MONOGRAPHS IN INEQUALITIES 13

Further Development of Hilbert-type Inequalities

Selected topics in Hilbert-type inequalities

Tserendorj Batbold, Mario Krnić, Josip Pečarić and Predrag Vuković

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ELEMENT

Zagreb, 2017

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1st edition

A CIP catalogue record for this book is available from the National and University Library in Zagreb under 000980469.

ISBN 978-953-197-629-9

Preface

The famous Hilbert inequality asserts that

$$\int_{\mathbb{R}_+^2} \frac{f(x)g(y)}{x+y} dxdy \leq \frac{\pi}{\sin\frac{\pi}{p}} \|f\|_p \|g\|_q,$$

where $f \in L^p(\mathbb{R}_+)$, $g \in L^q(\mathbb{R}_+)$ are non-negative functions, and p, q are mutually conjugate parameters, that is, $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$. The corresponding discrete version states that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} \leq \frac{\pi}{\sin\frac{\pi}{p}} \|a\|_p \|b\|_q,$$

where $a = (a_m)_{m\in\mathbb{N}} \in l^p$ and $b = (b_n)_{n\in\mathbb{N}} \in l^q$ are non-negative sequences. The constant $\frac{\pi}{\sin\frac{\pi}{p}}$ appearing on the right-hand sides of both inequalities is the best possible. This discrete version of the Hilbert inequality was initially studied by D. Hilbert at the end of the nineteenth century, hence, in his honor these inequalities are referred to as the integral and the discrete Hilbert inequalities.

After discovering, the Hilbert inequality was extensively studied by numerous mathematicians. A rich variety of generalizations included inequalities with more general kernels, weight functions and integration domains, extension to a multidimensional case, as well as refinements of the initial Hilbert inequality. The established inequalities are usually referred to as the Hilbert-type inequalities. In addition, Hardy, Littlewood and Pólya noticed in their monograph *Inequalities* (see [47]) that every Hilbert-type inequality possesses the equivalent Hardy-Hilbert-type form, closely connected to a famous Hardy inequality. For a detailed review of the starting development of the Hilbert inequality the reader is referred to monograph [47].

Nowadays, more than a century after discovering the Hilbert inequality, this topic is still of interest to numerous authors. In 2012, Krnić, Pečarić, Perić and Vuković published a monograph *Recent Advances in Hilbert-type inequalities* (see [63]) which was collection of decennial research of authors and their collaborators. That monograph provides a unified treatment of Hilbert-type inequalities, with integrals taken over $\sigma-$finite measure space, and with general kernel and weight functions. In addition, several new methods for improving Hilbert-type inequalities were also presented in [63].

The present book *Further Development of Hilber-type Inequalities* may be regarded as a continuation of *Recent Advances in Hilbert-type inequalities*. Namely, this book is a result of five-year research of authors in Hilbert-type inequalities. The book is based on some twenty significant papers published in the course of the last five years. Roughly speaking, we give some new generalizations, interpretations, refinements and applications of Hilbert-type inequalities. The book is divided into nine chapters.

An introductory part of this book is Chapter 1 in which we give definitions and basic results necessary for establishing the results that will follow. Namely, for the reader's convenience we present Hilbert-type inequalities with conjugate and non-conjugate exponents in most general forms. Most of these results have been taken from already mentioned book [63], which will be the starting point in establishing Hilbert-type inequalities in succeeding chapters.

In Chapter 2 we deal with some particular classes of Hilbert-type inequalities. First, we derive more accurate version of the discrete Hilbert-type inequality by means of the Hermite-Hadamard inequality. Then, we establish some particular multidimensional versions of the Hilbert inequality in both integral and discrete case. Finally, we give a unified treatment of half-discrete Hilbert-type inequalities. Such inequalities include both integral and sum. All results are given in two equivalent forms. Finally, we establish a condition under which the constants appearing on the right-hand sides of these inequalities are the best possible.

In Chapter 3 we derive Hilbert-type inequalities on time scales. After recalling essentials about time scales, we establish the corresponding results.

In Chapter 4 we present a new method for improving Hilbert-type inequalities, based on an improved form of the Young inequality, known from the literature. We obtain refined and reversed relations in a general multidimensional case. As an application, we also establish improved versions of the classical Hilbert and Hardy inequalities.

In Chapter 5 we establish several new Hilbert-type inequalities with a homogeneous kernel, involving arithmetic, geometric, and harmonic mean operators in integral, discrete and half-discrete case. A particular emphasis is placed on the problem of the best possible constants. Namely, it is interesting that the constants appearing on the right-hand sides of the established inequalities are also the best possible. Finally, some multidimensional extensions are also studied.

Several classes of Hilbert-type inequalities involving certain differential operators are studied in Chapter 6. We show that the constants appearing in derived inequalities are the best possible. Finally, the corresponding multidimensional extensions are also given.

Chapter 7 is dedicated to an operator interpretation of the Hilbert inequality. We give a general form of the Hilbert inequality for positive invertible operators on a Hilbert space. Special emphasis is placed on inequalities with a homogeneous kernel. In some general cases the best possible constants are also derived. Finally, some more accurate Hilbert-type inequalities are established by means of the Hermite-Hadamard inequality.

In Chapter 8 we study a more accurate class of the Hilbert inequality closely connected to the Carlson inequality. The established inequalities are given in both discrete and integral forms, and they include the best possible constants on their right-hand sides.

The main objective of Chapter 9 is a study of some generalizations of Hilbert-Pachpatte-type inequalities closely connected to the Hilbert inequality. A special emphasis is placed on inequalities with homogeneous kernels. Finally, we obtain a class of inequalities involving fractional derivatives.

Throughout the monograph, presented results are discussed and compared with previously known from the literature. Furthermore, at the end of a section or a chapter we cite the corresponding references for presented results. We also give some relevant references closely related to presented topics.

Since this book integrates the whole variety of results that were previously published by several authors in numerous papers, it was almost impossible, despite our great effort, to quite unify the terminology and the notation in the book. Nevertheless, starting from the introductory chapter, but also in each particular chapter, most of the used terminology is defined and explained for the reader's convenience. It is done, of course, on the assumption that the reader is familiar with the basis in real and in functional analysis.

Authors

Contents

Chapter 1

Definitions and Basic Results

1.1 Hilbert-type Inequalities with Conjugate Exponents

Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$. The Hilbert inequality asserts that

$$\int_{\mathbb{R}_+^2} \frac{f(x)g(y)}{x+y}dxdy \leq \frac{\pi}{\sin\frac{\pi}{p}}\|f\|_p\|g\|_q \tag{1.1}$$

holds for all non-negative measurable functions $f \in L^p(\mathbb{R}_+)$ and $g \in L^q(\mathbb{R}_+)$. After its discovery at the beginning of the 20th century, the Hilbert inequality was studied by numerous authors, who improved and generalized it in many different directions. This inequality is still of interest to numerous authors. The applications in diverse fields of mathematics have certainly contributed to its importance. For a comprehensive inspection of the initial development of the Hilbert inequality, the reader is referred to a classical monograph [47], while some recent results are collected in monograph [63].

In this book we refer to the following multidimensional extension of inequality (1.1) established by Krnić *et al.* (see [63], [99]).

Theorem 1.1 *Suppose* $(\Omega_i,\Sigma_i,\mu_i)$ *are* σ*-finite measure spaces,* $\sum_{i=1}^n \frac{1}{p_i}=1$, $p_i>1$, *and* $K:\mathbf{\Omega}\to\mathbb{R}$, $\phi_{ij}:\Omega_j\to\mathbb{R}$, $f_i:\Omega_i\to\mathbb{R}$, $i,j=1,2,\ldots,n$, *are non-negative measurable functions. If* $\prod_{i,j=1}^n \phi_{ij}(x_j)=1$, *then the following inequalities hold and are equivalent*

$$\int_{\mathbf{\Omega}} K(\mathbf{x})\prod_{i=1}^n f_i(x_i)d\mu(\mathbf{x}) \leq \prod_{i=1}^n \|\phi_{ii}\omega_i f_i\|_{p_i} \tag{1.2}$$

and

$$\left[\int_{\Omega_n}\left(\frac{1}{(\phi_{nn}\omega_n)(x_n)}\int_{\hat{\Omega}^n}K(\mathbf{x})\prod_{i=1}^{n-1}f_i(x_i)d\hat{\mu}^n(\mathbf{x})\right)^P d\mu(x_n)\right]^{\frac{1}{P}} \\ \leq \prod_{i=1}^{n-1}\|\phi_{ii}\omega_i f_i\|_{p_i}, \tag{1.3}$$

where $\frac{1}{P}=\sum_{i=1}^{n-1}\frac{1}{p_i}$, $\mathbf{\Omega}=\prod_{i=1}^{n}\Omega_i$, $\hat{\mathbf{\Omega}}^i=\prod_{j=1,j\neq i}^{n}\Omega_j$, $\mathbf{x}=(x_1,x_2,\ldots,x_n)$, $d\mu(\mathbf{x})=\prod_{i=1}^{n}d\mu_i(x_i)$, $d\hat{\mu}^i(\mathbf{x})=\prod_{j=1,j\neq i}^{n}d\mu_j(x_j)$, *and*

$$\omega_i(x_i)=\left[\int_{\hat{\mathbf{\Omega}}^i}K(\mathbf{x})\prod_{j=1,j\neq i}^{n}\phi_{ij}^{p_i}(x_j)d\hat{\mu}^i(\mathbf{x})\right]^{\frac{1}{p_i}}. \tag{1.4}$$

The above notation will be used throughout the whole monograph. In addition, $\|\cdot\|_r$ stands for the usual norm in $L^r(\Omega)$, that is $\|f\|_r=[\int_\Omega|f(x)|^r d\mu(x)]^{\frac{1}{r}}$, $r>1$. Inequalities following from (1.2) are usually referred to as the Hilbert-type inequalities since (1.1) is a particular case of (1.2). Further, inequalities related to (1.3) are usually called Hardy-Hilbert-type inequalities since (1.3) implies the classical Hardy inequality, which will be discussed later. Inequalities (1.2) and (1.3) are closely connected in the sense that one implies the other, hence they are sometimes both referred to as the Hilbert-type inequalities, for brevity.

Perić and Vuković [77], developed a unified treatment of the Hilbert and Hardy-Hilbert type inequalities with general homogeneous kernel. Further, regarding the notations from Theorem 1.1, we assume that $\Omega_i=\mathbb{R}_+$, equipped with the non-negative Lebesgue measures $d\mu_i(x_i)=dx_i$, $i=1,2,\ldots,n$. In addition, we have $\mathbf{\Omega}=\mathbb{R}_+^n$ and $d\mathbf{x}=dx_1dx_2\ldots dx_n$.

Recall that the function $K:\mathbb{R}_+^n\to\mathbb{R}$ is said to be homogeneous of degree $-s$, $s>0$, if $K(t\mathbf{x})=t^{-s}K(\mathbf{x})$ for all $t>0$. Furthermore, for $\mathbf{a}=(a_1,a_2,\ldots,a_n)\in\mathbb{R}_+^n$, we define

$$k_i(\mathbf{a})=\int_{\mathbb{R}_+^{n-1}}K(\hat{\mathbf{u}}^i)\prod_{j=1,j\neq i}^{n}u_j^{a_j}\hat{d}^i\mathbf{u},\quad i=1,2,\ldots,n, \tag{1.5}$$

where $\hat{\mathbf{u}}^i=(u_1,\ldots,u_{i-1},1,u_{i+1},\ldots,u_n)$, $\hat{d}^i\mathbf{u}=du_1\ldots du_{i-1}du_{i+1}\ldots du_n$, and provided that the above integral converges.

Utilizing Theorem 1.1 one obtains the following equivalent inequalities with general homogeneous kernel of degree $-s$:

$$\int_{\mathbb{R}_+^n}K(\mathbf{x})\prod_{i=1}^{n}f_i(x_i)d\mathbf{x}\leq\prod_{i=1}^{n}k_i^{1/p_i}(p_i\mathbf{A_i})\prod_{i=1}^{n}\|x_i^{(n-1-s)/p_i+\alpha_i}f_i\|_{p_i} \tag{1.6}$$

and

$$\left[\int_{\mathbb{R}_+}x_n^{(1-P)(n-1-s)-P\alpha_n}\left(\int_{\mathbb{R}_+^{n-1}}K(\mathbf{x})\prod_{i=1}^{n-1}f_i(x_i)\hat{d}^n\mathbf{x}\right)^P dx_n\right]^{\frac{1}{P}} \\ \leq\prod_{i=1}^{n}k_i^{1/p_i}(p_i\mathbf{A_i})\prod_{i=1}^{n-1}\|x_i^{(n-1-s)/p_i+\alpha_i}f_i\|_{p_i}, \tag{1.7}$$

where A_{ij}, $i,j=1,2,\dots,n$, are real parameters such that $\sum_{i=1}^{n}A_{ij}=0$ for $j=1,2,\dots,n$, $\alpha_i=\sum_{j=1}^{n}A_{ij}$, $\quad \mathbf{A_i}=(A_{i1},A_{i2},\dots,A_{in})$, $i=1,2,\dots,n$, and $k_i(\cdot)$, $i=1,2,\dots,n$, is defined by (1.5).
To obtain a case of the inequalities with the best possible constants it is natural to impose the following conditions on parameters A_{ij} :

$$p_jA_{ji}=s-n-p_i(\alpha_i-A_{ii}),\quad j\neq i,\quad i,j\in\{1,2,\dots,n\}. \tag{1.8}$$

In that case the constant factors from inequalities (1.6) and (1.7) are simplified to the following form:

$$L^*=k_1(\widetilde{\mathbf{A}}), \tag{1.9}$$

where $\widetilde{\mathbf{A}}=(\widetilde{A}_1,\widetilde{A}_2,\dots,\widetilde{A}_n)$ and

$$\widetilde{A}_i=p_1A_{1i}\quad\text{for}\quad i\neq 1\quad\text{and}\quad \widetilde{A}_1=p_nA_{n1}. \tag{1.10}$$

Further, by using (1.8) and (1.9), the inequalities (1.6) and (1.7) with the parameters A_{ij}, satisfying the relation (1.8) become

$$\int_{\mathbb{R}_+^n}K(\mathbf{x})\prod_{i=1}^{n}f_i(x_i)d\mathbf{x}\le L^*\prod_{i=1}^{n}\|x_i^{-\widetilde{A}_i-1/p_i}f_i\|_{p_i} \tag{1.11}$$

and

$$\begin{aligned}&\left[\int_{\mathbb{R}_+}x_n^{(1-P)(-1-p_n\widetilde{A}_n)}\left(\int_{\mathbb{R}_+^{n-1}}K(\mathbf{x})\prod_{i=1}^{n-1}f_i(x_i)\hat{d}^n\mathbf{x}\right)^P dx_n\right]^{\frac{1}{P}}\\&\le L^*\prod_{i=1}^{n-1}\|x_i^{-\widetilde{A}_i-1/p_i}f_i\|_{p_i}.\end{aligned} \tag{1.12}$$

Theorem 1.2 ([63]) *Let $K:\mathbb{R}_+^n\to\mathbb{R}$ be a non-negative measurable homogeneous function of degree $-s$, such that for every $i=2,3,\dots,n$,*

$$K(1,t_2,\dots,t_i,\dots,t_n)\le CK(1,t_2,\dots,0,\dots,t_n),\ -1\le t_i\le 1, \tag{1.13}$$

where C is a positive constant. Let the parameters $\widetilde{A}_i$, $i=1,\dots,n$, be defined by (1.10) *and $0<\varepsilon<\min_{1\le i\le n}\{p_i+p_i\widetilde{A}_i\}$. If the parameters A_{ij} satisfy the conditions $\sum_{i=1}^{n}A_{ij}=0$ for $j=1,2,\dots,n$, and* (1.8)*, then the constant L^* is the best possible in inequalities* (1.11) *and* (1.12).

The following result based on Theorem 1.1 can be seen in [88]. Let $K:\mathbb{R}_+^n\to\mathbb{R}$ and A_{ij}, $i,j=1,2,\dots,n$, be as in Theorem 1.2. If $u_i:(a_i,b_i)\to(0,\infty), i=1,\dots,n$ are strictly increasing differentiable functions such that $u_i(a_i)=0$ and $u_i(b_i)=\infty$, then the following inequalities hold and are equivalent

$$\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}K(u_1(t_1),\dots,u_n(t_n))\prod_{i=1}^{n}f_i(t_i)dt_1\cdots dt_n$$

$$< L\prod_{i=1}^{n}\left[\int_0^\infty (u_i(t_i))^{-1-p_i\widetilde{A}_i}(u_i'(t_i))^{1-p_i}f_i^{p_i}(t_i)dt_i\right]^{\frac{1}{p_i}} \tag{1.14}$$

and

$$\int_{a_n}^{b_n}(u_n(t_n))^{(1-P)(-1-p_n\widetilde{A}_n)}\left[\int_{a_1}^{b_1}\cdots\int_{a_{n-1}}^{b_{n-1}}K(u_1(t_1),\ldots,u_n(t_n))\prod_{i=1}^{n-1}f_i(t_i)dt_1\cdots dt_{n-1}\right]^P dt_n$$
$$< L^P\prod_{i=1}^{n-1}\left[\int_0^\infty (u_i(t_i))^{-1-p_i\widetilde{A}_i}(u_i'(t_i))^{1-p_i}f_i^{p_i}(t_i)dt_i\right]^{\frac{1}{p_i}}, \tag{1.15}$$

where the constants $L = k(\widetilde{A}_2,\ldots,\widetilde{A}_n)$ and L^P are the best possible in inequalities (1.14) and (1.15).

Since the case $n = 2$ of inequalities (1.2) and (1.3) will be of special interest to us, we state it as a separate result. The proof follows directly using substitutions $p_1 = p$, $p_2 = q$, $\phi_{11} = \varphi$ and $\phi_{22} = \psi$. Observe that from $\phi_{11}\phi_{21} = 1$ and $\phi_{12}\phi_{22} = 1$ we have $\phi_{21} = 1/\varphi$ and $\phi_{12} = 1/\psi$ (for more details see e.g. [66]).

Theorem 1.3 *Let $\frac{1}{p}+\frac{1}{q} = 1$, $p > 1$, and let Ω be a measure space with positive σ-finite measures μ_1 and μ_2. Let $K : \Omega\times\Omega \to \mathbb{R}$ and $\varphi,\psi : \Omega \to \mathbb{R}$ be non-negative measurable functions. If the functions F and G are defined by*

$$F^p(x) = \int_\Omega K(x,y)\psi^{-p}(y)d\mu_2(y), \quad G^q(y) = \int_\Omega K(x,y)\varphi^{-q}(x)d\mu_1(x), \tag{1.16}$$

then for all non-negative measurable functions f and g on Ω the inequalities

$$\int_\Omega\int_\Omega K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y) \le \|\varphi Ff\|_p\|\psi Gg\|_q \tag{1.17}$$

and

$$\int_\Omega G^{1-p}(y)\psi^{-p}(y)\left[\int_\Omega K(x,y)f(x)d\mu_1(x)\right]^p d\mu_2(y)$$
$$\le \|\varphi Ff\|_p^p \tag{1.18}$$

hold and are equivalent.
If $0 < p < 1$, then the reverse inequalities in (1.17) *and* (1.18) *are valid, as well as the inequality*

$$\int_\Omega F^{1-q}(x)\varphi^{-q}(x)\left[\int_\Omega K(x,y)g(y)d\mu_2(y)\right]^q d\mu_1(x)$$
$$\le \|\psi Gg\|_q^q. \tag{1.19}$$

Remark 1.1 The equality in the previous theorem is possible if and only if it holds in the Hölder inequality, that is, if

$$\left[f(x)\frac{\varphi(x)}{\psi(y)}\right]^p = C\left[g(y)\frac{\psi(y)}{\varphi(x)}\right]^q, \quad \text{a.e. on } \Omega,$$

where C is a positive constant. In that case we have

$$f(x) = C_1\varphi^{-q}(x) \quad \text{and} \quad g(y) = C_2\psi^{-p}(y) \quad \text{a.e. on } \Omega, \tag{1.20}$$

for some constants C_1 and C_2, which is possible if and only if

$$\int_\Omega F(x)\varphi^{-q}(x)d\mu_1(x) < \infty \quad \text{and} \quad \int_\Omega G(y)\psi^{-p}(y)d\mu_2(y) < \infty. \tag{1.21}$$

Otherwise, the inequalities in Theorem 1.3 are strict.

For homogeneous function $K(x,y)$ we define $k(\alpha)$ (see also definition (1.5)) as

$$k(\alpha) = \int_0^\infty K(1,u)u^{-\alpha}du, \tag{1.22}$$

provided that the above integral converges.
In the following theorem the integrals are taken over an arbitrary interval of non-negative real numbers, i.e. $(a,b) \subseteq \mathbb{R}_+$, $0 \le a < b \le \infty$, and the weight functions are chosen to be power functions.

Theorem 1.4 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $K:(a,b)\times(a,b)\to\mathbb{R}$ be a non-negative homogeneous function of degree $-s$, $s>0$, strictly decreasing in both variables. If A_1 and A_2 are real parameters such that $A_1 \in (\frac{1-s}{q},\frac{1}{q})$, $A_2 \in (\frac{1-s}{p},\frac{1}{p})$, then for all non-negative measurable functions $f,g:(a,b)\to\mathbb{R}$ the inequalities*

$$\begin{aligned}
&\int_a^b\int_a^b K(x,y)f(x)g(y)dxdy \\
&\le \left[\int_a^b \big(k(pA_2)-\varphi_1(pA_2,x)\big)x^{1-s+p(A_1-A_2)}f^p(x)dx\right]^{\frac{1}{p}} \\
&\quad\times\left[\int_a^b \big(k(2-s-qA_1)-\varphi_2(2-s-qA_1,y)\big)y^{1-s+q(A_2-A_1)}g^q(y)dy\right]^{\frac{1}{q}}
\end{aligned} \tag{1.23}$$

and

$$\begin{aligned}
&\int_a^b \big(k(2-s-qA_1)-\varphi_2(2-s-qA_1,y)\big)^{1-p}y^{(p-1)(s-1)+p(A_1-A_2)} \\
&\quad\times\left[\int_a^b K(x,y)f(x)dx\right]^p dy \\
&\le \int_a^b \big(k(pA_2)-\varphi_1(pA_2,x)\big)x^{1-s+p(A_1-A_2)}f^p(x)dx
\end{aligned} \tag{1.24}$$

hold and are equivalent, where

$$\varphi_1(\alpha,x) = \left(\frac{a}{x}\right)^{1-\alpha}\int_0^1 K(1,u)u^{-\alpha}du + \left(\frac{x}{b}\right)^{s+\alpha-1}\int_0^1 K(u,1)u^{s+\alpha-2}du,$$

$$\varphi_2(\alpha,y)=\left(\frac{a}{y}\right)^{s+\alpha-1}\int_0^1 K(u,1)u^{s+\alpha-2}du+\left(\frac{y}{b}\right)^{1-\alpha}\int_0^1 K(1,u)u^{-\alpha}du.$$

If $0<p<1$, $b=\infty$, and $K(x,y)$ is strictly decreasing in x and strictly increasing in y, then the reverse inequalities in (1.23) and (1.24) are valid for every $A_1\in(\frac{1}{q},\frac{1-s}{q})$ and $A_2\in(\frac{1-s}{p},\frac{1}{p})$, as well as the inequality

$$\int_a^\infty\left(k(pA_2)-\varphi_1(pA_2,x)\right)^{1-q}x^{(q-1)(s-1)+q(A_2-A_1)}\left[\int_a^\infty K(x,y)g(y)dy\right]^q dx$$
$$\leq\int_a^\infty\left(k(2-s-qA_1)-\varphi_2(2-s-qA_1,y)\right)y^{1-s+q(A_2-A_1)}g(y)^q dy.$$

Moreover, if $0<p<1$, $a=0$, and $K(x,y)$ is strictly increasing in x and strictly decreasing in y, then the reverse inequalities in (1.23) and (1.24) hold for every $A_1\in(\frac{1}{q},\frac{1-s}{q})$ and $A_2\in(\frac{1-s}{p},\frac{1}{p})$, as well as the inequality

$$\int_0^b\left(k(pA_2)-\varphi_1(pA_2,x)\right)^{1-q}x^{(q-1)(s-1)+q(A_2-A_1)}\left[\int_0^b K(x,y)g(y)dy\right]^q dx$$
$$\leq\int_0^b\left(k(2-s-qA_1)-\varphi_2(2-s-qA_1,y)\right)y^{1-s+q(A_2-A_1)}g(y)^q dy.$$

Setting $a=0$, $b=\infty$ in the previous theorem, one obtains the corresponding inequalities for an arbitrary non-negative homogeneous function of degree $-s$.

Corollary 1.1 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ be a non-negative homogeneous function of degree $-s$, $s>0$. If A_1 and A_2 are real parameters such that $A_1\in(\frac{1-s}{q},\frac{1}{q})$, $A_2\in(\frac{1-s}{p},\frac{1}{p})$, then for all non-negative measurable functions $f,g:\mathbb{R}_+\to\mathbb{R}$ the inequalities*

$$\int_0^\infty\int_0^\infty K(x,y)f(x)g(y)dxdy$$
$$\leq L\left[\int_0^\infty x^{1-s+p(A_1-A_2)}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{1-s+q(A_2-A_1)}g^q(y)dy\right]^{\frac{1}{q}} \tag{1.25}$$

and

$$\int_0^\infty y^{(p-1)(s-1)+p(A_1-A_2)}\left[\int_0^\infty K(x,y)f(x)dx\right]^p dy$$
$$\leq L^p\int_0^\infty x^{1-s+p(A_1-A_2)}f^p(x)dx \tag{1.26}$$

hold and are equivalent, where $L=k^{\frac{1}{p}}(pA_2)k^{\frac{1}{q}}(2-s-qA_1)$.
If $0<p<1$, then the reverse inequalities in (1.25) and (1.26) are valid for every $A_1\in(\frac{1}{q},\frac{1-s}{q})$ and $A_2\in(\frac{1-s}{p},\frac{1}{p})$, as well as the inequality

$$\int_0^\infty x^{(q-1)(s-1)+q(A_2-A_1)}\left[\int_0^\infty K(x,y)g(y)dy\right]^q dx$$

$$\leq L^q \int_0^\infty y^{1-s+q(A_2-A_1)} g^q(y)dy. \tag{1.27}$$

Inequalities (1.25) *and* (1.26)*, as well as their reverse inequalities are equivalent. Moreover, equality in the above relations holds if and only if $f=0$ or $g=0$ a.e. on $\mathbb{R}_+$.*

Considering inequalities in Corollary 1.1 with parameters A_1 and A_2 fulfilling condition

$$pA_2+qA_1=2-s, \tag{1.28}$$

the constant L reduces to $L=k(pA_2)$. It has been shown that such constant is the best possible in the corresponding inequalities.

The following result contains a generalized discrete Hilbert-type inequalities in both equivalent forms. Krnić *et al.* (see [65]) considered the weight functions involving real differentiable functions. By $H(r)$, $r>0$, is denoted the set of all non-negative differentiable functions $u:\mathbb{R}_+\to\mathbb{R}$ satisfying the following conditions:

(i) u is strictly increasing on $\mathbb{R}_+$ and there exists $x_0\in\mathbb{R}_+$ such that $u(x_0)=1$,

(ii) $\lim_{x\to\infty} u(x)=\infty$, $\frac{u'(x)}{[u(x)]^r}$ is decreasing on $\mathbb{R}_+$.

Theorem 1.5 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $s>0$. Further, suppose that $A_1\in\left(\max\{\frac{1-s}{q},0\},\frac{1}{q}\right)$, $A_2\in\left(\max\{\frac{1-s}{p},0\},\frac{1}{p}\right)$, $u\in H(qA_1)$ and $v\in H(pA_2)$. If $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ is a non-negative homogeneous function of degree $-s$, strictly decreasing in each argument, then the inequalities*

$$\begin{aligned}
&\sum_{m=1}^\infty\sum_{n=1}^\infty K(u(m),v(n))a_m b_n\\
&\leq L\left[\sum_{m=1}^\infty [u(m)]^{1-s+p(A_1-A_2)}[u'(m)]^{1-p}a_m^p\right]^{\frac{1}{p}}\\
&\quad\times\left[\sum_{n=1}^\infty [v(n)]^{1-s+q(A_2-A_1)}[v'(n)]^{1-q}b_n^q\right]^{\frac{1}{q}}
\end{aligned} \tag{1.29}$$

and

$$\begin{aligned}
&\sum_{n=1}^\infty [v(n)]^{(s-1)(p-1)+p(A_1-A_2)}v'(n)\left[\sum_{m=1}^\infty K(u(m),v(n))a_m\right]^p\\
&\leq L^p\sum_{m=1}^\infty [u(m)]^{(1-s)+p(A_1-A_2)}[u'(m)]^{1-p}a_m^p
\end{aligned} \tag{1.30}$$

hold for all non-negative sequences $(a_m)_{m\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, where

$$L=k^{\frac{1}{p}}(pA_2)k^{\frac{1}{q}}(2-s-qA_1). \tag{1.31}$$

Moreover, inequalities (1.29) *and* (1.30) *are equivalent.*

If the parameters A_1 and A_2 satisfy (1.28), that is, $pA_2 + qA_1 = 2 - s$, then the constant L from Theorem 1.5 becomes

$$L^* = k(pA_2). \tag{1.32}$$

Moreover, it has been shown that the constant L^* is the best possible in the following inequalities

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K(u(m),v(n))a_m b_n \le L^* \left[\sum_{m=1}^{\infty} [u(m)]^{-1+pqA_1}[u'(m)]^{1-p} a_m^p\right]^{\frac{1}{p}} \times \left[\sum_{n=1}^{\infty} [v(n)]^{-1+pqA_2}[v'(n)]^{1-q} b_n^q\right]^{\frac{1}{q}} \tag{1.33}$$

and

$$\sum_{n=1}^{\infty} [v(n)]^{(p-1)(1-pqA_2)} v'(n) \left[\sum_{m=1}^{\infty} K(u(m),v(n))a_m\right]^p \le (L^*)^p \sum_{m=1}^{\infty} [u(m)]^{-1+pqA_1}[u'(m)]^{1-p} a_m^p. \tag{1.34}$$

1.2 Hilbert-type Inequalities with Non-conjugate Exponents

First, we introduce n-dimensional extension of conjugate exponents. Let $i = 1,2,\ldots n$ and let p_i, p_i', q_i, λ satisfy

$$\begin{aligned}
&p_i > 1, \quad \frac{1}{p_i} + \frac{1}{p_i'} = 1,\\
&\sum_{i=1}^{n} \frac{1}{p_i} \ge 1,\\
&\lambda = \frac{1}{n-1}\sum_{i=1}^{n}\frac{1}{p_i'} \quad \text{and} \quad \frac{1}{q_i} = \lambda - \frac{1}{p_i'}, \quad i = 1,\ldots,n,\\
&\frac{1}{q_i} > 0, \quad i = 1,\ldots,n.
\end{aligned} \tag{1.35}$$

It follows from these conditions that

$$\frac{1}{q_i} + (1-\lambda) = \frac{1}{p_i}, \quad i = 1,\ldots,n, \tag{1.36}$$

and

$$\sum_{i=1}^{n} \frac{1}{q_i} + (1-\lambda) = 1. \tag{1.37}$$

Observe that for $\lambda = 1$ the above parameters reduce to the conjugate case, that is, $\sum_{i=1}^{n} \frac{1}{p_i} = 1$ and $p_i = q_i$, $i = 1,2,\ldots,n$.

The following extension from [27] may also be regarded as a non-conjugate version of Theorem 1.1.

Let Ω_i be a measure space with σ-finite measure μ_i, $i = 1,2,\ldots,n$. Further, suppose that $K : \Omega \to \mathbb{R}$ and $\phi_{ij} : \Omega \to \mathbb{R}$, $i, j = 1,\ldots,n$, are non-negative measurable functions such that $\prod_{i,j=1}^{n} \phi_{ij}(x_j) = 1$. If the functions ω_i, $i = 1,2,\ldots,n$, are defined by

$$\omega_i(x_i) = \left[\int_{\hat{\mathbf{\Omega}}^i} K(\mathbf{x}) \prod_{j=1, j\neq i}^{n} \phi_{ij}^{q_i}(x_j) d\hat{\mu}^i(\mathbf{x})\right]^{\frac{1}{q_i}} \tag{1.38}$$

then for all non-negative measurable functions $f_i : \Omega \to \mathbb{R}$, $i = 1,2,\ldots,n$, the inequalities

$$\int_{\mathbf{\Omega}} K^{\lambda}(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i) d\mu(\mathbf{x}) \leq \prod_{i=1}^{n} \|\phi_{ii}\omega_i f_i\|_{p_i} \tag{1.39}$$

and

$$\begin{aligned} &\left[\int_{\Omega_n} \left(\frac{1}{(\phi_{nn}\omega_n)(x_n)} \int_{\hat{\mathbf{\Omega}}^n} K(\mathbf{x}) \prod_{i=1}^{n-1} f_i(x_i) d\hat{\mu}^n(\mathbf{x})\right)^{p_n'} d\mu(x_n)\right]^{\frac{1}{p_n'}} \\ &\leq \prod_{i=1}^{n-1} \|\phi_{ii}\omega_i f_i\|_{p_i}, \end{aligned} \tag{1.40}$$

hold and are equivalent.

Remark 1.2 Equality in the previous inequalities is possible if and only if it holds in Hölder's inequality. It means that the functions

$$K(\mathbf{x}){\phi_{ii}}^{p_i}(x_i) \prod_{j=1, j\neq i}^{n} \phi_{ij}^{q_i}(x_j) {\omega_i}^{p_i - q_i}(x_i) {f_i}^{p_i}(x_i), \quad i = 1,2,\ldots n,$$

and $\prod_{i=1}^{n} (\phi_{ii}\omega_i f_i)^{p_i}(x_i)$ are proportional (see also [27]). Hence, we obtain that the equality in mentioned inequalities can be achieved only if the functions f_i and the kernel K are defined by $f_i(x_i) = C_i \phi_{ii}(x_i)^{\frac{q_i}{1-\lambda q_i}} \omega_i(x_i)^{(1-\lambda)q_i}$ and $K(\mathbf{x}) = C \prod_{i=1}^{n} {\omega_i}^{q_i}(x_i)$, $i = 1,2,\ldots n$, where C and C_i are arbitrary constants. It is possible only if the functions

$$\frac{\prod_{j=1, j\neq i}^{n} \phi_{jj}^{\frac{\lambda q_j}{1-\lambda q_j}}(x_j)}{\prod_{j=1, j\neq i}^{n} {\phi_{ij}}^{\lambda q_j}(x_j)}, \quad i = 1,2,\ldots,n$$

are adequate constants, and

$$\int_{\Omega} \omega_i^{q_i}(x_i)\phi_{ii}^{\frac{q_i}{1-\lambda q_i}}(x_i)d\mu_i(x_i) < \infty, \quad i = 1,2,\ldots n.$$

Otherwise, the inequalities (1.39) and (1.40) are strict.

Now, suppose that the kernel $K : \mathbb{R}_+^n \to \mathbb{R}$ is homogeneous of degree $-s$, $s > 0$. Taking into account the notation from Theorem 1.1, we assume that $\Omega_i = \mathbb{R}_+$, equipped with the non-negative Lebesgue measures $d\mu_i(x_i) = dx_i$, $i = 1,2,\ldots,n$. In addition, we have $\Omega = \mathbb{R}_+^n$ and $d\mathbf{x} = dx_1dx_2\ldots dx_n$. If the parameters A_{ij} appearing in functions $\phi_{ij}(x_j) = x_j^{A_{ij}}$ satisfy relations $\sum_{i=1}^n A_{ij} = 0$, $j = 1,\ldots,n$, then the condition $\prod_{i,j=1}^n \phi_{ij}(x_j) = 1$ is fulfilled. Setting the power weight functions in the inequalities (1.39) and (1.40), one obtains the following equivalent inequalities

$$\begin{aligned} &\int_{\mathbb{R}_+^n} K^\lambda(\mathbf{x})\prod_{i=1}^n f_i(x_i)d\mathbf{x} \\ &\leq \prod_{i=1}^n k_i^{\frac{1}{q_i}}(q_i\mathbf{A}_i)\prod_{i=1}^n \|x_i^{(n-1-s)/q_i+\alpha_i} f_i\|_{p_i}, \end{aligned} \tag{1.41}$$

and

$$\begin{aligned} &\left[\int_{\mathbb{R}_+} x_n^{(1-\lambda p_n')(n-1-s)-p_n'\alpha_n}\left(\int_{\mathbb{R}_+^{n-1}} K^\lambda(\mathbf{x})\prod_{i=1}^{n-1} f_i(x_i)dx_1\cdots dx_{n-1}\right)^{p_n'} dx_n\right]^{1/p_n'} \\ &\leq \prod_{i=1}^n k_i^{\frac{1}{q_i}}(q_i\mathbf{A}_i)\prod_{i=1}^{n-1} \|x_i^{(n-1-s)/q_i+\alpha_i} f_i\|_{p_i}, \end{aligned} \tag{1.42}$$

where $\alpha_i = \sum_{j=1}^n A_{ij}$, $q_i\mathbf{A}_i = (q_iA_{i1},\ldots,q_iA_{in})$ and $k_i(\cdot)$ is defined by (1.5).

To conclude this section, we restate conditions in (1.35) for the case when $n = 2$. Let p and q be real parameters, such that

$$p > 1,\ q > 1,\ \frac{1}{p} + \frac{1}{q} \geq 1, \tag{1.43}$$

and let p' and q' respectively be their conjugate exponents, that is, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Further, define

$$\lambda = \frac{1}{p'} + \frac{1}{q'} \tag{1.44}$$

and note that $0 < \lambda \leq 1$ for all p and q as in (1.43). Especially, $\lambda = 1$ holds if and only if $q = p'$, that is, only when p and q are mutually conjugate. Otherwise, we have $0 < \lambda < 1$.

The two-dimensional version of inequalities (1.39) and (1.40) can be found in [36].

Theorem 1.6 *Let p, q, and λ be real parameters as in* (1.43) *and* (1.44)*, and let Ω_1 and Ω_2 be measure spaces with positive σ-finite measures μ_1 and μ_2 respectively. Let K be a non-negative measurable function on $\Omega_1 \times \Omega_2$, φ a measurable, a.e. positive function on Ω_1, and ψ a measurable, a.e. positive function on Ω_2. If the functions F on Ω_1 and G on Ω_2 are defined by*

$$F(x) = \left[\int_{\Omega_2} K(x,y)\psi^{-q'}(y)\, d\mu_2(y)\right]^{\frac{1}{q'}}, \quad x \in \Omega_1, \tag{1.45}$$

and

$$G(y) = \left[\int_{\Omega_1} K(x,y)\varphi^{-p'}(x)\, d\mu_1(x)\right]^{\frac{1}{p'}}, \quad y \in \Omega_2, \tag{1.46}$$

then for all non-negative measurable functions f on Ω_1 and g on Ω_2 the inequalities

$$\int_{\Omega_1}\int_{\Omega_2} K^{\lambda}(x,y)f(x)g(y)\, d\mu_1(x)d\mu_2(y) \le \|\varphi F f\|_p \|\psi G g\|_q \tag{1.47}$$

and

$$\left\{\int_{\Omega_2}\left[(\psi G)^{-1}(y)\int_{\Omega_1} K^{\lambda}(x,y)f(x)\, d\mu_1(x)\right]^{q'} d\mu_2(y)\right\}^{\frac{1}{q'}} \le \|\varphi F f\|_p \tag{1.48}$$

hold and are equivalent.

Applying Theorem 1.6 to non-negative homogeneous functions $K : \Omega \subseteq \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ with a negative degree of homogeneity, one obtains the following result. In this way Theorem 1.4 from previous section can be extended to the case of non-conjugate exponents.

Theorem 1.7 *Let p, q, and λ be as in* (1.43) *and* (1.44)*, and let $K : (a,b) \times (a,b) \to \mathbb{R}$ be a non-negative homogeneous function of degree $-s$, $s > 0$, strictly decreasing in both arguments. Further, suppose that A_1 and A_2 are real parameters such that $A_1 \in \left(\frac{1-s}{p'}, \frac{1}{p'}\right)$, $A_2 \in \left(\frac{1-s}{q'}, \frac{1}{q'}\right)$. If the functions φ_1 and φ_2 are defined as in the statement of Theorem* 1.4*, then for all non-negative measurable functions f and g on (a,b) the inequalities*

$$\begin{aligned}
&\int_a^b\int_a^b K^{\lambda}(x,y)f(x)g(y)dxdy \\
&\le \left[\int_a^b \left(k(q'A_2) - \varphi_1(q'A_2,x)\right)^{\frac{p}{q'}} x^{\frac{p}{q'}(1-s)+p(A_1-A_2)} f^p(x)dx\right]^{\frac{1}{p}} \\
&\quad \times \left[\int_a^b \left(k(2-s-p'A_1) - \varphi_2(2-s-p'A_1,y)\right)^{\frac{q}{p'}} y^{\frac{q}{p'}(1-s)+q(A_2-A_1)} g^q(y)dy\right]^{\frac{1}{q}}
\end{aligned} \tag{1.49}$$

and

$$\left[\int_a^b y^{\frac{q'}{p'}(s-1)+q'(A_1-A_2)}\left(k(2-s-p'A_1)-\varphi_2(2-s-p'A_1,y)\right)^{-\frac{q'}{p'}}\right.$$
$$\left.\times\left(\int_a^b K^\lambda(x,y)f(x)dx\right)^{q'}dy\right]^{\frac{1}{q'}}$$
$$\leq\left[\int_a^b\left(k(q'A_2)-\varphi_1(q'A_2,x)\right)^{\frac{p}{q'}}x^{\frac{p}{q'}(1-s)+p(A_1-A_2)}f^p(x)dx\right]^{\frac{1}{p}} \tag{1.50}$$

hold and are equivalent. The function $k(\cdot)$ *is defined by* (1.22).

Setting $a=0$, $b=\infty$ in Theorem 1.7, one obtains the corresponding equivalent Hilbert-type and Hardy-Hilbert-type inequalities.

Corollary 1.2 *Assume that p, q, and λ are as in* (1.43) *and* (1.44)*, and $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ is a non-negative homogeneous function of degree $-s$, $s>0$. Then the inequalities*

$$\int_0^\infty\int_0^\infty K^\lambda(x,y)f(x)g(y)dxdy$$
$$\leq L'\left[\int_0^\infty x^{\frac{p}{q'}(1-s)+p(A_1-A_2)}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{\frac{q}{p'}(1-s)+q(A_2-A_1)}g^q(y)dy\right]^{\frac{1}{q}} \tag{1.51}$$

and

$$\left[\int_0^\infty y^{\frac{q'}{p'}(s-1)+q'(A_1-A_2)}\left(\int_0^\infty K^\lambda(x,y)f(x)dx\right)^{q'}dy\right]^{\frac{1}{q'}}$$
$$\leq L'\left[\int_0^\infty x^{\frac{p}{q'}(1-s)+p(A_1-A_2)}f^p(x)dx\right]^{\frac{1}{p}} \tag{1.52}$$

hold for all parameters $A_1\in\left(\frac{1-s}{p'},\frac{1}{p'}\right)$, $A_2\in\left(\frac{1-s}{q'},\frac{1}{q'}\right)$, and for all non-negative measurable functions f and g on $\mathbb{R}_+$, where $L'=k^{\frac{1}{q'}}(q'A_2)k^{\frac{1}{p'}}(2-s-p'A_1)$. Moreover, these inequalities are equivalent.

1.3 Hardy-type Inequalities

In 1925, Hardy stated and proved in [47] the following integral inequality:

$$\int_0^\infty\left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx<\left(\frac{p}{p-1}\right)^p\int_0^\infty f^p(x)dx, \tag{1.53}$$

which holds for $p > 1$ and for all non-negative functions $f : \mathbb{R}_+ \to \mathbb{R}$, provided that $0 < \|f\|_{L^p(\mathbb{R}_+)} < \infty$. This is the original form of the *Hardy integral inequality*.

Its discrete version asserts that

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} a_k \right)^p < \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p, \tag{1.54}$$

where $p > 1$ and $a = (a_n)_{n\in\mathbb{N}}$ is a non-negative sequence such that $0 < \|a\|_{l^p} < \infty$. It should be noticed here that the constant $\left(\frac{p}{p-1}\right)^p$ is the best possible in both inequalities.

The Hardy inequality plays an important role in various fields of mathematics, especially in functional and spectral analysis, where one investigates properties of the Hardy operator, like continuity and compactness, and also its behavior in more general function spaces. For comprehensive accounts on Hardy inequality including history, different proofs, refinements and diverse applications, we refer to a recent monograph [68] and references therein.

Observe that the Hardy inequality includes *arithmetic mean* in integral and discrete case. We shall also be occupied with the corresponding inequalities including a *geometric mean*. The integral version of such inequality is known as the *Knopp inequality*, i.e.

$$\int_0^{\infty} \exp\left(\frac{1}{x} \int_0^x \log f(t)dt \right) dx < e \int_0^{\infty} f(x)dx, \tag{1.55}$$

while its discrete version is known as the *Carleman inequality*:

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} a_k \right)^{\frac{1}{n}} < e \sum_{n=1}^{\infty} a_n. \tag{1.56}$$

Both Knopp and Carleman inequalities include the best possible constant e on their right-hand sides (for more details, see [74]).

In 2005, Yang [100], derived the corresponding inequalities equipped with a *generalized harmonic mean*. Namely, integral version asserts that

$$\int_0^{\infty} \left(\frac{x}{\int_0^x f^{-r}(t)dt} \right)^{\frac{1}{r}} dx < (1+r)^{\frac{1}{r}} \int_0^{\infty} f(x)dx \tag{1.57}$$

holds for $r > 0$, while its discrete analogue holds for $0 < r \le 1$:

$$\sum_{n=1}^{\infty} \left(\frac{n}{\sum_{k=1}^{n} a_k^{-r}} \right)^{\frac{1}{r}} < (1+r)^{\frac{1}{r}} \sum_{n=1}^{\infty} a_n. \tag{1.58}$$

Moreover, Yang also proved that inequalities (1.57) and (1.58) include the best possible constant $(1+r)^{\frac{1}{r}}$. In accordance to [100], inequalities (1.57) and (1.58) will be named respectively as integral and discrete *Hardy-Carleman inequality*.

For the reader's convenience, we define integral arithmetic, geometric and harmonic mean operators $\mathscr{A}, \mathscr{G}, \mathscr{H} : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$ by

$$(\mathscr{A}f)(x) = \frac{1}{x} \int_0^x f(t)dt,$$

$$(\mathscr{G}f)(x) = \exp\left(\frac{1}{x}\int_0^x \log f(t)dt\right),$$
$$(\mathscr{H}f)(x) = \frac{x}{\int_0^x f^{-1}(t)dt}.$$

Obviously, the above operators are well-defined since the Hardy inequality, the Knopp inequality and the integral Hardy-Carleman inequality may respectively be rewritten as

$$\|\mathscr{A}f\|_{L^p(\mathbb{R}_+)} < \frac{p}{p-1}\|f\|_{L^p(\mathbb{R}_+)}, \tag{1.59}$$
$$\|\mathscr{G}f\|_{L^p(\mathbb{R}_+)} < e^{\frac{1}{p}}\|f\|_{L^p(\mathbb{R}_+)}, \tag{1.60}$$
$$\|\mathscr{H}f\|_{L^p(\mathbb{R}_+)} < \left(1+\frac{1}{p}\right)\|f\|_{L^p(\mathbb{R}_+)}. \tag{1.61}$$

Moreover, since the above inequalities include the best possible constants on their right-hand sides, we are able to compute norms of the corresponding integral operators. Namely, since

$$\|\mathscr{A}\| = \sup_{f\neq 0}\frac{\|\mathscr{A}f\|_{L^p(\mathbb{R}_+)}}{\|f\|_{L^p(\mathbb{R}_+)}},$$

it follows that $\|\mathscr{A}\| = \frac{p}{p-1}$, and similarly $\|\mathscr{G}\| = e^{\frac{1}{p}}$, $\|\mathscr{H}\| = 1+\frac{1}{p}$.

Discrete versions of means operators $\mathscr{A},\mathscr{G},\mathscr{H} : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$, i.e. the operators $\overline{\mathscr{A}},\overline{\mathscr{G}},\overline{\mathscr{H}} : l^p \to l^p$, are defined by

$$(\overline{\mathscr{A}}a)_n = \frac{\sum_{k=1}^n a_k}{n},$$
$$(\overline{\mathscr{G}}a)_n = \left(\prod_{k=1}^n a_k\right)^{\frac{1}{n}},$$
$$(\overline{\mathscr{H}}a)_n = \frac{n}{\sum_{k=1}^n a_k^{-1}}.$$

With this notation, the discrete Hardy inequality, the Carleman inequality and the discrete Hardy-Carleman inequality respectively read

$$\|\overline{\mathscr{A}}a\|_{l^p} < \frac{p}{p-1}\|a\|_{l^p}, \tag{1.62}$$
$$\|\overline{\mathscr{G}}a\|_{l^p} < e^{\frac{1}{p}}\|a\|_{l^p}, \tag{1.63}$$
$$\|\overline{\mathscr{H}}a\|_{l^p} < \left(1+\frac{1}{p}\right)\|a\|_{l^p}. \tag{1.64}$$

Clearly, due to the best possible constants, above inequalities provide norms of the corresponding operators, that is, $\|\overline{\mathscr{A}}\| = \frac{p}{p-1}$, $\|\overline{\mathscr{G}}\| = e^{\frac{1}{p}}$, and $\|\overline{\mathscr{H}}\| = 1+\frac{1}{p}$.

In 1928, Hardy [48], proved the first weighted modification of the Hardy integral inequality, namely the inequality

$$\int_0^\infty x^{p-r}(\mathscr{A}f(x))^p dx < \left(\frac{p}{r-1}\right)^p \int_0^\infty x^{p-r} f^p(x)dx, \tag{1.65}$$

valid with $p > 1$, $r > 1$, $0 < \int_0^\infty x^{p-r} f^p(x)dx < \infty$, where the constant $\left(\frac{p}{r-1}\right)^p$ is the best possible. The *dual Hardy inequality*, accompanied with the dual integration operator or the *dual arithmetic mean operator*

$$\mathscr{A}^* f(x) = \frac{1}{x} \int_x^\infty f(t)dt,$$

asserts that

$$\int_0^\infty x^{p-r} \left(\mathscr{A}^* f(x)\right)^p dx < \left(\frac{p}{1-r}\right)^p \int_0^\infty x^{p-r} f^p(x)dx, \tag{1.66}$$

holds for $p > 1$ and $r < 1$, provided that $0 < \int_0^\infty x^{p-r} f^p(x)dx < \infty$.

In 2011, Čižmešija *et al.* investigated in the paper [35] general Hardy-type inequalities in the non-conjugate setting for $n = 2$. As a consequence, they obtained the inequality

$$\left[\int_0^\infty y^{(1-\lambda)q'} (\mathscr{A} f)^{q'} (y)\, dy\right]^{\frac{1}{q'}} \le \left(p'\lambda\right)^\lambda \|f\|_p. \tag{1.67}$$

This inequality coincides with the earlier Opic's estimate (see [69]). Clearly, for $\lambda = 1$, we obtain the Hardy inequality (1.53) in the original form.

In 1984, Cochran and Lee [34], obtained the following inequality

$$\int_0^\infty x^{\gamma-1} \exp\left[\frac{\alpha}{x^\alpha} \int_0^x t^{\alpha-1} \log f(t)dt\right] dx \le e^{\gamma/\alpha} \int_0^\infty x^{\gamma-1} f(x)dx, \tag{1.68}$$

with the best constant $e^{\gamma/\alpha}$, where $\alpha, \gamma \in \mathbb{R}, \alpha > 0$, and $\int_0^\infty x^{\gamma-1} f(x)dx < \infty$. Inequality (1.68) is known in the literature as the Levin-Cochran-Lee inequality and it includes the weighted geometric mean operator $\mathscr{G}_\alpha$ defined by

$$(\mathscr{G}_\alpha f)(x) = \exp\left[\frac{\alpha}{x^\alpha} \int_0^x t^{\alpha-1} \log f(t)dt\right]. \tag{1.69}$$

Clearly, if $\gamma = 1$, the above inequality may be rewritten as $\|\mathscr{G}_\alpha f\|_p \le e^{1/\alpha p} \|f\|_p$, $p > 1$, which means that the norm of operator $\mathscr{G}_\alpha : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$ is equal to $e^{1/\alpha p}$. It should be noticed here that for $\alpha = \gamma = 1$, inequality (1.68) reduces to the well-known Knopp inequality.

In order to define the weighted harmonic mean operator, we first cite the following inequality from [37]: Let $a, b, r, s \in \mathbb{R}, a < b, r < s, r, s \neq 0$, and f be a non-negative measurable function. Then,

$$\begin{aligned} &\left\{\frac{1}{(b-a)^\gamma} \int_a^b (x-a)^{\gamma-1} \left[\frac{1}{(x-a)^\alpha} \int_a^x (t-a)^{\alpha-1} f^r(t)dt\right]^{\frac{s}{r}} dx\right\}^{\frac{1}{s}} \\ &\le \left\{\frac{1}{(b-a)^\alpha} \int_a^b (x-a)^{\alpha-1} \left[\frac{1}{(x-a)^\gamma} \int_a^x (t-a)^{\gamma-1} f^s(t)dt\right]^{\frac{r}{s}} dx\right\}^{\frac{1}{r}}, \end{aligned} \tag{1.70}$$

where $\alpha, \gamma \in \mathbb{R}$. The above inequality is crucial in establishing the mixed means inequality (for more details see [37]).

The following generalization of inequality (1.57) has been established in [5].

Theorem 1.8 *Let α, γ, and $r > 0$ be real numbers such that $\alpha + \gamma r > 0$ and f be a non-negative measurable function. If $\int_0^\infty x^{\gamma-1} f(x)dx < \infty$, then*

$$\int_0^\infty x^{\gamma-1} \left[\frac{x^\alpha}{\int_0^x t^{\alpha-1} f^{-r}(t)dt} \right]^{\frac{1}{r}} dx \leq (\alpha+\gamma r)^{\frac{1}{r}} \int_0^\infty x^{\gamma-1} f(x)dx, \tag{1.71}$$

where the constant $(\alpha+\gamma r)^{\frac{1}{r}}$ is the best possible.

Proof. Setting $a = 0, s = 1$, and $r = -r$, inequality (1.70) reduces to

$$\int_0^b x^{\gamma-1} \left[\frac{x^\alpha}{\int_0^x t^{\alpha-1} f^{-r}(t)dt} \right]^{\frac{1}{r}} dx \leq b^{\frac{\alpha}{r}+\gamma} \left[\int_0^b \frac{x^{\alpha-1+\gamma r}}{\left(\int_0^x t^{\gamma-1} f(t)dt\right)^r} dx \right]^{-\frac{1}{r}}. \tag{1.72}$$

Further, since $\int_0^x t^{\gamma-1} f(t)dt \leq \int_0^b t^{\gamma-1} f(t)dt,\ 0 \leq x \leq b$, the right-hand side of (1.72) does not exceed

$$b^{\frac{\alpha}{r}+\gamma} \left(\int_0^b x^{\alpha-1+\gamma r} dx \right)^{-\frac{1}{r}} \left(\int_0^b x^{\gamma-1} f(x)dx \right) = (\alpha+\gamma r)^{\frac{1}{r}} \int_0^b x^{\gamma-1} f(x)dx.$$

Therefore we have

$$\int_0^b x^{\gamma-1} \left[\frac{x^\alpha}{\int_0^x t^{\alpha-1} f^{-r}(t)dt} \right]^{\frac{1}{r}} dx \leq (\alpha+\gamma r)^{\frac{1}{r}} \int_0^b x^{\gamma-1} f(x)dx,$$

so (1.71) follows by letting b to infinity.

In order to prove that (1.71) includes the best possible constant, we suppose that there exists a positive L, smaller than $(\alpha+\gamma r)^{\frac{1}{r}}$, such that the inequality

$$\int_0^\infty x^{\gamma-1} \left[\frac{x^\alpha}{\int_0^x t^{\alpha-1} f^{-r}(t)dt} \right]^{\frac{1}{r}} dx \leq L \int_0^\infty x^{\gamma-1} f(x)dx$$

holds for all non-negative functions $f : \mathbb{R}_+ \to \mathbb{R}$, provided $\int_0^\infty x^{\gamma-1} f(x)dx < \infty$. Considering the function

$$\widetilde{f}(x) = \begin{cases} x^{\varepsilon-\gamma}, & 0 < x \leq 1 \\ 0, & x > 1 \end{cases},$$

where $\varepsilon > 0$ is sufficiently small number, we have

$$\int_0^1 x^{\gamma-1} \left[\frac{x^\alpha}{\int_0^x t^{\alpha-1-r(\varepsilon-\gamma)}dt} \right]^{\frac{1}{r}} dx \leq L \int_0^1 x^{\varepsilon-1} dx = \frac{L}{\varepsilon}.$$

The above relation yields $(\alpha - r\varepsilon + \gamma r)^{\frac{1}{r}} \leq L$, and for $\varepsilon \to 0^+$, it follows that $(\alpha+\gamma r)^{\frac{1}{r}} \leq L$. This contradicts with $L < (\alpha+\gamma r)^{\frac{1}{r}}$, which means that $(\alpha+\gamma r)^{\frac{1}{r}}$ is the best possible constant in (1.71). $\square$

Motivated by Theorem 1.8, we define the weighted harmonic mean operator $\mathscr{H}_\alpha$ by

$$(\mathscr{H}_\alpha f)(x) = \frac{x^\alpha}{\int_0^x t^{\alpha-1} f^{-1}(t)dt}. \tag{1.73}$$

If $\gamma = 1$, the inequality (1.71) may be rewritten as $\|\mathscr{H}_\alpha f\|_p \le (\alpha + 1/p)\|f\|_p$, $p > 1$, so Theorem 1.8 implies that the norm of operator $\mathscr{H}_\alpha : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$ is equal to $\alpha + 1/p$.

Chapter 2

Some Classes of Hilbert-type Inequalities

2.1 More Accurate Discrete Hilbert-type Inequalities

A general form of Hilbert-type inequality with non-homogeneous kernel (Subsection 2.1.1) is established in [55]. On the other hand, nowadays, a particular attention is paid to developing various methods for improving the existing Hilbert-type inequalities. It turns out that the Hermite-Hadamard inequality is a quite useful tool for improving discrete Hilbert-type inequalities. Therefore, in Section 2.1.2, we establish a more accurate form of a Hilbert-type inequality based on the application of the Hermite-Hadamard inequality (see also [55]). For some related results, the reader can also consult the following papers: [49], [90] and [91].

In 2006, Yang [98], obtained the following discrete version of the Hilbert-type inequality: Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $u(t)$ be strictly increasing differentiable function on the interval (n_0-1,∞), $n_0\in\mathbb{N}$, such that $\lim_{t\to n_0-1}u(t)=0$ and $\lim_{t\to\infty}u(t)=\infty$. If the functions $[u(t)]^{\frac{s-2}{r}}u'(t)$, $r = p,q$, $s > 2-r$, are decreasing on (n_0-1,∞), then the inequality

$$\sum_{n=n_0}^{\infty}\sum_{m=n_0}^{\infty}\frac{a_m b_n}{(1+u(m)u(n))^s} \\ < B\left(\tfrac{p+s-2}{p},\tfrac{q+s-2}{q}\right)\left[\sum_{n=n_0}^{\infty}\frac{[u(n)]^{\frac{2}{q}(2-s)-1}}{[u'(n)]^{p-1}}a_n^p\right]^{\frac{1}{p}}\left[\sum_{n=n_0}^{\infty}\frac{[u(n)]^{\frac{2}{p}(2-s)-1}}{[u'(n)]^{q-1}}b_n^q\right]^{\frac{1}{q}} \tag{2.1}$$

holds for all non-negative real sequences $(a_n)_{n\geq n_0}$ and $(b_n)_{n\geq n_0}$, provided that the sums on the right-hand side converge and are not equal to zero. Here, $B(\cdot,\cdot)$ denotes the usual Beta function defined by $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$, $a > 0$, $b > 0$.

2.1.1 Extension to the Non-conjugate Case

Our intention here is to establish a more general Hilbert inequality that covers the inequality (2.1) with a non-homogeneous kernel. In addition, the results that follow, refer to the case of non-conjugate exponents (see Section 1.2).

In order to formulate and prove the corresponding extension, we first give some basic definitions. For a non-negative measurable function $h:\mathbb{R}_+\to\mathbb{R}$, we define

$$k(\eta) = \int_0^{\infty} h(t)t^{-\eta}dt. \tag{2.2}$$

If nothing else is explicitly stated, we assume that the integral $k(\eta)$ converges for considered values of η.

Besides, we consider the weight functions involving real differentiable functions. We denote by $H(a)$, $a \geq 1$, the set of all non-negative differentiable functions $u:\mathbb{R}_+\to\mathbb{R}$ such that u is strictly increasing on $(a-1,\infty)$ and $\lim_{t\to a-1}u(t) = 0$, $\lim_{t\to\infty}u(t) = \infty$.

Theorem 2.1 *Let p,q,p',q', and λ be as in (1.43) and (1.44), and let $u\in H(m_0)$, $v\in H(n_0)$, m_0, $n_0\in\mathbb{N}$. If $h:\mathbb{R}_+\to\mathbb{R}$ is a non-negative measurable function and A_1, A_2 are real parameters such that the functions $h(u(x)v(y))u'(x)[u(x)]^{-p'A_1}$ and $h(u(x)v(y))v'(y)[v(y)]^{-q'A_2}$ are decreasing on (m_0-1,∞) and (n_0-1,∞) for any fixed $y\in\mathbb{R}_+$ and $x\in\mathbb{R}_+$ respectively, then the inequality*

$$\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}h^{\lambda}(u(m)v(n))a_m b_n \\ \leq L\left[\sum_{m=m_0}^{\infty}[u(m)]^{p(A_1+A_2)-\frac{p}{q'}}[u'(m)]^{1-p}a_m^p\right]^{\frac{1}{p}} \\ \times\left[\sum_{n=n_0}^{\infty}[v(n)]^{q(A_1+A_2)-\frac{q}{p'}}[v'(n)]^{1-q}b_n^q\right]^{\frac{1}{q}}, \tag{2.3}$$

where $L = k^{\frac{1}{p'}}(p'A_1)k^{\frac{1}{q'}}(q'A_2)$, holds for all non-negative sequences $(a_m)_{m\geq m_0}$ and $(b_n)_{n\geq n_0}$.

Proof. Rewrite inequality (1.47) from Section 1.2 for the counting measures μ_1 and μ_2 on $\mathbb{N}$, the functions $K(x,y) = h(u(x)v(y))$, $(\varphi \circ u)(x) = [u(x)]^{A_1}[u'(x)]^{-\frac{1}{p'}}$ and $(\psi \circ v)(y) = [v(y)]^{A_2}[v'(y)]^{-\frac{1}{q'}}$, and the sequences $(a_m)_{m\geq m_0}$ and $(b_n)_{n\geq n_0}$. Clearly, the substitutions are well-defined, since u and v are injective functions. Then,

$$\begin{aligned}
&\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty} h^{\lambda}(u(m)v(n))a_m b_n \\
&\leq \left[\sum_{m=m_0}^{\infty} [u(m)]^{pA_1}[u'(x)]^{1-p}(F\circ u)^p(m)a_m^p\right]^{\frac{1}{p}} \\
&\times \left[\sum_{n=n_0}^{\infty} [v(n)]^{qA_2}[v'(y)]^{1-q}(G\circ v)^q(n)b_n^q\right]^{\frac{1}{q}},
\end{aligned} \tag{2.4}$$

where

$$(F\circ u)(m) = \Big[\sum_{n=n_0}^{\infty} h(u(m)v(n))v'(n)[v(n)]^{-q'A_2}\Big]^{\frac{1}{q'}},$$

$$(G\circ v)(n) = \Big[\sum_{m=m_0}^{\infty} h(u(m)v(n))u'(m)[u(m)]^{-p'A_1}\Big]^{\frac{1}{p'}}.$$

Taking into account that the function $h(u(x)v(y))[v(y)]^{-q'A_2}v'(y)$ is decreasing on (n_0-1,∞) for any fixed $x\in\mathbb{R}_+$, we have

$$(F\circ u)^{q'}(m) \leq \int_{n_0-1}^{\infty}\frac{h(u(m)v(y))}{[v(y)]^{q'A_2}}v'(y)dy,$$

since the sum on the left-hand side of this inequality represents the lower Darboux sum for the integral on the right-hand side. Now, passing to a new variable $t = u(m)v(y)$, we have

$$\int_{n_0-1}^{\infty}\frac{h(u(m)v(y))}{[v(y)]^{q'A_2}}v'(y)dy = [u(m)]^{q'A_2-1}\int_0^{\infty} h(t)t^{-q'A_2}dt,$$

that is,

$$(F\circ u)^{q'}(m) \leq [u(m)]^{q'A_2-1}k(q'A_2). \tag{2.5}$$

With the same arguments as above, it follows that

$$(G\circ v)^{p'}(n) \leq \int_{m_0-1}^{\infty}\frac{h(u(x)v(n))}{[u(x)]^{p'A_1}}u'(x)dx \leq [v(n)]^{p'A_1-1}k(p'A_1), \tag{2.6}$$

so relations (2.4), (2.5) and (2.6) yield (2.3). □

Remark 2.1 Considering inequality (2.3) with the function $h(t) = (1+t)^{-s}$ and the parameters $A_1 = A_2 = \frac{2-s}{\lambda p'q'}$, $s > 2-\lambda\min\{p',q'\}$, the constant L appearing on the right-hand side of (2.3) may be expressed in terms of the usual Beta function, i.e. $L = B^{\lambda}\left(\frac{s+\lambda p'-2}{\lambda p'}, \frac{s+\lambda q'-2}{\lambda q'}\right)$. Moreover, in the conjugate case, that is, when $\lambda = 1$, $p' = q$, and $q' = p$, inequality (2.3) reduces to the relation (2.1) from [98].

2.1.2 Applying the Hermite-Hadamard Inequality

While proving Theorem 2.1, we were establishing the integral bounds for the corresponding integral sums. Such sums were recognized as the lower Darboux sums for the corresponding integrals. This fact required monotonic decrease of the function that defines the integral sum.

In contrast to the previous section, we deal here with a slightly different method for estimating a sum with an integral, based on the well-known Hermite-Hadamard inequality. Clearly, this requires some extra assumptions regarding convexity, but as a consequence, we obtain an improvement of inequality (2.3).

Recall that $f:[a,b]\to\mathbb{R}$ is a convex function if

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y),$$

for all $x,y\in[a,b]$ and $t\in[0,1]$. The Hermite-Hadamard inequality asserts that

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(t)dt\leq\frac{f(a)+f(b)}{2}, \tag{2.7}$$

where $f:[a,b]\to\mathbb{R}$ is a convex function. In the sequel, we are going to adjust the left inequality in (2.7), to obtain a more precise estimates for integral sums.

Now, in order to present our main result, we define the integral

$$k(\alpha;r_1,r_2)=\int_{r_1}^{r_2}h(t)t^{-\alpha}dt,\quad 0\leq r_1<r_2\leq\infty, \tag{2.8}$$

where the arguments α, r_1 and r_2 are such that (2.8) converges. In addition, if $r_1=0$ and $r_2=\infty$, then the integral $k(\alpha;0,\infty)$ will be denoted by $k(\alpha)$, for short, as in previous section.

Theorem 2.2 *Let p,q,p',q', and λ be as in (1.43) and (1.44), and let $u\in H(m_0)$, $v\in H(n_0)$, m_0, $n_0\in\mathbb{N}$. If $h:\mathbb{R}_+\to\mathbb{R}$ is a non-negative measurable function and A_1, A_2 are real parameters such that the functions $h(u(x)v(y))[u(x)]^{-p'A_1}u'(x)$ and $h(u(x)v(y))[v(y)]^{-q'A_2}v'(y)$ are convex on $[m_0-\frac{1}{2},\infty)$ and $[n_0-\frac{1}{2},\infty)$ for any fixed $y\in\mathbb{R}_+$ and $x\in\mathbb{R}_+$ respectively, then the inequality*

$$\begin{aligned}&\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}h^{\lambda}(u(m)v(n))a_mb_n\\ &\leq\left[\sum_{m=m_0}^{\infty}[u(m)]^{p(A_1+A_2)-\frac{p}{q'}}[u'(m)]^{1-p}k^{\frac{p}{q'}}(q'A_2;u(m)v(n_0-\tfrac{1}{2}),\infty)a_m^p\right]^{\frac{1}{p}}\\ &\times\left[\sum_{n=n_0}^{\infty}[v(n)]^{q(A_1+A_2)-\frac{q}{p'}}[v'(n)]^{1-q}k^{\frac{q}{p'}}(p'A_1;u(m_0-\tfrac{1}{2})v(n),\infty)b_n^q\right]^{\frac{1}{q}}\end{aligned} \tag{2.9}$$

holds for all non-negative sequences $(a_m)_{m\geq m_0}$ and $(b_n)_{n\geq n_0}$.

