MONOGRAPHS IN INEQUALITIES 12

# General Linear Inequalities and Positivity

*Higher order convexity*

Asif R. Khan, Josip Pečarić, Marjan Praljak and Sanja Varošanec

ELEMENT

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# Preface

The goal of this book is to present recent results on general linear inequalities in discrete and integral form with an emphasis on applications of higher order convexity. Namely, we investigate inequalities containing the sums or integrals of the form $\sum_i p_i f(x_i)$, $\sum_{i,j} p_{ij} f(x_i, y_j)$, $\int p(x) f(x) dx$ or similar for different classes of functions such as convex functions of higher order, $\nabla$-convex functions of higher order, starshaped functions, functions convex at a point etc. Using a concept of higher order convexity introduced by T. Popoviciu seven decades ago, we give necessary and sufficient conditions for positivity of the weighted averages of function values. From the obtained inequalities we construct new linear functionals as the differences of their left-hand and right-hand sides, and study their properties. Corresponding mean value theorems and nontrivial classes of exponentially convex functions are given also.

The book is organized in six chapters.

The first chapter is devoted to results involving sequences. We obtain some identities which are used as the main tool for derivation of general linear inequalities and establishment of conditions under which the sums $\sum_i p_i a_i$ and $\sum_{i,j} p_{ij} a_{ij}$ are nonnegative. The sequences under consideration are convex, convex of higher order, $\nabla$-convex of higher order, starshaped of higher order, mean-convex and mean starshaped sequences. Mostly, we are interested in results involving sequences of various kinds of higher order convexity.

In Chapter 2 we consider discrete convex and convex functions of higher order with one variable. We give inequalities which involve the sum or integral and give identities and inequalities for $n$-convex and $\nabla - n$-convex functions, starshaped and $n$-convex functions at a point. New proofs and generalizations of some known inequalities are presented by using the new tools. The next chapter is devoted to functions with two or more variables which are convex of higher order.

The fourth chapter deals with another class of functions - functions with nondecreasing increments. Again we develop a concept of higher order of the characteristic property. In this case, it is the property "to have nondecreasing increments". By using the obtained results we get Levinson type inequalities and generalizations of Burkill-Mirsky-Pečarić result.

In the fifth chapter we investigate identities and corresponding inequalities involving discrete and integral weighted averages of $n$-convex functions by using of some interpolation formulae. Inequalities which are related to higher order convexity are usually called Popoviciu type inequalities due to the Romanian mathematician Tiberie Popoviciu who defined $n$-convex functions and gave first results of this type. We consider formulae based on the extended Montgomery identity, the Fink identity, the Taylor formula, and on the Lidstone, Hermite and Abel-Gontscharoff interpolation polynomials. Also, in each case we make a corresponding identity which involves the appropriate Green function. Using certain additional conditions we get that the sum $\sum p_i f(x_i)$ (or, analogously, the integral $\int p(x)f(g(x))dx$) is greater than a bound which depends only on the values of all higher order derivatives of the function $f$ at the boundary points $a$ and $b$ of the domain of $f$ and the values of some polynomials in points $a, x_1, \ldots, x_m, b$. From each of the considered identities we construct a linear functional and establish some of its properties. In particular, new families of exponentially convex functions are generated.

In the first part of the sixth chapter we investigate three functionals: the discrete and integral Čebyšev functionals and the Ky Fan functional. All of them involve a function of two variable defined on a square $[a,b] \times [a,b]$, and we find identities which, in general, have four parts. In the integral case, the first part of the formula is a sum of products where one factor is a partial derivative of the function $f$ in the point $(a,a)$ and the second factor depends on the weight. In the second and third parts of the formula partial derivatives of the function $f$ of all orders lower than the maximal one on the edges $(x,a)$ and $(a,x)$, $x \in [a,b]$, appear, respectively, while the fourth part contains only the highest order partial derivative of the function, i.e. the $(N+1, M+1)$-th partial derivative of the function $f$ on the whole square. The next step of investigation is to obtain an inequality for $(N+1, M+1)$-convex functions and, after that, to prove some mean value theorems.

The main motif of investigation in the second part of the sixth chapter is the well-known Montgomery identity for functions with one variable and its generalization for functions with two variables. While the basic Montgomery identity involves only the first derivative of the function under consideration, here we give an identity which involves function of two variables and its partial derivatives of order less than or equal to $N+M+2$. The main formula has a form which is described in the previous paragraph. We also give several estimations based on applications of the Hölder inequality.

Authors

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Chapter 1

# General Linear Inequalities for Sequences

In this chapter we prove several identities for sums $\sum p_k a_k$, $\sum p_{ij} a_i b_j$ involving finite forward or backward differences of higher order. Using these identities we obtain necessary and sufficient conditions under which the above-mentioned sums are nonnegative for different classes of sequences. We consider the classes of convex sequences of higher order, $\nabla$-convex sequences of higher order, starshaped sequences, the class of $p,q$-convex sequences etc.

## 1.1 Convex Sequences of Higher Order

This section is devoted to an identity for the sum $\sum p_k a_k$ and to necessary and sufficient conditions under which this sum is nonnegative for the class of convex sequences of higher order. Let us define and discuss some basic concepts. For a real sequence **a** we usually use notation $(a_i)$ or $(a_i)_{i=k}^{\infty}$ when we want to stress that the first element is $a_k$. Sometimes under the word ”sequence” we mean $n$-tuple also, but it is always clear from the context.

The finite forward difference of a sequence **a** (or, simple, $\Delta$-difference) is defined as

$$\Delta^1 a_i = \Delta a_i := a_{i+1} - a_i,$$

while the difference of order $m$ is defined as

$$\Delta^m a_i := \Delta(\Delta^{m-1} a_i),\ m \in \{2,3,\ldots\}.$$

Similarly, the finite backward difference ($\nabla$-difference) is defined as

$$\nabla^1 a_i = \nabla a_i := a_i - a_{i+1},$$

and the $\nabla$-difference of order $m$ as

$$\nabla^m a_i := \nabla(\nabla^{m-1} a_i).$$

For $m = 0$ we put $\Delta^0 a_i = a_i$, and $\nabla^0 a_i = a_i$. It is easy to see that

$$\Delta^m a_i = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} a_{i+k}.$$

We say that a sequence $\mathbf{a}$ is convex of order $m$ or $m$-convex if

$$\Delta^m a_i \geq 0$$

holds for any $i \in \mathbb{N}$. If $m = 1$, then $\mathbf{a}$ is nondecreasing, while if $m = 2$, then 2-convexity becomes the classical convexity, i.e. the following holds

$$a_{i+2} - 2a_{i+1} + a_i \geq 0,\ \ i \in \mathbb{N}.$$

We say that a sequence $\mathbf{a}$ is $\nabla$-convex of order $m$ if

$$\nabla^m a_i \geq 0$$

holds for any $i \in \mathbb{N}$.

Also, the following notation is frequently used: for some fixed real $a$ and $m \in \mathbb{N}$:

$$a^{(m)} = a(a-1)\cdots(a-m+1), \quad a^{(0)} = 1.$$

In the following Lemma, proved in [61], we give an identity on which all the results of this section are based. It can be observed as a generalization of the well-known Abel identity for an $n$-tuple $(a_1,\ldots,a_n)$ with weights $(p_1,\ldots,p_n)$, [51, p.334], given by

$$\sum_{i=1}^{n} p_i a_i = a_1 \sum_{i=1}^{n} p_i + \sum_{i=2}^{n} \left( \sum_{k=i}^{n} p_k \right) \Delta a_{i-1}. \tag{1.1}$$

The structure of the Abel identity can be described as following: the sum $\sum_{i=1}^{n} p_i a_i$ is represented as a sum of two sums. In the first sum the difference of the order 0 of element $a_1$ occures, while in the second sum the differences of the order 1 for the elements $a_1,\ldots,a_{n-m}$ occur. The Abel identity can be looked upon as a discrete analogue of the formula for integration by parts. The new identity has a similar structure: the right-hand side of it consists of two sums, in the first sum differences of order $0,1,\ldots,m-1$ of the first element $a_1$ appear, while in the second sum only the differences of order $m$ occur but for elements $a_1,\ldots,a_{n-m}$.

**Lemma 1.1** *Let $m,n \in \mathbb{N}$, $m < n$. Let $(p_1,\ldots,p_n)$, $(a_1,\ldots,a_n)$ be real $n$-tuples. Then*

$$\begin{aligned}\sum_{i=1}^{n} p_i a_i &= \sum_{k=0}^{m-1}\sum_{i=1}^{n} p_i (i-1)^{(k)} \frac{\Delta^k a_1}{k!} \\ &+ \sum_{k=m+1}^{n}\left(\sum_{i=k}^{n} p_i (i-k+m-1)^{(m-1)}\right)\frac{\Delta^m a_{k-m}}{(m-1)!}. \qquad (1.2)\end{aligned}$$

*Proof.* We prove it by using mathematical induction on $m$. If $m=1$, then we have

$$\sum_{i=1}^{n} p_i a_i = a_1 \sum_{i=1}^{n} p_i + \sum_{k=2}^{n}\left(\sum_{i=k}^{n} p_i\right)\Delta a_{k-1},$$

which is, in fact, the Abel identity. Suppose that (1.2) is valid. Writting the Abel identity for $(n-m)$-tuple $(\Delta^m a_1, \Delta^m a_2, \ldots, \Delta^m a_{n-m})$ with weights $(Q_{m+1}, Q_{m+2}, \ldots, Q_n)$, where

$$Q_k = \sum_{i=k}^{n} (i-k+m-1)^{(m-1)} p_i$$

we get

$$\sum_{k=m+1}^{n} Q_k \Delta^m a_{k-m} = \Delta^m a_1 \sum_{j=m+1}^{n} Q_j + \sum_{k=m+2}^{n}\left(\sum_{j=k}^{n} Q_j\right)\Delta^{m+1} a_{k-m-1}.$$

The sum $\sum_{j=k}^{n} Q_j$ is equal to

$$\sum_{j=k}^{n} Q_j = \sum_{j=k}^{n}\sum_{i=j}^{n} (i-j+m-1)^{(m-1)} p_i = \frac{1}{m}\sum_{i=k}^{n} (i-k+m)^{(m)} p_i.$$

For $k=m+1$ we have

$$\sum_{j=m+1}^{n} Q_j = \frac{1}{m}\sum_{j=m+1}^{n} (j-1)^{(m)} p_j = \frac{1}{m}\sum_{j=1}^{n} (j-1)^{(m)} p_j,$$

where we use the fact that for $j=1,\ldots,m$ the number $(j-1)^{(m)}$ is equal 0. So, we get

$$\begin{aligned}&\sum_{k=m+1}^{n} Q_k \Delta^m a_{k-m} \\ &= \frac{\Delta^m a_1}{m}\sum_{j=1}^{n} (j-1)^{(m)} p_j + \sum_{k=m+2}^{n}\left(\sum_{j=k}^{n} (j-k+m)^{(m)} p_j\right)\frac{\Delta^{m+1} a_{k-m-1}}{m}. \qquad (1.3)\end{aligned}$$

Let us write the right-hand side of identity (1.2) for $m+1$ instead of $m$:

$$\sum_{k=0}^{m}\sum_{i=1}^{n} p_i (i-1)^{(k)} \frac{\Delta^k a_1}{k!} + \sum_{k=m+2}^{n}\left(\sum_{i=k}^{n} p_i (i-k+m)^{(m)}\right)\frac{\Delta^{m+1} a_{k-m-1}}{m!}$$

$$
\begin{aligned}
&= \left( \sum_{k=0}^{m-1} \sum_{i=1}^{n} p_i (i-1)^{(k)} \frac{\Delta^k a_1}{k!} + \sum_{i=1}^{n} p_i (i-1)^{(m)} \frac{\Delta^m a_1}{m!} \right) \\
&+ \frac{1}{(m-1)!} \left( \sum_{k=m+1}^{n} Q_k \Delta^m a_{k-m} - \Delta^m a_1 \frac{1}{m} \sum_{i=1}^{n} (i-1)^{(m)} p_1 \right) \\
&= \sum_{k=0}^{m-1} \sum_{i=1}^{n} p_i (i-1)^{(k)} \frac{\Delta^k a_1}{k!} + \sum_{k=m+1}^{n} \left( \sum_{i=k}^{n} p_i (i-k+m-1)^{(m-1)} \right) \frac{\Delta^m a_{k-m}}{(m-1)!} \\
&= \sum_{i=1}^{n} p_i a_i,
\end{aligned}
$$

where we use (1.3) and the assumption of induction. So, by the principle of mathematical induction, identity (1.2) holds. □

**Remark 1.1** We use the above identity for $m = n$ also. In that case the second sum vanishes.

The following theorem about $m$-convex sequences is given in [61] by J. Pečarić (see also [77, p. 253]):

**Theorem 1.1** *Let $(p_1, \dots, p_n)$ be a real $n$-tuple and $m \in \mathbb{N}$, $m < n$. The inequality*

$$\sum_{i=1}^{n} p_i a_i \geq 0 \tag{1.4}$$

*holds for every $m$-convex $n$-tuple $(a_i)$ if and only if*

$$\sum_{i=1}^{n} (i-1)^{(k)} p_i = 0 \tag{1.5}$$

*holds for every $k \in \{0, 1, \dots, m-1\}$ and*

$$\sum_{i=k}^{n} (i-k+m-1)^{(m-1)} p_i \geq 0 \tag{1.6}$$

*holds for every $k \in \{m+1, \dots, n\}$.*

*Proof.* If equalities (1.5) and inequalities (1.6) are satisfied, then the first sum in identity (1.2) is equal to 0, the second sum is nonnegative and the inequality $\sum_{i=1}^{n} p_i a_i \geq 0$ holds.

Conversely, let us suppose that $\sum_{i=1}^{n} p_i a_i \geq 0$ holds for any $m$-convex sequence $(a_i)$. Since the sequence $a_i = (i-1)^{(k)}$, $i \in \{1, \dots, n\}$ is $m$-convex for every $k \in \{0, \dots, m-1\}$, we get $\sum_{i=1}^{n} p_i (i-1)^{(k)} \geq 0$. Convexity of the mentioned sequences are proved in Chapter 2 in detail. Similarly, since the sequence $a_i = -(i-1)^{(k)}$, $i \in \{1, \dots, n\}$ is $m$-convex for every $k \in \{0, \dots, m-1\}$, using (1.4) we get $-\sum_{i=1}^{n} p_i (i-1)^{(k)} \geq 0$. Hence, $\sum_{i=1}^{n} p_i (i-1)^{(k)} = 0$.

Also the sequence

$$a_i = \begin{cases} 0, & i \in \{1, \dots, k-1\}, \\ (i-k+m-1)^{(m-1)}, & i \in \{k, \dots, n\}, \end{cases} \tag{1.7}$$

is $m$-convex for every $k \in \{m+1,\dots,n\}$. Thus, by (1.4), we get (1.6). $\square$

**Remark 1.2** It is easy to see that condition (1.5) is equivalent to the following conditions:

$$\sum_{i=1}^{n}(i-1)^k p_i = 0, \quad k \in \{0,1,\dots,m-1\} \text{ with } 0^0 = 1 \tag{1.8}$$

or

$$\sum_{i=1}^{n} i^k p_i = 0, \quad k \in \{0,1,\dots,m-1\}. \tag{1.9}$$

Also, it is instructive to observe that

$$\frac{(i-1)^{(k)}}{k!} = \binom{i-1}{k}, \quad \frac{(i-k+m-1)^{(m-1)}}{(m-1)!} = \binom{i-k+m-1}{m-1}.$$

In the first sum of (1.2) the numbers $(i-1)^{(k)}$ are equal 0 for $i = 1,\dots,k$, so sometimes as a range for $i$ we use $i$ from $k+1$ till $n$.

If an $n$-tuple $(a_i)$ is convex of several consecutive orders we have the following theorem which is a consequence of Theorem 1.1. This result can be found in [71].

**Theorem 1.2** *Let $(p_1,\dots,p_n)$ be a real $n$-tuple and $m \in \mathbb{N}$, $m < n$, $j \in \{1,\dots,m\}$. Then inequality* (1.4) *holds for every $n$-tuple $(a_1,\dots,a_n)$ that is convex of order $j, j+1,\dots,m$ if and only if*

$$\sum_{i=1}^{n}(i-1)^{(k)} p_i = 0 \tag{1.10}$$

*holds for $k \in \{0,1,\dots,j-1\}$,*

$$\sum_{i=1}^{n}(i-1)^{(k)} p_i \geq 0 \tag{1.11}$$

*holds for $k \in \{j, j+1,\dots,m-1\}$ and*

$$\sum_{i=k}^{n}(i-k+m-1)^{(m-1)} p_i \geq 0 \tag{1.12}$$

*holds for $k \in \{m+1,\dots,n\}$.*

*Proof.* If $k \in \{0,1,\dots,j-1\}$, then the sequences $((i-1)^{(k)})_i$ and $(-(i-1)^{(k)})_i$ are convex of order $j, j+1,\dots,m$. So, for such $k$, $\sum_{i=1}^{n}(i-1)^{(k)} p_i = 0$ holds. If $k \in \{j, j+1,\dots,m-1\}$, then the sequence $((i-1)^{(k)})_i$ is convex of order $j, j+1,\dots,m$ and $\sum_{i=1}^{n}(i-1)^{(k)} p_i \geq 0$ for such $k$.

Since the sequence $(a_n)$ defined as in (1.7) is convex of order $j, j+1,\dots,m$, so (1.12) holds. This proves one implication of the theorem while the other follows from Lemma 1.1. $\square$

A sequence $(a_i)$ is called absolutely monotonic of order $m$ if all the lower order differences of that sequence are nonnegative, i.e. if

$$\Delta^k a_i \geq 0 \quad \text{for} \quad k \in \{1, 2, \ldots, m\}.$$

As a consequence of the previous Theorem 1.2 we get the following necessary and sufficient conditions for positivity of sum $\sum p_i a_i$ for an absolutely monotonic sequence of order $m$. Namely, we obtain the following theorem.

**Corollary 1.1** *Let $(p_1, \ldots, p_n)$ be a real $n$-tuple and $m \in \mathbb{N}$, $m < n$. Then inequality* (1.4) *holds for every $n$-tuple $(a_1, \ldots, a_n)$ that is absolutely monotonic of order $m$ if and only if*

$$\sum_{i=1}^{n} p_i = 0, \quad \sum_{i=1}^{n} (i-1)^{(k)} p_i \geq 0$$

*holds for $k \in \{1, \ldots, m-1\}$, and*

$$\sum_{i=k}^{n} (i-k+m-1)^{(m-1)} p_i \geq 0 \;\; \textit{for} \;\; k \in \{m+1, \ldots, n\}.$$

The following theorem describes how bounds for the sum $\sum p_i a_i$ depend on bounds of $\Delta^m a_k$, (see [71]). In fact, using that result we can strengthen the initial inequality.

**Theorem 1.3** *Let $m \in \mathbb{N}$, $m < n$ and $(a_1, \ldots, a_n)$, $(p_1, \ldots, p_n)$ be real $n$-tuples such that*

$$\sum_{i=1}^{n} (i-1)^{(k)} p_i = 0 \;\; \textit{for} \;\; k \in \{0, 1, \ldots, m-1\} \tag{1.13}$$

*and*

$$\sum_{i=k}^{n} (i-k+m-1)^{(m-1)} p_i \geq 0 \;\; \textit{for} \;\; k \in \{m+1, \ldots, n\}. \tag{1.14}$$

*If*

$$a \leq \Delta^m a_k \leq A \;\textit{for}\;\; k \in \{1, 2, \ldots, n-m\}, \tag{1.15}$$

*then*

$$\frac{a}{m!} \sum_{i=1}^{n} p_i i^{(m)} \leq \sum_{i=1}^{n} p_i a_i \leq \frac{A}{m!} \sum_{i=1}^{n} p_i i^{(m)}.$$

*Proof.* The sequences

$$b_k = a_k - \frac{a}{m!} k^{(m)} \quad \text{and} \quad c_k = \frac{A}{m!} k^{(m)} - a_k$$

have the following properties

$$\Delta^m b_k = \Delta^m a_k - a \quad \text{and} \quad \Delta^m c_k = A - \Delta^m a_k.$$

By (1.15), we get that the sequences $(b_k)$ and $(c_k)$ are $m$-convex. Since $(p_k)$ satisfies conditions (1.13) and (1.14), then using Theorem 1.1 we get that

$$\sum_{i=1}^{n} p_i b_i \geq 0 \quad \text{and} \quad \sum_{i=1}^{n} p_i c_i \geq 0$$

and desired inequalities hold. □

**Remark 1.3** For $a = -A$ condition (1.15) becomes $|\Delta^m a_k| \leq A$ and then the statement of the above theorem becomes

$$\left| \sum_{i=1}^{n} p_i a_i \right| \leq \frac{A}{m!} \sum_{i=1}^{n} p_i i^{(m)}.$$

***Example 1.1*** A nice application of Theorem 1.1 is a proof of the Nanson inequality. In [52] E.J. Nanson proved the following inequality: If a real $(2n+1)$-tuple $(a_1, \ldots, a_{2n+1})$ is convex, then

$$\frac{a_1 + a_3 + \ldots + a_{2n+1}}{n+1} \geq \frac{a_2 + a_4 + \ldots + a_{2n}}{n}. \tag{1.16}$$

The original proof of the Nanson inequality (1.16) and some historical remarks are given in $[49, pp.202-203]$. Here we give a proof of (1.16) based on Theorem 1.1.

Putting

$$N = 2n+1,\ p_1 = p_3 = \ldots = p_{2n+1} = \frac{1}{n+1},\ p_2 = p_4 = \ldots = p_{2n} = -\frac{1}{n}$$

we get

$$\sum_{i=1}^{N} p_i = \frac{1}{n+1} - \frac{1}{n} + \ldots + \frac{1}{n+1} - \frac{1}{n} + \frac{1}{n+1} = n\left(\frac{1}{n+1} - \frac{1}{n}\right) + \frac{1}{n+1} = 0,$$

$$\begin{aligned}\sum_{i=1}^{N} (i-1)p_i &= \frac{0}{n+1} - \frac{1}{n} + \frac{2}{n+1} - \frac{3}{n} \ldots + \frac{2n-2}{n+1} - \frac{2n-1}{n} + \frac{2n}{n+1} \\ &= \frac{2+4+\ldots+2n}{n+1} - \frac{1+3+\ldots+2n-1}{n} = \frac{n(n+1)}{n+1} - \frac{n^2}{n} = 0,\end{aligned}$$

and for $k \geq 3$

$$\begin{aligned}&\sum_{i=k}^{N} (i-k+1)p_i = p_k + 2p_{k+1} + 3p_{k+2} + \ldots + (N-k+1)p_N \\ &= \begin{cases} \frac{1}{n+1} + \left(-\frac{2}{n} + \frac{3}{n+1}\right) + \left(-\frac{4}{n} + \frac{5}{n+1}\right) + \ldots \left(-\frac{N-k}{n} + \frac{N-k+1}{n+1}\right), & k \text{ even} \\ \left(-\frac{1}{n} + \frac{2}{n+1}\right) + \left(-\frac{3}{n} + \frac{4}{n+1}\right) + \ldots + \left(-\frac{N-k}{n} + \frac{3}{N-k+1}\right), & k \text{ odd} \end{cases}\end{aligned}$$

$$= \begin{cases} \dfrac{\left(\frac{N-k}{2}+1\right)\left(n-\frac{N-k}{2}\right)}{n(n+1)} \geq 0, \ k \text{ even} \\ \dfrac{1}{n(n+1)} \dfrac{N-k+1}{2}\left(n-\dfrac{N-k+1}{2}\right) \geq 0, \ k \text{ odd.} \end{cases}$$

Applying Theorem 1.1 for $m=2$ we get that $\sum_{i=1}^{N} p_i a_i \geq 0$, i.e.

$$\frac{a_1}{n+1} - \frac{a_2}{n} + \frac{a_3}{n+1} - \frac{a_4}{n} + \ldots \frac{a_{2n-1}}{n+1} - \frac{a_{2n}}{n} + \frac{a_{2n+1}}{n+1} \geq 0$$

which is the desired inequality $(1.16)$.

Let us use Theorem 1.3 to get an estimate for the difference of the left-hand and the right-hand side of the Nanson inequality if the second differences are bounded. This result is proved in [3] using different approach.

Let us suppose that for sequence $(a_i)$ the following holds

$$a \leq \Delta^2 a_k \leq A, \ \ k \in \{1,2,\ldots 2n-1\}$$

for some $a, A \in \mathbf{R}$. Then

$$\frac{2n+1}{6}a \leq \frac{a_1+a_3+\ldots+a_{2n+1}}{n+1} - \frac{a_2+a_4+\ldots+a_{2n}}{n} \leq \frac{2n+1}{6}A. \tag{1.17}$$

From the previous calculation we have that $(1.13)$ holds for $k=0,1$ and $(1.14)$ holds for $k=2$. Let us calculate $\sum_{i=1}^{N} p_i i^{(2)}$.

$$\begin{aligned} \sum_{i=1}^{N} p_i i^{(2)} &= \sum_{i=1}^{N} p_i i^2 - \sum_{i=1}^{N} p_i i = \sum_{i=1}^{N} p_i i^2 \\ &= \frac{1}{n+1}(1^2+3^2+\ldots+(2n+1)^2) + \frac{1}{n}(2^2+4^2+\ldots+(2n)^2) \\ &= \frac{2n+1}{3}. \end{aligned}$$

From that result we get $(1.17)$.

***Example 1.2*** Let us illustrate an application of Theorem 1.1 to another inequality due to N. Ozeki. In [55], and also in [49, $p$.199], the following result is given: If $a_{n-1}+a_{n+1} \geq 2a_n$ for $n=2,3,\ldots$, then

$$A_{n-1}+A_{n+1} \geq 2A_n, \ \ n=2,3,\ldots, \tag{1.18}$$

where

$$A_n = \frac{a_1+\ldots+a_n}{n}.$$

In other words, if a sequence $(a_i)$ is convex, then the sequence $(A_i)$ of arithmetic means is also convex.

Putting

$$p_1 = p_2 = \ldots = p_{n-1} = \frac{1}{n-1} + \frac{1}{n+1} - \frac{2}{n}, \ \ p_n = \frac{1}{n+1} - \frac{2}{n}, \ \ p_{n+1} = \frac{1}{n+1},$$

we get

$$\sum_{i=1}^{n+1} p_i = 0, \ \sum_{i=1}^{n+1} (i-1)p_i = 0, \ \sum_{i=k}^{n+1} (i-k+1)p_i \geq 0.$$

Using Theorem 1.1 for $m = 2$ we get that $\sum_{i=1}^{n+1} p_i a_i \geq 0$, i.e.

$$a_1\left(\frac{1}{n-1}+\frac{1}{n+1}-\frac{2}{n}\right)+\ldots+a_{n-1}\left(\frac{1}{n-1}+\frac{1}{n+1}-\frac{2}{n}\right)$$
$$+a_n\left(\frac{1}{n+1}-\frac{2}{n}\right)+\frac{1}{n+1}a_{n+1} \geq 0,$$
$$\frac{a_1+a_2+\ldots+a_{n-1}}{n-1}+\frac{a_1+a_2+\ldots+a_{n+1}}{n+1}-2\frac{a_1+a_2+\ldots+a_n}{n} \geq 0$$

which is the desired inequality (1.18).

***Example 1.3*** If $(a_i)$ is convex, then for any $n \geq 1$

$$a_1+a_3+\ldots+a_{2n+1} \geq a_2+a_4+\ldots+a_{2n}+\frac{a_1+a_3+\ldots+a_{2n+1}}{n+1}. \tag{1.19}$$

This inequality for $a_i = a$ is due to Steinig ([3, 93]).

To prove this, we use Theorem 1.1 for $m = 2$. Putting

$$N = 2n+1, \ p_1 = p_3 = \ldots = p_{2n+1} = \frac{n}{n+1}, \ p_2 = p_4 = \ldots = p_{2n} = -1$$

we get that property (1.13) holds for $k = 0,1$ and (1.14) holds for $k = 2$. So, by Theorem 1.1 inequality (1.19) holds. Furthermore, if $(a_i)$ satisfies (1.13) for $k = 0,1$, (1.14) for $k = 2$ and if $a \leq \Delta^2 a_k \leq A$ $(k = 1,\ldots,2n-1)$, then

$$\frac{n(2n+1)}{6}a \leq a_1-a_2+a_3-\ldots+a_{2n+1}-\frac{a_1+a_3+\ldots+a_{2n+1}}{n+1} \leq \frac{n(2n+1)}{6}A.$$

Let us again consider a basic identity from Lemma 1.1, with slightly modified indexing in the first sum:

$$\begin{aligned}\sum_{i=1}^{n} p_i a_i &= \sum_{k=1}^{m}\sum_{i=1}^{n} p_i (i-1)^{(k-1)} \frac{\Delta^{k-1}a_1}{(k-1)!} \\ &+ \sum_{k=m+1}^{n}\left(\sum_{i=k}^{n} p_i (i-k+m-1)^{(m-1)}\right)\frac{\Delta^m a_{k-m}}{(m-1)!}.\end{aligned}$$

Putting $p_1 = \ldots = p_{n-1} = 0$ and $p_n = 1$ we obtain the following

$$a_n = \begin{cases} \displaystyle\sum_{k=1}^{m} (n-1)^{(k-1)} \frac{\Delta^{k-1}a_1}{(k-1)!} \\ \qquad \displaystyle + \sum_{k=m+1}^{n} (n-k+m-1)^{(m-1)} \frac{\Delta^m a_{k-m}}{(m-1)!}, & m < n, \\ \displaystyle\sum_{k=1}^{n} (n-1)^{(k-1)} \frac{\Delta^{k-1}a_1}{(k-1)!}, & m = n. \end{cases}$$

The above-mentioned identity can be considered as the Taylor formula for sequences.

The following theorem was published in [62] and it gives results about preservation of convexity of a sequence which is made from a sequence $(a_i)$.

Let $(a_0, a_1, a_2, \ldots)$ be a real sequence and $[p_{n,i}]$, $i = 0, 1, \ldots, n$; $n = 0, 1, 2, \ldots$ a lower triangular matrix of real numbers, i.e.

$$\begin{bmatrix} p_{00} & 0 & 0 & 0 & \ldots & 0 & \ldots \\ p_{10} & p_{11} & 0 & 0 & \ldots & 0 & \ldots \\ p_{20} & p_{21} & p_{22} & 0 & \ldots & 0 & \ldots \\ \vdots & \vdots & \vdots & & & & \\ p_{n0} & p_{n1} & p_{n2} & p_{n3} & \ldots & p_{nn} & 0\ldots \\ \vdots & \vdots & \vdots & & & & \end{bmatrix}.$$

Let $(\sigma_n)$ be a sequence defined as

$$\sigma_n = \sum_{k=0}^{n} p_{n,n-k} a_k, \quad n = 0, 1, 2, \ldots \tag{1.20}$$

**Theorem 1.4** *Let $\sigma_n$ be defined as in (1.20) and $s \in \mathbb{N}$. Then the implication*

$$\Delta^m a_n \geq 0 \Rightarrow \Delta^s \sigma_n \geq 0$$

*is valid for every sequence $(a_n)$ if and only if*

$$\Delta^s X_n(k+1, k) = 0 \quad \textit{for} \quad k \in \{0, 1, \ldots, m-1\}; \quad n \in \{0, 1, 2, \ldots\}$$

*and*

$$\Delta^s X_n(m, k) \geq 0 \quad \textit{for} \quad k \in \{m, \ldots, n+s\}; \quad n \in \{0, 1, 2, \ldots\}$$

*where*

$$X_n(m,k) = \begin{cases} 0 & \textit{for} \quad n < k \\ \displaystyle\sum_{j=0}^{n-k} \binom{n-k+m-1-j}{m-1} p_{n,j} & \textit{for} \quad n \geq k. \end{cases} \tag{1.21}$$

*Proof.* Let us write the difference $\Delta^s \sigma_n$ as a linear combination of the elements $a_j$. Using the notation:

$$q_n(j) = \begin{cases} 0 & \text{for} \quad n < j \\ p_{n,n-j} & \text{for} \quad n \geq j \end{cases}$$

we get the following

$$\begin{aligned} \Delta\sigma_n &= \sigma_{n+1} - \sigma_n = \sum_{j=0}^{n+1} p_{n+1,n+1-j} a_j - \sum_{j=0}^{n} p_{n,n-j} a_j \\ &= \sum_{j=0}^{n} (p_{n+1,n+1-j} - p_{n,n-j}) a_j + p_{n+1,0} a_{n+1} \end{aligned}$$

$$= \sum_{j=0}^{n} \Delta q_n(j)a_j + \Delta q_n(n+1)a_{n+1} = \sum_{j=0}^{n+1} \Delta q_n(j)a_j,$$

$$\begin{aligned}\Delta^2\sigma_n &= \Delta\sigma_{n+1} - \Delta\sigma_n = \sum_{j=0}^{n+2} \Delta q_{n+1}(j)a_j - \sum_{j=0}^{n+1} \Delta q_n(j)a_j \\ &= \sum_{j=0}^{n+1} \Delta(q_{n+1}(j) - q_n(j))a_j + \Delta q_{n+1}(n+2)a_{n+1} \\ &= \sum_{j=0}^{n+2} \Delta^2 q_n(j)a_j.\end{aligned}$$

Similarly, we get

$$\Delta^s\sigma_n = \sum_{j=0}^{n+s} \Delta^s q_n(j)\,a_j \quad \text{for every } s \tag{1.22}$$

and

$$\Delta^s X_n(m,k) = \sum_{i=k}^{n+s} \binom{i-k+m-1}{m-1} \Delta^s q_n(i).$$

Writting identity (1.2) for $n+s+1$-tuples $(a_0, a_1, \ldots, a_{n+s})$ and $(\Delta^s q_n(0), \Delta^s q_n(1), \ldots, \Delta^s q_n(n+s))$ and using the above results we get the identity

$$\Delta^s\sigma_n = \sum_{k=0}^{m-1} \Delta^k a_0\, \Delta^s X_n(k+1,k) + \sum_{k=m}^{n+s} \Delta^m a_{k-m}\, \Delta^s X_n(m,k). \tag{1.23}$$

Hence, the statement follows from Theorem 1.1. □

Theorem 1.4 is a generalization of several previously published results. Firstly, in [56] N. Ozeki obtained conditions on a matrix $[p_{n,i}]$ implying that for each convex sequence $(a_n)$ the sequence $(\sigma_n)$ is also convex, i.e. it is a particular case of Theorem 1.4 for $m = s = 2$. One decade later a particular case of Theorem 1.4 for $m = s$ was published in [34] and [41].

A result which is based on identity (1.23) is given as the following theorem, [62].

**Theorem 1.5** *Let $(a_n)$ be a real sequence and let $\sigma_n$ be defined as in (1.20). If $|\Delta^m a_n| \le N$ for $n \in \{0,1,2,\ldots\}$, and*

$$\Delta^s X_n(k+1,k) = 0 \quad \textit{for} \quad k \in \{0,1,\ldots,m-1\}; \quad n \in \{0,1,2,\ldots\} \tag{1.24}$$

*where $X_n(m,k)$ is given in (1.21), then*

$$|\Delta^s\sigma_n| \le N \sum_{k=m}^{n+s} |\Delta^s X_n(m,k)|.$$

*Proof.* This is an immediate consequence of (1.23). □

The following theorem also gives a bounds for $\Delta^s\sigma_n$, (see [71]).

**Theorem 1.6** *Let $m \in \mathbb{N}$, $m < n$, $(a_n)$ be a real sequence and let $\sigma_n$ be defined as in (1.20). Let $(p_{n,i})$ be reals such that*

$$\Delta^s X_n(k+1,k) = 0 \quad \text{for} \quad k \in \{0,1,\ldots,m-1\}; \quad n \in \{0,1,2,\ldots\} \tag{1.25}$$

*and*

$$\Delta^s X_n(m,k) \geq 0 \quad \text{for} \quad k \in \{m,\ldots,n+s\}; \quad n \in \{0,1,2,\ldots\}. \tag{1.26}$$

*If*

$$a \leq \Delta^m a_n \leq A \text{ for } n \in \{0,1,2,\ldots\}, \tag{1.27}$$

*then*

$$\frac{a}{m!}\Delta^s \alpha_n \leq \Delta^s \sigma_n \leq \frac{A}{m!}\Delta^s \alpha_n,$$

*where $\alpha_n = \sum_{k=0}^{n} p_{n,n-k}\, k^{(m)}$.*

*Proof.* For the sequences

$$b_n = a_n - \frac{a}{m!} n^{(m)} \text{ and } c_n = \frac{A}{m!} n^{(m)} - a_n$$

we obtain the following:

$$\Delta^m b_n = \Delta^m a_n - a \geq 0 \text{ and } \Delta^m c_n = A - \Delta^m a_n \geq 0.$$

Since $[p_{n,i}]$ satisfies conditions (1.25) and (1.26), using Theorem 1.4 for sequences $(b_n)$ and $(c_n)$ we get that

$$\Delta^s \sigma_n - \frac{a}{m!}\Delta^s \alpha_n \geq 0 \text{ and } \frac{A}{m!}\Delta^s \alpha_n - \Delta^s \sigma_n \geq 0$$

and the desired inequalities hold. □

**Remark 1.4** For $a = -A$ condition (1.27) becomes $|\Delta^m a_n| \leq A$ and then the statement of the above theorem becomes

$$|\Delta^s \sigma_n| \leq \frac{A}{m!}\Delta^s \alpha_n.$$

In [71] a general result which involves weighted arithmetic means is given.

**Corollary 1.2** *Let a sequence $\mathbf{a} = (a_n)$ satisfy*

$$a \leq \Delta^m a_n \leq A \text{ for } n = 0,1,2,\ldots$$

*and let the sequence $(p_n)$ be given by*

$$p_n = p_0 \binom{U+n-1}{n}, \quad U = \frac{p_1}{p_0}, \; n > 0.$$

*where $p_0, p_1$ are arbitrary positive numbers. Then*

$$\frac{a}{m!}\Delta^m A_n(\mathbf{b}) \leq \Delta A_n(\mathbf{a}) \leq \frac{A}{m!}\Delta^m A_n(\mathbf{b}),$$

*where $\mathbf{b} = (b_n)$ with $b_n = n^{(m)}$ and*

$$A_n(\mathbf{a}) = \frac{p_0 a_0 + p_1 a_1 + \ldots + p_n a_n}{p_0 + p_1 + \ldots + p_n}, \quad n = 0, 1, 2, \ldots$$

For a sequence which is convex of higher orders where these orders are consecutive integers the following theorem, published in [71], holds.

**Theorem 1.7** *Let $(a_n)$ be a real sequence, let $\sigma_n$ be defined as in $(1.20)$ and $j \in \{1, 2, \ldots, m\}$. The implication*

$$\Delta^k a_n \geq 0, \quad (k \in \{j, j+1, \ldots, m\}) \Rightarrow \Delta^s \sigma_n \geq 0 \ (s \in \mathbb{N})$$

*holds for every sequence $(a_n)$ if and only if*

$$\begin{aligned} \Delta^s X_n(k+1, k) &= 0, \quad && k \in \{0, 1, \ldots, j-1\}, \\ \Delta^s X_n(k+1, k) &\geq 0, \quad && k \in \{j, j+1, \ldots, m-1\}, \\ \Delta^s X_n(m, k) &\geq 0, \quad && k \in \{m, \ldots, n+s\}, n \in \mathbb{N}_0. \end{aligned}$$

*Proof.* The proof is similar to the proof of Theorem 1.2. □

## 1.2 ∇-Convex Sequences of Higher Order

Firstly, in this section we give formula for the sum $\sum p_i a_i$ involving ∇-differences. This result is given in [46] without proof. So, we give a detailed proof here.

**Lemma 1.2** *Let $(p_1, \ldots, p_n)$, $(a_1, \ldots, a_n)$ be real n-tuples, $m \in \mathbb{N}$ and $m < n$. Then*

$$\begin{aligned} \sum_{i=1}^{n} p_i a_i &= \sum_{k=0}^{m-1} \frac{1}{k!} \nabla^k a_{n-k} \sum_{i=1}^{n-k} (n-i)^{(k)} p_i \\ &+ \frac{1}{(m-1)!} \sum_{k=1}^{n-m} \left( \sum_{i=1}^{k} (k-i+m-1)^{(m-1)} p_i \right) \nabla^m a_k. \end{aligned} \tag{1.28}$$

*Proof.* For $m = 1$ identity (1.28) becomes

$$\sum_{i=1}^{n} p_i a_i = a_n \sum_{i=1}^{n} p_i + \sum_{k=1}^{n-1} \left( \sum_{i=1}^{k} p_i \right) \nabla a_k, \tag{1.29}$$

which is the second Abel identity, [51, p.334]. Let us suppose that for some $m$ identity (1.28) holds. We will prove that it holds for $m+1$ also. Using notation involving binomial coefficients the right-hand side of (1.28) for $m+1$ becomes

$$\sum_{k=0}^{m}\left(\sum_{i=1}^{n-k}\binom{n-i}{k}p_i\right)\nabla^k a_{n-k}+\sum_{k=1}^{n-m-1}\left(\sum_{i=1}^{k}\binom{k-i+m}{m}p_i\right)\nabla^{m+1}a_k$$
$$=\sum_{k=0}^{m}\left(\sum_{i=1}^{n-k}\binom{n-i}{k}p_i\right)\nabla^k a_{n-k}+\sum_{k=1}^{n-m-1}A_k\nabla^{m+1}a_k,$$

where $A_k=\sum_{i=1}^{k}\binom{k-i+m}{m}p_i$. Furthermore, it is equal to

$$\sum_{k=0}^{m-1}\left(\sum_{i=1}^{n-k}\binom{n-i}{k}p_i\right)\nabla^k a_{n-k}+\sum_{i=1}^{n-m}\binom{n-i}{m}p_i\nabla^m a_{n-m}$$
$$+\sum_{k=1}^{n-m-1}A_k\left(\nabla^m a_k-\nabla^m a_{k+1}\right)$$
$$=\sum_{k=0}^{m-1}\left(\sum_{i=1}^{n-k}\binom{n-i}{k}p_i\right)\nabla^k a_{n-k}+\sum_{i=1}^{n-m}\binom{n-i}{m}p_i\nabla^m a_{n-m}$$
$$+A_1\nabla^m a_1+\sum_{k=2}^{n-m-1}(A_k-A_{k-1})\nabla^m a_k-A_{n-m-1}\nabla^m a_{n-m}. \tag{1.30}$$

Since

$$A_1=\binom{m}{m}p_1=\binom{m-1}{m-1}p_1,$$

$$\sum_{i=1}^{n-m}\binom{n-i}{m}p_i-A_{n-m-1}$$
$$=\binom{m}{m}p_{n-m}+\sum_{i=1}^{n-m-1}\left(\binom{n-i}{m}-\binom{n-i-1}{m}\right)p_i$$
$$=\binom{m-1}{m-1}p_{n-m}+\sum_{i=1}^{n-m-1}\binom{n-i-1}{m-1}p_i=\sum_{i=1}^{n-m}\binom{n-i-1}{m-1}p_i$$

and

$$A_k-A_{k-1}=\sum_{i=1}^{k}\binom{k-i+m-1}{m-1}p_i$$

we get that (1.30) is equal to

$$\sum_{k=0}^{m-1}\sum_{i=1}^{n-k}\binom{n-i}{k}p_i\nabla^k a_{n-k}+\sum_{k=1}^{n-m}\sum_{i=1}^{k}\binom{k-i+m-1}{m-1}p_i\nabla^m a_k=\sum_{i=1}^{n}p_ia_i,$$

where in the last equality we use the assumption of mathematical induction. By the principle of mathematical induction, identity (1.28) is valid. □

Thus the following result is an analogue of Theorem 1.1 and it follows from identity (1.28).

**Theorem 1.8** *Let $(p_1,\dots,p_n)$ be a real n-tuple ($n > m$, $m \in \mathbb{N}$). Then the inequality*

$$\sum_{i=1}^{n} p_i a_i \geq 0 \tag{1.31}$$

*holds for every ∇-convex n-tuple $(a_i)$ of order m if and only if*

$$\sum_{i=1}^{n-k} (n-i)^{(k)} p_i = 0 \tag{1.32}$$

*holds for $k \in \{0,1,\dots,m-1\}$ and*

$$\sum_{i=1}^{k} (k-i+m-1)^{(m-1)} p_i \geq 0 \tag{1.33}$$

*holds for $k \in \{1,\dots,n-m\}$.*

*Proof.* If (1.32) and (1.33) are satisfied, then the first sum in identity (1.28) is equal to 0, the second sum is nonnegative and the inequality $\sum_{i=1}^{n} p_i a_i \geq 0$ holds.

Conversely, let us suppose that $\sum_{i=1}^{n} p_i a_i \geq 0$ holds for any ∇-convex $n$-tuple of order $m$. Since the sequence $a_i = (n-i)^{(k)}$, $i \in \{1,\dots,n\}$ is ∇-convex of order $m$ for every $k \in \{0,\dots,m-1\}$, we get $\sum_{i=1}^{n} p_i (n-i)^k \geq 0$. Similarly, since the sequence $a_i = -(n-i)^{(k)}$, $i \in \{1,\dots,n\}$ is ∇-convex of order $m$ for every $k \in \{0,\dots,m-1\}$, using (1.31) we get $-\sum_{i=1}^{n} p_i (n-i)^{(k)} \geq 0$. Hence, $\sum_{i=1}^{n} p_i (n-i)^{(k)} = 0$.

Also, since the sequence

$$a_i = \begin{cases} (k-i+m-1)^{(m-1)}, & i \in \{1,\dots,k\}, \\ 0, & i \in \{k+1,\dots,n\}, \end{cases} \tag{1.34}$$

is ∇-convex of order $m$ for every $k \in \{1,\dots,n-m\}$, we have by (1.31) that (1.33) is valid. □

If an $n$-tuple $(a_i)$ is ∇-convex of several consecutive orders, then, as in the previous section we have the following result, [71].

**Theorem 1.9** *Let $(p_1,\dots,p_n)$ be a real n-tuple and $m \in \mathbb{N}$, $m < n$. Then the inequality $\sum_{i=1}^{n} p_i a_i \geq 0$ holds for every n-tuple $(a_1,\dots,a_n)$ that is ∇-convex of orders $j, j+1,\dots,m$ ($j \in \{1,\dots,m\}$) if and only if*

$$\sum_{i=1}^{n-k} (n-i)^{(k)} p_i = 0, \quad k \in \{0,1,\dots,j-1\},$$

$$\sum_{i=1}^{n-k} (n-i)^{(k)} p_i \geq 0, \quad k \in \{j,j+1,\dots,m-1\},$$

*and*

$$\sum_{i=1}^{k} (k-i+m-1)^{(m-1)} p_i \geq 0, \quad k \in \{m+1,\dots,n\}.$$

*Proof.* The proof is similar to the proof of Theorem 1.2. □

A sequence $(a_i)$ is called totally monotonic of order $m$ if

$$\nabla^k a_i \geq 0 \quad \text{for} \quad k \in \{1, 2, \ldots, m\}.$$

As a consequence of the previous Theorem 1.9 we get the following necessary and sufficient conditions to positivity of the sum $\sum p_i a_i$ for totally monotonic sequences of order $m$. Namely, we obtain the following theorem.

**Corollary 1.3** *Let $(p_1, \ldots, p_n)$ be a real n-tuple and $m \in \mathbb{N}$, $m < n$. Then inequality* (1.4) *holds for every n-tuple $(a_1, \ldots, a_n)$ that is totally monotonic of order m if and only if*

$$\sum_{i=1}^{n} p_i = 0, \quad \sum_{i=1}^{n-k} (n-i)^{(k)} p_i \geq 0$$

*holds for $k \in \{1, \ldots, m-1\}$, and*

$$\sum_{i=1}^{n} (k-i+m-1)^{(m-1)} p_i \geq 0 \ \text{for} \ k \in \{1, \ldots, n-m\}.$$

In the rest of this section we pay attention to results about preservation of convexity. In a similar manner as in the first section we define a lower triangular matrix of real numbers $[p_{n,i}]$, $i = 0, 1, \ldots, n; n = 0, 1, 2, \ldots$ and a sequence $(\sigma_n)$ associated to the sequence $(a_n)_{n=0}^{\infty}$ by

$$\sigma_n = \sum_{k=0}^{n} p_{n,n-k} a_k, \quad n = 0, 1, 2, \ldots \tag{1.35}$$

**Theorem 1.10** *Let $\sigma_n$ be defined as in* (1.35)*, $m, s \in \mathbb{N}$. Then the implication*

$$\nabla^m a_n \geq 0 \Rightarrow \nabla^s \sigma_n \geq 0$$

*is valid for every sequence $(a_n)$ if and only if*

$$\nabla^s Y_n(k+1, n-k) = 0 \quad \text{for} \quad k \in \{0, 1, \ldots, m-1\}; \quad n \in \{0, 1, 2, \ldots\}$$

*and*

$$\nabla^s Y_n(m, k) \geq 0 \quad \text{for} \quad k \in \{0, \ldots, n+s-m\}; \quad n \in \{0, 1, 2, \ldots\},$$

*where*

$$Y_n(m,k) = \begin{cases} 0 & \text{for} \quad n < k \\ \displaystyle\sum_{j=0}^{k} \binom{k-j+m-1}{m-1} p_{n,n-j} & \text{for} \quad n \geq k. \end{cases}$$

*Proof.* We proceed as in the proof of Theorem 1.4. We prove that

$$\nabla^s \sigma_n = \sum_{j=0}^{n+s} \nabla^s q_n(j)\, a_j$$

and using the identity in (1.28) we obtain

$$\nabla^s \sigma = \sum_{k=0}^{m-1} \nabla^k a_{n+s-k} \nabla^s Y_n(k+1, n-k) + \sum_{k=0}^{n+s-m} \nabla^m a_k \nabla^s Y_n(m,k).$$

From this result the statement of the theorem follows directly. □

Finally, in the following theorem we get conditions on the numbers $p_{n,i}$ under which the sequence $(a_n)$ is ∇-convex of several consecutive orders, [71].

**Theorem 1.11** *Let $(a_n)$ be a real sequence, let $\sigma_n$ be defined as in (1.20) and $j \in \{1, 2, \ldots, m\}$. The implication*

$$\nabla^k a_n \geq 0, \ \ (k \in \{j, j+1, \ldots, m\}) \Rightarrow \nabla^s \sigma_n \geq 0 \ (s \in \mathbb{N})$$

*holds for every sequence $(a_n)$ if and only if*

$$\begin{aligned} \nabla^s Y_n(k+1, n-k) = 0, &\quad k \in \{0, 1, \ldots, j-1\}, \\ \nabla^s Y_n(k+1, n-k) \geq 0, &\quad k \in \{j, j+1, \ldots, m-1\}, \\ \nabla^s Y_n(m, k) \geq 0, &\quad k \in \{0, 1, \ldots, n+s-m\}, n \in \mathbb{N}_0. \end{aligned}$$

***Example 1.4*** In this example we connect Lemmas 1.1 and 1.2 with results from [16]. The following identity is due to M.P. Drazin.

For any sequence $(a_i)$ and any $y$

$$\sum_{i=0}^{n} \binom{n}{i} y^i a_i = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-1-y)^k \Delta^{n-k} a_k, \ \ n \geq 0. \tag{1.36}$$

By using this identity, Drazin proved the following inequalities:

$(i)$ If $(-1)^{n-k} \Delta^{n-k} a_k \geq 0$ $(k \in \{0, \ldots, n\})$ with at least one strict inequality, then

$$\sum_{i=0}^{n} \binom{n}{i} y^i a_i > 0 \ \text{ for } \ y > -1.$$

$(ii)$ If $\Delta^{n-k} a_k \geq 0$ $(k \in \{0, \ldots, n\})$ with at least one strict inequality, then

$$\sum_{i=0}^{n} \binom{n}{i} y^i a_i > 0 \ \text{ for } \ y < -1.$$

We give a proof of the above results using the basic identities (1.2) and (1.28), [78]. Namely, putting $n = m$ in (1.2) and (1.28) and starting numeration of sequences at 0, we get the following identities:

$$\sum_{i=0}^{n} p_i a_i = \sum_{k=0}^{n} \left( \sum_{i=k}^{n} p_i i^{(k)} \right) \frac{\Delta^k a_0}{k!} \tag{1.37}$$

$$\sum_{i=0}^{n} p_i a_i = \sum_{k=0}^{n} \left( \sum_{i=0}^{k} p_i (n-i)^{(n-k)} \right) \frac{\nabla^{n-k} a_k}{(n-k)!}. \tag{1.38}$$

Setting $p_i = \binom{n}{i} y^i$ in (1.38) we get

$$\begin{aligned}\sum_{i=0}^{n} \binom{n}{i} y^i a_i &= (-1)^n \sum_{k=0}^{n} (-1)^k \left( \sum_{i=0}^{k} \binom{n-i}{n-k} \binom{n}{i} y^i \right) \Delta^{n-k} a_k \\ &= (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-1-y)^k \Delta^{n-k} a_k,\end{aligned}$$

which is (1.36). Similarly, setting $p_i = \binom{n}{i} y^{n-i}$ in (1.37) we get

$$\sum_{i=0}^{n} \binom{n}{i} y^{n-i} a_i = \sum_{k=0}^{n} \binom{n}{k} (1+y)^{n-k} \Delta^k a_0.$$

In [78] the authors gave result for sequences $(a_k)$ whose $k$th differences have alternating signs. More precisely, the following theorem holds.

**Theorem 1.12** *Let $(r_k)$ be a given $(n+1)$-tuple, $r_k \in \{0,1\}$.*

*$(i)$ If $(p_0, p_1, \ldots, p_n)$ is a real $(n+1)$-tuple such that*

$$(-1)^{r_k} \sum_{i=k}^{n} p_i i^{(k)} \geq 0 \ \text{ for } \ 0 \leq k \leq n,$$

*then $\sum_{k=0}^{n} p_k a_k \geq 0$ for all $(n+1)$-tuples $(a_0, \ldots, a_n)$ such that $(-1)^{r_k} \Delta^k a_0 \geq 0$ for $0 \leq k \leq n$.*

*$(ii)$ If $(p_0, p_1, \ldots, p_n)$ satisfies*

$$(-1)^{r_k+n-k} \sum_{i=0}^{k} p_i (n-i)^{(n-k)} \geq 0 \ \text{ for } \ 0 \leq k \leq n,$$

*then $\sum_{k=0}^{n} p_k a_k \geq 0$ for all $(n+1)$-tuples $(a_0, \ldots, a_n)$ such that $(-1)^{r_k} \Delta^{n-k} a_k \geq 0$ for $0 \leq k \leq n$.*

*Proof.* $(i)$ Identity $(1.37)$ can be written as

$$\sum_{k=0}^{n} p_k a_k = \sum_{k=0}^{n} \left( \frac{1}{k!} \sum_{i=k}^{n} p_i i^{(k)} \right) \Delta^k a_0 = \sum_{k=0}^{n} \left( (-1)^{r_k} \frac{1}{k!} \sum_{i=k}^{n} p_i i^{(k)} \right) (-1)^{r_k} \Delta^k a_0.$$

Using assumptions of the theorem we get desired statement.

The proof of $(ii)$ is based on $(1.38)$ and it is similar to the previous one. □

## 1.3 Starshaped Sequences

In this section we investigate necessary and sufficient conditions for general linear inequalities for starshaped sequences of higher order.

The sequence $\mathbf{a} = (a_0, a_1, a_2, \ldots)$ is said to be starshaped of order $m$, $m \geq 2$, (or $m$-starshaped) if the sequence $\left(\dfrac{a_n - a_0}{n}\right)_n$ is $(m-1)$-convex.

The following lemma give us an identity on which further theorems are based, [70].

**Lemma 1.3** *For real sequences $(a_0, a_1, a_2, \ldots)$, $(p_0, p_1, p_2, \ldots)$ and $2 \leq m < n$ the folowing identity holds*

$$\sum_{i=0}^{n} p_i a_i = a_0 \sum_{i=0}^{n} p_i + \sum_{k=1}^{m-1} k T_k(a_1) \sum_{i=k}^{n} \binom{i}{k} p_i + \sum_{k=m}^{n} T_m(a_{k-m+1}) \sum_{i=k}^{n} i \binom{i-k+m-2}{m-2} p_i, \tag{1.39}$$

*where*

$$T_k(a_j) = \Delta^{k-1} \left(\frac{a_j - a_0}{j}\right).$$

*Proof.* Let us consider a new sequence $(b_1, b_2, \ldots)$ defined as

$$b_j = \frac{a_j - a_0}{j}.$$

Then $\Delta^{m-1} b_j = T_m(a_j)$. Let us write the basic identity from Lemma 1.1 for $m \to m-1$:

$$\sum_{i=1}^{n} p_i a_i = \sum_{k=0}^{m-2} \sum_{i=k+1}^{n} p_i (i-1)^{(k)} \frac{\Delta^k a_1}{k!} + \sum_{k=m}^{n} \left(\sum_{i=k}^{n} p_i (i-k+m-2)^{(m-2)}\right) \frac{\Delta^{m-1} a_{k-m+1}}{(m-2)!}.$$

Putting in the above identity substitutions

$$p_i \to i p_i, \;\; a_i \to b_i,$$

we get the equation

$$\sum_{i=1}^{n} i p_i b_i = \sum_{k=0}^{m-2} \sum_{i=k+1}^{n} i p_i (i-1)^{(k)} \frac{T_{k+1}(a_1)}{k!} + \sum_{k=m}^{n} \left(\sum_{i=k}^{n} i p_i (i-k+m-2)^{(m-2)}\right) \frac{T_m(a_{k-m+1})}{(m-2)!}.$$

Changing index $k$ in the first sum on the right-hand side $(k+1 \to k)$ we get that the right-hand side has a form

$$\sum_{k=1}^{m-1}\sum_{i=k}^{n} ip_i(i-1)^{(k-1)}\frac{T_k(a_1)}{(k-1)!} + \sum_{k=m}^{n}\left(\sum_{i=k}^{n} ip_i(i-k+m-2)^{(m-2)}\right)\frac{T_m(a_{k-m+1})}{(m-2)!}$$

which is equal to the right-hand side of (1.39) after using $\frac{(i-1)^{(k-1)}}{(k-1)!} = \binom{i-1}{k-1}$ and $\frac{(i-k+m-2)^{(m-2)}}{(m-2)!} = \binom{i-k+m-2}{m-2}$. Simple transformations of the left-hand side give:

$$\sum_{i=1}^{n} ip_ib_i = \sum_{i=1}^{n} ip_i\frac{a_i-a_0}{i} = \sum_{i=1}^{n} p_ia_i - a_0\sum_{i=1}^{n} p_i$$

which finishes the proof. □

The consequences of identity (1.39) are the following theorems.

**Theorem 1.13** *Let $(p_0, p_1, \ldots)$ be a real sequence and $2 \le m < n$. Then the inequality $\sum_{i=0}^{n} p_ia_i \ge 0$ holds for every real sequence $\mathbf{a}$ starshaped of order $m$ if and only if*

$$\sum_{i=0}^{n} p_i = 0, \tag{1.40}$$

$$\sum_{i=k}^{n}\binom{i}{k} p_i = 0 \qquad \textit{for } k = 1,2,\ldots,m-1, \tag{1.41}$$

$$\sum_{i=k}^{n} i\binom{i-k+m-2}{m-2} p_i \ge 0 \qquad \textit{for } k = m,\ldots,n. \tag{1.42}$$

*Proof.* If equalities (1.40), (1.41) and (1.42) are satisfied, then the first two sums in identity (1.39) are equal to 0, the third sum is nonnegative for starshaped sequence of order $m$ and inequality $\sum_{i=0}^{n} p_ia_i \ge 0$ holds.

Let us suppose that the inequality $\sum_{i=0}^{n} p_ia_i \ge 0$ holds for any starshaped sequence of order $m$. We consider sequences $\mathbf{a}^1$, $\mathbf{a}^2$, $\mathbf{a}^{k,3}$ and $\mathbf{a}^{k,4}$ defined as:

$$\mathbf{a}^1 = (1,1,1,\ldots),\ \ \mathbf{a}^2 = -\mathbf{a}^1,$$

$$\mathbf{a}^{k,3} = (0, 1^{(k)}, 2^{(k)}, \ldots, i^{(k)}, \ldots),\ \ \mathbf{a}^{k,4} = -\mathbf{a}^{k,3},$$

for $k = 1,2,\ldots,m-1$.

All these sequences are starshaped of order $m$, so we have the following inequalities

$$\sum_{i=0}^{n} p_i \ge 0,\ \ \sum_{i=0}^{n} p_i \le 0,$$

$$\sum_{i=0}^{n} p_i i^{(k)} \geq 0, \quad \sum_{i=0}^{n} p_i i^{(k)} \leq 0$$

from which conditions (1.40) and (1.41) follow. Condition (1.42) follows from the fact that the sequence

$$a_i = \begin{cases} 0, & i \in \{0,\dots,k-1\} \\ i(i-k+m-2)^{(m-2)}, & i \in \{k,\dots,n\} \end{cases}$$

is starshaped of order $m$ for every $k \in \{m,\dots,n\}$. □

**Theorem 1.14** *Let $(p_0, p_1, \dots)$ be a real sequence and $2 \leq m < n$, $2 \leq j \leq m$. Then the inequality $\sum_{i=0}^{n} p_i a_i \geq 0$ holds for every sequence* **a** *starshaped of orders $j, j+1, \dots, m$ if and only if*

$$\begin{aligned} &\sum_{i=0}^{n} p_i = 0, \\ &\sum_{i=k}^{n} \binom{i}{k} p_i = 0 \qquad \text{for } k = 1,2,\dots,j-1, \\ &\sum_{i=k}^{n} \binom{i}{k} p_i \geq 0 \qquad \text{for } k = j, j+1,\dots,m-1 \\ &\sum_{i=k}^{n} i \binom{i-k+m-2}{m-2} p_i \geq 0 \qquad \text{for } k = m,\dots,n. \end{aligned} \tag{1.43}$$

*For $j = m$, (1.43) is not necessary.*

## 1.4 Mean-convex and Mean-starshaped Sequences

In this section we consider mean-convex and mean-starshaped sequences. In general, we say that a sequence **a** has a mean-property P if the sequence $(A_n)$ has the property P, where

$$A_n = \frac{a_0 + a_1 + \dots + a_n}{n+1}. \tag{1.44}$$

We give results about positivity of the sum $\sum p_i a_i$ for mean-convex and a mean-starshaped sequences, [48], [92]. A real sequence $(a_0, a_1, \dots)$ is called mean-convex if the sequence $(A_n)$ defined as in (1.44) is convex.

**Theorem 1.15** *Let $(p_0, p_1, \dots)$ be a real sequence. The inequality*

$$\sum_{k=0}^{n} p_k a_k \geq 0$$

*holds for every mean-convex sequence* **a** *if and only if the following conditions*

$$\sum_{k=0}^{n} p_k = 0 \tag{1.45}$$

$$\sum_{k=1}^{n} kp_k = 0 \tag{1.46}$$

$$\sum_{j=k}^{n} (2j-k+1)p_j \geq 0 \qquad \text{for } k = 2,3,\ldots,n, \tag{1.47}$$

*are fulfilled.*

*Proof.* Let us suppose that $\sum_{k=0}^{n} p_k a_k \geq 0$ holds. Since the sequences $(c,c,c,\ldots)$ and $(-c,-c,-c,\ldots)$ are mean-convex, condition (1.45) is valid. Further, $(0,1,2,\ldots)$ and $(0,-1,-2,\ldots)$ are also mean-convex, so condition (1.46) holds. Finally, condition (1.47) follows from the fact that the sequence $(a_0,a_1,\ldots)$ where $a_0 = a_1 = \ldots = a_k = 0$, $a_j = 2j-k+1$, $j = k+1,k+2,\ldots$, is mean-convex.

The sufficiency of conditions (1.45), (1.46), and (1.47) is a consequence of the following identity:

$$\sum_{k=0}^{n} p_k a_k = a_0 \sum_{k=0}^{n} p_k + 2\Delta A_0 \sum_{k=0}^{n} kp_k + \sum_{k=2}^{n} \left( \sum_{j=k}^{n} p_j(2j-k+1) \right) \Delta^2 A_{k-2}.$$

□

The following result is given in [92] and it gives an answer to the question of necessity and sufficiency of $p_i$ if considered sequence is mean-starshaped. A real sequence $(a_0,a_1,\ldots)$ is called mean-starshaped if the sequence $(A_n)$ defined as in (1.44) is starshaped, i.e. if $\Delta \left( \frac{A_n(a_n)-a_0}{n} \right) \geq 0$ holds for $n \geq 2$.

**Theorem 1.16** *Let $(p_0,p_1,\ldots)$ be a real sequence. Then for a fixed $n \geq 2$ the inequality*

$$\sum_{k=0}^{n} p_k a_k \geq 0$$

*holds for every mean-starshaped sequence $(a_0,a_1,\ldots)$ if and only if the following conditions*

$$\sum_{k=0}^{n} p_k = 0 \tag{1.48}$$

$$\sum_{k=1}^{n} kp_k = 0 \tag{1.49}$$

$$k(k-1)p_k + \sum_{i=k}^{n} 2ip_i \geq 0 \qquad \text{for } k = 2,3,\ldots,n \tag{1.50}$$

*are fulfilled.*

*Proof.* Let us suppose that $\sum_{k=0}^{n} p_k a_k \geq 0$ holds. Like in the proof of the previous theorem, since the sequences $(c,c,c,\dots)$ and $(-c,-c,-c,\dots)$ are mean-starshaped, condition (1.48) is valid. Further, $(0,1,2,\dots)$ and $(0,-1,-2,\dots)$ are also mean-starshaped, so condition (1.49) holds. Since the sequence $(a_0,a_1,\dots)$, $a_0 = \dots = a_{k-1} = 0$, $a_k = k(k+1)$, $a_{k+1} = 2(k+1)$, $a_{k+2} = 2(k+2),\dots$, is mean-starshaped, then (1.50) holds.

The sufficiency of conditions (1.48), (1.49), and (1.50) is a consequence of the following identity:

$$\begin{aligned}\sum_{k=0}^{n} p_k a_k &= a_0 \sum_{k=0}^{n} p_k + (a_1 - a_0) \sum_{k=1}^{n} k p_k \\ &\quad + \sum_{k=2}^{n} \left( k(k-1) p_k + \sum_{i=k}^{n} 2 i p_i \right) \Delta \left( \frac{A_n(a_n) - a_0}{n} \right).\end{aligned}$$

□

The following theorem gives conditions under which the sequence $\sigma_n = \sum_{k=0}^{n} p_{n,k} a_k$ is also mean-starshaped when a sequence $(a_k)$ is mean-starshaped, (see [92]).

**Theorem 1.17** *Let $(a_0,a_1,\dots)$ be a mean-starshaped sequence and $(\sigma_0,\sigma_1,\dots)$ be the sequence $\sigma_n = \sum_{k=0}^{n} p_{n,k} a_k$, where $p_{n,k}$ are reals. Then the sequence $(\sigma_n)$ is mean-starshaped if and only if*

$$\begin{aligned}&\sum_{k=0}^{n} w_{n,k} = 0 \\ &\sum_{k=1}^{n} k w_{n,k} = 0 \\ &k(k+1) w_{n,k} + \sum_{i=k+1}^{n} 2 i w_{n,i} \geq 0 \qquad \text{for } k = 2,3,\dots,n,\end{aligned}$$

*where $w_{n,k}$ are defined as*

$$\begin{aligned}w_{n,0} &= \frac{1}{n(n+1)} (p_{n,0} + p_{0,0}) - \frac{2}{n(n^2-1)} \sum_{j=1}^{n-1} p_{j,0}, \\ w_{n,k} &= \frac{1}{n(n+1)} p_{n,k} - \frac{2}{n(n^2-1)} \sum_{j=k}^{n-1} p_{j,k}, \ k = 1,2,\dots,n-1 \\ w_{n,n} &= \frac{1}{n(n+1)} p_{n,n}.\end{aligned}$$

## 1.5 $p$-monotone and $p,q$-convex Sequences

In this section we consider somewhat different generalization of convex sequences. Let $p \neq 0$ be a real number. We define the operator $L_p$ by

$$L_p(a_i) = a_{i+1} - pa_i, \quad i \in \mathbb{N}.$$

For a sequence $(a_i)$, we say that it is $p$-monotone or that it belongs to the class $K_p$, if the inequality $L_p(a_i) \geq 0$ holds for all $i \in \mathbb{N}$, ([37]). It is obvious that for $p = 1$ we get nondecreasing sequence. Let us obtain an identity for the sum $\sum_{i=1}^{n} w_i a_i$ involving an operator $L_p$. From $L_p(a_i) = a_{i+1} - pa_i$ we get

$$\begin{aligned}
L_p(a_{k-1}) &= a_k - pa_{k-1} \\
pL_p(a_{k-2}) &= pa_{k-1} - p^2 a_{k-2} \\
p^2 L_p(a_{k-3}) &= p^2 a_{k-2} - p^3 a_{k-3} \\
&\vdots \\
p^{k-2} L_p(a_1) &= p^{k-2} a_2 - p^{k-1} a_1.
\end{aligned}$$

Summing all the above equalities we get

$$L_p(a_{k-1}) + pL_p(a_{k-2}) + \ldots + p^{k-2} L_p(a_1) = a_k - p^{k-1} a_1, \ \ k = 2, \ldots, n.$$

Multiplying the equality for $a_k$ with $w_k$ and writting them for $k = 2, \ldots, n$ we get

$$\begin{aligned}
w_2 L_p(a_1) &= w_2 a_2 - w_2 p a_1 \\
w_3 L_p(a_2) + w_3 p L_p(a_1) &= w_3 a_3 - w_3 p^2 a_1 \\
w_4 L_p(a_3) + w_4 p L_p(a_2) + w_4 p^2 L_p(a_1) &= w_4 a_4 - w_4 p^3 a_1 \\
&\vdots \\
w_n L_p(a_{n-1}) + w_n p L_p(a_{n-2}) + \ldots + w_n p^{n-2} L_p(a_1) &= w_n a_n - w_n p^{n-1} a_1.
\end{aligned}$$

Summing the above equalities we get

$$\begin{aligned}
L_p(a_1) \sum_{j=2}^{n} w_j p^{j-2} \ + \ L_p(a_2) \sum_{j=3}^{n} w_j p^{j-3} + \ldots + L_p(a_{n-2}) \sum_{j=n-1}^{n} w_j p^{j-n+1} \\
+ \ L_p(a_{n-1}) w_n = \sum_{i=2}^{n} w_i a_i - a_1 \sum_{j=2}^{n} w_i p^{i-1},
\end{aligned}$$

$$\sum_{k=2}^{n} \left( \sum_{j=k}^{n} w_j p^{j-k} \right) L_p(a_{k-1}) = \sum_{i=1}^{n} w_i a_i - a_1 \sum_{j=1}^{n} w_i p^{i-1},$$

i.e. we get the following identity involving the operator $L_p$:

$$\sum_{i=1}^{n} w_i a_i = a_1 \sum_{i=1}^{n} p^{i-1} w_i + \sum_{k=2}^{n} \left( \sum_{i=k}^{n} p^{i-k} w_i \right) L_p(a_{k-1}). \tag{1.51}$$

Thus, from (1.51) we can easily obtain the following theorem, ([60]).

**Theorem 1.18** *Let* $\mathbf{w} = (w_i)$ *be an arbitrary real sequence.*

$(i)$ *The inequality*

$$\sum_{i=1}^{n} w_i a_i \geq 0 \tag{1.52}$$

*holds for every* $p$*-monotone sequence* $(a_i)$ *if and only if*

$$\sum_{i=1}^{n} p^{i-1} w_i = 0 \tag{1.53}$$

*and*

$$\sum_{i=k}^{n} p^{i-k} w_i \geq 0, \ \ k \in \{2, \ldots, n\}. \tag{1.54}$$

$(ii)$ *Inequality* (1.52) *holds for every* $p$*-monotone sequence* $(a_i)$ *such that* $a_1 \geq 0$ *if and only if*

$$\sum_{i=1}^{n} p^{i-k} w_i \geq 0, \ \ k \in \{1, \ldots, n\}.$$

*Proof.* If conditions (1.53) and (1.54) hold, then from identity (1.51) we get that the sum $\sum w_i a_i$ is nonnegative for any $p$-monotone sequence.

On the other hand, since the sequences $(p^{i-1})_i$, $(-p^{i-1})_i$ and

$$a_i = \begin{cases} 0, & i \in \{1, \ldots, k-1\}, \\ p^{i-k}, & i \geq k, \end{cases}$$

are $p$-monotone, using (1.52) we get conditions (1.53) and (1.54). □

Let us consider a triangular matrix of real numbers $[p_{n,i}]$ $(i \in \{1, \ldots, n\};\ n \in \mathbb{N})$ Define the sequence $(\sigma_n)$ for a given $(a_n)$ with

$$\sigma_n = \sum_{i=1}^{n} p_{n,n+1-i} a_i. \tag{1.55}$$

The following preservation theorem is given in a slightly modified form in [60].

**Theorem 1.19** *Necessary and sufficient conditions such that the implication*

$$(a_n) \in K_p \Rightarrow (\sigma_n) \in K_q$$

*holds for every sequence* $(a_n)$*, where the sequence* $(\sigma_n)$ *is given by* (1.55)*, are that for every* $n$ *we have*

$$b_{n+1,1} - q b_{n,1} = 0, \quad b_{n+1,n+1} \geq 0,$$

$$b_{n+1,k} - qb_{n,k} \ge 0 \quad (k \in \{2,\dots,n\}),$$

*where* $b_{n,k} = \sum_{i=1}^{n-k+1} p^{i-1} p_{n,n-k-i+2}$.

*Proof.* We have

$$\begin{aligned} L_q(\sigma_n) &= \sigma_{n+1} - q\sigma_n = \sum_{j=1}^{n+1} p_{n+1,n+2-j} a_j - q \sum_{j=1}^{n} p_{n,n+1-j} a_j \\ & \sum_{j=1}^{n} (p_{n+1,n+2-j} - qp_{n,n+1-j}) a_j + p_{n+1,1} a_{n+1} = \sum_{j=1}^{n+1} w_j a_j, \end{aligned}$$

where $w_j = p_{n+1,n+2-j} - qp_{n,n+1-j}$ for $j = 1,\dots,n$ and $w_{n+1} = p_{n+1,1}$. Using Theorem 1.18 we get that the inequality $L_q(\sigma_n) \ge 0$ holds if and only if $\sum_{i=1}^{n+1} p^{i-1} w_i = 0$ and $\sum_{i=k}^{n+1} p^{i-k} w_i \ge 0$ for $k = 2,\dots,n+1$. The first equality is transformed to

$$\begin{aligned} 0 &= \sum_{i=1}^{n+1} p^{i-1} w_i = \sum_{i=1}^{n} p^{i-1} (p_{n+1,n+2-i} - qp_{n,n+1-i}) + p^n p_{n+1,1} \\ &= \left( \sum_{i=1}^{n} p^{i-1} p_{n+1,n+2-i} + p^n p_{n+1,1} \right) - q \sum_{i=1}^{n} p^{i-1} p_{n,n+1-i} = b_{n+1,1} - qb_{n,1}. \end{aligned}$$

The second condition splits to two cases: $k \in \{2,\dots,n\}$ and $k = n+1$. For $k = n+1$ we get $b_{n+1,n+1} \ge 0$ and for $k \in \{2,\dots,n\}$ we get that $\sum_{i=k}^{n+1} p^{i-k} w_i \ge 0$ are equivalent with $b_{n+1,k} - qb_{n,k} \ge 0$. □

The following type of sequences can be consider as a twofold generalization. Firstly, it is a generalization of the $p$-monotone sequences, and secondly, it is a generalization of the classical convexity of sequence.

Let $p, q$ be real numbers, $(a_i)$ be a real sequence and let us define an operator $L_{pq}$ as follows:

$$L_{pq}(a_i) = L_p(L_q(a_i)) = L_q(L_p(a_i)) = a_{i+2} - (p+q)a_{i+1} + pqa_i.$$

If $L_{pq}(a_i) \ge 0$ for any $i$, then a sequence $(a_i)$ is called $p,q$-convex sequence. Obviously, if $p = q = 1$, then a $1,1$-convex sequence becomes a convex sequence in the classical sense.

Necessary and sufficient conditions for validation of (1.4) for all $p,q$-convex sequences are given in [43].

**Theorem 1.20** *Let* $\mathbf{w} = (w_i)$ *be an arbitrary real sequence. The inequality*

$$\sum_{i=1}^{n} w_i a_i \ge 0 \tag{1.56}$$

*holds for every $p,q$-convex sequence $(a_i)$, $p \neq q$, $p,q \neq 0$, if and only if*

$$\sum_{i=1}^{n} p^{i-1} w_i = 0, \quad \sum_{i=2}^{n} \frac{p^{i-1} - q^{i-1}}{p-q} w_i = 0 \tag{1.57}$$

*and*

$$\sum_{i=r}^{n} \frac{p^{i-r+1} - q^{i-r+1}}{p-q} w_i \geq 0, \quad r \in \{3, \ldots, n\}. \tag{1.58}$$

*If $p = q$, then conditions* (1.57) *and* (1.58) *become*

$$\sum_{i=1}^{n} p^{i-1} w_i = 0, \quad \sum_{i=2}^{n} (i-1) p^{i-2} w_i = 0,$$

$$\sum_{i=r}^{n} (i-r+1) p^{i-r} w_i \geq 0, \quad r \in \{3, \ldots, n\}.$$

*Proof.* Let us obtain an identity for $\sum w_i a_i$ involving the operator $L_{pq}$. We start with an identity which involves $L_p$:

$$\sum_{i=1}^{n} w_i a_i = a_1 \left( \sum_{i=1}^{n} p^{i-1} w_i \right) + \sum_{k=2}^{n} W_k L_p(a_{k-1}), \tag{1.59}$$

where $W_k = \sum_{j=k}^{n} p^{j-k} w_j$. Using the same identity for the sum $\sum_{k=2}^{n} W_k b_k$ with the operator $L_q$, where $b_k = L_p(a_{k-1})$ we get

$$\sum_{k=2}^{n} W_k b_k \quad = b_2 \left( \sum_{j=2}^{n} q^{j-2} W_j \right) + \sum_{k=3}^{n} \overline{W}_k L_q(b_{k-1}), \tag{1.60}$$

where $\overline{W}_k = \displaystyle\sum_{j=k}^{n} q^{j-k} W_j$. Now, $L_q(b_{k-1}) = L_q(L_p(a_{k-2})) = L_{pq}(a_{k-2})$ and

$$\begin{aligned}
\overline{W}_k &= \sum_{j=k}^{n} q^{j-k} W_j = \sum_{j=k}^{n} q^{j-k} \left( \sum_{r=j}^{n} p^{r-j} w_r \right) = w_k + w_{k+1}(p+q) \\
&\qquad + w_{k+2}(p^2 + pq + q^2) + \ldots + w_n(p^{n-k} + qp^{n-k-1} + \ldots + q^{n-k}) \\
&= \begin{cases} w_k + w_{k+1} \dfrac{p^2 - q^2}{p-q} + \ldots + w_n \dfrac{p^{n-k+1} - q^{n-k+1}}{p-q}, & p \neq q \\ w_k + 2p w_{k+1} + 3p^2 w_{k+1} + \ldots + (n-k+1) p^{n-k} w_n, & p = q \end{cases} \\
&= \begin{cases} \displaystyle\sum_{i=k}^{n} w_i \frac{p^{i-k+1} - q^{i-k+1}}{p-q}, & p \neq q \\ \displaystyle\sum_{i=k}^{n} (i-k+1) p^{i-k} w_i, & p = q. \end{cases}
\end{aligned} \tag{1.61}$$

In particular, for $k=2$ we get

$$\sum_{j=2}^{n} q^{j-2} W_j = \begin{cases} \sum_{i=2}^{n} w_i \dfrac{p^{i-1}-q^{i-1}}{p-q}, & p \neq q \\ \sum_{i=2}^{n} (i-1) p^{i-2} w_i, & p = q. \end{cases} \tag{1.62}$$

Putting (1.61) and (1.62) in (1.60) and combining with (1.59) we obtain that for $p \neq q$

$$\begin{aligned} \sum_{i=1}^{n} w_i a_i &= a_1 \left( \sum_{i=1}^{n} p^{i-1} w_i \right) + L_p(a_1) \sum_{i=2}^{n} w_i \frac{p^{i-1}-q^{i-1}}{p-q} \\ &\quad + \sum_{k=3}^{n} \left( \sum_{i=k}^{n} w_i \frac{p^{i-k+1}-q^{i-k+1}}{p-q} \right) L_{pq}(a_{k-2}) \end{aligned} \tag{1.63}$$

and for $p=q$ the following holds

$$\begin{aligned} \sum_{i=1}^{n} w_i a_i &= a_1 \left( \sum_{i=1}^{n} p^{i-1} w_i \right) + L_p(a_1) \sum_{i=2}^{n} (i-1) p^{i-2} w_i \\ &\quad + \sum_{k=3}^{n} \left( \sum_{i=k}^{n} (i-k+1) p^{i-k} w_i \right) L_{pp}(a_{k-2}). \end{aligned} \tag{1.64}$$

Let us suppose that $p \neq q$. Now it is obvious that if $(a_i)$ is a $p,q$-convex sequence with properties (1.57) and (1.58), then using identity (1.63) we get that $\sum w_i a_i$ is nonnegative.

Conversely, let us suppose that $\sum w_i a_i$ is nonnegative for every $p,q$-convex sequence. Let us consider the sequence $(a_i)$ defined with

$$a_i = -q \frac{p^i - q^i}{p-q} + \frac{p^{i+1}-q^{i+1}}{p-q}.$$

Since $L_{pq}(a_i) = 0$, i.e. $(a_i)$ is $p,q$-convex, we have $\sum_{i=1}^{n} w_i a_i \geq 0$, i.e. from identity (1.63) we get $p \sum_{i=1}^{n} p^{i-1} w_i \geq 0$. Similarly, for a $p,q$-convex sequence $(-a_i)$ we obtain $-p \sum_{i=1}^{n} p^{i-1} w_i \geq 0$. Together, we conclude $\sum_{i=1}^{n} p^{i-1} w_i = 0$.

Furthermore, let us consider the sequence $(a_i)$ defined as

$$a_i = -(p+q) \frac{p^i - q^i}{p-q} + \frac{p^{i+1}-q^{i+1}}{p-q}.$$

It is $p,q$-convex. Since the sequence $(-a_i)$ is also $p,q$-convex, we get that $\sum_{i=2}^{n} \frac{p^{i-1}-q^{i-1}}{p-q} w_i = 0$.

And finally, fix $r \in \{3, \dots, n\}$ and let us consider the sequence $(a_i^r)$ defined as $a_1 = a_2 = \dots = a_{r-1} = 0$, $a_i = \dfrac{p^{i-r+1}-q^{i-r+1}}{p-q}$ for $i = r, \dots, n$. Since $(a_i^r)$ is $p,q$-convex condition (1.58) holds. □

## 1.6 Multilinear Forms

In the previous sections we consider a positivity of the sum $\sum p_i a_i$ for only one sequence $(a_i)$. What happens if we consider a sum in which two sequences appear? Results of such type are given in this section.

### 1.6.1 Results for $\Delta$- and $\nabla$-convex sequences

Bilinear sums were investigated by T. Popoviciu in [87] where monotone sequences $(a_i)$ and $(b_i)$ have the same number of members. The case when monotone sequences $(a_i)$ and $(b_i)$ have different numbers of elements was done by J. Pečarić in [57] and A. Kovačec in [35]. A generalization of the above mentioned results with several monotone sequences is given in [58]. As we know, a monotone sequence is in fact a 1-convex sequence, so we are interested in results which involve two or more sequences of higher ordered convexity. In the subsequent text we give a detailed proof of one of such results from [72]. Other related results will be given without proof.

**Theorem 1.21** *Let $x_{ij}$, $a_i$, $b_j$, $(1 \le i \le N; 1 \le j \le M)$, be real numbers, $n, m \in \mathbb{N}$, $n \le N, m \le M$. The inequality*

$$F(\mathbf{a},\mathbf{b}) \equiv \sum_{i=1}^{N}\sum_{j=1}^{M} x_{ij} a_i b_j \ge 0 \tag{1.65}$$

*holds for every n-convex sequence $(a_i)$ and m-convex sequence $(b_j)$ if and only if*

$$\sum_{r=1}^{N}\sum_{s=1}^{M}(r-1)^{(i)}(s-1)^{(j)}x_{rs} = 0 \tag{1.66}$$

$$\text{for} \quad i \in \{0,\ldots,n-1\}, j \in \{0,\ldots,m-1\},$$

$$\sum_{r=1}^{N}\sum_{s=1}^{M}(r-1)^{(i)}(s-j+m-1)^{(m-1)}x_{rs} = 0$$

$$\text{for} \quad i \in \{0,\ldots,n-1\}, j \in \{m+1,\ldots,M\},$$

$$\sum_{r=1}^{N}\sum_{s=1}^{M}(r-i+n-1)^{(n-1)}(s-1)^{(j)}x_{rs} = 0$$

$$\text{for} \quad i \in \{n+1,\ldots,N\}, j \in \{0,\ldots,m-1\},$$

$$\sum_{r=1}^{N}\sum_{s=1}^{M}(r-i+n-1)^{(n-1)}(s-j+m-1)^{(m-1)}x_{rs} \ge 0 \tag{1.67}$$

$$\text{for} \quad i \in \{n+1,\ldots,N\}, j \in \{m+1,\ldots,M\}.$$

*Proof.* The following identity has a crucial role in the proof:

$$\sum_{i=1}^{N}\sum_{j=1}^{M} x_{ij} a_i b_j = \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} \Delta^i a_1 \Delta^j b_1 \left(\sum_{r=1}^{N}\sum_{s=1}^{M}\binom{r-1}{i}\binom{s-1}{j}x_{rs}\right)$$

$$+\sum_{i=0}^{n-1}\sum_{j=m+1}^{M}\Delta^i a_1\Delta^m b_{j-m}\left(\sum_{r=1}^{N}\sum_{s=j}^{M}\binom{r-1}{i}\binom{s-j+m-1}{m-1}x_{rs}\right)$$
$$+\sum_{i=n+1}^{N}\sum_{j=0}^{m-1}\Delta^n a_{i-n}\Delta^j b_1\left(\sum_{r=i}^{N}\sum_{s=1}^{M}\binom{r-i+n-1}{n-1}\binom{s-1}{j}x_{rs}\right)$$
$$+\sum_{i=n+1}^{N}\sum_{j=m+1}^{M}\Delta^n a_{i-n}\Delta^m b_{j-m}\left(\sum_{r=i}^{N}\sum_{s=j}^{M}\binom{r-i+n-1}{n-1}\binom{s-j+m-1}{m-1}x_{rs}\right).$$

We have

$$\sum_{i=1}^{N}\sum_{j=1}^{M}x_{ij}a_ib_j=\sum_{i=1}^{N}w_ia_i,$$

where $w_i=\sum_{j=1}^{M}b_jx_{ij}$. So using basic identity (1.2) we have

$$\begin{aligned}\sum_{i=1}^{N}\sum_{j=1}^{M}x_{ij}a_ib_j &= \sum_{i=0}^{n-1}\Delta^i a_1\left(\sum_{r=1}^{N}\binom{r-1}{i}w_r\right)\\ &+\sum_{i=n+1}^{N}\Delta^n a_{i-m}\left(\sum_{r=i}^{N}\binom{r-i+n-1}{n-1}w_r\right).\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\sum_{r=1}^{N}\binom{r-1}{i}w_i=\sum_{r=1}^{N}\binom{r-1}{i}\sum_{j=1}^{M}b_jx_{rj}=\sum_{j=1}^{M}b_j\left(\sum_{r=1}^{N}\binom{r-1}{i}x_{rj}\right)\\
&=\sum_{j=0}^{m-1}\Delta^j b_1\left(\sum_{s=1}^{M}\binom{s-1}{j}\left(\sum_{r=1}^{N}\binom{r-1}{i}x_{rs}\right)\right)\\
&+\sum_{j=m+1}^{M}\Delta^m b_{j-m}\left(\sum_{s=j}^{M}\binom{s-j+m-1}{m-1}\left(\sum_{r=1}^{N}\binom{r-1}{i}x_{rs}\right)\right)\\
&=\sum_{j=0}^{m-1}\Delta^j b_1\left(\sum_{r=1}^{N}\sum_{s=1}^{M}\binom{r-1}{i}\binom{s-1}{j}x_{rs}\right)\\
&+\sum_{j=m+1}^{M}\Delta^m b_{j-m}\left(\sum_{r=1}^{N}\sum_{s=j}^{M}\binom{r-1}{i}\binom{s-j+m-1}{m-1}x_{rs}\right).\end{aligned}$$

Using the same idea we have

$$\begin{aligned}&\sum_{r=i}^{N}\binom{r-i+n-1}{n-i}w_r=\sum_{r=i}^{N}\binom{r-i+n-1}{n-i}\sum_{j=1}^{M}b_jx_{rj}\\
&=\sum_{j=1}^{M}b_j\left(\sum_{r=i}^{N}\binom{r-i+n-1}{n-i}x_{rj}\right)\end{aligned}$$

$$= \sum_{j=0}^{m-1} \Delta^j b_1 \left( \sum_{s=1}^{M} \binom{s-1}{j} \left( \sum_{r=1}^{N} \binom{r-i+n-1}{n-i} x_{rs} \right) \right)$$
$$+ \sum_{j=m+1}^{M} \Delta^m b_{j-m} \left( \sum_{s=j}^{M} \binom{s-j+m-1}{m-1} \left( \sum_{r=1}^{N} \binom{r-i+n-1}{n-i} x_{rs} \right) \right)$$
$$= \sum_{j=0}^{m-1} \Delta^j b_1 \left( \sum_{r=i}^{N} \sum_{s=1}^{M} \binom{r-i+n-1}{n-i} \binom{s-1}{j} x_{rs} \right)$$
$$+ \sum_{j=m+1}^{M} \Delta^m b_{j-m} \left( \sum_{r=i}^{N} \sum_{s=j}^{M} \binom{r-i+n-1}{n-i} \binom{s-j+m-1}{m-1} x_{rs} \right).$$

Using these two identities we obtain the desired identity. □

The identity from Theorem 1.21 has combinations of $\Delta$-differences of sequences $(a_i), (b_i)$. There exist identitites which involve combinations of $\nabla$-differences or $\Delta$-differences of the sequence $(a_i)$ with $\nabla$-differences of the sequence $(b_i)$. We write them here without a detailed proof.

**Theorem 1.22** *Let $x_{ij}$, $a_i$, $b_j$, $(1 \le i \le N; 1 \le j \le M)$, be real numbers, $m, n \in \mathbb{N}$, $m \le M$, $n \le N$.*

*The following identities hold*

$$\sum_{i=1}^{N} \sum_{j=1}^{M} x_{ij} a_i b_j = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \nabla^i a_{N-i} \nabla^j b_{M-j} \left( \sum_{r=1}^{N} \sum_{s=1}^{M} \binom{N-r}{i} \binom{M-s}{j} x_{rs} \right)$$
$$+ \sum_{i=0}^{n-1} \sum_{j=1}^{M-m} \nabla^i a_{N-i} \nabla^m b_j \left( \sum_{r=1}^{N} \sum_{s=1}^{j} \binom{N-r}{i} \binom{j-s+m-1}{m-1} x_{rs} \right)$$
$$+ \sum_{i=1}^{N-n} \sum_{j=0}^{m-1} \nabla^n a_i \nabla^j b_{M-j} \left( \sum_{r=1}^{i} \sum_{s=1}^{M} \binom{i-r+n-1}{n-1} \binom{M-s}{j} x_{rs} \right)$$
$$+ \sum_{i=1}^{N-n} \sum_{j=1}^{M-m} \nabla^n a_i \nabla^m b_j \left( \sum_{r=1}^{i} \sum_{s=1}^{j} \binom{i-r+n-1}{n-1} \binom{j-s+m-1}{m-1} x_{rs} \right),$$

*and*

$$\sum_{i=1}^{N} \sum_{j=1}^{M} x_{ij} a_i b_j = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \Delta^i a_1 \nabla^j b_{M-j} \left( \sum_{r=1}^{N} \sum_{s=1}^{M} \binom{r-1}{i} \binom{M-s}{j} x_{rs} \right)$$
$$+ \sum_{i=0}^{n-1} \sum_{j=1}^{M-m} \Delta^i a_1 \nabla^m b_j \left( \sum_{r=1}^{N} \sum_{s=1}^{j} \binom{r-1}{i} \binom{j-s+m-1}{m-1} x_{rs} \right)$$
$$+ \sum_{i=n+1}^{N} \sum_{j=0}^{m-1} \Delta^n a_{i-n} \nabla^j b_{M-j} \left( \sum_{r=i}^{N} \sum_{s=1}^{M} \binom{r-i+n-1}{n-1} \binom{M-s}{j} x_{rs} \right)$$
$$+ \sum_{i=n+1}^{N} \sum_{j=1}^{M-m} \Delta^n a_{i-n} \nabla^m b_j \left( \sum_{r=i}^{N} \sum_{s=1}^{j} \binom{r-i+n-1}{n-1} \binom{j-s+m-1}{m-1} x_{rs} \right).$$

A similar result, analogous to Theorem 1.65 based on the identities from the previous theorem can be given, see [72].

It is interesting to estimate the sum $F$ if the finite differences of the sequences $(a_i)$ and $(b_i)$ are bounded. Firstly we get a result with lower bounds.

**Corollary 1.4** *Let $x_{ij}$, $a_i$, $b_j$, $(1 \le i \le N; 1 \le j \le M)$, be real numbers satisfying conditions* (1.66)-(1.67) *of Theorem* 1.21 *and let*

$$\Delta^n a_i \ge \alpha,\ \Delta^m b_j \ge \beta,\ \ 1 \le i \le N-n,\ 1 \le j \le M-m.$$

*Then*

$$\sum_{i=1}^{N}\sum_{j=1}^{M} x_{ij} a_i b_j \ge \frac{\alpha\beta}{n!m!}\sum_{i=1}^{N}\sum_{j=1}^{M}(i-1)^n (j-1)^m x_{ij}.$$

*Proof.* Under the assumptions of Corollary, the identity from the proof of Theorem 1.21 becomes

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{j=1}^{M} x_{ij} a_i b_j &= \sum_{i=n+1}^{N}\sum_{j=m+1}^{M} x_{ij}\Delta^n a_{i-n}\Delta^m b_{j-m} \times \\
&\times \left(\sum_{r=i}^{N}\sum_{s=j}^{M}\binom{r-i+n-1}{n-1}\binom{s-j+m-1}{m-1}x_{rs}\right) \\
&\ge \alpha\beta\sum_{i=1}^{N}\sum_{j=1}^{M}\left(\sum_{r=i}^{N}\sum_{s=j}^{M}\binom{r-i+n-1}{n-1}\binom{s-j+m-1}{m-1}x_{rs}\right) \\
&= \frac{\alpha\beta}{n!m!}\sum_{i=1}^{N}\sum_{j=1}^{M}(i-1)^n (j-1)^m x_{ij}
\end{aligned}$$

since $\Delta^n (i-1)^n = n!$ and $\Delta^m (j-1)^m = m!$. □

Using the same method we obtain the following result, [72].

**Corollary 1.5** *Let $x_{ij}$, $a_i$, $b_j$, $(1 \le i \le N; 1 \le j \le M)$, be real numbers satisfying conditions* (1.66)-(1.67) *of Theorem* 1.21 *and let*

$$\Delta^n a_i \le R,\ \Delta^m b_j \le S,\ \ 1 \le i \le N-n,\ 1 \le j \le M-m. \tag{1.68}$$

*Then*

$$\left|\sum_{i=1}^{N}\sum_{j=1}^{M} x_{ij} a_i b_j\right| \le \frac{RS}{n!m!}\sum_{i=1}^{N}\sum_{j=1}^{M}(i-1)^n (j-1)^m x_{ij}.$$

**Remark 1.5** Theorem 1.21 can be generalized. Namely, the statement is true if instead of the two sequences $(a_i,\dots,a_N)$ and $(b_1,\dots,b_M)$ we involve a matrix $[a_{ij}]$, $j = 1,\dots,N$, $j = 1,\dots,M$ and the product $a_i b_j$ is substituted with $a_{ij}$. In that case, the phrase "holds for every $n$-convex sequence $(a_i)$ and $m$-convex sequence $(b_j)$" is substituted by "holds for every $(n,m)$-convex $(a_{ij})$", where the sequence $(a_{ij})_{i,j\in\mathbb{N}}$ is $(n,m)$-convex if $\Delta^{n,m}a_{ij} \geq 0$, $(i,j = 1,2,\ldots)$ with notations

$$\Delta^{n,m}a_{ij} = \Delta_1^n(\Delta_2^m a_{ij}), \quad \Delta_1 a_{ij} = a_{i+1,j} - a_{ij},$$

$$\Delta_2 a_{ij} = a_{i,j+1} - a_{ij}, \quad \Delta_1^0 a_{ij} = \Delta_2^0 a_{ij} = a_{ij}.$$

The particular case of Theorem 1.21 for $n = m = 1$, i.e. for monotone sequences $(a_i)$ and $(b_j)$ deserves special attention. Let us repeat the statement of Theorem 1.21 for the monotone sequences, [57].

**Corollary 1.6** *Let $x_{ij}$, $(i = 1,\ldots,n; j = 1,\ldots,m)$ be real numbers. For all sequences $(a_1,\ldots,a_n)$ and $(b_1,\ldots,b_m)$ monotone in the same sense the inequality*

$$\sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij}a_ib_j \geq 0$$

*holds if and only if*

$$X_{r,1} = 0 \ \ (r = 1,\ldots,n), \ \ X_{1,s} = 0 \ \ (s = 2,\ldots,m),$$

$$X_{r,s} \geq 0 \ \ (r = 2,\ldots,n; s = 2,\ldots,m),$$

*where*

$$X_{r,s} = \sum_{i=r}^{n}\sum_{j=s}^{m} x_{ij}.$$

In the subsequent text we show how this result leads to the discrete Čebyšev and Grüss inequalities. Let $\mathbf{a} = (a_1,\ldots,a_n)$, $\mathbf{b} = (b_1,\ldots,b_n)$ and $\mathbf{p} = (p_1,\ldots,p_n)$ be given $n$-tuples, $p_1,\ldots,p_n > 0$. Let us define the Čebyšev difference $D(\mathbf{a},\mathbf{b},\mathbf{p})$ as

$$D(\mathbf{a},\mathbf{b},\mathbf{p}) = \sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_ia_ib_i - \sum_{i=1}^{n} p_ia_i \sum_{i=1}^{n} p_ib_i.$$

Let $x_{ij}$, $(i,j = 1,\ldots,n)$ be defined as

$$x_{ij} = -p_ip_j, \ (i \neq j); \qquad x_{ii} = p_i\left(\sum_{k=1}^{n} p_k - p_i\right).$$

It is easy to see that for these particular numbers $x_{ij}$ the following equalities and inequalities hold:

$$X_{r,1} = 0 \ \ (r = 1,\ldots,n), \ \ X_{1,s} = 0 \ \ (s = 2,\ldots,n),$$

$$X_{r,s} \geq 0, \ \ r,s = 1,2,\ldots,n.$$

Using identity from the proof of Theorem 1.21 for $m = n = 1$ we get that for sequences $\mathbf{a}$ and $\mathbf{b}$ the following holds:

$$D(\mathbf{a},\mathbf{b},\mathbf{p}) \ = \ \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}a_ib_j = \sum_{r=2}^{n}\sum_{s=2}^{n} X_{rs}(a_r - a_{r-1})(b_s - b_{s-1}).$$

Furthermore, since $X_{r,s} \geq 0$, **a** and **b** are monotone in the same sense, then

$$D(\mathbf{a},\mathbf{b},\mathbf{p}) \geq 0,$$

i.e.

$$\sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i a_i b_i \geq \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i$$

which is, in fact, the discrete Čebyšev inequality. More results about the Čebyšev inequality will be given in Chapter 6.

If **a** and **b** are monotone, then

$$\begin{aligned} |D(\mathbf{a},\mathbf{b},\mathbf{p})| &= = \left| \sum_{r=2}^{n} \sum_{s=2}^{n} X_{rs}(a_r - a_{r-1})(b_s - b_{s-1}) \right| \\ &\leq \max_{1\leq r,s\leq n} |X_{rs}||a_n - a_1||b_n - b_1| \\ &= |a_n - a_1||b_n - b_1| \max_{1\leq k\leq n-1} \sum_{i=1}^{k} p_i \sum_{i=k+1}^{n} p_i. \end{aligned} \tag{1.69}$$

Now we can state and prove the discrete Grüss inequality.

**Proposition 1.1** *Let* **a**, **b** *be given real n-tuples such that*

$$a \leq a_i \leq A, \quad b \leq b_i \leq B, \quad (i = 1,2,\ldots,n).$$

*Then*

$$|D(\mathbf{a},\mathbf{b},1)| \leq (A-a)(B-b)\left[\frac{n}{2}\right]\left(n - \left[\frac{n}{2}\right]\right). \tag{1.70}$$

*Proof.* Let $\overline{\mathbf{a}} = (\overline{a_1},\ldots,\overline{a_n})$ be the increasing rearrangement of **a**, and $\underline{\mathbf{a}} = (\underline{a_1},\ldots,\underline{a_n})$ be the decreasing rearrangement of **a** and define $\overline{\mathbf{b}}$, $\underline{\mathbf{b}}$ similarly. Then

$$\sum_{i=1}^{n} \overline{a_i}\underline{b_i} \leq \sum_{i=1}^{n} a_i b_i \leq \sum_{i=1}^{n} \overline{a_i}\overline{b_i},$$

i.e.

$$D(\overline{\mathbf{a}},\underline{\mathbf{b}},1) \leq D(\mathbf{a},\mathbf{b},1) \leq D(\overline{\mathbf{a}},\overline{\mathbf{b}},1).$$

Using (1.69) on the pairs of the monotone sequences $\overline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ and on $\overline{\mathbf{a}}$ and $\overline{\mathbf{b}}$ we get

$$\begin{aligned} |D(\overline{\mathbf{a}},\underline{\mathbf{b}},1)|,|D(\overline{\mathbf{a}},\overline{\mathbf{b}},1)| &\leq (A-a)(B-b) \max_{1\leq k\leq n-1} \sum_{i=1}^{k} p_i \sum_{i=k+1}^{n} p_i \\ &= (A-a)(B-b)\left[\frac{n}{2}\right]\left(n - \left[\frac{n}{2}\right]\right), \end{aligned}$$

from which inequality (1.70) follows. □

Let us mention that the above discrete Grüss inequality was proved in 1950 by M. Biernacki, H. Pidek and C. Ryll-Nardzewski, [51, p.299].

The following theorem is a multilinear generalization of Theorem 1.6, (see [58]). We consider $m$ monotone sequences, give an identity for the sum $\sum_{i_1=1}^{n_1} \cdots \sum_{i_m=1}^{n_m} x_{i_1\dots i_m} a_{1i_1} \dots a_{mi_m}$ and necessary and sufficient conditions for positivity of that sum.

**Theorem 1.23** *Let* $x_{i_1\dots i_m}$ $(i_k = 1,\dots,n_k; k = 1,\dots,m)$ *be real numbers.*

$(i)$ *The inequality*

$$\sum_{i_1=1}^{n_1} \cdots \sum_{i_m=1}^{n_m} x_{i_1\dots i_m} a_{1i_1} \dots a_{mi_m} \geq 0 \tag{1.71}$$

*holds for all nondecreasing sequences* $\mathbf{a_j} = (a_{j1},\dots,a_{jn_j})$, $(j = 1,\dots,m)$ *if and only if*

$$X_{s_1\cdots s_m} \geq 0 \;\; (s_j = 2,\dots,n_j; j = 1,\dots,m),$$

$$X_{s_1\dots s_{k-1},1,s_{k+1},\dots s_m} = 0$$

*for* $s_j = 1,\dots,n_j;\ j = 1,\dots,k-1,k+1,\dots,m;\ k = 1,\dots,m$, *where*

$$X_{s_1\cdots s_m} = \sum_{i_1=s_1}^{n_1} \cdots \sum_{i_m=s_m}^{n_m} x_{i_1\dots i_m}.$$

$(ii)$ *Inequality* $(1.71)$ *holds for all nonnegative nondecreasing sequences* $\mathbf{a_j}$, $(j = 1,\dots,m)$ *if and only if*

$$X_{s_1\cdots s_m} \geq 0 \;\; (s_j = 1,\dots,n_j; j = 1,\dots,m).$$

*Proof.* The proof is based on the identity:

$$\begin{aligned}
&\sum_{i_1=1}^{n_1} \cdots \sum_{i_m=1}^{n_m} x_{i_1\dots i_m} a_{1i_1} \dots a_{mi_m} = a_{11} \dots a_{m1} X_{1\dots 1} \\
&+ \sum_{k=1}^{m-1} \sum_{\binom{m}{k}} \left\{ a_{i1} \dots a_{k1} \sum_{s_{k+1}=2}^{n_{k+1}} \cdots \sum_{s_m=2}^{n_m} X_{1\dots 1,s_{k+1},\dots s_m} \prod_{j=k+1}^{m} (a_{j,s_j} - a_{j,s_j-1}) \right\} \\
&+ \sum_{s_1=2}^{n_1} \cdots \sum_{s_m=2}^{n_m} X_{s_1\cdots s_m} \prod_{j=1}^{m} (a_{j,s_j} - a_{j,s_j-1}).
\end{aligned}$$

□

The above-mentioned article [58] contains some further results connected with lower and upper bounds for this sum also.

