MONOGRAPHS IN INEQUALITIES 11

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New methods for improving the Jensen-type inequalities in real and in operator cases

Mario Krnić, Neda Lovričević, Josip Pečarić and Jurica Perić

ELEMENT

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Preface

Due to the scientific contribution of the Danish mathematician J. L. W. Jensen from 1905, the theory of convex functions started to develop rapidly. The famous Jensen inequality, in one of its elementary forms:

$$f\left(\frac{1}{P_n}\sum_{1=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{1=1}^{n} p_i f(x_i) \tag{1}$$

holds for a convex function $f\colon I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, for an n-tuple $\mathbf{x} = (x_1,\ldots,x_n) \in I^n$, $n \geq 2$ and a nonnegative n-tuple $\mathbf{p} = (p_1,\ldots,p_n)$, such that $P_n = \sum_{i=1}^{n} p_i > 0$.

The Jensen inequality is one of the most important and most frequently applied in a diversity of mathematical fields, especially in mathematical analysis and statistics. It has been improved, generalized and adjusted to various environments in this process. Often referred to as "king of inequalities", it has been linked to other important inequalities, e.g. Young's, Hölder's, Minkowski's, Beckenbach-Dresher's, Hilbert's,... In this book, however, we present an aspect of development of Jensen's famous inequality considering the Jensen-type functionals in the first place, and then, in the second part, the results obtained via a specific monotonicity principle. In order to do so, we make an almost one hundred years shift in time, since the most important Jensen's result has been established. Namely, in 1996, S. S. Dragomir, J. E. Pečarić and L. E. Persson investigated the so called discrete Jensen functional, deduced from (1), by subtracting its left-hand side from its right-hand side. They proved that this functional was superadditive and increasing on the set of previously described n-tuples $\mathbf{p}$ in the case of a convex function f. This very result is the basis for the variety of improvements, generalizations and applications to the classical inequalities, presented in the first part of this book.

Thus in the second chapter, superadditivity on the space of real functions and monotonicity as its consequence, are proved to be possessed by Jensen-type functionals (Jessen's and McShane's functionals), both defined by means of a positive linear functional acting on the space of real functions.

In a similar manner, the third chapter deals with Jensen-Steffensen's and Jensen-Mercer's functionals, as well as with the Petrović-type functionals, in their discrete and integral forms, depending on which are superadditivity and monotonicity discussed on the set of real n-tuples or on the set of real functions, with their specific conditions involved.

Inequalities related to Jessen's functional that were analyzed in Chapter 2 are reexamined and improved in the fourth chapter of the book, under some new assumptions and due to some new results. In the same chapter, Jessen's functional served as a basis in studying superadditivity of the generalized form of the well known Levinson functional, as well.

What is of a special interest here is that for all of these functionals, their superadditivity and monotonicity on the described sets provide their specific both-sided bounds, expressed by means of the non-weight functionals of the same type. Presented refinements and converses of a variety of classical inequalities are immediate consequences of such obtained bounds.

Observe, for example, how widely this scope of investigation finally reached, that the whole class of refinements and converses of the Hilbert inequality is established (Chapter 2), all by means of superadditivity of the Jensen-type functionals.

The fifth chapter is specific in its structure since it starts with some of the first published results on superadditivity and then integrates a few different approaches to superadditivity. These concern several classes of functionals that were studied recently, independently of the previously described integrated research, but nevertheless significant in their contribution to this subject.

After this short digression, we go back to the unified approach to Jensen-type functionals employed in this area of research, which leads us to the sixth chapter. Here we have a transition from the domain of real analysis to the domain of the functional analysis. Thus we have bounded self-adjoint operators on a Hilbert space as the arguments of the observed Jensen-type functionals, and as applications – refinements and converses of the operator mean inequalities: arithmetic-geometric, arithmetic-harmonic, arithmetic-Heinz,... Additionally, integral operator Jensen's inequality with the correspondingly defined functional is also studied, as well as the multidimensional Jensen's functional for operators, with some interesting applications to connections, solidarities and multidimensional weight geometric means.

In the seventh chapter, several refinements of the Heinz norm inequalities are derived, by virtue of convexity of Heinz means and via the Jensen functional and, in the sequel, some improved majorization relations and eigenvalue inequalities for matrix versions of the Jensen inequality are also given.

A rich variety of the results of another group of authors is organized in the second part of the book (chapters 8 to 11) and is based on a related, still different basic motivation. Namely, in 1993 J. E. Pečarić investigated the method of interpolating inequalities which have reversed inequalities of Aczél type. Using Jensen's inequality and its reverse, he proved that

$$\sum_{i=1}^{n} p_i f(x_i) - P_n f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \geq \sum_{i=1}^{n} q_i f(x_i) - Q_n f\left(\frac{1}{Q_n}\sum_{i=1}^{n} q_i x_i\right) \geq 0, \tag{2}$$

where f is a convex function on an interval $I \subset \mathbb{R}$, $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$, $n \geq 2$, and $\mathbf{p}$ and $\mathbf{q}$ are positive n-tuples such that $\mathbf{p} \geq \mathbf{q}$, (i.e. $p_i \geq q_i$, $i = 1, \ldots, n$; $P_n = \sum_{i=1}^{n} p_i$, $Q_n = \sum_{i=1}^{n} q_i$).

By means of a simple consequence of this result (thoroughly described in Chapter 1), a whole series of results has been improved, as it is presented in the latter part of this book.

Only a few years after (2) had been established, in 1996, as we have already mentioned, S. S. Dragomir, J. E. Pečarić and L. E. Persson obtained the analogous result in their joint paper, but as a consequence of a quite different approach – via superadditivity.

Although it is evident that the same monotonicity property of the Jensen's functional is obtained twice – for the first time as a side-result incorporated in the interpolating inequalities, and for the second time – as a consequence of the superadditivity property, it is interesting that the first result was not even mentioned or referred to when the other one was published!

Now, let us outline the contents of the remained chapters.

In the eighth chapter, due to the monotonicity principle (2), various variants of the converse Jensen inequality are studied, improved and generalized. The first set of such results (generalizations are obtained for positive linear functionals and furthermore, on convex hulls and on k-simplices) is motivated by the Lah-Ribarič inequality, as the most important converse Jensen's inequality, and the second set is grouped around the Giaccardi-Petrović inequality. A large family of n-exponentially convex and exponentially convex functions is therefrom constructed, as it is similarly done in the following chapters, as well.

In the ninth chapter, we proceed with the applications of the monotonicity property (2) in a similar manner, with two improvements of the Jessen-Mercer inequality presented, as well as a generalization of the Jessen-Mercer inequality on convex hulls: the results accompanied with a k-dimensional variant of the Hammer-Bullen inequality and with an improvement of the classical Hermite-Hadamard inequality.

When mentioning the Hermite-Hadamard inequality, the improvements of its various forms (the ones of Fejèr, Lupaş, Brenner-Alzer, Beesack-Pečarić) are presented in the tenth chapter. These improvements, as it will be seen, imply the Hammer-Bullen inequality and are given in terms of positive linear functionals.

Finally, in eleventh chapter, several refinements of the Jensen operator inequality are presented, for n-tuples of self-adjoint operators, unital n-tuples of positive linear mappings and real valued continuous convex functions with the condition on the spectra of the operators. Using these refinements, the refinements of inequalities among quasi-arithmetic means, under similar conditions are obtained and, as an application of these results, a refinement of inequalities among power means is additionally provided. The chapter is concluded with the considerations on the converses of the generalized Jensen inequality for a continuous field of self-adjoint operators, a unital field of positive linear mappings and real valued convex functions, where new refined converses are presented using the Mond-Pečarić method improvements.

Since this monograph integrates the whole variety of results that were previously published by different authors in numerous papers, it was practically impossible, despite the great effort, to quite unify the terminology and the notation in the book. Nevertheless, starting with the introductory chapter, but also in each particular chapter, most of the used terminology is defined and explained for the reader's convenience. It is done, of course, on the assumption that the reader is familiar with the basis in real and in functional mathematical analysis.

Authors

Contents

Chapter 1

Basic notation and fundamental results

In this chapter, a brief review of some fundamental results on the topics in the sequel is given and a several basic motivating ideas are presented.

1.1 Jensen's inequality and its variants

Classical Jensen's inequality is the starting point for the variety of the results in this book. Therefore we give its discrete and integral forms in the first place, then some closely related inequalities like Jensen-Steffensen's, Jensen-Mercer's, Hölder's, Hermite-Hadamard's etc., as well as some of their numerous variants and generalizations (e.g. Jessen's inequality and its multidimensional form – McShane's inequality and, furthermore – their generalizations to the convex hulls.) Due to the close relation of Jensen's inequality to the class of convex functions, it is natural to start with the definition of convex functions. More on this topic one can find e.g. in the monographs *Convex Functions, Partial Orderings, and Statistical Applications* by J. E. Pečarić, F. Proschan and Y. L. Tong or *Convex Functions* by A. W. Roberts and D. E. Varberg.

Definition 1.1 *Let I be an interval in $\mathbb{R}$. Function $f : I \to \mathbb{R}$ is said to be a* ***convex function*** *on I if for all $x, y \in I$ and all $\lambda \in [0,1]$*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \tag{1.1}$$

holds. If (1.1) is strict for all $x, y \in I$, $x \neq y$ and for all $\lambda \in (0,1)$, then f is said to be ***strictly convex****. If the inequality in (1.1) is reversed, then f is said to be* ***concave****.*

In geometric terms, a function is convex (concave) if the part of the graph of the function between two points on that graph lies below (above) the chord which connects these two points, or, equivalently, if the epigraph of the function is a convex (concave) set.

In 1934 T. Popoviciu introduced the following generalization of the notion of convexity.

Definition 1.2 *Function $f : [a,b] \to \mathbb{R}$ is said to be* **n*****-convex****, $n \in \mathbb{N}_0$, if for every choice of mutually different points $y_0, \ldots, y_n \in [a,b]$*

$$[y_0, \ldots, y_n; f] \geq 0, \tag{1.2}$$

where $[y_0, \ldots, y_n; f]$ denotes the **n*****-th divided difference*** *of the function f in $y_0, \ldots, y_n$, inductively defined with*

$$[y_i; f] = f(y_i), \quad i = 0, \ldots, n,$$

$$[y_0, \ldots, y_k; f] = \frac{[y_0, \ldots, y_{k-1}; f] - [y_1, \ldots, y_k; f]}{y_0 - y_k}, \quad k = 1, \ldots, n. \tag{1.3}$$

If (1.2) is strict, then f is said to be a ***strictly*** **n*****-convex*** *function. If (1.2) is reversed, then f is said to be an* **n*****-concave*** *function.*

Remark 1.1 According to the definition, the notion of 0-convexity corresponds to nonnegativity of the function f, 1-convexity describes the increasing function f, whereas 2-convexity corresponds to convexity in the sense of Definition 1.1. Namely, $f[x_0, x_1, x_2] \geq 0$ if and only if f is a convex function.

As we previously announced, we finally quote the Jensen inequality which can also be viewed as an alternative way of defining convex functions.

Theorem 1.1 (JENSEN'S INEQUALITY) *Let I be an interval in $\mathbb{R}$, function $f : I \to \mathbb{R}$ be convex on I and let $\mathbf{p} = (p_1, \ldots, p_n)$ be a nonnegative n-tuple such that $P_n = \sum_{i=1}^n p_i > 0$. Then for any $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ the following inequality holds:*

$$f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^n p_i f(x_i). \tag{1.4}$$

If f is strictly convex, then (1.4) is strict, unless $x_i = c$ for all $i \in \{j : p_j > 0\}$. If f is concave, then (1.4) is reversed.

Here we also cite the accompanied reversed inequality for convex functions.

Theorem 1.2 (REVERSED JENSEN'S INEQUALITY) *Let I be an interval in $\mathbb{R}$, function $f: I \to \mathbb{R}$ be convex on I and let $p = (p_1, \dots, p_n)$ be a real n-tuple such that $p_1 > 0$, $p_i \leq 0$, $i = 2, \dots, n$, $P_n = \sum_{i=1}^n p_i > 0$. Then for $x_i \in I$ $(i = 1, \dots, n)$, such that $\frac{1}{P_n}\sum_{i=1}^n p_i x_i \in I$ the following inequality holds:*

$$f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) \geq \frac{1}{P_n}\sum_{i=1}^n p_i f(x_i). \tag{1.5}$$

It is worth mentioning that when Danish mathematician J. L. W. Jensen established the inequality (1.4) in 1905 (see [91]), he originally did it for the class of midconvex (Jensen-convex) functions, that is for the class of functions for which

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}. \tag{1.6}$$

Since comparison of means is at the core of the notion of convexity, let us firstly recall some of the basic related definitions, with an accent on means that we are going to use extensively in the following chapters. For more details on this subject, the reader may be referred e.g. to [165].

Definition 1.3 *Let $M: I \times I \to I$ be a continuous function, where I is an interval in $\mathbb{R}$. If M satisfies the condition*

$$\inf\{s,t\} \leq M(s,t) \leq \sup\{s,t\}, \quad \textit{for all} \quad s,t \in I,$$

then we say that M is a ***mean*** *on the interval I.*

The weight combinations $M(\mathbf{x},\mathbf{p})$, where $\mathbf{x}$ and $\mathbf{p}$ are positive real n-tuples, $\mathbf{x}=(x_1,\dots,x_n)$ and $\mathbf{p} = (p_1, \dots, p_n)$, such that $\sum_{i=1}^n p_i = 1$, can be defined in the same manner, with the condition

$$\inf\{x_1,\dots,x_n\} \leq M(\mathbf{x},\mathbf{p}) \leq \sup\{x_1,\dots,x_n\}, \quad \text{for all} \quad x_1,\dots,x_n \in I.$$

Let $n \in \mathbb{N}$ and let $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{p} = (p_1, \dots, p_n)$ be positive real n-tuples such that $\sum_{i=1}^n p_i = 1$. A **quasi-arithmetic mean** associated to a strictly monotonic continuous function $\varphi: I \to \mathbb{R}$ is defined by

$$M_\varphi(\mathbf{x};\mathbf{p}) = \varphi^{-1}\left(\sum_{i=1}^n p_i \varphi(x_i)\right).$$

For $n \in \mathbb{N}$ and for positive real n-tuples $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{p} = (p_1, \dots, p_n)$, such that $P_n = \sum_{i=1}^n p_i > 0$, a **weight power mean** of order r of $\mathbf{x}$ is defined by

$$M_r(\mathbf{x},\mathbf{p}) = \begin{cases} \left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i^r\right)^{\frac{1}{r}}, & r \in \mathbb{R},\ r \neq 0; \\ \left(\prod_{i=1}^n x_i^{p_i}\right)^{\frac{1}{P_n}}, & r = 0; \\ \min\{x_1,\dots,x_n\}, & r \to -\infty; \\ \max\{x_1,\dots,x_n\}, & r \to \infty. \end{cases} \tag{1.7}$$

Note that for $\varphi(x) = x^r$ the weight power mean can be obtained as a special case of the quasi-arithmetic mean. The following means are thus the special cases of the weight power mean:

(i) $M_1(\mathbf{x},\mathbf{p}) = A_n = \sum_{i=1}^{n} p_i x_i$ ... arithmetic mean,

(ii) $M_0(\mathbf{x},\mathbf{p}) = G_n = \prod_{i=1}^{n} x_i^{p_i}$ ... geometric mean,

(iii) $M_{-1}(\mathbf{x},\mathbf{p}) = H_n = \frac{1}{\sum_{i=1}^{n} \frac{p_i}{x_i}}$ ... harmonic mean.

Stolarsky's means are another class of means which are of interest for us in some of the following chapters. These are defined by

$$S_p(s,t) = \left[\frac{s^p - t^p}{ps - pt}\right]^{\frac{1}{p-1}}, \quad p \neq 0,1.$$

The limiting cases $p = 0$ and $p = 1$ provide the definitions of the **logarithmic** and the **identric** means, respectively:

$$\begin{aligned} S_0(s,t) &= \lim_{p\to 0} S_p(s,t) = \frac{s-t}{\log s - \log t} = L(s,t), \\ S_1(s,t) &= \lim_{p\to 1} S_p(s,t) = \frac{1}{e}\left(\frac{t^t}{s^s}\right)^{\frac{1}{t-s}} = I(s,t). \end{aligned}$$

After this short digression, we go back to analyzing the Jensen inequality. Notice that $M_1(\mathbf{x},\mathbf{p}) = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i$ represents a form of the weight arithmetic mean of $x_1,\ldots,x_n$. Hence Jensen's inequality (1.4) assumes the following form:

$$f(M_1(x,\mathbf{p})) \leq M_1(f(x),\mathbf{p}).$$

There are many integral variants of the Jensen inequality. The proof of the following theorem can be found e.g. in ([177, p. 45]).

Theorem 1.3 (INTEGRAL JENSEN'S INEQUALITY) *Let $(\Omega,\mathscr{A},\mu)$ be a measure space with $0 < \mu(\Omega) < \infty$ and let $\varphi : \Omega \to \mathbb{R}$ be a μ-integrable function. Let $f : I \to \mathbb{R}$ be a convex function such that $Im\,\varphi \subseteq I$ and $f \circ \varphi$ is a μ-integrable function. Then*

$$f\left(\frac{1}{\mu(\Omega)}\int_{\Omega} \varphi(x)d\mu(x)\right) \leq \frac{1}{\mu(\Omega)}\int_{\Omega} f(\varphi(x))d\mu(x), \tag{1.8}$$

or: $f(M_1(\varphi;\mu)) \leq M_1(f\circ\varphi;\mu)$, where $M_1(\varphi;\mu) = \frac{1}{\mu(\Omega)}\int_{\Omega} \varphi(x)d\mu(x)$, $M_1(\varphi;\mu) \in I$. If f is a strictly convex function, then (1.8) *becomes equality if and only if φ is a constant μ – almost everywhere on Ω. If f is a concave function, then* (1.8) *is reversed.*

Now the discrete Jensen inequality (1.4) is obtained by means of the discrete measure μ on $\Omega = \{1, \ldots, n\}$, with $\mu(\{i\}) = p_i$, and $\varphi(i) = x_i$.

Another integral variant of Jensen's inequality is based on the notion of the Riemann-Stieltjes integral, for which a brief outline is given here. One can find more information on the Riemann-Stieltjes integral in ([195]).

Let $[a,b] \subset \mathbb{R}$ and let $f, \varphi : [a,b] \to \mathbb{R}$ be bounded functions. To each decomposition $D = \{t_0, t_1, \ldots, t_n\}$ of $[a,b]$, such that $t_0 < t_1 < \cdots < t_{n-1} < t_n$ Stieltjes' integral sum

$$\sigma(f, \varphi; D, \gamma_1, \ldots, \gamma_n) = \sum_{i=1}^{n} f(\gamma_i)(\varphi(t_i) - \varphi(t_{i-1}))$$

is assigned, where γ_i are from $[t_{i-1}, t_i]$, $i \in \{1, 2, \ldots, n\}$. These sums will be denoted with $\sigma(f, \varphi; D)$ in the sequel.

Definition 1.4 *Let $f, \varphi : [a,b] \to \mathbb{R}$ be bounded functions. Function f is said to be* ***Riemann-Stieltjes integrable*** *regarding function φ if there exists $I \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exits a decomposition D_0 of $[a,b]$, such that for every decomposition $D \supseteq D_0$ of $[a,b]$ and for every sum $\sigma(f, \varphi; D)$*

$$|\sigma(f, \varphi; D) - I| < \varepsilon$$

holds. The unique number I is the ***Riemann-Stieltjes integral*** *of the function f regarding the function φ and is denoted with*

$$\int_a^b f(t)\, d\varphi(t). \tag{1.9}$$

The Riemann-Stieltjes integral is a generalization of the Riemann integral and coincides with it when φ is an identity.

The notion of the Riemann-Stieltjes integral is narrowly related to the class of the functions of bounded variation.

Let $\varphi : [a,b] \to \mathbb{R}$ be a real function. To each decomposition $D = \{t_0, t_1, \ldots, t_n\}$ of $[a,b]$ such that

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b \tag{1.10}$$

belongs the sum

$$V(\varphi; D) = \sum_{i=1}^{n} |\varphi(t_i) - \varphi(t_{i-1})|,$$

which is said to be a **variation of the function** φ regarding decomposition D.

Definition 1.5 *Function $\varphi : [a,b] \to \mathbb{R}$ is said to be a* ***function of bounded variation*** *if the set $\{V(\varphi; D) : D \in \mathscr{D}\}$ is bounded, where $\mathscr{D}$ is a family of all decompositions of the interval* (1.10). *Number*

$$V(\varphi) = \sup\{V(\varphi; D) : D \in \mathscr{D}\}$$

is called a ***total variation*** *of the function φ.*

Theorem 1.4 *The following assertions hold:*

a) Every monotonic function $f:[a,b]\to\mathbb{R}$ is a function of bounded variation on $[a,b]$ and $V_a^b(f)=|f(b)-f(a)|$;

b) Every function of bounded variation is a bounded function;

c) If f and g are functions of bounded variation on $[a,b]$, then $f+g$ is a function of bounded variation on $[a,b]$.

Theorem 1.5 *Let f be a function of bounded variation on $[a,b]$. Then*

a) f has at most countably many of step discontinuities on $[a,b]$;

b) f can be presented as $f=s_f+g$, where step-function s_f and continuous function g are both functions of bounded variation on $[a,b]$.

Theorem 1.6 *Let $f:[a,b]\to\mathbb{R}$ be a continuous function and $\varphi:[a,b]\to\mathbb{R}$ be a function of bounded variation. Then there exists the Riemann-Stieltjes integral* (1.9) *and*

$$\left|\int_a^b f(t)\,d\varphi(t)\right|\le V(\varphi)\cdot\max_{t\in[a,b]}|f(t)|.$$

Regarding the Riemann-Stieltjes integral, we now induce yet another integral form of Jensen's inequality, dealt with in one of the following chapters (for more details on this topic, the reader is referred to [177, p. 58]). It reads as follows:

$$f\left(\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta g(t)\mathrm{d}\lambda(t)\right)\le\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta f(g(t))\mathrm{d}\lambda(t),\tag{1.11}$$

where $g:[\alpha,\beta]\to(a,b)$ is a continuous function, $-\infty<\alpha<\beta<\infty$, $-\infty\le a<b\le\infty$, $f:(a,b)\to\mathbb{R}$ is a convex function and $\lambda:[\alpha,\beta]\to\mathbb{R}$ is an increasing function, such that $\lambda(\beta)\ne\lambda(\alpha)$.

In 1919 J. F. Steffensen proved that inequality (1.4) held when the condition on non-negativity of the n-tuple $\mathbf{p}$ was relaxed, but with simultaneously restricted choice on $\mathbf{x}$. In a more general form, Steffensen's theorem reads as follows.

Theorem 1.7 (JENSEN-STEFFENSEN'S INEQUALITY) *If $f:I\to\mathbb{R}$, $I\subseteq\mathbb{R}$, is a convex function, $\mathbf{x}\in I^n$ is a monotonic n-tuple and $\mathbf{p}$ is a real n-tuple such that*

$$P_n>0\quad and\quad 0\le P_k\le P_n,\quad 1\le k\le n-1,$$

where $P_k=\sum_{i=1}^k p_i$, $k=1,\dots,n$, then inequality (1.4) *holds.*

Recently, J. E. Pečarić provided yet another proof of Theorem 1.7 and one can find it in [177, p. 57]. Integral variants of the previous theorem will be discussed in one of the following chapters.

In 2003 A. McD. Mercer proved yet another variant of Jensen's inequality (see [134]). In a slightly generalized form his theorem is stated as below.

Theorem 1.8 (JENSEN-MERCER'S INEQUALITY) *Let $[a,b]$ be an interval in $\mathbb{R}$ and $\mathbf{x} = (x_1,\ldots,x_n) \in [a,b]^n$. Suppose $\mathbf{p} = (p_1,\ldots,p_n)$ is a nonnegative real n-tuple such that $P_n = \sum_{i=1}^{n} p_i > 0$. If $f : [a,b] \to \mathbb{R}$ is a convex function, then the following inequality holds:*

$$f\left(a+b-\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq f(a)+f(b)-\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i). \tag{1.12}$$

What is of an additional interest here is that in 2005 S. Abramovich *et.al.* in [2] proved another variant of Jensen-Steffensen's inequality, which included Mercer's original result as its special case. Jensen-Mercer's inequality was proved under the Steffensen's conditions as in Theorem 1.7.

Theorem 1.9 (see [2]) *Let $[a,b]$ be an interval in $\mathbb{R}$ and let $\mathbf{x} \in [a,b]^n$ be a monotonic n-tuple. Suppose $\mathbf{p}$ is a real n-tuple, such that*

$$P_n > 0 \quad and \quad 0 \leq P_k \leq P_n, \quad 1 \leq k \leq n-1, \tag{1.13}$$

where $P_k = \sum_{i=1}^{k} p_i$, $k = 1,2,\ldots,n$. If $f : [a,b] \to \mathbb{R}$ is a convex function, then inequality (1.12) holds.

An integral variant of the previous theorem will be discussed in one of the following chapters.

Strongly related to Jensen's inequality is the converse Jensen inequality. Although there are more variants of its converses, some of which are going to be explored in one of the following chapters, here we single out the Lah-Ribarič inequality as one of the most significant ones (see [125] or, for example, [151, p. 9]).

Theorem 1.10 (LAH-RIBARIČ) *Let $f\colon [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$, $x_i \in [a,b]$, $p_i \geq 0$, $i = 1,\ldots,n$ and $\sum\limits_{i=1}^{n} p_i = 1$. Then the following inequality holds:*

$$\sum_{i=1}^{n} p_i f(x_i) \leq \frac{b-\sum_{i=1}^{n} p_i x_i}{b-a} f(a) + \frac{\sum_{i=1}^{n} p_i x_i - a}{b-a} f(b). \tag{1.14}$$

If f is strictly convex, then (1.14) is strict unless $x_i \in \{a,b\}$, for all $i \in \{j : p_j > 0\}$.

In 1931 Jensen's inequality (1.4) was investigated by B. C. Jessen, who generalized it by means of the positive linear functional acting on a space of real functions.

Let E be a nonempty set and L a linear class of functions $f : E \to \mathbb{R}$ which possesses the following properties:

L1: If $f,g \in L$, then $\alpha f + \beta g \in L$, for all α, $\beta \in \mathbb{R}$;

L2: $1 \in L$, that is, if $f(x) = 1$, $x \in E$, then $f \in L$.

We consider positive linear functionals $A : L \to \mathbb{R}$, or in other words we assume:

A1: $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$, for $f,g \in L$ and $\alpha,\beta \in \mathbb{R}$;

A2: If $f(x) \geq 0$ for all $x \in E$, then $A(f) \geq 0$.

If additionally the condition

A3: $A(1) = 1$

is satisfied, we say that A is a normalized positive linear functional or that $A(f)$ is a linear mean on L.

In the described environment we cite the Jessen's result.

Theorem 1.11 (JESSEN'S INEQUALITY) *Let E be a nonempty set and let L be a linear class of functions $f : E \to \mathbb{R}$ which possesses the properties L1 and L2. Suppose $\Phi : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$ is a continuous and convex function. If $A : L \to \mathbb{R}$ is a normalized positive linear functional, then for all $f \in L$, such that $\Phi(f) \in L$ we have $A(f) \in I$ and the following inequality holds:*

$$\Phi(A(f)) \leq A(\Phi(f)). \tag{1.15}$$

In 1937 E. J. McShane gave an important generalization of Jessen's inequality, in his paper [133]. He observed Φ in (1.15) as a function of several variables. Namely, vector function $\mathbf{f} : E \to \mathbb{R}^n$ was defined with $\mathbf{f}(x) = (f_1(x), \dots, f_n(x))$, where $f_i \in L$, $i = 1, \dots, n$. Such multidimensional generalization of (1.15) is described in the following theorem.

Theorem 1.12 (MCSHANE'S INEQUALITY) *Let E be a nonempty set and let L be a linear class of real functions defined on E, which possesses the properties L1 and L2. Let $K \subseteq \mathbb{R}^n$ be a closed convex set and let $\Phi : K \to \mathbb{R}$ be a continuous convex function. If $A : L \to \mathbb{R}$ is a normalized positive linear functional, then for all functions $\mathbf{f} = (f_1, \dots, f_n) \in L^n$, such that $\Phi(\mathbf{f}) \in L$ we have $A(\mathbf{f}) \in K$ and the following inequality holds:*

$$\Phi(A(\mathbf{f})) \leq A(\Phi(\mathbf{f})). \tag{1.16}$$

In the previous theorem, acting of the functional A to the vector function $\mathbf{f} = (f_1, \dots, f_n)$ is defined with $A(\mathbf{f}) = (A(f_1), \dots, A(f_n))$.

One can find the proofs of the theorems 1.11 and 1.12 in [177, from p. 47].

When dealing with the positive normalized linear functionals, we need to mention that in 1985 J. Pečarić and P. R. Beesack presented a corresponding generalization of the Theorem 1.10. Namely, they proved that for a convex function f defined on an interval $I = [m, M] \subset \mathbb{R}$, $(-\infty < m < M < \infty)$ and for all $g \in L$ such that $g(E) \subset I$ and $f(g) \in L$ the following inequality holds:

$$A(f(g)) \leq \frac{M - A(g)}{M - m} f(m) + \frac{A(g) - m}{M - m} f(M). \tag{1.17}$$

As for the generalized forms of the Jensen-type inequalities in this setting, let us mention here the generalization of the Jensen-Mercer inequality (1.12) which involves positive normalized linear functionals and is called the Jessen-Mercer inequality.

Theorem 1.13 (JESSEN-MERCER'S INEQUALITY) *Let L satisfy L1, L2 on a nonempty set E, and let A be a positive normalized linear functional. If φ is a continuous convex* *function on* $[m,M]$, *then for all* $f \in L$ *such that* $\varphi(f), \varphi(m+M-f) \in L$ (*so that* $m \leq f(t) \leq M$ *for all* $t \in E$), *we have*

$$\varphi(m+M-A(f)) \leq \varphi(m)+\varphi(M)-A(\varphi(f)). \tag{1.18}$$

If the function φ *is concave, then* (1.18) *is reversed.*

In some of the following chapters we deal with the generalizations of Jensen's and related inequalities on convex hulls in $\mathbb{R}^k$ and, as a special case, on k-simplices in $\mathbb{R}^k$. For that purpose we define the mentioned notions.

The **convex hull** of the vectors $x_1,\ldots,x_n \in \mathbb{R}^k$ is the set

$$K = co(\{x_1,\ldots,x_n\}) = \left\{ \sum_{i=1}^{n} \alpha_i x_i | \alpha_i \in \mathbb{R}, \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$

Barycentric coordinates over K are continuous real functions $\lambda_1,\ldots,\lambda_n$ on K with the following properties:

$$\begin{aligned}
&\lambda_i(x) \geq 0, \quad i = 1,\ldots,n, \\
&\sum_{i=1}^{n} \lambda_i(x) = 1, \\
&x = \sum_{i=1}^{n} \lambda_i(x) x_i.
\end{aligned} \tag{1.19}$$

The **k-simplex** $S = [v_1,\ldots,v_{k+1}]$ is a convex hull of its vertices $v_1,\ldots,v_{k+1} \in \mathbb{R}^k$, where $v_2-v_1,\ldots,v_{k+1}-v_1 \in \mathbb{R}^k$ are linearly independent.

As an illustrative example serves a generalization of the result (1.17) that was obtained in [88], where for $x_1,\ldots,x_n \in \mathbb{R}^k$, $K = co(\{x_1,\ldots,x_n\})$, as well as for a convex function f on K, barycentric coordinates $\lambda_1,\ldots,\lambda_n$ over K and for all $g \in L^k$ such that $g(E) \subset K$ and $f(g), \lambda_i(g) \in L$, $i = 1,\ldots,n$, the inequality

$$A(f(g)) \leq \sum_{i=1}^{n} A(\lambda_i(g)) f(x_i) \tag{1.20}$$

holds.

1.1.1 n-exponentially and exponentially convex functions

Notions of n-exponentially and exponentially convex functions are going to be explored in some of the following chapters. For that purpose we define them here and provide some of their characterizations.

Definition 1.6 *A function* $\psi : I \to \mathbb{R}$ *is* **n-*exponentially convex in the Jensen sense*** *on* I *if*

$$\sum_{i,j=1}^{n} \xi_i \xi_j \psi\left(\frac{x_i+x_j}{2}\right) \geq 0$$

holds for all choices $\xi_i \in \mathbb{R}$ and $x_i \in I$, $i = 1, \dots, n$.

*A function $\psi\colon I \to \mathbb{R}$ is **n-exponentially convex** if it is n-exponentially convex in the Jensen sense and is continuous on I.*

Remark 1.2 It is clear from the definition that 1-exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $k \leq n$.

The following proposition follows by the definition of positive semi-definite matrices and by utilizing some basic linear algebra.

Proposition 1.1 *If ψ is an n-exponentially convex function in the Jensen sense, then the matrix $\left[\psi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k}$ is a positive semi-definite matrix for all $k \in \mathbb{N}$, $k \leq n$. Particularly,* $\det\left[\psi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k} \geq 0$, *for all $k \in \mathbb{N}$, $k \leq n$.*

Definition 1.7 *A function $\psi\colon I \to \mathbb{R}$ is **exponentially convex in the Jensen sense** on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.*

*A function $\psi\colon I \to \mathbb{R}$ is **exponentially convex** if it is exponentially convex in the Jensen sense and is continuous on I.*

Definition 1.8 *A positive function ψ is said to be **logarithmically convex** (or **log-convex**) on an interval $I \subseteq \mathbb{R}$ if* $\log\psi$ *is a convex function on I, or equivalently, if*

$$\psi(\lambda x + (1-\lambda)y) \leq \psi^{\lambda}(x)\psi^{1-\lambda}(y)$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

*A positive function ψ is **log-convex in the Jensen sense** if*

$$\psi^2\left(\frac{x+y}{2}\right) \leq \psi(x)\psi(y)$$

holds for all $x, y \in I$, i.e., if $\log\psi$ *is convex in the Jensen sense.*

Remark 1.3 It is known (and easy to show) that $\psi : I \to \mathbb{R}$ is a log-convex in the Jensen sense if and only if

$$\alpha^2\psi(x) + 2\alpha\beta\psi\left(\frac{x+y}{2}\right) + \beta^2\psi(y) \geq 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a function is log-convex in the Jensen-sense if and only if it is 2-exponentially convex in the Jensen sense.

Also, using basic convexity theory it follows that a function is log-convex if and only if it is 2-exponentially convex.

We will also need the following result (see for example [177]).

Proposition 1.2 *If Ψ is a convex function on an interval I and if $x_1 \le y_1$, $x_2 \le y_2$, $x_1 \ne x_2$, $y_1 \ne y_2$, then the following inequality is valid:*

$$\frac{\Psi(x_2)-\Psi(x_1)}{x_2-x_1} \le \frac{\Psi(y_2)-\Psi(y_1)}{y_2-y_1}. \tag{1.21}$$

If the function Ψ is concave, the inequality reverses.

When dealing with functions with different degree of smoothness, divided differences are found to be very useful.

Remark 1.4 Definition 1.2 provided the notion of the second order divided difference, needed in the sequel. The value $[y_0,y_1,y_2;f]$ is independent of the order of the points y_0,y_1 and y_2. This definition may be extended to include the case in which some or all the points coincide. Namely, taking the limit $y_1 \to y_0$, we get

$$\lim_{y_1\to y_0}[y_0,y_1,y_2;f] = [y_0,y_0,y_2;f] = \frac{f(y_2)-f(y_0)-f'(y_0)(y_2-y_0)}{(y_2-y_0)^2}, \quad y_2 \ne y_0,$$

provided f' exists, and furthermore, taking the limits $y_i \to y_0$, $i=1,2$ we get

$$\lim_{y_2\to y_0}\lim_{y_1\to y_0}[y_0,y_1,y_2;f] = [y_0,y_0,y_0;f] = \frac{f''(y_0)}{2},$$

provided that f'' exists.

We will use an idea from [90] to give an elegant method of producing n-exponentially convex functions and exponentially convex functions, applying some functionals to a given family with the same property.

1.2 Some classical inequalities

In this section we give an outline of some important classical inequalities to which we will often refer throughout the following chapters. Namely, refinements and converses of arithmetic-geometric, geometric-harmonic inequalities, as well as Young's, Hölder's, Minkowski's, Hilbert's and some other classical inequalities are going to be presented throughout this monograph. The reader can find more details on these topics, as well as the results with the corresponding proofs e.g. in [151], [165] or [177].

Theorem 1.14 (WEIGHT ARITHMETIC-GEOMETRIC MEAN INEQUALITY) *Let $n \in \mathbb{N}$, $n \ge 2$, $x_1,\dots,x_n > 0$, $\alpha_1,\dots,\alpha_n \in (0,1)$ such that $\sum_{i=1}^n \alpha_i = 1$. Then the inequality*

$$\sum_{i=1}^n \alpha_i x_i \ge \prod_{i=1}^n x_i^{\alpha_i} \tag{1.22}$$

holds. Equality holds for $x_1 = \cdots = x_n$.

Corollary 1.1 (WEIGHT GEOMETRIC-HARMONIC MEAN INEQUALITY) *Let $n \in \mathbb{N}$, $n \geq 2$, $x_1, \dots, x_n > 0$, $\alpha_1, \dots, \alpha_n \in (0,1)$ such that $\sum_{i=1}^n \alpha_i = 1$. Then*

$$\prod_{i=1}^n x_i^{\alpha_i} \geq \frac{1}{\sum_{i=1}^n \frac{\alpha_i}{x_i}} \tag{1.23}$$

holds. Equality holds for $x_1 = \cdots = x_n$.

Remark 1.5 From (1.22) and (1.23) and for $\alpha_1 = \cdots = \alpha_n = \frac{1}{n}$ we get the classical arithmetic-geometric-harmonic mean inequality:

$$\frac{1}{n}\sum_{i=1}^n x_i \geq \left(\prod_{i=1}^n x^i\right)^{\frac{1}{n}} \geq \frac{n}{\sum_{i=1}^n \frac{1}{x_i}}, \tag{1.24}$$

with corresponding equalities obtained for $x_1 = \cdots = x_n$.

Family of so called Heinz means, denoted with H_ν interpolates arithmetic and geometric mean of nonnegative real numbers a and b and is defined by

$$H_\nu(a,b) = \frac{a^\nu b^{1-\nu} + a^{1-\nu} b^\nu}{2}, \qquad \nu \in [0,1]. \tag{1.25}$$

Obviously,

$$\sqrt{ab} \leq H_\nu(a,b) \leq \frac{a+b}{2}. \tag{1.26}$$

Following inequality is closely related to the arithmetic-geometric mean inequality.

Theorem 1.15 (YOUNG'S INEQUALITY) *Let $f : [0,\infty] \to [0,\infty]$ be an increasing continuous function such that $f(0) = 0$ and $\lim_{x\to\infty} f(x) = \infty$. Then for all $a, b \geq 0$*

$$ab \leq \int_0^a f(x)\,dx + \int_0^b f^{-1}(x)\,dx, \tag{1.27}$$

holds. Equality holds if and only if $b = f(a)$.

Remark 1.6 If the function f in Theorem 1.15 is defined with $f(x) = x^{p-1}$, $p > 1$, we get

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \tag{1.28}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, and the connection with the arithmetic-geometric mean inequality becomes obvious.

Young's inequality is a starting point for Hölder's inequality.

Theorem 1.16 (DISCRETE HÖLDER'S INEQUALITY) *Let $a_i, b_i, i = 1, \ldots, n$ be complex numbers and for $p > 1$ let q be defined by $\frac{1}{p} + \frac{1}{q} = 1$. Then for all $n \in \mathbb{N}$ the following inequality holds:*

$$\sum_{i=1}^{n} |a_i b_i| \le \left(\sum_{i=1}^{n} |a_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |b_i|^q \right)^{\frac{1}{q}}. \tag{1.29}$$

Equality in (1.29) holds if and only if the n-tuples $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are proportional.

Remark 1.7 For $p = 2$ inequality (1.29) is the well known Cauchy-Schwarz inequality.

Hölder's inequality can be observed in a more general environment, involving the positive linear functionals acting on the space of real functions. For that purpose we refer to the notation induced in the previous Section 1.1 and cite the following result.

Theorem 1.17 *Let E be a nonempty set and L be a linear class of real functions defined on E, which satisfies properties L1 and L2. Suppose $p_i > 1$, $i = 1, \ldots, n$ are such that $\sum_{i=1}^{n} \frac{1}{p_i} = 1$. Let $f_i \in L$, $i = 1, \ldots, n$ be nonnegative functions, such that $\prod_{i=1}^{n} f_i^{\frac{1}{p_i}} \in L$ is a nonnegative function. If $A : L \to \mathbb{R}$ is a positive linear functional, then the following inequality holds:*

$$A\left(\prod_{i=1}^{n} f_i^{\frac{1}{p_i}} \right) \le \prod_{i=1}^{n} A^{\frac{1}{p_i}}(f_i). \tag{1.30}$$

The Minkowski inequality can also be observed in the discrete and in a more general setting.

Theorem 1.18 (DISCRETE MINKOWSKI'S INEQUALITY) *Let $a_i, b_i, i = 1, \ldots, n$ be complex numbers and let $p \ge 1$. Then for all $n \in \mathbb{N}$ the following inequality holds:*

$$\left(\sum_{i=1}^{n} |a_i + b_i|^p \right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{n} |a_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |b_i|^p \right)^{\frac{1}{p}}. \tag{1.31}$$

Theorem 1.19 *Let E be a nonempty set and L be a linear class of real functions defined on E, which satisfies properties L1 and L2. Suppose $p \ge 1$. Let $f_i \in L$, $i = 1, \ldots, n$ be nonnegative functions, such that f_i^p, $(\sum_{i=1}^{n} f_i)^p \in L$. If $A : L \to \mathbb{R}$ is a positive linear functional, then the following inequality holds:*

$$A^{\frac{1}{p}}\left[\left(\sum_{i=1}^{n} f_i \right)^p \right] \le \sum_{i=1}^{n} A^{\frac{1}{p}}(f_i^p). \tag{1.32}$$

In the early years of the last century two fundamental inequalities were proved. The first one was discrete.

Theorem 1.20 *Let $(a_m)_{m\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be nonnegative real sequences such that $\sum_{m=1}^{\infty} a_m^p < \infty$ and $\sum_{n=1}^{\infty} b_n^p < \infty$. Suppose that for $p > 1$ q is defined by $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\frac{\pi}{p})}\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}, \tag{1.33}$$

unless $(a_m)_{m\in\mathbb{N}}$ or $(b_n)_{n\in\mathbb{N}}$ is a null-sequence.

The second inequality was obtained in the integral form.

Theorem 1.21 *Let f and g be nonnegative integrable functions such that $\int_0^\infty f^p(x)dx < \infty$ and $\int_0^\infty g^q(y)dy < \infty$. Suppose that for $p > 1$ is q defined by $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\frac{\pi}{p})}\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}, \tag{1.34}$$

unless f or g is a null-function.

The bilinear form $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}$ from (1.33) was in the first place investigated and estimated by D. Hilbert in the nineteenth century, and thus both inequalities were named after him as Hilbert's inequalities. Their significance became obvious later in the 20th century, for its many generalizations, improvements and various proofs have been given by numerous famous mathematicians, for example, L. Fejér, G. H. Hardy, J. Littlewood, G. Polya, I. Schur and many others. The detailed approach to this subject was given in the monograph [83]. Recent results on Hilbert's inequality, including the following one that unifies its discrete and the integral case are presented in the monograph [122].

Theorem 1.22 *Let $\Omega \subseteq (0,\infty)$. Suppose p and q are conjugate parameters, i.e. $\frac{1}{p}+\frac{1}{q}=1$ such that $p>1$ and let $K\colon \Omega\times\Omega\to\mathbb{R}$, $\varphi\colon \Omega\to\mathbb{R}$ and $\psi\colon\Omega\to\mathbb{R}$ are nonnegative measurable functions. Let $F,G\colon\Omega\to\mathbb{R}$ be defined by*

$$F(x) = \int_\Omega \frac{K(x,y)}{\psi^p(y)}d\mu_2(y) \quad \text{and} \quad G(y) = \int_\Omega \frac{K(x,y)}{\varphi^q(x)}d\mu_1(x), \tag{1.35}$$

where μ_1 and μ_2 are σ-finite measures. Then for any choice of nonnegative measurable functions $f,g\colon\Omega\to\mathbb{R}$ the following inequality holds:

$$\begin{aligned}&\int_\Omega\int_\Omega K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y)\\ &\le \left[\int_\Omega \varphi^p(x)F(x)f^p(x)d\mu_1(x)\right]^{\frac{1}{p}}\left[\int_\Omega \psi^q(y)G(y)g^q(y)d\mu_2(y)\right]^{\frac{1}{q}}.\end{aligned} \tag{1.36}$$

Many important inequalities are established for the class of convex functions, but one of the most famous is the Hermite-Hadamard inequality. This double inequality, which was first discovered by C. Hermite in 1881, is stated as follows (see for example [177, p. 137]).

Theorem 1.23 (HERMITE-HADAMARD) *Let f be a convex function on $[a,b] \subset \mathbb{R}$, where $a < b$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1.37}$$

This result was later incorrectly attributed to J. S. Hadamard who apparently was not aware of Hermite's discovery and today, when relating to (1.37), we use both names. It is interesting to mention that the term *convex* also stems from a result obtained by Hermite in 1881.

Note that the first inequality in (1.37) is stronger than the second one.

The following inequality will be referred to as Hammer-Bullen's in the sequel. Its geometric proof was given in [79] and the analytic one in [46] (see also [177, p. 140]).

Theorem 1.24 (HAMMER-BULLEN) *Let f be a convex function on $[a,b] \subset \mathbb{R}$, where $a < b$. Then*

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x. \tag{1.38}$$

For the sake of further considerations throughout this monograph, we conclude this section with recalling two famous theorems: the Lagrange and the Cauchy mean value theorems, where the first one is a special case of the latter.

Theorem 1.25 (LAGRANGE MEAN VALUE THEOREM) *If a function f is continuous on the closed interval $[a,b]$, where $a < b$ and differentiable on the open interval (a,b), then there exists point c in (a,b) such that*

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Theorem 1.26 (CAUCHY MEAN VALUE THEOREM) *If f and g are continuous functions on the closed interval $[a,b]$, where $a < b$, if $g(a) \neq g(b)$, and both functions are differentiable on the open interval (a,b), then there exists at least one point c in (a,b), $a < c < b$, such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

1.3 C*-algebras. Operators on a Hilbert space

Dealing with the Jensen-type functionals whose real arguments are substituted with the bounded self-adjoint operators acting on a Hilbert space, the ones that we are also going to investigate in the sequel, requires additional overview of some basic notions and results. These concern C*-algebras and the theory on operators on a Hilbert space. For a more detailed analysis, the reader is referred to e.g. [15] and [74].

Recall that a linear space $\mathscr{A}$ over the field $\mathbf{F}$, together with the multiplication $(x,y) \mapsto xy$, $x,y \in \mathscr{A}$, constitutes an *algebra* if the mapping $(x,y) \mapsto xy$ possesses the following properties:

$$(xy)z = x(yz), \quad x(y+z) = xy + xz, \quad (x+y)z = xz + yz,$$
$$(\alpha x)y = \alpha(xy) = x(\alpha y),$$

for all $\alpha \in \mathbf{F}$ and for all $x,y \in \mathscr{A}$. If $\mathbf{F} = \mathbb{R}$, algebra is called a real algebra and if $\mathbf{F} = \mathbb{C}$, we call it a complex algebra. If there is an element 1 in $\mathscr{A}$ such that $x \cdot 1 = 1 \cdot x = x$, for all $x \in \mathscr{A}$, then we say that $\mathscr{A}$ is an algebra with the unit 1.

Mapping $x \mapsto \|x\|$ defined on an algebra $\mathscr{A}$ and with the values in $\mathbb{R}$ is a *norm* on $\mathscr{A}$ if the following conditions are satisfied: $x \mapsto \|x\|$ is a norm on the linear space of $\mathscr{A}$, $\|xy\| \leq \|x\|\|y\|$, $x,y \in \mathscr{A}$, and, if $\mathscr{A}$ has a unit 1, then $\|1\| = 1$. Ordered pair $(\mathscr{A}, \|\cdot\|)$ is called a *normed algebra*. Normed algebra is called a *Banach algebra* if the normed space of $\mathscr{A}$ is a Banach space, that is a complete normed vector space.

Involution on algebra $\mathscr{A}$ is a conjugate linear mapping $x \mapsto x^*$ on $\mathscr{A}$, such that $x^{**} = x$ and $(xy)^* = y^*x^*$, for all $x,y \in \mathscr{A}$. Ordered pair $(\mathscr{A}, *)$ is called an *involutive algebra* or a **-algebra*. Now, a *-algebra $\mathscr{A}$ equipped with the complete submultiplicative norm such that $\|x^*\| = \|x\|$, for all $x \in \mathscr{A}$ constitutes a *Banach *-algebra*.

Finally, Banach *-algebra is a *C*-algebra* if for every $x \in \mathscr{A}$, C*-identity $\|x^*x\| = \|x\|^2$ holds. We say that a C*-algebra is unital if it contains the unit 1. The element $a \in \mathscr{A}$ is called: self-adjoint if $a = a^*$, normal if $a^*a = aa^*$ and is called unitary (in the unital algebra) if $a^*a = aa^* = 1$. A standard example of the unital C*-algebra is the scalar field $\mathbb{C}$, where the involution is represented as the complex conjugation.

The environment that is of an interest for us in the sequel is the one of the bounded linear operators on a Hilbert space, which we therefore specify here.

A Hilbert space H is a (complex) vector space H that is complete regarding the metric $d(x,y) = \|x-y\|$ defined by the norm which is induced by an inner product $\|x\| := \langle x,x\rangle^{\frac{1}{2}}$. In other words, Hilbert space is a complete unitary space.

A linear operator A on a Hilbert space H is bounded if

$$\|A\| := \sup\{\|Ax\| : \|x\| \leq 1, x \in H\} < \infty.$$

We say that $\|A\|$ is an operator norm of A. The sum and the composition of the bounded linear operators is again a bounded linear operator. The mapping $(x,y) \mapsto \langle Ax,y\rangle$ is linear and continuous and according to the Riesz representation theorem (see e.g. [16]), it follows that

$$\langle x, A^*y\rangle = \langle Ax, y\rangle, \quad \text{for} \quad A^*y \in H.$$

Thus another bounded linear operator A^* on H is defined and $A^{**} = A$. Bounded (hence continuous) linear operators on H together with an operator norm and the corresponding involution constitute a C*-algebra denoted with $\mathscr{B}(H)$. *Spectrum* of an operator $A \in \mathscr{B}(H)$ is defined with

$$\sigma(A) := \{\lambda \in \mathbb{C} : A - \lambda 1_H \, \text{not invertible in} \, \mathscr{B}(H)\},$$

where 1_H is a unit operator on H. This set is non-empty and compact for operators in $\mathscr{B}(H)$. A bounded linear operator A on a Hilbert space H is self-adjoint if $A = A^*$. The following characterization holds: $A \in \mathscr{B}(H)$ is self-adjoint if and only if $\langle Ax, x\rangle \in \mathbb{R}$, for all $x \in H$.

Bounded self-adjoint operators constitute the subspace of the C*-algebra of all bounded linear operators and is denoted with $\mathscr{B}_h(H)$. We induce the partial ordering in $\mathscr{B}_h(H)$.

Definition 1.9 *Operator $A \in \mathscr{B}_h(H)$ is* ***positive semidefinite*** *or* ***positive*** *($A \geq 0$) if $\langle Ax, x\rangle \geq 0$, for all $x \in H$. Positive semidefinite operator $A \in \mathscr{B}_h(H)$ is* ***positive definite*** *or* ***strictly positive*** *($A > 0$) if there is a real number $m > 0$ such that $\langle Ax, x\rangle \geq m\langle x, x\rangle$, $x \in H$, that is $A \geq m1_H$. For operators $A, B \in \mathscr{B}_h(H)$ is $B \geq A$ or $A \leq B$ if $B - A \geq 0$, that is if $\langle Bx, x\rangle \geq \langle Ax, x\rangle$, for all $x \in H$. Such ordering is called an* ***operator ordering****. In particular, for scalars m and M is $m1_H \leq A \leq M1_H$ if $m \leq \langle Ax, x\rangle \leq M$, for every unit vector $x \in H$.*

If for a self-adjoint operator A is $\sigma(A) \subseteq [m, M]$, then $m1_H \leq A \leq M1_H$.

The set of all positive operators in $\mathscr{B}_h(H)$ is a convex cone in $\mathscr{B}_h(H)$ which defines the order "$\leq$" on $\mathscr{B}_h(H)$. This convex cone is denoted with $\mathscr{B}^+(H)$. The set of all strictly positive (or positive invertible) operators in $\mathscr{B}_h(H)$ is denoted with $\mathscr{B}^{++}(H)$.

The *continuous functional calculus* is based on the Gelfand mapping Φ which is a *-isometric isomorphism from the space $C(\sigma(A))$ of all continuous functions that act on the spectrum $\sigma(A)$ of a self-adjoint operator A on H, onto the C*-algebra $C^*(A)$ generated with A and the identity. This mapping has the following properties:

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$,

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\overline{f}) = \Phi(f)^*$,

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in \sigma(A)} |f(t)|$,

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$,

$f, g \in C(\sigma(A))$, $\alpha, \beta \in \mathbb{C}$.

Thus the continuous functional calculus

$$f(A) = \Phi(f) \tag{1.39}$$

provides acting of the function $f \in C(\sigma(A))$ on the self-adjoint operator A itself.

In that sense, if A is a positive operator and $f_{1/2}(t) = \sqrt{t}$, then $A^{1/2} = f_{1/2}(A)$.

Furthermore, if A is a self-adjoint operator and f is a real valued continuous function defined on $\sigma(A)$ such that $f(t) \geq 0$, for all $t \in \sigma(A)$, then $f(A) \geq 0$, i.e. $f(A)$ is a positive operator.

The following order preserving property is a consequence of the continuous functional calculus.

Lemma 1.1 *Let $A \in \mathscr{B}_h(H)$ and let $f, g \in C(\sigma(A))$.*

$$\text{If } f(t) \geq g(t), \text{ for all } t \in \sigma(A), \text{ then } f(A) \geq g(A). \tag{1.40}$$

Additionally, $f(A) = g(A)$ if and only if $f(t) = g(t)$, for all $t \in \sigma(A)$.

1.4 Operator monotone and operator convex functions

Denote with $C([m,M])$ the set of all real valued continuous functions on $[m,M] \subset \mathbb{R}$.

Definition 1.10 *We say that the function $f \in C([m,M])$ is* ***operator monotone*** *if it preserves the operator order: if $A \leq B$ then $f(A) \leq f(B)$, for all self-adjoint operators A, B on a Hilbert space H, such that $\sigma(A), \sigma(B) \subseteq [m,M]$.*

Definition 1.11 *We say that the function $f \in C([m,M])$ is* ***operator convex*** *if for all self-adjoint operators A, B on a Hilbert space H, such that $\sigma(A), \sigma(B) \subseteq [m,M]$ and for all $\lambda \in [0,1]$*

$$f((1-\lambda)A + \lambda B) \leq (1-\lambda) f(A) + \lambda f(B). \tag{1.41}$$

We say that the function $f \in C([m,M])$ is ***operator concave*** *if $-f$ is operator convex, that is if inequality* (1.41) *holds with the reverse sign.*

Example 1.1 Function $f\colon \mathbb{R} \to \mathbb{R}$, $f(t) = \alpha + \beta t$ is operator monotone function, for all $\alpha \in \mathbb{R}$ and $\beta \geq 0$ and is operator convex for all α, $\beta \in \mathbb{R}$, (see [74]).

Example 1.2 Function $f\colon (0,\infty) \to \mathbb{R}$, $f(t) = -\frac{1}{t}$ is operator monotone on $(0,\infty)$, (see [74]).

The following characterization holds.

Theorem 1.27 *Let $f\colon [0,\infty) \to [0,\infty)$ be a continuous function. Then f is operator monotone if and only if f is operator concave.*

The Löwner-Heinz inequality is a very important result which dates from 1934. For more details and the proof, see [74].

Theorem 1.28 (LÖWNER-HEINZ INEQUALITY) *Let A and B be positive operators on a Hilbert space H. If $A \geq B \geq 0$, then $A^r \geq B^r$, for all $r \in [0,1]$.*

In other words, function $f\colon [0,\infty) \to \mathbb{R}$, $f(t) = t^r$ is operator monotone on $[0,\infty)$, for all $r \in [0,1]$.

Remark 1.8 If we observe different examples of the operator functions and compare their monotonicity and convexity on $\mathbb{R}$ with their operator monotonicity (convexity), we see that there are some aberrations. Thus for example, the function $f\colon [0,\infty) \to \mathbb{R}$, $f(t) = t^2$ isn't operator monotone, although it is a monotone real valued function when defined on $\mathbb{R}$. Similarly, the function $f\colon [0,\infty) \to \mathbb{R}$, $f(t) = t^3$ isn't operator convex, although it is convex as a real valued function defined on $\mathbb{R}$. For more examples the reader is referred to [74].

Regarding the previous considerations and among many existing variants of operator Jensen's inequalities, we single out the one proved by B. Mond and J. Pečarić in [156]. It states that

$$f\left(\sum_{i=1}^{n} w_i \Phi_i(A_i)\right) \leq \sum_{i=1}^{n} w_i \Phi_i\left(f(A_i)\right), \tag{1.42}$$

for operator convex functions f defined on an interval I, where $\Phi_i : \mathscr{B}(H) \to \mathscr{B}(K)$, $i = 1, \ldots, n$, are unital positive linear mappings, $A_1, \ldots, A_n$ are self-adjoint operators with the spectra in I and $w_1, \ldots, w_n$ are non-negative real numbers with $\sum_{i=1}^{n} w_i = 1$.

1.5 Connections and solidarities. Operator means

The theory of operator means for positive linear operators on a Hilbert space was established and for most part developed by T. Ando and F. Kubo, (see [123]). In this process they used the results from the Löwner's theory on operator monotone functions. In the monograph [74] one can find more details on this topic.

Operator means are defined via connections.

Definition 1.12 *A binary operation $(A,B) \in \mathscr{B}^+(H) \times \mathscr{B}^+(H) \to A\,\sigma B \in \mathscr{B}^+(H)$ in the cone of positive operators on a Hilbert space H is called a* ***connection*** *if the following conditions are satisfied:*

(i) monotonicity: $A \leq C$ and $B \leq D$ imply $A\,\sigma\,B \leq C\,\sigma\,D$,

(ii) upper continuity: $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n\,\sigma\,B_n \downarrow A\,\sigma\,B$,

(iii) transformer inequality: $T^*(A\,\sigma\,B)T \leq (T^*AT)\,\sigma\,(T^*BT)$, for every operator T.

If the normalized condition

(iv) $1_H\,\sigma\,1_H = 1_H$,

is satisfied, then we say that the connection is an ***operator mean****.*

$A_n \downarrow A$ in condition (ii) denotes the convergence of $\{A_n\}$, $A_n \in \mathscr{B}_h(H)$, $A_1 \geq A_2 \geq \cdots$, to $A \in \mathscr{B}_h(H)$, in the strong operator topology.

If T is an invertible operator, then in (iii) the equality sign holds.

Furthermore, the following homogeneity property holds:

$$\alpha(A\,\sigma\,B) = (\alpha A)\,\sigma\,(\alpha B), \quad \text{for all} \quad \alpha > 0.$$

Connections posses the property of joint concavity. More precisely, the inequality

$$(\lambda A_1 + (1-\lambda)B_1)\ \sigma\ (\lambda A_2 + (1-\lambda)B_2) \geq \lambda(A_1\,\sigma\,A_2) + (1-\lambda)(B_1\,\sigma\,B_2) \tag{1.43}$$

holds for $\lambda \in [0,1]$ and $A_1, A_2, B_1, B_2 \in \mathscr{B}^+(H)$.

Let us recall the basic operator means. Arithmetic mean is defined by

$$A \nabla B = \frac{1}{2}(A+B), \tag{1.44}$$

where $A,\, B \in \mathscr{B}^+(H)$.

Parallel sum $A : B$ of operators $A,\, B \in \mathscr{B}^{++}(H)$ is defined by

$$A : B = \left(A^{-1} + B^{-1}\right)^{-1} \tag{1.45}$$

and is a connection. Harmonic mean is defined as a normalized parallel sum by

$$A \,!\, B = \left(\frac{1}{2}(A^{-1} + B^{-1})\right)^{-1}. \tag{1.46}$$

Geometric mean is defined by

$$A \sharp B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{\frac{1}{2}} A^{\frac{1}{2}}, \tag{1.47}$$

where $A \in \mathscr{B}^{++}(H)$ and $B \in \mathscr{B}^+(H)$.

Linear combination of two connections is defined as follows: if σ, τ are connections and a, b are nonnegative real numbers, then

$$A\,(a\sigma + b\tau)\,B = a(A\,\sigma\,B) + b(A\,\tau\,B). \tag{1.48}$$

Thus in particular the class of operator means is a convex set.

Furthermore, respecting the order properties, if $\sigma \geq \tau$ then $A\,\sigma\,B \geq A\,\tau\,B$, for all A, $B \in \mathscr{B}^+(H)$. Hence the following inequalities hold:

$$A \,!\, B \leq A \sharp B \leq A \nabla B. \tag{1.49}$$

The basic result of the theory developed by F. Kubo and T. Ando concerns the isomorphism between connections and the *nonnegative* operator monotone functions.

Theorem 1.29 (KUBO-ANDO) *Let σ be a connection. If for a function $f\colon [0,\infty) \to \mathbb{R}$ and $t \geq 0$*

$$f(t)1_H = 1_H\,\sigma\,(t1_H), \tag{1.50}$$

then f is nonnegative and operator monotone function on $[0,\infty)$ and the following statements are true:

(i) There is an isomorphism between the classes of all connections and all nonnegative operator monotone functions on $[0,\infty)$. Furthermore, for all $A, B \in \mathscr{B}^+(H)$ and $t \geq 0$ is

$$A\,\sigma_1\,B \leq A\,\sigma_2\,B \text{ if and only if } f_1(t) \leq f_2(t),$$

where $\sigma_1 \mapsto f_1$, and $\sigma_2 \mapsto f_2$ are the isomorphisms.

(ii) If A is an invertible operator, then $A\,\sigma\,B = A^{\frac{1}{2}} f(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}$.

(iii) The connection σ is an operator mean if and only if f is a normalized function, i.e. $f(1) = 1$.

The operator monotone function f is called a *representation function* for the connection σ. Since by Theorem 1.27 a continuous function $f\colon [0,\infty) \to [0,\infty)$ is operator monotone if and only if f is operator concave, it follows that the representation function f is an operator concave function.

Hence we have the following representation functions for the arithmetic, harmonic and geometric mean: $f_\nabla(t) = \frac{1+t}{2}$, $f_!(t) = \frac{2t}{1+t}$ and $f_\sharp(t) = \sqrt{t}$. Thus inequalities $f_\nabla(t) \geq f_\sharp(t) \geq f_!(t)$ imply the corresponding operator mean inequalities: $\nabla \geq \sharp \geq !$.

Operator means posses two important properties.

Theorem 1.30 *Operator mean σ possesses the property of subadditivity, that is*

$$A\,\sigma\,C + B\,\sigma\,D \leq (A+B)\,\sigma\,(C+D), \tag{1.51}$$

as well as of the joint concavity:

$$\lambda(A\,\sigma\,C) + (1-\lambda)(B\,\sigma\,D) \leq (\lambda A + (1-\lambda)B)\,\sigma\,(\lambda C + (1-\lambda)D), \tag{1.52}$$

where $A,B,C,D \in \mathscr{B}^+(H)$, $\lambda \in [0,1]$.

In the sequel we observe the weight operator arithmetic mean

$$A\,\nabla_\nu\,B = (1-\nu)A + \nu B, \tag{1.53}$$

$\nu \in [0,1]$ and $A,B \in \mathscr{B}^+(H)$, then the weight operator geometric mean

$$A\,\sharp_\nu\,B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\nu A^{\frac{1}{2}}, \tag{1.54}$$

where $A \in \mathscr{B}^{++}(H)$ and $B \in \mathscr{B}^+(H)$ and finally the weight operator harmonic mean

$$A\,!_\nu\,B = \left((1-\nu)A^{-1} + \nu B^{-1}\right)^{-1}, \tag{1.55}$$

$A,B \in \mathscr{B}^{++}(H)$.

For $A,B \in \mathscr{B}^{++}(H)$ and $\nu \in [0,1]$ the weight operator arithmetic-geometric-harmonic mean inequality holds:

$$A\,!_\nu\,B \leq A\,\sharp_\nu B \leq A\,\nabla_\nu\,B. \tag{1.56}$$

Particularly, we observe the weight operator Heinz means as a special class of means deduced from the operator geometric mean:

$$H_\nu(A,B) = \frac{A\sharp_\nu B + A\sharp_{1-\nu}B}{2}, \tag{1.57}$$

$\nu \in [0,1]$.

J. Fujii *et al.* generalized in [71] the results on connections from Kubo-Ando theory. They investigated the binary operation $s = s_f$ for an *arbitrary* operator monotone function f defined on $[0,\infty)$:

$$A\,s\,B = A^{\frac{1}{2}} f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}},$$

analogously as in (ii) in Theorem 1.29. Domain of s_f is the set of all ordered pairs (A,B) for which $A\,s\,B$ is a bounded operator, respecting f.

Definition 1.13 *A binary operation* $(A,B) \in \mathscr{D}_s \subseteq \mathscr{B}^+(H) \times \mathscr{B}^+(H) \to (A\,s\,B) \in \mathscr{B}_h(H)$ *is called a* ***solidarity*** *if it has the following properties:*

(i) $B \le C$ *implies* $A\,s\,B \le A\,s\,C$,

(ii) $B_n \downarrow B$ *implies* $A\,s\,B_n \downarrow A\,s\,B$,

(iii) $A_n \to A$ *implies* $A_n\,s\,1_H \to A\,s\,1_H$,

(iv) $T^*(A\,s\,B)T \le (T^*AT)\,s\,(T^*BT)$, *for all operators* T,

where $\mathscr{D}_s$ *is the domain of* s.

Solidarity s is defined for an arbitrary pair of positive invertible operators, but not for an arbitrary pair of positive operators. Thus $\mathscr{D}_s$ denotes the maximal subset of $\mathscr{B}^+(H) \times \mathscr{B}^+(H)$ on which s is a bounded operator. (For more details on the domain $\mathscr{D}_s$ the reader is referred to [71]).

In [71] the authors also proved that there was an isomorphism between the solidarities and the operator monotone functions. Furthermore, solidarities posses the joint concavity property. More precisely,

$$(\lambda A_1 + (1-\lambda)B_1)\; s\; (\lambda A_2 + (1-\lambda)B_2) \ge \lambda(A_1\,s\,A_2) + (1-\lambda)(B_1\,s\,B_2), \tag{1.58}$$

where $A_1, A_2,\ B_1, B_2 \in \mathscr{B}^{++}(H)$ and $\lambda \in [0,1]$ and under the assumption that all of the operators included exist as the bounded operators.

Relative operator entropy, here denoted by S, is a common example among solidarities and is defined by

$$S(X|Y) = X^{\frac{1}{2}} \left(\log X^{-\frac{1}{2}} Y X^{-\frac{1}{2}} \right) X^{\frac{1}{2}}, \tag{1.59}$$

where $X, Y \in \mathscr{B}^{++}(H)$.

Tsallis' relative operator entropy T_λ is defined by

$$T_\lambda(X|Y) = X^{\frac{1}{2}} \left(\log_\lambda X^{-\frac{1}{2}} Y X^{-\frac{1}{2}} \right) X^{\frac{1}{2}}, \quad X, Y \in \mathscr{B}^{++}(H),\ \lambda \in (0,1]. \tag{1.60}$$

More on this topic can be found in [73].

1.6 On some properties of the discrete Jensen's functional. Superadditivity

In this section we give the basic motivation for the variety of the obtained results presented in the sequel. Firstly, we recall a few notions that are of an interest in the following considerations.

Let C be a convex cone in the linear space X over F, where $F = \mathbb{R}$ or $\mathbb{C}$, namely:

a) if $x,y \in C$, then $x+y \in C$;

b) if $x \in C, \alpha > 0$, then $\alpha x \in C$.

Let L be a real number, $L \neq 0$. A functional $f : C \to \mathbf{R}$ is called ***L-superadditive*** (***L-subadditive***) on C if

$$f(x+y) \geq (\leq) L(f(x)+f(y)),$$

for any $x,y \in C$. If $L = 1$, then a functional f is simply called ***superadditive*** (***subadditive***).

Let K be a non-negative real function. We say that a functional f is ***K-positive homogeneous*** if

$$f(tx) = K(t)f(x),$$

for any $t \geq 0$ and $x \in C$. In particular, if $K(t) = t^k$, we simply say that f is ***positive homogeneous*** on C ***of order k***. If $k = 1$, we call it ***positive homogeneous***. It is easy to see that $K(1) = 1$ and K is multiplicative. Moreover, we have either $K \equiv 1$ or $K(0) = 0$.

In the monograph [151, p. 717] J. E. Pečarić investigated the method of interpolating inequalities which have reverse inequalities of Aczél type. Using Jensen's inequality and its reverse he proved the following result.

Theorem 1.31 *If f is a convex function on an interval $I \subset \mathbb{R}$, $\mathbf{x} = (x_1,\ldots,x_n) \in I^n$, $n \geq 2$, $\mathbf{p}$ and $\mathbf{q}$ are positive n-tuples such that $\mathbf{p} \geq \mathbf{q}$, (i.e. $p_i \geq q_i$, $i = 1,\ldots,n$; $P_n = \sum_{i=1}^n p_i$, $Q_n = \sum_{i=1}^n q_i$,) then*

$$\sum_{i=1}^n p_i f(x_i) - P_n f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) \geq \sum_{i=1}^n q_i f(x_i) - Q_n f\left(\frac{1}{Q_n}\sum_{i=1}^n q_i x_i\right) \geq 0. \tag{1.61}$$

Since Jensen's inequality is the starting point for many other well known inequalities, the same method of interpolating inequalities as in Theorem 1.31 was applied to Hölder's and Minkowski's inequalities, in [168] or [151, p. 718].

Theorem 1.32 *Let $a,b,u,v \in \mathbb{R}_+^n$ with $u \geq v$ and $p,q \in \mathbb{R}$ with $\frac{1}{p}+\frac{1}{q} = 1$. If $p > 1$, then*

$$\left(\sum_{i=1}^n u_i a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n u_i b_i^q\right)^{\frac{1}{q}} - \sum_{i=1}^n u_i a_i b_i \geq \left(\sum_{i=1}^n v_i a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n v_i b_i^q\right)^{\frac{1}{q}} - \sum_{i=1}^n v_i a_i b_i \geq 0. \tag{1.62}$$

If $p < 1$ $(\neq 0)$, then the reverse inequalities are valid.

Theorem 1.33 *Let $a,b,u,v \in \mathbb{R}_+^n$ with $u \geq v$ and $p \in \mathbb{R}$. If $p \geq 1$ or $p < 0$, then*

$$\left(\sum_{i=1}^n u_i a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n u_i b_i^q\right)^{\frac{1}{q}} - \sum_{i=1}^n u_i a_i b_i \geq \left(\sum_{i=1}^n v_i a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n v_i b_i^q\right)^{\frac{1}{q}} - \sum_{i=1}^n v_i a_i b_i \geq 0. \tag{1.63}$$

If $0 < p < 1$, then the reverse inequalities are valid.

Remark 1.9 For $p = 2$ theorems 1.32 and 1.33 had been proved earlier in [14].

Here we cite a simple, but very important consequence of Theorem 1.31. Since a whole series of existing results has been improved by means of it, it is given here as a lemma.

Lemma 1.2 *Let f be a convex function on an interval $I \subset \mathbb{R}$ and let $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$, $n \geq 2$. Suppose $\mathbf{p} = (p_1, \ldots, p_n)$ is a nonnegative n-tuple such that $P_n = \sum_{i=1}^n p_i > 0$. Then*

$$\min_{1 \leq i \leq n} \{p_i\} \left[\sum_{i=1}^n f(x_i) - nf\left(\frac{1}{n}\sum_{i=1}^n x_i\right)\right] \leq \sum_{i=1}^n p_i f(x_i) - P_n f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) \leq \max_{1 \leq i \leq n} \{p_i\} \left[\sum_{i=1}^n f(x_i) - nf\left(\frac{1}{n}\sum_{i=1}^n x_i\right)\right]. \tag{1.64}$$

Only a few years after Theorem 1.31 had been established, to be more precise, in 1996, the authors S. S. Dragomir, J. E. Pečarić and L. E. Persson obtained the analogous result in their joint paper [66], but as a consequence of a quite different approach. Namely, this time they observed the discrete Jensen's functional $J_n(f, \mathbf{x}, \mathbf{p})$, deduced from the discrete Jensen's inequality (1.4), by subtracting its left from the right side:

$$J_n(f, \mathbf{x}, \mathbf{p}) = \sum_{i=1}^n p_i f(x_i) - P_n f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right). \tag{1.65}$$

If we denote with $\mathscr{P}_n^0$ the set of all nonnegative real n-tuples $\mathbf{p} = (p_1, \ldots, p_n)$, such that $P_n = \sum_{i=1}^n p_i > 0$, and if we fix f and $\mathbf{x}$, then the functional $J_n(f, \mathbf{x}, \cdot)$ can be observed as a function on $\mathscr{P}_n^0$. So if f is a convex function, then $J_n(f, \mathbf{x}, \mathbf{p}) \geq 0$, for all $\mathbf{p} \in \mathscr{P}_n^0$.

The authors established the following result in [66].

Theorem 1.34 *Let I be an interval in $\mathbb{R}$ and let for $n \in \mathbb{N}$, $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$. Suppose $\mathbf{p}, \mathbf{q} \in \mathscr{P}_n^0$. If $f : I \to \mathbb{R}$ is a convex function, then*

$$J_n(f, \mathbf{x}, \mathbf{p} + \mathbf{q}) \geq J_n(f, \mathbf{x}, \mathbf{p}) + J_n(f, \mathbf{x}, \mathbf{q}), \tag{1.66}$$

that is, $J_n(f,\mathbf{x},\cdot)$ is superadditive on $\mathscr{P}_n^0$. If $\mathbf{p},\mathbf{q} \in P_n^0$ are such that $\mathbf{p} \geq \mathbf{q}$, $(p_i \geq q_i$, $i = 1,\ldots,n)$, then

$$J_n(f,\mathbf{x},\mathbf{p}) \geq J_n(f,\mathbf{x},\mathbf{q}) \geq 0, \tag{1.67}$$

that is, $J_n(f,\mathbf{x},\cdot)$ is increasing on $\mathscr{P}_n^0$.

Although it is evident that the same monotonicity property of the Jensen's functional is obtained twice – for the first time as a side-result incorporated in the interpolating inequalities, and for the second time – as a consequence of the superadditivity property, it is interesting that the first result was not even mentioned or referred to when the other one was published!

Dragomir continued the investigation of the properties of the normalized Jensen's functional and in his paper in 2006 he proved the following result for the comparative inequalities. We cite it here because we are going to use it repeatedly in some of the following chapters.

Theorem 1.35 *Let $f : C \to \mathbb{R}$ be a convex function defined on a convex set C in a real linear space X. Suppose $\mathbf{x} = (x_1,\ldots,x_n) \in C^n$. If $\mathbf{p} = (p_1,\ldots,p_n)$ and $\mathbf{q} = (q_1,\ldots,q_n)$ are real n-tuples such that $p_i \geq 0$, $q_i > 0$, $i = 1,\ldots,n$ and $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i = 1$, then the following inequalities hold:*

$$\min_{1\leq i\leq n}\left\{\frac{p_i}{q_i}\right\} J_n(f,\mathbf{x},\mathbf{q}) \leq J_n(f,\mathbf{x},\mathbf{p}) \leq \max_{1\leq i\leq n}\left\{\frac{p_i}{q_i}\right\} J_n(f,\mathbf{x},\mathbf{q}). \tag{1.68}$$

Chapter 2

On Jessen's and McShane's functionals

This chapter is focused on two related Jensen-type functionals, which are derived from Jessen's and McShane's inequalities. Since these inequalities represent the generalizations of Jensen's inequality by means of positive linear functionals acting on a space of real functions, the derived functionals are considered as the generalizations of the discrete Jensen's functional (1.65). It is interesting to comprehend a variety of applications to the classical inequalities, when generalizing the properties of superadditivity and monotonicity of the discrete Jensen's functional to Jessen's and McShane's functionals. Such extensiveness of the applications requires the whole separate section - on the related refinements and converses of Hilbert's inequality.

2.1 Properties of Jessen's functional and applications

We deduce the Jessen's functional from the weight Jessen's inequality and establish its properties of superadditivity and monotonicity. Consequently, we derive the lower and the upper bound for the functional, by means of the non-weight functional of the same type. These bounds are very applicable for they provide refinements and converses of numerous inequalities: starting with generalized weight means and, in particular, power means, as the final products we get difference and ratio type refinements and converses of the arithmetic-geometric mean inequality, then Young's and Hölder's inequalities, which corresponds for the most part to the contents of the published paper [108]. Finally, functionals that arise from the Minkowski inequality are investigated. It is of an importance to underline that the results on superadditivity and monotonicity of the corresponding Minkowski functionals served as a motivation for the investigation of all previously described general cases and as such were published earlier, see [87].

2.1.1 Jessen's functional

First we are going to describe the environment needed for our considerations in the sequel. Let E be a nonempty set and let L be a linear class of functions $f : E \to \mathbb{R}$ satisfying the following properties:

L1: $f, g \in L \Rightarrow \alpha f + \beta g \in L$, for all $\alpha, \beta \in \mathbb{R}$;

L2: $1 \in L$, that is, if $f(x) = 1$, for all $x \in E$, then $f \in L$.

By $L^+ \subseteq L$ the set of all nonnegative functions in L will be denoted. Moreover, let $A : L \to \mathbb{R}$ ba a positive linear functional, that is:

A1: $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$, for $f, g \in L$, $\alpha, \beta \in \mathbb{R}$;

A2: $f \in L$, $f(x) \geq 0$, for all $x \in E \Rightarrow A(f) \geq 0$.

If

A3: $A(1) = 1$,

A is said to be a normalized positive linear functional or we say that $A(f)$ is a *linear mean* on L. Common examples are

$$A(f) = \int_E f d\mu \quad \text{or} \quad A(f) = \sum_{k \in E} p_k f_k,$$

where μ is a positive measure on E in the first case, and in the other, $E = \{1, 2, \ldots\}$ is a countable set with the discrete measure $\mu(k) = p_k \geq 0$, $0 < \sum_{k \in E} p_k < \infty$, $f(k) = f_k$, defined on it.

Jessen's inequality (1.15) in its elementary form will not suffice for our further considerations, so we give here a reminder of its weight variant. For more details on the weight Jessen's inequality, see e.g. [177, p. 112].

Theorem 2.1 *Let E be a nonempty set and let L be a linear class of functions $f : E \to \mathbb{R}$ satisfying L1 and L2. Suppose $\Phi : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$, is a continuous convex function and let $p \in L^+$. If $A : L \to \mathbb{R}$ is a positive linear functional, $A(p) > 0$, then for all $f \in L$, such that pf, $p\Phi(f) \in L$, the following inequality holds:*

$$A(p)\Phi\left(\frac{A(pf)}{A(p)}\right) \leq A\left(p\Phi(f)\right). \tag{2.1}$$

Remark 2.1 Denote $A_1(f) = \frac{A(pf)}{A(p)}$, $A(p) > 0$. It follows that A_1 is a normalized (positive linear) functional, i.e. $A_1(1) = 1$, and inequality (1.15) can be applied.

Remark 2.2 We still need to show that $A_1(f) \in I$, for I is the domain of Φ. Let $I = [a,b]$. Obviously, $a \leq f(x) \leq b$, for all $x \in E$. Since $b - f(x) \geq 0$, by making use of A1, A2 and A3, it follows $b - A_1(f) = A_1(b) - A_1(f) = A_1(b-f) \geq 0$, so $A_1(f) \leq b$. Analogously, $a \leq A_1(f)$. Hence $A_1(f) = \frac{A(pf)}{A(p)}$ belongs to I.

We now deduce Jessen's functional $\mathscr{J}(\Phi, f, p; A)$ from inequality (2.1) as follows:

$$\mathscr{J}(\Phi, f, p; A) = A(p\Phi(f)) - A(p)\Phi\left(\frac{A(pf)}{A(p)}\right). \tag{2.2}$$

For fixed Φ, f and A, we consider $\mathscr{J}(\Phi, f, \cdot; A)$ as a function on L^+, as was previously done in [66], regarding discrete Jensen's functional (1.65). If Φ is a convex function, it follows from (2.1) that $\mathscr{J}(\Phi, f, p; A) \geq 0$, for all $p \in L^+$.

In the following section we work out the mentioned analogy with the discrete Jensen's functional in detail.

2.1.2 Superadditivity of Jessen's functional

In the following theorem we establish the superadditivity on L^+ of Jessen's functional (2.2), which then provides its other useful properties.

Theorem 2.2 *Let $A : L \to \mathbb{R}$ be a positive linear functional. Suppose $f \in L$ and $p, q \in L^+$. If $\Phi : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$, is a continuous and convex function, then*

$$\mathscr{J}(\Phi, f, p+q; A) \geq \mathscr{J}(\Phi, f, p; A) + \mathscr{J}(\Phi, f, q; A), \tag{2.3}$$

that is, $\mathscr{J}(\Phi, f, \cdot; A)$ is superadditive on L^+. Moreover, if $p, q \in L^+$ are such that $p \geq q$, then

$$\mathscr{J}(\Phi, f, p; A) \geq \mathscr{J}(\Phi, f, q; A) \geq 0, \tag{2.4}$$

that is, $\mathscr{J}(\Phi, f, \cdot; A)$ is increasing on L^+.

Proof. According to definition (2.2) and linearity of the functional A, it follows:

$$\begin{aligned}
\mathscr{J}(\Phi, f, p+q; A) &= A((p+q)\Phi(f)) - A(p+q)\phi\left(\frac{A((p+q)f)}{A(p+q)}\right) \\
&= A(p\Phi(f) + q\Phi(f)) - (A(p) + A(q))\Phi\left(\frac{A(pf+qf)}{A(p)+A(q)}\right) \\
&= A(p\Phi(f)) + A(q\Phi(f)) - (A(p) + A(q))\Phi\left(\frac{A(pf)+A(qf)}{A(p)+A(q)}\right).
\end{aligned} \tag{2.5}$$

On the other side, due to convexity of Φ we get

$$\begin{aligned}\Phi\left(\frac{A(pf)+A(qf)}{A(p)+A(q)}\right) &= \Phi\left(\frac{A(p)}{A(p)+A(q)}\cdot\frac{A(pf)}{A(p)}+\frac{A(q)}{A(p)+A(q)}\cdot\frac{A(qf)}{A(q)}\right)\\ &\leq \frac{A(p)}{A(p)+A(q)}\Phi\left(\frac{A(pf)}{A(p)}\right)+\frac{A(q)}{A(p)+A(q)}\Phi\left(\frac{A(qf)}{A(q)}\right),\end{aligned}$$

which can be rewritten as

$$(A(p)+A(q))\,\Phi\left(\frac{A(pf)+A(qf)}{A(p)+A(q)}\right)\leq A(p)\Phi\left(\frac{A(pf)}{A(p)}\right)+A(q)\Phi\left(\frac{A(qf)}{A(q)}\right). \tag{2.6}$$

Finally, (2.5) and (2.6) yield

$$\begin{aligned}\mathscr{J}\left(\Phi,f,p+q;A\right) &\geq A\left(p\Phi(f)\right)+A\left(q\Phi(f)\right)-A(p)\Phi\left(\frac{A(pf)}{A(p)}\right)-A(q)\Phi\left(\frac{A(qf)}{A(q)}\right)\\ &= \mathscr{J}\left(\Phi,f,p;A\right)+\mathscr{J}\left(\Phi,f,q;A\right),\end{aligned}$$

that is, functional $\mathscr{J}\left(\Phi,f,\cdot;A\right)$ is superadditive on L^+.

As for the increase on L^+ of the functional $\mathscr{J}\left(\Phi,f,\cdot;A\right)$, we write $p=(p-q)+q$ for $p\geq q\geq 0$. Hence (2.3) yields

$$\mathscr{J}\left(\Phi,f,p;A\right)=\mathscr{J}\left(\Phi,f,p-q+q;A\right)\geq \mathscr{J}\left(\Phi,f,p-q;A\right)+\mathscr{J}\left(\Phi,f,q;A\right).$$

Since $\mathscr{J}\left(\Phi,f,p-q;A\right)\geq 0$, it follows that $\mathscr{J}\left(\Phi,f,p;A\right)\geq \mathscr{J}\left(\Phi,f,q;A\right)$, which ends the proof. □

Remark 2.3 If Φ is a continuous and concave function, the signs of inequalities (2.3) and (2.4) are reversed, that is, functional $\mathscr{J}\left(\Phi,f,\cdot;A\right)$ is subadditive and decreasing on L^+. Namely, in the case of Φ being concave, the sign of Jensen's inequality is reversed and $\mathscr{J}\left(\Phi,f,p;A\right)\leq 0$, for all $p\in L^+$. Remark on the concavity of the function Φ is going to be taken into account in the sequel, in all the similar results of the kind, even if it is not accentuated.

The following corollary provides the lower and the upper bound for the functional (2.2), which are expressed by means of the non-weight functional of the same type.

Corollary 2.1 *Let function f and functional A be as in the Theorem 2.2. Suppose $p\in L^+$ attains its minimal and maximal value on E. If $\Phi: I\to\mathbb{R}$, $I\subseteq\mathbb{R}$ is a continuous and convex function, then the following inequalities hold:*

$$\left[\min_{x\in E}p(x)\right]\mathscr{J}\left(\Phi,f,1;A\right)\leq \mathscr{J}\left(\Phi,f,p;A\right)\leq\left[\max_{x\in E}p(x)\right]\mathscr{J}\left(\Phi,f,1;A\right), \tag{2.7}$$

where

$$\mathscr{J}\left(\Phi,f,1;A\right)=A\left(\Phi(f)\cdot 1\right)-A(1)\Phi\left(\frac{A(f)}{A(1)}\right). \tag{2.8}$$

Proof. The inequalities are proved by making use of (2.4). Namely, as $p \in L^+$ attains minimal and maximal value on its domain, it is clear that

$$\min_{x\in E} p(x) \le p(x) \le \max_{x\in E} p(x),$$

and we observe two constant functions

$$\underline{p} = \min_{x\in E} p(x) \quad \text{and} \quad \overline{p} = \max_{x\in E} p(x).$$

Double application of the property (2.4) yields (2.7), since

$$\mathscr{J}\left(\Phi, f, \underline{p}\cdot 1; A\right) = \underline{p}\,\mathscr{J}\left(\Phi, f, 1; A\right) \text{ and } \mathscr{J}\left(\Phi, f, \overline{p}\cdot 1; A\right) = \overline{p}\,\mathscr{J}\left(\Phi, f, 1; A\right).$$

□

Remark 2.4 If $p \in L^+$ is a bounded function on E, then infimum (supremum) in (2.7) may be observed. This fact is going to be taken into account in all the results on the non-weight bounds of the functionals.

A direct application of the monotonicity property (2.4) is an improvement of the Theorem 1.35 on comparative inequalities for the discrete normalized Jensen's functional. In order to present how the idea developed, we first cite a generalization of the Dragomir's result, obtained in 2009 in [53].

Theorem 2.3 (SEE [53]) *Let $\Phi : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$ be a continuous and convex function. Suppose A is a positive linear functional and m and M real constants, such that for p and $q \in L^+$ and for all $x \in E$*

$$p(x) - mq(x) \ge 0, \quad Mq(x) - p(x) \ge 0 \quad \textit{and}$$

$$A(p) - mA(q) > 0, \quad MA(q) - A(p) > 0.$$

Then the following inequalities hold:

$$M\mathscr{J}\left(\Phi, f, q; A\right) \ge \mathscr{J}\left(\Phi, f, p; A\right) \ge m\mathscr{J}\left(\Phi, f, q; A\right). \tag{2.9}$$

Remark 2.5 The proof of Theorem 2.3 is improved when Theorem 2.2 is applied. Namely, inequalities (2.9) follow easily from $mq(x) \le p(x) \le Mq(x)$ when the monotonicity property (2.4) is applied twice, since $\mathscr{J}\left(\Phi, f, mq; A\right) = m\mathscr{J}\left(\Phi, f, q; A\right)$ and $\mathscr{J}\left(\Phi, f, Mq; A\right) = M\mathscr{J}\left(\Phi, f, q; A\right)$.

In particular, the discrete form of Dragomir's comparative inequalities (1.68) is obtained directly in an analogous way when observing a linear space of n-tuples $\mathbf{x} = (x_1, \ldots, x_n)$, $n \in \mathbb{N}$, and a discrete functional $A : L \to \mathbb{R}$, such that $A(\mathbf{x}) = \sum_{i=1}^n x_i$, as well as $\mathbf{p}, \mathbf{q} \in L$, such that $p_i \ge 0$, $q_i > 0$, $i = 1, \ldots, n$ and $\sum_{i=1}^n p_i = 1$, $\sum_{i=1}^n q_i = 1$. If we denote

$$m = \min_{1\le i\le n} \left\{\frac{p_i}{q_i}\right\} \quad \text{and} \quad M = \max_{1\le i\le n} \left\{\frac{p_i}{q_i}\right\},$$

then relation $Mq_i \ge p_i \ge mq_i$, $i = 1, 2, \ldots, n$ is the starting point for the double application of the property (2.4).

Remark 2.6 Finally, as for the direct application of Corollary 2.1, let us demonstrate how a result from [196] is generalized. Using the discrete notation, inequalities (2.7) assume the following form:

$$\min_{1\le i\le n}\{p_i\}J_n(\mathbf{x}) \le J_n(f,\mathbf{x},\mathbf{p}) \le \max_{1\le i\le n}\{p_i\}J_n(\mathbf{x}), \tag{2.10}$$

where $J_n(f,\mathbf{x},\mathbf{p})$ is defined by (1.65) and $J_n(\mathbf{x}) = \sum_{i=1}^n f(x_i) - nf\left(\frac{\sum_{i=1}^n x_i}{n}\right)$. Namely, inequalities (2.10) were established in [196], but only for $n=2$ and the normalized functional $J_n(f,\mathbf{x},\mathbf{p})$.

2.1.3 Application to weight generalized means

A weight generalized mean of a function $f\in L$ is defined by means of a positive linear functional $A: L\to\mathbb{R}$ as

$$M_\chi(f,p;A) = \chi^{-1}\left(\frac{A(p\chi(f))}{A(p)}\right), \tag{2.11}$$

where $\chi: I\to\mathbb{R}$, $I\subseteq\mathbb{R}$ is a continuous and strictly monotonic function and $p\in L^+$ is a weight function. On the assumptions that $A(p)>0$ and $\chi(f)$, $p\chi(f)\in L$, it is easy to see that (2.11) is a mean (for more details, see [177, p. 107]).

Now, for a continuous and strictly monotonic function $\psi: I\to\mathbb{R}$, such that $\psi(f)$, $p\psi(f)\in L$ and motivated by (2.11), we deduce the following Jessen-type functional:

$$\mathscr{J}^{\mathscr{T}}(\chi\circ\psi^{-1},\psi(f),p;A) = A(p)\left[\chi\left(M_\chi(f,p;A)\right) - \chi\left(M_\psi(f,p;A)\right)\right]. \tag{2.12}$$

For fixed $\chi\circ\psi^{-1}$, $\psi(f)$ and A, we observe the functional $\mathscr{J}^{\mathscr{T}}(\chi\circ\psi^{-1},\psi(f),\cdot;A)$ as a function on L^+ and establish the following result.

Theorem 2.4 *Let $\chi,\psi: I\to\mathbb{R}$, $I\subseteq\mathbb{R}$, be continuous and strictly monotonic functions. Suppose $f\in L$ is such that $\psi(f)\in L$ and $A: L\to\mathbb{R}$ is a positive linear functional. If $\chi\circ\psi^{-1}$ is a convex function, then $\mathscr{J}^{\mathscr{T}}(\chi\circ\psi^{-1},\psi(f),\cdot;A)$, defined by* (2.12) *is superadditive and increasing on L^+.*

Proof. Making use of linearity of the functional A as well as (2.11), we may rearrange the expressions in (2.12):

$$\begin{aligned}
\mathscr{J}^{\mathscr{T}}\left(\chi\circ\psi^{-1},\psi(f),p;A\right) &= A(p)\left[\chi\left(M_\chi(f,p;A)\right) - \chi\left(M_\psi(f,p;A)\right)\right] \\
&= A(p)\chi\left(M_\chi(f,p;A)\right) - A(p)\chi\left(M_\psi(f,p;A)\right) \\
&= A\left(p\chi(f)\right) - A(p)\chi\left(M_\psi(f,p;A)\right) \\
&= A\left(p\cdot\left(\chi\circ\psi^{-1}(\psi(f))\right)\right) - A(p)\chi\left(\psi^{-1}\left(\frac{A(p\psi(f))}{A(p)}\right)\right).
\end{aligned}$$

This way it is obvious that (2.12) corresponds to the Jessen's functional (2.2), with function Φ substituted by $\chi\circ\psi^{-1}$, and $f\in L$ by $\psi(f)\in L$. Therefore, superadditivity and increase of the functional (2.12) follow directly from Theorem 2.2. □

Corollary 2.2 *Let χ, ψ, f and A be as in Theorem 2.4 and let $p \in L^+$ attain its minimal and maximal value on E. If $\chi \circ \psi^{-1}$ is a convex function and functional $\mathscr{J}^{\mathscr{T}}(\chi \circ \psi^{-1}, \psi(f), \cdot; A)$ is defined by (2.12), then the following inequalities hold:*

$$\begin{aligned}
\left[\min_{x \in E} p(x)\right] & \mathscr{J}^{\mathscr{T}}\left(\chi \circ \psi^{-1}, \psi(f), 1; A\right) \\
& \leq \mathscr{J}^{\mathscr{T}}\left(\chi \circ \psi^{-1}, \psi(f), p; A\right) \\
& \leq \left[\max_{x \in E} p(x)\right] \mathscr{J}^{\mathscr{T}}\left(\chi \circ \psi^{-1}, \psi(f), 1; A\right),
\end{aligned} \tag{2.13}$$

where $$\mathscr{J}^{\mathscr{T}}\left(\chi \circ \psi^{-1}, \psi(f), 1; A\right) = A(1)\left[\chi\left(M_\chi(f;A)\right) - \chi\left(M_\psi(f;A)\right)\right] \tag{2.14}$$

and $$M_\eta(f;A) = \eta^{-1}\left(\frac{A(\eta(f))}{A(1)}\right), \quad \eta = \chi, \psi. \tag{2.15}$$

Proof. Since functional (2.12) is increasing on L^+ according to Theorem 2.4, the proof follows the same lines as Corollary 2.1. □

Let $r \in \mathbb{R}$. We are going to observe the generalized weight power mean $M_r(f, p; A)$ of a function $f \in L^+$, $f(x) > 0$, $x \in E$, regarding the positive linear functional $A : L \to \mathbb{R}$. Function $\chi : I \to \mathbb{R}$ in (2.11) is here defined by $\chi_r(x) = x^r$, for $r \neq 0$ and $\chi_r(x) = \ln x$, for $r = 0$:

$$M_r(f, p; A) = \begin{cases} \left(\frac{A(pf^r)}{A(p)}\right)^{\frac{1}{r}}, & r \neq 0 \\ \exp\left(\frac{A(p\ln(f))}{A(p)}\right), & r = 0 \end{cases}, \tag{2.16}$$

$p \in L^+$ is a weight function. We additionally need to assume that $pf^r \in L^+$, $p\ln(f) \in L$ and $A(p) > 0$ in order to have the above expressions well defined.

Let $r, s \in \mathbb{R}$ and $s \neq 0$. Let us define the functional

$$\mathscr{J}^{\mathscr{P}}\left(\chi \circ \psi^{-1}, \psi(f), p; A\right) = A(p)\left(\left(M_s(f,p;A)\right)^s - \left(M_r(f,p;A)\right)^s\right), \tag{2.17}$$

where $\chi_s, \psi_r : I \to \mathbb{R}$ are defined by $\chi_s(x) = x^s$, $s \neq 0$, $\psi_r(x) = x^r$, for $r \neq 0$ and $\psi_r(x) = \ln x$, for $r = 0$.

The functional (2.17) possesses the analogous properties to those in Theorem 2.4.

Corollary 2.3 *Let $s \neq 0$ and r be real numbers. If $s > 0$, $s > r$ or $s < 0$, $s < r$ or $r = 0$, then functional $\mathscr{J}^{\mathscr{P}}\left(\chi \circ \psi^{-1}, \psi(f), \cdot; A\right)$, defined by (2.17), is superadditive and increasing on L^+.*

Proof. Follows directly from Theorem 2.4. If $r \neq 0$, then it follows from (2.17) that $\chi \circ \psi^{-1}(x) = x^{\frac{s}{r}}$, so $\left(\chi \circ \psi^{-1}\right)''(x) = \frac{s(s-r)}{r^2} x^{\frac{s}{r}-2}$. Hence $\chi \circ \psi^{-1}$ is convex for $s > 0$, $s > r$ or $s < 0$, $s < r$. If $r = 0$, then it follows from (2.17) that $\chi \circ \psi^{-1}(x) = e^{sx}$, which is convex for $s \neq 0$. Applying Theorem 2.4 we end the proof. □

Corollary 2.4 *Let $s \neq 0$ and r be real numbers, such that $s > 0$, $s > r$ or $s < 0$, $s < r$ or $r = 0$. If $p \in L^+$ attains its minimal and maximal value on E, then the following inequalities hold:*

$$\begin{aligned}\left[\min_{x\in E} p(x)\right] \mathscr{J}^{\mathscr{P}}\left(\chi\circ\psi^{-1},\psi(f),1;A\right) &\\ \leq \mathscr{J}^{\mathscr{P}}\left(\chi\circ\psi^{-1},\psi(f),p;A\right) &\\ \leq \left[\max_{x\in E} p(x)\right] \mathscr{J}^{\mathscr{P}}\left(\chi\circ\psi^{-1},\psi(f),1;A\right), &\qquad (2.18)\end{aligned}$$

where

$$\mathscr{J}^{\mathscr{P}}\left(\chi\circ\psi^{-1},\psi(f),1;A\right) = A(1)\left(\left(M_s(f;A)\right)^s - \left(M_r(f;A)\right)^s\right) \qquad (2.19)$$

and

$$M_t(f;A) = \begin{cases} \left(\frac{A(f^r)}{A(1)}\right)^{\frac{1}{t}}, & t \neq 0 \\ \exp\left(\frac{A(\ln(f))}{A(1)}\right), & t = 0 \end{cases}, \quad t = r,s. \qquad (2.20)$$

Proof. Since functional (2.17) is increasing on L^+ according to Corollary 2.3, the proof follows the same lines as in Corollary 2.1. □

Remark 2.7 If $s > 0$, $s < r$ or $s < 0$, $s > r$, then $\chi\circ\psi^{-1}$ is a concave function, and the functional $\mathscr{J}^{\mathscr{P}}\left(\chi\circ\psi^{-1},\psi(f),\cdot;A\right)$, defined by (2.17), is subadditive and decreasing on L^+, according to Remark 2.3. Inequalities (2.18) have the reverse signs in that setting.

Remark 2.8 In order to derive a converse and a refinement of the arithmetic-geometric mean inequality, as well as of closely related Young's inequality, we observe the discrete variant of the inequalities (2.18). For that purpose, let $E = \{1,2,\ldots,n\}$, $n \in \mathbb{N}$ and let L be a linear class of the real n-tuples $\mathbf{x} = (x_1,\ldots,x_n)$. In this setting, $A : L \to \mathbb{R}$ is a discrete functional, such that $A(\mathbf{x}) = \sum_{i=1}^n x_i$. For nonnegative n-tuples $\mathbf{p} \in L$ is $A(\mathbf{p}) = P_n = \sum_{i=1}^n p_i > 0$ and $A(\mathbf{1}) = \sum_{i=1}^n 1 = n$.

In the discrete notation, weight power mean (2.16) of $\mathbf{x} = (x_1,\ldots,x_n)$ reads

$$M_r(\mathbf{x},\mathbf{p}) = \begin{cases} \left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i^r\right)^{\frac{1}{r}}, & r \neq 0 \\ \left(\prod_{i=1}^n x_i^{p_i}\right)^{\frac{1}{P_n}}, & r = 0 \end{cases}. \qquad (2.21)$$

It is obvious that for $r = 1$ we get the expression for the arithmetic mean $A_n(\mathbf{x},\mathbf{p}) := M_1(\mathbf{x},\mathbf{p}) = \left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)$ and for $r = 0$ we get the expression for the geometric mean $G_n(\mathbf{x},\mathbf{p}) := M_0(\mathbf{x},\mathbf{p}) = \left(\prod_{i=1}^n x_i^{p_i}\right)^{\frac{1}{P_n}}$. If we insert the constant n-tuples

$$\overline{\mathbf{p}} = \left(\min_{1\leq i\leq n}\{p_i\},\ldots,\min_{1\leq i\leq n}\{p_i\}\right) \quad \text{or} \quad \underline{\mathbf{p}} = \left(\max_{1\leq i\leq n}\{p_i\},\ldots,\max_{1\leq i\leq n}\{p_i\}\right),$$

expressions for the arithmetic and geometric means are $G_n^0(\mathbf{x}) = (\prod_{i=1}^n x_i)^{\frac{1}{n}}$ and $A_n^0(\mathbf{x}) = \left(\frac{1}{n}\sum_{i=1}^n x_i\right)$, and inequalities (2.18), for $s=1$ and $r=0$ assume the following form:

$$\begin{aligned} 0 &\leq n \min_{1\leq i\leq n}\{p_i\}\left[A_n^0(\mathbf{x}) - G_n^0(\mathbf{x})\right] \leq P_n\left[A_n(\mathbf{x},\mathbf{p}) - G_n(\mathbf{x},\mathbf{p})\right] \\ &\leq n \max_{1\leq i\leq n}\{p_i\}\left[A_n^0(\mathbf{x}) - G_n^0(\mathbf{x})\right]. \end{aligned} \tag{2.22}$$

The first inequality in (2.22) is a refinement, and the second is a converse of the arithmetic-geometric mean inequality, obtained in the difference form. Thus this result generalizes the refinement and the converse obtained in [97] for $n=2$:

$$\begin{aligned} \min\{\nu, 1-\nu\}\left(\sqrt{a}-\sqrt{b}\right)^2 &\leq (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \\ &\leq \max\{\nu, 1-\nu\}\left(\sqrt{a}-\sqrt{b}\right)^2. \end{aligned} \tag{2.23}$$

Remark 2.9 Notice that the classical inequality

$$a^{\nu}b^{1-\nu} \leq \nu a + (1-\nu)b$$

implies that

$$(a^{\nu}b^{1-\nu})^m \leq (\nu a + (1-\nu)b)^m, \tag{2.24}$$

for positive real numbers a and b, $0 \leq \nu \leq 1$ and for $m = 1,2,3,\ldots$.

One can notice that in [97], Kittaneh and Manasrah gave a refinement of (2.24) in the form of relation (2.23), for $m=1$, by adding the expression $r_0(\sqrt{a}-\sqrt{b})^2$, where $r_0 = \min\{\nu, 1-\nu\}$.

On the other hand, Hirzallah and Kittaneh in [85] refined (2.24), for $m=2$, by adding the expression $r_0^2(a-b)^2$:

$$(a^{\nu}b^{1-\nu})^2 + r_0^2(a-b)^2 \leq (\nu a + (1-\nu)b)^2. \tag{2.25}$$

As it was mentioned in [97], there is no ordering between the refinements in (2.23) and (2.25).

Finally, very recently, a new generalization of the refinement (2.24) was obtained in [99], by adding the expression $r_0^m(a^{\frac{m}{2}} - b^{\frac{m}{2}})^2$, where $m = 1,2,3,\ldots$, as a natural extension of inequalities (2.23) and (2.25). Namely, the authors inductively proved that, if $a,b > 0$ and $0 \leq \nu \leq 1$, then for $m = 1,2,3,\ldots$, the following inequality holds:

$$(a^{\nu}b^{1-\nu})^m + r_0^m(a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 \leq (\nu a + (1-\nu)b)^m, \tag{2.26}$$

where $r_0 = \min\{\nu, 1-\nu\}$.

Remark 2.10 Young's inequality follows directly from the arithmetic-geometric inequality. Hence the expected converse and refinement of Young's inequality in the difference form, for positive n-tuples $\mathbf{x} = (x_1,\ldots,x_n)$ and $\mathbf{p} = (p_1,\ldots,p_n)$, such that $\sum_{i=1}^n \frac{1}{p_i} = 1$ are contained in the following inequalities:

$$n \min_{1\le i\le n}\left\{\frac{1}{p_i}\right\}\left[A_n^0(\mathbf{x^p}) - G_n^0(\mathbf{x^p})\right] \le A_n(\mathbf{x^p},\mathbf{p}^{-1}) - G_n(\mathbf{x^p},\mathbf{p}^{-1}) \tag{2.27}$$
$$\le n \max_{1\le i\le n}\left\{\frac{1}{p_i}\right\}\left[A_n^0(\mathbf{x^p}) - G_n^0(\mathbf{x^p})\right],$$

where

$$\mathbf{x^p} = \left(x_1^{p_1},\dots,x_n^{p_n}\right) \quad \text{and} \quad \mathbf{p}^{-1} = \left(\frac{1}{p_1},\dots,\frac{1}{p_n}\right).$$

The former considerations of the mean inequalities in the difference form motivate the analogous results in the ratio form. Namely, when (2.17) was defined by means of (2.16), the case $s = 0$ and $r \neq 0$ wasn't taken into consideration. By analyzing this case in the sequel, we get new interesting applications. Let $f \in L^+$ be such that $f(x) > 0$ for all $x \in E$, $A : L \to \mathbb{R}$ be a positive linear functional and let $p \in L^+$. We define the functional

$$\mathscr{J}^{\overline{\mathscr{P}}}\left(\chi\circ\psi^{-1},\psi(f),p;A\right) = A(p)\left(\frac{A\left(p\ln f\right)}{A(p)} - \ln\left(M_r(f,p;A)\right)\right), \tag{2.28}$$

where $\chi,\psi : I \to \mathbb{R}$ are defined by $\chi(x) = \ln x$, $\psi(x) = x^r$, $r \neq 0$. We additionally need to assume that $p\ln(f) \in L$ and $A(p) > 0$.

Corollary 2.5 *Suppose $r < 0$. Then functional $\mathscr{J}^{\overline{\mathscr{P}}}\left(\chi\circ\psi^{-1},\psi(f),\cdot;A\right)$, defined by* (2.28) *is superadditive and increasing on L^+.*

Proof. Follows directly from Theorem 2.4, since function $\chi\circ\psi^{-1}$, defined by $\chi\circ\psi^{-1}(x) = \frac{1}{r}\ln x$ is convex for $r < 0$. □

Corollary 2.6 *Suppose $r < 0$. If $p \in L^+$ attains its minimal and maximal value on E, then the following inequalities hold:*

$$\left[\min_{x\in E} p(x)\right]\mathscr{J}^{\overline{\mathscr{P}}}\left(\chi\circ\psi^{-1},\psi(f),1;A\right)$$
$$\le \mathscr{J}^{\overline{\mathscr{P}}}\left(\chi\circ\psi^{-1},\psi(f),p;A\right)$$
$$\le \left[\max_{x\in E} p(x)\right]\mathscr{J}^{\overline{\mathscr{P}}}\left(\chi\circ\psi^{-1},\psi(f),1;A\right), \tag{2.29}$$

where

$$\mathscr{J}^{\overline{\mathscr{P}}}\left(\chi\circ\psi^{-1},\psi(f),1;A\right) = A(1)\left(\frac{A\left(\ln f\right)}{A(1)} - \ln\left(M_r(f;A)\right)\right) \tag{2.30}$$

and $M_r(f;A)$ is defined by (2.20).

Proof. Follows the same lines as in Corollary 2.1. □

Remark 2.11 If $r > 0$, then $\chi \circ \psi^{-1}$ is a concave function and according to Remark 2.3 is functional $\mathscr{J}^{\overline{\mathscr{P}}}\left(\chi \circ \psi^{-1}, \psi(f), \cdot; A\right)$, defined by (2.28) subadditive and decreasing on L^+, while inequalities (2.29) have the reverse signs.

Remark 2.12 In order to derive the converse and the refinement of the arithmetic-geometric mean inequality in the ratio form, as well as of the closely related Young's inequality, we again use the discrete notation, as in Remark 2.8. Thus the expression $A\left(p \ln f\right)/A(p)$ becomes

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i \ln x_i = \ln\left(\prod_{i=1}^{n} x_i^{p_i}\right)^{\frac{1}{P_n}} = \ln G_n(\mathbf{x},\mathbf{p}).$$

For $r = 1$ we obtain the arithmetic mean and the inequalities (2.29) have the reverse signs. By rearranging we finally get

$$1 \leq \left[\frac{A_n^0(\mathbf{x})}{G_n^0(\mathbf{x})}\right]^{n \min\limits_{1\leq i\leq n}\{p_i\}} \leq \left[\frac{A_n(\mathbf{x},\mathbf{p})}{G_n(\mathbf{x},\mathbf{p})}\right]^{P_n} \leq \left[\frac{A_n^0(\mathbf{x})}{G_n^0(\mathbf{x})}\right]^{n \max\limits_{1\leq i\leq n}\{p_i\}}. \tag{2.31}$$

The first inequality in (2.31) is a refinement, and the second is a converse of the arithmetic-geometric mean inequality, in the ratio form.

According to the notation from Remark 2.10, the same is obtained for Young's inequality:

$$\left[\frac{A_n^0(\mathbf{x}^{\mathbf{p}})}{G_n^0(\mathbf{x}^{\mathbf{p}})}\right]^{n \min\limits_{1\leq i\leq n}\left\{\frac{1}{p_i}\right\}} \leq \frac{A_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1})}{G_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1})} \leq \left[\frac{A_n^0(\mathbf{x}^{\mathbf{p}})}{G_n^0(\mathbf{x}^{\mathbf{p}})}\right]^{n \max\limits_{1\leq i\leq n}\left\{\frac{1}{p_i}\right\}}. \tag{2.32}$$

2.1.4 Application to Hölder's inequality

It is well known that Young's inequality is a starting point in the proof of Hölder's inequality. Hence it is natural to expect the converses and refinements analogous to those obtained for Young's inequality to be obtained for Hölder's inequality as well. We are going to apply the previous results to Hölder's inequality expressed by means of a positive linear functional $A\colon L \to \mathbb{R}$ (see e.g. [151, p. 135] or [177, p. 113]):

$$A\left(\prod_{i=1}^{n} f_i^{\frac{1}{p_i}}\right) \leq \prod_{i=1}^{n} A^{\frac{1}{p_i}}(f_i), \tag{2.33}$$

where p_i, $i = 1,\ldots,n$, $n \in \mathbb{N}$, are such that $p_i > 1$, $\sum_{i=1}^{n}\frac{1}{p_i} = 1$ and $f_i \in L^+$ are such that $\prod_{i=1}^{n} f_i^{\frac{1}{p_i}} \in L^+$.

Just like in the previous section, we first deal with the converse and the refinement in the difference form.

Theorem 2.5 *Let p_i, $i = 1,\ldots,n$, $n \in \mathbb{N}$, be such that $p_i > 1$ and $\sum_{i=1}^{n}\frac{1}{p_i} = 1$. Suppose functions $f_i \in L^+$ are such that $\prod_{i=1}^{n} f_i^{\frac{1}{p_i}}$, $\prod_{i=1}^{n} f_i^{1/n} \in L^+$. If $A : L \to \mathbb{R}$ is a positive linear* *functional, then the following inequalities hold:*

$$
\begin{aligned}
& n \min_{1\le i\le n}\left\{\frac{1}{p_i}\right\}\left[\prod_{i=1}^{n} A^{\frac{1}{p_i}}(f_i) - \prod_{i=1}^{n} A^{\frac{1}{p_i}-\frac{1}{n}}(f_i)\cdot A\left(\prod_{i=1}^{n} f_i^{\frac{1}{n}}\right)\right] \\
& \le \prod_{i=1}^{n} A^{\frac{1}{p_i}}(f_i) - A\left(\prod_{i=1}^{n} f_i^{\frac{1}{p_i}}\right) \\
& \le n \max_{1\le i\le n}\left\{\frac{1}{p_i}\right\}\left[\prod_{i=1}^{n} A^{\frac{1}{p_i}}(f_i) - \prod_{i=1}^{n} A^{\frac{1}{p_i}-\frac{1}{n}}(f_i)\cdot A\left(\prod_{i=1}^{n} f_i^{\frac{1}{n}}\right)\right].
\end{aligned}
\tag{2.34}
$$

Proof. Follows directly from (2.27) so we use the notation from Remark 2.10. Let $\mathbf{x} = (x_1,\dots,x_n)$ be defined by $x_i = [f_i/A(f_i)]^{\frac{1}{p_i}}$, $i = 1,2,\dots,n$. Then the expressions for the difference between the arithmetic and the geometric mean in (2.27) become:

$$
\begin{aligned}
A_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1}) - G_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1}) &= \sum_{i=1}^{n}\frac{f_i}{p_i A(f_i)} - \prod_{i=1}^{n}\frac{f_i^{\frac{1}{p_i}}}{A^{\frac{1}{p_i}}(f_i)}, \\
A_n^0(\mathbf{x}^{\mathbf{p}}) - G_n^0(\mathbf{x}^{\mathbf{p}}) &= \frac{1}{n}\sum_{i=1}^{n}\frac{f_i}{A(f_i)} - \prod_{i=1}^{n}\frac{f_i^{\frac{1}{n}}}{A^{\frac{1}{n}}(f_i)}.
\end{aligned}
$$

If we act by a positive linear functional A on the expressions above, its linearity yields

$$
\begin{aligned}
A\left[A_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1}) - G_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1})\right] &= \sum_{i=1}^{n}\frac{A(f_i)}{p_i A(f_i)} - \frac{A\left(\prod_{i=1}^{n} f_i^{\frac{1}{p_i}}\right)}{\prod_{i=1}^{n} A^{\frac{1}{p_i}}(f_i)} \\
&= 1 - \frac{A\left(\prod_{i=1}^{n} f_i^{\frac{1}{p_i}}\right)}{\prod_{i=1}^{n} A^{\frac{1}{p_i}}(f_i)}
\end{aligned}
$$

and

$$
\begin{aligned}
A\left[A_n^0(\mathbf{x}^{\mathbf{p}}) - G_n^0(\mathbf{x}^{\mathbf{p}})\right] &= \frac{1}{n}\sum_{i=1}^{n}\frac{A(f_i)}{A(f_i)} - \frac{A\left(\prod_{i=1}^{n} f_i^{\frac{1}{n}}\right)}{\prod_{i=1}^{n} A^{\frac{1}{n}}(f_i)} \\
&= 1 - \frac{A\left(\prod_{i=1}^{n} f_i^{\frac{1}{n}}\right)}{\prod_{i=1}^{n} A^{\frac{1}{n}}(f_i)}.
\end{aligned}
$$

By application of A to inequalities (2.27), their signs are unchanged because of its positivity. Finally, by multiplying and rearranging of the above expressions, we easily conclude the proof. □

Remark 2.13 The first inequality in (2.34) is a refinement, and the other one is a converse of Hölder's inequality, in the difference form. For $n = 2$ some related (integral) results had been obtained earlier in [179].

In the sequel we give the analogues of the above results concerning the ratio form.

Theorem 2.6 *Let p_i, $i = 1,\dots,n$, $n \in \mathbb{N}$, be such that $p_i > 1$ and $\sum_{i=1}^n \frac{1}{p_i} = 1$. Suppose functions $f_i \in L^+$ are such that $\prod_{i=1}^n f_i^{\frac{1}{p_i}}$, $\prod_{i=1}^n f_i^{1/n} \in L^+$. If $A : L \to \mathbb{R}$ is a positive linear functional, then the following inequalities hold:*

$$\begin{aligned}
&\left[\frac{n^n}{\prod_{i=1}^n A(f_i)}\right]^{\max\limits_{1\le i\le n}\left\{\frac{1}{p_i}\right\}} A\left(\left[\sum_{i=1}^n \frac{f_i}{p_iA(f_i)}\right]\left[\frac{\prod_{i=1}^n f_i^{\frac{1}{n}}}{\sum_{i=1}^n \frac{f_i}{A(f_i)}}\right]^{n\max\limits_{1\le i\le n}\left\{\frac{1}{p_i}\right\}}\right)\\
&\le \frac{A\left(\prod_{i=1}^n f_i^{\frac{1}{p_i}}\right)}{\prod_{i=1}^n A^{\frac{1}{p_i}}(f_i)}\\
&\le \left[\frac{n^n}{\prod_{i=1}^n A(f_i)}\right]^{\min\limits_{1\le i\le n}\left\{\frac{1}{p_i}\right\}} A\left(\left[\sum_{i=1}^n \frac{f_i}{p_iA(f_i)}\right]\left[\frac{\prod_{i=1}^n f_i^{\frac{1}{n}}}{\sum_{i=1}^n \frac{f_i}{A(f_i)}}\right]^{n\min\limits_{1\le i\le n}\left\{\frac{1}{p_i}\right\}}\right).
\end{aligned} \tag{2.35}$$

Proof. Follows from (2.32). By inverting, the inequalities (2.32) can be rewritten in the following form:

$$\begin{aligned}
A_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1})\left[\frac{G_n^0(\mathbf{x}^{\mathbf{p}})}{A_n^0(\mathbf{x}^{\mathbf{p}})}\right]^{n\max\limits_{1\le i\le n}\left\{\frac{1}{p_i}\right\}} &\le G_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1})\\
&\le A_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1})\left[\frac{G_n^0(\mathbf{x}^{\mathbf{p}})}{A_n^0(\mathbf{x}^{\mathbf{p}})}\right]^{n\min\limits_{1\le i\le n}\left\{\frac{1}{p_i}\right\}}.
\end{aligned} \tag{2.36}$$

Let $\mathbf{x} = (x_1,\dots,x_n)$ be defined by $x_i = [f_i/A(f_i)]^{\frac{1}{p_i}}$, $i = 1,2,\dots,n$. The expressions from (2.36) become

$$A_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1}) = \sum_{i=1}^n \frac{f_i}{p_iA(f_i)}, \quad G_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1}) = \prod_{i=1}^n \frac{f_i^{\frac{1}{p_i}}}{A^{\frac{1}{p_i}}(f_i)},$$
$$A_n^0(\mathbf{x}^{\mathbf{p}}) = \frac{1}{n}\sum_{i=1}^n \frac{f_i}{A(f_i)}, \quad G_n^0(\mathbf{x}^{\mathbf{p}}) = \prod_{i=1}^n \frac{f_i^{\frac{1}{n}}}{A^{\frac{1}{n}}(f_i)}.$$

By acting of the functional A on inequalities (2.36) and rearranging, we get (2.35). □

Remark 2.14 In this case, the first inequality in (2.35) provides the converse and the second inequality is a refinement of Hölder's inequality, in the ratio form. Namely, if we make use of the notation from Theorem 2.6, then the ratio from (2.33) can be rewritten as $A\left(G_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1})\right)$. Since $G_n^0(\mathbf{x}^{\mathbf{p}}) \le A_n^0(\mathbf{x}^{\mathbf{p}})$, it follows from the arithmetic-geometric mean inequality that

$$G_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1}) \le A_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1})\left[\frac{G_n^0(\mathbf{x}^{\mathbf{p}})}{A_n^0(\mathbf{x}^{\mathbf{p}})}\right]^{n\min\limits_{1\le i\le n}\left\{\frac{1}{p_i}\right\}} \le A_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1}). \tag{2.37}$$

By acting of the positive linear functional A on (2.37), the arrangement among the inequalities remains unchanged. Since

$$A\left(A_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1})\right)=\sum_{i=1}^{n}\frac{A(f_i)}{p_iA(f_i)}=\sum_{i=1}^{n}\frac{1}{p_i}=1,$$

the middle expression in (2.37) yields the refinement of Hölder's inequality.

Yet another class of refinements and converses of Hölder's inequality can be obtained making use of Corollary 2.1, if we recall that Hölder's inequality can be deduced directly from Jensen's inequality (for details, see e.g. [151, p. 113]). For that purpose, we observe the following setting.

Let $r,s\in\mathbb{R}$ be such that $1/r+1/s=1$. Suppose $f,g\in L^+$ and $A:L\to\mathbb{R}$ is a positive linear functional. Define the functional

$$\mathscr{J}^{\mathscr{H}}\left(\Phi,\frac{g}{f},f;A\right)=rs\left[A^{\frac{1}{r}}(f)A^{\frac{1}{s}}(g)-A\left(f^{\frac{1}{r}}g^{\frac{1}{s}}\right)\right],\tag{2.38}$$

where $\Phi:I\to\mathbb{R}$, $I\subseteq\mathbb{R}$, is defined by $\Phi(x)=-rsx^{1/s}$.

Theorem 2.7 *Let $r,s\in\mathbb{R}$ be such that $1/r+1/s=1$. Suppose $A:L\to\mathbb{R}$ is a positive linear functional, $f,g\in L^+$ and let f attain its minimal and maximal value on E. If $r>1$, then the following inequalities hold:*

$$\begin{aligned}\left[\min_{x\in E}f(x)\right]\mathscr{J}^{\mathscr{H}}\left(\Phi,\frac{g}{f},1;A\right)&\le\mathscr{J}^{\mathscr{H}}\left(\Phi,\frac{g}{f},f;A\right)\\&\le\left[\max_{x\in E}f(x)\right]\mathscr{J}^{\mathscr{H}}\left(\Phi,\frac{g}{f},1;A\right),\end{aligned}\tag{2.39}$$

where

$$\mathscr{J}^{\mathscr{H}}\left(\Phi,\frac{g}{f},1;A\right)=rs\left[A^{\frac{1}{r}}(1)A^{\frac{1}{s}}\left(\frac{g}{f}\right)-A\left(\left(\frac{g}{f}\right)^{\frac{1}{s}}\right)\right].\tag{2.40}$$

Moreover, if $0<r<1$, then inequalities in (2.39) have the reverse signs.

Proof. If we substitute f and p in (2.7) by g/f and f respectively, we find out that functional (2.38) corresponds to Jessen's functional (2.7):

$$\begin{aligned}\mathscr{J}^{\mathscr{H}}\left(\Phi,\frac{g}{f},f;A\right)&=rs\left[A^{\frac{1}{r}}(f)A^{\frac{1}{s}}(g)-A\left(f^{\frac{1}{r}}g^{\frac{1}{s}}\right)\right]\\&=rs\left[A^{1-\frac{1}{s}}(f)A^{\frac{1}{s}}(g)-A\left(f^{1-\frac{1}{s}}g^{\frac{1}{s}}\right)\right]\\&=A\left(f\Phi\left(\frac{g}{f}\right)\right)-A(f)\Phi\left(\frac{A(g)}{A(f)}\right).\end{aligned}$$

If $r>1$, then $\Phi''(x)=x^{1/s-2}$, $x>0$, i.e. Φ is a convex function and inequalities (2.39) hold analogously as in Corollary 2.1. On the other hand, if $0<r<1$, then $rs<0$. Since the expressions $\mathscr{J}^{\mathscr{H}}(\Phi,g/f,f;A)$ and $\mathscr{J}^{\mathscr{H}}(\Phi,g/f,1;A)$ contain the factor rs, inequalities in (2.39) have the reverse signs. □

Similar considerations to those in the previous theorem lead to the following result.

Theorem 2.8 *Let $r,s \in \mathbb{R}$ be such that $1/r + 1/s = 1$. Suppose $A : L \to \mathbb{R}$ is a positive liner functional, $f,g \in L^+$, and let f attain its minimal and maximal value on E. If $r > 1$, then the following inequalities hold:*

$$\begin{aligned}
&\left[\min_{x\in E} f(x)\right]\left[A^{s-1}(f)A\left(\frac{g}{f}\right) - \left(\frac{A(f)}{A(1)}\right)^{s-1} A^s\left(\left(\frac{g}{f}\right)^{\frac{1}{s}}\right)\right] \\
&\quad \le \left[A^{\frac{1}{r}}(f)A^{\frac{1}{s}}(g)\right]^s - A^s\left(f^{\frac{1}{r}}g^{\frac{1}{s}}\right) \\
&\quad \le \left[\max_{x\in E} f(x)\right]\left[A^{s-1}(f)A\left(\frac{g}{f}\right) - \left(\frac{A(f)}{A(1)}\right)^{s-1} A^s\left(\left(\frac{g}{f}\right)^{\frac{1}{s}}\right)\right].
\end{aligned} \tag{2.41}$$

If $0 < r < 1$, then the inequalities in (2.41) have the reverse signs.

Proof. We observe the Jessen-type functional $\mathscr{J}^{\overline{\mathscr{H}}}\left(\Phi, (g/f)^{1/s}, f; A\right)$, where $\Phi(x) = x^s/(s(s-1))$. It is obvious that Φ is convex for $x > 0$ since $\Phi''(x) = x^{s-2}$. It follows that

$$\begin{aligned}
\mathscr{J}^{\overline{\mathscr{H}}}\left(\Phi, \left(\frac{g}{f}\right)^{\frac{1}{s}}, f; A\right) &= A\left(f\Phi\left(\left(\frac{g}{f}\right)^{\frac{1}{s}}\right)\right) - A(f)\Phi\left(\frac{A\left(f^{\frac{1}{r}}g^{\frac{1}{s}}\right)}{A(f)}\right) \\
&= \frac{1}{s(s-1)}\left[A(g) - A^{1-s}(f)A^s\left(f^{\frac{1}{r}}g^{\frac{1}{s}}\right)\right],
\end{aligned}$$

and

$$\begin{aligned}
\mathscr{J}^{\overline{\mathscr{H}}}\left(\Phi, \left(\frac{g}{f}\right)^{\frac{1}{s}}, 1; A\right) &= A\left(\Phi\left(\left(\frac{g}{f}\right)^{\frac{1}{s}}\right)\right) - A(1)\Phi\left(\frac{A\left(\left(\frac{g}{f}\right)^{\frac{1}{s}}\right)}{A(1)}\right) \\
&= \frac{1}{s(s-1)}\left[A\left(\frac{g}{f}\right) - A^{1-s}(1)A^s\left(\left(\frac{g}{f}\right)^{\frac{1}{s}}\right)\right].
\end{aligned}$$

Hence if we insert the above expressions in (2.7) and multiply obtained series of inequalities by $s(s-1)A^{s-1}(f)$, we get (2.41). If $0 < r < 1$, then $s(s-1)A^{s-1}(f) < 0$, which changes the signs of the inequalities in (2.41). $\square$

Remark 2.15 Inequalities (2.39) and (2.41) present the refinements and the converses of Hölder's inequality. Some related converses of this type can also be found in [13].

2.1.5 Properties of the functionals related to the Minkowski inequality

As for the functionals derived from the Minkowski inequality, the one that is closely related to Hölder's inequality, these had been formerly investigated in [87]. They had been in the first place derived from the integral form of the Minkowski inequality that we cite below and then proceed with the accompanied elaboration from [87].

Theorem 2.9 *Let (X,Σ_X,μ) and (Y,Σ_Y,ν) be measure spaces and let f be a non-negative function on $X\times Y$ which is integrable with respect to the measure $\mu\times\nu$. If $p\geq 1$, then*

$$\left[\int_X\left(\int_Y f(x,y)d\nu(y)\right)^p d\mu(x)\right]^{\frac{1}{p}} \leq \int_Y\left(\int_X f^p(x,y)d\mu(x)\right)^{\frac{1}{p}} d\nu(y).$$

If $0<p<1$ *and*

(i) $\int_X\left(\int_Y f(x,y)d\nu(y)\right)^p d\mu(x)>0,\ \int_Y f(x,y)d\nu(y)>0,$

then the reversed inequality holds.

If $p<0$ *and the above-mentioned assumptions (i) and the additional one*

(ii) $\int_X f^p(x,y)d\mu(x)>0$ *ν-a.e. hold,*

then the reversed inequality holds.

In specified settings, Theorem 2.9 also provides us with the discrete Minkowski inequality (1.18) and eventually, with its form for integrals.

Theorem 2.10 *Let (X,Σ_X,μ) be a measure space and $f_1,\ldots,f_n$ be non-negative integrable functions. If $p\geq 1$, then*

$$\left(\int_X (f_1+\ldots+f_n)^p d\mu\right)^{1/p} \leq \left(\int_X f_1^p d\mu\right)^{1/p}+\ldots+\left(\int_X f_n^p d\mu\right)^{1/p}. \tag{2.42}$$

If $0<p<1$ or if $p<0$ with $\int_X f_1^p d\mu>0$, …, $\int_X f_n^p d\mu>0$, then the reversed inequality in (2.42) holds.

Now let us suppose that p is a real number, $p\neq 0$, f is a non-negative function on $X\times Y$, $\mu\in Cone(\Sigma_X)$, $\nu\in Cone(\Sigma_Y)$. With $\mathsf{I}_X(f,\nu,p)$ we denote the set of all measures $\omega\in Cone(\Sigma_X)$, such that f is integrable with respect to $\omega\times\nu$ and

$$\left.\begin{array}{l}\left[\int_Y\left(\int_X f^p(x,y)d\omega(x)\right)^{\frac{1}{p}} d\nu(y)\right]^p \text{ and}\\ \int_X\left(\int_Y f(x,y)d\nu(y)\right)^p d\omega(x) \text{ are finite,}\\ \text{if } 0<p<1, \text{ condition (i) from Theorem 2.9 holds,}\\ \text{if } p<0, \text{ conditions (i) and (ii) from Theorem 2.9 hold.}\end{array}\right\} \tag{2.43}$$

With $\mathsf{J}_Y(f,\mu,p)$ we denote the set of all measures $\lambda \in Cone(\Sigma_Y)$, such that f is integrable with respect to $\mu \times \lambda$ and

$$\left.\begin{array}{l} \int_Y \left(\int_X f^p(x,y)d\mu(x)\right)^{\frac{1}{p}} d\lambda(y) \text{ and} \\ \left[\int_X \left(\int_Y f(x,y)d\lambda(y)\right)^p d\mu(x)\right]^{\frac{1}{p}} \text{ are finite,} \\ \text{if } 0<p<1, \text{ condition (i) from Theorem 2.9 holds,} \\ \text{if } p<0, \text{ conditions (i) and (ii) from Theorem 2.9 hold.} \end{array}\right\} \tag{2.44}$$

Let us consider the functionals $\mathsf{M}_1(f,\cdot,\nu,p) : \mathsf{I}_X(f,\nu,p) \to \mathbf{R}$ and $\mathsf{M}_2(f,\mu,\cdot,p) : \mathsf{J}_Y(f,\mu,p) \to \mathbf{R}$ defined as

$$\mathsf{M}_1(f,\omega,\nu,p)=\left[\int_Y\left(\int_X f^p(x,y)d\omega(x)\right)^{\frac{1}{p}} d\nu(y)\right]^p-\int_X\left(\int_Y f(x,y)d\nu(y)\right)^p d\omega(x)$$

and

$$\mathsf{M}_2(f,\mu,\lambda,p)=\int_Y\left(\int_X f^p(x,y)d\mu(x)\right)^{\frac{1}{p}} d\lambda(y)-\left[\int_X\left(\int_Y f(x,y)d\lambda(y)\right)^p d\mu(x)\right]^{\frac{1}{p}}.$$

Some properties of these functionals are obvious. We have the following:

(i) Mappings M_1 and M_2 are positive homogeneous, i.e. $\mathsf{M}_1(f,a\omega,\nu,p) = a\mathsf{M}_1(f,\omega,\nu,p)$, and $\mathsf{M}_2(f,\mu,a\lambda,p) = a\mathsf{M}_2(f,\mu,\lambda,p)$, for any $a>0$.

(ii) If $p \geq 1$ or $p<0$ and $\omega \in \mathsf{I}_X(f,\nu,p)$, then $\mathsf{M}_1(f,\omega,\nu,p) \geq 0$, and if $0<p<1$, then $\mathsf{M}_1(f,\omega,\nu,p) \leq 0$.

(iii) If $p \geq 1$, then $\mathsf{M}_2(f,\mu,\lambda,p) \geq 0$, and if $p<1$, $p \neq 0$, then $\mathsf{M}_2(f,\mu,\lambda,p) \leq 0$ for $\lambda \in \mathsf{J}_Y(f,\mu,p)$.

Theorem 2.11 *(i) If $p \geq 1$ or $p<0$, then $\mathsf{M}_1(f,\cdot,\nu,p)$ is superadditive on $\mathsf{I}_X(f,\nu,p)$. If $0<p<1$, then $\mathsf{M}_1(f,\cdot,\nu,p)$ is subadditive.*

If $p \geq 1$, then $\mathsf{M}_2(f,\mu,\cdot,p)$ is superadditive on $\mathsf{J}_Y(f,\mu,p)$. If $p<1$, $p \neq 0$, then $\mathsf{M}_2(f,\mu,\cdot,p)$ is subadditive.

(ii) If $\omega_1,\omega_2,\omega_2-\omega_1 \in \mathsf{I}_X(f,\nu,p)$, then

$$0 \leq \mathsf{M}_1(f,\omega_1,\nu,p) \leq \mathsf{M}_1(f,\omega_2,\nu,p), \quad \textit{for} \quad p \geq 1 \textit{ or } p<0, \tag{2.45}$$

and if $0<p<1$, then reversed signs in (2.45) hold. If $\lambda_1,\lambda_2,\lambda_2-\lambda_1 \in \mathsf{J}_Y(f,\mu,p)$, then

$$0 \leq \mathsf{M}_2(f,\mu,\lambda_1,p) \leq \mathsf{M}_2(f,\mu,\lambda_2,p), \quad \textit{for} \quad p \geq 1, \tag{2.46}$$

and if $p<1$, $p \neq 0$, then reversed signs in (2.46) hold.

Proof. (i) Let us transform $\mathsf{M}_1(f,\omega_1+\omega_2,\nu,p)-\mathsf{M}_1(f,\omega_1,\nu,p)-\mathsf{M}_1(f,\omega_2,\nu,p)$.

$$\begin{aligned}
&\mathsf{M}_1(f,\omega_1+\omega_2,\nu,p)-\mathsf{M}_1(f,\omega_1,\nu,p)-\mathsf{M}_1(f,\omega_2,\nu,p)\\
&=\left[\int_Y\left(\int_X f^p(x,y)d(\omega_1+\omega_2)(x)\right)^{\frac{1}{p}}d\nu(y)\right]^p-\int_X\left(\int_Y f(x,y)d\nu(y)\right)^p d(\omega_1+\omega_2)(x)\\
&\quad-\left[\int_Y\left(\int_X f^p(x,y)d\omega_1(x)\right)^{\frac{1}{p}}d\nu(y)\right]^p+\int_X\left(\int_Y f(x,y)d\nu(y)\right)^p d\omega_1(x)\\
&\quad-\left[\int_Y\left(\int_X f^p(x,y)d\omega_2(x)\right)^{\frac{1}{p}}d\nu(y)\right]^p+\int_X\left(\int_Y f(x,y)d\nu(y)\right)^p d\omega_2(x)\\
&=\left[\int_Y\left(\int_X f^p(x,y)d(\omega_1+\omega_2)(x)\right)^{\frac{1}{p}}d\nu(y)\right]^p\\
&\quad-\left[\int_Y\left(\int_X f^p(x,y)d\omega_1(x)\right)^{\frac{1}{p}}d\nu(y)\right]^p-\left[\int_Y\left(\int_X f^p(x,y)d\omega_2(x)\right)^{\frac{1}{p}}d\nu(y)\right]^p.
\end{aligned}$$

Using the Minkowski inequality for integrals (2.42) with p replaced by $\dfrac{1}{p}$ we have

$$\begin{aligned}
&\mathsf{M}_1(f,\omega_1+\omega_2,\nu,p)-\mathsf{M}_1(f,\omega_1,\nu,p)-\mathsf{M}_1(f,\omega_2,\nu,p)\\
&\qquad\begin{cases}\geq 0, \text{ if } p\geq 1 \text{ or } p<0\\ \leq 0, \text{ if } 0<p<1.\end{cases}
\end{aligned}\tag{2.47}$$

So, M_1 is superadditive for $p\geq 1$ or $p<0$ and it is subadditive for $0<p<1$. The proof for M_2 is similar. After simple transforming, we have

$$\begin{aligned}
&\mathsf{M}_2(f,\mu,\lambda_1+\lambda_2,p)-\mathsf{M}_2(f,\mu,\lambda_1,p)-\mathsf{M}_2(f,\mu,\lambda_2,p)\\
&=\left[\int_X\left(\int_Y f(x,y)d\lambda_1(y)\right)^p d\mu(x)\right]^{\frac{1}{p}}+\left[\int_X\left(\int_Y f(x,y)d\lambda_2(y)\right)^p d\mu(x)\right]^{\frac{1}{p}}\\
&\quad-\left[\int_X\left(\int_Y f(x,y)d(\lambda_1+\lambda_2)(y)\right)^p d\mu(x)\right]^{\frac{1}{p}}.
\end{aligned}$$

Using the Minkowski inequality for integrals (2.42) we have that the last sum is non-negative for $p\geq 1$ and it is non-positive for $p<1$, $p\neq 0$. So, the proof of case (i) is established.

(ii) If $p\geq 1$ or $p<0$, then using superadditivity and positivity of M_1 we obtain

$$\begin{aligned}
\mathsf{M}_1(f,\omega_2,\nu,p)&=\mathsf{M}_1(f,\omega_1+(\omega_2-\omega_1),\nu,p)\\
&\geq \mathsf{M}_1(f,\omega_1,\nu,p)+\mathsf{M}_1(f,\omega_2-\omega_1,\nu,p)\\
&\geq \mathsf{M}_1(f,\omega_1,\nu,p)
\end{aligned}$$

and the proof of (2.45) is established.

If $0 < p < 1$, then using subadditivity and negativity of M_1 we obtain

$$\begin{aligned}\mathsf{M}_1(f,\omega_2,\nu,p) &\le \mathsf{M}_1(f,\omega_1,\nu,p)+\mathsf{M}_1(f,\omega_2-\omega_1,\nu,p)\\ &\le \mathsf{M}_1(f,\omega_1,\nu,p).\end{aligned}$$

The proof for M_2 is similar. □

Corollary 2.7 *(i) Let $\omega_1,\ \omega_2 \in \mathsf{I}_X(f,\nu,p)$. If $c,C \in \mathbf{R}^+$ are such that $C\omega_2-\omega_1, \omega_1-c\omega_2 \in \mathsf{I}_X(f,\nu,p)$, then for $p \ge 1$ or $p < 0$*

$$\begin{aligned}&c\left\{\left[\int_Y\left(\int_X f^p(x,y)d\omega_2(x)\right)^{\frac{1}{p}}d\nu(y)\right]^p-\int_X\left(\int_Y f(x,y)d\nu(y)\right)^p d\omega_2(x)\right\}\\ &\le \left[\int_Y\left(\int_X f^p(x,y)d\omega_1(x)\right)^{\frac{1}{p}}d\nu(y)\right]^p-\int_X\left(\int_Y f(x,y)d\nu(y)\right)^p d\omega_1(x)\\ &\le C\left\{\left[\int_Y\left(\int_X f^p(x,y)d\omega_2(x)\right)^{\frac{1}{p}}d\nu(y)\right]^p\int_X\left(\int_Y f(x,y)d\nu(y)\right)^p d\omega_2(x)\right\}.\end{aligned}$$

If $0 < p < 1$, then the above inequalities hold in reversed direction.

(ii) Let $\lambda_1,\ \lambda_2 \in \mathsf{J}_Y(f,\mu,p)$. If $c,C \in \mathbf{R}^+$ are such that $C\lambda_2-\lambda_1, \lambda_1-c\lambda_2 \in \mathsf{J}_Y(f,\mu,p)$, then for $p \ge 1$

$$\begin{aligned}&c\left\{\int_Y\left(\int_X f^p(x,y)d\mu(x)\right)^{\frac{1}{p}}d\lambda_2(y)-\left[\int_X\left(\int_Y f(x,y)d\lambda_2(y)\right)^p d\mu(x)\right]^{\frac{1}{p}}\right\}\\ &\le \int_Y\left(\int_X f^p(x,y)d\mu(x)\right)^{\frac{1}{p}}d\lambda_1(y)-\left[\int_X\left(\int_Y f(x,y)d\lambda_1(y)\right)^p d\mu(x)\right]^{\frac{1}{p}}\\ &\le C\left\{\int_Y\left(\int_X f^p(x,y)d\mu(x)\right)^{\frac{1}{p}}d\lambda_2(y)\left[\int_X\left(\int_Y f(x,y)d\lambda_2(y)\right)^p d\mu(x)\right]^{\frac{1}{p}}\right\}.\end{aligned}$$

If $p < 1,\ p \ne 0$, then the above inequalities hold in reversed direction.

Let us consider two other mappings. Let $I \in \Sigma_X$ and $J \in \Sigma_Y$ be non-empty sets, $p \in \mathbf{R}$, $p \ne 0$, $\mu \in Cone(\Sigma_X)$, $\nu \in Cone(\Sigma_Y)$ and f be a non-negative function which is $(\mu\times\nu)$-integrable. As usual, the characteristic mapping of set S is denoted by χ_S.

Let us denote with $\mathsf{A} = \mathsf{A}(J,f,\mu,\nu,p) \subseteq \Sigma_X$ a family of sets $A \in \Sigma_X$ for which condition (2.43) holds when $f \to \chi_{A\times J}f$ and with $\mathsf{B} = \mathsf{B}(I,f,\mu,\nu,p) \subseteq \Sigma_Y$ a family of sets $B \in \Sigma_Y$ for which condition (2.44) holds when $f \to \chi_{I\times B}f$. We define functionals $\mathsf{M}_3(\cdot,J,f,\mu,\nu,p)$: $\mathsf{A} \to \mathbf{R}$ and $\mathsf{M}_4(I,\cdot,f,\mu,\nu,p) : \mathsf{B} \to \mathbf{R}$ as follows:

$$\mathsf{M}_3(A,J,f,\mu,\nu,p) = \mathsf{M}_1(\chi_{A\times J}f,\mu,\nu,p),\ \ A \in \mathsf{A}$$

and

$$\mathsf{M}_4(I,B,f,\mu,\nu,p) = \mathsf{M}_2(\chi_{I\times B}f,\mu,\nu,p),\ \ B \in \mathsf{B}.$$

The following theorem describes properties of superadditivity and monotonicity of M_3 and M_4.

Theorem 2.12 *(i) If $A_1, A_2 \in \mathsf{A}$ with $A_1 \cap A_2 = \emptyset$, then*

$$\mathsf{M}_3(A_1 \cup A_2, J, f, \mu, \nu, p) \geq \mathsf{M}_3(A_1, J, f, \mu, \nu, p) + \mathsf{M}_3(A_2, J, f, \mu, \nu, p),$$

for $p \geq 1$ or $p < 0$. If $0 < p < 1$, then the reverse inequality holds.
If $A_1, A_2 \in \mathsf{A}$ with $A_1 \subseteq A_2$, then

$$\mathsf{M}_3(A_1, J, f, \mu, \nu, p) \leq \mathsf{M}_3(A_2, J, f, \mu, \nu, p), \ \textit{for} \ \ p \geq 1 \ \ \textit{or} \ \ p < 0,$$

$$\mathsf{M}_3(A_1, J, f, \mu, \nu, p) \geq \mathsf{M}_3(A_2, J, f, \mu, \nu, p), \ \textit{for} \ \ 0 < p < 1.$$

(ii) If $B_1, B_2 \in \mathsf{B}$ with $B_1 \cap B_2 = \emptyset$, then

$$\mathsf{M}_4(I, B_1 \cup B_2, f, \mu, \nu, p) \geq \mathsf{M}_4(I, B_1, f, \mu, \nu, p) + \mathsf{M}_4(I, B_2, f, \mu, \nu, p),$$

for $p \geq 1$. If $p < 1$, $p \neq 0$, then the reverse inequality holds.
If $B_1, B_2 \in \mathsf{B}$ with $B_1 \subseteq B_2$, then

$$\mathsf{M}_4(I, B_1, f, \mu, \nu, p) \leq \mathsf{M}_4(I, B_2, f, \mu, \nu, p), \ \textit{for} \ \ p \geq 1,$$

and the reverse inequality holds for $p < 1$, $p \neq 0$.

Proof. Similar to the proof of the previous theorem and thus left to the reader. □

Theorem 2.13 *Let $\phi : [0,\infty) \to [0,\infty)$ be a concave function, f be a non-negative function on $X \times Y$, $\omega_1, \omega_2 \in Cone(\Sigma_X)$ and $\nu \in Cone(\Sigma_Y)$. If $\phi \circ \omega_1$, $\phi \circ \omega_2$ and $\phi \circ (a\omega_1 + (1-a)\omega_2)$ belong to $\mathsf{I}_X(f, \nu, p)$ for some $a \in [0,1]$, then*

$$\mathsf{M}_1(f, \phi \circ (a\omega_1 + (1-a)\omega_2), \nu, p) \geq a\mathsf{M}_1(f, \phi \circ \omega_1, \nu, p) + (1-a)\mathsf{M}_1(f, \phi \circ \omega_2, \nu, p), \tag{2.48}$$

where $p \geq 1$. If $0 < p < 1$, then the reversed sign in (2.48) *holds.*

Proof. For any $I \in \Sigma_X$ we have

$$\begin{aligned}
(\phi \circ (a\omega_1 + (1-a)\omega_2))(I) &= \phi(a\omega_1(I) + (1-a)\omega_2(I)) \\
&\geq a\phi(\omega_1(I)) + (1-a)\phi(\omega_2(I)) \\
&= (a(\phi \circ \omega_1) + (1-a)(\phi \circ \omega_2))(I),
\end{aligned}$$

where concavity of function ϕ is used. For measures $a(\phi \circ \omega_1) + (1-a)(\phi \circ \omega_2)$ and $\phi \circ (a\omega_1 + (1-a)\omega_2)$ it follows

$$\phi \circ (a\omega_1 + (1-a)\omega_2) \geq a(\phi \circ \omega_1) + (1-a)(\phi \circ \omega_2).$$

Using (2.47) and (2.45) we have the following

$$\begin{aligned}
\mathsf{M}_1(f, \phi \circ (a\omega_1 + (1-a)\omega_2), \nu, p) &\geq \mathsf{M}_1(f, a(\phi \circ \omega_1) + (1-a)(\phi \circ \omega_2), \nu, p) \\
&\geq \mathsf{M}_1(f, a(\phi \circ \omega_1), \nu, p) + \mathsf{M}_1(f, (1-a)(\phi \circ \omega_2), \nu, p) \\
&= a\mathsf{M}_1(f, \phi \circ \omega_1, \nu, p) + (1-a)\mathsf{M}_1(f, \phi \circ \omega_2, \nu, p)
\end{aligned}$$

and the proof is established. □

Similar result can be stated for the functional M_2.

Remark 2.16 If we employ various measures in theorems 2.11 and 2.12, we obtain various results concerning superadditivity and monotonicity of the functionals involved, related to the Minkowski inequality. Some of these results are well known ones, e.g., for the discrete measure, properties of M_1 and the corresponding refinements of the discrete Minkowski inequality are given in [66], [168], [182].