Proof. Since the function $h(u(x)v(y))[v(y)]^{-q'A_2}v'(y)$ is convex on $[n_0-\frac{1}{2},\infty)$ for any fixed $x\in\mathbb{R}_+$, applying the Hermite-Hadamard inequality, i.e. the left inequality in (2.7), to unit intervals $[n-\frac{1}{2},n+\frac{1}{2}]$, yields the following inequalities:

$$\frac{h(u(m)v(n))v'(n)}{[v(n)]^{q'A_2}}\leq\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}\frac{h(u(m)v(y))}{[v(y)]^{q'A_2}}v'(y)dy,\quad n=n_0,n_0+1,\ldots$$

Now, summing these inequalities we have

$$(F\circ u)^{q'}(m)=\sum_{n=n_0}^{\infty}\frac{h(u(m)v(n))v'(n)}{[v(n)]^{q'A_2}}\leq\int_{n_0-\frac{1}{2}}^{\infty}\frac{h(u(m)v(y))}{[v(y)]^{q'A_2}}v'(y)dy,$$

while the change of variable $t=u(m)v(y)$ and definition (2.8) yield

$$\begin{aligned}\int_{n_0-\frac{1}{2}}^{\infty}\frac{h(u(m)v(y))}{[v(y)]^{q'A_2}}v'(y)dy&=[u(m)]^{q'A_2-1}\int_{u(m)v(n_0-\frac{1}{2})}^{\infty}h(t)t^{-q'A_2}dt\\&=[u(m)]^{q'A_2-1}k(q'A_2;u(m)v(n_0-\tfrac{1}{2}),\infty).\end{aligned}$$

Clearly, the previous two relations yield the estimate

$$(F\circ u)^{q'}(m)\leq[u(m)]^{q'A_2-1}k(q'A_2;u(m)v(n_0-\tfrac{1}{2}),\infty).$$

With the same arguments as above and utilizing the convexity of the function $h(u(x)v(y))[u(x)]^{-p'A_1}u'(x)$ on $[m_0-\frac{1}{2},\infty)$, for any fixed $y\in\mathbb{R}_+$, we also have

$$(G\circ v)^{p'}(n)\leq[v(n)]^{p'A_1-1}k(p'A_1;u(m_0-\tfrac{1}{2})v(n),\infty),$$

where the function $(G\circ v)(n)$ is defined in the proof of Theorem 2.1. Now, the inequality (2.9) follows by virtue of the relation (2.4). □

Remark 2.2 According to the obvious estimates

$$k\big(q'A_2;u(m)v(n_0-\tfrac{1}{2}),\infty\big)\leq k\left(q'A_2\right)\ \text{and}\ k\big(p'A_1;u(m_0-\tfrac{1}{2})v(n),\infty\big)\leq k\left(p'A_1\right),$$

$m\geq m_0,n\geq n_0,m,n\in\mathbb{N}$, it follows that the right-hand side of inequality (2.9) does not exceed the right-hand side of (2.3) (see Theorem 2.1). In such a way we obtain the interpolating sequence of inequalities, that is, inequality (2.9) is an improvement of (2.3).

The following application of Theorem 2.2 refers to the kernel $h:\mathbb{R}_+\to\mathbb{R}$, defined by $h(t)=(1+t)^{-s}$, $s>0$. In this case, the weight functions appearing in (2.9) may be expressed in terms of the incomplete Beta function. Recall that the incomplete Beta function is defined by

$$B_r(a,b)=\int_0^r t^{a-1}(1-t)^{b-1}dt,\quad a,b>0.\tag{2.10}$$

If $r=1$, the incomplete Beta function coincides with the usual Beta function and obviously, $B_r(a,b)\leq B(a,b)$, $a,b>0$, $0\leq r\leq 1$.

Corollary 2.1 *Let p, q, p', q', and λ be as in (1.43) and (1.44), and let α, $\beta \in [1,2]$, $s > 0$. If A_1 and A_2 are real parameters such that* $\max\{2(1-\frac{1}{\beta}), 1-s)\} \leq p'A_1 \leq 1$ *and* $\max\{2(1-\frac{1}{\alpha}), 1-s)\} \leq q'A_2 \leq 1$, *then the inequality*

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(1+m^\alpha n^\beta)^{\lambda s}} \\
&\leq \alpha^{-\frac{1}{p'}}\beta^{-\frac{1}{q'}}\left[\sum_{m=1}^{\infty} m^{\alpha p(A_1+A_2-\lambda)+p-1} B^{\frac{p}{q'}}_{\frac{2\beta}{2\beta+m^\alpha}}(s+q'A_2-1, 1-q'A_2)a_m^p\right]^{\frac{1}{p}} \\
&\quad\times\left[\sum_{n=1}^{\infty} n^{\beta q(A_1+A_2-\lambda)+q-1} B^{\frac{q'}{p}}_{\frac{2\alpha}{2\alpha+n^\beta}}(s+p'A_1-1, 1-p'A_1)b_n^q\right]^{\frac{1}{q}} \qquad (2.11)
\end{aligned}$$

holds for all non-negative sequences $(a_m)_{m\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$.

Proof. In order to apply Theorem 2.2, we first show that a class of functions $f(x) = (1+x^\alpha y^\beta)^{-s} x^{\alpha(1-a)-1}$, $\alpha \geq 1$, $a \geq 2(1-\frac{1}{\alpha})$, is convex on $\mathbb{R}_+$ for any fixed $y \in \mathbb{R}_+$. By a straightforward computation, it follows that

$$\begin{aligned}
f''(x) =& \alpha^2 s(s+1) y^{2\beta}(1+x^\alpha y^\beta)^{-s-2} x^{\alpha(3-a)-3} \\
&+ \alpha s[3-\alpha(3-2a)] y^\beta (1+x^\alpha y^\beta)^{-s-1} x^{\alpha(2-a)-3} \\
&+ [\alpha(1-a)-1][\alpha(1-a)-2](1+x^\alpha y^\beta)^{-s} x^{\alpha(1-a)-3},
\end{aligned}$$

which means that f is convex on $\mathbb{R}_+$, since $s > 0$ and $\alpha(2a-3)-3 = [\alpha(2-a)-2] + [\alpha(1-a)-1]$.

Hence, the assumptions of Theorem 2.2 are fulfilled, and we utilize inequality (2.9) with functions $h(t) = (1+t)^{-s}$, $u(x) = x^\alpha$, and $v(y) = y^\beta$, $\alpha, \beta \in [1,2]$. From the definition of the incomplete Beta function and passing to the new variable $t = \frac{1}{u} - 1$, we have

$$\begin{aligned}
k\left(q'A_2; \frac{m^\alpha}{2^\beta}, \infty\right) &= \int_{\frac{m^\alpha}{2^\beta}}^{\infty} \frac{t^{-q'A_2}}{(1+t)^s} dt = \int_0^{\frac{2^\beta}{2^\beta+m^\alpha}} u^{s+q'A_2-2}(1-u)^{-q'A_2} du \\
&= B_{\frac{2^\beta}{2^\beta+m^\alpha}}\left(s+q'A_2-1, 1-q'A_2\right),
\end{aligned}$$

and the proof is completed. □

Remark 2.3 Setting $\alpha = \beta = 1$ in (2.11), we obtain the corresponding inequality from [59]. Moreover, if $A_1 = A_2 = \frac{2-s}{\lambda p'q'}$, $s > 2 - \lambda \min\{p', q'\}$, the arguments of the incomplete Beta functions appearing in (2.11) become $\frac{s+\lambda p'-2}{\lambda p'}$ and $\frac{s+\lambda q'-2}{\lambda q'}$, as in Remark 2.1.

2.2 Multidimensional Version of the Hilbert-type Inequality

2.2.1 Integral Case

In this section we shall be concerned with a recent version of the Hilbert-type inequality on certain weighted Lebesgue spaces, derived in paper [97]. In order to state the corresponding result, it is necessary to introduce some definitions.

Let $\|\cdot\|_{L^r_\rho(\mathbb{R}^n_+)}$ denotes the norm of non-negative measurable function $f:\mathbb{R}^n_+\to\mathbb{R}$, with respect to non-negative measurable weight function $\rho:\mathbb{R}^n_+\to\mathbb{R}$, that is,

$$\|f\|_{L^r_\rho(\mathbb{R}^n_+)}=\left[\int_{\mathbb{R}^n_+}\rho(x)f^r(x)dx\right]^{\frac{1}{r}}. \tag{2.12}$$

In the above relation, $L^r_\rho(\mathbb{R}^n_+)$ denotes the weighted measure space, that is,

$$L^r_\rho(\mathbb{R}^n_+)=\left\{f:\mathbb{R}^n_+\to\mathbb{R};\|f\|_{L^r_\rho(\mathbb{R}^n_+)}<\infty\right\}. \tag{2.13}$$

Further, let $k_s:\mathbb{R}^2_+\to\mathbb{R}$ be a non-negative, measurable homogeneous function of degree $-s$, $s>0$, and let $\Phi_A:\mathbb{R}^m_+\to\mathbb{R}$, $\Psi_A:\mathbb{R}^n_+\to\mathbb{R}$ be the power weight functions defined by

$$\Phi_A(x)=|x|_\alpha{}^{pqA_1-m}\quad\text{and}\quad\Psi_A(y)=|y|_\beta{}^{pqA_2-n}, \tag{2.14}$$

where $m,n\in\mathbb{N}$, $A_1,A_2\in\mathbb{R}$, p and q are conjugate exponents i.e. $1/p+1/q=1$, $p>1$. Here $|\cdot|_\alpha$ denotes $\alpha-$norm of the vector $t=(t_1,t_2,\dots,t_n)\in\mathbb{R}^n_+$, that is,

$$|t|_\alpha=(t_1^\alpha+t_2^\alpha+\cdots+t_n^\alpha)^{\frac{1}{\alpha}},\ \ \alpha>0. \tag{2.15}$$

In addition, let $\overline{C}_A$ denotes the constant defined by

$$\overline{C}_A=\left[\frac{\Gamma^m\left(\frac{1}{\alpha}\right)}{\alpha^{m-1}\Gamma\left(\frac{m}{\alpha}\right)}\right]^{\frac{1}{q}}\left[\frac{\Gamma^n\left(\frac{1}{\beta}\right)}{\beta^{n-1}\Gamma\left(\frac{n}{\beta}\right)}\right]^{\frac{1}{p}}c_s(pA_2+1-n), \tag{2.16}$$

where $c_s(\eta)=\int_0^\infty k_s(1,t)t^{-\eta}dt$ and $\Gamma(\cdot)$ is the usual Gamma function defined by $\Gamma(a)=\int_0^\infty t^{a-1}e^{-t}dt$, $a>0$.

Considering the above described setting and the real parameters A_1 and A_2 satisfying relation $qA_1+pA_2=m+n-s$, Yang et al. [97], obtained the following two equivalent inequalities

$$\int_{\mathbb{R}^m_+}\int_{\mathbb{R}^n_+}k_s\left(|x|_\alpha,|y|_\beta\right)f(x)g(y)dxdy\le\overline{C}_A\|f\|_{L^p_{\Phi_A}(\mathbb{R}^m_+)}\|g\|_{L^q_{\Psi_A}(\mathbb{R}^n_+)}, \tag{2.17}$$

$$\left\{\int_{\mathbb{R}^n_+}|y|_\beta{}^{pqA_1+p(s-m)-n}\left[\int_{\mathbb{R}^m_+}k_s\left(|x|_\alpha,|y|_\beta\right)f(x)dx\right]^p dy\right\}^{\frac{1}{p}}\le \overline{C}_A\|f\|_{L^p_{\Phi_A}(\mathbb{R}^m_+)},\quad (2.18)$$

provided that there exist $\delta>0$ such that $c_s(\eta)$ converges for all η belonging to interval $[pA_2+1-n-\delta, pA_2+1-n]$. Moreover, the authors also showed that the constant $\overline{C}_A$ is the best possible in both inequalities (2.17) and (2.18).

Our intention in this subsection is to extend the above two inequalities to the multidimensional case. More precisely, our result will include a general homogeneous kernel of the form $K_\beta(|x_1|_{\beta_1},\dots,|x_n|_{\beta_n})$, $\beta_i>0$, $i=1,\dots,n$.

Conventions 2.1 *Throughout this subsection we suppose that all the functions are non-negative and measurable, so that all integrals converge. In addition, $|\mathbb{S}^{n-1}|_\alpha$ denotes the area of the unit sphere in $\mathbb{R}^n$, with respect to $\alpha-$norm* (2.15)*, that is*

$$|\mathbb{S}^{n-1}|_\alpha=\frac{2^n\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}.\quad (2.19)$$

Further, in light of defininition (1.5) *(see Section* 1.1*) we define the integral $c(\gamma_1,\dots,\gamma_{n-1})$ by*

$$c(\gamma_1,\dots,\gamma_{n-1})=\int_{\mathbb{R}^{n-1}_+}K(1,t_1\dots,t_{n-1})t_1^{\gamma_1}\cdots t_{n-1}^{\gamma_{n-1}}dt_1\cdots dt_{n-1},\quad (2.20)$$

provided that $c(\gamma_1,\dots,\gamma_{n-1})<\infty$ for

$$\gamma_1,\dots,\gamma_{n-1}>-1\quad\text{and}\quad \gamma_1+\cdots+\gamma_{n-1}+n<s+1.\quad (2.21)$$

Moreover, in order to establish the corresponding multidimensional setting, we introduce real parameters A_{ij}, $i,j=1,\dots,n$, satisfying

$$\sum_{i=1}^n A_{ij}=0,\qquad j=1,2,\dots,n,\quad (2.22)$$

and also define

$$\alpha_i=\sum_{j=1}^n A_{ij},\qquad i=1,2,\dots,n.\quad (2.23)$$

In the above setting, the power weight functions $\varphi_i:\mathbb{R}^{k_i}_+\to\mathbb{R}$, defined by

$$\varphi_i(x_i)=|x_i|_{\beta_i}^{\sum_{j=1,j\ne i}^n k_i-s+p_i\alpha_i},\quad (2.24)$$

fulfill conditions as in the statement of Theorem 1.1.

The main objective of this subsection is to extend inequalities (2.17) and (2.18) by virtue of Theorem 1.1. Namely, in the sequel we consider measure spaces $\Omega_i=\mathbb{R}^{k_i}_+$, $k_i\in\mathbb{N}$, equipped with the Lebesgue measures, a non-negative homogeneous function K of degree $-s$, $s>0$, and the functions ϕ_{ij} of the form $\phi_{ij}(x_j)=|x_j|_{\beta_j}^{A_{ij}}$. $A_{ij}\in\mathbb{R}$, $i,j=1,\dots,n$. Since the weight functions are expressed by the norms, we shall deal with the so-called n-dimensional spherical coordinates. The following integral formula will be utilized in deriving our generalizations.

Lemma 2.1 *If $\alpha > 0$, $n \in \mathbb{N}$ and $t = (t_1, t_2, \dots, t_n) \in \mathbb{R}_+^n$, then the relation*

$$\int_{\mathbb{R}_+^n} \varphi\left(|t|_\alpha\right) dt_1 dt_2 \dots dt_n = \frac{|\mathbb{S}^{n-1}|_\alpha}{2^n} \int_{\mathbb{R}_+} \varphi(u) u^{n-1} du, \tag{2.25}$$

holds for all non-negative measurable functions $\varphi : \mathbb{R}_+ \to \mathbb{R}$.

Proof. We start with the following integral formula

$$\int_{\substack{|t|_\alpha < R \\ t_1,\dots,t_n > 0}} \varphi\left(\frac{|t|_\alpha^\alpha}{R^\alpha}\right) dt_1 dt_2 \dots dt_n = \frac{R^n \Gamma^n\left(\frac{1}{\alpha}\right)}{\alpha^n \Gamma\left(\frac{n}{\alpha}\right)} \int_0^1 \varphi(u) u^{\frac{n}{\alpha}-1} du,$$

where $R > 0$. The proof of the previous formula can be found in [43]. Now, considering $\varphi(Ru^{1/\alpha})$ instead of $\varphi(u)$, the previous formula becomes

$$\int_{\substack{|t|_\alpha < R \\ t_1,\dots,t_n > 0}} \varphi\left(|t|_\alpha\right) dt_1 dt_2 \dots dt_n = \frac{R^n \Gamma^n\left(\frac{1}{\alpha}\right)}{\alpha^n \Gamma\left(\frac{n}{\alpha}\right)} \int_0^1 \varphi\left(Ru^{\frac{1}{\alpha}}\right) u^{\frac{n}{\alpha}-1} du,$$

that is

$$\int_{\substack{|t|_\alpha < R \\ t_1,\dots,t_n > 0}} \varphi\left(|t|_\alpha\right) dt_1 dt_2 \dots dt_n = \frac{\Gamma^n\left(\frac{1}{\alpha}\right)}{\alpha^{n-1} \Gamma\left(\frac{n}{\alpha}\right)} \int_0^R \varphi(v) v^{n-1} dv,$$

after using the substitution $v = Ru^{1/\alpha}$. Finally, by letting $R \to \infty$ and by virtue of the formula for the area of the unit sphere (2.19), we obtain (2.25), as claimed. □

Now, the application of Theorem 1.1 in the above described setting yields the following result.

Theorem 2.3 *Let $n, k_1, \dots, k_n \in \mathbb{N}$, $n \geq 2$, and $\beta_1, \dots, \beta_n \in \mathbb{R}_+$. Further, suppose $K_\beta : \mathbb{R}_+^n \to \mathbb{R}$ is a non-negative measurable homogeneous function of degree $-s$, $s > 0$, and let A_{ij}, $i, j = 1, \dots, n$, and α_i, $i = 1, \dots, n$, be real parameters fulfilling conditions* (2.22) *and* (2.23)*. If $f_i : \mathbb{R}_+^{k_i} \to \mathbb{R}$, $f_i \neq 0$, $i = 1, \dots, n$, are non-negative measurable functions, then the inequalities*

$$\int_{\mathbb{R}_+^{k_1}} \cdots \int_{\mathbb{R}_+^{k_n}} K_\beta(|x_1|_{\beta_1}, \dots, |x_n|_{\beta_n}) \prod_{i=1}^n f_i(x_i) dx_1 \dots dx_n < M \prod_{i=1}^n \|f_i\|_{L_{\varphi_i}^{p_i}(\mathbb{R}_+^{k_i})}, \tag{2.26}$$

and

$$\int_{\mathbb{R}_+^{k_n}} |x_n|_{\beta_n}^{(1-P)(\sum_{j=1}^{n-1} k_j - s) - P\alpha_n} \times \left[\int_{\mathbb{R}_+^{k_1}} \cdots \int_{\mathbb{R}_+^{k_{n-1}}} K_\beta(|x_1|_{\beta_1}, \dots, |x_n|_{\beta_n}) \prod_{i=1}^{n-1} f_i(x_i) dx_1 \dots dx_{n-1}\right]^P dx_n$$
$$< M^P \prod_{i=1}^{n-1} \|f_i\|_{L_{\varphi_i}^{p_i}(\mathbb{R}_+^{k_i})}, \tag{2.27}$$

where

$$M = \prod_{i=1}^{n}\left(\frac{|\mathbb{S}^{k_i-1}|_{\beta_i}}{2^{k_i}}\right)^{1-\frac{1}{p_i}} c\,[k_2-1+p_1A_{12},\ldots,k_2-1+p_1A_{1n}]^{\frac{1}{p_1}} \tag{2.28}$$
$$\times c\left[s-\sum_{j=1,j\neq 2}^{n}k_j-1-p_2(\alpha_2-A_{22}),k_3-1+p_2A_{23},\ldots,k_n-1+p_2A_{2n}\right]^{\frac{1}{p_2}}$$
$$\cdots c\Big[k_2-1+p_nA_{n2},\ldots,k_{n-1}-1+p_nA_{n,n-1},s-\sum_{j=1}^{n-1}k_j-1-p_n(\alpha_n-A_{nn})\Big]^{\frac{1}{p_n}},$$

$1/P=\sum_{i=1}^{n-1}1/p_i$, $p_iA_{ij}>-k_j$, $i\neq j$, $p_i(\alpha_i-A_{ii})<s-\sum_{j=1,j\neq i}^{n}k_j$, *hold and are equivalent.*

Proof. We use Theorem 1.1 equipped with the kernel

$$K(x_1,\ldots,x_n)=K_\beta(|x_1|_{\beta_1},\ldots,|x_n|_{\beta_n})$$

and the weight functions $\phi_{ij}(x_j)=|x_j|_{\beta_j}^{A_{ij}}$, where $\sum_{i=1}^{n}A_{ij}=0$ for every $j=1,\ldots,n$. Obviously, it suffices to calculate the functions $\omega_i(x_i)$, $i=1,\ldots,n$ defined in Theorem 1.1. By utilizing formula (2.25), we find that

$$\omega_1(x_1)=\int_{\mathbb{R}_+^{k_2}}\cdots\int_{\mathbb{R}_+^{k_n}}K_\beta(|x_1|_{\beta_1},\ldots,|x_n|_{\beta_n})\prod_{j=2}^{n}|x_j|_{\beta_j}^{p_1A_{1j}}dx_2\cdots dx_n$$
$$=\prod_{j=2}^{n}\frac{|\mathbb{S}^{k_j-1}|_{\beta_j}}{2^{k_j}}\int_{\mathbb{R}_+^{n-1}}K_\beta(|x_1|_{\beta_1},t_2,\ldots,t_n)\prod_{j=2}^{n}t_j^{k_j-1+p_1A_{1j}}dt_2\cdots dt_n.$$

In addition, taking into account the homogeneity of function K_β and the substitutions $u_i=t_i/|x_1|_{\beta_1}$, $i=2,\ldots,n$, we have

$$\omega_1(x_1)=\prod_{j=2}^{n}\frac{|\mathbb{S}^{k_j-1}|_{\beta_j}}{2^{k_j}}\int_{\mathbb{R}_+^{n-1}}|x_1|_{\beta_1}^{-s}K_\beta(1,u_2,\ldots,u_n)$$
$$\times\prod_{j=2}^{n}(|x_1|_{\beta_1}u_j)^{k_j-1+p_1A_{1j}}|x_1|_{\beta_1}^{n-1}du_2\ldots du_n$$
$$=\prod_{j=2}^{n}\frac{|\mathbb{S}^{k_j-1}|_{\beta_j}}{2^{k_j}}|x_1|_{\beta_1}{}^{\sum_{j=2}^{n}k_j-s+p_1(\alpha_1-A_{11})}c(k_2-1+p_1A_{12},\ldots,k_n-1+p_1A_{1n}).$$

Similarly, yet another application of Lemma 2.1 and the homogeneity of the function K_β yields the relation

$$\omega_2(x_2)=\int_{\mathbb{R}_+^{k_1}}\int_{\mathbb{R}_+^{k_3}}\cdots\int_{\mathbb{R}_+^{k_n}}K_\beta(|x_1|_{\beta_1},\ldots,|x_n|_{\beta_n})\prod_{j=1,j\neq 2}^{n}|x_j|_{\beta_j}^{p_2A_{2j}}dx_1dx_3\cdots dx_n$$

$$= \prod_{j=1,j\neq 2}^{n} \frac{|\mathbb{S}^{k_j-1}|_{\beta_j}}{2^{k_j}} \int_{\mathbb{R}_+^{n-1}} t_1^{-s} K_\beta(1, \frac{|x_2|_{\beta_2}}{t_1}, \frac{t_3}{t_1}, \dots, \frac{t_n}{t_1}) \times \prod_{j=1,j\neq 2}^{n} t_j^{k_j-1+p_2A_{2j}} dt_1 dt_3 \dots dt_n.$$

Now, in order to express the previous formula in terms of the integral formula (2.20), we use the following change of variables

$$t_1 = |x_2|_{\beta_2} u_2^{-1},\ t_i = |x_2|_{\beta_2} u_2^{-1} u_i,\ i = 3, \dots, n,$$

so that

$$\frac{\partial(t_1, t_3, \dots, t_n)}{\partial(u_2, u_3, \dots, u_n)} = |x_2|_{\beta_2}^{n-1} u_2^{-n}.$$

Here, $\frac{\partial(t_1,t_3,\dots,t_n)}{\partial(u_2,u_3,\dots,u_n)}$ denotes the Jacobian of the transformation. Therefore, we have

$$\begin{aligned}
\omega_2(x_2) &= \prod_{j=1,j\neq 2}^{n} \frac{|\mathbb{S}^{k_j-1}|_{\beta_j}}{2^{k_j}} |x_2|_{\beta_2}^{\sum_{j=1,j\neq 2}^{n} k_j - s + p_2(\alpha_2 - A_{22})} \\
&\times \int_{\mathbb{R}_+^{n-1}} K_\beta(1, u_2, \dots, u_n) u_2^{s-1-\sum_{j=1,j\neq 2}^{n} k_j - p_2(\alpha_2 - A_{22})} \\
&\times \prod_{j=3}^{n} u_j^{k_j-1+p_2A_{2j}} du_2 \dots du_n \\
&= \left(\prod_{j=1,j\neq 2}^{n} \frac{|\mathbb{S}^{k_j-1}|_{\beta_j}}{2^{k_j}} \right) \cdot |x_2|_{\beta_2}^{\sum_{j=1,j\neq 2}^{n} k_j - s + p_2(\alpha_2 - A_{22})} \\
&\times c(s - \sum_{j=1,j\neq 2}^{n} k_j - 1 - p_2(\alpha_2 - A_{22}), k_3 - 1 + p_2A_{23}, \dots, k_n - 1 + p_2A_{2n}).
\end{aligned}$$

Clearly, the same procedure can be drawn in order to express $\omega_i(x_i)$, $i = 3, \dots, n$, in terms of the integral formula (2.20):

$$\begin{aligned}
\omega_i(x_i) &= \left(\prod_{j=1,j\neq i}^{n} \frac{|\mathbb{S}^{k_j-1}|_{\beta_j}}{2^{k_j}} \right) |x_i|_{\beta_i}^{\sum_{j=1,j\neq i}^{n} k_j - s + p_i(\alpha_i - A_{ii})} \\
&\times c(k_1 - 1 + p_iA_{i2}, \dots, k_{i-1} - 1 + p_iA_{i,i-1}, \\
&\quad s - \sum_{j=1,j\neq i}^{n} k_j - 1 - p_i(\alpha_i - A_{ii}), k_{i+1} - 1 + p_iA_{i,i+1}, \dots, k_n - 1 + p_iA_{in}).
\end{aligned}$$

This gives inequalities (2.26) and (2.27). The proof is now completed. □

Further, our attention will be focused on determining the conditions under which the constants on the right-hand sides of inequalities (2.26) and (2.27) are the best possible. For that sake, it is natural to impose the following conditions on the parameters A_{ij} :

$$k_i + p_jA_{ji} = s - \sum_{j=1,j\neq i}^{n} k_j - p_i(\alpha_i - A_{ii}), \quad j \neq i, \quad i, j \in \{1, 2, \dots, n\}. \tag{2.29}$$

In that case, the constant M from Theorem 2.3 reduces to the form

$$M^* = \prod_{i=1}^{n} \left[\frac{|\mathbb{S}^{k_i-1}|_{\beta_i}}{2^{k_i}} \right]^{1-\frac{1}{p_i}} c(k_2 - 1 + \widetilde{A}_2, \ldots, k_n - 1 + \widetilde{A}_n), \tag{2.30}$$

with the abbreviations

$$\widetilde{A}_i = p_1 A_{1i} \quad \text{for} \quad i \neq 1 \quad \text{and} \quad \widetilde{A}_1 = p_n A_{n1}. \tag{2.31}$$

Moreover, if the parameters A_{ij} fulfill conditions as in (2.29), the inequalities (2.26) and (2.27) from Theorem 2.3 respectively read

$$\int_{\mathbb{R}_+^{k_1}} \cdots \int_{\mathbb{R}_+^{k_n}} K_\beta(|x_1|_{\beta_1}, \ldots, |x_n|_{\beta_n}) \prod_{i=1}^{n} f_i(x_i) dx_1 \ldots dx_n < M^* \prod_{i=1}^{n} \|f_i\|_{L_{\Phi_i}^{p_i}(\mathbb{R}_+^{k_i})} \tag{2.32}$$

and

$$\begin{aligned} &\int_{\mathbb{R}_+^{k_n}} |x_n|_{\beta_n}{}^{(1-P)(-k_n - p_n \widetilde{A}_n)} \\ &\quad \times \left[\int_{\mathbb{R}_+^{k_1}} \cdots \int_{\mathbb{R}_+^{k_{n-1}}} K_\beta(|x_1|_{\beta_1}, \ldots, |x_n|_{\beta_n}) \prod_{i=1}^{n-1} f_i(x_i) dx_1 \ldots dx_{n-1} \right]^P dx_n \\ &\quad < (M^*)^P \prod_{i=1}^{n-1} \|f_i\|_{L_{\Phi_i}^{p_i}(\mathbb{R}_+^{k_i})}, \end{aligned} \tag{2.33}$$

where M^* is defined by (2.30) and

$$\Phi_i(x_i) = |x_i|_{\beta_i}^{-k_i - p_i \widetilde{A}_i}, \quad i = 1, \ldots, n. \tag{2.34}$$

The following result yields the best possible constants in the inequalities (2.32) and (2.33).

Theorem 2.4 *If the parameters A_{ij}, $i, j = 1, \ldots, n$, fulfill conditions* (2.22) *and* (2.29), *then the constants M^* and $(M^*)^P$ are the best possible in the inequalities* (2.32) *and* (2.33).

Proof. Suppose that the constant factor M^*, given by (2.30), is not the best possible in inequality (2.32). This means that there exists a positive constant $M_1 < M^*$, such that (2.32) still holds when replacing M^* with M_1.

Further, consider the real-valued functions $\widetilde{f}_{i,\varepsilon} : \mathbb{R}_+^{k_i} \mapsto \mathbb{R}$, defined by the formulas

$$\widetilde{f}_{i,\varepsilon}(x_i) = \begin{cases} 0, & |x_i|_{\beta_i} < 1 \\ |x_i|_{\beta_i}{}^{\widetilde{A}_i - \frac{\varepsilon}{p_i}}, & |x_i|_{\beta_i} \geq 1 \end{cases}, \quad i = 1, \ldots, n,$$

where $0 < \varepsilon < \min_{1 \leq i \leq n}\{p_i k_i + p_i \widetilde{A}_i\}$. Our next step is to substitute these functions in inequality (2.32) including the smaller constant M_1. By using the n-dimensional spherical coordinates, the right-hand side of the inequality (2.32) becomes

$$M_1 \prod_{i=1}^{n} \left[\int_{|x_i|_{\beta_i} \geq 1} |x_i|_{\beta_i}^{-k_i - \varepsilon} dx_i \right]^{\frac{1}{p_i}} = M_1 \prod_{i=1}^{n} \left(\frac{|\mathbb{S}^{k_i-1}|_{\beta_i}}{2^{k_i}} \right)^{\frac{1}{p_i}} \int_1^\infty t^{-1-\varepsilon} dt$$

$$= \frac{M_1}{\varepsilon} \prod_{i=1}^{n} \left(\frac{|\mathbb{S}^{k_i-1}|_{\beta_i}}{2^{k_i}} \right)^{\frac{1}{p_i}}. \tag{2.35}$$

Further, let I_ε denotes the left-hand side of the inequality (2.32) multiplied by ε, for the above choice of functions $\widetilde{f}_{i,\varepsilon}$. By applying the n-dimensional spherical coordinates and the substitutions $u_i = t_i/t_1$, $i \neq 1$, we find that

$$\begin{aligned}
I_\varepsilon &= \varepsilon \int_{|x_1|_{\beta_1} \geq 1} \cdots \int_{|x_n|_{\beta_i} \geq 1} K_\beta(|x_1|_{\beta_1}, \ldots, |x_n|_{\beta_n}) \prod_{i=1}^{n} |x_i|_{\beta_i}^{\widetilde{A}_i - \frac{\varepsilon}{p_i}} dx_1 \ldots dx_n \\
&= \varepsilon \prod_{i=1}^{n} \left(\frac{|\mathbb{S}^{k_i-1}|_{\beta_i}}{2^{k_i}} \right) \int_1^\infty \cdots \int_1^\infty K_\beta(t_1, \ldots, t_n) \prod_{i=1}^{n} t_i^{k_i - 1 + \widetilde{A}_i - \frac{\varepsilon}{p_i}} dt_1 \ldots dt_n \\
&= \varepsilon \prod_{i=1}^{n} \left(\frac{|\mathbb{S}^{k_i-1}|_{\beta_i}}{2^{k_i}} \right) \int_1^\infty t_1^{-1-\varepsilon} \left(\int_{\frac{1}{t_1}}^\infty \cdots \int_{\frac{1}{t_1}}^\infty K_\beta(1, u_2, \ldots, u_n) \right. \\
&\qquad \left. \times \prod_{i=2}^{n} u_i^{k_i - 1 + \widetilde{A}_i - \frac{\varepsilon}{p_i}} du_2 \ldots du_n \right) dt_1.
\end{aligned}$$

Now, it is easy to establish the following lower bound for I_ε, that is,

$$\begin{aligned}
I_\varepsilon \geq{}& \prod_{i=1}^{n} \left(\frac{|\mathbb{S}^{k_i-1}|_{\beta_i}}{2^{k_i}} \right) \int_{\mathbb{R}_+^{n-1}} K_\beta(1, u_2, \ldots, u_n) \prod_{i=2}^{n} u_i^{k_i - 1 + \widetilde{A}_i - \frac{\varepsilon}{p_i}} du_2 \ldots du_n \\
&- \varepsilon \prod_{i=1}^{n} \left(\frac{|\mathbb{S}^{k_i-1}|_{\beta_i}}{2^{k_i}} \right) \int_1^\infty t_1^{-1} \sum_{j=2}^{n} I_j(t_1) dt_1,
\end{aligned} \tag{2.36}$$

where for $j = 2, \ldots, n$, $I_j(t_1)$ is defined by

$$I_j(t_1) = \int_{D_j} K_\beta(1, u_2, \ldots, u_n) \prod_{i=2}^{n} u_i^{k_i - 1 + \widetilde{A}_i - \frac{\varepsilon}{p_i}} du_2 \ldots du_n,$$

$D_j = \{(u_2, \ldots, u_n); 0 < u_j < \frac{1}{t_1}, 0 < u_l < \infty, l \neq j\}$. Without losing generality, it is enough to estimate the integral $I_2(t_1)$. In fact, setting $\alpha > 0$ such that $k_2 + \widetilde{A}_2 > \varepsilon/p_2 + \alpha$, since $-u_2^\alpha \log u_2 \to 0$ $(u_2 \to 0^+)$, there exists $L \geq 0$ such that $-u_2^\alpha \log u_2 \leq L$ $(u_2 \in (0,1])$. On the other hand, it follows easily that the parameters $\gamma_1 = k_2 - 1 + \widetilde{A}_2 - (\varepsilon/p_2 + \alpha)$ and $\gamma_i = k_{i+1} - 1 + \widetilde{A}_{i+1} - (\varepsilon/p_{i+1})$, $i = 2, \ldots, n-1$, satisfy conditions as in (2.21). Then, by virtue of the Fubini theorem, we have

$$\begin{aligned}
0 &\leq \int_1^\infty t_1^{-1} I_2(t_1) dt_1 \\
&= \int_1^\infty t_1^{-1} \left[\int_{\mathbb{R}_+^{n-2}} \int_0^{\frac{1}{t_1}} K_\beta(1, u_2, \ldots, u_n) \prod_{i=2}^{n} u_i^{k_i - 1 + \widetilde{A}_i - \frac{\varepsilon}{p_i}} du_2 \ldots du_n \right] dt_1 \\
&= \int_{\mathbb{R}_+^{n-2}} \int_0^1 K_\beta(1, u_2, \ldots, u_n) \prod_{i=2}^{n} u_i^{k_i - 1 + \widetilde{A}_i - \frac{\varepsilon}{p_i}} \left(\int_1^{\frac{1}{u_2}} t_1^{-1} dt_1 \right) du_2 \ldots du_n
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}_+^{n-2}} \int_0^1 K_\beta(1,u_2,\ldots,u_n) \prod_{i=2}^n u_i^{k_i-1+\widetilde{A}_i-\frac{\varepsilon}{p_i}} (-\log u_2) du_2 \ldots du_n \\
&\leq L \int_{\mathbb{R}_+^{n-2}} \int_0^1 K_\beta(1,u_2,\ldots,u_n) u_2^{k_2-1+\widetilde{A}_2-(\frac{\varepsilon}{p_2}+\alpha)} \prod_{i=3}^n u_i^{k_i-1+\widetilde{A}_i-\frac{\varepsilon}{p_i}} du_2 \ldots du_n \\
&\leq L \int_{\mathbb{R}_+^{n-1}} K_\beta(1,u_2,\ldots,u_n) u_2^{k_2-1+\widetilde{A}_2-(\frac{\varepsilon}{p_2}+\alpha)} \prod_{i=3}^n u_i^{k_i-1+\widetilde{A}_i-\frac{\varepsilon}{p_i}} du_2 \ldots du_n \\
&= L \cdot c\left(k_2-1+\widetilde{A}_2-(\frac{\varepsilon}{p_2}+\alpha), k_3-1+\widetilde{A}_3-\frac{\varepsilon}{p_2}, \ldots, k_n-1+\widetilde{A}_n-\frac{\varepsilon}{p_n}\right) < \infty.
\end{aligned}
$$

Hence, considering (2.36), we obtain

$$
I_\varepsilon \geq \prod_{i=1}^n \left(\frac{|\mathbb{S}^{k_i-1}|_{\beta_i}}{2^{k_i}}\right) c\left(k_2-1+\widetilde{A}_2-\frac{\varepsilon}{p_2}, \ldots, k_n-1+\widetilde{A}_n-\frac{\varepsilon}{p_n}\right) - o(1). \tag{2.37}
$$

Finally, taking into account the relations (2.35) and (2.37), we have that $M^* \leq M_1$ when $\varepsilon \to 0^+$, which is an obvious contradiction. It follows that the constant M^* is the best possible in (2.32).

In addition, since the equivalence preserves the best possible constant, the proof is completed. □

Remark 2.4 It should be noticed here that our Theorems 2.3 and 2.4 extend the corresponding result from [97] (see inequalities (2.17) and (2.18)). More precisely, setting $\beta_1 = \alpha$, $\beta_2 = \beta$, $p_1 = p$, $p_2 = q$, $k_1 = m$ and $k_2 = n$, we obtain the mentioned Yang's result.

Now, we consider the application of our general result, i.e. Theorem 2.3, to a particular homogeneous kernel, defined by

$$
K_1(x_1,\ldots,x_n) = \frac{1}{(x_1+\ldots+x_n)^s}, \; s > 0.
$$

Utilizing the integral formula derived in [82], we have

$$
\begin{aligned}
c(\beta_1-1,\ldots,\beta_{n-1}-1) &= \int_{\mathbb{R}_+^{n-1}} \frac{\prod_{i=1}^{n-1} t_i^{\beta_i-1}}{(1+\sum_{i=1}^{n-1} t_i)^s} dt_1 \ldots dt_{n-1} \\
&= \frac{\Gamma(s-\sum_{i=1}^{n-1}\beta_i)\prod_{i=1}^{n-1}\Gamma(\beta_i)}{\Gamma(s)}.
\end{aligned} \tag{2.38}
$$

Now, in the above described setting, as an immediate consequence of Theorem 2.3, we obtain the following result.

Corollary 2.2 *Suppose the parameters P, p_i, A_{ij}, $i,j = 1,\ldots,n$, are defined as in the statement of Theorem* 1.1. *If the parameters A_{ij}, $i,j = 1,\ldots,n$, fulfill the conditions as in* (2.29)*, then the inequalities*

$$
\int_{\mathbb{R}_+^{k_1}} \cdots \int_{\mathbb{R}_+^{k_n}} \frac{\prod_{i=1}^n f_i(x_i)}{(\sum_{i=1}^n |x_i|_{\beta_i})^s} dx_1 \ldots dx_n < L_1 \prod_{i=1}^n \|f_i\|_{L_{\Phi_i}^{p_i}(\mathbb{R}_+^{k_i})}, \tag{2.39}
$$

and

$$\left[\int_{\mathbb{R}_+^{k_n}} |x_n|_{\beta_n}{}^{(1-P)(-k_n-p_n\widetilde{A}_n)} \right.$$
$$\left. \times \left(\int_{\mathbb{R}_+^{k_1}} \cdots \int_{\mathbb{R}_+^{k_{n-1}}} \frac{\prod_{i=1}^n f_i(x_i)}{(\sum_{i=1}^n |x_i|_{\beta_i})^s} \right)^P dx_n \right]^{\frac{1}{P}}$$
$$< L_1 \prod_{i=1}^{n-1} \|f_i\|_{L^{p_i}_{\Phi_i}(\mathbb{R}_+^{k_i})}, \tag{2.40}$$

where $L_1 = \frac{1}{\Gamma(s)} \prod_{i=1}^n \left[\frac{|\mathbb{S}^{k_i-1}|_{\beta_i}}{2^{k_i}}\right]^{1-\frac{1}{p_i}} \Gamma(\widetilde{A}_i + k_i)$, *hold for all non-negative measurable functions* f_i, $i = 1,\ldots,n$, *and are equivalent. Moreover, the constant* L_1 *is the best possible in both inequalities* (2.39) *and* (2.40).

2.2.2 Discrete Case

In this subsection we refer to the recent paper [51], where Huang obtained multidimensional discrete Hilbert-type inequality equipped with conjugate parameters. His result is contained in the following theorem.

Theorem 2.5 *Suppose that* $n \in \mathbb{N}\setminus\{1\}$, p_i, $r_i > 1$, $i = 1,\ldots,n$, $\sum_{i=1}^n \frac{1}{p_i} = \sum_{i=1}^n \frac{1}{r_i} = 1$, $\frac{1}{q_n} = 1 - \frac{1}{p_n}$, $s > 0$, $0 < \alpha < 2$, $\beta \geq -\frac{1}{2}$, $s\alpha \max\{\frac{1}{2-\alpha}, 1\} \leq \min_{1\leq i\leq n}\{r_i\}$, $a_{m_i}^{(i)} \geq 0$ $(m_i \in \mathbb{N})$, *so that*

$$0 < \sum_{m_i=1}^{\infty} (m_i+\beta)^{p_i(1-\frac{s\alpha}{r_i})-1} \left(a_{m_i}^{(i)}\right)^{p_i} < \infty \ \ (i = 1,\ldots,n).$$

Then the following two inequalities hold and are equivalent:

$$\sum_{m_n=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} \frac{1}{[\sum_{i=1}^n (m_i+\beta)^\alpha]^s} \prod_{i=1}^n a_{m_i}^{(i)}$$
$$< \frac{\alpha^{1-n}}{\Gamma(s)} \prod_{i=1}^n \Gamma\left(\frac{s}{r_i}\right) \left(\sum_{m_i=1}^{\infty} (m_i+\beta)^{p_i(1-\frac{s\alpha}{r_i})-1} \left(a_{m_i}^{(i)}\right)^{p_i} \right)^{\frac{1}{p_i}},$$

$$\left[\sum_{m_n=1}^{\infty} (m_n+\beta)^{\frac{s\alpha q_n}{r_n}-1} \left(\sum_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} \frac{\prod_{i=1}^{n-1} a_{m_i}^{(i)}}{[\sum_{i=1}^n (m_i+\beta)^\alpha]^s} \right)^{q_n} \right]^{\frac{1}{q_n}}$$
$$< \frac{\Gamma\left(\frac{s}{r_n}\right)}{\alpha^{n-1}\Gamma(s)} \prod_{i=1}^{n-1} \Gamma\left(\frac{s}{r_i}\right) \left(\sum_{m_i=1}^{\infty} (m_i+\beta)^{p_i(1-\frac{s\alpha}{r_i})-1} \left(a_{m_i}^{(i)}\right)^{p_i} \right)^{\frac{1}{p_i}}.$$

The constant $\frac{\alpha^{1-n}}{\Gamma(s)} \prod_{i=1}^n \Gamma\left(\frac{s}{r_i}\right)$ *is the best possible.*

The main purpose of the present improvement is to generalize Theorem 2.5 in a view of Theorem 1.1. More precisely, in the sequel we deduce the discrete forms of inequalities (1.2) and (1.3) containing the homogeneous kernel. Morever, much attention is given to the investigation of the best possible constants in obtained inequalities, which can be attained in some general settings. As an application, we also consider some particular settings of our general results which reduce to some recent results known from the literature.

In order to obtain the constants involved in the inequalities, we use the function $c(\gamma_1,\dots,\gamma_{n-1})$, parameters A_{ij} and α_i defined in Conventions 2.1. We consider the discrete weight functions involving real differentiable functions. More precisely, we have the following definition.

Definition 2.1 *Let $r \in \mathbb{R}$. We denote by $H(r)$ the set of all non-negative differentiable functions $u:(0,\infty)\to\mathbb{R}$ satisfying the following conditions.*

(i) u is strictly increasing on $(0,\infty)$ and there exists $x_0 \in (0,\infty)$ such that $u(x_0)=1$.

(ii) $\lim_{x\to\infty} u(x)=\infty$, $[u(x)]^r u'(x)$ is decreasing on $(0,\infty)$.

Now, taking into account the above definition and notations as in Conventions 2.1, we have the following general result.

Theorem 2.6 *Let $p_1,\dots,p_n$ be conjugate parameters such that $p_i>1$, $i=1,\dots,n$, and let $\frac{1}{P}=\sum_{i=1}^{n-1}\frac{1}{p_i}$. Let $K:(0,\infty)^n\to\mathbb{R}$ be non-negative homogeneous function of degree $-s$, $s>0$, strictly decreasing in each variable, and let A_{ij}, $i,j=1,\dots,n$, and α_i, $i=1,\dots,n$ be real parameters satisfying (2.22) and (2.23). If $a_{m_i}^{(i)}\ge 0$ ($m_i\in\mathbb{N}$) and $u_i\in H(p_iA_{ij})$, $i,j=1,\dots,n$, $i\neq j$, then we have the following equivalent inequalities*

$$\sum_{m_n=1}^{\infty}\cdots\sum_{m_1=1}^{\infty} K(u_1(m_1),\dots,u_n(m_n))\prod_{i=1}^{n} a_{m_i}^{(i)} \tag{2.41}$$

$$\le L\prod_{i=1}^{n}\left(\sum_{m_i=1}^{\infty}[u_i(m_i)]^{n-s-1+p_i\alpha_i}[u_i'(m_i)]^{1-p_i}\left(a_{m_i}^{(i)}\right)^{p_i}\right)^{\frac{1}{p_i}},$$

$$\Bigg[\sum_{m_n=1}^{\infty}[u_n(m_n)]^{(1-P)(n-1-s)-P\alpha_n}$$

$$\times\left(\sum_{m_{n-1}=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}K(u_1(m_1),\dots,u_n(m_n))\prod_{i=1}^{n-1}a_{m_i}^{(i)}\right)^{P}\Bigg]^{\frac{1}{P}}$$

$$\le L\prod_{i=1}^{n-1}\left(\sum_{m_i=1}^{\infty}[u_i(m_i)]^{n-s-1+p_i\alpha_i}[u_i'(m_i)]^{1-p_i}\left(a_{m_i}^{(i)}\right)^{p_i}\right)^{\frac{1}{p_i}}, \tag{2.42}$$

where

$$L=c(p_1A_{12},\dots,p_1A_{1n})^{\frac{1}{p_1}}\cdot c(s-n-p_2(\alpha_2-A_{22}),p_2A_{23},\dots,p_2A_{2n})^{\frac{1}{p_2}}$$

$$\cdots c(p_n A_{n2}, \ldots, p_n A_{n,n-1}, s-n-p_n(\alpha_n - A_{nn}))^{\frac{1}{p_n}}, \tag{2.43}$$

and $p_i A_{ij} > -1$, $i \neq j$, $p_i(A_{ii} - \alpha_i) > n - s - 1$.

Proof. Rewrite the inequality (1.2) from Section 1.1 for the counting measure on $\mathbb{N}$,

$$\begin{aligned}(\phi_{ij} \circ u_j)(m_j) &= [u_j(m_j)]^{A_{ij}} [u_j'(m_j)]^{1/p_i}, i \neq j, \\ (\phi_{ii} \circ u_i)(m_i) &= [u_i(m_i)]^{A_{ii}} [u_i'(m_i)]^{1/p_i - 1},\end{aligned}$$

and the sequences $\left(a_{m_i}^{(i)}\right)$, $i = 1, \ldots, n$. Obviously, these substitutions are well defined, since u_i, $i = 1, \ldots, n$ are injective functions. Thus, in the above setting, we get

$$\begin{aligned}&\sum_{m_n=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} K(u_1(m_1), \ldots, u_n(m_n)) \prod_{i=1}^{n} a_{m_i}^{(i)} \\ &\quad \leq \prod_{i=1}^{n} \left(\sum_{m_i=1}^{\infty} [u_i(m_i)]^{p_i A_{ii}} [u_i'(m_i)]^{1-p_i} (\omega_i \circ u_i)(m_i) \left(a_{m_i}^{(i)}\right)^{p_i} \right)^{\frac{1}{p_i}},\end{aligned} \tag{2.44}$$

where

$$\begin{aligned}(\omega_i \circ u_i)(m_i) = &\sum_{m_n=1}^{\infty} \cdots \sum_{m_{i+1}=1}^{\infty} \sum_{m_{i-1}=1}^{\infty} \sum_{m_1=1}^{\infty} K(u_1(m_1), \ldots, u_n(m_n)) \\ &\times \left(\prod_{j=1, j\neq i}^{n} [u_j(m_j)]^{p_i A_{ij}} u_j'(m_j) \right).\end{aligned}$$

Our next task is to estimate the functions $(\omega_i \circ u_i)(m_i)$, $i = 1, \ldots, n$. Since the kernel K is strictly decreasing in each variable and $u_i \in H(p_i A_{ij})$, $i \neq j$, we conclude that the functions $\omega_i \circ u_i$, $i = 1, \ldots, n$, are strictly decreasing. Hence, we have

$$\begin{aligned}(\omega_1 \circ u_1)(m_1) \leq &\int_{(0,\infty)^{n-1}} K(u_1(m_1), u_2(x_2), \ldots, u_n(x_n)) \\ &\times \prod_{j=2}^{n} \left([u_j(x_j)]^{p_1 A_{1j}} u_j'(x_j)\right) dx_2 \ldots dx_n,\end{aligned} \tag{2.45}$$

since the left-hand side of this inequality is obviously the lower Darboux sum for the integral on the right-hand side of inequality. Further, by using the substitution $t_i = u_i(x_i)$, $i = 2, \ldots, n$, from (2.45) we get

$$(\omega_1 \circ u_1)(m_1) \leq \int_{(0,\infty)^{n-1}} K(u_1(m_1), t_2, \ldots, t_n) \prod_{j=2}^{n} t_j^{p_1 A_{1j}} dt_2 \ldots dt_n,$$

whence, in view of the homogeneity of the kernel K and the obvious change of variables, we have

$$(\omega_1 \circ u_1)(m_1) \leq \int_{(0,\infty)^{n-1}} [u_1(m_1)]^{-s} K(1, t_2/u_1(m_1), \ldots, t_n/u_1(m_1)) \prod_{j=2}^{n} t_j^{p_1 A_{1j}} dt_2 \ldots dt_n$$

$$= [u_1(m_1)]^{n-1-s+p_1(\alpha_1-A_{11})}k(p_1A_{12},\dots,p_1A_{1n}).$$

By using the same arguments as for the function $\omega_1\circ u_1$, we also get

$$(\omega_2\circ u_2)(m_2)\le\int_{(0,\infty)^{n-1}}K(t_1,u_2(m_2),t_3,\dots,t_n)\prod_{j=1,j\ne 2}^{n}t_j^{p_2A_{2j}}dt_1dt_3\dots dt_n. \tag{2.46}$$

Now, let J denotes the right-hand side of the inequality (2.46). It is easy to see that the transformation of variables

$$t_1=u_2(m_2)\cdot\frac{1}{v_2},\ t_i=u_2(m_2)\cdot\frac{v_i}{v_2},\ i=3,\dots,n,$$

yields

$$\frac{\partial(t_1,t_3,\dots,t_n)}{\partial(v_2,v_3,\dots,v_n)}=[u_2(m_2)]^{n-1}v_2^{-n},$$

where $\frac{\partial(t_1,t_3,\dots,t_n)}{\partial(v_2,v_3,\dots,v_n)}$ is the Jacobian of the transformation.

Further, by using the homogeneity of the kernel K and the change of variables introduced above, we have

$$\begin{aligned}
J &= \int_{(0,\infty)^{n-1}}t_1^{-s}K(1,u_2(m_2)/t_1,t_3/t_1\dots,t_n/t_1)\prod_{j=1,j\ne 2}^{n}t_j^{p_2A_{2j}}dt_1dt_3\dots dt_n\\
&= \int_{(0,\infty)^{n-1}}[u_2(m_2)]^{-s}v_2^{s}K(1,v_2,\dots,v_n)[u_2(m_2)]^{p_2(\alpha_2-A_{22})}\\
&\qquad\times v_2^{-p_2(\alpha_2-A_{22})}v_2^{p_2A_{23}}\dots v_n^{p_2A_{2n}}[u_2(m_2)]^{n-1}v_2^{-n}dv_2dv_3\dots dv_n\\
&= [u_2(m_2)]^{n-1-s+p_2(\alpha_2-A_{22})}\int_{(0,\infty)^{n-1}}v_2^{s-n-p_2(\alpha_2-A_{22})}\prod_{j=3}^{n}v_j^{p_2A_{2j}}dv_2\dots dv_n\\
&= [u_2(m_2)]^{n-1-s+p_2(\alpha_2-A_{22})}k(s-n-p_2(\alpha_2-A_{22}),p_2A_{23},\dots,p_2A_{2n}).
\end{aligned}$$

Hence, inequality (2.46) and the equality established above imply that

$$(F_2\circ u_2)(m_2)\le[u_2(m_2)]^{n-1-s+p_2(\alpha_2-A_{22})}k(s-n-p_2(\alpha_2-A_{22}),p_2A_{23},\dots,p_2A_{2n}).$$

In a similar manner we obtain

$$\begin{aligned}
(F_i\circ u_i)(m_i)\le&[u_i(m_i)]^{n-1-s+p_i(\alpha_i-A_{ii})}\\
&\times k(p_iA_{i2},\dots,p_iA_{i,i-1},s-n-p_i(\alpha_i-A_{ii}),p_iA_{i,i+1},\dots,p_iA_{in}),
\end{aligned}$$

for $i=3,\dots,n$. This completes the proof of inequality (2.41).

The proof of the inequality (2.42) follows from the inequality (1.3), by using the same estimates as in the first part of the proof. □

The next problem we are dealing with in this section, is to determine the conditions under which the constant L, defined by (2.43), is the best possible in inequalities (2.41) and (2.42). Considering Theorem 2.5, we see that the appropriate constant does not include any exponent. Bearing in mind that fact, we shall find the conditions under which the constant L reduces to the form without any exponents.

In order to obtain the constant without exponents, we impose the following conditions on the parameters A_{ij} (see also Section 1.1):

$$p_j A_{ji} = s - n - p_i(\alpha_i - A_{ii}),\ i,j = 1,2,\dots,n,\ i \neq j. \tag{2.47}$$

If the parameters A_{ij} fulfill the set of conditions as in (2.47), then the constant L from Theorem 2.6 reduces to the form

$$L^* = k(\widetilde{A}_2,\dots,\widetilde{A}_n), \tag{2.48}$$

where we used the abbreviations

$$\widetilde{A}_i = p_j A_{ji},\ \ i,j = 1,2,\dots,n,\ i \neq j. \tag{2.49}$$

Taking into account the set of conditions (2.47), it is easy to see that the parameters $\widetilde{A}_i$ satisfy the relation

$$\sum_{i=1}^{n} \widetilde{A}_i = s - n. \tag{2.50}$$

Furthermore, by using (2.22) and (2.49), we have the following relationship between the parameters A_{ii} and $\widetilde{A}_i$, $i = 1,2,\dots,n$:

$$\begin{aligned} A_{ii} &= -A_{1i} - A_{2i} - \cdots - A_{i-1,i} - A_{i+1,i} - \cdots - A_{ni} \\ &= -\frac{\widetilde{A}_i}{p_1} - \frac{\widetilde{A}_i}{p_2} - \cdots - \frac{\widetilde{A}_i}{p_{i-1}} - \frac{\widetilde{A}_i}{p_{i+1}} - \cdots - \frac{\widetilde{A}_i}{p_n} \\ &= \widetilde{A}_i\left(\frac{1}{p_i} - 1\right). \end{aligned} \tag{2.51}$$

Now, taking into account the relations (2.48), (2.49), and (2.51), the inequalities (2.41) and (2.42) with the parameters A_{ij}, $i,j = 1,2,\dots,n$, satisfying the set of conditions (2.47), become

$$\begin{aligned} &\sum_{m_n=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} K(u_1(m_1),\dots,u_n(m_n)) \prod_{i=1}^{n} a_{m_i}^{(i)} \\ &\quad \leq L^* \prod_{i=1}^{n} \left(\sum_{m_i=1}^{\infty} [u_i(m_i)]^{-1-p_i\widetilde{A}_i} [u_i'(m_i)]^{1-p_i} \left(a_{m_i}^{(i)}\right)^{p_i} \right)^{\frac{1}{p_i}}, \end{aligned} \tag{2.52}$$

and

$$\begin{aligned} &\left[\sum_{m_n=1}^{\infty} [u_n(m_n)]^{(1-P)(-1-p_n\widetilde{A}_n)} \left(\sum_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} K(u_1(m_1),\dots,u_n(m_n)) \prod_{i=1}^{n-1} a_{m_i}^{(i)} \right)^P \right]^{\frac{1}{P}} \\ &\quad \leq L^* \prod_{i=1}^{n-1} \left(\sum_{m_i=1}^{\infty} [u_i(m_i)]^{-1-p_i\widetilde{A}_i} [u_i'(m_i)]^{1-p_i} \left(a_{m_i}^{(i)}\right)^{p_i} \right)^{\frac{1}{p_i}}, \end{aligned} \tag{2.53}$$

where the constant L^* is defined by (2.48).

Now we prove that the constant L^* is the best possible in both inequalities (2.52) and (2.53). That is the content of the following theorem.

Theorem 2.7 *If the parameters A_{ij}, $i,j=1,\dots,n$, satisfy the conditions* (2.22) *and* (2.47), *then the constant L^* is the best possible in both inequalities* (2.52) *and* (2.53).

Proof. It is enough to show that the constant L^* is the best possible in inequality (2.52), since (2.52) and (2.53) are equivalent. For that sake, we consider the real sequences $\widetilde{a}_{m_i}^{(i)} = [u_i(m_i)]^{\widetilde{A}_i-\frac{\varepsilon}{p_i}} u_i'(m_i)$, where $\varepsilon > 0$ is sufficiently small number. Since $u_i \in H(\widetilde{A}_i)$, $i=1,\dots,n$, we may assume that u_i is strictly increasing on $(0,\infty)$ and that there exists $x_0 \in (0,\infty)$ such that $u_i(x_0)=1$.

Therefore, by considering integral sums, we have

$$\begin{aligned}
\frac{1}{\varepsilon} &= \int_1^\infty [u_i(x)]^{-1-\varepsilon} d[u_i(x)] < \sum_{m_i=1}^\infty [u_i(m_i)]^{-1-\varepsilon} u_i'(m_i) \\
&= \sum_{m_i=1}^\infty [u_i(m_i)]^{-1-p_i\widetilde{A}_i} [u_i'(m_i)]^{1-p_i} \left(\widetilde{a}_{m_i}^{(i)}\right)^{p_i} \\
&< \vartheta_i(1) + \int_1^\infty [u_i(x)]^{-1-\varepsilon} d[u_i(x)] = \vartheta_i(1) + \frac{1}{\varepsilon},
\end{aligned}$$

where the function ϑ_i is defined by $\vartheta_i(x) = [u_i(x)]^{-1-\varepsilon} u_i'(x)$. In other words, the following relation is valid:

$$\sum_{m_i=1}^\infty [u_i(m_i)]^{-1-p_i\widetilde{A}_i} [u_i'(m_i)]^{1-p_i} \left(\widetilde{a}_{m_i}^{(i)}\right)^{p_i} = \frac{1}{\varepsilon} + O(1), \qquad i=1,\dots,n. \tag{2.54}$$

Now, let us suppose that there exists a positive constant M, smaller than L^*, such that the inequality (2.52) is still valid, if we replace L^* by M. Hence, if we insert relations (2.54) in the inequality (2.52), with the constant M instead of L^*, we get

$$\widetilde{I} := \sum_{m_n=1}^\infty \cdots \sum_{m_1=1}^\infty K(u_1(m_1),\dots,u_n(m_n)) \prod_{i=1}^n \widetilde{a}_{m_i}^{(i)} < \frac{1}{\varepsilon}(M+o(1)). \tag{2.55}$$

Now, let us estimate the left-hand side of inequality (2.52). Namely, by inserting the above defined sequences $\left(\widetilde{a}_{m_i}^{(i)}\right)_{m_i\in\mathbb{N}}$ in the left-hand side of inequality (2.52), we easily get the inequality

$$\begin{aligned}
\widetilde{I} > \int_1^\infty [u_1(x_1)]^{\widetilde{A}_1-\frac{\varepsilon}{p_1}} \Bigg(\int_1^\infty \cdots \int_1^\infty & K(u_1(x_1),\dots,u_n(x_n)) \\
& \times \prod_{i=2}^n [u_i(x_i)]^{\widetilde{A}_i-\frac{\varepsilon}{p_i}} d[u_2(x_2)]\dots d[u_n(x_n)] \Bigg) d[u_1(x_1)].
\end{aligned} \tag{2.56}$$

Further, let J denotes the right-hand side of the inequality (2.56). By using the substitution $t_i = \frac{u_i(x_i)}{u_1(x_1)}$, $i = 2,\dots,n$, we find that

$$J = \int_1^\infty [u_1(x_1)]^{-1-\varepsilon} \left[\int_{1/u_1(x_1)}^\infty \cdots \int_{1/u_1(x_1)}^\infty K(1,t_2,\dots,t_n) \right.$$
$$\left. \times \prod_{i=2}^n t_i^{\widetilde{A}_i - \frac{\varepsilon}{p_i}} dt_2 \dots dt_n \right] d[u_1(x_1)].$$

Now, considering the obtained expression for J, we easily get inequality

$$J \geq \int_1^\infty [u_1(x_1)]^{-1-\varepsilon} \left[\int_{(0,\infty)^{n-1}} K(1,t_2,\dots,t_n) \prod_{i=2}^n t_i^{\widetilde{A}_i - \frac{\varepsilon}{p_i}} dt_2 \dots dt_n \right] d[u_1(x_1)]$$
$$- \int_1^\infty [u_1(x_1)]^{-1-\varepsilon} \sum_{j=2}^n I_j(u_1) d[u_1(x_1)], \tag{2.57}$$

where for $j = 2,\dots,n$, $I_j(u_1)$ is defined by

$$I_j(u_1) = \int_{D_j} K(1,t_2,\dots,t_n) \prod_{i=2}^n t_i^{\widetilde{A}_i - \frac{\varepsilon}{p_i}} dt_2 \dots dt_n,$$

and $D_j = \{(t_2,t_3,\dots,t_n); 0 < t_j \leq \frac{1}{u_1(x_1)},\ 0 < t_k < \infty, k \neq j\}$.

Without losing generality, it is enough to estimate the integral $I_2(x_1)$. Obviously, since $1 - t_2^\varepsilon \to 1$ $(t_2 \to 0^+)$, there exists the constant $C \geq 0$ such that $1 - t_2^\varepsilon \leq C$ $(t_2 \in (0,1])$. Now, by using the well-known Fubini's theorem, it follows that

$$0 \leq \varepsilon \int_1^\infty [u_1(x_1)]^{-1-\varepsilon} I_2(u_1) d[u_1(x_1)]$$
$$= \varepsilon \int_1^\infty [u_1(x_1)]^{-1-\varepsilon} \left[\int_{(0,\infty)^{n-2}} \int_0^{1/u_1(x_1)} K(1,t_2,\dots,t_n) \right.$$
$$\left. \times \prod_{i=2}^n t_i^{\widetilde{A}_i - \frac{\varepsilon}{p_i}} dt_2 \dots dt_n \right] d[u_1(x_1)]$$
$$= \varepsilon \int_{(0,\infty)^{n-2}} \int_0^1 K(1,t_2,\dots,t_n) \prod_{i=2}^n t_i^{\widetilde{A}_i - \frac{\varepsilon}{p_i}} \left(\int_1^{\frac{1}{t_2}} t_1^{-1-\varepsilon} dt_1 \right) dt_2 \dots dt_n$$
$$= \varepsilon \int_{(0,\infty)^{n-2}} \int_0^1 K(1,t_2,\dots,t_n) \prod_{i=2}^n t_i^{\widetilde{A}_i - \frac{\varepsilon}{p_i}} \left(\frac{1}{\varepsilon}(1 - t_2^\varepsilon) \right) dt_2 \dots dt_n$$
$$\leq C \int_{(0,\infty)^{n-2}} \int_0^1 K(1,t_2,\dots,t_n) \prod_{i=2}^n t_i^{\widetilde{A}_i - \frac{\varepsilon}{p_i}} dt_2 \dots dt_n$$
$$\leq C \int_{(0,\infty)^{n-1}} K(1,t_2,\dots,t_n) \prod_{i=2}^n t_i^{\widetilde{A}_i - \frac{\varepsilon}{p_i}} dt_2 \dots dt_n$$

$$= C \cdot k\left(\widetilde{A}_2 - \frac{\varepsilon}{p_2}, \ldots, \widetilde{A}_n - \frac{\varepsilon}{p_n}\right) < \infty.$$

Further, considering the above derived relation and inequality (2.57), we have that

$$\widetilde{I} \geq \frac{1}{\varepsilon} k\left(\widetilde{A}_2 - \frac{\varepsilon}{p_2}, \ldots, \widetilde{A}_n - \frac{\varepsilon}{p_n}\right) - o(1). \tag{2.58}$$

Finally, by comparing relations (2.55) and (2.58), we conclude that $L^* \leq M$ when $\varepsilon \to 0^+$, which is an obvious contradiction. Hence, it follows that the constant L^* is the best possible in (2.52). Clearly, the constant L^* is also the best possible in the inequality (2.53) since the equivalence preserves the best possible constant. The proof is now completed. □

Here, we shall be concerned with the homogeneous function

$$K_1(x_1, \ldots, x_n) = \frac{1}{(x_1 + \ldots + x_n)^s}, \; s > 0.$$

Note that the kernel K_1 is symmetric, strictly decreasing in each variable, and

$$\begin{aligned} k(\beta_1 - 1, \ldots, \beta_{n-1} - 1) &= \int_{(0,\infty)^{n-1}} \frac{\prod_{i=1}^{n-1} t_i^{\beta_i - 1}}{(1 + \sum_{i=1}^{n-1} t_i)^s} dt_1 \ldots dt_{n-1} \\ &= \frac{\Gamma(s - \sum_{i=1}^{n-1} \beta_i) \prod_{i=1}^{n-1} \Gamma(\beta_i)}{\Gamma(s)}, \end{aligned} \tag{2.59}$$

where we used the integral formula derived in [82]. Now, in the above described setting, as an immediate consequence of Theorems 2.6 and 2.7, we get the following result.

Corollary 2.3 *Suppose the parameters P, p_i, A_{ij}, $i, j = 1, \ldots, n$, and the functions $u_i : (0,\infty) \to \mathbb{R}$, $i = 1, \ldots, n$, are defined as in statement of Theorem 2.6. If the parameters A_{ij}, $i, j = 1, \ldots, n$, fulfill the set of conditions as in (2.47), then the inequalities*

$$\begin{aligned} &\sum_{m_n=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} \frac{\prod_{i=1}^{n} a_{m_i}^{(i)}}{(\sum_{i=1}^{n} u_i(m_i))^s} \\ &\quad \leq L_1 \prod_{i=1}^{n} \left(\sum_{m_i=1}^{\infty} [u_i(m_i)]^{-1-p_i\widetilde{A}_i} [u_i'(m_i)]^{1-p_i} \left(a_{m_i}^{(i)}\right)^{p_i} \right)^{\frac{1}{p_i}} \end{aligned} \tag{2.60}$$

and

$$\begin{aligned} &\left[\sum_{m_n=1}^{\infty} [u_n(m_n)]^{(1-P)(-1-p_n\widetilde{A}_n)} \right. \\ &\quad \left. \times \left(\sum_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} \frac{\prod_{i=1}^{n} a_{m_i}^{(i)}}{(\sum_{i=1}^{n} u_i(m_i))^s} \right)^P \right]^{\frac{1}{P}} \\ &\quad \leq L_1 \prod_{i=1}^{n-1} \left(\sum_{m_i=1}^{\infty} [u_i(m_i)]^{-1-p_i\widetilde{A}_i} [u_i'(m_i)]^{1-p_i} \left(a_{m_i}^{(i)}\right)^{p_i} \right)^{\frac{1}{p_i}}, \end{aligned} \tag{2.61}$$

where $L_1 = \frac{\prod_{i=1}^{n}\Gamma(\widetilde{A}_i+1)}{\Gamma(s)}$, *hold for all non-negative real sequences* $\left(a_{m_i}^{(i)}\right)_{m_i\in\mathbb{N}}$ *and are equivalent. Moreover, the constant* L_1 *is the best possible in both inequalities* (2.60) *and* (2.61).

Remark 2.5 Note that inequalities (2.60) and (2.61) contain the parameters $\widetilde{A}_i$, $i = 1,2,\dots,n$, since the parameters A_{ij}, $i,j = 1,2,\dots,n$ fulfill the set of conditions as in (2.47).

The following remark describes the connection between our Corollary 2.3 and Theorem 2.5 in detail.

Remark 2.6 It is obvious that our Corollary 2.3 is the generalization of Theorem 2.5 (see also [51]). Namely if we substitute the power functions $u_i(x_i) = (x_i+\beta)^{\alpha}$ and the parameters $\widetilde{A}_i = \frac{s}{r_i} - 1$, $i = 1,\dots,n$, in Corollary 2.3 we get the inequalities from Theorem 2.5 with the best possible constant $\frac{\alpha^{1-n}}{\Gamma(s)}\prod_{i=1}^{n}\Gamma\left(\frac{s}{r_i}\right)$.

We conclude this section with one more consequence of Corollary 2.3, known from the literature.