## 1.6.2 Results for $p$-monotone, $p,q$-convex and Starshaped Sequences

As like for convex sequence of higher order, similar results about necessary and sufficient conditions for the numbers $x_{ij}$ such that inequality $\sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} a_i b_j \geq 0$ holds for $p$-monotone and $p,q$-convex sequences is stated in [60].

**Theorem 1.24** *(i) Let $(a_i),(b_i)$ be real sequences and $x_{ij}$, $(i \in \{1,2,\dots,n\}, j \in \{1,2,\dots,m\})$ be real numbers. The inequality*

$$\sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij}a_ib_j \geq 0$$

*holds for every p-monotone sequence $(a_i)$ and for every q-monotone sequence $(b_i)$ if and only if*

$$X_{1,s}=0,\ \ X_{r,1}=0,\ \ s\in\{1,2,\dots,m\}, r\in\{1,2,\dots,n\}, \tag{1.72}$$

$$X_{r,s}\geq 0,\quad s\in\{2,\dots,m\}, r\in\{2,\dots,n\}, \tag{1.73}$$

*where*

$$X_{r,s}=\sum_{i=r}^{n}\sum_{j=s}^{m} p^{i-r}q^{j-s}x_{ij}.$$

*(ii) The inequality $\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m} x_{ij}a_ib_j \geq 0$ holds for every p-monotone sequence $(a_i)$ and for every q-monotone sequence $(b_i)$ such that $a_1 \geq 0$ and $b_1 \geq 0$ if and only if*

$$X_{r,s}\geq 0,\quad s\in\{1,2,\dots,m\}, r\in\{1,2,\dots,n\}.$$

*Proof.* Let us prove case $(i)$. Let $s_j=\sum_{i=1}^{n} x_{ij}a_i$. Then

$$\sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij}a_ib_j=\sum_{j=1}^{m} s_jb_j=b_1\sum_{j=1}^{m} q^{j-1}s_j+\sum_{i=2}^{m}\left(\sum_{j=s}^{m} q^{j-s}s_j\right)L_q(b_{s-1}).$$

Let us write $x_i=\sum_{j=s}^{m} q^{j-s}x_{ij}$. Then

$$\begin{aligned}\sum_{j=s}^{m} q^{j-s}s_j &= \sum_{i=1}^{n}\left(\sum_{j=s}^{m} q^{j-s}x_{ij}\right)a_i=\sum_{i=1}^{n} x_ia_i\\ &= a_1\sum_{i=1}^{n} p^{i-1}x_i+\sum_{r=2}^{n}\left(\sum_{i=r}^{n} p^{i-r}x_i\right)L_p(a_{r-1})\\ &= a_1X_{1,s}+\sum_{r=2}^{n} X_{r,s}L_p(a_{r-1}).\end{aligned}$$

For $s=1$ we have

$$\sum_{j=1}^{m} q^{j-1}s_j=a_1X_{1,1}+\sum_{r=2}^{n} X_{r,1}L_p(a_{r-1}).$$

Hence,

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij}a_ib_j &= a_1b_1X_{1,1}+b_1\sum_{r=2}^{n} X_{r,1}L_p(a_{r-1})+a_1\sum_{s=2}^{m} X_{1,s}L_q(b_{s-1})\\ &\quad+\sum_{r=2}^{n}\sum_{s=2}^{m} X_{r,s}L_p(a_{r-1})L_q(b_{s-1}).\end{aligned} \tag{1.74}$$

From this identity we conclude that if $x_{ij}$ satisfy assumptions (1.72) and (1.73), then $\sum_{i=1}^n \sum_{j=1}^m x_{ij}a_ib_j \geq 0$.

Let us prove the second implication. Let us consider the $p$-monotone sequence $(a_i)$ defined as $a_i = 0$ for $i = 1,2,\ldots,r-1$, $a_i = p^{i-r}$ for $i = r,\ldots,n$, and the $q$-monotone sequences $(b_j)_{j=1}^m = (q^{j-1})_{j=1}^m$ and $(b_j)_{j=1}^m = (-q^{j-1})_{j=1}^m$. For these sequences the inequality $\sum_{i=1}^n \sum_{j=1}^m x_{ij}a_ib_j \geq 0$ holds, so, the condition $X_{r,1} = 0$ holds. Similarly, we get $X_{1,s} = 0$. Let us define the sequences $(a_i)$ and $(b_j)$:

$$a_i = 0,\ \ i = 1,2,\ldots,r-1,\ \ a_i = p^{i-r},\ \ i = r,\ldots,n,$$

$$b_j = 0,\ \ j = 1,2,\ldots,s-1,\ \ b_j = q^{j-s},\ \ j = s,\ldots,m.$$

These sequences are $p$-monotone and $q$-monotone respectively, so from $\sum_{i=1}^n \sum_{j=1}^m x_{ij}a_ib_j \geq 0$ we get $X_{r,s} \geq 0$. □

Also, a similar result for bilinear form for $p,q$-convex functions is valid, [47]. The key role plays identity (1.74).

**Theorem 1.25** *Let $(a_i),(b_i)$ be real sequences and $x_{ij}$, $(i \in \{1,2,\ldots,n\}, j \in \{1,2,\ldots,m\})$ be real numbers. The inequality*

$$\sum_{i=1}^n \sum_{j=1}^m x_{ij}a_ib_j \geq 0$$

*holds for every $p,t$-convex sequence $(a_i)$ and $q,r$-convex sequence $(b_i)$ if and only if*

$$\begin{aligned}
&X_{1,1} = 0, \quad B_{r,1}^1 = 0 \quad (r = 2,\ldots,n), \quad B_{1,s}^2 = 0 \quad (s = 2,\ldots,m),\\
&B_{r,2} = 0 \quad (r = 3,\ldots n), \quad B_{2,s} = 0 \quad (s = 3,\ldots,m),\\
&B_{r,s} \geq 0 \quad (r = 3,\ldots,n; \quad s = 3,\ldots,m),
\end{aligned}$$

*where*

$$X_{r,s} = \sum_{i=r}^n \sum_{j=s}^m p^{i-r}q^{j-s}x_{ij}, \qquad r = 1,\ldots,n; \quad s = 1,\ldots,m,$$

$$B_{r,1}^1 = \begin{cases} \displaystyle\sum_{i=r}^n \sum_{j=1}^m q^{j-1}\frac{p^{i-r+1}-t^{i-r+1}}{p-t}x_{ij}, & p \neq t\\ \displaystyle\sum_{i=r}^n \sum_{j=1}^m (i-r+1)p^{i-r}q^{j-1}x_{ij}, & p = t \end{cases}$$

*for $r = 2,\ldots,n$,*

$$B_{1,s}^2 = \begin{cases} \displaystyle\sum_{i=1}^n \sum_{j=s}^m p^{i-1}\frac{q^{j-s+1}-r^{j-s+1}}{q-r}x_{ij}, & q \neq r\\ \displaystyle\sum_{i=1}^n \sum_{j=s}^m (j-s+1)p^{i-1}q^{j-s}x_{ij}, & q = r \end{cases}$$

*for* $s = 2,\ldots,m$, *and*

$$B_{r,s} = \begin{cases} \sum_{i=r}^{n}\sum_{j=s}^{m} \frac{p^{i-r+1}-t^{i-r+1}}{p-t}\cdot\frac{q^{j-s+1}-r^{j-s+1}}{q-r}x_{ij}, & p\neq t, q\neq r \\ \sum_{i=r}^{n}\sum_{j=s}^{m}(i-r+1)(j-s+1)p^{i-r}q^{j-s}x_{ij}, & p=t, q=r \end{cases}$$

*for* $r = 2,\ldots,n$, $s = 2,\ldots,m$.

*Proof.* Using identities (1.51) and (1.74) we get the following

$$\sum_{r=2}^{n} X_{r,1}L_p(a_{r-1}) = B^1_{2,1}L_p(a_1) + \sum_{r=3}^{n} B^1_{r,1}L_{pt}(a_{r-2}) \tag{1.75}$$

$$\sum_{s=2}^{m} X_{1,s}L_q(b_{s-1}) = B^2_{1,2}L_q(b_1) + \sum_{s=3}^{m} B^2_{1,s}L_{qr}(b_{s-2}) \tag{1.76}$$

$$\begin{aligned} \sum_{r=2}^{n}\sum_{s=2}^{m} X_{r,s}L_p(a_{r-1})L_q(b_{s-1}) &= B_{2,2}L_p(a_1)L_q(b_1) \\ &+L_q(b_1)\sum_{r=3}^{n} B_{r,2}L_{pt}(a_{r-2}) + L_p(a_1)\sum_{s=3}^{m} B_{2,s}L_{qr}(b_{r-2}) \\ &+\sum_{r=3}^{n}\sum_{s=3}^{m} B_{r,s}L_{pt}(a_{r-2})L_{qr}(b_{s-2}). \end{aligned} \tag{1.77}$$

By inserting (1.75), (1.76) and (1.77) in (1.74) we obtain

$$\begin{aligned} \sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij}a_ib_j &= X_{1,1}a_1b_1 + b_1L_p(a_1)B^1_{2,1} + a_1L_q(b_1)B^2_{1,2} \\ &+B_{2,2}L_p(a_1)L_q(b_1) + b_1\sum_{r=3}^{n} B^1_{r,1}L_{pt}(a_{r-2}) + a_1\sum_{s=3}^{m} B^2_{1,s}L_{qr}(b_{s-2}) \\ &+L_q(b_1)\sum_{r=3}^{n} B_{r,2}L_{pt}(a_{r-2}) + L_p(a_1)\sum_{s=3}^{m} B_{2,s}L_{qr}(b_{r-2}) \\ &+\sum_{r=3}^{n}\sum_{s=3}^{m} B_{r,s}L_{pt}(a_{r-2})L_{qr}(b_{s-2}). \end{aligned} \tag{1.78}$$

It is the identity from which the statement of the theorem follows. □

Analogously, we can prove the following result which combines $p,t$-convex and $q$-monotone sequences, [47].

**Theorem 1.26** *Let* $(a_i),(b_i)$ *be real sequences and* $x_{ij}$, $(i\in\{1,2,\ldots,n\}, j\in\{1,2,\ldots,m\})$ *be real numbers. The inequality*

$$\sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij}a_ib_j \geq 0$$

*holds for every $p,t$-convex sequence $(a_i)$ and $q$-monotone sequence $(b_i)$ if and only if*

$$
\begin{aligned}
&X_{1,s}=0, \quad (s=1,\dots,m)\\
&R_{r,1}=0 \quad (r=2,\dots n), \qquad R_{2,s}=0 \quad (s=2,\dots,m)\\
&R_{r,s}\geq 0 \quad (r=3,\dots,n; \quad s=2,\dots,m),
\end{aligned}
$$

*where $X_{r,s}$ ia defined in the previous theorem and*

$$
R_{r,s}=\begin{cases}\sum_{i=r}^{n}\sum_{j=s}^{m} q^{j-s}\dfrac{p^{i-r+1}-t^{i-r+1}}{p-t}x_{ij}, & p\neq t\\ \sum_{i=r}^{n}\sum_{j=s}^{m}(i-r+1)p^{i-r}q^{j-s}x_{ij}, & p=t\end{cases}
$$

*for $r=2,\dots,n$, $s=1,\dots,m$.*

In [70] a bilinear form for a starshaped sequence is considered. Namely, the following theorem is given:

**Theorem 1.27** *Let $a_i,b_j$ and $x_{ij}$, $i=0,1,\dots N; j=0,1,\dots,M$, be real numbers. Then the inequality*

$$
\sum_{i=0}^{N}\sum_{j=0}^{M} a_i b_j x_{ij}\geq 0
$$

*holds for every sequence $(a_i)$ starshaped of order $n$ and sequence $(b_i)$ starshaped of order $m$ if and only if*

$$
\begin{aligned}
&\sum_{i=0}^{N}\sum_{j=0}^{M} x_{ij}=0,\\
&\sum_{i=r}^{N}\sum_{j=0}^{M}\binom{i}{r}x_{ij}=0, \ \sum_{i=0}^{N}\sum_{j=s}^{M}\binom{j}{s}x_{ij}=0, \ \ r=1,\dots,n-1; s=1,\dots,m-1,\\
&\sum_{i=r}^{N}\sum_{j=0}^{M} i\binom{i-r+n-2}{n-2}x_{ij}=0, \quad \sum_{i=0}^{N}\sum_{j=s}^{M} j\binom{j-s+m-2}{m-2}x_{ij}=0,\\
&\mathit{for} \ \ r=n,\dots,N; s=m,\dots,M,\\
&\sum_{i=r}^{N}\sum_{j=s}^{M}\binom{i}{r}\binom{j}{s}x_{ij}=0, r=1,\dots,n-1; s=1,\dots,m-1,\\
&\sum_{i=r}^{N}\sum_{j=s}^{M} i\binom{i-r+n-2}{n-2}\binom{j}{s}x_{ij}=0, r=n,\dots,N; s=1,\dots,m-1,\\
&\sum_{i=r}^{N}\sum_{j=s}^{M} j\binom{i}{r}\binom{j-s+m-2}{m-2}x_{ij}=0, r=1,\dots,n-1; s=m,\dots,M,\\
&\sum_{i=r}^{N}\sum_{j=s}^{M} ij\binom{i-r+n-2}{n-2}\binom{j-s+m-2}{m-2}x_{ij}\geq 0, r=n,\dots,N; s=m,\dots,M.
\end{aligned}
$$

*Proof.* The proof is based on the following identity:

$$\begin{aligned}
\sum_{i=0}^{N}\sum_{j=0}^{M} a_i b_j x_{ij} &= a_0 b_0 \sum_{i=0}^{N}\sum_{j=0}^{M} x_{ij} + \sum_{r=1}^{n-1}\sum_{s=1}^{m-1} rs T_r(a_1) T_s(b_1) \sum_{i=r}^{N}\sum_{j=s}^{M} \binom{i}{r}\binom{j}{s} x_{ij} \\
&+ \sum_{r=n}^{N}\sum_{s=m}^{M} T_n(a_{r-n+1}) T_m(b_{s-m+1}) \times \\
&\times \sum_{i=r}^{N}\sum_{j=s}^{M} ij \binom{i-r+n-2}{n-2}\binom{j-s+m-2}{m-2} x_{ij} \\
&+ a_0 \sum_{s=1}^{m-1} s T_s(b_1) \sum_{i=0}^{N}\sum_{j=s}^{M} \binom{j}{s} x_{ij} + b_0 \sum_{r=1}^{n-1} r T_r(a_1) \sum_{i=r}^{N}\sum_{j=0}^{M} \binom{i}{r} x_{ij} \\
&+ a_0 \sum_{s=m}^{M} T_m(b_{s-m+1}) \sum_{i=r}^{N}\sum_{j=s}^{M} j \binom{i}{r}\binom{j-s+m-2}{m-2} x_{ij} \\
&+ b_0 \sum_{r=n}^{N} T_n(a_{r-n+1}) \sum_{i=r}^{N}\sum_{j=s}^{M} i \binom{i-r+n-2}{n-2}\binom{j}{s} x_{ij} \\
&+ \sum_{r=1}^{n-1}\sum_{s=m}^{M} r T_r(a_1) T_m(b_{s-m+1}) \sum_{i=r}^{N}\sum_{j=s}^{M} j \binom{i}{r}\binom{j-s+m-2}{m-2} x_{ij} \\
&+ \sum_{r=n}^{N}\sum_{s=1}^{m-1} s T_r(a_{r-n+1}) T_s(b_1) \sum_{i=r}^{N}\sum_{j=s}^{M} i \binom{i-r+n-2}{n-2}\binom{j}{s} x_{ij},
\end{aligned}$$

where $T_m(a_n) = \Delta^{m-1}\left(\dfrac{a_n - a_0}{n}\right)$. □

Chapter 2

# General Linear Inequalities for Functions of One Variable

## 2.1 Basic Results on Convexity of Higher Order

While the previous chapter was devoted to different classes of sequences, this chapter brings results mostly about functions which are convex of higher order. The concept of higher convexity was introduced by T. Popoviciu in the forties of the previous century, [85]. We are interested in results connected with positivity of sum $\sum p_k f(x_k)$ or integral $\int p(x)f(x)dx$ where $f$ is a convex function of higher order. Such results which involve higher order convex function and which solve the question of necessary and sufficient conditions for positivity of the mentioned sum or integral are called Popoviciu type inequality.

Let $f$ be a real-valued function defined on $I = [a,b] \subset \mathbb{R}$. The $n$-th order divided difference of $f$ at distinct points $x_i, x_{i+1}, \ldots, x_{i+n}$ in $I$ is defined recursively by:

$$[x_j; f] = f(x_j), \quad i \leq j \leq i+n$$

$$[x_i, \ldots, x_{i+n}; f] = \frac{[x_{i+1}, \ldots, x_{i+n}; f] - [x_i, \ldots, x_{i+n-1}; f]}{x_{i+n} - x_i}.$$

It is easy to see that

$$[x_i, \ldots, x_{i+n}; f] = \sum_{k=0}^{n} \frac{f(x_{i+k})}{w'(x_{i+k})},$$

where $w(x) = \prod_{j=i}^{i+n}(x-x_j)$.

In this book we use notation $\Delta^n f(x_i)$ for $[x_i,\ldots,x_{i+n};f]$ also.

We say that $f : I \to \mathbb{R}$ is a convex function of order $n$ (or $n$-convex function or $\Delta$-convex function of order $n$) if for all choices of $n+1$ distinct points $x_i,\ldots,x_{i+n} \in I$ inequality

$$[x_i,\ldots,x_{i+n};f] \geq 0$$

holds. The function $f$ is said to be $\nabla$-convex of order $n$ if for all choices of $n+1$ distinct points $x_i,\ldots,x_{i+n}$ inequality

$$\nabla^n f(x_i) = (-1)^n \Delta^n f(x_i) \geq 0$$

holds.

If $n = 0$, then a convex function $f$ of order 0 is, in fact, a nonnegative function, 1-convex function is nondecreasing function, while a class of 2-convex functions coincides with a class of convex functions. It is well-known that if $f^{(n)}$ exists, then $f$ is $n$-convex if and only if $f^{(n)} \geq 0$. Furthermore, if $f$ is an $n$-convex function on $[a,b]$ for $n \geq 2$, then the function $f^{(k)}$ exists and is $(n-k)$-convex for $1 \leq k \leq n-2$; $f_{+}^{(n-1)}$ exists and is right continuous and increasing in $[a,b\rangle$.

Let $E = \{x_1,x_2,\ldots,x_N\} \subset \mathbb{R}$. A function $f : E \to \mathbb{R}$ is said to be a discrete $n$-convex function if inequality

$$[x_i,\ldots,x_{i+n};f] \geq 0$$

holds for all choices of $n+1$ distinct points $x_i,\ldots,x_{i+n} \in E$. Similarly, a discrete $\nabla$-convex function of order $n$ is defined.

If $\{x'_1,\ldots,x'_{k+1}\} \subset E$, then there exist nonnegative constants $A_1,\ldots, A_{N-k-1}$ such that $\sum_{i=1}^{N-k-1} A_i = 1$ and

$$[x'_1,\ldots,x'_{k+1};f] = \sum_{i=1}^{N-k-1} A_i[x_i,\ldots,x_{i+k};f].$$

We use also the following notation.

For $n$ real numbers $x_i$, $i \in \{1,\ldots,n\}$ and $m \geq 0$:

$$(x_k - x_i)^{(m+1)} = (x_k - x_i)(x_k - x_{i+1})\cdots(x_k - x_{i+m}), \quad (x_k - x_i)^{(0)} = 1$$

and

$$(x_k - x_i)^{\{m+1\}} = (x_k - x_i)(x_{k-1} - x_i)\cdots(x_{k-m} - x_i), \quad (x_k - x_i)^{\{0\}} = 1.$$

***Example 2.1*** Let $r \in \mathbb{N}$. The function $e_r : [a,b] \to \mathbb{R}$ defined as

$$e_r(x) = x^r$$

is $n$-convex for any $n \geq r$. If $[a,b] \subset \langle 0,\infty\rangle$, then $e_r$ is $n$-convex of any order $n$. This consideration can be expanded to general polynomials, i.e. a polynomial of degree $r$ is $n$-convex for $n \geq r$.

***Example 2.2*** Let $E = \{x_1, x_2, \ldots, x_N\} \subset \mathbb{R}$, $x_1 < x_2 < \ldots < x_N$. Let $i > n-1$ be a fixed number. A function $f : E \to \mathbb{R}$ defined as:

$$h(x) = \begin{cases} 0, & x \le x_{i-1} \\ (x - x_{i-n+1})(x - x_{i-n+2}) \ldots (x - x_{i-1}), & x > x_{i-1} \end{cases}$$

is discrete $n$-convex.

To see this, it is enough to prove $[x_k, \ldots, x_{k+n}; h] \ge 0$ for any $k \le N-n$. We consider several cases. If $k+n \le i-1$, then $h(x_j) = 0$ for all $j = k, \ldots, k+n$ and $[x_k, \ldots, x_{k+n}; h] = 0$.

If $k+n = i+r$, $0 \le r \le n$, then

$$[x_k, \ldots, x_{k+n}; h] = [x_{k+n-r}, \ldots, x_{k+n}; (x - x_{k-r+1}) \ldots (x - x_{k-1})] \ge 0,$$

because the function $g(x) = (x - x_{k-r+1}) \ldots (x - x_{k-1})$ is a polynomial of the $(r-1)$th degree and, hence, it is $n$-convex.

If $k+n = i+r$, $r > n$, then

$$[x_k, \ldots, x_{k+n}; h] = [x_k, \ldots, x_{k+n}; (x - x_{i-n+1}) \ldots (x - x_{i-1})] \ge 0,$$

because the function $g(x) = (x - x_{i-n+1}) \ldots (x - x_{i-1})$ is a polynomial of the $(n-1)$th degree and, hence, it is $n$-convex.

***Example 2.3*** Let $s \in [a,b]$, $n \in \mathbb{N}$. The function $w_n(\cdot, s) : [a,b] \to \mathbb{R}$ is defined as

$$w_n(x,s) = (x-s)_+^{n-1},$$

where

$$(x-s)_+ = \begin{cases} 0, & x < s \\ x-s, & x \ge s. \end{cases}$$

Particularly, for $n = 1$ we get

$$w_1(x,s) = \begin{cases} 0, & x < s \\ 1, & x \ge s. \end{cases}$$

Function $w_n(\cdot, s)$ is $n$-convex. If $n = 1$ it is obvious that $w_1(x,s)$ is nondecreasing, i.e. it is 1-convex. For $m \ge 2$, let us consider the $(n-2)$th derivative of $w_n(\cdot, s)$:

$$\frac{d^{n-2}}{dx^{n-2}} w_n(x,s) = \begin{cases} 0, & x < s \\ (n-1)!\,(x-s), & x \ge s. \end{cases}$$

It is 2-convex, so $w_n(\cdot, s)$ is $n$-convex.

One of the crucial results for studying general linear inequalities involving the higher order convex functions is the following theorem due to T. Popoviciu, [77].

**Theorem 2.1** *Let $x_k \in [a,b]$ and $p_k \in \mathbb{R}$, $k = 1, \ldots, N$ and $n \in \mathbb{N}$.*

*Then the inequality*

$$\sum_{k=1}^{N} p_k f(x_k) \ge 0 \tag{2.1}$$

*holds for every convex function f of order n if and only if*

$$\sum_{k=1}^{N} p_k x_k^i = 0, \quad i \in \{0,\ldots,n-1\}, \tag{2.2}$$

$$\sum_{k=i}^{N} p_k (x_k - s)_+^{n-1} \geq 0, \quad s \in [a,b]. \tag{2.3}$$

In further decades some generalizations of that results are done. Let us mention result which involve linear operator $A$ instead an operator of sum. In the next text we describe a method based on a representation of convex function of higher order as a limit of the sequence of functions which are equal to a sum of certain polynomial and a linear combination of very particular functions, i.e. splines $w_n(\cdot,x_j)$. Let us describe some new notations which are used in this section.

We consider operators $A$ of the following form $A : C([a,b]) \to S(D)$, where $S(D)$ is one of the normed subspaces of the space of all real functions defined on $D$, and where the norm of a function $f \in S(D)$ is denoted by $\|f\|_D$. We say that $A$ is continuous if $\lim_n \|f_n - f\| = 0$ implies $\lim_n \|Af_n - Af\|_D = 0$ as well. Also, we write $Af \geq 0$ if $Af(t) \geq 0$ holds for every $t \in D$, where $f$ is a given function in the space $C([a,b])$.

The family of the polynomials of degree at most $k$ is denoted by $\Pi_k$. The family of continuous $n$-convex functions on $[a,b]$ (i. e. right-continuous at $a$ and left-continuous at $b$) is denoted by $K_n([a,b])$. Monomials are denoted by $e_i$, i. e. $e_i(x) = x^i$ for $i = 0,1,2,\ldots$

T. Popoviciu proved the following representation of $n$-convex functions, [90].

**Lemma 2.1** *Let the function $F_n$ be of the form*

$$F_n(x) = P_{n-1}(x) + \sum_{i=1}^{k} \alpha_i w_n(x,x_i), \tag{2.4}$$

*where $P_{n-1} \in \Pi_{n-1}$, $\alpha_i$, $i = 1,\ldots,k$, are real constants and $a \leq x_1 < x_2 < \cdots < x_k \leq b$.*

(*a*) *A necessary and sufficient condition for $F_n$ to be n-convex is that $\alpha_i \geq 0$ $(i = 1,\ldots,k)$.*

(*b*) *Every continuous n-convex function on $[a,b]$ is the uniform limit of the sequence of functions $F_n$ $(n = 1,2,\ldots)$ where the $F_n$'s are of the form in (2.4) and $\alpha_i \geq 0$ $(i = 1,\ldots,k)$ are real constants.*

Using that representation of $n$-convex functions the following theorem was proved in [38], see also [77].

**Theorem 2.2** *Let $A : C([a,b]) \to S(D)$ be a linear and continuous operator and $n \geq 2$. Then, the inequality*

$$Af \geq 0 \tag{2.5}$$

*holds for every function $f \in K_n([a,b])$ if and only if the operator A satisfies:*

$$Ae_i = 0 \ \text{ for } \ i = 0,1,\ldots,n-1, \tag{2.6}$$

$$Aw_n(\cdot,s) \geq 0 \ \text{ for every } \ s \in [a,b]. \tag{2.7}$$

More related results can be found in monograph [77].

## 2.2 Approach via Taylor's Formula

In this section we focus our attention on two particular cases of a linear operator $A$ and reach necessary and sufficient conditions for the inequality $Af \geq 0$ by a different method. That method is based on using of the Taylor expansion of a function $f$.

Let $I$ be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous on $I \subseteq \mathbb{R}$, $a, b \in I$, $a < b$. Then for $c, x \in [a,b]$ the following formula holds

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{1}{(n-1)!} \int_c^x f^{(n)}(s)(x-s)^{n-1} ds. \tag{2.8}$$

It is called the Taylor expansion of a function $f$ around a point $c$.

The following theorem contains identities for the sum $\sum_{i=1}^N p_i f(x_i)$ and the integral $\int_\alpha^\beta p(x) f(g(x))\, dx$.

**Theorem 2.3** *(i) Let $N, n \in \mathbb{N}$ and $f : I \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous on $I \subset \mathbb{R}$, $a, b \in I$, $a < b$. Furthermore, let $x_i \in [a,b]$ and $p_i \in \mathbb{R}$ for $i \in \{1, 2, \ldots, N\}$. Then*

$$\begin{aligned} \sum_{i=1}^N p_i f(x_i) &= \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} \sum_{i=1}^N p_i (x_i - a)^k \\ &+ \frac{1}{(n-1)!} \int_a^b f^{(n)}(s) \left( \sum_{i=1}^N p_i (x_i - s)_+^{n-1} \right) ds \end{aligned} \tag{2.9}$$

*and*

$$\begin{aligned} \sum_{i=1}^N p_i f(x_i) &= \sum_{k=0}^{n-1} (-1)^k \frac{f^{(k)}(b)}{k!} \sum_{i=1}^N p_i (b - x_i)^k \\ &+ \frac{(-1)^n}{(n-1)!} \int_a^b f^{(n)}(s) \left( \sum_{i=1}^N p_i (s - x_i)_+^{n-1} \right) ds. \end{aligned}$$

*(ii) Let $p, g : [a,b] \to \mathbb{R}$ be integrable functions and let $f$ satisfy assumptions from part (i). Then*

$$\begin{aligned} \int_\alpha^\beta p(x) f(g(x))\, dx &= \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} \int_\alpha^\beta p(x)(g(x) - a)^k dx \\ &+ \frac{1}{(n-1)!} \int_a^b f^{(n)}(s) \int_\alpha^\beta p(x)(g(x) - s)_+^{n-1} dx ds, \end{aligned} \tag{2.10}$$

$$\int_\alpha^\beta p(x) f(g(x))\, dx = \sum_{k=0}^{n-1} (-1)^k \frac{f^{(k)}(b)}{k!} \int_\alpha^\beta p(x)(b - g(x))^k dx$$

$$+\frac{(-1)^n}{(n-1)!}\int_a^b f^{(n)}(s)\int_\alpha^\beta p(x)(s-g(x))_+^{n-1}dxds.$$

*Proof.* (i) We get

$$\int_a^x f^{(n)}(s)(x-s)^{n-1}ds=\int_a^b f^{(n)}(s)(x-s)_+^{n-1}ds$$

for $x\in[a,b]$ and applying the Taylor formula (2.8) for $c=a$ we get

$$f(x)=\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k+\frac{1}{(n-1)!}\int_a^b f^{(n)}(s)(x-s)_+^{n-1}ds. \tag{2.11}$$

Putting in (2.11) $x=x_i$, multiplying with $p_i$ and adding all identities for $i=1,\ldots,N$ we get (2.9).

The second statement in (i) is obtained by the Taylor formula for $c=b$ and using the fact that for $x\in[a,b]$

$$\int_b^x f^{(n)}(s)(x-s)^{n-1}ds=(-1)^n\int_a^b f^{(n)}(s)(s-x)_+^{n-1}ds.$$

(ii) Putting in (2.11) $x=g(x)$, multiplying with $p(x)$, integrating over $[\alpha,\beta]$ and using the Fubini theorem we get (2.10). The second identity is obtained in a similar manner as the second identity in (i). □

**Remark 2.1** The above theorem is given in [31]. The identity (2.9) for a sum is also given in [7].

Let us proceed with theorem which contains necessary and sufficient conditions that inequalities $\sum_{k=1}^N p_k f(x_k)\geq 0$ and $\int_\alpha^\beta p(x)f(g(x))\,dx\geq 0$ hold for every convex function of order $n$. As we can see, the first part is, in fact, known Popoviciu's result.

**Theorem 2.4** *(i) Let the assumptions of Theorem* 2.3*(i) be valid.*

*Then the inequality*

$$\sum_{k=1}^N p_k f(x_k)\geq 0 \tag{2.12}$$

*holds for every n-convex function* $f:[a,b]\to\mathbb{R}$ *if and only if*

$$\sum_{k=1}^N p_k x_k^i = 0,\quad i\in\{0,\ldots,n-1\}, \tag{2.13}$$

$$\sum_{k=i}^N p_k(x_k-s)_+^{n-1} \geq 0,\quad s\in[a,b]. \tag{2.14}$$

*(ii) Let the assumptions of Theorem* 2.3*(ii) hold. Then the inequality*

$$\int_\alpha^\beta p(x)f(g(x))\,dx\geq 0 \tag{2.15}$$

*holds for all n-convex functions $f : [a,b] \to \mathbb{R}$ if and only if*

$$\int_{\alpha}^{\beta} p(x)g^k(x)\,dx = 0, \quad \text{for all } k \in \{0,1,\ldots,n-1\} \tag{2.16}$$

$$\int_{\alpha}^{\beta} p(x)\,(g(x)-s)_+^{n-1}\,dx \geq 0, \quad \text{for every } s \in [a,b]. \tag{2.17}$$

*Proof.* (i) If (2.13) and (2.14) are valid, then from identity (2.9) we get that for any $n$-convex function $f$ inequality (2.12) holds.

Let us prove the opposite direction of equivalence. If inequality (2.12) holds for every $n$-convex continuous function $f$ on $[a,b]$, then, since $e_i$, $-e_i$, $i = 0,\ldots,n-1$ and $w_n(x,s)$, $s \in [a,b]$, are $n$-convex functions, we get conditions (2.13) and (2.14).

(ii) The proof is similar to the previous proof and it is based on identity (2.10). □

***Example 2.4*** In this example we prove one remarkable inequality for convex function, the well-known Hermite-Hadamard inequality. The statement is the following:

If $f$ is an integrable convex function on $[a,b]$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{2.18}$$

Proof. First we prove the right inequality. Let us define the linear operator $A$ as:

$$Af = \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx.$$

Easy calculation give us the following:

$$Ae_0 = \frac{1+1}{2} - \frac{1}{b-a}\int_a^b dx = 0,$$

$$Ae_1 = \frac{a+b}{2} - \frac{1}{b-a}\int_a^b xdx = \frac{a+b}{2} - \frac{1}{b-a}\frac{x^2}{2}\Big|_a^b = 0$$

and for fixed $s \in [a,b]$

$$A((x-s)_+) = \frac{0+(b-s)}{2} - \frac{1}{b-a}\int_a^b (x-s)_+dx = \frac{b-s}{2} - \frac{1}{b-a}\int_s^b (x-s)dx$$

$$= \frac{b-s}{2} - \frac{1}{b-a}\frac{(x-s)^2}{2}\Big|_s^b = \frac{b-s}{2}\left(1-\frac{b-s}{b-a}\right) \geq 0.$$

By the above-mentioned theorem, we get that $Af \geq 0$ for any convex function continuous on $[a,b]$. If $f$ is not continuous, then since $f$ is convex, we can define a continuous convex function $\tilde{f}$ such that: $\tilde{f}(x) = f(x)$ for $x \in \langle a,b\rangle$, $\tilde{f}(a) = f(a+) \leq f(a)$, $\tilde{f}(b) = f(b-) \leq f(b)$ and $Af \geq A\tilde{f} \geq 0$. So, the right Hermite-Hadamard inequality holds for any convex function on $[a,b]$.

The proof of the left Hermite-Hadamard inequality is done in a similar manner.

## 2.3 Discrete Convex Functions of Higher Order

In this section we focus our attention on inequality $\sum p_i f(x_i) \geq 0$ for real functions of one variable, defined on an interval $I$ or on a discrete set $E$. A result analogous to (1.2) for real functions was proved by T. Popoviciu in [84] and it is stated as:

**Lemma 2.2** *Let $p_k \in \mathbb{R}$ for $k \in \{1,\ldots,N\}$. If $f : I \to \mathbb{R}$ is a given real function and $x_k$, $k \in \{1,\ldots,N\}$ be mutually distinct points from I, then the following identity holds*

$$\begin{aligned}\sum_{k=1}^{N} p_k f(x_k) &= \sum_{i=0}^{n-1}\left(\sum_{k=i+1}^{N} p_k (x_k - x_1)^{(i)}\right)\Delta^i f(x_1) \\ &+ \sum_{i=n+1}^{N}\left(\sum_{k=i}^{N} p_k (x_k - x_{i-n+1})^{(n-1)}\right)\Delta^n f(x_{i-n})(x_i - x_{i-n}).\end{aligned} \tag{2.19}$$

*Proof.* For $n = 1$ we get that the right-hand side of (2.19) is equal to

$$\begin{aligned}&\left(\sum_{k=1}^{N} p_k\right) f(x_1) + \sum_{i=2}^{N}\left(\sum_{k=i}^{N} p_k\right)\Delta f(x_{i-1})(x_i - x_{i-1}) \\ &= \left(\sum_{k=1}^{N} p_k\right) f(x_1) + \sum_{i=2}^{N}\left(\sum_{k=i}^{N} p_k\right)(f(x_i) - f(x_{i-1})) \\ &= \left(\sum_{k=1}^{N} p_k\right) f(x_1) + \left(\sum_{k=2}^{N} p_k\right)(f(x_2) - f(x_1)) + \left(\sum_{k=3}^{N} p_k\right)(f(x_3) - f(x_2)) \\ &\quad + \ldots + \left(\sum_{k=N}^{N} p_N\right)(f(x_N) - f(x_{N-1})) = \sum_{k=1}^{N} p_k f(x_k).\end{aligned}$$

Let us suppose that (2.19) holds for $n$. For the second step of mathematical induction we have to prove that

$$\begin{aligned}&\sum_{i=n+1}^{N}\left(\sum_{k=i}^{N} p_k (x_k - x_{i-n+1})^{(n-1)}\right)\Delta^n f(x_{i-n})(x_i - x_{i-n}) \\ &= \sum_{k=n+1}^{N}\left(p_k (x_k - x_1)^{(n)}\right)\Delta^n f(x_1) \\ &\quad + \sum_{i=n+2}^{N}\left(\sum_{k=i}^{N} p_k (x_k - x_{i-n})^{(n)}\right)\Delta^{n+1} f(x_{i-n-1})(x_i - x_{i-n-1}).\end{aligned} \tag{2.20}$$

Using notation $B_i = \sum_{k=i}^{N} p_k (x_k - x_{i-n})^{(n)}$ the second sum in the right-hand side is equal to

$$\sum_{i=n+2}^{N} B_i \Delta^n f(x_{i-n}) - \sum_{i=n+2}^{N} B_i \Delta^n f(x_{i-n-1})$$

$$
\begin{aligned}
&= \sum_{i=n+2}^{N} B_i \Delta^n f(x_{i-n}) - \sum_{i=n+1}^{N-1} B_{i+1} \Delta^n f(x_{i-n}) \\
&= \sum_{i=n+2}^{N-1} (B_i - B_{i+1}) \Delta^n f(x_{i-n}) + p_N (x_N - x_{N-n})^{(n)} \Delta^n f(x_{N-n}) \\
&- \sum_{k=n+2}^{N} p_k (x_k - x_2)^{(n)} \Delta^n f(x_1).
\end{aligned}
$$

Using

$$B_i - B_{i+1} = \sum_{k=i}^{N} p_k (x_k - x_{i-n+1})^{(n)} (x_i - x_{i-n})$$

and the above identity we prove (2.20). By mathematical induction identity (2.19) holds. □

Necessary and sufficient conditions under which the inequality $\sum p_k f(x_k) \geq 0$ holds for every discrete convex function of order $n$ is given in the following theorem.

**Theorem 2.5** *Let $E = \{x_1, \ldots, x_N\} \subset \mathbb{R}$ with $x_1 < x_2 < \cdots < x_N$ and let $p_k \in \mathbb{R}$ for $k \in \{1, \ldots, N\}$. Then the inequality*

$$\sum_{k=1}^{N} p_k f(x_k) \geq 0 \tag{2.21}$$

*holds for every discrete n-convex function $f : E \to \mathbb{R}$ if and only if*

$$\sum_{k=i+1}^{N} p_k (x_k - x_1)^{(i)} = 0, \quad i \in \{0, \ldots, n-1\}, \tag{2.22}$$

$$\sum_{k=i}^{N} p_k (x_k - x_{i-n+1})^{(n-1)} \geq 0, \quad i \in \{n+1, \ldots, N\}. \tag{2.23}$$

*Proof.* If inequalities (2.22) and (2.23) are satisfied, then the first sum in identity (2.19) is equal to 0, the second sum is nonnegative and hence inequality (2.21) holds.

Conversely, if for each convex functions of order $n$ inequality (2.21) holds, then we consider the functions $h^1(x) = x^r$ and $h^2(x) = -x^r$, $0 \leq r \leq n-1$. Since functions $h^1$ and $h^2$ are convex functions of order $n$ for $0 \leq r \leq n-1$, for them (2.21) holds and we have

$$\sum_{k=1}^{N} p_k x_k^r = 0.$$

From this equality we obtain (2.22). For each $i \in \{n+1, \ldots, N\}$, the function

$$h^3(x) = \begin{cases} 0, & x \leq x_{i-1} \\ (x - x_{i-n+1}) \cdot \ldots \cdot (x - x_{i-1}), & x > x_{i-1} \end{cases}$$

is convex of order $n$ and using these facts we obtain (2.23). □

***Example 2.5*** This example is devoted to the Petrović inequality, [51, $p$. 11]. It states:

*Let* $x_1,\dots,x_m \in [0,a]$ *such that* $\sum_{j=1}^m x_j \in [0,a]$. *Then for any convex function* $f : [0,a] \to \mathbb{R}$ *the following inequality holds*

$$\sum_{j=1}^{m} f(x_j) \leq f(\sum_{j=1}^{m} x_j) + (m-1)f(0). \tag{2.24}$$

*Proof.* Without loss of generality we can assume that $0 < x_1 < x_2 < \ldots < x_m$. Then $x_m < \sum_{j=1}^m x_j$. The sequence is $(0, x_1, x_2, \dots, x_m, \sum_{j=1}^m x_j)$, the sequence of weights is $(m-1, -1, -1, \dots, -1, 1)$ and $N = m+2$. For $n = 2$ conditions (2.22) and (2.23) become

$$\sum_{k=1}^{N} p_k = 0, \tag{2.25}$$

$$\sum_{k=1}^{N} p_k x_k = 0, \tag{2.26}$$

$$\sum_{k=i}^{N} p_k (x_k - x_{i-1}) \geq 0, \ i = 3, \dots, N. \tag{2.27}$$

Since

$$(m-1)\underbrace{-1-1-\ldots-1}_{m \ \ summands}+1 = 0$$

and

$$(m-1)\cdot 0 - x_1 - x_2 - \ldots - x_m + \sum_{j=1}^{m} x_j = 0$$

the first two conditions are satisfied. Let us prove that (2.27) is valid.

$$\begin{aligned}
&\sum_{k=i}^{N} p_k (x_k - x_{i-1}) = p_i (x_i - x_{i-1}) + \ldots + p_N (x_N - x_{i-1}) \\
&= -(x_i - x_{i-1}) - (x_{i+1} - x_{i-1}) - \ldots - (x_N - x_{i-1}) + \left( \sum_{j=1}^{m} x_j - x_{i-1} \right) \\
&= -\sum_{j=i}^{m} x_j + (m-i+1)x_{i-1} + \left( \sum_{j=1}^{m} x_j - x_{i-1} \right) \\
&= \sum_{j=i+1}^{m} x_j + (m-i)x_{i-1} \geq 0.
\end{aligned}$$

Using Theorem 2.5 we get inequality (2.24). Another proof of this inequality is given in [77, $p$.154].

***Example 2.6*** Prove: if $f : I \to \mathbb{R}$ is a convex, $I$ is an interval, then for any $x, y, z \in I$

$$\frac{f(x)+f(y)+f(z)}{3} + f\left(\frac{x+y+z}{3}\right)$$

$$\geq \frac{2}{3}\left(f\left(\frac{x+y}{2}\right)+f\left(\frac{z+y}{2}\right)+f\left(\frac{x+z}{2}\right)\right). \tag{2.28}$$

The above inequality is due to T. Popoviciu, ([89]).

*Proof.* Inequality (2.28) is equivalent to the following

$$f(x)+f(y)+f(z)+3f\left(\frac{x+y+z}{3}\right)$$
$$\geq 2f\left(\frac{x+y}{2}\right)+2f\left(\frac{z+y}{2}\right)+2f\left(\frac{x+z}{2}\right).$$

Let $z<y<x$. Then $z<\frac{y+z}{2}<y<\frac{x+y}{2}<x$ and we have only two posibilities for orders: $y\leq\frac{x+z}{2}$ or $y>\frac{x+z}{2}$. Let us consider the first case, the proof of the second case is similar. Without loss of generality we can assume that $y<\frac{x+z}{2}$. Then $n=2, N=7$ and the sequence $(x_i)$ has a form:

$$z<\frac{y+z}{2}<y<\frac{x+y+z}{3}<\frac{x+z}{2}<\frac{x+y}{2}<x$$

and corresponding weights are: $1,-2,1,3,-2,-2,1$ respectively. The first condition (2.25) holds because $1-2+1+3-2-2+1=0$. Similarly, the second condition (2.26) becomes:

$$\sum_{k=1}^{7}p_kx_k=z-2\frac{y+z}{2}+y+3\frac{x+y+z}{3}-2\frac{x+z}{2}-2\frac{x+y}{2}+x=0.$$

Let us see that (2.27) holds for $i=3$.

$$\begin{aligned}\sum_{k=3}^{7}p_k(x_k-x_2) &= \sum_{k=1}^{7}p_k(x_k-x_2)-\sum_{k=1}^{2}p_k(x_k-x_2)\\ &= -\sum_{k=1}^{2}p_k(x_k-x_2)=-\left(z-\frac{y+z}{2}\right)\geq 0.\end{aligned}$$

Similarly, we prove (2.27) for $i=4,5,6,7$.

In the case when $n=2$, i.e. if $f$ is convex, then J. Pečarić ([68]) showed that a property of monotonicity of elements $x_1,x_2,\ldots,x_N$ can be omitted. In fact, he proved the following result.

**Theorem 2.6** *The inequality* $\sum_{k=1}^{N}p_kf(x_k)\geq 0$ *holds for all N-tuples* $\boldsymbol{x}=(x_1,\ldots,x_N)$, $\boldsymbol{p}=(p_1,\ldots,p_N)\in\mathbb{R}^N$ *and all discrete convex functions* $f$ *if and only if*

$$\sum_{k=1}^{N}p_k=0 \quad \textit{and} \quad \sum_{k=1}^{N}p_k|x_k-x_i|\geq 0 \quad \textit{for} \quad i\in\{1,\ldots,N\}. \tag{2.29}$$

Namely, since

$$\sum_{k=1}^{N}p_k|x_k-x_i|=2\sum_{k=1}^{N}p_k(x_k-x_i)_+-\sum_{k=1}^{N}p_k(x_k-x_i)$$

it is easy to see that (2.29) is equivalent to $\sum_{k=1}^{N} p_k = 0$, $\sum_{k=1}^{N} p_k x_k = 0$ and $\sum_{k=1}^{N} p_k (x_k - x_i)_+ \geq 0$, $i = 1, \ldots, N-1$ which is (2.22) and (2.23) for $m = 2$.

Now, let us state and prove an identity from [28] which involves $\nabla$-differences of function $f$.

**Lemma 2.3** *Let $p_k \in \mathbb{R}$ for $k \in \{1, \ldots, N\}$. If $f : I \to \mathbb{R}$ is a given function and $x_k$, $k \in \{1, \ldots, N\}$ are mutually distinct points from I, then the identity*

$$\begin{aligned}\sum_{k=1}^{N} p_k f(x_k) &= \sum_{i=0}^{n-1} \left( \sum_{k=1}^{N-i} p_k (x_N - x_k)^{\{i\}} \right) \nabla^i f(x_{N-i}) \\ &+ \sum_{i=1}^{N-n} \left( \sum_{k=1}^{i} p_k (x_{i+n-1} - x_k)^{\{n-1\}} \right) \nabla^n f(x_i)(x_{i+n} - x_i)\end{aligned} \tag{2.30}$$

*holds.*

*Proof.* Let us prove it by induction on $n$. For $n = 1$, we have

$$\sum_{k=1}^{N} p_k f(x_k) = \sum_{k=1}^{N} p_k f(x_N) + \sum_{i=1}^{N-1} \left( \sum_{k=1}^{i} p_k \right) (f(x_i) - f(x_{i+1}))$$

which is true.

Suppose that (2.30) is valid. Then

$$\begin{aligned}&\sum_{i=0}^{n} \left( \sum_{k=1}^{N-i} p_k (x_N - x_k)^{\{i\}} \right) \nabla^i f(x_{N-i}) \\ &+ \sum_{i=1}^{N-n-1} \left( \sum_{k=1}^{i} p_k (x_{i+n} - x_k)^{\{n\}} \right) \nabla^{n+1} f(x_i)(x_{i+n+1} - x_i) \\ &= A + \sum_{k=1}^{N-n} p_k (x_N - x_k)^{\{n\}} \nabla^n f(x_{N-n}) \\ &+ \sum_{i=1}^{N-n-1} B(-1)^{n+1} \left( [x_{i+1}, \ldots, x_{i+n+1}; f] - [x_i, \ldots, x_{i+n}; f] \right) \\ &= A + \sum_{k=1}^{N-n} p_k (x_N - x_k)^{\{n\}} \nabla^n f(x_{N-n}) \\ &+ \sum_{k=1}^{N-n-1} p_k (x_{N-1} - x_k)^{\{n\}} (-1)^{n+1} [x_{N-n}, \ldots, x_N; f] \\ &+ \sum_{i=1}^{N-n-2} B(-1)^{n+1} [x_{i+1}, \ldots, x_{i+n+1}; f] - \sum_{i=2}^{N-n-1} B(-1)^{n+1} [x_i, \ldots, x_{i+n}; f] \\ &- p_1 (x_{n+1} - x_1)^{\{n\}} (-1)^{n+1} [x_1, \ldots, x_{n+1}; f] \\ &= A + \sum_{k=1}^{N-n} p_k (x_{N-1} - x_k)^{\{n-1\}} \nabla^n f(x_{N-n})(x_N - x_{N-n})\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=2}^{N-n-1}(-1)^n[x_i,\dots,x_{i+n};f]\left(\sum_{k=1}^{i}p_k(x_{i+n}-x_k)^{\{n\}}\right.\\
&\left.-\sum_{k=1}^{i-1}p_k(x_{i+n-1}-x_k)^{\{n\}}\right)+p_1(x_{n+1}-x_1)^{\{n\}}\nabla^n f(x_1)\\
&=A+\sum_{k=1}^{N-n}p_k(x_{N-1}-x_k)^{\{n-1\}}\nabla^n f(x_{N-n})(x_N-x_{N-n})\\
&+\sum_{i=2}^{N-n-1}\left(\sum_{k=1}^{i}p_k(x_{i+n-1}-x_k)^{\{n-1\}}\right)\nabla^n f(x_i)(x_{i+n}-x_i)\\
&+p_1(x_n-x_1)^{\{n-1\}}\nabla^n f(x_1)(x_{n+1}-x_1)\\
&=A+\sum_{i=1}^{N-n}\left(\sum_{k=1}^{i}p_k(x_{i+n-1}-x_k)^{\{n-1\}}\right)\nabla^n f(x_i)(x_{i+n}-x_i)=\sum_{k=1}^{N}p_kf(x_k),
\end{aligned}
$$

where

$$A=\sum_{i=0}^{n-1}\left(\sum_{k=1}^{N-i}p_k(x_N-x_k)^{\{i\}}\right)\nabla^i f(x_{N-i}),$$

and

$$B=\sum_{k=1}^{i}p_k(x_{i+n}-x_k)^{\{n\}}.$$

Thus, identity (2.30) is proved. □

From identity (2.30) we can obtain the following result about necessary and sufficient conditions that inequality $\sum_{k=1}^{N}p_kf(x_k)\geq 0$ holds for every $\nabla$-convex function of order $m$.

**Theorem 2.7** *Let $E=\{x_1,\dots,x_N\}\subset\mathbb{R}$ with $x_1<x_2<\cdots<x_N$ and let $p_k\in\mathbb{R}$ for $k\in\{1,\dots,N\}$. Then the inequality*

$$\sum_{k=1}^{N}p_kf(x_k)\geq 0 \tag{2.31}$$

*holds for every discrete $\nabla$-convex function $f$ of order $n$ if and only if*

$$\sum_{k=1}^{N-i}p_k(x_N-x_k)^{\{i\}}\;=\;0,\quad i\in\{0,\dots,n-1\}, \tag{2.32}$$

$$\sum_{k=1}^{i}p_k(x_{i+n-1}-x_k)^{\{n-1\}}\;\geq\;0,\quad i\in\{1,\dots,N-n\}. \tag{2.33}$$

*Proof.* If inequalities (2.32) and (2.33) are satisfied, then the first sum in identity (2.30) is equal to 0, the second sum is nonnegative and hence the inequality (2.31) holds.

Conversely, if for each $\nabla$-convex functions of order $n$ inequality (2.31) holds, then we consider the functions $h^1(x)=x^r$ and $h^2(x)=-x^r$, $0\leq r\leq n-1$. Functions $h^1$ and $h^2$ are $\nabla-$convex functions of order $n$ and for $0\leq r\leq n-1$, we have

$$\sum_{k=1}^{N}p_kx_k^r=0.$$

From this equality we obtain (2.32). For each $i \in \{1, \dots, N-n\}$, $n > 1$, the function

$$h^3(x) = \begin{cases} (x_{i+1} - x) \cdot \ldots \cdot (x_{i+n-1} - x), & x < x_{i+1} \\ 0, & x \geq x_{i+1} \end{cases}$$

is $\nabla$-convex of order $n$ and using these facts we obtain (2.33). □

The generalization of Theorem 1.2, i.e. a result for function which are convex of orders $j, j+1, \dots, n$ is given in [28].

**Theorem 2.8** *Let $E = \{x_1, \dots, x_N\} \subset \mathbb{R}$ with $x_1 < x_2 < \cdots < x_N$ and let $p_k \in \mathbb{R}$ for $k \in \{1, \dots, N\}$.*

*(i) Inequality*

$$\sum_{i=1}^{N} p_i f(x_i) \geq 0$$

*holds for every discrete convex function $f$ of order $j, j+1, \dots, n$, $(j = 0, 1, 2, \dots, n)$ if and only if*

$$\sum_{i=k+1}^{N} p_i (x_i - x_1)^{(k)} = 0, \qquad k = 0, \dots, j-1, \tag{2.34}$$

$$\sum_{i=k+1}^{N} p_i (x_i - x_1)^{(k)} \geq 0, \qquad k = j, \dots, n-1, \tag{2.35}$$

$$\sum_{i=k}^{N} p_i (x_i - x_{k-n+1})^{(n-1)} \geq 0, \qquad k = n+1, \dots, N.$$

*If $j = 0$ (or $j = n$), condition (2.34) (or (2.35)) can be omitted.*

*(ii) Inequality*

$$\sum_{i=1}^{N} p_i f(x_i) \geq 0$$

*holds for every discrete $\nabla$-convex function $f$ of order $j, j+1, \dots, n$, $(j = 0, 1, \dots, n)$ if and only if*

$$\sum_{i=1}^{N-k} p_i (x_N - x_i)^{\{k\}} = 0, \qquad k = 0, \dots, j-1, \tag{2.36}$$

$$\sum_{i=1}^{N-k} p_i (x_N - x_i)^{\{k\}} \geq 0, \qquad k = j, \dots, n-1, \tag{2.37}$$

$$\sum_{i=1}^{k} p_i (x_{k+n-1} - x_i)^{\{n-1\}} \geq 0, \qquad k = 1, \dots, N-n.$$

*For $j = 0$ (or $j = n$), condition (2.36) (or (2.37)) can be omitted.*

**Remark 2.2** T. Popoviciu ([81], [85, $p$.35]) gave the following necessary and sufficient conditions for the positivity of considered sum instead of conditions (2.22) and (2.23):

$$\sum_{i=1}^{N} p_i x_i^k = 0 \quad \text{for} \quad k \in \{0,1,\dots,n-1\}$$

and

$$\sum_{i=1}^{r} p_i (x_i - x_{r+1})(x_i - x_{r+2}) \cdots (x_i - x_{r+n-1}) \leq 0 \quad \text{for} \quad r \in \{1,\dots,N-n\}.$$

## 2.4 Discrete Starshaped Functions of Higher Order

A function $f$ is called (discrete) $n$-starshaped or starshaped of order $n$ if $\frac{f(x)}{x}$ is (discrete) $(n-1)$-convex. Let us see how results from the previous sections are reflecting on starshaped functions of higher order.

Firstly, we state an identity involving divided differences of the function $\frac{f(x)}{x}$.

**Lemma 2.4** *Let $p_k \in \mathbb{R}$ for $k \in \{1,\dots,N\}$. If $f : I \to \mathbb{R}$ is a function and $x_k$, $k \in \{1,\dots,N\}$ are mutually distinct, non-zero points from $I$, then the following identity holds*

$$\sum_{k=1}^{N} p_k f(x_k) = \sum_{i=1}^{n-1} \left( \sum_{k=i}^{N} p_k x_k (x_k - x_1)^{(i-1)} \right) \Delta^i g(x_1)$$
$$+ \sum_{i=n}^{N} \left( \sum_{k=i}^{N} p_k x_k (x_k - x_{i-n+2})^{(n-2)} \right) \Delta^{n-1} g(x_{i-n+1})(x_i - x_{i-n+1}),$$

*where* $g(x) = \frac{f(x)}{x}$.

*Proof.* Putting in (2.19) $m \to m-1$, $p_k \to x_k p_k$ and $f \to g$ we get:

$$\sum_{k=1}^{N} p_k x_k g(x_k) = \sum_{i=0}^{n-2} \left( \sum_{k=i+1}^{N} p_k x_k (x_k - x_1)^{(i)} \right) \Delta^i g(x_1)$$
$$+ \sum_{i=n}^{N} \left( \sum_{k=i}^{N} p_k x_k (x_k - x_{i-n+2})^{(n-2)} \right) \Delta^{n-1} g(x_{i-n+1})(x_i - x_{i-n+1})$$
$$= \sum_{i=1}^{n-1} \left( \sum_{k=i}^{N} p_k x_k (x_k - x_1)^{(i-1)} \right) \Delta^i g(x_1)$$

$$+\sum_{i=n}^{N}\left(\sum_{k=i}^{N} p_k x_k (x_k - x_{i-n+2})^{(n-2)}\right)\Delta^{n-1} g(x_{i-n+1})(x_i - x_{i-n+1}).$$

□

**Theorem 2.9** *Let $E = \{x_1, \ldots, x_N\} \subset \mathbb{R}$ with $x_1 < x_2 < \ldots < x_N$ and let $p_k \in \mathbb{R}$ for $k \in \{1, \ldots, N\}$. Then the inequality*

$$\sum_{k=1}^{N} p_k f(x_k) \geq 0 \tag{2.38}$$

*holds for every discrete m-starshaped function $f : E \to \mathbb{R}$ if and only if*

$$\sum_{k=i}^{N} p_k x_k (x_k - x_1)^{(i-1)} = 0, \quad i \in \{1, \ldots, n-1\}, \tag{2.39}$$

$$\sum_{k=i}^{N} p_k x_k (x_k - x_{i-n+2})^{(n-2)} \geq 0, \ i \in \{n, \ldots, N\}. \tag{2.40}$$

*Proof.* Let us suppose that the sum $\sum_{i=1}^{N} p_k f(x_k)$ is nonnegative for every $n$-starshaped function $f$. Let us consider functions $h_1(x) = x(x - x_1)^{(i-1)}$ and $h_2(x) = -x(x - x_1)^{(i-1)}$. Since $\frac{h_1(x)}{x}$ and $\frac{h_2(x)}{x}$ are polynomials of degree $i$, $h_1$ and $h_2$ are $n$-starshaped for $i \in \{1, \ldots, n-1\}$. Hence (2.39) holds. For each $i \in \{n, \ldots, N\}$ the function

$$h_3(x) = \begin{cases} 0, & x \leq x_{i-1} \\ x(x - x_{i-n+2}) \ldots (x - x_{i-1}), & x > x_{i-1} \end{cases}$$

is starshaped of order $n$ and (2.40) is valid. □

***Example 2.7*** Let $f : \langle 0, a] \to \mathbb{R}$ be 3-starshaped. If $x, y, z, x + y + z \in \langle 0, a]$, then

$$f(x + y + z) - f(x + y) - f(x + z) - f(y + z) + f(x) + f(y) + f(z) \geq 0.$$

This inequality is given in [86] under assumptions that $f(0) = 0$, $f$ is continuous on $[0, a]$ and has the increasing second derivative. In [96] an increase of the second derivative was substituted by assumption that $f$ is convex of order 3. Here, we consider a 3-starshaped function.

*Proof.* Firstly, we examine the following order of elements: $x < y < x + y < z < x + z < y + z < x + y + z$. Other orderings are proved similarly. We get $N = 7$, $n = 3$, sequence $(x_k)_k$ is $(x, y, x + y, z, x + z, y + z, x + y + z)$ and sequence of weights is $(p_k)_k = (1, 1, -1, 1, -1, -1, 1)$. Let us prove that (2.39) and (2.40) are satisfied.

$$\sum_{k=1}^{7} p_k x_k = x + y - (x + y) + z - (x + z) - (y + z) + (x + y + z) = 0.$$

$$\sum_{k=2}^{7} p_k x_k (x_k - x_1) = \sum_{k=1}^{7} p_k x_k (x_k - x_1) = \sum_{k=1}^{7} p_k x_k^2$$

$$= x^2 + y^2 - (x+y)^2 + z^2 - (x+z)^2 - (y+z)^2 + (x+y+z)^2 = 0.$$

Further, we have to check (2.40) for $i = 3,4,\ldots,7$.

$$\sum_{k=3}^{7} p_k x_k (x_k - x_2) = -\sum_{k=1}^{2} p_k x_k^2 + x_2 \sum_{k=1}^{2} p_k x_k = x(y-x) \geq 0.$$

$$\sum_{k=4}^{7} p_k x_k (x_k - x_3) = 2xy \geq 0, \quad \sum_{k=5}^{7} p_k x_k (x_k - x_4) = 2xy \geq 0,$$

$$\sum_{k=6}^{7} p_k x_k (x_k - x_5) = 2xy + xz \geq 0, \quad \sum_{k=7}^{7} p_k x_k (x_k - x_6) = x + y + z > 0.$$

## 2.5 Results for $n$-convex Functions with $2n$ Nodes

In this section we consider inequality $\sum f(x_i) \geq \sum f(y_i)$ for $n$-convex function $f$ which is obviously a particular variant of a general linear inequality $\sum p_i f(a_i) \geq 0$ where weights $p_i$ are 1 or $-1$. The crucial moment in this consideration is that the numbers of nodes are exactly twice than the order of convexity.

Firstly, results of such type are given in paper [7] by Z. Brady, but here we give a variant from [23] in which an extra claim is appeared.

**Theorem 2.10** *Given real numbers $x_1, x_2, \ldots, x_n \in [a,b]$ and $y_1, y_2, \ldots, y_n \in [a,b]$ such that*

$$\sum_{i=1}^{n} x_i^j = \sum_{i=1}^{n} y_i^j, \quad j = 1,2,\ldots,n-1, \tag{2.41}$$

*the following claims are equivalent:*

$(i)$ $\sum_{i=1}^{n} x_i^n \geq \sum_{i=1}^{n} y_i^n$;

$(ii)$ $\max\{x_i : i = 1,\ldots,n\} \geq \max\{y_i : i = 1,\ldots,n\}$;

$(iii)$ $(-1)^n \prod_{i=1}^{n} x_i \leq (-1)^n \prod_{i=1}^{n} y_i$;

$(iv)$ $\sum_{i=1}^{n} f(x_i) \geq \sum_{i=1}^{n} f(y_i)$ *for all functions* $f : [a,b] \to \mathbb{R}$ *provided* $f^{(n)} \geq 0$.

*Proof.* Use the Taylor expansion in the form

$$f(x) = \sum_{j=1}^{n-1} \frac{f^{(j)}(a)}{j!}(x-a)^j + \frac{1}{(n-1)!}\int_a^b (x-t)_+^{n-1} f^{(n)}(t)\,dt,$$

which by using (2.41) gives

$$\sum_{i=1}^{n} f(x_i) - \sum_{i=1}^{n} f(y_i) = \frac{1}{(n-1)!}\int_a^b r_n(t) f^{(n)}(t)dt, \tag{2.42}$$

where $r_n(t) = \sum_{i=1}^{n}\left[(x_i-t)_+^{n-1} - (y_i-t)_+^{n-1}\right]$. Obviously $r_n^{(j)}(a) = r_n^{(j)}(b) = 0$, $j = 0, 1, \ldots, n-2$. Using Rolle's theorem it follows that $r_n^{(n-1)}$ has at least $(n-1)$-sign changes in the sense that there is a sequence $a_1, a_2, \ldots a_n$ in $[a,b]$ such that $r_n^{(n-1)}(a_j) r_n^{(n-1)}(a_{j+1}) < 0$, for $j = 1, \ldots, n-1$.

On the other hand, suppose (w.l.o.g.) that $x_1 \geq x_2 \geq \cdots \geq x_n$, $y_1 \geq y_2 \geq \cdots \geq y_n$ and that $z_1 \geq z_2 \geq \cdots \geq z_{2n}$ is the sequence formed by the previous two sequences. In this case $r_n(t) = \sum_{j=1}^{2n} w_j (z_j - t)_+^{n-1}$, where $w_j = +1$ or $-1$ depending on $w_j$ belongs to $x_i$'s or to $y_i$'s. Note that $r_n^{(n-1)}$ is a step function defined by $w_1$, $w_1 + w_2$,...., $w_1 + w_2 + \cdots + w_{2n}$. It is obvious that this sequence can change the sign in the worst case on every other interval, so it can change the sign at most $n-1$ times.

Finally, since it is clear that $r_n$ has a constant sign, that in the case $x_1 = \max\{z_j : j = 1, 2, \ldots, 2n\}$ it holds that $r_n(t) = (x_1 - t)^{n-1}$ for $t \in [z_2, x_1]$, so it follows that $r_n \geq 0$. This gives that $(ii)$ implies $(iv)$.

It follows from the proof that if $x_1 = \max\{z_j : j = 1, 2, \ldots, 2n\}$, then $x_n = \min\{z_j : j = 1, 2, \ldots, 2n\}$ for $n$ even and $y_n = \min\{z_j : j = 1, 2, \ldots, 2n\}$ for $n$ odd.

It is obvious that $(iv)$ implies $(i)$ using $f(x) = x^n$ and $(ii)$ using $f(x) = (x - \max\{x_i : i = 1, 2, \ldots, n\})_+^{n-1}$ (since in this case $\sum_{i=1}^{n} f(x_i) = 0$ and this implies $\sum_{i=1}^{n} f(y_i) = 0$ which gives $\max_i x_i \geq \max_i y_i$).

Set $P_n(x) = \prod_{i=1}^{n}(x - x_i)$, $Q_n(x) = \prod_{i=1}^{n}(x - y_i)$. Obviously

$$P_n(x) = \sum_{j=0}^{n}(-1)^j I_j(\mathbf{x}) x^{n-j}, \;\; Q_n(x) = \sum_{j=0}^{n}(-1)^j I_j(\mathbf{y}) x^{n-j},$$

where

$$I_j(\mathbf{x}) = I_j(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < i_2 < \cdots i_j \leq n} x_{i_1} \cdots x_{i_j}, \;\; j = 0, 1, \ldots, n, \;\; I_0(\mathbf{x}) = 1,$$

and analogously $I_j(\mathbf{y})$. $I_n$ polynomially depends on variables $t_j = \mathbf{x}^{(j)}$, $j = 1, \ldots, n$, where

$$\mathbf{x}^{(j)} = \sum_{i=1}^{n} x_i^j, \;\; j \in \mathbb{N}.$$

Using identities (2.41), it follows $I_j(\mathbf{x}) = I_j(\mathbf{y})$, $j = 1, \ldots, n-1$. We get

$$P_n(x) - Q_n(x) = (-1)^n \left(I_n(\mathbf{x}) - I_n(\mathbf{y})\right) = (-1)^n \left(\prod_{i=1}^{n} x_i - \prod_{i=1}^{n} y_i\right). \tag{2.43}$$

On the other hand, since $P_n(x_i) = 0, i = 1, \ldots, n$, it follows

$$x_i^n = \sum_{j=1}^{n}(-1)^{j+1} I_j(\mathbf{x}) x_i^{n-j}, \qquad i = 1, \ldots, n,$$

which by summation gives:

$$\mathbf{x}^{(n)} = \sum_{j=1}^{n}(-1)^{j+1}I_j(\mathbf{x})\mathbf{x}^{(n-j)} = (-1)^{n+1}nI_n(\mathbf{x}) + \sum_{j=1}^{n-1}(-1)^{j+1}I_j(\mathbf{x})\mathbf{x}^{(n-j)}.$$

Using analogous identity for $\mathbf{y}^{(n)}$, it follows

$$\begin{aligned}\sum_{i=1}^{n} x_i^n \; - \; \sum_{i=1}^{n} y_i^n &= \mathbf{x}^{(n)} - \mathbf{y}^{(n)} \\ &= (-1)^{n+1}n\left(I_n(\mathbf{x}) - I_n(\mathbf{y})\right) = (-1)^{n+1}n\left(\prod_{i=1}^{n}x_i - \prod_{i=1}^{n}y_i\right), \end{aligned} \tag{2.44}$$

from which the equivalence of $(i)$ and $(iii)$ is obvious. Using (2.43) and (2.44) we get

$$P_n(x) - Q_n(x) = -\frac{1}{n}\left[\sum_{i=1}^{n} x_i^n - \sum_{i=1}^{n} y_i^n\right]$$

from which the equivalence of $(i)$ and $(ii)$ is obvious. □

**Theorem 2.11** *Let $x_1,\ldots,x_n \in [a,b]$, $y_1,\ldots,y_n \in [a,b]$ be non-identical $n$-tuples such that* (2.41) *holds and such that (a) or (b) or (c) in Theorem 2.10 holds. If $f \in C^n([a,b])$, then there exists $\xi \in [a,b]$ such that*

$$\begin{aligned}\sum_{i=1}^{n} f(x_i) - \sum_{i=1}^{n} f(y_i) &= \frac{f^{(n)}(\xi)}{n!}\left(\sum_{i=1}^{n} x_i^n - \sum_{i=1}^{n} y_i^n\right) \\ &= (-1)^{n+1}\frac{f^{(n)}(\xi)}{(n-1)!}\left[\prod_{i=1}^{n}x_i - \prod_{i=1}^{n}y_i\right]. \end{aligned} \tag{2.45}$$

*Proof.* Set $m = \min_{x\in[a,b]} f^{(n)}(x)$, $M = \max_{x\in[a,b]} f^{(n)}(x)$. Obviously

$$f_1(x) = \frac{M}{n!}x^n - f(x), \;\; f_2(x) = f(x) - \frac{m}{n!}x^n$$

are $n$-convex functions. Applying Theorem 2.10 on $f_1$ and $f_2$ and rearranging, we get

$$\frac{m}{n!}\left(\sum_{i=1}^{n} x_i^n - \sum_{i=1}^{n} y_i^n\right) \le \sum_{i=1}^{n} f(x_i) - \sum_{i=1}^{n} f(y_i) \le \frac{M}{n!}\left(\sum_{i=1}^{n} x_i^n - \sum_{i=1}^{n} y_i^n\right),$$

from which (2.45) obviously follows. The second identity in (2.45) is given in (2.44). Notice that $\sum_{i=1}^{n} x_i^n > \sum_{i=1}^{n} y_i^n$ since the involved $n$-tuples are not identical. □

Similarly, Cauchy type mean-value theorem can be obtained, see [23].

Using the above theorem we can easily prove some non-trivial inequalities.

**Theorem 2.12** *Inequality*

$$f(t_1)+f(t_2)+f(2\sqrt{t_1t_2}) \le f(t_1+t_2)+2f(\sqrt{t_1t_2}) \tag{2.46}$$

*holds if $f$ is a 3-convex function. Reversed inequality holds if $f$ is a 3-concave function.*

*Proof.* Using

$$x_1=t_1, \quad x_2=t_2, \quad x_3=2\sqrt{t_1t_2},$$
$$y_1=t_1+t_2, \quad y_2=\sqrt{t_1t_2}, \quad y_3=\sqrt{t_1t_2}$$

it is easy to verify $x_1+x_2+x_3=y_1+2y_2$, $x_1^2+x_2^2+x_3^2=y_1^2+2y_2^2$ and $y_1=\max\{x_1,x_2,x_3,y_1,y_2\}$. So, the claim follows from Theorem 2.10. □

In (2.46) an improvement of AG inequality is hidden. Namely, plugging in (2.46) $f(x)=x^p$, $p>0$ we get

$$A(t_1,t_2)^p - G(t_1,t_2)^p \ge 2^{1-p}\left(A(t_1^p,t_2^p)-G(t_1^p,t_2^p)\right) \ge 0,$$

where $A(t_1,t_2)=\dfrac{t_1+t_2}{2}$, $G(t_1,t_2)=\sqrt{t_1t_2}$.

Using the same arguing as in the proof of of Theorem 2.12 it follows that the inequality

$$f\left(2\left(\frac{x_1+x_2}{2}\right)^2+x_3^2\right)+f\left(2\left(\frac{x_1+x_3}{2}\right)^2+x_2^2\right)+f\left(2\left(\frac{x_2+x_3}{2}\right)^2+x_1^2\right)$$
$$\le f\left(x_1^2+x_2^2+x_3^2\right)+2f\left(\left(\frac{x_1+x_2}{2}\right)^2+\left(\frac{x_1+x_3}{2}\right)^2+\left(\frac{x_2+x_3}{2}\right)^2\right)$$

holds for 3-convex functions and the opposite inequality holds for 3-concave functions.

## 2.6 $n$-Convex Functions at a Point

This section is devoted to the class of functions which are $n$-convex at a point. The particular case of this class was introduced by I. Baloch, J. Pečarić and M. Praljak in the paper [4] which contains some new results about the Levinson inequality. After this section, we will return to that remarkable inequality and show how the below-mentioned general results reflect to this particular inequality.

**Definition 2.1** *Let $I$ be an interval in $\mathbb{R}$, $c$ a point in the interior of $I$ and $n\in\mathbb{N}_0$. A function $f:I\to\mathbb{R}$ is said to be $(n+1)$-convex at point $c$ if there exists a constant $K_f$ such that the function*

$$F(x)=f(x)-\frac{K_f}{n!}x^n \tag{2.47}$$

*is $n$-concave on $I\cap(-\infty,c]$ and $n$-convex on $I\cap[c,\infty)$. We denote the family of $(n+1)$-convex functions at point $c$ by $\mathcal{K}_{n+1}^c(I)$. A function $f$ is said to be $(n+1)$-concave at point $c$ if the function $-f$ is $(n+1)$-convex at point $c$.*

The class of 3-convex functions at a point was introduced in [4] while the class of $n$-convex functions at a point was introduced in [76]. Let us prove several properties of these classes, [4, 76].

**Lemma 2.5** *If $f \in \mathscr{K}_{n+1}^{c}(I)$ and $f^{(n)}$ exists, then $f^{(n)}(c) = K_f$.*

*Proof.* Let us prove this for $n = 2$. Due to the concavity and convexity of $F(x) = f(x) - \frac{K_f}{2}x^2$ for every distinct points $x_1, x_2, x_3 \in \langle a, c]$ and $y_1, y_2, y_3 \in [c, b\rangle$ we have

$$[x_1, x_2, x_3; F] = [x_1, x_2, x_3; f] - \frac{K_f}{2} \leq 0 \leq [y_1, y_2, y_3; f] - \frac{K_f}{2} = [y_1, y_2, y_3; F].$$

Therefore, if $f''_-(c)$ and $f''_+(c)$ exist, letting $x_j \nearrow c$ and $y_j \searrow c$, we get

$$f''_-(c) \leq K_f \leq f''_+(c).$$

Therefore, if $f''(c)$ exists, then it is equal to $K_f$.

It is known that: a function $\varphi$ is $n$-convex ($n$-concave) if and only if $\varphi^{(k)}$ exists and $\varphi^{(k)}$ is $(n-k)$-convex ($(n-k)$-concave) for $0 \leq k \leq n-2$. From that property we can conclude that the statement of Lemma holds for any $n \geq 2$. □

**Lemma 2.6** *A function $f$ belongs to $\mathscr{K}_{n+1}^{c}(I)$ for every $c \in \langle a, b\rangle$, if and only if $f$ is $(n+1)$-convex.*

*Proof.* Let us prove this lemma for $n = 2$. Let us assume that $f$ is 3-convex. Then $f', f''_-$ and $f''_+$ exist and $f'$ is convex. Hence, for every $\alpha_1, \alpha_2 \in \langle a, c]$ and $\beta_1, \beta_2 \in [c, b\rangle$ it holds

$$\frac{f'(\alpha_2) - f'(\alpha_1)}{\alpha_2 - \alpha_1} \leq f''_-(c) \leq f''_+(c) \leq \frac{f'(\beta_2) - f'(\beta_1)}{\beta_2 - \beta_1}.$$

Therefore, for every $A \in [f''_-(c), f''_+(c)]$ the function $F(x) = f(x) - \frac{A}{2}x^2$ satisfies

$$\frac{F'(\alpha_2) - F'(\alpha_1)}{\alpha_2 - \alpha_1} \leq 0 \leq \frac{F'(\beta_2) - F'(\beta_1)}{\beta_2 - \beta_1},$$

so, $F'$ is nonincreasing on $\langle a, c]$ and nondecreasing on $[c, b\rangle$.

Let us asume that $f \in \mathscr{K}_{n+1}^{c}(I)$ for every $c \in \langle a, b\rangle$. It is enough to prove that $f'$ exists and is convex. For every $c \in \langle a, b\rangle$ there exists constant $A_c$ such that the function $F_c(x) = f(x) - \frac{A_c}{2}x^2$ is concave on $\langle a, c]$ and convex on $[c, b\rangle$. Therefore $F'_{c-}$ and $F'_{c+}$ exist and $F'_{c-}(x) \geq F'_{c+}(x)$ for $x \in \langle a, c\rangle$ and $F'_{c-}(x) \leq F'_{c+}(x)$ for $x \in \langle c, b\rangle$. Since the function $x \mapsto \frac{A_c}{2}x^2$ is differentiable, $f'_-$ and $f'_+$ also exist. Let $x \in \langle a, b\rangle$ be arbitrary and $c_1 < x < c_2$. We have $f'_-(x) \leq f'_+(x)$ due to convexity of $F_{c_1}$ and $f'_-(x) \geq f'_+(x)$ due to convexity of $F_{c_2}$, so $f'$ exists. Furthermore, due to convexity and concavity of $F_c$ we also have, for every $x_1 \neq x_2 \leq c \leq y_1 \neq y_2$

$$\frac{F'_c(x_2) - F'_c(x_1)}{x_2 - x_1} = \frac{f'_c(x_2) - f'_c(x_1)}{x_2 - x_1} - A_c \leq 0$$

$$\leq \frac{f_c'(y_2) - f_c'(y_1)}{y_2 - y_1} - A_c = \frac{F_c'(y_2) - F_c'(y_1)}{y_2 - y_1}.$$

In particular, for $z_1 < z_2 < z_3$

$$\frac{f'(z_2) - f'(z_1)}{z_2 - z_1} \leq A_{z_2} \leq \frac{f'(z_3) - f'(z_2)}{z_3 - z_2}. \tag{2.48}$$

Now, let $x_1, x_2, y \in \langle a, b \rangle$ be arbitrary. If $y < x_1 < x_2$, applying (2.48) we get

$$\frac{f'(x_1) - f'(y)}{x_1 - y} \leq A_{x_1} \leq \frac{f'(x_2) - f'(x_1)}{x_2 - x_1} = \frac{f'(x_2) - f'(y)}{x_2 - x_1} - \frac{f'(x_1) - f'(y)}{x_2 - x_1}.$$

By multiplying the above inequality with $\frac{x_2 - x_1}{x_2 - y} > 0$ and rearranging we get

$$\frac{f'(x_1) - f'(y)}{x_1 - y} \leq \frac{f'(x_2) - f'(y)}{x_2 - y}.$$

We can treat the cases $x_1 < y < x_2$ and $x_1 < x_2 < y$ similarly and conclude that the function $(x, y) \mapsto \frac{f'(x) - f'(y)}{x - y}$ is nondecreasing in $x$. By symmetry, the same thing holds for $y$ which means that $f'$ is convex.

By this, we finish the proof for the case $n = 2$. From the properties of $n$-convex functions we can deduce that this property transfers to $(n+1)$-convex functions. □

Let us also mention that an $n$-convex function $f$, $n \geq 2$, on the closed interval $[a, b]$ can have discontinuities only at the edges, $a$ and $b$, and only in a certain direction. More precisely, it holds $(-1)^n (f_+(a) - f(a)) \leq 0$ and $f_-(b) \leq f(b)$. Consequently, $f \in \mathscr{K}_{n+1}^c([a, b])$ can have discontinuities only at $a$, $c$ and $b$ and their directions can be derived from the aforementioned discontinuity properties of $n$-convex functions.

After this introduction about convexity at a point let us give necessary and sufficient conditions for inequalities of type (2.49) to hold. The result can be derived directly from Theorem 2.2, but we will derive it from Lemma 2.1 for an easier and more instructive comparison to Theorem 2.14. The following text is based on material from paper [76].

**Theorem 2.13** *Let $A : C([a, c]) \to S(D)$ and $B : C([c, b]) \to S(D)$ be two linear and continuous operators, $h : D \to \mathbb{R}$ and $n \geq 2$. Then, the inequalities*

$$Af \leq \frac{K_f}{n!} h \leq Bf \tag{2.49}$$

*hold for every continuous $f \in \mathscr{K}_{n+1}^c([a, b])$ (and arbitrary constant $K_f$ from Definition 2.1) if and only if the operators $A$ and $B$ satisfy:*

*(a) $Ae_i = Be_i = 0$ for $i = 0, 1, \ldots, n-1$, and $Ae_n = Be_n = h$,*

*(b) $A\rho_n(\cdot, d) \leq 0$ for every $d \in [a, c]$,*

*(c) $Bw_n(\cdot, d) \geq 0$ for every $d \in [c, b]$.*

*Proof.* Assume that $(a)-(c)$ hold and let $F = f - K_f e_n/n!$ be as in Definition 2.1. Since $F$ is $n$-concave on the segment $[a,c]$, by Lemma 2.1 it can be obtained as a uniform limit of functions $F_m$ of the form

$$F_m(x) = P_{n-1}(x) - \sum_{i=1}^{m} \alpha_i w_n(x,x_i) = \tilde{P}_{n-1}(x) + \sum_{i=1}^{m} \alpha_i \rho_n(x,x_i),$$

where $P_{n-1} \in \Pi_{n-1}$, $\alpha_i \geq 0$, $a \leq x_1 < \cdots < x_m \leq c$ and $\tilde{P}_{n-1}(x) = P_{n-1}(x) - \sum_{i=1}^{m} \alpha_i (x - x_i)^{n-1}$. Due to the assumptions,

$$AF_m = A\tilde{P}_{n-1} + \sum_{i=1}^{m} \alpha_i A\rho_n(\cdot,x_i) \leq 0$$

and

$$Af - \frac{K_f}{n!} Ae_n = AF = \lim_{m\to\infty} AF_m \leq 0.$$

Similarly, $F$ restricted to $[c,b]$ can be obtained as a uniform limit of the functions $G_k$ of the form

$$G_k(y) = Q_{n-1}(y) + \sum_{i=1}^{k} \beta_i w_n(y,y_i),$$

where $Q_{n-1} \in \Pi_{n-1}$, $\beta_i \geq 0$ and $c \leq y_1 < \cdots < y_k \leq b$ and we conclude that

$$Bf - \frac{K_f}{n!} Be_n = BG = \lim_{k\to\infty} BG_k \geq 0.$$

On the other hand, suppose that (2.49) holds for every continuous $f \in \mathscr{K}_{n+1}^{c}([a,b])$. Then property $(a)$ holds since both $e_i$ and $-e_i$ for $i = 0,1,\ldots,n-1$ belong to $\mathscr{K}_{n+1}^{c}([a,b])$ with $K_{e_i} = K_{-e_i} = 0$ and both $e_n$ and $-e_n$ belong to $\mathscr{K}_{n+1}^{c}([a,b])$ with $K_{e_n} = n! = -K_{-e_n}$. Moreover, since $\rho_n(\cdot,d)$ (resp. $w_n(\cdot,d)$) belongs to $\mathscr{K}_{n+1}^{c}([a,b])$ for $d \in [a,c]$ (resp. $d \in [c,b]$) and $B\rho_n(\cdot,d) = B0 = 0$ (resp. $Aw_n(\cdot,d) = A0 = 0$), we conclude that property $(b)$ (resp. $(c)$) holds. $\square$

**Remark 2.3** Theorem 2.13 is an extension of Theorem 2.2. For a linear and continuous operator $B : C([c,b]) \to S(D)$ let us define the linear operator $A$ with $Af = B(e_n)[x_0,x_1,\ldots,x_n]f$, where $x_i$, $i = 0,1,\ldots,n$, are some arbitrary distinct points in $[a,c]$. Notice that $Ae_i = 0$ for $i = 0,1,\ldots,n-1$ and $Ae_n = Be_n$, so $A$ satisfies assumption $(a)$ from Theorem 2.13. Moreover, if $B$ satisfies the same assumption, then $BP_{n-1} = 0$ for every $P_{n-1} \in \Pi_{n-1}$ and if, additionally, $B$ satisfies assumption $(c)$, then using the representation of Lemma 2.1 for the $n$-convex function $e_n$, we conclude that $Be_n \geq 0$. Now, since $\rho_n(\cdot,d)$ is an $n$-concave function, we conclude that $A\rho_n(\cdot,d) \leq 0$, *i. e.* $A$ satisfies assumption $(b)$ as well. In conclusion, for the given $A$ and $B$ conditions $(a)-(c)$ are equivalent to

$(i)$ $Be_i = 0$ for $i = 0,1,\ldots,n-1$,

$(ii)$ $Bw_n(\cdot,d) \geq 0$ for every $d \in [c,b]$,

i. e. the same conditions as for the linear operator $A$ in Theorem 2.2. An arbitrary continuous $n$-convex function $f$ on $[c,b]$ can be extended to a continuous function $f \in \mathscr{K}_{n+1}^c([a,b])$ with $K_f = 0$ by defining $f = g$ on $[a,c]$, where $g$ is an arbitrary $n$-concave function such that $g(c) = f(c)$. Then (2.49) yields $Af \leq 0 \leq Bf$, which gives the "if" part of Theorem 2.2. The "only if" part is immediate since $w_n(\cdot,d)$, $e_i$ and $-e_i$ for $i = 0,1,\ldots,n-1$ are all continuous $n$-convex functions.

As we can see from the proof of Theorem 2.13, the function $F$ is approximated well by functions $F_m$ on $[a,c]$ and by functions $G_k$ on $[c,b]$. The polynomials $\tilde{P}_{n-1}$ and $Q_{n-1}$ are different, but if $F$ (and, hence, $f$ as well) satisfies sufficiently strong regularity properties at $c$, then these two polynomials can be chosen equal, *i.e.* one polynomial can be used in approximation of $F$ over the whole interval $[a,b]$. If this is the case, then we can obtain a result similar to Theorem 2.13, but without the middle part in (2.49).

The next lemma shows that it is enough to assume that $F^{(n-2)}$ is continuous at $c$. Since $F$ is $n$-concave on $[a,c]$ and $n$-convex on $[c,b]$, $F^{(n-2)}$ exists and is continuous on the open intervals $(a,c)$ and $(c,b)$, so the additional requirement is that the same property holds at point $c$ as well.

**Lemma 2.7** *Let $n \geq 2$ and let the function $F_{m,k}$ be of the form*

$$F_{m,k}(x) = P_{n-1}(x) + \sum_{i=1}^{m} \alpha_i \rho_n(x,x_i) + \sum_{j=1}^{k} \beta_j w_n(x,y_j), \tag{2.50}$$

*where $P_{n-1} \in \Pi_{n-1}$, $\alpha_i$ ($i = 1,\ldots,m$) and $\beta_j$ ($j = 1,\ldots,k$) are real constants and $a \leq x_1 < \cdots < x_m < c < y_1 < \cdots y_k \leq b$.*

- $(a)$ *A necessary and sufficient condition for $F_{m,k}$ to be $n$-concave on $[a,c]$ and $n$-convex on $[c,b]$ is that $\alpha_i \geq 0$ ($i = 1,\ldots,m$) and $\beta_j \geq 0$ ($j = 1,\ldots,k$).*
- $(b)$ *Every function $F \in C([a,b]) \cap C^{n-2}(a,b)$ that is $n$-concave on $[a,c]$ and $n$-convex on $[c,b]$ is the uniform limit of a sequence of functions $F_{m,k}$ as $m \to \infty$ and $k \to \infty$, where the $F_{m,k}$'s are of the form (2.50) with real constants $\alpha_i \geq 0$ ($i = 1,\ldots,m$) and $\beta_j \geq 0$ ($j = 1,\ldots,k$).*

*Proof.* The intuitive idea of the proof is simple - the goal is to construct a step function that approximates $F_+^{(n-1)}$ well enough so that, after integrating it $n-1$ times, we get a uniformly good approximation of $F$.

Firstly, due to the assumptions, $F_+^{(n-1)}$ exists on $(a,c)$, where it is nonincreasing, and on $(c,b)$, where it is nondecreasing. Furthermore, for every $x,x' \in (a,c)$ and $y,y' \in (c,b)$ it holds

$$\int_x^{x'} F_+^{(n-1)}(t)\,dt = F^{(n-2)}(x') - F^{(n-2)}(x), \tag{2.51}$$

$$\int_{y'}^{y} F_+^{(n-1)}(t)\,dt = F^{(n-2)}(y) - F^{(n-2)}(y'). \tag{2.52}$$

Since $F^{(n-2)}$ is continuous at $c$, the limits $x' \to c$ in (2.51) and $y' \to c$ in (2.52) exist. The limit $\lim_{t \nearrow c} F_+^{(n-1)}(t)$ (resp. $\lim_{t \searrow c} F_+^{(n-1)}(t)$) can be $-\infty$, but then the integral (2.51) (resp. (2.52)) with $x' = c$ (resp. $y' = c$) exists and is finite as an improper integral. In conclusion,

$$\int_x^y F_+^{(n-1)}(t)\,dt = F^{(n-2)}(y) - F^{(n-2)}(x), \quad \text{for every } x, y \in (a,b), \tag{2.53}$$

with, potentially, improper integral(s) at $c$. Furthermore, due to the properties of $F_+^{(n-1)}$ mentioned above, it is easy to see that for arbitrary $\varepsilon_1 > 0$ there exist a constant $\gamma$ and points $\tilde{x} < c$ and $\tilde{y} > c$ sufficiently close to $c$ such that

$$\int_{\tilde{x}}^{\tilde{y}} \left| F_+^{(n-1)}(t) - \gamma \right| dt < \varepsilon_1, \tag{2.54}$$

where $\gamma = \min\{F_+^{(n-1)}(\tilde{x}), F_+^{(n-1)}(\tilde{y})\}$. Let us now define the step function

$$g_{n-1}(x) = \gamma + \sum_{i=1}^{m} \tilde{\alpha}_i \rho_1(x, x_i) + \sum_{j=1}^{k} \tilde{\beta}_j w_1(x, y_j), \tag{2.55}$$

where

$$\begin{aligned}
\tilde{\alpha}_i &= F_+^{(n-1)}(x_i) - F_+^{(n-1)}(x_{i+1}) \geq 0, \quad i = 1, \ldots, m-1 \\
\tilde{\alpha}_m &= F_+^{(n-1)}(x_m) - \gamma \geq 0 \\
\tilde{\beta}_1 &= F_+^{(n-1)}(y_1) - \gamma \geq 0 \\
\tilde{\beta}_j &= F_+^{(n-1)}(y_j) - F_+^{(n-1)}(y_{j-1}) \geq 0, \quad j = 2, \ldots, k.
\end{aligned}$$

The points $x_i$'s and $y_j$'s will suitably be chosen later (so that $g_{n-1}$ will be a "good" approximation of $F_+^{(n-1)}$). Furthermore, let us define, recursively, for $l = n-2, \ldots, 1, 0$:

$$\begin{aligned}
g_l(x) &= \int_c^x g_{l+1}(t)\,dt + F^{(l)}(c) \qquad (2.56) \\
&= P_{n-1-l}(x) + \frac{1}{(n-l)!} \left( \sum_{i=1}^{m} \tilde{\alpha}_i \rho_{n-l}(x, x_i) + \sum_{j=1}^{k} \tilde{\beta}_j w_{n-l}(x, y_j) \right).
\end{aligned}$$

Since $g_{n-1}$ will be a "good" approximation of $F_+^{(n-1)}$, by construction (2.56) the function $g_l$ will be a "good" approximation of $F^{(l)}$. Notice, also, that $g_l^{(j)} = g_{l+j}$ and $g_0$ is a function of the form (2.50) with $\alpha_i = \tilde{\alpha}_i/n!$ and $\beta_i = \tilde{\beta}_i/n!$.

Let $\varepsilon_2 > 0$ be arbitrary and let us now choose the points $y_1, y_2, \ldots$ recursively by the following algorithm: set $y_1 = \tilde{y}$, where $\tilde{y}$ is from (2.54). If $y_j$ is chosen, let

$$y_{j+1} = \inf_{y_i < y < b} \{y : F_+^{(n-1)}(y) - F_+^{(n-1)}(y_j) \geq \varepsilon_2\}. \tag{2.57}$$

Since $F_+^{(n-1)}$ is right-continuous and nonincreasing on $(c,b)$ we have

$$|F_+^{(n-1)}(y_j)-F_+^{(n-1)}(y)|\le \varepsilon_2 \text{ for all } y\in[y_j,y_{j+1}), \tag{2.58}$$

$$F_+^{(n-1)}(y_{j+1})-F_+^{(n-1)}(y_j)\ge \varepsilon_2. \tag{2.59}$$

Due to (2.59), if $\lim_{t\nearrow b}F_+^{(n-1)}(t)$ is finite, then the procedure (2.57) will stop after finitely many steps at some $y_{k-1}$ and, in that case, set $y_k=b$. Otherwise, if $\lim_{t\nearrow b}F_+^{(n-1)}(t)=\infty$, then for sufficiently large $k$ the point $y_k$ can be arbitrarily close to $b$.

If $\lim_{t\searrow a}F_+^{(n-1)}(t)$ is finite, then set $x_1=a$. Otherwise, if $\lim_{t\searrow a}F_+^{(n-1)}(t)=\infty$, the point $x_1$ will suitably be chosen later (and such to be sufficiently close to $a$). Let us now choose the points $x_2,x_3,\ldots$ recursively by the following rule: if $x_i$ is chosen, let

$$x_{i+1}=\inf_{x_i<x<c}\{x: F_+^{(n-1)}(x_i)-F_+^{(n-1)}(x)\ge\varepsilon_2\}. \tag{2.60}$$

Again, the following holds

$$|F_+^{(n-1)}(x_i)-F_+^{(n-1)}(x)|\le \varepsilon_2 \text{ for all } x\in[x_i,x_{i+1}), \tag{2.61}$$

$$F_+^{(n-1)}(x_i)-F_+^{(n-1)}(x_{i+1})\ge \varepsilon_2. \tag{2.62}$$

Due to (2.62), after finitely many steps of the form (2.60) we will reach a point $x_{m-1}$ such that $\tilde{x}$ from (2.54) satisfies $F_+^{(n-1)}(x_{m-1})-F_+^{(n-1)}(\tilde{x})\le\varepsilon_2$. Set $x_m=\tilde{x}$ and stop. Notice now that, due to (2.54) , (2.58) and (2.61), for every $x\in[x_1,y_k]$ one has

$$\begin{aligned}
|F^{(n-2)}\;\;(x)-g_{n-2}(x)| &= \left|\int_c^x\left(F_+^{(n-1)}(t)-g_{n-1}(t)\right)dt\right|\\
&\le \int_{x_1}^{x_m}\left|F_+^{(n-1)}(t)-g_{n-1}(t)\right|dt+\int_{x_m}^{y_1}\left|F_+^{(n-1)}(t)-g_{n-1}(t)\right|dt+\\
&+\int_{y_1}^{y_k}\left|F_+^{(n-1)}(t)-g_{n-1}(t)\right|dt\le\varepsilon_2(x_m-x_1+y_k-y_1)+\varepsilon_1\le\varepsilon_3,
\end{aligned}$$

where the last inequality holds for arbitrary $\varepsilon_3>0$ when we choose $\varepsilon_1$ and $\varepsilon_2$ sufficiently small. In a similar way, it can now be shown by induction that for every $x\in[x_1,y_k]$ and $i=n-3,n-4,\ldots,1,0$,

$$|F^{(i)}(x)-g_i(x)|=\left|\int_c^x\left(F^{(i+1)}(t)-g_{i+1}(t)\right)dt\right|\le\varepsilon_3(b-a)^{n-2-i}. \tag{2.63}$$

If $x_1=a$ and $y_k=b$, then $\|F-g_0\|\le\varepsilon_3(b-a)^{n-2}$ by (2.63) and this finishes the proof. Otherwise, if $\lim_{t\nearrow b}F_+^{(n-1)}(t)=\infty$ or $\lim_{t\searrow a}F_+^{(n-1)}(t)$, we will use some properties of Taylor's expansion and polynomials.

Let $P_{y_k,n-2}\in\Pi_{n-2}$ denote Taylor's polynomial of $F$ at $y_k$ of degree $n-2$, i. e. $P^{(i)}_{y_k,n-2}(y_k)=F^{(i)}(y_k)$ for $i=0,1,\ldots,n-2$. Due to (2.53), the remainder in Taylor's expansion can be written in the integral form, i. e. for every $x\in(a,b)$ it holds

$$F(x)=P_{y_k,n-2}(x)+\frac{1}{(n-2)!}\int_{y_k}^x(x-t)^{n-2}F_+^{(n-1)}(t)\,dt.$$

Let us also denote by $\overline{P}_{n-1}$ the polynomial

$$\begin{aligned}\overline{P}_{n-1}(x) &= P_{y_k,n-2}(x) + F_+^{(n-2)}(y_k)(x-y_k)^{n-1} \\ &= \sum_{i=0}^{n-2} \frac{F^{(i)}(y_k)}{i!}(x-y_k)^i + F_+^{(n-2)}(y_k)(x-y_k)^{n-1}.\end{aligned}$$

We have

$$F(x) = \overline{P}_{n-1}(x) + \frac{1}{(n-2)!}\int_{y_k}^{x}(x-t)^{n-2}(F_+^{(n-1)}(t) - F_+^{(n-1)}(y_k))\,dt \tag{2.64}$$

since $\int_{y_k}^{x}(x-t)^{n-2}\,dt = (n-2)!(x-y_k)^{n-1}$. It is easy to see that the mapping $h_{y_k}(x) = \int_{y_k}^{x}(x-t)^{n-2}(F_+^{(n-1)}(t) - F_+^{(n-1)}(y_k))\,dt$ is monotone on $[y_k,b]$ with $h_{y_k}(y_k) = 0$. Since $F$ is continuous at $b$, the limit $x \to b$ in (2.64) exists and the integral

$$\int_{y_k}^{b}(x-t)^{n-2}(F_+^{(n-1)}(t) - F_+^{(n-1)}(y_k))\,dt \tag{2.65}$$

is finite. Moreover, since we can choose $y_k$ arbitrarily close to $b$, by the dominated convergence theorem integral (2.65) can be arbitrarily small. Therefore, for arbitrary $\varepsilon_4 > 0$, we can choose $y_k$ such that for every $x \in [y_k,b]$ it holds

$$\begin{aligned}|F(x) - \overline{P}_{n-1}(x)| &= \left|\frac{1}{(n-2)!}\int_{y_k}^{x}(x-t)^{n-2}(F_+^{(n-1)}(t) - F_+^{(n-1)}(y_k))\,dt\right| \\ &\leq \left|\frac{1}{(n-2)!}\int_{y_k}^{b}(b-t)^{n-2}(F_+^{(n-1)}(t) - F_+^{(n-1)}(y_k))\,dt\right| < \varepsilon_4.\end{aligned}$$

By construction, for $x \in [y_k,b]$ we have $g_0(x) = P_{n-1}(x) + \sum_{j=1}^{k}\beta_j(x-y_j)^{n-1}$, i. e. $g_0$ on the interval $[y_k,b]$ is a polynomial in $\Pi_{n-1}$. Furthermore, by construction $g_0 \in C^{(n-2)}([a,b])$ and $g_0^{(n-1)}(y_k) = g_{n-1}(y_k) = F_+^{(n-1)}(y_k)$. Therefore, for $x \in [y_k,b]$ it holds

$$g_0(x) = \sum_{i=0}^{n-2}\frac{g_0^{(i)}(y_k)}{i!}(x-y_k)^i + F_+^{(n-1)}(y_k)(x-y_k)^{n-1}.$$

From (2.63) we conclude $|F^{(i)}(y_k) - g_0^{(i)}(y_k)| \leq \varepsilon_3(b-a)^{n-2-i}$. Therefore, for every $x \in [y_k,b]$ we have

$$|\overline{P}(x) - g_0(x)| \leq \varepsilon_3(b-a)^{n-2}\sum_{i=0}^{n-2}\frac{1}{i!}$$

and

$$|F(x) - g_0(x)| \leq |F(x) - \overline{P}(x)| + |\overline{P}(x) - g_0(x)| \leq \varepsilon_4 + \varepsilon_3(b-a)^{n-2}\sum_{i=0}^{n-2}\frac{1}{i!}. \tag{2.66}$$

In the same way we can show that we can choose $x_1$ sufficiently close to $a$ such that (2.66) holds for every $x \in [a,x_1]$. Finally, for sufficiently small $\varepsilon_3$ and $\varepsilon_4$, from (2.63) and (2.66) we conclude that for arbitrary $\varepsilon > 0$ we can construct $g_0$ of the form (2.50) such that

$$|F(x) - g_0(x)| \leq \varepsilon$$

for every $x \in [a,b]$. □

The following theorem gives necessary and sufficient conditions for inequality of type $Af \leq Bf$ to hold and it is based on Lemma 2.7.

**Theorem 2.14** *Let $n \geq 2$ and let $A : C([a,c]) \to S(D)$ and $B : C([c,b]) \to S(D)$ be two linear and continuous operators. Then, the inequality*

$$Af \leq Bf, \tag{2.67}$$

*holds for every continuous $f \in \mathscr{K}_{n+1}^c([a,b]) \cap C^{n-2}((a,b))$ if and only if the operators $A$ and $B$ satisfy:*

- $(\tilde{a})$ *$Ae_i = Be_i$ for $i = 0,1,\ldots,n$,*
- $(\tilde{b})$ *$A\rho_n(\cdot,d) \leq 0$ for every $d \in [a,c]$,*
- $(\tilde{c})$ *$Bw_n(\cdot,d) \geq 0$ for every $d \in [c,b]$.*

*Proof.* Assume that $(\tilde{a})-(\tilde{c})$ hold and let $f \in \mathscr{K}_{n+1}^c([a,b]) \cap C^{n-2}((a,b))$ be continuous with $F = f - K_f e_n/n!$ as in Definition 2.1. By Lemma 2.7, the function $F$ given by (2.47) can be obtained as a uniform limit of functions of the form (2.50) with $\alpha_i \geq 0$ and $\beta_j \geq 0$. Assumption $(\tilde{a})$ yields $AP_{n-1} = BP_{n-1}$. Moreover, since $Aw_n(\cdot,y_j) = A0 = 0$ for $y_j \in [c,b]$ and $B\rho_n(\cdot,x_i) = B0 = 0$ for $x_i \in [a,c]$, we have

$$\begin{aligned} AF_{m,k} &= AP_{n-1} + \sum_{i=1}^{m} \alpha_i A\rho_n(\cdot,x_i) \leq AP_{n-1} = BP_{n-1} \\ &\leq BP_{n-1} + \sum_{j=1}^{k} \beta_i Bw_n(\cdot,y_j) = BF_{m,k}. \end{aligned}$$

By taking limits we conclude $AF \leq BF$, so

$$Af = AF + \frac{K_f}{n!} Ae_n \leq BF + \frac{K_f}{n!} Be_n = Bf.$$

On the other hand, assume (2.67) holds for every continuous function $f \in \mathscr{K}_{n+1}^c([a,b]) \cap C^{n-2}((a,b))$. Since both $e_i$ and $-e_i$ for $i = 0,\ldots,n-1$ belong to $\mathscr{K}_{n+1}^c([a,b]) \cap C^{n-2}((a,b))$, we conclude that both $Ae_i \leq Be_i$ and $A(-e_i) \leq B(-e_i)$, so $(\tilde{a})$ holds. Furthermore, $\rho_n(\cdot,x_i)$, $w_n(\cdot,y_j) \in C^{n-2}((a,b))$ and analogously as in the proof of Theorem 2.13 we conclude that both $(\tilde{b})$ and $(\tilde{c})$ hold. □

**Remark 2.4** Condition $(a)$ is stronger than condition $(\tilde{a})$, which is reflected in inequalities (2.49) being stronger than inequality (2.67) with the middle term squeezed in between in (2.49). On the other hand, Theorems 2.13 and 2.14 represent separate results since it is possible to construct linear operators $A$ and $B$ that satisfy conditions $(\tilde{a})-(\tilde{c})$ and such that there exists an $i$, $0 \leq i \leq n-2$, such that $Ae_i = Be_i \neq 0$.

