{\frac{p}{q}}\,[G(x)+F(x)]^{\frac{p}{q}+1} - \frac{q}{p+q}\left[G(x)^{\frac{p}{q}+1} + F(x)^{\frac{p}{q}+1}\right] \\
&\le \frac{q}{p+q}\left(d_{\frac{p}{q}} - 2^{-\frac{p}{q}}\right)[G(x)+F(x)]^{\frac{p}{q}+1}. \qquad (6.131)
\end{aligned}$$

Hence from (6.130) and (6.131) we obtain

$$\begin{aligned}
&\int_a^x w_1(t)\left[|D^{\beta}_{a+}g(t)|^p\,|D^{\alpha}_{a+}f(t)|^q + |D^{\beta}_{a+}f(t)|^p\,|D^{\alpha}_{a+}g(t)|^q\right]dt \\
&\le 2^{1-\frac{q}{r}}\left(d_{\frac{p}{q}} - 2^{-\frac{p}{q}}\right)^{\frac{q}{r}}\left(\frac{q}{p+q}\right)^{\frac{q}{r}}\left(\int_a^x [h(t)]^{\frac{r}{r-q}}\,dt\right)^{\frac{r-q}{r}}[G(x)+F(x)]^{\frac{p+q}{r}} \\
&= \frac{2}{[\Gamma(\alpha-\beta)]^p}\left(d_{\frac{p}{q}} - 2^{-\frac{p}{q}}\right)^{\frac{q}{r}}\left[\frac{q}{2(p+q)}\right]^{\frac{q}{r}} S(x)\,[G(x)+F(x)]^{\frac{p+q}{r}}.
\end{aligned}$$

If $p \ge q$, then

$$\left(d_{\frac{p}{q}} - 2^{-\frac{p}{q}}\right)^{\frac{q}{r}} = \left(1 - 2^{-\frac{p}{q}}\right)^{\frac{q}{r}},$$

while for $p \le q$

$$\left(d_{\frac{p}{q}} - 2^{-\frac{p}{q}}\right)^{\frac{q}{r}} = \left(2^{1-\frac{p}{q}} - 2^{-\frac{p}{q}}\right)^{\frac{q}{r}} = 2^{-\frac{p}{r}},$$

which proves the theorem. □

The following result deals with the extreme case of the preceding theorem when $r = \infty$.

Proposition 6.19 *Let $\alpha > \beta_1, \beta_2 \ge 0$ and suppose that one of conditions $(i)-(vii)$ in Corollary 2.21 holds for $\{\alpha, \beta_i, f\}$ and $\{\alpha, \beta_i, g\}$, $i = 1, 2$. Let w be continuous nonnegative* *weight function on* $[a,x]$ *and let* $p,q_1,q_2 \geq 0$. *Let* $D^{\alpha}_{a+}f, D^{\alpha}_{a+}g \in L_\infty[a,b]$. *Then for a.e.* $x \in [a,b]$ *holds*

$$\begin{aligned}
&\int_a^x w(t)\left[|D^{\beta_1}_{a+}f(t)|^{q_1}\,|D^{\beta_2}_{a+}g(t)|^{q_2}\,|D^{\alpha}_{a+}f(t)|^{p}\right.\\
&\qquad\left.+|D^{\beta_2}_{a+}f(t)|^{q_2}\,|D^{\beta_1}_{a+}g(t)|^{q_1}\,|D^{\alpha}_{a+}g(t)|^{p}\right]dt\\
&\leq T_7\,(x-a)^{q_1(\alpha-\beta_1)+q_2(\alpha-\beta_2)+1}\,\|w\|_\infty\\
&\quad\cdot\left[\|D^{\alpha}_{a+}f\|_\infty^{2(q_1+p)}+\|D^{\alpha}_{a+}f\|_\infty^{2q_2}+\|D^{\alpha}_{a+}g\|_\infty^{2q_2}+\|D^{\alpha}_{a+}g\|_\infty^{2(q_1+p)}\right], \qquad (6.132)
\end{aligned}$$

where

$$T_7=\Big[2\,[\Gamma(\alpha-\beta_1+1)]^{q_1}\,[\Gamma(\alpha-\beta_2+1)]^{q_2}\,[q_1(\alpha-\beta_1)+q_2(\alpha-\beta_2)+1]\Big]^{-1}. \qquad (6.133)$$

Proof. Using Theorem 2.13, the triangle inequality and Hölder's inequality, for $i=1,2$ and $t\in[a,x]$ we have

$$\begin{aligned}
|D^{\beta_i}_{a+}f(t)|^{q_i} &\leq \frac{1}{[\Gamma(\alpha-\beta_i)]^{q_i}}\left(\int_a^t (t-\tau)^{\alpha-\beta_i-1}\,|D^{\alpha}_{a+}f(\tau)|\,d\tau\right)^{q_i}\\
&\leq \frac{1}{[\Gamma(\alpha-\beta_i)]^{q_i}}\left(\int_a^t (t-\tau)^{\alpha-\beta_i-1}\,d\tau\right)^{q_i}\|D^{\alpha}_{a+}f\|_\infty^{q_i}\\
&= \frac{(t-a)^{q_i(\alpha-\beta_i)}}{[\Gamma(\alpha-\beta_i+1)]^{q_i}}\|D^{\alpha}_{a+}f\|_\infty^{q_i}.
\end{aligned}$$

By analogy, for $i=1,2$ we get

$$|D^{\beta_i}_{a+}g(t)|^{q_i} \leq \frac{(t-a)^{q_i(\alpha-\beta_i)}}{[\Gamma(\alpha-\beta_i+1)]^{q_i}}\|D^{\alpha}_{a+}g\|_\infty^{q_i}.$$

Also,

$$|D^{\alpha}_{a+}f(t)|^{p}\leq \|D^{\alpha}_{a+}f\|_\infty^{p}, \quad |D^{\alpha}_{a+}g(t)|^{p}\leq \|D^{\alpha}_{a+}g\|_\infty^{p}.$$

Hence

$$\begin{aligned}
&|D^{\beta_1}_{a+}f(t)|^{q_1}\,|D^{\beta_2}_{a+}g(t)|^{q_2}\,|D^{\alpha}_{a+}f(t)|^{p}\\
&\leq \frac{(t-a)^{q_1(\alpha-\beta_1)+q_2(\alpha-\beta_2)}}{[\Gamma(\alpha-\beta_1+1)]^{q_1}\,[\Gamma(\alpha-\beta_2+1)]^{q_2}}\|D^{\alpha}_{a+}f\|_\infty^{q_1+p}\,\|D^{\alpha}_{a+}g\|_\infty^{q_2}, \qquad (6.134)
\end{aligned}$$

$$\begin{aligned}
&|D^{\beta_2}_{a+}f(t)|^{q_2}\,|D^{\beta_1}_{a+}g(t)|^{q_1}\,|D^{\alpha}_{a+}g(t)|^{p}\\
&\leq \frac{(t-a)^{q_2(\alpha-\beta_2)+q_1(\alpha-\beta_1)}}{[\Gamma(\alpha-\beta_1+1)]^{q_1}\,[\Gamma(\alpha-\beta_2+1)]^{q_2}}\|D^{\alpha}_{a+}f\|_\infty^{q_2}\,\|D^{\alpha}_{a+}g\|_\infty^{q_1+p}. \qquad (6.135)
\end{aligned}$$

Form (6.134) and (6.135) follow

$$\int_a^x w(t)\left[|D^{\beta_1}_{a+}f(t)|^{q_1}\,|D^{\beta_2}_{a+}g(t)|^{q_2}\,|D^{\alpha}_{a+}f(t)|^{p}+|D^{\beta_2}_{a+}f(t)|^{q_2}\,|D^{\beta_1}_{a+}g(t)|^{q_1}\,|D^{\alpha}_{a+}g(t)|^{p}\right]dt$$

$$\leq \frac{1}{[\Gamma(\alpha-\beta_1+1)]^{q_1}\,[\Gamma(\alpha-\beta_2+1)]^{q_2}} \int_a^x w(t)\,(t-a)^{q_1(\alpha-\beta_1)+q_2(\alpha-\beta_2)}\,dt$$
$$\cdot\left[\|D^{\alpha}_{a+}f\|_\infty^{q_1+p}\,\|D^{\alpha}_{a+}g\|_\infty^{q_2}+\|D^{\alpha}_{a+}f\|_\infty^{q_2}\,\|D^{\alpha}_{a+}g\|_\infty^{q_1+p}\right]$$
$$\leq \frac{\|w\|_\infty}{[\Gamma(\alpha-\beta_1+1)]^{q_1}\,[\Gamma(\alpha-\beta_2+1)]^{q_2}} \int_a^x (t-a)^{q_1(\alpha-\beta_1)+q_2(\alpha-\beta_2)}\,dt$$
$$\cdot\frac{1}{2}\left[\|D^{\alpha}_{a+}f\|_\infty^{2(q_1+p)}+\|D^{\alpha}_{a+}f\|_\infty^{2q_2}+\|D^{\alpha}_{a+}g\|_\infty^{2q_2}+\|D^{\alpha}_{a+}g\|_\infty^{2(q_1+p)}\right],$$

from which we obtain inequality (6.132). □

Now we present a counterpart of the previous theorem for the case $r<0$. Conditions on r and q allow us to apply reverse Hölder's inequalities, first with parameters $\{\frac{r}{r-1}\in(0,1), r<0\}$, then with $\{\frac{r}{r-q}\in(0,1), \frac{r}{q}<0\}$. Apart from using inequalities (6.129), we have to require similar inequalities for negative power, that is (6.142). Hence, instead of constant factor M we get $\widetilde{M}$.

Theorem 6.74 *Let $\alpha>\beta\geq 0$ and suppose that one of conditions $(i)-(vii)$ in Corollary 2.21 holds for $\{\alpha,\beta,f\}$ and $\{\alpha,\beta,g\}$. Let $w_1\geq 0$ and $w_2>0$ be continuous weight functions on $[a,x]$. Let $r<0$, $q>0$ and $p\geq 0$. Let $D^{\alpha}_{a+}f, D^{\alpha}_{a+}g\in L_r[a,b]$ be of fixed sign on $[a,b]$, with $1/D^{\alpha}_{a+}f, 1/D^{\alpha}_{a+}g\in L_r[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$\int_a^x w_1(t)\left[|D^{\beta}_{a+}g(t)|^p\,|D^{\alpha}_{a+}f(t)|^q+|D^{\beta}_{a+}f(t)|^p\,|D^{\alpha}_{a+}g(t)|^q\right]dt$$
$$\geq T_6\widetilde{M}\,S(x)\left(\int_a^x w_2(t)\left[|D^{\alpha}_{a+}f(t)|^r+|D^{\alpha}_{a+}g(t)|^r\right]dt\right)^{\frac{p+q}{r}}, \tag{6.136}$$

where S and T_6 are given by (6.117) *and* (6.120)*, respectively, and*

$$\widetilde{M}=\begin{cases} 2^{-\frac{p}{r}}, & p\geq q,\\ \left(1-2^{-\frac{p}{q}}\right)^{\frac{q}{r}}, & p\leq q.\end{cases} \tag{6.137}$$

Proof. Using Theorem 2.13, fixed sign of $D^{\alpha}_{a+}g$ on $[a,b]$, and reverse Hölder's inequality for $\{\frac{r}{r-1}, r\}$, for $t\in[a,x]$ we have

$$\begin{aligned}
&|D^{\beta}_{a+}g(t)|\\
&=\frac{1}{\Gamma(\alpha-\beta)}\int_a^t (t-\tau)^{\alpha-\beta-1}\,|D^{\alpha}_{a+}g(\tau)|\,d\tau\\
&=\frac{1}{\Gamma(\alpha-\beta)}\int_a^t (t-\tau)^{\alpha-\beta-1}\,[w_2(\tau)]^{-\frac{1}{r}}\,[w_2(\tau)]^{\frac{1}{r}}\,|D^{\alpha}_{a+}g(\tau)|\,d\tau\\
&\geq\frac{1}{\Gamma(\alpha-\beta)}\left(\int_a^t (t-\tau)^{\frac{r(\alpha-\beta-1)}{r-1}}\,[w_2(\tau)]^{\frac{1}{1-r}}\,d\tau\right)^{\frac{r-1}{r}}\left(\int_a^t w_2(\tau)\,|D^{\alpha}_{a+}g(\tau)|^r\,d\tau\right)^{\frac{1}{r}}\\
&=\frac{1}{\Gamma(\alpha-\beta)}\,[s(t)]^{\frac{r-1}{r}}\,[G(t)]^{\frac{1}{r}},
\end{aligned} \tag{6.138}$$

where G is defined by (6.122). Let F be defined by (6.123). Then (6.124) holds, and by (6.138) and (6.124) follows

$$w_1(t)\,|D^{\beta}_{a+}g(t)|^p\,|D^{\alpha}_{a+}f(t)|^q \geq h(t)\,[G(t)]^{\frac{p}{r}}\left[F'(t)\right]^{\frac{q}{r}}, \tag{6.139}$$

where h is defined by (6.126). Integrating (6.139) and applying reverse Hölder's inequality for $\{\frac{r}{r-q},\frac{r}{q}\}$, follows

$$\begin{aligned}
&\int_a^x w_1(t)\,|D^{\beta}_{a+}g(t)|^p\,|D^{\alpha}_{a+}f(t)|^q\,dt\\
&\geq \int_a^x h(t)\,[G(t)]^{\frac{p}{r}}\left[F'(t)\right]^{\frac{q}{r}}dt\\
&\geq \left(\int_a^x [h(t)]^{\frac{r}{r-q}}\,dt\right)^{\frac{r-q}{r}}\left(\int_a^x [G(t)]^{\frac{p}{q}}\,F'(t)\,dt\right)^{\frac{q}{r}}.
\end{aligned} \tag{6.140}$$

Similarly we get

$$\begin{aligned}
&\int_a^x w_1(t)\,|D^{\beta}_{a+}f(t)|^p\,|D^{\alpha}_{a+}g(t)|^q\,dt\\
&\geq \left(\int_a^x [h(t)]^{\frac{r}{r-q}}\,dt\right)^{\frac{r-q}{r}}\left(\int_a^x [F(t)]^{\frac{p}{q}}\,G'(t)\,dt\right)^{\frac{q}{r}}.
\end{aligned} \tag{6.141}$$

For negative power we use inequality

$$A^{\delta}+B^{\delta}\geq 2^{1-\delta}(A+B)^{\delta}, \quad (\delta<0 \text{ and } A,B>0), \tag{6.142}$$

since x^{δ} is convex function on $(0,\infty)$ for $\delta<0$. Using (6.142) for $\frac{q}{r}<0$, (6.140) and (6.141), we conclude

$$\begin{aligned}
&\int_a^x w_1(t)\left[|D^{\beta}_{a+}g(t)|^p\,|D^{\alpha}_{a+}f(t)|^q+|D^{\beta}_{a+}f(t)|^p\,|D^{\alpha}_{a+}g(t)|^q\right]dt\\
&\geq \left(\int_a^x [h(t)]^{\frac{r}{r-q}}\,dt\right)^{\frac{r-q}{r}}\left[\left(\int_a^x [G(t)]^{\frac{p}{q}}\,F'(t)\,dt\right)^{\frac{q}{r}}+\left(\int_a^x [F(t)]^{\frac{p}{q}}\,G'(t)\,dt\right)^{\frac{q}{r}}\right]\\
&\geq \left(\int_a^x [h(t)]^{\frac{r}{r-q}}\,dt\right)^{\frac{r-q}{r}}2^{1-\frac{q}{r}}\left(\int_a^x \left[[G(t)]^{\frac{p}{q}}\,F'(t)+[F(t)]^{\frac{p}{q}}\,G'(t)\right]dt\right)^{\frac{q}{r}}.
\end{aligned} \tag{6.143}$$

For $\frac{p}{q}>0$ we use (6.129), and with $G(a)=F(a)=0$, we get

$$\begin{aligned}
&\int_a^x \left[[G(t)]^{\frac{p}{q}}\,F'(t)+[F(t)]^{\frac{p}{q}}\,G'(t)\right]dt\\
&=\int_a^x \left[[G(t)]^{\frac{p}{q}}+[F(t)]^{\frac{p}{q}}\right]\left[G'(t)+F'(t)\right]dt-\int_a^x \left[[G(t)]^{\frac{p}{q}}\,G'(t)+[F(t)]^{\frac{p}{q}}\,F'(t)\right]dt\\
&\geq c_{\frac{p}{q}}\int_a^x [G(t)+F(t)]^{\frac{p}{q}}\,[G(t)+F(t)]'\,dt-\frac{q}{p+q}\left[G(x)^{\frac{p}{q}+1}+F(x)^{\frac{p}{q}+1}\right]
\end{aligned}$$

$$\geq \frac{q}{p+q}\left(c_{\frac{p}{q}}-2^{-\frac{p}{q}}\right)[G(x)+F(x)]^{\frac{p}{q}+1}. \tag{6.144}$$

Now from (6.143) and (6.144) follow

$$\begin{aligned}
&\int_a^x w_1(t)\left[|D_{a+}^{\beta}g(t)|^p\,|D_{a+}^{\alpha}f(t)|^q+|D_{a+}^{\beta}f(t)|^p\,|D_{a+}^{\alpha}g(t)|^q\right]dt\\
&\geq 2^{1-\frac{q}{r}}\left(c_{\frac{p}{q}}-2^{-\frac{p}{q}}\right)^{\frac{q}{r}}\left(\frac{q}{p+q}\right)^{\frac{q}{r}}\left(\int_a^x [h(t)]^{\frac{r}{r-q}}\,dt\right)^{\frac{r-q}{r}}[G(x)+F(x)]^{\frac{p+q}{r}}\\
&=\frac{2}{[\Gamma(\alpha-\beta)]^p}\left(c_{\frac{p}{q}}-2^{-\frac{p}{q}}\right)^{\frac{q}{r}}\left[\frac{q}{2(p+q)}\right]^{\frac{q}{r}}S(x)\,[G(x)+F(x)]^{\frac{p+q}{r}}.
\end{aligned}$$

If $p \geq q$, then

$$\left(c_{\frac{p}{q}}-2^{-\frac{p}{q}}\right)^{\frac{q}{r}}=\left(2^{1-\frac{p}{q}}-2^{-\frac{p}{q}}\right)^{\frac{q}{r}}=2^{-\frac{p}{r}},$$

while for $p \leq q$

$$\left(c_{\frac{p}{q}}-2^{-\frac{p}{q}}\right)^{\frac{q}{r}}=\left(1-2^{-\frac{p}{q}}\right)^{\frac{q}{r}},$$

which proves the theorem. □

Following inequalities include the right-sided Riemann-Liouville fractional derivatives.

Theorem 6.75 *Let $\alpha > \beta \geq 0$ and suppose that one of conditions $(i)-(vii)$ in Corollary 2.22 holds for $\{\alpha,\beta,f\}$ and $\{\alpha,\beta,g\}$. Let $w_1 \geq 0$ and $w_2 > 0$ be continuous weight functions on $[x,b]$. Let $r > 1$, $r > q > 0$ and $p \geq 0$. Let $D_{b-}^{\alpha}f, D_{b-}^{\alpha}g \in L_r[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\begin{aligned}
&\int_x^b w_1(t)\left[|D_{b-}^{\beta}g(t)|^p\,|D_{b-}^{\alpha}f(t)|^q+|D_{b-}^{\beta}f(t)|^p\,|D_{b-}^{\alpha}g(t)|^q\right]dt\\
&\leq T_6 M\widetilde{S}(x)\left(\int_x^b w_2(t)\left[|D_{b-}^{\alpha}f(t)|^r+|D_{b-}^{\alpha}g(t)|^r\right]dt\right)^{\frac{p+q}{r}},
\end{aligned} \tag{6.145}$$

where M and T_6 are given by (6.119) and (6.120), respectively, and

$$\widetilde{S}(x)=\left[\int_x^b [w_1(t)]^{\frac{r}{r-q}}\,[w_2(t)]^{\frac{q}{q-r}}\,[\tilde{s}(t)]^{\frac{p(r-1)}{r-q}}\,dt\right]^{\frac{r-q}{r}}, \tag{6.146}$$

$$\tilde{s}(t)=\int_t^b(\tau-t)^{\frac{r(\alpha-\beta-1)}{r-1}}\,[w_2(\tau)]^{\frac{1}{1-r}}\,d\tau. \tag{6.147}$$

Following cases are for $r=\infty$ and $r<0$.

Proposition 6.20 *Let $\alpha > \beta_1,\beta_2 \geq 0$ and suppose that one of conditions $(i)-(vii)$ in Corollary 2.22 holds for $\{\alpha,\beta_i,f\}$ and $\{\alpha,\beta_i,g\}$, $i=1,2$. Let w be continuous nonnegative* *weight function on* $[x,b]$ *and let* $p,q_1,q_2 \geq 0$. *Let* $D^{\alpha}_{b-}f, D^{\alpha}_{b-}g \in L_{\infty}[a,b]$. *Then for a.e.* $x \in [a,b]$ *holds*

$$\begin{aligned}
&\int_x^b w(t)\Big[|D^{\beta_1}_{b-}f(t)|^{q_1}\,|D^{\beta_2}_{b-}g(t)|^{q_2}\,|D^{\alpha}_{b-}f(t)|^{p} \\
&\qquad + |D^{\beta_2}_{b-}f(t)|^{q_2}\,|D^{\beta_1}_{b-}g(t)|^{q_1}\,|D^{\alpha}_{b-}g(t)|^{p}\Big]\,dt \\
&\leq T_7\,(b-x)^{q_1(\alpha-\beta_1)+q_2(\alpha-\beta_2)+1}\,\|w\|_{\infty} \\
&\quad\cdot\Big[\|D^{\alpha}_{b-}f\|_{\infty}^{2(q_1+p)} + \|D^{\alpha}_{b-}f\|_{\infty}^{2q_2} + \|D^{\alpha}_{b-}g\|_{\infty}^{2q_2} + \|D^{\alpha}_{b-}g\|_{\infty}^{2(q_1+p)}\Big], \qquad (6.148)
\end{aligned}$$

where T_7 *is given by* (6.133).

Theorem 6.76 *Let* $\alpha > \beta \geq 0$ *and suppose that one of conditions* $(i)-(vii)$ *in Corollary 2.22 holds for* $\{\alpha,\beta,f\}$ *and* $\{\alpha,\beta,g\}$. *Let* $w_1 \geq 0$ *and* $w_2 > 0$ *be continuous weight functions on* $[x,b]$. *Let* $r<0$, $q>0$ *and* $p \geq 0$. *Let* $D^{\alpha}_{b-}f, D^{\alpha}_{b-}g \in L_r[a,b]$ *be of fixed sign on* $[a,b]$, *with* $1/D^{\alpha}_{b-}f, 1/D^{\alpha}_{b-}g \in L_r[a,b]$. *Then for a.e.* $x \in [a,b]$ *holds*

$$\begin{aligned}
&\int_x^b w_1(t)\Big[|D^{\beta}_{b-}g(t)|^{p}\,|D^{\alpha}_{b-}f(t)|^{q} + |D^{\beta}_{b-}f(t)|^{p}\,|D^{\alpha}_{b-}g(t)|^{q}\Big]dt \\
&\geq T_6\widetilde{M}\widetilde{S}(x)\left(\int_x^b w_2(t)\Big[|D^{\alpha}_{b-}f(t)|^{r} + |D^{\alpha}_{b-}g(t)|^{r}\Big]dt\right)^{\frac{p+q}{r}}, \qquad (6.149)
\end{aligned}$$

where $\widetilde{S}$, $\widetilde{M}$ *and* T_6 *are given by* (6.146), (6.137) *and* (6.120), *respectively.*

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Theorem 6.77 *Let* $\alpha > \beta \geq 0$ *with n and m given by* (2.70). *Let* $f,g \in AC^n[a,b]$ *be such that* $f^{(i)}(a) = g^{(i)}(a) = 0$ *for* $i = m,\ldots,n-1$. *Let* $w_1 \geq 0$ *and* $w_2 > 0$ *be continuous weight functions on* $[a,x]$. *Let* $r>1$, $r>q>0$ *and* $p \geq 0$. *Let* ${}^{C}D^{\alpha}_{a+}f, {}^{C}D^{\alpha}_{a+}g \in L_r[a,b]$. *Then for a.e.* $x \in [a,b]$ *holds*

$$\begin{aligned}
&\int_a^x w_1(t)\Big[\left|{}^{C}D^{\beta}_{a+}g(t)\right|^{p}\left|{}^{C}D^{\alpha}_{a+}f(t)\right|^{q} + \left|{}^{C}D^{\beta}_{a+}f(t)\right|^{p}\left|{}^{C}D^{\alpha}_{a+}g(t)\right|^{q}\Big]dt \\
&\leq T_6 M S(x)\left(\int_a^x w_2(t)\Big[\left|{}^{C}D^{\alpha}_{a+}f(t)\right|^{r} + \left|{}^{C}D^{\alpha}_{a+}g(t)\right|^{r}\Big]dt\right)^{\frac{p+q}{r}}, \qquad (6.150)
\end{aligned}$$

where S, M and T_6 *are given by* (6.117), (6.119) *and* (6.120), *respectively.*

Proposition 6.21 *Let* $\alpha > \beta_1,\beta_2 \geq 0$ *with n,* m_1 *and* m_2 *given by* (2.70). *Let* $m = \min\{m_1,m_2\}$ *and* $f,g \in AC^n[a,b]$ *be such that* $f^{(i)}(a) = g^{(i)}(a) = 0$ *for* $i = m,\ldots,n-1$. *Let w be continuous nonnegative weight function on* $[a,x]$ *and let* $p,q_1,q_2 \geq 0$. *Let* ${}^{C}D^{\alpha}_{a+}f, {}^{C}D^{\alpha}_{a+}g \in L_{\infty}[a,b]$. *Then for a.e.* $x \in [a,b]$ *holds*

$$\int_a^x w(t)\Big[|{}^{C}D^{\beta_1}_{a+}f(t)|^{q_1}\,|{}^{C}D^{\beta_2}_{a+}g(t)|^{q_2}\,|{}^{C}D^{\alpha}_{a+}f(t)|^{p}$$

$$
\begin{aligned}
&+|{}^C D_{a+}^{\beta_2} f(t)|^{q_2}\,|{}^C D_{a+}^{\beta_1} g(t)|^{q_1}\,|{}^C D_{a+}^{\alpha} g(t)|^{p}\Big]\,dt \\
&\le T_7\,(x-a)^{q_1(\alpha-\beta_1)+q_2(\alpha-\beta_2)+1}\,\|w\|_\infty \\
&\quad\cdot\Big[\|{}^C D_{a+}^{\alpha} f\|_\infty^{2(q_1+p)}+\|{}^C D_{a+}^{\alpha} f\|_\infty^{2q_2}+\|{}^C D_{a+}^{\alpha} g\|_\infty^{2q_2}+\|{}^C D_{a+}^{\alpha} g\|_\infty^{2(q_1+p)}\Big],
\end{aligned}
\tag{6.151}
$$

where T_7 is given by (6.133).