Remark 2.17 Theorem 2.11 provides us nevertheless with another refinement of the Minkowski inequality. Namely, put $X,Y \subseteq \mathbf{N}$ and let μ be a measure on X and λ_1 and λ_2 be measures on Y such that $\mu(i) = u_i \geq 0$, $i \in X$, $\lambda_1(j) = n_j \geq 0$, $\lambda_2(j) = p_j \geq 0$, $j \in Y$. Then, for fixed f,μ,p, the functional M_2 has a form

$$\mathsf{M}_2(f,\mu,\lambda_1,p) = \sum_{j\in Y} n_j \left(\sum_{i\in X} u_i a_{ij}^p\right)^{1/p} - \left(\sum_{i\in X} u_i \left(\sum_{j\in Y} n_j a_{ij}\right)^p\right)^{1/p},$$

where $f(i,j) = a_{ij} \geq 0$. If $p \geq 1$, then $\mathsf{M}_2(f,\mu,\cdot,p)$ is superadditive and if $p_j \geq n_j$, $(j \in Y)$, then

$$\begin{aligned} 0 &\leq \sum_{j\in Y} n_j \left(\sum_{i\in X} u_i a_{ij}^p\right)^{1/p} - \left(\sum_{i\in X} u_i \left(\sum_{j\in Y} n_j a_{ij}\right)^p\right)^{1/p} \\ &\leq \sum_{j\in Y} p_j \left(\sum_{i\in X} u_i a_{ij}^p\right)^{1/p} - \left(\sum_{i\in X} u_i \left(\sum_{j\in Y} p_j a_{ij}\right)^p\right)^{1/p}, \end{aligned}$$

where we suppose that all sums are finite. It is a refinement of the discrete Minkowski inequality.

Let I,f,μ,λ_1,p be fixed objects described in the second section and in the introduction of this section. Then M_4 has the following form:

$$\begin{aligned} \mathsf{M}_4(B) &= \mathsf{M}_4(I,B,f,\nu,\lambda,p) \\ &= \sum_{j\in B} n_j \left(\sum_{i\in I} u_i a_{ij}^p\right)^{1/p} - \left(\sum_{i\in I} u_i \left(\sum_{j\in B} n_j a_{ij}\right)^p\right)^{1/p}, \end{aligned}$$

where $B \in \Sigma_Y$ is such that all sums are finite.

As a consequence of Theorem 2.12, for $p \geq 1$, we have the following:

(i) if $B_1,B_2 \in \Sigma_Y$, $B_1 \cap B_2 = \emptyset$, then

$$\mathsf{M}_4(B_1 \cup B_2) \geq \mathsf{M}_4(B_1) + \mathsf{M}_4(B_2).$$

(ii) if J_m is a notation for a set from Σ_Y with m elements, then for $J_m \supset J_{m-1} \supset \ldots \supset J_2$ we have

$$\mathsf{M}_4(J_m) \geq \mathsf{M}_4(J_{m-1}) \geq \ldots \geq \mathsf{M}_4(J_2) \geq 0$$

and

$$\mathsf{M}_4(J_m) \geq \max\{\mathsf{M}_4(J_2) : J_2 \subset J_m,\ \operatorname{card}(J_2) = 2\}.$$

2.2 Properties of McShane's functional and applications

Following the analogous procedure to the one employed in the case of Jessen's functional, we deduce McShane's functional from the weight McShane's inequality and establish its properties of superadditivity and monotonicity. These provide deriving the lower and upper bound for the functional, by means of the non-weight functional of the same type. Again, such bounds are the starting point for the applications to the variety of the existing inequalities. The contents of this section corresponds for the most part to the contents of the published paper [111].

2.2.1 McShane's functional

The environment of the positive linear functionals acting on a linear class of real valued functions, described in the Section 2.1.1 is implicitly understood in this section as well. Hence the previously used notation is still valid. Furthermore, the reminder of the needed *weight* form of McShane's inequality (1.16) is given in the following theorem.

Theorem 2.14 *Let E be a nonempty set and let L be a linear class of real-valued functions satisfying L1 and L2. Suppose $\Phi : K \to \mathbb{R}$ is a continuous and convex function defined on a closed convex set $K \subseteq \mathbb{R}^n$, and let $p \in L^+$. If $A : L \to \mathbb{R}$ is a positive linear functional with $A(p) > 0$, then for all functions $\mathbf{f} = (f_1, \ldots, f_n) \in L^n$, such that $p\mathbf{f} \in L^n$ and $p\Phi(\mathbf{f}) \in L$ the following inequality holds:*

$$A(p)\Phi\left(\frac{A(p\mathbf{f})}{A(p)}\right) \leq A\left(p\Phi(\mathbf{f})\right). \tag{2.49}$$

Remark 2.18 Denote $A_1(\mathbf{f}) = \frac{A(p\mathbf{f})}{A(p)}$, $A(p) > 0$. It follows that A_1 is a normalized (positive linear) functional, i.e. $A_1(\mathbf{1}) = 1$, and inequality (1.12) can be applied. One can find the proof of the statement $A_1(\mathbf{f}) \in K$ in [177, p. 48].

We now deduce McShane's functional which can be interpreted as a multidimensional generalization of Jessen's functional (2.2), as follows:

$$\mathscr{M}\left(\Phi, \mathbf{f}, p; A\right) = A\left(p\Phi(\mathbf{f})\right) - A(p)\Phi\left(\frac{A(p\mathbf{f})}{A(p)}\right). \tag{2.50}$$

For fixed Φ, $\mathbf{f}$ and A, we consider $\mathscr{M}(\Phi, \mathbf{f}, \cdot; A)$ as a function on L^+. If Φ is a convex function, it follows from (2.49) that $\mathscr{M}\left(\Phi, \mathbf{f}, p; A\right) \geq 0$, for all $p \in L^+$.

2.2.2 Superadditivity of McShane's functional

The following theorem is a generalization of Theorem 2.2.

Theorem 2.15 *Let $A : L \to \mathbb{R}$ be a positive linear functional. Suppose $\mathbf{f} = (f_1, \ldots, f_n)$ is a function in L^n and p, q are functions in L^+. If $\Phi : K \to \mathbb{R}$ is a continuous and convex function defined on a closed convex set $K \subseteq \mathbb{R}^n$, then*

$$\mathscr{M}(\Phi, \mathbf{f}, p+q; A) \geq \mathscr{M}(\Phi, \mathbf{f}, p; A) + \mathscr{M}(\Phi, \mathbf{f}, q; A), \tag{2.51}$$

that is, $\mathscr{M}(\Phi, \mathbf{f}, \cdot; A)$ is superadditive on L^+. Moreover, if $p, q \in L^+$ are such that $p \geq q$, then

$$\mathscr{M}(\Phi, \mathbf{f}, p; A) \geq \mathscr{M}(\Phi, \mathbf{f}, q; A) \geq 0, \tag{2.52}$$

that is, $\mathscr{M}(\Phi, \mathbf{f}, \cdot; A)$ is increasing on L^+.

Proof. According to definition (2.50) and linearity of the functional A, it follows:

$$\begin{aligned}
\mathscr{M}(\Phi, \mathbf{f}, p+q; A) &= A((p+q)\Phi(\mathbf{f})) - A(p+q)\phi\left(\frac{A((p+q)\mathbf{f})}{A(p+q)}\right) \\
&= A(p\Phi(\mathbf{f}) + q\Phi(\mathbf{f})) - (A(p)+A(q))\,\Phi\left(\frac{A(p\mathbf{f}+q\mathbf{f})}{A(p)+A(q)}\right) \\
&= A(p\Phi(\mathbf{f})) + A(q\Phi(\mathbf{f})) - (A(p)+A(q))\,\Phi\left(\frac{A(p\mathbf{f})+A(q\mathbf{f})}{A(p)+A(q)}\right).
\end{aligned} \tag{2.53}$$

On the other side, due to convexity of Φ we get

$$\begin{aligned}
\Phi\left(\frac{A(p\mathbf{f})+A(q\mathbf{f})}{A(p)+A(q)}\right) &= \Phi\left(\frac{A(p)}{A(p)+A(q)} \cdot \frac{A(p\mathbf{f})}{A(p)} + \frac{A(q)}{A(p)+A(q)} \cdot \frac{A(q\mathbf{f})}{A(q)}\right) \\
&\leq \frac{A(p)}{A(p)+A(q)}\Phi\left(\frac{A(p\mathbf{f})}{A(p)}\right) + \frac{A(q)}{A(p)+A(q)}\Phi\left(\frac{A(q\mathbf{f})}{A(q)}\right),
\end{aligned}$$

which can be rewritten as

$$(A(p)+A(q))\,\Phi\left(\frac{A(p\mathbf{f})+A(q\mathbf{f})}{A(p)+A(q)}\right) \leq A(p)\Phi\left(\frac{A(p\mathbf{f})}{A(p)}\right) + A(q)\Phi\left(\frac{A(q\mathbf{f})}{A(q)}\right). \tag{2.54}$$

Finally, (2.53) and (2.54) yield

$$\begin{aligned}
\mathscr{M}(\Phi, \mathbf{f}, p+q; A) &\geq A(p\Phi(\mathbf{f})) + A(q\Phi(\mathbf{f})) - A(p)\Phi\left(\frac{A(p\mathbf{f})}{A(p)}\right) - A(q)\Phi\left(\frac{A(q\mathbf{f})}{A(q)}\right) \\
&= \mathscr{M}(\Phi, \mathbf{f}, p; A) + \mathscr{M}(\Phi, \mathbf{f}, q; A),
\end{aligned}$$

that is, functional $\mathscr{M}(\Phi, \mathbf{f}, \cdot; A)$ is superadditive on L^+.

As for increase on L^+ of the functional $\mathscr{M}(\Phi, \mathbf{f}, \cdot; A)$, we write $p = (p-q)+q$, for $p \geq q \geq 0$. Hence (2.51) yields

$$\mathscr{M}(\Phi, \mathbf{f}, p; A) = \mathscr{M}(\Phi, \mathbf{f}, p-q+q; A) \geq \mathscr{M}(\Phi, \mathbf{f}, p-q; A) + \mathscr{M}(\Phi, \mathbf{f}, q; A).$$

Since $\mathscr{M}(\Phi,\mathbf{f},p-q;A)\geq 0$, it follows that $\mathscr{M}(\Phi,\mathbf{f},p;A)\geq \mathscr{M}(\Phi,\mathbf{f},q;A)$, which ends the proof. □

Remark 2.19 If Φ is a continuous and concave function, the signs of inequalities (2.51) and (2.52) are reversed, that is, functional $\mathscr{M}(\Phi,\mathbf{f},\cdot;A)$ is subadditive and decreasing on L^+. Namely, in the case of Φ being concave, the sign of Jensen's inequality is reversed and $\mathscr{M}(\Phi,\mathbf{f},p;A)\leq 0$, for all $p\in L^+$. This remark on the concavity of the function Φ is going to be taken into account in the sequel, in all the similar results, even if it is not accentuated.

The following corollary provides the lower and the upper bound for the functional (2.50), which are expressed by means of the non-weight functional of the same type.

Corollary 2.8 *Let function* $\mathbf{f}$ *and functional* A *be as in Theorem 2.15. Suppose* $p\in L^+$ *attains its minimal and maximal value on* E. *If* $\Phi: K\to\mathbb{R}$, *where* $K\subseteq\mathbb{R}^n$ *is a closed convex set, is a continuous and convex function, then the following inequalities hold:*

$$\left[\min_{x\in E}p(x)\right]\mathscr{M}(\Phi,\mathbf{f},1;A)\leq \mathscr{M}(\Phi,\mathbf{f},p;A)\leq \left[\max_{x\in E}p(x)\right]\mathscr{M}(\Phi,\mathbf{f},1;A), \tag{2.55}$$

where

$$\mathscr{M}(\Phi,\mathbf{f},1;A)=A(\Phi(\mathbf{f})\cdot 1)-A(1)\Phi\left(\frac{A(\mathbf{f})}{A(1)}\right). \tag{2.56}$$

Proof. The inequalities are proved by making use of (2.52). Namely, as $p\in L^+$ attains its minimal and maximal value on E, it is clear that

$$\min_{x\in E}p(x)\leq p(x)\leq \max_{x\in E}p(x),$$

and we observe two constant functions

$$\underline{p}=\min_{x\in E}p(x)\quad\text{and}\quad \overline{p}=\max_{x\in E}p(x).$$

Double application of the property (2.52) yields (2.55), since

$$\mathscr{M}\left(\Phi,\mathbf{f},\underline{p}\cdot 1;A\right)=\underline{p}\mathscr{M}(\Phi,\mathbf{f},1;A)\text{ and }\mathscr{M}(\Phi,\mathbf{f},\overline{p}\cdot 1;A)=\overline{p}\mathscr{M}(\Phi,\mathbf{f},1;A).$$

□

Remark 2.20 If $p\in L^+$ is a bounded function on E, then infimum (supremum) in (2.55) are observed. This fact is going to be taken into account in all the results on the non-weight bounds of the functionals of this type.

As for the first application of the monotonicity property (2.52), we recall Theorem 1.35 on comparative inequalities for the discrete Jensen's functional, proved by S. S. Dragomir and its generalization – Theorem 2.3 from [53], for which the multidimensional generalization is given in the same paper. We cite it here.

Theorem 2.16 (SEE [53]) *Let $\Phi : K \to \mathbb{R}$ be a continuous and convex function defined on a closed convex set $K \subseteq \mathbb{R}^n$. Suppose A is a positive linear functional and m and M are real constants, such that for p and $q \in L^+$ and for all $x \in E$*

$$p(x) - mq(x) \geq 0, \quad Mq(x) - p(x) \geq 0 \quad \text{and}$$

$$A(p) - mA(q) > 0, \quad MA(q) - A(p) > 0.$$

Then the following inequalities hold:

$$m\mathscr{M}(\Phi, \mathbf{f}, q; A) \leq \mathscr{M}(\Phi, \mathbf{f}, p; A) \leq M\mathscr{M}(\Phi, \mathbf{f}, q; A). \tag{2.57}$$

Remark 2.21 The proof of Theorem 2.16 is improved when Theorem 2.15 is applied. Namely, inequalities (2.57) follow easily from $mq(x) \leq p(x) \leq Mq(x)$ when the monotonicity property (2.52) is applied twice, since $\mathscr{M}(\Phi, \mathbf{f}, mq; A) = m\mathscr{M}(\Phi, \mathbf{f}, q; A)$ and $\mathscr{M}(\Phi, \mathbf{f}, Mq; A) = M\mathscr{M}(\Phi, \mathbf{f}, q; A)$.

For the sake of another application of the obtained results, we induce the discrete notation. For $n \in \mathbb{N}$, $E = \{1, 2, \ldots, n\}$ and linear space L of real n-tuples $\mathbf{x} = (x_1, \ldots, x_n)$, the functional $A : L \to \mathbb{R}$ becomes a discrete functional, such that $A(\mathbf{x}) = \sum_{i=1}^n x_i$. In particular, for nonnegative n-tuples $\mathbf{p} \in L$ is $A(\mathbf{p}) = P_n = \sum_{i=1}^n p_i > 0$ and $A(\mathbf{1}) = \sum_{i=1}^n 1 = n$.

According to the discrete notation, the multidimensional functional (2.50) assumes the following form

$$\mathbf{M}(\Phi, \mathbf{X}, \mathbf{p}) = \sum_{i=1}^n p_i \Phi(\mathbf{x_i}) - P_n \Phi\left(\frac{1}{P_n} \sum_{i=1}^n p_i \mathbf{x_i}\right), \tag{2.58}$$

where Φ is a convex function, $\mathbf{X} = (\mathbf{x_1}, \ldots, \mathbf{x_n})$, with $\mathbf{x_i} = (x_{i_1}, \ldots, x_{i_n}) \in I^n$, $I \subseteq \mathbb{R}$, for $i = 1, \ldots, n$. Furthermore, $\mathbf{p} = (p_1, \ldots, p_n)$, where $p_i \geq 0$, $i = 1, \ldots, n$, and $\sum_{i=1}^n p_i = P_n > 0$.

Now, let $f : [a, b] \to \mathbb{R}$ be an $(n+1)$-convex function. By means of the divided difference of the function f in $x_{i_1}, \ldots, x_{i_n}$, we define the function $G : [a, b]^n \to \mathbb{R}$ by $G(\mathbf{x_i}) = f[x_{i_1}, \ldots, x_{i_n}]$. It follows from [177, p. 74, Theorem 2.52.] that G is a convex function.

Substituting Φ with G, the functional (2.58) becomes

$$\mathbf{M}(G, \mathbf{X}, \mathbf{p}) = \sum_{i=1}^n p_i f\left[x_{i_1}, \ldots, x_{i_n}\right] - P_n f\left[\frac{1}{P_n} \sum_{i=1}^n p_i x_{i_1}, \ldots, \frac{1}{P_n} \sum_{i=1}^n p_i x_{i_n}\right] \tag{2.59}$$

and hence possesses the analogous properties to those in Theorem 2.15 and Corollary 2.8, as is described below.

Corollary 2.9 *Suppose $f : [a, b] \to \mathbb{R}$ is an $(n+1)$-convex function. Let $\mathbf{X} = (\mathbf{x_1}, \ldots, \mathbf{x_n})$, where $\mathbf{x_i} = (x_{i_1}, \ldots, x_{i_n}) \in [a, b]^n$, $i = 1, \ldots, n$, and let $G : [a, b]^n \to \mathbb{R}$, defined by $G(x_{i_1}, \ldots, x_{i_n}) = f[x_{i_1}, \ldots, x_{i_n}]$ be a convex function. Then the functional $\mathbf{M}(G, \mathbf{X}, \cdot)$, defined by (2.59) is superadditive and increasing on the set of all n-tuples $\mathbf{p} = (p_1, \ldots, p_n)$, such that $p_i \geq 0$, $i = 1, \ldots, n$, $\sum_{i=1}^n p_i = P_n > 0$. Moreover, the following inequalities hold*

$$\left[\min_{1 \leq i \leq n} \{p_i\}\right] \mathbf{M}(G, \mathbf{X}) \leq \mathbf{M}(G, \mathbf{X}, \mathbf{p}) \leq \left[\max_{1 \leq i \leq n} \{p_i\}\right] \mathbf{M}(G, \mathbf{X}), \tag{2.60}$$

where

$$\mathbf{M}(G, \mathbf{X}) = \sum_{i=1}^n f\left[x_{i_1}, \ldots, x_{i_n}\right] - nf\left[\frac{1}{n} \sum_{i=1}^n x_{i_1}, \ldots, \frac{1}{n} \sum_{i=1}^n x_{i_n}\right].$$

2.2.3 Application to weight generalized means

Weight generalized mean of a real valued function $f \in L$ is defined by (2.11). In a similar way, by means of a positive linear functional $A : L \to \mathbb{R}$, the same notion, but regarding vector functions $\mathbf{f} = (f_1, \ldots, f_n) \in L^n$ is defined as

$$M_\chi(\varphi(\mathbf{f}), p; A) = \chi^{-1}\left(\frac{A(p\chi(\varphi(\mathbf{f})))}{A(p)}\right), \tag{2.61}$$

where $\chi : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$, is a continuous and strictly monotonic function, $\varphi : K \to \mathbb{R}$, $K \subseteq \mathbb{R}^n$ is such that $\varphi(\mathbf{f}) \in I$ and $p \in L^+$ is a weight function. On the assumptions that $A(p) > 0$ and $\chi(\varphi(\mathbf{f}))$, $p\chi(\varphi(\mathbf{f})) \in L$, it is easy to see that (2.61) is a mean.

Making use of (2.61) we deduce the following McShane-type functional:

$$\begin{aligned} &\mathscr{M}^{\mathscr{T}}(H, \Psi(\mathbf{f}), p; A) \\ &\quad = A(p)\left[\chi\left(M_\chi(\varphi(\mathbf{f}), p; A)\right) - \chi\left(\varphi\left(M_{\psi_1}(f_1, p; A), \ldots, M_{\psi_n}(f_n, p; A)\right)\right)\right], \end{aligned} \tag{2.62}$$

where $\psi_i : I \to \mathbb{R}$, $i = 1, \ldots, n$, are continuous and strictly monotonic functions, such that $\psi_i(f_i)$, $p\psi_i(f_i) \in L$, $M_{\psi_i}(f_i, p; A)$ are defined by (2.11) and function H

$$H(s_1, s_2, \ldots, s_n) = \chi \circ \varphi(\psi_1^{-1}(s_1), \ldots, \psi_n^{-1}(s_n))$$

is assumed to be well defined. For the sake of an abbreviated notation, the acting of functions $\psi_i : I \to \mathbb{R}$ on f_i, $i = 1, \ldots, n$ is denoted by $\Psi(\mathbf{f})$. For fixed H, $\Psi(\mathbf{f})$ and A, functional $\mathscr{M}^{\mathscr{T}}(H, \Psi(\mathbf{f}), \cdot; A)$ is observed as a function on L^+. In this setting the following result is valid.

Theorem 2.17 *Suppose $A : L \to \mathbb{R}$ is a positive linear functional and $\mathbf{f} = (f_1, \ldots, f_n)$ is a function in L^n. Let $\varphi : K \to \mathbb{R}$, $K \subseteq \mathbb{R}^n$, be such that $\varphi(\mathbf{f}) \in I$, where $I \subseteq \mathbb{R}$ is domain of real-valued, continuous and strictly monotonic functions χ and ψ_i, $i = 1, \ldots, n$, such that $\psi_i(f_i)$, $p\psi_i(f_i)$, $\chi(\varphi(\mathbf{f}))$, $p\chi(\varphi(\mathbf{f})) \in L$. If $H(s_1, s_2, \ldots, s_n) = \chi \circ \varphi(\psi_1^{-1}(s_1), \ldots, \psi_n^{-1}(s_n))$ is a convex function, then the functional $\mathscr{M}^{\mathscr{T}}(H, \Psi(\mathbf{f}), \cdot; A)$, defined by* (2.62) *is superadditive and increasing on L^+.*

Proof. Let $p \in L^+$. Making use of linearity of the functional A as well as of (2.11) and (2.61), we may rearrange the expressions in (2.62):

$$\begin{aligned} &\mathscr{M}^{\mathscr{T}}(H, \Psi(\mathbf{f}), p; A) \\ &= A(p)\left[\chi\left(M_\chi(\varphi(\mathbf{f}), p; A)\right) - \chi\left(\varphi\left(M_{\psi_1}(f_1, p; A), \ldots, M_{\psi_n}(f_n, p; A)\right)\right)\right] \\ &= A(p)\chi\left(M_\chi(\varphi(\mathbf{f}), p; A)\right) - A(p)\chi\left(\varphi\left(M_{\psi_1}(f_1, p; A), \ldots, M_{\psi_n}(f_n, p; A)\right)\right) \\ &= A\left(p\chi\left(\varphi(f_1, \ldots, f_n)\right)\right) \\ &\quad - A(p)\chi\left(\varphi\left(\psi_1^{-1}\left(\frac{A(p\psi_1(f_1))}{A(p)}\right), \ldots, \psi_n^{-1}\left(\frac{A(p\psi_n(f_n))}{A(p)}\right)\right)\right). \end{aligned}$$

Functional (2.62) obviously corresponds to McShane's functional (2.50), where Φ is substituted by H and functions $f_i \in L$ by $\psi_i(f_i) \in L$, $i = 1, \ldots, n$. We see that $\Phi(\mathbf{f}) = H(\psi_1(f_1), \ldots, \psi_n(f_n)) = \chi(\varphi(f_1, \ldots, f_n)) \in L$. Hence superadditivity and monotonicity of (2.62) follow directly from Theorem 2.15. □

Corollary 2.10 *Suppose functions* $\mathbf{f}$, φ, χ, ψ_i, $i=1,\dots,n$ *and functional* A *are as in Theorem 2.17 and let* $p \in L^+$ *assumes its minimal and maximal value on* E. *If* $H(s_1,s_2,\dots,s_n) = \chi \circ \varphi(\psi_1^{-1}(s_1),\dots,\psi_n^{-1}(s_n))$ *is a convex function and functional* $\mathscr{M}^{\mathscr{T}}(H,\Psi(\mathbf{f}),p;A)$ *is defined by (2.62), then the following inequalities hold:*

$$\begin{aligned}\left[\min_{x\in E} p(x)\right]\mathscr{M}^{\mathscr{T}}(H,\Psi(\mathbf{f}),1;A) &\le \mathscr{M}^{\mathscr{T}}(H,\Psi(\mathbf{f}),p;A)\\ &\le \left[\max_{x\in E} p(x)\right]\mathscr{M}^{\mathscr{T}}(H,\Psi(\mathbf{f}),1;A),\end{aligned} \tag{2.63}$$

where

$$\mathscr{M}^{\mathscr{T}}(H,\Psi(\mathbf{f}),1;A) = A(1)\left[\chi\left(M_\chi(\varphi(\mathbf{f});A)\right) - \chi\left(\varphi\left(M_{\psi_1}(f_1;A),\dots,M_{\psi_n}(f_n;A)\right)\right)\right], \tag{2.64}$$

$$M_\chi(\varphi(\mathbf{f});A) = \chi^{-1}\left(\frac{A(\chi(\varphi(\mathbf{f})))}{A(1)}\right), M_{\psi_i}(f_i;A) = \psi_i^{-1}\left(\frac{A(\psi_i(f_i))}{A(1)}\right), \quad i=1,\dots,n. \tag{2.65}$$

Proof. Since the functional (2.62) is increasing on L^+, according to Theorem 2.17, the proof follows the same lines as in Corollary 2.8. □

Let $\chi : I \to \overline{\mathbb{R}}$, $I \subseteq \overline{\mathbb{R}}$, $\overline{\mathbb{R}} = \{\mathbb{R} \cup \pm\infty\}$, be a continuous and strictly monotonic function, $\mathbf{a} = (a_1,\dots,a_n)$, $a_k \in I$, $k=1,\dots,n$, and $\mathbf{w} = (w_1,\dots,w_n)$, $w_k \ge 0$ with $\sum_{k=1}^n w_k = 1$. Weight quasiarithmetic mean (more details can be found in the monograph [151, p. 193]) is defined by

$$M_\chi(\mathbf{a},\mathbf{w}) = \chi^{-1}\left(\sum_{k=1}^n w_k \chi(a_k)\right). \tag{2.66}$$

In the same monograph [151, p. 197, Theorem 1], a characterization of the convexity of (2.66) by means of the concavity of the function χ'/χ'' is given, provided χ is a strictly increasing and strictly convex function with the continuous second derivative. Now, if we substitute the function Φ with M_χ in the definition of the discrete McShane's functional (2.58), we deduce the following type of functional, making use of (2.66):

$$\mathbf{M}(M_\chi,\mathbf{X},\mathbf{p}) = \sum_{i=1}^n p_i \chi^{-1}\left(\sum_{k=1}^n w_k \chi(x_{i_k})\right) - P_n \chi^{-1}\left(\sum_{k=1}^n w_k \chi\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_{i_k}\right)\right), \tag{2.67}$$

where, as we previously had, $\mathbf{X} = (\mathbf{x_1},\dots,\mathbf{x_n})$, $\mathbf{x_i} = (x_{i_1},\dots,x_{i_n}) \in I^n$, $I \subseteq \mathbb{R}$, $i=1,\dots,n$ and $\mathbf{p} = (p_1,\dots,p_n)$, $p_i \ge 0$, $i=1,\dots,n$, $\sum_{i=1}^n p_i = P_n > 0$.

Corollary 2.11 *Suppose* $\chi : I \to \overline{\mathbb{R}}$, $I \subseteq \overline{\mathbb{R}}$ *is a strictly increasing and strictly convex function with the continuous second derivative, such that* χ'/χ'' *is concave. Let* $\mathbf{X} = (\mathbf{x_1},\dots,\mathbf{x_n})$, $\mathbf{x_i} = (x_{i_1},\dots,x_{i_n}) \in I^n$, $i=1,\dots,n$ *and let* M_χ *be a quasiarithmetic mean defined by* (2.66). *Then the functional* $\mathbf{M}(M_\chi,\mathbf{X},\cdot)$, *defined by* (2.67) *is superadditive and increasing on the set of all n-tuples* $\mathbf{p} = (p_1,\dots,p_n)$, *such that* $p_i \ge 0$, $i=1,\dots,n$ *and* $\sum_{i=1}^n p_i = P_n > 0$. *Moreover, the following inequalities hold:*

$$\left[\min_{1\le i\le n}\{p_i\}\right]\mathbf{M}(M_\chi,\mathbf{X}) \le \mathbf{M}(M_\chi,\mathbf{X},\mathbf{p}) \le \left[\max_{1\le i\le n}\{p_i\}\right]\mathbf{M}(M_\chi,\mathbf{X}), \tag{2.68}$$

where

$$\mathbf{M}(M_\chi,\mathbf{X}) = \sum_{i=1}^{n} \chi^{-1}\left(\sum_{k=1}^{n} w_k \chi(x_{i_k})\right) - n\chi^{-1}\left(\sum_{k=1}^{n} w_k \chi\left(\frac{1}{n}\sum_{i=1}^{n} x_{i_k}\right)\right).$$

Proof. According to [151, p. 197, Theorem 1], M_χ is convex. Therefore, superadditivity and increase of the functional (2.67) follow directly from Theorem 2.17. Moreover, the inequalities (2.68) are obtained by double application of the monotonicity property of the functional to the relation $\mathbf{p}_{\min} \le \mathbf{p} \le \mathbf{p}_{\max}$, where

$$\mathbf{p}_{\min} = \left(\min_{1\le i\le n}\{p_i\},\ldots,\min_{1\le i\le n}\{p_i\}\right) \quad \text{and} \quad \mathbf{p}_{\max} = \left(\max_{1\le i\le n}\{p_i\},\ldots,\max_{1\le i\le n}\{p_i\}\right).$$

□

In [151, p. 193], a weight quasiarithmetic mean

$$\widetilde{M}_\chi(\mathbf{a},\mathbf{w}) = \chi^{-1}\left(\sum_{k=1}^{n} w_k \chi(a_k)\right) \tag{2.69}$$

with altered conditions on χ and $\mathbf{w}$ is also observed, and these conditions are:

(i) $w_i \ge 1$, $i = 1,\ldots,n$,

(ii) $\chi : \mathbb{R}^+ \to \mathbb{R}^+$,

(iii) $\lim_{x\to 0} \chi(x) = +\infty$ or $\lim_{x\to\infty} \chi(x) = +\infty$.

This type of means is provided with an analogous result to the previous one. It is based on [151, p. 197, Theorem 2], which for a strictly increasing and strictly convex function χ with the continuous second derivative, such that χ/χ' is convex, provides convexity of (2.69). When (2.69) is observed, the functional (2.67) is denoted by $\mathbf{M}(\widetilde{M}_\chi,\mathbf{X})$.

Corollary 2.12 *Suppose* $\mathbf{w}$ *and* χ *are defined as in* (2.69) *and let* χ *be a strictly increasing and strictly convex function with the continuous second derivative, such that* χ/χ' *is convex. Let* $\mathbf{X} = (\mathbf{x_1},\ldots,\mathbf{x_n})$, $\mathbf{x_i} = (x_{i_1},\ldots,x_{i_n}) \in I^n$, $i = 1,\ldots,n$ *and let* $\widetilde{M}_\chi$ *be a quasiarithmetic mean defined by* (2.69). *Then the functional* $\mathbf{M}(\widetilde{M}_\chi,\mathbf{X},\cdot)$, *defined by* (2.67) *is superadditive and increasing on the set of all n-tuples* $\mathbf{p} = (p_1,\ldots,p_n)$, *such that* $p_i \ge 0$, $i = 1,\ldots,n$ *and* $\sum_{i=1}^n p_i = P_n > 0$. *Moreover, the following inequalities hold:*

$$\left[\min_{1\le i\le n}\{p_i\}\right]\mathbf{M}(\widetilde{M}_\chi,\mathbf{X}) \le \mathbf{M}(\widetilde{M}_\chi,\mathbf{X},\mathbf{p}) \le \left[\max_{1\le i\le n}\{p_i\}\right]\mathbf{M}(\widetilde{M}_\chi,\mathbf{X}), \tag{2.70}$$

where

$$\mathbf{M}(\widetilde{M}_\chi,\mathbf{X}) = \sum_{i=1}^{n} \chi^{-1}\left(\sum_{k=1}^{n} w_k \chi(x_{i_k})\right) - n\chi^{-1}\left(\sum_{k=1}^{n} w_k \chi\left(\frac{1}{n}\sum_{i=1}^{n} x_{i_k}\right)\right).$$

Proof. According to [151, p. 197, Theorem 2], $\widetilde{M}_\chi$ is convex. The proof is analogous to the proof of Corollary 2.11. □

2.2.4 Application to the additive and multiplicative-type inequalities

We apply the results on generalized weight means which were presented in the previous section to the additive and multiplicative-type inequalities, in order to obtain their refinements and converses. The starting point are the well known results of E. Beck from [25] that were recently used in [53], where the authors obtained the series of inequalities of Hölder's type, concerning multiplicative type inequalities and of Minkowski's type, concerning additive type inequalities, for the case of a vector function of two positive variables. According to Remark 2.21, we improve here the results from [53] that were obtained for the weight quasiarithmetic means generalized by means of positive linear functional, in light of the analyzed properties of McShane's functional from the previous section. We first induce the needed notions and basic results from Beck's paper [25].

Suppose M_{ψ_1}, M_{ψ_2} and M_χ are quasiarithmetic means defined by (2.66). Beck observed the inequalities

$$\varphi(M_{\psi_1}(\mathbf{x};\mathbf{w}), M_{\psi_2}(\mathbf{y};\mathbf{w})) \geq M_\chi(\varphi(\mathbf{x},\mathbf{y});\mathbf{w}), \tag{2.71}$$

where $\chi(\varphi(\mathbf{x},\mathbf{y});\mathbf{w}) = \chi^{-1}\left(\sum_{i=1}^n w_i \chi(\varphi(x_i,y_i))\right)$, functions χ, ψ_1 and ψ_2 are strictly increasing and continuous function $z := \varphi(\mathbf{x},\mathbf{y})$ has continuous partial derivatives of the first and the second order on $I \times I$. Beck investigated the impact of the convexity (concavity) of the function $\chi \circ \varphi(\psi_1^{-1}(s_1), \psi_2^{-1}(s_2))$ on the inequalities (2.71). What follows is his basic result.

Theorem 2.18 *Inequalities* (2.71) *hold if and only if the function* $H(s_1,s_2) = \chi \circ \varphi(\psi_1^{-1}(s_1), \psi_2^{-1}(s_2))$ *is concave. Inequalities* (2.71) *are reverse if and only if* H *is a convex function.*

Theorem (2.18) is applied to additive and to multiplicative type of inequalities (2.71). For $\varphi(\mathbf{x},\mathbf{y}) \equiv \mathbf{x}+\mathbf{y}$ the relation (2.71) yields additive inequalities

$$M_{\psi_1}(\mathbf{x};\mathbf{w}) + M_{\psi_2}(\mathbf{y};\mathbf{w}) \geq M_\chi(\mathbf{x}+\mathbf{y};\mathbf{w}). \tag{2.72}$$

Minkowski's inequality belongs to this class of inequalities (compare to (1.31) or (1.32)). The related result from [25] is given in the following corollary.

Corollary 2.13 *Let* $E = \frac{\psi_1'}{\psi_1''}$, $F = \frac{\psi_2'}{\psi_2''}$ *and* $G = \frac{\chi'}{\chi''}$. *If* χ', ψ_1', ψ_2' *and* χ'', ψ_1'', ψ_2'' *are all positive functions, then inequalities* (2.72) *hold if and only if*

$$G(x+y) \geq E(x) + F(y).$$

If χ', ψ_1', ψ_2' *are positive and* χ'', ψ_1'', ψ_2'' *are negative, then* (2.72) *are reversed if and only if*

$$G(x+y) \leq E(x) + F(y).$$

For $\varphi(\mathbf{x},\mathbf{y}) \equiv \mathbf{x}\cdot\mathbf{y}$ the relation (2.71) yields the multiplicative inequalities

$$M_{\psi_1}(\mathbf{x};\mathbf{w}) \cdot M_{\psi_2}(\mathbf{y};\mathbf{w}) \geq M_\chi(\mathbf{x}\cdot\mathbf{y};\mathbf{w}). \tag{2.73}$$

Hölder's inequality belongs to this class of inequality (compare to (1.29) or (1.30)). The related result from [25] is given in the following corollary.

Corollary 2.14 *Let*

$$A(t) = \frac{\psi_1'(t)}{\psi_1'(t) + t\psi_1''(t)}, \qquad B(t) = \frac{\psi_2'(t)}{\psi_2'(t) + t\psi_2''(t)}, \qquad C(t) = \frac{\chi'(t)}{\chi'(t) + t\chi''(t)},$$

$t \in I$. If χ', ψ_1', ψ_2' and A, B, C are all positive functions, then inequalities (2.73) *hold if and only if*

$$C(x \cdot y) \geq A(x) + B(y).$$

If χ', ψ_1', ψ_2' are positive and A, B, C are negative, then (2.73) *are reversed if and only if*

$$C(x \cdot y) \leq A(x) + B(y).$$

We now go back to our considerations of the weight means generalized by means of positive linear functional. We first observe the functional (2.62) when $\varphi(\mathbf{f}) = f_1 + f_2$. Acting of the functions ψ_1 i ψ_2 on f_1 and f_2, is denoted with $\Psi(\mathbf{f})$. Function H is defined by $H(s_1,s_2) = \chi(\psi_1^{-1}(s_1) + \psi_2^{-1}(s_2))$. Functional (2.62) then becomes

$$\mathscr{M}^{\mathscr{B}}(H,\Psi(\mathbf{f}),p;A) = A(p)\left[\chi\left(M_\chi(f_1+f_2,p;A)\right) - \chi\left(M_{\psi_1}(f_1,p;A) + M_{\psi_2}(f_2,p;A)\right)\right]. \tag{2.74}$$

Theorem 2.17 and Corollary 2.10 are adjusted to the functional (2.74) in the following way.

Corollary 2.15 *Suppose functional A and functions χ, ψ_1, ψ_2, $\mathbf{f} = (f_1, f_2)$ and φ, with $\varphi(\mathbf{f}) = f_1 + f_2$, are as in Theorem 2.17 and let*

$$E = \frac{\psi_1'}{\psi_1''}, \qquad F = \frac{\psi_2'}{\psi_2''}, \qquad G = \frac{\chi'}{\chi''}.$$

If χ', ψ_1', ψ_2' are positive and χ'', ψ_1'', ψ_2'' are negative, then functional $\mathscr{M}^{\mathscr{B}}(H,\Psi(\mathbf{f}),\cdot;A)$, defined by (2.74) *is superadditive and increasing on L^+ if and only if $G(x+y) \leq E(x) + F(y)$, where $H(s_1,s_2) = \chi(\psi_1^{-1}(s_1) + \psi_2^{-1}(s_2))$. Moreover, if $p \in L^+$ attains its minimal and maximal value on E, then the following inequalities hold:*

$$\begin{aligned}\left[\min_{x\in E} p(x)\right]\mathscr{M}^{\mathscr{B}}(H,\Psi(\mathbf{f}),1;A) &\leq \mathscr{M}_{\mathscr{B}}(H,\Psi(\mathbf{f}),p;A)\\ &\leq \left[\max_{x\in E} p(x)\right]\mathscr{M}^{\mathscr{B}}(H,\Psi(\mathbf{f}),1;A),\end{aligned} \tag{2.75}$$

where

$$\mathscr{M}^{\mathscr{B}}(H,\Psi(\mathbf{f}),1;A) = A(1)\left[\chi\left(M_\chi(f_1+f_2;A)\right) - \chi\left(M_{\psi_1}(f_1;A) + M_{\psi_2}(f_2;A)\right)\right], \tag{2.76}$$

$$M_\chi(f_1+f_2;A) = \chi^{-1}\left(\frac{A(\chi(f_1+f_2))}{A(1)}\right), M_{\psi_i}(f_i;A) = \psi_i^{-1}\left(\frac{A(\psi_i(f_i))}{A(1)}\right), \quad i = 1,2. \tag{2.77}$$

Proof. The functional $\mathscr{M}^{\mathscr{B}}(H,\Psi(\mathbf{f}),\cdot;A)$ is superadditive and increasing on L^+ according to Theorem 2.17 in the case of convex function H, for this functional is a special case of the functional (2.62), for the choice of an additive function φ. On the other side, convexity of H is equivalent to the condition $G(x+y) \leq E(x) + F(y)$, after Theorem 2.18 and Corollary 2.13. Inequalities (2.75) follow directly from Corollary 2.10. $\square$

Remark 2.22 According to Theorem 2.18 and Corollary 2.13, for positive χ', ψ_1', ψ_2', as well as χ'', ψ_1'', ψ_2'', the condition $G(x+y) \geq E(x)+F(y)$ is equivalent to concavity of the function H, which then again, after Theorem 2.17 and Corollary 2.10 corresponds to subadditivity and decrease on L^+ of the functional $\mathscr{M}^{\mathscr{B}}(H,\Psi(\mathbf{f}),\cdot;A)$ and changes the signs in inequalities (2.75).

If we now observe the functional (2.62) for $\varphi(\mathbf{f}) = f_1 \cdot f_2$, then H is defined by $H(s_1,s_2) = \chi(\psi_1^{-1}(s_1)\cdot\psi_2^{-1}(s_2))$, and the functional (2.62) becomes

$$\mathscr{M}^{\overline{\mathscr{B}}}(H,\Psi(\mathbf{f}),p;A) = A(p)\left[\chi\left(M_\chi(f_1\cdot f_2,p;A)\right) - \chi\left(M_{\psi_1}(f_1,p;A)\cdot M_{\psi_2}(f_2,p;A)\right)\right]. \tag{2.78}$$

The corresponding results are valid for this functional, too.

Corollary 2.16 *Suppose A, χ, ψ_1, ψ_2, $\mathbf{f} = (f_1,f_2)$ and φ, $\varphi(\mathbf{f}) = f_1\cdot f_2$, are as in Theorem 2.17 and let*

$$A(t) = \frac{\psi_1'(t)}{\psi_1'(t)+t\psi_1''(t)}, \qquad B(t) = \frac{\psi_2'(t)}{\psi_2'(t)+t\psi_2''(t)}, \qquad C(t) = \frac{\chi'(t)}{\chi'(t)+t\chi''(t)},$$

$t \in I$. If χ', ψ_1', ψ_2' are positive and A, B, C are negative, then the functional $\mathscr{M}^{\overline{\mathscr{B}}}(H,\Psi(\mathbf{f}),\cdot;A)$, defined by (2.78) is superadditive and increasing on L^+ if and only if $C(x\cdot y) \leq A(x)+B(y)$, where $H(s_1,s_2) = \chi(\psi_1^{-1}(s_1)\cdot\psi_2^{-1}(s_2))$. Moreover, if $p \in L^+$ attains its minimal and maximal value on E, then the following inequalities hold:

$$\begin{aligned}\left[\min_{x\in E} p(x)\right]\mathscr{M}^{\overline{\mathscr{B}}}(H,\Psi(\mathbf{f}),1;A) &\leq \mathscr{M}^{\overline{\mathscr{B}}}(H,\Psi(\mathbf{f}),p;A)\\ &\leq \left[\max_{x\in E} p(x)\right]\mathscr{M}^{\overline{\mathscr{B}}}(H,\Psi(\mathbf{f}),1;A),\end{aligned} \tag{2.79}$$

where

$$\mathscr{M}^{\overline{\mathscr{B}}}(H,\Psi(\mathbf{f}),1;A) = A(1)\left[\chi\left(M_\chi(f_1\cdot f_2;A)\right) - \chi\left(M_{\psi_1}(f_1;A)\cdot M_{\psi_2}(f_2;A)\right)\right], \tag{2.80}$$

$$M_\chi(f_1\cdot f_2;A) = \chi^{-1}\left(\frac{A(\chi(f_1\cdot f_2))}{A(1)}\right), M_{\psi_i}(f_i;A) = \psi_i^{-1}\left(\frac{A(\psi_i(f_i))}{A(1)}\right), \quad i=1,2. \tag{2.81}$$

Proof. Similarly as in the previous corollary, the functional $\mathscr{M}^{\overline{\mathscr{B}}}(H,\Psi(\mathbf{f}),\cdot;A)$ is superadditive and increasing on L^+ in the case of a convex function H, according to Theorem 2.17. On the other hand, convexity of H is equivalent to the condition $C(x\cdot y) \leq A(x)+B(y)$, after Theorem 2.18 and Corollary 2.14. Inequalities (2.79) follow directly from Corollary 2.10. □

Remark 2.23 According to Theorem 2.18 and Corollary 2.14, for positive χ', ψ_1', ψ_2', as well as A, B, C, the condition $C(x\cdot y) \geq A(x)+B(y)$ is equivalent to concavity of H, which corresponds to subadditivity and decrease on L^+ of the functional $\mathscr{M}^{\overline{\mathscr{B}}}(H,\Psi(\mathbf{f}),\cdot;A)$, after Theorem 2.17 and Corollary 2.10 and changes the signs in inequalities (2.79).

In the sequel we consider an application of the previous corollary for χ, ψ_1, ψ_2 being power functions, in order to obtain a refinement and a converse of a multiplicative-type inequality.

Corollary 2.17 *Suppose A and $\mathbf{f} = (f_1, f_2)$ are as in Theorem 2.17 and let λ, μ and ν be real numbers. Let $p \in L^+$ attain its minimal and maximal value on E. If the following conditions are satisfied:*

1° $\lambda < 0 < \mu, \nu$ *or* $\mu, \nu < 0 < \lambda$;

2° $\lambda < \mu, \nu < 0$ *or* $\nu < 0 < \mu < \lambda$ *or* $\mu < 0 < \nu < \lambda$, *for* $\frac{1}{\lambda} \leq \frac{1}{\mu} + \frac{1}{\nu}$;

3° $\lambda < \mu < 0 < \nu$ *or* $\lambda < \nu < 0 < \mu$, *for* $\frac{1}{\lambda} \geq \frac{1}{\mu} + \frac{1}{\nu}$,

then the following inequalities hold:

$$\begin{aligned}
&\min_{x \in E} p(x) \left[A\left(f_1^\lambda \cdot f_2^\lambda\right) - A(1) \left(\left(\frac{A\left(f_1^\mu\right)}{A(1)} \right)^{\frac{1}{\mu}} \cdot \left(\frac{A\left(f_2^\nu\right)}{A(1)} \right)^{\frac{1}{\nu}} \right)^\lambda \right] \\
&\leq A\left(p \cdot f_1^\lambda \cdot f_2^\lambda\right) - A(p) \left[\left(\frac{A\left(p f_1^\mu\right)}{A(p)} \right)^{\frac{1}{\mu}} \cdot \left(\frac{A\left(p f_2^\nu\right)}{A(p)} \right)^{\frac{1}{\nu}} \right]^\lambda \\
&\leq \max_{x \in E} p(x) \left[A\left(f_1^\lambda \cdot f_2^\lambda\right) - A(1) \left(\left(\frac{A\left(f_1^\mu\right)}{A(1)} \right)^{\frac{1}{\mu}} \cdot \left(\frac{A\left(f_2^\nu\right)}{A(1)} \right)^{\frac{1}{\nu}} \right)^\lambda \right].
\end{aligned} \tag{2.82}$$

Proof. If we define φ, χ ψ_1 and ψ_2 as: $\varphi(\mathbf{f}) = f_1 \cdot f_2$, $\chi(t) = t^\lambda$, $\psi_1(t) = t^\mu$ and $\psi_2(t) = t^\nu$, then the functional (2.78) becomes

$$A\left(p \cdot f_1^\lambda \cdot f_2^\lambda\right) - A(p) \left[\left(\frac{A\left(p f_1^\mu\right)}{A(p)} \right)^{\frac{1}{\mu}} \cdot \left(\frac{A\left(p f_2^\nu\right)}{A(p)} \right)^{\frac{1}{\nu}} \right]^\lambda \tag{2.83}$$

and function H is then defined by $H(s_1, s_2) = \chi(\psi_1^{-1}(s_1) \cdot \psi_2^{-1}(s_2)) = (s_1^{\frac{1}{\mu}} \cdot s_2^{\frac{1}{\nu}})^\lambda$. Functional (2.83) is superadditive and increasing when H is convex. On the other hand, H is convex if $d^2 H \geq 0$, that is, $\frac{\lambda}{\mu}\left(\frac{\lambda}{\mu} - 1\right) \geq 0$, $\frac{\lambda}{\nu}\left(\frac{\lambda}{\nu} - 1\right) \geq 0$ and $\frac{\lambda^3}{\mu\nu}\left(\frac{1}{\lambda} - \frac{1}{\mu} - \frac{1}{\nu}\right) \geq 0$, which corresponds to the conditions 1°, 2° and 3° on λ, μ and ν. Then inequalities (2.82) follow from Corollary 2.10. □

Remark 2.24 Inequalities (2.82) have the reversed signs when functional is subadditive and decreasing on L^+, that is, when H is concave. Function H is concave when $d^2 H \leq 0$, that is, when $\frac{\lambda}{\mu}\left(\frac{\lambda}{\mu} - 1\right) \leq 0$, $\frac{\lambda}{\nu}\left(\frac{\lambda}{\nu} - 1\right) \leq 0$ and $\frac{\lambda^3}{\mu\nu}\left(\frac{1}{\lambda} - \frac{1}{\mu} - \frac{1}{\nu}\right) \geq 0$. These conditions are satisfied if

1° $\mu, \nu > \lambda > 0$, for $\frac{1}{\lambda} \geq \frac{1}{\mu} + \frac{1}{\nu}$;

2° $\mu, \nu < \lambda < 0$, for $\frac{1}{\lambda} \leq \frac{1}{\mu} + \frac{1}{\nu}$.

Yet another application of Corollary 2.14 is obtained for exponential functions χ, ψ_1, and ψ_2.

Corollary 2.18 *Suppose A and $\mathbf{f} = (f_1, f_2)$ are as in Theorem 2.17 and let λ, μ and ν be positive real numbers different from 1. Let $p \in L^+$ attain its minimal and maximal value on E. If the following conditions are satisfied:*

1° $\lambda < 1 < \mu, \nu$ *or* $\mu, \nu < 1 < \lambda$;

2° $\lambda < \mu, \nu < 1$ *or* $\nu < 1 < \mu < \lambda$ *or* $\mu < 1 < \nu < \lambda$, *for* $\frac{1}{\log \lambda} \leq \frac{1}{\log \mu} + \frac{1}{\log \nu}$;

3° $\lambda < \mu < 1 < \nu$ *or* $\lambda < \nu < 1 < \mu$, *for* $\frac{1}{\log \lambda} \geq \frac{1}{\log \mu} + \frac{1}{\log \nu}$,

then the following inequalities hold:

$$\begin{aligned}
&\min_{x \in E} p(x) \left(A\left(\lambda^{f_1+f_2}\right) - A(1)\lambda^{\log_\mu \frac{A\left(\mu^{f_1}\right)}{A(1)} + \log_\nu \frac{A\left(\nu^{f_2}\right)}{A(1)}} \right) \\
&\leq A\left(p\lambda^{f_1+f_2}\right) - A(p)\lambda^{\log_\mu \frac{A\left(p \cdot \mu^{f_1}\right)}{A(p)} + \log_\nu \frac{A\left(p \cdot \nu^{f_2}\right)}{A(p)}} \\
&\leq \max_{x \in E} p(x) \left(A\left(\lambda^{f_1+f_2}\right) - A(1)\lambda^{\log_\mu \frac{A\left(\mu^{f_1}\right)}{A(1)} + \log_\nu \frac{A\left(\nu^{f_2}\right)}{A(1)}} \right).
\end{aligned} \tag{2.84}$$

Proof. If functions φ, χ ψ_1 and ψ_2 are defined by: $\varphi(\mathbf{f}) = f_1 \cdot f_2$, $\chi(t) = \lambda^t$, $\psi_1(t) = \mu^t$ and $\psi_2(t) = \nu^t$, then the function H becomes $H(s_1, s_2) = (s_1^{\frac{1}{\log \mu}} \cdot s_2^{\frac{1}{\log \nu}})^{\log \lambda}$. The statement of the corollary follows easily from Corollary 2.17, when the following substitutions are taken into account: $\mu \leftrightarrow \log \mu$, $\nu \leftrightarrow \log \nu$ and $\lambda \leftrightarrow \log \lambda$. □

Remark 2.25 Analogously as in Remark 2.24, inequalities (2.84) have reversed signs if:

1° $\mu, \nu > \lambda > 1$, for $\frac{1}{\log \lambda} \geq \frac{1}{\log \mu} + \frac{1}{\log \nu}$;

2° $\mu, \nu < \lambda < 1$, for $\frac{1}{\log \lambda} \leq \frac{1}{\log \mu} + \frac{1}{\log \nu}$.

Now we observe the additive-type inequalities, defining again functions χ, ψ_1, and ψ_2 as power functions. The following result is thus obtained.

Corollary 2.19 *Suppose A and $\mathbf{f} = (f_1, f_2)$ are as in Theorem 2.17 and let λ, μ and ν are real numbers. Let $p \in L^+$ attain its minimal and maximal value on E. If the following conditions are satisfied:*

1° $0 < \mu, \nu \leq \lambda < 1$, *for all* $f_1, f_2 > 0$;

2° $0 < \nu \leq \lambda \leq \mu < 1$, *for* $f_2 \geq \dfrac{(\mu-\lambda)(1-\nu)}{(\lambda-\nu)(1-\mu)} f_1 \geq 0$;

3° $0 < \mu \leq \lambda \leq \nu < 1$, *for* $\dfrac{(\lambda-\mu)(1-\nu)}{(\nu-\lambda)(1-\mu)} f_1 \geq f_2 \geq 0$,

then the following inequalities hold:

$$\begin{aligned}
&\min_{x\in E} p(x) \left[A\left((f_1+f_2)^\lambda\right) - A(1)\left(\left(\frac{A\left(f_1^\mu\right)}{A(1)}\right)^{\frac{1}{\mu}} + \left(\frac{A\left(f_2^\nu\right)}{A(1)}\right)^{\frac{1}{\nu}}\right)^\lambda \right] \\
&\leq A\left(p\cdot(f_1+f_2)^\lambda\right) - A(p)\left(\left(\frac{A\left(pf_1^\mu\right)}{A(p)}\right)^{\frac{1}{\mu}} + \left(\frac{A\left(pf_2^\nu\right)}{A(p)}\right)^{\frac{1}{\nu}}\right)^\lambda \\
&\leq \max_{x\in E} p(x) \left[A\left((f_1+f_2)^\lambda\right) - A(1)\left(\left(\frac{A\left(f_1^\mu\right)}{A(1)}\right)^{\frac{1}{\mu}} + \left(\frac{A\left(f_2^\nu\right)}{A(1)}\right)^{\frac{1}{\nu}}\right)^\lambda \right].
\end{aligned} \tag{2.85}$$

Proof. If functions χ, ψ_1, and ψ_2 are defined by $\varphi(\mathbf{f}) = f_1 + f_2$, $\chi(t) = t^\lambda$, $\psi_1(t) = t^\mu$ and $\psi_2(t) = t^\nu$, then the functional (2.74) assumes the following form:

$$A\left(p\cdot(f_1+f_2)^\lambda\right) - A(p)\left(\left(\frac{A\left(pf_1^\mu\right)}{A(p)}\right)^{\frac{1}{\mu}} + \left(\frac{A\left(pf_2^\nu\right)}{A(p)}\right)^{\frac{1}{\nu}}\right)^\lambda, \tag{2.86}$$

and function H is defined by $H(s_1,s_2) = \chi(\psi_1^{-1}(s_1) + \psi_2^{-1}(s_2)) = (s_1^{\frac{1}{\mu}} + s_2^{\frac{1}{\nu}})^\lambda$. Functional (2.86) is superadditive and increasing when H is convex. According to Theorem 2.18 and Corollary 2.13, H is convex if and only if $G(x+y) \leq E(x) + F(y)$, for positive χ', ψ_1', ψ_2' and negative $\chi'', \psi_1'', \psi_2''$, that is, for $0 < \lambda, \mu, \nu < 1$. If we rewrite this inequality, taking the definitions of E, F and G into account, we get the inequality: $\dfrac{x+y}{\lambda-1} \leq \dfrac{x}{\mu-1} + \dfrac{y}{\nu-1}$, which holds if the following conditions are satisfied:

1° $0 < \mu, \nu \leq \lambda < 1$, for all $x, y > 0$;

2° $0 < \nu \le \lambda \le \mu < 1$, for $y \ge \dfrac{(\mu-\lambda)(1-\nu)}{(\lambda-\nu)(1-\mu)} x \ge 0$;

3° $0 < \mu \le \lambda \le \nu < 1$, for $\dfrac{(\lambda-\mu)(1-\nu)}{(\nu-\lambda)(1-\mu)} x \ge y \ge 0$.

Since convexity of H yields superadditivity and increase of the functional (2.86) on L^+, inequalities (2.85) hold after Corollary 2.2. □

Remark 2.26 Inequalities (2.85) have reversed signs if the functional is subadditive and decreasing on L^+, that is, if H is concave. According to Theorem 2.18 and Corollary 2.14, H is concave if and only if $G(x+y) \ge E(x) + F(y)$, for positive χ', ψ_1', ψ_2' as well as positive $\chi'', \psi_1'', \psi_2''$, that is, for $\lambda, \mu, \nu > 1$. If we rewrite this inequality, taking the definitions of E, F and G into account, we get the inequality: $\dfrac{x+y}{\lambda-1} \ge \dfrac{x}{\mu-1} + \dfrac{y}{\nu-1}$, which holds if the following conditions are satisfied:

1° $1 < \lambda \le \mu, \nu$, for all $x, y > 0$;

2° $1 < \nu \le \lambda \le \mu$, for $0 \le y \le \dfrac{(\mu-\lambda)(\nu-1)}{(\lambda-\nu)(\mu-1)} x$;

3° $1 < \mu \le \lambda \le \nu$, for $y \ge \dfrac{(\lambda-\mu)(\nu-1)}{(\nu-\lambda)(\mu-1)} x \ge 0$.

In order to bring this considerations to a close, we cite an example which illustrates the application of the refinement and the converse of the generalized additive-type inequality.

Example 2.1 Let $\varphi(\mathbf{f}) = f_1 + f_2$ and $\chi(t) = \psi_1(t) = \psi_2(t) = -\cos t$. Function H then becomes $H(s_1, s_2) = -\cos(\arccos(-s_1) + (\arccos(-s_2))$. Functions $\chi', \psi_1', \psi_2', \chi'', \psi_1'', \psi_2''$ are all positive for $0 \le t \le \frac{\pi}{2}$. In that case, $\operatorname{tg}(x+y) \ge \operatorname{tg} x + \operatorname{tg} y$ if and only if H is concave. The last inequality is satisfied if $0 \le x, y \le \frac{\pi}{4}$. Hence on the interval $[0, \dfrac{\pi}{4}]$, according to Remark 2.26 the following inequalities hold:

$$
\begin{aligned}
\max_{x\in E} p(x) & \left[A(1)\cdot\cos\left[\arccos\left(\frac{A(\cos f_1)}{A(1)}\right) + \arccos\left(\frac{A(\cos f_2)}{A(1)}\right)\right]\right. \\
& \quad \left. - A(\cos(f_1+f_2))\right] \\
& \le A(p)\cdot\cos\left[\arccos\left(\frac{A(p\cdot\cos f_1)}{A(p)}\right) + \arccos\left(\frac{A(p\cdot\cos f_2)}{A(p)}\right)\right] \\
& \quad - A(p\cdot\cos(f_1+f_2)) \\
& \le \min_{x\in E} p(x)\left[A(1)\cdot\cos\left[\arccos\left(\frac{A(\cos f_1)}{A(1)}\right) + \arccos\left(\frac{A(\cos f_2)}{A(1)}\right)\right]\right. \\
& \quad \left. - A(\cos(f_1+f_2))\right].
\end{aligned}
$$

2.2.5 Application to Hölder's inequality

In view of the presented results on McShane's functional, we again observe Hölder's inequality, generalized by means of positive linear functional $A : L \to \mathbb{R}$:

$$A\left(\prod_{i=1}^{n} f_i^{p_i}\right) \le \prod_{i=1}^{n} A^{p_i}(f_i), \tag{2.87}$$

where $p_i \ge 0$, $i = 1,\dots,n$ are such that $\sum_{i=1}^{n} p_i = 1$ and $f_1,\dots,f_n$, $\prod_{i=1}^{n} f_i^{p_i} \in L^+$.

V. Čuljak et al. stated in [53] the following theorem.

Theorem 2.19 (SEE [53]) *Suppose $p_i, q_i > 0$, $i = 1,\dots,n$ are such that $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$. Let $m < 1$ and $M > 1$ be real constants such that $Mq_i \ge p_i \ge mq_i$, $i = 1,\dots,n$. If $f_i \in L$ are such that $f_i^{p_i}$, $f_i^{q_i}$, $f_i^{\frac{p_i - mq_i}{1-m}}$, $f_i^{\frac{Mq_i - p_i}{M-1}}$, $\prod_{i=1}^{n} f_i^{p_i}$, $\prod_{i=1}^{n} f_i^{q_i} \in L$, then the following inequalities hold:*

$$\left[\frac{\prod_{i=1}^{n} A^{q_i}(f_i)}{A\left(\prod_{i=1}^{n} f_i^{q_i}\right)}\right]^m \le \frac{\prod_{i=1}^{n} A^{p_i}(f_i)}{A\left(\prod_{i=1}^{n} f_i^{p_i}\right)} \le \left[\frac{\prod_{i=1}^{n} A^{q_i}(f_i)}{A\left(\prod_{i=1}^{n} f_i^{q_i}\right)}\right]^M. \tag{2.88}$$

If we generalize (2.87) by having $\sum_{i=1}^{n} p_i = P_n > 0$, Hölder's inequality assumes the form

$$A\left(\prod_{i=1}^{n} f_i^{\frac{p_i}{P_n}}\right) \le \prod_{i=1}^{n} A^{\frac{p_i}{P_n}}(f_i), \tag{2.89}$$

i.e.

$$A^{P_n}\left(\prod_{i=1}^{n} f_i^{\frac{p_i}{P_n}}\right) \le \prod_{i=1}^{n} A^{p_i}(f_i). \tag{2.90}$$

This provides us with the definition of the following functional:

$$\mathscr{H}(\mathbf{f},\mathbf{p};A) = \frac{\prod_{i=1}^{n} A^{p_i}(f_i)}{A^{P_n}\left(\prod_{i=1}^{n} f_i^{\frac{p_i}{P_n}}\right)}, \tag{2.91}$$

where $A : L \to \mathbb{R}$, $\mathbf{f} = (f_1,\dots,f_n)$, $f_i \in L^+$ and $\mathbf{p} = (p_1,\dots,p_n)$, $p_i \ge 0$, $\sum_{i=1}^{n} p_i = P_n > 0$.

Our aim here is to improve Theorem 2.19, by establishing the non-weight bounds for the functional (2.91), for which the first step is proving this functional to be increasing on L^+, when observed as a function on the set of all described n-tuples $\mathbf{p}$.

Theorem 2.20 *Suppose $\mathbf{p} = (p_1,\dots,p_n)$ and $\mathbf{q} = (q_1,\dots,q_n)$ are such that $p_i, q_i \ge 0$, $i = 1,\dots,n$ and $\sum_{i=1}^{n} p_i = P_n > 0$, $\sum_{i=1}^{n} q_i = Q_n > 0$. Let $\mathbf{f} = (f_1,\dots,f_n)$ be an n-tuple from L^+ and let $\prod_{i=1}^{n} f_i^{\frac{p_i}{P_n}}$, $\prod_{i=1}^{n} f_i^{\frac{q_i}{Q_n}} \in L^+$. If $A : L \to \mathbb{R}$ is a positive linear functional, then the following inequality holds:*

$$\mathscr{H}(\mathbf{f},\mathbf{p}+\mathbf{q};A) \ge \mathscr{H}(\mathbf{f},\mathbf{p};A) \cdot \mathscr{H}(\mathbf{f},\mathbf{q};A). \tag{2.92}$$

Moreover, if $\mathbf{p} \ge \mathbf{q}$, then

$$\mathscr{H}(\mathbf{f},\mathbf{p};A) \ge \mathscr{H}(\mathbf{f},\mathbf{q};A). \tag{2.93}$$

Proof. It follows from the definition of the functional (2.91) that

$$\mathscr{H}(\mathbf{f},\mathbf{p}+\mathbf{q};A)=\frac{\prod_{i=1}^{n}A^{p_i+q_i}(f_i)}{A^{P_n+Q_n}\left(\prod_{i=1}^{n}f_i^{\frac{p_i+q_i}{P_n+Q_n}}\right)}. \tag{2.94}$$

On the other hand, we have

$$\begin{aligned}A^{P_n+Q_n}\left(\prod_{i=1}^{n}f_i^{\frac{p_i+q_i}{P_n+Q_n}}\right) &= A^{P_n+Q_n}\left[\left(\prod_{i=1}^{n}f_i^{\frac{p_i}{P_n}}\right)^{\frac{P_n}{P_n+Q_n}}\cdot\left(\prod_{i=1}^{n}f_i^{\frac{q_i}{Q_n}}\right)^{\frac{Q_n}{P_n+Q_n}}\right]\\ &\leq A^{P_n}\left(\prod_{i=1}^{n}f_i^{\frac{p_i}{P_n}}\right)\cdot A^{Q_n}\left(\prod_{i=1}^{n}f_i^{\frac{q_i}{Q_n}}\right).\end{aligned} \tag{2.95}$$

Now (2.94) and (2.95) yield

$$\begin{aligned}\mathscr{H}(\mathbf{f},\mathbf{p}+\mathbf{q};A) &= \frac{\prod_{i=1}^{n}A^{p_i}(f_i)\cdot\prod_{i=1}^{n}A^{q_i}(f_i)}{A^{P_n+Q_n}\left(\prod_{i=1}^{n}f_i^{\frac{p_i+q_i}{P_n+Q_n}}\right)}\\ &\geq \frac{\prod_{i=1}^{n}A^{p_i}(f_i)\cdot\prod_{i=1}^{n}A^{q_i}(f_i)}{A^{P_n}\left(\prod_{i=1}^{n}f_i^{\frac{p_i}{P_n}}\right)\cdot A^{Q_n}\left(\prod_{i=1}^{n}f_i^{\frac{q_i}{Q_n}}\right)}\\ &= \mathscr{H}(\mathbf{f},\mathbf{p};A)\cdot\mathscr{H}(\mathbf{f},\mathbf{q};A),\end{aligned}$$

whence (2.92) is proved. Since $\mathbf{p}=(\mathbf{p}-\mathbf{q})+\mathbf{q}$, the proved inequality (2.92) yields

$$\begin{aligned}\mathscr{H}(\mathbf{f},\mathbf{p};A)=\mathscr{H}(\mathbf{f},\mathbf{p}-\mathbf{q}+\mathbf{q};A) &\geq \mathscr{H}(\mathbf{f},\mathbf{p}-\mathbf{q};A)\cdot\mathscr{H}(\mathbf{f},\mathbf{q};A)\\ &\geq \mathscr{H}(\mathbf{f},\mathbf{q};A),\end{aligned}$$

where the last inequality follows from (2.90), that is, from $\mathscr{H}(\mathbf{f},\mathbf{p}-\mathbf{q};A)\geq 1$. □

Corollary 2.20 *Suppose* $\mathbf{f}$, *n-tuple* $\mathbf{p}$ *and A are as in Theorem 2.20. Then the following inequalities hold:*

$$\left[\frac{\prod_{i=1}^{n}A(f_i)}{A^n\left(\prod_{i=1}^{n}f_i^{\frac{1}{n}}\right)}\right]^{\min\limits_{1\leq i\leq n}\{p_i\}}\leq\mathscr{H}(\mathbf{f},\mathbf{p};A)\leq\left[\frac{\prod_{i=1}^{n}A(f_i)}{A^n\left(\prod_{i=1}^{n}f_i^{\frac{1}{n}}\right)}\right]^{\max\limits_{1\leq i\leq n}\{p_i\}}. \tag{2.96}$$

Proof. If we insert

$$\mathbf{p}_{\min}=\left(\min_{1\leq i\leq n}\{p_i\},\ldots,\min_{1\leq i\leq n}\{p_i\}\right),$$

$$\mathbf{p}_{\max}=\left(\max_{1\leq i\leq n}\{p_i\},\ldots,\max_{1\leq i\leq n}\{p_i\}\right)$$

in (2.91), we have

$$\mathscr{H}(\mathbf{f},\mathbf{p}_{\min};A)=\left[\frac{\prod_{i=1}^{n}A(f_i)}{A^n\left(\prod_{i=1}^{n}f_i^{\frac{1}{n}}\right)}\right]^{\min\limits_{1\le i\le n}\{p_i\}},$$

$$\mathscr{H}(\mathbf{f},\mathbf{p}_{\max};A)=\left[\frac{\prod_{i=1}^{n}A(f_i)}{A^n\left(\prod_{i=1}^{n}f_i^{\frac{1}{n}}\right)}\right]^{\max\limits_{1\le i\le n}\{p_i\}}.$$

Since $\mathbf{p}_{\min}\le\mathbf{p}\le\mathbf{p}_{\max}$, according to (2.91) the statement is proved. □

Remark 2.27 Inequalities (2.88) from [53] are now easily obtained by applying Theorem 2.20. Namely, if m, M, $\mathbf{p}$ and $\mathbf{q}$ are as in Theorem 2.19 and such that $m\mathbf{q}\le\mathbf{p}\le M\mathbf{q}$, then by applying (2.93) it follows that

$$\mathscr{H}(\mathbf{f},m\mathbf{q};A)\le\mathscr{H}(\mathbf{f},\mathbf{p};A)\le\mathscr{H}(\mathbf{f},M\mathbf{q};A), \tag{2.97}$$

which are actually inequalities (2.88).

2.3 Related results on Hilbert's inequality

Hilbert's inequality, although considered as a classical one, still presents a challenge to mathematicians in providing its new improvements, generalizations and consequently its various applications. Generalizations include inequalities with more general kernels, weight functions and integration sets, extension to a multidimensional case, and so forth. The resulting relations are usually called the Hilbert-type inequalities. Among the variety of recent articles dealing with this problem area, we single out the following references: [57, 58, 104, 115, 116, 117, 148, 149, 150, 191]. For a comprehensive inspection of the initial development of the Hilbert-type inequalities, the reader is referred to classical monographs [83] and [151]. Recent investigations on Hilbert-type inequalities are contained in the monograph [122].

2.3.1 On more accurate Hilbert-type inequalities in finite measure spaces

One of the earliest versions of the Hilbert inequality, established at the beginning of the 20th century, asserts that

$$\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{x+y}dxdy\le\frac{\pi}{\sin\frac{\pi}{p}}\|f\|_{L^p(\mathbb{R}_+)}\|g\|_{L^q(\mathbb{R}_+)}, \tag{2.98}$$

where $f \in L^p(\mathbb{R}_+)$ and $g \in L^q(\mathbb{R}_+)$ are non-negative measurable functions. Here, parameters p and q are mutually conjugate, that is, they fulfill the condition $\frac{1}{p}+\frac{1}{q}=1$, where $p>1$. In addition, the constant $\pi/\sin\frac{\pi}{p}$, appearing in (2.98), is the best possible.

Recent paper [117] provides a unified treatment of Hilbert-type inequalities with conjugate parameters. The corresponding result regards σ-finite measure spaces $(\Omega_1,\Sigma_1,\mu_1)$ and $(\Omega_2,\Sigma_2,\mu_2)$, a non-negative measurable kernel $K:\Omega_1\times\Omega_2\to\mathbb{R}$, a measurable, a.e. positive function $\varphi:\Omega_1\to\mathbb{R}$, and a measurable, a.e. positive function $\psi:\Omega_2\to\mathbb{R}$. Then, if $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, the inequality

$$\int_{\Omega_1}\int_{\Omega_2} K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y) \le \|\varphi F f\|_{L^p(\mu_1)}\|\psi G g\|_{L^q(\mu_2)} \tag{2.99}$$

holds for all non-negative measurable functions $f:\Omega_1\to\mathbb{R}$ and $g:\Omega_2\to\mathbb{R}$, where

$$F(x)=\left[\int_{\Omega_2}\frac{K(x,y)}{\psi^p(y)}d\mu_2(y)\right]^{\frac{1}{p}},\quad x\in\Omega_1, \tag{2.100}$$

and

$$G(y)=\left[\int_{\Omega_1}\frac{K(x,y)}{\varphi^q(x)}d\mu_1(x)\right]^{\frac{1}{q}},\quad y\in\Omega_2. \tag{2.101}$$

Observe that the general Hilbert-type inequality (2.99) extends the classical Hilbert inequality (2.98). Namely, setting $\Omega_1=\Omega_2=\mathbb{R}_+$, $d\mu_1(x)=dx$, $d\mu_2(y)=dy$, $K(x,y)=(x+y)^{-1}$, $\varphi(x)=x^{\frac{1}{pq}}$, $\psi(y)=y^{\frac{1}{pq}}$ in (2.99), we get (2.98).

In this section some more accurate versions of the general Hilbert's inequality (2.99) are derived. Since the crucial step in proving Hilbert-type inequalities is in applying the well-known Hölder inequality, new results are based on several new improvements of the Hölder inequality established in [108] (see also Section 2.1) in a more general environment. However, improvements that are given here require the setting provided with finite measure spaces. These are also contained in the paper [121].

More on refinements and converses of Hölder's inequality

Generally speaking, the starting point in proving Hilbert-type inequalities is well-known Hölder's inequality which asserts that

$$\|fg\|_{L^1(\mu)} \le \|f\|_{L^p(\mu)}\|g\|_{L^q(\mu)} \tag{2.102}$$

holds for all non-negative measurable functions $f,g:\Omega\to\mathbb{R}$ such that $f\in L^p(\mu)$ and $g\in L^q(\mu)$, where (Ω,Σ,μ) is a σ-finite measure space and p,q are mutually conjugate parameters with $p>1$. In the sequel, $L^r(\mu)$, $r\ge 1$, denotes the space of all measurable functions $f:\Omega\to\mathbb{R}$ such that $\|f\|_{L^r(\mu)}=[\int_\Omega |f(x)|^r d\mu(x)]^{\frac{1}{r}}<\infty$.

Refinements and converses of Hölder's inequality, expressed in a more general form – by means of a positive linear functional, were established in [108] (see also Theorem 2.7), whereas here, assuming that (Ω,Σ,μ) is a finite measure space (i.e. $\mu(\Omega)<\infty$) and that $f:\Omega\to\mathbb{R}$ is a non-negative bounded function, these inequalities become

$$f_{\sup}^p\mathscr{I}_{p,q}(f,g) \ge \|f\|_{L^p(\mu)}\|g\|_{L^q(\mu)} - \|fg\|_{L^1(\mu)} \ge f_{\inf}^p\mathscr{I}_{p,q}(f,g), \tag{2.103}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and the corresponding Jensen-type functional is defined by

$$\mathscr{J}_{p,q}(f,g)=\mu^{\frac{1}{p}}(\Omega)\|gf^{1-p}\|_{L^q(\mu)}-\|gf^{1-p}\|_{L^1(\mu)}.$$

In the above relation, $f^p_{\sup}$ denotes a supremum of the function f^p on Ω, that is, $f^p_{\sup}=\sup_{x\in\Omega}f^p(x)$, while $f^p_{\inf}$ denotes its infimum on Ω, i.e. $f^p_{\inf}=\inf_{x\in\Omega}f^p(x)$. This notation is used in the sequel, as well.

Remark 2.28 Since $\mu^{\frac{1}{p}}(\Omega)=\|1\|_{L^p(\mu)}$, it follows that $\mathscr{J}_{p,q}(f,g)\geq 0$, due to Hölder's inequality (2.102) applied to the functions 1 and gf^{1-p}. In addition, having in mind that $f^p_{\inf}gf^{1-p}\leq fg\leq f^p_{\sup}gf^{1-p}$, the inequality $\mathscr{J}_{p,q}(f,g)\geq 0$ may be interpreted as the non-weight Hölder inequality.

Clearly, the right inequality in (2.103) provides the refinement of (2.102), while the left one yields the converse of (2.102). Moreover, the double inequality (2.103) is usually referred to as the refinement and the converse of Hölder's inequality, in a difference form. In addition, if $0<p<1$, then the inequality signs in (2.103) are reversed.

In [108], (see also Theorem 2.8), yet another accurate version of Hölder's inequality, expressed by means of a positive linear functional, was established. Here the double inequality of the type becomes

$$f^p_{\sup}\mathscr{J}^*_{p,q}(f,g)\geq\|f\|^q_{L^p(\mu)}\|g\|^q_{L^q(\mu)}-\|fg\|^q_{L^1(\mu)}\geq f^p_{\inf}\mathscr{J}^*_{p,q}(f,g), \tag{2.104}$$

where

$$\mathscr{J}^*_{p,q}(f,g)=\|f\|^q_{L^p(\mu)}\left[\|gf^{1-p}\|^q_{L^q(\mu)}-\frac{\|gf^{1-p}\|^q_{L^1(\mu)}}{\mu^{q-1}(\Omega)}\right]$$

refers to a finite measure space (Ω,Σ,μ) and a bounded non-negative function $f:\Omega\to\mathbb{R}$.

According to Remark 2.28, it follows that $\mathscr{J}^*_{p,q}(f,g)\geq 0$, so inequalities in (2.104) also provide the refinement and the converse of (2.102).

Refinements and converses of the general Hilbert-type inequality

Hölder-type inequalities in (2.103) and (2.104) lead to some improvements of the inequality (2.99), having in mind some extra conditions concerning boundedness of the appropriate functions appearing in it. We firstly derive a refinement and a converse of the general Hilbert-type inequality (2.99) making use of the Hölder-type inequalities in (2.103).

Theorem 2.21 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $(\Omega_1,\Sigma_1,\mu_1)$, $(\Omega_2,\Sigma_2,\mu_2)$ be finite measure spaces. Let K be a non-negative measurable function on $\Omega_1\times\Omega_2$, φ a measurable, a.e. positive function on Ω_1, ψ a measurable, a.e. positive function on Ω_2, and let the functions F on Ω_1 and G on Ω_2 be defined by* (2.100) *and* (2.101)*, respectively. If $f:\Omega_1\to\mathbb{R}$ and $g:\Omega_2\to\mathbb{R}$ are non-negative measurable functions, and the function*

$$L(x,y)=K(x,y)\frac{(\varphi f)^p(x)}{\psi^p(y)},\quad (x,y)\in\Omega_1\times\Omega_2,$$

is bounded on $\Omega_1 \times \Omega_2$*, then the inequalities*

$$\begin{aligned}
&L_{\sup}\mathscr{H}_{p,q}(f,g,\varphi,\psi) \\
\geq &\|\varphi F f\|_{L^p(\mu_1)}\|\psi G g\|_{L^q(\mu_2)} - \int_{\Omega_1}\int_{\Omega_2} K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y) \\
\geq &L_{\inf}\mathscr{H}_{p,q}(f,g,\varphi,\psi)
\end{aligned} \tag{2.105}$$

hold, where

$$\begin{aligned}
\mathscr{H}_{p,q}(f,g,\varphi,\psi) = &\mu_1^{\frac{1}{p}}(\Omega_1)\mu_2^{\frac{1}{p}}(\Omega_2)\|\varphi^{-p}f^{1-p}\|_{L^q(\mu_1)}\|\psi^p g\|_{L^q(\mu_2)} \\
&- \|\varphi^{-p}f^{1-p}\|_{L^1(\mu_1)}\|\psi^p g\|_{L^1(\mu_2)}.
\end{aligned}$$

Proof. Let us rewrite the left-hand side of (2.99) in a form more suitable for applying Hölder-type inequalities in (2.103). Thus, we start with an obvious relation

$$\int_{\Omega_1}\int_{\Omega_2} K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y) = \int_{\Omega_1}\int_{\Omega_2}(h_1h_2)(x,y)d\mu_1(x)d\mu_2(y),$$

where the functions $h_1 : \Omega_1 \times \Omega_2 \to \mathbb{R}$ and $h_2 : \Omega_1 \times \Omega_2 \to \mathbb{R}$ are defined by

$$h_1(x,y) = K^{\frac{1}{p}}(x,y)\frac{(\varphi f)(x)}{\psi(y)} \quad \text{and} \quad h_2(x,y) = K^{\frac{1}{q}}(x,y)\frac{(\psi g)(y)}{\varphi(x)}.$$

The further step is to utilize inequalities in (2.103) with the above functions h_1 and h_2, with respect to product measure $\mu_1 \times \mu_2$ on $\Omega_1 \times \Omega_2$. Making use of the Fubini theorem and taking into account definition (2.100) of the function F, we have

$$\begin{aligned}
\|h_1\|_{L^p(\mu_1\times\mu_2)} &= \left[\int_{\Omega_1}\int_{\Omega_2} K(x,y)\frac{(\varphi f)^p(x)}{\psi^p(y)}d\mu_1(x)d\mu_2(y)\right]^{\frac{1}{p}} \\
&= \left[\int_{\Omega_1}(\varphi f)^p(x)\left(\int_{\Omega_2}\frac{K(x,y)}{\psi^p(y)}d\mu_2(y)\right)d\mu_1(x)\right]^{\frac{1}{p}} \\
&= \left[\int_{\Omega_1}(\varphi F f)^p(x)d\mu_1(x)\right]^{\frac{1}{p}} = \|\varphi F f\|_{L^p(\mu_1)},
\end{aligned} \tag{2.106}$$

and similarly, $\|h_2\|_{L^q(\mu_1\times\mu_2)} = \|\psi G g\|_{L^q(\mu_2)}$. In the same way, it follows that

$$\begin{aligned}
\|h_2h_1^{1-p}\|_{L^q(\mu_1\times\mu_2)} &= \left[\int_{\Omega_1}\int_{\Omega_2}(h_2^q h_1^{-p})(x,y)d\mu_1(x)d\mu_2(y)\right]^{\frac{1}{q}} \\
&= \left[\int_{\Omega_1}\int_{\Omega_2}(\varphi^{-pq}f^{-p})(x)(\psi^{pq}g^q)(y)d\mu_1(x)d\mu_2(y)\right]^{\frac{1}{q}} \\
&= \left[\int_{\Omega_1}(\varphi^{-pq}f^{-p})(x)d\mu_1(x)\right]^{\frac{1}{q}}\left[\int_{\Omega_2}(\psi^{pq}g^q)(y)d\mu_2(y)\right]^{\frac{1}{q}} \\
&= \|\varphi^{-p}f^{1-p}\|_{L^q(\mu_1)}\|\psi^p g\|_{L^q(\mu_2)}
\end{aligned} \tag{2.107}$$

and

$$\begin{aligned}
\|h_2 h_1^{1-p}\|_{L^1(\mu_1\times\mu_2)} &= \int_{\Omega_1}\int_{\Omega_2}(h_2 h_1^{1-p})(x,y)d\mu_1(x)d\mu_2(y)\\
&= \int_{\Omega_1}\int_{\Omega_2}(\varphi^{-p}f^{1-p})(x)(\psi^p g)(y)d\mu_1(x)d\mu_2(y)\\
&= \int_{\Omega_1}(\varphi^{-p}f^{1-p})(x)d\mu_1(x)\int_{\Omega_2}(\psi^p g)(y)d\mu_2(y)\\
&= \|\varphi^{-p}f^{1-p}\|_{L^1(\mu_1)}\|\psi^p g\|_{L^1(\mu_2)},
\end{aligned} \tag{2.108}$$

where we have used the fact that p and q are mutually conjugate parameters.

Finally, since $(\mu_1\times\mu_2)(\Omega_1\times\Omega_2)=\mu_1(\Omega_1)\mu_2(\Omega_2)$, the result follows from (2.103), (2.106), (2.107) and (2.108). □

Remark 2.29 The right inequality in (2.105) provides the refinement of the general inequality (2.99), while the left one yields the converse of (2.105), in a difference form. On the other hand, if $0<p<1$, then the inequality signs in (2.105) are reversed.

The following theorem also provides the refinement and the converse of the Hilbert-type inequality (2.99), this time by virtue of (2.104).

Theorem 2.22 *Suppose that the assumptions of Theorem* 2.21 *are fulfilled. Then the inequalities*

$$\begin{aligned}
&L_{\sup}\mathscr{H}^*_{p,q}(f,g,\varphi,\psi)\\
\geq&\|\varphi Ff\|^q_{L^p(\mu_1)}\|\psi Gg\|^q_{L^q(\mu_2)}-\left[\int_{\Omega_1}\int_{\Omega_2}K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y)\right]^q\\
\geq&L_{\inf}\mathscr{H}^*_{p,q}(f,g,\varphi,\psi)
\end{aligned} \tag{2.109}$$

hold, where

$$\begin{aligned}
\mathscr{H}^*_{p,q}(f,g,\varphi,\psi) =&\|\varphi Ff\|^q_{L^p(\mu_1)}\Bigg[\|\varphi^{-p}f^{1-p}\|^q_{L^q(\mu_1)}\|\psi^p g\|^q_{L^q(\mu_2)}\\
&-\frac{\|\varphi^{-p}f^{1-p}\|^q_{L^1(\mu_1)}\|\psi^p g\|^q_{L^1(\mu_2)}}{\mu_1^{q-1}(\Omega_1)\mu_2^{q-1}(\Omega_2)}\Bigg].
\end{aligned}$$

If $0<p<1$, *then the inequality signs in* (2.109) *are reversed.*

Proof. In order to derive (2.109), we utilize inequalities in (2.104) equipped with the product measure space $\Omega_1\times\Omega_2$, and the functions $h_1,h_2:\Omega_1\times\Omega_2\to\mathbb{R}$, defined in the proof of Theorem 2.21. Now, the result follows by virtue of (2.106), (2.107) and (2.108). □

Applications to homogeneous kernels

General results are now applied to homogeneous kernels and power weight functions, with respect to Lebesgue measure spaces. Since theorems 2.21 and 2.22 regard finite measure spaces, we investigate inequalities with integrals taken over bounded intervals in $\mathbb{R}_+$. More precisely, we consider here the intervals $\Omega_1 = [a,A]$ and $\Omega_2 = [b,B]$, where $0 < a < A < \infty$, $0 < b < B < \infty$, with respective Lebesgue measures $d\mu_1(x) = dx$ and $d\mu_2(y) = dy$.

Recall that a function $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is said to be homogeneous of degree $-s$, $s > 0$, if $K(tx,ty) = t^{-s}K(x,y)$, for every $x,y,t \in \mathbb{R}_+$. In addition, for such a function we define

$$k(\eta; r_1, r_2) = \int_{r_1}^{r_2} K(1,t)t^{-\eta}\,dt,$$

where $0 < r_1 < r_2 < \infty$. If nothing else is explicitly stated, we assume that the integral $k(\eta)$ converges for considered values of η. In addition, for the sake of simplicity, we also assume convergence of all the integrals appearing in the sequel.

In order to summarize further discussion, we utilize Theorem 2.21 only. Namely, the corresponding results following from Theorem 2.22 are derived similarly and are left to the reader.

Theorem 2.23 *Let $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, α, $\beta \in \mathbb{R}$, and let $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be a non-negative homogeneous function of degree $-s$, $s > 0$. If $f : [a,A] \to \mathbb{R}$ and $g : [b,B] \to \mathbb{R}$ are non-negative measurable functions, and the function*

$$L(x,y) = K(x,y)\frac{x^{\alpha p} f^p(x)}{y^{\beta p}}, \quad (x,y) \in [a,A] \times [b,B], \tag{2.110}$$

is bounded on $[a,A] \times [b,B]$, then the inequalities

$$\begin{aligned} L_{\sup}\mathscr{M}_{p,q}(f,g;\alpha,\beta) \geq & \left[\int_a^A k\left(\beta p; \tfrac{b}{x}, \tfrac{B}{x}\right) x^{1-s+(\alpha-\beta)p} f^p(x)\,dx\right]^{\frac{1}{p}} \\ & \times \left[\int_b^B k\left(2-s-\alpha q; \tfrac{y}{A}, \tfrac{y}{a}\right) y^{1-s+(\beta-\alpha)q} g^q(y)\,dy\right]^{\frac{1}{q}} \\ & - \int_a^A \int_b^B K(x,y)f(x)g(y)\,dx\,dy \\ \geq & L_{\inf}\mathscr{M}_{p,q}(f,g;\alpha,\beta) \end{aligned} \tag{2.111}$$

hold, where

$$\begin{aligned} \mathscr{M}_{p,q}(f,g;\alpha,\beta) = & [(A-a)(B-b)]^{\frac{1}{p}} \left[\int_a^A x^{-\alpha pq} f^{-p}(x)\,dx\right]^{\frac{1}{q}} \left[\int_b^B y^{\beta pq} g^q(y)\,dy\right]^{\frac{1}{q}} \\ & - \int_a^A x^{-\alpha p} f^{1-p}(x)\,dx \int_b^B y^{\beta p} g(y)\,dy. \end{aligned}$$

Proof. We utilize Theorem 2.21 with finite measure spaces $\Omega_1 = [a,A]$, $0 < a < A < \infty$, and $\Omega_2 = [b,B]$, $0 < b < B < \infty$, with respective Lebesgue measures $d\mu_1(x) = dx$,

$d\mu_2(y) = dy$, and with power weight functions $\varphi(x) = x^\alpha$, $\psi(y) = y^\beta$. Then, making use of the homogeneity of the kernel K, and passing to the new variable $t = \frac{y}{x}$, we have

$$\begin{aligned} F^p(x) &= \int_b^B K(x,y)y^{-\beta p}dy = x^{-s}\int_b^B K\left(1,\frac{y}{x}\right)y^{-\beta p}dy \\ &= x^{1-s-\beta p}\int_{\frac{b}{x}}^{\frac{B}{x}} K(1,t)t^{-\beta p}dt = x^{1-s-\beta p}k\left(\beta p;\frac{b}{x},\frac{B}{x}\right), \end{aligned}$$

and similarly,

$$G^q(y) = y^{1-s-\alpha q}k\left(2-s-\alpha q;\frac{y}{A},\frac{y}{a}\right).$$

Now, the double inequality (2.111) follows by virtue of (2.105). □

Remark 2.30 Assume that $\alpha, \beta > 0$ and that the kernel K from Theorem 2.23 is decreasing on $[a,A]\times[b,B]$ in each argument. This means that the function $k_1(x) = K(x,y)$ is decreasing on $[a,A]$, for any fixed $y \in [b,B]$, as well as that the function $k_2(y) = K(x,y)$ is decreasing on $[b,B]$, for any fixed $x \in [a,A]$. Then, it follows that the double inequality

$$K(A,B)\frac{a^{\alpha p}}{B^{\beta p}}f^p(x) \le L(x,y) \le K(a,b)\frac{A^{\alpha p}}{b^{\beta p}}f^p(x)$$

holds for all $(x,y)\in[a,A]\times[b,B]$. Moreover, if the function f is bounded on the interval $[a,A]$, then, it follows that

$$K(A,B)\frac{a^{\alpha p}}{B^{\beta p}}f_{\inf}^p \le L(x,y) \le K(a,b)\frac{A^{\alpha p}}{b^{\beta p}}f_{\sup}^p,$$

which means that

$$L_{\inf} \ge K(A,B)\frac{a^{\alpha p}}{B^{\beta p}}f_{\inf}^p \quad \text{and} \quad L_{\sup} \le K(a,b)\frac{A^{\alpha p}}{b^{\beta p}}f_{\sup}^p.$$

In the sequel, Hilbert-type inequalities with some particular homogeneous kernels that are decreasing in each argument are encountered. Having in mind Remark 2.30, it suffices to require boundedness of the non-negative function $f : [a,A] \to \mathbb{R}$ instead of boundedness of the function $L : [a,A]\times[b,B] \to \mathbb{R}$, defined by (2.110).

Our first application of Theorem 2.23 deals with the homogeneous function $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, defined by $K(x,y) = (x+y)^{-s}$, $s > 0$. In this case, the weight function $k(\eta;r_1,r_2)$ can be expressed in terms of the incomplete Beta function. Recall that the incomplete Beta function is defined by

$$B_r(a,b) = \int_0^r t^{a-1}(1-t)^{b-1}dt, \quad a,b > 0.$$

If $r = 1$, the incomplete Beta function coincides with the common Beta function. For more details about the incomplete Beta function, the reader is referred to [3]. Now, making use of the change of variable $t = \frac{1}{u} - 1$, it follows that

$$\begin{aligned} k(\eta; r_1, r_2) &= \int_{r_1}^{r_2} (1+t)^{-s} t^{-\eta} dt \\ &= \int_{\frac{1}{r_2+1}}^{\frac{1}{r_1+1}} u^{s+\eta-2} (1-u)^{-\eta} du \\ &= B_{\frac{1}{r_1+1}}(s+\eta-1, 1-\eta) - B_{\frac{1}{r_2+1}}(s+\eta-1, 1-\eta), \end{aligned} \tag{2.112}$$

where $1 - s < \eta < 1$.

For the sake of the simplicity, we provide the corresponding consequence of Theorem 2.23 with parameters $s = 1$ and $\alpha = \beta = \frac{1}{pq}$.

Corollary 2.21 *Let $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, and let $f : [a, A] \to \mathbb{R}$, $g : [b, B] \to \mathbb{R}$ be non-negative measurable functions such that f is bounded on $[a, A]$. Then the inequalities*

$$\begin{aligned} &\frac{1}{a+b} \left(\frac{A}{b}\right)^{\frac{1}{q}} f_{\sup}^{p} \mu_{p,q}(f, g) \\ &\geq \left[\int_a^A \left(B_{\frac{x}{b+x}}\left(\tfrac{1}{q}, \tfrac{1}{p}\right) - B_{\frac{x}{B+x}}\left(\tfrac{1}{q}, \tfrac{1}{p}\right)\right) f^p(x) dx\right]^{\frac{1}{p}} \\ &\times \left[\int_b^B \left(B_{\frac{A}{A+y}}\left(\tfrac{1}{q}, \tfrac{1}{p}\right) - B_{\frac{a}{a+y}}\left(\tfrac{1}{q}, \tfrac{1}{p}\right)\right) g^q(y) dy\right]^{\frac{1}{q}} - \int_a^A \int_b^B \frac{f(x)g(y)}{x+y} dx dy \\ &\geq \frac{1}{A+B} \left(\frac{a}{B}\right)^{\frac{1}{q}} f_{\inf}^{p} \mu_{p,q}(f, g) \end{aligned} \tag{2.113}$$

hold, where

$$\begin{aligned} \mu_{p,q}(f, g) = &[(A-a)(B-b)]^{\frac{1}{p}} \left[\int_a^A x^{-1} f^{-p}(x) dx\right]^{\frac{1}{q}} \left[\int_b^B y g^q(y) dy\right]^{\frac{1}{q}} \\ &- \int_a^A x^{-\frac{1}{q}} f^{1-p}(x) dx \int_b^B y^{\frac{1}{q}} g(y) dy. \end{aligned}$$

Proof. We utilize inequalities in (2.111) with the kernel $K(x, y) = (x+y)^{-1}$, and with the parameters $s = 1$, $\alpha = \beta = \frac{1}{pq}$. Now, observing that $\mu_{p,q}(f, g) = \mathscr{M}_{p,q}(f, g; \frac{1}{pq}, \frac{1}{pq})$, the inequalities in (2.113) hold by virtue of (2.112) and Remark 2.30, since the kernel K is decreasing in each argument. □

Remark 2.31 It should be noticed here that the double inequality (2.113) represents an improved version of the Hilbert inequality (2.98), for the case of bounded intervals in $\mathbb{R}_+$.

Remark 2.32 If $p = q = 2$, then the weight functions appearing in (2.113) can be expressed in terms of the inverse tangent function. More precisely, since $B_r\left(\frac{1}{2}, \frac{1}{2}\right) = 2 \arctan \sqrt{\frac{r}{1-r}}, 0 \leq r \leq 1$, it follows that

$$B_{\frac{x}{b+x}}\left(\frac{1}{2}, \frac{1}{2}\right) - B_{\frac{x}{B+x}}\left(\frac{1}{2}, \frac{1}{2}\right) = 2 \arctan \frac{\sqrt{Bx} - \sqrt{bx}}{x + \sqrt{Bb}}, \quad x > 0,$$

and

$$B_{\frac{A}{A+y}}\left(\frac{1}{2},\frac{1}{2}\right)-B_{\frac{a}{a+y}}\left(\frac{1}{2},\frac{1}{2}\right)=2\arctan\frac{\sqrt{Ay}-\sqrt{ay}}{y+\sqrt{Aa}},\quad y>0.$$

Utilizing Theorem 2.23 with the above kernel, for $s=1$ and with parameters $\alpha=\beta=\frac{1}{pq}$, we obtain the following consequence:

Corollary 2.22 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $f:[a,A]\to\mathbb{R}$, $g:[b,B]\to\mathbb{R}$ be non-negative measurable functions such that f is bounded on $[a,A]$. Then the inequalities*

$$\begin{aligned}
&\frac{1}{\max\{a,b\}}\left(\frac{A}{b}\right)^{\frac{1}{q}} f_{\sup}^{p}\mu_{p,q}(f,g)\\
&\geq\left[\int_a^A \kappa_{b,B}(x)f^p(x)dx\right]^{\frac{1}{p}}\left[\int_b^B \kappa_{a,A}(y)g^q(y)dy\right]^{\frac{1}{q}}-\int_a^A\int_b^B\frac{f(x)g(y)}{\max\{x,y\}}dxdy\\
&\geq\frac{1}{\max\{A,B\}}\left(\frac{a}{B}\right)^{\frac{1}{q}} f_{\inf}^{p}\mu_{p,q}(f,g)
\end{aligned}\tag{2.114}$$

hold, where $\mu_{p,q}(\cdot,\cdot)$ is defined in Corollary 4.8, and $\kappa_{a,A},\kappa_{b,B}:\mathbb{R}_+\to\mathbb{R}$ are functions defined by

$$\kappa_{b,B}(x)=\begin{cases} q\left[\left(\frac{x}{b}\right)^{\frac{1}{q}}-\left(\frac{x}{B}\right)^{\frac{1}{q}}\right], & x\leq b\\ p\left[1-\left(\frac{b}{x}\right)^{\frac{1}{p}}\right]+q\left[1-\left(\frac{x}{B}\right)^{\frac{1}{q}}\right], & b<x\leq B\\ p\left[\left(\frac{B}{x}\right)^{\frac{1}{p}}-\left(\frac{b}{x}\right)^{\frac{1}{p}}\right], & x>B \end{cases},$$

and

$$\kappa_{a,A}(y)=\begin{cases} p\left[\left(\frac{y}{a}\right)^{\frac{1}{p}}-\left(\frac{y}{A}\right)^{\frac{1}{p}}\right], & y\leq a\\ p\left[1-\left(\frac{y}{A}\right)^{\frac{1}{p}}\right]+q\left[1-\left(\frac{a}{y}\right)^{\frac{1}{q}}\right], & a<y\leq A\\ q\left[\left(\frac{A}{y}\right)^{\frac{1}{q}}-\left(\frac{a}{y}\right)^{\frac{1}{q}}\right], & y>A \end{cases}.$$

Proof. Utilizing (2.111) with the kernel $K(x,y)=\max^{-1}\{x,y\}$, and with the parameters $s=1$, $\alpha=\beta=\frac{1}{pq}$, it follows that

$$k\left(\frac{1}{q};\frac{b}{x},\frac{B}{x}\right)=\kappa_{b,B}(x)\quad\text{and}\quad k\left(\frac{1}{q};\frac{y}{A},\frac{y}{a}\right)=\kappa_{a,A}(y).$$

Now, the double inequality (2.114) holds due to Remark 2.30, since the function $K(x,y)=\max^{-1}\{x,y\}$ is decreasing in each argument. $\square$

A non-homogeneous example

Note that developed method for improving Hilbert-type inequalities may also be utilized for kernels which are not homogeneous. Of course, in that case the starting point is Theorem 2.21.

Similarly to the previous section, the following example deals with a non-homogeneous kernel $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$, defined by $K(x,y)=(1+xy)^{-1}$, and with integrals taken over bounded intervals in $\mathbb{R}_+$. In this case, the corresponding weight functions are also expressed in terms of the incomplete Beta function.

Corollary 2.23 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $f:[a,A]\to\mathbb{R}$, $g:[b,B]\to\mathbb{R}$ be non-negative measurable functions such that f is bounded on $[a,A]$. Then the inequalities*

$$\begin{aligned}
&\frac{1}{1+ab}\left(\frac{A}{b}\right)^{\frac{1}{q}} f_{\sup}^{p}\mu_{p,q}(f,g)\\
&\geq \left[\int_a^A x^{\frac{p-q}{pq}}\left(B_{\frac{1}{1+bx}}\left(\tfrac{1}{q},\tfrac{1}{p}\right)-B_{\frac{1}{1+Bx}}\left(\tfrac{1}{q},\tfrac{1}{p}\right)\right)f^p(x)dx\right]^{\frac{1}{p}}\\
&\quad\times\left[\int_b^B y^{\frac{q-p}{pq}}\left(B_{\frac{1}{1+ay}}\left(\tfrac{1}{p},\tfrac{1}{q}\right)-B_{\frac{1}{1+Ay}}\left(\tfrac{1}{p},\tfrac{1}{q}\right)\right)g^q(y)dy\right]^{\frac{1}{q}}-\int_a^A\int_b^B\frac{f(x)g(y)}{1+xy}dxdy\\
&\geq \frac{1}{1+AB}\left(\frac{a}{B}\right)^{\frac{1}{q}} f_{\inf}^{p}\mu_{p,q}(f,g)
\end{aligned}\tag{2.115}$$

hold, where $\mu_{p,q}(\cdot,\cdot)$ is defined in Corollary 2.21.

Proof. Rewrite Theorem 2.21 for the finite measure spaces $\Omega_1=[a,A]$, $0<a<A<\infty$, $\Omega_2=[b,B]$, $0<b<B<\infty$, with respective Lebesgue measures $d\mu_1(x)=dx$, $d\mu_2(y)=dy$, and for $K(x,y)=(1+xy)^{-1}$, $\varphi(x)=x^{\frac{1}{pq}}$, $\psi(y)=y^{\frac{1}{pq}}$. Then, passing to the new variable $t=xy$ and utilizing (2.112), it follows that

$$\begin{aligned}
F^p(x)&=\int_b^B(1+xy)^{-1}y^{-\frac{1}{q}}dy\\
&=x^{-\frac{1}{p}}\int_{bx}^{Bx}(1+t)^{-1}t^{-\frac{1}{q}}dt\\
&=x^{-\frac{1}{p}}\left(B_{\frac{1}{1+bx}}\left(\frac{1}{q},\frac{1}{p}\right)-B_{\frac{1}{1+Bx}}\left(\frac{1}{q},\frac{1}{p}\right)\right),
\end{aligned}$$

and similarly,

$$G^q(y)=y^{-\frac{1}{q}}\left(B_{\frac{1}{1+ay}}\left(\frac{1}{p},\frac{1}{q}\right)-B_{\frac{1}{1+Ay}}\left(\frac{1}{p},\frac{1}{q}\right)\right).$$

Moreover, since $K(x,y)=(1+xy)^{-1}$ is decreasing on $\mathbb{R}_+\times\mathbb{R}_+$ in each argument, the result follows by virtue of (2.105) and Remark 2.30. □

Remark 2.33 Similarly to Remark 2.32, in the case of $p = q = 2$, the weight functions appearing in (2.115) are also expressed in terms of the inverse tangent function, that is, we have

$$B_{\frac{1}{1+bx}}\left(\frac{1}{2},\frac{1}{2}\right) - B_{\frac{1}{1+Bx}}\left(\frac{1}{2},\frac{1}{2}\right) = 2\arctan\frac{\sqrt{Bx}-\sqrt{bx}}{1+x\sqrt{Bb}}, \quad x > 0,$$

and

$$B_{\frac{1}{1+ay}}\left(\frac{1}{2},\frac{1}{2}\right) - B_{\frac{1}{1+Ay}}\left(\frac{1}{2},\frac{1}{2}\right) = 2\arctan\frac{\sqrt{Ay}-\sqrt{ay}}{1+y\sqrt{Aa}}, \quad y > 0.$$

2.3.2 Related results on multidimensional Hilbert's inequality

Refinements and converses of multidimensional Hilbert's inequality are mainly obtained with the help of the previously presented improvements of Hölder's inequality. These are worked out for the case of the conjugate, as well as the non-conjugate exponents, in the difference and/or in the ratio form. Corresponding results are then applied to homogeneous kernels with the negative degree of homogeneity. The conditions on the best possible constant factors in the obtained inequalities are also established and some particular settings with homogeneous kernels and weight functions are considered. Finally, the comparison to the existing results known from the literature is given.

The contents of this section corresponds for the most part to the contents of papers [105] and [106].

Some extra notes on Hölder's inequality

As it was previously pointed out, Hölder's inequality is the starting point in obtaining Hilbert's inequality. In order to apply the presented improvements of Hölder's inequality to multidimensional Hilbert's inequality, we present its following form, as the most convenient for this purpose:

$$\int_{\Omega}\prod_{i=1}^{n} F_i^{\alpha_i}(x)d\mu(x) \le \prod_{i=1}^{n} ||F_i^{\alpha_i}||_{1/\alpha_i}, \tag{2.116}$$

where $F_i : \Omega \to \mathbb{R}$, $i = 1,2,\ldots,n$, are non-negative measurable functions on σ-finite measure space (Ω,Σ,μ) and α_i are positive real numbers such that $\sum_{i=1}^{n}\alpha_i = 1$. The improvement of the inequality (2.116) in the ratio form is given in the following lemma.

Lemma 2.1 *Let (Ω,Σ,μ) be a σ-finite measure space and let $F_i : \Omega \to \mathbb{R}$ be non-negative measurable functions, $i = 1,2,\ldots,n$. If $\sum_{i=1}^{n}\alpha_i = 1$, $\alpha_i > 0$, then the following series of inequalities holds:*

$$\begin{aligned}&\left[\frac{\int_{\Omega}\prod_{i=1}^{n}F_i^{1/n}(x)d\mu(x)}{\prod_{i=1}^{n}||F_i^{1/n}||_n}\right]^{n\max_{1\le i\le n}\{\alpha_i\}}\\ &\qquad\le \frac{\int_{\Omega}\prod_{i=1}^{n}F_i^{\alpha_i}(x)d\mu(x)}{\prod_{i=1}^{n}||F_i^{\alpha_i}||_{1/\alpha_i}} \le \left[\frac{\int_{\Omega}\prod_{i=1}^{n}F_i^{1/n}(x)d\mu(x)}{\prod_{i=1}^{n}||F_i^{1/n}||_n}\right]^{n\min_{1\le i\le n}\{\alpha_i\}}.\end{aligned} \tag{2.117}$$

Proof. The left-hand side of Hölder's inequality (2.116) can be rewritten as

$$\int_\Omega \prod_{i=1}^n F_i^{\alpha_i}(x)d\mu(x) = \int_\Omega \left[\prod_{i=1}^n F_i^{\beta_i}(x)\right]^{1-nm} \cdot \left[\prod_{i=1}^n F_i^{1/n}(x)\right]^{nm} d\mu(x),$$

where $m = \min_{1\le i\le n}\{\alpha_i\}$ and $\beta_i = (\alpha_i - m)/(1-nm)$, $i = 1,2,\ldots,n$.

Since $1-nm \ge 0$, the application of Hölder's inequality to the previous relation yields inequality

$$\int_\Omega \prod_{i=1}^n F_i^{\alpha_i}(x)d\mu(x) \le \left[\int_\Omega \prod_{i=1}^n F_i^{\beta_i}(x)d\mu(x)\right]^{1-nm} \cdot \left[\int_\Omega \prod_{i=1}^n F_i^{1/n}(x)d\mu(x)\right]^{nm}. \tag{2.118}$$

On the other hand, the right-hand side of Hölder's inequality (2.116) can be rewritten as

$$\prod_{i=1}^n ||F_i^{\alpha_i}||_{1/\alpha_i} = \left[\prod_{i=1}^n ||F_i^{\beta_i}||_{1/\beta_i}\right]^{1-nm} \cdot \left[\prod_{i=1}^n ||F_i^{1/n}||_n\right]^{nm}. \tag{2.119}$$

Now, relations (2.118) and (2.119) imply inequality

$$\begin{aligned} &\frac{\int_\Omega \prod_{i=1}^n F_i^{\alpha_i}(x)d\mu(x)}{\prod_{i=1}^n ||F_i^{\alpha_i}||_{1/\alpha_i}} \\ &\le \left[\frac{\int_\Omega \prod_{i=1}^n F_i^{\beta_i}(x)d\mu(x)}{\prod_{i=1}^n ||F_i^{\beta_i}||_{1/\beta_i}}\right]^{1-nm} \cdot \left[\frac{\int_\Omega \prod_{i=1}^n F_i^{1/n}(x)d\mu(x)}{\prod_{i=1}^n ||F_i^{1/n}||_n}\right]^{nm}. \end{aligned} \tag{2.120}$$

Note that $\sum_{i=1}^n \beta_i = 1$, $\beta_i \ge 0$, so yet another application of Hölder's inequality implies

$$\frac{\int_\Omega \prod_{i=1}^n F_i^{\beta_i}(x)d\mu(x)}{\prod_{i=1}^n ||F_i^{\beta_i}||_{1/\beta_i}} \le 1,$$

that is, from (2.120) we get the right inequality in (2.117).

The left inequality in (2.117) is proved in a similar way. Namely, we use decomposition

$$\int_\Omega \prod_{i=1}^n F_i^{1/n}(x)d\mu(x) = \int_\Omega \left[\prod_{i=1}^n F_i^{\alpha_i}(x)\right]^{1/(nM)} \cdot \left[\prod_{i=1}^n F_i^{\gamma_i}(x)\right]^{1-1/(nM)} d\mu(x),$$

where $M = \max_{1\le i\le n}\{\alpha_i\}$, $\gamma_i = (M-\alpha_i)/(nM-1)$, $i = 1,2,\ldots,n$, and apply Hölder's inequality as in the first part of the proof. □

Clearly, the quotient between the left-hand side and the right-hand side of Hölder's inequality (2.116) is mutually bounded via the quotient of the same type involving equal exponents. Moreover, since $\int_\Omega \prod_{i=1}^n F_i^{1/n}(x)d\mu(x) \le \prod_{i=1}^n ||F_i^{1/n}||_n$, the right inequality in (2.117) yields a refinement, while the left one yields a converse of Hölder's inequality (2.116). The interpolating series of inequalities (2.117) is referred to as the refinement and the converse of Hölder's inequality in the ratio form.

On the other hand, the following lemma provides a refinement and a converse of Hölder's inequality in the difference form.

Lemma 2.2 *Let (Ω,Σ,μ) be a σ-finite measure space and let $F_i:\Omega\to\mathbb{R}$ be non-negative measurable functions, $i=1,2,\ldots,n$. If $\sum_{i=1}^n \alpha_i=1$, $\alpha_i>0$, then the following series of inequalities holds:*

$$\begin{aligned}
& n\min_{1\le i\le n}\{\alpha_i\}\prod_{i=1}^n ||F_i^{\alpha_i}||_{1/\alpha_i}\left[1-\frac{\int_\Omega \prod_{i=1}^n F_i^{1/n}(x)d\mu(x)}{\prod_{i=1}^n ||F_i^{1/n}||_n}\right]\\
&\le \prod_{i=1}^n ||F_i^{\alpha_i}||_{1/\alpha_i}-\int_\Omega \prod_{i=1}^n F_i^{\alpha_i}(x)d\mu(x) \qquad (2.121)\\
&\le n\max_{1\le i\le n}\{\alpha_i\}\prod_{i=1}^n ||F_i^{\alpha_i}||_{1/\alpha_i}\left[1-\frac{\int_\Omega \prod_{i=1}^n F_i^{1/n}(x)d\mu(x)}{\prod_{i=1}^n ||F_i^{1/n}||_n}\right].
\end{aligned}$$

Proof. The series of inequalities (2.121) can be derived throughout the refinement and the converse of the classical arithmetic-geometric mean inequality. Namely, the difference between the weight arithmetic and geometric mean can be rewritten as

$$\sum_{i=1}^n \alpha_i t_i-\prod_{i=1}^n t_i^{\alpha_i}=\sum_{i=1}^n(\alpha_i-m)t_i+m\sum_{i=1}^n t_i-\left[\prod_{i=1}^n t_i^{\beta_i}\right]^{1-nm}\cdot\left[\prod_{i=1}^n t_i^{1/n}\right]^{nm}, \qquad (2.122)$$

where $m=\min_{1\le i\le n}\{\alpha_i\}$, $\beta_i=(\alpha_i-m)/(1-nm)$, $t_i>0$, $i=1,2,\ldots,n$. In addition, the arithmetic-geometric mean inequality yields

$$\left[\prod_{i=1}^n t_i^{\beta_i}\right]^{1-nm}\cdot\left[\prod_{i=1}^n t_i^{1/n}\right]^{nm}\le(1-nm)\prod_{i=1}^n t_i^{\beta_i}+nm\prod_{i=1}^n t_i^{1/n}, \qquad (2.123)$$

thus relations (2.122) and (2.123) provide inequality

$$\sum_{i=1}^n \alpha_i t_i-\prod_{i=1}^n t_i^{\alpha_i}\ge(1-nm)\left[\sum_{i=1}^n\beta_i t_i-\prod_{i=1}^n t_i^{\beta_i}\right]+nm\left[\frac{\sum_{i=1}^n x_i}{n}-\prod_{i=1}^n x_i^{1/n}\right],$$

that is,

$$\sum_{i=1}^n \alpha_i t_i-\prod_{i=1}^n t_i^{\alpha_i}\ge nm\left[\frac{\sum_{i=1}^n x_i}{n}-\prod_{i=1}^n x_i^{1/n}\right], \qquad (2.124)$$

since $\sum_{i=1}^n\beta_i=1$ and $\sum_{i=1}^n\beta_i t_i-\prod_{i=1}^n t_i^{\beta_i}\ge 0$.

Now, if we replace t_i with $F_i(x)/\int_\Omega F_i(x)d\mu(x)$ and taking into account that $\int_\Omega F_i(x)d\mu(x)$ $=||F_i^{\alpha_i}||_{1/\alpha_i}^{1/\alpha_i}=||F_i^{1/n}||_n^n$, inequality (2.124) takes form

$$\sum_{i=1}^n\frac{\alpha_i f_i(x)}{\int_\Omega F_i(x)d\mu(x)}-\frac{\prod_{i=1}^n F_i^{\alpha_i}(x)}{\prod_{i=1}^n ||F_i^{\alpha_i}||_{1/\alpha_i}}\ge nm\left[\frac{f_i(x)}{n\int_\Omega F_i(x)d\mu(x)}-\frac{\prod_{i=1}^n F_i^{1/n}(x)}{\prod_{i=1}^n ||F_i^{1/n}||_n}\right],$$

that is,

$$1-\frac{\int_\Omega\prod_{i=1}^n F_i^{\alpha_i}(x)d\mu(x)}{\prod_{i=1}^n ||F_i^{\alpha_i}||_{1/\alpha_i}}\ge nm\left[1-\frac{\int_\Omega\prod_{i=1}^n F_i^{1/n}(x)d\mu(x)}{\prod_{i=1}^n ||F_i^{1/n}||_n}\right],$$

after integrating over Ω with respect to measure μ.

To prove the right inequality in (2.121) we start with the relation

$$\begin{aligned} nM&\left[\frac{\sum_{i=1}^n x_i}{n}-\prod_{i=1}^n x_i^{1/n}\right]\\ &=\sum_{i=1}^n (M-\alpha_i)t_i+\sum_{i=1}^n \alpha_i t_i - nM\left[\prod_{i=1}^n t_i^{\alpha_i}\right]^{1/(nM)}\cdot\left[\prod_{i=1}^n t_i^{\gamma_i}\right]^{1-1/(nM)}, \end{aligned}\tag{2.125}$$

where $M=\max_{1\le i\le n}\{\alpha_i\}$ and $\gamma_i=(M-\alpha_i)/(nM-1)$, $i=1,2,\dots,n$. Further, the arithmetic-geometric mean inequality yields inequality

$$nM\left[\prod_{i=1}^n t_i^{\alpha_i}\right]^{1/(nM)}\cdot\left[\prod_{i=1}^n t_i^{\gamma_i}\right]^{1-1/(nM)} \le \prod_{i=1}^n t_i^{\alpha_i}+(nM-1)\left[\prod_{i=1}^n t_i^{\gamma_i}\right]. \tag{2.126}$$

Therefore, the relations (2.125) and (2.126) imply inequality

$$nM\left[\frac{\sum_{i=1}^n x_i}{n}-\prod_{i=1}^n x_i^{1/n}\right]\ge\sum_{i=1}^n \alpha_i t_i-\prod_{i=1}^n t_i^{\alpha_i}+(nM-1)\left[\sum_{i=1}^n \gamma_i t_i-\prod_{i=1}^n t_i^{\gamma_i}\right],$$

that is,

$$nM\left[\frac{\sum_{i=1}^n x_i}{n}-\prod_{i=1}^n x_i^{1/n}\right]\ge\sum_{i=1}^n \alpha_i t_i-\prod_{i=1}^n t_i^{\alpha_i},$$

since $\sum_{i=1}^n \gamma_i t_i \ge \prod_{i=1}^n t_i^{\gamma_i}$, $\sum_{i=1}^n \gamma_i=1$. The rest of the proof follows the same lines as the proof of the left inequality in (2.121). □

Obviously, since $\int_\Omega \prod_{i=1}^n F_i^{1/n}(x)d\mu(x)\le\prod_{i=1}^n ||F_i^{1/n}||_n$, the left inequality in (2.121) yields the refinement, while the right one provides the converse of Hölder's inequality. The interpolating series of inequalities (2.121) is referred to as the refinement and the converse of Hölder's inequality in the difference form.

A refinement and a converse in ratio form: conjugate exponents

Some of the recent results concerning Hilbert's inequality (1.36) include an extension to multidimensional case, equipped with n conjugate exponents p_i, that is, $\sum_{i=1}^n 1/p_i=1$, $p_i>1$, $n\ge 2$. For more details on the subject, the reader is referred to [37], [38], [118], [184], [208] and the monograph [122]. Here we refer to paper [37], which provides a unified treatment of the multidimensional Hilbert-type inequality in the setting with conjugate exponents. Suppose $(\Omega_i,\Sigma_i,\mu_i)$ are σ-finite measure spaces and $K:\prod_{i=1}^n\Omega_i\to\mathbb{R}$, $\phi_{ij}:\Omega_j\to\mathbb{R}$, $f_i:\Omega_i\to\mathbb{R}$, $i,j=1,2,\dots,n$, are non-negative measurable functions. If $\prod_{i,j=1}^n \phi_{ij}(x_j)=1$, then

$$\int_{\mathbf{\Omega}} K(\mathbf{x})\prod_{i=1}^n f_i(x_i)d\mu(\mathbf{x})\le\prod_{i=1}^n ||\phi_{ii}\omega_i f_i||_{p_i}, \tag{2.127}$$

where

$$\omega_i(x_i) = \left[\int_{\hat{\mathbf{\Omega}}^i} K(\mathbf{x}) \prod_{j=1, j\neq i}^{n} \phi_{ij}^{p_i}(x_j) d\hat{\mu}^i(\mathbf{x}) \right]^{1/p_i} \tag{2.128}$$

and

$$\begin{aligned} \mathbf{\Omega} &= \prod_{i=1}^{n} \Omega_i, \quad \hat{\mathbf{\Omega}}^i = \prod_{j=1, j\neq i}^{n} \Omega_j, \quad \mathbf{x} = (x_1, x_2, \ldots, x_n), \\ d\mu(\mathbf{x}) &= \prod_{i=1}^{n} d\mu_i(x_i), \quad d\hat{\mu}^i(\mathbf{x}) = \prod_{j=1, j\neq i}^{n} d\mu_j(x_j). \end{aligned} \tag{2.129}$$

The abbreviations as in (2.129) will be valid in the sequel. Also note that $||\cdot||_{p_i}$ denotes the usual norm in $L^{p_i}(\Omega_i)$, that is

$$||\phi_{ii}\omega_i f_i||_{p_i} = \left[\int_{\Omega_i} (\phi_{ii}\omega_i f_i)^{p_i}(x_i) d\mu_i(x_i) \right]^{1/p_i}, \quad i = 1, 2, \ldots, n.$$

We are going to consider the ratio between the left-hand side and the right-hand side of the inequality (2.127), in order to establish the lower and the upper bound for the above mentioned quotient, expressed in terms of a similar quotient. By means of the lower bound we get the converse of the Hilbert-type inequality (2.127), while the upper bound provides its refinement. Such improvements will be referred to as the refinement and the converse of the Hilbert-type inequality in the ratio form. Since Hilbert-type inequality is derived by means of Hölder's inequality, the main results are derived with the help of a sophisticated use of Hölder's inequality. The following theorem provides the refinement of the inequality (2.127).

Theorem 2.24 *Let $(\Omega_i, \Sigma_i, \mu_i)$ be σ-finite measure spaces and let $K : \mathbf{\Omega} \to \mathbb{R}$, $\phi_{ij} : \Omega_j \to \mathbb{R}$, $f_i : \Omega_i \to \mathbb{R}$, $i, j = 1, 2, \ldots, n$ be non-negative measurable functions. If $\prod_{i,j=1}^{n} \phi_{ij}(x_j) = 1$, then*

$$\begin{aligned} &\frac{\int_{\mathbf{\Omega}} K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i) d\mu(\mathbf{x})}{\prod_{i=1}^{n} ||\phi_{ii}\omega_i f_i||_{p_i}} \\ &\leq \frac{\left[\int_{\mathbf{\Omega}} K(\mathbf{x}) \prod_{i=1}^{n} f_i^{p_i/n}(x_i) \prod_{i,j=1}^{n} \phi_{ij}^{p_i/n}(x_j) d\mu(\mathbf{x}) \right]^{n/\max_{1\leq i\leq n}\{p_i\}}}{\prod_{i=1}^{n} ||\phi_{ii}\omega_i f_i||_{p_i}^{p_i/\max_{1\leq i\leq n}\{p_i\}}}, \end{aligned} \tag{2.130}$$

where $p_i > 1$ are conjugate exponents and $\omega_i : \Omega_i \to \mathbb{R}$ are defined by (2.128)*, $i = 1, 2, \ldots, n$.*

Proof. The left-hand side of the Hilbert-type inequality (2.127) can be rewritten in the form

$$\int_{\mathbf{\Omega}} K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i) d\mu(\mathbf{x}) = \int_{\mathbf{\Omega}} \left[\prod_{i=1}^{n} F_i^{1/q_i}(\mathbf{x}) \right]^{1-n/M} \cdot \left[\prod_{i=1}^{n} F_i^{1/n}(\mathbf{x}) \right]^{n/M} d\mu(\mathbf{x}),$$

where the functions $F_i : \mathbf{\Omega} \to \mathbb{R}$ are defined by

$$F_i(\mathbf{x}) = K(\mathbf{x}) f_i^{p_i}(x_i) \prod_{j=1}^{n} \phi_{ij}^{p_i}(x_j), \quad i = 1,2,\ldots,n, \tag{2.131}$$

$M = \max_{1 \le i \le n}\{p_i\}$, and

$$q_i = \frac{p_i(M-n)}{M-p_i}, \quad i = 1,2,\ldots,n.$$

Clearly, the above relation is meaningful because if $M = p_l$ for some $l \in \{1,2,\ldots,n\}$, then $1/q_l = 0$. Further, the application of Hölder's inequality to the above form of the left-hand side of inequality (2.127) yields inequality

$$\begin{aligned} &\int_{\mathbf{\Omega}} K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i) d\mu(\mathbf{x}) \\ &\le \left[\int_{\mathbf{\Omega}} \prod_{i=1}^{n} F_i^{1/q_i}(\mathbf{x}) d\mu(\mathbf{x})\right]^{1-n/M} \cdot \left[\int_{\mathbf{\Omega}} \prod_{i=1}^{n} F_i^{1/n}(\mathbf{x}) d\mu(\mathbf{x})\right]^{n/M}. \end{aligned} \tag{2.132}$$

On the other hand, by using the well-known Fubini's theorem we have

$$\begin{aligned} ||F_i^{1/t}||_t &= \left[\int_{\mathbf{\Omega}} K(\mathbf{x}) (\phi_{ii} f_i)^{p_i}(x_i) \prod_{j=1, j\neq i}^{n} \phi_{ij}^{p_i}(x_j) d\mu(\mathbf{x})\right]^{1/t} \\ &= \left[\int_{\Omega_i} (\phi_{ii} f_i)^{p_i}(x_i) \left(\int_{\hat{\mathbf{\Omega}}^i} K(\mathbf{x}) \prod_{j=1, j\neq i}^{n} \phi_{ij}^{p_i}(x_j) d\hat{\mu}^i(\mathbf{x})\right) d\mu_i(x_i)\right]^{1/t} \\ &= \left[\int_{\Omega_i} (\phi_{ii}\omega_i f_i)^{p_i}(x_i) d\mu_i(x_i)\right]^{1/t} \\ &= ||\phi_{ii}\omega_i f_i||_{p_i}^{p_i/t}, \quad i = 1,2,\ldots,n, \ t > 0, \end{aligned} \tag{2.133}$$

and the right-hand side of Hilbert-type inequality (2.127) can be rewritten in the form

$$\prod_{i=1}^{n} ||\phi_{ii}\omega_i f_i||_{p_i} = \left[\prod_{i=1}^{n} ||F_i^{1/q_i}||_{q_i}\right]^{1-n/M} \cdot \left[\prod_{i=1}^{n} ||F_i^{1/n}||_n\right]^{n/M}.$$

Therefore, inequality (2.132) can be expressed in the following form:

$$\begin{aligned} &\frac{\int_{\mathbf{\Omega}} K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i) d\mu(\mathbf{x})}{\prod_{i=1}^{n} ||\phi_{ii}\omega_i f_i||_{p_i}} \\ &\le \left[\frac{\int_{\mathbf{\Omega}} \prod_{i=1}^{n} F_i^{1/q_i}(\mathbf{x}) d\mu(\mathbf{x})}{\prod_{i=1}^{n} ||F_i^{1/q_i}||_{q_i}}\right]^{1-n/M} \left[\frac{\int_{\mathbf{\Omega}} \prod_{i=1}^{n} F_i^{1/n}(\mathbf{x}) d\mu(\mathbf{x})}{\prod_{i=1}^{n} ||F_i^{1/n}||_n}\right]^{n/M}. \end{aligned} \tag{2.134}$$

Obviously $M \geq n$. If $M > n$, then $q_i > 0$ and

$$\sum_{i=1}^{n} \frac{1}{q_i} = \sum_{i=1}^{n} \frac{M-p_i}{p_i(M-n)} = \frac{1}{M-n}\left[M\sum_{i=1}^{n}\frac{1}{p_i} - n\right] = 1,$$

that is, q_i are also conjugate exponents and Hölder's inequality yields inequality $\int_{\mathbf{\Omega}} \prod_{i=1}^{n} F_i^{1/q_i}(\mathbf{x})d\mu(\mathbf{x}) \leq \prod_{i=1}^{n} ||F_i^{1/q_i}||_{q_i}$. Hence, relation (2.134) implies inequality

$$\frac{\int_{\mathbf{\Omega}} K(\mathbf{x})\prod_{i=1}^{n} f_i(x_i)d\mu(\mathbf{x})}{\prod_{i=1}^{n} ||\phi_{ii}\omega_i f_i||_{p_i}} \leq \left[\frac{\int_{\mathbf{\Omega}} \prod_{i=1}^{n} F_i^{1/n}(\mathbf{x})d\mu(\mathbf{x})}{\prod_{i=1}^{n} ||F_i^{1/n}||_n}\right]^{n/M},$$

which is also valid if $M = n$. Finally, by substituting the functions F_i in the last inequality, we get (2.130) as required. □

Remark 2.34 Bearing in mind the notation as in the proof of Theorem 2.24, by Hölder's inequality we have $\int_{\mathbf{\Omega}} \prod_{i=1}^{n} F_i^{1/n}(\mathbf{x})d\mu(\mathbf{x}) \leq \prod_{i=1}^{n} ||F_i^{1/n}||_n$. Therefore, the quotient on the right-hand side of inequality (2.130) is not greater than 1, which means that (2.130) represents the refinement of inequality (2.127).

In a similar way, the converse of inequality (2.127) is obtained, which is the contents of the following theorem.

Theorem 2.25 *Let* $(\Omega_i, \Sigma_i, \mu_i)$ *be* σ*-finite measure spaces and let* $K : \mathbf{\Omega} \to \mathbb{R}$, $\phi_{ij} : \Omega_j \to \mathbb{R}$, $f_i : \Omega_i \to \mathbb{R}$, $i, j = 1, 2, \ldots, n$, *be non-negative measurable functions. If* $\prod_{i,j=1}^{n} \phi_{ij}(x_j) = 1$, *then*

$$\begin{aligned}
&\frac{\int_{\mathbf{\Omega}} K(\mathbf{x})\prod_{i=1}^{n} f_i(x_i)d\mu(\mathbf{x})}{\prod_{i=1}^{n} ||\phi_{ii}\omega_i f_i||_{p_i}} \\
&\geq \frac{\left[\int_{\mathbf{\Omega}} K(\mathbf{x})\prod_{i=1}^{n} f_i^{p_i/n}(x_i)\prod_{i,j=1}^{n} \phi_{ij}^{p_i/n}(x_j)d\mu(\mathbf{x})\right]^{n/\min_{1\leq i\leq n}\{p_i\}}}{\prod_{i=1}^{n} ||\phi_{ii}\omega_i f_i||_{p_i}^{p_i/\min_{1\leq i\leq n}\{p_i\}}},
\end{aligned} \tag{2.135}$$

where $p_i > 1$ *are conjugate exponents and* $\omega_i : \Omega_i \to \mathbb{R}$ *are defined by* (2.128), $i = 1, 2, \ldots, n$.

Proof. The starting point in obtaining (2.135) is the relation

$$\int_{\mathbf{\Omega}} \prod_{i=1}^{n} F_i^{1/n}(\mathbf{x})d\mu(\mathbf{x}) = \int_{\mathbf{\Omega}} \left[K(\mathbf{x})\prod_{i=1}^{n} f_i(x_i)\right]^{m/n} \left[\prod_{i=1}^{n} F_i^{1/r_i}(\mathbf{x})\right]^{1-m/n} d\mu(\mathbf{x}),$$

where the functions $F_i : \mathbf{\Omega} \to \mathbb{R}$ are defined by (2.131), $m = \min_{1\leq i\leq n}\{p_i\}$ and

$$r_i = \frac{p_i(n-m)}{p_i - m}, \quad i = 1, 2, \ldots, n.$$

If $m = p_l$ for some $l \in \{1,2,\dots,n\}$, then $1/r_l = 0$, which means that the above decomposition is meaningful. Now, the application of Hölder's inequality yields relation

$$\begin{aligned} &\int_{\mathbf{\Omega}} \prod_{i=1}^{n} F_i^{1/n}(\mathbf{x})d\mu(\mathbf{x}) \\ &\leq \left[\int_{\mathbf{\Omega}} K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i)d\mu(\mathbf{x})\right]^{m/n} \left[\int_{\mathbf{\Omega}} \prod_{i=1}^{n} F_i^{1/r_i}(\mathbf{x})d\mu(\mathbf{x})\right]^{1-m/n}. \end{aligned} \tag{2.136}$$

On the other hand, regarding relation (2.133) we have

$$\prod_{i=1}^{n} ||F_i^{1/n}||_n = \left[\prod_{i=1}^{n} ||\phi_{ii}\omega_i f_i||_{p_i}\right]^{m/n} \cdot \left[\prod_{i=1}^{n} ||F_i^{1/r_i}||_{r_i}\right]^{1-m/n}.$$

If we divide inequality (2.136) with the previous relation, we get inequality

$$\begin{aligned} &\frac{\int_{\mathbf{\Omega}} \prod_{i=1}^{n} F_i^{1/n}(\mathbf{x})d\mu(\mathbf{x})}{\prod_{i=1}^{n} ||F_i^{1/n}||_n} \\ &\leq \left[\frac{\int_{\mathbf{\Omega}} K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i)d\mu(\mathbf{x})}{\prod_{i=1}^{n} ||\phi_{ii}\omega_i f_i||_{p_i}}\right]^{m/n} \left[\frac{\int_{\mathbf{\Omega}} \prod_{i=1}^{n} F_i^{1/r_i}(\mathbf{x})d\mu(\mathbf{x})}{\prod_{i=1}^{n} ||F_i^{1/r_i}||_{r_i}}\right]^{1-m/n}. \end{aligned} \tag{2.137}$$

Obviously $m \leq n$. If $m < n$, then $r_i > 0$ and

$$\sum_{i=1}^{n} \frac{1}{r_i} = \sum_{i=1}^{n} \frac{p_i - m}{p_i(n-m)} = \frac{1}{n-m}\left[n - m\sum_{i=1}^{n} \frac{1}{p_i}\right] = 1,$$

that is, r_i are conjugate exponents. Hence, yet another application of Hölder's inequality implies that

$$\frac{\int_{\mathbf{\Omega}} \prod_{i=1}^{n} F_i^{1/r_i}(\mathbf{x})d\mu(\mathbf{x})}{\prod_{i=1}^{n} ||F_i^{1/r_i}||_{r_i}} \leq 1.$$

Therefore, inequality (2.137) yields

$$\frac{\int_{\mathbf{\Omega}} \prod_{i=1}^{n} F_i^{1/n}(\mathbf{x})d\mu(\mathbf{x})}{\prod_{i=1}^{n} ||F_i^{1/n}||_n} \leq \left[\frac{\int_{\mathbf{\Omega}} K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i)d\mu(\mathbf{x})}{\prod_{i=1}^{n} ||\phi_{ii}\omega_i f_i||_{p_i}}\right]^{m/n},$$

that is,

$$\left[\frac{\int_{\mathbf{\Omega}} \prod_{i=1}^{n} F_i^{1/n}(\mathbf{x})d\mu(\mathbf{x})}{\prod_{i=1}^{n} ||F_i^{1/n}||_n}\right]^{n/m} \leq \frac{\int_{\mathbf{\Omega}} K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i)d\mu(\mathbf{x})}{\prod_{i=1}^{n} ||\phi_{ii}\omega_i f_i||_{p_i}}.$$

Note that the last inequality also holds for $m = n$. Finally, making use of the definition (2.131) of F_i, the last inequality yields (2.135). □

Remark 2.35 The proofs of theorems 2.24 and 2.25 are taken from paper [105]. Note that they follow directly from Lemma 2.1, i.e. its relation (2.117).

Application to homogeneous kernels

General results are now applied to homogeneous functions with the negative degree of homogeneity. Further, regarding the notation from the previous considerations, we assume that $\Omega_i = \mathbb{R}_+$, equipped with the non-negative Lebesgue measures $d\mu_i(x_i) = dx_i$, $i = 1,2,\dots,n$. In addition, we have $\mathbf{\Omega} = \mathbb{R}^n_+$ and $d\mathbf{x} = dx_1 dx_2 \dots dx_n$.

We introduce the real parameters A_{ij}, $i,j = 1,2,\dots,n$, such that $\sum_{i=1}^n A_{ij} = 0$, $j = 1,2,\dots,n$, and denote $\alpha_i = \sum_{j=1}^n A_{ij}$, $i = 1,2,\dots,n$. Next, we consider the set of power functions $\phi_{ij} : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$\phi_{ij}(x_j) = x_j^{A_{ij}}. \tag{2.138}$$

Clearly, above defined power functions satisfy the condition

$$\prod_{i,j=1}^n \phi_{ij}(x_j) = \prod_{j=1}^n \prod_{i=1}^n x_j^{A_{ij}} = \prod_{j=1}^n x_j^{\sum_{i=1}^n A_{ij}} = 1,$$

since $\sum_{i=1}^n A_{ij} = 0$. Therefore, functions ϕ_{ij}, $i,j = 1,2,\dots,n$ satisfy the conditions as in Theorems 2.24 and 2.25.

Recall that function $K : \mathbb{R}^n_+ \to \mathbb{R}$ is said to be homogeneous of degree $-s$, $s > 0$, if $K(t\mathbf{x}) = t^{-s}K(\mathbf{x})$ for all $t > 0$. Furthermore, for $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$, we define

$$k_i(\mathbf{a}) = \int_{\mathbb{R}^{n-1}_+} K(\hat{\mathbf{u}}^i) \prod_{j=1, j\neq i}^n u_j^{a_j} \hat{d}^i\mathbf{u}, \quad i = 1,2,\dots,n, \tag{2.139}$$

where $\hat{\mathbf{u}}^i = (u_1, \dots, u_{i-1}, 1, u_{i+1}, \dots, u_n)$, $\hat{d}^i\mathbf{u} = du_1 \dots du_{i-1} du_{i+1} \dots du_n$, and provided that the above integral converges. Note that the constant factor $k_i(\mathbf{a})$ does not depend on the component a_i. Thus, the component a_i can be replaced by an arbitrary real number, which will be used in the sequel, for the sake of simplicity. Further, in the described setting we can find the explicit formula for the weight function (2.128) including the constant factor $k_i(\mathbf{a})$. More precisely, we use the substitution $x_j = u_j x_i$, $j \neq i$, that is, $\hat{d}^i\mathbf{x} = x_i^{n-1}\hat{d}^i\mathbf{u}$, while the homogeneity of the kernel K yields relation $K(\mathbf{x}) = x_i^{-s}K(\hat{\mathbf{u}}^i)$. Moreover, regarding definition (2.139) we have

$$\begin{aligned}
\omega_i(x_i) &= \left[\int_{\mathbb{R}^{n-1}_+} K(\mathbf{x}) \prod_{j=1, j\neq i}^n x_j^{p_i A_{ij}} \hat{d}^i\mathbf{x} \right]^{1/p_i} \\
&= \left[x_i^{n-1-s+\sum_{j=1, j\neq i}^n p_i A_{ij}} \int_{\mathbb{R}^{n-1}_+} K(\hat{\mathbf{u}}^i) \prod_{j=1, j\neq i}^n u_j^{p_i A_{ij}} \hat{d}^i\mathbf{u} \right]^{1/p_i} \\
&= x_i^{(n-1-s)/p_i + \alpha_i - A_{ii}} k_i^{1/p_i}(p_i \mathbf{A_i}),
\end{aligned} \tag{2.140}$$

where $\mathbf{A_i} = (A_{i1}, A_{i2}, \dots, A_{in})$, $i = 1,2,\dots,n$.

What follows is a simple consequence of theorems 2.24 and 2.25, in the described setting with homogeneous kernels. Note that inequalities (2.130) and (2.135) can be interpreted as the interpolating series of inequalities for the quotient between the left-hand side and the right-hand side of inequality (2.127).

Corollary 2.24 *Let $p_i > 1$, $i = 1,2,\dots,n$ be conjugate exponents and let A_{ij}, $i,j = 1,2,\dots,n$ be the real parameters such that $\sum_{i=1}^n A_{ij} = 0$, $j = 1,2,\dots,n$. If $K : \mathbb{R}_+^n \to \mathbb{R}$ is a non-negative measurable homogeneous function of degree $-s$, $s > 0$, and $f_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1,2,\dots,n$ are non-negative measurable functions, then*

$$\frac{\left[\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^n x_i^{\sum_{j=1}^n p_j A_{ji}/n} f_i^{p_i/n}(x_i)d\mathbf{x}\right]^{n/\min_{1\le i\le n}\{p_i\}}}{\left[\prod_{i=1}^n k_i(p_i\mathbf{A_i})\right]^{1/\min_{1\le i\le n}\{p_i\}} \prod_{i=1}^n ||x_i^{(n-1-s)/p_i+\alpha_i} f_i||_{p_i}^{p_i/\min_{1\le i\le n}\{p_i\}}}$$
$$\le \frac{\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^n f_i(x_i)d\mathbf{x}}{\prod_{i=1}^n k_i^{1/p_i}(p_i\mathbf{A_i}) \prod_{i=1}^n ||x_i^{(n-1-s)/p_i+\alpha_i} f_i||_{p_i}} \tag{2.141}$$
$$\le \frac{\left[\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^n x_i^{\sum_{j=1}^n p_j A_{ji}/n} f_i^{p_i/n}(x_i)d\mathbf{x}\right]^{n/\max_{1\le i\le n}\{p_i\}}}{\left[\prod_{i=1}^n k_i(p_i\mathbf{A_i})\right]^{1/\max_{1\le i\le n}\{p_i\}} \prod_{i=1}^n ||x_i^{(n-1-s)/p_i+\alpha_i} f_i||_{p_i}^{p_i/\max_{1\le i\le n}\{p_i\}}},$$

where $\alpha_i = \sum_{j=1}^n A_{ij}$, $i = 1,2,\dots,n$ and $k_i(\cdot)$, $i = 1,2,\dots,n$ is defined by (2.139).

Proof. The proof is a direct consequence of theorems 2.24 and 2.25. Namely, if we substitute functions ϕ_{ij} and ω_i, $i,j = 1,2,\dots,n$, defined respectively by (2.138) and (2.140), in relations (2.130) and (2.135), we get the series of inequalities (2.141) by a straightforward computation. □

Remark 2.36 The left-hand side inequality in (2.141) yields the converse, while the right-hand side inequality provides the refinement of the general Hilbert-type inequality from paper [184]. Moreover, by using $x_i = u_i(t_i)$, where $u_i : (a_i,b_i) \to \mathbb{R}$ are strictly increasing differentiable functions satisfying $u_i(a_i) = 0$, $u_i(b_i) = \infty$, the interpolating series (2.141) also yields a refinement and a converse of the corresponding multidimensional Hilbert-type inequality from paper [208].

In papers [37], [38], [114] and [208] the authors investigated the conditions under which the constant factors involved in appropriate Hilbert-type inequalities were the best possible in the sense that they could not be replaced with the smaller constants.

In the sequel we consider the problem of the best possible constant factors involved in the interpolating series of inequalities (2.141). By the similar reasoning as in the above mentioned papers and within the same problem area, the best possible constant factors can be obtained if they don't contain conjugate parameters p_i in the exponents. For that reason, we assume

$$k_1(p_1\mathbf{A_1}) = k_2(p_2\mathbf{A_2}) = \cdots = k_n(p_n\mathbf{A_n}). \tag{2.142}$$

If we use the change of variables $u_1 = 1/t_2$, $u_3 = t_3/t_2$, $u_4 = t_4/t_2,\dots,u_n = t_n/t_2$, which provides the Jacobian of the transformation

$$\left|\frac{\partial(u_1,u_3,\dots,u_n)}{\partial(t_2,t_3,\dots,t_n)}\right| = t_2^{-n},$$

we have

$$\begin{aligned} k_2(p_2\mathbf{A_2}) &= \int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{t}}^1) t_2^{s-n-p_2(\alpha_2-A_{22})} \prod_{j=3}^{n} t_j^{p_2A_{2j}} \hat{d}^1\mathbf{t} \\ &= k_1(p_1A_{11}, s-n-p_2(\alpha_2-A_{22}), p_2A_{23}, \ldots, p_2A_{2n}). \end{aligned}$$

According to (2.142), we have $p_1A_{12} = s-n-p_2(\alpha_2-A_{22})$, $p_1A_{13} = p_2A_{23}, \ldots, p_1A_{1n} = p_2A_{2n}$. In a similar manner we express $k_i(p_i\mathbf{A_i})$, $i = 3, \ldots, n$, in terms of $k_1(\cdot)$. In such a way we see that (2.142) is fulfilled if

$$p_jA_{ji} = s-n-p_i(\alpha_i - A_{ii}), i, j = 1, 2, \ldots, n, \ i \neq j. \tag{2.143}$$

The above set of conditions also implies that $p_iA_{ik} = p_jA_{jk}$, when $k \neq i, j$. Hence, we use abbreviations $\widetilde{A}_1 = p_nA_{n1}$ and $\widetilde{A}_i = p_1A_{1i}$, $i \neq 1$. Since $\sum_{i=1}^n A_{ij} = 0$, one easily obtains that $p_jA_{jj} = \widetilde{A}_j(1-p_j)$ and $\sum_{i=1}^n \widetilde{A}_i = s-n$ (see also paper [208]).

In order to obtain the best possible constant factors, we establish some more specific conditions about the convergence of the integral $k_1(\mathbf{a})$, $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, defined by (2.139). More precisely, we assume that $k_1(\mathbf{a}) < \infty$ for $a_2, \ldots, a_n > -1$, $\sum_{i=2}^n a_i < s-n+1$, and $n \in \mathbb{N}$.

Hence, in the described setting, the interpolating series of inequalities (2.141) can be rewritten as

$$\begin{aligned} & k_1^{1-n/m}(\widetilde{\mathbf{A}}) \frac{\left[\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^n x_i^{\widetilde{A}_i(1-p_i/n)} f_i^{p_i/n}(x_i) d\mathbf{x}\right]^{n/m}}{\prod_{i=1}^n ||x_i^{-\widetilde{A}_i-1/p_i} f_i||_{p_i}^{p_i/m}} \\ \leq & \frac{\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^n f_i(x_i) d\mathbf{x}}{\prod_{i=1}^n ||x_i^{-\widetilde{A}_i-1/p_i} f_i||_{p_i}} \\ \leq & k_1^{1-n/M}(\widetilde{\mathbf{A}}) \frac{\left[\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^n x_i^{\widetilde{A}_i(1-p_i/n)} f_i^{p_i/n}(x_i) d\mathbf{x}\right]^{n/M}}{\prod_{i=1}^n ||x_i^{-\widetilde{A}_i-1/p_i} f_i||_{p_i}^{p_i/M}}, \end{aligned} \tag{2.144}$$

where $m = \min_{1 \leq i \leq n}\{p_i\}$, $M = \max_{1 \leq i \leq n}\{p_i\}$ and $\widetilde{\mathbf{A}} = (\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_n)$. In the sequel, we show that the constant factors involved in the series of inequalities (2.144) are the best possible under certain assumptions on the homogeneous kernel.

Theorem 2.26 *Let $K : \mathbb{R}_+^n \to \mathbb{R}$ be a non-negative measurable homogeneous function of degree $-s$, $s > 0$, such that for every $i = 2, 3, \ldots, n$*

$$K(1, t_2, \ldots, t_i, \ldots, t_n) \leq CK(1, t_2, \ldots, 0, \ldots, t_n), \ 0 \leq t_i \leq 1, \tag{2.145}$$

where C is a positive constant. Then the constant factors $k_1^{1-n/M}(\widetilde{\mathbf{A}})$ and $k_1^{1-n/m}(\widetilde{\mathbf{A}})$ are the best possible in the series of inequalities (2.144).

Proof. Suppose $k_1^{1-n/M}(\widetilde{\mathbf{A}})$ is not the best possible constant factor in (2.144), that is, suppose there exists a positive constant $\alpha < k_1^{1-n/M}(\widetilde{\mathbf{A}})$ such that the right-hand side inequality in (2.144) holds if we replace $k_1^{1-n/M}(\widetilde{\mathbf{A}})$ with α. In other words,

$$\begin{aligned} &\frac{\int_{\mathbb{R}_+^n} K(\mathbf{x})\prod_{i=1}^n f_i(x_i)d\mathbf{x}}{\prod_{i=1}^n ||x_i^{-\widetilde{A}_i-1/p_i} f_i||_{p_i}} \\ \le\alpha &\frac{\left[\int_{\mathbb{R}_+^n} K(\mathbf{x})\prod_{i=1}^n x_i^{\widetilde{A}_i(1-p_i/n)} f_i^{p_i/n}(x_i)d\mathbf{x}\right]^{n/M}}{\prod_{i=1}^n ||x_i^{-\widetilde{A}_i-1/p_i} f_i||_{p_i}^{p_i/M}} \end{aligned} \tag{2.146}$$

holds for all non-negative measurable functions $f_i : \mathbb{R}_+ \to \mathbb{R}$, provided that all the integrals in the inequality converge. For this purpose, let's substitute the functions

$$\widetilde{f}_i(x_i) = \begin{cases} 0, & 0 < x < 1, \\ x_i^{\widetilde{A}_i-\varepsilon/p_i}, & x \ge 1, \end{cases} \tag{2.147}$$

where $0 < \varepsilon < \min_{1\le i\le n}\{p_i + p_i\widetilde{A}_i\}$, in the previous inequality.