For example, let $n = 3$, $b = -a > 0$, $c = 0$, $x_i \in [a,0]$ $(i = 1,\dots,m)$, $y_i = -x_i$ and let the operators $A$ and $B$ be given by

$$Af = \sum_{i=1}^{m} p_i f(x_i), \qquad Bf = \sum_{i=1}^{m} p_i f(y_i).$$

Notice that

$$Ae_0 = Be_0 = \sum_{i=1}^{m} p_i, \quad Ae_1 = -Be_1 = \sum_{i=1}^{m} p_i x_i, \quad Ae_2 = Be_2 = \sum_{i=1}^{m} p_i x_i^2.$$

If $p_i$'s are such that $Ae_1 = 0$, then $(\tilde{a})$ holds. Furthermore, if $p_i$'s and $x_i$'s are such that the condition

$$\sum_{i=1}^{m} p_i (x_i - d)_- \le 0 \qquad \text{for every } d \in [a,0]$$

holds, then also $(\tilde{b})$ and $(\tilde{c})$ hold. For example, all this holds for $m = 2$, $p_1 = 1$, $x_1 = -3$, $p_2 = -3$ and $x_2 = -1$. Therefore, the linear operators

$$\begin{aligned} Af &= f(-3) - 3f(-1) \\ Bf &= f(3) - 3f(1) \end{aligned}$$

satisfy $(\tilde{a})$–$(\tilde{c})$, but $Ae_0 = Be_0 = -2 \neq 0$. Thus, $Af \le Bf$ for every continuous $f \in \mathscr{K}_1^{n,c}([a,b])$, but there exists such an $f$ such that (2.49) doesn't hold. For example, the constant function $f(x) = u$, where $0 \neq u \in \mathbb{R}$, satisfies $K_f = f''(0) = 0$, so the middle term in (2.49) is zero, while $Af = Bf = -2u \neq 0$.

Similar results hold for $n = 1$ as well, but with minor technical modifications. Firstly, the functions $\rho_n$ and $w_n$ for $n = 1$ are not continuous, so we need to require that the linear operators $A$ and $B$ are defined on a larger class of functions that contains them. More importantly, we also loose the "only if" parts of Theorems 2.13 and 2.14. Secondly, the representation in Lemma 2.7 assumes that $F \in C^{n-2}((a,b))$, an assumption that is mute for $n = 1$ and can, actually, be ignored. Therefore, for simplicity of presentation, we state the result for $n = 1$ in a separate theorem. As for notation, let $\overline{C}[a,b]$ denote a linear space of functions such that $C([a,b]) \subset \overline{C}([a,b])$ and $w_1(\cdot,d) \in \overline{C}[a,b]$ for $d \in [a,b]$ (for example, $\overline{C}[a,b] = \{f + \sum_{i=1}^{m} \alpha_i w_1(\cdot,x_i) : f \in C([a,b]), \alpha_i \in \mathbb{R}, x_i \in [a,b]\}$).

**Theorem 2.15** *Let $A : \overline{C}[a,c] \to S(D)$ and $B : \overline{C}[c,b] \to S(D)$ be two linear and continuous operators. If*

*(i)* $Ae_0 = Be_0$ *and* $Ae_1 = Be_1$,

*(ii)* $A\rho_1(\cdot,d) \le 0$ *for every* $d \in [a,c]$,

*(iii)* $Bw_1(\cdot,d) \ge 0$ *for every* $d \in (c,b]$,

*then for every continuous $f \in \mathscr{K}_2^c([a,b])$ the following inequality holds*

$$Af \le Bf. \tag{2.68}$$

*If, additionally,*

$(iv)$ $Ae_0 = Be_0 = 0$,

*then for every constant* $K_f$ *from Definition* 2.1 *the following inequalities hold*

$$Af \leq K_f Ae_1 = K_f Be_1 \leq Bf.$$

*Proof.* The function $F = f - K_f e_1$ is continuous and it is nonincreasing on $[a,c]$ and nondecreasing on $[c,b]$. Since a continuous function on a closed interval is uniformly continuous, for arbitrary $\varepsilon > 0$ there exist points $a \leq x_1 < x_2 < \ldots < x_m \leq c < y_1 < \ldots < y_k \leq b$ such that the step function

$$g(x) = F(c) + \sum_{i=1}^{m} \alpha_i \rho_1(x, x_i) + \sum_{j=1}^{k} \beta_j w_1(x, y_j)$$

satisfies

$$\max_{a \leq x \leq b} |F(x) - g(x)| \leq \varepsilon,$$

where

$$\begin{aligned}
\alpha_m &= F(x_m) - F(c) \geq 0, \\
\alpha_i &= F(x_{i-1}) - F(x_i) \geq 0, \quad i = m-1, m-2, \ldots, 2, \\
\beta_1 &= F(y_1) - F(c) \geq 0, \\
\beta_j &= F(y_j) - F(y_{j-1}) \geq 0, \quad j = 2, 3, \ldots, k.
\end{aligned}$$

The rest of the proof follows the same lines as the proof of Theorem 2.14. □

**Remark 2.5** We, indeed, do not have the "only if" part in Theorem 2.15. For example, if $A$ and $B$ are the linear operators

$$Af = 0 \quad \text{and} \quad Bf = f_-(d) - f(d) \text{ for some fixed } d \in (c,b),$$

then $Af = Bf = 0$ for every continuous $f$ (so (2.68) holds), but $Bw_1(\cdot, d) = -1 < 0$ (*i. e.* $(iii)$ of Theorem 2.15 doesn't hold).

**Remark 2.6** If $(i)$–$(iii)$ of Theorem 2.15 hold and $Aw_1(\cdot, \tilde{x}) = 0$ for some $a < \tilde{x} < c$ and $B\rho_1(\cdot, \tilde{y}) = 0$ for some $c < \tilde{y} < b$, then $(iv)$ of Theorem 2.15 holds also. Indeed, for every $d$ we have $w_1(\cdot, d) + \rho_1(\cdot, d) = e_0$, so

$$0 \geq A\rho_1(\cdot, \tilde{x}) = Ae_0 = Be_0 = Bw_1(\cdot, \tilde{y}) \geq 0.$$

## 2.6.1 Levinson Type Inequality as an Application

### Historical overwiev about the Levinson inequality

For the first time, an inequality which is later called the Levinson inequality appeared in paper [39] in the following form:

**Theorem 2.16** *If $f : \langle 0,c\rangle \to \mathbb{R}$ satisfies $f''' \geq 0$ and $p_i, x_i, y_i$, $i = 1,2,\ldots,n$, are such that $p_i > 0$, $\sum_{i=1}^{n} p_i = 1$, $0 \leq x_i \leq c$ and*

$$x_1 + y_1 = x_2 + y_2 = \ldots = x_n + y_n = 2c, \tag{2.69}$$

*then the inequality*

$$\sum_{i=1}^{n} p_i f(x_i) - f(\overline{x}) \leq \sum_{i=1}^{n} p_i f(y_i) - f(\overline{y}) \tag{2.70}$$

*holds, where $\overline{x} = \sum_{i=1}^{n} p_i x_i$ and $\overline{y} = \sum_{i=1}^{n} p_i y_i$ denote the weighted arithmetic means.*

If $a = \frac{1}{2}$, $p_1 = \ldots = p_n = 1$ and $f(x) = \log x$, then the Levinson inequality (2.70) becomes the famous Ky-Fan inequality

$$\frac{G_n}{G'_n} \leq \frac{A_n}{A'_n},$$

where $A_n = \frac{1}{n}\sum_{k=1}^{n} x_k$, $A'_n = \frac{1}{n}\sum_{k=1}^{n}(1-x_k)$, $G_n = \left(\prod_{k=1}^{n} x_k\right)^{1/n}$ and $G'_n = \left(\prod_{k=1}^{n}(1-x_k)\right)^{1/n}$.

The assumptions on the differentiability of $f$ can be weakened by working with the divided differences. Namely, T. Popoviciu [88] showed that in Theorem 2.16 it is enough to assume that $f$ is 3-convex. P. Bullen [9] gave another proof of Popoviciu's result, as well as a converse of the Levinson inequaliy rescaled to a general interval $[a,b]$. Bullen's result is the following:

**Theorem 2.17** *$(a)$ If $f : [a,b] \to \mathbb{R}$ is 3-convex and $p_i, x_i, y_i$, $i = 1,2,\ldots,n$, are such that $p_i > 0$, $\sum_{i=1}^{n} p_i = 1$, $a \leq x_i, y_i \leq b$, (2.69) holds for some $c \in [a,b]$, and*

$$\max\{x_1,\ldots,x_n\} \leq \min\{y_1,\ldots,y_n\}, \tag{2.71}$$

*then (2.70) holds.*

*$(b)$ If for a continuous function $f$ inequality (2.70) holds for all $n$, all $c \in [a,b]$, all $2n$ distinct points satisfying (2.69) and (2.71) and all weights $p_i > 0$ such that $\sum_{i=1}^{n} p_i = 1$, then $f$ is 3-convex.*

In [59] J. Pečarić proved that one can weaken assumption (2.71) substituting it with the following condition:

$$x_i + x_{n-i+1} \leq 2c, \quad \frac{p_i x_i + p_{n-i+1} x_{n-i+1}}{p_i + p_{n-i+1}} \leq c, \; i = 1,2,\ldots,n.$$

In [63], J. Pečarić showed that in Theorem 2.16 instead of variables with sum equal to $2c$, we can use variables with constant difference.

**Theorem 2.18** *If $f : [a,b] \to \mathbb{R}$ is 3-convex and $p_i, x_i, y_i$, $i = 1,2,\ldots,n$, are such that $p_i > 0$, $a \leq x_i, y_i \leq b$, and*

$$y_1 - x_1 = y_2 - x_2 = \ldots = y_n - x_n > 0,$$

*then*

$$\frac{\sum_{i=1}^{n} p_i f(x_i)}{\sum_{i=1}^{n} p_i} - f\left(\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i}\right) \leq \frac{\sum_{i=1}^{n} p_i f(y_i)}{\sum_{i=1}^{n} p_i} - f\left(\frac{\sum_{i=1}^{n} p_i y_i}{\sum_{i=1}^{n} p_i}\right) \tag{2.72}$$

*holds.*

*Proof.* We use mathematical induction by $n$. Let $n = 2$. Firstly, let us prove that for 3-convex function $f$

$$[z_0, z_1, z_2; f] \geq [z_3, z_4, z_5; f]$$

holds for $z_0 > z_3$, $z_1 > z_4$, $z_2 > z_5$. Namely, we have the following

$$\begin{aligned}
(z_0 - z_3)[z_0, z_1, z_2, z_3; f] &= [z_0, z_1, z_2; f] - [z_1, z_2, z_3; f] \\
(z_1 - z_4)[z_1, z_2, z_3, z_4; f] &= [z_1, z_2, z_3; f] - [z_2, z_3, z_4; f] \\
(z_2 - z_5)[z_2, z_3, z_4, z_5; f] &= [z_2, z_3, z_4; f] - [z_3, z_4, z_5; f].
\end{aligned}$$

From 3-convexity we get

$$[z_0, z_1, z_2; f] \geq [z_1, z_2, z_3; f] \geq [z_2, z_3, z_4; f] \geq [z_3, z_4, z_5; f],$$

i.e.

$$[z_0, z_1, z_2; f] \geq [z_3, z_4, z_5; f]. \tag{2.73}$$

Putting in (2.73)

$$z_0 = y_1,\ z_2 = y_2,\ z_1 = \frac{p_1 y_1 + p_2 y_2}{p_1 + p_2},\ z_3 = x_1,\ z_5 = x_2,\ z_4 = \frac{p_1 x_1 + p_2 x_2}{p_1 + p_2}$$

the denominator of $[z_0, z_1, z_2; f]$ is equal to $(y_1 - y_2)^2$, while the denominator of $[z_3, z_4, z_5; f]$ is equal to $(x_1 - x_2)^2$, i.e. the denominators are the same and the rest of that inequality is, in fact, inequality (2.72) for $n = 2$.

Let us suppose that (2.72) holds for $n \leq m - 1$. Putting

$$x_1 \to \frac{1}{P_{m-1}} \sum_{i=1}^{m-1} p_i x_i, \quad x_2 \to x_m, \quad p_1 \to P_{m-1}, \quad p_2 \to p_m,$$

$$y_1 \to \frac{1}{P_{m-1}} \sum_{i=1}^{m-1} p_i y_i, \quad y_2 \to y_m$$

in (2.72) for $n = 2$ and using the assumption of induction for $n = m - 1$ we obtain

$$f\left(\frac{\sum_{i=1}^{m} p_i y_i}{P_m}\right) - f\left(\frac{\sum_{i=1}^{m} p_i x_i}{P_m}\right)$$

$$\leq \frac{1}{P_m}\left\{P_{m-1}f\left(\frac{1}{P_{m-1}}\sum_{i=1}^{m-1}p_iy_i\right)-f\left(\frac{1}{P_m}\sum_{i=1}^{m-1}p_ix_i\right)+p_m(f(y_m)-f(x_m))\right\}$$
$$\leq \frac{1}{P_m}\sum_{i=1}^{m}p_i(f(y_i)-f(x_i)),$$

where $P_m = \sum_{i=1}^{m} p_i$. So, the inequality (2.72) holds for any $n \geq 2$ by the principle of mathematical induction. □

Some other generalizations of the Levinson inequality will be given in the following chapter.

### Levinson type inequality for 3-convex function at a point

In the previous text we described the history of the Levinson inequality. Here it is continued by further investigation. At 2010 A.McD. Mercer ([45]) made a significant improvement by replacing the condition of symetric distribution of points $x_i$'s and $y_i$'s around $c$ by the weaker one that the variances of two sets of points are equal. Namely, he proved the following result.

**Theorem 2.19** *If $f : [a,b] \to \mathbb{R}$ satisfies $f''' \geq 0$ and $p_i, x_i, y_i$, $i = 1,2,\ldots,n$ are such that $p_i \geq 0$, $\sum_{i=1}^{n} p_i = 1$, $a \leq x_i, y_i \leq b$,*

$$\max\{x_1,\ldots,x_n\} \leq \min\{y_1,\ldots,y_n\}$$

*and*

$$\sum_{i=1}^{n} p_i(x_i-\overline{x})^2 = \sum_{i=1}^{n} p_i(y_i-\overline{y})^2,$$

*then*

$$\sum_{i=1}^{n} p_i f(x_i) - f(\overline{x}) = \sum_{i=1}^{n} p_i f(y_i) - f(\overline{y}),$$

*where $\overline{x} = \sum_{i=1}^{n} p_ix_i$, $\overline{y} = \sum_{i=1}^{n} p_iy_i$.*

A. Witkowski extended this result to probabilistic settings, [102], while I. Baloch, J. Pečarić and M. Praljak in [4] showed that under the equal-variances assumption the Levinson inequality holds for a larger class of functions, namely for the 3-convex functions at a point. A probabilistic version of Levinson's inequality under the equal-variances assumption for the class of 3-convex functions at a point was proved by J. Pečarić, M. Praljak and A. Witkowski [76]. It is a consequence of Theorem 2.13. Also, other consequences of the above-proved results from paper [76] are given.

**Corollary 2.1** *Let $X, Y : \Omega \to [a,c]$ be two random variables such that* $\mathrm{Var}[X] = \mathrm{Var}[Y] = C$. *Then, for every continuous $f \in \mathscr{K}_3^c([a,b])$ the inequalities*

$$\mathrm{E}[f(X)] - f(\mathrm{E}[X]) \leq \frac{K_f}{2}C \leq \mathrm{E}[f(Y)] - f(\mathrm{E}[Y])$$

*hold.*

*Proof.* Apply Theorem 2.13 to the linear operators

$$\begin{aligned} Af &= \mathrm{E}[f(X)] - f(\mathrm{E}[X]), \\ Bf &= \mathrm{E}[f(Y)] - f(\mathrm{E}[Y]). \end{aligned}$$

Since continuous functions on a segment are bounded, by the dominated convergence theorem the linear operators $A$ and $B$ are continuous. Condition $(a)$ holds since $Ae_0 = Be_0 = \mathrm{E}[1] - 1 = 0$, $Ae_1 = \mathrm{E}[X] - \mathrm{E}[X] = 0 = \mathrm{E}[Y] - \mathrm{E}[Y] = Bf$, $Ae_2 = \mathrm{Var}[X]$ and $Be_2 = \mathrm{Var}[Y]$. Furthermore, the functions $w_2(\cdot,d)$ (resp. $\rho_2(\cdot,d)$) for $d \in [a,c]$ (resp. $d \in [c,b]$) are convex (resp. concave), so $(b)$ (resp. $(c)$) hold due to Jensen's inequality. □

We can get a generalization of the probabilistic Levinson type inequality from Corollary 2.1 without the middle term as a corollary of Theorem 2.14.

**Corollary 2.2** *Let $\lambda : [a,c] \to \mathbb{R}$ and $\mu : [c,b] \to \mathbb{R}$ be two functions of bounded variation such that*

$$\bar{x}_\lambda = \int_a^c x\, d\lambda(x) \in [a,c] \quad \text{and} \quad \bar{x}_\mu = \int_c^b x\, d\mu(x) \in [c,b].$$

*Then, the inequality*

$$\int_a^c f(x)\, d\lambda(x) - f(\bar{x}_\lambda) \leq \int_c^b f(x)\, d\mu(x) - f(\bar{x}_\mu)$$

*holds for every continuous $f \in \mathscr{K}_3^c([a,b])$ if and only if $\lambda$ and $\mu$ satisfy:*

*(i)* $\displaystyle\int_a^c d\lambda(x) = \int_c^b d\mu(x)$ *and* $\displaystyle\int_a^c x^2\, d\lambda(x) - \bar{x}_\lambda^2 = \int_c^b x^2\, d\mu(x) - \bar{x}_\mu^2,$

*(ii)* $\displaystyle\int_a^d (x-d)\, d\lambda(x) \leq (\bar{x}_\lambda - d)_- = \min\{\bar{x}_\lambda - d, 0\}$ *for every $d \in [a,c]$,*

*(iii)* $\displaystyle\int_d^b (x-d)\, d\mu(x) \geq (\bar{x}_\mu - d)_+ = \max\{\bar{x}_\mu - d, 0\}$ *for every $d \in [c,b]$.*

*Proof.* Apply Theorem 2.14 to the linear operators $A$ and $B$ given by

$$Af = \int_a^c f(x)\, d\lambda(x) - f(\bar{x}_\lambda),$$

$$Bf = \int_c^b f(x)\, d\mu(x) - f(\bar{x}_\mu).$$

By the same argument as in the proof of Corollary 2.1, the operators $A$ and $B$ are continuous. Conditions $(\tilde{a})-(\tilde{c})$ for these particular operators correspond to conditions $(i)-(iii)$. □

Since the functions $\lambda$ and $\mu$ in Corollary 2.3 do not need to generate probability measures, that corollary is, indeed, a generalization of Corollary 2.1. For example, if $\lambda$ and $\mu$ satisfy the Jensen-Steffensen conditions (*i. e.* $\lambda(a) \leq \lambda(x) \leq \lambda(c)$ for every $x \in [a,c]$ and $\lambda(a) < \lambda(c)$; $\mu(c) \leq \mu(x) \leq \mu(b)$ for every $x \in [c,b]$ and $\mu(c) < \mu(b)$), then the Jensen-type inequality still holds for convex functions (see, e. g., [77]). Hence, the convex functions $w_2(\cdot,d)$ and $-\rho_2(\cdot,d)$ satisfy the inequalities in $(ii)$ and $(iii)$ of Corollary 2.3.

The following is another corollary of Theorem 2.14.

**Corollary 2.3** *Let $\lambda : [a,c] \to \mathbb{R}$ and $\mu : [c,b] \to \mathbb{R}$ be two functions of bounded variation and $n \geq 2$. Then, the inequality*

$$\int_a^c f(x)\,d\lambda(x) \leq \int_c^b f(x)\,d\mu(x)$$

*holds for every continuous $f \in \mathscr{K}_{n+1}^c([a,b]) \cap C^{n-2}((a,b))$ if and only if $\lambda$ and $\mu$ satisfy:*

*(i)* $\int_a^c x^i\,d\lambda(x) = \int_c^b x^i\,d\mu(x)$, *for every $i = 0,1,\dots,n$,*

*(ii)* $\int_a^d (x-d)^{n-1}\,d\lambda(x) \leq 0$ *for every $d \in [a,c]$,*

*(iii)* $\int_d^b (x-d)^{n-1}\,d\mu(x) \geq 0$ *for every $d \in [c,b]$.*

*Proof.* Apply Theorem 2.14 to the linear operators $A$ and $B$ given by

$$\begin{aligned} Af &= \int_a^c f(x)\,d\lambda(x), \\ Bf &= \int_c^b f(x)\,d\mu(x). \end{aligned}$$

By the same argument as in the proof of Corollary 2.1, the operators $A$ and $B$ are continuous. Conditions $(\tilde{a})-(\tilde{c})$ for these particular operators correspond to conditions $(i)-(iii)$. □

The following corollary is the discrete version of Corollary 2.3.

**Corollary 2.4** *Let $n \in \mathbb{N}$, $n \geq 2$, and $a \leq x_1 < \cdots < x_m \leq c \leq y_1 < \cdots < y_k \leq b$. Then, the inequality*

$$\sum_{i=1}^{m} p_i f(x_i) \leq \sum_{j=1}^{k} q_j f(y_j)$$

*holds for every continuous $f \in \mathscr{K}_{n+1}^c([a,b]) \cap C^{n-2}((a,b))$ if and only if the sequences $p$ and $q$ satisfy:*

*(i)* $\sum_{j=1}^m p_j x_j^i = \sum_{j=1}^k q_j y_j^i$ *for every $i = 0,1,\dots,n$,*

*(ii)* $\sum_{i=1}^m p_i (x_i - d)_-^{n-1} \leq 0$ *for every $d \in [a,c]$,*

*(iii)* $\sum_{j=1}^k q_j (y_j - d)_+^{n-1} \geq 0$ *for every $d \in [c,b]$.*

*Proof.* Apply Theorem 2.14 to the linear operators

$$Af = \sum_{i=1}^{m} p_i f(x_i) \qquad \text{and} \qquad Bf = \sum_{j=1}^{k} q_j f(y_j). \tag{2.74}$$

□

**Remark 2.7** Popoviciu studied necessary and sufficient conditions on points $x_i$ and weights $p_i$ for the inequality $\sum_{i=1}^{m} p_m f(x_m) \geq 0$ to be valid for every $n$-convex function $f$ (see [77], [85]). In light of Remark 2.3, Corollary 2.4 is an extension of Popoviciu's results.

The version of Corollary 2.4 for $n = 1$ can be obtained as a corollary of Theorem 2.15.

**Corollary 2.5** *Let* $a \leq x_1 \leq \cdots < x_m < c < y_1 < \cdots < y_k \leq b$, $\overline{x}_p = \sum_{i=1}^{m} p_i x_i$, $\overline{y}_q = \sum_{j=1}^{k} q_j y_j$ *be such that:*

(*i*) $\overline{x}_p = \overline{y}_q$,

(*ii*) $\sum_{i=1}^{m_1} p_i \leq 0$ *for every* $m_1 = 1, \ldots, m-1$, *and* $\sum_{i=1}^{m} p_i = 0$,

(*iii*) $\sum_{j=k_1}^{k} q_j \geq 0$ *for every* $k_1 = 2, \ldots, k$, *and* $\sum_{j=1}^{k} q_j = 0$.

*Then, the inequality*

$$\sum_{i=1}^{m} p_i f(x_i) \leq K_f \overline{x}_p = K_f \overline{y}_q \leq \sum_{j=1}^{k} q_j f(y_j) \tag{2.75}$$

*holds for every continuous* $f \in \mathscr{K}_2^c([a,b])$.

*Proof.* Follows by applying Theorem 2.15 to the linear operators $A$ and $B$ given by (2.74). Notice that $Aw_1(\cdot,d) = 0$ for $d \in (x_m, c)$ and $B\rho_1(\cdot,d) = 0$ for $d \in (c, y_1)$, so, by Remark 2.6, $Ae_0 = \sum_{i=1}^{m} p_i = 0$ and $Be_0 = \sum_{j=1}^{k} q_j = 0$ are implied by the other assumptions. □

A simple example when $p_i$'s and $q_j$'s satisfy assumptions (*ii*) and (*iii*) of Corollary 2.5 is when $m = 2m'$ and $k = 2k'$ are even and $p_i = (-1)^i$, $q_j = (-1)^j$. Furthermore, if $x_i$'s and $y_j$'s are such that $\overline{x}_p = \sum_{i=1}^{m'} (x_{2i} - x_{2i-1}) = \sum_{j=1}^{k'} \left(y_{2j} - y_{2j-1}\right) = \overline{y}_q$, then (*i*) holds as well and inequality (2.75) states

$$\sum_{i=1}^{m'} \left(f(x_{2i}) - f(x_{2i-1})\right) \leq K_f \overline{x}_p = K_f \overline{y}_q \leq \sum_{j=1}^{k'} \left(f(y_{2j}) - f(y_{2j-1})\right).$$

# Chapter 3

# General Linear Inequalities for Multivariate Functions

This chapter is based on recent results given in [28] and [30] by A.R. Khan, J. Pečarić and S. Varošanec.

Firstly we give results for a function with two variables. This approach is chosen just because in the case of two variable reader can easily pointed out a method. The same method is used also in more general case for functions with several variables, but in that general case complicated notations and calculations are fogging up mathematical ideas.

Let $f$ be a real-valued function defined on $I \times J$, $I = [a,b]$, $J = [c,d]$. Then the $(n,m)$th divided difference of a function $f$ at distinct points $x_i, \dots, x_{i+n} \in I$, $y_j, \dots, y_{j+m} \in J$ is defined by

$$\Delta_{(n,m)} f(x_i, y_j) = [x_i, \dots, x_{i+n}; [y_j, \dots, y_{j+m}; f]].$$

Sometimes, notation $\Delta_m^n f(x_i, y_j)$ is used in literature for the $(n,m)$th divided difference of a function $f$. A function $f : I \times J \to \mathbb{R}$ is said to be convex of order $(n,m)$ or $(n,m)$-convex if inequality

$$\Delta_{(n,m)} f(x_i, y_j) \geq 0$$

holds for all distinct points $x_i, \dots, x_{i+n} \in I$, $y_j, \dots, y_{j+m} \in J$.

It is known that if the partial derivative $\dfrac{\partial^{n+m} f}{\partial x^n \partial y^m}$ exists, then $f$ is $(n,m)$-convex if and only if $\dfrac{\partial^{n+m} f}{\partial x^n \partial y^m} \geq 0$.

Similarly, we can extend the above-mentioned definition of divided difference up to order $(m_1, \dots, m_n)$ as follows: Let $f$ be a function of $n$ variables defined on $I_1 \times \ldots \times I_n =$

$[a_1,b_1]\times\ldots\times[a_n,b_n]$ where $[a_i,b_i]\subset\mathbb{R}$ for $i=1,2,\ldots,n$. Then the $(m_1,\ldots,m_n)$th divided difference of the function $f$ at distinct points $x_{ji_j},\ldots,x_{j(i_j+m_j)}\in I_j$, for $j=1,\ldots,n$ is given as

$$\Delta_{(m_1,\ldots,m_n)}f(x_{1i_1},\ldots,x_{ni_n})=$$
$$[x_{1i_1},\ldots,x_{1(i_1+m_1)};[x_{2i_2},\ldots,x_{2(i_2+m_2)};[\ldots;[x_{ni_n},\ldots,x_{n(i_n+m_n)};f]\ldots]]].$$

We say that $f:I_1\times\ldots\times I_n\to\mathbb{R}$ is a convex function of order $(m_1,\ldots,m_n)$ (or $(m_1,\ldots,m_n)$-convex function) if

$$\Delta_{(m_1,\ldots,m_n)}f(x_{1j},\ldots,x_{nj})\geq 0$$

holds, where $x_{ji_j},\ldots,x_{j(i_j+m_j)}\in I_j$, for $j=1,\ldots,n$.

If all partial derivatives $\dfrac{\partial^{m_1+\ldots+m_n}f}{\partial x_1^{m_1}\ldots\partial x_n^{m_n}}$ (denoted by $f_{(m_1,\ldots,m_n)}$) exist, then $f$ is $(m_1,\ldots,m_n)$-convex if and only if $f_{(m_1,\ldots,m_n)}\geq 0$.

Similarly as in the case of one variable, we can talk about discrete $(n,m)$-convex function or a discrete $(m_1,\ldots,m_n)$-convex function if a domain of a function is not a product of intervals but a product of discrete sets of real numbers.

## 3.1 Discrete Results for Functions of Two Variables

Let us now consider a real-valued function of two variables defined on $I\times J$, $I,J$ are segments in $\mathbb{R}$. Firstly, we obtain an identity for $\sum_{i=1}^N\sum_{j=1}^M p_{ij}f(x_i,y_j)$ which involves divided differences of $f$ and then, in the next theorem, we consider necessary and sufficient conditions that inequality

$$\sum_{i=1}^N\sum_{j=1}^M p_{ij}f(x_i,y_j)\geq 0$$

holds for every discrete convex function of order $(n,m)$, $m,n\in\mathbb{N}$.

**Theorem 3.1** *Let $x_1,\ldots,x_N$ be mutually distinct numbers from $I=[a,b]$ and $y_1,\ldots,y_M$ be mutually distinct numbers from $J=[c,d]$ and let $f$ be a real-valued functions on $I\times J$. Let $p_{ij}$, $(i=1,\ldots,N,\ j=1,\ldots,M)$, be real numbers.*

*Then the following identity holds:*

$$\sum_{i=1}^N\sum_{j=1}^M p_{ij}f(x_i,y_j) \tag{3.1}$$

$$=\sum_{k=0}^{m-1}\sum_{t=0}^{n-1}\sum_{s=t+1}^{N}\sum_{r=k+1}^{M} p_{sr}(x_s-x_1)^{(t)}(y_r-y_1)^{(k)}\Delta_{(t,k)}f(x_1,y_1)$$

$$+\sum_{k=0}^{m-1}\sum_{t=n+1}^{N}\sum_{s=t}^{N}\sum_{r=k+1}^{M} p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_r-y_1)^{(k)}\times$$

$$\times\Delta_{(n,k)}f(x_{t-n},y_1)(x_t-x_{t-n})$$
$$+\sum_{k=m+1}^{M}\sum_{t=0}^{n-1}\sum_{s=t+1}^{N}\sum_{r=k}^{M}p_{sr}(x_s-x_1)^{(t)}(y_r-y_{k-m+1})^{(m-1)}\times$$
$$\times\Delta_{(t,m)}f(x_1,y_{k-m})(y_k-y_{k-m})$$
$$+\sum_{k=m+1}^{M}\sum_{t=n+1}^{N}(\sum_{s=t}^{N}\sum_{r=k}^{M}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_r-y_{k-m+1})^{(m-1)})\times$$
$$\times\Delta_{(n,m)}f(x_{t-n},y_{k-m})(x_t-x_{t-n})(y_k-y_{k-m}).$$

*Proof.* We have

$$\sum_{i=1}^{N}\sum_{j=1}^{M}p_{ij}f(x_i,y_j)=\sum_{i=1}^{N}\left(\sum_{j=1}^{M}q_jG_i(y_j)\right),$$

where $p_{ij}=q_j$ and $G_i: y\mapsto f(x_i,y)$. Using (2.19) on the inner sum we have

$$\sum_{i=1}^{N}\sum_{j=1}^{M}p_{ij}f(x_i,y_j)=\sum_{i=1}^{N}\sum_{k=0}^{m-1}\left(\sum_{j=k+1}^{M}q_j(y_j-y_1)^{(k)}\right)\Delta^kG_i(y_1)$$
$$+\sum_{i=1}^{N}\sum_{k=m+1}^{M}\left(\sum_{j=k}^{M}q_j(y_j-y_{k-m+1})^{(m-1)}\right)\Delta^mG_i(y_{k-m})(y_k-y_{k-m})$$
$$=\sum_{k=0}^{m-1}\left(\sum_{i=1}^{N}\left(\sum_{j=k+1}^{M}q_j(y_j-y_1)^{(k)}\right)\Delta^kG_i(y_1)\right)$$
$$+\sum_{k=m+1}^{M}\left(\sum_{i=1}^{N}\left(\sum_{j=k}^{M}q_j(y_j-y_{k-m+1})^{(m-1)}(y_k-y_{k-m})\right)\Delta^mG(y_{k-m})\right)$$
$$=\sum_{k=0}^{m-1}\left(\sum_{i=1}^{N}w_iF(x_i)\right)+\sum_{k=m+1}^{M}\left(\sum_{i=1}^{N}v_iH(x_i)\right)$$

where

$$w_i=\sum_{j=k+1}^{M}q_j(y_j-y_1)^{(k)}=\sum_{j=k+1}^{M}p_{ij}(y_j-y_1)^{(k)},\ F(x_i)=\Delta^kG_i(y_1),$$
$$v_i=\sum_{j=k}^{M}q_j(y_j-y_{k-m-1})^{(m-1)}(y_k-y_{k-m}),\ H(x_i)=\Delta^mG(y_{k-m}).$$

Applying again (2.19) on the inner sums we obtain

$$\sum_{i=1}^{N}\sum_{j=1}^{M}p_{ij}f(x_i,y_j)=$$

$$=\sum_{k=0}^{m-1}\sum_{r=0}^{n-1}\left(\sum_{i=r+1}^{N}w_i(x_i-x_1)^{(r)}\right)\Delta^rF(x_1)$$

$$+\sum_{k=0}^{m-1}\sum_{r=n+1}^{N}\left(\sum_{i=r}^{N} w_i(x_i-x_{r-n+1})^{(n-1)}\right)\Delta^n F(x_{r-n})(x_r-x_{r-n})$$
$$+\sum_{k=m+1}^{M}\sum_{t=0}^{n-1}\sum_{i=t+1}^{N} v_i(x_i-x_1)^{(t)}\Delta^t H(x_1)$$
$$+\sum_{k=m+1}^{M}\sum_{t=n+1}^{N}\sum_{i=t}^{N} v_i(x_i-x_{t-n+1})^{(n-1)}\Delta^n H(x_{t-n})(x_t-x_{t-n})$$
$$=\sum_{k=0}^{m-1}\sum_{r=0}^{n-1}\sum_{i=r+1}^{N}\left(\sum_{j=k+1}^{M} p_{ij}(y_j-y_1)^{(k)}\right)(x_i-x_1)^{(r)}\Delta_{(r,k)}f(x_1,y_1)$$
$$+\sum_{k=0}^{m-1}\sum_{r=n+1}^{N}\sum_{i=r}^{N}\left(\sum_{j=k+1}^{M} p_{ij}(y_j-y_1)^{(k)}\right)(x_i-x_{r-n+1})^{(n-1)}\times$$
$$\times\Delta_{(n,k)}f(x_{r-n},y_1)(x_r-x_{r-n})$$
$$+\sum_{k=m+1}^{M}\sum_{t=0}^{n-1}\sum_{i=t+1}^{N}\sum_{j=k}^{M} p_{ij}(y_j-y_{k-m-1})^{(m-1)}(y_k-y_{k-m})(x_i-x_1)^{(t)}\times$$
$$\times\Delta_{(t,m)}f(x_1,y_{k-m})$$
$$+\sum_{k=m+1}^{M}\sum_{t=n+1}^{N}\sum_{i=t}^{N}\sum_{j=k}^{M} p_{ij}(y_j-y_{k-m-1})^{(m-1)}(y_k-y_{k-m})\times$$
$$\times(x_i-x_{t-n+1})^{(n-1)}\Delta_{(n,m)}f(x_{t-n},y_{k-m})(x_t-x_{t-n}).$$

If we substitute in the first and in the second sums $r\to t$, and in all sums change $i\to s$, $j\to r$, we get the identity (3.1). □

If in Theorem 3.1 we simply put $f(x_i,y_j)=f(x_i)g(y_j)$, where $f$ and $g$ are real-valued functions on $I$ and $J$, respectively, then we get the following consequence of the previously mentioned theorem.

**Corollary 3.1** *Let $f: I\to\mathbb{R}$ and $g: J\to\mathbb{R}$ be two functions and let $p_{ij}\in\mathbb{R}$ for $i\in\{1,\dots,N\}$ and $j\in\{1,\dots,M\}$. Let $x_i$, $i\in\{1,\dots,N\}$ be mutually distinct points from $I$ and $y_j$, $j\in\{1,\dots,M\}$ be mutually distinct points from $J$. Then the following identity holds*

$$\sum_{i=1}^{N}\sum_{j=1}^{M} p_{ij}f(x_i)g(y_j)$$
$$=\sum_{k=0}^{m-1}\sum_{t=0}^{n-1}\sum_{s=t+1}^{N}\sum_{r=k+1}^{M} p_{sr}(x_s-x_1)^{(t)}\Delta^t f(x_1)(y_r-y_1)^{(k)}\Delta^k g(y_1)$$
$$+\sum_{k=0}^{m-1}\sum_{t=n+1}^{N}\sum_{s=t}^{N}\sum_{r=k+1}^{M} p_{sr}(x_s-x_{t-n+1})^{(n-1)}\Delta^n f(x_{t-n})(x_t-x_{t-n})\times$$
$$\times(y_r-y_1)^{(k)}\Delta^k g(y_1)+\sum_{k=m+1}^{M}\sum_{t=0}^{n-1}\sum_{s=t+1}^{N}\sum_{r=k}^{M} p_{sr}(x_s-x_1)^{(t)}\times$$

$$\times \Delta^t f(x_1)(y_r - y_{k-m+1})^{(m-1)} \Delta^m g(y_{k-m})(y_k - y_{k-m})$$
$$+ \sum_{k=m+1}^{M} \sum_{t=n+1}^{N} \sum_{s=t}^{N} \sum_{r=k}^{M} p_{sr}(x_s - x_{t-n+1})^{(n-1)} \Delta^n f(x_{t-n})(x_t - x_{t-n}) \times$$
$$\times (y_r - y_{k-m+1})^{(m-1)} \Delta^m g(y_{k-m})(y_k - y_{k-m}).$$

**Theorem 3.2** *Let $p_{ij}$, $(i = 1,\dots,N,\ j = 1,\dots,M)$ be real numbers, $E = \{x_1,x_2,\dots,x_N\} \subset \mathbb{R}$, $F = \{y_1,y_2,\dots,y_M\} \subset \mathbb{R}$, with $x_1 < x_2 < \dots < x_N$, $y_1 < y_2 < \dots < y_M$ and $n < N, m < M$.*
*Inequality*

$$\sum_{i=1}^{N} \sum_{j=1}^{M} p_{ij} f(x_i,y_j) \geq 0$$

*holds for every discrete convex function $f : E \times F \to \mathbb{R}$ of order $(n,m)$ if and only if*

$$\begin{aligned}
\sum_{s=t+1}^{N} \sum_{r=k+1}^{M} p_{sr}(x_s - x_1)^{(t)}(y_r - y_1)^{(k)} &= 0, && k = 0,\dots,m-1 \\ &&& t = 0,\dots,n-1 \\
\sum_{s=t}^{N} \sum_{r=k+1}^{M} p_{sr}(x_s - x_{t-n+1})^{(n-1)}(y_r - y_1)^{(k)} &= 0, && k = 0,\dots,m-1 \\ &&& t = n+1,\dots,N \\
\sum_{s=t+1}^{N} \sum_{r=k}^{M} p_{sr}(x_s - x_1)^{(t)}(y_r - y_{k-m+1})^{(m-1)} &= 0, && k = m+1,\dots,M \\ &&& t = 0,\dots,n-1 \\
\sum_{s=t}^{N} \sum_{r=k}^{M} p_{sr}(x_s - x_{t-n+1})^{(n-1)}(y_r - y_{k-m+1})^{(m-1)} &\geq 0, && k = m+1,\dots,M \\ &&& t = n+1,\dots,N.
\end{aligned}$$

**Remark 3.1** A version of Theorem 3.2 for $x_i \to i$, $y_j \to j$, $f(x_i,y_j) \to a_i b_j$ where $(a_i)$ is an $n$-convex sequence and $(b_j)$ is an $m$-convex sequence is given in [72].

A version of Theorem 3.2 for $x_i \to i$, $y_j \to j$, $f(x_i,y_j) = a_{ij}$ and $m = n = 1$ was considered in [67].

## 3.2 *P*-convex functions

A class of *P*-convex functions of order $k$ was introduced by J. Pečarić in [69]. Here we give some theorems from that paper.

**Definition 3.1** *Let $f$ be a real-valued function defined on $I \times J$, $I$ and $J$ are intervals. We say that $f$ is $P$-convex of order $k$ if*

$$[x_0,\dots,x_i;[y_0,\dots,y_{k-i};f]] \geq 0, \quad i = 0,1,\dots,k,$$

*is valid for all different choices $(x_j)_{j=0}^{i}$ from $I$ and different choices $(y_j)_{j=0}^{k-i}$ from $J$, i.e. $f$ is convex of order $(i,k-i)$ for all $i = 0,1,\dots,k$.*

If $I$ and $J$ are not intervals but discrete sets $I_N, J_N$, then we are talking about a discrete $P$-convex function. A $P$-convex function of order $k$ is not necessarily continuous. If the $k$th partial derivatives of a function $f$ exist, then $f$ is a $P$-convex function of order $k$ if and only if these partial derivatives are nonnegative.

If the $(k-1)$th partial derivatives of a function $f$ exist, then $f$ is a $P$-convex function of order $k$ iff these partial derivatives are nondecreasing in each argument.

If $f$ is a $P$-convex of order $k$, then the function $g$ defined by $g(t) = f(a_1t+b_1, a_2t+b_2)$, $a_1, a_2 > 0$, is $k$-convex.

A very interesting $P$-convex function is $f(x,y) = xy$, $x, y \in \mathbb{R}$.

**Theorem 3.3** *Let $p_i, x_i \in I_N, y_i \in J_N$, $i = 1, \ldots, n$, be real numbers such that $x_1 \leq \ldots \leq x_n$, $y_1 \leq \ldots y_n$. The inequality*

$$\sum_{i=1}^{n} p_i f(x_i, y_i) \geq 0 \tag{3.2}$$

*holds for every discrete $P$-convex function $f$ on $I_N \times J_N$ if and only if*

$$\sum_{i=1}^{n} p_i = 0 \tag{3.3}$$

$$\sum_{i=1}^{n} p_i x_i = 0, \quad \sum_{i=1}^{n} p_i y_i = 0 \tag{3.4}$$

$$\sum_{i=k}^{n} p_i (x_i - x_{k-1}) \geq 0, \quad \sum_{i=k}^{n} p_i (y_i - y_{k-1}) \geq 0, \quad k = 3, \ldots, n. \tag{3.5}$$

*Proof.* Using the Abel identity we obtain corresponding identity for $\sum_{i=1}^{n} p_i f(x_i, y_i)$ from which the statement of theorem follows.

$$\begin{aligned}
&\sum_{i=1}^{n} p_i f(x_i, y_i) = f(x_1, y_1)P_1 + \sum_{k=2}^{n} P_k \Big( f(x_k, y_k) - f(x_{k-1}, y_{k-1}\Big) = f(x_1, y_1)P_1 \\
&+\sum_{k=2}^{n} P_k \Big\{ (x_k - x_{k-1})[x_{k-1}, x_k; f(x, y_k)] + (y_k - y_{k-1})[y_{k-1}, y_k; f(x_{k-1}, y)] \Big\} \\
&= f(x_1, y_1)P_1 + [x_1, x_2; f(x, y_2)] \sum_{k=2}^{n} P_k (x_k - x_{k-1}) \\
&+ \sum_{k=3}^{n} \left( \sum_{i=k}^{n} P_i (x_i - x_{i-1}) \right) \Big\{ [x_{k-1}, x_k; f(x, y_k)] - [x_{k-2}, x_{k-1}; f(x, y_{k-1})] \Big\} \\
&+ [y_1, y_2; f(x_1, y)] \sum_{k=2}^{n} P_k (y_k, y_{k-1}) + \sum_{k=3}^{n} \left( \sum_{i=k}^{n} P_i (y_i - y_{i-1}) \right) \times \\
&\times \Big\{ ([y_{k-1}, y_k; f(x_{k-1}, y)] - [y_{k-2}, y_{k-1}; f(x_{k-2}, y)]) \Big\},
\end{aligned}$$

i.e.

$$\sum_{i=1}^{n} p_i f(x_i, y_i) = f(x_1, y_1)P_1 + [x_1, x_2; f(x, y_2)] \sum_{i=1}^{n} p_i (x_i - x_1)$$

$$
\begin{aligned}
&+[y_1,y_2;f(x_1,y)]\sum_{i=1}^{n}p_i(y_i-y_1)\\
&+\sum_{k=3}^{n}\left(\sum_{i=k}^{n}p_i(x_i-x_{k-1})\right)\Big\{(x_k-x_{k-2})[x_{k-2},x_{k-1},x_k;f(x,y_k)]\\
&+(y_k-y_{k-1})[x_{k-2},x_{k-1};[y_{k-1},y_k;f]]\Big\}\\
&+\sum_{k=3}^{n}\left(\sum_{i=k}^{n}p_i(y_i-y_{k-1})\right)\Big((y_k-y_{k-2})[y_{k-2},y_{k-1},y_k;f(x_{k-1},y)]\\
&+(x_{k-1}-x_{k-2})[y_{k-2},y_{k-1};[x_{k-1},x_{k-2};f]]\Big),
\end{aligned}
\tag{3.6}
$$

where $P_k=\sum_{i=k}^{n}p_i$. Sufficiency of conditions given in (3.3), (3.4) and (3.5) follows from the identity (3.6).

Since functions $f_1=1,\ f_2=-1,\ f_3(x,y)=x,\ f_4=-f_3,\ f_5(x,y)=y,\ f_6=-f_5$ are $P$-convex, conditions (3.3) and (3.4) are valid. The inequality (3.5) follows from (3.2) by setting for fixed $k$, $(k=3,\dots,n)$, $f(x,y)=(x-x_{k-1})_+$ or $f(x,y)=(y-y_{k-1})_+$. □

## 3.3 Discrete Results for Functions of $n$ Variables

For our main theorems of this section we define some notations to be used as follows.

Let for $r\in\{0,\dots,n\},\ j\in\{1,\dots,n\}$, ${}^nC_r(i_j,m_j)$ be the set of all $n$-tuples in which on the $k$th place we put $m_k$ or $i_k$ and $r$ places are filled with constants from the set $\{m_1,\dots,m_n\}$ while on the rest $n-r$ places we put variables from the set $\{i_1,\dots,i_n\}$. For example:

$$
\begin{aligned}
{}^nC_1(i_j,m_j) &= \{(m_1,i_2,\dots,i_n),(i_1,m_2,\dots,i_n),\dots,(i_1,i_2,\dots,i_{n-1},m_n)\},\\
{}^nC_2(i_j,m_j) &= \{(m_1,m_2,i_3,\dots,i_n),(m_1,i_2,m_3,i_4,\dots,i_n),\dots,\\
&\quad (m_1,i_2,\dots,i_{n-1},m_n),(i_1,m_2,m_3,i_4,\dots,i_n),\\
&\quad \dots,(i_1,m_2,i_3,\dots,i_{n-1},m_n),\dots,(i_1,i_2,\dots,i_{n-2},m_{n-1},m_n)\}.
\end{aligned}
$$

Note that the number of elements of the class ${}^nC_r(i_j,m_j)$ are equal to the binomial coefficient $\binom{n}{r}$. We introduce $\overline{\Delta}$ involving variables $i_1,\dots,i_n$ and constants $m_1,\dots,m_n$ as follows. For $(i_1,\dots,i_n)\in^n C_0(i_j,m_j)$, we define

$$
\begin{aligned}
\overline{\Delta}(i_1,\dots,i_n) &= \sum_{i_n=0}^{m_n-1}\cdots\sum_{i_1=0}^{m_1-1}\sum_{k_1=i_1+1}^{N_1}\cdots\sum_{k_n=i_n+1}^{N_n}p_{k_1\cdots k_n}\prod_{j=1}^{n}(x_{jk_j}-x_{j1})^{(i_j)}\times\\
&\quad\times\ \Delta_{(i_1,\dots,i_n)}f(x_{11},\dots,x_{n1}),
\end{aligned}
$$

For $(i_1,\dots,i_{t-1},m_t,i_{t+1},\dots,i_n)\in{}^nC_1(i_j,m_j)$, we have

$$
\overline{\Delta}(i_1,\dots,i_{t-1},m_t,i_{t+1},\dots,i_n)
$$

$$
\begin{aligned}
&= \sum_{i_n=0}^{m_n-1} \cdots \sum_{i_{t+1}=0}^{m_{t+1}-1} \sum_{i_t=m_t+1}^{N_t} \sum_{i_{t-1}=0}^{m_{t-1}-1} \cdots \sum_{i_1=0}^{m_1-1} \sum_{k_1=i_1+1}^{N_1} \cdots \sum_{k_{t-1}=i_{t-1}+1}^{N_{t-1}} \sum_{k_t=i_t}^{N_t} \\
&\sum_{k_{t+1}=i_{t+1}+1}^{N_{t+1}} \cdots \sum_{k_n=i_n+1}^{N_n} p_{k_1\cdots k_n} \Big( \prod_{j=1, j\neq t}^{n} (x_{jk_j} - x_{j1})^{(i_j)} \Big) \times \\
&(x_{tk_t} - x_{t(i_t-m_t+1)})^{(m_t-1)} (x_{ti_t} - x_{t(i_t-m_t)}) \times \\
&\times \Delta_{(i_1,\ldots,i_{t-1},m_t,i_{t+1},\ldots,i_n)} f(x_{11},\ldots,x_{(t-1)1},x_{t(i_t-m_t)},x_{(t+1)1},\ldots,x_{n1}).
\end{aligned}
$$

In general, for $(i_1,\ldots,i_{s-1},m_s,i_{s+1},\ldots,i_{t-1},m_t,i_{t+1},\ldots,i_n) \in {}^nC_r(i_j,m_j)$, we have

$$
\begin{aligned}
&\overline{\Delta}(i_1,\ldots,i_{s-1},m_s,i_{s+1},\ldots,i_{t-1},m_t,i_{t+1},\ldots,i_n) \\
&= \sum_{i_n=0}^{m_n-1} \cdots \sum_{i_{t+1}=0}^{m_{t+1}-1} \sum_{i_t=m_t+1}^{N_t} \sum_{i_{t-1}=0}^{m_{t-1}-1} \cdots \sum_{i_{s+1}=0}^{m_{s+1}-1} \sum_{i_t=m_s+1}^{N_s} \sum_{i_{s-1}=0}^{m_{s-1}-1} \cdots \sum_{i_1=0}^{m_1-1} \\
&\sum_{k_1=i_1+1}^{N_1} \cdots \sum_{k_{s-1}=i_{s-1}+1}^{N_{s-1}} \sum_{k_s=i_s}^{N_s} \sum_{k_{s+1}=i_{s+1}+1}^{N_{s+1}} \cdots \sum_{k_{t-1}=i_{t-1}+1}^{N_{t-1}} \sum_{k_t=i_t}^{N_t} \sum_{k_{t+1}=i_{t+1}+1}^{N_{t+1}} \cdots \\
&\sum_{k_n=i_n+1}^{N_n} p_{k_1\cdots k_n} \prod_{j=1, j\notin S_r}^{n} (x_{jk_j} - x_{j1})^{(i_j)} \prod_{j\in S_r} (x_{jk_j} - x_{j(i_j-m_j+1)})^{(m_j-1)} \times \\
&\times (x_{ji_j} - x_{j(i_j-m_j)}) \Delta_{(i_1,\ldots,i_{s-1},m_s,i_{s+1},\ldots,i_{t-1},m_t,i_{t+1},\ldots,i_n)} \\
&f(x_{11},\ldots,x_{(s-1)1},x_{s(i_s-m_s)},x_{(s+1)1},\ldots,x_{(t-1)1},x_{t(i_t-m_t)},x_{(t+1)1},\ldots,x_{n1})
\end{aligned}
$$

where $S_r$ is a set of all $r$ indices $s,\ldots,t$ of used constants $m_s,\ldots,m_t$.

Finally, for $(m_1,\ldots,m_n) \in {}^nC_n(i_j,m_j)$, we have

$$
\begin{aligned}
\overline{\Delta}(m_1,\ldots,m_n) = &\sum_{i_n=m_n+1}^{N_n} \cdots \sum_{i_1=m_1+1}^{N_1} \sum_{k_1=i_1}^{N_1} \cdots \sum_{k_n=i_n}^{N_n} \\
&p_{k_1\cdots k_n} \Delta_{(m_1,\ldots,m_n)} f(x_{1(k_1-m_1)},\ldots,x_{n(k_n-m_n)}) \times \\
&\times \prod_{j=1}^{n} \Big( (x_{jk_j} - x_{j(i_j-m_j+1)})^{(m_j-1)} (x_{ji_j} - x_{j(i_j-m_j)}) \Big)
\end{aligned}
$$

The following theorem gives an identity for sum $\sum \cdot \sum p_{k_1\cdots k_n} f(x_{1k_1},\ldots,x_{nk_n})$ involving $n$ variables.

**Theorem 3.4** *Let $f : I_1 \times \cdots \times I_n \to \mathbb{R}$ be a function. Let $p_{k_1\ldots k_n} \in \mathbb{R}$ and let $x_{jk_j} \in I_j$ be distinct real numbers for $k_j \in \{1,\ldots,N_j\}$, $j \in \{1,\ldots,n\}$, where $I_j = [a_j,b_j] \subset \mathbb{R}$. Then, we have*

$$
\sum_{k_1=1}^{N_1} \cdots \sum_{k_n=1}^{N_n} p_{k_1\cdots k_n} f(x_{1k_1},\ldots,x_{nk_n}) = \sum_{r=0}^{n} \sum_{(p_1,\ldots,p_n)\in {}^nC_r(i_j,m_j)} \overline{\Delta}(p_1,\ldots,p_n). \tag{3.7}
$$

*Proof.* We start with considering

$$
\sum_{k_1=1}^{N_1} \cdots \sum_{k_n=1}^{N_n} p_{k_1\cdots k_n} f(x_{1k_1},\ldots,x_{nk_n}) = \sum_{k_1=1}^{N_1} \cdots \sum_{k_{n-1}=1}^{N_{n-1}} \Big[ \sum_{k_n=1}^{N_n} Q_{k_n}^{(1,1)} F_{x_{nk_n}}^{(1,1)}(x_{nk_n}) \Big]
$$

where $Q_{k_n}^{(1,1)} = p_{k_1\cdots k_n}$ and $F_{x_{nk_n}}^{(1,1)}(x_{nk_n}) = f(x_{1k_1},\ldots,x_{nk_n})$ where $Q_{k_n}^{(1,1)}$ represents that this function only depends on $k_n$ and independent of other $n-1$ variables. Similarly, $F_{x_{nk_n}}^{(1,1)}$ represents that this is only function of variable $x_{nk_n}$ and independent of other $n-1$ variables. So using Theorem 2.2 we get,

$$
\begin{aligned}
&\sum_{k_1=1}^{N_1}\cdots\sum_{k_n=1}^{N_n} p_{k_1\cdots k_n} f(x_{1k_1},\ldots,x_{nk_n})\\
&=\sum_{k_1=1}^{N_1}\cdots\sum_{k_{n-1}=1}^{N_{n-1}}\Bigg[\sum_{i_n=0}^{m_n-1}\Bigg(\sum_{k_n=i_n+1}^{N_n} Q_{k_n}^{(1,1)}(x_{nk_n}-x_{n1})^{(i_n)}\Delta_{(i_n)}F_{x_{nk_n}}^{(1,1)}(x_{n1})\\
&+\sum_{i_n=m_n+1}^{N_n}\sum_{k_n=i_n}^{N_n} Q_{k_n}^{(1,1)}(x_{nk_n}-x_{n(i_n-m_n+1)})^{(m_n-1)}\Bigg)\times\\
&\times\Delta_{(m_n)}F_{x_{nk_n}}^{(1,1)}(x_{n(i_n-m_n)})(x_{ni_n}-x_{n(i_n-m_n)})\Bigg]\\
&=\sum_{k_1=1}^{N_1}\cdots\sum_{k_{n-2}=1}^{N_{n-2}}\sum_{i_n=0}^{m_n-1}\Bigg[\sum_{k_{n-1}=1}^{N_{n-1}}\Bigg(\sum_{k_n=i_n+1}^{N_n} p_{k_1\cdots k_n}(x_{nk_n}-x_{n1})^{(i_n)}\Bigg)\times\\
&\times\Delta_{(i_n)}f(x_{1k_1},\ldots,x_{(n-1)k_{n-1}},x_{n1})\Bigg]\\
&+\sum_{k_1=1}^{N_1}\cdots\sum_{k_{n-2}=1}^{N_{n-2}}\sum_{i_n=m_n+1}^{N_n}\Bigg[\sum_{k_{n-1}=1}^{N_{n-1}}\Bigg(\sum_{k_n=i_n}^{N_n} p_{k_1\cdots k_n}(x_{nk_n}-x_{n(i_n-m_n+1)})^{(m_n-1)}\times\\
&\times(x_{ni_n}-x_{n(i_n-m_n)})\Bigg)\Delta_{(m_n)}f(x_{1k_1},\ldots,x_{(n-1)k_{n-1}},x_{n(i_n-m_n)})\Bigg]\\
&=\sum_{k_1=1}^{N_1}\cdots\sum_{k_{n-2}=1}^{N_{n-2}}\sum_{i_n=0}^{m_n-1}\Bigg[\sum_{k_{n-1}=1}^{N_{n-1}} Q_{k_{n-1}}^{(2,1)}F_{x_{(n-1)k_{n-1}}}^{(2,1)}\Bigg]\\
&+\sum_{k_1=1}^{N_1}\cdots\sum_{k_{n-2}=1}^{N_{n-2}}\sum_{i_n=m_n+1}^{N_n}\Bigg[\sum_{k_{n-1}=1}^{N_{n-1}} Q_{k_{n-1}}^{(2,2)}F_{x_{n-1k_{n-1}}}^{(2,2)}\Bigg],
\end{aligned}
$$

where

$$
\begin{aligned}
Q_{k_{n-1}}^{(2,1)} &= \sum_{k_n=i_n+1}^{N_n} p_{k_1\cdots k_n}(x_{nk_n}-x_{n1})^{(i_n)},\\
Q_{k_{n-1}}^{(2,2)} &= \sum_{k_n=i_n}^{N_n} p_{k_1\cdots k_n}(x_{nk_n}-x_{n(i_n-m_n+1)})^{(m_n-1)}(x_{ni_n}-x_{n(i_n-m_n)}),\\
F_{x_{(n-1)k_{n-1}}}^{(2,1)}(x_{(n-1)k_{n-1}}) &= \Delta_{(i_n)}f(x_{1k_1},\ldots,x_{(n-1)k_{n-1}},x_{n1}),\\
F_{x_{(n-1)k_{n-1}}}^{(2,2)}(x_{(n-1)k_{n-1}}) &= \Delta_{(m_n)}f(x_{1k_1},\ldots,x_{(n-1)k_{n-1}},x_{n(i_n-m_n)}).
\end{aligned}
$$

Note that, this time we assume $Q_{k_{n-1}}^{(2,1)}$ to be only dependent on $k_{n-1}$, whereas $F_{x_{(n-1)k_{n-1}}}^{(2,1)}$ is considered to be a function of variable $x_{(n-1)k_{n-1}}$ as far as $Q_{k_{n-1}}^{(2,2)}$ is concerned, it only depends on $k_{n-1}$ and $F_{x_{(n-1)k_{n-1}}}^{(2,2)}$ is a function of one variable $x_{(n-1)k_{n-1}}$.

So, again applying Theorem 2.2, we have

$$\sum_{k_1=1}^{N_1}\cdots\sum_{k_n=1}^{N_n} p_{k_1\cdots k_n} f(x_{1k_1},\ldots,x_{nk_n})$$

$$=\sum_{k_1=1}^{N_1}\cdots\sum_{k_{n-2}=1}^{N_{n-2}}\sum_{i_n=0}^{m_n-1}\Bigg[\sum_{i_{n-1}=0}^{m_{n-1}-1}\sum_{k_{n-1}=i_{n-1}+1}^{N_{n-1}} Q_{k_{n-1}}^{(2,1)}(x_{(n-1)k_{n-1}}-x_{(n-1)1})^{(i_{n-1})}\times$$

$$\times\Delta_{(i_{n-1})}F^{(2,1)}_{x_{(n-1)k_{n-1}}}(x_{(n-1)1})$$

$$+\sum_{i_{n-1}=m_{n-1}+1}^{N_{n-1}}\sum_{k_{n-1}=i_{n-1}}^{N_{n-1}} Q_{k_{n-1}}^{(2,1)}(x_{(n-1)k_{n-1}}-x_{(n-1)(i_{n-1}-m_{n-1}+1)})^{(m_{n-1}-1)}\times$$

$$\times\Delta_{(i_{n-1})}F^{(2,1)}_{x_{(n-1)k_{n-1}}}(x_{(n-1)(i_{n-1}-m_{n-1})})(x_{(n-1)(i_{n-1})}-x_{(n-1)(i_{n-1}-m_{n-1})})\Bigg]$$

$$+\sum_{k_1=1}^{N_1}\cdots\sum_{k_{n-2}=1}^{N_{n-2}}\sum_{i_n=m_n+1}^{N_n}\Bigg[\sum_{i_{n-1}=0}^{m_{n-1}-1}\sum_{k_{n-1}=i_{n-1}+1}^{N_{n-1}}$$

$$Q_{k_{n-1}}^{(2,2)}(x_{(n-1)k_{n-1}}-x_{(n-1)1})^{(i_{n-1})}\Delta_{(i_{n-1})}F^{(2,2)}_{x_{(n-1)k_{n-1}}}(x_{(n-1)1})$$

$$+\sum_{i_{n-1}=m_{n-1}+1}^{N_{n-1}}\sum_{k_{n-1}=i_{n-1}}^{N_{n-1}} Q_{k_{n-1}}^{(2,2)}(x_{(n-1)k_{n-1}}-x_{(n-1)(i_{n-1}-m_{n-1}+1)})^{(m_{n-1}-1)}\times$$

$$\times\Delta_{(m_{n-1})}F^{(2,2)}_{x_{(n-1)k_{n-1}}}(x_{(n-1)(i_{n-1}-m_{n-1})})(x_{(n-1)(i_{n-1})}-x_{(n-1)(i_{n-1}-m_{n-1})})\Bigg]$$

$$=\sum_{k_1=1}^{N_1}\cdots\sum_{k_{n-3}=1}^{N_{n-3}}\sum_{i_n=0}^{m_n-1}\sum_{i_{n-1}=0}^{m_{n-1}-1}\Bigg[\sum_{k_{n-2}=1}^{N_{n-2}}\sum_{k_{n-1}=i_{n-1}+1}^{N_{n-1}} Q_{k_{n-1}}^{(2,1)}(x_{(n-1)k_{n-1}}-x_{(n-1)1})^{(i_{n-1})}$$

$$\times\Delta_{(i_{n-1})}F^{(2,1)}_{x_{(n-1)k_{n-1}}}(x_{(n-1)1})\Bigg]+\sum_{k_1=1}^{N_1}\cdots\sum_{k_{n-3}=1}^{N_{n-3}}\sum_{i_n=0}^{m_n-1}\sum_{i_{n-1}=m_{n-1}+1}^{N_{n-1}}$$

$$\Bigg[\sum_{k_{n-2}=1}^{N_{n-2}}\sum_{k_{n-1}=i_{n-1}}^{N_{n-1}} Q_{k_{n-1}}^{(2,1)}(x_{(n-1)k_{n-1}}-x_{(n-1)(i_{n-1}-m_{n-1}+1)})^{(m_{n-1}-1)}\times$$

$$\times\Delta_{(m_{n-1})}F^{(2,1)}_{x_{(n-1)k_{n-1}}}(x_{(n-1)(i_{n-1}-m_{n-1})})(x_{(n-1)(i_{n-1})}-x_{(n-1)(i_{n-1}-m_{n-1})})\Bigg]$$

$$+\sum_{k_1=1}^{N_1}\cdots\sum_{k_{n-3}=1}^{N_{n-3}}\sum_{i_n=m_n+1}^{N_n}\sum_{i_{n-1}=0}^{m_{n-1}-1}\Bigg[\sum_{k_{n-2}=1}^{N_{n-2}}\sum_{k_{n-1}=i_{n-1}+1}^{N_{n-1}}$$

$$Q_{k_{n-1}}^{(2,2)}(x_{(n-1)k_{n-1}}-x_{(n-1)1})^{(i_{n-1})}\Delta_{(i_{n-1})}F^{(2,2)}_{x_{(n-1)k_{n-1}}}(x_{(n-1)1})\Bigg]$$

$$+\sum_{k_1=1}^{N_1}\cdots\sum_{k_{n-3}=1}^{N_{n-3}}\sum_{i_n=m_n+1}^{N_n}\sum_{i_{n-1}=m_{n-1}+1}^{N_{n-1}}$$

$$\Bigg[\sum_{k_{n-2}=1}^{N_{n-2}}\sum_{k_{n-1}=i_{n-1}}^{N_{n-1}} Q_{k_{n-1}}^{(2,2)}(x_{(n-1)k_{n-1}}-x_{(n-1)(i_{n-1}-m_{n-1}+1)})^{(m_{n-1}-1)}\times$$

$$\times\Delta_{(m_{n-1})}F^{(2,2)}_{x_{(n-1)k_{n-1}}}(x_{(n-1)(i_{n-1}-m_{n-1})})(x_{(n-1)(i_{n-1})}-x_{(n-1)(i_{n-1}-m_{n-1})})\Big]$$

$$= \sum_{k_1=1}^{N_1}\cdots\sum_{k_{n-3}=1}^{N_{n-3}}\sum_{i_n=0}^{m_n-1}\sum_{i_{n-1}=0}^{m_{n-1}-1}\sum_{k_{n-2}=1}^{N_{n-2}}\sum_{k_{n-1}=i_{n-1}+1}^{N_{n-1}}\sum_{k_n=i_n+1}^{N_n} p_{k_1\cdots k_n}\times$$
$$\times(x_{nk_n}-x_{n1})^{(i_n)}(x_{(n-1)k_{n-1}}-x_{(n-1)1})^{(i_{n-1})}\times$$
$$\times\Delta_{(i_{n-1},i_n)}f(x_{1k_1},\dots,x_{(n-2)k_{n-2}},x_{(n-1)1},x_{n1})$$
$$+\sum_{k_1=1}^{N_1}\cdots\sum_{k_{n-3}=1}^{N_{n-3}}\sum_{i_n=0}^{m_n-1}\sum_{i_{n-1}=m_{n-1}+1}^{N_{n-1}}\sum_{k_{n-2}=1}^{N_{n-2}}\sum_{k_{n-1}=i_{n-1}}^{N_{n-1}}\sum_{k_n=i_n+1}^{N_n} p_{k_1\cdots k_n}\times$$
$$\times(x_{nk_n}-x_{n1})^{(i_n)}(x_{(n-1)k_{n-1}}-x_{(n-1)(i_{n-1}-m_{n-1}+1)})^{(m_{n-1}-1)}\times$$
$$\times(x_{(n-1)(i_{n-1})}-x_{(n-1)(i_{n-1}-m_{n-1})})\times$$
$$\times\Delta_{(m_{n-1},i_n)}f(x_{1k_1},\dots,x_{(n-2)k_{n-2}},x_{(n-1)(i_{n-1}-m_{n-1})},x_{n1})$$
$$+\sum_{k_1=1}^{N_1}\cdots\sum_{k_{n-3}=1}^{N_{n-3}}\sum_{i_n=m_n+1}^{N_n}\sum_{i_{n-1}=0}^{m_{n-1}-1}\sum_{k_{n-2}=1}^{N_{n-2}}\sum_{k_{n-1}=i_{n-1}+1}^{N_{n-1}}\sum_{k_n=i_n}^{N_n} p_{k_1\cdots k_n}\times$$
$$\times(x_{nk_n}-x_{n(i_n-m_n+1)})^{(m_n-1)}(x_{ni_n}-x_{n(i_n-m_n)})(x_{(n-1)k_{n-1}}-x_{(n-1)1})^{(i_{n-1})}\times$$
$$\times\Delta_{(i_{n-1},m_n)}f(x_{1k_1},\dots,x_{(n-2)k_{n-2}},x_{(n-1)1},x_{n(i_n-m_n)})$$
$$+\sum_{k_1=1}^{N_1}\cdots\sum_{k_{n-3}=1}^{N_{n-3}}\sum_{i_n=m_n+1}^{N_n}\sum_{i_{n-1}=m_{n-1}+1}^{N_{n-1}}\sum_{k_{n-2}=1}^{N_{n-2}}\sum_{k_{n-1}=i_{n-1}}^{N_{n-1}}\sum_{k_n=i_n}^{N_n} p_{k_1\cdots k_n}\times$$
$$\times(x_{nk_n}-x_{n(i_n-m_n+1)})^{(m_n-1)}(x_{(n-1)k_{n-1}}-x_{(n-1)(i_{n-1}-m_{n-1}+1)})^{(m_{n-1}-1)}\times$$
$$\times\Delta_{(m_{n-1},m_n)}f(x_{1k_1},\dots,x_{(n-2)k_{n-2}},x_{(n-1)(i_{n-1}-m_{n-1})},x_{n(i_n-m_n)})\times$$
$$\times(x_{ni_n}-x_{n(i_n-m_n)})(x_{(n-1)(i_{n-1})}-x_{(n-1)(i_{n-1}-m_{n-1})}).$$

Continuing in the similar fashion we finally get identity (3.7). □

**Theorem 3.5** *Let $E_j=\{x_{j1},x_{j2},\dots,x_{jN_j}\}\subset\mathbb{R}$, $x_{j1}<x_{j2}<\dots<x_{jN_j}$ for $j=1,\dots,n$. Let $p_{k_1\dots k_n}\in\mathbb{R}$ be real numbers for $k_j\in\{1,\dots,N_j\}$ and let $m_j<N_j$, $j\in\{1,\dots,n\}$. Then the inequality*

$$\sum_{k_1=1}^{N_1}\cdots\sum_{k_n=1}^{N_n} p_{k_1\cdots k_n}f(x_{1k_1},\dots,x_{nk_n})\ge 0 \tag{3.8}$$

*holds for every discrete convex function $f:E_1\times\cdots\times E_n\to\mathbb{R}$ of order $(m_1,\dots,m_n)$ if and only if*

$$\sum_{k_1=i_1+1}^{N_1}\cdots\sum_{k_n=i_n+1}^{N_n} p_{k_1\cdots k_n}\prod_{j=1}^{n}(x_{jk_j}-x_{j1})^{(i_j)}=0, \tag{3.9}$$
$$i_1\in\{0,\dots,m_1-1\},\ \dots,\ i_n\in\{0,\dots,m_n-1\},$$
$$\sum_{k_1=i_1+1}^{N_1}\cdots\sum_{k_n=i_n+1}^{N_n} p_{k_1\cdots k_n}(x_{1k_1}-x_{1(i_1-m_1+1)})^{(m_1-1)}\times$$

$$\times \prod_{j=2}^{n} (x_{jk_j} - x_{j1})^{(i_j)} = 0, \tag{3.10}$$

$$i_1 \in \{m_1+1,\ldots,N_1\},\ i_2 \in \{0,\ldots,m_2-1\},\ \ldots,\ i_n \in \{0,\ldots,m_n-1\},$$

$$\vdots$$

$$\sum_{i_1=m_1+1}^{N_1} \cdots \sum_{i_{n-1}=m_{n-1}+1}^{N_{n-1}} \sum_{k_n=i_n}^{N_n} p_{k_1\cdots k_n} \times$$
$$\times \prod_{j=1}^{n-1} (x_{jk_j} - x_{j1})^{(i_j)} (x_{nk_n} - x_{n(i_n-m_n+1)})^{(m_n-1)} = 0, \tag{3.11}$$

$$i_1 \in \{0,\ldots,m_1-1\},\ldots,i_{n-1} \in \{0,\ldots,m_{n-1}-1\}, i_n \in \{m_n+1,\ldots,N_n\},$$

$$\vdots$$

$$\sum_{k_1=i_1+1}^{N_1} \cdots \sum_{k_n=i_n+1}^{N_n} p_{k_1\cdots k_n} \prod_{j=1}^{n} (x_{jk_j} - x_{j(i_j-m_j+1)})^{(m_j-1)} \geq 0, \tag{3.12}$$

$$i_1 \in \{m_1+1,\ldots,N_1\},\ \ldots,\ i_n \in \{m_n+1,\ldots,N_n\}.$$

*Proof.* If (3.9), (3.10), $\ldots$, (3.11) hold then all these sums are zero in (3.7) and the required inequality (3.8) holds by using (3.12). Conversely, let (3.8) holds for every convex function $f$ of order $(m_1,\ldots,m_n)$. Let us consider the following functions

$$f^1(x_{1k_1},\ldots,x_{nk_n}) = \prod_{j=1}^{n} (x_{jk_j} - x_{j1})^{(i_j)} \quad \text{and} \quad f^2 = -f^1,$$

for $i_1 \in \{0,\ldots,m_1-1\}$, $\ldots$, $i_n \in \{0,\ldots,m_n-1\}$. Since these functions are convex of order $(m_1,\ldots,m_n)$, so by (3.8) the inequalities

$$\sum_{k_1=1}^{N_1} \cdots \sum_{k_n=1}^{N_n} p_{k_1\cdots k_n} f^k(x_{1k_1},\ldots,x_{nk_n}) \geq 0 \quad \text{for} \quad k \in \{1,2\}$$

hold and we get required equality (3.9). In the same way if we consider the following functions for $i_1 \in \{m_1+1,\ldots,N_1\}$, $i_2 \in \{0,\ldots,m_2-1\}$, $\ldots$, $i_n \in \{0,\ldots,m_n-1\}$

$$f^3(x_{1k_1},\ldots,x_{nk_n})$$
$$= \begin{cases} (x_{1k_1} - x_{1(i_1-m_1+1)})^{(m_1-1)} \prod_{j=2}^{n} (x_{jk_j} - x_{j1})^{(i_j)}, & x_{1(i_1-1)} < x_{1k_1} \\ 0, & x_{1(i_1-1)} \geq x_{1k_1}, \end{cases}$$
$$\text{and} \qquad f^4 = -f^3$$

such that $\Delta_{(m_1,\ldots,m_n)} f^k \geq 0$ for $k \in \{3,4\}$, then we get the required equality (3.10).

Similarly, if we consider in (3.8) the following functions for $i_1 \in \{0,\ldots,m_1-1\}$, $\ldots$, $i_{n-1} \in \{0,\ldots,m_{n-1}-1\}$, $i_n \in \{m_n+1,\ldots,N_n\}$

$$f^5(x_{1k_1},\ldots,x_{nk_n})$$

$$= \begin{cases} (x_{nk_n} - x_{n(i_n-m_n+1)})^{(m_n-1)} \prod_{j=1}^{n-1} (x_{jk_j} - x_{j1})^{(i_j)}, & x_{n(i_n-1)} < x_{nk_n} \\ 0, & x_{n(i_n-1)} \geq x_{nk_n} \end{cases}$$

and $$f^6 = -f^5$$

such that $\Delta_{(m_1,\ldots,m_n)} f_k \geq 0$ for $k \in \{5,6\}$, then we get the required equality (3.11) and so on.

The last inequality (3.12) is followed by considering the following function in (3.8) for $i_1 \in \{m_1+1,\ldots,N_1\}$, ..., $i_n \in \{m_n+1,\ldots,N_n\}$

$$f^7(x_{1k_1},\ldots,x_{nk_n}) = \begin{cases} \prod_{j=1}^{n} (x_{jk_j} - x_{j(i_j-m_j+1)})^{(m_j-1)}, & x_{1(i_1-1)} < x_{1k_1},\ldots,x_{n(i_n-1)} < x_{nk_n}, \\ 0, & \text{otherwise.} \end{cases}$$

□

## 3.4 Results for Integral of Function of Two Variables

In this section we pay attention to functions defined on a product of intervals. In [67] J. Pečarić gave the following identity for a function $f$ with continuous partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x \partial y}$.

**Theorem 3.6** *Let $P, f : [a,b] \times [a,b] \to R$ be integrable functions, if $f$ has the continuous partial derivatives $f_{(1,0)}$, $f_{(0,1)}$ and $f_{(1,1)}$ on $[a,b] \times [a,b]$ then*

$$\int_a^b \int_a^b P(x,y) f(x,y) dx dy = f(a,a) P_1(a,a) + \int_a^b P_1(x,a) f_{(1,0)}(x,a) dx + \int_a^b P_1(a,y) f_{(0,1)}(a,y) dy + \int_a^b \int_a^b P_1(x,y) f_{(1,1)}(x,y) dx dy$$

*where*

$$P_1(x,y) = \int_x^b \int_y^b P(s,t) dt ds,$$

$$f_{(1,0)} = \frac{\partial f}{\partial x}, \ f_{(0,1)} = \frac{\partial f}{\partial y} \ \text{and} \ f_{(1,1)} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

Generalization of the previous identity for functions with higher partial derivatives is given in [28]. Throughout this section a notation for a partial derivative $\frac{\partial^{n+m} f}{\partial x^n \partial y^m}$ is $f_{(n,m)}$.

**Theorem 3.7** *Let $P, f : I\times J \to R$ be integrable functions, $I=[a,b]$, $J=[c,d]$, and $f$ has the continuous partial derivatives $f_{(i,j)}$ on $I\times J$ for $i=0,1,\dots,n+1$ and $j=0,1,\dots,m+1$, then we have*

$$\begin{aligned}
&\int_a^b\int_c^d P(x,y)f(x,y)\,dy\,dx \qquad (3.13)\\
&=\sum_{i=0}^{n}\sum_{j=0}^{m}\int_a^b\int_c^d P(s,t)f_{(i,j)}(a,c)\frac{(s-a)^i}{i!}\frac{(t-c)^j}{j!}\,dt\,ds\\
&+\sum_{j=0}^{m}\int_a^b\int_x^b\int_c^d P(s,t)f_{(n+1,j)}(x,c)\frac{(s-x)^n}{n!}\frac{(t-c)^j}{j!}\,dt\,ds\,dx\\
&+\sum_{i=0}^{n}\int_c^d\int_a^b\int_y^d P(s,t)f_{(i,m+1)}(a,y)\frac{(s-a)^i}{i!}\frac{(t-y)^m}{m!}\,dt\,ds\,dy\\
&+\int_a^b\int_c^d\int_x^b\int_y^d P(s,t)f_{(n+1,m+1)}(x,y)\frac{(s-x)^n}{n!}\frac{(t-y)^m}{m!}\,dt\,ds\,dy\,dx.
\end{aligned}$$

*Proof.* Let $G(y)=f(x,y)$, i.e. we consider a function $f(x,y)$ as a function of variable $y$. Then a function $G$ can be represented as

$$\begin{aligned}
f(x,y) &= G(y)=\sum_{j=0}^{m}G^{(j)}(c)\frac{(y-c)^j}{j!}+\int_c^y G^{m+1}(t)\frac{(y-t)^m}{m!}\,dt\\
&=\sum_{j=0}^{m}f_{(0,j)}(x,c)\frac{(y-c)^j}{j!}+\int_c^y f_{(0,m+1)}(x,t)\frac{(y-t)^m}{m!}\,dt,
\end{aligned}$$

where we use the facts that $G^{(j)}(c)=f_{(0,j)}(x,c)$ and $G^{(m+1)}(t)=f_{(0,m+1)}(x,t)$.

Multiply the above formula with $P(x,y)$ and integrate it over $[c,d]$ by variable $y$. Then we have

$$\begin{aligned}
\int_c^d P(x,y)f(x,y)\,dy &= \sum_{j=0}^{m}f_{(0,j)}(x,c)\int_c^d P(x,y)\frac{(y-c)^j}{j!}\,dy \qquad (3.14)\\
&+\int_c^d\left(\int_c^y P(x,y)f_{(0,m+1)}(x,t)\frac{(y-t)^m}{m!}\,dt\right)dy.
\end{aligned}$$

Let us represent the functions $x\mapsto f_{(0,j)}(x,c)$ and $x\mapsto f_{(0,m+1)}(x,t)$ using Taylor expansions:

$$\begin{aligned}
f_{(0,j)}(x,c) &= \sum_{i=0}^{n}f_{(i,j)}(a,c)\frac{(x-a)^i}{i!}+\int_a^x f_{(n+1,j)}(s,c)\frac{(x-s)^n}{n!}\,ds,\\
f_{(0,m+1)}(x,t) &= \sum_{i=0}^{n}f_{(i,m+1)}(a,t)\frac{(x-a)^i}{i!}+\int_a^x f_{(n+1,m+1)}(s,t)\frac{(x-s)^n}{n!}\,ds.
\end{aligned}$$

Putting these two formulae in (3.14) we get

$$\int_c^d P(x,y)f(x,y)\,dy$$

$$
\begin{aligned}
&= \sum_{j=0}^{m}\left(\sum_{i=0}^{n} f_{(i,j)}(a,c)\frac{(x-a)^i}{i!}+\int_a^x f_{(n+1,j)}(s,c)\frac{(x-s)^n}{n!}ds\right)\times \\
&\quad \times \int_c^d P(x,y)\frac{(y-c)^j}{j!}dy+\int_c^d\left(\int_c^y P(x,y)\left(\sum_{i=0}^{n} f_{(i,m+1)}(a,t)\frac{(x-a)^i}{i!}\right.\right. \\
&+ \left.\left.\int_a^x f_{(n+1,m+1)}(s,t)\frac{(x-s)^n}{n!}ds\right)\frac{(y-t)^m}{m!}dt\right)dy \\
&= \sum_{j=0}^{m}\left(\sum_{i=0}^{n} f_{(i,j)}(a,c)\frac{(x-a)^i}{i!}\right)\int_c^d P(x,y)\frac{(y-c)^j}{j!}dy \\
&+ \sum_{j=0}^{m}\left(\int_a^x f_{(n+1,j)}(s,c)\frac{(x-s)^n}{n!}ds\right)\int_c^d P(x,y)\frac{(y-c)^j}{j!}dy \\
&+ \int_c^d\int_c^y P(x,y)\left(\sum_{i=0}^{n} f_{(i,m+1)}(a,t)\frac{(x-a)^i}{i!}\right)\frac{(y-t)^m}{m!}dt\,dy \\
&+ \int_c^d\int_c^y\left(\int_a^x P(x,y)f_{(n+1,m+1)}(s,t)\frac{(x-s)^n}{n!}ds\right)\frac{(y-t)^m}{m!}dt\,dy.
\end{aligned}
$$

Now, we integrate over $[a,b]$ by variable $x$ and get:

$$
\begin{aligned}
&\int_a^b\int_c^d P(x,y)f(x,y)dy\,dx \\
&= \int_a^b\left[\sum_{j=0}^{m}\left(\sum_{i=0}^{n} f_{(i,j)}(a,c)\frac{(x-a)^i}{i!}\right)\int_c^d P(x,y)\frac{(y-c)^j}{j!}dy\right]dx \\
&+ \int_a^b\left[\sum_{j=0}^{m}\left(\int_a^x f_{(n+1,j)}(s,c)\frac{(x-s)^n}{n!}ds\right)\int_c^d P(x,y)\frac{(y-c)^j}{j!}dy\right]dx \\
&+ \int_a^b\left[\int_c^d\int_c^y P(x,y)\left(\sum_{i=0}^{n} f_{(i,m+1)}(a,t)\frac{(x-a)^i}{i!}\right)\frac{(y-t)^m}{m!}dt\,dy\right]dx \\
&+ \int_a^b\left[\int_c^d\int_c^y\left(\int_a^x P(x,y)f_{(n+1,m+1)}(s,t)\frac{(x-s)^n}{n!}ds\right)\frac{(y-t)^m}{m!}dt\,dy\right]dx.
\end{aligned}
$$

In the first summand we change the order of summation, use the linearity of the integral and get

$$
\sum_{i=0}^{n}\sum_{j=0}^{m}\int_a^b\int_c^d P(x,y)f_{(i,j)}(a,c)\frac{(x-a)^i}{i!}\frac{(y-c)^j}{j!}dy\,dx.
$$

The second summand is rewriten as

$$
\int_a^b\left[\sum_{j=0}^{m}\left(\int_a^x f_{(n+1,j)}(s,c)\frac{(x-s)^n}{n!}ds\right)\int_c^d P(x,y)\frac{(y-c)^j}{j!}dy\right]dx
$$

$$
\begin{aligned}
&= \int_a^b \left[ \sum_{j=0}^{m} \left( \int_a^x \int_c^d P(x,y) \frac{(y-c)^j}{j!} f_{(n+1,j)}(s,c) \frac{(x-s)^n}{n!} dy\, ds \right) \right] dx \\
&= \sum_{j=0}^{m} \int_a^b \int_a^x \int_c^d P(x,y) f_{(n+1,j)}(s,c) \frac{(x-s)^n}{n!} \frac{(y-c)^j}{j!} dy\, ds\, dx \\
&= \sum_{j=0}^{m} \int_a^b \int_s^b \int_c^d P(x,y) f_{(n+1,j)}(s,c) \frac{(x-s)^n}{n!} \frac{(y-c)^j}{j!} dy\, dx\, ds,
\end{aligned}
$$

where in the last equation we use the Fubini theorem for the variables $s$ and $x$. Let us point out, that firstly, the variable $x$ is changed from $a$ to $b$ while the variable $s$ is changed from $a$ to $x$. After changing the order of integration we have that variable $s$ is changed from $a$ to $b$ while the variable $x$ is changed from $s$ to $b$.

Similarly, the third summand is rewriten as:

$$
\begin{aligned}
&\int_a^b \left[ \int_c^d \int_c^y P(x,y) \left( \sum_{i=0}^{n} f_{(i,m+1)}(a,t) \frac{(x-a)^i}{i!} \right) \frac{(y-t)^m}{m!} dt\, dy \right] dx \\
&= \sum_{i=0}^{n} \int_a^b \int_c^d \int_c^y P(x,y) f_{(i,m+1)}(a,t) \frac{(x-a)^i}{i!} \frac{(y-t)^m}{m!} dt\, dy\, dx \\
&= \sum_{i=0}^{n} \int_a^b \int_c^d \int_t^d P(x,y) f_{(i,m+1)}(a,t) \frac{(x-a)^i}{i!} \frac{(y-t)^m}{m!} dy\, dt\, dx \\
&= \sum_{i=0}^{n} \int_c^d \int_a^b \int_t^d P(x,y) f_{(i,m+1)}(a,t) \frac{(x-a)^i}{i!} \frac{(y-t)^m}{m!} dy\, dx\, dt,
\end{aligned}
$$

where we use the Fubini theorem twice, firstly for changing $t$ and $y$, and then for $t$ and $x$.

The fourth summand is rewriten as:

$$
\begin{aligned}
&\int_a^b \left[ \int_c^d \int_c^y \left( \int_a^x P(x,y) f_{(n+1,m+1)}(s,t) \frac{(x-s)^n}{n!} ds \right) \frac{(y-t)^m}{m!} dt\, dy \right] dx \\
&= \int_a^b \int_c^d \int_c^y \int_a^x P(x,y) f_{(n+1,m+1)}(s,t) \frac{(x-s)^n}{n!} \frac{(y-t)^m}{m!} ds\, dt\, dy\, dx \\
&= \int_a^b \int_c^d \int_s^b \int_t^d P(x,y) f_{(n+1,m+1)}(s,t) \frac{(x-s)^n}{n!} \frac{(y-t)^m}{m!} dy\, dx\, dt\, ds,
\end{aligned}
$$

where we use the Fubini theorem several times. Firstly, we change $t$ and $y$, then $y$ and $s$, then $s$ and $t$, then $s$ and $x$, then $t$ and $x$.

Using all these results we get

$$
\begin{aligned}
&\int_a^b \int_c^d P(x,y) f(x,y) dy\, dx \\
&= \sum_{i=0}^{n} \sum_{j=0}^{m} \int_a^b \int_c^d P(x,y) f_{(i,j)}(a,c) \frac{(x-a)^i}{i!} \frac{(y-c)^j}{j!} dy\, dx
\end{aligned}
$$

$$+\sum_{j=0}^{m}\int_a^b\int_s^b\int_c^d P(x,y)f_{(n+1,j)}(s,c)\frac{(x-s)^n}{n!}\frac{(y-c)^j}{j!}dy\,dx\,ds$$
$$+\sum_{i=0}^{n}\int_c^d\int_a^b\int_t^d P(x,y)f_{(i,m+1)}(a,t)\frac{(x-a)^i}{i!}\frac{(y-t)^m}{m!}dy\,dx\,dt$$
$$+\int_a^b\int_c^d\int_s^b\int_t^d P(x,y)f_{(n+1,m+1)}(s,t)\frac{(x-s)^n}{n!}\frac{(y-t)^m}{m!}dy\,dx\,dt\,ds.$$

It is, in fact, the statement of Theorem 3.7 when we change the names of variables on the right side: $x\leftrightarrow s$, $y\leftrightarrow t$. □

Using result of the previous theorem we obtain necessary and sufficient conditions that inequality $\int_a^b\int_c^d P(x,y)f(x,y)dy\,dx\geq 0$ holds for every $(n+1,m+1)$-convex two-variables function.

**Theorem 3.8** *The inequality*

$$\int_a^b\int_c^d P(x,y)f(x,y)dy\,dx\geq 0 \tag{3.15}$$

*holds for every function whose continuous partial derivative $f_{(n+1,m+1)}\geq 0$ on $[a,b]\times[c,d]$ if and only if*

$$\int_a^b\int_c^d P(s,t)\frac{(s-a)^i}{i!}\frac{(t-c)^j}{j!}dt\,ds=0,\quad i=0,\dots,n;\, j=0,\dots,m \tag{3.16}$$
$$\int_x^b\int_c^d P(s,t)\frac{(s-x)^n}{n!}\frac{(t-c)^j}{j!}dt\,ds=0,\quad j=0,\dots,m;\, x\in[a,b] \tag{3.17}$$
$$\int_a^b\int_y^d P(s,t)\frac{(s-a)^i}{i!}\frac{(t-y)^m}{m!}dt\,ds=0,\quad i=0,\dots,n;\, y\in[c,d] \tag{3.18}$$
$$\int_x^b\int_y^d P(s,t)\frac{(s-x)^n}{n!}\frac{(t-y)^m}{m!}dt\,ds\geq 0,\quad x\in[a,b];\, y\in[c,d]. \tag{3.19}$$

*Proof.* If (3.16), (3.17) and (3.18) hold then the first three sums are zero in (3.13) and the required inequality (3.15) holds by using (3.19).

Conversely, if we consider in (3.15) the following functions

$$f^{(1)}(s,t)=\frac{(s-a)^i}{i!}\frac{(t-c)^j}{j!},\qquad f^{(2)}=-f^{(1)}$$

for $0\leq i\leq n$ and $0\leq j\leq m$ such that $f^{(k)}_{(n+1,m+1)}(s,t)\geq 0$, $k=1,2$; then we get the required equality (3.16). In the same way if we consider in (3.15) the following functions for $0\leq j\leq m$, $x\in[a,b]$ and $t\in[c,d]$

$$f^{(3)}(s,t)=\begin{cases}\dfrac{(s-x)^n}{n!}\dfrac{(t-c)^j}{j!}, & x<s\\ 0, & x\geq s\end{cases},\qquad f^{(4)}=-f^{(3)}$$

such that $f^{(k)}_{(n+1,m+1)}(s,t) \geq 0$, $k = 3,4$, then we get the required equality (3.17). Similarly, if we consider in (3.15) the following functions for $0 \leq i \leq n$, $y \in [c,d]$ and $s \in [a,b]$

$$f^{(5)}(s,t) = \begin{cases} \dfrac{(s-a)^i}{i!}\dfrac{(t-y)^m}{m!}, & y < t \\ 0, & y \geq t \end{cases}, \qquad f^{(6)} = -f^{(5)}$$

such that $f^{(k)}_{(n+1,m+1)}(s,t) \geq 0$, $k = 5,6$, then we get the required equality (3.18). The last inequality (3.19) is followed by considering the following function in (3.15) for $x \in [a,b]$, $y \in [c,d]$

$$f(s,t) = \begin{cases} \dfrac{(s-x)^n}{n!}\dfrac{(t-y)^m}{m!}, & x < s \text{ and } y < t \\ 0, & x \geq s \text{ or } y \geq t. \end{cases}$$

□

## 3.5 Results for Integral of Function of $n$ Variables

As we done in previous sections, for the present section we also introduce some notations to simplify the statement of our main theorems as follows. Results of this section are based on paper [30].

For variables $i_1,\ldots,i_n$ and constants $m_1+1,\ldots,m_n+1$ we define $\tilde{\Delta}$ in the following way:

$$\tilde{\Delta}(i_1,\ldots,i_n) = \sum_{i_1=0}^{m_1}\cdots\sum_{i_n=0}^{m_n}\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} p(\mathbf{x}) f_{(i_1,\ldots,i_n)}(a_1,\ldots,a_n) \times$$
$$\times \prod_{j=1}^{n}\frac{(y_j-a_j)^{i_j}}{i_j!}dy_n\cdots dy_1,$$

where $p(x_1,\ldots,x_n) = p(\mathbf{x})$ and

$$\tilde{\Delta}(i_1,\ldots,i_{k-1},m_k,i_{k+1},\ldots,i_n) =$$
$$\sum_{i_1=0}^{m_1}\cdots\sum_{i_{k-1}=0}^{m_{k-1}}\sum_{i_{k+1}=0}^{m_{k+1}}\cdots\sum_{i_n=0}^{m_n}\int_{a_k}^{b_k}\int_{a_1}^{b_1}\cdots\int_{a_{k-1}}^{b_{k-1}}\int_{x_k}^{b_k}\int_{a_{k+1}}^{b_{k+1}}\cdots\int_{a_n}^{b_n} p(\mathbf{x}) \times$$
$$\times f_{(i_1,\ldots,i_{k-1},m_k+1,i_{k+1},\ldots,i_n)}\frac{(y_k-x_k)^{m_k}}{m_k!}\prod_{j=1,\, j\neq k}^{n}\frac{(y_j-a_j)^{i_j}}{i_j!}dy_n\cdots dy_1 dx_k.$$

Similarly, we can define $\tilde{\Delta}$ for any $n$-tuple from ${}^nC_r(i_j,m_j)$ (where ${}^nC_r(i_j,m_j)$ was introduced in the start of previous section) for some $j \in \{1,\ldots,n\}$ and finally we define

$$\tilde{\Delta}(m_1,\ldots,m_n) = \int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\int_{x_1}^{b_1}\cdots\int_{x_n}^{b_n} p(\mathbf{x}) \times$$

$$\times\ f_{(m_1+1,\ldots,m_n+1)}(x_1,\ldots,x_n)\prod_{j=1}^{n}\frac{(y_j-x_j)^{m_j}}{m_j!}dy_n\cdots dy_1 dx_n\cdots dx_1.$$

Now we are ready to state main theorems of this section.

**Theorem 3.9** *Let $p, f : I_1\times\cdots\times I_n\to\mathbb{R}$ be integrable functions, $I_i=[a_i,b_i], i=1,\ldots n$, and let $f\in C^{(m_1+1,\ldots,m_n+1)}(I_1\times\cdots\times I_n)$. Then the identity*

$$\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} p(\mathbf{x})f(x_1,\ldots,x_n)dx_n\cdots dx_1 = \sum_{r=0}^{n}\sum_{(p_1,\ldots,p_n)\in{}^nC_r(i_j,m_j+1)}\tilde{\Delta}(p_1,\ldots,p_n) \tag{3.20}$$

*holds.*

*Proof.* We consider the Taylor expansion:

$$\begin{aligned} f(x_1,\ldots,x_n) &= \sum_{i_n=0}^{m_n} f_{(0,\ldots,0,i_n)}(x_1,\ldots,x_{n-1},a_n)\frac{(x_n-a_n)^{i_n}}{i_n!} \\ &\quad + \int_{a_n}^{x_n} f_{(0,\ldots,0,m_n+1)}(x_1,\ldots,x_{n-1},y_n)\frac{(x_n-y_n)^{m_n}}{m_n!}dy_n. \end{aligned}$$

Multiply the above formula with $p(\mathbf{x})$ and integrate it over $[a_n,b_n]$ by variable $x_n$. Then we have

$$\begin{aligned} &\int_{a_n}^{b_n} p(\mathbf{x})f(x_1,\ldots,x_n)dx_n \\ &= \sum_{i_n=0}^{m_n} f_{(0,\ldots,0,i_n)}(x_1,\ldots,x_{n-1},a_n)\int_{a_n}^{b_n} p(\mathbf{x})\frac{(x_n-a_n)^{i_n}}{i_n!}dx_n \\ &+ \int_{a_n}^{b_n}\Big(\int_{a_n}^{x_n} p(\mathbf{x})f_{(0,\ldots,0,m_n+1)}(x_1,\ldots,x_{n-1},y_n)\frac{(x_n-y_n)^{m_n}}{m_n!}dy_n\Big)dx_n. \end{aligned} \tag{3.21}$$

Let us use the following Taylor expansions:

$$\begin{aligned} &f_{(0,\ldots,0,i_n)}(x_1,\ldots,x_{n-1},a_n) \\ &= \sum_{i_{n-1}=0}^{m_{n-1}} f_{(0,\ldots,0,i_{n-1},i_n)}(x_1,\ldots,x_{n-2},a_{n-1},a_n)\frac{(x_{n-1}-a_{n-1})^{i_{n-1}}}{i_{n-1}!} \\ &+ \int_{a_{n-1}}^{x_{n-1}} f_{(0,\ldots,0,m_{n-1}+1,i_n)}(x_1,\ldots,x_{n-2},y_{n-1},a_n)\frac{(x_{n-1}-y_{n-1})^{m_{n-1}}}{m_{n-1}!}dy_{n-1}, \end{aligned}$$

$$\begin{aligned} &f_{(0,\ldots,0,m_n+1)}(x_1,\ldots,x_{n-1},y_n) \\ &= \sum_{i_{n-1}=0}^{m_{n-1}} f_{(0,\ldots,0,i_{n-1},m_n+1)}(x_1,\ldots,x_{n-2},a_{n-1},y_n)\frac{(x_{n-1}-a_{n-1})^{i_{n-1}}}{i_{n-1}!} \end{aligned}$$

$$+\int_{a_{n-1}}^{x_{n-1}} f_{(0,\ldots,0,m_{n-1}+1,m_n+1)}(x_1,\ldots,x_{n-2},y_{n-1},y_n)\times$$
$$\times\frac{(x_{n-1}-y_{n-1})^{m_{n-1}}}{m_{n-1}!}dy_{n-1}.$$

Put these two formulae in (3.21) and integrate over $[a_{n-1},b_{n-1}]$ by variable $x_{n-1}$. Then, we have

$$\int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n} p(\mathbf{x})f(x_1,\ldots,x_n)dx_n dx_{n-1}$$
$$=\int_{a_{n-1}}^{b_{n-1}}\Big[\sum_{i_n=0}^{m_n}\sum_{i_{n-1}=0}^{m_{n-1}} f_{(0,\ldots,0,i_{n-1},i_n)}(x_1,\ldots,x_{n-2},a_{n-1},a_n)\frac{(x_{n-1}-a_{n-1})^{i_{n-1}}}{i_{n-1}!}$$
$$\times\int_{a_n}^{b_n} p(\mathbf{x})\frac{(x_n-a_n)^{i_n}}{i_n!}dx_n\Big]dx_{n-1}$$
$$+\int_{a_{n-1}}^{b_{n-1}}\Big[\sum_{i_n=0}^{m_n}\int_{a_{n-1}}^{x_{n-1}} f_{(0,\ldots,0,m_{n-1}+1,i_n)}(x_1,\ldots,x_{n-2},y_{n-1},a_n)\times$$
$$\times\frac{(x_{n-1}-y_{n-1})^{m_{n-1}}}{m_{n-1}!}dy_{n-1}\int_{a_n}^{b_n} p(\mathbf{x})\frac{(x_n-a_n)^{i_n}}{i_n!}dx_n\Big]dx_{n-1}$$

$$+\int_{a_{n-1}}^{b_{n-1}}\Big[\int_{a_n}^{b_n}\int_{a_n}^{x_n} p(\mathbf{x})\sum_{i_{n-1}=0}^{m_{n-1}} f_{(0,\ldots,0,i_{n-1},m_n+1)}(x_1,\ldots,x_{n-2},a_{n-1},y_n)\times$$
$$\times\frac{(x_{n-1}-a_{n-1})^{i_{n-1}}}{i_{n-1}!}\frac{(x_n-y_n)^{m_n}}{m_n!}dy_n dx_n\Big]dx_{n-1}$$
$$+\int_{a_{n-1}}^{b_{n-1}}\Big[\int_{a_n}^{b_n}\int_{a_n}^{x_n} p(\mathbf{x})\times$$
$$\times\int_{a_{n-1}}^{x_{n-1}} f_{(0,\ldots,0,m_{n-1}+1,m_n+1)}(x_1,\ldots,x_{n-2},y_{n-1},y_n)\times$$
$$\times\frac{(x_{n-1}-y_{n-1})^{m_{n-1}}}{m_{n-1}!}\frac{(x_n-y_n)^{m_n}}{m_n!}dy_{n-1}dy_n dx_n\Big]dx_{n-1}.$$

In the first summand we change the order of summation, use linearity of integral and get

$$\sum_{i_n=0}^{m_n}\sum_{i_{n-1}=0}^{m_{n-1}}\int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n} p(\mathbf{x})f_{(0,\ldots,0,i_{n-1},i_n)}(x_1,\ldots,x_{n-2},a_{n-1},a_n)\times$$
$$\times\frac{(x_{n-1}-a_{n-1})^{i_{n-1}}}{i_{n-1}!}\frac{(x_n-a_n)^{i_n}}{i_n!}dx_n dx_{n-1}.$$

The second summand is rewritten as

$$\int_{a_{n-1}}^{b_{n-1}}\Big[\sum_{i_n=0}^{m_n}\int_{a_{n-1}}^{x_{n-1}} f_{(0,\ldots,0,m_{n-1}+1,i_n)}(x_1,\ldots,x_{n-2},y_{n-1},a_n)\times$$

$$\times \frac{(x_{n-1}-y_{n-1})^{m_{n-1}}}{m_{n-1}!}dy_{n-1}\int_{a_n}^{b_n} p(\mathbf{x})\frac{(x_n-a_n)^{i_n}}{i_n!}dx_n\Big]dx_{n-1}$$

$$=\int_{a_{n-1}}^{b_{n-1}}\Big[\sum_{i_n=0}^{m_n}\int_{a_{n-1}}^{x_{n-1}}\int_{a_n}^{b_n} p(\mathbf{x})\frac{(x_n-a_n)^{i_n}}{i_n!}\frac{(x_{n-1}-y_{n-1})^{m_{n-1}}}{m_{n-1}!}\times$$

$$\times f_{(0,\ldots,0,m_{n-1}+1,i_n)}(x_1,\ldots,x_{n-2},y_{n-1},a_n)dx_ndy_{n-1}\Big]dx_{n-1}$$

$$=\sum_{i_n=0}^{m_n}\int_{a_{n-1}}^{b_{n-1}}\int_{a_{n-1}}^{x_{n-1}}\int_{a_n}^{b_n} p(\mathbf{x})f_{(0,\ldots,0,m_{n-1}+1,i_n)}(x_1,\ldots,x_{n-2},y_{n-1},a_n)\times$$

$$\times\frac{(x_n-a_n)^{i_n}}{i_n!}\frac{(x_{n-1}-y_{n-1})^{m_{n-1}}}{m_{n-1}!}dx_ndy_{n-1}dx_{n-1}$$

$$=\sum_{i_n=0}^{m_n}\int_{a_{n-1}}^{b_{n-1}}\int_{y_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n} p(\mathbf{x})f_{(0,\ldots,0,m_{n-1}+1,i_n)}(x_1,\ldots,x_{n-2},y_{n-1},a_n)\times$$

$$\times\frac{(x_n-a_n)^{i_n}}{i_n!}\frac{(x_{n-1}-y_{n-1})^{m_{n-1}}}{m_{n-1}!}dx_ndx_{n-1}dy_{n-1}$$

where in the last equation we used the Fubini theorem for variables $y_{n-1}$ and $x_{n-1}$. Let us point out that firstly, the variable $x_{n-1}$ is changed from $a_{n-1}$ to $b_{n-1}$ while the variable $y_{n-1}$ is changed from $a_{n-1}$ to $x_{n-1}$. After changing the order of integration we have that variable $y_{n-1}$ is changed from $a_{n-1}$ to $b_{n-1}$ while the variable $x_{n-1}$ is changed from $y_{n-1}$ to $b_{n-1}$.