Theorem 6.78 *Let $\alpha>\beta\ge 0$ with n and m given by* (2.70)*. Let $f,g\in AC^n[a,b]$ be such that $f^{(i)}(a)=g^{(i)}(a)=0$ for $i=m,\dots,n-1$. Let $w_1\ge 0$ and $w_2>0$ be continuous weight functions on $[a,x]$. Let $r<0$, $q>0$ and $p\ge 0$. Let ${}^C D_{a+}^{\alpha} f, {}^C D_{a+}^{\alpha} g\in L_r[a,b]$ be of fixed sign on $[a,b]$, with $1/{}^C D_{a+}^{\alpha} f, 1/{}^C D_{a+}^{\alpha} g\in L_r[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$
\begin{aligned}
&\int_a^x w_1(t)\left[\left|{}^C D_{a+}^{\beta} g(t)\right|^p\left|{}^C D_{a+}^{\alpha} f(t)\right|^q+\left|{}^C D_{a+}^{\beta} f(t)\right|^p\left|{}^C D_{a+}^{\alpha} g(t)\right|^q\right]dt \\
&\ge T_6\,\widetilde{M}\,S(x)\left(\int_a^x w_2(t)\left[\left|{}^C D_{a+}^{\alpha} f(t)\right|^r+\left|{}^C D_{a+}^{\alpha} g(t)\right|^r\right]dt\right)^{\frac{p+q}{r}},
\end{aligned}
\tag{6.152}
$$

where S, $\widetilde{M}$ and T_6 are given by (6.117)*,* (6.137) *and* (6.120)*, respectively.*

Following inequalities include the right-sided Caputo fractional derivatives.

Theorem 6.79 *Let $\alpha>\beta\ge 0$ with n and m given by* (2.70)*. Let $f,g\in AC^n[a,b]$ be such that $f^{(i)}(b)=g^{(i)}(b)=0$ for $i=m,\dots,n-1$. Let $w_1\ge 0$ and $w_2>0$ be continuous weight functions on $[x,b]$. Let $r>1$, $r>q>0$ and $p\ge 0$. Let ${}^C D_{b-}^{\alpha} f, {}^C D_{b-}^{\alpha} g\in L_r[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$
\begin{aligned}
&\int_x^b w_1(t)\left[\left|{}^C D_{b-}^{\beta} g(t)\right|^p\left|{}^C D_{b-}^{\alpha} f(t)\right|^q+\left|{}^C D_{b-}^{\beta} f(t)\right|^p\left|{}^C D_{b-}^{\alpha} g(t)\right|^q\right]dt \\
&\le T_6\,M\,\widetilde{S}(x)\left(\int_x^b w_2(t)\left[\left|{}^C D_{b-}^{\alpha} f(t)\right|^r+\left|{}^C D_{b-}^{\alpha} g(t)\right|^r\right]dt\right)^{\frac{p+q}{r}},
\end{aligned}
\tag{6.153}
$$

where $\widetilde{S}$, M and T_6 are given by (6.146)*,* (6.119) *and* (6.120)*, respectively.*

Proposition 6.22 *Let $\alpha>\beta_1,\beta_2\ge 0$ with n, m_1 and m_2 given by* (2.70)*. Let $m=\min\{m_1,m_2\}$ and $f,g\in AC^n[a,b]$ be such that $f^{(i)}(b)=g^{(i)}(b)=0$ for $i=m,\dots,n-1$. Let w be continuous nonnegative weight function on $[x,b]$ and let $p,q_1,q_2\ge 0$. Let ${}^C D_{b-}^{\alpha} f, {}^C D_{b-}^{\alpha} g\in L_\infty[a,b]$. Then for a.e. $x\in[a,b]$ holds*

$$
\begin{aligned}
&\int_x^b w(t)\Big[|{}^C D_{b-}^{\beta_1} f(t)|^{q_1}\,|{}^C D_{b-}^{\beta_2} g(t)|^{q_2}\,|{}^C D_{b-}^{\alpha} f(t)|^{p} \\
&\quad+|{}^C D_{b-}^{\beta_2} f(t)|^{q_2}\,|{}^C D_{b-}^{\beta_1} g(t)|^{q_1}\,|{}^C D_{b-}^{\alpha} g(t)|^{p}\Big]\,dt \\
&\le T_7\,(b-x)^{q_1(\alpha-\beta_1)+q_2(\alpha-\beta_2)+1}\,\|w\|_\infty
\end{aligned}
$$

$$\cdot \left[\|{}^C D^{\alpha}_{b-} f\|_{\infty}^{2(q_1+p)} + \|{}^C D^{\alpha}_{b-} f\|_{\infty}^{2q_2} + \|{}^C D^{\alpha}_{b-} g\|_{\infty}^{2q_2} + \|{}^C D^{\alpha}_{b-} g\|_{\infty}^{2(q_1+p)} \right], \tag{6.154}$$

where T_7 is given by (6.133).

Theorem 6.80 *Let $\alpha > \beta \geq 0$ with n and m given by* (2.70). *Let $f, g \in AC^n[a,b]$ be such that $f^{(i)}(b) = g^{(i)}(b) = 0$ for $i = m, \ldots, n-1$. Let $w_1 \geq 0$ and $w_2 > 0$ be continuous weight functions on $[x,b]$. Let $r < 0$, $q > 0$ and $p \geq 0$. Let ${}^C D^{\alpha}_{b-} f, {}^C D^{\alpha}_{b-} g \in L_r[a,b]$ be of fixed sign on $[a,b]$, with $1/{}^C D^{\alpha}_{b-} f, 1/{}^C D^{\alpha}_{b-} g \in L_r[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\begin{aligned} &\int_x^b w_1(t) \left[\left|{}^C D^{\beta}_{b-} g(t)\right|^p \left|{}^C D^{\alpha}_{b-} f(t)\right|^q + \left|{}^C D^{\beta}_{b-} f(t)\right|^p \left|{}^C D^{\alpha}_{b-} g(t)\right|^q \right] dt \\ &\geq T_6 \widetilde{M} \widetilde{S}(x) \left(\int_x^b w_2(t) \left[\left|{}^C D^{\alpha}_{b-} f(t)\right|^r + \left|{}^C D^{\alpha}_{b-} g(t)\right|^r \right] dt \right)^{\frac{p+q}{r}}, \end{aligned} \tag{6.155}$$

where $\widetilde{S}$, $\widetilde{M}$ and T_6 are given by (6.146), (6.137) *and* (6.120), *respectively.*

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Theorem 6.81 *Let $\alpha > \beta \geq 0$, $n = [\alpha]+1$ and $m = [\beta]+1$. Let $f, g \in C^{\alpha}_{a+}[a,b]$ be such that $f^{(i)}(a) = g^{(i)}(a) = 0$ for $i = m-1, \ldots, n-2$. Let $w_1 \geq 0$ and $w_2 > 0$ be continuous weight functions on $[a,x]$. Let $r > 1$, $r > q > 0$ and $p \geq 0$. Let ${}^{C_1} D^{\alpha}_{a+} f, {}^{C_1} D^{\alpha}_{a+} g \in L_r[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\begin{aligned} &\int_a^x w_1(t) \left[\left|{}^{C_1} D^{\beta}_{a+} g(t)\right|^p \left|{}^{C_1} D^{\alpha}_{a+} f(t)\right|^q + \left|{}^{C_1} D^{\beta}_{a+} f(t)\right|^p \left|{}^{C_1} D^{\alpha}_{a+} g(t)\right|^q \right] dt \\ &\leq T_6 M S(x) \left(\int_a^x w_2(t) \left[\left|{}^{C_1} D^{\alpha}_{a+} f(t)\right|^r + \left|{}^{C_1} D^{\alpha}_{a+} g(t)\right|^r \right] dt \right)^{\frac{p+q}{r}}, \end{aligned} \tag{6.156}$$

where S, M and T_6 are given by (6.117), (6.119) *and* (6.120), *respectively.*

Proposition 6.23 *Let $\alpha > \beta_1, \beta_2 \geq 0$, $n = [\alpha]+1$ and $m_i = [\beta_i]+1$, $i = 1,2$. Let $m = \min\{m_1, m_2\}$ and $f, g \in C^{\alpha}_{a+}[a,b]$ be such that $f^{(i)}(a) = g^{(i)}(a) = 0$ for $i = m-1, \ldots, n-2$. Let w be continuous nonnegative weight function on $[a,x]$ and let $p, q_1, q_2 \geq 0$. Let ${}^{C_1} D^{\alpha}_{a+} f, {}^{C_1} D^{\alpha}_{a+} g \in L_{\infty}[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\begin{aligned} &\int_a^x w(t) \left[|{}^{C_1} D^{\beta_1}_{a+} f(t)|^{q_1} \, |{}^{C_1} D^{\beta_2}_{a+} g(t)|^{q_2} \, |{}^{C_1} D^{\alpha}_{a+} f(t)|^p \right. \\ &\qquad \left. + |{}^{C_1} D^{\beta_2}_{a+} f(t)|^{q_2} \, |{}^{C_1} D^{\beta_1}_{a+} g(t)|^{q_1} \, |{}^{C_1} D^{\alpha}_{a+} g(t)|^p \right] dt \\ &\leq T_7 \, (x-a)^{q_1(\alpha-\beta_1)+q_2(\alpha-\beta_2)+1} \, \|w\|_{\infty} \\ &\quad \cdot \left[\|{}^{C_1} D^{\alpha}_{a+} f\|_{\infty}^{2(q_1+p)} + \|{}^{C_1} D^{\alpha}_{a+} f\|_{\infty}^{2q_2} + \|{}^{C_1} D^{\alpha}_{a+} g\|_{\infty}^{2q_2} + \|{}^{C_1} D^{\alpha}_{a+} g\|_{\infty}^{2(q_1+p)} \right], \end{aligned} \tag{6.157}$$

where T_7 is given by (6.133).

Theorem 6.82 *Let $\alpha > \beta \geq 0$, $n = [\alpha]+1$ and $m = [\beta]+1$. Let $f, g \in C^{\alpha}_{a+}[a,b]$ be such that $f^{(i)}(a) = g^{(i)}(a) = 0$ for $i = m-1, \dots, n-2$. Let $w_1 \geq 0$ and $w_2 > 0$ be continuous weight functions on $[a,x]$. Let $r < 0$, $q > 0$ and $p \geq 0$. Let ${}^{C_1}D^{\alpha}_{a+}f, {}^{C_1}D^{\alpha}_{a+}g \in L_r[a,b]$ be of fixed sign on $[a,b]$, with $1/{}^{C_1}D^{\alpha}_{a+}f, 1/{}^{C_1}D^{\alpha}_{a+}g \in L_r[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\begin{aligned}
&\int_a^x w_1(t) \left[\left| {}^{C_1}D^{\beta}_{a+}g(t) \right|^p \left| {}^{C_1}D^{\alpha}_{a+}f(t) \right|^q + \left| {}^{C_1}D^{\beta}_{a+}f(t) \right|^p \left| {}^{C_1}D^{\alpha}_{a+}g(t) \right|^q \right] dt \\
&\geq T_6 \widetilde{M} S(x) \left(\int_a^x w_2(t) \left[\left| {}^{C_1}D^{\alpha}_{a+}f(t) \right|^r + \left| {}^{C_1}D^{\alpha}_{a+}g(t) \right|^r \right] dt \right)^{\frac{p+q}{r}}, \qquad (6.158)
\end{aligned}$$

where S, $\widetilde{M}$ and T_6 are given by (6.117), (6.137) *and* (6.120)*, respectively.*

Following inequalities include the right-sided Canavati fractional derivatives.

Theorem 6.83 *Let $\alpha > \beta \geq 0$, $n = [\alpha]+1$ and $m = [\beta]+1$. Let $f, g \in C^{\alpha}_{b-}[a,b]$ be such that $f^{(i)}(b) = g^{(i)}(b) = 0$ for $i = m-1, \dots, n-2$. Let $w_1 \geq 0$ and $w_2 > 0$ be continuous weight functions on $[x,b]$. Let $r > 1$, $r > q > 0$ and $p \geq 0$. Let ${}^{C_1}D^{\alpha}_{b-}f, {}^{C_1}D^{\alpha}_{b-}g \in L_r[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\begin{aligned}
&\int_x^b w_1(t) \left[\left| {}^{C_1}D^{\beta}_{b-}g(t) \right|^p \left| {}^{C_1}D^{\alpha}_{b-}f(t) \right|^q + \left| {}^{C_1}D^{\beta}_{b-}f(t) \right|^p \left| {}^{C_1}D^{\alpha}_{b-}g(t) \right|^q \right] dt \\
&\leq T_6 M \widetilde{S}(x) \left(\int_x^b w_2(t) \left[\left| {}^{C_1}D^{\alpha}_{b-}f(t) \right|^r + \left| {}^{C_1}D^{\alpha}_{b-}g(t) \right|^r \right] dt \right)^{\frac{p+q}{r}}, \qquad (6.159)
\end{aligned}$$

where $\widetilde{S}$, M and T_6 are given by (6.146), (6.119) *and* (6.120)*, respectively.*

Proposition 6.24 *Let $\alpha > \beta_1, \beta_2 \geq 0$, $n = [\alpha]+1$ and $m_i = [\beta_i]+1$, $i = 1,2$. Let $m = \min\{m_1, m_2\}$ and $f, g \in C^{\alpha}_{b-}[a,b]$ be such that $f^{(i)}(b) = g^{(i)}(b) = 0$ for $i = m-1, \dots, n-2$. Let w be continuous nonnegative weight function on $[x,b]$ and let $p, q_1, q_2 \geq 0$. Let ${}^{C_1}D^{\alpha}_{b-}f, {}^{C_1}D^{\alpha}_{b-}g \in L_\infty[a,b]$. Then for a.e. $x \in [a,b]$ holds*

$$\begin{aligned}
&\int_x^b w(t) \Big[|{}^{C_1}D^{\beta_1}_{b-}f(t)|^{q_1} \, |{}^{C_1}D^{\beta_2}_{b-}g(t)|^{q_2} \, |{}^{C_1}D^{\alpha}_{b-}f(t)|^p \\
&\quad + |{}^{C_1}D^{\beta_2}_{b-}f(t)|^{q_2} \, |{}^{C_1}D^{\beta_1}_{b-}g(t)|^{q_1} \, |{}^{C_1}D^{\alpha}_{b-}g(t)|^p \Big] dt \\
&\leq T_7 \, (x-a)^{q_1(\alpha-\beta_1)+q_2(\alpha-\beta_2)+1} \, \|w\|_\infty \\
&\quad \cdot \left[\|{}^{C_1}D^{\alpha}_{b-}f\|_\infty^{2(q_1+p)} + \|{}^{C_1}D^{\alpha}_{b-}f\|_\infty^{2q_2} + \|{}^{C_1}D^{\alpha}_{b-}g\|_\infty^{2q_2} + \|{}^{C_1}D^{\alpha}_{b-}g\|_\infty^{2(q_1+p)} \right], \\
&\qquad (6.160)
\end{aligned}$$

where T_7 is given by (6.133).

Theorem 6.84 *Let $\alpha > \beta \geq 0$, $n = [\alpha]+1$ and $m = [\beta]+1$. Let $f, g \in C^{\alpha}_{b-}[a,b]$ be such that $f^{(i)}(b) = g^{(i)}(b) = 0$ for $i = m-1, \dots, n-2$. Let $w_1 \geq 0$ and $w_2 > 0$ be continuous* *weight functions on* $[x,b]$. *Let* $r<0$, $q>0$ *and* $p\geq 0$. *Let* ${}^{C_1}D^{\alpha}_{b-}f, {}^{C_1}D^{\alpha}_{b-}g \in L_r[a,b]$ *be of fixed sign on* $[a,b]$, *with* $1/{}^{C_1}D^{\alpha}_{b-}f, 1/{}^{C_1}D^{\alpha}_{b-}g \in L_r[a,b]$. *Then for a.e.* $x\in[a,b]$ *holds*

$$\begin{aligned}&\int_x^b w_1(t)\left[\left|{}^{C_1}D^{\beta}_{b-}g(t)\right|^p\left|{}^{C_1}D^{\alpha}_{b-}f(t)\right|^q+\left|{}^{C_1}D^{\beta}_{b-}f(t)\right|^p\left|{}^{C_1}D^{\alpha}_{b-}g(t)\right|^q\right]dt\\&\geq T_6\widetilde{M}\widetilde{S}(x)\left(\int_x^b w_2(t)\left[\left|{}^{C_1}D^{\alpha}_{b-}f(t)\right|^r+\left|{}^{C_1}D^{\alpha}_{b-}g(t)\right|^r\right]dt\right)^{\frac{p+q}{r}},\end{aligned}\tag{6.161}$$

where $\widetilde{S}$, $\widetilde{M}$ *and* T_6 *are given by* (6.146), (6.137) *and* (6.120), *respectively.*

Remark 6.6 Comparing theorems with left-sided fractional derivatives with ones from [12] we conclude: With relaxed restrictions and smaller constant M, defined by (6.119), Theorem 6.73 including the Riemann-Liouville derivatives improves [12, Theorem 7.5], Theorem 6.77 including the Caputo derivatives improves [12, Theorem 16.31] and Theorem 6.81 including the Canavati derivatives improves [12, Theorem 6.6]. In theorems from [12] the role of constant M has

$$\delta_3^{\frac{q}{r}}=\begin{cases}\left(2^{\frac{p}{q}}-1\right)^{\frac{q}{r}}, & p\geq q,\\ 1, & p\leq q.\end{cases}$$

Obviously, $\delta_3^{q/r}\geq 1$, while $M\leq 1$. Since $\lim_{p\to\infty}\delta_3^{q/r}=\infty$, for all sufficiently large p we obtain a substantial improvement of inequality.

Further, with relaxed restrictions Theorem 6.19 improves [12, Theorem 7.18], Theorem 6.21 improves [12, Theorem 16.38] and Theorem 6.23 improves [12, Theorem 6.18].

Theorems 6.74, 6.78 and 6.82 are newly presented.

Chapter 7

Inequalities for integral operators with a kernel and applications to a Green function

In this chapter we give general Opial-type inequalities on a measure space (Ω,Σ,μ), for two functions, convex and concave. Integrals in these inequalities contain function and its integral representation. Results are applied to numerous symmetric functions and new results involving the Green function, the Lidstone polynomials and the Hermite interpolating polynomials are obtained by Barbir, Krulić Himmelreich and Pečarić in [29, 30].

7.1 Inequalities for integral operators with a kernel

In [54] (see also [55, Chapter II, p. 15]), Krulić-Pečarić-Persson studied measure spaces $(\Omega_1,\Sigma_1,\mu_1)$, $(\Omega_2,\Sigma_2,\mu_2)$, and the general integral operator A_k defined by

$$A_k f(x) = \frac{1}{K(x)}\int_{\Omega_2} k(x,y)f(y)\,d\mu(y),\ x\in\Omega_1, \tag{7.1}$$

where $f : \Omega_2 \longrightarrow \mathbb{R}$ is a measurable function, $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is measurable and nonnegative, and

$$K(x) = \int_{\Omega_2} k(x,y)\, d\mu(y) > 0,\ x \in \Omega_1. \tag{7.2}$$

Just by using Jensen's inequality and Fubini's theorem, they proved the weighted inequality

$$\int_{\Omega_1} u(x)\Phi(A_k f(x))\, d\mu(x) \leq \int_{\Omega_2} v(y)\Phi(f(y))\, d\mu(y), \tag{7.3}$$

where $u : \Omega_1 \to \mathbb{R}$ is a nonnegative measurable function, $x \mapsto u(x)\frac{k(x,y)}{K(x)}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$, v is defined on Ω_2 by

$$v(y) = \int_{\Omega_1} u(x)\frac{k(x,y)}{K(x)}\, d\mu(x), \tag{7.4}$$

Φ is a convex function on an interval $I \subseteq \mathbb{R}$, and $f : \Omega_2 \to \mathbb{R}$ is such that $f(y) \in I$, for all $y \in \Omega_2$.

In particular, inequality (7.3) unifies and generalizes most of results of this type (including the classical ones by Hardy, Hilbert and Godunova).

In the sequel let (Ω, Σ, μ) be a measure space and let $k : \Omega \times \Omega \to \mathbb{R}$ be a symmetric nonnegative or nonpositive function such that

$$K(x) := \int_{\Omega} k(x,y) d\mu(y),\ K(x) \neq 0,\ a.e.\, x \in \Omega, \tag{7.5}$$

where $|K(x)| < \infty$. We assume that all integrals are well defined.

Theorem 7.1 *Let $k : \Omega \times \Omega \to \mathbb{R}$ be a symmetric nonnegative or nonpositive function. If f is a positive convex function, g a positive concave function on an interval $I \subseteq \mathbb{R}$, $v : \Omega \to \mathbb{R}$ is either nonnegative or nonpositive, such that $Im|v| \subseteq I$ and u defined by*

$$u(x) := \int_{\Omega} k(x,y) v(y) d\mu(y) < \infty, \tag{7.6}$$

then the following inequality

$$\begin{aligned} &\int_{\Omega} |K(x)| f\left(\left| \frac{u(x)}{K(x)} \right| \right) g(|v(x)|)\, d\mu(x) \\ &\leq \int_{\Omega} |K(x)| f(|v(x)|) g\left(\left| \frac{u(x)}{K(x)} \right| \right) d\mu(x) \end{aligned} \tag{7.7}$$

holds, where K is defined by (7.5).

Proof. We notice that $\left|\frac{u(x)}{K(x)}\right| \in I$, for all $x \in \Omega$. The motivation for this is that $\left|\frac{u(x)}{K(x)}\right|$ is simply a generalized mean and since $|v(y)| \in I$ for all $y \in \Omega$ (by assumption), then also the mean $\left|\frac{u(x)}{K(x)}\right| \in I$.

Now, let us prove the inequality (7.7). By using Jensen's inequality and the Fubini theorem we find that

$$\begin{aligned}
&\int_\Omega |K(x)|\, f\left(\left|\frac{u(x)}{K(x)}\right|\right) g(|v(x)|)\, d\mu(x) \\
&= \int_\Omega |K(x)|\, f\left(\frac{|\int_\Omega k(x,t)\, v(t)\, d\mu(t)|}{|K(x)|}\right) g(|v(x)|)\, d\mu(x) \\
&= \int_\Omega |K(x)|\, f\left(\frac{\int_\Omega |k(x,t)\, v(t)|\, d\mu(t)}{|K(x)|}\right) g(|v(x)|)\, d\mu(x) \\
&\le \int_\Omega \left(\int_\Omega |k(x,t)|\, f(|v(t)|)\, d\mu(t)\right) g(|v(x)|)\, d\mu(x) \\
&= \int_\Omega f(|v(t)|) \left(\int_\Omega |k(x,t)|\, g(|v(x)|)\, d\mu(x)\right) d\mu(t).
\end{aligned}$$

Since k is a symmetric function we get that $|k(x,t)| = |k(t,x)|$, and by using Jensen's inequality we obtain that

$$\begin{aligned}
&\int_\Omega f(|v(t)|) \left(\int_\Omega |k(t,x)|\, g(|v(x)|)\, d\mu(x)\right) d\mu(t) \\
&\le \int_\Omega f(|v(t)|)\, |K(t)|\, g\left(\frac{1}{|K(t)|}\int_\Omega |k(t,x)\, v(x)|\, d\mu(x)\right) d\mu(t) \\
&= \int_\Omega f(|v(t)|)\, |K(t)|\, g\left(\frac{|u(t)|}{|K(t)|}\right) d\mu(t)
\end{aligned}$$

and the proof is complete. □

Remark 7.1 By applying (7.3) with $\Omega_1 = \Omega_2 = \Omega$ and $u(x) = K(x)$, inequality (7.3) reduces to

$$\begin{aligned}
&\int_\Omega K(x)\, \Phi\left(\frac{1}{K(x)}\int_\Omega k(x,y)\, f(y)\, d\mu(y)\right) d\mu(x) \\
&\le \int_\Omega \Phi(f(y)) \int_\Omega k(x,y)\, d\mu(x)\, d\mu(y).
\end{aligned} \tag{7.8}$$

Notice that if we apply Theorem 7.1 with function $g(x) = 1$, then inequality (7.7) reduces to

$$\begin{aligned}
&\int_\Omega |K(x)|\, f\left(\frac{1}{|K(x)|}\int_\Omega |k(x,y)\, v(y)|\, d\mu(y)\right) d\mu(x) \\
&\le \int_\Omega f(|v(y)|) \int_\Omega |k(x,y)|\, d\mu(x)\, d\mu(y),
\end{aligned} \tag{7.9}$$

that is, inequalities (7.8) and (7.9) are equivalent.