Since $||x_i^{-\widetilde{A}_i-1/p_i}\widetilde{f}_i||_{p_i} = ||x_i^{-(1+\varepsilon)/p_i}||_{p_i} = \varepsilon^{-1/p_i}$, the left-hand side of inequality (2.146) becomes

$$I = \varepsilon\int_{[1,\infty\rangle^n} K(\mathbf{x})\prod_{i=1}^n x_i^{\widetilde{A}_i-\varepsilon/p_i}d\mathbf{x},$$

while the right-hand side becomes

$$I_M = \alpha\left[\varepsilon\int_{[1,\infty\rangle^n} K(\mathbf{x})\prod_{i=1}^n x_i^{\widetilde{A}_i-\varepsilon/n}d\mathbf{x}\right]^{n/M}.$$

Obviously, by using the variable changes $u_i = x_i/x_1$, $i = 2,\dots,n$, and the homogeneity of the kernel K, the left-hand side I can be rewritten as

$$I = \varepsilon\int_1^\infty x_1^{-1-\varepsilon}\left[\int_{[1/x_1,\infty\rangle^{n-1}} K(\hat{\mathbf{u}}^1)\prod_{i=2}^n u_i^{\widetilde{A}_i-\varepsilon/p_i}\hat{d}^1\mathbf{u}\right]dx_1,$$

providing the inequality

$$\begin{aligned} I \ge&\varepsilon\int_1^\infty x_1^{-1-\varepsilon}\left[\int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{u}}^1)\prod_{i=2}^n u_i^{\widetilde{A}_i-\varepsilon/p_i}\hat{d}^1\mathbf{u}\right]dx_1 \\ &-\varepsilon\int_1^\infty x_1^{-1-\varepsilon}\left[\sum_{i=2}^n\int_{\mathbb{D}_i} K(\hat{\mathbf{u}}^1)\prod_{j=2}^n u_j^{\widetilde{A}_j-\varepsilon/p_j}\hat{d}^1\mathbf{u}\right]dx_1 \\ =&k_1\left(\widetilde{\mathbf{A}}-\varepsilon\mathbf{1}/\mathbf{p}\right)-\varepsilon\int_1^\infty x_1^{-1-\varepsilon}\left[\sum_{i=2}^n\int_{\mathbb{D}_i} K(\hat{\mathbf{u}}^1)\prod_{j=2}^n u_j^{\widetilde{A}_j-\varepsilon/p_j}\hat{d}^1\mathbf{u}\right]dx_1, \end{aligned} \tag{2.148}$$

where $\mathbb{D}_i = \{(u_2, u_3, \dots, u_n); 0 < u_i \le 1/x_1, u_j > 0, j \ne i\}$ and $\mathbf{1/p} = (1/p_1, \dots, 1/p_n)$. Without loss of generality, it is enough to find the upper bound for the integral $\int_{\mathbb{D}_2} K(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{\widetilde{A}_j - \varepsilon/p_j} \hat{d}^1\mathbf{u}$. Regarding (2.145), we have

$$\begin{aligned}
&\int_{\mathbb{D}_2} K(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{\widetilde{A}_j - \varepsilon/p_j} \hat{d}^1\mathbf{u} \\
\le & C \left[\int_{\mathbb{R}_+^{n-2}} K(1, 0, u_3, \dots, u_n) \prod_{j=3}^n u_j^{\widetilde{A}_j - \varepsilon/p_j} du_3 \dots du_n \right] \int_0^{1/x_1} u_2^{\widetilde{A}_2 - \varepsilon/p_2} du_2 \\
= & C(1 - \varepsilon/p_2 + \widetilde{A}_2)^{-1} x_1^{\varepsilon/p_2 - \widetilde{A}_2 - 1} k_1(\widetilde{A}_1 - \varepsilon/p_1, \widetilde{A}_3 - \varepsilon/p_3, \dots, \widetilde{A}_n - \varepsilon/p_n),
\end{aligned}$$

where $k_1(\widetilde{A}_1 - \varepsilon/p_1, \widetilde{A}_3 - \varepsilon/p_3, \dots, \widetilde{A}_n - \varepsilon/p_n)$ is well defined since obviously $\sum_{i=3}^n \widetilde{A}_i < s - n + 2$. Hence, we have

$$\int_{\mathbb{D}_i} K(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{\widetilde{A}_j - \varepsilon/p_j} \hat{d}^1\mathbf{u} = x_1^{\varepsilon/p_i - \widetilde{A}_i - 1} O(1), \quad i = 2, 3, \dots, n,$$

and consequently

$$\int_1^\infty x_1^{-1-\varepsilon} \left[\sum_{i=2}^n \int_{\mathbb{D}_i} K(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{\widetilde{A}_j - \varepsilon/p_j} \hat{d}^1\mathbf{u} \right] dx_1 = O(1).$$

Thus, by using (2.148), we have

$$I \ge k_1\left(\widetilde{\mathbf{A}} - \varepsilon \mathbf{1/p}\right) - o(1), \quad \text{when} \quad \varepsilon \to 0^+. \tag{2.149}$$

On the other hand, by using the fact that $\sum_{i=1}^n \widetilde{A}_i = s - n$, the expression I_M can be bounded from above in the following way:

$$\begin{aligned}
I_M &= \alpha \left[\varepsilon \int_1^\infty x_1^{-1-\varepsilon} \left[\int_{[1/x_1, \infty\rangle^{n-1}} K(\hat{\mathbf{u}}^1) \prod_{i=2}^n u_i^{\widetilde{A}_i - \varepsilon/n} \hat{d}^1\mathbf{u} \right] dx_1 \right]^{n/M} \\
&\le \alpha \left[\varepsilon \int_1^\infty x_1^{-1-\varepsilon} \left[\int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{u}}^1) \prod_{i=2}^n u_i^{\widetilde{A}_i - \varepsilon/n} \hat{d}^1\mathbf{u} \right] dx_1 \right]^{n/M} \\
&= \alpha k_1^{n/M}\left(\widetilde{\mathbf{A}} - \varepsilon/n\mathbf{1}\right), \quad \text{when} \quad \varepsilon \to 0^+.
\end{aligned} \tag{2.150}$$

Here, $\mathbf{1}$ denotes the constant n-tuple $(1, 1, \dots, 1)$. Finally, relations (2.149) and (2.150) yield inequality

$$k_1\left(\widetilde{\mathbf{A}} - \varepsilon \mathbf{1/p}\right) - o(1) \le \alpha k_1^{n/M}\left(\widetilde{\mathbf{A}} - \varepsilon/n\mathbf{1}\right), \quad \text{when} \quad \varepsilon \to 0^+,$$

i.e. $k_1^{1-n/M}(\widetilde{\mathbf{A}}) \le \alpha$, which is obviously opposite to our assumption.

It remains to prove that $k_1^{1-n/m}(\widetilde{\mathbf{A}})$ is the best possible constant factor in the left-hand side inequality in (2.144). Suppose, on the contrary, that there exists a constant $\beta > k_1^{1-n/m}(\widetilde{\mathbf{A}})$, such that the inequality

$$\begin{aligned}
&\frac{\int_{\mathbb{R}_+^n} K(\mathbf{x})\prod_{i=1}^n f_i(x_i)d\mathbf{x}}{\prod_{i=1}^n ||x_i^{-\widetilde{A}_i-1/p_i} f_i||_{p_i}} \\
\geq \beta &\frac{\left[\int_{\mathbb{R}_+^n} K(\mathbf{x})\prod_{i=1}^n x_i^{\widetilde{A}_i(1-p_i/n)} f_i^{p_i/n}(x_i)d\mathbf{x}\right]^{n/m}}{\prod_{i=1}^n ||x_i^{-\widetilde{A}_i-1/p_i} f_i||_{p_i}^{p_i/m}}
\end{aligned} \tag{2.151}$$

holds for all non-negative measurable functions $f_i : \mathbb{R}_+ \to \mathbb{R}$, provided that all the integrals in the inequality converge. For the above choice of functions $\widetilde{f}_i$ defined by (2.147), the left-hand side of inequality (2.151) becomes I as before, while the right-hand side, denoted here by I_m, can be rewritten as

$$I_m = \beta\left[\varepsilon\int_1^\infty x_1^{-1-\varepsilon}\left[\int_{[1/x_1,\infty\rangle^{n-1}} K(\hat{\mathbf{u}}^1)\prod_{i=2}^n u_i^{\widetilde{A}_i-\varepsilon/n}\hat{d}^1\mathbf{u}\right]dx_1\right]^{n/m}.$$

Now, similarly as in the first part of the proof, we get the estimates

$$\begin{aligned}
I &\leq k_1\left(\widetilde{\mathbf{A}}-\varepsilon\mathbf{1}/\mathbf{p}\right), \\
I_m &\geq \beta k_1^{\frac{n}{m}}\left(\widetilde{\mathbf{A}}-\varepsilon/n\mathbf{1}\right)-o(1),
\end{aligned} \tag{2.152}$$

i.e. $k_1(\widetilde{\mathbf{A}}-\varepsilon\mathbf{1}/\mathbf{p}) \geq \beta k_1^{\frac{n}{m}}(\widetilde{\mathbf{A}}-\varepsilon/n\mathbf{1})-o(1)$, when $\varepsilon \to 0^+$. Finally, by letting $\varepsilon \to 0^+$ we get $k_1^{1-n/m}(\widetilde{\mathbf{A}}) \geq \beta$, which is a contradiction. The proof is now completed. □

If we consider these results in some particular settings, we obtain the refinements and the converses of some results previously known from the literature.

Example 2.2 Let

$$A_{ii} = \frac{(n-s)(p_i-1)}{p_i^2} \text{ and } A_{ij} = \frac{s-n}{p_ip_j},\ i,j=1,2,\ldots,n,\ i\neq j. \tag{2.153}$$

These parameters are symmetric and

$$\sum_{i=1}^n A_{ij} = \sum_{j=1}^n A_{ij} = \frac{(n-s)(p_i-1)}{p_i^2} + \sum_{j=1,j\neq i}^n \frac{s-n}{p_ip_j} = \frac{n-s}{p_i}\left(1-\sum_{j=1}^n\frac{1}{p_j}\right) = 0.$$

Moreover, the above defined parameters satisfy conditions (2.143), so the resulting relations will include the best possible constant factors. More precisely, in the described setting, the interpolating series of inequalities (2.144) reads

$$\begin{aligned}
&[k_1((s-n)\mathbf{1}/\mathbf{p})]^{1-n/m}\,\frac{\left[\int_{\mathbb{R}^n_+}K(\mathbf{x})\prod_{i=1}^n x_i^{(s-n)(n-p_i)/(np_i)}f_i^{p_i/n}(x_i)d\mathbf{x}\right]^{n/m}}{\prod_{i=1}^n\|x_i^{(n-1-s)/p_i}f_i\|_{p_i}^{p_i/m}}\\
&\leq\frac{\int_{\mathbb{R}^n_+}K(\mathbf{x})\prod_{i=1}^n f_i(x_i)d\mathbf{x}}{\prod_{i=1}^n\|x_i^{(n-1-s)/p_i}f_i\|_{p_i}} \qquad (2.154)\\
&\leq[k_1((s-n)\mathbf{1}/\mathbf{p})]^{1-n/M}\,\frac{\left[\int_{\mathbb{R}^n_+}K(\mathbf{x})\prod_{i=1}^n x_i^{(s-n)(n-p_i)/(np_i)}f_i^{p_i/n}(x_i)d\mathbf{x}\right]^{n/M}}{\prod_{i=1}^n\|x_i^{(n-1-s)/p_i}f_i\|_{p_i}^{p_i/M}},
\end{aligned}$$

where $m=\min_{1\leq i\leq n}\{p_i\}$, $M=\max_{1\leq i\leq n}\{p_i\}$. However, under assumption (2.145), the constant factors $[k_1((s-n)\mathbf{1}/\mathbf{p})]^{1-n/m}$ and $[k_1((s-n)\mathbf{1}/\mathbf{p})]^{1-n/M}$ are the best possible in the interpolating series (2.154).

A typical example of a homogeneous kernel with the negative degree of homogeneity is the function $K:\mathbb{R}^n_+\to\mathbb{R}$, defined by

$$K(x)=\frac{1}{\left(\sum_{i=1}^n x_i\right)^s},\quad s>0. \qquad (2.155)$$

Clearly, K is a homogeneous function of degree $-s$ and the constant (2.139) can be expressed in terms of the usual Gamma function Γ. For that reason, we use the well-known formula

$$\int_{\mathbb{R}^n_+}\frac{\prod_{i=1}^{n-1}u_i^{a_i-1}}{\left(1+\sum_{i=1}^{n-1}u_i\right)^{\sum_{i=1}^n a_i}}\hat{d}^n\mathbf{u}=\frac{\prod_{i=1}^n\Gamma(a_i)}{\Gamma\left(\sum_{i=1}^n a_i\right)}, \qquad (2.156)$$

which holds for $a_i>0$, $i=1,2,\dots n$ (see, e.g. [38]). In such a way, the constant factors $k_i(p_i\mathbf{A_i})$, $i=1,2,\dots,n$, involved in the series of inequalities (2.141) become

$$k_i(p_i\mathbf{A_i})=\frac{\Gamma(s-n+1-p_i\alpha_i+p_iA_{ii})}{\Gamma(s)}\prod_{j=1,j\neq i}^n\Gamma(1+p_iA_{ij}),\quad i=1,2,\dots,n,$$

provided that $A_{ij}>-1/p_i$, $i\neq j$ and $A_{ii}-\alpha_i>(n-s-1)/p_i$.

It is easy to see that the kernel (2.155) satisfies the relation (2.145). Hence, according to Theorem 2.26, the interpolating series of inequalities (2.141), equipped with the kernel (2.155) and the parameters A_{ij} satisfying conditions (2.143), contains the best possible constant factors.

Remark 2.37 If $K:\mathbb{R}^n_+\to\mathbb{R}$ is defined by (2.155), then, regarding (2.156), we easily compute the constant factor $k_1((s-n)\mathbf{1}/\mathbf{p})$ included in the interpolating series of inequalities in (2.154). Namely, we have

$$k_1((s-n)\mathbf{1}/\mathbf{p})=\frac{1}{\Gamma(s)}\prod_{i=1}^n\Gamma\left(\frac{p_i+s-n}{p_i}\right),$$

provided that $s > n-m$. This constant factor appears in paper [38] as the best possible in the Hilbert-type inequality determined with the middle quotient in the interpolating series (2.154). Hence, relations as in (2.154) represent the refinement and the converse of the corresponding Hilbert-type inequality from paper [38].

Example 2.3 Suppose A_i, $i = 1,2,\dots,n$ are the real parameters satisfying relations $(n-s-1)/p_{i-1} < A_i < 1/p_{i-1}$, provided that $s > n-2$. Of course, we use convention $p_0 = p_n$. Now, we define parameters A_{ij}, $i, j = 1,2,\dots,n$, by

$$A_{ij} = \begin{cases} A_i, & j = i, \\ -A_{i+1}, & j = i+1, \\ 0 & \text{otherwise}, \end{cases} \tag{2.157}$$

where the indices are taken modulo n from the set $\{1,2,\dots,n\}$. Now, if the kernel $K : \mathbb{R}^n_+ \to \mathbb{R}$ is defined by (2.155), then the series of inequalities (2.141) becomes

$$\begin{aligned} &\frac{\left\{\int_{\mathbb{R}^n_+} (\sum_{i=1}^n x_i)^{-s} \prod_{i=1}^n x_i^{(p_i-p_{i-1})A_i/n} f_i^{p_i/n}(x_i)d\mathbf{x}\right\}^{n/\min_{1\le i\le n}\{p_i\}}}{\mathscr{R}_m \prod_{i=1}^n ||x_i^{(n-1-s)/p_i+A_i-A_{i+1}} f_i||_{p_i}^{p_i/\min_{1\le i\le n}\{p_i\}}} \\ \le &\frac{\int_{\mathbb{R}^n_+} (\sum_{i=1}^n x_i)^{-s} \prod_{i=1}^n f_i(x_i)d\mathbf{x}}{\mathscr{R} \prod_{i=1}^n ||x_i^{(n-1-s)/p_i+A_i-A_{i+1}} f_i||_{p_i}} \\ \le &\frac{\left\{\int_{\mathbb{R}^n_+} (\sum_{i=1}^n x_i)^{-s} \prod_{i=1}^n x_i^{(p_i-p_{i-1})A_i/n} f_i^{p_i/n}(x_i)d\mathbf{x}\right\}^{n/\max_{1\le i\le n}\{p_i\}}}{\mathscr{R}_M \prod_{i=1}^n ||x_i^{(n-1-s)/p_i+A_i-A_{i+1}} f_i||_{p_i}^{p_i/\max_{1\le i\le n}\{p_i\}}}, \end{aligned} \tag{2.158}$$

where the constant factors $\mathscr{R}_m$, $\mathscr{R}$, and $\mathscr{R}_M$ are given by

$$\begin{aligned} \mathscr{R}_m &= \left[\frac{\prod_{i=1}^n \Gamma(s-n+1+p_iA_{i+1})\Gamma(1-p_iA_{i+1})}{\Gamma(s)}\right]^{1/\min_{1\le i\le n}\{p_i\}}, \\ \mathscr{R} &= \frac{\prod_{i=1}^n \Gamma(s-n+1+p_iA_{i+1})^{1/p_i}\Gamma(1-p_iA_{i+1})^{1/p_i}}{\Gamma(s)}, \\ \mathscr{R}_M &= \left[\frac{\prod_{i=1}^n \Gamma(s-n+1+p_iA_{i+1})\Gamma(1-p_iA_{i+1})}{\Gamma(s)}\right]^{1/\max_{1\le i\le n}\{p_i\}}. \end{aligned}$$

Remark 2.38 The interpolating series of inequalities (2.158) provides the refinement and the converse of the multidimensional Hilbert-type inequality from [43] (see also [118]). Moreover, the parameters A_{ij} defined by (2.157), can satisfy the set of conditions as in (2.143) only for $n = 2$. In this case, the set of conditions (2.143) reduces to the relation $p_1A_2 + p_2A_1 = 2-s$, providing the best possible constant factors

$$\left[\frac{\Gamma(1-p_1A_2)\Gamma(1-p_2A_1)}{\Gamma(s)}\right]^{1-2/\min\{p_1,p_2\}}$$

and

$$\left[\frac{\Gamma(1-p_1A_2)\Gamma(1-p_2A_1)}{\Gamma(s)}\right]^{1-2/\max\{p_1,p_2\}}$$

in (2.158) for $n = 2$.

Refinements and converses: non-conjugate exponents

In paper [42], a unified treatment of multidimensional Hilbert-type inequality in the setting with non-conjugate exponents was provided. Thus we firstly recall the definition of non-conjugate parameters.

Let p_i be real parameters satisfying

$$\sum_{i=1}^{n} \frac{1}{p_i} > 1, \quad p_i > 1, \ i = 1,2,\ldots,n. \tag{2.159}$$

The parameters p_i' are defined as associated conjugates, that is

$$\frac{1}{p_i} + \frac{1}{p_i'} = 1, \quad i = 1,2,\ldots,n. \tag{2.160}$$

Since $p_i > 1$, it follows that $p_i' > 1$, $i = 1,2,\ldots,n$. In addition, we define

$$\lambda = \frac{1}{n-1} \sum_{i=1}^{n} \frac{1}{p_i'}. \tag{2.161}$$

Clearly, relations (2.159) and (2.160) imply that $0 < \lambda < 1$. Finally, we introduce the parameters q_i defined by

$$\frac{1}{q_i} = \lambda - \frac{1}{p_i'}, \quad i = 1,2,\ldots,n, \tag{2.162}$$

assuming $q_i > 0$, $i = 1,2,\ldots,n$. The conditions (2.159)–(2.162) establish the n-tuple of non-conjugate exponents and were given by Bonsall in [41], more than half of a century ago. The above conditions also imply relations $\lambda = \sum_{i=1}^{n} 1/q_i$ and $1/q_i + 1 - \lambda = 1/p_i$, $i = 1,2,\ldots,n$. Of course, if $\lambda = 1$, then $\sum_{i=1}^{n} 1/p_i = 1$, which represents the setting with conjugate parameters.

General multidimensional Hilbert-type inequality (2.127) with accompanied abbreviated notation (2.129) is provided in the above described setting with

$$\omega_i(x_i) = \left[\int_{\hat{\mathbf{\Omega}}^i} K(\mathbf{x}) \prod_{j=1, j\neq i}^{n} \phi_{ij}^{q_i}(x_j) d\hat{\mu}^i(\mathbf{x}) \right]^{1/q_i}. \tag{2.163}$$

In such a way, results from papers [38], [43], [44] and [118] are extended to the case of non-conjugate exponents. For more details, the reader is referred to [42].

Similarly as it was done in the case of conjugate exponents, the ratio and also the difference between the left-hand side and the right-hand side of the inequality (2.127) are considered here, in its non-conjugate exponents setting. In such a way, two pairs of refinements and converses of this inequality are obtained, all by means of the appropriate improvements of related Hölder's inequality. However, Hölder's inequality includes conjugate parameters. We use Bonsall's idea from paper [41] about reduction of non-conjugate exponents to the setting which includes conjugate exponents. Regarding the definitions (2.159)–(2.162) of non-conjugate exponents, the essence of the above mentioned idea is the apparently trivial observation that $\sum_{i=1}^{n} 1/q_i + (1-\lambda) = 1$ and the application of Hölder's inequality to conjugate exponents q_i, $i = 1,2,\ldots,n$, and $1/(1-\lambda)$. In such a way, by using refinements and converses of Hölder's inequality from Section 2.3.2, we obtain refinements and converses of multidimensional Hilbert-type inequality (2.127) as well.

Hence the following result provides the refinement and the converse of Hilbert-type inequality (2.127) in the non-conjugate exponents setting, in the ratio form.

Theorem 2.27 *Suppose that p_i, p_i', q_i, $i = 1,2,\ldots,n$ and λ are real parameters satisfying conditions* (2.159)–(2.162). *Let $(\Omega_i, \Sigma_i, \mu_i)$ be σ-finite measure spaces, and let $K : \mathbf{\Omega} \to \mathbb{R}$, $\phi_{ij} : \Omega_j \to \mathbb{R}$, $f_i : \Omega_i \to \mathbb{R}$, $i, j = 1,2,\ldots,n$ be non-negative measurable functions. If $\prod_{i,j=1}^n \phi_{ij}(x_j) = 1$, then*

$$
\begin{aligned}
&\frac{\left\{\int_{\mathbf{\Omega}} \left[K^n(\mathbf{x}) \prod_{i=1}^n (\phi_{ii}\omega_i f_i)^{2p_i}(x_i)\,\omega_i^{-q_i}(x_i) \prod_{i,j=1, j\neq i}^n \phi_{ij}^{q_i}(x_j)\right]^{1/(n+1)} d\mu(\mathbf{x})\right\}^{(n+1)M}}{\prod_{i=1}^n ||\phi_{ii}\omega_i f_i||_{p_i}^{2Mp_i}} \\
&\leq \frac{\int_{\mathbf{\Omega}} K^\lambda(\mathbf{x}) \prod_{i=1}^n f_i(x_i)\, d\mu(\mathbf{x})}{\prod_{i=1}^n ||\phi_{ii}\omega_i f_i||_{p_i}} \\
&\leq \frac{\left\{\int_{\mathbf{\Omega}} \left[K^n(\mathbf{x}) \prod_{i=1}^n (\phi_{ii}\omega_i f_i)^{2p_i}(x_i)\,\omega_i^{-q_i}(x_i) \prod_{i,j=1, j\neq i}^n \phi_{ij}^{q_i}(x_j)\right]^{1/(n+1)} d\mu(\mathbf{x})\right\}^{(n+1)m}}{\prod_{i=1}^n ||\phi_{ii}\omega_i f_i||_{p_i}^{2mp_i}},
\end{aligned}
\tag{2.164}
$$

where $\omega_i : \Omega_i \to \mathbb{R}$ are defined by (2.163) *and* $m = \min\{1/q_1, 1/q_2, \ldots, 1/q_n, 1-\lambda\}$, $M = \max\{1/q_1, 1/q_2, \ldots, 1/q_n, 1-\lambda\}$.

Proof. The left-hand side of the Hilbert-type inequality (2.127) can be rewritten in the form

$$
\int_{\mathbf{\Omega}} K^\lambda(\mathbf{x}) \prod_{i=1}^n f_i(x_i)\, d\mu(\mathbf{x}) = \int_{\mathbf{\Omega}} \prod_{i=1}^n F_i^{1/q_i}(\mathbf{x}) \cdot F_{n+1}^{1-\lambda}(\mathbf{x})\, d\mu(\mathbf{x}),
$$

where

$$
F_i(\mathbf{x}) = K(\mathbf{x}) \frac{(\phi_{ii}\omega_i f_i)^{p_i}(x_i)}{\omega_i^{q_i}(x_i)} \prod_{j=1, j\neq i}^n \phi_{ij}^{q_i}(x_j), \quad i = 1,2,\ldots,n, \tag{2.165}
$$

and

$$
F_{n+1}(\mathbf{x}) = \prod_{i=1}^n (\phi_{ii}\omega_i f_i)^{p_i}(x_i). \tag{2.166}
$$

Now we apply the interpolating series of inequalities (2.117) with $n+1$ instead of n and the parameters $\alpha_i = 1/q_i$, $i = 1,2,\ldots,n$, and $\alpha_{n+1} = 1-\lambda$. Clearly, due to definitions of non-conjugate exponents, we have $\sum_{i=1}^{n+1} \alpha_i = 1$. Moreover, by using Fubini's theorem we have

$$||F_i^{1/q_i}||_{q_i} = \left[\int_{\mathbf{\Omega}} K(\mathbf{x}) \frac{(\phi_{ii}\omega_i f_i)^{p_i}(x_i)}{\omega_i^{q_i}(x_i)} \prod_{j=1, j\neq i}^{n} \phi_{ij}^{q_i}(x_j) d\mu(\mathbf{x})\right]^{1/q_i}$$
$$= \left[\int_{\Omega_i} \frac{(\phi_{ii}\omega_i f_i)^{p_i}(x_i)}{\omega_i^{q_i}(x_i)} \left(\int_{\hat{\mathbf{\Omega}}^i} K(\mathbf{x}) \prod_{j=1, j\neq i}^{n} \phi_{ij}^{q_i}(x_j) d\hat{\mu}^i(\mathbf{x})\right) d\mu_i(x_i)\right]^{1/q_i}$$
$$= \left[\int_{\Omega_i} (\phi_{ii}\omega_i f_i)^{p_i}(x_i) d\mu_i(x_i)\right]^{1/q_i} = ||\phi_{ii}\omega_i f_i||_{p_i}^{p_i/q_i}, \quad i = 1, 2, \ldots, n,$$

and

$$||F_{n+1}^{1-\lambda}||_{1/1-\lambda} = \left[\int_{\mathbf{\Omega}} \prod_{i=1}^{n} (\phi_{ii}\omega_i f_i)^{p_i}(x_i) d\mu(\mathbf{x})\right]^{1-\lambda}$$
$$= \prod_{i=1}^{n} \left[\int_{\Omega_i} (\phi_{ii}\omega_i f_i)^{p_i}(x_i) d\mu_i(x_i)\right]^{1-\lambda} = \prod_{i=1}^{n} ||\phi_{ii}\omega_i f_i||_{p_i}^{p_i(1-\lambda)},$$

which yields relation

$$\prod_{i=1}^{n} ||F_i^{1/q_i}||_{q_i} \cdot ||F_{n+1}^{1-\lambda}||_{1/1-\lambda} = \prod_{i=1}^{n} ||\phi_{ii}\omega_i f_i||_{p_i}^{p_i(1/q_i+1-\lambda)} = \prod_{i=1}^{n} ||\phi_{ii}\omega_i f_i||_{p_i}. \tag{2.167}$$

Similarly, we have

$$||F_i^{1/n+1}||_{n+1} = \left[\int_{\Omega_i} (\phi_{ii}\omega_i f_i)^{p_i}(x_i) d\mu_i(x_i)\right]^{1/n+1} = ||\phi_{ii}\omega_i f_i||_{p_i}^{p_i/n+1}, \quad i = 1, 2, \ldots, n,$$

and

$$||F_{n+1}^{1/n+1}||_{n+1} = \prod_{i=1}^{n} \left[\int_{\Omega_i} (\phi_{ii}\omega_i f_i)^{p_i}(x_i) d\mu_i(x_i)\right]^{1/n+1} = \prod_{i=1}^{n} ||\phi_{ii}\omega_i f_i||_{p_i}^{p_i/n+1},$$

that is

$$\prod_{i=1}^{n+1} ||F_i^{1/n+1}||_{n+1} = \prod_{i=1}^{n} ||\phi_{ii}\omega_i f_i||_{p_i}^{2p_i/n+1}. \tag{2.168}$$

It remains to compute the product of functions F_i, $i = 1, 2, \ldots, n+1$. We have

$$\prod_{i=1}^{n+1} F_i(\mathbf{x}) = K^n(\mathbf{x}) \prod_{i=1}^{n} \frac{(\phi_{ii}\omega_i f_i)^{2p_i}(x_i)}{\omega_i^{q_i}(x_i)} \prod_{i,j=1, j\neq i}^{n} \phi_{ij}^{q_i}(x_j). \tag{2.169}$$

Finally, if we substitute the expressions (2.167), (2.168) and (2.169) in the series of inequalities (2.117) where n is replaced with $n+1$, we get (2.164) and the proof is complete. $\square$

Remark 2.39 According to the interpolating series of inequalities (2.117), we conclude that the left inequality in (2.164) yields the converse, while the right one yields the refinement of Hilbert-type inequality (2.127) with non-conjugate exponents, in the ratio form.

On the other hand, regarding the series of inequalities (2.121), we also obtain the refinement and the converse of Hilbert-type inequality (2.127) with non-conjugate exponents, in the difference form, as is presented in the following theorem.

Theorem 2.28 *Let p_i, p_i', q_i, $i = 1,2,\ldots,n$, and λ be real parameters satisfying conditions* (2.159)–(2.162)*, and let $(\Omega_i, \Sigma_i, \mu_i)$, $i = 1,2,\ldots,n$ be σ-finite measure spaces. If $K : \mathbf{\Omega} \to \mathbb{R}$, $\phi_{ij} : \Omega_j \to \mathbb{R}$, $f_i : \Omega_i \to \mathbb{R}$, $i, j = 1,2,\ldots,n$ are non-negative measurable functions with $\prod_{i,j=1}^n \phi_{ij}(x_j) = 1$, then*

$$
\begin{aligned}
&(n+1)m\prod_{i=1}^{n}||\phi_{ii}\omega_i f_i||_{p_i} \\
&\quad\times\left[1-\frac{\int_{\mathbf{\Omega}}\left[K^n(\mathbf{x})\prod_{i=1}^n(\phi_{ii}\omega_i f_i)^{2p_i}(x_i)\omega_i^{-q_i}(x_i)\prod_{i,j=1,j\neq i}^n\phi_{ij}^{q_i}(x_j)\right]^{1/(n+1)}d\mu(\mathbf{x})}{\prod_{i=1}^n||\phi_{ii}\omega_i f_i||_{p_i}^{2p_i/(n+1)}}\right] \\
&\leq \prod_{i=1}^{n}||\phi_{ii}\omega_i f_i||_{p_i} - \int_{\mathbf{\Omega}}K^{\lambda}(\mathbf{x})\prod_{i=1}^{n}f_i(x_i)d\mu(\mathbf{x}) \\
&\leq (n+1)M\prod_{i=1}^{n}||\phi_{ii}\omega_i f_i||_{p_i} \\
&\quad\times\left[1-\frac{\int_{\mathbf{\Omega}}\left[K^n(\mathbf{x})\prod_{i=1}^n(\phi_{ii}\omega_i f_i)^{2p_i}(x_i)\omega_i^{-q_i}(x_i)\prod_{i,j=1,j\neq i}^n\phi_{ij}^{q_i}(x_j)\right]^{1/(n+1)}d\mu(\mathbf{x})}{\prod_{i=1}^n||\phi_{ii}\omega_i f_i||_{p_i}^{2p_i/(n+1)}}\right],
\end{aligned}
\tag{2.170}
$$

where $\omega_i : \Omega_i \to \mathbb{R}$ are defined by (2.163) *and $m = \min\{1/q_1, 1/q_2, \ldots, 1/q_n, 1-\lambda\}$, $M = \max\{1/q_1, 1/q_2, \ldots, 1/q_n, 1-\lambda\}$.*

Proof. The proof is very similar to the proof of Theorem 2.27. Namely, we use the same decomposition of the left-hand side of multidimensional Hilbert-type inequality (2.127), involving functions F_i, $i = 1,2,\ldots,n+1$, defined by (2.165) and (2.166). Now, the result follows after substituting the expressions (2.167), (2.168) and (2.169) in the interpolating series of inequalities (2.121) with $n+1$ instead of n, and the parameters $\alpha_i = 1/q_i$, $i = 1,2,\ldots,n$, and $\alpha_{n+1} = 1-\lambda$. □

Remark 2.40 Considering the interpolating series of inequalities (2.121), we conclude that the left inequality in (2.170) yields the refinement, while the right one provides the converse of Hilbert-type inequality (2.127) with non-conjugate exponents, in the difference form.

Applications to homogeneous functions

Now the general results are applied to homogenous functions with negative degree of homogeneity. Hence the same notation is still valid, except for the weight functions (2.140) which in the non-conjugate setting and regarding the definition (2.139) assume the following form:

$$\begin{aligned}
\omega_i(x_i) &= \left[\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x}) \prod_{j=1, j\neq i}^{n} x_j^{q_i A_{ij}} \hat{d}^i\mathbf{x}\right]^{1/q_i} \\
&= \left[x_i^{n-1-s+\sum_{j=1,j\neq i}^n q_i A_{ij}} \int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{u}}^i) \prod_{j=1, j\neq i}^{n} u_j^{q_i A_{ij}} \hat{d}^i\mathbf{u}\right]^{1/q_i} \\
&= x_i^{(n-1-s)/q_i+\alpha_i-A_{ii}} k_i^{1/q_i}(q_i\mathbf{A_i}).
\end{aligned} \tag{2.171}$$

What follows are the consequences of theorems 2.27 and 2.28, in the described setting with homogeneous kernels. The first one is the interpolating series of inequalities in the ratio form.

Corollary 2.25 *Let p_i, p_i', q_i, $i = 1,2,\ldots,n$ and λ be as in* (2.159)–(2.162)*, and let A_{ij}, $i,j = 1,2,\ldots,n$ be the parameters such that $\sum_{i=1}^n A_{ij} = 0$. If $K : \mathbb{R}_+^n \to \mathbb{R}$ is a non-negative measurable homogeneous function of degree $-s$, $s > 0$, and $f_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1,2,\ldots,n$ are non-negative measurable functions, then*

$$\begin{aligned}
&\frac{\left\{\int_{\mathbb{R}_+^n}\left[K^n(\mathbf{x})\prod_{i=1}^n x_i^{(2p_i/q_i-1)(n-1-s)+(2p_i-q_i)\alpha_i+\sum_{j=1}^n q_j A_{ji}} f_i^{2p_i}(x_i)\right]^{1/(n+1)} d\mathbf{x}\right\}^{(n+1)M}}{\left[\prod_{i=1}^n k_i(q_i\mathbf{A_i})\right]^M \prod_{i=1}^n ||x_i^{(n-1-s)/q_i+\alpha_i} f_i||_{p_i}^{2Mp_i}} \\
&\le \frac{\int_{\mathbb{R}_+^n} K^\lambda(\mathbf{x})\prod_{i=1}^n f_i(x_i)d\mathbf{x}}{\prod_{i=1}^n k_i^{1/q_i}(q_i\mathbf{A_i})\prod_{i=1}^n ||x_i^{(n-1-s)/q_i+\alpha_i} f_i||_{p_i}} \\
&\le \frac{\left\{\int_{\mathbb{R}_+^n}\left[K^n(\mathbf{x})\prod_{i=1}^n x_i^{(2p_i/q_i-1)(n-1-s)+(2p_i-q_i)\alpha_i+\sum_{j=1}^n q_j A_{ji}} f_i^{2p_i}(x_i)\right]^{1/(n+1)} d\mathbf{x}\right\}^{(n+1)m}}{\left[\prod_{i=1}^n k_i(q_i\mathbf{A_i})\right]^m \prod_{i=1}^n ||x_i^{(n-1-s)/q_i+\alpha_i} f_i||_{p_i}^{2mp_i}},
\end{aligned} \tag{2.172}$$

where $\alpha_i = \sum_{j=1}^n A_{ij}$, $i = 1,2,\ldots,n$, $m = \min\{1/q_1, 1/q_2, \ldots, 1/q_n, 1-\lambda\}$, $M = \max\{1/q_1, 1/q_2, \ldots, 1/q_n, 1-\lambda\}$, and $k_i(\cdot)$, $i = 1,2,\ldots,n$ are defined by (2.139).

Proof. The proof is a direct consequence of Theorem 2.27. Namely, if we substitute the functions ϕ_{ij} and ω_i, $i,j = 1,2,\ldots,n$, defined respectively by (2.138) and (2.171), in the relation (2.164), we get (2.172) after straightforward computation. □

The following result yields the interpolating series of inequalities in the difference form.

Corollary 2.26 *Suppose p_i, p_i', q_i, $i = 1,2,\dots,n$ and λ are as in* (2.159)–(2.162), *and A_{ij}, $i,j = 1,2,\dots,n$ are real parameters satisfying $\sum_{i=1}^n A_{ij} = 0$. If $K : \mathbb{R}_+^n \to \mathbb{R}$ is a non-negative measurable homogeneous function of degree $-s$, $s > 0$, and $f_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1,2,\dots,n$ are non-negative measurable functions, then*

$$
\begin{aligned}
&(n+1)m\prod_{i=1}^n k_i^{1/q_i}(q_i\mathbf{A_i})\prod_{i=1}^n ||x_i^{(n-1-s)/q_i+\alpha_i} f_i||_{p_i} \\
&\times\left[1-\frac{\int_{\mathbb{R}_+^n}\left[K^n(\mathbf{x})\prod_{i=1}^n x_i^{(2p_i/q_i-1)(n-1-s)+(2p_i-q_i)\alpha_i+\sum_{j=1}^n q_jA_{ji}} f_i^{2p_i}(x_i)\right]^{1/(n+1)}d\mathbf{x}}{\left[\prod_{i=1}^n k_i(q_i\mathbf{A_i})\right]^{1/(n+1)}\prod_{i=1}^n ||x_i^{(n-1-s)/q_i+\alpha_i} f_i||_{p_i}^{2p_i/(n+1)}}\right] \\
&\le \prod_{i=1}^n k_i^{1/q_i}(q_i\mathbf{A_i})\prod_{i=1}^n ||x_i^{(n-1-s)/q_i+\alpha_i} f_i||_{p_i} - \int_{\mathbb{R}_+^n} K^\lambda(\mathbf{x})\prod_{i=1}^n f_i(x_i)d\mathbf{x} \\
&\le (n+1)M\prod_{i=1}^n k_i^{1/q_i}(q_i\mathbf{A_i})\prod_{i=1}^n ||x_i^{(n-1-s)/q_i+\alpha_i} f_i||_{p_i} \\
&\times\left[1-\frac{\int_{\mathbb{R}_+^n}\left[K^n(\mathbf{x})\prod_{i=1}^n x_i^{(2p_i/q_i-1)(n-1-s)+(2p_i-q_i)\alpha_i+\sum_{j=1}^n q_jA_{ji}} f_i^{2p_i}(x_i)\right]^{1/(n+1)}d\mathbf{x}}{\left[\prod_{i=1}^n k_i(q_i\mathbf{A_i})\right]^{1/(n+1)}\prod_{i=1}^n ||x_i^{(n-1-s)/q_i+\alpha_i} f_i||_{p_i}^{2p_i/(n+1)}}\right],
\end{aligned}
\tag{2.173}
$$

where $\alpha_i = \sum_{j=1}^n A_{ij}$, $i = 1,2,\dots,n$, $m = \min\{1/q_1, 1/q_2, \dots, 1/q_n, 1-\lambda\}$, $M = \max\{1/q_1, 1/q_2, \dots, 1/q_n, 1-\lambda\}$, and $k_i(\cdot)$, $i = 1,2,\dots,n$ are defined by (2.139).

Proof. We use Theorem 2.28. More precisely, if we insert the functions ϕ_{ij} and ω_i, $i,j = 1,2,\dots,n$, defined respectively by (2.138) and (2.171), in the relation (2.170), we get (2.173) after straightforward computation. □

Now we discuss these results in some particular settings. More precisely, we consider the homogeneous function $K : \mathbb{R}_+^n \to \mathbb{R}$ defined by

$$K(x) = \left(\sum_{i=1}^n x_i\right)^{-s}, \quad s > 0. \tag{2.174}$$

Clearly, K is a homogeneous function of degree $-s$. In this setting, the constant (2.139) can be expressed in the terms of a gamma function Γ. For that sake, we use the formula (2.156). In such a way, the constant factors $k_i(q_i\mathbf{A_i})$, $i = 1,2,\dots,n$, involved in the series of inequalities (2.172) and (2.173) become

$$k_i(q_i\mathbf{A_i}) = \frac{\Gamma(s-n+1-q_i\alpha_i+q_iA_{ii})}{\Gamma(s)}\prod_{j=1,j\neq i}^n \Gamma(1+q_iA_{ij}), \quad i = 1,2,\dots,n,$$

provided that $A_{ij} > -1/q_i$, $i \neq j$ and $A_{ii} - \alpha_i > (n-s-1)/q_i$. In the sequel we consider some special choices of the parameters A_{ij}, $i,j = 1,2,\dots,n$, which will bring us to some results known from the literature.

Example 2.4 Let $A_{ii}=(n-s)(\lambda q_i-1)/q_i^2$ and $A_{ij}=(s-n)/(q_iq_j)$, $i,j=1,2,\dots,n$, $i\neq j$. One easily verifies that $\sum_{i=1}^n A_{ij}=\sum_{j=1}^n A_{ij}=0$, so the interpolating series of inequalities (2.172) reduces to

$$\begin{aligned}
&\frac{\left\{\int_{\mathbb{R}_+^n}\left[(\sum_{i=1}^n x_i)^{-ns}\prod_{i=1}^n x_i^{(2p_i/q_i-1)(n-1-s)+(n-s)(\lambda q_i-n)/q_i} f_i^{2p_i}(x_i)\right]^{1/(n+1)}d\mathbf{x}\right\}^{(n+1)M}}{\mathscr{K}_M\prod_{i=1}^n ||x_i^{(n-1-s)/q_i}f_i||_{p_i}^{2Mp_i}}\\
&\quad\leq \frac{\int_{\mathbb{R}_+^n}(\sum_{i=1}^n x_i)^{-\lambda s}\prod_{i=1}^n f_i(x_i)d\mathbf{x}}{\mathscr{K}\prod_{i=1}^n ||x_i^{(n-1-s)/q_i}f_i||_{p_i}}\\
&\quad\leq \frac{\left\{\int_{\mathbb{R}_+^n}\left[(\sum_{i=1}^n x_i)^{-ns}\prod_{i=1}^n x_i^{(2p_i/q_i-1)(n-1-s)+(n-s)(\lambda q_i-n)/q_i} f_i^{2p_i}(x_i)\right]^{1/(n+1)}d\mathbf{x}\right\}^{(n+1)m}}{\mathscr{K}_m\prod_{i=1}^n ||x_i^{(n-1-s)/q_i}f_i||_{p_i}^{2mp_i}},
\end{aligned}\tag{2.175}$$

where

$$\begin{aligned}
\mathscr{K}_M&=\frac{1}{\Gamma(s)^M}\prod_{i=1}^n\Gamma\left(\frac{p_i+s-n}{p_i}\right)^M\prod_{i=1}^n\Gamma\left(\frac{q_i+s-n}{q_i}\right)^{(n-1)M},\\
\mathscr{K}&=\frac{1}{\Gamma(s)^\lambda}\prod_{i=1}^n\Gamma\left(\frac{p_i+s-n}{p_i}\right)^{1/q_i}\prod_{i=1}^n\Gamma\left(\frac{q_i+s-n}{q_i}\right)^{\lambda-(1/q_i)},\\
\mathscr{K}_m&=\frac{1}{\Gamma(s)^m}\prod_{i=1}^n\Gamma\left(\frac{p_i+s-n}{p_i}\right)^m\prod_{i=1}^n\Gamma\left(\frac{q_i+s-n}{q_i}\right)^{(n-1)m},
\end{aligned}$$

assuming $s>n-\min_{1\leq i\leq n}\{p_i,q_i\}$.

Remark 2.41 If p_i are conjugate exponents, that is $p_i=q_i$, $i=1,2,\dots,n$, and $\lambda=1$, then the constant $\mathscr{K}$ from relation (2.175) reduces to $\mathscr{K}=1/(\Gamma(s))\prod_{i=1}^n\Gamma((p_i+s-n)/p_i)$, that is, the middle term in (2.175) represents the quotient between the left-hand and right-hand side of multidimensional inequality from [38]. Therefore, the interpolating series of inequalities (2.175) can be regarded as a non-conjugate extension of the result from [38].

Example 2.5 If $A_{ii}=(\lambda q_i-1)/(\lambda q_i^2)$ and $A_{ij}=-1/(\lambda q_iq_j)$, $i,j=1,2,\dots,n$, $i\neq j$, then it follows that $\sum_{i=1}^n A_{ij}=\sum_{j=1}^n A_{ij}=0$. Now, if the degree of homogeneity of kernel (2.174) is $1-n$, that is $s=n-1$, the interpolating series of inequalities (2.172) becomes

$$\begin{aligned}
&\frac{\left\{\int_{\mathbb{R}_+^n}\left[(\sum_{i=1}^n x_i)^{-n(n-1)}\prod_{i=1}^n x_i^{1-n/(\lambda q_i)} f_i^{2p_i}(x_i)\right]^{1/(n+1)}d\mathbf{x}\right\}^{(n+1)M}}{\mathscr{L}_M\prod_{i=1}^n ||f_i||_{p_i}^{2Mp_i}}\\
&\quad\leq \frac{\int_{\mathbb{R}_+^n}(\sum_{i=1}^n x_i)^{-\lambda(n-1)}\prod_{i=1}^n f_i(x_i)d\mathbf{x}}{\mathscr{L}\prod_{i=1}^n ||f_i||_{p_i}}\\
&\quad\leq \frac{\left\{\int_{\mathbb{R}_+^n}\left[(\sum_{i=1}^n x_i)^{-n(n-1)}\prod_{i=1}^n x_i^{1-n/(\lambda q_i)} f_i^{2p_i}(x_i)\right]^{1/(n+1)}d\mathbf{x}\right\}^{(n+1)m}}{\mathscr{L}_m\prod_{i=1}^n ||f_i||_{p_i}^{2mp_i}},
\end{aligned}\tag{2.176}$$

where

$$\mathscr{L}_M = \frac{1}{[(n-2)!]^M}\left[\prod_{i=1}^{n}\Gamma\left(\frac{1}{\lambda p_i'}\right)\right]^M,$$

$$\mathscr{L} = \frac{1}{[(n-2)!]^\lambda}\left[\prod_{i=1}^{n}\Gamma\left(\frac{1}{\lambda p_i'}\right)\right]^\lambda,$$

$$\mathscr{L}_m = \frac{1}{[(n-2)!]^m}\left[\prod_{i=1}^{n}\Gamma\left(\frac{1}{\lambda p_i'}\right)\right]^m.$$

Remark 2.42 The middle term in (2.176) represents the quotient between the left-hand and right-hand side of the non-conjugate Hilbert-type inequality which was proved by Bonsall [41], in the case $n=3$.

Note that the parameters A_{ij}, $i,j=1,2,\dots,n$ were symmetric in the previous two examples, which is not the case in the following one.

Example 2.6 Suppose A_i, $i=1,2,\dots,n$, are real parameters satisfying relations $(n-s-1)/q_{i-1} < A_i < 1/q_{i-1}$, provided that $s > n-2$. Of course, we use the convention $q_0 = q_n$. We define parameters A_{ij}, $i,j=1,2,\dots,n$, by

$$A_{ij} = \begin{cases} A_i, & j=i, \\ -A_{i+1}, & j=i+1, \\ 0 & \text{otherwise,} \end{cases}$$

where the indices are taken modulo n from the set $\{1,2,\dots,n\}$. In the described setting equipped with homogeneous kernel (2.174), the series of inequalities in (2.172) reads

$$\begin{aligned}
&\frac{\left\{\int\limits_{\mathbb{R}_+^n}\left[(\sum_{i=1}^{n}x_i)^{-ns}\prod_{i=1}^{n}x_i^{(2p_i/q_i-1)(n-1-s)+(2p_i-q_{i-1})A_i-(2p_i-q_i)A_{i+1}}f_i^{2p_i}(x_i)\right]^{1/(n+1)}d\mathbf{x}\right\}^{(n+1)M}}{\mathscr{R}_M\prod_{i=1}^{n}||x_i^{(n-1-s)/q_i+A_i-A_{i+1}}f_i||_{p_i}^{2Mp_i}} \\
&\le \frac{\int\limits_{\mathbb{R}_+^n}(\sum_{i=1}^{n}x_i)^{-\lambda s}\prod_{i=1}^{n}f_i(x_i)d\mathbf{x}}{\mathscr{R}\prod_{i=1}^{n}||x_i^{(n-1-s)/q_i+A_i-A_{i+1}}f_i||_{p_i}} \\
&\le \frac{\left\{\int\limits_{\mathbb{R}_+^n}\left[(\sum_{i=1}^{n}x_i)^{-ns}\prod_{i=1}^{n}x_i^{(2p_i/q_i-1)(n-1-s)+(2p_i-q_{i-1})A_i-(2p_i-q_i)A_{i+1}}f_i^{2p_i}(x_i)\right]^{1/(n+1)}d\mathbf{x}\right\}^{(n+1)m}}{\mathscr{R}_m\prod_{i=1}^{n}||x_i^{(n-1-s)/q_i+A_i-A_{i+1}}f_i||_{p_i}^{2mp_i}},
\end{aligned} \tag{2.177}$$

where

$$\mathscr{R}_M = \frac{1}{\Gamma(s)^M}\left[\prod_{i=1}^{n}\Gamma(s-n+1+q_iA_{i+1})\Gamma(1-q_iA_{i+1})\right]^M,$$

$$\mathscr{R} = \frac{1}{\Gamma(s)^\lambda}\prod_{i=1}^{n}\Gamma(s-n+1+q_iA_{i+1})^{1/q_i}\Gamma(1-q_iA_{i+1})^{1/q_i},$$

$$\mathscr{R}_m = \frac{1}{\Gamma(s)^m}\left[\prod_{i=1}^{n}\Gamma(s-n+1+q_iA_{i+1})\Gamma(1-q_iA_{i+1})\right]^m.$$

Remark 2.43 The middle term in (2.177) represents the multidimensional inequality from [43] and [44] in the conjugate setting.

Remark 2.44 It is important to emphasize that the multidimensional inequalities (in non-conjugate setting) represented by the middle terms in relations (2.175), (2.176) and (2.177), were also derived in [42]. Therefore, our relations (2.175), (2.176) and (2.177) represent refinements and converses of the appropriate results from [42]. Refinements and converses of these relations, although omitted here, can be obtained in the difference form, in an analogous way.

Chapter 3

Jensen-type functionals under the Steffensen's conditions. Petrović-type functionals

In this chapter we make use of two variants of the Jensen inequality: Jensen-Steffensen's inequality (Theorem 1.7) and Jensen-Mercer's inequality (Theorem 1.8), in order to deduce and investigate two corresponding Jensen-type functionals, that is, Jensen-Steffensen's and Jensen-Mercer's functional, where the latter is additionally observed under the Steffensen's conditions. For both functionals, the properties of superadditivity and increase on certain sets of the real n-tuples are expressed and investigated in the discrete and in the integral cases as well, by using the Riemann-Stieltjes' integral.

The functionals of this type were investigated earlier in [23] and [24]. The results that were obtained there are here improved.

In the last part of the chapter, Petrović-type functionals, derived from the corresponding Petrović and related inequalities are considered in a similar sense, also referring to their superadditivity established on certain sets of the real n-tuples.

3.1 Superadditivity of Jensen-Steffensen's functional

We first consider Jensen-Steffensen's functional, starting with its discrete form. After proving its superadditivity property, we apply it to establishing its increase property on the set of the real n-tuples that satisfy the Steffensen's conditions. The inequalities of the form

$$M\mathfrak{J}(f,\mathbf{x},\mathbf{q}) \geq \mathfrak{J}(f,\mathbf{x},\mathbf{p}) \geq m\mathfrak{J}(f,\mathbf{x},\mathbf{q}),$$

that were observed by S. S. Dragomir in [60] (see Theorem 1.35) and subsequently in [23], are thus improved by obtaining the new form of the non-weight bounds for Jensen-Steffensen's functional.

The integral analogues of the discrete results are obtained making use of the Boas' variant of the integral Jensen-Steffensen's inequality. Some related additional integral results of the Boas' type are also established.

At the end of the considerations on the Jensen-Steffensen's functional, we give an application to the functional defined in [155], by means of the weight quasiarithmetic mean.

The contents of the Section 3.1 corresponds for the most part to the contents of the published paper [113].

3.1.1 On discrete Jensen-Steffensen's functional

As we already have mentioned (Theorem 1.7), J. F. Steffensen proved that if $\mathbf{x} \in I^n$, $I \subseteq \mathbb{R}$ is a monotonic (increasing or decreasing) n-tuple and $\mathbf{p}$ is a real n-tuple such that

$$P_n > 0 \quad \text{and} \quad 0 \leq P_k \leq P_n, \quad 1 \leq k \leq n-1, \tag{3.1}$$

where $P_k = \sum_{i=1}^{k} p_i$, $k = 1,\ldots,n$, then Jensen's inequality

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \tag{3.2}$$

holds for every convex function $f : I \to \mathbb{R}$. Under Steffensen's conditions (3.1), inequality (3.2) is referred to as Jensen-Steffensen's inequality. In the sequel, the set of all real n-tuples $\mathbf{p}$ that satisfy (3.1) will be denoted by $\mathscr{P}_n$. Furthermore, $\mathscr{P}_n^0 \subseteq \mathscr{P}_n$, where $\mathscr{P}_n^0$ is (as in the previous considerations) the set of all nonnegative n-tuples $\mathbf{p}$, such that $\sum_{i=1}^{n} p_i = P_n > 0$.

Discrete Jensen-Steffensen's functional $\mathfrak{J}(f,\mathbf{x},\mathbf{p})$ is now deduced from the inequality (3.2), analogously as it was done with Jensen's functional (1.65):

$$\mathfrak{J}(f,\mathbf{x},\mathbf{p}) = \sum_{i=1}^{n} p_i f(x_i) - P_n f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right). \tag{3.3}$$

For fixed f and $\mathbf{x}$, functional $\mathfrak{J}(f,\mathbf{x},\cdot)$ can be observed as a function on $\mathscr{P}_n$. Furthermore, when f is a convex function, then (3.2) yields $\mathfrak{J}(f,\mathbf{x},\mathbf{p}) \geq 0$, for all $\mathbf{p} \in \mathscr{P}_n$.

Nevertheless, superadditivity property of the functional (3.3) holds in a more general environment of the previously described. Therefore we state it this time in a separate result.

Theorem 3.1 *Let I be an interval in $\mathbb{R}$ and let $\mathbf{x} = (x_1,\ldots,x_n) \in I^n$. Suppose $\mathbf{p} = (p_1,\ldots,p_n)$ and $\mathbf{q} = (q_1,\ldots,q_n)$ are real n-tuples, such that $\sum_{i=1}^n p_i = P_n > 0$ and $\sum_{i=1}^n q_i = Q_n > 0$ and $\frac{1}{P_n}\sum_{i=1}^n p_i x_i$, $\frac{1}{Q_n}\sum_{i=1}^n q_i x_i \in I$. If $f : I \to \mathbb{R}$ is a convex function, then functional* (3.3) *is superadditive on the set of the described real n-tuples, that is*

$$\mathfrak{J}(f,\mathbf{x},\mathbf{p}+\mathbf{q}) \geq \mathfrak{J}(f,\mathbf{x},\mathbf{p}) + \mathfrak{J}(f,\mathbf{x},\mathbf{q}). \tag{3.4}$$

Proof. It follows from the definition of the functional (3.3) that

$$\begin{aligned}\mathfrak{J}(f,\mathbf{x},\mathbf{p}+\mathbf{q}) &= \sum_{i=1}^n (p_i+q_i) f(x_i) - (P_n+Q_n) f\left(\frac{\sum_{i=1}^n (p_i+q_i)x_i}{P_n+Q_n}\right)\\ &= \sum_{i=1}^n p_i f(x_i) + \sum_{i=1}^n q_i f(x_i) - (P_n+Q_n) f\left(\frac{\sum_{i=1}^n (p_i+q_i)x_i}{P_n+Q_n}\right),\end{aligned} \tag{3.5}$$

whereas convexity of f yields

$$\begin{aligned} f\left(\frac{\sum_{i=1}^n (p_i+q_i)x_i}{P_n+Q_n}\right) &= f\left(\frac{\sum_{i=1}^n p_i x_i + \sum_{i=1}^n q_i x_i}{P_n+Q_n}\right)\\ &= f\left(\frac{P_n}{P_n+Q_n}\frac{\sum_{i=1}^n p_i x_i}{P_n} + \frac{Q_n}{P_n+Q_n}\frac{\sum_{i=1}^n q_i x_i}{Q_n}\right)\\ &\leq \frac{P_n}{P_n+Q_n} f\left(\frac{\sum_{i=1}^n p_i x_i}{P_n}\right) + \frac{Q_n}{P_n+Q_n} f\left(\frac{\sum_{i=1}^n q_i x_i}{Q_n}\right).\end{aligned} \tag{3.6}$$

Finally, (3.5) and (3.6) yield

$$\begin{aligned}\mathfrak{J}(f,\mathbf{x},\mathbf{p}+\mathbf{q}) &\geq \sum_{i=1}^n p_i f(x_i) + \sum_{i=1}^n q_i f(x_i) - P_n f\left(\frac{\sum_{i=1}^n p_i x_i}{P_n}\right) - Q_n f\left(\frac{\sum_{i=1}^n q_i x_i}{Q_n}\right)\\ &= \mathfrak{J}(f,\mathbf{x},\mathbf{p}) + \mathfrak{J}(f,\mathbf{x},\mathbf{q}),\end{aligned}$$

whereby the proof is concluded. □

Remark 3.1 The superadditivity of Jensen-Steffensen's functional $\mathfrak{J}(f,\mathbf{x},\cdot)$ is proved as it would be done for discrete Jensen's functional $J(f,\mathbf{x},\cdot)$ in Theorem 1.34. Namely, the classical Jensen's inequality is applied in both proofs, where the only needed assumptions are $P_n, Q_n > 0$, $\frac{1}{P_n}\sum_{i=1}^n p_i x_i$ and $\frac{1}{Q_n}\sum_{i=1}^n q_i x_i \in I$. On the contrary, the property of increase (3.3) of the functional needs to be proved depending on the choice of the real n-tuples from $\mathscr{P}_n$, as it will be done in the sequel.

Theorem 3.2 *Suppose $\mathbf{q}$ and $\mathbf{p}-\mathbf{q}$ are n-tuples in $\mathscr{P}_n$, where $\mathbf{p}$ is a real n-tuple, such that $\sum_{i=1}^n p_i = P_n > 0$. Let $\mathbf{x} \in I^n$, $I \subseteq \mathbb{R}$, be a monotonic (either nondecreasing or nonincreasing) n-tuple. If $f : I \to \mathbb{R}$ is a convex function, then the following inequalities hold:*

$$\mathfrak{J}(f,\mathbf{x},\mathbf{p}) \geq \mathfrak{J}(f,\mathbf{x},\mathbf{q}) \geq 0. \tag{3.7}$$

Proof. Since $\mathfrak{J}(f,\mathbf{x},\mathbf{p}) = \mathfrak{J}(f,\mathbf{x},\mathbf{p}-\mathbf{q}+\mathbf{q})$, applying (3.3) to $\mathbf{p}-\mathbf{q}$ and $\mathbf{q}$, we get

$$\mathfrak{J}(f,\mathbf{x},\mathbf{p}) = \mathfrak{J}(f,\mathbf{x},\mathbf{p}-\mathbf{q}+\mathbf{q}) \geq \mathfrak{J}(f,\mathbf{x},\mathbf{p}-\mathbf{q}) + \mathfrak{J}(f,\mathbf{x},\mathbf{q}) \geq \mathfrak{J}(f,\mathbf{x},\mathbf{q}),$$

where the last inequality holds since $\mathfrak{J}(f,\mathbf{x},\mathbf{p}-\mathbf{q}) \geq 0$, according to Jensen-Steffensen's inequality, applied to a convex function f and $\mathbf{p}-\mathbf{q} \in \mathscr{P}_n$. Analogously, for a convex function f and $\mathbf{q} \in \mathscr{P}_n$, the last inequality in (3.7) holds. □

Dragomir's Theorem 1.35 on comparative inequalities for normalized Jensen's functional was given an alternative proof in [23]. An analogous result, which we cite in the sequel was established for n-tuples $\mathbf{p}$ and $\mathbf{q}$ satisfying the Steffensen's conditions, where normalized Jensen-Steffensen's functional is denoted by $\mathfrak{J}_n(f,\mathbf{x},\cdot)$.

Theorem 3.3 (SEE [23]) *Let $\mathbf{p} = (p_1,\dots,p_n)$ and $\mathbf{q} = (q_1,\dots,q_n)$ be two n-tuples satisfying the following conditions:*

$$0 \leq P_k, Q_k \leq 1, \quad k = 1,\dots,n-1, \quad P_n = Q_n = 1.$$

For $k \in \{1,\dots,n\}$ denote $P_k := \sum_{i=1}^k p_i$, $Q_k := \sum_{i=1}^k q_i$. Let m and M be any real constants such that

$$P_k - mQ_k \geq 0, \quad (1-P_k) - m(1-Q_k) \geq 0, \quad k = 1,\dots,n-1$$

and

$$MQ_k - P_k \geq 0, \quad M(1-Q_k) - (1-P_k) \geq 0, \quad k = 1,\dots,n-1.$$

If $f : I \to \mathbb{R}$ is a convex function defined on an interval $I \subseteq \mathbb{R}$ and if $\mathbf{x} = (x_1,\dots,x_n) \in I^n$ is any monotonic n-tuple, then

$$M\mathfrak{J}_n(f,\mathbf{x},\mathbf{q}) \geq \mathfrak{J}_n(f,\mathbf{x},\mathbf{p}) \geq m\mathfrak{J}_n(f,\mathbf{x},\mathbf{q}). \tag{3.8}$$

We cite here the accompanied corollary from [23], where the normalized Jensen-Steffensen's functional is bounded by means of the functional

$$\mathfrak{J}_n(f,\mathbf{x}) := \mathfrak{J}_n(f,\mathbf{x},\mathbf{u}) = \frac{1}{n}\sum_{i=1}^n f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^n x_i\right),$$

regarding the uniform distribution $\mathbf{u} = (\frac{1}{n},\dots,\frac{1}{n})$.

Corollary 3.1 (SEE [23]) *Let $\mathbf{p} = (p_1,\dots,p_n)$ be an n-tuple that satisfies*

$$0 \leq P_k \leq 1, \quad k = 1,\dots,n-1, \quad P_n = 1.$$

For $k \in \{1,\dots,n\}$ denote $P_k := \sum_{i=1}^k p_i$ and define

$$\tilde{m}_0 := n \cdot \min\left\{\frac{P_k}{k}, \frac{1-P_k}{n-k} : k = 1,\dots,n-1\right\},$$

$$\tilde{M}_0 := n \cdot \max\left\{\frac{P_k}{k}, \frac{1-P_k}{n-k} : k = 1,\dots,n-1\right\}.$$

If $f : I \to \mathbb{R}$ is a convex function defined on an interval $I \subseteq \mathbb{R}$ and if $\mathbf{x} = (x_1,\dots,x_n) \in I^n$ is any monotonic n-tuple, then

$$\tilde{M}_0\mathfrak{J}_n(f,\mathbf{x}) \geq \mathfrak{J}_n(f,\mathbf{x},\mathbf{p}) \geq \tilde{m}_0\mathfrak{J}_n(f,\mathbf{x}). \tag{3.9}$$

Remark 3.2 We now show that the statement of Theorem 3.3 from [23] can easily be deduced from Theorem 3.2. Let m and M be real constants and let $\mathbf{p}$ be a real n-tuple such that $\sum_{i=1}^{n} p_i = P_n > 0$. Suppose $\mathbf{q}$, $\mathbf{p}-m\mathbf{q}$ and $M\mathbf{q}-\mathbf{p} \in \mathscr{P}_n$. If $f : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$, is a convex function and if $\mathbf{x} = (x_1,\dots,x_n) \in I^n$ is a monotonic n-tuple, then, according to Theorem 3.2:

$$\mathfrak{J}(f,\mathbf{x},\mathbf{p}) = \mathfrak{J}(f,\mathbf{x},\mathbf{p}-m\mathbf{q}+m\mathbf{q}) \geq \mathfrak{J}(f,\mathbf{x},\mathbf{p}-m\mathbf{q}) + \mathfrak{J}(f,\mathbf{x},m\mathbf{q}) \geq m\mathfrak{J}(f,\mathbf{x},\mathbf{q}).$$

Similarly,

$$\mathfrak{J}(f,\mathbf{x},\mathbf{p}) \leq M\mathfrak{J}(f,\mathbf{x},\mathbf{q}),$$

that is

$$M\mathfrak{J}(f,\mathbf{x},\mathbf{q}) \geq \mathfrak{J}(f,\mathbf{x},\mathbf{p}) \geq m\mathfrak{J}(f,\mathbf{x},\mathbf{q}). \tag{3.10}$$

Since $\mathbf{p}-m\mathbf{q} \in \mathscr{P}_n$ implies $P_k \geq mQ_k$ and $(P_n - P_k) \geq m(Q_n - Q_k)$, and $M\mathbf{q}-\mathbf{p} \in \mathscr{P}_n$ implies $P_k \leq MQ_k$ and $(P_n - P_k) \leq M(Q_n - Q_k)$, $k = 1,\dots,n-1$, which are the assumptions of Theorem 3.3, only written in a slightly generalized, non-normalized form, the starting assumptions yield the statement of Theorem 3.3.

Another application of Theorem 3.2 is deducing the lower and the upper bound for the functional (3.3), by means of the non-weight functional of the same type. However, almost identical result which slightly differs from this, for it was given in the normalized form, was obtained in Corollary 3.1. Thus this proof is comprehended as an alternative one and is consequently given within a remark.

Remark 3.3 Let us write Corollary 3.1 in a slightly generalized way:

Let $\mathbf{p} = (p_1,\dots,p_n)$ be an n-tuple in $\mathscr{P}_n$. Define

$$m := \min_{1\leq k\leq n-1}\left\{\frac{P_k}{k}, \frac{P_n - P_k}{n-k}\right\}, \quad M := \max_{1\leq k\leq n-1}\left\{\frac{P_k}{k}, \frac{P_n - P_k}{n-k}\right\},$$

where $P_k = \sum_{i=1}^{k} p_i$ and $P_n = \sum_{i=1}^{n} p_i > 0$. If $f : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$, is a convex function and if $\mathbf{x} = (x_1,\dots,x_n) \in I^n$ is a monotonic n-tuple, then

$$M\mathfrak{J}(f,\mathbf{x}) \geq \mathfrak{J}(f,\mathbf{x},\mathbf{p}) \geq m\mathfrak{J}(f,\mathbf{x}), \tag{3.11}$$

where $\mathfrak{J}(f,\mathbf{x}) = \sum_{i=1}^{n} f(x_i) - nf\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)$.

Alternative proof of Corollary 3.1. Let us firstly prove the right-hand side inequality in (3.11). Let $\mathbf{q}_{\min} \in \mathscr{P}_n^0$ be a constant n-tuple, $q_{\min} = (\alpha,\alpha,\dots,\alpha)$, where $\alpha > 0$ since $Q_n := \sum_{i=1}^{n} q_i > 0$. The proof is based on the application of Theorem 3.2 to $\mathbf{p}$ and $\mathbf{q}_{\min}$ because

$$\mathfrak{J}(f,\mathbf{x},\mathbf{q}_{\min}) = m\left(\sum_{i=1}^{n} f(x_i) - nf\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)\right) = m\mathfrak{J}(f,\mathbf{x}).$$

However, we firstly need to hold an argumentation on the choice of α regarding the defined m. In order to apply Theorem 3.2, one must provide the following conditions satisfied: $P_k \geq Q_k = k\alpha$, $P_n - P_k \geq Q_n - Q_k = (n-k)\alpha$, $k = 1,\dots,n-1$, $P_n > Q_n = n\alpha$, which means:

(i) $\alpha \leq \frac{P_k}{k}, \quad k = 1, \ldots, n-1,$

(ii) $\alpha \leq \frac{P_n - P_k}{n-k}, \quad k = 1, \ldots, n-1,$

(iii) $\alpha < \frac{P_n}{n}.$

Obviously, m satisfies the conditions (i) and (ii) by definition and is a candidate for the choice of α. Fix $k \in \{1, \ldots, n\}$. It easily follows from (i) and (ii) that $nm \leq P_n$, i.e. $m \leq \frac{P_n}{n}$ and m is a good choice for α. The left-hand side inequality is proved analogously, by exchanging the roles of **p** and **q**. □

3.1.2 Integral variants of Jensen-Steffensen's functional

Although Steffensen established the integral variant of Jensen's inequality too, we take here as the starting point the integral variant of Jensen-Stefensen's inequality that was proved by R. P. Boas in [39], but can be also found in e.g. [177, p. 59]. As for the following considerations, we need to refer to the properties of Riemann-Stieltjes' integral, which are described in the first chapter and regarding to which we already cited the integral form of Jensen's inequality (1.11). Now we proceed in the similar way regarding Jensen's inequality under Steffensen's conditions.

Theorem 3.4 (JENSEN-STEFFENSEN) *Suppose $x : [\alpha, \beta] \to (a, b)$ is a continuous and monotonic (either nondecreasing or nonincreasing) function, where $-\infty < \alpha < \beta < \infty$ and $-\infty \leq a < b \leq \infty$, and let $f : (a, b) \to \mathbb{R}$ be a convex function. If $\lambda : [\alpha, \beta] \to \mathbb{R}$ is either continuous or of bounded variation, satisfying*

$$\lambda(\alpha) \leq \lambda(t) \leq \lambda(\beta), \quad \textit{for all} \quad t \in [\alpha, \beta] \quad \textit{and} \quad \lambda(\beta) - \lambda(\alpha) > 0, \tag{3.12}$$

then the following inequality holds:

$$f\left(\frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_{\alpha}^{\beta} x(t) d\lambda(t)\right) \leq \frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_{\alpha}^{\beta} f(x(t)) d\lambda(t). \tag{3.13}$$

The condition (3.12) on λ can be regarded as a very weak monotonicity condition. Still, the monotonicity condition required for x is quite restrictive. So Boas proved in [39] that if one strengthens the hypothesis on λ and correspondingly weakens the hypothesis on x, the inequality (3.13) still holds.

Theorem 3.5 (BOAS) *Let $\lambda : [\alpha, \beta] \to \mathbb{R}$ be either continuous or of bounded variation and such that there exist $k \geq 2$ points $\alpha = \gamma_0 < \gamma_1 < \cdots < \gamma_k = \beta$, so that*

$$\begin{aligned} &\lambda(\alpha) \leq \lambda(t_1) \leq \lambda(\gamma_1) \leq \lambda(t_2) \leq \cdots \leq \lambda(\gamma_{k-1}) \leq \lambda(t_k) \leq \lambda(\beta), \\ &\textit{for all} \quad t_i \in [\gamma_{i-1}, \gamma_i], \quad i = 1, \ldots, k, \quad \lambda(\beta) - \lambda(\alpha) > 0. \end{aligned} \tag{3.14}$$

If $x : [\alpha, \beta] \to (a, b)$ is a continuous function and monotonic (either nondecreasing or nonincreasing) on each of the intervals $[\gamma_{i-1}, \gamma_i]$, $i = 1, \ldots, k$, then (3.13) holds for any convex function $f : (a, b) \to \mathbb{R}$.

When the inequality (3.13) is observed under the Steffensen's conditions (3.12), it is called Jensen-Steffensen's inequality. Otherwise, when it is observed under the conditions (3.14), it is called Jensen-Steffensen-Boas' inequality.

Finally, using the integral variants of the inequality (3.13), we can observe the corresponding functionals $\mathfrak{J}(f,x,\lambda)$, which are deduced as follows:

$$\mathfrak{J}(f,x,\lambda)=\int_{\alpha}^{\beta} f(x(t))d\lambda(t)-(\lambda(\beta)-\lambda(\alpha))f\left(\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_{\alpha}^{\beta} x(t)d\lambda(t)\right). \quad (3.15)$$

The common form of the functional (3.15) is called (integral) Jensen-Steffensen's or Jensen-Steffensen-Boas' functional, depending on the conditions under which it is observed: the ones from Theorem 3.4 or those of Theorem 3.5. For the sake of simplicity and shortness, we use the following notation:

$\Lambda_{[\alpha,\beta]}, -\infty<\alpha<\beta<\infty$, for the class of all functions $\lambda:[\alpha,\beta]\to\mathbb{R}$ that are either continuous or of bounded variation and satisfy the conditions (3.12);

$\tilde{\Lambda}_{[\alpha,\beta]}$, for subclass of $\Lambda_{[\alpha,\beta]}$, that contains all $\lambda\in\Lambda_{[\alpha,\beta]}$ satisfying the conditions (3.14).

We notice here that every nondecreasing function $\lambda:[\alpha,\beta]\to\mathbb{R}$, such that $\lambda(\beta)\neq\lambda(\alpha)$, belongs to the class $\Lambda_{[\alpha,\beta]}$ (see inequality (1.11)). Moreover, notice that if f is convex, then after (3.13) is $\mathfrak{J}(f,x,\lambda)\geq 0$, for all λ that are in $\in\Lambda_{[\alpha,\beta]}$ or $\tilde{\Lambda}_{[\alpha,\beta]}$, which follows from (3.13). When proving the properties of the functionals (3.15) that are analogous to those in the discrete case, we observe $\mathfrak{J}(f,x,\cdot)$ as a function on $\Lambda_{[\alpha,\beta]}$ or $\tilde{\Lambda}_{[\alpha,\beta]}$, with fixed f and x.

Similarly as in the discrete case, superadditivity of both functionals defined by (3.15) does not depend on the special classes of the functions λ. Hence the result on the superadditivity is proved as a common result for both functionals and only after that their properties are analyzed separately.

Theorem 3.6 *Let $x:[\alpha,\beta]\to(a,b)$, $a,b\in\overline{\mathbb{R}}$, be a continuous function and let $\lambda,\mu:[\alpha,\beta]\to\mathbb{R}$ be such that $\lambda(\beta)-\lambda(\alpha)>0$, $\mu(\beta)-\mu(\alpha)>0$. If $f:(a,b)\to\mathbb{R}$ is a convex function, then functional $\mathfrak{J}(f,x,\cdot)$ defined by* (3.15) *is superadditive on the set of the described functions λ and μ, that is*

$$\mathfrak{J}(f,x,\lambda+\mu)\geq\mathfrak{J}(f,x,\lambda)+\mathfrak{J}(f,x,\mu). \quad (3.16)$$

Proof. Let us first denote $\lambda(\beta)-\lambda(\alpha):=\lambda_{\alpha}^{\beta}$ and $\mu(\beta)-\mu(\alpha):=\mu_{\alpha}^{\beta}$. It follows from the definition of the functional (3.15) that

$$\begin{aligned}\mathfrak{J}(f,x,\lambda+\mu) &= \int_{\alpha}^{\beta} f(x(t))d(\lambda+\mu)(t)-\left(\lambda_{\alpha}^{\beta}+\mu_{\alpha}^{\beta}\right)f\left(\frac{1}{\lambda_{\alpha}^{\beta}+\mu_{\alpha}^{\beta}}\int_{\alpha}^{\beta} x(t)d(\lambda+\mu)(t)\right)\\ &= \int_{\alpha}^{\beta} f(x(t))d\lambda(t)+\int_{\alpha}^{\beta} f(x(t))d\mu(t)-\left(\lambda_{\alpha}^{\beta}+\mu_{\alpha}^{\beta}\right)\cdot\\ &\quad\times f\left(\frac{1}{\lambda_{\alpha}^{\beta}+\mu_{\alpha}^{\beta}}\int_{\alpha}^{\beta} x(t)d(\lambda+\mu)(t)\right), \end{aligned} \quad (3.17)$$

whereas convexity of f yields

$$\begin{aligned}
& f\left(\frac{1}{\lambda_\alpha^\beta+\mu_\alpha^\beta}\int_\alpha^\beta x(t)d(\lambda+\mu)(t)\right) \\
&= f\left(\frac{\lambda_\alpha^\beta}{\lambda_\alpha^\beta+\mu_\alpha^\beta}\cdot\frac{\int_\alpha^\beta x(t)d\lambda(t)}{\lambda_\alpha^\beta}+\frac{\mu_\alpha^\beta}{\lambda_\alpha^\beta+\mu_\alpha^\beta}\cdot\frac{\int_\alpha^\beta x(t)d\mu(t)}{\mu_\alpha^\beta}\right) \\
&\leq \frac{\lambda_\alpha^\beta}{\lambda_\alpha^\beta+\mu_\alpha^\beta}\cdot f\left(\frac{\int_\alpha^\beta x(t)d\lambda(t)}{\lambda_\alpha^\beta}\right)+\frac{\mu_\alpha^\beta}{\lambda_\alpha^\beta+\mu_\alpha^\beta}\cdot f\left(\frac{\int_\alpha^\beta x(t)d\mu(t)}{\mu_\alpha^\beta}\right). \qquad (3.18)
\end{aligned}$$

Finally, (3.17) and (3.18) yield

$$\begin{aligned}
\mathfrak{J}(f,x,\lambda+\mu) \geq & \int_\alpha^\beta f(x(t))d\lambda(t)+\int_\alpha^\beta f(x(t))d\mu(t)-\lambda_\alpha^\beta\cdot f\left(\frac{\int_\alpha^\beta x(t)d\lambda(t)}{\lambda_\alpha^\beta}\right) \\
& -\mu_\alpha^\beta\cdot f\left(\frac{\int_\alpha^\beta x(t)d\mu(t)}{\mu_\alpha^\beta}\right)=\mathfrak{J}(f,x,\lambda)+\mathfrak{J}(f,x,\mu),
\end{aligned}$$

which concludes the proof. □

We now focus on the integral variant of Jensen-Steffensen's functional, starting with the integral variant of Theorem 3.2.

Theorem 3.7 *Suppose μ and $\lambda-\mu$ are functions in $\Lambda_{[\alpha,\beta]}$, either continuous or of bounded variation, where $\lambda:[\alpha,\beta]\to\mathbb{R}$ is such that $\lambda(\beta)-\lambda(\alpha)>0$. Let $x:[\alpha,\beta]\to(a,b)$, $a,b\in\overline{\mathbb{R}}$ be a continuous and monotonic function. If $f:(a,b)\to\mathbb{R}$ is a convex function, then the following inequalities hold:*

$$\mathfrak{J}(f,x,\lambda)\geq\mathfrak{J}(f,x,\mu)\geq 0. \qquad (3.19)$$

Proof. Since $\mathfrak{J}(f,x,\lambda)=\mathfrak{J}(f,x,\lambda-\mu+\mu)$, applying (3.16) to $\lambda-\mu$ and μ, we get

$$\mathfrak{J}(f,x,\lambda)=\mathfrak{J}(f,x,\lambda-\mu+\mu)\geq\mathfrak{J}(f,x,\lambda-\mu)+\mathfrak{J}(f,x,\mu)\geq\mathfrak{J}(f,x,\mu),$$

where the last inequality holds since $\mathfrak{J}(f,x,\lambda-\mu)\geq 0$, according to Jensen-Steffensen's inequality, applied to a convex function f and $\lambda-\mu\in\Lambda_{[\alpha,\beta]}$. Analogously, for a convex function f and $\mu\in\Lambda_{[\alpha,\beta]}$, the last inequality in (3.19) holds. □

Remark 3.4 Theorem 3.3 from [23] is accompanied with its integral variant in [23]. Now, following the same lines as in Remark 3.2, its statement follows easily from Theorem 3.7.

Theorem 3.7 provides us with the lower and the upper bound for Jensen-Steffensen's functional (3.15), by means of the non-weight functional of the same type. However, almost identical result which slightly differs from this, for it was given in the normalized form, was obtained in [23]. Thus this proof is comprehended as an alternative one and is consequently given within a remark.

Remark 3.5 Let us rewrite [23, Corollary 6] in a slightly generalized way:

Let λ be a function in $\Lambda_{[\alpha,\beta]}$. Suppose $x : [\alpha,\beta] \to (a,b)$, $a,b \in \overline{\mathbb{R}}$ is a continuous and monotonic function and $f : (a,b) \to \mathbb{R}$ is a convex function. If m and M are real constants defined by

$$m := \inf_{\alpha<t<\beta} \left\{ \frac{\lambda(t)-\lambda(\alpha)}{t-\alpha}, \frac{\lambda(\beta)-\lambda(t)}{\beta-t} \right\},$$

$$M := \sup_{\alpha<t<\beta} \left\{ \frac{\lambda(t)-\lambda(\alpha)}{t-\alpha}, \frac{\lambda(\beta)-\lambda(t)}{\beta-t} \right\},$$

then

$$M\mathfrak{J}(f,x) \geq \mathfrak{J}(f,x,\lambda) \geq m\mathfrak{J}(f,x), \tag{3.20}$$

where $\mathfrak{J}(f,x) := \int_\alpha^\beta f(x(t))dt - (\beta-\alpha)f\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta x(t)dt\right)$.

Alternative proof of [23, Corollary 6]. Let us prove the right-hand side inequality in (3.20). Let $\mu : [\alpha,\beta] \to \mathbb{R}$ be defined by $\mu(t) = mt$. The proof is based on the application of Theorem 3.7 to λ and μ because

$$\begin{aligned} \mathfrak{J}(f,x,\mu) &= \int_\alpha^\beta f(x(t))d(mt) - (m\beta - m\alpha)f\left(\frac{1}{m\beta - m\alpha}\int_\alpha^\beta x(t)d(mt)\right) \\ &= m\left(\int_\alpha^\beta f(x(t))dt - (\beta-\alpha)f\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta x(t)dt\right)\right) = m\mathfrak{J}(f,x). \end{aligned}$$

Namely, according to the definition of m is $m \leq \dfrac{\lambda(t)-\lambda(\alpha)}{t-\alpha}$ and $m \leq \dfrac{\lambda(t)-\lambda(\beta)}{t-\beta}$, which is equivalent to $\lambda(\alpha) - m\alpha \leq \lambda(t) - mt \leq \lambda(\beta) - m\beta$, i.e. $\lambda(\alpha) - \mu(\alpha) \leq \lambda(t) - \mu(t) \leq \lambda(\beta) - \mu(\beta)$, $t \in [\alpha,\beta]$, so $\lambda - \mu \in \Lambda_{[\alpha,\beta]}$. Furthermore, $m \geq 0$ implies $\mu(\alpha) \leq \mu(t) \leq \mu(\beta)$, $t \in [\alpha,\beta]$, hence $\mu \in \Lambda_{[\alpha,\beta]}$. The left-hand side inequality is proved analogously, by exchanging the roles of λ and μ. □

In the sequel we observe the functional (3.15) under the conditions of Theorem 3.5, that is, we obtain the results for (integral) Jensen-Steffensen-Boas' functional. We start with the integral variant of Theorem 3.2 under the conditions (3.14).

Theorem 3.8 *Let μ and $\lambda - \mu$ be functions in $\tilde{\Lambda}_{[\alpha,\beta]}$, either continuous or of bounded variation. Let $\alpha = \gamma_0 < \gamma_1 < \cdots < \gamma_k = \beta$, $k \geq 2$, be points in $[\alpha,\beta]$. Suppose $x : [\alpha,\beta] \to (a,b)$, $a,b \in \overline{\mathbb{R}}$ is a continuous function that is monotonic on each of the intervals $[\gamma_{i-1},\gamma_i]$, $i = 1,\ldots,k$. If $f : (a,b) \to \mathbb{R}$ is a convex function, then the following inequalities hold:*

$$\mathfrak{J}(f,x,\lambda) \geq \mathfrak{J}(f,x,\mu) \geq 0. \tag{3.21}$$

Proof. Follows the same lines as in the proof of Theorem 3.7, with Jensen-Steffensen-Boas' inequality applied instead of Jensen-Steffensen's, for $\mu, \lambda - \mu \in \tilde{\Lambda}_{[\alpha,\beta]}$. □

Remark 3.6 The existing corresponding integral result [23, Theorem 6] can now easily be proved by applying Theorem 3.8, according to Remark 3.2.

Theorem 3.8 provides us with the lower and upper bound for Jensen-Steffensen-Boas' functional, by means of the non-weight functional of the same type. Again, almost identical result [23, Corollary 8] is supplemented by an alternative proof.

Remark 3.7 Let us rewrite [23, Corollary 8] in a slightly generalized form:

> *Let λ be a function in $\tilde{\Lambda}_{[\alpha,\beta]}$. Let $\alpha=\gamma_0<\gamma_1<\cdots<\gamma_k=\beta$, $k\geq 2$, be points in $[\alpha,\beta]$. Suppose $x:[\alpha,\beta]\to(a,b)$, $a,b\in\overline{\mathbb{R}}$ is a continuous function that is monotonic on each of the intervals $[\gamma_{i-1},\gamma_i]$, $i=1,\dots,k$, and $f:(a,b)\to\mathbb{R}$ is a convex function. If m and M are real constants defined by*
>
> $$m:=\min_{i=1,\dots,k}\left\{\inf\left\{\frac{\lambda(t)-\lambda(\gamma_{i-1})}{t-\gamma_{i-1}},\frac{\lambda(\gamma_i)-\lambda(t)}{\gamma_i-t}:\gamma_{i-1}<t<\gamma_i\right\}\right\},$$
>
> $$M:=\max_{i=1,\dots,k}\left\{\sup\left\{\frac{\lambda(t)-\lambda(\gamma_{i-1})}{t-\gamma_{i-1}},\frac{\lambda(\gamma_i)-\lambda(t)}{\gamma_i-t}:\gamma_{i-1}<t<\gamma_i\right\}\right\},$$
>
> *then*
>
> $$M\mathfrak{J}(f,x)\geq\mathfrak{J}(f,x,\lambda)\geq m\mathfrak{J}(f,x),\tag{3.22}$$
>
> *where* $\mathfrak{J}(f,x):=\int_\alpha^\beta f(x(t))dt-(\beta-\alpha)f\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta x(t)dt\right)$.

Alternative proof of [23, Corollary 8]. Let us prove the right-hand side inequality in (3.22). Let $\mu:[\alpha,\beta]\to\mathbb{R}$ be defined by $\mu(t)=mt$. The proof is based on the application of Theorem 3.8 to λ and μ because

$$\begin{aligned}\mathfrak{J}(f,x,\mu)&=\int_\alpha^\beta f(x(t))d(mt)-(m\beta-m\alpha)f\left(\frac{1}{m\beta-m\alpha}\int_\alpha^\beta x(t)d(mt)\right)\\&=m\left(\int_\alpha^\beta f(x(t))dt-(\beta-\alpha)f\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta x(t)dt\right)\right)=m\mathfrak{J}(f,x).\end{aligned}$$

Namely, according to the definition of m is $m\leq\dfrac{\lambda(t)-\lambda(\gamma_{i-1})}{t-\gamma_{i-1}}$ and $m\leq\dfrac{\lambda(t)-\lambda(\gamma_i)}{t-\gamma_i}$, $i=1,\dots,k$, which is equivalent to $\lambda(\gamma_{i-1})-m\gamma_{i-1}\leq\lambda(t)-mt\leq\lambda(\gamma_i)-m\gamma_i$, i.e. $\lambda(\gamma_{i-1})-\mu(\gamma_{i-1})\leq\lambda(t)-\mu(t)\leq\lambda(\gamma_i)-\mu(\gamma_i)$, for all $t\in[\gamma_{i-1},\gamma_i]$, $i=1,\dots,k$. Hence $\lambda-\mu\in\tilde{\Lambda}_{[\alpha,\beta]}$. Furthermore, $m\geq 0$ implies $\mu(\gamma_{i-1})\leq\mu(t)\leq\mu(\gamma_i)$, for all $\gamma_{i-1}\leq t\leq\gamma_i$ hence $\mu\in\tilde{\Lambda}_{[\alpha,\beta]}$. The left-hand side inequality is proved analogously, by substituting the roles of λ and μ. □

3.1.3 Application to quasiarithmetic means

Here we illustrate the application of the established properties of Jensen-Steffensen's functional to improvement of one of the results from [155]. Namely, in that paper a functional of type (3.3) was investigated as well, only defined by means of weight quasiarithmetic mean. In order to deduce it, recall that for a continuous and strictly monotonic function $\chi : I \to J$, $I,J \subseteq \overline{\mathbb{R}}$, for $\mathbf{x} = (x_1,\dots,x_n) \in I^n$ and $\mathbf{p} = (p_1,\dots,p_n)$, $p_i \geq 0$ and $\sum_{i=1}^n p_i = P_n > 0$, a weight quasiarithmetic mean is defined by

$$M_\chi(\mathbf{x},\mathbf{p}) = \chi^{-1}\left(\sum_{i=1}^n p_i\chi(x_i)\right). \tag{3.23}$$

In particular, if $\chi(x) = x$, then (3.23) defines a weight arithmetic mean, whereas for $\chi(x) = \log x$ we have a weight geometric mean defined.

Let $\chi : I \to I$ and $\psi : J \to J$, with $I,J \subseteq \overline{\mathbb{R}}$ be continuous and strictly monotonic functions. Function $f : I \to J$ is said to be (M_χ, M_ψ)-convex if for all $x,y \in I$ and $\lambda \in [0,1]$ the following inequality holds:

$$f(\chi^{-1}((1-\lambda)\chi(x)+\lambda\chi(y))) \leq \psi^{-1}((1-\lambda)\psi(f(x))+\lambda\psi(f(y))). \tag{3.24}$$

In particular, when $\chi(x) = \psi(x) = x$, inequality (3.24) defines a convex function f, whereas for $\chi(x) = x$ and $\psi(x) = \log x$ a logarithmic convex function f is defined. Moreover, if f is an (M_χ, M_ψ)-convex function, then $g := \psi \circ f \circ \chi^{-1}$ is convex (for details, see [164]).

If we observe ψ as an identity function, then $\psi(x) = x$ implies that $f \circ \chi^{-1}$ is a convex function. Function f is then simply said to be an M_χ-convex function. Furthermore, if $\mathbf{x} = (x_1,\dots,x_n)$ is a monotonic n-tuple and $\chi(\mathbf{x}) := (\chi(x_1),\dots,\chi(x_n))$, then $\chi(\mathbf{x})$ is also a monotonic n-tuple. If additionally the n-tuple $\mathbf{p} = (p_1,\dots,p_n)$ satisfies the Steffensen's conditions (3.1), then inequality

$$(f\circ\chi^{-1})\left(\frac{1}{P_n}\sum_{i=1}^n p_i\chi(x_i)\right) \leq \frac{1}{P_n}\sum_{i=1}^n p_i(f\circ\chi^{-1})(\chi(x_i)) = \frac{1}{P_n}\sum_{i=1}^n p_i f(x_i) \tag{3.25}$$

corresponds to Jensen-Steffensen's inequality (3.2). It was proved in [155] that $\min\{x_1,\dots, x_n\} \leq \frac{1}{P_n}\sum_{i=1}^n p_i\chi(x_i) \leq \max\{x_1,\dots,x_n\}$, so the left-hand side expression in (3.25) is well defined. Similarly as before, inequality (3.25) provides the following functional:

$$\mathfrak{J}_\chi(f,\mathbf{x},\mathbf{p}) = \sum_{i=1}^n p_i f(x_i) - P_n f\left(\chi^{-1}\left(\frac{1}{P_n}\sum_{i=1}^n p_i\chi(x_i)\right)\right), \tag{3.26}$$

that corresponds to Jensen-Steffensen's functional (3.3) when the substitutions $f \leftrightarrow f\circ\chi^{-1}$ and $x_i \leftrightarrow \chi(x_i)$ are taken into consideration.

Hence for the functional (3.26) the following results are established.

Corollary 3.2 *Let I be an interval in $\mathbb{R}$ and let $\mathbf{x} = (x_1,\dots,x_n) \in I^n$. Suppose $\mathbf{p} = (p_1,\dots,p_n)$ and $\mathbf{q} = (q_1,\dots,q_n)$ are real n-tuples, such that $\sum_{i=1}^n p_i = P_n > 0$, $\sum_{i=1}^n q_i = Q_n > 0$ and $\frac{1}{P_n}\sum_{i=1}^n p_i x_i$, $\frac{1}{Q_n}\sum_{i=1}^n q_i x_i \in I$. Assume $\chi : I \to I$ is a continuous and strictly monotonic function. If $f : I \to \mathbb{R}$ is an M_χ-convex function, then functional* (3.26) *is superadditive on the set of the described n-tuples* $\mathbf{p}$ *and* $\mathbf{q}$, *that is*

$$\mathfrak{J}_\chi(f,\mathbf{x},\mathbf{p}+\mathbf{q}) \geq \mathfrak{J}_\chi(f,\mathbf{x},\mathbf{p}) + \mathfrak{J}_\chi(f,\mathbf{x},\mathbf{q}) \geq 0. \tag{3.27}$$

Proof. Since $f\circ\chi^{-1}$ is a convex function, the proof follows the same lines as in Theorem 3.1. □

Corollary 3.3 *Let* $\mathbf{q}$ *and* $\mathbf{p}-\mathbf{q}$ *be two n-tuples in* $\mathscr{P}_n$, *where* $\mathbf{p}$ *is a real n-tuple, such that* $\sum_{i=1}^{n} p_i = P_n > 0$. *Let* $\mathbf{x} \in I^n$, $I \subseteq \mathbb{R}$ *be a monotonic n-tuple. Suppose* $\chi : I \to I$ *is a continuous and strictly monotonic function. If* $f : I \to \mathbb{R}$ *is an* M_χ*-convex function, then the following inequalities hold:*

$$\mathfrak{J}_\chi(f,\mathbf{x},\mathbf{p}) \geq \mathfrak{J}_\chi(f,\mathbf{x},\mathbf{q}) \geq 0. \tag{3.28}$$

Proof. Since $\chi(\mathbf{x})$ is a monotonic n-tuple and $f \circ \chi^{-1}$ is a convex function, the proof follows the same lines as in Theorem 3.2. □

Functional (3.26) possesses the lower and the upper bound expressed by the non-weight functional of the same type.

Corollary 3.4 *Let* $\mathbf{p}$, *m and M be as in Remark 3.3. Suppose* $\chi : I \to I$, $I \subseteq \mathbb{R}$ *is a continuous and strictly monotonic function. If* $f : I \to \mathbb{R}$ *is an* M_χ*-convex function and* $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ *is a monotonic n-tuple, then*

$$M\mathfrak{J}_\chi(f,\mathbf{x}) \geq \mathfrak{J}_\chi(f,\mathbf{x},\mathbf{p}) \geq m\mathfrak{J}_\chi(f,\mathbf{x}), \tag{3.29}$$

where $\mathfrak{J}_\chi(f,\mathbf{x}) = \sum_{i=1}^{n} f(x_i) - nf\left(\chi^{-1}\left(\frac{\sum_{i=1}^{n}\chi(x_i)}{n}\right)\right)$.

Proof. Follows from the previous two corollaries, analogously as in Remark 3.3. □

Remark 3.8 Let us observe the left inequality in (3.29). As in Remark 3.3, it is defined: $M = \max\limits_{1\leq k\leq n-1}\left\{\frac{P_k}{k}, \frac{P_n-P_k}{n-k}\right\}$. Let $\alpha = \max\{p_1,\ldots,p_n\}$. Obviously, $P_k \leq k\alpha$, so $\frac{P_k}{k} \leq \alpha$ and $\frac{P_n-P_k}{n-k} \leq \alpha$. It follows that $M \leq \alpha$, which is a better upper bound than the one obtained in [155, Teorem 3.1.]

In order to obtain analogous integral results, we need to transform (3.15) by means of M_χ-convex function f and the substitutions of convex functions $f \leftrightarrow f\circ\chi^{-1}$, as well as of monotonic functions $x \leftrightarrow \chi\circ x$. The following form of the functional is obtained:

$$\begin{aligned}\mathfrak{J}_\chi(f,x,\lambda) := & \int_\alpha^\beta f(x(t))d\lambda(t) \\ & -(\lambda(\beta)-\lambda(\alpha))f\left(\chi^{-1}\left(\frac{1}{\lambda(\beta)-\lambda(\alpha)}\int_\alpha^\beta \chi(x(t))d\lambda(t)\right)\right).\end{aligned} \tag{3.30}$$

Remark 3.9 By applying the afore described substitutions ($f \leftrightarrow f\circ\chi^{-1}$ and $x \leftrightarrow \chi\circ x$) for χ being a continuous and strictly monotonic function and f being an M_χ-convex function, all integral results from Section 3.1.2 can then be applied to the functional (3.30).

3.2 Superadditivity of Jensen-Mercer's functional

In the sequel we prove superadditivity of discrete Jensen-Mercer's functional which we then apply to prove that this functional is increasing on the set of nonnegative real n-tuples, as well as on the set of the real n-tuples that satisfy the Steffensen's conditions. We then take into consideration the inequalities of the form

$$M\mathfrak{M}(f,\mathbf{x},\mathbf{q}) \geq \mathfrak{M}(f,\mathbf{x},\mathbf{p}) \geq m\mathfrak{M}(f,\mathbf{x},\mathbf{q}),$$

which were recently observed in [24], give a comparison with the results obtained in [24], and finally obtain a new type of the bounds for Jensen-Mercer's functional, expressed by means of the non-weight functional of the same type.

All of the obtained discrete results are accompanied by their integral variants deduced from the integral form of Jensen-Mercer's inequality established in [49], in the first place, and then the integral form of Jensen-Mercer's inequality under the conditions of the Steffensen-Boas' theorem which was proved in [24].

The contents of Section 3.2 corresponds for the most part to the contents of the published paper [112].

3.2.1 Discrete Jensen-Mercer's functional

As we already have mentioned (Theorem 1.8), A. McD. Mercer proved that if $\mathbf{x} = (x_1,\ldots,x_n) \in [a,b]^n$, $[a,b] \subset \mathbb{R}$ and $\mathbf{p} = (p_1,\ldots,p_n)$ is such that $p_i \geq 0$ and $P_n = \sum_{i=1}^n p_i > 0$, then the inequality

$$f\left(a+b-\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) \leq f(a)+f(b)-\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i) \tag{3.31}$$

holds for every convex function $f:[a,b]\to\mathbb{R}$. As usual, the set of the described nonnegative real n-tuples $\mathbf{p}$ will be denoted by $\mathscr{P}_n^0$.

Discrete Jensen-Mercer's functional $\mathfrak{M}(f,\mathbf{x},\mathbf{p})$ is assigned to inequality (3.31) as follows:

$$\mathfrak{M}(f,\mathbf{x},\mathbf{p}) = P_n[f(a)+f(b)] - \sum_{i=1}^n p_i f(x_i) - P_n f\left(a+b-\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right). \tag{3.32}$$

If f is convex, then (3.31) implies $\mathfrak{M}(f,\mathbf{x},\mathbf{p}) \geq 0$, for all $\mathbf{p} \in \mathscr{P}_n^0$. For fixed f and $\mathbf{x}$, the functional $\mathfrak{M}(f,\mathbf{x},\cdot)$ can be observed as a function on $\mathscr{P}_n^0$.

Nevertheless, superadditivity property of (3.32), that we are about to prove here, holds in a more general environment.

Theorem 3.9 *Let $[a,b]$ be an interval in $\mathbb{R}$ and let $\mathbf{x} = (x_1,\ldots,x_n) \in [a,b]^n$. Suppose $\mathbf{p}$ and $\mathbf{q}$ are real n-tuples, such that $\sum_{i=1}^n p_i = P_n > 0$, $\sum_{i=1}^n q_i = Q_n > 0$ and $a+b-$*

$\frac{1}{P_n}\sum_{i=1}^n p_i x_i$, $a+b-\frac{1}{Q_n}\sum_{i=1}^n q_i x_i \in [a,b]$. *If* $f:[a,b]\to\mathbb{R}$ *is a convex function, then functional* (3.32) *is superadditive on the set of the described real n-tuples* $\mathbf{p}$ *and* $\mathbf{q}$, *that is*

$$\mathfrak{M}(f,\mathbf{x},\mathbf{p}+\mathbf{q}) \geq \mathfrak{M}(f,\mathbf{x},\mathbf{p})+\mathfrak{M}(f,\mathbf{x},\mathbf{q}). \tag{3.33}$$

Proof. It follows from the definition of the functional (3.32) that

$$\begin{aligned}\mathfrak{M}(f,\mathbf{x},\mathbf{p}+\mathbf{q}) &= (P_n+Q_n)[f(a)+f(b)]-\sum_{i=1}^n (p_i+q_i)f(x_i)\\ &\quad -(P_n+Q_n)f\left(a+b-\frac{\sum_{i=1}^n (p_i+q_i)x_i}{P_n+Q_n}\right)\\ &= P_n[f(a)+f(b)]+Q_n[f(a)+f(b)]-\sum_{i=1}^n p_i f(x_i)-\sum_{i=1}^n q_i f(x_i)\\ &\quad -(P_n+Q_n)f\left(a+b-\frac{\sum_{i=1}^n (p_i+q_i)x_i}{P_n+Q_n}\right),\end{aligned} \tag{3.34}$$

whereas convexity of f yields

$$\begin{aligned}&f\left(a+b-\frac{\sum_{i=1}^n (p_i+q_i)x_i}{P_n+Q_n}\right)=f\left(\frac{\sum_{i=1}^n (p_i+q_i)(a+b-x_i)}{P_n+Q_n}\right)\\ &= f\left(\frac{P_n}{P_n+Q_n}\frac{\sum_{i=1}^n p_i(a+b-x_i)}{P_n}+\frac{Q_n}{P_n+Q_n}\frac{\sum_{i=1}^n q_i(a+b-x_i)}{Q_n}\right)\\ &\leq \frac{P_n}{P_n+Q_n}f\left(a+b-\frac{\sum_{i=1}^n p_i x_i}{P_n}\right)+\frac{Q_n}{P_n+Q_n}f\left(a+b-\frac{\sum_{i=1}^n q_i x_i}{Q_n}\right).\end{aligned} \tag{3.35}$$

Finally, (3.34) and (3.35) yield

$$\begin{aligned}\mathfrak{M}(f,\mathbf{x},\mathbf{p}+\mathbf{q}) &\geq P_n[f(a)+f(b)]+Q_n[f(a)+f(b)]-\sum_{i=1}^n p_i f(x_i)-\sum_{i=1}^n q_i f(x_i)\\ &\quad -P_n f\left(a+b-\frac{\sum_{i=1}^n p_i x_i}{P_n}\right)-Q_n f\left(a+b-\frac{\sum_{i=1}^n q_i x_i}{Q_n}\right)\\ &= \mathfrak{M}(f,\mathbf{x},\mathbf{p})+\mathfrak{M}(f,\mathbf{x},\mathbf{q}),\end{aligned}$$

and the proof is concluded. □

Functional (3.32) is increasing on the set $\mathscr{P}_n^0$, which we prove in the following theorem.

Theorem 3.10 *Let* $[a,b]$ *be an interval in* $\mathbb{R}$ *and* $\mathbf{x}=(x_1,\ldots,x_n)\in[a,b]^n$. *Suppose* $\mathbf{p}=(p_1,\ldots,p_n)$ *and* $\mathbf{q}=(q_1,\ldots,q_n)$ *are two n-tuples in* $\mathscr{P}_n^0$, *such that* $\mathbf{p}\geq\mathbf{q}$, *(i.e.* $p_i\geq q_i$, $i=1,\ldots,n$.*) If* $f:[a,b]\to\mathbb{R}$ *is a convex function, then the following inequalities hold:*

$$\mathfrak{M}(f,\mathbf{x},\mathbf{p})\geq\mathfrak{M}(f,\mathbf{x},\mathbf{q})\geq 0, \tag{3.36}$$

that is, $\mathfrak{M}(f,\mathbf{x},\cdot)$ *is increasing on* $\mathscr{P}_n^0$.

Proof. If we write $\mathfrak{M}(f,\mathbf{x},\mathbf{p})$ as $\mathfrak{M}(f,\mathbf{x},\mathbf{p}-\mathbf{q}+\mathbf{q})$ and then apply the superadditivity property of (3.32) to $\mathbf{p}-\mathbf{q}$ and $\mathbf{q}$, we get

$$\mathfrak{M}(f,\mathbf{x},\mathbf{p})=\mathfrak{M}(f,\mathbf{x},\mathbf{p}-\mathbf{q}+\mathbf{q})\geq\mathfrak{M}(f,\mathbf{x},\mathbf{p}-\mathbf{q})+\mathfrak{M}(f,\mathbf{x},\mathbf{q})\geq\mathfrak{M}(f,\mathbf{x},\mathbf{q}),$$

where we applied Jensen-Mercer's inequality in order to prove the last inequality: convexity of f and $\mathbf{p}-\mathbf{q}\in\mathscr{P}_n^0$ imply $\mathfrak{M}(f,\mathbf{x},\mathbf{p}-\mathbf{q})\geq 0$. In a similar way, convexity of f and $\mathbf{q}\in\mathscr{P}_n^0$ imply the last inequality in (3.36). □

In the following corollary the lower and the upper bound for (3.32) by means of the non-weight functional of the same type are obtained.

Corollary 3.5 *Let* $\mathbf{p}$, $\mathbf{x}$, f *and functional* $\mathfrak{M}$ *be as in Theorem 3.10. Then the following inequalities hold:*

$$\max_{1\leq i\leq n}\{p_i\}\mathfrak{M}(f,\mathbf{x})\geq\mathfrak{M}(f,\mathbf{x},\mathbf{p})\geq\min_{1\leq i\leq n}\{p_i\}\mathfrak{M}(f,\mathbf{x}),\tag{3.37}$$

where $\mathfrak{M}(f,\mathbf{x})=n[f(a)+f(b)]-\sum_{i=1}^n f(x_i)-nf\left(a+b-\frac{1}{n}\sum_{i=1}^n x_i\right)$.

Proof. Let $\mathbf{p}_{\min}=\left(\min_{1\leq i\leq n}\{p_i\},\ldots,\min_{1\leq i\leq n}\{p_i\}\right)$. Then $\mathbf{p}\geq\mathbf{p}_{\min}$, for all $\mathbf{p}\in\mathscr{P}_n^0$. Applying Theorem 3.10 it follows

$$\mathfrak{M}(f,\mathbf{x},\mathbf{p})\geq\mathfrak{M}(f,\mathbf{x},\mathbf{p}_{\min}).$$

On the other hand,

$$\mathfrak{M}(f,\mathbf{x},\mathbf{p}_{\min})=\min_{1\leq i\leq n}\{p_i\}\left\{n[f(a)+f(b)]-\sum_{i=1}^n f(x_i)-nf\left(a+b-\frac{1}{n}\sum_{i=1}^n x_i\right)\right\},$$

that is, $\mathfrak{M}(f,\mathbf{x},\mathbf{p}_{\min})=\min_{1\leq i\leq n}\{p_i\}\mathfrak{M}(f,\mathbf{x})$, which was to obtain. The left inequality in (3.37) is proved analogously, by substituting the roles of minimum and maximum. □

In [24], the following result concerning the normalized Jensen-Mercer's functional $\mathfrak{M}_n(f,\mathbf{x},\cdot)$ was proved.

Theorem 3.11 (SEE [24]) *Let* $\mathbf{p}=(p_1,\ldots,p_n)$ *and* $\mathbf{q}=(q_1,\ldots,q_n)$ *be two n-tuples in* $\mathscr{P}_n^0$. *Suppose m and M are real constants such that*

$$m\geq 0,\quad p_i-mq_i\geq 0,\quad Mq_i-p_i\geq 0,\quad i=1,\ldots,n.$$

If $f:[a,b]\to\mathbb{R}$ *is a convex function and if* $\mathbf{x}=(x_1,\ldots,x_n)$ *is an n-tuple in* $[a,b]^n$, *then the following inequalities hold:*

$$M\mathfrak{M}_n(f,\mathbf{x},\mathbf{q})\geq\mathfrak{M}_n(f,\mathbf{x},\mathbf{p})\geq m\mathfrak{M}_n(f,\mathbf{x},\mathbf{q}).$$

Remark 3.10 The statement of Theorem 3.11 follows easily by applying Theorem 3.10, as we explain in the sequel. Namely, for $\mathbf{p},\mathbf{q}\in\mathscr{P}_n^0$ and real constants m and M, such that $\mathbf{p}-m\mathbf{q}$ and $M\mathbf{q}-\mathbf{p}\in\mathscr{P}_n^0$, for a convex function $f:[a,b]\to\mathbb{R}$, $[a,b]\subseteq\mathbb{R}$ and $\mathbf{x}=(x_1,\dots,x_n)\in[a,b]^n$, Theorem 3.10 yields

$$\mathfrak{M}(f,\mathbf{x},\mathbf{p})\geq\mathfrak{M}(f,\mathbf{x},\mathbf{p}-m\mathbf{q})+\mathfrak{M}(f,\mathbf{x},m\mathbf{q})\geq m\mathfrak{M}(f,\mathbf{x},\mathbf{q}).$$

Similarly,

$$\mathfrak{M}(f,\mathbf{x},\mathbf{p})\leq M\mathfrak{M}(f,\mathbf{x},\mathbf{q}),$$

i.e.

$$M\mathfrak{M}(f,\mathbf{x},\mathbf{q})\geq\mathfrak{M}(f,\mathbf{x},\mathbf{p})\geq m\mathfrak{M}(f,\mathbf{x},\mathbf{q})$$

which is indeed the statement of Theorem 3.11, only in a slightly generalized – non-normalized form.

3.2.2 Integral Jensen-Mercer's functional

In order to deduce the integral form of Jensen-Mercer's functional, we use as a starting point the integral variant of Jensen-Mercer's inequality. It was proved in [49] that under the assumption of $(\Omega,\mathscr{A},\mu)$ being a probability measure space and $x:\Omega\to[a,b]$, $-\infty<a<b<\infty$ being a measurable function the inequality

$$f\left(a+b-\int_{\Omega}x\,\mathrm{d}\mu\right)\leq f(a)+f(b)-\int_{\Omega}f(x)\,\mathrm{d}\mu \tag{3.38}$$

holds for any continuous and convex function $f:[a,b]\to\mathbb{R}$.

It can be proved analogously that for a measure space $(\Omega,\mathscr{A},\mu)$, where $0<\mu(\Omega)<\infty$, the integral Jensen-Mercer's inequality

$$f\left(a+b-\frac{1}{\mu(\Omega)}\int_{\Omega}x\,\mathrm{d}\mu\right)\leq f(a)+f(b)-\frac{1}{\mu(\Omega)}\int_{\Omega}f(x)\,\mathrm{d}\mu \tag{3.39}$$

holds. In particular, for $\Omega=[\alpha,\beta]$, $-\infty<\alpha<\beta<\infty$ and $\lambda:[\alpha,\beta]\to\mathbb{R}$ is an increasing function, such that $\lambda(\beta)\neq\lambda(\alpha)$, inequality (3.39) assumes the following form:

$$f\left(a+b-\tfrac{1}{\lambda(\beta)-\lambda(\alpha)}\int_{\alpha}^{\beta}x(t)\,\mathrm{d}\lambda(t)\right)\leq f(a)+f(b)-\tfrac{1}{\lambda(\beta)-\lambda(\alpha)}\int_{\alpha}^{\beta}f(x(t))\,\mathrm{d}\lambda(t). \tag{3.40}$$

If we write $\lambda(\beta)-\lambda(\alpha):=\lambda_{\alpha}^{\beta}$, inequality (3.40) can be rewritten as

$$f\left(\tfrac{1}{\lambda_{\alpha}^{\beta}}\int_{\alpha}^{\beta}(a+b-x(t))\,\mathrm{d}\lambda(t)\right)\leq\tfrac{1}{\lambda_{\alpha}^{\beta}}\int_{\alpha}^{\beta}(f(a)+f(b)-f(x(t)))\,\mathrm{d}\lambda(t). \tag{3.41}$$

We now deduce the integral Jensen-Mercer's functional $\mathfrak{M}(f,x,\lambda)$ as follows:

$$\mathfrak{M}(f,x,\lambda)=\lambda_{\alpha}^{\beta}[f(a)+f(b)]-\int_{\alpha}^{\beta}f(x(t))\,\mathrm{d}\lambda(t)-\lambda_{\alpha}^{\beta}f\left(a+b-\frac{1}{\lambda_{\alpha}^{\beta}}\int_{\alpha}^{\beta}x(t)\,\mathrm{d}\lambda(t)\right). \tag{3.42}$$

If f is a convex function, then $\mathfrak{M}(f,x,\lambda) \geq 0$, which is inherited from (3.41). In order to obtain the results on the bounds of the functional $\mathfrak{M}(f,x,\cdot)$, we are going to observe it as a function on the set of the (increasing) functions λ, such that $\lambda(\beta) \neq \lambda(\alpha)$. Nevertheless, superadditivity of (3.42) is again proved in a more generalized environment.

Theorem 3.12 *Let $\lambda,\mu : [\alpha,\beta] \to \mathbb{R}$, $-\infty < \alpha < \beta < \infty$ be such that $\lambda(\beta) > \lambda(\alpha)$ and $\mu(\beta) > \mu(\alpha)$ and let $x : [\alpha,\beta] \to [a,b]$, $-\infty < a < b < \infty$ be a continuous function. If $f : [a,b] \to \mathbb{R}$ is a continuous and convex function, then functional $\mathfrak{M}(f,x,\cdot)$, defined by* (3.42) *is superadditive on the set of the described functions λ and μ, that is*

$$\mathfrak{M}(f,x,\lambda+\mu) \geq \mathfrak{M}(f,x,\lambda) + \mathfrak{M}(f,x,\mu). \tag{3.43}$$

Proof. It follows from the definition of (3.42) that

$$\begin{aligned}\mathfrak{M}(f,x,\lambda+\mu) &= \left(\lambda_\alpha^\beta + \mu_\alpha^\beta\right)[f(a)+f(b)] - \int_\alpha^\beta f(x(t))\,\mathrm{d}(\lambda+\mu)(t) \\ &\quad - \left(\lambda_\alpha^\beta + \mu_\alpha^\beta\right) \cdot f\left(a+b-\frac{1}{\lambda_\alpha^\beta+\mu_\alpha^\beta}\int_\alpha^\beta x(t)\,\mathrm{d}(\lambda+\mu)(t)\right),\end{aligned} \tag{3.44}$$

whereas convexity of f yields

$$\begin{aligned}& f\left(a+b-\frac{1}{\lambda_\alpha^\beta+\mu_\alpha^\beta}\int_\alpha^\beta x(t)\,\mathrm{d}(\lambda+\mu)(t)\right) \\ &= f\left(\frac{\lambda_\alpha^\beta}{\lambda_\alpha^\beta+\mu_\alpha^\beta}\cdot\frac{\int_\alpha^\beta (a+b-x(t))\,\mathrm{d}\lambda(t)}{\lambda_\alpha^\beta} + \frac{\mu_\alpha^\beta}{\lambda_\alpha^\beta+\mu_\alpha^\beta}\cdot\frac{\int_\alpha^\beta (a+b-x(t))\,\mathrm{d}\mu(t)}{\mu_\alpha^\beta}\right) \\ &\leq \frac{\lambda_\alpha^\beta}{\lambda_\alpha^\beta+\mu_\alpha^\beta}\cdot f\left(\frac{\int_\alpha^\beta (a+b-x(t))\,\mathrm{d}\lambda(t)}{\lambda_\alpha^\beta}\right) + \frac{\mu_\alpha^\beta}{\lambda_\alpha^\beta+\mu_\alpha^\beta}\cdot f\left(\frac{\int_\alpha^\beta (a+b-x(t))\,\mathrm{d}\mu(t)}{\mu_\alpha^\beta}\right).\end{aligned} \tag{3.45}$$

Finally, (3.44) and (3.45) yield

$$\begin{aligned}&\mathfrak{M}(f,x,\lambda+\mu) \\ &\geq \lambda_\alpha^\beta[f(a)+f(b)] - \int_\alpha^\beta f(x(t))\,\mathrm{d}\lambda(t) - \lambda_\alpha^\beta \cdot f\left(a+b-\frac{\int_\alpha^\beta x(t)\,\mathrm{d}\lambda(t)}{\lambda_\alpha^\beta}\right) \\ &\quad + \mu_\alpha^\beta[f(a)+f(b)] - \int_\alpha^\beta f(x(t))\,\mathrm{d}\mu(t) - \mu_\alpha^\beta \cdot f\left(a+b-\frac{\int_\alpha^\beta x(t)\,\mathrm{d}\mu(t)}{\mu_\alpha^\beta}\right) \\ &= \mathfrak{M}(f,x,\lambda) + \mathfrak{M}(f,x,\mu),\end{aligned}$$

which concludes the proof. □

The extra conditions on functions λ and μ are needed in the sequel.

Theorem 3.13 *Let $\lambda,\mu:[\alpha,\beta]\to\mathbb{R}$, $-\infty<\alpha<\beta<\infty$, be such that $\lambda(\beta)>\lambda(\alpha)$ and $\mu(\beta)>\mu(\alpha)$. Suppose $x:[\alpha,\beta]\to[a,b]$, $-\infty<a<b<\infty$, is a convex function and $f:[a,b]\to\mathbb{R}$ is a continuous and convex function. If μ and $\rho:=\lambda-\mu$ are increasing functions, then the following inequalities hold:*

$$\mathfrak{M}(f,x,\lambda)\geq\mathfrak{M}(f,x,\mu)\geq 0. \tag{3.46}$$

Proof. Since μ and $\rho:=\lambda-\mu$ are increasing functions, the same holds for λ. If we write $\mathfrak{M}(f,x,\lambda)=\mathfrak{M}(f,x,\lambda-\mu+\mu)$, by applying Theorem 3.12 we get

$$\mathfrak{M}(f,x,\lambda)=\mathfrak{M}(f,x,\lambda-\mu+\mu)\geq\mathfrak{M}(f,x,\lambda-\mu)+\mathfrak{M}(f,x,\mu).$$

For a convex f and for an increasing $\lambda-\mu$, it follows from (3.41) that $\mathfrak{M}(f,x,\lambda-\mu)\geq 0$. Hence $\mathfrak{M}(f,x,\lambda)\geq\mathfrak{M}(f,x,\mu)$. For the same reason is for an increasing function μ the right-hand side inequality in (3.46) satisfied. □

Remark 3.11 Theorem 3.11 from [24] that we concerned in the previous subsection has its integral version [24, Theorem 5]. According to Remark 3.10, when applying Theorem 3.13, the statement of this integral theorem can be easily obtained.

Another application of Theorem 3.13 is obtaining the lower and the upper bound for the functional (3.42), by means of the non-weight functional of the same type. However, almost identical result was obtained in [24, Corollary 5], except for its normalized form. Our proof can be considered as an alternative one and is thus given within a remark.

Remark 3.12 Regarding the previous considerations, we adjust the statement of the [24, Corollary 5].

Let $\lambda:[\alpha,\beta]\to\mathbb{R}$ be a nondecreasing function, such that $\lambda(\beta)\neq\lambda(\alpha)$. Suppose $x:[\alpha,\beta]\to[a,b]$, $-\infty<a<b<\infty$, is a continuous and $f:[a,b]\to\mathbb{R}$ is a continuous and convex function. If m and M are real constants defined by

$$m:=\inf_{\alpha<t<\beta}\left\{\inf\left\{\frac{\lambda(t)-\lambda(s)}{t-s},\ \alpha\leq s\leq\beta,\ s\neq t\right\}\right\},$$

$$M:=\sup_{\alpha<t<\beta}\left\{\sup\left\{\frac{\lambda(t)-\lambda(s)}{t-s},\ \alpha\leq s\leq\beta,\ s\neq t\right\}\right\},$$

then the following inequalities hold:

$$M\mathfrak{M}(f,x)\geq\mathfrak{M}(f,x,\lambda)\geq m\mathfrak{M}(f,x), \tag{3.47}$$

where

$$\mathfrak{M}(f,x):=(\beta-\alpha)[f(a)+f(b)]-\int_\alpha^\beta f(x(t))\,\mathrm{d}t-(\beta-\alpha)f\left(a+b-\tfrac{1}{\beta-\alpha}\int_\alpha^\beta x(t)\,\mathrm{d}t\right).$$

Alternative proof. Let us prove the right-hand side inequality in (3.47). Let $\mu : [\alpha, \beta] \to \mathbb{R}$ be defined by $\mu(z) = mz$. Then Theorem 3.13 can be applied to functions λ and μ because

$$\begin{aligned} \mathfrak{M}(f,x,\mu) &= (m\beta - m\alpha)[f(a)+f(b)] - \int_\alpha^\beta f(x(t))\,\mathrm{d}(mt) \\ &\quad -(m\beta - m\alpha) f\left(a+b-\frac{1}{m\beta - m\alpha}\int_\alpha^\beta x(t)\,\mathrm{d}(mt)\right) \\ &= m\mathfrak{M}(f,x). \end{aligned} \tag{3.48}$$

Namely, according to the definition of m is $m \le \dfrac{\lambda(t)-\lambda(s)}{t-s}$. Function $\rho := \lambda - \mu$ is nondecreasing since $\lambda(t) - mt - (\lambda(s) - ms) \ge 0$, for $t > s$, and $\lambda(s) - ms - (\lambda(t) - mt) \ge 0$, for $s > t$. Theorem 3.13 then implies

$$\mathfrak{M}(f,x,\lambda) = \mathfrak{M}(f,x,\lambda - \mu + \mu) \ge \mathfrak{M}(f,x,\mu). \tag{3.49}$$

The left-hand side inequality in (3.47) is proved similarly, by exchanging the roles of λ and μ. □

3.2.3 Discrete Jensen-Mercer's functional under the Steffensen's conditions

As it was pointed out by Theorem 1.9, Jensen-Mercer's inequality (3.31) holds under the Steffensen's conditions as well. Precisely, if $[a,b]$ is an interval in $\mathbb{R}$, $\mathbf{x} \in [a,b]^n$ is a monotonic n-tuple, either nondecreasing or nonincreasing and $\mathbf{p}$ is a real n-tuple such that

$$P_n > 0 \quad \text{and} \quad 0 \le P_k \le P_n, \quad 1 \le k \le n-1, \tag{3.50}$$

where $P_k = \sum_{i=1}^k p_i$, $k = 1,2,\dots,n$, then inequality (3.31) holds for any convex function $f : [a,b] \to \mathbb{R}$. Jensen-Mercer's functional (3.32) is now observed under the Steffensen's conditions. As in previous considerations of this type, the set of all real n-tuples $\mathbf{p}$ that satisfy (3.50) is denoted by $\mathscr{P}_n$. For fixed f and $\mathbf{x}$, functional $\mathfrak{M}(f,\mathbf{x},\cdot)$ is observed as a function on $\mathscr{P}_n$. If f is convex, then $\mathfrak{M}(f,\mathbf{x},\mathbf{p}) \ge 0$, for all $\mathbf{p} \in \mathscr{P}_n$.

The superadditivity property of the functional (3.32), established in Theorem 3.9 is now applied to prove its increase on $\mathscr{P}_n$.

Theorem 3.14 *Let* $\mathbf{q}$ *and* $\mathbf{p}-\mathbf{q}$ *be two n-tuples in* $\mathscr{P}_n$, *where* $\mathbf{p}$ *is a real n-tuple, such that* $\sum_{i=1}^n p_i = P_n > 0$. *Suppose* $\mathbf{x} \in [a,b]^n$, $[a,b] \subseteq \mathbb{R}$, *is a monotonic, either nondecreasing or nonincreasing n-tuple. If* $f : [a,b] \to \mathbb{R}$ *is a convex function, then the following inequalities hold:*

$$\mathfrak{M}(f,\mathbf{x},\mathbf{p}) \ge \mathfrak{M}(f,\mathbf{x},\mathbf{q}) \ge 0. \tag{3.51}$$

Proof. If we write $\mathfrak{M}(f,\mathbf{x},\mathbf{p}) = \mathfrak{M}(f,\mathbf{x},\mathbf{p}-\mathbf{q}+\mathbf{q})$ and then apply (3.32) to $\mathbf{p}-\mathbf{q}$ and $\mathbf{q}$, we get

$$\mathfrak{M}(f,\mathbf{x},\mathbf{p}) = \mathfrak{M}(f,\mathbf{x},\mathbf{p}-\mathbf{q}+\mathbf{q}) \ge \mathfrak{M}(f,\mathbf{x},\mathbf{p}-\mathbf{q}) + \mathfrak{M}(f,\mathbf{x},\mathbf{q}) \ge \mathfrak{M}(f,\mathbf{x},\mathbf{q}),$$

where the last inequality follows from $\mathfrak{M}(f,\mathbf{x},\mathbf{p}-\mathbf{q}) \geq 0$, which is satisfied due to Jensen-Mercer's inequality under the Steffensen's conditions. The same argument proves the last inequality in (3.51), too. □

In [24], the following result on the comparative inequalities for the normalized Jensen-Mercer's functional $\mathfrak{M}_n(f,\mathbf{x},\cdot)$, under the Steffensen's conditions was proved.

Theorem 3.15 (SEE [24]) *Let $\mathbf{p}=(p_1,\ldots,p_n)$ and $\mathbf{q}=(q_1,\ldots,q_n)$ be two n-tuples such that*

$$0 \leq P_k, Q_k \leq 1, \quad k=1,\ldots,n-1, \quad P_n = Q_n = 1.$$

Let m and M be any real constants such that

$$m \geq 0, \quad P_k - mQ_k \geq 0, \quad (1-P_k) - m(1-Q_k) \geq 0, \quad k=1,\ldots,n-1$$

and

$$MQ_k - P_k \geq 0, \quad M(1-Q_k) - (1-P_k) \geq 0, \quad k=1,\ldots,n-1.$$

If $f:[a,b] \to \mathbb{R}$, $[a,b] \subseteq \mathbb{R}$ is a convex function and $\mathbf{x}=(x_1,\ldots,x_n) \in [a,b]^n$ is a monotonic n-tuple, then

$$M\mathfrak{M}_n(f,\mathbf{x},\mathbf{q}) \geq \mathfrak{M}_n(f,\mathbf{x},\mathbf{p}) \geq m\mathfrak{M}_n(f,\mathbf{x},\mathbf{q}).$$

The accompanied result in [24] was on the bounding of the normalized Jensen-Mercer's functional by means of the non-weight functional of the same type, defined by

$$\mathfrak{M}_n(f,\mathbf{x}) := \mathfrak{M}_n(f,\mathbf{x},\mathbf{u}) = f(a)+f(b) - \frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\left(a+b-\frac{1}{n}\sum_{i=1}^{n} x_i\right),$$

regarding the uniform distribution $\mathbf{u}=(\frac{1}{n},\ldots,\frac{1}{n})$.

Corollary 3.6 (SEE [24]) *Let $\mathbf{p}=(p_1,\ldots,p_n)$ be an n-tuple such that*

$$0 \leq P_k \leq 1, \quad k=1,\ldots,n-1, \quad P_n = 1.$$

For $k \in \{1,\ldots,n\}$ denote $P_k := \sum_{i=1}^{k} p_i$ and define

$$\tilde{m}_0 := n \cdot \min\left\{\frac{P_k}{k}, \frac{1-P_k}{n-k} : k=1,\ldots,n-1\right\},$$

$$\tilde{M}_0 := n \cdot \max\left\{\frac{P_k}{k}, \frac{1-P_k}{n-k} : k=1,\ldots,n-1\right\}.$$

If $f:[a,b] \to \mathbb{R}$ is a convex function and if $\mathbf{x}=(x_1,\ldots,x_n) \in [a,b]^n$ is any monotonic n-tuple, then

$$\tilde{M}_0\mathfrak{M}_n(f,\mathbf{x}) \geq \mathfrak{M}_n(f,\mathbf{x},\mathbf{p}) \geq \tilde{m}_0\mathfrak{M}_n(f,\mathbf{x}). \tag{3.52}$$

Remark 3.13 We now show that the statement of Theorem 3.15 from [24] can easily be deduced from Theorem 3.14. Let m and M be real constants and let $\mathbf{p}$ be a real n-tuple such that $\sum_{i=1}^{n} p_i = P_n > 0$. Suppose $\mathbf{q}$, $\mathbf{p}-m\mathbf{q}$ and $M\mathbf{q}-\mathbf{p} \in \mathscr{P}_n$. If $f:[a,b]\to\mathbb{R}$, $[a,b]\subseteq\mathbb{R}$ is a convex function and if $\mathbf{x}=(x_1,\dots,x_n)\in[a,b]^n$ is a monotonic n-tuple, then, according to Theorem 3.14:

$$\mathfrak{M}(f,\mathbf{x},\mathbf{p}) \geq \mathfrak{M}(f,\mathbf{x},\mathbf{p}-m\mathbf{q}) + \mathfrak{M}(f,\mathbf{x},m\mathbf{q}) \geq m\mathfrak{M}(f,\mathbf{x},\mathbf{q}).$$

Similarly,

$$\mathfrak{M}(f,\mathbf{x},\mathbf{p}) \leq M\mathfrak{M}(f,\mathbf{x},\mathbf{q}),$$

that is

$$M\mathfrak{M}(f,\mathbf{x},\mathbf{q}) \geq \mathfrak{M}(f,\mathbf{x},\mathbf{p}) \geq m\mathfrak{M}(f,\mathbf{x},\mathbf{q}). \tag{3.53}$$

Since $\mathbf{p}-m\mathbf{q}\in\mathscr{P}_n$ implies $P_k \geq mQ_k$ and $(P_n-P_k)\geq m(Q_n-Q_k)$, and $M\mathbf{q}-\mathbf{p}\in\mathscr{P}_n$ implies $P_k \leq MQ_k$ and $(P_n-P_k)\leq M(Q_n-Q_k)$, $k=1,\dots,n-1$, which are the assumptions of Theorem 3.15, only written in a slightly generalized, non-normalized form, the starting assumptions yield the statement of Theorem 3.15.

Another application of Theorem 3.14 is deducing the lower and the upper bound for the functional (3.32), by means of the non-weight functional of the same type. However, almost identical result which slightly differs from this, for it was given in the normalized form, was obtained in Corollary 3.6. Thus this proof is comprehended as an alternative one and is therefore presented within a remark.

Remark 3.14 Let us write Corollary 3.6 in a slightly generalized way:

Let $\mathbf{p}=(p_1,\dots,p_n)$ be an n-tuple in $\mathscr{P}_n$. Define

$$m := \min_{1\leq k\leq n-1}\left\{\frac{P_k}{k},\frac{P_n-P_k}{n-k}\right\}, \quad M := \max_{1\leq k\leq n-1}\left\{\frac{P_k}{k},\frac{P_n-P_k}{n-k}\right\},$$

where $P_k=\sum_{i=1}^{k} p_i$ and $P_n=\sum_{i=1}^{n} p_i>0$. If $f:[a,b]\to\mathbb{R}$, $[a,b]\subseteq\mathbb{R}$ is a convex function and if $\mathbf{x}=(x_1,\dots,x_n)\in[a,b]^n$ is a monotonic n-tuple, then

$$M\mathfrak{M}(f,\mathbf{x}) \geq \mathfrak{M}(f,\mathbf{x},\mathbf{p}) \geq m\mathfrak{M}(f,\mathbf{x}), \tag{3.54}$$

where $\mathfrak{M}(f,\mathbf{x}) = n[f(a)+f(b)] - \sum_{i=1}^{n} f(x_i) - nf\left(a+b-\frac{1}{n}\sum_{i=1}^{n} x_i\right)$.

Alternative proof of Corollary 3.6. Let us firstly prove the right-hand side inequality in (3.54). Let $\mathbf{q}_{\min}\in\mathscr{P}_n^0$ be a constant n-tuple, $q_{\min}=(\alpha,\alpha,\dots,\alpha)$, where $\alpha>0$ since $Q_n:=\sum_{i=1}^{n} q_i>0$. The proof is based on the application of Theorem 3.14 to $\mathbf{p}$ and $\mathbf{q}_{\min}$, because

$$\begin{aligned}\mathfrak{M}(f,\mathbf{x},\mathbf{q}_{\min}) &= m\left(n[f(a)+f(b)] - \sum_{i=1}^{n} f(x_i) - nf\left(a+b-\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right)\\ &= m\mathfrak{M}(f,\mathbf{x}).\end{aligned}$$

However, we firstly need to hold argumentation on the choice of α regarding the defined m. In order to apply Theorem 3.14, one should provide the following conditions satisfied: $P_k \geq Q_k = k\alpha$, $P_n - P_k \geq Q_n - Q_k = (n-k)\alpha$, $k = 1, \ldots, n-1$, and $P_n > Q_n = n\alpha$, which means:

(i) $\alpha \leq \frac{P_k}{k}$, $\quad k = 1, \ldots, n-1$,

(ii) $\alpha \leq \frac{P_n - P_k}{n-k}$, $\quad k = 1, \ldots, n-1$,

(iii) $\alpha < \frac{P_n}{n}$.

Obviously, m satisfies the conditions (i) and (ii) by definition and is a candidate for the choice of α. Fix $k \in \{1, \ldots, n\}$. It easily follows from (i) and (ii) that $nm \leq P_n$, i.e. $m \leq \frac{P_n}{n}$ and m is a good choice for α.

The left-hand side inequality is proved analogously, by exchanging the roles of $\mathbf{p}$ and $\mathbf{q}$. $\square$

3.2.4 Integral Jensen-Mercer's functional under the Steffensen's conditions

Integral variants of the results from the previous section make use of [24] where it was proved that integral Jensen-Mercer's inequality (3.40) was satisfied under the conditions of Theorem 3.4. Recall, the condition on λ being an increasing function was weakened by a stricter choice of the function x. Precisely, the following theorem was established in [24].

Theorem 3.16 (SEE [24]) *Let $x : [\alpha, \beta] \to [a, b]$ be a continuous and monotonic function, where $-\infty < \alpha < \beta < \infty$ and $-\infty < a < b < \infty$. Let function $\lambda : [\alpha, \beta] \to \mathbb{R}$ be either continuous or of bounded variation, such that*

$$\lambda(\alpha) \leq \lambda(t) \leq \lambda(\beta), \quad \text{for all} \quad t \in [\alpha, \beta] \quad \text{and} \quad \lambda(\beta) - \lambda(\alpha) > 0. \tag{3.55}$$

Then for any continuous convex function $f : [a, b] \to \mathbb{R}$ the inequality (3.40) *holds.*

As in Section 3.1.2, with $\Lambda_{[\alpha,\beta]}$, $-\infty < \alpha < \beta < \infty$, is denoted the class of all functions $\lambda : [\alpha, \beta] \to \mathbb{R}$ that are either continuous or of bounded variations and satisfy the conditions (3.55).

Integral Jensen-Mercer's functional $\mathfrak{M}(f, x, \lambda)$ defined by (3.42) will now be observed under the conditions of Theorem 3.4.

We firstly prove that this functional is increasing on $\Lambda_{[\alpha,\beta]}$. As superadditivity of (3.42) does not depend on the particularly chosen functions from this class, we simply apply Theorem 3.12, by which this property was proved in the first place.

Theorem 3.17 *Let μ and $\lambda - \mu$ be functions in $\Lambda_{[\alpha,\beta]}$, either both continuous or of bounded variation where $\lambda : [\alpha, \beta] \to \mathbb{R}$ is such that $\lambda(\beta) - \lambda(\alpha) > 0$. Suppose $x : [\alpha, \beta] \to [a, b]$, $-\infty < a < b < \infty$, is a continuous and monotonic function. If $f : [a, b] \to \mathbb{R}$ is a continuous and convex function, then the following inequalities hold:*

$$\mathfrak{M}(f, x, \lambda) \geq \mathfrak{M}(f, x, \mu) \geq 0. \tag{3.56}$$

Proof. If we write $\mathfrak{M}(f,x,\lambda) = \mathfrak{M}(f,x,\lambda-\mu+\mu)$ and then apply the property (3.43) of the functional (3.42) to $\lambda-\mu$ and μ, we get

$$\mathfrak{M}(f,x,\lambda) = \mathfrak{M}(f,x,\lambda-\mu+\mu) \geq \mathfrak{M}(f,x,\lambda-\mu)+\mathfrak{M}(f,x,\mu) \geq \mathfrak{M}(f,x,\mu),$$

where the last inequality follows from $\mathfrak{M}(f,x,\lambda-\mu) \geq 0$, which is again satisfied due to the inequality (3.40) under the conditions of Theorem 3.16, for a convex f and $\lambda-\mu \in \Lambda_{[\alpha,\beta]}$. For a convex f and $\mu \in \Lambda_{[\alpha,\beta]}$ is then in an analogous way proved the last inequality in (3.56). □

Remark 3.15 Since Theorem 3.16 from [24] was given an integral variant [24, Theorem 6], following the analogy with the Remark 3.13 and applying Theorem 3.17, we can easily obtain its statement as well.

Theorem 3.17 provides the lower and the upper bound for the functional (3.42) under the conditions of Theorem 3.16. Since almost identical result was obtained in [24], only in a normalized form, this alternative proof is given within the following remark.

Remark 3.16 We write [24, Corollary 7] in a more generalized (non-normalized) form:

Let λ be a function in $\Lambda_{[\alpha,\beta]}$. Let $x:[\alpha,\beta]\to[a,b]$ be a monotonic function (either nondecreasing or nonincreasing) and let $f:[a,b]\to\mathbb{R}$ be a continuous convex function. If m and M are real constants defined by

$$m := \inf_{\alpha<t<\beta}\left\{\frac{\lambda(t)-\lambda(\alpha)}{t-\alpha}, \frac{\lambda(\beta)-\lambda(t)}{\beta-t}\right\},$$

$$M := \sup_{\alpha<t<\beta}\left\{\frac{\lambda(t)-\lambda(\alpha)}{t-\alpha}, \frac{\lambda(\beta)-\lambda(t)}{\beta-t}\right\},$$

then

$$M\mathfrak{M}(f,x) \geq \mathfrak{M}(f,x,\lambda) \geq m\mathfrak{M}(f,x), \tag{3.57}$$

where

$$\mathfrak{M}(f,x) := (\beta-\alpha)[f(a)+f(b)] - \int_\alpha^\beta f(x(t))\,\mathrm{d}t - (\beta-\alpha)f\left(a+b-\tfrac{1}{\beta-\alpha}\int_\alpha^\beta x(t)\,\mathrm{d}t\right).$$

Alternative proof. Let us prove the right-hand side inequality in (3.57). Let $\mu:[\alpha,\beta]\to\mathbb{R}$ be defined by $\mu(t)=mt$. The proof is based on the application of Theorem 3.17 to λ and μ because

$$\begin{aligned}\mathfrak{M}(f,x,\mu) &= (m\beta-m\alpha)[f(a)+f(b)] - \int_\alpha^\beta f(x(t))\,\mathrm{d}(mt)\\ &\quad -(m\beta-m\alpha)f\left(a+b-\frac{1}{m\beta-m\alpha}\int_\alpha^\beta x(t)\,\mathrm{d}(mt)\right)\\ &= m\mathfrak{M}(f,x).\end{aligned}\tag{3.58}$$

Namely, after the definition of m is $m \leq \dfrac{\lambda(t)-\lambda(\alpha)}{t-\alpha}$ and $m \leq \dfrac{\lambda(t)-\lambda(\beta)}{t-\beta}$, which is equivalent with $\lambda(\alpha)-m\alpha \leq \lambda(t)-mt \leq \lambda(\beta)-m\beta$, that is $\lambda(\alpha)-\mu(\alpha) \leq \lambda(t)-$

$\mu(t) \leq \lambda(\beta) - \mu(\beta)$, $t \in [\alpha, \beta]$. Hence $\lambda - \mu \in \Lambda_{[\alpha,\beta]}$. Furthermore, $m \geq 0$ yields $\mu(\alpha) \leq \mu(t) \leq \mu(\beta)$, $t \in [\alpha, \beta]$, and so $\mu \in \Lambda_{[\alpha,\beta]}$. The left-hand side inequality is proved analogously, by exchanging the roles of λ and μ. □

3.3 Superadditivity of the Petrović-type functionals

Here we present the Petrović and some related inequalities (see [177], p. 152–159) and define the corresponding functionals whose properties of superadditivity and monotonicity will be the subject in the sequel. The presented results were previously published in [48]. For the sake of simplicity these inequalities will be referred to as the Petrović-type inequalities, while the corresponding functionals will be referred to as the Petrović-type functionals. We start with the following inequality.

Theorem 3.18 *Let $I = (0, a] \subseteq \mathbb{R}_+$ be an interval, $(x_1, \ldots, x_n) \in I^n$, and let $(p_1, \ldots, p_n) \in \mathbb{R}^n_+$ be a non-negative real n-tuple such that*

$$\sum_{i=1}^{n} p_i x_i \in I \quad \text{and} \quad \sum_{i=1}^{n} p_i x_i \geq x_j \text{ for } j = 1, \ldots, n. \tag{3.59}$$

If $f : I \to \mathbb{R}$ is such that the function $f(x)/x$ is decreasing on I, then

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i f(x_i). \tag{3.60}$$

In addition, if $f(x)/x$ is increasing on I, then the sign of inequality in (3.60) *is reversed.*

Remark 3.17 It should be noticed here that if $f(x)/x$ is strictly increasing function on I, then the equality in (3.60) is valid if and only if we have equalities in (3.59) instead of inequalities, that is, if $x_1 = \cdots = x_n$ and $\sum_{i=1}^{n} p_i = 1$.

Motivated by the above theorem, we define the Petrović-type functional $\mathscr{P}_1$, as a difference between the right-hand side and the left-hand side of inequality (3.60), that is,

$$\mathscr{P}_1(\mathbf{x}, \mathbf{p}; f) = \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right), \tag{3.61}$$

where $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$, $I = (0, a]$, $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{R}^n_+$, and f is defined on the interval I.

Remark 3.18 If (3.59) holds and $f(x)/x$ is decreasing on I, then

$$\mathscr{P}_1(\mathbf{x}, \mathbf{p}; f) \geq 0. \tag{3.62}$$

On the other hand, if (3.59) is valid and $f(x)/x$ is increasing on I, then

$$\mathscr{P}_1(\mathbf{x}, \mathbf{p}; f) \leq 0. \tag{3.63}$$

The above functional (3.61) will also be considered under slightly altered assumptions on real n-tuples $\mathbf{x}$ and $\mathbf{p}$. For that sake, the following result from [177] will be used in due course.

Theorem 3.19 *Suppose $I = (0,a] \subseteq \mathbb{R}_+$, $(x_1,\dots,x_n) \in I^n$ is a real n-tuple such that $0 < x_1 \le \cdots \le x_n$, and let $(p_1,\dots,p_n) \in \mathbb{R}_+^n$. Further, let $f : I \to \mathbb{R}$ be such that $f(x)/x$ is increasing on I.*

(i) If there exists m $(\le n)$ such that

$$\overline{P}_1 \ge \overline{P}_2 \ge \cdots \ge \overline{P}_m \ge 1, \quad \overline{P}_{m+1} = \cdots = \overline{P}_n = 0, \tag{3.64}$$

where $P_k = \sum_{i=1}^k p_i$, $\overline{P}_k = P_n - P_{k-1}$, $k = 2,\dots,n$, and $\overline{P}_1 = P_n$, then (3.60) holds.

(ii) If there exists m $(\le n)$ such that

$$0 \le \overline{P}_1 \le \overline{P}_2 \le \cdots \le \overline{P}_m \le 1, \quad \overline{P}_{m+1} = \cdots = \overline{P}_n = 0, \tag{3.65}$$

then the reverse inequality in (3.60) holds.

Remark 3.19 If $f(x)/x$ is increasing on I and (3.64) holds, then the Petrović-type functional $\mathscr{P}_1$ is non-negative, i.e. inequality (3.62) is valid. Conversely, if $f(x)/x$ is increasing on I and conditions as in (3.65) are fulfilled, then relation (3.63) holds.

In order to define another Petrović-type functional, we cite the following Petrović-type inequality involving a convex function.

Theorem 3.20 *Let $I = [0,a] \subseteq \mathbb{R}_+$, $(x_1,\dots,x_n) \in I^n$ and let $(p_1,\dots,p_n) \in \mathbb{R}_+^n$ fulfill conditions as in (3.59). If $f : I \to \mathbb{R}$ is a convex function, then*

$$f\left(\sum_{i=1}^n p_i x_i\right) \ge \sum_{i=1}^n p_i f(x_i) + \left(1 - \sum_{i=1}^n p_i\right) f(0). \tag{3.66}$$

Remark 3.20 If f is a concave function then $-f$ is convex, hence replacing f by $-f$ in Theorem 3.20, we obtain inequality

$$f\left(\sum_{i=1}^n p_i x_i\right) \le \sum_{i=1}^n p_i f(x_i) + \left(1 - \sum_{i=1}^n p_i\right) f(0). \tag{3.67}$$

Remark 3.21 If the function f from Theorem 3.20 is strictly convex, then the inequality in (3.66) is strict, if all x_i's are not equal or $\sum_{i=1}^n p_i \ne 1$.

Now, regarding inequality (3.66) we define another Petrović-type functional $\mathscr{P}_2$ by the formula

$$\mathscr{P}_2(\mathbf{x},\mathbf{p};f) = f\left(\sum_{i=1}^n p_i x_i\right) - \sum_{i=1}^n p_i f(x_i) - \left(1 - \sum_{i=1}^n p_i\right) f(0), \tag{3.68}$$

provided that $\mathbf{x} = (x_1,\dots,x_n) \in I^n$, $I = [0,a]$, $\mathbf{p} = (p_1,\dots,p_n) \in \mathbb{R}_+^n$, and f is defined on I.

Remark 3.22 If (3.59) holds and $f : I \to \mathbb{R}$ is a convex function, then

$$\mathscr{P}_2(\mathbf{x},\mathbf{p};f) \geq 0. \tag{3.69}$$

If (3.59) holds and $f : I \to \mathbb{R}$ is a concave function, then

$$\mathscr{P}_2(\mathbf{x},\mathbf{p};f) \leq 0. \tag{3.70}$$

Finally, we shall also be concerned with an integral form of the Petrović-type functional, based on the following integral Petrović-type inequality.

Theorem 3.21 *Let $I \subseteq \mathbb{R}$ be an interval, $0 \in I$, and let $f : I \to \mathbb{R}$ be a convex function. Further, suppose $h : [a,b] \to I$ is continuous and monotone with $h(t_0) = 0$, where $t_0 \in [a,b]$ is fixed, and g is a function of bounded variation with*

$$G(t) := \int_a^t dg(x), \quad \overline{G}(t) := \int_t^b dg(x).$$

(a) *If $\int_a^b h(t)dg(t) \in I$ and*

$$0 \leq G(t) \leq 1 \quad for \;\; a \leq t \leq t_0, \; 0 \leq \overline{G}(t) \leq 1 \; for \; t_0 \leq t \leq b, \tag{3.71}$$

then

$$\int_a^b f(h(t))dg(t) \geq f\left(\int_a^b h(t)dg(t)\right) + \left(\int_a^b dg(t) - 1\right) f(0). \tag{3.72}$$

(b) *If $\int_a^b h(t)dg(t) \in I$ and either there exists an $s \leq t_0$ such that $G(t) \leq 0$ for $t < s$,*

$$G(t) \geq 1 \;\; for \;\; s \leq t \leq t_0, \;\; and \; \overline{G}(t) \leq 0 \;\; for \; t > t_0 \tag{3.73}$$

or there exists an $s \geq t_0$ such that $G(t) \leq 0$ for $t < t_0$,

$$\overline{G}(t) \geq 1 \; for \; t_0 < t < s, \; and \; \overline{G}(t) \leq 0 \; for \; t \geq s, \tag{3.74}$$

then the reverse inequality in (3.72) holds.

In view of Theorem 3.21, we define the functional

$$\mathscr{P}_3(h,g;f) = \int_a^b f(h(t))dg(t) - f\left(\int_a^b h(t)dg(t)\right) - \left(\int_a^b dg(t) - 1\right) f(0), \tag{3.75}$$

which represents the integral form of the Petrović-type functional.

Remark 3.23 If the functions f, g, and h are defined as in the statement of Theorem 3.21 and (3.71) holds, then the functional $\mathscr{P}_3$ is non-negative, i.e.

$$\mathscr{P}_3(h,g;f) \geq 0. \tag{3.76}$$

Moreover, if either (3.73) or (3.74) holds then

$$\mathscr{P}_3(h,g;f) \leq 0. \tag{3.77}$$

For a comprehensive inspection on the Petrović-type inequalities including proofs and diverse applications, the reader is referred to [177].

In the sequel we establish the conditions under which the appropriate functional is superadditive (subadditive) and increasing (decreasing), with respect to the corresponding n-tuple of real numbers. Our first result refers to the Petrović-type functional $\mathscr{P}_1$ defined by (3.61).