Similarly, the third summand is rewritten as:

$$\int_{a_{n-1}}^{b_{n-1}}\Big[\int_{a_n}^{b_n}\int_{a_n}^{x_n} p(\mathbf{x})\sum_{i_{n-1}=0}^{m_{n-1}} f_{(0,\ldots,0,i_{n-1},m_n+1)}(x_1,\ldots,x_{n-2},a_{n-1},y_n)\times$$

$$\times\frac{(x_{n-1}-a_{n-1})^{i_{n-1}}}{i_{n-1}!}\frac{(x_n-y_n)^{m_n}}{m_n!}dy_ndx_n\Big]dx_{n-1}$$

$$=\sum_{i_{n-1}=0}^{m_{n-1}}\int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n}\int_{a_n}^{x_n} p(\mathbf{x})f_{(0,\ldots,0,i_{n-1},m_n+1)}(x_1,\ldots,x_{n-2},a_{n-1},y_n)\times$$

$$\times\frac{(x_{n-1}-a_{n-1})^{i_{n-1}}}{i_{n-1}!}\frac{(x_n-y_n)^{m_n}}{m_n!}dy_ndx_ndx_{n-1}$$

$$=\sum_{i_{n-1}=0}^{m_{n-1}}\int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n}\int_{y_n}^{b_n} p(\mathbf{x})f_{(0,\ldots,0,i_{n-1},m_n+1)}(x_1,\ldots,x_{n-2},a_{n-1},y_n)\times$$

$$\times\frac{(x_{n-1}-a_{n-1})^{i_{n-1}}}{i_{n-1}!}\frac{(x_n-y_n)^{m_n}}{m_n!}dx_ndy_ndx_{n-1}$$

$$=\sum_{i_{n-1}=0}^{m_{n-1}}\int_{a_n}^{b_n}\int_{a_{n-1}}^{b_{n-1}}\int_{y_n}^{b_n} p(\mathbf{x})f_{(0,\ldots,0,i_{n-1},m_n+1)}(x_1,\ldots,x_{n-2},a_{n-1},y_n)\times$$

$$\times\frac{(x_{n-1}-a_{n-1})^{i_{n-1}}}{i_{n-1}!}\frac{(x_n-y_n)^{m_n}}{m_n!}dx_ndx_{n-1}dy_n$$

where we use the Fubini theorem twice, firstly for changing $y_n$ and $x_n$ and then for $y_n$ and $x_{n-1}$.

The fourth summand is rewritten as

$$\int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n}\int_{a_n}^{x_n}\int_{a_{n-1}}^{x_{n-1}} p(\mathbf{x})f_{(0,\ldots,0,m_{n-1}+1,m_n+1)}(x_1,\ldots,x_{n-2},y_{n-1},y_n)\times$$
$$\times\frac{(x_{n-1}-y_{n-1})^{m_{n-1}}}{m_{n-1}!}\frac{(x_n-y_n)^{m_n}}{m_n!}dy_{n-1}dy_ndx_ndx_{n-1}$$
$$=\int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n}\int_{y_{n-1}}^{b_{n-1}}\int_{y_n}^{b} p(\mathbf{x})f_{(0,\ldots,0,m_{n-1}+1,m_n+1)}(x_1,\ldots,x_{n-2},y_{n-1},y_n)\times$$
$$\times\frac{(x_{n-1}-y_{n-1})^{m_{n-1}}}{m_{n-1}!}\frac{(x_n-y_n)^{m_n}}{m_n!}dx_ndx_{n-1}dy_ndy_{n-1},$$

where we use the Fubini theorem several times. Firstly, we change $y_n$ and $x_n$, then $x_n$ and $y_{n-1}$, then $y_{n-1}$ and $y_n$, then $y_{n-1}$ and $x_{n-1}$, then $y_n$ and $x_{n-1}$. Using all these results we get

$$\int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n} p(\mathbf{x})f(x_1,\ldots,x_n)dx_ndx_{n-1}$$
$$=\sum_{i_n=0}^{m_n}\sum_{i_{n-1}=0}^{m_{n-1}}\int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n} p(\mathbf{x})f_{(0,\ldots,0,i_{n-1},i_n)}(x_1,\ldots,x_{n-2},a_{n-1},a_n)\times$$
$$\times\frac{(x_{n-1}-a_{n-1})^{i_{n-1}}}{i_{n-1}!}\frac{(x_n-a_n)^{i_n}}{i_n!}dx_ndx_{n-1}$$

$$+\sum_{i_n=0}^{m_n}\int_{a_{n-1}}^{b_{n-1}}\int_{y_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n} p(\mathbf{x})f_{(0,\ldots,0,m_{n-1}+1,i_n)}(x_1,\ldots,x_{n-2},y_{n-1},a_n)\times$$
$$\times\frac{(x_{n-1}-y_{n-1})^{m_{n-1}}}{m_{n-1}!}\frac{(x_n-a_n)^{i_n}}{i_n!}dx_ndx_{n-1}dy_{n-1}$$
$$+\sum_{i_{n-1}=0}^{m_{n-1}}\int_{a_n}^{b_n}\int_{a_{n-1}}^{b_{n-1}}\int_{y_n}^{b_n} p(\mathbf{x})f_{(0,\ldots,0,i_{n-1},m_n+1)}(x_1,\ldots,x_{n-2},a_{n-1},y_n)\times$$
$$\times\frac{(x_{n-1}-a_{n-1})^{i_{n-1}}}{i_{n-1}!}\frac{(x_n-y_n)^{m_n}}{m_n!}dx_ndx_{n-1}dy_n$$
$$+\int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n}\int_{y_{n-1}}^{b_{n-1}}\int_{y_{n-1}}^{b_{n-1}}\int_{y_n}^{b_n} p(\mathbf{x})\times$$
$$\times f_{(0,\ldots,0,m_{n-1}+1,m_n+1)}(x_1,\ldots,x_{n-2},y_{n-1},y_n)\frac{(x_{n-1}-y_{n-1})^{m_{n-1}}}{m_{n-1}!}\times$$
$$\times\frac{(x_n-y_n)^{m_n}}{m_n!}dx_ndx_{n-1}dy_ndy_{n-1}.$$

Now, use the Taylor expansion again and integrate over $[a_{n-2},b_{n-2}]$ by variable $x_{n-2}$. If we proceed in the similar fashion as we done before, then we finally get:

$$\int_{a_{n-2}}^{b_{n-2}}\int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n} p(\mathbf{x})f(x_1,\ldots,x_n)dx_ndx_{n-1}dx_{n-2}$$

$$= \sum_{i_n=0}^{m_n} \sum_{i_{n-1}=0}^{m_{n-1}} \sum_{i_{n-2}=0}^{m_{n-2}} \int_{a_{n-2}}^{b_{n-2}} \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} p(\mathbf{x}) \times$$
$$\times f_{(0,\ldots,0,i_{n-2},i_{n-1},i_n)}(x_1,\ldots,x_{n-3},a_{n-2},a_{n-1},a_n) \times$$
$$\times \frac{(x_{n-2}-a_{n-2})^{i_{n-2}}}{i_{n-2}!} \frac{(x_{n-1}-a_{n-1})^{i_{n-1}}}{i_{n-1}!} \frac{(x_n-a_n)^{i_n}}{i_n!} dx_n dx_{n-1} dx_{n-2}$$
$$+ \sum_{i_n=0}^{m_n} \sum_{i_{n-1}=0}^{m_{n-1}} \int_{a_{n-2}}^{b_{n-2}} \int_{y_{n-2}}^{b_{n-2}} \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} p(\mathbf{x}) \times$$
$$\times f_{(0,\ldots,0,m_{n-2}+1,i_{n-1}+1,i_n)}(x_1,\ldots,x_{n-3},y_{n-2},a_{n-1},a_n) \times$$
$$\times \frac{(x_{n-2}-y_{n-2})^{m_{n-2}}}{m_{n-2}!} \frac{(x_{n-1}-a_{n-1})^{i_{n-1}}}{i_{n-1}!} \frac{(x_n-a_n)^{i_n}}{i_n!} dx_n dx_{n-1} dx_{n-2} dy_{n-2}$$
$$+ \sum_{i_n=0}^{m_n} \sum_{i_{n-2}=0}^{m_{n-2}} \int_{a_{n-1}}^{b_{n-1}} \int_{a_{n-2}}^{b_{n-2}} \int_{y_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} p(\mathbf{x}) \times$$
$$\times f_{(0,\ldots,0,i_{n-2},m_{n-1}+1,i_n)}(x_1,\ldots,x_{n-3},a_{n-2},y_{n-1},a_n) \times$$
$$\times \frac{(x_{n-2}-a_{n-2})^{i_{n-2}}}{i_{n-2}!} \frac{(x_{n-1}-y_{n-1})^{m_{n-1}}}{m_{n-1}!} \frac{(x_n-a_n)^{i_n}}{i_n!} dx_n dx_{n-1} dx_{n-2} dy_{n-1}$$

$$+ \sum_{i_n=0}^{m_n} \int_{a_{n-2}}^{b_{n-2}} \int_{a_{n-1}}^{b_{n-1}} \int_{y_{n-2}}^{b_{n-2}} \int_{y_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} p(\mathbf{x}) \times$$
$$\times f_{(0,\ldots,0,m_{n-2}+1,m_{n-1}+1,i_n)}(x_1,\ldots,x_{n-3},y_{n-2},y_{n-1},a_n) \times$$
$$\times \frac{(x_{n-2}-y_{n-2})^{m_{n-2}}}{m_{n-2}!} \frac{(x_{n-1}-y_{n-1})^{m_{n-1}}}{m_{n-1}!} \frac{(x_n-a_n)^{i_n}}{i_n!} \times$$
$$\times dx_n dx_{n-1} dx_{n-2} dy_{n-1} dy_{n-2}$$
$$+ \sum_{i_{n-1}=0}^{m_{n-1}} \sum_{i_{n-2}=0}^{m_{n-2}} \int_{a_n}^{b_n} \int_{a_{n-2}}^{b_{n-2}} \int_{a_{n-1}}^{b_{n-1}} \int_{y_n}^{b_n} p(\mathbf{x}) \times$$
$$\times f_{(0,\ldots,0,i_{n-2},i_{n-1},m_n+1)}(x_1,\ldots,x_{n-3},a_{n-2},a_{n-1},y_n) \times$$
$$\times \frac{(x_{n-2}-a_{n-2})^{i_{n-2}}}{i_{n-2}!} \frac{(x_{n-1}-a_{n-1})^{i_{n-1}}}{i_{n-1}!} \frac{(x_n-y_n)^{m_n}}{m_n!} dx_n dx_{n-1} dx_{n-2} dy_n$$
$$+ \sum_{i_{n-1}=0}^{m_{n-1}} \int_{a_{n-2}}^{b_{n-2}} \int_{a_n}^{b_n} \int_{y_{n-2}}^{b_{n-2}} \int_{a_{n-1}}^{b_{n-1}} \int_{y_n}^{b_n} p(\mathbf{x}) \times$$
$$\times f_{(0,\ldots,0,m_{n-2}+1,i_{n-1},m_n+1)}(x_1,\ldots,x_{n-3},y_{n-2},a_{n-1},y_n) \times$$
$$\times \frac{(x_{n-2}-y_{n-2})^{m_{n-2}}}{m_{n-2}!} \frac{(x_{n-1}-a_{n-1})^{i_{n-1}}}{i_{n-1}!} \frac{(x_n-y_n)^{m_n}}{m_n!} \times$$
$$\times dx_n dx_{n-1} dx_{n-2} dy_n dy_{n-2}$$
$$+ \sum_{i_{n-2}=0}^{m_{n-2}} \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \int_{a_{n-2}}^{b_{n-2}} \int_{y_{n-1}}^{b_{n-1}} \int_{y_n}^{b_n} p(\mathbf{x}) \times$$
$$\times f_{(0,\ldots,0,i_{n-2},m_{n-1}+1,m_n+1)}(x_1,\ldots,x_{n-3},a_{n-2},y_{n-1},y_n) \times$$

$$
\begin{aligned}
&\times \frac{(x_{n-2}-a_{n-2})^{i_{n-2}}}{i_{n-2}!} \frac{(x_{n-1}-y_{n-1})^{m_{n-1}}}{m_{n-1}!} \frac{(x_n-y_n)^{m_n}}{m_n!} \times \\
&\times dx_n dx_{n-1} dx_{n-2} dy_n dy_{n-1} \\
&+ \int_{a_{n-2}}^{b_{n-2}} \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \int_{y_{n-2}}^{b_{n-2}} \int_{y_{n-1}}^{b_{n-1}} \int_{y_n}^{b_n} p(\mathbf{x}) \times \\
&\times f_{(0,\ldots,0,m_{n-2}+1,m_{n-1}+1,m_n+1)}(x_1,\ldots,x_{n-3},y_{n-2},y_{n-1},y_n) \times \\
&\times \frac{(x_{n-2}-y_{n-2})^{m_{n-2}}}{m_{n-2}!} \frac{(x_{n-1}-y_{n-1})^{m_{n-1}}}{m_{n-1}!} \frac{(x_n-y_n)^{m_n}}{m_n!} \times \\
&\times dx_n dx_{n-1} dx_{n-2} dy_n dy_{n-1} dy_{n-2}.
\end{aligned}
$$

Then we use the Taylor expansion again and integrate the result over interval $[a_{n-3}, b_{n-3}]$ by variable $x_{n-3}$. If we continue this process, we get required identity. □

**Corollary 3.2** *Let the assumptions of Theorem* 3.9 *be valid and let* $p \equiv 1$. *Then the following identity holds*

$$
\begin{aligned}
&\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1,\ldots,x_n) dx_n \cdots dx_1 \\
&= \sum_{i_1=0}^{m_1} \cdots \sum_{i_n=0}^{m_n} \prod_{j=1}^{n} \frac{(b_j-a_j)^{i_j+1}}{(i_i+1)!} f_{(i_1,\ldots,i_n)}(a_1,\ldots,a_n) \\
&+ \sum_{i_2=0}^{m_2} \cdots \sum_{i_n=0}^{m_n} \int_{a_1}^{b_1} \frac{(b_1-y_1)^{m_1+1}}{(m_1+1)!} \times \\
&\times \prod_{j=2}^{n} \frac{(b_j-a_j)^{i_j+1}}{(i_j+1)!} f_{(m_1+1,i_2,\ldots,i_n)}(y_1,a_2,\ldots,a_n) dy_1 \\
&\qquad + \cdots + \\
&+ \sum_{i_1=0}^{m_1} \cdots \sum_{i_{n-1}=0}^{m_{n-1}} \int_{a_n}^{b_n} \frac{(b_n-y_n)^{m_n+1}}{(m_n+1)!} \prod_{j=1}^{n-1} \frac{(b_j-a_j)^{i_j+1}}{(i_j+1)!} \times \\
&\times f_{(i_1,\ldots,i_{n-1},m_n+1)}(a_1,\ldots,a_{n-1},y_n) dy_n \\
&\qquad + \cdots + \\
&+ \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \prod_{j=1}^{n} \frac{(b_j-y_j)^{m_j+1}}{(m_j+1)!} f_{(m_1+1,\ldots,m_n+1)}(y_1,\ldots,y_n) dy_n \cdots dy_1.
\end{aligned}
$$

**Remark 3.2** For $n = 2$ in the above corollary we get Theorem 6.16 in the book [15] by simply putting $x = a$ and $y = c$.

**Theorem 3.10** *Let the assumptions of Theorem* 3.9 *be valid. Then the inequality*

$$
\Lambda(f) = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} p(\mathbf{x}) f(x_1,\ldots,x_n) dx_n \cdots dx_1 \geq 0 \tag{3.22}
$$

*holds for every* $(m_1+1,\ldots,m_n+1)$*-convex function* $f$ *on* $I_1 \times \cdots \times I_n$ *if and only if*

$$
\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} p(\mathbf{y}) \prod_{j=1}^{n} \frac{(y_j-a_j)^{i_j}}{i_j!} dy_n \cdots dy_1 = 0, \tag{3.23}
$$

$$i_1 \in \{0,1,\ldots,m_1\},\ \ldots,\ i_n \in \{0,1,\ldots,m_n\},$$

$$\int_{x_1}^{b_1} \cdots \int_{a_n}^{b_n} p(\mathbf{y}) \frac{(y_1 - x_1)^{m_1}}{m_1!} \prod_{j=2}^{n} \frac{(y_j - a_j)^{i_j}}{i_j!} \, dy_n \cdots dy_1 = 0, \tag{3.24}$$

$$i_2 \in \{0,1,\ldots,m_2\},\ldots,i_n \in \{0,1,\ldots,m_n\}, \forall x_1 \in [a_1,b_1],$$

$$\vdots$$

$$\int_{a_1}^{b_1} \cdots \int_{a_{n-1}}^{b_{n-1}} \int_{x_n}^{b_n} p(\mathbf{y}) \prod_{j=1}^{n-1} \frac{(y_j - a_j)^{i_j}}{i_j!} \frac{(y_n - x_n)^{m_n}}{m_n!} \, dy_n \cdots dy_1 = 0, \tag{3.25}$$

$$i_1 \in \{0,1,\ldots,m_1\},\ldots,i_{n-1} \in \{0,1,\ldots,m_{n-1}\}, x_n \in [a_n,b_n],$$

$$\vdots$$

$$\int_{x_1}^{b_1} \cdots \int_{x_n}^{b_n} p(\mathbf{y}) \prod_{j=1}^{n} \frac{(y_j - x_j)^{m_j}}{m_j!} \, dy_n \cdots dy_1 \geq 0, \tag{3.26}$$

$$x_1 \in [a_1,b_1],\ldots,x_n \in [a_n,b_n].$$

*Proof.* If (3.23), (3.24), $\ldots$, (3.25) hold, then all these sums are zero in (3.20) and the required inequality (3.22) holds by using (3.26).

Conversely, if we consider in (3.22) the following functions

$$g^1(y_1,\ldots,y_n) = \prod_{j=1}^{n} \frac{(y_j - a_j)^{i_j}}{i_j!} \quad \text{and} \quad g^2 = -g^1$$

for $i_1 \in \{0,1,\ldots,m_1\},\ \ldots,\ i_n \in \{0,1,\ldots,m_n\}$ such that $g^k_{(m_1+1,\ldots,m_n+1)} \geq 0, \quad k \in \{1,2\}$, then we get the required equality (3.23).

In the same way, if we consider in (3.22) the following functions for $i_2 \in \{0,1,\ldots,m_2\}$, $\ldots, i_n \in \{0,1,\ldots,m_n\}, \forall x_1 \in [a_1,b_1]$

$$g^3(y_1,\ldots,y_n) = \begin{cases} \dfrac{(y_1 - x_1)^{m_1}}{m_1!} \displaystyle\prod_{j=2}^{n} \frac{(y_j - a_j)^{i_j}}{i_j!}, & x_1 < y_1, \\ 0, & x_1 \geq y_1, \end{cases} \quad \text{and} \quad g^4 = -g^3$$

such that $g^k_{(m_1+1,\ldots,m_n+1)} \geq 0, \quad k \in \{3,4\}$, then we get the required equality (3.24). Similarly, if we consider in (3.22) the following functions for $i_1 \in \{0,1,\ldots,m_1\}, \ldots,\ i_{n-1} \in \{0,1,\ldots,m_{n-1}\}, \forall x_n \in [a_n,b_n]$

$$g^5(y_1,\ldots,y_n) = \begin{cases} \displaystyle\prod_{j=1}^{n-1} \frac{(y_j - a_j)^{i_j}}{i_j!} \frac{(y_n - x_n)^{m_n}}{m_n!}, & x_n < y_n, \\ 0, & x_n \geq y_n, \end{cases} \quad \text{and} \quad g^6 = -g^5$$

such that $g^k_{(m_1+1,\ldots,m_n+1)} \geq 0, \quad k \in \{5,6\}$, then we get the required equality (3.25) and so on.

The last inequality (3.26) is followed by considering the following function in (3.22) for any $x_1 \in [a_1, b_1], \ldots, x_n \in [a_n, b_n]$,

$$g^7(y_1, \ldots, y_n) = \begin{cases} \prod_{j=1}^{n} \frac{(y_j - x_j)^{m_j}}{m_j!}, & x_1 < y_1, \ldots, x_n < y_n, \\ 0, & \text{otherwise.} \end{cases}$$

□

## 3.6 Mean Value Theorems and Exponential Convexity

It is a well known fact that many results of classical real analysis are a consequence of the mean value theorem. Lagrange's and Cauchy's mean value theorems are among the most important theorems of differential calculus. Here we state some generalized mean value theorems of Lagrange and of Cauchy-type. These results are given in [30].

**Theorem 3.11** *Let $\Lambda : C^{(m_1+1,\ldots,m_n+1)}(I_1 \times \cdots \times I_n) \to \mathbb{R}$ be the linear functional defined in* (3.22)*. Let $p : I_1 \times \cdots \times I_n \to \mathbb{R}$ be an integrable function and $f \in C^{(m_1+1,\ldots,m_n+1)}(I_1 \times \ldots I_n)$, $I_i = [a_i, b_i], i = 1, \ldots n$, such that the conditions* (3.23), (3.24), …, (3.25), …, (3.26) *of Theorem* 3.10 *are satisfied. Then there exists $(\xi_1, \ldots, \xi_n) \in I_1 \times \cdots \times I_n$ such that*

$$\Lambda(f) = f_{(m_1+1,\ldots,m_n+1)}(\xi_1, \ldots, \xi_n)\Lambda(f_0) \tag{3.27}$$

*where* $f_0(x_1, \ldots, x_n) = \prod_{j=1}^{n} \frac{x_j^{m_j+1}}{(m_j+1)!}$.

*Proof.* Since $f_{(m_1+1,\ldots,m_n+1)}$ is continuous on $I_1 \times \cdots \times I_n$, so it attains its maximum and minimum values on $I_1 \times \cdots \times I_n$. Let $L = \min f_{(m_1+1,\ldots,m_n+1)}$ and $U = \max f_{(m_1+1,\ldots,m_n+1)}$. Then the function $G = U f_0 - f$ satisfies

$$G_{(m_1+1,\ldots,m_n+1)}(x_1, \ldots, x_n) = U - f_{(m_1+1,\ldots,m_n+1)}(x_1, \ldots, x_n) \geq 0,$$

i.e., $G$ is an $(m_1+1, \ldots, m_n+1)$-convex function. Hence $\Lambda(G) \geq 0$ by Theorem 3.10 and we conclude that

$$\Lambda(f) \leq U\Lambda(f_0).$$

Similarly, we have

$$L\Lambda(f_0) \leq \Lambda(f).$$

Combining the two inequalities we get

$$L\Lambda(f_0) \leq \Lambda(f) \leq U\Lambda(f_0).$$

If $\Lambda(f_0) = 0$, then $\Lambda(f) = 0$ and the statement obviously holds. If $\Lambda(f_0) \neq 0$, then $\dfrac{\Lambda(f)}{\Lambda(f_0)} \in [L,U]$ and hence, there exists $(\xi_1,\ldots,\xi_n) \in I_1 \times \cdots \times I_n$ such that

$$\frac{\Lambda(f)}{\Lambda(f_0)} = f_{(m_1+1,\ldots,m_n+1)}(\xi_1,\ldots,\xi_n)$$

which gives us (3.27). □

**Theorem 3.12** *Let all the assumptions of Theorem* 3.11 *be valid. Then there exists* $(\xi_1,\ldots,\xi_n) \in I_1 \times \cdots \times I_n$ *such that*

$$\frac{\Lambda(f)}{\Lambda(g)} = \frac{f_{(m_1+1,\ldots,m_n+1)}(\xi_1,\ldots,\xi_n)}{g_{(m_1+1,\ldots,m_n+1)}(\xi_1,\ldots,\xi_n)}$$

*provided that the denominator of the left-hand side is nonzero.*

*Proof.* Let $h \in C^{(m_1+1,\ldots,m_n+1)}(I_1 \times \cdots \times I_n)$ be defined as

$$h = \Lambda(g)f - \Lambda(f)g.$$

Using Theorem 3.11 there exists $(\xi_1,\ldots,\xi_n)$ such that

$$0 = \Lambda(h) = h_{(m_1+1,\ldots,m_n+1)}(\xi_1,\ldots,\xi_n)\Lambda(f_0)$$

or

$$\Big[\Lambda(g)f_{(m_1+1,\ldots,m_n+1)}(\xi_1,\ldots,\xi_n) - \Lambda(f)g_{(m_1+1,\ldots,m_n+1)}(\xi_1,\ldots,\xi_n)\Big]\Lambda(f_0) = 0$$

which gives us required result. □

**Corollary 3.3** *Let all the assumptions of Theorem* 3.12 *be satisfied with* $m = m_1 = m_2 = \ldots = m_n$. *Then there exists* $(\xi_1,\ldots,\xi_n) \in I_1 \times \cdots \times I_n$ *such that*

$$(\xi_1 \cdots \xi_n)^{q-q'} = \frac{[(q'+1)q'\cdots(q'-n+1)]^n \Lambda((x_1\cdots x_n)^{q+1})}{[(q+1)q\cdots(q-n+1)]^n \Lambda((x_1\cdots x_n)^{q'+1})}$$

*for* $-\infty < q \neq q' < +\infty$ *and* $q,q' \notin \{-1,0,1,\ldots,n-1\}$.

*Proof.* If we put

$$f(x_1,\ldots,x_n) = (x_1\cdots x_n)^{q+1}$$

and

$$g(x_1,\ldots,x_n) = (x_1\cdots x_n)^{q'+1}$$

in Theorem 3.12, then we get the required result. □

Bernstein [9] and Widder [98] independently introduced an important sub-class of convex functions, which is called class of exponentially convex functions on a given open interval and studied some properties of this newly defined class.

Let $J \subset \mathbb{R}$ be an open interval. Here we give some definitions and properties related to exponential convexity. For further reading we refer to [9], [22].

**Definition 3.2** [9] *A function $\psi : J \to \mathbb{R}$ is exponentially convex on $J$ if it is continuous and*

$$\sum_{i,j=1}^{n} \xi_i \xi_j \, \psi(x_i + x_j) \geq 0$$

*$\forall n \in \mathbb{N}$ and all choices $\xi_i, \xi_j \in \mathbb{R}$; $i, j = 1, \dots, n$ such that $x_i + x_j \in J$; $1 \leq i, j \leq n$.*

***Example 3.1*** [22] For constant $c \geq 0$ and $k \in \mathbb{R}$, $x \mapsto ce^{kx}$ is an example of exponentially convex function.

The following proposition and two corollaries are given in [22].

**Proposition 3.1** *Let $\psi : J \to \mathbb{R}$, the following propositions are equivalent:*

*(i) $\psi$ is exponentially convex on $J$.*

*(ii) $\psi$ is continuous and $\sum_{i,j=1}^{n} \xi_i \xi_j \, \psi\left(\frac{x_i + x_j}{2}\right) \geq 0$, for all $\xi_i, \xi_j \in \mathbb{R}$ and every $x_i, x_j \in J$; $1 \leq i, j \leq n$.*

**Corollary 3.4** *If $\psi$ is an exponentially convex function on $J$, then the matrix*

$$\left[\psi\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{n}$$

*is a positive semi-definite matrix. Particularly*

$$det\left[\psi\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{n} \geq 0,$$

*$\forall n \in \mathbb{N}$, $x_i, x_j \in J$; $i, j = 1, \dots, n$.*

**Corollary 3.5** *If $\psi : J \to \langle 0, \infty \rangle$ is an exponentially convex function, then $\psi$ is a log-convex function, i.e. for every $x, y \in J$ and every $\lambda \in [0, 1]$, we have*

$$\psi(\lambda x + (1 - \lambda) y) \leq \psi^{\lambda}(x) \psi^{1-\lambda}(y).$$

Let $I = [a, b] \subset \mathbb{R}_+$ and $\Omega = \{\varphi^{(t)} : I^n \to \mathbb{R} : t \in \mathbb{R}\}$ be a family of functions defined as:

$$\varphi^{(t)}(x_1, \dots, x_n) = \begin{cases} \dfrac{(x_1 \cdots x_n)^t}{[t(t-1)\cdots(t-m)]^n}, & t \notin \{0, \dots, m\} \\ \dfrac{(x_1 \cdots x_n)^t \log^n(x_1 \cdots x_n)}{(-1)^{m-t} n! [t!(m-t)!]^n}, & t \in \{0, \dots, m\}. \end{cases}$$

Clearly $\varphi^{(t)}_{(m+1,\dots,m+1)}(x_1, \dots, x_n) = (x_1 \cdots x_n)^{t-m-1} = e^{(t-m-1)\log(x_1 \cdots x_n)}$ for $(x_1, \dots, x_n) \in I^n$ so $\varphi^{(t)}$ is an $(m+1, \dots, m+1)$-convex function and $t \mapsto \varphi^{(t)}_{(m+1,\dots,m+1)}(x_1, \dots, x_n)$ is an exponentially convex function on $\mathbb{R}$. From Corollary 3.5 we know that every positive function which is exponentially convex is log-convex. So, we state our next theorem.

**Theorem 3.13** *Let $\Lambda : C^{(m+1,\ldots,m+1)}(I^n) \to \mathbb{R}$ be a linear functional as defined in (3.22) and let the conditions (3.23), (3.24), (3.25), (3.26) of Theorem 3.10 for function $p$ be satisfied and $\varphi^{(t)}$ be a function defined above. Then the following statements hold:*

$(a)$ *The function $t \mapsto \Lambda(\varphi^{(t)})$ is continuous on $\mathbb{R}$.*

$(b)$ *The function $t \mapsto \Lambda(\varphi^{(t)})$ is exponentially convex on $\mathbb{R}$.*

$(c)$ *If the function $t \mapsto \Lambda(\varphi^{(t)})$ is positive on $\mathbb{R}$, then $t \mapsto \Lambda(\varphi^{(t)})$ is log-convex on $\mathbb{R}$. Moreover, the following Lyapunov inequality holds for $r < s < t$*

$$(\Lambda(\varphi^{(s)}))^{t-r} \leq (\Lambda(\varphi^{(r)}))^{t-s}(\Lambda(\varphi^{(t)}))^{s-r}. \tag{3.28}$$

$(d)$ *The matrix $\left[\Lambda(\varphi^{(\frac{t_i+t_j}{2})})\right]_{i,j=1}^{m}$ is positive-semidefinite. Particularly,*

$$\det\left[\Lambda(\varphi^{(\frac{t_i+t_j}{2})})\right]_{i,j=1}^{m} \geq 0$$

*for each $t_i \in \mathbb{R}$ and $m \in \mathbb{N}$ for $i \in \{1,\ldots,m\}$.*

$(e)$ *If the function $t \mapsto \Lambda(\varphi^{(t)})$ is differentiable on $\mathbb{R}$. Then for every $s,t,u,v \in \mathbb{R}$ such that $s \leq u$ and $t \leq v$, we have*

$$\mu_{s,t}(\Lambda,\Omega) \leq \mu_{u,v}(\Lambda,\Omega), \tag{3.29}$$

*where*

$$\mu_{s,t}(\Lambda,\Omega) = \begin{cases} \left(\dfrac{\Lambda(\varphi^{(s)})}{\Lambda(\varphi^{(t)})}\right)^{\frac{1}{s-t}}, & s \neq t \\ \exp\left(\dfrac{\frac{d}{ds}\Lambda(\varphi^{(s)})}{\Lambda(\varphi^{(s)})}\right), & s = t. \end{cases} \tag{3.30}$$

*Proof.* (a) For fixed $n \in \mathbb{N} \cup \{0\}$, using the L'Hôpital rule $n$-times and applying limit, we get

$$\begin{aligned} \lim_{t\to 0}\Lambda(\varphi^{(t)}) &= \lim_{t\to 0}\frac{\int_a^b \cdots \int_a^b p(\mathbf{x})(x_1\cdots x_n)^t dx_n \cdots dx_1}{[t(t-1)\cdots(t-m)]^n} \\ &= \frac{\int_a^b \cdots \int_a^b p(\mathbf{x})\log^n(x_1\cdots x_n)dx_n\cdots dx_1}{(-1)^m n!\left(m!\right)^n} \\ &= \Lambda(\varphi^{(0)}). \end{aligned}$$

In the similar fashion we can get

$$\lim_{t\to k}\Lambda(\varphi^{(t)}) = \Lambda(\varphi^{(k)}), \quad k \in \{1,\ldots,m\}.$$

So we conclude that the function $t \mapsto \Lambda(\varphi^{(t)})$ is continuous on $\mathbb{R}$.

(b) Let us define the function

$$\omega = \sum_{i,j=1}^{k} u_i u_j \varphi^{(\frac{t_i+t_j}{2})},$$

where $t_i, u_i \in \mathbb{R}$, $i \in \{1,\dots,k\}$. Since the function $t \mapsto \varphi^{(t)}_{(m+1,\dots,m+1)}$ is exponentially convex, we have

$$\omega_{(m+1,\dots,m+1)} = \sum_{i,j=1}^{k} u_i u_j \varphi^{(\frac{t_i+t_j}{2})}_{(m+1,\dots,m+1)} \geq 0,$$

which implies that $\omega$ is an $(m+1,\dots,m+1)$-convex function on $I^n$ and therefore we have $\Lambda(\omega) \geq 0$. Hence $\sum_{i,j=1}^{k} u_i u_j \Lambda(\varphi^{(\frac{t_i+t_j}{2})}) \geq 0$ and we conclude that the function $t \mapsto \Lambda(\varphi^{(t)})$ is exponentially convex on $\mathbb{R}$.

(c) It is a direct consequence of $(b)$ by using Corollary 3.5. As the function $t \mapsto \Lambda(\varphi^{(t)})$ is log-convex, i.e. $\log(\Lambda(\varphi^{(t)}))$ is convex, so we have

$$\log(\Lambda(\varphi^{(s)}))^{t-r} \leq \log(\Lambda(\varphi^{(r)}))^{t-s} + \log(\Lambda(\varphi^{(t)}))^{s-r},$$

which gives us (3.28).

(d) This is a consequence of Corollary 3.4.

(e) For any convex function $\phi$ the inequality

$$\frac{\phi(s)-\phi(t)}{s-t} \leq \frac{\phi(u)-\phi(v)}{u-v} \tag{3.31}$$

holds for $s,t,u,v \in I \subset \mathbb{R}$ such that $s \leq u$, $t \leq v$, $s \neq t$, $u \neq v$, [77, p.2]. Since by $(c)$, $\Lambda(\varphi^{(t)})$ is log-convex, so set $\phi(x) = \log(\Lambda(\varphi^{(x)}))$ in (3.31) we have

$$\frac{\log(\Lambda(\varphi^{(s)})) - \log(\Lambda(\varphi^{(t)}))}{s-t} \leq \frac{\log(\Lambda(\varphi^{(u)})) - \log(\Lambda(\varphi^{(v)}))}{u-v} \tag{3.32}$$

for $s \leq u$, $t \leq v$, $s \neq t$, $u \neq v$, which is equivalent to (3.29). The cases for $s = t$ and / or $u = v$ are easily followed from (3.32) by taking respective limits. □

Chapter 4

# Functions with Nondecreasing Increments

This chapter is devoted to recent results about functions with nondecreasing increments of higher order. In the first section we list definitions and basic properties of functions with nondecreasing increments, in the second section we give results for functions with nondecresing increments of higher order, while the third section contains new results about the Levinson inequality which connect that inequality with functions with nondecreasing increments of the third order.

## 4.1 Inequalities for Functions with Nondecreasing Increments

Let $\mathbb{R}^k$ denote the $k$-dimensional vector lattice of points $\mathbf{x} = (x_1,\ldots,x_k), x_i$ real for $i = 1,\ldots,k$, with the partial ordering $\mathbf{x} = (x_1,\ldots,x_k) \leq (y_1,\ldots,y_k) = \mathbf{y}$ if and only if $x_i \leq y_i$ for $i = 1,\ldots,k$. For $\mathbf{a},\mathbf{b} \in \mathbb{R}^k$, $\mathbf{a} \leq \mathbf{b}$, a set $\{\mathbf{x} \in \mathbb{R}^k \ : \ \mathbf{a} \leq \mathbf{x} \leq \mathbf{b}\}$ is called an interval $[\mathbf{a},\mathbf{b}]$. We also use a simbol $\mathbf{I}$ for an interval in $\mathbb{R}^k$.

By $\mathbf{X}(t) = (X_1(t),\ldots,X_k(t))$ we denote a mapping of an interval from $\mathbb{R}$ into an interval $\mathbf{I} \subset \mathbb{R}^k$. If all components $X_i$, $i = 1,\ldots,k$, satisfy any property we say that $\mathbf{X}$ has this property. Further, by $\int_J \mathbf{X} dH$ we, mean the vector $(\int_J X_1 dH,\ldots,\int_J X_k dH)$. Also $\int_J H d\mathbf{X}$ $= (\int_J H dX_1, \ldots, \int_J H dX_k)$.

**Definition 4.1** *A real valued function $f$ on an interval $\mathbf{I} \subset \mathbb{R}^k$ will be said to have nondecreasing increments if*

$$f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) \leq f(\mathbf{b}+\mathbf{h}) - f(\mathbf{b}) \tag{4.1}$$

*whenever* $\mathbf{a} \in \mathbf{I}$, $\mathbf{b}+\mathbf{h} \in \mathbf{I}$, $\mathbf{0} \leq \mathbf{h} \in \mathbb{R}^k$, $\mathbf{a} \leq \mathbf{b}$.

This class of functions was introduced by Brunk in [8] where some basic properties are also given.

For example, a function with nondecreasing increments is not necessarily continuous. If the first partial derivatives of a function $f : \mathbf{I} \to \mathbb{R}$ exist, then $f$ has nondecreasing increments if and only if each of these partial derivatives is nondecreasing in each arguments, in other words, if and only if the gradient $\nabla f$ is nonnegative on $\mathbf{I}$.

If the second partial derivatives of a function $f : \mathbf{I} \to \mathbb{R}$ exist, then $f$ has nondecreasing increments if and only if each of these partial derivatives is nonnegative. If a function $f$ with nondecreasing increments is continuous for $\mathbf{b} \leq \mathbf{x} \leq \mathbf{a}+\mathbf{b}$, where $\mathbf{0} \leq \mathbf{a} \in \mathbb{R}^k$, then the function $\varphi : [0,1] \to \mathbb{R}$ defined by $\varphi(t) = f(t\mathbf{a}+\mathbf{b})$ is convex.

More corresponding results about functions with nondecreasing increments are collected in [51, pp. 351-362]. We are interested in the following theorem in which the main inequality is similar to general linear inequalities which are investigated in the first two chapters.

**Theorem 4.1** ([8]) *Let $\mathbf{I}$ denote an interval in $\mathbb{R}^k$, let $\mathbf{X} : [a,b\rangle \to \mathbf{I}$ be a nondecreasing continuous map and let $H$ be a function of bounded variation and continuous from the left on $[a,b\rangle$ with $H(a) = 0$. Then*

$$\int_{[a,b\rangle} f(\mathbf{X}(t))\, dH(t) \geq 0 \tag{4.2}$$

*for every continuous function $f : \mathbf{I} \to \mathbb{R}$ with nondecreasing increments if and only if*

$$H(b) = 0, \tag{4.3}$$

$$\int_{[a,b)} H(u)d\mathbf{X}(u) = 0, \tag{4.4}$$

*and*

$$\int_{[a,t\}} H(u)d\mathbf{X}(u) \geq 0 \quad \textit{for} \quad [a,t\} \subset [a,b\rangle, \tag{4.5}$$

*where $\int H\, d\mathbf{X} = (\int H\, dX_1, \ldots, \int H\, dX_k)$ and the symbol $[a,t\}$ refer to either of the intervals $[a,t]$ or $[a,t\rangle$.*

*Proof.* Let us prove a necessity, i.e. let inequality (4.2) holds for every continuous function $f : \mathbf{I} \to \mathbb{R}$ with nondecreasing increments. Putting in (4.2) $f \equiv 1$ and $f \equiv -1$ we get (4.3). Putting in (4.2)

$$f(\mathbf{x}) = x_j, \quad f(\mathbf{x}) = -x_j, \quad j = 1,2,\ldots,k$$

respectively, where $\mathbf{x} = (x_1, \ldots, x_k)$, we get

$$\int_{[a,b\rangle} X_j(t)\, dH(t) = 0, \quad j = 1, 2, \ldots, k,$$

i.e. $\int_{[a,b)} H(t) d\mathbf{X}(t) = 0$, which is (4.4). Inequality (4.5) follows from (4.2) after integration by parts for fixed $j(j = 1, 2, \ldots, k)$ and fixed $t \in [a,b\rangle$, $f(\mathbf{x}) = (X_j(t^+) - x_j)_+$ or $f(x) = (X_j(t^-) - x_j)_+$.

Let us suppose that (4.3), (4.4) and (4.5) hold. Since $f$ may be approximated uniformly on $\mathbf{I}$ by functions with continuous nonnegative second partial derivatives, we may assume that the second partials $f_{(i,j)}$ exist and are continuous and nonnegative. Then, using (4.3) and (4.4) we get

$$\begin{aligned}
\int_{[a,b\rangle} f(\mathbf{X}(t))\, dH(t) &= -\int_{[a,b\rangle} H(t) \nabla f(\mathbf{X}(t))\, d\mathbf{X}(t) \\
&= -\sum_{j=1}^{k} \int_{[a,b\rangle} H(t) f_j(\mathbf{X}(t))\, dX_j(t) \\
&= \sum_{j=1}^{k} \sum_{i=1}^{k} \int_{[a,b\rangle} f_{ij}(\mathbf{X}(t))\, dX_i(t) \int_{[0,t\rangle} H(u)\, dX_j(u).
\end{aligned}$$

Since (4.5) holds each term in the last sum is nonnegative, so inequality (4.2) is verified. □

A majorization theorem is valid also for functions with nondecreasing increments. Before its proof we state and prove a helpful lemma from [65]. We use the following notation:

$$\Delta_{h_i}^{x_i} f = \Delta_{h_i}^{x_i} f(x_1, \ldots, x_k) := f(x_1, \ldots, x_i + h_i, \ldots, x_k) - f(x_1, \ldots, x_i, \ldots, x_k),$$

$$\Delta_{h_i}^{x_i} \Delta_{h_j}^{x_j} f := \Delta_{h_i}^{x_i} \left( \Delta_{h_j}^{x_j} f(x_1, \ldots, x_k) \right).$$

**Lemma 4.1** *Let $H$ be a continuous real function depending on $t$ and $u_1, \ldots, u_k$, defined for $t \in [a,b]$, $u_i \in [a_i, b_i]$, for $i = 1, 2, \ldots, k$ such that*

$$\Delta_p^t \Delta_{h_i}^{u_i} H \le 0, \quad i \in \{1, \ldots, r\}, \qquad \Delta_p^t \Delta_{h_i}^{u_i} H \ge 0, \quad i \in \{r+1, \ldots, k\},$$

$$\Delta_{h_i}^{u_i} \Delta_{h_j}^{u_j} H \ge 0, \quad i, j \in \{1, \ldots, r\} \text{ or } i, j \in \{r+1, \ldots, k\}$$

$$\begin{aligned}
\Delta_{h_i}^{u_i} \Delta_{h_j}^{u_j} H \le 0, \quad &\Big( i \in \{1, \ldots, r\} \text{ and } j \in \{r+1, \ldots, k\} \Big) \\
&\text{or } \Big( i \in \{r+1, \ldots, k\} \text{ and } j \in \{1, \ldots, r\} \Big)
\end{aligned}$$

*hold for all $u_i, h_i \ge 0, t, p \ge 0$, $t + p \in [a,b]$, $u_i + h_i \in [a_i, b_i]$ for $i = 1, \ldots, k$.*

*Let $f_i, g_i : [a,b] \to [a_i, b_i]$, $(i = 1, \ldots, k)$ be real continuous functions, nonincreasing for $i = 1, \ldots, r$ and nondecreasing for $i = r+1, \ldots k$, and let $G : [a,b] \to \mathbb{R}$ be a function of bounded variation.*

*Let us define a function* $F:[0,1]\to\mathbb{R}$ *as*

$$F(\lambda)=\int_a^b H(t;\lambda g_1(t)+(1-\lambda)f_1(t),\dots,\lambda g_k(t)+(1-\lambda)f_k(t))dG(t).$$

*(a) If*

$$\begin{aligned}
&\int_a^x f_i(t)dG(t)\le\int_a^x g_i(t)dG(t)\quad(a\le x\le b, i\in\{1,\dots,r\}),\\
&\int_x^b f_i(t)dG(t)\le\int_x^b g_i(t)dG(t)\quad(a\le x\le b, i\in\{r+1,\dots,k\}),\\
&\int_a^b f_i(t)dG(t)=\int_a^b g_i(t)dG(t)\quad(i\in\{1,\dots,k\}),
\end{aligned}\tag{4.6}$$

*then* $F$ *is nondecreasing on* $[0,1]$. *Particularly,*

$$\int_a^b H(t;f_1(t),\dots,f_k(t))dG(t)\le\int_a^b H(t;g_1(t),\dots,g_k(t))dG(t).\tag{4.7}$$

*(b) If* $H$ *is a nondecreasing function on variables* $u_1,\dots,u_k$*, and if, instead of* (4.6),

$$\begin{aligned}
&\int_a^x f_i(t)dG(t)\le\int_a^x g_i(t)dG(t)\quad(a\le x\le b, i\in\{1,\dots,r\}),\\
&\int_x^b f_i(t)dG(t)\le\int_x^b g_i(t)dG(t)\quad(a\le x\le b, i\in\{r+1,\dots,k\})
\end{aligned}\tag{4.8}$$

*hold, then* $F$ *is also nondecreasing and* (4.7) *is valid too.*

*Proof.* The function $H$ may be approximated uniformly by polynomials which satisfy the conditions

$$\frac{\partial^2H}{\partial t\partial u_i}\le 0,\ \ i\in\{1,\dots,r\},\qquad \frac{\partial^2H}{\partial t\partial u_i}\ge 0,\ \ i\in\{r+1,\dots,k\},$$

$$\frac{\partial^2H}{\partial u_i\partial u_j}\ge 0,\ \ i,j\in\{1,\dots,r\}\text{ or }\ i,j\in\{r+1,\dots,k\}$$

$$\frac{\partial^2H}{\partial u_i\partial u_j}\le 0,\ \ \Big(i\in\{1,\dots,r\}\text{ and }\ j\in\{r+1,\dots,k\}\Big)$$
$$\text{or }\Big(i\in\{r+1,\dots,k\}\text{ and }\ j\in\{1,\dots,r\}\Big).$$

So, there is no loss in generality in assuming that the second partial derivatives exist. Putting $u_i(t)=\lambda g_i(t)+(1-\lambda)f_i(t)$ we get

$$F'(\lambda)=\sum_{i=1}^k\int_a^b\lambda(g_i(t)-f_i(t))\frac{\partial H(t;u_1(t),\dots,u_k(t))}{\partial u_i}dG(t)$$

$$
\begin{aligned}
&= \sum_{i=1}^{r} \int_a^b \lambda(g_i(t) - f_i(t)) \frac{\partial H}{\partial u_i} dG(t) + \sum_{i=r+1}^{k} \int_a^b \lambda(g_i(t) - f_i(t)) \frac{\partial H}{\partial u_i} dG(t) \\
&= - \sum_{i=1}^{r} \int_a^b \left( \int_a^x \lambda(g_i(t) - f_i(t)) dG(t) \right) d\left( \frac{\partial H}{\partial u_i} \right) \\
&+ \sum_{i=r+1}^{k} \int_a^b \left( \int_x^b \lambda(g_i(t) - f_i(t)) dG(t) \right) d\left( \frac{\partial H}{\partial u_i} \right).
\end{aligned}
$$

Since

$$
\frac{\partial}{\partial t}\left(\frac{\partial H}{\partial t}\right) = \frac{\partial^2 H}{\partial t \partial u_i} + \sum_{j=1}^{r} \frac{\partial^2 H}{\partial u_i \partial u_j} \cdot \frac{\partial u_j}{\partial t} + \sum_{j=r+1}^{k} \frac{\partial^2 H}{\partial u_i \partial u_j} \cdot \frac{\partial u_j}{\partial t} \le 0
$$

for $i \in \{1, \dots, r\}$ and $\frac{\partial}{\partial t}\left(\frac{\partial H}{\partial t}\right) \ge 0$ for $i \in \{r+1, \dots, k\}$ we get

$$
F'(\lambda) \ge 0
$$

and $F$ is nondecreasing. Since $0 < 1$ we get $F(0) \le F(1)$, so, (4.7) holds. □

Now, we can prove the Majorization theorem for a function with nondecreasing increments, [65].

**Theorem 4.2** *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two mappings of a real interval* $[a,b]$ *into an interval* $\mathbf{I}$*, continuous and nondecreasing, and let* $G : [a,b] \to \mathbb{R}$ *be a function of bounded variation.*

*(a) If*

$$
\begin{aligned}
\int_u^b \mathbf{X}(t)\, dG(t) &\le \int_u^b \mathbf{Y}(t)\, dG(t), \quad \text{for each} \quad u \in (a,b), \\
\int_a^b \mathbf{X}(t)\, dG(t) &= \int_a^b \mathbf{Y}(t)\, dG(t),
\end{aligned}
\tag{4.9}
$$

*then for every continuous function* $f : \mathbf{I} \to \mathbb{R}$ *with nondecreasing increments we have*

$$
\int_a^b f(\mathbf{X}(t))\, dG(t) = \int_a^b f(\mathbf{Y}(t))\, dG(t). \tag{4.10}
$$

*(b) If*

$$
\int_u^b f(\mathbf{X}(t))\, dG(t) \le \int_u^b f(\mathbf{Y}(t))\, dG(t), \quad \text{for each} \quad u \in (a,b), \tag{4.11}
$$

*then* (4.10) *holds for every continuous nondecreasing function* $f : \mathbf{I} \to \mathbb{R}$ *with nondecreasing increments.*

*Proof.* Putting in Lemma 4.1:

$$
H(t; u_1, \dots, u_k) = f(u_1, \dots, u_k)
$$

we get statements of Theorem. □

Let the function $G$ be nondecreasing. Then (4.10) holds for every continuous function $f$ with nondecreasing increments if and only if (4.9) holds; analogously (4.10) holds for every continuous nondecreasing function $f$ with nondecreasing increments on $I$ if and only if (4.11) holds.

In a special case we get not only the Jensen-Steffensen inequality but also its reverse inequality.

**Theorem 4.3** *Let $\mathbf{X} : [a,b] \to \mathbf{I}$ be a nondecreasing continuous map and let $G$ be a function of bounded variation on $[a,b]$.*

*(a) If*

$$G(a) \leq G(x) \leq G(b), \quad G(a) < G(b) \tag{4.12}$$

*and if $f : \mathbf{I} \to \mathbb{R}$ is a continuous function with nondecreasing increments, then*

$$f\left(\frac{\int_a^b \mathbf{X}(t)\,dG(t)}{\int_a^b dG(t)}\right) \leq \frac{\int_a^b f(\mathbf{X}(t))\,dG(t)}{\int_a^b dG(t)}. \tag{4.13}$$

*(b) If $\frac{\int_a^b \mathbf{X}(t)\,dG(t)}{\int_a^b dG(t)} \in \mathbf{I}$, and if for each $x \in (a,b)$ we have either $G(x) \leq G(a)$ or $G(x) \geq G(b)$ then the reverse inequality in (4.13) holds.*

*(c) If for a continuous function $f : \mathbf{I} \to \mathbb{R}$ inequality (4.13) holds for every nondecreasing $\mathbf{X}$ and for every function of bounded variation $G$ which satisfies (4.12), then $f$ is a function with nondecreasing increments.*

*Proof.* (c) Putting $a \leq t_1 < t_2 < t_3 \leq b$, $\mathbf{X}(t_1) = A$, $\mathbf{X}(t_2) = B$, $\mathbf{X}(t_3) = B + H$, $0 \leq H \in \mathbb{R}^k$), $G(t) = 0$ $(a \leq t \leq t_1, t_2 < t \leq t_3)$ and $G(t) = 1$ $(t_1 < t \leq t_2, t_3 < t \leq b)$, then inequality (4.13) reduces to $f(A+H) \leq f(A) - f(B) + f(B+H)$. Therefore, $f$ is a function with nondecreasing increments.

(a) and (b) Using substitutions

$$\mathbf{X}(t) \to \frac{\int_a^b \mathbf{X}(t)\,dG(t)}{\int_a^b dG(t)}, \quad \mathbf{Y}(t) \to \mathbf{X}(t)$$

into Theorem 4.2 we have that (4.13) is valid if

$$\int_a^b X_j(t)dG(t) \int_x^b dG(t) \leq \int_x^b X_j(t)dG(t) \int_a^b dG(t) \tag{4.14}$$

holds for any $x \in [a,b]$, $j = 1, \ldots, k$. Also, we have that the reverse inequality in the above inequality holds, then the reverse inequality in (4.13) holds too. It is worth to mention that (4.14) is proved using integration by parts. Namely we have

$$\begin{aligned} &\int_x^b X_j(t)dG(t) \int_a^b dG(t) - \int_a^b X_j(t)dG(t) \int_x^b dG(t) \\ &= \int_x^b X_j(t)dG(t) \left(\int_a^x dG(t) + \int_x^b dG(t)\right) \end{aligned}$$

$$
\begin{aligned}
&- \left( \int_a^x X_j(t)dG(t) + \int_x^b X_j(t)dG(t) \right) \int_x^b dG(t) \\
&= \int_x^b X_j(t)dG(t) \int_a^x dG(t) - \int_x^b dG(t) \int_a^x X_j(t)dG(t) \\
&= (G(b)-G(x)) \int_a^x (G(t)-G(a))df(t) \\
&+ (G(x)-G(a)) \int_x^b (G(b)-G(t))df(t) \geq 0.
\end{aligned}
$$

□

From the previous results we can simply obtain analogous discrete results. Here, we shall consider only a special case of a corresponding generalization of the Jensen-Steffensen inequality with nonnegative weights.

**Theorem 4.4** *Let $f : \mathbf{I} \to \mathbb{R}$ be a continuous function with nondecreasing increments and let $(\mathbf{X_1}, \dots, \mathbf{X_n})$ be a monotonic sequence with elements from $\mathbf{I}$.*

*(a) If $w_i$ $(i \in \{1, \dots, n\})$ are nonnegative numbers, then*

$$
f\left( \frac{1}{W_n} \sum_{i=1}^n w_i \mathbf{X_i} \right) \leq \frac{1}{W_n} \sum_{i=1}^n w_i f(\mathbf{X_i}) \tag{4.15}
$$

*where $W_j = \sum_{i=1}^j w_i \neq 0$.*

*(b) If $w_i$, $(i \in \{1, \dots, n\})$ satisfy*

$$
w_1 > 0, \quad w_i \leq 0 \quad (i \in \{2, \dots, n\}), \quad W_n > 0,
$$

*and $A_n(\mathbf{X}; \mathbf{w}) = \frac{1}{W_n} \sum_{i=1}^n w_i \mathbf{X_i} \in \mathbf{I}$, then the reverse inequality in (4.15) holds.*

Now, let us describe monotonicity in means which we use in upcoming results, [66].

**Definition 4.2** *A finite sequence $(\mathbf{X_1}, \dots, \mathbf{X_n}) \in \mathbf{I}^n$ is said to be nondecreasing in means with respect to weights $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{R}_+^n$ if the inequalities*

$$
\mathbf{X_1} \leq A_2(\mathbf{X}; \mathbf{w}) \leq \cdots \leq A_n(\mathbf{X}; \mathbf{w}) \tag{4.16}
$$

*hold, where*

$$
A_j(\mathbf{X}; \mathbf{w}) = \frac{1}{W_j} \sum_{i=1}^j w_i \mathbf{X_i}, \quad W_j = \sum_{i=1}^j w_i.
$$

*If the inequalities in (4.16) holds in reverse order, then the sequence $(\mathbf{X_1}, \dots, \mathbf{X_n})$ is said to be nonincreasing in means.*

The following theorems gives us Jensen type and reverse Jensen type inequalities for functions with nondecreasing increments when the finite sequence of $k$-tuples $(\mathbf{X_1}, \dots, \mathbf{X_n})$ is monotone in means, [66].

**Theorem 4.5** *Let $f$ and $\mathbf{w}$ be defined as in Theorem* 4.4$(a)$ *and let the sequence* $(A_k(\mathbf{X};\mathbf{w}))_k$ *be monotonic. Then*

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i \mathbf{X_i}\right) \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i f(\mathbf{X_i}). \tag{4.17}$$

*If the assumptions of Theorem* 4.4$(b)$ *are satisfied and the sequence* $(A_k(\mathbf{X};\mathbf{w}))$ *is monotonic, then the reverse inequality in* (4.17) *holds.*

The proof of this theorem is based on the property of subadditivity of index set function $F$ defined as follows:

$$F(J) = W_J f\left(\frac{1}{W_J}\sum_{i\in J} w_i \mathbf{X_i}\right) - \sum_{i\in J} w_i f(\mathbf{X_i}), \tag{4.18}$$

where $J$ is a finite nonempty subset of $\mathbb{N}$, $\mathbf{X_i}'s$ are sequences with elements from $\mathbf{I}$ and

$$W_j = \sum_{i\in J} w_i, \;\; A_J(\mathbf{X};\mathbf{w}) = \frac{1}{W_J}\sum_{i\in J} w_i \mathbf{X_i}.$$

The above-mentioned property of subadditivity is proved in the following theorem, [66].

**Theorem 4.6** *Let $f : \mathbf{I} \to \mathbb{R}$ be a continuous function with nondecreasing increments, let $J$ and $K$ be finite nonempty sets of positive integers such that $J \cap K = \emptyset$, $\mathbf{w} = (w_i)_{i\in J\cup K}$ is a real sequence with $W_{J\cup K} > 0$, and $\mathbf{X}_i \in \mathbf{I}$, $i \in J\cup K$, $A_J(\mathbf{X};\mathbf{w}), A_K(\mathbf{X};\mathbf{w}), A_{J\cup K}(\mathbf{X};\mathbf{w}) \in \mathbf{I}$.*

$(a)$ *Let $W_J > 0$ and $W_K > 0$. If*

$$A_J(\mathbf{X};\mathbf{w}) \leq A_K(\mathbf{X};\mathbf{w}) \quad or \quad A_J(\mathbf{X};\mathbf{w}) \geq A_K(\mathbf{X};\mathbf{w}), \tag{4.19}$$

*i.e. if*

$$A_J(\mathbf{X};\mathbf{w}) \leq A_{J\cup K}(\mathbf{X};\mathbf{w}) \quad or \quad A_J(\mathbf{X};\mathbf{w}) \geq A_{J\cup K}(\mathbf{X};\mathbf{w}), \tag{4.20}$$

*then*

$$F(J\cup K) \leq F(J) + F(K). \tag{4.21}$$

*(b) If $W_J > 0$ and $W_K < 0$, and* (4.19) *(*i.e. (4.20) *holds, then the inequality in* (4.21) *holds in reverse order.*

*Proof.* Putting in (4.15) for $n = 2$

$$X_1 \to A_J(\mathbf{X};\mathbf{w}), \;\; w_1 \to W_J, \;\; X_2 \to A_K(\mathbf{X};\mathbf{w}), \;\; w_2 \to W_K,$$

we get

$$W_{J\cup K} F(A_{J\cup K}(\mathbf{X};\mathbf{w})) \leq W_J f(A_J(\mathbf{X};\mathbf{w})) + W_K f(A_K(\mathbf{X};\mathbf{w})),$$

i.e. (4.21) holds if (4.19) is valid. Since

$$A_{J\cup K}(\mathbf{X};\mathbf{w}) - A_J(\mathbf{X};\mathbf{w}) = \frac{W_K}{W_{J\cup K}}\Big(A_K(\mathbf{X};\mathbf{w}) - A_J(\mathbf{X};\mathbf{w})\Big)$$

we have that the conditions (4.19) and (4.20) are equivalent. □

The following results give refinements of Theorem 4.4 and therefore these results are generalizations of some refinements of the Čebyšev inequality as well as the corresponding results of H. Burkill and L. Mirsky, see Example 4.1.

**Corollary 4.1** *If the conditions of Theorem* 4.4$(a)$ *are fulfilled, then*

$$F(I_n) \le F(I_{n-1}) \le \cdots \le F(I_2) \le 0 \quad (I_j = \{1,\ldots,j\}). \tag{4.22}$$

*If the conditions of Theorem* 4.4$(b)$ *are valid, then the reverse inequalities in* (4.22) *are valid.*

*Proof.* Let us suppose that the conditions of Theorem 4.4$(a)$ are fulfilled. Let us define $J = I_{n-1}$ and $K = \{n\}$. Since $F(K) = 0$, from (4.21) we get

$$F(I_n) = F(J \cup K) \le F(J) + F(K) = F(J) = F(I_{n-1}).$$

□

*Proof of Theorem 4.5.* Let $f$ and $\mathbf{w}$ be defined as in Theorem 4.4$(a)$. By the Corollary 4.1 we get

$$\begin{aligned} F(I_n) &\le F(I_{n-1}) \\ W_n f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i \mathbf{X_i}\right) - \sum_{i=1}^{n} w_i f(\mathbf{X_i}) &\le W_{n-1} f\left(\frac{1}{W_{n-1}}\sum_{i=1}^{n-1} w_i \mathbf{X_i}\right) - \sum_{i=1}^{n-1} w_i f(\mathbf{X_i}) \\ W_n f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i \mathbf{X_i}\right) &\le w_n f(\mathbf{X_n}) + W_{n-1} f\left(\frac{1}{W_{n-1}}\sum_{i=1}^{n-1} w_i \mathbf{X_i}\right). \end{aligned}$$

Following in a similar manner we get

$$\begin{aligned} & W_n f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i \mathbf{X_i}\right) \\ & \qquad \le w_n f(\mathbf{X_n}) + \left(w_{n-1} f(\mathbf{X_{n-1}}) + W_{n-2} f\left(\frac{1}{W_{n-2}}\sum_{i=1}^{n-2} w_i \mathbf{X_i}\right)\right) \\ & \qquad \le \ldots \le \sum_{i=1}^{n} w_i f(\mathbf{X_i}) \end{aligned}$$

and the statement is established. □

***Example 4.1*** Let $f : [0,\infty\rangle^k \to \mathbb{R}$ be defined as

$$f(x_1,\ldots,x_k) = x_1 \cdot \ldots \cdot x_k.$$

Since $\dfrac{\partial^2 f}{\partial x_i \partial x_j} \ge 0$ for all $i, j \in \{1,\ldots,k\}$ the function $f$ is a function with nondecreasing increments.

Let us define

$$\mathbf{X_j} = (x_{j1}, x_{j2}, \ldots, x_{jk}), \quad j = 1, \ldots, n.$$

If sequences $(x_{j1})_j, (x_{j2})_j, \ldots, (x_{jk})_j$ are nondecreasing (nonincreasing), then $\mathbf{X_1} \leq \mathbf{X_2} \leq \ldots \leq \mathbf{X_n}$ ($\mathbf{X_1} \geq \mathbf{X_2} \geq \ldots \geq \mathbf{X_n}$ and by Theorem 4.4, if $w_i \geq 0$, the following holds

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i \mathbf{X_i}\right) \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i f(\mathbf{X_i})$$

which becomes

$$\frac{1}{W_n^k}\left(\sum_{j=1}^{n} w_j x_{j1}\right) \cdots \left(\sum_{j=1}^{n} w_j x_{k1}\right) \leq \frac{1}{W_n}\sum_{j=1}^{n} w_j x_{j1} x_{j2} \ldots x_{jk},$$

i.e. we get the classical Čebyšev inequality for $k$ sequences monotonic in the same sense. But, using the result of Theorem 4.5 we get that the same inequality holds if the finite sequence of $k$-tuples $(\mathbf{X_1}, \ldots, \mathbf{X_n})$ is monotone in means. This result was obtained by H. Burkill and L. Mirsky in [10].

## 4.2 Functions with Nondecreasing Increments of Order $n$

The aim of the present section is to give generalization of Theorem 4.1. It is based on paper [29] due to A. Khan, J. Pečarić and S. Varošanec. Let us introduce some further notations.

Let us write $\Delta_{\mathbf{h_1}} f(\mathbf{x}) = f(\mathbf{x} + \mathbf{h_1}) - f(\mathbf{x})$ and inductively,

$$\Delta_{\mathbf{h_1}}\Delta_{\mathbf{h_2}} \cdots \Delta_{\mathbf{h_n}} f(\mathbf{x}) = \Delta_{\mathbf{h_1}}(\Delta_{\mathbf{h_2}} \cdots \Delta_{\mathbf{h_n}} f(\mathbf{x})) \quad \text{for} \quad n \geq 2,$$

where $\mathbf{x}, \mathbf{x} + \mathbf{h_1} + \cdots + \mathbf{h_n} \in \mathbf{I}$, $\mathbf{h_i} \in \mathbb{R}_*^k$ for $i \in \{1, \ldots, n\}$. Using this notation with $n = 2$, $\mathbf{h} = \mathbf{h_1}$, $\mathbf{s} = \mathbf{h_2}$, $\mathbf{b} = \mathbf{a} + \mathbf{s}$, condition (4.1) becomes

$$\Delta_{\mathbf{h_1}}\Delta_{\mathbf{h_2}} f(\mathbf{a}) \geq 0.$$

Let us extend Definition 4.1 to the following.

**Definition 4.3** *$f : \mathbf{I} \to \mathbb{R}$ is said to be a function with nondecreasing increments of order $n$ if*

$$\Delta_{\mathbf{h_1}} \cdots \Delta_{\mathbf{h_n}} f(\mathbf{x}) \geq 0$$

*holds whenever $\mathbf{x}, \mathbf{x} + \mathbf{h_1} + \cdots + \mathbf{h_n} \in \mathbf{I}$, $\mathbf{0} \leq \mathbf{h_i} \in \mathbb{R}^k$ for $i \in \{1, \ldots, n\}$.*

Every solution $f$ of the Cauchy equation $f(\mathbf{x_1}+\mathbf{x_2})=f(\mathbf{x_1})+f(\mathbf{x_2})$ is a function with nondecreasing increments of order $n$ with null increments.

If the $n$th partial derivatives $f_{i_1\cdots i_n}(\mathbf{x})=\frac{\partial^n}{\partial x_{i_1}\cdots\partial x_{i_n}}f(\mathbf{x})$ exist, they are nonnegative. If $f$ is a continuous function with nondecreasing increments of order $n$, it may be approximated uniformly on $\mathbf{I}$ by polynomials having nonnegative $n$th partial derivatives. To see this, we set, for convenience, $\mathbf{I}=[\mathbf{0},\mathbf{1}]$ where $\mathbf{1}=(1,\ldots,1)$. It is known that the Bernstein polynomials

$$\sum_{i_1=0}^{n_1}\cdots\sum_{i_k=0}^{n_k} f\left(\frac{i_1}{n_1},\ldots,\frac{i_k}{n_k}\right)\prod_{j=1}^{k}\binom{n_j}{i_j}x_j^{i_j}\,(1-x_j)^{n_j-i_j}$$

converge uniformly to $f$ on $\mathbf{I}$ as $n_1\to\infty,\ldots,n_k\to\infty$, if $f$ is continuous. Furthermore, if $f$ is a function with nondecreasing increments of order $n$, these polynomials have nonnegative $n$th partial derivatives, as may be shown by repeated application of the formula (see [8] and [29])

$$\frac{d}{dx}\sum_{i=0}^{n}\binom{n}{i}a_ix^i(1-x)^{n-i}=n\sum_{i=0}^{n-1}\binom{n-1}{i}(a_{i+1}-a_i)x^i(1-x)^{n-1-i}.$$

Let $p_1,\ldots,p_r$ be positive integers such that $p_1+\cdots+p_r=w$. Let $(i_1^{p_1}\cdots i_r^{p_r})_p$ be a set of all permutations with repetitions whose elements are from the multiset

$$S=\{\underbrace{i_1,\ldots,i_1}_{p_1-times},\underbrace{i_2,\ldots,i_2}_{p_2-times},\ldots,\underbrace{i_r,\ldots,i_r}_{p_r-times}\},\ i_1<\cdots<i_r,\ i_1,\ldots,i_r\in\{1,\ldots,k\}.$$

There are $\dfrac{w!}{p_1!p_2!\cdots p_r!}$ elements in the class $(i_1^{p_1}\cdots i_r^{p_r})_p$.

For $0<p_1\le p_2\le\cdots\le p_r,\ p_1+\cdots+p_r=w$, let $(p_1\cdots p_r)_c$ be a set whose elements are described in the following way. We say that permutation $j_1\cdots j_w$ belongs to the set $(p_1\cdots p_r)_c$ if and only if there exist $i_1,i_2,\ldots,i_r\in\{1,\ldots,k\}$, $i_1<i_2<\cdots<i_r$ and permutation $\sigma$ of the multiset $\{p_1\cdots p_r\}$ such that $j_1\cdots j_w\in(i_1^{\sigma(p_1)}\cdots i_r^{\sigma(p_r)})_p$. Family of all classes $(p_1\cdots p_r)_c$ is denoted with $C_w^k$.

For illustration, we describe the above notation on one example. Let $k=5$ and $w=4$. Classes $(p_1\cdots p_r)_c$ are the following: $(1,1,1,1)_c$, $(1,1,2)_c$, $(1,3)_c$, $(2,2)_c$ and $(4)_c$. Let us describe the elements of the set $(1,1,2)_c$. There are three different permutations of the multiset $\{1,1,2\}$. These are

$$\begin{pmatrix}1&1&2\\1&1&2\end{pmatrix},\ \begin{pmatrix}1&1&2\\1&2&1\end{pmatrix},\ \begin{pmatrix}1&1&2\\2&1&1\end{pmatrix}.$$

So, $(i_1^{\sigma(p_1)}\cdots i_r^{\sigma(p_r)})_p$ are $(i_1,i_2,i_3,i_3)_p$, $(i_1,i_2,i_2,i_3)_p$, $(i_1,i_1,i_2,i_3)_p$, where $i_1<i_2<i_3$ and $i_1,i_2,i_3\in\{1,2,3,4,5\}$. If, for example, $(i_1,i_2,i_3,i_3)_p=(2,3,5,5)_p$, then it contains all permutations with repetitions of elements $2,3,5,5$, i.e. $(2,3,5,5)_p=\{2355,2535,2533,\ldots,5532\}$ and it has $\frac{4!}{2!}=12$ elements.

In the following text, $H\in BV[a,b]$ with $H(a)=0$ and $i_1,\ldots,i_n\in\{1,\ldots,k\}$. Let $K^n_{i_1\cdots i_n}$ be a function such that

$$K^n_{i_1\cdots i_n}(t)\;=\;\int_a^t K^{n-1}_{i_1\cdots i_{n-1}}(x_n)\,dX_{i_n}(x_n),\qquad n\ge 2$$

$$\text{and} \qquad K^1_{i_1}(t) \; = \; \int_a^t H(x_1)\,dX_{i_1}(x_1).$$

Further, we write

$$\prod(S)(x) \; = \; \prod_{j\in S}(X_j(t)-X_j(x)),$$
$$\text{and} \qquad \prod(\phi)(x) \; = \; 1,$$

where $S$ is a multiset with elements from $\{1,\ldots,k\}$. Clearly

$$d\left\{\prod(S)(x)\right\} \; = \; -\sum_{j\in S} dX_j(x)\prod(S\setminus\{j\})(x)$$
$$\text{and} \qquad dK^n_{i_1\cdots i_n}(t) \; = \; K^{n-1}_{i_1\cdots i_{n-1}}(t)\,dX_{i_n}(t).$$

Now, the following result holds.

**Lemma 4.2** *Let $w$ be a fixed positive integer. Then*

$$\int_a^t \prod(\{i_1,\ldots,i_w\})\,(x)dH(x)$$

$$= \sum_{j_1=1}^{w} \sum_{\substack{j_2=1\\ j_2\neq j_1}}^{w} \cdots \sum_{\substack{j_m=1\\ j_m\neq j_k\\ k<m}}^{w} \int_a^t \prod(\{i_1,\ldots,i_w\}\setminus\{i_{j_1},\ldots,i_{j_m}\})\,(x)\,dK^m_{i_{j_1}\cdots i_{j_m}}(x)$$

*holds for each $m\in\{1,\ldots,w\}$.*

*Proof.* We prove it using induction on $m$. For $m=1$, using integration by parts, we have

$$\int_a^t \prod(\{i_1,\ldots,i_w\})\,(x)\,dH(x) = -\int_a^t H(x)\,d\left(\prod(\{i_1,\ldots,i_w\})\,(x)\right)$$
$$= \int_a^t H(x)\sum_{j_1=1}^{w} dX_{j_1}(x)\prod(\{i_1,\ldots,i_m\}\setminus\{i_{j_1}\})\,(x)$$
$$= \sum_{j_1=1}^{w}\int_a^t \prod(\{i_1,\ldots,i_w\}\setminus\{i_{j_1}\})\,(x)\,dK^1_{i_{j_1}}(x).$$

Let us suppose that the statement holds for $m-1$ and let us apply integration by parts on the right-hand side of the formula.

$$\int_a^t \prod(\{i_1,\ldots,i_w\})\,(x)\,dH(x)$$
$$= \sum_{j_1=1}^{w}\cdots\sum_{\substack{j_{m-1}=1\\ j_{m-1}\neq j_k\\ k<m-1}}^{w}\int_a^t \prod\left(\{i_1,\ldots,i_w\}\setminus\{i_{j_1},\ldots,i_{j_{m-1}}\}\right)(x)\,dK^{m-1}_{i_{j_1}\cdots i_{j_{m-1}}}(x)$$

$$= \sum_{j_1=1}^{w} \cdots \sum_{\substack{j_{m-1}=1\\ j_{m-1}\neq j_k\\ k<m-1}}^{w} (-1) \int_a^t K^{m-1}_{i_{j_1}\cdots i_{j_{m-1}}}(x) \times$$

$$\times d\left(\prod\left(\{i_1,\ldots,i_w\}\setminus\{i_{j_1},\ldots,i_{j_{m-1}}\}\right)(x)\right)$$

$$= \sum_{j_1=1}^{w} \cdots \sum_{\substack{j_{m-1}=1\\ j_{m-1}\neq j_k\\ k<m-1}}^{w} (-1) \int_a^t K^{m-1}_{i_{j_1}\cdots i_{j_{m-1}}}(x) \times$$

$$\times(-1) \sum_{\substack{j_m=1\\ j_m\neq j_k\\ k<m}}^{w} dX_{i_{j_m}}(x) \prod\left(\{i_1,\ldots,i_w\}\setminus\{i_{j_1},\ldots,i_{j_m}\}\right)(x)$$

$$= \sum_{j_1=1}^{w} \cdots \sum_{\substack{j_m=1\\ j_m\neq j_k\\ k<m}}^{w} \int_a^t \prod\left(\{i_1,\ldots,i_w\}\setminus\{i_{j_1},\ldots,i_{j_m}\}\right)(x)\, K^{m-1}_{i_{j_1}\cdots i_{j_{m-1}}}(x)\, dX_{i_{j_m}}(x)$$

$$= \sum_{j_1=1}^{w} \cdots \sum_{\substack{j_m=1\\ j_m\neq j_k\\ k<m}}^{w} \int_a^t \prod\left(\{i_1,\ldots,i_w\}\setminus\{i_{j_1},\ldots,i_{j_m}\}\right)(x)\, dK^{m}_{i_{j_1}\ldots i_{j_m}}(x).$$

$\square$

Especially for $m = w$, we have

$$\int_a^t \prod\left(\{i_1,\ldots,i_w\}\right)(x)dH(x) = \sum_{j_1=1}^{w} \cdots \sum_{\substack{j_w=1\\ j_w\neq j_k\\ k<w}}^{w} \int_a^t dK^{w}_{i_{j_1}\cdots i_{j_w}}(x)$$

$$= \sum_{j_1=1}^{w} \cdots \sum_{\substack{j_w=1\\ j_w\neq j_k\\ k<w}}^{w} K^{w}_{i_{j_1}\cdots i_{j_w}}(t) = p_1!\cdots p_r! \sum_{i_{j_1}\cdots i_{j_w}\in(i_1^{p_1}\cdots i_r^{p_r})_p} K^{w}_{i_{j_1}\cdots i_{j_w}}(t) \tag{4.23}$$

where $\{i_{j_1},\ldots,i_{j_w}\} = \{\underbrace{i_1,\ldots,i_1}_{p_1-times},\ldots,\underbrace{i_r,\ldots,i_r}_{p_r-times}\},\ i_1<i_2<\cdots<i_r,\ i_1,i_2,\ldots,i_r\in\{1,\ldots,k\},$ $p_1+\cdots+p_r = w.$

***Example 4.2*** If $w=3,\ i_1=i_2=1,\ i_3=2$, then

$$\int_a^t \prod(\{1,1,2\})(x)dH(x) = \sum_{j_1=1}^{3} \sum_{\substack{j_2=1\\ j_2\neq j_1}}^{3} \sum_{\substack{j_3=1\\ j_3\neq j_1,j_2}}^{3} K^3_{i_{j_1}i_{j_2}i_{j_3}}(t)$$
$$= 2!\,1!\,(K^3_{112}+K^3_{121}+K^3_{211}).$$

□

Furthermore, if we suppose

$$\int_a^b X_{j_1}(u)\cdots X_{j_s}(u)\,dH(u)=0 \quad \text{for} \quad j_1,\ldots,j_s\in\{1,\ldots,k\},\ s\in\{0,\ldots,w\},$$

then

$$p_1!\cdots p_r!\sum K^w_{i_{j_1}\cdots i_{j_w}}(b)=\int_a^b \prod(\{i_1,\ldots,i_w\})(x)\,dH(x)$$
$$=\sum(-1)^s\int_a^b X_{j_1}(x)\cdots X_{j_s}(x)X_{j_{s+1}}(b)\cdots X_{j_w}(b)\,dH(x)=0. \tag{4.24}$$

Now, we state our main theorems of this section:

**Theorem 4.7** *Let $\mathbf{X}:[a,b]\to\mathbf{I}$ be a continuous function and let $H\in BV[a,b]$ with $H(a)=H(b)=0$. Further, assume that $f$ has continuous $(n-1)$th partial derivatives for $n\geq 2$. If*

$$\int_a^b X_{i_1}(u)\cdots X_{i_m}(u)\,dH(u)=0$$

*for $i_1,\ldots,i_m\in\{1,\ldots,k\},\ m\in\{1,\ldots,n-1\}$, then*

$$\int_a^b f(\mathbf{X}(t))\,dH(t)=(-1)^{n-1}\sum_{(p_1\cdots p_r)_c\in C^k_{n-1}}\frac{1}{p_1!\cdots p_r!}\sum_{(i_1^{p_1}\cdots i_r^{p_r})_p\subset(p_1\cdots p_r)_c}\times$$
$$\times\int_a^b f_{\underbrace{i_1\cdots i_1}_{p_1-times}\cdots\underbrace{i_r\cdots i_r}_{p_r-times}}(\mathbf{X}(t))d\left(\int_a^t\prod(\{i_1^{p_1},\ldots,i_r^{p_r}\})(x)\,dH(x)\right). \tag{4.25}$$

*Proof.* The proof follows from induction on $n$. Let $n=2$,

$$\int_a^b f(\mathbf{X}(t))\,dH(t)=-\sum_{i=1}^k\int_a^b f_i(\mathbf{X}(t))H(t)dX_i(t)$$
$$=-\sum_{i=1}^k\int_a^b f_i(\mathbf{X}(t))\,dK_i^1(t)=-\sum_{i=1}^k\int_a^b f_i(\mathbf{X}(t))\,d\left(\int_a^t H(x)dX_i(x)\right)$$
$$=-\sum_{i=1}^k\int_a^b f_i(\mathbf{X}(t))\,d\left(\int_a^t H(x)d(X_i(x)-X_i(t))\right)$$

$$= \sum_{i=1}^{k} \int_a^b f_i(\mathbf{X}(t))\, d\left(\int_a^t H(x) d(X_i(t) - X_i(x))\right)$$

$$= -\sum_{i=1}^{k} \int_a^b f_i(\mathbf{X}(t))\, d\left(\int_a^t (X_i(t) - X_i(x)) dH(x)\right)$$

$$= -\sum_{i=1}^{k} \int_a^b f_i(\mathbf{X}(t))\, d\left(\int_a^t \prod(\{i\})(x)\, dH(x)\right).$$

If we have $\int_a^b X_{i_1}(u)\cdots X_{i_m}(u)\, dH(u) = 0$ for $i_1, \ldots, i_m \in \{1, \ldots, k\}$, $m \in \{1, \ldots, n-2\}$ and if we suppose that (4.25) holds for $(n-1)$, then

$$\int_a^b f(\mathbf{X}(t))\, dH(t)$$

$$= (-1)^{n-2} \sum_{(p_1\cdots p_r)_c \in C_{n-2}^k} \frac{1}{p_1! \cdots p_r!} \sum_{(i_1^{p_1}\cdots i_r^{p_r})_p \subset (p_1\cdots p_r)_c} \int_a^b f_{i_1^{p_1}\cdots i_r^{p_r}}(\mathbf{X}(t)) \times$$

$$\times d\left(\int_a^t \prod\left(\{i_1^{p_1}, \ldots, i_r^{p_r}\}\right)(x)\, dH(x)\right)$$

$$= (-1)^{n-2} \sum_{(p_1\cdots p_r)_c \in C_{n-2}^k} \frac{1}{p_1! \cdots p_r!} \sum_{(i_1^{p_1}\cdots i_r^{p_r})_p} \int_a^b f_{i_1^{p_1}\cdots i_r^{p_r}}(\mathbf{X}(t)) \times$$

$$\times d\left(p_1! \cdots p_r! \sum_{i_{j_1}\cdots i_{j_{n-2}} \in (i_1^{p_1}\cdots i_r^{p_r})_p} K_{i_{j_1}\cdots i_{j_{n-2}}}^{n-2}(t)\right)$$

$$= (-1)^{n-1} \sum_{(p_1\cdots p_r)_c \in C_{n-2}^k} \sum_{(i_1^{p_1}\cdots i_r^{p_r})_p} \int_a^b d f_{i_1^{p_1}\cdots i_r^{p_r}}(\mathbf{X}(t)) \times$$

$$\times \sum_{i_{j_1}\cdots i_{j_{n-2}} \in (i_1^{p_1}\cdots i_r^{p_r})_p} K_{i_{j_1}\cdots i_{j_{n-2}}}^{n-2}(t)$$

$$= (-1)^{n-1} \sum_{(p_1\cdots p_r)_c \in C_{n-2}^k} \sum_{(i_1^{p_1}\cdots i_r^{p_r})_p} \int_a^b \sum_{i_{n-1}=1}^{k} f_{i_1^{p_1}\cdots i_r^{p_r} i_{n-1}}(\mathbf{X}(t)) \times$$

$$\times dX_{i_{n-1}}(t) \left(\sum_{i_{j_1}\cdots i_{j_{n-2}}} K_{i_{j_1}\cdots i_{j_{n-2}}}^{n-2}(t)\right)$$

$$= (-1)^{n-1} \sum_{\substack{(s_1\cdots s_g)_c \in C_{n-1}^k \\ s_1 + \cdots + s_g = n-1}} \sum_{(i_1^{s_1}\cdots i_g^{s_g})_p \subset (s_1\cdots s_g)_c} \int_a^b f_{i_1^{s_1}\cdots i_g^{s_g}}(\mathbf{X}(t)) \times$$

$$\times \sum_{l_1\cdots l_{n-1} \in (i_1^{s_1}\cdots i_g^{s_g})_p} K_{l_1\cdots l_{n-2}}^{n-2}(t)\, dX_{l_{n-1}}(t)$$

$$= (-1)^{n-1} \sum_{(s_1\cdots s_g)_c\in C^k_{n-1}} \sum_{(i_1^{s_1}\cdots i_g^{s_g})_p} \int_a^b f_{i_1^{s_1}\cdots i_g^{s_g}}(\mathbf{X}(t))d\Big(\sum_{l_1\cdots l_{n-1}} K^{n-1}_{l_1\cdots l_{n-1}}(t)\Big)$$

$$= (-1)^{n-1} \sum_{(s_1\cdots s_g)_c\in C^k_{n-1}} \sum_{(i_1^{s_1}\cdots i_g^{s_g})_p} \int_a^b f_{i_1^{s_1}\cdots i_g^{s_g}}(\mathbf{X}(t)) \times$$

$$\times d\left(\frac{1}{s_1!\cdots s_g!}\int_a^t \prod\left(\{i_1^{s_1}\cdots i_g^{s_g}\}\right)(x)dH(x)\right)$$

by (4.23) and (4.24). Hence we have (4.25). □

**Theorem 4.8** *Let $X : [a,b] \to \mathbf{I}$ be a nondecreasing continuous map and let $H \in BV[a,b]$ with $H(a) = 0$. Then*

$$\int_a^b f(\mathbf{X}(t))\,dH(t) \ge 0 \tag{4.26}$$

*holds for every continuous function $f$ with nondecreasing increments of order $n$ on $\mathbf{I}$ if and only if*

$$H(b) = 0, \tag{4.27}$$

$$\int_a^b X_{i_1}(t)\cdots X_{i_m}(t)\,dH(t) = 0, \tag{4.28}$$

*for $i_1,\dots,i_m \in \{1,\dots,k\}$, $m \in \{1,\dots,n-1\}$ and*

$$(-1)^n \int_a^t \prod(\{i_1,\dots,i_{n-1}\})(u)\,dH(u) \ge 0 \tag{4.29}$$

*for each $t \in [a,b]$, $i_1,\dots,i_{n-1} \in \{1,\dots,k\}$.*

*Proof.* Necessity: The validity of (4.26) for constant functions $f^1 \equiv 1$ and $f^2 \equiv -1$ implies (4.27). From (4.26) for $f^3(\mathbf{x}) = x_{i_1}\cdots x_{i_s}$ and $f^4(\mathbf{x}) = -x_{i_1}\cdots x_{i_s}$, for $i_1,\dots,i_s \in \{1,\dots,k\}$, $s \in \{1,\dots,n-1\}$, we have (4.28).

Inequality (4.29) is obtained from (4.26) on setting, for fixed $t \in [a,b]$ and fixed $i_1,\dots,i_{n-1} \in \{1,\dots,k\}$,

$$f^5(x) = -[x_{i_1} - X_{i_1}(t)]_- \cdots [x_{i_{n-1}} - X_{i_{n-1}}(t)]_- \quad \text{where} \quad c_- = \min\{c,0\},\ c \in \mathbb{R}.$$

Sufficiency: Since $f$ may be approximated uniformly on $\mathbf{I}$ by functions with continuous and nonnegative $n$th partial derivatives, we may assume that the $n$th partials $f_{i_1,\dots,i_n}$ exist and are continuous and nonnegative. By Theorem 4.7 and (4.28), we have

$$\int_a^b f(\mathbf{X}(t))\,dH(t)$$

$$= (-1)^n \sum_{(p_1\cdots p_r)_c\in C^k_{n-1}} \frac{1}{p_1!\cdots p_r!} \sum_{(i_1^{p_1}\cdots i_r^{p_r})_p\subset(p_1\cdots p_r)_c} \sum_{i_n=1}^{k} \int_a^b f_{i_1^{p_1}\cdots i_r^{p_r} i_n}(\mathbf{X}(t)) \times$$

$$\times dX_{i_n}(t)\int_a^t \prod\left(\{i_1^{p_1}\cdots i_r^{p_r}\}\right)(x)\,dH(x).$$

By (4.29), each term in the sum is nonnegative so that (4.26) is verified. □

## 4.3 Arithmetic Integral Mean

It is known that if $f : [0,a] \to \mathbb{R}$, $a > 0$, is a nonnegative and nondecreasing function, then the function $F$, defined as

$$F(x) = \frac{1}{x}\int_0^x f(t)dt$$

is also a nondecreasing function on $[0,a]$. Let us observe that $F$ is an arithmetic integral mean of a function $f$ on an interval $[0,a]$. This result was generalized in [36] by considering a real-valued function $f$ for which $\Delta_h^n f(x) \geq 0$ holds for any $h > 0$, where $\Delta_h^n$ is defined as follows:

$$\Delta_h^0 f(x) = f(x), \qquad \Delta_h^n f(x) = \Delta_h^{n-1} f(x+h) - \Delta_h^{n-1} f(x).$$

Here, we extend the above-mentioned result to functions with nondecreasing increments of higher order ([29]).

**Theorem 4.9** *Let the function $f : [\mathbf{a},\mathbf{b}] \to \mathbb{R}$ be continuous and with nondecreasing increments of order $n$. Then the function $F$, defined as*

$$F(\mathbf{x}) = \left(\prod_{i=1}^{k}(x_i - a_i)\right)^{-1} \int_{a_1}^{x_1} \cdots \int_{a_k}^{x_k} f(\mathbf{u})\mathbf{du},$$

*is a function with nondecreasing increments of order $n$ on $[\mathbf{a},\mathbf{b}]$, where $\mathbf{u} = (u_1,\ldots,u_k)$ and $\mathbf{du} = du_1 \cdots du_k$.*

*Proof.* Let $\mathbf{x} > \mathbf{a} = (a_1,\ldots,a_k)$. Then

$$F(\mathbf{x}) = \int_0^1 \cdots \int_0^1 f(\mathbf{a} + \mathbf{s}(\mathbf{x} - \mathbf{a}))\mathbf{ds},$$

where we used the substitutions $u_i = a_i + s_i(x_i - a_i)$, $i \in \{1,\ldots,k\}$, $0 \leq s_i \leq 1$, where $\mathbf{a} + \mathbf{s}(\mathbf{x} - \mathbf{a}) = (a_1 + s_1(x_1 - a_1),\ldots,a_k + s_k(x_i - a_k))$ and $\mathbf{ds} = ds_1 \cdots ds_k$. Now, we have

$$\begin{aligned}
\Delta_{\mathbf{h_1}} \cdots \Delta_{\mathbf{h_n}} F(\mathbf{x}) &= \Delta_{\mathbf{h_1}} \cdots \Delta_{\mathbf{h_n}} \int_0^1 \cdots \int_0^1 f(\mathbf{a} + \mathbf{s}(\mathbf{x} - \mathbf{a}))\mathbf{ds} \\
&= \int_0^1 \cdots \int_0^1 \Delta_{\mathbf{h_1}} \cdots \Delta_{\mathbf{h_n}} f(\mathbf{a} + \mathbf{s}(\mathbf{x} - \mathbf{a}))\mathbf{ds} \geq 0
\end{aligned}$$

because if $f(\mathbf{x})$ is a function with nondecreasing increments of order $n$, then the function $f(\mathbf{a} + \mathbf{s}(\mathbf{x} - \mathbf{a}))$ is also a function with nondecreasing increments of order $n$. □

## 4.4 Functions with Nondecreasing Increments of Order 3

### 4.4.1 Levinson Type Inequality for Functions with Nondecreasing Increments of the Third Order

In this subsection we give two generalizations of the Levinson inequality based on results about functions with nondecreasing increments of the third order, [29].

**Theorem 4.10** *Let $H : [a,b] \to \mathbb{R}$ be a function of bounded variation such that* (4.12) *holds and let $\mathbf{X} : [a,b] \to [\mathbf{0},\mathbf{d}]$, $(\mathbf{d} > \mathbf{0})$ be a nondecreasing continuous map. If $f$ is a continuous function with nondecreasing increments of order three on $\mathbf{J} = [\mathbf{0}, 2\mathbf{d}]$, then*

$$\frac{\int_a^b f(\mathbf{X}(t))\, dH(t)}{\int_a^b dH(t)} - f\left(\frac{\int_a^b \mathbf{X}(t)\, dH(t)}{\int_a^b dH(t)}\right) \leq \frac{\int_a^b f(2\mathbf{d} - \mathbf{X}(t))\, dH(t)}{\int_a^b dH(t)} - f\left(\frac{\int_a^b (2\mathbf{d} - \mathbf{X}(t))\, dH(t)}{\int_a^b dH(t)}\right).$$

*Proof.* If $f$ is a function with nondecreasing increments of order three on $\mathbf{J}$, then the following inequality holds

$$\Delta_{\mathbf{h_1}} \Delta_{\mathbf{h_2}} \Delta_{\mathbf{h_3}} f(\mathbf{x}) \geq 0 \quad \text{for} \quad \mathbf{x}, \mathbf{x} + \mathbf{h_1} + \mathbf{h_2} + \mathbf{h_3} \in \mathbf{J}, \quad \mathbf{0} \leq \mathbf{h_1}, \mathbf{h_2}, \mathbf{h_3} \in \mathbb{R}^k,$$

i.e.,

$$\Delta_{\mathbf{h_1}} \Delta_{\mathbf{h_2}} \left(f(\mathbf{x} + \mathbf{h_3}) - f(\mathbf{x})\right) \geq 0. \tag{4.30}$$

If $\mathbf{x} \in \mathbf{I}$ and $\mathbf{h_3} = 2\mathbf{d} - 2\mathbf{x}$, we have

$$\Delta_{\mathbf{h_1}} \Delta_{\mathbf{h_2}} \left(f(2\mathbf{d} - \mathbf{x}) - f(\mathbf{x})\right) \geq 0,$$

i.e., the function $\mathbf{x} \mapsto f(2\mathbf{d} - \mathbf{x}) - f(\mathbf{x})$ is a function with nondecreasing increments of order two, i.e. it is a function with nondecreasing increments. Now, using Theorem 4.3, we obtain Theorem 4.10. □

**Theorem 4.11** *Let Let $H : [a,b] \to \mathbb{R}$ be a function of bounded variation such that* (4.12) *holds and let $f$ be a continuous function with nondecreasing increments of order three on $[\mathbf{c}, \mathbf{d}] \subset \mathbb{R}^k$. Let $\mathbf{0} < \mathbf{a} < \mathbf{d} - \mathbf{c}$. If $\mathbf{X} : [a,b] \to [\mathbf{c}, \mathbf{d} - \mathbf{a}]$ is a nondecreasing continuous map, then*

$$\frac{\int_a^b f(\mathbf{X}(t))\, dH(t)}{\int_a^b dH(t)} - f\left(\frac{\int_a^b \mathbf{X}(t)\, dH(t)}{\int_a^b dH(t)}\right) \leq \frac{\int_a^b f(\mathbf{a} + \mathbf{X}(t))\, dH(t)}{\int_a^b dH(t)} - f\left(\frac{\int_a^b (\mathbf{a} + \mathbf{X}(t))\, dH(t)}{\int_a^b dH(t)}\right).$$

*Proof.* Using (4.30) for $\mathbf{h_3} = \mathbf{a} = \text{constant} \in \mathbb{R}^k$, we have that $\mathbf{x} \mapsto f(\mathbf{a}+\mathbf{x}) - f(\mathbf{x})$ is a function with nondecreasing increments, so from Theorem 4.3, we obtain Theorem 4.11. □

**Remark 4.1** For $k=1$, Theorem 4.11 gives us a result from [63].

**Corollary 4.2** *(a) Let* $\mathbf{X}$ *satisfy the assumptions of Theorem* 4.10*. Then the inequalities*

$$\begin{aligned} 0 &\le \left(\int_a^b dH(t)\right)^{k-1} \int_a^b \prod_{i=1}^k X_i(t)\,dH(t) - \prod_{i=1}^k \int_a^b X_i(t)\,dH(t) \\ &\le \left(\int_a^b dH(t)\right)^{k-1} \int_a^b \prod_{i=1}^k (2d_i - X_i(t))\,dH(t) - \prod_{i=1}^k \int_a^b (2d_i - X_i(t))\,dH(t) \end{aligned}$$

*hold.*

*(b) If* $\mathbf{X}$ *satisfies the assumptions of Theorem* 4.11*, then the inequalities*

$$\begin{aligned} 0 &\le \left(\int_a^b dH(t)\right)^{k-1} \int_a^b \prod_{i=1}^k X_i(t)\,dH(t) - \prod_{i=1}^k \int_a^b X_i(t)\,dH(t) \\ &\le \left(\int_a^b dH(t)\right)^{k-1} \int_a^b \prod_{i=1}^k (a_i + X_i(t))\,dH(t) - \prod_{i=1}^k \int_a^b (a_i + X_i(t))\,dH(t) \end{aligned}$$

*hold, where all components of* $\mathbf{X}$ *are nonnegative.*

*Proof.* The function $f(\mathbf{x}) = x_1 \cdots x_k$ is a function with nondecreasing increments of orders two and three for $\mathbf{0} \le \mathbf{x} \in \mathbb{R}^k$. So, using Theorems 4.3, 4.10 and 4.11, we obtain Corollary 4.2. □

### 4.4.2 Generalizations of Burkill-Mirsky-Pečarić's Result

In the current subsection, we consider a sequence of $k$-tuples $(\mathbf{X_1}, \ldots, \mathbf{X_n})$ which is monotone in means, [29].

**Theorem 4.12** *Let* $(\mathbf{X_1}, \ldots, \mathbf{X_n}) \in [\mathbf{0}, \mathbf{d}]^n$, $(\mathbf{d} > \mathbf{0})$ *be nondecreasing or nonincreasing in means with respect to positive weights* $w_i$ *for* $i \in \{1, \ldots, n\}$*. If* $f$ *is a continuous function with nondecreasing increments of order three on* $\mathbf{J} = [\mathbf{0}, 2\mathbf{d}]$*, then the inequality*

$$\begin{aligned} &\frac{1}{W_n}\sum_{i=1}^n w_i f(\mathbf{X_i}) - f\left(\frac{1}{W_n}\sum_{i=1}^n w_i \mathbf{X_i}\right) \\ &\qquad \le \frac{1}{W_n}\sum_{i=1}^n w_i f(2\mathbf{d} - \mathbf{X_i}) - f\left(\frac{1}{W_n}\sum_{i=1}^n w_i (2\mathbf{d} - \mathbf{X_i})\right) \end{aligned}$$

*holds.*

*Proof.* By following the proof of Theorem 4.10, we obtain Theorem 4.12 by simply replacing "Theorem 4.3" by "Theorem 4.5". □

**Theorem 4.13** *Let* $(\mathbf{X_1}, \ldots, \mathbf{X_n}) \in [\mathbf{c}, \mathbf{d} - \mathbf{a}]^n$, $(\mathbf{0} < \mathbf{a} < \mathbf{d} - \mathbf{c})$ *be nondecreasing or nonincreasing in means with respect to positive weights* $w_i$ *for* $i \in \{1, \ldots, n\}$. *If* $f$ *is a continuous function with nondecreasing increments of order three on* $\mathbf{J} = [\mathbf{c}, \mathbf{d}]$, *then*

$$\frac{1}{W_n}\sum_{i=1}^n w_i f(\mathbf{X_i}) - f\left(\frac{1}{W_n}\sum_{i=1}^n w_i \mathbf{X_i}\right) \leq \frac{1}{W_n}\sum_{i=1}^n w_i f(\mathbf{a} + \mathbf{X_i}) - f\left(\frac{1}{W_n}\sum_{i=1}^n w_i (\mathbf{a} + \mathbf{X_i})\right).$$

*Proof.* By following the proof of Theorem 4.11, we obtain Theorem 4.13 by simply replacing "Theorem 4.3" by "Theorem 4.5". □

**Corollary 4.3** $(a)$ *Let* $\mathbf{X}$ *satisfy the assumptions of Theorem* 4.12. *Then the inequalities*

$$0 \leq W_n^{k-1}\sum_{i=1}^n w_i^k \left(\prod_{j=1}^k x_{ij}\right) - \prod_{j=1}^k \left(\sum_{i=1}^n w_i x_{ij}\right) \leq W_n^{k-1}\sum_{i=1}^n w_i^k \left(\prod_{j=1}^k (2d_j - x_{ij})\right) - \prod_{j=1}^k \left(\sum_{i=1}^n w_i (2d_j - x_{ij})\right)$$

*hold.*

$(b)$ *If* $\mathbf{X}$ *satisfies the assumptions of Theorem* 4.13. *Then the inequalities*

$$0 \leq W_n^{k-1}\sum_{i=1}^n w_i^k \left(\prod_{j=1}^k x_{ij}\right) - \prod_{j=1}^k \left(\sum_{i=1}^n w_i x_{ij}\right) \leq W_n^{k-1}\sum_{i=1}^n w_i^k \left(\prod_{j=1}^k (a_j + x_{ij})\right) - \prod_{j=1}^k \left(\sum_{i=1}^n w_i (a_j + x_{ij})\right)$$

*hold, where all components of* $\mathbf{X}$ *are nonnegative.*

*Proof.* We again consider the function $f(\mathbf{x}) = x_1 \cdots x_k$ which is a function with nondecreasing increments of orders two and three for $\mathbf{x} \geq 0$. So, using Theorems 4.5, 4.12 and 4.13, we obtain Corollary 4.3. □

# Chapter 5

# Linear Inequalities via Interpolation Polynomials

Let us recall some basic inequalities which are proved in the previous chapters and which are the framework of this book. They are known under the name "the Popoviciu inequalities" since they follow from the Popoviciu's work on $n$-convex functions and general linear inequalities in the forties of the twentieth century ([81, 82, 83, 85]).

**Theorem 5.1** *Let $n \geq 2$. The inequality*

$$\sum_{i=1}^{m} p_i f(x_i) \geq 0 \tag{5.1}$$

*holds for all $n$-convex functions $f : [a,b] \to \mathbb{R}$ if and only if the $m$-tuples $\mathbf{x} \in [a,b]^m$, $\mathbf{p} \in \mathbb{R}^m$ satisfy*

$$\sum_{i=1}^{m} p_i x_i^k = 0, \quad \text{for all } k = 0,1,\ldots,n-1, \tag{5.2}$$

$$\sum_{i=1}^{m} p_i (x_i - t)_+^{n-1} \geq 0, \quad \text{for every } t \in [a,b]. \tag{5.3}$$

The integral analogue is given in the next theorem.

**Theorem 5.2** *Let $n \geq 2$, $p : [\alpha,\beta] \to \mathbb{R}$ and $g : [\alpha,\beta] \to [a,b]$. Then, the inequality*

$$\int_{\alpha}^{\beta} p(x) f(g(x))\, dx \geq 0 \tag{5.4}$$

*holds for all n-convex functions* $f : [a,b] \to \mathbb{R}$ *if and only if*

$$\int_{\alpha}^{\beta} p(x)g(x)^k \, dx = 0, \quad \text{for all } k = 0,1,\ldots,n-1, \tag{5.5}$$

$$\int_{\alpha}^{\beta} p(x)\,(g(x)-t)_+^{n-1} \, dx \geq 0, \quad \text{for every } t \in [a,b]. \tag{5.6}$$

**Remark 5.1** As we discussed in Proposition 2.6 from Chapter 2, if $n = 2$, then conditions (5.2) and (5.3), i.e.

$$\sum_{k=1}^{m} p_k = 0, \qquad \sum_{k=1}^{m} p_k x_k = 0$$

and

$$\sum_{k=1}^{m} p_k (x_k - x_i)_+ \geq 0, \qquad i = 1,\ldots,m-1$$

can be replaced by

$$\sum_{k=1}^{m} p_k = 0 \quad \text{and} \quad \sum_{k=1}^{m} p_k |x_k - x_i| \geq 0 \quad \text{for} \quad i = 1,\ldots,m,$$

and vice versa.

As it is shown in Chapter 2, these results can be reached using the Taylor formula. Following that idea we use other interpolation formulae and identities to obtain Popoviciu type inequalities.

## 5.1 Inequalities via Extension of the Montgomery Identity

In this section we use extension of the Montgomery identity to obtain inequalities of type (5.1) and (5.4) for $n$-convex functions. The mentioned extension of the Montgomery identity via Taylor's formula was obtained in paper [2] and we give it in the following text.

**Theorem 5.3** *Let* $n \in \mathbb{N}$, $f : I \to \mathbb{R}$ *be such that* $f^{(n-1)}$ *is absolutely continuous,* $I \subset \mathbb{R}$ *an open interval,* $a,b \in I$, $a < b$. *Then the following identity holds*

$$\begin{aligned} f(x) &= \frac{1}{b-a}\int_a^b f(t)\,dt + \sum_{k=0}^{n-2} \frac{f^{(k+1)}(a)}{k!\,(k+2)} \frac{(x-a)^{k+2}}{b-a} \\ &\quad - \sum_{k=0}^{n-2} \frac{f^{(k+1)}(b)}{k!\,(k+2)} \frac{(x-b)^{k+2}}{b-a} + \frac{1}{(n-1)!}\int_a^b T_n(x,s)\, f^{(n)}(s)\,ds, \end{aligned} \tag{5.7}$$

*where*

$$T_n(x,s) = \begin{cases} -\dfrac{(x-s)^n}{n(b-a)} + \dfrac{x-a}{b-a}(x-s)^{n-1}, & a \le s \le x, \\[2ex] -\dfrac{(x-s)^n}{n(b-a)} + \dfrac{x-b}{b-a}(x-s)^{n-1}, & x < s \le b. \end{cases} \tag{5.8}$$

In case $n = 1$ the sum $\sum_{k=0}^{n-2} \cdots$ is empty, so identity (5.7) reduces to the well-known Montgomery identity

$$f(x) = \frac{1}{b-a} \int_a^b f(t)\,dt + \int_a^b P(x,s) f'(s)\,ds,$$

where $P(x,s)$ is the Peano kernel, defined by

$$P(x,s) = \begin{cases} \dfrac{s-a}{b-a}, & a \le s \le x, \\[1ex] \dfrac{s-b}{b-a}, & x < s \le b. \end{cases}$$

In fact, the previous theorem is a particular case of a more general extension of the Montgomery identity which is proved in the following text. The proof can be find in [2].