Related interesting result is given in the following theorem.

Theorem 7.2 *Let $0 < q < 1$, $k : \Omega \times \Omega \to \mathbb{R}$ be a symmetric nonnegative or nonpositive function, $S : \Omega \to \mathbb{R}_+$. If f is a positive convex function on an interval $I \subseteq \mathbb{R}$, function $v : \Omega \to \mathbb{R}$ is either nonnegative or nonpositive, such that $Im|v| \subseteq I$ and u defined by* (7.6), *then the inequality*

$$\int_\Omega S(x) f\left(\left|\frac{u(x)}{K(x)}\right|\right) |v(x)|^q d\mu(x) \le \int_\Omega R(x) f(|v(x)|)|u(x)|^q d\mu(x) \tag{7.10}$$

holds, where K is defined by (7.5) and

$$R(t) = \left[\int_\Omega \left(\frac{S(x)}{|K(x)|}\right)^{\frac{1}{1-q}} |k(x,t)| d\mu(x)\right]^{1-q}.$$

Proof.

$$\begin{aligned}
&\int_\Omega S(x) f\left(\left|\frac{u(x)}{K(x)}\right|\right) |v(x)|^q \, d\mu(x) \\
&= \int_\Omega S(x) f\left(\left|\frac{\int_\Omega k(x,t) v(t)\, d\mu(t)}{K(x)}\right|\right) |v(x)|^q \, d\mu(x) \\
&\le \int_\Omega S(x) \frac{|v(x)|^q}{|K(x)|} \left(\int_\Omega |k(x,t)| f(|v(t)|)\, d\mu(t)\right) d\mu(x) \\
&= \int_\Omega f(|v(t)|) \left(\int_\Omega \frac{S(x)}{|K(x)|} |k(x,t)||v(x)|^q \, d\mu(x)\right) d\mu(t) \\
&= \int_\Omega f(|v(t)|) \left(\int_\Omega \frac{S(x)}{|K(x)|} |k(x,t)|^{1-q} |k(x,t) v(x)|^q \, d\mu(x)\right) d\mu(t).
\end{aligned}$$

By Hölder's inequality we get

$$\begin{aligned}
&\int_\Omega S(x) f\left(\left|\frac{u(x)}{K(x)}\right|\right) |v(x)|^q \, d\mu(x) \\
&\le \int_\Omega f(|v(t)|) \left[\int_\Omega \left(\frac{S(x)}{|K(x)|} |k(x,t)|^{1-q}\right)^{\frac{1}{1-q}} d\mu(x)\right]^{1-q} \\
&\quad \times \left[\int_\Omega |k(x,t) v(x)| d\mu(x)\right]^q d\mu(t).
\end{aligned}$$

Since k is symmetric function, i.e. $k(t,x) = k(x,t)$, we have

$$\left[\int_\Omega |k(x,t) v(x)| d\mu(x)\right]^q = \left[\int_\Omega |k(t,x) v(x)| d\mu(x)\right]^q = |u(t)|^q,$$

from which follows (7.10). □

Remark 7.2 Notice that if $S(x) = |K(x)|$, then we obtain

$$\int_\Omega |K(x)| f\left(\left|\frac{u(x)}{K(x)}\right|\right) |v(x)|^q d\mu(x) \le \int_\Omega |K(x)|^{1-q} f(|v(x)|)|u(x)|^q d\mu(x),$$

which is a special case of Theorem 7.1 for concave function $g(x) = x^q$, $0 < q < 1$.

Finally, we give a result for the quotient of functions.

Theorem 7.3 *Let $k : \Omega \times \Omega \to \mathbb{R}$ be a symmetric nonnegative or nonpositive function. If f is a positive convex function, g a positive concave function on an interval $I \subseteq \mathbb{R}$, $v, v_1 : \Omega \to \mathbb{R}$ are either nonnegative or nonpositive, such that $Im|v|, Im|\frac{v}{v_1}| \subseteq I$, u defined by* (7.6) *and u_1 defined by*

$$u_1(x) := \int_\Omega k(x,y)v_1(y)d\mu(y) < \infty, \tag{7.11}$$

then the following inequality

$$\begin{aligned} &\int_\Omega |u_1(x)| f\left(\left|\frac{u(x)}{u_1(x)}\right|\right) g(|v(x)|)\, d\mu(x) \\ &\leq \int_\Omega |v_1(x)||K(x)| f\left(\left|\frac{v(x)}{v_1(x)}\right|\right) g\left(\left|\frac{u(x)}{K(x)}\right|\right) d\mu(x) \end{aligned} \tag{7.12}$$

holds, where K is defined by (7.5).

Proof. By using Jensen's inequality and the Fubini theorem we find that

$$\begin{aligned} &\int_\Omega |u_1(x)| f\left(\left|\frac{u(x)}{u_1(x)}\right|\right) g(|v(x)|)\, d\mu(x) \\ &= \int_\Omega |u_1(x)| f\left(\frac{|\int_\Omega k(x,t)v(t)d\mu(t)|}{|u_1(x)|}\right) g(|v(x)|)\, d\mu(x) \\ &= \int_\Omega |u_1(x)| f\left(\int_\Omega \frac{|k(x,t)v_1(t)|}{|u_1(x)|}\frac{|v(t)|}{|v_1(t)|} d\mu(t)\right) g(|v(x)|)\, d\mu(x) \\ &\leq \int_\Omega \left(\int_\Omega |k(x,t)v_1(t)| f\left(\frac{|v(t)|}{|v_1(t)|}\right) d\mu(t)\right) g(|v(x)|)\, d\mu(x) \\ &= \int_\Omega f\left(\frac{|v(t)|}{|v_1(t)|}\right) |v_1(t)| \left(\int_\Omega |k(x,t)| g(|v(x)|)\, d\mu(x)\right) d\mu(t). \end{aligned}$$

Since k is a symmetric function we get that $|k(x,t)| = |k(t,x)|$, so by using Jensen's inequality we obtain

$$\begin{aligned} &f\left(\frac{|v(t)|}{|v_1(t)|}\right) |v_1(t)| \left(\int_\Omega |k(t,x)| g(|v(x)|) d\mu(x)\right) d\mu(t) \\ &\leq \int_\Omega |v_1(t)| f\left(\frac{|v(t)|}{|v_1(t)|}\right) |K(t)| g\left(\frac{1}{|K(t)|}\int_\Omega |k(t,x)v(x)| d\mu(x)\right) d\mu(t) \\ &= \int_\Omega |v_1(t)| f\left(\frac{|v(t)|}{|v_1(t)|}\right) |K(t)| g\left(\frac{|u(t)|}{|K(t)|}\right) d\mu(t) \end{aligned}$$

and the proof is complete. □

Remark 7.3 If $v_1(t) = 1$, then $u_1(x) = K(x)$ and inequality (7.12) reduces to (7.7).

7.2 Opial-type inequalities using a Green function

Now we consider special symmetric functions k and obtain new results involving the Green function, the Lidstone polynomials and the Hermite interpolating polynomials.

The Green function G is defined on $[a,b]\times[a,b]$ by

$$G(t,s)=\begin{cases}\frac{(t-b)(s-a)}{b-a}, & a\le s\le t;\\ \frac{(t-a)(s-b)}{b-a}, & t\le s\le b.\end{cases} \tag{7.13}$$

The function G is convex under s and t, continuous under s and t and it is symmetric nonpositive function. For any function $\varphi\in C^2[a,b]$, we can easily show integrating by parts that the following is valid

$$\varphi(x)=\frac{b-x}{b-a}\varphi(a)+\frac{x-a}{b-a}\varphi(b)+\int_a^b G(x,s)\varphi''(s)ds, \tag{7.14}$$

where the function G is defined as above in (7.13).

In the following, by using identities where the Green function G is a kernel, we obtain several Opial-type inequalities. First, since the function G is a nonpositive symmetric function, we can apply Theorem 7.1 and obtain the following corollary.

Corollary 7.1 *If f is a positive convex function and g a positive concave function on an interval $I\subseteq\mathbb{R}$, then the inequality*

$$\begin{aligned}&\int_a^b (b-x)(x-a)\,f\left(\frac{2|\varphi(x)-\frac{b-x}{b-a}\varphi(a)-\frac{x-a}{b-a}\varphi(b)|}{(b-x)(x-a)}\right)g(|\varphi''(x)|)\,dx\\ &\le\int_a^b (b-x)(x-a)\,f(|\varphi''(x)|)\,g\left(\frac{2|\varphi(x)-\frac{b-x}{b-a}\varphi(a)-\frac{x-a}{b-a}\varphi(b)|}{(b-x)(x-a)}\right)dx\end{aligned} \tag{7.15}$$

holds for all functions $\varphi\in C^2[a,b]$.

Proof. For the function G we can apply Theorem 7.1. Let $\Omega=[a,b]$, $k(x,s)=G(x,s)$, $v(s)=\varphi''(s)$. Then

$$|K(x)|=\int_a^b|G(x,s)|\,ds=\frac{(b-x)(x-a)}{2}, \tag{7.16}$$

$$u(x)=\int_a^b G(x,s)\varphi''(s)\,ds=\varphi(x)-\frac{b-x}{b-a}\varphi(a)-\frac{x-a}{b-a}\varphi(b) \tag{7.17}$$

and inequality (7.7) becomes (7.15). □

Remark 7.4 If $\varphi(a)=\varphi(b)=0$, then (7.15) reduces to

$$\begin{aligned}&\int_a^b (b-x)(x-a)\,f\left(\frac{2|\varphi(x)|}{(b-x)(x-a)}\right)g(|\varphi''(x)|)\,dx\\ &\le \int_a^b (b-x)(x-a)\,f(|\varphi''(x)|)\,g\left(\frac{2|\varphi(x)|}{(b-x)(x-a)}\right)dx.\end{aligned}$$

Next is a special case of Theorem 7.2 for Green's function.

Corollary 7.2 *Let $0<q<1$, $S:[a,b]\to\mathbb{R}_+$ and $\varphi\in C^2[a,b]$. If f is a positive convex function on an interval $I\subseteq\mathbb{R}$, then the inequality*

$$\begin{aligned}&\int_a^b S(x)f\left(\frac{2|\varphi(x)-\frac{b-x}{b-a}\varphi(a)-\frac{x-a}{b-a}\varphi(b)|}{(b-x)(x-a)}\right)|\varphi''(x)|^q dx\\ &\le \int_a^b R(x)f(|\varphi''(x)|)\left(\frac{2|\varphi(x)-\frac{b-x}{b-a}\varphi(a)-\frac{x-a}{b-a}\varphi(b)|}{(b-x)(x-a)}\right)^q dx \qquad (7.18)\end{aligned}$$

holds, where

$$R(t)=\left[\int_a^b\left(\frac{2S(x)}{(b-x)(x-a)}\right)^{\frac{1}{1-q}}|G(x,t)|dx\right]^{1-q}.$$

Proof. For the function G we can apply Theorem 7.2 with $\Omega=[a,b]$, $k(x,s)=G(x,s)$, $v(s)=\varphi''(s)$. Then by (7.16) and (7.17) inequality (7.10) becomes (7.18). □

Remark 7.5 If $\varphi(a)=\varphi(b)=0$, then (7.18) reduces to

$$\begin{aligned}&\int_a^b S(x)f\left(\frac{2|\varphi(x)|}{(b-x)(x-a)}\right)|\varphi''(x)|^q dx\\ &\le \int_a^b R(x)f(|\varphi''(x)|)\left(\frac{2|\varphi(x)|}{(b-x)(x-a)}\right)^q dx.\end{aligned}$$

We proceed with a special case of Theorem 7.3 for Green's function.

Corollary 7.3 *If f is a positive convex function and g a positive concave function on an interval $I\subseteq\mathbb{R}$, then the inequality*

$$\begin{aligned}&\int_a^b \left|\psi(x)-\frac{b-x}{b-a}\psi(a)-\frac{x-a}{b-a}\psi(b)\right| f\left(\frac{|\varphi(x)-\frac{b-x}{b-a}\varphi(a)-\frac{x-a}{b-a}\varphi(b)|}{|\psi(x)-\frac{b-x}{b-a}\psi(a)-\frac{x-a}{b-a}\psi(b)|}\right)\\ &\quad\times g(|\varphi''(x)|)\,dx\\ &\le \frac{1}{2}\int_a^b |\psi''(x)|(b-x)(x-a)\,f\left(\frac{|\varphi''(x)|}{|\psi''(x)|}\right)\\ &\quad\times g\left(\frac{2|\varphi(x)-\frac{b-x}{b-a}\varphi(a)-\frac{x-a}{b-a}\varphi(b)|}{(b-x)(x-a)}\right)dx \qquad (7.19)\end{aligned}$$

holds for all functions $\varphi,\psi\in C^2[a,b]$ such that φ'' is either nonnegative or nonpositive function, and ψ'' is either nonnegative or nonpositive function.

Proof. Since the function G defined by (7.13) is a nonpositive symmetric function we can apply Theorem 7.3. Let $\Omega = [a,b]$, $k(x,s) = G(x,s), v(s) = \varphi''(s)$ and $v_1(x) = \psi''(x)$. Then by (7.16), (7.17) and

$$\psi(x) = \int_a^b G(x,s)\psi''(s)ds = \psi(x) - \frac{b-x}{b-a}\psi(a) - \frac{x-a}{b-a}\psi(b)$$

inequality (7.12) becomes (7.19). □

Remark 7.6 If $\varphi(a) = \varphi(b) = 0$, then (7.19) reduces to

$$\int_a^b |\psi(x) - \frac{b-x}{b-a}\psi(a) - \frac{x-a}{b-a}\psi(b)| f\left(\frac{|\varphi(x|}{|\psi(x) - \frac{b-x}{b-a}\psi(a) - \frac{x-a}{b-a}\psi(b)|}\right)$$
$$\times g(|\varphi''(x)|)dx$$
$$\leq \frac{1}{2}\int_a^b |\psi''(x)|(b-x)(x-a) f\left(\frac{|\varphi''(x)|}{|\psi''(x)|}\right) g\left(\frac{2|\varphi(x)|}{(b-x)(x-a)}\right) dx.$$

Remark 7.7 If $\psi''(x) = 1$, then $\psi(x) = K(x)$ and (7.19) reduces to

$$\int_a^b (b-x)(x-a) f\left(\frac{2|\varphi(x) - \frac{b-x}{b-a}\varphi(a) - \frac{x-a}{b-a}\varphi(b)|}{(b-x)(x-a)}\right) g(|\varphi''(x)|)dx$$
$$\leq \int_a^b (b-x)(x-a) f(|\varphi''(x)|) g\left(\frac{2|\varphi(x) - \frac{b-x}{b-a}\varphi(a) - \frac{x-a}{b-a}\varphi(b)|}{(b-x)(x-a)}\right) dx$$

which is given in Corollary 7.1.

We continue with the definition of the Lidstone polynomials, that is a generalization of the Taylor polynomials. It approximates to a given function in the neighborhood of two points (instead of one). Such polynomials have been studied by G. J. Lidstone (1929), H. Poritsky (1932), J. M. Whittaker (1934) and others: Let $\varphi \in C^\infty[0,1]$, then the Lidstone polynomial has the form

$$\sum_{k=0}^{\infty} \left(\varphi^{(2k)}(0)\Lambda_k(1-x) + \varphi^{(2k)}(1)\Lambda_k(x)\right),$$

where Λ_n is a polynomial of degree $2n+1$ defined by the relations

$$\begin{aligned} &\Lambda_0(t) = t, \\ &\Lambda_n''(t) = \Lambda_{n-1}(t), \\ &\Lambda_n(0) = \Lambda_n(1) = 0, \quad n \geq 1. \end{aligned} \tag{7.20}$$

Another explicit representations of the Lidstone polynomial are given by [9] and [73]:

$$\Lambda_n(t) = (-1)^n \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2n+1}} \sin k\pi t,$$

$$\Lambda_n(t) = -\sum_{k=0}^{n} \frac{2(2^{2k-1}-1)}{(2k)!} \frac{B_{2k}}{(2n-2k+1)!} t^{2n-2k+1},\ n=1,2,\ldots,$$

$$\Lambda_n(t) = \frac{2^{2n+1}}{(2n+1)!} B_{2n+1}\left(\frac{1+t}{2}\right),\ n=1,2\ldots,$$

where B_{2k} is the $2k$-th Bernoulli number and $B_{2n+1}\left(\frac{1+t}{2}\right)$ is the Bernoulli polynomial.

Widder proved next fundamental lemma in [74].

Lemma 7.1 *If $\varphi \in C^{2n}[0,1]$, then*

$$\varphi(t) = \sum_{k=0}^{n-1}\left[\varphi^{(2k)}(0)\Lambda_k(1-t) + \varphi^{(2k)}(1)\Lambda_k(t)\right] + \int_0^1 G_n(t,s)\varphi^{(2n)}(s)ds, \tag{7.21}$$

where

$$G_1(t,s) = G(t,s) = \begin{cases} (t-1)s, & s \le t, \\ (s-1)t, & t \le s, \end{cases} \tag{7.22}$$

is the homogeneous Green's function of the differential operator $\frac{d^2}{ds^2}$ on $[0,1]$, and with the successive iterates of $G(t,s)$

$$G_n(t,s) = \int_0^1 G_1(t,p)G_{n-1}(p,s)dp,\ \ n \ge 2. \tag{7.23}$$

We can see that the equation (7.21) is the generalization of (7.14).

Lidstone's polynomial can be expressed in terms of $G_n(t,s)$ as

$$\Lambda_n(t) = \int_0^1 G_n(t,s)s\,ds.$$

Notice that $G_n(t,s)$ is a symmetric function. Now we give the following special case of Theorem 7.1.

Corollary 7.4 *Let $\varphi \in C^{2n}[a,b]$, $n \ge 1$. If f is a positive convex function and g a positive concave function on an interval $I \subseteq \mathbb{R}$, then the inequality*

$$\begin{aligned}
&\int_a^b E_{2n}(x) f\left(\frac{|\varphi(x) - \sum_{k=0}^{n-1}(b-a)^{2k}\left[\varphi^{(2k)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right) + \varphi^{(2k)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right]|}{E_{2n}(x)}\right) \\
&\quad \times g(|\varphi^{(2n)}(x)|)\,dx \\
&\le \int_a^b E_{2n}(x) g\left(\frac{|\varphi(x) - \sum_{k=0}^{n-1}(b-a)^{2k}\left[\varphi^{(2k)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right) + \varphi^{(2k)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right]|}{E_{2n}(x)}\right) \\
&\quad \times f(|\varphi^{(2n)}(x)|)\,dx
\end{aligned} \tag{7.24}$$

holds, where E_{2n} is the Euler polynomial.

Proof. By Widder's lemma we can represent every function $\varphi \in C^{2n}[a,b]$ in the form

$$\varphi(x) - \sum_{k=0}^{n-1} (b-a)^{2k} \left[\varphi^{(2k)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right) + \varphi^{(2k)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right) \right]$$
$$= (b-a)^{2n-1} \int_a^b G_n\left(\frac{x-a}{b-a}, \frac{s-a}{b-a}\right) \varphi^{(2n)}(s)\, ds. \tag{7.25}$$

Now by Theorem 7.1 with $k(x,s) = (b-a)^{2n-1} G_n\left(\frac{x-a}{b-a}, \frac{s-a}{b-a}\right)$ and $v(s) = \varphi^{(2n)}(s)$ follows

$$K(x) = (b-a)^{2n-1} \int_a^b G_n\left(\frac{x-a}{b-a}, \frac{s-a}{b-a}\right) ds = E_{2n}(x), \tag{7.26}$$

$$u(x) = (b-a)^{2n-1} \int_a^b G_n\left(\frac{x-a}{b-a}, \frac{s-a}{b-a}\right) \varphi^{(2n)}(s)\, ds$$
$$= \varphi(x) - \sum_{k=0}^{n-1} (b-a)^{2k} \left[\varphi^{(2k)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right) + \varphi^{(2k)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right) \right] \tag{7.27}$$

and inequality (7.7) becomes (7.24), which completes the proof. □

Remark 7.8 If $\varphi^{(2k)}(a) = \varphi^{(2k)}(b) = 0$, then (7.24) reduces to

$$\int_a^b E_{2n}(x) f\left(\frac{|\varphi(x)|}{E_{2n}(x)}\right) g(|\varphi^{(2n)}(x)|)\, dx$$
$$\leq \int_a^b E_{2n}(x) f(|\varphi^{(2n)}(x)|)\, g\left(\frac{|\varphi(x)|}{E_{2n}(x)}\right) dx.$$

Following result is a special case of Theorem 7.2 involving the Lidstone polynomials.

Corollary 7.5 *Let $0 < q < 1$, $S : [a,b] \to \mathbb{R}_+$ and $\varphi \in C^{2n}[a,b]$, $n \geq 1$. If f is a positive convex function on an interval $I \subseteq \mathbb{R}$, then the inequality*

$$\int_a^b S(x) f\left(\frac{|\varphi(x) - \sum_{k=0}^{n-1}(b-a)^{2k}\left[\varphi^{(2k)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right) + \varphi^{(2k)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right]|}{E_{2n}(x)} \right)$$
$$\times |\varphi^{(2n)}(x)|^q\, dx$$
$$\leq \int_a^b R(x) \left(\frac{|\varphi(x) - \sum_{k=0}^{n-1}(b-a)^{2k}\left[\varphi^{(2k)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right) + \varphi^{(2k)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right]|}{E_{2n}(x)} \right)^q$$
$$\times f(|\varphi^{(2n)}(x)|) dx, \tag{7.28}$$

holds, where

$$R(t) = \left[(b-a)^{2n-1} \int_a^b \left(\frac{S(x)}{E_{2n}(x)}\right)^{\frac{1}{1-q}} |G_n\left(\frac{x-a}{b-a}, \frac{t-a}{b-a}\right)| dx \right]^{1-q}.$$

Proof. We apply Theorem 7.2 with $k(x,s) = (b-a)^{2n-1}G_n\left(\frac{x-a}{b-a}, \frac{s-a}{b-a}\right)$, $v(s) = \varphi^{(2n)}(s)$, $n \geq 1$. Then by (7.26) and (7.27) inequality (7.10) becomes (7.28). □

Remark 7.9 If $\varphi^{(2k)}(a) = \varphi^{(2k)}(b) = 0$, then (7.28) reduces to

$$\begin{aligned}
&\int_a^b S(x) f\left(\frac{|\varphi(x)|}{E_{2n}(x)}\right) |\varphi^{(2n)}(x)|^q dx \\
&\leq \int_a^b R(x) \left(\frac{|\varphi(x)|}{E_{2n}(x)}\right)^q f(|\varphi^{(2n)}(x)|) dx.
\end{aligned}$$

We also give a special case of Theorem 7.3.

Corollary 7.6 *Let $\varphi, \psi \in C^{2n}[a,b]$, $n \geq 1$, such that $\varphi^{(2n)}$ is either nonnegative or nonpositive function, and $\psi^{(2n)}$ is either nonnegative or nonpositive function. If f is a positive convex function and g a positive concave function on an interval $I \subseteq \mathbb{R}$, then the inequality*

$$\begin{aligned}
&\int_a^b |\psi(x) - \sum_{k=0}^{n-1} (b-a)^{2k} \left[\psi^{(2k)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right) + \psi^{(2k)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right]| \\
&\quad \times f\left(\frac{|\varphi(x) - \sum_{k=0}^{n-1}(b-a)^{2k}\left[\varphi^{(2k)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right) + \varphi^{(2k)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right]|}{|\psi(x) - \sum_{k=0}^{n-1}(b-a)^{2k}\left[\psi^{(2k)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right) + \psi^{(2k)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right]|}\right) \\
&\quad \times g(|\varphi^{(2n)}(x)|) dx \\
&\leq \int_a^b E_{2n}(x) |\psi^{(2n)}(x)| \\
&\quad \times g\left(\frac{|\varphi(x) - \sum_{k=0}^{n-1}(b-a)^{2k}\left[\varphi^{(2k)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right) + \varphi^{(2k)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right]|}{E_{2n}(x)}\right) \\
&\quad \times f\left(\left|\frac{\varphi^{(2n)}(x)}{\psi^{(2n)}(x)}\right|\right) dx
\end{aligned} \tag{7.29}$$

holds, where E_{2n} is Euler polynomial.

Proof. By Widder's lemma we can represent every function $\varphi \in C^{2n}[a,b]$ in the form (7.25). Now we apply Theorem 7.3 with $k(x,s) = (b-a)^{2n-1}G_n\left(\frac{x-a}{b-a}, \frac{s-a}{b-a}\right)$, $v(s) = \varphi^{(2n)}(s)$ and $v_1(x) = \psi^{(2n)}(x)$, from which follows (7.26), (7.27) and

$$\begin{aligned}
\psi(x) &= (b-a)^{2n-1} \int_a^b G_n\left(\frac{x-a}{b-a}, \frac{s-a}{b-a}\right) \psi^{(2n)}(s) ds \\
&= \psi(x) - \sum_{k=0}^{n-1} (b-a)^{2k} \left[\psi^{(2k)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right) + \psi^{(2k)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right].
\end{aligned}$$

Hence, inequality (7.12) becomes (7.29). □

We continue with the following special case of inequality (7.29).