Theorem 3.22 *Let $I = (0,a] \subseteq \mathbb{R}_+$, $\mathbf{x} \in I^n$, and let non-negative n-tuples $\mathbf{p}$, $\mathbf{q}$ fulfill conditions as in* (3.59). *If $f : I \to \mathbb{R}$ is such that the function $f(x)/x$ is decreasing on I, then the functional* (3.61) *possesses the following properties:*

(i) *$\mathscr{P}_1(\mathbf{x}, .; f)$ is superadditive on non-negative n-tuples, i.e.*

$$\mathscr{P}_1(\mathbf{x},\mathbf{p}+\mathbf{q};f) \geq \mathscr{P}_1(\mathbf{x},\mathbf{p};f) + \mathscr{P}_1(\mathbf{x},\mathbf{q};f), \tag{3.78}$$

provided that $\sum_{i=1}^{n}(p_i+q_i)x_i \in I$.

(ii) *If $\mathbf{p},\mathbf{q} \in \mathbb{R}_+^n$ are such that $\mathbf{p} \geq \mathbf{q}$ and $\sum_{i=1}^{n}(p_i-q_i)x_i \geq x_j$, $j = 1,\ldots,n$, then*

$$\mathscr{P}_1(\mathbf{x},\mathbf{p};f) \geq \mathscr{P}_1(\mathbf{x},\mathbf{q};f) \geq 0, \tag{3.79}$$

that is, $\mathscr{P}_1(\mathbf{x}, .; f)$ is increasing on non-negative n-tuples.

(iii) *If $f(x)/x$ is increasing on I, then the signs of inequalities in* (3.78) *and* (3.79) *are reversed, i.e. $\mathscr{P}_1(\mathbf{x}, .; f)$ is subadditive and decreasing on non-negative n-tuples.*

Proof. (i) Using definition (3.61) of the Petrović-type functional $\mathscr{P}_1$ and utilizing the linearity of the sum, we have

$$\begin{aligned}\mathscr{P}_1(\mathbf{x},\mathbf{p}+\mathbf{q};f) &= \sum_{i=1}^{n}(p_i+q_i)f(x_i) - f\left(\sum_{i=1}^{n}(p_i+q_i)x_i\right)\\ &= \sum_{i=1}^{n}p_if(x_i) + \sum_{i=1}^{n}q_if(x_i) - f\left(\sum_{i=1}^{n}p_ix_i + \sum_{i=1}^{n}q_ix_i\right).\end{aligned} \tag{3.80}$$

On the other hand, since $f(x)/x$ is decreasing function, Theorem 3.18 in the non-weight case (for $n = 2$), yields inequality

$$f\left(\sum_{i=1}^{n}p_ix_i + \sum_{i=1}^{n}q_ix_i\right) \leq f(\sum_{i=1}^{n}p_ix_i) + f(\sum_{i=1}^{n}q_ix_i). \tag{3.81}$$

Finally, combining relations (3.80) and (3.81), we obtain

$$\mathscr{P}_1(\mathbf{x},\mathbf{p}+\mathbf{q};f) \geq \sum_{i=1}^{n}p_if(x_i) + \sum_{i=1}^{n}q_if(x_i) - f(\sum_{i=1}^{n}p_ix_i) - f(\sum_{i=1}^{n}q_ix_i).$$

Therefore we have

$$\mathscr{P}_1(\mathbf{x},\mathbf{p}+\mathbf{q};f) \geq \mathscr{P}_1(\mathbf{x},\mathbf{p};f)+\mathscr{P}_1(\mathbf{x},\mathbf{q};f),$$

as claimed.

(ii) Monotonicity follows easily from the superadditivity property. Since $\mathbf{p} \geq \mathbf{q} \geq 0$, we can represent $\mathbf{p}$ as the sum of two non-negative n-tuples, namely $\mathbf{p} = (\mathbf{p}-\mathbf{q})+\mathbf{q}$. Now, from relation (3.78) we get

$$\mathscr{P}_1(\mathbf{x},\mathbf{p};f) = \mathscr{P}_1(\mathbf{x},\mathbf{p}-\mathbf{q}+\mathbf{q};f) \geq \mathscr{P}_1(\mathbf{x},\mathbf{p}-\mathbf{q};f)+\mathscr{P}_1(\mathbf{x},\mathbf{q};f).$$

Finally, if the conditions as in (ii) are fulfilled, then, taking into account Theorem 3.18 we have that $\mathscr{P}_1(\mathbf{x},\mathbf{p}-\mathbf{q};f) \geq 0$, which implies that $\mathscr{P}_1(\mathbf{x},\mathbf{p};f) \geq \mathscr{P}_1(\mathbf{x},\mathbf{q};f)$.

(iii) The case of increasing function $f(x)/x$ is treated in the same way as in (i) and (ii), taking into account that the sign of the corresponding Petrović-type inequality is reversed. □

By virtue of Theorem 3.19, the above properties of the functional $\mathscr{P}_1$ can also be derived in a slightly different setting.

Theorem 3.23 *Let $I = (0,a] \subseteq \mathbb{R}_+$, $\mathbf{x} \in I^n$, and let real n-tuples $\mathbf{p}$, $\mathbf{q}$ fulfill conditions as in (3.64). If $f : I \to \mathbb{R}$ is such that the function $f(x)/x$ is increasing on I, then the functional $\mathscr{P}_1$ has the following properties:*

(i) $\mathscr{P}_1(\mathbf{x},.;f)$ is superadditive on real n-tuples, i.e.

$$\mathscr{P}_1(\mathbf{x},\mathbf{p}+\mathbf{q};f) \geq \mathscr{P}_1(\mathbf{x},\mathbf{p};f)+\mathscr{P}_1(\mathbf{x},\mathbf{q};f), \tag{3.82}$$

provided that $\sum_{i=1}^{n}(p_i+q_i)x_i \in I$ and $0 < \sum_{i=1}^{n} p_i x_i \leq \sum_{i=1}^{n} q_i x_i$.

(ii) If $0 < x_1 \leq \cdots \leq x_n$, $\mathbf{p} \geq \mathbf{q}$, and there exist m $(\leq n)$ such that

$$\begin{aligned} &\bar{P}_1-\bar{Q}_1 \geq \bar{P}_2-\bar{Q}_2 \geq \cdots \geq \bar{P}_m-\bar{Q}_m \geq 1, \\ &\bar{P}_{m+1} = \bar{Q}_{m+1} = \cdots = \bar{P}_n = \bar{Q}_n = 0, \end{aligned} \tag{3.83}$$

where $P_k = \sum_{i=1}^{k} p_i$, $Q_k = \sum_{i=1}^{k} q_i$, $\bar{P}_k-\bar{Q}_k = (P_n+Q_n)-(P_{k-1}+Q_{k-1})$, $k = 2,\ldots,n$, $\bar{P}_1 = P_n$, and $\bar{Q}_1 = Q_n$, then

$$\mathscr{P}_1(\mathbf{x},\mathbf{p};f) \geq \mathscr{P}_1(\mathbf{x},\mathbf{q};f) \geq 0, \tag{3.84}$$

i.e. $\mathscr{P}_1(\mathbf{x},.;f)$ is increasing on real n-tuples.

(iii) If real n-tuples $\mathbf{p}$ and $\mathbf{q}$ fulfill conditions as in (3.65), then the signs of inequalities in (3.82) and (3.84) are reversed, that is, $\mathscr{P}_1(\mathbf{x},.;f)$ is subadditive and decreasing on real n-tuples.

Proof. (i) The proof follows the same lines as the proof of the previous theorem. Namely, the left-hand side of (3.82) can be rewritten as

$$\begin{aligned}\mathscr{P}_1(\mathbf{x},\mathbf{p}+\mathbf{q};f) &= \sum_{i=1}^{n}(p_i+q_i)f(x_i) - f\left(\sum_{i=1}^{n}(p_i+q_i)x_i\right)\\ &= \sum_{i=1}^{n}p_i f(x_i) + \sum_{i=1}^{n}q_i f(x_i) - f\left(\sum_{i=1}^{n}p_i x_i + \sum_{i=1}^{n}q_i x_i\right). \end{aligned} \tag{3.85}$$

Moreover, $f(x)/x$ is increasing, hence Theorem 3.19 for $n=2$ yields inequality

$$f\left(\sum_{i=1}^{n}p_i x_i + \sum_{i=1}^{n}q_i x_i\right) \le f(\sum_{i=1}^{n}p_i x_i) + f(\sum_{i=1}^{n}q_i x_i). \tag{3.86}$$

Finally, relations (3.85) and (3.86) imply inequality

$$\mathscr{P}_1(\mathbf{x},\mathbf{p}+\mathbf{q};f) \ge \sum_{i=1}^{n}p_i f(x_i) + \sum_{i=1}^{n}q_i f(x_i) - f(\sum_{i=1}^{n}p_i x_i) - f(\sum_{i=1}^{n}q_i x_i),$$

i.e. we obtain (3.82).

(ii) Considering $\mathbf{p} \ge \mathbf{q} \ge 0$, the real n-tuple $\mathbf{p}$ can be rewritten as $\mathbf{p} = (\mathbf{p}-\mathbf{q})+\mathbf{q}$. Now, regarding relation (3.82) we have

$$\mathscr{P}_1(\mathbf{x},\mathbf{p};f) = \mathscr{P}_1(\mathbf{x},\mathbf{p}-\mathbf{q}+\mathbf{q};f) \ge \mathscr{P}_1(\mathbf{x},\mathbf{p}-\mathbf{q};f) + \mathscr{P}_1(\mathbf{x},\mathbf{q};f).$$

Finally, taking into account conditions as in (3.83), it follows by Theorem 3.19 that $\mathscr{P}_1(\mathbf{x},\mathbf{p}-\mathbf{q};f) \ge 0$, that is, $\mathscr{P}_1(\mathbf{x},\mathbf{p};f) \ge \mathscr{P}_1(\mathbf{x},\mathbf{q};f)$, which completes the proof.

(iii) This case is treated in the same way as in (i) and (ii), taking into account that the sign of the corresponding Petrović-type inequality is reversed. □

Superadditivity and monotonicity properties stated in Theorem 3.22 play an important role in numerous applications of the Petrović-type inequalities. In the sequel we utilize the monotonicity property of the Petrović-type functional $\mathscr{P}_1$. More precisely, we derive some bounds for this functional, expressed in terms of the non-weight functional of the same type.

Corollary 3.7 *Let $I=(0,a] \subseteq \mathbb{R}_+$, $\mathbf{x} \in I^n$, and let $f: I \to \mathbb{R}$ be such that $f(x)/x$ is decreasing on I. Further, suppose $\mathbf{p} \in \mathbb{R}_+^n$ is such that $\sum_{i=1}^{n}(p_i-m)x_i \ge x_j$ and $\sum_{i=1}^{n}(M-p_i)x_i \ge x_j$, $j=1,2,\ldots,n$, where $m=\min_{1\le i\le n}\{p_i\}$ and $M=\max_{1\le i\le n}\{p_i\}$.*

If $m>1$ then the Petrović-type functional $\mathscr{P}_1$ fulfills inequality

$$\mathscr{P}_1(\mathbf{x},\mathbf{p};f) \ge m\mathscr{P}_1^0(\mathbf{x};f), \tag{3.87}$$

while for $M<1$ we have

$$\mathscr{P}_1(\mathbf{x},\mathbf{p};f) \le M\mathscr{P}_1^0(\mathbf{x};f), \tag{3.88}$$

where

$$\mathscr{P}_1^0(\mathbf{x};f) = \sum_{i=1}^{n}f(x_i) - f\left(\sum_{i=1}^{n}x_i\right). \tag{3.89}$$

Moreover, if $f(x)/x$ is increasing on I, then the signs of inequalities in (3.87) *and* (3.88) *are reversed.*

Proof. Since $\mathbf{p} = (p_1,\dots,p_n) \geq \mathbf{m} = (m,m,\dots,m)$, monotonicity of the Petrović-type functional implies that $\mathscr{P}_1(\mathbf{x},\mathbf{p};f) \geq \mathscr{P}_1(\mathbf{x},\mathbf{m};f)$.

On the other hand, if $f(x)/x$ is decreasing function, we have

$$f(au) \leq af(u), \quad a > 1 \quad \text{and} \quad f(au) \geq af(u), \quad a < 1. \tag{3.90}$$

Now, regarding (3.90) we have

$$\mathscr{P}_1(\mathbf{x},\mathbf{m};f) = m\sum_{i=1}^{n} f(x_i) - f\left(m\sum_{i=1}^{n} x_i\right) \geq m\sum_{i=1}^{n} f(x_i) - mf\left(\sum_{i=1}^{n} x_i\right),$$

that is, we obtain (3.87). Inequality (3.88) is derived in a similar way, by using the second inequality in (3.90). □

Our next result provides superadditivity and monotonicity properties of the Petrović-type functional defined by (3.68).

Theorem 3.24 *Let $I = [0,a] \subseteq \mathbb{R}_+$, $\mathbf{x} \in I^n$, and let $\mathbf{p},\mathbf{q} \in \mathbb{R}_+^n$ fulfill conditions as in* (3.59)*. If $f : I \to \mathbb{R}$ is a convex function, then the functional* (3.68) *has the following properties:*

(i) $\mathscr{P}_2(\mathbf{x},.;f)$ *is superadditive on non-negative n-tuples, i.e.*

$$\mathscr{P}_2(\mathbf{x},\mathbf{p}+\mathbf{q};f) \geq \mathscr{P}_2(\mathbf{x},\mathbf{p};f) + \mathscr{P}_2(\mathbf{x},\mathbf{q};f), \tag{3.91}$$

provided that $\sum_{i=1}^{n}(p_i+q_i)x_i \in I$.

(ii) If $\mathbf{p},\mathbf{q}$ *are such that* $\mathbf{p} \geq \mathbf{q}$ *and* $\sum_{i=1}^{n}(p_i-q_i)x_i \geq x_j$, $j = 1,\dots,n$, *then*

$$\mathscr{P}_2(\mathbf{x},\mathbf{p};f) \geq \mathscr{P}_2(\mathbf{x},\mathbf{q};f) \geq 0, \tag{3.92}$$

that is, $\mathscr{P}_2(\mathbf{x},.;f)$ *is increasing on non-negative n-tuples.*

(iii) If $f : I \to \mathbb{R}$ is a concave function, then the signs of inequalities in (3.91) *and* (3.92) *are reversed, i.e.* $\mathscr{P}_2(\mathbf{x},.;f)$ *is subadditive and decreasing on non-negative n-tuples.*

Proof. (i) The left-hand side of inequality (3.91) can be rewritten as

$$\begin{aligned}
\mathscr{P}_2(\mathbf{x},\mathbf{p}+\mathbf{q};f) &= f\left(\sum_{i=1}^{n}(p_i+q_i)x_i\right) - \sum_{i=1}^{n}(p_i+q_i)f(x_i) - \left(1-\sum_{i=1}^{n}(p_i+q_i)\right)f(0) \\
&= f\left(\sum_{i=1}^{n} p_ix_i + \sum_{i=1}^{n} q_ix_i\right) - \sum_{i=1}^{n} p_if(x_i) - \sum_{i=1}^{n} q_if(x_i) \\
&\quad - \left(1-(\sum_{i=1}^{n} p_i + \sum_{i=1}^{n} q_i)\right)f(0).
\end{aligned} \tag{3.93}$$

Further, Theorem 3.20 in the non-weight case (for $n = 2$) yields inequality

$$f\left(\sum_{i=1}^{n} p_i x_i + \sum_{i=1}^{n} q_i x_i\right) \geq f(\sum_{i=1}^{n} p_i x_i) + f(\sum_{i=1}^{n} q_i x_i) - f(0), \tag{3.94}$$

hence combining relations (3.93) and (3.94), we get

$$\begin{aligned} \mathscr{P}_2(\mathbf{x}, \mathbf{p}+\mathbf{q}; f) \ \geq \ & f(\sum_{i=1}^{n} p_i x_i) - \sum_{i=1}^{n} p_i f(x_i) - \left(1 - \sum_{i=1}^{n} p_i\right) f(0) \\ & + f(\sum_{i=1}^{n} p_i x_i) - \sum_{i=1}^{n} q_i f(x_i) - \left(1 - \sum_{i=1}^{n} q_i\right) f(0). \end{aligned} \tag{3.95}$$

Thus, considering definition (3.68) we obtain (3.91), as claimed.

(ii) Monotonicity property follows from the corresponding superadditivity property (3.91), as in Theorem 3.23.

(iii) The case of concave function f follows from the fact that the sign of the corresponding Petrović-type inequality is reversed. □

Finally, we derive the properties of the integral Petrović-type functional, defined by (3.75).

Theorem 3.25 *Suppose $f : I = [0,a] \to \mathbb{R}$ is a convex function, $h : [a,b] \to I$ is continuous and monotone with $h(t_0) = 0$, where $t_0 \in [a,b]$ is fixed, and let g_1, g_2 be functions of bounded variation with*

$$G_i(t) := \int_a^t dg_i(x), \ \ \overline{G}_i(t) := \int_t^b dg_i(x) \ \ for \ i = 1,2.$$

Then the functional $\mathscr{P}_3$, defined by (3.75), has the following properties:

(i) $\mathscr{P}_3(h, .; f)$ is subadditive with respect to functions of bounded variation, i.e.

$$\mathscr{P}_3(h, g_1+g_2; f) \leq \mathscr{P}_3(h, g_1; f) + \mathscr{P}_3(h, g_2; f), \tag{3.96}$$

where $\int_a^b h(t)dg_1(t) \geq 0$, $\int_a^b h(t)dg_2(t) \geq 0$, and $\int_a^b h(t)dg_1(t) + \int_a^b h(t)dg_2(t) \in I$.

(ii) If $\int_a^b h(t)d(g_1)(t) - \int_a^b h(t)d(g_2)(t) \in I$ and either there exists an $s \leq t_0$ such that $G_1(t) \leq G_2(t)$ for $t < s$, $G_1(t) - G_2(t) \geq 1$ for $s \leq t \leq t_0$, and $\overline{G}_1(t) \leq \overline{G}_2(t)$ for $t > t_0$, or there exists an $s \geq t_0$ such that $G_1(t) \leq G_2(t)$ for $t < t_0$, $\overline{G}_1(t) - \overline{G}_2(t) \geq 1$ for $t_0 < t < s$, and $\overline{G}_1(t) \leq \overline{G}_2(t)$ for $t \geq s$, then

$$\mathscr{P}_3(h, g_1; f) \leq \mathscr{P}_3(h, g_2; f). \tag{3.97}$$

Proof. (i) Regarding definition (3.75) of the Petrović-type integral functional, we have

$$\begin{aligned} \mathscr{P}_3(h, g_1+g_2; f) \ = \ & \int_a^b f(h(t))d(g_1+g_2)(t) - f\left(\int_a^b h(t)d(g_1+g_2)(t)\right) \\ & - \left(\int_a^b d(g_1+g_2)(t) - 1\right) f(0), \end{aligned}$$

that is,

$$\begin{aligned}\mathscr{P}_3(h,g_1+g_2;f) = & \int_a^b f(h(t))dg_1(t)+\int_a^b f(h(t))dg_2(t) \\ & -f\left(\int_a^b h(t)dg_1(t)+\int_a^b h(t)dg_2(t)\right) \\ & -\left(\int_a^b dg_1(t)+\int_a^b dg_2(t)-1\right)f(0),\end{aligned} \tag{3.98}$$

by the linearity of the differential. Now, applying inequality (3.66) to term

$$f\left(\int_a^b h(t)dg_1(t)+\int_a^b h(t)dg_2(t)\right),$$

we obtain

$$f\left(\int_a^b h(t)dg_1(t)+\int_a^b h(t)dg_2(t)\right)\geq f\left(\int_a^b h(t)dg_1(t)\right)+f\left(\int_a^b h(t)dg_2(t)\right)-f(0). \tag{3.99}$$

Further, inserting (3.99) in (3.98), we have

$$\begin{aligned}\mathscr{P}_3(h,g_1+g_2;f) \leq & \int_a^b f(h(t))dg_1(t)+\int_a^b f(h(t))dg_2(t) \\ & -f\left(\int_a^b h(t)dg_1(t)\right)-f\left(\int_a^b h(t)dg_2(t)\right)+f(0) \\ & -\left(\int_a^b dg_1(t)+\int_a^b dg_2(t)-1\right)f(0),\end{aligned}$$

i.e. by rearranging,

$$\mathscr{P}_3(h,g_1+g_2;f)\leq \mathscr{P}_3(h,g_1;f)+\mathscr{P}_3(h,g_2;f).$$

(ii) Monotonicity follows from the subadditivity property (3.96). Namely, representing g_1 as $g_1=(g_1-g_2)+g_2$, we have

$$\mathscr{P}_3(h,g_1;f)=\mathscr{P}_3(h,(g_1-g_2)+g_2;f)\leq \mathscr{P}_3(h,g_1-g_2;f)+\mathscr{P}_3(h,g_2;f).$$

Clearly, under assumptions as in the statement of theorem, we have $\mathscr{P}_3(h,g_1-g_2;f)\leq 0$ (see also Remark 3.23), hence it follows that $\mathscr{P}_3(h,g_1;f)\leq \mathscr{P}_3(h,g_2;f)$, which completes the proof. □

Chapter 4

Some further improvements. Levinson's functional

In the first part of this chapter some additional refinements concerning Jessen's functional (2.2) (see also [22]) are presented. Inequalities observed in Chapter 2 are in this part reexamined under new assumptions and by means of new results involved. It is interesting to see how non-weight bounds for discrete Jensen's functional, given in Lemma 1.2 served as a tool in obtaining these new refinements.

In the second part of the chapter, superadditivity of Levinson's functional is studied, as it was previously done in [109]. Closely related to the Jensen inequality, this functional is here observed in a more general setting that again belongs to the Jessen's variant of this inequality.

In view of this facts, this chapter can be regarded as a continuation of Chapter 2 and thus inherits its environment of the positive linear functionals acting on the space of real valued functions.

4.1 Refinements of the inequalities related to Jessen's functional

Properties of superadditivity and increase of Jessen's functional (2.2) proved in Theorem 2.2 are improved in the following theorem. Remarks 2.1 and 2.2 concerning conditions on well-defining of Jessen's functional need to be taken into account in the sequel, as well.

Theorem 4.1 *Let $A: L \to \mathbb{R}$ be a positive linear functional. Suppose f, p and $q \in L$. If $\Phi: I \to \mathbb{R}$, $I \subseteq \mathbb{R}$ is a continuous and convex function, then*

$$\begin{aligned}
&\min\{A(p),A(q)\}\left[\Phi\left(\frac{A(pf)}{A(p)}\right)+\Phi\left(\frac{A(qf)}{A(q)}\right)-2\Phi\left(\frac{A(pf)}{2A(p)}+\frac{A(qf)}{2A(q)}\right)\right]\\
&\leq \mathscr{J}(\Phi,f,p+q;A)-\mathscr{J}(\Phi,f,p;A)-\mathscr{J}(\Phi,f,q;A)\\
&\leq \max\{A(p),A(q)\}\left[\Phi\left(\frac{A(pf)}{A(p)}\right)+\Phi\left(\frac{A(qf)}{A(q)}\right)-2\Phi\left(\frac{A(pf)}{2A(p)}+\frac{A(qf)}{2A(q)}\right)\right]. \qquad (4.1)
\end{aligned}$$

Moreover, if $p \geq q$, $A(p) \neq A(q)$ and $\frac{A(pf)-A(qf)}{A(p)-A(q)} \in I$, then

$$\begin{aligned}
&\mathscr{J}(\Phi,f,p;A)-\mathscr{J}(\Phi,f,q;A)\\
&\geq \min\{A(p)-A(q),A(q)\}\left[\Phi\left(\frac{A(pf)-A(qf)}{A(p)-A(q)}\right)+\Phi\left(\frac{A(qf)}{A(q)}\right)\right.\\
&\quad\left.-2\Phi\left\{\frac{1}{2}\left[\frac{A(pf)-A(qf)}{A(p)-A(q)}+\frac{A(qf)}{A(q)}\right]\right\}\right]. \qquad (4.2)
\end{aligned}$$

Proof. In order to prove relation (4.1), we use non-weight bounds from Lemma 1.2. For convex function Φ and in case $n=2$ relation (1.64) reads:

$$\begin{aligned}
&\min\{\bar{p},\bar{q}\}\left[\Phi(x)+\Phi(y)-2\Phi\left(\frac{x+y}{2}\right)\right]\\
&\leq \bar{p}\Phi(x)+\bar{q}\Phi(y)-(\bar{p}+\bar{q})\Phi\left(\frac{\bar{p}x+\bar{q}y}{\bar{p}+\bar{q}}\right)\\
&\leq \max\{\bar{p},\bar{q}\}\left[\Phi(x)+\Phi(y)-2\Phi\left(\frac{x+y}{2}\right)\right]. \qquad (4.3)
\end{aligned}$$

If we substitute $\bar{p}$ with $A(p)$, $\bar{q}$ with $A(q)$, x with $\frac{A(pf)}{A(p)}$ and y with $\frac{A(qf)}{A(q)}$ in (4.3) we obtain

$$\begin{aligned}
&\min\{A(p),A(q)\}\left[\Phi\left(\frac{A(pf)}{A(p)}\right)+\Phi\left(\frac{A(qf)}{A(q)}\right)-2\Phi\left(\frac{A(pf)}{2A(p)}+\frac{A(qf)}{2A(q)}\right)\right]\\
&\leq A(p)\Phi\left(\frac{A(pf)}{A(p)}\right)+A(q)\Phi\left(\frac{A(qf)}{A(q)}\right)-(A(p)+A(q))\Phi\left(\frac{A(pf)+A(qf)}{A(p)+A(q)}\right)\\
&\leq \max\{A(p),A(q)\}\left[\Phi\left(\frac{A(pf)}{A(p)}\right)+\Phi\left(\frac{A(qf)}{A(q)}\right)-2\Phi\left(\frac{A(pf)}{2A(p)}+\frac{A(qf)}{2A(q)}\right)\right]. \qquad (4.4)
\end{aligned}$$

From the definition (2.2) of Jessen's functional it follows that

$$\begin{aligned}
&\mathscr{J}(\Phi,f,p+q;A)-\mathscr{J}(\Phi,f,p;A)-\mathscr{J}(\Phi,f,q;A)\\
&=A(p)\Phi\left(\frac{A(pf)}{A(p)}\right)+A(q)\Phi\left(\frac{A(qf)}{A(q)}\right)-(A(p)+A(q))\Phi\left(\frac{A(pf)+A(qf)}{A(p)+A(q)}\right). \qquad (4.5)
\end{aligned}$$

Combining relations (4.4) and (4.5) we get (4.1).

Functional $\mathscr{J}(\Phi, f, \cdot, A)$ is superadditive and increasing on L and satisfies relation (4.1). Hence for $p \geq q$, $A(p) \neq A(q)$ and $\frac{A(pf)-A(gf)}{A(p)-A(q)} \in I$ the following holds:

$$\begin{aligned}
&\mathscr{J}(\Phi, f, p; A) - \mathscr{J}(\Phi, f, p-q; A) - \mathscr{J}(\Phi, f, q; A) \\
&\geq \min\{A(p-q), A(q)\} \left[\Phi\left(\frac{A((p-q)f)}{A(p-q)}\right) + \Phi\left(\frac{A(qf)}{A(q)}\right)\right. \\
&\quad \left. - 2\Phi\left(\frac{A((p-q)f)}{2A(p-q)} + \frac{A(qf)}{2A(q)}\right)\right] \\
&= \min\{A(p)-A(q), A(q)\} \left[\Phi\left(\frac{A(pf)-A(qf)}{A(p)-A(q)}\right) + \Phi\left(\frac{A(qf)}{A(q)}\right)\right. \\
&\quad \left. - 2\Phi\left\{\frac{1}{2}\left[\frac{A(pf)-A(qf)}{A(p)-A(q)} + \frac{A(qf)}{A(q)}\right]\right\}\right].
\end{aligned} \tag{4.6}$$

Since $\mathscr{J}(\Phi, f, p-q; A) \geq 0$, we obtain (4.2). This completes the proof. □

Theorem 4.1 provides the refinement and the converse of the superadditivity property (2.3) from Theorem 2.2 and refines the monotonicity property (2.4) from the same theorem. Note also that the statements of Theorem 4.1 are formulated in a more general setting with $p, q \in L$, since for $n = 2$ this condition is as valid as with $p, q \in L^+$.

Corollary 4.1 *Let $A\colon L \to \mathbb{R}$ be a positive linear functional and $\Phi\colon I \to \mathbb{R}$, $I \subset \mathbb{R}$ be a continuous and convex function. Suppose $f \in L$ and $p, q \in L^+$. Then the inequality* (4.1) *holds. If $p \geq q$ and $A(p) \neq A(q) > 0$, then* (4.2) *holds.*

Corollary 4.2 *Let $A\colon L \to \mathbb{R}$ be a positive linear functional, $f \in L$ and let $\Phi\colon I \to \mathbb{R}$, $I \subset \mathbb{R}$ be a continuous and convex function. If $p \in L^+$ attains its minimal value $\underline{p} = \min_{x\in E} p(x)$ and its maximal value $\overline{p} = \max_{x\in E} p(x)$, then the following series of inequalities holds:*

$$\begin{aligned}
&\overline{p}\mathscr{J}(\Phi, f, 1; A) - \mathscr{J}(\Phi, f, p; A) \\
&\geq \min\{\overline{p}A(1) - A(p), A(p)\} \left[\Phi\left(\frac{\overline{p}A(f) - A(pf)}{\overline{p}A(1) - A(p)}\right) + \Phi\left(\frac{A(pf)}{A(p)}\right)\right. \\
&\quad \left. - 2\Phi\left\{\frac{1}{2}\left[\frac{\overline{p}A(f) - A(pf)}{\overline{p}A(1) - A(p)} + \frac{A(pf)}{A(p)}\right]\right\}\right],
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
&\mathscr{J}(\Phi, f, p; A) - \underline{p}\mathscr{J}(\Phi, f, 1; A) \\
&\geq \min\{A(p) - \underline{p}A(1), \underline{p}A(1)\} \left[\Phi\left(\frac{A(pf) - \underline{p}A(f)}{A(p) - \underline{p}A(1)}\right) + \Phi\left(\frac{A(f)}{A(1)}\right)\right. \\
&\quad \left. - 2\Phi\left\{\frac{1}{2}\left[\frac{A(pf) - \underline{p}A(f)}{A(p) - \underline{p}A(1)} + \frac{A(f)}{A(1)}\right]\right\}\right],
\end{aligned} \tag{4.8}$$

where

$$\mathscr{J}(\Phi, f, 1; A) = A(\Phi(f)) - A(1)\Phi\left(\frac{A(f)}{A(1)}\right). \tag{4.9}$$

Proof. Since

$$\overline{p} \geq p(x) \geq \underline{p},$$

double application of property (4.2) yields required result since

$$\mathscr{J}\left(\Phi, f, \overline{p}; A\right) = \overline{p}\,\mathscr{J}\left(\Phi, f, 1; A\right) \text{ and } \mathscr{J}\left(\Phi, f, \underline{p}; A\right) = \underline{p}\,\mathscr{J}\left(\Phi, f, 1; A\right).$$

□

Remark 4.1 Let's rewrite relations (4.7) and (4.8) from Corollary 4.2 in the discrete form. We suppose $E = \{1, 2, \ldots, n\}$ and L is the class of real n-tuples. If we consider discrete functional A defined by $A(\mathbf{x}) = \sum_{i=1}^{n} x_i$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, then we deal with the discrete Jensen's functional (1.65) and the relation (4.7) takes form

$$\begin{aligned}
&\max_{1\leq i\leq n}\{p_i\} J_n(\Phi, \mathbf{x}) - J_n(\Phi, \mathbf{x}, \mathbf{p}) \\
&\geq \min\{n \max_{1\leq i\leq n}\{p_i\} - P_n, P_n\} \left[\Phi\left(\frac{\max_{1\leq i\leq n}\{p_i\} \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} p_i x_i}{n \max_{1\leq i\leq n}\{p_i\} - P_n}\right) + \Phi\left(\frac{\sum_{i=1}^{n} p_i x_i}{P_n}\right)\right. \\
&\quad \left. - 2\Phi\left\{\frac{1}{2}\left[\frac{\max_{1\leq i\leq n}\{p_i\} \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} p_i x_i}{n \max_{1\leq i\leq n}\{p_i\} - P_n} + \frac{\sum_{i=1}^{n} p_i x_i}{P_n}\right]\right\}\right],
\end{aligned} \tag{4.10}$$

and the relation (4.8) takes form

$$\begin{aligned}
&J_n(\Phi, \mathbf{x}, \mathbf{p}) - \min_{1\leq i\leq n}\{p_i\} J_n(\Phi, \mathbf{x}) \\
&\geq \min\{P_n - n \min_{1\leq i\leq n}\{p_i\}, n \min_{1\leq i\leq n}\{p_i\}\} \left[\Phi\left(\frac{\sum_{i=1}^{n} p_i x_i - \min_{1\leq i\leq n}\{p_i\} \sum_{i=1}^{n} x_i}{P_n - n \min_{1\leq i\leq n}\{p_i\}}\right)\right. \\
&\quad \left. + \Phi\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) - 2\Phi\left\{\frac{1}{2}\left[\frac{\sum_{i=1}^{n} p_i x_i - \min_{1\leq i\leq n}\{p_i\} \sum_{i=1}^{n} x_i}{P_n - n \min_{1\leq i\leq n}\{p_i\}} + \frac{\sum_{i=1}^{n} x_i}{n}\right]\right\}\right],
\end{aligned} \tag{4.11}$$

where the functional $J_n(\Phi, \mathbf{x}, \mathbf{p})$ is defined by (1.65) and $J_n(\Phi, \mathbf{x}) = \sum_{i=1}^{n} \Phi(x_i) - n\Phi\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)$.

4.1.1 Application to weight generalized means

As expected, basic results from the previous section are applied to weight generalized means (2.11) with respect to positive linear functional $A\colon L \to \mathbb{R}$ and thus the corresponding results from Section 2.1.3 are improved. We firstly establish the improvement of Theorem 2.4 concerning the Jensen-type functional (2.12). Again, notation and the definition conditions induced in Section 2.1.3 remain valid in the sequel, as well.

Theorem 4.2 *Let $A\colon L \to \mathbb{R}$ be a positive linear functional and $\chi, \psi\colon I \to \mathbb{R}$, $I \subseteq \mathbb{R}$ be continuous and strictly monotonic functions. Suppose $f \in L$ is such that $\psi(f) \in L$ and $p, q \in L^+$. If $\chi \circ \psi^{-1}$ is a convex function, then functional $\mathscr{J}^\tau(\chi \circ \psi^{-1}, \psi(f), \cdot; A)$ defined by (2.12) possesses the following properties*

$$
\begin{aligned}
&\min\{A(p),A(q)\}\left[\chi\circ\psi^{-1}\left(\frac{A(p\psi(f))}{A(p)}\right)+\chi\circ\psi^{-1}\left(\frac{A(q\psi(f))}{A(q)}\right)\right.\\
&\quad\left.-2\chi\circ\psi^{-1}\left(\frac{A(p\psi(f))}{2A(p)}+\frac{A(q\psi(f))}{2A(q)}\right)\right]\\
&\le \mathscr{J}^\tau(\chi\circ\psi^{-1},\psi(f),p+q;A)-\mathscr{J}^\tau(\chi\circ\psi^{-1},\psi(f),p;A)-\mathscr{J}^\tau(\chi\circ\psi^{-1},\psi(f),q;A)\\
&\le \max\{A(p),A(q)\}\left[\chi\circ\psi^{-1}\left(\frac{A(p\psi(f))}{A(p)}\right)+\chi\circ\psi^{-1}\left(\frac{A(q\psi(f))}{A(q)}\right)\right.\\
&\quad\left.-2\chi\circ\psi^{-1}\left(\frac{A(p\psi(f))}{2A(p)}+\frac{A(q\psi(f))}{2A(q)}\right)\right].
\end{aligned}\tag{4.12}
$$

Moreover, if $p \ge q$, then

$$
\begin{aligned}
&\mathscr{J}^\tau(\chi\circ\psi^{-1},\psi(f),p;A)-\mathscr{J}^\tau(\chi\circ\psi^{-1},\psi(f),q;A)\\
&\ge \min\{A(p)-A(q),A(q)\}\left[\chi\circ\psi^{-1}\left(\frac{A(p\psi(f))-A(q\psi(f))}{A(p)-A(q)}\right)+\chi\circ\psi^{-1}\left(\frac{A(q\psi(f))}{A(q)}\right)\right.\\
&\quad\left.-2\chi\circ\psi^{-1}\left\{\frac{1}{2}\left[\frac{A(p\psi(f))-A(q\psi(f))}{A(p)-A(q)}+\frac{A(q\psi(f))}{A(q)}\right]\right\}\right].
\end{aligned}\tag{4.13}
$$

Proof. We consider Jessen's functional (2.2) where the convex function Φ is replaced with $\chi \circ \psi^{-1}$ and $f \in L$ with $\psi(f) \in L$. Functional (2.12) can be rewritten in the following way:

$$
\begin{aligned}
\mathscr{J}^\tau\left(\chi\circ\psi^{-1},\psi(f),p;A\right) &= A\left(p\cdot\left(\chi\circ\psi^{-1}\left(\psi(f)\right)\right)\right)-A(p)\chi\left(\psi^{-1}\left(\frac{A\left(p\psi(f)\right)}{A(p)}\right)\right)\\
&= A\left(p\chi(f)\right)-A(p)\chi\left(M_\psi(f,p;A)\right)\\
&= A(p)\chi\left(M_\chi(f,p;A)\right)-A(p)\chi\left(M_\psi(f,p;A)\right)\\
&= A(p)\left[\chi\left(M_\chi(f,p;A)\right)-\chi\left(M_\psi(f,p;A)\right)\right].
\end{aligned}
$$

Now, inequalities (4.12) and (4.13) follow from Theorem 4.1. □

Using the similar substitutions as in the previous theorem, we give its consequence in the form of the following corollary, which is also an improvement of the corresponding Corollary 2.2.

Corollary 4.3 *Let functions χ, ψ, f and functional A be defined as in Theorem 4.2 and let $p \in L^+$ attain its minimal value $\underline{p} = \min_{x\in E} p(x)$ and its maximal value $\overline{p} = \max_{x\in E} p(x)$.*

If $\chi\circ\psi^{-1}$ is a convex function, then the following series of inequalities holds:

$$\begin{aligned}
&\overline{p}\mathscr{J}^{\tau}\left(\chi\circ\psi^{-1},\psi(f),1;A\right)-\mathscr{J}^{\tau}\left(\chi\circ\psi^{-1},\psi(f),p;A\right)\\
&\geq \min\{\overline{p}A(1)-A(p),A(p)\}\left[\chi\circ\psi^{-1}\left(\frac{\overline{p}A(\psi(f))-A(p\psi(f))}{\overline{p}A(1)-A(p)}\right)\right.\\
&\quad\left.+\chi\circ\psi^{-1}\left(\frac{A(p\psi(f))}{A(p)}\right)-2\chi\circ\psi^{-1}\left\{\frac{1}{2}\left[\frac{\overline{p}A(\psi(f))-A(p\psi(f))}{\overline{p}A(1)-A(p)}+\frac{A(p\psi(f))}{A(p)}\right]\right\}\right],
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
&\mathscr{J}^{\tau}\left(\chi\circ\psi^{-1},\psi(f),p;A\right)-\underline{p}\mathscr{J}^{\tau}\left(\chi\circ\psi^{-1},\psi(f),1;A\right)\\
&\geq \min\{A(p)-\underline{p}A(1),\underline{p}A(1)\}\left[\chi\circ\psi^{-1}\left(\frac{A(p\psi(f))-\underline{p}A(\psi(f))}{A(p)-\underline{p}A(1)}\right)\right.\\
&\quad\left.+\chi\circ\psi^{-1}\left(\frac{A(\psi(f))}{A(1)}\right)-2\chi\circ\psi^{-1}\left\{\frac{1}{2}\left[\frac{A(p\psi(f))-\underline{p}A(\psi(f))}{A(p)-\underline{p}A(1)}+\frac{A(\psi(f))}{A(1)}\right]\right\}\right],
\end{aligned} \tag{4.15}$$

where

$$\mathscr{J}^{\tau}\left(\chi\circ\psi^{-1},\psi(f),1;A\right)=A(1)\left[\chi\left(M_{\chi}(f;A)\right)-\chi\left(M_{\psi}(f;A)\right)\right] \tag{4.16}$$

and

$$M_{\eta}(f;A)=\eta^{-1}\left(\frac{A(\eta(f))}{A(1)}\right),\quad \eta=\chi,\psi. \tag{4.17}$$

We now observe the Jensen-type functional (2.17) defined by means of a generalized weight power mean (2.16).

Corollary 4.4 *Let r and $s\neq 0$ be real numbers and functions $f,p,q\in L$, $f(x)>0$, $x\in E$. Suppose $A\colon L\to\mathbb{R}$ is a positive linear functional such that $A(p),A(q)>0$. Then functional (2.17) possesses the following properties:*

(i) *If $r\neq 0$ and $s>0$, $s>r$ or $s<0$, $s<r$, then*

$$\begin{aligned}
&\min\{A(p),A(q)\}\left[\left(\frac{A(pf^r)}{A(p)}\right)^{\frac{s}{r}}+\left(\frac{A(qf^r)}{A(q)}\right)^{\frac{s}{r}}-2\left(\frac{A(pf^r)}{2A(p)}+\frac{A(qf^r)}{2A(q)}\right)^{\frac{s}{r}}\right]\\
&\leq \mathscr{J}^{\mathscr{P}}(\chi\circ\psi^{-1},\psi(f),p+q;A)-\mathscr{J}^{\mathscr{P}}(\chi\circ\psi^{-1},\psi(f),p;A)-\mathscr{J}^{\mathscr{P}}(\chi\circ\psi^{-1},\psi(f),q;A)\\
&\leq \max\{A(p),A(q)\}\left[\left(\frac{A(pf^r)}{A(p)}\right)^{\frac{s}{r}}+\left(\frac{A(qf^r)}{A(q)}\right)^{\frac{s}{r}}-2\left(\frac{A(pf^r)}{2A(p)}+\frac{A(qf^r)}{2A(q)}\right)^{\frac{s}{r}}\right].
\end{aligned} \tag{4.18}$$

(ii) *If $r\neq 0$ and $s>0$, $s>r$ or $s<0$, $s<r$, then for $p,q\in L^{+}$ such that $p\geq q$ the*

following inequality holds:

$$\begin{aligned}
&\mathscr{J}^{\mathscr{P}}(\chi\circ\psi^{-1},\psi(f),p;A)-\mathscr{J}^{\mathscr{P}}(\chi\circ\psi^{-1},\psi(f),q;A)\\
&\geq \min\{A(p-q),A(q)\}\left[\left(\frac{A(pf^r)-A(qf^r)}{A(p)-A(q)}\right)^{\frac{s}{r}}+\left(\frac{A(qf^r)}{A(q)}\right)^{\frac{s}{r}}\right.\\
&\quad\left.-2^{1-\frac{s}{r}}\left(\frac{A(pf^r)-A(qf^r)}{A(p)-A(q)}+\frac{A(qf^r)}{A(q)}\right)^{\frac{s}{r}}\right]. \qquad (4.19)
\end{aligned}$$

(iii) *If $r=0$, then*

$$\begin{aligned}
&\min\{A(p),A(q)\}\left[\exp\left(\frac{sA(p\ln f)}{A(p)}\right)+\exp\left(\frac{sA(q\ln f)}{A(q)}\right)\right.\\
&\quad\left.-2\exp\left(\frac{sA(p\ln f)}{2A(p)}+\frac{sA(q\ln f)}{2A(q)}\right)\right]\\
&\leq \mathscr{J}^{\mathscr{P}}(\chi\circ\psi^{-1},\psi(f),p+q;A)-\mathscr{J}^{\mathscr{P}}(\chi\circ\psi^{-1},\psi(f),p;A)-\mathscr{J}^{\mathscr{P}}(\chi\circ\psi^{-1},\psi(f),q;A)\\
&\leq \max\{A(p),A(q)\}\left[\exp\left(\frac{sA(p\ln f)}{A(p)}\right)+\exp\left(\frac{sA(q\ln f)}{A(q)}\right)\right.\\
&\quad\left.-2\exp\left(\frac{sA(p\ln f)}{2A(p)}+\frac{sA(q\ln f)}{2A(q)}\right)\right]. \qquad (4.20)
\end{aligned}$$

(iv) *If $r=0$, then for $p,q\in L^+$ such that $p\geq q$ the following inequality holds:*

$$\begin{aligned}
&\mathscr{J}^{\mathscr{P}}(\chi\circ\psi^{-1},\psi(f),p;A)-\mathscr{J}^{\mathscr{P}}(\chi\circ\psi^{-1},\psi(f),q;A)\\
&\geq \min\{A(p-q),A(q)\}\left[\exp\left(\frac{sA(p\ln f)-sA(q\ln f)}{A(p)-A(q)}\right)+\exp\left(\frac{sA(q\ln f)}{A(q)}\right)\right.\\
&\quad\left.-2\exp\left\{\frac{s}{2}\left[\frac{A(p\ln f)-A(q\ln f)}{A(p)-A(q)}+\frac{A(q\ln f)}{A(q)}\right]\right\}\right]. \qquad (4.21)
\end{aligned}$$

Proof. Follows directly from Theorem 4.2. We have to consider two cases depending on whether $r\neq 0$ or $r=0$.

If $r\neq 0$, we define $\chi(x)=x^s$ and $\psi(x)=x^r$. Then, $\chi\circ\psi^{-1}(x)=x^{\frac{s}{r}}$ and $\left(\chi\circ\psi^{-1}\right)''(x)$ $=\frac{s(s-r)}{r^2}x^{\frac{s}{r}-2}$. Thus, $\chi\circ\psi^{-1}$ is convex if $s>0$, $s>r$ or $s<0$, $s<r$. On the other hand, $\chi\circ\psi^{-1}$ is concave if $s>0$, $s<r$ or $s<0$, $s>r$.

If $r=0$, we put $\chi(x)=x^s$ and $\psi(x)=\ln x$. Then, $\chi\circ\psi^{-1}(x)=e^{sx}$ is convex under assumption $s\neq 0$. Results follow immediately from Theorem 4.2. □

Corollary 4.5 *Let $s\neq 0$ and r be real numbers such that $r\neq 0$, $s>0$, $s>r$ or $s<0$, $s<r$ and let $p\in L$ attain its minimal value $\underline{p}=\min_{x\in E}p(x)$ and its maximal value $\overline{p}=$*

$\max_{x\in E} p(x)$. *Then the following series of inequalities holds:*

$$\begin{aligned}
&\overline{p}\mathscr{J}^{\mathscr{P}}\left(\chi\circ\psi^{-1},\psi(f),1;A\right)-\mathscr{J}^{\mathscr{P}}\left(\chi\circ\psi^{-1},\psi(f),p;A\right)\\
&\geq \min\{\overline{p}A(1)-A(p),A(p)\}\left[\left(\frac{\overline{p}A(f^r)-A(pf^r)}{\overline{p}A(1)-A(p)}\right)^{\frac{s}{r}}\right.\\
&\quad\left.+\left(\frac{A(pf^r)}{A(p)}\right)^{\frac{s}{r}}-2^{1-\frac{s}{r}}\left(\frac{\overline{p}A(f^r)-A(pf^r)}{\overline{p}A(1)-A(p)}+\frac{A(pf^r)}{A(p)}\right)^{\frac{s}{r}}\right],
\end{aligned}\tag{4.22}$$

$$\begin{aligned}
&\mathscr{J}^{\mathscr{P}}\left(\chi\circ\psi^{-1},\psi(f),p;A\right)-\underline{p}\mathscr{J}^{\mathscr{P}}\left(\chi\circ\psi^{-1},\psi(f),1;A\right)\\
&\geq \min\{A(p)-\underline{p}A(1),\underline{p}A(1)\}\left[\left(\frac{A(pf^r)-\underline{p}A(f^r)}{A(p)-\underline{p}A(1)}\right)^{\frac{s}{r}}\right.\\
&\quad\left.+\left(\frac{A(f^r)}{A(1)}\right)^{\frac{s}{r}}-2^{1-\frac{s}{r}}\left(\frac{A(pf^r)-\underline{p}A(f^r)}{A(p)-\underline{p}A(1)}+\frac{A(f^r)}{A(1)}\right)^{\frac{s}{r}}\right].
\end{aligned}\tag{4.23}$$

If $r=0$, *then*

$$\begin{aligned}
&\overline{p}\mathscr{J}^{\mathscr{P}}\left(\chi\circ\psi^{-1},\psi(f),1;A\right)-\mathscr{J}^{\mathscr{P}}\left(\chi\circ\psi^{-1},\psi(f),p;A\right)\\
&\geq \min\{\overline{p}A(1)-A(p),A(p)\}\left[\exp\left(s\frac{\overline{p}A(\ln f)-A(p\ln f)}{\overline{p}A(1)-A(p)}\right)\right.\\
&\quad\left.+\exp\left(\frac{sA(p\ln f)}{A(p)}\right)-2\exp\left\{\frac{s}{2}\left[\frac{\overline{p}A(\ln f)-A(p\ln f)}{\overline{p}A(1)-A(p)}+\frac{A(p\ln f)}{A(p)}\right]\right\}\right],
\end{aligned}\tag{4.24}$$

$$\begin{aligned}
&\mathscr{J}^{\mathscr{P}}\left(\chi\circ\psi^{-1},\psi(f),p;A\right)-\underline{p}\mathscr{J}^{\mathscr{P}}\left(\chi\circ\psi^{-1},\psi(f),1;A\right)\\
&\geq \min\{A(p)-\underline{p}A(1),\underline{p}A(1)\}\left[\exp\left(s\frac{A(p\ln f)-\underline{p}A(\ln f)}{A(p)-\underline{p}A(1)}\right)\right.\\
&\quad\left.+\exp\left(\frac{sA(\ln f)}{A(1)}\right)-2\exp\left\{\frac{s}{2}\left[\frac{A(p\ln f)-\underline{p}A(\ln f)}{A(p)-\underline{p}A(1)}+\frac{A(\ln f)}{A(1)}\right]\right\}\right].
\end{aligned}\tag{4.25}$$

where

$$\mathscr{J}^{\mathscr{P}}\left(\chi\circ\psi^{-1},\psi(f),1;A\right)=A(1)\left\{[M_s(f;A)]^s-[M_r(f;A)]^s\right\}\tag{4.26}$$

and

$$M_t\left(f;A\right)=\begin{cases}\left(\frac{A(f^r)}{A(1)}\right)^{\frac{1}{t}}, & t\neq 0\\ \exp\left(\frac{A(\ln(f))}{A(1)}\right), & t=0\end{cases},\quad t=r,s.\tag{4.27}$$

Now, we consider discrete variants of relations (4.22)–(4.25). We suppose $E=\{1,2,\ldots,n\}$, $n\in\mathbb{N}$ and L is a class of real n-tuples. Hence we consider discrete functional defined by $A(\mathbf{x})=\sum_{i=1}^n x_i$, where $\mathbf{x}=(x_1,x_2,\ldots,x_n)$. Clearly, $A(\mathbf{1})=\sum_{i=1}^n 1=n$.

Recall the discrete form

$$M_r(\mathbf{x},\mathbf{p}) = \begin{cases} \left(\frac{\sum_{i=1}^n p_i x_i^r}{P_n}\right)^{\frac{1}{r}}, & r \neq 0 \\ \left(\prod_{i=1}^n x_i^{p_i}\right)^{\frac{1}{P_n}}, & r = 0 \end{cases} \tag{4.28}$$

that generalized weight power mean (2.16) assumes in this environment. For $r = 1$ we obtain arithmetic mean $A_n(\mathbf{x},\mathbf{p}) = M_1(\mathbf{x},\mathbf{p}) = \left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)$, while for $r = 0$ geometric mean $G_n(\mathbf{x},\mathbf{p}) = M_0(\mathbf{x},\mathbf{p}) = \left(\prod_{i=1}^n x_i^{p_i}\right)^{\frac{1}{P_n}}$ is obtained. Now, if we insert constant n-tuples

$$\overline{\mathbf{p}} = \left(\max_{1\le i\le n}\{p_i\},\ldots,\max_{1\le i\le n}\{p_i\}\right) \text{ or } \underline{\mathbf{p}} = \left(\min_{1\le i\le n}\{p_i\},\ldots,\min_{1\le i\le n}\{p_i\}\right),$$

expressions for arithmetic and geometric mean reduce to

$$A_n^0(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^n x_i \quad \text{and} \quad G_n^0(\mathbf{x}) = \left(\prod_{i=1}^n x_i\right)^{\frac{1}{n}} \tag{4.29}$$

and inequalities (4.24) and (4.25) for $s = 1$ and $r = 0$ can be rewritten as

$$\begin{aligned} &n\max_{1\le i\le n}\{p_i\}\left[A_n^0(\mathbf{x}) - G_n^0(\mathbf{x})\right] - P_n\left[A_n(\mathbf{x},\mathbf{p}) - G_n(\mathbf{x},\mathbf{p})\right] \\ &\ge \min\{n\max_{1\le i\le n}\{p_i\} - P_n, P_n\}\left[\exp\left(\frac{\max_{1\le i\le n}\{p_i\}\ln(G_n^0(\mathbf{x}))^n - \ln(G_n(\mathbf{x},\mathbf{p}))^{P_n}}{n\max_{1\le i\le n}\{p_i\} - P_n}\right)\right. \\ &\quad \left. + G_n(\mathbf{x},\mathbf{p}) - 2\exp\left\{\frac{1}{2}\left[\frac{\max_{1\le i\le n}\{p_i\}\ln(G_n^0(\mathbf{x}))^n - \ln(G_n(\mathbf{x},\mathbf{p}))^{P_n}}{n\max_{1\le i\le n}\{p_i\} - P_n} + \ln G_n(\mathbf{x},\mathbf{p})\right]\right\}\right], \end{aligned} \tag{4.30}$$

$$\begin{aligned} &P_n\left[A_n(\mathbf{x},\mathbf{p}) - G_n(\mathbf{x},\mathbf{p})\right] - n\min_{1\le i\le n}\{p_i\}\left[A_n^0(\mathbf{x}) - G_n^0(\mathbf{x})\right] \\ &\ge \min\{P_n - n\min_{1\le i\le n}\{p_i\}, n\min_{1\le i\le n}\{p_i\}\}\left[\exp\left(\frac{\ln(G_n(\mathbf{x},\mathbf{p}))^{P_n} - \min_{1\le i\le n}\{p_i\}\ln(G_n^0(\mathbf{x}))^n}{P_n - n\min_{1\le i\le n}\{p_i\}}\right)\right. \\ &\quad \left. + G_n^0(\mathbf{x}) - 2\exp\left\{\frac{1}{2}\left[\frac{\ln(G_n(\mathbf{x},\mathbf{p}))^{P_n} - \min_{1\le i\le n}\{p_i\}\ln(G_n^0(\mathbf{x}))^n}{P_n - n\min_{1\le i\le n}\{p_i\}} + \ln G_0(\mathbf{x})\right]\right\}\right]. \end{aligned} \tag{4.31}$$

Some variants of inequalities (4.30) and (4.31) were studied in papers [4]–[9].

Remark 4.2 As we previously discussed (see Chapter 2.1), Young's inequality follows directly from arithmetic-geometric mean inequality, so relations (4.30) and (4.31) provide refinements of Young's inequality, as well. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ be positive n-tuples such that $\sum_{i=1}^n \frac{1}{p_i} = 1$. We denote

$$\mathbf{x}^{\mathbf{p}} = \left(x_1^{p_1}, x_2^{p_2}, \ldots, x_n^{p_n}\right) \quad \text{and} \quad \mathbf{p}^{-1} = \left(\frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_n}\right).$$

Series of inequalities (4.30) and (4.31) can be rewritten in the form

$$
\begin{aligned}
& n \max_{1\le i\le n}\left\{\frac{1}{p_i}\right\}\left[A_n^0(\mathbf{x}^{\mathbf{p}})-G_n^0(\mathbf{x}^{\mathbf{p}})\right]-\left[A_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1})-G_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1})\right] \\
\ge\ & \min\left\{n \max_{1\le i\le n}\left\{\frac{1}{p_i}\right\}-1,1\right\}\left[\exp\left(\frac{\max_{1\le i\le n}\left\{\frac{1}{p_i}\right\}\ln(G_n^0(\mathbf{x}^{\mathbf{p}}))^n-\ln(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1})}{n\max_{1\le i\le n}\left\{\frac{1}{p_i}\right\}-1}\right)\right. \\
& +G_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1}) \\
& \left.-2\exp\left\{\frac{1}{2}\left[\frac{\max_{1\le i\le n}\left\{\frac{1}{p_i}\right\}\ln(G_n^0(\mathbf{x}^{\mathbf{p}}))^n-\ln G_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1})}{n\max_{1\le i\le n}\left\{\frac{1}{p_i}\right\}-1}+\ln G_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1})\right]\right\}\right],
\end{aligned}
\tag{4.32}
$$

$$
\begin{aligned}
& A_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1})-G_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1})-n\min_{1\le i\le n}\left\{\frac{1}{p_i}\right\}\left[A_n^0(\mathbf{x}^{\mathbf{p}})-G_n^0(\mathbf{x}^{\mathbf{p}})\right] \\
\ge\ & \min\left\{1-n\min_{1\le i\le n}\left\{\frac{1}{p_i}\right\},n\min_{1\le i\le n}\left\{\frac{1}{p_i}\right\}\right\} \\
& \times\left[\exp\left(\frac{\ln G_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1})-\min_{1\le i\le n}\left\{\frac{1}{p_i}\right\}\ln(G_n^0(\mathbf{x}^{\mathbf{p}}))^n}{1-n\min_{1\le i\le n}\left\{\frac{1}{p_i}\right\}}\right)\right. \\
& \left.+G_n^0(\mathbf{x}^{\mathbf{p}})-2\exp\left\{\frac{1}{2}\left[\frac{\ln G_n(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1})-\min_{1\le i\le n}\left\{\frac{1}{p_i}\right\}\ln(G_n^0(\mathbf{x}^{\mathbf{p}}))^n}{1-n\min_{1\le i\le n}\left\{\frac{1}{p_i}\right\}}+\ln(G_0(\mathbf{x}^{\mathbf{p}}))\right]\right\}\right].
\end{aligned}
\tag{4.33}
$$

Since corollaries 4.4 and 4.5 do not cover the case with $s=0$ and $r\neq 0$, this case ought to be considered separately, as it was done in Chapter 2.1, where for that purpose functional (2.28) was defined. The refinements that correspond to this functional are given in the following corollary.

Corollary 4.6 *Let $r<0$ and let $f,p,q\in L$, $f(x)>0$, $x\in E$. Suppose $A\colon L\to\mathbb{R}$ is a positive linear functional such that $A(p),A(q)>0$. Then functional* (2.28) *possesses the following properties:*

(i)

$$
\begin{aligned}
& \min\{A(p),A(q)\}\frac{1}{r}\left[\ln\left(\frac{A(pf^r)}{A(p)}\right)+\ln\left(\frac{A(qf^r)}{A(q)}\right)-2\ln\left(\frac{A(pf^r)}{2A(p)}+\frac{A(qf^r)}{2A(q)}\right)\right] \\
\le\ & \mathscr{J}^{\overline{\mathscr{P}}}(\chi\circ\psi^{-1},\psi(f),p+q;A)-\mathscr{J}^{\overline{\mathscr{P}}}(\chi\circ\psi^{-1},\psi(f),p;A)-\mathscr{J}^{\overline{\mathscr{P}}}(\chi\circ\psi^{-1},\psi(f),q;A) \\
\le\ & \max\{A(p),A(q)\}\frac{1}{r}\left[\ln\left(\frac{A(pf^r)}{A(p)}\right)+\ln\left(\frac{A(qf^r)}{A(q)}\right)-2\ln\left(\frac{A(pf^r)}{2A(p)}+\frac{A(qf^r)}{2A(q)}\right)\right].
\end{aligned}
\tag{4.34}
$$

(ii) *If* $p,q \in L^+$ *with* $p \geq q$ *and* $A(p) \geq A(q) \geq 0$, *then*

$$\begin{aligned}&\mathscr{J}^{\overline{\mathscr{P}}}(\chi\circ\psi^{-1},\psi(f),p;A)-\mathscr{J}^{\overline{\mathscr{P}}}(\chi\circ\psi^{-1},\psi(f),q;A)\\ &\geq \min\{A(p)-A(q),A(q)\}\frac{1}{r}\left[\ln\left(\frac{A(pf^r)-A(qf^r)}{A(p)-A(q)}\right)+\ln\left(\frac{A(qf^r)}{A(q)}\right)\right.\\ &\quad\left.-2\ln\left\{\frac{1}{2}\left[\frac{A(pf^r)-A(qf^r)}{A(p)-A(q)}+\frac{A(qf^r)}{A(q)}\right]\right\}\right].\end{aligned} \tag{4.35}$$

Proof. The proof is direct consequence of Theorem 4.2. We define $\chi(x)=\ln x$ and $\psi(x)=x^r$. Then function $\chi\circ\psi^{-1}(x)=\frac{1}{r}\ln x$ is convex if $r<0$ and concave if $r>0$. That completes the proof. □

The analogue of Corollary 4.5, that covers the case $s=0$ and $r\neq 0$ is contained in the following result.

Corollary 4.7 *Let* $r<0$, $f\in L^+$, $f(x)>0$, $x\in E$ *and* $A\colon L\to\mathbb{R}$ *be a positive linear functional. Suppose* $p\in L^+$ *attains its minimal value* $\underline{p}=\min_{x\in E}p(x)$ *and its maximal value* $\overline{p}=\max_{x\in E}p(x)$. *Then the following series of inequalities holds:*

$$\begin{aligned}&\overline{p}\mathscr{J}^{\overline{\mathscr{P}}}(\chi\circ\psi^{-1},\psi(f),1;A)-\mathscr{J}^{\overline{\mathscr{P}}}(\chi\circ\psi^{-1},\psi(f),p;A)\\ &\geq\min\{\overline{p}A(1)-A(p),A(p)\}\frac{1}{r}\left[\ln\left(\frac{\overline{p}A(f^r)-A(pf^r)}{\overline{p}A(1)-A(p)}\right)\right.\\ &\quad\left.+\ln\left(\frac{A(pf^r)}{A(p)}\right)-2\ln\left\{\frac{1}{2}\left[\frac{\overline{p}A(f^r)-A(pf^r)}{\overline{p}A(1)-A(p)}+\frac{A(f^r)}{A(p)}\right]\right\}\right],\end{aligned} \tag{4.36}$$

$$\begin{aligned}&\mathscr{J}^{\overline{\mathscr{P}}}(\chi\circ\psi^{-1},\psi(f),p;A)-\underline{p}\mathscr{J}^{\overline{\mathscr{P}}}(\chi\circ\psi^{-1},\psi(f),1;A)\\ &\geq\min\{A(p)-\underline{p}A(1),\underline{p}A(1)\}\frac{1}{r}\left[\ln\left(\frac{A(pf^r)-\underline{p}A(f^r)}{A(p)-\underline{p}A(1)}\right)\right.\\ &\quad\left.+\ln\left(\frac{A(f^r)}{A(1)}\right)-2\ln\left\{\frac{1}{2}\left[\frac{A(pf^r)-\underline{p}A(f^r)}{A(p)-\underline{p}A(1)}+\frac{A(f^r)}{A(1)}\right]\right\}\right],\end{aligned} \tag{4.37}$$

where

$$\mathscr{J}^{\overline{\mathscr{P}}}(\chi\circ\psi^{-1},\psi(f),1;A)=A(1)\left(\frac{A(\ln f)}{A(1)}-\ln[M_r(f;A)]\right). \tag{4.38}$$

4.1.2 Application to Hölder's inequality

Refinements presented in the previous section, considering arithmetic-geometric and Young's inequality imply that the analogous ones can be obtained for Hölder's inequality (2.33), expressed by means of a positive linear functional, in a continuation to the corresponding results from Chapter 2.1 An improvement of Theorem 2.5 is thus given in the following theorem. Here it involves inequality (4.32), although an analogous result can be obtained for inequality (4.33).

Theorem 4.3 *Let $p_i > 1$, $i = 1,2,\dots,n$ be such that $\sum_{i=1}^n \frac{1}{p_i} = 1$. Suppose $f_i \in L^+$, $i = 1,2,\dots,n$ are such that $\prod_{i=1}^n f_i^{1/p_i}$, $\prod_{i=1}^n f_i^{1/n} \in L^+$. If $A\colon L \to \mathbb{R}$ is a positive linear functional, then the following inequality holds:*

$$
\begin{aligned}
& n \max_{1\le i\le n}\left\{\frac{1}{p_i}\right\}\left[\prod_{i=1}^n A^{\frac{1}{p_i}}(f_i) - \prod_{i=1}^n A^{\frac{1}{p_i}-\frac{1}{n}}(f_i)\cdot A\left(\prod_{i=1}^n f_i^{\frac{1}{n}}\right) - \prod_{i=1}^n A^{\frac{1}{p_i}}(f_i) - A\left(\prod_{i=1}^n f_i^{\frac{1}{p_i}}\right)\right] \\
& \ge \min\{n \max_{1\le i\le n}\left\{\frac{1}{p_i}\right\} - 1, 1\} \prod_{i=1}^n A^{\frac{1}{p_i}}(f_i) \\
& \times \left[A\left\{ \exp\left(\frac{n \max_{1\le i\le n}\left\{\frac{1}{p_i}\right\} \ln \prod_{i=1}^n \frac{f_i^{\frac{1}{n}}}{A^{\frac{1}{n}}(f_i)} - \ln \prod_{i=1}^n \frac{f_i^{\frac{1}{p_i}}}{A^{\frac{1}{p_i}}(f_i)}}{n \max_{1\le i\le n}\left\{\frac{1}{p_i}\right\} - 1} \right)\right\} + A\left(\prod_{i=1}^n f_i^{\frac{1}{p_i}}\right) \right. \\
& \left. - 2A\left\{ \exp\left(\frac{n \max_{1\le i\le n}\left\{\frac{1}{p_i}\right\} \ln \prod_{i=1}^n \frac{f_i^{\frac{1}{n}}}{A^{\frac{1}{n}}(f_i)} - \ln \prod_{i=1}^n \frac{f_i^{\frac{1}{p_i}}}{A^{\frac{1}{p_i}}(f_i)}}{2(n \max_{1\le i\le n}\left\{\frac{1}{p_i}\right\} - 1)} + \frac{1}{2}\ln \prod_{i=1}^n \frac{f_i^{\frac{1}{p_i}}}{A^{\frac{1}{p_i}}(f_i)} \right)\right\}\right].
\end{aligned} \tag{4.39}
$$

Proof. If we consider n-tuple $\mathbf{x} = (x_1, x_2, \dots, x_n)$, where $x_i = [f_i/A(f_i)]^{\frac{1}{p_i}}$, $i = 1,2,\dots,n$, the expressions in (4.32) that represent the difference between arithmetic and geometric mean, become

$$
\begin{aligned}
A_n(\mathbf{x^p}, \mathbf{p}^{-1}) - G_n(\mathbf{x^p}, \mathbf{p}^{-1}) &= \sum_{i=1}^n \frac{f_i}{p_i A(f_i)} - \prod_{i=1}^n \frac{f_i^{\frac{1}{p_i}}}{A^{\frac{1}{p_i}}(f_i)}, \\
A_n^0(\mathbf{x^p}) - G_n^0(\mathbf{x^p}) &= \frac{1}{n}\sum_{i=1}^n \frac{f_i}{A(f_i)} - \prod_{i=1}^n \frac{f_i^{\frac{1}{n}}}{A^{\frac{1}{n}}(f_i)}.
\end{aligned}
$$

Now, if we apply functional A to the above expressions, and use its linearity, we get

$$
A\left[A_n(\mathbf{x^p}, \mathbf{p}^{-1}) - G_n(\mathbf{x^p}, \mathbf{p}^{-1})\right] = \sum_{i=1}^n \frac{A(f_i)}{p_i A(f_i)} - \frac{A\left(\prod_{i=1}^n f_i^{\frac{1}{p_i}}\right)}{\prod_{i=1}^n A^{\frac{1}{p_i}}(f_i)} = 1 - \frac{A\left(\prod_{i=1}^n f_i^{\frac{1}{p_i}}\right)}{\prod_{i=1}^n A^{\frac{1}{p_i}}(f_i)}
$$

and

$$
A\left[A_n^0(\mathbf{x^p}) - G_n^0(\mathbf{x^p})\right] = \frac{1}{n}\sum_{i=1}^n \frac{A(f_i)}{A(f_i)} - \frac{A\left(\prod_{i=1}^n f_i^{\frac{1}{n}}\right)}{\prod_{i=1}^n A^{\frac{1}{n}}(f_i)} = 1 - \frac{A\left(\prod_{i=1}^n f_i^{\frac{1}{n}}\right)}{\prod_{i=1}^n A^{\frac{1}{n}}(f_i)}.
$$

Applying functional A to the inequality (4.32), the sign of the inequality does not change, since A is linear and positive. □

Yet another improvement of the previously presented results is motivated by recalling the fact that Hölder's inequality can be deduced directly from Jensen's inequality, which was described in detail in Chapter 2.1 where functional (2.38) was deduced and Theorem 2.7 was established. We now give its refined form.

Theorem 4.4 *Let r and $s \in \mathbb{R}$ be such that $1/r+1/s=1$. Suppose $A\colon L \to \mathbb{R}$ is a positive linear functional, $f,g \in L^+$ and f attains its minimal and its maximal value on E. If $r>1$, then*

$$\begin{aligned}
&[\max_{x\in E} f(x)]\left[A^{\frac{1}{r}}(1)A^{\frac{1}{s}}\left(\frac{g}{f}\right)-A\left(\left(\frac{g}{f}\right)^{\frac{1}{s}}\right)\right]-A^{\frac{1}{r}}(f)A^{\frac{1}{s}}(g)-A\left(f^{\frac{1}{r}}g^{\frac{1}{s}}\right)\\
&\geq \min\{[\max_{x\in E} f(x)]A(1)-A(f),A(f)\}\left[2^{1-\frac{1}{s}}\left(\frac{[\max_{x\in E} f(x)]A\left(\frac{g}{f}\right)-A(g)}{[\max_{x\in E} f(x)]A(1)-A(f)}+\frac{A(g)}{A(f)}\right)^{\frac{1}{s}}\right.\\
&\left.\quad-\left(\frac{[\max_{x\in E} f(x)]A\left(\frac{g}{f}\right)-A(g)}{[\max_{x\in E} f(x)]A(1)-A(f)}\right)^{\frac{1}{s}}+A^{\frac{1}{r}}(f)A^{\frac{1}{s}}(g)\right]. \qquad (4.40)
\end{aligned}$$

Proof. We consider relation (4.7) from Corollary 4.2 with arguments f and p respectively replaced with g/f and f, where $\Phi(x)=-rsx^{1/s}$. Clearly, $\Phi''(x)=x^{1/s-2}$, so Φ is convex function if $x>0$. In this setting, Jessen's functional (2.2) reads

$$\begin{aligned}
\mathscr{J}^{\mathscr{H}}\left(\Phi,\frac{g}{f},f;A\right) &= A\left(f\Phi\left(\frac{g}{f}\right)\right)-A(f)\Phi\left(\frac{A(g)}{A(f)}\right)\\
&= rs\left[A^{1-\frac{1}{s}}(f)A^{\frac{1}{s}}(g)-A\left(f^{1-\frac{1}{s}}g^{\frac{1}{s}}\right)\right]\\
&= rs\left[A^{\frac{1}{r}}(f)A^{\frac{1}{s}}(g)-A\left(f^{\frac{1}{r}}g^{\frac{1}{s}}\right)\right].
\end{aligned}$$

Further,

$$\begin{aligned}
\mathscr{J}^{\mathscr{H}}\left(\Phi,\frac{g}{f},1;A\right) &= A\left(\Phi\left(\frac{g}{f}\right)\right)-A(1)\Phi\left(\frac{A\left(\frac{g}{f}\right)}{A(1)}\right)\\
&= rs\left[A^{1-\frac{1}{s}}(1)A^{\frac{1}{s}}\left(\frac{g}{f}\right)-A\left(\left(\frac{g}{f}\right)^{\frac{1}{s}}\right)\right]\\
&= rs\left[A^{\frac{1}{r}}(1)A^{\frac{1}{s}}\left(\frac{g}{f}\right)-A\left(\left(\frac{g}{f}\right)^{\frac{1}{s}}\right)\right].
\end{aligned}$$

Now, we insert obtained expressions $\mathscr{J}^{\mathscr{H}}(\Phi,g/f,f;A)$ and $\mathscr{J}^{\mathscr{H}}(\Phi,g/f,1;A)$ in (4.7) and obtain (4.40). □

4.2 Superadditivity of Levinson's functional

In 1964, N. Levinson [126], proved the following:

If $f:(0,2c)\to\mathbb{R}$ has a non-negative third derivative and p_i, x_i, y_i, $i=1,2,\dots,n$, are such that $p_i>0$, $\sum_{i=1}^n p_i=1$, $0\le x_i\le c$, and

$$x_1+y_1=x_2+y_2=\cdots=x_n+y_n=2c, \tag{4.41}$$

then the inequality

$$\sum_{i=1}^n p_i f(x_i)-f(\overline{x})\le\sum_{i=1}^n p_i f(y_i)-f(\overline{y}) \tag{4.42}$$

holds, where $\overline{x}=\sum_{i=1}^n p_i x_i$ and $\overline{y}=\sum_{i=1}^n p_i y_i$ are the weight arithmetic means.

During decades, Levinson's result has been generalized and extended in several directions. Popoviciu [189], noted that the assumptions on the differentiability of f can be weakened and for the inequality (4.42) it suffices to assume that f is 3-convex.

Recall that $f:I\to\mathbb{R}$ is n-convex if its nth order divided difference is non-negative, that is, if $[x_0,x_1,\dots,x_n]f\ge 0$, for all choices of $n+1$ distinct points $x_0,x_1,\dots,x_n\in I$. The nth order divided difference of a function $f:I\to\mathbb{R}$ at distinct points $x_0,x_1,\dots,x_n\in I$ is defined inductively by

$$[x_i]f=f(x_i),\quad i=0,1,2,\dots,n,$$

and

$$[x_0,x_1,\dots,x_n]f=\frac{[x_1,\dots,x_n]f-[x_0,\dots,x_{n-1}]f}{x_n-x_0}.$$

If the nth derivative $f^{(n)}$ of an n-convex function exists, then $f^{(n)}\ge 0$, but $f^{(n)}$ may not exist (for more details, see [177]).

Bullen [45], gave another proof of the Popoviciu result rescaled to a general interval: if $f:[a,b]\to\mathbb{R}$ is 3-convex and p_i,x_i,y_i, $i=1,2,\dots,n$, are such that $p_i>0$, $\sum_{i=1}^n p_i=1$, $a\le x_i,y_i\le b$, (4.41) holds for some $c\in[a,b]$ and

$$\max\{x_1,\dots,x_n\}\le\min\{y_1,\dots,y_n\}, \tag{4.43}$$

then (4.42) holds.

The aforementioned generalizations of the Levinson inequality assume that the distribution of the points x_i is equal to the distribution of the points y_i reflected around the point $c\in[a,b]$. A few years ago, Mercer [135], gave an important extension of the Levinson inequality by replacing the condition of symmetric distribution with the weaker one that the variances of the corresponding sequences are equal. More precisely, he showed that if a function f has a non-negative third derivative and (4.43) holds, then the inequality (4.42) is valid when the condition (4.41) is replaced by a weaker assumption

$$\sum_{i=1}^n p_i(x_i-\overline{x})^2=\sum_{i=1}^n p_i(y_i-\overline{y})^2. \tag{4.44}$$

Witkowski [213], extended the result of Mercer to hold for a 3-convex function and he further weakened the condition (4.44) by replacing equality of variances with the inequality in a certain direction.

Motivated by the ideas of Witkowski, Pečarić *et.al.* [176] (see also [21]), showed that the Levinson inequality under the Mercer assumption (4.44) holds for a more general class of functions described in the following definition.

Definition 4.1 *Let $\Phi\colon I \to \mathbb{R}$ and $c \in I^0$, where I^0 is the interior of the interval I. We say that $\Phi \in \mathscr{K}_1^c(I)$ ($\Phi \in \mathscr{K}_2^c(I)$) if there exists a constant α such that the function $F(x) = \Phi(x) - \frac{\alpha}{2}x^2$ is concave (resp. convex) on $I \cap (-\infty, c]$ and convex (resp. concave) on $I \cap [c, \infty)$.*

Remark 4.3 It should be noticed here that the constant α appearing in Definition 4.1 is not necessarily unique. For example, it has been shown in [21] that if $\Phi \in \mathscr{K}_1^c(I)$ and there exist one-sided second order derivatives $\Phi''_-(c), \Phi''_+(c)$, then α can be chosen to be any real number from the interval $\left[\Phi''_-(c), \Phi''_+(c)\right]$. In particular, if there exists the second order derivative $\Phi''(c)$, then $\alpha = \Phi''(c)$, which yields the uniqueness of the constant in this case.

A function $\Phi \in \mathscr{K}_1^c(I)$ is said to be 3-convex at point c and $\mathscr{K}_1^c(I)$ generalizes 3-convex functions in the following sense: a function is 3-convex on I if and only if it is 3-convex at every $c \in I^0$. For example, $\Phi(x) = x^4$ is an example of a function that belongs to $\mathscr{K}_1^2(-1,3)$, but is not 3-convex on $(-1,3)$. Moreover, function $\Phi(x) = |x|$ belongs to $\mathscr{K}_1^0(-1,1)$, but is not differentiable at 0 (for more details, see [21]).

Pečarić *et.al.* [176], proved a more general probabilistic version of the Levinson inequality under the assumption of equality of variances. In particular, they showed that in the discrete Levinson inequality the number of the points of two sequences and associated weights do not need to be same. More precisely, they showed that if $x_i \in I \cap (-\infty, c]$, $y_j \in I \cap [c, \infty)$, $p_i > 0$, $q_j > 0$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, are such that

$$\sum_{i=1}^{n} p_i = \sum_{j=1}^{m} q_j = 1 \qquad \text{and} \qquad \sum_{i=1}^{n} p_i (x_i - \overline{x})^2 = \sum_{j=1}^{m} q_j (y_j - \overline{y})^2,$$

where $\overline{x} = \sum_{i=1}^{n} p_i x_i$ and $\overline{y} = \sum_{j=1}^{m} q_j y_j$, then the inequality

$$\sum_{i=1}^{n} p_i \Phi(x_i) - \Phi(\overline{x}) \leq \sum_{j=1}^{m} q_j \Phi(y_j) - \Phi(\overline{y}) \tag{4.45}$$

holds for every $\Phi \in \mathscr{K}_1^c(I)$.

Note also that paper [176] shows that $\mathscr{K}_1^c(I)$ is the largest class of the functions for which the Levinson inequality holds. In other words, class $\mathscr{K}_1^c(I)$ characterizes the Levinson inequality.

The recent investigation on this topic ([109]) related to the previously presented results on superadditivity of Jessen's functional in Chapter 2, again takes place in the linear space of the real valued functions and the positive linear functionals acting on it. Thus Levinson's functional is established in the above setting, and its corresponding properties are analyzed.

Similarly to the concept of the Jessen functional, the Levinson functional may be regarded as the difference between the right-hand side and the left-hand side of the Levinson inequality. We deal with the difference between two Jessen functionals. The associated linear spaces and functionals need not to be the same.

Let L_1 and L_2 be the linear spaces of real-valued functions defined on nonempty sets E_1 and E_2, respectively. Further, let A_1 and A_2 be positive linear functionals defined on L_1 and L_2, respectively. In this setting, the Levinson functional $\mathscr{L}_{A_1,A_2}$ is defined as the difference between the corresponding Jessen functionals, i.e.

$$\mathscr{L}_{A_1,A_2}(\Phi,f_1,f_2,p_1,p_2) = \mathscr{J}_{A_2}(\Phi,f_2,p_2) - \mathscr{J}_{A_1}(\Phi,f_1,p_1), \tag{4.46}$$

where $A_i : L_i \to \mathbb{R}$ are positive linear functionals, $f_i \in L_i$, $p_i \in L_i^+$, $i=1,2$, and $\Phi \in \mathscr{K}_1^c(I)$ (or $\mathscr{K}_2^c(I)$). Recall that L_i^+ stands for the subset of L_i, $i=1,2$, consisting of all non-negative functions.

In order to show that the Levinson functional possesses the properties of superadditivity and monotonicity, Jessen functional's properties of the same kind are employed.

Remark 4.4 While the Jessen functional is accompanied with a convex or a concave function, the Levinson functional is related to a class $\mathscr{K}_1^c(I)$ or $\mathscr{K}_2^c(I)$. Thus, if $\Phi \in \mathscr{K}_1^c(I)$ (or $\mathscr{K}_2^c(I)$), the expression $\mathscr{J}_A(\Phi,f,p)$ should be transformed so that it contains a convex (or a concave) function as an argument. More precisely, let $F(x) = \Phi(x) - \frac{\alpha}{2}x^2$, where α is an arbitrary constant fulfilling conditions as in Definition 4.1. Then, utilizing the definition of the Jessen functional, we have

$$\begin{aligned}
\mathscr{J}_A(\Phi,f,p) &= A(pF(f)) - A(p)F\left(\frac{A(pf)}{A(p)}\right) + \frac{\alpha}{2}\left[A(pf^2) - \frac{(A(pf))^2}{A(p)}\right] \\
&= \mathscr{J}_A(F,f,p) + \frac{\alpha}{2}A\left(p\left(f - \frac{A(pf)}{A(p)}\right)^2\right).
\end{aligned}$$

It should be noticed here that if $A(p)=1$, then the quantity $A\big(p(f - \frac{A(pf)}{A(p)})^2\big)$ represents a variance of the function f. In order to summarize our discussion, we use the abbreviation

$$\Delta_A(f,p) = A\left(p\left(f - \frac{A(pf)}{A(p)}\right)^2\right),$$

so that the previous relation can be rewritten as

$$\mathscr{J}_A(\Phi,f,p) = \mathscr{J}_A(F,f,p) + \frac{\alpha}{2}\Delta_A(f,p). \tag{4.47}$$

This relation will be extensively used in deriving the properties of the Levinson functional. Clearly, the relation (4.47) depends on the chosen constant α.

The following proposition yields conditions under which the Levinson functional is positive (or negative). It corresponds to a probabilistic version of the Levinson inequality derived in [176] (see Theorem 2.3).

Proposition 4.1 *Suppose $A_i : L_i \to \mathbb{R}$ are positive linear functionals and let $f_i \in L_i$, $p_i \in L_i^+$, $i = 1,2$. Further, let $f_i(E_i) \subseteq I$, $i = 1,2$, where I is an interval, and suppose that there exists $c \in I^0$ such that*

$$\sup_{x\in E_1} f_1(x) \le c \le \inf_{x\in E_2} f_2(x). \tag{4.48}$$

If $\Delta_{A_1}(f_1,p_1) = \Delta_{A_2}(f_2,p_2)$, then the inequality

$$\mathscr{L}_{A_1,A_2}(\Phi, f_1, f_2, p_1, p_2) \ge 0 \tag{4.49}$$

holds for every continuous function $\Phi \in \mathscr{K}_1^c(I)$, provided that $p_i f_i, p_i f_i^2$, $p_i\Phi(f_i) \in L_i$, and $A_i(p_i) > 0$, $i = 1,2$. If $\Phi \in \mathscr{K}_2^c(I)$ is a continuous function, then the sign of inequality (4.49) *is reversed.*

Proof. Since $\Delta_{A_1}(f_1,p_1) = \Delta_{A_2}(f_2,p_2)$, taking into account the relation (4.47), we have

$$\begin{aligned}\mathscr{L}_{A_1,A_2}(\Phi, f_1, f_2, p_1, p_2) &= \mathscr{J}_{A_2}(\Phi, f_2, p_2) - \mathscr{J}_{A_1}(\Phi, f_1, p_1)\\ &= \mathscr{J}_{A_2}(F, f_2, p_2) - \mathscr{J}_{A_1}(F, f_1, p_1) + \frac{\alpha}{2}\left(\Delta_{A_2}(f_2,p_2) - \Delta_{A_1}(f_1,p_1)\right)\\ &= \mathscr{J}_{A_2}(F, f_2, p_2) - \mathscr{J}_{A_1}(F, f_1, p_1),\end{aligned}$$

where F and α are as in Definition 4.1. Moreover, since F is convex on $I \cap [c,\infty)$ and concave on $I \cap (-\infty, c]$, utilizing the Jensen inequality (2.1) we have $\mathscr{J}_{A_2}(F, f_2, p_2) \ge 0$ and $\mathscr{J}_{A_1}(F, f_1, p_1) \le 0$, which completes the proof. □

Rewriting inequality (4.49) in its expanded form yields

$$A_2(p_2\Phi(f_2)) - A_2(p_2)\Phi\left(\frac{A_2(p_2f_2)}{A_2(p_2)}\right) \ge A_1(p_1\Phi(f_1)) - A_1(p_1)\Phi\left(\frac{A_1(p_1f_1)}{A_1(p_1)}\right), \tag{4.50}$$

which represents the Levinson inequality in this setting.

Remark 4.5 It is obvious from the proof of the Proposition 4.1 that the inequality (4.49) holds if the condition $\Delta_{A_1}(f_1,p_1) = \Delta_{A_2}(f_2,p_2)$ is replaced by the weaker condition

$$\alpha\left(\Delta_{A_2}(f_2,p_2) - \Delta_{A_1}(f_1,p_1)\right) \ge 0,$$

where α is any constant fulfilling the conditions as in Definition 4.1. Since $\Phi''_-(c) \le \alpha \le \Phi''_+(c)$ (for more details, see [21]), if, additionally, Φ is convex (resp. concave), this condition can be further weakened to $\Delta_{A_2}(f_2,p_2) - \Delta_{A_1}(f_1,p_1) \ge 0$ (resp. ≤ 0).

The following theorem provides the superadditivity property of the Levinson functional which is the crucial result in further investigation.

Theorem 4.5 *Suppose $A_i : L_i \to \mathbb{R}$ are positive linear functionals and let $f_i \in L_i$, $p_i, q_i \in L_i^+$, $i = 1,2$. Further, let $f_i(E_i) \subseteq I$, $i = 1,2$, where I is an interval, and suppose that there exists $c \in I^0$ such that the condition* (4.48) *is fulfilled. If*

$$\begin{aligned}&\Delta_{A_1}(f_1,p_1) + \Delta_{A_1}(f_1,q_1) - \Delta_{A_1}(f_1,p_1+q_1)\\ =&\Delta_{A_2}(f_2,p_2) + \Delta_{A_2}(f_2,q_2) - \Delta_{A_2}(f_2,p_2+q_2),\end{aligned} \tag{4.51}$$

then for every continuous function $\Phi \in \mathcal{K}_1^c(I)$ *one has*

$$\mathcal{L}_{A_1,A_2}(\Phi, f_1, f_2, p_1+q_1, p_2+q_2) \geq \mathcal{L}_{A_1,A_2}(\Phi, f_1, f_2, p_1, p_2) + \mathcal{L}_{A_1,A_2}(\Phi, f_1, f_2, q_1, q_2), \tag{4.52}$$

provided that $p_i f_i, q_i f_i, p_i f_i^2, q_i f_i^2, p_i\Phi(f_i), q_i\Phi(f_i) \in L_i$, *and* $A_i(p_i) > 0$, $A_i(q_i) > 0$, $i = 1,2$. *If* $\Phi \in \mathcal{K}_2^c(I)$ *is a continuous function, then the sign of inequality* (4.52) *is reversed.*

Proof. Let $\Phi \in \mathcal{K}_1^c(I)$ and $F(x) = \Phi(x) - \frac{\alpha}{2}x^2$, where α is any constant fulfilling conditions as in Definition 4.1. Then, taking into account the relation (4.47) and the fact that the function F is convex on $I \cap [c, \infty)$, it follows that

$$\begin{aligned}
\mathcal{J}_{A_2}(\Phi, f_2, p_2+q_2) &= \mathcal{J}_{A_2}(F, f_2, p_2+q_2) + \frac{\alpha}{2}\Delta_{A_2}(f_2, p_2+q_2) \\
&\geq \mathcal{J}_{A_2}(F, f_2, p_2) + \mathcal{J}_{A_2}(F, f_2, q_2) + \frac{\alpha}{2}\Delta_{A_2}(f_2, p_2+q_2) \\
&= \mathcal{J}_{A_2}(\Phi, f_2, p_2) + \mathcal{J}_{A_2}(\Phi, f_2, q_2) - \frac{\alpha}{2}\Delta_{A_2}(f_2, p_2) - \frac{\alpha}{2}\Delta_{A_2}(f_2, q_2) \\
&\quad + \frac{\alpha}{2}\Delta_{A_2}(f_2, p_2+q_2),
\end{aligned}$$

due to the superadditivity of the Jessen functional. Similarly, since F is concave on $I \cap (-\infty, c]$, utilizing the subadditivity property of the Jessen functional, we have

$$\begin{aligned}
\mathcal{J}_{A_1}(\Phi, f_1, p_1+q_1) \leq \mathcal{J}_{A_1}(\Phi, f_1, p_1) + \mathcal{J}_{A_1}(\Phi, f_1, q_1) - \frac{\alpha}{2}\Delta_{A_1}(f_1, p_1) - \frac{\alpha}{2}\Delta_{A_1}(f_1, q_1) \\
+ \frac{\alpha}{2}\Delta_{A_1}(f_1, p_1+q_1).
\end{aligned}$$

Finally, since

$$\mathcal{L}_{A_1,A_2}(\Phi, f_1, f_2, p_1+q_1, p_2+q_2) = \mathcal{J}_{A_2}(\Phi, f_2, p_2+q_2) - \mathcal{J}_{A_1}(\Phi, f_1, p_1+q_1),$$

subtracting the previous two inequalities and taking into account the assumption (4.51), we obtain (4.52), as required. □

Remark 4.6 It should be noticed here that the properties of positivity and superadditivity of the Levinson functional are not directly related. Namely, the functionals appearing in relation (4.52) are not positive in general. They are positive if in addition $\Delta_{A_1}(f_1, p_1) = \Delta_{A_2}(f_2, p_2)$ and $\Delta_{A_1}(f_1, q_1) = \Delta_{A_2}(f_2, q_2)$.

Corollary 4.8 *Suppose* $A_i : L_i \to \mathbb{R}$ *are positive linear functionals and let* $f_i \in L_i$, $p_i, q_i \in L_i^+$, $i = 1,2$. *Further, let* $f_i(E_i) \subseteq I$, $i = 1,2$, *where* I *is an interval, and suppose that there exists* $c \in I^0$ *such that* (4.48) *holds. If* $p_i \geq q_i$, $i = 1,2$,

$$\Delta_{A_1}(f_1, p_1 - q_1) = \Delta_{A_2}(f_2, p_2 - q_2), \tag{4.53}$$

and

$$\Delta_{A_1}(f_1, p_1) - \Delta_{A_1}(f_1, q_1) = \Delta_{A_2}(f_2, p_2) - \Delta_{A_2}(f_2, q_2), \tag{4.54}$$

then the inequality

$$\mathscr{L}_{A_1,A_2}(\Phi, f_1, f_2, p_1, p_2) \geq \mathscr{L}_{A_1,A_2}(\Phi, f_1, f_2, q_1, q_2) \tag{4.55}$$

holds for every continuous function $\Phi \in \mathscr{K}_1^c(I)$, *provided that* $p_i f_i$, $q_i f_i$, $p_i f_i^2$, $q_i f_i^2$, $p_i \Phi(f_i)$, $q_i \Phi(f_i) \in L_i$, *and* $A_i(p_i) > 0$, $A_i(q_i) > 0$, $i = 1, 2$. *If* $\Phi \in \mathscr{K}_2^c(I)$ *is a continuous function, then the sign of inequality* (4.55) *is reversed.*

Proof. Due to superadditivity property (4.52), it follows that

$$\mathscr{L}_{A_1,A_2}(\Phi, f_1, f_2, p_1, p_2) \geq \mathscr{L}_{A_1,A_2}(\Phi, f_1, f_2, p_1 - q_1, p_2 - q_2) + \mathscr{L}_{A_1,A_2}(\Phi, f_1, f_2, q_1, q_2).$$

In addition, since $\Delta_{A_1}(f_1, p_1 - q_1) = \Delta_{A_2}(f_2, p_2 - q_2)$, it follows that

$$\mathscr{L}_{A_1,A_2}(\Phi, f_1, f_2, p_1 - q_1, p_2 - q_2) \geq 0,$$

so (4.52) holds. $\square$

Remark 4.7 Similarly to Remark 4.5, it should be noticed here that the superadditivity and monotonicity properties of the Levinson functional hold under some weaker conditions. Namely, taking into account the proof of Theorem 4.5, it follows that the inequality (4.52) holds if the condition (4.51) is replaced by

$$\begin{aligned} &\alpha \left(\Delta_{A_2}(f_2, p_2 + q_2) - \Delta_{A_2}(f_2, p_2) - \Delta_{A_2}(f_2, q_2)\right) \\ &\geq \alpha \left(\Delta_{A_1}(f_1, p_1 + q_1) - \Delta_{A_1}(f_1, p_1) - \Delta_{A_1}(f_1, q_1)\right), \end{aligned}$$

where α is any constant fulfilling conditions as in Definition 4.1. In the same way the inequality (4.55) holds if the conditions (4.53) and (4.54) are respectively replaced by weaker conditions

$$\alpha \left(\Delta_{A_2}(f_2, p_2 - q_2) - \Delta_{A_1}(f_1, p_1 - q_1)\right) \geq 0$$

and

$$\begin{aligned} &\alpha \left(\Delta_{A_2}(f_2, p_2) - \Delta_{A_2}(f_2, p_2 - q_2) - \Delta_{A_2}(f_2, q_2)\right) \\ &\geq \alpha \left(\Delta_{A_1}(f_1, p_1) - \Delta_{A_1}(f_1, p_1 - q_1) - \Delta_{A_1}(f_1, q_1)\right). \end{aligned}$$

Monotonicity of the Levinson functional can be employed in obtaining bounds for the corresponding functional. Namely, if the weights p_1 and p_2 are bounded functions, the Levinson functional $\mathscr{L}_{A_1,A_2}(\Phi, f_1, f_2, p_1, p_2)$ can mutually be bounded by a non-weight functional of the same type. The non-weight Levinson functional is defined by

$$\mathscr{L}^0_{A_1,A_2}(\Phi, f_1, f_2) = \mathscr{L}_{A_1,A_2}(\Phi, f_1, f_2, 1, 1), \tag{4.56}$$

that is, when $p_1(x) = 1$ and $p_2(y) = 1$, for all $x \in E_1$ and $y \in E_2$. The importance of the following result lies in the fact that it provides both a refinement and a converse of the Levinson inequality. Of course, in order to obtain such relations, it is necessary to assume positivity of the Levinson functionals appearing there.

Theorem 4.6 *Suppose $A_i : L_i \to \mathbb{R}$ are positive linear functionals and $f_i \in L_i$, $i = 1,2$. Let $f_i(E_i) \subseteq I$, $i = 1,2$, where I is an interval, and suppose that there exists $c \in I^0$ such that* (4.48) *holds. Further, suppose $p_i \in L_i^+$, $i = 1,2$, are bounded functions and let $m_i = \inf_{x\in E_i} p_i(x)$, $M_i = \sup_{x\in E_i} p_i(x)$, $i = 1,2$. If*

$$\Delta_{A_1}(f_1, p_1) = \Delta_{A_2}(f_2, p_2), \tag{4.57}$$

$$\Delta_{A_1}(f_1, p_1 - \min\{m_1, m_2\}) = \Delta_{A_2}(f_2, p_2 - \min\{m_1, m_2\}), \tag{4.58}$$

and

$$\Delta_{A_1}(f_1, 1) = \Delta_{A_2}(f_2, 1), \tag{4.59}$$

then the inequality

$$\mathscr{L}_{A_1,A_2}(\Phi, f_1, f_2, p_1, p_2) \geq \min\{m_1, m_2\}\mathscr{L}^0_{A_1,A_2}(\Phi, f_1, f_2) \tag{4.60}$$

holds for every continuous function $\Phi \in \mathscr{K}_1^c(I)$, provided that $p_i f_i$, $p_i f_i^2$, f_i^2, $\Phi(f_i)$, $p_i\Phi(f_i)$ $\in L_i$, and $A_i(p_i)$, $A_i(1) > 0$, $i = 1,2$. In addition, if the condition (4.58) *is replaced by*

$$\Delta_{A_1}(f_1, \max\{m_1, m_2\} - p_1) = \Delta_{A_2}(f_2, \max\{m_1, m_2\} - p_2), \tag{4.61}$$

then

$$\mathscr{L}_{A_1,A_2}(\Phi, f_1, f_2, p_1, p_2) \leq \max\{m_1, m_2\}\mathscr{L}^0_{A_1,A_2}(\Phi, f_1, f_2). \tag{4.62}$$

If $\Phi \in \mathscr{K}_2^c(I)$ is a continuous function, then the signs of inequalities (4.60) *and* (4.62) *are reversed.*

Proof. Since $p_i(x) \geq \min\{m_1, m_2\}$, for all $x \in E_i$, $i = 1,2$, we utilize the relation (4.55) with constant functions $q_i(x) = \min\{m_1, m_2\}$, $x \in E_i$, $i = 1,2$. Now, since

$$\mathscr{L}_{A_1,A_2}(\Phi, f_1, f_2, \min\{m_1, m_2\}, \min\{m_1, m_2\}) = \min\{m_1, m_2\}\mathscr{L}^0_{A_1,A_2}(\Phi, f_1, f_2),$$

the inequality (4.60) holds. The remaining inequality (4.62) is proved in the same way. □

It should be noticed here that the relation (4.60) provides a refinement of the Levinson inequality (4.50), while (4.62) yields its converse. Rewriting these inequalities in the expanded forms, we have

$$\begin{aligned} &A_2(p_2\Phi(f_2)) - A_2(p_2)\Phi\left(\frac{A_2(p_2 f_2)}{A_2(p_2)}\right) - A_1(p_1\Phi(f_1)) + A_1(p_1)\Phi\left(\frac{A_1(p_1 f_1)}{A_1(p_1)}\right) \\ &\geq \min\{m_1, m_2\}\left[A_2(\Phi(f_2)) - A_2(1)\Phi\left(\frac{A_2(f_2)}{A_2(1)}\right) - A_1(\Phi(f_1)) + A_1(1)\Phi\left(\frac{A_1(f_1)}{A_1(1)}\right)\right] \end{aligned} \tag{4.63}$$

and

$$\begin{aligned} &A_2(p_2\Phi(f_2)) - A_2(p_2)\Phi\left(\frac{A_2(p_2 f_2)}{A_2(p_2)}\right) - A_1(p_1\Phi(f_1)) + A_1(p_1)\Phi\left(\frac{A_1(p_1 f_1)}{A_1(p_1)}\right) \\ &\leq \max\{m_1, m_2\}\left[A_2(\Phi(f_2)) - A_2(1)\Phi\left(\frac{A_2(f_2)}{A_2(1)}\right) - A_1(\Phi(f_1)) + A_1(1)\Phi\left(\frac{A_1(f_1)}{A_1(1)}\right)\right]. \end{aligned} \tag{4.64}$$

We say that the relations (4.63) and (4.64) represent the refinement and the converse of the Levinson inequality in the difference form.

Remark 4.8 Similarly to Remark 2.29 and Remark 4.7, the inequalities (4.60) and (4.62) hold if the assumptions (4.57), (4.58), (4.59) and (4.61) are respectively replaced by weaker conditions

$$\begin{aligned}
&\alpha\,(\Delta_{A_2}(f_2,p_2)-\Delta_{A_1}(f_1,p_1))\geq 0,\\
&\alpha\,(\Delta_{A_2}(f_2,p_2-\min\{m_1,m_2\})-\Delta_{A_1}(f_1,p_1-\min\{m_1,m_2\}))\geq 0,\\
&\alpha\,(\Delta_{A_2}(f_2,1)-\Delta_{A_1}(f_1,1))\geq 0,\\
&\alpha\,(\Delta_{A_2}(f_2,\max\{m_1,m_2\}-p_2)-\Delta_{A_1}(f_1,\max\{m_1,m_2\}-p_1))\geq 0,
\end{aligned}$$

where α is any constant fulfilling conditions as in Definition 4.1.

4.2.1 Applications

In this section we give a refinement and a converse of the Levinson inequality in the discrete form. Let L_1 and L_2 be linear spaces of real-valued functions defined on finite sets $E_1=\{1,2,\ldots,n\}$ and $E_2=\{1,2,\ldots,m\}$. Further, suppose that the corresponding functionals are defined as the sums of coordinates, that is, $A_1(x)=\sum_{i=1}^n x_i$ for $x=(x_1,x_2,\ldots,x_n)$, and $A_2(y)=\sum_{j=1}^m y_j$ for $y=(y_1,y_2,\ldots,y_m)$. Notice that in this setting we have $A_1(1)=n$ and $A_2(1)=m$. Now, employing Theorem 4.6 and Remark 4.8, as well as the relations (4.63) and (4.64), we obtain the following consequence.

Corollary 4.9 *Let $c\in I^0$, where I is an interval, $x_i\in I\cap(-\infty,c]$, $y_j\in I\cap[c,\infty)$, $i=1,2,\ldots,n$, $j=1,2,\ldots,m$, and $p_i>0$, $q_j>0$ be such that $\sum_{i=1}^n p_i=\sum_{j=1}^m q_j=1$. Further, suppose $\Phi\in\mathscr{K}_1^c(I)$ and let α be any constant fulfilling conditions as in Definition* 4.1. *If*

$$\alpha\sum_{i=1}^n p_i(x_i-\overline{x})^2\leq\alpha\sum_{j=1}^m q_j(y_j-\overline{y})^2, \tag{4.65}$$

$$\alpha\sum_{i=1}^n(p_i-\mu)\left(x_i-\frac{\overline{x}-\mu n x_0}{1-\mu n}\right)^2\leq\alpha\sum_{j=1}^m(q_j-\mu)\left(y_j-\frac{\overline{y}-\mu m y_0}{1-\mu m}\right)^2 \tag{4.66}$$

and

$$\alpha\sum_{i=1}^n(x_i-x_0)^2\leq\alpha\sum_{j=1}^m(y_j-y_0)^2, \tag{4.67}$$

where $\mu=\min\{p_i,q_j;\ i=1,2,\ldots,n, j=1,2,\ldots,m\}$, $\overline{x}=\sum_{i=1}^n p_ix_i$, $\overline{y}=\sum_{j=1}^m q_jy_j$, $x_0=\frac{1}{n}\sum_{i=1}^n x_i$, $y_0=\frac{1}{m}\sum_{j=1}^m y_j$, then the inequality

$$\begin{aligned}
&\sum_{j=1}^m q_j\Phi(y_j)-\Phi(\overline{y})-\sum_{i=1}^n p_i\Phi(x_i)+\Phi(\overline{x})\\
&\geq\mu\left[\sum_{j=1}^m\Phi(y_j)-m\Phi(y_0)-\sum_{i=1}^n\Phi(x_i)+n\Phi(x_0)\right]
\end{aligned} \tag{4.68}$$

holds. In addition, if

$$\alpha\sum_{i=1}^n(\nu-p_i)\left(x_i-\frac{\nu n x_0-\overline{x}}{\nu n-1}\right)^2\leq\alpha\sum_{j=1}^m(\nu-q_j)\left(y_j-\frac{\nu m y_0-\overline{y}}{\nu m-1}\right)^2, \tag{4.69}$$

where $\nu = \max\{p_i, q_j;\ i = 1,2,\dots,n, j = 1,2,\dots,m\}$*, then*

$$\begin{aligned} &\sum_{j=1}^{m} q_j\Phi(y_j) - \Phi(\overline{y}) - \sum_{i=1}^{n} p_i\Phi(x_i) + \Phi(\overline{x}) \\ &\leq \nu\left[\sum_{j=1}^{m} \Phi(y_j) - m\Phi(y_0) - \sum_{i=1}^{n} \Phi(x_i) + n\Phi(x_0)\right]. \end{aligned} \tag{4.70}$$

If $\Phi \in \mathscr{K}_2^c(I)$*, then the signs of inequalities* (4.68) *and* (4.70) *are reversed.*

Notice that inequalities (4.68) and (4.70) yield a refinement and a converse of the inequality (4.45).

Remark 4.9 With notation as in Corollary 4.9, it follows that

$$\mu \leq \min\left\{\frac{1}{m}, \frac{1}{n}\right\} \leq \max\left\{\frac{1}{m}, \frac{1}{n}\right\} \leq \nu.$$

For example, if $\mu = \frac{1}{n}$, then the expression $\dfrac{\overline{x} - \mu n x_0}{1 - \mu n}$ appearing in (4.66) is taken to be zero. The remaining limiting cases are treated in the same way.

Remark 4.10 Suppose that $m = n$ and $p_i = q_i$, $i = 1,2,\dots,n$, in Corollary 4.9. It is easy to see that in the case of a symmetric distribution of points x_i, y_i around the point c, that is, when the assumption (4.41) is fulfilled, the conditions (4.65), (4.66), (4.67) and (4.69) hold trivially. In fact, we have equality signs in these relations.

Corollary 4.9 can be employed in order to obtain more precise estimates for some known inequalities involving the arithmetic, geometric and harmonic means. In particular – a refinement of the famous Ky Fan inequality.

Example 4.1 Let us consider Corollary 4.9 for the case of the function $\Phi(x) = \log x$, defined on the interval $I = (0,\infty)$, and let $c \in I$. Obviously, $\Phi \in \mathscr{K}_1^c(I)$ for every $c \in I$ and the corresponding parameter α from Definition 4.1 is $\alpha = f''(c) = -\frac{1}{c^2} < 0$. Moreover, in this case the inequality (4.68) reads

$$\frac{G_m(y,q)}{G_n(x,p)} \geq \left[\frac{G_m(y)}{A_m(y)}\right]^{m\mu} \left[\frac{A_n(x)}{G_n(y)}\right]^{n\mu} \frac{A_m(y,q)}{A_n(x,p)}, \tag{4.71}$$

where $A_n(x,p)$, $G_n(x,p)$, $A_n(x)$, $G_n(x)$ are the arithmetic and geometric means in both weight and non-weight forms, i.e. $A_n(x,p) = \sum_{i=1}^{n} p_i x_i$, $G_n(x,p) = \prod_{i=1}^{n} x_i^{p_i}$, $A_n(x) = \frac{1}{n}\sum_{i=1}^{n} x_i$, $G_n(x) = \prod_{i=1}^{n} x_i^{\frac{1}{n}}$, and the parameter μ is defined in Corollary 4.9. Note that, due to the Levinson inequality, we have

$$\left[\frac{G_m(y)}{A_m(y)}\right]^{m} \left[\frac{A_n(x)}{G_n(y)}\right]^{n} \geq 1. \tag{4.72}$$

If $m = n$, the inequality (4.72) represents the Ky Fan inequality for arithmetic and geometric means in a non-weight form. Originally, the Ky Fan inequality was proved on the interval

$(0,1)$, with a symmetric distribution of points x_i and y_i around the point $\frac{1}{2}$ (for more details, see [126]). Therefore, the inequality (4.71) represents the refinement of the weight Ky Fan inequality. Similarly, utilizing the relation (4.70), we obtain

$$\frac{G_m(y,q)}{G_n(x,p)} \leq \left[\frac{G_m(y)}{A_m(y)}\right]^{m\nu} \left[\frac{A_n(x)}{G_n(y)}\right]^{n\nu} \frac{A_m(y,q)}{A_n(x,p)}, \tag{4.73}$$

which represents the converse of the weight Ky Fan inequality. Finally, taking into account the Remark 4.5 and by virtue of concavity of the function $\Phi(x) = \log x$, we conclude that inequalities (4.71) and (4.73) hold when in conditions (4.65), (4.66), (4.67) and (4.69) the parameter α is removed and the signs of inequalities are reversed.

Example 4.2 In this example, we refer to the concave function $\Phi(x) = -\frac{1}{x}$ defined on $I = (0,\infty)$. In this case, Corollary 4.9 yields more precise estimates for differences of reciprocals of arithmetic and harmonic means. More precisely, with notation as in Corollary 4.9, we obtain the refinement

$$\begin{aligned}
&\frac{1}{A_m(y,q)} - \frac{1}{H_m(y,q)} - \frac{1}{A_n(x,p)} + \frac{1}{H_n(x,p)} \\
&\geq \mu\left[\frac{m}{A_m(y)} - \frac{m}{H_m(y)} - \frac{n}{A_n(x)} + \frac{n}{H_n(x)}\right]
\end{aligned}$$

and the converse

$$\begin{aligned}
&\frac{1}{A_m(y,q)} - \frac{1}{H_m(y,q)} - \frac{1}{A_n(x,p)} + \frac{1}{H_n(x,p)} \\
&\leq \nu\left[\frac{m}{A_m(y)} - \frac{m}{H_m(y)} - \frac{n}{A_n(x)} + \frac{n}{H_n(x)}\right],
\end{aligned}$$

where $H_n(x,p) = \left(\sum_{i=1}^n \frac{p_i}{x_i}\right)^{-1}$, $H_n(x) = \left(\frac{1}{n}\sum_{i=1}^n \frac{1}{x_i}\right)^{-1}$ is the harmonic mean in its weight and non-weight form, respectively. Like in the previous example, these inequalities hold with the conditions (4.65), (4.66), (4.67) and (4.69) reduced to the forms without the parameter α and with reversed signs of inequalities.

Chapter 5

Different approaches to superadditivity

This chapter integrates the results on superadditivity for several classes of functionals, each having its own specificities and thus requiring a specific approach developed. In the first place, we use the opportunity to present here a few not so recent, but nevertheless basic important results on superadditivity, developed by D. S. Mitrinović, J. E. Pečarić and L. E. Persson in 1992, see [152].

In the second part of the chapter, quasilinearity of the functional $(h \circ v) \cdot \left(\Phi \circ \frac{g}{v}\right)$ is analyzed, where Φ is a monotone h-concave (h-convex) function, v and g are functionals with certain super(sub)additivity properties. General results of the type are then applied to the functionals generated with the Jensen, the Jensen-Mercer, the Beckenbach, the Chebyshev and the Milne inequality. This approach was firstly developed in [167]. Finally, in the third section, superadditivity of the functionals associated with the Gauss-Winckler and the Gauss-Polya inequalities is proved, as it was previously done in [207].

5.1 On a general inequality with applications

The authors in [152] proved a general set-valued inequality in two analogical forms. As applications, they obtained some simple inequalities for convex, concave, subadditive and superadditive functions, also pointing out that some classical inequalities (e.g. those by

Minkowski, Beckenbach and Dresher, as well as those by Pečarić and Beesack (see [31]) or by Peetre and Persson (see [183])) are the special cases of their obtained results. To be more specific, those results are the special cases of the following proposition. The details are worked out in the sequel.

Proposition 5.1 *Let $F\colon I \to \mathbb{R}$, $g\colon D \to \mathbb{R}_+$ and $f\colon D \to I$, where D is an additive Abelian semigroup and I is a subset of $\mathbb{R}^n$.*

1° *Assume that F is convex and that one of the following conditions holds:*

- (i) *f is affine;*
- (ii) *F is non-increasing and f is superadditive;*
- (iii) *F is non-decreasing and f is subadditive.*

If g is affine or if g is superadditive and $F(0) \leq 0$, then

$$g(x+y)F\left(\frac{f(x+y)}{g(x+y)}\right) \leq g(x)F\left(\frac{f(x)}{g(x)}\right) + g(y)F\left(\frac{f(y)}{g(y)}\right). \tag{5.1}$$

2° *Suppose that F is concave and that one of the following conditions holds:*

- (i) *f is affine;*
- (iv) *F is non-increasing and f is subadditive;*
- (v) *F is non-decreasing and f is superadditive.*

If g is affine or if g is superadditive and $F(0) \geq 0$, then

$$g(x+y)F\left(\frac{f(x+y)}{g(x+y)}\right) \geq g(x)F\left(\frac{f(x)}{g(x)}\right) + g(y)F\left(\frac{f(y)}{g(y)}\right). \tag{5.2}$$

Let $\mathscr{P}(\Omega)$ denote the power set of the set Ω, i.e., the set of all subsets of Ω. The following "set-valued" versions of the inequalities from Proposition 5.1 are proved in two analogical forms, as follows.

Theorem 5.1 *Let $F\colon I \to \mathbb{R}$ be a convex function and let $G\colon D \to \mathscr{P}(\mathbb{R}_+)$ and $f\colon D \to I$ be arbitrary functions. Then the function*

$$f_1(x) = \inf_{a \in G(x)} aF\left(\frac{f(x)}{a}\right), \quad x \in D, \tag{5.3}$$

is subadditive if one of the conditions (i), (ii) or (iii) holds, and if, for all $a \in G(x)$ and $b \in G(y,)$ $a+b \in G(x+y)$ or if there exists $c \geq a+b$, such that $c \in G(x+y)$ and $F(0) \leq 0$.

Proof. First we assume that $F(0) \leq 0$, f is non-decreasing, f is subadditive and, for all $a \in G(x)$ and $b \in G(y)$, there exists $c \geq a+b$, such that $c \in G(x+y)$. Consider $a \in G(x)$ and $b \in G(y)$. We note that the function $H(t) = F(t\,f(x))$, $t \geq 0$, is convex and (since also $F(0) \leq 0$) we conclude that the function $H(t)/t$ is non-decreasing. Therefore, by using the assumption that f is subadditive and F is convex and non-decreasing, we obtain that, for some $c \geq a+b$, such that $c \in G(x+y)$,

$$\begin{aligned} cF\left(\frac{f(x+y)}{c}\right) &\leq (a+b)F\left(\frac{f(x+y)}{a+b}\right) \leq (a+b)F\left(\frac{f(x)+f(y)}{a+b}\right) \\ &\leq aF\left(\frac{f(x)}{a}\right)+bF\left(\frac{f(y)}{b}\right). \end{aligned} \tag{5.4}$$

Therefore, for any ε, $0<\varepsilon<\frac{1}{2}$, there exists $c \in G(x+y)$, such that

$$cF\left(\frac{f(x+y)}{c}\right) \leq (1+\varepsilon)f_1(x)+(1+\varepsilon)f_1(y). \tag{5.5}$$

By taking infimum once more and letting $\varepsilon \to 0$, we obtain

$$f_1(x+y) \leq f_1(x)+f_1(y).$$

The proofs of the remaining cases consist of making obvious modifications of the proof above, so the details are omitted. □

Theorem 5.2 *Let $F\colon I \to \mathbb{R}$ be a concave function and let $G\colon D \to \mathscr{P}(\mathbb{R}_+)$ and $f\colon D \to I$ be arbitrary functions. Then the function*

$$f_2(x) = \sup_{a \in G(x)} aF\left(\frac{f(x)}{a}\right), \quad x \in D, \tag{5.6}$$

is superadditive if one of the conditions (i), (iv) or (v) holds, and if, for all $a \in G(x)$ and $b \in G(y)$, $a+b \in G(x+y)$ or if there exists $c \geq a+b$, such that $c \in G(x+y)$ and $F(0) \geq 0$.

Proof. Suppose that f is superadditive, $F(0) \geq 0$, F is non-decreasing and, for all $a \in G(x)$ and $b \in G(y)$, there exists $c \geq a+b$, such that $c \in G(x+y)$. Then, in particular, we find that $H(t) = F(tf(x))$, $t \geq 0$, is a concave function and, thus, that the function $H(t)/t$ is non-increasing. Hence, by arguing in a similar way as in the proof of Theorem 5.1, we find that, for any ε, $0<\varepsilon<\frac{1}{2}$, and some $c \in G(x+y)$,

$$(1-\varepsilon)f_2(x)+(1-\varepsilon)f_2(y) \leq cF\left(\frac{f(x+y)}{c}\right), \tag{5.7}$$

and, by taking supremum once more and letting $\varepsilon \to 0$, we find that the function f_2 is superadditive. The proofs of the other cases are similar. □

Remark 5.1 Theorem 5.1 may be seen as a further generalization of results in [183] and [186]. Moreover, Theorem 5.2 generalizes the corresponding result from [186] in a similar way.

Proof of Proposition 5.1. Follows by applying theorems 5.1 and 5.2 with $G(x) = [g(x)]$ (the singleton case). □

Concluding remarks and examples

When Proposition 5.1 is applied with $F(u) = u^p$, $p = \frac{\alpha}{\alpha-\beta}$, $f(x) = (\int_\Omega x^\alpha d\mu)^{\frac{1}{\alpha}}$, $g(x) = (\int_\Omega x^\beta d\mu)^{\frac{1}{\beta}}$, the following form of the Beckenbach-Dresher inequality (see e.g. [26, 28, 47, 183]) is obtained.

Example 5.1 Let $x, y > 0$ a.e. on Ω. If $0 \le \alpha \le 1 \le \beta$ or if $0 \le \beta \le 1 \le \alpha$, $\alpha \ne \beta$, then

$$\left(\frac{\int_\Omega (x+y)^\alpha d\mu}{\int_\Omega (x+y)^\beta d\mu}\right)^{\frac{1}{\alpha-\beta}} \le \left(\frac{\int_\Omega x^\alpha d\mu}{\int_\Omega x^\beta d\mu}\right)^{\frac{1}{\alpha-\beta}} + \left(\frac{\int_\Omega y^\alpha d\mu}{\int_\Omega y^\beta d\mu}\right)^{\frac{1}{\alpha-\beta}}. \tag{5.8}$$

If $\beta \le 0 \le \alpha \le 1$ or if $\alpha \le 0 \le \beta \le 1$, then (5.8) holds in the reversed direction.

Remark 5.2 In view of the discussion above, it is obvious that Example 5.1 can be easily generalized in various directions. Here are a few such generalizations (complements):

(i) By using a positive linear functional A acting on the space of real functions, instead of the special cases $A(x) = \int_\Omega x d\mu$, we obtain (generalized forms of) some versions of the Beckenbach-Dresher inequality, previously proved by Pečarić and Beesack (see [31]) and by Peetre and Persson (see [185, 186]).

(ii) The inequality (5.8), in its turn, is a subadditivity condition and the reversed inaquality is a superadditivity condition. Therefore, we can use Proposition 5.1 and iterate the procedure. After the first step, we obtain the following generalization of Example 5.1: If $0 \le \beta \le 1 \le \alpha$, $\gamma \le 0 \le \delta \le 1$, $\alpha \ne \beta$, $\gamma \ne \delta$, $\alpha - \beta - \gamma + \delta \ge 0$, then

$$\left(\frac{\int_\Omega (x+y)^\alpha d\mu \int_\Omega (x+y)^\delta d\mu}{\int_\Omega (x+y)^\beta d\mu \int_\Omega (x+y)^\gamma d\mu}\right)^{\frac{1}{\alpha-\beta-\gamma+\delta}} \le \left(\frac{\int_\Omega x^\alpha d\mu \int_\Omega x^\delta d\mu}{\int_\Omega x^\beta d\mu \int_\Omega x^\gamma d\mu}\right)^{\frac{1}{\alpha-\beta-\gamma+\delta}} + \left(\frac{\int_\Omega y^\alpha d\mu \int_\Omega y^\delta d\mu}{\int_\Omega y^\beta d\mu \int_\Omega y^\gamma d\mu}\right)^{\frac{1}{\alpha-\beta-\gamma+\delta}}.$$

Moreover, if $\beta \le 0 \le \alpha \le 1$, $\gamma \le 0 \le \delta \le 1$, $\alpha \ne \beta$, $\gamma \ne \delta$, $\alpha - \beta - \delta + \gamma \ge 0$, then

$$\left(\frac{\int_\Omega (x+y)^\alpha d\mu \int_\Omega (x+y)^\gamma d\mu}{\int_\Omega (x+y)^\beta d\mu \int_\Omega (x+y)^\delta d\mu}\right)^{\frac{1}{\alpha-\beta-\delta+\gamma}} \ge \left(\frac{\int_\Omega x^\alpha d\mu \int_\Omega x^\gamma d\mu}{\int_\Omega x^\beta d\mu \int_\Omega x^\delta d\mu}\right)^{\frac{1}{\alpha-\beta-\delta+\gamma}} + \left(\frac{\int_\Omega y^\alpha d\mu \int_\Omega y^\gamma d\mu}{\int_\Omega y^\beta d\mu \int_\Omega y^\delta d\mu}\right)^{\frac{1}{\alpha-\beta-\delta+\gamma}}.$$

(iii) By using the continuity property of generalized Gini means (see [183]), we obtain the inequalities corresponding to the exceptional cases in Example 5.1 (and the inequalities in case 2 above). For example, for the extremal case: $\alpha = \beta = 1$, the inequality (5.8) reads:

$$\exp\left(\frac{\int_\Omega (x+y)\ln(x+y)d\mu}{\int_\Omega (x+y)d\mu}\right) \le \exp\left(\frac{\int_\Omega x\ln x d\mu}{\int_\Omega x d\mu}\right) + \exp\left(\frac{\int_\Omega y\ln y d\mu}{\int_\Omega y d\mu}\right)$$

and the corresponding inequality for the other limiting case: $\alpha=\beta=0$ reads:

$$\exp\left(\frac{1}{\mu(\Omega)}\int_\Omega \ln(x+y)d\mu\right)\geq \exp\left(\frac{1}{\mu(\Omega)}\int_\Omega \ln x d\mu\right)+\exp\left(\frac{1}{\mu(\Omega)}\int_\Omega \ln y d\mu\right).$$

A special case of this inequality is the following well-known inequality for positive sequences (see e.g. [28, p. 26]):

$$\left(\prod_{i=1}^n (x_k+y_k)\right)^{\frac{1}{n}} \geq \left(\prod_{i=1}^n x_k\right)^{\frac{1}{n}} + \left(\prod_{i=1}^n y_k\right)^{\frac{1}{n}}.$$

So far, the only applications of the general theorems have been given for the single-valued case presented in Proposition 5.1. An application for another extremal case, when $G(x)=\mathbb{R}_+$, for all $x\in D$, is given within the following example.

Example 5.2 Let $D=\mathbb{R}^n$, $f(\mathbf{x})=\mathbf{x}=(x_1,\ldots,x_n)$, $G(\mathbf{x})=\mathbb{R}_+$, for all $\mathbf{x}\in D$ and consider the (Amemiya) norm

$$\|\mathbf{x}\|_\Phi=\inf_{a\in\mathbb{R}_+} a\left(1+\sum_{k=1}^n \Phi\left(\frac{x_k}{a}\right)\right),$$

where $\Phi\colon \mathbb{R}_+\to\mathbb{R}_+$ is a convex function. By applying Theorem 5.1 with

$$F(\mathbf{u})=1+\sum_{k=1}^n \Phi(|u_k|),$$

we find that

$$\|\mathbf{x}+\mathbf{y}\|_\Phi\leq \|\mathbf{x}\|_\Phi+\|\mathbf{y}\|_\Phi,$$

and thus we have obtained another proof of the Minkowski inequality for Orlicz sequence spaces. Moreover, by using Theorem 5.2 in a similar way, we find that the inequality

$$\|\mathbf{x}+\mathbf{y}\|_\Psi\geq \|\mathbf{x}\|_\Psi+\|\mathbf{y}\|_\Psi$$

holds, where $\Psi\colon \mathbb{R}_+\to\mathbb{R}_+$ is a concave function and

$$\|\mathbf{x}\|_\Psi=\sup_{a\in\mathbb{R}_+} a\left(1+\sum_{k=1}^n \Psi\left(\frac{x_k}{a}\right)\right).$$

Finally, we remark that the Beckenbach-Dresher inequality (Example 5.1) means that (integral forms of) the classical Gini means (investigated e.g. in [183] and [187]) are subadditive or superadditive, with certain restrictions on the parameters involved.

5.2 Properties of some functionals associated with h-concave and quasilinear functions with applications to inequalities

In [61], [62], [63], [64] S.S. Dragomir researched functionals which arise from quasilinear functionals related to the classical inequalities. For example, he considered the functionals $v\log(\frac{g}{v})$ (in [61]), $v^{q-\frac{q}{p}}g^q$, $\frac{v^{q-\frac{q}{p}}}{g^q}$ (both in [62]), $v^{\frac{p-q}{p}}g^q$ (in [63]), and finally, $v\cdot\left(\Phi\circ\frac{g}{v}\right)$ (in [64]), where v is additive, g is super(sub)additive, Φ is a concave (convex) function and p and q are real numbers with some properties. In each paper he applied the given results about composite functional to some of the classical inequalities such are the Jensen, the Hölder or the Minkowski inequality. L. Nikolova and S. Varošanec in [167] generalized his results. They investigated similar functionals related to an h-convex function Φ under assumptions which are weaker than the assumptions in the above mentioned papers. More specifically, they investigated quasilinearity of the functional $(h\circ v)\cdot\left(\Phi\circ\frac{g}{v}\right)$, where Φ is a monotone h-concave (h-convex) function, v and g are functionals with certain super(sub)additivity properties. They applied those general results to some special functionals generated with several inequalities such as the Jensen, the Jensen-Mercer, the Beckenbach, the Chebyshev and the Milne inequality.

In the sequel, I and J are intervals in $\mathbb{R}$, $(0,1)\subseteq J$ and functions h and f are non-negative functions defined on J and I, respectively.

Let us recall that function $h: J\subseteq\mathbb{R}\to\mathbb{R}$ is said to be a supermultiplicative function if

$$h(xy)\geq h(x)h(y), \tag{5.9}$$

for all $x,y\in J$. If the inequality (5.9) is reverse, then h is said to be a submultiplicative function. If equality holds in (5.9), then h is said to be a multiplicative function.

Definition 5.1 (SEE [206]) *Let $h: J\to\mathbb{R}$ be a non-negative function, $h\not\equiv 0$. Function $f: I\to\mathbb{R}$ is an h-convex function if f is non-negative and for all $x,y\in I,\alpha\in(0,1)$,*

$$f(\alpha x+(1-\alpha)y)\leq h(\alpha)f(x)+h(1-\alpha)f(y).$$

If the inequality is reverse, then f is an h-concave function.

5.2.1 Functionals associated with monotone h-concave and h-convex functions

Lemma 5.1 *Let $x,y\in C$ and $f: C\to\mathbb{R}$ be a non-negative, L-superadditive and K-positive homogeneous functional on C. If $M\geq m>0$ are such that $x-my$ and $My-x\in C$, then*

$$\frac{1}{L}K(M)f(y)\geq f(x)\geq LK(m)f(y).$$

Proof. Using L-superadditivity and K-positive homogenity of f we have

$$\begin{aligned} f(x) &= f(x-my+my) \geq L(f(x-my)+f(my)) \\ &\geq Lf(my) = LK(m)f(y), \end{aligned}$$

giving the second inequality. Similarly, we get the first inequality. Namely, using homogenity and L-superadditivity we get

$$\frac{1}{L}K(M)f(y) = \frac{1}{L}f(My) = \frac{1}{L}f(My-x+x) \geq f(My-x)+f(x) \geq f(x).$$

□

The above-proved lemma is a generalization of a result from [61] in which f is superadditive and positive homogeneous of order s.

Theorem 5.3 *Let h be a non-negative function which is k_1-positive homogeneous. Let C be a convex cone in the linear space X and $v : C \to (0,\infty)$ be an L-superadditive functional on C.*

(i) If h is submultiplicative, $g : C \to [0,\infty)$ is an L-superadditive (L-subadditive) functional on C and $\Phi : [0,\infty) \to [0,\infty)$ is h-concave and non-decreasing (non-increasing), then the functional $\eta_\Phi : C \to \mathbb{R}$ defined by

$$\eta_\Phi(x) := h(v(x))\Phi\left(\frac{g(x)}{v(x)}\right)$$

is $k_1(L)$-superadditive on C.

(ii) If h is supermultiplicative, g is L-subadditive, Φ is h-convex and non-decreasing with $\Phi(0) = 0$, then η_Φ is $k_1(L)$-subadditive.

Proof. (i) Let us suppose that h is submultiplicative, g is L-superadditive, Φ is h-concave and non-decreasing. Let $\alpha = L\frac{v(x)}{v(x+y)}$, $\beta = L\frac{v(y)}{v(x+y)}$. Since v is L-superadditive, we have $\alpha+\beta \leq 1$ and

$$\begin{aligned} \Phi\left(\frac{g(x+y)}{v(x+y)}\right) &\geq \Phi\left(\frac{Lg(x)+Lg(y)}{v(x+y)}\right) \\ &= \Phi\left(\frac{Lv(x)}{v(x+y)}\frac{g(x)}{v(x)} + \frac{Lv(y)}{v(x+y)}\frac{g(y)}{v(y)}\right) \\ &\geq h\left(\frac{Lv(x)}{v(x+y)}\right)\Phi\left(\frac{g(x)}{v(x)}\right) + h\left(\frac{Lv(y)}{v(x+y)}\right)\Phi\left(\frac{g(y)}{v(y)}\right) \\ &= k_1(L)\left[h\left(\frac{v(x)}{v(x+y)}\right)\Phi\left(\frac{g(x)}{v(x)}\right) + h\left(\frac{v(y)}{v(x+y)}\right)\Phi\left(\frac{g(y)}{v(y)}\right)\right] \\ &\geq k_1(L)\left[\frac{h(v(x))}{h(v(x+y))}\Phi\left(\frac{g(x)}{v(x)}\right) + \frac{h(v(y))}{h(v(x+y))}\Phi\left(\frac{g(y)}{v(y)}\right)\right]. \end{aligned}$$

The first inequality holds because Φ is non-decreasing and because of the L-superadditivity of g. The second inequality follows h-concavity of Φ. Next we use k_1-positive homogeneity of h and finally the submultiplicativity of h. Multiplying with $h(v(x+y))$ we have

$$h(v(x+y))\Phi\left(\frac{g(x+y)}{v(x+y)}\right) \geq k_1(L)\left[h(v(x)\Phi\left(\frac{g(x)}{v(x)}\right)+h(v(y)\Phi\left(\frac{g(y)}{v(y)}\right)\right].$$

Hence η_Φ is $k_1(L)$-superadditive. The proofs of the other cases follow in a similar manner. □

A superadditive and non-negative functional has the following property.

Corollary 5.1 *Let h be a non-negative submultiplicative function which is k_1-positive homogeneous. Let C be a convex cone in the linear space X and $v : C \to (0,\infty)$ be L-superadditive and k_2-positive homogeneous on C. Let $x,y \in C$ and assume that there exist $M \geq m > 0$ such that $x - my$ and $My - x \in C$. Let $K(t) = k_1(k_2(t))$. If $g : C \to [0,\infty)$ is an L-superadditive (L-subadditive) and k_2-positive homogeneous functional on C and $\Phi : [0,\infty) \to [0,\infty)$ is h-concave and non-decreasing (non-increasing), then*

$$\frac{1}{k_1(L)}K(M)\eta_\Phi(y) \geq \eta_\Phi(x) \geq k_1(L)K(m)\eta_\Phi(y).$$

Proof. Note that $h(v(\alpha x)) = h(k_2(\alpha)v(x)) = k_1(k_2(\alpha))h(v(x)) = K(\alpha)h(v(x))$. We observe that if v and g are k_2-positive homogeneous functionals, then $\eta_\Phi(x) = h(v(x))$ $\Phi\left(\frac{g(x)}{v(x)}\right)$ is a K-positive homogeneous functional and, by Theorem 5.3, it follows that η_Φ is a $k_1(L)$-superadditive functional on C. By applying Lemma 5.1 we get the result. □

Corollary 5.2 *Let h be a non-negative submultiplicative function which is positive homogeneous of order s_1. Let C be a convex cone in the linear space X and $v : C \to [0,\infty)$ be L-superadditive and positive homogeneous of order s_2 on C. Let $x,y \in C$ and assume that there exist $M \geq m > 0$ such that $x - my$ and $My - x \in C$. If $g : C \to [0,\infty)$ is an L-superadditive and positive homogeneous functional of order s_2 on C and $\Phi : [0,\infty) \to [0,\infty)$ is h-concave and non-decreasing, then*

$$\frac{M^s}{L^{s_1}}\eta_\Phi(y) \geq \eta_\Phi(x) \geq m^s L^{s_1}\eta_\Phi(y)$$

where $s = s_1 s_2$.

Proof. Put in the previous corollary $k_1(t) = t^{s_1}$, $k_2(t) = t^{s_2}$, and $K(t) = t^{s_1 s_2} = t^s$. □

Remark 5.3 If $L = 1$, then the assumption about homogeneity of h can be omitted and the statement of Theorem 5.3 still holds, namely we get superadditivity (subadditivity) of η_Φ.

If we consider the additive function v, then using the same proof ($L = 1$ and the first inequality is just equality) we get the following statements:

(i) If g is superadditive (subadditive), Φ is h-concave and non-decreasing (non-increasing), where h is submultiplicative, then the functional η_Φ is superadditive.

(ii) If g is superadditive (subadditive), Φ is h-convex and non-increasing (non-decreasing), where h is supermultiplicative, then the functional η_Φ is subadditive.

Comparing these statements with the results of Theorem 5 from the paper [60] we see that if Φ is a non-negative function, then we have results for wider class of functions Φ, i.e. for h-concave or h-convex functions.

The case $s_1 = 1$, $h(t) = t$ gives results for concave Φ, as it is in [60], but for v and g superadditive and s_2-positive homogeneous. The case when v is only superadditive is important for applications – see the application to the Chebyshev and Milne functionals.

Moreover, Corollary 5.1 under assumptions that v is additive and $L = 1$, $k_1(t) = k_2(t) = t$, becomes the same as Corollary 1 a) from [60].

More about Corollary 5.1: If $h(t) = t^s$, $s_2 = 1$, $L = 1$ and we use as an example $\Phi(x) = \Phi_1^s(x)$, $s \geq 1$, Φ_1 is concave non-decreasing, then we get the result of Corollary 1 from [60].

5.2.2 Case 1: function v is additive

Application to Jensen-type inequalities

Let f be a real mapping on a convex subset C_1 of a linear space. Let us fix $n \in \mathbb{N}$ and $x_i \in C_1$, $(i = 1, \ldots, n)$, and let $S_+(n) := \{\overline{p} = (p_1, \ldots, p_n) \ : \ p_i \geq 0, i = 1, \ldots, n \text{ and } P_n = \sum_{i=1}^n p_i > 0\}$. $S_+(n)$ is a convex cone.

As usual, the Jensen functional $J : S_+(n) \to \mathbb{R}$ is given by

$$J(\overline{p}) = \sum_{i=1}^{n} p_i f(x_i) - P_n f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right),$$

i.e. is a difference between the right-hand and the left-hand sides of the Jensen inequality for the convex function. As we know, J is positive homogeneous; if f is convex, then J is non-negative and superadditive, while, if f is concave, then J is non-positive and subadditive. The comparative inequalities for the Jensen functional are derived in [61]: if $M \geq m > 0$ such that $M\overline{p} \geq \overline{q} \geq m\overline{p}$ (i.e. $Mp_i \geq q_i \geq mp_i$ for each $i = 1, \ldots, n$), then

$$MJ(\overline{p}) \geq J(\overline{q}) \geq mJ(\overline{p}).$$

As an application of the results from the previous section we have the following theorem.

Theorem 5.4 *Let h be a non-negative submultiplicative function and f be convex. Suppose Φ is h-concave and non-decreasing on $[0, \infty)$. Then the composite functional η_Φ : $S_+(n) \to \mathbb{R}$ defined by*

$$\eta_\Phi(\overline{p}) = h(P_n)\Phi\left(\sum_{i=1}^{n} \frac{p_i}{P_n} f(x_i) - f\left(\sum_{i=1}^{n} \frac{p_i}{P_n} x_i\right)\right) \tag{5.10}$$

is superadditive. Let, furthermore, h be k-positive homogeneous. Suppose $\overline{p}$, $\overline{q} \in S_+(n)$ and let $M \geq m > 0$ be such that $M\overline{p} \geq \overline{q} \geq m\overline{p}$. Then

$$k(M)h(P_n)\Phi\left(\frac{J(\overline{p})}{P_n}\right) \geq h(Q_n)\Phi\left(\frac{J(\overline{q})}{Q_n}\right) \geq k(m)h(P_n)\Phi\left(\frac{J(\overline{p})}{P_n}\right).$$

Proof. Take $v(\overline{p}) = P_n$ and $g(\overline{p}) = J(\overline{p})$. The functionals v and g are positive homogeneous, v is additive and g is superadditive. Using Theorem 5.3 we get that the composite functional η_Φ is superadditive on $S_+(n)$ and k-positive homogeneous. Hence, we apply Lemma 5.1 and get the wanted inequalities. □

Remark 5.4 If $h(t) = t$, then we get results from [60].

On the Jensen-Steffensen conditions

Now, let f be a real function on an interval $I \subseteq \mathbb{R}$. In the previous theorem weights p_i are non-negative and considered cone C is the cone $S_+(n)$. As we have already discussed in the previous chapters, for some choices of points $x_1, \dots, x_n \in I$ this cone can be substituted with a larger cone. Let $\overline{x} = (x_1, \dots, x_n)$ be fixed monotonic n-tuple of elements from I and let us define the set $S(\overline{x}, n)$ as the set of all $\overline{p} \in \mathbb{R}^n$ such that $P_n > 0$, $0 \le P_k \le P_n$, where $P_k = \sum_{i=1}^k p_i$, $\sum_{i=1}^n \frac{p_i}{P_n} x_i \in I$.

The set $S(\overline{x}, n)$ is a cone. By the Jensen-Steffensen inequality [177, p. 57], the difference $J(\overline{p}) = \sum_{i=1}^n p_i f(x_i) - P_n f\left(\sum_{i=1}^n \frac{p_i}{P_n} x_i\right)$, where f is convex on I, is non-negative for each $\overline{p} \in S(\overline{x}, n)$. Using a similar proof as for the Jensen functional on $S_+(n)$ we get that J is superadditive for convex function f and applying Theorem 5.3 we obtain that the functional η_Φ given by (5.10) is superadditive and corresponding comparative inequalities hold. Let us recall that these comparative inequalities under the Jensen-Steffensen conditions with an additional normalizing property $P_n = 1$ were proved in [23], only by using a different method.

Applications to the Jensen-Mercer functional

As we have already considered, A. McD. Mercer in the paper [134] proved the Jensen-type inequality which includes boundary points of an interval – the Jensen-Mercer inequality (1.12).

The Jensen-Mercer functional $JM : S_+(n) \to \mathbb{R}$ defined by

$$JM(\overline{p}) = P_n(f(a) + f(b)) - \sum p_i f(x_i) - P_n f(a + b - \frac{1}{P_n} \sum p_1 x_i)$$

is positive homogeneous, non-negative for a convex function f and non-positive for concave function f. We have already established its superadditivity in Chapter 3, and now, applying the results from the previous section, we have the following theorem.

Theorem 5.5 *Let h be a non-negative submultiplicative function, f be a convex function and let $\Phi : [0, \infty) \to [0, \infty)$ be a h-concave non-decreasing function. Then the functional $\zeta : S_+(n) \to \mathbb{R}$ defined by*

$$\zeta(\overline{p}) = h(P_n)\Phi\left(f(a) + f(b) - \frac{1}{P_n}\sum_{i=1}^n p_i f(x_i) - f(a + b - \frac{1}{P_n}\sum_{i=1}^n p_i x_i)\right)$$

is superadditive on $S_+(n)$.

Proof. Consider the functionals $v(\overline{p}) = P_n$ and $g(\overline{p}) = JM(\overline{p})$. The functional v is additive and g is superadditive and

$$\eta_\Phi(\overline{p}) = h(v(\overline{p}))\Phi\left(\frac{g(\overline{p})}{v(\overline{p})}\right) = \zeta(\overline{p}).$$

Hence, by applying Theorem 5.3 we get the desired result. □

Corollary 5.3 *Let us suppose that the assumptions of Theorem 5.5 are fulfilled and let h be k-positive homogeneous. If $\overline{p}$, $\overline{q} \in S_+(n)$ and $M \geq m > 0$ are such that $M\overline{p} \geq \overline{q} \geq m\overline{p}$, then*

$$k(M)h(P_n)\Phi\left(\frac{JM(\overline{p})}{P_n}\right) \geq h(Q_n)\Phi\left(\frac{JM(\overline{q})}{Q_n}\right) \geq k(m)h(P_n)\Phi\left(\frac{JM(\overline{p})}{P_n}\right).$$

The proof follows from Corollary 5.1.

Applications to the Beckenbach functional

As [27], Theorem 5, shows: if f is convex for $x \in [0,a]$ and starshaped in $[0,b]$, (i.e. $f(\alpha x) \leq \alpha f(x)$ for any $\alpha \in (0,1)$,) $b > a$, then for $x_i \in [0,b]$ and $\alpha_i \in (0,1)$, $\sum_{i=1}^n \alpha_i = 1$, we have

$$f(\frac{a}{b}\sum_{i=1}^n \alpha_i x_i) \leq \frac{a}{b}\sum_{i=1}^n \alpha_i f(x_i).$$

That inequality is known as the Beckenbach inequality. Let us consider the Beckenbach functional $J_{a,b}$:

$$J_{a,b}(\overline{p}) = \frac{a}{b}\sum_{i=1}^n p_i f(x_i) - P_n f\left(\frac{a}{b}\sum_{i=1}^n \frac{p_i}{P_n}x_i\right),$$

where $\overline{p}, \overline{q} \in S_+(n)$ and a, b, f satisfy assumptions of the Beckenbach inequality. The above-mentioned theorem shows that $J_{a,b}(\overline{p}) \geq 0$.

Proposition 5.2 *The functional $J_{a,b}$ is superadditive.*

Proof. It yields that

$$\begin{aligned}
&J_{a,b}(\overline{p}+\overline{q}) - J_{a,b}(\overline{p}) - J_{a,b}(\overline{q}) \\
= &-(P_n+Q_n)f\left(\frac{P_n}{P_n+Q_n}\frac{a}{b}\sum_{i=1}^n \frac{p_i}{P_n}x_i + \frac{Q_n}{P_n+Q_n}\frac{a}{b}\sum_{i=1}^n \frac{q_i}{Q_n}x_i\right) \\
&+P_n f\left(\frac{a}{b}\sum_{i=1}^n \frac{p_i}{P_n}x_i\right) + P_n f\left(\frac{a}{b}\sum_{i=1}^n \frac{q_i}{Q_n}x_i\right) \\
\geq &-(P_n+Q_n)\left[\frac{P_n}{P_n+Q_n}f\left(\frac{a}{b}\sum_{i=1}^n \frac{p_i}{P_n}x_i\right) + \frac{Q_n}{P_n+Q_n}f\left(\frac{a}{b}\sum_{i=1}^n \frac{q_i}{Q_n}x_i\right)\right] \\
&+P_n f\left(\frac{a}{b}\sum_{i=1}^n \frac{p_i}{P_n}x_i\right) + P_n f\left(\frac{a}{b}\sum_{i=1}^n \frac{q_i}{Q_n}x_i\right) = 0,
\end{aligned}$$

because f is convex on $[0,a]$ and

$$\frac{a}{b}\sum_{i=1}^{n}\frac{p_i}{P_n}x_i \le a, \quad \frac{a}{b}\sum_{i=1}^{n}\frac{q_i}{Q_n}x_i \le a.$$

□

Theorem 5.6 *Let h be a non-negative submultiplicative function, f be a convex function and let $\Phi : [0,\infty) \to [0,\infty)$ be an h-concave non-decreasing function. Then the functional $\eta_\Phi : S_+(n) \to \mathbb{R}$ defined by*

$$\eta_\Phi(\overline{p}) = h(P_n)\Phi\left(\frac{a}{b}\sum_{i=1}^{n}\frac{p_i}{P_n}f(x_i) - f\left(\frac{a}{b}\sum_{i=1}^{n}\frac{p_i}{P_n}x_i\right)\right)$$

is superadditive on $S_+(n)$. Furthermore, if h is k-positive homogeneous, $\overline{p}$, $\overline{q} \in S_+(n)$ and $M \ge m > 0$ such that $M\overline{p} \ge \overline{q} \ge m\overline{p}$, then

$$k(M)h(P_n)\Phi\left(\frac{J_{a,b}(\overline{p})}{P_n}\right) \ \ge \ h(Q_n)\Phi\left(\frac{J_{a,b}(\overline{q})}{Q_n}\right) \ge k(m)h(P_n)\Phi\left(\frac{J_{a,b}(\overline{p})}{P_n}\right).$$

Proof. Consider the functionals $v(\overline{p}) = P_n$ and $g(\overline{p}) = J_{a,b}(\overline{p})$. The functional v is additive and g is superadditive, and applying Theorem 5.3 we get that η_Φ is superadditive. The comparative inequalities follow from Corollary 5.1. □

5.2.3 Case 2: function v is superadditive

What follows in this section are some applications concerning the superadditive function v.

Applications to the Chebyshev functional for sums

Let $\overline{a}$ and $\overline{b}$ be two real n-tuples. We call it similarly ordered if

$$(a_i - a_j)(b_i - b_j) \ge 0$$

for any $i, j = 1, \dots, n$. If the above inequality is reversed, then n-tuples are called oppositely ordered.

Let us denote

$$T(\overline{a},\overline{b},\overline{p}) = \sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i.$$

The statement of the classical Chebyshev inequality is the following (see [177, p. 197–204]).

The Chebyshev inequality. Let $\overline{a} = (a_1, \dots, a_n)$ and $\overline{b} = (b_1, \dots, b_n)$ be two n-tuples of real numbers and $\overline{p} = (p_1, \dots, p_n)$ be a non-negative n-tuple. If $\overline{a}$ and $\overline{b}$ are similarly ordered, then the Chebyshev inequality

$$T(\overline{a},\overline{b},\overline{p}) \ge 0$$

holds. If $\overline{a}$ and $\overline{b}$ are oppositely ordered, then the reverse inequality holds.

In the following theorem we consider quasilinear property of the Chebyshev functional $\overline{p} \mapsto T(\overline{a},\overline{b},\overline{p})$.

Theorem 5.7 *If $\overline{a}$ and $\overline{b}$ are similarly ordered real n-tuples, $\overline{p} \geq 0$, then the functional $T(\overline{a},\overline{b},\overline{p})$ is superadditive in the variable $\overline{p}$. If $\overline{a}$ and $\overline{b}$ are oppositely ordered real n-tuples, then the functional $T(\overline{a},\overline{b},\overline{p})$ is subadditive.*

Proof. Let us suppose that $\overline{a}$ and $\overline{b}$ are similarly ordered n-tuples and let us consider the sum $T(\overline{a},\overline{b},\overline{p}+\overline{q}) - T(\overline{a},\overline{b},\overline{p}) - T(\overline{a},\overline{b},\overline{q})$. We have

$$\begin{aligned} & T(\overline{a},\overline{b},\overline{p}+\overline{q}) - T(\overline{a},\overline{b},\overline{p}) - T(\overline{a},\overline{b},\overline{q}) \\ = & \sum_{i=1}^{n} p_i \sum_{i=1}^{n} q_i a_i b_i + \sum_{i=1}^{n} q_i \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} q_i b_i - \sum_{i=1}^{n} q_i a_i \sum_{i=1}^{n} p_i b_i = L_n. \end{aligned}$$

After simple calculation we get

$$L_{n+1} = L_n + \sum_{j=1}^{n} (p_{n+1}q_j + q_{n+1}p_j)(a_j - a_{n+1})(b_j - b_{n+1}).$$

Since $\overline{p}$ and $\overline{q}$ are non-negative and $\overline{a}$ and $\overline{b}$ are similarly ordered, we have

$$L_{n+1} \geq L_n \geq L_{n-1} \geq \ldots \geq L_1 = 0,$$

which means that $T(\overline{a},\overline{b},\overline{p})$ is superadditive. If $\overline{a}$ and $\overline{b}$ are oppositely ordered, the proof is similar. $\square$

Let us apply the previously obtained results to the functional $T(\overline{a},\overline{b},\overline{p})$.

Theorem 5.8 *Let h be a non-negative submultiplicative function, and $\Phi : [0,\infty) \to [0,\infty)$ is h-concave and non-decreasing.*

(i) If $\overline{a}$ and $\overline{b}$ are similarly ordered, then the functional $\eta_\Phi(\overline{p}) = h(P_n^2)\Phi\left(\frac{T(\overline{a},\overline{b},\overline{p})}{P_n^2}\right)$ is superadditive on $S_+(n)$.