Let us suppose $w : [a,b] \to [0,\infty\rangle$ is some probability density function, i.e. an integrable function satisfying $\int_a^b w(t)dt = 1$ and $W(t) = \int_a^t w(x)dx$ for $t \in [a,b]$, $W(t) = 0$ for $t < a$ and $W(t) = 1$ for $t > b$. Let us by $P_w$ denote the weighted Peano kernel

$$P_w(x,t) = \begin{cases} W(t), & a \le t \le x, \\ W(t) - 1, & x < t \le b. \end{cases}$$

**Theorem 5.4** *Let $n \in \mathbb{N}, n \ge 2$, $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$, $w : [a,b] \to [0,\infty\rangle$ be a probability density function. Then the following identity holds*

$$\begin{aligned} f(x) &= \int_a^b w(t) f(t)dt + \sum_{k=0}^{n-2} \frac{f^{(k+1)}(a)}{(k+1)!} \int_a^x w(s)\left((x-a)^{k+1} - (s-a)^{k+1}\right) ds \\ &- \sum_{k=0}^{n-2} \frac{f^{(k+1)}(b)}{(k+1)!} \int_x^b w(s)\left((x-b)^{k+1} - (s-b)^{k+1}\right) ds \\ &+ \frac{1}{(n-1)!} \int_a^b T_{w,n}(x,s) f^{(n)}(s)ds, \end{aligned} \tag{5.9}$$

*where*

$$T_{w,n}(x,s) = \begin{cases} \displaystyle\int_x^s w(u)(u-s)^{n-1}du + W(x)(x-s)^{n-1}, & a \le s \le x, \\ \displaystyle\int_x^s w(u)(u-s)^{n-1}du + (W(x)-1)(x-s)^{n-1}, & x < s \le b. \end{cases}$$

*Proof.* If we apply the Taylor formula for $f'(t)$ and replace $n$ with $n-1$ we have

$$\begin{aligned} f'(t) &= \sum_{k=0}^{n-2} \frac{f^{(k+1)}(a)}{k!}(t-a)^k + \int_a^t f^{(n)}(s)\frac{(t-s)^{n-2}}{(n-2)!}ds \\ &= \sum_{k=0}^{n-2} \frac{f^{(k+1)}(b)}{k!}(t-b)^k - \int_t^b f^{(n)}(s)\frac{(t-s)^{n-2}}{(n-2)!}ds. \end{aligned}$$

By putting these two formulae in the weighted Montgomery identity $f(x) = \int_a^b w(t)f(t)dt + \int_a^b P_w(x,t)f'(t)dt$, we obtain

$$\begin{aligned} f(t) &= \int_a^b w(t)f(t)dt \\ &+ \sum_{k=0}^{n-2} \frac{f^{(k+1)}(a)}{k!} \int_a^x (t-a)^k W(t)dt + \sum_{k=0}^{n-2} \frac{f^{(k+1)}(b)}{k!} \int_x^b (t-b)^k (W(t)-1)dt \\ &+ \int_a^x W(t)\left(\int_a^t f^{(n)}(s)\frac{(t-s)^{n-2}}{(n-2)!}ds\right)dt \\ &- \int_x^b (W(t)-1)\left(\int_t^b f^{(n)}(s)\frac{(t-s)^{n-2}}{(n-2)!}ds\right)dt. \end{aligned}$$

Now, we have

$$\begin{aligned} \int_a^x (t-a)^k W(t)dt &= \int_a^x (t-a)^k \left(\int_a^t w(s)ds\right)dt \\ &= \int_a^x w(s)\left(\int_s^x (t-a)^k dt\right)ds \\ &= \frac{1}{k+1}\int_a^x w(s)\Big((x-a)^{k+1} - (s-a)^{k+1}\Big)ds, \\ \int_x^b (t-b)^k (W(t)-1)dt &= \frac{1}{k+1}\int_x^b w(s)\Big((x-b)^{k+1} - (s-b)^{k+1}\Big)ds. \end{aligned}$$

Also, we obtain

$$\int_a^x W(t)\left(\int_a^t f^{(n)}(s)(t-s)^{n-2}ds\right)dt = \int_a^x f^{(n)}(s)\left(\int_s^x W(t)(t-s)^{n-2}dt\right)ds$$

with

$$\begin{aligned} &\int_s^x W(t)(t-s)^{n-2}dt = \int_s^x \left(\int_a^t w(u)du\right)(t-s)^{n-2}dt \\ &= \int_a^s w(u)\left(\int_s^x (t-s)^{n-2}dt\right)du + \int_s^x w(u)\left(\int_u^x (t-s)^{n-2}dt\right)du \end{aligned}$$

$$
\begin{aligned}
&= \int_a^s w(u)\frac{(x-s)^{n-1}}{n-1}du + \int_s^x w(u)\frac{(x-s)^{n-1}-(u-s)^{n-1}}{n-1}du \\
&= \frac{(x-s)^{n-1}}{n-1}W(x) - \int_s^x w(u)\frac{(u-s)^{n-1}}{n-1}du.
\end{aligned}
$$

By similar calculation we get

$$
\begin{aligned}
&-\int_x^b (W(t)-1)\left(\int_t^b f^{(n)}(s)(t-s)^{n-2}ds\right)dt \\
&\qquad = \int_x^b f^{(n)}(s)\left(\int_x^s (1-W(t))(t-s)^{n-2}dt\right)ds
\end{aligned}
$$

and

$$
\int_x^s (1-W(t))(t-s)^{n-2}dt = (W(x)-1)\frac{(x-s)^{n-1}}{n-1} + \int_x^s w(u)\frac{(u-s)^{n-1}}{n-1}du.
$$

Now, the reminder in the weighted Taylor formula becomes

$$
\begin{aligned}
&\frac{1}{(n-1)!}\left[\int_a^b f^{(n)}(s)\left(\int_x^s w(u)(u-s)^{n-1}du\right)ds\right. \\
&\quad \left. + W(x)\int_a^x f^{(n)}(s)(x-s)^{n-1}ds + (W(x)-1)\int_x^b f^{(n)}(s)(x-s)^{n-1}ds\right].
\end{aligned}
$$

□

In the particular case when $w(t) = \frac{1}{b-a}$, $t \in [a,b]$, then identity (5.9) reduces to (5.7).

Now we recall the definition of the Green function $G$ which we use in some of our results. The function $G : [a,b] \times [a,b]$ is defined by

$$
G(t,s) = \begin{cases} \dfrac{(t-b)(s-a)}{b-a} & \text{for } a \le s \le t, \\ \dfrac{(s-b)(t-a)}{b-a} & \text{for } t \le s \le b. \end{cases} \tag{5.10}
$$

The function $G$ is convex and continuous with respect to both $s$ and $t$.

For any function $f : [a,b] \to \mathbb{R}$, $f \in C^2[a,b]$, it can be easily shown by using integration by parts that the following is valid

$$
f(x) = \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b) + \int_a^b G(x,s)f''(s)ds, \tag{5.11}
$$

where the function $G$ is defined as above in (5.10) (see also [98]).

### 5.1.1 Inequalities Obtained by the Montgomery Identity

In this section we give identities for $\sum_{i=1}^m p_i f(x_i)$ and $\int_\alpha^\beta p(x)f(g(x))dx$ using the extension of the Montgomery identity, then obtain Popoviciu type inequalities and describe some consequences of it. These results are published in [26].

**Theorem 5.5** *Suppose all the assumptions from Theorem 5.3 hold and let $T_n$ be given by (5.8). Furthermore, let $m \in \mathbb{N}$, $x_i \in [a,b]$ and $p_i \in \mathbb{R}$ for $i \in \{1,2,\ldots,m\}$ be such that $\sum_{i=1}^{m} p_i = 0$. Then*

$$\begin{aligned}
\sum_{i=1}^{m} p_i f(x_i) &= \frac{1}{b-a}\left[\sum_{k=0}^{n-2} \frac{1}{k!\,(k+2)} f^{(k+1)}(a) \sum_{i=1}^{m} p_i (x_i-a)^{k+2}\right. \\
&\quad \left. - \sum_{k=0}^{n-2} \frac{1}{k!\,(k+2)} f^{(k+1)}(b) \sum_{i=1}^{m} p_i (x_i-b)^{k+2}\right] \\
&\quad + \frac{1}{(n-1)!} \int_a^b \left(\sum_{i=1}^{m} p_i T_n(x_i,s)\right) f^{(n)}(s)\, ds. \qquad (5.12)
\end{aligned}$$

*Proof.* Putting in the extension of the Montgomery identity (5.7) $x_i$, $i = 1,\ldots m$, multiplying with $p_i$ and summing all the identities we obtain

$$\begin{aligned}
\sum_{i=1}^{m} p_i f(x_i) &= \frac{1}{b-a} \int_a^b f(t)\, dt \sum_{i=1}^{m} p_i \\
&\quad + \sum_{i=1}^{m} p_i \left(\sum_{k=0}^{n-2} \frac{f^{(k+1)}(a)}{k!\,(k+2)} \frac{(x_i-a)^{k+2}}{b-a} - \sum_{k=0}^{n-2} \frac{f^{(k+1)}(b)}{k!\,(k+2)} \frac{(x_i-b)^{k+2}}{b-a}\right) \\
&\quad + \frac{1}{(n-1)!} \sum_{i=1}^{m} p_i \int_a^b T_n(x_i,s) f^{(n)}(s)\, ds.
\end{aligned}$$

By simplifying this expressions we obtain (5.12). □

We may state its integral version as follows.

**Theorem 5.6** *Let $g : [\alpha,\beta] \to [a,b]$ and $p : [\alpha,\beta] \to \mathbb{R}$ be integrable functions such that $\int_\alpha^\beta p(x)dx = 0$. Let $n \in \mathbb{N}$, $I \subset \mathbb{R}$ be an open interval, $a,b \in I$, $a < b$, $T_n$ be given by (5.8) and $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous. Then*

$$\begin{aligned}
\int_\alpha^\beta p(x) f(g(x))\, dx &= \frac{1}{b-a}\left[\sum_{k=0}^{n-2} \frac{1}{k!\,(k+2)} f^{(k+1)}(a) \int_\alpha^\beta p(x)(g(x)-a)^{k+2} dx\right. \\
&\quad \left. - \sum_{k=0}^{n-2} \frac{1}{k!\,(k+2)} f^{(k+1)}(b) \int_\alpha^\beta p(x)(g(x)-b)^{k+2} dx\right] \\
&\quad + \frac{1}{(n-1)!} \int_a^b \left(\int_\alpha^\beta p(x) T_n(g(x),s)\, dx\right) f^{(n)}(s)\, ds. \qquad (5.13)
\end{aligned}$$

*Proof.* Putting in the extension of the Montgomery identity (5.7) $x = g(x)$, multiplying with $p(x)$, integrating over $[\alpha,\beta]$, and using some transformations and the Fubini theorem we get the required identity. □

Now we state inequalities derived from the obtained identities.

**Theorem 5.7** *Let all the assumptions of Theorem* 5.5 *hold with the additional condition*

$$\sum_{i=1}^{m} p_i T_n(x_i,s) \geq 0, \quad \text{for all } s \in [a,b]. \tag{5.14}$$

*Then, for every n-convex function $f : I \to \mathbb{R}$ the following inequality holds*

$$\begin{aligned}\sum_{i=1}^{m} p_i f(x_i) \geq \frac{1}{b-a} &\left[\sum_{k=0}^{n-2} \frac{1}{k!\,(k+2)} f^{(k+1)}(a) \sum_{i=1}^{m} p_i (x_i-a)^{k+2}\right. \\ &\left. - \sum_{k=0}^{n-2} \frac{1}{k!\,(k+2)} f^{(k+1)}(b) \sum_{i=1}^{m} p_i (x_i-b)^{k+2}\right]. \end{aligned} \tag{5.15}$$

*If the inequality in* (5.14) *is reversed, then* (5.15) *holds with the reversed sign of inequality.*

*Proof.* The function $f$ is $n$-convex, so without lossing the generality we can assume that $f$ is $n$-times differentiable and $f^{(n)} \geq 0$. Using this fact and (5.14) in (5.12) we easily arrive at our required result. □

Now we state an important consequence.

**Theorem 5.8** *Suppose all the assumptions from Theorem* 5.5 *hold. Additionally, let $j \in \mathbb{N}$, $2 \leq j \leq n$ and let $\mathbf{x} = (x_1,\ldots,x_m) \in [a,b]^m$, $\mathbf{p} = (p_1,\ldots,p_m) \in \mathbb{R}^m$ satisfy* (5.2) *and* (5.3) *with n replaced by j. If f is n-convex and $n-j$ is even, then*

$$\begin{aligned}\sum_{i=1}^{m} p_i f(x_i) \geq \frac{1}{b-a} &\left[\sum_{k=j-2}^{n-2} \frac{1}{k!\,(k+2)} f^{(k+1)}(a) \sum_{i=1}^{m} p_i (x_i-a)^{k+2}\right. \\ &\left. - \sum_{k=j-2}^{n-2} \frac{1}{k!\,(k+2)} f^{(k+1)}(b) \sum_{i=1}^{m} p_i (x_i-b)^{k+2}\right]. \end{aligned} \tag{5.16}$$

*Proof.* Let $s \in [a,b]$ be fixed. Notice that

$$(b-a)T_n(x,s) = L_n(x) + (b-a)(x-s)_+^{n-1}, \tag{5.17}$$

where

$$L_n(x) = -\frac{(x-s)^n}{n} + (x-b)(x-s)^{n-1}.$$

Using the Pochhammer symbol $(y)_k = y(y-1)\cdots(y-k+1)$ we have

$$\begin{aligned} L_n^{(j)}(x) &= -(n-1)_{j-1}(x-s)^{n-j} + \binom{j}{0}(x-b)(n-1)_j(x-s)^{n-j-1} \\ &+ \binom{j}{1}(n-1)_{j-1}(x-s)^{n-j} \\ &= (n-1)_{j-1}(x-s)^{n-j-1}\left[(j-1)(x-s) + (n-j)(x-b)\right]. \end{aligned} \tag{5.18}$$

Therefore, (5.17) and (5.18) for $s < x \leq b$ yield

$$\begin{aligned}
\frac{d^j}{dx^j} T_n(x,s) &= \frac{1}{b-a} L_n^{(j)}(x) + (n-1)_j (x-s)^{n-j-1} \\
&= \frac{(n-1)_{j-1}}{b-a} (x-s)^{n-j-1} \left[ (j-1)(x-s) + (n-j)(x-a) \right] \geq 0,
\end{aligned} \tag{5.19}$$

while for $a \leq x < s$ we have

$$\begin{aligned}
(-1)^{n-j} \frac{d^j}{dx^j} T_n(x,s) &= (-1)^{n-j} \frac{1}{b-a} L_n^{(j)}(x) \\
&= \frac{(n-1)_{j-1}}{b-a} (s-x)^{n-j-1} \left[ (j-1)(s-x) + (n-j)(b-x) \right] \geq 0.
\end{aligned} \tag{5.20}$$

From (5.17) it is clear that $x \mapsto \dfrac{d^j}{dx^j} T_n(x,s)$ is continuous for $j \leq n-2$. Hence, if $j \leq n-2$ and $n-j$ is even, from (5.19) and (5.20) we can conclude that the function $x \mapsto T_n(x,s)$ is $j$-convex. Moreover, the conclusion extends to the case $j = n$, i. e. the mapping $x \mapsto T_n(x,s)$ is $n$-convex, since the mapping $x \mapsto \dfrac{d^{n-2}}{dx^{n-2}} T_n(x,s)$ is 2-convex.

Now, by Theorem 5.1, we see that assumption (5.14) is satisfied, so inequality (5.15) holds. Moreover, due to assumption (5.2), $\sum_{i=1}^m p_i P(x_i) = 0$ for every polynomial $P$ of degree $\leq j-1$, so the first $j-2$ terms in the inner sum in (5.15) vanish, i. e., the right-hand side of (5.15) under the assumptions of this theorem is equal to the right-hand side of (5.16). □

**Corollary 5.1** *Suppose all the assumptions from Theorem* 5.5 *hold. Additionally, let* $j \in \mathbb{N}$, $2 \leq j \leq n$, *let* $\mathbf{x} = (x_1, \ldots, x_m) \in [a,b]^m$, $\mathbf{p} = (p_1, \ldots, p_m) \in \mathbb{R}^m$ *satisfy* (5.2) *and* (5.3) *with n replaced by j and denote*

$$\begin{aligned}
H(x) := \frac{1}{b-a} \Bigg[ & \sum_{k=j-2}^{n-2} \frac{1}{k!\,(k+2)} f^{(k+1)}(a)\,(x-a)^{k+2} \\
- & \sum_{k=j-2}^{n-2} \frac{1}{k!\,(k+2)} f^{(k+1)}(b)\,(x-b)^{k+2} \Bigg].
\end{aligned} \tag{5.21}$$

*If H is j-convex on* $[a,b]$ *and* $n-j$ *is even, then*

$$\sum_{i=1}^m p_i f(x_i) \geq 0.$$

*Proof.* Applying Theorem 5.1 we conclude that the right-hand side of (5.16) is non-negative for the $j$-convex function $H$. □

**Remark 5.2** For example, since the functions $x \mapsto (x-a)^{k+2}$ and $x \mapsto (-1)^{k-j}(x-b)^{k+2}$ are $j$-convex on $[a,b]$, the function $H$ given by (5.21) is $j$-convex if $f^{(k+1)}(a) \geq 0$ and $(-1)^{k+1-j} f^{(k+1)}(b) \geq 0$ for $k \in \{j-2, \ldots, n-2\}$.

In the remainder of the section we state integral versions of the previous results, the proofs of which are analogous to the discrete case.

**Theorem 5.9** *Let all the assumptions of Theorem 5.6 hold with the additional condition*

$$\int_{\alpha}^{\beta} p(x)\, T_n(g(x),s)\, dx \geq 0, \quad \text{for all } s \in [a,b]. \tag{5.22}$$

*Then, for every n-convex function $f : I \to \mathbb{R}$ the following inequality holds*

$$\begin{aligned}\int_{\alpha}^{\beta} p(x)\, f(g(x))dx \geq \frac{1}{b-a}\Bigg[&\sum_{k=0}^{n-2}\frac{1}{k!\,(k+2)} f^{(k+1)}(a)\int_{\alpha}^{\beta} p(x)\,(g(x)-a)^{k+2}\,dx \\ &- \sum_{k=0}^{n-2}\frac{1}{k!\,(k+2)} f^{(k+1)}(b)\int_{\alpha}^{\beta} p(x)\,(g(x)-b)^{k+2}\,dx\Bigg].\end{aligned} \tag{5.23}$$

**Theorem 5.10** *Suppose all the assumptions from Theorem 5.6 hold. Additionally, let $j \in \mathbb{N}$, $2 \leq j \leq n$ and let $p : [\alpha,\beta] \to \mathbb{R}$ and $g : [\alpha,\beta] \to [a,b]$ satisfy (5.5) with n replaced by j. If f is n-convex and $n-j$ is even, then*

$$\begin{aligned}\int_{\alpha}^{\beta} p(x) f(g(x))dx \geq \frac{1}{b-a}\Bigg[&\sum_{k=j-2}^{n-2}\frac{1}{k!\,(k+2)} f^{(k+1)}(a)\int_{\alpha}^{\beta} p(x)\,(g(x)-a)^{k+2}\,dx \\ &- \sum_{k=j-2}^{n-2}\frac{1}{k!\,(k+2)} f^{(k+1)}(b)\int_{\alpha}^{\beta} p(x)\,(g(x)-b)^{k+2}\,dx\Bigg].\end{aligned}$$

**Corollary 5.2** *Let $j,n,f,p$ and $g$ be as in Theorem 5.10 and let H be given by (5.21). If H is j-convex and $n-j$ is even, then*

$$\int_{\alpha}^{\beta} p(x) f(g(x))\, dx \geq 0.$$

### 5.1.2 Inequalities for *n*-convex Functions at a Point

In Chapter 2 we gave the definition and some results about *n*-convex functions at a point. Here we improve those results using results from the previous subsection. These results can be found in [26].

Let $T_n^{[a,c]}$ and $T_n^{[c,b]}$ denote the equivalent of (5.8) on these intervals, i. e.,

$$T_n^{[a,c]}(x,s) = \begin{cases} -\dfrac{(x-s)^n}{n(c-a)} + \dfrac{x-a}{c-a}(x-s)^{n-1}, & a \leq s \leq x, \\[2ex] -\dfrac{(x-s)^n}{n(c-a)} + \dfrac{x-c}{c-a}(x-s)^{n-1}, & x < s \leq c, \end{cases} \tag{5.24}$$

$$T_n^{[c,b]}(x,s) = \begin{cases} -\dfrac{(x-s)^n}{n(b-c)} + \dfrac{x-c}{b-c}(x-s)^{n-1}, & c \leq s \leq x, \\[2ex] -\dfrac{(x-s)^n}{n(b-c)} + \dfrac{x-b}{b-c}(x-s)^{n-1}, & x < s \leq b. \end{cases} \tag{5.25}$$

Let $\mathbf{x} \in [a,c]^m$, $\mathbf{p} \in \mathbb{R}^m$, $\mathbf{y} \in [c,b]^l$ and $\mathbf{q} \in \mathbb{R}^l$ and denote

$$\begin{aligned} A_1(f) \;=\; & \sum_{i=1}^{m} p_i f(x_i) - \frac{1}{c-a}\left[\sum_{k=0}^{n-2} \frac{1}{k!\,(k+2)} f^{(k+1)}(a) \sum_{i=1}^{m} p_i (x_i - a)^{k+2} \right. \\ & \left. - \sum_{k=0}^{n-2} \frac{1}{k!\,(k+2)} f^{(k+1)}(c) \sum_{i=1}^{m} p_i (x_i - c)^{k+2} \right], \end{aligned} \tag{5.26}$$

$$\begin{aligned} B_1(f) \;=\; & \sum_{i=1}^{l} q_i f(y_i) - \frac{1}{b-c}\left[\sum_{k=0}^{n-2} \frac{1}{k!\,(k+2)} f^{(k+1)}(c) \sum_{i=1}^{l} q_i (y_i - c)^{k+2} \right. \\ & \left. - \sum_{k=0}^{n-2} \frac{1}{k!\,(k+2)} f^{(k+1)}(b) \sum_{i=1}^{l} q_i (y_i - b)^{k+2} \right]. \end{aligned} \tag{5.27}$$

Notice that, using the newly introduced functionals $A$ and $B$, identity (5.12) applied to the intervals $[a,c]$ and $[c,b]$ can be written as

$$A_1(f) = \frac{1}{(n-1)!} \int_a^c \left( \sum_{i=1}^{m} p_i T_n^{[a,c]}(x_i,s) \right) f^{(n)}(s)\, ds, \tag{5.28}$$

$$B_1(f) = \frac{1}{(n-1)!} \int_c^b \left( \sum_{i=1}^{l} q_i T_n^{[c,b]}(y_i,s) \right) f^{(n)}(s)\, ds. \tag{5.29}$$

**Theorem 5.11** *Let $\mathbf{x} \in [a,c]^m$, $\mathbf{p} \in \mathbb{R}^m$, $\mathbf{y} \in [c,b]^l$ and $\mathbf{q} \in \mathbb{R}^l$ be such that*

$$\sum_{i=1}^{m} p_i T_n^{[a,c]}(x_i,s) \geq 0, \quad \text{for every } s \in [a,c], \tag{5.30}$$

$$\sum_{i=1}^{l} q_i T_n^{[c,b]}(y_i,s) \geq 0, \quad \text{for every } s \in [c,b], \tag{5.31}$$

$$\int_a^c \left( \sum_{i=1}^{m} p_i T_n^{[a,c]}(x_i,s) \right) f^{(n)}(s)\, ds = \int_c^b \left( \sum_{i=1}^{l} q_i T_n^{[c,b]}(y_i,s) \right) f^{(n)}(s)\, ds, \tag{5.32}$$

*where $T_n^{[a,c]}$, $T_n^{[c,b]}$, $A_1$ and $B_1$ are given by (5.24), (5.25), (5.26) and (5.27) respectively. If $f : [a,b] \to \mathbb{R}$ is $(n+1)$-convex at point $c$, then*

$$A_1(f) \leq B_1(f). \tag{5.33}$$

*If the inequalities in (5.30) and (5.31) are reversed, then (5.33) holds with the reversed sign of inequality.*

*Proof.* Let $F = f - \frac{K}{n!}e_n$ be as in the definition of a function $n$-convex at a point, i. e., the function $F$ is $n$-concave on $[a,c]$ and $n$-convex on $[c,b]$. Applying Theorem 5.7 to $F$ on the interval $[a,c]$ we have

$$0 \geq A_1(F) = A_1(f) - \frac{K}{n!}A_1(e_n) \tag{5.34}$$

and applying Theorem 5.7 to $F$ on the interval $[c,b]$ we have

$$0 \leq B_1(F) = B_1(f) - \frac{K}{n!}B_1(e_n). \tag{5.35}$$

Identities (5.28) and (5.29) applied to the function $e_n$ yield

$$\begin{aligned} A_1(e_n) &= \frac{1}{(n-1)!}\int_a^c \left(\sum_{i=1}^m p_i T_n^{[a,c]}(x_i,s)\right) ds, \\ B_1(e_n) &= \frac{1}{(n-1)!}\int_c^b \left(\sum_{i=1}^l q_i T_n^{[c,b]}(y_i,s)\right) ds. \end{aligned}$$

Therefore, assumption (5.32) is equivalent to $A_1(e_n) = B_1(e_n)$. Now, from (5.34) and (5.35) we obtain the stated inequality. □

**Remark 5.3** In the proof of Theorem 5.11 we have, actually, shown that

$$A_1(f) \leq \frac{K}{n!}A_1(e_n) = \frac{K}{n!}B_1(e_n) \leq B_1(f).$$

In fact, inequality (5.33) still holds if we replace assumption (5.32) with the weaker assumption that $K(B_1(e_n) - A_1(e_n)) \geq 0$.

**Corollary 5.3** *Let $j_1, j_2, n \in \mathbb{N}$, $\leq j_1, j_2 \leq n$, let $f : [a,b] \to \mathbb{R}$ be $(n+1)$-convex at point $c$, let $m$-tuples $\mathbf{x} \in [a,c]^m$ and $\mathbf{p} \in \mathbb{R}^m$ satisfy* (5.2) *and* (5.3) *with $n$ replaced by $j_1$, let $l$-tuples $\mathbf{y} \in [c,b]^l$ and $\mathbf{q} \in \mathbb{R}^l$ satisfy*

$$\begin{cases} \sum_{i=1}^{l} q_i y_i^k = 0 & \textit{for all } k = 0,1,\ldots,j_2-1 \\ \sum_{i=1}^{l} q_i (y_i - t)_+^{j_2-1} \geq 0 & \textit{for every } t \in [y_{(1)}, y_{(l-n+1)}] \end{cases}$$

*and let* (5.32) *hold. If $n - j_1$ and $n - j_2$ are even, then*

$$A_1(f) \leq B_1(f).$$

*Proof.* See the proof of Theorem 5.8. □

### 5.1.3 Bounds for Remainders and Functionals

By using the aforementioned results we get bounds for the remainders appearing in the identities obtained in this section, [26]. Let $f,h:[a,b]\to\mathbb{R}$ be two Lebesgue integrable functions. We consider the Čebyšev functional as follows:

$$T(f,h)=\frac{1}{b-a}\int_a^b f(x)h(x)\,dx-\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b h(x)\,dx\right). \tag{5.36}$$

As usual the symbol $L_p[a,b]$ $(1\le p<\infty)$ denotes the space of $p$-power integrable functions on the interval $[a,b]$ equipped with the norm

$$\|f\|_p=\left(\int_a^b |f(t)|^p\,dt\right)^{\frac{1}{p}}<\infty$$

and $L_\infty[a,b]$ denotes the space of essentially bounded functions on $[a,b]$ with the norm

$$\|f\|_\infty=\operatorname{ess}\sup_{t\in[a,b]}|f(t)|.$$

The following results can be found in [11].

**Theorem 5.12** *Let $f:[a,b]\to\mathbb{R}$ be a Lebesgue integrable function and $h:[a,b]\to\mathbb{R}$ be an absolutely continuous function with $(\cdot-a)(b-\cdot)[h']^2\in L[a,b]$. Then we have the inequality*

$$|T(f,h)|\le\frac{1}{\sqrt{2}}\left(\frac{1}{b-a}|T(f,f)|\int_a^b (x-a)(b-x)[h'(x)]^2\,dx\right)^{\frac{1}{2}}. \tag{5.37}$$

*The constant $\frac{1}{\sqrt{2}}$ in (5.37) is the best possible.*

**Theorem 5.13** *Let $h:[a,b]\to\mathbb{R}$ be a monotonic nondecreasing function and let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function such that $f'\in L_\infty[a,b]$. Then we have the inequality*

$$|T(f,h)|\le\frac{1}{2(b-a)}\|f'\|_\infty\int_a^b (x-a)(b-x)dh(x). \tag{5.38}$$

*The constant $\frac{1}{2}$ in (5.38) is the best possible.*

For $m$-tuples $\mathbf{p}=(p_1,\dots,p_m)$, $\mathbf{x}=(x_1,\dots,x_m)$ with $x_i\in[a,b]$, $p_i\in\mathbb{R}$ $(i=1,\dots,m)$ such that $\sum_{i=0}^m p_i=0$ and the function $T_n$ defined as in (5.8), denote

$$\delta_1(s)=\sum_{i=1}^m p_iT_n(x_i,s),\quad \text{for } s\in\langle a,b]. \tag{5.39}$$

Similarly for functions $g:[\alpha,\beta]\to[a,b]$ and $p:[\alpha,\beta]\to\mathbb{R}$ such that $\int_\alpha^\beta p(x)dx=0$, denote

$$\Delta_1(s)=\int_\alpha^\beta p(x)\,T_n(g(x),s)\,dx,\qquad \text{for } s\in[a,b]. \tag{5.40}$$

Now, we are ready to state the main results of this section.

**Theorem 5.14** *Let $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ is an absolutely continuous function with $(\cdot - a)(b - \cdot)[f^{(n+1)}]^2 \in L[a,b]$, $x_i \in [a,b]$ and $p_i \in \mathbb{R}$ ($i \in \{1,\dots,m\}$) such that $\sum_{i=0}^{m} p_i = 0$ and let the functions $T_n$, $T$ and $\delta_1$ be defined in (5.8), (5.36) and (5.39) respectively. Then the remainder $R_n^1(f;a,b)$ given by the following identity*

$$\begin{aligned}\sum_{i=1}^{m} p_i f(x_i) &= \frac{1}{b-a}\left[\sum_{k=0}^{n-2}\frac{1}{k!\,(k+2)}f^{(k+1)}(a)\sum_{i=1}^{m}p_i(x_i-a)^{k+2}\right.\\ &\quad\left.-\sum_{k=0}^{n-2}\frac{1}{k!\,(k+2)}f^{(k+1)}(b)\sum_{i=1}^{m}p_i(x_i-b)^{k+2}\right]\\ &\quad+\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(n-1)!(b-a)}\int_a^b \delta_1(s)ds + R_n^1(f;a,b) \end{aligned} \tag{5.41}$$

*satisfies the estimation*

$$|R_n^1(f;a,b)| \le \frac{1}{(n-1)!}\left(\frac{b-a}{2}T(\delta_1,\delta_1)\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2 ds\right)^{1/2}.$$

*Proof.* By identity (5.12) we have

$$\begin{aligned}\sum_{i=1}^{m} p_i f(x_i) &- \frac{1}{b-a}\left[\sum_{k=0}^{n-2}\frac{1}{k!\,(k+2)}f^{(k+1)}(a)\sum_{i=1}^{m}p_i(x_i-a)^{k+2}\right.\\ &\quad\left.-\sum_{k=0}^{n-2}\frac{1}{k!\,(k+2)}f^{(k+1)}(b)\sum_{i=1}^{m}p_i(x_i-b)^{k+2}\right]\\ &= \frac{1}{(n-1)!}\int_a^b\left(\sum_{i=1}^{m}p_iT_n(x_i,s)\right)f^{(n)}(s)ds\\ &= \frac{1}{(n-1)!}\int_a^b \delta_1(s)f^{(n)}(s)ds\\ &= \frac{1}{(n-1)!(b-a)}\int_a^b f^{(n)}(s)ds\int_a^b \delta_1(s)ds + R_n^1(f;a,b)\\ &= \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(n-1)!(b-a)}\int_a^b \delta_1(s)ds + R_n^1(f;a,b), \end{aligned}$$

where

$$R_n^1(f;a,b) = \frac{1}{(n-1)!}\left(\int_a^b \delta_1(s)f^{(n)}(s)ds - \frac{1}{b-a}\int_a^b f^{(n)}(s)ds\int_a^b \delta_1(s)ds\right).$$

If we apply Theorem 5.12 for $f \to \delta_1$ and $h \to f^{(n)}$, then we obtain

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b \delta_1(s)f^{(n)}(s)ds - \left(\frac{1}{b-a}\int_a^b \delta_1(s)ds\right)\left(\frac{1}{b-a}\int_a^b f^{(n)}(s)ds\right)\right|\\ &\qquad\le \frac{1}{\sqrt{2}}\left(\frac{1}{b-a}T(\delta_1,\delta_1)\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2ds\right)^{1/2}\end{aligned}$$

from where desired estimation follows. □

Here we state the integral version of the previous theorem.

**Theorem 5.15** *Let $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ is an absolutely continuous function with $(\cdot - a)(b - \cdot)[f^{(n+1)}]^2 \in L[a,b]$, let $g : [\alpha,\beta] \to [a,b]$ and $p : [\alpha,\beta] \to \mathbb{R}$ be functions such that $\int_\alpha^\beta p(x)dx = 0$ and let the functions $T_n$, $T$ and $\Delta_1$ be defined in* (5.8), (5.36) *and* (5.40) *respectively. Then the remainder $R_n^2(f;a,b)$ given by the following identity*

$$\begin{aligned}\int_\alpha^\beta p(x) f(g(x))dx &= \frac{1}{b-a}\left[\sum_{k=0}^{n-2} \frac{1}{k!\,(k+2)} f^{(k+1)}(a) \int_\alpha^\beta p(x)\,(g(x)-a)^{k+2}\,dx\right. \\ &\quad \left. - \sum_{k=0}^{n-2} \frac{1}{k!\,(k+2)} f^{(k+1)}(b) \int_\alpha^\beta p(x)\,(g(x)-b)^{k+2}\,dx\right] \\ &\quad + \frac{\left[f^{(n-1)}(b) - f^{(n-1)}(a)\right]}{(n-1)!(b-a)} \int_a^b \Delta_1(s)ds + R_n^2(f;a,b) \end{aligned} \tag{5.42}$$

*satisfies the estimation*

$$|R_n^2(f;a,b)| \le \frac{1}{(n-1)!}\left(\frac{b-a}{2} T(\Delta_1,\Delta_1) \int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2 ds\right)^{1/2}.$$

By using Theorem 5.13 we obtain the following Grüss type inequality.

**Theorem 5.16** *Let $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ is an absolutely continuous function with $f^{(n+1)} \ge 0$ on $[a,b]$, $x_i \in [a,b]$ and $p_i \in \mathbb{R}$ ($i \in \{1,\dots,m\}$) such that $\sum_{i=0}^m p_i = 0$. Also, let the functions $T$ and $\delta_1$ be defined in* (5.36) *and* (5.39) *respectively. Then in representation* (5.41) *the remainder $R_n^1(f;a,b)$ satisfies the following estimation*

$$\begin{aligned}|R_n^1(f;a,b)| \le \frac{1}{(n-1)!}\|\delta_1'\|_\infty &\left[\frac{b-a}{2}\left[f^{(n-1)}(b) + f^{(n-1)}(a)\right]\right. \\ &\left. - \left[f^{(n-2)}(b) - f^{(n-2)}(a)\right]\right].\end{aligned} \tag{5.43}$$

*Proof.* If we apply Theorem 5.13 for $f \to \delta$ and $h \to f^{(n)}$, then we obtain

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b \delta_1(s) f^{(n)}(s)ds - \left(\frac{1}{b-a}\int_a^b \delta_1(s)ds\right)\left(\frac{1}{b-a}\int_a^b f^{(n)}(s)ds\right)\right| \\ &\qquad \le \frac{1}{2(b-a)}\|\delta'\|_\infty \int_a^b (s-a)(b-s) f^{(n+1)}(s)ds.\end{aligned}$$

Since

$$\int_a^b (s-a)(b-s) f^{(n+1)}(s)ds = \int_a^b (2s-a-b) f^{(n)}(s)ds$$

$$= (b-a)\left[f^{(n-1)}(b)+f^{(n-1)}(a)\right]-2\left[f^{(n-2)}(b)-f^{(n-2)}(a)\right], \tag{5.44}$$

by using identities (5.41) and (5.44) we deduce (5.43). □

Next we give the integral version of the above theorem.

**Theorem 5.17** *Let $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ is an absolutely continuous function with $f^{(n+1)} \geq 0$ on $[a,b]$, let $g : [\alpha,\beta] \to [a,b]$ and $p : [\alpha,\beta] \to \mathbb{R}$ be functions such that $\int_\alpha^\beta p(x)dx = 0$. Also, let the functions $T$ and $\Delta_1$ be defined in* (5.36) *and* (5.40) *respectively. Then in representation* (5.42) *the remainder $R_n^2(f;a,b)$ satisfies the following estimation*

$$|R_n^2(f;a,b)| \leq \frac{1}{(n-1)!}\|\Delta_1'\|_\infty \left[\frac{b-a}{2}\left[f^{(n-1)}(b)+f^{(n-1)}(a)\right] - \left[f^{(n-2)}(b)-f^{(n-2)}(a)\right]\right].$$

Now we state some Ostrowski-type inequalities related to the obtained identities.

**Theorem 5.18** *Let all the assumptions of Theorem* 5.5 *hold. Furthermore, let $(q,r)$ be a pair of conjugate exponents, that is $1 \leq q,r \leq \infty$, $\frac{1}{q}+\frac{1}{r}=1$. Let $f^{(n)} \in L_q[a,b]$ for some $n \in \mathbb{N}$, $n>1$. Then we have*

$$\left|\sum_{i=1}^m p_i f(x_i) - \frac{1}{b-a}\left[\sum_{k=0}^{n-2}\frac{1}{k!\,(k+2)}f^{(k+1)}(a)\sum_{i=1}^m p_i (x_i-a)^{k+2} - \sum_{k=0}^{n-2}\frac{1}{k!\,(k+2)}f^{(k+1)}(b)\sum_{i=1}^m p_i(x_i-b)^{k+2}\right]\right| \leq \frac{1}{(n-1)!}\|f^{(n)}\|_q \left\|\sum_{i=1}^m p_i T_n(x_i,\cdot)\right\|_r. \tag{5.45}$$

*The constant on the right-hand side of* (5.45) *is sharp for $1<q\leq\infty$ and the best possible for $q=1$.*

*Proof.* Let us denote

$$\lambda(s) = \frac{1}{(n-1)!}\sum_{i=1}^m p_i T_n(x_i,s).$$

Now, by using identity (5.12) and applying Hölder's inequality we obtain

$$\left|\sum_{i=1}^m p_i f(x_i) - \frac{1}{b-a}\left[\sum_{k=0}^{n-2}\frac{1}{k!\,(k+2)}f^{(k+1)}(a)\sum_{i=1}^m p_i (x_i-a)^{k+2} - \sum_{k=0}^{n-2}\frac{1}{k!\,(k+2)}f^{(k+1)}(b)\sum_{i=1}^m p_i(x_i-b)^{k+2}\right]\right| \leq \|f^{(n)}\|_q\|\lambda\|_r. \tag{5.46}$$

For the proof of the sharpness of the constant $\left(\int_a^b |\lambda(s)|^r ds\right)^{1/r}$, let us find a function $f$ for which the equality in (5.46) is obtained.

For $1 < q < \infty$ take $f$ to be such that

$$f^{(n)}(s) = \operatorname{sgn} \lambda(s) \cdot |\lambda(s)|^{1/(q-1)}.$$

For $q = \infty$, take $f$ such that

$$f^{(n)}(s) = \operatorname{sgn} \lambda(s).$$

Finally, for $q = 1$, we prove that

$$\left| \int_a^b \lambda(s) f^{(n)}(s) ds \right| \le \max_{s \in [a,b]} |\lambda(s)| \int_a^b f^{(n)}(s) ds \tag{5.47}$$

is the best possible inequality.

Function $T_n(x,\cdot)$ for $n = 1$ has a jump of $-1$ at point $x$. But, for $n \ge 2$ it is continuous, and thus $\lambda(s)$ is continuous. Suppose that $|\lambda(s)|$ attains its maximum at $s_0 \in [a,b]$. First we consider the case $\lambda(s_0) > 0$. For $\varepsilon$ small enough we define $f_\varepsilon(s)$ by

$$f_\varepsilon(s) = \begin{cases} 0, & a \le s \le s_0, \\ \frac{1}{\varepsilon n!}(s-s_0)^n, & s_0 \le s \le s_0 + \varepsilon, \\ \frac{1}{(n-1)!}(s-s_0)^{n-1}, & s_0 + \varepsilon \le s \le b. \end{cases}$$

We have

$$\left| \int_a^b \lambda(s) f_\varepsilon^{(n)}(s) ds \right| = \left| \int_{s_0}^{s_0+\varepsilon} \lambda(s) \frac{1}{\varepsilon} ds \right| = \frac{1}{\varepsilon} \int_{s_0}^{s_0+\varepsilon} \lambda(s) ds.$$

Now, from inequality (5.47) we have

$$\frac{1}{\varepsilon} \int_{s_0}^{s_0+\varepsilon} \lambda(s) ds \le \lambda(s_0) \frac{1}{\varepsilon} \int_{s_0}^{s_0+\varepsilon} ds = \lambda(s_0).$$

Since

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{s_0}^{s_0+\varepsilon} \lambda(s) ds = \lambda(s_0),$$

the statement follows.

In the case $\lambda(s_0) < 0$, we define $f_\varepsilon(s)$ by

$$f_\varepsilon(s) = \begin{cases} \frac{1}{(n-1)!}(s-s_0-\varepsilon)^{n-1}, & a \le s \le s_0, \\ -\frac{1}{\varepsilon n!}(s-s_0-\varepsilon)^n, & s_0 \le s \le s_0 + \varepsilon, \\ 0, & s_0 + \varepsilon \le s \le b \end{cases}$$

and the rest of the proof is the same as above. □

Now we give the integral case of the above theorem.

**Theorem 5.19** *Let all the assumptions of Theorem 5.6 hold. Furthermore, let $(q,r)$ be a pair of conjugate exponents, that is $1 \le q,r \le \infty$, $\frac{1}{q}+\frac{1}{r}=1$. Let $f^{(n)} \in L_q[a,b]$ for some $n \in \mathbb{N}$, $n>1$. Then we have*

$$\begin{aligned}
&\Bigg|\int_{\alpha}^{\beta} p(x) f(g(x))\, dx \\
&-\frac{1}{b-a}\left[\sum_{k=0}^{n-2} \frac{1}{k!\,(k+2)} f^{(k+1)}(a) \int_{\alpha}^{\beta} p(x)(g(x)-a)^{k+2}\, dx\right. \\
&\left.-\sum_{k=0}^{n-2} \frac{1}{k!\,(k+2)} f^{(k+1)}(b) \int_{\alpha}^{\beta} p(x)(g(x)-b)^{k+2}\, dx\right]\Bigg| \\
&\le \frac{1}{(n-1)!}\|f^{(n)}\|_q \left\| \int_{\alpha}^{\beta} p(x) T_n(g(x),s)\, dx \right\|_r . \qquad (5.48)
\end{aligned}$$

*The constant on the right-hand side of* (5.48) *is sharp for $1<q\le\infty$ and the best possible for $q=1$.*

### 5.1.4 Results Obtained by the Green Function and the Montgomery Identity

In this section we give several identities and inequalities based on application of the extension of the Montgomery identity involving the Green function. These results can be find in paper [27].

**Theorem 5.20** *Let $n \in \mathbb{N}$, $n \ge 3$, $f: I \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $a,b \in I$, $a<b$. Let $\mathbf{x}=(x_1,\ldots,x_m) \in [a,b]^m$, $\mathbf{p}=(p_1,\ldots,p_m) \in \mathbb{R}^m$ satisfy*

$$\sum_{i=1}^{m} p_i = 0, \qquad \sum_{i=1}^{m} p_i x_i = 0 \qquad (5.49)$$

*and let $G$ be given by* (5.10)*. Then*

$$\begin{aligned}
\sum_{i=1}^{m} p_i f(x_i) &= \frac{f'(a)-f'(b)}{b-a} \int_a^b \sum_{i=1}^{m} p_i G(x_i,s)\, ds \\
&+\sum_{k=2}^{n-1} \frac{k}{(k-1)!} \int_a^b \sum_{i=1}^{m} p_i G(x_i,s) \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a}\, ds \\
&+\frac{1}{(n-3)!} \int_a^b f^{(n)}(t) \left( \int_a^b \sum_{i=1}^{m} p_i G(x_i,s) \tilde{T}_{n-2}(s,t)\, ds \right) dt, \qquad (5.50)
\end{aligned}$$

*where*

$$\tilde{T}_{n-2}(s,t) = \begin{cases} \dfrac{1}{b-a}\left[\dfrac{(s-t)^{n-2}}{(n-2)} + (s-a)(s-t)^{n-3}\right], & a \le t \le s \le b, \\[2ex] \dfrac{1}{b-a}\left[\dfrac{(s-t)^{n-2}}{(n-2)} + (s-b)(s-t)^{n-3}\right], & a \le s < t \le b. \end{cases}$$

*and G is as defined in* (5.10). *Moreover, the following identity holds*

$$\begin{aligned}
\sum_{i=1}^{m} p_i f(x_i) &= \frac{f'(b)-f'(a)}{b-a}\int_a^b \sum_{i=1}^{m} p_i G(x_i,s)ds \\
&+\sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\int_a^b \sum_{i=1}^{m} p_i G(x_i,s)\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a}ds \\
&+\frac{1}{(n-3)!}\int_a^b f^{(n)}(t)\left(\int_a^b \sum_{i=1}^{m} p_i G(x_i,s)T_{n-2}(s,t)ds\right)dt, \qquad (5.51)
\end{aligned}$$

*where $T_n$ is as defined in* (5.8).

*Proof.* Using (5.11) in $\sum_{i=1}^{m} p_i f(x_i)$, and condition (5.49) we get

$$\sum_{i=1}^{m} p_i f(x_i) = \int_a^b \sum_{i=1}^{m} p_i G(x_i,s) f''(s)ds. \qquad (5.52)$$

Differentiating (5.7) twice with respect to s, we get

$$\begin{aligned}
f''(s) &= \frac{f'(a)-f'(b)}{b-a}+\sum_{k=2}^{n-1}\frac{k}{(k-1)!}\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a} \\
&+ \frac{1}{(n-3)!}\int_a^b \tilde{T}_{n-2}(s,t)f^{(n)}(t)dt. \qquad (5.53)
\end{aligned}$$

Now using (5.53) in (5.52) we get

$$\begin{aligned}
\sum_{i=1}^{m} p_i f(x_i) &= \frac{f'(a)-f'(b)}{b-a}\int_a^b \sum_{i=1}^{m} p_i G(x_i,s)ds \\
&+\sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b \sum_{i=1}^{m} p_i G(x_i,s)\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a}ds \\
&+\frac{1}{(n-3)!}\int_a^b \sum_{i=1}^{m} p_i G(x_i,s)\left(\int_a^b \tilde{T}_{n-2}(s,t)f^{(n)}(t)dt\right)ds
\end{aligned}$$

and then using Fubini's theorem in the last term we get (5.50).

Also, by using formula (5.7) on the function $f''$, replacing $n$ by $n-2$ ($n \geq 3$) and rearranging the indices we get

$$\begin{aligned}
f''(s) &= \frac{f'(a)-f'(b)}{b-a}+\sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a} \\
&+ \frac{1}{(n-3)!}\int_a^b T_{n-2}(s,t)f^{(n)}(t)dt. \qquad (5.54)
\end{aligned}$$

Similarly, using (5.54) in (5.52) and applying Fubini's Theorem, we get (5.51). □

**Theorem 5.21** *Let all the assumptions of Theorem* 5.20 *hold with additional condition*

$$\int_a^b \sum_{i=1}^m p_i G(x_i,s)\tilde{T}_{n-2}(s,t)ds \geq 0, \quad \forall\, t \in [a,b] \tag{5.55}$$

*where $G$ is defined in* (5.10) *and $\tilde{T}_n$ is defined in Theorem* 5.20*. Then for every $n$-convex function $f : I \to \mathbb{R}$ the following inequality holds*

$$\begin{aligned}\sum_{i=1}^m p_i f(x_i) \geq\ & \frac{f'(a)-f'(b)}{b-a}\int_a^b \sum_{i=1}^m p_i G(x_i,s)ds \\ &+\sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b \sum_{i=1}^m p_i G(x_i,s)\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a}ds.\end{aligned} \tag{5.56}$$

*Proof.* Since the function $f$ is $n$-convex we have $f^{(n)} \geq 0$. Using this fact and (5.55) in (5.50) we easily arrive at our required result. □

**Theorem 5.22** *Let all the assumptions of Theorem* 5.20 *hold with additional condition*

$$\int_a^b \sum_{i=1}^m p_i G(x_i,s)T_{n-2}(s,t)ds \geq 0, \quad \forall\, t \in [a,b], \tag{5.57}$$

*where $G$ is defined in* (5.10) *and $T_n$ is defined in Theorem* 5.3*. If $f$ is $n$-convex then*

$$\begin{aligned}\sum_{i=1}^m p_i f(x_i) \geq\ & \frac{f'(b)-f'(a)}{b-a}\int_a^b \sum_{i=1}^m p_i G(x_i,s)ds \\ &+\sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\int_a^b \sum_{i=1}^m p_i G(x_i,s)\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a}ds.\end{aligned} \tag{5.58}$$

*Proof.* Since the function $f$ is $n$-convex we have $f^{(n)} \geq 0$. Using this fact and (5.57) in (5.51) we easily arrive at our required result. □

Now we state one important consequences.

**Theorem 5.23** *Let all the assumptions from Theorem* 5.20 *hold with*

$$\sum_{i=1}^m p_i = 0, \quad \sum_{i=1}^m p_i|x_i - x_k| \geq 0, \quad \text{for } k \in \{1,\ldots,m\}. \tag{5.59}$$

*If $f$ is $n$-convex and $n$ is even, then inequalities* (5.56) *and* (5.58) *hold.*

*Proof.* Since the Green function $G(s,t)$ is convex with respect to $t$ for every $s \in [a,b]$ and $\mathbf{x} = (x_1,\ldots,x_m)$ and $\mathbf{p} = (p_1,\ldots,p_m)$ satisfy conditions from Remark 5.1 we have

$$\sum_{i=1}^m p_i G(x_i,s) \geq 0 \quad \text{for} \quad s \in [a,b]. \tag{5.60}$$

Also note that for even $n$ $\tilde{T}_{n-2}(s,t) \geq 0$ and $T_{n-2}(s,t) \geq 0$. Therefore, combining this fact with (5.60) we get inequalities (5.55) and (5.57). As $f$ is $n$-convex, so results follows from Theorems 5.21 and 5.22. □

The integral version of our main results may be stated as follows. Since the proofs are of similar nature, we omit the details.

**Theorem 5.24** *Let $n \in \mathbb{N}$, $n \geq 3$, $f : I \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$. Furthermore, let $g : [\alpha, \beta] \to [a,b]$ and $p : [\alpha, \beta] \to \mathbb{R}$ satisfy $\int_\alpha^\beta p(x)dx = 0$ and $\int_\alpha^\beta p(x)g(x)dx = 0$, and let $G$, $T_n$ and $\tilde{T}_n$ be given by* (5.10), (5.8) *and in Theorem* 5.20*. Then the following two identities hold:*

$$\int_\alpha^\beta p(x) f(g(x))\, dx = \frac{f'(a) - f'(b)}{b-a} \int_a^b \int_\alpha^\beta p(x) G(g(x),s)\, dx\, ds$$
$$+ \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \int_a^b \left( \int_\alpha^\beta p(x) G(g(x),s) dx \right) \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} ds$$
$$+ \frac{1}{(n-3)!} \int_a^b f^{(n)}(t) \left( \int_a^b \left( \int_\alpha^\beta p(x) G(g(x),s)\, dx \right) \tilde{T}_{n-2}(s,t) ds \right) dt$$

*and*

$$\int_\alpha^\beta p(x) f(g(x))\, dx = \frac{f'(b) - f'(a)}{b-a} \int_a^b \int_\alpha^\beta p(x) G(g(x),s)\, dx\, ds$$
$$+ \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \int_a^b \left( \int_\alpha^\beta p(x) G(g(x),s)\, dx \right) \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} ds$$
$$+ \frac{1}{(n-3)!} \int_a^b f^{(n)}(t) \left( \int_a^b \left( \int_\alpha^\beta p(x) G(g(x),s)\, dx \right) T_{n-2}(s,t) ds \right) dt.$$

**Theorem 5.25** *Let all the assumptions of Theorem* 5.24 *hold with additional condition*

$$\int_a^b \int_\alpha^\beta p(x) G(g(x),s)\, \tilde{T}_{n-2}(s,t)\, dx\, ds \geq 0, \quad \forall\, t \in [a,b], \tag{5.61}$$

*where $G$ is defined in* (5.10) *and $\tilde{T}_n$ is defined in Theorem* 5.20*. If $f$ is n-convex, then*

$$\int_\alpha^\beta p(x) f(g(x))\, dx \geq \frac{f'(a) - f'(b)}{b-a} \int_a^b \int_\alpha^\beta p(x) G(g(x),s)\, dx\, ds \tag{5.62}$$
$$+ \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \times$$
$$\times \int_a^b \left( \int_\alpha^\beta p(x) G(g(x),s)\, dx \right) \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} ds.$$

**Theorem 5.26** *Let all the assumptions of Theorem* 5.24 *hold with additional condition*

$$\int_a^b \int_\alpha^\beta p(x) G(g(x),s)\, T_{n-2}(s,t) dx\, ds \geq 0, \quad \forall\, t \in [a,b]. \tag{5.63}$$

*If $f$ is $n$-convex, then*

$$
\begin{aligned}
\int_\alpha^\beta p(x)f(g(x))\,dx \geq \frac{f'(b)-f'(a)}{b-a}\int_a^b\int_\alpha^\beta p(x)\,G(g(x),s)\,dx\,ds \\
+\sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\times \\
\int_a^b\left(\int_\alpha^\beta p(x)\,G(g(x),s)\,dx\right)\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a}ds
\end{aligned}
\tag{5.64}
$$

**Theorem 5.27** *Let all the assumptions from Theorem* 5.24 *hold with the additional assumption that $g:[\alpha,\beta]\to[a,b]$ and $p:[\alpha,\beta]\to\mathbb{R}$ satisfy*

$$
\begin{aligned}
\int_\alpha^\beta p(x)g^k(x)dx &= 0 \ \textit{for all}\ k=0,1,\ldots,n-1, \\
\int_\alpha^\beta p(x)(g(x)-t)_+^{n-1}dx &\geq 0 \ \textit{for every}\ \ t\in[a,b].
\end{aligned}
$$

*If $f$ is $n$-convex and $n$ is even, then inequalities* (5.62) *and* (5.64) *hold.*

### 5.1.5 Bounds for the Remainders

Theorems in this section are devoted to estimations of the remainders which occur in certain representations of the sum $\sum_{i=1}^m p_i f(x_i)$ and the integral $\int_\alpha^\beta p(x)f(g(x))dx$. Namely, we give some Grüss and Ostrowski type inequalities, [27].

Under the assumptions of Theorems 5.20 and 5.24 respectively, we define the following functions

$$\Omega_1(t) = \int_a^b \sum_{i=1}^m p_i G(x_i,s)\tilde{T}_{n-2}(s,t)ds, \quad t\in[a,b] \tag{5.65}$$

$$\Omega_2(t) = \int_a^b \sum_{i=1}^m p_i G(x_i,s)T_{n-2}(s,t)ds, \quad t\in[a,b] \tag{5.66}$$

$$\Omega_3(t) = \int_a^b\int_\alpha^\beta p(x)\,G(g(x),s)\,\tilde{T}_{n-2}(s,t)\,dx\,ds, \quad t\in[a,b] \tag{5.67}$$

$$\Omega_4(t) = \int_a^b\int_\alpha^\beta p(x)\,G(g(x),s)\,T_{n-2}(s,t)\,dx\,ds, \quad t\in[a,b]. \tag{5.68}$$

**Theorem 5.28** *Let $n\in\mathbb{N}$, $n\geq 3$, $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ is an absolutely continuous function with $(\cdot-a)(b-\cdot)[f^{(n+1)}]^2\in L[a,b]$ and let $\mathbf{x}\in[a,b]^m$ and $\mathbf{p}\in\mathbb{R}^m$ satisfy $\sum_{i=1}^m p_i=0$ and $\sum_{i=1}^m p_ix_i=0$. Then the remainders $R_n^1(f;a,b)$ and $R_n^2(f;a,b)$ given by the following identities*

$$\sum_{i=1}^m p_i f(x_i) = \frac{f'(a)-f'(b)}{b-a}\int_a^b\sum_{i=1}^m p_i G(x_i,s)ds$$

$$+\sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b\sum_{i=1}^m p_iG(x_i,s)\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a}ds$$
$$+\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(n-3)!(b-a)}\int_a^b\Omega_1(s)ds+R_n^1(f;a,b) \tag{5.69}$$

*and*

$$\sum_{i=1}^m p_if(x_i)=\frac{f'(b)-f'(a)}{b-a}\int_a^b\sum_{i=1}^m p_iG(x_i,s)ds$$
$$+\sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\int_a^b\sum_{i=1}^m p_iG(x_i,s)\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a}ds$$
$$+\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(n-3)!(b-a)}\int_a^b\Omega_2(s)ds+R_n^2(f;a,b) \tag{5.70}$$

*satisfy estimations*

$$|R_n^k(f;a,b)|\leq\frac{1}{(n-3)!}\left(\frac{b-a}{2}T(\Omega_k,\Omega_k)\int_a^b(s-a)(b-s)[f^{(n+1)}(s)]^2ds\right)^{1/2},$$

*for* $k=1,2$.

*Proof.* We will prove the claim for $k=1$, while the proof for $k=2$ is analogous. Proposition 5.12 with $f\to\Omega_1$ and $h\to f^{(n)}$ yields

$$\left|\frac{1}{b-a}\int_a^b\Omega_1(t)f^{(n)}(t)dt-\left(\frac{1}{b-a}\int_a^b\Omega_1(t)dt\right)\left(\frac{1}{b-a}\int_a^b f^{(n)}(t)dt\right)\right|$$
$$\leq\frac{1}{\sqrt{2}}\left(\frac{1}{b-a}|T(\Omega_1,\Omega_1)|\int_a^b(t-a)(b-t)[f^{(n+1)}(t)]^2dt\right)^{1/2}. \tag{5.71}$$

By identity (5.50) from Theorem 5.20

$$\sum_{i=1}^m p_if(x_i)-\frac{f'(a)-f'(b)}{b-a}\int_a^b\sum_{i=1}^m p_iG(x_i,s)ds$$
$$-\sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b\sum_{i=1}^m p_iG(x_i,s)\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a}ds$$
$$=\frac{1}{(n-3)!}\int_a^b\Omega_1(t)f^{(n)}(t)dt$$

and, since

$$\frac{1}{(n-3)!}\int_a^b\Omega_1(t)f^{(n)}(t)dt=\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(n-3)!(b-a)}\int_a^b\Omega_1(t)dt+R_n^1(f;a,b),$$

the bound for the remainder $R_n^1(f;a,b)$ follows from (5.71). □

By using Theorem 5.13 we obtain the following Grüss type inequality.

**Theorem 5.29** *Let $n \in \mathbb{N}$, $n \geq 3$, $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ is an absolutely continuous function with $f^{(n+1)} \geq 0$ and let $\mathbf{x} \in [a,b]^m$ and $\mathbf{p} \in \mathbb{R}^m$ satisfy $\sum_{i=1}^m p_i = 0$ and $\sum_{i=1}^m p_i x_i = 0$. Then remainders $R_n^i(f;a,b), i = 1,2,$ from representations* (5.69) *and* (5.70) *satisfy*

$$|R_n^i(f;a,b)| \leq \frac{1}{(n-3)!}\|\Omega_i'\|_\infty \left\{ \frac{b-a}{2}\left[f^{(n-1)}(b) + f^{(n-1)}(a)\right] - \left[f^{(n-2)}(b) - f^{(n-2)}(a)\right] \right\}.$$

Now we state some Ostrowski-type inequalities related to the generalized linear inequalities. The proofs of the below-mentioned two theorems are done in a similar manner as the proof of Theorem 5.18.

**Theorem 5.30** *Let $n \in \mathbb{N}$, $n \geq 3$, $1 \leq q, r \leq \infty$, $\frac{1}{q} + \frac{1}{r} = 1$, $f^{(n)} \in L_q[a,b]$ and let $\mathbf{x} \in [a,b]^m$ and $\mathbf{p} \in \mathbb{R}^m$ satisfy $\sum_{i=1}^m p_i = 0$ and $\sum_{i=1}^m p_i x_i = 0$. Then*

$$\left| \sum_{i=1}^m p_i f(x_i) - \frac{f'(a) - f'(b)}{b-a} \int_a^b \sum_{i=1}^m p_i G(x_i,s)ds - \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \int_a^b \sum_{i=1}^m p_i G(x_i,s) \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} ds \right|$$
$$\leq \frac{1}{(n-3)!}\|f^{(n)}\|_q \left\| \int_a^b \sum_{i=1}^m p_i G(x_i,s)\tilde{T}_{n-2}(s,t)ds \right\|_r$$

*and*

$$\left| \sum_{i=1}^m p_i f(x_i) - \frac{f'(b) - f'(a)}{b-a} \int_a^b \sum_{i=1}^m p_i G(x_i,s)ds - \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \int_a^b \sum_{i=1}^m p_i G(x_i,s) \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} ds \right|$$
$$\leq \frac{1}{(n-3)!}\|f^{(n)}\|_q \left\| \int_a^b \sum_{i=1}^m p_i G(x_i,s)T_{n-2}(s,t)ds \right\|_r.$$

*The constant on the right-hand side of the above inequalities is sharp for $1 < q \leq \infty$ and the best possible for $q = 1$.*

The integral analogous of stated results are as follow. Since the proofs are of similar nature we omit the details.

**Theorem 5.31** *Let $n \in \mathbb{N}$, $n \geq 3$, $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ is an absolutely continuous function with $(\cdot - a)(b - \cdot)[f^{(n+1)}]^2 \in L[a,b]$ and let $g : [\alpha,\beta] \to [a,b]$ and $p : [\alpha,\beta] \to \mathbb{R}$ satisfy $\int_\alpha^\beta p(x)dx = 0$ and $\int_\alpha^\beta p(x)g(x)dx = 0$. Then the remainders $R_n^k(f;a,b)$, $k = 3,4$, given by the following identities*

$$\int_\alpha^\beta p(x) f(g(x))\, dx = \frac{f'(a) - f'(b)}{b-a} \int_a^b \int_\alpha^\beta p(x)\, G(g(x),s)\, dx ds$$

$$
\begin{aligned}
&+\sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b\left(\int_\alpha^\beta p(x)\,G(g(x),s)\,dx\right)\times\\
&\times\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a}ds\\
&+\frac{\left[f^{(n-1)}(b)-f^{(n-1)}(a)\right]}{(n-3)!(b-a)}\int_a^b\Omega_3(s)ds+R_n^3(f;a,b),
\end{aligned}
\tag{5.72}
$$

*and*

$$
\begin{aligned}
\int_\alpha^\beta p(x)\,f(g(x))\,dx=&\frac{f'(b)-f'(a)}{b-a}\int_a^b\int_\alpha^\beta p(x)\,G(g(x),s)\,dx\,ds\\
&+\sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\int_a^b\left(\int_\alpha^\beta p(x)\,G(g(x),s)\,dx\right)\times\\
&\times\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a}ds\\
&+\frac{\left[f^{(n-1)}(b)-f^{(n-1)}(a)\right]}{(n-3)!(b-a)}\int_a^b\Omega_4(s)ds+R_n^4(f;a,b),
\end{aligned}
\tag{5.73}
$$

*have a bound*

$$
|R_n^k(f;a,b)|\le\frac{1}{(n-3)!}\left(\frac{b-a}{2}T(\Omega_k,\Omega_k)\int_a^b(s-a)(b-s)[f^{(n+1)}(s)]^2ds\right)^{1/2}.
$$

By using Theorem 5.13 we obtain the following Grüss type inequality.

**Theorem 5.32** *Let $n\in\mathbb{N}$, $n\ge 3$, $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ is an absolutely continuous function with $f^{(n+1)}\ge 0$ and let $g:[\alpha,\beta]\to[a,b]$ and $p:[\alpha,\beta]\to\mathbb{R}$ satisfy $\int_\alpha^\beta p(x)dx=0$ and $\int_\alpha^\beta p(x)g(x)dx=0$. Then the remainders $R_n^i(f;a,b)$, $i=3,4$ from representations* (5.72) *and* (5.73) *satisfy*

$$
\begin{aligned}
&|R_n^i(f;a,b)|\\
&\le\frac{1}{(n-3)!}\|\Omega_i'\|_\infty\left\{\frac{b-a}{2}\left[f^{(n-1)}(b)+f^{(n-1)}(a)\right]-\left[f^{(n-2)}(b)-f^{(n-2)}(a)\right]\right\}.
\end{aligned}
$$

The Ostrowski-type inequalities related to the generalized integral linear inequalities are given below.

**Theorem 5.33** *Let $n\in\mathbb{N}$, $n\ge 3$, $1\le q,r\le\infty$, $\frac{1}{q}+\frac{1}{r}=1$, $f^{(n)}\in L_q[a,b]$ and let $g:[\alpha,\beta]\to[a,b]$ and $p:[\alpha,\beta]\to\mathbb{R}$ satisfy $\int_\alpha^\beta p(x)dx=0$ and $\int_\alpha^\beta p(x)g(x)dx=0$. Then*

$$
\begin{aligned}
&\left|\int_\alpha^\beta p(x)\,f(g(x))\,dx-\frac{f'(a)-f'(b)}{b-a}\int_a^b\int_\alpha^\beta p(x)\,G(g(x),s)\,dx\,ds\right.\\
&-\sum_{k=2}^{n-1}\frac{k}{(k-1)!}\times
\end{aligned}
$$

$$\times \int_a^b \left(\int_\alpha^\beta p(x)\,G(g(x),s)\,dx\right) \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} ds$$
$$\leq \frac{1}{(n-3)!}\|f^{(n)}\|_q\|\Omega_3\|_r$$

*and*

$$\left|\int_\alpha^\beta p(x)\,f(g(x))\,dx - \frac{f'(b)-f'(a)}{b-a}\int_a^b\int_\alpha^\beta p(x)\,G(g(x),s)\,dx\,ds\right.$$
$$-\sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\int_a^b\left(\int_\alpha^\beta p(x)\,G(g(x),s)\,dx\right)\times$$
$$\left.\times\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a}ds\right| \leq \frac{1}{(n-3)!}\|f^{(n)}\|_q\|\Omega_4\|_r.$$

*The constant on the right-hand side of the above inequalities is sharp for* $1<q\leq\infty$ *and the best possible for* $q=1$.

### 5.1.6 Mean Value Theorems and Exponential Convexity

In this section we prove some properties of linear functionals associated with the inequalities obtained in earlier sections. Under the assumptions of Theorem 5.7 using (5.15), Theorem 5.9 using (5.23), Theorem 5.21 using (5.56), Theorem 5.22 using (5.58), Theorem 5.25 using (5.62) and Theorem 5.26 using (5.64) we define the following functionals respectively:

$$\begin{aligned}\Lambda_1(f) &= \sum_{i=1}^m p_i f(x_i) - \frac{1}{b-a}\left[\sum_{k=0}^{n-2}\frac{1}{k!\,(k+2)}f^{(k+1)}(a)\sum_{i=1}^m p_i(x_i-a)^{k+2}\right.\\ &\quad\left.-\sum_{k=0}^{n-2}\frac{1}{k!\,(k+2)}f^{(k+1)}(b)\sum_{i=1}^m p_i(x_i-b)^{k+2}\right],\end{aligned} \tag{5.74}$$

$$\begin{aligned}\Lambda_2(f) &= \int_\alpha^\beta p(x)f(g(x))dx\\ &\quad-\frac{1}{b-a}\left[\sum_{k=0}^{n-2}\frac{1}{k!\,(k+2)}f^{(k+1)}(a)\int_\alpha^\beta p(x)\,(g(x)-a)^{k+2}\,dx\right.\\ &\quad\left.-\sum_{k=0}^{n-2}\frac{1}{k!\,(k+2)}f^{(k+1)}(b)\int_\alpha^\beta p(x)\,(g(x)-b)^{k+2}\,dx\right],\end{aligned} \tag{5.75}$$

$$\begin{aligned}\Lambda_3(f) &= \sum_{i=1}^m p_i f(x_i) - \frac{f'(a)-f'(b)}{b-a}\int_a^b\sum_{i=1}^m p_i G(x_i,s)ds\\ &\quad-\sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b\sum_{i=1}^m p_i G(x_i,s)\times\\ &\quad\quad\times\frac{f^{(k)}(a)(s-a)^{k-1}-f^{(k)}(b)(s-b)^{k-1}}{b-a}ds,\end{aligned} \tag{5.76}$$

$$
\begin{aligned}
\Lambda_4(f) &= \sum_{i=1}^{m} p_i f(x_i) - \frac{f'(b)-f'(a)}{b-a}\int_a^b \sum_{i=1}^{m} p_i G(x_i,s)ds \\
&\quad - \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!}\int_a^b \sum_{i=1}^{m} p_i G(x_i,s) \times \\
&\qquad\qquad \times \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} ds, \qquad (5.77) \\
\Lambda_5(f) &= \int_\alpha^\beta p(x) f(g(x))\, dx - \frac{f'(a)-f'(b)}{b-a}\int_a^b \int_\alpha^\beta p(x) G(g(x),s)\, dx ds \\
&\quad - \sum_{k=2}^{n-1} \frac{k}{(k-1)!}\int_a^b \left(\int_\alpha^\beta p(x) G(g(x),s)\, dx\right) \times \\
&\qquad\qquad \times \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} ds, \qquad (5.78) \\
\Lambda_6(f) &= \int_\alpha^\beta p(x) f(g(x))\, dx - \frac{f'(b)-f'(a)}{b-a}\int_a^b \int_\alpha^\beta p(x) G(g(x),s)\, dx ds \\
&\quad - \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!}\int_a^b \left(\int_\alpha^\beta p(x) G(g(x),s)\, dx\right) \times \\
&\qquad\qquad \times \frac{f^{(k)}(a)(s-a)^{k-1} - f^{(k)}(b)(s-b)^{k-1}}{b-a} ds. \qquad (5.79)
\end{aligned}
$$

**Remark 5.4** In the following text until the end of this section we use an agreement that if $k$ is a fixed number from the set $\{1,2,3,4,5,6\}$, then assumptions of Theorem 5.7, Theorem 5.9, Theorem 5.21, Theorem 5.22, Theorem 5.25, Theorem 5.26 hold respectively.

Now we give mean value theorems for $\Lambda_k$, $k \in \{1,\ldots,6\}$.

**Theorem 5.34** *Let $k \in \{1,\ldots,6\}$ and let $\Lambda_k : C^n([a,b]) \to \mathbb{R}$ be a linear functional as defined in* (5.74)$,\ldots,$(5.79) *under the agreement described in Remark* 5.4, *respectively. Then for $f \in C^n([a,b])$ there exists $\xi_k \in [a,b]$ such that*

$$
\Lambda_k(f) = f^{(n)}(\xi_k)\Lambda_k(f_0), \qquad (5.80)
$$

*where* $f_0(x) = \dfrac{x^n}{n!}$.

*Proof.* Fix $k \in \{1,\ldots,6\}$. Since $f^{(n)}$ is continuous on $[a,b]$, we have $L \le f^{(n)}(x) \le M$ for $x \in [a,b]$ where $L = \min_{x\in[a,b]} f^{(n)}(x)$ and $M = \max_{x\in[a,b]} f^{(n)}(x)$.

Therefore the function

$$
F(x) = M\frac{x^n}{n!} - f(x) = Mf_0(x) - f(x)
$$

satisfies

$$
F^{(n)}(x) = M - f^{(n)}(x) \ge 0,
$$

i.e. $F$ is $n$-convex function. Hence $\Lambda_k(F) \geq 0$ and we conclude that

$$\Lambda_k(f) \leq M\Lambda_k(f_0).$$

Similarly, we have

$$L\Lambda_k(f_0) \leq \Lambda_k(f).$$

Combining the two inequalities we get

$$L\Lambda_k(f_0) \leq \Lambda_k(f) \leq M\Lambda_k(f_0).$$

If $\Lambda_k(f_0) = 0$, then $\Lambda_k(f) = 0$ and statement (5.80) obviously holds. If $\Lambda_k(f_0) \neq 0$, then $\dfrac{\Lambda_k(f)}{\Lambda_k(f_0)} \in [L,M]$. Hence there exists $\xi_k \in [a,b]$ such that $\dfrac{\Lambda_k(f)}{\Lambda_k(f_0)} = f^{(n)}(\xi_k)$, i.e. the statement of the theorem is proved. □

**Theorem 5.35** *Let $k \in \{1,\ldots,6\}$ and let $\Lambda_k : C^n([a,b]) \to \mathbb{R}$ be a linear functional as defined in* (5.74)$,\ldots,$(5.79) *under the agreement described in Remark* 5.4*, respectively. Then for $f,h \in C^n([a,b])$ exists $\xi_k \in [a,b]$ such that*

$$\frac{\Lambda_k(f)}{\Lambda_k(h)} = \frac{f^{(n)}(\xi_k)}{h^{(n)}(\xi_k)}$$

*assuming that both denominators are non-zero.*

*Proof.* Fix $k \in \{1,\ldots,6\}$. For $f,h \in C^n([a,b])$ define $\omega \in C^n([a,b])$ as

$$\omega = \Lambda_k(h)f - \Lambda_k(f)h.$$

Using Theorem 5.34 there exists $\xi_k$ such that

$$\Lambda_k(\omega) = \omega^{(n)}(\xi_k)\Lambda_k(f_0).$$

Obviously, $\Lambda_k(\omega) = 0$ and $\omega^{(n)}(\xi_k) = \Lambda_k(h)f^{(n)}(\xi_k) - \Lambda_k(f)h^{(n)}(\xi_k)$. Since $\Lambda_k(h) \neq 0$ by Theorem 5.34 we conclude that $\Lambda_k(f_0) \neq 0$. So

$$\Lambda_k(h)f^{(n)}(\xi_k) - \Lambda_k(f)h^{(n)}(\xi_k) = 0$$

which gives us the required result. □

**Remark 5.5** If the inverse of $\frac{f^{(n)}}{h^{(n)}}$ exists, then from the above mean value theorems we can give generalized means

$$\xi_k = \left(\frac{f^{(n)}}{h^{(n)}}\right)^{-1}\left(\frac{\Lambda_k(f)}{\Lambda_k(h)}\right), \qquad k \in \{1,\ldots,6\}. \tag{5.81}$$

A number of important inequalities arises from the logarithmic convexity of some functions. In the following definitions $I$ is an interval in $\mathbb{R}$.

**Definition 5.1** *A function $f : I \to \langle 0, \infty \rangle$ is called* log*-convex in $J$-sense if the inequality*

$$f^2\left(\frac{x_1+x_2}{2}\right) \leq f(x_1)\, f(x_2)$$

*holds for each $x_1, x_2 \in I$.*

A function log-convex in the $J$-sense is log-convex if it is continuous as well.

Some results about exponentially convex functions are already given in Section 3.6. Here we wide that concept to $n$-exponentially convex functions. J. Pečarić and J. Perić in [73] introduced the notion of $n$-exponentially convex functions which is in fact a generalization of the concept of exponentially convex functions. In the present subsection, we discuss the concept of $n$-exponential convexity by describing related definitions and some important results with some remarks from [73].

**Definition 5.2** *A function $f : I \to \mathbb{R}$ is $n$-exponentially convex in the $J$-sense if the inequality*

$$\sum_{i,j=1}^{n} u_i u_j f\left(\frac{t_i+t_j}{2}\right) \geq 0$$

*holds for each $t_i \in I$ and $u_i \in \mathbb{R}$, $i \in \{1, \ldots, n\}$.*

**Definition 5.3** *A function $f : I \to \mathbb{R}$ is $n$-exponentially convex if it is $n$-exponentially convex in the $J$-sense and continuous on $I$.*

**Remark 5.6** We can see from the definition that 1-exponentially convex functions in the $J$-sense are in fact nonnegative functions. Also, $n$-exponentially convex functions in the $J$-sense are $k$-exponentially convex in the $J$-sense for every $k \in \mathbb{N}$ such that $k \leq n$.

**Definition 5.4** *A function $f : I \to \mathbb{R}$ is exponentially convex in the $J$-sense, if it is $n$-exponentially convex in the $J$-sense for each $n \in \mathbb{N}$.*

**Remark 5.7** A function $f : I \to \mathbb{R}$ is exponentially convex if it is $n$-exponentially convex in the $J$-sense and continuous on $I$.

Here we state without proof a proposition from [73].

**Proposition 5.1** *If function $f : I \to \mathbb{R}$ is $n$-exponentially convex in the $J$-sense, then the matrix*

$$\left[ f\left(\frac{t_i+t_j}{2}\right) \right]_{i,j=1}^{m}$$

*is positive-semidefinite. Particularly*

$$\det \left[ f\left(\frac{t_i+t_j}{2}\right) \right]_{i,j=1}^{m} \geq 0$$

*for each $m \in \mathbb{N}, m \leq n$ and $t_i \in I$ for $i \in \{1, \ldots, m\}$.*

**Remark 5.8** A function $f : I \to (0,\infty)$ is log-convex in the $J$-sense if and only if the inequality

$$u_1^2 f(t_1) + 2u_1u_2 f\left(\frac{t_1+t_2}{2}\right) + u_2^2 f(t_2) \geq 0$$

holds for each $t_1,t_2 \in I$ and $u_1,u_2 \in \mathbb{R}$. It follows that a positive function is log-convex in the $J$-sense if and only if it is 2-exponentially convex in the $J$-sense. Also, using basic convexity theory it follows that a positive function is log-convex if and only if it is 2-exponentially convex.

Here, we get our results concerning the $n$-exponential convexity and exponential convexity for our functionals $\Lambda_k$, $k \in \{1,\ldots,6\}$. Throughout the section $I$ is an interval in $\mathbb{R}$.

**Theorem 5.36** *Let $D_1 = \{f_t : t \in I\}$ be a class of functions such that the function $t \mapsto [z_0,z_1,\ldots,z_n;f_t]$ is $n$-exponentially convex in the $J$-sense on $I$ for any $n+1$ mutually distinct points $z_0,z_1,\ldots,z_n \in [a,b]$. Let $\Lambda_k$ for $k \in \{1,\ldots,6\}$ be the linear functionals as defined in* (5.74)$,\ldots,$(5.79) *under the agreement described in Remark* 5.4*, respectively. Then the following statements are valid:*

$(a)$ *The function $t \mapsto \Lambda_k(f_t)$ is $n$-exponentially convex function in the $J$-sense on $I$.*

$(b)$ *If the function $t \mapsto \Lambda_k(f_t)$ is continuous on $I$, then the function $t \mapsto \Lambda_k(f_t)$ is $n$-exponentially convex on $I$.*

*Proof.*

$(a)$ Fix $k \in \{1,\ldots,6\}$. Let us define the function $\omega$ for $t_i \in I$, $u_i \in \mathbb{R}$, $i \in \{1,\ldots,n\}$ as follows

$$\omega = \sum_{i,j=1}^{n} u_i u_j f_{\frac{t_i+t_j}{2}},$$

Since the function $t \mapsto [z_0,z_1,\ldots,z_n;f_t]$ is $n$-exponentially convex in the $J$-sense, we have

$$[z_0,z_1,\ldots,z_n;\omega] = \sum_{i,j=1}^{n} u_i u_j [z_0,z_1,\ldots,z_n;f_{\frac{t_i+t_j}{2}}] \geq 0$$

which implies that $\omega$ is $n$-convex function on $I$ and therefore $\Lambda_k(\omega) \geq 0$. Hence

$$\sum_{i,j=1}^{n} u_i u_j \Lambda_k(f_{\frac{t_i+t_j}{2}}) \geq 0.$$

We conclude that the function $t \mapsto \Lambda_k(f_t)$ is an $n$-exponentially convex function on $I$ in the $J$-sense.

$(b)$ This part easily follows from the definition of $n$-exponentially convex functions.

□

As a consequence of the above theorem we give the following corollaries.

**Corollary 5.4** *Let $D_2 = \{f_t : t \in I\}$ be a class of functions such that the function $t \mapsto [z_0, z_1, \ldots, z_n; f_t]$ is exponentially convex in the J-sense on I for any $n+1$ mutually distinct points $z_0, z_1, \ldots, z_n \in [a,b]$. Let $\Lambda_k$ for $k \in \{1, \ldots, 6\}$ be the linear functionals as defined in* (5.74), …, (5.79) *under the agreement described in Remark* 5.4*, respectively. Then the following statements are valid:*

$(a)$ *The function $t \mapsto \Lambda_k(f_t)$ is exponentially convex in the J-sense on I.*

$(b)$ *If the function $t \mapsto \Lambda_k(f_t)$ is continuous on I, then the function $t \mapsto \Lambda_k(f_t)$ is exponentially convex on I.*

$(c)$ *The matrix $\left[\Lambda_k\left(f_{\frac{t_i+t_j}{2}}\right)\right]_{i,j=1}^m$ is positive-semidefinite. Particularly,*

$$\det\left[\Lambda_k\left(f_{\frac{t_i+t_j}{2}}\right)\right]_{i,j=1}^m \geq 0$$

*for each $m \in \mathbb{N}$ and $t_i \in I$ where $i \in \{1, \ldots, m\}$.*

*Proof.* Proof follows directly from Theorem 5.36 by using the definition of exponential convexity and Corollary 3.4. □

**Corollary 5.5** *Let $D_3 = \{f_t : t \in I\}$ be a class of functions such that the function $t \mapsto [z_0, z_1, \ldots, z_n; f_t]$ is 2-exponentially convex in the J-sense on I for any $n+1$ mutually distinct points $z_0, z_1, \ldots, z_n \in [a,b]$. Let $\Lambda_k$ for $k \in \{1, \ldots, 6\}$ be the linear functionals as defined in* (5.74), …, (5.79) *under the agreement described in Remark* 5.4*, respectively. Then the following statements are valid:*

$(a)$ *If the function $t \mapsto \Lambda_k(f_t)$ is continuous on I, then it is 2-exponentially convex on I. If the function $t \mapsto \Lambda_k(f_t)$ is additionally positive, then it is also log-convex on I. Moreover, the following Lyapunov's inequality holds for $r < s < t$, $r, s, t \in I$*

$$[\Lambda_k(f_s)]^{t-r} \leq [\Lambda_k(f_r)]^{t-s}\,[\Lambda_k(f_t)]^{s-r}. \tag{5.82}$$

$(b)$ *If the function $t \mapsto \Lambda_k(f_t)$ is positive and differentiable on I, then for every $s,t,u,v \in I$ such that $s \leq u$ and $t \leq v$, we have*

$$\mu_{s,t}(\Lambda_k, D_3) \leq \mu_{u,v}(\Lambda_k, D_3), \tag{5.83}$$

*where $\mu_{s,t}$ is defined as*

$$\mu_{s,t}(\Lambda_k, D_3) = \begin{cases} \left(\dfrac{\Lambda_k(f_s)}{\Lambda_k(f_t)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(\dfrac{\frac{d}{ds}\Lambda_k(f_s)}{\Lambda_k(f_s)}\right), & s = t \end{cases} \tag{5.84}$$

*for $f_s, f_t \in D_3$.*

*Proof.* (a) It follows directly form Theorem 5.36 and Remark 5.8. As the function $t \mapsto \Lambda_k(f_t)$ is log-convex, i.e., $\log \Lambda_k(f_t)$ is convex we have

$$\log[\Lambda_k(f_s)]^{t-r} \leq \log[\Lambda_k(f_r)]^{t-s} + \log[\Lambda_k(f_t)]^{s-r}, \quad k \in \{1, \ldots, 6\}$$

which gives us (5.82).

(b) For a convex function $f$, the inequality

$$\frac{f(s) - f(t)}{s - t} \leq \frac{f(u) - f(v)}{u - v} \tag{5.85}$$

holds for all $s,t,u,v \in I \subset \mathbb{R}$ such that $s \leq u$, $t \leq v$, $s \neq t$, $u \neq v$.
Since $\Lambda(f_t)$ is log-convex, setting $f(t) = \log \Lambda(f_t)$ in (5.85) we have

$$\frac{\log \Lambda_k(f_s) - \log \Lambda_k(f_t)}{s-t} \leq \frac{\log \Lambda_k(f_u) - \log \Lambda_k(f_v)}{u-v} \tag{5.86}$$

for $s \leq u$, $t \leq v$, $s \neq t$, $u \neq v$, which is equivalent to (5.83). The cases for $s = t$ and/or $u = v$ are easily derived from (5.86) by taking respective limits. □

**Remark 5.9** The results from Theorem 5.36 and Corollaries 5.4 and 5.5 still hold when any two (all) points $z_0, z_1, \ldots, z_n \in [a,b]$ coincide for a family of differentiable ($n$-times differentiable) functions $f_t$ such that the function $t \mapsto [z_0, z_1, \ldots, z_n; f_t]$ is $n$-exponentially convex, exponentially convex and 2-expoenetially convex in the $J$-sense respectively.

Now, we give two important remarks and one useful corollary from [22], which we will use in some examples in the next section.

**Remark 5.10** We say that $\mu_{s,t}(\Lambda_k, \Omega)$ defined with (5.84) is a mean if

$$a \leq \mu_{s,t}(\Lambda_k, \Omega) \leq b$$

for $s,t \in I$ and $k \in \{1, \ldots, 6\}$, where $\Omega = \{f_t : t \in I\}$ is a family of functions and $[a,b] \subseteq Dom(f_t)$. □

Theorem 5.36 give us the following corollary.

**Corollary 5.6** *Let $a,b \in \mathbb{R}$ and $\Lambda_k$ for $k \in \{1, \ldots, 6\}$ be the linear functionals as defined in* (5.74), ..., (5.79). *Let $\Omega = \{f_t : t \in I\}$ be a family of functions in $C^2([a,b])$. If*

$$a \leq \left( \frac{\frac{d^2 f_s}{dx^2}}{\frac{d^2 f_t}{dx^2}} \right)^{\frac{1}{s-t}} (\xi) \leq b,$$

*for $\xi \in [a,b]$, $s,t \in I$, then $\mu_{s,t}(\Lambda_k, \Omega)$ is a mean for $k \in \{1, \ldots, 6\}$.*

**Remark 5.11** In some examples, we will get means of this type:

$$\left( \frac{\frac{d^2 f_s}{dx^2}}{\frac{d^2 f_t}{dx^2}} \right)^{\frac{1}{s-t}} (\xi) = \xi, \;\; \xi \in [a,b], \quad s \neq t.$$

### 5.1.7 Examples with Applications

In this section, we use various classes of functions $\Omega = \{f_t : t \in I\}$, where $I$ is an interval in $\mathbb{R}$, to construct different examples of exponentially convex functions and Stolarsky-type means. Let us consider some examples.

***Example 5.1*** Let $F_1 = \{\psi_t : \mathbb{R} \to [0,\infty\rangle : t \in \mathbb{R}\}$ be a family of functions defined by

$$\psi_t(x) = \begin{cases} \dfrac{e^{tx}}{t^n}, & t \neq 0, \\ \dfrac{x^n}{n!}, & t = 0. \end{cases}$$

Since $\frac{d^n}{dx^n}\psi_t(x) = e^{tx} > 0$, the function $\psi_t(x)$ is $n$-convex on $\mathbb{R}$ for every $t \in \mathbb{R}$ and $t \to \frac{d^n}{dx^n}\psi_t(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorems 5.36, we have that $t \mapsto [z_0, z_1, \ldots, z_n; \psi_t]$ is exponentially convex (and so exponentially convex in the $J$-sense). Using Corollary 5.4 we conclude that $t \mapsto \Lambda_k(\psi_t)$, $k \in \{1,\ldots,6\}$ are exponentially convex in the $J$-sense. It is easy to see that these mappings are continuous, so they are exponentially convex.

Assume that $t \mapsto \Lambda_k(\psi_t) > 0$ for $k \in \{1,\ldots,6\}$. By inserting functions $\psi_t$ and $\psi_s$ in (5.81), we obtain the following means: for $k \in \{1,\ldots,6\}$

$$\mathfrak{M}_{s,t}(\Lambda_k, F_1) = \begin{cases} \dfrac{1}{s-t}\log\left(\dfrac{\Lambda_k(\psi_s)}{\Lambda_k(\psi_t)}\right), & s \neq t, \\ \dfrac{\Lambda_k(id \cdot \psi_s)}{\Lambda_k(\psi_s)} - \dfrac{n}{s}, & s = t \neq 0, \\ \dfrac{\Lambda_k(id \cdot \psi_0)}{(n+1)\Lambda_k(\psi_0)}, & s = t = 0, \end{cases}$$

where $id$ stands for the identity function on $\mathbb{R}$. Here $\mathfrak{M}_{s,t}(\Lambda_k, F_1) = \log(\mu_{s,t}(\Lambda_k, F_1))$, $k \in \{1,\ldots,6\}$ are in fact means.

We observe here that $\left(\dfrac{\frac{d^n\psi_s}{dx^n}}{\frac{d^n\psi_t}{dx^n}}\right)^{\frac{1}{s-t}}(\log\xi) = \xi$ is a mean for $\xi \in [a,b]$ where $a, b \in \mathbb{R}_+$.

***Example 5.2*** Let $F_2 = \{\varphi_t : \langle 0,\infty\rangle \to \mathbb{R} : t \in \mathbb{R}\}$ be a family of functions defined as

$$\varphi_t(x) = \begin{cases} \dfrac{x^t}{t(t-1)\cdots(t-n+1)}, & t \notin \{0,\ldots,n-1\}, \\ \dfrac{(x)^j \log x}{(-1)^{n-1-j} j!(n-1-j)!}, & t = j \in \{0,\ldots,n-1\}. \end{cases}$$

Since $\varphi_t(x)$ is an $n$-convex function for $x \in (0,\infty)$ and $t \mapsto \frac{d^2}{dx^2}\varphi_t(x)$ is exponentially convex, so by the same arguments given in the previous example we conclude that $\Lambda_k(\varphi_t)$, $k \in \{1,\ldots,6\}$ are exponentially convex.

We assume that $\Lambda_k(\varphi_t) > 0$ for $k \in \{1,\ldots,6\}$. For this family of $n$-convex functions we obtain the following means: for $k \in \{1,\ldots,6\}$, $J = \{0,1,\ldots,n-1\}$

$$\mathfrak{M}_{s,t}(\Lambda_k, F_2) = \begin{cases} \left(\dfrac{\Lambda_k(\varphi_s)}{\Lambda_k(\varphi_t)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left((-1)^{n-1}(n-1)!\dfrac{\Lambda_k(\varphi_0\varphi_s)}{\Lambda_k(\varphi_s)} + \displaystyle\sum_{k=0}^{n-1}\dfrac{1}{k-t}\right), & s = t \notin J, \\ \exp\left((-1)^{n-1}(n-1)!\dfrac{\Lambda_k(\varphi_0\varphi_s)}{2\Lambda_k(\varphi_s)} + \displaystyle\sum_{k=0,k\neq t}^{n-1}\dfrac{1}{k-t}\right), & s = t \in J. \end{cases}$$

Here $\mathfrak{M}_{s,t}(\Lambda_k, F_2) = \mu_{s,t}(\Lambda_k, F_2)$, $k \in \{1,\ldots,6\}$ are in fact means.

**Remark 5.12** Further, in this choice of family $F_2$, we have

$$\left(\frac{\frac{d^n\varphi_s}{dx^n}}{\frac{d^n\varphi_t}{dx^n}}\right)^{\frac{1}{s-t}}(\xi) = \xi,\ \ \xi \in [a,b],\ s \neq t,\ \text{where } a,b \in \langle 0,\infty\rangle.$$

So, using Remark 5.11 we have an important conclusion that $\mu_{s,t}(\Lambda_k, F_2)$ is in fact a mean for $k \in \{1,\ldots,6\}$.

***Example 5.3*** Let $F_3 = \{\theta_t : \langle 0,\infty\rangle \to \langle 0,\infty\rangle : t \in \langle 0,\infty\rangle\}$ be a family of functions defined by

$$\theta_t(x) = \frac{e^{-x\sqrt{t}}}{t^{n/2}}.$$

The function $t \mapsto \frac{d^n}{dx^n}\theta_t(x) = e^{-x\sqrt{t}}$ is exponentially convex for $x > 0$, being the Laplace transform of a nonnegative function [22]. So, by the same argument as in Example 5.1 we conclude that $\Lambda_k(\theta_t)$, $k \in \{1,\ldots,6\}$ are exponentially convex.

We assume that $\Lambda_k(\theta_t) > 0$ for $k \in \{1,\ldots,6\}$. For this family of functions we have the following possible cases of $\mu_{s,t}(\Lambda_k, F_3)$: for $k \in \{1,\ldots,6\}$

$$\mathfrak{M}_{s,t}(\Lambda_k, F_3) = \begin{cases} \left(\dfrac{\Lambda_k(\theta_s)}{\Lambda_k(\theta_t)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(-\dfrac{\Lambda_k(id\cdot\theta_s)}{2\sqrt{s}\,\Lambda_k(\theta_s)} - \dfrac{n}{2s}\right), & s = t. \end{cases}$$

By (5.81), $\mathfrak{M}_{s,t}(\Lambda_k, F_3) = -(\sqrt{s}+\sqrt{t})\log\mu_{s,t}(\Lambda_k, F_3)$, $k \in \{1,\ldots,6\}$ defines a class of means.

***Example 5.4*** Let $F_4 = \{\phi_t : (0,\infty) \to (0,\infty) : t \in (0,\infty)\}$ be a family of functions defined by

$$\phi_t(x) = \begin{cases} \dfrac{t^{-x}}{\log^n t}, & t \neq 1, \\ \dfrac{x^n}{n}, & t = 1. \end{cases}$$

Since $\frac{d^n}{dx^n}\phi_t(x) = t^{-x} = e^{-x\log t} > 0$ for $x > 0$, by the same argument as in Example 5.1 we conclude that $t \mapsto \Lambda_k(\phi_t)$, $k \in \{1,\ldots,6\}$ are exponentially convex.

We assume that $\Lambda_k(\phi_t) > 0$ for $k \in \{1,\ldots,6\}$. For this family of functions we have the following possible cases of $\mu_{s,t}(\Lambda_k, F_4)$: for $k \in \{1,\ldots,6\}$

$$\mathfrak{M}_{s,t}(\Lambda_k, F_4) = \begin{cases} \left(\dfrac{\Lambda_k(\phi_s)}{\Lambda_k(\phi_t)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(-\dfrac{\Lambda_k(id\cdot\phi_s)}{s\Lambda_k(\phi_s)} - \dfrac{n}{s\log s}\right), & s = t \neq 1, \\ \exp\left(-\dfrac{1}{(n+1)}\dfrac{\Lambda_k(id\cdot\phi_1)}{\Lambda_k(\phi_1)}\right), & s = t = 1. \end{cases}$$

By (5.81), $\mathfrak{M}_{s,t}(\Lambda_k, F_4) = -L(s,t)\log\mu_{s,t},(\Lambda_k, F_4)$, $\quad k \in \{1,\ldots,6\}$ defines a class of means, where $L(s,t)$ is the logarithmic mean defined as:

$$L(s,t) = \begin{cases} \dfrac{s-t}{\log s - \log t}, & s \neq t, \\ s, & s = t. \end{cases} \tag{5.87}$$

Monotonicity of $\mu_{s,t}(\Lambda_k, F_j)$ follow form (5.83) for $j \in \{1,2,3,4\}$ $k \in \{1,\ldots,6\}$.

## 5.2 Linear Inequalities via the Taylor formula

While in the previous section we use an extension of the Montgomery identity to make new identities for the sum $\sum_{i=1}^{m} p_i f(x_i)$ and the integral $\int_\alpha^\beta p(x)f(g(x))dx$, in this section the Taylor formula has a crucial role in our attempts to get new identities for the above-mentioned sum and integral. The results of this section are given in [31].

### 5.2.1 Inequalities via the Taylor Formula

Our first result is an identity which is a basic tool for our subsequent investigation. In fact this identity is given in Chapter 2 but here we repeat it because it is a base for further results in this section.

**Theorem 5.37** *Let $n,m \in \mathbb{N}$ and $f : I \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous on $I \subset \mathbb{R}$, $a,b \in I$, $a < b$. Furthermore, let $x_i \in [a,b]$ and $p_i \in \mathbb{R}$ for $i \in \{1,2,\ldots,m\}$. Then*

$$\sum_{i=1}^{m} p_i f(x_i) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} \sum_{i=1}^{m} p_i (x_i - a)^k$$

$$+\frac{1}{(n-1)!}\int_a^b f^{(n)}(s)\left(\sum_{i=1}^m p_i(x_i-s)_+^{n-1}\right)ds \tag{5.88}$$

*and*

$$\begin{aligned}\sum_{i=1}^m p_i f(x_i) &= \sum_{k=0}^{n-1}(-1)^k\frac{f^{(k)}(b)}{k!}\sum_{i=1}^m p_i(b-x_i)^k\\ &+\frac{(-1)^n}{(n-1)!}\int_a^b f^{(n)}(s)\left(\sum_{i=1}^m p_i(s-x_i)_+^{n-1}\right)ds.\end{aligned}$$

We may state its integral version as follows.

**Theorem 5.38** *Let $g:[\alpha,\beta]\to[a,b]$ and $p:[\alpha,\beta]\to\mathbb{R}$ be integrable functions. Let $n\in\mathbb{N}$ and $f:I\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous on $I\subset\mathbb{R}$, $a,b\in I$, $a<b$. Then*

$$\begin{aligned}\int_\alpha^\beta p(x)f(g(x))\,dx &= \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}\int_\alpha^\beta p(x)(g(x)-a)^k dx\\ &+\frac{1}{(n-1)!}\int_a^b f^{(n)}(s)\int_\alpha^\beta p(x)\,(g(x)-s)_+^{n-1}\,dxds,\end{aligned}$$

$$\begin{aligned}\int_\alpha^\beta p(x)f(g(x))\,dx &= \sum_{k=0}^{n-1}(-1)^k\frac{f^{(k)}(b)}{k!}\int_\alpha^\beta p(x)\,(b-g(x))^k dx\\ &+\frac{(-1)^n}{(n-1)!}\int_a^b f^{(n)}(s)\int_\alpha^\beta p(x)\,(s-g(x))_+^{n-1}\,dxds.\end{aligned}$$

Now we state inequalities derived from the obtained identities. In the rest of the section we use the following notation:

$$\Omega_1^{[a,b]}(m,\mathbf{x},\mathbf{p},s):=\sum_{i=1}^m p_i\,(x_i-s)_+^{n-1}, \tag{5.89}$$

$$\Omega_2^{[a,b]}(m,\mathbf{x},\mathbf{p},s):=(-1)^n\sum_{i=1}^m p_i\,(s-x_i)_+^{n-1}, \tag{5.90}$$

$$A_1^{[a,b]}(m,\mathbf{x},\mathbf{p},f):=\sum_{i=1}^m p_i f(x_i)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}\sum_{i=1}^m p_i(x_i-a)^k, \tag{5.91}$$

$$A_2^{[a,b]}(m,\mathbf{x},\mathbf{p},f):=\sum_{i=1}^m p_i f(x_i)-\sum_{k=0}^{n-1}(-1)^k\frac{f^{(k)}(b)}{k!}\sum_{i=1}^m p_i(b-x_i)^k. \tag{5.92}$$

**Theorem 5.39** *Let $n,m\in\mathbb{N}$, $x_i\in[a,b]$, $I$ is an interval, $[a,b]\subset I$ and $p_i\in\mathbb{R}$ for $i\in\{1,2,\ldots,m\}$.*

*(i) If*

$$(U_1) \qquad \Omega_1^{[a,b]}(m,\mathbf{x},\mathbf{p},s) \geq 0, \quad \text{for all } s \in [a,b],$$

*then for every n-convex function* $f : I \to \mathbb{R}$ *such that* $f^{(n-1)}$ *is absolutely continuous on I the following inequality holds*

$$A_1^{[a,b]}(m,\mathbf{x},\mathbf{p},f) \geq 0. \tag{5.93}$$

*If in* $(U_1)$ *reversed sign of inequality holds, then inequality* (5.93) *is also reversed.*

*(ii) If*

$$(U_2) \qquad \Omega_2^{[a,b]}(m,\mathbf{x},\mathbf{p},s) \geq 0, \quad \text{for all } s \in [a,b],$$

*then for every n-convex function* $f : I \to \mathbb{R}$ *such that* $f^{(n-1)}$ *is absolutely continuous on I the following inequality holds*

$$A_2^{[a,b]}(m,\mathbf{x},\mathbf{p},f) \geq 0. \tag{5.94}$$

*If in* $(U_2)$ *reversed sign of inequality holds, then inequality* (5.94) *is also reversed.*

*If the condition "f is n-convex" is replaced by "f is n-concave", then under the same assumptions about* $\Omega_1^{[a,b]}$ *and* $\Omega_2^{[a,b]}$*, inequalities* (5.93) *and* (5.94) *hold in the reversed direction.*

*Proof.* We prove (i). Let $\Omega_1^{[a,b]}(m,\mathbf{x},\mathbf{p},s) \geq 0$ for all $s \in [a,b]$ and let $f$ be $n$-convex. Then, $f^{(n)} \geq 0$ and

$$\int_a^b f^{(n)}(s)\left(\sum_{i=1}^m p_i(x_i-s)_+^{n-1}\right) ds \geq 0.$$

By Theorem 5.37

$$A_1^{[a,b]}(m,\mathbf{x},\mathbf{p},f) = \frac{1}{(n-1)!}\int_a^b f^{(n)}(s)\left(\sum_{i=1}^m p_i(x_i-s)_+^{n-1}\right) ds \geq 0$$

and we get (5.93). Other cases are proved in a similar manner. □

Now we state an important consequence.

**Theorem 5.40** *Let* $n \in \mathbb{N}$, $n \geq 2$, $[a,b] \subset I \subseteq \mathbb{R}$ *and* $f : I \to \mathbb{R}$ *be a function such that* $f^{(n-1)}$ *is absolutely continuous. Additionally, let* $j \in \mathbb{N}$ *be fixed,* $2 \leq j \leq n$ *and let* $(x_1,\ldots,x_m) \in [a,b]^m$, $(p_1,\ldots,p_m) \in \mathbb{R}^m$ *satisfy*

$$\sum_{i=1}^m p_i x_i^k = 0 \;\; for \;\; k = 0,1,\ldots,j-1, \tag{5.95}$$

$$\sum_{i=1}^m p_i(x_i-s)_+^{j-1} \geq 0, \;\; for\; s \in [a,b]. \tag{5.96}$$

*If f is n-convex, then*

$$\sum_{i=1}^m p_i f(x_i) \geq \sum_{k=j}^{n-1} \frac{f^{(k)}(a)}{k!} \sum_{i=1}^m p_i (x_i-a)^k \tag{5.97}$$

*with agreement that for $j = n$, we put $\sum_{k=j}^{n-1} = 0$.*

*Furthermore, if $n - j$ is even, then*

$$\sum_{i=1}^{m} p_i f(x_i) \geq \sum_{k=j}^{n-1} (-1)^k \frac{f^{(k)}(b)}{k!} \sum_{i=1}^{m} p_i (b - x_i)^k \tag{5.98}$$

*while if $n - j$ is odd, then the reversed inequality in* (5.98) *holds.*

*Proof.* Let $s \in [a,b]$ be fixed. Notice that for $j = n$ we just get Theorem 5.1. For $j \leq n-2$ we get

$$\frac{d^j}{dx^j}(x-s)_+^{n-1} = \begin{cases} (n-1)(n-2)\cdots(n-j)(x-s)^{n-j-1}, & s \leq x \leq b, \\ 0, & a \leq x < s, \end{cases}$$

and

$$(-1)^j \frac{d^j}{dx^j}(s-x)_+^{n-1} = \begin{cases} (n-1)(n-2)\cdots(n-j)(s-x)^{n-j-1}, & a \leq x \leq s, \\ 0, & s < x \leq b, \end{cases}$$

The functions $x \mapsto \frac{d^j}{dx^j}(x-s)_+^{n-1}$ and $x \mapsto (-1)^j \frac{d^j}{dx^j}(s-x)_+^{n-1}$ are nonnegative. Hence the functions $x \mapsto (x-s)_+^{n-1}$ and $x \mapsto (-1)^j (s-x)_+^{n-1}$ are $j$-convex.