Remark 7.10 If $\varphi^{(2k)}(a)=\varphi^{(2k)}(b)=0$, then (7.29) reduces to

$$\int_a^b |\psi(x)-\sum_{k=0}^{n-1}(b-a)^{2k}\left[\psi^{(2k)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right)+\psi^{(2k)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right]|$$
$$\times f\left(\frac{|\varphi(x|}{|\psi(x)-\sum_{k=0}^{n-1}(b-a)^{2k}\left[\psi^{(2k)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right)+\psi^{(2k)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right]|}\right)$$
$$\times g(|\varphi^{(2n)}(x)|)dx$$
$$\leq \int_a^b E_{2n}(x)|\psi^{(2n)}(x)|g\left(\frac{|\varphi(x)|}{E_{2n}(x)}\right)f\left(\left|\frac{\varphi^{(2n)}(x)}{\psi^{(2n)}(x)}\right|\right)dx.$$

Remark 7.11 If $v_1(x)=1$, then $\psi(x)=K(x)$ and (7.29) reduces to

$$\int_a^b E_{2n}(x)f\left(\frac{|\varphi(x)-\sum_{k=0}^{n-1}(b-a)^{2k}\left[\varphi^{(2k)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right)+\varphi^{(2k)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right]|}{E_{2n}(x)}\right)$$
$$\times g(|\varphi^{(2n)}(x)|)dx$$
$$\leq \int_a^b E_{2n}(x)g\left(\frac{|\varphi(x)-\sum_{k=0}^{n-1}(b-a)^{2k}\left[\varphi^{(2k)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right)+\varphi^{(2k)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right]|}{E_{2n}(x)}\right)$$
$$\times f(|\varphi^{(2n)}(x)|)dx$$

which is given in Corollary 7.4.

Next we present several results involving the Hermite interpolating polynomial. Following lemma for two-point Taylor conditions comes from [9].

Lemma 7.2 *Let $\varphi\in C^n[a,b]$, $(n\geq 2, n=2m)$. Then*

$$\varphi(x) = \sum_{i=0}^{m-1}\sum_{k=0}^{m-1-i}\binom{m+k-1}{k}\left[\varphi^{(i)}(a)\tau_i(x)+\varphi^{(i)}(b)\nu_i(x)\right] + \int_a^b \varphi^{(2m)}(s)G_{2T}(x,s)\,ds, \tag{7.30}$$

where τ_i and ν_i are defined on $[a,b]$ with

$$\tau_i(x)=\frac{(x-a)^i}{i!}\left(\frac{x-b}{a-b}\right)^m\left(\frac{x-a}{b-a}\right)^k, \tag{7.31}$$

$$\nu_i(x)=\frac{(x-b)^i}{i!}\left(\frac{x-a}{b-a}\right)^m\left(\frac{x-b}{a-b}\right)^k, \tag{7.32}$$

G_{2T} is Green's function of the two-point Taylor problem

$$G_{2T}(t,s)=\begin{cases}\frac{(-1)^m}{(2m-1)!}p^m(t,s)\sum_{j=0}^{m-1}\binom{m-1+j}{j}(t-s)^{m-1-j}q^j(t,s), & s\leq t,\\[2ex] \frac{(-1)^m}{(2m-1)!}q^m(t,s)\sum_{j=0}^{m-1}\binom{m-1+j}{j}(s-t)^{m-1-j}p^j(t,s), & s\geq t,\end{cases} \tag{7.33}$$

and for all $t,s \in [a,b]$

$$p(t,s) = \frac{(s-a)(b-t)}{b-a}, \qquad q(t,s) = p(s,t).$$

Now, we give a special case of Theorem 7.1 that involves the Hermite interpolating polynomial.

Corollary 7.7 *Let* $\varphi \in C^{2m}[a,b]$*. If* f *is a positive convex function and* g *a positive concave function on an interval* $I \subseteq \mathbb{R}$*, then the inequality*

$$\begin{aligned}
&\int_a^b (x-a)^m (b-x)^m \\
&\quad \times f\left(\frac{(2m)!|\varphi(x) - \sum_{i=0}^{m-1}\sum_{k=0}^{m-1-i}\binom{m+k-1}{k}[\varphi^{(i)}(a)\tau_i(x) + \varphi^{(i)}(b)\nu_i(x)]|}{(x-a)^m(b-x)^m} \right) \\
&\quad \times g\left(|\varphi^{(2m)}(x)|\right) dx \\
&\leq \int_a^b (x-a)^m (b-x)^m \\
&\quad \times g\left(\frac{(2m)!|\varphi(x) - \sum_{i=0}^{m-1}\sum_{k=0}^{m-1-i}\binom{m+k-1}{k}[\varphi^{(i)}(a)\tau_i(x) + \varphi^{(i)}(b)\nu_i(x)]|}{(x-a)^m(b-x)^m} \right) \\
&\quad \times f\left(|\varphi^{(2m)}(x)|\right) dx \qquad (7.34)
\end{aligned}$$

holds.

Proof. Since $G_{2T}(t,s)$ is a symmetric function, we define

$$\begin{aligned}
&\varphi(x) - \sum_{i=0}^{m-1}\sum_{k=0}^{m-1-i}\binom{m+k-1}{k} \cdot [\varphi^{(i)}(a)\tau_i(x) + \varphi^{(i)}(b)\nu_i(x)] \qquad (7.35) \\
&= \int_a^b \varphi^{(2m)}(s) G_{2T}(x,s)\, ds.
\end{aligned}$$

Now by Theorem 7.1 with $k(x,s) = G_{2T}(x,s)$ and $v(s) = \varphi^{(2m)}(s)$ follows

$$\begin{aligned}
|K(x)| &= \int_a^b |G_{2T}(x,s)| ds = \frac{1}{(2m)!}|(x-a)^m(x-b)^m| \\
&= \frac{1}{(2m)!}(x-a)^m(b-x)^m, \qquad (7.36)
\end{aligned}$$

$$\begin{aligned}
u(x) &= \int_a^b G_{2T}(x,s)\varphi^{(2m)}(s)\, ds \\
&= \varphi(x) - \sum_{i=0}^{m-1}\sum_{k=0}^{m-1-i}\binom{m+k-1}{k} \cdot [\varphi^{(i)}(a)\tau_i(x) + \varphi^{(i)}(b)\nu_i(x)] \qquad (7.37)
\end{aligned}$$

and inequality (7.7) becomes (7.34). □

Remark 7.12 If $\varphi^{(i)}(a) = \varphi^{(i)}(b) = 0$, then inequality (7.34) reduces to

$$\int_a^b (x-a)^m(b-x)^m f\left(\frac{(2m)!|\varphi(x)|}{(x-a)^m(b-x)^m}\right) g(|\varphi^{(2m)}(x)|)\,dx$$
$$\leq \int_a^b (x-a)^m(b-x)^m g\left(\frac{(2m)!|\varphi(x)|}{(x-a)^m(b-x)^m}\right) f(|\varphi^{(2m)}(x)|)\,dx. \tag{7.38}$$

We proceed with a special case of Theorem 7.2.

Corollary 7.8 *Let* $0 < q < 1$, $S : [a,b] \to \mathbb{R}_+$ *and* $\varphi \in C^{2m}[a,b]$. *If* f *is a positive convex function on an interval* $I \subseteq \mathbb{R}$, *then the inequality*

$$\int_a^b S(x) f\left(\frac{(2m)!|\varphi(x) - \sum_{i=0}^{m-1}\sum_{k=0}^{m-1-i}\binom{m+k-1}{k}[\varphi^{(i)}(a)\tau_i(x)+\varphi^{(i)}(b)\nu_i(x)]|}{(x-a)^m(b-x)^m}\right)$$
$$\times |\varphi^{(2m)}(x)|^q\,dx$$
$$\leq \int_a^b S(x)\left(\frac{(2m)!|\varphi(x) - \sum_{i=0}^{m-1}\sum_{k=0}^{m-1-i}\binom{m+k-1}{k}[\varphi^{(i)}(a)\tau_i(x)+\varphi^{(i)}(b)\nu_i(x)]|}{(x-a)^m(b-x)^m}\right)^q$$
$$\times f(|\varphi^{(2m)}(x)|)\,dx \tag{7.39}$$

holds, where

$$R(t) = \left[\int_\Omega \left(\frac{(2m)!S(x)}{(x-a)^m(b-x)^m}\right)^{\frac{1}{1-q}} |G_{2T}(x,t)|dx\right]^{1-q}.$$

Proof. By Theorem 7.2 with $k(x,s) = G_{2T}(x,s)$ and $v(s) = \varphi^{(2m)}(s)$ follows (7.36), (7.37), hence, inequality (7.10) becomes (7.39). □

Remark 7.13 If $\varphi^{(i)}(a) = \varphi^{(i)}(b) = 0$, then (7.39) becomes

$$\int_a^b S(x) f\left(\frac{(2m)!|\varphi(x)|}{(x-a)^m(b-x)^m}\right) |\varphi^{(2m)}(x)|^q\,dx$$
$$\leq \int_a^b S(x)\left(\frac{(2m)!|\varphi(x)|}{(x-a)^m(b-x)^m}\right)^q f(|\varphi^{(2m)}(x)|)\,dx.$$

Finally, we conclude with a special case of Theorem 7.3 for the Hermite interpolating polynomial.

Corollary 7.9 *Let* $\varphi, \psi \in C^{2m}[a,b]$ *such that* $\varphi^{(2m)}$ *is either nonnegative or nonpositive function, and* $\psi^{(2m)}$ *is either nonnegative or nonpositive function. If* f *is a positive convex function and* g *a positive concave function on an interval* $I \subseteq \mathbb{R}$, *then the inequality*

$$\int_a^b |\psi(x) - \sum_{i=0}^{m-1}\sum_{k=0}^{m-1-i}\binom{m+k-1}{k}[\psi^{(i)}(a)\tau_i(x)+\psi^{(i)}(b)\nu_i(x)])|$$

$$\times f\left(\frac{|\varphi(x)-\sum_{i=0}^{m-1}\sum_{k=0}^{m-1-i}\binom{m+k-1}{k}[\varphi^{(i)}(a)\tau_i(x)+\varphi^{(i)}(b)\nu_i(x)]|}{|\psi(x)-\sum_{i=0}^{m-1}\sum_{k=0}^{m-1-i}\binom{m+k-1}{k}[\psi^{(i)}(a)\tau_i(x)+\psi^{(i)}(b)\nu_i(x)])|}\right)$$
$$\times g(|\varphi^{(2m)}(x)|)\,dx$$
$$\leq \int_a^b \frac{1}{(2m)!}(x-a)^m(b-x)^m|\psi^{(2m)}(x)|\,f\left(\left|\frac{\varphi^{(2m)}(x)}{\psi^{(2m)}(x)}\right|\right)$$
$$\times g\left(\frac{(2m)!|\varphi(x)-\sum_{i=0}^{m-1}\sum_{k=0}^{m-1-i}\binom{m+k-1}{k}[\varphi^{(i)}(a)\tau_i(x)+\varphi^{(i)}(b)\nu_i(x)]|}{(x-a)^m(b-x)^m}\right)dx \tag{7.40}$$

holds.

Proof. Since $G_{2T}(t,s)$ is a symmetric function, (7.35) holds. Now from Theorem 7.3 with $k(x,s)=G_{2T}(x,s)$, $v(s)=\varphi^{(2m)}(s)$ and $v_1(x)=\psi^{(2m)}(x)$, we obtain (7.36), (7.37) and

$$\psi(x)=\int_a^b G_{2T}(x,s)\psi^{(2m)}(s)ds$$
$$=\psi(x)-\sum_{i=0}^{m-1}\sum_{k=0}^{m-1-i}\binom{m+k-1}{k}\cdot[\psi^{(i)}(a)\tau_i(x)+\psi^{(i)}(b)\nu_i(x)]\,.$$

Hence, inequality (7.12) becomes (7.40), which completes the proof. □

Remark 7.14 If $\varphi^{(i)}(a)=\varphi^{(i)}(b)=0$, then inequality (7.40) reduces to

$$\int_a^b |\psi(x)-\sum_{i=0}^{m-1}\sum_{k=0}^{m-1-i}\binom{m+k-1}{k}[\psi^{(i)}(a)\tau_i(x)+\psi^{(i)}(b)\nu_i(x)])|$$
$$\times f\left(\frac{|\varphi(x)|}{|\psi(x)-\sum_{i=0}^{m-1}\sum_{k=0}^{m-1-i}\binom{m+k-1}{k}[\psi^{(i)}(a)\tau_i(x)+\psi^{(i)}(b)\nu_i(x)])}\right)$$
$$\times g(|\varphi^{(2m)}(x)|)dx$$
$$\leq \int_a^b \frac{1}{(2m)!}(x-a)^m(b-x)^m|\psi^{(2m)}(x)|f\left(\left|\frac{\varphi^{(2m)}(x)}{\psi^{(2m)}(x)}\right|\right)$$
$$\times g\left(\frac{(2m)!|\varphi(x)|}{(x-a)^m(b-x)^m}\right)dx\,.$$

Remark 7.15 If $v_1(x)=1$, then $\psi(x)=K(x)$ and (7.40) reduces to

$$\int_a^b (x-a)^m(b-x)^m$$
$$\times f\left(\frac{(2m)!|\varphi(x)-\sum_{i=0}^{m-1}\sum_{k=0}^{m-1-i}\binom{m+k-1}{k}[\varphi^{(i)}(a)\tau_i(x)+\varphi^{(i)}(b)\nu_i(x)]|}{(x-a)^m(b-x)^m}\right)$$
$$\times g(|\varphi^{(2m)}(x)|)\,dx$$

$$\leq \int_a^b (x-a)^m (b-x)^m$$
$$\times g\left(\frac{(2m)!|\varphi(x) - \sum_{i=0}^{m-1}\sum_{k=0}^{m-1-i}\binom{m+k-1}{k}[\varphi^{(i)}(a)\tau_i(x) + \varphi^{(i)}(b)\nu_i(x)]|}{(x-a)^m(b-x)^m}\right)$$
$$\times f(|\varphi^{(2m)}(x)|)\,dx$$

which is given in Corollary 7.7.

Chapter 8

Weighted integral and discrete Opial-type inequalities

We will start this chapter with improvements of some Opial-type inequalities in one variable due to Agarwal and Pang ([3]). In [8], Agarwal and Sheng proved a total of 25 results on integral inequalities in n variables. These inequalities are similar to those of Nirenberg, Opial, Poincaré, Serrin, Sobolev and Wirtinger. Furthermore, in [4], Agarwal and Pang proved several Opial and Wirtinger type discrete inequalities, and also multidimensional generalizations. The sharpness as well as the unification of several known results, mainly of Pachpatte, was shown in the numerous remarks.

In this chapter we will use some elementary techniques such as appropriate integral representations of functions, appropriate summation representations of discrete functions and inequalities involving means to establish integral, and discrete, multidimensional inequalities. The obtained results are sharper than those known in the literature [3, 4, 5, 8]. This chapter is based on our results: Agarwal, Andrić, Brnetić, Pečarić and Perić [6, 7, 22, 36].

8.1 Integral inequalities in one variable

We will improve following results by Agarwal and Pang from [3].

Theorem 8.1 *Let $\lambda \geq 1$ be a given real number and let p be a nonnegative and continuous function on $[0,h]$. Further, let x be an absolutely continuous function on $[0,h]$, with*

$x(0)=x(h)=0$. *Then, the following inequality holds*

$$\int_0^h p(t)|x(t)|^\lambda\,dt \le \frac{1}{2}\left(\int_0^h (t(h-t))^{\frac{\lambda-1}{2}} p(t)\,dt\right)\int_0^h |x'(t)|^\lambda\,dt\,. \tag{8.1}$$

For $p(t)=const.$ *the inequality* (8.1) *reduces to*

$$\int_0^h |x(t)|^\lambda\,dt \le \frac{h^\lambda}{2} B\left(\frac{\lambda+1}{2},\frac{\lambda+1}{2}\right)\int_0^h |x'(t)|^\lambda\,dt\,, \tag{8.2}$$

where B *is the beta function.*

Theorem 8.2 *Assume that*

(i) *l, m, μ and λ are nonnegative real numbers such that* $\frac{1}{\mu}+\frac{1}{\lambda}=1$ *and* $l\mu\ge 1$,

(ii) *q is a nonnegative and continuous function on* $[0,h]$,

(iii) *x_1 and x_2 are absolutely continuous functions on* $[0,h]$, *with* $x_1(0)=x_1(h)=x_2(0)=x_2(h)=0$.

Then, the following inequality holds

$$\begin{aligned}\int_0^h q(t)\big(|x_1(t)|^l|x_2'(t)|^m+|x_2(t)|^l|x_1'(t)|^m\big)\,dt \;\le\; &\left(\frac{1}{2}\int_0^h (t(h-t))^{\frac{l\mu-1}{2}} q^\mu(t)\,dt\right)^{\frac{1}{\mu}}\\ &\times\int_0^h \Big(\frac{1}{\mu}(|x_1'(t)|^{l\mu}+|x_2'(t)|^{l\mu})+\frac{1}{\lambda}(|x_1'(t)|^{m\lambda}+|x_2'(t)|^{m\lambda})\Big)\,dt\,.\end{aligned}$$

Theorem 8.3 *Let* r_k, $k=0,\dots,n-1$, *and l be nonnegative real numbers such that* $l\alpha\ge 1$, *where* $\alpha=\sum_{k=0}^{n-1} r_k$ *and let p be a nonnegative continuous function on* $[0,h]$. *Further, let* $x\in C^{(n-1)}[0,h]$ *be such that* $x^{(i)}(0)=x^{(i)}(h)=0$, $i=0,\dots,n-1$, *and* $x^{(n-1)}$ *is absolutely continuous. Then, the following inequality holds*

$$\begin{aligned}\int_0^h p(t)\left(\prod_{k=0}^{n-1}|x^{(k)}(t)|^{r_k}\right)^l dt \;\le\; &\frac{1}{2\alpha}\left(\int_0^h (t(h-t))^{\frac{l\alpha-1}{2}} p(t)\,dt\right)\\ &\times\sum_{k=0}^{n-1} r_k\int_0^h |x^{(k+1)}(t)|^{l\alpha}\,dt\,.\end{aligned}$$

First we give a generalization of Theorem 8.1 involving submultiplicative convex function. Recall that function $f:[0,\infty)\to[0,\infty)$ is called *submultiplicative* function if it satisfies the inequality

$$f(xy)\;\le\;f(x)f(y)\,,\quad \text{for all } x,y\in[0,\infty)\,.$$

In a special case this theorem will improve Theorem 8.1 (see Remark 8.2).

Theorem 8.4 *Let f be an increasing, submultiplicative convex function on $[0,\infty)$ and let p be a nonnegative and integrable function on $[0,h]$. Further, let $x \in AC[0,h]$ be such that $x(0) = x(h) = 0$. Then the following inequality holds*

$$\int_0^h p(t) f(|x(t)|)\, dt \le \left(\int_0^h \left(\frac{t}{f(t)} + \frac{h-t}{f(h-t)} \right)^{-1} p(t)\, dt \right) \int_0^h f\left(|x'(t)|\right) dt\,. \tag{8.3}$$

Proof. As in [3] from the hypotheses of theorem we have

$$x(t) = \int_0^t x'(s) ds\,,$$

$$x(t) = -\int_t^h x'(s) ds\,.$$

Since f is an increasing and convex function, we use Jensen's inequality to obtain

$$f(|x(t)|) \le f\left(\frac{1}{t} \int_0^t t\, |x'(s)| ds \right) \le \frac{1}{t} \int_0^t f\left(t\, |x'(s)|\right) ds$$

and by submultiplicativity of f follows

$$f(|x(t)|) \le \frac{1}{t} \int_0^t f(t) f(|x'(s)|)\, ds = \frac{f(t)}{t} \int_0^t f(|x'(s)|)\, ds\,. \tag{8.4}$$

Analogously we obtain

$$\begin{aligned} f(|x(t)|) &\le f\left(\frac{1}{h-t} \int_t^h (h-t)\, |x'(s)| ds \right) \\ &\le \frac{1}{h-t} \int_t^h f\left((h-t)\, |x'(s)|\right) ds \\ &\le \frac{1}{h-t} \int_t^h f(h-t) f(|x'(s)|)\, ds \\ &= \frac{f(h-t)}{h-t} \int_t^h f(|x'(s)|)\, ds\,. \end{aligned} \tag{8.5}$$

Multiplying (8.4) by $\frac{t}{f(t)}$ and (8.5) by $\frac{h-t}{f(h-t)}$ and adding these inequalities, we find

$$\left(\frac{t}{f(t)} + \frac{h-t}{f(h-t)} \right) f(|x(t)|) \le \int_0^h f\left(|x'(s)|\right) ds\,,$$

i.e.

$$f(|x(t)|) \le \left(\frac{t}{f(t)} + \frac{h-t}{f(h-t)} \right)^{-1} \int_0^h f\left(|x'(s)|\right) ds\,. \tag{8.6}$$

Now multiplying (8.6) by p and integrating on $[0,h]$ we obtain the inequality (8.3). □

Remark 8.1 For a special class of a submultiplicative convex functions f on $[0,\infty)$ with $f(0)=0$, Theorem 8.4 also holds. Namely, submultiplicativity of a function implies its positivity, and if f is a convex, nonnegative function on $[0,\infty)$ with $f(0)=0$, then f is obviously an increasing function.

Corollary 8.1 *Let f be an increasing, submultiplicative convex function on $[0,\infty)$ and let p be a nonnegative and integrable function on $[0,h]$. Further, let $x\in AC[0,h]$ be such that $x(0)=x(h)=0$. Then the following inequality holds*

$$\int_0^h p(t)f(|x(t)|)\,dt \le \frac{1}{2}\left(\int_0^h \left(\frac{f(t)f(h-t)}{t(h-t)}\right)^{\frac{1}{2}} p(t)\,dt\right)\int_0^h f(|x'(t)|)\,dt\,. \tag{8.7}$$

Proof. The inequality (8.7) follows by the harmonic-geometric inequality

$$2\left(\frac{t}{f(t)}+\frac{h-t}{f(h-t)}\right)^{-1} \le \left(\frac{f(t)f(h-t)}{t(h-t)}\right)^{\frac{1}{2}}.$$

□

Next is a special case of Theorem 8.4.

Corollary 8.2 *Let $\lambda\ge 1$ be a given real number and let p be a nonnegative and continuous function on $[0,h]$. Further, let $x\in AC[0,h]$ be such that $x(0)=x(h)=0$. Then, the following inequality holds*

$$\int_0^h p(t)|x(t)|^\lambda\,dt \le \left(\int_0^h \left(t^{1-\lambda}+(h-t)^{1-\lambda}\right)^{-1}p(t)\,dt\right)\int_0^h |x'(t)|^\lambda\,dt\,. \tag{8.8}$$

Proof. The inequality (8.8) will follow if we use function $f(t)=t^\lambda$ and apply Theorem 8.4. □

Remark 8.2 By the harmonic-geometric inequality, we have

$$2\left(t^{1-\lambda}+(h-t)^{1-\lambda}\right)^{-1} \le \left(t^{\lambda-1}(h-t)^{\lambda-1}\right)^{\frac{1}{2}}.$$

Hence, it is clear that (8.8) improves (8.1).

Corollary 8.3 *Let $\lambda\ge 1$ be a given real number and let $x\in AC[0,h]$ be such that $x(0)=x(h)=0$. Then, the following inequality holds*

$$\int_0^h |x(t)|^\lambda\,dt \le h^\lambda I(\lambda)\int_0^h |x'(t)|^\lambda\,dt\,, \tag{8.9}$$

where

$$I(\lambda)=\int_0^1 \left(t^{1-\lambda}+(1-t)^{1-\lambda}\right)^{-1}dt\,, \tag{8.10}$$

Proof. By putting $p(t) = const.$ in (8.8) we obtain

$$\int_0^h |x(t)|^\lambda dt \le \left(\int_0^h \left(t^{1-\lambda} + (h-t)^{1-\lambda}\right)^{-1} dt \right) \int_0^h |x'(t)|^\lambda dt .$$

The inequality (8.9) is now clear. □

Remark 8.3 By applying some Wirtinger-type inequalities, Saker established new large spaces between the zeros of the Riemann zeta-function in [71]. Our inequality (8.9) is used in [71, Theorem 2.2].

Corollary 8.4 *Let $\lambda \ge 1$ be a given real number and let $x \in AC[0,h]$ be such that $x(0) = x(h) = 0$. Then, the following inequalities hold:*

$$\int_0^h |x(t)|^2 dt \le \frac{h^2}{6} \int_0^h |x'(t)|^2 dt , \tag{8.11}$$

$$\int_0^h |x(t)|^3 dt \le \frac{3\pi - 8}{24} h^3 \int_0^h |x'(t)|^3 dt , \tag{8.12}$$

$$\int_0^h |x(t)|^4 dt \le \frac{20\sqrt{3}\pi - 81}{1215} h^4 \int_0^h |x'(t)|^4 dt . \tag{8.13}$$

Proof. It is a special case of Corollary 8.3 for $\lambda = 2,3,4$. The integrals $I(2) = \frac{1}{6}$, $I(3) = \frac{3\pi-8}{24}$ and $I(4) = \frac{20\sqrt{3}\pi-81}{1215}$ are computed easily. □

Remark 8.4 It is interesting to compare this results with the inequalities which follow from Theorem 8.1 by taking $p(t) = const.$ and $\lambda = 2,3,4$. From (8.2), the inequalities corresponding to (8.11) – (8.13) will have corresponding constants $\frac{\pi h^2}{16}$, $\frac{h^3}{12}$ and $\frac{3\pi h^4}{256}$.

Now we will improve the result of Theorem 8.2.