Furthermore, if h is k-positive homogeneous, $\overline{p},\overline{q} \in S_+(n)$ and $M \geq m > 0$ are such that $M\overline{p} \geq \overline{q} \geq m\overline{p}$, then

$$\begin{aligned} k(M^2)h(P_n^2)\Phi\left(\frac{T(\overline{a},\overline{b},\overline{p})}{P_n^2}\right) &\geq h(Q_n^2)\Phi\left(\frac{T(\overline{a},\overline{b},\overline{q})}{Q_n^2}\right) \\ &\geq k(m^2)h(P_n^2)\Phi\left(\frac{T(\overline{a},\overline{b},\overline{p})}{P_n^2}\right). \end{aligned} \tag{5.11}$$

(ii) If $\overline{a}$ and $\overline{b}$ are oppositely ordered, then the functional $\eta_\Phi(\overline{p}) = h(P_n^2)\Phi\left(\frac{-T(\overline{a},\overline{b},\overline{p})}{P_n^2}\right)$ is superadditive on $S_+(n)$. If additionally, the assumptions on h, $\overline{p},\overline{q},M$ and m are satisfied as in case (i), then the inequalities (5.11) hold with substitution $T \to -T$.

Proof. If $\overline{a}$ and $\overline{b}$ are similarly ordered, let us define v and g as $v(\overline{p}) = P_n^2$ and $g(\overline{p}) = T(\overline{a},\overline{b},\overline{p})$. These functionals are positive homogeneous of order 2 and superadditive. By Theorem 5.3 with $L = 1$ we have that η_Φ is superadditive, and by Corollary 5.1 for the functional η_Φ, we obtain the inequality (5.11).

If $\overline{a}$ and $\overline{b}$ are oppositely ordered, then the functional $-T(\overline{a},\overline{b},\overline{p})$ is superadditive and non-negative and we proceed as in the proof of case (i). □

Remark 5.5 If $\Phi(x) = x$, i.e. $h(t) = t$, $k(t) = t$, and if $\overline{a}$ and $\overline{b}$ are similarly ordered n-tuples, then for $\overline{p},\overline{q}$ such that $M\overline{p} \geq \overline{q} \geq m\overline{p}$, we get

$$M^2 T(\overline{a},\overline{b},\overline{p}) \geq T(\overline{a},\overline{b},\overline{q}) \geq m^2 T(\overline{a},\overline{b},\overline{p}). \tag{5.12}$$

If $\overline{p} \geq \overline{q}$, i.e. $M = 1$, then from the above inequalities we get the following property of monotonicity:

$$T(\overline{a},\overline{b},\overline{p}) \geq T(\overline{a},\overline{b},\overline{q}). \tag{5.13}$$

If $\overline{a}$ and $\overline{b}$ are oppositely ordered, then the reversed inequalities in (5.12) and (5.13) hold.

Let us take $\overline{p} = \overline{p^{(n)}} = (p_1, p_2, \ldots, p_n)$, $\overline{p^{(n-1)}} = (p_1, p_2, \ldots, p_{n-1}, 0)$, $\overline{p^{(n-2)}} = (p_1, p_2, \ldots, p_{n-2}, 0, 0), \ldots, \overline{p^{(2)}} = (p_1, p_2, 0, \ldots, 0, 0)$. Since $\overline{p^{(n)}} \geq \overline{p^{(n-1)}} \geq \ldots \geq \overline{p^{(2)}}$ we can use the above monotonicity to obtain the following result.

Corollary 5.4 *If $\overline{a}$ and $\overline{b}$ are similarly ordered n-tuples and $\overline{p} \geq 0$, then*

$$T(\overline{a},\overline{b},\overline{p^{(n)}}) \geq T(\overline{a},\overline{b},\overline{p^{(n-1)}}) \geq T(\overline{a},\overline{b},\overline{p^{(n-2)}}) \geq \ldots \geq T(\overline{a},\overline{b},\overline{p^{(2)}}) \geq 0$$

and

$$T(\overline{a},\overline{b},\overline{p}) \geq \max_{1 \leq i < j \leq n} [(p_i + p_j)(p_i a_i b_i + p_j a_j b_j) - (p_i a_i + p_j a_j)(p_i b_i + p_j b_j)].$$

If $\overline{a}$ and $\overline{b}$ are oppositely ordered, then the reversed inequalities in the above inequalities hold with the substitution max $\to$ min *in the second result.*

Chebyshev functional for integrals

Let f, g be real functions on $I = [a,b]$. Let $S_+(I)$ be the cone of non-negative functions p on I such that p, pf, pg and pfg are integrable. Denote

$$T(f,g,p) = \int_a^b p(x)\,dx \int_a^b p(x)f(x)g(x)\,dx - \int_a^b p(x)f(x)\,dx \int_a^b p(x)g(x)\,dx.$$

The Chebyshev inequality for integrals states that $T(f,g,p) \geq 0$ when f and g are similarly ordered, i.e.

$$(f(x) - f(y))(g(x) - g(y)) \geq 0.$$

If f and g are oppositely ordered, then $T(f,g,p) \leq 0$. It is known that the following identity holds:

$$T(f,g,p) = \frac{1}{2} \int_a^b \int_a^b p(x)p(y)(f(x) - f(y))(g(x) - g(y))\,dx\,dy.$$

Using that identity we obtain that

$$\begin{aligned}&T(f,g,p+q)-T(f,g,p)-T(f,g,q)\\&=\frac{1}{2}\int_a^b\int_a^b[p(x)q(y)+q(x)p(y)](f(x)-f(y))(g(x)-g(y))\,dx\,dy\geq 0\end{aligned}$$

when f and g are similarly ordered. Thus follows superadditivity of $p\mapsto T(f,g,p)$ on the cone $S_+(I)$. If f and g are oppositely ordered, then the functional $-T(f,g,p)$ is superadditive. Corollary 5.1 refers to this case.

Corollary 5.5 *Let h be a non-negative submultiplicative function, which is k-positive homogeneous and $\Phi:[0,\infty)\to[0,\infty)$ is h-concave and non-decreasing. Let f and g be similarly ordered. If $p,q\in S_+(I)$ such that $P=\int_a^b p(x)\,dx>0$, $Q=\int_a^b q(x)\,dx>0$ and $M\geq m>0$ are such that $Mp(x)\geq q(x)\geq mp(x)$, then*

$$k(M^2)h(P^2)\Phi\left(\frac{T(f,g,p)}{P^2}\right)\ \geq\ Q^2\Phi\left(\frac{T(f,g,q)}{Q^2}\right)\geq k(m^2)h(P^2)\Phi\left(\frac{T(f,g,p)}{P^2}\right).$$

Proof. Let the function v be defined by $v(p)=\left(\int_a^b p(x)\,dx\right)^2$. It is superadditive and positive homogeneous of order $s_2=2$. The function g will be the Chebyshev functional $T(f,g,p)$. It is also positive homogeneous of order $s_2=2$, superadditive and non-negative. By Corollary 5.1 for the functional $\eta_\Phi(p)=h(v(p))\Phi\left(\frac{g(p)}{v(p)}\right)=h(P^2)\Phi\left(\frac{T(f,g,p)}{P^2}\right)$ with $L=1$, $K(t)=k(t^2)$, the wanted inequality is obtained. □

Remark 5.6 If $\Phi(x)=x$, i.e. $h(t)=t$, $k(t)=t$, and if the functions f and g are similarly ordered, then for $p,q\in S_+(I)$ such that such that $Mp(x)\geq q(x)\geq mp(x)$ we get

$$M^2T(f,g,p)\geq T(f,g,q)\geq m^2T(f,g,p).$$

If $p(x)\geq q(x)$, i.e. $M=1$, then from the above inequalities we get the following property of monotonicity:

$$T(f,g,p)\geq T(f,g,q).$$

If f and g are oppositely ordered, then the reversed inequalities hold.

Applications to the Milne functional

Here we consider the Milne inequality (see [83, p. 61–62]). Let a_i,b_i, $i=1,\dots,n$, be positive real numbers. Then

$$\sum_{i=1}^n(a_i+b_i)\sum_{i=1}^n\frac{a_ib_i}{a_i+b_i}\leq\sum_{i=1}^n a_i\sum_{i=1}^n b_i.$$

It is easy to get a weight version of the Milne inequality using substitutions

$$a_i\to p_ia_i,\ \ b_i\to p_ib_i,$$

where $p_1,\dots,p_n$ are positive real numbers. Of course, it can be improved to non-negative weights.

The Milne functional is defined as follows:

$$J_{Mi}(\overline{p}) = \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i - \sum_{i=1}^{n} p_i(a_i+b_i) \sum_{i=1}^{n} \frac{p_i a_i b_i}{a_i+b_i}.$$

The weight Milne inequality means that $J_{Mi}(\overline{p}) \geq 0$. Also, it is easy to see that $J_{Mi}(\alpha\overline{p}) = \alpha^2 J_{Mi}(\overline{p})$, i.e. J_{Mi} is positive homogeneous of order 2.

Theorem 5.9 *The functional $J_{Mi}(\overline{p})$ is superadditive on $S_+(n)$.*

Proof. It yields that

$$\begin{aligned}
&J_{Mi}(\overline{p}+\overline{q}) - J_{Mi}(\overline{p}) - J_{Mi}(\overline{q}) \\
&= \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} q_i b_i + \sum_{i=1}^{n} q_i a_i \sum_{i=1}^{n} p_i b_i - \left(\sum_{i=1}^{n} p_i(a_i+b_i) \sum_{i=1}^{n} \frac{q_i a_i b_i}{a_i+b_i} + \sum_{i=1}^{n} q_i(a_i+b_i) \sum_{i=1}^{n} \frac{p_i a_i b_i}{a_i+b_i} \right) \\
&= L_n.
\end{aligned}$$

After some (not so short, but simple) calculations we get

$$\begin{aligned}
L_{n+1} - L_n = {} & p_n \left(a_n \sum_{i=1}^{n} q_i b_i + b_n \sum_{i=1}^{n} q_i a_i - \frac{a_n b_n}{a_n+b_n} \sum_{i=1}^{n} q_i(a_i+b_i) - (a_n+b_n) \sum_{i=1}^{n} \frac{q_i a_i b_i}{a_i+b_i} \right) \\
& + q_n \left(a_n \sum_{i=1}^{n} p_i b_i + b_n \sum_{i=1}^{n} p_i a_i - \frac{a_n b_n}{a_n+b_n} \sum_{i=1}^{n} p_i(a_i+b_i) - (a_n+b_n) \sum_{i=1}^{n} \frac{p_i a_i b_i}{a_i+b_i} \right).
\end{aligned}$$

The term in the first bracket can be written as:

$$\frac{1}{a_n+b_n} \sum_{i=1}^{n} \frac{q_i}{a_i+b_i} (a_n b_i - a_i b_n)^2.$$

Thus

$$L_{n+1} \geq L_n \geq L_{n-1} \geq \ldots \geq L_1 = 0,$$

which means that J_{Mi} is superadditive and the proof is complete. □

Let $v(\overline{p}) = P_n^2$ and $g(\overline{p}) = J_{Mi}(\overline{p})$. Then the functional η_Φ defined by

$$\eta_\Phi(\overline{p}) = h(P_n^2)\Phi\left(\frac{J_{Mi}(\overline{p}}{P_n^2}\right)$$

is superadditive and it has boundedness property which follows from Corollary 5.1. The following chain of inequalities hold.

Corollary 5.6 *If $\overline{a},\overline{b},\overline{p} \geq 0$, then*

$$J_{Mi}(\overline{p^{(n)}}) \geq J_{Mi}(\overline{p^{(n-1)}}) \geq J_{Mi}(\overline{p^{(n-2)}}) \geq \ldots \geq J_{Mi}(\overline{p^{(2)}}) \geq 0$$

and

$$J_{Mi}(\overline{p}) \geq \max_{1\leq i<j\leq n} \left[(p_i a_i + p_j a_j)(p_i b_i + p_j b_j) - (p_i(a_i+b_i) + p_j(a_j+b_j)) \left(\frac{p_i a_i b_i}{a_i+b_i} + \frac{p_j a_j b_j}{a_j+b_j} \right) \right].$$

5.3 Superadditivity of functionals related to Gauss' type inequalities

In [207] superadditivity of some functionals associated with the Gauss-Winckler and the Gauss-Pólya inequalities was investigated. In [75] C. F. Gauss mentioned the following inequality between the second and the fourth absolute moments.

If f is a non-negative and decreasing function, then

$$\left(\int_0^\infty x^2 f(x)dx\right)^2 \leq \frac{5}{9}\int_0^\infty f(x)dx \int_0^\infty x^4 f(x)dx. \tag{5.14}$$

There have been many generalizations, sharpenings and improvements of inequality (5.14). One of the major lines of generalization is due to A. Winckler and the other due to the pair of the results of G. Pólya.

A. Winckler, [212], gave the following result which is known as the Gauss-Winckler inequality in the recent literature. More about it and its history one can find in [29].

Theorem 5.10 *If f is a non-negative, continuous and non-increasing function on $[0,\infty)$ such that $\int_0^\infty f(x)dx = 1$, then for $m \leq r$*

$$\left((m+1)\int_0^\infty x^m f(x)dx\right)^{\frac{1}{m}} \leq \left((r+1)\int_0^\infty x^r f(x)dx\right)^{\frac{1}{r}}. \tag{5.15}$$

Another generalization was done by G. Pólya and today all of the inequalities of the type are called the Gauss-Pólya inequalites. Namely, in the book "Problems and Theorems in Analysis" (see [188, Vol I, p. 83, Vol II, p. 129] one can find the following results.

Theorem 5.11 *(i) Let $f:[0,\infty)\to\mathbb{R}$ be a non-negative and decreasing function. If a and b are non-negative real numbers, then*

$$\left(\int_0^\infty x^{a+b} f(x)dx\right)^2 \leq \left(1-\left(\frac{a-b}{a+b+1}\right)^2\right)\int_0^\infty x^{2a} f(x)dx \int_0^\infty x^{2b} f(x)dx$$

if all the integrals exist.

(ii) Let $f:[0,1]\to\mathbb{R}$ be a non-negative and increasing function. If a and b are non-negative real numbers, then

$$\left(\int_0^1 x^{a+b} f(x)dx\right)^2 \geq \left(1-\left(\frac{a-b}{a+b+1}\right)^2\right)\int_0^1 x^{2a} f(x)dx \int_0^1 x^{2b} f(x)dx.$$

J. Pečarić and S. Varošanec treated the above mentioned inequalities in a unified way and proved the following generalizations (see [180], [181]).

Theorem 5.12 *Let $g : [a,b] \to \mathbb{R}$ be a non-negative increasing differentiable function and let $f : [a,b] \to \mathbb{R}$, be a non-negative function such that $x \mapsto \frac{f(x)}{g'(x)}$ is a non-decreasing function. Let p_i $(i = 1,\dots,n)$ be positive real numbers such that $\sum_{i=1}^{n} \frac{1}{p_i} = 1$. If a_i $(i = 1,\dots,n)$ are real numbers such that $a_i > -\frac{1}{p_i}$, then*

$$\int_a^b g(x)^{a_1+\dots+a_n} f(x)dx \geq \frac{\prod_{i=1}^{n}(a_i p_i + 1)^{\frac{1}{p_i}}}{1+\sum_{i=1}^{n} a_i} \prod_{i=1}^{n} \left(\int_a^b g(x)^{a_i p_i} f(x)dx \right)^{\frac{1}{p_i}}. \tag{5.16}$$

If $g(a) = 0$ and if the quotient function $\frac{f}{g'}$ is non-increasing, then the reverse inequality in (5.16) holds.

As a consequence of the above results one can conclude that if f and g satisfy the assumptions of Theorem 5.12, then the function

$$Q(r) = (r+1) \int_a^b g^r(x) f(x)dx$$

is log-concave when $\frac{f}{g'}$ is a non-decreasing function and the function Q is log-convex when $g(a) = 0$ and $\frac{f}{g'}$ is non-increasing.

Using that property, the following generalization of the Gauss-Winckler inequality was proved in [180]:

Theorem 5.13 *Let f and g be defined as in Theorem 5.12, $\frac{f}{g'}$ be a non-decreasing function and p, q, r, s be real numbers from the domain of definition of the function Q.*

If $p \leq q$, $r \leq s$ and $p > r$, $q > s$, then

$$\left(\frac{(p+1)\int_a^b g^p(x) f(x)dx}{(r+1)\int_a^b g^r(x) f(x)dx} \right)^{\frac{1}{p-r}} \geq \left(\frac{(q+1)\int_a^b g^q(x) f(x)dx}{(s+1)\int_a^b g^s(x) f(x)dx} \right)^{\frac{1}{q-s}}. \tag{5.17}$$

If $g(a) = 0$ and $\frac{f}{g'}$ is non-increasing, then the reverse inequality holds.

Remark 5.7 In [180] authors considered the case when $g(x) = x$, f is non-increasing and $a = 0$. In that case inequalities (5.16) and (5.17) hold with $b = \infty$ and thus the results for moments follow.

Investigation of the properties of the mapping which arises from Gauss-Pólya's inequalities or Gauss-Winckler inequality requires the specific tool which is here the following type of the Hölder inequality, [151]:

Proposition 5.3 *Let a_i, b_i, p_i, $(i = 1,\dots,n)$ be non-negative real numbers such that $\sum_{i=1}^{n} \frac{1}{p_i} = 1$. Then*

$$a_1^{\frac{1}{p_1}} \cdots a_n^{\frac{1}{p_n}} + b_1^{\frac{1}{p_1}} \cdots b_n^{\frac{1}{p_n}} \leq \prod_{i=1}^{n} (a_i + b_i)^{\frac{1}{p_i}}. \tag{5.18}$$

It is a simple consequence of the weight arithmetic-geometric mean inequality:

$$\frac{a_1^{\frac{1}{p_1}}\cdots a_n^{\frac{1}{p_n}}}{(a_1+b_1)^{\frac{1}{p_1}}\cdots(a_n+b_n)^{\frac{1}{p_n}}}+\frac{b_1^{\frac{1}{p_1}}\cdots b_n^{\frac{1}{p_n}}}{(a_1+b_1)^{\frac{1}{p_1}}\cdots(a_n+b_n)^{\frac{1}{p_n}}}$$
$$\leq \frac{a_1}{p_1(a_1+b_1)}+\cdots+\frac{a_n}{p_n(a_n+b_n)}+\frac{b_1}{p_1(a_1+b_1)}+\cdots+\frac{b_n}{p_n(a_n+b_n)}=1.$$

5.3.1 Functionals related to Gauss-Pólya inequalites

Throughout this section functions $f,g:[a,b]\to\mathbb{R}$ are non-negative, g is increasing differentiable, numbers p_i $(i=1,\ldots,n)$ are positive reals such that $\sum_{i=1}^n \frac{1}{p_i}=1$ and a_i $(i=1,\ldots,n)$ are real numbers such that $a_i>-\frac{1}{p_i}$.

Let us consider the functional G defined as

$$G(f)=\prod_{i=1}^n(a_ip_i+1)^{\frac{1}{p_i}}\prod_{i=1}^n\left(\int_a^b g(x)^{a_ip_i}f(x)dx\right)^{\frac{1}{p_i}}-(1+\sum_{i=1}^n a_i)\int_a^b g(x)^{a_1+\cdots+a_n}f(x)dx.$$

It is obvious that $f\mapsto G(f)$ is positive homogeneous, i.e. $G(\lambda f)=\lambda G(f)$, for any $\lambda\geq 0$. As a consequence of Theorem 5.12, if f/g' is a non-decreasing function, then $G(f)\leq 0$, while if f/g' is non-increasing and $g(a)=0$, then $G(f)\geq 0$.

The following theorem provides the superadditivity property of the functional G.

Theorem 5.14 *Let $f_1,f_2,g:[a,b]\to\mathbb{R}$ be non-negative functions, g increasing differentiable, numbers p_i $(i=1,\ldots,n)$ be positive reals such that $\sum_{i=1}^n\frac{1}{p_i}=1$ and a_i $(i=1,\ldots,n)$ be real numbers such that $a_i>-\frac{1}{p_i}$. Then*

$$G(f_1+f_2)\geq G(f_1)+G(f_2),$$

i.e. G is a superadditive functional. Furthermore, if $f_1\geq f_2$ are such that $\frac{f_1-f_2}{g'}$ is non-increasing, $g(a)=0$, then

$$G(f_1)\geq G(f_2),$$

i.e. G is non-decreasing.

Proof. Let us consider the difference $G(f_1+f_2)-G(f_1)-G(f_2)$.

$$\begin{aligned}
&G(f_1+f_2)-G(f_1)-G(f_2)\\
&=\prod_{i=1}^n(a_ip_i+1)^{\frac{1}{p_i}}\prod_{i=1}^n\left(\int_a^b g(x)^{a_ip_i}(f_1+f_2)(x)dx\right)^{\frac{1}{p_i}}\\
&\quad-(1+\sum_{i=1}^n a_i)\int_a^b g(x)^{a_1+\cdots+a_n}(f_1+f_2)(x)dx-\prod_{i=1}^n(a_ip_i+1)^{\frac{1}{p_i}}\prod_{i=1}^n\left(\int_a^b g(x)^{a_ip_i}f_1(x)dx\right)^{\frac{1}{p_i}}
\end{aligned}$$

$$+(1+\sum_{i=1}^{n}a_i)\int_a^b g(x)^{a_1+\cdots+a_n}f_1(x)dx-\prod_{i=1}^{n}(a_ip_i+1)^{\frac{1}{p_i}}\prod_{i=1}^{n}\left(\int_a^b g(x)^{a_ip_i}f_2(x)dx\right)^{\frac{1}{p_i}}$$
$$+(1+\sum_{i=1}^{n}a_i)\int_a^b g(x)^{a_1+\cdots+a_n}f_2(x)dx$$
$$=\prod_{i=1}^{n}(a_ip_i+1)^{\frac{1}{p_i}}\left[\prod_{i=1}^{n}\left(\int_a^b g(x)^{a_ip_i}(f_1+f_2)(x)dx\right)^{\frac{1}{p_i}}\right.$$
$$\left.-\prod_{i=1}^{n}\left(\int_a^b g(x)^{a_ip_i}f_1(x)dx\right)^{\frac{1}{p_i}}-\prod_{i=1}^{n}\left(\int_a^b g(x)^{a_ip_i}f_2(x)dx\right)^{\frac{1}{p_i}}\right].$$

Setting in (5.18):

$$a_i=\int_a^b g(x)^{a_ip_i}f_1(x)dx,\;\; b_i=\int_a^b g(x)^{a_ip_i}f_2(x)dx,\;\; i=1,2,\ldots,n$$

and using the Hölder inequality we have that $G(f_1+f_2)-G(f_1)-G(f_2)\geq 0$, so G is superadditive.

If $f_1\geq f_2$, $\frac{f_1-f_2}{g'}$ is non-increasing and $g(a)=0$, then $G(f_1-f_2)\geq 0$. Thus

$$G(f_1)\;=\;G(f_2+(f_1-f_2))\geq G(f_2)+G(f_1-f_2)\geq G(f_2).$$ □

Corollary 5.7 *Let f_1,f_2,g be non-negative functions on $[a,b]$, g increasing differentiable, $g(a)=0$, numbers p_i $(i=1,\ldots,n)$ be positive reals such that $\sum_{i=1}^{n}\frac{1}{p_i}=1$, a_i $(i=1,\ldots,n)$ be real numbers such that $a_i>-\frac{1}{p_i}$ and $c,C\in\mathbb{R}$ such that Cf_2-f_1, f_1-cf_2 are non-negative and $\frac{Cf_2-f_1}{g'}$, $\frac{f_1-cf_2}{g'}$ are non-negative non-increasing functions. Then*

$$C\left\{\prod_{i=1}^{n}(a_ip_i+1)^{\frac{1}{p_i}}\prod_{i=1}^{n}\left(\int_a^b g(x)^{a_ip_i}f_2(x)dx\right)^{\frac{1}{p_i}}-(1+\sum_{i=1}^{n}a_i)\int_a^b g(x)^{a_1+\cdots+a_n}f_2(x)dx\right\}$$
$$\geq\prod_{i=1}^{n}(a_ip_i+1)^{\frac{1}{p_i}}\prod_{i=1}^{n}\left(\int_a^b g(x)^{a_ip_i}f_1(x)dx\right)^{\frac{1}{p_i}}-(1+\sum_{i=1}^{n}a_i)\int_a^b g(x)^{a_1+\cdots+a_n}f_1(x)dx$$
$$\geq c\left\{\prod_{i=1}^{n}(a_ip_i+1)^{\frac{1}{p_i}}\prod_{i=1}^{n}\left(\int_a^b g(x)^{a_ip_i}f_2(x)dx\right)^{\frac{1}{p_i}}-(1+\sum_{i=1}^{n}a_i)\int_a^b g(x)^{a_1+\cdots+a_n}f_2(x)dx\right\}.$$

Proof. Using the previous results we have

$$CG(f_2)=G(Cf_2)=G((Cf_2-f_1)+f_1)\geq G(Cf_2-f_1)+G(f_1)\geq G(f_1)$$

and

$$G(f_1)=G((f_1-cf_2)+cf_2)\geq G(f_1-cf_2)+G(cf_2)\geq G(cf_2)=cG(f_2)$$

which concludes the proof. □

The following theorem contains a result on concavity of function $G\circ\phi$, where ϕ is a concave function.

Theorem 5.15 *Let $\phi : [0,\infty) \to [0,\infty)$ be a concave function, f_1, f_2, g be non-negative functions on $[a,b]$ such that $(\phi \circ (\alpha f_1 + (1-\alpha) f_2) - [\alpha(\phi \circ f_1) + (1-\alpha)(\phi \circ f_2)])/g'$ is non-increasing for some $\alpha \in [0,1]$, $g(a) = 0$. Then*

$$G \circ \phi \circ (\alpha f_1 + (1-\alpha) f_2) \geq \alpha(G \circ \phi \circ f_1) + (1-\alpha)(G \circ \phi \circ f_2).$$

Proof. For any $x \in [a,b]$ we have

$$\begin{aligned}(\phi \circ (\alpha f_1 + (1-\alpha) f_2))(x) &= \phi(\alpha f_1(x) + (1-\alpha) f_2(x)) \\ &\geq \alpha\phi(f_1(x)) + (1-\alpha)\phi(f_2(x)) \\ &= (\alpha(\phi \circ f_1) + (1-\alpha)(\phi \circ f_2))(x),\end{aligned}$$

where concavity of the function ϕ is used. So, we have $\phi \circ (\alpha f_1 + (1-\alpha) f_2) \geq \alpha(\phi \circ f_1) + (1-\alpha)(\phi \circ f_2)$. Using properties of G and the above-proved inequality we have

$$\begin{aligned}G(\phi \circ (\alpha f_1 + (1-\alpha) f_2)) &\geq G(\alpha(\phi \circ f_1) + (1-\alpha)(\phi \circ f_2)) \\ &\geq G(\alpha(\phi \circ f_1)) + G((1-\alpha)(\phi \circ f_2)) \\ &= \alpha G(\phi \circ f_1) + (1-\alpha) G(\phi \circ f_2)\end{aligned}$$

and the proof is established. □

Remark 5.8 Let us consider the case when $g(x) = x$, $a = 0$, $b = \infty$ and f is non-increasing. Let us denote by $\mu_r(f)$ a moment of the order r, i.e.

$$\mu_r(f) = \int_0^\infty x^r f(x)dx.$$

Then the functional G has a form

$$G(f) = \prod_{i=1}^n (a_i p_i + 1)^{\frac{1}{p_i}} \prod_{i=1}^n \mu_{a_i p_i}^{\frac{1}{p_i}}(f) - (1 + \sum_{i=1}^n a_i)\mu_{a_1+\cdots+a_n}(f)$$

and G is superadditive. Also, if $f_1 \geq f_2$ such that $f_1 - f_2$ is non-increasing, then $G(f_1) \geq G(f_2)$.

5.3.2 Functionals related to the Gauss-Winckler inequality

Putting in (5.17) $r = s = 0$ we get the Gauss-Winckler inequality for f/g' non-decreasing function:

$$\left(\frac{(p+1)\int_a^b g^p(x)f(x)dx}{\int_a^b f(x)dx}\right)^{\frac{1}{p}} \geq \left(\frac{(q+1)\int_a^b g^q(x)f(x)dx}{\int_a^b f(x)dx}\right)^{\frac{1}{q}},$$

where $0 < p \leq q$. If f/g' is non-increasing and $g(a) = 0$, then the reversed inequality holds.

Let us consider the functional W defined as

$$W(f) = \left(\int_a^b f(x)dx\right)^{1-\frac{p}{q}} \left((q+1)\int_a^b g^q(x)f(x)dx\right)^{\frac{p}{q}} - (p+1)\int_a^b g^p(x)f(x)dx.$$

The following theorem provides its superadditivity and monotonicity.

Theorem 5.16 *Let $f_1, f_2, g : [a,b] \to \mathbb{R}$ be non-negative functions, g increasing differentiable function and numbers p, q be positive reals such that $p \le q$. Then*

$$W(f_1+f_2) \ge W(f_1)+W(f_2).$$

Additionaly, if $f_1 \ge f_2$ are such that $\frac{f_1-f_2}{g'}$ is non-increasing, $g(a)=0$, then

$$W(f_1) \ge W(f_2).$$

Proof. Let us transform $W(f_1+f_2)-W(f_1)-W(f_2)$.

$$\begin{aligned}
& W(f_1+f_2)-W(f_1)-W(f_2) \\
={}& \left(\int_a^b (f_1+f_2)(x)dx\right)^{1-\frac{p}{q}} \left((q+1)\int_a^b g^q(x)(f_1+f_2)(x)dx\right)^{\frac{p}{q}} \\
& -(p+1)\int_a^b g^p(x)(f_1+f_2)(x)dx \\
& -\left(\int_a^b f_1(x)dx\right)^{1-\frac{p}{q}} \left((q+1)\int_a^b g^q(x)f_1(x)dx\right)^{\frac{p}{q}} \\
& +(p+1)\int_a^b g^p(x)f_1(x)dx - \left(\int_a^b f_2(x)dx\right)^{1-\frac{p}{q}} \left((q+1)\int_a^b g^q(x)f_2(x)dx\right)^{\frac{p}{q}} \\
& +(p+1)\int_a^b g^p(x)f_2(x)dx \\
={}& \left(\int_a^b (f_1+f_2)(x)dx\right)^{1-\frac{p}{q}} \left((q+1)\int_a^b g^q(x)(f_1+f_2)(x)dx\right)^{\frac{p}{q}} \\
& -\left(\int_a^b f_1(x)dx\right)^{1-\frac{p}{q}} \left((q+1)\int_a^b g^q(x)f_1(x)dx\right)^{\frac{p}{q}} \\
& -\left(\int_a^b f_2(x)dx\right)^{1-\frac{p}{q}} \left((q+1)\int_a^b g^q(x)f_2(x)dx\right)^{\frac{p}{q}} \ge 0,
\end{aligned}$$

where in the last inequality we use the Hölder inequality with

$$n=2,\ \frac{1}{p_1}=1-\frac{p}{q}>0,\ \frac{1}{p_2}=\frac{p}{q}>0,\ a_1=\int_a^b f_1(x)dx,\ b_1=\int_a^b f_2(x)dx,$$

$$a_2=(q+1)\int_a^b g^q(x)f_1(x)dx,\ b_2=(q+1)\int_a^b g^q(x)f_2(x)dx.$$

So, superadditivity of the functional W is established.

If $\frac{f_1-f_2}{g'}$ is non-increasing, $g(a)=0$, then from Theorem 5.13 we obtain $W(f_1-f_2)\ge 0$ and

$$W(f_1)=W(f_2+(f_1-f_2)) \ge W(f_2)+W(f_1-f_2) \ge W(f_2).$$

□

Remark 5.9 Let us consider the case when $g(x) = x$, $a = 0$, $b = \infty$ and f is non-increasing. Now the functional W has the form

$$W(f) = (q+1)^{\frac{p}{q}} \left(\mu_0(f)\right)^{1-\frac{p}{q}} \mu_q^{\frac{p}{q}}(f) - (p+1)\mu_p(f)$$

and W is superadditive. Also, if $f_1 \geq f_2$ are such that $f_1 - f_2$ is non-increasing, then $W(f_1) \geq W(f_2)$.

The following result is an interesting inequality for the Beta function.

Corollary 5.8 *Let $0 < p \leq q$, $y_1, y_2 > -1$. Then*

$$\left(\frac{1}{y_1+1} + \frac{1}{y_2+1}\right)^{1-\frac{p}{q}} \left[B(q+1, y_1+1) + B(q+1, y_2+1)\right]^{\frac{p}{q}}$$

$$\geq \left(\frac{1}{y_1+1}\right)^{1-\frac{p}{q}} B^{\frac{p}{q}}(q+1, y_1+1) + \left(\frac{1}{y_2+1}\right)^{1-\frac{p}{q}} B^{\frac{p}{q}}(q+1, y_2+1)$$

where B is the Beta function defined as $B(x+1, y+1) = \int_0^1 t^x (1-t)^y dt$.

Proof. It is a consequence of the previous theorem with $[a,b] = [0,1]$, $f_i(t) = (1-t)^{y_i}$, $i = 1, 2$, $g(x) = x$. □

Chapter 6

Jensen-type functionals for the operators on a Hilbert space

In this chapter we present the refinements and the converses of the operator mean inequalities (arithmetic-geometric, arithmetic-harmonic, arithmetic-Heinz,...), all of which are deduced from the superadditivity of the Jensen functional for the operators on a Hilbert space, in its several variants. These improvements are obtained due to the inventive method developed by J. Pečarić, which is employed on the discrete Jensen functional (1.65) whose real arguments are now substituted by operators on a Hilbert space. Some of the references used in this chapter are e.g. [97], [98], [145], [147], [215] and the contents is for the most part included within the published papers [96] and [107].

In the second part of the chapter, integral operator Jensen's inequality is the base for defining the corresponding functional, whose superadditivity and monotonicity are then proved. Apart from this, but published in the same paper [110], an analysis of the multidimensional Jensen's functional for operators is presented, accompanied with several interesting applications.

6.1 Motivation

F. Kittaneh i Y. Manasrah in [97] obtained the following improvement of the classical arithmetic-geometric mean inequality:

$$\begin{aligned} a^{\nu}b^{1-\nu} + \max\{\nu, 1-\nu\}\left(\sqrt{a}-\sqrt{b}\right)^2 &\geq \nu a + (1-\nu)b \\ &\geq a^{\nu}b^{1-\nu} + \min\{\nu, 1-\nu\}\left(\sqrt{a}-\sqrt{b}\right)^2, \end{aligned} \tag{6.1}$$

$a, b \geq 0$, $\nu \in [0,1]$, with its converse contained in the left inequality and the refinement in the right one. It is well known that Heinz means

$$H_{\nu}(a,b) = \frac{a^{\nu}b^{1-\nu} + a^{1-\nu}b^{\nu}}{2}, \tag{6.2}$$

$\nu \in [0,1]$, interpolate the geometric and the arithmetic mean of $a, b \geq 0$, wherefrom authors in [97] managed to obtain the improvement in the following form:

$$H_{\nu}(a,b) + \min\{\nu, 1-\nu\}\left(\sqrt{a}-\sqrt{b}\right)^2 \leq \frac{a+b}{2}. \tag{6.3}$$

On the other hand, recall that in not so recent paper [136] of R. Merris and S. Pierce the matrix variant of the arithmetic-geometric inequality was given:

$$A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\nu}A^{\frac{1}{2}} \leq A \nabla_{\nu} B, \tag{6.4}$$

$A, B \in M_n(\mathbb{C})$, such that B is positive semidefinite and A is positive definite, $\nu \in [0,1]$.

It was by means of (6.4) that Kittaneh and Manasrah in their another paper [98] obtained new matrix generalizations of the relations (6.1) and (6.3). In that sense, matrix variant of (6.1) reads

$$\begin{aligned} &2\max\{\nu, 1-\nu\}\left[A \nabla B - A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}\right] \\ \geq\ &A \nabla_{\nu} B - A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\nu}A^{\frac{1}{2}} \\ \geq\ &2\min\{\nu, 1-\nu\}\left[A \nabla B - A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}\right], \end{aligned} \tag{6.5}$$

where $A \in M_n(\mathbb{C})$ is positive definite and $B \in M_n(\mathbb{C})$ is positive semidefinite, $\nu \in [0,1]$.

These mean inequalities are of a special interest here because in the sequel we present the results published in [96] and [107], where authors obtained the corresponding operator (infinite dimensional) variant of these (finite dimensional) matrix inequalities.

For the sake of the further analysis concerning operators on a Hilbert space, let's mention here that H. Zuo *et al.* obtained in [215] the refinement of the weight *operator* arithmetic-harmonic mean inequality in the difference form:

$$A \nabla_{\nu} B - A\,!_{\nu} B \geq 2\min\{\nu, 1-\nu\}\,[A \nabla B - A\,!\,B], \tag{6.6}$$

where A and B are positive invertible operators and $\nu \in [0,1]$.

6.2 Superadditivity of the Jensen functional for the operators on a Hilbert space

Here we define the functional for the operators on a Hilbert space whose properties will provide us with the improvements of the operator mean inequalities.

Recall that the self-adjoint operators constitute the subspace of the C*-algebra of all bounded linear operators and is denoted with $\mathscr{B}_h(H)$, the set of all positive operators in $\mathscr{B}_h(H)$ is a convex cone in $\mathscr{B}_h(H)$ which defines the order " $\leq$ " on $\mathscr{B}_h(H)$. This convex cone is denoted with $\mathscr{B}^+(H)$. The set of all strictly positive (or positive invertible) operators in $\mathscr{B}_h(H)$ is denoted with $\mathscr{B}^{++}(H)$.

The idea on which the method developed by J. Pečarić was based has as a starting point the discrete Jensen functional $J(f,\mathbf{x},\mathbf{p})$ defined by (1.65) and analysed in [66]. We write $\mathbf{x}=(x,\delta)$ and the first coordinate substitute with the self-adjoint operator D.

Let $[a,b]$ be an interval in $\mathbb{R}$ and $\delta\in[a,b]$. Suppose $\mathbf{p}=(p_1,p_2)$ is a nonnegative pair of real numbers such that $p_1+p_2>0$. The set of all such pairs $\mathbf{p}$ will be denoted with $\mathscr{P}^0$. Now, let $D\in\mathscr{B}_h(H)$ be such that $a1_H\leq D\leq b1_H$, where 1_H is a unit operator on H and let $f\colon[a,b]\to\mathbb{R}$ be a continuous function. Then Jensen's functional $\mathscr{J}(f,D,\delta,\mathbf{p})$ for the operators on a Hilbert space is defined by

$$\mathscr{J}(f,D,\delta,\mathbf{p})=p_1f(D)+p_2f(\delta)1_H-(p_1+p_2)f\left(\frac{p_1D+p_2\delta 1_H}{p_1+p_2}\right). \tag{6.7}$$

If f is a convex function, then $\mathscr{J}(f,D,\delta,\mathbf{p})\geq 0$, for all $\mathbf{p}\in\mathscr{P}^0$, as a consequence of the Jensen inequality and the monotonicity property (1.40) valid for the operator functions.

Furthermore, if we fix f, D and δ, then we can observe $\mathscr{J}(f,D,\delta,\cdot)$ as a function on $\mathscr{P}^0$.

Theorem 6.1 *Let $[a,b]$ be an interval in $\mathbb{R}$ and let $\delta\in[a,b]$. Suppose D is an operator in $\mathscr{B}_h(H)$ such that $a1_H\leq D\leq b1_H$ and $\mathbf{p}=(p_1,p_2),\mathbf{q}=(q_1,q_2)\in\mathscr{P}^0$. If $f\colon[a,b]\to\mathbb{R}$ is a continuous and convex function, then*

$$\mathscr{J}(f,D,\delta,\mathbf{p}+\mathbf{q})\geq\mathscr{J}(f,D,\delta,\mathbf{p})+\mathscr{J}(f,D,\delta,\mathbf{q}), \tag{6.8}$$

that is, $\mathscr{J}(f,D,\delta,\cdot)$ is superadditive on $\mathscr{P}^0$. Furthermore, if $\mathbf{p},\mathbf{q}\in\mathscr{P}^0$ are such that $\mathbf{p}\geq\mathbf{q}$, (i.e. $p_1\geq q_1$, $p_2\geq q_2$), then

$$\mathscr{J}(f,D,\delta,\mathbf{p})\geq\mathscr{J}(f,D,\delta,\mathbf{q})\geq 0, \tag{6.9}$$

that is, $\mathscr{J}(f,D,\delta,\cdot)$ is increasing on $\mathscr{P}^0$.

Proof. Discrete Jensen's functional $J(f,\mathbf{x},\mathbf{p})$ defined by (1.65) for $n=2$ and $\mathbf{x}=(x,\delta)$ assumes the following form:

$$j(f,x,\delta,\mathbf{p})=p_1f(x)+p_2f(\delta)-(p_1+p_2)f\left(\frac{p_1x+p_2\delta}{p_1+p_2}\right). \tag{6.10}$$

Since the functional (1.65) is superadditive and increasing on $\mathscr{P}^0$, the functional $j(f,x,\delta,\mathbf{p})$ possesses the corresponding properties:

$$j(f,x,\delta,\mathbf{p}+\mathbf{q}) \geq j(f,x,\delta,\mathbf{p}) + j(f,x,\delta,\mathbf{q}), \tag{6.11}$$

$$j(f,x,\delta,\mathbf{p}) \geq j(f,x,\delta,\mathbf{q}), \qquad \mathbf{p} \geq \mathbf{q}. \tag{6.12}$$

Continuous functional calculus (1.39) provides for the function f, which is continuous on the spectrum of the operator D, to act on the self-adjoint operator D. Order preserving property (1.40) for operator functions provides that inequalities (6.11) and (6.12) hold if we substitute x by D, $a1_H \leq D \leq b1_H$. Hence the statement of the theorem is true. □

The lower and the upper bound for the functional $J(f,D,\delta,\mathbf{p})$ are expressed by means of the non-weight functional of the same type.

Corollary 6.1 *Let f, D, δ, $\mathbf{p}$ and functional $\mathscr{J}$ be as in Theorem 6.1. Then the following inequalities hold:*

$$2\max\{p_1,p_2\}\mathscr{J}(f,D,\delta) \geq \mathscr{J}(f,D,\delta,\mathbf{p}) \geq 2\min\{p_1,p_2\}\mathscr{J}(f,D,\delta), \tag{6.13}$$

where

$$\mathscr{J}(f,D,\delta) = \frac{f(D)+f(\delta)1_H}{2} - f\left(\frac{D+\delta 1_H}{2}\right).$$

Proof. If we compare the ordered pair $\mathbf{p} = (p_1,p_2) \in \mathscr{P}^0$ with the constant pairs

$$\mathbf{p}_{\max} = \big(\max\{p_1,p_2\},\max\{p_1,p_2\}\big) \text{ and } \mathbf{p}_{\min} = \big(\min\{p_1,p_2\},\min\{p_1,p_2\}\big),$$

we see that $\mathbf{p}_{\max} \geq \mathbf{p} \geq \mathbf{p}_{\min}$, so by applying (6.9) the following inequalities hold:

$$\mathscr{J}(f,D,\delta,\mathbf{p}_{\max}) \geq \mathscr{J}(f,D,\delta,\mathbf{p}) \geq \mathscr{J}(f,D,\delta,\mathbf{p}_{\min}).$$

Finally, since $\mathscr{J}(f,D,\delta,\mathbf{p}_{\max}) = 2\max\{p_1,p_2\}\mathscr{J}(f,D,\delta)$ and $\mathscr{J}(f,D,\delta,\mathbf{p}_{\min}) = 2\min\{p_1,p_2\}\mathscr{J}(f,D,\delta)$, inequalities (6.13) hold. □

With f, δ, $\mathbf{p}$ and $\mathbf{q}$ as in Theorem 6.1, let $A \in \mathscr{B}^{++}(H)$ and $B \in \mathscr{B}_h(H)$. Suppose that $aA \leq B \leq bA$. Now from the Jensen functional (6.7) we deduce the functional $A^{\frac{1}{2}}\mathscr{J}\left(f,A^{-\frac{1}{2}}BA^{-\frac{1}{2}},\delta,\mathbf{p}\right)A^{\frac{1}{2}}$ defined as

$$\begin{aligned} &A^{\frac{1}{2}}\mathscr{J}\left(f,A^{-\frac{1}{2}}BA^{-\frac{1}{2}},\delta,\mathbf{p}\right)A^{\frac{1}{2}} \\ &= p_1A^{\frac{1}{2}}f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} + p_2f(\delta)A - (p_1+p_2)A^{\frac{1}{2}}f\left(\frac{p_1A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + p_2\delta 1_H}{p_1+p_2}\right)A^{\frac{1}{2}}. \end{aligned} \tag{6.14}$$

Remark 6.1 The functional (6.14) is well defined because the condition $aA \leq B \leq bA$ implies $a1_H \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq b1_H$, that is, the spectrum of the operator $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ does belong to the domain of the function f.

The functional (6.14) also possesses the properties described in Theorem 6.1.

Theorem 6.2 *Let $A \in \mathscr{B}^{++}(H)$ and $B \in \mathscr{B}_h(H)$. Suppose $aA \le B \le bA$. If f, δ, $\mathbf{p}$ and $\mathbf{q}$ and the functional $\mathscr{J}$ are as in Theorem 6.1, then*

$$\begin{aligned} &A^{\frac{1}{2}} \mathscr{J}\left(f, A^{-\frac{1}{2}} B A^{-\frac{1}{2}}, \delta, \mathbf{p}+\mathbf{q}\right) A^{\frac{1}{2}} \\ &\quad \ge A^{\frac{1}{2}} \mathscr{J}\left(f, A^{-\frac{1}{2}} B A^{-\frac{1}{2}}, \delta, \mathbf{p}\right) A^{\frac{1}{2}} + A^{\frac{1}{2}} \mathscr{J}\left(f, A^{-\frac{1}{2}} B A^{-\frac{1}{2}}, \delta, \mathbf{q}\right) A^{\frac{1}{2}}, \end{aligned} \tag{6.15}$$

that is, $A^{\frac{1}{2}} \mathscr{J}\left(f, A^{-\frac{1}{2}} B A^{-\frac{1}{2}}, \delta, \cdot\right) A^{\frac{1}{2}}$ is superadditive on $\mathscr{P}^0$. Furthermore, if $\mathbf{p}, \mathbf{q} \in \mathscr{P}^0$ are such that $\mathbf{p} \ge \mathbf{q}$, then

$$A^{\frac{1}{2}} \mathscr{J}\left(f, A^{-\frac{1}{2}} B A^{-\frac{1}{2}}, \delta, \mathbf{p}\right) A^{\frac{1}{2}} \ge A^{\frac{1}{2}} \mathscr{J}\left(f, A^{-\frac{1}{2}} B A^{-\frac{1}{2}}, \delta, \mathbf{q}\right) A^{\frac{1}{2}} \ge 0, \tag{6.16}$$

that is, $A^{\frac{1}{2}} \mathscr{J}\left(f, A^{-\frac{1}{2}} B A^{-\frac{1}{2}}, \delta, \cdot\right) A^{\frac{1}{2}}$ is increasing on $\mathscr{P}^0$.

Proof. According to Remark 6.1, functional $\mathscr{J}\left(f, A^{-\frac{1}{2}} B A^{-\frac{1}{2}}, \delta, \mathbf{p}\right)$ is well defined. Because of the superadditivity in (6.8), the functional

$$\mathscr{J}\left(f, A^{-\frac{1}{2}} B A^{-\frac{1}{2}}, \delta, \mathbf{p}+\mathbf{q}\right) - \mathscr{J}\left(f, A^{-\frac{1}{2}} B A^{-\frac{1}{2}}, \delta, \mathbf{p}\right) - \mathscr{J}\left(f, A^{-\frac{1}{2}} B A^{-\frac{1}{2}}, \delta, \mathbf{q}\right) \tag{6.17}$$

is non-negative. Multiplicating this functional by $A^{\frac{1}{2}}$ both-sidedly, its non-negativity stays preserved. Thus superadditivity in (6.15) follows directly, as well as monotonicity property in (6.16). □

Both sided bounds, analogous to those of the functional (6.7) are obtained for (6.14), as a consequence of Theorem 6.2. These bounds are of a special interest for they allow us to observe some refinements and converses of the operator mean inequalities, which are analyzed in the sequel.

Corollary 6.2 *Let $A \in \mathscr{B}^{++}(H)$ and $B \in \mathscr{B}_h(H)$ with $aA \le B \le bA$. Suppose $\delta \in [a,b]$, $[a,b] \subseteq \mathbb{R}$, $\mathbf{p} = (p_1, p_2) \in \mathscr{P}^0$ and let $f\colon [a,b] \to \mathbb{R}$ be a continuous and convex function. Then the following inequalities hold:*

$$\begin{aligned} &2\max\{p_1,p_2\} A^{\frac{1}{2}} \mathscr{J}\left(f, A^{-\frac{1}{2}} B A^{-\frac{1}{2}}, \delta\right) A^{\frac{1}{2}} \ge A^{\frac{1}{2}} \mathscr{J}\left(f, A^{-\frac{1}{2}} B A^{-\frac{1}{2}}, \delta, \mathbf{p}\right) A^{\frac{1}{2}} \\ &\quad \ge 2\min\{p_1,p_2\} A^{\frac{1}{2}} \mathscr{J}\left(f, A^{-\frac{1}{2}} B A^{-\frac{1}{2}}, \delta\right) A^{\frac{1}{2}}, \end{aligned} \tag{6.18}$$

where

$$\mathscr{J}(f, A^{-\frac{1}{2}} B A^{-\frac{1}{2}}, \delta) = \frac{f(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) + f(\delta) 1_H}{2} - f\left(\frac{A^{-\frac{1}{2}} B A^{-\frac{1}{2}} + \delta 1_H}{2}\right).$$

6.3 Application to operator means

Let us observe the functional (6.14) with the continuous convex function $f\colon \mathbb{R} \to \mathbb{R}_+$ defined by $f(x) = \exp x$, for $\delta = 0$ and for the operator $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ substituted by $\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})$, where therefore $B \in \mathscr{B}^{++}(H)$. With these definition adjustments made, we deduce the functional that is the difference between the operator arithmetic and geometric mean:

$$\mathscr{L}(A,B,\mathbf{p}) = (p_1+p_2)\left[A\,\nabla_{\frac{p_1}{p_1+p_2}}B - A\sharp_{\frac{p_1}{p_1+p_2}}B\right], \tag{6.19}$$

where $aA \le B \le bA$.

The statements of the Theorem 6.2 and Corollary 6.2 referring to the functional (6.14) are valid for its specified form - functional (6.19). Thus the relation (6.18) assumes the following form:

$$\begin{aligned} 2\max\{p_1,p_2\}\,[A\,\nabla B - A\sharp B] \;&\ge\; (p_1+p_2)\left[A\,\nabla_{\frac{p_1}{p_1+p_2}}B - A\sharp_{\frac{p_1}{p_1+p_2}}B\right] \\ &\ge\; 2\min\{p_1,p_2\}\,[A\,\nabla B - A\sharp B], \end{aligned} \tag{6.20}$$

which actually presents the converse and the refinement of the operator arithmetic-geometric mean inequality. In that sense, inequalities (6.20) are the operator generalization of the matrix inequalities (6.5) from [98].

In order to analyze and improve the operator arithmetic-Heinz inequality, let us firstly show how operator Heinz mean interpolates operator arithmetic and geometric means, considering the operator order. Recall that the operator Heinz mean is defined by

$$H_\nu(A,B) = \frac{A\sharp_\nu B + A\sharp_{1-\nu}B}{2}, \tag{6.21}$$

$\nu \in [0,1]$, $A,B \in \mathscr{B}^{++}(H)$.

Proposition 6.1 *Let H be a Hilbert space and $A,B \in \mathscr{B}^{++}(H)$. If $\nu \in [0,1]$, then*

$$A\sharp B \le H_\nu(A,B) \le A\,\nabla B. \tag{6.22}$$

Proof. Since

$$A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\nu}A^{\frac{1}{2}} + A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{1-\nu}A^{\frac{1}{2}} \le A\,\nabla_\nu B + A\,\nabla_{1-\nu}B = A+B,$$

the second inequality in (6.22) holds.

With the aim of proving the first inequality in (6.22), we observe the scalar inequality

$$x^\nu + x^{1-\nu} \ge 2\sqrt{x}$$

which holds for all $x \in \mathbb{R}_+$. On the other hand, the operator $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ has a positive spectrum. Thus according to the order preservation property (1.40) we can write

$$\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\nu} + \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{1-\nu} \geq 2\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}. \tag{6.23}$$

By multiplying the inequality (6.23) both sidedly by $A^{\frac{1}{2}}$, the first inequality in (6.22) is also proved. □

Inequality (6.3) described in the introductory part of this chapter improves the classical arithmetic-Heinz inequality. Related results for the finite dimensional matrix variant also exist and here we deal with the analogous results for the means defined by the operators on a Hilbert space. In order to measure the difference between the arithmetic and the weight Heinz mean on a Hilbert space, according to relation (6.22), we deduce the functional

$$\mathscr{M}(A,B,\mathbf{p}) = \frac{1}{2}(p_1+p_2)\left[2A\,\nabla\,B - A\sharp_{\frac{p_1}{p_1+p_2}}B - A\sharp_{\frac{p_2}{p_1+p_2}}B\right], \tag{6.24}$$

where $\mathbf{p} = (p_1,p_2) \in \mathscr{P}^0$ and $A,B \in \mathscr{B}^{++}(H)$. From (6.22) it follows that $\mathscr{M}(B,C,\mathbf{p}) \geq 0$. Superadditivity and monotonicity of this functional are the contents of the following theorem.

Theorem 6.3 *Let H be a Hilbert space and $A,B \in \mathscr{B}^{++}(H)$. Suppose $\mathbf{p},\mathbf{q} \in \mathscr{P}^0$. Then*

$$\mathscr{M}(A,B,\mathbf{p}+\mathbf{q}) \geq \mathscr{M}(A,B,\mathbf{p}) + \mathscr{M}(A,B,\mathbf{q}), \tag{6.25}$$

that is, $\mathscr{M}(A,B,\cdot)$ is superadditive on $\mathscr{P}^0$. Furthermore, if $\mathbf{p},\mathbf{q} \in \mathscr{P}^0$ are such that $\mathbf{p} \geq \mathbf{q}$ $(p_1 \geq q_1,\ p_2 \geq q_2)$, then

$$\mathscr{M}(A,B,\mathbf{p}) \geq \mathscr{M}(A,B,\mathbf{q}) \geq 0, \tag{6.26}$$

that is, $\mathscr{M}(A,B,\cdot)$ is increasing on $\mathscr{P}^0$.

Proof. We make use of superadditivity of the functional $\mathscr{L}$, defined by (6.19). For ordered tuples $\mathbf{p} = (p_1,p_2)$ and $\mathbf{q} = (q_1,q_2)$ is

$$\mathscr{L}(A,B,\mathbf{p}+\mathbf{q}) \geq \mathscr{L}(A,B,\mathbf{p}) + \mathscr{L}(A,B,\mathbf{q}). \tag{6.27}$$

For ordered tuples $\widetilde{\mathbf{p}} = (p_2,p_1)$ and $\widetilde{\mathbf{q}} = (q_2,q_1)$ is also

$$\mathscr{L}(A,B,\widetilde{\mathbf{p}}+\widetilde{\mathbf{q}}) \geq \mathscr{L}(A,B,\widetilde{\mathbf{p}}) + \mathscr{L}(A,B,\widetilde{\mathbf{q}}). \tag{6.28}$$

Since

$$\begin{aligned}
&\mathscr{L}(A,B,\mathbf{p}) + \mathscr{L}(A,B,\widetilde{\mathbf{p}}) \\
&= (p_1+p_2)\left[A\,\nabla_{\frac{p_1}{p_1+p_2}}B - A\sharp_{\frac{p_1}{p_1+p_2}}B\right] + (p_1+p_2)\left[A\,\nabla_{\frac{p_2}{p_1+p_2}}B - A\sharp_{\frac{p_2}{p_1+p_2}}B\right] \\
&= (p_1+p_2)\left[2A\,\nabla\,B - A\sharp_{\frac{p_1}{p_1+p_2}}B - A\sharp_{\frac{p_2}{p_1+p_2}}B\right] \\
&= 2\mathscr{M}(A,B,\mathbf{p}),
\end{aligned}$$

by adding up inequalities (6.27) and (6.28) is (6.25) proved.

Since $\mathbf{p} \geq \mathbf{q}$, the ordered tuples $\mathbf{p}-\mathbf{q}$ i $\mathbf{q}$ have nonnegative coordinates and thus

$$\mathscr{M}(A,B,\mathbf{p}) = \mathscr{M}(A,B,\mathbf{p}-\mathbf{q}+\mathbf{q}) \geq \mathscr{M}(A,B,\mathbf{p}-\mathbf{q}) + \mathscr{M}(A,B,\mathbf{q}).$$

As $\mathscr{M}(A,B,\mathbf{p}-\mathbf{q}) \geq 0$, it follows that $\mathscr{M}(A,B,\mathbf{p}) \geq \mathscr{M}(A,B,\mathbf{q})$, which proves the monotonicity property. □

Corollary 6.3 *Let H be a Hilbert space. Suppose $A,B \in \mathscr{B}^{++}(H)$ and $\mathbf{p} \in \mathscr{P}^0$. Then*

$$(p_1+p_2)[A \nabla B - A \sharp B] \geq \mathscr{M}(A,B,\mathbf{p}) \geq 2\min\{p_1,p_2\}[A \nabla B - A \sharp B]. \tag{6.29}$$

Proof. The first inequality in (6.29) is proved trivially by means of (6.22). When proving the second inequality in (6.29), we compare the ordered pair $\mathbf{p}$ with the constant ordered pair $\mathbf{p}_{\min}$. Since $\mathbf{p} \geq \mathbf{p}_{\min}$, due to the monotonicity property of the functional $\mathscr{M}$ we have

$$\mathscr{M}(A,B,\mathbf{p}) \geq \mathscr{M}(A,B,\mathbf{p}_{\min}).$$

Now, since

$$\begin{aligned}\mathscr{M}(A,B,\mathbf{p}_{\min}) &= \min\{p_1,p_2\}[2A \nabla B - 2A \sharp B] \\ &= 2\min\{p_1,p_2\}[A \nabla B - A \sharp B],\end{aligned}$$

the second inequality in (6.29) is proved. □

Remark 6.2 The first inequality in (6.29) stands for the converse of the arithmetic-Heinz inequality and the second one is its refinement. Recall that the second inequality also generalizes the improvement obtained for the classical arithmetic-Heinz inequality (6.3) to the case of the inequality for the operators on a Hilbert space.

Remark 6.3 In Corollary 6.3 the constant ordered tuple $\mathbf{p}_{\max}$ wasn't observed. Namely, $\mathbf{p}_{\max} \geq \mathbf{p}$ and monotonicity of the functional $\mathscr{M}$ yield

$$2\max\{p_1,p_2\}[A \nabla B - A \sharp B] \geq \mathscr{M}(A,B,\mathbf{p}). \tag{6.30}$$

If we write $r = 2\max\{p_1,p_2\}/(p_1+p_2)$, inequality (6.30) assumes the form

$$H_{\frac{p_1}{p_1+p_2}}(A,B) \geq A \sharp B - (r-1)[A \nabla B - A \sharp B]. \tag{6.31}$$

Since $r \geq 1$ and $A \nabla B - A \sharp B \geq 0$, the last inequality (6.31) is weaker result than the starting inequality $H_{\frac{p_1}{p_1+p_2}}(A,B) \geq A \sharp B$. This justifies omitting this part from consideration.

Finally, let us refer to the inequality (6.6) obtained in [215] which represents the refinement of the operator arithmetic-harmonic inequality. Using the tool developed here, we supplement it with its converse as follows.

Remark 6.4 Observe the inequality sequence (6.18). Define the continuous convex function $f\colon (0,\infty) \to \mathbb{R}$ by $f(x) = 1/x$ and let $\delta = 1$. If we substitute the operators $A, B \in \mathscr{B}^{++}(H)$ with A^{-1} and B^{-1}, the right inequality in (6.18) is actually the inequality (6.6) of refinement of the operator arithmetic-harmonic inequality. The left inequality obtained in (6.18) is then its converse in the following form:

$$2\max\{p_1,p_2\}\left[A\nabla B - A\,!\,B\right] \geq (p_1+p_2)\left[A\nabla_{\frac{p_1}{p_1+p_2}} B - A\,!_{\frac{p_1}{p_1+p_2}} B\right]. \tag{6.32}$$

6.4 Integral Jensen's functional for the operators on a Hilbert space

Suppose T is a locallly compact Hausdorff space and $\mathscr{A}$ is a C^*-algebra of bounded operators on a Hilbert space H. A field $(X_t)_{t\in T}$ of operators in $\mathscr{A}$ is continuous if the function $t \to \|X_t\|$ is continuous on T. Moreover, by introducing bounded Radon measure μ on T and assuming the function $t \to \|X_t\|$ is integrable, we can form the Bochner integral $\int_T X_t d\mu(t)$. Recall, the Bochner integral is the unique element in $\mathscr{A}$ which satisfies the relation

$$\varphi\left(\int_T X_t d\mu(t)\right) = \int_T \varphi\left(X_t\right) d\mu(t),$$

for every linear functional φ in the norm dual $\mathscr{A}^*$ of $\mathscr{A}$ (see [82]).

In addition, we consider a field $(\phi_t)_{t\in T}$ of positive linear mappings $\phi_t : \mathscr{A} \to \mathscr{B}$ from $\mathscr{A}$ to another C^*-algebra $\mathscr{B}$ of bounded operators on Hilbert space K. Such field is assumed to be continuous if the function $t \to \phi_t(X)$ is continuous for all $X \in \mathscr{A}$. Now, in the described setting the authors in [144] obtained the following operator integral variant of Jensen's inequality.

Theorem 6.4 (INTEGRAL OPERATOR JENSEN'S INEQUALITY) *Suppose $\mathscr{A}$ and $\mathscr{B}$ are unital C^*-algebras on H and K respectively and $(X_t)_{t\in T}$ is a bounded continuous field of self-adjoint elements in $\mathscr{A}$ with spectra in an interval I defined on a locally compact Hausdorff space T, equipped with a bounded Radon measure μ. Furthermore, let $(\phi_t)_{t\in T}$ be a field of positive linear maps $\phi_t : \mathscr{A} \to \mathscr{B}$, such that the field $t \to \phi_t(1_H)$ is integrable and $\int_T \phi_t(1_H)\,d\mu(t) = k1_K$ for some positive constant k. If $f : I \to \mathbb{R}$ is operator convex function, then*

$$f\left(\frac{1}{k}\int_T \phi_t\left(X_t\right) d\mu(t)\right) \leq \frac{1}{k}\int_T \phi_t\left(f\left(X_t\right)\right) d\mu(t). \tag{6.33}$$

In addition, if $f : I \to \mathbb{R}$ is operator concave, then the sign of inequality in (6.33) is reversed.

If $k = 1$, one obtains the non-weight form of the above mentioned inequality (6.33) (see [80]).

Integral Jensen's functional $\mathscr{J}_{\phi_t}(f, X_t, \mu)$ is deduced from (6.33) as follows:

$$\mathscr{J}_{\phi_t}(f, X_t, \mu) = \int_T \phi_t(f(X_t))\, d\mu(t) - kf\left(\frac{1}{k}\int_T \phi_t(X_t)\, d\mu(t)\right). \tag{6.34}$$

If $f : I \to \mathbb{R}$ is operator convex, then $\mathscr{J}_{\phi_t}(f, X_t, \mu) \geq 0$. If $f : I \to \mathbb{R}$ is operator concave function, then $\mathscr{J}_{\phi_t}(f, X_t, \mu) \leq 0$.

In the sequel we investigate the properties of the above defined Jensen's integral operator, dependent on bounded Radon measure. For that sake, we define $\mathbb{M}_+$ to be the set of all bounded Radon measures μ on T such that the field $t \to \phi_t(1_H)$ is integrable and $\int_T \phi_t(1_H)\, d\mu(t) = k_\mu 1_K$, for some positive constant k_μ.

Further, suppose that $\mu, \nu \in \mathbb{M}_+$ are bounded Radon measures such that

$$\int_T \phi_t(1_H)\, d\mu(t) = k_\mu 1_K,\ k_\mu > 0 \quad \text{and} \quad \int_T \phi_t(1_H)\, d\nu(t) = k_\nu 1_K,\ k_\nu > 0. \tag{6.35}$$

If $\xi = \mu - \nu$ is bounded Radon measure in $\mathbb{M}_+$, then

$$\int_T \phi_t(1_H)\, d\xi(t) = (k_\mu - k_\nu)1_K,$$

which implies $k_\mu > k_\nu$.

When Jensen's functional (6.34) is observed for a convex function f, we get the following result.

Theorem 6.5 *Suppose $\mathscr{J}_{\phi_t}(f, X_t, \mu)$ is the functional defined by* (6.34), *in the setting as in Theorem 6.4. If $f : I \to \mathbb{R}$ is an operator convex function, then*

$$\mathscr{J}_{\phi_t}(f, X_t, \mu + \nu) \geq \mathscr{J}_{\phi_t}(f, X_t, \mu) + \mathscr{J}_{\phi_t}(f, X_t, \nu), \tag{6.36}$$

i.e. $\mathscr{J}_{\phi_t}(f, X_t, \cdot)$ is superadditive on $\mathbb{M}_+$. Furthermore, if $\mu, \nu \in \mathbb{M}_+$ are such that $\mu - \nu \in \mathbb{M}_+$, then

$$\mathscr{J}_{\phi_t}(f, X_t, \mu) \geq \mathscr{J}_{\phi_t}(f, X_t, \nu) \geq 0, \tag{6.37}$$

i.e. $\mathscr{J}_{\phi_t}(f, X_t, \cdot)$ is increasing on $\mathbb{M}_+$.

Proof. Suppose μ and ν are bounded Radon measures satisfying relation (6.35). According to the definition of the functional (6.34) it follows

$$\begin{aligned}
&\mathscr{J}_{\phi_t}(f, X_t, \mu + \nu) \\
&= \int_T \phi_t(f(X_t))\, d(\mu + \nu)(t) - (k_\mu + k_\nu) f\left(\frac{1}{k_\mu + k_\nu}\int_T \phi_t(X_t)\, d(\mu + \nu)(t)\right) \\
&= \int_T \phi_t(f(X_t))\, d\mu(t) + \int_T \phi_t(f(X_t))\, d\nu(t) - (k_\mu + k_\nu) \\
&\quad \times f\left(\frac{k_\mu}{k_\mu + k_\nu}\cdot\frac{1}{k_\mu}\int_T \phi_t(X_t)\, d\mu(t) + \frac{k_\nu}{k_\mu + k_\nu}\cdot\frac{1}{k_\nu}\int_T \phi_t(X_t)\, d\nu(t)\right).
\end{aligned} \tag{6.38}$$

Operator convexity of f implies

$$\begin{aligned}
& f\left(\frac{k_\mu}{k_\mu+k_\nu}\cdot\frac{1}{k_\mu}\int_T \phi_t(X_t)\,d\mu(t)+\frac{k_\nu}{k_\mu+k_\nu}\cdot\frac{1}{k_\nu}\int_T \phi_t(X_t)\,d\nu(t)\right)\\
&\quad\le \frac{k_\mu}{k_\mu+k_\nu}f\left(\frac{1}{k_\mu}\int_T \phi_t(X_t)\,d\mu(t)\right)+\frac{k_\nu}{k_\mu+k_\nu}f\left(\frac{1}{k_\nu}\int_T \phi_t(X_t)\,d\nu(t)\right).
\end{aligned} \tag{6.39}$$

Now (6.38) and (6.39) imply

$$\begin{aligned}
\mathscr{J}_{\phi_t}(f,X_t,\mu+\nu) &\ge \int_T \phi_t(f(X_t))\,d\mu(t)-k_\mu f\left(\frac{1}{k_\mu}\int_T \phi_t(X_t)\,d\mu(t)\right)\\
&\quad+\int_T \phi_t(f(X_t))\,d\nu(t)-k_\nu f\left(\frac{1}{k_\nu}\int_T \phi_t(X_t)\,d\nu(t)\right)\\
&= \mathscr{J}_{\phi_t}(f,X_t,\mu)+\mathscr{J}_{\phi_t}(f,X_t,\nu),
\end{aligned}$$

that is, superadditivity of $\mathscr{J}_{\phi_t}(f,X_t,\cdot)$ on $\mathbb{M}_+$.

Since $\mu-\nu\in\mathbb{M}_+$, the measure μ can be represented as the sum of two Radon measures in $\mathbb{M}_+$, i.e. $\mu=(\mu-\nu)+\nu$. Thus, superadditivity property (6.36) yields inequality

$$\mathscr{J}_{\phi_t}(f,X_t,\mu)=\mathscr{J}_{\phi_t}(f,X_t,(\mu-\nu)+\nu)\ge \mathscr{J}_{\phi_t}(f,X_t,\mu-\nu)+\mathscr{J}_{\phi_t}(f,X_t,\nu).$$

Since $\mathscr{J}_{\phi_t}(f,X_t,\mu-\nu)\ge 0$, it follows that $\mathscr{J}_{\phi_t}(f,X_t,\mu)\ge \mathscr{J}_{\phi_t}(f,X_t,\nu)\ge 0$. □

Remark 6.5 When $f: I\to\mathbb{R}$ is an operator concave function, inequalities (6.36) and (6.37) are with the reversed sign, that is, functional (6.34) is subadditive and decreasing on $\mathbb{M}_+$.

Let $\mu,\nu\in\mathbb{M}_+$ be the bounded Radon measures on a locally compact Hausdorff space T such that μ is absolutely continuous with respect to ν. The Radon-Nikodym theorem (see [195, p. 122]) states that there exists a non-negative integrable function $p: T\to\mathbb{R}$, the so called Radon-Nikodym derivative, such that $d\mu(t)=p(t)d\nu(t)$.

In the described setting, we have the following result.

Corollary 6.4 *Supppose $\mu,\nu\in\mathbb{M}_+$ are bounded Radon measures on a locally compact Hausdorff space T, such that μ is absolutely continuous with respect to ν and let $p: T\to\mathbb{R}$ be the Radon-Nikodym derivative, i.e. $d\mu(t)=p(t)d\nu(t)$. If p is a bounded function, then*

$$\left[\sup_{t\in T}p(t)\right]\mathscr{J}_{\phi_t}(f,X_t,\nu)\ge \mathscr{J}_{\phi_t}(f,X_t,\mu)\ge\left[\inf_{t\in T}p(t)\right]\mathscr{J}_{\phi_t}(f,X_t,\nu), \tag{6.40}$$

where $\mathscr{J}_{\phi_t}(f,X_t,\cdot)$ is defined by (6.34).

Proof. Let us define the measures $\mu_{\sup}$ and $\mu_{\inf}$ by

$$d\mu_{\sup}(t)=\left[\sup_{t\in T}p(t)\right]d\nu(t)\quad\text{and}\quad d\mu_{\inf}(t)=\left[\inf_{t\in T}p(t)\right]d\nu(t).$$

Since $\int_T \phi_t(1_H)\,d\nu(t) = k_\nu 1_H$, $k_\nu > 0$, we have

$$\int_T \phi_t(1_H)\,d\mu_{\sup}(t) = \left[\sup_{t\in T} p(t)\right] k_\nu 1_K$$

and

$$\int_T \phi_t(1_H)\,d\mu_{\inf}(t) = \left[\inf_{t\in T} p(t)\right] k_\nu 1_K,$$

that is, $\mu_{\sup}, \mu_{\inf} \in \mathbb{M}_+$. Moreover, it is obvious that $\mu_{\sup} - \mu$ and $\mu - \mu_{\inf} \in \mathbb{M}_+$, so the double use of (6.37) yields the interpolating series of inequalities

$$\mathscr{J}_{\phi_t}\left(f, X_t, \mu_{\sup}\right) \geq \mathscr{J}_{\phi_t}\left(f, X_t, \mu\right) \geq \mathscr{J}_{\phi_t}\left(f, X_t, \mu_{\inf}\right).$$

Finally, since

$$\mathscr{J}_{\phi_t}\left(f, X_t, \mu_{\sup}\right) = \left[\sup_{t\in T} p(t)\right] \mathscr{J}_{\phi_t}\left(f, X_t, \nu\right)$$

and

$$\mathscr{J}_{\phi_t}\left(f, X_t, \mu_{\inf}\right) = \left[\inf_{t\in T} p(t)\right] \mathscr{J}_{\phi_t}\left(f, X_t, \nu\right),$$

we get the lower and the upper bound for Jensen's integral functional for the operators, as in (6.40). □

Remark 6.6 The left inequality in (6.40) is a converse and the right one is a refinement of the inequality (6.33).

6.5 Multidimensional Jensen's functional for the operators on a Hilbert space

After observing the operator convex (concave) functions of several variables, we introduce the multidimensional Jensen's functional for the operators on a Hilbert space and then prove its superadditivity and monotonicity on the set of real nonnegative n-tuples. Accompanied both-sided bounds expressed by the non-weight functional of the same type provide us with the converse and the refinement of the multidimensional Jensen inequality. The general results of this type are then applied to the weight multidimensional geometric means which had been defined in [70]. A special type of the Jensen functional is derived for connections and solidarities, which then leads to new types of converses and refinements of Hölder's inequalitiy and the inequality of Minkowski.

The contents of this section is published in [110].

6.5.1 Multidimensional operator convexity and concavity

Let $H_j,\ j=1,\dots,m$ be Hilbert spaces and let $X \subseteq \prod_{j=1}^{m} \mathscr{B}_h(H_j)$ be a convex set.

Definition 6.1 *Function $F : X \to \mathbb{R}$ is **operator convex in m variables** if for all $\mathbf{A} = (A_1, A_2, \dots, A_m)$, $\mathbf{B} = (B_1, B_2, \dots, B_m) \in X$ and for $0 \le \lambda \le 1$ is*

$$F(\lambda \mathbf{A} + (1-\lambda)\mathbf{B}) \le \lambda F(\mathbf{A}) + (1-\lambda) F(\mathbf{B}). \tag{6.41}$$

If the reverse inequality holds in (6.41), then the function F is operator concave in m variables.

Inequality (6.41) is going to be referred to as the multidimensional operator Jensen's inequality. Similarly as was the case with the classical Jensen inequality, we can observe (6.41) for n operators, $n \in \mathbb{N}$.

Proposition 6.2 *Let $\mathbf{p} = (p_1, p_2, \dots, p_n)$ be a nonnegative real n-tuple such that $P_n = \sum_{i=1}^{n} p_i > 0$ and let $\mathbf{A_i} \in X$, $i = 1, 2, \dots, n$. If $F : X \to \mathbb{R}$ is an operator convex function in m variables, then the folowing inequality holds:*

$$F\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i \mathbf{A_i}\right) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i F(\mathbf{A_i}). \tag{6.42}$$

If F is an operator concave function in m variables, then (6.42) holds with the reverse sign.

Proof. We are going to prove (6.42) by means of mathematical induction. For $n = 2$ inequality (6.42) holds according to Definition 6.1. Suppose that inequality (6.42) holds for $n \in \mathbb{N}$. Using $P_{n+1} = \sum_{i=1}^{n+1} p_i$, and applying (6.41) it follows that

$$\begin{aligned}
F\left(\frac{1}{P_{n+1}}\sum_{i=1}^{n+1} p_i \mathbf{A_i}\right) &= F\left(\frac{P_n}{P_{n+1}} \cdot \frac{1}{P_n}\sum_{i=1}^{n} p_i \mathbf{A_i} + \frac{p_{n+1}}{P_{n+1}}\mathbf{A_{n+1}}\right) \\
&\le \frac{P_n}{P_{n+1}} F\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i \mathbf{A_i}\right) + \frac{p_{n+1}}{P_{n+1}} F(\mathbf{A_{n+1}}) \\
&\le \frac{P_n}{P_{n+1}} \cdot \frac{1}{P_n}\sum_{i=1}^{n} p_i F(\mathbf{A_i}) + \frac{p_{n+1}}{P_{n+1}} F(\mathbf{A_{n+1}}) \\
&= \frac{1}{P_{n+1}}\sum_{i=1}^{n+1} p_i F(\mathbf{A_i}),
\end{aligned}$$

which was to prove. The reverse inequality for the case of a concave function F is proved in a similar manner. □

The set of all nonnegative real n-tuples $\mathbf{p} = (p_1, p_2, \dots, p_n)$ such that $P_n = \sum_{i=1}^{n} p_i > 0$ is going to be denoted with $\mathscr{P}_n^0$ in the sequel.

6.5.2 Superadditivity of the multidimensional Jensen functional for the operators on a Hilbert space

Multidimensional Jensen's functional $\mathscr{J}(F,\mathbf{A_1},\ldots,\mathbf{A_n};\mathbf{p})$ for the operators on a Hilbert space is deduced from (6.42) as follows:

$$\mathscr{J}(F,\mathbf{A_1},\ldots,\mathbf{A_n};\mathbf{p})=\sum_{i=1}^{n}p_iF(\mathbf{A_i})-P_nF\left(\frac{1}{P_n}\sum_{i=1}^{n}p_i\mathbf{A_i}\right). \tag{6.43}$$

By fixing the first $n+1$ parameters, functional (6.43) can be observed as a function on $\mathscr{P}_n^0$. If the function F is operator convex, then due to Proposition 6.2 is $\mathscr{J}(F,\mathbf{A_1},\ldots,\mathbf{A_n};\mathbf{p})\geq 0$, for all $\mathbf{p}\in\mathscr{P}_n^0$. If F is operator concave, then $\mathscr{J}(F,\mathbf{A_1},\ldots,\mathbf{A_n};\mathbf{p})\leq 0$, for all $\mathbf{p}\in\mathscr{P}_n^0$.

Properties of the functional (6.43) are established in the following theorem.

Theorem 6.6 *Let $\mathbf{A_i}\in X$, $i=1,2,\ldots,n$, and $\mathbf{p}=(p_1,p_2,\ldots,p_n)$, $\mathbf{q}=(q_1,q_2,\ldots,q_n)\in\mathscr{P}_n^0$. If $F:X\to\mathbb{R}$ is an operator convex function, then*

$$\mathscr{J}(F,\mathbf{A_1},\ldots,\mathbf{A_n};\mathbf{p}+\mathbf{q})\geq\mathscr{J}(F,\mathbf{A_1},\ldots,\mathbf{A_n};\mathbf{p})+\mathscr{J}(F,\mathbf{A_1},\ldots,\mathbf{A_n};\mathbf{q}), \tag{6.44}$$

that is, $\mathscr{J}(F,\mathbf{A_1},\ldots,\mathbf{A_n};\cdot)$ is superadditive on $\mathscr{P}_n^0$. Furthermore, if $\mathbf{p}$ and $\mathbf{q}\in\mathscr{P}_n^0$ are such that $\mathbf{p}\geq\mathbf{q}$, then

$$\mathscr{J}(F,\mathbf{A_1},\ldots,\mathbf{A_n};\mathbf{p})\geq\mathscr{J}(F,\mathbf{A_1},\ldots,\mathbf{A_n};\mathbf{q})\geq 0, \tag{6.45}$$

that is, $\mathscr{J}(F,\mathbf{A_1},\ldots,\mathbf{A_n};\cdot)$ is increasing on $\mathscr{P}_n^0$.

Proof. Let us write

$$\begin{aligned}
&\mathscr{J}(F,\mathbf{A_1},\ldots,\mathbf{A_n};\mathbf{p}+\mathbf{q})\\
&=\sum_{i=1}^{n}(p_i+q_i)F(\mathbf{A_i})-(P_n+Q_n)F\left(\frac{1}{P_n+Q_n}\sum_{i=1}^{n}(p_i+q_i)\mathbf{A_i}\right)\\
&=\sum_{i=1}^{n}p_iF(\mathbf{A_i})+\sum_{i=1}^{n}q_iF(\mathbf{A_i})-(P_n+Q_n)F\left(\frac{P_n}{P_n+Q_n}\cdot\frac{1}{P_n}\sum_{i=1}^{n}p_i\mathbf{A_i}+\frac{Q_n}{P_n+Q_n}\cdot\frac{1}{Q_n}\sum_{i=1}^{n}q_i\mathbf{A_i}\right).
\end{aligned} \tag{6.46}$$

On the other hand, operator convexity of the function F yields

$$\begin{aligned}
&F\left(\frac{P_n}{P_n+Q_n}\cdot\frac{1}{P_n}\sum_{i=1}^{n}p_i\mathbf{A_i}+\frac{Q_n}{P_n+Q_n}\cdot\frac{1}{Q_n}\sum_{i=1}^{n}q_i\mathbf{A_i}\right)\\
&\qquad\leq\frac{P_n}{P_n+Q_n}F\left(\frac{1}{P_n}\sum_{i=1}^{n}p_i\mathbf{A_i}\right)+\frac{Q_n}{P_n+Q_n}F\left(\frac{1}{Q_n}\sum_{i=1}^{n}q_i\mathbf{A_i}\right).
\end{aligned} \tag{6.47}$$

From (6.46) and (6.47) we prove superadditivity:

$$\begin{aligned}
\mathscr{J}(F,\mathbf{A_1},\ldots,\mathbf{A_n};\mathbf{p}+\mathbf{q})\geq&\sum_{i=1}^{n}p_iF(\mathbf{A_i})-P_nF\left(\frac{1}{P_n}\sum_{i=1}^{n}p_i\mathbf{A_i}\right)+\sum_{i=1}^{n}q_iF(\mathbf{A_i})\\
&-Q_nF\left(\frac{1}{Q_n}\sum_{i=1}^{n}q_i\mathbf{A_i}\right).
\end{aligned}$$

If $\mathbf{p} = \mathbf{q}$, then the relation (6.45) obviously holds. If $\mathbf{p} > \mathbf{q}$, we can write $\mathbf{p} = (\mathbf{p} - \mathbf{q}) + \mathbf{q}$. Applying (6.44) it follows:

$$\begin{aligned} \mathscr{J}(F, \mathbf{A_1}, \ldots, \mathbf{A_n}; \mathbf{p}) &= \mathscr{J}(F, \mathbf{A_1}, \ldots, \mathbf{A_n}; \mathbf{p} - \mathbf{q} + \mathbf{q}) \\ &\geq \mathscr{J}(F, \mathbf{A_1}, \ldots, \mathbf{A_n}; \mathbf{p} - \mathbf{q}) + \mathscr{J}(F, \mathbf{A_1}, \ldots, \mathbf{A_n}; \mathbf{q}). \end{aligned}$$

Since $\mathbf{p} - \mathbf{q} \in \mathscr{P}_n^0$, it follows $\mathscr{J}(F, \mathbf{A_1}, \ldots, \mathbf{A_n}; \mathbf{p} - \mathbf{q}) \geq 0$ and finally

$$\mathscr{J}(F, \mathbf{A_1}, \ldots, \mathbf{A_n}; \mathbf{p}) \geq \mathscr{J}(F, \mathbf{A_1}, \ldots, \mathbf{A_n}; \mathbf{q}) \geq 0,$$

which was to prove. □

Remark 6.7 When F is an operator concave function, the inequalities (6.44) and (6.45) change their sign, i.e. $\mathscr{J}(F, \mathbf{A_1}, \ldots, \mathbf{A_n}; \cdot)$ is subadditive and decreasing on $\mathscr{P}_n^0$. Namely, concavity of F changes the sign in Jensen's inequality (6.42) and $\mathscr{J}(F, \mathbf{A_1}, \ldots, \mathbf{A_n}; \mathbf{p}) \leq 0$, for all $\mathbf{p} \in \mathscr{P}_n^0$.

The property (6.45) of Jensen's functional yields the both-sided bounds of the functional expressed by means of the non-weight functional of the same type.

Corollary 6.5 *Suppose* $\mathbf{A_i} \in X$, $i = 1, 2, \ldots, n$ *and* $\mathbf{p} = (p_1, p_2, \ldots, p_n) \in \mathscr{P}_n^0$. *If* $F : X \to \mathbb{R}$ *is an operator convex function, then*

$$\max_{1 \leq i \leq n} \{p_i\} \mathscr{J}(F, \mathbf{A_1}, \ldots, \mathbf{A_n}) \geq \mathscr{J}(F, \mathbf{A_1}, \ldots, \mathbf{A_n}; \mathbf{p}) \geq \min_{1 \leq i \leq n} \{p_i\} \mathscr{J}(F, \mathbf{A_1}, \ldots, \mathbf{A_n}), \tag{6.48}$$

where

$$\mathscr{J}(F, \mathbf{A_1}, \ldots, \mathbf{A_n}) = \sum_{i=1}^{n} F(\mathbf{A_i}) - nF\left(\frac{1}{n} \sum_{i=1}^{n} \mathbf{A_i}\right). \tag{6.49}$$

Proof. If we compare the n-tuple $\mathbf{p} = (p_1, p_2, \ldots, p_n) \in \mathscr{P}_n^0$ with the constant n-tuples

$$\mathbf{p}_{\max} = \big(\max_{1 \leq i \leq n} \{p_i\}, \ldots, \max_{1 \leq i \leq n} \{p_i\}\big) \text{ and } \mathbf{p}_{\min} = \big(\min_{1 \leq i \leq n} \{p_i\}, \ldots, \min_{1 \leq i \leq n} \{p_i\}\big),$$

it is obvious that $\mathbf{p}_{\max} \geq \mathbf{p} \geq \mathbf{p}_{\min}$, and convexity of F as well as the property (6.45) yield

$$\mathscr{J}(F, \mathbf{A_1}, \ldots, \mathbf{A_n}; \mathbf{p}_{\max}) \geq \mathscr{J}(F, \mathbf{A_1}, \ldots, \mathbf{A_n}; \mathbf{p}) \geq \mathscr{J}(F, \mathbf{A_1}, \ldots, \mathbf{A_n}; \mathbf{p}_{\min}).$$

Since

$$\mathscr{J}(F, \mathbf{A_1}, \ldots, \mathbf{A_n}; \mathbf{p}_{\max}) = \max_{1 \leq i \leq n} \{p_i\} \mathscr{J}(F, \mathbf{A_1}, \ldots, \mathbf{A_n})$$

and

$$\mathscr{J}(F, \mathbf{A_1}, \ldots, \mathbf{A_n}; \mathbf{p}_{\min}) = \min_{1 \leq i \leq n} \{p_i\} \mathscr{J}(F, \mathbf{A_1}, \ldots, \mathbf{A_n}),$$

we proved the inequalities (6.48). □

Remark 6.8 When F is operator concave, inequalities (6.48) change their sign.

Remark 6.9 Bounds (6.48) for the functional (6.43) are the converse and the refinement of the multidimensional operator Jensen's inequality (6.42).

6.5.3 Connections

As we analyzed in Section 1.5, there is an isomorphism established between connections and the operator monotone nonnegative functions defined on $[0,\infty)$, which are in that case, as we also remarked – operator concave. These are called the representation functions for connections. Having this in mind, we apply our previously derived results to an arbitrary connection σ.

For $\mathbf{A_i} = (X_i, Y_i) \in \mathscr{B}^+(H) \times \mathscr{B}^+(H)$, $i = 1, \dots, n$, the functional (6.43) assumes the following form:

$$\mathscr{J}^{\sigma}(\mathbf{X},\mathbf{Y};\mathbf{p}) = \sum_{i=1}^{n} p_i\,(X_i \sigma Y_i) - P_n \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i X_i\right) \sigma \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i Y_i\right),$$

where $\mathbf{X}$ and $\mathbf{Y}$ stay for the ordered n-tuples $(X_1, X_2, \dots, X_n)$ and $(Y_1, Y_2, \dots, Y_n)$ of the positive operators. Furthermore, the property of positive homogeneity $\alpha\,(X\sigma Y) = (\alpha X)\,\sigma\,(\alpha Y)$, $X, Y \in \mathscr{B}^+(H)$, $\alpha > 0$, yields

$$\mathscr{J}^{\sigma}(\mathbf{X},\mathbf{Y};\mathbf{p}) = \sum_{i=1}^{n} p_i\,(X_i \sigma Y_i) - \left(\sum_{i=1}^{n} p_i X_i\right) \sigma \left(\sum_{i=1}^{n} p_i Y_i\right). \tag{6.50}$$

Functional (6.50) is Jensen's functional for connections.

According to Proposition 6.2, $\mathscr{J}^{\sigma}(\mathbf{X},\mathbf{Y};\mathbf{p}) \le 0$, for all $\mathbf{p} \in \mathscr{P}_n^0$ and acccording to Theorem 6.6 is $\mathscr{J}^{\sigma}(\mathbf{X},\mathbf{Y};\cdot)$ subadditive and decreasing on $\mathscr{P}_n^0$.

If we apply Corollary 6.5 to functional $\mathscr{J}^{\sigma}(\mathbf{X},\mathbf{Y};\mathbf{p})$, we obtain its both-sided non-weight bounds expressed by the functional of the same type.

Corollary 6.6 *Let H be a Hilbert space, $\mathbf{X} = (X_1, X_2, \dots, X_n)$ and $\mathbf{Y} = (Y_1, Y_2, \dots, Y_n)$ $\in [\mathscr{B}^+(H)]^n$, $\mathbf{p} = (p_1, p_2, \dots, p_n) \in \mathscr{P}_n^0$. If σ is a connection, then*

$$\max_{1\le i\le n}\{p_i\}\,\mathscr{J}^{\sigma}(\mathbf{X},\mathbf{Y}) \le \mathscr{J}^{\sigma}(\mathbf{X},\mathbf{Y};\mathbf{p}) \le \min_{1\le i\le n}\{p_i\}\,\mathscr{J}^{\sigma}(\mathbf{X},\mathbf{Y}), \tag{6.51}$$

where

$$\mathscr{J}^{\sigma}(\mathbf{X},\mathbf{Y}) = \sum_{i=1}^{n} (X_i \sigma Y_i) - \left(\sum_{i=1}^{n} X_i\right) \sigma \left(\sum_{i=1}^{n} Y_i\right). \tag{6.52}$$

Proof. Follows directly from Corollary 6.5, relation (6.50) and positive homogeneity of the connection σ. □

Remark 6.10 We are going to analyze some special cases of connections by means of which we are going to derive the operator variants of some well known operator inequalities. Let s and t be conjugate exponents, i.e. such that $1/s + 1/t = 1$, $s > 1$. If we consider σ a geometric mean $\sharp_{1/s}$ and substitute the operators X_i and Y_i with X_i^s and Y_i^t, where

$X_i, Y_i \in \mathscr{B}^{++}(H)$, the inequalities (6.51) are given in the following form:

$$\begin{aligned}
&\max_{1\le i\le n}\{p_i\}\left[\sum_{i=1}^{n}\left(X_i^s\sharp_{1/s}Y_i^t\right)-\left(\sum_{i=1}^{n}X_i^s\right)\sharp_{1/s}\left(\sum_{i=1}^{n}Y_i^t\right)\right]\\
&\le \sum_{i=1}^{n}p_i\left(X_i^s\sharp_{1/s}Y_i^t\right)-\left(\sum_{i=1}^{n}p_iX_i^s\right)\sharp_{1/s}\left(\sum_{i=1}^{n}p_iY_i^t\right)\\
&\le \min_{1\le i\le n}\{p_i\}\left[\sum_{i=1}^{n}\left(X_i^s\sharp_{1/s}Y_i^t\right)-\left(\sum_{i=1}^{n}X_i^s\right)\sharp_{1/s}\left(\sum_{i=1}^{n}Y_i^t\right)\right]. \qquad (6.53)
\end{aligned}$$

We clearly see that in (6.53) we have the refinement and the converse of the weight operator Hölder inequality, expressed via its non-weight form. In particular, for $s=t=2$, we have the same for the Cauchy inequality.

Another application of Corollary 6.6 refers to the parallel sum. Recall that for $X,Y \in \mathscr{B}^{++}(H)$ a parallel sum is defined by $X:Y=\left(X^{-1}+Y^{-1}\right)^{-1}$. If we substitute invertible operators X_i, Y_i, $i=1,2,\dots,n$ with X_i^{-1}, Y_i^{-1}, $i=1,2,\dots,n$, inequalities in (6.51) become

$$\begin{aligned}
&\max_{1\le i\le n}\{p_i\}\left[\sum_{i=1}^{n}(X_i+Y_i)^{-1}-\left(\left(\sum_{i=1}^{n}X_i^{-1}\right)^{-1}+\left(\sum_{i=1}^{n}Y_i^{-1}\right)^{-1}\right)^{-1}\right]\\
&\le \sum_{i=1}^{n}p_i(X_i+Y_i)^{-1}-\left(\left(\sum_{i=1}^{n}p_iX_i^{-1}\right)^{-1}+\left(\sum_{i=1}^{n}p_iY_i^{-1}\right)^{-1}\right)^{-1}\\
&\le \min_{1\le i\le n}\{p_i\}\left[\sum_{i=1}^{n}(X_i+Y_i)^{-1}-\left(\left(\sum_{i=1}^{n}X_i^{-1}\right)^{-1}+\left(\sum_{i=1}^{n}Y_i^{-1}\right)^{-1}\right)^{-1}\right]. \qquad (6.54)
\end{aligned}$$

Similarly as before, in (6.54) we have the refinement and the converse of the operator Minkowski inequality. Formerly, the idea of relating connections and the operator inequalities of Hölder and Minkowski had been presented in [160]. The *scalar* inequalities that correspond to the relations (6.53) and (6.54) were obtained in [168] and are also contained in the monograph [151, pp. 718].

6.5.4 Solidarities

In Section 1.5 we introduced solidarities, which, being understood as generalized connections, also possess the property of joint concavity described in relation (1.58).

For $\mathbf{A_i}=(X_i,Y_i)\in\mathscr{B}^{++}(H)\times\mathscr{B}^{++}(H)$, $i=1,\dots,n$, the functional (6.43) assumes the following form:

$$\mathscr{J}^s(\mathbf{X},\mathbf{Y};\mathbf{p})=\sum_{i=1}^{n}p_i\left(X_isY_i\right)-P_n\left(\frac{1}{P_n}\sum_{i=1}^{n}p_iX_i\right)s\left(\frac{1}{P_n}\sum_{i=1}^{n}p_iY_i\right),$$

where $\mathbf{X} = (X_1, X_2, \dots, X_n)$ and $\mathbf{Y} = (Y_1, Y_2, \dots, Y_n)$. Since solidarities also possess the property of positive homogeneity, (see [71] for more details), it follows that

$$\mathscr{J}^s(\mathbf{X}, \mathbf{Y}; \mathbf{p}) = \sum_{i=1}^{n} p_i (X_i s Y_i) - \left(\sum_{i=1}^{n} p_i X_i \right) s \left(\sum_{i=1}^{n} p_i Y_i \right). \tag{6.55}$$

According to Proposition 6.2 is $\mathscr{J}^s(\mathbf{X}, \mathbf{Y}; \mathbf{p}) \leq 0$, for $\mathbf{X}, \mathbf{Y} \in [\mathscr{B}^{++}(H)]^n$, $\mathbf{p} \in \mathscr{P}_n^0$. According to Theorem 6.6 is $\mathscr{J}^s(\mathbf{X}, \mathbf{Y}; \cdot)$ subadditive and decreasing on $\mathscr{P}_n^0$. Functional $\mathscr{J}^s(\mathbf{X}, \mathbf{Y}; \mathbf{p})$ also possesses the non-weight bounds, analogous to those described for the functional for connections.

Corollary 6.7 *Let H be a Hilbert space, $\mathbf{X} = (X_1, X_2, \dots, X_n)$ and $\mathbf{Y} = (Y_1, Y_2, \dots, Y_n) \in [\mathscr{B}^{++}(H)]^n$ and $\mathbf{p} = (p_1, p_2, \dots, p_n) \in \mathscr{P}_n^0$. If s is a solidarity, then*

$$\max_{1 \leq i \leq n} \{p_i\} \mathscr{J}^s(\mathbf{X}, \mathbf{Y}) \leq \mathscr{J}^s(\mathbf{X}, \mathbf{Y}; \mathbf{p}) \leq \min_{1 \leq i \leq n} \{p_i\} \mathscr{J}^s(\mathbf{X}, \mathbf{Y}), \tag{6.56}$$

where

$$\mathscr{J}^s(\mathbf{X}, \mathbf{Y}) = \sum_{i=1}^{n} (X_i s Y_i) - \left(\sum_{i=1}^{n} X_i \right) s \left(\sum_{i=1}^{n} Y_i \right). \tag{6.57}$$

Remark 6.11 We are going to apply the functional (6.55) to a special type of solidarities – the relative operator entropy defined by (1.59). To be more precise, for $\mathbf{X}, \mathbf{Y} \in [\mathscr{B}^{++}(H)]^n$ functional (6.55) becomes

$$\mathscr{J}^S(\mathbf{X}, \mathbf{Y}; \mathbf{p}) = \sum_{i=1}^{n} p_i S(X_i | Y_i) - S\left(\sum_{i=1}^{n} p_i X_i \middle| \sum_{i=1}^{n} p_i Y_i \right), \tag{6.58}$$

and is called Jensen's functional for the relative operator entropy. In particular, $\mathscr{J}^S(\mathbf{X}, \mathbf{Y}; \mathbf{p}) \leq 0$, for $\mathbf{X}, \mathbf{Y} \in [\mathscr{B}^{++}(H)]^n$, $\mathbf{p} \in \mathscr{P}_n^0$ and $\mathscr{J}^S(\mathbf{X}, \mathbf{Y}; \cdot)$ is subadditive and decreasing on $\mathscr{P}_n^0$. Analogous non-weight bounds are inherited from the previously analyzed functional for solidarities:

$$\max_{1 \leq i \leq n} \{p_i\} \mathscr{J}^S(\mathbf{X}, \mathbf{Y}) \leq \mathscr{J}^S(\mathbf{X}, \mathbf{Y}; \mathbf{p}) \leq \min_{1 \leq i \leq n} \{p_i\} \mathscr{J}^S(\mathbf{X}, \mathbf{Y}), \tag{6.59}$$

where

$$\mathscr{J}^S(\mathbf{X}, \mathbf{Y}) = \sum_{i=1}^{n} S(X_i | Y_i) - S\left(\sum_{i=1}^{n} X_i \middle| \sum_{i=1}^{n} Y_i \right). \tag{6.60}$$

Inequalities in (6.56) and (6.59) represent the weight forms of the operator inequalities which had been earlier obtained in [160], but only in its non-weight form.

Let us just mention here that Tsallis' relative operator entropy defined by (1.60) could also be analyzed in view of a special type of the functional defined as a parametric extension of (6.58) and this functional also possesses all of the properties of the previously observed functionals for the operators.

6.5.5 Multidimensional weight geometric means

Multidimensional weight geometric mean $G[n,t]$, $0 \leq t \leq 1$, of positive invertible operators $A_1, A_2, \ldots, A_n$ was defined inductively in [70], using (1.54).

Let $G[2,t](A_1,A_2) = A_1 \sharp_t A_2 = A_1^{\frac{1}{2}} \left(A_1^{-\frac{1}{2}} A_2 A_1^{-\frac{1}{2}}\right)^t A_1^{\frac{1}{2}}$. For $n \geq 3$ we have the following consideration. Let $A_i^{(1)} = A_i$, for all $i = 1,2,\ldots,n$ and inductively for r

$$A_i^{(r)} = G[n-1,t]\left(A_1^{(r-1)},\ldots,A_{i-1}^{(r-1)},A_{i+1}^{(r-1)},\ldots,A_n^{(r-1)}\right).$$

Then, there exists the limit $\lim_{r\to\infty} A_i^{(r)}$ in the Thompson metric on the convex cone $\mathscr{B}^{++}(H)$ and it does not depend on i (see [70]). Thus one can define

$$G[n,t](A_1,A_2,\ldots,A_n) = \lim_{r\to\infty} A_i^{(r)}.$$

Multidimensional weight geometric mean possesses the property of joint concavity (which can be proved by induction, considering (1.52), for $n = 2$):

$$G[n,t]\left(\sum_{i=1}^{n} \lambda_i \mathbf{A_i}\right) \geq \sum_{i=1}^{n} \lambda_i G[n,t](\mathbf{A_i}), \tag{6.61}$$

where $\lambda_i \geq 0$, $\sum_{i=1}^{n} \lambda_i = 1$ and $\mathbf{A_i} = (A_{i1}, A_{i2}, \ldots, A_{in})$, $i = 1,2,\ldots,n$ are the ordered n-tuples of positive invertible operators on a Hilbert space.

Now, suppose $\mathbf{A_i} = (A_{i1}, A_{i2}, \ldots, A_{in}) \in [\mathscr{B}^{++}(H)]^n$, $i = 1,2,\ldots,n$, $\mathbf{p} = (p_1,p_2,\ldots,p_n) \in \mathscr{P}_n^0$, $0 \leq t \leq 1$ and $G[n,t]$ is a multidimensional weight geometric mean. Motivated by (6.61) we deduce the Jensen functional for multidimensional weight geometric means:

$$\mathscr{J}^{G[n,t]}(\mathbf{A_1},\ldots,\mathbf{A_n};\mathbf{p}) = \sum_{i=1}^{n} p_i G[n,t](\mathbf{A_i}) - P_n G[n,t]\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i \mathbf{A_i}\right). \tag{6.62}$$

Since $G[n,t]\left(1/P_n \sum_{i=1}^{n} p_i \mathbf{A_i}\right) = 1/P_n G[n,t]\left(\sum_{i=1}^{n} p_i \mathbf{A_i}\right)$, functional (6.62) becomes

$$\mathscr{J}^{G[n,t]}(\mathbf{A_1},\ldots,\mathbf{A_n};\mathbf{p}) = \sum_{i=1}^{n} p_i G[n,t](\mathbf{A_i}) - G[n,t]\left(\sum_{i=1}^{n} p_i \mathbf{A_i}\right). \tag{6.63}$$

As a consequence of the joint concavity property of $G[n,t]$ it follows that the functional (6.63) is non-positive and after Theorem 6.6 is subadditive and decreasing on $\mathscr{P}_n^0$. Both-sided bounds of the functional (6.63) are given in the following corollary.

Corollary 6.8 *Let H be a Hilbert space, $\mathbf{A_i} = (A_{i1}, A_{i2}, \ldots, A_{in}) \in [\mathscr{B}^{++}(H)]^n$, $i = 1,2,\ldots,n$, $\mathbf{p} = (p_1,p_2,\ldots,p_n) \in \mathscr{P}_n^0$ and $0 \leq t \leq 1$. If $G[n,t]$ is a multidimensional weight geometric mean, then*

$$\max_{1\leq i\leq n}\{p_i\}\,\mathscr{J}^{G[n,t]}(\mathbf{A_1},\ldots,\mathbf{A_n}) \leq \mathscr{J}^{G[n,t]}(\mathbf{A_1},\ldots,\mathbf{A_n};\mathbf{p})$$
$$\leq \min_{1\leq i\leq n}\{p_i\}\,\mathscr{J}^{G[n,t]}(\mathbf{A_1},\ldots,\mathbf{A_n}), \tag{6.64}$$

where

$$\mathscr{J}^{G[n,t]}(\mathbf{A_1},\ldots,\mathbf{A_n}) = \sum_{i=1}^{n} G[n,t](\mathbf{A_i}) - G[n,t]\left(\sum_{i=1}^{n}\mathbf{A_i}\right). \tag{6.65}$$

Proof. Follows directly from Corollary 6.5, relation (6.63) and positive homogeneity of the geometric mean $G[n,t]$. □

Remark 6.12 Bounds in (6.64) represent the refinement and the converse of the inequality (6.61).

Chapter 7

Improvements of some matrix and operator inequalities via the Jensen functional

In this chapter, according to published paper [119], several refinements of Heinz norm inequalities are derived by virtue of convexity of Heinz means and with the help of the Jensen functional. Operator analogues of refined Heinz norm inequalities are also derived.

In the second part of the chapter, as it was published in [120] some improved weak majorization relations and eigenvalue inequalities for matrix versions of the Jensen inequalities regarding convexity are derived, with corresponding applications to log convex functions and the refinements of some related matrix inequalities.

7.1 Improved Heinz inequalities via the Jensen functional

Recall that Heinz mean in parameter $t \in [0,1]$, defined by

$$h_t(a,b) = \frac{a^t b^{1-t} + a^{1-t} b^t}{2}, \quad a,b \geq 0, \tag{7.1}$$

interpolates between the geometric mean and the arithmetic mean, i.e.

$$\sqrt{ab} \leq h_t(a,b) \leq \frac{a+b}{2}, \quad t \in [0,1].$$

Clearly, $h_0(a,b) = h_1(a,b) = \frac{a+b}{2}$ and $h_{1/2}(a,b) = \sqrt{ab}$. Moreover, it is easy to see that the Heinz mean, considered as a function of t, $t \in [0,1]$, is convex, attains its minimum at $t = 1/2$, and attains its maximum at $t = 0$ and $t = 1$. Moreover, $h_t(a,b)$ is symmetric with respect to $t = 1/2$, that is, $h_t(a,b) = h_{1-t}(a,b)$, $t \in [0,1]$.

The previous discussion is likewise extended to the operator level in the following sense: If A, B, and X are operators on a complex separable Hilbert space such that A and B are positive, then for every unitarily invariant norm $|||\cdot|||$, the function

$$f_h(t) = |||A^t X B^{1-t} + A^{1-t} X B^t||| \tag{7.2}$$

is also convex on $[0,1]$, attains minimum at $t = 1/2$, attains maximum at $t = 0$ and $t = 1$, and is symmetric with respect to $t = 1/2$, i.e. $f_h(t) = f_h(1-t)$.

Remark 7.1 Considering the Hilbert space $M_n(\mathbb{C})$ of $n \times n$ complex matrices, the unitarily invariance of the norm $|||\cdot|||$ means that $|||UAV||| = |||A|||$ for all $A \in M_n(\mathbb{C})$ and for all unitary matrices $U, V \in M_n(\mathbb{C})$. Examples of unitarily invariant norms are the Hilbert-Schmidt norm, the trace norm, and the spectral norm defined respectively by $\|A\|_2 = \left(\sum_{i=1}^n s_i^2(A)\right)^{1/2}$, $\|A\|_1 = \sum_{i=1}^n s_i(A)$, and $\|A\| = s_1(A)$, where $s_1(A) \geq \ldots \geq s_n(A)$ are the singular values of $A \in M_n(\mathbb{C})$, that is, the eigenvalues of the positive semi-definite matrix $|A| = \left(A^*A\right)^{1/2}$. For more details the reader is referred to [34]. By a slight modification, the notion of unitarily invariant norms can also be extended to operators on a complex separable Hilbert space (see e.g. [202]). In such setting, when considering $|||A|||$, the operator A is implicitly assumed to belong to the norm ideal associated with $|||\cdot|||$.

Hence, for every unitarily invariant norm, the Heinz norm inequalities (see [34]) read

$$2|||A^{\frac{1}{2}} X B^{\frac{1}{2}}||| \leq |||A^t X B^{1-t} + A^{1-t} X B^t||| \leq |||AX + XB|||. \tag{7.3}$$

The first inequality in (7.3) will be referred to as the left Heinz norm inequality, while the second one will be referred to as the right Heinz norm inequality. For a comprehensive inspection of the results concerning the above norm inequalities the reader is referred to [34], [35], [36] and [84], where one can also find proofs of the properties of function f_h defined by (7.2).

The following presented results were motivated by [95], where Kittaneh had obtained several refinements of Heinz norm inequalities (7.3) by utilizing convexity of function f_h and the well-known Hermite-Hadamard inequality (see e.g. [151]). The above norm inequalities are here improved via convexity of function f_h, by means of the properties of the discrete Jensen functional

$$\mathscr{J}^n(f, \mathbf{x}, \mathbf{p}) = \sum_{i=1}^n p_i f(x_i) - P_n f\left(\frac{\sum_{i=1}^n p_i x_i}{P_n}\right), \tag{7.4}$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function, $\mathbf{x} = (x_1, x_2, \dots, x_n) \in I^n$, $n \geq 2$, and $\mathbf{p} = (p_1, p_2, \dots, p_n)$ is a positive n-tuple of real numbers with $P_n = \sum_{i=1}^n p_i$, whose properties of superadditivity (1.66) and monotonicity (1.67) were extensively exploited in the previous chapters of this monograph.

7.1.1 Improved Heinz norm inequalities via boundedness of the Jensen functional

In order to improve Heinz norm inequalities (7.3), we are going to exploit properties of the Jensen functional (7.4) and convexity of function f_h defined by (7.2). For that sake we consider the Jensen functional in the form that will be more suitable in our research. More precisely, we consider the two-dimensional normalized Jensen functional, that is,

$$\mathscr{J}(f, \mathbf{x}, t) = (1-t)f(x_1) + tf(x_2) - f((1-t)x_1 + tx_2), \tag{7.5}$$

where $t \in [0,1]$ and $\mathbf{x} = (x_1, x_2)$. Taking into account order relation as in (1.67), the ordered pair $(t, 1-t)$ can mutually be bounded with the constant pairs, i.e. we have

$$(\min\{t, 1-t\}, \min\{t, 1-t\}) \leq (t, 1-t) \leq (\max\{t, 1-t\}, \max\{t, 1-t\}).$$

Therefore, by virtue of the monotonicity property (1.67), the above relation yields bounds for the two-dimensional normalized Jensen functional (7.5), that is,

$$2\min\{t, 1-t\} \mathscr{J}_0(f, \mathbf{x}) \leq \mathscr{J}(f, \mathbf{x}, t) \leq 2\max\{t, 1-t\} \mathscr{J}_0(f, \mathbf{x}), \tag{7.6}$$

where

$$\mathscr{J}_0(f, \mathbf{x}) = \frac{f(x_1) + f(x_2)}{2} - f\left(\frac{x_1 + x_2}{2}\right).$$

In other words, functional (7.5) is mutually bounded by a non-weight functional of the same type. The double inequality (7.6) will be crucial in deriving improvements of Heinz norm inequalities. Namely, we are going to study relations (7.6) equipped with the convex function (7.2) on certain subintervals of $[0,1]$. This will provide various improvements of Heinz norm inequalities.

In the sequel let $\mathscr{H}_s$ denote a complex separable Hilbert space, while $\mathfrak{B}^+(\mathscr{H}_s)$ denotes the set of positive operators on $\mathscr{H}_s$. For the sake of a simpler notation, we here use the abbreviation

$$H_t(A, B, X) = A^t X B^{1-t} + A^{1-t} X B^t,$$

where A, B, X are operators on $\mathscr{H}_s$ such that $A, B \in \mathfrak{B}^+(\mathscr{H}_s)$ and $t \in [0,1]$. Clearly, operator $H_t(A, B, X)/2$ has the meaning of Heinz operator mean. The following result is an immediate consequence of the first inequality in (7.6), for convex function $f_h(t) = |||H_t(A, B, X)|||$ considered on the interval $[0,1]$.

Theorem 7.1 *Let A, B and X be operators on a complex separable Hilbert space $\mathscr{H}_s$ such that $A, B \in \mathfrak{B}^+(\mathscr{H}_s)$. Then for every unitarily invariant norm $|||\cdot|||$ the inequality*

$$|||AX + XB||| - |||H_t(A, B, X)||| \geq 2\min\{t, 1-t\} \left[|||AX + XB||| - 2|||A^{\frac{1}{2}} X B^{\frac{1}{2}}|||\right] \geq 0 \tag{7.7}$$

holds for every $t \in [0,1]$.

Proof. Let $\mathbf{x_0} = (0,1)$ and let $f_h(t) = |||H_t(A,B,X)|||$. Considering the first inequality in (7.6), that is, $\mathscr{J}(f_h, \mathbf{x_0}, t) \geq 2\min\{t, 1-t\}\mathscr{J}_0(f_h, \mathbf{x_0})$, we have

$$(1-t)f_h(0) + tf_h(1) - f_h(t) \geq 2\min\{t, 1-t\}\left[\frac{f_h(0)+f_h(1)}{2} - f_h\left(\frac{1}{2}\right)\right].$$

Now, since $f_h(0) = f_h(1)$, the previous inequality reduces to

$$f_h(0) - f_h(t) \geq 2\min\{t, 1-t\}\left[f_h(0) - f_h\left(\frac{1}{2}\right)\right],$$

which represents (7.7). □

The inequality (7.7) was obtained in [95] with a slightly different technique (see also [97] where this result was derived for the Hilbert-Schmidt norm). On the other hand, since the function $f_h(t) = |||H_t(A,B,X)|||$ is symmetric with respect to $t = 1/2$, i.e. $f_h(t) = f_h(1-t), t \in [0,1]$, it suffices to consider the Jensen functional on the interval $[0, 1/2]$.

Theorem 7.2 *Let A, B and X be operators on a complex separable Hilbert space $\mathscr{H}_s$ such that $A, B \in \mathfrak{B}^+(\mathscr{H}_s)$. If $t \in [0,1/2]$, then the inequality*

$$\begin{aligned}
&|||AX+XB||| - |||H_t(A,B,X)||| \\
&\quad \geq \min\{2t, 1-2t\}\left[|||AX+XB||| + 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| - 2|||H_{\frac{1}{4}}(A,B,X)|||\right] \\
&\quad\quad +2t\left[|||AX+XB||| - 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||\right] \geq 0 \qquad (7.8)
\end{aligned}$$

holds for every unitarily invariant norm $|||\cdot|||$. Moreover, if $t \in [1/2, 1]$, then

$$\begin{aligned}
&|||AX+XB||| - |||H_t(A,B,X)||| \\
&\quad \geq \min\{2-2t, 2t-1\}\left[|||AX+XB||| + 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| - 2|||H_{\frac{1}{4}}(A,B,X)|||\right] \\
&\quad\quad +(2-2t)\left[|||AX+XB||| - 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||\right] \geq 0. \qquad (7.9)
\end{aligned}$$

Proof. Considering the first inequality in (7.6) with $f_h(t) = |||H_t(A,B,X)|||$ and $\mathbf{x_1} = (0, 1/2)$, we have $\mathscr{J}(f_h, \mathbf{x_1}, t) \geq 2\min\{t, 1-t\}\mathscr{J}_0(f_h, \mathbf{x_1})$, that is,

$$(1-t)f_h(0) + tf_h\left(\frac{1}{2}\right) - f_h\left(\frac{t}{2}\right) \geq \min\{t, 1-t\}\left[f_h(0) + f_h\left(\frac{1}{2}\right) - 2f_h\left(\frac{1}{4}\right)\right].$$

The previous inequality can be rewritten in the form

$$f_h(0) - f_h\left(\frac{t}{2}\right) \geq \min\{t, 1-t\}\left[f_h(0) + f_h\left(\frac{1}{2}\right) - 2f_h\left(\frac{1}{4}\right)\right] + t\left[f_h(0) - f_h\left(\frac{1}{2}\right)\right],$$

that is,

$$f_h(0) - f_h(t) \geq \min\{2t, 1-2t\}\left[f_h(0) + f_h\left(\frac{1}{2}\right) - 2f_h\left(\frac{1}{4}\right)\right] + 2t\left[f_h(0) - f_h\left(\frac{1}{2}\right)\right], \qquad (7.10)$$

where $t \in [0,1/2]$. Hence, we get inequality (7.8).

On the other hand, if $t \in [1/2,1]$, then $1-t \in [0,1/2]$. Hence, replacing t with $1-t$ in (7.10) and using the fact that $f_h(t) = f_h(1-t)$, relation (7.10) yields inequality

$$\begin{aligned} f_h(0) - f_h(t) \geq \min\{2-2t, 2t-1\} &\left[f_h(0) + f_h\left(\frac{1}{2}\right) - 2f_h\left(\frac{1}{4}\right)\right] \\ &+ (2-2t)\left[f_h(0) - f_h\left(\frac{1}{2}\right)\right], \quad t \in [1/2,1], \end{aligned}$$

that is, we obtain (7.9). The proof is now completed. □

Remark 7.2 It should be noticed here that inequalities (7.8) and (7.9) yield the refinement of inequality (7.7). More precisely, due to convexity of function $f_h(t) = |||H_t(A,B,X)|||$ one has

$$|||AX + XB||| + 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| - 2|||H_{\frac{1}{4}}(A,B,X)||| \geq 0,$$

so the right-hand sides of inequalities (7.8) and (7.9) are not less than the right-hand side of inequality (7.7).

In order to obtain the refinement of the left Heinz norm inequality, we have to apply the second inequality in (7.6), together with a convex function $f_h(t) = |||H_t(A,B,X)|||$.

Remark 7.3 By considering the upper bound for the Jensen functional, that is, the second inequality in (7.6), and $\mathbf{x_0} = (0,1)$, we have

$$f_h(0) - f_h(t) \leq 2\max\{t, 1-t\}\left[f_h(0) - f_h\left(\frac{1}{2}\right)\right].$$

The above inequality is equivalent to

$$f_h(t) - f_h\left(\frac{1}{2}\right) \geq (1 - 2\max\{t,1-t\})\left[f_h(0) - f_h\left(\frac{1}{2}\right)\right]. \tag{7.11}$$

Since $\max_{t\in[0,1]}\{t, 1-t\} \geq 1/2$, the right-hand side of inequality (7.11) is non-positive, which means that in this case we actually obtain worse result than the original inequality in (7.3).

Although the application of the second inequality in (7.6) on interval $[0,1]$ does not provide refinement of the left Heinz norm inequality, we can obtain some refinements by considering certain subintervals of $[0,1]$.

Theorem 7.3 *Let A, B and X be operators on a complex separable Hilbert space $\mathscr{H}_s$ such that $A, B \in \mathfrak{B}^+(\mathscr{H}_s)$, and let $0 \leq a < 1/2$. If $|||\cdot|||$ is unitarily invariant norm, then the inequality*

$$|||H_t(A,B,X)||| - 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| \geq \frac{2|1-2t|}{1-2a}\left[|||H_{\frac{2a+1}{4}}(A,B,X)||| - 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||\right] \geq 0 \tag{7.12}$$

holds for all $t \in \left[a, \frac{1+2a}{4}\right] \cup \left[\frac{2a+3}{4}, 1-a\right]$.

Proof. Considering the second inequality in (7.6) equipped with $\mathbf{x} = \left(a, \frac{1}{2}\right)$ and $f_h(t) = |||H_t(A,B,X)|||$, we have $\mathscr{J}(f_h, \mathbf{x}, t) \leq 2\max\{t, 1-t\}\mathscr{J}_0(f_h, \mathbf{x})$, that is,

$$\begin{aligned}(1-t)f_h(a) + tf_h\left(\frac{1}{2}\right) - f_h\left(\frac{(1-2a)t+2a}{2}\right)\\ \leq 2\max\{t,1-t\}\left[\frac{f_h(a)+f_h(\frac{1}{2})}{2} - f_h\left(\frac{2a+1}{4}\right)\right].\end{aligned}$$

The previous inequality can be rewritten in the form

$$\begin{aligned}&f_h\left(\frac{(1-2a)t+2a}{2}\right) - f_h\left(\frac{1}{2}\right)\\ &\geq (1-t-\max\{t,1-t\})f_h(a) + ((t-1-\max\{t,1-t\})f_h\left(\frac{1}{2}\right)\\ &\quad +2\max\{t,1-t\}f_h\left(\frac{2a+1}{4}\right).\end{aligned} \tag{7.13}$$

Generally speaking, the right-hand side of inequality (7.13) is not always non-negative. Hence, we are going to find a restricted set of parameters t such that the right-hand side of (7.13) is non-negative.

For that sake we assume that $1-t-\max\{t,1-t\} \geq 0$, that is, $\max\{t,1-t\} \leq 1-t$. In that case $t \leq 1/2$ and also $\max\{t,1-t\} = 1-t$. Under such conditions, the above inequality (7.13) reduces to

$$f_h\left(\frac{(1-2a)t+2a}{2}\right) - f_h\left(\frac{1}{2}\right) \geq 2(1-t)\left[f_h\left(\frac{2a+1}{4}\right) - f_h\left(\frac{1}{2}\right)\right],\ t \in \left[0, \frac{1}{2}\right],$$

which is equivalent to

$$f_h(t) - f_h\left(\frac{1}{2}\right) \geq \frac{2(1-2t)}{1-2a}\left[f_h\left(\frac{2a+1}{4}\right) - f_h\left(\frac{1}{2}\right)\right],\ t \in \left[a, \frac{1+2a}{4}\right]. \tag{7.14}$$

On the other hand, if $t \in \left[\frac{2a+3}{4}, 1-a\right]$, then $1-t \in \left[a, \frac{1+2a}{4}\right]$, so replacing t with $1-t$ in (7.14), together with the symmetry argument, yields inequality

$$f_h(t) - f_h\left(\frac{1}{2}\right) \geq \frac{2(2t-1)}{1-2a}\left[f_h\left(\frac{2a+1}{4}\right) - f_h\left(\frac{1}{2}\right)\right],\ t \in \left[\frac{2a+3}{4}, 1-a\right]. \tag{7.15}$$

Finally, inequalities (7.14) and (7.15) provide (7.12) and the proof is completed. □

Note that the previous theorem yields the refinement of the left Heinz norm inequality on the subset of $[0,1]$ which consists of two symmetric intervals with respect to the midpoint of $[0,1]$. On the other hand, the interval $[0,1]$ can be covered with such symmetric intervals, hence we can obtain refinements of the Heinz inequality on each symmetric subset. This procedure is described in the following result.

Theorem 7.4 *Let A, B and X be operators on a complex separable Hilbert space $\mathscr{H}_s$ such that $A, B \in \mathfrak{B}^+(\mathscr{H}_s)$, and let $n \in \mathbb{N}$. If $|||\cdot|||$ is unitarily invariant norm, then the inequality*

$$|||H_t(A,B,X)||| - 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| \geq 2^n|1-2t|\left[|||H_{\frac{2^n-1}{2^{n+1}}}(A,B,X)||| - 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||\right] \geq 0 \tag{7.16}$$

holds for all $t \in \left[\frac{2^{n-1}-1}{2^n}, \frac{2^n-1}{2^{n+1}}\right] \cup \left[\frac{2^n+1}{2^{n+1}}, \frac{2^{n-1}+1}{2^n}\right]$.

Proof. In order to cover the interval $[0,1]$, we repeatedly utilize Theorem 7.3. At the first step, we use inequality (7.12) with $a = 0$, which yields inequality

$$|||H_t(A,B,X)||| - 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| \geq 2|1-2t|\left[|||H_{\frac{1}{4}}(A,B,X)||| - 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||\right],$$

valid for all $t \in \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]$.

At the second step we apply the previous theorem to the endpoint of the interval that establishes the inequality in the previous step, that is, $a = 1/4$. In this case relation (7.12) yields inequality

$$|||H_t(A,B,X)||| - 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| \geq 2^2|1-2t|\left[|||H_{\frac{3}{8}}(A,B,X)||| - 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||\right],$$

which is valid for the values $t \in \left[\frac{1}{4}, \frac{3}{8}\right] \cup \left[\frac{5}{8}, \frac{3}{4}\right]$.

Now, the further construction is explicitly determined. At the n-th step we consider inequality (7.12) for $a = a_n$, where a_n is the solution of the recurrence relation

$$a_n = \frac{2a_{n-1}+1}{4}, \tag{7.17}$$

with the initial condition $a_1 = 0$. The above recurrence relation (7.17) is linear non-homogeneous of the first degree. Using the usual methods for solving recurrence relations or by a mathematical induction principle, we easily obtain solution of (7.17), that is, $a_n = \frac{2^{n-1}-1}{2^n}$. Finally, considering (7.12) with this solution a_n, we exactly obtain inequality (7.16) as required. The proof is now completed. □

7.1.2 Operator analogues of the Heinz norm inequalities

Although it was previously explored (Chapter 6), it is not redundant to come up with the introduced notation once more: $\mathfrak{B}(\mathscr{H})$ denotes the semi-space of all bounded linear self-adjoint operators on Hilbert space $\mathscr{H}$, while $\mathfrak{B}^{++}(\mathscr{H})$ denote the set of all positive invertible operators in $\mathfrak{B}(\mathscr{H})$. The weight operator arithmetic mean ∇_t and geometric mean $\sharp_t$, for $t \in [0,1]$ and $A, B \in \mathfrak{B}^{++}(\mathscr{H})$, are defined as follows:

$$A\nabla_t B = (1-t)A + tB,$$

$$A\sharp_t B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^t A^{\frac{1}{2}}.$$

If $t = 1/2$, the arithmetic and the geometric mean are denoted respectively by ∇ and $\sharp$, for brevity. Recall e.g. (6.21) that the definition of Heinz means in real case can be raised up to the level of operators, by

$$\widetilde{H}_t(A,B) = \frac{A\sharp_t B + A\sharp_{1-t} B}{2}$$

and, as we have already seen, Heinz mean $\widetilde{H}_t(\cdot,\cdot)$ interpolates between the non-weight arithmetic and geometric mean, that is,

$$A\sharp B \leq \widetilde{H}_t(A,B) \leq A\nabla B. \tag{7.18}$$

The first inequality in (7.18) will be referred to as the left Heinz operator inequality, while the second one will be referred to as the right Heinz operator inequality. As we have already discussed in Chapter 6, authors in [96] obtained the following refinement of the right Heinz operator inequality:

$$A\nabla B - \widetilde{H}_t(A,B) \geq 2\min\{t, 1-t\}\left(A\nabla B - A\sharp B\right), \tag{7.19}$$

which corresponds to the norm inequality (7.7). The same result had been obtained in paper [98], for matrices.

The previous refinement (in general case) was established with the help of the appropriate scalar inequality and the well-known monotonicity principle (1.40).

Remark 7.4 Inequality (7.19) is a simple consequence of a more general method developed in [107]. More precisely, recall (Chapter 6) that the authors in [107] established the Jensen functional for bounded self-adjoint operators. Such functional is defined by

$$\widetilde{\mathscr{J}}(f,X,\delta,t) = (1-t)f(X) + tf(\delta)1_H - f\left((1-t)X + t\delta 1_{\mathscr{H}}\right), \tag{7.20}$$

where $f : [a,b] \to \mathbb{R}$ is a convex function, $X \in \mathfrak{B}(\mathscr{H})$, $a1_{\mathscr{H}} \leq X \leq b1_{\mathscr{H}}$, and $1_{\mathscr{H}}$ denotes identity operator on Hilbert space $\mathscr{H}$. The above defined functional possesses both monotonicity (1.67) and superadditivity (1.66) properties as in the real case, which implies mutually boundedness of the functional (7.20) via the non-weight functional:

$$2\min\{t,1-t\}\,\widetilde{\mathscr{J}_0}(f,X,\delta) \leq \widetilde{\mathscr{J}}(f,X,\delta,t) \leq 2\max\{t,1-t\}\,\widetilde{\mathscr{J}_0}(f,X,\delta), \tag{7.21}$$

where

$$\widetilde{\mathscr{J}_0}(f,X,\delta) = \frac{f(X) + f(\delta)1_H}{2} - f\left(\frac{X+\delta 1_H}{2}\right).$$

Hence, considering the first inequality in (7.21) equipped with the exponential mapping $f(x) = \exp x$, $\delta = 0$, and $X = \log\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)$, [107] provides inequality

$$A\nabla_{1-t}B - A\sharp_{1-t}B \geq 2\min\{t,1-t\}\left(A\nabla B - A\sharp B\right). \tag{7.22}$$

Finally, considering inequality (7.22) with $1-t$ instead of t and adding these two inequalities, we get (7.19).

It should be noticed here that the second inequality in (7.21), i.e. the method developed in [107], *did not provide the refinement of the left* Heinz operator inequality, similarly as in Section 7.1.1 (see Remark 7.3).

On the other hand, such improvement can be established by virtue of the following classical inequality. Namely, it is well-known that the logarithmic mean $L(x,y)=\frac{x-y}{\log x-\log y}$, $x\neq y$, $L(x,x)=x$, $x,y>0$, interpolates between the non-weight arithmetic and geometric mean:

$$\sqrt{xy}\leq L(x,y)\leq \frac{x+y}{2},\quad x,y>0. \tag{7.23}$$

The above interpolating series of inequalities can be used in obtaining a refinement of the left Heinz operator mean.

Theorem 7.5 *Let $A,B\in\mathfrak{B}^{++}(\mathscr{H})$ and let $t\in[0,1]\setminus\{1/2\}$. Then*

$$A\sharp B\leq \frac{1}{2t-1}A^{\frac{1}{2}}F_t\big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big)A^{\frac{1}{2}}\leq \widetilde{H}_t(A,B), \tag{7.24}$$

where

$$F_t(x)=\begin{cases}\frac{x^t-x^{1-t}}{\log x}, & x>0, x\neq 1\\ 2t-1, & x=1.\end{cases}$$

Proof. Starting from (7.23), we have

$$\sqrt{x}\leq \frac{x^t-x^{1-t}}{(2t-1)\log x}\leq \frac{x^t+x^{1-t}}{2},\quad x>0, x\neq 1,$$

that is,

$$\sqrt{x}\leq \frac{1}{2t-1}F_t(x)\leq \frac{x^t+x^{1-t}}{2},\quad x>0.$$

Since $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\geq 0$, monotonicity principle (1.40) for operator functions yields inequality

$$\big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big)^{\frac{1}{2}}\leq \frac{1}{2t-1}F_t\big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big)\leq \frac{\big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big)^t+\big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big)^{1-t}}{2}.$$

Moreover, multiplying both sides of the previous series of inequalities by $A^{\frac{1}{2}}$, we have (7.24), as claimed. □

Remark 7.5 If $t=1/2$, the above inequality (7.24) reduces to a trivial equality.

Note that the improvements (7.19) and (7.24) of the Heinz operator inequalities were derived without utilizing convexity of the Heinz means. If this fact is taken into account, we can derive operator analogues of all results from Sections 7.1.1 and 7.1.2

For that sake, we here define the function $f_x:[0,1]\to\mathbb{R}$, $x>0$, by

$$f_x(t)=\frac{x^t+x^{1-t}}{2}. \tag{7.25}$$

Clearly, the above function represents the Heinz mean in the real case, that is, we have $f_x(t) = h_t(1,x)$. Recall that f_x is convex on [0,1], attains minimum at $t = 1/2$, attains maximum at $t = 0$ and $t = 1$, and is symmetric with respect to the midpoint of interval $[0,1]$.

On the other hand, the same properties of function f_h defined by (7.2), were exploited in Sections 7.1.1 and 7.1.2, in order to obtain improved Heinz norm inequalities.

Therefore, in the sequel we consider the Jensen functional (7.5) equipped with the above function f_x. In addition, using the same technique as in the previous section, we obtain appropriate scalar inequalities, which will, by virtue of monotonicity principle (1.40) for operator functions, yield the corresponding operator analogues of the Heinz norm inequalities.

What follows is an operator analogue of Theorem 7.2.

Theorem 7.6 *Suppose $A,B \in \mathfrak{B}^{++}(\mathscr{H})$. If $t \in [0,1/2]$, then the inequality*

$$\begin{aligned} A\nabla B - \widetilde{H}_t(A,B) &\geq \min\{2t,1-2t\}\left[A\nabla B + A\sharp B - 2\widetilde{H}_{\frac{1}{4}}(A,B)\right] \\ &\quad +2t\left[A\nabla B - A\sharp B\right] \geq 0 \end{aligned} \tag{7.26}$$

holds. Moreover, if $t \in [1/2,1]$, then

$$\begin{aligned} A\nabla B - \widetilde{H}_t(A,B) &\geq \min\{2-2t,2t-1\}\left[A\nabla B + A\sharp B - 2\widetilde{H}_{\frac{1}{4}}(A,B)\right] \\ &\quad +(2-2t)\left[A\nabla B - A\sharp B\right] \geq 0. \end{aligned} \tag{7.27}$$

Proof. Considering the first inequality in (7.6) with $f_x(t) = \frac{x^t+x^{1-t}}{2}$, $x > 0$, and $\mathbf{x_1} = \left(0,\frac{1}{2}\right)$, we have $\mathscr{J}(f_x,\mathbf{x_1},t) \geq 2\min\{t,1-t\}\mathscr{J}_0(f_x,\mathbf{x_1})$, that is,

$$(1-t)f_x(0) + tf_x\left(\frac{1}{2}\right) - f_x\left(\frac{t}{2}\right) \geq \min\{t,1-t\}\left[f_x(0) + f_x\left(\frac{1}{2}\right) - 2f_x\left(\frac{1}{4}\right)\right].$$

The previous inequality can be rewritten in the form

$$f_x(0) - f_x(t) \geq \min\{2t,1-2t\}\left[f_x(0) + f_x\left(\frac{1}{2}\right) - 2f_x\left(\frac{1}{4}\right)\right] + 2t\left[f_x(0) - f_x\left(\frac{1}{2}\right)\right],$$

where $t \in [0,1/2]$, that is,

$$\frac{1+x}{2} - \frac{x^t+x^{1-t}}{2} \geq \min\{2t,1-2t\}\left[\frac{1+x}{2} + x^{\frac{1}{2}} - x^{\frac{1}{4}} - x^{\frac{3}{4}}\right] + 2t\left[\frac{1+x}{2} - x^{\frac{1}{2}}\right], \tag{7.28}$$

where $t \in [0,1/2]$. Since inequality (7.28) holds for all $x > 0$, using the monotonicity principle (1.40) and the fact that $A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \geq 0$, we have

$$\begin{aligned} &\frac{1_{\mathscr{H}} + A^{-\frac{1}{2}}BA^{-\frac{1}{2}}}{2} - \frac{\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^t + \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{1-t}}{2} \\ &\geq \min\{2t,1-2t\}\left[\frac{1_{\mathscr{H}} + A^{-\frac{1}{2}}BA^{-\frac{1}{2}}}{2} + \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}} - \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{4}}\right. \\ &\quad \left. - \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{3}{4}}\right] + 2t\left[\frac{1_{\mathscr{H}} + A^{-\frac{1}{2}}BA^{-\frac{1}{2}}}{2} - \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}\right]. \end{aligned} \tag{7.29}$$

Finally, multiplying both sides of inequality (7.29) by $A^{\frac{1}{2}}$, we obtain (7.26). Obviously, the inequality (7.27) follows by the symmetry argument, since $\widetilde{H}_t(A,B) = \widetilde{H}_{1-t}(A,B)$. □

Remark 7.6 Since f_x is a convex function, by virtue of monotonicity principle (1.40) for operator functions, we have $A\nabla B + A\sharp B - 2\widetilde{H}_{\frac{1}{4}}(A,B) \geq 0$. Therefore, the above Theorem 7.6 yields an improvement of inequality (7.19).

Note that the proof of Theorem 7.6 follows the same lines as the proof of Theorem 7.2. This is meaningful since the functions f_h and f_x, defined respectively by (7.2) and (7.25), have the same properties concerning monotonicity, symmetry and extrema points.

Therefore, the proofs of theorems 7.3 and 7.4 together with the monotonicity principle (1.40) for operator functions can also be utilized in deriving the operator versions of inequalities (7.12) and (7.16). More precisely, we consider the mentioned proofs with a function f_x instead of f_h, and use the monotonicity property (1.40) in the same way as in the proof of Theorem 7.6.

Theorem 7.7 *Suppose $A,B \in \mathfrak{B}^{++}(\mathscr{H})$ and let $0 \leq a < 1/2$. Then the inequality*

$$\widetilde{H}_t(A,B) - A\sharp B \geq \frac{2|1-2t|}{1-2a}\left[\widetilde{H}_{\frac{2a+1}{4}}(A,B) - A\sharp B\right] \geq 0$$

holds for all $t \in \left[a, \frac{1+2a}{4}\right] \cup \left[\frac{2a+3}{4}, 1-a\right]$.

Theorem 7.8 *Let $A,B \in \mathfrak{B}^{++}(\mathscr{H})$ and let $n \in \mathbb{N}$. Then the inequality*

$$\widetilde{H}_t(A,B) - A\sharp B \geq 2^n|1-2t|\left[\widetilde{H}_{\frac{2^n-1}{2^{n+1}}}(A,B) - A\sharp B\right] \geq 0$$

holds for all $t \in \left[\frac{2^{n-1}-1}{2^n}, \frac{2^n-1}{2^{n+1}}\right] \cup \left[\frac{2^n+1}{2^{n+1}}, \frac{2^{n-1}+1}{2^n}\right]$.

Of course, theorems 7.7 and 7.8 provide refinements of the first Heinz operator inequality.

7.2 More accurate weak majorization relations for the Jensen and some related inequalities

Throughout this section we deal with the algebra $\mathscr{M}_n$ of all $n \times n$ complex matrices, where $\mathscr{H}_n$ stands for the set of all Hermitian matrices in $\mathscr{M}_n$. For an interval $J \subseteq \mathbb{R}$, we denote by $\mathscr{H}_n(J)$ the set of all Hermitian matrices in $\mathscr{M}_n$ whose spectrum is contained in J. We denote by $\mathscr{S}_n$ the set of all positive semi-definite matrices in $\mathscr{M}_n$, while $\mathscr{P}_n$ stands for the set of all positive definite matrices in $\mathscr{M}_n$. For column vectors $x,y \in \mathbb{C}^n$ their inner product is denoted by $\langle x,y\rangle = y^*x$.

For Hermitian matrices A and B we define an operator order, i.e. $A \leq B$ if $B - A \in \mathscr{S}_n$. Further, for $A \in \mathscr{H}_n$ we denote by $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ the eigenvalues of A arranged in a decreasing order with their multiplicities counted. The notation $\lambda(A)$ stands for the row vector $(\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A))$. The eigenvalue inequality $\lambda(A) \leq \lambda(B)$ means that $\lambda_j(A) \leq \lambda_j(B)$ for all $1 \leq j \leq n$. The weak majorization inequality $\lambda(A) \prec_w \lambda(B)$ means $\sum_{j=1}^k \lambda_j(A) \leq \sum_{j=1}^k \lambda_j(B)$, $k = 1, 2, \ldots, n$. The above three kinds of ordering satisfy $A \leq B \Rightarrow \lambda(A) \leq \lambda(B) \Rightarrow \lambda(A) \prec_w \lambda(B)$. Note that the first implication is the Weyl monotonicity theorem (see, e.g. [34, p. 63]), while the second holds trivially.

On the other hand, $f : J \to \mathbb{R}$ is operator convex if

$$f(tA + (1-t)B) \leq tf(A) + (1-t)f(B), \tag{7.30}$$

for all $0 \leq t \leq 1$ and $A, B \in \mathscr{H}_n(J)$. Recall that for a Hermitian matrix $H \in \mathscr{H}_n(J)$, $f(H)$ is defined by familiar functional calculus.

One of the numerous operator versions of the Jensen inequality asserts that if $f : J \to \mathbb{R}$ is an operator convex function such that $0 \in J$ and $f(0) \leq 0$, then

$$f(X^*AX) \leq X^* f(A) X \tag{7.31}$$

holds for all $A \in \mathscr{H}_n(J)$ and contractions $X \in \mathscr{M}_n$. Recall that $X \in \mathscr{M}_n$ is called contraction if $||X|| \leq 1$, considering spectral norm $||\cdot||$. For some related versions of the Jensen operator inequality, the reader is referred to [74].

Some ten years ago, Aujla and Silva [20] proved that if $f : J \to \mathbb{R}$ is a convex function, then the eigenvalues of $f(tA + (1-t)B)$ are weakly majorized by the eigenvalues of $tf(A) + (1-t)f(B)$, that is,

$$\lambda\left(f(tA + (1-t)B)\right) \prec_w \lambda\left(tf(A) + (1-t)f(B)\right), \tag{7.32}$$

where $A, B \in \mathscr{H}_n(J)$ and $0 \leq t \leq 1$. In addition, if $0 \in J$ and $f(0) \leq 0$, they also showed that

$$\lambda\left(f(X^*AX)\right) \prec_w \lambda\left(X^* f(A) X\right) \tag{7.33}$$

holds for all $A \in \mathscr{H}_n(J)$ and contractions $X \in \mathscr{M}_n$. It has also been shown in [20] that if f is additionally a monotone function, then the relations (7.32) and (7.33) become the eigenvalue inequalities.

Here we derive the improvements of weak majorization inequalities (7.32) and (7.33), as well as the improvements of their eigenvalue counterparts, as it was previously published in paper [120]. Firstly, we cite several auxiliary results, needed for this study.

The following two lemmas were utilized in proving weak majorization inequalities (7.32) and (7.33). First of them is an operator version of the Jensen inequality with regard to an inner product $\langle \cdot, \cdot \rangle$.

Lemma 7.1 (SEE [158]) *Let $A \in \mathscr{H}_n(J)$ and let $f : J \to \mathbb{R}$ be a convex function. Then the inequality*

$$f(\langle Au, u \rangle) \leq \langle f(A)u, u \rangle \tag{7.34}$$

holds for every unit vector $u \in \mathbb{C}^n$.

What follows is the extremal representation for eigenvalues which is known in literature as the Ky Fan maximum principle.

Lemma 7.2 (SEE [34], P. 35) *If $A \in \mathscr{H}_n$, then*

$$\sum_{i=1}^{k} \lambda_i(A) = \max \sum_{i=1}^{k} \langle Au_i, u_i \rangle, \quad k = 1, 2, \ldots, n, \tag{7.35}$$

where the maximum is taken over all choices of orthonormal vectors $u_1, u_2, \ldots, u_k$.

In the sequel, in order to improve weak majorization inequalities (7.32) and (7.33), we again employ the well known properties of the discrete Jensen functional

$$\mathscr{J}^m(f, x, p) = \sum_{i=1}^{m} p_i f(x_i) - P_m f\left(\frac{\sum_{i=1}^{m} p_i x_i}{P_m}\right),$$

where $f : J \to \mathbb{R}$ is a convex function, $x = (x_1, x_2, \ldots, x_m) \in J^m$, $m \geq 2$, and $p = (p_1, p_2, \ldots, p_m)$ is a positive m-tuple of real numbers with $P_m = \sum_{i=1}^{m} p_i$. These properties were analyzed in detail in Chapter 1 and Chapter 2, but we cite them, nevertheless, repeatedly whenever a new environment with its specific conditions and applications is involved.

Since the above functional is increasing on the set of positive real m-tuples in the sense that

$$\mathscr{J}^m(f, x, p) \geq \mathscr{J}^m(f, x, q) \geq 0,$$

whenever $p \geq q$. i.e. $p_i \geq q_i$, $i = 1, 2, \ldots, m$ the following converse and the refinement of the corresponding discrete Jensen inequality are immediate consequences:

$$m \max_{1 \leq i \leq m} \{p_i\} \mathscr{J}_0^m(f, x) \geq \mathscr{J}^m(f, x, p) \geq m \min_{1 \leq i \leq m} \{p_i\} \mathscr{J}_0^m(f, x), \tag{7.36}$$

where $\mathscr{J}_0^m(f, x) = \frac{\sum_{i=1}^{m} f(x_i)}{m} - f\left(\frac{\sum_{i=1}^{m} x_i}{m}\right)$.

7.2.1 Improved weak majorization inequalities for the Jensen inequality

The crucial step in obtaining weak majorization inequalities that are more precise than (7.32) and (7.33) is an improvement of the Jensen inequality described in the previous section (the second inequality in (7.36)).

Observe that, due to the discrete Jensen inequality (1.4), the relation (7.32) can be stated in the following multivariate form:

$$\lambda\left(f\left(\sum_{i=1}^{m} p_i A_i\right)\right) \prec_w \lambda\left(\sum_{i=1}^{m} p_i f(A_i)\right), \tag{7.37}$$

where $f : J \to \mathbb{R}$ is a convex function, $A_1, A_2, \ldots, A_m \in \mathscr{H}_n(J)$, and $\sum_{i=1}^{m} p_i = 1$, $p_i \geq 0$, $i = 1, 2, \ldots, m$.

The following theorem provides an improvement of inequality (7.37) (I_n stands for a unit matrix in $\mathscr{M}_n$.)

Theorem 7.9 *Let $f : J \to \mathbb{R}$ be a continuous convex function and $A_1, A_2, \ldots, A_m \in \mathscr{H}_n(J)$. If $\sum_{i=1}^m p_i = 1$, $p_i \geq 0$, $i = 1, 2, \ldots, m$, then*

$$\lambda\left(f\left(\sum_{i=1}^m p_i A_i\right) + \mu\nu I_n\right) \prec_w \lambda\left(\sum_{i=1}^m p_i f(A_i)\right), \tag{7.38}$$

where $\mu = \min\{p_1, p_2, \ldots, p_m\}$ and

$$\nu = \min_{\substack{\|u\|=1 \\ u\in\mathbb{C}^n}} \sum_{i=1}^m f(\langle A_i u, u\rangle) - mf\left(\langle \frac{1}{m}(\sum_{i=1}^m A_i)u, u\rangle\right). \tag{7.39}$$

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $\sum_{i=1}^m p_i A_i$ arranged so that $f(\lambda_1) \geq f(\lambda_2) \geq \cdots \geq f(\lambda_n)$, and let $u_1, u_2, \ldots, u_n$ be the corresponding orthonormal eigenvectors. Then, utilizing (7.39) and the second inequality in (7.36), we have

$$\begin{aligned}
&\sum_{j=1}^k \lambda_j\left(f\left(\sum_{i=1}^m p_i A_i\right)\right) + \mu\nu k \\
&= \sum_{j=1}^k f\left(\langle (\sum_{i=1}^m p_i A_i) u_j, u_j\rangle\right) + \mu\nu k \\
&\leq \sum_{j=1}^k f\left(\sum_{i=1}^m p_i \langle A_i u_j, u_j\rangle\right) + \mu\sum_{j=1}^k\left(\sum_{i=1}^m f(\langle A_i u_j, u_j\rangle) - mf\left(\langle \frac{1}{m}(\sum_{i=1}^m A_i)u_j, u_j\rangle\right)\right) \\
&\leq \sum_{j=1}^k\left(\sum_{i=1}^m p_i f(\langle A_i u_j, u_j\rangle)\right).
\end{aligned}$$

Moreover, by virtue of the inequality (7.34) and the Ky Fan maximum principle (7.35), it follows that

$$\begin{aligned}
\sum_{j=1}^k\left(\sum_{i=1}^m p_i f(\langle A_i u_j, u_j\rangle)\right) &\leq \sum_{j=1}^k\left(\sum_{i=1}^m p_i \langle f(A_i) u_j, u_j\rangle\right) \\
&= \sum_{j=1}^k \langle (\sum_{i=1}^m p_i f(A_i)) u_j, u_j\rangle \\
&\leq \sum_{j=1}^k \lambda_j\left(\sum_{i=1}^m p_i f(A_i)\right),
\end{aligned}$$

that is,

$$\sum_{j=1}^k \lambda_j\left(f\left(\sum_{i=1}^m p_i A_i\right) + \mu\nu I_n\right) \leq \sum_{j=1}^k \lambda_j\left(\sum_{i=1}^m p_i f(A_i)\right),$$

for $k = 1, 2, \ldots, n$. This proves our assertion. □

It should be noticed here that the scalars μ and ν appearing in Theorem 7.9 are non-negative. Of course, it is possible to achieve that they are positive.

Example 7.1 Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=|x|$ and let

$$A=\begin{pmatrix} -6 & 5 \\ 5 & -6 \end{pmatrix}, \qquad B=\begin{pmatrix} 11 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, $\lambda\left(|\frac{A+B}{2}|\right)=\left(\frac{5\sqrt{2}}{2},\frac{5\sqrt{2}}{2}\right)$ and $\lambda\left(\frac{1}{2}|A|+\frac{1}{2}|B|\right)=\left(\frac{12+5\sqrt{2}}{2},\frac{12-5\sqrt{2}}{2}\right)$, that is, we have $\left(\frac{5\sqrt{2}}{2},\frac{5\sqrt{2}}{2}\right)\prec_w\left(\frac{12+5\sqrt{2}}{2},\frac{12-5\sqrt{2}}{2}\right)$, by virtue of (7.32).