If $j = n-1$, then we consider the functions $x \mapsto \frac{d^{n-3}}{dx^{n-3}}(x-s)_+^{n-1}$ and $x \mapsto (-1)^{n-1} \frac{d^{n-3}}{dx^{n-3}}(s-x)_+^{n-1}$. They are 2-convex, so $x \mapsto (x-s)_+^{n-1}$ and $x \mapsto (-1)^{n-1}(s-x)_+^{n-1}$ are $(n-1)$-convex. Hence if $2 \leq j \leq n-1$, functions $x \mapsto (x-s)_+^{n-1}$ and $x \mapsto (-1)^j (s-x)_+^{n-1}$ are $j$-convex.

Using Theorem 5.1 for $j$-convex functions $x \mapsto (x-s)_+^{n-1}$ and $x \mapsto (-1)^j (s-x)_+^{n-1}$, we get that

$$\sum_{i=1}^{m} p_i (x_i - s)_+^{n-1} \geq 0 \tag{5.99}$$

and

$$(-1)^j \sum_{i=1}^{m} p_i (s - x_i)_+^{n-1} \geq 0.$$

Multiplying the last inequality with $(-1)^{n-j}$ (it is positive for even $n-j$) we get

$$(-1)^n \sum_{i=1}^{m} p_i (s - x_i)_+^{n-1} \geq 0. \tag{5.100}$$

Inequalities (5.99) and (5.100) mean that assumptions of Theorem 5.39 (i) and (ii) are satisfied, hence inequalities (5.93) and (5.94) hold respectively. Moreover, due to assumption (5.95), $\sum_{i=1}^{m} p_i P(x_i) = 0$ for every polynomial $P$ of degree $\leq j-1$, so the first $j$ terms in the inner sum in (5.91) and (5.92) vanish, i.e. we get inequalities (5.97) and (5.98). □

**Theorem 5.41** *Let $n \in \mathbb{N}, n \geq 3$. Let $j \in \{2,3,\ldots,n-1\}$ be a fixed number and let $m$-tuples $\mathbf{x} = (x_1,\ldots,x_m) \in [a,b]^m$, $\mathbf{p} = (p_1,\ldots,p_m) \in \mathbb{R}^m$ satisfy*

$$\sum_{i=1}^{m} p_i x_i^k = 0, \quad \textit{for all } k \in \{0,1,\ldots,j-1\} \tag{5.101}$$

$$\sum_{i=1}^{m} p_i(x_i - s)_+^{j-1} \geq 0, \quad \text{for every } s \in [a,b]. \tag{5.102}$$

*If* $[a,b] \subset I \subseteq \mathbb{R}$ *and* $f : I \to \mathbb{R}$ *is n-convex such that* $f^{(n-1)}$ *is absolutely continuous with at least one of the following two properties*

(*i*) $\displaystyle\sum_{k=j}^{n-1} \frac{f^{(k)}(a)}{(k-j)!}(x-a)^{k-j} \geq 0$ *for all* $x \in [a,b]$

(*ii*) $\displaystyle\sum_{k=j}^{n-1} (-1)^{k-j} \frac{f^{(k)}(b)}{(k-j)!}(b-x)^{k-j} \geq 0$ *for all* $x \in [a,b]$ *with even* $n-j$, *then the inequality*

$$\sum_{i=1}^{m} p_i f(x_i) \geq 0 \tag{5.103}$$

*holds.*

*Proof.* Let us suppose that $f$ satisfies property (i). Define $H$ by

$$H(x) = \sum_{k=j}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Then

$$H^{(j)}(x) = \sum_{k=j}^{n-1} \frac{f^{(k)}(a)}{(k-j)!}(x-a)^{k-j}$$

and $H^{(j)}(x) \geq 0$, $x \in [a,b]$. Hence $H$ is $j$-convex. Using Theorem 5.1 for the $j$-convex function $H$ we obtain

$$\sum_{i=1}^{m} p_i H(x_i) \geq 0.$$

That conclusion and the previous theorem give

$$\sum_{i=1}^{m} p_i f(x_i) \geq \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} \sum_{i=1}^{m} p_i(x_i - a)^k = \sum_{i=1}^{m} p_i H(x_i) \geq 0$$

which is the desired inequality (5.103). If $f$ satisfies property $(ii)$, then we consider the function $H(x) = \displaystyle\sum_{k=j}^{n-1} (-1)^k \frac{f^{(k)}(b)}{k!}(b-x)^k$ and proceed in a similar manner. □

**Remark 5.13** Let us consider the case: $j = n-1$. Then for an $n$-convex $f$ under the assumptions $f^{(n-1)}(a) \geq 0$, (5.101) and (5.102) we get $\sum_{i=1}^{m} p_i f(x_i) \geq 0$. In comparison with Theorem 5.1, we see that one condition is added and (5.2), (5.3) are valid not for $n$, but for $n-1$. So, this result is an improvement of one direction given in Theorem 5.1.

In the rest of the section we state integral versions of the previous results, the proofs of which are analogous to the discrete case.

**Theorem 5.42** *Let $g : [\alpha,\beta] \to [a,b]$ and $p : [\alpha,\beta] \to \mathbb{R}$ be integrable functions and let $f : I \to \mathbb{R}$, $[a,b] \subset I$, be such that $f^{(n-1)}$ is absolutely continuous.*

*If*

$(U_3)$ $\qquad \Omega_3^{[a,b]}([\alpha,\beta],g,p,s) := \int_\alpha^\beta p(x)\,(g(x)-s)_+^{(n-1)}\,dx \geq 0$, *for all* $s \in [a,b]$, *then for every n-convex function f the following inequality holds*

$$A_3^{[a,b]}\left([\alpha,\beta],g,p,f\right) := \int_\alpha^\beta p\,(x)\,f(g(x))\,dx - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} \int_\alpha^\beta p(x)(g(x)-a)^k dx \geq 0, \tag{5.104}$$

*If in $(U_3)$ reversed sign of inequality holds, then inequality (5.104) is also reversed.*

*If*

$(U_4)$ $\qquad \Omega_4^{[a,b]}([\alpha,\beta],g,p,s) := (-1)^n \int_\alpha^\beta p(x)\,(s-g(x))_+^{(n-1)}\,dx \geq 0$, *for all* $s \in [a,b]$, *then for every n-convex function f the following inequality holds*

$$A_4^{[a,b]}\left([\alpha,\beta],g,p,f\right) := \int_\alpha^\beta p\,(x)\,f(g(x))\,dx - \sum_{k=0}^{n-1} (-1)^k \frac{f^{(k)}(b)}{k!} \int_\alpha^\beta p(x)(b-g(x))^k dx \geq 0. \tag{5.105}$$

*If in $(U_4)$ reversed sign of inequality holds, then inequality (5.105) is also reversed.*

*If the condition "f is n-convex" is replaced by "f is n-concave", then under the same assumptions about $\Omega_3^{[a,b]}$ and $\Omega_4^{[a,b]}$, inequalities (5.104) and (5.105) hold in the reversed direction.*

**Theorem 5.43** *Suppose all the assumptions from Theorem 5.38 hold. Additionally, let $j \in \mathbb{N}$, $2 \leq j \leq n$ and let $p : [\alpha,\beta] \to \mathbb{R}$ and $g : [\alpha,\beta] \to [a,b]$ satisfy*

$$\int_\alpha^\beta p(x)g(x)^k\,dx = 0, \qquad \text{for all } k \in \{0,1,\ldots,j-1\}$$

$$\int_\alpha^\beta p(x)\,(g(x)-s)_+^{j-1}\,dx \geq 0, \qquad \text{for every } s \in [a,b].$$

*If f is n-convex, then*

$$\int_\alpha^\beta p\,(x)\,f(g(x))\,dx \geq \sum_{k=j}^{n-1} \frac{f^{(k)}(a)}{k!} \int_\alpha^\beta p\,(x)\,(g(x)-a)_+^k\,dx.$$

*If, in addition $n-j$ is even, then*

$$\int_\alpha^\beta p\,(x)\,f(g(x))\,dx \geq \sum_{k=j}^{n-1} (-1)^k \frac{f^{(k)}(b)}{k!} \int_\alpha^\beta p\,(x)\,(b-g(x))_+^k\,dx \tag{5.106}$$

*while if $n-j$ is odd, then the reversed sign of inequality holds in (5.106)*

### 5.2.2 Inequalities via the Green Function

In this section we obtain other identities and the corresponding linear inequality using the Green function defined by (5.10) and applying again the Taylor formula. The next theorem contains two identities in which the sum $\sum_{i=1}^{m} p_i f(x_i)$ is expressed as a relation involving the $n$-th derivative of the function $f$ and the values of first $n-3$ derivatives of $f$ only in points $a$ or $b$. The whole subsection is based on results given in [31].

**Theorem 5.44** *Let $n \in \mathbb{N}$, $n \geq 3$, and $f : I \to \mathbb{R}$, $[a,b] \subset I$, be a function such that $f^{(n-1)}$ is absolutely continuous. Furthermore, let $m \in \mathbb{N}$, $x_i \in [a,b]$ and $p_i \in \mathbb{R}$ for $i \in \{1,2,\dots,m\}$ be such that*

$$\sum_{i=1}^{m} p_i = 0, \quad \sum_{i=1}^{m} p_i x_i = 0.$$

*Then*

$$\sum_{i=1}^{m} p_i f(x_i) = \sum_{k=0}^{n-3} \frac{f^{(k+2)}(a)}{k!} \int_a^b \sum_{i=1}^{m} p_i G(x_i,t)(t-a)^k dt$$
$$+ \frac{1}{(n-3)!} \int_a^b f^{(n)}(s) \left( \int_s^b \sum_{i=1}^{m} p_i G(x_i,t)(t-s)^{n-3} dt \right) ds \tag{5.107}$$

*and*

$$\sum_{i=1}^{m} p_i f(x_i) = \sum_{k=0}^{n-3} (-1)^k \frac{f^{(k+2)}(b)}{k!} \int_a^b \sum_{i=1}^{m} p_i G(x_i,t)(b-t)^k dt$$
$$- \frac{1}{(n-3)!} \int_a^b f^{(n)}(s) \left( \int_a^s \sum_{i=1}^{m} p_i G(x_i,t)(t-s)^{n-3} dt \right) ds. \tag{5.108}$$

*Proof.* Using integration by parts the following is valid

$$f(x) = \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b) + \int_a^b G(x,t) f''(t) dt.$$

Putting in the above equality $x = x_i$, multiplying with $p_i$, adding all equalities for $i = 1,\dots,m$ and using conditions that $\sum_{i=1}^{m} p_i = 0$, $\sum_{i=1}^{m} p_i x_i = 0$ we get

$$\sum_{i=1}^{m} p_i f(x_i) = \int_a^b \left( \sum_{i=1}^{m} p_i G(x_i,t) \right) f''(t) dt.$$

Differentiating the Taylor formula twice we get

$$f''(x) = \sum_{k=0}^{n-3} \frac{f^{(k+2)}(c)}{k!} (x-c)^k + \frac{1}{(n-3)!} \int_c^x f^{(n)}(s)(x-s)^{n-3} ds. \tag{5.109}$$

Putting in (5.109) $c = a$ and $c = b$ respectively we get

$$\sum_{i=1}^{m} p_i f(x_i) = \sum_{k=0}^{n-3} \frac{f^{(k+2)}(a)}{k!} \int_a^b \left( \sum_{i=1}^{m} p_i G(x_i,t) \right) (t-a)^k dt$$

$$+\frac{1}{(n-3)!}\int_a^b\int_a^t f^{(n)}(s)(t-s)^{n-3}\left(\sum_{i=1}^m p_i G(x_i,t)\right)dsdt$$

and

$$\begin{aligned}\sum_{i=1}^m p_i f(x_i) &= \sum_{k=0}^{n-3}\frac{f^{(k+2)}(b)}{k!}\int_a^b\left(\sum_{i=1}^m p_i G(x_i,t)\right)(t-b)^k dt\\ &+\frac{1}{(n-3)!}\int_a^b\int_b^t f^{(n)}(s)(t-s)^{n-3}\left(\sum_{i=1}^m p_i G(x_i,t)\right)dsdt.\end{aligned}$$

Using the Fubini theorem we obtain identities (5.107) and (5.108). □

**Theorem 5.45** *Let $n,m\in\mathbb{N}$, $n\geq 3$, $\mathbf{x}=(x_1,\ldots,x_m)\in[a,b]^m$ and $\mathbf{p}=(p_1,\ldots,p_m)\in\mathbb{R}^m$ be such that*

$$\sum_{i=1}^m p_i=0,\quad \sum_{i=1}^m p_i x_i=0. \tag{5.110}$$

*(i) If*

$$(U_5)\qquad \Omega_5^{[a,b]}(m,\mathbf{x},\mathbf{p},s):=\int_s^b\sum_{i=1}^m p_i G(x_i,t)(t-s)^{n-3}dt\geq 0 \text{ for all } s\in[a,b],$$

*then for every $n$-convex function $f:I\to\mathbb{R}$ such that $f^{(n-1)}$ is absolutely continuous on $I\subseteq[a,b]$ the following inequality holds*

$$A_5^{[a,b]}(m,\mathbf{x},\mathbf{p},f):=$$
$$\sum_{i=1}^m p_i f(x_i)-\sum_{k=0}^{n-3}\frac{f^{(k+2)}(a)}{k!}\int_a^b\sum_{i=1}^m p_i G(x_i,t)(t-a)^k dt\geq 0. \tag{5.111}$$

*If in $(U_5)$ reversed sign of inequality holds, then inequality $(5.111)$ is also reversed.*

*(ii) If*

$$(U_6)\qquad \Omega_6^{[a,b]}(m,\mathbf{x},\mathbf{p},s):=\int_a^s\sum_{i=1}^m p_i G(x_i,t)(t-s)^{n-3}dt\leq 0 \text{ for all } s\in[a,b],$$

*then for every $n$-convex function $f:I\to\mathbb{R}$ such that $f^{(n-1)}$ is absolutely continuous on $I\subseteq[a,b]$ the following inequality holds*

$$A_6^{[a,b]}(m,\mathbf{x},\mathbf{p},f):=$$
$$\sum_{i=1}^m p_i f(x_i)-\sum_{k=0}^{n-3}(-1)^k\frac{f^{(k+2)}(b)}{k!}\int_a^b\sum_{i=1}^m p_i G(x_i,t)(b-t)^k dt\geq 0. \tag{5.112}$$

*If in $(U_6)$ reversed sign of inequality holds, then inequality $(5.112)$ is also reversed.*

*(iii) If the condition "$f$ is $n$-convex" is replaced by "$f$ is $n$-concave", then under the same assumptions about $\Omega_5^{[a,b]}$ and $\Omega_6^{[a,b]}$, inequalities $(5.111)$ and $(5.112)$ hold in the reversed direction.*

*Proof.* If $f$ is $n$-convex, without lossing of generality we can assume that $f$ is $n$-times differentiable and $f^{(n)} \geq 0$. Using this fact and the identities from Theorem 5.44 we get the required results. □

If we add a new condition on $\mathbf{x}$, then in the previous statements we can remove assumptions about $\Omega_5$ and $\Omega_6$. More precisely, we have the following result.

**Theorem 5.46** *Let $n \in \mathbb{N}$, $n \geq 3$, and $f : I \to \mathbb{R}$, $[a,b] \subset I$, be a function such that $f^{(n-1)}$ is absolutely continuous. Furthermore, let $m \in \mathbb{N}$, $x_i \in [a,b]$ and $p_i \in \mathbb{R}$ for $i \in \{1,2,\ldots,m\}$ be such that*

$$\sum_{i=1}^{m} p_i = 0, \quad \sum_{i=1}^{m} p_i |x_i - x_k| \geq 0 \text{ for } k = 1,2,\ldots,m. \tag{5.113}$$

*If $f$ is $n$-convex, then (5.111) holds. If $n$ is even, then (5.112) is valid, while if $n$ is odd, then a reversed sign in inequality (5.112) holds.*

*If $f$ is $n$-concave, then reversed (5.111) holds. If $n$ is even, then reversed (5.112), while if $n$ is odd, then inequality (5.112) holds.*

*Proof.* Since

$$\sum_{i=1}^{m} p_i |x_i - x_k| = 2\sum_{i=1}^{m} p_i (x_i - x_k)_+ - \sum_{i=1}^{m} p_i (x_i - x_k),$$

condition (5.113) is equivalent to

$$\sum_{i=1}^{m} p_i = 0, \quad \sum_{i=1}^{m} p_i x_i = 0, \quad \sum_{i=1}^{m} p_i (x_i - x_k)_+ \geq 0$$

for $k \in \{1,\ldots,m-1\}$ which means that $m$-tuples $\mathbf{x}$, $\mathbf{p}$ satisfy assumptions of Theorem 5.1. Since $G$ is convex with respect to the first variable, using Theorem 5.1 we conclude that

$$\sum_{i=1}^{m} p_i G(x_i,t) \geq 0 \quad \text{for } t \in [a,b].$$

Note that $(t-s)^{n-3} \geq 0$ for $t \in [s,b]$ so we get $\Omega_5^{[a,b]}(m,\mathbf{x},\mathbf{p},s) \geq 0$. By Theorem 5.45 (i), we have that $A_5^{[a,b]}(m,\mathbf{x},\mathbf{p},f) \geq 0$. Other parts are proved in a similar manner. □

The integral versions of the previous three theorems may also be stated. Since the proofs of these results are similar, we omit the details.

**Theorem 5.47** *Let $g : [\alpha,\beta] \to [a,b]$, $p : [\alpha,\beta] \to \mathbb{R}$ be integrable functions such that*

$$\int_{\alpha}^{\beta} p(x)dx = 0, \quad \int_{\alpha}^{\beta} p(x)g(x)dx = 0. \tag{5.114}$$

*Let $n \geq 3$ and $f : I \to \mathbb{R}$, $[a,b] \subset I$, be a function such that $f^{(n-1)}$ is absolutely continuous. Then we get the following identities*

$$\int_{\alpha}^{\beta} p(x) f(g(x))\, dx$$

$$
\begin{aligned}
&= \sum_{k=0}^{n-3} \frac{f^{(k+2)}(a)}{k!} \int_a^b \left( \int_\alpha^\beta p(x) G(g(x),t)\,dx \right) (t-a)^k\,dt \qquad (5.115)\\
&+ \frac{1}{(n-3)!} \int_a^b f^{(n)}(s) \left( \int_s^b \left( \int_\alpha^\beta p(x) G(g(x),t)\,dx \right) (t-s)^{n-3}\,dt \right) ds,
\end{aligned}
$$

$$
\begin{aligned}
&\int_\alpha^\beta p(x) f(g(x))\,dx \\
&= \sum_{k=0}^{n-3} (-1)^k \frac{f^{(k+2)}(b)}{k!} \int_a^b \left( \int_\alpha^\beta p(x) G(g(x),t)\,dx \right) (b-t)^k dt \qquad (5.116)\\
&- \frac{1}{(n-3)!} \int_a^b f^{(n)}(s) \left( \int_a^s \left( \int_\alpha^\beta p(x) G(g(x),t)\,dx \right) (t-s)^{n-3} dt \right) ds.
\end{aligned}
$$

**Theorem 5.48** *Let $g, p, n$ satisfy assumptions of Theorem 5.47 hold.*

*(i) If*

$(U_7)$ $\Omega_7^{[a,b]}([\alpha,\beta], g, p, s) := \int_s^b \left( \int_\alpha^\beta p(x) G(g(x),t)\,dx \right) (t-s)^{n-3} dt \geq 0$ *for all $s \in [a,b]$, then for every n-convex function $f : I \to \mathbb{R}$, $[a,b] \subset I$, such that $f^{(n-1)}$ is absolutely continuous, the following inequality holds*

$$
\begin{aligned}
A_7^{[a,b]}([\alpha,\beta], g, p, f) &:= \int_\alpha^\beta p(x) f(g(x))\,dx \\
&- \sum_{k=0}^{n-3} \frac{f^{(k+2)}(a)}{k!} \int_a^b \left( \int_\alpha^\beta p(x) G(g(x),t)\,dx \right) (t-a)^k\,dt \geq 0. \qquad (5.117)
\end{aligned}
$$

*If in $(U_7)$ reversed sign of inequality holds, then inequality (5.117) is also reversed.*

*(ii) If*

$(U_8)$ $\Omega_8^{[a,b]}([\alpha,\beta], g, p, s) := \int_a^s \left( \int_\alpha^\beta p(x) G(g(x),t)\,dx \right) (t-s)^{n-3} dt \leq 0$ *for all $s \in [a,b]$, then for every n-convex function $f : I \to \mathbb{R}$, $[a,b] \subset I$, such that $f^{(n-1)}$ is absolutely continuous, the following inequality holds*

$$
\begin{aligned}
A_8^{[a,b]}([\alpha,\beta], g, p, f)) &:= \int_\alpha^\beta p(x) f(g(x))\,dx \\
&- \sum_{k=0}^{n-3} (-1)^k \frac{f^{(k+2)}(b)}{k!} \int_a^b \left( \int_\alpha^\beta p(x) G(g(x),t)\,dx \right) (b-t)^k dt \geq 0. \qquad (5.118)
\end{aligned}
$$

*If in $(U_8)$ reversed sign of inequality holds, then inequality (5.118) is also reversed.*

*(iii) If the condition "f is n-convex" is replaced by "f is n-concave", then under the same assumptions about $\Omega_7^{[a,b]}$ and $\Omega_8^{[a,b]}$, inequalities (5.117) and (5.118) hold in the reversed direction.*

**Theorem 5.49** *Let all the assumptions of Theorem 5.47 hold. Additionally, let*

$$
\int_\alpha^\beta p(x) (g(x)-t)_+ dx \geq 0 \text{ for all } t \in [a,b].
$$

*If $f$ is n-convex, then* (5.117) *holds. If n is even, then* (5.118)*, while if n is odd, then a reversed sign in inequality* (5.118) *holds.*

*If $f$ is n-concave, then reversed* (5.117) *holds. If n is even, then reversed* (5.118)*, while if n is odd, then inequality* (5.118) *holds.*

### 5.2.3 Inequalities for $n$-convex Functions at a Point

In this section we give related results for the class of $n$-convex functions at a point which is introduced in [76] and described in Chapter 2. First we state our main theorem of this section for the discrete case, [31].

**Theorem 5.50** *Let $c \in \langle a,b \rangle$, $\mathbf{x} \in [a,c]^m$, $\mathbf{y} \in [c,b]^l$, $\mathbf{p} \in \mathbb{R}^m$, $\mathbf{q} \in \mathbb{R}^l$ and $f : [a,b] \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous.*

*(i) For $k = 1,2$ let $A_k^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,f)$ and $\Omega_k^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,s)$ be defined as in* (5.89)−−(5.92) *and satisfy the following conditions:*

$$\Omega_k^{[a,c]}(m,\mathbf{x},\mathbf{p},s) \geq 0, \quad \text{for every } s \in [a,c], \tag{5.119}$$

$$\Omega_k^{[c,b]}(l,\mathbf{y},\mathbf{q},s) \geq 0, \quad \text{for every } s \in [c,b], \tag{5.120}$$

*and*

$$A_k^{[a,c]}(m,\mathbf{x},\mathbf{p},e_n) = A_k^{[c,b]}(l,\mathbf{y},\mathbf{q},e_n). \tag{5.121}$$

*If $f$ is $(n+1)$-convex at point c, then*

$$A_k^{[a,c]}(m,\mathbf{x},\mathbf{p},f) \leq A_k^{[c,b]}(l,\mathbf{y},\mathbf{q},f). \tag{5.122}$$

*If inequalities in* (5.119) *and* (5.120) *are reversed, then* (5.122) *holds with the reversed sign of inequality.*

*(ii) For $k = 5,6$ let $A_k^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,f)$ and $\Omega_k^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,s)$ be defined as in Theorem* 5.45 *and let assumption* (5.110) *hold. For $k = 5$, if* (5.119), (5.120) *and* (5.121) *are valid, then for an $(n+1)$-convex function $f$ at point c, $(n \geq 3)$, inequality* (5.122) *holds. For $k = 6$, if* (5.121) *holds and reversed* (5.119), (5.120) *are valid, then inequality* (5.122) *holds.*

*Proof.* (i) Let $k \in \{1,2\}$ and (5.119), (5.120), (5.121) hold. Since $f$ is $(n+1)$-convex at point $c$ there exists a constant $K$ such that the function $F = f - \frac{K}{n!}e_n$ is $n$-concave on $[a,c]$ and $n$-convex on $[c,b]$.

Applying Theorem 5.39 to $F$ on the interval $[a,c]$ and on the interval $[c,b]$ we have

$$A_k^{[a,c]}(m,\mathbf{x},\mathbf{p},F) \leq 0 \leq A_k^{[c,b]}(l,\mathbf{y},\mathbf{q},F).$$

Using the definition of $F$ we obtain that

$$A_k^{[a,c]}(m,\mathbf{x},\mathbf{p},f) - \frac{K}{n!}A_k^{[a,c]}(m,\mathbf{x},\mathbf{p},e_n) \leq A_k^{[c,b]}(l,\mathbf{y},\mathbf{q},f) - \frac{K}{n!}A_k^{[c,b]}(l,\mathbf{y},\mathbf{q},e_n)$$

$$A_k^{[a,c]}(m,\mathbf{x},\mathbf{p},f) \leq A_k^{[c,b]}(l,\mathbf{y},\mathbf{q},f) - \frac{K}{n!}\Big[A_k^{[c,b]}(l,\mathbf{y},\mathbf{q},e_n) - A_k^{[a,c]}(m,\mathbf{x},\mathbf{p},e_n)\Big].$$

Since equality (5.121) is valid we get

$$A_k^{[a,c]}(m,\mathbf{x},\mathbf{p},f) \le A_k^{[c,b]}(l,\mathbf{y},\mathbf{q},f).$$

$\square$

A closer look at the proof of Theorem 5.50 give us that similar result holds if instead equality (5.121) we consider the positivity of the difference

$$K\left(A_k^{[c,b]}(l,\mathbf{y},\mathbf{q},e_n) - A_k^{[a,c]}(m,\mathbf{x},\mathbf{p},e_n)\right) \ge 0.$$

**Corollary 5.7** *Let $j_1, j_2, n \in \mathbb{N}$, $2 \le j_1, j_2 \le n$ and let $f : [a,b] \to \mathbb{R}$ be $(n+1)$-convex at point $c$. Let $m$-tuples $\mathbf{x} \in [a,c]^m$ and $\mathbf{p} \in \mathbb{R}^m$ satisfy*

$$\sum_{i=1}^{l} p_i x_i^k = 0, \quad \text{for all } k \in \{0,1,\ldots,j_1-1\}$$

$$\sum_{i=1}^{l} p_i (x_i - s)_+^{j_1-1} \ge 0, \quad \text{for every } s \in [a,c].$$

*Let $l$-tuples $\mathbf{y} \in [c,b]^l$ and $\mathbf{q} \in \mathbb{R}^l$ satisfy*

$$\sum_{i=1}^{l} q_i y_i^k = 0, \quad \text{for all } k \in \{0,1,\ldots,j_2-1\}$$

$$\sum_{i=1}^{l} q_i (y_i - s)_+^{j_2-1} \ge 0, \quad \text{for every } s \in [c,b]$$

*and let identity* (5.121) *holds.*

*Then*

$$A_1^{[a,c]}(m,\mathbf{x},\mathbf{p},f) \le A_1^{[c,b]}(l,\mathbf{y},\mathbf{q},f)$$

*and if $n-j_1, n-j_2$ are even, then*

$$A_2^{[a,c]}(m,\mathbf{x},\mathbf{p},f) \le A_2^{[c,b]}(l,\mathbf{y},\mathbf{q},f).$$

*Proof.* Since $f$ is $(n+1)$-convex at point $c$ there exists a constant $K$ such that function $F = f - \dfrac{K}{n!}e_n$ is $n$-concave on $[a,c]$ and $n$-convex on $[c,b]$. The number $j_1$ and $m$-tuples $\mathbf{x},\mathbf{p}$ satisfy the assumptions of Theorem 5.40 and for concave $F$ on $[a,c]$ we get

$$A_1^{[a,c]}(m,\mathbf{x},\mathbf{p},F) \le 0.$$

Also, the number $j_2$ and $l$-tuples $\mathbf{y},\mathbf{q}$ satisfy the assumptions of Theorem 5.40 and for convex $F$ on $[c,b]$ we get

$$A_1^{[c,b]}(l,\mathbf{y},\mathbf{q},F) \ge 0.$$

So we have

$$A_1^{[a,c]}(m,\mathbf{x},\mathbf{p},F) \leq A_1^{[c,b]}(l,\mathbf{y},\mathbf{q},F)$$

which is equivalent to

$$A_1^{[a,c]}(m,\mathbf{x},\mathbf{p},f) - \frac{K}{n!}A_1^{[a,c]}(m,\mathbf{x},\mathbf{p},e_n) \leq A_1^{[c,b]}(l,\mathbf{y},\mathbf{q},f) - \frac{K}{n!}A_1^{[c,b]}(l,\mathbf{y},\mathbf{q},e_n)$$

and using condition (5.121) we get the desired inequality. The second statement is proved in a similar manner. □

The integral analogous of previous theorem may be stated as:

**Theorem 5.51** *Let $\alpha \leq \beta, \gamma \leq \delta$, $a < c < b$, $g : [\alpha,\beta] \to [a,c]$, $p : [\alpha,\beta] \to \mathbb{R}$, $h : [\gamma,\delta] \to [c,b]$, $q : [\gamma,\delta] \to \mathbb{R}$ be integrable. Let $f : I \to \mathbb{R}$, $[a,b] \subset I$, be a function such that $f^{(n-1)}$ is absolutely continuous.*

*(i) For $k = 3,4$ let $A_k^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,f)$ and $\Omega_k^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,s)$ be defined as in Theorem 5.42 and satisfy the following conditions:*

$$\Omega_k^{[a,c]}([\alpha,\beta],g,p,s) \geq 0, \quad \textit{for every } s \in [a,c], \tag{5.123}$$

$$\Omega_k^{[c,b]}([\gamma,\delta],h,q,s) \geq 0, \quad \textit{for every } s \in [c,b], \tag{5.124}$$

$$A_k^{[a,c]}([\alpha,\beta],g,p,e_n) = A_k^{[c,b]}([\gamma,\delta],h,q,e_n). \tag{5.125}$$

*If $f$ is $(n+1)$-convex at point $c$, then*

$$A_k^{[a,c]}([\alpha,\beta],g,p,f) \leq A_k^{[c,b]}([\gamma,\delta],h,q,f). \tag{5.126}$$

*If the inequalities in (5.123) and 5.124 are reversed, then the reversed sign in (5.126) holds.*

*(ii) For $k = 7,8$ let $A_k^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,f)$ and $\Omega_k^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,s)$ be defined as in Theorem 5.47 and let assumption (5.114) holds. For $k = 7$, if (5.123), (5.124) and (5.125) are valid, then for an $(n+1)$-convex function $f$ at point $c$, $(n \geq 3)$, inequality (5.126) holds. For $k = 8$, if (5.125) holds and reversed (5.123), (5.124) are valid, then inequality (5.126) holds.*

**Corollary 5.8** *Let $j_1, j_2, n \in \mathbb{N}$, $2 \leq j_1, j_2 \leq n$, let $f : I \to \mathbb{R}$, $[a,b] \subset I$, be $(n+1)$-convex at point $c$, let integrable $p : [\alpha,\beta] \to \mathbb{R}$ and $g : [\alpha,\beta] \to [a,c]$ satisfy (5.5) with $n$ replaced by $j_1$, let $q : [\gamma,\delta] \to \mathbb{R}$ and $h : [\gamma,\delta] \to [c,b]$ satisfy*

$$\int_\gamma^\delta q(x)h^k(x) = 0, \quad \textit{for all } k \in \{0,1,\ldots,j_2-1\}$$

$$\int_\gamma^\delta q(x)(h(x)-s)_+^{j_2-1}dx \geq 0, \quad \textit{for every } s \in [c,b]$$

*and let (5.125) holds. Then*

$$A_3^{[a,c]}([\alpha,\beta],g,p,f) \leq A_3^{[c,b]}([\gamma,\delta],h,q,f).$$

*If $n - j_1$ and $n - j_2$ are even, then*

$$A_4^{[a,c]}([\alpha,\beta],g,p,f) \leq A_4^{[c,b]}([\gamma,\delta],h,q,f).$$

### 5.2.4 Bounds for the Remainders and Functionals

Here we give several estimations connected with the functionals $A_k^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,f)$, $k\in\{1,\ldots,8\}$. We use the well-known Hölder inequality and a bound for the Čebyšev functional $T(f,h)$ which is defined by (5.36). This bound is given in the following proposition in which the pre-Grüss inequality is given.

**Proposition 5.2** ([42]) *Let $f,h:[a,b]\to\mathbb{R}$ be integrable such that $fh\in L(a,b)$. If*

$$\gamma\le h(x)\le\Gamma \ \ \textit{for}\ \ x\in[a,b],$$

*then*

$$|T(f,h)|\le\frac{1}{2}(\Gamma-\gamma)\sqrt{T(f,f)}.$$

Now by using the aforementioned result, we are going to obtain formula for $A_k^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,f)$ and estimations of remainders which appear in this formula. For the sake of brevity, in present and next two sections we use the notations $A_k(f)=A_k^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,f)$ and $\Omega_k(t)=\Omega_k^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,t)$ for $k\in\{1,2,\ldots,8\}$. Now, we are ready to state the main results of this section.

**Theorem 5.52** *(i) Let $k\in\{1,2,3,4\}$. Let $f:I\to\mathbb{R}$, $[a,b]\subset I$, be such that $f^{(n-1)}$ is an absolutely continuous function and*

$$\gamma\le f^{(n)}(x)\le\Gamma \ \ \textit{for}\ x\in[a,b].$$

*Then in the representation*

$$A_k(f)=\frac{\left[f^{n-1}(b)-f^{n-1}(a)\right]}{(n-1)!(b-a)}\int_a^b\Omega_k(s)ds+R_n^k(f;a,b), \tag{5.127}$$

*the remainder $R_n^k(f;a,b)$ satisfies the estimation*

$$|R_n^k(f;a,b)|\le\frac{b-a}{2(n-1)!}(\Gamma-\gamma)\sqrt{T(\Omega_k,\Omega_k)}. \tag{5.128}$$

*(ii) Let $k\in\{5,6,7,8\}$. Let us assume that condition (5.110) holds if $k=5,6$, or condition (5.114) holds if $k=7,8$.*

*If the assumptions of (i) hold with $n\ge 3$, then (5.127) and (5.128) hold with $(n-3)!$ instead of $(n-1)!$ in the denominator of $A_k(f)$ and in the bound for $R_n^k$.*

*Proof.* Fix $k\in\{1,2,3,4\}$. Using the definition of $A_k$ and results from the previous subsection we have

$$\begin{aligned}A_k(f)&=\frac{1}{(n-1)!}\int_a^b f^{(n)}(s)\Omega_k(s)ds\\&=\frac{1}{(n-1)!(b-a)}\int_a^b f^{(n)}(s)ds\int_a^b\Omega_k(s)ds+R_n^k(f;a,b)\end{aligned}$$

$$= \frac{f^{n-1}(b) - f^{n-1}(a)}{(n-1)!(b-a)} \int_a^b \Omega_k(s)ds + R_n^k(f;a,b),$$

where

$$R_n^k(f;a,b) = \frac{1}{(n-1)!} \left( \int_a^b f^{(n)}(s)\Omega_k(s)ds - \frac{1}{b-a} \int_a^b f^{(n)}(s)ds \int_a^b \Omega_k(s)ds \right).$$

Applying Proposition 5.2 for $f \to \Omega_k$ and $h \to f^{(n)}$, we obtain

$$|R_n^k(f;a,b)| = |T(\Omega_k, f^{(n)}| \leq \frac{b-a}{2(n-1)!}(\Gamma - \gamma)\sqrt{T(\Omega_k, \Omega_k)}.$$

The proof for $k \in \{5,6,7,8\}$ is done in a similar manner. □

Using the same method as in the previous theorem and other type of bounds for the Čebyšev functional, for example, the bounds given in Theorems 5.12 and 5.13, we are able to give other estimations for the remainder. Now we state some Ostrowski-type inequalities related to the generalized linear inequalities.

**Theorem 5.53** *(i) Let $k \in \{1,2,3,4\}$. Let $(q,r)$ be a pair of conjugate exponents, i.e., $1 \leq q,r \leq \infty$, $\frac{1}{q} + \frac{1}{r} = 1$. Let $f^{(n)} \in L_q[a,b]$ for some $n \geq 2$. Then*

$$|A_k(f)| \leq \frac{1}{(n-1)!} \|f^{(n)}\|_q \|\Omega_k\|_r. \tag{5.129}$$

*The constant on the right-hand side of (5.129) is sharp for $1 < q \leq \infty$ and the best possible for $q = 1$.*

*(ii) Let $k \in \{5,6,7,8\}$. Let us assume that condition (5.110) holds if $k = 5,6$, or condition (5.114) holds if $k = 7,8$.*

*If assumptions of (i) hold with $n \geq 3$, then the statement holds with $(n-3)!$ instead of $(n-1)!$ in the denominator of the bound for $A_k$.*

*Proof.* Fix $k \in \{1,2,3,4\}$. From the definition of $A_k$ and results from the second section, applying the Hölder inequality we get

$$|A_k(f)| = \left| \frac{1}{(n-1)!} \int_a^b f^{(n)}(s)\Omega_k(s)ds \right| \leq \|f^{(n)}\|_q \|\lambda_k\|_r.$$

Let us denote the quotient $\frac{1}{(n-1)!}\Omega_k$ by $\lambda_k$. For the proof of the sharpness of $\left(\int_a^b |\lambda_k(t)|^r dt\right)^{1/r}$, let us find a function $f$ for which the equality in (5.129) is obtained.

For $1 < q < \infty$ take $f$ to be such that

$$f^{(n)}(t) = \operatorname{sgn} \lambda_k(t) \cdot |\lambda_k(t)|^{1/(q-1)}.$$

For $q = \infty$, take $f$ such that

$$f^{(n)}(t) = \operatorname{sgn} \lambda_k(t).$$

The fact that (5.129) is the best possible for $q = 1$ can be proved as in Theorem 5.18. The proof for $k \in \{5,6,7,8\}$ is done in a similar manner. □

### 5.2.5 Mean Value Theorems and Exponential Convexity

In this section we give several mean-value theorems and apply a general method for obtaining new exponentially convex functions related to the functionals $A_k$ defined in previous sections. As we said in the previous section we use notation $A_k(f) := A_k^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,f)$, $k \in \{1,\dots,8\}$. Since theorems in this section contain results for $k = 1,\dots,8$, we use this agreement throughout this section: if $k \in \{1,2,3,4\}$, then $n \in \mathbb{N}$; if $k \in \{5,6\}$, then $n \geq 3$ and (5.110) holds; if $k \in \{7,8\}$, then $n \geq 3$ and (5.114) holds.

**Theorem 5.54** *Let $k \in \{1,\dots,8\}$ and let us consider $A_k$ as a functional on $C^n([a,b])$. If $(U_k)$ holds, then for $f \in C^n([a,b])$ exists $\xi_k \in [a,b]$ such that*

$$A_k(f) = f^{(n)}(\xi_k)A_k(f_0),$$

*where $f_0(x) = \frac{x^n}{n!}$.*

*Proof.* The proof is similar as the proof of Theorem 5.34. □

From Theorem 5.54 we can conclude some refinements of the basic inequalities $A_k(f) \geq 0$. We write it in detail for $k = 1$. Let image of $f^{(n)}$ be an interval $[L,M] \subseteq [0,\infty\rangle$, where $f \in C^n([a,b])$. Then from $A_1(f) \geq LA_1(f_0) \geq 0$ we get

$$\begin{aligned}
\sum_{i=1}^{m} p_i f(x_i) \geq & \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} \sum_{i=1}^{m} p_i (x_i - a)^k \\
& + L\left( \sum_{i=1}^{m} \frac{p_i x_i^n}{n!} - \sum_{k=0}^{n-1} \binom{n}{k} \frac{a^{n-k}}{n!} \sum_{i=1}^{m} p_i (x_i - a)^k \right) \\
\geq & \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} \sum_{i=1}^{m} p_i (x_i - a)^k
\end{aligned}$$

which is a refinement of $A_1(f) \geq 0$.

**Theorem 5.55** *Let $k \in \{1,\dots,8\}$. Let $f,h \in C^n([a,b])$. If $(U_k)$ holds, then there exists $\xi_k \in [a,b]$ such that*

$$\frac{A_k(f)}{A_k(h)} = \frac{f^{(n)}(\xi_k)}{h^{(n)}(\xi_k)}$$

*assuming that both denominators are non-zero.*

*Proof.* The proof is similar as the proof of Theorem 5.35. □

**Remark 5.14** If the inverse of $\frac{f^{(n)}}{h^{(n)}}$ exists, then for $k \in \{1,\dots,8\}$ from the above mean value theorem we can define a generalized mean

$$\xi_k = \left( \frac{f^{(n)}}{h^{(n)}} \right)^{-1} \left( \frac{A_k(f)}{A_k(h)} \right). \tag{5.130}$$

Here, we get new results concerning $n$-exponential convexity and exponential convexity for the functionals $A_k$, $k \in \{1,\dots,8\}$.

**Theorem 5.56** *Let $D_1 = \{f_t : t \in I\}$ be a class of functions such that the function $t \mapsto [z_0, z_1, \dots, z_n; f_t]$ is $r$-exponentially convex in the $J$-sense on $I$ for any mutually distinct points $z_0, z_1, \dots, z_n \in [a,b]$, $n \geq 2$. Let $k \in \{1,\dots,8\}$.*

*If condition $(U_k)$ holds, then the following statements are valid:*

(a) *The function $t \mapsto A_k(f_t)$ is $r$-exponentially convex function in the $J$-sense on $I$.*

(b) *If the function $t \mapsto A_k(f_t)$ is continuous on $I$, then the function $t \mapsto A_k(f_t)$ is $r$-exponentially convex on $I$.*

*If the phrase "$r$-exponentially convex" is replaced with "exponentially convex", then statements also hold.*

*Proof.* (a) Fix $k \in \{1,2\}$. Let us define the function $\omega$ for $t_i, t_j \in I$, $u_i u_j \in \mathbb{R}$, $i,j \in \{1,\dots,r\}$ as follows

$$\omega = \sum_{i,j=1}^{r} u_i u_j f_{\frac{t_i+t_j}{2}},$$

Since the function $t \mapsto [z_0, z_1, \dots, z_n; f_t]$ is $r$-exponentially convex in the $J$-sense, we have

$$[z_0, z_1, \dots, z_n; \omega] = \sum_{i,j=1}^{r} u_i u_j [z_0, z_1, \dots, z_n; f_{\frac{t_i+t_j}{2}}] \geq 0$$

which implies that $\omega$ is $n$-convex function on $I$ and using Theorem 5.39 we get $A_k(\omega) \geq 0$. Hence

$$\sum_{i,j=1}^{r} u_i u_j A_k(f_{\frac{t_i+t_j}{2}}) \geq 0.$$

We conclude that the function $t \mapsto A_k(f_t)$ is an $r$-exponentially convex function on $I$ in the $J$-sense. Other cases are proved in a similar manner.

(b) This part easily follows from the definition of $n$-exponentially convex functions. □

**Remark 5.15** The condition "$D_1 = \{f_t : t \in I\}$ be a class of functions such that the function $t \mapsto [z_0, z_1, \dots, z_n; f_t]$ is $r$-exponentially convex" can be replaced with "$D_1 = \{f_t : t \in I\}$ be a class of $n$-time differentiable functions such that the function $t \mapsto f_t^{(n)}$ is $r$-exponentially convex".

As a consequence of the above theorem we give the following theorem which connects $A_k$ with log-convexity.

**Theorem 5.57** *Let $D_2 = \{f_t : t \in I\}$ be a class of functions such that the function $t \mapsto [z_0, z_1, \dots, z_n; f_t]$ is 2-exponentially convex in the $J$-sense on $I$ for any mutually distinct points $z_0, z_1, \dots, z_n \in [a,b]$, $n \geq 2$. Let $k \in \{1,\dots,8\}$.*

*If condition $(U_k)$ holds, then the following statements are valid:*

$(a)$ *If the function $t \mapsto A_k(f_t)$ is positive and continuous, then it is log-convex on $I$. Moreover, the following Lyapunov type inequality holds for $r < s < t$, $r, s, t \in I$*

$$[A_k(f_s)]^{t-r} \leq [A_k(f_r)]^{t-s}\,[A_k(f_t)]^{s-r}. \tag{5.131}$$

$(b)$ *If the function $t \mapsto A_k(f_t)$ is positive and differentiable on $I$, then for every $s,t,u,v \in I$ such that $s \leq u$ and $t \leq v$, we have*

$$\mu_{s,t}(A_k, D_2) \leq \mu_{u,v}(A_k, D_2) \tag{5.132}$$

*where $\mu_{s,t}$ is defined as*

$$\mu_{s,t}(A_k, D_2) = \begin{cases} \left(\dfrac{A_k(f_s)}{A_k(f_t)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(\dfrac{\frac{d}{ds}A_k(f_s)}{A_k(f_s)}\right), & s = t \end{cases} \tag{5.133}$$

*for $f_s, f_t \in D_2$.*

*Furthermore, if $r, r_1, \ldots, r_l, r+r_1, \ldots, r+r_l, r+r_1+\ldots+r_l \in I$, then*

$$A_k(f_r)^{n-1} A_k(f_{r+r_1+\ldots+r_l}) \geq A_k(f_{r+r_1}) \cdot \ldots \cdot A_k(f_{r+r_l}). \tag{5.134}$$

*Particularly, if $0 \in I$, then we get the Čebyšev type inequality*

$$A_k(f_0)^{n-1} A_k(f_{r_1+\ldots+r_l}) \geq A_k(f_{r_1}) \cdot \ldots \cdot A_k(f_{r_l}).$$

*Proof.* (a) Applying Theorem 5.56 for $r = 2$ we get that $t \mapsto A_k(f_t)$ is 2-exponentially convex in the $J$-sense i.e. for any $t_1, t_2 \in I$, $u_1, u_2 \in \mathbb{R}$

$$u_1^2 A_k(f_{t_1}) + 2u_1u_2 A_k(f_{\frac{t_1+t_2}{2}}) + u_2^2 A_k(f_{t_2}) \geq 0.$$

If we consider the left-hand side as a nonnegative quadratic polynomial, then its discriminant is nonpositive, i.e.

$$[A_k(f_{\frac{t_1+t_2}{2}})]^2 - A_k(f_{t_1}) \cdot A_k(f_{t_2}) \leq 0.$$

This means that $t \mapsto A_k(f_t)$ is log-convex in $J$-sense. From continuity we conclude that $t \mapsto A_k(f_t)$ is log-convex. Using the Jensen inequality for convex combination $s = \dfrac{t-s}{t-r}r + \dfrac{s-r}{t-r}t$ we get

$$\log A_k(f_s) \leq \frac{t-s}{t-r} \log A_k(f_r) + \frac{s-r}{t-r} \log A_k(f_t)$$
$$\log[A_k(f_s)]^{t-r} \leq \log[A_k(f_r)]^{t-s} + \log[A_k(f_t)]^{s-r},$$

which gives (5.131).

(b) For a convex function $\varphi$, the inequality

$$\frac{\varphi(s)-\varphi(t)}{s-t} \leq \frac{\varphi(u)-\varphi(v)}{u-v} \tag{5.135}$$

holds for all $s,t,u,v \in I$ such that $s \leq u$, $t \leq v$, $s \neq t$, $u \neq v$.
Since by $(a)$, $A_k(f_t)$ is log-convex, so setting $\varphi(t) = \log A_k(f_t)$ in (5.135) we have

$$\frac{\log A_k(f_s) - \log A_k(f_t)}{s-t} \leq \frac{\log A_k(f_u) - \log A_k(f_v)}{u-v}, \tag{5.136}$$

for $s \leq u$, $t \leq v$, $s \neq t$, $u \neq v$, which is equivalent to (5.132) i.e. to

$$\left(\frac{A_k(f_s)}{A_k(f_t)}\right)^{\frac{1}{s-t}} \leq \left(\frac{A_k(f_u)}{A_k(f_v)}\right)^{\frac{1}{u-v}}. \tag{5.137}$$

The cases for $s=t$ and / or $u=v$ are obtained by taking respective limits.

Putting in (5.137) $t=v=r$, $s=r+r_1+\ldots+r_l$, $u=r+r_i$ we get

$$\left(\frac{A_k(f_{r+r_1+\ldots+r_l})}{A_k(f_r)}\right)^{\frac{1}{r_1+\ldots+r_l}} \leq \left(\frac{A_k(f_{r+r_i})}{A_k(f_r)}\right)^{\frac{1}{r_i}}$$

$$\left(\frac{A_k(f_{r+r_1+\ldots+r_l})}{A_k(f_r)}\right)^{\frac{r_i}{r_1+\ldots+r_l}} \leq \frac{A_k(f_{r+r_i})}{A_k(f_r)}.$$

Multiplying all inequalities for $i=1,2,\ldots,l$ we get (5.134). □

Let us consider some examples:

***Example 5.5*** Let $\tilde{F}_1 = \{\psi_t : [a,b] \subset \mathbb{R} \to [0,\infty\rangle : t \in \mathbb{R}\}$ be a family of functions defined by

$$\psi_t(x) = \begin{cases} \dfrac{e^{tx}}{t^n}, & t \neq 0, \\ \dfrac{x^n}{n!}, & t = 0. \end{cases}$$

Since $\frac{d^n}{dx^n}\psi_t(x) = e^{tx}$, the function $t \mapsto \frac{d^n}{dx^n}\psi_t(x)$ is exponentially convex. Using Theorem 5.56, we have that $t \mapsto A_k(\psi_t)$, $k \in \{1,\ldots,8\}$ are exponentially convex.

Assume that $t \mapsto A_k(\psi_t) > 0$ for $k \in \{1,\ldots,8\}$. By introducing functions $\psi_t$ and $\psi_s$ in (5.130), we obtain the following means:

$$\mathfrak{M}_{s,t}(A_k,\tilde{F}_1) = \begin{cases} \dfrac{1}{s-t}\log\left(\dfrac{A_k(\psi_s)}{A_k(\psi_t)}\right), & s \neq t, \\ \dfrac{A_k(id\cdot\psi_s)}{A_k(\psi_s)} - \dfrac{n}{s}, & s = t \neq 0, \\ \dfrac{A_k(id\cdot\psi_0)}{(n+1)A_k(\psi_0)}, & s = t = 0. \end{cases}$$

where $id$ stands for the identity function on $[a,b] \subset \mathbb{R}$.

In particular, for $k=1$ we have

$$\mathfrak{M}_{s,t}(A_1,\tilde{F}_1) = \frac{1}{s-t}\log\left(\frac{t^n}{s^n}\frac{\sum_{i=1}^m p_i e^{sx_i} - \sum_{k=0}^{n-1}\frac{s^k e^{sa}}{k!}\sum_{i=1}^m p_i(x_i-a)^k}{\sum_{i=1}^m p_i e^{tx_i} - \sum_{k=0}^{n-1}\frac{t^k e^{ta}}{k!}\sum_{i=1}^m p_i(x_i-a)^k}\right),$$

for $s \neq t; s,t \neq 0$,

$$\mathfrak{M}_{s,s}(A_1,\tilde{F}_1) = \frac{\sum_{i=1}^m p_i x_i e^{sx_i} - \sum_{k=0}^{n-1}\frac{(ks^{k-1}+as^k)e^{sa}}{k!}\sum_{i=1}^m p_i(x_i-a)^k}{\sum_{i=1}^m p_i e^{sx_i} - \sum_{k=0}^{n-1}\frac{s^k e^{sa}}{k!}\sum_{i=1}^m p_i(x_i-a)^k} - \frac{n}{s},$$

for $s \neq 0$; and

$$\mathfrak{M}_{0,0}(A_1,\tilde{F}_1) = \frac{\sum_{i=1}^m p_i \frac{x_i^{n+1}}{n!} - \sum_{k=0}^{n-1}\frac{(n+1)a^{n-k+1}}{(n-k+1)!k!}\sum_{i=1}^m p_i(x_i-a)^k}{(n+1)\left(\sum_{i=1}^m p_i \frac{x_i^n}{n!} - \sum_{k=0}^{n-1}\frac{a^{n-k}}{(n-k)!k!}\sum_{i=1}^m p_i(x_i-a)^k\right)}.$$

Here $\mathfrak{M}_{s,t}(A_k,\tilde{F}_1) = \log(\mu_{s,t}(A_k,\tilde{F}_1))$, $k \in \{1,\ldots,8\}$ are in fact means.

**Remark 5.16** We observe here that $\left(\frac{\frac{d^n\psi_s}{dx^n}}{\frac{d^n\psi_t}{dx^n}}\right)^{\frac{1}{s-t}}(\log\xi) = \xi$ is a mean for $\xi \in [a,b]$ where $a,b \in \mathbb{R}_+$.

***Example 5.6*** Let $n \in \mathbb{N}$, $\tilde{F}_2 = \{\varphi_t : [0,\infty\rangle \to \mathbb{R} : t \in \mathbb{R}, t > n\}$ be a family of functions defined as

$$\varphi_t(x) = \frac{x^t}{t(t-1)\cdot\ldots\cdot(t-n+1)}.$$

Since $t \mapsto \frac{d^n}{dx^n}\varphi_t(x) = x^{t-n} = e^{(t-n)\log x}$ is exponentially convex, by Theorem 5.56 we conclude that $t \mapsto A_k(\varphi_t)$, $k \in \{1,\ldots,8\}$ are exponentially convex.

We assume that $A_k(\varphi_t) > 0$ for $k \in \{1,\ldots,8\}$. For this family of functions we obtain the following means:

$$\mathfrak{M}_{s,t}(A_k,\tilde{F}_2) = \begin{cases} \left(\frac{A_k(\varphi_s)}{A_k(\varphi_t)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left((-1)^{n-1}(n-1)!\frac{A_k(\varphi_0\varphi_s)}{A_k(\varphi_s)} + \sum_{k=0}^{n-1}\frac{1}{k-s}\right), & s = t. \end{cases}$$

In particular, for $k = 1$ we have

$$\mathfrak{M}_{s,t}(A_1,\tilde{F}_2) = \left( \frac{t(t-1)\cdots(t-n+1)}{s(s-1)\cdots(s-n+1)} \frac{\sum_{i=1}^{m} p_i x_i^s}{\sum_{i=1}^{m} p_i x_i^t} \right)^{\frac{1}{s-t}}, \qquad s \neq t$$

$$\mathfrak{M}_{s,s}(A_1,\tilde{F}_2) = \exp\left( \frac{\sum_{i=1}^{m} p_i x_i{}^s \log x_i}{\sum_{i=1}^{m} p_i x_i{}^s} + \sum_{k=0}^{n-1} \frac{1}{k-s} \right).$$

For other examples see paper [22].

## 5.3 Linear Inequalities and Lidstone Interpolation Polynomials

Here we pay attention to inequalities of type (5.1) and (5.4) for $n$-convex functions by making use of the Lidstone interpolation. This section is based on the paper [91]. The Lidstone series is a generalization of the Taylor series and it approximates a given function in the neighborhood of two points (instead of one). For $f \in C^{(2n)}([0,1])$ there exists a unique polynomial $P_L$ of degree $2n-1$ such that

$$P_L^{(2i)}(0) = f^{(2i)}(0), \quad P_L^{(2i)}(1) = f^{(2i)}(1), \quad 0 \le i \le n-1.$$

The polynomial $P_L$ can be expressed with the Lidstone polynomials. The Lidstone polynomials $\Lambda_n$ are polynomials of degree $2n+1$ defined by the relations

$$\begin{aligned} &\Lambda_0(t) = t, \\ &\Lambda_n''(t) = \Lambda_{n-1}(t), \\ &\Lambda_n(0) = \Lambda_n(1) = 0, n \ge 1. \end{aligned} \tag{5.138}$$

Some explicit expressions of the Lidstone polynomials are (see [1])

$$\Lambda_n(t) = (-1)^n \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2n+1}} \sin k\pi t,$$

$$\begin{aligned} \Lambda_n(t) = {} & \frac{1}{6}\left[ \frac{6t^{2n+1}}{(2n+1)!} - \frac{t^{2n-1}}{(2n-1)!} \right] \\ & - \sum_{k=0}^{n-2} \frac{2(2^{2k+3}-1)}{(2k+4)!} B_{2k+4} \frac{t^{2n-2k-3}}{(2n-2k-3)!}, \end{aligned}$$

$$\Lambda_n(t) = \frac{2^{2n+1}}{(2n+1)!} B_{2n+1}\left(\frac{1+t}{2}\right),$$

where $B_{2k+4}$ is the $(2k+4)$-th Bernoulli number and $B_{2n+1}\left(\frac{1+t}{2}\right)$ is the Bernoulli polynomial. The error term $e_L(t) = f(t) - P_L(t)$ of the interpolation can be expressed in the integral form using the Green function. Widder [99] proved the following lemma.

**Lemma 5.1** *If $f \in C^{(2n)}([0,1])$, then*

$$\begin{aligned} f(t) &= P_L(t) + e_L(t) \\ &= \sum_{k=0}^{n-1} \left[ f^{(2k)}(0)\Lambda_k(1-t) + f^{(2k)}(1)\Lambda_k(t) \right] + \int_0^1 G_n(t,s) f^{(2n)}(s)\,ds, \end{aligned} \tag{5.139}$$

*where*

$$G_1(t,s) = G(t,s) = \begin{cases} (t-1)s, & s \le t, \\ (s-1)t, & t \le s. \end{cases} \tag{5.140}$$

*is the homogeneous Green function of the differential operator $\frac{d^2}{ds^2}$ on $[0,1]$, and with the successive iterates of $G(t,s)$*

$$G_n(t,s) = \int_0^1 G_1(t,u) G_{n-1}(u,s)\,du, \quad n \ge 2. \tag{5.141}$$

The Lidstone polynomial can be expressed in terms of $G_n(t,s)$ as

$$\Lambda_n(t) = \int_0^1 G_n(t,s)s\,ds. \tag{5.142}$$

For more on the Lidstone polynomials and interpolation see [1].

**Theorem 5.58** *Let $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ be $2n$-convex and let $\mathbf{x} \in [a,b]^m$ and $\mathbf{p} \in \mathbb{R}^m$ be $m$-tuples such that*

$$\sum_{i=1}^m p_i G_n\left(\frac{x_i - a}{b-a}, \frac{s-a}{b-a}\right) \ge 0, \quad \textit{for every } s \in [a,b], \tag{5.143}$$

*where $G_n$ is the Green function given by (5.141). Then*

$$\begin{aligned} \sum_{i=1}^m p_i f(x_i) \ge \sum_{i=1}^m \sum_{k=0}^{n-1} (b-a)^{2k} \Bigg[ & p_i f^{(2k)}(a) \Lambda_k\left(\frac{b-x_i}{b-a}\right) \\ & + p_i f^{(2k)}(b) \Lambda_k\left(\frac{x_i - a}{b-a}\right) \Bigg]. \end{aligned} \tag{5.144}$$

*If the inequality in (5.143) is reversed, then (5.144) holds with the reversed sign of inequality.*

*Proof.* Let us first assume $f \in C^{(2n)}([a,b])$ . By Lemma 5.1 we have

$$\begin{aligned} f(x) &= \sum_{k=0}^{n-1}(b-a)^{2k}\left[f^{(2k)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right)+f^{(2k)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right] \\ &+ (b-a)^{2n-1}\int_a^b G_n\left(\frac{x-a}{b-a},\frac{s-a}{b-a}\right)f^{(2n)}(s)ds. \end{aligned} \tag{5.145}$$

Applying (5.145) at $x_i$, multiplying the obtained identity by $p_i$ and adding up we get

$$\begin{aligned} \sum_{i=1}^m p_i f(x_i) &= \sum_{i=1}^m\sum_{k=0}^{n-1}(b-a)^{2k}\left[p_i f^{(2k)}(a)\Lambda_k\left(\frac{b-x_i}{b-a}\right)\right. \\ &+ \left. p_i f^{(2k)}(b)\Lambda_k\left(\frac{x_i-a}{b-a}\right)\right] \\ &+ (b-a)^{2n-1}\int_a^b\sum_{i=1}^m p_i G_n\left(\frac{x_i-a}{b-a},\frac{s-a}{b-a}\right)f^{(2n)}(s)\,ds. \end{aligned} \tag{5.146}$$

Assumption (5.143) and $f^{(2n)} \geq 0$ yield the stated inequality. The inequality for general $f$ follows since every $2n$-convex function can be obtained, by making use of Bernstein polynomials, as a uniform limit of $2n$-convex functions with a continuous $2n$-th derivative (see [77]). □

**Corollary 5.9** *Let $j,n \in \mathbb{N}$, $1 \leq j \leq n$, let $f : [a,b] \to \mathbb{R}$ be $2n$-convex and let $m$-tuples $\mathbf{x} \in [a,b]^m$ and $\mathbf{p} \in \mathbb{R}^m$ satisfy (5.2) and (5.3) with $n$ replaced by $2j$. If $n-j$ is even, then*

$$\begin{aligned} \sum_{i=1}^m p_i f(x_i) \geq \sum_{i=1}^m\sum_{k=j}^{n-1}(b-a)^{2k}&\left[p_i f^{(2k)}(a)\Lambda_k\left(\frac{b-x_i}{b-a}\right)\right. \\ &\left.+p_i f^{(2k)}(b)\Lambda_k\left(\frac{x_i-a}{b-a}\right)\right], \end{aligned} \tag{5.147}$$

*while the reversed inequality holds if $n-j$ is odd.*

*Proof.* From (5.140) and (5.141) by induction one can conclude that $(-1)^n G_n \geq 0$. Furthermore, from (5.141) one can get $\frac{\partial^2}{\partial t^2}G_n(t,s) = G_{n-1}(t,s)$ and, hence, by induction $\frac{\partial^{2i}}{\partial t^{2i}}G_n(t,s) = G_{n-i}(t,s)$ for $0 \leq i \leq n-1$. Therefore, the function $t \mapsto G_n(t,s)$ is $2j$-convex if $n-j$ is even and $2j$-concave if $n-j$ is odd for $0 \leq j \leq n-1$, while the statement for $j=n$ follows since $t \mapsto G_1(t,s)$ is convex.

By Theorem 5.1, assumption (5.143) in Theorem 5.58 is satisfied, so (5.144) holds. Moreover, due to assumption (5.2), $\sum_{i=1}^m p_i P(x_i) = 0$ for every polynomial $P$ of degree $\leq 2j-1$ and since $\Lambda_k$ is a polynomial of degree $2k+1$, the first $j+1$ terms in the inner sum in (5.144) vanish, i. e., the right-hand side of (5.144) under the assumptions of this corollary is equal to the right-hand side of (5.147). □

When $j=n$ in (5.147), the notation means that the inner sum is void, i. e. $\sum_{k=n}^{n-1}\cdots = 0$. In particular, inequality (5.147) with $j=n$ is inequality $\sum_{i=1}^m p_i f(x_i) \geq 0$.

**Corollary 5.10** *Let $j,n \in \mathbb{N}$, $1 \le j \le n$, let $f : [a,b] \to \mathbb{R}$ be $2n$-convex, let $m$-tuples $\mathbf{x} \in [a,b]^m$ and $\mathbf{p} \in \mathbb{R}^m$ satisfy (5.2) and (5.3) with $n$ replaced by $2j$ and denote*

$$H(x) = \sum_{k=j}^{n-1} (b-a)^{2k} \left[ f^{(2k)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right) + f^{(2k)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right]. \tag{5.148}$$

*If $n-j$ is even and $H$ is $2j$-convex, then*

$$\sum_{i=1}^{m} p_i f(x_i) \ge 0,$$

*while the reversed inequality holds if $n-j$ is odd and $H$ is $2j$-concave.*

*Proof.* Applying Theorem 5.1 we conclude that the right-hand side of (5.147) is nonnegative for $2j$-convex $H$ and nonpositive for $2j$-concave $H$. □

**Remark 5.17** Due to (5.139) we have $\Lambda_k^{(2l)} = \Lambda_{k-l}$ and, furthermore, $(-1)^n\Lambda_n \ge 0$ due to (5.142). Therefore, if the function $f$ satisfies $(-1)^{k-j} f^{(2k)}(a) \ge 0$ and $(-1)^{k-j} f^{(2k)}(b) \ge 0$ for $j \le k \le n-1$, then the function $H$ given by (5.148) is $2j$-convex, while if $(-1)^{k-j} f^{(2k)}(a) \le 0$ and $(-1)^{k-j} f^{(2k)}(b) \le 0$ for $j \le k \le n-1$, then $H$ is $2j$-concave.

As already mentioned before, the inequality in Corollaries 5.9 and 5.10 with $j = n$ is the same as the inequality in Theorem 5.1. Of course, in the proof of Corollary 5.9 we have used Theorem 5.1 to prove that assumption (5.143) holds, so, due to circularity, we didn't obtain another proof of the Popoviciu result. But, it is possible, as we will show in the next lemma, to prove directly that conditions (5.2) and (5.3) imply (5.143), i. e. it is possible to prove Corollary 5.9 independently of Theorem 5.1 and, thus, provide a new proof of the Popoviciu result for even $n$.

**Lemma 5.2** *Let $n \ge 2$ and let $m$-tuples $\mathbf{x} \in [a,b]^m$ and $\mathbf{p} \in \mathbb{R}^m$ satisfy*

$$\sum_{i=1}^{m} p_i x_i^k = 0, \quad \textit{for all } k = 0,1,\ldots,2n-1 \tag{5.149}$$

$$\sum_{i=1}^{m} p_i (x_i - t)_+^{2n-1} \ge 0, \quad \textit{for every } t \in [a,b]. \tag{5.150}$$

*Then (5.143) holds.*

*Proof.* Let $s \in [a,b]$ be fixed and $y = (s-a)/(b-a)$. We will show, by induction, that $G_n$ is of the form

$$G_n(x,y) = P_{s,2n-1}(x) + \frac{1}{(2n-1)!}(x-y)_+^{2n-1}, \tag{5.151}$$

where $P_{s,2n-1}$ is a polynomial of degree $2n-1$. Hence, similarly as in the proof of Corollary 5.9, from (5.149) we can conclude that

$$\sum_{i=1}^{m} p_i P_{s,2n-1}\left(\frac{x_i-a}{b-a}\right) = 0,$$

while (5.150) yields

$$\sum_{i=1}^{m}\frac{p_i}{(2n-1)!}\left(\frac{x_i-a}{b-a}-\frac{s-a}{b-a}\right)_+^{2n-1}=\frac{1}{(2n-1)!(b-a)^{2n-1}}\sum_{i=1}^{m}p_i(x_i-s)_+^{2n-1}\geq 0.$$

Therefore, it is enough to show that (5.151) holds. From (5.140) we have

$$G_1(x,y)=xy-\min(x,y)=x(y-1)+(x-y)_+,$$

so (5.151) holds for $n=1$. Now, assume that (5.151) holds. Then (5.141) yields

$$\begin{aligned} G_{n+1}(x,y) &= \textstyle\int_0^1 (x(u-1)+(x-u)_+)\left(P_{s,2n-1}(u)+\frac{1}{(2n-1)!}(u-y)_+^{2n-1}\right)du \\ &= I+II+III, \end{aligned}$$

where

$$\begin{aligned} I &= x\int_0^1 (u-1)G_n(u,y)\,du = x\cdot \text{ constant} \\ II &= \int_0^1 (x-u)_+P_{s,2n-1}(u)\,du \\ III &= \frac{1}{(2n-1)!}\int_0^1 (x-u)_+(u-y)_+^{2n-1}\,du. \end{aligned}$$

Integration by parts yields

$$\begin{aligned} II &= \int_0^x (x-u)P_{s,2n-1}(u)\,du \\ &= (x-u)\int_0^u P_{s,2n-1}(z)\,dz\Big|_{u=0}^{u=x}+\int_0^x\int_0^u P_{s,2n-1}(z)\,dz=\tilde{P}_{s,2n+1}(x), \end{aligned}$$

where $\tilde{P}_{s,2n+1}$ is a polynomial of degree $2n+1$. Notice that

$$I+II=P_{s,2n-1}$$

is a polynomial of degree $2n+1$ in the variable $x$. Clearly $III=0$ for $x\leq y$, while for $x>y$

$$\begin{aligned} III &= \frac{1}{(2n-1)!}\int_y^x (x-u)(u-y)^{2n-1}\,du \\ &= \frac{1}{(2n)!}(x-u)(u-y)\Big|_{u=y}^{u=x}+\frac{1}{(2n)!}\int_y^x (u-y)^{2n}\,du=\frac{1}{(2n+1)!}(x-y)^{2n+1}. \end{aligned}$$

Therefore, $III=(x-y)_+^{2n+1}/(2n+1)!$, so (5.151) holds for $n+1$ as well, which finishes the proof. □

Lemma 5.2 together with Theorem 5.58 gives the "if" part of Theorem 5.1. On the other hand, the "only if" part is straightforward: since the functions $e_k(x)=x^k$ are both $2n$-convex and $2n$-concave for $k\in\{0,1,\ldots,2n-1\}$, inequality (5.1) yields that $\sum_{i=1}^m p_ie_k(x_i)$ is both $\geq 0$ and $\leq 0$, so (5.149) holds. Similarly, the function $w_{2n}(x)=(x-t)_+^{2n-1}$ is $2n$-convex and inequality (5.1) applied to $w_{2n}$ yields (5.150).

In the remainder of this section we will give integral versions of the results. The proofs are analogous to the discrete case and we will omit them.

**Theorem 5.59** *Let $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ be $2n$-convex and let the functions $p : [\alpha,\beta] \to \mathbb{R}$ and $g : [\alpha,\beta] \to [a,b]$ be such that*

$$\int_\alpha^\beta p(x) G_n\left(\frac{g(x)-a}{b-a}, \frac{s-a}{b-a}\right) dx \geq 0, \quad \text{for every } s \in [a,b], \tag{5.152}$$

*where $G_n$ is the Green function given by* (5.141). *Then*

$$\int_\alpha^\beta p(x) f(g(x))\, dx \geq \int_\alpha^\beta p(x) \sum_{k=0}^{n-1} (b-a)^{2k} \left[ f^{(2k)}(a) \Lambda_k\left(\frac{b-g(x)}{b-a}\right) + f^{(2k)}(b) \Lambda_k\left(\frac{g(x)-a}{b-a}\right)\right] dx. \tag{5.153}$$

*If the inequality in* (5.152) *is reversed, then* (5.153) *holds with the reversed sign of inequality.*

**Corollary 5.11** *Let $j,n \in \mathbb{N}$, $1 \leq j \leq n$, let $f : [a,b] \to \mathbb{R}$ be $2n$-convex and let the functions $p : [\alpha,\beta] \to \mathbb{R}$ and $g : [\alpha,\beta] \to [a,b]$ satisfy* (5.5) *with $n$ replaced by $2j$. If $n-j$ is even, then*

$$\int_\alpha^\beta p(x) f(g(x))\, dx \geq \int_\alpha^\beta p(x) \sum_{k=j}^{n-1} (b-a)^{2k} \left[ f^{(2k)}(a) \Lambda_k\left(\frac{b-g(x)}{b-a}\right) + f^{(2k)}(b) \Lambda_k\left(\frac{g(x)-a}{b-a}\right)\right] dx,$$

*while the reversed inequality holds if $n-j$ is odd.*

**Corollary 5.12** *Let $j,n,f,p$ and $g$ be as in Corollary* 5.11 *and let $H$ be given by* (5.148). *If $n-j$ is even and $H$ is $2j$-convex, then*

$$\int_\alpha^\beta p(x) f(g(x))\, dx \geq 0,$$

*while the reversed inequality holds if $n-j$ is odd and $H$ is $2j$-concave.*

**Lemma 5.3** *Let $n \geq 2$ and let the functions $p : [\alpha,\beta] \to \mathbb{R}$ and $g : [\alpha,\beta] \to [a,b]$ satisfy*

$$\int_\alpha^\beta p(x) g(x)^k\, dx = 0, \quad \text{for all } k = 0,1,\ldots,2n-1$$

$$\int_\alpha^\beta p(x) \left(g(x)-t\right)_+^{2n-1} dx \geq 0, \quad \text{for every } t \in [a,b].$$

*Then* (5.152) *holds.*

### 5.3.1 Inequalities for $n$-convex Functions at a Point

In this section we will give related results for the class of $n$-convex functions at a point introduced in Chapter 2.

Let $e_i$ denote the monomials $e_i(x) = x^i$, $i \in \mathbb{N}_0$. For the rest of this section, $A$ and $B$ will denote the linear functionals obtained as the difference of the left and right-hand sides of inequality (5.144) applied to the intervals $[a,c]$ and $[c,b]$, respectively, i. e., for $\mathbf{x} \in [a,c]^m$, $\mathbf{p} \in \mathbb{R}^m$, $\mathbf{y} \in [c,b]^l$ and $\mathbf{q} \in \mathbb{R}^l$ let

$$\begin{aligned} A(f) \;=\; & \sum_{i=1}^{m} p_i f(x_i) - \sum_{i=1}^{m} \sum_{k=0}^{n-1} (c-a)^{2k} \left[ p_i f^{(2k)}(a) \Lambda_k \left( \frac{c-x_i}{c-a} \right) \right. \\ & \left. + p_i f^{(2k)}(c) \Lambda_k \left( \frac{x_i-a}{c-a} \right) \right], \end{aligned} \tag{5.154}$$

$$\begin{aligned} B(f) \;=\; & \sum_{i=1}^{l} q_i f(y_i) - \sum_{i=1}^{l} \sum_{k=0}^{n-1} (b-c)^{2k} \left[ q_i f^{(2k)}(c) \Lambda_k \left( \frac{b-y_i}{b-c} \right) \right. \\ & \left. + q_i f^{(2k)}(b) \Lambda_k \left( \frac{y_i-c}{b-c} \right) \right]. \end{aligned} \tag{5.155}$$

Notice that, using the newly introduced functionals $A$ and $B$, identity (5.146) applied to the intervals $[a,c]$ and $[c,b]$ can be written as

$$A(f) = (c-a)^{2n-1} \int_a^c \sum_{i=1}^{m} p_i G_n \left( \frac{x_i-a}{c-a}, \frac{s-a}{c-a} \right) f^{(2n)}(s)\, ds, \tag{5.156}$$

$$B(f) = (b-c)^{2n-1} \int_c^b \sum_{i=1}^{l} q_i G_n \left( \frac{y_i-c}{b-c}, \frac{s-c}{b-c} \right) f^{(2n)}(s)\, ds. \tag{5.157}$$

**Theorem 5.60** *Let $\mathbf{x} \in [a,c]^m$, $\mathbf{p} \in \mathbb{R}^m$, $\mathbf{y} \in [c,b]^l$ and $\mathbf{q} \in \mathbb{R}^l$ be such that*

$$\sum_{i=1}^{m} p_i G_n \left( \frac{x_i-a}{c-a}, \frac{s-a}{c-a} \right) \geq 0, \quad \textit{for every } s \in [a,c], \tag{5.158}$$

$$\sum_{i=1}^{l} q_i G_n \left( \frac{y_i-c}{b-c}, \frac{s-c}{b-c} \right) \geq 0, \quad \textit{for every } s \in [c,b], \tag{5.159}$$

$$\begin{aligned} & \int_a^c \sum_{i=1}^{m} p_i G_n \left( \frac{x_i-a}{c-a}, \frac{s-a}{c-a} \right) ds \\ & \quad = \left( \frac{b-c}{c-a} \right)^{2n-1} \int_c^b \sum_{i=1}^{l} q_i G_n \left( \frac{y_i-c}{b-c}, \frac{s-c}{b-c} \right) ds, \end{aligned} \tag{5.160}$$

*where $G_n$ is the Green function given by (5.141), and let $A$ and $B$ be the linear functionals given by (5.154) and (5.155). If $f : [a,b] \to \mathbb{R}$ is $(2n+1)$-convex at point $c$, then*

$$A(f) \leq B(f). \tag{5.161}$$

*If the inequalities in* (5.158) *and* (5.159) *are reversed, then* (5.161) *holds with the reversed sign of inequality.*

*Proof.* Let $F = f - \frac{K}{(2n)!}e_{2n}$ be as in definition i. e., the function $F$ is $2n$-concave on $[a,c]$ and $2n$-convex on $[c,b]$. Applying Theorem 5.58 to $F$ on the interval $[a,c]$ we have

$$0 \geq A(F) = A(f) - \frac{K}{(2n)!}A(e_{2n}) \tag{5.162}$$

and applying Theorem 5.58 to $F$ on the interval $[c,b]$ we have

$$0 \leq B(F) = B(f) - \frac{K}{(2n)!}B(e_{2n}). \tag{5.163}$$

Identities (5.156) and (5.157) applied to the function $e_{2n}$ yield

$$A(e_{2n}) = (2n)!(c-a)^{2n-1}\int_a^c \sum_{i=1}^m p_i G_n\left(\frac{x_i - a}{c-a}, \frac{s-a}{c-a}\right) ds,$$

$$B(e_{2n}) = (2n)!(b-c)^{2n-1}\int_c^b \sum_{i=1}^l q_i G_n\left(\frac{y_i - c}{b-c}, \frac{s-c}{b-c}\right) ds.$$

Therefore, assumption (5.160) is equivalent to $A(e_{2n}) = B(e_{2n})$. Now, from (5.162) and (5.163) we obtain the stated inequality. □

In the proof of Theorem 5.60 we have, actually, shown that

$$A(f) \leq \frac{K}{(2n)!}A(e_{2n}) = \frac{K}{(2n)!}B(e_{2n}) \leq B(f).$$

In fact, inequality (5.161) still holds if we replace assumption (5.160) with the weaker assumption that $K\left(B(e_{2n}) - A(e_{2n})\right) \geq 0$.

**Corollary 5.13** *Let $j_1, j_2, n \in \mathbb{N}$, $1 \leq j_1, j_2 \leq n$, let $f : [a,b] \to \mathbb{R}$ be $(2n+1)$-convex at point c, let m-tuples $\mathbf{x} \in [a,c]^m$ and $\mathbf{p} \in \mathbb{R}^m$ satisfy* (5.2) *and* (5.3) *with n replaced by $2j_1$, let l-tuples $\mathbf{y} \in [c,b]^l$ and $\mathbf{q} \in \mathbb{R}^l$ satisfy*

$$\sum_{i=1}^l q_i y_i^k = 0, \quad \textit{for all } k = 0,1,\ldots,2j_2 - 1$$

$$\sum_{i=1}^l q_i (y_i - t)_+^{2j_2-1} \geq 0, \quad \textit{for every } t \in [y_{(1)}, y_{(l-n+1)}]$$

*and let* (5.160) *holds. If $n - j_1$ and $n - j_2$ are even, then*

$$A(f) \leq B(f),$$

*while the reversed inequality holds if $n - j_1$ and $n - j_2$ are odd.*

*Proof.* Similar to the proof of Corollary 5.9. □

### 5.3.2 Bounds for the Remainders and Functionals

For $m$-tuples $\mathbf{p} = (p_1, \ldots, p_m) \in \mathbb{R}^m$, $\mathbf{x} = (x_1, \ldots, x_m) \in [a,b]^m$ and the function $G_n$ given be (5.141), denote

$$\delta(s) = \sum_{i=1}^{m} p_i G_n\left(\frac{x_i - a}{b-a}, \frac{s-a}{b-a}\right), \quad \text{for } s \in [a,b]. \tag{5.164}$$

Similarly, for functions $g : [\alpha,\beta] \to [a,b]$ and $p : [\alpha,\beta] \to \mathbb{R}$ denote

$$\Delta(s) = \int_{\alpha}^{\beta} p(x) G_n\left(\frac{g(x)-a}{b-a}, \frac{s-a}{b-a}\right) dx, \quad \text{for } s \in [a,b]. \tag{5.165}$$

**Theorem 5.61** *Let $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ be such that $f^{(2n)}$ is an absolutely continuous function with $(\cdot - a)(b - \cdot)[f^{(2n+1)}]^2 \in L[a,b]$ and let $G_n$, $T$ and $\delta$ be given by* (5.141), (5.36) *and* (5.164) *respectively. Then the remiander $R_n^1(f;a,b)$ given in*

$$\begin{aligned} \sum_{i=1}^{m} p_i f(x_i) \;=\; & \sum_{i=1}^{m}\sum_{k=0}^{n-1} (b-a)^{2k} \Bigg[ p_i f^{(2k)}(a) \Lambda_k\left(\frac{b-x_i}{b-a}\right) \\ & \qquad\qquad + p_i f^{(2k)}(b) \Lambda_k\left(\frac{x_i-a}{b-a}\right)\Bigg] \\ & + (b-a)^{2n-2}\left(f^{(2n-1)}(b) - f^{(2n-1)}(a)\right)\int_a^b \delta(s)\,ds + R_n^1(f;a,b) \end{aligned} \tag{5.166}$$

*satisfies the estimation*

$$|R_n^1(f;a,b)| \le \frac{(b-a)^{2n-\frac{1}{2}}}{\sqrt{2}} \left(T(\delta,\delta)\int_a^b (s-a)(b-s)[f^{(2n+1)}(s)]^2\,ds\right)^{\frac{1}{2}}.$$

*Proof.* If we apply Theorem 5.12 for $f \to \delta$ and $h \to f^{(2n)}$, then we obtain

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b \delta(s) f^{(2n)}(s)\,ds - \left(\frac{1}{b-a}\int_a^b \delta(s)\,ds\right)\left(\frac{1}{b-a}\int_a^b f^{(2n)}(s)\,ds\right)\right| \\ &\qquad \le \frac{1}{\sqrt{2}}\left(\frac{1}{b-a} T(\delta,\delta)\int_a^b (s-a)(b-s)[f^{(2n+1)}(s)]^2\,ds\right)^{\frac{1}{2}}. \end{aligned} \tag{5.167}$$

From (5.146) and (5.166) we obtain

$$\begin{aligned} &(b-a)^{2n-1}\int_a^b \delta(s) f^{(2n)}(s)\,ds \\ &\qquad = (b-a)^{2n-2}\left(f^{(2n-1)}(b) - f^{(2n-1)}(a)\right)\int_a^b \delta(s)\,ds + R_n^1(f;a,b), \end{aligned}$$

where the estimate (5.233) follows from (5.167). □

The following integral version of the previous theorem is proven analogously.

**Theorem 5.62** *Let $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ be such that $f^{(2n)}$ is an absolutely continuous function with $(\cdot - a)(b - \cdot)[f^{(2n+1)}]^2 \in L[a,b]$ and let $G_n$, $T$ and $\Delta$ be given by* (5.141), (5.36) *and* (5.165) *respectively. Then the remainder $R_n^2(f;a,b)$ from representation*

$$\int_\alpha^\beta p(x)f(g(x))\,dx = \tag{5.168}$$
$$\int_\alpha^\beta p(x)\sum_{k=j}^{n-1}(b-a)^{2k}\left[f^{(2k)}(a)\Lambda_k\left(\frac{b-g(x)}{b-a}\right)+f^{(2k)}(b)\Lambda_k\left(\frac{g(x)-a}{b-a}\right)\right]dx$$
$$+(b-a)^{2n-2}\left(f^{(2n-1)}(b)-f^{(2n-1)}(a)\right)\int_a^b \Delta(s)\,ds+R_n^2(f;a,b),$$

*satisfies the estimation*

$$|R_n^2(f;a,b)| \le \frac{(b-a)^{2n-\frac{1}{2}}}{\sqrt{2}}\left(T(\Delta,\Delta)\int_a^b (s-a)(b-s)[f^{(2n+1)}(s)]^2\,ds\right)^{\frac{1}{2}}.$$

By using Theorem 5.13 we obtain the following Grüss type inequality.

**Theorem 5.63** *Let $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ be such that $f^{(2n)}$ is an absolutely continuous function with $f^{(2n+1)} \ge 0$ and let $\delta$ be given by* (5.164). *Then we have the representation* (5.166) *and the remainder $R_n^1(f;a,b)$ satisfies the bound*

$$|R_n^1(f;a,b)| \le (b-a)^{2n}\|\delta'\|_\infty\left[\frac{f^{(2n-1)}(b)+f^{(2n-1)}(a)}{2}\right.$$
$$\left.-\frac{f^{(2n-2)}(b)-f^{(2n-2)}(a)}{b-a}\right]. \tag{5.169}$$

*Proof.* If we apply Theorem 5.13 for $f \to \delta$ and $h \to f^{(2n)}$ we obtain

$$\left|\frac{1}{b-a}\int_a^b \delta(s)f^{(2n)}(s)\,ds-\left(\frac{1}{b-a}\int_a^b \delta(s)\,ds\right)\left(\frac{1}{b-a}\int_a^b f^{(2n)}(s)\,ds\right)\right|$$
$$\le \frac{1}{2(b-a)}\|\delta'\|_\infty\int_a^b (s-a)(b-s)f^{(2n+1)}(s)\,ds.$$

Since

$$\int_a^b (s-a)(b-s)f^{(2n+1)}(s)\,ds = \int_a^b (2s-a-b)f^{(2n)}(s)\,ds \tag{5.170}$$
$$=(b-a)\left[f^{(2n-1)}(b)+f^{(2n-1)}(a)\right]-2\left[f^{(2n-2)}(b)-f^{(n-2)}(a)\right],$$

using identities (5.146) and (5.170) we deduce (5.169). □

Again, we only state the integral version of the previous result.

**Theorem 5.64** *Let $n \in \mathbb{N}$, $f : [a,b] \to \mathbb{R}$ be such that $f^{(2n)}$ is an absolutely continuous function with $f^{(2n+1)} \geq 0$ and let $\Delta$ be given by* (5.165). *Then we have the representation* (5.168) *and the remainder $R_n^2(f;a,b)$ satisfies the bound*

$$|R_n^2(f;a,b)| \leq (b-a)^{2n}\|\Delta'\|_\infty \left[\frac{f^{(2n-1)}(b)+f^{(2n-1)}(a)}{2} - \frac{f^{(2n-2)}(b)-f^{(2n-2)}(a)}{b-a}\right].$$

## 5.4 Linear Inequalities and Hermite Interpolation Polynomials

In this section we derive inequalities of type (5.1) and (5.4) for $n$-convex functions by making use of the Hermite interpolation. These results are contained in paper [75]. Let $-\infty < a \leq a_1 < a_2 < \cdots < a_r \leq b < \infty$, $r \geq 2$. The Hermite interpolation of a function $f \in C^n[a,b]$ is of the form

$$f(x) = P_H(x) + e_H(x)$$

where $P_H$ is the unique polynomial of degree $n-1$, called the Hermite interpolating polynomial of $f$, satisfying

$$P_H^{(i)}(a_j) = f^{(i)}(a_j), \quad 0 \leq i \leq k_j,\ 1 \leq j \leq r,\ \sum_{j=1}^{r} k_j + r = n.$$

The associated error $e_H(x)$ can be represented in terms of the Green function $G_{H,n}(x,s)$ for the multipoint boundary value problem

$$z^{(n)}(x) = 0, \quad z^{(i)}(a_j) = 0, \quad 0 \leq i \leq k_j, \quad 1 \leq j \leq r,$$

that is, the following result holds (see [1]):

**Theorem 5.65** *Let $f \in C^n[a,b]$, and let $P_H$ be its Hermite interpolating polynomial. Then*

$$\begin{aligned} f(x) &= P_H(x) + e_H(x) \\ &= \sum_{j=1}^{r}\sum_{i=0}^{k_j} H_{ij}(x) f^{(i)}(a_j) + \int_a^b G_{H,n}(x,s) f^{(n)}(s)\,ds, \end{aligned} \tag{5.171}$$

*where $H_{ij}$ are the fundamental polynomials of the Hermite basis defined by*

$$H_{ij}(x) = \frac{1}{i!}\frac{w(x)}{(x-a_j)^{k_j+1-i}}\sum_{k=0}^{k_j-i}\frac{1}{k!}\frac{d^k}{dx^k}\left(\frac{(x-a_j)^{k_j+1}}{w(x)}\right)\Big|_{x=a_j}(x-a_j)^k, \tag{5.172}$$

*where*

$$w(x) = \prod_{j=1}^{r}(x-a_j)^{k_j+1} \tag{5.173}$$

*and $G_{H,n}$ is the Green function defined by*

$$G_{H,n}(x,s) = \begin{cases} \sum_{j=1}^{l}\sum_{i=0}^{k_j} \frac{(a_j-s)^{n-i-1}}{(n-i-1)!} H_{ij}(x), & s \le x, \\ -\sum_{j=l+1}^{r}\sum_{i=0}^{k_j} \frac{(a_j-s)^{n-i-1}}{(n-i-1)!} H_{ij}(x), & s \ge x \end{cases} \tag{5.174}$$

*for all $a_l \le s \le a_{l+1}$, $l = 0,1,\ldots,r$ ($a_0 = a, a_{r+1} = b$).*

The following are some special cases of the Hermite interpolation of functions:

(i) $(m,n-m)$ conditions: $r = 2$, $a_1 = a$, $a_2 = b$, $1 \le m \le n-1$, $k_1 = m-1$ and $k_2 = n-m-1$. In this case

$$f(x) = \sum_{i=0}^{m-1} \tau_i(x) f^{(i)}(a) + \sum_{i=0}^{n-m-1} \eta_i(x) f^{(i)}(b) + \int_a^b G_{m,n}(x,s) f^{(n)}(s)ds,$$

where

$$\tau_i(x) = \frac{1}{i!}(x-a)^i \left(\frac{x-b}{a-b}\right)^{n-m} \sum_{k=0}^{m-1-i} \binom{n-m+k-1}{k} \left(\frac{x-a}{b-a}\right)^k, \tag{5.175}$$

$$\eta_i(x) = \frac{1}{i!}(x-b)^i \left(\frac{x-a}{b-a}\right)^{m} \sum_{k=0}^{n-m-1-i} \binom{m+k-1}{k} \left(\frac{x-b}{a-b}\right)^k, \tag{5.176}$$

and the Green function $G_{m,n}$ is of the form

$$G_{m,n}(x,s) = \begin{cases} \sum_{j=0}^{m-1}\left[\sum_{p=0}^{m-1-j} \binom{n-m+p-1}{p} \left(\frac{x-a}{b-a}\right)^p\right] \\ \qquad \times \frac{(x-a)^j(a-s)^{n-j-1}}{j!(n-j-1)!} \left(\frac{b-x}{b-a}\right)^{n-m}, & s \le x, \\ -\sum_{i=0}^{n-m-1}\left[\sum_{q=0}^{n-m-1-i} \binom{m+q-1}{q} \left(\frac{b-x}{b-a}\right)^q\right] \\ \qquad \times \frac{(x-b)^i(b-s)^{n-i-1}}{i!(n-i-1)!} \left(\frac{x-a}{b-a}\right)^{m}, & s \ge x. \end{cases}$$

$(ii)$ Taylor's two-point condition: $m \in \mathbb{N}$, $n = 2m$, $r = 2$, $a_1 = a$, $a_2 = b$ and $k_1 = k_2 = m-1$. In this case

$$f(x) = \sum_{i=0}^{m-1}\sum_{k=0}^{m-i-1} \binom{m+k-1}{k} \left[\frac{(x-a)^i}{i!} \left(\frac{x-b}{a-b}\right)^m \left(\frac{x-a}{b-a}\right)^k f^{(i)}(a)\right.$$

$$+ \frac{(x-b)^i}{i!}\left(\frac{x-a}{b-a}\right)^m\left(\frac{x-b}{a-b}\right)^k f^{(i)}(b)\Big] + \int_a^b G_{2T,m}(x,s)f^{(2m)}(s)ds,$$

where the Green function $G_{2T,m}$ is of the form

$$G_{2T,m}(x,s)$$

$$= \frac{(-1)^m}{(2m-1)!}\begin{cases} p^m(x,s)\sum_{k=0}^{m-1}\binom{m+k-1}{k}(x-s)^{m-1-k}q^k(x,s), & s \le x, \\ q^m(x,s)\sum_{k=0}^{m-1}\binom{m+k-1}{k}(s-x)^{m-1-k}p^k(x,s), & x \le s, \end{cases}$$

where $p(x,s) = \dfrac{(s-a)(b-x)}{(b-a)}$ and $q(x,s) = p(s,x)$.

The following lemma yields the sign of the Green function (5.174) on certain intervals (see Lemma 2.3.3, page 75, in [1]).

**Lemma 5.4** *The Green function $G_{H,n}$ given by* (5.174) *and $w$ given by* (5.173) *satisfy*

$$\frac{G_{H,n}(x,s)}{w(x)} > 0, \quad \text{for } a_1 \le x \le a_r,\ a_1 < s < a_r.$$

Integration by parts easily yields that for any function $f \in C^2[a,b]$ the following holds

$$f(x) = \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b) + \int_a^b G(x,s)f''(s)ds, \tag{5.177}$$

where the function $G : [a,b] \times [a,b] \to \mathbb{R}$ is the Green function of the boundary value problem

$$z''(x) = 0,\ z(a) = z(b) = 0$$

and is given by

$$G(x,s) = \begin{cases} \dfrac{(x-b)(s-a)}{b-a}, & \text{for } a \le s \le x, \\ \dfrac{(s-b)(x-a)}{b-a}, & \text{for } x \le s \le b. \end{cases} \tag{5.178}$$

The function $G$ is continuous, symmetric and convex with respect to both variables $x$ and $s$.

## 5.4.1 Inequalities Obtained via Hermite Interpolating Polynomials

We will start this section with several identities.

**Theorem 5.66** *Let $-\infty < a \le a_1 < a_2 < \cdots < a_r \le b < \infty$, $r \ge 2$, $\sum_{j=1}^r k_j + r = n$, $f \in C^n[a,b]$, $\mathbf{x} \in [a,b]^m$, $\mathbf{p} \in \mathbb{R}^m$ and let $H_{ij}$ and $G_{H,n}$ be given by* (5.172) *and* (5.174). *Then*

$$\sum_{k=1}^m p_k f(x_k) = \sum_{j=1}^r \sum_{i=0}^{k_j} \sum_{k=1}^m p_k H_{ij}(x_k) f^{(i)}(a_j)$$

$$+\int_a^b \sum_{k=1}^m p_k G_{H,n}(x_k,s) f^{(n)}(s)\,ds. \tag{5.179}$$

*Proof.* By applying identity (5.171) at $x_k$, multiplying it by $p_k$ and summing up we obtained the required identity. □

The integral version of the previous theorem is the following:

**Theorem 5.67** *Let* $-\infty < a \le a_1 < a_2 < \cdots < a_r \le b < \infty$, $r \ge 2$, $\sum_{j=1}^r k_j + r = n$, $f \in C^n[a,b]$, $g : [\alpha,\beta] \to [a,b]$, $p : [\alpha,\beta] \to \mathbb{R}$ *and let* $H_{ij}$ *and* $G_{H,n}$ *be given by* (5.172) *and* (5.174). *Then*

$$\int_\alpha^\beta p(x) f(g(x))\,dx = \sum_{j=1}^r \sum_{i=0}^{k_j} f^{(i)}(a_j) \int_\alpha^\beta p(x) H_{ij}(x)\,dx$$
$$+\int_a^b \left( \int_\alpha^\beta p(x) G_{H,n}(g(x),s)\,dx \right) f^{(n)}(s)\,ds.$$

**Theorem 5.68** *Let* $-\infty < a \le a_1 < a_2 < \cdots < a_r \le b < \infty$, $r \ge 2$, $\sum_{j=1}^r k_j + r = n-2$, $f \in C^n[a,b]$, $\mathbf{x} \in [a,b]^m$, $\mathbf{p} \in \mathbb{R}^m$ *and let* $H_{ij}$ *and* $G_{H,n-2}$ *be given by* (5.172) *and* (5.174). *Then*

$$\begin{aligned}\sum_{k=1}^m p_k f(x_k) &= \frac{f(b)-f(a)}{b-a} \sum_{k=1}^m p_k x_k + \frac{bf(a)-af(b)}{b-a} \sum_{k=1}^m p_k \\ &+ \sum_{j=1}^r \sum_{i=0}^{k_j} f^{(i+2)}(a_j) \int_a^b \sum_{k=1}^m p_k G(x_k,s) H_{ij}(s)\,ds \\ &+ \int_a^b \int_a^b \sum_{k=1}^m p_k G(x_k,s) G_{H,n-2}(s,t) f^{(n)}(t)\,dt\,ds. \end{aligned} \tag{5.180}$$

*Proof.* Applying identity (5.177) at $x_k$, multiplying it by $p_k$ and summing up we obtain

$$\sum_{k=1}^m p_k f(x_k) = \frac{f(b)-f(a)}{b-a} \sum_{k=1}^m p_k x_k + \frac{bf(a)-af(b)}{b-a} \sum_{k=1}^m p_k$$
$$+\int_a^b \sum_{k=1}^m p_k G(x_k,s) f''(s)\,ds. \tag{5.181}$$

By Theorem 5.65, $f''(s)$ can be expressed as

$$f''(s) = \sum_{j=1}^r \sum_{i=0}^{k_j} H_{ij}(s) f^{(i+2)}(a_j) + \int_a^b G_{H,n-2}(s,t) f^{(n)}(t)\,dt. \tag{5.182}$$

Inserting (5.182) in (5.181) we get (5.180). □

We also state the integral version of the previous theorem.

**Theorem 5.69** *Let $-\infty < a \leq a_1 < a_2 < \cdots < a_r \leq b < \infty$, $r \geq 2$, $\sum_{j=1}^{r} k_j + r = n-2$, $f \in C^n[a,b]$, $g : [\alpha,\beta] \to [a,b]$, $p : [\alpha,\beta] \to \mathbb{R}$ and let $H_{ij}$ and $G_{H,n-2}$ be given by* (5.172) *and* (5.174). *Then*

$$\begin{aligned}\int_\alpha^\beta p(x) f(g(x))\,dx &= \frac{f(b)-f(a)}{b-a}\int_\alpha^\beta p(x)g(x)\,dx + \frac{bf(a)-af(b)}{b-a}\int_\alpha^\beta p(x)\,dx \\ &+ \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+2)}(a_j)\int_a^b \left(\int_\alpha^\beta p(x)G(g(x),s)\,dx\right) H_{ij}(s)\,ds \\ &+ \int_a^b\int_a^b\left(\int_\alpha^\beta p(x)G(g(x),s)\,dx\right) G_{H,n-2}(s,t) f^{(n)}(t)\,dt\,ds.\end{aligned}$$

Next we will use the identities proven above to derive inequalities.

**Theorem 5.70** *Let $-\infty < a \leq a_1 < a_2 < \cdots < a_r \leq b < \infty$, $r \geq 2$, $\sum_{j=1}^{r} k_j + r = n$, $\mathbf{x} \in [a,b]^m$, $\mathbf{p} \in \mathbb{R}^m$ and let $H_{ij}$ and $G_{H,n}$ be given by* (5.172) *and* (5.174). *If $f : [a,b] \to \mathbb{R}$ is n-convex and*

$$\sum_{k=1}^{m} p_k G_{H,n}(x_k,s) \geq 0 \quad \text{for all } s \in [a,b], \tag{5.183}$$

*then*

$$\sum_{k=1}^{m} p_k f(x_k) \geq \sum_{j=1}^{r}\sum_{i=0}^{k_j}\sum_{k=1}^{m} p_k H_{ij}(x_k) f^{(i)}(a_j). \tag{5.184}$$

*If the inequality in* (5.183) *is reversed, then the inequality in* (5.184) *is reversed also.*

*Proof.* If (5.183) holds, then the second term on the right-hand side (5.179) is nonnegative. □

**Theorem 5.71** *Let $-\infty < a \leq a_1 < a_2 < \cdots < a_r \leq b < \infty$, $r \geq 2$, $\sum_{j=1}^{r} k_j + r = n$, $\mathbf{x} \in [a,b]^m$, $p : [\alpha,\beta] \to \mathbb{R}$ and let $H_{ij}$ and $G_{H,n}$ be given by* (5.172) *and* (5.174). *If $f : [a,b] \to \mathbb{R}$ is n-convex and*

$$\int_\alpha^\beta p(x) G_{H,n}(g(x),s)\,dx \geq 0 \quad \text{for all } s \in [a,b], \tag{5.185}$$

*then*

$$\int_\alpha^\beta p(x) f(g(x))\,dx \geq \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i)}(a_j)\int_\alpha^\beta p(x) H_{ij}(x)\,dx. \tag{5.186}$$

*If the inequality in* (5.185) *is reversed, then the inequality in* (5.186) *is reversed also.*

**Theorem 5.72** *Let $-\infty < a = a_1 < a_2 < \cdots < a_r = b < \infty$, $r \geq 2$, $\sum_{j=1}^{r} k_j + r = n-2$, $\mathbf{x} \in [a,b]^m$, $\mathbf{p} \in \mathbb{R}^m$ and let $H_{ij}$ and $G_{H,n-2}$ be given by* (5.172) *and* (5.174). *Let $f : [a,b] \to \mathbb{R}$ be n-convex and*

$$\sum_{k=1}^{m} p_k G(x_k,s) \geq 0 \quad \text{for all } s \in [a,b], \tag{5.187}$$

*and consider the inequality*

$$\sum_{k=1}^{m} p_k f(x_k) \geq \frac{f(b)-f(a)}{b-a} \sum_{k=1}^{m} p_k x_k + \frac{bf(a)-af(b)}{b-a} \sum_{k=1}^{m} p_k + \sum_{j=1}^{r} \sum_{i=0}^{k_j} f^{(i+2)}(a_j) \int_a^b \sum_{k=1}^{m} p_k G(x_k,s) H_{ij}(s)\, ds. \tag{5.188}$$

$(i)$ *If $k_j$ for $j=2,\ldots,r$ are odd, then* (5.188) *holds.*

$(ii)$ *If $k_j$ for $j=2,\ldots,r-1$ are odd and $k_r$ is even, then the reverse of* (5.188) *holds.*

*Proof.* $(i)$ Assume first that $f \in C^n[a,b]$. Due to the assumptions $w$ given by (5.173) satisfies $w(x) \geq 0$ for all $x$ and, hence, by Lemma 5.4, $G_{H,n-2}(s,t) \geq 0$ for all $s,t \in [a,b]$. Therefore, the last term on the right-hand side of (5.180) is nonnegative, so inequality (5.188) holds. The inequality for general $f$ follows since every $n$-convex function can be obtained, by making use of the Bernstein polynomials, as a uniform limit of $n$-convex functions with a continuous $n$-th derivative (see [77]).
$(ii)$ Under these assumptions $w(x) \leq 0$, so $G_{H,n-2}(s,t) \leq 0$. The rest of the proof is the same as in $(i)$. □

**Theorem 5.73** *Let $-\infty < a = a_1 < a_2 < \cdots < a_r = b < \infty$, $r \geq 2$, $\sum_{j=1}^{r} k_j + r = n-2$, $g:[\alpha,\beta] \to \mathbb{R}$, $p:[\alpha,\beta] \to \mathbb{R}$ and let $H_{ij}$ be given by* (5.172). *Let $f:[a,b] \to \mathbb{R}$ be $n$-convex and*

$$\int_\alpha^\beta p(x) G(g(x),s)\, dx \geq 0 \quad \textit{for all } s \in [a,b],$$

*and consider the inequality*

$$\int_\alpha^\beta p(x) f(g(x))\, dx \geq \frac{f(b)-f(a)}{b-a} \int_\alpha^\beta p(x) g(x)\, dx + \frac{bf(a)-af(b)}{b-a} \int_\alpha^\beta p(x)\, dx + \sum_{j=1}^{r} \sum_{i=0}^{k_j} f^{(i+2)}(a_j) \int_a^b \left( \int_\alpha^\beta p(x) G(g(x),s)\, dx \right) H_{ij}(s)\, ds. \tag{5.189}$$

$(i)$ *If $k_j$ for $j=2,\ldots,r$ are odd, then* (5.189) *holds.*

$(ii)$ *If $k_j$ for $j=2,\ldots,r-1$ are odd and $k_r$ is even, then the reverse of* (5.189) *holds.*

In the case of the $(m,n-m)$ conditions we have the following corollary.

**Corollary 5.14** *Let $\tau_i$ and $\eta_i$ be given by* (5.175) *and* (5.176) *and let $\mathbf{x} \in [a,b]^m$ and $\mathbf{p} \in \mathbb{R}^m$ be such that* (5.187) *holds. Let $f:[a,b] \to \mathbb{R}$ be $n$-convex and consider the inequality*

$$\sum_{k=1}^{m} p_k f(x_k) \geq \frac{f(b)-f(a)}{b-a} \sum_{k=1}^{m} p_k x_k + \frac{bf(a)-af(b)}{b-a} \sum_{k=1}^{m} p_k + \int_a^b \left( \sum_{k=1}^{m} p_k G(x_k,s) \right) \left( \sum_{i=0}^{l-1} \tau_i(s) f^{(i+2)}(a) + \sum_{i=0}^{n-l-1} \eta_i(s) f^{(i+2)}(b) \right) ds. \tag{5.190}$$

(*i*) *If* $n-l$ *is even, then* (5.190) *holds.*

(*ii*) *If* $n-l$ *is odd, then the reverse of* (5.190) *holds.*

In the case of Taylor's two point conditions we have the following corollary.