Theorem 8.5 *Assume that*

(*i*) *l, m, μ and λ are nonnegative real numbers such that $\frac{1}{\mu} + \frac{1}{\lambda} = 1$ and $l\mu \ge 1$,*

(*ii*) *q is a nonnegative and continuous function on $[0,h]$,*

(*iii*) *$x_1, x_2 \in AC[0,h]$ be such that $x_1(0) = x_1(h) = x_2(0) = x_2(h) = 0$.*

Then, the following inequality holds

$$\begin{aligned}
&\int_0^h q(t)\left(|x_1(t)|^l |x_2'(t)|^m + |x_2(t)|^l |x_1'(t)|^m\right) dt \\
&\le \left(\int_0^h \left(t^{1-l\mu} + (h-t)^{1-l\mu}\right)^{-1} q^\mu(t)\, dt \right)^{\frac{1}{\mu}} \\
&\quad \times \int_0^h \left(\frac{1}{\mu}\left(|x_1'(t)|^{l\mu} + |x_2'(t)|^{l\mu}\right) + \frac{1}{\lambda}\left(|x_1'(t)|^{m\lambda} + |x_2'(t)|^{m\lambda}\right)\right) dt .
\end{aligned} \tag{8.14}$$

Proof. From Hölder's inequality with indices μ and λ we have

$$\int_0^h q(t)|x_1(t)|^l|x_2'(t)|^m dt \le \left(\int_0^h q^\mu(t)|x_1(t)|^{l\mu}dt\right)^{\frac{1}{\mu}}\left(\int_0^h |x_2'(t)|^{m\lambda}dt\right)^{\frac{1}{\lambda}}.$$

Now, from (8.8) we find

$$\int_0^h q(t)|x_1(t)|^l|x_2'(t)|^m dt \le \left(\int_0^h \left(t^{1-l\mu}+(h-t)^{1-l\mu}\right)^{-1}q^\mu(t)\,dt\right)^{\frac{1}{\mu}} \times\left(\int_0^h |x_1'(t)|^{l\mu}dt\right)^{\frac{1}{\mu}}\left(\int_0^h |x_2'(t)|^{m\lambda}dt\right)^{\frac{1}{\lambda}},$$

and from Young's inequality it follows that

$$\int_0^h q(t)|x_1(t)|^l|x_2'(t)|^m dt \le \left(\int_0^h \left(t^{1-l\mu}+(h-t)^{1-l\mu}\right)^{-1}q^\mu(t)\,dt\right)^{\frac{1}{\mu}} \times\int_0^h \left(\frac{1}{\mu}|x_1'(t)|^{l\mu}+\frac{1}{\lambda}|x_2'(t)|^{m\lambda}\right)dt\,. \tag{8.15}$$

Similarly, we obtain

$$\int_0^h q(t)|x_1'(t)|^m|x_2(t)|^l dt \le \left(\int_0^h \left(t^{1-l\mu}+(h-t)^{1-l\mu}\right)^{-1}q^\mu(t)\,dt\right)^{\frac{1}{\mu}} \times\int_0^h \left(\frac{1}{\lambda}|x_1'(t)|^{m\lambda}+\frac{1}{\mu}|x_2'(t)|^{l\mu}\right)dt\,. \tag{8.16}$$

An addition of (8.15) and (8.16) gives the inequality (8.14). □

Now we will establish Opial-type inequality involving higher order derivatives which improves the result of Theorem 8.3.

Theorem 8.6 *Let r_k, $k=0,\dots,n-1$, and l be nonnegative real numbers such that $l\alpha \ge 1$, where $\alpha=\sum_{k=0}^{n-1} r_k$ and let p be a nonnegative continuous function on $[0,h]$. Further, let $x\in AC^n[0,h]$ be such $x^{(i)}(0)=x^{(i)}(h)=0$, $i=0,\dots,n-1$, $n\ge 1$. Then, the following inequality holds*

$$\int_0^h p(t)\left(\prod_{k=0}^{n-1}|x^{(k)}(t)|^{r_k}\right)^l dt \le \frac{1}{\alpha}\left(\int_0^h \left(t^{1-l\alpha}+(h-t)^{1-l\alpha}\right)^{-1}p(t)\,dt\right) \times\sum_{k=0}^{n-1} r_k\int_0^h |x^{(k+1)}(t)|^{l\alpha}dt\,. \tag{8.17}$$

Proof. Using some well-known elementary inequalities, we have

$$\left(\prod_{k=0}^{n-1}|x^{(k)}(t)|^{r_k}\right)^l = \left(\prod_{k=0}^{n-1}|x^{(k)}(t)|^{\frac{r_k}{\alpha}}\right)^{l\alpha} \le \left(\sum_{k=0}^{n-1}\frac{r_k}{\alpha}|x^{(k)}(t)|\right)^{l\alpha}$$

$$\leq \sum_{k=0}^{n-1} \frac{r_k}{\alpha} |x^{(k)}(t)|^{l\alpha}. \tag{8.18}$$

The inequality (8.17) now follows from (8.18) and (8.8). □

8.2 Multidimensional integral inequalities

Let Ω be a bounded domain in $\mathbb{R}^m$ defined by $\Omega = \prod_{j=1}^{m}[a_j, b_j]$. Let $x = (x_1, \ldots, x_m)$ be a general point in Ω and $dx = dx_1 \ldots dx_m$. For any continuous real-valued function u defined on Ω we denote $\int_\Omega u(x)\,dx$ the m-fold integral $\int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} u(x_1, \ldots, x_m)\,dx_1 \ldots dx_m$. Let $D_k u(x_1, \ldots, x_m) = \frac{\partial}{\partial x_k} u(x_1, \ldots, x_m)$ and $D^k u(x_1, \ldots, x_m) = D_1 \cdots D_k u(x_1, \ldots, x_m)$, $1 \leq k \leq m$. We denote by $G(\Omega)$ the class of continuous functions $u : \Omega \to \mathbb{R}$ for which $D^m u(x)$ exists with $u(x)|_{x_j = a_j} = u(x)|_{x_j = b_j} = 0$, $1 \leq j \leq m$.
Further, let $u(x; s_j) = u(x_1, \ldots, x_{j-1}, s_j, x_{j+1}, \ldots, x_m)$, and

$$\|\mathrm{grad}\, u(x)\|_\mu = \left(\sum_{j=1}^{m} \left| \frac{\partial}{\partial x_j} u(x) \right|^\mu \right)^{\frac{1}{\mu}}.$$

Also let $\alpha = (\alpha_1, \ldots, \alpha_m)$ and $\alpha^\lambda = (\alpha_1^\lambda, \ldots, \alpha_m^\lambda)$, $\lambda \in \mathbb{R}$. In particular, $(b-a) = (b_1 - a_1, \ldots, b_m - a_m)$ and $(b-a)^\lambda = ((b_1 - a_1)^\lambda, \ldots, (b_m - a_m)^\lambda)$. For the geometric and the harmonic means of $\alpha_1, \ldots, \alpha_m$ we will use $G_m(\alpha)$ and $H_m(\alpha)$, respectively. Let $M^{[k]}(\alpha)$ denote the mean of order k of $\alpha_1, \ldots, \alpha_m$.

We will improve the following results by Agarwal and Sheng from [8]. First theorem includes Poincaré-type inequality.

Theorem 8.7 *Let $\lambda, \mu \geq 1$ and let $u \in G(\Omega)$. Then, the following inequality holds*

$$\int_\Omega |u(x)|^\lambda\,dx \leq K(\lambda, \mu) \int_\Omega \|\mathrm{grad}\, u(x)\|_\mu^\lambda\,dx,$$

where

$$K(\lambda, \mu) = \frac{1}{2m} B\left(\frac{1+\lambda}{2}, \frac{1+\lambda}{2}\right) C\left(\frac{\lambda}{\mu}\right) G_m\left((b-a)^\lambda\right), \tag{8.19}$$

$$C(\alpha) = \begin{cases} 1, & \alpha \geq 1, \\ m^{1-\alpha}, & 0 \leq \alpha \leq 1 \end{cases} \tag{8.20}$$

and B is the beta function.

Next result is involving certain Writinger-type inequality.

Theorem 8.8 *Let $p,\lambda,\mu \geq 1$ with $\frac{1}{\mu}+\frac{1}{\lambda}=1$ and let $u \in G(\Omega)$. Then, the following inequality holds*

$$\int_\Omega |u(x)|^p\,dx \leq \frac{M^{[\mu]}(b-a)}{2m^{\frac{1}{\lambda}}}\left(\int_\Omega |u(x)|^{\mu(p-1)}\,dx\right)^{\frac{1}{\mu}}\left(\int_\Omega \|\operatorname{grad} u(x)\|_\lambda^\lambda\,dx\right)^{\frac{1}{\lambda}}.$$

We continue with Sobolev-type inequality.

Theorem 8.9 *Let $p \geq 1$ and let $u \in G(\Omega)$. Then, the following inequality holds*

$$\left(\int_\Omega |u(x)|^{2p}\,dx\right)^{\frac{1}{p}} \leq \frac{\pi p^2\alpha^2}{16m}\left(\int_\Omega \|\operatorname{grad} u(x)\|_p^{2p}\,dx\right)^{\frac{1}{p}},$$

where $\alpha = \max\{(b_1-a_1),\ldots,(b_m-a_m)\}$.

The following two theorems are derived from the inequalities due to Pachpatte.

Theorem 8.10 *Let $l \geq 0$, $n \geq 1$ and let $u \in G(\Omega)$. Then, the following inequality holds*

$$\int_\Omega |u(x)|^{l+n}\,dx \leq \frac{1}{m}\left(\frac{l+n}{2n}\right)^n \sum_{i=1}^m (b_i-a_i)^n \int_\Omega |u(x)|^l \left|\frac{\partial}{\partial x_i}u(x)\right|^n dx.$$

Theorem 8.11 *Let $p,n \geq 0$ be such that $p+n \geq 1$ and let $u \in G(\Omega)$. Then, the following inequality holds*

$$\int_\Omega |u(x)|^p \|\operatorname{grad} u(x)\|_2^n\,dx \leq (K(p+n,2))^{\frac{p}{p+n}}\int_\Omega \|\operatorname{grad} u(x)\|_2^{p+n}\,dx,$$

where K is defined by (8.19).

8.2.1 Improvements of Poincaré-type inequality

We start with a weighted extension of Theorem 8.7 involving submultiplicative convex function.

Theorem 8.12 *Let f be an increasing, submultiplicative convex function on $[0,\infty)$. Let p be a nonnegative and integrable function on Ω and $u \in G(\Omega)$. Then, the following inequality holds*

$$\int_\Omega p(x)\,f(|u(x)|)\,dx \leq \frac{1}{m}H_m(\alpha)\int_\Omega\left(\sum_{i=1}^m f\left(\left|\frac{\partial}{\partial x_i}u(x)\right|\right)\right)dx, \tag{8.21}$$

where $\alpha = (\alpha_1,\ldots,\alpha_m)$ and

$$\alpha_i = \int_{a_i}^{b_i}\left(\frac{x_i-a_i}{f(x_i-a_i)}+\frac{b_i-x_i}{f(b_i-x_i)}\right)^{-1} p(x)\,dx_i, \quad i=1,\ldots,m.$$

Proof. For each fixed i, $i = 1,\ldots,m$, we have

$$u(x) = \int_{a_i}^{x_i} \frac{\partial}{\partial s_i} u(x;s_i)\, ds_i$$

and

$$u(x) = -\int_{x_i}^{b_i} \frac{\partial}{\partial s_i} u(x;s_i)\, ds_i\,.$$

First we use Jensen's inequality (since f is an increasing convex function) and then submultiplicativity of f, to obtain

$$\begin{aligned} f(|u(x)|) &\leq f\left(\frac{1}{x_i-a_i}\int_{a_i}^{x_i}(x_i-a_i)\left|\frac{\partial}{\partial s_i}u(x;s_i)\right| ds_i\right) \\ &\leq \frac{1}{x_i-a_i}\int_{a_i}^{x_i} f\left((x_i-a_i)\left|\frac{\partial}{\partial s_i}u(x;s_i)\right|\right) ds_i \\ &\leq \frac{1}{x_i-a_i}\int_{a_i}^{x_i} f(x_i-a_i) f\left(\left|\frac{\partial}{\partial s_i}u(x;s_i)\right|\right) ds_i \\ &= \frac{f(x_i-a_i)}{x_i-a_i}\int_{a_i}^{x_i} f\left(\left|\frac{\partial}{\partial s_i}u(x;s_i)\right|\right) ds_i \end{aligned} \tag{8.22}$$

and analogously

$$f(|u(x)|) \leq \frac{f(b_i-x_i)}{b_i-x_i}\int_{x_i}^{b_i} f\left(\left|\frac{\partial}{\partial s_i}u(x;s_i)\right|\right) ds_i\,, \tag{8.23}$$

for $i = 1,\ldots,m$. Multiplying (8.22) by $\frac{x_i-a_i}{f(x_i-a_i)}$ and (8.23) by $\frac{b_i-x_i}{f(b_i-x_i)}$ and adding these inequalities, we find

$$\left(\frac{x_i-a_i}{f(x_i-a_i)}+\frac{b_i-x_i}{f(b_i-x_i)}\right) f(|u(x)|) \leq \int_{a_i}^{b_i} f\left(\left|\frac{\partial}{\partial s_i}u(x;s_i)\right|\right) ds_i\,,$$

i.e.

$$f(|u(x)|) \leq \left(\frac{x_i-a_i}{f(x_i-a_i)}+\frac{b_i-x_i}{f(b_i-x_i)}\right)^{-1} \int_{a_i}^{b_i} f\left(\left|\frac{\partial}{\partial s_i}u(x;s_i)\right|\right) ds_i\,, \tag{8.24}$$

for $i = 1,\ldots,m$. Now multiplying (8.24) by p and integrating on Ω we obtain

$$\begin{aligned} \int_\Omega p(x) f(|u(x)|)\, dx \leq \int_{a_i}^{b_i} &\left(\frac{x_i-a_i}{f(x_i-a_i)}+\frac{b_i-x_i}{f(b_i-x_i)}\right)^{-1} p(x)\, dx_i \\ &\times \int_\Omega f\left(\left|\frac{\partial}{\partial x_i}u(x)\right|\right) dx, \end{aligned} \tag{8.25}$$

i.e.

$$\left(\int_{a_i}^{b_i}\left(\frac{x_i-a_i}{f(x_i-a_i)}+\frac{b_i-x_i}{f(b_i-x_i)}\right)^{-1} p(x)\, dx_i\right)^{-1} \int_\Omega p(x) f(|u(x)|)\, dx$$

$$\leq \int_\Omega f\left(\left|\frac{\partial}{\partial x_i}u(x)\right|\right)dx, \tag{8.26}$$

for $i = 1,\ldots,m$. Notice that

$$\alpha_i^{-1} = \left(\int_{a_i}^{b_i}\left(\frac{x_i-a_i}{f(x_i-a_i)}+\frac{b_i-x_i}{f(b_i-x_i)}\right)^{-1}p(x)\,dx_i\right)^{-1}, \quad i = 1,\ldots,m. \tag{8.27}$$

Now, by summing these m inequalities (8.26), we find

$$\sum_{i=1}^m \alpha_i^{-1}\int_\Omega p(x)\,f(|u(x)|)\,dx \leq \sum_{i=1}^m \int_\Omega f\left(\left|\frac{\partial}{\partial x_i}u(x)\right|\right)dx,$$

which is the same as the inequality (8.21). □

Corollary 8.5 *Let f be an increasing, submultiplicative convex function on $[0,\infty)$. Let p be a nonnegative and integrable function on Ω and let $u\in G(\Omega)$. Then, the following inequality holds*

$$\int_\Omega p(x)\,f(|u(x)|)\,dx \leq \frac{1}{2m}H_m(\beta)\int_\Omega\left(\sum_{i=1}^m f\left(\left|\frac{\partial}{\partial x_i}u(x)\right|\right)\right)dx. \tag{8.28}$$

where $\beta = (\beta_1,\ldots,\beta_m)$ and

$$\beta_i = \int_{a_i}^{b_i}\left(\frac{f(x_i-a_i)\,f(b_i-x_i)}{(x_i-a_i)\,(b_i-x_i)}\right)^{\frac12}p(x)\,dx_i, \quad i = 1,\ldots,m.$$

Proof. By harmonic-geometric inequality we have

$$2\left(\frac{x_i-a_i}{f(x_i-a_i)}+\frac{b_i-x_i}{f(b_i-x_i)}\right)^{-1} \leq \left(\frac{f(x_i-a_i)\,f(b_i-x_i)}{(x_i-a_i)\,(b_i-x_i)}\right)^{\frac12}.$$

Applying this and using $H_m(\frac12\gamma) = \frac12 H_m(\gamma)$, the inequality (8.28) follows. □

Next we have an improvement of Theorem 8.7.

Corollary 8.6 *Let $\lambda,\mu \geq 1$ and let $u\in G(\Omega)$. Then, the following inequality holds*

$$\int_\Omega |u(x)|^\lambda\,dx \leq K_1(\lambda,\mu)\int_\Omega \|\mathrm{grad}\,u(x)\|_\mu^\lambda\,dx, \tag{8.29}$$

where

$$K_1(\lambda,\mu) = \frac1m I(\lambda)\,C\left(\frac{\lambda}{\mu}\right)H_m\left((b-a)^\lambda\right), \tag{8.30}$$

I is defined by (8.10) *and C is defined by* (8.20)*.*

Proof. We follow steps from the proof of Theorem 8.12, using function $f(t) = t^\lambda$, up to the inequality (8.25), which is now equal to

$$\int_\Omega |u(x)|^\lambda \, dx \le \int_{a_i}^{b_i} \left((x_i - a_i)^{1-\lambda} + (b_i - x_i)^{1-\lambda} \right)^{-1} dx_i \int_\Omega \left| \frac{\partial}{\partial x_i} u(x) \right|^\lambda dx \tag{8.31}$$

for $i = 1, \ldots, m$. However, since

$$\begin{aligned} \int_{a_i}^{b_i} \left((x_i - a_i)^{1-\lambda} + (b_i - x_i)^{1-\lambda} \right)^{-1} dx_i \; &= \; (b_i - a_i)^\lambda \int_0^1 \left(t^{1-\lambda} + (1-t)^{1-\lambda} \right)^{-1} dt \\ &= \; (b_i - a_i)^\lambda \, I(\lambda) \, , \end{aligned}$$

the inequality (8.31) can be written as

$$\int_\Omega |u(x)|^\lambda \, dx \le (b_i - a_i)^\lambda \, I(\lambda) \int_\Omega \left| \frac{\partial}{\partial x_i} u(x) \right|^\lambda dx \, . \tag{8.32}$$

Multiplying both sides of the inequality (8.32) by $(b_i - a_i)^{-\lambda}$, $i = 1, \ldots, m$, and then summing these inequalities, we obtain

$$\sum_{i=1}^m (b_i - a_i)^{-\lambda} \int_\Omega |u(x)|^\lambda \, dx \le I(\lambda) \int_\Omega \left(\sum_{i=1}^m \left| \frac{\partial}{\partial x_i} u(x) \right|^\lambda \right) dx \, ,$$

i.e.

$$\int_\Omega |u(x)|^\lambda \, dx \le \frac{1}{m} I(\lambda) \, H_m\left((b-a)^\lambda \right) \int_\Omega \left(\sum_{i=1}^m \left| \frac{\partial}{\partial x_i} u(x) \right|^\lambda \right) dx \, . \tag{8.33}$$

Our result now follows from (8.33) and the elementary inequality

$$\sum_{i=1}^m a_i^\alpha \le C(\alpha) \left(\sum_{i=1}^m a_i \right)^\alpha , \quad a_i \ge 0 \, . \tag{8.34}$$

□

Remark 8.5 By the harmonic-geometric means inequality, we have

$$2 \left(s^{1-\lambda} + (1-s)^{1-\lambda} \right)^{-1} \le \left(s^{\lambda-1} (1-s)^{\lambda-1} \right)^{\frac{1}{2}} ,$$

and hence,

$$I(\lambda) \le \frac{1}{2} \int_0^1 t^{\frac{\lambda-1}{2}} (1-t)^{\frac{\lambda-1}{2}} \, dt = \frac{1}{2} B\left(\frac{\lambda+1}{2}, \frac{\lambda+1}{2} \right) . \tag{8.35}$$

Thus, again from the harmonic-geometric means inequality, and (8.35), it follows that Corollary 8.6 improves Theorem 8.7.

Corollary 8.7 *For $u \in G(\Omega)$, the following Poincaré type inequalities hold*

$$\int_\Omega |u(x)|^2\,dx \le \frac{1}{6m} H_m\left((b-a)^2\right) \int_\Omega \|\operatorname{grad} u(x)\|_2^2\,dx, \tag{8.36}$$

$$\int_\Omega |u(x)|^3\,dx \le \frac{3\pi-8}{24m} H_m\left((b-a)^3\right) \int_\Omega \|\operatorname{grad} u(x)\|_2^3\,dx, \tag{8.37}$$

$$\int_\Omega |u(x)|^4\,dx \le \frac{20\sqrt{3}\pi-81}{1215m} H_m\left((b-a)^4\right) \int_\Omega \|\operatorname{grad} u(x)\|_2^4\,dx. \tag{8.38}$$

Remark 8.6 From Theorem 8.7, the inequalities corresponding to (8.36)-(8.38) will contain geometric means instead of harmonic means, moreover, the corresponding constants are $(\frac{\pi}{16m})$, $(\frac{1}{12m})$ and $(\frac{3\pi}{256m})$.

Corollary 8.8 *Let $\mu_k > 0$, $\lambda_k \ge 1$ be given real numbers such that $\sum_{k=1}^n (\mu_k/\lambda_k) = 1$, and let $u_k \in G(\Omega)$, $k = 1,\ldots,n$. Then, the following inequality holds*

$$\int_\Omega \prod_{k=1}^n |u_k(x)|^{\mu_k}\,dx \le \sum_{k=1}^n \frac{\mu_k}{\lambda_k} K_1(\lambda_k, 2) \int_\Omega \|\operatorname{grad} u_k(x)\|_2^{\lambda_k}\,dx, \tag{8.39}$$

where the constant K_1 is defined by (8.30).

Proof. Using weighted arithmetic-geometric means inequality, we find

$$\prod_{k=1}^n |u_k(x)|^{\mu_k}\,dx = \prod_{k=1}^n \left(|u_k(x)|^{\lambda_k}\right)^{\frac{\mu_k}{\lambda_k}} dx \le \sum_{k=1}^n \frac{\mu_k}{\lambda_k} |u_k(x)|^{\lambda_k}\,dx.$$

The inequality (8.39) now follows from Corollary 8.6 with $\lambda = \lambda_k$ and $\mu = 2$. □

Remark 8.7 Inequality (8.39) improves Corollary 7 proved in [8].

In the following remarks, we will state several important particular cases of the inequality (8.39). The obtained inequalities improve corresponding results established in [8].

Remark 8.8 Let $\mu_1, \mu_2 > 0$ be such that $\mu_1 + \mu_2 \ge 2$ and let $u_1, u_2 \in G(\Omega)$. Then, for $n = 2$ and $\lambda_1 = \lambda_2 = \mu_1 + \mu_2$ the inequality (8.39) reduces to

$$\begin{aligned}\int_\Omega |u_1(x)|^{\mu_1}\,|u_2(x)|^{\mu_2}\,dx \le\ & \frac{1}{m} H_m\left((b-a)^{\mu_1+\mu_2}\right) I(\mu_1+\mu_2)\\ &\times\left(\frac{\mu_1}{\mu_1+\mu_2}\int_\Omega \|\operatorname{grad} u_1(x)\|_2^{\mu_1+\mu_2}\,dx + \frac{\mu_2}{\mu_1+\mu_2}\int_\Omega \|\operatorname{grad} u_2(x)\|_2^{\mu_1+\mu_2}\,dx\right).\end{aligned} \tag{8.40}$$

In particular, for $\mu_1 = \mu_2 = 1$, the inequality (8.40) is the same as

$$\begin{aligned}\int_\Omega |u_1(x)|\,|u_2(x)|\,dx \le\ & \frac{H_m\left((b-a)^2\right)}{12m}\\ &\times\left(\int_\Omega \|\operatorname{grad} u_1(x)\|_2^2\,dx + \int_\Omega \|\operatorname{grad} u_2(x)\|_2^2\,dx\right).\end{aligned}$$

Remark 8.9 Let $1 \le \lambda_1, \lambda_2 < 2$ be such that $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 1$, and let $u_1, u_2 \in G(\Omega)$. Then, for $n = 2$ and $\mu_1 = \mu_2 = 1$ the inequality (8.39) leads to

$$\begin{aligned}\int_\Omega |u_1(x)|\,|u_2(x)|\,dx \le\ & \frac{1}{\lambda_1 m^{\frac{\lambda_1}{2}}} H_m\left((b-a)^{\lambda_1}\right) I(\lambda_1) \int_\Omega \|\mathrm{grad}\, u_1(x)\|_2^{\lambda_1}\,dx \\ & + \frac{1}{\lambda_2 m^{\frac{\lambda_2}{2}}} H_m\left((b-a)^{\lambda_2}\right) I(\lambda_2) \int_\Omega \|\mathrm{grad}\, u_2(x)\|_2^{\lambda_2}\,dx.\end{aligned}$$

Remark 8.10 Let $\mu_1, \mu_2 > 0$ be such that $1 \le \mu_1 + \mu_2 < 2$ and let $u_1, u_2 \in G(\Omega)$. Then, for $n = 2$ and $\lambda_1 = \lambda_2 = \mu_1 + \mu_2$ the inequality (8.39) gives

$$\begin{aligned}\int_\Omega |u_1(x)|^{\mu_1}\,|u_2(x)|^{\mu_2}\,dx \le\ & \frac{1}{m^{\frac{\mu_1+\mu_2}{2}}} H_m\left((b-a)^{\mu_1+\mu_2}\right) I(\mu_1+\mu_2) \\ \times \Bigg(& \frac{\mu_1}{\mu_1+\mu_2} \int_\Omega \|\mathrm{grad}\, u_1(x)\|_2^{\mu_1+\mu_2}\,dx + \frac{\mu_2}{\mu_1+\mu_2} \int_\Omega \|\mathrm{grad}\, u_2(x)\|_2^{\mu_1+\mu_2}\,dx\Bigg).\end{aligned}$$

Remark 8.11 Let $\mu_k > 0$, $\lambda_k \ge 1$, $k = 1,2,3$, be such that $\frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_2} = 1$, $\frac{\mu_2}{\lambda_2} + \frac{\mu_3}{\lambda_3} = 1$, $\frac{\mu_3}{\lambda_3} + \frac{\mu_1}{\lambda_1} = 1$, and let $u_k \in G(\Omega)$, $k = 1,2,3$. Then, the following inequality holds

$$\begin{aligned}\int_\Omega \Big(|u_1(x)|^{\mu_1}\,|u_2(x)|^{\mu_2} + |u_2(x)|^{\mu_2}\,|u_3(x)|^{\mu_3} + |u_3(x)|^{\mu_3}\,|u_1(x)|^{\mu_1}\Big)dx \\ \le \sum_{k=1}^{3} \frac{2\mu_k}{\lambda_k} K_1(\lambda_k, 2) \int_\Omega \|\mathrm{grad}\, u_k(x)\|_2^{\lambda_k}\,dx, \qquad (8.41)\end{aligned}$$

where K_1 is defined by (8.30).