Remark 2.7 Let

$$A_{ii} = \frac{(n-s)(p_i-1)}{p_i^2} \text{ and } A_{ij} = \frac{s-n}{p_i p_j},\ i,j = 1,2,\dots,n,\ i \neq j, \tag{2.62}$$

where p_i, $i = 1,2,\dots,n$, are conjugate exponents. These parameters are symmetric and

$$\sum_{i=1}^{n} A_{ij} = \sum_{j=1}^{n} A_{ij} = \frac{(n-s)(p_i-1)}{p_i^2} + \sum_{j=1, j\neq i}^{n} \frac{s-n}{p_i p_j} = \frac{n-s}{p_i}\left(1 - \sum_{j=1}^{n}\frac{1}{p_j}\right) = 0.$$

Moreover, the above defined parameters satisfy the set of conditions as in (2.47), so the resulting relations will include the best possible constants.

Now, for the above choice of parameters A_{ij} defined by (2.62), and the functions $u_i(x_i) = x_i$, the inequalities (2.60) and (2.61) respectively read

$$\sum_{m_n=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}\frac{\prod_{i=1}^{n}a_{m_i}^{(i)}}{(\sum_{i=1}^{n}m_i)^s} \le L_2\prod_{i=1}^{n}\left(\sum_{m_i=1}^{\infty}m_i^{n-1-s}\left(a_{m_i}^{(i)}\right)^{p_i}\right)^{\frac{1}{p_i}} \tag{2.63}$$

and

$$\left[\sum_{m_n=1}^{\infty} m_n^{(1-P)(p_n-s-1)}\left(\sum_{m_{n-1}=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}\frac{\prod_{i=1}^{n}a_{m_i}^{(i)}}{(\sum_{i=1}^{n}m_i)^s}\right)^P\right]^{\frac{1}{P}}$$

$$\le L_2\prod_{i=1}^{n-1}\left(\sum_{m_i=1}^{\infty}m_i^{n-1-s}\left(a_{m_i}^{(i)}\right)^{p_i}\right)^{\frac{1}{p_i}}, \tag{2.64}$$

where $L_2 = \frac{1}{\Gamma(s)} \prod_{i=1}^{n} \Gamma\left(\frac{p_i+s-n}{p_i}\right)$. Note that the condition $s \leq \min_{1\leq i\leq n}\{p_i\}$ must be satisfied, so that the function u_i belongs to the set $H(p_iA_{ij})$, $i, j = 1,2,\ldots,n$ (see the statement of Theorem 2.6). Moreover, since we consider the Gamma function with positive argument, inequalities (2.63) and (2.64) hold under condition $n - \min_{1\leq i\leq n}\{p_i\} \leq s \leq \min_{1\leq i\leq n}\{p_i\}$. Finally, let us mention that our inequality (2.63) is a discrete variant of the appropriate integral result established in [82].

Remark 2.8 The multidimensional integral Hilbert-type inequalities are proved in [64] (Subsection 2.2.1), while the multidimensional discrete Hilbert-type inequalities are obtained in [58] (Subsection 2.2.2). Related results can found in [52], [93] and [101].

2.3 A Unified Treatment of Half-discrete Hilbert-type Inequalities

In this section we deal with the so-called half-discrete Hilbert-type inequalities, including both integral and sum. Recently, He and Yang [23], obtained the following result: Let $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, and let α, β, λ_1, λ_2 be real parameters such that $\lambda_1 + \lambda_2 = \alpha - \beta$, $-\beta < \lambda_1 < \alpha$, and $\lambda_2 \leq 1 - \beta$. Then the inequalities

$$\begin{aligned}\sum_{n=1}^{\infty} a_n \int_0^{\infty} \frac{\min^{\beta}\{x,n\}}{\max^{\alpha}\{x,n\}} f(x)dx &= \int_0^{\infty} f(x) \sum_{n=1}^{\infty} \frac{\min^{\beta}\{x,n\}}{\max^{\alpha}\{x,n\}} a_n dx \\ &< C \left[\int_0^{\infty} x^{p(1-\lambda_1)-1} f^p(x)dx\right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} a_n^q\right]^{\frac{1}{q}},\end{aligned} \tag{2.65}$$

$$\sum_{n=1}^{\infty} n^{p\lambda_2-1} \left[\int_0^{\infty} \frac{\min^{\beta}\{x,n\}}{\max^{\alpha}\{x,n\}} f(x)dx\right]^p < C^p \int_0^{\infty} x^{p(1-\lambda_1)-1} f^p(x)dx, \tag{2.66}$$

and

$$\int_0^{\infty} x^{q\lambda_1-1} \left[\sum_{n=1}^{\infty} \frac{\min^{\beta}\{x,n\}}{\max^{\alpha}\{x,n\}} a_n\right]^q dx < C^q \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} a_n^q, \tag{2.67}$$

hold for any non-negative measurable function $f : \mathbb{R}_+ \to \mathbb{R}$ and a non-negative sequence $a = (a_n)_{n\in\mathbb{N}}$, provided that $0 < \int_0^{\infty} x^{p(1-\lambda_1)-1} f^p(x)dx < \infty$ and $0 < \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} a_n^q < \infty$. Moreover, these inequalities are equivalent, and $C = \frac{1}{\alpha-\lambda_1} + \frac{1}{\beta+\lambda_1}$, C^p, and C^q are the best constants in the corresponding inequalities.

Observe that the kernel $K(x,y) = \frac{\min^{\beta}\{x,y\}}{\max^{\alpha}\{x,y\}}$, appearing in the above inequalities, is a homogeneous function. The main objective of the paper [57] was to provide a unified treatment of half-discrete Hilbert-type inequalities with a general homogeneous kernel.

The corresponding results will be presented throughout this section. For some related half-discrete Hilbert-type inequalities, concerning some particular classes of kernels and weight functions, the reader is referred to the following references: [50], [81], [89], [94] and [95].

Now, the first step is to reformulate Theorem 1.6 (see Section 1.2) for a half-discrete case. Namely, rewriting inequalities (1.47) and (1.48) for a Lebesgue measure $\mu_1 = dx$ on $\mathbb{R}_+$ and a counting measure μ_2 on $\mathbb{N}$, we have

$$\begin{aligned} &\sum_{n=1}^{\infty} a_n \int_0^{\infty} K^{\lambda}(x,n) f(x)\, dx = \int_0^{\infty} f(x) \left(\sum_{n=1}^{\infty} K^{\lambda}(x,n) a_n \right) dx \\ &\leq \|\varphi F f\|_{L^p(\mathbb{R}_+)} \|\psi G a\|_{l^q} \end{aligned} \tag{2.68}$$

and

$$\left[\sum_{n=1}^{\infty} \left(\frac{1}{\psi_n G_n} \int_0^{\infty} K^{\lambda}(x,n) f(x)\, dx \right)^{q'} \right]^{\frac{1}{q'}} \leq \|\varphi F f\|_{L^p(\mathbb{R}_+)}, \tag{2.69}$$

where p, q, and λ are real parameters as in (1.43) and (1.44). Clearly, in this form $\psi = (\psi_n)_{n\in\mathbb{N}}$, $a = (a_n)_{n\in\mathbb{N}}$ are non-negative sequences,

$$F(x) = \left[\sum_{n=1}^{\infty} K(x,n) \psi_n^{-q'} \right]^{\frac{1}{q'}}, \quad x \in \mathbb{R}_+, \tag{2.70}$$

$$G_n = \left[\int_0^{\infty} K(x,n) \varphi^{-p'}(x)\, dx \right]^{\frac{1}{p'}}, \quad n \in \mathbb{N}, \tag{2.71}$$

and we assume the convergence of integrals and series appearing in (2.68) and (2.69). Note also that the equality sign in (2.68) holds due to the Fubini theorem. In addition, relations (2.68) and (2.69) will be referred to as the general half-discrete Hilbert-type and Hardy-Hilbert-type inequalities, respectively. Moreover, interchanging the roles of parameters p and q, as well as making use of (1.48) with a counting measure μ_1 on $\mathbb{N}$ and a Lebesgue measure $\mu_2 = dx$ on $\mathbb{R}_+$, we obtain yet another half-discrete Hardy-Hilbert-type inequality:

$$\left[\int_0^{\infty} \left(\frac{1}{(\varphi F)(x)} \sum_{n=1}^{\infty} K^{\lambda}(x,n) a_n \right)^{p'} dx \right]^{\frac{1}{p'}} \leq \|\psi G a\|_{l^q}. \tag{2.72}$$

Of course, inequality (2.72) is equivalent to relations (2.68) and (2.69).

Now, our further step is to derive the corresponding inequalities for a homogeneous kernel with a negative degree of homogeneity. In order to establish the main result for the case of a homogeneous kernel, we give the following lemma:

Lemma 2.2 *If $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a non-negative homogeneous function of degree $-s$, $s > 0$, then*

$$\int_0^{\infty} K(x,t) t^{-\alpha} dt = x^{1-s-\alpha} k(\alpha), \quad x \in \mathbb{R}_+, \tag{2.73}$$

and

$$\int_0^\infty K(x,n)x^{-\alpha}dx = n^{1-s-\alpha}k(2-s-\alpha), \quad n \in \mathbb{N}, \tag{2.74}$$

where the function $k(\cdot)$ *is defined by* (1.22).

Proof. Making use of the homogeneity of K and the change of variables $t = ux$, we obtain (2.73). Similarly, utilizing $x = nu$ and $u = \frac{1}{t}$, we have (2.74). □

Now, exploiting inequalities (2.68), (2.69), and (2.72) in the context of a homogeneous kernel, we have:

Theorem 2.8 *Let* p, q, *and* λ *be real parameters as in* (1.43) *and* (1.44), *and let* $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ *be a non-negative measurable homogeneous function of degree* $-s$, $s > 0$. *If* A_1 *and* A_2 *are real parameters such that the function* $K(x,y)y^{-q'A_2}$ *is decreasing on* $\mathbb{R}_+$ *for any fixed* $x \in \mathbb{R}_+$, *then the inequalities*

$$\begin{aligned}
&\sum_{n=1}^\infty a_n \int_0^\infty K^\lambda(x,n)f(x)\,dx = \int_0^\infty f(x)\left(\sum_{n=1}^\infty K^\lambda(x,n)a_n\right)dx \\
&\leq L\left[\int_0^\infty x^{\frac{p}{q'}(1-s)+p(A_1-A_2)} f^p(x)dx\right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty n^{\frac{q}{p'}(1-s)+q(A_2-A_1)} a_n^q\right]^{\frac{1}{q}},
\end{aligned} \tag{2.75}$$

$$\begin{aligned}
&\left[\sum_{n=1}^\infty n^{\frac{q'}{p'}(s-1)+q'(A_1-A_2)}\left(\int_0^\infty K^\lambda(x,n)f(x)\,dx\right)^{q'}\right]^{\frac{1}{q'}} \\
&\leq L\left[\int_0^\infty x^{\frac{p}{q'}(1-s)+p(A_1-A_2)} f^p(x)dx\right]^{\frac{1}{p}},
\end{aligned} \tag{2.76}$$

and

$$\begin{aligned}
&\left[\int_0^\infty x^{\frac{p'}{q'}(s-1)+p'(A_2-A_1)}\left(\sum_{n=1}^\infty K^\lambda(x,n)a_n\right)^{p'} dx\right]^{\frac{1}{p'}} \\
&\leq L\left[\sum_{n=1}^\infty n^{\frac{q}{p'}(1-s)+q(A_2-A_1)} a_n^q\right]^{\frac{1}{q}},
\end{aligned} \tag{2.77}$$

hold for any non-negative measurable function $f : \mathbb{R}_+ \to \mathbb{R}$ *and a non-negative sequence* $a = (a_n)_{n\in\mathbb{N}}$, *where* $L = k^{\frac{1}{q'}}(q'A_2)k^{\frac{1}{p'}}(2-s-p'A_1)$.

Proof. Rewrite inequality (2.68) for the function $\varphi(x) = x^{A_1}$ and the sequence $\psi_n = n^{A_2}$. Further, making use of (2.70) and (2.71), it follows that

$$F(x) = \left[\sum_{n=1}^\infty K(x,n)n^{-q'A_2}\right]^{\frac{1}{q'}}, \quad x \in \mathbb{R}_+, \tag{2.78}$$

and

$$G_n = \left[\int_0^\infty K(x,n) x^{-p'A_1}\, dx \right]^{\frac{1}{p'}}, \quad n \in \mathbb{N}. \tag{2.79}$$

In addition, since the function $K(x,y)y^{-q'A_2}$ is decreasing on $\mathbb{R}_+$ for any fixed $x \in \mathbb{R}_+$, we have

$$F(x) \le \left[\int_0^\infty K(x,t) t^{-q'A_2} dt \right]^{\frac{1}{q'}},$$

since the sum on the left-hand side of this inequality represents the lower Darboux sum for the integral on the right-hand side. Now, Lemma 2.2 provides relations

$$F(x) \le x^{\frac{1}{q'}(1-s)-A_2} k^{\frac{1}{q'}}(q'A_2) \tag{2.80}$$

and

$$\begin{aligned} G_n =& \left[\int_0^\infty K(x,n) x^{-p'A_1} dx \right]^{\frac{1}{p'}} \\ =& n^{\frac{1}{p'}(1-s)-A_1} k^{\frac{1}{p'}}(2-s-p'A_1). \end{aligned} \tag{2.81}$$

Finally, utilizing (2.68), (2.80), and (2.81), we get the inequality (2.75). Similarly, inequalities (2.76) and (2.77) follow from (2.69) and (2.72) respectively, by virtue of relations (2.80) and (2.81). □

The main problem in connection with Theorem 2.8 is whether or not L is the best possible constant in inequalities (2.75), (2.76), and (2.77) for some choices of parameters A_1 and A_2. Unfortunately, there is still no evidence that L is the best constant in the corresponding inequalities. This problem seems to be very hard in the non-conjugate case and remains still open. Luckily, we can solve the mentioned problem for some choices of A_1 and A_2 in the conjugate case.

2.3.1 The Conjugate Case and the Best Constants

We start with the conjugate version of Theorem 2.8, that is, when $q' = p$, $p' = q$ and $\lambda = 1$.

Corollary 2.4 *Let $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, and let $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be a non-negative measurable homogeneous function of degree $-s$, $s > 0$. If A_1 and A_2 are real parameters such that the function $K(x,y)y^{-pA_2}$ is decreasing on $\mathbb{R}_+$ for any fixed $x \in \mathbb{R}_+$, then the inequalities*

$$\begin{aligned} &\sum_{n=1}^\infty a_n \int_0^\infty K(x,n) f(x)\, dx = \int_0^\infty f(x) \left(\sum_{n=1}^\infty K(x,n) a_n \right) dx \\ &\le \overline{L} \left[\int_0^\infty x^{1-s+p(A_1-A_2)} f^p(x) dx \right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty n^{1-s+q(A_2-A_1)} a_n^q \right]^{\frac{1}{p}}, \end{aligned} \tag{2.82}$$

$$\left[\sum_{n=1}^{\infty} n^{\frac{p}{q}(s-1)+p(A_1-A_2)} \left(\int_0^{\infty} K(x,n)f(x)\,dx\right)^p\right]^{\frac{1}{p}} \leq \overline{L}\left[\int_0^{\infty} x^{1-s+p(A_1-A_2)} f^p(x)dx\right]^{\frac{1}{p}}, \tag{2.83}$$

and

$$\left[\int_0^{\infty} x^{\frac{q}{p}(s-1)+q(A_2-A_1)} \left(\sum_{n=1}^{\infty} K(x,n)a_n\right)^q dx\right]^{\frac{1}{q}} \leq \overline{L}\left[\sum_{n=1}^{\infty} n^{1-s+q(A_2-A_1)} a_n^q\right]^{\frac{1}{q}} \tag{2.84}$$

hold for any non-negative measurable function $f : \mathbb{R}_+ \to \mathbb{R}$ *and a non-negative sequence* $a = (a_n)_{n\in\mathbb{N}}$*, where*

$$\overline{L} = k^{\frac{1}{p}}(pA_2)k^{\frac{1}{q}}(2-s-qA_1). \tag{2.85}$$

Now, our intention is to determine conditions under which the constant $\overline{L} = k^{\frac{1}{p}}(pA_2)k^{\frac{1}{q}}(2-s-qA_1)$ is the best possible in inequalities (2.82), (2.83) and (2.84). Observe that the constants appearing in (2.65) contain no exponents dependent on p and q. Guided by that fact we are going to simplify the constant $\overline{L}$. Similarly to the previous sections we impose the condition

$$pA_2 + qA_1 = 2 - s, \tag{2.86}$$

since in this case relation $k(pA_2) = k(2-s-qA_1)$ holds. Moreover, $\overline{L}$ reduces to

$$L^* = k(pA_2), \tag{2.87}$$

so that inequalities (2.82), (2.83) and (2.84) read respectively as follows:

$$\sum_{n=1}^{\infty} a_n \int_0^{\infty} K(x,n)f(x)\,dx = \int_0^{\infty} f(x)\left(\sum_{n=1}^{\infty} K(x,n)a_n\right)dx \leq L^*\left[\int_0^{\infty} x^{-1+pqA_1} f^p(x)dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty} n^{-1+pqA_2}\, a_n^q\right]^{\frac{1}{q}}, \tag{2.88}$$

$$\left[\sum_{n=1}^{\infty} n^{(p-1)(1-pqA_2)} \left(\int_0^{\infty} K(x,n)f(x)\,dx\right)^p\right]^{\frac{1}{p}} \leq L^*\left[\int_0^{\infty} x^{-1+pqA_1} f^p(x)dx\right]^{\frac{1}{p}}, \tag{2.89}$$

and

$$\left[\int_0^\infty x^{(q-1)(1-pqA_1)}\left(\sum_{n=1}^\infty K(x,n)a_n\right)^q dx\right]^{\frac{1}{q}} \\ \leq L^*\left[\sum_{n=1}^\infty n^{-1+pqA_2}a_n^q\right]^{\frac{1}{q}}. \tag{2.90}$$

In the sequel, we show that L^* is the best constant in (2.88), (2.89), and (2.90), provided that $pA_2+qA_1=2-s$. In order to prove our assertions, we first provide the following auxiliary result.

Lemma 2.3 *Let the function K and parameters p, q, s, A_1, A_2 fulfill conditions as in Corollary 2.4 and let $pA_2+qA_1=2-s$, $pA_2<1$. For $0<\varepsilon<pq\left(\frac{1}{p}-A_2\right)$, define the function $\widetilde{f}:\mathbb{R}_+\to\mathbb{R}$ and the sequence $(\widetilde{a}_n)_{n\in\mathbb{N}}$ by*

$$\widetilde{f}(x)=x^{-qA_1-\frac{\varepsilon}{p}}\cdot\chi_{[1,\infty)}(x),\quad \widetilde{a}_n=n^{-pA_2-\frac{\varepsilon}{q}},$$

respectively, where χ_A is the characteristic function of a set A. If $\sup_{t\in(0,1)}K(1,t)<\infty$, then

$$\varepsilon\int_0^\infty \widetilde{f}(x)\left(\sum_{n=1}^\infty K(x,n)\widetilde{a}_n\right)dx \\ \geq k\left(pA_2+\frac{\varepsilon}{q}\right)-\frac{\varepsilon\sup_{t\in(0,1)}K(1,t)}{\left(1-pA_2+\frac{\varepsilon}{p}\right)\left(1-pA_2-\frac{\varepsilon}{q}\right)}, \tag{2.91}$$

where $k(\cdot)$ is defined by (1.22).

Proof. Let I_ε denote the left-hand side of relation (2.91). Then, it follows that

$$I_\varepsilon=\varepsilon\int_1^\infty\left[\sum_{n=1}^\infty K(x,n)n^{-pA_2-\frac{\varepsilon}{q}}\right]x^{-qA_1-\frac{\varepsilon}{p}}dx \\ \geq\varepsilon\int_1^\infty\left[\int_1^\infty K(x,y)y^{-pA_2-\frac{\varepsilon}{q}}dy\right]x^{-qA_1-\frac{\varepsilon}{p}}dx, \tag{2.92}$$

since the function $K(x,y)y^{-pA_2-\frac{\varepsilon}{q}}$ is decreasing on $\mathbb{R}_+$ for any fixed $x\in\mathbb{R}_+$ and for $0<\varepsilon<pq\left(\frac{1}{p}-A_2\right)$. Now, exploiting the change of variables $y=xt$, the homogeneity of the function K, and the condition $pA_2+qA_1=2-s$, the right-hand side of (2.92) can be transformed in the following way:

$$\varepsilon\int_1^\infty x^{-1-\varepsilon}\left(\int_{\frac{1}{x}}^\infty K(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dt\right)dx. \tag{2.93}$$

Further, since the function $K(1,t)$ is bounded on $(0,1)$, denoting $\alpha=\sup_{t\in(0,1)}K(1,t)$, it follows that

$$\int_{\frac{1}{x}}^\infty K(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dt\geq\int_0^\infty K(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dt-\alpha\int_0^{\frac{1}{x}}t^{-pA_2-\frac{\varepsilon}{q}}dt$$

$$= k\left(pA_2+\frac{\varepsilon}{q}\right)-\frac{\alpha}{1-pA_2-\frac{\varepsilon}{q}}x^{-1+pA_2+\frac{\varepsilon}{q}}, \quad x\geq 1,$$

and consequently,

$$\begin{aligned}\varepsilon\int_1^\infty x^{-1-\varepsilon}&\left[\int_{\frac{1}{x}}^\infty K(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dt\right]dx\\ &\geq k\left(pA_2+\frac{\varepsilon}{q}\right)-\frac{\varepsilon\alpha}{\left(1-pA_2+\frac{\varepsilon}{p}\right)\left(1-pA_2-\frac{\varepsilon}{q}\right)}.\end{aligned} \tag{2.94}$$

Finally, making use of (2.92), (2.93), and (2.94), we obtain inequality (2.91). □

The following theorem asserts that L^* is the best constant in (2.88), (2.89), and (2.90), assuming some weak conditions on the kernel.

Theorem 2.9 *Let the function K and parameters p,q,s,A_1,A_2 fulfill conditions of Corollary* 2.4 *and let $pA_2+qA_1=2-s$, $pA_2<1$. If $\sup_{t\in(0,1)}K(1,t)<\infty$, then L^* is the best possible constant in* (2.88), (2.89), *and* (2.90).

Proof. Due to the equivalence, it suffices to show that L^* is the best constant in inequality (2.88). In order to prove our assertion, suppose that there exists a positive constant L' smaller than L^*, such that inequality

$$\begin{aligned}\sum_{n=1}^\infty a_n\int_0^\infty K(x,n)f(x)dx&=\int_0^\infty f(x)\left(\sum_{n=1}^\infty K(x,n)a_n\right)dx\\ &\leq L'\left[\int_0^\infty x^{-1+pqA_1}f^p(x)dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty n^{-1+pqA_2}a_n^q\right]^{\frac{1}{p}}\end{aligned}$$

holds for all non-negative measurable functions $f:\mathbb{R}_+\to\mathbb{R}$ and non-negative sequences $a=(a_n)_{n\in\mathbb{N}}$. Now, considering the above inequality with the function $\widetilde{f}$ and the sequence $(\widetilde{a}_n)_{n\in\mathbb{N}}$, defined in the statement of Lemma 2.3, it follows that

$$\int_0^\infty \widetilde{f}(x)\left(\sum_{n=1}^\infty K(x,n)\widetilde{a}_n\right)dx\leq L'\left[\int_1^\infty x^{-1-\varepsilon}dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty n^{-1-\varepsilon}\right]^{\frac{1}{q}}. \tag{2.95}$$

Moreover, since the function $h(t)=t^{-1-\varepsilon}$ is decreasing on $\mathbb{R}_+$, we obtain the following estimate for the sequence appearing on the right-hand side of (2.95):

$$\sum_{n=1}^\infty n^{-1-\varepsilon}=1+\sum_{n=2}^\infty n^{-1-\varepsilon}<1+\int_1^\infty t^{-1-\varepsilon}dt=\frac{\varepsilon+1}{\varepsilon}. \tag{2.96}$$

Hence, making use of (2.95) and (2.96) yields the inequality

$$\varepsilon\int_0^\infty \widetilde{f}(x)\left(\sum_{n=1}^\infty K(x,n)\widetilde{a}_n\right)dx\leq L'(1+\varepsilon)^{\frac{1}{q}}. \tag{2.97}$$

Finally, utilizing relation (2.91), it follows that

$$k\left(pA_2+\frac{\varepsilon}{q}\right)-\frac{\varepsilon\,\sup_{t\in(0,1)}K(1,t)}{\left(1-pA_2+\frac{\varepsilon}{p}\right)\left(1-pA_2-\frac{\varepsilon}{q}\right)}\leq L'(1+\varepsilon)^{\frac{1}{q}},$$

which implies that $L^*=k(pA_2)\leq L'$, after letting $\varepsilon\to 0^+$. This contradiction shows that L^* is the best constant in (2.88). □

Remark 2.9 It should be noticed here that the integral version of Theorem 2.9 was proved in [77], while the corresponding discrete analogue can be found in [65].

2.3.2 Some Examples and Applications

In this subsection we deal with some particular choices of homogeneous kernels and real parameters A_1,A_2. In such a way we shall obtain Hilbert-type inequalities with the best constants expressed in terms of some well-known special functions.

Example 2.1 *Our first example refers to a homogeneous kernel $K_1:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ defined by $K_1(x,y)=(x+y)^{-s}$, $s>0$. Moreover, assume that A_1 and A_2 are real parameters such that $0\leq pA_2<1$, $qA_1<1$, and $pA_2+qA_1=2-s$. In this case, the function $K_1(x,y)y^{-pA_2}$ is decreasing on $\mathbb{R}_+$ for any fixed $x\in\mathbb{R}_+$, and $K_1(1,t)=(1+t)^{-s}$ is bounded on $(0,1)$ as well, so that the assumptions from Theorem 2.9 are fulfilled. Hence, in this setting we obtain inequalities (2.88), (2.89), and (2.90) with the best constant expressed in terms of a usual Beta function:*

$$L_1^*=k(pA_2)=\int_0^\infty\frac{u^{-pA_2}}{(1+u)^s}du=B(1-pA_2,s+pA_2-1)=B(1-pA_2,1-qA_1).$$

In particular, if $A_1=A_2=\frac{2-s}{pq}$, where $2-\min\{p,q\}<s\leq 2$, the above constant reduces to $B\left(\frac{p+s-2}{p},\frac{q+s-2}{q}\right)$, that is, to $B\left(\frac{1}{p},\frac{1}{q}\right)=\frac{\pi}{\sin\frac{\pi}{p}}$, when $s=1$. In this case we have a half-discrete version of the basic Hilbert inequality (1.1):

$$\sum_{n=1}^{\infty}a_n\int_0^\infty\frac{f(x)}{x+n}dx=\int_0^\infty f(x)\left(\sum_{n=1}^{\infty}\frac{a_n}{x+n}\right)dx\leq\frac{\pi}{\sin\frac{\pi}{p}}\|f\|_{L^p(\mathbb{R}_+)}\|a\|_{l^q}. \tag{2.98}$$

Example 2.2 *The constant appearing in our second example is expressed in terms of a Gaussian hypergeometric function. Recall that the Gaussian hypergeometric function is a formal power series, but we are interested here in its integral representation (see [1] and [46]):*

$$F(\alpha,\beta;\gamma;z)=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}dt,\ \gamma>\beta>0,|z|<1.$$

Here Γ denote the usual Gamma function, i.e. $\Gamma(a)=\int_0^\infty t^{a-1}e^{-t}dt$, $a>0$.

In order to obtain the corresponding constant, let $K_2 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ *be defined by* $K(x,y) = (x+y+\max\{x,y\})^{-s}$, $s>0$, *and let* $0 \le pA_2 < 1$, $qA_1 < 1$, *and* $pA_2 + qA_1 = 2-s$. *In this case the function* $K_2(x,y)y^{-pA_2}$ *is decreasing on* $\mathbb{R}_+$ *for any fixed* $x \in \mathbb{R}_+$, *and* $K_2(1,t) = (2+t)^{-s}$ *is bounded on* $(0,1)$, *so Theorem* 2.9 *provides the corresponding inequalities with the best possible constant*

$$\frac{2^{-s}}{1-qA_1}F\left(s,1-qA_1;2-qA_1;-\frac{1}{2}\right)+\frac{2^{-s}}{1-pA_2}F\left(s,1-pA_2;2-pA_2;-\frac{1}{2}\right),$$

that is, with the constant

$$L_2^* = \frac{q}{2}F\left(1,\frac{1}{q};1+\frac{1}{q};-\frac{1}{2}\right)+\frac{p}{2}F\left(1,\frac{1}{p};1+\frac{1}{p};-\frac{1}{2}\right),$$

when $s=1$ *and* $A_1 = A_2 = \frac{1}{pq}$. *In this case inequalities* (2.88), (2.89), *and* (2.90) *reduce respectively to*

$$\begin{aligned}\sum_{n=1}^{\infty} a_n \int_0^{\infty} \frac{f(x)}{x+n+\max\{x,n\}}\,dx &= \int_0^{\infty} f(x)\left(\sum_{n=1}^{\infty}\frac{a_n}{x+n+\max\{x,n\}}\right)dx\\ &\le L_2^*\|f\|_{L^p(\mathbb{R}_+)}\|a\|_{l^q},\end{aligned}$$

$$\left[\sum_{n=1}^{\infty}\left(\int_0^{\infty}\frac{f(x)}{x+n+\max\{x,n\}}dx\right)^p\right]^{\frac{1}{p}} \le L_2^*\|f\|_{L^p(\mathbb{R}_+)},$$

and

$$\left[\int_0^{\infty}\left(\sum_{n=1}^{\infty}\frac{a_n}{x+n+\max\{x,n\}}\right)^q dx\right]^{\frac{1}{q}} \le L_2^*\|a\|_{l^p}.$$

Example 2.3 *In order to complete the previous discussion, consider the kernel* $K_3(x,y) = \frac{\min^{\beta}\{x,y\}}{\max^{\alpha}\{x,y\}}$, $\alpha > \beta \ge 0$, *from the begining of this section, and parameters* A_1, A_2 *such that* $pA_2 + qA_1 = 2-\alpha+\beta$ *and* $\max\{1-\alpha,\beta\} < pA_2 < \beta+1$. *Since*

$$K_3(x,y)y^{-pA_2} = \begin{cases} x^{-\alpha}y^{\beta-pA_2}, & y \le x\\ x^{\beta}y^{-\alpha-pA_2}, & y > x\end{cases}$$

is decreasing function on $\mathbb{R}_+$ *for any fixed* $x \in \mathbb{R}_+$, *and* $K_3(1,t) = t^{\beta}$, $\beta > 0$, *is bounded on* $(0,1)$, *Theorem* 2.9 *provides the inequalities with the best constant*

$$\begin{aligned}L_3^* &= \int_0^{\infty}\frac{\min^{\beta}\{1,t\}}{\max^{\alpha}\{1,t\}}t^{-pA_2}dt\\ &= \int_0^1 t^{\beta-pA_2}dt + \int_1^{\infty} t^{-\alpha-pA_2}dt = \frac{1}{\beta-pA_2+1}+\frac{1}{\alpha+pA_2-1}.\end{aligned}$$

Moreover, with parameters $A_1 = \frac{1-\lambda_1}{q}$ *and* $A_2 = \frac{1-\lambda_2}{p}$, *where* $\lambda_1+\lambda_2 = \alpha-\beta$ *and* $\max\{\alpha-1,-\beta\} < \lambda_1 < \alpha$, *we obtain inequalities* (2.65), (2.66), *and* (2.67) *(see also* [23]*).*

Another interesting feature in connection with the best constants appears when considering certain operator expressions closely connected to Hardy-Hilbert-type inequalities (2.89) and (2.90). In order to simplify our discussion, we deal here with inequality (2.89) for $A_1 = \frac{1}{pq}$ (then, $A_2 = \frac{q+1-qs}{pq}$), and with (2.90) for $A_2 = \frac{1}{pq}$ (then, $A_1 = \frac{p+1-ps}{pq}$). In this context, inequalities (2.89) and (2.90) reduce respectively to

$$\|\mathscr{L}_1 f\|_{l^p} \leq k\Big(1+\frac{1}{q}-s\Big)\|f\|_{L^p(\mathbb{R}_+)} \tag{2.99}$$

and

$$\|\mathscr{L}_2 a\|_{L^q(\mathbb{R}_+)} \leq k\Big(\frac{1}{q}\Big)\|a\|_{l^q}, \tag{2.100}$$

where $\mathscr{L}_1 : L^p(\mathbb{R}_+) \to l^p$ and $\mathscr{L}_2 : l^q \to L^q(\mathbb{R}_+)$ are linear operators

$$(\mathscr{L}_1 f)_n = n^{s-1}\int_0^\infty K(x,n)f(x)dx,\ n \in \mathbb{N},$$

and

$$(\mathscr{L}_2 a)(x) = x^{s-1}\sum_{n=1}^{\infty} K(x,n)a_n,\ x > 0.$$

Due to inequalities (2.99) and (2.100), the operators $\mathscr{L}_1$ and $\mathscr{L}_2$ are well-defined and they are bounded, as well. Moreover, since $k\big(1+\frac{1}{q}-s\big)$ and $k\big(\frac{1}{q}\big)$ are the best constants in (2.99) and (2.100), we are able to determine norms of $\mathscr{L}_1$ and $\mathscr{L}_2$. Namely, exploiting this fact, it follows that

$$\|\mathscr{L}_1\| = \sup_{f\neq 0}\frac{\|\mathscr{L}_1 f\|_{l^p}}{\|f\|_{L^p(\mathbb{R}_+)}} = k\Big(1+\frac{1}{q}-s\Big)$$

and

$$\|\mathscr{L}_2\| = \sup_{a\neq 0}\frac{\|\mathscr{L}_2 a\|_{L^q(\mathbb{R}_+)}}{\|a\|_{l^q}} = k\Big(\frac{1}{q}\Big).$$

2.3.3 Refined Half-discrete Hilbert-type Inequalities

While proving half-discrete Hilbert-type inequalities, we were establishing integral bounds for the corresponding discrete sums. Such sums were recognized as the lower Darboux sums for the corresponding integrals. This fact required monotonic decrease of the function that defines the integral sum.

Similarly to the Subsection 2.1.2, we deal here with a slightly different approach in estimating a sum with an integral, based on the Hermite-Hadamard inequality. Of course, this requires some extra assumptions concerning convexity, but as a consequence, we shall obtain improvements of the corresponding half-discrete Hilbert-type inequalities in Theorem 2.8.

Theorem 2.10 *Let p, q, and λ be real parameters as in* (1.43) *and* (1.44), *and let* $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ *be a non-negative measurable homogeneous function of degree* $-s$, $s > 0$.

If A_1 and A_2 are real parameters such that the function $K(x,y)y^{-q'A_2}$ is convex on $\mathbb{R}_+$ for any fixed $x \in \mathbb{R}_+$, then the inequalities

$$\sum_{n=1}^{\infty} a_n \int_0^{\infty} K^{\lambda}(x,n) f(x)dx = \int_0^{\infty} f(x) \left(\sum_{n=1}^{\infty} K^{\lambda}(x,n) a_n \right) dx$$
$$\leq k^{\frac{1}{p'}}(2-p'A_1-s) \left[\int_0^{\infty} x^{\frac{p}{q'}(1-s)+p(A_1-A_2)} k^{\frac{p}{q'}}\left(q'A_2; \tfrac{1}{2x}, \infty\right) f^p(x)dx \right]^{\frac{1}{p}} \tag{2.101}$$
$$\times \left[\sum_{n=1}^{\infty} n^{\frac{q}{p'}(1-s)+q(A_2-A_1)} a_n^q \right]^{\frac{1}{q}},$$

$$\left[\sum_{n=1}^{\infty} n^{\frac{q'}{p'}(s-1)+q'(A_1-A_2)} \left(\int_0^{\infty} K^{\lambda}(x,n) f(x)\, dx \right)^{q'} \right]^{\frac{1}{q'}}$$
$$\leq k^{\frac{1}{p'}}(2-p'A_1-s) \left[\int_0^{\infty} x^{\frac{p}{q'}(1-s)+p(A_1-A_2)} k^{\frac{p}{q'}}\left(q'A_2; \tfrac{1}{2x}, \infty\right) f^p(x)dx \right]^{\frac{1}{p}}, \tag{2.102}$$

and

$$\left[\int_0^{\infty} x^{\frac{p'}{q'}(s-1)+p'(A_2-A_1)} k^{-\frac{p'}{q'}}\left(q'A_2; \tfrac{1}{2x}, \infty\right) \left(\sum_{n=1}^{\infty} K^{\lambda}(x,n) a_n \right)^{p'} dx \right]^{\frac{1}{p'}}$$
$$\leq k^{\frac{1}{p'}}(2-p'A_1-s) \left[\sum_{n=1}^{\infty} n^{\frac{q}{p'}(1-s)+q(A_2-A_1)} a_n^q \right]^{\frac{1}{q}} \tag{2.103}$$

hold for any non-negative measurable function $f : \mathbb{R}_+ \to \mathbb{R}$ and a non-negative sequence $a = (a_n)_{n\in\mathbb{N}}$.

Proof. We prove (2.101) only. To show this, we follow the same procedure as in the proof of Theorem 2.8, except that we provide a more precise estimate for the function $F(x)$ defined by (2.78) (see Theorem 2.8).

More precisely, since the function $K(x,y)y^{-q'A_2}$ is convex on interval $\mathbb{R}_+$ for any fixed $x \in \mathbb{R}_+$, applying the Hermite-Hadamard inequality, i.e. the left inequality in (2.7), to intervals $[n-\frac{1}{2}, n+\frac{1}{2}]$, yields the following inequalities:

$$\frac{K(x,n)}{n^{q'A_2}} \leq \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{K(x,t)}{t^{q'A_2}} dt, \quad n \in \mathbb{N}.$$

Now, summing these inequalities we have

$$\sum_{n=1}^{\infty} \frac{K(x,n)}{n^{q'A_2}} \leq \int_{\frac{1}{2}}^{\infty} \frac{K(x,t)}{t^{q'A_2}} dt.$$

In addition, making use of the homogeneity of the kernel K, it follows that

$$\int_{\frac{1}{2}}^{\infty} K(x,t)t^{-q'A_2}dt = x^{1-s-q'A_2} \int_{\frac{1}{2x}}^{\infty} K(1,u)u^{-q'A_2}du$$

$$=x^{1-s-q'A_2}k\left(q'A_2;\tfrac{1}{2x},\infty\right),$$

and consequently,

$$F(x) \leq x^{\frac{1}{q'}(1-s)-A_2}k^{\frac{1}{q'}}\left(q'A_2;\tfrac{1}{2x},\infty\right). \tag{2.104}$$

Finally, utilizing (2.68), (2.81), and (2.104), we obtain (2.101). □

Remark 2.10 According to an obvious estimate $k\left(q'A_2;\frac{1}{2x},\infty\right) \leq k\left(q'A_2\right)$, which holds for all $x \in \mathbb{R}_+$, it follows that the right-hand side of inequality (2.101) does not exceed the right-hand side of (2.75) (see Theorem 2.8). In such a way we get the interpolating sequence of inequalities, that is, inequality (2.101) refines (2.75). In the same way inequalities (2.102) and (2.103) represent improvements of (2.76) and (2.77), respectively. Therefore, the convexity assumptions in Theorem 2.10 yield a better result than the monotonicity assumptions of the kernel in Theorem 2.8.

Remark 2.11 Observe that in Theorem 2.10, it suffices to require the convexity of functions $K(x,y)y^{-q'A_2}$ on the interval $\left[\frac{1}{2},\infty\right)$, for any fixed $x \in \mathbb{R}_+$.

The following application of Theorem 2.10 refers to the homogeneous kernel $K: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, defined by $K(x,y) = (x+y)^{-s}$, $s > 0$. In such a way, we shall obtain the weight function expressed in terms of the incomplete Beta function. Recall that the incomplete Beta function (see Section 2.1) is defined by (2.10).

For $r = 1$ the incomplete Beta function coincides with the usual Beta function and obviously, $B_r(a,b) \leq B(a,b)$, $a,b > 0$, $0 \leq r \leq 1$. Due to the simplicity, we provide this consequence in Hilbert-type form only.

Corollary 2.5 *Let p, q, and λ satisfy* (1.43) *and* (1.44), *and let $s > 0$. If A_1 and A_2 are real parameters such that $p'A_1 \in (1-s,1)$ and $q'A_2 \in (\max\{1-s,0\},1)$, then the inequality*

$$\begin{aligned}
&\sum_{n=1}^{\infty} a_n \int_0^{\infty} \frac{f(x)}{(x+n)^{\lambda s}}\,dx = \int_0^{\infty} f(x)\left(\sum_{n=1}^{\infty}\frac{a_n}{(x+n)^{\lambda s}}\right)dx \\
&\leq B^{\frac{1}{p'}}\left(s+p'A_1-1,1-p'A_1\right) \\
&\quad\times\left[\int_0^{\infty} x^{\frac{p}{q'}(1-s)+p(A_1-A_2)}B^{\frac{p}{q'}}_{\frac{2x}{2x+1}}\left(s+q'A_2-1,1-q'A_2\right)f^p(x)dx\right]^{\frac{1}{p}} \\
&\quad\times\left[\sum_{n=1}^{\infty} n^{\frac{q}{p'}(1-s)+q(A_2-A_1)}\,a_n^q\right]^{\frac{1}{q}}
\end{aligned} \tag{2.105}$$

holds for any non-negative measurable function $f: \mathbb{R}_+ \to \mathbb{R}$ and a non-negative sequence $a = (a_n)_{n\in\mathbb{N}}$.

Proof. In order to apply Theorem 2.10, we first show that a class of functions $f_x(y) = (x+y)^{-s}y^{-a}$ is convex on $\mathbb{R}_+$ for any fixed $x \in \mathbb{R}_+$ and $a > 0$. By a straightforward computation, it follows that

$$f_1''(y) = \frac{(s+a)(s+a+1)y^2 + 2a(s+a+1)y + a(a+1)}{y^{a+2}(1+y)^{s+2}},$$

which means that f_1 is convex on $\mathbb{R}_+$, since $s > 0$ and $a > 0$. In addition, since $f_x''(y) = x^{-a-s-2} f_1''\left(\frac{y}{x}\right)$, it follows that f_x is convex on $\mathbb{R}_+$ for any fixed $x \in \mathbb{R}_+$.

Since the assumptions of Theorem 2.10 are fulfilled, we are able to apply inequality (2.101) in the case of homogeneous kernel $K(x,y) = (x+y)^{-s}$. From the definition of the incomplete Beta function and passing to the new variable $t = \frac{1}{u} - 1$, we have

$$\begin{aligned} k\left(q'A_2; \tfrac{1}{2x}, \infty\right) &= \int_{\frac{1}{2x}}^{\infty} \frac{t^{-q'A_2}}{(1+t)^s} dt = \int_0^{\frac{2x}{2x+1}} u^{s+q'A_2-2}(1-u)^{-q'A_2} du \\ &= B_{\frac{2x}{2x+1}}\left(s + q'A_2 - 1, 1 - q'A_2\right), \end{aligned}$$

while the definition of the usual Beta function yields

$$k(2 - p'A_1 - s) = \int_0^{\infty} \frac{t^{s+p'A_1-2}}{(1+t)^s} dt = B(s + p'A_1 - 1, 1 - p'A_1).$$

Now, the result follows from (2.101).

Note also that the intervals defining the parameters A_1 and A_2 are established due to the domain of the incomplete Beta function and the convexity of a class of functions f_x. □

Remark 2.12 Considering the parameters $A_1 = A_2 = \frac{1}{pq}$ and the kernel of degree -1 in the conjugate case, relation (2.105) provides the following interpolating set of inequalities

$$\begin{aligned} \sum_{n=1}^{\infty} a_n \int_0^{\infty} \frac{f(x)}{x+n} dx &= \int_0^{\infty} f(x) \left(\sum_{n=1}^{\infty} \frac{a_n}{x+n} \right) dx \\ &\le B^{\frac{1}{q}}\left(\frac{1}{p}, \frac{1}{q}\right) \left[\int_0^{\infty} B_{\frac{2x}{2x+1}}\left(\frac{1}{q}, \frac{1}{p}\right) f^p(x) dx \right]^{\frac{1}{p}} \|a\|_{l^q} \\ &\le \frac{\pi}{\sin \frac{\pi}{p}} \|f\|_{L^p(\mathbb{R}_+)} \|a\|_{l^q}, \end{aligned}$$

since $B_{\frac{2x}{2x+1}}\left(\frac{1}{q}, \frac{1}{p}\right) \le B\left(\frac{1}{q}, \frac{1}{p}\right) = B\left(\frac{1}{p}, \frac{1}{q}\right) = \frac{\pi}{\sin \frac{\pi}{p}}$. Observe that the above set of inequalities refines the half-discrete inequality (2.98).

Although we provided a unified treatment of half-discrete Hilbert-type inequalities with a homogeneous kernel, the described method regarding convexity can also be applied to non-homogeneous kernels. The following example refers to a homogeneous kernel $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, $K(x,y) = (1+xy)^{-s}$, $s > 0$, studied in [98].

Corollary 2.6 *Let p, q, and λ satisfy* (1.43) *and* (1.44)*, and let $s > 0$. If A_1 and A_2 are real parameters such that $p'A_1 \in (1-s,1)$ and $q'A_2 \in (\max\{1-s,0\},1)$, then the inequality*

$$\begin{aligned}
&\sum_{n=1}^{\infty} a_n \int_0^{\infty} \frac{f(x)}{(1+xn)^{\lambda s}}\,dx = \int_0^{\infty} f(x)\left(\sum_{n=1}^{\infty} \frac{a_n}{(1+xn)^{\lambda s}}\right)dx \\
&\leq B^{\frac{1}{p'}}(s+p'A_1-1,1-p'A_1) \\
&\times \left[\int_0^{\infty} x^{-\frac{p}{q'}+p(A_1+A_2)} B^{\frac{p}{q'}}_{\frac{2}{x+2}}\left(s+q'A_2-1,1-q'A_2\right) f^p(x)dx\right]^{\frac{1}{p}} \\
&\times \left[\sum_{n=1}^{\infty} n^{-\frac{q}{p'}+q(A_1+A_2)}\, a_n^q\right]^{\frac{1}{q}}
\end{aligned} \tag{2.106}$$

holds for any non-negative measurable function $f:\mathbb{R}_+ \to \mathbb{R}$ and a non-negative sequence $a = (a_n)_{n\in\mathbb{N}}$.

Proof. Similarly to the proof of Theorem 2.8, we start by exploiting inequality (2.68) with the function $\varphi(x) = x^{A_1}$ and the sequence $\psi_n = n^{A_2}$. Further, making use of (2.70) and (2.71), it follows that

$$F(x) = \left[\sum_{n=1}^{\infty}(1+xn)^{-s}n^{-q'A_2}\right]^{\frac{1}{q'}}, \quad x \in \mathbb{R}_+,$$

and

$$G_n = \left[\int_0^{\infty}(1+xn)^{-s}x^{-p'A_1}\,dx\right]^{\frac{1}{p'}}, \quad n \in \mathbb{N}.$$

From the definition of the usual Beta function, we have

$$\int_0^{\infty}(1+xn)^{-s}x^{-p'A_1}\,dx = n^{p'A_1-1}B(s+A_1p'-1,1-A_1p'),$$

i.e.

$$G_n = n^{A_1-\frac{1}{p'}}B^{\frac{1}{p'}}B(s+A_1p'-1,1-A_1p'), \quad n \in \mathbb{N}. \tag{2.107}$$

Now, in order to find the appropriate estimate for the function $F(x)$, we first show that a class of functions $g_x(y) = (1+xy)^{-s}y^{-a}$ is convex on $\mathbb{R}_+$ for any fixed $x \in \mathbb{R}_+$ and for $a,s > 0$. Namely, since $g_x(y) = x^a f_x(xy)$, where f_x, $x \in \mathbb{R}_+$, is a convex class of functions defined in the proof of Corollary 2.5, it follows that $g_x''(y) = x^{a+2}f_x''(xy)$, so that g_x is convex on $\mathbb{R}_+$ for any $x \in \mathbb{R}_+$.

Now, applying the Hermite-Hadamard inequality to intervals $[n-\frac{1}{2}, n+\frac{1}{2}]$ yields

$$(1+xn)^{-s}n^{-q'A_2} \leq \int_{n-\frac{1}{2}}^{n+\frac{1}{2}}(1+xt)^{-s}t^{-q'A_2}dt, \quad n \in \mathbb{N},$$

that is
$$\sum_{n=1}^{\infty}(1+xn)^{-s}n^{-q'A_2} \le \int_{\frac{1}{2}}^{\infty}(1+xt)^{-s}t^{-q'A_2}dt,$$
after summing these inequalities. Moreover, from the definition of the incomplete Beta function, it follows that

$$\begin{aligned}\int_{\frac{1}{2}}^{\infty}(1+xt)^{-s}t^{-q'A_2}dt =& x^{q'A_2-1}\int_0^{\frac{2}{x+2}} u^{s+q'A_2-2}(1-u)^{-q'A_2}du\\ =& x^{q'A_2-1}B_{\frac{2}{x+2}}(s+q'A_2-1,1-q'A_2),\end{aligned}$$

and consequently,

$$F(x) \le x^{A_2-\frac{1}{q'}}B_{\frac{2}{x+2}}^{\frac{1}{q'}}(s+q'A_2-1,1-q'A_2) \;\; x\in\mathbb{R}_+. \tag{2.108}$$

Finally, the result follows from (2.68), (2.107) and (2.108). □

Remark 2.13 All the results from this section are established in paper [57].

Chapter 3

Hilbert-type Inequalities on Time Scales

3.1 On Time Scales

Let us recall essentials about time scales. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Let $a,b \in \mathbb{T}$. The interval $[a,b]$ in time scale $\mathbb{T}$ is defined by $[a,b] := \{t \in \mathbb{T} : a \le t \le b\}$. We define the forward jump operator σ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, and the graininess μ of the time scale $\mathbb{T}$ by $\mu(t) := \sigma(t) - t$. A point $t \in \mathbb{T}$ is said to be right-dense, right-scattered, if $\sigma(t) = t$, $\sigma(t) > t$, respectively. We define $f^\sigma := f \circ \sigma$. For a function $f : \mathbb{T} \to \mathbb{R}$ the delta derivative is defined by

$$f^\Delta(t) := \lim_{s \to t, \sigma(s) \neq t} \frac{f^\sigma(s) - f(t)}{\sigma(s) - t}.$$

Here are some basic formulas involving delta derivatives: $f^\sigma = f + \mu f^\Delta$, $(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = f^\Delta g^\sigma + f g^\Delta$, $(f/g)^\Delta = (f^\Delta g - f g^\Delta)/(g g^\sigma)$, where f, g are delta differentiable and $g g^\sigma \neq 0$ in the last formula. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. The classes of real rd-continuous functions on an interval I will be denoted by $C_{rd}(I,\mathbb{R})$. For a, $b \in \mathbb{T}$ and a delta differentiable function f, the Cauchy integral is defined by $\int_a^b f^\Delta(t)\Delta t = f(b) - f(a)$. For the concept of the Riemann delta integral and the Lebesgue delta integral, see [25]. Note that the definition of the Riemann delta integrability is similar to the classical one of a real variable, and that the Lebesgue delta integral is the Lebesgue integral associated with the so-called Lebesgue delta measure. Every rd-continuous function is Riemann delta integrable, and every Riemann delta integrable function is Lebesgue delta integrable. Throughout, for convenience, when we speak about a delta integrability, we mean the integrability in some of the above senses. The integration by parts formula is given by:

$$\int_a^b u(t)v^{\Delta}(t)\Delta t = [u(t)v(t)]_a^b - \int_a^b u^{\Delta}(t)v^{\sigma}(t)\Delta t. \tag{3.1}$$

The chain rule formula (see [26], Theorem 1.90) that we will use in this chapter reads

$$(u^{\gamma}(t))^{\Delta} = \gamma\left(\int_0^1 [hu^{\sigma}(t) + (1-h)u(t)]^{\gamma-1}dh\right)u^{\Delta}(t), \tag{3.2}$$

where $\gamma > 1$ and $u : \mathbb{T} \to \mathbb{R}$ is delta differentiable function. For more details about time scales the reader is referred to [25], [26] and references therein.

3.2 Hilbert-type Inequalities

The results we present here are based on the mentioned results of Krnić and Pečarić obtained in [66]. First step is to reformulate the inequalities (1.17) and (1.18) for time scales. Namely, rewriting inequalities (1.17) and (1.18) for Lebesgue delta measures Δx, Δy and time scale interval $[a,b]$, we have

$$\begin{aligned} &\int_a^b\int_a^b K(x,y)f(x)g(y)\Delta x\Delta y \\ &\le \left[\int_a^b \varphi^p(x)F(x)f^p(x)\Delta x\right]^{\frac{1}{p}}\left[\int_a^b \psi^q(y)G(y)g^q(y)\Delta y\right]^{\frac{1}{q}} \end{aligned} \tag{3.3}$$

and

$$\int_a^b G^{1-p}(y)\psi^{-p}(y)\left[\int_a^b K(x,y)f(x)\Delta x\right]^p\Delta y \le \int_a^b \varphi^p(x)F(x)f^p(x)\Delta x, \tag{3.4}$$

where $p > 1$, $K : [a,b]\times[a,b] \to \mathbb{R}$, $f,g,\varphi,\psi : [a,b] \to \mathbb{R}$ are delta measurable, non-negative functions and

$$F(x) = \int_a^b \frac{K(x,y)}{\psi^p(y)}\Delta y \quad \text{and} \quad G(y) = \int_a^b \frac{K(x,y)}{\varphi^q(x)}\Delta x. \tag{3.5}$$

In what follows, without further explanation, we assume that all integrals exist on the respective domains of their definitions. By applying the inequalities (3.3) and (3.4) we obtain the following result.

Theorem 3.1 *Let $\mathbb{T}$ be a time scale with $a \in \mathbb{T}$. Let $\lambda \geq 2$, $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$, and define*

$$\Lambda(x) := \int_a^\infty \frac{1}{\sigma(y)} \left(\frac{1}{(x+\sigma(y))^\lambda (x+y)} + \frac{1}{(x+y)^\lambda (x+\sigma(y))} \right) \Delta y, \qquad x \in [a,\infty).$$

Then the following inequality

$$\begin{aligned}
&\int_a^\infty \int_a^\infty \frac{f(x)g(y)}{(x+y)^\lambda} \Delta x \Delta y \\
&\leq \left(\int_a^\infty [x\sigma(x)]^{p-1} \left(\frac{1}{a(a+x)^\lambda} - \Lambda(x) \right) f^p(x) \Delta x \right)^{\frac{1}{p}} \\
&\times \left(\int_a^\infty [y\sigma(y)]^{q-1} \left(\frac{1}{a(a+y)^\lambda} - \Lambda(y) \right) g^q(y) \Delta y \right)^{\frac{1}{q}}
\end{aligned} \tag{3.6}$$

holds for all non-negative and delta measurable functions $f, g : \mathbb{T} \to \mathbb{R}$.

Proof. Rewrite the inequality (3.3) for the functions $K(x,y) = (x+y)^{-\lambda}$, $\lambda \geq 2$, $\varphi(x) = [x\sigma(x)]^{1/q}$, $\psi(y) = [y\sigma(y)]^{1/p}$, $x, y \in [a,\infty)$. Further, making use of (3.5), it follows that

$$F(x) = G(x) = \int_a^\infty \frac{1}{y\sigma(y)} \frac{1}{(x+y)^\lambda} \Delta y, \qquad x \in [a,\infty). \tag{3.7}$$

Using the integration by parts formula (3.1) on the term $F(x)$ with

$$u^\lambda(y) = \frac{1}{(x+y)^\lambda} \qquad \text{and} \qquad v^\Delta(y) = \frac{1}{y\sigma(y)},$$

we have

$$F(x) = u^\lambda v|_a^\infty - \int_a^\infty (u^\lambda(y))^\Delta v^\sigma(y) \Delta y, \tag{3.8}$$

where

$$v(y) = -\frac{1}{y} \qquad \text{and} \qquad v^\sigma(y) = -\frac{1}{\sigma(y)}.$$

Applying the chain rule (3.2) we obtain

$$(u^\lambda(y))^\Delta = \lambda \left(\int_0^1 [hu^\sigma + (1-h)u]^{\lambda-1} dh \right) u^\Delta(y), \tag{3.9}$$

where

$$u^\Delta(y) = -\frac{1}{(x+y)(x+\sigma(y))}. \tag{3.10}$$

Taking into account (3.9) and an obvious inequality

$$(a+b)^\gamma \geq a^\gamma + b^\gamma, \;\; a, b \geq 0, \; \gamma \geq 1,$$

we have

$$
\begin{aligned}
&\lambda \int_0^1 \left[\frac{h}{x+\sigma(y)}+\frac{1-h}{x+y}\right]^{\lambda-1} dh \\
&\geq \lambda \int_0^1 \left[\left(\frac{h}{x+\sigma(y)}\right)^{\lambda-1}+\left(\frac{1-h}{x+y}\right)^{\lambda-1}\right] dh \\
&= \frac{1}{(x+\sigma(y))^{\lambda-1}}+\frac{1}{(x+y)^{\lambda-1}},
\end{aligned}
$$

and consequently,

$$
\begin{aligned}
F(x) &\leq u^\lambda v|_a^\infty - \int_a^\infty \frac{1}{\sigma(y)}\left(\frac{1}{(x+\sigma(y))^{\lambda-1}}+\frac{1}{(x+y)^{\lambda-1}}\right)\frac{1}{(x+y)(x+\sigma(y))}\Delta y \\
&= \frac{1}{a(a+x)^\lambda} - \Lambda(x).
\end{aligned} \tag{3.11}
$$

Finally, using (3.3) and (3.11) we obtain (3.6). □

The Hardy-Hilbert type inequality is proved in the following theorem.

Theorem 3.2 *Let $\mathbb{T}$ be a time scale with $a \in \mathbb{T}$. Let $\lambda \geq 2$, $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$, and let Λ be defined as in the statement of Theorem* 3.1. *Then the inequality*

$$
\begin{aligned}
&\int_a^\infty \frac{1}{y\sigma(y)}\left(\frac{1}{a(a+y)^\lambda}-\Lambda(y)\right)^{1-p}\left[\frac{f(x)}{(x+y)^\lambda}\Delta x\right]^p \Delta y \\
&\leq \int_a^\infty [x\sigma(x)]^{p-1}\left(\frac{1}{a(a+x)^\lambda}-\Lambda(x)\right) f^p(x)\Delta x
\end{aligned} \tag{3.12}
$$

holds for all non-negative and delta measurable functions $f : \mathbb{T} \to \mathbb{R}$.

Proof. The proof follows directly from the inequalities (3.4) and (3.11). Namely, if $p>1$, then we have

$$
\left[\frac{1}{a(a+y)^\lambda}-\Lambda(y)\right]^{1-p} \leq G^{1-p}(y),
$$

where $G(y)$ is defined by (3.7). Now, the inequality (3.12) follows easily from (3.4). □

Remark 3.1 For $\mathbb{T}=\mathbb{R}$, we have $\sigma(y)=y$, $y \in \mathbb{R}$, and the term $\Lambda(x)$ defined in Theorem 3.1 takes form

$$
\Lambda(x) = 2\int_a^\infty \frac{dy}{y(x+y)^{\lambda+1}}, \qquad x \in [a,\infty),\ a \in \mathbb{R}_+.
$$

For example, if $a=1$, $\lambda \geq 2$, then, applying the inequality (3.6) we obtain the following result

$$
\int_1^\infty\int_1^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy \leq \left(\int_1^\infty x^{2(p-1)}\left(\frac{1}{(x+1)^\lambda}-2F(1+\lambda,1+\lambda;2+\lambda;-x)\right) f^p(x)dx\right)^{\frac{1}{p}}
$$

$$\times\left(\int_1^\infty y^{2(q-1)}\left(\frac{1}{(y+1)^\lambda}-2F(1+\lambda,1+\lambda;2+\lambda;-y)\right)g^q(y)dy\right)^{\frac{1}{q}},$$

where $F(\alpha,\beta;\gamma;z)$ stands for the Gaussian hypergeometric function defined by

$$F(\alpha,\beta;\gamma;z)=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}dt,\ \gamma>\beta>0, z<1.$$

Remark 3.2 Similarly, for $\mathbb{T}=\mathbb{N}$, $a\in\mathbb{N}$, we obtain

$$\Lambda(n)=\sum_{s=a}^\infty\frac{1}{s+1}\left(\frac{1}{(n+s+1)^\lambda(n+s)}+\frac{1}{(n+s)^\lambda(n+s+1)}\right),\ n\in\mathbb{N},$$

and the inequalities (3.6) and (3.12) become

$$\begin{aligned}&\sum_{m=a}^\infty\sum_{n=a}^\infty\frac{f(m)g(n)}{(m+n)^\lambda}\\ &\leq\left(\sum_{m=a}^\infty[m(m+1)]^{p-1}\left(\frac{1}{a(a+m)^\lambda}-\Lambda(m)\right)f^p(m)\right)^{\frac{1}{p}}\\ &\times\left(\sum_{n=a}^\infty[n(n+1)]^{q-1}\left(\frac{1}{a(a+n)^\lambda}-\Lambda(n)\right)g^q(n)\right)^{\frac{1}{q}}\end{aligned}$$

and

$$\begin{aligned}&\sum_{n=a}^\infty\frac{1}{n+1}\left(\frac{1}{a(a+n)^\lambda}-\Lambda(n)\right)^{1-p}\left[\sum_{m=a}^\infty\frac{f(m)}{(m+n)^\lambda}\right]^p\\ &\leq\sum_{m=a}^\infty[m(m+1)]^{p-1}\left(\frac{1}{a(a+m)^\lambda}-\Lambda(m)\right)f^p(m).\end{aligned}$$

Now, our further step is to derive corresponding inequalities for the kernel $K(x,y)=(1+xy)^{-\lambda}$, $\lambda>0$, and the weight functions $\varphi^q(x)=\psi^p(x)=x\sqrt{\sigma(x)}+\sigma(x)\sqrt{x}$.

Acting as in the proof of Theorem 3.1, we can establish the following result.

Theorem 3.3 *Let $\mathbb{T}$ be a time scale with $a\in\mathbb{T}$. Let $\lambda\geq 0$, $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$, and define*

$$\Lambda(x):=\int_a^\infty\frac{x}{\sqrt{\sigma(y)}}\left(\frac{1}{(1+x\sigma(y))^\lambda(1+xy)}+\frac{1}{(1+xy)^\lambda(1+x\sigma(y))}\right)\Delta y,\ \ x\in[a,\infty).$$

Then the inequality

$$\begin{aligned}&\int_a^\infty\int_a^\infty\frac{f(x)g(y)}{(1+xy)^\lambda}\Delta x\Delta y\\ &\leq\left(\int_a^\infty[x\sqrt{\sigma(x)}+\sigma(x)\sqrt{x}]^{p-1}\left(\frac{1}{\sqrt{a}(1+ax)^\lambda}-\Lambda(x)\right)f^p(x)\Delta x\right)^{\frac{1}{p}}\\ &\times\left(\int_a^\infty[y\sqrt{\sigma(y)}+\sigma(y)\sqrt{y}]^{q-1}\left(\frac{1}{\sqrt{a}(1+ay)^\lambda}-\Lambda(y)\right)g^q(y)\Delta y\right)^{\frac{1}{q}}\end{aligned}\tag{3.13}$$

holds for all non-negative and delta measurable functions $f,g:\mathbb{T}\to\mathbb{R}$.

In what follows, instead of formula (3.2) we use the chain rule (see [26], Theorem 1.87):

$$(f \circ g)^{\Delta}(t) = f'(g(c))g^{\Delta}(t), \text{ for some } c \in [t, \sigma(t)], \tag{3.14}$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous, $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable function.

Theorem 3.4 *Let $\mathbb{T}$ be a time scale with $a \in \mathbb{T}$. Let $\lambda \geq 0$, $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$, and define*

$$\Lambda(x) := \lambda \int_a^{\infty} \frac{1}{\sigma(y)} \frac{1}{(x+\sigma(y))^{\lambda+2}(x+\sigma(y))} \Delta y, \qquad x \in [a, \infty).$$

Then the inequalities

$$\begin{aligned} &\int_a^{\infty}\int_a^{\infty} \frac{f(x)g(y)}{(x+y)^{\lambda}} \Delta x \Delta y \\ &\leq \left(\int_a^{\infty} [x\sigma(x)]^{p-1} \left(\frac{1}{a(a+x)^{\lambda}} + \Lambda(x) \right) f^p(x) \Delta x \right)^{\frac{1}{p}} \\ &\times \left(\int_a^{\infty} [y\sigma(y)]^{q-1} \left(\frac{1}{a(a+y)^{\lambda}} + \Lambda(y) \right) g^q(y) \Delta y \right)^{\frac{1}{q}} \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} &\int_a^{\infty} \frac{1}{y\sigma(y)} \left(\frac{1}{a(a+y)^{\lambda}} + \Lambda(y) \right)^{1-p} \left[\int_a^{\infty} \frac{f(x)}{(x+y)^{\lambda}} \Delta x \right]^p \Delta y \\ &\leq \int_a^{\infty} [x\sigma(x)]^{p-1} \left(\frac{1}{a(a+x)^{\lambda}} + \Lambda(x) \right) f^p(x) \Delta x \end{aligned} \tag{3.16}$$

hold for all non-negative and delta measurable functions $f, g : \mathbb{T} \to \mathbb{R}$.

Proof. We prove (3.15) only. To show this, we follow the same procedure as in the proof of Theorem 3.1 except that we provide a new estimate for the functions $F(x)$ and $G(x)$ defined by (3.7).

More precisely, from the inequality (3.8) we get

$$F(x) = \frac{1}{a(a+x)^{\lambda}} + \int_a^{\infty} (u^{\lambda}(y))^{\Delta} \frac{1}{\sigma(y)} \Delta y, \;\; x \in [a, \infty), \tag{3.17}$$

where $u(y) = 1/(x+y)$. Using (3.10) and (3.14) we have

$$(u^{\lambda}(y))^{\Delta} = \frac{\lambda}{(x+c)^{\lambda+1}(x+y)(x+\sigma(y))}, \text{ for some } c \in [y, \sigma(y)],$$

and therefore

$$(u^{\lambda}(y))^{\Delta} \leq \frac{\lambda}{(x+y)^{\lambda+2}(x+\sigma(y))}. \tag{3.18}$$

Finally, making use of (3.3), (3.17) and (3.18) we obtain (3.15). □

Remark 3.3 Hilbert-type inequalities presented in this chapter are taken from [79]. However, similar Hilbert-type and Hardy-type inequalities can also be derived for homogeneous kernels of arbitrary degree of homogeneity. For more details about similar results, the reader is referred to [14], [24] and [40].

Chapter 4

A Class of Hilbert-type Inequalities Obtained via the Improved Young Inequality

4.1 Preliminaries

Nowadays, considerable attention is focused on establishing methods for improving Hilbert-type inequalities. The main objective of this chapter is to present improved versions of Hilbert-type inequalities (1.2) and (1.3), based on the improved form of the well-known Young inequality

$$\prod_{i=1}^{n} x_i \leq \sum_{i=1}^{n} \frac{x_i^{p_i}}{p_i}, \tag{4.1}$$

where $x_i > 0$, $p_i > 1$, and $\sum_{i=1}^{n} \frac{1}{p_i} = 1$. The results that follow are established in [53]. We first give refined and reversed Hilbert-type relations in a general multidimensional case. As an application, we give improved versions of the classical Hilbert and Hardy inequalities.

The starting point in our research is the following improvement of the Young inequality (4.1) established in [61]: If $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, $p_i > 1$, $x_i > 0$, $i = 1,2,\ldots,n$, then

$$\left(\frac{\prod_{i=1}^{n} x_i^{\frac{p_i}{n}}}{\frac{1}{n}\sum_{i=1}^{n} x_i^{p_i}} \right)^{\frac{n}{m}} \leq \frac{\prod_{i=1}^{n} x_i}{\sum_{i=1}^{n} \frac{x_i^{p_i}}{p_i}} \leq \left(\frac{\prod_{i=1}^{n} x_i^{\frac{p_i}{n}}}{\frac{1}{n}\sum_{i=1}^{n} x_i^{p_i}} \right)^{\frac{n}{M}}, \tag{4.2}$$

where $m = \min_{1\le i\le n} p_i$ and $M = \max_{1\le i\le n} p_i$. The first inequality in (4.2) provides the reverse, while the second yields the refinement of the Young inequality. This improved form of the Young inequality relies on the improved version of the Jensen inequality obtained in [61] (see also [41]).

Considering the second inequality in (4.2) with $\frac{f_i}{\|f_i\|_{p_i}}$ instead of x_i, $i = 1,2,\dots,n$, where $f_i \in L^{p_i}(\Omega)$, and integrating over Ω, it follows that

$$\int_\Omega \prod_{i=1}^n f_i(x)d\mu(x) \le n^{\frac{n}{M}} G(f_1, f_2, \dots, f_n) \prod_{i=1}^n \|f_i\|_{p_i}^{1-\frac{p_i}{M}}, \tag{4.3}$$

where

$$G(f_1, f_2, \dots, f_n) = \int_\Omega \left[\sum_{i=1}^n \frac{f_i^{p_i}(x)}{p_i\|f_i\|_{p_i}^{p_i}}\right] \left[\frac{\prod_{i=1}^n f_i^{\frac{p_i}{n}}(x)}{\sum_{i=1}^n \frac{f_i^{p_i}(x)}{\|f_i\|_{p_i}^{p_i}}}\right]^{\frac{n}{M}} d\mu(x). \tag{4.4}$$

It should be noticed here that the inequality (4.3) provides the refinement of the Hölder inequality. Namely, by the arithmetic-geometric mean inequality we have

$$\frac{\prod_{i=1}^n f_i^{\frac{p_i}{n}}(x)}{\sum_{i=1}^n \frac{f_i^{p_i}(x)}{\|f_i\|_{p_i}^{p_i}}} \le \frac{\prod_{i=1}^n \|f_i\|_{p_i}^{\frac{p_i}{n}}}{n},$$

and consequently,

$$G(f_1, f_2, \dots, f_n) \le \left(\frac{\prod_{i=1}^n \|f_i\|_{p_i}^{p_i}}{n^n}\right)^{\frac{1}{M}}. \tag{4.5}$$

Now, combining (4.3) and (4.5) yields $\int_\Omega \prod_{i=1}^n f_i(x)d\mu(x) \le \prod_{i=1}^n \|f_i\|_{p_i}$, i.e. the Hölder inequality.