**Corollary 5.15** *Let* $\mathbf{x} \in [a,b]^m$ *and* $\mathbf{p} \in \mathbb{R}^m$ *be such that* (5.187) *holds. Let* $f : [a,b] \to \mathbb{R}$ *be n-convex and consider the inequality*

$$\begin{aligned}\sum_{k=1}^{m} p_k f(x_k) \geq& \frac{f(b)-f(a)}{b-a}\sum_{k=1}^{m} p_k x_k + \frac{bf(a)-af(b)}{b-a}\sum_{k=1}^{m} p_k \\ &+ \int_a^b \left(\sum_{k=1}^{m} p_k G(x_k,s)\right)\left(\sum_{i=0}^{l-1}\sum_{k=0}^{l-i-1}\binom{l+k-1}{k}\right. \\ &\qquad\times\left[\frac{(s-a)^i}{i!}\left(\frac{s-b}{a-b}\right)^l\left(\frac{s-a}{b-a}\right)^k f^{(i+2)}(a)\right. \\ &\qquad\left.\left.+\frac{(s-b)^i}{i!}\left(\frac{s-a}{b-a}\right)^l\left(\frac{s-b}{a-b}\right)^k f^{(i+2)}(b)\right]\right) ds. \qquad (5.191)\end{aligned}$$

(*i*) *If* $l$ *is even, then* (5.191) *holds.*

(*ii*) *If* $l$ *is odd, then the reverse of* (5.191) *holds.*

**Theorem 5.74** *Let* $-\infty < a = a_1 < a_2 < \cdots < a_r = b < \infty$, $r \geq 2$, $\sum_{j=1}^{r} k_j + r = n-2$, *let* $\mathbf{x} \in [a,b]^m$ *and* $\mathbf{p} \in \mathbb{R}^m$ *satisfy*

$$\sum_{k=1}^{m} p_k = 0, \qquad \sum_{k=1}^{m} p_k |x_k - x_i| \geq 0 \text{ for } i \in \{1,\ldots,m\}$$

*and let* $H_{ij}$ *and* $G_{H,n-2}$ *be given by* (5.172) *and* (5.174). *Let* $f : [a,b] \to \mathbb{R}$ *be n-convex and consider the inequality*

$$\sum_{k=1}^{m} p_k f(x_k) \geq \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+2)}(a_j)\int_a^b \sum_{k=1}^{m} p_k G(x_k,s) H_{ij}(s)\, ds \qquad (5.192)$$

*and the function*

$$F(x) = \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+2)}(a_j)\int_a^b G(x,s) H_{ij}(s)\, ds. \qquad (5.193)$$

(*i*) *If* $k_j$ *for* $j = 2,\ldots,r$ *are odd, then* (5.192) *holds. Furthermore, if the function* $F$ *is convex, then inequality* $\sum_{k=1}^{m} p_k f(x_k) \geq 0$ *holds.*

(*ii*) *If* $k_j$ *for* $j = 2,\ldots,r-1$ *are odd and* $k_r$ *is even, then the reverse of* (5.192) *holds. Furthermore, if the function* $F$ *is concave, then inequality* $\sum_{k=1}^{m} p_k f(x_k) \leq 0$ *holds.*

*Proof.* The function $G(x,s)$ is convex in the first variable, so assumption (5.187) is satisfied by Remark 5.1. Now, the claims of the theorem follow from Theorem 5.72. □

**Theorem 5.75** *Let $-\infty < a = a_1 < a_2 < \cdots < a_r = b < \infty$, $r \geq 2$, $\sum_{j=1}^{r} k_j + r = n-2$, let $g : [\alpha,\beta] \to \mathbb{R}$ and $p : [\alpha,\beta] \to \mathbb{R}$ satisfy* (5.5) *and* (5.6). *Let $H_{ij}$ and $G_{H,n-2}$ be given by* (5.172) *and* (5.174). *Let $f : [a,b] \to \mathbb{R}$ be n-convex and consider the inequality*

$$\int_{\alpha}^{\beta} p(x)f(x)\,dx \geq \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+2)}(a_j)\int_a^b\left(\int_{\alpha}^{\beta} p(x)G(g(x),s)\,dx\right)H_{ij}(s)\,ds \tag{5.194}$$

*and the function $F$ given by* (5.193).

(*i*) *If $k_j$ for $j = 2,\ldots,r$ are odd, then* (5.194) *holds. Furthermore, if the function $F$ is convex, then inequality $\int_{\alpha}^{\beta} p(x)f(g(x))dx \geq 0$ holds.*

(*ii*) *If $k_j$ for $j = 2,\ldots,r-1$ are odd and $k_r$ is even, then the reverse of* (5.194) *holds. Furthermore, if the function $F$ is concave, then inequality $\int_{\alpha}^{\beta} p(x)f(g(x))dx \leq 0$ holds.*

## 5.4.2 Bounds for the Remainders and Functionals

For $m$-tuples $\mathbf{p} = (p_1,\ldots,p_m) \in \mathbb{R}^m$, $\mathbf{x} = (x_1,\ldots,x_m) \in [a,b]^m$ and the functions $G$ and $G_{H,n}$ given by (5.178) and (5.174) denote

$$\theta_1(t) = \sum_{k=1}^{m} p_k G_{H,n}(x_k,t), \qquad \text{for } t \in [a,b]. \tag{5.195}$$

$$\theta_2(t) = \int_a^b \sum_{k=1}^{m} p_k G(x_k,s)G_{H,n-2}(s,t)\,ds, \qquad \text{for } t \in [a,b]. \tag{5.196}$$

**Theorem 5.76** *Let $-\infty < a \leq a_1 < a_2 < \cdots < a_r \leq b < \infty$, $r \geq 2$, let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ is an absolutely continuous function with $(\cdot - a)(b - \cdot)[f^{(n+1)}]^2 \in L[a,b]$, $\mathbf{x} \in [a,b]^m$, $\mathbf{p} \in \mathbb{R}^m$ and let $H_{ij}$, $\delta_1$ and $\delta_2$ be given by* (5.172), (5.195) *and* (5.196).

(*i*) *If $\sum_{j=1}^{r} k_j + r = n$, then*

$$\sum_{k=1}^{m} p_k f(x_k) = \sum_{j=1}^{r}\sum_{i=0}^{k_j}\sum_{k=1}^{m} p_k H_{ij}(x_k) f^{(i)}(a_j) + \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}\int_a^b \theta_1(s)\,ds + R_n^1(f;a,b), \tag{5.197}$$

*where the remainder $R_n^1(f;a,b)$ satisfies the estimation*

$$|R_n^1(f;a,b)| \leq \left(\frac{b-a}{2}T(\theta_1,\theta_1)\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2\,ds\right)^{\frac{1}{2}}. \tag{5.198}$$

*(ii) If* $\sum_{j=1}^{r} k_j + r = n-2$, *then*

$$\begin{aligned}\sum_{k=1}^{m} p_k f(x_k) &= \frac{f(b)-f(a)}{b-a}\sum_{k=1}^{m} p_k x_k + \frac{bf(a)-af(b)}{b-a}\sum_{k=1}^{m} p_k \\ &+ \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i+2)}(a_j)\int_a^b \sum_{k=1}^{m} p_k G(x_k,s)H_{ij}(s)\,ds \\ &+ \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\int_a^b \theta_2(s)\,ds + R_n^2(f;a,b), \end{aligned} \tag{5.199}$$

*where the remainder* $R_n^2(f;a,b)$ *satisfies the estimation*

$$|R_n^2(f;a,b)| \le \left(\frac{b-a}{2}T(\theta_2,\theta_2)\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2\,ds\right)^{\frac{1}{2}}.$$

*Proof.* $(i)$ Applying Theorem 5.12 with $f \to \theta_1$ and $h \to f^{(n)}$ we get

$$\begin{aligned}&\left|\int_a^b \theta_1(s)f^{(n)}(s)\,ds - \frac{1}{b-a}\int_a^b \theta_1(s)\,ds\int_a^b f^{(n)}(s)\,ds\right| \\ &\le \left(\frac{b-a}{2}T(\theta_1,\theta_1)\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2\,ds\right)^{\frac{1}{2}}.\end{aligned} \tag{5.200}$$

From identities (5.179) and (5.197) we obtain

$$\int_a^b \theta_1(s)f^{(n)}(s)\,ds = \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\int_a^b \theta_1(s)\,ds + R_n^1(f;a,b),$$

where the estimate (5.198) follows from (5.200).
$(ii)$ Analogous as in $(i)$. □

By using Theorem 5.13 we obtain the following Grüss type inequality.

**Theorem 5.77** *Let* $-\infty < a \le a_1 < a_2 < \cdots < a_r \le b < \infty$, $r \ge 2$, *let* $\mathbf{x}$, $\mathbf{p}$, $H_{ij}$, $\delta_1$, $\delta_2$ *and* $n$ *be as in Theorem* 5.76 *and let* $f:[a,b]\to\mathbb{R}$ *be such that* $f^{(n)}$ *is an absolutely continuous function with* $f^{(n+1)} \ge 0$. *Then the remainders* $R_n^i(f;a,b)$, $i=1,2$, *from representations* (5.197) *and* (5.199) *satisfy the bounds*

$$|R_n^i(f;a,b)| \le \|\theta_i'\|_\infty\left[\frac{b-a}{2}\left(f^{(n-1)}(b)+f^{(n-1)}(a)\right) - f^{(2n-2)}(b)+f^{(2n-2)}(a)\right].$$

*Proof.* This results easily follows by proceeding as in the proof of Theorem 5.16. □

We can construct linear functionals by taking differences of the left and right-hand sides of the inequalities from Theorems 5.70, 5.71, 5.72 and 5.73. By using similar methods as in [26, 31] (or in the first section of this chapter) we can prove mean value results for these functionals, as well as construct new families of exponentially convex functions and Cauchy-type means. Then, by using some known properties of exponentially convex functions, we can derive new inequalities and prove monotonicity of the obtained Cauchy-type means analogously as in [26, 31].

## 5.5 Linear Inequalities and the Fink Identity

Let us recall the Fink identity on which we base further results. The following theorem is proved by A. M. Fink in [18].

**Proposition 5.3** *Let $a,b \in \mathbb{R}$, $f:[a,b] \to \mathbb{R}$, $n \geq 1$ and $f^{(n-1)}$ is absolutely continuous on $[a,b]$. Then*

$$\begin{aligned} f(x) &= \frac{n}{b-a}\int_a^b f(t)\,dt \\ &- \sum_{k=1}^{n-1}\frac{n-k}{k!}\left(\frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a}\right) \\ &+ \frac{1}{(n-1)!(b-a)}\int_a^b (x-t)^{n-1}k^{[a,b]}(t,x)f^{(n)}(t)\,dt, \end{aligned} \tag{5.201}$$

*where*

$$k^{[a,b]}(t,x) = \begin{cases} t-a, & a \leq t \leq x \leq b, \\ t-b, & a \leq x < t \leq b. \end{cases} \tag{5.202}$$

We follow with identities for $\sum_{i=1}^n p_i f(x_i)$ and $\int_a^b p(x)f(g(x))dx$ constructed by using the Fink identity and the Green function. Also we consider inequalities for $n$-convex functions which are based on these identities.

### 5.5.1 Inequalities via the Fink Identity

**Theorem 5.78** *Let $n \in \mathbb{N}$ and $f:[a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous. Let $x_i \in [a,b]$, $p_i \in \mathbb{R}$ $(i \in \{1,\ldots,m\})$ be reals such that $\sum_{i=0}^m p_i = 0$ and let $k^{[a,b]}$ be the function as defined in* (5.202)*. Then we have*

$$\begin{aligned} &\sum_{i=1}^m p_i f(x_i) \\ &= \sum_{k=1}^{n-1}\frac{n-k}{k!(b-a)}\left(f^{(k-1)}(b)\sum_{i=1}^m p_i(x_i-b)^k - f^{(k-1)}(a)\sum_{i=1}^m p_i(x_i-a)^k\right) \\ &+ \frac{1}{(n-1)!(b-a)}\int_a^b f^{(n)}(t)\left(\sum_{i=1}^m p_i(x_i-t)^{n-1}k^{[a,b]}(t,x_i)\right)dt. \end{aligned} \tag{5.203}$$

*Proof.* By using the Fink identity (5.201) for $x = x_i$, multiplying it with $p_i$ and taking the sum over $i$ from 1 to $m$, we have

$$\sum_{i=1}^m p_i f(x_i) = \frac{n}{b-a}\int_a^b f(t)\,dt \sum_{i=0}^m p_i$$

$$+\sum_{i=1}^{m} p_i \sum_{k=1}^{n-1} \frac{n-k}{k!} \frac{f^{(k-1)}(b)(x_i-b)^k - f^{(k-1)}(a)(x_i-a)^k}{b-a}$$
$$+\sum_{i=1}^{m} p_i \frac{\int_a^b f^{(n)}(t)(x_i-t)^{n-1} k^{[a,b]}(t,x_i)\,dt}{(n-1)!(b-a)}.$$

After some rearrangement we get our required result. □

The following theorem is the integral version of Theorem 5.78.

**Theorem 5.79** *Let $n \in \mathbb{N}$ and $f:[a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and let $k^{[a,b]}(t,x)$ be the same as defined in* (5.202). *Let $g:[\alpha,\beta] \to [a,b]$ and $p:[\alpha,\beta] \to \mathbb{R}$ be integrable functions such that $\int_\alpha^\beta p(x)dx = 0$. Then we have*

$$\int_\alpha^\beta p(x) f(g(x))\,dx = \sum_{k=1}^{n-1} \frac{n-k}{k!(b-a)}$$
$$\times \left( f^{(k-1)}(b) \int_\alpha^\beta p(x)(g(x)-b)^k dx - f^{(k-1)}(a) \int_\alpha^\beta p(x)(g(x)-a)^k dx \right)$$
$$+\frac{1}{(n-1)!(b-a)} \int_a^b f^{(n)}(t) \left( \int_\alpha^\beta p(x)(g(x)-t)^{n-1} k^{[a,b]}(t,g(x))\,dx \right) dt.$$

*Proof.* Putting $x \to g(x)$ in (5.201), multiplying it by $p(x)$ and integrating with respect on $x$, we get an identity from which after using the Fubini theorem, we obtain the desired identity. □

Let us now introduce some notations which will be used in rest of the paper:

$$\mathscr{M}_1^{[a,b]}(m,\mathbf{x},\mathbf{p},t) = \sum_{i=1}^{m} p_i (x_i-t)^{n-1} k^{[a,b]}(t,x_i), \tag{5.204}$$
$$\mathscr{M}_2^{[a,b]}([\alpha,\beta],g,p,t) = \int_\alpha^\beta p(x)(g(x)-t)^{n-1} k^{[a,b]}(t,g(x))\,dx, \tag{5.205}$$

$$B_1^{[a,b]}(m,\mathbf{x},\mathbf{p},f) = \sum_{i=1}^{m} p_i f(x_i) - \sum_{k=1}^{n-1} \frac{n-k}{k!(b-a)}$$
$$\times \left( f^{(k-1)}(b) \sum_{i=1}^{m} p_i (x_i-b)^k - f^{(k-1)}(a) \sum_{i=1}^{m} p_i (x_i-a)^k \right), \tag{5.206}$$

$$B_2^{[a,b]}([\alpha,\beta],g,p,f) = \int_\alpha^\beta p(x) f(g(x))\,dx - \sum_{k=1}^{n-1} \frac{n-k}{k!(b-a)}$$
$$\times \left( f^{(k-1)}(b) \int_\alpha^\beta p(x)(g(x)-b)^k dx - f^{(k-1)}(a) \int_\alpha^\beta p(x)(g(x)-a)^k dx \right).$$

**Theorem 5.80** *Let all the assumptions of Theorem* 5.78 *be satisfied and let*

$$\mathscr{M}_1^{[a,b]}(m,\mathbf{x},\mathbf{p},t) \geq 0, \quad \textit{for all } t \in [a,b]. \tag{5.207}$$

*If $f$ is n-convex, then we have*

$$B_1^{[a,b]}(m,\mathbf{x},\mathbf{p},f) \geq 0. \tag{5.208}$$

*If opposite inequality holds in* (5.207)*, then* (5.208) *holds in the reverse direction.*

*Proof.* Since $f^{(n-1)}$ is absolutely continuous on $[a,b]$, $f^{(n)}$ exists almost everywhere. As $f$ is $n$-convex, by definition of $n$-convex functions we have $f^{(n)}(x) \geq 0$ for all $x \in [a,b]$. Now by using $f^{(n)} \geq 0$ and (5.207) in (5.203), we have (5.208). □

A consequence of the previous theorem is the following:

**Theorem 5.81** *Suppose all the assumptions from Theorem* 5.78 *hold. Additionally, let $j \in \mathbb{N}$, $2 \leq j \leq n$ and let $\mathbf{x} = (x_1,\ldots,x_m) \in [a,b]^m$, $\mathbf{p} = (p_1,\ldots,p_m) \in \mathbb{R}^m$ satisfy* (5.2) *and* (5.3) *with n replaced by j. If f is n-convex and $n-j$ is even, then*

$$\begin{aligned}\sum_{i=1}^{m} p_i f(x_i) \geq \sum_{k=j}^{n-1} \frac{n-k}{k!\,(b-a)} \Bigg( & f^{(k-1)}(b)\left(\sum_{i=1}^{m} p_i (x_i-b)^k\right) \\ & - f^{(k-1)}(a)\left(\sum_{i=1}^{m} p_i (x_i-a)^k\right)\Bigg).\end{aligned} \tag{5.209}$$

*Proof.* Let $t \in [a,b]$ be fixed. For $j \leq n-2$ we get

$$\frac{d^j}{dx^j}(x-t)^{n-1} = (n-1)(n-2)\cdots(n-j)(x-t)^{n-j-1}. \tag{5.210}$$

Therefore, (5.210) for $a \leq t \leq x \leq b$ yields

$$(t-a)\frac{d^j}{dx^j}(x-t)^{n-1} \geq 0, \tag{5.211}$$

while for $a \leq x < t \leq b$ we have

$$(-1)^{n-j}(t-b)\frac{d^j}{dx^j}(x-t)^{n-1} \geq 0. \tag{5.212}$$

It is clear that $x \mapsto \frac{d^j}{dx^j}(x-t)^{n-1}k^{[a,b]}(t,x)$ is continuous for $j \leq n-2$. Hence, if $j \leq n-2$ and $n-j$ is even, from (5.211) and (5.212) we can conclude that the function $x \mapsto (x-t)^{n-1}k^{[a,b]}(t,x)$ is $j$-convex. Moreover, the conclusion extends to the case $j=n$, i. e. the mapping $x \mapsto (x-t)^{n-1}k^{[a,b]}(t,x)$ is $n$-convex, since the mapping $x \mapsto \frac{d^{n-2}}{dx^{n-2}}(x-t)^{n-1}k^{[a,b]}(t,x)$ is 2-convex.

Using Theorem 5.1 for $j$-convex function $x \mapsto (x-t)^{n-1}k^{[a,b]}(t,x)$ with assumptions (5.2) and (5.3) where $n$ is replaced with $j$, we get $\sum_{i=1}^{m} p_i(x_i-t)^{n-1}k^{[a,b]}(t,x) \geq 0$. It means that (5.207) is satisfied and by Theorem 5.80 inequality (5.208) holds. Moreover, due to assumption (5.2), $\sum_{i=1}^{m} p_i P(x_i) = 0$ for every polynomial $P$ of degree $\leq j-1$, so the first $j-2$ terms in the inner sum in (5.206) vanish, i.e. we get inequality (5.209). □

When $j = n$ in (5.209), the notation means that the inner sum is void, i. e. $\sum_{k=n}^{n-1} \cdots = 0$. In particular, inequality (5.209) with $j = n$ is inequality (5.1).

**Corollary 5.16** *Let all the assumptions of Theorem 5.78 be satisfied and let the function $f : [a,b] \to \mathbb{R}$ be n-convex for even n. Let m-tuples $\mathbf{x} = (x_1, \ldots, x_m)$, $\mathbf{p} = (p_1, \ldots, p_m) \in \mathbb{R}^m$ satisfy the conditions*

$$\sum_{k=1}^{m} p_k = 0, \quad \sum_{k=1}^{m} p_k |x_k - x_i| \geq 0 \textit{ for } i \in \{1, \ldots, m\}.$$

*Then* (5.208) *holds.*

*Furthermore, if $f^{(k-1)}(a) \leq 0$ and $(-1)^k f^{(k-1)}(b) \geq 0$, for $k \in \{2,3,\ldots,n-1\}$, then*

$$\sum_{i=1}^{m} p_i f(x_i) \geq 0. \tag{5.213}$$

*Proof.* Inequality (5.208) holds by Theorem 5.81 applied for $j = 2$.

Moreover, the functions $x \mapsto (x-a)^k$ and $x \mapsto (-1)^k (x-b)^k$ are convex, so Remark 5.1 yields

$$\sum_{i=1}^{m} p_i (x_i - a)^k \geq 0, \tag{5.214}$$

and

$$(-1)^k \sum_{i=1}^{m} p_i (x_i - b)^k \geq 0. \tag{5.215}$$

Therefore, if $f^{(k-1)}(a) \leq 0$ and $(-1)^k f^{(k-1)}(b) \geq 0$, then (5.214) and (5.215) together with (5.206) yield inequality (5.213). □

**Corollary 5.17** *Suppose all the assumptions from Theorem 5.78 hold and let the function $f : [a,b] \to \mathbb{R}$ be n-convex. Additionally, let $j \in \mathbb{N}$, $2 \leq j \leq n$, let $\mathbf{x} = (x_1, \ldots, x_m) \in [a,b]^m$, $\mathbf{p} = (p_1, \ldots, p_m) \in \mathbb{R}^m$ satisfy* (5.2) *and* (5.3) *with n replaced by j and denote*

$$H(x) = \sum_{k=j}^{n-1} \frac{n-k}{k!(b-a)} \left( f^{(k-1)}(b)(x-b)^k - f^{(k-1)}(a)(x-a)^k \right). \tag{5.216}$$

*If H is j-convex on $[a,b]$ and $n-j$ is even, then*

$$\sum_{i=1}^{m} p_i f(x_i) \geq 0.$$

*Proof.* Applying Theorem 5.1 we conclude that $\sum_{i=1}^{m} p_i H(x_i) \geq 0$, so the right-hand side of inequality (5.209) is nonnegative and we get desired result. □

**Remark 5.18** For example, since the functions $x \mapsto (x-a)^k$ and $x \mapsto (-1)^{k-j}(x-b)^k$ are $j$-convex on $[a,b]$, the function $H$ given by (5.216) is $j$-convex if $f^{(k-1)}(a) \leq 0$ and $(-1)^{k-1-j} f^{(k-1)}(b) \geq 0$ for $k \in \{j, \dots, n-1\}$.

As we already mentioned, the inequality in Theorem 5.81 and Corollary 5.17 with $j = n$ is the same as inequality (5.1) from Popoviciu's Proposition 5.1. Of course, in the proof of Theorem 5.81 we have used Proposition 5.1 to prove that assumption (5.207) holds, so, due to circularity, we didn't obtain another proof of Popoviciu's result. But, it is possible, as we will show in the next lemma, to prove directly that conditions (5.2) and (5.3) imply (5.207), i. e. it is possible to prove Theorem 5.81 with $j = n$ independently of Proposition 5.1 and, thus, provide a new proof of Popoviciu's result.

**Lemma 5.5** *Let $n \geq 2$ and let $m$-tuples $\mathbf{x} \in [a,b]^m$ and $\mathbf{p} \in \mathbb{R}^m$ satisfy (5.2) and (5.3). Then (5.207) holds.*

*Proof.* Let $t \in [a,b]$ be fixed. Notice that

$$\Omega_1^{[a,b]}(m, \mathbf{x}, \mathbf{p}, t) = \sum_{i=1}^{m} p_i \varphi_t(x_i),$$

where $\varphi_t$ is the function

$$\varphi_t(x) = (x-t)^{n-1} k^{[a,b]}(t,x) = (t-b)(x-t)^{n-1} + (b-a)(x-t)_+^{n-1}.$$

As in the proof of Theorem 5.81 we conclude that (5.2) implies $\sum_{i=1}^m p_i P(x_i) = 0$ for every polynomial $P$ of degree $\leq n-1$. In particular, for $P(x) = (x-t)^{n-1}$ we have $\sum_{i=1}^m p_i (x_i - t)^{n-1} = 0$. Therefore,

$$\sum_{i=1}^{m} p_i \varphi_t(x_i) = (b-a) \sum_{i=1}^{m} p_i (x_i - t)_+^{n-1} \geq 0,$$

where the last inequalities holds due to (5.3). Since the previous inequality holds for every $t \in [a,b]$, we conclude that (5.207) holds. $\square$

Lemma 5.5 together with Theorem 5.80 gives the "if" part of Popoviciu's Proposition 5.1. On the other hand, the "only if" part is straightforward: since the functions $e_j(x) = x^j$ are both $n$-convex and $n$-concave for $j = 0, 1, \dots, n-1$, inequality (5.1) yields that $\sum_{i=1}^m p_i e_k(x_i)$ is both $\geq 0$ and $\leq 0$, so (5.2) holds. Similarly, the function $x \mapsto (x-t)_+^{n-1}$ is $n$-convex and applying inequality (5.1) yields (5.3).

In the remainder of the section we will state integral versions of the previous results, the proofs of which are analogous to the discrete case.

**Theorem 5.82** *Let all the assumptions of Theorem 5.79 be satisfied and*

$$\mathscr{M}_2^{[a,b]}([\alpha, \beta], g, p, t) \geq 0, \quad \textit{for all } t \in [a,b]. \tag{5.217}$$

*If $f$ is $n$-convex, then we have*

$$B_2^{[a,b]}([\alpha, \beta], g, p, f) \geq 0. \tag{5.218}$$

*If opposite inequality holds in (5.217), then (5.218) holds in the reverse direction.*

*Proof.* The idea of the proof is the same as that of Theorem 5.80. □

A result analogous to Corollary 5.16 can be stated for integrals.

**Theorem 5.83** *Suppose all the assumptions from Theorem* 5.79 *hold. Additionally, let* $j \in \mathbb{N}$, $2 \le j \le n$ *and let* $p : [\alpha, \beta] \to \mathbb{R}$ *and* $g : [\alpha, \beta] \to [a, b]$ *satisfy* (5.5) *with* $n$ *replaced by* $j$. *If* $f$ *is* $n$*-convex and* $n - j$ *is even, then*

$$\begin{aligned}
&\int_{\alpha}^{\beta} p(x) f(g(x))\,dx \\
&\qquad \ge \frac{1}{b-a}\left[\sum_{k=j}^{n-1} \frac{n-k}{k!} f^{(k-1)}(b) \int_{\alpha}^{\beta} p(x)(g(x)-b)^{k+2}\,dx \right. \\
&\qquad \left. - \sum_{k=j}^{n-1} \frac{n-k}{k!} f^{(k-1)}(a) \int_{\alpha}^{\beta} p(x)(g(x)-a)^{k}\,dx \right].
\end{aligned}$$

**Corollary 5.18** *Let* $j, n, f, p$ *and* $g$ *be as in Theorem* 5.83 *and let* $H$ *be given by* (5.216). *If* $H$ *is* $j$*-convex,* $n - j$ *is even and* $f$ *is* $n$*-convex, then*

$$\int_{\alpha}^{\beta} p(x) f(g(x))\,dx \ge 0.$$

## 5.5.2 Inequalities via the Fink Identity and the Green Function

In this section we will obtain another identity and the corresponding linear inequality by using the Green function (5.10) and applying again the Fink identity.

**Theorem 5.84** *Let* $n \in \mathbb{N}$, $n \ge 3$, *and* $f : [a, b] \to \mathbb{R}$ *be such that* $f^{(n-1)}$ *is absolutely continuous. Let* $x_i, y_i \in [a, b]$, $p_i \in \mathbb{R}$ *for* $i \in \{1, \ldots, m\}$ *be such that* $\sum_{i=1}^{m} p_i = 0$ *and* $\sum_{i=1}^{m} p_i x_i = 0$ *and let* $k^{[a,b]}$ *be as defined in* (5.202). *If* $G$ *is the Green function, then*

$$\begin{aligned}
\sum_{i=1}^{m} p_i f(x_i) &= \sum_{k=0}^{n-3} \frac{n-k-2}{k!\,(b-a)} \int_a^b \left( \sum_{i=1}^{m} p_i G(x_i, s) \right) \\
&\quad \times \left( f^{(k+1)}(b)(s-b)^k - f^{(k+1)}(a)(s-a)^k \right) ds + \frac{1}{(n-3)!\,(b-a)} \\
&\quad \times \int_a^b f^{(n)}(t) \left( \int_a^b \sum_{i=1}^{m} p_i G(x_i, s)(s-t)^{n-3} k^{[a,b]}(t, s)\,ds \right) dt.
\end{aligned} \tag{5.219}$$

*Proof.* Putting $x = x_i$ in (5.11), multiplying it with $p_i$, adding all the identities and using the properties $\sum_{i=1}^{m} p_i = 0$ and $\sum_{i=1}^{m} p_i x_i = 0$, we get

$$\sum_{i=1}^{m} p_i f(x_i) = \int_a^b \left( \sum_{i=1}^{m} p_i G(x_i, s) \right) f''(s)\,ds. \tag{5.220}$$

Applying the Fink identity with $f \to f''$ and $n \to n-2$, it is easy to see that

$$f''(x) = \sum_{k=0}^{n-3} \frac{n-k-2}{k!} \frac{f^{(k+1)}(b)(x-b)^k - f^{(k+1)}(a)(x-a)^k}{b-a} + \frac{1}{(n-3)!(b-a)} \int_a^b (x-t)^{n-3} k^{[a,b]}(t,x) f^{(n)}(t)\,dt, \tag{5.221}$$

and by using (5.221) in (5.220), we have

$$\sum_{i=1}^m p_i f(x_i) = \int_a^b \left( \sum_{i=1}^m p_i G(x_i,s) \right) \times \sum_{k=0}^{n-3} \frac{n-k-2}{k!} \frac{f^{(k+1)}(b)(s-b)^k - f^{(k+1)}(a)(s-a)^k}{b-a} ds + \frac{1}{(n-3)!(b-a)} \int_a^b \sum_{i=1}^m p_i G(x_i,s) \left( \int_a^b (s-t)^{n-3} k^{[a,b]}(t,s) f^{(n)}(t)\,dt \right) ds.$$

Now by interchanging the integral and summation in the second term and by applying Fubini's theorem in the last term, we have (5.219). □

The following theorem is the integral version of Theorem 5.84.

**Theorem 5.85** *Let $n \in \mathbb{N}$, $n \geq 3$, and let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$, let $p : [\alpha,\beta] \to \mathbb{R}$ and $g : [\alpha,\beta] \to [a,b]$ be integrable functions such that $\int_\alpha^\beta p(x)dx = 0$ and $\int_\alpha^\beta p(x)g(x)dx = 0$. Let $k^{[a,b]}$ be as defined in (5.202). If G is the Green function, then*

$$\int_\alpha^\beta p(x) f(g(x))\,dx = \sum_{k=0}^{n-3} \frac{n-k-2}{k!(b-a)} \int_a^b \left( \int_\alpha^\beta p(x) G(g(x),s)\,dx \right) \left( f^{(k+1)}(b)(s-b)^k - f^{(k+1)}(a)(s-a)^k \right) ds + \frac{1}{(n-3)!(b-a)} \times \int_a^b f^{(n)}(t) \left( \int_a^b \left( \int_\alpha^\beta p(x) G(g(x),s)\,dx \right) (s-t)^{n-3} k^{[a,b]}(t,s)\,ds \right) dt.$$

*Proof.* Since proof is similar to the proof of the previous theorem we omit the details. □

Again we introduce some notations here which will be used in rest of the section:

$$\mathscr{M}_3^{[a,b]}(m,\mathbf{x},\mathbf{p},t) = \int_a^b \sum_{i=1}^m p_i G(x_i,s)(s-t)^{n-3} k^{[a,b]}(t,s)\,ds, \tag{5.222}$$

$$\mathscr{M}_4^{[a,b]}([\alpha,\beta],g,p,t) = \int_a^b \left( \int_\alpha^\beta p(x) G(g(x),s)\,dx \right) (s-t)^{n-3} k^{[a,b]}(t,s)\,ds.$$

$$B_3^{[a,b]}(m,\mathbf{x},\mathbf{p},f) = \sum_{i=1}^m p_i f(x_i) - \sum_{k=0}^{n-3} \frac{n-k-2}{k!(b-a)} \int_a^b \sum_{i=1}^m p_i G(x_i,s)$$

$$\begin{aligned}
&\times \left(f^{(k+1)}(b)(s-b)^k - f^{(k+1)}(a)(s-a)^k\right) ds \\
B_4^{[a,b]}([\alpha,\beta],g,p,f) &= \int_\alpha^\beta p(x) f(g(x))\,dx \\
&- \sum_{k=0}^{n-3} \frac{n-k-2}{k!\,(b-a)} \int_a^b \left(\int_\alpha^\beta p(x) G(g(x),s)\,dx\right) \\
&\times \left(f^{(k+1)}(b)(s-b)^k - f^{(k+1)}(a)(s-a)^k\right) ds.
\end{aligned}$$

The following theorem is our second main result of this section:

**Theorem 5.86** *Let all the assumptions of Theorem* 5.84 *be satisfied and let*

$$\mathscr{M}_3^{[a,b]}(m,\mathbf{x},\mathbf{p},t) \geq 0 \quad \text{for all} \quad t \in [a,b]. \tag{5.223}$$

*If $f$ is n-convex, then we have*

$$B_3^{[a,b]}(m,\mathbf{x},\mathbf{p},f) \geq 0. \tag{5.224}$$

*If opposite inequality holds in* (5.223)*, then* (5.224) *holds in the reverse direction.*

*Proof.* The proof is done in a similar manner as in Theorem 5.80. □

**Corollary 5.19** *Let all the assumptions of Theorem* 5.84 *be satisfied. In addition, let $n$ be even and*

$$\sum_{i=1}^m p_i (x_i - x_k)_+ \geq 0 \quad \text{for} \quad k \in \{1,\ldots,m\}.$$

*If the function $f:[a,b] \to \mathbb{R}$ is n-convex, then inequality* (5.224) *is satisfied, i.e.*

$$\begin{aligned}
\sum_{i=1}^m p_i f(x_i) \geq \sum_{k=0}^{n-3} \frac{n-k-2}{k!\,(b-a)} \int_a^b \sum_{i=1}^m p_i G(x_i,s) \\
\times \left(f^{(k+1)}(b)(s-b)^k - f^{(k+1)}(a)(s-a)^k\right) ds.
\end{aligned} \tag{5.225}$$

*Furthermore, if $f^{(k+1)}(a) \leq 0$ and $(-1)^k f^{(k+1)}(b) \geq 0$ for $k = 0,1,\ldots,n-3$, then $\sum_{i=1}^m p_i f(x_i) \geq 0$.*

*Proof.* Since $x \mapsto G(x,s)$ is a convex function, applying Theorem 5.1 we get

$$\sum_{i=1}^m p_i G(x_i,s) \geq 0. \tag{5.226}$$

The assumptions of the corollary for even $n$ imply

$$(s-t)^{n-3} k^{[a,b]}(t,s) \geq 0$$

for all $s,t \in [a,b]$. Therefore,

$$\int_a^t \sum_{i=1}^m p_i G(x_i,s)(s-t)^{n-3} k^{[a,b]}(t,s)\,ds \geq 0 \tag{5.227}$$

and applying Theorem 5.86 when $f$ is $n$-convex gives inequality (5.225).

Moreover, if $f^{(k+1)}(a) \leq 0$ and $(-1)^k f^{(k+1)}(b) \geq 0$, then

$$f^{(k+1)}(b)(s-b)^k - f^{(k+1)}(a)(s-a)^k \geq 0, \tag{5.228}$$

so from inequalities (5.225), (5.226) and (5.228) we obtain $\sum_{i=1}^m p_i f(x_i) \geq 0$. □

An integral version of our second main result states that:

**Theorem 5.87** *Let all the assumptions of Theorem 5.85 be satisfied and let*

$$\mathscr{M}_4^{[a,b]}([\alpha,\beta],g,p,t) \geq 0 \quad \textit{for all } t \in [a,b]. \tag{5.229}$$

*If $f$ is $n$-convex, then we have*

$$B_4^{[a,b]}([\alpha,\beta],g,p,f) \geq 0. \tag{5.230}$$

*If opposite inequality holds in (5.229), then (5.230) holds in the reverse direction.*

*Proof.* The idea of the proof is the same as that of the proof of Theorem 5.80. □

**Corollary 5.20** *Let all the assumptions of Theorem 5.85 be satisfied. In addition, let $n$ be even and*

$$\int_\alpha^\beta p(x)(g(x)-t)_+^{n-1}\,dx \geq 0, \quad \textit{for every } t \in [a,b].$$

*If the function $f:[a,b]\to\mathbb{R}$ is $n$-convex, then*

$$\begin{aligned}\int_\alpha^\beta p(x) f(g(x))\,dx &\geq \sum_{k=0}^{n-3} \frac{n-k-2}{k!(b-a)} \int_a^b \left(\int_\alpha^\beta p(x) G(g(x),s)\,dx\right) \\ &\quad \times \left(f^{(k+1)}(b)(s-b)^k - f^{(k+1)}(a)(s-a)^k\right) ds. \end{aligned} \tag{5.231}$$

*Furthermore if $f^{(k+1)}(a) \leq 0$ and $(-1)^k f^{(k+1)}(b) \geq 0$ for $k = 0,\ldots,n-3$, then the right-hand side of (5.231) is nonnegative.*

*Proof.* The proof is analogous to the proof of Corollary 5.19 but instead of Theorem 5.86, we apply Theorem 5.87. □

### 5.5.3 Bounds for the Remainders and Functionals

In the present section we give several estimations related to the functionals $B_k^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,f)$, for $k \in \{1,2,3,4\}$. For the sake of brevity, in the present and the next section we use the notation $B_k(f) := B_k^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,f)$ and $\mathscr{M}_k(t) := \mathscr{M}_k^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,t)$ for $k \in \{1,2,3,4\}$. By using the well-known Hölder inequality and bound for the Čebyšev functional $T(f,h)$ given in Proposition 5.2 we are going to obtain a formula for $B_k$ and estimate the remainder which occurs in this formula.

**Theorem 5.88** *Let $n \in \mathbb{N}$ and let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is an absolutely continuous function and*

$$\gamma \le f^{(n)}(x) \le \Gamma \quad \text{for} \quad x \in [a,b].$$

*(i) Let $k \in \{1,2\}$ and let $\sum_{i=1}^{m} p_i = 0$ (for $k=1$) or $\int_\alpha^\beta p(x)dx = 0$ (for $k=2$). Then the remainder $R_n^k(f;a,b)$ given by the following identity*

$$B_k(f) = \frac{\left[f^{(n-1)}(b) - f^{(n-1)}(a)\right]}{(n-1)!(b-a)^2} \int_a^b \mathscr{M}_k(t)dt + R_n^k(f;a,b), \tag{5.232}$$

*satisfies the estimation*

$$|R_n^k(f;a,b)| \le \frac{b-a}{2(n-1)!}(\Gamma - \gamma)\sqrt{T(\mathscr{M}_k,\mathscr{M}_k)}. \tag{5.233}$$

*(ii) Let $k \in \{3,4\}$ and $n \ge 3$. Let the assumptions stated in Theorem 5.84 for $\mathbf{p}$ and $\mathbf{x}$ (for $k=3$) and in Theorem 5.85 for $p$ and $g$ (for $k=4$) hold. Then (5.232) and (5.233) hold with $(n-3)!$ instead of $(n-1)!$ in the denominator of $B_k(f)$ and in the bound of $R_n^k$.*

*Proof.* Fix $k \in \{1,2\}$. Using the definition of $B_k$ and result from the previous subsection we have

$$\begin{aligned} B_k(f) &= \frac{1}{(n-1)!(b-a)} \int_a^b f^{(n)}(t)\mathscr{M}_k(t)dt \\ &= \frac{1}{(n-1)!(b-a)^2} \int_a^b f^{(n)}(t)dt \int_a^b \mathscr{M}_k(t)dt + R_n^k(f;a,b) \\ &= \frac{\left[f^{(n-1)}(b) - f^{(n-1)}(a)\right]}{(n-1)!(b-a)^2} \int_a^b \mathscr{M}_k(t)dt + R_n^k(f;a,b), \end{aligned}$$

where

$$R_n^k(f;a,b) = \frac{1}{(n-1)!(b-a)}\left(\int_a^b f^{(n)}(t)\mathscr{M}_k(t)dt - \frac{1}{b-a}\int_a^b f^{(n)}(s)ds \int_a^b \mathscr{M}_k(t)dt\right).$$

If we apply Proposition 5.2 for $f \to \mathscr{M}_k$ and $h \to f^{(n)}$, then we obtain

$$|R_n^k(f;a,b)| = \frac{1}{(n-1)!}|T(\mathscr{M}_k, f^{(n)})| \le \frac{b-a}{2(n-1)!}(\Gamma-\gamma)\sqrt{T(\mathscr{M}_k,\mathscr{M}_k)}.$$

The proof for $k \in \{3,4\}$ is done in a similar manner. □

Using the same method as we used in the previous theorem and other type of bounds for the Čebyšev functional given in Theorems 5.12 and 5.13 we are able to give another estimation for the remainder. The following theorem gives us some Ostrowski-type inequalities.

**Theorem 5.89** *Let $f^{(n)} \in L_q[a,b]$ for some $n \in \mathbb{N}$ and let $(q,r)$ be a pair of conjugate exponents, that is, $1 \le q,r \le \infty$, $\frac{1}{q}+\frac{1}{r}=1$.*
*(i) Let $k \in \{1,2\}$ and let $\sum_{i=1}^{m} p_i = 0$ (for $k=1$) or $\int_{\alpha}^{\beta} p(x)dx = 0$ (for $k=2$). Then*

$$|B_k(f)| \le \frac{1}{(n-1)!}\|f^{(n)}\|_q \|\mathscr{M}_k\|_r. \tag{5.234}$$

*The constant on the right-hand side of (5.234) is sharp for $1 < q \le \infty$ and the best possible for $q=1$.*
*(ii) Let $k \in \{3,4\}$ and $n \ge 3$. For $k=3$ we assume that $\mathbf{x}$ and $\mathbf{p}$ satisfy the assumptions of Theorem 5.84 and for $k=4$ we assume that $p$ and $g$ satisfy the assumptions of Theorem 5.85. Then the statement holds with $(n-3)!$ instead of $(n-1)!$ in the denominator of the bound for $B_k$.*

*Proof.* The proof is similar to the proof of Theorem 5.18. □

### 5.5.4 Mean Value Theorems

In this subsection we consider mean value theorems involving $B_k$. Throughout the section we use the agreement that if $k \in \{1,2\}$, then $n \in \mathbb{N}$; if $k \in \{3,4\}$, then $n \ge 3$. Furthermore, for $k=1$ we assume that $\sum_{i=1}^{m} p_i = 0$, for $k=2$ we assume that $\int_{\alpha}^{\beta} p(x)dx = 0$, for $k=3$ we assume that $\mathbf{x}$ and $\mathbf{p}$ satisfy the assumptions of Theorem 5.84 and for $k=4$ we assume that $\mathbf{x}$ and $\mathbf{p}$ satisfy the assumptions of Theorem 5.85.

**Theorem 5.90** *Let $k \in \{1,2,3,4\}$ and let us consider $B_k$ as a functional on $C^n([a,b])$. If the corresponding conditions from the set $\{(5.204),(5.90),(5.222),(5.229)\}$ related to the fixed k hold, then there exists $\xi_k \in [a,b]$ such that*

$$B_k(f) = f^{(n)}(\xi_k) B_k(f_0),$$

*where* $f_0(x) = \dfrac{x^n}{n!}$.

*Proof.* The proof is similar to the proof of Theorem 5.34. □

Applying Theorem 5.90 on function $\omega = B_k(h)f - B_k(f)h$, we get the following result.

**Theorem 5.91** *Let $k \in \{1,2,3,4\}$ and let us consider $B_k$ as a functional on $C^n([a,b])$. If the corresponding conditions from the set $\{(5.204),(5.90),(5.222),(5.229)\}$ related to the fixed k hold, then there exists $\xi_k \in [a,b]$ such that*

$$\frac{B_k(f)}{B_k(h)} = \frac{f^{(n)}(\xi_k)}{h^{(n)}(\xi_k)}$$

*assuming that both the denominators are non-zero.*

**Remark 5.19** If the inverse of $\frac{f^{(n)}}{h^{(n)}}$ exists, then from the above mean value theorems we can give generalized means

$$\xi_k = \left(\frac{f^{(n)}}{h^{(n)}}\right)^{-1}\left(\frac{B_k(f)}{B_k(h)}\right). \tag{5.235}$$

Using the same method as in the subsection 5.1.6, we can construct new families of exponentially convex functions and Cauchy type means. Also, using the idea described in the subsection 5.1.2 we can obtain results for $n$-convex functions at point.

## 5.6 Linear Inequalities and the Abel-Gontscharoff Interpolation Polynomial

The Abel-Gontscharoff interpolation problem in the real case was introduced in 1935 by Whittaker [100] and subsequently by Gontscharoff [20] and Davis [12].

Let us recall results from [1] for representation of a function $f$ via the Abel-Gontscharoff interpolating polynomial for two points with integral remainder.

**Theorem 5.92** *Let $n,k \in \mathbb{N}$, $n \geq 2$, $0 \leq k \leq n-1$ and $f \in C^n([a,b])$. Then*

$$f(t) = Q_{n-1}(f,t) + R(f,t), \tag{5.236}$$

*where $Q_{n-1}$ is the Abel-Gontscharoff interpolating polynomial for two-points of degree $n-1$, i.e.*

$$\begin{aligned} Q_{n-1}(f,t) &= \sum_{i=0}^{k} \frac{(t-a)^i}{i!} f^{(i)}(a) \\ &+ \sum_{j=0}^{n-k-2}\left(\sum_{i=0}^{j} \frac{(t-a)^{k+1+i}(a-b)^{j-i}}{(k+1+i)!\,(j-i)!}\right) f^{(k+1+j)}(b) \end{aligned}$$

*and the remainder is given by*

$$R(f,t) = \int_a^b G_n(t,s) f^{(n)}(s)ds,$$

*where $G_n(t,s)$ is the Green function given by*

$$G_n(t,s) = \frac{1}{(n-1)!}\begin{cases} \displaystyle\sum_{i=0}^{k} \binom{n-1}{i}(t-a)^i(a-s)^{n-i-1}, & a \leq s \leq t \\ \displaystyle -\sum_{i=k+1}^{n-1} \binom{n-1}{i}(t-a)^i(a-s)^{n-i-1}, & t \leq s \leq b. \end{cases} \tag{5.237}$$

Further, for $a \leq s,t \leq b$ the following inequalities hold

$$(-1)^{n-k-1}\frac{\partial^i G_n(t,s)}{\partial t^i} \geq 0, \quad 0 \leq i \leq k, \tag{5.238}$$

$$(-1)^{n-i}\frac{\partial^i G_n(t,s)}{\partial t^i} \geq 0, \quad k+1 \leq i \leq n-1. \tag{5.239}$$

### 5.6.1 Inequalities Obtained via the Abel-Gontscharoff Interpolating Polynomials

We start this section with identities for the sum $\sum p_r f(x_r)$ and the integral $\int p(t)f(x(t))dt$ using the Abel-Gontscharoff interpolating polynomial for two points. These results are given in paper [33].

**Theorem 5.93** *Let $n,k \in \mathbb{N}$, $n \geq 2$, $0 \leq k \leq n-1$, and let $\boldsymbol{x} = (x_1,\ldots,x_m) \in [a,b]^m$, $\mathbf{p} = (p_1,\ldots,p_m) \in \mathbb{R}^m$ be m-tuples. Let $f \in C^n([a,b])$ and $G_n$ be the Green function defined as in* (5.237). *Then*

$$\sum_{r=1}^{m} p_r f(x_r) = \theta_1(f) + \int_a^b \left(\sum_{r=1}^{m} p_r G_n(x_r,s)\right) f^{(n)}(s)ds, \tag{5.240}$$

*where*

$$\begin{aligned}\theta_1(f) &= \sum_{i=0}^{k}\frac{f^{(i)}(a)}{i!}\sum_{r=1}^{m} p_r (x_r-a)^i \\ &+ \sum_{j=0}^{n-k-2}\sum_{i=0}^{j}\left(\sum_{r=1}^{m} p_r (x_r-a)^{k+1+i}\right)\frac{(-1)^{j-i}(b-a)^{j-i}}{(k+1+i)!(j-i)!} f^{(k+1+j)}(b).\end{aligned} \tag{5.241}$$

*Proof.* Putting $t = x_r$ in (5.236), multiplying it with $p_r$, $r = 1,2,\ldots,m$, and adding all the identities we get (5.240). □

Similarly, we get an integral version of the above theorem.

**Theorem 5.94** *Let $n,k \in \mathbb{N}$, $n \geq 2$, $0 \leq k \leq n-1$, and $x : [\alpha,\beta] \to [a,b]$, $p : [\alpha,\beta] \to \mathbb{R}$ be continuous functions. Let $f \in C^n([a,b])$ and $G_n$ be the Green function defined as in* (5.237). *Then*

$$\int_\alpha^\beta p(t) f(x(t))\, dt = \theta_2(f) + \int_a^b \left(\int_\alpha^\beta p(t) G_n(x(t),s)\, dt\right) f^{(n)}(s)ds, \tag{5.242}$$

*where*

$$\begin{aligned}\theta_2(f) &= \sum_{i=0}^{k}\frac{f^{(i)}(a)}{i!}\int_\alpha^\beta p(t)(x(t)-a)^i\, dt \\ &+ \sum_{j=0}^{n-k-2}\sum_{i=0}^{j}\left(\int_\alpha^\beta p(t)(x(t)-a)^{k+1+i}\, dt\right)\frac{(-1)^{j-i}(b-a)^{j-i}}{(k+1+i)!(j-i)!} f^{(k+1+j)}(b).\end{aligned} \tag{5.243}$$

If $\mathbf{x}$ and $\mathbf{p}$ satisfy additional conditions, then we get a generalization of a Popoviciu type inequality for $n$-convex functions, i.e. we give a lower bound for the sum $\sum p_r f(x_r)$ which depends only on the nodes $x_1,\ldots,x_m$, the weights $p_1,\ldots,p_m$ and values of higher derivatives of a function $f$ at points $a$ and $b$.

**Theorem 5.95** *Let $n,k \in \mathbb{N}$, $n \geq 2$, $0 \leq k \leq n-1$, $\mathbf{x} = (x_1,\ldots,x_m)$ and $\mathbf{p} = (p_1,\ldots,p_m)$ be $m$-tuples such that $x_r \in [a,b]$ and $p_r \in \mathbb{R}$ $(r \in \{1,\ldots,m\})$ and let $G_n$ be the Green function defined as in* (5.237).

*If for all $s \in [a,b]$*

$$\sum_{r=1}^{m} p_r G_n(x_r,s) \geq 0, \tag{5.244}$$

*then for every $n$-convex function $f : [a,b] \to \mathbb{R}$,*

$$\sum_{r=1}^{m} p_r f(x_r) \geq \theta_1(f), \tag{5.245}$$

*where $\theta_1(f)$ is given in* (5.241).

*If the reverse inequality in* (5.244) *holds, then also the reverse inequality in* (5.245) *holds.*

*Proof.* Since the function $f$ is $n$-convex, therefore without loss of generality we can assume that $f$ is $n$-times differentiable and $f^{(n)}(x) \geq 0$, for all $x \in [a,b]$. Hence we apply Theorem 5.93 to get (5.245). □

Integral version of the above theorem is stated as:

**Theorem 5.96** *Let $n,k \in \mathbb{N}$, $n \geq 2$, $0 \leq k \leq n-1$, and $x : [\alpha,\beta] \to [a,b]$, $p : [\alpha,\beta] \to \mathbb{R}$ be continuous functions and let $G_n$ be the Green function defined as in* (5.237).

*If for all $s \in [a,b]$*

$$\int_{\alpha}^{\beta} p(t)\, G_n(x(t),s)\, dt \geq 0, \tag{5.246}$$

*then for every $n$-convex function $f : [a,b] \to \mathbb{R}$*

$$\int_{\alpha}^{\beta} p(t)\, f(x(t))\, dt \geq \theta_2(f), \tag{5.247}$$

*where $\theta_2(f)$ is defined in* (5.243).

*If the reverse inequality in* (5.246) *holds, then also the reverse inequality in* (5.247) *holds.*

In some cases the assumption $\sum_{r=1}^{m} p_r G_n(x_r,s) \geq 0$, $\ s \in [a,b]$ can be replaced with more simpler condition in which we recognize assumptions from Popoviciu's theorem about positivity of sum $\sum p_r f(x_r)$ for a convex function $f$. Namely we have the following statement.

**Theorem 5.97** *Let $n,k \in \mathbb{N}$, $n \geq 2$, $1 \leq k \leq n-1$, $\mathbf{x} \in [a,b]^m$ $\mathbf{p} \in \mathbb{R}^m$ be m-tuples such that*

$$\sum_{r=1}^{m} p_r = 0, \qquad \sum_{r=1}^{m} p_r|x_r - x_s| \geq 0, \text{ for } s = 1,2,\ldots,m$$

*and let $G_n$ be the Green function defined as in (5.237).*

*(i) If k is odd and n is even or k is even and n is odd, then for every n-convex function $f:[a,b] \to \mathbb{R}$, it holds*

$$\sum_{r=1}^{m} p_r f(x_r) \geq \theta_1(f), \tag{5.248}$$

*where $\theta_1(f)$ is given in (5.241).*

*Moreover, if $f^{(i)}(a) \geq 0$ for $i = 2,\ldots,k$ and $(-1)^{j-i} f^{(k+1+j)}(b) \geq 0$ for $i \in \{0,\ldots,j\}$ and $j \in \{0,\ldots,n-k-2\}$, then $\sum_{r=1}^{m} p_r f(x_r) \geq 0$.*

*(ii) If k and n are both even or odd, then for every n-convex function $f:[a,b] \to \mathbb{R}$, the reverse inequality in (5.248) holds.*

*Moreover, if $f^{(i)}(a) \leq 0$ for $i = 0,\ldots,k$ and $(-1)^{j-i} f^{(k+1+j)}(b) \leq 0$ for $i \in \{0,\ldots,j\}$ and $j \in \{0,\ldots,n-k-2\}$, then $\sum_{r=1}^{m} p_r f(x_r) \leq 0$.*

*Proof.* $(i)$ Let us consider properties (5.238) and (5.239) for $i=2$. If $k$ is odd and $n$ is even, then for $k=1$ we get $(-1)^{n-2}\dfrac{\partial^2 G_n(t,s)}{\partial t^2} \geq 0$ from (5.239), i.e. $\dfrac{\partial^2 G_n(t,s)}{\partial t^2} \geq 0$, i.e. $G_n$ is convex. For $k>1$, from (5.238) we get the same inequality. If $k$ is even and $n$ is odd, then $k \geq 2$ and from (5.238) we get that $G_n$ is convex in the first variable. By Remark 5.1, applied on the function $G_n$ we get

$$\sum_{r=1}^{m} p_r G_n(x_r,s) \geq 0,$$

i.e. the assumptions of Theorem 5.95 are fullfilled and inequality (5.248) holds. If further assumptions on $f^{(i)}(a)$ and $f^{(k+1+j)}(b)$ are valid, then the right-hand side of (5.248) is nonnegative.

The case $(ii)$ is proved in a similar manner. □

An integral analogue of the previous theorem is the following theorem.

**Theorem 5.98** *Let $n,k \in \mathbb{N}$, $n \geq 2$, $1 \leq k \leq n-1$, $x:[\alpha,\beta] \to [a,b]$ and $p:[\alpha,\beta] \to \mathbb{R}$ be continuous functions satisfying*

$$\int_{\alpha}^{\beta} p(t) = 0, \qquad \int_{\alpha}^{\beta} p(t)x(t) = 0,$$

*and*

$$\int_{\alpha}^{\beta} p(t)(x(t)-s)_+ \geq 0 \quad \text{for } s \in [a,b],$$

*and let $G_n$ be the Green function defined as in* (5.237).

$(i)$ *If $k$ is odd and $n$ is even or $k$ is even and $n$ is odd, then for every $n$-convex function $f:[a,b]\to\mathbb{R}$, then*

$$\int_{\alpha}^{\beta} p(t)\, f(x(t))\, dt \geq \theta_2(f). \tag{5.249}$$

*Moreover, if $f^{(i)}(a)\geq 0$ for $i=0,\ldots,k$ and $(-1)^{j-i}f^{(k+1+j)}(b)\geq 0$ for $i\in\{0,\ldots,j\}$ and $j\in\{0,\ldots,n-k-2\}$, then $\int_{\alpha}^{\beta} p(t) f(x(t))\, dt\geq 0$.*

$(ii)$ *If $k$ and $n$ are both even or odd, then for every $n$-convex function $f:[a,b]\to\mathbb{R}$, then the reverse inequality holds in* (5.249).

*Moreover, if $f^{(i)}(a)\leq 0$ for $i=0,\ldots,k$ and $(-1)^{j-i}f^{(k+1+j)}(b)\leq 0$ for $i\in\{0,\ldots,j\}$ and $j\in\{0,\ldots,n-k-2\}$, then $\int_{\alpha}^{\beta} p(t) f(x(t))\, dt\leq 0$.*

### 5.6.2 Results Obtained by the Green Function

In this subsection we obtain results using the Green function $G$, (5.10), together with the Abel-Gontscharoff polynomials.

We begin the subsection with some identities related to generalizations of a Popoviciu type inequality.

**Theorem 5.99** *Let $n,k\in\mathbb{N}$, $n\geq 4$, $0\leq k\leq n-1$, $f\in C^n[\alpha,\beta]$ and $\mathbf{x}\in[a,b]^m$, $\mathbf{p}\in\mathbb{R}^m$. Also let $G$ and $G_n$ be defined by* (5.10) *and* (5.237) *respectively. Then*

$$\sum_{l=1}^{m} p_l\, f(x_l) = \theta_3(f) + + \int_a^b \int_a^b \left(\sum_{l=1}^{m} p_l G(x_l,s)\right) G_{n-2}(s,t) f^{(n)}(t)\, dt\, ds,$$

*where $\theta_3(f)$ is defined as*

$$\begin{aligned}
\theta_3(f) &= \frac{f(b)-f(a)}{b-a}\sum_{l=1}^{m} p_l x_l + \frac{bf(a)-af(b)}{b-a}\sum_{l=1}^{m} p_l \\
&\quad + \sum_{i=0}^{k} \frac{f^{(i+2)}(a)}{i!} \int_a^b \sum_{l=1}^{m} p_l G(x_l,s)\,(s-a)^i\, ds \qquad (5.250)\\
&\quad + \sum_{j=0}^{n-k-4}\sum_{i=0}^{j} \frac{(-1)^{j-i}(b-a)^{j-i} f^{(k+3+j)}(b)}{(k+1+i)!\,(j-i)!} \int_a^b \sum_{l=1}^{m} p_l G(x_l,s)\,(s-a)^{k+1+i}\, ds.
\end{aligned}$$

*Proof.* The proof is similar to the proof of Theorem 5.84 using representation which follows from Theorem 5.92:

$$\begin{aligned}
f''(s) &= \sum_{i=0}^{k} \frac{(s-a)^i}{i!} f^{(i+2)}(a) + \sum_{j=0}^{n-k-4}\sum_{i=0}^{j} \frac{(s-a)^{k+1+i}(a-b)^{j-i}}{(k+1+i)!\,(j-i)!} f^{(k+3+j)}(b) \\
&\quad + \int_a^b G_{n-2}(s,\tau) f^{(n)}(\tau)\, d\tau.
\end{aligned}$$

□

**Theorem 5.100** *Let $n,k \in \mathbb{N}$, $n \geq 4$, $0 \leq k \leq n-1$, $f \in C^n[a,b]$, and let $x : [\alpha,\beta] \to [a,b]$, $p : [\alpha,\beta] \to \mathbb{R}$ be continuous functions and $G$, $G_n$ be defined by* (5.10) *and* (5.237) *respectively. Then*

$$\int_\alpha^\beta p(\tau)f(x(\tau))d\tau = \theta_4(f) + \int_a^b \int_a^b \int_\alpha^\beta p(\tau)G(x(\tau),s)G_{n-2}(s,t)f^{(n)}(t)d\tau\,dt\,ds,$$

*where*

$$\begin{aligned}
\theta_4(f) &= \frac{f(b)-f(a)}{b-a}\int_\alpha^\beta p(\tau)x(\tau)d\tau + \frac{bf(a)-af(b)}{b-a}\int_\alpha^\beta p(\tau)d\tau \\
&+ \sum_{i=0}^{k} \frac{f^{(i+2)}(a)}{i!}\int_a^b \int_\alpha^\beta p(\tau)G(x(\tau),s)(s-a)^i d\tau ds \qquad (5.251)\\
&+ \sum_{j=0}^{n-k-4}\sum_{i=0}^{j} \frac{(-1)^{j-i}(b-a)^{j-i} f^{(k+3+j)}(b)}{(k+1+i)!\,(j-i)!} \times \\
&\qquad \times \int_a^b \int_\alpha^\beta p(\tau)G(x(\tau),s)\,(s-a)^{k+1+i}d\tau\,ds.
\end{aligned}$$

**Theorem 5.101** *Let $n,k \in \mathbb{N}$, $n \geq 4$, $0 \leq k \leq n-1$, $\boldsymbol{p} \in \mathbb{R}^m$, $\boldsymbol{x} \in [a,b]^m$. Also let $G$ and $G_n$ be defined by* (5.10) *and* (5.237) *respectively.*

*If $f : [a,b] \to \mathbb{R}$ is $n$-convex, and*

$$\int_a^b \left( \sum_{l=1}^{m} p_l G(x_l,s) \right) G_{n-2}(s,t)ds \geq 0, \qquad t \in [a,b], \tag{5.252}$$

*then*

$$\sum_{l=1}^{m} p_l f(x_l) \geq \theta_3(f). \tag{5.253}$$

*If the reverse inequality in* (5.252) *holds, then also the reverse inequality in* (5.253) *holds.*

*Proof.* It follows from $n$-convexity of a function $f$ and from Theorem 5.99. □

As from (5.238) we have $(-1)^{n-k-3}G_{n-2}(s,t) \geq 0$, therefore for the case when $n$ is even and $k$ is odd or $n$ is odd and $k$ is even, it is enough to assume that $\sum_{l=1}^{m} p_l G(x_l,s) \geq 0, s \in [\alpha,\beta]$, instead of the assumption (5.252) in Theorem 5.101. Similarly we can discuss for the reverse inequality in (5.253).

Integral version of the above theorem can be stated as:

**Theorem 5.102** *Let $n,k \in \mathbb{N}$, $n \geq 4$, $0 \leq k \leq n-1$, $x : [\alpha,\beta] \to [a,b]$, $p : [\alpha,\beta] \to \mathbb{R}$ be continuous functions and $G$, $G_n$ be defined by* (5.10) *and* (5.237) *respectively. If $f : [a,b] \to \mathbb{R}$ is $n$-convex, and*

$$\int_a^b \int_\alpha^\beta p(\tau)G(x(\tau),s)G_{n-2}(s,t)d\tau\,ds \geq 0, \tag{5.254}$$

*then*

$$\int_{\alpha}^{\beta} p(\tau) f(x(\tau)) d\tau \geq \theta_4(f). \tag{5.255}$$

*If the reverse inequality in* (5.254) *holds, then also the reverse inequality in* (5.255) *holds.*

As from (5.238) we have $(-1)^{n-k-3} G_{n-2}(s,t) \geq 0$, therefore for the case when $n$ is even and $k$ is odd or $n$ is odd and $k$ is even, it is enough to assume that $\int_a^b p(\tau) G(x(\tau), s) d\tau \geq 0, s \in [\alpha, \beta]$, instead of the assumption (5.254) in Theorem 5.100. Similarly we can discuss for the reverse inequality in (5.255).

If we deal with the assumptions from Remark 5.1, which are equivalent to the Popoviciu's conditions for positivity of the sum involving convex function $f$, then for some combinations of $n$ and $k$ we get a result for a $n$-convex function $f$. More precisely, we get the following theorem.

**Theorem 5.103** *Let* $n, k \in \mathbb{N}$, $n \geq 4$, $0 \leq k \leq n-1$. *Let* $G$ *be defined by* (5.10) *and let* $f : [a,b] \to \mathbb{R}$ *be* $n$*-convex. Let* $\mathbf{x} \in [a,b]^m$ *and* $\mathbf{p} \in \mathbb{R}$ *satisfy*

$$\sum_{r=1}^{m} p_r = 0, \qquad \sum_{r=1}^{m} p_r |x_r - x_s| \geq 0, \text{ for } s = 1, 2, \ldots, m.$$

$(i)$ *If* $n$ *is even and* $k$ *is odd or* $n$ *is odd and* $k$ *is even, then*

$$\begin{aligned}\sum_{l=1}^{m} p_l f(x_l) \geq & \sum_{i=0}^{k} \frac{f^{(i+2)}(a)}{i!} \int_a^b \sum_{l=1}^{m} p_l G(x_l, s)(s-a)^i ds \\ & + \sum_{j=0}^{n-k-4} \sum_{i=0}^{j} \frac{(-1)^{j-i}(b-a)^{j-i} f^{(k+3+j)}(b)}{(k+1+i)!(j-i)!} \\ & \times \int_a^b \sum_{l=1}^{m} p_l G(x_l, s)(s-a)^{k+1+i} ds.\end{aligned} \tag{5.256}$$

*Moreover if* $f^{(i+2)}(a) \geq 0$ *for* $i = 0, \ldots, k$ *and* $(-1)^{j-i} f^{(k+3+j)}(b) \geq 0$ *for* $i \in \{0, \ldots, j\}$ *and* $j \in \{0, \ldots, n-k-4\}$, *then* $\sum_{l=1}^{m} p_l f(x_l) \geq 0$.

$(ii)$ *If* $n$ *and* $k$ *both are even or both are odd, then the reverse inequality holds in* (5.256).

*Moreover if* $f^{(i+2)}(a) \leq 0$ *for* $i = 0, \ldots, k$ *and* $(-1)^{j-i} f^{(k+3+j)}(b) \leq 0$ *for* $i \in \{0, \ldots, j\}$ *and* $j \in \{0, \ldots, n-k-4\}$, *then* $\sum_{l=1}^{m} p_l f(x_l) \leq 0$.

*Proof.* $(i)$ By using (5.238) we have $(-1)^{n-k-3} G_{n-2}(s,t) \geq 0$, $a \leq s, t \leq b$, therefore if $n$ is even and $k$ is odd or $n$ is odd and $k$ is even then $G_{n-2}(s,t) \geq 0$. Since $G$ is convex and $G_{n-2}$ is nonnegative, inequality (5.252) holds. Hence by Theorem 5.101 inequality (5.256) holds. By using the other conditions the nonnegativity of the right-hand side of (5.256) is obvious.

Similarly we prove $(ii)$. □

The integral version of Theorem 5.103 can be stated as:

**Theorem 5.104** *Let $n,k \in \mathbb{N}$, $n \geq 4$, $0 \leq k \leq n-1$, $x : [\alpha,\beta] \to [a,b]$ and $p : [\alpha,\beta] \to \mathbb{R}$ be any continuous functions. Also let $G$ be defined by (5.10). Let $f : [a,b] \to \mathbb{R}$ a $n$-convex and*

$$\int_\alpha^\beta p(\tau)\,d\tau \geq 0, \qquad \int_\alpha^\beta p(\tau)(x(\tau)-t)_+\,d\tau \geq 0 \text{ for } t \in [a,b], \tag{5.257}$$

*$(i)$ If $n$ is even and $k$ is odd or $n$ is odd and $k$ is even, then*

$$\begin{aligned}\int_\alpha^\beta p(\tau)f(x(\tau))d\tau \geq{}& \sum_{i=0}^{k} \frac{f^{(i+2)}(a)}{i!} \int_a^b \int_\alpha^\beta p(\tau)G(x(\tau),s)(s-a)^i d\tau\,ds \\ &+ \sum_{j=0}^{n-k-4} \sum_{i=0}^{j} \frac{(-1)^{j-i}(b-a)^{j-i} f^{(k+3+j)}(b)}{(k+1+i)!\,(j-i)!} \\ &\times \int_a^b \int_\alpha^\beta p(\tau)G(x(\tau),s)\,(s-a)^{k+1+i} d\tau\,ds. \end{aligned} \tag{5.258}$$

*Moreover if $f^{(i+2)}(a) \geq 0$ for $i = 0,\ldots,k$ and $(-1)^{j-i}f^{(k+3+j)}(b) \geq 0$ for $i \in \{0,\ldots,j\}$ and $j \in \{0,\ldots,n-k-4\}$, then the right-hand side of (5.258) is nonnegative, that is integral version of (5.1) holds.*

*$(ii)$ If $n$ and $k$ both are even or both are odd, then reverse inequality holds in (5.258).*

*Moreover if $f^{(i+2)}(a) \leq 0$ for $i = 0,\ldots,k$ and $(-1)^{j-i}f^{(k+3+j)}(b) \leq 0$ for $i \in \{0,\ldots,j\}$ and $j \in \{0,\ldots,n-k-4\}$, then the right-hand side of the reverse inequality in (5.258) is nonpositive, that is the reverse inequality in the integral version of (5.1) holds.*

Using the same method as in the first section of this Chapter, we can state mean value theorems and results connected with exponentially convexity.

Chapter 6

# Čebyšev-Popoviciu Type Inequalities

## 6.1 Generalized Čebyšev and Ky Fan Identities and Inequalities

One of the classical, well-known inequalities is the Čebyšev inequality. For more about this inequality, its history, variants and generalizations we refer to the books [51], [77] and paper [44]. Here we recall only few facts about it and then we proceed to our main aim: to give a Popoviciu type inequality for the difference which arises from the Čebyšev inequality.

The discrete Čebyšev inequality is given as follows ([77, p. 197]).

**Theorem 6.1** *Let* **a** *and* **b** *be two real $N$-tuples monotonic in the same sense and* **p** *be a nonnegative $N$-tuple. Then the inequality*

$$\sum_{i=1}^{N} p_i \sum_{i=1}^{N} p_i a_i b_i - \sum_{i=1}^{N} p_i a_i \sum_{j=1}^{N} p_i b_i \geq 0 \tag{6.1}$$

*holds. If* **a** *and* **b** *are monotonic in the opposite sense, then the reverse of the inequality in* $(6.1)$ *holds. In both cases equality in* $(6.1)$ *holds if and only if either* $a_1 = a_2 = \ldots = a_N$ *or* $b_1 = b_2 = \ldots = b_N$.

The integral Čebyšev inequality is given as follows ([77, p. 197]).

**Theorem 6.2** *Let $f,g : [a,b] \to \mathbb{R}$ and $p : [a,b] \to [0,\infty\rangle$ be integrable functions. If $f$ and $g$ are monotonic in the same direction, then*

$$\int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx \geq \int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx, \tag{6.2}$$

*provided that the integrals exist. If $f$ and $g$ are monotonic in opposite direction, then the reverse of inequality* (6.2) *is valid. In both cases, equality holds if and only if eighter $f$ or $g$ is constant almost everywhere.*

A.M. Ostrowski [54] gave the following result related to Čebyšev's inequality:

**Theorem 6.3** *Let $f,g \in C^{(1)}(I)$ be two monotonic functions and let $p : I \to \mathbb{R}_+$ be an integrable function. Then there exist $\xi$, $\eta \in I$ such that*

$$T(f,g,p) = f'(\xi)g'(\eta)T(x-a,x-a,p), \tag{6.3}$$

*where*

$$T(f,g,p) = \int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx - \int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx. \tag{6.4}$$

A term $T(f,g,p)$ defined by (6.4) is called the Čebyshev difference or the Čebyšev functional. For other generalizations of Theorem 6.3, [64] can be seen. In [67], J. Pečarić gave the following generalization of Theorem 6.3 by using the functional

$$C(f,p) = \int_a^b \int_a^b p(x,y)f(y,y)dy\,dx - \int_a^b \int_a^b p(x,y)f(x,y)dy\,dx, \tag{6.5}$$

where $p$ and $f$ are integrable functions. It is clear that for the particular case when $f(x,y) = f(x)g(y)$ and $p(x,y) = p(x)p(y)$, then $C(f,p)$ collapses to $T(f,g,p)$. In fact, Theorems 6.4 and 6.7 from [67] are devoted to the functions with continuous second partial derivatives, while in Theorem 6.5 from the same article we find a result for sequences involving the second ordered differences.

**Theorem 6.4** *Let $p : I^2 \to \mathbb{R}$ be an integrable function such that*

$$X(x,x) = \overline{X}(x,x) \quad \forall x \in I$$

*and let either*

$$X(x,y) \geq 0, \quad a \leq y \leq x \leq b, \quad \overline{X}(x,y) \geq 0, \quad a \leq x \leq y \leq b$$

*or its reverse inequalities be valid, where*

$$X(x,y) = \int_x^b \int_a^y p(s,t)\,dt\,ds \quad \text{and} \quad \overline{X}(x,y) = \int_a^x \int_y^b p(s,t)\,dt\,ds.$$

*If $f \in C^2(I^2)$, then there exists $(\xi,\eta) \in I^2$ such that*

$$C(f,p) = f_{(1,1)}(\xi,\eta)C\left((x-a)(y-a),p\right), \quad \text{where} \quad f_{(1,1)}(x,y) = \frac{\partial^2 f}{\partial x \partial y}.$$

*Proof.* Using notation

$$\alpha(x) = -\int_a^x \left(\int_a^y p(x,t)dt\right)(f_1(y,y) + f_2(y,y) - f_2(x,y))dy,$$

$$\beta(x) = \int_x^b \left(\int_y^b p(x,t)dt\right)(f_1(y,y) + f_2(y,y) - f_2(x,y))dy$$

we get

$$\int_a^b p(x,y)(f(y,y) - f(x,y))dy = \alpha(x) + \beta(x), \quad \text{where} \quad f_1 = \frac{\partial f}{\partial x} \quad \text{and} \quad f_2 = \frac{\partial f}{\partial y}.$$

So, we obtain

$$\begin{aligned}
C(f,p) &= \int_a^b \alpha(x)dx + \int_a^b \beta(x)dx \\
&= -\int_a^b \left(\int_a^x \left(\int_a^y p(x,t)dt\right)(f_1(y,y) + f_2(y,y) - f_2(x,y))dy\right)dx \\
&+ \int_a^b \left(\int_x^b \left(\int_y^b p(x,t)dt\right)(f_1(y,y) + (f_2(y,y) - f_2(x,y))dy\right)dx \\
&= -\int_a^b \left(\int_y^b \left(\int_a^y p(x,t)dt\right)(f_1(y,y) + f_2(y,y) - f_2(x,y))dx\right)dy \\
&+ \int_a^b \left(\int_a^y \left(\int_y^b p(x,t)dt\right)(f_1(y,y) + (f_2(y,y) - f_2(x,y))dx\right)dy \\
&= -\int_a^b \left(-X(y,y)f_1(y,y) - \int_y^b X(x,y)f_{(1,1)}(x,y)dx\right)dy \\
&+ \int_a^b \left(\overline{X}(y,y)f_1(y,y) + \int_a^y \overline{X}(x,y)f_{(1,1)}(x,y)dx\right)dy.
\end{aligned}$$

So,

$$C(f,p) = \int_a^b \int_y^b X(x,y)f_{(1,1)}(x,y)dxdy + \int_a^b \int_a^y \overline{X}(x,y)f_{(1,1)}(x,y)dxdy \tag{6.6}$$

If we put

$$\tilde{X}(x,y) = \overline{X}(x,y), \quad (x \le y) \quad \text{and} \quad \tilde{X}(x,y) = X(x,y) \quad (x \ge y)$$

we have

$$C(f,p) = \int_a^b \int_a^b \tilde{X}(x,y)f_{(1,1)}(x,y)dxdy,$$

and using the mean value theorem for double integrals there exist $\xi,\eta \in I$ such that

$$C(f,p) = f_{(1,1)}(\xi,\eta)\int_a^b \tilde{X}(x,y)dxdy.$$

If instead of $f$ we put the function $(x,y) \mapsto (x-a)(y-a)$, then

$$C((x-a)(y-a),p) = \int_a^b \tilde{X}(x,y)dxdy$$

and combining the last two equalities we get the statement of the theorem. □

Let us consider a discrete analogue of the difference $C(f,p)$. If $a_{ij}, p_{ij}$, $(i,j = 2,\ldots,N)$, are reals, then we define $C_\Delta(a,p)$ as following

$$C_\Delta(a,p) = \sum_{i=1}^{N}\sum_{j=1}^{N} p_{ij}a_{jj} - \sum_{i=1}^{N}\sum_{j=1}^{N} p_{ij}a_{ij},$$

The following theorem gives us necessary and sufficient conditions under which $C_\Delta(a,p)$ is nonnegative. In short, it is Popoviciu type inequality for $C_\Delta(a,p)$.

**Theorem 6.5** *The inequality*

$$C_\Delta(a,p) \geq 0 \tag{6.7}$$

*holds for each real numbers $a_{ij}$ for $i,j \in \{1,\ldots,N\}$ such that $\Delta^{(1,1)}\, a_{ij} \geq 0$ for $i,j \in \{1,\ldots,N-1\}$ if and only if*

$$X_{j+1,j} = \overline{X}_{j,j+1}, \quad j \in \{1,\ldots,N-1\}$$

*and*

$$X_{ij} \geq 0, \quad i \in \{j+1,\ldots,n\} \quad \textit{for} \quad j \in \{1,\ldots,N-1\} \tag{6.8}$$

$$\overline{X}_{ij} \geq 0, \quad i \in \{1,\ldots,j-1\} \quad \textit{for} \quad j \in \{2,\ldots,N\} \tag{6.9}$$

*hold. If $\Delta^{(1,1)}a_{ij} \leq 0$ for $i,j \in \{1,\ldots,N-1\}$, then the reverse inequality in* (6.7) *is valid, where*

$$X_{ij} = \sum_{r=i}^{N}\sum_{s=1}^{j} p_{rs} \quad \textit{and} \quad \overline{X}_{ij} = \sum_{r=1}^{i}\sum_{s=j}^{N} p_{rs}.$$

*Proof.* The following identity holds

$$\begin{aligned} C_\Delta(a,p) \;&=\; \sum_{j=1}^{N-1}\left(-X_{j+1,j}\Delta_1 a_{jj} + \sum_{i=j+2}^{N} X_{ij}\Delta^{(1,1)}a_{i-1,j}\right) \\ &+ \sum_{j=1}^{N-1}\left(-\overline{X}_{j,j+1}\Delta_1 a_{j,j+1} + \sum_{i=1}^{j-1}\overline{X}_{i,j+1}\Delta^{(1,1)}a_{ij}\right). \end{aligned}$$

Since $X_{j+1,j} = \overline{X}_{j,j+1}$ for all $j$ we get

$$C_\Delta(a,p) \;=\; \sum_{j=1}^{N-1}\sum_{i=j+2}^{N} X_{ij}\Delta^{(1,1)}a_{i-1,j} + \sum_{j=1}^{N-1}\sum_{i=1}^{j-1}\overline{X}_{i,j+1}\Delta^{(1,1)}a_{ij}.$$

Using other assumptions of Theorem we get inequality $C_\Delta(a,p) \geq 0$.

Take $a_{rs} = -1$ if $i \le r \le N$ and $1 \le s \le j$, and $a_{rs} = 0$ if either $1 \le r \le i-1$ or $j+1 \le s \le N$. Then one can verify that $\Delta^{(1,1)} a_{rs} \ge 0$ for $1 \le r,s \le N-1$, so inequality $C_\Delta(a,p) \ge 0$ holds for this sequence $(a_{rs})$. But this inequality reduces to $0 \le X_{ij}$ proving inequality (6.8). Considering the sequence $(a_{rs})$ defined by $a_{rs} = -1$ if $1 \le r \le i$ and $j \le s \le N$, and $a_{rs} = 0$ otherwise, inequality $C_\Delta(a,p) \ge 0$ reduces to (6.9). To show remaining conditions we choose two sequences $(a_{rs})$, $(b_{rs})$ such that for $1 \le j \le N-1$,

$$C_\Delta(a,p) = X_{j+1,j} - \overline{X}_{j,j+1},$$

$$C_\Delta(b,p) = \overline{X}_{j,j+1} - X_{j+1,j}.$$

For the first of these, $(a_{rs})$ must satisfies

$$\sum_{r=1}^{N}\sum_{s=1}^{N} p_{rs}(a_{ss} - a_{rs}) = \sum_{r=j+1}^{N}\sum_{s=1}^{j} p_{rs} - \sum_{r=1}^{j}\sum_{s=j+1}^{N} p_{rs},$$

and this requires that

$$a_{ss} - a_{rs} = \begin{cases} 1 & \text{if } j+1 \le r \le N, 1 \le s \le j \\ -1 & \text{if } 1 \le r \le j, j+1 \le s \le N \\ 0 & \text{otherwise} . \end{cases} \tag{6.10}$$

For arbitrary choice of $a_{ss}$ $(1 \le s \le N)$, if one defines $a_{rs}$ by (6.10), then $\Delta^{(1,1)} a_{rs} \ge 0$ for $1 \le r,s \le N-1$. Hence we obtain $X_{j+1,j} \ge \overline{X}_{j,j+1}$ for $1 \le j \le N-1$. A similar analysis using arbitrary $b_{ss}$ and

$$b_{rs} = \begin{cases} b_{ss} + 1 & \text{if } j+1 \le r \le N, 1 \le s \le j \\ b_{ss} - 1 & \text{if } 1 \le r \le j, j+1 \le s \le N \\ b_{ss} & \text{otherwise} , \end{cases}$$

also gives $\Delta^{(1,1)} b_{rs} \ge 0$ for $1 \le r,s \le N-1$ and $X_{j+1,j} \le \overline{X}_{j,j+1}$ for $1 \le j \le N-1$. □

In 1952, Fan [17] proposed as a problem the following result (see also [44]):

**Theorem 6.6** *Let $(x,y) \mapsto w(x,y)$ be a nonnegative Lebesgue integrable function over the square $\{(x,y) : a \le x \le b \quad \text{and} \quad a \le y \le b\}$. Suppose that $B$ is a positive constant such that $\int_a^b w(x,y)dy \le B$ for almost all $x \in [a,b]$ and also $\int_a^b w(x,y)dx \le B$ for almost all $y \in [a,b]$. If two finite-valued functions $f$ and $g$ are both nonnegative and nonincreasing on $[a,b]$, then the following inequality holds*

$$\int_a^b \int_a^b w(x,y) f(x) g(y) dx dy \le B \int_a^b f(x) g(x) dx. \tag{6.11}$$

For a generalization of Fan's result, Pečarić in [67] considered the following expression for integrable functions $f$, $p$ and $q$,

$$K(f,p,q) = \int_a^b q(x) f(x,x) dx - \int_a^b \int_a^b p(x,y) f(x,y) dx dy \tag{6.12}$$

and gave the following result.

**Theorem 6.7** *Let $p : I^2 \to \mathbb{R}$ and $q : I \to \mathbb{R}$ be two integrable functions such that $P(x,a) = Q(x)$, $P(a,y) = Q(y)$, $\quad P(x,y) \le Q(max\{x,y\})$, $\quad \forall x, y \in I$,*

$$\text{where} \quad Q(x) = \int_x^b q(t)dt \quad \text{and} \quad P(x,y) = \int_x^b \int_y^b p(s,t)dt\,ds.$$

*If $f : I^2 \to \mathbb{R}$ has the continuous partial derivatives $f_1$, $f_2$ and $f_{(1,1)}$, then there exists $(\xi,\eta) \in I^2$ such that*

$$K(f,p,q) = f_{(1,1)}(\xi,\eta)K\left((x-a)(y-a),p,q\right).$$

*Proof.* The following identities hold:

$$\int_a^b q(x)f(x,x)dx \;=\; f(a,a)Q(a) + \int_a^b Q(x)f_1(x,x)dx + \int_a^b Q(x)f_2(x)(x,x)dx,$$

$$\begin{aligned}\int_a^b \int_x^b Q(y)f_{(1,1)}(x,y)dy\,dx \;&=\; \int_a^b \int_a^y Q(y)f_{(1,1)}(x,y)dx\,dy \\ &=\; \int_a^b Q(y)f_2(y,y)dy - \int_a^b Q(y)f_2(a,y)dy,\end{aligned}$$

and

$$\int_a^b \int_a^x Q(x)f_{(1,1)}(x,y)dy\,dx \;=\; \int_a^b Q(x)f_1(x,x)dx - \int_a^b Q(x)f_1(x,a)dx,$$

where $f_1, f_2$ and $f_{(1,1)}$ are partial derivatives of $f$, i.e. $f_1 = \frac{\partial f}{\partial x}$, $f_2 = \frac{\partial f}{\partial y}$ and $f_{(1,1)} = \frac{\partial^2 f}{\partial x \partial y}$. Using the above-mentioned identities and the following

$$\begin{aligned}\int_a^b \int_a^b p(x,y)f(x,y)\;dx\;dy = f(a,a)P(a,a) + \int_a^b P(x,a)f_1(x,a)dx \\ + \int_a^b P(a,y)f_2(a,y)dx + \int_a^b \int_a^b P(x,y)f_{(1,1)}(x,y)dx\,dy,\end{aligned}$$

where $P(x,y) = \displaystyle\int_x^b \int_y^b p(s,t)dt\,ds$ we get

$$\begin{aligned}K(f,p,q) \;&=\; f(a,a)(Q(a) - P(a,a)) + \int_a^b (Q(x) - P(x,a))f_1(x,a)dx \\ &+ \int_a^b (Q(y) - P(a,y))f_2(a,y)dx \\ &+ \int_a^b \int_a^b (Q(\max(x,y)) - P(x,y))f_{(1,1)}(x,y)dx\,dy,\end{aligned}$$

i.e. in our case

$$K(f,p,q) \;=\; \int_a^b \int_a^b (Q(\max(x,y)) - P(x,y))f_{(1,1)}(x,y)dx\,dy$$

$$
\begin{aligned}
&= f_{(1,1)}(\xi,\eta)\int_a^b\int_a^b (Q(\max(x,y)) - P(x,y))dxdy \\
&= f_{(1,1)}(\xi,\eta)K((x-a)(y-a),p,q).
\end{aligned}
$$

□

Under the assumptions of Theorem 6.7, we introduce the following notations for simplification of statements of the upcoming theorems:

$$
P^{(i,j)}(x,y) = \int_x^b\int_y^b p(s,t)\frac{(s-x)^i}{i!}\frac{(t-y)^j}{j!}dtds, \tag{6.13}
$$

$$
\overline{P}^{(i,j)}(x,y) = \int_x^b\int_y^b p(s,t)\frac{(s-x)^i}{i!}\frac{(s-y)^j}{j!}dtds, \tag{6.14}
$$

$$
Q^{(i,j)}(x) = \int_x^b q(s)\frac{(s-x)^i}{i!}\frac{(s-a)^j}{j!}ds, \tag{6.15}
$$

$$
\begin{aligned}
R(x,y) &= \int_{max\{x,y\}}^b\int_a^b p(s,t)\frac{(s-x)^N}{N!}\frac{(s-y)^M}{M!}dt\,ds \\
&- \int_x^b\int_y^b p(s,t)\frac{(s-x)^N}{N!}\frac{(t-y)^M}{M!}dt\,ds,
\end{aligned} \tag{6.16}
$$

$$
\begin{aligned}
\overline{R}(x,y) &= \int_{max\{x,y\}}^b q(s)\frac{(s-x)^N}{N!}\frac{(s-y)^M}{M!}ds \\
&- \int_x^b\int_y^b p(s,t)\frac{(s-x)^N}{N!}\frac{(t-y)^M}{M!}dt\,ds,
\end{aligned} \tag{6.17}
$$

$$
f_0(x,y) = \frac{(x-a)^{N+1}(y-a)^{M+1}}{(N+1)!(M+1)!}. \tag{6.18}
$$

In the following text an absolutely continuity of a function $u$ means an absolutely continuity in the sense of Charathéodory described in [94].

If $u : D \to \mathbb{R}$ is absolutely continuous in the sense of Carathéodory, then for every $(x,y) \in D$ it admits the integral representation

$$
u(x,y) = u(a,c) + \int_a^x u_{(1,0)}(s,c)\,ds + \int_c^y u_{(0,1)}(a,t)\,dt + \int_a^x\int_c^y u_{(1,1)}(s,t)\,dtds, \tag{6.19}
$$

where the partial derivatives in (6.19) exist almost everywhere.

Let $f,p : I^2 \to \mathbb{R}$ and $q : I \to \mathbb{R}$ be three functions such that $p,q$ are integrable and $f_{(N,M)}$ exists and is absolutely continuous (in the sense of Carathéodory). The values $\overline{C}$ and $\overline{K}$ are defined as follows:

$$
\begin{aligned}
\overline{C}(f,p) &= C(f,p) - \sum_{i=0}^N\sum_{j=0}^M f_{(i,j)}(a,a)\left[\overline{P}^{(i,j)}(a,a) - P^{(i,j)}(a,a)\right] \\
&- \sum_{j=0}^M\int_a^b f_{(N+1,j)}(x,a)\left[\overline{P}^{(N,j)}(x,a) - P^{(N,j)}(x,a)\right]dx
\end{aligned}
$$

$$- \sum_{i=0}^{N} \int_a^b f_{(i,M+1)}(a,y) \left[ \overline{P}^{(i,M)}(a,y) - P^{(i,M)}(a,y) \right] dy, \tag{6.20}$$

where $C$ is defined in (6.5).

$$\begin{aligned} \overline{K}(f,p,q) &= K(f,p,q) - \sum_{j=0}^{M} \sum_{i=0}^{N} f_{(i,j)}(a,a) \left[ Q^{(i,j)}(a) - P^{(i,j)}(a,a) \right] \\ &- \sum_{j=0}^{M} \int_a^b f_{(N+1,j)}(x,a) \left[ Q^{(N,j)}(x) - P^{(N,j)}(x,a) \right] dx \\ &- \sum_{i=0}^{N} \int_a^b f_{(i,M+1)}(a,y) \left[ Q^{(M,i)}(y) - P^{(i,M)}(a,y) \right] dy, \end{aligned} \tag{6.21}$$

where $K$ is defined in (6.12).

### 6.1.1 Generalized Discrete Čebyšev's Identity and Inequality

In this section we consider the difference $C_\Delta(f,p)$ defined as follows:

$$C_\Delta(f,p) = \sum_{i=1}^{N} \sum_{j=1}^{N} p_{ij} f(x_i,y_i) - \sum_{i=1}^{N} \sum_{j=1}^{N} p_{ij} f(x_i,y_j).$$

Our aim is to obtain a Popoviciu type inequality connected with this difference. We will get an identity for $C_\Delta(f,p)$ which involves higher ordered differences $\Delta_{(t,k)}$ and then we consider necessary conditions for positivity of $C_\Delta(f,p)$ when $f$ is an $(n,m)$-convex function. The following results are given in [24].

**Theorem 6.8** *Let $(x_i,y_j) \in I^2$ for $i,j \in \{1,\dots,N\}$ be mutually distinct points and let $f : I^2 \to \mathbb{R}$ be a function and $p_{ij} \in \mathbb{R}$ for $i,j \in \{1,\dots,N\}$. Then,*

$$\begin{aligned} &C_\Delta(f,p) \\ &= \sum_{k=0}^{m-1} \sum_{t=0}^{n-1} \Delta_{(t,k)} f(x_1,y_1) \left[ \sum_{s=max\{t,k\}+1}^{N} \sum_{r=1}^{N} p_{sr}(x_s - x_1)^{(t)} (y_s - y_1)^{(k)} \right. \\ &\left. - \sum_{s=t+1}^{N} \sum_{r=k+1}^{N} p_{sr}(x_s - x_1)^{(t)} (y_r - y_1)^{(k)} \right] \\ &+ \sum_{k=0}^{m-1} \sum_{t=n+1}^{N} \Delta_{(n,k)} f(x_{t-n},y_1)(x_t - x_{t-n}) \times \\ &\times \left[ \sum_{s=max\{t,k+1\}}^{N} \sum_{r=1}^{N} p_{sr}(x_s - x_{t-n+1})^{(n-1)} (y_s - y_1)^{(k)} \right. \\ &\left. - \sum_{s=t}^{N} \sum_{r=k+1}^{N} p_{sr}(x_s - x_{t-n+1})^{(n-1)} (y_r - y_1)^{(k)} \right] \end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=m+1}^{N}\sum_{t=0}^{n-1}\Delta_{(t,m)}f(x_1,y_{k-m})(y_k-y_{k-m})\times\\
&\times\left[\sum_{s=max\{t+1,k\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_1)^{(t)}(y_s-y_{k-m+1})^{(m-1)}\right.\\
&\left.-\sum_{s=t+1}^{N}\sum_{r=k}^{N}p_{sr}(x_s-x_1)^{(t)}(y_r-y_{k-m+1})^{(m-1)}\right]\\
&+\sum_{k=m+1}^{N}\sum_{t=n+1}^{N}\Delta_{(n,m)}f(x_{t-n},y_{k-m})(x_t-x_{t-n})(y_k-y_{k-m})\times\\
&\times\left[\sum_{s=max\{t,k\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_s-y_{k-m+1})^{(m-1)}\right.\\
&\left.-\sum_{s=t}^{N}\sum_{r=k}^{N}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_r-y_{k-m+1})^{(m-1)}\right]
\end{aligned}
\tag{6.22}
$$

*holds, where $a^{(k)}=a(a-1)\ldots(a-k+1)$ and $a^{(0)}=1$.*

*Proof.* We start the proof by considering the expression

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\tilde{p}_{ij}f(x_i,y_i)$$

where $\tilde{p}_{ij}$ is defined as

$$\tilde{p}_{ij}=\begin{cases}\sum_{r=1}^{N}p_{ir}\ , & i=j,\\ 0\ , & i\neq j.\end{cases}$$

We get

$$
\begin{aligned}
&\sum_{i=1}^{N}\sum_{j=1}^{N}\tilde{p}_{ij}f(x_i,y_i)=\sum_{i=1}^{N}\sum_{j=1}^{N}p_{ij}f(x_i,y_i)\\
&=\sum_{k=0}^{m-1}\sum_{t=0}^{n-1}\Delta_{(t,k)}f(x_1,y_1)\sum_{s=max\{t+1,k+1\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_1)^{(t)}(y_s-y_1)^{(k)}\\
&+\sum_{k=0}^{m-1}\sum_{t=n+1}^{N}\Delta_{(n,k)}f(x_{t-n},y_1)(x_t-x_{t-n})\times\\
&\times\sum_{s=max\{t,k+1\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_s-y_1)^{(k)}\\
&+\sum_{k=m+1}^{N}\sum_{t=0}^{n-1}\Delta_{(t,m)}f(x_1,y_{k-m})(y_k-y_{k-m})\times\\
&\times\sum_{s=max\{t+1,k\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_1)^{(t)}(y_s-y_{k-m+1})^{(m-1)}
\end{aligned}
$$

$$+\sum_{k=m+1}^{N}\sum_{t=n+1}^{N}\Delta_{(n,m)}f(x_{t-n},y_{k-m})(x_t-x_{t-n})(y_k-y_{k-m})\times$$
$$\times\sum_{s=max\{t,k\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_s-y_{k-m+1})^{(m-1)}.$$

So, we get our required result by putting the expressions $\sum_{i=1}^{N}\sum_{j=1}^{N}p_{ij}f(x_i,y_i)$ and $\sum_{i=1}^{N}\sum_{j=1}^{N}p_{ij}f(x_i,y_j)$ in $C_\Delta(f,p)=\sum_{i=1}^{N}\sum_{j=1}^{N}p_{ij}f(x_i,y_i)-\sum_{i=1}^{N}\sum_{j=1}^{N}p_{ij}f(x_i,y_j)$. □

If we put $x_i=i$, $y_j=j$ and $f(x_i,y_j)=f(i,j)=a_{ij}$ in Theorem 6.8, then we get the following corollary.

**Corollary 6.1** *Let $p_{ij}$, $a_{ij}\in\mathbb{R}$ for $i,j\in\{1,\ldots,N\}$. Then, the following identity holds*

$$C_\Delta(a,p)=\sum_{i=1}^{N}\sum_{j=1}^{N}p_{ij}a_{ii}-\sum_{i=1}^{N}\sum_{j=1}^{N}p_{ij}a_{ij}$$

$$=\sum_{k=0}^{m-1}\sum_{t=0}^{n-1}\Delta^{(t,k)}a_{11}\left[\sum_{s=max\{t,k\}+1}^{N}\sum_{r=1}^{N}p_{sr}\binom{s-1}{t}\binom{s-1}{k}\right.$$
$$\left.-\sum_{s=t+1}^{N}\sum_{r=k+1}^{N}p_{sr}\binom{s-1}{t}\binom{r-1}{k}\right]+\sum_{k=0}^{m-1}\sum_{t=n+1}^{N}\Delta^{(n,k)}a_{(t-n)1}\times$$
$$\times\left[\sum_{s=max\{t,k+1\}}^{N}\sum_{r=1}^{N}p_{sr}\binom{s-t+n-1}{n-1}\binom{s-1}{k}\right.$$
$$\left.-\sum_{s=t}^{N}\sum_{r=k+1}^{N}p_{sr}\binom{s-t+n-1}{n-1}\binom{r-1}{k}\right]+\sum_{k=m+1}^{N}\sum_{t=0}^{n-1}\Delta^{(t,m)}a_{1(k-m)}\times$$
$$\times\left[\sum_{s=max\{t+1,k\}}^{N}\sum_{r=1}^{N}p_{sr}\binom{s-1}{t}\binom{s-k+m-1}{m-1}\right.$$
$$\left.-\sum_{s=t+1}^{N}\sum_{r=k}^{N}p_{sr}\binom{s-1}{t}\binom{r-k+m-1}{m-1}\right]$$
$$+\sum_{k=m+1}^{N}\sum_{t=n+1}^{N}\Delta^{(n,m)}a_{(t-n)(k-m)}$$
$$\times\left[\sum_{s=max\{t,k\}}^{N}\sum_{r=1}^{N}p_{sr}\binom{s-t+n-1}{n-1}\binom{s-k+m-1}{m-1}\right.$$
$$\left.-\sum_{s=t}^{N}\sum_{r=k}^{N}p_{sr}\binom{s-t+n-1}{n-1}\binom{r-k+m-1}{m-1}\right],$$

*where $\Delta^{(t,k)}a_{ij}$ represents finite difference of order $(t,k)$ of the sequence $(a_{ij})$.*

**Remark 6.1** If we put $n = m = 1$ in Corollary 6.1, then we get Theorem 3 of [54].

Before we state our next theorem, under the assumptions of Theorem 6.8 we introduce some notations as follows:

$$\begin{aligned}
\overline{C}_\Delta(f,p) = C_\Delta(f,p) - \sum_{k=0}^{m-1}\sum_{t=0}^{n-1}\Delta_{(t,k)}f(x_1,y_1)\times \\
\times\Bigg[\sum_{s=max\{t+1,k+1\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_1)^{(t)}(y_s-y_1)^{(k)} \\
-\sum_{s=t+1}^{N}\sum_{r=k+1}^{N}p_{sr}(x_s-x_1)^{(t)}(y_r-y_1)^{(k)}\Bigg]
\end{aligned}$$

$$\begin{aligned}
&-\sum_{k=0}^{m-1}\sum_{t=n+1}^{N}\Delta_{(n,k)}f(x_{t-n},y_1)(x_t-x_{t-n})\times \\
&\times\Bigg[\sum_{s=max\{t,k+1\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_s-y_1)^{(k)} \\
&-\sum_{s=t}^{N}\sum_{r=k+1}^{N}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_r-y_1)^{(k)}\Bigg] \\
&-\sum_{k=m+1}^{N}\sum_{t=0}^{n-1}\Delta_{(t,m)}f(x_1,y_{k-m})(y_k-y_{k-m})\times \\
&\times\Bigg[\sum_{s=max\{t+1,k\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_1)^{(t)}(y_s-y_{k-m+1})^{(m-1)} \\
&-\sum_{s=t+1}^{N}\sum_{r=k}^{N}p_{sr}(x_s-x_1)^{(t)}(y_r-y_{k-m+1})^{(m-1)}\Bigg],
\end{aligned} \tag{6.23}$$

$$\begin{aligned}
R_\Delta(t,k) \;=\; &\Bigg[\sum_{s=max\{t,k\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_s-y_{k-m+1})^{(m-1)} \\
&-\sum_{s=t}^{N}\sum_{r=k}^{N}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_r-y_{k-m+1})^{(m-1)}\Bigg].
\end{aligned} \tag{6.24}$$

**Theorem 6.9** *let $p_{ij}\in\mathbb{R}$ for $i,j\in\{1,\ldots,N\}$ and let $(x_i)$ and $(y_j)$ for $i,j\in\{1,\ldots,N\}$ be two real sequences that are monotonic in the same sense. We also assume that $f$ is an $(n,m)$-convex function. If*

$$R_\Delta(t,k)\geq 0,\quad t\in\{n+1,\ldots,N\},\quad k\in\{m+1,\ldots,N\},$$

*then*

$$\overline{C}_\Delta(f,p) \geq 0,$$

*where* $\overline{C}_\Delta$ *and* $R_\Delta$ *are defined in* (6.23) *and* (6.24) *respectively.*

*Proof.* The result follows easily by using identity (6.22). □

**Remark 6.2** If we put $x_i = i$, $y_j = j$ and $f(x_i,y_j) = f(i,j) = a_{ij}$ in Theorem 6.9 for $n=m=1$, then we get Theorem 3 of paper [67] and hence in this theorem for $a_{ij} = f(a_i,b_j)$ we get Corollary 2 of paper [67].

**Theorem 6.10** *Let* $p_{ij} \in \mathbb{R}$ *and let* $(x_i,y_j) \in I^2$ *be the distinct points, where* $i,j \in \{1,\dots,N\}$. *If* $f,g : I^2 \to \mathbb{R}$ *are two functions such that the inequalities*

$$R_\Delta(t,k) \geq 0, \quad t \in \{n+1,\dots,N\}, \quad k \in \{m+1,\dots,N\} \tag{6.25}$$

*and*

$$L\Delta_{(n,m)}g(x_i,y_j) \leq \Delta_{(n,m)}f(x_i,y_j) \leq U\Delta_{(n,m)}g(x_i,y_j) \tag{6.26}$$

*hold, then the following inequalities are valid*

$$L\overline{C}_\Delta(g,p) \leq \overline{C}_\Delta(f,p) \leq U\overline{C}_\Delta(g,p), \tag{6.27}$$

*where* $R_\Delta$ *is defined in* (6.24) *and* $L$ *and* $U$ *are some real constants.*

*Proof.* Let $F_1(x_i,y_j) = f(x_i,y_j) - Lg(x_i,y_j)$ and $F_2(x_i,y_j) = Ug(x_i,y_j) - f(x_i,y_j)$. Then $\Delta_{(n,m)}F_1(x_i,y_j) \geq 0$ and $\Delta_{(n,m)}F_2(x_i,y_j) \geq 0$. So, from Theorem 6.9 we easily obtain Theorem 6.10. □

**Remark 6.3** If the reverse inequalities hold in (6.25) and (6.26), then the inequalities in (6.27) still hold. Moreover, if the reverse inequality holds in (6.25), then the reverse inequalities in (6.27) are valid.

**Remark 6.4** If we put $x_i = i$, $y_j = j$ and $f(x_i,y_j) = f(i,j) = a_{ij}$ and $g(i,j) = b_{ij}$ in the previous theorem then we get Theorem 4 of paper [67].

### 6.1.2 Generalized Integral Čebyšev's Identity and Inequality

This subsection has the same structure as the previous one, only here we consider integrals instead of sums. Also, it is based on paper [24].