Indeed, for $n = 2$, Corollary 8.8 gives

$$\begin{aligned}\int_\Omega |u_1(x)|^{\mu_1}\,|u_2(x)|^{\mu_2}\,dx \le\ & \frac{\mu_1}{\lambda_1} K_1(\lambda_1, 2) \int_\Omega \|\mathrm{grad}\, u_1(x)\|_2^{\lambda_1}\,dx \\ & + \frac{\mu_2}{\lambda_2} K_1(\lambda_2, 2) \int_\Omega \|\mathrm{grad}\, u_2(x)\|_2^{\lambda_2}\,dx.\end{aligned}$$

Similar to this inequality, we have two more inequalities involving μ_2, μ_3, λ_2, λ_3, u_2, u_3 and μ_3, μ_1, λ_3, λ_1, u_3, u_1. An addition of these three inequalities gives (8.41).

In particular, for $\lambda_k = 2\mu_k$, $k = 1,2,3$, where $\mu_k \ge 1$, the inequality (8.41) reduces to

$$\begin{aligned}\int_\Omega \Big(|u_1(x)|^{\mu_1}\,|u_2(x)|^{\mu_2} + |u_2(x)|^{\mu_2}\,|u_3(x)|^{\mu_3} + |u_3(x)|^{\mu_3}\,|u_1(x)|^{\mu_1}\Big)dx \\ \le \frac{1}{m} \sum_{k=1}^{3} I(2\mu_k) H_m\left((b-a)^{2\mu_k}\right) \int_\Omega \|\mathrm{grad}\, u_k(x)\|_2^{2\mu_k}\,dx. \qquad (8.42)\end{aligned}$$

Further, when $\frac{1}{2} \le \mu_k < 1$, $k = 1,2,3$, the inequality (8.41) gives

$$\begin{aligned}\int_\Omega \Big(|u_1(x)|^{\mu_1}\,|u_2(x)|^{\mu_2} + |u_2(x)|^{\mu_2}\,|u_3(x)|^{\mu_3} + |u_3(x)|^{\mu_3}\,|u_1(x)|^{\mu_1}\Big)dx \\ \le \sum_{k=1}^{3} \frac{1}{m^{\mu_k}} I(2\mu_k) H_m\left((b-a)^{2\mu_k}\right) \int_\Omega \|\mathrm{grad}\, u_k(x)\|_2^{2\mu_k}\,dx. \qquad (8.43)\end{aligned}$$

Remark 8.12 Let $\mu_k > 0$, $\lambda_k \geq 1$, $k = 1,2,3$, be such that $\frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_2} = 1$, $\frac{\mu_2}{\lambda_2} + \frac{\mu_3}{\lambda_3} = 1$, $\frac{\mu_3}{\lambda_3} + \frac{\mu_1}{\lambda_1} = 1$, and let $u_k \in G(\Omega)$, $k = 1,2,3$. Then, the following inequality holds

$$\int_\Omega |u_1(x)|^{\mu_1} |u_2(x)|^{\mu_2} |u_3(x)|^{\mu_3} \left(|u_1(x)|^{\mu_1} + |u_2(x)|^{\mu_2} + |u_3(x)|^{\mu_3}\right) dx$$
$$\leq \frac{2}{m} \sum_{k=1}^{3} \frac{\mu_k}{\lambda_k} I(2\lambda_k) H_m\left((b-a)^{2\lambda_k}\right) \int_\Omega \|\operatorname{grad} u_k(x)\|_2^{2\lambda_k}\, dx. \tag{8.44}$$

Indeed, the inequality (8.44) follows from the elementary inequality $\alpha_1\alpha_2\alpha_3(\alpha_1 + \alpha_2 + \alpha_3) \leq \alpha_1^2\alpha_2^2 + \alpha_2^2\alpha_3^2 + \alpha_3^2\alpha_1^2$, and the inequality (8.41) with μ_k and λ_k replaced by $2\mu_k$ and $2\lambda_k$.

In particular, for $\lambda_k = 2\mu_k$, $k = 1,2,3$, where $\mu_k \geq 2^{-1}$, the inequality (8.44) reduces to

$$\int_\Omega |u_1(x)|^{\mu_1} |u_2(x)|^{\mu_2} |u_3(x)|^{\mu_3} \left(|u_1(x)|^{\mu_1} + |u_2(x)|^{\mu_2} + |u_3(x)|^{\mu_3}\right) dx$$
$$\leq \frac{1}{m} \sum_{k=1}^{3} I(4\mu_k) H_m\left((b-a)^{4\mu_k}\right) \int_\Omega \|\operatorname{grad} u_k(x)\|_2^{4\mu_k}\, dx.$$

The following theorem is a consequence of Corollary 8.6.

Theorem 8.13 *Let $\mu_k > 0$, $\lambda_k \geq 1$, $k = 1,\ldots,n$, be real numbers such that $\sum_{k=1}^n \frac{\mu_k}{\lambda_k} = 1$ and let $u, u_k \in G(\Omega)$, $k = 1,\ldots,n$. Then, the following inequality holds*

$$\int_\Omega \prod_{k=1}^{n} |u_k(x)|^{\mu_k} dx \leq \frac{M^{[-2]}(b-a)}{\sqrt{6m}} \sum_{k=1}^{n} \frac{\mu_k}{\lambda_k} \left(\int_\Omega |u(x)|^{2(\lambda_k-1)} dx\right)^{\frac{1}{2}}$$
$$\times \left(\int_\Omega \|\operatorname{grad} u(x)\|_2^2\, dx\right)^{\frac{1}{2}}. \tag{8.45}$$

Proof. By applying Cauchy-Schwarz inequality and the result of Corollary 8.6 for $\lambda = 2$ and $\mu = 2$, we find

$$\int_\Omega |u(x)|^{\lambda_k} dx \leq \left(\int_\Omega |u(x)|^{2(\lambda_k-1)} dx\right)^{\frac{1}{2}} \left(\int_\Omega |u(x)|^2 dx\right)^{\frac{1}{2}}$$
$$\leq \frac{M^{[-2]}(b-a)}{\sqrt{6m}} \left(\int_\Omega |u(x)|^{2(\lambda_k-1)} dx\right)^{\frac{1}{2}} \left(\int_\Omega \|\operatorname{grad} u(x)\|_2^2\, dx\right)^{\frac{1}{2}}. \tag{8.46}$$

The inequality (8.45) now follows from (8.46) by using weighted arithmetic-geometric inequality. □

Remark 8.13 In Remark 8.17 a special case of Theorem 8.13 (for $n = 2$ and $\mu_1 = \mu_2 = 1$) is proved. In [8] Agarwal and Sheng obtained the same type of inequality for $n = 2$ and with the right-hand side of the inequality (8.45) multiplied by $\sqrt{\frac{3\pi}{8}}$ and the term $G_m(b-a)$ instead of $M^{[-2]}(b-a)$. On the other hand, in [40], Cheung obtained the same type of inequality with the right-hand side of the inequality (16) multiplied by $\frac{\sqrt{6}}{2}$, the term $max\{b_i - a_i : i = 1,\ldots,n\}$ instead of $M^{[-2]}(b-a)$ and the term μ_k instead of μ_k/λ_k.

8.2.2 Improvements of Wirtinger-type inequality

In our next result we will improve Theorem 8.8.

Theorem 8.14 *Let $p,\lambda,\mu \geq 1$ with $\frac{1}{\mu}+\frac{1}{\lambda}=1$ and let $u \in G(\Omega)$. Then, the following inequality holds*

$$\int_\Omega |u(x)|^p\,dx \;\leq\; \frac{H_m(b-a)}{2m^{\frac{1}{\lambda}}}\left(\int_\Omega |u(x)|^{\mu(p-1)}\,dx\right)^{\frac{1}{\mu}}\left(\int_\Omega \|\operatorname{grad} u(x)\|_\lambda^\lambda\,dx\right)^{\frac{1}{\lambda}}. \qquad (8.47)$$

Proof. From the hypotheses, we have

$$u^p(x) = u^{p-1}(x)\int_{a_i}^{x_i}\frac{\partial}{\partial s_i}u(x;s_i)\,ds_i$$

and

$$u^p(x) = -u^{p-1}(x)\int_{x_i}^{b_i}\frac{\partial}{\partial s_i}u(x;s_i)\,ds_i$$

for $i=1,\ldots,m$. Using Hölder's inequality in the above inequalities with indices μ and λ, and summing, we obtain

$$\begin{aligned}|u(x)|^p \;&\leq\; \frac{1}{2}|u(x)|^{p-1}\int_{a_i}^{b_i}\left|\frac{\partial}{\partial s_i}u(x;s_i)\right|ds_i\\ &\leq\; \frac{1}{2}|u(x)|^{p-1}(b_i-a_i)^{\frac{1}{\mu}}\left(\int_{a_i}^{b_i}\left|\frac{\partial}{\partial s_i}u(x;s_i)\right|^\lambda ds_i\right)^{\frac{1}{\lambda}}.\end{aligned}$$

Now, multiplying both sides of the above inequality by $A_i=(b_i-a_i)^{-1}$, $i=1,\ldots,m$, summing these inequalities, and then integrating both sides on Ω, we get

$$\begin{aligned}\int_\Omega |u(x)|^p\,dx \leq\; &\frac{1}{2\sum_{i=1}^m A_i}\sum_{i=1}^m A_i(b_i-a_i)^{\frac{1}{\mu}}\\ &\times\int_\Omega |u(x)|^{p-1}\left(\int_{a_i}^{b_i}\left|\frac{\partial}{\partial s_i}u(x;s_i)\right|^\lambda ds_i\right)^{\frac{1}{\lambda}}dx.\end{aligned}$$

Applying Hölder's inequality with indices λ and μ, we find

$$\begin{aligned}\int_\Omega |u(x)|^p\,dx \leq\; &\frac{1}{2\sum_{i=1}^m A_i}\left(\int_\Omega |u(x)|^{\mu(p-1)}\,dx\right)^{\frac{1}{\mu}}\sum_{i=1}^m A_i(b_i-a_i)^{\frac{1}{\mu}}\\ &\times\left(\int_\Omega\int_{a_i}^{b_i}\left|\frac{\partial}{\partial s_i}u(x;s_i)\right|^\lambda ds_i\,dx\right)^{\frac{1}{\lambda}}\\ \leq\; &\frac{1}{2\sum_{i=1}^m A_i}\left(\int_\Omega |u(x)|^{\mu(p-1)}\,dx\right)^{\frac{1}{\mu}}\sum_{i=1}^m A_i(b_i-a_i)^{\frac{1}{\mu}+\frac{1}{\lambda}}\end{aligned}$$

$$\times \left(\int_\Omega \left| \frac{\partial}{\partial x_i} u(x) \right|^\lambda dx \right)^{\frac{1}{\lambda}}.$$

Obviously, $A_i(b_i - a_i)^{\frac{1}{\mu}+\frac{1}{\lambda}} = 1$, $i = 1, \ldots, m$. Thus, it follows that

$$\int_\Omega |u(x)|^p\, dx \le \frac{1}{2\sum_{i=1}^m A_i} \left(\int_\Omega |u(x)|^{\mu(p-1)}\, dx \right)^{\frac{1}{\mu}} \sum_{i=1}^m \left(\int_\Omega \left| \frac{\partial}{\partial x_i} u(x) \right|^\lambda dx \right)^{\frac{1}{\lambda}}. \quad (8.48)$$

Again from Hölder's inequality with indices λ and μ, we have

$$\sum_{i=1}^m \left(\int_\Omega \left| \frac{\partial}{\partial x_i} u(x) \right|^\lambda dx \right)^{\frac{1}{\lambda}} \le m^{\frac{1}{\mu}} \left(\int_\Omega \|\mathrm{grad}\, u(x)\|_\lambda^\lambda \right)^{\frac{1}{\lambda}}. \quad (8.49)$$

On combining (8.48) and (8.49) the required inequality (8.47) follows. □

Remark 8.14 By the harmonic-geometric means inequality it is clear that (8.47) is an improvement over the inequality in Theorem 8.8.

Theorem 8.15 *Let $p, \lambda, \mu \ge 1$ with $\frac{1}{\mu} + \frac{1}{\lambda} = 1$ and let $u \in G(\Omega)$. Let $r \ge 1$. Then, the following inequality holds*

$$\int_\Omega |u(x)|^p\, dx \le (K_1(\lambda, r))^{\frac{1}{\lambda}} \left(\int_\Omega |u(x)|^{\mu(p-1)}\, dx \right)^{\frac{1}{\mu}} \left(\int_\Omega \|\mathrm{grad}\, u(x)\|_r^\lambda\, dx \right)^{\frac{1}{\lambda}}. \quad (8.50)$$

Proof. Applying Hölder's inequality with indices λ and μ, we find

$$\begin{aligned}\int_\Omega |u(x)|^p\, dx &= \int_\Omega |u(x)|^{p-1} |u(x)|\, dx \\ &\le \left(\int_\Omega |u(x)|^{\mu(p-1)}\, dx \right)^{\frac{1}{\mu}} \left(\int_\Omega |u(x)|^\lambda\, dx \right)^{\frac{1}{\lambda}}.\end{aligned}$$

The inequality (8.50) now follows from Corollary 8.6. □

Remark 8.15 For $\lambda \ge r$, the inequality (8.50) can be written as

$$\begin{aligned}\int_\Omega |u(x)|^p\, dx \le{}& \frac{M^{[-\lambda]}(b-a)}{m^{\frac{1}{\lambda}}} (I(\lambda))^{\frac{1}{\lambda}} \\ &\times \left(\int_\Omega |u(x)|^{\mu(p-1)}\, dx \right)^{\frac{1}{\mu}} \left(\int_\Omega \|\mathrm{grad}\, u(x)\|_r^\lambda\, dx \right)^{\frac{1}{\lambda}}.\end{aligned}$$

Thus, in view of $I(\lambda) \le 2^{-\lambda}$ and the harmonic-geometric means inequality, it is clear that for $r = \lambda$, the inequality (8.50) improves (8.47).

By letting $\lambda = \mu = r = 2$ in Theorem 8.15, we obtain the following interesting Sobolev type inequality.

Corollary 8.9 *Let $p \geq 1$ and let $u \in G(\Omega)$. Then, the following inequality holds*

$$\int_\Omega |u(x)|^p \, dx \;\leq\; \frac{M^{[-2]}(b-a)}{\sqrt{6m}} \left(\int_\Omega |u(x)|^{2(p-1)} \, dx\right)^{\frac{1}{2}} \left(\int_\Omega \|\mathrm{grad}\, u(x)\|_2^2 \, dx\right)^{\frac{1}{2}}.$$

8.2.3 Improvements of Sobolev-type inequality

The following result generalizes as well as improves Theorem 8.9.

Theorem 8.16 *Let $p, \lambda \geq 1$ and let $u \in G(\Omega)$. Then, the following inequality holds*

$$\left(\int_\Omega |u(x)|^{\lambda p} \, dx\right)^{\frac{1}{p}} \;\leq\; \frac{p^\lambda H_m\left((b-a)^\lambda\right)}{m} I(\lambda) \left(\int_\Omega \|\mathrm{grad}\, u(x)\|_\lambda^{\lambda p} \, dx\right)^{\frac{1}{p}}. \tag{8.51}$$

Proof. From the hypotheses, we have

$$u^p(x) = p \int_{a_i}^{x_i} u^{p-1}(x; s_i) \frac{\partial}{\partial s_i} u(x; s_i) \, ds_i$$

for $i = 1, \ldots, m$, which gives

$$|u(x)|^{p\lambda} \leq p^\lambda \left(\int_{a_i}^{x_i} \left|u^{p-1}(x; s_i)\right| \left|\frac{\partial}{\partial s_i} u(x; s_i)\right| ds_i\right)^\lambda.$$

Applying Hölder's inequality with indices λ and $\frac{\lambda}{\lambda-1}$, it follows that

$$|u(x)|^{p\lambda} \leq p^\lambda (x_i - a_i)^{\lambda-1} \int_{a_i}^{x_i} |u(x; s_i)|^{\lambda(p-1)} \left|\frac{\partial}{\partial s_i} u(x; s_i)\right|^\lambda ds_i. \tag{8.52}$$

Similarly, we obtain

$$|u(x)|^{p\lambda} \leq p^\lambda (b_i - x_i)^{\lambda-1} \int_{x_i}^{b_i} |u(x; s_i)|^{\lambda(p-1)} \left|\frac{\partial}{\partial s_i} u(x; s_i)\right|^\lambda ds_i. \tag{8.53}$$

Multiplying (8.52) by $(x_i - a_i)^{1-\lambda}$ and (8.53) by $(b_i - x_i)^{1-\lambda}$, then adding the resulting inequalities, we get

$$\left((x_i - a_i)^{1-\lambda} + (b_i - x_i)^{1-\lambda}\right) |u(x)|^{p\lambda} \leq p^\lambda \int_{a_i}^{b_i} |u(x; s_i)|^{\lambda(p-1)} \left|\frac{\partial}{\partial s_i} u(x; s_i)\right|^\lambda ds_i,$$

and hence

$$|u(x)|^{p\lambda} \leq p^\lambda \left((x_i - a_i)^{1-\lambda} + (b_i - x_i)^{1-\lambda}\right)^{-1} \int_{a_i}^{b_i} |u(x; s_i)|^{\lambda(p-1)} \left|\frac{\partial}{\partial s_i} u(x; s_i)\right|^\lambda ds_i. \tag{8.54}$$

Integrating (8.54) on Ω, we get

$$\int_\Omega |u(x)|^{p\lambda}\,dx \le p^\lambda \int_{a_i}^{b_i} \left((x_i-a_i)^{1-\lambda}+(b_i-x_i)^{1-\lambda}\right)^{-1} dx_i \times \int_\Omega |u(x)|^{\lambda(p-1)} \left|\frac{\partial}{\partial x_i}u(x)\right|^\lambda dx.$$

It is clear that this inequality is the same as

$$\int_\Omega |u(x)|^{p\lambda}\,dx \le p^\lambda (b_i-a_i)^\lambda\, I(\lambda) \int_\Omega |u(x)|^{\lambda(p-1)} \left|\frac{\partial}{\partial x_i}u(x)\right|^\lambda dx,$$

where $I(\lambda)$ is the same as in Corollary 8.6.

Multiplying each of the above inequalities by $(b_i-a_i)^{-\lambda}$, $i=1,\ldots,m$, and summing these inequalities, we find

$$\sum_{i=1}^m (b_i-a_i)^{-\lambda} \int_\Omega |u(x)|^{p\lambda}\,dx \le p^\lambda I(\lambda) \int_\Omega |u(x)|^{\lambda(p-1)} \left(\sum_{i=1}^m \left|\frac{\partial}{\partial x_i}u(x)\right|^\lambda\right) dx.$$

Thus, we have

$$\int_\Omega |u(x)|^{p\lambda}\,dx \le \frac{p^\lambda H_m\left((b-a)^\lambda\right)}{m} I(\lambda) \int_\Omega |u(x)|^{\lambda(p-1)} \left(\sum_{i=1}^m \left|\frac{\partial}{\partial x_i}u(x)\right|^\lambda\right) dx. \quad (8.55)$$

Finally, the inequality (8.51) follows from (8.55) on applying Hölder's inequality with indices p and $\frac{p}{p-1}$. $\square$

Remark 8.16 From (8.55), for $\lambda \ge 2$, instead of the inequality (8.51), we can establish the following weaker inequality

$$\left(\int_\Omega |u(x)|^{\lambda p}\,dx\right)^{\frac{1}{p}} \le \frac{p^\lambda H_m\left((b-a)^\lambda\right)}{m} I(\lambda) \left(\int_\Omega \|\operatorname{grad} u(x)\|_2^{\lambda p}\,dx\right)^{\frac{1}{p}}.$$

Corollary 8.10 *For $p \ge 1$ and let $u \in G(\Omega)$. Then, the following inequalities hold*

$$\left(\int_\Omega |u(x)|^{2p}\,dx\right)^{\frac{1}{p}} \le \frac{p^2 H_m\left((b-a)^2\right)}{6m} \left(\int_\Omega \|\operatorname{grad} u(x)\|_2^{2p}\,dx\right)^{\frac{1}{p}}, \quad (8.56)$$

$$\left(\int_\Omega |u(x)|^{3p}\,dx\right)^{\frac{1}{p}} \le \frac{p^3 (3\pi-8) H_m\left((b-a)^3\right)}{24m} \left(\int_\Omega \|\operatorname{grad} u(x)\|_2^{3p}\,dx\right)^{\frac{1}{p}}, \quad (8.57)$$

$$\left(\int_\Omega |u(x)|^{4p}\,dx\right)^{\frac{1}{p}} \le \frac{p^4 \left(20\sqrt{3}\pi-81\right) H_m\left((b-a)^4\right)}{1215m} \left(\int_\Omega \|\operatorname{grad} u(x)\|_2^{4p}\,dx\right)^{\frac{1}{p}}. \quad (8.58)$$

Corollary 8.11 *Let $p,\lambda \geq 1$ and let $u \in G(\Omega)$. Then, the following inequality holds*

$$\int_\Omega |u(x)|^{\lambda p}\,dx \leq \frac{I(\lambda p)\,H_m\left((b-a)^{\lambda p}\right)}{m} \int_\Omega \|\mathrm{grad}\,u(x)\|_\lambda^{\lambda p}\,dx. \tag{8.59}$$

Proof. From Corollary 8.6 we have

$$\int_\Omega |u(x)|^{\lambda p}\,dx \;\leq\; K_1(\lambda p,\lambda) \int_\Omega \|\mathrm{grad}\,u(x)\|_\lambda^{\lambda p}\,dx,$$

which gives the inequality (8.59). □

In the following remarks, we will obtain Sobolev type inequalities involving two and three functions. These inequalities improve several known results in the literature.

Remark 8.17 Let $\lambda_1,\lambda_2 > 1$ be such that $\frac{1}{\lambda_1}+\frac{1}{\lambda_2}=1$ and let $u_l,u_2 \in G(\Omega)$. Then, from the arithmetic-geometric means inequality and Corollary 8.9 with $p=\lambda_k$, $u=u_k$, $k=1,2$, the following inequality holds

$$\begin{aligned}\int_\Omega |u_1(x)|\,|u_2(x)|\,dx \;&\leq\; \frac{M^{[-2]}(b-a)}{\sqrt{6m}} \sum_{k=1}^{2} \frac{1}{\lambda_k}\left(\int_\Omega |u_k(x)|^{2(\lambda_k-1)}\,dx\right)^{\frac{1}{2}}\\ &\quad\times\left(\int_\Omega \|\mathrm{grad}\,u_k(x)\|_2^2\,dx\right)^{\frac{1}{2}}.\end{aligned}$$

Remark 8.18 Let $\mu_1,\mu_2 \geq 1$ and let $u_l,u_2 \in G(\Omega)$. Then, from the arithmetic-geometric means inequality and Corollary 8.9 with $p=\mu_1+\mu_2$, $u=u_k$, $k=1,2$, the following inequality holds

$$\begin{aligned}\int_\Omega |u_1(x)|^{\mu_1}\,|u_2(x)|^{\mu_2}\,dx \;&\leq\; \frac{M^{[-2]}(b-a)}{\sqrt{6m}(\mu_1+\mu_2)} \sum_{k=1}^{2} \mu_k\left(\int_\Omega |u_k(x)|^{2(\mu_1+\mu_2-1)}\,dx\right)^{\frac{1}{2}}\\ &\quad\times\left(\int_\Omega \|\mathrm{grad}\,u_k(x)\|_2^2\,dx\right)^{\frac{1}{2}}.\end{aligned}$$

Remark 8.19 Let $\mu_k \geq 1$, $u_k \in G(\Omega)$, $k=1,2,3$. Then, from the elementary inequality $\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1 \leq \alpha_1^2+\alpha_2^2+\alpha_3^2$ and (8.56) with $p=\mu_k$, $u=u_k$, $k=1,2,3$, the following inequality holds

$$\begin{aligned}&\int_\Omega \left(|u_1(x)|^{\mu_1}\,|u_2(x)|^{\mu_2}+|u_2(x)|^{\mu_2}\,|u_3(x)|^{\mu_3}+|u_3(x)|^{\mu_3}\,|u_1(x)|^{\mu_1}\right)dx\\ &\leq \frac{1}{6m}H_m\left((b-a)^2\right)\sum_{k=1}^{3}\mu_k^2\left(\int_\Omega |u_k(x)|^{2\mu_k}\,dx\right)^{\frac{\mu_k-1}{\mu_k}}\left(\int_\Omega \|\mathrm{grad}\,u_k(x)\|_2^{2\mu_k}\,dx\right)^{\frac{1}{\mu_k}}.\end{aligned}$$

Remark 8.20 Let $\mu_k \geq 1$, $u_k \in G(\Omega)$, $k=1,2,3$. Then, from the elementary inequalities used in Remark 8.12 and 8.19, (8.58) with $p=\mu_k$, $u=u_k$, $k=1,2,3$, the following inequality holds

$$\int_\Omega |u_1(x)|^{\mu_1}\,|u_2(x)|^{\mu_2}\,|u_3(x)|^{\mu_3}\left(|u_1(x)|^{\mu_1}+|u_2(x)|^{\mu_2}+|u_3(x)|^{\mu_3}\right)dx$$

$$\leq \frac{20\sqrt{3}\pi - 81}{1215m} H_m\left((b-a)^4\right) \sum_{k=1}^{3} \mu_k^4 \left(\int_\Omega |u_k(x)|^{4\mu_k}\, dx\right)^{\frac{\mu_k - 1}{\mu_k}}$$
$$\times \left(\int_\Omega \|\operatorname{grad} u_k(x)\|_2^{4\mu_k}\, dx\right)^{\frac{1}{\mu_k}}.$$

8.2.4 Improvements of inequalities due to Pachpatte

Our next result improves Theorem 8.10.