Utilizing Theorem 7.9 we can get even a more precise estimate. Namely, defining $g(u)=|\langle Au,u\rangle|+|\langle Bu,u\rangle|-|\langle (A+B)u,u\rangle|$, where $u=\binom{u_1}{u_2}\in\mathbb{C}^2$, $||u||=1$, it follows that

$$g(u)=7+10|u_1|^2-10\mathrm{Re}(u_1\overline{u}_2)-\left|10|u_1|^2+10\mathrm{Re}(u_1\overline{u}_2)-5\right|.$$

Now, if $10|u_1|^2+10\mathrm{Re}(u_1\overline{u}_2)-5\geq 0$ then $g(u)=12-20\mathrm{Re}(u_1\overline{u}_2)\geq 2$ since $\mathrm{Re}(u_1\overline{u}_2)\leq \frac{|u_1|^2+|u_2|^2}{2}\leq\frac{1}{2}$. Otherwise, if $10|u_1|^2+10\mathrm{Re}(u_1\overline{u}_2)-5<0$, then $g(u)=2+20|u_1|^2\geq 2$. Moreover, since $g(u_0)=2$, where $u_0=\binom{0}{1}$, it follows that

$$\nu=\min_{\substack{\|u\|=1\\ u\in\mathbb{C}^2}}|\langle Au,u\rangle|+|\langle Bu,u\rangle|-|\langle (A+B)u,u\rangle|=2.$$

Therefore, the relation (7.38) yields a more accurate estimate

$$\left(\frac{5\sqrt{2}}{2},\frac{5\sqrt{2}}{2}\right)+\frac{1}{2}\cdot 2(1,1)=\left(\frac{2+5\sqrt{2}}{2},\frac{2+5\sqrt{2}}{2}\right)\prec_w\left(\frac{12+5\sqrt{2}}{2},\frac{12-5\sqrt{2}}{2}\right).$$

Remark 7.7 If $f:J\to\mathbb{R}$ is a strictly convex function and there exist indices $i,j\in\{1,2,\dots,m\}$ such that $A_i<A_j$, then

$$g(u):=\sum_{i=1}^{m}f(\langle A_iu,u\rangle)-mf\left(\langle\frac{1}{m}(\sum_{i=1}^{m}A_i)u,u\rangle\right)>0,\quad u\in\mathbb{C}^n,\ \|u\|=1.$$

Since the set $S=\{u\in\mathbb{C}^n;\|u\|=1\}$ is compact, the function $g:S\to\mathbb{R}$ attains its minimum on S which implies that $\nu>0$. Hence, in this case the relation (7.38) represents the refinement of (7.37).

Our next intention is to give the corresponding refinement of the inequality (7.33). Clearly, the multivariate version of (7.33) asserts that if $f:J\to\mathbb{R}$ is a convex function such that $0\in J$ and $f(0)\leq 0$, then

$$\lambda\left(f\left(\sum_{i=1}^{m}X_i^*A_iX_i\right)\right)\prec_w\lambda\left(\sum_{i=1}^{m}X_i^*f(A_i)X_i\right), \tag{7.40}$$

for $A_1,A_2,\dots,A_m\in\mathscr{H}_n(J)$ and $X_1,X_2,\dots,X_m\in\mathscr{M}_n$ such that $\sum_{i=1}^{m}X_i^*X_i\leq I_n$.

Theorem 7.10 *Let $f : J \to \mathbb{R}$ be a continuous convex function and $A_1, A_2, \ldots, A_m \in \mathscr{H}_n(J)$. Further, let $X_1, X_2, \ldots, X_m \in \mathscr{M}_n$ be such that $\sum_{i=1}^m X_i^* X_i \le I_n$. If $0 \in J$ and $f(0) \le 0$, then*

$$\lambda\left(f\left(\sum_{i=1}^m X_i^* A_i X_i\right) + \mu' \nu' I_n\right) \prec_w \lambda\left(\sum_{i=1}^m X_i^* f(A_i) X_i\right), \tag{7.41}$$

where the scalars μ' and ν' are defined by

$$\mu' = 1 - \max\left\{\|I_n - X_1^* X_1\|, \|I_n - X_2^* X_2\|, \ldots, \|I_n - X_m^* X_m\|, \|\sum_{i=1}^m X_i^* X_i\|\right\}$$

and

$$\nu' = \min_{\substack{\|u\|=1 \\ u \in \mathbb{C}^n}} \sum_{i=1}^m f(\langle A_i u, u\rangle) + f(0) - (m+1) f\left(\langle \frac{1}{m+1}\left(\sum_{i=1}^m A_i\right) u, u\rangle\right).$$

Proof. We give only the sketch of the proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $\sum_{i=1}^m X_i^* A_i X_i$ arranged so that $f(\lambda_1) \ge f(\lambda_2) \ge \cdots \ge f(\lambda_n)$, and let $u_1, u_2, \ldots, u_n$ be the corresponding orthonormal eigenvectors.

Without loss of generality we can assume that $\|X_i u_j\| \neq 0$ for all indices $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$. Then, it follows that

$$\lambda_j\left(f\left(\sum_{i=1}^m X_i^* A_i X_i\right)\right) = f\left(\sum_{i=1}^m \|X_i u_j\|^2 \left\langle A_i \frac{X_i u_j}{\|X_i u_j\|}, \frac{X_i u_j}{\|X_i u_j\|}\right\rangle + \left(1 - \sum_{i=1}^m \|X_i u_j\|^2\right) \cdot 0\right),$$

for $j = 1, 2, \ldots, n$. Now, taking into account the above relation, the obvious inequality $\mu' \le \min\{\|X_1 u_j\|^2, \|X_2 u_j\|^2, \ldots, \|X_m u_j\|^2, 1 - \sum_{i=1}^m \|X_i u_j\|^2\}$, $j = 1, 2, \ldots, n$, and the condition $f(0) \le 0$, the proof follows the lines of the proof of Theorem 7.9 with scalars μ' and ν' instead of μ and ν. □

Example 7.2 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^4$ and let

$$A = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$$

In this setting, $\lambda\left((X^* A X)^4\right) = \left(\frac{1}{4}, \frac{1}{64}\right) \prec_w \left(2, \frac{1}{8}\right) = \lambda\left(X^* A^4 X\right)$, due to (7.33). Now, utilizing Theorem 7.10, it follows that $\mu' = \frac{1}{2}$ since $||X^* X|| = ||I_2 - X^* X|| = \frac{1}{2}$, and

$$\nu' = \frac{7}{8} \min_{\substack{\|u\|=1 \\ u \in \mathbb{C}^2}} \langle Au, u\rangle^4 = \frac{7}{32}.$$

Finally, the inequality (7.41) yields more precise relation

$$\left(\frac{1}{4}, \frac{1}{64}\right) + \frac{7}{64}(1, 1) = \left(\frac{23}{64}, \frac{1}{8}\right) \prec_w \left(2, \frac{1}{8}\right).$$

Remark 7.8 The previous example shows that it is possible to achieve that the scalars μ' and ν' appearing in Theorem 7.10 are positive. More generally, if $X_i^*X_i \in \mathscr{P}_n$, $i = 1,2,\dots,m$, and $\sum_{i=1}^m X_i^*X_i < I_n$, then $\mu' > 0$. In addition, if $f : J \to \mathbb{R}$ is a strictly convex function and if either there exist indices $i, j \in \{1,2,\dots,m\}$ such that $A_i < A_j$, or there exists index $i \in \{1,2,\dots,m\}$ such that $A_i \in \mathscr{P}_n$ or $-A_i \in \mathscr{P}_n$, then $\nu' > 0$.

If f is a non-negative function, then the relations (7.38) and (7.41) entail the corresponding inequalities for unitarily invariant norms. Recall that a norm $|||\cdot|||$ on $\mathscr{M}_n$ is called unitarily invariant if $|||UAV||| = |||A|||$ for all $A \in \mathscr{M}_n$ and for all unitary matrices $U, V \in \mathscr{M}_n$. One of the most basic classes of unitarily invariant norms are the Ky Fan norms $||\cdot||_{(k)}$, $k = 1,2,\dots,n$, defined as

$$||A||_{(k)} = \sum_{j=1}^{k} s_j(A), \quad k = 1,2,\dots,n,$$

where $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$ are the singular values of A, that is, the eigenvalues of $|A| = (A^*A)^{1/2}$. The Fan Dominance Theorem (see, e.g. [34, p. 93]) asserts that if $||A||_{(k)} \leq ||B||_{(k)}$ for $k = 1,2,\dots,n$, then $|||A||| \leq |||B|||$ for all unitarily invariant norms.

If f is a non-negative function, then the eigenvalues appearing in relations (7.38) and (7.41) coincide with the corresponding singular values. Thus, by virtue of the Fan Dominance Theorem, theorems 7.9 and 7.10 entail the inequalities for unitarily invariant norms.

Corollary 7.1 *Let $f : J \to \mathbb{R}$ be a non-negative continuous convex function and $A_1, A_2, \dots, A_m \in \mathscr{H}_n(J)$. If $\sum_{i=1}^m p_i = 1$, $p_i \geq 0$, $i = 1,2,\dots,m$, then the inequality*

$$|||f\Big(\sum_{i=1}^{m} p_iA_i\Big) + \mu\nu I_n||| \leq |||\sum_{i=1}^{m} p_i f(A_i)|||$$

holds for any unitarily invariant norm $|||\cdot|||$, where the scalars μ and ν are defined as in Theorem 7.9. In addition, if $0 \in J$ and $f(0) = 0$, then

$$|||f\Big(\sum_{i=1}^{m} X_i^*A_iX_i\Big) + \mu'\nu' I_n||| \leq |||\sum_{i=1}^{m} X_i^* f(A_i)X_i|||,$$

where $X_1, X_2, \dots, X_m \in \mathscr{M}_n$ are such that $\sum_{i=1}^m X_i^*X_i \leq I_n$, and the scalars μ' and ν' are defined as in Theorem 7.10.

Observe that the relations derived in Example 7.2 are eigenvalue inequalities since the corresponding function $f(x) = x^4$ is increasing on $\mathbb{R}^+$. Namely, it has been shown in [20] that if $f : J \to \mathbb{R}$ is a monotone convex function, then there is a stronger result with regard to weak majorization inequalities (7.32) and (7.33), i.e (7.37) and (7.40). More precisely, if $f : J \to \mathbb{R}$ is in addition a monotone function, then

$$\lambda\left(f\left(\sum_{i=1}^{m} p_iA_i\right)\right) \leq \lambda\left(\sum_{i=1}^{m} p_i f(A_i)\right)$$

and

$$\lambda\left(f\left(\sum_{i=1}^{m} X_i^* A_i X_i\right)\right) \leq \lambda\left(\sum_{i=1}^{m} X_i^* f(A_i) X_i\right),$$

provided that the assumptions of Theorem 7.9 and Theorem 7.10 are fulfilled. The above eigenvalue inequalities have been derived by virtue of the following minimax characterization of eigenvalues (see, e.g. [34, p. 58]):

$$\begin{aligned}\lambda_j(H) &= \max_{\dim \mathcal{M}=j} \min\{\langle Hu,u\rangle; u \in \mathcal{M}, \|u\|=1\} \\ &= \min_{\dim \mathcal{M}=n-j+1} \max\{\langle Hu,u\rangle; u \in \mathcal{M}, \|u\|=1\},\ j=1,2,\ldots,n,\end{aligned} \tag{7.42}$$

where H is a Hermitian matrix and $\mathcal{M}$ is a subspace of $\mathbb{C}^n$.

Having in mind the results of theorems 7.9 and 7.10, we can also derive refinements of these eigenvalue inequalities, again by virtue of the above minimax principle.

Theorem 7.11 *Let $f : J \to \mathbb{R}$ be a monotone continuous convex function and $A_1, A_2, \ldots, A_m \in \mathcal{H}_n(J)$. If $\sum_{i=1}^{m} p_i = 1$, $p_i \geq 0$, $i = 1,2,\ldots,m$, then*

$$\lambda\left(f\left(\sum_{i=1}^{m} p_i A_i\right) + \mu\nu I_n\right) \leq \lambda\left(\sum_{i=1}^{m} p_i f(A_i)\right), \tag{7.43}$$

where the scalars μ and ν are defined as in Theorem 7.9. In addition, if $0 \in J$ and $f(0) \leq 0$, then

$$\lambda\left(f\left(\sum_{i=1}^{m} X_i^* A_i X_i\right) + \mu'\nu' I_n\right) \leq \lambda\left(\sum_{i=1}^{m} X_i^* f(A_i) X_i\right), \tag{7.44}$$

where $X_1, X_2, \ldots, X_m \in \mathcal{M}_n$ are such that $\sum_{i=1}^{m} X_i^* X_i \leq I_n$, and the scalars μ' and ν' are defined as in Theorem 7.10.

Proof. We prove (7.43) only. The proof of the inequality (7.44) is similar.

If $f : J \to \mathbb{R}$ is increasing, then $\lambda_j(f(H)) = f(\lambda_j(H))$, $j = 1,2,\ldots,n$, for any $H \in \mathcal{H}_n(J)$. Moreover, since f is increasing, it follows that

$$\begin{aligned}\lambda_j(f(H)) = f(\lambda_j(H)) &= f\left(\max_{\dim \mathcal{M}=j} \min\{\langle Hu,u\rangle; u \in \mathcal{M}, \|u\|=1\}\right) \\ &= \max_{\dim \mathcal{M}=j} \min\{f(\langle Hu,u\rangle); u \in \mathcal{M}, \|u\|=1\}.\end{aligned}$$

Now, since the function $g : J \to \mathbb{R}$, defined by $g(x) = f(x) + \mu\nu$ is also increasing, we have that

$$\lambda_j\left(f\left(\sum_{i=1}^{m} p_i A_i\right) + \mu\nu I_n\right) = \max_{\dim \mathcal{M}=j} \min\left\{f\left(\left\langle\left(\sum_{i=1}^{m} p_i A_i\right)u,u\right\rangle\right) + \mu\nu; u \in \mathcal{M}, \|u\|=1\right\}. \tag{7.45}$$

Now, utilizing (7.39), the second inequality in (7.36), and the inequality (7.34), we have

$$\begin{aligned}
&f\left(\langle(\sum_{i=1}^{m} p_iA_i)u,u\rangle\right)+\mu\nu\\
\leq &f\left(\sum_{i=1}^{m} p_i\langle A_iu,u\rangle\right)+\mu\left(\sum_{i=1}^{m} f\left(\langle A_iu,u\rangle\right)-mf\left(\langle\frac{1}{m}(\sum_{i=1}^{m} A_i)u,u\rangle\right)\right)\\
\leq &\sum_{i=1}^{m} p_if\left(\langle A_iu,u\rangle\right)\leq\sum_{i=1}^{m} p_i\langle f(A_i)u,u\rangle=\langle\sum_{i=1}^{m} p_if(A_i)u,u\rangle,\quad \|u\|=1.
\end{aligned}$$

Combining this with (7.45) and the minimax principle (7.42), we have

$$\lambda_j\left(f\left(\sum_{i=1}^{m} p_iA_i\right)+\mu\nu I_n\right)\leq\lambda_j\left(\sum_{i=1}^{m} p_if(A_i)\right),$$

which proves our assertion in the case of increasing function. The same conclusion can be drawn for the case of decreasing function f since in that case $\lambda_j(f(H))=f(\lambda_{n-j+1}(H))$, $j=1,2,\ldots,n$, $H\in\mathscr{H}_n(J)$. □

As an application of Theorem 7.11, we give the following consequence.

Corollary 7.2 *Let $A_i\in\mathscr{S}_n$, $i=1,2,\ldots,m$, $r\geq 1$ or $A_i\in\mathscr{P}_n$, $i=1,2,\ldots,m$, $r\leq 0$. Then,*

$$\lambda\left(\left(\sum_{i=1}^{m} A_i\right)^r+\nu m^{r-1}I_n\right)\leq\lambda\left(m^{r-1}\sum_{i=1}^{m} A_i^r\right), \tag{7.46}$$

where

$$\nu=\min_{\substack{\|u\|=1\\ u\in\mathbb{C}^n}}\sum_{i=1}^{m}\langle A_iu,u\rangle^r-m^{1-r}\langle(\sum_{i=1}^{m} A_i)u,u\rangle^r.$$

Further, if $A_i\in\mathscr{S}_n$, $i=1,2,\ldots,m$, and $0<r<1$, then

$$\lambda\left(\left(\sum_{i=1}^{m} A_i\right)^r-\nu_0 m^{r-1}I_n\right)\geq\lambda\left(m^{r-1}\sum_{i=1}^{m} A_i^r\right), \tag{7.47}$$

where

$$\nu_0=\min_{\substack{\|u\|=1\\ u\in\mathbb{C}^n}} m^{1-r}\langle(\sum_{i=1}^{m} A_i)u,u\rangle^r-\sum_{i=1}^{m}\langle A_iu,u\rangle^r.$$

Proof. If $r\geq 1$, then $f(x)=x^r$ is increasing convex function. On the other hand, if $r\leq 0$, then $f(x)=x^r$ is decreasing convex function. Therefore, the application of the eigenvalue inequality (7.43) yields (7.46) in both cases.

Similarly, to obtain (7.47) we utilize (7.43) equipped with the decreasing convex function $f(x)=-x^r$, $0<r<1$. □

Remark 7.9 It should be noticed here that the Corollary 7.2 provides the refinements of the corresponding results from [11], [17], and [20]. Observe that for $r=-1$ and for operators A_i replaced by their inverses A_i^{-1}, relation (7.46) represents a strengthened version of a harmonic-arithmetic mean inequality.

Let us now consider the case of a concave function. To do this, we need the notion of a weak supermajorization. Let $x,y \in \mathbb{R}^n$ be the vectors with coordinates arranged in an increasing order, i.e. $x_1 \le x_2 \le \cdots \le x_n$ and $y_1 \le y_2 \le \cdots \le y_n$. We say that x is weakly supermajorized by y, in symbols $x \prec^w y$, if $\sum_{j=1}^k x_j \ge \sum_{j=1}^k y_j$, $k=1,2,\ldots,n$.

Remark 7.10 Let $f : J \to \mathbb{R}$ be a concave function. Applying Theorem 7.9 to the convex function $-f$ and utilizing that $x \prec_w y$ implies $-x \prec^w -y$ (see [34, p. 30]), it follows that

$$\lambda^{\uparrow}\left(f\left(\sum_{i=1}^m p_iA_i\right)-\mu\nu_1 I_n\right) \prec^w \lambda^{\uparrow}\left(\sum_{i=1}^m p_i f(A_i)\right),$$

where $\mu=\min\{p_1,p_2,\ldots,p_m\}$,

$$\nu_1 = \min_{\substack{\|u\|=1\\ u\in\mathbb{C}^n}} mf\left(\langle\frac{1}{m}\big(\sum_{i=1}^m A_i\big)u,u\rangle\right)-\sum_{i=1}^m f\left(\langle A_iu,u\rangle\right),$$

and $\uparrow$ means that the eigenvalues of the corresponding matrix are arranged in an increasing order. If f is in addition a monotone function, then, utilizing Theorem 7.11, we obtain the corresponding eigenvalue relation, that is,

$$\lambda\left(f\left(\sum_{i=1}^m p_iA_i\right)-\mu\nu_1 I_n\right) \ge \lambda\left(\sum_{i=1}^m p_i f(A_i)\right).$$

The same conclusion can be drawn for the weak majorization relation (7.41) and its eigenvalue counterpart (7.44).

Our next application of weak majorization inequalities deals with a version of the Jensen inequality due to Mercer [134]. Recall, if $f : [\alpha,\beta] \to \mathbb{R}$ is a convex function, then

$$f\big(\alpha+\beta-\sum_{i=1}^m p_ix_i\big) \le f(\alpha)+f(\beta)-\sum_{i=1}^m p_i f(x_i), \tag{7.48}$$

where $\sum_{i=1}^m p_i = 1$, $p_i \ge 0$, and $x_i \in [\alpha,\beta]$, $i=1,2,\ldots,m$. This inequality has been derived as a consequence of the original Jensen inequality (see Chapter 1 for details). Having in mind this fact, we derive here the weak majorization inequality that corresponds to (7.48).

Corollary 7.3 *Suppose that $f : [\alpha,\beta] \to \mathbb{R}$ is a continuous convex function and let $A_1,A_2,\ldots,A_m \in \mathscr{H}_n([\alpha,\beta])$. If $\sum_{i=1}^m p_i = 1$, $p_i \ge 0$, $i=1,2,\ldots,m$, then*

$$\lambda\left(f\left((\alpha+\beta)I_n-\sum_{i=1}^m p_iA_i\right)+\mu\eta I_n\right) \prec_w \lambda\left(\big(f(\alpha)+f(\beta)\big)I_n-\sum_{i=1}^m p_i f(A_i)\right), \tag{7.49}$$

where $\mu = \min\{p_1, p_2, \ldots, p_m\}$ *and*

$$\eta = \min_{\substack{\|u\|=1 \\ u\in\mathbb{C}^n}} \sum_{i=1}^{m} f\left(\alpha+\beta-\langle A_i u, u\rangle\right) - mf\left(\alpha+\beta-\langle \frac{1}{m}\Big(\sum_{i=1}^{m} A_i\Big)u, u\rangle\right).$$

Proof. Rewriting relation (7.38) with operators $(\alpha+\beta)I_n - A_i$ instead of A_i, $i = 1, 2, \ldots, m$, we obtain

$$\lambda\left(f\left((\alpha+\beta)I_n - \sum_{i=1}^{m} p_i A_i\right) + \mu\eta I_n\right) \prec_w \lambda\left(\sum_{i=1}^{m} p_i f\big((\alpha+\beta)I_n - A_i\big)\right). \tag{7.50}$$

On the other hand, since f is convex on $[\alpha, \beta]$, it follows that

$$f(\alpha+\beta-x) \le f(\alpha) + f(\beta) - f(x), \quad x \in [\alpha, \beta],$$

and consequently,

$$p_i f\big((\alpha+\beta)I_n - A_i\big) \le p_i\big(f(\alpha)+f(\beta)\big)I_n - p_i f(A_i), \quad i = 1, 2, \ldots, m.$$

Here, $\le$ means the operator order. Now, adding these m inequalities, we have

$$\sum_{i=1}^{m} p_i f\big((\alpha+\beta)I_n - A_i\big) \le \left(f(\alpha)+f(\beta)\right) I_n - \sum_{i=1}^{m} p_i f(A_i),$$

that is,

$$\lambda\left(\sum_{i=1}^{m} p_i f\big((\alpha+\beta)I_n - A_i\big)\right) \le \lambda\left(\big(f(\alpha)+f(\beta)\big)I_n - \sum_{i=1}^{m} p_i f(A_i)\right), \tag{7.51}$$

by the Weyl monotonicity principle. Now, relations (7.50) and (7.51) entail the inequality (7.49). □

If f is in addition a non-negative function, Corollary 7.3 entails the corresponding inequality for unitarily invariant norms, while in the case of a monotone function f the relation (7.49) becomes the eigenvalue inequality. These relations are omitted here.

7.2.2 Applications to log convex functions

A positive function $f : J \to \mathbb{R}$ is called log convex if

$$f(tx + (1-t)y) \le f(x)^t f(y)^{1-t},$$

for all $0 \le t \le 1$ and $x, y \in J$. Obviously, f is a log convex function if and only if $\log f$ is a convex function.

By virtue of (7.32), Aujla and Silva [20] proved that if $f : J \to \mathbb{R}$ is a log convex function, $A, B \in \mathscr{H}_n(J)$, and $0 \le t \le 1$, then the eigenvalues of $f(tA + (1-t)B)$ are weakly majorized by the eigenvalues of $f(A)^t f(B)^{1-t}$, that is,

$$\lambda\left(f(tA + (1-t)B)\right) \prec_w \lambda\left(f(A)^t f(B)^{1-t}\right). \tag{7.52}$$

Taking into account Theorem 7.9, we can improve relation (7.52). In order to do this, we first cite the following lemma.

Lemma 7.3 (SEE [18]) *Let $A,B \in \mathscr{P}_n$. Then,*

$$\lambda\left(\log A + \log B\right) \prec_w \lambda\left(\log\left(A^{\frac{1}{2}} B A^{\frac{1}{2}}\right)\right).$$

The following result yields a general refinement of the inequality (7.52).

Theorem 7.12 *Let $f : J \to \mathbb{R}$ be a continuous log convex function. Then*

$$\lambda\left(f(tA + (1-t)B)\right) \prec_w \xi^{-\mu} \lambda\left(f(A)^t f(B)^{1-t}\right) \tag{7.53}$$

for all $A,B \in \mathscr{H}_n(J)$ and $0 \le t \le 1$, where $\mu = \min\{t, 1-t\}$ and

$$\xi = \min_{\substack{\|u\|=1 \\ u\in\mathbb{C}^n}} \frac{f(\langle Au,u\rangle) f(\langle Bu,u\rangle)}{\left[f(\langle \frac{A+B}{2}u,u\rangle)\right]^2}.$$

Proof. The function $\log f$ is convex on interval J. Therefore, by Theorem 7.9 and Lemma 7.3, we obtain

$$\begin{aligned}
\lambda\left(\log\left(\xi^{\mu} f(tA + (1-t)B)\right)\right) &= \lambda\left(\log f(tA + (1-t)B) + \mu \log \xi I_n\right) \\
&\prec_w \lambda\left(t \log f(A) + (1-t)\log f(B)\right) \\
&= \lambda\left(\log f(A)^t + \log f(B)^{1-t}\right) \\
&\prec_w \lambda\left(\log\left(f(A)^{\frac{t}{2}} f(B)^{1-t} f(A)^{\frac{t}{2}}\right)\right).
\end{aligned}$$

Now, since the function $x \to e^x$ is increasing and convex (see also [34, p. 42]), it follows that

$$\begin{aligned}
\lambda\left(\xi^{\mu} f(tA + (1-t)B)\right) &\prec_w \lambda\left(f(A)^{\frac{t}{2}} f(B)^{1-t} f(A)^{\frac{t}{2}}\right) \\
&= \lambda\left(f(A)^t f(B)^{1-t}\right),
\end{aligned}$$

which entails the relation (7.53). $\square$

By the Weyl Majorant Theorem (see [34, p. 42]) it follows that $|\lambda(X)| \prec_w \lambda(|X|)$ holds for any $X \in \mathscr{M}_n$. Consequently, the Theorem 7.12 entails the corresponding inequality for unitarily invariant norms.

Corollary 7.4 *Let $f : J \to \mathbb{R}$ be a continuous log convex function and $A,B \in \mathscr{H}_n(J)$. If $0 \le t \le 1$, then the inequality*

$$|||f(tA + (1-t)B)||| \le \xi^{-\mu} |||f(A)^t f(B)^{1-t}|||$$

holds for any unitarily invariant norm $|||\cdot|||$, where the scalars μ and ξ are defined as in Theorem 7.12.

Our last application of Theorem 7.12 refers to the function $x \to x^{-r}$, $r \ge 0$. Clearly, this function is log convex on $\mathbb{R}^+$.

Corollary 7.5 *Let $A, B \in \mathscr{P}_n$ and $0 \le t \le 1$. Then*

$$\lambda\left((tA^{-1} + (1-t)B^{-1})^{-r}\right) \prec_w \xi_1^{-\mu} \lambda\left(A^{tr} B^{(1-t)r}\right), \tag{7.54}$$

where $r \ge 0$, $\mu = \min\{t, 1-t\}$, and

$$\xi_1 = \frac{1}{4^r} \min_{\substack{\|u\|=1 \\ u \in \mathbb{C}^n}} \frac{\langle (A+B)u, u\rangle^{2r}}{\langle Au, u\rangle^r \langle Bu, u\rangle^r}.$$

Proof. It follows from (7.53) by setting $f(x) = x^{-r}$, $r \ge 0$, and replacing A and B by A^{-1} and B^{-1} respectively. □

The relation (7.54) improves the corresponding result from [20]. In particular, putting $t = \frac{1}{2}$ and $r = 1$ in (7.54), we obtain the inequality

$$\lambda\left(\left(\frac{A^{-1} + B^{-1}}{2}\right)^{-1}\right) \prec_w \xi_2^{-\frac{1}{2}} \lambda\left(A^{\frac{1}{2}} B^{\frac{1}{2}}\right),$$

where

$$\xi_2 = \frac{1}{4} \min_{\substack{\|u\|=1 \\ u \in \mathbb{C}^n}} \frac{\langle (A+B)u, u\rangle^2}{\langle Au, u\rangle \langle Bu, u\rangle}.$$

The previous relation represents a more accurate form of a harmonic-geometric mean inequality.

Chapter 8

The converse Jensen inequality: variants, improvements and generalizations

In this chapter, various variants of the converse Jensen inequality are studied. The Lah-Ribarič inequality, as the most important converse Jensen's inequality, motivates the first group of its accompanied generalizations and improvements. Generalizations are obtained for positive linear functionals and furthermore, on convex hulls and on k-simplices. The main tool for improvements in this part is Lemma 1.2. All of these results were published in [102]. From these results are then derived k-dimensional variant of the Hammer-Bullen inequality and an improvement of the classical Hermite-Hadamard inequality. Another important variant of the converse Jensen inequality is the Giaccardi-Petrović inequality. Its improvements are also derived by means of Lemma 1.2, as well as from some previous improvements. Motivated by the obtained results, two functionals (the Giaccardi-Petrović differences) are defined and for those are Lagrange and Cauchy type mean value theorems proved. A large family of n-exponentially convex and exponentially convex functions is constructed. Most of the results regarding the Giaccardi-Petrović inequality correspond to the contents of the paper [173].

Finally, we present results from paper [175], where the generalizations and the improvements of the converse Hölder and Minkowski inequalities are studied, with their accompanied applications to mixed means.

8.1 Variants of the converse Jensen inequality

Strongly related to Jensen's inequality is the converse Jensen inequality. As it was described (Theorem 1.10) in the introductory chapter, one of its most important variants is the Lah-Ribarič inequality, which for a real convex function defined on an interval $[a,b]$ and for $x_i \in [a,b], p_i \geq 0, i = 1,\ldots,n$, with $\sum_{i=1}^{n} p_i = 1$, claims that

$$\sum_{i=1}^{n} p_i f(x_i) \leq \frac{b - \sum_{i=1}^{n} p_i x_i}{b-a} f(a) + \frac{\sum_{i=1}^{n} p_i x_i - a}{b-a} f(b), \tag{8.1}$$

where (8.1) is strict for a strictly convex function f, unless $x_i \in \{a,b\}$, $i \in \{j : p_j > 0\}$.

Another related result can be found in [154] (see also [151, p. 690]) where the authors proved the following theorem.

Theorem 8.1 *Let $x = (x_1,\ldots,x_n)$ be an n-tuple in I^n, $p = (p_1,\ldots,p_n)$ a nonnegative n-tuple such that $P_n = \sum_{i=1}^{n} p_i > 0$, $m = \min\{x_1,\ldots,x_n\}$ and $M = \max\{x_1,\ldots,x_n\}$. If $f : I \to \mathbb{R}$ is differentiable and f' strictly increasing, then the inequalities*

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \leq \lambda + f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \tag{8.2}$$

hold, where

$$\begin{aligned}\lambda \ = \ & \frac{f(M) - f(m)}{M-m}\left(f'\right)^{-1}\left(\frac{f(M)-f(m)}{M-m}\right) \\ & + \frac{Mf(m) - mf(M)}{M-m} - f\left(\left(f'\right)^{-1}\left(\frac{f(M)-f(m)}{M-m}\right)\right).\end{aligned}$$

In the sequel, we again observe the environment of E being a nonempty set and L a linear class of functions $f : E \to \mathbb{R}$ which possesses the following properties:

L1: If $f,g \in L$, then $\alpha f + \beta g \in L$, for all α, $\beta \in \mathbb{R}$;

L2: $1 \in L$, that is, if $f(x) = 1$, $x \in E$, then $f \in L$.

We consider positive linear functionals $A : L \to \mathbb{R}$, or in other words we assume:

A1: $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$, for $f,g \in L$ and $\alpha,\beta \in \mathbb{R}$;

A2: If $f(x) \geq 0$ for all $x \in E$, then $A(f) \geq 0$.

If additionally the condition

A3: $A(1) = 1$

is satisfied, we say that A is a normalized positive linear functional or that $A(f)$ is a linear mean on L.

In [30] (or see [177, p. 98]), Beesack and Pečarić gave the following generalization of the Lah-Ribarič inequality, which involves positive normalized linear functionals, leaning on the appropriate variant of Jensen's inequality, i.e. Jessen's inequality (1.15).

Theorem 8.2 *Let L and A be as in Theorem 1.11. If $\phi : [m,M] \to \mathbb{R}$ is a convex function, then for all $g \in L$ such that $\phi(g) \in L$ the inequality*

$$A(\phi(g)) \leq \frac{M - A(g)}{M - m}\phi(m) + \frac{A(g) - m}{M - m}\phi(M) \tag{8.3}$$

holds.

Remark 8.1 The right hand side of (8.3) is an increasing function of M and a decreasing function of m. This follows by writing it in the form

$$\phi(m) + (A(g) - m)\frac{\phi(M) - \phi(m)}{M - m} = \phi(M) - (M - A(g))\frac{\phi(M) - \phi(m)}{M - m}$$

and noting that $m \leq A(g) \leq M$, while both functions $m \mapsto (\phi(M) - \phi(m))/(M - m)$ and $M \mapsto (\phi(M) - \phi(m))/(M - m)$ are increasing by the convexity of ϕ.

In the same paper [30] (or see [177, p. 100–101]), the authors also proved the following theorem.

Theorem 8.3 *Let L, A and g be as in Theorem 1.11 and let $\phi : [m,M] \to \mathbb{R}$ be a differentiable function.*

(i) If ϕ' is strictly increasing on $[m,M]$ then

$$A(\phi(g)) \leq \lambda + \phi(A(g)) \tag{8.4}$$

for some λ satisfying $0 < \lambda < (M - m)(\mu - \phi'(m))$, where

$$\mu = \frac{\phi(M) - \phi(m)}{M - m}.$$

More precisely, λ may be determined as follows: Let $\tilde{x}$ be the (unique) solution of the equation $\phi'(x) = \mu$. Then

$$\lambda = \phi(m) + \mu(\tilde{x} - m) - \phi(\tilde{x})$$

satisfies (8.4).

(ii) If ϕ' is strictly decreasing on $[m,M]$ then

$$\phi(A(g)) \leq \lambda + A(\phi(g)) \tag{8.5}$$

for some λ satisfying $0 < \lambda < (M - m)(\phi'(m) - \mu)$, where μ is defined as in (i). More precisely, for $\tilde{x}$ defined as in (i) we have that

$$\lambda = \phi(\tilde{x}) - \phi(m) - \mu(\tilde{x} - m)$$

satisfies (8.5).

It can easily be verified that the right hand side of (8.2) from Theorem 8.1 can be obtained as a special case of (8.4): we just have to consider $E=[m,M]$, $L=\mathbb{R}^E=\{f|f\colon E\to\mathbb{R}\}$, $g=id_E$ and $A(f)=\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i)$, where $p_i\geq 0$, $x_i\in[m,M]$, $i=1,\dots,n$, and $P_n=\sum_{i=1}^n p_i\neq 0$. After a few simple steps we obtain the same λ. The left hand side of (8.2) is obviously a special case of (1.15).

In [32] (or see [177, p. 101]) Beesack and Pečarić gave the following generalization of Theorem 8.3, which at the same time presents a generalization of Knopp's inequality for convex functions (see [103]).

Theorem 8.4 *Let L and A be as in Theorem 1.11. Let $\phi\colon[m,M]\to\mathbb{R}$ be a convex function and J an interval in $\mathbb{R}$ such that $\phi([m,M])\subset J$. If $F\colon J\times J\to\mathbb{R}$ is increasing in the first variable then for all $g\in L$ such that $\phi(g)\in L$ the following inequality holds*

$$\begin{aligned} F(A(\phi(g)),\phi(A(g))) &\leq \max_{x\in[m,M]} F\left(\frac{M-x}{M-m}\phi(m)+\frac{x-m}{M-m}\phi(M),\phi(x)\right) \\ &= \max_{\theta\in[0,1]} F(\theta\phi(m)+(1-\theta)\phi(M),\phi(\theta m+(1-\theta)M)). \end{aligned} \tag{8.6}$$

Furthermore, the right-hand side of (8.6) *is an increasing function of M and a decreasing function of m.*

It is quite simple to prove that Theorem 8.3 is a special case of Theorem 8.4. Namely, if we apply Theorem 8.4 to ϕ which is differentiable and strictly convex on $[m,M]$ (at the end points $x=m$ and $x=M$ we may consider the right and the left derivatives respectively) in a few easy steps we obtain (8.4). Here we give the proof only for the first case (i); (ii) can be obtained in a similar way if in Theorem 8.4 we take $-F$ instead of F and consider ϕ which is differentiable and strictly concave.

If F is defined by $F(x,y)=x-y$, the inequality in (8.6) becomes

$$A(\phi(g))-\phi(A(g))\leq \max_{x\in[m,M]} f(x;m,M,\phi), \tag{8.7}$$

where $f:[m,M]\to\mathbb{R}$ is defined by

$$f(x):=f(x;m,M,\phi)=\frac{(M-x)\phi(m)+(x-m)\phi(M)}{M-m}-\phi(x).$$

Note that $f(m)=f(M)=0$ and

$$f'(x)=\frac{\phi(M)-\phi(m)}{M-m}-\phi'(x)=\mu-\phi'(x).$$

Since ϕ is strictly convex, f' is strictly decreasing on $[m,M]$ and the equation $f'(x)=0$ (that is, $\phi'(x)=\mu$) holds for a unique $x=\tilde{x}\in(m,M)$. It follows that $f(x)\geq 0$ for all $x\in[m,M]$ with equality for $x\in\{m,M\}$. Consequently, the maximum value on the right hand side of (8.7) is attained at $x=\tilde{x}$ and thus for

$$\begin{aligned} \lambda &= f(\tilde{x})=\frac{(M-\tilde{x})\phi(m)+(\tilde{x}-m)\phi(M)}{(M-m)}-\phi(\tilde{x}) \\ &= \phi(m)+\mu(\tilde{x}-m)-\phi(\tilde{x}) \end{aligned}$$

we have that

$$A(\phi(g)) \leq \lambda + \phi(A(g)).$$

The discrete version of Theorem 8.4 can be found in [151, Theorem 8, p. 9–10]. Some results of this type were considered in [74], where generalizations for positive linear operators were obtained. Further generalizations for positive operators are given in [147]. Some related results for convex functions of higher order can be found in [52], [54] and [55]. Recently, S. Ivelić and J. Pečarić [88] have obtained generalizations of Theorem 8.4 for convex functions defined on convex hulls.

Throughout this section, without further noticing, when using $[m,M]$, we assume that $-\infty < m < M < \infty$. We will also need to equip our linear class L with the additional *lattice* property:

L3: $(\forall f,g \in L)(\min\{f,g\} \in L \wedge \max\{f,g\} \in L)$.

Obviously, $\left(\mathbb{R}^E, \leq\right)$ (with the standard ordering) is a lattice. It can also be easily verified that a subspace $X \subseteq \mathbb{R}^E$ is a lattice if and only if $x \in X$ implies $|x| \in X$. This is a simple consequence of the fact that for every $x \in X$ the functions $|x|$, x^- and x^+ can be defined by

$$|x|(t) = |x(t)|, \quad x^+(t) = \max\{0, x(t)\}, \quad x^-(t) = -\min\{0, x(t)\}, \qquad t \in E,$$

and

$$x^+ + x^- = |x|, \qquad x^+ - x^- = x,$$
$$\min\{x,y\} = \frac{1}{2}(x+y-|x-y|), \quad \max\{x,y\} = \frac{1}{2}(x+y+|x-y|). \tag{8.8}$$

8.2 Improvements and generalizations of the Lah-Ribarič inequality

In order to prove our main result related to the Lah-Ribarič inequality, we make use of Lemma 1.2, which we, for the reader's convenience, cite again for $n=2$. From its described monotonicity property, which is the main tool for all improvements presented in the second part of the book, also follow the improvements of some related inequalities from the previous section.

Lemma 8.1 *Let ϕ be a convex function on D_ϕ, $x,y \in D_\phi$ and $p,q \in [0,1]$ such that $p+q=1$. Then*

$$\begin{aligned} &\min\{p,q\}\left[\phi(x)+\phi(y)-2\phi\left(\frac{x+y}{2}\right)\right] \\ \leq\ & p\phi(x)+q\phi(y)-\phi(px+qy) \\ \leq\ & \max\{p,q\}\left[\phi(x)+\phi(y)-2\phi\left(\frac{x+y}{2}\right)\right]. \end{aligned} \tag{8.9}$$

The following theorem [102] is an improvement of Theorem 8.2.

Theorem 8.5 *Let L satisfy L1, L2, L3 on a nonempty set E and let A be a positive normalized linear functional. If ϕ is a convex function on $[m,M]$, then for all $g \in L$ such that $\phi(g) \in L$ we have $A(g) \in [m,M]$ and*

$$A(\phi(g)) \leq \frac{M-A(g)}{M-m}\phi(m) + \frac{A(g)-m}{M-m}\phi(M) - A(\tilde{g})\delta_\phi, \tag{8.10}$$

where

$$\tilde{g} = \frac{1}{2}1 - \frac{\left|g - \frac{m+M}{2}1\right|}{M-m}, \quad \delta_\phi = \phi(m) + \phi(M) - 2\phi\left(\frac{m+M}{2}\right).$$

Proof. First note that $\phi(g) \in L$ also means that the composition $\phi(g)$ is well defined, hence $g(E) \subseteq [m,M]$. Now we have $m1 \leq g \leq M1$ and

$$m = A(m1) \leq A(g) \leq A(M1) = M.$$

Let the functions $p,q : [m,M] \to \mathbb{R}$ be defined by

$$p(x) = \frac{M-x}{M-m}, \quad q(x) = \frac{x-m}{M-m}.$$

For any $x \in [m,M]$ we can write

$$\phi(x) = \phi\left(\frac{M-x}{M-m}m + \frac{x-m}{M-m}M\right) = \phi(p(x)m + q(x)M).$$

By Lemma 8.1 we get

$$\phi(x) \leq p(x)\phi(m) + q(x)\phi(M) - \min\{p(x), q(x)\}\left[\phi(m) + \phi(M) - 2\phi\left(\frac{m+M}{2}\right)\right].$$

Let $g \in L$ be such that $\phi(g) \in L$. In this case we have $p(g)$, $q(g) \in L$ and applying A to the above inequality with $x \leftrightarrow g(x)$ we obtain

$$A(\phi(g)) \leq A(p(g))\phi(m) + A(q(g))\phi(M) - A(\tilde{g})\left[\phi(m) + \phi(M) - 2\phi\left(\frac{m+M}{2}\right)\right],$$

where the function $\tilde{g}$ is defined on E by

$$\tilde{g}(x) = (\min\{p(g), q(g)\})(x) = \frac{1}{2} - \frac{\left|g(x) - \frac{m+M}{2}\right|}{M-m}$$

and by $L3$ it belongs to L. Since p and q are linear functions, we have $A(p(g)) = p(A(g))$ and $A(q(g)) = q(A(g))$, hence

$$A(\phi(g)) \leq p(A(g))\phi(m) + q(A(g))\phi(M) - A(\tilde{g})\delta_\phi,$$

which is (8.10). $\square$

Remark 8.2 Obviously, if applied to an appropriate L, Theorem 8.5 is an improvement of Theorem 8.2, since under the required assumptions we have

$$A(\tilde{g})\,\delta_\phi = A\left(\frac{1}{2}1 - \frac{\left|g - \frac{m+M}{2}1\right|}{M-m}\right)\left(\phi(m) + \phi(M) - 2\phi\left(\frac{m+M}{2}\right)\right) \geq 0.$$

Corollary 8.1 *Let p be a nonnegative n-tuple with $P_n = \sum_{i=1}^{n} p_i \neq 0$ and $x \in [m,M]^n$. If $\phi : [m,M] \to \mathbb{R}$ is a convex function then*

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i \phi(x_i) \leq \frac{M-\overline{x}}{M-m}\phi(m) + \frac{\overline{x}-m}{M-m}\phi(M) - \frac{\delta_\phi}{P_n}\sum_{i=1}^{n} p_i\left(\frac{1}{2} - \frac{\left|x_i - \frac{m+M}{2}\right|}{M-m}\right), \quad (8.11)$$

where $\overline{x} = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i$ and δ_ϕ is defined as in Theorem 8.5.

Proof. If we consider $E = [m,M]$, $L = \mathbb{R}^{[m,M]}$, $g = id_E$, $A(f) = \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i)$, then the inequality in (8.10) becomes (8.11). □

We can use Theorem 8.5 to obtain refinements of some other inequalities mentioned previously. First we give an improvement of Theorem 8.4 in the special case of $F(x,y) = x - y$.

Theorem 8.6 *Under the assumptions of Theorem 8.5, the following inequality holds:*

$$\begin{aligned} & A(\phi(g)) - \phi(A(g)) \\ \leq\ & \max_{x\in[m,M]}\left\{\frac{M-x}{M-m}\phi(m) + \frac{x-m}{M-m}\phi(M) - \phi(x)\right\} - A(\tilde{g})\,\delta_\phi \\ =\ & \max_{\theta\in[0,1]}\{\theta\phi(m) + (1-\theta)\phi(M) - \phi(\theta m + (1-\theta)M)\} - A(\tilde{g})\,\delta_\phi, \end{aligned}$$

where $\tilde{g}$ and δ_ϕ are defined as in Theorem 8.5.

Proof. This is an immediate consequence of Theorem 8.5. The identity follows from the change of variable $\theta = (M-x)/(M-m)$, so that for $x \in [m,M]$ we have $\theta \in [0,1]$ and $x = \theta m + (1-\theta)M$. □

Next we give an improvement of Theorem 8.3. We will consider only the case when ϕ' is strictly increasing (and therefore is ϕ convex), since an analogous result for ϕ' being strictly decreasing can be obtained in a similar way.

Theorem 8.7 *Let L and A be as in Theorem 8.5. If $\phi : [m,M] \to \mathbb{R}$ is a differentiable function such that ϕ' is strictly increasing on $[m,M]$, then for all $g \in L$ such that $\phi(g) \in L$ the inequality*

$$A(\phi(g)) \leq \lambda + \phi(A(g)) - A(\tilde{g})\,\delta_\phi \quad (8.12)$$

holds for some λ satisfying $0 < \lambda < (M-m)(\mu - \phi'(m))$, where

$$\mu = \frac{\phi(M) - \phi(m)}{M-m}$$

and $\tilde{g}$, δ_ϕ *are defined as in Theorem 8.5.*

More precisely, λ *may be determined as follows: let* $\tilde{x}$ *be the (unique) solution of the equation* $\phi'(x) = \mu$. *Then*

$$\lambda = \phi(m) + \mu(\tilde{x} - m) - \phi(\tilde{x})$$

satisfies (8.12).

Proof. The proof follows by the same argument we used in the previous section, since by Theorem 8.6 we have

$$A(\phi(g)) - \phi(A(g)) \leq \max_{x \in [m,M]} f(x; m, M, \phi) - A(\tilde{g})\,\delta_\phi,$$

where f is defined as in (8.7). □

Another interesting consequence of Theorem 8.5 is the following Hadamard type inequality which is an improvement of a result from [31].

Theorem 8.8 *Let L and A be as in Theorem 8.5. If* $\phi : [m,M] \to \mathbb{R}$ *is a continuous convex function, then for all* $g \in L$ *such that* $\phi(g) \in L$ *the inequalities*

$$\phi\left(\frac{pm + qM}{p+q}\right) \leq A(\phi(g)) \leq \frac{p\phi(m) + q\phi(M)}{p+q} - A(\tilde{g})\,\delta_\phi \tag{8.13}$$

hold, where p and q are any nonnegative real numbers such that

$$A(g) = \frac{pm + qM}{p+q} \tag{8.14}$$

and $\tilde{g}$, δ_ϕ *are defined as in Theorem 8.5.*

Proof. First note that $\phi(g) \in L$ implies $A(g) \in [m,M]$. Hence there exist a unique nonnegative real number $\lambda \in [0,1]$ such that $A(g) = \lambda m + (1-\lambda)M$. If p, q are nonnegative real numbers satisfying (8.14) then obviously

$$\frac{p}{p+q} = \lambda, \quad \frac{q}{p+q} = 1 - \lambda.$$

From Theorem 1.11 we have

$$\phi\left(\frac{pm + qM}{p+q}\right) = \phi(A(g)) \leq A(\phi(g)),$$

which is the first inequality in (8.13).

Since

$$\frac{M - A(g)}{M - m}\phi(m) + \frac{A(g) - m}{M - m}\phi(M) = \frac{p}{p+q}\phi(m) + \frac{q}{p+q}\phi(M),$$

by (8.10) we obtain

$$A(\phi(g)) \leq \frac{p\phi(m)+q\phi(M)}{p+q} - A(\tilde{g})\,\delta_\phi,$$

which is the second inequality in (8.13). □

As a corollary of Theorem 8.8 we obtain a result from [46] where it was proved that trapezoid rule produces a bigger error than the mid-point rule.

Corollary 8.2 *If $\phi:[m,M]\to\mathbb{R}$ is a continuous convex function, then the inequalities*

$$\begin{aligned}&\frac{\phi(m)+\phi(M)}{2}-\frac{1}{M-m}\int_m^M \phi(x)\,\mathrm{d}x \\ &\geq \frac{1}{M-m}\int_m^M \phi(x)\,\mathrm{d}x-\phi\left(\frac{m+M}{2}\right)\geq 0\end{aligned} \tag{8.15}$$

hold.

Proof. This is a special case of Theorem 8.8 attained for $E=[m,M]$, $L=C([m,M])$, $g=id_E$, $A(f)=\frac{1}{M-m}\int_m^M f(x)\,\mathrm{d}x$, $p=q=1$. We obtain

$$\begin{aligned}\phi\left(\frac{m+M}{2}\right) &\leq \frac{1}{M-m}\int_m^M \phi(x)\,\mathrm{d}x \\ &\leq \frac{\phi(m)+\phi(M)}{2}-\frac{\delta_\phi}{M-m}\int_m^M \tilde{g}(x)\,\mathrm{d}x.\end{aligned} \tag{8.16}$$

A simple calculation gives

$$\frac{1}{M-m}\int_m^M \tilde{g}(x)\,\mathrm{d}x=\frac{1}{4},$$

hence

$$\begin{aligned}&\frac{\phi(m)+\phi(M)}{2}-\frac{\delta_\phi}{M-m}\int_m^M \tilde{g}(x)\,\mathrm{d}x \\ &=\frac{\phi(m)+\phi(M)}{2}-\frac{1}{4}\left[\phi(m)+\phi(M)-2\phi\left(\frac{m+M}{2}\right)\right] \\ &=\frac{1}{2}\phi\left(\frac{m+M}{2}\right)+\frac{\phi(m)+\phi(M)}{4}.\end{aligned} \tag{8.17}$$

From (8.16) and (8.17) we obtain

$$\begin{aligned}2\phi\left(\frac{m+M}{2}\right) &\leq \frac{2}{M-m}\int_m^M \phi(x)\,\mathrm{d}x \\ &\leq \phi\left(\frac{m+M}{2}\right)+\frac{\phi(m)+\phi(M)}{2},\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\phi\left(\frac{m+M}{2}\right)-\frac{1}{M-m}\int_m^M \phi(x)\,\mathrm{d}x\\
\le\; &\frac{1}{M-m}\int_m^M \phi(x)\,\mathrm{d}x-\phi\left(\frac{m+M}{2}\right)\\
\le\; &\frac{\phi(m)+\phi(M)}{2}-\frac{1}{M-m}\int_m^M \phi(x)\,\mathrm{d}x.
\end{aligned}$$

By the Hermite-Hadamard inequality for convex functions we know that

$$0\le \frac{1}{M-m}\int_m^M \phi(x)\,\mathrm{d}x-\phi\left(\frac{m+M}{2}\right),$$

hence (8.15) is proved. □

What follows are some further applications of our main result.

Corollary 8.3 *Let L and A be as in Theorem 8.5. If $g\in L$ is such that $\log g$ belongs to L and $g(E)\subseteq[m,M]\subset\mathbb{R}_+$, then*

$$A(g)\le \exp(A(\log g))\frac{\exp S\left(\frac{M}{m}\right)}{\left[\frac{(m+M)^2}{4mM}\right]^{A(\tilde{g})}}, \tag{8.18}$$

where $S(\cdot)$ is Specht ratio and $\tilde{g}$ is defined as in Theorem 8.5.

Proof. This is a special case of Theorem 8.7 for $\phi=-\log$. In this case (8.12) becomes

$$-A(\log g)\le \lambda-\log A(g)-A(\tilde{g})\,\delta_{-\log},$$

that is,

$$\begin{aligned}
\exp\log A(g) &= A(g)\le \exp\left(A(\log g)+\lambda-A(\tilde{g})\,\delta_{-\log}\right)\\
&= \exp(A(\log g))\frac{\exp\lambda}{\exp\left(A(\tilde{g})\,\delta_{-\log}\right)},
\end{aligned}$$

where

$$\delta_{-\log}=-\log m-\log M+2\log\frac{m+M}{2}=\log\frac{(m+M)^2}{4mM},$$

$$\mu=\frac{\log m-\log M}{M-m},\quad \tilde{x}=-\frac{1}{\mu}=\frac{M-m}{\log M-\log m},$$

hence

$$\begin{aligned}
\lambda &= -\log m+\mu(\tilde{x}-m)+\log\tilde{x}\\
&= \log\frac{\left(\frac{M}{m}\right)^{\frac{m}{M-m}}}{e\log\left(\frac{M}{m}\right)^{\frac{m}{M-m}}}=S\left(\frac{M}{m}\right),
\end{aligned}$$

where $S(\cdot)$ is Specht ratio (see for example [74, p. 71]), defined by

$$S(h) = \frac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}}, \quad h \in \mathbb{R}_+ \setminus \{1\}.$$

Considering all this, we obtain (8.18). □

Corollary 8.4 *Let L and A be as in Theorem 8.5. If $p \in L$ is such that $\log(p)$ belongs to L and $p(E) \subseteq [m,M] \subset \mathbb{R}_+$, then*

$$A(p) \leq \exp A(\log p) + \frac{M-m}{\log \frac{M}{m}} S\left(\frac{M}{m}\right) - A(\tilde{p})\left(m+M-2\sqrt{mM}\right), \tag{8.19}$$

where $S(\cdot)$ is Specht ratio and $\tilde{p}$ is defined by

$$\tilde{p} = \frac{1}{2}1 - \frac{\left|\log p - \log\sqrt{mM}1\right|}{\log M - \log m}.$$

Proof. This is a special case of Theorem 8.7 for $\phi = \exp$ and $g = \log p$. In this case (8.12) becomes

$$A(\exp\log p) \leq \lambda + \exp A(\log p) - A(\tilde{p})\,\delta_{\exp},$$

where

$$\delta_{\exp} = \exp\log m + \exp\log M - 2\exp\frac{\log m + \log M}{2} = m + M - 2\sqrt{mM},$$

$$\mu = \frac{M-m}{\log M - \log m}, \quad \tilde{x} = \log\mu = \log\frac{M-m}{\log M - \log m},$$

hence

$$\begin{aligned}
\lambda &= \exp\log m + \mu(\tilde{x} - \log m) - \exp\tilde{x} \\
&= m + \frac{M-m}{\log M - \log m}\left(\log\frac{M-m}{\log M - \log m} - \log m - 1\right) \\
&= \frac{M-m}{\log \frac{M}{m}} S\left(\frac{M}{m}\right).
\end{aligned}$$

Considering all this we obtain (8.19). □

Now we give generalizations and improvements of Lah-Ribarič and related inequalities for convex functions on convex hulls in $\mathbb{R}^k$ and, analogously, for convex functions on k-simplices in $\mathbb{R}^k$. We also verify that this is a generalization and an improvement of the Hermite-Hadamard inequality for simplices and generalization of the Hammer-Bullen inequality.

Theorem 8.9 *Let U be a convex subset of $\mathbb{R}^k$ and $n \in \mathbb{N}$. If $f\colon U \to \mathbb{R}$ is a convex function, $x_1,\ldots,x_n \in U$ and $p_1,\ldots,p_n$ nonnegative real numbers with $P_n = \sum_{i=1}^n p_i > 0$, then Jensen's inequality*

$$f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) \le \frac{1}{P_n}\sum_{i=1}^n p_i f(x_i) \tag{8.20}$$

holds.

At this point, the reader is referred to Section 1.1, in order to recall the notions of a convex hull, k-simplex and barycentric coordinates, which define the new environment in the sequel.

With L^k we denote the linear class of functions $g\colon E \to \mathbb{R}^k$ defined by

$$g(t) = (g_1(t),\ldots,g_k(t)), \quad g_i \in L, \quad i = 1,\ldots,k.$$

For a given linear functional A, we also consider linear operator $\widetilde{A} = (A,\ldots,A)\colon L^k \to \mathbb{R}^k$ defined by

$$\widetilde{A}(g) = (A(g_1),\ldots,A(g_k)) \tag{8.21}$$

If $A3$ is satisfied, then using $A1$ we also have:

A4: $A(f(g)) = f\left(\widetilde{A}(g)\right)$ for every linear function f on $\mathbb{R}^k$.

Remark 8.3 If we choose $F(x,y) = x - y$, as a simple consequence of Theorem 8.4 it follows

$$A(f(g)) - f(A(g)) \le \max_{\theta\in[0,1]} \left[\theta f(m) + (1-\theta)f(M) - f(\theta m + (1-\theta)M)\right]. \tag{8.22}$$

Taking $F(x,y) = \frac{x}{y}$, for $f > 0$, it follows

$$\frac{A(f(g))}{f(A(g))} \le \max_{\theta\in[0,1]} \left[\frac{\theta f(m) + (1-\theta)f(M)}{f(\theta m + (1-\theta)M)}\right]. \tag{8.23}$$

As already mentioned in the introductory part, (see Theorem 1.12), McShane proved an additional generalization of Jessen's inequality (1.15). According to the above described setting, this theorem states that for a continuous convex function f defined on a closed convex set $U \subset \mathbb{R}^k$ and for all $g \in L^k$ such that $g(E) \subset U$ and $f(g) \in L$, with A being a positive normalized linear functional on L and $\widetilde{A}$ defined as in (8.21), the following inequality holds:

$$f(\widetilde{A}(g)) \le A(f(g)). \tag{8.24}$$

J. Pečarić and S. Ivelić proved in [88] the following generalization of Theorem 8.2.

Theorem 8.10 *Let L satisfy properties $L1$ and $L2$ on a nonempty set E and A be a positive normalized linear functional on L. Let $x_1,\ldots,x_n \in \mathbb{R}^k$ and $K = co(\{x_1,\ldots,x_n\})$. Let f be* *a convex function on K and $\lambda_1,\dots,\lambda_n$ barycentric coordinates over K. Then for all $g\in L^k$ such that $g(E)\subset K$ and $f(g)$, $\lambda_i(g)\in L$, $i=1,\dots,n$ we have*

$$A(f(g))\leq\sum_{i=1}^{n}A(\lambda_i(g))\,f(x_i)\,.$$

We give generalizations and improvements of theorems 8.4 and 8.10 which will be obtained using the following lemma, which is a generalization of Lemma 1.2 on convex sets.

Lemma 8.2 *Let ϕ be a convex function on U where U is a convex set in $\mathbb{R}^k$, $(x_1,\dots,x_n)\in U^n$ and $p=(p_1,\dots,p_n)$ be nonnegative n-tuple such that $\sum_{i=1}^{n}p_i=1$. Then*

$$\begin{aligned}&\min\{p_1,\dots,p_n\}\left[\sum_{i=1}^{n}\phi(x_i)-n\phi\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)\right]\\ &\leq\sum_{i=1}^{n}p_i\phi(x_i)-\phi\left(\sum_{i=1}^{n}p_ix_i\right)\\ &\leq\max\{p_1,\dots,p_n\}\left[\sum_{i=1}^{n}\phi(x_i)-n\phi\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)\right].\end{aligned}$$

For $n\in\mathbb{N}$ we denote

$$\Delta_{n-1}=\left\{(\mu_1,\dots,\mu_n):\ \mu_i\geq 0, i\in\{1,\dots,n\},\sum_{i=1}^{n}\mu_i=1\right\}.$$

If f is a function defined on a convex subset $U\subseteq\mathbb{R}^k$ and $x_1,x_2,\dots,x_n\in U$, we denote

$$S_f^n(x_1,\dots,x_n)=\sum_{i=1}^{n}f(x_i)-nf\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right).$$

Obviously, if f is convex, $S_f^n(x_1,\dots,x_n)\geq 0$.

The following theorem presents an improvement of Theorem 8.10.

Theorem 8.11 *Let L satisfy properties L1, L2 and L3 on a nonempty set E and A be a positive normalized linear functional on L. Let $x_1,\dots,x_n\in\mathbb{R}^k$ and $K=co(\{x_1,\dots,x_n\})$. Let f be a convex function on K and $\lambda_1,\dots,\lambda_n$ barycentric coordinates over K. Then for all $g\in L^k$ such that $g(E)\subset K$ and $f(g)$, $\lambda_i(g)\in L$, $i=1,\dots,n$, we have*

$$A(f(g))\leq\sum_{i=1}^{n}A(\lambda_i(g))\,f(x_i)-A(\min\{\lambda_i(g)\})\,S_f^n(x_1,\dots,x_n).\tag{8.25}$$

Proof. For each $t \in E$ we have $g(t) \in K$. Using barycentric coordinates we have $\lambda_i(g(t)) \geq 0$, $i = 1, \ldots, n$, $\sum_{i=1}^n \lambda_i(g(t)) = 1$ and

$$g(t) = \sum_{i=1}^n \lambda_i(g(t))x_i.$$

Since f is convex, we can apply Lemma 8.2, and then

$$\begin{aligned} f(g(t)) &= f\left(\sum_{i=1}^n \lambda_i(g(t))x_i\right) \\ &\leq \sum_{i=1}^n \lambda_i(g(t))f(x_i) - \min\{\lambda_i(g(t))\}\left[\sum_{i=1}^n f(x_i) - nf\left(\frac{1}{n}\sum_{i=1}^n x_i\right)\right]. \end{aligned} \tag{8.26}$$

Now, applying the functional A to (8.26), we get

$$\begin{aligned} A(f(g)) &\leq A\left(\sum_{i=1}^n \lambda_i(g)f(x_i) - \min\{\lambda_i(g)\}S_f^n(x_1,\ldots,x_n)\right) \\ &= \sum_{i=1}^n A(\lambda_i(g))f(x_i) - A(\min\{\lambda_i(g)\})S_f^n(x_1,\ldots,x_n). \end{aligned}$$

□

Remark 8.4 Theorem 8.11 is an improvement of Theorem 8.10, since under the required assumptions we have

$$A(\min\{\lambda_i(g)\})S_f^n(x_1,\ldots,x_n) \geq 0.$$

Remark 8.5 If all the assumptions of Theorem 8.11 are satisfied and additionally is f a continuous function, then

$$f(\widetilde{A}(g)) \leq A(f(g)) \leq \sum_{i=1}^n A(\lambda_i(g))f(x_i) - A(\min\{\lambda_i(g)\})S_f^n(x_1,\ldots,x_n),$$

where the first inequality is actually McShane's inequality (8.24) and the second one is the statement of Theorem 8.11.

Remark 8.6 We know that under the assumptions of Theorem 8.11 we have

$$A(f(g)) \leq \sum_{i=1}^n A(\lambda_i(g))f(x_i) - A(\min\{\lambda_i(g)\})S_f^n(x_1,\ldots,x_n).$$

Dividing by $f(g(t)) = f\left(\sum_{i=1}^n \lambda_i(g(t))x_i\right)$, when $f > 0$, we obtain

$$\begin{aligned} \frac{A(f(g))}{f\left(\widetilde{A}(g)\right)} &\leq \frac{\sum_{i=1}^n A(\lambda_i(g))f(x_i)}{f(\sum_{i=1}^n A(\lambda_i(g))x_i)} - \frac{A(\min\{\lambda_i(g) \colon i=1,\ldots,n\})}{f\left(\widetilde{A}(g)\right)}S_f^n(x_1,\ldots,x_n) \\ &\leq \max_{\Delta_{n-1}} \frac{\sum_{i=1}^n \mu_i f(x_i)}{f(\sum_{i=1}^n \mu_i x_i)} - \frac{A(\min\{\lambda_i(g) \colon i=1,\ldots,n\})}{f\left(\widetilde{A}(g)\right)}S_f^n(x_1,\ldots,x_n), \end{aligned}$$

which is equivalent to

$$A\left(f\left(g\right)\right)\leq\max_{\Delta_{n-1}}\frac{\sum_{i=1}^{n}\mu_i f(x_i)}{f\left(\sum_{i=1}^{n}\mu_i x_i\right)}f\left(\widetilde{A}\left(g\right)\right)-A\left(\min\left\{\lambda_i(g)\colon i=1,\ldots,n\right\}\right)S_f^n(x_1,\ldots,x_n). \tag{8.27}$$

This is an improvement of the inequality (2.6) from [88].

Now, making use of Theorem 8.11 we are able to obtain a generalization and an improvement of Theorem 8.4.

Theorem 8.12 *Let L satisfy properties L1, L2 and L3 on a nonempty set E, A be a positive normalized linear functional on L and $\widetilde{A}$ defined as in (8.21). Let $x_1,\ldots,x_n\in\mathbb{R}^k$ and $K=co(\{x_1,\ldots,x_n\})$. Let f be a convex function on K and $\lambda_1,\ldots,\lambda_n$ barycentric coordinates over K. If J is an interval in $\mathbb{R}$ such that $f(K)\subset J$ and $F\colon J\times J\to\mathbb{R}$ is an increasing function in the first variable, then for all $g\in L^k$ such that $g(E)\subset K$ and $f(g),\lambda_i(g)\in L, i=1,\ldots,n$ the following inequalities hold:*

$$\begin{aligned}
&F\left(A(f(g)),f(\widetilde{A}(g))\right)\\
&\leq F\left(\sum_{i=1}^{n}A\left(\lambda_i(g)\right)f\left(x_i\right)-A\left(\min\left\{\lambda_i(g)\right\}\right)S_f^n(x_1,\ldots,x_n),f(\widetilde{A}(g))\right)\\
&\leq \max_{\Delta_{n-1}}F\left(\sum_{i=1}^{n}\mu_i f(x_i)-A\left(\min\left\{\lambda_i(g)\right\}\right)S_f^n(x_1,\ldots,x_n),f\left(\sum_{i=1}^{n}\mu_i x_i\right)\right).
\end{aligned} \tag{8.28}$$

Proof. For each $t\in E$ is $g(t)\in K$. Using barycentric coordinates it follows that $\lambda_i(g(t))\geq 0$, $i=1,\ldots,n$, $\sum_{i=1}^{n}\lambda_i(g(t))=1$ and

$$g(t)=\sum_{i=1}^{n}\lambda_i(g(t))x_i.$$

Since A is a positive normalized linear functional on L and $\widetilde{A}$ a linear operator on L^k, it follows that

$$\widetilde{A}\left(g\right)=\left(A(g_1),\ldots,A(g_k)\right)=\sum_{i=1}^{n}A\left(\lambda_i(g)\right)x_i,$$

where $A\left(\lambda_i(g)\right)\geq 0$, $i=1,\ldots,n$ and $\sum_{i=1}^{n}A\left(\lambda_i(g)\right)=A\left(\sum_{i=1}^{n}\lambda_i(g)\right)=A(1)=1$. Therefore, $\widetilde{A}\left(g\right)\in K$.

Since $F\colon J\times J\to\mathbb{R}$ is an increasing function in the first variable, using (8.25) we have

$$\begin{aligned}
&F\left(A(f(g)),f(\widetilde{A}(g))\right)\\
&\leq F\left(\sum_{i=1}^{n}A\left(\lambda_i(g)\right)f(x_i)-A\left(\min\left\{\lambda_i(g(t))\right\}\right)S_f^n(x_1,\ldots,x_n),f(\widetilde{A}(g))\right).
\end{aligned} \tag{8.29}$$

By substitutions

$$A(\lambda_i(g)) = \mu_i, i = 1, \ldots, n,$$

it follows

$$\widetilde{A}(g) = \sum_{i=1}^{n} \mu_i x_i.$$

Now we have

$$\begin{aligned}
&F\left(\sum_{i=1}^{n} A(\lambda_i(g)) f(x_i) - A(\min\{\lambda_i(g(t))\}) S_f^n(x_1,\ldots,x_n), f(\widetilde{A}(g))\right) \\
&= F\left(\sum_{i=1}^{n} \mu_i f(x_i) - A(\min\{\lambda_i(g(t))\}) S_f^n(x_1,\ldots,x_n), f\left(\sum_{i=1}^{n} \mu_i x_i\right)\right) \\
&\leq \max_{\Delta_{n-1}} F\left(\sum_{i=1}^{n} \mu_i f(x_i) - A(\min\{\lambda_i(g(t))\}) S_f^n(x_1,\ldots,x_n), f\left(\sum_{i=1}^{n} \mu_i x_i\right)\right).
\end{aligned}$$

By combining (8.29) and the last inequality we get (8.28). □

Remark 8.7 If we choose $F(x,y) = x - y$, as a simple consequence of Theorem 8.12 it follows

$$A(f(g)) - f(\widetilde{A}(g)) \leq \max_{\Delta_{n-1}} \left(\sum_{i=1}^{n} \mu_i f(x_i) - f\left(\sum_{i=1}^{n} \mu_i x_i\right) - A(\min\{\lambda_i(g)\}) S_f^n(x_1,\ldots,x_n)\right). \tag{8.30}$$

Taking $F(x,y) = \frac{x}{y}$, for $f > 0$, it follows

$$\frac{A(f(g))}{f(\widetilde{A}(g))} \leq \max_{\Delta_{n-1}} \left(\frac{\sum_{i=1}^{n} \mu_i f(x_i) - A(\min\{\lambda_i(g)\}) S_f^n(x_1,\ldots,x_n)}{f\left(\sum_{i=1}^{n} \mu_i x_i\right)}\right). \tag{8.31}$$

The inequalities (8.30) and (8.31) present generalizations and improvements of (8.22) and (8.23).

If we replace F by $-F$ in Theorem 8.12 we obtain the following result.

Theorem 8.13 *Let L satisfy properties L1, L2 and L3 on a nonempty set E, A be a positive normalized linear functional on L and $\widetilde{A}$ defined as in (8.21). Let $x_1, \ldots, x_n \in \mathbb{R}^k$ and $K = co(\{x_1, \ldots, x_n\})$. Let f be a convex function on K and $\lambda_1, \ldots, \lambda_n$ barycentric coordinates over K. If J is an interval in $\mathbb{R}$ such that $f(K) \subset J$ and $F\colon J \times J \to \mathbb{R}$ is a decreasing function in the first variable, then for all $g \in L^k$ such that $g(E) \subset K$ and $f(g), \lambda_i(g) \in L, i = 1, \ldots, n$ the following inequalities hold:*

$$\begin{aligned}
&F\left(A(f(g)), f(\widetilde{A}(g))\right) \\
&\geq F\left(\sum_{i=1}^{n} A(\lambda_i(g)) f(x_i) - A(\min\{\lambda_i(g)\}) S_f^n(x_1,\ldots,x_n), f(\widetilde{A}(g))\right) \\
&\geq \min_{\Delta_{n-1}} F\left(\sum_{i=1}^{n} \mu_i f(x_i) - A(\min\{\lambda_i(g)\}) S_f^n(x_1,\ldots,x_n), f\left(\sum_{i=1}^{n} \mu_i x_i\right)\right).
\end{aligned} \tag{8.32}$$

Now we consider the special case where the convex hull is a k-simplex.

Let S be a k-simplex in $\mathbb{R}^k$ with vertices $v_1, v_2, \ldots, v_{k+1} \in \mathbb{R}^k$. The barycentric coordinates $\lambda_1, \ldots \lambda_{k+1}$ over S are nonnegative linear polynomials which satisfy Lagrange's property

$$\lambda_i(v_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

It is known (see [33]) that for each $x \in S$ barycentric coordinates $\lambda_1(x), \ldots, \lambda_{k+1}(x)$ have the form

$$\begin{aligned}
\lambda_1(x) &= \frac{\mathrm{Vol}_k([x, v_2, \ldots, v_{k+1}])}{\mathrm{Vol}_k(S)}, \\
\lambda_2(x) &= \frac{\mathrm{Vol}_k([v_1, x, v_3, \ldots, v_{k+1}])}{\mathrm{Vol}_k(S)}, \\
&\vdots \\
\lambda_{k+1}(x) &= \frac{\mathrm{Vol}_k([v_1, \ldots, v_k, x])}{\mathrm{Vol}_k(S)},
\end{aligned} \tag{8.33}$$

where $\mathrm{Vol}_k(F)$ denotes the k-dimensional Lebesgue measure of a measurable set $F \subset \mathbb{R}^k$. Here, for example, $[v_1, x, \ldots, v_{k+1}]$ denotes the subsimplex obtained by replacing v_2 by x, i.e. the subsimplex opposite to v_2, when adding x as a new vertex.

The signed volume $\mathrm{Vol}_k(S)$ is given by $(k+1) \times (k+1)$ determinant

$$\mathrm{Vol}_k(S) = \frac{1}{k!} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ v_{11} & v_{21} & & v_{k+11} \\ v_{12} & v_{22} & & v_{k+12} \\ \vdots & \vdots & & \vdots \\ v_{1k} & v_{2k} & \cdots & v_{k+1k} \end{vmatrix},$$

where $v_1 = (v_{11}, v_{12}, \ldots, v_{1k}), \ldots, v_{k+1} = (v_{k+11}, v_{k+12}, \ldots, v_{k+1k})$ (see [193]).

Since vectors $v_2 - v_1, \ldots, v_{k+1} - v_1$ are linearly independent, then each $x \in S$ can be written as a convex combination of $v_1, \ldots, v_{k+1}$ in the form

$$x = \frac{\mathrm{Vol}_k([x, v_2, \ldots, v_{k+1}])}{\mathrm{Vol}_k(S)} v_1 + \cdots + \frac{\mathrm{Vol}_k([v_1, \ldots, v_k, x])}{\mathrm{Vol}_k(S)} v_{k+1}. \tag{8.34}$$

Now we present an analogue of Theorem 8.11 for convex functions defined on k-simplices in $\mathbb{R}^k$.

Theorem 8.14 *Let L satisfy properties L1, L2 and L3 on a nonempty set E, A be a positive normalized linear functional on L and $\widetilde{A}$ defined as in (8.21). Let f be a convex function on a k-simplex $S = [v_1, v_2, \ldots, v_{k+1}]$ in $\mathbb{R}^k$ and $\lambda_1, \ldots, \lambda_{k+1}$ be barycentric coordinates over*

S. Then for all $g \in L^k$ such that $g(E) \subset S$ and $f(g) \in L$ we have

$$
\begin{aligned}
A(f(g)) &\leq \sum_{i=1}^{k+1} A(\lambda_i(g)) f(v_i) - A(\min\{\lambda_i(g)\}) S_f^{k+1}(v_1,\ldots,v_{k+1}) \\
&= \frac{\mathrm{Vol}_k\left(\left[\widetilde{A}(g), v_2, \ldots, v_{k+1}\right]\right)}{\mathrm{Vol}_k(S)} f(v_1) + \cdots + \frac{\mathrm{Vol}_k\left(\left[v_1, v_2, \ldots, \widetilde{A}(g)\right]\right)}{\mathrm{Vol}_k(S)} f(v_{k+1}) \\
&\quad - A(\min\{\lambda_i(g)\}) S_f^{k+1}(v_1,\ldots,v_{k+1}).
\end{aligned} \tag{8.35}
$$

Proof. Analogous to the proof of Theorem 8.11, with

$$
\lambda_1(g(t)) = \frac{\mathrm{Vol}_k([g(t), v_2, \ldots, v_{k+1}])}{\mathrm{Vol}_k(S)} = \frac{\frac{1}{k!}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ g_1(t) & v_{21} & & v_{k+11} \\ \vdots & \vdots & & \vdots \\ g_k(t) & v_{2k} & \cdots & v_{k+1k} \end{vmatrix}}{\frac{1}{k!}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ v_{11} & v_{21} & & v_{k+11} \\ \vdots & \vdots & & \vdots \\ v_{1k} & v_{2k} & \cdots & v_{k+1k} \end{vmatrix}},
$$

$$\vdots$$

$$
\lambda_{k+1}(g(t)) = \frac{\mathrm{Vol}_k([v_1, \ldots, v_k, g(t)])}{\mathrm{Vol}_k(S)} = \frac{\frac{1}{k!}\begin{vmatrix} 1 & \cdots & 1 & 1 \\ v_{11} & & v_{k1} & g_1(t) \\ \vdots & & \vdots & \vdots \\ v_{1k} & \cdots & v_{kk} & g_k(t) \end{vmatrix}}{\frac{1}{k!}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ v_{11} & v_{21} & & v_{k+11} \\ \vdots & \vdots & & \vdots \\ v_{1k} & v_{2k} & \cdots & v_{k+1k} \end{vmatrix}},
$$

and

$$
A(\lambda_1(g)) = \frac{\frac{1}{k!}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ A(g_1) & v_{21} & & v_{k+11} \\ \vdots & \vdots & & \vdots \\ A(g_k) & v_{2k} & \cdots & v_{k+1k} \end{vmatrix}}{\frac{1}{k!}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ v_{11} & v_{21} & & v_{k+11} \\ \vdots & \vdots & & \vdots \\ v_{1k} & v_{2k} & \cdots & v_{k+1k} \end{vmatrix}} = \frac{\mathrm{Vol}_k\left(\left[\widetilde{A}(g), v_2, \ldots, v_{k+1}\right]\right)}{\mathrm{Vol}_k(S)},
$$

$$\vdots \tag{8.36}$$

$$A\left(\lambda_{k+1}\left(g\right)\right)=\frac{\frac{1}{k!}\begin{vmatrix}1&\cdots&1&1\\v_{11}&&v_{k1}&A(g_1)\\\vdots&&\vdots&\vdots\\v_{1k}&\cdots&v_{kk}&A(g_k)\end{vmatrix}}{\frac{1}{k!}\begin{vmatrix}1&1&\cdots&1\\v_{11}&v_{21}&&v_{k+11}\\\vdots&\vdots&&\vdots\\v_{1k}&v_{2k}&\cdots&v_{k+1k}\end{vmatrix}}=\frac{\mathrm{Vol}_k\left(\left[v_1,\ldots,v_k,\widetilde{A}(g)\right]\right)}{\mathrm{Vol}_k(S)},$$

□

Using Theorem 8.14 we prove an analogue of Theorem 8.12 for k-simplices in $\mathbb{R}^k$.

Theorem 8.15 *Let L satisfy properties L1, L2 and L3 on a nonempty set E, A be a positive normalized linear functional on L and $\widetilde{A}$ defined as in (8.21). Let f be a convex function on a k-simplex $S=[v_1,v_2,\ldots,v_{k+1}]$ in $\mathbb{R}^k$ and $\lambda_1,\ldots,\lambda_{k+1}$ be barycentric coordinates over S. If J is an interval in $\mathbb{R}$ such that $f(S)\subset J$ and $F: J\times J\to\mathbb{R}$ an increasing function in the first variable, then for all $g\in L^k$ such that $g\left(E\right)\subset S$ and $f(g)\in L$ we have*

$$\begin{aligned}&F\left(A(f(g)),f(\widetilde{A}(g))\right) \qquad (8.37)\\&\leq \max_{x\in S}F\Big(\frac{\mathrm{Vol}_k\left([x,v_2,\ldots,v_{k+1}]\right)}{\mathrm{Vol}_k(S)}f(v_1)+\cdots+\frac{\mathrm{Vol}_k\left([v_1,\ldots,v_k,x]\right)}{\mathrm{Vol}_k(S)}f(v_{k+1})\\&\quad -A\left(\min\left\{\lambda_i(g)\right\}\right)S_f^{k+1}(v_1,\ldots,v_{k+1}),f\left(x\right)\Big)\\&=\max_{\Delta_k}F\left(\sum_{i=1}^{k+1}\mu_i f(v_i)-A\left(\min\left\{\lambda_i(g)\right\}\right)S_f^{k+1}(v_1,\ldots,v_{k+1}),f\left(\sum_{i=1}^{k+1}\mu_i v_i\right)\right).\end{aligned}$$

Proof. Analogous to the proof of Theorem 8.12, with substitutions

$$\mu_1=\frac{\mathrm{Vol}_k\left([x,v_2,\ldots,v_{k+1}]\right)}{\mathrm{Vol}_k(S)},\ldots,\mu_{k+1}=\frac{\mathrm{Vol}_k\left([v_1,\ldots,v_k,x]\right)}{\mathrm{Vol}_k(S)},$$

and

$$x=\sum_{i=1}^{k+1}\mu_i v_i.$$

□

Remark 8.8 If all the assumptions of Theorem 8.14 are satisfied and in addition f is continuous, then

$$\begin{aligned}f(\widetilde{A}(g))&\leq A(f(g))\\&\leq\sum_{i=1}^{k+1}A\left(\lambda_i(g)\right)f\left(v_i\right)-A\left(\min\left\{\lambda_i(g)\right\}\right)S_f^{k+1}(v_1,\ldots,v_{k+1})\end{aligned}$$

$$= \frac{\mathrm{Vol}_k\left(\left[\widetilde{A}(g), v_2, \ldots, v_{k+1}\right]\right)}{\mathrm{Vol}_k(S)} f(v_1) + \cdots + \frac{\mathrm{Vol}_k\left(\left[v_1, v_2, \ldots, \widetilde{A}(g)\right]\right)}{\mathrm{Vol}_k(S)} f(v_{k+1})$$
$$-A\left(\min\{\lambda_i(g)\}\right) S_f^{k+1}(v_1, \ldots, v_{k+1}). \tag{8.38}$$

The first inequality is McShane's (8.24) and the second one is Theorem 8.14.

Example 8.1 Let $S = [v_1, v_2, \ldots, v_{k+1}]$ be a k-simplex in $\mathbb{R}^k$ and f a continuous convex function on S. Let $(E, \mathscr{A}, \lambda)$ be a measure space with positive measure λ such that $\lambda(E) < \infty$. Let L be a linear class of measurable real functions on E. We define the functional $A\colon L \to \mathbb{R}$ by

$$A(g) = \frac{1}{\lambda(E)} \int_E g(t) d\lambda(t).$$

It is obvious that A is a positive normalized linear functional on L. Then the linear operator $\widetilde{A}$ is defined by

$$\widetilde{A}(g) = \frac{1}{\lambda(E)} \int_E g(t) d\lambda(t).$$

We denote $\overline{g} = \frac{1}{\lambda(E)} \int_E g(t) d\lambda(t)$. If $g(E) \subset S$ and $f(g) \in L$, then from (8.38) it follows

$$f(\overline{g}) \leq A(f(g)) \tag{8.39}$$
$$\leq \frac{\mathrm{Vol}_k([\overline{g}, v_2, \ldots, v_{k+1}])}{\mathrm{Vol}_k(S)} f(v_1) + \cdots + \frac{\mathrm{Vol}_k([v_1, \ldots, v_k, \overline{g}])}{\mathrm{Vol}_k(S)} f(v_{k+1})$$
$$-\left(\frac{1}{\lambda(E)} \int_E \min\{\lambda_i(g(t))\colon i = 1, \ldots, k+1\} d\lambda(t)\right) S_f^{k+1}(v_1, \ldots, v_{k+1}).$$

Remark 8.9 Let $S = [v_1, \ldots, v_{k+1}]$ be a k-simplex in $\mathbb{R}^k$. If we put $E = S, g = id_S$ and λ is a Lebesgue measure on S from Example 8.1, we get

$$\overline{id_S} = \frac{1}{|S|} \int_S t dt = v^* = \frac{1}{k+1} \sum_{i=1}^{k+1} v_i$$
$$A(f(id_S)) = \frac{1}{|S|} \int_S f(t) dt,$$

where v^* is the barycenter of S. Now we have

$$f(v^*) \leq \frac{1}{|S|} \int_S f(t) dt$$
$$\leq \frac{\mathrm{Vol}_k([v^*, v_2, \ldots, v_{k+1}])}{|S|} f(v_1) + \cdots + \frac{\mathrm{Vol}_k([v_1, \ldots, v_k, v^*])}{|S|} f(v_{k+1})$$
$$-\left(\frac{1}{|S|} \int_S \min\{\lambda_i(t)\colon i = 1, \ldots, k+1\} dt\right) \left[\sum_{i=1}^{k+1} f(v_i) - (k+1) f(v^*)\right]$$
$$= \frac{1}{k+1} \left(\sum_{i=1}^{k+1} f(v_i)\right) - \left(\frac{1}{|S|} \int_S \min\{\lambda_i(t)\colon i = 1, \ldots, k+1\} dt\right)$$
$$\times \left[\sum_{i=1}^{k+1} f(v_i) - (k+1) f(v^*)\right]. \tag{8.40}$$

For $i = 1, \ldots, k+1$, let S_i be the simplex whose vertices are v^* and all vertices of S except v_i. Denote by v_i^* the barycentre of $S_i, i = 1, \ldots, k+1$. Since $\mathrm{Vol}_k(S_i) = \mathrm{Vol}_k(S_j)$, $i, j = 1, \ldots, k+1$, it follows from (8.33) that $t \in S_j$ implies $\min_i \lambda_i(t) = \lambda_j(t)$. It follows

$$\int_S \min_i \lambda_i(t)dt = \sum_{j=1}^{k+1} \int_{S_j} \lambda_j(t)dt. \tag{8.41}$$

We have

$$\begin{aligned}
\int_{S_j} \lambda_j(t)dt &= \frac{1}{|S|} \int_{S_j} \mathrm{Vol}_k[v_1, \ldots, t, \ldots, v_{k+1}]\, dt \\
&= \frac{1}{|S|} \mathrm{Vol}_k \left[v_1, \ldots, \int_{S_j} t\,dt, \ldots, v_{k+1} \right] \\
&= \frac{|S_j|}{|S|} \mathrm{Vol}_k \left[v_1, \ldots, v_j^*, \ldots, v_{k+1} \right] = \frac{1}{k+1} \mathrm{Vol}_k \left[v_1, \ldots, v_j^*, \ldots, v_{k+1} \right] \\
&= \frac{1}{(k+1)^2} \mathrm{Vol}_k [v_1, \ldots, v^*, \ldots, v_{k+1}] = \frac{1}{(k+1)^3} |S|.
\end{aligned} \tag{8.42}$$

Using (8.41) and (8.42) we get

$$\int_S \min_i \lambda_i(t)dt = \frac{1}{(k+1)^2} |S|. \tag{8.43}$$

Now, putting (8.43) in (8.40), we have

$$\begin{aligned}
f(v^*) &\leq \frac{1}{|S|} \int_S f(t)dt \\
&\leq \frac{k}{(k+1)^2} \sum_{i=1}^{k+1} f(v_i) + \frac{1}{k+1} f(v^*),
\end{aligned}$$

which is obtained in [78, Theorem 4.1].

It can be easily verified that the right-hand side of this inequality is equivalent to the k-dimensional version of the Hammer-Bullen inequality, namely

$$\frac{1}{|S|} \int_S f(t)dt - f(v^*) \leq \frac{k}{k+1} \sum_{i=1}^{k+1} f(v_i) - \frac{k}{|S|} \int_S f(t)dt,$$

which is proved, for example in [210].

In one-dimensional case, this is an improvement of classical Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2} - \frac{1}{4} S_f^2(a,b),$$

where $S_f^2(a,b) = f(a) + f(b) - 2f\left(\frac{a+b}{2}\right)$.

8.3 Improvements of the Giaccardi and the Petrović inequality

Another variant of converse Jensen's inequality is the Giaccardi inequality, the special case of which is the Petrović inequality (for the latter, see also Chapter 3). Improvements of these inequalities are again obtained by means of Lemma 1.2 and are given in the sequel.

Theorem 8.16 (Giaccardi's inequality) *Let ϕ be a convex function on an interval I, $\mathbf{p}$ a nonnegative n-tuple with $\sum_{i=1}^n p_i = P_n \neq 0$ and $\mathbf{x}$ a real n-tuple. If $\mathbf{x} \in I^n$ and $x_0 \in I$ are such that $\sum_{i=1}^n p_i x_i = \tilde{x} \in I, \tilde{x} \neq x_0$ and*

$$(x_i - x_0)(\tilde{x} - x_i) \geq 0, \;\; i = 1, \ldots, n,$$

then

$$\sum_{i=1}^n p_i \phi(x_i) \leq A\phi(\tilde{x}) + B\left(\sum_{i=1}^n p_i - 1\right)\phi(x_0),$$

where

$$A = \frac{\sum_{i=1}^n p_i(x_i - x_0)}{\sum_{i=1}^n p_i x_i - x_0}, \qquad B = \frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i x_i - x_0}.$$

A simple consequence of the Giaccardi inequality is the Petrović inequality:

Corollary 8.5 (Petrović' inequality) *Let ϕ be a convex function on $[0,a], 0 < a < \infty$. Then for every nonnegative n-tuple $\mathbf{p}$ and every $\mathbf{x} \in [0,a]^n$ such that $\sum_{i=1}^n p_i x_i = \tilde{x} \in (0,a]$ and*

$$\sum_{i=1}^n p_i x_i \geq x_j, \;\; j = 1, \ldots, n,$$

the following inequality holds:

$$\sum_{i=1}^n p_i \phi(x_i) \leq \phi(\tilde{x}) + \left(\sum_{i=1}^n p_i - 1\right)\phi(0).$$

For further details on the Giaccardi and the Petrović inequality see [177].

In order to improve these two inequalities, we use the left-hand side of Lemma 1.2, which is here observed for $n = 2$ and for a convex function ϕ defined on an interval I, it states that

$$\min\{p,q\}\left[\phi(x) + \phi(y) - 2\phi\left(\frac{x+y}{2}\right)\right] \leq p\phi(x) + q\phi(y) - \phi(px + qy), \qquad (8.44)$$

where $x, y \in I$ and $p, q \in [0,1]$ are such that $p + q = 1$.

Furthermore, we also prove the Lagrange and the Cauchy-type mean value theorems, which we then use in studying Stolarsky-type means defined by the Giaccardi and the

Petrović differences. By means of these differences, n-exponentially convex and exponentially convex functions are produced, making use of some known families of functions of the same type.

The following theorem is our main result in this scope.

Theorem 8.17 *Let ϕ be a convex function on an interval I, $\mathbf{p}$ a nonnegative n-tuple with $\sum_{i=1}^n p_i = P_n \neq 0$ and $\mathbf{x}$ a real n-tuple. If $\mathbf{x} \in I^n$ and $x_0 \in I$ are such that $\sum_{i=1}^n p_i x_i = \tilde{x} \in I, \tilde{x} \neq x_0$ and*

$$(x_i - x_0)(\tilde{x} - x_i) \geq 0, \quad i = 1, \dots, n, \tag{8.45}$$

then

$$\sum_{i=1}^n p_i \phi(x_i) \leq A\phi(\tilde{x}) + B\left(\sum_{i=1}^n p_i - 1\right)\phi(x_0) - \frac{\delta_\phi}{2}P_n + \delta_\phi \sum_{i=1}^n p_i \left| \frac{x_i - \frac{x_0+\tilde{x}}{2}}{\tilde{x} - x_0} \right|, \tag{8.46}$$

where

$$A = \frac{\sum_{i=1}^n p_i(x_i - x_0)}{\sum_{i=1}^n p_i x_i - x_0}, \qquad B = \frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i x_i - x_0}, \qquad \delta_\phi = \phi(x_0) + \phi(\tilde{x}) - 2\phi\left(\frac{x_0+\tilde{x}}{2}\right).$$

Proof. The condition $(x_i - x_0)(\tilde{x} - x_i) \geq 0$, $i = 1, \dots, n$, means that either $x_0 \leq x_i \leq \tilde{x}$ or $\tilde{x} \leq x_i \leq x_0$, $i = 1, \dots, n$. Consider the first case (the second is analogous).

Let the functions $p,\ q : [x_0, \tilde{x}] \to [0,1]$ be defined by

$$p(x) = \frac{\tilde{x} - x}{\tilde{x} - x_0}, \quad q(x) = \frac{x - x_0}{\tilde{x} - x_0}.$$

For any $x \in [x_0, \tilde{x}]$ we can write

$$\phi(x) = \phi\left(\frac{\tilde{x} - x}{\tilde{x} - x_0}x_0 + \frac{x - x_0}{\tilde{x} - x_0}\tilde{x}\right) = \phi(p(x)x_0 + q(x)\tilde{x}).$$

By inequality (8.44) we get for $x \in [x_0, \tilde{x}]$

$$\begin{aligned} &\min\{p(x), q(x)\}\left[\phi(x_0) + \phi(\tilde{x}) - 2\phi\left(\frac{x_0+\tilde{x}}{2}\right)\right] \\ &\leq p(x)\phi(x_0) + q(x)\phi(\tilde{x}) - \phi(p(x)x_0 + q(x)\tilde{x}) \end{aligned}$$

and then,

$$\begin{aligned} \phi(x) &= \phi(p(x)x_0 + q(x)\tilde{x}) \\ &\leq \phi(x_0) + q(x)\phi(\tilde{x}) - \min\{p(x), q(x)\}\left[\phi(x_0) + \phi(\tilde{x}) - 2\phi\left(\frac{x_0+\tilde{x}}{2}\right)\right]. \end{aligned}$$

Multiplying $\phi(x_i)$ by p_i and summing, we get

$$\begin{aligned} \sum_{i=1}^n p_i \phi(x_i) &\leq \sum_{i=1}^n p_i \Bigg[p(x_i)\phi(x_0) + q(x_i)\phi(\tilde{x}) - \min\{p(x_i), q(x_i)\} \\ &\qquad \times \left[\phi(x_0) + \phi(\tilde{x}) - 2\phi\left(\frac{x_0+\tilde{x}}{2}\right)\right]\Bigg] \end{aligned}$$

$$= \phi(\tilde{x}) \sum_{i=1}^{n} p_i \frac{x_i - x_0}{\tilde{x} - x_0} + \phi(x_0) \sum_{i=1}^{n} p_i \frac{\tilde{x} - x_i}{\tilde{x} - x_0} - \delta_\phi \sum_{i=1}^{n} p_i \min\{p(x_i), q(x_i)\}$$
$$= A\phi(\tilde{x}) + B(\sum_{i=1}^{n} p_i - 1)\phi(x_0) - \frac{\delta_\phi}{2} P_n + \delta_\phi \sum_{i=1}^{n} p_i \left| \frac{x_i - \frac{x_0+\tilde{x}}{2}}{\tilde{x} - x_0} \right|.$$

□

Remark 8.10 Obviously, Theorem 8.17 is an improvement of Theorem 8.16, since under the required assumptions we have

$$\delta_\phi \sum_{i=1}^{n} p_i \min\{p(x_i), q(x_i)\} \geq 0.$$

What follows is an improvement of the Petrović inequality.

Theorem 8.18 *Let ϕ be a convex function on $[0,a]$, $0 < a < \infty$. Then for every nonnegative n-tuple $\mathbf{p}$ and every $\mathbf{x} \in [0,a]^n$ such that $\sum_{i=1}^{n} p_i x_i = \tilde{x} \in (0,a]$ and*

$$\sum_{i=1}^{n} p_i x_i \geq x_j, \quad j = 1, \ldots, n, \tag{8.47}$$

the following inequality holds

$$\sum_{i=1}^{n} p_i \phi(x_i) \leq \phi(\tilde{x}) + \left(\sum_{i=1}^{n} p_i - 1 \right) \phi(0) - \frac{\delta_\phi}{2} P_n + \delta_\phi \sum_{i=1}^{n} p_i \left| \frac{x_i}{\tilde{x}} - \frac{1}{2} \right|, \tag{8.48}$$

where $\delta_\phi = \phi(0) + \phi(\tilde{x}) - 2\phi\left(\frac{\tilde{x}}{2}\right)$.

Proof. This is a special case of Theorem 8.17; choose $x_0 = 0$. □

Remark 8.11 The Giaccardi inequality can also be improved by means of Theorem 8.5, viewed as a special case, similarly as in Corollary 8.1.

Assume $x_0 < \tilde{x}$. For $m = x_0$ and $M = \tilde{x}$, from (8.45) we have $x \in [m,M]^n$ and Corollary 8.1 implies

$$\sum_{i=1}^{n} p_i \phi(x_i) \leq \frac{P_n \tilde{x} - \tilde{x}}{\tilde{x} - x_0} \phi(x_0) + \frac{\tilde{x} - x_0 P_n}{\tilde{x} - x_0} \phi(\tilde{x}) - \frac{\delta_\phi}{2} P_n + \delta_\phi \sum_{i=1}^{n} p_i \left| \frac{x_i - \frac{x_0+\tilde{x}}{2}}{\tilde{x} - x_0} \right|$$
$$= A\phi\left(\sum_{i=1}^{n} p_i x_i \right) + B\left(\sum_{i=1}^{n} p_i - 1 \right) \phi(x_0) - \frac{\delta_\phi}{2} P_n + \delta_\phi \sum_{i=1}^{n} p_i \left| \frac{x_i - \frac{x_0+\tilde{x}}{2}}{\tilde{x} - x_0} \right|.$$

For $x_0 > \sum_{i=1}^{n} p_i x_i$ we define $m = \sum_{i=1}^{n} p_i x_i$, $M = x_0$, so the rest of the proof is similar to the one above.

8.3.1 Giaccardi-Petrović differences

At the very beginning of the current section, the reader is referred to Section 1.1 in order to study or recall the notions of n–exponentially convex and exponentially convex functions, as well as some other related notions.

Motivated by inequalities (8.46) and (8.48), we define two functionals:

$$\Phi_1(\mathbf{x},\mathbf{p},f) = Af(\tilde{x}) + B\left(\sum_{i=1}^{n} p_i - 1\right) f(x_0) - \frac{\delta_f}{2}P_n + \delta_f \sum_{i=1}^{n} p_i \left|\frac{x_i - \frac{x_0+\tilde{x}}{2}}{\tilde{x}-x_0}\right| - \sum_{i=1}^{n} p_i f(x_i), \tag{8.49}$$

where f is a function on an interval I, $\mathbf{p}$ is a nonnegative n-tuple, $\mathbf{x}$ is a real n-tuple, $\tilde{x}$, P_n, δ_f, A, B are as in Theorem 8.17, and

$$\Phi_2(\mathbf{x},\mathbf{p},f) = f(\tilde{x}) + \left(\sum_{i=1}^{n} p_i - 1\right) f(0) - \frac{\delta_\phi}{2}P_n + \delta_f \sum_{i=1}^{n} p_i \left|\frac{x_i}{\tilde{x}} - \frac{1}{2}\right| - \sum_{i=1}^{n} p_i f(x_i), \tag{8.50}$$

where f is a function on an interval $[0,a]$, $\mathbf{p}$ is a nonnegative n-tuple, $\mathbf{x}$ is a real n-tuple and $\tilde{x}$, P_n, δ_f are as in Corollary 8.18.

If f is a convex function, then Theorem 8.17 and Corollary 8.18 imply that $\Phi_i(\mathbf{x},\mathbf{p},f) \geq 0, i = 1,2.$

Now, we present Lagrange and Cauchy type mean value theorems for the functionals Φ_i, $i = 1,2$.

Theorem 8.19 *Let $I = [a,b]$, $\mathbf{p}$ be a nonnegative n-tuple with $\sum_{i=1}^{n} p_i = P_n \neq 0$ and $\mathbf{x}$ a real n-tuple. Let $\mathbf{x} \in I^n$ and $x_0 \in I$ be such that $\sum_{i=1}^{n} p_i x_i = \tilde{x} \in I$, $\tilde{x} \neq x_0$ and (8.45) holds. Let $f \in C^2(I)$. Then there exists $\xi \in I$ such that*

$$\Phi_1(\mathbf{x},\mathbf{p},f) = \frac{f''(\xi)}{2}\Phi_1(x,p,f_0), \tag{8.51}$$

where $f_0(x) = x^2$.

Proof. Since $f \in C^2(I)$, there exist real numbers $m = \min\limits_{x\in[a,b]} f''(x)$ and $M = \max\limits_{x\in[a,b]} f''(x)$. It is easy to show that the functions f_1 and f_2 defined by

$$\begin{aligned} f_1(x) &= \frac{M}{2}x^2 - f(x), \\ f_2(x) &= f(x) - \frac{m}{2}x^2 \end{aligned}$$

are convex. Therefore

$$\begin{aligned} \Phi_1(\mathbf{x},\mathbf{p},f_1) \geq 0, \\ \Phi_1(\mathbf{x},\mathbf{p},f_2) \geq 0, \end{aligned}$$

and we get

$$\Phi_1(\mathbf{x},\mathbf{p},f) \leq \frac{M}{2}\Phi_1(\mathbf{x},\mathbf{p},f_0), \tag{8.52}$$

$$\Phi_1(\mathbf{x},\mathbf{p},f) \geq \frac{m}{2}\Phi_1(\mathbf{x},\mathbf{p},f_0). \tag{8.53}$$

From (8.52) and (8.53) we get

$$\frac{m}{2}\Phi_1(\mathbf{x},\mathbf{p},f_0) \leq \Phi_1(\mathbf{x},\mathbf{p},f) \leq \frac{M}{2}\Phi_1(\mathbf{x},\mathbf{p},f_0).$$

If $\Phi_1(\mathbf{x},\mathbf{p},x^2) = 0$ there is nothing left to prove. Suppose $\Phi_1(\mathbf{x},\mathbf{p},x^2) > 0$. Then

$$m \leq \frac{2\Phi_1(\mathbf{x},\mathbf{p},f)}{\Phi_1(\mathbf{x},\mathbf{p},x^2)} \leq M.$$

Hence, there exists $\xi \in I$ such that

$$\Phi_1(\mathbf{x},\mathbf{p},f) = \frac{f''(\xi)}{2}\Phi_1(\mathbf{x},\mathbf{p},f_0).$$

□

Theorem 8.20 *Let $I = [0,a]$, $\mathbf{p}$ be a nonnegative n-tuple and $\mathbf{x}$ a real n-tuple. Let $\mathbf{x} \in [0,a]^n$ such that $\sum_{i=1}^n p_i x_i = \tilde{x} \in I$ and (8.47) holds. Let $f \in C^2(I)$. Then there exists $\xi \in I$ such that*

$$\Phi_2(\mathbf{x},\mathbf{p},f) = \frac{f''(\xi)}{2}\Phi_2(\mathbf{x},\mathbf{p},f_0), \tag{8.54}$$

where $f_0(x) = x^2$.

Proof. Analogous to the proof of Theorem 8.19. □

Theorem 8.21 *Let $I = [a,b]$, $\mathbf{p}$ be a nonnegative n-tuple with $\sum_{i=1}^n p_i = P_n \neq 0$ and $\mathbf{x}$ be a real n-tuple. Let $\mathbf{x} \in I^n$ and $x_0 \in I$ be such that $\sum_{i=1}^n p_i x_i = \tilde{x} \in I$, $\tilde{x} \neq x_0$ and (8.45) holds. Let $f,g \in C^2(I)$. Then there exists $\xi \in I$ such that*

$$\frac{\Phi_1(\mathbf{x},\mathbf{p},f)}{\Phi_1(\mathbf{x},\mathbf{p},g)} = \frac{f''(\xi)}{g''(\xi)}, \tag{8.55}$$

provided that the denominators are non-zero.

Proof. Define $h \in C^2([a,b])$ by

$$h = c_1 f - c_2 g,$$

where

$$c_1 = \Phi_1(\mathbf{x},\mathbf{p},g), \;\; c_2 = \Phi_1(\mathbf{x},\mathbf{p},f).$$

Now by Theorem 8.19 there exists $\xi \in [a,b]$ such that

$$\left(c_1\frac{f''(\xi)}{2} - c_2\frac{g''(\xi)}{2}\right)\Phi_1(\mathbf{x},\mathbf{p},f_0) = 0.$$

Since $\Phi_1(\mathbf{x},\mathbf{p},f_0) \neq 0$ (otherwise we have a contradiction with $\Phi_1(\mathbf{x},\mathbf{p},g) \neq 0$, by Theorem 8.19), we get

$$\frac{\Phi_1(\mathbf{x},\mathbf{p},f)}{\Phi_1(\mathbf{x},\mathbf{p},g)} = \frac{f''(\xi)}{g''(\xi)}.$$

□

Theorem 8.22 *Let $I = [0,a]$, $\mathbf{p}$ be a nonnegative n-tuple and $\mathbf{x}$ be a real n-tuple. Let $\mathbf{x} \in [0,a]^n$ be such that $\sum_{i=1}^n p_i x_i = \tilde{x} \in I$ and (8.47) holds. Let $f,g \in C^2(I)$. Then there exists $\xi \in I$ such that*

$$\frac{\Phi_2(\mathbf{x},\mathbf{p},f)}{\Phi_2(\mathbf{x},\mathbf{p},g)} = \frac{f''(\xi)}{g''(\xi)}, \tag{8.56}$$

provided that the denominators are non zero.

Proof. Analogous to the proof of Theorem 8.21. □

We use an idea from [90] to give an elegant method of producing an n-exponentially convex functions and exponentially convex functions applying the functionals Φ_1 and Φ_2 to a given family with the same property.

Theorem 8.23 *Let $\Upsilon = \{f_s : s \in J\}$, J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $s \mapsto [y_0,y_1,y_2;f_s]$ is n-exponentially convex in the Jensen sense on J for every three mutually different points $y_0,y_1,y_2 \in I$. Let Φ_i $(i = 1,2)$ be linear functionals defined as in (8.49) and (8.50). Then $s \mapsto \Phi_i(\mathbf{x},\mathbf{p},f_s)$ is an n-exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Phi_i(\mathbf{x},\mathbf{p},f_s)$ is continuous on J, then it is n-exponentially convex on J.*

Proof. For $\xi_i \in \mathbb{R}$ and $s_i \in J$, $i = 1,\ldots,n$, we define the function

$$g(y) = \sum_{i,j=1}^n \xi_i \xi_j f_{\frac{s_i+s_j}{2}}(y).$$

Using the assumption that the function $s \mapsto [y_0,y_1,y_2;f_s]$ is n-exponentially convex in the Jensen sense, we have

$$[y_0,y_1,y_2;g] = \sum_{i,j=1}^n \xi_i \xi_j [y_0,y_1,y_2;f_{\frac{s_i+s_j}{2}}] \geq 0,$$

which in turn implies that g is a convex function on I and therefore we have $\Phi_i(\mathbf{x},\mathbf{p},g) \geq 0$, $i = 1,2$. Hence

$$\sum_{i,j=1}^n \xi_i \xi_j \Phi_i(\mathbf{x},\mathbf{p},f_{\frac{s_i+s_j}{2}}) \geq 0.$$

We conclude that the function $s \mapsto \Phi_i(\mathbf{x},\mathbf{p},f_s)$ is n-exponentially convex on J in the Jensen sense.

If the function $s \mapsto \Phi_i(\mathbf{x},\mathbf{p},f_s)$ is also continuous on J, then $s \mapsto \Phi_i(\mathbf{x},\mathbf{p},f_s)$ is n-exponentially convex by definition. □

The following corollary is an immediate consequence of the above theorem.

Corollary 8.6 *Let $\Upsilon = \{f_s : s \in J\}$, J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Let Φ_i $(i = 1, 2)$ be linear functionals defined as in (8.49) and (8.50). Then $s \mapsto \Phi_i(\mathbf{x}, \mathbf{p}, f_s)$ is an exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Phi_i(\mathbf{x}, \mathbf{p}, f_s)$ is continuous on J, then it is exponentially convex on J.*

Corollary 8.7 *Let $\Omega = \{f_s : s \in J\}$, where J an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is 2-exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Let Φ_i, $i = 1, 2$, be linear functionals defined as in (8.49) and (8.50). Then the following statements hold:*

(i) If the function $s \mapsto \Phi_i(\mathbf{x}, \mathbf{p}, f_s)$ is continuous on J, then it is 2-exponentially convex function on J, and thus log-convex function.

(ii) If the function $s \mapsto \Phi_i(\mathbf{x}, \mathbf{p}, f_s)$ is strictly positive and differentiable on J, then for every $s, q, u, v \in J$, such that $s \leq u$ and $q \leq v$, we have

$$\mu_{s,q}(\mathbf{x}, \Phi_i, \Omega) \leq \mu_{u,v}(\mathbf{x}, \Phi_i, \Omega), \ \ i = 1, 2, \tag{8.57}$$

where

$$\mu_{s,q}(\mathbf{x}, \Phi_i, \Omega) = \begin{cases} \left(\frac{\Phi_i(\mathbf{x}, \mathbf{p}, f_s)}{\Phi_i(\mathbf{x}, \mathbf{p}, f_q)} \right)^{\frac{1}{s-q}} & , \ s \neq q, \\ \exp\left(\frac{\frac{d}{ds}\Phi_i(\mathbf{x}, \mathbf{p}, f_s)}{\Phi_i(\mathbf{x}, \mathbf{p}, f_s)} \right), & s = q, \end{cases} \tag{8.58}$$

for $f_s, f_q \in \Omega$.

Proof. (i) This is an immediate consequence of Theorem 8.23 and Remark 1.3.

(ii) Since by (i) the function $s \mapsto \Phi_i(\mathbf{x}, \mathbf{p}, f_s)$ is log-convex on J, that is, the function $s \mapsto \log \Phi_i(\mathbf{x}, \mathbf{p}, f_s)$ is convex on J. Applying Proposition 1.2 we get

$$\frac{\log \Phi_i(\mathbf{x}, \mathbf{p}, f_s) - \log \Phi_i(\mathbf{x}, \mathbf{p}, f_q)}{s - q} \leq \frac{\log \Phi_i(\mathbf{x}, \mathbf{p}, f_u) - \log \Phi_i(\mathbf{x}, \mathbf{p}, f_v)}{u - v}, \tag{8.59}$$

for $s \leq u$, $q \leq v$, $s \neq q$, $u \neq v$, and therefrom conclude that

$$\mu_{s,q}(\mathbf{x}, \Phi_i, \Omega) \leq \mu_{u,v}(\mathbf{x}, \Phi_i, \Omega), \quad i = 1, 2.$$

Cases $s = q$ and $u = v$ follow from (8.59) as limit cases. □

Remark 8.12 Note that the results from Theorem 8.23, Corollary 8.6, Corollary 8.7 still hold when two of the points $y_0, y_1, y_2 \in I$ coincide, say $y_1 = y_0$, for a family of differentiable functions f_s, such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, they still hold when all three points coincide for a family of twice differentiable functions with the same property. The proofs are obtained by recalling Remark 1.4 and suitable characterization of convexity.

We present several families of functions which fulfil the conditions of Theorem 8.23, Corollary 8.6 and Corollary 8.7 (and Remark 8.12). This enable us to construct a large family of functions which are exponentially convex. For a discussion related to this problem see [68].

Example 8.2 Consider a family of functions

$$\Omega_1 = \{g_s : \mathbb{R} \to [0,\infty) : s \in \mathbb{R}\}$$

defined by

$$g_s(x) = \begin{cases} \frac{1}{s^2} e^{sx}, & s \neq 0, \\ \frac{1}{2} x^2, & s = 0. \end{cases}$$

We have $\frac{d^2 g_s}{dx^2}(x) = e^{sx} > 0$ which shows that g_s is convex on $\mathbb{R}$ for every $s \in \mathbb{R}$ and $s \mapsto \frac{d^2 g_s}{dx^2}(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 8.23 we also have that $s \mapsto [y_0, y_1, y_2; g_s]$ is exponentially convex (and so exponentially convex in the Jensen sense). Using Theorem 8.6 we conclude that $s \mapsto \Phi_i(\mathbf{x}, \mathbf{p}, g_s)$, $i = 1,2$, are exponentially convex in the Jensen sense. It is easy to verify that these mappings are continuous (although mapping $s \mapsto g_s$ is not continuous for $s = 0$), so they are exponentially convex.

For this family of functions, $\mu_{s,q}(\mathbf{x}, \Phi_i, \Omega_1)$, $i = 1,2$, from (8.58) becomes

$$\mu_{s,q}(\mathbf{x}, \Phi_i, \Omega_1) = \begin{cases} \left(\frac{\Phi_i(\mathbf{x},\mathbf{p},g_s)}{\Phi_i(\mathbf{x},\mathbf{p},g_q)} \right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{\Phi_i(\mathbf{x},\mathbf{p},id \cdot g_s)}{\Phi_i(\mathbf{x},\mathbf{p},g_s)} - \frac{2}{s} \right), & s = q \neq 0, \\ \exp\left(\frac{\Phi_i(\mathbf{x},\mathbf{p},id \cdot g_0)}{3\Phi_i(\mathbf{x},\mathbf{p},g_0)} \right), & s = q = 0, \end{cases}$$

and using (8.57) they are monotonic functions in parameters s and q.

Using theorems 8.21 and 8.22 it follows that for $i = 1,2$

$$M_{s,q}(\mathbf{x}, \Phi_i, \Omega_1) = \log \mu_{s,q}(\mathbf{x}, \Phi_i, \Omega_1)$$

satisfy $\min\{x_0, \tilde{x}\} \leq M_{s,q}(\mathbf{x}, \Phi_i, \Omega_1) \leq \max\{x_0, \tilde{x}\}$, which shows that $M_{s,q}(\mathbf{x}, \Phi_i, \Omega_1)$ are means (of $x_0, x_1, \ldots x_n, \tilde{x}$). Notice that by (8.57) they are monotonic means.

Example 8.3 Consider a family of functions

$$\Omega_2 = \{f_s : (0,\infty) \to \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$f_s(x) = \begin{cases} \frac{x^s}{s(s-1)}, & s \neq 0,1, \\ -\log x, & s = 0, \\ x \log x, & s = 1. \end{cases}$$

Here, $\frac{d^2 f_s}{dx^2}(x) = x^{s-2} = e^{(s-2)\log x} > 0$ which shows that f_s is convex for $x > 0$ and $s \mapsto \frac{d^2 f_s}{dx^2}(x)$ is exponentially convex by definition. Arguing as in Example 8.2 we get that the mapping $s \mapsto \Phi_1(\mathbf{x},\mathbf{p},g_s)$ is exponentially convex. In this case we assume $x_j > 0$, $j = 0,1\ldots,n$. Notice that the functional Φ_2 is not defined in this case (of course it can be defined for $s \geq 0$). Functions (8.58) in this case are equal to:

$$\mu_{s,q}(\mathbf{x},\Phi_1,\Omega_2) = \begin{cases} \left(\frac{\Phi_1(\mathbf{x},\mathbf{p},f_s)}{\Phi_1(\mathbf{x},\mathbf{p},f_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{1-2s}{s(s-1)} - \frac{\Phi_1(\mathbf{x},\mathbf{p},f_s f_0)}{\Phi_1(\mathbf{x},\mathbf{p},f_s)}\right), & s = q \neq 0,1, \\ \exp\left(1 - \frac{\Phi_1(\mathbf{x},\mathbf{p},f_0^2)}{2\Phi_1(\mathbf{x},\mathbf{p},f_0)}\right), & s = q = 0, \\ \exp\left(-1 - \frac{\Phi_1(\mathbf{x},\mathbf{p},f_0 f_1)}{2\Phi_1(\mathbf{x},\mathbf{p},f_1)}\right), & s = q = 1. \end{cases}$$

If Φ_1 is positive, then Theorem 8.21 and Theorem 8.22, applied for $f = f_s \in \Omega_2$ and $g = f_q \in \Omega_2$ yield that there exists $\xi \in [\min\{x_0,\tilde{x}\},\max\{x_0,\tilde{x}\}]$ such that

$$\xi^{s-q} = \frac{\Phi_1(\mathbf{x},\mathbf{p},f_s)}{\Phi_1(\mathbf{x},\mathbf{p},f_q)}.$$

Since the function $\xi \mapsto \xi^{s-q}$ is invertible for $s \neq q$, we have

$$\min\{x_0,\tilde{x}\} \leq \left(\frac{\Phi_1(\mathbf{x},\mathbf{p},f_s)}{\Phi_1(\mathbf{x},\mathbf{p},f_q)}\right)^{\frac{1}{s-q}} \leq \max\{x_0,\tilde{x}\}, \tag{8.60}$$

which together with the fact that $\mu_{s,q}(\mathbf{x},\Phi_1,\Omega_2)$ is continuous, symmetric and monotonic (by (8.57)), shows that $\mu_{s,q}(\mathbf{x},\Phi_1,\Omega_2)$ is a mean. Now, by substitutions $x_i \to x_i^t$, $s \to \frac{s}{t}$, $q \to \frac{q}{t}$ ($t \neq 0$, $s \neq q$) from (8.60) we get

$$\min\left\{x_0^t,\tilde{x}^t\right\} \leq \left(\frac{\Phi_1(\mathbf{x}^t,\mathbf{p},f_{s/t})}{\Phi_1(\mathbf{x}^t,\mathbf{p},f_{q/t}}\right)^{\frac{t}{s-q}} \leq \max\left\{x_0^t,\tilde{x}^t\right\},$$

where $\mathbf{x}^t = (x_1^t,\ldots,x_n^t)$. We define a new mean as follows:

$$\mu_{s,q;t}(\mathbf{x},\Phi_1,\Omega_2) = \begin{cases} \left(\mu_{\frac{s}{t},\frac{q}{t}}(\mathbf{x}^t,\Phi_1,\Omega_2)\right)^{1/t}, & t \neq 0 \\ \mu_{s,q}(\log\mathbf{x},\Phi_1,\Omega_1), & t = 0. \end{cases} \tag{8.61}$$

These new means are also monotonic. More precisely, for $s,q,u,v \in \mathbb{R}$, such that $s \leq u$, $q \leq v$, $s \neq u$, $q \neq v$, we have

$$\mu_{s,q;t}(\mathbf{x},\Phi_1,\Omega_2) \leq \mu_{u,v;t}(\mathbf{x},\Phi_1,\Omega_2). \tag{8.62}$$

We know that

$$\mu_{\frac{s}{t},\frac{q}{t}}(\mathbf{x},\Phi_1,\Omega_2) = \left(\frac{\Phi_1(\mathbf{x},\mathbf{p},f_{s/t})}{\Phi_1(\mathbf{x},\mathbf{p},f_{q/t})}\right)^{\frac{t}{s-q}} \leq \mu_{\frac{u}{t},\frac{v}{t}}(\mathbf{x},\Phi_1,\Omega_2) = \left(\frac{\Phi_1(\mathbf{x},\mathbf{p},f_{s/t})}{\Phi_1(\mathbf{x},\mathbf{p},f_{q/t})}\right)^{\frac{t}{s-q}},$$

for $s,q,u,v \in I$, such that $s/t \le u/t$, $q/t \le v/t$ and $t \neq 0$. Since $\mu_{s,q}(\mathbf{x},\Phi_1,\Omega_2)$ are monotonic in both parameters, the claim follows. For $t = 0$, we obtain the required result by taking the limit $t \to 0$.

Example 8.4 Consider a family of functions

$$\Omega_3 = \{h_s : (0,\infty) \to (0,\infty) : s \in (0,\infty)\}$$

defined by

$$h_s(x) = \begin{cases} \frac{s^{-x}}{\log^2 s}, & s \neq 1, \\ \frac{x^2}{2}, & s = 1. \end{cases}$$

Since $s \mapsto \frac{d^2 h_s}{dx^2}(x) = s^{-x}$ is the Laplace transform of a non-negative function (see [211]), it is exponentially convex. Obviously h_s are convex functions for every $s > 0$.

For this family of functions, $\mu_{s,q}(\mathbf{x},\Phi_1,\Omega_3)$, in this case for $x_j > 0$, $j = 0,1,\ldots,n$, from (8.58) becomes

$$\mu_{s,q}(\mathbf{x},\Phi_1,\Omega_3) = \begin{cases} \left(\frac{\Phi_1(\mathbf{x},\mathbf{p},h_s)}{\Phi_1(\mathbf{x},\mathbf{p},h_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(-\frac{\Phi_1(\mathbf{x},\mathbf{p},id\cdot h_s)}{s\Phi_1(\mathbf{x},\mathbf{p},h_s)} - \frac{2}{s\log s}\right), & s = q \neq 1, \\ \exp\left(-\frac{\Phi_1(\mathbf{x},\mathbf{p},id\cdot h_1)}{3\Phi_1(\mathbf{x},\mathbf{p},h_1)}\right), & s = q = 1, \end{cases}$$

and it is monotonic in parameters s and q by (8.57).

Using Theorem 8.21, it follows that

$$M_{s,q}(\mathbf{x},\Phi_1,\Omega_3) = -L(s,q)\log \mu_{s,q}(\mathbf{x},\Phi_1,\Omega_3),$$

satisfies $\min\{x_0,\tilde{x}\} \le M_{s,q}(\mathbf{x},\Phi_1,\Omega_3) \le \max\{x_0,\tilde{x}\}$, which shows that $M_{s,q}(\mathbf{x},\Phi_1,\Omega_3)$ is a mean (of $x_0,x_1,\ldots x_n,\tilde{x}$). $L(s,q)$ is the logarithmic mean defined by $L(s,q) = \frac{s-q}{\log s - \log q}$, $s \neq q$, $L(s,s) = s$.

Example 8.5 Consider a family of functions

$$\Omega_4 = \{k_s : (0,\infty) \to (0,\infty) : s \in (0,\infty)\}$$

defined by

$$k_s(x) = \frac{e^{-x\sqrt{s}}}{s}.$$

Since $s \mapsto \frac{d^2 k_s}{dx^2}(x) = e^{-x\sqrt{s}}$ is the Laplace transform of a non-negative function (see [211]), it is exponentially convex. Obviously k_s are convex functions for every $s > 0$.

For this family of functions, $\mu_{s,q}(\mathbf{x},\Phi_1,\Omega_4)$, in this case for $x_j > 0$, $j = 0,1,\ldots,n$, from (8.58) becomes

$$\mu_{s,q}(\mathbf{x},\Phi_1,\Omega_4) = \begin{cases} \left(\frac{\Phi_1(\mathbf{x},\mathbf{p},k_s)}{\Phi_1(\mathbf{x},\mathbf{p},k_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(-\frac{\Phi_1(\mathbf{x},\mathbf{p},id\cdot k_s)}{2\sqrt{s}\Phi(\mathbf{x},\mathbf{p},k_s)} - \frac{1}{s}\right), & s = q, \end{cases}$$

and it is monotonic function in parameters s and q by (8.57).

Using Theorem 8.21, it follows that

$$M_{s,q}(\mathbf{x},\Phi_1,\Omega_4) = -\left(\sqrt{s}+\sqrt{q}\right)\log\mu_{s,q}(\mathbf{x},\Phi_1,\Omega_4)$$

satisfies $\min\{x_0,\tilde{x}\} \leq M_{s,q}(\mathbf{x},\Phi_1,\Omega_4) \leq \max\{x_0,\tilde{x}\}$, which shows that $M_{s,q}(\mathbf{x},\Phi_1,\Omega_4)$ is a mean (of $x_0,x_1,\dots x_n,\tilde{x}$).

8.4 Refinements of the converse Hölder and Minkowski inequalities

Most of the classical inequalities have their variants involving positive linear functionals. Among others, in [177, p. 115] we can find the following generalization of this type for the converse Hölder inequality.

Theorem 8.24 *Let L satisfy conditions* L1 *and* L2 *and let A be an isotonic linear functional. Let $p>1$, $q=p/(p-1)$, and $w,f,g\geq 0$ on E with wf^p, wg^q, $wfg\in L$. If $0<m\leq f(x)g^{-q/p}(x)\leq M$ for $x\in E$, then*

$$K(p,m,M)A^{\frac{1}{p}}(wf^p)A^{\frac{1}{q}}(wg^q)\leq A(wfg) \tag{8.63}$$

where $K(p,m,M)$ is a constant defined as

$$K(p,m,M)=|p|^{\frac{1}{p}}|q|^{\frac{1}{q}}\frac{(M-m)^{\frac{1}{p}}|mM^p-Mm^p|^{\frac{1}{q}}}{|M^p-m^p|}. \tag{8.64}$$

If $p<0$ or $0<p<1$, then the reverse inequality in (8.63) holds, provided either $A(wf^p)>0$ or $A(wg^q)>0$.

In the sequel, we present a refinement of the converse Hölder inequality, and, as its consequence – a refinement of the converse Beckenbach inequality. We consider the Minkowski inequality for infinitely many functions and for functionals, state its converse and give refinements of both variants of the converse Minkowski inequality. Finally, obtained results are applied to integral mixed means.

The starting point of this consideration is Theorem 1.31, cited in the introductory part and for the sake of simplicity is again cited here, in a more suitable form.

Theorem 8.25 *If ϕ is a convex function on an interval $I\subseteq\mathbf{R}$, $x=(x_1,\dots,x_n)\in I^n$ $(n\geq 2)$, p and q are positive n-tuples such that $p_i\geq q_i$ for all $i=1,2,\dots,n$, $P_n=\sum_{i=1}^n p_i$, $Q_n=\sum_{i=1}^n q_i$, then*

$$\sum_{i=1}^n p_i\phi(x_i)-P_n\phi\left(\frac{1}{P_n}\sum_{i=1}^n p_ix_i\right)\geq\sum_{i=1}^n q_i\phi(x_i)-Q_n\phi\left(\frac{1}{Q_n}\sum_{i=1}^n q_ix_i\right)\geq 0. \tag{8.65}$$

Furthermore, let us recall the AG inequality in the following form.

Proposition 8.1 (AG inequality) *Let a,b be positive real numbers. If α, β are positive real numbers such that $\alpha+\beta=1$, then*

$$\alpha a+\beta b \geq a^{\alpha} b^{\beta}. \tag{8.66}$$

If $\alpha<0$ or $\alpha>1$, then the reversed inequality in (8.66) holds.

The following theorem contains, as the main result here, a refinement of the converse Hölder inequality.

Theorem 8.26 *Let L satisfy L1, L2 on a nonempty set E and let A be a positive linear functional. Let $p\in\mathbb{R}$, $q=\frac{p}{p-1}$, and $w,f,g\geq 0$ on E with wf^p, wg^q, $wfg\in L$.*

Let m,M be such that $0<m\leq f(x)g^{-q/p}(x)\leq M$ for $x\in E$.

If $p>1$, then

$$\begin{aligned} A(wfg) &\geq K(p,m,M)A^{\frac{1}{p}}(wf^p)A^{\frac{1}{q}}(wg^q)+\Delta(g^q,fg)N(p,m,M) && (8.67)\\ &\geq K(p,m,M)A^{\frac{1}{p}}(wf^p)A^{\frac{1}{q}}(wg^q), && (8.68)\end{aligned}$$

where

$$K(p,m,M)=|p|^{\frac{1}{p}}|q|^{\frac{1}{q}}\frac{(M-m)^{\frac{1}{p}}|mM^p-Mm^p|^{\frac{1}{q}}}{|M^p-m^p|},$$

$$N(p,m,M)=\frac{m^p+M^p-2\left(\frac{m+M}{2}\right)^p}{M^p-m^p}$$

and

$$\Delta(g^q,fg)=A\left(w\left(\frac{M-m}{2}g^q-\left|fg-\frac{m+M}{2}g^q\right|\right)\right).$$

If $0<p<1$ and $A(wg^q)>0$, or $p<0$ and $A(wf^p)>0$, then the reversed inequalities in (8.67) and (8.68) hold.

Proof. Putting in (8.65) $p_1=\alpha$, $p_2=\beta$ where α and β are positive real numbers such that $\alpha+\beta=1$, $q_1=q_2=\min\{\alpha,\beta\}$, $\phi(x)=x^p$, $p>1$, we have:

$$(\alpha x+\beta y)^p\leq \alpha x^p+\beta y^p-\min\{\alpha,\beta\}\left(x^p+y^p-2\left(\frac{x+y}{2}\right)^p\right). \tag{8.69}$$

Let h be a function in L such that $0<m\leq h(x)\leq M$ for $x\in E$, $m\neq M$, and define α and β as follows:

$$\alpha(x)=\frac{M-h(x)}{M-m},\quad \beta(x)=\frac{h(x)-m}{M-m}.$$

Obviously, $\alpha(x)+\beta(x)=1$, $h(x)=\alpha(x)m+\beta(x)M$. Putting in (8.69): $x=m$, $y=M$, and above-defined $\alpha(x)$ and $\beta(x)$, we have

$$h^p(x)\leq\frac{M-h(x)}{M-m}m^p+\frac{h(x)-m}{M-m}M^p-\min\{\alpha(x),\beta(x)\}\left(m^p+M^p-2\left(\frac{m+M}{2}\right)^p\right).$$

Multiplying that inequality with $k(x) \geq 0$ and using linear functional A we obtain:

$$\begin{aligned} A(kh^p) \leq\ & \frac{m^p}{M-m}(MA(k)-A(kh)) + \frac{M^p}{M-m}(A(kh)-mA(k)) \\ & -A(k\min\{\alpha,\beta\})\left(m^p+M^p-2\left(\frac{m+M}{2}\right)^p\right). \end{aligned}$$

Using formula $\min\{\alpha,\beta\} = \frac{1}{2}(\alpha+\beta-|\beta-\alpha|)$, putting $h = fg^{-\frac{q}{p}}$, $k = wg^q$, where $\frac{1}{p}+\frac{1}{q}=1$ after multiplying with $M-m$ we get

$$\begin{aligned} & (M-m)A(wf^p) + (mM^p - Mm^p)A(wg^q) \\ & \quad + A\left(w\left(\frac{M-m}{2}g^q - \left|fg - \frac{m+M}{2}g^q\right|\right)\right)\left[m^p+M^p-2\left(\frac{m+M}{2}\right)^p\right] \\ & \leq (M^p-m^p)A(wfg). \end{aligned} \tag{8.70}$$

In the following text, the term $A\left(w\left(\frac{M-m}{2}F - \left|G-\frac{m+M}{2}F\right|\right)\right)$ is denoted by $\Delta(F,G)$.

Using AG inequality (8.66) with $\alpha = \frac{1}{p} > 0$, $\beta = \frac{1}{q} > 0$, $a = p(M-m)A(wf^p) \geq 0$ and $b = q(mM^p - Mm^p)A(wg^q) \geq 0$ we obtain:

$$\begin{aligned} & (M-m)A(wf^p) + (mM^p - Mm^p)A(wg^q) \\ & = \frac{p}{p}(M-m)A(wf^p) + \frac{q}{q}(mM^p - Mm^p)A(wg^q) \\ & \geq p^{\frac{1}{p}}q^{\frac{1}{q}}(M-m)^{\frac{1}{p}}(mM^p-Mm^p)^{\frac{1}{q}}A^{\frac{1}{p}}(wf^p)A^{\frac{1}{q}}(wg^q). \end{aligned} \tag{8.71}$$

Combining (8.70) and (8.71) and rearranging, we finally have

$$\begin{aligned} & p^{\frac{1}{p}}q^{\frac{1}{q}}(M-m)^{\frac{1}{p}}(mM^p-Mm^p)^{\frac{1}{q}}A^{\frac{1}{p}}(wf^p)A^{\frac{1}{q}}(wg^q) \\ & \quad + \Delta(g^q, fg)\left[m^p+M^p-2\left(\frac{m+M}{2}\right)^p\right] \leq (M^p-m^p)A(wfg). \end{aligned}$$

If $p > 1$, then $M^p - m^p > 0$, and after dividing with $M^p - m^p$ we get

$$K(p,m,M)A^{\frac{1}{p}}(wf^p)A^{\frac{1}{q}}(wg^q) + \Delta(g^q, fg)N(p,m,M) \leq A(wfg), \tag{8.72}$$

where $K(p,m,M)$ is a constant from (8.64) and $N(p,m,M)$ is a constant defined as

$$N(p,m,M) = \frac{m^p + M^p - 2\left(\frac{m+M}{2}\right)^p}{M^p - m^p}. \tag{8.73}$$

Since the term $\Delta(g^q, fg)N(p,m,M)$ is non-negative for $p > 1$, inequality (8.72) is an improvement of the converse Hölder inequality (8.63).

Let us discuss other cases for exponent p.

Let $p<0$. Then the function $x\mapsto x^p$ is also convex on $(0,\infty)$, so inequality (8.70) holds. Also we want to use AG inequality, but now, $\alpha<0$, $a<0$ and $b\leq 0$ since in this case $mM^p-Mm^p\leq 0$. So, we have $\alpha a+\beta b=-(\alpha|a|+\beta|b|)\geq -|a|^{\alpha}|b|^{\beta}$ and

$$\begin{aligned}
&(M-m)A(wf^p)+(mM^p-Mm^p)A(wg^q)\\
&=-\left(\frac{1}{p}|p(M-m)A(wf^p)|+\frac{1}{q}|q(mM^p-Mm^p)A(wg^q)|\right)\\
&\geq -|p|^{\frac{1}{p}}|q|^{\frac{1}{q}}(M-m)^{\frac{1}{p}}|mM^p-Mm^p|^{\frac{1}{q}}A^{\frac{1}{p}}(wf^p)A^{\frac{1}{q}}(wg^q).
\end{aligned}$$

Combining above inequality with (8.70) and multiplying with -1 we obtain

$$\begin{aligned}
&|p|^{\frac{1}{p}}q^{\frac{1}{q}}(M-m)^{\frac{1}{p}}|mM^p-Mm^p|^{\frac{1}{q}}A^{\frac{1}{p}}(wf^p)A^{\frac{1}{q}}(wg^q)\\
&\quad-\Delta(g^q,fg)\left[m^p+M^p-2\left(\frac{m+M}{2}\right)^p\right]\\
&\geq -(M^p-m^p)A(wfg)=|M^p-m^p|A(wfg).
\end{aligned}$$

A term $m^p+M^p-2\left(\frac{m+M}{2}\right)^p$ is positive because it is a consequence of the Jensen inequality for a strictly convex function $x\mapsto x^p$, $p<0$. After dividing with $|M^p-m^p|=-(M^p-m^p)$ we obtain

$$K(p,m,M)A^{\frac{1}{p}}(wf^p)A^{\frac{1}{q}}(wg^q)+\Delta(g^q,fg)N(p,m,M)\geq A(wfg). \tag{8.74}$$

Let us point out that in this case the factor $N(p,m,M)$ is negative.

If $0<p<1$, then $x\mapsto x^p$ is concave on $[0,\infty)$ and in (8.70) reversed sign holds. Using AG inequality for $\alpha=\frac{1}{p}>1$, $\beta=\frac{1}{q}<0$, $a=p(M-m)A(wf^p)\geq 0$ and $b=q(mM^p-Mm^p)A(wg^q)=|q|\cdot|mM^p-Mm^p|A(wg^q)\geq 0$ we obtain:

$$\begin{aligned}
&p^{\frac{1}{p}}|q|^{\frac{1}{q}}(M-m)^{\frac{1}{p}}|mM^p-Mm^p|^{\frac{1}{q}}A^{\frac{1}{p}}(wf^p)A^{\frac{1}{q}}(wg^q)\\
&\quad+\ \Delta(g^q,fg)\left[m^p+M^p-2\left(\frac{m+M}{2}\right)^p\right]\geq (M^p-m^p)A(wfg).
\end{aligned}$$

In this case $M^p-m^p>0$ and dividing above inequality with M^p-m^p we obtain (8.74). Let us mention that in this case $m^p+M^p-2\left(\frac{m+M}{2}\right)^p$ is negative, so the factor $N(p,m,M)$ is negative. □

One of the numerous generalizations of the Hölder inequality is the well-known Beckenbach inequality ([214]). Here we pay attention to the converse Beckenbach inequality. In [157] the following result (slightly modified) is given.

Theorem 8.27 *Suppose that $\frac{1}{p}+\frac{1}{q}=1$, $a,b,c,x_i,y_i>0$ and $z_i=\left(\frac{ay_i}{b}\right)^{q/p}$, $(i=1,2,\dots,n)$. Let there exist positive numbers m and M such that*

$$m\leq\left(\frac{a}{b}\right)^{q/p}\leq M \ \text{ and } \ m\leq\frac{x_i}{y_i^{q/p}}\leq M,\ \ i=1,2,\dots,n.$$

If $p > 1$, then

$$\frac{\left(a+c\sum_{i=1}^{n}x_i^p\right)^{\frac{1}{p}}}{b+c\sum_{i=1}^{n}x_iy_i} \le \frac{1}{K(p,m,M)}\frac{\left(a+c\sum_{i=1}^{n}z_i^p\right)^{\frac{1}{p}}}{b+c\sum_{i=1}^{n}z_iy_i}. \tag{8.75}$$

If $p < 1$ $(p \neq 0)$, the reverse inequality holds in (8.75).

The following theorem provides a refinement of the above-mentioned converse Beckenbach inequality.

Theorem 8.28 *Suppose that assumptions of Theorem 8.27 hold. If $p > 1$, then*

$$\begin{aligned}\frac{\left(a+c\sum_{i=1}^{n}x_i^p\right)^{\frac{1}{p}}}{b+c\sum_{i=1}^{n}x_iy_i} &\le \frac{1}{K(p,m,M)}\frac{\left(a+c\sum_{i=1}^{n}z_i^p\right)^{\frac{1}{p}}}{b+c\sum_{i=1}^{n}z_iy_i}\left(1-\frac{N(p,m,M)\Delta}{b+c\sum_{i=1}^{n}x_iy_i}\right)\\ &\le \frac{1}{K(p,m,M)}\frac{\left(a+c\sum_{i=1}^{n}z_i^p\right)^{\frac{1}{p}}}{b+c\sum_{i=1}^{n}z_iy_i},\end{aligned}$$

where

$$\Delta = \frac{M-m}{2}\left(a^{-\frac{q}{p}}b^q + c\sum_{i=1}^{n}y_i^q\right) - \left|b - \frac{m+M}{2a^{q/p}b^{-q}}\right| - c\sum_{i=1}^{n}\left|x_iy_i - \frac{m+M}{2}y_i^q\right|$$

and $K(p,m,M)$ is defined as in (8.64). If $p < 1$ $(p \neq 0)$, the reverse inequalities hold.

Proof. Let $p > 1$. From equality $\frac{q}{p}+1 = q$ we have

$$\left(\frac{ay_i}{b}\right)^{\frac{q}{p}}y_i = \left(\frac{a}{b}\right)^{\frac{q}{p}}y_i^q,$$

and using that equality we obtain

$$\begin{aligned}\frac{\left(a+c\sum_{i=1}^{n}z_i^p\right)^{\frac{1}{p}}}{b+c\sum_{i=1}^{n}z_iy_i} &= \frac{\left(a+\left(\frac{a}{b}\right)^q c\sum_{i=1}^{n}y_i^q\right)^{\frac{1}{p}}}{b+\left(\frac{a}{b}\right)^{\frac{q}{p}}c\sum_{i=1}^{n}y_i^q}\\ &= \left(a^{-\frac{q}{p}}b^q + c\sum_{i=1}^{n}y_i^q\right)^{-\frac{1}{q}}.\end{aligned} \tag{8.76}$$

The product $\left(a+c\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}}\left(a^{-\frac{q}{p}}b^q+c\sum_{i=1}^{n} y_i^q\right)^{\frac{1}{q}}$ is one side of the Hölder inequality for two sequences: $(a^{\frac{1}{p}},x_1,\ldots,x_n)$ and $(a^{-\frac{1}{p}}b,y_1,\ldots,y_n)$ with weights $(1,c,\ldots,c)$. Using inequality (8.67) we get

$$\left(a+c\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}}\left(a^{-\frac{q}{p}}b^q+c\sum_{i=1}^{n} y_i^q\right)^{\frac{1}{q}} \leq \frac{1}{K(p,m,M)}\left(b+c\sum_{i=1}^{n} x_iy_i-\Delta\cdot N(p,m,M)\right),$$

where K and N are defined in Theorem 8.26 and Δ is defined in Theorem 8.28. Dividing the above inequality with $(b+c\sum_{i=1}^{n} x_iy_i)(a^{-q/p}b^q+c\sum_{i=1}^{n} y_i^q)^{1/q}$ and using result (8.76), we get the desired improvement. □

Now we investigate the converse Minkowski inequality for functionals and the converse of the continuous form of the Minkowski inequality. In [177, p. 116] one can find the following form of the converse Minkowski inequality for functionals.

Theorem 8.29 *Let A,p,q,w,f,g be as in Theorem 8.26 with additional property $w(f+g)^p\in L$. Let m and M be such that $0<m<f(x)(f(x)+g(x))^{-1}\leq M$ and $0<m<g(x)(f(x)+g(x))^{-1}\leq M$, for $x\in E$.*

If $p>1$, then

$$A^{\frac{1}{p}}(w(f+g)^p)\geq K(p,m,M)\cdot\left(A^{\frac{1}{p}}(wf^p)+A^{\frac{1}{p}}(wg^p)\right), \tag{8.77}$$

where $K(p,m,M)$ is defined as in (8.64).

If $0<p<1$ or if $p<0$, then the reverse inequality in (8.77) holds, provided that $A(w(f+g)^p)>0$, for $p<0$.

Using the improvement of the converse Hölder inequality, we can prove the following improvement of the converse Minkowski inequality for functionals.

Theorem 8.30 *Let the assumptions of Theorem 8.29 be satisfied. Then for $p>1$*

$$\begin{aligned} A^{\frac{1}{p}}(w(f+g)^p) \;\geq\; & K(p,m,M)\left(A^{\frac{1}{p}}(wf^p)+A^{\frac{1}{p}}(wg^p)\right) \\ & +N(p,m,M)\frac{\Delta((f+g)^p,f(f+g)^{p-1})+\Delta((f+g)^p,g(f+g)^{p-1})}{A^{1-\frac{1}{p}}(w(f+g)^p)}, \end{aligned} \tag{8.78}$$

and for $p<1$ ($p\neq 0$) the reversed inequality holds.

Proof. Let $p>1$. Writing $A(w(f+g)^p)$ as

$$A(w(f+g)(f+g)^{p-1})=A(wf(f+g)^{p-1}+wg(f+g)^{p-1})$$

and using inequality (8.67) we obtain

$$\begin{aligned} &A(w(f+g)^p)=A(wf(f+g)^{p-1})+A(wg(f+g)^{p-1}) \\ &\geq K(p,m,M)A^{\frac{1}{p}}(wf^p)A^{\frac{1}{q}}(w(f+g)^p)+\Delta((f+g)^p,f(f+g)^{p-1})N(p,m,M) \\ &\quad +K(p,m,M)A^{\frac{1}{p}}(wg^p)A^{\frac{1}{q}}(w(f+g)^p)+\Delta((f+g)^p,g(f+g)^{p-1})N(p,m,M) \end{aligned}$$

$$= K(p,m,M)A^{\frac{1}{q}}(w(f+g)^p)\left(A^{\frac{1}{p}}(wf^p)+A^{\frac{1}{p}}(wg^p)\right)$$
$$+N(p,m,M)\left(\Delta((f+g)^p,f(f+g)^{p-1})+\Delta((f+g)^p,g(f+g)^{p-1})\right).$$

Dividing by $A^{\frac{1}{q}}(w(f+g)^p)$ we get desired result.

If $p>1$, then the second term in the sum on the right-hand side in (8.78) is non-negative and inequality (8.78) is a refinement of the known converse (8.77). Similar proof holds for $p<1$, $(p \neq 0)$. □

The previous investigation does not cover the so-called Minkowski integral inequality. Let (X,Σ_X,μ) and (Y,Σ_Y,ν) be two measure spaces with σ-finite measures μ and ν respectively. Let f be a non-negative function on $X\times Y$ which is integrable with respect to the measure $\mu\times\nu$.

If $p\geq 1$, then

$$\left[\int_X\left(\int_Y f(x,y)d\nu(y)\right)^p d\mu(x)\right]^{\frac{1}{p}} \leq \int_Y\left(\int_X f^p(x,y)d\mu(x)\right)^{\frac{1}{p}} d\nu(y). \tag{8.79}$$

The above result is also called “the continuous form of the Minkowski inequality” or “the Minkowski inequality for infinitely many functions” and, for example, it can be found in [128, p. 41]. Considering the proof of this inequality we can conclude that there exist a related result for other values of the exponent p (see [87]).

If $0<p<1$ and

$$\int_X\left(\int_Y f(x,y)d\nu(y)\right)^p d\mu(x)>0, \int_Y f(x,y)d\nu(y)>0, (\mu-a.e.) \tag{8.80}$$

then the reverse inequality holds.

If $p<0$ and the above-mentioned assumptions (8.80) hold as well as the additional one:

$$\int_X f^p(x,y)d\mu(x)>0 \quad (\nu-a.e.), \tag{8.81}$$

then the reverse inequality holds.

As we know, in the literature there is no result corresponding to the converse of the above mentioned results. In the following theorem we state a converse of that variant of the Minkowski inequality.

Theorem 8.31 *Let (X,Σ_X,μ) and (Y,Σ_Y,ν) be two measure spaces with σ-finite measures μ and ν respectively. Let f be a non-negative function on $X\times Y$ which is integrable with respect to the measure $\mu\times\nu$.*

If $0 < m \leq \dfrac{f(x,y)}{\int_Y f(x,y)d\nu(y)} \leq M$ *for all* $x \in X$, $y \in Y$, *then for* $p \geq 1$

$$\left[\int_X \left(\int_Y f(x,y)d\nu(y)\right)^p d\mu(x)\right]^{\frac{1}{p}}$$

$$\geq K(p,m,M)\int_Y \left(\int_X f^p(x,y)d\mu(x)\right)^{\frac{1}{p}} d\nu(y) + N(p,m,M)\left[\int_X H^p(x)d\mu(x)\right]^{\frac{1-p}{p}} \Delta_1 \quad (8.82)$$

$$\geq K(p,m,M)\int_Y \left(\int_X f^p(x,y)d\mu(x)\right)^{\frac{1}{p}} d\nu(y), \quad (8.83)$$

where $K(p,m,M)$ *is defined with (8.64),* $N(p,m,M)$ *is defined with (8.73),* $H(x) = \int_Y f(x,y)d\nu(y)$ *and*

$$\Delta_1 = \int_Y \left(\int_X \left(\frac{m-M}{2}H^p(x) - \left|f(x,y)H^{p-1}(x) - \frac{m+M}{2}H^p(x)\right|\right) d\mu(x)\right) d\nu(y).$$

If $0 < p < 1$ *with (8.80) or* $p < 0$ *with (8.80) and (8.81), then the reversed inequality holds.*

Proof. Let us denote

$$H(x) = \int_Y f(x,y)d\nu(y).$$

Using Fubini's theorem we get

$$\begin{aligned}
\int_X \left(\int_Y f(x,y)d\nu(y)\right)^p d\mu(x) &= \int_X H^p(x)d\mu(x) \\
&= \int_X \left(\int_Y f(x,y)d\nu(y)\right) H^{p-1}(x)d\mu(x) \\
&= \int_Y \left(\int_X f(x,y)H^{p-1}(x)d\mu(x)\right) d\nu(y).
\end{aligned}$$

Using (8.68) for functional $A(\phi) = \int_X \phi(x)d\mu(x)$ we get

$$\int_Y \left(\int_X f(x,y)H^{p-1}(x)d\mu(x)\right) d\nu(y)$$

$$\geq K(p,m,M)\int_Y \left(\int_X f^p(x,y)d\mu(x)\right)^{1/p} \left(\int_X H^p(x)d\mu(x)\right)^{\frac{p-1}{p}} d\nu(y) + N(p,m,M)\Delta_1$$

$$\geq K(p,m,M)\int_Y \left(\int_X f^p(x,y)d\mu(x)\right)^{1/p} \left(\int_X H^p(x)d\mu(x)\right)^{\frac{p-1}{p}} d\nu(y).$$

Dividing by $\left(\int_X H^p(x)d\mu(x)\right)^{\frac{p-1}{p}}$ we get inequalities (8.82) and (8.83). $\square$

8.4.1 Application to mixed means

Let r,s be two positive numbers, $r<s$. Let us put $p=\frac{s}{r}$ and $f\to f^r$ in inequalities (8.79) and (8.83). After powering by $\frac{1}{r}$ and dividing with $(\mu(X))^{1/s}$ and $(\nu(Y))^{1/r}$ we get the following relations

$$\left[\frac{\int_X\left(\frac{\int_Y f^r(x,y)d\nu(y)}{\nu(Y)}\right)^{\frac{s}{r}}d\mu(x)}{\mu(X)}\right]^{\frac{1}{s}}\le\left[\frac{\int_Y\left(\frac{\int_X f^s(x,y)d\mu(x)}{\mu(X)}\right)^{\frac{r}{s}}d\nu(y)}{\nu(Y)}\right]^{\frac{1}{r}}\tag{8.84}$$

and

$$\left[\frac{\int_X\left(\frac{\int_Y f^r(x,y)d\nu(y)}{\nu(Y)}\right)^{\frac{s}{r}}d\mu(x)}{\mu(X)}\right]^{\frac{1}{s}}\ge K^{\frac{1}{r}}\left(\frac{s}{r},m,M\right)\left[\frac{\int_Y\left(\frac{\int_X f^s(x,y)d\mu(x)}{\mu(X)}\right)^{\frac{r}{s}}d\nu(y)}{\nu(Y)}\right]^{\frac{1}{r}},\tag{8.85}$$

where in (8.85) m and M are real numbers such that $0<m\le\dfrac{f^r(x,y)}{\int_Y f^r(x,y)d\nu(y)}\le M$. Using notation

$$M^{[r]}(f,\mu)=\begin{cases}\left(\frac{\int_X f^r(x)d\mu(x)}{\mu(X)}\right)^{\frac{1}{r}}; & r\neq 0\\ \exp\left(\frac{\int_X\log f(x)d\mu(x)}{\mu(X)}\right); & r=0\end{cases}$$

in inequalities (8.84) and (8.85) we obtain the following theorem.