In the same way, the first inequality in (4.2) yields the reverse of the Hölder inequality, that is,

$$\int_\Omega \prod_{i=1}^n f_i(x)d\mu(x) \ge n^{\frac{n}{m}} H(f_1, f_2, \dots, f_n) \prod_{i=1}^n \|f_i\|_{p_i}^{1-\frac{p_i}{m}}, \tag{4.6}$$

where

$$H(f_1, f_2, \dots, f_n) = \int_\Omega \left[\sum_{i=1}^n \frac{f_i^{p_i}(x)}{p_i\|f_i\|_{p_i}^{p_i}}\right] \left[\frac{\prod_{i=1}^n f_i^{\frac{p_i}{n}}(x)}{\sum_{i=1}^n \frac{f_i^{p_i}(x)}{\|f_i\|_{p_i}^{p_i}}}\right]^{\frac{n}{m}} d\mu(x). \tag{4.7}$$

The improved Hölder-type inequalities (4.3) and (4.6) were established in paper [61], in a more general setting with positive isotonic linear functionals. We will utilize them in obtaining improved versions of Hilbert-type inequalities presented in Section 1.1.

4.2 Improved Hilbert-type Inequalities

In this section we give a class of Hilbert-type inequalities based on more precise Hölder-type inequalities stated in the previous section. First we give a refinement of the inequality (1.2) which relies on the refined Hölder inequality (4.3).

Theorem 4.1 *Let $\sum_{i=1}^{n}\frac{1}{p_i}=1$, $p_i>1$, let $(\Omega_i,\Sigma_i,\mu_i)$ be σ-finite measure spaces, and let $K:\mathbf{\Omega}\to\mathbb{R}$, $\phi_{ij}:\Omega_j\to\mathbb{R}$, $f_i:\Omega_i\to\mathbb{R}$, $i,j=1,2,\ldots,n$, be non-negative measurable functions. If $\prod_{i,j=1}^{n}\phi_{ij}(x_j)=1$ and the functions $F_i:\mathbf{\Omega}\to\mathbb{R}$ are defined by*

$$F_i(\mathbf{x})=K^{\frac{1}{p_i}}(\mathbf{x})f_i(x_i)\prod_{j=1}^{n}\phi_{ij}(x_j),\quad i=1,2,\ldots,n,$$

then

$$\int_{\mathbf{\Omega}}K(\mathbf{x})\prod_{i=1}^{n}f_i(x_i)d\mu(\mathbf{x})\le n^{\frac{n}{M}}G(F_1,F_2,\ldots,F_n)\prod_{i=1}^{n}\|\phi_{ii}\omega_i f_i\|_{p_i}^{1-\frac{p_i}{M}},\tag{4.8}$$

where $M=\max_{1\le i\le n}p_i$, ω_i is defined by (1.4)*, $\phi_{ii}\omega_i f_i\in L^{p_i}(\Omega_i)$, $i=1,2,\ldots,n$, and G is defined by* (4.4).

Proof. Rewriting the left-hand side of inequality (4.8) and utilizing the improved Hölder inequality (4.3) we have

$$\begin{aligned}
&\int_{\mathbf{\Omega}}K(\mathbf{x})\prod_{i=1}^{n}f_i(x_i)d\mu(\mathbf{x})\\
&=\int_{\mathbf{\Omega}}\prod_{i=1}^{n}\left(K^{1/p_i}(\mathbf{x})f_i(x_i)\prod_{j=1}^{n}\phi_{ij}(x_j)\right)d\mu(\mathbf{x})\\
&=\int_{\mathbf{\Omega}}\prod_{i=1}^{n}F_i(\mathbf{x})d\mu(\mathbf{x})\\
&\le n^{\frac{n}{M}}G(F_1,F_2,\ldots,F_n)\prod_{i=1}^{n}\|F_i\|_{p_i}^{1-\frac{p_i}{M}}.
\end{aligned}$$

In addition, since $\phi_{ii}\omega_i f_i\in L^{p_i}(\Omega_i)$, it follows that $F_i\in L^{p_i}(\mathbf{\Omega})$, $i=1,2,\ldots,n$. In other words, we have

$$\begin{aligned}
\|F_i\|_{p_i}&=\left[\int_{\mathbf{\Omega}}K(\mathbf{x})(\phi_{ii}f_i)^{p_i}(x_i)\prod_{j=1,j\neq i}^{n}\phi_{ij}^{p_i}(x_j)d\mu(\mathbf{x})\right]^{\frac{1}{p_i}}\\
&=\left[\int_{\Omega_i}(\phi_{ii}f_i)^{p_i}(x_i)\left(\int_{\hat{\mathbf{\Omega}}^i}K(\mathbf{x})\prod_{j=1,j\neq i}^{n}\phi_{ij}^{p_i}(x_j)d\hat{\mu}^i(\mathbf{x})\right)d\mu_i(x_i)\right]^{\frac{1}{p_i}}\qquad(4.9)\\
&=\left[\int_{\Omega_i}(\phi_{ii}\omega_i f_i)^{p_i}(x_i)d\mu_i(x_i)\right]^{\frac{1}{p_i}}\\
&=\|\phi_{ii}\omega_i f_i\|_{p_i},\quad i=1,2,\ldots,n,
\end{aligned}$$

which completes the proof. □

Remark 4.1 The inequality (4.8) provides the improvement of inequality (1.2), due to relation (4.5). More precisely, utilizing (4.5) and (4.9), it follows that

$$G(F_1,F_2,\dots,F_n) \le n^{-\frac{n}{M}} \prod_{i=1}^{n} \|\phi_{ii}\omega_i f_i\|_{p_i}^{\frac{p_i}{M}}.$$

This means that the right-hand side of the inequality (4.8) is not greater than the right-hand side of (1.2), that is, not greater than $\prod_{i=1}^{n} \|\phi_{ii}\omega_i f_i\|_{p_i}$.

Now, as a consequence of Theorem 4.1 we also obtain the refinement of the Hardy-Hilbert-type inequality (1.3).

Theorem 4.2 *Suppose that the assumptions as in Theorem* 4.1 *are fulfilled. Then,*

$$\begin{aligned} &\left[\int_{\Omega_n}\left(\frac{1}{(\phi_{nn}\omega_n)(x_n)}\int_{\hat{\Omega}^n} K(\mathbf{x})\prod_{i=1}^{n-1} f_i(x_i)d\hat{\mu}^n(\mathbf{x})\right)^P d\mu(x_n)\right]^{\frac{1}{P}+\frac{1}{M}} \\ &\le n^{\frac{n}{M}} G(F_1,F_2,\dots,F_{n-1},\widetilde{F}_n)\prod_{i=1}^{n-1}\|\phi_{ii}\omega_i f_i\|_{p_i}^{1-\frac{p_i}{M}}, \end{aligned} \tag{4.10}$$

where $\frac{1}{P} = \sum_{i=1}^{n-1}\frac{1}{p_i}$ *and*

$$\widetilde{F}_n(\mathbf{x}) = \frac{K^{\frac{1}{p_n}}(\mathbf{x})}{(\phi_{nn}\omega_n)^P(x_n)}\left(\int_{\hat{\Omega}^n} K(\mathbf{x})\prod_{i=1}^{n-1} f_i(x_i)d\hat{\mu}^n(\mathbf{x})\right)^{P-1}\prod_{j=1}^{n}\phi_{nj}(x_j).$$

Proof. It follows from Theorem 4.1, by substituting the function

$$\widetilde{f}_n(x_n) = (\phi_{nn}\omega_n)^{-P}(x_n)\left(\int_{\hat{\Omega}^n} K(\mathbf{x})\prod_{i=1}^{n-1} f_i(x_i)d\hat{\mu}^n(\mathbf{x})\right)^{P-1}$$

in inequality (4.8). In that case, (4.8) reduces to

$$I \le n^{\frac{n}{M}} G(F_1,F_2,\dots,\widetilde{F}_n) I^{\frac{1}{p_n}-\frac{1}{M}} \prod_{i=1}^{n-1}\|F_i\|_{p_i}^{1-\frac{p_i}{M}},$$

where $I = \int_{\Omega_n}\left(\frac{1}{(\phi_{nn}\omega_n)(x_n)}\int_{\hat{\Omega}^n} K(\mathbf{x})\prod_{i=1}^{n-1} f_i(x_i)d\hat{\mu}^n(\mathbf{x})\right)^P d\mu(x_n)$. Finally, rearranging we obtain (4.10). □

Remark 4.2 It should be noticed here that the inequality (4.10) is more accurate than (1.3). In order to show this, note that $\|\widetilde{F}_n\|_{p_n} = I^{\frac{1}{p_n}}$, where I is as in the proof of Theorem 4.2. Therefore we have

$$G(F_1,F_2,\dots,\widetilde{F}_n) \le n^{-\frac{n}{M}} I^{\frac{1}{M}} \prod_{i=1}^{n-1}\|\phi_{ii}\omega_i f_i\|_{p_i}^{\frac{p_i}{M}},$$

and the relation (4.10) implies the inequality

$$I^{\frac{1}{P}+\frac{1}{M}} \leq I^{\frac{1}{M}} \prod_{i=1}^{n-1} \|\phi_{ii}\omega_i f_i\|_{p_i},$$

which provides (1.3) after dividing by $I^{\frac{1}{M}}$.

In the same way as in theorems 4.1 and 4.2, we can also derive reverses of inequalities (1.2) and (1.3). These reverses rely on the reverse Hölder inequality (4.6). The following theorem is established in the same way as theorems 4.1 and 4.2, except that we use relation (4.6) instead of (4.3).

Theorem 4.3 *Suppose that the assumptions as in Theorems* 4.1 *and* 4.2 *are fulfilled. Then,*

$$\int_{\mathbf{\Omega}} K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i) d\mu(\mathbf{x}) \geq n^{\frac{n}{m}} H(F_1, F_2, \ldots, F_n) \prod_{i=1}^{n} \|\phi_{ii}\omega_i f_i\|_{p_i}^{1-\frac{p_i}{m}}$$

and

$$\left[\int_{\Omega_n} \left(\frac{1}{(\phi_{nn}\omega_n)(x_n)} \int_{\hat{\mathbf{\Omega}}^n} K(\mathbf{x}) \prod_{i=1}^{n-1} f_i(x_i) d\hat{\mu}^n(\mathbf{x})\right)^P d\mu(x_n)\right]^{\frac{1}{P}+\frac{1}{m}}$$
$$\geq n^{\frac{n}{m}} H(F_1, F_2, \ldots, F_{n-1}, \widetilde{F_n}) \prod_{i=1}^{n-1} \|\phi_{ii}\omega_i f_i\|_{p_i}^{1-\frac{p_i}{m}},$$

where $m = \min_{1 \leq i \leq n} p_i$ *and* H *is defined by* (4.7).

4.3 Applications

Now, our intention is to apply results from the previous section to obtain the improvements of the classical Hilbert and Hardy inequalities. Here we deal with real measure spaces $\Omega_i = \mathbb{R}_+$, accompanied with the non-negative Lebesgue measures $d\mu_i(x_i) = dx_i$, $i = 1, 2, \ldots, n$. In this particular setting we have $\mathbf{\Omega} = \mathbb{R}_+^n$, $\hat{\mathbf{\Omega}}^i = \mathbb{R}_+^{n-1}$, $d\mathbf{x} = dx_1 dx_2 \ldots dx_n$, and $\hat{d}^n\mathbf{x} = dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n$, $i = 1, 2, \ldots, n$.

4.3.1 Connection with the classical Hilbert inequality

We first give an improved form of the inequality (1.1) in a multidimensional case. In order to do this, we consider the kernel $K_0 : \mathbb{R}_+^n \to \mathbb{R}$ defined by $K_0(\mathbf{x}) = (\sum_{i=1}^n x_i)^{-\lambda}$, $\lambda > 0$, and the power weight functions $\phi_{ij}(x_j) = x_j^{A_{ij}}$, where $A_{ij} = \frac{\lambda - n}{p_i p_j}$, $i \neq j$, and $A_{ii} = \frac{(n-\lambda)(p_i - 1)}{p_i^2}$, $n - \lambda < m$, $m = \min_{1 \leq i \leq n} p_i$, $i, j = 1, 2, \ldots, n$. These power weight functions fulfill condition $\prod_{i,j=1}^{n}\phi_{ij}(x_j)=1$ as in theorems 4.1, 4.2, and 4.3. In addition, by means of the formula

$$\int_{\mathbb{R}_+^{n-1}}\frac{\prod_{i=1}^{n-1}u_i^{a_i-1}}{(1+\sum_{i=1}^{n-1}u_i)^{\sum_{i=1}^{n}a_i}}\hat{d}^n\mathbf{u}=\frac{\prod_{i=1}^{n}\Gamma(a_i)}{\Gamma(\sum_{i=1}^{n}a_i)},$$

where $\Gamma(a)=\int_0^\infty t^{a-1}\exp(-t)dt$, $a>0$, is the usual Gamma function (see, e.g. [1]), we have

$$\begin{aligned}\omega_i^{p_i}(x_i)&=\int_{\mathbb{R}_+^{n-1}}\frac{\prod_{j=1,j\neq i}^{n}x_j^{\frac{\lambda-n}{p_j}}}{\left(\sum_{j=1}^{n}x_j\right)^{\lambda}}\hat{d}^i\mathbf{x}\\&=\frac{1}{\Gamma(\lambda)}\prod_{j=1}^{n}\Gamma\left(\frac{\lambda-n+p_j}{p_j}\right)x_i^{\frac{n-\lambda-p_i}{p_i}},\quad x_i>0,\end{aligned}$$

and consequently, $\|\phi_{ii}\omega_i f_i\|_{p_i}^{p_i}=\frac{1}{\Gamma(\lambda)}\prod_{j=1}^{n}\Gamma(\frac{\lambda-n+p_j}{p_j})\|x_i^{\frac{n-\lambda-1}{p_i}}f_i\|_{p_i}^{p_i}$, $i=1,2,\ldots,n$. Therefore, in this case Theorem 4.1 takes the following form:

Corollary 4.1 *Let $\sum_{i=1}^{n}\frac{1}{p_i}=1$, $p_i>1$, $f_i:\mathbb{R}_+\to\mathbb{R}$, $i=1,2,\ldots,n$, are non-negative measurable functions, and let $\lambda>n-m$, where $m=\min_{1\leq i\leq n}p_i$. If the functions $F_i:\mathbb{R}_+^n\to\mathbb{R}$ are defined by*

$$F_i(\mathbf{x})=(\sum_{j=1}^{n}x_j)^{-\frac{\lambda}{p_i}}f_i(x_i)x_i^{\frac{(n-\lambda)(p_i-1)}{p_i^2}}\prod_{j=1,j\neq i}^{n}x_j^{\frac{\lambda-n}{p_ip_j}},\ i=1,2,\ldots,n,$$

then

$$\int_{\mathbb{R}_+^n}\frac{\prod_{i=1}^{n}f_i(x_i)}{(\sum_{i=1}^{n}x_i)^{\lambda}}d\mathbf{x}\leq n^{\frac{n}{M}}A^{1-\frac{n}{M}}G(F_1,\ldots,F_n)\prod_{i=1}^{n}\|x_i^{\frac{n-1-\lambda}{p_i}}f_i\|_{p_i}^{1-\frac{p_i}{M}},\tag{4.11}$$

where $A=\frac{1}{\Gamma(\lambda)}\prod_{j=1}^{n}\Gamma(\frac{\lambda-n+p_j}{p_j})$, $M=\max_{1\leq i\leq n}p_i$, $x_i^{\frac{n-1-\lambda}{p_i}}f_i\in L^{p_i}(\mathbb{R}_+)$, $i=1,2,\ldots,n$, and G is defined by (4.4).

Remark 4.3 Since the functions F_i, defined in Corollary 4.1, fulfill relation $\|F_i\|_{p_i}=A^{\frac{1}{p_i}}\|x_i^{\frac{n-\lambda-1}{p_i}}f_i\|_{p_i}^{p_i}$, it follows by (4.5) that

$$G(F_1,F_2,\ldots,F_n)\leq n^{-\frac{n}{M}}A^{\frac{n}{M}}\prod_{i=1}^{n}\|x_i^{\frac{n-1-\lambda}{p_i}}f_i\|_{p_i}^{\frac{p_i}{M}}.$$

The above relation implies that the right-hand side of the inequality (4.11) is not greater than $A\prod_{i=1}^{n}\|x_i^{\frac{n-1-\lambda}{p_i}}f_i\|_{p_i}$, which provides the right-hand side of the corresponding Hilbert-type inequality derived in [99].

In the same way, as a consequence of Theorem 4.2, we obtain the Hardy-Hilbert form of the inequality (4.11).

Corollary 4.2 *Suppose that the assumptions as in Corollary* 4.1 *are fulfilled. Then,*

$$\left[\int_{\mathbb{R}_+} x_n^{(1-P)(n-\lambda-1)}\left(\int_{\mathbb{R}_+^{n-1}} \frac{\prod_{i=1}^{n-1} f_i(x_i)}{(\sum_{i=1}^n x_i)^\lambda}\hat{d}^n\mathbf{x}\right)^P dx_n\right]^{\frac{1}{P}+\frac{1}{M}} \\ \leq n^{\frac{n}{M}} A^{1+\frac{P-n}{M}} G(F_1,F_2,\ldots,F_{n-1},\widetilde{F}_n)\prod_{i=1}^{n-1}\|x_i^{\frac{n-1-\lambda}{p_i}} f_i\|_{p_i}^{1-\frac{p_i}{M}}, \tag{4.12}$$

where $\frac{1}{P}=\sum_{i=1}^{n-1}\frac{1}{p_i}$ *and*

$$\widetilde{F}_n(\mathbf{x}) = \frac{(\sum_{i=1}^n x_i)^{-\frac{\lambda}{p_n}}}{A^{P-1}x_n^{(P-1)(n-\lambda-1)}}\left(\int_{\mathbb{R}_+^{n-1}} \frac{\prod_{i=1}^{n-1} f_i(x_i)}{(\sum_{i=1}^n x_i)^\lambda}\hat{d}^n\mathbf{x}\right)^{P-1} x_n^{\frac{(n-\lambda)(p_n-1)}{p_n^2}}\prod_{j=1}^{n-1} x_j^{\frac{\lambda-n}{p_n p_j}}.$$

Taking into account remarks 4.2 and 4.3, the relation (4.12) provides an improvement of the corresponding Hardy-Hilbert-type inequality from [99].

Remark 4.4 Note that the kernel $K_0(\mathbf{x})=(\sum_{i=1}^n x_i)^{-\lambda}$, $\lambda>0$, appearing in corollaries 4.1 and 4.2 is a homogeneous function of degree $-\lambda$. The same conclusion, as in corollaries 4.1 and 4.2, can be drawn for an arbitrary homogeneous function of degree $-\lambda$, $\lambda > 0$. More precisely, let $K:\mathbb{R}_+^n\to\mathbb{R}$ be a homogeneous function of degree $-\lambda$, such that $k\big(\frac{\lambda-n}{p_2},\ldots,\frac{\lambda-n}{p_n}\big)<\infty$, where the function $k(\cdot)$ is defined by (1.5) (see Section 1.1). Now, if the kernel K_0 is replaced by the kernel K, we obtain the same inequalities as (4.11) and (4.12), except that the constant A is replaced by $k\big(\frac{\lambda-n}{p_2},\ldots,\frac{\lambda-n}{p_n}\big)$.

Remark 4.5 Considering relations (4.11) and (4.12) with $m=\min_{1\leq i\leq n} p_i$ instead of $M=\max_{1\leq i\leq n} p_i$ provides inequalities with reversed sign of inequality, due to Theorem 4.3.

In order to end our discussion regarding the Hilbert inequality, we give the two-dimensional version of Corollary 4.1, that is, when $n=2$ and $\lambda=1$. With a more suitable notation $x_1=x$, $x_2=y$, $p_1=p$, $p_2=q$, $f_1=f$, $f_2=g$, we have $A=\Gamma(\frac{1}{p})\Gamma(\frac{1}{q})=\frac{\pi}{\sin\frac{\pi}{p}}$, $F_1(x,y)=f(x)(x+y)^{-\frac{1}{p}}\left(\frac{x}{y}\right)^{\frac{1}{pq}}$, $F_2(x,y)=g(y)(x+y)^{-\frac{1}{q}}\left(\frac{y}{x}\right)^{\frac{1}{pq}}$, so in this case relation (4.11) reduces to

$$\int_{\mathbb{R}_+^2}\frac{f(x)g(y)}{x+y}dxdy \leq 4^{\frac{1}{M}}\left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^{1-\frac{2}{M}} G(F_1,F_2)\|f\|_p^{1-\frac{p}{M}}\|g\|_q^{1-\frac{q}{M}}, \tag{4.13}$$

where $M=\max\{p,q\}$. In addition, since $\|F_1\|_p^p=\frac{\pi}{\sin\frac{\pi}{p}}\|f\|_p^p$ and $\|F_2\|_q^q=\frac{\pi}{\sin\frac{\pi}{p}}\|g\|_q^q$, it follows that $G(F_1,F_2)=\left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^{\frac{2}{M}-1}\gamma(f,g)$, where

$$\gamma(f,g)=\int_{\mathbb{R}_+^2}\frac{\frac{f^p(x)}{p\|f\|_p^p}\left(\frac{x}{y}\right)^{\frac{1}{q}}+\frac{g^q(y)}{q\|g\|_q^q}\left(\frac{y}{x}\right)^{\frac{1}{p}}}{x+y}\left[\frac{f^{\frac{p}{2}}(x)g^{\frac{q}{2}}(y)}{\frac{f^p(x)}{\|f\|_p^p}\left(\frac{x}{y}\right)^{\frac{1}{2p}-\frac{1}{q}}+\frac{g^q(y)}{\|g\|_q^q}\left(\frac{y}{x}\right)^{\frac{1}{2q}-\frac{1}{p}}}\right]^{\frac{2}{M}}dxdy.$$

Therefore, the relation (4.13) can also be rewritten in the following form:

$$\int_{\mathbb{R}_+^2} \frac{f(x)g(y)}{x+y} dxdy \leq 4^{\frac{1}{M}} \gamma(f,g) \|f\|_p^{1-\frac{p}{M}} \|g\|_q^{1-\frac{q}{M}}. \tag{4.14}$$

Remark 4.6 We give a trivial example which shows that the relation (4.14) yields a better estimate for the integral $\int_{\mathbb{R}_+^2} \frac{f(x)g(y)}{x+y} dxdy$, than the original Hilbert inequality (1.1). To see this, put $f = \chi_{(0,1)}$ and $g = \exp(-\frac{x}{q})\chi_{(1,\infty)}$, where χ stands for a characteristic function of the corresponding interval. Then, $\|f\|_p = 1$, $\|g\|_q = \exp(\frac{1}{q})$, and $fg \equiv 0$, so the inequality (1.1) reduces to $0 < \frac{\pi}{\sin\frac{\pi}{p}} \exp(\frac{1}{q})$. On the other hand, in this case we have $\gamma(f,g) = 0$, so the inequality (4.14) reduces to a trivial equality, providing a more accurate estimate than (1.1).

4.3.2 A few examples with the classical Hardy inequality

Now we deal with another famous classical inequality closely connected to the Hilbert inequality, i.e. the Hardy inequality. The Hardy inequality (1.53) can be rewritten in the following form:

$$\left[\int_{\mathbb{R}_+} \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx\right]^{\frac{1}{p}} \leq \frac{p}{p-1}\|f\|_p, \tag{4.15}$$

where $p > 1$ and $f \in L^p(\mathbb{R}_+)$. For comprehensive accounts on Hardy inequality including history, different proofs, refinements and diverse applications, the reader is referred to monographs [47] and [68].

As we have mentioned, the inequality (1.3) is usually referred to as the Hardy-Hilbert-type inequality since it is a multiple generalization of (4.15). To see this, let us consider (1.3) with $n = 2$, $\phi_{11}(x_1) = x_1^{\frac{1}{p_1p_2}}$, $\phi_{21}(x_1) = x_1^{-\frac{1}{p_1p_2}}$, $\phi_{12}(x_2) = x_2^{-\frac{1}{p_1p_2}}$, $\phi_{22}(x_2) = x_2^{\frac{1}{p_1p_2}}$ and the Hardy kernel $K(x_1,x_2) = \frac{1}{x_2}\chi_T(x_1,x_2)$, where χ stands for the characteristic function of $T = \{(x_1,x_2) \in \mathbb{R}_+^2;\ x_1 \leq x_2\}$. Then, it follows that $\omega_1(x_1) = p_2^{\frac{1}{p_1}} x_1^{-\frac{1}{p_1p_2}}$ and $\omega_2(x_2) = p_2^{\frac{1}{p_2}} x_2^{-\frac{1}{p_1p_2}}$, so (1.3) reduces to (4.15), after using a more suitable notation $x_1 = t$, $x_2 = x$, $p_1 = p$, and $f_1 = f$.

Clearly, our Theorem 4.2 provides a refinement of the Hardy inequality. In fact, utilizing the fact that the Hardy kernel $K(x_1,x_2) = \frac{1}{x_2}\chi_T(x_1,x_2)$ is a homogeneous function of degree -1, we can apply Corollary 4.2 and Remark 4.4. Since $n = 2$ and $\lambda = 1$, the constant $k(\cdot)$ in Remark 4.4 becomes $k(-\frac{1}{p_2}) = p_2$, after a straightforward computation. Therefore, with $x_1 = t$, $x_2 = x$, $p_1 = p$, $p_2 = q$, and $f_1 = f$, we obtain more strengthened version of the Hardy inequality (4.15), that is, we have

$$\left[\int_{\mathbb{R}_+} \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx\right]^{\frac{1}{p}+\frac{1}{M}} \leq 4^{\frac{1}{M}} q^{1+\frac{p-2}{M}} G(F_1,\widetilde{F}_2)\|f\|_p^{1-\frac{p}{M}}, \tag{4.16}$$

where $M = \max\{p,q\}$,

$$F_1(x,y) = x^{\frac{1}{pq}} y^{-\frac{1+q}{pq}} f(x) \chi_T(x,y),$$
$$\widetilde{F}_2(x,y) = q^{-\frac{p}{q}} x^{-\frac{1}{pq}} y^{\frac{1+q-p^2 q}{pq}} \left(\int_0^y f(t)dt \right)^{p-1} \chi_T(x,y),$$

and G is defined by (4.4).

Remark 4.7 Our inequality (4.16) is an improvement of the classical Hardy inequality, due to Remark 4.2. More precisely, since $\|F_1\|_p = q^{\frac{1}{p}} \|f\|_p$ and $\|\widetilde{F}_2\|_q = q^{\frac{1-p}{q}} J^{\frac{1}{q}}$, where $J = \int_{\mathbb{R}_+} \left(\frac{1}{x} \int_0^x f(t)dt \right)^p dx$, we have

$$G(F_1, \widetilde{F}_2) \leq 4^{-\frac{1}{M}} \|F_1\|_p^{\frac{p}{M}} \|\widetilde{F}_2\|_q^{\frac{q}{M}} = 4^{-\frac{1}{M}} q^{\frac{2-p}{M}} J^{\frac{1}{M}} \|f\|_p^{\frac{p}{M}},$$

which implies that the right-hand side of inequality (4.16) is not greater than $qJ^{\frac{1}{M}} \|f\|_p$. This yields the Hardy inequality (4.15).

Our last example refers to the so called dual Hardy inequality. The corresponding result can not be derived directly from Corollary 4.2, but employing Theorem 4.2 with $n = 2$, $\phi_{11}(x_1) = x_1^{1-\frac{1}{p_1^2}}$, $\phi_{21}(x_1) = x_1^{\frac{1}{p_1^2}-1}$, $\phi_{12}(x_2) = x_2^{\frac{1}{p_1^2}}$, $\phi_{22}(x_2) = x_2^{-\frac{1}{p_1^2}}$, and the dual Hardy kernel $K(x_1,x_2) = \frac{1}{x_2} \chi_S(x_1,x_2)$, where χ stands for the characteristic function of $S = \{(x_1,x_2) \in \mathbb{R}_+^2;\ x_1 \geq x_2\}$, it follows that $\omega_1(x_1) = p_1^{\frac{1}{p_1}} x_1^{\frac{1}{p_1^2}}$ and $\omega_2(x_2) = p_1^{\frac{1}{p_2}} x_2^{-\frac{p_1+1}{p_1 p_2}}$. Consequently, with $x_1 = t$, $x_2 = x$, $p_1 = p$, $p_2 = q$, and $f_1 = f$, relation (4.10) becomes

$$\left[\int_{\mathbb{R}_+} \left(\int_x^\infty f(t)dt \right)^p dx \right]^{\frac{1}{p}+\frac{1}{M}} \leq 4^{\frac{1}{M}} p^{1+\frac{p-2}{M}} G(F_1, \widetilde{F}_2) \|xf\|_p^{1-\frac{p}{M}}, \tag{4.17}$$

where

$$F_1(x,y) = x^{1-\frac{1}{p^2}} y^{-\frac{1}{pq}} f(x) \chi_S(x,y),$$
$$\widetilde{F}_2(x,y) = p^{-\frac{p}{q}} x^{1-\frac{1}{p^2}} y^{\frac{1}{pq}} \left(\int_y^\infty f(t)dt \right)^{p-1} \chi_S(x,y).$$

Similarly to the previous example, we have $\|F_1\|_p = p^{\frac{1}{p}} \|xf\|_p$ and $\|\widetilde{F}_2\|_q = p^{\frac{1-p}{q}} J^{\frac{1}{q}}$, where $J = \int_{\mathbb{R}_+} (\int_x^\infty f(t)dt)^p dx$, and consequently, $G(F_1, \widetilde{F}_2) \leq 4^{-\frac{1}{M}} p^{\frac{2-p}{M}} J^{\frac{1}{M}} \|xf\|_p^{\frac{p}{M}}$. Therefore, the right-hand side of (4.17) is not greater than $pJ^{\frac{1}{M}} \|xf\|_p$, which in turn yields the dual Hardy inequality

$$\left[\int_{\mathbb{R}_+} \left(\int_x^\infty f(t)dt \right)^p dx \right]^{\frac{1}{p}} \leq p\|xf\|_p.$$

Remark 4.8 Multidimensional refinements of Hilbert-type inequalities via the improved Young inequality, presented in this chapter, are derived in [53] by Krnić and Vuković. Some related refinements of Hilbert-type inequalities based on the improved Jensen inequality are present in recent monograph [54]. For some other refinements the reader can also consult the following papers: [28], [31], [33] and [104].

Chapter 5

Hilbert-type Inequalities Involving Some Means Operators

In this chapter, we provide several Hilbert-type inequalities with a homogeneous kernel, involving arithmetic, geometric and harmonic mean operators in two-dimensional, half-discrete and multidimensional cases.

5.1 Two-dimensional Inequalities

In this section we deal with two-dimensional Hilbert-type inequalities, in both integral and discrete case, involving arithmetic, geometric, and harmonic operators.

In 2010, based on the Hardy integral inequality, Das and Sahoo [38], obtained the following pair of Hilbert-type inequalities involving the arithmetic mean operator $\mathscr{A}$: $L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$ defined by $(\mathscr{A} f)(x) = \frac{1}{x}\int_0^x f(t)dt$ (see Section 1.3, Chapter 1).

Theorem 5.1 *If ν, μ, s are positive real parameters such that $s = \nu + \mu$, then the inequalities*

$$\int_0^\infty \int_0^\infty \frac{x^{\nu-\frac{1}{q}} y^{\mu-\frac{1}{p}}}{(x+y)^s} (\mathscr{A} f)(x)(\mathscr{A} g)(y)dxdy < pqB(\nu,\mu)\|f\|_{L^p(\mathbb{R}_+)}\|g\|_{L^q(\mathbb{R}_+)} \tag{5.1}$$

and

$$\left[\int_0^\infty y^{p\mu-1}\left(\int_0^\infty \frac{x^{\nu-\frac{1}{q}}}{(x+y)^s}(\mathscr{A}f)(x)dx\right)^p dy\right]^{\frac{1}{p}} < qB(\nu,\mu)\|f\|_{L^p(\mathbb{R}_+)} \tag{5.2}$$

hold for all non-negative functions $f,g:\mathbb{R}_+\to\mathbb{R}$ *such that* $0<\|f\|_{L^p(\mathbb{R}_+)}<\infty$ *and* $0<\|g\|_{L^q(\mathbb{R}_+)}<\infty$. *In addition, the constants* $pqB(\nu,\mu)$ *and* $qB(\nu,\mu)$ *are the best possible in the corresponding inequalities.*

It should be noticed here that some particular cases of inequality (5.1) were studied in [86], few years earlier. Furthermore, with the assumption $s>2$, Das and Sahoo also proved a discrete version of Theorem 5.1.

Theorem 5.2 *Let* $\nu,\mu>0$ *and* $s>2$ *be real parameters such that* $s=\nu+\mu$. *Then the inequalities*

$$\sum_{m=1}^\infty\sum_{n=1}^\infty \frac{m^{\nu-\frac{1}{q}}n^{\mu-\frac{1}{p}}}{(m+n)^s}(\overline{\mathscr{A}}a)_m(\overline{\mathscr{A}}b)_n < pqB(\nu,\mu)\|a\|_{l^p}\|b\|_{l^q} \tag{5.3}$$

and

$$\left[\sum_{n=1}^\infty n^{p\mu-1}\left(\sum_{m=1}^\infty \frac{m^{\nu-\frac{1}{q}}}{(m+n)^s}(\overline{\mathscr{A}}a)_m\right)^p\right]^{\frac{1}{p}} < qB(\nu,\mu)\|a\|_{l^p} \tag{5.4}$$

hold for all non-negative sequences $a=(a_m)_{m\in\mathbb{N}}$ *and* $b=(b_n)_{n\in\mathbb{N}}$ *satisfying* $0<\|a\|_{l^p}<\infty$ *and* $0<\|b\|_{l^q}<\infty$. *In addition, the constants* $pqB(\nu,\mu)$ *and* $qB(\nu,\mu)$ *are the best possible in the corresponding inequalities.*

In the previous theorem $\overline{\mathscr{A}}$ stands for a discrete version of operator $\mathscr{A}$ (see Section 1.3, Chapter 1). Observe also that the paper [39] provides the corresponding result for the kernel $\frac{1}{\max\{x^s,y^s\}}$, with the best possible constant.

Considering the kernels $1/(x+y)^s$ and $1/\max\{x^s,y^s\}$, we see that they have homogeneity of degree $-s$ in common. The purpose of this section is to derive an extension of Theorems 5.1 and 5.2 to a general homogeneous case. Furthermore, we establish inequalities related to it, which include other classical means (geometric and harmonic) in both integral and discrete case.

It should be noticed here that Sulaiman (see [85, 87]), Du and Miao [42] investigated some related results with a homogeneous kernel, without considering the problem of the best constants.

5.1.1 Integral Inequalities

To present the main results we first establish the following two lemmas.

Lemma 5.1 *Let* p *and* q *be conjugate parameters with* $p>1$, *and let* $s,\mu,\nu>0$ *such that* $\mu+\nu=s$. *If* $K:\mathbb{R}_+^2\to\mathbb{R}$ *is a non-negative homogeneous function of degree* $-s$, *then*

$$\omega_s(\mu,x)=\varpi_s(\nu,y)=k(1-\mu), \tag{5.5}$$

where

$$\omega_s(\mu,x) := \int_0^\infty K(x,y) y^{\mu-1} x^\nu dy,$$

and

$$\varpi_s(\nu,y) := \int_0^\infty K(x,y) x^{\nu-1} y^\mu dx.$$

Proof. Setting $u = \frac{y}{x}$, we find

$$\omega_s(\mu,x) = \int_0^\infty K(1,u) u^{\mu-1} du = k(1-\mu),$$

and for $y > 0$ letting $x = \frac{y}{u}$, it follows that

$$\varpi_s(\nu,y) = \int_0^\infty K\left(\frac{y}{u}, y\right) y^\mu \frac{y^{\nu-1}}{u^{\nu-1}} \frac{y}{u^2} du = \int_0^\infty K(1,u) u^{\mu-1} du = k(1-\mu),$$

so (5.5) holds. □

Lemma 5.2 *If* $q > \frac{1}{\beta}, 0 < \beta \leq 1, n > \frac{1}{\beta q - 1}$ *for* $x \geq 1$, *then*

$$(x^{\frac{\beta q - (1+(1/n))}{\beta q}} - 1)^\beta \geq x^{\frac{\beta q - (1+(1/n))}{q}} - 1. \tag{5.6}$$

Proof. For $x \geq 1$, set

$$F(x) = (x^{\frac{\beta q - (1+(1/n))}{\beta q}} - 1)^\beta - x^{\frac{\beta q - (1+(1/n))}{q}} + 1.$$

Simple computations yield for $x > 1$

$$F'(x) = \frac{\beta q - (1+(1/n))}{q} x^{\frac{(\beta-1)q - (1+(1/n))}{q}} \left((1 - x^{\frac{1+(1/n) - \beta q}{\beta q}})^{\beta - 1} - 1 \right) > 0.$$

F is increasing function on $(1,\infty)$ and continuous on $[1,\infty)$. In particular, we have $F(x) \geq F(1) = 0$, which gives the desired inequality. □

Now we give the first result of this section.

Theorem 5.3 *Let* $\frac{1}{p} + \frac{1}{q} = 1$ *with* $p > \frac{1}{\alpha}, q > \frac{1}{\beta}, 0 < \alpha, \beta \leq 1$, *and let* ν, μ, s *be non-negative real parameters such that* $\mu + \nu = s$. *Further, suppose* $K : \mathbb{R}_+^2 \to \mathbb{R}$ *is a non-negative homogeneous function of degree* $-s$. *If* $0 < \int_0^\infty K(1,u) u^{\mu - \frac{1}{p} - \beta} du < \infty, 0 < \int_0^\infty K(1,u) u^{\nu - \frac{1}{q} - \alpha} du < \infty$, *then the inequalities*

$$\int_0^\infty \int_0^\infty K(x,y) x^{\nu - \frac{1}{q}} y^{\mu - \frac{1}{p}} (\mathscr{A} f)^\alpha (x) (\mathscr{A} g)^\beta (y) dx dy$$
$$< k(1-\mu) \left(\frac{\alpha p}{\alpha p - 1}\right)^\alpha \left(\frac{\beta q}{\beta q - 1}\right)^\beta \|f^\alpha\|_{L^p(\mathbb{R}_+)} \|g^\beta\|_{L^q(\mathbb{R}_+)} \tag{5.7}$$

and

$$\left[\int_0^\infty\left(\int_0^\infty K(x,y)x^{\nu-\frac{1}{q}}y^{\mu-\frac{1}{p}}(\mathscr{A}f)^\alpha(x)dx\right)^p dy\right]^{\frac{1}{p}}$$
$$< k(1-\mu)\left(\frac{\alpha p}{\alpha p-1}\right)^\alpha \|f^\alpha\|_{L^p(\mathbb{R}_+)}, \tag{5.8}$$

hold for all non-negative functions $f,g:\mathbb{R}_+\to\mathbb{R}$ *such that* $0<\|f^\alpha\|_{L^p(\mathbb{R}_+)}<\infty$ *and* $0<\|g^\beta\|_{L^q(\mathbb{R}_+)}<\infty$. *In addition, constants* $k(1-\mu)\left(\frac{\alpha p}{\alpha p-1}\right)^\alpha\left(\frac{\beta q}{\beta q-1}\right)^\beta$ *and* $k(1-\mu)\left(\frac{\alpha p}{\alpha p-1}\right)^\alpha$ *are the best possible.*

Proof. By the Hölder inequality and Lemma 5.1, we have

$$\begin{aligned}
&\int_0^\infty\int_0^\infty K(x,y)x^{\nu-\frac{1}{q}}y^{\mu-\frac{1}{p}}(\mathscr{A}f)^\alpha(x)(\mathscr{A}g)^\beta(y)dxdy\\
&=\int_0^\infty\int_0^\infty K(x,y)(y^{\frac{\mu-1}{p}}x^{\frac{\nu}{p}}(\mathscr{A}f)^\alpha(x))(x^{\frac{\nu-1}{q}}y^{\frac{s}{q}}(\mathscr{A}g)^\beta(y))dxdy\\
&\le\left\{\int_0^\infty\int_0^\infty K(x,y)y^{\mu-1}x^\nu(\mathscr{A}f)^{\alpha p}(x)dxdy\right\}^{\frac{1}{p}}\\
&\quad\times\left\{\int_0^\infty\int_0^\infty K(x,y)x^{\nu-1}y^\mu(\mathscr{A}g)^{\beta q}(y)dxdy\right\}^{\frac{1}{q}}\\
&=k(1-\mu)\left\{\int_0^\infty(\mathscr{A}f)^{\alpha p}(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty(\mathscr{A}g)^{\beta q}(y)dy\right\}^{\frac{1}{q}}.
\end{aligned}$$

Then by the Hardy inequality, (5.7) is valid.

Supposing that there exists a positive constant $C<k(1-\mu)\left(\frac{\alpha p}{\alpha p-1}\right)^\alpha\left(\frac{\beta q}{\beta q-1}\right)^\beta$, such that (5.7) is still valid when $k(1-\mu)\left(\frac{\alpha p}{\alpha p-1}\right)^\alpha\left(\frac{\beta q}{\beta q-1}\right)^\beta$ is replaced by C and for $n>\max\left\{\frac{1}{\alpha p-1},\frac{1}{\beta q-1}\right\}$, $n\in\mathbb{N}$, setting $\widetilde{f}(x),\widetilde{g}(y)$ as follows:

$$\widetilde{f}(x)=\begin{cases}0, & \text{for } x\in(0,1)\\ x^{-\frac{1+(1/n)}{\alpha p}}, & \text{for } x\in[1,\infty)\end{cases},\quad \widetilde{g}(y)=\begin{cases}0, & \text{for } y\in(0,1)\\ y^{-\frac{1+(1/n)}{\beta q}}, & \text{for } y\in[1,\infty)\end{cases},$$

we have

$$C\|\widetilde{f}^\alpha\|_{L^p(\mathbb{R}_+)}\|\widetilde{g}^\beta\|_{L^q(\mathbb{R}_+)}=nC, \tag{5.9}$$

and

$$x(\mathscr{A}\widetilde{f})(x)=\begin{cases}0, & \text{for } x\in(0,1)\\ \frac{\alpha p}{\alpha p-(1+(1/n))}(x^{\frac{\alpha p-(1+(1/n))}{\alpha p}}-1), & \text{for } x\in[1,\infty)\end{cases},$$

$$y(\mathscr{A}\widetilde{g})(y)=\begin{cases}0, & \text{for } y\in(0,1)\\ \frac{\beta q}{\beta q-(1+(1/n))}(y^{\frac{\beta q-(1+(1/n))}{\beta q}}-1), & \text{for } y\in[1,\infty)\end{cases}.$$

Denote $\phi(n) = \left(\frac{\alpha p}{\alpha p-(1+(1/n))}\right)^{\alpha}\left(\frac{\beta q}{\beta q-(1+(1/n))}\right)^{\beta}$. Then $\phi(n) \to \left(\frac{\alpha p}{\alpha p-1}\right)^{\alpha}\left(\frac{\beta q}{\beta q-1}\right)^{\beta}$, as $n \to \infty$ and for $x, y \geq 1$, by Lemma 5.2, we have

$$\begin{aligned} x^{\alpha}y^{\beta}(\mathscr{A}\widetilde{f})^{\alpha}(x)(\mathscr{A}\widetilde{g})^{\beta}(y) &= \phi(n)(x^{\frac{\alpha p-(1+(1/n))}{\alpha p}}-1)^{\alpha}(y^{\frac{\beta q-(1+(1/n))}{\beta q}}-1)^{\beta} \\ &\geq \phi(n)(x^{\frac{\alpha p-(1+(1/n))}{p}}-1)(y^{\frac{\beta q-(1+(1/n))}{q}}-1) \\ &> \phi(n)(x^{\frac{\alpha p-(1+(1/n))}{p}}y^{\frac{\beta q-(1+(1/n))}{q}}-x^{\frac{\alpha p-(1+(1/n))}{p}}-y^{\frac{\beta q-(1+(1/n))}{q}}). \end{aligned}$$

Then

$$\begin{aligned} &\int_0^{\infty}\int_0^{\infty}K(x,y)x^{\nu-\frac{1}{q}}y^{\mu-\frac{1}{p}}(\mathscr{A}\widetilde{f})^{\alpha}(x)(\mathscr{A}\widetilde{g})^{\beta}(y)dxdy \\ &> \phi(n)\int_1^{\infty}\int_1^{\infty}K(x,y)\left(x^{\nu-\frac{1}{np}-1}y^{\mu-\frac{1}{nq}-1}-x^{\nu-\frac{1}{np}-1}y^{\mu-\frac{1}{p}-\beta}-x^{\nu-\frac{1}{q}-\alpha}y^{\mu-\frac{1}{nq}-1}\right)dxdy \\ &= \phi(n)(I_1-I_2-I_3). \end{aligned}$$

Taking $u = \frac{y}{x}$ and by the Fubini theorem, we obtain

$$\begin{aligned} I_1 &= \int_1^{\infty}\int_1^{\infty}K(x,y)x^{\nu-\frac{1}{np}-1}y^{\mu-\frac{1}{nq}-1}dxdy \\ &= \int_1^{\infty}x^{-1-\frac{1}{n}}\left(\int_1^{\infty}K(x,y)y^{\mu-\frac{1}{nq}-1}x^{\nu+\frac{1}{nq}}dy\right)dx \\ &= \int_1^{\infty}x^{-1-\frac{1}{n}}\left(\int_{1/x}^{1}K(1,u)u^{\mu-\frac{1}{nq}-1}du+\int_1^{\infty}K(1,u)u^{\mu-\frac{1}{nq}-1}du\right)dx \\ &= n\int_1^{\infty}K(1,u)u^{\mu-\frac{1}{nq}-1}du+\int_1^{\infty}x^{-1-\frac{1}{n}}dx\int_{1/x}^{1}K(1,u)u^{\mu-\frac{1}{nq}-1}du \\ &= n\int_1^{\infty}K(1,u)u^{\mu-\frac{1}{nq}-1}du+\int_0^{1}K(1,u)u^{\mu-\frac{1}{nq}-1}du\int_{1/u}^{\infty}x^{-1-\frac{1}{n}}dx \\ &= n\left(\int_1^{\infty}K(1,u)u^{\mu-\frac{1}{nq}-1}du+\int_0^{1}K(1,u)u^{\mu+\frac{1}{np}-1}du\right). \end{aligned}$$

Again taking $u = \frac{y}{x}$, we have

$$\begin{aligned} I_2 &= \int_1^{\infty}\int_1^{\infty}K(x,y)x^{\nu-\frac{1}{np}-1}y^{\mu-\frac{1}{p}-\beta}dxdy \\ &= \int_1^{\infty}\int_0^{\infty}K(x,y)x^{\nu-\frac{1}{np}-1}y^{\mu-\frac{1}{p}-\beta}dxdy-\int_1^{\infty}\int_0^{1}K(x,y)x^{\nu-\frac{1}{np}-1}y^{\mu-\frac{1}{p}-\beta}dxdy \\ &< \int_1^{\infty}x^{-1-\left(\beta-\frac{1}{q}+\frac{1}{np}\right)}dx\int_0^{\infty}K(1,u)u^{\mu-\frac{1}{p}-\beta}du \\ &= \frac{1}{\beta-\frac{1}{q}+\frac{1}{np}}\int_0^{\infty}K(1,u)u^{\mu-\frac{1}{p}-\beta}du < \infty. \end{aligned}$$

Similarly, we get

$$I_3 = \int_1^{\infty}\int_1^{\infty}K(x,y)x^{\nu-\frac{1}{q}-\alpha}y^{\mu-\frac{1}{nq}-1}dxdy$$

$$< \frac{1}{\alpha - \frac{1}{p} + \frac{1}{nq}} \int_0^\infty K(1,u) u^{\nu - \frac{1}{q} - \alpha} du < \infty.$$

Hence by (5.9), we have

$$\int_1^\infty \phi(n) K(1,u) u^{\mu - \frac{1}{nq} - 1} du + \int_0^1 \phi(n) K(1,u) u^{\mu + \frac{1}{np} - 1} du - \frac{\phi(n)}{n} O(1) < C.$$

Then, by Fatou lemma (see e.g. [84]), we have

$$\begin{aligned}
k(1-\mu)\left(\frac{\alpha p}{\alpha p - 1}\right)^\alpha \left(\frac{\beta q}{\beta q - 1}\right)^\beta &= \left(\frac{\alpha p}{\alpha p - 1}\right)^\alpha \left(\frac{\beta q}{\beta q - 1}\right)^\beta \int_0^\infty K(1,u) u^{\mu - 1} du \\
&= \int_1^\infty \lim_{n\to\infty} \phi(n) K(1,u) u^{\mu - \frac{1}{nq} - 1} du \\
&\quad + \int_0^1 \lim_{n\to\infty} \phi(n) K(1,u) u^{\mu + \frac{1}{np} - 1} du - \lim_{n\to\infty} \frac{\phi(n)}{n} O(1) \\
&\le \varliminf_{n\to\infty} \left(\int_1^\infty \phi(n) K(1,u) u^{\mu - \frac{1}{nq} - 1} du \right. \\
&\quad \left. + \int_0^1 \phi(n) K(1,u) u^{\mu + \frac{1}{np} - 1} du - \frac{\phi(n)}{n} O(1) \right) < C.
\end{aligned}$$

Hence, the constant $C = k(1-\mu)\left(\frac{\alpha p}{\alpha p-1}\right)^\alpha \left(\frac{\beta q}{\beta q-1}\right)^\beta$ is the best possible.

By the Hölder inequality and Lemma 5.1, we get

$$\begin{aligned}
L(y) &:= \int_0^\infty K(x,y) x^{\nu - \frac{1}{q}} y^{\mu - \frac{1}{p}} (\mathscr{A} f)^\alpha (x) dx \\
&= \int_0^\infty K(x,y) (x^{\frac{\nu}{p}} y^{\frac{\mu-1}{p}} (\mathscr{A} f)^\alpha (x))(x^{\frac{\nu-1}{q}} y^{\frac{\mu}{q}}) dx \\
&\le \left\{ \int_0^\infty K(x,y) x^\nu y^{\mu-1} (\mathscr{A} f)^{\alpha p}(x) dx \right\}^{1/p} \left\{ \int_0^\infty K(x,y) x^{\nu-1} y^\mu dx \right\}^{1/q} \\
&= (k(1-\mu))^{\frac{1}{q}} \left\{ \int_0^\infty K(x,y) x^\nu y^{\mu-1} (\mathscr{A} f)^{\alpha p}(x) dx \right\}^{1/p}.
\end{aligned}$$

Hence, applying Lemma 5.1 again, we have

$$\begin{aligned}
\int_0^\infty L^p(y) dy &\le (k(1-\mu))^{\frac{p}{q}} \int_0^\infty \left(\int_0^\infty K(x,y) x^\nu y^{\mu-1} dy \right) (\mathscr{A} f)^{\alpha p}(x) dx \\
&= (k(1-\mu))^p \int_0^\infty (\mathscr{A} f)^{\alpha p}(x) dx.
\end{aligned}$$

Then by the Hardy inequality, (5.8) is valid.

Assuming that the constant $k(1-\mu)\left(\frac{\alpha p}{\alpha p-1}\right)^\alpha$ in (5.8) is not the best possible, then there exists a positive constant K such that $K < k(1-\mu)\left(\frac{\alpha p}{\alpha p-1}\right)^\alpha$ and (5.8) still remains valid if $k(1-\mu)\left(\frac{\alpha p}{\alpha p-1}\right)^{\alpha}$ is replaced by K. Then, utilizing the Hölder inequality, (5.8) and the Hardy inequality, we obtain

$$\begin{aligned}
J &= \int_0^\infty \left(\int_0^\infty K(x,y)x^{\nu-\frac{1}{q}}y^{\mu-\frac{1}{p}}(\mathscr{A}f)^{\alpha}(x)dx\right)(\mathscr{A}g)^{\beta}(y)dy \\
&\leq \left\{\int_0^\infty \left(\int_0^\infty K(x,y)x^{\nu-\frac{1}{q}}y^{\mu-\frac{1}{p}}(\mathscr{A}f)^{\alpha}(x)\right)^p dy\right\}^{1/p}\left\{\int_0^\infty (\mathscr{A}g)^{\beta q}(y)dy\right\}^{1/q} \\
&< \left(\frac{\beta q}{\beta q-1}\right)^{\beta} K\left\{\int_0^\infty f^{\alpha p}(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty g^{\beta q}(x)dx\right\}^{\frac{1}{q}},
\end{aligned}$$

which results that the constant $k(1-\mu)\left(\frac{\alpha p}{\alpha p-1}\right)^{\alpha}\left(\frac{\beta q}{\beta q-1}\right)^{\beta}$ in (5.7) is not the best possible. This contradiction shows that the constant $k(1-\mu)\left(\frac{\alpha p}{\alpha p-1}\right)^{\alpha}$ in (5.8) is the best possible. The theorem is proved. □

Now we obtain Hilbert-type inequalities with a homogeneous kernel, involving geometric mean operators, in the different way as in the proof of the previous theorem. It is established by virtue of the general Hilbert-type and Knopp inequalities. Note that these operators and the corresponding inequalities are presented in Section 1.3 of Chapter 1.

Theorem 5.4 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let ν, μ, s be non-negative real parameters such that $s=\nu+\mu$. Further, suppose $K:\mathbb{R}_+^2\to\mathbb{R}$ is a non-negative homogeneous function of degree $-s$. Then the inequalities*

$$\begin{aligned}
&\int_0^\infty\int_0^\infty K(x,y)x^{\nu-\frac{1}{q}}y^{\mu-\frac{1}{p}}(\mathscr{G}f)(x)(\mathscr{G}g)(y)dxdy \\
&\qquad < e\cdot k(1-\mu)\|f\|_{L^p(\mathbb{R}_+)}\|g\|_{L^q(\mathbb{R}_+)}
\end{aligned} \tag{5.10}$$

and

$$\left[\int_0^\infty y^{p\mu-1}\left(\int_0^\infty K(x,y)x^{\nu-\frac{1}{q}}(\mathscr{G}f)(x)dx\right)^p dy\right]^{\frac{1}{p}} < e^{\frac{1}{p}}k(1-\mu)\|f\|_{L^p(\mathbb{R}_+)} \tag{5.11}$$

hold for all non-negative functions $f,g:\mathbb{R}_+\to\mathbb{R}$ such that $0<\|f\|_{L^p(\mathbb{R}_+)}<\infty$ and $0<\|g\|_{L^q(\mathbb{R}_+)}<\infty$. In addition, constants $e\cdot k(1-\mu)$ and $e^{\frac{1}{p}}k(1-\mu)$ are the best possible in inequalities (5.10) *and* (5.11).

Proof. The starting point in this proof is Hilbert-type inequality (1.25) with parameters $A_1=\frac{1-\nu}{q}$, $A_2=\frac{1-\mu}{p}$, and with functions f and g respectively replaced with $x^{\nu-\frac{1}{q}}(\mathscr{G}f)(x)$ and $y^{\mu-\frac{1}{p}}(\mathscr{G}g)(y)$, that is, the inequality

$$\begin{aligned}
&\int_0^\infty\int_0^\infty K(x,y)x^{\nu-\frac{1}{q}}y^{\mu-\frac{1}{p}}(\mathscr{G}f)(x)(\mathscr{G}g)(y)dxdy \\
&\qquad < k(1-\mu)\|\mathscr{G}f\|_{L^p(\mathbb{R}_+)}\|\mathscr{G}g\|_{L^q(\mathbb{R}_+)}.
\end{aligned}$$

Now, by virtue of the Knopp inequality (1.60), it follows that $\|\mathscr{G}f\|_{L^p(\mathbb{R}_+)} < e^{\frac{1}{p}}\|f\|_{L^p(\mathbb{R}_+)}$ and $\|\mathscr{G}g\|_{L^q(\mathbb{R}_+)} < e^{\frac{1}{q}}\|g\|_{L^q(\mathbb{R}_+)}$, which yields inequality (5.10).

In order to prove inequality (5.11), we consider Hardy-Hilbert-type inequality (1.26) with parameters $A_1 = \frac{1-\nu}{q}$, $A_2 = \frac{1-\mu}{p}$, and with function $x^{\nu-\frac{1}{q}}(\mathscr{G}f)(x)$. This yields inequality

$$\left[\int_0^\infty y^{p\mu-1}\left(\int_0^\infty K(x,y)x^{\nu-\frac{1}{q}}(\mathscr{G}f)(x)dx\right)^p dy\right]^{\frac{1}{p}} < k(1-\mu)\|\mathscr{G}f\|_{L^p(\mathbb{R}_+)},$$

which becomes (5.11) after applying the Knopp inequality (1.60) on its right-hand side.

Now, we prove that inequalities (5.10) and (5.11) involve the best possible constants on their right-hand sides. First, suppose that there exists a positive constant C smaller than $e\cdot k(1-\mu)$ such that the inequality

$$\int_0^\infty\int_0^\infty K(x,y)x^{\nu-\frac{1}{q}}y^{\mu-\frac{1}{p}}(\mathscr{G}f)(x)(\mathscr{G}g)(y)dxdy < C\|f\|_{L^p(\mathbb{R}_+)}\|g\|_{L^q(\mathbb{R}_+)} \tag{5.12}$$

holds for all non-negative functions $f,g:\mathbb{R}_+\to\mathbb{R}$, provided that $0<\|f\|_{L^p(\mathbb{R}_+)}<\infty$ and $0<\|g\|_{L^q(\mathbb{R}_+)}<\infty$.

Considering the above inequality with functions $\widetilde{f},\widetilde{g}:\mathbb{R}_+\to\mathbb{R}$ defined by

$$\widetilde{f}(x) = \begin{cases} 1, & 0<x<1 \\ e^{-\frac{1}{p}}x^{\frac{-\varepsilon-1}{p}}, & x\ge 1 \end{cases}, \quad \widetilde{g}(y) = \begin{cases} 1, & 0<y<1 \\ e^{-\frac{1}{q}}y^{\frac{-\varepsilon-1}{q}}, & y\ge 1 \end{cases},$$

where $\varepsilon>0$ is sufficiently small number, the right-hand side reduces to

$$C\|\widetilde{f}\|_{L^p(\mathbb{R}_+)}\|\widetilde{g}\|_{L^q(\mathbb{R}_+)} = \frac{C}{\varepsilon}(\varepsilon+\frac{1}{e}). \tag{5.13}$$

On the other hand, since

$$(\mathscr{G}\widetilde{f})(x) = \begin{cases} 1, & 0<x<1 \\ e^{\frac{\varepsilon}{p}-\frac{\varepsilon}{xp}}x^{\frac{-\varepsilon-1}{p}}, & x\ge 1 \end{cases}$$

and

$$(\mathscr{G}\widetilde{g})(y) = \begin{cases} 1, & 0<y<1 \\ e^{\frac{\varepsilon}{q}-\frac{\varepsilon}{yq}}y^{\frac{-\varepsilon-1}{q}}, & y\ge 1 \end{cases},$$

the Fubini theorem and the change of variables $t = \frac{y}{x}$ imply the following series of relations:

$$\begin{aligned}
&\int_0^\infty \int_0^\infty K(x,y) x^{\nu-\frac{1}{q}} y^{\mu-\frac{1}{p}} \big(\mathscr{G}\widetilde{f}\big)(x) \big(\mathscr{G}\widetilde{g}\big)(y) dxdy \\
&\quad > \int_1^\infty \int_1^\infty K(x,y) x^{\nu-\frac{1}{q}} y^{\mu-\frac{1}{p}} \big(\mathscr{G}\widetilde{f}\big)(x) \big(\mathscr{G}\widetilde{g}\big)(y) dxdy \\
&\quad = \int_1^\infty \int_1^\infty K(x,y) x^{\nu-\frac{\varepsilon}{p}-1} y^{\mu-\frac{\varepsilon}{q}-1} e^{\varepsilon-\frac{\varepsilon}{xp}-\frac{\varepsilon}{yq}} dxdy \\
&\quad \ge \int_1^\infty \int_1^\infty K(x,y) x^{\nu-\frac{\varepsilon}{p}-1} y^{\mu-\frac{\varepsilon}{q}-1} dxdy \\
&\quad = \int_1^\infty x^{-\varepsilon-1} \int_{\frac{1}{x}}^\infty K(1,t) t^{\mu-\frac{\varepsilon}{q}-1} dtdx \qquad (5.14) \\
&\quad = \frac{1}{\varepsilon} \int_1^\infty K(1,t) t^{\mu-\frac{\varepsilon}{q}-1} dt + \int_1^\infty x^{-\varepsilon-1} \int_{\frac{1}{x}}^1 K(1,t) t^{\mu-\frac{\varepsilon}{q}-1} dtdx \\
&\quad = \frac{1}{\varepsilon} \int_1^\infty K(1,t) t^{\mu-\frac{\varepsilon}{q}-1} dt + \int_0^1 K(1,t) t^{\mu-\frac{\varepsilon}{q}-1} \int_{\frac{1}{t}}^\infty x^{-\varepsilon-1} dxdt \\
&\quad = \frac{1}{\varepsilon} \left(\int_1^\infty K(1,t) t^{\mu-\frac{\varepsilon}{q}-1} dt + \int_0^1 K(1,t) t^{\mu+\frac{\varepsilon}{p}-1} dt \right).
\end{aligned}$$

Now, multiplying both sides of inequality (5.12) by ε, relations (5.13) and (5.14) yield inequality

$$\int_1^\infty K(1,t) t^{\mu-\frac{\varepsilon}{q}-1} dt + \int_0^1 K(1,t) t^{\mu+\frac{\varepsilon}{p}-1} dt < C\left(\varepsilon + \frac{1}{e}\right).$$

Finally, when ε goes to 0, it follows that $e \cdot k(1-\mu) \le C$, which is in contrast to our hypothesis. Therefore, the constant $e \cdot k(1-\mu)$, on the right-hand side of (5.10), is the best possible.

It remains to show that $e^{\frac{1}{p}} k(1-\mu)$ is the best possible constant factor in inequality (5.11). Similarly to above discussion, suppose that there exists a constant C' smaller than $e^{\frac{1}{p}} k(1-\mu)$ such that inequality

$$\left[\int_0^\infty y^{p\mu-1} \left(\int_0^\infty K(x,y) x^{\nu-\frac{1}{q}} (\mathscr{G}f)(x) dx \right)^p dy \right]^{\frac{1}{p}} < C' \|f\|_{L^p(\mathbb{R}_+)}$$

holds for all non-negative functions $f : \mathbb{R}_+ \to \mathbb{R}$ such that $0 < \|f\|_{L^p(\mathbb{R}_+)} < \infty$. Then, utilizing the Hölder and the Knopp inequality, we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty K(x,y) x^{\nu-\frac{1}{q}} y^{\mu-\frac{1}{p}} (\mathscr{G}f)(x) (\mathscr{G}g)(y) dxdy \\
&= \int_0^\infty \left(\int_0^\infty K(x,y) x^{\nu-\frac{1}{q}} y^{\mu-\frac{1}{p}} (\mathscr{G}f)(x) dx \right) (\mathscr{G}g)(y) dy \\
&\le \left[\int_0^\infty y^{p\mu-1} \left(\int_0^\infty K(x,y) x^{\nu-\frac{1}{q}} (\mathscr{G}f)(x) dx \right)^p dy \right]^{\frac{1}{p}} \|\mathscr{G}g\|_{L^q(\mathbb{R}_+)} \\
&< C' e^{\frac{1}{q}} \|f\|_{L^p(\mathbb{R}_+)} \|g\|_{L^q(\mathbb{R}_+)},
\end{aligned}$$

which results that the constant $e \cdot k(1-\mu)$ is not the best possible in (5.10), since $C' e^{\frac{1}{q}} < k(1-\mu) e^{\frac{1}{p}} e^{\frac{1}{q}} = e \cdot k(1-\mu)$. This contradiction completes the proof. □

Hence, inserting geometric operator $\mathscr{G}$ in appropriate Hilbert-type inequalities, we also obtain relations with the best possible constants. The same conclusion may be derived for the integral harmonic operator $\mathscr{H}$.

Theorem 5.5 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let ν, μ, s be non-negative parameters such that $s=\nu+\mu$. Further, let $K:\mathbb{R}_+^2 \to \mathbb{R}$ be a non-negative homogeneous function of degree $-s$. Then the inequalities*

$$\begin{aligned}\int_0^\infty \int_0^\infty & K(x,y) x^{\nu-\frac{1}{q}} y^{\mu-\frac{1}{p}} (\mathscr{H} f)(x) (\mathscr{H} g)(y) dx dy \\ & < \left(2+\frac{1}{pq}\right) k(1-\mu) \|f\|_{L^p(\mathbb{R}_+)} \|g\|_{L^q(\mathbb{R}_+)}\end{aligned} \tag{5.15}$$

and

$$\begin{aligned}\left[\int_0^\infty y^{p\mu-1} \left(\int_0^\infty K(x,y) x^{\nu-\frac{1}{q}} (\mathscr{H} f)(x) dx\right)^p dy\right]^{\frac{1}{p}} \\ < \left(1+\frac{1}{p}\right) k(1-\mu) \|f\|_{L^p(\mathbb{R}_+)}\end{aligned} \tag{5.16}$$

hold for all non-negative functions $f,g:\mathbb{R}_+ \to \mathbb{R}$ such that $0<\|f\|_{L^p(\mathbb{R}_+)}<\infty$ and $0<\|g\|_{L^q(\mathbb{R}_+)}<\infty$. In addition, the constants $\left(2+\frac{1}{pq}\right)k(1-\mu)$ and $\left(1+\frac{1}{p}\right)k(1-\mu)$ are the best possible in the corresponding inequalities.

Proof. Similarly to the proof of Theorem 5.4, we consider inequality (1.25) with parameters $A_1=\frac{1-\nu}{q}$, $A_2=\frac{1-\mu}{p}$, and with functions $x^{\nu-\frac{1}{q}}(\mathscr{H} f)(x)$ and $y^{\mu-\frac{1}{p}}(\mathscr{H} g)(y)$ instead of f and g, that is,

$$\begin{aligned}\int_0^\infty \int_0^\infty & K(x,y) x^{\nu-\frac{1}{q}} y^{\mu-\frac{1}{p}} (\mathscr{H} f)(x) (\mathscr{H} g)(y) dx dy \\ & < k(1-\mu) \|\mathscr{H} f\|_{L^p(\mathbb{R}_+)} \|\mathscr{H} g\|_{L^q(\mathbb{R}_+)}.\end{aligned}$$

Now, utilizing the integral Hardy-Carleman inequality (1.61), it follows that $\|\mathscr{H} f\|_{L^p(\mathbb{R}_+)} < \left(1+\frac{1}{p}\right)\|f\|_{L^p(\mathbb{R}_+)}$ and $\|\mathscr{H} g\|_{L^q(\mathbb{R}_+)} < \left(1+\frac{1}{q}\right)\|g\|_{L^q(\mathbb{R}_+)}$, which yields inequality (5.15).

In addition, considering Hardy-Hilbert-type inequality (1.26) in the same setting as above, it follows that

$$\left[\int_0^\infty y^{p\mu-1} \left(\int_0^\infty K(x,y) x^{\nu-\frac{1}{q}} (\mathscr{H} f)(x) dx\right)^p dy\right]^{\frac{1}{p}} < k(1-\mu) \|\mathscr{H} f\|_{L^p(\mathbb{R}_+)},$$

which becomes (5.16) after applying the integral Hardy-Carleman inequality on its right-hand side.

In order to prove that inequality (5.15) includes the best possible constant, we suppose that there exists a positive constant M smaller than $\left(2+\frac{1}{pq}\right)k(1-\mu)$, such that the inequality

$$\int_0^\infty\int_0^\infty K(x,y)x^{\nu-\frac{1}{q}}y^{\mu-\frac{1}{p}}(\mathscr{H}f)(x)(\mathscr{H}g)(y)dxdy < M\|f\|_{L^p(\mathbb{R}_+)}\|g\|_{L^q(\mathbb{R}_+)} \tag{5.17}$$

holds for all non-negative functions $f,g:\mathbb{R}_+\to\mathbb{R}$, provided that $0<\|f\|_{L^p(\mathbb{R}_+)}<\infty$ and $0<\|g\|_{L^q(\mathbb{R}_+)}<\infty$.