Theorem 8.17 *Let $l \geq 0, n \geq 1$ and let $u \in G(\Omega)$. Then, the following inequality holds*

$$\int_\Omega |u(x)|^{l+n}\, dx \leq \frac{1}{m}\left(\frac{l+n}{n}\right)^n I(n) \sum_{i=1}^{m} (b_i - a_i)^n \int_\Omega |u(x)|^l \left|\frac{\partial}{\partial x_i} u(x)\right|^n dx. \tag{8.60}$$

Proof. For each fixed i, $i = 1, \ldots, m$, we have

$$(u(x))^{l+n} = \frac{l+n}{n}\,(u(x))^{\frac{(n-1)(l+n)}{n}} \int_{a_i}^{x_i} (u(x;s_i))^{\frac{l}{n}} \frac{\partial}{\partial s_i} u(x;s_i)\, ds_i\,.$$

Thus, on applying Hölder's inequality with indices n and $\frac{n}{n-1}$, it follows that

$$|u(x)|^{l+n} \leq \frac{l+n}{n}\,|u(x)|^{\frac{(n-1)(l+n)}{n}}$$
$$\times (x_i - a_i)^{\frac{n-1}{n}} \left(\int_{a_i}^{x_i} |u(x;s_i)|^l \left|\frac{\partial}{\partial s_i} u(x;s_i)\right|^n ds_i\right)^{\frac{1}{n}}.$$

Now, since

$$|u(x)|^{(l+n)\left(1-\frac{n-1}{n}\right)} = |u(x)|^{\frac{l+n}{n}}$$

we get

$$|u(x)|^{\frac{l+n}{n}} \leq \frac{l+n}{n}(x_i - a_i)^{\frac{n-1}{n}} \left(\int_{a_i}^{x_i} |u(x;s_i)|^l \left|\frac{\partial}{\partial s_i} u(x;s_i)\right|^n ds_i\right)^{\frac{1}{n}},$$

i.e.,

$$|u(x)|^{l+n} \leq \left(\frac{l+n}{n}\right)^n (x_i - a_i)^{n-1} \int_{a_i}^{x_i} |u(x;s_i)|^l \left|\frac{\partial}{\partial s_i} u(x;s_i)\right|^n ds_i. \tag{8.61}$$

Similarly, we have

$$|u(x)|^{l+n} \leq \left(\frac{l+n}{n}\right)^n (b_i - x_i)^{n-1} \int_{x_i}^{b_i} |u(x;s_i)|^l \left|\frac{\partial}{\partial s_i} u(x;s_i)\right|^n ds_i. \tag{8.62}$$

Multiplying (8.61) by $(x_i - a_i)^{1-n}$ and (8.62) by $(b_i - x_i)^{1-n}$, then adding the resulting inequalities, we get

$$\left((x_i - a_i)^{1-n} + (b_i - x_i)^{1-n}\right) |u(x)|^{l+n} \leq \left(\frac{l+n}{n}\right)^n$$

$$\times \int_{a_i}^{b_i} |u(x;s_i)|^l \left| \frac{\partial}{\partial s_i} u(x;s_i) \right|^n ds_i \,,$$

which is the same as

$$|u(x)|^{l+n} \le \left((x_i-a_i)^{1-n} + (b_i-x_i)^{1-n}\right)^{-1} \left(\frac{l+n}{n}\right)^n \times \int_{a_i}^{b_i} |u(x;s_i)|^l \left| \frac{\partial}{\partial s_i} u(x;s_i) \right|^n ds_i \,. \tag{8.63}$$

Finally, integrating (8.63) on Ω, we get

$$\int_\Omega |u(x)|^{l+n}\,dx \le \left(\frac{l+n}{n}\right)^n \int_{a_i}^{b_i} \left((x_i-a_i)^{1-n} + (b_i-x_i)^{1-n}\right)^{-1} dx_i \times \int_\Omega |u(x)|^l \left| \frac{\partial}{\partial x_i} u(x) \right|^n dx \,,$$

i.e.,

$$\int_\Omega |u(x)|^{l+n}\,dx \le \left(\frac{l+n}{n}\right)^n I(n)\,(b_i-a_i)^n \int_\Omega |u(x)|^l \left| \frac{\partial}{\partial x_i} u(x) \right|^n dx \,. \tag{8.64}$$

The required inequality (8.60) follows on adding these m inequalities. □

Remark 8.21 Since $I(n) \le 2^{-n}$ the inequality (8.60) improves Theorem 8.10.

Theorem 8.18 *Let $l \ge 0, n \ge 1$ and let $u \in G(\Omega)$. Then, the following inequality holds*

$$\int_\Omega |u(x)|^{l+n}\,dx \le \frac{H_m\left((b-a)^n\right)}{m} \left(\frac{l+n}{n}\right)^n I(n) \sum_{i=1}^m \int_\Omega |u(x)|^l \left| \frac{\partial}{\partial x_i} u(x) \right|^n dx. \tag{8.65}$$

Proof. The main part of the proof is the same as that of Theorem 8.17. We multiply (8.64) by $(b_i-a_i)^{-n}$, $i = 1,\ldots,m$ and add these m inequalities, to obtain

$$\sum_{i=1}^m (b_i-a_i)^{-n} \int_\Omega |u(x)|^{l+n}\,dx \le \left(\frac{l+n}{n}\right)^n I(n) \sum_{i=1}^m \int_\Omega |u(x)|^l \left| \frac{\partial}{\partial x_i} u(x) \right|^n dx \,.$$

This inequality is the same as (8.65). □

Now we will prove another interesting inequality which improves Theorem 8.11.

Theorem 8.19 *Let $p, n \ge 0$ be such that $p+n \ge 1$ and let $u \in G(\Omega)$. Then, the following inequality holds*

$$\int_\Omega |u(x)|^p \,\|\mathrm{grad}\, u(x)\|_2^n\,dx \le \left(K_1(p+n,2)\right)^{\frac{p}{p+n}} \int_\Omega \|\mathrm{grad}\, u(x)\|_2^{p+n}\,dx. \tag{8.66}$$

Proof. Inequality (8.66) is an easy consequence of the result of Corollary 8.6. Indeed, first we apply Hölder's inequality with indices $\frac{p+n}{p}$ and $\frac{p+n}{n}$ in the left side of (8.66) and then apply (8.29) with $\lambda = p+n$ and $\mu = 2$. □

Finally, in the following remarks we will deduce some inequalities from (8.66). These inequalities improve several results known in the literature.

Remark 8.22 For $p = 1$ and $n = l-1$, $l \geq 2$, the inequality (8.66) reduces to

$$\int_\Omega |u(x)|\,\|\operatorname{grad} u(x)\|_2^{l-1}\,dx \;\leq\; \frac{M^{[-l]}(b-a)}{m^{\frac{1}{l}}} I(l)^{\frac{1}{l}} \int_\Omega \|\operatorname{grad} u(x)\|_2^l\,dx.$$

Remark 8.23 For $p = n = 1$, the inequality (8.66) reduces to

$$\int_\Omega |u(x)|\,\|\operatorname{grad} u(x)\|_2\,dx \;\leq\; \frac{M^{[-2]}(b-a)}{\sqrt{6m}} \int_\Omega \|\operatorname{grad} u(x)\|_2^2\,dx.$$

Remark 8.24 Let $u_1, u_2 \in G(\Omega)$. Then, the following inequality holds

$$\int_\Omega \Big(|u_1(x)|\,\|\operatorname{grad} u_2(x)\|_2 + |u_2(x)|\,\|\operatorname{grad} u_1(x)\|_2\Big)\,dx$$
$$\leq \frac{M^{[-2]}(b-a)}{\sqrt{6m}} \int_\Omega \left(\|\operatorname{grad} u_1(x)\|_2^2 + \|\operatorname{grad} u_2(x)\|_2^2\right)dx.$$

Indeed, it follows on applying Cauchy-Schwarz inequality for each term of the left side, and then on applying the inequality (8.36), and finally using the arithmetic-geometric means inequality.

Remark 8.25 Let $p_k, n_k \geq 0$ be such that $r(p_k+n_k) \geq 1$ and let $u_k \in G(\Omega)$, $k = 1,\dots,r$. Then, the following inequality holds

$$\int_\Omega \prod_{k=1}^r |u_k(x)|^{p_k}\,\|\operatorname{grad} u_k(x)\|_2^{n_k}\,dx$$
$$\leq \frac{1}{r}\sum_{k=1}^r \left(K_1(r(p_k+n_k),2)\right)^{\frac{p_k}{p_k+n_k}} \int_\Omega \|\operatorname{grad} u_k(x)\|_2^{r(p_k+n_k)}\,dx.$$

8.3 Multidimensional discrete inequalities

Let $x, X \in \mathbb{N}_0^m$ be such that $x \leq X$, i.e., $x_i \leq X_i$, $i = 1,\dots,m$. Let $\Omega = [0,X]$, where $[0,X] \subset \mathbb{N}_0^m$. We denote by $G(\Omega)$ the class of functions $u : \Omega \to \mathbb{R}$, which satisfy conditions $u(x)|_{x_i=0} = u(x)|_{x_i=X_i} = 0$, $i = 1,\dots,m$. For u we define forward difference operators Δ_i, $i = 1,\dots,m$, as

$$\Delta_i u(x) \;=\; u(x_1,\dots,x_{i-1},x_i+1,x_{i+1},\dots,x_m) - u(x)\,.$$

As in a previous section, let $u(x;s_i)$ stand for $u(x_1,\ldots,x_{i-1},s_i,x_{i+1},\ldots,x_m)$, $\alpha=(\alpha_1,\ldots,\alpha_m)$ and let $H_m(\alpha)$ denote the harmonic mean of $\alpha_1,\ldots,\alpha_m$. Also, let $\sum_{x=1}^{X-1}$ denote $\sum_{j=1}^{m}\sum_{x_j=1}^{X_j-1}$.

The following results by Agarwal and Pang are given in [4]. We will present obtained extensions and improvements.

Theorem 8.20 *Let $\lambda \geq 1$ and let $u \in G(\Omega)$. Then, the following inequality holds*

$$\sum_{x=1}^{X-1} |u(x)|^{\lambda} \leq K(\lambda) \sum_{x=0}^{X-1} \left(\sum_{i=1}^{m} |\Delta_i u(x)|^2 \right)^{\frac{\lambda}{2}},$$

where

$$K(\lambda) = \frac{1}{m} C\left(\frac{\lambda}{2}\right) \prod_{i=1}^{m} \left(\sum_{x_i=1}^{X_i-1} \frac{1}{2} \left(x_i(X_i-x_i)\right)^{\frac{\lambda-1}{2}} \right)^{\frac{1}{m}} \tag{8.67}$$

and C is defined by (8.20)*.*

Theorem 8.21 *Let $\mu_k \geq 0$, $\lambda_k \geq 1$, $k=1,2$ be such that $\frac{\mu_1}{\lambda_1}+\frac{\mu_2}{\lambda_2}=1$ and let $u_1,u_2 \in G(\Omega)$. Then, the following inequality holds*

$$\begin{aligned}
&\sum_{x=1}^{X-1} |u_1(x)|^{\mu_1} |u_2(x)|^{\mu_2} \\
&\leq \sum_{k=1}^{2} \frac{\mu_k}{m\lambda_k} \max_{1\leq i\leq m} \left(\sum_{x_i=1}^{X_i-1} \frac{1}{2} \left(x_i(X_i-x_i)\right)^{\frac{\lambda_k-1}{2}} \right) \sum_{x=0}^{X-1} \sum_{i=1}^{m} |\Delta_i u_k(x)|^{\lambda_k}.
\end{aligned}$$

Theorem 8.22 *Let $p,n \geq 0$ be such that $p+n \geq 1$ and let $u \in G(\Omega)$. Then, the following inequality holds*

$$\sum_{x=1}^{X-1} |u(x)|^{p} \left(\sum_{i=1}^{m} |\Delta_i u(x)|^2 \right)^{\frac{n}{2}} \leq \left(K(p+n)\right)^{\frac{p}{p+n}} \sum_{x=0}^{X-1} \left(\sum_{i=1}^{m} |\Delta_i u(x)|^2 \right)^{\frac{p+n}{2}},$$

where K is defined by (8.67)*.*

8.3.1 Improvements of the Agarwal-Pang inequality I

We start with a weighted extension of Theorem 8.20 involving submultiplicative convex function.

Theorem 8.23 *Let f be a submultiplicative convex function on $[0,\infty)$ with $f(0)=0$. Let p be a nonnegative function on Ω and $u \in G(\Omega)$. Then, the following inequality holds*

$$\sum_{x=1}^{X-1} p(x) f(|u(x)|) \leq \frac{1}{m} H_m(\alpha) \sum_{x=0}^{X-1} \sum_{i=1}^{m} f(|\Delta_i u(x)|), \tag{8.68}$$

where $\alpha = (\alpha_1, \dots, \alpha_m)$ *and*

$$\alpha_i = \sum_{x_i=1}^{X_i-1} \left(\frac{x_i}{f(x_i)} + \frac{X_i - x_i}{f(X_i - x_i)} \right)^{-1} p(x), \quad i = 1, \dots, m.$$

Proof. For each fixed i, $i = 1, \dots, m$, we have

$$u(x) = \sum_{s_i=0}^{x_i-1} \Delta_i u(x; s_i), \qquad u(x) = -\sum_{s_i=x_i}^{X_i-1} \Delta_i u(x; s_i).$$

From the discrete case of Jensen's inequality (since f is an increasing convex function) and the submultiplicativity of f, we have

$$\begin{aligned} f(|u(x)|) &\le f\left(\frac{1}{x_i} \sum_{s_i=0}^{x_i-1} x_i \,|\Delta_i u(x; s_i)| \right) \\ &\le \frac{1}{x_i} \sum_{s_i=0}^{x_i-1} f\left(x_i \,|\Delta_i u(x; s_i)|\right) \\ &\le \frac{1}{x_i} \sum_{s_i=0}^{x_i-1} f(x_i)\, f\left(|\Delta_i u(x; s_i)|\right) \\ &= \frac{f(x_i)}{x_i} \sum_{s_i=0}^{x_i-1} f\left(|\Delta_i u(x; s_i)|\right) \end{aligned} \tag{8.69}$$

and analogously

$$f(|u(x)|) \le \frac{f(X_i - x_i)}{X_i - x_i} \sum_{s_i=x_i}^{X_i-1} f\left(|\Delta_i u(x; s_i)|\right) \tag{8.70}$$

for $i = 1, \dots, m$. We multiply (8.69) by $\frac{x_i}{f(x_i)}$ and (8.70) by $\frac{X_i - x_i}{f(X_i - x_i)}$. Then we add these resulting inequalities, to obtain

$$\left(\frac{x_i}{f(x_i)} + \frac{X_i - x_i}{f(X_i - x_i)} \right) f(|u(x)|) \le \sum_{s_i=0}^{X_i-1} f\left(|\Delta_i u(x; s_i)|\right),$$

i.e.

$$f(|u(x)|) \le \left(\frac{x_i}{f(x_i)} + \frac{X_i - x_i}{f(X_i - x_i)} \right)^{-1} \sum_{s_i=0}^{X_i-1} f\left(|\Delta_i u(x; s_i)|\right) \tag{8.71}$$

for $i = 1, \dots, m$. Now multiplying (8.71) by p and summing from $x = 1$ to $x = X - 1$, we get

$$\begin{aligned} \sum_{x=1}^{X-1} p(x) f\left(|u(x)|\right) \le & \sum_{x_i=1}^{X_i-1} \left(\frac{x_i}{f(x_i)} + \frac{X_i - x_i}{f(X_i - x_i)} \right)^{-1} p(x) \\ & \times \sum_{x=0}^{X-1} f\left(|\Delta_i u(x)|\right) \end{aligned} \tag{8.72}$$

for $i = 1, \ldots, m$. Multiplying both sides of the inequality (8.72) by α_i^{-1} and then adding these m inequalities , we obtain

$$\sum_{i=1}^{m} \alpha_i^{-1} \sum_{x=1}^{X-1} p(x) f(|u(x)|) \le \sum_{i=1}^{m} \sum_{x=0}^{X-1} f(|\Delta_i u(x)|),$$

which is the same as the inequality (8.68). □

Corollary 8.12 *Let f be a submultiplicative convex function on $[0,\infty)$ with $f(0) = 0$. Let p be a nonnegative function on Ω and $u \in G(\Omega)$. Then, the following inequality holds*

$$\sum_{x=1}^{X-1} p(x) f(|u(x)|) \le \frac{1}{2m} H_m(\beta) \sum_{x=0}^{X-1} \sum_{i=1}^{m} f(|\Delta_i u(x)|), \tag{8.73}$$

where $\beta = (\beta_1, \ldots, \beta_m)$ and

$$\beta_i = \sum_{x_i=1}^{X_i-1} \left(\frac{f(x_i) f(X_i - x_i)}{x_i (X_i - x_i)} \right)^{\frac{1}{2}} p(x), \quad i = 1, \ldots, m.$$

Proof. By harmonic-geometric inequality we have

$$2 \left(\frac{x_i}{f(x_i)} + \frac{X_i - x_i}{f(X_i - x_i)} \right)^{-1} \le \left(\frac{f(x_i) f(X_i - x_i)}{x_i (X_i - x_i)} \right)^{\frac{1}{2}}.$$

Applying this and using $H_m(\frac{1}{2}\gamma) = \frac{1}{2} H_m(\gamma)$, the inequality (8.73) follows. □

In the following inequalities let

$$h(x, X, \lambda) = (h_1(x, X, \lambda), \ldots, h_m(x, X, \lambda)) \tag{8.74}$$

and

$$h_i(x, X, \lambda) = \sum_{x_i=1}^{X_i-1} \left(x_i^{1-\lambda} + (X_i - x_i)^{1-\lambda} \right)^{-1}, \quad i = 1, \ldots, m.$$

Next we have an improvement of Theorem 8.20.

Corollary 8.13 *Let $\lambda, \mu \ge 1$ and let $u \in G(\Omega)$. Then, the following inequality holds*

$$\sum_{x=1}^{X-1} |u(x)|^{\lambda} \le K_1(\lambda, \mu) \sum_{x=0}^{X-1} \left(\sum_{i=1}^{m} |\Delta_i u(x)|^{\mu} \right)^{\frac{\lambda}{\mu}}, \tag{8.75}$$

where

$$K_1(\lambda, \mu) = \frac{1}{m} C\left(\frac{\lambda}{\mu} \right) H_m(h(x, X, \lambda)), \tag{8.76}$$

C is defined by (8.20) *and h by* (8.74)*.*

Proof. From Theorem 8.23 using function $f(t) = t^{\lambda}$ we have

$$\sum_{x=1}^{X-1} |u(x)|^{\lambda} \le \frac{1}{m} H_m\left(h(x,X,\lambda)\right) \sum_{x=0}^{X-1} \sum_{i=1}^{m} |\Delta_i u(x)|^{\lambda}. \tag{8.77}$$

The inequality (8.75) now follows from (8.77) and the elementary inequality (8.34). □

Remark 8.26 By the harmonic-geometric means inequality, we have

$$2\left(s^{1-\lambda} + (1-s)^{1-\lambda}\right)^{-1} \le \left(s^{\lambda-1}(1-s)^{\lambda-1}\right)^{\frac{1}{2}}.$$

Thus, again from the harmonic-geometric means inequality it follows that Corollary 8.13 for $\mu = 2$ improves Theorem 8.20.

8.3.2 Improvements of the Agarwal-Pang inequalities II and III

In our next result, we will improve Theorem 8.21.

Theorem 8.24 *Let $\mu \ge 1$ and $\mu_k \ge 0$, $\lambda_k \ge 1$ be such that $\sum_{k=1}^{n} \mu_k/\lambda_k = 1$ and let $u_k \in G(\Omega)$, $k = 1,\dots,n$. Then, the following inequality holds*

$$\sum_{x=1}^{X-1} \prod_{k=1}^{n} |u_k(x)|^{\mu_k} \le \sum_{k=1}^{n} \frac{\mu_k}{\lambda_k} K_1(\lambda_k,\mu) \sum_{x=0}^{X-1} \left(\sum_{i=1}^{m} |\Delta_i u_k(x)|^{\mu}\right)^{\frac{\lambda_k}{\mu}}, \tag{8.78}$$

where K_1 is defined by (8.76)*.*

Proof. Using weighted arithmetic-geometric means inequality, we find

$$\prod_{k=1}^{n} |u_k(x)|^{\mu_k} = \prod_{k=1}^{n} \left(|u_k(x)|^{\lambda_k}\right)^{\frac{\mu_k}{\lambda_k}} \le \sum_{k=1}^{n} \frac{\mu_k}{\lambda_k} |u_k(x)|^{\lambda_k}.$$

The inequality (8.78) now follows from Corollary 8.13 with $\lambda = \lambda_k$. □

In the following remarks, we will state several important particular cases of the inequality (8.78). The obtained inequalities improve corresponding results established in [4].

Remark 8.27 Let $\mu_1, \mu_2 > 0$ be such that $\mu_1 + \mu_2 \ge 2$ and let $u_1, u_2 \in G(\Omega)$. Then, for $n = \mu = 2$ and $\lambda_1 = \lambda_2 = \mu_1 + \mu_2$ the inequality (8.78) reduces to

$$\sum_{x=1}^{X-1} |u_1(x)|^{\mu_1} |u_2(x)|^{\mu_2}$$
$$\le \frac{1}{m} H_m\left(h(x,X,\mu_1+\mu_2)\right) \left[\frac{\mu_1}{\mu_1+\mu_2} \sum_{x=0}^{X-1} \left(\sum_{i=1}^{m} |\Delta_i u_1(x)|^2\right)^{\frac{\mu_1+\mu_2}{2}} \right.$$

$$+\frac{\mu_2}{\mu_1+\mu_2}\sum_{x=0}^{X-1}\left(\sum_{i=1}^{m}|\Delta_i u_2(x)|^2\right)^{\frac{\mu_1+\mu_2}{2}}\Bigg]. \tag{8.79}$$

In particular, for $\mu_1=\mu_2=1$, the inequality (8.79) is the same as

$$\begin{aligned}&\sum_{x=1}^{X-1}|u_1(x)|\,|u_2(x)|\\ &\quad\le \frac{1}{2m}H_m(h(x,X,2))\left[\sum_{x=0}^{X-1}\left(\sum_{i=1}^{m}|\Delta_i u_1(x)|^2\right)+\sum_{x=0}^{X-1}\left(\sum_{i=1}^{m}|\Delta_i u_2(x)|^2\right)\right].\end{aligned}$$

Remark 8.28 Let $1\le\lambda_1,\lambda_2<2$ be such that $\frac{1}{\lambda_1}+\frac{1}{\lambda_2}=1$, and let $u_1,u_2\in G(\Omega)$. Then, for $n=\mu=2$ and $\mu_1=\mu_2=1$ the inequality (8.78) leads to

$$\begin{aligned}\sum_{x=1}^{X-1}|u_1(x)|\,|u_2(x)| \le\ & \frac{1}{\lambda_1 m^{\frac{\lambda_1}{2}}}H_m(h(x,X,\lambda_1))\sum_{x=0}^{X-1}\left(\sum_{i=1}^{m}|\Delta_i u_1(x)|^2\right)^{\frac{\lambda_1}{2}}\\ &+\frac{1}{\lambda_2 m^{\frac{\lambda_2}{2}}}H_m(h(x,X,\lambda_2))\sum_{x=0}^{X-1}\left(\sum_{i=1}^{m}|\Delta_i u_2(x)|^2\right)^{\frac{\lambda_2}{2}}.\end{aligned}$$

Remark 8.29 Let $\mu_1,\mu_2>0$ be such that $1\le\mu_1+\mu_2<2$ and let $u_1,u_2\in G(\Omega)$. Then, for $n=\mu=2$ and $\lambda_1=\lambda_2=\mu_1+\mu_2$ the inequality (8.78) gives

$$\begin{aligned}\sum_{x=1}^{X-1}|u_1(x)|^{\mu_1}|u_2(x)|^{\mu_2} \le\ & \frac{1}{m^{\frac{\mu_1+\mu_2}{2}}}H_m(h(x,X,\mu_1+\mu_2))\\ &\times\left[\frac{\mu_1}{\mu_1+\mu_2}\sum_{x=0}^{X-1}\left(\sum_{i=1}^{m}|\Delta_i u_1(x)|^2\right)^{\frac{\mu_1+\mu_2}{2}}\right.\\ &\left.+\frac{\mu_2}{\mu_1+\mu_2}\sum_{x=0}^{X-1}\left(\sum_{i=1}^{m}|\Delta_i u_2(x)|^2\right)^{\frac{\mu_1+\mu_2}{2}}\right].\end{aligned}$$

Remark 8.30 Let $\mu_k>0$, $\lambda_k\ge1$, $k=1,2,3$, be such that $\frac{\mu_1}{\lambda_1}+\frac{\mu_2}{\lambda_2}=1$, $\frac{\mu_2}{\lambda_2}+\frac{\mu_3}{\lambda_3}=1$, $\frac{\mu_3}{\lambda_3}+\frac{\mu_1}{\lambda_1}=1$, and let $\mu\ge1$, $u_k\in G(\Omega)$, $k=1,2,3$. Then, the following inequality holds

$$\begin{aligned}&\sum_{x=1}^{X-1}\left(|u_1(x)|^{\mu_1}|u_2(x)|^{\mu_2}+|u_2(x)|^{\mu_2}|u_3(x)|^{\mu_3}+|u_3(x)|^{\mu_3}|u_1(x)|^{\mu_1}\right)\\ &\qquad\le\sum_{k=1}^{3}\frac{2\mu_k}{\lambda_k}K_1(\lambda_k,\mu)\sum_{x=0}^{X-1}\left(\sum_{i=1}^{m}|\Delta_i u_k(x)|^{\mu}\right)^{\frac{\lambda_k}{\mu}}\end{aligned} \tag{8.80}$$

where K_1 is defined by (8.76).