Theorem 8.32 *Let the assumptions of Theorem 8.31 be valid. If $r<s$, $r,s\neq 0$, then*

$$M^{[s]}\left(M^{[r]}(f,\nu),\mu\right)\le M^{[r]}\left(M^{[s]}(f,\mu),\nu\right).\tag{8.86}$$

If m and M are real numbers such that $0<m\le\dfrac{f^r(x,y)}{\int_Y f^r(x,y)d\nu(y)}\le M$, then

$$M^{[s]}\left(M^{[r]}(f,\nu),\mu\right)\ge K^{\frac{1}{r}}\left(\frac{s}{r},m,M\right)\cdot M^{[r]}\left(M^{[s]}(f,\mu),\nu\right),\tag{8.87}$$

where K is defined by (8.64)

Making use of (8.82), the refinement of the above mixed mean inequality can be obtained. These are inequalities for mixed means, the second one is a converse of the first inequality. Discrete version of (8.86) is given in [151, p. 109], while its converse is a new result. It is instructive to calculate mixed means for some special spaces and measures.

Corollary 8.8 *Let $a,b,\alpha,\gamma,r,s\in\mathbb{R}$ be such that $a<b$, $r<s$, $\alpha,\gamma>0$, $r,s\neq 0$. If $g:[a,b]\to\mathbb{R}$ is a non-negative measurable function, then the following inequality holds:*

$$\left[\frac{1}{(b-a)^{\gamma}}\int_a^b(y-a)^{\gamma-1}\left(\frac{1}{(y-a)^{\alpha}}\int_a^y g^r(t)(t-a)^{\alpha-1}dt\right)^{\frac{s}{r}}dy\right]^{\frac{1}{s}}\tag{8.88}$$
$$\le\left[\frac{1}{(b-a)^{\alpha}}\int_a^b(y-a)^{\alpha-1}\left(\frac{1}{(y-a)^{\gamma}}\int_a^y g^s(t)(t-a)^{\gamma-1}dt\right)^{\frac{r}{s}}dy\right]^{\frac{1}{r}}.$$

Furthermore, if m and M are real numbers such that for $x \in [0,1]$, $y \in [a,b]$

$$0 < m \leq \frac{g^r(a+x(y-a))}{\int_a^b g^r(a+x(y-a))(y-a)^{\alpha-1}dy} \leq M,$$

then

$$\left[\frac{1}{(b-a)^\gamma}\int_a^b (y-a)^{\gamma-1}\left(\frac{1}{(y-a)^\alpha}\int_a^y g^r(t)(t-a)^{\alpha-1}dt\right)^{\frac{s}{r}} dy\right]^{\frac{1}{s}} \tag{8.89}$$
$$\geq K^{\frac{1}{r}}(\frac{s}{r},m,M)\cdot\left[\frac{1}{(b-a)^\alpha}\int_a^b (y-a)^{\alpha-1}\left(\frac{1}{(y-a)^\gamma}\int_a^y g^s(t)(t-a)^{\gamma-1}dt\right)^{\frac{r}{s}} dy\right]^{\frac{1}{r}}.$$

Proof. Let us put in (8.86) the following: $X=[0,1]$, $Y=[a,b]$, $d\mu(x)=x^{\gamma-1}dx$ and $d\nu(y)=(y-a)^{\alpha-1}dy$, $\alpha,\gamma\in\mathbb{R}\backslash\{0\}$, $f(x,y)=g(a+x(y-a))$ where g is a non-negative measurable function. Then $\nu(Y)=\frac{1}{\alpha}(b-a)^\alpha$ and $\mu(X)=\frac{1}{\gamma}$.

After substitutions, inequality (8.84) becomes

$$\left[\frac{\gamma\alpha^{s/r}}{(b-a)^{\alpha s/r}}\int_0^1\left(\int_a^b g^r(a+x(y-a))(y-a)^{\alpha-1}dy\right)^{\frac{s}{r}} x^{\gamma-1}dx\right]^{\frac{1}{s}} \tag{8.90}$$
$$\leq\left[\frac{\gamma^{r/s}\alpha}{(b-a)^\alpha}\int_a^b\left(\int_0^1 g^s(a+x(y-a))x^{\gamma-1}dx\right)^{\frac{r}{s}}(y-a)^{\alpha-1}dy\right]^{\frac{1}{r}}.$$

Putting in the right-hand side of the inequality a new variable $t=a+x(y-a)$, we get that the right-hand side has a form

$$\left[\frac{\gamma^{r/s}\alpha}{(b-a)^\alpha}\int_a^b\left(\frac{1}{(y-a)^\gamma}\int_a^y g^s(t)(t-a)^{\gamma-1}dt\right)^{\frac{r}{s}}(y-a)^{\alpha-1}dy\right]^{\frac{1}{r}}. \tag{8.91}$$

The same substitution is done in the left-hand side of (8.90) and we get that the left-hand side is equal to:

$$\left[\frac{\gamma\alpha^{s/r}}{(b-a)^{\alpha s/r}}\int_0^1 x^{\gamma-1}\left(\frac{1}{x^\alpha}\int_a^{a+x(b-a)} g^r(t)(t-a)^{\alpha-1}dt\right)^{\frac{s}{r}} dx\right]^{\frac{1}{s}}.$$

By the change of the variable $y=a+x(b-a)$ in outer integral, it is further equal to

$$\left[\frac{\gamma\alpha^{s/r}}{(b-a)^{\alpha s/r}}\int_a^b\frac{(y-a)^{\gamma-1}}{(b-a)^{\gamma-1}}\left(\frac{(b-a)^\alpha}{(y-a)^\alpha}\int_a^y g^r(t)(t-a)^{\alpha-1}dt\right)^{\frac{s}{r}}\frac{dy}{b-a}\right]^{\frac{1}{s}}.$$

Finally, we get

$$\left[\frac{1}{(b-a)^{\gamma}}\int_a^b (y-a)^{\gamma-1}\left(\frac{1}{(y-a)^{\alpha}}\int_a^y g^r(t)(t-a)^{\alpha-1}dt\right)^{\frac{s}{r}} dy\right]^{\frac{1}{s}} \tag{8.92}$$

$$\leq \left[\frac{1}{(b-a)^{\alpha}}\int_a^b (y-a)^{\alpha-1}\left(\frac{1}{(y-a)^{\gamma}}\int_a^y g^s(t)(t-a)^{\gamma-1}dt\right)^{\frac{r}{s}} dy\right]^{\frac{1}{r}},$$

where $\alpha,\gamma > 0$, $r < s$, $r,s \neq 0$.

From (8.85), using the same substitutions, the converse of (8.88) follows. □

Let us point out that inequality (8.88) was firstly obtained in [51, Theorem 3] and it was used for proving the well-known Hardy's inequality. Also, let us mention that the above inequalities about mixed means can be refined like it was done with the inequalities in previous sections.

Chapter 9

Further improvements and generalizations of the Jessen-Mercer inequality

As we have already comprehended, Lemma 1.2 has a great impact considering the improvements and generalizations of a variety of the Jensen-type inequalities. We proceed with the applications of this important monotonicity property in a similar manner. Thus in the first section of this chapter, two improvements of the Jessen-Mercer inequality are presented, and in the second section - a generalization of the Jessen-Mercer inequality (as a consequence of the analogous one for the Jensen inequality) on convex hulls, with an improvement obtained by means of Lemma 1.2. What follows are the k-dimensional variant of the Hammer-Bullen inequality, as well as an improvement of the classical Hermite-Hadamard inequality. Two functionals (Jessen-Mercer differences) are consequently defined and Lagrange and Cauchy type mean value theorems are proved in this setting, too. Hence it was possible to construct a large family of functions which are n-exponentially convex and exponentially convex, which is finally presented at the end of the chapter.

Most of the results presented in this chapter were previously published in [132].

9.1 Improvements of the Jessen-Mercer inequality

Jensen-Mercer inequality (1.12) generalized by means of a positive linear functional is called Jessen-Mercer's inequality ([49]) and was cited in Theorem 1.13 in the introductory part of the monograph: for a positive normalized linear functional A, for L satisfying $L1$ and $L2$ and for a continuous convex function φ defined on $[m,M]$, the inequality

$$\varphi(m+M-A(f)) \leq \varphi(m)+\varphi(M)-A(\varphi(f)) \tag{9.1}$$

holds for all $f\in L$, such that $\varphi(f), \varphi(m+M-f)\in L$ (so that $m\leq f(t)\leq M$ for all $t\in E$), and is reversed if φ is concave.

Remark 9.1 In fact, to be more specific, the following series of inequalities was proved:

$$\begin{aligned}\varphi(m+M-A(f)) &\leq A(\varphi(m+M-f))\\ &\leq \frac{M-A(f)}{M-m}\varphi(M)+\frac{A(f)-m}{M-m}\varphi(m)\\ &\leq \varphi(m)+\varphi(M)-A(\varphi(f)).\end{aligned} \tag{9.2}$$

Furthermore, according to [102] and already cited Theorem 8.5, it was proved that if φ is a convex function defined on $[m,M]$, then for all $g\in L$ such that $\varphi(g)\in L$ we have $A(g)\in[m,M]$ and the following inequality holds:

$$A(\varphi(g)) \leq \frac{M-A(g)}{M-m}\varphi(m)+\frac{A(g)-m}{M-m}\varphi(M)-A(\tilde{g})\,\delta_{\varphi}, \tag{9.3}$$

where

$$\tilde{g}=\frac{1}{2}-\frac{1}{M-m}\left|g-\frac{m+M}{2}\right|,\quad \delta_{\varphi}=\varphi(m)+\varphi(M)-2\varphi\left(\frac{m+M}{2}\right).$$

Utilizing inequality (9.3) and Lemma 8.2, which is the generalization of Lemma 1.2 on convex sets, we will refine the series of inequalities (9.2).

The following two theorems are the main results.

Theorem 9.1 *Let L satisfy $L1$, $L2$, $L3$ on a nonempty set E, and let A be a positive normalized linear functional. If φ is a continuous convex function on $[m,M]$, then for all $f\in L$ such that $\varphi(f), \varphi(m+M-f)\in L$, we have*

$$\begin{aligned}\varphi(m+M-A(f)) &\leq A(\varphi(m+M-f))\\ &\leq \frac{M-A(f)}{M-m}\varphi(M)+\frac{A(f)-m}{M-m}\varphi(m)-A\left(\frac{1}{2}-\frac{1}{M-m}\left|f-\frac{m+M}{2}\right|\right)\delta_{\varphi}\\ &\leq \varphi(m)+\varphi(M)-A(\varphi(f))-\left[1-\frac{2}{M-m}A\left(\left|f-\frac{m+M}{2}\right|\right)\right]\delta_{\varphi}\\ &\leq \varphi(m)+\varphi(M)-A(\varphi(f)),\end{aligned}$$

where

$$\delta_\varphi = \varphi(m) + \varphi(M) - 2\varphi\left(\frac{m+M}{2}\right). \tag{9.4}$$

Proof. Using the first inequality from the series (9.2) and applying inequality (9.3), firstly to the function $g = m + M - f$, and then to the function f, we obtain

$$\begin{aligned}
&\varphi(m+M-A(f)) \\
&\leq A(\varphi(m+M-f)) \\
&\leq \frac{M-A(f)}{M-m}\varphi(M) + \frac{A(f)-m}{M-m}\varphi(m) - A\left(\frac{1}{2} - \frac{1}{M-m}\left|f - \frac{m+M}{2}\right|\right)\delta_\varphi \\
&= \varphi(m) + \varphi(M) - \left[\frac{M-A(f)}{M-m}\varphi(m) + \frac{A(f)-m}{M-m}\varphi(M)\right] - A\left(\frac{1}{2} - \frac{1}{M-m}\left|f - \frac{m+M}{2}\right|\right)\delta_\varphi \\
&\leq \varphi(m) + \varphi(M) - A(\varphi(f)) - 2A\left(\frac{1}{2} - \frac{1}{M-m}\left|f - \frac{m+M}{2}\right|\right)\delta_\varphi \\
&= \varphi(m) + \varphi(M) - A(\varphi(f)) - \left[1 - \frac{2}{M-m}A\left(\left|f - \frac{m+M}{2}\right|\right)\right]\delta_\varphi \\
&\leq \varphi(m) + \varphi(M) - A(\varphi(f)).
\end{aligned}$$

The last inequality is a simple consequence of the easily provable facts that $\delta_\varphi = \varphi(m) + \varphi(M) - 2\varphi\left(\frac{m+M}{2}\right) \geq 0$ and $1 - \frac{2}{M-m}A\left(\left|f - \frac{m+M}{2}\right|\right) \geq 0$. □

Theorem 9.2 *Let L satisfy L1, L2, L3 on a nonempty set E, and let A be a positive normalized linear functional. If φ is a continuous convex function on $[m,M]$, then for all $f \in L$ such that $\varphi(f), \varphi(m+M-f) \in L$, we have*

$$\begin{aligned}
&\varphi(m+M-A(f)) \\
&\leq \frac{M-A(f)}{M-m}\varphi(M) + \frac{A(f)-m}{M-m}\varphi(m) - \left(\frac{1}{2} - \frac{1}{M-m}\left|A(f) - \frac{m+M}{2}\right|\right)\delta_\varphi \\
&\leq \varphi(m) + \varphi(M) - A(\varphi(f)) - \left[1 - \frac{1}{M-m}\left(A\left(\left|f - \frac{m+M}{2}\right|\right) + \left|A(f) - \frac{m+M}{2}\right|\right)\right]\delta_\varphi \\
&\leq \varphi(m) + \varphi(M) - A(\varphi(f)) - \left[1 - \frac{2}{M-m}A\left(\left|f - \frac{m+M}{2}\right|\right)\right]\delta_\varphi \\
&\leq \varphi(m) + \varphi(M) - A(\varphi(f)),
\end{aligned}$$

where δ_φ is defined as in (9.4).

Proof. Inequality (9.3) provides:

$$A(\varphi(f)) \leq \frac{M-A(f)}{M-m}\varphi(m) + \frac{A(f)-m}{M-m}\varphi(M) - A\left(\frac{1}{2} - \frac{1}{M-m}\left|f - \frac{m+M}{2}\right|\right)\delta_\varphi. \tag{9.5}$$

Let the functions $p, q : [m,M] \to \mathbb{R}$ be defined by

$$p(t) = \frac{M-t}{M-m}, \qquad q(t) = \frac{t-m}{M-m}.$$

For any $t \in [m,M]$ we can write

$$\varphi(t) = \varphi(\frac{M-t}{M-m}m + \frac{t-m}{M-m}M) = \varphi(p(t)m + q(t)M).$$

By Lemma 8.2, for $n = 2$, it follows:

$$\varphi(t) \leq p(t)\varphi(m) + q(t)\varphi(M) - \min\{p(t), q(t)\}\delta_{\varphi},$$

where $\delta_{\varphi} = \varphi(m) + \varphi(M) - 2\varphi\left(\frac{m+M}{2}\right)$. Using (8.8) we can write it in the form

$$\varphi(t) \leq \frac{M-t}{M-m}\varphi(m) + \frac{t-m}{M-m}\varphi(M) - \left(\frac{1}{2} - \frac{1}{M-m}\left|t - \frac{m+M}{2}\right|\right)\delta_{\varphi}.$$

Substituting $t \leftrightarrow A(g)$, where $g \in L$ such that $A(g) \in [m,M]$, we get

$$\varphi(A(g)) \leq \frac{M-A(g)}{M-m}\varphi(m) + \frac{A(g)-m}{M-m}\varphi(M) - \left(\frac{1}{2} - \frac{1}{M-m}\left|A(g) - \frac{m+M}{2}\right|\right)\delta_{\varphi}. \tag{9.6}$$

Now, applying inequality (9.6) to $g = m + M - f$ (and using linearity and normality of A), and then using inequality (9.5), we have

$$\begin{aligned}
&\varphi(m+M-A(f)) \\
\leq\ & \frac{M-A(f)}{M-m}\varphi(M) + \frac{A(f)-m}{M-m}\varphi(m) - \left(\frac{1}{2} - \frac{1}{M-m}\left|A(f) - \frac{m+M}{2}\right|\right)\delta_{\varphi} \\
=\ & \varphi(m) + \varphi(M) - \left[\frac{M-A(f)}{M-m}\varphi(m) + \frac{A(f)-m}{M-m}\varphi(M)\right] - \left(\frac{1}{2} - \frac{1}{M-m}\left|A(f) - \frac{m+M}{2}\right|\right)\delta_{\varphi} \\
\leq\ & \varphi(m) + \varphi(M) - A(\varphi(f)) - A\left(\frac{1}{2} - \frac{1}{M-m}\left|f - \frac{m+M}{2}\right|\right)\delta_{\varphi} \\
& - \left(\frac{1}{2} - \frac{1}{M-m}\left|A(f) - \frac{m+M}{2}\right|\right)\delta_{\varphi} \\
=\ & \varphi(m) + \varphi(M) - A(\varphi(f)) - \left[1 - \frac{1}{M-m}\left(A\left(\left|f - \frac{m+M}{2}\right|\right) + \left|A(f) - \frac{m+M}{2}\right|\right)\right]\delta_{\varphi} \\
\leq\ & \varphi(m) + \varphi(M) - A(\varphi(f)) - \left[1 - \frac{2}{M-m}A\left(\left|f - \frac{m+M}{2}\right|\right)\right]\delta_{\varphi}.
\end{aligned}$$

The last inequality is obtained applying Jessen's inequality to the continuous and convex function $|x|$ so that

$$\left|A(f) - \frac{m+M}{2}\right| = \left|A\left(f - \frac{m+M}{2}\right)\right| \leq A\left(\left|f - \frac{m+M}{2}\right|\right).$$

□

Using Theorem 9.2 we can get an upper bound for the difference $A(\varphi(f)) - \varphi(A(f))$, obtained in [174].

Corollary 9.1 *Let L satisfy L1, L2, L3 on a nonempty set E, and let A be a positive normalized linear functional. If φ is a continuous convex function on $[m,M]$, then for all $f \in L$ such that $\varphi(f), \varphi(m+M-f) \in L$, we have*

$$A(\varphi(f)) - \varphi(A(f)) \leq \frac{1}{M-m}\left(A\left(\left|f - \frac{m+M}{2}\right|\right) + \left|A(f) - \frac{m+M}{2}\right|\right)\delta_{\varphi},$$

where δ_{φ} is defined as in (9.4).

Proof. Theorem 9.2 gives us

$$\begin{aligned} A(\varphi(f)) &\leq \varphi(m) + \varphi(M) - \varphi(A(f) - m + M) \\ &\quad - \left[1 - \frac{1}{M-m}\left(A\left(\left|f - \frac{m+M}{2}\right|\right) + \left|A(f) - \frac{m+M}{2}\right|\right)\right]\delta_{\varphi}. \end{aligned} \tag{9.7}$$

Since the function φ is convex, it follows that

$$\varphi(m+M-A(f)) + \varphi(A(f)) \geq 2\varphi\left(\frac{m+M}{2}\right). \tag{9.8}$$

Combining inequalities (9.7) and (9.8) we obtain

$$\begin{aligned} & A(\varphi(f)) - \varphi(A(f)) \\ \leq\ & \varphi(m) + \varphi(M) - [\varphi(m+M-A(f)) + \varphi(A(f))] \\ & - \left[1 - \frac{1}{M-m}\left(A\left(\left|f - \frac{m+M}{2}\right|\right) + \left|A(f) - \frac{m+M}{2}\right|\right)\right]\delta_{\varphi} \\ \leq\ & \varphi(m) + \varphi(M) - 2\varphi\left(\frac{m+M}{2}\right) - \left[1 - \frac{1}{M-m}\left(A\left(\left|f - \frac{m+M}{2}\right|\right) + \left|A(f) - \frac{m+M}{2}\right|\right)\right]\delta_{\varphi} \\ =\ & \frac{1}{M-m}\left(A\left(\left|f - \frac{m+M}{2}\right|\right) + \left|A(f) - \frac{m+M}{2}\right|\right)\delta_{\varphi}. \end{aligned}$$

□

9.2 Generalization on convex hulls

We present a generalization of the Jessen-Mercer inequality for convex functions on convex hulls in $\mathbb{R}^k$ and give its improvement. From these results we obtain a k-dimensional variant of the Hammer-Bullen inequality and an improvement of the classical Hermite-Hadamard inequality.

Let's notice that δ_{φ} from (9.3) is equal to $S_{\varphi}^{2}(m,M) = f(m) + f(M) - 2f\left(\frac{m+M}{2}\right)$.

The following variant of Jensen's inequality (generalization on convex hulls) was proved by A. Matković and J. Pečarić in [136].

Theorem 9.3 *Let U be a convex subset in $\mathbb{R}^k, x_1, \ldots, x_n \in U$ and $y_1, \ldots, y_m \in co(\{x_1, \ldots, x_n\})$. If φ is a convex function on U, then the inequality*

$$\varphi\left(\frac{\sum_{i=1}^n p_i x_i - \sum_{j=1}^m w_j y_j}{P_n - W_m}\right) \leq \frac{\sum_{i=1}^n p_i \varphi(x_i) - \sum_{j=1}^m w_j \varphi(y_j)}{P_n - W_m} \tag{9.9}$$

holds for all positive real numbers $p_1, \ldots, p_n$ and $w_1, \ldots, w_m$ satisfying the condition

$$p_i \geq W_m, \quad i = 1, \ldots, n,$$

where $P_n = \sum_{i=1}^n p_i$ and $W_m = \sum_{j=1}^m w_j$.

Our following theorem generalizes and improves Theorem 9.3.

Theorem 9.4 *Let L satisfy properties L1, L2, L3 on a nonempty set E, A be a positive linear functional on L and $\widetilde{A}$ be defined as in (8.21). Let $x_1, \ldots, x_n \in \mathbb{R}^k$ and $K = co(\{x_1, \ldots, x_n\})$. Let φ be a convex function on K and $\lambda_1, \ldots, \lambda_n$ be barycentric coordinates over K. Then for all $g \in L^k$ such that $g(E) \subset K$ and $\varphi(g), \lambda_i(g) \in L, i = 1, \ldots, n$, and positive real numbers $p_1, \ldots, p_n$, with $P_n = \sum_{i=1}^n p_i$, satisfying the condition*

$$p_i \geq A(\mathbf{1}), \quad i = 1, \ldots, n, \tag{9.10}$$

we have

$$\begin{aligned}
&\varphi\left(\frac{\sum_{i=1}^n p_i x_i - \widetilde{A}(g)}{P_n - A(\mathbf{1})}\right) \\
&\leq \frac{\sum_{i=1}^n p_i \varphi(x_i) - \sum_{i=1}^n A(\lambda_i(g))\varphi(x_i) - \min_i\{p_i - A(\lambda_i(g))\} S_\varphi^n(x_1, \ldots, x_n)}{P_n - A(\mathbf{1})} \\
&\leq \frac{\sum_{i=1}^n p_i \varphi(x_i) - A(\varphi(g)) - S_\varphi^n(x_1, \ldots, x_n)[\min_i\{p_i - A(\lambda_i(g))\} + A(\min_i\{\lambda_i(g)\})]}{P_n - A(\mathbf{1})}.
\end{aligned} \tag{9.11}$$

Proof. For each $t \in E$ we have $g(t) \in K$. Using barycentric coordinates we have $\lambda_i(g(t)) \geq 0$, $i = 1, \ldots, n$, $\sum_{i=1}^n \lambda_i(g(t)) = 1$ and

$$g(t) = \sum_{i=1}^n \lambda_i(g(t)) x_i.$$

Since φ is convex on K, it follows that

$$\varphi(g(t)) \leq \sum_{i=1}^n \lambda_i(g(t))\varphi(x_i) - \min_i\{\lambda_i(g(t))\} S_\varphi^n(x_1, \ldots, x_n). \tag{9.12}$$

Applying positive linear functional A to (9.12) we get

$$A(\varphi(g)) \leq \sum_{i=1}^{n} A(\lambda_i(g))\varphi(x_i) - A\left(\min_i \{\lambda_i(g)\}\right) S_\varphi^n(x_1,\ldots,x_n),$$

where

$$\sum_{i=1}^{n} A(\lambda_i(g)) = A\left(\sum_{i=1}^{n} \lambda_i(g)\right) = A(\mathbf{1})$$

and

$$A(\mathbf{1}) \geq A(\lambda_i(g)) \geq 0, \quad i = 1,\ldots,n.$$

Also we have

$$\widetilde{A}(g) = \sum_{i=1}^{n} A(\lambda_i(g))\, x_i.$$

Now we can write

$$\begin{aligned}
\frac{\sum_{i=1}^{n} p_i x_i - \widetilde{A}(g)}{P_n - A(\mathbf{1})} &= \frac{1}{P_n - A(\mathbf{1})}\left(\sum_{i=1}^{n} p_i x_i - \sum_{i=1}^{n} A(\lambda_i(g))\, x_i\right) \\
&= \frac{1}{P_n - A(\mathbf{1})} \sum_{i=1}^{n} (p_i - A(\lambda_i(g)))\, x_i.
\end{aligned}$$

We have

$$\frac{1}{P_n - A(\mathbf{1})} \sum_{i=1}^{n} (p_i - A(\lambda_i(g))) = 1$$

and

$$\frac{1}{P_n - A(\mathbf{1})} (p_i - A(\lambda_i(g))) \geq 0, \quad i = 1,\ldots,n,$$

since

$$p_i \geq A(\mathbf{1}) \geq A(\lambda_i(g)), \quad i = 1,\ldots,n.$$

Therefore, expression $\dfrac{\sum_{i=1}^{n} p_i x_i - \widetilde{A}(g)}{P_n - A(\mathbf{1})}$ is a convex combination of vectors $x_1,\ldots,x_n$ and belongs to K.

Since φ is convex on K, we have

$$\begin{aligned}
&\varphi\left(\frac{\sum_{i=1}^{n} p_i x_i - \widetilde{A}(g)}{P_n - A(\mathbf{1})}\right) \\
&= \varphi\left(\frac{1}{P_n - A(\mathbf{1})} \sum_{i=1}^{n} (p_i - A(\lambda_i(g)))\, x_i\right) \\
&\leq \frac{1}{P_n - A(\mathbf{1})} \sum_{i=1}^{n} (p_i - A(\lambda_i(g)))\, \varphi(x_i) - \min_i \left\{\frac{p_i - A(\lambda_i(g))}{P_n - A(\mathbf{1})}\right\} S_\varphi^n(x_1,\ldots,x_n) \\
&= \frac{\sum_{i=1}^{n} p_i \varphi(x_i) - \sum_{i=1}^{n} A(\lambda_i(g))\, \varphi(x_i) - \min_i \{p_i - A(\lambda_i(g))\} S_\varphi^n(x_1,\ldots,x_n)}{P_n - A(\mathbf{1})}
\end{aligned}$$

$$\leq \frac{\sum_{i=1}^{n} p_i \varphi(x_i) - A(\varphi(g)) - S_{\varphi}^{n}(x_1,\ldots,x_n)\left[\min_i\{p_i - A(\lambda_i(g))\} + A(\min_i\{\lambda_i(g)\})\right]}{P_n - A(\mathbf{1})}.$$

□

The following corollary shows that Theorem 9.4 is also a generalization of Theorem 9.2 on convex hulls.

Corollary 9.2 *Let L satisfy properties L1, L2, L3 on a nonempty set E, and A be a positive normalized linear functional on L. Let φ be a convex function on an interval $I = [m,M] \subset \mathbb{R}$. Then for all $g \in L$ such that $g(E) \subset I$ and $\varphi(g) \in L$, we have*

$$\begin{aligned}
&\varphi(m+M-A(g))\\
&\leq \frac{A(g)-m}{M-m}\varphi(m)+\frac{M-A(g)}{M-m}\varphi(M)-\left(\frac{1}{2}-\frac{1}{M-m}\left|A(g)-\frac{m+M}{2}\right|\right)S_{\varphi}^{2}(m,M)\\
&\leq \varphi(m)+\varphi(M)-A(\varphi(g))-\left[1-\frac{1}{M-m}\left(\left|A(g)-\frac{m+M}{2}\right|+A\left(\left|g-\frac{m+M}{2}\right|\right)\right)\right]S_{\varphi}^{2}(m,M).
\end{aligned} \tag{9.13}$$

Proof. For each $t \in E$ we have $g(t) \in I$.

Since interval $I = [m,M]$ is 1-simplex with vertices m and M, then the barycentric coordinates have the special form:

$$\lambda_1(g(t)) = \frac{M-g(t)}{M-m} \quad \text{and} \quad \lambda_2(g(t)) = \frac{g(t)-m}{M-m}.$$

Applying functional A we have

$$A(\lambda_1(g)) = \frac{M-A(g)}{M-m} \quad \text{and} \quad A(\lambda_2(g)) = \frac{A(g)-m}{M-m}. \tag{9.14}$$

Choosing $n=2$, $p_1 = p_2 = 1$, $x_1 = m$, $x_2 = M$, it follows from (9.11) that

$$\begin{aligned}
&\varphi(m+M-A(g))\\
&\leq \varphi(m)+\varphi(M)-\left[\frac{M-A(g)}{M-m}\varphi(m)+\frac{A(g)-m}{M-m}\varphi(M)\right]\\
&\quad -\left(\frac{1}{2}-\frac{1}{M-m}\left|A(g)-\frac{m+M}{2}\right|\right)\left[\varphi(m)+\varphi(M)-2\varphi\left(\frac{m+M}{2}\right)\right]\\
&= \frac{A(g)-m}{M-m}\varphi(m)+\frac{M-A(g)}{M-m}\varphi(M)-\left(\frac{1}{2}-\frac{1}{M-m}\left|A(g)-\frac{m+M}{2}\right|\right)S_{\varphi}^{2}(m,M)\\
&\leq \varphi(m)+\varphi(M)-A(\varphi(g))\\
&\quad -\left[\frac{1}{2}-\frac{1}{M-m}\left|A(g)-\frac{m+M}{2}\right|+A\left(\frac{1}{2}-\frac{1}{M-m}\left|g-\frac{m+M}{2}\right|\right)\right]S_{\varphi}^{2}(m,M)\\
&= \varphi(m)+\varphi(M)-A(\varphi(g))-\left[1-\frac{1}{M-m}\left(\left|A(g)-\frac{m+M}{2}\right|+A\left(\left|g-\frac{m+M}{2}\right|\right)\right)\right]S_{\varphi}^{2}(m,M).
\end{aligned}$$

□

Theorem 9.5 *Let L satisfy properties L1, L2, L3 on a nonempty set E, A be a positive linear functional on L and $\widetilde{A}$ be defined as in (8.21). Let $x_1,\ldots,x_n \in \mathbb{R}^k$ and $K = co(\{x_1,\ldots,x_n\})$. Let φ be a convex function on K and $\lambda_1,\ldots,\lambda_n$ barycentric coordinates over K. Then for all $g \in L^k$ such that $g(E) \subset K$ and $\varphi(g), \lambda_i(g) \in L, i = 1,\ldots,n$ and positive real numbers $p_1,\ldots,p_n$ satisfying the conditions $P_n - A(\mathbf{1}) > 0$, where $P_n = \sum_{i=1}^n p_i$, and*

$$\frac{\sum_{i=1}^n p_i x_i - \widetilde{A}(g)}{P_n - A(1)} \in K, \tag{9.15}$$

we have

$$\begin{aligned}
&\varphi\left(\frac{\sum_{i=1}^n p_i x_i - \widetilde{A}(g)}{P_n - A(\mathbf{1})}\right) \\
\geq\ & \frac{P_n \varphi\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) - A(\mathbf{1})\varphi\left(\frac{1}{A(\mathbf{1})}\widetilde{A}(g)\right)}{P_n - A(\mathbf{1})} \\
\geq\ & \frac{P_n \varphi\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) - \sum_{i=1}^n A(\lambda_i(g))\varphi(x_i) + \min_i\{A(\lambda_i(g))\} S_\varphi^n(x_1,\ldots,x_n)}{P_n - A(\mathbf{1})}.
\end{aligned} \tag{9.16}$$

Proof. For each $t \in E$ we have $g(t) \in K$. Using barycentric coordinates we have $\lambda_i(g(t)) \geq 0, i = 1,\ldots,n$, $\sum_{i=1}^n \lambda_i(g(t)) = 1$ and

$$g(t) = \sum_{i=1}^n \lambda_i(g(t)) x_i.$$

Also we have

$$\widetilde{A}(g) = \sum_{i=1}^n A(\lambda_i(g)) x_i.$$

We can easily see that

$$\frac{1}{A(\mathbf{1})}\widetilde{A}(g) = \frac{1}{A(\mathbf{1})}\sum_{i=1}^n A(\lambda_i(g)) x_i \in K,$$

since

$$\frac{1}{A(\mathbf{1})}\sum_{i=1}^n A(\lambda_i(g)) = 1 \text{ and } \frac{1}{A(\mathbf{1})} A(\lambda_i(g)) \geq 0, \quad i = 1,\ldots,n.$$

Since φ is convex on K, then using Lemma 8.2

$$\varphi\left(\frac{1}{A(\mathbf{1})}\widetilde{A}(g)\right) \leq \frac{1}{A(\mathbf{1})}\sum_{i=1}^n A(\lambda_i(g))\varphi(x_i) - \min_i\left\{\frac{A(\lambda_i(g))}{A(\mathbf{1})}\right\} S_\varphi^n(x_1,\ldots,x_n). \tag{9.17}$$

Using reversed Jensen's inequality (see [177, p. 83]) and (9.17) we have

$$\varphi\left(\frac{P_n\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)-A(\mathbf{1})\left(\frac{1}{A(\mathbf{1})}\widetilde{A}(g)\right)}{P_n-A(\mathbf{1})}\right)$$

$$\geq \frac{P_n\varphi\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)-A(\mathbf{1})\varphi\left(\frac{1}{A(\mathbf{1})}\widetilde{A}(g)\right)}{P_n-A(\mathbf{1})}$$

$$\geq \frac{P_n\varphi\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)-A(\mathbf{1})\frac{1}{A(\mathbf{1})}\sum_{i=1}^n A(\lambda_i(g))\varphi(x_i)+\min_i\{A(\lambda_i(g))\}S_\varphi^n(x_1,\ldots,x_n)}{P_n-A(\mathbf{1})}.$$

□

Remark 9.2 If positive real numbers $p_1,\ldots,p_n$ satisfy condition (9.10), then condition (9.15) is also satisfied since K is a convex set. Hence (9.11) can be extended as follows:

$$\frac{P_n\varphi\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)-\sum_{i=1}^n A(\lambda_i(g))\varphi(x_i)+\min_i\{A(\lambda_i(g))\}S_\varphi^n(x_1,\ldots,x_n)}{P_n-A(\mathbf{1})}$$

$$\leq \frac{P_n\varphi\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)-A(\mathbf{1})\varphi\left(\frac{1}{A(\mathbf{1})}\widetilde{A}(g)\right)}{P_n-A(\mathbf{1})}$$

$$\leq \varphi\left(\frac{\sum_{i=1}^n p_i x_i-\widetilde{A}(g)}{P_n-A(\mathbf{1})}\right)$$

$$\leq \frac{\sum_{i=1}^n p_i\varphi(x_i)-\sum_{i=1}^n A(\lambda_i(g))\varphi(x_i)-\min_i\{p_i-A(\lambda_i(g))\}S_\varphi^n(x_1,\ldots,x_n)}{P_n-A(\mathbf{1})}$$

$$\leq \frac{\sum_{i=1}^n p_i\varphi(x_i)-A(\varphi(g))-S_\varphi^n(x_1,\ldots,x_n)\left[\min_i\{p_i-A(\lambda_i(g))\}+A(\min_i\{\lambda_i(g)\})\right]}{P_n-A(\mathbf{1})}.$$

Corollary 9.3 *Let L satisfy properties L1, L2, L3 on a nonempty set E, and A be a positive normalized linear functional on L. Let φ be a convex function on an interval $I=[m,M]\subset\mathbb{R}$. Then for all $g\in L$ such that $g(E)\subset I$ and $\varphi(g)\in L$, we have*

$$\begin{aligned}\varphi(m+M-A(g)) &\geq 2\varphi\left(\frac{m+M}{2}\right)-\varphi(A(g))\\ &\geq 2\varphi\left(\frac{m+M}{2}\right)-\left[\frac{M-A(g)}{M-m}\varphi(m)+\frac{A(g)-m}{M-m}\varphi(M)\right]\\ &\quad+\left(\frac{1}{2}-\frac{1}{M-m}\left|A(g)-\frac{m+M}{2}\right|\right)S_\varphi^2(m,M). \qquad (9.18)\end{aligned}$$

Proof. Choosing $n=2$, $x_1=m$, $x_2=M$, $p_1=p_2=1$ and using (9.14), the inequalities in (9.18) easily follow from (9.16). □

Corollary 9.4 *Let L satisfy properties L1, L2, L3 on a nonempty set E, A be a positive normalized linear functional on L and $\widetilde{A}$ be defined as in (8.21). Let φ be a convex function on k-simplex $S = [v_1, v_2, \ldots, v_{k+1}]$ in $\mathbb{R}^k$ and $\lambda_1, \ldots, \lambda_{k+1}$ barycentric coordinates over S. Then for all $g \in L^k$ such that $g(E) \subset S$ and $\varphi(g) \in L$ we have*

$$\begin{aligned}
&\frac{(k+1)\varphi\left(\frac{1}{k+1}\sum_{i=1}^{k+1} v_i\right) - \sum_{i=1}^{k+1} \lambda_i(\widetilde{A}(g))\varphi(v_i) + \min_i\left\{\lambda_i(\widetilde{A}(g))\right\} S_{\varphi}^{k+1}(v_1,\ldots,v_{k+1})}{k} \\
&\le \frac{(k+1)\varphi\left(\frac{1}{k+1}\sum_{i=1}^{k+1} v_i\right) - \varphi(\widetilde{A}(g))}{k} \\
&\le \varphi\left(\frac{\sum_{i=1}^{k+1} v_i - \widetilde{A}(g)}{k}\right) \\
&\le \frac{\sum_{i=1}^{k+1}\varphi(v_i) - \sum_{i=1}^{k+1}\lambda_i(\widetilde{A}(g))\varphi(v_i) - \min_i\left\{1-\lambda_i(\widetilde{A}(g))\right\} S_{\varphi}^{k+1}(v_1,\ldots,v_{k+1})}{k} \\
&\le \frac{\sum_{i=1}^{k+1}\varphi(v_i) - A(\varphi(g)) - S_{\varphi}^{k+1}(v_1,\ldots,v_{k+1})\left[\min_i\left\{1-\lambda_i(\widetilde{A}(g))\right\} + A\left(\min_i\{\lambda_i(g)\}\right)\right]}{k}.
\end{aligned} \tag{9.19}$$

Proof. Since barycentric coordinates $\lambda_1, \ldots, \lambda_{k+1}$ over k-simplex S in $\mathbb{R}^k$ are nonnegative linear polynomials, then $A(\lambda_i(g)) = \lambda_i(\widetilde{A}(g)), i = 1, \ldots, k+1$.

Choosing $x_i = v_i, i = 1, \ldots, k+1$ and $p_1 = p_2 = \cdots = p_{k+1} = 1$, the inequalities in (9.19) easily follow from (9.11) and (9.16). □

Remark 9.3 As a special case of Corollary 9.4, for $k = 1$, and if we take p and q as nonnegative real numbers such that $A(g) = \dfrac{pm+qM}{p+q}$, we get right hand side of the inequality (2.3) in [101].

Remark 9.4 Using the same technique and the same special case as in Example 8.1 and Remark 8.9, from (9.19) we get the same results, that is, the k-dimensional version of the Hammer-Bullen inequality, namely

$$\frac{1}{|S|}\int_S f(t)dt - f(v^*) \le \frac{k}{k+1}\sum_{i=1}^{k+1} f(v_i) - \frac{k}{|S|}\int_S f(t)dt,$$

and, as a special case in one dimension, an improvement of the classical Hermite-Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2} - \frac{1}{4}S_f^2(a,b),$$

where $S_f^2(a,b) = f(a) + f(b) - 2f\left(\dfrac{a+b}{2}\right)$.

9.2.1 Jessen-Mercer differences

Motivated by theorems 9.1 and 9.2, we define two functionals $\Phi_i\colon L_f \to \mathbb{R}$, $i = 1,2$, by

$$\begin{aligned}\Phi_1(\varphi) &= \varphi(m)+\varphi(M)-\varphi(m+M-A(f))-A(\varphi(f))\\ &\quad -\left[1-\frac{2}{M-m}A\left(\left|f-\frac{m+M}{2}\right|\right)\right]\delta_\varphi\end{aligned} \tag{9.20}$$

and

$$\begin{aligned}\Phi_2(\varphi) &= \varphi(m)+\varphi(M)-\varphi(m+M-A(f))-A(\varphi(f))\\ &\quad -\left[1-\frac{1}{M-m}\left(A\left(\left|f-\frac{m+M}{2}\right|\right)+\left|\frac{m+M}{2}-A(f)\right|\right)\right]\delta_\varphi,\end{aligned} \tag{9.21}$$

where A, f and δ_φ are as in Theorem 9.1, $L_f = \{\varphi\colon I \to \mathbb{R}\colon \varphi(f), \varphi(m+M-f) \in L\}$, $[m,M] \subseteq I$. Obviously, Φ_1 and Φ_2 are linear.

If φ is additionally continuous and convex, then theorems 9.1 and 9.2 imply $\Phi_i(f) \geq 0, i = 1,2$.

In the following, with φ_0 we denote the function defined by $\varphi_0(x) = x^2$, on any domain we need.

Now we provide Lagrange and Cauchy mean value type theorems regarding the functionals Φ_i, $i = 1,2$.

Theorem 9.6 *Let L satisfy L1, L2 and L3 on a nonempty set E, and let A be a positive normalized linear functional on L. Let $f \in L$ be such that $\varphi_0 \in L_f$, $f(E) \subseteq [m,M] \subseteq I$ and let $\varphi \in C^2(I)$ be such that $\varphi \in L_f$. If Φ_1 and Φ_2 are linear functionals defined as in* (9.20) *and* (9.21), *then there exist $\xi_i \in [m,M], i = 1,2$ such that*

$$\Phi_i(\varphi) = \frac{\varphi''(\xi_i)}{2}\Phi_i(\varphi_0), \quad i = 1,2.$$

Proof. We give the proof for the functional Φ_1. Since $\varphi \in C^2(I)$, there exist real numbers $a = \min_{x\in[m,M]} \varphi''(x)$ and $b = \max_{x\in[m,M]} \varphi''(x)$. It is easy to show that the functions φ_1, φ_2 defined by

$$\varphi_1(x) = \frac{b}{2}x^2 - \varphi(x), \quad \varphi_2(x) = f(x) - \frac{a}{2}x^2$$

are continuous and convex, hence $\Phi_1(\varphi_1) \geq 0$, $\Phi_1(\varphi_2) \geq 0$. This implies

$$\frac{a}{2}\Phi_1(\varphi_0) \leq \Phi_1(\varphi) \leq \frac{b}{2}\Phi_1(\varphi_0).$$

If $\Phi_1(\varphi_0) = 0$, there is nothing to prove. Suppose $\Phi_1(\varphi_0) > 0$. We have

$$a \leq \frac{2\Phi_1(\varphi)}{\Phi_1(\varphi_0)} \leq b.$$

Hence, there exists $\xi_1 \in [m,M]$ such that

$$\Phi_1(\varphi) = \frac{\varphi''(\xi_1)}{2}\Phi_1(\varphi_0).$$

□

Theorem 9.7 *Let L satisfy L1, L2 and L3 on a non-empty set E, and let A be a positive normalized linear functional on L. Let $f \in L$ be such that $\varphi_0 \in L_f$, $f(E) \subseteq [m,M] \subseteq I$ and $\varphi_1, \varphi_2 \in C^2(I)$ such that $\varphi_1, \varphi_2 \in L_f$. If Φ_1 and Φ_2 are linear functionals defined as in* (9.20) *and* (9.21), *then there exist $\xi_i \in [m,M]$, $i = 1,2$ such that*

$$\frac{\Phi_i(\varphi_1)}{\Phi_i(\varphi_2)} = \frac{\varphi_1''(\xi_i)}{\varphi_2''(\xi_i)}, \quad i = 1,2,$$

provided that the denominators are non-zero.

Proof. We give the proof for the functional Φ_1. Define $\varphi_3 \in C^2([m,M])$ by

$$\varphi_3 = c_1\varphi_1 - c_2\varphi_2, \text{ where } c_1 = \Phi_1(\varphi_2),\ c_2 = \Phi_1(\varphi_1).$$

Using Theorem 9.6, we get that there exists $\xi_1 \in [m,M]$ such that

$$\left(c_1\frac{\varphi_1''(\xi_1)}{2} - c_2\frac{\varphi_2''(\xi_1)}{2}\right)\Phi_1(\varphi_0) = 0.$$

Since $\Phi_1(\varphi_0) \neq 0$, (otherwise we have a contradiction with $\Phi_1(\varphi_2) \neq 0$, by Theorem 9.6), we obtain

$$\frac{\Phi_1(\varphi_1)}{\Phi_1(\varphi_2)} = \frac{\varphi_1''(\xi_1)}{\varphi_2''(\xi_1)}.$$

□

We use an idea from [90] to give an elegant method of producing n-exponentially convex and exponentially convex functions, applying the functionals Φ_1 and Φ_2 to a given family of functions with the same property.

Theorem 9.8 *Let Φ_i, $i = 1,2$, be linear functionals defined as in (9.20) and (9.21). Let $\Upsilon = \{\varphi_s \colon s \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an open interval I such that $\Upsilon \subseteq L_f$ and that the function $s \mapsto [y_0, y_1, y_2; \varphi_s]$ is n-exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Then $s \mapsto \Phi_i(\varphi_s)$ is an n-exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Phi_i(\varphi_s)$ is also continuous on J, then it is n-exponentially convex on J.*

Proof. For $\xi_i \in \mathbb{R}$, $i = 1,\ldots,n$ and $s_i \in J$, $i = 1,\ldots,n$, we define the function $\chi \colon I \to \mathbb{R}$ by

$$\chi(y) = \sum_{i,j=1}^{n} \xi_i\xi_j\varphi_{\frac{s_i+s_j}{2}}(y).$$

Using the assumption that the function $s \mapsto [y_0, y_1, y_2; \varphi_s]$ is n-exponentially convex in the Jensen sense, we obtain

$$[y_0, y_1, y_2; \chi] = \sum_{i,j=1}^{n} \xi_i\xi_j[y_0, y_1, y_2; \varphi_{\frac{s_i+s_j}{2}}] \geq 0,$$

which in turn implies that χ is a convex (and continuous) function on I, and therefore $\Phi_i(\chi) \geq 0, i = 1,2$. Hence

$$\sum_{i,j=1}^{n} \xi_i \xi_j \Phi_i(\varphi_{\frac{s_i+s_j}{2}}) \geq 0.$$

We conclude that the function $s \mapsto \Phi_i(\varphi_s)$ is n-exponentially convex on J in the Jensen sense. If the function $s \mapsto \Phi_i(\varphi_s)$ is also continuous on J, then $s \mapsto \Phi_i(\varphi_s)$ is n-exponentially convex by definition. □

The first of the following two corollaries is an immediate consequence of Theorem 9.8.

Corollary 9.5 *Let Φ_i, $i = 1,2$, be linear functionals defined as in (9.20) and (9.21). Let $\Upsilon = \{\varphi_s \colon s \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an open interval I such that $\Upsilon \subseteq L_f$ and that the function $s \mapsto [y_0, y_1, y_2; \varphi_s]$ is exponentially convex in the Jensen sense on J, for every three mutually different points $y_0, y_1, y_2 \in I$. Then $s \mapsto \Phi_i(\varphi_s)$ is an exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Phi_i(\varphi_s)$ is continuous on J, then it is exponentially convex on J.*

Corollary 9.6 *Let Φ_i, $i = 1,2$, be linear functionals defined as in (9.20) and (9.21). Let $\Omega = \{\varphi_s \colon s \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an open interval I such that $\Omega \subseteq L_f$ and that the function $s \mapsto [y_0, y_1, y_2; \varphi_s]$ is 2-exponentially convex in the Jensen sense on J, for every three mutually different points $y_0, y_1, y_2 \in I$. Then the following statements hold:*

(*i*) *If the function $s \mapsto \Phi_i(\varphi_s)$ is continuous on J, then it is 2-exponentially convex function on J. If $s \mapsto \Phi_i(\varphi_s)$ is additionally strictly positive, then it is also log-convex on J.*

(*ii*) *If the function $s \mapsto \Phi_i(\varphi_s)$ is strictly positive and differentiable on J, then for every $s,q,u,v \in J$, such that $s \leq u$ and $q \leq v$, we have*

$$\mu_{s,q}(\Phi_i, \Omega) \leq \mu_{u,v}(\Phi_i, \Omega), \quad i = 1,2, \tag{9.22}$$

where

$$\mu_{s,q}(\Phi_i, \Omega) = \begin{cases} \left(\dfrac{\Phi_i(\varphi_s)}{\Phi_i(\varphi_q)} \right)^{\frac{1}{s-q}} & , s \neq q, \\ \exp\left(\dfrac{\frac{d}{ds}\Phi_i(\varphi_s)}{\Phi_i(\varphi_s)} \right) & , s = q, \end{cases} \tag{9.23}$$

for $\varphi_s, \varphi_q \in \Omega$ ($\mu_{s,q}(\Phi_i, \Omega), i = 1,2$, are the Stolarsky type means.)

Proof. (*i*) This is an immediate consequence of Theorem 9.8 and Remark 1.3.

(*ii*) Since by (*i*) the function $s \mapsto \Phi_i(\varphi_s)$ is log-convex on J, that is, the function $s \mapsto \log \Phi_i(\varphi_s)$ is convex on J, applying Proposition 1.2 we get

$$\frac{\log \Phi_i(\varphi_s) - \log \Phi_i(\varphi_q)}{s-q} \leq \frac{\log \Phi_i(\varphi_u) - \log \Phi_i(\varphi_v)}{u-v} \tag{9.24}$$

for $s \le u$, $q \le v$, $s \ne q$, $u \ne v$, and therefrom conclude that

$$\mu_{s,q}(\Phi_i, \Omega) \le \mu_{u,v}(\Phi_i, \Omega), \quad i = 1, 2.$$

Cases $s = q$ and $u = v$ follow from (9.24) as limit cases. □

Remark 9.5 Note that the results from Theorem 9.8, Corollary 9.5, Corollary 9.6 still hold when two of the points $y_0, y_1, y_2 \in I$ coincide, say $y_1 = y_0$, for a family of differentiable functions φ_s, such that the function $s \mapsto [y_0, y_1, y_2; \varphi_s]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, they still hold when all three points coincide for a family of twice differentiable functions with the same property. The proofs are obtained by recalling Remark 1.4 and suitable characterization of convexity.

Now, we present several families of functions which fulfil the conditions of Theorem 9.8, Corollary 9.5 and Corollary 9.6 (and Remark 9.5). This enables us to construct a large family of functions which are exponentially convex. For a discussion related to this problem see [68].

In the rest of the section we consider only Φ_1 and Φ_2 defined as in (9.20) and (9.21), with A being continuous and f such that compositions with any function from the chosen family Ω_i, as well as with other functions which appear as arguments of Φ_1 and Φ_2, remain in L.

Example 9.1 Consider a family of functions

$$\Omega_1 = \{g_s \colon \mathbb{R} \to [0, \infty) \colon s \in \mathbb{R}\}$$

defined by

$$g_s(x) = \begin{cases} \frac{1}{s^2} e^{sx}, & s \ne 0, \\ \frac{1}{2} x^2, & s = 0. \end{cases}$$

We have $\frac{d^2 g_s}{dx^2}(x) = e^{sx} > 0$, which shows that g_s is convex on $\mathbb{R}$, for every $s \in \mathbb{R}$ and $s \mapsto \frac{d^2 g_s}{dx^2}(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 9.8 we also have that $s \mapsto [y_0, y_1, y_2; g_s]$ is exponentially convex (and thus exponentially convex in the Jensen sense). Using Corollary 9.5 we conclude that $s \mapsto \Phi_i(g_s)$, $i = 1, 2$, are exponentially convex in the Jensen sense. It is easy to verify that these mappings are continuous (although mapping $s \mapsto g_s$ is not continuous for $s = 0$), so they are exponentially convex.

For this family of functions, $\mu_{s,q}(\Phi_i, \Omega_1)$, $i = 1, 2$, from (9.23) become

$$\mu_{s,q}(\Phi_i, \Omega_1) = \begin{cases} \left(\frac{\Phi_i(g_s)}{\Phi_i(g_q)} \right)^{\frac{1}{s-q}}, & s \ne q, \\ \exp\left(\frac{\Phi_i(id \cdot g_s)}{\Phi_i(g_s)} - \frac{2}{s} \right), & s = q \ne 0, \\ \exp\left(\frac{\Phi_i(id \cdot g_0)}{3\Phi_i(g_0)} \right), & s = q = 0, \end{cases}$$

and, by (9.22), they are monotonic functions in parameters s and q.

Using Theorem 9.7, it follows that for $i = 1,2$

$$M_{s,q}(\Phi_i,\Omega_1) = \log \mu_{s,q}(\Phi_i,\Omega_1)$$

satisfy $m \leq M_{s,q}(\Phi_i,\Omega_1) \leq M$, which shows that $M_{s,q}(\Phi_i,\Omega_1)$ are means (of a function g). Notice that by (9.22) they are monotonic.

Example 9.2 Consider a family of functions

$$\Omega_2 = \{f_s\colon (0,\infty) \to \mathbb{R}\colon s \in \mathbb{R}\}$$

defined by

$$f_s(x) = \begin{cases} \frac{x^s}{s(s-1)}, & s \neq 0,1, \\ -\log x, & s = 0, \\ x\log x, & s = 1. \end{cases}$$

Here, $\frac{d^2 f_s}{dx^2}(x) = x^{s-2} = e^{(s-2)\ln x} > 0$, which shows that f_s is convex for $x > 0$ and $s \mapsto \frac{d^2 f_s}{dx^2}(x)$ is exponentially convex by definition. Arguing as in Example 9.1 we get that the mappings $s \mapsto \Phi_i(g_s), i = 1,2$ are exponentially convex. Functions (9.23) in this case are equal to:

$$\mu_{s,q}(\Phi_i,\Omega_2) = \begin{cases} \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{1-2s}{s(s-1)} - \frac{\Phi_i(f_s f_0)}{\Phi_i(f_s)}\right), & s = q \neq 0,1, \\ \exp\left(1 - \frac{\Phi_i(f_0^2)}{2\Phi_i(f_0)}\right), & s = q = 0, \\ \exp\left(-1 - \frac{\Phi_i(f_0 f_1)}{2\Phi_i(f_1)}\right), & s = q = 1. \end{cases}$$

If Φ_i is positive, then Theorem 9.7 applied for $f = f_s \in \Omega_2$ and $g = f_q \in \Omega_2$ yields that there exists $\xi \in [m,M]$ such that

$$\xi^{s-q} = \frac{\Phi_i(f_s)}{\Phi_i(f_q)}.$$

Since the function $\xi \mapsto \xi^{s-q}$ is invertible for $s \neq q$, we then have

$$m \leq \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}} \leq M, \tag{9.25}$$

which together with the fact that $\mu_{s,q}(\Phi_i,\Omega_2)$ is continuous, symmetric and monotonic (by (9.22)), shows that $\mu_{s,q}(\Phi_i,\Omega_2)$ is a mean (of a function f).

Example 9.3 Consider a family of functions

$$\Omega_3 = \{h_s\colon (0,\infty) \to (0,\infty)\colon s \in (0,\infty)\}$$

defined by

$$h_s(x) = \begin{cases} \frac{s^{-x}}{\ln^2 s}, & s \neq 1, \\ \frac{x^2}{2}, & s = 1. \end{cases}$$

Since $s \mapsto \frac{d^2 h_s}{dx^2}(x) = s^{-x}$ is the Laplace transform of a nonnegative function (see [211]), it is exponentially convex. Obviously, h_s are convex functions for every $s > 0$.

For this family of functions, $\mu_{s,q}(\Phi_i, \Omega_3)$, $i = 1,2$, from (9.23) become

$$\mu_{s,q}(\Phi_i, \Omega_3) = \begin{cases} \left(\frac{\Phi_i(h_s)}{\Phi_i(h_q)} \right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(-\frac{\Phi_i(id \cdot h_s)}{s\Phi_i(h_s)} - \frac{2}{s \ln s} \right), & s = q \neq 1, \\ \exp\left(-\frac{2\Phi_i(id \cdot h_1)}{3\Phi_i(h_1)} \right), & s = q = 1, \end{cases}$$

and are monotonic in parameters s and q by (9.22).

Using Theorem 9.7, it follows that

$$M_{s,q}(\Phi_i, \Omega_3) = -L(s,q) \log \mu_{s,q}(\Phi_i, \Omega_3)$$

satisfies $m \leq M_{s,q}(\Phi_i, \Omega_3) \leq M$, which shows that $M_{s,q}(\Phi_i, \Omega_3)$ is a mean (of a function h). $L(s,q)$ is the logarithmic mean defined by $L(s,q) = \frac{s-q}{\log s - \log q}$, $s \neq q$, $L(s,s) = s$.

Example 9.4 Consider a family of functions

$$\Omega_4 = \{k_s : (0,\infty) \to (0,\infty) : s \in (0,\infty)\}$$

defined by

$$k_s(x) = \frac{e^{-x\sqrt{s}}}{s}.$$

Since $s \mapsto \frac{d^2 k_s}{dx^2}(x) = e^{-x\sqrt{s}}$ is the Laplace transform of a nonnegative function (see [211]), it is exponentially convex. Obviously, k_s are convex functions for every $s > 0$.

For this family of functions, $\mu_{s,q}(\Phi_i, \Omega_4)$, $i = 1,2$, from (9.23) become

$$\mu_{s,q}(\Phi_i, \Omega_4) = \begin{cases} \left(\frac{\Phi_i(k_s)}{\Phi_i(k_q)} \right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(-\frac{\Phi_i(id \cdot k_s)}{2\sqrt{s}\Phi_i(k_s)} - \frac{1}{s} \right), & s = q, \end{cases}$$

and are monotonic in parameters s and q, by (9.22).

Using Theorem 9.7, it follows that

$$M_{s,q}(\Phi_i, \Omega_4) = -\left(\sqrt{s} + \sqrt{q}\right) \log \mu_{s,q}(\Phi_i, \Omega_4)$$

satisfies $m \leq M_{s,q}(\Phi_i, \Omega_4) \leq M$, which shows that $M_{s,q}(\Phi_i, \Omega_4)$ is a mean (of a function k).

Chapter 10

New improved forms of the Hermite-Hadamard-type inequalities

In this chapter, improvements of various forms of the Hermite-Hadamard inequality (the ones of Fejèr, Lupaş, Brenner-Alzer, Beesack-Pečarić) are presented. It is interesting that these improvements also imply the Hammer-Bullen inequality which deals with a comparison of the left-hand and the right-hand side of the Hermite-Hadamard inequality. These improvements are given in terms of positive linear functionals and are again obtained by means of the monotonicity property of Lemma 1.2, adjusted to this environment. Obtained results are used in constructing new families of exponentially convex functions.

All new results in this chapter are contained in paper [101].

10.1 More on the Hermite-Hadamard inequality

The Hermite-Hadamard inequality was discussed in detail in Chapter 1, (see Theorem 1.23). Namely, for a convex function f defined on $[a,b] \subset \mathbb{R}$, where $a < b$, is

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{10.1}$$

The Hammer-Bullen inequality, (see Theorem 1.24 and the related considerations in Chapter 1 for details), proves that the first inequality in (10.1) is stronger than the second one:

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - f\left(\frac{a+b}{2}\right) \le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x. \tag{10.2}$$

In 1906, Fejér, while studying trigonometric polynomials, obtained inequalities which generalized those of Hermite. He proved that if $w:[a,b]\to\mathbb{R}$ is a nonnegative integrable function such that the curve $y=w(x)$ is symmetric with respect to the straight line $x=(a+b)/2$, then for every convex function $f:[a,b]\to\mathbb{R}$ the following inequalities hold (see [177, p. 138]):

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)\mathrm{d}x \le \int_a^b w(x)f(x)\mathrm{d}x \le \frac{f(a)+f(b)}{2}\int_a^b w(x)\mathrm{d}x. \tag{10.3}$$

Obviously, for $w=1$, the inequalities in (10.3) become the Hermite-Hadamard inequality. Another generalization of the Hermite-Hadamard inequality was given in [124] and [130] (or see [177, p. 143]).

Theorem 10.1 *Let p,q be given positive numbers and $[a,b]\subseteq I$, $a<b$. Then the inequalities*

$$f\left(\frac{pa+qb}{p+q}\right) \le \frac{1}{2y}\int_{T-y}^{T+y} f(x)\mathrm{d}x \le \frac{pf(a)+qf(b)}{p+q} \tag{10.4}$$

hold for $T=(pa+qb)/(p+q)$, $y>0$ and all continuous convex functions $f:I\to\mathbb{R}$ if

$$y\le \frac{b-a}{p+q}\min\{p,q\}.$$

It can be easily verified that for $p=q=1$ and $y=(b-a)/2$ the inequalities in (10.4) become the Hermite-Hadamard inequalities. Using the same technique as in the proof of (10.2) (see [170]) it can be proved that the first inequality in (10.4) is stronger than the second one, that is,

$$\frac{1}{2y}\int_{T-y}^{T+y} f(x)\mathrm{d}x - f\left(\frac{pa+qb}{p+q}\right) \le \frac{pf(a)+qf(b)}{p+q} - \frac{1}{2y}\int_{T-y}^{T+y} f(x)\mathrm{d}x. \tag{10.5}$$

In [10] Brenner and Alzer proved the following generalization of the Hermite-Hadamard inequality which is in fact a Fejér-type variant of (10.4).

Theorem 10.2 *Let p,q be given positive numbers and let $w:[a,b]\to\mathbb{R}_0^+$ be integrable and symmetric with respect to the line $x=(pa+qb)/(p+q)=T$, in the sense that $w(T+t)$ $=w(T-t)$, for all $t\in\left[0,\frac{b-a}{p+q}\min\{p,q\}\right]$. If $f:[a,b]\to\mathbb{R}$ is a convex function, then for all $y\in\mathbb{R}$ such that*

$$0<y\le \frac{b-a}{p+q}\min\{p,q\} \tag{10.6}$$

the following inequalities hold:

$$f\left(\frac{pa+qb}{p+q}\right)\int_{T-y}^{T+y} w(x)\mathrm{d}x \le \int_{T-y}^{T+y} w(x)f(x)\mathrm{d}x \le \frac{pf(a)+qf(b)}{p+q}\int_{T-y}^{T+y} w(x)\mathrm{d}x. \tag{10.7}$$

Theorem 10.1 was generalized for positive linear functionals in [31].

Theorem 10.3 *Let L satisfy L1, L2 on a nonempty set E and let A be a positive normalized linear functional. If $f: I \to \mathbb{R}$ is a continuous convex function and $[a,b] \subseteq I$, where $a < b$, then for all $g \in L$ such that $f(g) \in L$ the inequalities*

$$f\left(\frac{pa+qb}{p+q}\right) \leq A(f(g)) \leq \frac{pf(a)+qf(b)}{p+q} \tag{10.8}$$

hold, where p and q are any nonnegative real numbers such that

$$A(g) = \frac{pa+qb}{p+q}. \tag{10.9}$$

Remark 10.1 It can be easily verified that Theorem 10.2 (and therefore Theorem 10.1) can be obtained as a special case of Theorem 10.3. Namely, for given positive numbers p and q, T and w as in Theorem 10.2 and y satisfying (10.6) such that $\overline{w} = \int_{T-y}^{T+y} w(x)\,\mathrm{d}x \neq 0$, we define $E = [a,b]$, $L = \mathscr{R}(E)$, $g = id_E$ and

$$A(f) = \frac{1}{\overline{w}} \int_{T-y}^{T+y} w(x) f(x)\mathrm{d}x.$$

Here $\mathscr{R}(E)$ denotes the subspace of all (bounded) R-integrable functions on $E = [a,b]$. Observe that A is a positive normalized linear functional and

$$A(g) = A(id_E) = \frac{1}{\overline{w}} \int_{T-y}^{T+y} w(x)x\mathrm{d}x = T = \frac{pa+qb}{p+q}.$$

By Theorem 10.3 we immediately obtain (10.7).

10.2 Improvements

We now provide the improvements of the forms of the Hermite-Hadamard inequality, which were analyzed in the previous section. By means of these, new families of exponentially convex functions are constructed.

With I we denote an interval in $\mathbb{R}$ and with $[a,b]$ an interval in $\mathbb{R}$ such that $-\infty < a < b < \infty$.

We make use of Lemma 8.1, as a a special case of Lemma 1.2 for $n = 2$.

Our main result in this scope is the following improvement of Theorem 10.3.

Theorem 10.4 *Let L satisfy L1, L2 and L3 on a nonempty set E and let A be a positive normalized linear functional. If $f: I \to \mathbb{R}$ is a continuous convex function and $[a,b] \subseteq I$, then for all $g \in L$ such that $g(E) \subseteq [a,b]$ and $f(g) \in L$ we have $A(g) \in [a,b]$ and*

$$f\left(\frac{pa+qb}{p+q}\right) \leq A(f(g)) \leq \frac{pf(a)+qf(b)}{p+q} - A(\tilde{g})\,\delta_f, \tag{10.10}$$

where p and q are any nonnegative real numbers such that

$$A(g) = \frac{pa+qb}{p+q} \tag{10.11}$$

and $\tilde{g}$, δ_f are defined by

$$\tilde{g} = \frac{1}{2}1 - \frac{\left|g - \frac{a+b}{2}1\right|}{b-a}, \quad \delta_f = f(a) + f(b) - 2f\left(\frac{a+b}{2}\right).$$

Proof. Firstly, note that $g(E) \subseteq [a,b]$ implies

$$a = A(a1) \leq A(g) \leq A(b1) = b,$$

hence there exists a unique nonnegative real number $\lambda \in [0,1]$ such that $A(g) = \lambda a + (1-\lambda)b$. If p,q are nonnegative real numbers satisfying (10.11), then

$$\frac{p}{p+q} = \lambda, \quad \frac{q}{p+q} = 1-\lambda.$$

From Jessen's inequality we have

$$f\left(\frac{pa+qb}{p+q}\right) = f(A(g)) \leq A(f(g)),$$

which is the first inequality in (10.10).

By Lemma 8.1, for $n = 2$, we have

$$\begin{aligned} f(g(x)) &= f\left(\frac{b-g(x)}{b-a}a + \frac{g(x)-a}{b-a}b\right) \\ &\leq \frac{b-g(x)}{b-a}f(a) + \frac{g(x)-a}{b-a}f(b) \\ &\quad - \min\left\{\frac{b-g(x)}{b-a}, \frac{g(x)-a}{b-a}\right\}\left[f(a)+f(b)-2f\left(\frac{a+b}{2}\right)\right]. \end{aligned}$$

Applying A to the above inequality, we obtain

$$A(f(g)) \leq \frac{b-A(g)}{b-a}f(a) + \frac{A(g)-a}{b-a}f(b) - A(\tilde{g})\left[f(a)+f(b)-2f\left(\frac{a+b}{2}\right)\right],$$

where $\tilde{g}$ is defined on E by

$$\tilde{g}(x) = \min\left\{\frac{b-g(x)}{b-a}, \frac{g(x)-a}{b-a}\right\} = \frac{1}{2} - \frac{\left|g(x) - \frac{a+b}{2}\right|}{b-a}$$

and by $L3$ it belongs to L. By (10.11) we obtain

$$A(f(g)) \leq \frac{pf(a)+qf(b)}{p+q} - A(\tilde{g})\,\delta_f,$$

which is the second inequality in (10.10). □

Remark 10.2 Theorem 10.4 is an improvement of Theorem 10.3, since under the required assumptions we have

$$A(\tilde{g})\,\delta_f = A\left(\frac{1}{2}1 - \frac{\left|g - \frac{a+b}{2}1\right|}{b-a}\right)\left(f(a) + f(b) - 2f\left(\frac{a+b}{2}\right)\right) \geq 0.$$

Furthermore (this will be important in the following considerations)

$$0 \leq A\left(\frac{1}{2}1 - \frac{\left|g - \frac{a+b}{2}1\right|}{b-a}\right) \leq \frac{1}{2}.$$

The following theorem is another improvement of Theorem 10.3.

Theorem 10.5 *Let L satisfy L1, L2 and L3 on a nonempty set E and let A be a positive normalized linear functional. If $f : I \to \mathbb{R}$ is a continuous convex function and $[a,b] \subseteq I$, then for all $g \in L$ such that*

$$g(E) \subseteq [a,b] \quad \text{and} \quad f(g) \in L$$

and for all y such that

$$0 < y \leq \frac{b-a}{p+q}\min\{p,q\}, \tag{10.12}$$

we have

$$\begin{aligned} f\left(\frac{pa+qb}{p+q}\right) &\leq A(f(g)) \qquad (10.13)\\ &\leq \frac{pf(a)+qf(b)}{p+q} - 2A(\tilde{g})\left[\frac{pf(a)+qf(b)}{p+q} - f\left(\frac{pa+qb}{p+q}\right)\right], \end{aligned}$$

where p and q are any nonnegative real numbers such that

$$A(g) = \frac{pa+qb}{p+q} \tag{10.14}$$

and $\tilde{g}$ is defined by

$$\tilde{g} = \frac{1}{2}1 - \frac{|g - A(g)1|}{2y}.$$

Proof. First note that from $g(E) \subseteq [a,b]$ follows $A(g) \in [a,b]$ and by (10.12) we have

$$a \leq A(g) - y < A(g) + y \leq b.$$

If we apply Theorem 10.4 to $a_1 = A(g) - y,\ b_1 = A(g) + y$, we have that

$$A(g) = \frac{A(g) - y + A(g) + y}{2} = \frac{a_1 + b_1}{2},$$

which implies that we can set $p = q = 1$ and by (10.10) we obtain

$$f(A(g)) \le A(f(g))$$

and

$$\begin{aligned} A(f(g)) &\le \frac{f(A(g)-y)+f(A(g)+y)}{2} - A(\tilde{g})[f(A(g)-y)+f(A(g)+y)-2f(A(g))] \\ &= (1-2A(\tilde{g}))\frac{f(A(g)-y)+f(A(g)+y)}{2} + 2A(\tilde{g})f(A(g)). \end{aligned}$$

Since f is convex on $[a,b]$, we know that

$$\begin{aligned} f(A(g)-y) &\le \frac{b-(A(g)-y)}{b-a}f(a) + \frac{A(g)-y-a}{b-a}f(b), \\ f(A(g)+y) &\le \frac{b-(A(g)+y)}{b-a}f(a) + \frac{A(g)+y-a}{b-a}f(b), \end{aligned}$$

hence

$$\frac{f(A(g)-y)+f(A(g)+y)}{2} \le \frac{b-A(g)}{b-a}f(a) + \frac{A(g)-a}{b-a}f(b).$$

If p and q are any nonnegative numbers such that (10.14) holds (note that they are different from those we started with), we obtain

$$\frac{f(A(g)-y)+f(A(g)+y)}{2} \le \frac{pf(a)+qf(b)}{p+q}.$$

Considering all this and the fact that $1-2A(\tilde{g}) \ge 0$ (see Remark 10.2), we deduce

$$\begin{aligned} A(f(g)) &\le (1-2A(\tilde{g}))\frac{pf(a)+qf(b)}{p+q} + 2A(\tilde{g})f(A(g)) \\ &= \frac{pf(a)+qf(b)}{p+q} - 2A(\tilde{g})\left[\frac{pf(a)+qf(b)}{p+q} - f\left(\frac{pa+qb}{p+q}\right)\right]. \end{aligned}$$

□

From (10.13) we can easily obtain a Hammer-Bullen type inequality for positive linear functionals.

Corollary 10.1 *Under the conditions of Theorem 10.5 the following inequality holds:*

$$(1-2A(\tilde{g}))\left[\frac{pf(a)+qf(b)}{p+q} - A(f(g))\right] \ge 2A(\tilde{g})\left[A(f(g)) - f\left(\frac{pa+qb}{p+q}\right)\right].$$

In the sequel, we show how these results can be used to obtain refinements of the previously given inequalities as well as the related Hammer-Bullen type inequalities.

Corollary 10.2 *Let p,q be given positive numbers and let $w:[a,b]\to\mathbb{R}_0^+$ be an integrable function which is symmetric with respect to the line $x=(pa+qb)/(p+q)=T$, in the sense that*

$$\left(\forall t\in\left[0,\frac{b-a}{p+q}\min\{p,q\}\right]\right)w(T+t)=w(T-t).$$

If $f:[a,b]\to\mathbb{R}$ is a convex function, then for all $y\in\mathbb{R}$ such that

$$0<y\le\frac{b-a}{p+q}\min\{p,q\} \tag{10.15}$$

and

$$\overline{w}=\int_{T-y}^{T+y}w(x)\mathrm{d}x\neq 0$$

the following inequalities hold:

$$f\left(\frac{pa+qb}{p+q}\right)\le\frac{1}{\overline{w}}\int_{T-y}^{T+y}w(x)f(x)\mathrm{d}x\le\frac{pf(a)+qf(b)}{p+q}-\Delta_w\delta_f, \tag{10.16}$$

where

$$\begin{aligned}\Delta_w&=\frac{1}{2}-\frac{1}{\overline{w}}\int_{T-y}^{T+y}w(x)\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\mathrm{d}x,\\ \delta_f&=f(a)+f(b)-2f\left(\frac{a+b}{2}\right).\end{aligned}$$

Proof. This is a special case of Theorem 10.4. First note that for some given positive numbers p,q and $T=(pa+qb)/(p+q)$ the assumptions on y imply $a\le T-y<T+y\le b$, hence f is defined on $[T-y,T+y]$. If we choose E, L, A and g as in Remark 10.1, all the conditions of Theorem 10.4 will be satisfied and (10.10) accordingly becomes

$$f\left(\frac{pa+qb}{p+q}\right)\le\frac{1}{\overline{w}}\int_{T-y}^{T+y}w(x)f(x)\mathrm{d}x\le\frac{pf(a)+qf(b)}{p+q}-A(\tilde{g})\,\delta_f,$$

where

$$\begin{aligned}A(\tilde{g})&=\frac{1}{\overline{w}}\int_{T-y}^{T+y}w(x)\tilde{g}(x)\mathrm{d}x \qquad (10.17)\\ &=\frac{1}{2}-\frac{1}{\overline{w}}\int_{T-y}^{T+y}w(x)\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\mathrm{d}x=\Delta_w.\end{aligned}$$

The condition on continuity of f on $[a,b]$, required in Theorem 10.4 for an arbitrary chosen A, just like in Jessen's inequality, can be omitted in this special case. □

Remark 10.3 Let us emphasize here that under the conditions of Corollary 10.2 we have $\Delta_w\delta_f>0$, hence (10.16) is a refinement of (10.7).

If we want to simplify Δ_w from the previous theorem we have to consider four cases:

(i) $T \in (a, (3a+b)/4]$ and y satisfying (10.15) or $T \in ((3a+b)/4, (a+b)/2]$ and $0 < y \leq (a+b)/2 - T$.
For such T and y we have $x - (a+b)/2 \leq 0$, for all $x \in [T-y, T+y]$, hence

$$\begin{aligned}\Delta_w &= \frac{1}{2} + \frac{1}{\overline{w}}\int_{T-y}^{T+y} w(x)\frac{x-\frac{a+b}{2}}{b-a}\mathrm{d}x \\ &= \frac{1}{2} + \frac{T}{b-a} - \frac{a+b}{2(b-a)} = \frac{T-a}{b-a}.\end{aligned}$$

Here we used the fact that symmetry of w yields

$$\frac{1}{\overline{w}}\int_{T-y}^{T+y} w(x)x\mathrm{d}x = T.$$

(ii) $T \in ((3a+b)/4, (a+b)/2]$ and $y > (a+b)/2 - T$, but still satisfying (10.15).
For such T and y the function defined by $v = x - (a+b)/2$ changes sign on $[T-y, T+y]$, hence we leave Δ_w in the form (10.17).

(iii) $T \in ((a+b)/2, (a+3b)/4,]$ and $y > T - (a+b)/2$, but still satisfying (10.15).
For such T and y the function v defined by $v = x - (a+b)/2$ changes sign on $[T-y, T+y]$, hence we again leave Δ_w in the form (10.17).

(iv) $T \in ((a+b)/2, (a+3b)/4,]$ and $0 < y \leq T - (b+a)/2$ or $T \in [(a+3b)/4, b)$ and y satisfying (10.15).
For such T and y we have $x - (a+b)/2 \geq 0$, for all $x \in [T-y, T+y]$, hence in a similar way as in (ii) we obtain

$$\Delta_w = \frac{b-T}{b-a}.$$

As a special case of Corollary 10.2, we obtain the Hammer-Bullen inequality (10.2).

Corollary 10.3 *If $f : [a,b] \to \mathbb{R}$ is a convex function, then the inequality*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \geq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - f\left(\frac{a+b}{2}\right) \tag{10.18}$$

holds.

Proof. This is a special case of Corollary 10.2 for $w = 1$, $p = q = 1$, $y = (b-a)/2$. In this case we have

$$\int_{T-y}^{T+y} w(x)\mathrm{d}x = \int_a^b \mathrm{d}x = b-a,$$

so it follows from (10.16) that

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2} - \Delta_w\delta_f. \tag{10.19}$$

A simple calculation gives $\Delta_w = 1/4$, hence

$$\begin{aligned}\frac{f(a)+f(b)}{2} - \Delta_w \delta_f &= \frac{f(a)+f(b)}{2} - \frac{1}{4}\left[f(a)+f(b)-2f\left(\frac{a+b}{2}\right)\right] \\ &= \frac{1}{2} f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{4}. \qquad (10.20)\end{aligned}$$

From (10.19) and (10.20) we obtain

$$2f\left(\frac{a+b}{2}\right) \le \frac{2}{b-a}\int_a^b f(x)\,\mathrm{d}x \le f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2},$$

which implies

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \ge \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - f\left(\frac{a+b}{2}\right).$$

□

In a similar way, as a special case of Corollary 10.2 we obtain (10.5), but we skip its proof here.

Corollary 10.4 *Let p,q be given positive numbers and let $w : [a,b] \to \mathbb{R}_0^+$ be an integrable function, symmetric with respect to the line $x = (pa+qb)/(p+q) = T$, in the sense that*

$$(\forall t \in [0, \min\{T-a, b-T\}])\, w(T+t) = w(T-t).$$

If $f : [a,b] \to \mathbb{R}$ is a convex function, then for all y such that

$$0 < y \le \frac{b-a}{p+q}\min\{p,q\} \text{ and } \overline{w} = \int_{T-y}^{T+y} w(x)\mathrm{d}x \ne 0$$

the following inequalities hold

$$\begin{aligned} f\left(\frac{pa+qb}{p+q}\right) &\le \frac{1}{\overline{w}}\int_{T-y}^{T+y} w(x) f(x)\mathrm{d}x \qquad (10.21)\\ &\le \frac{pf(a)+qf(b)}{p+q} - \Delta_w\left(\frac{pf(a)+qf(b)}{p+q} - f\left(\frac{pa+qb}{p+q}\right)\right),\end{aligned}$$

where

$$\Delta_w = 1 - \frac{1}{y\overline{w}}\left[\int_T^{T+y} w(x)x\mathrm{d}x - \int_{T-y}^{T} w(x)x\mathrm{d}x\right].$$

Proof. This is a special case of Theorem 10.5, for E, L, A and g as in Remark 10.1. In this case (10.13) becomes

$$\begin{aligned} f\left(\frac{pa+qb}{p+q}\right) &\le \frac{1}{\overline{w}}\int_{T-y}^{T+y} w(x) f(x)\mathrm{d}x \\ &\le \frac{pf(a)+qf(b)}{p+q} - 2A(\tilde{g})\left[\frac{pf(a)+qf(b)}{p+q} - f\left(\frac{pa+qb}{p+q}\right)\right],\end{aligned}$$

where

$$\begin{aligned}
A(\tilde{g}) &= \frac{1}{2} - \frac{1}{2y\overline{w}} \int_{T-y}^{T+y} w(x)\,|x-T|\,\mathrm{d}x \\
&= \frac{1}{2} - \frac{1}{2y\overline{w}} \left[T \int_{T-y}^{T} w(x)\,\mathrm{d}x - T \int_{T}^{T+y} w(x)\,\mathrm{d}x - \int_{T-y}^{T} w(x)\,x\mathrm{d}x + \int_{T}^{T+y} w(x)\,x\mathrm{d}x \right] \\
&= \frac{1}{2} - \frac{1}{2y\overline{w}} \left[\int_{T}^{T+y} w(x)\,x\mathrm{d}x - \int_{T-y}^{T} w(x)\,x\mathrm{d}x \right] = \frac{1}{2}\Delta_w .
\end{aligned}$$

□

Remark 10.4 A Hammer-Bullen type inequality follows easily from (10.21). Namely, under the conditions of Corollary 10.4, the following inequality holds:

$$\begin{aligned}
&(1-\Delta_w) \left[\frac{pf(a)+qf(b)}{p+q} - \frac{1}{\overline{w}} \int_{T-y}^{T+y} w(x)\,f(x)\mathrm{d}x \right] \\
&\geq \Delta_w \left[\frac{1}{\overline{w}} \int_{T-y}^{T+y} w(x)\,f(x)\mathrm{d}x - f\left(\frac{pa+qb}{p+q} \right) \right].
\end{aligned}$$

In the following corollary we give a refinement of the discrete analogue of the Hermite-Hadamard inequality (see [177, p. 145]).

Corollary 10.5 *Let $x_1 < x_2 < \cdots < x_n$ be equidistant points in I. Then for every convex function $f : I \to \mathbb{R}$ the following inequalities are valid:*

$$\begin{aligned}
f\left(\frac{x_1+x_n}{2}\right) &\leq \frac{1}{n}\sum_{i=1}^{n} f(x_i) \\
&\leq \frac{f(x_1)+f(x_n)}{2} - \Delta_n \left(\frac{f(x_1)+f(x_n)}{2} - f\left(\frac{x_1+x_n}{2}\right) \right),
\end{aligned}$$

where

$$\Delta_n = \begin{cases} 1 - \frac{k+1}{2k+1}, & n = 2k+1 \\ 1 - \frac{k}{2k-1}, & n = 2k \end{cases}, \ k \in \mathbb{N}_0.$$

Proof. This is a special case of Theorem 10.4 for $E = [a,b] = [x_1, x_n]$, $L = \mathbb{R}^E$, $g = id_E$ and A defined by

$$A(f) = \frac{1}{n}\sum_{i=1}^{n} f(x_i).$$

Since

$$(\forall i \in \{1, \ldots, n-1\}) \ \ x_{i+1} - x_i = h,$$

we have

$$\begin{aligned}
A(g) = A(id_E) &= \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{nx_1 + \frac{(n-1)(h+(n-1)h)}{2}}{n} \\
&= \frac{2x_1 + (n-1)h}{2} = \frac{x_1+x_n}{2},
\end{aligned}$$

that is, we can choose $p = q = 1$ and (10.10) becomes

$$f\left(\frac{x_1+x_n}{2}\right) \leq \frac{1}{n}\sum_{i=1}^{n} f(x_i) \leq \frac{f(x_1)+f(x_n)}{2} - A(\tilde{g})\,\delta_f,$$

where

$$\delta_f = 2\left(\frac{f(x_1)+f(x_n)}{2} - f\left(\frac{x_1+x_n}{2}\right)\right)$$

and

$$\begin{aligned} A(\tilde{g}) &= \frac{1}{2} - \frac{1}{n(x_n-x_1)}\sum_{i=1}^{n}\left|x_i - \frac{x_1+x_n}{2}\right| \\ &= \frac{1}{2} - \frac{1}{n(n-1)h}\sum_{i=1}^{n}\left|x_1+(i-1)h - \frac{2x_1+(n-1)h}{2}\right| \\ &= \frac{1}{2} - \frac{1}{2n(n-1)}\sum_{i=1}^{n}|2i-n-1|. \end{aligned}$$

Considering the parity of n, we obtain

$$\begin{aligned} A(\tilde{g}) &= \begin{cases} \frac{1}{2} - \frac{1}{2k(2k+1)}\sum_{i=1}^{k} 2i, & n = 2k+1 \\ \frac{1}{2} - \frac{1}{2k(2k-1)}\sum_{i=1}^{k}(2i-1), & n = 2k \end{cases} \\ &= \begin{cases} \frac{1}{2}\left(1-\frac{k+1}{2k+1}\right), & n = 2k+1 \\ \frac{1}{2}\left(1-\frac{k}{2k-1}\right), & n = 2k \end{cases} = \frac{1}{2}\Delta_n. \end{aligned}$$

Note that for $n = 1$ and $n = 2$ we have $\Delta_n = 0$. □

In order to provide another result, we need to add yet another property to the linear class L.

Let $\mathscr{A}$ be an algebra of subsets of E and let L be a class of functions $f\colon E \to \mathbb{R}$ having the properties $L1, L2, L3$ and

L4: $(\forall f \in L)\,(\forall E_1 \in \mathscr{A})\ fC_{E_1} \in L$,

where C_{E_1} is the characteristic function of E_1, that is,

$$C_{E_1}(t) = \begin{cases} 1, & t \in E_1 \\ 0, & t \in E \setminus E_1 \end{cases}.$$

It can be easily seen that for every $E_1 \in \mathscr{A}$ the following assertions are true:

(i) $C_{E_1} \in L$;

(ii) If A is a positive linear functional on L such that $A(C_{E_1}) > 0$ and $g \in L$, then A_1 defined by

$$A_1(g) = \frac{A(gC_{E_1})}{A(C_{E_1})}$$

is a positive normalized linear functional;

(iii) If A is a positive linear functional on L and $g \in L$, then

$$A(C_{E_1}) + A\left(C_{E \setminus E_1}\right) = 1$$

and

$$A(gC_{E_1}) + A\left(gC_{E \setminus E_1}\right) = A(g).$$

Theorem 10.6 *Let L satisfy $L1 - L4$ on a nonempty set E and let $f : I \to \mathbb{R}$ be a continuous convex function while $g, h \in L$ are such that $f(g), f(h) \in L$. Let A, B be two positive normalized linear functionals on L such that $A(h) = B(g)$. If $E_1 \in \mathscr{A}$ satisfies $A(C_{E_1}) > 0$, $A\left(C_{E \setminus E_1}\right) > 0$ and*

$$(\forall t \in E) \quad a \le g(t) \le b,$$

where

$$a = \min\left\{\frac{A(hC_{E_1})}{A(C_{E_1})}, \frac{A\left(hC_{E \setminus E_1}\right)}{A\left(C_{E \setminus E_1}\right)}\right\},$$

$$b = \max\left\{\frac{A(hC_{E_1})}{A(C_{E_1})}, \frac{A\left(hC_{E \setminus E_1}\right)}{A\left(C_{E \setminus E_1}\right)}\right\},$$

then

$$f(A(h)) \le B(f(g)) \le A(f(h)) - B(\tilde{g})\,\delta_f, \tag{10.22}$$

where $\tilde{g}$ and δ_f are defined as in Theorem 10.4. In the limiting case $a = b$, (10.22) becomes

$$f(A(h)) = B(f(g)) \le A(f(h)).$$

Proof. By Jessen's inequality we have

$$f\left(\frac{A(hC_{E_1})}{A(C_{E_1})}\right) \le \frac{A(f(h)C_{E_1})}{A(C_{E_1})}$$

and

$$f\left(\frac{A\left(hC_{E \setminus E_1}\right)}{A\left(C_{E \setminus E_1}\right)}\right) \le \frac{A\left(f(h)C_{E \setminus E_1}\right)}{A\left(C_{E \setminus E_1}\right)}.$$

Without loss of generality we may assume

$$a = \min\left\{\frac{A(hC_{E_1})}{A(C_{E_1})}, \frac{A\left(hC_{E \setminus E_1}\right)}{A\left(C_{E \setminus E_1}\right)}\right\} = \frac{A(hC_{E_1})}{A(C_{E_1})},$$

$$b = \max\left\{\frac{A(hC_{E_1})}{A(C_{E_1})}, \frac{A\left(hC_{E \setminus E_1}\right)}{A\left(C_{E \setminus E_1}\right)}\right\} = \frac{A\left(hC_{E \setminus E_1}\right)}{A\left(C_{E \setminus E_1}\right)}.$$

If $a < b$ and

$$p = A(C_{E_1}), \quad q = A\left(C_{E \setminus E_1}\right),$$

we have

$$p+q=A(C_E)=A(1)=1,$$

$$B(g)=A(h)=A(hC_{E_1})+A\left(hC_{E\setminus E_1}\right)=pa+qb,$$

and applying Theorem 10.4 to B and g, by (10.10) we obtain

$$\begin{aligned}
f(A(h)) &= f(B(g)) \le B(f(g)) \le pf(a)+qf(b)-B(\tilde{g})\,\delta_f \\
&= A(C_{E_1})\,f\left(\frac{A(hC_{E_1})}{A(C_{E_1})}\right)+A\left(C_{E\setminus E_1}\right)f\left(\frac{A\left(hC_{E\setminus E_1}\right)}{A\left(C_{E\setminus E_1}\right)}\right)-B(\tilde{g})\,\delta_f \\
&\le A(f(h)\,C_{E_1})+A\left(f(h)\,C_{E\setminus E_1}\right)-B(\tilde{g})\,\delta_f \\
&= A(f(h))-B(\tilde{g})\,\delta_f.
\end{aligned}$$

If $a=b$, it follows that g is a constant function and the limiting case follows immediately. □

Theorem 10.6 is an improvement of [177, Theorem 5.14] and at the same time it gives a refinement of Jessen's inequality. We also give the following improvement of [177, Theorem 5.14].

Theorem 10.7 *Suppose that the assumptions of Theorem 10.6 hold. If $a<b$, then for all y such that*

$$0<y\le \min\{B(g)-a,\,b-B(g)\} \tag{10.23}$$

the following inequalities are valid:

$$\begin{aligned}
f(A(h)) &\le B(f(g)) \\
&\le A(f(h))-2B(\tilde{g})\left[A(C_{E_1})\,f(a)+A\left(C_{E\setminus E_1}\right)f(b)-f(B(g))\right],
\end{aligned}$$

where

$$\tilde{g}=\frac{1}{2}1-\frac{|g-B(g)\,1|}{2y}.$$

Proof. This proof is almost identical to the proof of Theorem 10.6, except that we use Theorem 10.5 instead of Theorem 10.4, hence for $a<b$ and y satisfying (10.23), using (10.13) we obtain

$$\begin{aligned}
f(A(h)) &\le B(f(g)) \\
&\le A(f(h))-2B(\tilde{g})\left[A(C_{E_1})\,f(a)+A\left(C_{E\setminus E_1}\right)f(b)-f(B(g))\right].
\end{aligned}$$

□

10.3 Hammer-Bullen differences

Motivated by theorems 10.4 and 10.5, we define two functionals $\Phi_i : L_g \to \mathbb{R}$, $i = 1,2$, by

$$\Phi_1(f) = \frac{pf(a)+qf(b)}{p+q} - A(f(g)) - A(\tilde{g})\delta_f, \tag{10.24}$$

where $A, g, \tilde{g}, p$ and q are as in Theorem 10.4, $L_g = \{f : I \to \mathbb{R} : f(g) \in L\}$, $[a,b] \subseteq I$ and

$$\Phi_2(f) = \frac{pf(a)+qf(b)}{p+q} - A(f(g)) - 2A(\tilde{g})\left[\frac{pf(a)+qf(b)}{p+q} - f\left(\frac{pa+qb}{p+q}\right)\right], \tag{10.25}$$

where $A, g, \tilde{g}, p$ and q are as in Theorem 10.5, L_g as above and $[a,b] \subseteq I$. Obviously, Φ_1 and Φ_2 are linear.

If f is additionally continuous and convex, then theorems 10.4 and 10.5 imply $\Phi_i(f) \geq 0$, $i = 1,2$.

In the sequel, with f_0 we denote the function defined by $f_0(x) = x^2$ on any domain we might need.

Now, we give Lagrange and Cauchy type mean value theorems for the functionals Φ_i, $i = 1,2$.

Theorem 10.8 *Let L satisfy L1, L2 and L3 on a nonempty set E and let A be a positive normalized linear functional on L. Let $g \in L$ be such that $f_0 \in L_g$, $g(E) \in [a,b], [a,b] \subseteq I$ and let $f \in C^2(I)$ be such that $f \in L_g$. If Φ_1 and Φ_2 are linear functionals defined as in* (10.24) *and* (10.25), *then there exist $\xi_i \in [a,b]$ such that*

$$\Phi_i(f) = \frac{f''(\xi_i)}{2}\Phi_i(f_0), \quad i = 1,2.$$

Proof. We give a proof for the functional Φ_1. Since $f \in C^2(I)$, there exist real numbers $m = \min_{x\in[a,b]} f''(x)$ and $M = \max_{x\in[a,b]} f''(x)$. It is easy to show that the functions f_1, f_2 defined by

$$f_1(x) = \frac{M}{2}x^2 - f(x), \quad f_2(x) = f(x) - \frac{m}{2}x^2$$

are continuous and convex, therefore $\Phi_1(f_1) \geq 0, \Phi_1(f_2) \geq 0$. This implies

$$\frac{m}{2}\Phi_1(f_0) \leq \Phi_1(f) \leq \frac{M}{2}\Phi_1(f_0).$$

If $\Phi_1(f_0) = 0$, there is nothing left to prove. Suppose $\Phi_1(f_0) > 0$. We have

$$m \leq \frac{2\Phi_1(f)}{\Phi_1(x^2)} \leq M.$$

Hence, there exists $\xi_1 \in [a,b]$ such that

$$\Phi_1(f) = \frac{f''(\xi_1)}{2}\Phi_1(f_0).$$

$\square$

Theorem 10.9 *Let L satisfy L1, L2 and L3 on a non-empty set E and let A be a positive normalized linear functional on L. Let $g \in L$ be such that $f_0 \in L_g$, $g(E) \in [a,b], [a,b] \subseteq I$ and $f_1, f_2 \in C^2(I)$ such that $f_1, f_2 \in L_g$. If Φ_1 and Φ_2 are linear functionals defined as in* (10.24) *and* (10.25)*, then there exist $\xi_i \in [a,b]$ such that*

$$\frac{\Phi_i(f_1)}{\Phi_i(f_2)} = \frac{f_1''(\xi_i)}{f_2''(\xi_i)}, \quad i = 1,2,$$

provided that the denominators are non-zero.

Proof. We give a proof for the functional Φ_1. Define $f_3 \in C^2([a,b])$ by

$$f_3 = c_1 f_1 - c_2 f_2, \text{ where } c_1 = \Phi_1(f_2), \; c_2 = \Phi_1(f_1).$$

Using Theorem 10.8 we get that there exists $\xi_1 \in [a,b]$ such that

$$\left(c_1 \frac{f_1''(\xi_1)}{2} - c_2 \frac{f_2''(\xi_1)}{2} \right) \Phi_1(f_0) = 0.$$

Since $\Phi_1(f_0) \neq 0$, (otherwise we have a contradiction with $\Phi_1(f_2) \neq 0$, by Theorem 10.8), we obtain

$$\frac{\Phi_1(f_1)}{\Phi_1(f_2)} = \frac{f_1''(\xi_1)}{f_2''(\xi_1)}.$$

□

As we did in the previous chapters when we considered the similar subjects, we make use of an idea from [90] in employing an elegant method of producing an n-exponentially convex functions and exponentially convex functions, applying the functionals Φ_1 and Φ_2 to a given family with the same property.

Theorem 10.10 *Let Φ_i, $\quad i = 1,2$, be linear functionals defined as in (10.24) and (10.25). Let $\Upsilon = \{f_s : s \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an open interval I such that $\Upsilon \subseteq L_g$ and that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is n-exponentially convex in the Jensen sense on J, for every three mutually different points $y_0, y_1, y_2 \in I$. Then $s \mapsto \Phi_i(f_s)$ is an n-exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Phi_i(f_s)$ is continuous on J, then it is n-exponentially convex on J.*

Proof. For $\xi_i \in \mathbb{R}$, $i = 1, \ldots, n$ and $s_i \in J$, $i = 1, \ldots, n$, we define the function $h : I \to \mathbb{R}$ by

$$h(y) = \sum_{i,j=1}^{n} \xi_i \xi_j f_{\frac{s_i+s_j}{2}}(y).$$

Using the assumption that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is n-exponentially convex in the Jensen sense we obtain

$$[y_0, y_1, y_2; h] = \sum_{i,j=1}^{n} \xi_i \xi_j [y_0, y_1, y_2; f_{\frac{s_i+s_j}{2}}] \geq 0,$$

which in turn implies that h is a convex (and continuous) function on I, therefore $\Phi_i(h) \geq 0, \quad i = 1,2$. Hence

$$\sum_{i,j=1}^{n} \xi_i \xi_j \Phi_i(f_{\frac{s_i+s_j}{2}}) \geq 0.$$

We conclude that the function $s \mapsto \Phi_i(f_s)$ is n-exponentially convex on J in the Jensen sense. If the function $s \mapsto \Phi_i(f_s)$ is also continuous on J, then $s \mapsto \Phi_i(f_s)$ is n-exponentially convex by definition. □

The following corollary is an immediate consequence of the above theorem.

Corollary 10.6 *Let $\Phi_i, \quad i = 1,2$, be linear functionals defined as in (10.24) and (10.25). Let $\Upsilon = \{f_s : s \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an open interval I, such that $\Upsilon \subseteq L_g$ and that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is exponentially convex in the Jensen sense on J, for every three mutually different points $y_0, y_1, y_2 \in I$. Then $s \mapsto \Phi_i(f_s)$ is an exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Phi_i(f_s)$ is continuous on J, then it is exponentially convex on J.*

Corollary 10.7 *Let $\Phi_i, \quad i = 1,2$, be linear functionals defined as in (10.24) and (10.25). Let $\Omega = \{f_s : s \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an open interval I such that $\Omega \subseteq L_g$ and that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is 2-exponentially convex in the Jensen sense on J, for every three mutually different points $y_0, y_1, y_2 \in I$. Then the following statements hold:*

(i) If the function $s \mapsto \Phi_i(f_s)$ is continuous on J, then it is 2-exponentially convex function on J. If $s \mapsto \Phi_i(f_s)$ is additionally strictly positive, then it is also log-convex on J.

(ii) If the function $s \mapsto \Phi_i(f_s)$ is strictly positive and differentiable on J, then for every $s, q, u, v \in J$, such that $s \leq u$ and $q \leq v$, we have

$$\mu_{s,q}(\Phi_i, \Omega) \leq \mu_{u,v}(\Phi_i, \Omega), \;\; i = 1,2, \tag{10.26}$$

where

$$\mu_{s,q}(\Phi_i, \Omega) = \begin{cases} \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)} \right)^{\frac{1}{s-q}} & , s \neq q, \\ \exp\left(\frac{\frac{d}{ds}\Phi_i(f_s)}{\Phi_i(f_s)} \right) & , s = q, \end{cases} \tag{10.27}$$

for $f_s, f_q \in \Omega$.

Proof. (i) This is an immediate consequence of Theorem 10.10 and Remark 1.3.

(ii) Since by (i) the function $s \mapsto \Phi_i(f_s)$ is log-convex on J, that is, the function $s \mapsto \log \Phi_i(f_s)$ is convex on J, applying Proposition 1.2 we get

$$\frac{\log \Phi_i(f_s) - \log \Phi_i(f_q)}{s-q} \leq \frac{\log \Phi_i(f_u) - \log \Phi_i(f_v)}{u-v}, \tag{10.28}$$

for $s \leq u$, $q \leq v$, $s \neq q$, $u \neq v$, and therefrom conclude that

$$\mu_{s,q}(\Phi_i, \Omega) \leq \mu_{u,v}(\Phi_i, \Omega), \quad i = 1, 2.$$

Cases $s = q$ and $u = v$ follow from (10.28) as limit cases. □

Remark 10.5 Note that the results from Theorem 10.10, Corollary 10.6, Corollary 10.7 still hold when two of the points $y_0, y_1, y_2 \in I$ coincide, say $y_1 = y_0$, for a family of differentiable functions f_s such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, they still hold when all three points coincide for a family of twice differentiable functions with the same property. The proofs are obtained by recalling Remark 1.4 and the suitable characterization of convexity.

Now, we present several families of functions which fulfil the conditions of Theorem 10.10, Corollary 10.6 and Corollary 10.7 (and Remark 10.5). This enable us to construct a large family of functions which are exponentially convex. For a discussion related to this problem see [68].

In the rest of the section we consider only Φ_1 and Φ_2 defined as in (10.24) and (10.25) with A which is continuous and g such that compositions with any function from the chosen familly Ω_i, as well as with other functions which appear as arguments of Φ_1 and Φ_2, remain in L.

Example 10.1 Consider a family of functions

$$\Omega_1 = \{g_s : \mathbb{R} \to [0, \infty) : s \in \mathbb{R}\}$$

defined by

$$g_s(x) = \begin{cases} \frac{1}{s^2} e^{sx}, & s \neq 0, \\ \frac{1}{2} x^2, & s = 0. \end{cases}$$

We have $\frac{d^2 g_s}{dx^2}(x) = e^{sx} > 0$ which shows that g_s is convex on $\mathbb{R}$ for every $s \in \mathbb{R}$ and $s \mapsto \frac{d^2 g_s}{dx^2}(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 10.10 we also have that $s \mapsto [y_0, y_1, y_2; g_s]$ is exponentially convex (and so exponentially convex in the Jensen sense). Using Theorem 10.6 we conclude that $s \mapsto \Phi_i(g_s)$, $i = 1, 2$, are exponentially convex in the Jensen sense. It is easy to verify that these mappings are continuous (although mapping $s \mapsto g_s$ is not continuous for $s = 0$), so they are exponentially convex.

For this family of functions, $\mu_{s,q}(\Phi_i, \Omega_1)$, $\quad i = 1, 2$, from (10.27) become

$$\mu_{s,q}(\Phi_i, \Omega_1) = \begin{cases} \left(\frac{\Phi_i(g_s)}{\Phi_i(g_q)} \right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{\Phi_i(id \cdot g_s)}{\Phi_i(g_s)} - \frac{2}{s} \right), & s = q \neq 0, \\ \exp\left(\frac{\Phi_i(id \cdot g_0)}{3\Phi_i(g_0)} \right), & s = q = 0, \end{cases}$$

and using (10.26) they are monotonic functions in parameters s and q.

Using Theorem 10.9 it follows that for $i = 1,2$

$$M_{s,q}(\Phi_i, \Omega_1) = \log \mu_{s,q}(\Phi_i, \Omega_1)$$

satisfy $a \leq M_{s,q}(\Phi_i, \Omega_1) \leq b$, which shows that $M_{s,q}(\Phi_i, \Omega_1)$ are means (of a function g). Notice that by (10.26) they are monotonic.

Example 10.2 Consider a family of functions

$$\Omega_2 = \{f_s : (0,\infty) \to \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$f_s(x) = \begin{cases} \frac{x^s}{s(s-1)}, & s \neq 0,1, \\ -\log x, & s = 0, \\ x \log x, & s = 1. \end{cases}$$

Here, $\frac{d^2 f_s}{dx^2}(x) = x^{s-2} = e^{(s-2)\ln x} > 0$ which shows that f_s is convex for $x > 0$ and $s \mapsto \frac{d^2 f_s}{dx^2}(x)$ is exponentially convex by definition. Arguing as in Example 10.1 we get that the mappings $s \mapsto \Phi_i(g_s), \quad i = 1,2$ are exponentially convex. Functions (10.27) in this case are equal to:

$$\mu_{s,q}(\Phi_i, \Omega_2) = \begin{cases} \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{1-2s}{s(s-1)} - \frac{\Phi_i(f_s f_0)}{\Phi_i(f_s)}\right), & s = q \neq 0,1, \\ \exp\left(1 - \frac{\Phi_i(f_0^2)}{2\Phi_i(f_0)}\right), & s = q = 0, \\ \exp\left(-1 - \frac{\Phi_i(f_0 f_1)}{2\Phi_i(f_1)}\right), & s = q = 1. \end{cases}$$

If Φ_i is positive, then Theorem 10.9 applied for $f = f_s \in \Omega_2$ and $g = f_q \in \Omega_2$ yields that there exists $\xi \in [a,b]$ such that

$$\xi^{s-q} = \frac{\Phi_i(f_s)}{\Phi_i(f_q)}.$$

Since the function $\xi \mapsto \xi^{s-q}$ is invertible for $s \neq q$, we then have

$$a \leq \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}} \leq b, \tag{10.29}$$

which together with the fact that $\mu_{s,q}(\Phi_i, \Omega_2)$ is continuous, symmetric and monotonic (by (10.26)), shows that $\mu_{s,q}(\Phi_i, \Omega_2)$ is a mean (of a function h).

Example 10.3 Consider a family of functions

$$\Omega_3 = \{h_s : (0,\infty) \to (0,\infty) : s \in (0,\infty)\}$$

defined by

$$h_s(x) = \begin{cases} \frac{s^{-x}}{\ln^2 s}, & s \neq 1, \\ \frac{x^2}{2}, & s = 1. \end{cases}$$

Since $s \mapsto \frac{d^2 h_s}{dx^2}(x) = s^{-x}$ is the Laplace transform of a non-negative function (see [211]), it is exponentially convex. Obviously h_s are convex functions for every $s > 0$.

For this family of functions, $\mu_{s,q}(\Phi_i, \Omega_3)$, from (10.27) becomes

$$\mu_{s,q}(\Phi_i, \Omega_3) = \begin{cases} \left(\frac{\Phi_i(h_s)}{\Phi_i(h_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(-\frac{\Phi_i(id \cdot h_s)}{s\Phi_i(h_s)} - \frac{2}{s \ln s}\right), & s = q \neq 1, \\ \exp\left(-\frac{2\Phi_i(id \cdot h_1)}{3\Phi_i(h_1)}\right), & s = q = 1, \end{cases}$$

and it is monotonic in parameters s and q by (10.26).

Using Theorem 10.9, it follows that

$$M_{s,q}(\Phi_i, \Omega_3) = -L(s,q) \log \mu_{s,q}(\Phi_i \Omega_3)$$

satisfies $a \leq M_{s,q}(\Phi_i, \Omega_3) \leq b$, which shows that $M_{s,q}(\Phi_i, \Omega_3)$ is a mean (of a function h). $L(s,q)$ is the logarithmic mean defined by $L(s,q) = \frac{s-q}{\log s - \log q}$, $s \neq q$, $L(s,s) = s$.

Example 10.4 Consider a family of functions

$$\Omega_4 = \{k_s : (0,\infty) \to (0,\infty) : s \in (0,\infty)\}$$

defined by

$$k_s(x) = \frac{e^{-x\sqrt{s}}}{s}.$$

Since $s \mapsto \frac{d^2 k_s}{dx^2}(x) = e^{-x\sqrt{s}}$ is the Laplace transform of a non-negative function (see [211]), it is exponentially convex. Obviously k_s are convex functions for every $s > 0$.

For this family of functions, $\mu_{s,q}(\Phi_i, \Omega_4)$ from (10.27) becomes

$$\mu_{s,q}(\Phi_i, \Omega_4) = \begin{cases} \left(\frac{\Phi_i(k_s)}{\Phi_i(k_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(-\frac{\Phi_i(id \cdot k_s)}{2\sqrt{s}\Phi_i(k_s)} - \frac{1}{s}\right), & s = q, \end{cases}$$

and it is monotonic function in parameters s and q by (10.26).

Using Theorem 10.9, it follows that

$$M_{s,q}(\Phi_i, \Omega_4) = -\left(\sqrt{s} + \sqrt{q}\right) \log \mu_{s,q}(\Phi_i, \Omega_4)$$

satisfies $a \leq M_{s,q}(\Phi_i, \Omega_4) \leq b$, which shows that $M_{s,q}(\Phi_i, \Omega_4)$ is a mean (of a function h).

Chapter 11

On the refinements of the Jensen operator inequality

In this chapter, several refinements of the Jensen operator inequality are presented, for n-tuples of self-adjoint operators, unital n-tuples of positive linear mappings and real valued continuous convex functions with the condition on the spectra of the operators. Using these refinements, the refinements of inequalities among quasi-arithmetic means, under similar conditions are obtained and, as an application of these results, a refinement of inequalities among power means is additionally provided.

The chapter is concluded with the considerations on the converses of the generalized Jensen inequality for a continuous field of self-adjoint operators, a unital field of positive linear mappings and real valued continuous convex functions, where new refined converses are presented using the Mond-Pečarić method improvements.

The reader can find the presented results published in [141], [142] and [143].

11.1 Jensen's operator inequality

At the very start of our consideration, we recall the basic notions and definitions, some of which have already been used throughout the monograph. Let $\mathscr{B}(H)$ be a C^*-algebra of all bounded linear operators on a Hilbert space H, where 1_H stands for the identity operator.

We define bounds of a self-adjoint operator $A \in \mathscr{B}(H)$ by

$$m_A = \inf_{\|x\|=1} \langle Ax, x\rangle \quad \text{and} \quad M_A = \sup_{\|x\|=1} \langle Ax, x\rangle,$$

for $x \in H$. If $\mathsf{Sp}(A)$ denotes the spectrum of A, then $\mathsf{Sp}(A)$ is real and $\mathsf{Sp}(A) \subseteq [m_A, M_A]$.

Furthermore, for an operator $A \in \mathscr{B}(H)$ we define operators $|A|$, A^+, A^- by

$$|A| = (A^*A)^{1/2}, \qquad A^+ = (|A| + A)/2, \qquad A^- = (|A| - A)/2.$$

Obviously, if A is self-adjoint, then $|A| = (A^2)^{1/2}$ and $A^+, A^- \geq 0$ (called positive and negative parts of $A = A^+ - A^-$).

In [156], B. Mond and J. Pečarić proved the following version of the Jensen operator inequality:

$$f\left(\sum_{i=1}^n w_i \Phi_i(A_i)\right) \leq \sum_{i=1}^n w_i \Phi_i\left(f(A_i)\right), \tag{11.1}$$

for an operator convex function f defined on an interval I, where $\Phi_i : \mathscr{B}(H) \to \mathscr{B}(K)$, $i = 1, \ldots, n$, are unital positive linear mappings, $A_1, \ldots, A_n$ are self-adjoint operators with the spectra in I and $w_1, \ldots, w_n$ are non-negative real numbers with $\sum_{i=1}^n w_i = 1$.

In [80], F. Hansen, J. Pečarić and I. Perić gave a generalization of (11.1) for a unital field of positive linear mappings. The following discrete version of their inequality:

$$f\left(\sum_{i=1}^n \Phi_i(A_i)\right) \leq \sum_{i=1}^n \Phi_i\left(f(A_i)\right) \tag{11.2}$$

holds for an operator convex function f defined on an interval I, where $\Phi_i : \mathscr{B}(H) \to \mathscr{B}(K)$, $i = 1, \ldots, n$, are unital fields of positive linear mappings (i.e. $\sum_{i=1}^n \Phi_i(1_H) = 1_K$), $A_1, \ldots, A_n$ are self-adjoint operators with the spectra in I.

Very recently, in [139, Theorem 1], J. Mićić, Z. Pavić and J. Pečarić provided the form of the Jensen operator inequality without operator convexity, as follows.

Theorem 11.1 *Let $(A_1, \ldots, A_n)$ be an n-tuple of self-adjoint operators $A_i \in \mathscr{B}(H)$ with bounds m_i and M_i, $m_i \leq M_i$, $i = 1, \ldots, n$. Let $(\Phi_1, \ldots, \Phi_n)$ be an n-tuple of positive linear mappings $\Phi_i : \mathscr{B}(H) \to \mathscr{B}(K)$, $i = 1, \ldots, n$, such that $\sum_{i=1}^n \Phi_i(1_H) = 1_K$. If*

$$(m_A, M_A) \cap [m_i, M_i] = \emptyset, \quad \textit{for } i = 1, \ldots, n, \tag{11.3}$$

where m_A and M_A, $m_A \leq M_A$, are bounds of the self-adjoint operator $A = \sum_{i=1}^n \Phi_i(A_i)$, then

$$f\left(\sum_{i=1}^n \Phi_i(A_i)\right) \leq \sum_{i=1}^n \Phi_i\left(f(A_i)\right) \tag{11.4}$$

holds for every continuous convex function $f : I \to \mathbb{R}$ provided that the interval I contains all m_i, M_i.

If $f : I \to \mathbb{R}$ is concave, then the reverse inequality is valid in (11.4).

Furthermore, they considered in [138, Theorem 2.1] the case when $(m_A, M_A) \cap [m_i, M_i] = \emptyset$ is valid for several $i \in \{1, \dots, n\}$, but not for all $i = 1, \dots, n$ and thus obtained an extension of (11.2), as follows.

Theorem 11.2 *Let $(A_1, \dots, A_n)$ be an n-tuple of self-adjoint operators $A_i \in B(H)$ with the bounds m_i and M_i, $m_i \le M_i$, $i = 1, \dots, n$. Let $(\Phi_1, \dots, \Phi_n)$ be an n-tuple of positive linear mappings $\Phi_i : \mathscr{B}(H) \to \mathscr{B}(K)$, such that $\sum_{i=1}^{n_1} \Phi_i(1_H) = \alpha\, 1_K$, $\sum_{i=n_1+1}^{n} \Phi_i(1_H) = \beta\, 1_K$, where $1 \le n_1 < n$, $\alpha, \beta > 0$ and $\alpha + \beta = 1$. Let $m = \min\{m_1, \dots, m_{n_1}\}$ and $M = \max\{M_1, \dots, M_{n_1}\}$. If*

$$(m, M) \cap [m_i, M_i] = \emptyset \qquad \text{for} \quad i = n_1 + 1, \dots, n,$$

and one of two equalities

$$\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(A_i) = \sum_{i=1}^{n} \Phi_i(A_i) = \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(A_i)$$

is valid, then

$$\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(f(A_i)) \le \sum_{i=1}^{n} \Phi_i(f(A_i)) \le \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) \tag{11.5}$$

holds for every continuous convex function $f : I \to \mathbb{R}$ provided that the interval I contains all m_i, M_i, $i = 1, \dots, n$.

If $f : I \to \mathbb{R}$ is concave, then the reverse inequality is valid in (11.5).

In order to obtain our main result, we make use of the left hand side of Lemma 1.2, for $n = 2$, which for a convex function f defined on an interval I and for $x, y \in I$, $p_1, p_2 \in [0, 1]$, such that $p_1 + p_2 = 1$ provides

$$\begin{aligned} \min\{p_1, p_2\} & \left[f(x) + f(y) - 2f\left(\frac{x+y}{2}\right) \right] \\ & \le p_1 f(x) + p_2 f(y) - f(p_1 x + p_2 y). \end{aligned} \tag{11.6}$$

In Theorem 11.1 it was shown that Jensen's operator inequality holds for every continuous convex function and for every n-tuple of self-adjoint operators $(A_1, \dots, A_n)$, for every n-tuple of positive linear mappings $(\Phi_1, \dots, \Phi_n)$ in the case when the interval with bounds of the operator $A = \sum_{i=1}^{n} \Phi_i(A_i)$ has no intersection points with the interval with bounds of the operator A_i for each $i = 1, \dots, n$. Now, by means of (11.6) we obtain a refinement of this inequality. We still need another result, utilizing the idea previously moderated in [102, Theorem 12].

Lemma 11.1 *Let A be a self-adjoint operator $A \in \mathscr{B}(H)$ with $Sp(A) \subseteq [m, M]$, for some scalars $m < M$. Then*

$$f(A) \le \frac{M 1_H - A}{M - m} f(m) + \frac{A - m 1_H}{M - m} f(M) - \delta_f \widetilde{A} \tag{11.7}$$

$$\left(\textit{resp.} \quad f(A) \ge \frac{M 1_H - A}{M - m} f(m) + \frac{A - m 1_H}{M - m} f(M) + \delta_f \widetilde{A}\right)$$

holds for every continuous convex (resp. concave) function $f\colon [m,M] \to \mathbb{R}$*, where*

$$\delta_f = f(m) + f(M) - 2f\left(\tfrac{m+M}{2}\right) \quad (\textit{resp. } \delta_f = 2f\left(\tfrac{m+M}{2}\right) - f(m) - f(M)),$$
$$\textit{and} \quad \widetilde{A} = \tfrac{1}{2}1_H - \tfrac{1}{M-m}\left|A - \tfrac{m+M}{2}1_H\right|.$$

Proof. We prove only the convex case. Putting $x = m, y = M$ in (11.6) it follows that

$$\begin{aligned} f(p_1 m + p_2 M) \ \le\ & p_1 f(m) + p_2 f(M) \\ & - \min\{p_1,p_2\}\left(f(m) + f(M) - 2f\left(\tfrac{m+M}{2}\right)\right) \end{aligned} \tag{11.8}$$

holds for every $p_1, p_2 \in [0,1]$ such that $p_1 + p_2 = 1$. For any $t \in [m,M]$ we can write

$$f(t) = f\left(\frac{M-t}{M-m}m + \frac{t-m}{M-m}M\right).$$

Then by using (11.8) for $p_1 = \frac{M-t}{M-m}$ and $p_2 = \frac{t-m}{M-m}$ we get

$$\begin{aligned} f(t) \ \le\ & \frac{M-t}{M-m}f(m) + \frac{t-m}{M-m}f(M) \\ & - \left(\frac{1}{2} - \frac{1}{M-m}\left|t - \frac{m+M}{2}\right|\right)\left(f(m) + f(M) - 2f\left(\frac{m+M}{2}\right)\right), \end{aligned} \tag{11.9}$$

since

$$\min\left\{\frac{M-t}{M-m}, \frac{t-m}{M-m}\right\} = \frac{1}{2} - \frac{1}{M-m}\left|t - \frac{m+M}{2}\right|.$$

Finally, we use the continuous functional calculus for a self-adjoint operator A: $f, g \in \mathscr{C}(I), Sp(A) \subseteq I$ and $f \ge g$ implies $f(A) \ge g(A)$; and $h(t) = |t|$ implies $h(A) = |A|$. Then, by using (11.9), we obtain the desired inequality (11.7). $\square$

Theorem 11.3 *Let* $(A_1,\dots,A_n)$ *be an n-tuple of self-adjoint operators* $A_i \in \mathscr{B}(H)$ *with the bounds* m_i *and* M_i, $m_i \le M_i$, $i = 1,\dots,n$. *Let* $(\Phi_1,\dots,\Phi_n)$ *be an n-tuple of positive linear mappings* $\Phi_i : \mathscr{B}(H) \to \mathscr{B}(K)$, $i = 1,\dots,n$, *such that* $\sum_{i=1}^n \Phi_i(1_H) = 1_K$. *Let*

$$(m_A, M_A) \cap [m_i, M_i] = \emptyset \quad \textit{for } i = 1,\dots,n, \qquad \textit{and} \qquad m < M,$$

where m_A *and* M_A, $m_A \le M_A$, *are the bounds of the operator* $A = \sum_{i=1}^n \Phi_i(A_i)$ *and*

$$m = \max\{M_i \colon M_i \le m_A, i \in \{1,\dots,n\}\}, \quad M = \min\{m_i \colon m_i \ge M_A, i \in \{1,\dots,n\}\}.$$

If $f\colon I \to \mathbb{R}$ *is a continuous convex (resp. concave) function provided that the interval* I *contains all* m_i, M_i*, then*

$$f\left(\sum_{i=1}^n \Phi_i(A_i)\right) \le \sum_{i=1}^n \Phi_i(f(A_i)) - \delta_f \widetilde{A} \le \sum_{i=1}^n \Phi_i(f(A_i)) \tag{11.10}$$

$$(\textit{resp.} \quad f\left(\sum_{i=1}^n \Phi_i(A_i)\right) \ge \sum_{i=1}^n \Phi_i(f(A_i)) + \delta_f \widetilde{A} \ge \sum_{i=1}^n \Phi_i(f(A_i))) \tag{11.11}$$

holds, where

$$\begin{aligned} \delta_f \equiv \delta_f(\bar{m},\bar{M}) &= f(\bar{m}) + f(\bar{M}) - 2f\left(\tfrac{\bar{m}+\bar{M}}{2}\right) \\ (\textit{resp.}\quad \delta_f \equiv \delta_f(\bar{m},\bar{M}) &= 2f\left(\tfrac{\bar{m}+\bar{M}}{2}\right) - f(\bar{m}) - f(\bar{M})\ ), \\ \widetilde{A} \equiv \widetilde{A}_A(\bar{m},\bar{M}) &= \tfrac{1}{2}1_K - \tfrac{1}{\bar{M}-\bar{m}}\left|A - \tfrac{\bar{m}+\bar{M}}{2}1_K\right| \end{aligned} \tag{11.12}$$

and $\bar{m} \in [m, m_A]$, $\bar{M} \in [M_A, M]$, $\bar{m} < \bar{M}$, *are arbitrary numbers.*

Proof. We prove only the convex case.

Since $A = \sum_{i=1}^n \Phi_i(A_i) \in \mathscr{B}(K)$ is the self-adjoint operator such that $\bar{m}1_K \le m_A 1_K \le \sum_{i=1}^n \Phi_i(A_i) \le M_A 1_K \le \bar{M}1_K$ and f is convex on $[\bar{m},\bar{M}] \subseteq I$, then by Lemma 11.1 we obtain

$$f\left(\sum_{i=1}^n \Phi_i(A_i)\right) \le \frac{\bar{M}1_K - \sum_{i=1}^n \Phi_i(A_i)}{\bar{M}-\bar{m}} f(\bar{m}) + \frac{\sum_{i=1}^n \Phi_i(A_i) - \bar{m}1_K}{\bar{M}-\bar{m}} f(\bar{M}) - \delta_f \widetilde{A}, \tag{11.13}$$

where δ_f and $\widetilde{A}$ are defined by (11.12).

But since f is convex on $[m_i, M_i]$ and since $(m_A, M_A) \cap [m_i, M_i] = \emptyset$ implies $(\bar{m},\bar{M}) \cap [m_i, M_i] = \emptyset$, then

$$f(A_i) \ge \frac{\bar{M}1_H - A_i}{\bar{M}-\bar{m}} f(\bar{m}) + \frac{A_i - \bar{m}1_H}{\bar{M}-\bar{m}} f(\bar{M}), \qquad i = 1,\ldots,n$$

holds. Applying a positive linear mapping Φ_i, summing and adding $-\delta_f \widetilde{A}$, we obtain

$$\sum_{i=1}^n \Phi_i(f(A_i)) - \delta_f \widetilde{A} \ge \frac{\bar{M}1_K - \sum_{i=1}^n \Phi_i(A_i)}{\bar{M}-\bar{m}} f(\bar{m}) + \frac{\sum_{i=1}^n \Phi_i(A_i) - \bar{m}1_K}{\bar{M}-\bar{m}} f(\bar{M}) - \delta_f \widetilde{A}, \tag{11.14}$$

since $\sum_{i=1}^n \Phi_i(1_H) = 1_K$. Combining inequalities (11.13) and (11.14), we have the left hand side of (11.10). Since $\delta_f \ge 0$ and $\widetilde{A} \ge 0$, we also have the right hand side of (11.10). □

Remark 11.1 In particular, if $m_A < M_A$, then Theorem 11.3 in the convex case yields

$$f\left(\sum_{i=1}^n \Phi_i(A_i)\right) \le \sum_{i=1}^n \Phi_i(f(A_i)) - \bar{\delta}_f \bar{A} \le \sum_{i=1}^n \Phi_i(f(A_i)),$$

where

$$\bar{\delta}_f \equiv \delta_f(m_A, M_A) = f(m_A) + f(M_A) - 2f\left(\frac{m_A + M_A}{2}\right)$$

and

$$\bar{A} \equiv \widetilde{A}_A(m_A, M_A) = \frac{1}{2}1_K - \frac{1}{M_A - m_A}\left|A - \frac{m_A + M_A}{2}1_K\right|.$$

Note that if $m < M$ and $m_A = M_A$, then inequality (11.10) holds, but $\bar{\delta}_f \bar{A}$ is not defined. This case is worked out in Example 11.1 (I) and (II).

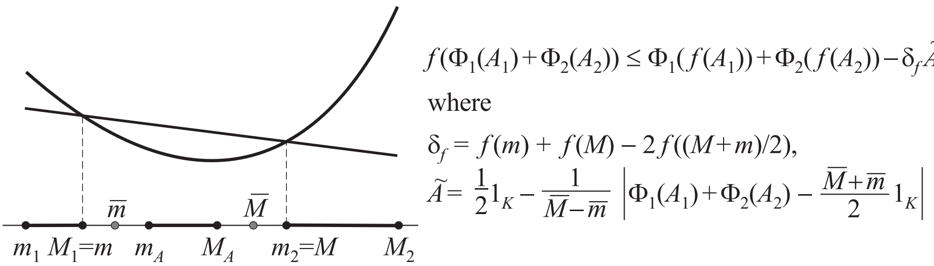

Figure 11.1: *Refinement for two operators and a convex function f*

Example 11.1 We give three examples for the matrix case and $n = 2$.

We put $f(t) = t^4$ which is convex, but not operator convex in (11.10) (see [74]). Also, we define mappings $\Phi_1, \Phi_2 : M_3(\mathbb{C}) \to M_2(\mathbb{C})$ as follows: $\Phi_1((a_{ij})_{1\le i,j\le 3}) = \frac{1}{2}(a_{ij})_{1\le i,j\le 2}$, $\Phi_2 = \Phi_1$ (then $\Phi_1(I_3) + \Phi_2(I_3) = I_2$).

I) Firstly, we observe an example when $\delta_f\widetilde{A}$ is equal to the difference of the right hand side and the left hand side of Jensen's inequality. If $A_1 = -3I_3$ and $A_2 = 2I_3$, then $A = \Phi_1(A_1) + \Phi_2(A_2) = -0.5I_2$, so $m = -3$, $M = 2$. We also put $\overline{m} = -3$ and $\overline{M} = 2$ and obtain

$$(\Phi_1(A_1) + \Phi_2(A_2))^4 = 0.0625I_2 \le 48.5I_2 = \Phi_1\left(A_1^4\right) + \Phi_2\left(A_2^4\right)$$

and its improvement

$$(\Phi_1(A_1) + \Phi_2(A_2))^4 = 0.0625I_2 = \Phi_1\left(A_1^4\right) + \Phi_2\left(A_2^4\right) - 48.4375I_2,$$

since $\delta_f = 96.875$, $\widetilde{A} = 0.5I_2$.

II) Next, we observe an example when $\delta_f\widetilde{A}$ is not equal to the difference of the right hand side and the left hand side of Jensen's inequality. If

$$A_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad \text{then} \quad A = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so $m = -1$, $M = 2$. We put $\overline{m} = -1$ and $\overline{M} = 2$ and obtain

$$(\Phi_1(A_1) + \Phi_2(A_2))^4 = \frac{1}{16}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \le \begin{pmatrix} \frac{17}{2} & 0 \\ 0 & \frac{97}{2} \end{pmatrix} = \Phi_1\left(A_1^4\right) + \Phi_2\left(A_2^4\right)$$

and its improvement

$$\begin{aligned}(\Phi_1(A_1) + \Phi_2(A_2))^4 &= \frac{1}{16}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \le \frac{1}{16}\begin{pmatrix} 1 & 0 \\ 0 & 641 \end{pmatrix} \\ &= \Phi_1\left(A_1^4\right) + \Phi_2\left(A_2^4\right) - \frac{135}{16}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\end{aligned}$$

since $\delta_f = 135/8$, $\widetilde{A} = I_2/2$.

III) Next, we observe another example with matrices A_1 and A_2. If

$$A_1 = \begin{pmatrix} -4 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & -1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 5 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 3, \end{pmatrix} \quad \text{then} \quad A = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

so $m_1 = -4.8662$, $M_1 = -0.3446$, $m_2 = 1.3446$, $M_2 = 5.8662$, $m = -0.3446$, $M = 1.3446$ and we put $\overline{m} = m$, $\overline{M} = M$ (rounded to four decimal places). We have

$$(\Phi_1(A_1) + \Phi_2(A_2))^4 = \frac{1}{16}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \leq \begin{pmatrix} \frac{1283}{2} & -255 \\ -255 & \frac{237}{2} \end{pmatrix} = \Phi_1\left(A_1^4\right) + \Phi_2\left(A_2^4\right)$$

and its improvement

$$\begin{aligned} (\Phi_1(A_1) + \Phi_2(A_2))^4 &= \frac{1}{16}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \leq \begin{pmatrix} 639.9213 & -255 \\ -255 & 117.8559 \end{pmatrix} \\ &= \Phi_1\left(A_1^4\right) + \Phi_2\left(A_2^4\right) - \begin{pmatrix} 1.5787 & 0 \\ 0 & 0.6441 \end{pmatrix} \end{aligned}$$

(rounded to four decimal places), since

$$\delta_f = 3.1574, \qquad \widetilde{A} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.2040 \end{pmatrix}.$$

But, if we put $\overline{m} = m_A = 0$, $\overline{M} = M_A = 0.5$, then $\tilde{A} = \mathbf{0}$, so we do not have an improvement of Jensen's inequality. Also, if we put $\overline{m} = 0$, $\overline{M} = 1$, then $\tilde{A} = 0.5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\delta_f = 7/8$ and $\delta_f \tilde{A} = 0.4375\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, which is worse than the above improvement.

What follows is a corollary of Theorem 11.3 involving the convex combination of operators A_i, $i = 1, \ldots, n$.

Corollary 11.1 *Let $(A_1, \ldots, A_n)$ be an n-tuple of self-adjoint operators $A_i \in \mathscr{B}(H)$ with the bounds m_i and M_i, $m_i \leq M_i$, $i = 1, \ldots, n$. Let $(\alpha_1, \ldots, \alpha_n)$ be an n-tuple of nonnegative real numbers such that $\sum_{i=1}^n \alpha_i = 1$. Let*

$$(m_A, M_A) \cap [m_i, M_i] = \emptyset \quad \textit{for } i = 1, \ldots, n, \qquad \textit{and} \qquad m < M,$$

where m_A and M_A, $m_A \leq M_A$, are the bounds of $A = \sum_{i=1}^n \alpha_i A_i$ and

$$m = \max\{M_i \leq m_A, i \in \{1, \ldots, n\}\}, \; M = \min\{m_i \geq M_A, i \in \{1, \ldots, n\}\}.$$

If $f \colon I \to \mathbb{R}$ is a continuous convex (resp. concave) function provided that the interval I contains all m_i, M_i, then

$$\begin{aligned} f\left(\sum_{i=1}^n \alpha_i A_i\right) &\leq \textstyle\sum_{i=1}^n \alpha_i f(A_i) - \delta_f \tilde{\tilde{A}} \leq \sum_{i=1}^n \alpha_i f(A_i) \\ (\textit{resp.} \quad f\left(\sum_{i=1}^n \alpha_i A_i\right) &\geq \textstyle\sum_{i=1}^n \alpha_i f(A_i) + \delta_f \tilde{\tilde{A}} \geq \sum_{i=1}^n \alpha_i f(A_i)) \end{aligned}$$

holds, where δ_f *is defined by* (11.12), $\tilde{\tilde{A}} = \frac{1}{2}1_H - \frac{1}{\bar{M}-\bar{m}}\left|\sum_{i=1}^n \alpha_i A_i - \frac{\bar{m}+\bar{M}}{2}1_H\right|$ *and* $\bar{m} \in [m, m_A]$, $\bar{M} \in [M_A, M]$, $\bar{m} < \bar{M}$, *are arbitrary numbers.*

Proof. We apply Theorem 11.3 for positive linear mappings $\Phi_i : \mathscr{B}(H) \to \mathscr{B}(H)$ defined by $\Phi_i : B \mapsto \alpha_i B$, $i = 1, \dots, n$. □

In the following theorem we give an extension of Jensen's operator inequality given in Theorem 11.1 and a refinement of Theorem 11.2.

Theorem 11.4 *Let* $(A_1, \dots, A_n)$ *be an n-tuple of self-adjoint operators* $A_i \in \mathscr{B}(H)$ *with the bounds* m_i *and* M_i, $m_i \le M_i$, $i = 1, \dots, n$. *Let* $(\Phi_1, \dots, \Phi_n)$ *be an n-tuple of positive linear mappings* $\Phi_i : \mathscr{B}(H) \to \mathscr{B}(K)$, *such that* $\sum_{i=1}^{n_1} \Phi_i(1_H) = \alpha\, 1_K$, $\sum_{i=n_1+1}^{n} \Phi_i(1_H) = \beta\, 1_K$, *where* $1 \le n_1 < n$, $\alpha, \beta > 0$ *and* $\alpha + \beta = 1$. *Let* $m_L = \min\{m_1, \dots, m_{n_1}\}$, $M_R = \max\{M_1, \dots, M_{n_1}\}$ *and*

$$m = \begin{cases} m_L, & \text{if } \{M_i \colon M_i \le m_L, i \in \{n_1+1, \dots, n\}\} = \emptyset, \\ \max\{M_i \colon M_i \le m_L, i \in \{n_1+1, \dots, n\}\}, & \text{otherwise,} \end{cases}$$
$$M = \begin{cases} M_R, & \text{if } \{m_i \colon m_i \ge M_R, i \in \{n_1+1, \dots, n\}\} = \emptyset, \\ \min\{m_i \colon m_i \ge M_R, i \in \{n_1+1, \dots, n\}\}, & \text{otherwise.} \end{cases}$$

If

$$(m_L, M_R) \cap [m_i, M_i] = \emptyset \quad \textit{for} \quad i = n_1+1, \dots, n, \qquad m < M,$$

and one of two equalities

$$\frac{1}{\alpha}\sum_{i=1}^{n_1} \Phi_i(A_i) = \sum_{i=1}^{n} \Phi_i(A_i) = \frac{1}{\beta}\sum_{i=n_1+1}^{n} \Phi_i(A_i)$$

is valid, then

$$\begin{aligned} \frac{1}{\alpha}\sum_{i=1}^{n_1} \Phi_i(f(A_i)) &\le \frac{1}{\alpha}\sum_{i=1}^{n_1} \Phi_i(f(A_i)) + \beta\delta_f \widetilde{A} \le \sum_{i=1}^{n} \Phi_i(f(A_i)) \\ &\le \frac{1}{\beta}\sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \alpha\delta_f \widetilde{A} \le \frac{1}{\beta}\sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) \end{aligned} \tag{11.15}$$

holds for every continuous convex function $f : I \to \mathbb{R}$ *provided that the interval I contains all* m_i, M_i, $i = 1, \dots, n$, *where*

$$\begin{aligned} \delta_f \equiv \delta_f(\bar{m}, \bar{M}) &= f(\bar{m}) + f(\bar{M}) - 2f\left(\frac{\bar{m}+\bar{M}}{2}\right) \\ \widetilde{A} \equiv \widetilde{A}_{A,\Phi,n_1,\alpha}(\bar{m}, \bar{M}) &= \frac{1}{2}1_K - \frac{1}{\alpha(\bar{M}-\bar{m})}\sum_{i=1}^{n_1} \Phi_i\left(\left|A_i - \frac{\bar{m}+\bar{M}}{2}1_H\right|\right) \end{aligned} \tag{11.16}$$

and $\bar{m} \in [m, m_L]$, $\bar{M} \in [M_R, M]$, $\bar{m} < \bar{M}$, *are arbitrary numbers.*

If $f : I \to \mathbb{R}$ *is concave, then the reverse inequality is valid in* (11.15).

Proof. We prove only the convex case.

Let us denote

$$A = \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i), \qquad B = \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(A_i), \qquad C = \sum_{i=1}^{n}\Phi_i(A_i).$$

It is easy to verify that $A = B$ or $B = C$ or $A = C$ implies $A = B = C$.

Since f is convex on $[\bar{m},\bar{M}]$ and $\mathsf{Sp}(A_i) \subseteq [m_i, M_i] \subseteq [\bar{m},\bar{M}]$ for $i = 1,\dots,n_1$, it follows from Lemma 11.1 that

$$f(A_i) \le \frac{\bar{M}1_H - A_i}{\bar{M}-\bar{m}} f(\bar{m}) + \frac{A_i - \bar{m}1_H}{\bar{M}-\bar{m}} f(\bar{M}) - \delta_f \widetilde{A}_i, \qquad i = 1,\dots,n_1$$

holds, where $\delta_f = f(\bar{m}) + f(\bar{M}) - 2f\left(\frac{\bar{m}+\bar{M}}{2}\right)$ and $\widetilde{A}_i = \frac{1}{2}1_H - \frac{1}{\bar{M}-\bar{m}}\left|A_i - \frac{\bar{m}+\bar{M}}{2}1_H\right|$. Applying a positive linear mapping Φ_i and summing, we obtain

$$\begin{aligned}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) \le\ & \frac{\bar{M}\alpha 1_K - \sum_{i=1}^{n_1}\Phi_i(A_i)}{\bar{M}-\bar{m}} f(\bar{m}) + \frac{\sum_{i=1}^{n_1}\Phi_i(A_i) - \bar{m}\alpha 1_K}{\bar{M}-\bar{m}} f(\bar{M}) \\ & -\delta_f\left(\frac{\alpha}{2}1_K - \frac{1}{\bar{M}-\bar{m}}\sum_{i=1}^{n_1}\Phi_i\left(\left|A_i - \frac{\bar{m}+\bar{M}}{2}1_H\right|\right)\right),\end{aligned}$$

since $\sum_{i=1}^{n_1}\Phi_i(1_H) = \alpha 1_K$. It follows that

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) \le \frac{\bar{M}1_K - A}{\bar{M}-\bar{m}} f(\bar{m}) + \frac{A - \bar{m}1_K}{\bar{M}-\bar{m}} f(\bar{M}) - \delta_f \widetilde{A}, \tag{11.17}$$

where $\widetilde{A} = \frac{1}{2}1_K - \frac{1}{\alpha(\bar{M}-\bar{m})}\sum_{i=1}^{n_1}\Phi_i\left(\left|A_i - \frac{\bar{m}+\bar{M}}{2}1_H\right|\right)$.

In addition, since f is convex on all $[m_i, M_i]$ and $(\bar{m},\bar{M}) \cap [m_i, M_i] = \emptyset$ for $i = n_1 + 1,\dots,n$, then

$$f(A_i) \ge \frac{\bar{M}1_H - A_i}{\bar{M}-\bar{m}} f(\bar{m}) + \frac{A_i - \bar{m}1_H}{\bar{M}-\bar{m}} f(\bar{M}), \qquad i = n_1+1,\dots,n.$$

It follows

$$\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) - \delta_f\widetilde{A} \ge \frac{\bar{M}1_K - B}{\bar{M}-\bar{m}} f(\bar{m}) + \frac{B - \bar{m}1_K}{\bar{M}-\bar{m}} f(\bar{M}) - \delta_f\widetilde{A}. \tag{11.18}$$

Combining (11.17) and (11.18) and taking into account that $A = B$, we obtain

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) \le \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) - \delta_f\widetilde{A}. \tag{11.19}$$

Next, we obtain

$$\begin{aligned}&\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) \\ &= \sum_{i=1}^{n_1}\Phi_i(f(A_i)) + \frac{\beta}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) \quad (\text{by } \alpha+\beta = 1)\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{i=1}^{n_1} \Phi_i(f(A_i)) + \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \beta\delta_f \widetilde{A} && \text{(by (11.19))} \\
&\le \frac{\alpha}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \alpha\delta_f \widetilde{A} + \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \beta\delta_f \widetilde{A} && \text{(by (11.19))} \\
&= \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \delta_f \widetilde{A} \quad (\text{by } \alpha+\beta=1),
\end{aligned}
$$

which gives the following double inequality

$$
\frac{1}{\alpha}\sum_{i=1}^{n_1} \Phi_i(f(A_i)) \le \sum_{i=1}^{n} \Phi_i(f(A_i)) - \beta\delta_f \widetilde{A} \le \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \delta_f \widetilde{A}.
$$

Adding $\beta\delta_f\widetilde{A}$ in the above inequalities, we get

$$
\frac{1}{\alpha}\sum_{i=1}^{n_1} \Phi_i(f(A_i)) + \beta\delta_f \widetilde{A} \le \sum_{i=1}^{n} \Phi_i(f(A_i)) \le \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \alpha\delta_f \widetilde{A}. \tag{11.20}
$$

Now, we prove that $\delta_f \ge 0$ and $\widetilde{A} \ge 0$.

Indeed, since f is convex, then $f\left((\bar{m}+\bar{M})/2\right) \le (f(\bar{m})+f(\bar{M}))/2$, which implies that $\delta_f \ge 0$. Also, since

$$
\mathsf{Sp}(A_i) \subseteq [\bar{m},\bar{M}] \quad \Rightarrow \quad \left| A_i - \frac{\bar{M}+\bar{m}}{2} 1_H \right| \le \frac{\bar{M}-\bar{m}}{2} 1_H, \qquad \text{for } i = 1,\dots,n_1,
$$

then

$$
\sum_{i=1}^{n_1} \Phi_i\left(\left| A_i - \frac{\bar{M}+\bar{m}}{2} 1_H \right| \right) \le \frac{\bar{M}-\bar{m}}{2} \alpha 1_K,
$$

which gives

$$
0 \le \frac{1}{2} 1_K - \frac{1}{\alpha(\bar{M}-\bar{m})} \sum_{i=1}^{n_1} \Phi_i\left(\left| A_i - \frac{\bar{M}+\bar{m}}{2} 1_H \right| \right) = \widetilde{A}.
$$

Consequently, the following inequalities

$$
\begin{aligned}
&\frac{1}{\alpha}\sum_{i=1}^{n_1} \Phi_i(f(A_i)) \le \frac{1}{\alpha}\sum_{i=1}^{n_1} \Phi_i(f(A_i)) + \beta\delta_f \widetilde{A}, \\
&\frac{1}{\beta}\sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \alpha\delta_f \widetilde{A} \le \frac{1}{\beta}\sum_{i=n_1+1}^{n} \Phi_i(f(A_i))
\end{aligned}
$$

hold, which together with (11.20) proves inequalities (11.15). □

Example 11.2 We observe the matrix case of Theorem 11.4 for $f(t) = t^4$, which is the convex function but not operator convex, $n = 4$, $n_1 = 2$. We present an example such that

$$\begin{aligned}\frac{1}{\alpha}\left(\Phi_1(A_1^4)+\Phi_2(A_2^4)\right) &< \frac{1}{\alpha}\left(\Phi_1(A_1^4)+\Phi_2(A_2^4)\right)+\beta\delta_f\widetilde{A}\\ &< \Phi_1(A_1^4)+\Phi_2(A_2^4)+\Phi_3(A_3^4)+\Phi_4(A_4^4) \qquad (11.21)\\ &< \frac{1}{\beta}\left(\Phi_3(A_3^4)+\Phi_4(A_4^4)\right)-\alpha\delta_f\widetilde{A} < \frac{1}{\beta}\left(\Phi_3(A_3^4)+\Phi_4(A_4^4)\right)\end{aligned}$$

holds, where $\delta_f = \bar{M}^4 + \bar{m}^4 - (\bar{M} + \bar{m})^4/8$ and

$$\widetilde{A} = \frac{1}{2}I_2 - \frac{1}{\alpha(\bar{M}-\bar{m})}\left(\Phi_1\left(\left|A_1 - \frac{\bar{M}+\bar{m}}{2}I_h\right|\right) + \Phi_2\left(\left|A_2 - \frac{\bar{M}+\bar{m}}{2}I_3\right|\right)\right).$$

We define mappings $\Phi_i : M_3(\mathbb{C}) \to M_2(\mathbb{C})$ as follows: $\Phi_i((a_{jk})_{1\le j,k\le 3}) = \frac{1}{4}(a_{jk})_{1\le j,k\le 2}$, $i = 1,\dots,4$. Then $\sum_{i=1}^{4}\Phi_i(I_3) = I_2$ and $\alpha = \beta = \frac{1}{2}$.

Let

$$A_1 = 2\begin{pmatrix} 2 & 9/8 & 1\\ 9/8 & 2 & 0\\ 1 & 0 & 3\end{pmatrix}, \qquad A_2 = 3\begin{pmatrix} 2 & 9/8 & 0\\ 9/8 & 1 & 0\\ 0 & 0 & 2\end{pmatrix},$$
$$A_3 = -3\begin{pmatrix} 4 & 1/2 & 1\\ 1/2 & 4 & 0\\ 1 & 0 & 2\end{pmatrix}, \qquad A_4 = 12\begin{pmatrix} 5/3 & 1/2 & 0\\ 1/2 & 3/2 & 0\\ 0 & 0 & 3\end{pmatrix}.$$

Then $m_1 = 1.28607, M_1 = 7.70771, m_2 = 0.53777, M_2 = 5.46221, m_3 = -14.15050, M_3 = -4.71071$, $m_4 = 12.91724$, $M_4 = 36$. Hence $m_L = m_2$, $M_R = M_1$, $m = M_3$ and $M = m_4$ (rounded to five decimal places). Also,

$$\frac{1}{\alpha}\left(\Phi_1(A_1)+\Phi_2(A_2)\right) = \frac{1}{\beta}\left(\Phi_3(A_3)+\Phi_4(A_4)\right) = \begin{pmatrix} 4 & 9/4\\ 9/4 & 3\end{pmatrix},$$

and

$$\begin{aligned} A_f \equiv \frac{1}{\alpha}\left(\Phi_1(A_1^4)+\Phi_2(A_2^4)\right) &= \begin{pmatrix} 989.00391 & 663.46875\\ 663.46875 & 526.12891\end{pmatrix},\\ C_f \equiv \Phi_1(A_1^4)+\Phi_2(A_2^4)+\Phi_3(A_3^4)+\Phi_4(A_4^4) &= \begin{pmatrix} 68093.14258 & 48477.98437\\ 48477.98437 & 51335.39258\end{pmatrix},\\ B_f \equiv \frac{1}{\beta}\left(\Phi_3(A_3^4)+\Phi_4(A_4^4)\right) &= \begin{pmatrix} 135197.28125 & 96292.5\\ 96292.5 & 102144.65625\end{pmatrix}.\end{aligned}$$

Then

$$A_f < C_f < B_f \qquad (11.22)$$

holds (which is consistent with (11.5)).

We will choose three pairs of numbers $(\bar{m}, \bar{M})$, $\bar{m} \in [-4.71071, 0.53777]$, $\bar{M} \in [7.70771, 12.91724]$ as follows:

i) $\overline{m} = m_L = 0.53777$, $\overline{M} = M_R = 7.70771$, then

$$\widetilde{\Delta}_1 = \beta\delta_f\widetilde{A} = 0.5\cdot 2951.69249\cdot\begin{pmatrix}0.15678 & 0.09030\\ 0.09030 & 0.15943\end{pmatrix} = \begin{pmatrix}231.38908 & 133.26139\\ 133.26139 & 235.29515\end{pmatrix},$$

ii) $\overline{m} = m = -4.71071$, $\overline{M} = M = 12.91724$, then

$$\widetilde{\Delta}_2 = \beta\delta_f\widetilde{A} = 0.5\cdot 27766.07963\cdot\begin{pmatrix}0.36022 & 0.03573\\ 0.03573 & 0.36155\end{pmatrix} = \begin{pmatrix}5000.89860 & 496.04498\\ 496.04498 & 5019.50711\end{pmatrix},$$

iii) $\overline{m} = -1$, $\overline{M} = 10$, then

$$\widetilde{\Delta}_3 = \beta\delta_f\widetilde{A} = 0.5\cdot 9180.875\cdot\begin{pmatrix}0.28203 & 0.08975\\ 0.08975 & 0.27557\end{pmatrix} = \begin{pmatrix}1294.66 & 411.999\\ 411.999 & 1265.\end{pmatrix}.$$

Next, we obtain the following improvement of (11.22) (see (11.21)):

i) $A_f < A_f + \widetilde{\Delta}_1 = \begin{pmatrix}1220.39299 & 796.73014\\ 796.73014 & 761.42406\end{pmatrix} < C_f < \begin{pmatrix}134965.89217 & 96159.23861\\ 96159.23861 & 101909.36110\end{pmatrix}$
$= B_f - \widetilde{\Delta}_1 < B_f$,

ii) $A_f < A_f + \widetilde{\Delta}_2 = \begin{pmatrix}5989.90251 & 1159.51373\\ 1159.51373 & 5545.63601\end{pmatrix} < C_f < \begin{pmatrix}130196.38265 & 95796.45502\\ 95796.45502 & 97125.14914\end{pmatrix}$
$= B_f - \widetilde{\Delta}_2 < B_f$,

iii) $A_f < A_f + \widetilde{\Delta}_3 = \begin{pmatrix}2283.66362 & 1075.46746\\ 1075.46746 & 1791.12874\end{pmatrix} < C_f < \begin{pmatrix}133902.62153 & 95880.50129\\ 95880.50129 & 100879.65641\end{pmatrix}$
$= B_f - \widetilde{\Delta}_3 < B_f$.