Considering functions $\widetilde{f},\widetilde{g}:\mathbb{R}_+\to\mathbb{R}$ defined by

$$\widetilde{f}(x)=\begin{cases} x^{\frac{\varepsilon-1}{p}}, & 0<x\le 1\\ 0, & x>1\end{cases},\quad \widetilde{g}(y)=\begin{cases} y^{\frac{\varepsilon-1}{q}}, & 0<y\le 1\\ 0, & y>1\end{cases},$$

where $\varepsilon>0$ is sufficiently small number, the right-hand side of the above inequality becomes

$$M\|\widetilde{f}\|_{L^p(\mathbb{R}_+)}\|\widetilde{g}\|_{L^q(\mathbb{R}_+)}=\frac{M}{\varepsilon}. \tag{5.18}$$

In addition, since

$$\left(\mathscr{H}\widetilde{f}\right)(x)=\begin{cases} \frac{1+p-\varepsilon}{p}x^{\frac{\varepsilon-1}{p}}, & 0<x\le 1\\ 0, & x>1\end{cases}$$

and

$$\left(\mathscr{H}\widetilde{g}\right)(y)=\begin{cases} \frac{1+q-\varepsilon}{q}y^{\frac{\varepsilon-1}{q}}, & 0<y\le 1\\ 0, & y>1\end{cases},$$

utilizing a suitable variable changes and the Fubini theorem, the left-hand side of inequality (5.17) can be rewritten as

$$\begin{aligned}
&\int_0^\infty\int_0^\infty K(x,y)x^{\nu-\frac{1}{q}}y^{\mu-\frac{1}{p}}\left(\mathscr{H}\widetilde{f}\right)(x)\left(\mathscr{H}\widetilde{g}\right)(y)dxdy\\
&\quad=\left(2-\varepsilon+\frac{(1-\varepsilon)^2}{pq}\right)\int_0^1\int_0^1 K(x,y)x^{\nu+\frac{\varepsilon}{p}-1}y^{\mu+\frac{\varepsilon}{q}-1}dxdy\\
&\quad=\left(2-\varepsilon+\frac{(1-\varepsilon)^2}{pq}\right)\int_0^1 x^{\varepsilon-1}dx\int_0^{\frac{1}{x}}K(1,t)t^{\mu+\frac{\varepsilon}{q}-1}dt\\
&\quad=\left(2-\varepsilon+\frac{(1-\varepsilon)^2}{pq}\right)\left(\frac{1}{\varepsilon}\int_0^1 K(1,t)t^{\mu+\frac{\varepsilon}{q}-1}dt+\int_0^1 x^{\varepsilon-1}dx\int_1^{\frac{1}{x}}K(1,t)t^{\mu+\frac{\varepsilon}{q}-1}dt\right)\\
&\quad=\left(2-\varepsilon+\frac{(1-\varepsilon)^2}{pq}\right)\left(\frac{1}{\varepsilon}\int_0^1 K(1,t)t^{\mu+\frac{\varepsilon}{q}-1}dt+\int_1^\infty K(1,t)t^{\mu+\frac{\varepsilon}{q}-1}dt\int_0^{\frac{1}{t}}x^{\varepsilon-1}dx\right)\\
&\quad=\frac{1}{\varepsilon}\left(2-\varepsilon+\frac{(1-\varepsilon)^2}{pq}\right)\left(\int_0^1 K(1,t)t^{\mu+\frac{\varepsilon}{q}-1}dt+\int_1^\infty K(1,t)t^{\mu-\frac{\varepsilon}{p}-1}dt\right).
\end{aligned}$$

Now, multiplying both sides of inequality (5.17) by ε, the above relation and (5.18) yield inequality

$$\left(2-\varepsilon+\frac{(1-\varepsilon)^2}{pq}\right)\left(\int_0^1 K(1,t)t^{\mu+\frac{\varepsilon}{q}-1}dt+\int_1^\infty K(1,t)t^{\mu-\frac{\varepsilon}{p}-1}dt\right)<M,$$

which implies that $\left(2+\frac{1}{pq}\right)k(1-\mu)\leq M$, after letting $\varepsilon\searrow 0$. This contradiction shows that $\left(2+\frac{1}{pq}\right)k(1-\mu)$ is the best possible constant in (5.15).

In order to show that $\left(1+\frac{1}{p}\right)k(1-\mu)$ is the best possible constant in (5.16), suppose that there exists a constant M' smaller than $\left(1+\frac{1}{p}\right)k(1-\mu)$ such that inequality

$$\left[\int_0^\infty y^{p\mu-1}\left(\int_0^\infty K(x,y)x^{\nu-\frac{1}{q}}(\mathscr{H}f)(x)dx\right)^p dy\right]^{\frac{1}{p}}<M'\|f\|_{L^p(\mathbb{R}_+)}$$

holds for all non-negative functions $f:\mathbb{R}_+\to\mathbb{R}$, provided that $0<\|f\|_{L^p(\mathbb{R}_+)}<\infty$. Then, utilizing the Hölder and the integral Hardy-Carleman inequality, we have

$$\begin{aligned}
&\int_0^\infty\int_0^\infty K(x,y)x^{\nu-\frac{1}{q}}y^{\mu-\frac{1}{p}}(\mathscr{H}f)(x)(\mathscr{H}g)(y)dxdy\\
&\quad=\int_0^\infty\left(\int_0^\infty K(x,y)x^{\nu-\frac{1}{q}}y^{\mu-\frac{1}{p}}(\mathscr{H}f)(x)dx\right)(\mathscr{H}g)(y)dy\\
&\quad\leq\left[\int_0^\infty y^{p\mu-1}\left(\int_0^\infty K(x,y)x^{\nu-\frac{1}{q}}(\mathscr{H}f)(x)dx\right)^p dy\right]^{\frac{1}{p}}\|\mathscr{H}g\|_{L^q(\mathbb{R}_+)}\\
&\quad<M'\left(1+\frac{1}{q}\right)\|f\|_{L^p(\mathbb{R}_+)}\|g\|_{L^q(\mathbb{R}_+)},
\end{aligned}$$

which results that the constant $\left(2+\frac{1}{pq}\right)k(1-\mu)$ is not the best possible in (5.15), since $M'\left(1+\frac{1}{q}\right)<\left(1+\frac{1}{p}\right)\left(1+\frac{1}{q}\right)k(1-\mu)=\left(2+\frac{1}{pq}\right)k(1-\mu)$. This contradiction completes the proof. □

In Section 1.3 we have defined a class of operators representing arithmetic, geometric, and harmonic mean in both integral and discrete case. Their norms were deduced as a simple consequences of the corresponding inequalities. With the same reasoning, Hardy-Hilbert-type inequalities established in this section, enable us to define another class of integral operators and to determine their norms.

Remark 5.1 Regarding notations from Section 1.3, we define integral operators $\mathbf{A},\mathbf{G},\mathbf{H}:L^p(\mathbb{R}_+)\to L^p(\mathbb{R}_+)$ by

$$\begin{aligned}
(\mathbf{A}f)(y)&=y^{\mu-\frac{1}{p}}\int_0^\infty K(x,y)x^{\nu-\frac{1}{q}}(\mathscr{A}f)(x)dx,\\
(\mathbf{G}f)(y)&=y^{\mu-\frac{1}{p}}\int_0^\infty K(x,y)x^{\nu-\frac{1}{q}}(\mathscr{G}f)(x)dx,\\
(\mathbf{H}f)(y)&=y^{\mu-\frac{1}{p}}\int_0^\infty K(x,y)x^{\nu-\frac{1}{q}}(\mathscr{H}f)(x)dx.
\end{aligned}$$

Due to inequalities (5.8) with $\alpha = 1$, (5.11), and (5.16), the above operators are well-defined. Moreover, since the corresponding inequalities include the best possible constants, it follows that $\|\mathbf{A}\| = qk(1-\mu)$, $\|\mathbf{G}\| = e^{\frac{1}{p}}k(1-\mu)$, and $\|\mathbf{H}\| = \left(1+\frac{1}{p}\right)k(1-\mu)$.

5.1.2 Discrete Inequalities

Ideas of proving are similar to integral case, except that we use the corresponding discrete Hilbert-type inequalities and discrete versions of means operators.

Theorem 5.6 *Suppose that $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and ν, μ, s are real parameters such that $0<\nu,\mu\le 1$ and $s=\nu+\mu$. Further, let $K:\mathbb{R}_+^2\to\mathbb{R}$ be a non-negative homogeneous function of degree $-s$, strictly decreasing in each argument. Then the inequalities*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K(m,n)m^{\nu-\frac{1}{q}}n^{\mu-\frac{1}{p}}(\overline{\mathscr{A}}a)_m(\overline{\mathscr{A}}b)_n < pqk(1-\mu)\|a\|_{l^p}\|b\|_{l^q} \tag{5.19}$$

and

$$\left[\sum_{n=1}^{\infty} n^{p\mu-1}\left(\sum_{m=1}^{\infty} K(m,n)m^{\nu-\frac{1}{q}}(\overline{\mathscr{A}}a)_m\right)^p\right]^{\frac{1}{p}} < qk(1-\mu)\|a\|_{l^p} \tag{5.20}$$

hold for all non-negative sequences $a=(a_m)_{m\in\mathbb{N}}$ and $b=(b_n)_{n\in\mathbb{N}}$ satisfying $0<\|a\|_{l^p}<\infty$ and $0<\|b\|_{l^q}<\infty$. In addition, constants $pqk(1-\mu)$ and $qk(1-\mu)$ are the best possible in the corresponding inequalities.

Proof. Considering discrete Hilbert-type inequality (1.33) with sequences $m^{\nu-\frac{1}{q}}(\overline{\mathscr{A}}a)_m$, $n^{\mu-\frac{1}{p}}(\overline{\mathscr{A}}b)_n$, with $u(m)=m, v(n)=n$ and with parameters $A_1=\frac{1-\nu}{q}$, $A_2=\frac{1-\mu}{p}$, it follows that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K(m,n)m^{\nu-\frac{1}{q}}n^{\mu-\frac{1}{p}}(\overline{\mathscr{A}}a)_m(\overline{\mathscr{A}}b)_n < k(1-\mu)\|\overline{\mathscr{A}}a\|_{l^p}\|\overline{\mathscr{A}}b\|_{l^q}.$$

Now, double use of discrete Hardy inequality (1.62) yields (5.19).

In order to obtain (5.20), we consider discrete Hardy-Hilbert-type inequality (1.34) in the same setting as in the proof of inequality (5.19). This yields inequality

$$\left[\sum_{n=1}^{\infty} n^{p\mu-1}\left(\sum_{m=1}^{\infty} K(m,n)m^{\nu-\frac{1}{q}}(\overline{\mathscr{A}}a)_m\right)^p\right]^{\frac{1}{p}} < k(1-\mu)\|\overline{\mathscr{A}}a\|_{l^p},$$

which together with inequality (1.62) yields (5.20).

Now, we prove that inequalities (5.19) and (5.20) include the best possible constants on their right-hand sides. First, suppose that there exists a positive constant K smaller than $pqk(1-\mu)$ such that relation

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K(m,n)m^{\nu-\frac{1}{q}}n^{\mu-\frac{1}{p}}(\overline{\mathscr{A}}a)_m(\overline{\mathscr{A}}b)_n < K\|a\|_{l^p}\|b\|_{l^q} \tag{5.21}$$

holds for all non-negative sequences $a=(a_m)_{m\in\mathbb{N}}$ and $b=(b_n)_{n\in\mathbb{N}}$ such that $0<\|a\|_{l^p}<\infty$ and $0<\|b\|_{l^q}<\infty$. Let $\widetilde{L}$ and $\widetilde{R}$ respectively denote the left-hand side and the right-hand side of inequality (5.21) equipped with the sequences

$$\widetilde{a}_m=\begin{cases} m^{-\frac{1}{p}}, & m\le N\\ 0, & \text{otherwise}\end{cases} \quad \text{and} \quad \widetilde{b}_n=\begin{cases} n^{-\frac{1}{q}}, & n\le N\\ 0, & \text{otherwise}\end{cases}, \tag{5.22}$$

where $N\in\mathbb{N}$ is fixed. Then, the right-hand side of (5.21) may be bounded from above with a natural logarithm function:

$$\begin{aligned}\widetilde{R}=K\|\widetilde{a}\|_{l^p}\|\widetilde{b}\|_{l^q}=&K\left(\sum_{m=1}^{N}\frac{1}{m}\right)=K\left(1+\sum_{m=2}^{N}\frac{1}{m}\right)\\ <&K\left(1+\int_1^N\frac{dx}{x}\right)=K(1+\log N).\end{aligned} \tag{5.23}$$

Our next intention is to estimate the left-hand side of inequality (5.21) from below. More precisely, considering $\sum_{k=1}^{m}k^{-\frac{1}{p}}$ as the upper Darboux sum for the function $h(x)=x^{-\frac{1}{p}}$ on segment $[1,m+1]$, we have

$$\sum_{k=1}^{m}k^{-\frac{1}{p}}>\int_1^{m+1}x^{-\frac{1}{p}}dx>\int_1^{m}x^{-\frac{1}{p}}dx=q(m^{\frac{1}{q}}-1),$$

and consequently,

$$\begin{aligned}(\overline{\mathscr{A}}\widetilde{a})_m &> \frac{q(m^{\frac{1}{q}}-1)}{m}=qm^{-\frac{1}{p}}(1-m^{-\frac{1}{q}}), \quad m\le N,\\ (\overline{\mathscr{A}}\widetilde{b})_n &> \frac{p(n^{\frac{1}{p}}-1)}{n}=pn^{-\frac{1}{q}}(1-n^{-\frac{1}{p}}), \quad n\le N.\end{aligned}$$

Therefore, $\widetilde{L}$ may be estimated as follows:

$$\widetilde{L}>pq\sum_{m=1}^{N}\sum_{n=1}^{N}K(m,n)m^{\nu-1}n^{\mu-1}(1-m^{-\frac{1}{q}})(1-n^{-\frac{1}{p}}).$$

Moreover, since

$$(1-m^{-\frac{1}{q}})(1-n^{-\frac{1}{p}})=1-m^{-\frac{1}{q}}-n^{-\frac{1}{p}}+m^{-\frac{1}{q}}n^{-\frac{1}{p}}>1-m^{-\frac{1}{q}}-n^{-\frac{1}{p}},$$

the above relation implies inequality

$$\begin{aligned}\frac{\widetilde{L}}{pq}>&\sum_{m=1}^{N}\sum_{n=1}^{N}K(m,n)m^{\nu-1}n^{\mu-1}\\ &-\sum_{m=1}^{N}\sum_{n=1}^{N}K(m,n)m^{\nu-1-\frac{1}{q}}n^{\mu-1}\\ &-\sum_{m=1}^{N}\sum_{n=1}^{N}K(m,n)m^{\nu-1}n^{\mu-1-\frac{1}{p}}.\end{aligned} \tag{5.24}$$

Our next aim is to establish suitable estimates for double sums on the right-hand side of inequality (5.24). The first double sum may be regarded as the upper Darboux sum for the function $K(x,y)x^{\nu-1}y^{\mu-1}$ defined on square $[1,N+1]\times[1,N+1]$, since this two-variable function is strictly decreasing in each argument. Hence, utilizing suitable variable changes and the Fubini theorem, we have

$$\begin{aligned}
&\sum_{m=1}^{N}\sum_{n=1}^{N}K(m,n)m^{\nu-1}n^{\mu-1}\\
&\quad>\int_1^{N+1}\int_1^{N+1}K(x,y)x^{\nu-1}y^{\mu-1}dxdy\\
&\quad>\int_1^{N}\int_1^{N}K(x,y)x^{\nu-1}y^{\mu-1}dxdy\\
&\quad=\int_1^{N}\frac{dx}{x}\int_{\frac{1}{x}}^{\frac{N}{x}}K(1,t)t^{\mu-1}dt\\
&\quad=\int_{\frac{1}{N}}^{1}\left(\int_{\frac{1}{t}}^{N}\frac{dx}{x}\right)K(1,t)t^{\mu-1}dt+\int_1^{N}\left(\int_1^{\frac{N}{t}}\frac{dx}{x}\right)K(1,t)t^{\mu-1}dt\\
&\quad=\log N\int_{\frac{1}{N}}^{1}K(1,t)t^{\mu-1}\left(1+\frac{\log t}{\log N}\right)dt\\
&\qquad+\log N\int_1^{N}K(1,t)t^{\mu-1}\left(1-\frac{\log t}{\log N}\right)dt.
\end{aligned}\tag{5.25}$$

The second sum on the right-hand side of (5.24) may be rewritten as

$$\sum_{m=1}^{N}\sum_{n=1}^{N}K(m,n)m^{\nu-1-\frac{1}{q}}n^{\mu-1}=\sum_{n=1}^{N}K(1,n)n^{\mu-1}+\sum_{m=2}^{N}\sum_{n=1}^{N}K(m,n)m^{\nu-1-\frac{1}{q}}n^{\mu-1},$$

and both sums on the right-hand side of this relation may be regarded as the lower Darboux sums for the corresponding functions. More precisely, we have

$$\sum_{n=1}^{N}K(1,n)n^{\mu-1}<\int_0^{N}K(1,t)t^{\mu-1}dt<\int_0^{\infty}K(1,t)t^{\mu-1}dt=k(1-\mu)$$

and

$$\begin{aligned}
\sum_{m=2}^{N}\sum_{n=1}^{N}K(m,n)m^{\nu-1-\frac{1}{q}}n^{\mu-1}&<\int_1^{N}\int_0^{N}K(x,y)x^{\nu-1-\frac{1}{q}}y^{\mu-1}dxdy\\
&=\int_1^{N}\frac{dx}{x^{1+\frac{1}{q}}}\int_0^{\frac{N}{x}}K(1,t)t^{\mu-1}dt\\
&<\int_1^{N}\frac{dx}{x^{1+\frac{1}{q}}}\int_0^{\infty}K(1,t)t^{\mu-1}dt\\
&=\left(q-\frac{q}{N^{\frac{1}{q}}}\right)k(1-\mu),
\end{aligned}$$

so that

$$\sum_{m=1}^{N}\sum_{n=1}^{N} K(m,n)m^{\nu-1-\frac{1}{q}}n^{\mu-1} < \left(1+q-\frac{q}{N^{\frac{1}{q}}}\right)k(1-\mu). \tag{5.26}$$

In a similar manner we also estimate the third sum on the right-hand side of relation (5.24):

$$\begin{aligned}
&\sum_{m=1}^{N}\sum_{n=1}^{N} K(m,n)m^{\nu-1}n^{\mu-1-\frac{1}{p}} \\
&\quad = \sum_{m=1}^{N} K(m,1)m^{\nu-1} + \sum_{m=1}^{N}\sum_{n=2}^{N} K(m,n)m^{\nu-1}n^{\mu-1-\frac{1}{p}} \\
&\quad < \int_0^N K(t,1)t^{\nu-1}dt + \int_0^N\int_1^N K(x,y)x^{\nu-1}y^{\mu-1-\frac{1}{p}}dxdy \\
&\quad = \int_{\frac{1}{N}}^{\infty} K(1,t)t^{\mu-1}dt + \int_1^N \frac{dy}{y^{1+\frac{1}{p}}}\int_{\frac{y}{N}}^{\infty} K(1,t)t^{\mu-1}dt \\
&\quad < k(1-\mu) + \left(p-\frac{p}{N^{\frac{1}{p}}}\right)k(1-\mu) \\
&\quad = \left(1+p-\frac{p}{N^{\frac{1}{p}}}\right)k(1-\mu).
\end{aligned} \tag{5.27}$$

Now, relations (5.21), (5.23), (5.24), (5.25), (5.26), and (5.27) yield inequality

$$\begin{aligned}
\frac{K(1+\log N)}{pq} > &\log N\int_{\frac{1}{N}}^{1} K(1,t)t^{\mu-1}\left(1+\frac{\log t}{\log N}\right)dt \\
&+\log N\int_1^N K(1,t)t^{\mu-1}\left(1-\frac{\log t}{\log N}\right)dt \\
&-\left(2+pq-\frac{p}{N^{\frac{1}{p}}}-\frac{q}{N^{\frac{1}{q}}}\right)k(1-\mu).
\end{aligned} \tag{5.28}$$

Dividing inequality (5.28) by $\log N$ and letting N to infinity, it follows that

$$\frac{K}{pq} \geq k(1-\mu),$$

which contradicts with the assumption that K is smaller than $pqk(1-\mu)$. Therefore, the constant $pqk(1-\mu)$ is the best possible in inequality (5.19).

It remains to prove that $qk(1-\mu)$ is the best possible constant in inequality (5.20). For this reason, suppose that there exists a positive constant K' smaller than $qk(1-\mu)$, such that inequality

$$\left[\sum_{n=1}^{\infty} n^{p\mu-1}\left(\sum_{m=1}^{\infty} K(m,n)m^{\nu-\frac{1}{q}}(\overline{\mathscr{A}}a)_m\right)^p\right]^{\frac{1}{p}} < K'\|a\|_{l^p}$$

holds for all non-negative sequences $a = (a_m)_{m\in\mathbb{N}}$, provided that $0 < \|a\|_{l^p} < \infty$. Then, utilizing the Hölder and the Hardy inequality, we have

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K(m,n)m^{\nu-\frac{1}{q}}n^{\mu-\frac{1}{p}}(\overline{\mathscr{A}}a)_m(\overline{\mathscr{A}}b)_n \\
&\quad = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} K(m,n)m^{\nu-\frac{1}{q}}n^{\mu-\frac{1}{p}}(\overline{\mathscr{A}}a)_m\right)(\overline{\mathscr{A}}b)_n \\
&\quad \le \left[\sum_{n=1}^{\infty} n^{p\mu-1}\left(\sum_{m=1}^{\infty} K(m,n)m^{\nu-\frac{1}{q}}(\overline{\mathscr{A}}a)_m\right)^p\right]^{\frac{1}{p}} \|\overline{\mathscr{A}}b\|_{l^q} < K'p\|a\|_{l^p}\|b\|_{l^q},
\end{aligned}$$

which is impossible since $K'p < pqk(1-\mu)$ and $pqk(1-\mu)$ is the best possible constant in (5.19). $\square$

Remark 5.2 Yang and Xie [96], derived a pair of Hilbert-type inequalities similar to those in Theorem 5.6, which are closely connected to the so-called dual Hardy inequality (for more details, see e.g. [48], [68], and [74]).

The following two theorems respectively represent discrete analogues of Theorems 5.4 and 5.5. The first one includes discrete geometric operator $\overline{\mathscr{G}}$, while the second one includes harmonic operator $\overline{\mathscr{H}}$.

Theorem 5.7 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let ν, μ, s be real parameters such that $0 < \nu,\mu \le 1$ and $s = \nu+\mu$. Further, suppose $K : \mathbb{R}_+^2 \to \mathbb{R}$ is a non-negative homogeneous function of degree $-s$, strictly decreasing in each argument. Then the inequalities*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K(m,n)m^{\nu-\frac{1}{q}}n^{\mu-\frac{1}{p}}(\overline{\mathscr{G}}a)_m(\overline{\mathscr{G}}b)_n < e\cdot k(1-\mu)\|a\|_{l^p}\|b\|_{l^q} \tag{5.29}$$

and

$$\left[\sum_{n=1}^{\infty} n^{p\mu-1}\left(\sum_{m=1}^{\infty} K(m,n)m^{\nu-\frac{1}{q}}(\overline{\mathscr{G}}a)_m\right)^p\right]^{\frac{1}{p}} < e^{\frac{1}{p}}k(1-\mu)\|a\|_{l^p} \tag{5.30}$$

hold for all non-negative sequences $a = (a_m)_{m\in\mathbb{N}}$ and $b = (b_n)_{n\in\mathbb{N}}$, $0 < \|a\|_{l^p} < \infty$, $0 < \|b\|_{l^q} < \infty$. In addition, constants $e\cdot k(1-\mu)$ and $e^{\frac{1}{p}}k(1-\mu)$ are the best possible in the corresponding inequalities.

Proof. Similarly to the proof of Theorem 5.6, we start with inequality (1.33) equipped with sequences $m^{\nu-\frac{1}{q}}(\overline{\mathscr{G}}a)_m$, $n^{\mu-\frac{1}{p}}(\overline{\mathscr{G}}b)_n$, with $u(m)=m, v(n)=n$, and with parameters $A_1 = \frac{1-\nu}{q}$, $A_2 = \frac{1-\mu}{p}$, which yields inequality

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K(m,n)m^{\nu-\frac{1}{q}}n^{\mu-\frac{1}{p}}(\overline{\mathscr{G}}a)_m(\overline{\mathscr{G}}b)_n < k(1-\mu)\|\overline{\mathscr{G}}a\|_{l^p}\|\overline{\mathscr{G}}b\|_{l^q}.$$

Moreover, by virtue of the Carleman inequality (1.63), it follows that $\|\overline{\mathscr{G}}a\|_{l^p} < e^{\frac{1}{p}}\|a\|_{l^p}$ and $\|\overline{\mathscr{G}}b\|_{l^q} < e^{\frac{1}{q}}\|b\|_{l^q}$, i.e. we get inequality (5.29).

In addition, a similar application of discrete inequality (1.34) yields relation

$$\left[\sum_{n=1}^{\infty} n^{p\mu-1}\left(\sum_{m=1}^{\infty} K(m,n)m^{\nu-\frac{1}{q}}(\overline{\mathscr{G}}a)_m\right)^p\right]^{\frac{1}{p}} < k(1-\mu)\|\overline{\mathscr{G}}a\|_{l^p},$$

which together with Carleman inequality yields (5.30).

Our next intention is to show that derived inequalities include the best possible constants on their right-hand sides. First, suppose that there exists a positive constant C smaller than $e\cdot k(1-\mu)$ such that inequality

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K(m,n)m^{\nu-\frac{1}{q}}n^{\mu-\frac{1}{p}}(\overline{\mathscr{G}}a)_m(\overline{\mathscr{G}}b)_n < C\|a\|_{l^p}\|b\|_{l^q} \tag{5.31}$$

holds for all non-negative sequences $a=(a_m)_{m\in\mathbb{N}}$ and $b=(b_n)_{n\in\mathbb{N}}$, $0<\|a\|_{l^p}<\infty$, $0<\|b\|_{l^q}<\infty$. Let $\widetilde{L}$ and $\widetilde{R}$ respectively denote the left-hand side and the right-hand side of inequality (5.31) equipped with sequences

$$\widetilde{a}_m=\begin{cases}1 & m=1\\ \left(\frac{(m-1)^{m-1}}{m^m}\right)^{\frac{1}{p}}, & 2\le m\le N\\ 0, & m>N\end{cases},\quad \widetilde{b}_n=\begin{cases}1 & n=1\\ \left(\frac{(n-1)^{n-1}}{n^n}\right)^{\frac{1}{q}}, & 2\le n\le N\\ 0, & n>N\end{cases},$$

where $N>2$ is fixed integer. Making use of the well-known estimate

$$\left(1+\frac{1}{m}\right)^{m+1} > e,$$

which holds for every positive integer m, it follows that

$$\|\widetilde{a}\|_{l^p}^p = 1+\sum_{m=1}^{N-1}\left(\frac{m}{m+1}\right)^{m+1}\cdot\frac{1}{m} < 1+\frac{1}{e}\sum_{m=1}^{N-1}\frac{1}{m}.$$

Moreover, since $\sum_{m=2}^{N}\frac{1}{m}$ may be regarded as the lower Darboux sum for the function $h(x)=\frac{1}{x}$ on segment $[1,N]$, i.e.

$$\sum_{m=2}^{N-1}\frac{1}{m} < \sum_{m=2}^{N}\frac{1}{m} < \int_1^N\frac{dx}{x} = \log N,$$

we have

$$\|\widetilde{a}\|_{l^p} < \left(1+\frac{1}{e}+\frac{1}{e}\log N\right)^{\frac{1}{p}},$$

and similarly

$$\|\widetilde{b}\|_{l^q} < \left(1+\frac{1}{e}+\frac{1}{e}\log N\right)^{\frac{1}{q}}.$$

The above discussion yields the following estimate for the right-hand side of inequality (5.31):

$$\widetilde{R} = C\|\widetilde{a}\|_{l^p}\|\widetilde{b}\|_{l^q} < C\left(1+\frac{1}{e}+\frac{1}{e}\log N\right). \tag{5.32}$$

On the other hand, since

$$(\overline{\mathscr{G}}\widetilde{a})_m = \begin{cases} m^{-\frac{1}{p}}, & m \leq N \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad (\overline{\mathscr{G}}\widetilde{b})_n = \begin{cases} n^{-\frac{1}{q}}, & n \leq N \\ 0, & \text{otherwise} \end{cases},$$

the left-hand side of (5.31) in the above setting becomes

$$\widetilde{L} = \sum_{m=1}^{N}\sum_{n=1}^{N} K(m,n)m^{\nu-1}n^{\mu-1}.$$

It should be noticed here that relation (5.25) (see Theorem 5.6) provides lower bound for this double sum. Therefore, from (5.25), (5.31), and (5.32) it follows that

$$\begin{aligned} C\left(1+\frac{1}{e}+\frac{1}{e}\log N\right) > \log N &\int_{\frac{1}{N}}^{1} K(1,t)t^{\mu-1}\left(1+\frac{\log t}{\log N}\right)dt \\ &+ \log N\int_{1}^{N} K(1,t)t^{\mu-1}\left(1-\frac{\log t}{\log N}\right)dt. \end{aligned}$$

From dividing the above inequality with $\log N$ and letting N to infinity it follows that

$$\frac{C}{e} \geq k(1-\mu),$$

which is in contrast to $C < e\cdot k(1-\mu)$. Therefore, the constant $e\cdot k(1-\mu)$ is the best possible in inequality (5.29).

To conclude the proof, we show that $e^{\frac{1}{p}}k(1-\mu)$ is the best possible constant in inequality (5.30). Hence, suppose that there exists a positive constant $C' < e^{\frac{1}{p}}k(1-\mu)$ such that inequality

$$\left[\sum_{n=1}^{\infty} n^{p\mu-1}\left(\sum_{m=1}^{\infty} K(m,n)m^{\nu-\frac{1}{q}}(\mathscr{G}a)_m\right)^p\right]^{\frac{1}{p}} < C'\|a\|_{l^p}$$

holds for all non-negative sequences $a = (a_m)_{m\in\mathbb{N}}$, provided that $0 < \|a\|_{l^p} < \infty$. Then, making use of the Hölder and the Carleman inequality, we have

$$\begin{aligned} &\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K(m,n)m^{\nu-\frac{1}{q}}n^{\mu-\frac{1}{p}}(\mathscr{G}a)_m(\mathscr{G}b)_n \\ &\quad = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} K(m,n)m^{\nu-\frac{1}{q}}n^{\mu-\frac{1}{p}}(\overline{\mathscr{G}}a)_m\right)(\overline{\mathscr{G}}b)_n \\ &\quad \leq \left[\sum_{n=1}^{\infty} n^{p\mu-1}\left(\sum_{m=1}^{\infty} K(m,n)m^{\nu-\frac{1}{q}}(\overline{\mathscr{G}}a)_m\right)^p\right]^{\frac{1}{p}} \|\overline{\mathscr{G}}b\|_{l^q} \\ &\quad < C'e^{\frac{1}{q}}\|a\|_{l^p}\|b\|_{l^q}. \end{aligned}$$

Clearly, obtained inequality is impossible since $C'e^{\frac{1}{q}} < e \cdot k(1-\mu)$ and $e \cdot k(1-\mu)$ is the best possible constant in (5.29). □

Theorem 5.8 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let ν, μ, s be real parameters such that $0<\nu,\mu\leq 1$ and $s=\nu+\mu$. Further, let $K:\mathbb{R}_+^2\to\mathbb{R}$ be a non-negative homogeneous function of degree $-s$, strictly decreasing in each argument. Then the inequalities*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}K(m,n)m^{\nu-\frac{1}{q}}n^{\mu-\frac{1}{p}}(\overline{\mathscr{H}}a)_m(\overline{\mathscr{H}}b)_n < \left(2+\frac{1}{pq}\right)k(1-\mu)\|a\|_{l^p}\|b\|_{l^q} \tag{5.33}$$

and

$$\left[\sum_{n=1}^{\infty}n^{p\mu-1}\left(\sum_{m=1}^{\infty}K(m,n)m^{\nu-\frac{1}{q}}(\overline{\mathscr{H}}a)_m\right)^p\right]^{\frac{1}{p}} < \left(1+\frac{1}{p}\right)k(1-\mu)\|a\|_{l^p} \tag{5.34}$$

hold for all non-negative sequences $a=(a_m)_{m\in\mathbb{N}}$ and $b=(b_n)_{n\in\mathbb{N}}$, provided that $0<\|a\|_{l^p}<\infty$ and $0<\|b\|_{l^q}<\infty$. In addition, constants $\left(2+\frac{1}{pq}\right)k(1-\mu)$ and $\left(1+\frac{1}{p}\right)k(1-\mu)$ are the best possible in the corresponding inequalities.

Proof. Rewriting inequality (1.33) with sequences $m^{\nu-\frac{1}{q}}(\overline{\mathscr{H}}a)_m$, $n^{\mu-\frac{1}{p}}(\overline{\mathscr{H}}b)_n$, with $u(m)=m, v(n)=n$, and with parameters $A_1=\frac{1-\nu}{q}$, $A_2=\frac{1-\mu}{p}$, we get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}K(m,n)m^{\nu-\frac{1}{q}}n^{\mu-\frac{1}{p}}(\overline{\mathscr{H}}a)_m(\overline{\mathscr{H}}b)_n < k(1-\mu)\|\overline{\mathscr{H}}a\|_{l^p}\|\overline{\mathscr{H}}b\|_{l^q},$$

so (5.33) follows by virtue of discrete Hardy-Carleman inequality (1.64).

Further, making use of inequality (1.34) with the sequence $m^{\nu-\frac{1}{q}}(\overline{\mathscr{H}}a)_m$, and $u(m)=m, v(n)=n$, and parameters $A_1=\frac{1-\nu}{q}$, $A_2=\frac{1-\mu}{p}$, we have

$$\left[\sum_{n=1}^{\infty}n^{p\mu-1}\left(\sum_{m=1}^{\infty}K(m,n)m^{\nu-\frac{1}{q}}(\overline{\mathscr{H}}a)_m\right)^p\right]^{\frac{1}{p}} < k(1-\mu)\|\overline{\mathscr{H}}a\|_{l^p},$$

which together with the Hardy-Carleman inequality yields (5.34).

In order to prove that (5.33) includes the best possible constant, we suppose that there exists a positive constant $M<\left(2+\frac{1}{pq}\right)k(1-\mu)$ such that inequality

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}K(m,n)m^{\nu-\frac{1}{q}}n^{\mu-\frac{1}{p}}(\overline{\mathscr{H}}a)_m(\overline{\mathscr{H}}b)_n < M\|a\|_{l^p}\|b\|_{l^q} \tag{5.35}$$

holds for all non-negative sequences $a=(a_m)_{m\in\mathbb{N}}$ and $b=(b_n)_{n\in\mathbb{N}}$, provided that $0<\|a\|_{l^p}<\infty$ and $0<\|b\|_{l^q}<\infty$. Let $\widetilde{L}$ and $\widetilde{R}$ respectively denote the left-hand side and the right-hand side of inequality (5.35) equipped with the sequences $(\widetilde{a}_m)_{m\in\mathbb{N}}$ and $(\widetilde{b}_n)_{n\in\mathbb{N}}$ defined by (5.22) (see Theorem 5.6). Then, taking into account relation (5.23), we get

$$\widetilde{R}=M\|\widetilde{a}\|_{l^p}\|\widetilde{b}\|_{l^q} < M(1+\log N), \tag{5.36}$$

where $N \in \mathbb{N}$ is a fixed positive integer.

Guided by ideas from previous proofs, we establish now the lower bound for the left-hand side of inequality (5.35). Obviously, $(\overline{\mathscr{H}}\widetilde{a})_m = (\overline{\mathscr{H}}\widetilde{b})_n = 0$ for $m,n > N$. Moreover, considering $\sum_{k=1}^m k^{\frac{1}{p}}$ as the lower Darboux sum for the function $h(x) = x^{\frac{1}{p}}$ on segment $[1, m+1]$, we have

$$\sum_{k=1}^{m} k^{\frac{1}{p}} < \int_1^{m+1} x^{\frac{1}{p}}\,dx < \int_0^{m+1} x^{\frac{1}{p}}\,dx = \frac{p}{p+1}(m+1)^{1+\frac{1}{p}},$$

and consequently,

$$(\overline{\mathscr{H}}\widetilde{a})_m > \left(1+\frac{1}{p}\right)\frac{m}{(m+1)^{1+\frac{1}{p}}} = \frac{p+1}{p}\left(\frac{m}{m+1}\right)^{1+\frac{1}{p}} m^{-\frac{1}{p}}, \quad m \le N.$$

Therefore, $\widetilde{L}$ may be estimated as

$$\begin{aligned} \frac{pq\widetilde{L}}{2pq+1} &> \sum_{m=1}^{N}\sum_{n=1}^{N} K(m,n) m^{\nu-1} n^{\mu-1} \left(\frac{m}{m+1}\right)^{1+\frac{1}{p}} \left(\frac{n}{n+1}\right)^{1+\frac{1}{q}} \\ &= \sum_{m=1}^{N}\sum_{n=1}^{N} K(m,n) m^{\nu-1} n^{\mu-1} (1-\varphi_m)(1-\psi_n), \end{aligned} \tag{5.37}$$

where the sequences $(\varphi_m)_{m\in\mathbb{N}}$ and $(\psi_n)_{n\in\mathbb{N}}$ are defined by

$$\varphi_m = 1 - \left(\frac{m}{m+1}\right)^{1+\frac{1}{p}} \quad \text{and} \quad \psi_n = 1 - \left(\frac{n}{n+1}\right)^{1+\frac{1}{q}}.$$

In addition, since

$$\left(\frac{m}{m+1}\right)^2 < \left(\frac{m}{m+1}\right)^{1+\frac{1}{p}} < \frac{m}{m+1},$$

it follows that

$$\frac{1}{m+1} < \varphi_m < \frac{2m+1}{(m+1)^2},$$

i.e.

$$\frac{1}{2m} < \varphi_m < \frac{2}{m},\ m \in \mathbb{N},$$

and similarly,

$$\frac{1}{2n} < \psi_n < \frac{2}{n},\ n \in \mathbb{N}.$$

Hence, we have

$$(1-\varphi_m)(1-\psi_n) = 1 - \varphi_m - \psi_n + \varphi_m\psi_n > 1 - \varphi_m - \psi_n > 1 - \frac{2}{m} - \frac{2}{n},$$

so relation (5.37) implies inequality

$$\begin{aligned}\frac{pq\widetilde{L}}{2pq+1} > &\sum_{m=1}^{N}\sum_{n=1}^{N} K(m,n)m^{\nu-1}n^{\mu-1}\\ &-2\sum_{m=1}^{N}\sum_{n=1}^{N} K(m,n)m^{\nu-2}n^{\mu-1}\\ &-2\sum_{m=1}^{N}\sum_{n=1}^{N} K(m,n)m^{\nu-1}n^{\mu-2}.\end{aligned} \tag{5.38}$$

In the sequel, we use estimate (5.25) for the first double sum on the right-hand side of relation (5.38). Remark also that estimates (5.26) and (5.27) hold respectively for $q=1$ and $p=1$, i.e. we have

$$\sum_{m=1}^{N}\sum_{n=1}^{N} K(m,n)m^{\nu-2}n^{\mu-1} < \left(2-\frac{1}{N}\right)k(1-\mu) \tag{5.39}$$

and

$$\sum_{m=1}^{N}\sum_{n=1}^{N} K(m,n)m^{\nu-1}n^{\mu-2} < \left(2-\frac{1}{N}\right)k(1-\mu). \tag{5.40}$$

Now, relations (5.25), (5.35), (5.36), (5.38), (5.39), and (5.40) yield inequality

$$\begin{aligned}&\frac{pq}{2pq+1}M\left(1+\frac{1}{\log N}\right)\\ &\quad > \int_{\frac{1}{N}}^{1} K(1,t)t^{\mu-1}\left(1+\frac{\log t}{\log N}\right)dt\\ &\qquad + \int_{1}^{N} K(1,t)t^{\mu-1}\left(1-\frac{\log t}{\log N}\right)dt - \left(2-\frac{1}{N}\right)\frac{4k(1-\mu)}{\log N},\end{aligned}$$

after dividing by $\log N$. Moreover, when N goes to infinity, the above relation reduces to

$$\frac{pqM}{2pq+1} \geq k(1-\mu),$$

which is in contrast to our assumption that M is smaller than $\left(2+\frac{1}{pq}\right)k(1-\mu)$. Hence, $\left(2+\frac{1}{pq}\right)k(1-\mu)$ is the best possible constant in inequality (5.33).

Finally, assuming that there exists a positive constant $M' < \left(1+\frac{1}{p}\right)k(1-\mu)$ such that inequality

$$\left[\sum_{n=1}^{\infty} n^{p\mu-1}\left(\sum_{m=1}^{\infty} K(m,n)m^{\nu-\frac{1}{q}}(\overline{\mathscr{H}}a)_m\right)^{p}\right]^{\frac{1}{p}} < M'\|a\|_{l^p}$$

holds for all non-negative sequences $a = (a_m)_{m\in\mathbb{N}}$, $0 < \|a\|_p < \infty$, it follows that

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K(m,n)m^{\nu-\frac{1}{q}}n^{\mu-\frac{1}{p}}(\overline{\mathscr{H}}a)_m(\overline{\mathscr{H}}b)_n \\
&= \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} K(m,n)m^{\nu-\frac{1}{q}}n^{\mu-\frac{1}{p}}(\overline{\mathscr{H}}a)_m\right)(\overline{\mathscr{H}}b)_n \\
&\le \left[\sum_{n=1}^{\infty} n^{ps-1}\left(\sum_{m=1}^{\infty} K(m,n)m^{\nu-\frac{1}{q}}(\overline{\mathscr{H}}a)_m\right)^p\right]^{\frac{1}{p}} \|\overline{\mathscr{H}}b\|_{l^q} \\
&< M'\left(1+\frac{1}{q}\right)\|a\|_{l^p}\|b\|_{l^q}.
\end{aligned}$$

Clearly, the above inequality is impossible since $M'\left(1+\frac{1}{q}\right) < \left(2+\frac{1}{pq}\right)k(1-\mu)$ and $\left(2+\frac{1}{pq}\right)k(1-\mu)$ is the best possible constant in (5.33). □

To conclude this subsection, we discuss operator expressions arising from Hardy-Hilbert-type inequalities involving discrete arithmetic, geometric, and harmonic mean operators.

Remark 5.3 Similarly to Remark 5.1, we define discrete operators $\overline{\mathbf{A}}, \overline{\mathbf{G}}, \overline{\mathbf{H}} : l^p \to l^p$ by

$$(\overline{\mathbf{A}}a)_n = n^{\mu-\frac{1}{p}}\sum_{m=1}^{\infty} K(m,n)m^{\nu-\frac{1}{q}}(\overline{\mathscr{A}}a)_m,$$

$$(\overline{\mathbf{G}}a)_n = n^{\mu-\frac{1}{p}}\sum_{m=1}^{\infty} K(m,n)m^{\nu-\frac{1}{q}}(\overline{\mathscr{G}}a)_m,$$

$$(\overline{\mathbf{H}}a)_n = n^{\mu-\frac{1}{p}}\sum_{m=1}^{\infty} K(m,n)m^{\nu-\frac{1}{q}}(\overline{\mathscr{H}}a)_m.$$

Due to inequalities (5.20), (5.30), and (5.34), the above operators are well-defined. Moreover, since the corresponding inequalities include the best possible constants, it follows that $\|\overline{\mathbf{A}}\| = qk(1-\mu)$, $\|\overline{\mathbf{G}}\| = e^{\frac{1}{p}}k(1-\mu)$, and $\|\overline{\mathbf{H}}\| = \left(1+\frac{1}{p}\right)k(1-\mu)$.

5.1.3 Applications and Concluding Remarks

Our first example refers to the function $K(x,y) = (x+y)^{-s}$, where $s > 0$. Obviously, this function is homogeneous with degree $-s$, so in this case we have

$$k(1-\mu) = \int_0^{\infty}(1+t)^{-s}t^{\mu-1}dt = B(\mu, s-\mu) = B(\mu,\nu),$$

where B is the usual Beta function. Now, with this kernel, Theorem 5.6 reduces to the main result from [38] (see Theorem 3.1.). Thus, our Theorem 5.6 may be regarded as an extension to the case of a general homogeneous kernel.

Moreover, considering Theorems 5.3, 5.4, 5.5 with the kernel $K(x,y) = (x+y)^{-s}$ and parameters $\nu = \frac{1}{q}$, $\mu = \frac{1}{p}$, $\alpha = \beta = 1$, it follows that $k\left(\frac{1}{q}\right) = B\left(\frac{1}{p},\frac{1}{q}\right) = \frac{\pi}{\sin\frac{\pi}{p}}$, so we obtain the following result.

Corollary 5.1 *If $p,q>1$ are conjugate parameters, then the series of inequalities*

$$\int_0^\infty\int_0^\infty \frac{(\mathscr{A}f)(x)(\mathscr{A}g)(y)}{x+y}dxdy < \frac{pq\pi}{\sin\frac{\pi}{p}}\|f\|_{L^p(\mathbb{R}_+)}\|g\|_{L^q(\mathbb{R}_+)},$$

$$\int_0^\infty\int_0^\infty \frac{(\mathscr{G}f)(x)(\mathscr{G}g)(y)}{x+y}dxdy < \frac{e\pi}{\sin\frac{\pi}{p}}\|f\|_{L^p(\mathbb{R}_+)}\|g\|_{L^q(\mathbb{R}_+)},$$

$$\int_0^\infty\int_0^\infty \frac{(\mathscr{H}f)(x)(\mathscr{H}g)(y)}{x+y}dxdy < \left(2+\frac{1}{pq}\right)\frac{\pi}{\sin\frac{\pi}{p}}\|f\|_{L^p(\mathbb{R}_+)}\|g\|_{L^q(\mathbb{R}_+)},$$

and

$$\left[\int_0^\infty\left(\int_0^\infty \frac{(\mathscr{A}f)(x)}{x+y}dx\right)^p dy\right]^{\frac{1}{p}} < \frac{q\pi}{\sin\frac{\pi}{p}}\|f\|_{L^p(\mathbb{R}_+)},$$

$$\left[\int_0^\infty\left(\int_0^\infty \frac{(\mathscr{G}f)(x)}{x+y}dx\right)^p dy\right]^{\frac{1}{p}} < \frac{e^{\frac{1}{p}}\pi}{\sin\frac{\pi}{p}}\|f\|_{L^p(\mathbb{R}_+)},$$

$$\left[\int_0^\infty\left(\int_0^\infty \frac{(\mathscr{H}f)(x)}{x+y}dx\right)^p dy\right]^{\frac{1}{p}} < \left(1+\frac{1}{p}\right)\frac{\pi}{\sin\frac{\pi}{p}}\|f\|_{L^p(\mathbb{R}_+)},$$

hold for all non-negative functions $f,g:\mathbb{R}_+\to\mathbb{R}$ such that $0<\|f\|_{L^p(\mathbb{R}_+)}<\infty$ and $0<\|g\|_{L^q(\mathbb{R}_+)}<\infty$. In addition, the above inequalities include the best possible constants on their right-hand sides.

Remark 5.4 Since the kernel $K(x,y)=(x+y)^{-s}$ is strictly decreasing in each argument and since parameters $\nu=\frac{1}{q}$ and $\mu=\frac{1}{p}$ fulfill conditions as in the statements of Theorems 5.6, 5.7, and 5.8, Corollary 5.1 is also valid in discrete setting. Moreover, such discrete versions with arithmetic mean operator were also discussed in [38].

Our next example deals with the homogeneous kernel $K:\mathbb{R}_+^2\to\mathbb{R}$ given by $K(x,y)=\max\{x,y\}^{-s}$, $s>0$. A straightforward computation shows that $k(1-\mu)=\frac{s}{\nu\mu}$, that is, $k\left(\frac{1}{q}\right)=pq$. Therefore, an analogue of Corollary 5.1 in this setting reads:

Corollary 5.2 *If $p,q>1$ are conjugate parameters, then the series of inequalities*

$$\int_0^\infty\int_0^\infty \frac{(\mathscr{A}f)(x)(\mathscr{A}g)(y)}{\max\{x,y\}}dxdy < p^2q^2\|f\|_{L^p(\mathbb{R}_+)}\|g\|_{L^q(\mathbb{R}_+)},$$

$$\int_0^\infty\int_0^\infty \frac{(\mathscr{G}f)(x)(\mathscr{G}g)(y)}{\max\{x,y\}}dxdy < epq\|f\|_{L^p(\mathbb{R}_+)}\|g\|_{L^q(\mathbb{R}_+)},$$

$$\int_0^\infty\int_0^\infty \frac{(\mathscr{H}f)(x)(\mathscr{H}g)(y)}{\max\{x,y\}}dxdy < (2pq+1)\|f\|_{L^p(\mathbb{R}_+)}\|g\|_{L^q(\mathbb{R}_+)},$$

and

$$\left[\int_0^\infty\left(\int_0^\infty \frac{(\mathscr{A}f)(x)}{\max\{x,y\}}dx\right)^p dy\right]^{\frac{1}{p}} < pq^2\|f\|_{L^p(\mathbb{R}_+)},$$

$$\left[\int_0^\infty \left(\int_0^\infty \frac{(\mathscr{G}f)(x)}{\max\{x,y\}}dx\right)^p dy\right]^{\frac{1}{p}} < e^{\frac{1}{p}} pq\|f\|_{L^p(\mathbb{R}_+)},$$
$$\left[\int_0^\infty \left(\int_0^\infty \frac{(\mathscr{H}f)(x)}{\max\{x,y\}}dx\right)^p dy\right]^{\frac{1}{p}} < (p+2q)\,\|f\|_{L^p(\mathbb{R}_+)},$$

hold for all non-negative functions $f,g:\mathbb{R}_+\to\mathbb{R}$ *such that* $0<\|f\|_{L^p(\mathbb{R}_+)}<\infty$ *and* $0<\|g\|_{L^q(\mathbb{R}_+)}<\infty$. *In addition, the above inequalities include the best possible constants on their right-hand sides.*

Remark 5.5 Similarly to Remark 5.4, Corollary 5.2 also holds in discrete case.

We conclude this subsection with the function $K:\mathbb{R}_+^2\to\mathbb{R}$ defined by $K(x,y)=\frac{\log y-\log x}{y-x}$. Evidently, it is homogeneous of degree -1 and strictly decreasing in both arguments, $k(1-\mu)$ converges for all $\mu\in(0,1)$, and we have

$$k(1-\mu)=\int_0^\infty \frac{\log u}{u-1}u^{\mu-1}du=\int_{-\infty}^\infty \frac{te^{\mu t}}{e^t-1}\,dt=\psi'(s)+\psi'(1-\mu)=\frac{\pi^2}{\sin^2\pi\mu},$$

where $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$, $x>0$, is the Digamma function and where we used the well-known identity $\psi(1-x)=\psi(x)+\pi\cot\pi x$, $x\in(0,1)$ (for details on ψ see [1]). Now, considering Theorems 5.6, 5.7, and 5.8 equipped with this kernel and parameters $\nu=\frac{1}{q}$, $\mu=\frac{1}{p}$, we obtain the following result in discrete form:

Corollary 5.3 *If* $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, *then the series of inequalities*

$$\sum_{m=1}^\infty\sum_{n=1}^\infty \frac{\log\frac{m}{n}}{m-n}(\overline{\mathscr{A}}a)_m(\overline{\mathscr{A}}b)_n < \frac{pq\pi^2}{\sin^2\frac{\pi}{p}}\|a\|_{l^p}\|b\|_{l^q},$$
$$\sum_{m=1}^\infty\sum_{n=1}^\infty \frac{\log\frac{m}{n}}{m-n}(\overline{\mathscr{G}}a)_m(\overline{\mathscr{G}}b)_n < \frac{e\pi^2}{\sin^2\frac{\pi}{p}}\|a\|_{l^p}\|b\|_{l^q},$$
$$\sum_{m=1}^\infty\sum_{n=1}^\infty \frac{\log\frac{m}{n}}{m-n}(\overline{\mathscr{H}}a)_m(\overline{\mathscr{H}}b)_n < \left(2+\frac{1}{pq}\right)\frac{\pi^2}{\sin^2\frac{\pi}{p}}\|a\|_{l^p}\|b\|_{l^q},$$

and

$$\left[\sum_{n=1}^\infty\left(\sum_{m=1}^\infty \frac{\log\frac{m}{n}}{m-n}(\overline{\mathscr{A}}a)_m\right)^p\right]^{\frac{1}{p}} < \frac{q\pi^2}{\sin^2\frac{\pi}{p}}\|a\|_{l^p},$$
$$\left[\sum_{n=1}^\infty\left(\sum_{m=1}^\infty \frac{\log\frac{m}{n}}{m-n}(\overline{\mathscr{G}}a)_m\right)^p\right]^{\frac{1}{p}} < \frac{e^{\frac{1}{p}}\pi^2}{\sin^2\frac{\pi}{p}}\|a\|_{l^p},$$
$$\left[\sum_{n=1}^\infty\left(\sum_{m=1}^\infty \frac{\log\frac{m}{n}}{m-n}(\overline{\mathscr{H}}a)_m\right)^p\right]^{\frac{1}{p}} < \left(1+\frac{1}{p}\right)\frac{\pi^2}{\sin^2\frac{\pi}{p}}\|a\|_{l^p}$$

hold for all non-negative sequences $a = (a_m)_{m\in\mathbb{N}}$ *and* $b = (b_n)_{n\in\mathbb{N}}$, *provided that* $0 < \|a\|_{l^p} < \infty$ *and* $0 < \|b\|_{l^q} < \infty$. *Moreover, above inequalities include the best possible constants on their right-hand sides.*

It should be noticed here that Yang and Chen [92], investigated some particular generalizations of Theorem 5.3. They proved the equality

$$\|T_1 \circ T_2\| = \|T_1\| \cdot \|T_2\| \tag{5.41}$$

under some strong conditions for Hilbert-type integral operators $T_1, T_2 : L^p(\mathbb{R}_+, \phi) \to L^p(\mathbb{R}_+, \phi)$. So, it is natural to ask how to extend their result. In particular, the following two problems are naturally raised.

Open problem 1 *Find a necessary and sufficient condition for the equality in* (5.41).

Open problem 2 *Under which conditions does the equality* (5.41) *holds for discrete Hilbert-type operators* T_1 *and* T_2?

5.2 Half-discrete Versions

Our goal in this section is to derive half-discrete Hilbert-type inequalities with arithmetic, geometric and harmonic mean operators.

5.2.1 Half-discrete Inequalities in the Non-conjugate Case

The starting point for this direction is the set of half-discrete inequalities (2.75), (2.76), and (2.77). Our first result is a half-discrete analogue of relations (5.7) and (5.8), extended to the case of non-conjugate exponents.

Theorem 5.9 *Let* p, q, p', q', *and* λ *be as in* (1.43) *and* (1.44), *and let* $K : \mathbb{R}_+^2 \to \mathbb{R}$ *be a non-negative measurable homogeneous function of degree* $-s, s > 0$. *If* A_1 *and* A_2 *are real parameters such that the function* $K(x,y)y^{-q'A_2}$ *is decreasing on* $\mathbb{R}_+$ *for any fixed* $x \in \mathbb{R}_+$, *then the inequalities*

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\frac{s-1}{p'}+A_1-A_2} (\overline{\mathscr{A}}a)_n \int_0^{\infty} K^{\lambda}(x,n) x^{\frac{s-1}{q'}+A_2-A_1} (\mathscr{A}f)(x)dx \\
&= \int_0^{\infty} x^{\frac{s-1}{q'}+A_2-A_1} (\mathscr{A}f)(x) \left(\sum_{n=1}^{\infty} K^{\lambda}(x,n) n^{\frac{s-1}{p'}+A_1-A_2} (\overline{\mathscr{A}}a)_n \right) dx \\
&< Lp'q' \|f\|_{L^p(\mathbb{R}_+)} \|a\|_{l^q},
\end{aligned} \tag{5.42}$$

$$\left[\sum_{n=1}^{\infty}\left(n^{\frac{s-1}{p'}+A_1-A_2}\int_0^{\infty}K^{\lambda}(x,n)x^{\frac{s-1}{q'}+A_2-A_1}(\mathscr{A}f)(x)dx\right)^{q'}\right]^{\frac{1}{q'}}$$
$$< Lp'\|f\|_{L^p(\mathbb{R}_+)}, \tag{5.43}$$

and

$$\left[\int_0^{\infty}\left(x^{\frac{s-1}{q'}+A_2-A_1}\sum_{n=1}^{\infty}K^{\lambda}(x,n)n^{\frac{s-1}{p'}+A_1-A_2}(\overline{\mathscr{A}}a)_n\right)^{p'}dx\right]^{\frac{1}{p'}}$$
$$< Lq'\|a\|_{l^q}, \tag{5.44}$$

where $0 < L = k^{\frac{1}{q'}}(q'A_2)k^{\frac{1}{p'}}(2-s-p'A_1) < \infty$, *hold for any non-negative measurable function* $f:\mathbb{R}_+\to\mathbb{R}$ *and a non-negative sequence* $a=(a_n)_{n\in\mathbb{N}}$, *provided* $0<\|f\|_{L^p(\mathbb{R}_+)}<\infty$ *and* $0<\|a\|_{l^q}<\infty$.

Proof. The result is an easy consequence of half-discrete Hilbert-type inequalities (2.75), (2.76), and (2.77). Namely, if the function f and the sequence a_n are respectively replaced by $x^{\frac{s-1}{q'}+A_2-A_1}(\mathscr{A}f)(x)$ and $n^{\frac{s-1}{p'}+A_1-A_2}(\overline{\mathscr{A}}a)_n$, then, applying the Hardy integral and discrete inequalities to the right-hand side of (2.75) yields

$$\begin{aligned}
&\sum_{n=1}^{\infty}n^{\frac{s-1}{p'}+A_1-A_2}(\overline{\mathscr{A}}a)_n\int_0^{\infty}K^{\lambda}(x,n)x^{\frac{s-1}{q'}+A_2-A_1}(\mathscr{A}f)(x)dx\\
&=\int_0^{\infty}x^{\frac{s-1}{q'}+A_2-A_1}(\mathscr{A}f)(x)\left(\sum_{n=1}^{\infty}K^{\lambda}(x,n)n^{\frac{s-1}{p'}+A_1-A_2}(\overline{\mathscr{A}}a)_n\right)dx\\
&< L\|\mathscr{A}f\|_{L^p(\mathbb{R}_+)}\|\overline{\mathscr{A}}a\|_{l^q} < Lp'q'\|f\|_{L^p(\mathbb{R}_+)}\|a\|_{l^q}.
\end{aligned}$$

Due to the same reasoning as above, inequalities (5.43) and (5.44) follow from (2.76) and (2.77), respectively, which completes the proof. □

The following two theorems are the corresponding analogues of Theorem 5.9, where the arithmetic mean operator is replaced by geometric and harmonic mean operator, respectively.

Theorem 5.10 *Let* p, q, p', q', *and* λ *be as in* (1.43) *and* (1.44), *and let* $K:\mathbb{R}_+^2\to\mathbb{R}$ *be a non-negative measurable homogeneous function of degree* $-s, s>0$. *If* A_1 *and* A_2 *are real parameters such that the function* $K(x,y)y^{-q'A_2}$ *is decreasing on* $\mathbb{R}_+$ *for any fixed* $x\in\mathbb{R}_+$, *then the inequalities*

$$\begin{aligned}
&\sum_{n=1}^{\infty}n^{\frac{s-1}{p'}+A_1-A_2}(\overline{\mathscr{G}}a)_n\int_0^{\infty}K^{\lambda}(x,n)x^{\frac{s-1}{q'}+A_2-A_1}(\mathscr{G}f)(x)dx\\
&=\int_0^{\infty}x^{\frac{s-1}{q'}+A_2-A_1}(\mathscr{G}f)(x)\left(\sum_{n=1}^{\infty}K^{\lambda}(x,n)n^{\frac{s-1}{p'}+A_1-A_2}(\overline{\mathscr{G}}a)_n\right)dx \qquad (5.45)\\
&< Le^{2-\lambda}\|f\|_{L^p(\mathbb{R}_+)}\|a\|_{l^q},
\end{aligned}$$

$$\left[\sum_{n=1}^{\infty}\left(n^{\frac{s-1}{p'}+A_1-A_2}\int_0^{\infty}K^{\lambda}(x,n)x^{\frac{s-1}{q'}+A_2-A_1}(\mathscr{G}f)(x)dx\right)^{q'}\right]^{\frac{1}{q'}} \\ < Le^{\frac{1}{p}}\|f\|_{L^p(\mathbb{R}_+)}, \tag{5.46}$$

and

$$\left[\int_0^{\infty}\left(x^{\frac{s-1}{q'}+A_2-A_1}\sum_{n=1}^{\infty}K^{\lambda}(x,n)n^{\frac{s-1}{p'}+A_1-A_2}(\overline{\mathscr{G}}a)_n\right)^{p'}dx\right]^{\frac{1}{p'}} \\ < Le^{\frac{1}{q}}\|a\|_{l^q}, \tag{5.47}$$

where $0 < L = k^{\frac{1}{q'}}(q'A_2)k^{\frac{1}{p'}}(2-s-p'A_1) < \infty$, *hold for any non-negative measurable function* $f:\mathbb{R}_+\to\mathbb{R}$ *and a non-negative sequence* $a=(a_n)_{n\in\mathbb{N}}$, *provided* $0<\|f\|_{L^p(\mathbb{R}_+)}<\infty$ *and* $0<\|a\|_{l^q}<\infty$.

Proof. We follow the same procedure as in the proof of the previous theorem, except that we use the Knopp inequality (1.55) and the Carleman inequality (1.56) instead of the integral and discrete Hardy inequality.

More precisely, considering (2.75) with the function $x^{\frac{s-1}{q'}+A_2-A_1}(\mathscr{G}f)(x)$ and the sequence $n^{\frac{s-1}{p'}+A_1-A_2}(\overline{\mathscr{G}}a)_n$, instead of f and a_n, it follows that

$$\sum_{n=1}^{\infty}n^{\frac{s-1}{p'}+A_1-A_2}(\overline{\mathscr{G}}a)_n\int_0^{\infty}K^{\lambda}(x,n)x^{\frac{s-1}{q'}+A_2-A_1}(\mathscr{G}f)(x)dx \\ =\int_0^{\infty}x^{\frac{s-1}{q'}+A_2-A_1}(\mathscr{G}f)(x)\left(\sum_{n=1}^{\infty}K^{\lambda}(x,n)n^{\frac{s-1}{p'}+A_1-A_2}(\overline{\mathscr{G}}a)_n\right)dx \\ < L\|\mathscr{G}f\|_{L^p(\mathbb{R}_+)}\|\overline{\mathscr{G}}a\|_{l^q} < Le^{\frac{1}{p}+\frac{1}{q}}\|f\|_{L^p(\mathbb{R}_+)}\|a\|_{l^q} = Le^{2-\lambda}\|f\|_{L^p(\mathbb{R}_+)}\|a\|_{l^q},$$

and the proof is completed. □

Theorem 5.11 *Let p, q, p′, q′, and λ be as in* (1.43) *and* (1.44)*, and let* $K:\mathbb{R}_+^2\to\mathbb{R}$ *be a non-negative measurable homogeneous function of degree* $-s, s>0$. *If* A_1 *and* A_2 *are real parameters such that the function* $K(x,y)y^{-q'A_2}$ *is decreasing on* $\mathbb{R}_+$ *for any fixed* $x\in\mathbb{R}_+$, *then the inequalities*

$$\sum_{n=1}^{\infty}n^{\frac{s-1}{p'}+A_1-A_2}(\overline{\mathscr{H}}a)_n\int_0^{\infty}K^{\lambda}(x,n)x^{\frac{s-1}{q'}+A_2-A_1}(\mathscr{H}f)(x)dx \\ =\int_0^{\infty}x^{\frac{s-1}{q'}+A_2-A_1}(\mathscr{H}f)(x)\left(\sum_{n=1}^{\infty}K^{\lambda}(x,n)n^{\frac{s-1}{p'}+A_1-A_2}(\overline{\mathscr{H}}a)_n\right)dx \\ < L\left(3-\lambda+\frac{1}{pq}\right)\|f\|_{L^p(\mathbb{R}_+)}\|a\|_{l^q}, \tag{5.48}$$

$$\left[\sum_{n=1}^{\infty}\left(n^{\frac{s-1}{p'}+A_1-A_2}\int_0^{\infty}K^{\lambda}(x,n)x^{\frac{s-1}{q'}+A_2-A_1}(\mathscr{H}f)(x)dx\right)^{q'}\right]^{\frac{1}{q'}}
< L\left(1+\frac{1}{p}\right)\|f\|_{L^p(\mathbb{R}_+)}, \tag{5.49}$$

and

$$\left[\int_0^{\infty}\left(x^{\frac{s-1}{q'}+A_2-A_1}\sum_{n=1}^{\infty}K^{\lambda}(x,n)n^{\frac{s-1}{p'}+A_1-A_2}(\overline{\mathscr{H}}a)_n\right)^{p'}dx\right]^{\frac{1}{p'}}
< L\left(1+\frac{1}{q}\right)\|a\|_{l^q}, \tag{5.50}$$

where $0 < L = k^{\frac{1}{q'}}(q'A_2)k^{\frac{1}{p'}}(2-s-p'A_1) < \infty$, *hold for any non-negative measurable function* $f:\mathbb{R}_+\to\mathbb{R}$ *and a non-negative sequence* $a=(a_n)_{n\in\mathbb{N}}$, *provided* $0<\|f\|_{L^p(\mathbb{R}_+)}<\infty$ *and* $0<\|a\|_{l^q}<\infty$.

Proof. Similarly to the previous two proofs, we utilize half-discrete inequalities (2.75), (2.76), and (2.77), and inequalities (1.57) and (1.58).

Namely, considering (2.75) with the function $x^{\frac{s-1}{q'}+A_2-A_1}(\mathscr{H}f)(x)$ and the sequence $n^{\frac{s-1}{p'}+A_1-A_2}(\overline{\mathscr{H}}a)_n$, instead of f and a_n, it follows that

$$\begin{aligned}
&\sum_{n=1}^{\infty}n^{\frac{s-1}{p'}+A_1-A_2}(\overline{\mathscr{H}}a)_n\int_0^{\infty}K^{\lambda}(x,n)x^{\frac{s-1}{q'}+A_2-A_1}(\mathscr{H}f)(x)dx\\
&=\int_0^{\infty}x^{\frac{s-1}{q'}+A_2-A_1}(\mathscr{H}f)(x)\left(\sum_{n=1}^{\infty}K^{\lambda}(x,n)n^{\frac{s-1}{p'}+A_1-A_2}(\overline{\mathscr{H}}a)_n\right)dx\\
&<L\|\mathscr{H}f\|_{L^p(\mathbb{R}_+)}\|\overline{\mathscr{H}}a\|_{l^q}<L\left(1+\frac{1}{p}\right)\left(1+\frac{1}{q}\right)\|f\|_{L^p(\mathbb{R}_+)}\|a\|_{l^q}\\
&=L\left(3-\lambda+\frac{1}{pq}\right)\|f\|_{L^p(\mathbb{R}_+)}\|a\|_{l^q}.
\end{aligned}$$

□

5.2.2 Reduction to Conjugate Case and the Best Constants

Now, our intention is to determine conditions under which the constants appearing in the established half-discrete inequalities from the previous subsection are the best possible. As we have already discussed, there is still no evidence that these constants are the best possible in the non-conjugate case. This problem seems to be very hard in the non-conjugate case and remains still open. Luckily, we can solve the mentioned problem for some particular settings in the conjugate case.

Therefore, here we consider conjugate parameters p and q. In this case $p'=q$, $q'=p$, and $\lambda=1$.

Observe that the constants appearing in (5.7) and (5.8) contain no exponents. Guided by that fact, we are going to simplify the part of the constant regarding a homogeneous kernel. Hence, it is natural to impose the condition

$$pA_2 + qA_1 = 2 - s, \tag{5.51}$$

since in this case relation $k(pA_2) = k(2 - s - qA_1)$ holds. In this case, L reduces to $L^* = k(pA_2)$.