Indeed, for $n=2$, Theorem 8.24 gives

$$\sum_{x=1}^{X-1}|u_1(x)|^{\mu_1}|u_2(x)|^{\mu_2} \le \frac{\mu_1}{\lambda_1}K_1(\lambda_1,\mu)\sum_{x=0}^{X-1}\left(\sum_{i=1}^{m}|\Delta_i u_1(x)|^{\mu}\right)^{\frac{\lambda_1}{\mu}} + \frac{\mu_2}{\lambda_2}K_1(\lambda_2,\mu)\sum_{x=0}^{X-1}\left(\sum_{i=1}^{m}|\Delta_i u_2(x)|^{\mu}\right)^{\frac{\lambda_2}{\mu}}.$$

Similar to this inequality, we have two more inequalities involving μ_2, μ_3, λ_2, λ_3, u_2, u_3 and μ_3, μ_1, λ_3, λ_1, u_3, u_1. An addition of these three inequalities gives (8.80).

In particular, for $\mu=2$, $\lambda_k=2\mu_k$, $k=1,2,3$, where $\mu_k\ge 1$, the inequality (8.80) reduces to

$$\sum_{x=1}^{X-1}\left(|u_1(x)|^{\mu_1}|u_2(x)|^{\mu_2}+|u_2(x)|^{\mu_2}|u_3(x)|^{\mu_3}+|u_3(x)|^{\mu_3}|u_1(x)|^{\mu_1}\right) \le \frac{1}{m}\sum_{k=1}^{3}H_m(h(x,X,2\mu_k))\sum_{x=0}^{X-1}\left(\sum_{i=1}^{m}|\Delta_i u_k(x)|^2\right)^{\mu_k}.$$

Further, when $\frac{1}{2}\le\mu_k<1$, $k=1,2,3$, the inequality (8.80) gives

$$\sum_{x=1}^{X-1}\left(|u_1(x)|^{\mu_1}|u_2(x)|^{\mu_2}+|u_2(x)|^{\mu_2}|u_3(x)|^{\mu_3}+|u_3(x)|^{\mu_3}|u_1(x)|^{\mu_1}\right) \le \sum_{k=1}^{3}\frac{1}{m^{\mu_k}}H_m(h(x,X,2\mu_k))\sum_{x=0}^{X-1}\left(\sum_{i=1}^{m}|\Delta_i u_k(x)|^2\right)^{\mu_k}.$$

Remark 8.31 Let $\mu_k>0$, $\lambda_k\ge 1$, $k=1,2,3$, be such that $\frac{\mu_1}{\lambda_1}+\frac{\mu_2}{\lambda_2}=1$, $\frac{\mu_2}{\lambda_2}+\frac{\mu_3}{\lambda_3}=1$, $\frac{\mu_3}{\lambda_3}+\frac{\mu_1}{\lambda_1}=1$, and let $\mu\ge 1$, $u_k\in G(\Omega)$, $k=1,2,3$. Then, the following inequality holds

$$\sum_{x=1}^{X-1}|u_1(x)|^{\mu_1}|u_2(x)|^{\mu_2}|u_3(x)|^{\mu_3}\left(|u_1(x)|^{\mu_1}+|u_2(x)|^{\mu_2}+|u_3(x)|^{\mu_3}\right) \le \frac{2}{m}\sum_{k=1}^{3}\frac{\mu_k}{\lambda_k}H_m(h(x,X,2\lambda_k))\sum_{x=0}^{X-1}\left(\sum_{i=1}^{m}|\Delta_i u_k(x)|^2\right)^{\lambda_k}. \tag{8.81}$$

Indeed, the inequality (8.81) follows from the elementary inequality $\alpha_1\alpha_2\alpha_3(\alpha_1+\alpha_2+\alpha_3)\le\alpha_1^2\alpha_2^2+\alpha_2^2\alpha_3^2+\alpha_3^2\alpha_1^2$, and the inequality (8.80) for $\mu=2$ with μ_k and λ_k replaced by $2\mu_k$ and $2\lambda_k$.

In particular, for $\lambda_k=2\mu_k$, $k=1,2,3$, where $\mu_k\ge 2^{-1}$, the inequality (8.81) reduces to

$$\sum_{x=1}^{X-1}|u_1(x)|^{\mu_1}|u_2(x)|^{\mu_2}|u_3(x)|^{\mu_3}\left(|u_1(x)|^{\mu_1}+|u_2(x)|^{\mu_2}+|u_3(x)|^{\mu_3}\right) \le \frac{1}{m}\sum_{k=1}^{3}H_m(h(x,X,4\mu_k))\sum_{x=0}^{X-1}\left(\sum_{i=1}^{m}|\Delta_i u_k(x)|^2\right)^{2\mu_k}.$$

Now we will prove two results which complement Theorem 8.20.

Theorem 8.25 *Let $p,\lambda,\mu \geq 1$ with $\frac{1}{\mu}+\frac{1}{\lambda}=1$ and let $u \in G(\Omega)$. Then, the following inequality holds*

$$\sum_{x=1}^{X-1} |u(x)|^p \leq \frac{m^{\frac{1}{\mu}}}{2\sum_{i=1}^m X_i^{-1}} \left(\sum_{x=1}^{X-1} |u(x)|^{\mu(p-1)}\right)^{\frac{1}{\mu}} \left(\sum_{x=0}^{X-1}\sum_{i=1}^{m} |\Delta_i u(x)|^{\lambda}\right)^{\frac{1}{\lambda}}. \tag{8.82}$$

Proof. From the hypotheses, we have

$$u^p(x) = u^{p-1}(x) \sum_{s_i=0}^{x_i-1} \Delta_i u(x;s_i)$$

and

$$u^p(x) = -u^{p-1}(x) \sum_{s_i=x_i}^{X_i-1} \Delta_i u(x;s_i)$$

for $i=1,\ldots,m$. Using Hölder's inequality in the above inequalities with indices μ and λ, we obtain

$$\begin{aligned} |u(x)|^p &\leq \frac{1}{2}|u(x)|^{p-1} \sum_{s_i=0}^{X_i-1} |\Delta_i u(x;s_i)| \\ &\leq \frac{1}{2}|u(x)|^{p-1} X_i^{\frac{1}{\mu}} \left(\sum_{s_i=0}^{X_i-1} |\Delta_i u(x;s_i)|^{\lambda}\right)^{\frac{1}{\lambda}}. \end{aligned}$$

Now, multiplying both sides of the above inequality by X_i^{-1}, $i=1,\ldots,m$, summing these inequalities, and then summing both sides from $x=1$ to $x=X-1$, we get

$$\sum_{x=1}^{X-1} |u(x)|^p \leq \frac{1}{2\sum_{i=1}^m X_i^{-1}} \sum_{i=1}^{m} X_i^{-\frac{1}{\lambda}} \sum_{x=1}^{X-1} |u(x)|^{p-1} \left(\sum_{s_i=0}^{X_i-1} |\Delta_i u(x;s_i)|^{\lambda}\right)^{\frac{1}{\lambda}}.$$

Applying Hölder's inequality with indices λ and μ, we find

$$\begin{aligned} \sum_{x=1}^{X-1} |u(x)|^p &\leq \frac{1}{2\sum_{i=1}^m X_i^{-1}} \left(\sum_{x=1}^{X-1} |u(x)|^{\mu(p-1)}\right)^{\frac{1}{\mu}} \sum_{i=1}^{m} X_i^{-\frac{1}{\lambda}} \\ &\quad \times \left(\sum_{x=1}^{X-1}\sum_{s_i=0}^{X_i-1} |\Delta_i u(x;s_i)|^{\lambda}\right)^{\frac{1}{\lambda}} \\ &\leq \frac{1}{2\sum_{i=1}^m X_i^{-1}} \left(\sum_{x=1}^{X-1} |u(x)|^{\mu(p-1)}\right)^{\frac{1}{\mu}} \sum_{i=1}^{m} \left(\sum_{x=0}^{X-1} |\Delta_i u(x)|^{\lambda}\right)^{\frac{1}{\lambda}}. \end{aligned}$$

Finally, once again on applying Hölder's inequality with indices λ and μ, we obtained the required inequality (8.82). □

Theorem 8.26 *Let $p,\lambda,\mu \geq 1$ with $\frac{1}{\mu}+\frac{1}{\lambda}=1$ and let $u \in G(\Omega)$. Let $r \geq 1$. Then, the following inequality holds*

$$\sum_{x=1}^{X-1}|u(x)|^p \leq (K_1(\lambda,r))^{\frac{1}{\lambda}}\left(\sum_{x=1}^{X-1}|u(x)|^{\mu(p-1)}\right)^{\frac{1}{\mu}}\left(\sum_{x=0}^{X-1}\left(\sum_{i=1}^{m}|\Delta_i u(x)|^r\right)^{\frac{\lambda}{r}}\right)^{\frac{1}{\lambda}}. \tag{8.83}$$

Proof. Applying Hölder's inequality with indices λ and μ, we find

$$\begin{aligned}\sum_{x=1}^{X-1}|u(x)|^p &= \sum_{x=1}^{X-1}|u(x)|^{p-1}|u(x)| \\ &\leq \left(\sum_{x=1}^{X-1}|u(x)|^{\mu(p-1)}\right)^{\frac{1}{\mu}}\left(\sum_{x=1}^{X-1}|u(x)|^{\lambda}\right)^{\frac{1}{\lambda}}.\end{aligned}$$

The inequality (8.83) now follows as an application of Corollary 8.13. □

Next we give another interesting inequality which improves Theorem 8.22.

Theorem 8.27 *Let $p,n \geq 0$ be such that $p+n \geq 1$ and let $u \in G(\Omega)$. Then, the following inequality holds*

$$\begin{aligned}&\sum_{x=1}^{X-1}|u(x)|^p\left(\sum_{i=1}^{m}|\Delta_i u(x)|^2\right)^{\frac{n}{2}} \\ &\leq (K_1(p+n,2))^{\frac{p}{p+n}}\sum_{x=0}^{X-1}\left(\sum_{i=1}^{m}|\Delta_i u(x)|^2\right)^{\frac{p+n}{2}},\end{aligned} \tag{8.84}$$

where K_1 is defined by (8.76)*.*

Proof. Inequality (8.84) is an easy consequence of the result of Corollary 8.13. Indeed, first we apply Hölder's inequality with indices $\frac{p+n}{p}$ and $\frac{p+n}{n}$ in the left side of (8.84) and then apply (8.75) with $\lambda = p+n$ and $\mu = 2$. □

Remark 8.32 For $p=1$ and $n=l-1$, $l \geq 2$, the inequality (8.84) reduces to

$$\begin{aligned}&\sum_{x=1}^{X-1}|u(x)|\left(\sum_{i=1}^{m}|\Delta_i u(x)|^2\right)^{\frac{l-1}{2}} \\ &\leq \frac{1}{m^{\frac{1}{l}}}H_m(h(x,X,l))^{\frac{1}{l}}\sum_{x=0}^{X-1}\left(\sum_{i=1}^{m}|\Delta_i u(x)|^2\right)^{\frac{l}{2}}.\end{aligned}$$

Finally, in the following we state a generalization of Theorem 8.27. which improves Corollary 11 established in [4].

Remark 8.33 Let $p_k, n_k \geq 0$ be such that $r(p_k + n_k) \geq 1$ and let $u_k \in G(\Omega)$, $k = 1, \ldots, r$. Then, the following inequality holds

$$\sum_{x=1}^{X-1} \prod_{k=1}^{r} |u_k(x)|^{p_k} \left(\sum_{i=1}^{m} |\Delta_i u_k(x)|^2 \right)^{\frac{n_k}{2}}$$
$$\leq \frac{1}{r} \sum_{k=1}^{r} (K_1(r(p_k + n_k), 2))^{\frac{p_k}{p_k+n_k}} \sum_{x=0}^{X-1} \left(\sum_{i=1}^{m} |\Delta_i u_k(x)|^2 \right)^{\frac{r(p_k+n_k)}{2}} .$$

Bibliography

[1] R. P. Agarwal, *Sharp Opial-type inequalities involving r-derivatives and their applications*, Tâhoku Math. J. **47** (1995), 567–593.

[2] R. P. Agarwal, P. Y. H. Pang, *Opial-type inequalities involving high order derivatives*, J. Math. Anal. Appl. **189** (1995), 85–103.

[3] R. P. Agarwal, P. Y. H. Pang, *Remarks on the generalization of Opial's inequality*, J. Math. Anal. Appl. **190** (1995), 559–577.

[4] R. P. Agarwal, P. Y. H. Pang, *Sharp discrete inequalities in n independent variables*, Appl. Math. Comp. **72** (1995), 97–112.

[5] R. P. Agarwal, P. Y. H. Pang, *Opial Inequalities with Applications in Differential and Difference Equations*, Kluwer Academic Publishers, Dordrecht, Boston, London (1995).

[6] R. P. Agarwal, J. Pečarić, I. Brnetić, *Improved integral inequalities in n independent variables*, Computers Math. Applic. **33**(8) (1997), 27–38.

[7] R. P. Agarwal, J. Pečarić, I. Brnetić, *Improved discrete inequalities in n independent variables*, Appl. Math. Lett. **11**(2) (1998), 91–97.

[8] R.P. Agarwal, Q. Sheng, *Sharp integral inequalities in n independent variables*, Nonlinear Anal. **26** (1996), 179–210.

[9] R. P. Agarwal, P. J. Y. Wong, *Error Inequalities in Polynomial Interpolation and Their Applications*, Kluwer Academic Publishers, Dordrecht, Boston, London (1993).

[10] H. Alzer, *On some inequalities of Opial-type*, Arch. Math. **63** (1994), 431–436.

[11] H. Alzer, *An Opial-type inequality involving higher-order derivativess ot two functions*, Appl. Math. Lett. **10** (4) (1997), 123–128.

[12] G. A. Anastassiou, *Fractional Differentiation Inequalities*, Springer (2009).

[13] G. A. Anastassiou, *Riemann-Liouville fractional multivariate Opial type inequalities over spherical shells*, Bulletin of Allahabad Math. Soc. India **23** (2008), 65–140.

[14] G. A. Anastassiou, J. J. Koliha, J. Pečarić, *Opial inequalities for fractional derivatives*, Dynam. Systems Appl. **10** (2001), 395–406.

[15] G. A. Anastassiou, J. J. Koliha, J. Pečarić, *Opial type L_p-inequalities for fractional derivatives*, Inter. J. Math & Math. Sci. **31** (2) (2002), 85–95.

[16] M. Andrić, *Composition Identities for Fractional Derivatives and Opial-type Inequalities*, Ph.D. dissertation (2011).

[17] M. Andrić, A. Barbir, G. Farid, J. Pečarić, *Opial-type inequality due to Agarwal-Pang and fractional differential inequalities*, Integral Transforms Spec. Funct. **25** (4) (2014), 324–335.

[18] M. Andrić, A. Barbir, G. Farid, J. Pečarić, *More on certain Opial-type inequality for fractional derivatives and exponentially convex functions*, Nonlinear. Funct. Anal. Appl. **19** (4) (2014), 563–583.

[19] M. Andrić, A. Barbir, S. Iqbal, J. Pečarić, *An Opial-type integral inequality and exponentially convex functions*, Fractional Differ. Calc. **5** (1) (2015), 25–42.

[20] M. Andrić, A. Barbir, J. Pečarić, *On Willett's, Godunova-Levin's and Rozanova's Opial-type inequalities with related Stolarsky type means*, Math. Notes **96** (6) (2014), 841–854.

[21] M. Andrić, A. Barbir, J. Pečarić, G. Roqia, *Generalizations of Opial-type inequalities in several independent variables*, Demonstratio Math. **47** (4) (2014), 839–847.

[22] M. Andrić, J. Pečarić, I. Perić, *On weighted integral and discrete Opial-type inequalities*.

[23] M. Andrić, J. Pečarić, I. Perić, *Improvements of composition rule for the Canavati fractional derivatives and applications to Opial-type inequalities*, Dynam. Systems Appl. **20** (2011), 383–394.

[24] M. Andrić, J. Pečarić, I. Perić, *General multiple Opial-type inequalities for the Canavati fractional derivatives*, Ann. Funct. Anal. **4** (1) (2013), 149–162.

[25] M. Andrić, J. Pečarić, I. Perić, *A multiple Opial type inequality for the Riemann-Liouville fractional derivatives*, J. Math. Inequal. **7** (1) (2013), 139–150.

[26] M. Andrić, J. Pečarić, I. Perić, *Composition identities for the Caputo fractional derivatives and applications to Opial-type inequalities*, Math. Inequal. Appl. **16** (3) (2013), 657–670.

[27] M. Andrić, J. Pečarić, I. Perić, *An Opial-type inequality for fractional derivatives of two functions*, Fractional Differ. Calc. **3** (1) (2013), 55–68.

[28] M. Anwar, J. Jakšetić, J. Pečarić, Atiq ur Rehman, *Exponential convexity, positive semi-definite matrices and fundamental inequalities*, J. Math. Inequal. **4** (2) (2010), 171–189.

[29] A. Barbir, K. Krulić Himmelreich, J. Pečarić, *General Opial type inequality*, Aequat. Math. (2014), to appear.

[30] A. Barbir, K. Krulić Himmelreich, J. Pečarić, *General Opial type inequality for quotient of functions*.

[31] P. R. Beesack, *On an integral inequality of Z. Opial*, Trans. Amer. Math. Soc. **104** (1962), 470–475.

[32] S. N. Bernstein, *Sur les fonctions absolument monotones*, Acta Math. **52** (1929), 1–66.

[33] D. W. Boyd, *Best constants in a class of integral inequalities*, Pacific J. Math. **30** (2) (1969), 367–383.

[34] I. Brnetić, J. Pečarić, *Note on generalization of Godunova-Levin-Opial inequality*, Demonstratio Math. **3**(30) (1997), 545–549.

[35] I. Brnetić, J. Pečarić, *Note on the generalization of the Godunova-Levin-Opial type inequality in several independent variables*, J. Math. Anal. Appl. **215** (1997), 274–283.

[36] I. Brnetić, J. Pečarić, *Some new Opial-type inequalities*, Math. Inequal. Appl. **1**(3) (1998), 385–390.

[37] I. Brnetić, J. Pečarić, *Generalization of Hwang-Yang inequality*, Sec. Math. Tech. Sci., MANU, **22**(1-2) (2001), 5–20.

[38] J. A. Canavati, *The Riemann-Liouville integral*, Nieuw Archief Voor Wiskunde **5** (1) (1987), 53–75.

[39] M.-P. Chen, H. M. Srivastava, *Fractional calculus operators and their applications involving power functions and summation series*, Appl. Math. Comput. **81** (1997), 287–304.

[40] W.-S. Cheung, *On integral inequalities of the Sobolev type*, Aequat. Math. **49** (1995), 153–159.

[41] W.-S. Cheung, Z. Dandan, J. Pečarić, *Opial-type inequalities for differential operators*, Nonlinear Anal. **66** (9) (2007), 2028–2039.

[42] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer, Berlin (2010).

[43] G. Farid, J. Pečarić, *Opial type integral inequalities for fractional derivatives*, Fractional Differ. Calc. **2** (1) (2012), 31–54.

[44] G. Farid, J. Pečarić, *Opial type integral inequalities for fractional derivatives II*, Fractional Differ. Calc. **2** (2) (2012), 139–155.

[45] A. M. Fink, *On Opial's inequality for $f^{(n)}$*, Proc. Amer. Math. Soc. **115** (1992), 177–181.

[46] E. K. Godunova, V. I. Levin, *On an inequality of Maroni*, Mat. Zametki **2** (1967), 221–224.

[47] G. D. Handley, J. J. Koliha, J. Pečarić, *Hilbert-Pachpatte type integral inequalities for fractional derivatives*, Fract. Calc. Appl. Anal. **4** (1) (2001), 37–46.

[48] G. H. Hardy, *Notes on some points in the integral calculus*, Messenger of Mathematics **47** (10) (1918), 145–150.

[49] S. Iqbal, K. Krulić, J. Pečarić, *On an inequality of H. G. Hardy*, J. Inequal. Appl. **10** (2010), 1–23.

[50] J. Jakšetić, J. Pečarić, *Exponential Convexity Method*, J. Convex Anal. **20** (1) (2013), 181–197.

[51] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, Netherlands (2006).

[52] J. J. Koliha, J. Pečarić, *Weighted Opial inequalities*, Tamkang J. Math. **33** (2002), 93–102.

[53] A. N. Kolmogorov, S. V. Fomin, R. A. Silverman, *Introductory Real Analysis*, Dover Publication, New York (1975).

[54] K. Krulić, J. Pečarić, L.-E. Persson, *A new Hardy-type inequality with kernel*, Math. Inequal. Appl. **12**(3) (2009), 473–485.

[55] K. Krulić Himmelreich, J. Pečarić, D. Pokaz, *Inequalities of Hardy and Jensen*, Element, Zagreb, 2013.

[56] C. L. Mallows, *An even simpler proof of Opial's inequality*, Proc. Amer. Math. Soc. **16** (1965), 173.

[57] S. Mardešić, *Matematička analiza u n-dimanzionalnom realnom prostoru, I i II*, Školska knjiga, Zagreb (1977).

[58] K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, Inc., New York (1993).

[59] D. S. Mitrinović, *Analytic Inequalities*, Springe-Verlag, Berlin and New York (1970).

[60] D. S. Mitrinović, J. E. Pečarić, *Generalizations of two inequalities of Godunova and Levin*, Bull. Polish Acad. Sci. Math. **36** (1988), 645–648.

[61] D. S. Mitrinović, J. E. Pečarić, A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Acad. Publ., Dordrecht (1993).

[62] C. P. Niculescu, L.-E. Persson, *Convex Functions and Their Applications, A Contemporary Approach*, CMS Books in Mathematics/Ouvrages de mathématiques de la SMC 23, Springer, New York (2006).

[63] K. B. Oldham, J. Spanier, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, Dover Publication, New York (2006).

[64] Z. Opial, *Sur une inégalité*, Ann. Polon. Math. **8** (1960), 29–32.

[65] P. Y. H. Pang, R. P. Agarwal, *On an Opial type inequality due to Fink*, J. Math. Anal. Appl. **196** (2) (1995), 748–753.

[66] J. Pečarić, J. Perić, *Improvements of the Giaccardi and Petrović inequality and related Stolarsky type means*, An. Univ. Craiova Ser. Mat. Inform. **39** (1) (2012), 65–75.

[67] J. E. Pečarić, F. Proschan , Y.C. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Inc. (1992).

[68] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego (1999).

[69] G. I. Rozanova, *Integral'nye neravenstva s proizvodnysi i proizvol'nymi vypuklymi funkcijami*, Uc. Zap. Mosk. Gos. Ped. In-ta im. Lenina **460** (1972), 58–65.

[70] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York (1970).

[71] S. H. Saker, *Applications of Wirtinger inequalities on the distribution of zeros of the Riemann zeta-function*, J. Inequal. Appl. **2010** (2010), ID 215416.

[72] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Reading (1993).

[73] J. M. Whittaker, *On Lidstone series and two-point expansions of analytic functions*, Proc. Lond. Math. Soc. **36** (1933–1934), 451–469.

[74] D. V. Widder, *The Laplace Transform*, Princeton University Press (1941).

[75] D. Willett, *The existence-uniqueness theorem for an n-th order linear ordinary differential equation*, Amer. Math. Monthly **75** (1968), 174–178.

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