By means of Theorem 11.4, we get the following result.

Corollary 11.2 *Let the assumptions of Theorem 11.4 hold. Then*

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) \le \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) + \gamma_1\delta_f\widetilde{A} \le \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) \tag{11.23}$$

and

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) \le \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) - \gamma_2\delta_f\widetilde{A} \le \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) \tag{11.24}$$

holds for every γ_1,γ_2 in the closed interval $[\alpha,\beta]$, where δ_f and $\widetilde{A}$ are defined by (11.16)*.*

Proof. Adding $\alpha\delta_f\widetilde{A}$ in (11.15) and noticing $\delta_f\widetilde{A} \ge 0$, we obtain

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) \le \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) + \alpha\delta_f\widetilde{A} \le \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)).$$

Taking into account the above inequality and the left hand side of (11.15) we obtain (11.23).
Similarly, subtracting $\beta\delta_f\widetilde{A}$ in (11.15) we obtain (11.24). □

Remark 11.2 *Let the assumptions of Theorem 11.4 be valid.*

1) We observe that the following inequality

$$f\left(\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(A_i)\right) \leq \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) - \delta_f \widetilde{A}_\beta \leq \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i))$$

holds for every continuous convex function $f : I \to \mathbb{R}$ provided that the interval I contains all m_i, M_i, $i = 1,\ldots,n$, where δ_f is defined by (11.16),

$$\widetilde{A}_\beta \equiv \widetilde{A}_{\beta,A,\Phi,n_1}(\bar{m},\bar{M}) = \frac{1}{2}1_K - \frac{1}{\bar{M}-\bar{m}}\left|\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i A_i - \frac{\bar{m}+\bar{M}}{2}1_K\right|$$

and $\bar{m} \in [m, m_L]$, $\bar{M} \in [M_R, M]$, $\bar{m} < \bar{M}$, are arbitrary numbers.

Indeed, by the assumptions of Theorem 11.4 we have

$$m_L \alpha 1_H \leq \sum_{i=1}^{n_1}\Phi_i(A_i) \leq M_R \alpha 1_H \quad \text{and} \quad \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i) = \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(A_i),$$

which implies

$$m_L 1_H \leq \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(A_i) \leq M_R 1_H.$$

Also $(m_L, M_R) \cap [m_i, M_i] = \emptyset$ for $i = n_1+1,\ldots,n$ and $\sum_{i=n_1+1}^{n}\frac{1}{\beta}\Phi_i(1_H) = 1_K$. Hence we can apply Theorem 11.3 to operators $A_{n_1+1},\ldots,A_n$ and mappings $\frac{1}{\beta}\Phi_i$ wherefrom we obtain the desired inequality.

2) We denote by m_C and M_C the bounds of $C = \sum_{i=1}^{n}\Phi_i(A_i)$. If $(m_C, M_C) \cap [m_i, M_i] = \emptyset$, $i = 1,\ldots,n_1$, then series of inequalities (11.15) can be extended from the left side if we use refined Jensen's operator inequality (11.10):

$$\begin{aligned}
f\left(\sum_{i=1}^{n}\Phi_i(A_i)\right) &= f\left(\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i)\right) \leq \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) - \delta_f \widetilde{A}_\alpha \\
\leq \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) &\leq \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) + \beta\delta_f \widetilde{A} \leq \sum_{i=1}^{n}\Phi_i(f(A_i)) \\
\leq \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)) - \alpha\delta_f \widetilde{A} &\leq \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i)),
\end{aligned}$$

where δ_f and $\widetilde{A}$ are defined by (11.16),

$$\widetilde{A}_\alpha \equiv \widetilde{A}_{\alpha,A,\Phi,n_1}(\bar{m},\bar{M}) = \frac{1}{2}1_K - \frac{1}{\bar{M}-\bar{m}}\left|\frac{1}{\alpha}\sum_{i=n_1+1}^{n}\Phi_i A_i - \frac{\bar{m}+\bar{M}}{2}1_K\right|.$$

Remark 11.3 We obtain the equivalent inequalities to the ones in Theorem 11.4 in the case when $\sum_{i=1}^{n}\Phi_i(1_H)=\gamma 1_K$, for some positive scalar γ. If $\alpha+\beta=\gamma$ and one of two equalities

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i)=\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(A_i)=\frac{1}{\gamma}\sum_{i=1}^{n}\Phi_i(A_i)$$

is valid, then

$$\begin{aligned}\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i)) &\le \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i))+\frac{\beta}{\gamma}\delta_f\widetilde{A}\le\frac{1}{\gamma}\sum_{i=1}^{n}\Phi_i(f(A_i))\\ &\le \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i))-\frac{\alpha}{\gamma}\delta_f\widetilde{A}\le\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(A_i))\end{aligned}$$

holds for every continuous convex function $f:I\to\mathbb{R}$ provided that the interval I contains all m_i,M_i, $i=1,\dots,n$, where δ_f and $\widetilde{A}$ are defined by (11.16).

With respect to Remark 11.3, we obtain the following obvious corollary of Theorem 11.4 with the convex combination of operators A_i, $i=1,\dots,n$.

Corollary 11.3 *Let $(A_1,\dots,A_n)$ be an n-tuple of self-adjoint operators $A_i\in\mathscr{B}(H)$ with the bounds m_i and M_i, $m_i\le M_i$, $i=1,\dots,n$. Let $(p_1,\dots,p_n)$ be an n-tuple of non-negative numbers such that $0<\sum_{i=1}^{n_1}p_i=\mathbf{p_{n_1}}<\mathbf{p_n}=\sum_{i=1}^{n}p_i$, where $1\le n_1<n$. Let $m_L=\min\{m_1,\dots,m_{n_1}\}$, $M_R=\max\{M_1,\dots,M_{n_1}\}$ and*

$$m=\begin{cases}m_L, & \text{if } \{M_i\colon M_i\le m_L, i\in\{n_1+1,\dots,n\}\}=\emptyset,\\ \max\{M_i\colon M_i\le m_L, i\in\{n_1+1,\dots,n\}\}, & \text{otherwise,}\end{cases}$$
$$M=\begin{cases}M_R, & \text{if } \{m_i\colon m_i\ge M_R, i\in\{n_1+1,\dots,n\}\}=\emptyset,\\ \min\{m_i\colon m_i\ge M_R, i\in\{n_1+1,\dots,n\}\}, & \text{otherwise.}\end{cases}$$

If

$$(m_L,M_R)\cap[m_i,M_i]=\emptyset \quad \textit{for} \quad i=n_1+1,\dots,n, \qquad m<M,$$

and one of two equalities

$$\frac{1}{\mathbf{p_{n_1}}}\sum_{i=1}^{n_1}p_iA_i=\frac{1}{\mathbf{p_n}}\sum_{i=1}^{n}p_iA_i=\frac{1}{\mathbf{p_n}-\mathbf{p_{n_1}}}\sum_{i=n_1+1}^{n}p_iA_i$$

is valid, then

$$\begin{aligned}\frac{1}{\mathbf{p_{n_1}}}\sum_{i=1}^{n_1}p_if(A_i) &\le \frac{1}{\mathbf{p_{n_1}}}\sum_{i=1}^{n_1}p_if(A_i)+\left(1-\frac{\mathbf{p_{n_1}}}{\mathbf{p_n}}\right)\delta_f\widetilde{A}\le\frac{1}{\mathbf{p_n}}\sum_{i=1}^{n}p_if(A_i)\\ &\le \frac{1}{\mathbf{p_n}-\mathbf{p_{n_1}}}\sum_{i=n_1+1}^{n}p_if(A_i)-\frac{\mathbf{p_{n_1}}}{\mathbf{p_n}}\delta_f\widetilde{A}\le\frac{1}{\mathbf{p_n}-\mathbf{p_{n_1}}}\sum_{i=n_1+1}^{n}p_if(A_i)\end{aligned} \tag{11.25}$$

holds for every continuous convex function $f:I\to\mathbb{R}$ provided that the interval I contains all m_i,M_i, $i=1,\dots,n$, where δ_f is defined by (11.16)*,*

$$\widetilde{A}\equiv\widetilde{A}_{A,p,n_1}(\bar{m},\bar{M})=\frac{1}{2}1_H-\frac{1}{\mathbf{p_{n_1}}(M-\bar{m})}\sum_{i=1}^{n_1}p_i\left(\left|A_i-\frac{\bar{m}+\bar{M}}{2}1_H\right|\right)$$

and $\bar{m} \in [m, m_L]$, $\bar{M} \in [M_R, M]$, $\bar{m} < \bar{M}$, *are arbitrary numbers.*

If $f : I \to \mathbb{R}$ *is concave, then the reverse inequality is valid in* (11.25)*.*

As a special case of Corollary 11.3 we obtain an extension of Corollary 11.1.

Corollary 11.4 *Let* $(A_1, \ldots, A_n)$ *be an n-tuple of self-adjoint operators* $A_i \in \mathscr{B}(H)$ *with the bounds* m_i *and* M_i, $m_i \le M_i$, $i = 1, \ldots, n$. *Let* $(p_1, \ldots, p_n)$ *be an n-tuple of non-negative numbers such that* $\sum_{i=1}^n p_i = 1$. *Let*

$$(m_A, M_A) \cap [m_i, M_i] = \emptyset \quad \text{for } i = 1, \ldots, n, \qquad \text{and} \qquad m < M,$$

where m_A *and* M_A, $m_A \le M_A$, *are the bounds of* $A = \sum_{i=1}^n p_i A_i$ *and*

$$m = \max\{M_i \le m_A, i \in \{1, \ldots, n\}\}, \quad M = \min\{m_i \ge M_A, i \in \{1, \ldots, n\}\}.$$

If $f : I \to \mathbb{R}$ *is a continuous convex function provided that the interval* I *contains all* m_i, M_i, *then*

$$\begin{aligned} f\Big(\sum_{i=1}^n p_i A_i\Big) &\le f\Big(\sum_{i=1}^n p_i A_i\Big) + \frac{1}{2}\delta_f \tilde{\tilde{A}} \le \frac{1}{2} f\Big(\sum_{i=1}^n p_i A_i\Big) + \frac{1}{2}\sum_{i=1}^n p_i f(A_i) \\ &\le \sum_{i=1}^n p_i f(A_i) - \frac{1}{2}\delta_f \tilde{\tilde{A}} \le \sum_{i=1}^n p_i f(A_i) \end{aligned} \tag{11.26}$$

holds, where δ_f *is defined by* (11.16)*,* $\tilde{\tilde{A}} = \frac{1}{2}1_H - \frac{1}{\bar{M}-\bar{m}}\left|\sum_{i=1}^n p_i A_i - \frac{\bar{m}+\bar{M}}{2}1_H\right|$ *and* $\bar{m} \in [m, m_A]$, $\bar{M} \in [M_A, M]$, $\bar{m} < \bar{M}$, *are arbitrary numbers.*

If $f : I \to \mathbb{R}$ *is concave, then the reverse inequality is valid in* (11.26)*.*

Proof. We prove only the convex case.

We define $(n+1)$-tuple of operators $(B_1, \ldots, B_{n+1})$, $B_i \in \mathscr{B}(H)$, by $B_1 = A = \sum_{i=1}^n p_i A_i$ and $B_i = A_{i-1}$, $i = 2, \ldots, n+1$. Then $m_{B_1} = m_A$, $M_{B_1} = M_A$ are the bounds of B_1 and $m_{B_i} = m_{i-1}$, $M_{B_i} = M_{i-1}$ are the ones of B_i, $i = 2, \ldots, n+1$. Also, we define $(n+1)$-tuple of non-negative numbers $(q_1, \ldots, q_{n+1})$ by $q_1 = 1$ and $q_i = p_{i-1}$, $i = 2, \ldots, n+1$. We have that $\sum_{i=1}^{n+1} q_i = 2$ and

$$(m_{B_1}, M_{B_1}) \cap [m_{B_i}, M_{B_i}] = \emptyset, \ \text{ for } i = 2, \ldots, n+1 \quad \text{and} \quad m < M \tag{11.27}$$

holds. Since

$$\sum_{i=1}^{n+1} q_i B_i = B_1 + \sum_{i=2}^{n+1} q_i B_i = \sum_{i=1}^n p_i A_i + \sum_{i=1}^n p_i A_i = 2B_1,$$

then

$$q_1 B_1 = \frac{1}{2}\sum_{i=1}^{n+1} q_i B_i = \sum_{i=2}^{n+1} q_i B_i. \tag{11.28}$$

Taking into account (11.27) and (11.28), we can apply Corollary 11.3 for $n_1 = 1$ and B_i, q_i as above, and we get

$$q_1 f(B_1) \le q_1 f(B_1) + \frac{1}{2}\delta_f \widetilde{B} \le \frac{1}{2}\sum_{i=1}^{n+1} q_i f(B_i) \le \sum_{i=2}^{n+1} q_i f(B_i) - \frac{1}{2}\delta_f \widetilde{B} \le \sum_{i=2}^{n+1} q_i f(B_i),$$

where $\widetilde{B} = \frac{1}{2}1_H - \frac{1}{\bar{M}-\bar{m}}\left|B_1 - \frac{\bar{m}+\bar{M}}{2}1_H\right|$, which yields the desired inequality (11.26). □

11.2 Application to quasi-arithmetic and power means

In this section we study an application of the results obtained in the previous section to quasi-arithmetic operator means as well as to their special case – power means.

A quasi-arithmetic operator mean is defined by

$$\mathscr{M}_\varphi(\mathbf{A},\mathbf{\Phi},n) = \varphi^{-1}\left(\sum_{i=1}^{n} \Phi_i\left(\varphi(A_i)\right)\right), \tag{11.29}$$

where $(A_1,\ldots,A_n)$ is an n-tuple of self-adjoint operators in $\mathscr{B}(H)$ with the spectra in I, $(\Phi_1,\ldots,\Phi_n)$ is an n-tuple of positive linear mappings $\Phi_i : \mathscr{B}(H) \to \mathscr{B}(K)$, such that $\sum_{i=1}^{n} \Phi_i(1_H) = 1_K$, and $\varphi : I \to \mathbb{R}$ is a continuous strictly monotone function.

The following result about the monotonicity of this mean was proven in [139, Theorem 3].

Theorem 11.5 *Let $(A_1,\ldots,A_n)$ and $(\Phi_1,\ldots,\Phi_n)$ be as in the definition of the quasi-arithmetic mean* (11.29)*. Let m_i and M_i, $m_i \le M_i$ be the bounds of A_i, $i = 1,\ldots,n$. Let $\varphi,\psi : I \to \mathbb{R}$ be continuous strictly monotone functions on an interval I which contains all m_i, M_i. Let m_φ and M_φ, $m_\varphi \le M_\varphi$, be the bounds of the mean $\mathscr{M}_\varphi(\mathbf{A},\mathbf{\Phi},n)$, such that*

$$\left(m_\varphi, M_\varphi\right) \cap [m_i, M_i] = \emptyset, \qquad \textit{for } i = 1,\ldots,n. \tag{11.30}$$

If one of the following conditions

(i) $\psi\circ\varphi^{-1}$ is convex and ψ^{-1} is operator monotone,

(i') $\psi\circ\varphi^{-1}$ is concave and $-\psi^{-1}$ is operator monotone,

is satisfied, then

$$\mathscr{M}_\varphi(\mathbf{A},\mathbf{\Phi},n) \le \mathscr{M}_\psi(\mathbf{A},\mathbf{\Phi},n). \tag{11.31}$$

If one of the following conditions

(ii) $\psi\circ\varphi^{-1}$ is concave and ψ^{-1} is operator monotone,

(ii') $\psi\circ\varphi^{-1}$ is convex and $-\psi^{-1}$ is operator monotone,

is satisfied, then the reverse inequality is valid in (11.31).

For convenience, we introduce the following notation:

$$\begin{aligned}
\delta_{\varphi,\psi}(m,M) &= \psi(m) + \psi(M) - 2\psi\circ\varphi^{-1}\left(\tfrac{\varphi(m)+\varphi(M)}{2}\right), \\
\widetilde{A}_\varphi(m,M) &= \tfrac{1}{2}1_K - \tfrac{1}{|\varphi(M)-\varphi(m)|}\left|\textstyle\sum_{i=1}^{n} \Phi_i(\varphi(A_i)) - \tfrac{\varphi(M)+\varphi(m)}{2}1_K\right|,
\end{aligned} \tag{11.32}$$

where $(A_1,\dots,A_n)$ is an n-tuple of self-adjoint operators in $\mathscr{B}(H)$ with the spectra in I, $(\Phi_1,\dots,\Phi_n)$ is an n-tuple of positive linear mappings $\Phi_i : \mathscr{B}(H) \to \mathscr{B}(K)$ such that $\sum_{i=1}^n \Phi_i(1_H) = 1_K$, $\varphi,\psi : I \to \mathbb{R}$ are continuous strictly monotone functions and $m,M \in I$, $m < M$. We include implicitly that $\widetilde{A}_\varphi(m,M) \equiv \widetilde{A}_{\varphi,A}(m,M)$, where $A = \sum_{i=1}^n \Phi_i(\varphi(A_i))$.

In the following theorem we make use of Theorem 11.3 and give a refinement of the results presented in Theorem 11.5.

Theorem 11.6 *Let $(A_1,\dots,A_n)$ and $(\Phi_1,\dots,\Phi_n)$ be as in the definition of the quasi-arithmetic mean* (11.29)*. Let $\varphi,\psi : I \to \mathbb{R}$ be continuous strictly monotone functions on an interval I which contains all m_i,M_i. Let*

$$\big(m_\varphi,M_\varphi\big)\cap[m_i,M_i]=\emptyset \qquad \textit{for } i=1,\dots,n, \qquad \textit{and} \qquad m<M,$$

where m_φ and M_φ, $m_\varphi \le M_\varphi$, are the bounds of the mean $\mathscr{M}_\varphi(\mathbf{A},\mathbf{\Phi},n)$ and $m = \max\left\{M_i\colon M_i \le m_\varphi, i \in \{1,\dots,n\}\right\}$, $M = \min\left\{m_i\colon m_i \ge M_\varphi, i \in \{1,\dots,n\}\right\}$.

(i) *If $\psi\circ\varphi^{-1}$ is convex and ψ^{-1} is operator monotone, then*

$$\mathscr{M}_\varphi(\mathbf{A},\mathbf{\Phi},n) \le \psi^{-1}\left(\sum_{i=1}^n \Phi_i\left(\psi(A_i)\right) - \delta_{\varphi,\psi}\widetilde{A}_\varphi\right) \le \mathscr{M}_\psi(\mathbf{A},\mathbf{\Phi},n) \tag{11.33}$$

holds, where $\delta_{\varphi,\psi} \ge 0$ and $\widetilde{A}_\varphi \ge 0$.

(i′) *If $\psi\circ\varphi^{-1}$ is convex and $-\psi^{-1}$ is operator monotone, then the reverse inequality is valid in* (11.33)*, where $\delta_{\varphi,\psi} \ge 0$ and $\widetilde{A}_\varphi \ge 0$.*

(ii) *If $\psi\circ\varphi^{-1}$ is concave and $-\psi^{-1}$ is operator monotone, then* (11.33) *holds, where $\delta_{\varphi,\psi} \le 0$ and $\widetilde{A}_\varphi \ge 0$.*

(ii′) *If $\psi\circ\varphi^{-1}$ is concave and ψ^{-1} is operator monotone, then the reverse inequality is valid in* (11.33)*, where $\delta_{\varphi,\psi} \le 0$ and $\widetilde{A}_\varphi \ge 0$.*

In all the above cases, we assume that $\delta_{\varphi,\psi} \equiv \delta_{\varphi,\psi}(\bar{m},\bar{M})$, $\widetilde{A}_\varphi \equiv \widetilde{A}_\varphi(\bar{m},\bar{M})$ are defined by (11.32) *and $\bar{m} \in [m,m_\varphi]$, $\bar{M} \in [M_\varphi,M]$, $\bar{m} < \bar{M}$, are arbitrary numbers.*

Proof. We only prove the case (i). Suppose that φ is a strictly increasing function. Since $m_i 1_H \le A_i \le M_i 1_H$, $i=1,\dots,n$, and $m_\varphi 1_K \le \mathscr{M}_\varphi(\mathbf{A},\mathbf{\Phi},n) \le M_\varphi 1_K$, then

$$\varphi(m_i)1_H \le \varphi(A_i) \le \varphi(M_i)1_H, \quad i=1,\dots,n,$$
$$\varphi(m_\varphi)1_K \le \textstyle\sum_{i=1}^n \Phi_i(\varphi(A_i)) \le \varphi(M_\varphi)1_K.$$

Also

$$\big(m_\varphi,M_\varphi\big)\cap[m_i,M_i]=\emptyset \quad \text{for } i=1,\dots,n$$

implies

$$\big(\varphi(m_\varphi),\varphi(M_\varphi)\big)\cap[\varphi(m_i),\varphi(M_i)]=\emptyset \quad \text{for } i=1,\dots,n. \tag{11.34}$$

Replacing A_i by $\varphi(A_i)$ in (11.10) and taking into account (11.34), we obtain that

$$f\left(\sum_{i=1}^n \Phi_i(\varphi(A_i))\right) \le \sum_{i=1}^n \Phi_i\left(f(\varphi(A_i))\right) - \delta_f\widetilde{A}_\varphi \le \sum_{i=1}^n \Phi_i\left(f(\varphi(A_i))\right) \tag{11.35}$$

holds for every convex function $f\colon J \to \mathbb{R}$ on an interval J which contains all $[\varphi(m_i),\varphi(M_i)] = \varphi([m_i,M_i])$, where

$$\delta_f = f(\varphi(\bar{m})) + f(\varphi(\bar{M})) - 2f\left(\frac{\varphi(\bar{m})+\varphi(\bar{M})}{2}\right) \geq 0 \tag{11.36}$$

and $\widetilde{A}_\varphi = \frac{1}{2}1_K - \frac{1}{\varphi(\bar{M})-\varphi(\bar{m})}\left|\sum_{i=1}^n \Phi_i(\varphi(A_i)) - \frac{\varphi(\bar{M})+\varphi(\bar{m})}{2}1_K\right| \geq 0$.

Also, if φ is strictly decreasing, then we check that (11.35) holds for convex function $f\colon J \to \mathbb{R}$ on J which contains all $[\varphi(M_i),\varphi(m_i)] = \varphi([m_i,M_i])$, where δ_f is defined by (11.36) and $\widetilde{A}_\varphi = \frac{1}{2}1_K - \frac{1}{\varphi(\bar{m})-\varphi(\bar{M})}\left|\sum_{i=1}^n \Phi_i(\varphi(A_i)) - \frac{\varphi(\bar{M})+\varphi(\bar{m})}{2}1_K\right| \geq 0$.

Putting $f = \psi \circ \varphi^{-1}$ in (11.35) and then applying an operator monotone function ψ^{-1}, we obtain (11.33).

The proof of the case (ii) is similar to the above case with the inequality (11.11) instead of (11.10). □

Now, we give a special case of the above theorem. It is a refinement of [139, Corollary 5].

Corollary 11.5 *Let* $(A_1,\ldots,A_n)$ *and* $(\Phi_1,\ldots,\Phi_n)$ *be as in the definition of the quasi-arithmetic mean* (11.29)*. Let* m_i *and* M_i*,* $m_i \leq M_i$ *be the bounds of* A_i*,* $i=1,\ldots,n$*. Let* $\varphi,\psi\colon I \to \mathbb{R}$ *be continuous strictly monotone functions on an interval* I *which contains all* m_i,M_i *and* $\mathscr{I}$ *be the identity function on* I*.*

(i) *If* φ^{-1} *is convex and*

$$\left(m_\varphi,M_\varphi\right)\cap[m_i,M_i]=\emptyset \qquad \text{for } i=1,\ldots,n, \qquad \text{and} \qquad m_{[\varphi]} < M_{[\varphi]} \tag{11.37}$$

is valid, where m_φ *and* M_φ*,* $m_\varphi \leq M_\varphi$ *are the bounds of* $M_\varphi(\mathbf{A},\mathbf{\Phi},n)$ *and* $m_{[\varphi]} = \max\left\{M_i\colon M_i \leq m_\varphi, i\in\{1,\ldots,n\}\right\}$, $M_{[\varphi]} = \min\left\{m_i\colon m_i \geq M_\varphi, i\in\{1,\ldots,n\}\right\}$*, then*

$$M_\varphi(\mathbf{A},\mathbf{\Phi},n) \leq M_{\mathscr{I}}(\mathbf{A},\mathbf{\Phi},n) - \delta_{\varphi,\mathscr{I}}(\bar{m},\bar{M})\widetilde{A}_\varphi(\bar{m},\bar{M}) \leq M_{\mathscr{I}}(\mathbf{A},\mathbf{\Phi},n) \tag{11.38}$$

holds for every $\bar{m}\in[m_{[\varphi]},m_\varphi]$*,* $\bar{M}\in[M_\varphi,M_{[\varphi]}]$*,* $\bar{m}<\bar{M}$*, where* $\delta_{\varphi,\mathscr{I}}(\bar{m},\bar{M}) \geq 0$ *and* $\widetilde{A}_\varphi(\bar{m},\bar{M}) \geq 0$ *are defined by* (11.32)*.*

(ii) *If* φ^{-1} *is concave and* (11.37) *is valid, then*

$$M_\varphi(\mathbf{A},\mathbf{\Phi},n) \geq M_{\mathscr{I}}(\mathbf{A},\mathbf{\Phi},n) - \delta_{\varphi,\mathscr{I}}(\bar{m},\bar{M})\widetilde{A}_\varphi(\bar{m},\bar{M}) \geq M_{\mathscr{I}}(\mathbf{A},\mathbf{\Phi},n) \tag{11.39}$$

holds for every $\bar{m}\in[m_{[\varphi]},m_\varphi]$*,* $\bar{M}\in[M_\varphi,M_{[\varphi]}]$*,* $\bar{m}<\bar{M}$*, where* $\delta_{\varphi,\mathscr{I}}(\bar{m},\bar{M}) \leq 0$ *and* $\widetilde{A}_\varphi(\bar{m},\bar{M}) \geq 0$ *are defined by* (11.32)*.*

(iii) *If* φ^{-1} *is convex and* (11.37) *is valid and if* ψ^{-1} *is concave, and*

$$\left(m_\psi,M_\psi\right)\cap[m_i,M_i]=\emptyset \qquad \text{for } i=1,\ldots,n, \qquad \text{and} \qquad m_{[\psi]} < M_{[\psi]}$$

is valid, where m_ψ *and* M_ψ*,* $m_\psi \leq M_\psi$ *are the bounds of* $M_\psi(\mathbf{A},\mathbf{\Phi},n)$ *and* $m_{[\psi]} = \max\left\{M_i\colon M_i \leq m_\psi, i\in\{1,\ldots,n\}\right\}$, $M_{[\psi]} = \min\left\{m_i\colon m_i \geq M_\psi, i\in\{1,\ldots,n\}\right\}$*, then*

$$\begin{aligned} M_\varphi(\mathbf{A},\mathbf{\Phi},n) &\leq M_{\mathscr{I}}(\mathbf{A},\mathbf{\Phi},n) - \delta_{\varphi,\mathscr{I}}(\bar{m},\bar{M})\widetilde{A}_\varphi(\bar{m},\bar{M}) \leq M_{\mathscr{I}}(\mathbf{A},\mathbf{\Phi},n) \\ &\leq M_{\mathscr{I}}(\mathbf{A},\mathbf{\Phi},n) - \delta_{\psi,\mathscr{I}}(\bar{\bar{m}},\bar{\bar{M}})\widetilde{A}_\psi(\bar{\bar{m}},\bar{\bar{M}}) \leq M_\psi(\mathbf{A},\mathbf{\Phi},n) \end{aligned} \tag{11.40}$$

holds for every $\bar{m} \in [m_{[\varphi]}, m_\varphi]$, $\bar{M} \in [M_\varphi, M_{[\varphi]}]$, $\bar{m} < \bar{M}$ *and every* $\bar{\bar{m}} \in [m_{[\psi]}, m_\psi]$, $\bar{\bar{M}} \in [M_\psi, M_{[\psi]}]$, $\bar{\bar{m}} < \bar{\bar{M}}$, *where* $\delta_{\varphi,\mathscr{I}}(\bar{m},\bar{M}) \geq 0$, $\widetilde{A}_\varphi(\bar{m},\bar{M}) \geq 0$ *and* $\delta_{\psi,\mathscr{I}}(\bar{\bar{m}},\bar{\bar{M}}) \leq 0$, $\widetilde{A}_\psi(\bar{\bar{m}},\bar{\bar{M}}) \geq 0$ *are defined by* (11.32).

Proof. (i)–(ii): Putting $\psi = \mathscr{I}$ in Theorem 11.6 (i) and (ii'), we obtain (11.38) and (11.39), respectively.

(iii): Replacing ψ by φ in (ii) and combining this with (i), we obtain the desired inequality (11.40). □

Remark 11.4 Let the assumptions of Corollary 11.5 (iii) be valid. We get the following refinement of the inequalities between quasi-arithmetic means

$$M_\varphi(\mathbf{A},\mathbf{\Phi},n) \leq M_\varphi(\mathbf{A},\mathbf{\Phi},n) + \Delta_{\varphi,\psi}(\bar{m},\bar{M},\bar{\bar{m}},\bar{\bar{M}}) \leq M_\psi(\mathbf{A},\mathbf{\Phi},n),$$

where

$$\Delta_{\varphi,\psi}(\bar{m},\bar{M},\bar{\bar{m}},\bar{\bar{M}}) = \delta_{\varphi,\mathscr{I}}(\bar{m},\bar{M})\widetilde{A}_\varphi(\bar{m},\bar{M}) - \delta_{\psi,\mathscr{I}}(\bar{\bar{m}},\bar{\bar{M}})\widetilde{A}_\psi(\bar{\bar{m}},\bar{\bar{M}}) \geq 0.$$

In particular,

$$M_\varphi(\mathbf{A},\mathbf{\Phi},n) \leq M_\varphi(\mathbf{A},\mathbf{\Phi},n) + \bar{\delta}_\varphi(\bar{m},\bar{M})\widetilde{A}_\varphi(\bar{m},\bar{M}) + \bar{\delta}_\psi(\bar{m},\bar{M})\widetilde{A}_\psi(\bar{m},\bar{M}) \leq M_\psi(\mathbf{A},\mathbf{\Phi},n),$$

where

$$\begin{aligned}\bar{\delta}_\varphi(\bar{m},\bar{M}) &= \bar{m} + \bar{M} - 2\varphi^{-1}\left(\tfrac{\varphi(\bar{m})+\varphi(\bar{M})}{2}\right) \geq 0,\\ \bar{\delta}_\psi(\bar{m},\bar{M}) &= 2\psi^{-1}\left(\tfrac{\psi(\bar{m})+\psi(\bar{M})}{2}\right) - \bar{m} - \bar{M} \geq 0.\end{aligned}$$

It is interesting to study a refinement of (11.31) under the condition placed only on the bounds of operators whose means we are considering. We study it in the following corollary. It is a refinement of the result given in [140, Theorem 2.1].

Corollary 11.6 *Let* A_i, Φ_i, m_i, M_i, $i = 1,\ldots,n$, *and* $\varphi, \psi, \mathscr{I}$ *as in the assumptions of Corollary 11.5.*

Let

$$(m_A, M_A) \cap [m_i, M_i] = \emptyset \quad \textit{for } i = 1,\ldots,n, \qquad \textit{and} \qquad m < M$$

be valid, where m_A *and* M_A, $m_A \leq M_A$, *are the bounds of* $A = \sum_{i=1}^n \Phi_i(A_i)$ *and*

$$m = \max\{M_i \colon M_i \leq m_A, i \in \{1,\ldots,n\}\},\ M = \min\{m_i \colon m_i \geq M_A, i \in \{1,\ldots,n\}\}.$$

If ψ *is convex,* ψ^{-1} *is operator monotone,* φ *is concave,* φ^{-1} *is operator monotone, then*

$$\begin{aligned}\mathscr{M}_\varphi(\mathbf{A},\mathbf{\Phi},n) &\leq \varphi^{-1}\left(\textstyle\sum_{i=1}^n \Phi_i(\varphi(A_i)) + \delta_\varphi \widetilde{A}\right) \leq M_{\mathscr{I}}(\mathbf{A},\mathbf{\Phi},n)\\ &\leq \psi^{-1}\left(\textstyle\sum_{i=1}^n \Phi_i(\psi(A_i)) - \delta_\psi \widetilde{A}\right) \leq \mathscr{M}_\psi(\mathbf{A},\mathbf{\Phi},n)\end{aligned} \tag{11.41}$$

holds, where

$$\delta_\varphi = 2\varphi\left(\tfrac{\bar{m}+\bar{M}}{2}\right) - \varphi(\bar{m}) - \varphi(\bar{M}) \geq 0, \quad \delta_\psi = \psi(\bar{\bar{m}}) + \psi(\bar{\bar{M}}) - 2\psi\left(\tfrac{\bar{\bar{m}}+\bar{\bar{M}}}{2}\right) \geq 0,$$

$$\widetilde{A} = \tfrac{1}{2}1_K - \tfrac{1}{\bar{M}-\bar{m}}\left|A - \tfrac{\bar{m}+\bar{M}}{2}1_K\right|, \quad \bar{A} = \tfrac{1}{2}1_K - \tfrac{1}{\bar{\bar{M}}-\bar{\bar{m}}}\left|A - \tfrac{\bar{\bar{m}}+\bar{\bar{M}}}{2}1_K\right|$$

and $\bar{m}, \bar{\bar{m}} \in [m, m_A]$, $\bar{M}, \bar{\bar{M}} \in [M_A, M]$, $\bar{m} < \bar{M}$, $\bar{\bar{m}} < \bar{\bar{M}}$ *are arbitrary numbers.*

If ψ is convex, $-\psi^{-1}$ is operator monotone, φ is concave, $-\varphi^{-1}$ is operator monotone, then the reverse inequality is valid in (11.41)*.*

Proof. We only prove (11.41). By replacing φ by $\mathscr{I}$ and next ψ by φ in Theorem 11.6 (ii') we obtain left hand side of (11.41). Also, by replacing φ by $\mathscr{I}$ in Theorem 11.6 (i) we obtain right hand side of (11.41). □

Now we illustrate an application of Theorem 11.6 and Remark 11.4 to power functions. Results for arbitrary power means are given in Corollary 11.8 in the next section.

Example 11.3 We put $\varphi(t) = t^{1/3}$, $\psi(t) = t^5$ and we define $\Phi_1, \Phi_2 : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ by $\Phi_1(B) = \Phi_2(B) = \frac{1}{2}B$, for $B \in M_2(\mathbb{C})$ (then $\Phi_1(I_2) + \Phi_2(I_2) = I_2$).

If $A_1 = \begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix}$ and $A_2 = 125\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then

$$\begin{aligned} \mathscr{M}_{1/3} &\equiv \mathscr{M}_{1/3}(\mathbf{A}, \mathbf{\Phi}, 2) = \left(\tfrac{1}{2}\sqrt[3]{A_1} + \tfrac{1}{2}\sqrt[3]{A_2}\right)^3 = \begin{pmatrix} 45.375 & 16 \\ 16 & 29.375 \end{pmatrix}, \\ \mathscr{M}_5 &\equiv \mathscr{M}_5(\mathbf{A}, \mathbf{\Phi}, 2) = \sqrt[5]{\tfrac{1}{2}A_1^5 + \tfrac{1}{2}A_2^5} = \begin{pmatrix} 108.81978 & 0.00059 \\ 0.00059 & 108.81919 \end{pmatrix} \end{aligned}$$

and we can take $m = 17.94427$, $M = 125$. We put also that $\bar{m} = \bar{\bar{m}} = 17.94427$, $\bar{M} = \bar{\bar{M}} = 125$. It follows $\delta_{1/3,5} \equiv \delta_{\varphi,\psi} = 2.94885 \times 10^{10}$, $\delta_{1/3} \equiv \delta_{\varphi,\mathscr{I}} = 32.41718$, $\delta_5 \equiv \delta_{\psi,\mathscr{I}} = 74.69602$, $\widetilde{A}_{1/3} \equiv \widetilde{A}_\varphi = \begin{pmatrix} 0.37027 & 0.20991 \\ 0.20991 & 0.16036 \end{pmatrix}$, $\widetilde{A}_5 \equiv \widetilde{A}_\psi = \begin{pmatrix} 0.49999 & 0.00001 \\ 0.00001 & 0.49998 \end{pmatrix}$ (rounded to five decimal places).

Then the following inequality holds:

$$\mathscr{M}_{1/3} \leq \sqrt[5]{\frac{1}{2}A_1^5 + \frac{1}{2}A_2^5 - \delta_{1/3,5}\widetilde{A}_{1/3}} = \begin{pmatrix} 69.70109 & -23.36045 \\ -23.36045 & 93.06154 \end{pmatrix} \leq \mathscr{M}_5,$$

which is in accordance with Theorem 11.6, and

$$\mathscr{M}_{1/3} \leq \mathscr{M}_{1/3} + \delta_{1/3}\widetilde{A}_{1/3} + \delta_5\widetilde{A}_5 = \begin{pmatrix} 94.72543 & 22.80573 \\ 22.80573 & 71.91970 \end{pmatrix} \leq \mathscr{M}_5,$$

which is accordance with the special case of Remark 11.4.

As a special case of the quasi-arithmetic mean (11.29) we can study the operator power mean

$$\mathscr{M}_n^{[r]}(\mathbf{A}, \mathbf{\Phi}) = \begin{cases} \left(\sum_{i=1}^n \Phi_i\left(A_i^r\right)\right)^{1/r}, & r \in \mathbb{R}\backslash\{0\}, \\ \exp\left(\sum_{i=1}^n \Phi_i\left(\ln\left(A_i\right)\right)\right), & r = 0, \end{cases} \tag{11.42}$$

where $(A_1, \ldots, A_n)$ is an n-tuple of strictly positive operators in $\mathscr{B}(H)$ and $(\Phi_1, \ldots, \Phi_n)$ is an n-tuple of positive linear mappings $\Phi_i : \mathscr{B}(H) \to \mathscr{B}(K)$ such that $\sum_{i=1}^n \Phi_i(1_H) = 1_K$.

For convenience, we introduce notation as special cases of (11.32), as follows:

$$\begin{aligned}
\delta_{r,s}(m,M) &= \begin{cases} m^s+M^s-2\left(\frac{m^r+M^r}{2}\right)^{s/r}, & r\neq 0,\\ m^s+M^s-2\,(mM)^{s/2}, & r=0,\end{cases}\\
\widetilde{A}_r(m,M) &= \begin{cases} \frac{1}{2}1_K-\frac{1}{|M^r-m^r|}\left|\sum_{i=1}^n\Phi_i(A_i^r)-\frac{M^r+m^r}{2}1_K\right|, & r\neq 0,\\ \frac{1}{2}1_K-\left|\ln\left(\frac{M}{m}\right)\right|^{-1}\left|\sum_{i=1}^n\Phi_i(\ln A_i)-\ln\sqrt{Mm}1_K\right|, & r=0,\end{cases}
\end{aligned}\tag{11.43}$$

where $m,M\in\mathbb{R}$, $0<m<M$ and $r,s\in\mathbb{R}$, $r\leq s$. We include implicitly that $\widetilde{A}_r(m,M)\equiv\widetilde{A}_{r,A}(m,M)$, where $A=\sum_{i=1}^n\Phi_i(A_i^r)$ for $r\neq 0$ and $A=\sum_{i=1}^n\Phi_i(\ln A_i)$ for $r=0$.

Applying Theorem 11.6 to the operator power means, we obtain the following refinement of inequalities among power means given in [139, Corollary 7].

Corollary 11.7 *Let* $(A_1,\ldots,A_n)$ *and* $(\Phi_1,\ldots,\Phi_n)$ *be as in the definition of the power mean* (11.42)*. Let m_i and M_i, $0<m_i\leq M_i$ be the bounds of A_i, $i=1,\ldots,n$.*

(i) *If $r\leq s$, $s\geq 1$ or $r\leq s\leq -1$,*

$$\left(m^{[r]},M^{[r]}\right)\cap[m_i,M_i]=\emptyset,\quad i=1,\ldots,n,\qquad\text{and}\qquad \underline{m}<\overline{M},$$

where $m^{[r]}$ and $M^{[r]}$, $m^{[r]}\leq M^{[r]}$ are the bounds of $\mathscr{M}_n^{[r]}(\mathbf{A},\mathbf{\Phi})$ and $\underline{m}=\max\left\{M_i\colon M_i\leq m^{[r]},i\in\{1,\ldots,n\}\right\}$, $\overline{M}=\min\left\{m_i\colon m_i\geq M^{[r]},i\in\{1,\ldots,n\}\right\}$, then

$$\mathscr{M}_n^{[r]}(\mathbf{A},\mathbf{\Phi})\leq\left(\sum_{i=1}^n\Phi_i\left(A_i^s\right)-\delta_{r,s}\widetilde{A}_r\right)^{1/s}\leq\mathscr{M}_n^{[s]}(\mathbf{A},\mathbf{\Phi})\tag{11.44}$$

holds, where $\delta_{r,s}\geq 0$, for $s\geq 1$, $\delta_{r,s}\leq 0$ for $s\leq -1$ and $\widetilde{A}_r\geq 0$. Here we assume that $\delta_{r,s}\equiv\delta_{r,s}(\bar{m},\bar{M})$, $\widetilde{A}_r\equiv\widetilde{A}_r(\bar{m},\bar{M})$ are defined by (11.43) *and $\bar{m}\in[\underline{m},m^{[r]}]$, $\bar{M}\in[M^{[r]},\overline{M}]$, $\bar{m}<\bar{M}$, are arbitrary numbers.*

(ii) *If $r\leq s$, $r\leq -1$ or $1\leq r\leq s$,*

$$\left(m^{[s]},M^{[s]}\right)\cap[m_i,M_i]=\emptyset,\quad i=1,\ldots,n,\qquad\text{and}\qquad \underline{m}<\overline{M},$$

where $m^{[s]}$ and $M^{[s]}$, $m^{[s]}\leq M^{[s]}$ are the bounds of $\mathscr{M}_n^{[s]}(\mathbf{A},\mathbf{\Phi})$ and $\underline{m}=\max\left\{M_i\colon M_i\leq m^{[s]},i\in\{1,\ldots,n\}\right\}$, $\overline{M}=\min\left\{m_i\colon m_i\geq M^{[s]},i\in\{1,\ldots,n\}\right\}$, then

$$\mathscr{M}_n^{[r]}(\mathbf{A},\mathbf{\Phi})\leq\left(\sum_{i=1}^n\Phi_i\left(A_i^r\right)-\delta_{s,r}\widetilde{A}_s\right)^{1/r}\leq\mathscr{M}_n^{[s]}(\mathbf{A},\mathbf{\Phi})$$

holds, where $\delta_{s,r}\geq 0$ for $r\leq -1$, $\delta_{s,r}\leq 0$ for $r\geq 1$ and $\widetilde{A}_s\geq 0$. Here we assume that $\delta_{s,r}\equiv\delta_{s,r}(\bar{m},\bar{M})$, $\widetilde{A}_s\equiv\widetilde{A}_s(\bar{m},\bar{M})$ are defined by (11.43) *and $\bar{m}\in[\underline{m},m^{[s]}]$, $\bar{M}\in[M^{[s]},\overline{M}]$, $\bar{m}<\bar{M}$, are arbitrary numbers.*

Proof. We prove only the case (i) by putting $\varphi(t) = t^r$ and $\psi(t) = t^s$, for $t > 0$.

Then $\psi \circ \varphi^{-1}(t) = t^{s/r}$ is concave for $r \leq s$, $s \leq 0$ and $r \neq 0$. Since $-\psi^{-1}(t) = -t^{1/s}$ is operator monotone for $s \leq -1$ and $\left(m^{[r]}, M^{[r]}\right) \cap [m_i, M_i] = \emptyset$ is satisfied, then, by applying Theorem 11.6 (ii) we obtain (11.44) for $r \leq s \leq -1$.

But, $\psi \circ \varphi^{-1}(t) = t^{s/r}$ is convex for $r \leq s$, $s \geq 0$ and $r \neq 0$. Since $\psi^{-1}(t) = t^{1/s}$ is operator monotone for $s \geq 1$, then by applying Theorem 11.6 (i) we obtain (11.44) for $r \leq s$, $s \geq 1$, $r \neq 0$.

If $r = 0$ and $s \geq 1$, we put $\varphi(t) = \ln t$ and $\psi(t) = t^s$, $t > 0$. Since $\psi \circ \varphi^{-1}(t) = \exp(st)$ is convex, then, similarly as above, we obtain the desired inequality.

In the case (ii) we put $\varphi(t) = t^s$ and $\psi(t) = t^r$, for $t > 0$ and we use the same technique as in the case (i). $\square$

Example 11.4 Figure 11.2 shows regions (1), (2), (4), (6), (7) in which the monotonicity of the power mean holds true [139, Corollary 6]. On the other hand, Figure 11.2 also shows regions (1)–(7) in which the same holds true, but with the condition on spectra [139, Corollary 7]. In [139, Example 2], it was shown that the order among power means does not hold generally without the condition on spectra in regions (3), (5). Now, by using Corollary 11.7, we give a refinement of inequalities among power means in the regions (2)–(6) (see Remark 11.5).

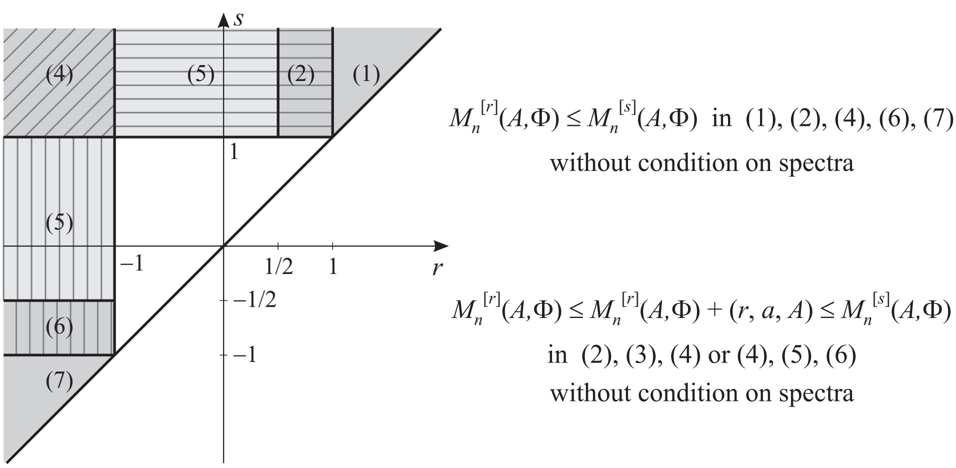

Figure 11.2: *Regions describing inequalities among power means*

Corollary 11.8 *Let $(A_1,\dots,A_n)$ and $(\Phi_1,\dots,\Phi_n)$ be as in the definition of the power mean* (11.42)*. Let m_i and M_i, $0 < m_i \leq M_i$ be the bounds of A_i, $i = 1,\dots,n$. Let*

$$\left(m^{[r]}, M^{[r]}\right) \cap [m_i, M_i] = \emptyset, \quad i = 1,\dots,n, \qquad m_{[r]} < M_{[r]},$$
$$\left(m^{[s]}, M^{[s]}\right) \cap [m_i, M_i] = \emptyset, \quad i = 1,\dots,n, \qquad m_{[s]} < M_{[s]},$$

where $m^{[r]}$, $M^{[r]}$, $m^{[r]} \le M^{[r]}$ *and* $m^{[s]}$, $M^{[s]}$, $m^{[s]} \le M^{[s]}$ *are the bounds of* $\mathscr{M}_n^{[r]}(\mathbf{A},\mathbf{\Phi})$ *and* $\mathscr{M}_n^{[r]}(\mathbf{A},\mathbf{\Phi})$*, respectively, and*

$$\begin{aligned} m_{[r]} &= \max\left\{M_i \le m^{[r]}, i\in\{1,\dots,n\}\right\},\ M_{[r]} = \min\left\{m_i \ge M^{[r]}, i\in\{1,\dots,n\}\right\},\\ m_{[s]} &= \max\left\{M_i \le m^{[s]}, i\in\{1,\dots,n\}\right\},\ M_{[s]} = \min\left\{m_i \ge M^{[s]}, i\in\{1,\dots,n\}\right\}. \end{aligned}$$

Let $\overline{m}\in[m_{[r]},m^{[r]}]$, $\overline{M}\in[M^{[r]},M_{[r]}]$, $\overline{m}<\overline{M}$, *and* $\overline{\overline{m}}\in[m_{[s]},m^{[s]}]$, $\overline{\overline{M}}\in[M^{[s]},M_{[s]}]$, $\overline{\overline{m}}<\overline{\overline{M}}$ *be arbitrary numbers.*

(**i**) *If* $r\le 1\le s$*, then*

$$\begin{aligned} \mathscr{M}_n^{[r]}(\mathbf{A},\mathbf{\Phi}) &\le \textstyle\sum_{i=1}^n \Phi_i(A_i) - \delta_{r,1}(\overline{m},\overline{M})\widetilde{A}_r(\overline{m},\overline{M}) \le \mathscr{M}_n^{[1]}(\mathbf{A},\mathbf{\Phi})\\ &\le \textstyle\sum_{i=1}^n \Phi_i(A_i) - \delta_{s,1}(\overline{\overline{m}},\overline{\overline{M}})\widetilde{A}_s(\overline{\overline{m}},\overline{\overline{M}}) \le \mathscr{M}_n^{[s]}(\mathbf{A},\mathbf{\Phi}) \end{aligned} \tag{11.45}$$

holds, where $\delta_{r,1}(\overline{m},\overline{M})\ge 0$, $\widetilde{A}_r(\overline{m},\overline{M})\ge 0$, $\delta_{s,1}(\overline{\overline{m}},\overline{\overline{M}})\le 0$ *and* $\widetilde{A}_s(\overline{\overline{m}},\overline{\overline{M}})\ge 0$ *are defined by* (11.43)*.*

(**ii**) *Furthermore if* $r\le -1\le s$*, then*

$$\begin{aligned} \mathscr{M}_n^{[r]}(\mathbf{A},\mathbf{\Phi}) &\le \left(\textstyle\sum_{i=1}^n \Phi_i\left(A_i^{-1}\right) - \delta_{r,-1}(\overline{m},\overline{M})\widetilde{A}_r(\overline{m},\overline{M})\right)^{-1} \le \mathscr{M}_n^{[-1]}(\mathbf{A},\mathbf{\Phi})\\ &\le \left(\textstyle\sum_{i=1}^n \Phi_i\left(A_i^{-1}\right) - \delta_{s,-1}(\overline{\overline{m}},\overline{\overline{M}})\widetilde{A}_s(\overline{\overline{m}},\overline{\overline{M}})\right)^{-1} \le \mathscr{M}_n^{[s]}(\mathbf{A},\mathbf{\Phi}) \end{aligned} \tag{11.46}$$

holds, where $\delta_{r,-1}(\overline{m},\overline{M})\le 0$, $\widetilde{A}_r(\overline{m},\overline{M})\ge 0$, $\delta_{s,-1}(\overline{\overline{m}},\overline{\overline{M}})\ge 0$ *and* $\widetilde{A}_s(\overline{\overline{m}},\overline{\overline{M}})\ge 0$ *are defined by* (11.43)*.*

(**iii**) *Furthermore if* $r\le -1$, $s\ge 1$*, then*

$$\begin{aligned} \mathscr{M}_n^{[r]}(\mathbf{A},\mathbf{\Phi}) &\le \left(\textstyle\sum_{i=1}^n \Phi_i\left(A_i^{-1}\right) - \delta_{r,-1}(\overline{m},\overline{M})\widetilde{A}_r(\overline{m},\overline{M})\right)^{-1} \le \mathscr{M}_n^{[-1]}(\mathbf{A},\mathbf{\Phi})\\ &\le \mathscr{M}_n^{[1]}(\mathbf{A},\mathbf{\Phi}) \le \textstyle\sum_{i=1}^n \Phi_i(A_i) - \delta_{s,1}(\overline{\overline{m}},\overline{\overline{M}})\widetilde{A}_s(\overline{\overline{m}},\overline{\overline{M}}) \le \mathscr{M}_n^{[s]}(\mathbf{A},\mathbf{\Phi}) \end{aligned} \tag{11.47}$$

holds, where $\delta_{r,-1}(\overline{m},\overline{M})\le 0$, $\widetilde{A}_r(\overline{m},\overline{M})\ge 0$, $\delta_{s,1}(\overline{\overline{m}},\overline{\overline{M}})\le 0$, $\widetilde{A}_s(\overline{\overline{m}},\overline{\overline{M}})\ge 0$ *are defined by* (11.43)*.*

Proof. We prove only (11.45). If $r\le 1$, then putting $s=1$ in Corollary 11.7 (i) we get the left hand side of (11.45). Also, if $s\ge 1$, then putting $r=1$ in Corollary 11.7 (ii) we get the right hand side of (11.45). □

Remark 11.5 Let the assumptions of Corollary 11.8 be valid. We get a refinement of inequalities among power means as follows.

If $r\le 1\le s$, then

$$\mathscr{M}_n^{[r]}(\mathbf{A},\mathbf{\Phi}) \le \mathscr{M}_n^{[r]}(\mathbf{A},\mathbf{\Phi}) + \delta_{r,1}(\overline{m},\overline{M})\widetilde{A}_r(\overline{m},\overline{M}) - \delta_{s,1}(\overline{\overline{m}},\overline{\overline{M}})\widetilde{A}_s(\overline{\overline{m}},\overline{\overline{M}}) \le \mathscr{M}_n^{[s]}(\mathbf{A},\mathbf{\Phi}).$$

If $r\le -1\le s$, then

$$\begin{aligned} \mathscr{M}_n^{[r]}(\mathbf{A},\mathbf{\Phi}) &\le \mathscr{M}_n^{[r]}(\mathbf{A},\mathbf{\Phi}) + \left(\textstyle\sum_{i=1}^n \Phi_i\left(A_i^{-1}\right) - \delta_{s,-1}(\overline{\overline{m}},\overline{\overline{M}})\widetilde{A}_s(\overline{\overline{m}},\overline{\overline{M}})\right)^{-1}\\ &\quad - \left(\textstyle\sum_{i=1}^n \Phi_i\left(A_i^{-1}\right) - \delta_{r,-1}(\overline{m},\overline{M})\widetilde{A}_r(\overline{m},\overline{M})\right)^{-1} \le \mathscr{M}_n^{[s]}(\mathbf{A},\mathbf{\Phi}). \end{aligned}$$

If $r \leq -1$, $s \geq 1$, then

$$\begin{aligned}\mathscr{M}_n^{[r]}(\mathbf{A},\mathbf{\Phi}) &\leq \mathscr{M}_n^{[r]}(\mathbf{A},\mathbf{\Phi}) + \mathscr{M}_n^{[1]}(\mathbf{A},\mathbf{\Phi}) - \delta_{s,1}(\bar{\bar{m}},\bar{\bar{M}})\widetilde{A}_s(\bar{\bar{m}},\bar{\bar{M}}) \\ &\quad - \left(\sum_{i=1}^n \Phi_i\left(A_i^{-1}\right) - \delta_{r,-1}(\bar{m},\bar{M})\widetilde{A}_r(\bar{m},\bar{M})\right)^{-1} \leq \mathscr{M}_n^{[s]}(\mathbf{A},\mathbf{\Phi}).\end{aligned}$$

Finally, we give a refinement of inequalities among power means under the condition placed only on the bounds of operators whose means we are considering.

Corollary 11.9 *Let $(A_1,\ldots,A_n)$ and $(\Phi_1,\ldots,\Phi_n)$ be as in the definition of the quasi-arithmetic mean* (11.29). *Let m_i and M_i, $m_i \leq M_i$ be the bounds of A_i, $i=1,\ldots,n$. Let*

$$\begin{aligned}(m^{[1]},M^{[1]}) \cap [m_i,M_i] = \emptyset, \quad i=1,\ldots,n, \qquad m_{[1]} < M_{[1]}, \\ (m^{[-1]},M^{[-1]}) \cap [m_i,M_i] = \emptyset, \quad i=1,\ldots,n, \qquad m_{[-1]} < M_{[-1]},\end{aligned}$$

where $m^{[1]}$, $M^{[1]}$, $m^{[1]} \leq M^{[1]}$, and $m^{[-1]}$, $M^{[-1]}$, $m^{[-1]} \leq M^{[-1]}$, are the bounds of $\mathscr{M}_n^{[1]}(\mathbf{A},\mathbf{\Phi})$ and $\mathscr{M}_n^{[-1]}(\mathbf{A},\mathbf{\Phi})$, respectively, and

$$\begin{aligned}m_{[1]} &= \max\left\{M_i \leq m^{[1]}, i \in \{1,\ldots,n\}\right\}, \quad M_{[1]} = \min\left\{m_i \geq M^{[1]}, i \in \{1,\ldots,n\}\right\}, \\ m_{[-1]} &= \max\left\{M_i \leq m^{[-1]}, i \in \{1,\ldots,n\}\right\}, \quad M_{[-1]} = \min\left\{m_i \geq M^{[-1]}, i \in \{1,\ldots,n\}\right\}.\end{aligned}$$

Let $\bar{m} \in [m_{[1]}, m^{[1]}]$, $\bar{M} \in [M^{[1]}, M_{[1]}]$, $\bar{m} < \bar{M}$, and $\bar{\bar{m}} \in [m_{[-1]}, m^{[-1]}]$, $\bar{\bar{M}} \in [M^{[-1]}, M_{[-1]}]$, $\bar{\bar{m}} < \bar{\bar{M}}$ be arbitrary numbers.

If $r \leq -1$, $s \geq 1$, then

$$\begin{aligned}\mathscr{M}_n^{[r]}(\mathbf{A},\mathbf{\Phi}) &\leq \left(\sum_{i=1}^n \Phi_i\left(A_i^r\right) - \delta_{-1,r}(\bar{m},\bar{M})\widetilde{A}_{-1}(\bar{m},\bar{M})\right)^{1/r} \leq \mathscr{M}_n^{[-1]}(\mathbf{A},\mathbf{\Phi}) \\ \leq \mathscr{M}_1^{[r]}(\mathbf{A},\mathbf{\Phi}) &\leq \left(\sum_{i=1}^n \Phi_i\left(A_i^s\right) - \delta_{1,s}(\bar{\bar{m}},\bar{\bar{M}})\widetilde{A}_1(\bar{\bar{m}},\bar{\bar{M}})\right)^{1/s} \leq \mathscr{M}_n^{[s]}(\mathbf{A},\mathbf{\Phi})\end{aligned} \tag{11.48}$$

holds, where $\delta_{-1,r}(\bar{m},\bar{M}) \geq 0$, $\widetilde{A}_{-1}(\bar{m},\bar{M}) \geq 0$, $\delta_{1,s}(\bar{\bar{m}},\bar{\bar{M}}) \geq 0$ and $\widetilde{A}_1(\bar{\bar{m}},\bar{\bar{M}}) \geq 0$ are defined by (11.43)*.*

Proof. If $r \leq -1$, then by putting $s=-1$ in Corollary 11.7 (ii) we obtain left hand side of (11.48). Also, if $s \geq 1$, then putting $r=1$ in Corollary 11.7 (i) we obtain right hand side of (11.48). Finally, we apply the order $\mathscr{M}_n^{[-1]}(\mathbf{A},\mathbf{\Phi}) \leq \mathscr{M}_n^{[1]}(\mathbf{A},\mathbf{\Phi})$. □

Now we give an application of Theorem 11.4 to the quasi-arithmetic mean with weights. For a subset $\{A_{n_1},\ldots,A_{n_2}\}$ of $\{A_1,\ldots,A_n\}$, we denote the quasi-arithmetic mean by

$$\mathscr{M}_\varphi(\gamma,\mathbf{A},\mathbf{\Phi},n_1,n_2) = \varphi^{-1}\left(\frac{1}{\gamma}\sum_{i=n_1}^{n_2}\Phi_i\left(\varphi(A_i)\right)\right), \tag{11.49}$$

where $(A_{n_1},\ldots,A_{n_2})$ are self-adjoint operators in $\mathscr{B}(H)$ with the spectra in I, $(\Phi_{n_1},\ldots,\Phi_{n_2})$ are positive linear mappings $\Phi_i : \mathscr{B}(H) \to \mathscr{B}(K)$ such that $\sum_{i=n_1}^{n_2}\Phi_i(1_H) = \gamma 1_K$ and $\varphi : I \to \mathbb{R}$ is a continuous strictly monotone function.

Under the same conditions, we introduce, for convenience, the following notation:

$$\begin{aligned}
\delta_{\varphi,\psi}(m,M) &= \psi(m)+\psi(M)-2\psi\circ\varphi^{-1}\left(\frac{\varphi(m)+\varphi(M)}{2}\right),\\
\widetilde{A}_{\varphi,n_1,\gamma}(m,M) &= \frac{1}{2}1_K-\frac{1}{\gamma(M-m)}\sum_{i=1}^{n_1}\Phi_i\left(\left|\varphi(A_i)-\frac{\varphi(M)+\varphi(m)}{2}1_H\right|\right),
\end{aligned} \tag{11.50}$$

where $\varphi,\psi : I\to\mathbb{R}$ are continuous strictly monotone functions and $m,M\in I$, $m<M$. We include implicitly that $\widetilde{A}_{\varphi,n_1,\gamma}(m,M)\equiv\widetilde{A}_{\varphi,A,\Phi,n_1,\gamma}(m,M)$.

The following theorem is an extension of Theorem 11.6 and a refinement of [138, Theorem 3.1].

Theorem 11.7 *Let $(A_1,\dots,A_n)$ be an n-tuple of self-adjoint operators $A_i\in\mathscr{B}(H)$ with the bounds m_i and M_i, $m_i\le M_i$, $i=1,\dots,n$. Let $\varphi,\psi : I\to\mathbb{R}$ be continuous strictly monotone functions on an interval I which contains all m_i,M_i. Let $(\Phi_1,\dots,\Phi_n)$ be an n-tuple of positive linear mappings $\Phi_i:\mathscr{B}(H)\to\mathscr{B}(K)$, such that $\sum_{i=1}^{n_1}\Phi_i(1_H)=\alpha\,1_K$, $\sum_{i=n_1+1}^{n}\Phi_i(1_H)=\beta\,1_K$, where $1\le n_1<n$, $\alpha,\beta>0$ and $\alpha+\beta=1$. Let one of two equalities*

$$\mathscr{M}_\varphi(\alpha,\mathbf{A},\mathbf{\Phi},1,n_1)=\mathscr{M}_\varphi(1,\mathbf{A},\mathbf{\Phi},1,n)=\mathscr{M}_\varphi(\beta,\mathbf{A},\mathbf{\Phi},n_1+1,n) \tag{11.51}$$

be valid and let

$$(m_L,M_R)\cap[m_i,M_i]=\emptyset \quad \textit{for} \quad i=n_1+1,\dots,n, \qquad m<M,$$

where $m_L=\min\{m_1,\dots,m_{n_1}\}$, $M_R=\max\{M_1,\dots,M_{n_1}\}$,

$$\begin{aligned}
m &= \begin{cases} m_L, & \text{if } \{M_i\colon M_i\le m_L, i\in\{n_1+1,\dots,n\}\}=\emptyset,\\ \max\{M_i\colon M_i\le m_L, i\in\{n_1+1,\dots,n\}\}, & \text{otherwise,}\end{cases}\\
M &= \begin{cases} M_R, & \text{if } \{m_i\colon m_i\ge M_R, i\in\{n_1+1,\dots,n\}\}=\emptyset,\\ \min\{m_i\colon m_i\ge M_R, i\in\{n_1+1,\dots,n\}\}, & \text{otherwise.}\end{cases}
\end{aligned}$$

(i) *If $\psi\circ\varphi^{-1}$ is convex and ψ^{-1} is operator monotone, then*

$$\begin{aligned}
\mathscr{M}_\psi(\alpha,\mathbf{A},\mathbf{\Phi},1,n_1) &\le \psi^{-1}\left(\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i\left(\psi(A_i)\right)+\beta\delta_{\varphi,\psi}\widetilde{A}_{\varphi,n_1,\alpha}\right)\le\mathscr{M}_\psi(1,\mathbf{A},\mathbf{\Phi},1,n)\\
&\le \psi^{-1}\left(\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i\left(\psi(A_i)\right)-\alpha\delta_{\varphi,\psi}\widetilde{A}_{\varphi,n_1,\alpha}\right)\\
&\le \mathscr{M}_\psi(\beta,\mathbf{A},\mathbf{\Phi},n_1+1,n)
\end{aligned} \tag{11.52}$$

holds, where $\delta_{\varphi,\psi}\ge 0$ and $\widetilde{A}_{\varphi,n_1,\alpha}\ge 0$.

(i′) *If $\psi\circ\varphi^{-1}$ is convex and $-\psi^{-1}$ is operator monotone, then the reverse inequality is valid in* (11.52)*, where $\delta_{\varphi,\psi}\ge 0$ and $\widetilde{A}_{\varphi,n_1,\alpha}\ge 0$.*

(ii) *If* $\psi \circ \varphi^{-1}$ *is concave and* $-\psi^{-1}$ *is operator monotone, then* (11.52) *holds, where* $\delta_{\varphi,\psi} \le 0$ *and* $\widetilde{A}_{\varphi,n_1,\alpha} \ge 0$.

(ii′) *If* $\psi \circ \varphi^{-1}$ *is concave and* ψ^{-1} *is operator monotone, then the reverse inequality is valid in* (11.52)*, where* $\delta_{\varphi,\psi} \le 0$ *and* $\widetilde{A}_{\varphi,n_1,\alpha} \ge 0$.

In all the above cases, we assume that $\delta_{\varphi,\psi} \equiv \delta_{\varphi,\psi}(\overline{m},\overline{M})$, $\widetilde{A}_{\varphi,n_1,\alpha} \equiv \widetilde{A}_{\varphi,n_1,\alpha}(\overline{m},\overline{M})$ *are defined by* (11.50) *and* $\overline{m} \in [m,m_L]$, $\overline{M} \in [M_R,M]$, $\overline{m} < \overline{M}$, *are arbitrary numbers.*

Proof. We only prove the case (i). Suppose that φ is a strictly increasing function. Then

$$(m_L,M_R) \cap [m_i,M_i] = \emptyset \qquad \text{for} \quad i = n_1+1,\dots,n$$

implies

$$(\varphi(m_L),\varphi(M_R)) \cap [\varphi(m_i),\varphi(M_i)] = \emptyset \quad \text{for } i = n_1+1,\dots,n. \tag{11.53}$$

Also, by using (11.51), we have

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i\left(\varphi(A_i)\right) = \sum_{i=1}^{n}\Phi_i\left(\varphi(A_i)\right) = \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i\left(\varphi(A_i)\right).$$

Taking into account (11.53) and the above double equality, we obtain by Theorem 11.4 that

$$\begin{aligned}\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(\varphi(A_i))) &\le \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(\varphi(A_i))) + \beta\delta_f\widetilde{A}_{\varphi,n_1,\alpha} \le \sum_{i=1}^{n}\Phi_i(f(\varphi(A_i)))\\ &\le \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(\varphi(A_i))) - \alpha\delta_f\widetilde{A}_{\varphi,n_1,\alpha} \le \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(f(\varphi(A_i))),\end{aligned} \tag{11.54}$$

for every continuous convex function $f : J \to \mathbb{R}$ on an interval J which contains all $[\varphi(m_i),\varphi(M_i)] = \varphi([m_i,M_i])$, $i = 1,\dots,n$, where $\delta_f = f(\varphi(m)) + f(\varphi(M)) - 2f\left(\frac{\varphi(m)+\varphi(M)}{2}\right)$.

Also, if φ is strictly decreasing, then we check that (11.54) holds for convex function $f : J \to \mathbb{R}$ on J which contains all $[\varphi(M_i),\varphi(m_i)] = \varphi([m_i,M_i])$.

Putting $f = \psi \circ \varphi^{-1}$ in (11.54), we obtain

$$\begin{aligned}\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i\left(\psi(A_i)\right) &\le \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i\left(\psi(A_i)\right) + \beta\delta_{\varphi,\psi}\widetilde{A}_{\varphi,n_1,\alpha} \le \sum_{i=1}^{n}\Phi_i\left(\psi(A_i)\right)\\ &\le \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i\left(\psi(A_i)\right) - \alpha\delta_{\varphi,\psi}\widetilde{A}_{\varphi,n_1,\alpha} \le \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i\left(\psi(A_i)\right).\end{aligned}$$

Applying an operator monotone function ψ^{-1} to the above double inequality, we obtain the desired inequality (11.52). □

We now give some results that can be derived from Theorem 11.7, which are extensions of Corollary 11.5, Corollary 11.6 and a refinement of [138, Corollary 3.3].

Corollary 11.10 *Let* $(A_1,\ldots,A_n)$ *and* $(\Phi_1,\ldots,\Phi_n)$, m_i, M_i, m, M, m_L, M_R, α *and* β *be as in Theorem 11.7. Let* I *be an interval which contains all* m_i, M_i *and*

$$(m_L, M_R) \cap [m_i, M_i] = \emptyset \quad \text{for} \quad i = n_1+1,\ldots,n, \qquad m < M.$$

I) *If one of two equalities*

$$\mathscr{M}_\varphi(\alpha,\mathbf{A},\mathbf{\Phi},1,n_1) = \mathscr{M}_\varphi(1,\mathbf{A},\mathbf{\Phi},1,n) = \mathscr{M}_\varphi(\beta,\mathbf{A},\mathbf{\Phi},n_1+1,n)$$

is valid, then

$$\begin{aligned} \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i) &\le \frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i) + \beta\delta_{\varphi^{-1}}\widetilde{A}_{\varphi,n_1,\alpha} \le \sum_{i=1}^{n}\Phi_i(A_i) \\ &\le \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(A_i) - \alpha\delta_{\varphi^{-1}}\widetilde{A}_{\varphi,n_1,\alpha} \le \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i\Phi_i(A_i) \end{aligned} \tag{11.55}$$

holds for every continuous strictly monotone function $\varphi : I \to \mathbb{R}$ *such that* φ^{-1} *is convex on* I*, where* $\delta_{\varphi^{-1}} = \bar{m} + \bar{M} - 2\,\varphi^{-1}\left(\frac{\varphi(\bar{m})+\varphi(\bar{M})}{2}\right) \ge 0$,

$$\widetilde{A}_{\varphi,n_1,\alpha} = \frac{1}{2}1_K - \frac{1}{\alpha(\bar{M}-\bar{m})}\sum_{i=1}^{n_1}\Phi_i\left(\left|\varphi(A_i) - \frac{\varphi(\bar{M})+\varphi(\bar{m})}{2}1_H\right|\right)$$

and $\bar{m} \in [m, m_L]$, $\bar{M} \in [M_R, M]$, $\bar{m} < \bar{M}$, *are arbitrary numbers.*

But, if φ^{-1} *is concave, then the reverse inequality is valid in* (11.55) *for* $\delta_{\varphi^{-1}} \le 0$.

II) *If one of two equalities*

$$\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(A_i) = \sum_{i=1}^{n}\Phi_i(A_i) = \frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i(A_i)$$

is valid, then

$$\begin{aligned} \mathscr{M}_\varphi(\alpha,\mathbf{A},\mathbf{\Phi},1,n_1) &\le \varphi^{-1}\left(\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i\left(\varphi(A_i)\right) + \beta\delta_\varphi\widetilde{A}_{n_1}\right) \le \mathscr{M}_\varphi(1,\mathbf{A},\mathbf{\Phi},1,n) \\ &\le \varphi^{-1}\left(\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i\left(\varphi(A_i)\right) - \alpha\delta_\varphi\widetilde{A}_{n_1}\right) \le \mathscr{M}_\varphi(\beta,\mathbf{A},\mathbf{\Phi},n_1+1,n) \end{aligned} \tag{11.56}$$

holds for every continuous strictly monotone function $\varphi : I \to \mathbb{R}$ *such that one of the following conditions*

(i) φ *is convex and* φ^{-1} *is operator monotone,*

(i') φ *is concave and* $-\varphi^{-1}$ *is operator monotone,*

is satisfied, where $\delta_\varphi = \varphi(\bar{m}) + \varphi(\bar{M}) - 2\varphi\left(\frac{\bar{m}+\bar{M}}{2}\right)$,

$$\widetilde{A}_{n_1} = \frac{1}{2}1_K - \frac{1}{\alpha(M-\bar{m})} \cdot \sum_{i=1}^{n_1} \Phi_i\left(\left|A_i - \frac{\bar{m}+\bar{M}}{2}1_H\right|\right)$$

and $\bar{m} \in [m, m_L]$, $\bar{M} \in [M_R, M]$, $\bar{m} < \bar{M}$, *are arbitrary numbers. But, if one of the following conditions*

(ii) φ *is concave and* φ^{-1} *is operator monotone,*

(ii') φ *is convex and* $-\varphi^{-1}$ *is operator monotone,*

is satisfied, then the reverse inequality is valid in (11.56).

Proof. The inequalities (11.55) follow from Theorem 11.7, by replacing ψ with the identity function, while the inequalities (11.56) follow by replacing φ with the identity function and ψ with φ. □

Remark 11.6 Let the assumptions of Theorem 11.7 be valid.

1) Note that if one of the following conditions

(i) $\psi \circ \varphi^{-1}$ is convex and ψ^{-1} is operator monotone,

(i') $\psi \circ \varphi^{-1}$ is concave and $-\psi^{-1}$ is operator monotone,

is satisfied, then the following obvious inequality

$$\mathscr{M}_\varphi(\beta, \mathbf{A}, \mathbf{\Phi}, n_1+1, n) \le \psi^{-1}\left(\frac{1}{\beta}\sum_{i=n_1+1}^{n} \Phi_i(\psi(A_i)) - \delta_\varphi \widetilde{A}_\beta\right) \le \mathscr{M}_\psi(\beta, \mathbf{A}, \mathbf{\Phi}, n_1+1, n)$$

holds, (see Remark 11.2), where $\delta_\varphi = \varphi(\bar{m}) + \varphi(\bar{M}) - 2\varphi\left(\frac{\bar{m}+\bar{M}}{2}\right)$,

$$\widetilde{A}_\beta = \frac{1}{2}1_K - \frac{1}{\bar{M}-\bar{m}}\left|\frac{1}{\beta}\sum_{i=n_1+1}^{n} \Phi_i A_i - \frac{\bar{m}+\bar{M}}{2}1_K\right|$$

and $\bar{m} \in [m, m_L]$, $\bar{M} \in [M_R, M]$, $\bar{m} < \bar{M}$, are arbitrary numbers.

2) We denote by m_φ and M_φ the bounds of $\mathscr{M}_\varphi(1, \mathbf{A}, \mathbf{\Phi}, 1, n)$. If $(m_\varphi, M_\varphi) \cap [m_i, M_i] = \emptyset$, $i = 1, \ldots, n_1$, and one of two following conditions

(i) $\psi \circ \varphi^{-1}$ is convex and ψ^{-1} is operator monotone

(ii) $\psi \circ \varphi^{-1}$ is concave and $-\psi^{-1}$ is operator monotone is satisfied, then the double inequality (11.52) can be extended from the left side as follows:

$$\begin{aligned}
&\mathscr{M}_{\varphi}(1,\mathbf{A},\mathbf{\Phi},1,n)=\mathscr{M}_{\varphi}(1,\mathbf{A},\mathbf{\Phi},1,n_1)\le\psi^{-1}\left(\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i(f(A_i))-\delta_{\varphi,\psi}\widetilde{A}_{\alpha}\right)\\
&\le\mathscr{M}_{\psi}(\alpha,\mathbf{A},\mathbf{\Phi},1,n_1)\le\psi^{-1}\left(\frac{1}{\alpha}\sum_{i=1}^{n_1}\Phi_i\left(\psi(A_i)\right)+\beta\delta_{\varphi,\psi}\widetilde{A}_{\varphi,n_1,\alpha}\right)\le\mathscr{M}_{\psi}(1,\mathbf{A},\mathbf{\Phi},1,n)\\
&\le\psi^{-1}\left(\frac{1}{\beta}\sum_{i=n_1+1}^{n}\Phi_i\left(\psi(A_i)\right)-\alpha\delta_{\varphi,\psi}\widetilde{A}_{\varphi,n_1,\alpha}\right)\le\mathscr{M}_{\psi}(\beta,\mathbf{A},\mathbf{\Phi},n_1+1,n),
\end{aligned}$$

where $\delta_{\varphi,\psi}$ and $\widetilde{A}_{\varphi,n_1,\alpha}$ are defined by (11.50),

$$\widetilde{A}_{\alpha}=\frac{1}{2}1_K-\frac{1}{\bar{M}-\bar{m}}\left|\frac{1}{\alpha}\sum_{i=n_1+1}^{n}\Phi_iA_i-\frac{\bar{m}+\bar{M}}{2}1_K\right|.$$

As a special case of the quasi-arithmetic mean (11.49), we can study the weight power mean as follows. For a subset $\{A_{p_1},\dots,A_{p_2}\}$ of $\{A_1,\dots,A_n\}$ we define this mean by

$$M^{[r]}(\gamma,\mathbf{A},\mathbf{\Phi},p_1,p_2)=\begin{cases}\left(\dfrac{1}{\gamma}\displaystyle\sum_{i=p_1}^{p_2}\Phi_i\left(A_i^r\right)\right)^{1/r}, & r\in\mathbb{R}\backslash\{0\},\\ \exp\left(\dfrac{1}{\gamma}\displaystyle\sum_{i=p_1}^{p_2}\Phi_i\left(\ln\left(A_i\right)\right)\right), & r=0,\end{cases}$$

where $(A_{p_1},\dots,A_{p_2})$ are strictly positive operators, $(\Phi_{p_1},\dots,\Phi_{p_2})$ are positive linear mappings $\Phi_i:\mathscr{B}(H)\to\mathscr{B}(K)$ such that $\sum_{i=p_1}^{p_2}\Phi_i(1_H)=\gamma 1_K$.

Under the same conditions, we introduce for convenience denotations as special cases of (11.50), as follows:

$$\begin{aligned}
\delta_{r,s}(m,M)&=\begin{cases}m^s+M^s-2\left(\frac{m^r+M^r}{2}\right)^{s/r}, & r\neq 0,\\ m^s+M^s-2\left(mM\right)^{s/2}, & r=0,\end{cases}\\
\widetilde{A}_r(m,M)&=\begin{cases}\frac{1}{2}1_K-\frac{1}{|M^r-m^r|}\left|\sum_{i=1}^{n}\Phi_i(A_i^r)-\frac{M^r+m^r}{2}1_K\right|, & r\neq 0,\\ \frac{1}{2}1_K-\left|\ln\left(\frac{M}{m}\right)\right|^{-1}\left|\sum_{i=1}^{n}\Phi_i(\ln A_i)-\ln\sqrt{Mm}1_K\right|, & r=0,\end{cases}
\end{aligned}\tag{11.57}$$

where $m,M\in\mathbb{R}$, $0<m<M$ and $r,s\in\mathbb{R}$, $r\le s$. We include implicitly that $\widetilde{A}_r(m,M)\equiv\widetilde{A}_{r,A}(m,M)$, where $A=\sum_{i=1}^{n}\Phi_i(A_i^r)$ for $r\neq 0$ and $A=\sum_{i=1}^{n}\Phi_i(\ln A_i)$ for $r=0$.

We obtain the following corollary by applying Theorem 11.7 to the above mean. This is an extension of Corollary 11.8 and a refinement of [138, Corollary 3.4].

Corollary 11.11 *Let $(A_1,\dots,A_n)$ be an n-tuple of self-adjoint operators $A_i\in\mathscr{B}(H)$ with the bounds m_i and M_i, $m_i\le M_i$, $i=1,\dots,n$. Let $(\Phi_1,\dots,\Phi_n)$ be an n-tuple of positive* *linear mappings* $\Phi_i : \mathscr{B}(H) \to \mathscr{B}(K)$, *such that* $\sum_{i=1}^{n_1} \Phi_i(1_H) = \alpha\, 1_K$, $\sum_{i=n_1+1}^{n} \Phi_i(1_H) = \beta\, 1_K$, *where* $1 \le n_1 < n$, $\alpha, \beta > 0$ *and* $\alpha + \beta = 1$. *Let*

$$(m_L, M_R) \cap [m_i, M_i] = \emptyset \quad \text{for} \quad i = n_1+1, \ldots, n, \qquad m < M,$$

where $m_L = \min\{m_1, \ldots, m_{n_1}\}$, $M_R = \max\{M_1, \ldots, M_{n_1}\}$ *and*

$$m = \begin{cases} m_L, & \text{if } \{M_i \colon M_i \le m_L, i \in \{n_1+1, \ldots, n\}\} = \emptyset, \\ \max\{M_i \colon M_i \le m_L, i \in \{n_1+1, \ldots, n\}\}, & \text{otherwise,} \end{cases}$$
$$M = \begin{cases} M_R, & \text{if } \{m_i \colon m_i \ge M_R, i \in \{n_1+1, \ldots, n\}\} = \emptyset, \\ \min\{m_i \colon m_i \ge M_R, i \in \{n_1+1, \ldots, n\}\}, & \text{otherwise.} \end{cases}$$

(i) *If either* $r \le s$, $s \ge 1$ *or* $r \le s \le -1$ *and also one of two equalities*

$$\mathscr{M}^{[r]}(\alpha, \mathbf{A}, \mathbf{\Phi}, 1, n_1) = \mathscr{M}^{[r]}(1, \mathbf{A}, \mathbf{\Phi}, 1, n) = \mathscr{M}^{[r]}(\beta, \mathbf{A}, \mathbf{\Phi}, n_1+1, n)$$

is valid, then

$$\begin{aligned} \mathscr{M}^{[s]}(\alpha, \mathbf{A}, \mathbf{\Phi}, 1, n_1) &\le \left(\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(A_i^s) + \beta \delta_{r,s} \widetilde{A}_{s,n_1,\alpha} \right)^{1/s} \le \mathscr{M}^{[s]}(1, \mathbf{A}, \mathbf{\Phi}, 1, n) \\ &\le \left(\frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(A_i^s) - \alpha \delta_{r,s} \widetilde{A}_{s,n_1,\alpha} \right)^{1/s} \le \mathscr{M}^{[s]}(\beta, \mathbf{A}, \mathbf{\Phi}, n_1+1, n) \end{aligned}$$

holds, where $\delta_{r,s} \ge 0$ *and* $\widetilde{A}_{s,n_1,\alpha} \ge 0$.

In this case, we assume that $\delta_{r,s} \equiv \delta_{r,s}(\bar{m}, \bar{M})$, $\widetilde{A}_{s,n_1,\alpha} \equiv \widetilde{A}_{s,n_1,\alpha}(\bar{m}, \bar{M})$ *are defined by* (11.57) *and* $\bar{m} \in [m, m_L]$, $\bar{M} \in [M_R, M]$, $\bar{m} < \bar{M}$, *are arbitrary numbers.*

(ii) *If either* $r \le s$, $r \le -1$ *or* $1 \le r \le s$ *and also one of two equalities*

$$\mathscr{M}^{[s]}(\alpha, \mathbf{A}, \mathbf{\Phi}, 1, n_1) = \mathscr{M}^{[s]}(1, \mathbf{A}, \mathbf{\Phi}, 1, n) = \mathscr{M}^{[s]}(\beta, \mathbf{A}, \mathbf{\Phi}, n_1+1, n)$$

is valid, then

$$\begin{aligned} \mathscr{M}^{[r]}(\alpha, \mathbf{A}, \mathbf{\Phi}, 1, n_1) &\ge \left(\frac{1}{\alpha} \sum_{i=1}^{n_1} \Phi_i(A_i^r) + \beta \delta_{s,r} \widetilde{A}_{r,n_1,\alpha} \right)^{1/r} \ge \mathscr{M}^{[r]}(1, \mathbf{A}, \mathbf{\Phi}, 1, n) \\ &\ge \left(\frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(A_i^r) - \alpha \delta_{s,r} \widetilde{A}_{r,n_1,\alpha} \right)^{1/r} \ge \mathscr{M}^{[r]}(\beta, \mathbf{A}, \mathbf{\Phi}, n_1+1, n) \end{aligned}$$

holds, where $\delta_{s,r} \le 0$ *and* $\widetilde{A}_{s,n_1,\alpha} \ge 0$.

In this case, we assume that $\delta_{s,r} \equiv \delta_{s,r}(\bar{m}, \bar{M})$, $\widetilde{A}_{r,n_1,\alpha} \equiv \widetilde{A}_{r,n_1,\alpha}(\bar{m}, \bar{M})$ *are defined by* (11.57) *and* $\bar{m} \in [m, m_L]$, $\bar{M} \in [M_R, M]$, $\bar{m} < \bar{M}$, *are arbitrary numbers.*

Proof. In the case (i) we put $\psi(t) = t^s$ and $\varphi(t) = t^r$ if $r \ne 0$ or $\varphi(t) = \ln t$ if $r \ne 0$ in Theorem 11.7. In the case (ii) we put $\psi(t) = t^r$ and $\varphi(t) = t^s$ if $s \ne 0$ or $\varphi(t) = \ln t$ if $s \ne 0$. The details are here omitted. □

11.3 Converses of Jensen's operator inequality

In the sequel, converses of a generalized Jensen's inequality for a continuous field of self-adjoint operators, a unital field of positive linear mappings and real valued continuous convex functions are studied. New refined converses are presented by using the Mond-Pečarić method improvement. Obtained results are then applied in order to refine some selected inequalities that include power functions.

We firstly recall some definitions needed in the sequel. Let T be a locally compact Hausdorff space and let $\mathscr{A}$ be a C^*-algebra of operators on some Hilbert space H. We say that a field $(x_t)_{t\in T}$ of operators in $\mathscr{A}$ is continuous if the function $t \mapsto x_t$ is norm continuous on T. If in addition μ is a Radon measure on T and the function $t \mapsto \|x_t\|$ is integrable, then we can form *the Bochner integral* $\int_T x_t \, d\mu(t)$, which is the unique element in $\mathscr{A}$ such that

$$\varphi\left(\int_T x_t \, d\mu(t)\right) = \int_T \varphi(x_t) \, d\mu(t),$$

for every linear functional φ in the norm dual $\mathscr{A}^*$.

Assume further that there is a field $(\phi_t)_{t\in T}$ of positive linear mappings $\phi_t : \mathscr{A} \to \mathscr{B}$ from $\mathscr{A}$ to another $\mathscr{C}^*$-algebra $\mathscr{B}$ of operators on a Hilbert space K. We recall that a linear mapping $\phi_t : \mathscr{A} \to \mathscr{B}$ is said to be a positive mapping if $\phi_t(x_t) \geq 0$, for all $x_t \geq 0$. We say that such a field is continuous if the function $t \mapsto \phi_t(x)$ is continuous for every $x \in \mathscr{A}$. Let the $\mathscr{C}^*$-algebras include the identity operators and the function $t \mapsto \phi_t(1_H)$ be integrable with $\int_T \phi_t(1_H) \, d\mu(t) = k1_K$, for some positive scalar k. Specially, if $\int_T \phi_t(1_H) \, d\mu(t) = 1_K$, we say that a field $(\phi_t)_{t\in T}$ is unital.

Let f be an operator convex function defined on an interval I. Davis [56] proved the Schwarz inequality

$$f(\phi(x)) \leq \phi(f(x)), \tag{11.58}$$

where $\phi \colon \mathscr{A} \to \mathscr{B}(K)$ is a unital completely positive linear mapping from a C^*-algebra $\mathscr{A}$ to linear operators on a Hilbert space K, and x is a self-adjoint element in $\mathscr{A}$ with its spectrum in I. Subsequently, Choi [50] noted that it is enough to assume that ϕ is unital and positive.

The authors in [156]–[160] and [74] observed converses of Jensen's inequality. In order to present these results, we introduce some abbreviations. Let $f : [m,M] \to \mathbb{R}$, $m < M$. Then a linear function through $(m, f(m))$ and $(M, f(M))$ has the form $h(z) = k_f z + l_f$, where

$$k_f := \frac{f(M) - f(m)}{M - m} \qquad \text{and} \qquad l_f := \frac{Mf(m) - mf(M)}{M - m}. \tag{11.59}$$

Using the Mond-Pečarić method, in [146] the following generalized converse of a Schwarz inequality (11.58) is presented:

$$F\left[\phi\left(f(A)\right), g\left(\phi(A)\right)\right] \leq \max_{m\leq z\leq M} F\left[k_f z + l_f, g(z)\right] 1_{\tilde{n}}, \tag{11.60}$$

for convex functions f defined on an interval $[m,M]$, $m<M$, where g is a real valued continuous function on $[m,M]$, $F(u,v)$ is a real valued function defined on $U\times V$, matrix non-decreasing in u, $U\supset f[m,M]$, $V\supset g[m,M]$, $\phi : H_n\to H_{\tilde{n}}$ is a unital positive linear mapping and A is a Hermitian matrix with its spectrum contained in $[m,M]$.

A continuous version of (11.60) for operators and in the case of $\int_T \phi_t(1_H)\,d\mu(t)=k1_K$, for some positive scalar k, was presented in [145]. Recently, Mićić, Pavić and Pečarić [138] obtained a better bound than the one given in (11.60), as is cited below.

Theorem 11.8 *Let $(x_t)_{t\in T}$ be a bounded continuous field of self-adjoint elements in a unital C^*-algebra $\mathscr{A}$ with the spectra in $[m,M]$, $m<M$, defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ, and let $(\phi_t)_{t\in T}$ be a unital field of positive linear maps $\phi_t:\mathscr{A}\to\mathscr{B}$ from $\mathscr{A}$ to another unital C^*-algebra $\mathscr{B}$. Let m_x and M_x, $m_x\le M_x$, be the bounds of the self-adjoint operator $x=\int_T\phi_t(x_t)\,d\mu(t)$ and $f\colon[m,M]\to\mathbb{R}$, $g\colon[m_x,M_x]\to\mathbb{R}$, $F\colon U\times V\to\mathbb{R}$, where $f([m,M])\subseteq U$, $g([m_x,M_x])\subseteq V$ and F be bounded.*

If f is convex and F is operator monotone in the first variable, then

$$F\left[\int_T\phi_t(f(x_t))\,d\mu(t)\,,\,g(\int_T\phi_t(x_t)\,d\mu(t))\right]\le C_1\,1_K\le C\,1_K,\tag{11.61}$$

where constants $C_1\equiv C_1(F,f,g,m,M,m_x,M_x)$ and $C\equiv C(F,f,g,m,M)$ are

$$\begin{aligned}C_1 &= \sup_{m_x\le z\le M_x} F\left[k_f z+l_f, g(z)\right]\\ &= \sup_{\frac{M-M_x}{M-m}\le p\le\frac{M-m_x}{M-m}} F[pf(m)+(1-p)f(M)\,,\,g(pm+(1-p)M)],\\ C &= \sup_{m\le z\le M} F\left[k_f z+l_f, g(z)\right]\\ &= \sup_{0\le p\le 1} F[pf(m)+(1-p)f(M)\,,\,g(pm+(1-p)M)].\end{aligned}$$

If f is concave, then the reverse inequalities are valid in (11.61) *with* inf *instead of* sup *in bounds C_1 and C.*

In the sequel, we assume that $(x_t)_{t\in T}$ is a bounded continuous field of self-adjoint elements in a unital C^*-algebra $\mathscr{A}$ defined on a locally compact Hausdorff space T equipped with a bounded Radon measure μ and $(\phi_t)_{t\in T}$ is a unital field of positive linear mappings $\phi_t:\mathscr{A}\to\mathscr{B}$ from $\mathscr{A}$ to another unital C^*-algebra $\mathscr{B}$.

For convenience, we introduce abbreviations $\widetilde{x}$ and δ_f as follows:

$$\widetilde{x}\equiv\widetilde{x}_{x_t,\phi_t}(m,M):=\frac{1}{2}1_K-\frac{1}{M-m}\int_T\phi_t\left(\left|x_t-\frac{m+M}{2}1_H\right|\right)d\mu(t),\tag{11.62}$$

where m,M, $m<M$, are some scalars such that the spectra of x_t, $t\in T$, are in $[m,M]$;

$$\delta_f\equiv\delta_f(m,M):=f(m)+f(M)-2f\left(\frac{m+M}{2}\right),\tag{11.63}$$

where $f : [m,M] \to \mathbb{R}$ is a continuous function.

We remark that $m1_H \le x_t \le M1_H$, $t \in T$ implies $\int_T \phi_t\left(\left|x_t - \frac{m+M}{2}1_H\right|\right) d\mu(t) \le \frac{M-m}{2}1_K$. It follows that $\widetilde{x} \ge 0$. Also, if f is *convex* (resp. *concave*), then easily follows $\delta_f \ge 0$ (resp. $\delta_f \le 0$).

In order to prove our main result related to the converse Jensen's inequality, we again make use of Lemma 1.2, which has already served as our main tool throughout the second part of this monograph.

What follows is the main result. We use the Mond-Pečarić method improvement.

Lemma 11.2 *Let $(x_t)_{t\in T}$ and $(\phi_t)_{t\in T}$ be as above. If the spectra of x_t, $t \in T$ are in $[m,M]$, for some scalars $m < M$, then*

$$\int_T \phi_t(f(x_t))\,d\mu(t) \le k_f \int_T \phi_t(x_t)\,d\mu(t) + l_f 1_K - \delta_f \widetilde{x} \le k_f \int_T \phi_t(x_t)\,d\mu(t) + l_f 1_K, \quad (11.64)$$

for every continuous convex function $f\colon [m,M] \to \mathbb{R}$, where $\widetilde{x}$ and δ_f are defined by (11.62) *and* (11.63)*, respectively.*

If f is concave, then the reverse inequality is valid in (11.64)*.*

Proof. We prove the convex case only. By using Lemma 1.2 we get

$$\begin{aligned} & f\left(p_1 m + p_2 M\right) \\ \le\ & p_1 f(m) + p_2 f(M) - \min\{p_1,p_2\}\left[f(m)+f(M)-2f\left(\frac{m+M}{2}\right)\right], \end{aligned} \quad (11.65)$$

for every $p_1, p_2 \in [0,1]$, such that $p_1 + p_2 = 1$. Let functions $p_1, p_2\colon [m,M] \to \mathbb{R}$ be defined by

$$p_1(z) = \frac{M-z}{M-m}, \qquad p_2(z) = \frac{z-m}{M-m}.$$

Then for any $z \in [m,M]$ we can write

$$f(z) = f\left(\frac{M-z}{M-m}m + \frac{z-m}{M-m}M\right) = f\left(p_1(z)m + p_2(z)M\right).$$

By (11.65) we get

$$f(z) \le \frac{M-z}{M-m}f(m) + \frac{z-m}{M-m}f(M) - \tilde{z}\left[f(m)+f(M)-2f\left(\frac{m+M}{2}\right)\right], \quad (11.66)$$

where

$$\tilde{z} = \frac{1}{2} - \frac{1}{M-m}\left|z - \frac{m+M}{2}\right|,$$

since

$$\min\left\{\frac{M-z}{M-m}, \frac{z-m}{M-m}\right\} = \frac{1}{2} - \frac{1}{M-m}\left|z - \frac{m+M}{2}\right|.$$

Now, we use the following properties of a functional calculus for a self-adjoint operator x_t: $f,g \in \mathscr{C}([m,M]), Sp(x_t) \subseteq [m,M]$ and $f \le g$ on $[m,M]$ implies $f(x_t) \le g(x_t)$; $h(z) = |z|$ implies $h(x_t) = |x_t|$, $t \in T$. By using (11.66) we obtain

$$f(x_t) \le \frac{M - x_t}{M-m} f(m) + \frac{x_t - m}{M-m} f(M) - \widetilde{x}_t \left[f(m) + f(M) - 2f\left(\frac{m+M}{2}\right)\right],$$

where

$$\widetilde{x}_t = \frac{1}{2} 1_H - \frac{1}{M-m}\left| x_t - \frac{m+M}{2} 1_H \right|.$$

Applying a positive linear mapping ϕ_t, integrating and using $\int_T \phi_t(1_H)\,d\mu(t) = 1_K$, we get the first inequality in (11.64), since

$$\widetilde{x} = \int_T \phi_t(\widetilde{x}_t)\,d\mu(t) = \frac{1}{2} 1_K - \frac{1}{M-m} \int_T \phi_t\left(\left| x_t - \frac{m+M}{2} 1_H \right| \right) d\mu(t).$$

The fact that $\delta_f \widetilde{x} \ge 0$ yields the second inequality in (11.64). □

At this point, Lemma 11.2 may provide the refinements of some other, previously mentioned inequalities. In the first place, we present a refinement of Theorem 11.8.

Theorem 11.9 *Let the assumptions be as in Lemma 11.2. Let m_x and M_x, $m_x \le M_x$ be the bounds of the operator $x = \int_T \phi_t(x_t)\,d\mu(t)$ and $m_{\widetilde{x}}$ be the lower bound of the operator $\widetilde{x}$.*

If f is convex and F is operator monotone in the first variable, then

$$\begin{aligned} &F\left[\int_T \phi_t(f(x_t))\,d\mu(t),\, g\left(\int_T \phi_t(x_t)\,d\mu(t)\right)\right] \\ &\le \sup_{m_x \le z \le M_x} F\left[k_f z + l_f - \delta_f m_{\widetilde{x}}, g(z)\right] 1_K \le \sup_{m_x \le z \le M_x} F\left[k_f z + l_f, g(z)\right] 1_K. \end{aligned} \tag{11.67}$$

If f is concave, then the reverse inequality is valid in (11.67) *with* inf *instead of* sup.

Proof. We only prove the case when f is convex. Then $\delta_f \ge 0$ implies $0 \le \delta_f m_{\widetilde{x}} 1_K \le \delta_f \widetilde{x}$. By using (11.64) it follows that

$$\begin{aligned} &\int_T \phi_t(f(x_t))\,d\mu(t) \le k_f \int_T \phi_t(x_t)\,d\mu(t) + l_f - \delta_f \widetilde{x} \\ &\le k_f \int_T \phi_t(x_t)\,d\mu(t) + l_f - \delta_f m_{\widetilde{x}} 1_K \le k_f \int_T \phi_t(x_t)\,d\mu(t) + l_f. \end{aligned}$$

Taking into account operator monotonicity of $F(\cdot, v)$ in the first variable, we obtain (11.67). □

11.3.1 Difference type converse inequalities

By using Jensen's inequality, we obtain that

$$\alpha g\left(\int_T \phi_t(x_t) d\mu(t)\right) \le \int_T \phi_t(f(x_t))\,d\mu(t) \tag{11.68}$$

holds for every operator convex function f on $[m,M]$, every function g and real number α such that $\alpha g \leq f$ on $[m,M]$. Further, applying Lemma 11.2 we obtain the following converse of (11.68). It is also a refinement of [138, Theorem 3.1].

Theorem 11.10 *Let the assumptions be as in Lemma 11.2. Let m_x and M_x, $m_x \leq M_x$, be the bounds of the operator $x = \int_T \phi_t(x_t)\,d\mu(t)$ and $f\colon [m,M] \to \mathbb{R}$, $g\colon [m_x,M_x] \to \mathbb{R}$ be continuous functions.*

If f is convex and $\alpha \in \mathbb{R}$, then

$$\int_T \phi_t(f(x_t))\,d\mu(t) - \alpha g\left(\int_T \phi_t(x_t)\,d\mu(t)\right) \leq \max_{m_x \leq z \leq M_x} \left\{k_f z + l_f - \alpha g(z)\right\} 1_K - \delta_f \widetilde{x}, \tag{11.69}$$

where $\widetilde{x}$ and δ_f are defined by (11.62) *and* (11.63)*, respectively.*

If f is concave, then the reverse inequality with min *instead of* max *is valid in* (11.69)*.*

Proof. We only prove the convex case. By using the first inequality in (11.64), we obtain

$$\begin{aligned}
&\int_T \phi_t(f(x_t))\,d\mu(t) - \alpha g\left(\int_T \phi_t(x_t)\,d\mu(t)\right) \\
&\leq k_f \int_T \phi_t(x_t)\,d\mu(t) + l_f 1_K - \delta_f \widetilde{x} - \alpha g\left(\int_T \phi_t(x_t)\,d\mu(t)\right) \\
&\leq \max_{m_x \leq z \leq M_x} \left\{k_f z + l_f - \alpha g(z)\right\} 1_K - \delta_f \widetilde{x}.
\end{aligned}$$

The function $z \mapsto k_f z + l_f - \alpha g(z)$ is continuous on $[m_x,M_x]$, so the above global extremes exist. □

Remark 11.7 1) We remark that by using (11.69) and Theorem 11.9 the following inequalities:

$$\begin{aligned}
&\int_T \phi_t(f(x_t))\,d\mu(t) - \alpha g\left(\int_T \phi_t(x_t)\,d\mu(t)\right) \\
&\leq \max_{m_x \leq z \leq M_x} \left\{k_f z + l_f - \alpha g(z)\right\} 1_K - \delta_f \widetilde{y} \leq \max_{m_x \leq z \leq M_x} \left\{k_f z + l_f - \alpha g(z)\right\} 1_K
\end{aligned}$$

hold for every convex function f, every $\alpha \in \mathbb{R}$, and $m_{\widetilde{x}} 1_K \leq \widetilde{y} \leq \widetilde{x}$, where $m_{\widetilde{x}}$ is the lower bound of $\widetilde{x}$.

2) According to [138, Corollary 3.2] we can determine the constant in the right hand side of (11.69).

i) Let f be convex. We can determine the bound C_α in

$$\int_T \phi_t(f(x_t))\,d\mu(t) - \alpha g\left(\int_T \phi_t(x_t)\,d\mu(t)\right) \leq C_\alpha 1_K - \delta_f \widetilde{x}$$

more precisely as follows:

- if $\alpha \leq 0$ and g is convex, then

$$C_\alpha = \max\left\{k_f m_x + l_f - \alpha g(m_x),\, k_f M_x + l_f - \alpha g(M_x)\right\}; \tag{11.70}$$

• if $\alpha \le 0$ and g is concave, then

$$C_\alpha = \begin{cases} k_f m_x + l_f - \alpha g(m_x) & \text{if } \alpha g'_-(z) \ge k_f \text{ for every } z \in (m_x, M_x), \\ k_f z_0 + l_f - \alpha g(z_0) & \text{if } \alpha g'_-(z_0) \le k_f \le \alpha g'_+(z_0) \text{ for some } z_0 \in (m_x, M_x), \\ k_f M_x + l_f - \alpha g(M_x) & \text{if } \alpha g'_+(z) \le k_f \text{ for every } z \in (m_x, M_x), \end{cases} \tag{11.71}$$

• if $\alpha \ge 0$ and g is convex, then C_α is defined by (11.71);
• if $\alpha \ge 0$ and g is concave, then C_α is defined by (11.70).
ii) Let f be concave. We can determine the bound c_α in

$$c_\alpha 1_K - \delta_f \widetilde{x} \le \int_T \phi_t(f(x_t))\, d\mu(t) - \alpha g\left(\int_T \phi_t(x_t)\, d\mu(t)\right)$$

more precisely as follows:
• if $\alpha \le 0$ and g is convex, then c_α is equal to the right side in (11.71) with reverse inequality signs;
• if $\alpha \le 0$ and g is concave, then c_α is equal to the right side in (11.70) with min instead of max;
• if $\alpha \ge 0$ and g is convex, then c_α is equal to the right side in (11.70) with min instead of max;
• if $\alpha \ge 0$ and g is concave, then c_α is equal to the right side in (11.71) with reverse inequality signs.

Theorem 11.10 and Remark 11.7-2 applied to functions $f(z) = z^p$ and $g(z) = z^q$ provide the following refinement of [138, Corollary 3.3].

Corollary 11.12 *Let the assumptions be as in Lemma 11.2. Let m_x and M_x, $m_x \le M_x$, be the bounds of the operator $x = \int_T \phi_t(x_t)\, d\mu(t)$ and additionally let operators x_t be strictly positive. Let $\widetilde{x}$ be defined by* (11.62)*.*

(i) *Let $p \in (-\infty, 0] \cup [1, \infty)$. Then*

$$\int_T \phi_t(x_t^p)\, d\mu(t) - \alpha \left(\int_T \phi_t(x_t)\, d\mu(t)\right)^q \le C_\alpha^\star 1_K - \left(m^p + M^p - 2^{1-p}(m+M)^p\right) \widetilde{x},$$

where the bound $C_\alpha^\star$ is determined as follows:
• *if $\alpha \le 0$ and $q \in (-\infty, 0] \cup [1, \infty)$, then*

$$C_\alpha^\star = \max\{k_{t^p} m_x + l_{t^p} - \alpha m_x^q, k_{t^p} M_x + l_{t^p} - \alpha M_x^q\}; \tag{11.72}$$

• *if $\alpha \le 0$ and $q \in (0,1)$, then*

$$C_\alpha^\star = \begin{cases} k_{t^p} m_x + l_{t^p} - \alpha m_x^q & \text{if } (\alpha q / k_{t^p})^{1/(1-q)} \le m_x, \\ l_{t^p} + \alpha(q-1)(\alpha q / k_{t^p})^{q/(1-q)} & \text{if } m_x \le (\alpha q / k_{t^p})^{1/(1-q)} \le M_x, \\ k_{t^p} M_x + l_{t^p} - \alpha M_x^q & \text{if } (\alpha q / k_{t^p})^{1/(1-q)} \ge M_x, \end{cases} \tag{11.73}$$

where $k_{t^p} := (M^p - m^p)/(M-m)$ *and* $l_{t^p} := (Mm^p - mM^p)/(M-m)$ *(i.e. replacing* f *with* z^p *in* (11.59)*);*
- *if* $\alpha \geq 0$ *and* $q \in (-\infty, 0] \cup [1, \infty)$*, then* $C_\alpha^\star$ *is defined by* (11.73)*;*
- *if* $\alpha \geq 0$ *and* $q \in (0,1)$*, then* $C_\alpha^\star$ *is defined by* (11.72)*.*

(ii) *Let* $p \in (0,1)$*. Then*

$$c_\alpha^\star 1_K + \left(2^{1-p}(m+M)^p - m^p - M^p\right)\widetilde{x} \leq \int_T \phi_t(x_t^p)\,d\mu(t) - \alpha\left(\int_T \phi_t(x_t)\,d\mu(t)\right)^q,$$

where the bound $c_\alpha^\star$ *is determined as follows:*
- *if* $\alpha \leq 0$ *and* $q \in (-\infty, 0] \cup [1, \infty)$*, then* $c_\alpha^\star$ *is equal to the right side in* (11.73)*;*
- *if* $\alpha \leq 0$ *and* $q \in (0,1)$*, then* $c_\alpha^\star$ *is equal to the right side in* (11.72) *with* min *instead of* max*;*
- *if* $\alpha \geq 0$ *and* $q \in (-\infty, 0] \cup [1, \infty)$*, then* $c_\alpha^\star$ *is equal to the right side in* (11.72) *with* min *instead of* max*;*
- *if* $\alpha \geq 0$ *and* $q \in (0,1)$*, then* $c_\alpha^\star$ *is equal to the right side in* (11.73)*.*

Using Theorem 11.10 and Remark 11.7 for $g \equiv f$ and $\alpha = 1$ we obtain the following result.

Theorem 11.11 *Let the assumptions be as in Lemma 11.2. Let* m_x *and* M_x*,* $m_x \leq M_x$*, be the bounds of the operator* $x = \int_T \phi_t(x_t)\,d\mu(t)$ *and* $f\colon [m,M] \to \mathbb{R}$ *be a continuous function.*

If f *is convex, then*

$$0 \leq \int_T \phi_t(f(x_t))\,d\mu(t) - f\left(\int_T \phi_t(x_t)\,d\mu(t)\right) \leq \overline{C}1_K - \delta_f\widetilde{x}, \tag{11.74}$$

where $\widetilde{x}$ *and* δ_f *are defined by* (11.62) *and* (11.63)*, respectively, and*

$$\overline{C} = \max_{m_x \leq z \leq M_x} \left\{k_f z + l_f - f(z)\right\}. \tag{11.75}$$

Furthermore, if f *is strictly convex and differentiable, then the bound* $\overline{C}1_K - \delta_f\widetilde{x}$ *satisfies the following condition:*

$$0 \leq \overline{C}1_K - \delta_f\widetilde{x} \leq \left\{f(M) - f(m) - f'(m)(M-m) - \delta_f m_{\widetilde{x}}\right\} 1_K, \tag{11.76}$$

where $m_{\widetilde{x}}$ *is the lower bound of the operator* $\widetilde{x}$*. We can determine more precisely the value* $\overline{C} \equiv \overline{C}(m, M, m_x, M_x, f)$ *in* (11.75)*, as follows:*

$$\overline{C} = k_f z_0 + l_f - f(z_0), \tag{11.77}$$

where

$$z_0 = \begin{cases} m_x & \text{if } f'(m_x) \geq k_f, \\ f'^{-1}\left(k_f\right) & \text{if } f'(m_x) \leq k_f \leq f'(M_x), \\ M_x & \text{if } f'(M_x) \leq k_f. \end{cases} \tag{11.78}$$

In the dual case, when f *is concave, then the reverse inequality is valid in* (11.74) *with* min *instead of* max *in* (11.75)*. Furthermore, if* f *is strictly concave differentiable, then the bound* $\overline{C}1_K - \delta_f\widetilde{x}$ *satisfies the following condition:*

$$\left\{f(M) - f(m) - f'(m)(M-m) - \delta_f m_{\widetilde{x}}\right\} 1_K \leq \overline{C}1_K - \delta_f\widetilde{x} \leq 0.$$

We can determine more precisely the value $\bar{C}$ *in* (11.77)*, with* z_0 *which equals the right side in* (11.78) *with reverse inequality signs.*

Proof. We only prove the right hand side of (11.76). Let the maximum value of a continuous function $z \mapsto k_f z + l_f - f(z)$ on $[m_x, M_x]$ be attained in z_0. Since f is a strictly convex function, it follows that $f(m) - f(z_0) \le f'(m)(m - z_0)$. Then

$$\begin{aligned}\bar{C} &= \max_{m_x \le z \le M_x} \{k_f z + l_f - f(z)\} \\ &= k_f z_0 + l_f - f(z_0) = f(m) - f(z_0) + k_f(z_0 - m) \\ &\le \left(-f'(m) + k_f\right)(z_0 - m) \le \left(-f'(m) + k_f\right)(M - m).\end{aligned}$$

Taking into account that $\delta_f m_{\tilde{x}} 1_K \le \delta_f \tilde{x}$ and the above inequalities, we obtain (11.76). □

Example 11.5 We illustrate examples for matrix case and $T = \{1,2\}$ by putting $f(t) = t^4$, which is convex, but not operator convex. Also, we define mappings $\Phi_1, \Phi_2 : M_3(\mathbb{C}) \to M_2(\mathbb{C})$ as follows: $\Phi_1((a_{ij})_{1\le i,j\le 3}) = \frac{1}{2}(a_{ij})_{1\le i,j\le 2}$, $\Phi_2 = \Phi_1$ (then $\Phi_1(I_3) + \Phi_2(I_3) = I_2$).

I) Firstly, we observe an example without the spectra condition (see Figure 11.3). Then we obtain a refined inequality as in (11.74), but don't have a refined Jensen's inequality.

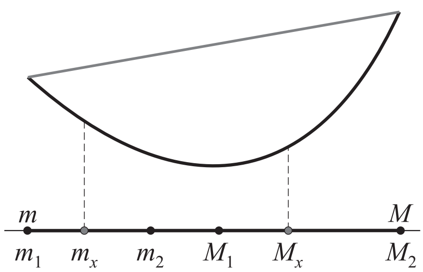

Figure 11.3: *Refinement for two operators and a convex function f*

$$\text{If} \quad X_1 = 2\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad X_2 = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{then } X = 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and $m_1 = -1.604$, $M_1 = 4.494$, $m_2 = 0$, $M_2 = 2$, $m = -1.604$, $M = 4.494$ (rounded to three decimal places). We have

$$(\Phi_1(X_1) + \Phi_2(X_2))^4 = \begin{pmatrix} 16 & 0 \\ 0 & 0 \end{pmatrix} \gneqq\!\!\!\!\!\lneqq \begin{pmatrix} 80 & 40 \\ 40 & 24 \end{pmatrix} = \Phi_1\left(X_1^4\right) + \Phi_2\left(X_2^4\right)$$

and

$$
\begin{aligned}
\Phi_1\left(X_1^4\right)+\Phi_2\left(X_2^4\right) &= \begin{pmatrix} 80 & 40 \\ 40 & 24 \end{pmatrix} \\
< \begin{pmatrix} 111.742 & 39.327 \\ 39.327 & 142.858 \end{pmatrix} &= \Phi_1\left(X_1^4\right)+\Phi_2\left(X_2^4\right)+\overline{C}I_2-\delta_f\widetilde{X} \\
< \begin{pmatrix} 243.758 & 0 \\ 0 & 227.758 \end{pmatrix} &= (\Phi_1(X_1)+\Phi_2(X_2))^4+\overline{C}I_2,
\end{aligned}
$$

since $\overline{C}=227.758$, $\delta_f=405.762$, $\widetilde{X}=\begin{pmatrix} 0.325 & -0.097 \\ -0.097 & 0.2092 \end{pmatrix}$.

II) Next, we observe an example with the spectra condition (see Figure 11.3). Then we obtain a series of inequalities involving the refined Jensen's inequality and its converses.

$$
\text{If } X_1=\begin{pmatrix} -4 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & -1 \end{pmatrix} \text{ and } X_2=\begin{pmatrix} 5 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 3 \end{pmatrix}, \text{ then } X=\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
$$

and $m_1=-4.866$, $M_1=-0.345$, $m_2=1.345$, $M_2=5.866$, $m=-4.866$, $M=5.866$, $a=-0.345$, $b=1.345$ and we put $\overline{m}=a$, $\overline{M}=b$ (rounded to three decimal places). We have

$$
\begin{aligned}
(\Phi_1(X_1)+\Phi_2(X_2))^4 &= \begin{pmatrix} 0.0625 & 0 \\ 0 & 0 \end{pmatrix} \\
< \begin{pmatrix} 639.921 & -255 \\ -255 & 117.856 \end{pmatrix} &= \Phi_1\left(X_1^4\right)+\Phi_2\left(X_2^4\right)-\delta_f(a,b)\overline{X} \\
< \begin{pmatrix} 641.5 & -255 \\ -255 & 118.5 \end{pmatrix} &= \Phi_1\left(X_1^4\right)+\Phi_2\left(X_2^4\right) \\
< \begin{pmatrix} 731.649 & -162.575 \\ -162.575 & 325.15 \end{pmatrix} &= (\Phi_1(X_1)+\Phi_2(X_2))^4+\overline{C}I_2-\delta_f(m,M)\widetilde{X} \\
< \begin{pmatrix} 872.471 & 0 \\ 0 & 872.409 \end{pmatrix} &= (\Phi_1(X_1)+\Phi_2(X_2))^4+\overline{C}I_2,
\end{aligned}
$$

since $\delta_f(a,b)=3.158$, $\overline{X}=\begin{pmatrix} 0.5 & 0 \\ 0 & 0.204 \end{pmatrix}$, $\delta_f(m,M)=1744.82$, $\widetilde{X}=\begin{pmatrix} 0.325 & -0.097 \\ -0.097 & 0.2092 \end{pmatrix}$ and $\overline{C}=872.409$.

Applying Theorem 11.11 to $f(t)=t^p$, we obtain the following refinement of [138, Corollary 3.6].

Corollary 11.13 *Let the assumptions be as in Lemma 11.2. Let m_x and M_x, $m_x\le M_x$, be the bounds of the operator $x=\int_T\phi_t(x_t)\,d\mu(t)$ and additionally let operators x_t be strictly positive. Let $\widetilde{x}$ is defined by* (11.62)*. Then*

$$
\begin{aligned}
0 &\le \int_T \phi_t(x_t^p)\,d\mu(t)-\left(\int_T \phi_t(x_t)\,d\mu(t)\right)^p \\
&\le \overline{C}(m_x,M_x,m,M,p)1_K-\left(m^p+M^p-2^{1-p}(m+M)^p\right)\widetilde{x} \\
&\le \overline{C}(m_x,M_x,m,M,p)1_K\le C(m,M,p)1_K
\end{aligned}
$$

for $p \notin (0,1)$*, and*

$$\begin{aligned} C(m,M,p)1_K &\leq \bar{c}(m_x,M_x,m,M,p)1_K \\ &\leq \bar{c}(m_x,M_x,m,M,p)1_K + \left(2^{1-p}(m+M)^p - m^p - M^p\right)\widetilde{x} \\ &\leq \int_T \phi_t(x_t^p)\,d\mu(t) - \left(\int_T \phi_t(x_t)\,d\mu(t)\right)^p \leq 0 \end{aligned}$$

for $p \in (0,1)$*, where*

$$\bar{C}(m_x,M_x,m,M,p) = \begin{cases} k_{t^p}\, m_x + l_{t^p} - m_x^p & \text{if } pm_x^{p-1} \geq k_{t^p}, \\ C(m,M,p) & \text{if } pm_x^{p-1} \leq k_{t^p} \leq pM_x^{p-1}, \\ k_{t^p}\, M_x + l_{t^p} - M_x^p & \text{if } pM_x^{p-1} \leq k_{t^p}, \end{cases} \tag{11.79}$$

and $\bar{c}(m_x,M_x,m,M,p)$ *equals the right side in* (11.79) *with reverse inequality signs.* $C(m,M,p)$ *is the well known Kantorovich type constant for difference (see e.g. [74, §2.7]):*

$$C(m,M,p) = (p-1)\left(\frac{M^p - m^p}{p(M-m)}\right)^{1/(p-1)} + \frac{Mm^p - mM^p}{M-m}, \qquad \text{for } p \in \mathbb{R}.$$

11.3.2 Ratio type converse inequalities

In [138, Theorem 4.1], the following ratio type converse of (11.68) was given:

$$\int_T \phi_t(f(x_t))\,d\mu(t) \leq \max_{m_x \leq z \leq M_x} \left\{\frac{k_f z + l_f}{g(z)}\right\} g\left(\int_T \phi_t(x_t)\,d\mu(t)\right), \tag{11.80}$$

where f is convex and $g > 0$. Applying Theorem 11.9 and Theorem 11.10, we obtain the following two refinements of (11.80).

Theorem 11.12 *Let the assumptions be as in Lemma 11.2. Let* m_x *and* M_x*,* $m_x \leq M_x$ *be the bounds of the operator* $x = \int_T \phi_t(x_t)\,d\mu(t)$ *and* $f\colon [m,M] \to \mathbb{R}$*,* $g\colon [m_x,M_x] \to \mathbb{R}$ *be continuous functions.*

If f *is convex and* $g > 0$*, then*

$$\int_T \phi_t(f(x_t))\,d\mu(t) \leq \max_{m_x \leq z \leq M_x} \left\{\frac{k_f z + l_f}{g(z)}\right\} g\left(\int_T \phi_t(x_t)\,d\mu(t)\right) - \delta_f \widetilde{x} \tag{11.81}$$

and

$$\int_T \phi_t(f(x_t))\,d\mu(t) \leq \max_{m_x \leq z \leq M_x} \left\{\frac{k_f z + l_f - \delta_f m_{\widetilde{x}}}{g(z)}\right\} g\left(\int_T \phi_t(x_t)\,d\mu(t)\right), \tag{11.82}$$

where $\widetilde{x}$ *and* δ_f *are defined by* (11.62) *and* (11.63) *respectively and* $m_{\widetilde{x}}$ *is the lower bound of the operator* $\widetilde{x}$*. If* f *is concave, then the reverse inequalities are valid in* (11.81) *and* (11.82) *with* min *instead of* max.

Proof. We only prove the convex case. Let

$$\alpha_1 = \max_{m_x \le z \le M_x} \left\{ \frac{k_f z + l_f}{g(z)} \right\}.$$

Then there is $z_0 \in [m_x, M_x]$ such that $\alpha_1 = \frac{k_f z_0 + l_f}{g(z_0)}$ and $\frac{k_f z + l_f}{g(z)} \le \alpha_1$, for all $z \in [m_x, M_x]$. It follows that $k_f z_0 + l_f - \alpha_1 g(z_0) = 0$ and $k_f z + l_f - \alpha_1 g(z) \le 0$, for all $z \in [m_x, M_x]$, since $g > 0$. Hence

$$\max_{m_x \le z \le M_x} \left\{ k_f z + l_f - \alpha_1 g(z) \right\} = 0.$$

By using (11.69) we obtain (11.81). The inequality (11.82) follows directly from Theorem 11.9 by putting $F(u,v) = v^{-1/2} u v^{-1/2}$. Finally, functions $z \mapsto \frac{k_f z + l_f}{g(z)}$ and $z \mapsto \frac{k_f z + l_f - \delta_f m_{\widetilde{x}}}{g(z)}$ are continuous on $[m_x, M_x]$, so the global extremes exist in (11.81) and (11.82). □

Remark 11.8 1) Inequality (11.81) is a refinement of (11.80) since $\delta_f \widetilde{x} \ge 0$. Also, (11.82) is a refinement of (11.80) since $m_{\widetilde{x}} \ge 0$ and $g > 0$ imply

$$\max_{m_x \le z \le M_x} \left\{ \frac{k_f z + l_f - \delta_f m_{\widetilde{x}}}{g(z)} \right\} \le \max_{m_x \le z \le M_x} \left\{ \frac{k_f z + l_f}{g(z)} \right\}.$$

2) Let the assumptions of Theorem 11.12 hold. Generally, there is no relation between the right sides of the inequalities (11.81) and (11.82) under the operator order (see Example 11.3). But, e.g. if $g(\int_T \phi_t(x_t)\, d\mu(t)) \le g(z_0) 1_K$, where $z_0 \in [m_x, M_x]$ is the point where it attains $\max\limits_{m_x \le z \le M_x} \left\{ \frac{k_f z + l_f}{g(z)} \right\}$, then the following order

$$\begin{aligned} \int_T \phi_t(f(x_t))\, d\mu(t) &\le \max_{m_x \le z \le M_x} \left\{ \frac{k_f z + l_f}{g(z)} \right\} g\left(\int_T \phi_t(x_t)\, d\mu(t) \right) - \delta_f \widetilde{x} \\ &\le \max_{m_x \le z \le M_x} \left\{ \frac{k_f z + l_f - \delta_f m_{\widetilde{x}}}{g(z)} \right\} g\left(\int_T \phi_t(x_t)\, d\mu(t) \right) \end{aligned}$$

holds.

Example 11.6 Let $T = \{1,2\}$, $f(t) = g(t) = t^4$, $\Phi_k((a_{ij})_{1 \le i,j \le 3}) = \frac{1}{2}(a_{ij})_{1 \le i,j \le 2}$, $k = 1,2$.

$$\text{If } X_1 = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \text{ and } X_2 = \begin{pmatrix} 5 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 3 \end{pmatrix}, \quad \text{then } X = \begin{pmatrix} 4.5 & 0 \\ 0 & 2 \end{pmatrix}$$

and $m_1 = 0.623$, $M_1 = 4.651$, $m_2 = 1.345$, $M_2 = 5.866$, $m = 0.623$, $M = 5.866$ (rounded to three decimal places). We have

$$\begin{aligned} \Phi_1\left(X_1^4\right) + \Phi_2\left(X_2^4\right) &= \begin{pmatrix} 629.5 & -87.5 \\ -87.5 & 99 \end{pmatrix} \\ < \alpha_1 \left(\Phi_1(X_1) + \Phi_2(X_2)\right)^4 - \delta_f \widetilde{x} &= \begin{pmatrix} 7823.449 & -53.737 \\ -53.737 & 139.768 \end{pmatrix} \\ < \alpha_1 \left(\Phi_1(X_1) + \Phi_2(X_2)\right)^4 &= \begin{pmatrix} 7974.38 & 0 \\ 0 & 311.148 \end{pmatrix}, \end{aligned} \tag{11.83}$$

since $\alpha_1 = 19.447$ (defined as in the right hand side of (11.81)), $\delta_f = 962.73$, $\widetilde{x} = \begin{pmatrix} 0.157 & 0.056 \\ 0.056 & 0.178 \end{pmatrix}$. Further,

$$\begin{aligned} \Phi_1\left(X_1^4\right) + \Phi_2\left(X_2^4\right) &= \begin{pmatrix} 629.5 & -87.5 \\ -87.5 & 99 \end{pmatrix} \\ < \alpha_2\left(\Phi_1(X_1) + \Phi_2(X_2)\right)^4 &= \begin{pmatrix} 5246.13 & 0 \\ 0 & 204.696 \end{pmatrix} \\ < \alpha_1\left(\Phi_1(X_1) + \Phi_2(X_2)\right)^4 &= \begin{pmatrix} 7974.38 & 0 \\ 0 & 311.148 \end{pmatrix}, \end{aligned} \tag{11.84}$$

since $\alpha_2 = 12.794$ (defined as in the right hand side of (11.82)). We see that in this example there is no relation between matrices in the right sides of the equalities (11.83) and (11.84).

Remark 11.9 Similarly as in [138, Corollary 4.2], we can determine the constant in the right hand side of (11.82).

(i) Let f be convex. We can determine the bound C in

$$\int_T \phi_t(f(x_t))\,d\mu(t) \le C g\left(\int_T \phi_t(x_t)\,d\mu(t)\right)$$

more precisely as follows:

- if g is convex, then

$$C_\alpha = \begin{cases} \dfrac{k_f m_x + l_f - \delta_f m_{\widetilde{x}}}{g(m_x)} & \text{if } g'_-(z) \ge \dfrac{k_f g(z)}{k_f z + l_f - \delta_f m_{\widetilde{x}}} \text{ for every } z \in (m_x, M_x), \\ \dfrac{k_f z_0 + l_f - \delta_f m_{\widetilde{x}}}{g(z_0)} & \text{if } g'_-(z_0) \le \dfrac{k_f g(z_0)}{k_f z_0 + l_f - \delta_f m_{\widetilde{x}}} \le g'_+(z_0) \text{ for some } z_0 \in (m_x, M_x), \\ \dfrac{k_f M_x + l_f - \delta_f m_{\widetilde{x}}}{g(M_x)} & \text{if } g'_+(z) \le \dfrac{k_f g(z)}{k_f z + l_f - \delta_f m_{\widetilde{x}}} \text{ for every } z \in (m_x, M_x); \end{cases} \tag{11.85}$$

- if g is concave, then

$$C = \max\left\{ \frac{k_f m_x + l_f - \delta_f m_{\widetilde{x}}}{g(m_x)}, \frac{k_f M_x + l_f - \delta_f m_{\widetilde{x}}}{g(M_x)} \right\}. \tag{11.86}$$

Also, we can determine the bound D in

$$\int_T \phi_t(f(x_t))\,d\mu(t) \le D g\left(\int_T \phi_t(x_t)\,d\mu(t)\right) - \delta_f \widetilde{x}$$

in the same way as the above bound C, but without $m_{\widetilde{x}}$.

(ii) Let f be concave. We can determine the bound c in

$$c g\left(\int_T \phi_t(x_t)\,d\mu(t)\right) \le \int_T \phi_t(f(x_t))\,d\mu(t)$$

more precisely, as follows:

- if g is convex, then c is equal to the right side in (11.86) with min instead of max;
- if g is concave, then c is equal to the right side in (11.85) with reverse inequality signs.

Also, we can determine the bound d in

$$d\, g\left(\int_T \phi_t(x_t)\, d\mu(t)\right) - \delta_f \widetilde{x} \le \int_T \phi_t(f(x_t))\, d\mu(t)$$

in the same way as the above bound c, but without $m_{\widetilde{x}}$.

Theorem 11.12 and Remark 11.9 applied to functions $f(z) = z^p$ and $g(z) = z^q$ provide the following corollary, which is a refinement of [138, Corollary 4.4].

Corollary 11.14 *Let the assumptions be as in Lemma 11.2. Let m_x and M_x, $m_x \le M_x$ be the bounds of the operator $x = \int_T \phi_t(x_t)\, d\mu(t)$ and additionally let operators x_t be strictly positive. Let $\widetilde{x}$ be defined by* (11.62)*, $\delta_p := m^p + M^p - 2^{1-p}(m+M)^p$ and $m_{\widetilde{x}}$ be the lower bound of the operator $\widetilde{x}$.*

(i) *Let $p \in (-\infty, 0] \cup [1, \infty)$. Then*

$$\int_T \phi_t(x_t^p)\, d\mu(t) \le C^\star \left(\int_T \phi_t(x_t)\, d\mu(t)\right)^q,$$

where the bound $C^\star$ is determined as follows:

- *if $q \in (-\infty, 0] \cup [1, \infty)$, then*

$$C^\star = \begin{cases} \dfrac{k_{t^p}\, m_x + l_{t^p} - \delta_p m_{\widetilde{x}}}{m_x^q} & \text{if } \dfrac{q}{1-q}\dfrac{l_{t^p} - \delta_p m_{\widetilde{x}}}{k_{t^p}} \le m_x, \\ \dfrac{l_{t^p} - \delta_p m_{\widetilde{x}}}{1-q}\left(\dfrac{1-q}{q}\dfrac{k_{t^p}}{l_{t^p} - \delta_p m_{\widetilde{x}}}\right)^q & \text{if } m_x \le \dfrac{q}{1-q}\dfrac{l_{t^p} - \delta_p m_{\widetilde{x}}}{k_{t^p}} \le M_x, \\ \dfrac{k_{t^p}\, M_x + l_{t^p} - \delta_p m_{\widetilde{x}}}{M_x^q} & \text{if } \frac{q}{1-q}\frac{l_{t^p} - \delta_p m_{\widetilde{x}}}{k_{t^p}} \ge M_x; \end{cases} \tag{11.87}$$

- *if $q \in (0,1)$, then*

$$C^\star = \max\left\{\frac{k_{t^p}\, m_x + l_{t^p} - \delta_p m_{\widetilde{x}}}{m_x^q}, \frac{k_{t^p} q, M_x + l_{t^p} - \delta_p m_{\widetilde{x}}}{M_x^q}\right\}. \tag{11.88}$$

Also,

$$\int_T \phi_t(x_t^p)\, d\mu(t) \le D^\star \left(\int_T \phi_t(x_t)\, d\mu(t)\right)^q - \delta_p \widetilde{x}$$

holds, where $D^\star$ is determined in the same way as the above bound $C^\star$, but without $m_{\widetilde{x}}$.

(ii) *Let $p \in (0,1)$. Then*

$$c^\star \left(\int_T \phi_t(x_t)\, d\mu(t)\right)^q \le \int_T \phi_t(x_t^p)\, d\mu(t),$$

where the bound $c^\star$ is determined as follows:

- *if $q \in (-\infty, 0] \cup [1, \infty)$, then $c^\star$ is equal to the right side in* (11.88) *with* min *instead of* max*;*

- *if* $q \in (0,1)$*, then* $c_\alpha^\star$ *is equal to the right side in* (11.87)*.*

Also,

$$d^\star \left(\int_T \phi_t(x_t)\,d\mu(t) \right)^q - \delta_p \widetilde{x} \le \int_T \phi_t(x_t^p)\,d\mu(t)$$

holds, where $\delta_p \le 0$*,* $\widetilde{x} \ge 0$ *and* $d^\star$ *is determined in the same way as the above bound* $d^\star$*, but without* $m_{\widetilde{x}}$*.*

Applying Theorem 11.12 and Remark 11.9 for $g \equiv f$, we obtain the following result.

Theorem 11.13 *Let the assumptions be as in Lemma 11.2. Let* m_x *and* M_x*,* $m_x \le M_x$ *be the bounds of the operator* $x = \int_T \phi_t(x_t)\,d\mu(t)$ *and* $f\colon [m,M] \to \mathbb{R}$ *be a continuous function.*

If $f\colon [m,M] \to \mathbb{R}$ *is a continuous convex function and strictly positive on* $[m_x, M_x]$*, then*

$$\int_T \phi_t(f(x_t))\,d\mu(t) \le \max_{m_x \le z \le M_x} \left\{ \frac{k_f z + l_f - \delta_f m_{\widetilde{x}}}{f(z)} \right\} f\left(\int_T \phi_t(x_t)\,d\mu(t) \right) \tag{11.89}$$

and

$$\int_T \phi_t(f(x_t))\,d\mu(t) \le \max_{m_x \le z \le M_x} \left\{ \frac{k_f z + l_f}{f(z)} \right\} f\left(\int_T \phi_t(x_t)\,d\mu(t) \right) - \delta_f \widetilde{x}, \tag{11.90}$$

where $\widetilde{x}$ *and* δ_f *are defined by* (11.62) *and* (11.63)*, respectively, and* $m_{\widetilde{x}}$ *is the lower bound of the operator* $\widetilde{x}$*.*

In the dual case, if f *is concave, then the reverse inequalities are valid in* (11.89) *and* (11.90)*, with* min *instead of* max*.*

Furthermore, if f *is convex and differentiable on* $[m_x, M_x]$*, we can determine the bound*

$$\alpha_1 \equiv \alpha_1(m, M, m_x, M_x, f) = \max_{m_x \le z \le M_x} \left\{ \frac{k_f z + l_f - \delta_f m_{\widetilde{x}}}{f(z)} \right\}$$

in (11.89) *more precisely, as follows:*

$$\alpha_1 = \begin{cases} \dfrac{k_f m_x + l_f - \delta_f m_{\widetilde{x}}}{f(m_x)} & \text{if } f'(z) \ge \dfrac{k_f f(z)}{k_f z + l_f - \delta_f m_{\widetilde{x}}} \text{ for every } z \in (m_x, M_x), \\ \dfrac{k_f z_0 + l_f - \delta_f m_{\widetilde{x}}}{f(z_0)} & \text{if } f'(z_0) = \dfrac{k_f f(z_0)}{k_f z_0 + l_f - \delta_f m_{\widetilde{x}}} \text{ for some } z_0 \in (m_x, M_x), \\ \dfrac{k_f M_x + l_f - \delta_f m_{\widetilde{x}}}{f(M_x)} & \text{if } f'(z) \le \dfrac{k_f f(z)}{k_f z + l_f - \delta_f m_{\widetilde{x}}} \text{ for every } z \in (m_x, M_x). \end{cases} \tag{11.91}$$

Also, if f *is strictly convex and twice differentiable on* $[m_x, M_x]$*, then we can determine the bound*

$$\alpha_2 \equiv \alpha_2(m, M, m_x, M_x, f) = \max_{m_x \le z \le M_x} \left\{ \frac{k_f z + l_f}{f(z)} \right\}$$

in (11.90) *more precisely, as follows:*

$$\alpha_2 = \frac{k_f z_0 + l_f}{f(z_0)}, \tag{11.92}$$

where $z_0 \in (m_x, M_x)$ *is defined as the unique solution of the equation* $k_f f(z) = (k_f z + l_f) f'(z)$ *provided* $(k_f m_x + l_f) f'(m_x)/f(m_x) \le k_f \le (k_f M_x + l_f) f'(M_x)\,/f(M_x)$*, otherwise* z_0 *is defined as* m_x *or* M_x *provided* $k_f \le (k_f m_x + l_f) f'(m_x)\,/f(m_x)$ *or* $k_f \ge (k_f M_x + l_f) f'(M_x)\,/f(M_x)$*, respectively.*

In the dual case, if f *is concave differentiable, then the value* α_1 *is equal to the right side in* (11.91) *with reverse inequality signs. Also, if* f *is strictly concave twice differentiable, then we can determine more precisely the value* α_2 *in* (11.92)*, with* z_0 *which equals the right side in* (11.92) *with reverse inequality signs.*

Proof. The value α_1 follows from Remark 11.9. The value α_2 follows from [138, Corollary 4.7]. □

Remark 11.10 If f is convex and strictly negative on $[m_x, M_x]$, then (11.89) and (11.90) are valid with min instead of max. If f is concave and strictly negative, then the reverse inequalities are valid in (11.89) and (11.90).

Applying Theorem 11.13 to $f(t) = t^p$, we obtain the following refinement of [138, Corollary 4.8].

Corollary 11.15 *Let the assumptions be as in Lemma 11.2. Let* m_x *and* M_x*,* $m_x \le M_x$ *be the bounds of the operator* $x = \int_T \phi_t(x_t)\,d\mu(t)$ *and additionally let operators* x_t *be strictly positive. Let* $\widetilde{x}$ *be defined by* (11.62)*,* $\delta_p := m^p + M^p - 2^{1-p}(m+M)^p$ *and* $m_{\widetilde{x}}$ *be the lower bound of the operator* $\widetilde{x}$*.*

If $p \notin (0,1)$*, then*

$$\begin{aligned} 0 &\le \int_T \phi_t(x_t^p)\,d\mu(t) \le \bar{K}(m_x, M_x, m, M, p, 0)\left(\int_T \phi_t(x_t)\,d\mu(t)\right)^p - \delta_p \\ &\le \bar{K}(m_x, M_x, m, M, p, 0)\left(\int_T \phi_t(x_t)\,d\mu(t)\right)^p \\ &\le K(m, M, p)\left(\int_T \phi_t(x_t)\,d\mu(t)\right)^p \end{aligned} \tag{11.93}$$

and

$$\begin{aligned} 0 &\le \int_T \phi_t(x_t^p)\,d\mu(t) \le \bar{K}(m_x, M_x, m, M, p, m_{\widetilde{x}})\left(\int_T \phi_t(x_t)\,d\mu(t)\right)^p \\ &\le \bar{K}(m_x, M_x, m, M, p, 0)\left(\int_T \phi_t(x_t)\,d\mu(t)\right)^p \\ &\le K(m, M, p)\left(\int_T \phi_t(x_t)\,d\mu(t)\right)^p, \end{aligned} \tag{11.94}$$

where $\bar{K}(m_x, M_x, m, M, p, c)$

$$= \begin{cases} \dfrac{k_{t^p}\, m_x + l_{t^p} - c\delta_p}{m_x^p} & \text{if } \dfrac{p\,(l_{t^p} - c\delta_p)}{m_x} \ge (1-p)\,k_{t^p}, \\ K(m, M, p, c) & \text{if } \dfrac{p\,(l_{t^p} - c\delta_p)}{m_x} < (1-p)\,k_{t^p} < \dfrac{p\,(l_{t^p} - c\delta_p)}{M_x}, \\ \dfrac{k_{t^p}\, M_x + l_{t^p} - c\delta_p}{M_x^p} & \text{if } \dfrac{p\,(l_{t^p} - c\delta_p)}{M_x} \le (1-p)\,k_{t^p}. \end{cases} \tag{11.95}$$

$K(m,M,p,c)$ is a generalization of the well known Kantorovich constant $K(m,M,p) \equiv K(m,M,p,0)$ (defined in [74, §2.7]), as follows

$$K(m,M,p,c) := \frac{mM^p - Mm^p + c\delta_p(M-m)}{(p-1)(M-m)} \left(\frac{p-1}{p} \frac{M^p - m^p}{mM^p - Mm^p + c\delta_p(M-m)} \right)^p, \tag{11.96}$$

for $p \in \mathbb{R}$ and $0 \le c \le 0.5$.

If $p \in (0,1)$, then

$$\begin{aligned}\int_T \phi_t(x_t^p)\,d\mu(t) &\ge \bar{k}(m_x,M_x,m,M,p,0)\left(\int_T \phi_t(x_t)\,d\mu(t)\right)^p - \delta_p\widetilde{x} \\ &\ge \bar{k}(m_x,M_x,m,M,p,0)\left(\int_T \phi_t(x_t)\,d\mu(t)\right)^p \\ &\ge K(m,M,p)\left(\int_T \phi_t(x_t)\,d\mu(t)\right)^p \ge 0\end{aligned}$$

and

$$\begin{aligned}\int_T \phi_t(x_t^p)\,d\mu(t) &\ge \bar{k}(m_x,M_x,m,M,p,m_{\widetilde{x}})\left(\int_T \phi_t(x_t)\,d\mu(t)\right)^p \\ &\ge \bar{k}(m_x,M_x,m,M,p,0)\left(\int_T \phi_t(x_t)\,d\mu(t)\right)^p \\ &\ge K(m,M,p)\left(\int_T \phi_t(x_t)\,d\mu(t)\right)^p \ge 0,\end{aligned}$$

where $\bar{k}(m_x,M_x,m,M,p,c)$ equals the right side in (11.95) *with reverse inequality signs.*

Proof. The second inequalities in (11.93) and (11.94) follow directly from (11.90) and (11.89) by using (11.92) and (11.91), respectively. The last inequality in (11.93) follows from

$$\begin{aligned}\bar{K}(m_x,M_x,m,M,p,0) &= \max_{m_x \le z \le M_x}\left\{\frac{k_{t^p}\,z + l_{t^p}}{z^p}\right\} \\ &\le \max_{m \le z \le M}\left\{\frac{k_{t^p}\,z + l_{t^p}}{z^p}\right\} = K(m,M,p).\end{aligned}$$

The third inequality in (11.94) follows from

$$\begin{aligned}\bar{K}(m_x,M_x,m,M,p,m_{\widetilde{x}}) &= \max_{m_x \le z \le M_x}\left\{\frac{k_{t^p}\,z + l_{t^p} - \delta_p m_{\widetilde{x}}}{z^p}\right\} \\ &\le \bar{K}(m_x,M_x,m,M,p,0),\end{aligned}$$

since $\delta_p m_{\widetilde{x}} \ge 0$, for $p \notin (0,1)$ and $M_x \ge m_x \ge 0$. □

11.3.3 A new generalization of the Kantorovich constant

Definition 11.1 *Let $h > 0$. Further generalization of the Kantorovich constant $K(h,p)$ (given in [74, Definition 2.2]) is defined by*

$$\begin{aligned} K(h,p,c) &:= \frac{h^p - h + c(h^p + 1 - 2^{1-p}(h+1)^p)(h-1)}{(p-1)(h-1)} \\ &\quad \times \left(\frac{p-1}{p} \frac{h^p - 1}{h^p - h + c(h^p + 1 - 2^{1-p}(h+1)^p)(h-1)} \right)^p, \end{aligned}$$

for any real number $p \in \mathbb{R}$ and any $0 \le c \le 0.5$. The constant $K(h,p,c)$ is sometimes briefly denoted by $K(p,c)$.

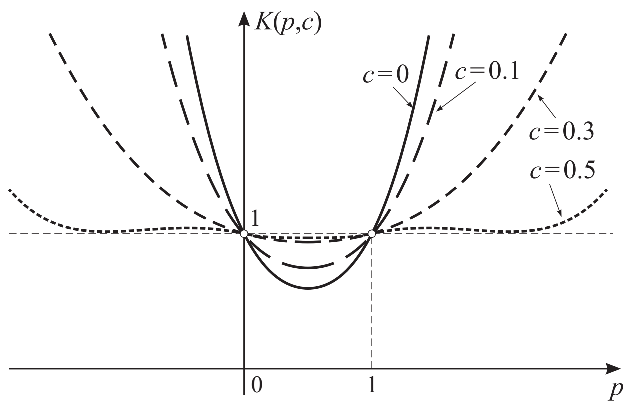

Figure 11.4: *Relation between $K(p,c)$ for $p \in \mathbb{R}$ and $0 \le c \le 0.5$*

By inserting $c = 0$ in $K(h,p,c)$ we obtain the Kantorovich constant $K(h,p)$. The constant $K(m,M,p,c)$ defined by (11.96) coincides with $K(h,p,c)$ when putting $h = M/m > 1$.

Lemma 11.3 *Let $h > 0$. The generalized Kantorovich constant $K(h,p,c)$ has the following properties:*

(i) $K(h,p,c) = K(\frac{1}{h},p,c)$, *for all* $p \in \mathbb{R}$,

(ii) $K(h,0,c) = K(h,1,c) = 1$, *for all* $0 \le c \le 0.5$ *and* $K(1,p,c) = 1$, *for all* $p \in \mathbb{R}$,

(iii) $K(h,p,c)$ *is decreasing in* c *for* $p \notin (0,1)$ *and increasing for* $p \in (0,1)$,

(iv) $K(h,p,c) \ge 1$, *for all* $p \notin (0,1)$ *and* $0 < K(h,0.5,0) \le K(h,p,c) \le 1$, *for all* $p \in (0,1)$,

(v) $K(h,p,c) \le h^{p-1}$, *for all* $p \ge 1$.

Proof. (i): We use an easy calculation:

$$\begin{aligned}
K\left(\frac{1}{h},p,c\right) &= \frac{h^{-p}-h^{-1}+c(h^{-p}+1-2^{1-p}(h^{-1}+1)^p)(h^{-1}-1)}{(p-1)(h^{-1}-1)}\\
&\quad\times\left(\frac{p-1}{p}\frac{h^{-p}-1}{h^{-p}-h^{-1}+c(h^{-p}+1-2^{1-p}(h^{-1}+1)^p)(h^{-1}-1)}\right)^p\\
&= \frac{h-h^p+c(1+h^p-2^{1-p}(h+1)^p)(1-h)}{(p-1)(1-h)}\\
&\quad\times\left(\frac{p-1}{p}\frac{1-h^p}{h-h^p+c(1+h^p-2^{1-p}(h+1)^p)(1-h)}\right)^p\\
&= K(h,p,c).
\end{aligned}$$

(ii): Let $h>1$. The logarithms calculation and the L'Hospital's rule give $K(h,p,b)\to 1$ as $p\to 1$, $K(h,p,b)\to 1$ as $p\to 0$ and $K(h,p,b)\to 1$ as $h\to 1+$. Now, using (i) we obtain (ii).

(iii): Let $h>0$ and $0\le c\le 0.5$. We have:

$$\begin{aligned}
\frac{\mathrm{d}K(h,p,c)}{\mathrm{d}c} &= 2\left(\left(\frac{h+1}{2}\right)^p-\frac{h^p+1}{2}\right)\\
&\quad\times\left(\frac{p-1}{p}\frac{h^p-1}{h-h^p+c(h^p+1-2^{1-p}(h+1)^p)(h-1)}\right)^p.
\end{aligned}$$

Since the function $z\to z^p$ is convex (resp. concave) on $(0,\infty)$ if $p\notin(0,1)$ (resp. $p\in(0,1)$), then $(\frac{h+1}{2})^p\le\frac{h^p+1}{2}$ (resp. $(\frac{h+1}{2})^p\ge\frac{h^p+1}{2}$), for every $h>0$. Then $\frac{\mathrm{d}K(h,p,c)}{\mathrm{d}c}\le 0$ if $p\notin(0,1)$ and $\frac{\mathrm{d}K(h,p,c)}{\mathrm{d}c}\ge 0$ if $p\in(0,1)$, which gives that $K(h,p,c)$ is decreasing in c if $p\notin(0,1)$ and increasing if $p\in(0,1)$.

(iv): Let $h>1$ and $0\le c\le 0.5$. If $p>1$ then

$$\begin{aligned}
0 &< \frac{(p-1)(h-1)}{h^p-h+c(h^p+1-2^{1-p}(h+1)^p)(h-1)}\\
&\le \frac{p-1}{p}\frac{h^p-1}{h^p-h+c(h^p+1-2^{1-p}(h+1)^p)(h-1)}
\end{aligned}$$

implies

$$\begin{aligned}
&\frac{(p-1)(h-1)}{h^p-h+c(h^p+1-2^{1-p}(h+1)^p)(h-1)}\\
&\le\left(\frac{p-1}{p}\frac{h^p-1}{h^p-h+c(h^p+1-2^{1-p}(h+1)^p)(h-1)}\right)^p,
\end{aligned}$$

which gives $K(h,p,c)\ge 1$. Similarly, $K(h,p,c)\ge 1$ if $p<0$ and $K(h,p,c)\le 1$ if $p\in(0,1)$. Hence (iii) and [74, Theorem 2.54 (iv)] yield $K(h,p,c)\ge K(h,p,0)\ge K(h,0.5,0)$, for $p\in(0,1)$.

(v): Let $p\ge 1$. Then (iii) and [74, Theorem 2.54 (vi)] yield $K(h,p,c)\le K(h,p,0)\le h^{p-1}$. □

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