Now, if the condition (5.51) is fulfilled, then, $n^{\frac{s-1}{q}+A_1-A_2} = n^{\frac{1-pqA_2}{q}}$, $x^{\frac{s-1}{p}+A_2-A_1} = x^{\frac{1-pqA_1}{p}}$, so that inequalities (5.42), (5.43), and (5.44), in the conjugate case, reduce to

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{\frac{1-pqA_2}{q}} (\overline{\mathscr{A}}a)_n \int_0^{\infty} K(x,n) x^{\frac{1-pqA_1}{p}} (\mathscr{A}f)(x)dx \\ &= \int_0^{\infty} x^{\frac{1-pqA_1}{p}} (\mathscr{A}f)(x) \left(\sum_{n=1}^{\infty} K(x,n) n^{\frac{1-pqA_2}{q}} (\overline{\mathscr{A}}a)_n \right) dx \\ &< L^* pq \|f\|_{L^p(\mathbb{R}_+)} \|a\|_{l^q}, \end{aligned} \tag{5.52}$$

$$\left[\sum_{n=1}^{\infty} \left(n^{\frac{1-pqA_2}{q}} \int_0^{\infty} K(x,n) x^{\frac{1-pqA_1}{p}} (\mathscr{A}f)(x)dx \right)^p \right]^{\frac{1}{p}} < L^* q \|f\|_{L^p(\mathbb{R}_+)}, \tag{5.53}$$

and

$$\left[\int_0^{\infty} \left(x^{\frac{1-pqA_1}{p}} \sum_{n=1}^{\infty} K(x,n) n^{\frac{1-pqA_2}{q}} (\overline{\mathscr{A}}a)_n \right)^q dx \right]^{\frac{1}{q}} < L^* p \|a\|_{l^q}. \tag{5.54}$$

In the same setting, inequalities (5.45), (5.46) and (5.47) read respectively

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{\frac{1-pqA_2}{q}} (\overline{\mathscr{G}}a)_n \int_0^{\infty} K(x,n) x^{\frac{1-pqA_1}{p}} (\mathscr{G}f)(x)dx \\ &= \int_0^{\infty} x^{\frac{1-pqA_1}{p}} (\mathscr{G}f)(x) \left(\sum_{n=1}^{\infty} K(x,n) n^{\frac{1-pqA_2}{q}} (\overline{\mathscr{G}}a)_n \right) dx \\ &< L^* e \|f\|_{L^p(\mathbb{R}_+)} \|a\|_{l^q}, \end{aligned} \tag{5.55}$$

$$\left[\sum_{n=1}^{\infty} \left(n^{\frac{1-pqA_2}{q}} \int_0^{\infty} K(x,n) x^{\frac{1-pqA_1}{p}} (\mathscr{G}f)(x)dx \right)^p \right]^{\frac{1}{p}} < L^* e^{\frac{1}{p}} \|f\|_{L^p(\mathbb{R}_+)}, \tag{5.56}$$

and

$$\left[\int_0^{\infty} \left(x^{\frac{1-pqA_1}{p}} \sum_{n=1}^{\infty} K(x,n) n^{\frac{1-pqA_2}{q}} (\overline{\mathscr{G}}a)_n \right)^q dx \right]^{\frac{1}{q}} < L^* e^{\frac{1}{q}} \|a\|_{l^q}, \tag{5.57}$$

while inequalities (5.48), (5.49) and (5.50) become

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\frac{1-pqA_2}{q}} (\overline{\mathscr{H}}a)_n \int_0^{\infty} K(x,n) x^{\frac{1-pqA_1}{p}} (\mathscr{H}f)(x)dx \\
&= \int_0^{\infty} x^{\frac{1-pqA_1}{p}} (\mathscr{H}f)(x) \left(\sum_{n=1}^{\infty} K(x,n) n^{\frac{1-pqA_2}{q}} (\overline{\mathscr{H}}a)_n \right) dx \qquad (5.58)\\
&< L^* \left(2 + \frac{1}{pq}\right) \|f\|_{L^p(\mathbb{R}_+)} \|a\|_{l^q},
\end{aligned}$$

$$\left[\sum_{n=1}^{\infty} \left(n^{\frac{1-pqA_2}{q}} \int_0^{\infty} K(x,n) x^{\frac{1-pqA_1}{p}} (\mathscr{H}f)(x)dx \right)^p \right]^{\frac{1}{p}} < L^* \big(1 + \frac{1}{p}\big) \|f\|_{L^p(\mathbb{R}_+)}, \qquad (5.59)$$

and

$$\left[\int_0^{\infty} \left(x^{\frac{1-pqA_1}{p}} \sum_{n=1}^{\infty} K(x,n) n^{\frac{1-pqA_2}{q}} (\overline{\mathscr{H}}a)_n \right)^q dx \right]^{\frac{1}{q}} < L^* \big(1 + \frac{1}{q}\big) \|a\|_{l^q}. \qquad (5.60)$$

In the rest of this subsection we show that the constants appearing on the right-hand sides of inequalities (5.52)–(5.60) are the best possible. The corresponding proofs are the substance of the following three theorems.

Theorem 5.12 *Let $p,q > 1$ be conjugate exponents and $K : \mathbb{R}_+^2 \to \mathbb{R}$ be a non-negative measurable homogeneous function of degree $-s, s > 0$. If A_1 and A_2 are real parameters such that the condition* (5.51) *is fulfilled and the function $K(x,y)y^{-pA_2}$ is decreasing on $\mathbb{R}_+$ for any fixed $x \in \mathbb{R}_+$, then the constants L^*pq, L^*q, and L^*p are the best possible in* (5.52)*,* (5.53)*, and* (5.54)*, respectively.*

Proof. In order to prove that inequality (5.52) includes the best constant on its right-hand side, we suppose that there exists a positive constant C_1, smaller than L^*pq, such that the relation

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\frac{1-pqA_2}{q}} (\overline{\mathscr{A}}a)_n \int_0^{\infty} K(x,n) x^{\frac{1-pqA_1}{p}} (\mathscr{A}f)(x)dx \\
&= \int_0^{\infty} x^{\frac{1-pqA_1}{p}} (\mathscr{A}f)(x) \left(\sum_{n=1}^{\infty} K(x,n) n^{\frac{1-pqA_2}{q}} (\overline{\mathscr{A}}a)_n \right) dx \qquad (5.61)\\
&< C_1 \|f\|_{L^p(\mathbb{R}_+)} \|a\|_{l^q}
\end{aligned}$$

holds for all non-negative $f : \mathbb{R}_+ \to \mathbb{R}$ and $a = (a_n)_{n \in \mathbb{N}}$, provided $0 < \|f\|_{L^p(\mathbb{R}_+)} < \infty$ and $0 < \|a\|_{l^q} < \infty$.

Let $\widetilde{L}$ and $\widetilde{R}$ respectively denote the left-hand side and the right-hand side of (5.61) equipped with

$$\widetilde{f}(x) = \begin{cases} x^{-\frac{1}{p}}, & 1 \le x \le N \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \widetilde{a}_n = \begin{cases} n^{-\frac{1}{q}}, & n \le N \\ 0, & \text{otherwise} \end{cases}, \qquad (5.62)$$

where $N \in \mathbb{N}$, $N > 2$, is fixed. Then, the right-hand side of (5.61) may be bounded from above with a natural logarithm function as follows:

$$\begin{aligned}\widetilde{R} =& C_1 \|\widetilde{f}\|_{L^p(\mathbb{R}_+)} \|\widetilde{a}\|_{l^q} = C_1 \left[\int_0^N x^{-1} dx\right]^{\frac{1}{p}} \left[\sum_{n=1}^{N} n^{-1}\right]^{\frac{1}{q}} \\ <& C_1 (\log N)^{\frac{1}{p}} \left(1 + \int_1^N x^{-1} dx\right)^{\frac{1}{q}} < C_1 (1 + \log N).\end{aligned} \tag{5.63}$$

Our next task is to estimate the left-hand side of inequality (5.61) from below. More precisely, considering $\sum_{k=1}^{n} k^{-\frac{1}{q}}$ as the upper Darboux sum for the function $h(x) = x^{-\frac{1}{q}}$ on the segment $[1, n+1]$, we have

$$\sum_{k=1}^{n} k^{-\frac{1}{q}} > \int_1^{n+1} x^{-\frac{1}{q}} dx > \int_1^{n} x^{-\frac{1}{q}} dx = p(n^{\frac{1}{p}} - 1),$$

and consequently,

$$(\overline{\mathscr{A}}\widetilde{a})_n > \frac{p(n^{\frac{1}{p}} - 1)}{n} = pn^{-\frac{1}{q}}(1 - n^{-\frac{1}{p}}), \quad n \le N.$$

In addition,

$$(\mathscr{A}\widetilde{f})(x) = \frac{1}{x}\int_0^x \widetilde{f}(t)dt = \frac{1}{x}\int_1^x t^{-\frac{1}{p}} dt = qx^{-\frac{1}{p}}(1 - x^{-\frac{1}{q}}), \quad 1 \le x \le N,$$

hence, $\widetilde{L}$ may be estimated as follows:

$$\widetilde{L} > pq \int_1^N \sum_{n=1}^{N} K(x,n) x^{-qA_1} n^{-pA_2} (1 - x^{-\frac{1}{q}})(1 - n^{-\frac{1}{p}}) dx.$$

Moreover, since $(1 - x^{-\frac{1}{q}})(1 - n^{-\frac{1}{p}}) > 1 - x^{-\frac{1}{q}} - n^{-\frac{1}{p}}$, the above relation implies inequality

$$\begin{aligned}\frac{\widetilde{L}}{pq} >& \int_1^N \sum_{n=1}^{N} K(x,n) x^{-qA_1} n^{-pA_2} dx \\ &- \int_1^N \sum_{n=1}^{N} K(x,n) x^{-qA_1 - \frac{1}{q}} n^{-pA_2} dx \\ &- \int_1^N \sum_{n=1}^{N} K(x,n) x^{-qA_1} n^{-pA_2 - \frac{1}{p}} dx.\end{aligned} \tag{5.64}$$

Our next intention is to establish suitable estimates for the integrals on the right-hand side of (5.64). In the first integral, utilizing suitable variable changes and the Fubini theorem, we have

$$\begin{aligned}
&\int_1^N \sum_{n=1}^N K(x,n)x^{-qA_1}n^{-pA_2}dx \\
&\quad > \int_1^N \int_1^{N+1} K(x,y)x^{-qA_1}y^{-pA_2}dydx \\
&\quad > \int_1^N \int_1^N K(x,y)x^{-qA_1}y^{-pA_2}dydx \\
&\quad = \int_1^N \frac{dx}{x}\int_{\frac{1}{x}}^{\frac{N}{x}} K(1,t)t^{-pA_2}dt \\
&\quad = \int_{\frac{1}{N}}^1 \left(\int_{\frac{1}{t}}^N \frac{dx}{x}\right)K(1,t)t^{-pA_2}dt + \int_1^N \left(\int_1^{\frac{N}{t}} \frac{dx}{x}\right)K(1,t)t^{-pA_2}dt \\
&\quad = \log N \int_{\frac{1}{N}}^1 K(1,t)t^{-pA_2}\left(1+\frac{\log t}{\log N}\right)dt \\
&\qquad + \log N \int_1^N K(1,t)t^{-pA_2}\left(1-\frac{\log t}{\log N}\right)dt,
\end{aligned} \tag{5.65}$$

since the function $K(x,y)y^{-pA_2}$ is decreasing on $\mathbb{R}_+$ for any fixed $x \in \mathbb{R}_+$. Similarly, for the remaining two integrals on the right-hand side of (5.64) we have:

$$\begin{aligned}
\int_1^N \sum_{n=1}^N K(x,n)x^{-qA_1-\frac{1}{q}}n^{-pA_2}dx &< \int_1^N \int_0^N K(x,y)x^{-qA_1-\frac{1}{q}}y^{-pA_2}dydx \\
&= \int_1^N \frac{dx}{x^{1+\frac{1}{q}}}\int_0^{\frac{N}{x}} K(1,t)t^{-pA_2}dt \\
&< \int_1^N \frac{dx}{x^{1+\frac{1}{q}}}\int_0^{\infty} K(1,t)t^{-pA_2}dt \\
&= \left(q-\frac{q}{N^{\frac{1}{q}}}\right)k(pA_2)
\end{aligned} \tag{5.66}$$

and

$$\begin{aligned}
\int_1^N \sum_{n=1}^N K(x,n)x^{-qA_1}n^{-pA_2-\frac{1}{p}}dx &< \int_1^N \int_0^N K(x,y)x^{-qA_1}y^{-pA_2-\frac{1}{p}}dydx \\
&= \int_1^N \frac{dx}{x^{1+\frac{1}{p}}}\int_0^{\frac{N}{x}} K(1,t)t^{-pA_2-\frac{1}{p}}dt \\
&< \int_1^N \frac{dx}{x^{1+\frac{1}{p}}}\int_0^{\infty} K(1,t)t^{-pA_2-\frac{1}{p}}dt \\
&= \left(p-\frac{p}{N^{\frac{1}{p}}}\right)k\left(pA_2+\frac{1}{p}\right).
\end{aligned} \tag{5.67}$$

Now, taking into account (5.61), (5.63), (5.64), (5.65), (5.66), and (5.67), we have the following inequality

$$\begin{aligned}\frac{C_1(1+\log N)}{pq} &> \log N \int_{\frac{1}{N}}^{1} K(1,t)t^{-pA_2}\left(1+\frac{\log t}{\log N}\right)dt \\ &\quad + \log N \int_{1}^{N} K(1,t)t^{-pA_2}\left(1-\frac{\log t}{\log N}\right)dt \\ &\quad - \left(q-\frac{q}{N^{\frac{1}{q}}}\right)k(pA_2) - \left(p-\frac{p}{N^{\frac{1}{p}}}\right)k\left(pA_2+\frac{1}{p}\right).\end{aligned} \tag{5.68}$$

Finally, dividing both sides of (5.68) with $\log N$, and letting N to infinity, it follows that

$$\frac{C_1}{pq} \geq k(pA_2) = L^*,$$

which contradicts with the assumption that C_1 is smaller than L^*pq. Thus, L^*pq is the best constant in inequality (5.52).

Now, we prove that L^*q is the best constant in inequality (5.53). For this reason, suppose that there exists a positive constant C_1' smaller than L^*q such that inequality

$$\left[\sum_{n=1}^{\infty}\left(n^{\frac{1-pqA_2}{q}}\int_0^{\infty} K(x,n)x^{\frac{1-pqA_1}{p}}(\mathscr{A}f)(x)dx\right)^p\right]^{\frac{1}{p}} < C_1'\|f\|_{L^p(\mathbb{R}_+)},$$

holds for all non-negative functions $f:\mathbb{R}_+\to\mathbb{R}$, provided $0<\|f\|_{L^p(\mathbb{R}_+)}<\infty$. Then, utilizing the Hölder inequality and the discrete Hardy inequality, we have

$$\begin{aligned}&\sum_{n=1}^{\infty} n^{\frac{1-pqA_2}{q}}(\overline{\mathscr{A}}a)_n\int_0^{\infty}K(x,n)x^{\frac{1-pqA_1}{p}}(\mathscr{A}f)(x)dx \\ &= \sum_{n=1}^{\infty}\left(n^{\frac{1-pqA_2}{q}}\int_0^{\infty}K(x,n)x^{\frac{1-pqA_1}{p}}(\mathscr{A}f)(x)dx\right)(\overline{\mathscr{A}}a)_n \\ &\leq \left[\sum_{n=1}^{\infty}\left(n^{\frac{1-pqA_2}{q}}\int_0^{\infty}K(x,n)x^{\frac{1-pqA_1}{p}}(\mathscr{A}f)(x)dx\right)^p\right]^{\frac{1}{p}}\|\overline{\mathscr{A}}a\|_{l^q} \\ &< C_1'p\|f\|_{L^p(\mathbb{R}_+)}\|a\|_{l^q},\end{aligned} \tag{5.69}$$

which is impossible since $C_1'p < L^*pq$ and L^*pq is the best constant in (5.52).

Finally, with the assumption that there exists a positive constant $C_1'' < L^*p$ such that the inequality

$$\left[\int_0^{\infty}\left(x^{\frac{1-pqA_1}{p}}\sum_{n=1}^{\infty}K(x,n)n^{\frac{1-pqA_2}{q}}(\overline{\mathscr{A}}a)_n\right)^q dx\right]^{\frac{1}{q}} < C_1''\|a\|_{l^q}$$

holds for all non-negative sequences $a=(a_n)_{n\in\mathbb{N}}$, $0<\|a\|_{l^q}<\infty$, we obtain (in the same way as in (5.69)) that inequality (5.52) holds with the constant $C_1''q$, smaller than L^*pq. This contradiction completes the proof. □

Theorem 5.13 *Under the same assumptions as in Theorem* 5.12*, the constants* L^*e*,* $L^*e^{\frac{1}{p}}$*, and* $L^*e^{\frac{1}{q}}$ *are the best possible in* (5.55)*,* (5.56)*, and* (5.57)*, respectively.*

Proof. We follow the same procedure as in the proof of Theorem 5.12, that is, we assume that the inequality

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\frac{1-pqA_2}{q}} (\overline{\mathscr{G}}a)_n \int_0^{\infty} K(x,n) x^{\frac{1-pqA_1}{p}} (\mathscr{G}f)(x)dx \\
&= \int_0^{\infty} x^{\frac{1-pqA_1}{p}} (\mathscr{G}f)(x) \left(\sum_{n=1}^{\infty} K(x,n) n^{\frac{1-pqA_2}{q}} (\overline{\mathscr{G}}a)_n \right) dx \\
&< C_2 \|f\|_{L^p(\mathbb{R}_+)} \|a\|_{l^q},
\end{aligned} \tag{5.70}$$

holds with a positive constant C_2, smaller than L^*e. Let $\widetilde{L}$ and $\widetilde{R}$ respectively denote the left-hand side and the right-hand side of inequality (5.70) equipped with

$$\widetilde{f}(x) = \begin{cases} 1, & 0 < x < 1 \\ e^{-\frac{1}{p}} x^{-\frac{1}{p}}, & 1 \le x \le N \\ x^{-\frac{2}{p}}, & x > N \end{cases}, \quad \widetilde{a}_n = \begin{cases} 1, & n = 1 \\ \left(\frac{(n-1)^{n-1}}{n^n} \right)^{\frac{1}{q}}, & 2 \le n \le N \\ 0, & n > N \end{cases},$$

where $N > e$ is a fixed integer. Making use of the estimate $\left(1 + \frac{1}{n}\right)^{n+1} > e$, $n \in \mathbb{N}$, it follows that

$$\|\widetilde{a}\|_{l^q}^q = 1 + \sum_{n=1}^{N-1} \left(\frac{n}{n+1} \right)^{n+1} \cdot \frac{1}{n} < 1 + \frac{1}{e} \sum_{n=1}^{N-1} \frac{1}{n}.$$

Moreover, considering $\sum_{n=2}^{N} \frac{1}{n}$ as the lower Darboux sum for the function $h(x) = \frac{1}{x}$ on the segment $[1,N]$, it follows that

$$\sum_{n=2}^{N-1} \frac{1}{n} < \sum_{n=2}^{N} \frac{1}{n} < \int_1^N \frac{dx}{x} = \log N,$$

that is,

$$\|\widetilde{a}\|_{l^q} < \left(1 + \frac{1}{e} + \frac{1}{e} \log N \right)^{\frac{1}{q}}.$$

In addition, since

$$\|\widetilde{f}\|_{L^p(\mathbb{R}_+)} = \left(1 + \frac{1}{e} \log N + \frac{1}{N} \right)^{\frac{1}{p}} < \left(1 + \frac{1}{e} + \frac{1}{e} \log N \right)^{\frac{1}{p}},$$

we obtain the following estimate for the right-hand side of (5.70):

$$\widetilde{R} = C_2 \|\widetilde{f}\|_{L^p(\mathbb{R}_+)} \|\widetilde{a}\|_{l^q} < C_2 \left(1 + \frac{1}{e} + \frac{1}{e} \log N \right). \tag{5.71}$$

On the other hand, since

$$(\mathscr{G}\widetilde{f})(x) = \begin{cases} 1, & 0 < x < 1 \\ x^{-\frac{1}{p}}, & 1 \le x \le N \\ x^{-\frac{2}{p}} e^{\frac{2}{px}(1+N\log N - N)}, & x > N \end{cases} \quad \text{and} \quad (\overline{\mathscr{G}}\widetilde{a})_n = \begin{cases} n^{-\frac{1}{q}}, & n \le N \\ 0, & \text{otherwise} \end{cases},$$

the left-hand side of (5.70) is greater than $\int_1^N \sum_{n=1}^N K(x,n) x^{-qA_1} n^{-pA_2} dx$. It should be noticed here that relation (5.65) (see Theorem 5.12) provides lower bound for this double sum. Therefore, utilizing (5.65), (5.70), and (5.71), it follows that

$$C_2 \left(1 + \frac{1}{e} + \frac{1}{e}\log N\right) > \log N \int_{\frac{1}{N}}^1 K(1,t) t^{-pA_2} \left(1 + \frac{\log t}{\log N}\right) dt$$
$$+ \log N \int_1^N K(1,t) t^{-pA_2} \left(1 - \frac{\log t}{\log N}\right) dt.$$

Dividing the above inequality by $\log N$ and letting N to infinity, it follows that

$$\frac{C_2}{e} \ge L^*,$$

which is in contrast to $C_2 < L^* e$. Therefore, the constant $L^* e$ is the best possible in inequality (5.55).

To conclude the proof, we show that $L^* e^{\frac{1}{p}}$ is the best possible in (5.56). Hence, suppose that there exists a positive constant $C_2' < L^* e^{\frac{1}{p}}$ such that inequality

$$\left[\sum_{n=1}^{\infty} \left(n^{\frac{1-pqA_2}{q}} \int_0^{\infty} K(x,n) x^{\frac{1}{p}(1-pqA_1)} (\mathscr{G}f)(x) dx\right)^p\right]^{\frac{1}{p}} < C_2 \|f\|_{L^p(\mathbb{R}_+)},$$

holds for all non-negative measurable functions $f : \mathbb{R}_+ \to \mathbb{R}$. Then, utilizing the Hölder and the Carleman inequality, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\frac{1-pqA_2}{q}} (\overline{\mathscr{G}}a)_n \int_0^{\infty} K(x,n) x^{\frac{1-pqA_1}{p}} (\mathscr{G}f)(x) dx \\
&= \sum_{n=1}^{\infty} \left(n^{\frac{1-pqA_2}{q}} \int_0^{\infty} K(x,n) x^{\frac{1-pqA_1}{p}} (\mathscr{G}f)(x) dx\right) (\overline{\mathscr{G}}a)_n \\
&\le \left[\sum_{n=1}^{\infty} \left(n^{\frac{1-pqA_2}{q}} \int_0^{\infty} K(x,n) x^{\frac{1-pqA_1}{p}} (\mathscr{G}f)(x) dx\right)^p\right]^{\frac{1}{p}} \|\overline{\mathscr{G}}a\|_{l^q} \\
&< C_2' e^{\frac{1}{q}} \|f\|_{L^p(\mathbb{R}_+)} \|a\|_{l^q},
\end{aligned}$$

which is impossible since $C_2' e^{\frac{1}{q}} < L^* e$ and $L^* e$ is the best possible constant in (5.55). In the same way we show that $L^* e^{\frac{1}{q}}$ is the best constant in (5.57). □

Theorem 5.14 *Under the same assumptions as in Theorem* 5.12, *the constants* $L^*\left(2+\frac{1}{pq}\right)$, $L^*\left(1+\frac{1}{p}\right)$, *and* $L^*\left(1+\frac{1}{q}\right)$ *are the best possible in* (5.58), (5.59), *and* (5.60), *respectively.*

Proof. We first show that $L^*\left(2+\frac{1}{pq}\right)$ is the best constant in (5.58), as in the previous two theorems. Hence, suppose that the inequality

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{\frac{1-pqA_2}{q}} (\overline{\mathscr{H}} a)_n \int_0^{\infty} K(x,n) x^{\frac{1-pqA_1}{p}} (\mathscr{H} f)(x) dx \\ &= \int_0^{\infty} x^{\frac{1-pqA_1}{p}} (\mathscr{H} f)(x) \left(\sum_{n=1}^{\infty} K(x,n) n^{\frac{q-pqA_2}{q}} (\overline{\mathscr{H}} a)_n \right) dx \\ &< C_3 \|f\|_{L^p(\mathbb{R}_+)} \|a\|_{l^q} \end{aligned} \tag{5.72}$$

holds with a positive constant C_3, smaller than $L^*\left(2+\frac{1}{pq}\right)$. Let $\widetilde{L}$ and $\widetilde{R}$ respectively denote the left-hand side and the right-hand side of (5.72) equipped with the function $\widetilde{f}:\mathbb{R}_+\to\mathbb{R}$ and the sequence $(\widetilde{a}_n)_{n\in\mathbb{N}}$ defined by (5.62) (see Theorem 5.12). Then, taking into account relation (5.63), it follows that

$$\widetilde{R} = C_3 \|\widetilde{f}\|_{L^p(\mathbb{R}_+)} \|\widetilde{a}\|_{l^q} < C_3(1+\log N), \tag{5.73}$$

where $N\in\mathbb{N}$, $N>2$, is a fixed positive integer.

Guided by the ideas from previous two proofs, we establish now the lower bound for the left-hand side of inequality (5.72). Obviously, $(\mathscr{H}\widetilde{f})(x) = (\overline{\mathscr{H}}\widetilde{a})_n = 0$, for $x,n>N$. Moreover, considering $\sum_{k=1}^{n} k^{\frac{1}{q}}$ as the lower Darboux sum for the function $h(x)=x^{\frac{1}{q}}$ on segment $[1,n+1]$, we have

$$\sum_{k=1}^{n} k^{\frac{1}{q}} < \int_1^{n+1} x^{\frac{1}{q}} dx < \int_0^{n+1} x^{\frac{1}{q}} dx = \frac{q}{q+1}(n+1)^{1+\frac{1}{q}},$$

and consequently,

$$(\overline{\mathscr{H}}\widetilde{a})_n > \left(1+\frac{1}{q}\right) \frac{n}{(n+1)^{1+\frac{1}{q}}} = \frac{q+1}{q} \left(\frac{n}{n+1}\right)^{1+\frac{1}{q}} n^{-\frac{1}{q}}, \quad n\le N.$$

On the other hand, since

$$(\mathscr{H}\widetilde{f})(x) = \left(\frac{p+1}{p}\right) \frac{x}{x^{1+\frac{1}{p}}-1} > \frac{p+1}{p} x^{-\frac{1}{p}},$$

it follows that

$$\begin{aligned} \frac{pq\widetilde{L}}{2pq+1} &> \int_1^N \sum_{n=1}^{N} K(x,n) x^{-qA_1} n^{-pA_2} \left(\frac{n}{n+1}\right)^{1+\frac{1}{q}} dx \\ &= \int_1^N \sum_{n=1}^{N} K(x,n) x^{-qA_1} n^{-pA_2} (1-\varphi_n) dx, \end{aligned} \tag{5.74}$$

where the sequence $(\varphi_n)_{n\in\mathbb{N}}$ is defined by

$$\varphi_n = 1 - \left(\frac{n}{n+1}\right)^{1+\frac{1}{q}}.$$

In addition, since $\left(\frac{n}{n+1}\right)^2 < \left(\frac{n}{n+1}\right)^{1+\frac{1}{q}} < \frac{n}{n+1}$, it follows that $\frac{1}{n+1} < \varphi_n < \frac{2n+1}{(n+1)^2}$, i.e.

$$\frac{1}{2n} < \varphi_n < \frac{2}{n},\ n \in \mathbb{N}.$$

Hence, utilizing (5.74), we have

$$\begin{aligned} \frac{pq\widetilde{L}}{2pq+1} > &\int_1^N \sum_{n=1}^N K(x,n)x^{-qA_1}n^{-pA_2}dx \\ &-2\int_1^N \sum_{n=1}^N K(x,n)x^{-qA_1}n^{-pA_2-1}dx. \end{aligned} \tag{5.75}$$

In the sequel, we use estimate (5.65) for the first double sum on the right-hand side of (5.75). Moreover, similarly to (5.67), we have

$$\int_1^N \sum_{n=1}^N K(x,n)x^{-qA_1}n^{-pA_2-1}dx < \left(1-\frac{1}{N}\right)k(pA_2+1), \tag{5.76}$$

so that relations (5.65), (5.72), (5.73), (5.75), and (5.76) yield inequality

$$\begin{aligned} &\frac{pq}{2pq+1}C_3\left(1+\frac{1}{\log N}\right) \\ &> \int_{\frac{1}{N}}^1 K_\lambda(1,t)t^{s-1}\left(1+\frac{\log t}{\log N}\right)dt \\ &+\int_1^N K_\lambda(1,t)t^{s-1}\left(1-\frac{\log t}{\log N}\right)dt - \left(1-\frac{1}{N}\right)\frac{k(pA_2+1)}{\log N}, \end{aligned}$$

after dividing by $\log N$. Clearly, letting N to infinity, the above relation reduces to

$$\frac{pqC_3}{2pq+1} \geq L^*,$$

which is in contrast to our assumption that C_3 is smaller than $L^*\left(2+\frac{1}{pq}\right)$. Hence, $L^*\left(2+\frac{1}{pq}\right)$ is the best constant in inequality (5.58).

In order to conclude the proof, we only show that $L^*\left(1+\frac{1}{p}\right)$ is the best constant in (5.59). Namely, assuming that there exist a positive constant $0 < C_3' < L^*\left(1+\frac{1}{p}\right)$ such that inequality

$$\left[\sum_{n=1}^\infty \left(n^{\frac{1-pqA_2}{q}} \int_0^\infty K(x,n)x^{\frac{1-pqA_1}{p}}(\mathscr{H}f)(x)dx\right)^p\right]^{\frac{1}{p}} < C_3'\|f\|_{L^p(\mathbb{R}_+)}$$

holds for all non-negative measurable functions $f:\mathbb{R}_+ \to \mathbb{R}$, then by virtue of the Hölder and the Hardy-Carleman inequality, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\frac{1-pqA_2}{q}} (\overline{\mathscr{H}}a)_n \int_0^{\infty} K(x,n) x^{\frac{1-pqA_1}{p}} (\mathscr{H}f)(x)dx \\
&= \sum_{n=1}^{\infty} \left(n^{\frac{1-pqA_2}{q}} \int_0^{\infty} K(x,n) x^{\frac{1-pqA_1}{p}} (\mathscr{H}f)(x)dx \right) (\overline{\mathscr{H}}a)_n \\
&\leq \left[\sum_{n=1}^{\infty} \left(n^{\frac{1-pqA_2}{q}} \int_0^{\infty} K(x,n) x^{\frac{1-pqA_1}{p}} (\mathscr{H}f)(x)dx \right)^p \right]^{\frac{1}{p}} \|\overline{\mathscr{H}}a\|_{l^q} \\
&< C_3' \left(1+\frac{1}{q}\right) \|f\|_{L^p(\mathbb{R}_+)} \|a\|_{l^q},
\end{aligned}$$

which is impossible since $C_3'\left(1+\frac{1}{q}\right) < L^*\left(2+\frac{1}{pq}\right)$ and $L^*\left(2+\frac{1}{pq}\right)$ is the best constant in (5.58). □

Remark 5.6 Considering inequalities (5.52)–(5.60) with a homogeneous kernel of degree $-(\nu+\mu)$, $\nu,\mu>0$, such that the function $K(x,y)y^{\mu-1}$ is decreasing on $\mathbb{R}_+$ for any fixed $x\in\mathbb{R}_+$, and with parameters $A_1=\frac{1-\nu}{q}$, $A_2=\frac{1-\mu}{p}$, we obtain half-discrete versions of Hilbert-type inequalities derived in the previous section. For example, a half-discrete version of inequality (5.7) reads

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\mu-\frac{1}{p}} (\overline{\mathscr{A}}a)_n \int_0^{\infty} K(x,n) x^{\nu-\frac{1}{q}} (\mathscr{A}f)(x)dx \\
&= \int_0^{\infty} x^{\nu-\frac{1}{q}} (\mathscr{A}f)(x) \left(\sum_{n=1}^{\infty} K(x,n) n^{\mu-\frac{1}{p}} (\overline{\mathscr{A}}a)_n \right) dx \\
&< k(1-\mu)pq \|f\|_{L^p(\mathbb{R}_+)} \|a\|_{l^q},
\end{aligned}$$

where $k(1-\mu)pq$ is the best possible constant.

Remark 5.7 A typical example of a homogeneous kernel with a negative degree is the function $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$, defined by $K(x,y)=(x+y)^{-s}$, $s>0$. In this case, the parts of the constants in (5.52)–(5.60), with respect to a homogeneous kernel, may be expressed in terms of the Beta function since

$$k(\eta)=\int_0^{\infty}(1+t)^{-s}t^{-\eta}dt = B(1-\eta, s+\eta-1),\ 1-s<\eta<1.$$

Another interesting example of a homogeneous kernel with a negative degree is the function $K(x,y)=\max\{x,y\}^{-s}$, $s>0$. In this case,

$$k(\eta)=\int_0^{\infty}\max\{1,t\}^{-s}t^{-\eta}dt = \frac{s}{(1-\eta)(s+\eta-1)},\ 1-s<\eta<1.$$

5.2.3 On Half-discrete Operators Arising from Hardy-Hilbert Type Inequalities

As an application, we shall take a closer look to inequalities (5.53), (5.54), (5.56), (5.57), (5.59), and (5.60).

By virtue of the half-discrete Hardy-Hilbert type inequalities from the previous subsection, we establish here the corresponding half-discrete operators between Lebesgue spaces $L^p(\mathbb{R}_+)$ and l^p, and vice versa. In addition, since the above mentioned inequalities include the best constants on their right-hand sides, we shall be able to determine their norms.

First, with the assumptions of Theorem 5.12, we introduce a pair of arithmetic half-discrete Hilbert-type operators $\mathbf{A_1} : L^p(\mathbb{R}_+) \to l^p$ and $\mathbf{A_2} : l^q \to L^q(\mathbb{R}_+)$, by

$$\begin{aligned}(\mathbf{A_1}f)_n &= n^{\frac{1-pqA_2}{q}} \int_0^\infty K(x,n)x^{\frac{1-pqA_1}{p}}(\mathscr{A}f)(x)dx,\\ (\mathbf{A_2}a)(x) &= x^{\frac{1-pqA_1}{p}} \sum_{n=1}^\infty K(x,n)n^{\frac{1-pqA_2}{q}}(\overline{\mathscr{A}}a)_n.\end{aligned}$$

Clearly, $\mathbf{A_1}$ and $\mathbf{A_2}$ are well-defined, due to inequalities (5.53) and (5.54). Moreover, inequalities (5.53) and (5.54) may be rewritten as $\|\mathbf{A_1}f\|_{l^p} < L^*q\|f\|_{L^p(\mathbb{R}_+)}$ and $\|\mathbf{A_2}a\|_{L^q(\mathbb{R}_+)} < L^*p\|a\|_{l^q}$. Due to the best constants established in Theorem 5.12, we can calculate the norms of $\mathbf{A_1}$ and $\mathbf{A_2}$, that is, we have $\|\mathbf{A_1}\| = L^*q$ and $\|\mathbf{A_2}\| = L^*p$.

Similarly, utilizing Theorem 5.13, that is, by virtue of relations (5.56) and (5.57), we define a pair of geometric half-discrete Hilbert-type operators $\mathbf{G_1} : L^p(\mathbb{R}_+) \to l^p$ and $\mathbf{G_2} : l^q \to L^q(\mathbb{R}_+)$, by

$$\begin{aligned}(\mathbf{G_1}f)_n &= n^{\frac{1-pqA_2}{q}} \int_0^\infty K(x,n)x^{\frac{1-pqA_1}{p}}(\mathscr{G}f)(x)dx,\\ (\mathbf{G_2}a)(x) &= x^{\frac{1-pqA_1}{p}} \sum_{n=1}^\infty K(x,n)n^{\frac{1-pqA_2}{q}}(\overline{\mathscr{G}}a)_n.\end{aligned}$$

With this notation, inequalities (5.56) and (5.57) read $\|\mathbf{G_1}f\|_{l^p} < L^*e^{\frac{1}{p}}\|f\|_{L^p(\mathbb{R}_+)}$ and $\|\mathbf{G_2}a\|_{L^q(\mathbb{R}_+)} < L^*e^{\frac{1}{q}}\|a\|_{l^q}$, that is, it follows that $\|\mathbf{G_1}\| = L^*e^{\frac{1}{p}}$ and $\|\mathbf{G_2}\| = L^*e^{\frac{1}{q}}$, due to the best constants $L^*e^{\frac{1}{p}}$ and $L^*e^{\frac{1}{q}}$.

It remains to define the corresponding pair of harmonic half-discrete Hilbert-type operators arising from relations (5.59) and (5.60). More precisely, the operators $\mathbf{H_1} : L^p(\mathbb{R}_+) \to l^p$ and $\mathbf{H_2} : l^q \to L^q(\mathbb{R}_+)$, defined by

$$\begin{aligned}(\mathbf{H_1}f)_n &= n^{\frac{1-pqA_2}{q}} \int_0^\infty K(x,n)x^{\frac{1-pqA_1}{p}}(\mathscr{H}f)(x)dx,\\ (\mathbf{H_2}a)(x) &= x^{\frac{1-pqA_1}{p}} \sum_{n=1}^\infty K(x,n)n^{\frac{1-pqA_2}{q}}(\overline{\mathscr{H}}a)_n,\end{aligned}$$

are meaningful since $\|\mathbf{H_1}f\|_{l^p} < L^*\left(1+\frac{1}{p}\right)\|f\|_{L^p(\mathbb{R}_+)}$ and $\|\mathbf{H_2}a\|_{L^q(\mathbb{R}_+)} < L^*\left(1+\frac{1}{q}\right)\|a\|_{l^q}$. Finally, due to Theorem 5.14, the constants $L^*\left(1+\frac{1}{p}\right)$ and $L^*\left(1+\frac{1}{p}\right)$ are the best possible in the corresponding inequalities, which means that $\|\mathbf{H_1}\| = L^*\left(1+\frac{1}{p}\right)$ and $\|\mathbf{H_2}\| = L^*\left(1+\frac{1}{q}\right)$.

Similarly to Open problem 2, we may propose the following open problem.

Open problem 3 *Find conditions so that the equality* (5.41) *holds for the corresponding half-discrete Hilbert-type operators T_1 and T_2.*

5.3 Extension to a Multidimensional Case

The main goal of this section is to present extensions of Theorems 5.3, 5.4, and 5.5 to a multidimensional case. Such results are consequences of multidimensional Hilbert-type inequalities.

Theorem 5.15 *Suppose p_i, p_i', q_i, $i = 1,2,\dots,n$, and λ are as in* (1.35), and A_{ij}, $i,j = 1,2,\dots,n$, are the real parameters satisfying $\sum_{i=1}^n A_{ij} = 0$. Further, let $\alpha_i = \sum_{j=1}^n A_{ij}$, $i = 1,2,\dots,n$, and let ν_i, μ_i be real parameters satisfying $\alpha_i + \nu_i + \frac{1}{p_i} < \frac{s+1-n}{q_i} \le \alpha_i + \nu_i + \mu_i$. If $K : \mathbb{R}_+^n \to \mathbb{R}$ is a non-negative measurable homogeneous function of degree $-s$, $s > 0$, and $f_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1,2,\dots,n$, are non-negative measurable functions, then

$$\int_{\mathbb{R}_+^n} K^\lambda(\mathbf{x}) \prod_{i=1}^n x_i^{\nu_i+\mu_i} (\mathscr{A} f_i)^{\mu_i}(x_i) d\mathbf{x}$$
$$\le c_n^s(\mathbf{p},\mathbf{q},\mathbf{A},\boldsymbol{\nu}) \prod_{i=1}^n \left|\left| f_i^{\frac{q_i\mu_i}{p_iq_i(\alpha_i+\nu_i+\mu_i)+p_i(n-1-s)+q_i}} \right|\right|_{p_i}^{p_i(\alpha_i+\nu_i+\mu_i)+p_i(n-1-s)/q_i+1}, \tag{5.77}$$

and

$$\left[\int_{\mathbb{R}_+} x_n^{(1-\lambda p_n')(n-1-s)-p_n'\alpha_n} \left(\int_{\mathbb{R}_+^{n-1}} K^\lambda(\mathbf{x}) \prod_{i=1}^{n-1} x_i^{\nu_i+\mu_i} (\mathscr{A} f_i)^{\mu_i}(x_i) d\hat{}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}$$
$$\le c_{n-1}^s(\mathbf{p},\mathbf{q},\mathbf{A},\boldsymbol{\nu}) \prod_{i=1}^{n-1} \left|\left| f_i^{\frac{q_i\mu_i}{p_iq_i(\alpha_i+\nu_i+\mu_i)+p_i(n-1-s)+q_i}} \right|\right|_{p_i}^{p_i(\alpha_i+\nu_i+\mu_i)+p_i(n-1-s)/q_i+1}, \tag{5.78}$$

where

$$c_n^s(\mathbf{p},\mathbf{q},\mathbf{A},\boldsymbol{\nu})$$
$$= \prod_{i=1}^n k_i^{1/q_i}(q_i\mathbf{A_i}) \prod_{i=1}^n \left[\frac{p_iq_i(\alpha_i+\nu_i)+p_i(n-1-s)+q_i}{p_iq_i(\alpha_i+\nu_i)+p_i(n-1-s)}\right]^{\alpha_i+\nu_i+(n-1-s)/q_i},$$

$$c_{n-1}^{s}(\mathbf{p},\mathbf{q},\mathbf{A},\boldsymbol{\nu}) = \prod_{i=1}^{n} k_i^{1/q_i}(q_i\mathbf{A_i}) \prod_{i=1}^{n-1}\left[\frac{p_iq_i(\alpha_i+\nu_i)+p_i(n-1-s)+q_i}{p_iq_i(\alpha_i+\nu_i)+p_i(n-1-s)}\right]^{\alpha_i+\nu_i+(n-1-s)/q_i},$$

$\mathbf{A_i} = (A_{i1}, A_{i2}, \dots, A_{in})$, $k_i(q_i\mathbf{A_i}) < \infty$, $i = 1,2,\dots,n$.

Proof. The result follows easily from relations (1.41) and (1.42) for the appropriate choice of non-negative measurable functions f_i, $i = 1,2,\dots,n$.

Namely, if the functions $f_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1,2,\dots,n$, are respectively replaced with $x_i^{\nu_i+\mu_i}(\mathscr{A} f_i)^{\mu_i}(x_i)$, then the terms on the right-hand side of inequality (1.41) become

$$\begin{aligned}\|x_i^{(n-1-s)/q_i+\alpha_i} f_i\|_{p_i}^{p_i} &= \|x_i^{(n-1-s)/q_i+\alpha_i+\nu_i+\mu_i}(\mathscr{A} f_i)^{\mu_i}(x_i)\|_{p_i}^{p_i}\\ &= \int_{\mathbb{R}_+} x_i^{p_i\mu_i\left[\frac{n-1-s}{q_i\mu_i}+\frac{\alpha_i+\nu_i}{\mu_i}+1\right]}(\mathscr{A} f_i)^{p_i\mu_i}(x_i)dx_i\\ &= \int_{\mathbb{R}_+} x_i^{(1-\lambda)q'}(\mathscr{A} f_i)^{q'}(x_i)dx_i = \|x_i^{1-\lambda}(\mathscr{A} f_i)(x_i)\|_{q'}^{q'}, \end{aligned}\tag{5.79}$$

where $q' = p_i\mu_i$ and

$$\lambda = -\frac{q_i(\alpha_i+\nu_i)+n-1-s}{q_i\mu_i}.\tag{5.80}$$

Moreover, considering the two-dimensional setting with non-conjugate exponents, the expression $\|x_i^{1-\lambda}(\mathscr{A} f_i)(x_i)\|_{q'}$ represents the left-hand side of the Hardy-type inequality (1.67), that is, we have inequality

$$\|x_i^{1-\lambda}(\mathscr{A} f_i)(x_i)\|_{q'}^{q'} \le \left(p'\lambda\right)^{q'\lambda}\|f_i\|_p^{q'},\tag{5.81}$$

with abbreviated

$$p = \frac{p_iq_i\mu_i}{p_iq_i(\alpha_i+\nu_i+\mu_i)+p_i(n-1-s)+q_i}$$

and

$$p' = -\frac{p_iq_i\mu_i}{p_iq_i(\alpha_i+\nu_i)+p_i(n-1-s)+q_i}.$$

In other words, the right-hand side of inequality (5.82) reads

$$\begin{aligned}&\left[\frac{p_iq_i(\alpha_i+\nu_i)+p_i(n-1-s)+q_i}{p_iq_i(\alpha_i+\nu_i)+p_i(n-1-s)}\right]^{p_i(\alpha_i+\nu_i)+p_i(n-1-s)/q_i}\\ &\times\left|\left|f^{\frac{q_i\mu_i}{p_iq_i(\alpha_i+\nu_i+\mu_i)+p_i(n-1-s)+q_i}}\right|\right|_{p_i}^{p_i^2(\alpha_i+\nu_i+\mu_i)+p_i^2(n-1-s)/q_i+p_i}.\end{aligned}\tag{5.82}$$

Hence, relations (5.79), (5.81) and (5.82) yield the series of inequalities

$$\begin{aligned}&\|x_i^{(n-1-s)/q_i+\alpha_i} f_i\|_{p_i}\\ &\le \left[\frac{p_iq_i(\alpha_i+\nu_i)+p_i(n-1-s)+q_i}{p_iq_i(\alpha_i+\nu_i)+p_i(n-1-s)}\right]^{\alpha_i+\nu_i+(n-1-s)/q_i}\end{aligned}$$

$$\times \left|\left| f^{\frac{q_i\mu_i}{p_i q_i(\alpha_i+\nu_i+\mu_i)+p_i(n-1-s)+q_i}} \right|\right|_{p_i}^{p_i(\alpha_i+\nu_i+\mu_i)+p_i(n-1-s)/q_i+1}, \ i=1,2,\dots,n,$$

so the inequality (5.77) follows immediately from (1.41).

Obviously the same reasoning is used to establish inequality (5.78) from (1.42), which completes the proof. □

The next result refers to the weighted geometric mean operator $\mathscr{G}_\alpha$ defined by (1.69).

Theorem 5.16 *Suppose p_i, p_i', q_i, $i=1,2,\dots,n$, and λ are as in (1.35), and A_{ij}, $i,j=1,2,\dots,n$, are real parameters satisfying $\sum_{i=1}^n A_{ij}=0$. Further, let $\alpha_i=\sum_{j=1}^n A_{ij}$, $i=1,2,\dots,n$, and let ν_i,μ_i, and $\alpha>0$ be real parameters. If $K:\mathbb{R}_+^n\to\mathbb{R}$ is a non-negative measurable homogeneous function of degree $-s$, $s>0$, and $f_i:\mathbb{R}_+\to\mathbb{R}$, $i=1,2,\dots,n$, non-negative measurable functions, then*

$$\int_{\mathbb{R}_+^n} K^\lambda(\mathbf{x})\prod_{i=1}^n x_i^{\nu_i}(\mathscr{G}_\alpha f_i)^{\mu_i}(x_i)d\mathbf{x} \le k_n^s(\mathbf{p},\mathbf{q},\mathbf{A},\nu)\prod_{i=1}^n \|x_i^{\frac{(n-1-s)}{q_i}+\alpha_i+\nu_i} f_i^{\mu_i}\|_{p_i}, \tag{5.83}$$

and

$$\left[\int_{\mathbb{R}_+} x_n^{(1-\lambda p_n')(n-1-s)-p_n'\alpha_n}\left(\int_{\mathbb{R}_+^{n-1}} K^\lambda(\mathbf{x})\prod_{i=1}^{n-1} x_i^{\nu_i}(\mathscr{G}_\alpha f_i)^{\mu_i}(x_i)\hat{d}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}$$
$$\le k_{n-1}^s(\mathbf{p},\mathbf{q},\mathbf{A},\nu)\prod_{i=1}^{n-1}\|x_i^{\frac{(n-1-s)}{q_i}+\alpha_i+\nu_i} f_i^{\mu_i}\|_{p_i}, \tag{5.84}$$

where

$$\begin{aligned} k_n^s(\mathbf{p},\mathbf{q},\mathbf{A},\nu) &= e^{\frac{1}{\alpha}\left[-\lambda_n s+n+\sum_{i=1}^n \nu_i\right]}\prod_{i=1}^n k_i^{1/q_i}(q_i\mathbf{A_i}),\\ k_{n-1}^s(\mathbf{p},\mathbf{q},\mathbf{A},\nu) &= e^{\frac{1}{\alpha}\left[\lambda(1-s)+n-1-\alpha_n-\frac{n-s}{qn}+\sum_{i=1}^{n-1}\nu_i\right]}\prod_{i=1}^n k_i^{1/q_i}(q_i\mathbf{A_i}), \end{aligned}$$

$\mathbf{A_i}=(A_{i1},A_{i2},\dots,A_{in})$, *and* $k_i(q_i\mathbf{A_i})<\infty$, $i=1,2,\dots,n$.

Proof. The result is an immediate consequence of general Hilbert-type inequalities (1.41) and (1.42) equipped with the functions $x_i^{\nu_i}(\mathscr{G}_\alpha f_i)^{\mu_i}(x_i)$ instead of $f_i:\mathbb{R}_+\to\mathbb{R}$, $i=1,2,\dots,n$, and the Levin-Cochran-Lee inequality (1.68). Namely, applying (1.68) to the right-hand sides of (1.41) and (1.42) yields

$$\|x_i^{(n-1-s)/q_i+\alpha_i+\nu_i}(\mathscr{G}_\alpha f_i)^{\mu_i}(x_i)\|_{p_i} \le e^{\frac{1}{\alpha}\left[\frac{n-s}{q_i}+\alpha_i+\nu_i-\lambda+1\right]}\|x_i^{\frac{(n-1-s)}{q_i}+\alpha_i+\nu_i} f_i^{\mu_i}\|_{p_i},$$

which completes the proof. □

The following pair of Hilbert-type inequalities deals with the weighted harmonic mean operator $\mathscr{H}_\alpha$, defined by (1.73).

Theorem 5.17 *Suppose p_i, p_i', q_i, $i = 1,2,\dots,n$, and λ are as in (1.35), and A_{ij}, $i,j = 1,2,\dots,n$, are real parameters such that $\sum_{i=1}^n A_{ij} = 0$. Further, let $\alpha_i = \sum_{j=1}^n A_{ij}$, $i = 1,2,\dots,n$, and let α, ν_i and $\mu_i > 0$ be real parameters such that $\alpha + \frac{1}{\mu_i}\big(1-\lambda+\alpha_i+\nu_i+\frac{n-s}{q_i}\big) > 0$. If $K : \mathbb{R}_+^n \to \mathbb{R}$ is a non-negative measurable homogeneous function of degree $-s$, $s > 0$, and $f_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1,2,\dots,n$, non-negative measurable functions, then*

$$\int_{\mathbb{R}_+^n} K^\lambda(\mathbf{x}) \prod_{i=1}^n x_i^{\nu_i} (\mathscr{H}_\alpha f_i)^{\mu_i}(x_i) d\mathbf{x} \le l_n^s(\mathbf{p},\mathbf{q},\mathbf{A},\nu,\mu) \prod_{i=1}^n \|x_i^{\frac{(n-1-s)}{q_i}+\alpha_i+\nu_i} f_i^{\mu_i}\|_{p_i}, \tag{5.85}$$

and

$$\left[\int_{\mathbb{R}_+} x_n^{(1-\lambda p_n')(n-1-s)-p_n'\alpha_n} \left(\int_{\mathbb{R}_+^{n-1}} K^\lambda(\mathbf{x}) \prod_{i=1}^{n-1} x_i^{\nu_i} (\mathscr{H}_\alpha f_i)^{\mu_i}(x_i) d\hat{}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}$$
$$\le l_{n-1}^s(\mathbf{p},\mathbf{q},\mathbf{A},\nu,\mu) \prod_{i=1}^{n-1} \|x_i^{\frac{(n-1-s)}{q_i}+\alpha_i+\nu_i} f_i^{\mu_i}\|_{p_i}, \tag{5.86}$$

where

$$l_n^s(\mathbf{p},\mathbf{q},\mathbf{A},\nu,\mu) = \prod_{i=1}^n k_i^{1/q_i}(q_i\mathbf{A_i}) \prod_{i=1}^n \left[\alpha + \frac{1}{\mu_i}\left(1-\lambda+\alpha_i+\nu_i+\frac{n-s}{q_i}\right)\right]^{\mu_i},$$
$$l_{n-1}^s(\mathbf{p},\mathbf{q},\mathbf{A},\nu,\mu) = \prod_{i=1}^n k_i^{1/q_i}(q_i\mathbf{A_i}) \prod_{i=1}^{n-1} \left[\alpha + \frac{1}{\mu_i}\left(1-\lambda+\alpha_i+\nu_i+\frac{n-s}{q_i}\right)\right]^{\mu_i},$$

$\mathbf{A_i} = (A_{i1}, A_{i2}, \dots, A_{in})$, *$k_i(q_i\mathbf{A_i}) < \infty$, and $i = 1,2,\dots,n$.*

Proof. We follow the same procedure as in the proof of the previous theorem, except that we use inequality (1.71) instead of the Levin-Cochran-Lee inequality.

More precisely, considering (1.41) and (1.42) with the functions $x_i^{\nu_i}(\mathscr{H}_\alpha f_i)^{\mu_i}(x_i)$, $i = 1,2,\dots,n$, it follows that

$$\|x_i^{(n-1-s)/q_i+\alpha_i+\nu_i}(\mathscr{H}_\alpha f_i)^{\mu_i}(x_i)\|_{p_i}$$
$$\le \left[\alpha + \frac{1}{\mu_i}\left(1-\lambda+\alpha_i+\nu_i+\frac{n-s}{q_i}\right)\right]^{\mu_i} \|x_i^{\frac{(n-1-s)}{q_i}+\alpha_i+\nu_i} f_i^{\mu_i}\|_{p_i},$$

and the proof is completed. □

Our next step is to determine conditions under which the constants $c_n^s(\mathbf{p},\mathbf{q},\mathbf{A},\nu)$, $c_{n-1}^s(\mathbf{p},\mathbf{q},\mathbf{A},\nu)$, $k_n^s(\mathbf{p},\mathbf{q},\mathbf{A},\nu)$, $k_{n-1}^s(\mathbf{p},\mathbf{q},\mathbf{A},\nu)$, $l_n^s(\mathbf{p},\mathbf{q},\mathbf{A},\nu,\mu)$, and $l_{n-1}^s(\mathbf{p},\mathbf{q},\mathbf{A},\nu,\mu)$ are the best possible in the corresponding inequalities. This happens in the case of conjugate exponents.

5.3.1 Reduction to the Conjugate Case and the Best Constants

In order to obtain the best possible constants in inequalities (5.77), (5.78), (5.83), (5.84), (5.85), and (5.86), we consider here their conjugate forms. Namely, if $p_i > 1$, $i = 1,2,\dots,n$, is the set of conjugate exponents, then inequalities (5.77) and (5.78) with $\nu_i = (s+1-n)/q_i - \alpha_i$, $i = 1,2,\ldots,n$ become respectively

$$\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^{n} x_i^{\frac{s+1-n}{p_i}-\alpha_i} (\mathscr{A} f_i)^{\mu_i}(x_i) d\mathbf{x} \le \overline{l_n^s}(\mathbf{p},\mathbf{A},\boldsymbol{\mu}) \prod_{i=1}^{n} \|f_i^{\mu_i}\|_{p_i}, \tag{5.87}$$

$$\left[\int_{\mathbb{R}_+} x_n^{(1-p_n')(n-1-s)-p_n'\alpha_n} \left(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x}) \prod_{i=1}^{n-1} x_i^{\frac{s+1-n}{p_i}-\alpha_i} (\mathscr{A} f_i)^{\mu_i}(x_i) \hat{d}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}$$
$$\le \overline{l_{n-1}^s}(\mathbf{p},\mathbf{A},\boldsymbol{\mu}) \prod_{i=1}^{n-1} \|f_i^{\mu_i}\|_{p_i}, \tag{5.88}$$

where

$$\overline{c_n^s}(\mathbf{p},\mathbf{A},\boldsymbol{\mu}) = \prod_{i=1}^{n} k_i^{1/p_i}(p_i\mathbf{A_i}) \prod_{i=1}^{n} \left(\frac{p_i\mu_i}{p_i\mu_i-1}\right)^{\mu_i},$$
$$\overline{c_{n-1}^s}(\mathbf{p},\mathbf{A},\boldsymbol{\mu}) = \prod_{i=1}^{n} k_i^{1/p_i}(p_i\mathbf{A_i}) \prod_{i=1}^{n-1} \left(\frac{p_i\mu_i}{p_i\mu_i-1}\right)^{\mu_i}.$$

Similarly, the conjugate forms of inequalities (5.83), (5.84), (5.85) and (5.86) read respectively

$$\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^{n} x_i^{\nu_i} (\mathscr{G}_\alpha f_i)^{\mu_i}(x_i) d\mathbf{x} \le \overline{k_n^s}(\mathbf{p},\mathbf{A},\boldsymbol{\nu}) \prod_{i=1}^{n} \|x_i^{\frac{(n-1-s)}{p_i}+\alpha_i+\nu_i} f_i^{\mu_i}\|_{p_i}, \tag{5.89}$$

$$\left[\int_{\mathbb{R}_+} x_n^{(1-p_n')(n-1-s)-p_n'\alpha_n} \left(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x}) \prod_{i=1}^{n-1} x_i^{\nu_i} (\mathscr{G}_\alpha f_i)^{\mu_i}(x_i) \hat{d}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}$$
$$\le \overline{k_{n-1}^s}(\mathbf{p},\mathbf{A},\boldsymbol{\nu}) \prod_{i=1}^{n-1} \|x_i^{\frac{(n-1-s)}{p_i}+\alpha_i+\nu_i} f_i^{\mu_i}\|_{p_i}, \tag{5.90}$$

and

$$\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^{n} x_i^{\nu_i} (\mathscr{H}_\alpha f_i)^{\mu_i}(x_i) d\mathbf{x} \le \overline{l_n^s}(\mathbf{p},\mathbf{A},\boldsymbol{\nu},\boldsymbol{\mu}) \prod_{i=1}^{n} \|x_i^{\frac{(n-1-s)}{p_i}+\alpha_i+\nu_i} f_i^{\mu_i}\|_{p_i}, \tag{5.91}$$

$$\left[\int_{\mathbb{R}_+} x_n^{(1-p_n')(n-1-s)-p_n'\alpha_n} \left(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x}) \prod_{i=1}^{n-1} x_i^{\nu_i} (\mathscr{H}_\alpha f_i)^{\mu_i}(x_i) \hat{d}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}$$
$$\le \overline{l_{n-1}^s}(\mathbf{p},\mathbf{A},\boldsymbol{\nu},\boldsymbol{\mu}) \prod_{i=1}^{n-1} \|x_i^{\frac{(n-1-s)}{p_i}+\alpha_i+\nu_i} f_i^{\mu_i}\|_{p_i}, \tag{5.92}$$

where

$$\overline{k_n^s}(\mathbf{p},\mathbf{A},\boldsymbol{\nu}) \;=\; e^{\frac{1}{\alpha}\left[-s+n+\sum_{i=1}^{n}\nu_i\right]} \prod_{i=1}^{n} k_i^{1/p_i}(p_i\mathbf{A_i}),$$

$$\begin{aligned}
\overline{k}^s_{n-1}(\mathbf{p},\mathbf{A},\boldsymbol{\nu}) &= e^{\frac{1}{\alpha}\left[-s+n-\alpha_n-\frac{n-s}{p_n}+\sum_{i=1}^{n-1}\nu_i\right]}\prod_{i=1}^{n} k_i^{1/p_i}(p_i\mathbf{A_i}),\\
\overline{l}^s_{n}(\mathbf{p},\mathbf{A},\boldsymbol{\nu},\boldsymbol{\mu}) &= \prod_{i=1}^{n} k_i^{1/p_i}(p_i\mathbf{A_i})\prod_{i=1}^{n}\left[\alpha+\frac{1}{\mu_i}\left(\alpha_i+\nu_i+\frac{n-s}{p_i}\right)\right]^{\mu_i},\\
\overline{l}^s_{n-1}(\mathbf{p},\mathbf{A},\boldsymbol{\nu},\boldsymbol{\mu}) &= \prod_{i=1}^{n} k_i^{1/p_i}(p_i\mathbf{A_i})\prod_{i=1}^{n-1}\left[\alpha+\frac{1}{\mu_i}\left(\alpha_i+\nu_i+\frac{n-s}{p_i}\right)\right]^{\mu_i}.
\end{aligned}$$

In the sequel we determine the conditions under which the inequalities (5.87), (5.88), (5.89), (5.90), (5.91), and (5.92) include the best possible constants on their right-hand sides. To do this, we establish some more specific conditions about the convergence of the integral $k_1(\mathbf{a})$, $\mathbf{a}=(a_1,a_2,\dots,a_n)$, defined by (1.5). More precisely, we assume that

$$k_1(\mathbf{a})<\infty \text{ for } a_2,\dots,a_n>-1, \sum_{i=2}^{n} a_i < s-n+1, n\in\mathbb{N}, n\geq 2. \tag{5.93}$$

By the similar reasoning as in the previous chapters, the best possible constants can be obtained if their parts regarding homogeneous kernel contain no exponents. For that sake, assume that

$$k_1(p_1\mathbf{A_1})=k_2(p_2\mathbf{A_2})=\cdots=k_n(p_n\mathbf{A_n}). \tag{5.94}$$

Utilizing the change of variables $u_1=1/t_2, u_3=t_3/t_2, u_4=t_4/t_2,\dots,u_n=t_n/t_2$, which provides the Jacobian of the transformation

$$\left|\frac{\partial(u_1,u_3,\dots,u_n)}{\partial(t_2,t_3,\dots,t_n)}\right| = t_2^{-n},$$

we have

$$\begin{aligned}
k_2(p_2\mathbf{A_2}) &= \int_{\mathbb{R}^{n-1}_+} K(\hat{\mathbf{t}}^1)t_2^{s-n-p_2(\alpha_2-A_{22})}\prod_{j=3}^{n} t_j^{p_2A_{2j}}\hat{d}^1\mathbf{t}\\
&= k_1(p_1A_{11}, s-n-p_2(\alpha_2-A_{22}), p_2A_{23},\dots,p_2A_{2n}).
\end{aligned}$$

According to (5.94), we have $p_1A_{12}=s-n-p_2(\alpha_2-A_{22})$, $p_1A_{13}=p_2A_{23}$, …, $p_1A_{1n}=p_2A_{2n}$. In a similar manner we express $k_i(p_i\mathbf{A_i})$, $i=3,\dots,n$, in terms of $k_1(\cdot)$. In such a way we see that (5.94) is fulfilled if

$$p_jA_{ji}=s-n-p_i(\alpha_i-A_{ii}), i,j=1,2,\dots,n,\ i\neq j. \tag{5.95}$$

The above set of conditions also implies that $p_iA_{ik}=p_jA_{jk}$, when $k\neq i,j$. Hence, we use abbreviations $\widetilde{A}_1=p_nA_{n1}$ and $\widetilde{A}_i=p_1A_{1i}$, $i\neq 1$. Since $\sum_{i=1}^n A_{ij}=0$, one easily obtains that $p_jA_{jj}=\widetilde{A}_j(1-p_j)$. Moreover, $\sum_{i=1}^n \widetilde{A}_i=s-n$ (see also [88]).

Now, if the set of conditions (5.95) is fulfilled, then, with the above abbreviations, inequalities (5.87) and (5.88) become respectively

$$\int_{\mathbb{R}^n_+} K(\mathbf{x})\prod_{i=1}^{n} x_i^{\frac{1}{p_i}+\widetilde{A}_i}(\mathscr{A}f_i)^{\mu_i}(x_i)d\mathbf{x} \leq \tilde{m}^s_n(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\mu})\prod_{i=1}^{n}\|f_i^{\mu_i}\|_{p_i}, \tag{5.96}$$

$$\left[\int_{\mathbb{R}_+} x_n^{(p_n'-1)(1+p_n\widetilde{A}_n)} \left(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x}) \prod_{i=1}^{n-1} x_i^{\frac{1}{p_i}+\widetilde{A}_i} (\mathscr{A} f_i)^{\mu_i}(x_i) \hat{d}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}$$
$$\leq \tilde{m}_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\mu}) \prod_{i=1}^{n-1} \|f_i^{\mu_i}\|_{p_i}, \tag{5.97}$$

where

$$\tilde{m}_n^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\mu}) = k_1(\widetilde{\mathbf{A}}) \prod_{i=1}^{n} \left(\frac{p_i\mu_i}{p_i\mu_i - 1}\right)^{\mu_i},$$
$$\tilde{m}_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\mu}) = k_1(\widetilde{\mathbf{A}}) \prod_{i=1}^{n-1} \left(\frac{p_i\mu_i}{p_i\mu_i - 1}\right)^{\mu_i},$$

and $\widetilde{\mathbf{A}} = (\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_n)$.

In the same way, inequalities (5.89), (5.90), (5.91) and (5.92) read respectively

$$\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^{n} x_i^{\nu_i} (\mathscr{G}_\alpha f_i)^{\mu_i}(x_i) d\mathbf{x} \leq m_n^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\nu}) \prod_{i=1}^{n} \|x_i^{\nu_i - \frac{1}{p_i} - \widetilde{A}_i} f_i^{\mu_i}\|_{p_i}, \tag{5.98}$$

$$\left[\int_{\mathbb{R}_+} x_n^{(p_n'-1)(1+p_n\widetilde{A}_n)} \left(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x}) \prod_{i=1}^{n-1} x_i^{\nu_i} (\mathscr{G}_\alpha f_i)^{\mu_i}(x_i) \hat{d}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}$$
$$\leq m_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\nu}) \prod_{i=1}^{n-1} \|x_i^{\nu_i - \frac{1}{p_i} - \widetilde{A}_i} f_i^{\mu_i}\|_{p_i}, \tag{5.99}$$

and

$$\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^{n} x_i^{\nu_i} (\mathscr{H}_\alpha f_i)^{\mu_i}(x_i) d\mathbf{x} \leq \overline{m}_n^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\nu},\boldsymbol{\mu}) \prod_{i=1}^{n} \|x_i^{\nu_i - \frac{1}{p_i} - \widetilde{A}_i} f_i^{\mu_i}\|_{p_i}, \tag{5.100}$$

$$\left[\int_{\mathbb{R}_+} x_n^{(p_n'-1)(1+p_n\widetilde{A}_n)} \left(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x}) \prod_{i=1}^{n-1} x_i^{\nu_i} (\mathscr{H}_\alpha f_i)^{\mu_i}(x_i) \hat{d}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}$$
$$\leq \overline{m}_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\nu},\boldsymbol{\mu}) \prod_{i=1}^{n-1} \|x_i^{\nu_i - \frac{1}{p_i} - \widetilde{A}_i} f_i^{\mu_i}\|_{p_i}, \tag{5.101}$$

where

$$m_n^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\nu}) = k_1(\widetilde{\mathbf{A}}) e^{\frac{1}{\alpha}\left[-s+n+\sum_{i=1}^n \nu_i\right]},$$
$$m_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\nu}) = k_1(\widetilde{\mathbf{A}}) e^{\frac{1}{\alpha}\left[-s+n+\widetilde{A}_n+\sum_{i=1}^{n-1} \nu_i\right]},$$
$$\overline{m}_n^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\nu},\boldsymbol{\mu}) = k_1(\widetilde{\mathbf{A}}) \prod_{i=1}^{n} \left[\alpha + \frac{1}{\mu_i}\left(\nu_i - \widetilde{A}_i\right)\right]^{\mu_i},$$

$$\overline{m}_{n-1}^{s}(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\nu},\boldsymbol{\mu}) = k_1(\widetilde{\mathbf{A}})\prod_{i=1}^{n-1}\left[\alpha+\frac{1}{\mu_i}\left(\nu_i-\widetilde{A}_i\right)\right]^{\mu_i}.$$

Finally, we show that the constants $\tilde{m}_n^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\mu})$, $\tilde{m}_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\mu})$ $m_n^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\nu})$, $m_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\nu})$, $\overline{m}_n^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\nu},\boldsymbol{\mu})$, and $\overline{m}_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\nu},\boldsymbol{\mu})$ are the best possible in the corresponding inequalities.

Theorem 5.18 *Let $\mu_i p_i > 1$, $i=1,2,\ldots,n$, and let the parameters $\widetilde{A}_i$, $i=2,\ldots,n$, fulfill conditions as in* (5.93)*. Then, the constant $\tilde{m}_n^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\mu})$ is the best possible in the inequality* (5.96).

Proof. Suppose to the contrary that there exists a positive constant C_n, $0<C_n<\tilde{m}_n^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\mu})$, such that inequality

$$\int_{\mathbb{R}_+^n} K(\mathbf{x})\prod_{i=1}^{n} x_i^{\frac{1}{p_i}+\widetilde{A}_i}\,(\mathscr{A}f_i)^{\mu_i}(x_i)d\mathbf{x} < C_n\prod_{i=1}^{n}\|f_i^{\mu_i}\|_{p_i}, \tag{5.102}$$

holds for non-negative measurable functions $f_i:\mathbb{R}_+\to\mathbb{R}, i=1,\ldots,n$. Let us set

$$K_N(\mathbf{x}) = \min(N,K(\mathbf{x}))\times\chi_{(N^{-1},N)^{n-1}}\left(\frac{x_2}{x_1},\frac{x_3}{x_1},\ldots,\frac{x_n}{x_1}\right). \tag{5.103}$$

Considering this inequality with the function

$$\widetilde{f}_i(x_i) = x_i^{\frac{\varepsilon-1}{\mu_i p_i}}\chi_{(0,1)}(x_i),\;\; i=1,\ldots,n,$$

where ε is a positive sufficiently small number, its right-hand side becomes

$$C_n\prod_{i=1}^{n}\|\widetilde{f}_i^{\mu_i}\|_{p_i} = C_n\prod_{i=1}^{n}\left(\int_0^1 x_i^{\varepsilon-1}dx_i\right)^{\frac{1}{p_i}} = \frac{C_n}{\varepsilon}.$$

On the other hand, since

$$\begin{aligned}0<x_i\le 1,\;\left(\mathscr{A}\widetilde{f}_i\right)(x_i) &= \frac{1}{x_i}\int_0^{x_i}\widetilde{f}_i(t)dt\\ &= \frac{1}{x_i}\int_0^{x_i} t^{\frac{\varepsilon-1}{\mu_i p_i}}dt\\ &= x_i^{\frac{\varepsilon-1}{\mu_i p_i}}\frac{\mu_i p_i}{\mu_i p_i-1+\varepsilon},\end{aligned}$$

the left-hand side of (5.102), can be estimated as

$$\begin{aligned}&\int_{\mathbb{R}_+^n} K_N(\mathbf{x})\prod_{i=1}^{n} x_i^{\frac{1}{p_i}+\widetilde{A}_i}\left(\mathscr{A}\widetilde{f}_i\right)^{\mu_i}(x_i)d\mathbf{x} \\ &> \prod_{i=1}^{n}\left(\frac{\mu_i p_i}{\mu_i p_i-1+\varepsilon}\right)^{\mu_i}\int_{(0,1]^n} K_N(\mathbf{x})\prod_{i=1}^{n} x_i^{\widetilde{A}_i+\frac{\varepsilon}{p_i}}d\mathbf{x}\end{aligned} \tag{5.104}$$

$$
\begin{aligned}
&= \prod_{i=1}^{n}\left(\frac{\mu_i p_i}{\mu_i p_i - 1 + \varepsilon}\right)^{\mu_i} \int_0^1 x_1^{\varepsilon-1}\left[\int_{(0,x_1]^{n-1}} K_N(\hat{\mathbf{u}}^1)\prod_{i=2}^{n} u_i^{\widetilde{A}_i+\frac{\varepsilon}{p_i}}\, \hat{d}^1\mathbf{u}\right] dx_1 \\
&\geq \prod_{i=1}^{n}\left(\frac{\mu_i p_i}{\mu_i p_i - 1 + \varepsilon}\right)^{\mu_i}\left[\int_0^1 x_1^{\varepsilon-1}\left(\int_{\mathbb{R}_+^{n-1}} K_N(\hat{\mathbf{u}}^1)\prod_{i=2}^{n} u_i^{\widetilde{A}_i+\frac{\varepsilon}{p_i}}\, \hat{d}^1\mathbf{u}\right) dx_1\right. \\
&\quad \left. - \int_0^1 x_1^{\varepsilon-1}\left(\sum_{i=2}^{n}\int_{\mathbb{D}_i} K_N(\hat{\mathbf{u}}^1)\prod_{j=2}^{n} u_j^{\widetilde{A}_j+\frac{\varepsilon}{p_j}}\, \hat{d}^1\mathbf{u}\right) dx_1\right].
\end{aligned}
$$

Let $\mathbb{D}_i = \{(u_2,\ldots,u_n):\ u_i > \frac{1}{x_1}, u_j > 0, j \neq i\}$. Then we have

$$
\begin{aligned}
&\int_{\mathbb{R}_+^n} K_N(\mathbf{x})\prod_{i=1}^{n} x_i^{\frac{1}{p_i}+\widetilde{A}_i}\left(\mathscr{A}\widetilde{f}_i\right)^{\mu_i}(x_i)d\mathbf{x} \qquad (5.105)\\
&\quad \geq \prod_{i=1}^{n}\left(\frac{\mu_i p_i}{\mu_i p_i - 1 + \varepsilon}\right)^{\mu_i}\left[\frac{1}{\varepsilon}\int_{\mathbb{R}_+^{n-1}} K_N(\hat{\mathbf{u}}^1)\prod_{i=2}^{n} u_i^{\widetilde{A}_i+\frac{\varepsilon}{p_i}}\, \hat{d}^1\mathbf{u}\right. \\
&\quad \left. - \int_0^1 x_1^{-1}\left(\sum_{i=2}^{n}\int_{\mathbb{D}_i} K_N(\hat{\mathbf{u}}^1)\prod_{j=2}^{n} u_j^{\widetilde{A}_j+\frac{\varepsilon}{p_j}}\, \hat{d}^1\mathbf{u}\right) dx_1\right].
\end{aligned}
$$

Without loss of generality, it suffices to find the appropriate estimate for the integral $\int_{\mathbb{D}_2} K_N(\hat{\mathbf{u}}^1)\prod_{j=2}^{n} u_j^{\widetilde{A}_j+\frac{\varepsilon}{p_j}}\, \hat{d}^1\mathbf{u}$: We plan to find a constant M_N independent of $\varepsilon > 0$ such that

$$
\int_{\mathbb{D}_2} K_N(\hat{\mathbf{u}}^1)\prod_{j=2}^{n} u_j^{\widetilde{A}_j+\frac{\varepsilon}{p_j}}\, \hat{d}^1\mathbf{u} \leq M_N
$$

for all $0 < \varepsilon < 1$. It should be noticed that M_N depends on N.