**Theorem 6.11** *Let* $p,f : I^2 \to \mathbb{R}$ *be two functions such that* $p$ *is integrable,* $f_{(N+1,M)}$ *and* $f_{(N,M+1)}$ *exist and are absolutely continuous. Then, we have*

$$C(f,p) = \int_a^b \int_a^b p(x,y)f(x,x)\,dy\,dx - \int_a^b \int_a^b p(x,y)f(x,y)\,dy\,dx$$

$$
\begin{aligned}
&= \sum_{i=0}^{N}\sum_{j=0}^{M} f_{(i,j)}(a,a)\left[\overline{P}^{(i,j)}(a,a) - P^{(i,j)}(a,a)\right] \\
&+ \sum_{j=0}^{M}\int_a^b f_{(N+1,j)}(x,a)\left[\overline{P}^{(N,j)}(x,a) - P^{(N,j)}(x,a)\right] dx \\
&+ \sum_{i=0}^{N}\int_a^b f_{(i,M+1)}(a,y)\left[\overline{P}^{(i,M)}(a,y) - P^{(i,M)}(a,y)\right] dy \\
&+ \int_a^b\int_a^b f_{(N+1,M+1)}(x,y)R(x,y)dy\,dx,
\end{aligned}
$$

*where* $\overline{P}^{(i,j)}$*,* $P^{(i,j)}$ *and* $R$ *are defined in* (6.14)*,* (6.13) *and* (6.16) *respectively.*

*Proof.* To prove this identity, first we find an expression for

$$\int_a^b\int_a^b p(x,y)f(x,x)dy\,dx$$

as follows. First we expand $f(x,x)$ in the Taylor expansion of two variables and multiply it with $p(x,y)$ and integrate it over $I^2$ by variables $x$ and $y$ to get

$$
\begin{aligned}
&\int_a^b\int_a^b p(x,y)f(x,x)dy\,dx \\
&= \int_a^b\left[\sum_{j=0}^{M}\left(\sum_{i=0}^{N} f_{(i,j)}(a,a)\frac{(x-a)^i}{i!}\right)\int_a^b p(x,y)\frac{(x-a)^j}{j!}dy\right]dx \\
&+ \int_a^b\left[\sum_{j=0}^{M}\left(\int_a^x f_{(N+1,j)}(s,a)\frac{(x-s)^N}{N!}ds\right)\int_a^b p(x,y)\frac{(x-a)^j}{j!}dy\right]dx \\
&+ \int_a^b\left[\int_a^b\int_a^x p(x,y)\left(\sum_{i=0}^{N} f_{(i,M+1)}(a,t)\frac{(x-a)^i}{i!}\right)\frac{(x-t)^M}{M!}dt\,dy\right]dx \\
&+ \int_a^b\left[\int_a^b\int_a^x\left(\int_a^x p(x,y)f_{(N+1,M+1)}(s,t)\frac{(x-s)^N}{N!}ds\right)\frac{(x-t)^M}{M!}dt\,dy\right]dx
\end{aligned}
$$

In the first summand, we change the order of summation, use linearity of integral to obtain

$$\sum_{i=0}^{N}\sum_{j=0}^{M}\int_a^b\int_a^b p(x,y)f_{(i,j)}(a,a)\frac{(x-a)^i}{i!}\frac{(x-a)^j}{j!}dydx.$$

By using Fubini's theorem, the second summand is rewritten as:

$$
\begin{aligned}
&\int_a^b\left[\sum_{j=0}^{M}\left(\int_a^x f_{(N+1,j)}(s,a)\frac{(x-s)^N}{N!}ds\right)\int_a^b p(x,y)\frac{(x-a)^j}{j!}dy\right]dx \\
&= \int_a^b\left[\sum_{j=0}^{M}\left(\int_a^x\int_a^b p(x,y)\frac{(x-a)^j}{j!}f_{(N+1,j)}(s,a)\frac{(x-s)^N}{N!}dy\,ds\right)\right]dx
\end{aligned}
$$

$$= \sum_{j=0}^{M} \int_a^b \int_a^x \int_a^b p(x,y) f_{(N+1,j)}(s,a) \frac{(x-s)^N}{N!} \frac{(x-a)^j}{j!} dy\, ds\, dx$$
$$= \sum_{j=0}^{M} \int_a^b \int_s^b \int_a^b p(x,y) f_{(N+1,j)}(s,a) \frac{(x-s)^N}{N!} \frac{(x-a)^j}{j!} dy\, dx\, ds,$$

Similarly, the third summand is rewritten as:

$$\int_a^b \left[ \int_a^b \int_a^x p(x,y) \left( \sum_{i=0}^{N} f_{(i,M+1)}(a,t) \frac{(x-a)^i}{i!} \right) \frac{(x-t)^M}{M!} dt\, dy \right] dx$$
$$= \sum_{i=0}^{N} \int_a^b \int_a^b \int_a^x p(x,y) f_{(i,M+1)}(a,t) \frac{(x-a)^i}{i!} \frac{(x-t)^M}{M!} dt\, dy\, dx$$
$$= \sum_{i=0}^{N} \int_a^b \int_a^b \int_t^b p(x,y) f_{(i,M+1)}(a,t) \frac{(x-a)^i}{i!} \frac{(x-t)^M}{M!} dy\, dx\, dt,$$

Finally, the fourth summand is rewritten as:

$$\int_a^b \left[ \int_a^b \int_a^x \left( \int_a^x p(x,y) f_{(N+1,M+1)}(s,t) \frac{(x-s)^N}{N!} ds \right) \frac{(x-t)^M}{M!} dt\, dy \right] dx$$
$$= \int_a^b \int_a^b \int_a^x \int_a^x p(x,y) f_{(N+1,M+1)}(s,t) \frac{(x-s)^N}{N!} \frac{(x-t)^M}{M!} ds\, dt\, dy\, dx$$
$$= \int_a^b \int_a^b \int_{max\{s,t\}}^b \int_a^b p(x,y) f_{(N+1,M+1)}(s,t) \frac{(x-s)^N}{N!} \frac{(x-t)^M}{M!} dy\, dx\, dt\, ds.$$

Now, we add up all these results to get

$$\int_a^b \int_a^b p(x,y) f(x,x) dy\, dx$$
$$= \sum_{i=0}^{N} \sum_{j=0}^{M} \int_a^b \int_a^b p(x,y) f_{(i,j)}(a,a) \frac{(x-a)^i}{i!} \frac{(x-a)^j}{j!} dy\, dx$$
$$= \sum_{j=0}^{M} \int_a^b \int_s^b \int_a^b p(x,y) f_{(N+1,j)}(s,a) \frac{(x-s)^N}{N!} \frac{(x-a)^j}{j!} dy\, dx\, ds$$
$$= \sum_{i=0}^{N} \int_a^b \int_a^b \int_t^b p(x,y) f_{(i,M+1)}(a,t) \frac{(x-a)^i}{i!} \frac{(x-t)^M}{M!} dy\, dx\, dt$$
$$= \int_a^b \int_a^b \int_{max\{s,t\}}^b \int_a^b p(x,y) f_{(N+1,M+1)}(s,t) \frac{(x-s)^N}{N!} \frac{(x-t)^M}{M!} dy\, dx\, dt\, ds,$$

when we change the names of variables on the right-hand side $x \leftrightarrow s$, $y \leftrightarrow t$, then we have,

$$\int_a^b \int_a^b p(x,y) f(x,x) dy\, dx$$

$$= \sum_{i=0}^{N}\sum_{j=0}^{M}\int_a^b\int_a^b p(s,t)f_{(i,j)}(a,a)\frac{(s-a)^{i+j}}{i!j!}dt\,ds$$
$$+\sum_{j=0}^{M}\int_a^b\int_x^b\int_a^b p(s,t)f_{(N+1,j)}(x,a)\frac{(s-x)^N}{N!}\frac{(s-a)^j}{j!}dt\,ds\,dx$$
$$+\sum_{i=0}^{N}\int_a^b\int_a^b\int_x^b p(s,t)f_{(i,M+1)}(a,y)\frac{(s-a)^i}{i!}\frac{(s-y)^M}{M!}dt\,ds\,dy$$
$$+\int_a^b\int_a^b\int_{max\{x,y\}}^b\int_a^b p(s,t)f_{(N+1,M+1)}(x,y)\frac{(s-x)^N}{N!}\frac{(s-y)^M}{M!}dt\,ds\,dy\,dx,$$

by using defined notations we finally obtain

$$\int_a^b\int_a^b p(x,y)f(x,x)dy\,dx = \sum_{i=0}^{N}\sum_{j=0}^{M}f_{(i,j)}(a,a)\overline{P}^{(i,j)}(a,a)$$
$$+\sum_{j=0}^{M}\int_a^b f_{(N+1,j)}(x,a)\overline{P}^{(N,j)}(x,a)\,dx+\sum_{i=0}^{N}\int_a^b f_{(i,M+1)}(a,y)\overline{P}^{(i,M)}(a,y)\,dy$$
$$+\int_a^b\int_a^b f_{(N+1,M+1)}(x,y)\int_{max\{x,y\}}^b\int_a^b p(s,t)\frac{(s-x)^N}{N!}\frac{(s-y)^M}{M!}dt\,ds\,dy\,dx,$$

where $\overline{P}^{(i,j)}$ is defined in (6.13).

Using the above expression for $\int_a^b\int_a^b p(x,y)f(x,x)dy\,dx$ and Theorem 3.7 in

$$C(f,p) = \int_a^b\int_a^b p(x,y)f(x,x)dy\,dx - \int_a^b\int_a^b p(x,y)f(x,y)dy\,dx,$$

we get the required identity. □

If in Theorem 6.11 we put $f(x,y) = f(x)g(y)$ and $p(x,y) = p(x)p(y)$, then we may state the following corollary.

**Corollary 6.2** *Let $p,f,g: I\to\mathbb{R}$ be three functions such that $p$ is integrable and $f_{(N)}$ and $g_{(M)}$ exist and are absolutely continuous. Then, we have*

$$T(f,g,p) = T(P_N(f),P_M(g),p)+T(R_N(f),P_M(g),p)+T(P_N(f),R_M(g),p)$$
$$+\int_a^b p(x)\,dx\times$$
$$\times\int_a^b\int_a^b\int_{max\{x,y\}}^b\frac{f_{(N+1)}(x)(s-x)^N}{N!}\frac{g_{(M+1)}(y)(s-y)^M}{M!}p(s)\,ds\,dy\,dx$$
$$-\int_a^b R_N(f)(x)p(x)\,dx\int_a^b R_M(g)(x)p(x)\,dx \tag{6.28}$$

*where $P_k(h)(x) = \sum_{i=0}^{k}\frac{h^{(i)}(a)(x-a)^i}{i!}$, $R_k(h)(x) = \int_a^x\frac{h^{(N+1)}(s)(x-s)^N}{N!}ds$, $k\in\mathbb{N}$ for a function $h$ and $T$ is defined in* (6.4).

**Corollary 6.3** *Let the assumptions of Theorem* 6.11 *be valid. Then for* $s,t > 1$ *such that* $1/s + 1/t = 1$, *we have*

$$\left|\overline{C}(f,p)\right| \leq \left(\int_a^b \int_a^b \left|f_{(N+1,M+1)}(x,y)\right|^s dy\,dx\right)^{1/s} \left(\int_a^b \int_a^b |R(x,y)|^t dy\,dx\right)^{1/t} \tag{6.29}$$

*where* $\overline{C}$ *and* $R$ *are defined in* (6.20) *and* (6.16) *respectively.*

*Proof.* We can get (6.29) by using Hölder's inequality for the integrals in Theorem 6.11. □

**Theorem 6.12** *Let the assumptions of Theorem* 6.11 *be valid. If the inequality*

$$R(x,y) \geq 0$$

*holds for every* $(x, y) \in I^2$, *then there exists* $(\xi, \eta) \in I^2$ *such that*

$$\overline{C}(f,p) = f_{(N+1,M+1)}(\xi,\eta)\, C(f_0,p),$$

*where* $R$, $f_0$ *and* $\overline{C}$ *are defined in* (6.16), (6.18) *and* (6.20), *respectively.*

*Proof.* We have

$$\overline{C}(f,p) = \int_a^b \int_a^b f_{(N+1,M+1)}(x,y) R(x,y)\, dy\, dx, \tag{6.30}$$

using the mean value theorem for double integrals we get

$$\overline{C}(f,p) = f_{(N+1,M+1)}(\xi,\eta) \int_a^b \int_a^b R(x,y)\, dy\, dx.$$

If we put $f = f_0$ in the above expression, then we obtain

$$\overline{C}(f_0,p) = C(f_0,p) = \int_a^b \int_a^b R(x,y)\, dy\, dx$$

and hence we get what we wanted. □ □

**Remark 6.5** (a) For $N = M = 0$, Theorem 6.12 is equivalent to Theorem 6.4.

(b) If we take $f(x,y) = f(x)g(y)$ and $p(x,y) = p(x)p(y)$ in Theorem 6.12 with $N = M = 0$, then we get (6.3).

**Theorem 6.13** *Let the assumptions of Theorem* 6.11 *be valid and let* $g_{(N+1,M+1)} \neq 0$ *on* $I^2$ *where* $g \in C^{(N+1,M+1)}(I^2)$. *If the inequality*

$$R(x,y) \geq 0$$

*holds for every* $(x, y) \in I^2$, *then there exists* $(\xi, \eta) \in I^2$ *such that*

$$\overline{C}(f,p) = \frac{f_{(N+1,M+1)}(\xi,\eta)}{g_{(N+1,M+1)}(\xi,\eta)} \overline{C}(g,p),$$

*where* $R$ *and* $\overline{C}$ *are defined in* (6.16) *and* (6.20) *respectively.*

*Proof.* Using (6.30) and the integral mean value theorem we have

$$\begin{aligned}
\overline{C}(f,p) &= \int_a^b \int_a^b \frac{f_{(N+1,M+1)}(x,y)}{g_{(N+1,M+1)}(x,y)} g_{(N+1,M+1)}(x,y)R(x,y)dydx \\
&= \frac{f_{(N+1,M+1)}(\xi,\eta)}{g_{(N+1,M+1)}(\xi,\eta)} \int_a^b \int_a^b g_{(N+1,M+1)}(x,y)R(x,y)dydx \\
&= \frac{f_{(N+1,M+1)}(\xi,\eta)}{g_{(N+1,M+1)}(\xi,\eta)} \overline{C}(g,p).
\end{aligned}$$

□

**Remark 6.6** For $N = M = 0$, Theorem 6.13 becomes Theorem 2 of [67].

**Theorem 6.14** *Let $p, f : I^2 \to \mathbb{R}$ be two functions such that $p$ is integrable and $f$ is $(N+1,M+1)$-convex. If the inequality*

$$R(x,y) \geq 0$$

*holds for every $(x, y) \in I^2$, then the following inequality is valid*

$$\overline{C}(f,p) \geq 0,$$

*where $R$ and $\overline{C}$ are defined in (6.16) and (6.20) respectively.*

*Proof.* If $f$ is $(N+1,M+1)$-convex function it may be approximated uniformly on $I^2$ by polynomials having nonnegative partial derivatives of order $(N+1,M+1)$. It is known that the Bernstein polynomials $B^{n,m}$ defined as

$$B^{n,m}(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{m} \binom{n}{i}\binom{m}{j} f(a_i,b_j)(x-a)^i(b-x)^{n-i}(y-a)^j(b-y)^{m-j},$$

where $a_i = a + i\frac{b-a}{n}$, $b_j = a + j\frac{b-a}{m}$, converge uniformly to $f$ on $I^2$ as $n,m \to \infty$ provided that $f$ is continuous. Further, if $f$ is $(N+1,M+1)$-convex function these polynomials have nonnegative partial derivatives of order $(N+1,M+1)$, *i.e.*, $B^{n,m}_{(N+1,M+1)} \geq 0$ which can be prove by induction by using the following formula:

$$\begin{aligned}
B^{n,m}_{(N+1,M+1)}(x,y) &= (N+1)!(M+1)!\binom{n}{N+1}\binom{m}{M+1} \times \\
&\times \sum_{i=0}^{n-N-1}\sum_{j=0}^{m-M-1} \binom{n-N-1}{i}\binom{m-M-1}{j} \times \\
&\times \Delta^{(N+1,M+1)} f(a_i,b_j)(x-a)^i(b-x)^{n-N-1-i}(y-a)^j(b-y)^{m-M-1-j}.
\end{aligned}$$

As $(a_i)$ and $(b_j)$ are increasing sequences and $f$ is $(N+1,M+1)$-convex function, so we have $\Delta^{(N+1,M+1)} f(a_i,b_j) \geq 0$. Since $R$ is continuous and $B^{n,m}_{(N+1,M+1)} \geq 0$ on $I^2$ so by (6.20) we obtain

$$\overline{C}(B^{n,m},p) = \int_a^b \int_a^b B^{n,m}_{(N+1,M+1)}(x,y) \times$$

$$\times \left[ \int_{max\{s,t\}}^{b} \int_{a}^{b} p(s,t) \frac{(x-s)^N}{N!} \frac{(x-t)^M}{M!} dt\, ds \right.$$
$$\left. - \int_{x}^{b} \int_{y}^{b} p(s,t) \frac{(s-x)^N}{N!} \frac{(t-y)^M}{M!} dt\, ds \right] dy\, dx \geq 0,$$

or we can write $\overline{C}(B^{n,m}, p)$ as

$$\overline{C}(B^{n,m}, p) = \int_a^b \int_a^b B^{n,m}_{(N+1,M+1)}(x,y) R(x,y)\, dy\, dx. \tag{6.31}$$

Now by letting $n, m \to \infty$ through an appropriate sequence, the uniform convergence of $B^{n,m}_{(N+1,M+1)}$ to $f_{(N+1,M+1)}$ provides our desired result. □

**Theorem 6.15** *Let the assumptions of Theorem* 6.14 *be valid. Then there exists* $(\xi, \eta) \in I^2$ *such that*

$$\overline{C}(f,p) = R(\xi,\eta) \left( f_{(N,M)}(b,b) - f_{(N,M)}(a,b) - f_{(N,M)}(b,a) + f_{(N,M)}(a,a) \right),$$

*where $R$ and $\overline{C}$ are defined in* (6.16) *and* (6.20) *respectively.*

*Proof.* Since $R$ is continuous and $B^{n,m}_{(N+1,M+1)} \geq 0$ on $I^2$, where $B^{n,m}$ is Bernstien polynomial, by the same arguments used in proof of Theorem 6.12, starting from (6.31), we obtain

$$\begin{aligned}
&\overline{C}(B^{n,m}, p) \\
&= \int_a^b \int_a^b R(x,y) B^{n,m}_{(N+1,M+1)}(x,y)\, dy\, dx \\
&= R(\xi_{n,m}, \eta_{n,m}) \int_a^b \int_a^b B^{n,m}_{(N+1,M+1)}(x,y)\, dy\, dx \\
&= R(\xi_{n,m}, \eta_{n,m}) \left( B^{n,m}_{(N,M)}(b,b) - B^{n,m}_{(N,M)}(a,b) - B^{n,m}_{(N,M)}(b,a) + B^{n,m}_{(N,M)}(a,a) \right).
\end{aligned}$$

The points $\mathbf{x}_{n,m} = (\xi_{n,m}, \eta_{n,m})$ have a limit point $(\xi, \eta)$ in $I^2$ as $n, m \to \infty$, so letting $n, m \to \infty$ through an appropriate sequence, the uniform convergence of $B^{n,m}_{(N,M)}$ to $f_{(N,M)}$ provides our desired result. □

**Remark 6.7** For $N = M = 0$, Theorem 6.15 becomes Theorem 6 of [67].

### 6.1.3 Generalized Integral Ky Fan's Identity and Inequality

**Theorem 6.16** *Let the assumptions of Theorem* 6.11 *be valid and let $q : I \to \mathbb{R}$ be an integrable function. Then the following identity holds*

$$K(f,p,q) = \sum_{j=0}^{M} \sum_{i=0}^{N} f_{(i,j)}(a,a) \left[ Q^{(i,j)}(a) - P^{(i,j)}(a,a) \right]$$

$$
\begin{aligned}
&+\sum_{j=0}^{M}\int_a^b f_{(N+1,j)}(x,a)\left[Q^{(N,j)}(x)-P^{(N,j)}(x,a)\right]dx \\
&+\sum_{i=0}^{N}\int_a^b f_{(i,M+1)}(a,y)\left[Q^{(M,i)}(y)-P^{(i,M)}(a,y)\right]dy \\
&+\int_a^b\int_a^b f_{(N+1,M+1)}(x,y)\overline{R}(x,y)\,dy\,dx,
\end{aligned}
$$

*where* $P^{(i,j)}$, $Q^{(i,j)}$ *and* $\overline{R}$ *are defined in* (6.13), (6.15) *and* (6.17) *respectively.*

*Proof.* The proof of this theorem is analogous to the proof of Theorem 6.11. We only need the following substitution $\int_a^b p(x,y)dy = q(x)$. □

The above and all results in this section is given in [24]. If in Theorem 6.16 we put $f(x,y) = f(x)g(y)$ and $p(x,y) = \dfrac{q(x)q(y)}{\int_a^b q(t)\,dt}$ where $q$ is an integrable function such that $\int_a^b q(t)\,dt \neq 0$, then we state the following corollary.

**Corollary 6.4** *Let the assumptions of Corollary* 6.2 *be valid for functions* $f,g$ *and* $p$ *and let* $q : I \to \mathbb{R}$ *be an integrable function such that* $\int_a^b q(t)\,dt \neq 0$*. Then the identity*

$$
\begin{aligned}
&T(f,g,q) = T(P_N(f),P_M(g),q) + T(R_N(f),P_M(g),q) + T(P_N(f),R_M(g),q) \\
&+\int_a^b\int_a^b\int_{max\{x,y\}}^b \frac{f_{(N+1)}(x)(s-x)^N}{N!}\frac{g_{(M+1)}(y)(s-y)^M}{M!}q(s)\,ds\,dy\,dx \\
&-\int_a^b R_N(f)(x)q(x)\,dx\int_a^b R_M(g)(x)q(x)\,dx
\end{aligned}
$$

*holds, where* $P_k(h)(x) = \sum_{i=0}^{k}\dfrac{h^{(i)}(a)(x-a)^i}{i!}$, $R_k(h)(x) = \int_a^x \dfrac{h^{(N+1)}(s)(x-s)^N}{N!}ds$, $k \in \mathbb{N}$ *for a function* $h$ *and* $T$ *is defined in* (6.4).

**Corollary 6.5** *Let the assumptions of Theorem* 6.16 *be valid. Then for* $s,t > 1$ *such that* $1/s + 1/t = 1$, *we have*

$$
|\overline{K}(f,p,q)| \leq \left(\int_a^b\int_a^b |f_{(N+1,M+1)}(x,y)|^s\,dy\,dx\right)^{1/s}\left(\int_a^b\int_a^b \left|\overline{R}(x,y)\right|^t\,dy\,dx\right)^{1/t},
$$

*where* $\overline{R}$ *and* $\overline{K}$ *are defined in* (6.17) *and* (6.21) *respectively.*

**Theorem 6.17** *Let the assumptions of Theorem* 6.14 *be valid for* $f,p$*. If the inequality*

$$\overline{R}(x,y) \geq 0$$

*holds for every* $(x,y) \in I^2$, *then there exists* $(\xi,\eta) \in I^2$ *such that*

$$\overline{K}(f,p,q) = f_{(N+1,M+1)}(\xi,\eta)K(f_0,p,q),$$

*where* $\overline{R}$, $f_0$ *and* $\overline{K}$ *are defined in* (6.17), (6.18) *and* (6.21) *respectively.*

**Theorem 6.18** *Let the assumptions of Theorem* 6.16 *be valid. If the inequality*

$$\overline{R}(x,y) \geq 0$$

*holds for every* $(x,y) \in I^2$*, then there exists* $(\xi,\eta) \in I^2$ *such that*

$$\overline{K}(f,p,q) = \frac{f_{(N+1,M+1)}(\xi,\eta)}{g_{(N+1,M+1)}(\xi,\eta)}\overline{K}(g,p,q),$$

*where* $\overline{R}$*,* $f_0$ *and* $\overline{K}$ *are defined in* (6.17)*,* (6.18) *and* (6.21) *respectively.*

**Theorem 6.19** *Let the assumptions of Theorem* 6.14 *be valid for functions* $p$ *and* $f$ *and let* $q : I \to \mathbb{R}$ *be an integrable function. If the inequality*

$$\overline{R}(x,y) \geq 0$$

*holds for every* $(x,y) \in I^2$*, then the following inequality holds*

$$\overline{K}(f,p,q) \geq 0,$$

*where* $\overline{R}$ *and* $\overline{K}$ *are defined in* (6.17) *and* (6.21) *respectively.*

*Proof.* The proof is analogous to the proof of Theorem 6.14 so we omit the details. □

## 6.2 Montgomery Identities for Higher Order Differentiable Functions of Two Variables

Ostrowski type inequalities have many applications in the field of numerical integrations and in probability theory. We can also obtain special means with the help of such inequalities. The celebrated Čebyšev inequality is also a special case of the Ostrowski type inequalities. As far as we are concerned with the Grüss-type inequalities, these inequalities play a paramount role in numerical integrations and in other fields. In recent years a rapid advancement in generalizations and improvements of these type of inequalities has been observed. In present chapter we have also proposed certain generalizations of the Montgomery identities and hence generalizations of Ostrowski and Grüs type inequalities by using higher order differentiable functions.

The results presented in this section are taken from [25].

Let us recall the weighted Montgomery identity which we already used in Chapter 5.

**Theorem 6.20** *Let* $f \in C^{(1)}[a,b]$*. Then the identity*

$$f(x) = \int_a^b w(s)f(s)ds + \int_a^b p_w(x,s)f'(s)ds,$$

*holds for weighted Peano kernel $p_w$ defined as*

$$p_w(x,s) = \begin{cases} W(s) & , \quad a \le s \le x, \\ W(s) - 1 & , \quad x < s \le b, \end{cases}$$

*where $w : [a,b] \to \mathbb{R}_*$ is such that $\int_a^b w(s)ds = 1$ and*

$$W(s) = \begin{cases} 0 & , \quad s < a, \\ \int_a^s w(\xi)d\xi & , \quad s \in [a,b], \\ 1 & , \quad s > b. \end{cases}$$

For functions of two variables the following generalized identities were obtained by authors in [5].

**Theorem 6.21** *Let $f \in C^{(1,1)}([a,b] \times [c,d])$. Then identities*

$$(b-a)(d-c)f(x,y) = -\int_a^b \int_c^d f(s,t)\,dt\,ds + (d-c)\int_a^b f(s,y)\,ds$$
$$+(b-a)\int_c^d f(x,t)\,dt + \int_a^b \int_c^d p(x,s)q(y,t)f_{(1,1)}(s,t)\,dt\,ds,$$

*and*

$$(b-a)(d-c)f(x,y) = \int_a^b \int_c^d f(s,t)\,dt\,ds + \int_a^b \int_c^d q(x,s)f_{(1,0)}(s,t)\,dt\,ds$$
$$+\int_a^b \int_c^d r(y,t)f_{(0,1)}(s,t)\,dt\,ds + \int_a^b \int_c^d q(x,s)\,r(y,t)f_{(1,1)}(s,t)\,dt\,ds,$$

*hold, where $p$ and $q$ are the Peano kernals.*

J. Pečarić and A. Vukelić in [80] gave the following weighted Montgomery identities for functions of two variables.

**Theorem 6.22** *Let $p : [a,b] \times [c,d] \to \mathbb{R}$ be an integrable function and $P$ be defined as*

$$P(x,y) = \int_x^b \int_y^d p(\xi,\eta)\,d\eta\,d\xi. \tag{6.32}$$

*If $f \in C^{(1,1)}([a,b] \times [c,d])$, then the following identity holds*

$$\begin{aligned} P(a,c)f(x,y) &= \int_a^b \int_c^d p(s,t)f(s,t)dtds + \int_a^b \hat{P}(x,s)f_{(1,0)}(s,y)\,ds \\ &+ \int_c^d \tilde{P}(y,t)f_{(0,1)}(x,t)dt - \int_a^b \int_c^d \overline{P}^{(N,M)}(x,s,y,t)f_{(1,1)}(s,t)dtds, \end{aligned} \tag{6.33}$$

*where*

$$\hat{P}(x,s) = \begin{cases} \int_a^s \int_c^d p(\xi,\eta)d\eta\,d\xi & , \quad a \le s \le x, \\ -P(s,c) & , \quad x < s \le b, \end{cases}$$

$$\tilde{P}^{(i,M)}(x,y,t) = \begin{cases} \int_a^b \int_c^t p(\xi,\eta)d\eta\, d\xi\ , & c \le t \le y, \\ -P(a,t) & , \quad y < t \le d, \end{cases}$$

$$\overline{P}(x,s,y,t) = \begin{cases} \int_a^s \int_c^t p(\xi,\eta)d\eta d\xi, & a \le s \le x, \quad c \le t \le y, \\ -\int_s^b \int_c^t p(\xi,\eta)d\eta d\xi, & x < s \le b, \quad c \le t \le y, \\ -\int_a^s \int_t^d p(\xi,\eta)d\eta d\xi, & a \le s \le x, \quad y < t \le d, \\ P(s,t), & x < s \le b, \quad y < t \le d. \end{cases}$$

**Theorem 6.23** *Let the assumptions of Theorem* 6.22 *be valid. Then the identity*

$$\begin{aligned} P(a,c)f(x,y) &= -\int_a^b \int_c^d p(s,t)f(s,t)\,dt\,ds + \int_a^b \int_c^d p(s,t)f(s,y)\,dt\,ds \\ &+ \int_a^b \int_c^d p(s,t)f(x,t)\,dt\,ds + \int_a^b \int_c^d \overline{P}(x,s,y,t)f_{(1,1)}(s,t)\,dt\,ds, \end{aligned} \tag{6.34}$$

*holds, where $\overline{P}$ is as defined in Theorem* 6.22.

**Theorem 6.24** *Let the assumptions of Theorem* 6.22 *be valid. Then the identity*

$$\begin{aligned} [P(a,c)]^2 f(x,y) &= P(a,c)\int_a^b \int_c^d p(s,t)f(s,t)\,dt\,ds \\ &+ \int_a^b \left(\int_a^b \int_c^d p(\xi,t)\hat{P}(x,s)f_{(1,0)}(s,t)\,dt\,ds\right) d\xi \\ &+ \int_c^d \left(\int_a^b \int_c^d p(s,\eta)\tilde{P}(y,t)f_{(0,1)}(s,t)\,dt\,ds\right) d\eta \\ &+ \int_a^b \int_c^d \check{P}(x,s,y,t)f_{(1,1)}(s,t)\,dt\,ds, \end{aligned}$$

*holds, where $\hat{P}$, $\tilde{P}$ and $\overline{P}$ are defined in Theorem* 6.22 *and*

$$\check{P}(x,s,y,t) = 2\hat{P}(x,s)\tilde{P}(y,t) - P(a,c)\overline{P}(x,s,y,t).$$

### 6.2.1 Montgomery Identities for Double Weighted Integrals of Higher Order Differentiable Functions

In the start of this section, we introduce some notations to reduce our lengthy expressions as follows:

$$P^{(i,j)}_{(a,c)\to(b,d)}(x,y) = \int_a^b \int_c^d p(\xi,\eta)\frac{(\xi-x)^i}{i!}\frac{(\eta-y)^j}{j!}\,d\eta\,d\xi, \tag{6.35}$$

$$P^{(0,j)}_{(a,c)\to(b,d)}(y) = \int_a^b \int_c^d p(\xi,\eta)\frac{(\eta-y)^j}{j!}\,d\eta\,d\xi, \tag{6.36}$$

$$P^{(i,0)}_{(a,c)\to(b,d)}(x) = \int_a^b \int_c^d p(\xi,\eta)\frac{(\xi-x)^i}{i!}\,d\eta\,d\xi, \tag{6.37}$$

$$\begin{aligned} R(x,y;f) &= -\sum_{i=1}^{N}\sum_{j=1}^{M} f_{(i,j)}(x,y)P^{(i,j)}_{(a,c)\to(b,d)}(x,y) - \sum_{j=1}^{M} f_{(0,j)}(x,y)P^{(0,j)}_{(a,c)\to(b,d)}(y) \\ &- \sum_{i=1}^{N} f_{(i,0)}(x,y)P^{(i,0)}_{(a,c)\to(b,d)}(x). \end{aligned} \tag{6.38}$$

For our next theorem we restate Theorem 3.7 using our notations as follows.

**Theorem 6.25** *Let $p:[a,b]\times[c,d]\to\mathbb{R}$ be an integrable function and let $f\in C^{(N+1,M+1)}([a,b]\times[c,d])$. Then the following identity holds*

$$\begin{aligned} \int_a^b\int_c^d p(x,y)f(x,y)dy\,dx &= \sum_{i=0}^{N}\sum_{j=0}^{M} P^{(i,j)}_{(a,c)\to(b,d)}(a,c)f_{(i,j)}(a,c) \\ &+ \sum_{j=0}^{M}\int_a^b P^{(N,j)}_{(x,c)\to(b,d)}(x,c)f_{(N+1,j)}(x,c)\,dx \\ &+ \sum_{i=0}^{N}\int_c^d P^{(i,M)}_{(a,y)\to(b,d)}(a,y)f_{(i,M+1)}(a,y)\,dy \\ &+ \int_a^b\int_c^d P^{(N,M)}_{(x,y)\to(b,d)}(x,y)f_{(N+1,M+1)}(x,y)\,dy\,dx. \end{aligned}$$

Now we give generalizations of Theorems 6.22, 6.23 and 6.24 respectively as follows:

**Theorem 6.26** *Let the assumptions of Theorem* 6.25 *be valid. Then the identity*

$$\begin{aligned} &P(a,c)f(x,y) = R(x,y;f) + \int_a^b\int_c^d p(s,t)f(s,t)dt\,ds \\ &+ \sum_{j=0}^{M}\int_a^b \hat{P}^{(N,j)}(x,s,y)f_{(N+1,j)}(s,y)\,ds + \sum_{i=0}^{N}\int_c^d \tilde{P}^{(i,M)}(x,y,t)f_{(i,M+1)}(x,t)\,dt \\ &- \int_a^b\int_c^d \overline{P}^{(N,M)}(x,s,y,t)f_{(N+1,M+1)}(s,t)\,dt\,ds, \end{aligned} \tag{6.39}$$

*holds, where*

$$\hat{P}^{(N,j)}(x,s,y) = \begin{cases} P^{(N,j)}_{(a,c)\to(s,d)}(s,y), & a\le s\le x, \\ -P^{(N,j)}_{(s,c)\to(b,d)}(s,y), & x<s\le b, \end{cases}$$

$$\tilde{P}^{(i,M)}(x,y,t) = \begin{cases} P^{(i,M)}_{(a,c)\to(b,t)}(x,t), & c\le t\le y, \\ -P^{(i,M)}_{(a,t)\to(b,d)}(x,t), & y<t\le d, \end{cases}$$

*and*

$$\overline{P}^{(N,M)}(x,s,y,t) = \begin{cases} P^{(N,M)}_{(a,c)\to(s,t)}(s,t), & a \le s \le x \quad , \quad c \le t \le y, \\ -P^{(N,M)}_{(s,c)\to(b,t)}(s,t), & x < s \le b \quad , \quad c \le t \le y, \\ -P^{(N,M)}_{(a,t)\to(s,d)}(s,t), & a \le s \le x \quad , \quad y < t \le d, \\ P^{(N,M)}_{(s,t)\to(b,d)}(s,t), & x < s \le b \quad , \quad y < t \le d, \end{cases}$$

*where* $P^{(i,j)}_{(.,.)\to(.,.)}$ *for* $i,j \in \{N,M\}$ *is defined in* (6.35)*, and* $P$ *and* $R$ *are as defined in* (6.32) *and* (6.38) *respectively.*

*Proof.* Using Theorem 6.25 for $[a,x]\times[c,y]$, we get

$$\int_a^x \int_c^y p(s,t)f(s,t)dt\,ds = \int_x^a \int_y^c p(s,t)f(s,t)dt\,ds$$
$$= \sum_{i=0}^{N}\sum_{j=0}^{M} P^{(i,j)}_{(x,y)\to(a,c)}(x,y)f_{(i,j)}(x,y) + \sum_{j=0}^{M}\int_x^a P^{(N,j)}_{(s,y)\to(a,c)}(s,y)f_{(N+1,j)}(s,y)\,ds$$
$$+\sum_{i=0}^{N}\int_y^c P^{(i,M)}_{(x,t)\to(a,c)}(x,t)f_{(i,M+1)}(x,t)\,dt$$
$$+\int_x^a\int_y^c P^{(N,M)}_{(s,t)\to(a,c)}(s,t)f_{(N+1,M+1)}(s,t)\,dt\,ds$$
$$= \sum_{i=0}^{N}\sum_{j=0}^{M}\Big[P^{(i,j)}_{(x,y)\to(b,d)}(x,y) - P^{(i,j)}_{(x,c)\to(b,d)}(x,y) - P^{(i,j)}_{(a,y)\to(b,d)}(x,y)$$
$$+P^{(i,j)}_{(a,c)\to(b,d)}(x,y)\Big]f_{(i,j)}(x,y)$$
$$-\sum_{j=0}^{M}\int_a^x\Big[P^{(N,j)}_{(s,y)\to(b,d)}(s,y) - P^{(N,j)}_{(s,c)\to(b,d)}(s,y) - P^{(N,j)}_{(a,y)\to(b,d)}(s,y)$$
$$+P^{(N,j)}_{(a,c)\to(b,d)}(s,y)\Big]f_{(N+1,j)}(s,y)\,ds$$
$$-\sum_{i=0}^{N}\int_c^y\Big[P^{(i,M)}_{(x,t)\to(b,d)}(x,t) - P^{(i,M)}_{(x,c)\to(b,d)}(x,t) - P^{(i,M)}_{(a,t)\to(b,d)}(x,t)$$
$$+P^{(i,M)}_{(a,c)\to(b,d)}(x,t)\Big]\times f_{(i,M+1)}(x,t)\,dt$$
$$+\int_a^x\int_c^y\Big[P^{(N,M)}_{(s,t)\to(b,d)}(s,t) - P^{(N,M)}_{(s,c)\to(b,d)}(s,t) - P^{(N,M)}_{(a,t)\to(b,d)}(s,t)$$
$$+P^{(N,M)}_{(a,c)\to(b,d)}(s,t)\Big]f_{(N+1,M+1)}(s,t)\,dt\,ds.$$

Similarly for $[x,b]\times[c,y]$, we have

$$\int_x^b \int_c^y p(s,t)f(s,t)dt\,ds = -\int_x^b \int_y^c p(s,t)f(s,t)dt\,ds$$

$$= -\sum_{i=0}^{N}\sum_{j=0}^{M} f_{(i,j)}(x,y)\left[P^{(i,j)}_{(x,y)\to(b,d)}(x,y) - P^{(i,j)}_{(x,c)\to(b,d)}(x,y)\right]$$
$$-\sum_{j=0}^{M}\int_x^b f_{(N+1,j)}(s,y)\left[P^{(N,j)}_{(s,y)\to(b,d)}(s,y) - P^{(N,j)}_{(s,c)\to(b,d)}(s,y)\right] ds$$
$$+\sum_{i=0}^{N}\int_c^y f_{(i,M+1)}(x,t)\left[P^{(i,M)}_{(x,t)\to(b,d)}(x,t) - P^{(i,M)}_{(x,c)\to(b,d)}(x,t)\right] dt$$
$$+\int_x^b\int_c^y f_{(N+1,M+1)}(s,t)\left[P^{(N,M)}_{(s,t)\to(b,d)}(s,t) - P^{(N,M)}_{(s,c)\to(b,d)}(s,t)\right] dt\,ds.$$

For $[a,x]\times[y,d]$, we obtain

$$\int_a^x\int_y^d p(s,t)f(s,t)dt\,ds = -\int_x^a\int_y^d p(s,t)f(s,t)dt\,ds$$
$$= -\sum_{i=0}^{N}\sum_{j=0}^{M} f_{(i,j)}(x,y)\left[P^{(i,j)}_{(x,y)\to(b,d)}(x,y) - P^{(i,j)}_{(a,y)\to(b,d)}(x,y)\right]$$
$$+\sum_{j=0}^{M}\int_a^x f_{(N+1,j)}(s,y)\left[P^{(N,j)}_{(s,y)\to(b,d)}(s,y) - P^{(N,j)}_{(a,y)\to(b,d)}(s,y)\right] ds$$
$$-\sum_{i=0}^{N}\int_y^d f_{(i,M+1)}(x,t)\left[P^{(i,M)}_{(x,t)\to(b,d)}(x,t) - P^{(i,M)}_{(a,t)\to(b,d)}(x,t)\right] dt$$
$$+\int_a^x\int_y^d f_{(N+1,M+1)}(s,t)\left[P^{(N,M)}_{(s,t)\to(b,d)}(s,t) - P^{(N,M)}_{(a,t)\to(b,d)}(s,t)\right] dt\,ds.$$

Finally for $[x,b]\times[y,d]$, we have

$$\int_x^b\int_y^d p(s,t)f(s,t)dt\,ds = \sum_{i=0}^{N}\sum_{j=0}^{M} f_{(i,j)}(x,y)P^{(i,j)}_{(x,y)\to(b,d)}(x,y)$$
$$+\sum_{j=0}^{M}\int_x^b f_{(N+1,j)}(s,y)P^{(N,j)}_{(s,y)\to(b,d)}(s,y)\,ds$$
$$+\sum_{i=0}^{N}\int_y^d f_{(i,M+1)}(x,t)P^{(i,M)}_{(x,t)\to(b,d)}(x,t)\,dt$$
$$+\int_x^b\int_y^d f_{(N+1,M+1)}(s,t)P^{(N,M)}_{(s,t)\to(b,d)}(s,t)\,dt\,ds.$$

Adding up the four expressions we get our required result. □

**Theorem 6.27** *Let the assumptions of Theorem* 6.25 *be valid. Then the identity*

$$P(a,c)f(x,y) = R(x,y;f) + \sum_{j=1}^{M}\int_a^b\int_c^d p(s,\eta)\frac{(\eta-y)^j}{j!}f_{(0,j)}(s,y)\,d\eta\,ds$$

$$+\sum_{i=1}^{N}\int_a^b\int_c^d p(\xi,t)\frac{(\xi-x)^i}{i!}f_{(i,0)}(x,t)\,dt\,d\xi-\int_a^b\int_c^d p(s,t)f(s,t)dt\,ds$$
$$+\int_a^b\int_c^d p(s,t)f(s,y)\,dt\,ds+\int_a^b\int_c^d p(s,t)f(x,t)\,dt\,ds$$
$$+\int_a^b\int_c^d \overline{P}^{(N,M)}(x,s,y,t)f_{(N+1,M+1)}(s,t)\,dt\,ds \tag{6.40}$$

*holds, where* $\overline{P}^{(N,M)}$ *is as in Theorem* 6.26*, P and R are defined in* (6.32) *and* (6.38) *respectively.*

*Proof.* First we find an expression for

$$\int_a^b \hat{P}^{(N,j)}(x,s,y)f_{(N+1,j)}(s,y)\,ds$$

by using integration by parts as follows:

$$\int_a^b \hat{P}^{(N,j)}(x,s,y)f_{(N+1,j)}(s,y)\,ds$$
$$=\int_a^x P^{(N,j)}_{(a,c)\to(s,d)}(s,y)f_{(N+1,j)}(s,y)\,ds-\int_x^b P^{(N,j)}_{(s,c)\to(b,d)}(s,y)f_{(N+1,j)}(s,y)\,ds$$
$$=\int_a^x P^{(N,j)}_{(a,c)\to(s,d)}(s,y)f_{(N+1,j)}(s,y)\,ds+\int_x^b P^{(N,j)}_{(b,c)\to(s,d)}(s,y)f_{(N+1,j)}(s,y)\,ds$$
$$=P^{(N,j)}_{(a,c)\to(x,d)}(x,y)f_{(N,j)}(x,y)+\int_a^x P^{(N-1,j)}_{(a,c)\to(s,d)}(s,y)f_{(N,j)}(s,y)\,ds$$
$$+P^{(N,j)}_{(x,c)\to(b,d)}(x,y)f_{(N,j)}(x,y)+\int_x^b P^{(N-1,j)}_{(b,c)\to(s,d)}(s,y)f_{(N,j)}(s,y)\,ds$$
$$=P^{(N,j)}_{(a,c)\to(b,d)}(x,y)f_{(N,j)}(x,y)+\int_a^x P^{(N-1,j)}_{(a,c)\to(s,d)}(s,y)f_{(N,j)}(s,y)\,ds$$
$$+\int_x^b P^{(N-1,j)}_{(b,c)\to(s,d)}(s,y)f_{(N,j)}(s,y)\,ds$$
$$=P^{(N,j)}_{(a,c)\to(b,d)}(x,y)f_{(N,j)}(x,y)+\int_a^b P^{(N-1,j)}_{(a,c)\to(s,d)}(s,y)f_{(N,j)}(s,y)\,ds,$$

continuing in similar fashion, we finally get

$$\int_a^b \hat{P}^{(N,j)}(x,s,y)f_{(N+1,j)}(s,y)\,ds$$
$$=\int_a^b\int_c^d p(\xi,\eta)\frac{(\eta-y)^j}{j!}\Big[\sum_{k=0}^{N}\frac{(\xi-x)^k}{k!}f_{(k,j)}(x,y)\Big]\,d\eta\,d\xi$$
$$-\int_a^b\int_c^d p(s,\eta)\frac{(\eta-y)^j}{j!}f_{(0,j)}(s,y)\,d\eta\,ds. \tag{6.41}$$

Similarly

$$\int_c^d \tilde{P}^{(i,M)}(x,y,t)f_{(i,M+1)}(x,t)\,dt$$

$$= \int_a^b \int_c^d p(\xi,\eta)\frac{(\xi-x)^i}{i!}\Big[\sum_{l=0}^{M}\frac{(\eta-y)^l}{l!}f_{(i,l)}(x,y)\Big]\,d\eta\,d\xi$$
$$-\int_a^b \int_c^d p(\xi,t)\frac{(\xi-x)^i}{i!}f_{(i,0)}(x,t)\,d\xi\,dt. \tag{6.42}$$

If we put all these values in (6.39), then after some cancelation and some rearrangements we get our required identity. □

**Theorem 6.28** *Let $f \in C^{(2N+1,2M+1)}([a,b]\times[c,d])$. Then the identity*

$$[P(a,c)]^2 f(x,y) = P(a,c)R(x,y;f) + P(a,c)\int_a^b\int_c^d p(s,t)f(s,t)dt\,ds$$
$$+\sum_{j=0}^{M}\int_a^b \hat{P}^{(N,j)}(x,s,y)R(s,y;f_{(N+1,j)})\,ds$$
$$+\sum_{i=0}^{N}\int_c^d \tilde{P}^{(i,M)}(x,y,t)R(x,t;f_{(i,M+1)})\,dt$$
$$+\sum_{i=0}^{N}\sum_{j=0}^{M}\int_a^b\int_a^b\int_c^d \hat{P}^{(N,j)}(x,s,y)p(\xi,t)\frac{(\xi-x)^i}{i!}f_{(N+1+i,j)}(s,t)\,dt\,ds\,d\xi$$
$$+\sum_{i=0}^{N}\sum_{j=0}^{M}\int_c^d\int_a^b\int_c^d \tilde{P}^{(i,M)}(x,y,t)p(s,\eta)\frac{(\eta-y)^j}{j!}f_{(i,M+1+j)}(s,t)\,dt\,ds\,d\eta$$
$$+\int_a^b\int_c^d\Big[2\sum_{i=0}^{N}\sum_{j=0}^{M}\hat{P}^{(N,j)}(x,s,y)\tilde{P}^{(i,M)}(x,y,t)f_{(N+1+i,M+1+j)}(s,t)$$
$$-\overline{P}^{(N,M)}(x,s,y,t)f_{(N+1,M+1)}(s,t)\Big]\,dt\,ds, \tag{6.43}$$

*holds, where p, P, $\hat{P}^{(N,j)}$, $\tilde{P}^{(i,M)}$ are $\overline{P}^{(N,M)}$ are as in Theorem* 6.26.

*Proof.* Summing (6.41) for $j \in \{0,\ldots,M\}$ and (6.42) for $i \in \{0,\ldots,N\}$, we get respectively for each $(x,y)\in[a,b]\times[c,d]$.

$$P(a,c)f(x,y) = R(x,y;f)+\sum_{j=0}^{M}\int_a^b\int_c^d p(s,\eta)\frac{(\eta-y)^j}{j!}f_{(0,j)}(s,y)\,d\eta\,ds$$
$$+\sum_{j=0}^{M}\int_a^b \hat{P}^{(N,j)}(x,s,y)f_{(N+1,j)}(s,y)\,ds, \tag{6.44}$$

and

$$P(a,c)f(x,y) = R(x,y;f)+\sum_{i=0}^{N}\int_a^b\int_c^d p(\xi,t)\frac{(\xi-x)^i}{i!}f_{(i,0)}(x,t)\,dt\,d\xi$$
$$+\sum_{i=0}^{N}\int_c^d \tilde{P}^{(i,M)}(x,y,t)f_{(i,M+1)}(x,t)\,dt. \tag{6.45}$$

By using formula (6.44) for partial derivatives $f_{(i,M+1)}$ for $i \in \{0,\dots,N\}$, we obtain

$$\begin{aligned}
&P(a,c)f_{(i,M+1)}(x,t) \\
&= R(x,t;f_{(i,M+1)}) + \sum_{j=0}^{M} \int_a^b \int_c^d p(s,\eta)\frac{(\eta-t)^j}{j!} f_{(i,M+1+j)}(s,t)\,d\eta\,ds \\
&+ \sum_{j=0}^{M} \int_a^b \hat{P}^{(N,j)}(x,s,t) f_{(N+1+i,M+1+j)}(s,t)\,ds.
\end{aligned} \tag{6.46}$$

Similarly, by using formula (6.45) for partial derivatives $f_{(N+1,j)}$ for $j = 0,\dots,M$ we have

$$\begin{aligned}
&P(a,c)f_{(N+1,j)}(s,y) \\
&= R(s,y;f_{(N+1,j)}) + \sum_{i=0}^{N} \int_a^b \int_c^d p(\xi,t)\frac{(\xi-s)^i}{i!} f_{(N+1+i,j)}(s,t)\,dt\,d\xi \\
&+ \sum_{i=0}^{N} \int_c^d \tilde{P}^{(i,M)}(s,y,t) f_{(N+1+i,M+1+j)}(s,t)\,dt.
\end{aligned} \tag{6.47}$$

Substituting (6.46) and (6.47) into (6.39), we get

$$\begin{aligned}
P(a,c)f(x,y) \;=\; & R(x,y;f) + \int_a^b \int_c^d p(s,t) f(s,t)\,dt\,ds \\
&+ \frac{1}{P(a,c)} \sum_{j=0}^{M} \int_a^b \hat{P}^{(N,j)}(x,s,y) \Big[ R(s,y;f_{(N+1,j)}) \\
&+ \sum_{i=0}^{N} \int_a^b \int_c^d p(\xi,t) \frac{(\xi-s)^i}{i!} f_{(N+1+i,j)}(s,t)\,dt\,d\xi \\
&+ \sum_{i=0}^{N} \int_c^d \tilde{P}^{(i,M)}(s,y,t) f_{(N+1+i,M+1+j)}(s,t)\,dt \Big]\,ds \\
&+ \frac{1}{P(a,c)} \sum_{i=0}^{N} \int_c^d \tilde{P}^{(i,M)}(x,y,t) \Big[ R(x,t;f_{(i,M+1)}) \\
&+ \sum_{j=0}^{M} \int_a^b \int_c^d p(s,\eta) \frac{(\eta-t)^j}{j!} f_{(i,M+1+j)}(s,t)\,d\eta\,ds \\
&+ \sum_{j=0}^{M} \int_a^b \hat{P}^{(N,j)}(x,s,t) f_{(N+1+i,M+1+j)}(s,t)\,ds \Big]\,dt \\
&- \int_a^b \int_c^d \overline{P}^{(N,M)}(x,s,y,t) f_{(N+1,M+1)}(s,t)\,dt\,ds.
\end{aligned}$$

After some rearrangements and using Fubini's Theorem we obtain our required result. □

**Remark 6.8** For $N = M = 0$, Theorems 6.22, 6.23 and 6.24 become special cases of Theorems 6.26, 6.27 and 6.28 respectively.

**Special Cases**:

If $p(s,t) = q(s)r(t)$ in identities (6.39), (6.40) and (6.43), then we get respectively the following special cases:

$$
\begin{aligned}
& f(x,y)P_{a\to b}(q)P_{c\to d}(r) = Q(x,y;f) + \int_a^b \int_c^d q(s)r(t)f(s,t)dt\,ds \\
& + \sum_{j=0}^{M} \int_a^b \hat{Q}^{(N,j)}(x,s,y) f_{(N+1,j)}(s,y)\,ds \\
& + \sum_{i=0}^{N} \int_c^d \tilde{Q}^{(i,M)}(x,y,t) f_{(i,M+1)}(x,t)\,dt \\
& - \int_a^b \int_c^d \bar{Q}^{(N,M)}(x,s,y,t) f_{(N+1,M+1)}(s,t)\,dt\,ds, \\
& f(x,y)P_{a\to b}(q)P_{c\to d}(r) = Q(x,y;f) + \sum_{j=1}^{M} \int_a^b q(s) f_{(0,j)}(s,y)\,ds\, Q^{(j)}_{c\to d}(r,y) \\
& + \sum_{i=1}^{N} Q^{(i)}_{a\to b}(q,x) \int_c^d r(t) f_{(i,0)}(x,t)\,dt - \int_a^b \int_c^d q(s)r(t)f(s,t)dt\,ds \\
& + \int_a^b \int_c^d q(s)r(t)f(s,y)\,dt\,ds + \int_a^b \int_c^d q(s)r(t)f(x,t)\,dt\,ds \\
& - \int_a^b \int_c^d \bar{Q}^{(N,M)}(x,s,y,t) f_{(N+1,M+1)}(s,t)\,dt\,ds, \\
& f(x,y)[P_{a\to b}(q)P_{c\to d}(r)]^2 = P_{a\to b}(q)P_{c\to d}(r)Q(x,y;f) \\
& + \sum_{j=0}^{M} \int_a^b \hat{Q}^{(N,j)}(x,s,y) Q(s,y;f_{(N+1,j)})\,ds \\
& + \sum_{i=0}^{N} \int_c^d \tilde{Q}^{(i,M)}(x,y,t) Q(x,t;f_{(i,M+1)})\,dt \\
& + P_{a\to b}(q)P_{c\to d}(r) \int_a^b \int_c^d q(s)r(t)f(s,t)dt\,ds \\
& + \sum_{i=0}^{N} \sum_{j=0}^{M} Q^{(i)}_{a\to b}(q,x) \int_a^b \int_c^d \hat{Q}^{(N,j)}(x,s,y) r(t) f_{(N+1+i,j)}(s,t)\,dt\,ds \\
& + \sum_{i=0}^{N} \sum_{j=0}^{M} Q^{(j)}_{c\to d}(r,y) \int_a^b \int_c^d \tilde{Q}^{(i,M)}(x,y,t) q(s)\, f_{(i,M+1+j)}(s,t)\,dt\,ds \\
& + \int_a^b \int_c^d \Big[ 2 \sum_{i=0}^{N} \sum_{j=0}^{M} \hat{Q}^{(N,j)}(x,s,y) \tilde{Q}^{(i,M)}(x,y,t) f_{(N+1+i,M+1+j)}(s,t) \\
& - \bar{Q}^{(N,M)}(x,s,y,t) f_{(N+1,M+1)}(s,t) \Big]\,dt\,ds,
\end{aligned}
$$

where $\quad P_{a\to b}(q) = \displaystyle\int_a^b q(s)\,ds, \qquad Q^{(i)}_{a\to b}(q,x) = \int_a^b q(\xi) \frac{(\xi - x)^i}{i!}\,d\xi,$

$$\begin{aligned} Q^{(i,j)}_{(a,c)\to(b,d)}(x,y) &= Q^{(i)}_{a\to b}(q,x)Q^{(j)}_{c\to d}(r,y),\\ Q^{(0,j)}_{(a,c)\to(b,d)}(y) &= P_{a\to b}(q)\,Q^{(j)}_{c\to d}(r,y),\\ Q^{(i,0)}_{(a,c)\to(b,d)}(x) &= Q^{(i)}_{a\to b}(q,x)\,P_{c\to d}(r), \end{aligned}$$

$$\begin{aligned} Q(x,y;f) &= -\sum_{i=1}^{N}\sum_{j=1}^{M} f_{(i,j)}(x,y)Q^{(i,j)}_{(a,c)\to(b,d)}(x,y)\\ &- \sum_{j=1}^{M} f_{(0,j)}(x,y)Q^{(0,j)}_{(a,c)\to(b,d)}(y) - \sum_{i=1}^{N} f_{(i,0)}(x,y)Q^{(i,0)}_{(a,c)\to(b,d)}(x), \end{aligned}$$

$$\hat{Q}^{(N,j)}(x,s,y) = \begin{cases} Q^{(N,j)}_{(a,c)\to(s,d)}(s,y)\ , & a\le s\le x,\\ -Q^{(N,j)}_{(s,c)\to(b,d)}(s,y)\ , & x< s\le b, \end{cases}$$

$$\tilde{Q}^{(i,M)}(x,y,t) = \begin{cases} Q^{(i,M)}_{(a,c)\to(b,t)}(x,t)\ , & c\le t\le y,\\ -Q^{(i,M)}_{(a,t)\to(b,d)}(x,t)\ , & y< t\le d, \end{cases}$$

and $$\bar{Q}^{(N,M)}(x,s,y,t) = \begin{cases} Q^{(N,M)}_{(a,c)\to(s,t)}(s,t)\ , & a\le s\le x, \quad c\le t\le y,\\ -Q^{(N,M)}_{(s,c)\to(b,t)}(s,t)\ , & x< s\le b, \quad c\le t\le y,\\ -Q^{(N,M)}_{(a,t)\to(s,d)}(s,t)\ , & a\le s\le x, \quad y< t\le d,\\ Q^{(N,M)}_{(s,t)\to(b,d)}(s,t)\ , & x< s\le b, \quad y< t\le d. \end{cases}$$

Particularly, if $p\equiv 1$ in identities (6.39), (6.40) and (6.43) then the expressions will look like

$$P_{a\to b} = b-a, \qquad Q^{(i)}_{a\to b}(x) = \frac{(b-x)^{i+1}-(a-x)^{i+1}}{(i+1)!},$$

$$\begin{aligned} Q(x,y;f) &= -\sum_{i=1}^{N}\sum_{j=1}^{M}\frac{(b-x)^{i+1}-(a-x)^{i+1}}{(i+1)!}\ \frac{(d-y)^{j+1}-(c-y)^{j+1}}{(j+1)!}\ f_{(i,j)}(x,y)\\ &- (b-a)\sum_{j=1}^{M}\frac{(d-y)^{j+1}-(c-y)^{j+1}}{(j+1)!}\ f_{(0,j)}(x,y)\\ &- (d-c)\sum_{i=1}^{N}\frac{(b-x)^{i+1}-(a-x)^{i+1}}{(i+1)!}\ f_{(i,0)}(x,y), \end{aligned}$$

$$\hat{Q}^{(N,j)}(x,s,y) = \begin{cases} -\dfrac{(a-s)^{N+1}}{(N+1)!}\ \dfrac{(d-y)^{j+1}-(c-y)^{j+1}}{(j+1)!}, & a\le s\le x,\\ -\dfrac{(b-s)^{N+1}}{(N+1)!}\ \dfrac{(d-y)^{j+1}-(c-y)^{j+1}}{(j+1)!}, & x< s\le b, \end{cases}$$

$$\tilde{Q}^{(i,M)}(x,y,t) = \begin{cases} -\dfrac{(c-t)^{M+1}}{(M+1)!} \dfrac{(b-x)^{i+1} - (a-x)^{i+1}}{(i+1)!}, & c \le t \le y, \\ -\dfrac{(d-t)^{M+1}}{(M+1)!} \dfrac{(b-x)^{i+1} - (a-x)^{i+1}}{(i+1)!}, & y < t \le d, \end{cases}$$

and $$\overline{Q}^{(N,M)}(x,s,y,t) = \begin{cases} \dfrac{(a-s)^{N+1}}{(N+1)!} \dfrac{(c-t)^{M+1}}{(M+1)!}, & a \le s \le x \quad , \quad c \le t \le y, \\ \dfrac{(b-s)^{N+1}}{(N+1)!} \dfrac{(c-t)^{M+1}}{(M+1)!}, & x < s \le b \quad , \quad c \le t \le y, \\ \dfrac{(a-s)^{N+1}}{(N+1)!} \dfrac{(d-t)^{M+1}}{(M+1)!}, & a \le s \le x \quad , \quad y < t \le d, \\ \dfrac{(b-s)^{N+1}}{(N+1)!} \dfrac{(d-t)^{M+1}}{(M+1)!}, & x < s \le b \quad , \quad y < t \le d. \end{cases}$$

### 6.2.2 Ostrowski Inequalities for Double Weighted Integrals of Higher Order Differentiable Functions

In [80], J. Pečarić and A. Vukelić also have given some generalizations of Ostrowski's inequality by using identities (6.33) and (6.34). By using identities (6.39) and (6.40) we can give generalized results of Ostrowski type for higher order differentiable functions of two independent variables as follows:

**Theorem 6.29** *Let $f \in C^{(N+1,M+1)}([a,b] \times [c,d])$. Then the inequality*

$$\left| f(x,y) - \frac{1}{P(a,c)} \int_a^b \int_c^d p(s,t) f(s,t)\, dt\, ds \right| \le D(x,y) + \sum_{j=0}^{M} \hat{D}^{(0,j)}(x,y)$$
$$+ \sum_{i=0}^{N} \tilde{D}^{(i,0)}(x,y) + \overline{D}(x,y), \tag{6.48}$$

*holds for each $(x,y) \in [a,b] \times [c,d]$, where*

$$D(x,y) = \frac{1}{|P(a,c)|} |R(x,y;f)|,$$

$$\hat{D}^{(0,j)}(x,y) = \frac{1}{|P(a,c)|} \left( \sum_{j=0}^{M} \int_a^b |\hat{P}^{(N,j)}(x,s,y)|^{\hat{q}_j} ds \right)^{1/\hat{q}_j} \|f_{(N+1,j)}\|_{\hat{p}_j},$$

*provided that $f_{(N+1,j)} \in L_{\hat{p}_j}([a,b] \times [c,d])$, $1/\hat{p}_j + 1/\hat{q}_j = 1$,*

$$\tilde{D}^{(i,0)}(x,y) = \frac{1}{|P(a,c)|} \left( \sum_{i=0}^{N} \int_c^d |\tilde{P}^{(i,M)}(x,y,t)|^{\tilde{q}_i} dt \right)^{1/\tilde{q}_i} \|f_{(i,M+1)}\|_{\tilde{p}_i},$$

*provided that $f_{(i,M+1)} \in L_{\tilde{p}_i}([a,b] \times [c,d])$, $1/\tilde{p}_i + 1/\tilde{q}_i = 1$,*

$$\overline{D}(x,y) = \frac{1}{|P(a,c)|} \left( \int_a^b \int_c^d |\overline{P}^{(N,M)}(x,s,y,t)|^{\overline{q}} dt\, ds \right)^{1/\overline{q}} \|f_{(N+1,M+1)}\|_{\overline{p}},$$

*provided that* $f_{(N+1,M+1)} \in L_{\bar{p}}([a,b]\times[c,d])$, $1/\bar{p}+1/\bar{q}=1$,

*where* $p$, $P$, $\hat{P}^{(N,j)}$, $\tilde{P}^{(i,M)}$ *and* $\overline{P}^{(N,M)}$ *are as in Theorem* 6.26 *whereas* $R$ *is defined in* (6.38).

*Proof.* Identity (6.39) can be rewritten as

$$\begin{aligned}
f(x,y) &- \frac{1}{P(a,c)}\int_a^b\int_c^d p(s,t)f(s,t)\,dt\,ds \\
&= \frac{1}{P(a,c)}\Big[R(x,y;f)+\sum_{j=0}^{M}\int_a^b \hat{P}^{(N,j)}(x,s,y)f_{(N+1,j)}(s,y)\,ds \\
&+ \sum_{i=0}^{N}\int_c^d \tilde{P}^{(i,M)}(x,y,t)f_{(i,M+1)}(x,t)\,dt \\
&- \int_a^b\int_c^d \overline{P}^{(N,M)}(x,s,y,t)f_{(N+1,M+1)}(s,t)\,dt\,ds\Big].
\end{aligned}$$

Now, taking absolute value and applying the Hölder inequality for double integrals, we easily obtain our required inequality. □

**Remark 6.9** For $N=M=0$, Theorem 4 of [80] becomes special case of Theorem 6.29 and we also retrieve results of [14] by simply putting $p\equiv 1$.

**Theorem 6.30** *Let* $f:[a,b]\times[c,d]\to\mathbb{R}$ *be a continuous function such that* $f\in C^{(N+1,M+1)}((a,b)\times(c,d))$ *and* $|f_{(N+1,M+1)}|^q$ *be an integrable function such that*

$$\left\|f_{(N+1,M+1)}\right\|_q := \left(\int_a^b\int_c^d |f_{(N+1,M+1)}(s,t)|^q\,dt\,ds\right)^{1/q} < \infty.$$

*Then the inequality*

$$\begin{aligned}
&\Big|\int_a^b\int_c^d p(s,t)f(x,t)\,dt\,ds - \Big[R(x,y;f) \\
&+\sum_{j=1}^{M}\int_a^b\int_c^d p(s,\eta)\frac{(\eta-y)^j}{j!}f_{(0,j)}(s,y)\,d\eta\,ds \\
&+\sum_{i=1}^{N}\int_a^b\int_c^d p(\xi,t)\frac{(\xi-x)^i}{i!}f_{(i,0)}(x,t)\,dt\,d\xi \\
&+\int_a^b\int_c^d p(s,t)f(x,t)\,dt\,ds \\
&+\int_a^b\int_c^d p(s,t)f(s,y)dt\,ds - P(a,c)f(x,y)\Big]\Big| \\
&\le \left(\int_a^b\int_c^d |\overline{P}^{(N,M)}(x,s,y,t)|\,dt\,ds\right)^{1/q}\|f_{(N+1,M+1)}\|_{q'}.
\end{aligned}$$

*holds for each* $(x,y)\in[a,b]\times[c,d]$, *where* $1/q+1/q'=1$; $q,q'>1$ *and* $P$, $\overline{P}^{(N,M)}$ *are as in Theorem* 6.26.

*Proof.* Identity (6.40) may be rewritten as

$$\begin{aligned}
&\int_a^b \int_c^d p(s,t) f(s,t)\, dt\, ds - \Big[ R(x,y;f) \\
&+ \int_a^b \int_c^d p(s,t) f(s,y)\, dt\, ds + \int_a^b \int_c^d p(s,t) f(x,t) dt\, ds \\
&+ \sum_{j=1}^{M} \int_a^b \int_c^d p(s,\eta) \frac{(\eta - y)^j}{j!} f_{(0,j)}(s,y)\, d\eta\, ds \\
&+ \sum_{i=1}^{N} \int_a^b \int_c^d p(\xi,t) \frac{(\xi - x)^i}{i!} f_{(i,0)}(x,t)\, dt\, d\xi - P(a,c) f(x,y) \Big] \\
&= \int_a^b \int_c^d \overline{P}^{(N,M)}(x,s,y,t) f_{(N+1,M+1)}(s,t)\, dt\, ds.
\end{aligned}$$

Now, taking absolute value and applying Hölder's inequality for double integrals, we easily obtain our required inequality. □

**Remark 6.10** For $N = M = 0$, Theorem 5 of [80] becomes special case of Theorem 6.30 and we also retrieve results of [5] and [13] by simply putting $p \equiv 1$.

### 6.2.3 Grüss' Inequalities for Double Weighted Integrals of Higher Order Differentiable Functions

In [80], J. Pečarić and A. Vukelić gave new Grüss-type inequalities for double weighted integrals by using identities (6.33) and (6.34). Now, we give more generalized results by using higher order differentiable functions of two independent variables but in order to simplify the details of the presentations we define the following notations.

$$\begin{aligned}
A^{(i,j)}(x,y) &= p(x,y)[f_{(i,j)}(x,y) g(x,y) \\
&+ g_{(i,j)}(x,y) f(x,y)] P^{(i,j)}_{(a,c)\to(b,d)}(x,y),
\end{aligned} \tag{6.49}$$

$$A(x,y) = p(x,y) \int_a^b \int_c^d p(s,t)[f(s,t) g(x,y) + g(s,t) f(x,y)]\, dt\, ds, \tag{6.50}$$

$$\begin{aligned}
\hat{A}^{(N,j)}(x,y) &= p(x,y) \int_a^b [f_{(N+1,j)}(s,y) g(x,y) + g_{(N+1,j)}(s,y) f(x,y)] \times \\
&\times \hat{P}^{(N,j)}(x,s,y)\, ds,
\end{aligned} \tag{6.51}$$

$$\begin{aligned}
\tilde{A}^{(i,M)}(x,y) &= p(x,y) \int_c^d [f_{(i,M+1)}(x,t) g(x,y) + g_{(i,M+1)}(x,t) f(x,y)] \times \\
&\times \tilde{P}^{(i,M)}(x,y,t)\, dt,
\end{aligned} \tag{6.52}$$

$$\begin{aligned}
\overline{A}^{(N,M)}(x,y) &= p(x,y) \int_a^b \int_c^d \overline{P}^{(N,M)}(x,s,y,t) \times \\
&\times [f_{(N+1,M+1)}(s,t) g(x,y) + g_{(N+1,M+1)}(s,t) f(x,y)]\, dt\, ds,
\end{aligned} \tag{6.53}$$

$$
\begin{aligned}
B^{(i,j)}(x,y) &= |p(x,y)g(x,y)|\,\|f_{(i,j)}(x,y)\|_\infty \\
&+ |p(x,y)f(x,y)|\,\|g_{(i,j)}(x,y)\|_\infty,
\end{aligned} \tag{6.54}
$$

$$
C^{(i,j)}(x,y) = \frac{(\max\{b-x,x-a\})^{i+1}}{(i+1)!}\,\frac{(\max\{d-y,y-c\})^{j+1}}{(j+1)!}\times \tag{6.55}
$$

$$
\times \int_a^b \int_c^d |p(\xi,\eta)|\,d\eta\,d\xi, \tag{6.56}
$$

$$
C^{(0,j)}(y) = (b-a)\frac{(\max\{d-y,y-c\})^{j+1}}{(j+1)!}\int_a^b \int_c^d |p(\xi,\eta)|\,d\eta\,d\xi, \tag{6.57}
$$

$$
C^{(i,0)}(x) = (d-c)\frac{(\max\{b-x,x-a\})^{i+1}}{(i+1)!}\int_a^b \int_c^d |p(\xi,\eta)|\,d\eta\,d\xi, \tag{6.58}
$$

$$
\hat{C}^{(N,j)}(x,y) = \int_a^b |\hat{P}^{(N,j)}(x,s,y)|\,ds, \tag{6.59}
$$

$$
\tilde{C}^{(i,M)}(x,y) = \int_c^d |\tilde{P}^{(i,M)}(x,y,t)|\,dt, \tag{6.60}
$$

$$
\overline{C}^{(N,M)}(x,y) = \int_a^b \int_c^d |\overline{P}^{(N,M)}(x,s,y,t)|\,dt\,ds, \tag{6.61}
$$

$$
\begin{aligned}
F(x,y) &= R(x,y;f) + \int_a^b \int_c^d p(s,t)f(s,y)dtds + \int_a^b \int_c^d p(s,t)f(x,t)dtds \\
&+ \sum_{j=1}^{M} \int_a^b \int_c^d p(s,\eta)\frac{(\eta-y)^j}{j!} f_{(0,j)}(s,y)\,d\eta\,ds \\
&+ \sum_{i=1}^{N} \int_a^b \int_c^d p(\xi,t)\frac{(\xi-x)^i}{i!} f_{(i,0)}(x,t)\,dt\,d\xi,
\end{aligned} \tag{6.62}
$$

$$
\begin{aligned}
G(x,y) &= R(x,y;g) + \int_a^b \int_c^d p(s,t)g(s,y)dtds + \int_a^b \int_c^d p(s,t)g(x,t)dtds \\
&+ \sum_{j=1}^{M} \int_a^b \int_c^d p(s,\eta)\frac{(\eta-y)^j}{j!} g_{(0,j)}(s,y)\,d\eta\,ds \\
&+ \sum_{i=1}^{N} \int_a^b \int_c^d p(\xi,t)\frac{(\xi-x)^i}{i!} g_{(i,0)}(x,t)\,dt\,d\xi,
\end{aligned} \tag{6.63}
$$

where $f,g \in C^{(N+1,M+1)}([a,b]\times[c,d])$ and $p$, $P$, $\hat{P}^{(N,j)}$, $\tilde{P}^{(i,M)}$ and $\overline{P}^{(N,M)}$ are as in Theorem 6.26 whereas $R$ is defined in (6.38).

Now, we present our main results of this section by using notations introduced earlier in this section, which are as follows:

**Theorem 6.31** *Let $p:[a,b]\times[c,d]\to\mathbb{R}$ be an integrable function and let $f$, $g \in C^{(N+1,M+1)}([a,b]\times[c,d])$. Then the inequality*

$$
\left| \frac{1}{P(a,c)} \int_a^b \int_c^d p(x,y)f(x,y)g(x,y)\,dy\,dx \right.
$$

$$- \left( \frac{1}{P(a,c)} \int_a^b \int_c^d p(x,y) f(x,y)\, dy\, dx \right) \left( \frac{1}{P(a,c)} \int_a^b \int_c^d p(x,y) g(x,y)\, dy\, dx \right) \Bigg|$$

$$\leq \frac{1}{2[P(a,c)]^2} \int_a^b \int_c^d \Big[ \sum_{i=1}^N \sum_{j=1}^M B^{(i,j)}(x,y) C^{(i,j)}(x,y)$$
$$+ \sum_{j=1}^M B^{(0,j)}(y) C^{(0,j)}(y) + \sum_{i=1}^N B^{(i,0)}(x) C^{(i,0)}(x) + B^{(N+1,j)}(x,y) \hat{C}^{(N,j)}(x,y)$$
$$+ B^{(i,M+1)}(x,y) \tilde{C}^{(i,M)}(x,y) + B^{(N+1,M+1)}(x,y) \overline{C}^{(N,M)}(x,y) \Big]\, dy\, dx$$

*holds, where P is defined in* (6.32).

*Proof.* From (6.39) for $(x,y) \in [a,b] \times [c,d]$, we have

$$\begin{aligned} P(a,c) f(x,y) &= R(x,y;f) + \int_a^b \int_c^d p(s,t) f(s,t) dt\, ds \\ &+ \sum_{j=0}^M \int_a^b \hat{P}^{(N,j)}(x,s,y) f_{(N+1,j)}(s,y)\, ds \\ &+ \sum_{i=0}^N \int_c^d \tilde{P}^{(i,M)}(x,y,t) f_{(i,M+1)}(x,t)\, dt \\ &- \int_a^b \int_c^d \overline{P}^{(N,M)}(x,s,y,t) f_{(N+1,M+1)}(s,t)\, dt\, ds, \end{aligned} \tag{6.64}$$

$$\begin{aligned} P(a,c) g(x,y) &= R(x,y;g) + \int_a^b \int_c^d p(s,t) g(s,t) dt\, ds \\ &+ \sum_{j=0}^M \int_a^b \hat{P}^{(N,j)}(x,s,y) g_{(N+1,j)}(s,y)\, ds \\ &+ \sum_{i=0}^N \int_c^d \tilde{P}^{(i,M)}(x,y,t) g_{(i,M+1)}(x,t)\, dt \\ &- \int_a^b \int_c^d \overline{P}^{(N,M)}(x,s,y,t) g_{(N+1,M+1)}(s,t)\, dt\, ds. \end{aligned} \tag{6.65}$$

Now, if we multiply (6.64) by $p(x,y)g(x,y)$ and (6.65) by $p(x,y)f(x,y)$ and add them, then we obtain

$$\begin{aligned} 2P(a,c) p(x,y) f(x,y) g(x,y) &= - \sum_{i=1}^N \sum_{j=1}^M A^{(i,j)}(x,y) - \sum_{j=1}^M A^{(0,j)}(y) \\ &- \sum_{i=1}^N A^{(i,0)}(x) + A(x,y) + \hat{A}^{(N,j)}(x,y) \\ &+ \tilde{A}^{(i,M)}(x,y) - \overline{A}^{(N,M)}(x,y). \end{aligned} \tag{6.66}$$

If we integrate (6.66) over $[a,b]\times[c,d]$ and divide both sides by $2P(a,c)$, then we get

$$\begin{aligned}
&\int_a^b\int_c^d p(x,y)f(x,y)g(x,y)\,dy\,dx\\
&=\frac{1}{2P(a,c)}\int_a^b\int_c^d\Big[-\sum_{i=1}^{N}\sum_{j=1}^{M}A^{(i,j)}(x,y)-\sum_{j=1}^{M}A^{(0,j)}(y)\\
&-\sum_{i=1}^{N}A^{(i,0)}(x)+A(x,y)+\hat{A}^{(N,j)}(x,y)+\tilde{A}^{(i,M)}(x,y)-\overline{A}^{(N,M)}(x,y)\Big]\,dy\,dx.
\end{aligned}$$

It can be rewritten as

$$\begin{aligned}
&\frac{1}{P(a,c)}\int_a^b\int_c^d p(x,y)f(x,y)g(x,y)\,dy\,dx\\
&-\left(\frac{1}{P(a,c)}\int_a^b\int_c^d p(x,y)f(x,y)\,dy\,dx\right)\left(\frac{1}{P(a,c)}\int_a^b\int_c^d p(x,y)g(x,y)\,dy\,dx\right)\\
&=\frac{1}{2[P(a,c)]^2}\int_a^b\int_c^d\Big[-\sum_{i=1}^{N}\sum_{j=1}^{M}A^{(i,j)}(x,y)-\sum_{j=1}^{M}A^{(0,j)}(y)\\
&-\sum_{i=1}^{N}A^{(i,0)}(x)+\hat{A}^{(N,j)}(x,y)+\tilde{A}^{(i,M)}(x,y)-\overline{A}^{(N,M)}(x,y)\Big]\,dy\,dx. \qquad (6.67)
\end{aligned}$$

Using $(6.49),\ldots,(6.61)$, we have the following inequalities for all $(x,y)\in[a,b]\times[c,d]$

$$\begin{aligned}
|A^{(i,j)}(x,y)| &\le B^{(i,j)}(x,y)\,C^{(i,j)}(x,y),\\
|A^{(0,j)}(y)| &\le B^{(0,j)}(y)\,C^{(0,j)}(y),\\
|A^{(i,0)}(x)| &\le B^{(i,0)}(x)\,C^{(i,0)}(x),\\
|\hat{A}^{(N,j)}(x,y)| &\le B^{(N+1,j)}(x,y)\,\hat{C}^{(N,j)}(x,y),\\
|\tilde{A}^{(i,M)}(x,y)| &\le B^{(i,M+1)}(x,y)\,\tilde{C}^{(i,M)}(x,y),\\
|\overline{A}^{(N,M)}(x,y)| &\le B^{(N+1,M+1)}(x,y)\,\overline{C}^{(N,M)}(x,y).
\end{aligned}$$

Taking absolute value on both sides in (6.67) and using all these inequalities in it, we get our required result. □

**Theorem 6.32** *Let the assumptions of Theorem* 6.31 *be valid. Then the inequality*

$$\begin{aligned}
&\Bigg|\frac{1}{P(a,c)}\int_a^b\int_c^d p(x,y)f(x,y)g(x,y)\,dy\,dx\\
&+\left(\frac{1}{P(a,c)}\int_a^b\int_c^d p(x,y)f(x,y)\,dy\,dx\right)\left(\frac{1}{P(a,c)}\int_a^b\int_c^d p(x,y)g(x,y)\,dy\,dx\right)\\
&-\frac{1}{2[P(a,c)]^2}\int_a^b\int_c^d p(x,y)[g(x,y)F(x,y)+f(x,y)G(x,y)]\,dy\,dx\Bigg|\\
&\le\frac{1}{2[P(a,c)]^2}\int_a^b\int_c^d B^{(N+1,M+1)}(x,y)\,\overline{C}^{(N,M)}(x,y)\,dy\,dx
\end{aligned}$$

*holds, where P is defined in* (6.32).

*Proof.* From (6.40) for $(x,y) \in [a,b] \times [c,d]$ we have

$$\begin{aligned} P(a,c)f(x,y) &= F(x,y) - \int_a^b \int_c^d p(s,t)f(s,t)dt\,ds \\ &+ \int_a^b \int_c^d \overline{P}^{(N,M)}(x,s,y,t)f_{(N+1,M+1)}(s,t)\,dt\,ds, \quad (6.68) \\ P(a,c)g(x,y) &= G(x,y) - \int_a^b \int_c^d p(s,t)g(s,t)dt\,ds \\ &+ \int_a^b \int_c^d \overline{P}^{(N,M)}(x,s,y,t)g_{(N+1,M+1)}(s,t)\,dt\,ds. \quad (6.69) \end{aligned}$$

If we multiply (6.68) by $p(x,y)g(x,y)$ and (6.69) by $p(x,y)f(x,y)$ and add them, then we get

$$\begin{aligned} 2P(a,c)p(x,y)f(x,y)g(x,y) &= p(x,y)g(x,y)F(x,y) + p(x,y)f(x,y)G(x,y) \\ &- A(x,y) + \overline{A}^{(N,M)}(x,y). \quad (6.70) \end{aligned}$$

If we integrate (6.70) over $[a,b] \times [c,d]$ and divide both sides by $2P(a,c)$, then we get

$$\begin{aligned} &\int_a^b \int_c^d p(x,y)f(x,y)g(x,y)\,dy\,dx \\ &= \frac{1}{2P(a,c)} \int_a^b \int_c^d p(x,y)[g(x,y)F(x,y) + f(x,y)G(x,y)]\,dy\,dx \\ &- \frac{1}{P(a,c)} \left( \int_a^b \int_c^d p(x,y)f(x,y)\,dy\,dx \right) \left( \int_a^b \int_c^d p(x,y)g(x,y)\,dy\,dx \right) \\ &+ \frac{1}{2P(a,c)} \int_a^b \int_c^d \overline{A}^{(N,M)}(x,y)\,dy\,dx. \quad (6.71) \end{aligned}$$

Also we have

$$|\overline{A}^{(N,M)}(x,y)| \leq B^{(N+1,M+1)}(x,y)\,\overline{C}^{(N,M)}(x,y). \quad (6.72)$$

From (6.71) and (6.72), we obtain our required inequality. □

**Remark 6.11** For $N = M = 0$, Theorems 6 and 7 of [80] become special cases of Theorems 6.31 and 6.32 respectively and we also retrieve results of [53] by simply putting $p \equiv 1$. For $N = M = 0$, we can also find similar results as given in [21].

## 6.3 Inequalities for the Čebyšev Functional Involving Higher Order Derivatives

Suppose that $\mu$ is normalized (signed) measure on the interval $[0,1]$ and that $L^1(\mu)$ is a space of integrable functions with respect to the measure $\mu$. For $f,g,fg \in L^1(\mu)$, the Čebyšev functional is defined by

$$T(f,g;\mu) = \int_0^1 fg\,d\mu - \int_0^1 f\,d\mu \int_0^1 g\,d\mu. \tag{6.73}$$

Majority of problems involving the Čebyšev functional are to give a lower bound or an upper bound for $T$ under various assumptions (Čebyšev inequalities, Grüss inequality, etc.)(see [51]). Usually the main step in obtaining such a type of estimation is to prove an appropriate identity for $T$ and one of the basic properties of the functional $T$ is abundance of identities (Korkine's identity, Sonin's identity etc.).

Our main goal is to give a general form of the identity which started with J. Pečarić in [64], which in our notation can be formulated in the following form:

$$T(f,g;\mu) = \int_0^1 \left[ R_1(x)\int_0^x L_1(t)g'(t)dt + L_1(x)\int_x^1 R_1(t)g'(t)dt \right] f'(x)dx \tag{6.74}$$

where $f'$ and $g'$ are integrable on $[0,1]$ and $L_1(x) = \int_0^x d\mu$, $R_1(x) = \int_x^1 d\mu$. The second step was done by A. M. Fink in [19] where he showed that

$$\begin{aligned} T(f,g;\mu) &= \int_0^1 f'(x)g'(x)\left(R_1(x)L_2(x) + L_1(x)R_2(x)\right)dx \\ &- \int_0^1 \left( R_2(x)f''(x)\int_0^x g''(t)L_2(t)dt + R_2(x)g''(x)\int_0^x f''(t)L_2(t)dt \right) dx \end{aligned} \tag{6.75}$$

where $f''$, $g''$ are integrable on $[0,1]$ and $L_2(x) = \int_0^x L_1(t)dt$, $R_2(x) = \int_x^1 R_1(t)dt$.

The applications are mostly inspired by A. M. Fink's expository paper [19]. We give a unified approach to establishing upper bounds of the Čebyšev functional of functions, derivatives which belong to $L^p$ spaces. These results are given in [74].

### 6.3.1 The Main Identity

In the following we assume that $\mu$ is the (signed) normalized measure on $[0,1]$. We define sequences $(L_n)$, $(R_n)$ of functions on $[0,1]$ by:

$$\begin{aligned} L_1(x) &= \int_0^x d\mu,\ \ L_n(x) = \int_0^x L_{n-1}(x)dx,\ \ n \geq 2 \\ R_1(x) &= \int_x^1 d\mu,\ \ R_n(x) = \int_x^1 R_{n-1}(x)dx,\ \ n \geq 2. \end{aligned} \tag{6.76}$$

We also use the following kernels for $n,\ m \geq 1$:

$$k_{n,m}(x,t) = \begin{cases} R_n(x)L_m(t), \ 0 \leq x \leq t \\ L_n(x)R_m(t), \ x < t \leq 1 \end{cases}$$

$$K_{n,m}(x,t) = \begin{cases} R_n(x)L_m(t), \ 0 \leq x \leq t \\ -L_n(x)R_m(t), \ x < t \leq 1 \end{cases}$$

The following lemma contains the key technical identities.

**Lemma 6.1** *Suppose that $f$ and $g$ are differentiable functions on $(0,1)$, such that $f'$ and $g'$ are integrable on $[0,1]$. The following identities hold:*

$$\int_0^1 \int_0^1 k_{n,m}(x,t) f(x) g(t) dt dx$$
$$= \int_0^1 [R_n L_{m+1} + L_n R_{m+1}] fg - \int_0^1 \int_0^1 K_{n,m+1}(x,t) f(x) g'(t) dt dx,$$

$$\int_0^1 \int_0^1 k_{n,m}(x,t) f(x) g(t) dt dx$$
$$= \int_0^1 [R_{n+1} L_m + L_{n+1} R_m] fg + \int_0^1 \int_0^1 K_{n+1,m}(x,t) f'(x) g(t) dt dx,$$

$$\int_0^1 \int_0^1 K_{n,m}(x,t) f(x) g(t) dt dx$$
$$= \int_0^1 [R_n L_{m+1} - L_n R_{m+1}] fg - \int_0^1 \int_0^1 k_{n,m+1}(x,t) f(x) g'(t) dt dx,$$

$$\int_0^1 \int_0^1 K_{n,m}(x,t) f(x) g(t) dt dx$$
$$= \int_0^1 [R_{n+1} L_m - L_{n+1} R_m] fg + \int_0^1 \int_0^1 k_{n+1,m}(x,t) f'(x) g(t) dt dx.$$

*Proof.* Using integration by parts, we have

$$\int_0^1 K_{n,m+1}(x,t) g'(t) dt = -\int_0^1 g(t) dK_{n,m+1}(x,t). \tag{6.77}$$

Obviously $\frac{\partial K_{n,m+1}}{\partial t} = k_{n,m}(x,t)$ for $t \neq x$ and $K_{n,m+1}(x,x+0) - K_{n,m+1}(x,x-0) = -L_n(x)R_{m+1}(x)$ $-R_n(x)L_{m+1}(x)$, so by decomposition of the second integral in (6.77) in the (absolutely) continuous part and the singular (discrete) part, we obtain

$$\int_0^1 K_{n,m+1}(x,t) g'(t) dt$$

$$= [R_n(x)L_{m+1}(x) + L_n(x)R_{m+1}(x)]\, g(x) - \int_0^1 k_{n,m}(x,t)g(t)dt. \tag{6.78}$$

The first identity now follows by multiplying (6.78) by $f(x)$ and by integration.

The proofs of the second, third and fourth identity are analogous. □

Our main identity, which is a generalization of the Pečarić identity (6.74) and the Fink identity (6.75), is contained in the following theorem.

**Theorem 6.33** *Let $m$ and $n$ be natural numbers. Let $((m_i,n_i))$, $i=1,\ldots,m+n-1$ be a sequence of pairs of natural numbers such that $(m_i)$ and $(n_i)$ are nondecreasing sequences, $m_1=n_1=1$, $m_{m+n-1}=m$, $n_{m+n-1}=n$ and $m_i+n_i=m_{i-1}+n_{i-1}+1$, $i=2,\ldots,m+n-1$. If $f$ and $g$ are functions such that $g^{(m)}$ and $f^{(n)}$ are integrable on $[0,1]$, then*

$$\begin{aligned} &T(f,g;\mu) \\ &= \sum_{i=1}^{m+n-2} (-1)^{m_i+1} \int_0^1 \left[R_{n_{i+1}}L_{m_{i+1}} + (-1)^{m_i+n_i} L_{n_{i+1}}R_{m_{i+1}}\right] g^{(m_i)} f^{(n_i)} \\ &+(-1)^{m+1} \int_0^1\int_0^1 \overline{k}_{n,m}(x,t)g^{(m)}(t)f^{(n)}(x)dtdx, \end{aligned} \tag{6.79}$$

*where $\overline{k}_{n,m}=k_{n,m}$ for $m+n$ even and $\overline{k}_{n,m}=K_{n,m}$ for $m+n$ odd.*

*Proof.* The proof is by induction. For $m+n=2$ identity (6.79) is equal to $T(f,g;\mu)=\int_0^1\int_0^1 k_{1,1}(x,t)f'(x)g'(t)dtdx$ and this is the Pečarić identity (6.74). For the induction step, we consider two cases: $m+n$ even, and $m+n$ odd. Suppose that $m+n$ is even. In this case also, we consider two subcases: $(m_{n+m},n_{n+m})=(m+1,n)$, and $(m_{n+m},n_{n+m})=(m,n+1)$. Suppose that $(m_{n+m},n_{n+m})=(m+1,n)$. Assume that identity (6.79) holds for the sequence $((m_i,n_i))$ with $(m_{m+n-1},n_{m+n-1})=(m,n)$. Then

$$\begin{aligned} &T(f,g;\mu) \\ &= \sum_{i=1}^{m+n-2} (-1)^{m_i+1} \int_0^1 \left[R_{n_{i+1}}L_{m_{i+1}} + (-1)^{m_i+n_i} L_{n_{i+1}}R_{m_{i+1}}\right] g^{(m_i)} f^{(n_i)} \\ &+(-1)^{m+1} \int_0^1\int_0^1 k_{n,m}(x,t)g^{(m)}(t)f^{(n)}(x)dtdx \\ &= \sum_{i=1}^{m+n-2} (-1)^{m_i+1} \int_0^1 \left[R_{n_{i+1}}L_{m_{i+1}} + (-1)^{m_i+n_i} L_{n_{i+1}}R_{m_{i+1}}\right] g^{(m_i)} f^{(n_i)} \\ &+(-1)^{m+1} \left[\int_0^1 [R_nL_{m+1}+L_nR_{m+1}]\, g^{(m)} f^{(n)} \right. \\ &\left. - \int_0^1\int_0^1 K_{n,m+1}(x,t)g^{(m+1)}(t)f^{(n)}(x)dtdx\right] \\ &= \sum_{i=1}^{m+n-1} (-1)^{m_i+1} \int_0^1 \left[R_{n_{i+1}}L_{m_{i+1}} + (-1)^{m_i+n_i} L_{n_{i+1}}R_{m_{i+1}}\right] g^{(m_i)} f^{(n_i)} \\ &+(-1)^{m+2} \int_0^1\int_0^1 K_{n,m+1}(x,t)g^{(m+1)}(t)f^{(n)}(x)dtdx, \end{aligned} \tag{6.80}$$

where the second equality follows from the first identity in Lemma 6.1 and the last equality follows from the properties of the sequence $((m_i,n_i))$ ($m+n$ even number, $m_{m+n-1}=m$, $m_{m+n}=m+1$).

The proofs of the remaining cases follow the analogous arguing. □

The reason for introducing the sequence $((m_i,n_i))$ in Theorem 6.33 is that the $i$-th member of the summation in (6.80) is determined by $i$-th member (derivates) and $(i+1)$-th member (expressions in square brackets) of the sequence $((m_i,n_i))$.

The simplest ("diagonal") case is given in the following corollary.

**Corollary 6.6** *Let n be a natural number. If f and g are functions such that $f^{(n)}$ and $g^{(n)}$ are integrable on $[0,1]$, then*

$$T(f,g;\mu)=\sum_{i=1}^{n-1}(-1)^{i+1}\int_0^1 [R_iL_{i+1}+L_iR_{i+1}]\,g^{(i)}f^{(i)}$$
$$+(-1)^{n+1}\int_0^1\int_0^1 k_{n,n}(x,t)g^{(n)}(t)f^{(n)}(x)dtdx \tag{6.81}$$

*Proof.* Apply Theorem 6.33 using the sequence $((m_i,n_i)))$ defined by: $m_{2i-1}=n_{2i-1}=i,\ i=1,...,n,\ m_{2i}=i+1,\ n_{2i}=i,\ i=1,...,n-1$. □

Notice that one can obtain identity (6.80) defined by one sequence from the identity defined by some other sequence using identities ($i$, $j$ natural numbers)

$$\begin{aligned}
&\int_0^1 \left[R_iL_{j+1}+(-1)^{i+j}L_iL_{j+1}\right] f^{(i)}g^{(j)}\\
&-\int_0^1 \left[R_{i+1}L_{j+1}+(-1)^{i+j+1}L_{i+1}L_{j+1}\right] f^{(i)}g^{(j+1)}\\
&=\int_0^1 \left[R_{i+1}L_j+(-1)^{i+j}L_{i+1}L_j\right] f^{(i)}g^{(j)}\\
&+\int_0^1 \left[R_{i+1}L_{j+1}+(-1)^{i+j+1}L_{i+1}L_{j+1}\right] f^{(i+1)}g^{(j)}
\end{aligned} \tag{6.82}$$

which can be easily established by integration by parts.

### 6.3.2 Applications

In this section we mostly follow the ideas given in [19] and [64]. For simplicity we give applications in "diagonal" case, i.e. where (6.81) holds. Set:

$$I_n=(-1)^{n-1}\left[T(f,g;\mu)-\sum_{i=1}^{n-1}(-1)^{i+1}\int_0^1 [R_iL_{i+1}+L_iR_{i+1}]\,f^{(i)}g^{(i)}\right]. \tag{6.83}$$

Notice that, according to Corollary 6.6, $I_n=\int_0^1\int_0^1 k_{n,n}(x,t)f^{(n)}(x)g^{(n)}(t)dtdx$.

**Theorem 6.34** *Suppose that $f^{(n-1)}$ and $g^{(n-1)}$ are monotone in the same sense and concave functions and $L_{n-1},\ R_n \geq 0$. Then*

$$I_n \leq M_n \left[f^{(n-1)}(1) - f^{(n-1)}(0)\right]\left[g^{(n-1)}(1) - g^{(n-1)}(0)\right] \tag{6.84}$$

*where $M_n = \max_{x\in[0,1]} R_n(x)L_{n+1}(x)/x$.*

*Proof.* Notice that

$$\begin{aligned}\int_0^1\int_0^1 k_{n,n}(x,t)f^{(n)}(x)g^{(n)}(t)dxdt \;&=\; \int_0^1 R_n(x)\left[f^{(n)}(x)\int_0^x L_n(t)g^{(n)}(t)dt\right.\\ &+\; \left. g^{(n)}(x)\int_0^x L_n(t)f^{(n)}(t)dt\right]dx.\end{aligned} \tag{6.85}$$

The rest of the proof is as in [19, Th.12]. □

In the case of the Lebesgue measure $d\mu = dx$

$$M_n = \max_{x\in[0,1]} R_n(x)L_{n+1}(x)/x = \frac{1}{4^n}\frac{1}{n!(n+1)!}.$$

Notice that $M_1 = 1/8$ which is not the best possible estimation. We give the following proof for obtaining the best possible estimation in the case $n=1$. If $f^{(n)}$ and $g^{(n)}$ are nonnegative and decreasing, then

$$\begin{aligned}I_n &= \int_0^1\int_0^1 k_{n,n}(x,t)g^{(n)}(t)f^{(n)}(x)dtdx\\ &= \int_0^1\left[R_n(x)f^{(n)}(x)\int_0^x L_n(t)g^{(n)}(t)dt + L_n(x)f^{(n)}(x)\int_x^1 R_n(t)g^{(n)}(t)dt\right]dx\\ &\leq \left[g^{(n-1)}(1) - g^{(n-1)}(0)\right]\int_0^1 L_n(x)R_n(x)f^{(n)}(x)dx.\end{aligned} \tag{6.86}$$

Since $f^{(n)}$ is decreasing and $P_{2n}(x) = (n!)^2L_n(x)R_n(x) = x^n(1-x)^n$ is symmetric with respect to $x = 1/2$, we have

$$\begin{aligned}&\int_0^1 P_{2n}(x)f^{(n)}(x)dx \leq 2\int_0^{1/2} P_{2n}(x)f^{(n)}(x)dx\\ &\leq \int_0^{1/2} P_{2n}(x)dx\int_0^{1/2} f^{(n)}(x)dx = \frac{1}{2}\frac{(n!)^2}{(2n+1)!}\left[f^{(n-1)}(1/2) - f^{(n-1)}(0)\right].\end{aligned} \tag{6.87}$$

Using (6.86) and (6.87),

$$I_n \leq M_n'\left[g^{(n-1)}(1) - g^{(n-1)}(0)\right]\left[f^{(n-1)}(1/2) - f^{(n-1)}(0)\right],$$

where $M_n' = \frac{1}{2(2n+1)!}$. Notice that $M_1' = 1/12$ which gives the best possible estimation [19, Th.13]. It is easy to see that $M_n' < M_n$ for $n = 1,2,3$ and $M_n' > M_n$ for $n \geq 4$.

Our second application is in estimating $|I_n|$ using Hölder's inequality. This can be done in various ways.The Lebesgue measure $d\mu = dx$ is assumed, although analogous estimations can be given for a general measure, but without explicit calculations. Set: $\|f\|_p = \left(\int_0^1 |f(x)|^p\right)^{1/p}$.

**Theorem 6.35** *The following inequalities hold:*

$$|I_n| \leq \frac{1}{4^n}\frac{1}{n!(n+1)!}\|g^{(n)}\|_p\|f^{(n)}\|_q,\ \ p,q \geq 1,\ 1/p+1/q=1, \tag{6.88}$$

$$\begin{aligned}|I_n| \leq \frac{1}{(n!)^2}\frac{1}{(nq+1)^{\frac{1}{q}}}\left[\frac{\Gamma^2(np_1+1)}{\Gamma(2np_1+2)}\right]^{\frac{1}{p_1}}|g^{(n)}|_p|f^{(n)}|_{q_1},\\ p,q,p_1,q_1 \geq 1,\ 1/p+1/q=1, 1/p_1+1/q_1=1.\end{aligned} \tag{6.89}$$

*Proof.* Using the (weighted) Hölder inequality we have

$$\begin{aligned}|I_n| &= \left|\int_0^1\int_0^1 g^{(n)}(t)f^{(n)}(x)k_{n,n}(x,t)dtdx\right|\\ &\leq \left(\int_0^1\int_0^1 \left|g^{(n)}(t)\right|^p k_{n,n}(x,t)dtdx\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1 \left|f^{(n)}(x)\right|^q k_{n,n}(x,t)dtdx\right)^{\frac{1}{q}}\\ &= \left(\int_0^1 \left|g^{(n)}(t)\right|^p\left(\int_0^1 k_{n,n}(x,t)dx\right)dt\right)^{\frac{1}{p}}\\ &\qquad\left(\int_0^1 \left|f^{(n)}(x)\right|^q\left(\int_0^1 k_{n,n}(x,t)dt\right)dx\right)^{\frac{1}{q}}.\end{aligned}$$

Since $\max_{t\in[0,1]}\int_0^1 k_{n,n}(x,t)dx = \max_{x\in[0,1]}\int_0^1 k_{n,n}(x,t)dt = \frac{1}{4^n n!(n+1)!}$, inequality (6.88) follows.

Using the Hölder inequality and the integral Minkowski inequality we have:

$$\begin{aligned}|I_n| &= \left|\int_0^1 g^{(n)}(t)\left(\int_0^1 f^{(n)}(x)k_{n,n}(x,t)dx\right)dt\right|\\ &\leq \left\|g^{(n)}\right\|_p\left(\int_0^1\left(\int_0^1\left|f^{(n)}(x)\right|k_{n,n}(x,t)dx\right)^q dt\right)^{\frac{1}{q}}\\ &\leq \left\|g^{(n)}\right\|_p\int_0^1\left(\int_0^1\left|f^{(n)}(x)\right|^q k_{n,n}^q(x,t)dt\right)^{\frac{1}{q}}dx\\ &= \frac{1}{(nq+1)^{\frac{1}{q}}(n!)^2}\left\|g^{(n)}\right\|_p\int_0^1\left|f^{(n)}(x)\right|(1-x)^n x^n\,dx.\end{aligned} \tag{6.90}$$

Inequality (6.89) follows by applying again Hölder's inequality, now with conjugate exponents $p_1, q_1$. □

We compare estimates (6.88) and (6.89) with some known estimates in the case $n = 1$. Notice that (6.88) gives

$$|I_1| \leq (1/8)\|g'\|_p \|f'\|_q. \tag{6.91}$$

This general (in the sense that $p$ and $q$ are arbitrary conjugate exponents) estimate is the best possible one. To prove this, we prove that $1/8$ is the best possible constant in (6.91) in the case $p = \infty$, $q = 1$. To do this take $g(x) = x$, $x \in [0,1]$ and $f_\varepsilon(x) = \int_0^x \tilde{f}_\varepsilon(t)dt$, where $\tilde{f}_\varepsilon(t) = 1/(2\varepsilon)$ for $t \in (1/2-\varepsilon, 1/2+\varepsilon)$ and $\tilde{f}_\varepsilon(t) = 0$ otherwise, $0 < \varepsilon < 1/2$. It is easy to see that $\lim_{\varepsilon\to 0} I_1 = \lim_{\varepsilon\to 0} T(f_\varepsilon, g; dx) = 1/8$, $|g'|_\infty = |f'_\varepsilon|_1 = 1$, which gives the optimality of the estimate (6.91).

Two classes of the estimate (6.89) are of a special interest:

$$q_1 = p,\ \ |I_1| \leq C_1(p)\, \|g'\|_p \|f'\|_p,\ C_1(p) = \frac{1}{(q+1)^{\frac{1}{q}}} \left[\frac{\Gamma^2(q+1)}{\Gamma(2q+2)}\right]^{\frac{1}{q}}, \tag{6.92}$$

$$q_1 = q,\ \ |I_1| \leq C_2(p)\, \|g'\|_p \|f'\|_q,\ \ C_2(p) = \frac{1}{(q+1)^{\frac{1}{q}}} \left[\frac{\Gamma^2(p+1)}{\Gamma(2p+2)}\right]^{\frac{1}{p}}. \tag{6.93}$$

The following cases can be easily checked: $p = 1$ ($f$ and $g$ are of bounded variation) $C_1(1) = 1/4$, which is the best possible estimation (compare [19]); $p = \infty$ ($f$ and $g$ are Lipshitzian), $C_1(\infty) = 1/12$, which is the best possible estimation ( [19]); $p = 2$, $C_1(2) = C_2(2) = 1/\sqrt{90}$, which is remarkably close to the best possible constant $1/\pi^2$ (compare [40]). Note that $C_2(1) = 1/6$ and $C_2(\infty) = 1/8$.

As a final application of identity (6.81), we give a series expansion of $T(f,g;dx)$. Since in this case $\displaystyle\int_0^1 \int_0^1 k_{n,n}(x,t)dxdt = \frac{1}{(n+1)(2n+1)!}$, it is obvious from (6.81), that if $|f^{(n)}| \leq F\ (n+1)M^n$ and $|g^{(n)}| \leq G\ (n+1)N^n$, for some $F, G, M, N > 0$ and every natural $n$, then

$$T(f,g;dx) = \sum_{i=1}^{\infty} (-1)^{i+1} \int_0^1 [R_i L_{i+1} + L_i R_{i+1}]\, f^{(i)} g^{(i)}. \tag{6.94}$$

Using this and series expansions of hyperbolic functions, it is easy to see that

$$|T(f,g;dx)| \leq FG\left(\frac{\sinh\sqrt{MN}}{\sqrt{MN}} - 1\right).$$

Analogously, if $|f^{(n)}| \leq F\,M^n$ and $|g^{(n)}| \leq G\,N^n$, for some $F, G, M, N > 0$ and every natural number $n$, then

$$|T(f,g;dx)| \leq FG\left(2\frac{\cosh\sqrt{MN} - 1}{\sqrt{MN}} - 1\right).$$

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