By virtue of the Fubini theorem, we have

$$
\begin{aligned}
&\int_0^1 x_1^{-1}\left(\int_{\mathbb{D}_2} K_N(\hat{\mathbf{u}}^1)\prod_{j=2}^{n} u_j^{\widetilde{A}_j+\frac{\varepsilon}{p_j}}\, \hat{d}^1\mathbf{u}\right) dx_1 \qquad (5.106)\\
&\quad = \int_0^1 x_1^{-1}\left(\int_{\mathbb{R}_+^{n-2}}\int_{1/x_1}^{\infty} K_N(\hat{\mathbf{u}}^1)\prod_{j=2}^{n} u_j^{\widetilde{A}_j+\frac{\varepsilon}{p_j}}\, \hat{d}^1\mathbf{u}\right) dx_1.
\end{aligned}
$$

Observe $u_2^{-1}\log u_2 \leq e^{-1} \leq 1$ $(u_2 \in [1,\infty))$. By enlarging the domain of integration, we obtain

$$
\begin{aligned}
&\int_0^1 x_1^{-1}\left(\int_{\mathbb{D}_2} K_N(\hat{\mathbf{u}}^1)\prod_{j=2}^{n} u_j^{\widetilde{A}_j+\frac{\varepsilon}{p_j}}\, \hat{d}^1\mathbf{u}\right) dx_1 \qquad (5.107)\\
&\quad \leq \int_{(1,\infty)\times\mathbb{R}_+^{n-2}} K_N(\hat{\mathbf{u}}^1)\prod_{j=2}^{n} u_j^{\widetilde{A}_j+\frac{\varepsilon}{p_j}}\left(\int_{1/u_2}^{1} x_1^{-1}dx_1\right)\hat{d}^1\mathbf{u} \\
&\quad = \int_{(1,\infty)\times\mathbb{R}_+^{n-2}} K_N(\hat{\mathbf{u}}^1)u_2^{\widetilde{A}_2+1+\frac{\varepsilon}{p_2}}\prod_{j=3}^{n} u_j^{\widetilde{A}_j+\frac{\varepsilon}{p_j}}(u_2^{-1}\log u_2)\hat{d}^1\mathbf{u}
\end{aligned}
$$

$$\leq \int_{(1,\infty)\times\mathbb{R}_+^{n-2}} K_N(\hat{\mathbf{u}}^1) u_2^{\widetilde{A}_2+1+\frac{\varepsilon}{p_2}} \prod_{j=3}^{n} u_j^{\widetilde{A}_j+\frac{\varepsilon}{p_j}} \hat{d}^1\mathbf{u}$$
$$\leq \int_{\mathbb{R}_+^{n-1}} K_N(\hat{\mathbf{u}}^1) u_2^{\widetilde{A}_2+1+\frac{\varepsilon}{p_2}} \prod_{j=3}^{n} u_j^{\widetilde{A}_j+\frac{\varepsilon}{p_j}} \hat{d}^1\mathbf{u} < \infty,$$

where for the last inequality we have used the fact that K_N is given by (5.103). Hence, we have

$$\prod_{i=1}^{n}\left(\frac{\mu_i p_i}{\mu_i p_i - 1+\varepsilon}\right)^{\mu_i}\left[\int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{u}}^1)\prod_{i=2}^{n} u_i^{\widetilde{A}_i+\frac{\varepsilon}{p_i}}\hat{d}^1\mathbf{u} - O(1)\right] < C_n.$$

Obviously, if $\varepsilon \to 0^+$, then

$$C_n \geq \tilde{m}_n^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\mu}) = \prod_{i=1}^{n}\left(\frac{\mu_i p_i}{\mu_i p_i - 1}\right)^{\mu_i}\int_{\mathbb{R}_+^{n-1}} K_N(\hat{\mathbf{u}}^i)\prod_{j=1,j\neq i}^{n} u_j^{\widetilde{A}_j}\hat{d}^i\mathbf{u}$$

for all $N = 1,2,\ldots$, which contradicts to our assumption $0 < C_n < \tilde{m}_n^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\mu})$. Hence, $\tilde{m}_n^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\mu})$ is the best possible. □

Theorem 5.19 *Let $\mu_i p_i > 1$, $i = 1,2,\ldots,n$, and let parameters $\widetilde{A}_i$, $i = 2,\ldots,n$, fulfill conditions as in* (5.93). *Then, the constant $\tilde{m}_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\mu})$ is the best possible in* (5.97).

Proof. Suppose, on the contrary, that there exist a positive constant C_{n-1}, $0 < C_{n-1} < \tilde{m}_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\mu})$ such that the inequality (5.97) holds for all non-negative measurable functions $f_i : \mathbb{R}_+ \to \mathbb{R}$, when replacing $\tilde{m}_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\mu})$ with C_{n-1}.

In that case, the left-hand side of inequality (5.96), denoted here with L, can be rewritten in the following form:

$$L = \int_{\mathbb{R}_+}\left(x_n^{\frac{1}{p_n}+\widetilde{A}_n}\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x})\prod_{i=1}^{n-1} x_i^{\frac{1}{p_i}+\widetilde{A}_i}(\mathscr{A} f_i)^{\mu_i}(x_i)\hat{d}^n\mathbf{x}\right)(\mathscr{A} f_n)^{\mu_n}(x_n)dx_n.$$

Now, the application of the Hölder's inequality with conjugate exponents p_n and p_n' yields inequality

$$L \leq L'\|(\mathscr{A} f_n)^{\mu_n}\|_{p_n}, \tag{5.108}$$

where L' denotes the left-hand side of inequality (5.97).

Furthermore, $L' \leq C_{n-1}\prod_{i=1}^{n-1}\|f_i^{\mu_i}\|_{p_i}$, while the Hardy inequality yields inequality

$$\|(\mathscr{A} f_n)^{\mu_n}\|_{p_n} \leq \left(\frac{p_n\mu_n}{p_n\mu_n - 1}\right)^{\mu_n}\|f_n^{\mu_n}\|_{p_n}.$$

Hence, the relation (5.108) yields inequality

$$L \leq C_{n-1}\left(\frac{p_n\mu_n}{p_n\mu_n - 1}\right)^{\mu_n}\prod_{i=1}^{n}\|f_i^{\mu_i}\|_{p_i}. \tag{5.109}$$

Finally, taking into account our assumption $0 < C_{n-1} < \tilde{m}_{n-1}^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\mu})$, we have

$$0 < C_{n-1}\left(\frac{p_n\mu_n}{p_n\mu_n - 1}\right)^{\mu_n} < \tilde{m}_{n-1}^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\mu})\left(\frac{p_n\mu_n}{p_n\mu_n - 1}\right)^{\mu_n} = \tilde{m}_n^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\mu}).$$

Hence, inequality (5.109) contradicts with the fact that $\tilde{m}_n^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\mu})$ is the best possible constant in inequality (5.96).

Therefore, the assumption that $\tilde{m}_{n-1}^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\mu})$ is not the best possible was false. The proof is now completed. □

Theorem 5.20 *Let* $\alpha > 0$, $\widetilde{A}_i \le \nu_i \le \frac{\alpha}{p_i} + \widetilde{A}_i$, $i = 1, 2, \ldots, n$, *and let the parameters* $\widetilde{A}_i$, $i = 2, \ldots, n$, *fulfill conditions as in* (5.93). *Then, the constant* $m_n^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\nu})$ *is the best possible in the inequality* (5.98).

Proof. Suppose that there exists a positive constant C_n, $0 < C_n < m_n^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\nu})$, such that inequality

$$\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^n x_i^{\nu_i} (\mathscr{G}_\alpha f_i)^{\mu_i}(x_i) d\mathbf{x} \le C_n \prod_{i=1}^n \|x_i^{\nu_i - \frac{1}{p_i} - \widetilde{A}_i} f_i^{\mu_i}\|_{p_i} \tag{5.110}$$

holds for all non-negative measurable functions $f_i : \mathbb{R}_+ \to \mathbb{R}$. Considering this inequality with the functions

$$f_i^\varepsilon(x_i) = \begin{cases} 1, & 0 < x_i < 1, \\ e^{-\frac{1}{p_i\mu_i}} x_i^{\frac{\widetilde{A}_i - \nu_i}{\mu_i} - \frac{\varepsilon}{p_i\mu_i}}, & x \ge 1, \end{cases}$$

where ε is sufficiently small number, its right-hand side becomes

$$C_n \prod_{i=1}^n \|x_i^{\nu_i - \frac{1}{p_i} - \widetilde{A}_i} (f_i^\varepsilon)^{\mu_i}\|_{p_i} = \frac{C_n}{\varepsilon} \prod_{i=1}^n \left(\frac{1}{e} - \frac{\varepsilon}{p_i(\nu_i - \widetilde{A}_i)}\right)^{\frac{1}{p_i}}. \tag{5.111}$$

On the other hand, since

$$(\mathscr{G}_\alpha f_i^\varepsilon)(x_i) = \begin{cases} 0, & 0 < x_i < 1, \\ e^{-\frac{1}{\mu_i p_i} - \frac{\widetilde{A}_i - \nu_i}{\alpha\mu_i}} x_i^{\frac{\widetilde{A}_i - \nu_i}{\mu_i} - \frac{\varepsilon}{\mu_i p_i}} e^{\frac{\varepsilon}{\mu_i p_i \alpha} + \frac{1}{x_i^\alpha}\left(\frac{1}{\mu_i p_i} + \frac{\widetilde{A}_i - \nu_i}{\alpha\mu_i} - \frac{\varepsilon}{\mu_i p_i \alpha}\right)}, & x \ge 1, \end{cases}$$

the left-hand side of (5.110), denoted here by L, can be estimated as

$$\begin{aligned} L &= \int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^n x_i^{\nu_i} (\mathscr{G}_\alpha f_i^\varepsilon)^{\mu_i}(x_i) d\mathbf{x} \\ &> e^{-1 + \frac{1}{\alpha}\left(n - s + \sum_{i=1}^n \nu_i\right)} \int_{[1,\infty)^n} K(\mathbf{x}) \prod_{i=1}^n x_i^{\widetilde{A}_i - \frac{\varepsilon}{p_i}} e^{\frac{\varepsilon}{p_i\alpha} + \frac{1}{x_i^\alpha}\left(\frac{1}{p_i} + \frac{\widetilde{A}_i - \nu_i}{\alpha} - \frac{\varepsilon}{p_i\alpha}\right)} d\mathbf{x} \\ &\ge e^{-1 + \frac{1}{\alpha}\left(n - s + \sum_{i=1}^n \nu_i\right)} \cdot I, \end{aligned}$$

where $I=\int_{[1,\infty\rangle^n}K(\mathbf{x})\prod_{i=1}^n x_i^{\widetilde{A}_i-\frac{\varepsilon}{p_i}}d\mathbf{x}$. Obviously, the integral I can be rewritten as

$$I=\int_1^\infty x_1^{-1-\varepsilon}\left[\int_{[1/x_1,\infty\rangle^{n-1}}K(\hat{\mathbf{u}}^1)\prod_{i=2}^n u_i^{\widetilde{A}_i-\varepsilon/p_i}\hat{d}^1\mathbf{u}\right]dx_1,$$

providing the inequality

$$\begin{aligned}I&\geq\int_1^\infty x_1^{-1-\varepsilon}\left[\int_{\mathbb{R}_+^{n-1}}K(\hat{\mathbf{u}}^1)\prod_{i=2}^n u_i^{\widetilde{A}_i-\varepsilon/p_i}\hat{d}^1\mathbf{u}\right]dx_1\\&\quad-\int_1^\infty x_1^{-1-\varepsilon}\left[\sum_{i=2}^n\int_{\mathbb{D}_i}K(\hat{\mathbf{u}}^1)\prod_{j=2}^n u_j^{\widetilde{A}_j-\varepsilon/p_j}\hat{d}^1\mathbf{u}\right]dx_1\\&\geq\frac{1}{\varepsilon}\int_{\mathbb{R}_+^{n-1}}K(\hat{\mathbf{u}}^1)\prod_{i=2}^n u_i^{\widetilde{A}_i-\varepsilon/p_i}\hat{d}^1\mathbf{u}\\&\quad-\int_1^\infty x_1^{-1}\left[\sum_{i=2}^n\int_{\mathbb{D}_i}K(\hat{\mathbf{u}}^1)\prod_{j=2}^n u_j^{\widetilde{A}_j-\varepsilon/p_j}\hat{d}^1\mathbf{u}\right]dx_1,\end{aligned}\tag{5.112}$$

where $\mathbb{D}_i=\{(u_2,u_3,\ldots,u_n);0<u_i\leq 1/x_1,u_j>0,j\neq i\}$, $\mathbf{1/p}=(1/p_1,\ldots,1/p_n)$.

Without loss of generality, it suffices to find the appropriate estimate for the integral $\int_{\mathbb{D}_2}K(\hat{\mathbf{u}}^1)\prod_{j=2}^n u_j^{\widetilde{A}_j-\varepsilon/p_j}\hat{d}^1\mathbf{u}$. In fact, setting $\alpha>0$ such that $\widetilde{A}_2+1>\varepsilon/p_2+\alpha$, since $-u_2^\alpha\log u_2\to 0$ $(u_2\to 0^+)$, there exists $M\geq 0$ such that $-u_2^\alpha\log u_2\leq M$ $(u_2\in(0,1])$. On the other hand, it follows easily that the parameters $a_2=\widetilde{A}_2-(\varepsilon/p_2+\alpha)$ and $a_i=\widetilde{A}_i-\varepsilon/p_i$, $i=3,\ldots,n$ satisfy conditions as in (5.93). Then, by virtue of the Fubini theorem, we have

$$\begin{aligned}0&\leq\int_1^\infty x_1^{-1}\int_{\mathbb{D}_2}K(\hat{\mathbf{u}}^1)\prod_{j=2}^n u_j^{\widetilde{A}_j-\varepsilon/p_j}\hat{d}^1\mathbf{u}dx_1\\&=\int_1^\infty x_1^{-1}\left[\int_{\mathbb{R}_+^{n-2}}\int_0^{1/x_1}K(\hat{\mathbf{u}}^1)\prod_{j=2}^n u_j^{\widetilde{A}_j-\varepsilon/p_j}\hat{d}^1\mathbf{u}\right]dx_1\\&=\int_{\mathbb{R}_+^{n-2}}\int_0^1K(\hat{\mathbf{u}}^1)\prod_{j=2}^n u_j^{\widetilde{A}_j-\varepsilon/p_j}\left(\int_1^{1/u_2}x_1^{-1}dx_1\right)\hat{d}^1\mathbf{u}\\&=\int_{\mathbb{R}_+^{n-2}}\int_0^1K(\hat{\mathbf{u}}^1)\prod_{j=2}^n u_j^{\widetilde{A}_j-\varepsilon/p_j}(-\log u_2)\hat{d}^1\mathbf{u}\\&\leq M\int_{\mathbb{R}_+^{n-2}}\int_0^1K(\hat{\mathbf{u}}^1)u_2^{\widetilde{A}_2-(\varepsilon/p_2+\alpha)}\prod_{j=3}^n u_j^{\widetilde{A}_j-\varepsilon/p_j}\hat{d}^1\mathbf{u}\\&\leq M\int_{\mathbb{R}_+^{n-1}}K(\hat{\mathbf{u}}^1)u_2^{\widetilde{A}_2-(\varepsilon/p_2+\alpha)}\prod_{j=3}^n u_j^{\widetilde{A}_j-\varepsilon/p_j}\hat{d}^1\mathbf{u}\\&=M\cdot k_1(\widetilde{A}_2-(\varepsilon/p_2+\alpha),\widetilde{A}_3-\varepsilon/p_3,\ldots,\widetilde{A}_n-\varepsilon/p_n)<\infty.\end{aligned}\tag{5.113}$$

Hence, taking into account (5.112), we obtain

$$L \geq e^{-1+\frac{1}{\alpha}\left(n-s+\sum_{i=1}^{n}\nu_i\right)}\left(\frac{1}{\varepsilon}k_1\left(\widetilde{\mathbf{A}}-\varepsilon\mathbf{1}/\mathbf{p}\right)-O(1)\right).$$

Moreover, the relation (5.111) implies that

$$\frac{C_n}{\varepsilon}\prod_{i=1}^{n}\left(\frac{1}{e}-\frac{\varepsilon}{p_i(\nu_i-\widetilde{A}_i)}\right)^{\frac{1}{p_i}} \geq e^{-1+\frac{1}{\alpha}\left(n-s+\sum_{i=1}^{n}\nu_i\right)}\left(\frac{1}{\varepsilon}k_1\left(\widetilde{\mathbf{A}}-\varepsilon\mathbf{1}/\mathbf{p}\right)-O(1)\right),$$

that is,

$$C_n\prod_{i=1}^{n}\left(\frac{1}{e}-\frac{\varepsilon}{p_i(\nu_i-\widetilde{A}_i)}\right)^{\frac{1}{p_i}} \geq e^{-1+\frac{1}{\alpha}\left(n-s+\sum_{i=1}^{n}\nu_i\right)}\left(k_1\left(\widetilde{\mathbf{A}}-\varepsilon\mathbf{1}/\mathbf{p}\right)-\varepsilon O(1)\right).$$

Obviously, if $\varepsilon \to 0^+$, then $C_n \geq m_n^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\nu})$, which contradicts with our assumption $0 < C_n < m_n^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\nu})$. Hence, $m_n^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\nu})$ is the best possible in (5.98). $\square$

Theorem 5.21 *Let $\alpha > 0$, $\widetilde{A}_i \leq \nu_i \leq \frac{\alpha}{p_i}+\widetilde{A}_i$, $i = 1,2,\ldots,n$, and let parameters $\widetilde{A}_i$, $i = 2,\ldots,n$, fulfill conditions as in* (5.93). *Then, the constant $m_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\nu})$ is the best possible in* (5.99).

Proof. Assume that there exists a positive constant C_{n-1}, smaller than $m_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\nu})$, such that the inequality (5.99) holds when replacing $m_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\nu})$ by C_{n-1}.

The left-hand side of inequality (5.98), denoted here by L, can be rewritten in the following form:

$$L = \int_{\mathbb{R}_+}\left(x_n^{\frac{1}{p_n}+\widetilde{A}_n}\int_{\mathbb{R}_+^{n-1}}K(\mathbf{x})\prod_{i=1}^{n-1}x_i^{\nu_i}\left(\mathscr{G}_\alpha f_i\right)^{\mu_i}(x_i)\hat{d}^n\mathbf{x}\right)x_n^{\nu_n-\frac{1}{p_n}-\widetilde{A}_n}\left(\mathscr{G}_\alpha f_n\right)^{\mu_n}(x_n)dx_n.$$

Now, applying the Hölder inequality with conjugate exponents p_n and p_n' to the above expression yields inequality

$$L \leq L'\|x_n^{\nu_n-\frac{1}{p_n}-\widetilde{A}_n}\left(\mathscr{G}_\alpha f_n\right)^{\mu_n}\|_{p_n}, \tag{5.114}$$

where L' denotes the left-hand side of (5.99).

Moreover, $L' \leq C_{n-1}\prod_{i=1}^{n-1}\|x_i^{\nu_i-\frac{1}{p_i}-\widetilde{A}_i}f_i^{\mu_i}\|_{p_i}$, while the Levin-Cochran-Lee inequality (1.68) yields

$$\|x_n^{\nu_n-\frac{1}{p_n}-\widetilde{A}_n}\left(\mathscr{G}_\alpha f_n\right)^{\mu_n}\|_{p_n} \leq e^{\frac{\nu_n-\widetilde{A}_n}{\alpha}}\cdot\|x_n^{\nu_n-\frac{1}{p_n}-\widetilde{A}_n}f_n^{\mu_n}\|_{p_n}.$$

Therefore relation (5.114) yields the inequality

$$L \leq C_{n-1}e^{\frac{\nu_n-\widetilde{A}_n}{\alpha}}\cdot\prod_{i=1}^{n}\|x_i^{\nu_i-\frac{1}{p_i}-\widetilde{A}_i}f_i^{\mu_i}\|_{p_i}. \tag{5.115}$$

Finally, taking into account our assumption $0 < C_{n-1} < m_{n-1}^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\mu})$, we have

$$0 < C_{n-1} e^{\frac{\nu_n - \widetilde{A}_n}{\alpha}} < m_{n-1}^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\mu}) e^{\frac{\nu_n - \widetilde{A}_n}{\alpha}} = m_n^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\mu}).$$

Hence, relation (5.115) contradicts with the fact that $m_n^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\mu})$ is the best possible constant in inequality (5.98). Thus, the assumption that $m_{n-1}^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\mu})$ is not the best possible is false. The proof is now completed. □

Theorem 5.22 *Let α, ν_i, and $\mu_i > 0$ be real parameters such that $\alpha + \frac{1}{\mu_i}(\nu_i - \widetilde{A}_i) > 0$, $i = 1, 2, \ldots, n$, and let parameters $\widetilde{A}_i$, $i = 2, \ldots, n$, fulfill conditions as in* (5.93). *Then, the constant $\overline{m}_n^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\nu}, \boldsymbol{\mu})$ is the best possible in* (5.100).

Proof. We follow the same procedure as in the proof of Theorem 5.20, that is, we suppose that the inequality

$$\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^n x_i^{\nu_i} (\mathscr{H}_\alpha f_i)^{\mu_i}(x_i) d\mathbf{x} \leq C_n \prod_{i=1}^n \| x_i^{\nu_i - \frac{1}{p_i} - \widetilde{A}_i} f_i^{\mu_i} \|_{p_i}, \tag{5.116}$$

holds with a positive constant C_n, smaller than $\overline{m}_n^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\nu}, \boldsymbol{\mu})$. Considering this inequality with the functions

$$f_i^\varepsilon(x_i) = \begin{cases} x_i^{\frac{\widetilde{A}_i - \nu_i}{\mu_i} + \frac{\varepsilon}{p_i \mu_i}}, & 0 < x_i \leq 1, \\ 0, & x > 1, \end{cases}$$

where ε is sufficiently small number, its right-hand side reduces to

$$C_n \prod_{i=1}^n \| x_i^{\nu_i - \frac{1}{p_i} - \widetilde{A}_i} (f_i^\varepsilon)^{\mu_i} \|_{p_i} = \frac{C_n}{\varepsilon}. \tag{5.117}$$

Moreover, since

$$(\mathscr{H}_\alpha f_i^\varepsilon)(x_i) = \begin{cases} \left[\alpha + \frac{\nu_i - \widetilde{A}_i}{\mu_i} - \frac{\varepsilon}{\mu_i p_i}\right] x_i^{\frac{\widetilde{A}_i - \nu_i}{\mu_i} + \frac{\varepsilon}{\mu_i p_i}}, & 0 < x_i \leq 1, \\ 0, & x_i > 1, \end{cases}$$

the left-hand side of (5.116), denoted here by L, reads

$$\begin{aligned} L &= \int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^n x_i^{\nu_i} (\mathscr{H}_\alpha f_i^\varepsilon)^{\mu_i}(x_i) d\mathbf{x} \\ &= \varphi(\varepsilon) \cdot I, \end{aligned}$$

where

$$\varphi(\varepsilon) = \prod_{i=1}^n \left[\alpha + \frac{\nu_i - \widetilde{A}_i}{\mu_i} - \frac{\varepsilon}{\mu_i p_i}\right]^{\mu_i} \quad \text{and} \quad I = \int_{\langle 0,1]^n} K(\mathbf{x}) \prod_{i=1}^n x_i^{\widetilde{A}_i + \frac{\varepsilon}{p_i}} d\mathbf{x}.$$

Obviously, the integral I can be rewritten as

$$I = \int_0^1 x_1^{\varepsilon-1}\left[\int_{\langle 0,1/x_1]^{n-1}} K(\hat{\mathbf{u}}^1)\prod_{i=2}^n u_i^{\widetilde{A}_i+\varepsilon/p_i}\hat{d}^1\mathbf{u}\right]dx_1,$$

providing the estimate

$$\begin{aligned}
I &\geq \int_0^1 x_1^{\varepsilon-1}\left[\int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{u}}^1)\prod_{i=2}^n u_i^{\widetilde{A}_i+\varepsilon/p_i}\hat{d}^1\mathbf{u}\right]dx_1 \\
&\quad - \int_0^1 x_1^{\varepsilon-1}\left[\sum_{i=2}^n\int_{\mathbb{E}_i} K(\hat{\mathbf{u}}^1)\prod_{j=2}^n u_j^{\widetilde{A}_j+\varepsilon/p_j}\hat{d}^1\mathbf{u}\right]dx_1 \\
&\geq \frac{1}{\varepsilon}\int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{u}}^1)\prod_{i=2}^n u_i^{\widetilde{A}_i+\varepsilon/p_i}\hat{d}^1\mathbf{u} \\
&\quad - \int_0^1 x_1^{-1}\left[\sum_{i=2}^n\int_{\mathbb{E}_i} K(\hat{\mathbf{u}}^1)\prod_{j=2}^n u_j^{\widetilde{A}_j+\varepsilon/p_j}\hat{d}^1\mathbf{u}\right]dx_1,
\end{aligned} \tag{5.118}$$

where $\mathbb{E}_i = \{(u_2,u_3,\ldots,u_n); 1/x_1 \leq u_i < \infty, u_j > 0, j \neq i\}$, $\mathbf{1/p} = (1/p_1,\ldots,1/p_n)$.

Clearly, it suffices to estimate the integral $\int_{\mathbb{E}_2} K(\hat{\mathbf{u}}^1)\prod_{j=2}^n u_j^{\widetilde{A}_j+\varepsilon/p_j}\hat{d}^1\mathbf{u}$. Namely, choosing $\alpha > 0$ such that $\widetilde{A}_2 + 1 > -\varepsilon/p_2 - \alpha$, since $-u_2^{-\alpha}\log\frac{1}{u_2} \to 0$ $(u_2 \to \infty)$, there exists $M \geq 0$ such that $-u_2^{-\alpha}\log\frac{1}{u_2} \leq M$ $(u_2 \in [1,\infty))$. Further, the parameters $a_2 = \widetilde{A}_2 + (\varepsilon/p_2 + \alpha)$ and $a_i = \widetilde{A}_i + \varepsilon/p_i$, $i = 3,\ldots,n$, fulfill conditions as in (5.93). Then, similarly to (5.113), we have

$$\begin{aligned}
&\int_0^1 x_1^{-1}\int_{\mathbb{E}_2} K(\hat{\mathbf{u}}^1)\prod_{j=2}^n u_j^{\widetilde{A}_j+\varepsilon/p_j}\hat{d}^1\mathbf{u}dx_1 \\
&\quad \leq M\cdot k_1(\widetilde{A}_2+(\varepsilon/p_2+\alpha),\widetilde{A}_3+\varepsilon/p_3,\ldots,\widetilde{A}_n+\varepsilon/p_n) < \infty,
\end{aligned}$$

and utilizing (5.118), it follows that

$$L \geq \varphi(\varepsilon)\cdot\left(\frac{1}{\varepsilon}k_1\left(\widetilde{\mathbf{A}}+\varepsilon\mathbf{1/p}\right) - O(1)\right). \tag{5.119}$$

Finally, taking into account (5.117) and (5.119), we have that $\overline{m}_n^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\nu},\boldsymbol{\mu}) \leq C_n$ when $\varepsilon \to 0^+$, which is an obvious contradiction. This means that the constant $\overline{m}_n^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\nu},\boldsymbol{\mu})$ is the best possible in (5.100). □

Theorem 5.23 *Let α, ν_i, and $\mu_i > 0$ be real parameters such that $\alpha + \frac{1}{\mu_i}(\nu_i - \widetilde{A}_i) > 0$, $i = 1,2,\ldots,n$, and let parameters $\widetilde{A}_i$, $i = 2,\ldots,n$, fulfill conditions as in* (5.93). *Then, the constant $\overline{m}_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}},\boldsymbol{\nu},\boldsymbol{\mu})$ is the best possible in* (5.101).

Proof. Suppose, on the contrary, that there exists a positive constant C_{n-1}, $0 < C_{n-1} < \overline{m}^s_{n-1}(\mathbf{p}, \widetilde{\mathbf{A}}, \nu, \mu)$, such that the inequality (5.101) holds with the constant C_{n-1} instead of $\overline{m}^s_{n-1}(\mathbf{p}, \widetilde{\mathbf{A}}, \nu, \mu)$.

Now, rewriting the left-hand side of inequality (5.100) in the form

$$\int_{\mathbb{R}_+} \left(x_n^{\frac{1}{p_n}+\widetilde{A}_n} \int_{\mathbb{R}_+^{n-1}} K(\mathbf{x}) \prod_{i=1}^{n-1} x_i^{\nu_i} \left(\mathscr{H}_\alpha f_i\right)^{\mu_i}(x_i) \hat{d}^n\mathbf{x} \right) x_n^{\nu_n - \frac{1}{p_n} - \widetilde{A}_n} \left(\mathscr{H}_\alpha f_n\right)^{\mu_n}(x_n) dx_n,$$

and applying the Hölder inequality with conjugate exponents p_n and p_n', we have

$$L \leq L' \| x_n^{\nu_n - \frac{1}{p_n} - \widetilde{A}_n} \left(\mathscr{H}_\alpha f_n\right)^{\mu_n} \|_{p_n}, \tag{5.120}$$

where L and L' respectively denote the left-hand sides of inequalities (5.100) and (5.101). In addition, $L' \leq C_{n-1} \prod_{i=1}^{n-1} \| x_i^{\nu_i - \frac{1}{p_i} - \widetilde{A}_i} f_i^{\mu_i} \|_{p_i}$, while (1.71) yields the inequality

$$\| x_n^{\nu_n - \frac{1}{p_n} - \widetilde{A}_n} \left(\mathscr{H}_\alpha f_n\right)^{\mu_n} \|_{p_n} \leq \left(\alpha + \frac{\nu_n - \widetilde{A}_n}{\mu_n} \right)^{\mu_n} \cdot \| x_n^{\nu_n - \frac{1}{p_n} - \widetilde{A}_n} f_n^{\mu_n} \|_{p_n}.$$

Hence, relation (5.120) provides the inequality

$$L \leq C_{n-1} \left(\alpha + \frac{\nu_n - \widetilde{A}_n}{\mu_n} \right)^{\mu_n} \cdot \prod_{i=1}^{n} \| x_i^{\nu_i - \frac{1}{p_i} - \widetilde{A}_i} f_i^{\mu_i} \|_{p_i}. \tag{5.121}$$

Finally, with our assumption $0 < C_{n-1} < m^s_{n-1}(\mathbf{p}, \widetilde{\mathbf{A}}, \mu)$, we have

$$C_{n-1} \left(\alpha + \frac{\nu_n - \widetilde{A}_n}{\mu_n} \right)^{\mu_n} < \overline{m}^s_{n-1}(\mathbf{p}, \widetilde{\mathbf{A}}, \nu, \mu) \left(\alpha + \frac{\nu_n - \widetilde{A}_n}{\mu_n} \right)^{\mu_n} = \overline{m}^s_n(\mathbf{p}, \widetilde{\mathbf{A}}, \nu, \mu).$$

Therefore, inequality (5.121) contradicts with the fact that $\overline{m}^s_n(\mathbf{p}, \widetilde{\mathbf{A}}, \nu, \mu)$ is the best possible constant in (5.100). The proof is now completed. □

5.3.2 Some Examples and Remarks

Now, we derive here several new Hilbert-type inequalities with arithmetic, geometric and harmonic mean operators and with some particular homogeneous kernels. In this subsection we deal with the case of conjugate exponents and the inequalities that follow include the best possible constants on their right-hand sides.

First Example

Our first example refers to the kernel $K_1 : \mathbb{R}_+^n \to \mathbb{R}$, defined by

$$K_1(\mathbf{x}) = \frac{1}{\left(\sum_{i=1}^n x_i\right)^s}, \; s > 0.$$

Clearly, K_1 is a homogeneous function of degree $-s$, and the constant $k_1(\widetilde{\mathbf{A}})$, appearing in inequalities (5.96), (5.97), (5.98), (5.99), (5.100), and (5.101), can be expressed in terms of the usual Gamma function Γ. Namely, utilizing the formula

$$\int_{\mathbb{R}_+^{n-1}} \frac{\prod_{i=1}^{n-1} u_i^{a_i-1}}{\left(1+\sum_{i=1}^{n-1} u_i\right)^{\sum_{i=1}^n a_i}} \hat{d}^n\mathbf{u} = \frac{\prod_{i=1}^n \Gamma(a_i)}{\Gamma\left(\sum_{i=1}^n a_i\right)},$$

which holds for $a_i > 0$, $i = 1,2,\ldots n$, it follows that

$$k_1(\widetilde{\mathbf{A}}) = \frac{1}{\Gamma(s)} \prod_{i=1}^n \Gamma(1+\widetilde{A}_i), \quad i = 1,2,\ldots,n,$$

provided that $\widetilde{A}_i > -1$, $i = 1,2,\ldots,n$, and $\sum_{i=1}^n \widetilde{A}_i = s-n$. In addition, considering the parameters $\widetilde{A}_i = r_i - 1$, $\mu_i = 1$, $v_i = r_i - 1/p_i'$, $i = 1,2,\ldots,n$, where $r_i > 0$ and $\sum_{i=1}^n r_i = s$, inequalities (5.96), (5.97), (5.98), (5.99), (5.100), and (5.101) reduce respectively to

$$\int_{\mathbb{R}_+^n} \frac{1}{\left(\sum_{i=1}^n x_i\right)^s} \prod_{i=1}^n x_i^{r_i-\frac{1}{p_i'}} (\mathscr{A} f_i)(x_i) d\mathbf{x} \le \frac{\prod_{i=1}^n p_i'}{\Gamma(s)} \prod_{i=1}^n \Gamma(r_i) \prod_{i=1}^n \|f_i\|_{p_i},$$

$$\left[\int_{\mathbb{R}_+} x_n^{r_n p_n'-1} \left(\int_{\mathbb{R}_+^{n-1}} \frac{1}{\left(\sum_{i=1}^n x_i\right)^s} \prod_{i=1}^{n-1} x_i^{r_i-\frac{1}{p_i'}} (\mathscr{A} f_i)(x_i) \hat{d}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}$$
$$\le \frac{\prod_{i=1}^{n-1} p_i'}{\Gamma(s)} \prod_{i=1}^n \Gamma(r_i) \prod_{i=1}^{n-1} \|f_i^{\mu_i}\|_{p_i},$$

$$\int_{\mathbb{R}_+^n} \frac{1}{\left(\sum_{i=1}^n x_i\right)^s} \prod_{i=1}^n x_i^{r_i-\frac{1}{p_i'}} (\mathscr{G}_\alpha f_i)(x_i) d\mathbf{x} \le \frac{e^{1/\alpha}}{\Gamma(s)} \prod_{i=1}^n \Gamma(r_i) \prod_{i=1}^n \|f_i\|_{p_i},$$

$$\left[\int_{\mathbb{R}_+} x_n^{r_n p_n'-1} \left(\int_{\mathbb{R}_+^{n-1}} \frac{1}{\left(\sum_{i=1}^n x_i\right)^s} \prod_{i=1}^{n-1} x_i^{r_i-\frac{1}{p_i'}} (\mathscr{G}_\alpha f_i)(x_i) \hat{d}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}$$
$$\le \frac{e^{1/(\alpha p_n')}}{\Gamma(s)} \prod_{i=1}^n \Gamma(r_i) \prod_{i=1}^{n-1} \|f_i\|_{p_i},$$

$$\int_{\mathbb{R}_+^n} \frac{1}{\left(\sum_{i=1}^n x_i\right)^s} \prod_{i=1}^n x_i^{r_i-\frac{1}{p_i'}} (\mathscr{H}_\alpha f_i)(x_i) d\mathbf{x} \le \prod_{i=1}^n \left(\alpha + \frac{1}{p_i}\right) \frac{\prod_{i=1}^n \Gamma(r_i)}{\Gamma(s)} \prod_{i=1}^n \|f_i\|_{p_i},$$

and

$$\left[\int_{\mathbb{R}_+} x_n^{r_n p_n'-1} \left(\int_{\mathbb{R}_+^{n-1}} \frac{1}{\left(\sum_{i=1}^n x_i\right)^s} \prod_{i=1}^{n-1} x_i^{r_i-\frac{1}{p_i'}} (\mathscr{H}_\alpha f_i)(x_i) \hat{d}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}$$
$$\le \prod_{i=1}^{n-1} \left(\alpha + \frac{1}{p_i}\right) \frac{\prod_{i=1}^n \Gamma(r_i)}{\Gamma(s)} \prod_{i=1}^{n-1} \|f_i\|_{p_i}.$$

Clearly, the constants appearing on their right-hand sides are the best possible.

Second Example

Another example of a homogeneous kernel with degree $-s$, is the function

$$K_2(\mathbf{x}) = \frac{1}{\max\{x_1^s,\ldots,x_n^s\}}, \quad s>0.$$

In order to derive analogues of the inequalities from the previous example, we utilize the integral formula

$$\int_{\mathbb{R}_+^{n-1}} \frac{\prod_{i=1}^{n-1} x_i^{a_i}}{\max\{1,x_1^s,\ldots,x_{n-1}^s\}} \hat{d}^n\mathbf{u} = \frac{s}{\prod_{i=1}^{n}(1+a_i)},$$

where $a_i > -1$ and $\sum_{i=1}^n a_i = s-n$. Hence, with this kernel and parameters $\widetilde{A}_i = r_i - 1$, $\mu_i = 1$, $\nu_i = r_i - 1/p_i'$, $i=1,2,\ldots,n$, where $r_i>0$ and $\sum_{i=1}^n r_i = s$, inequalities (5.96), (5.97), (5.98), (5.99), (5.100), and (5.101) become respectively

$$\int_{\mathbb{R}_+^n} \frac{1}{\max\{x_1^s,\ldots,x_n^s\}} \prod_{i=1}^n x_i^{r_i-\frac{1}{p_i'}} (\mathscr{A} f_i)(x_i) d\mathbf{x} \le s \prod_{i=1}^n \frac{p_i'}{r_i} \prod_{i=1}^n \|f_i\|_{p_i},$$

$$\left[\int_{\mathbb{R}_+} x_n^{r_n p_n'-1} \left(\int_{\mathbb{R}_+^{n-1}} \frac{1}{\max\{x_1^s,\ldots,x_n^s\}} \prod_{i=1}^{n-1} x_i^{r_i-\frac{1}{p_i'}} (\mathscr{A} f_i)(x_i) \hat{d}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}$$
$$\le \frac{s}{p_n'} \prod_{i=1}^n \frac{p_i'}{r_i} \prod_{i=1}^{n-1} \|f_i\|_{p_i},$$

$$\int_{\mathbb{R}_+^n} \frac{1}{\max\{x_1^s,\ldots,x_n^s\}} \prod_{i=1}^n x_i^{r_i-\frac{1}{p_i'}} (\mathscr{G}_\alpha f_i)(x_i) d\mathbf{x} \le \frac{s e^{1/\alpha}}{\prod_{i=1}^n r_i} \prod_{i=1}^n \|f_i\|_{p_i},$$

$$\left[\int_{\mathbb{R}_+} x_n^{r_n p_n'-1} \left(\int_{\mathbb{R}_+^{n-1}} \frac{1}{\max\{x_1^s,\ldots,x_n^s\}} \prod_{i=1}^{n-1} x_i^{r_i-\frac{1}{p_i'}} (\mathscr{G}_\alpha f_i)(x_i) \hat{d}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}$$
$$\le \frac{s e^{1/(\alpha p_n')}}{\prod_{i=1}^n r_i} \prod_{i=1}^{n-1} \|f_i\|_{p_i},$$

$$\int_{\mathbb{R}_+^n} \frac{1}{\max\{x_1^s,\ldots,x_n^s\}} \prod_{i=1}^n x_i^{r_i-\frac{1}{p_i'}} (\mathscr{H}_\alpha f_i)(x_i) d\mathbf{x} \le s \prod_{i=1}^n \frac{\alpha+1/p_i}{r_i} \prod_{i=1}^n \|f_i\|_{p_i},$$

and

$$\left[\int_{\mathbb{R}_+} x_n^{r_n p_n'-1} \left(\int_{\mathbb{R}_+^{n-1}} \frac{1}{\max\{x_1^s,\ldots,x_n^s\}} \prod_{i=1}^{n-1} x_i^{r_i-\frac{1}{p_i'}} (\mathscr{H}_\alpha f_i)(x_i) \hat{d}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}$$

$$\leq \frac{s}{\alpha+1/p_n}\prod_{i=1}^{n}\frac{\alpha+1/p_i}{r_i}\prod_{i=1}^{n-1}\|f_i\|_{p_i},$$

where the constants appearing on their right-hand sides are the best possible.

Finally, we propose the following open problem.

Open problem 4 *Find conditions so that the discrete versions of multidimensional inequalities from Section* 5.3 *(with best constants) hold.*

Remark 5.8 The Hilbert-type inequalities involving some mean operators in this chapter, as well as their consequences, are established by authors of this monograph and their collaborators in papers [5], [8], [9], [10], [12], [22], and [60]. For related results and some other forms of Hilbert-type inequalities involving some mean operators, the reader is referred to [11], [13], [20], [38], [39], [42], [71], [75], [85], [86], [87], [92], and [96].

Chapter 6

Hilbert-type Inequalities Involving Differential Operators

In this chapter, we derive several integral, half-discrete and multidimensional Hilbert inequalities with a differential operator, and a general homogeneous kernel. Moreover, we show that the constants appearing on the right-hand sides of these inequalities are the best possible.

Recently, Azar [17, 18], obtained two new forms of half-discrete and integral Hilbert-type inequalities including a *differential operator*. In order to state these results and summarize our further discussion, we start by giving some notation. We denote by $\mathscr{D}_+^n$, $n \geq 0$, a differential operator defined by $\mathscr{D}_+^n f(x) = f^{(n)}(x)$, where $f^{(n)}$ stands for the n-th derivative of a function $f : \mathbb{R}_+ \to \mathbb{R}$. In addition, throughout this chapter, Λ_+^n denotes the set of non-negative measurable functions $f : \mathbb{R}_+ \to \mathbb{R}$ such that $f^{(n)}$ exists a.e. on $\mathbb{R}_+$, $f^{(n)}(x) > 0$, a.e. on $\mathbb{R}_+$, and $f^{(k)}(0) = 0$, $k = 0, 1, 2, \ldots, n-1$.

Now, the above mentioned form of the Hilbert inequality obtained in [18] reads as follows: Let p and q be non-negative mutually conjugate parameters, $p > 1$, let $s > n\max\{p,q\}$, and let $A = \frac{\Gamma(\frac{s}{p}-n)\Gamma(\frac{s}{q}-n)}{\Gamma(s)}$, where Γ is a usual Gamma function. Then the inequality

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^s} dxdy$$
$$< A \left[\int_0^\infty x^{p(n+1)-s-1} \left(\mathscr{D}_+^n f(x) \right)^p dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(n+1)-s-1} \left(\mathscr{D}_+^n g(y) \right)^q dy \right]^{\frac{1}{q}} \tag{6.1}$$

holds for all $f, g \in \Lambda_+^n$, provided that the integrals on its right-hand side converge. In addition, the constant A is the best possible in (6.1). The above inequality may be regarded as a generalization of a classical Hilbert inequality since for $n = 0$, $p = q = 2$, and $s = 1$, we obtain the non-weighted inequality with the previously known sharp constant $A = \pi$.

Now, a differential form of the half-discrete Hilbert inequality derived in [17] can be stated as follows: Let p and q be non-negative conjugate parameters, i.e. $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, let $pm < s \leq q$, where m is a fixed non-negative integer, and let $C = \frac{\Gamma(\frac{s}{p}-m)\Gamma(\frac{s}{q})}{\Gamma(s)}$, where Γ is a usual Gamma function. Then the inequality

$$\int_0^\infty f(x) \sum_{n=1}^\infty \frac{a_n}{(x+n)^s} dx$$
$$< C \left[\int_0^\infty x^{p(m+1)-s-1} \left(\mathscr{D}_+^m f(x) \right)^p dx \right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty n^{q-s-1} a_n^q \right]^{\frac{1}{q}} \tag{6.2}$$

holds for all $f \in \Lambda_+^m$, $f \neq 0$, and for all non-negative sequences $a = (a_n)_{n \in \mathbb{N}}$, $a \neq 0$, provided that the integral and the series on the right-hand side converge. Moreover, the constant C is the best possible in (6.2). Similarly, the above inequality is an extension of a classical half-discrete Hilbert inequality.

6.1 Integral Forms

In this section, we present the extension of inequality (6.1) for the case of an arbitrary homogeneous kernel. The corresponding inequalities will be given in both equivalent forms, as (1.25) and (1.26).

In contrast to the proof of inequality (6.1) (see [18]), the following inequalities will be carried out by virtue of the weighted Hardy inequality. Moreover, we shall also derive appropriate complementary relations, based on the application of the dual weighted Hardy inequality.

It is interesting that the constants appearing in our extended inequalities are also expressed in terms of the Gamma function. Therefore, it is necessary to introduce the concept of rising and falling factorial powers.

The *rising factorial power* $x^{\overline{n}}$, where n is a non-negative integer, also known as a *Pochhammer symbol*, is defined by

$$x^{\overline{n}} = x(x+1)(x+2)\cdots(x+n-1),$$

while the *falling factorial power* $x^{\underline{n}}$ is given by

$$x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1).$$

The rising and falling factorial powers may be expressed in terms of the Gamma function, i.e.

$$x^{\overline{n}} = \frac{\Gamma(x+n)}{\Gamma(x)} \quad \text{and} \quad x^{\underline{n}} = \frac{\Gamma(x+1)}{\Gamma(x-n+1)}.$$

It should be noticed here that the above relations hold for complex arguments of the Gamma function which are not negative integers (for more details, see e.g. [1] or [46]).

With this notation, we are able to state and prove our main result in this section which is an extension of inequality (6.1).

Theorem 6.1 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let α_1^*, α_2^* be real parameters such that $\alpha_1^*, \alpha_2^* \in (n-1, s-1)$ and $\alpha_1^*+\alpha_2^*=s-2$, where n is a fixed non-negative integer and $s>n$. If $K:\mathbb{R}_+^2 \to \mathbb{R}$ is a non-negative measurable homogeneous function of degree $-s$, then the inequalities*

$$\begin{aligned} &\int_0^\infty \int_0^\infty K(x,y)f(x)g(y)dxdy \\ &\quad < M\left[\int_0^\infty x^{p(n-\alpha_1^*)-1}\left(\mathscr{D}_+^n f(x)\right)^p dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(n-\alpha_2^*)-1}\left(\mathscr{D}_+^n g(y)\right)^q dy\right]^{\frac{1}{q}} \end{aligned} \tag{6.3}$$

and

$$\begin{aligned} &\left[\int_0^\infty y^{(p-1)(1+q\alpha_2^*)}\left(\int_0^\infty K(x,y)f(x)dx\right)^p dy\right]^{\frac{1}{p}} \\ &\quad < m\left[\int_0^\infty x^{p(n-\alpha_1^*)-1}\left(\mathscr{D}_+^n f(x)\right)^p dx\right]^{\frac{1}{p}} \end{aligned} \tag{6.4}$$

hold for all non-negative functions $f,g \in \Lambda_+^n$. In addition, the constants $M = k_1(-\alpha_2^*)\frac{\Gamma(\alpha_1^*-n+1)\Gamma(\alpha_2^*-n+1)}{\Gamma(\alpha_1^*+1)\Gamma(\alpha_2^*+1)}$ and $m = k_1(-\alpha_2^*)\frac{\Gamma(\alpha_1^*-n+1)}{\Gamma(\alpha_1^*+1)}$ are the best possible in the corresponding inequalities.

Proof. Obviously, if $n=0$ inequalities (6.3) and (6.4) become respectively (1.25) and (1.26). Now, our first step is to rewrite the right-hand side of inequality (1.25) with $-qA_1 = \alpha_1^*, -pA_2 = \alpha_2^*$ in a form that is more suitable for the application of the Hardy inequality (1.65). Namely, since

$$x\mathscr{A}(\mathscr{D}_+ f)(x) = \int_0^x f'(t)dt = f(x)-f(0) = f(x),$$

we have that

$$
\begin{aligned}
k_1(-\alpha_2^*)&\left[\int_0^\infty x^{-p\alpha_1^*-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{-q\alpha_2^*-1}g^q(y)dy\right]^{\frac{1}{q}}\\
&=k_1(-\alpha_2^*)\left[\int_0^\infty x^{p-(p\alpha_1^*+1)}(\mathscr{A}(\mathscr{D}_+f)(x))^p dx\right]^{\frac{1}{p}}\\
&\times\left[\int_0^\infty y^{q-(q\alpha_2^*+1)}(\mathscr{A}(\mathscr{D}_+g)(y))^q dy\right]^{\frac{1}{q}}.
\end{aligned}
\tag{6.5}
$$

Moreover, due to the weighted Hardy inequality, it follows that

$$
\left[\int_0^\infty x^{p-(p\alpha_1^*+1)}(\mathscr{A}(\mathscr{D}_+f)(x))^p dx\right]^{\frac{1}{p}}<\frac{1}{\alpha_1^*}\left[\int_0^\infty x^{p(1-\alpha_1^*)-1}(\mathscr{D}_+f(x))^p dx\right]^{\frac{1}{p}}
$$

and

$$
\left[\int_0^\infty y^{q-(q\alpha_2^*+1)}(\mathscr{A}(\mathscr{D}_+g)(y))^q dy\right]^{\frac{1}{q}}<\frac{1}{\alpha_2^*}\left[\int_0^\infty y^{q(1-\alpha_2^*)-1}(\mathscr{D}_+g(y))^q dy\right]^{\frac{1}{q}}.
$$

In addition, applying the weighted Hardy inequality to the right-hand sides of the last two inequalities $n-1$ times, yields relations

$$
\left[\int_0^\infty x^{p-(p\alpha_1^*+1)}(\mathscr{A}(\mathscr{D}_+f)(x))^p dx\right]^{\frac{1}{p}}<\frac{1}{\alpha_1^{*\underline{n}}}\left[\int_0^\infty x^{p(n-\alpha_1^*)-1}(\mathscr{D}_+^n f(x))^p dx\right]^{\frac{1}{p}}
\tag{6.6}
$$

and

$$
\left[\int_0^\infty y^{q-(q\alpha_2^*+1)}(\mathscr{A}(\mathscr{D}_+g)(y))^q dy\right]^{\frac{1}{q}}<\frac{1}{\alpha_2^{*\underline{n}}}\left[\int_0^\infty y^{q(n-\alpha_2^*)-1}(\mathscr{D}_+^n g(y))^q dy\right]^{\frac{1}{q}}.
\tag{6.7}
$$

Finally, since $\alpha_1^{*\underline{n}}=\frac{\Gamma(\alpha_1^*+1)}{\Gamma(\alpha_1^*-n+1)}$ and $\alpha_2^{*\underline{n}}=\frac{\Gamma(\alpha_2^*+1)}{\Gamma(\alpha_2^*-n+1)}$, the inequality (6.3) holds due to (1.25), (6.5), (6.6), and (6.7). In the same way the inequality (6.4) holds by virtue of (1.26) and (6.6).

The next step is to prove that the constants M and m, appearing on the right-hand sides of the inequalities (6.3) and (6.4), are the best possible. For this reason, suppose that there exists a positive constant C smaller than M such that the inequality

$$
\begin{aligned}
&\int_0^\infty\int_0^\infty K(x,y)f(x)g(y)dxdy\\
&<C\left[\int_0^\infty x^{p(n-\alpha_1^*)-1}\left(\mathscr{D}_+^n f(x)\right)^p dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(n-\alpha_2^*)-1}\left(\mathscr{D}_+^n g(y)\right)^q dy\right]^{\frac{1}{q}}
\end{aligned}
\tag{6.8}
$$

holds for all non-negative functions $f,g:\mathbb{R}_+\to\mathbb{R}$ fulfilling conditions as in the statement of the Theorem.

Considering the above inequality with functions $\widetilde{f},\widetilde{g}:\mathbb{R}_+\to\mathbb{R}$ defined by

$$\widetilde{f}(x)=\begin{cases}0, & 0<x<1\\ \frac{\Gamma\left(1+\alpha_1^*-\frac{\varepsilon}{p}-n\right)}{\Gamma\left(1+\alpha_1^*-\frac{\varepsilon}{p}\right)}x^{\alpha_1^*-\frac{\varepsilon}{p}}, & x\geq 1\end{cases},$$

$$\widetilde{g}(y)=\begin{cases}0, & 0<y<1\\ \frac{\Gamma\left(1+\alpha_2^*-\frac{\varepsilon}{q}-n\right)}{\Gamma\left(1+\alpha_2^*-\frac{\varepsilon}{q}\right)}y^{\alpha_2^*-\frac{\varepsilon}{q}}, & y\geq 1\end{cases},$$

where $\varepsilon>0$ is a sufficiently small number, the Fubini theorem and the change of variables $t=\frac{y}{x}$ imply that

$$\begin{aligned}
&\int_0^\infty\int_0^\infty K(x,y)\widetilde{f}(x)\widetilde{g}(y)dxdy\\
&=\varphi(\varepsilon)\int_1^\infty\int_1^\infty K(x,y)x^{\alpha_1^*-\frac{\varepsilon}{p}}y^{\alpha_2^*-\frac{\varepsilon}{q}}dxdy\\
&=\varphi(\varepsilon)\int_1^\infty x^{-\varepsilon-1}\int_{\frac{1}{x}}^\infty K(1,t)t^{\alpha_2^*-\frac{\varepsilon}{q}}dtdx\\
&=\frac{\varphi(\varepsilon)}{\varepsilon}\int_1^\infty K(1,t)t^{\alpha_2^*-\frac{\varepsilon}{q}}dt+\varphi(\varepsilon)\int_1^\infty x^{-\varepsilon-1}\int_{\frac{1}{x}}^1 K(1,t)t^{\alpha_2^*-\frac{\varepsilon}{q}}dtdx\\
&=\frac{\varphi(\varepsilon)}{\varepsilon}\int_1^\infty K(1,t)t^{\alpha_2^*-\frac{\varepsilon}{q}}dt+\varphi(\varepsilon)\int_0^1 K(1,t)t^{\alpha_2^*-\frac{\varepsilon}{q}}\int_{\frac{1}{t}}^\infty x^{-\varepsilon-1}dxdt\\
&=\frac{\varphi(\varepsilon)}{\varepsilon}\left(\int_1^\infty K(1,t)t^{\alpha_2^*-\frac{\varepsilon}{q}}dt+\int_0^1 K(1,t)t^{\alpha_2^*+\frac{\varepsilon}{p}}dt\right),
\end{aligned}\tag{6.9}$$

where $\varphi(\varepsilon)=\frac{\Gamma\left(1+\alpha_1^*-\frac{\varepsilon}{p}-n\right)\Gamma\left(1+\alpha_2^*-\frac{\varepsilon}{q}-n\right)}{\Gamma\left(1+\alpha_1^*-\frac{\varepsilon}{p}\right)\Gamma\left(1+\alpha_2^*-\frac{\varepsilon}{q}\right)}$. On the other hand, since the n-th derivative of the function $x^{\alpha_1^*-\frac{\varepsilon}{p}}$ is equal to $\frac{\Gamma\left(1+\alpha_1^*-\frac{\varepsilon}{p}\right)}{\Gamma\left(1+\alpha_1^*-\frac{\varepsilon}{p}-n\right)}x^{\alpha_1^*-\frac{\varepsilon}{p}-n}$, it follows that

$$\mathscr{D}_+^n\widetilde{f}(x)=\begin{cases}0, & 0<x<1\\ x^{\alpha_1^*-\frac{\varepsilon}{p}-n}, & x>1\end{cases},\quad \mathscr{D}_+^n\widetilde{g}(y)=\begin{cases}0, & 0<y<1\\ y^{\alpha_2^*-\frac{\varepsilon}{q}-n}, & y>1\end{cases},$$

and the right-hand side of (6.8) reduces to

$$C\left[\int_0^\infty x^{p(n-\alpha_1^*)-1}\left(\mathscr{D}_+^n\widetilde{f}(x)\right)^p dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(n-\alpha_2^*)-1}\left(\mathscr{D}_+^n\widetilde{g}(y)\right)^q dy\right]^{\frac{1}{q}}=\frac{C}{\varepsilon}.\tag{6.10}$$

Now, multiplying both sides of relation (6.8) by ε, and taking into account relations (6.9) and (6.10), we have that

$$\varphi(\varepsilon)\left(\int_1^\infty K(1,t)t^{\alpha_2^*-\frac{\varepsilon}{q}}dt+\int_0^1 K(1,t)t^{\alpha_2^*+\frac{\varepsilon}{p}}dt\right)<C.$$

Finally, as $\varepsilon \to 0$, it follows that $M \leq C$, which is in contrast to our hypothesis. Therefore, the constant M is the best possible in (6.3).

It remains to show that m is the best constant in (6.4). Similarly to above discussion, suppose that there exists a positive constant c smaller than m such that inequality

$$\left[\int_0^\infty y^{(p-1)(1+q\alpha_2^*)}\left(\int_0^\infty K(x,y)f(x)dx\right)^p dy\right]^{\frac{1}{p}} < c\left[\int_0^\infty x^{p(n-\alpha_1^*)-1}\left(\mathscr{D}_+^n f(x)\right)^p dx\right]^{\frac{1}{p}}$$

holds for all non-negative functions $f : \mathbb{R}_+ \to \mathbb{R}$ as in the statement of Theorem. Then, utilizing the Hölder inequality and relation (6.7), we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty K(x,y)f(x)g(y)dxdy\\
&= \int_0^\infty \left[y^{\frac{q\alpha_2^*+1}{q}} \int_0^\infty K(x,y)f(x)dx\right] \cdot [y^{-\frac{q\alpha_2^*+1}{q}} g(y)]dy\\
&\leq \left[\int_0^\infty y^{(p-1)(1+q\alpha_2^*)}\left(\int_0^\infty K(x,y)f(x)dx\right)^p dy\right]^{\frac{1}{p}} \left[\int_0^\infty y^{-q\alpha_2^*-1} g^q(y)dy\right]^{\frac{1}{q}}\\
&< c\frac{\Gamma(\alpha_2^*-n+1)}{\Gamma\left(\alpha_2^*+1\right)}\left[\int_0^\infty x^{p(n-\alpha_1^*)-1}\left(\mathscr{D}_+^n f(x)\right)^p dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(n-\alpha_2^*)-1}\left(\mathscr{D}_+^n g(y)\right)^q dy\right]^{\frac{1}{q}},
\end{aligned}$$

which results that the constant M is not the best possible in (6.3), since

$$c\frac{\Gamma(\alpha_2^*-n+1)}{\Gamma\left(\alpha_2^*+1\right)} < m\frac{\Gamma(\alpha_2^*-n+1)}{\Gamma\left(\alpha_2^*+1\right)} = M.$$

With this contradiction, the proof is completed. □

Remark 6.1 Since for $n = 0$ inequalities (6.3) and (6.4) reduce respectively to (1.25) and (1.26), Theorem 6.1 may be regarded as an extension of relations (1.25) and (1.26). However, if $n \geq 1$, the relations (6.3) and (6.4) are less precise than (1.25) and (1.26), since the right-hand sides of (1.25) and (1.26) interpolate between the left-hand side and the right-hand side of inequalities (6.3) and (6.4).

Observe that the Theorem 6.1 covers the case when the degree of homogeneity of the kernel, i.e. $-s$ is less than $-n$, for a fixed non-negative integer n. Our next intention is to derive the corresponding relations that cover the case $0 < s \leq 1$. Such result is in some way complementary to Theorem 6.1 and it may be derived by virtue of the weighted dual Hardy inequality (1.66).

In order to state the next result, we define a differential operator $\mathscr{D}_\pm^n$ by

$$\mathscr{D}_\pm^n f(x) = (-1)^n f^{(n)}(x),$$

where n is a non-negative integer. Moreover, the following theorem holds for all non-negative functions $f:\mathbb{R}_+\to\mathbb{R}$ such that the n-th derivative $f^{(n)}$ exists a.e. on $\mathbb{R}_+$, $\mathscr{D}^n_\pm f(x)>0$, a.e. on $\mathbb{R}_+$, and $\lim_{x\to\infty} f^{(k)}(x)=0$ for $k=0,1,2,\ldots,n-1$. This set of functions will be denoted by $\Lambda^n_\pm$.

Theorem 6.2 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let α_1^*, α_2^* be real parameters such that $\alpha_1^*,\alpha_2^*\in(-1,s-1)$ and $\alpha_1^*+\alpha_2^*=s-2$, where $0<s\le 1$. If $K:\mathbb{R}^2_+\to\mathbb{R}$ is a non-negative homogeneous function of degree $-s$, then the inequalities*

$$\begin{aligned}&\int_0^\infty\int_0^\infty K(x,y)f(x)g(y)dxdy\\&\quad< M^*\left[\int_{\mathbb{R}_+} x^{p(n-\alpha_1^*)-1}\left(\mathscr{D}^n_\pm f(x)\right)^p dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(n-\alpha_2^*)-1}\left(\mathscr{D}^n_\pm g(y)\right)^q dy\right]^{\frac{1}{q}}\end{aligned}\tag{6.11}$$

and

$$\begin{aligned}&\left[\int_0^\infty y^{(p-1)(1+q\alpha_2^*)}\left(\int_0^\infty K(x,y)f(x)dx\right)^p dy\right]^{\frac{1}{p}}\\&\quad< m^*\left[\int_0^\infty x^{p(n-\alpha_1^*)-1}\left(\mathscr{D}^n_\pm f(x)\right)^p dx\right]^{\frac{1}{p}}\end{aligned}\tag{6.12}$$

hold for all non-negative functions $f,g\in\Lambda^n_\pm$, where n is a fixed non-negative integer. In addition, the constants $M^*=k_1(-\alpha_2^*)\frac{\Gamma(-\alpha_1^*)\Gamma(-\alpha_2^*)}{\Gamma(n-\alpha_1^*)\Gamma(n-\alpha_2^*)}$ and $m^*=k_1(-\alpha_2^*)\frac{\Gamma(-\alpha_1^*)}{\Gamma(n-\alpha_1^*)}$, appearing in (6.11) *and* (6.12)*, are the best possible.*

Proof. We follow the lines as in the proof of Theorem 6.1, this time accompanied with the dual Hardy inequality (1.66). In this setting, the right-hand side of inequality (1.25) with $-qA_1=\alpha_1^*,-pA_2=\alpha_2^*$ may be rewritten as

$$\begin{aligned}&k_1(-\alpha_2^*)\left[\int_0^\infty x^{-p\alpha_1^*-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{-q\alpha_2^*-1}g^q(y)dy\right]^{\frac{1}{q}}\\&=k_1(-\alpha_2^*)\left[\int_0^\infty x^{p-(p\alpha_1^*+1)}(\mathscr{A}^*(\mathscr{D}_\pm f)(x))^p dx\right]^{\frac{1}{p}}\\&\quad\times\left[\int_0^\infty y^{q-(q\alpha_2^*+1)}(\mathscr{A}^*(\mathscr{D}_\pm g)(y))^q dy\right]^{\frac{1}{q}},\end{aligned}\tag{6.13}$$

since

$$x\mathscr{A}^*(\mathscr{D}_\pm f)(x)=-\int_x^\infty f'(t)dt=f(x).$$

Moreover, by applying the dual Hardy inequality to the expressions on right-hand side of relation (6.13) n times, it follows that

$$\begin{aligned}&\left[\int_0^\infty x^{p-(p\alpha_1^*+1)}(\mathscr{A}^*(\mathscr{D}_\pm f)(x))^p dx\right]^{\frac{1}{p}}\\&\quad< \frac{1}{(-\alpha_1^*)^{\overline{n}}}\left[\int_0^\infty x^{p(n-\alpha_1^*)-1}(\mathscr{D}_\pm^n f(x))^p dx\right]^{\frac{1}{p}}\end{aligned} \tag{6.14}$$

and

$$\begin{aligned}&\left[\int_0^\infty y^{q-(q\alpha_2^*+1)}(\mathscr{A}^*(\mathscr{D}_\pm g)(y))^q dy\right]^{\frac{1}{q}}\\&\quad< \frac{1}{(-\alpha_2^*)^{\overline{n}}}\left[\int_0^\infty y^{q(n-\alpha_2^*)-1}(\mathscr{D}_\pm^n g(y))^q dy\right]^{\frac{1}{q}}.\end{aligned} \tag{6.15}$$

Now, since $(-\alpha_1^*)^{\overline{n}} = \frac{\Gamma(n-\alpha_1^*)}{\Gamma(-\alpha_1^*)}$ and $(-\alpha_2^*)^{\overline{n}} = \frac{\Gamma(n-\alpha_2^*)}{\Gamma(-\alpha_2^*)}$, the inequality (6.11) holds due to (1.25), (6.13), (6.14), and (6.15). In addition, inequality (6.12) holds by virtue of (1.26) and (6.14).

In order to show that M^* is the best constant in (6.11), we suppose that there exists a positive constant C^* smaller than M^* such that the inequality

$$\begin{aligned}&\int_0^\infty\int_0^\infty K(x,y)f(x)g(y)dxdy\\&\quad< C^*\left[\int_0^\infty x^{p(n-\alpha_1^*)-1}\left(\mathscr{D}_\pm^n f(x)\right)^p dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(n-\alpha_2^*)-1}\left(\mathscr{D}_\pm^n g(y)\right)^q dy\right]^{\frac{1}{q}}\end{aligned} \tag{6.16}$$

holds for all non-negative functions $f,g\in\Lambda_\pm^n$.

Similarly to the proof of Theorem 6.1, we consider the above inequality with the appropriate choice of functions f and g. It is easy to see that the functions $\widetilde{f}^*,\widetilde{g}^*:\mathbb{R}_+\to\mathbb{R}$, defined by

$$\widetilde{f}^*(x) = \begin{cases}0, & 0<x<1\\ \frac{\Gamma\left(-\alpha_1^*+\frac{\varepsilon}{p}\right)}{\Gamma\left(n-\alpha_1^*+\frac{\varepsilon}{p}\right)}x^{\alpha_1^*-\frac{\varepsilon}{p}}, & x\geq 1\end{cases},$$

$$\widetilde{g}^*(y) = \begin{cases}0, & 0<y<1\\ \frac{\Gamma\left(-\alpha_2^*+\frac{\varepsilon}{q}\right)}{\Gamma\left(n-\alpha_2^*+\frac{\varepsilon}{q}\right)}y^{\alpha_2^*-\frac{\varepsilon}{q}}, & y\geq 1\end{cases},$$

$\varepsilon>0$, belong to $\Lambda_\pm^n$. With regard to functions $\widetilde{f}^*,\widetilde{g}^*$, the left-hand side of (6.16) may be rewritten as

$$\begin{aligned}&\int_0^\infty\int_0^\infty K(x,y)\widetilde{f}^*(x)\widetilde{g}^*(y)dxdy\\&\quad=\frac{\varphi^*(\varepsilon)}{\varepsilon}\left(\int_1^\infty K(1,t)t^{\alpha_2^*-\frac{\varepsilon}{q}}dt+\int_0^1 K(1,t)t^{\alpha_2^*+\frac{\varepsilon}{p}}dt\right),\end{aligned} \tag{6.17}$$

where $\varphi^*(\varepsilon) = \frac{\Gamma\left(-\alpha_1^*+\frac{\varepsilon}{p}\right)\Gamma\left(-\alpha_2^*+\frac{\varepsilon}{q}\right)}{\Gamma\left(n-\alpha_1^*+\frac{\varepsilon}{p}\right)\Gamma\left(n-\alpha_2^*+\frac{\varepsilon}{q}\right)}$. Clearly, this follows immediately from relation (6.9).

On the other hand, since the n-th derivative of the function $x^{\alpha_1^*-\frac{\varepsilon}{p}}$ is equal to $(-1)^n \frac{\Gamma\left(n-\alpha_1^*+\frac{\varepsilon}{p}\right)}{\Gamma\left(-\alpha_1^*+\frac{\varepsilon}{p}\right)} x^{\alpha_1^*-\frac{\varepsilon}{p}-n}$, it follows that

$$\mathscr{D}_\pm^n \widetilde{f}^*(x) = \begin{cases} 0, & 0<x<1 \\ x^{\alpha_1^*-\frac{\varepsilon}{p}-n}, & x>1 \end{cases}, \quad \mathscr{D}_\pm^n \widetilde{g}^*(y) = \begin{cases} 0, & 0<y<1 \\ y^{\alpha_2^*-\frac{\varepsilon}{q}-n}, & y>1 \end{cases},$$

which means that the right-hand side of inequality (6.16) reads

$$C^* \left[\int_0^\infty x^{p(n-\alpha_1^*)-1} \left(\mathscr{D}_\pm^n \widetilde{f}^*(x)\right)^p dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(n-\alpha_2^*)-1} \left(\mathscr{D}_\pm^n \widetilde{g}^*(y)\right)^q dy\right]^{\frac{1}{q}} = \frac{C^*}{\varepsilon}. \quad (6.18)$$

Consequently, comparing (6.16), (6.17), and (6.18), it follows that

$$\varphi^*(\varepsilon)\left(\int_1^\infty K(1,t)t^{\alpha_2^*-\frac{\varepsilon}{q}}dt + \int_0^1 K(1,t)t^{\alpha_2^*+\frac{\varepsilon}{p}}dt\right) < C^*.$$

Therefore, as $\varepsilon \to 0$, it follows that $M^* \leq C^*$, which contradicts with our assumption. This means that the constant M^* is the best possible in (6.11).

To conclude the proof, we suppose that, contrary to our claim, there exists a constant $0 < c^* < m^*$ such that the inequality

$$\left[\int_0^\infty y^{(p-1)(1+q\alpha_2^*)} \left(\int_0^\infty K(x,y)f(x)dx\right)^p dy\right]^{\frac{1}{p}} < c^* \left[\int_0^\infty x^{p(n-\alpha_1^*)-1} \left(\mathscr{D}_+^n f(x)\right)^p dx\right]^{\frac{1}{p}}$$

holds for all non-negative functions $f \in \Lambda_\pm^n$, as in the statement of Theorem. In addition, employing the Hölder inequality as well as relation (6.15), we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty K(x,y)f(x)g(y)dxdy \\
&= \int_0^\infty \left[y^{\frac{q\alpha_2^*+1}{q}} \int_0^\infty K(x,y)f(x)dx\right] \cdot [y^{-\frac{q\alpha_2^*+1}{q}} g(y)]dy \\
&\leq \left[\int_0^\infty y^{(p-1)(1+q\alpha_2^*)} \left(\int_0^\infty K(x,y)f(x)dx\right)^p dy\right]^{\frac{1}{p}} \left[\int_0^\infty y^{-q\alpha_2^*-1} g^q(y)dy\right]^{\frac{1}{q}} \\
&< c^* \frac{\Gamma(-\alpha_2^*)}{\Gamma(n-\alpha_2^*)} \left[\int_0^\infty x^{p(n-\alpha_1^*)-1} \left(\mathscr{D}_\pm^n f(x)\right)^p dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(n-\alpha_2^*)-1} \left(\mathscr{D}_\pm^n g(y)\right)^q dy\right]^{\frac{1}{q}}.
\end{aligned}$$

Now, according to our assumption, it follows that $c^* \frac{\Gamma\left(-\alpha_2^*\right)}{\Gamma\left(n-\alpha_2^*\right)} < m^* \frac{\Gamma\left(-\alpha_2^*\right)}{\Gamma\left(n-\alpha_2^*\right)} = M^*$, which means that M^* is not the best constant in (6.11). This is a clear contradiction of our assumption and the proof is completed. $\square$

Remark 6.2 It should be noticed here that Theorem 6.2 may also be regarded as an extension of inequalities (1.25) and (1.26). Similarly to Remark 6.1, the relations (6.11) and (6.12), for $n \geq 1$, are less precise than (1.25) and (1.26), since the right-hand sides of (1.25) and (1.26) interpolate between the left-hand side and the right-hand side of inequalities (6.11) and (6.12).

6.1.1 Applications

In this subsection, we discuss our main results with regard to some particular choices of kernels and parameters α_1^* and α_2^*.

First example

Our first example refers to the homogeneous kernel $K(x,y) = (x+y)^{-s}$, $s > 0$, with a degree of homogeneity $-s$, and in this case the constant $k_1(-\alpha_2^*)$, appearing in inequalities (6.3), (6.4), (6.11), and (6.12) is expressed in terms of the Beta function. More precisely, we have

$$k_1(-\alpha_2^*) = \int_0^\infty (1+t)^{-s} t^{\alpha_2^*} dt = B(1+\alpha_2^*, s-1-\alpha_2^*) = B(\alpha_1^*+1, \alpha_2^*+1),$$

since $\alpha_1^* + \alpha_2^* = s-2$. Moreover, employing the well-known relationship between the Beta and the Gamma function, i.e. the formula $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, the constants M and m appearing in (6.3) and (6.4) (denoted here by M_1 and m_1, respectively) reduce to

$$\begin{aligned} M_1 &= \frac{\Gamma(\alpha_1^*-n+1)\Gamma(\alpha_2^*-n+1)}{\Gamma(s)} \\ m_1 &= \frac{\Gamma(\alpha_1^*-n+1)\Gamma(\alpha_2^*+1)}{\Gamma(s)}, \end{aligned}$$

where $\alpha_1^*, \alpha_2^* \in (n-1, s-1)$ and $s > n$. Now, considering the parameters $\alpha_1^* = \frac{s}{p} - 1$ and $\alpha_2^* = \frac{s}{q} - 1$, where $s > n\max\{p,q\}$, the above constants reduce respectively to $A = \frac{\Gamma\left(\frac{s}{p}-n\right)\Gamma\left(\frac{s}{q}-n\right)}{\Gamma(s)}$ and $a = \frac{\Gamma\left(\frac{s}{p}-n\right)\Gamma\left(\frac{s}{q}\right)}{\Gamma(s)}$. The constant A provides inequality (6.1), while its equivalent form asserts that

$$\left[\int_0^\infty y^{ps-s-1}\left(\int_0^\infty \frac{f(x)}{(x+y)^s}dx\right)^p dy\right]^{\frac{1}{p}} < a\left[\int_0^\infty x^{p(n+1)-s-1}\left(\mathscr{D}_+^n f(x)\right)^p dx\right]^{\frac{1}{p}} \tag{6.19}$$

holds for all non-negative functions $f \in \Lambda_+^n$.

On the other hand, the constants M^* and m^* appearing in dual inequalities (6.11) and (6.12) (denoted here by M_1^* and m_1^*, respectively) accompanied with the kernel $K(x,y) = (x+y)^{-s}$, become

$$\begin{aligned} M_1^* &= \frac{\pi^2}{\sin(\alpha_1^*\pi)\sin(\alpha_2^*\pi)} \cdot \frac{1}{\Gamma(s)\Gamma(n-\alpha_1^*)\Gamma(n-\alpha_2^*)} \\ m_1^* &= -\frac{\pi}{\sin(\alpha_1^*\pi)} \cdot \frac{\Gamma(\alpha_2^*+1)}{\Gamma(s)\Gamma(n-\alpha_1^*)}, \quad \alpha_1^*, \alpha_2^* \in (-1, s-1), 0 < s \leq 1, \end{aligned}$$

after applying the Euler reflection formula $\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin x\pi}$. In addition, with parameters $\alpha_1^*=\frac{s}{p}-1$ and $\alpha_2^*=\frac{s}{q}-1$, and this time with condition $s<\min\{p,q\}$, Theorem 6.2 yields dual forms of inequalities (6.1) and (6.19).

Corollary 6.1 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $s<\min\{p,q\}$. Then the inequalities*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^s}dxdy$$
$$< A^*\left[\int_{\mathbb{R}_+} x^{p(n+1)-s-1}\left(\mathscr{D}_\pm^n f(x)\right)^p dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(n+1)-s-1}\left(\mathscr{D}_\pm^n g(y)\right)^q dy\right]^{\frac{1}{q}} \tag{6.20}$$

and

$$\left[\int_0^\infty y^{ps-s-1}\left(\int_0^\infty \frac{f(x)}{(x+y)^s}dx\right)^p dy\right]^{\frac{1}{p}} < a^*\left[\int_0^\infty x^{p(n+1)-s-1}\left(\mathscr{D}_\pm^n f(x)\right)^p dx\right]^{\frac{1}{p}} \tag{6.21}$$

hold for all non-negative functions $f,g\in\Lambda_\pm^n$, where n is a non-negative integer. Moreover, the constants $A^*=\frac{\pi^2}{\sin(\frac{s\pi}{p})\sin(\frac{s\pi}{q})}\cdot\frac{1}{\Gamma(s)\Gamma(n+1-\frac{s}{p})\Gamma(n+1-\frac{s}{q})}$ and $a^*=\frac{\pi}{\sin(\frac{s\pi}{p})}\cdot\frac{\Gamma(\frac{s}{q})}{\Gamma(s)\Gamma(n+1-\frac{s}{p})}$ appearing in (6.20) *and* (6.21) *are the best possible.*

Second example

For the function $K:\mathbb{R}_+^2\to\mathbb{R}$ given by $K(x,y)=\max\{x,y\}^{-s}$, $s>0$, we have

$$k_1(-\alpha_2^*)=\int_0^\infty \max\{1,t\}^{-s}t^{\alpha_2^*}=\frac{s}{(\alpha_2^*+1)(s-\alpha_2^*-1)}$$
$$=\frac{s}{(\alpha_1^*+1)(\alpha_2^*+1)},\quad \alpha_1^*,\alpha_2^*\in(-1,s-1),$$

since $\alpha_1^*+\alpha_2^*=s-2$.

This time, the constants M and m on the right-hand sides of (6.3) and (6.4) (denoted here by M_2 and m_2, respectively) read

$$M_2=s\cdot\frac{\Gamma(\alpha_1^*-n+1)\Gamma(\alpha_2^*-n+1)}{\Gamma\left(\alpha_1^*+2\right)\Gamma\left(\alpha_2^*+2\right)}$$
$$m_2=\frac{s}{\alpha_2^*+1}\cdot\frac{\Gamma(\alpha_1^*-n+1)}{\Gamma\left(\alpha_1^*+2\right)},\quad \alpha_1^*,\alpha_2^*\in(n-1,s-1),s>n,$$

since $\Gamma(x+1)=x\Gamma(x)$. In this setting, dual inequalities (6.11) and (6.12) include the constants

$$M_2^*=\frac{s}{(\alpha_1^*+1)(\alpha_2^*+1)}\cdot\frac{\Gamma(-\alpha_1^*)\Gamma(-\alpha_2^*)}{\Gamma\left(n-\alpha_1^*\right)\Gamma\left(n-\alpha_2^*\right)}$$
$$m_2^*=\frac{s}{(\alpha_1^*+1)(\alpha_2^*+1)}\cdot\frac{\Gamma(-\alpha_1^*)}{\Gamma\left(n-\alpha_1^*\right)},\quad \alpha_1^*,\alpha_2^*\in(-1,s-1),0<s\le 1.$$

Third example

To conclude this section, we also consider the kernel $K:\mathbb{R}_+^2 \to \mathbb{R}$ given by $K(x,y) = \frac{\log y - \log y}{y-x}$. Evidently, it is homogeneous of degree -1, $k_1(-\alpha_2^*)$ converges for all $\alpha_2^* \in (-1,0)$ and

$$k_1(-\alpha_2^*) = \int_0^\infty \frac{\log t}{t-1} t^{\alpha_2^*} dt = \frac{\pi^2}{\sin^2 \alpha_2^* \pi}$$

(for more details, see [1] and [46]). Since Theorem 6.1 refers to homogeneous kernels with $s > n$, it can not be applied to the above kernel for the case when $n \geq 1$. On the other hand, the corresponding dual result follows directly from Theorem 6.2:

Corollary 6.2 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $\alpha_1^*, \alpha_2^* \in (-1,0)$ be real parameters such that $\alpha_1^*+\alpha_2^* = -1$. Then the inequalities*

$$\begin{aligned}&\int_0^\infty \int_0^\infty \frac{\log y - \log y}{y-x} f(x)g(y)dxdy \\ &< M_3^* \left[\int_{\mathbb{R}_+} x^{p(n-\alpha_1^*)-1} \left(\mathscr{D}_\pm^n f(x)\right)^p dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(n-\alpha_2^*)-1} \left(\mathscr{D}_\pm^n g(y)\right)^q dy\right]^{\frac{1}{q}}\end{aligned} \tag{6.22}$$

and

$$\begin{aligned}&\left[\int_0^\infty y^{(p-1)(1+q\alpha_2^*)} \left(\int_0^\infty \frac{\log y - \log y}{y-x} f(x)dx\right)^p dy\right]^{\frac{1}{p}} \\ &< m_3^* \left[\int_0^\infty x^{p(n-\alpha_1^*)-1} \left(\mathscr{D}_\pm^n f(x)\right)^p dx\right]^{\frac{1}{p}}\end{aligned} \tag{6.23}$$

hold for all non-negative functions $f,g \in \Lambda_\pm^n$, where n is a non-negative integer. In addition, the constants $M_3^* = -\frac{\pi^3}{\sin^3 \alpha_2^* \pi} \cdot \frac{1}{\Gamma(n-\alpha_1^*)\Gamma(n-\alpha_2^*)}$ and $m_3^* = \frac{\pi^2}{\sin^2 \alpha_2^* \pi} \cdot \frac{\Gamma(-\alpha_1^*)}{\Gamma(n-\alpha_1^*)}$ are the best possible.

6.2 Associated Half-discrete Forms

In this section we first give an extension of inequality (6.2) to the case of non-conjugate exponents and a general homogeneous kernel.

6.2.1 Half-discrete Inequalities in the Non-conjugate Case

Having in mind relations (2.75) and (2.76), our results will be given in two equivalent forms.

Theorem 6.3 *Let p, q, p', q', and λ be as in* (1.43) *and* (1.44), *and let $K : \mathbb{R}_+^2 \to \mathbb{R}$ be a non-negative measurable homogeneous function of degree $-s, s > 0$. If A_1 and A_2 are real parameters such that the function $K(x,y)y^{-q'A_2}$ is decreasing on $\mathbb{R}_+$ for any fixed $x \in \mathbb{R}_+$ and $\beta := \frac{1}{q'}(s-1) + A_2 - A_1 - \frac{1}{p} > m-1$, where m is a fixed non-negative integer, then the inequalities*

$$\begin{aligned}\sum_{n=1}^{\infty} a_n \int_0^{\infty} K^{\lambda}(x,n) f(x)dx &= \int_0^{\infty} f(x) \left(\sum_{n=1}^{\infty} K^{\lambda}(x,n) a_n \right) dx \\ &< L \cdot \frac{\Gamma(\beta - m + 1)}{\Gamma(\beta + 1)} \left[\int_0^{\infty} x^{p(m-\beta)-1} \left(\mathscr{D}_+^m f(x) \right)^p dx \right]^{\frac{1}{p}} \\ &\times \left[\sum_{n=1}^{\infty} n^{\frac{q}{p'}(1-s)+q(A_2-A_1)} a_n^q \right]^{\frac{1}{q}}\end{aligned} \tag{6.24}$$

and

$$\begin{aligned}&\left[\sum_{n=1}^{\infty} n^{\frac{q'}{p'}(s-1)+q'(A_1-A_2)} \left(\int_0^{\infty} K^{\lambda}(x,n) f(x)dx \right)^{q'} \right]^{\frac{1}{q'}} \\ &\quad < L \cdot \frac{\Gamma(\beta - m + 1)}{\Gamma(\beta + 1)} \left[\int_0^{\infty} x^{p(m-\beta)-1} \left(\mathscr{D}_+^m f(x) \right)^p dx \right]^{\frac{1}{p}},\end{aligned} \tag{6.25}$$

where $0 < L := k_1^{\frac{1}{q'}}(q'A_2) k_1^{\frac{1}{p'}}(2 - s - p'A_1) < \infty$, hold for a non-negative function $f \in \Lambda_+^m$ and a non-negative sequence $a = (a_n)_{n \in \mathbb{N}}$, provided that the integral and series on their right-hand sides converge to positive numbers.

Proof. Clearly, if $m = 0$, inequalities (6.24) and (6.25) coincide with (2.75) and (2.76) respectively. Otherwise, rewrite the right-hand side of (2.75) in a form that is more suitable for the application of the Hardy inequality. Namely, since

$$x\mathscr{A}(\mathscr{D}_+ f)(x) = \int_0^x f'(t)dt = f(x) - f(0) = f(x),$$

we have that

$$\left[\int_0^{\infty} x^{\frac{p}{q'}(1-s)+p(A_1-A_2)} f^p(x)dx \right]^{\frac{1}{p}} = \left[\int_0^{\infty} x^{p-p\beta-1} (\mathscr{A}(\mathscr{D}_+ f)(x))^p dx \right]^{\frac{1}{p}}. \tag{6.26}$$

Moreover, due to the weighted Hardy inequality (1.65), it follows that

$$\left[\int_0^{\infty} x^{p-p\beta-1} (\mathscr{A}(\mathscr{D}_+ f)(x))^p dx \right]^{\frac{1}{p}} < \frac{1}{\beta} \left[\int_0^{\infty} x^{p(1-\beta)-1} (\mathscr{D}_+ f(x))^p dx \right]^{\frac{1}{p}}.$$

Now, by applying the Hardy inequality to the right-hand side of the last inequality $m-1$ times, we get the relation

$$\left[\int_0^{\infty} x^{p-p\beta-1} (\mathscr{A}(\mathscr{D}_+ f)(x))^p dx \right]^{\frac{1}{p}} < \frac{1}{\beta^{\underline{m}}} \left[\int_0^{\infty} x^{p(m-\beta)-1} \left(\mathscr{D}_+^m f(x) \right)^p dx \right]^{\frac{1}{p}}. \tag{6.27}$$

Finally, the inequality (6.24) holds due to (2.75), (6.26), and (6.27). In the same way the inequality (6.25) follows by virtue of (2.76) and (6.27) which completes the proof. □

The previous theorem is derived by virtue of the Hardy inequality and covers the case when $\beta > m-1$, where m is a fixed non-negative integer. Our next result is in some way complementary to Theorem 6.3 since it covers the case when $\beta < 0$.

Theorem 6.4 *Let p, q, p', q', and λ be as in* (1.43) *and* (1.44)*, and let $K : \mathbb{R}_+^2 \to \mathbb{R}$ be a non-negative measurable homogeneous function of degree $-s, s > 0$. Further, let A_1 and A_2 be real parameters such that the function $K(x,y)y^{-q'A_2}$ is decreasing on $\mathbb{R}_+$ for any fixed $x \in \mathbb{R}_+$ and $\beta := \frac{1}{q'}(s-1)+A_2-A_1-\frac{1}{p} < 0$. If $0 < L := k_1^{\frac{1}{q'}}(q'A_2)k_1^{\frac{1}{p'}}(2-s-p'A_1) < \infty$, then the inequalities*

$$\begin{aligned}
&\sum_{n=1}^{\infty} a_n \int_0^\infty K^\lambda(x,n) f(x)dx = \int_0^\infty f(x)\left(\sum_{n=1}^{\infty} K^\lambda(x,n)a_n\right)dx \\
&\quad < L\cdot \frac{\Gamma(-\beta)}{\Gamma(-\beta+m)}\left[\int_0^\infty x^{p(m-\beta)-1}\left(\mathscr{D}_\pm^m f(x)\right)^p dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty} n^{\frac{q}{p'}(1-s)+q(A_2-A_1)} a_n^q\right]^{\frac{1}{q}}
\end{aligned} \tag{6.28}$$

and

$$\begin{aligned}
&\left[\sum_{n=1}^{\infty} n^{\frac{q'}{p'}(s-1)+q'(A_1-A_2)}\left(\int_0^\infty K^\lambda(x,n)f(x)dx\right)^{q'}\right]^{\frac{1}{q'}} \\
&\quad < L\cdot \frac{\Gamma(-\beta)}{\Gamma(-\beta+m)}\left[\int_0^\infty x^{p(m-\beta)-1}\left(\mathscr{D}_\pm^m f(x)\right)^p dx\right]^{\frac{1}{p}}
\end{aligned} \tag{6.29}$$

hold for any non-negative function $f \in \Lambda_\pm^m$ and a non-negative sequence $a = (a_n)_{n\in\mathbb{N}}$, provided that the integral and series on their right-hand sides converge to positive numbers.

Proof. We follow the same procedure as in the proof of Theorem 6.3, this time accompanied with the dual Hardy inequality (1.66). We have

$$\left[\int_0^\infty x^{\frac{p}{q'}(1-s)+p(A_1-A_2)} f^p(x)dx\right]^{\frac{1}{p}} = \left[\int_0^\infty x^{p-p\beta-1}(\mathscr{A}^*(\mathscr{D}_\pm f)(x))^p dx\right]^{\frac{1}{p}} \tag{6.30}$$

since

$$x\mathscr{A}^*(\mathscr{D}_\pm f)(x) = -\int_x^\infty f'(t)dt = f(x).$$

Moreover, utilizing the dual Hardy inequality m times, it follows that

$$\begin{aligned}
&\left[\int_0^\infty x^{p-p\beta-1}(\mathscr{A}^*(\mathscr{D}_\pm f)(x))^p dx\right]^{\frac{1}{p}} \\
&\quad < \frac{1}{(-\beta)^{\overline{m}}}\left[\int_0^\infty x^{p(m-\beta)-1}(\mathscr{D}_\pm^m f(x))^p dx\right]^{\frac{1}{p}}.
\end{aligned} \tag{6.31}$$

Now, the relations (2.75), (6.30), and (6.31) entail the desired inequality (6.28). Similarly, the inequality (6.29) follows by virtue of (2.76) and (6.31). □

Remark 6.3 It should be noticed here that Theorem 6.3 and Theorem 6.4 coincide in the case of $m = 0$. Therefore, presented results may be regarded as the differential extensions of inequalities (2.75) and (2.76).

6.2.2 Reduction to Conjugate Case and the Best Constants

Now, our goal is to determine conditions under which the constants appearing on the right-hand sides of inequalities (6.24), (6.25), (6.28), and (6.29) are the best possible.

Therefore, in this subsection we deal with non-negative conjugate exponents p,q, that is, with parameters p and q such that $\frac{1}{p}+\frac{1}{q}=1$, $p>1$. In this case $p'=q$, $q'=p$, and $\lambda=1$.

It should be noticed here that the constant appearing in the inequality (6.2) does not contain any exponent. Keeping in mind this fact, we are going to simplify the constants appearing in (6.24), (6.25), (6.28), and (6.29) so that they do not contain exponents. Therefore, we set

$$pA_2+qA_1=2-s, \tag{6.32}$$

since in this case relation $k_1(pA_2)=k_1(2-s-qA_1)$ holds. With this assumption, the constant L appearing in Theorem 6.3 and Theorem 6.4 reduces to $L^*=k_1(pA_2)$.

Thus, if the condition (6.32) is fulfilled, the conjugate forms of inequalities (6.24) and (6.25) become respectively

$$\begin{aligned}\sum_{n=1}^{\infty} a_n \int_0^{\infty} K(x,n)f(x)dx &= \int_0^{\infty} f(x)\left(\sum_{n=1}^{\infty} K(x,n)a_n\right)dx \\ &< L^*\cdot\frac{\Gamma(\beta-m+1)}{\Gamma(\beta+1)}\left[\int_0^{\infty} x^{p(m-\beta)-1}\left(\mathscr{D}_+^m f(x)\right)^p dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty} n^{-1+pqA_2}a_n^q\right]^{\frac{1}{q}}\end{aligned} \tag{6.33}$$

and

$$\begin{aligned}&\left[\sum_{n=1}^{\infty} n^{(p-1)(1-pqA_2)}\left(\int_0^{\infty} K(x,n)f(x)dx\right)^p\right]^{\frac{1}{p}} \\ &\qquad< L^*\cdot\frac{\Gamma(\beta-m+1)}{\Gamma(\beta+1)}\left[\int_0^{\infty} x^{p(m-\beta)-1}\left(\mathscr{D}_+^m f(x)\right)^p dx\right]^{\frac{1}{p}},\end{aligned} \tag{6.34}$$

where $\beta=-qA_1$. In the same setting, inequalities (6.28) and (6.29) read respectively

$$\begin{aligned}\sum_{n=1}^{\infty} a_n \int_0^{\infty} K(x,n)f(x)dx &= \int_0^{\infty} f(x)\left(\sum_{n=1}^{\infty} K(x,n)a_n\right)dx \\ &< L^*\cdot\frac{\Gamma(-\beta)}{\Gamma(-\beta+m)}\left[\int_0^{\infty} x^{p(m-\beta)-1}\left(\mathscr{D}_\pm^m f(x)\right)^p dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty} n^{-1+pqA_2}a_n^q\right]^{\frac{1}{q}}\end{aligned} \tag{6.35}$$

and

$$
\begin{aligned}
&\left[\sum_{n=1}^{\infty} n^{(p-1)(1-pqA_2)}\left(\int_0^{\infty} K(x,n)f(x)dx\right)^p\right]^{\frac{1}{p}} \\
&\quad < L^* \cdot \frac{\Gamma(-\beta)}{\Gamma(-\beta+m)}\left[\int_0^{\infty} x^{p(m-\beta)-1}\left(\mathscr{D}_{\pm}^m f(x)\right)^p dx\right]^{\frac{1}{p}},
\end{aligned}
\tag{6.36}
$$

where $\beta = -qA_1$.

Remark 6.4 Let $K(x,y) = (x+y)^{-s}$, $s > 0$, and $A_1 = \frac{p-s}{pq}$, $A_2 = \frac{q-s}{pq}$. In this case the constant L^* appearing in inequalities (6.33), (6.34), (6.35), and (6.36) becomes $L^* = k_1\left(1-\frac{s}{q}\right) = B\left(\frac{s}{p}, \frac{s}{q}\right)$. Then, utilizing the relationship between the Beta and the Gamma function, we have $L^* \cdot \frac{\Gamma(\beta-m+1)}{\Gamma(\beta+1)} = \frac{\Gamma(\frac{s}{p}-m)\Gamma(\frac{s}{q})}{\Gamma(s)}$, that is, the relation (6.33) becomes the inequality (6.2) from the beginning of this chapter, with a weaker condition $pm < s$. Thus, the dual form of (6.2) includes the constant which reduces to $L^* \cdot \frac{\Gamma(-\beta)}{\Gamma(-\beta+m)} = \frac{\pi}{\sin\frac{s\pi}{p}} \cdot \frac{\Gamma(\frac{s}{q})}{\Gamma(s)\Gamma(m+1-\frac{s}{p})}$, after applying the Euler reflection formula.

Now, our aim is to show that the constants appearing in (6.33), (6.34), (6.35), and (6.36) are the best possible. The corresponding proofs are the substance of the following two theorems.

Theorem 6.5 *Let $p, q > 1$ be conjugate parameters and $K : \mathbb{R}_+^2 \to \mathbb{R}$ be a non-negative measurable homogeneous function of degree $-s, s > 0$. Further, let A_1 and A_2 be real parameters fulfilling condition* (6.32) *and $\beta = -qA_1 \in (m-1, s-1)$, $s > m$, where m is a fixed non-negative integer. If the function $K(x,y)y^{-pA_2}$ is decreasing on $\mathbb{R}_+$ for any fixed $x \in \mathbb{R}_+$, then the constant $L^* \cdot \frac{\Gamma(\beta-m+1)}{\Gamma(\beta+1)}$ is the best possible in* (6.33) *and* (6.34).

Proof. In order to prove that the inequality (6.33) includes the best constant on its right-hand side, suppose that there exists a positive constant C_1, smaller than $L^* \cdot \frac{\Gamma(\beta-m+1)}{\Gamma(\beta+1)}$, such that the relation

$$
\begin{aligned}
&\sum_{n=1}^{\infty} a_n \int_0^{\infty} K(x,n)f(x)dx = \int_0^{\infty} f(x)\left(\sum_{n=1}^{\infty} K(x,n)a_n\right)dx \\
&\quad < C_1\left[\int_0^{\infty} x^{p(m-\beta)-1}\left(\mathscr{D}_{+}^m f(x)\right)^p dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty} n^{-1+pqA_2}a_n^q\right]^{\frac{1}{q}}
\end{aligned}
\tag{6.37}
$$

holds for any non-negative function $f \in \Lambda_+^m$ and a non-negative sequence $a = (a_n)_{n\in\mathbb{N}}$, provided that the integral and series on its right-hand side converge.

Now, let $\widetilde{L}$ and $\widetilde{R}$ respectively denote the left-hand side and the right-hand side of (6.37) accompanied with

$$
\widetilde{f}(x) = \frac{\Gamma\left(1+\beta-\frac{\varepsilon}{p}-m\right)}{\Gamma\left(1+\beta-\frac{\varepsilon}{p}\right)} \cdot x^{\beta-\frac{\varepsilon}{p}} \cdot \chi_{[1,\infty)}(x) \quad \text{and} \quad \widetilde{a}_n = n^{-pA_2-\frac{\varepsilon}{q}},
$$

where $\varepsilon > 0$ is a sufficiently small number. Here, χ stands for a characteristic function of the corresponding set. Since the m-th derivative of the function $x^{\beta-\frac{\varepsilon}{p}}$ is equal to $\frac{\Gamma\left(1+\beta-\frac{\varepsilon}{p}\right)}{\Gamma\left(1+\beta-\frac{\varepsilon}{p}-m\right)}x^{\beta-\frac{\varepsilon}{p}-m}$, it follows that

$$\mathscr{D}_+^m\widetilde{f}(x) = x^{\beta-\frac{\varepsilon}{p}-m}\cdot\chi_{(1,\infty)}(x).$$

Thus, the left-hand side of (6.37) may be bounded from above as follows:

$$\begin{aligned}\widetilde{R} &= C_1\left[\int_1^\infty x^{-1-\varepsilon}dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty n^{-1-\varepsilon}\right]^{\frac{1}{q}}\\ &< \frac{C_1}{\varepsilon^{\frac{1}{p}}}\left[1+\int_1^\infty x^{-1-\varepsilon}dx\right]^{\frac{1}{q}} = \frac{C_1(\varepsilon+1)^{\frac{1}{q}}}{\varepsilon}.\end{aligned}\tag{6.38}$$

On the other hand, utilizing the Fubini theorem and the suitable change of variables, it follows that

$$\begin{aligned}\widetilde{L} &= \varphi(\varepsilon)\int_1^\infty x^{\beta-\frac{\varepsilon}{p}}\left(\sum_{n=1}^\infty K(x,n)n^{-pA_2-\frac{\varepsilon}{q}}\right)dx\\ &> \varphi(\varepsilon)\int_1^\infty x^{-qA_1-\frac{\varepsilon}{p}}\left(\int_1^\infty K(x,y)y^{-pA_2-\frac{\varepsilon}{q}}dy\right)dx\\ &= \varphi(\varepsilon)\int_1^\infty x^{-\varepsilon-1}\int_{\frac{1}{x}}^\infty K(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dtdx\\ &= \frac{\varphi(\varepsilon)}{\varepsilon}\int_1^\infty K(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dt+\varphi(\varepsilon)\int_1^\infty x^{-\varepsilon-1}\int_{\frac{1}{x}}^1 K(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dtdx\\ &= \frac{\varphi(\varepsilon)}{\varepsilon}\int_1^\infty K(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dt+\varphi(\varepsilon)\int_0^1 K(1,t)t^{-pA_2-\frac{\varepsilon}{q}}\int_{\frac{1}{t}}^\infty x^{-\varepsilon-1}dxdt\\ &= \frac{\varphi(\varepsilon)}{\varepsilon}\left(\int_1^\infty K(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dt+\int_0^1 K(1,t)t^{-pA_2+\frac{\varepsilon}{p}}dt\right),\end{aligned}\tag{6.39}$$

since the function $K(x,y)y^{-pA_2}$ is decreasing on $\mathbb{R}_+$ for any fixed $x\in\mathbb{R}_+$. Here, φ stands for the function $\varphi(\varepsilon) = \frac{\Gamma\left(1+\beta-\frac{\varepsilon}{p}-m\right)}{\Gamma\left(1+\beta-\frac{\varepsilon}{p}\right)}$.

Now, relations (6.37), (6.38), and (6.39) entail the inequality

$$\varphi(\varepsilon)\left(\int_1^\infty K(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dt+\int_0^1 K(1,t)t^{-pA_2+\frac{\varepsilon}{p}}dt\right) < C_1(\varepsilon+1)^{\frac{1}{q}}.$$

Therefore, by Fatou lemma, as $\varepsilon\to 0$, it follows that $L^*\cdot\frac{\Gamma(\beta-m+1)}{\Gamma(\beta+1)}\le C_1$, which is in contrast to our assumption. Hence, $L^*\cdot\frac{\Gamma(\beta-m+1)}{\Gamma(\beta+1)}$ is the best constant in (6.33).

It remains to show that $L^* \cdot \frac{\Gamma(\beta-m+1)}{\Gamma(\beta+1)}$ is the best constant in (6.34). Similarly to above discussion, suppose that there exists a positive constant c_1 smaller than $L^* \cdot \frac{\Gamma(\beta-m+1)}{\Gamma(\beta+1)}$ such that

$$\begin{aligned}
&\left[\sum_{n=1}^{\infty} n^{(p-1)(1-pqA_2)} \left(\int_0^\infty K(x,n)f(x)dx\right)^p\right]^{\frac{1}{p}} \\
&\quad < c_1 \left[\int_0^\infty x^{p(m-\beta)-1} \left(\mathscr{D}_+^m f(x)\right)^p dx\right]^{\frac{1}{p}}
\end{aligned} \tag{6.40}$$

holds for all non-negative functions $f \in \Lambda_+^m$. Then, utilizing the Hölder inequality, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} a_n \int_0^\infty K(x,n)f(x)dx \\
&\quad = \sum_{n=1}^{\infty} \left[n^{\frac{1-pqA_2}{q}} \int_0^\infty K(x,n)f(x)dx\right] \cdot [n^{\frac{-1+pqA_2}{q}} a_n] \\
&\quad \le \left[\sum_{n=1}^{\infty} n^{(p-1)(1-pqA_2)} \left(\int_0^\infty K(x,n)f(x)dx\right)^p\right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} n^{-1+pqA_2} a_n^q\right]^{\frac{1}{q}} \\
&\quad < c_1 \left[\int_0^\infty x^{p(m-\beta)-1} \left(\mathscr{D}_+^m f(x)\right)^p dx\right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} n^{-1+pqA_2} a_n^q\right]^{\frac{1}{q}},
\end{aligned}$$

which results that $L^* \cdot \frac{\Gamma(\beta-m+1)}{\Gamma(\beta+1)}$ is not the best possible constant in (6.33). With this contradiction, the proof is completed. □

Theorem 6.6 *Let $p,q > 1$ be conjugate parameters and $K : \mathbb{R}_+^2 \to \mathbb{R}$ be a non-negative measurable homogeneous function of degree $-s$, $0 < s \le 1$. Further, let A_1 and A_2 be real parameters fulfilling condition* (6.32) *and $\beta = -qA_1 \in (-1,-1)$. If the function $K(x,y)y^{-pA_2}$ is decreasing on $\mathbb{R}_+$ for any fixed $x \in \mathbb{R}_+$, then $L^* \cdot \frac{\Gamma(-\beta)}{\Gamma(-\beta+m)}$ is the best constant in* (6.35) *and* (6.36).

Proof. We follow the lines of the proof of Theorem 6.5, that is, we assume that the inequality

$$\begin{aligned}
&\sum_{n=1}^{\infty} a_n \int_0^\infty K(x,n)f(x)dx = \int_0^\infty f(x) \left(\sum_{n=1}^{\infty} K(x,n)a_n\right) dx \\
&\quad < C_2 \left[\int_0^\infty x^{p(m-\beta)-1} \left(\mathscr{D}_\pm^m f(x)\right)^p dx\right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} n^{-1+pqA_2} a_n^q\right]^{\frac{1}{q}}
\end{aligned} \tag{6.41}$$

holds with a positive constant C_2, smaller than $L^* \cdot \frac{\Gamma(-\beta)}{\Gamma(-\beta+m)}$. Now, let $\widetilde{L}$ and $\widetilde{R}$ respectively denote the left-hand side and the right-hand side of inequality (6.41) accompanied with

$$\widetilde{f}^*(x) = \frac{\Gamma\left(-\beta+\frac{\varepsilon}{p}\right)}{\Gamma\left(m-\beta+\frac{\varepsilon}{p}\right)} x^{\beta-\frac{\varepsilon}{p}} \cdot \chi_{[1,\infty)}(x) \quad \text{and} \quad \widetilde{a}_n^* = n^{-pA_2-\frac{\varepsilon}{q}},$$

where $\varepsilon > 0$ is a sufficiently small number. Then, taking into account (6.39), we have

$$\widetilde{L} > \frac{\varphi^*(\varepsilon)}{\varepsilon}\left(\int_1^\infty K(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dt + \int_0^1 K(1,t)t^{-pA_2+\frac{\varepsilon}{p}}dt\right), \tag{6.42}$$

where $\varphi^*(\varepsilon) = \frac{\Gamma\left(-\beta+\frac{\varepsilon}{p}\right)}{\Gamma\left(m-\beta+\frac{\varepsilon}{p}\right)}$.

On the other hand, since the m-th derivative of the function $x^{\beta-\frac{\varepsilon}{p}}$ is equal to $(-1)^m \frac{\Gamma\left(m-\beta+\frac{\varepsilon}{p}\right)}{\Gamma\left(-\beta+\frac{\varepsilon}{p}\right)} x^{\beta-\frac{\varepsilon}{p}-m}$, it follows that $\mathscr{D}_\pm^m \widetilde{f}(x) = x^{\beta-\frac{\varepsilon}{p}-m} \cdot \chi_{(1,\infty)}(x)$, and so

$$\widetilde{R} < \frac{C_2(\varepsilon+1)^{\frac{1}{q}}}{\varepsilon}. \tag{6.43}$$

Now, comparing (6.41), (6.42), and (6.43), it follows that

$$\varphi^*(\varepsilon)\left(\int_1^\infty K(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dt + \int_0^1 K(1,t)t^{-pA_2+\frac{\varepsilon}{p}}dt\right) < C_2(\varepsilon+1)^{\frac{1}{q}},$$

and consequently, $L^* \cdot \frac{\Gamma(-\beta)}{\Gamma(-\beta+m)} \le C_2$, after letting $\varepsilon \to 0$. This means that the constant $L^* \cdot \frac{\Gamma(-\beta)}{\Gamma(-\beta+m)}$ is the best possible in (6.35).

To conclude the proof, we suppose that, contrary to our claim, there exists a constant $0 < c_2 < L^* \cdot \frac{\Gamma(-\beta)}{\Gamma(-\beta+m)}$ such that the inequality

$$\left[\sum_{n=1}^\infty n^{(p-1)(1-pqA_2)}\left(\int_0^\infty K(x,n)f(x)dx\right)^p\right]^{\frac{1}{p}} < c_2\left[\int_0^\infty x^{p(m-\beta)-1}\left(\mathscr{D}_\pm^m f(x)\right)^p dx\right]^{\frac{1}{p}}$$

holds for all non-negative functions $f \in \Lambda_\pm^m$, as in the statement of theorem. Finally, employing the Hölder inequality, we obtain

$$\begin{aligned}
&\sum_{n=1}^\infty a_n \int_0^\infty K(x,n)f(x)dx \\
&= \sum_{n=1}^\infty \left[n^{\frac{1-pqA_2}{q}}\int_0^\infty K(x,n)f(x)dx\right]\cdot[n^{\frac{-1+pqA_2}{q}}a_n] \\
&\le \left[\sum_{n=1}^\infty n^{(p-1)(1-pqA_2)}\left(\int_0^\infty K(x,n)f(x)dx\right)^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty n^{-1+pqA_2}a_n^q\right]^{\frac{1}{q}} \\
&< c_2\left[\int_0^\infty x^{p(m-\beta)-1}\left(\mathscr{D}_\pm^m f(x)\right)^p dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty n^{-1+pqA_2}a_n^q\right]^{\frac{1}{q}}
\end{aligned}$$

which is impossible since $L^* \cdot \frac{\Gamma(-\beta)}{\Gamma(-\beta+m)}$ is the best constant in (6.35). With this contradiction, the proof is completed. □

6.3 Multidimensional Cases

Now we give the multidimensional extension of inequalities (6.3) and (6.4) in the case of non-conjugate parameters.

Theorem 6.7 *Suppose p_i, p_i', q_i, $i=1,2,\dots,n$, and λ are as in* (1.35)*, and A_{ij}, $i,j=1,2,\dots,n$, are real parameters satisfying $\sum_{i=1}^n A_{ij}=0$. Further, let $\alpha_i=\sum_{j=1}^n A_{ij}$, and let $s>0$ be real parameter such that $\frac{s-n}{q_i}+\lambda-\alpha_i>m_i$, $m_i\in\mathbb{N}\cup\{0\}$, $i=1,2,\dots,n$. If $K:\mathbb{R}_+^n\to\mathbb{R}$ is a non-negative measurable homogeneous function of degree $-s$, and $f_i\in\Lambda_+^{m_i}$, $i=1,2,\dots,n$, then*

$$\int_{\mathbb{R}_+^n} K^\lambda(\mathbf{x})\prod_{i=1}^n f_i(x_i)d\mathbf{x}\le C_n^s(\mathbf{p},\mathbf{q},\mathbf{A})\prod_{i=1}^n \|x_i^{(n-1-s)/q_i+\alpha_i+m_i}\mathscr{D}_+^{m_i}f_i\|_{p_i}, \tag{6.44}$$

and

$$\left[\int_{\mathbb{R}_+} x_n^{(1-\lambda p_n')(n-1-s)-p_n'\alpha_n}\left(\int_{\mathbb{R}_+^{n-1}} K^\lambda(\mathbf{x})\prod_{i=1}^{n-1} f_i(x_i)d\hat{}^n\mathbf{x}\right)^{p_n'}dx_n\right]^{1/p_n'}$$
$$\le C_{n-1}^s(\mathbf{p},\mathbf{q},\mathbf{A})\prod_{i=1}^{n-1}\|x_i^{(n-1-s)/q_i+\alpha_i+m_i}\mathscr{D}_+^{m_i}f_i\|_{p_i}, \tag{6.45}$$

where

$$C_n^s(\mathbf{p},\mathbf{q},\mathbf{A}) = \prod_{i=1}^n k_i^{1/q_i}(q_i\mathbf{A_i})\prod_{i=1}^n\frac{\Gamma\left(\frac{s-n}{q_i}+\lambda-\alpha_i-m_i\right)}{\Gamma\left(\frac{s-n}{q_i}+\lambda-\alpha_i\right)},$$

$$C_{n-1}^s(\mathbf{p},\mathbf{q},\mathbf{A}) = \prod_{i=1}^n k_i^{1/q_i}(q_i\mathbf{A_i})\prod_{i=1}^{n-1}\frac{\Gamma\left(\frac{s-n}{q_i}+\lambda-\alpha_i-m_i\right)}{\Gamma\left(\frac{s-n}{q_i}+\lambda-\alpha_i\right)},$$

$\mathbf{A_i}=(A_{i1},A_{i2},\dots,A_{in})$, $x_i^{(n-1-s)/q_i+\alpha_i+m_i}\mathscr{D}_+^{m_i}f_i\in L^{p_i}(\mathbb{R}_+)$, *and* $k_i(q_i\mathbf{A_i})<\infty$, $i=1,2,\dots,n$.

Proof. First suppose that $m_i\in\mathbb{N}$, $i=1,2,\dots,n$. In order to prove (6.44) we will rewrite the right-hand side of inequality (1.41) in a form that is more suitable for the application of the Hardy inequality. Namely, since

$$\mathscr{A}(\mathscr{D}_+f)(x)=\int_0^x f'(t)dt=f(x)-f(0)=f(x),$$

we have that

$$\begin{aligned}&\prod_{i=1}^{n} k_i^{1/q_i}(q_i\mathbf{A_i})\prod_{i=1}^{n}\|x_i^{(n-1-s)/q_i+\alpha_i}f_i\|_{p_i}\\ =&\prod_{i=1}^{n} k_i^{1/q_i}(q_i\mathbf{A_i})\prod_{i=1}^{n}\|x_i^{(n-1-s)/q_i+\alpha_i}\mathscr{A}(\mathscr{D}_+f_i)\|_{p_i}.\end{aligned} \tag{6.46}$$

Now, due to the weighted Hardy inequality (1.65), it follows that

$$\|x_i^{(n-1-s)/q_i+\alpha_i}\mathscr{A}(\mathscr{D}_+f_i)\|_{p_i}\leq\frac{1}{\frac{s-n}{q_i}+\lambda-\alpha_i-1}\|x_i^{(n-1-s)/q_i+\alpha_i+1}\mathscr{D}_+f_i\|_{p_i},$$

$i=1,2,\ldots,n$. Moreover, applying the Hardy inequality to the right-hand side of the above inequality m_i-1 times, yields relation

$$\begin{aligned}&\|x_i^{(n-1-s)/q_i+\alpha_i}\mathscr{A}(\mathscr{D}_+f_i)\|_{p_i}\\ \leq&\frac{1}{\left(\frac{s-n}{q_i}+\lambda-\alpha_i-1\right)^{\overline{m_i}}}\cdot\|x_i^{(n-1-s)/q_i+\alpha_i+m_i}\mathscr{D}_+^{m_i}f_i\|_{p_i}.\end{aligned} \tag{6.47}$$

Finally, taking into account that $\left(\frac{s-n}{q_i}+\lambda-\alpha_i\right)^{\overline{m_i}}=\frac{\Gamma\left(\frac{s-n}{q_i}+\lambda-\alpha_i\right)}{\Gamma\left(\frac{s-n}{q_i}+\lambda-\alpha_i-m_i\right)}$, the inequality (6.44) holds due to (1.41), (6.46), and (6.47). It remains to consider the case when $m_i=0$ for some $i\in\{1,2,\ldots,n\}$. In that case the relation (6.47) reduces to a trivial equality, so (6.44) holds.

In the same way the inequality (6.45) holds by virtue of (1.42) and (6.47). The proof is completed. □

The Theorem 6.7 may be regarded as an extension of (1.41) and (1.42) since for $m_1=m_2=\ldots=m_n=0$ it reduces to relations (1.41) and (1.42).

The previous theorem holds when the corresponding parameters fulfill the set of conditions $\frac{s-n}{q_i}+\lambda-\alpha_i>m_i$, $i=1,2,\ldots,n$. If $\frac{s-n}{q_i}+\lambda-\alpha_i<1$, $i=1,2,\ldots,n$, we can also derive a pair of inequalities which are in some way dual to inequalities (6.44) and (6.45). Namely, this result relies on the dual Hardy inequality (1.66).

Theorem 6.8 *Suppose p_i, p_i', q_i, $i=1,2,\ldots,n$, and λ are as in* (1.35), and let A_{ij}, $i,j=1,2,\ldots,n$, be real parameters satisfying $\sum_{i=1}^n A_{ij}=0$. Further, let $\alpha_i=\sum_{j=1}^n A_{ij}$, and let $s>0$ be real parameter such that $\frac{s-n}{q_i}+\lambda-\alpha_i<1$, $i=1,2,\ldots,n$. If $K:\mathbb{R}_+^n\to\mathbb{R}$ is a non-negative measurable homogeneous function of degree $-s$ and $f_i\in\Lambda_\pm^{m_i}$, $m_i\in\mathbb{N}\cup\{0\}$, $i=1,2,\ldots,n$, then

$$\int_{\mathbb{R}_+^n}K^\lambda(\mathbf{x})\prod_{i=1}^n f_i(x_i)d\mathbf{x}\leq E_n^s(\mathbf{p},\mathbf{q},\mathbf{A})\prod_{i=1}^n\|x_i^{(n-1-s)/q_i+\alpha_i+m_i}\mathscr{D}_\pm^{m_i}f_i\|_{p_i}, \tag{6.48}$$

and

$$\left[\int_{\mathbb{R}_+}x_n^{(1-\lambda p_n')(n-1-s)-p_n'\alpha_n}\left(\int_{\mathbb{R}_+^{n-1}}K^\lambda(\mathbf{x})\prod_{i=1}^{n-1}f_i(x_i)d\hat{}^n\mathbf{x}\right)^{p_n'}dx_n\right]^{1/p_n'}$$

$$\leq E^s_{n-1}(\mathbf{p},\mathbf{q},\mathbf{A}) \prod_{i=1}^{n-1} \|x_i^{(n-1-s)/q_i+\alpha_i+m_i}\mathscr{D}^{m_i}_{\pm} f_i\|_{p_i}, \tag{6.49}$$

where

$$E^s_n(\mathbf{p},\mathbf{q},\mathbf{A}) = \prod_{i=1}^{n} k_i^{1/q_i}(q_i\mathbf{A_i}) \prod_{i=1}^{n} \frac{\Gamma\left(\frac{n-s}{q_i}-\lambda+\alpha_i+1\right)}{\Gamma\left(\frac{n-s}{q_i}-\lambda+\alpha_i+m_i+1\right)},$$

$$E^s_{n-1}(\mathbf{p},\mathbf{q},\mathbf{A}) = \prod_{i=1}^{n} k_i^{1/q_i}(q_i\mathbf{A_i}) \prod_{i=1}^{n-1} \frac{\Gamma\left(\frac{n-s}{q_i}-\lambda+\alpha_i+1\right)}{\Gamma\left(\frac{n-s}{q_i}-\lambda+\alpha_i+m_i+1\right)},$$

$\mathbf{A_i} = (A_{i1}, A_{i2}, \ldots, A_{in})$, $x_i^{(n-1-s)/q_i+\alpha_i+m_i}\mathscr{D}^{m_i}_{+} f_i \in L^{p_i}(\mathbb{R}_+)$, and $k_i(q_i\mathbf{A_i}) < \infty$, $i = 1,2,\ldots,n$.

Proof. The proof is similar to the proof of the previous theorem, except that we use the dual Hardy inequality (1.66) this time. In this regard, the right-hand side of (1.41) can be rewritten as

$$\begin{aligned}&\prod_{i=1}^{n} k_i^{1/q_i}(q_i\mathbf{A_i}) \prod_{i=1}^{n} \|x_i^{(n-1-s)/q_i+\alpha_i} f_i\|_{p_i}\\ =&\prod_{i=1}^{n} k_i^{1/q_i}(q_i\mathbf{A_i}) \prod_{i=1}^{n} \|x_i^{(n-1-s)/q_i+\alpha_i}\mathscr{H}^*(\mathscr{D}_{\pm} f_i)\|_{p_i},\end{aligned} \tag{6.50}$$

since

$$\mathscr{H}^*(\mathscr{D}_{\pm} f)(x) = -\int_x^{\infty} f'(t)dt = f(x).$$

Now, by applying the dual Hardy inequality to the expressions on the right-hand side of (6.50) m_i times (when $m_i \in \mathbb{N}$), it follows that

$$\begin{aligned}&\|x_i^{(n-1-s)/q_i+\alpha_i}\mathscr{H}^*(\mathscr{D}_{\pm} f_i)\|_{p_i}\\ \leq&\frac{1}{\left(\frac{n-s}{q_i}-\lambda+\alpha_i+1\right)^{\overline{m_i}}} \cdot \|x_i^{(n-1-s)/q_i+\alpha_i+m_i}\mathscr{D}^{m_i}_{\pm} f_i\|_{p_i},\end{aligned} \tag{6.51}$$

$i = 1,2,\ldots,n$. Further, since $\left(\frac{n-s}{q_i}-\lambda+\alpha_i+1\right)^{\overline{m_i}} = \frac{\Gamma\left(\frac{n-s}{q_i}-\lambda+\alpha_i+m_i+1\right)}{\Gamma\left(\frac{n-s}{q_i}-\lambda+\alpha_i+1\right)}$, the inequality (6.48) holds due to (1.41), (6.50), and (6.51). In the same way, inequality (6.49) holds by virtue of (1.42) and (6.51). The trivial case when $m_i = 0$ for some $i \in \{1,2,\ldots,n\}$ is treated in the same way as in Theorem 6.7. □

It should be noticed here that if $m_1 = m_2 = \ldots = m_n = 0$, inequalities (6.48) and (6.49) reduce to (1.41) and (1.42) respectively.

Our next step is to determine conditions under which the constants $C^s_n(\mathbf{p},\mathbf{q},\mathbf{A})$, $C^s_{n-1}(\mathbf{p},\mathbf{q},\mathbf{A})$, $E^s_n(\mathbf{p},\mathbf{q},\mathbf{A})$, and $E^s_{n-1}(\mathbf{p},\mathbf{q},\mathbf{A})$ appearing in Theorems 6.7 and 6.8 are the best possible. This happens in the case of conjugate parameters.

6.3.1 Inequalities with Conjugate Parameters. The Best Possible Constants

In order to obtain the best possible constants in inequalities (6.44), (6.45), (6.48), and (6.49), in this subsection we deal with their conjugate forms. Namely, if $p_i > 1$, $i = 1,2,\dots,n$, is the set of conjugate parameters, then inequalities (6.44) and (6.45) become respectively

$$\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^n f_i(x_i) d\mathbf{x} \le \overline{C}_n^s(\mathbf{p},\mathbf{A}) \prod_{i=1}^n \|x_i^{(n-1-s)/p_i+\alpha_i+m_i} \mathscr{D}_+^{m_i} f_i\|_{p_i}, \tag{6.52}$$

and

$$\left[\int_{\mathbb{R}_+} x_n^{(1-p_n')(n-1-s)-p_n'\alpha_n} \left(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x}) \prod_{i=1}^{n-1} f_i(x_i) \hat{d}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}$$
$$\le \overline{C}_{n-1}^s(\mathbf{p},\mathbf{A}) \prod_{i=1}^{n-1} \|x_i^{(n-1-s)/p_i+\alpha_i+m_i} \mathscr{D}_+^{m_i} f_i\|_{p_i}, \tag{6.53}$$

where

$$\overline{C}_n^s(\mathbf{p},\mathbf{A}) = \prod_{i=1}^n k_i^{1/p_i}(p_i\mathbf{A_i}) \prod_{i=1}^n \frac{\Gamma\left(\frac{s-n}{p_i} - \alpha_i - m_i + 1\right)}{\Gamma\left(\frac{s-n}{p_i} - \alpha_i + 1\right)},$$
$$\overline{C}_{n-1}^s(\mathbf{p},\mathbf{A}) = \prod_{i=1}^n k_i^{1/p_i}(p_i\mathbf{A_i}) \prod_{i=1}^{n-1} \frac{\Gamma\left(\frac{s-n}{p_i} - \alpha_i - m_i + 1\right)}{\Gamma\left(\frac{s-n}{p_i} - \alpha_i + 1\right)}.$$

In the same way, the conjugate forms of inequalities (6.48) and (6.49) read

$$\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^n f_i(x_i) d\mathbf{x} \le \overline{E}_n^s(\mathbf{p},\mathbf{A}) \prod_{i=1}^n \|x_i^{(n-1-s)/p_i+\alpha_i+m_i} \mathscr{D}_\pm^{m_i} f_i\|_{p_i}, \tag{6.54}$$

and

$$\left[\int_{\mathbb{R}_+} x_n^{(1-p_n')(n-1-s)-p_n'\alpha_n} \left(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x}) \prod_{i=1}^{n-1} f_i(x_i) \hat{d}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}$$
$$\le \overline{E}_{n-1}^s(\mathbf{p},\mathbf{A}) \prod_{i=1}^{n-1} \|x_i^{(n-1-s)/p_i+\alpha_i+m_i} \mathscr{D}_\pm^{m_i} f_i\|_{p_i}, \tag{6.55}$$

with the constants

$$\overline{E}_n^s(\mathbf{p},\mathbf{A}) = \prod_{i=1}^n k_i^{1/p_i}(p_i\mathbf{A_i}) \prod_{i=1}^n \frac{\Gamma\left(\frac{n-s}{p_i} + \alpha_i\right)}{\Gamma\left(\frac{n-s}{p_i} + \alpha_i + m_i\right)},$$

$$\overline{E}^{s}_{n-1}(\mathbf{p},\mathbf{A}) = \prod_{i=1}^{n} k_i^{1/p_i}(p_i\mathbf{A_i}) \prod_{i=1}^{n-1} \frac{\Gamma\left(\frac{n-s}{p_i}+\alpha_i\right)}{\Gamma\left(\frac{n-s}{p_i}+\alpha_i+m_i\right)}.$$

Now, our goal is to determine the conditions under which the inequalities (6.52), (6.53), (6.54), and (6.55) include the best possible constants on their right-hand sides. If the set of conditions (5.95) is fulfilled, then, with abbreviations as in Subsection 5.3.1, inequalities (6.52) and (6.53) become respectively

$$\int_{\mathbb{R}^n_+} K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i) d\mathbf{x} \le L^s_n(\mathbf{p},\mathbf{A}) \prod_{i=1}^{n} \| x_i^{m_i - \frac{1}{p_i} - \widetilde{A}_i} \mathscr{D}^{m_i}_{+} f_i \|_{p_i}, \tag{6.56}$$

and

$$\left[\int_{\mathbb{R}_+} x_n^{(p_n'-1)(1+p_n\widetilde{A}_n)} \left(\int_{\mathbb{R}^{n-1}_+} K(\mathbf{x}) \prod_{i=1}^{n-1} f_i(x_i) d\hat{}^n\mathbf{x} \right)^{p_n'} dx_n \right]^{1/p_n'}$$
$$\le L^s_{n-1}(\mathbf{p},\mathbf{A}) \prod_{i=1}^{n-1} \| x_i^{m_i - \frac{1}{p_i} - \widetilde{A}_i} \mathscr{D}^{m_i}_{+} f_i \|_{p_i}, \tag{6.57}$$

where

$$L^s_n(\mathbf{p},\widetilde{\mathbf{A}}) = k_1(\widetilde{\mathbf{A}}) \prod_{i=1}^{n} \frac{\Gamma\left(\widetilde{A}_i - m_i + 1\right)}{\Gamma\left(\widetilde{A}_i + 1\right)},$$

$$L^s_{n-1}(\mathbf{p},\widetilde{\mathbf{A}}) = k_1(\widetilde{\mathbf{A}}) \prod_{i=1}^{n-1} \frac{\Gamma\left(\widetilde{A}_i - m_i + 1\right)}{\Gamma\left(\widetilde{A}_i + 1\right)}.$$

In the same regard, the inequalities (6.54) and (6.55) read respectively

$$\int_{\mathbb{R}^n_+} K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i) d\mathbf{x} \le M^s_n(\mathbf{p},\widetilde{\mathbf{A}}) \prod_{i=1}^{n} \| x_i^{m_i - \frac{1}{p_i} - \widetilde{A}_i} \mathscr{D}^{m_i}_{\pm} f_i \|_{p_i}, \tag{6.58}$$

and

$$\left[\int_{\mathbb{R}_+} x_n^{(p_n'-1)(1+p_n\widetilde{A}_n)} \left(\int_{\mathbb{R}^{n-1}_+} K(\mathbf{x}) \prod_{i=1}^{n-1} f_i(x_i) d\hat{}^n\mathbf{x} \right)^{p_n'} dx_n \right]^{1/p_n'}$$
$$\le M^s_{n-1}(\mathbf{p},\widetilde{\mathbf{A}}) \prod_{i=1}^{n-1} \| x_i^{m_i - \frac{1}{p_i} - \widetilde{A}_i} \mathscr{D}^{m_i}_{\pm} f_i \|_{p_i}, \tag{6.59}$$

with the corresponding constants

$$M^s_n(\mathbf{p},\widetilde{\mathbf{A}}) = k_1(\widetilde{\mathbf{A}}) \prod_{i=1}^{n} \frac{\Gamma(-\widetilde{A}_i)}{\Gamma(-\widetilde{A}_i + m_i)},$$

$$M_{n-1}^{s}(\mathbf{p},\widetilde{\mathbf{A}}) \;=\; k_1(\widetilde{\mathbf{A}})\prod_{i=1}^{n-1}\frac{\Gamma(-\widetilde{A}_i)}{\Gamma(-\widetilde{A}_i+m_i)}.$$

Now, we show that the constants $L_n^s(\mathbf{p},\widetilde{\mathbf{A}})$, $L_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}})$, $M_n^s(\mathbf{p},\widetilde{\mathbf{A}})$, and $M_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}})$ appearing on the right-hand sides of the above inequalities are the best possible.

Theorem 6.9 *Let $m_i \in \mathbb{N}\cup\{0\}$, $\widetilde{A}_i > m_i - 1$, $i = 1,2,\dots,n$, and let the parameters $\widetilde{A}_i$, $i = 2,\dots,n$, fulfill conditions as in* (5.93). *Then, the constants $L_n^s(\mathbf{p},\widetilde{\mathbf{A}})$ and $L_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}})$ are the best possible in the inequalities* (6.56) *and* (6.57) *respectively.*

Proof. Suppose that the constant $L_n^s(\mathbf{p},\widetilde{\mathbf{A}})$ is not the best possible in (6.56). Then, there exists a positive constant C_n, smaller than $L_n^s(\mathbf{p},\widetilde{\mathbf{A}})$, such that the inequality (6.56) is still valid if we replace $L_n^s(\mathbf{p},\widetilde{\mathbf{A}})$ by C_n. Now, consider the functions

$$\widetilde{f}_i(x_i) = \begin{cases} 0, & 0 < x_i < 1 \\ \frac{\Gamma\left(1+\widetilde{A}_i-\frac{\varepsilon}{p_i}-m_i\right)}{\Gamma\left(1+\widetilde{A}_i-\frac{\varepsilon}{p_i}\right)} x_i^{\widetilde{A}_i-\frac{\varepsilon}{p_i}}, & x_i \geq 1 \end{cases}, i = 1,\dots,n,$$

where $\varepsilon > 0$ is a sufficiently small number. Since the m_i-th derivative of the function $x_i^{\widetilde{A}_i-\frac{\varepsilon}{p_i}}$ is equal to $\frac{\Gamma\left(1+\widetilde{A}_i-\frac{\varepsilon}{p_i}\right)}{\Gamma\left(1+\widetilde{A}_i-\frac{\varepsilon}{p_i}-m_i\right)} x_i^{\widetilde{A}_i-\frac{\varepsilon}{p_i}-m_i}$, it follows that

$$\mathscr{D}_+^{m_i}\widetilde{f}_i(x_i) = \begin{cases} 0, & 0 < x_i < 1 \\ x_i^{\widetilde{A}_i-\frac{\varepsilon}{p_i}-m_i}, & x_i > 1 \end{cases}, i = 1,\dots,n,$$

so in this setting the right-hand side of (6.56) reduces to

$$\begin{aligned} & C_n\prod_{i=1}^{n}\|x_i^{m_i-\frac{1}{p_i}-\widetilde{A}_i}\mathscr{D}_+^{m_i}\widetilde{f}_i\|_{p_i} \\ &= C_n\prod_{i=1}^{n}\left[\int_{\mathbb{R}_+} x_i^{p_i(m_i-\widetilde{A}_i)-1}(\mathscr{D}_+^{m_i}\widetilde{f}_i(x_i))^{p_i}dx_i\right]^{\frac{1}{p_i}} \\ &= \frac{C_n}{\varepsilon}. \end{aligned} \tag{6.60}$$

On the other hand, the left-hand side of (6.56), can be rewritten as

$$\int_{\mathbb{R}_+^n} K(\mathbf{x})\prod_{i=1}^{n}\widetilde{f}_i(x_i)d\mathbf{x} = I\cdot\prod_{i=1}^{n}\frac{\Gamma\left(1+\widetilde{A}_i-\frac{\varepsilon}{p_i}-m_i\right)}{\Gamma\left(1+\widetilde{A}_i-\frac{\varepsilon}{p_i}\right)}$$

where $I = \int_{[1,\infty)^n} K(\mathbf{x})\prod_{i=1}^{n} x_i^{\widetilde{A}_i-\frac{\varepsilon}{p_i}}d\mathbf{x}$. From the inequalities (5.112) and (5.113), we obtain

$$\int_{\mathbb{R}_+^n} K(\mathbf{x})\prod_{i=1}^{n}\widetilde{f}_i(x_i)d\mathbf{x} \geq \left(\frac{1}{\varepsilon}k_1\left(\widetilde{\mathbf{A}}-\varepsilon\mathbf{1}/\mathbf{p}\right)-O(1)\right)\prod_{i=1}^{n}\frac{\Gamma\left(1+\widetilde{A}_i-\frac{\varepsilon}{p_i}-m_i\right)}{\Gamma\left(1+\widetilde{A}_i-\frac{\varepsilon}{p_i}\right)},$$

where $\mathbf{1/p} = (1/p_1, \dots, 1/p_n)$. Moreover, the relation (6.60) implies that

$$C_n \geq \left(k_1\left(\widetilde{\mathbf{A}} - \varepsilon\mathbf{1/p}\right) - \varepsilon O(1)\right) \prod_{i=1}^{n} \frac{\Gamma\left(1+\widetilde{A}_i - \frac{\varepsilon}{p_i} - n\right)}{\Gamma\left(1+\widetilde{A}_i - \frac{\varepsilon}{p_i}\right)}.$$

Obviously, letting $\varepsilon \to 0^+$, it follows that $C_n \geq L_n^s(\mathbf{p}, \widetilde{\mathbf{A}})$, which contradicts with our assumption $0 < C_n < L_n^s(\mathbf{p}, \widetilde{\mathbf{A}})$. Hence, $L_n^s(\mathbf{p}, \widetilde{\mathbf{A}})$ is the best possible in (6.56).

It remains to show that $L_{n-1}^s(\mathbf{p}, \widetilde{\mathbf{A}})$ is the best possible constant in (6.57). Assume that there exists a positive constant C_{n-1}, smaller than $L_{n-1}^s(\mathbf{p}, \widetilde{\mathbf{A}})$, such that the inequality (6.57) holds when $L_{n-1}^s(\mathbf{p}, \widetilde{\mathbf{A}})$ is replaced by C_{n-1}. Then, utilizing the Hölder inequality and the inequality (6.47), we have

$$\begin{aligned}
&\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i) d\mathbf{x} \\
&= \int_{\mathbb{R}_+} \left[x_n^{\frac{1+p_n\widetilde{A}_n}{p_n}} \int_{\mathbb{R}_+^{n-1}} K(\mathbf{x}) \prod_{i=1}^{n-1} f_i(x_i) d\hat{}^n\mathbf{x} \right] \cdot [x_n^{-\frac{1+p_n\widetilde{A}_n}{p_n}} f_n(x_n)] dx_n \\
&\leq \left[\int_{\mathbb{R}_+} x_n^{(p_n'-1)(1+p_n\widetilde{A}_n)} \left(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x}) \prod_{i=1}^{n-1} f_i(x_i) d\hat{}^n\mathbf{x} \right)^{p_n'} dx_n \right]^{1/p_n'} \\
&\times \left[\int_{\mathbb{R}_+} x_n^{-1-p_n\widetilde{A}_n} f_n^{p_n}(x_n) dx_n \right]^{1/p_n} \\
&\leq C_{n-1} \frac{\Gamma(\widetilde{A}_n - m_n + 1)}{\Gamma(\widetilde{A}_n + 1)} \prod_{i=1}^{n} \| x_i^{m_i - \frac{1}{p_i} - \widetilde{A}_i} \mathscr{D}_+^{m_i} f_i \|_{p_i}.
\end{aligned} \tag{6.61}$$

Finally, taking into account our assumption $0 < C_{n-1} < L_{n-1}^s(\mathbf{p}, \widetilde{\mathbf{A}})$, we have

$$0 < C_{n-1} \frac{\Gamma(\widetilde{A}_n - m_n + 1)}{\Gamma(\widetilde{A}_n + 1)} < L_{n-1}^s(\mathbf{p}, \widetilde{\mathbf{A}}) \frac{\Gamma(\widetilde{A}_n - m_n + 1)}{\Gamma(\widetilde{A}_n + 1)} = L_n^s(\mathbf{p}, \widetilde{\mathbf{A}}).$$

Therefore, relation (6.61) contradicts with the fact that $L_n^s(\mathbf{p}, \widetilde{\mathbf{A}})$ is the best possible constant in inequality (6.56). Thus, the assumption that $L_{n-1}^s(\mathbf{p}, \widetilde{\mathbf{A}})$ is not the best possible is false. □

Theorem 6.10 *Let $m_i \in \mathbb{N} \cup \{0\}$, $\widetilde{A}_i < 0$, $i = 1, 2, \dots, n$, and let the parameters $\widetilde{A}_i$, $i = 2, \dots, n$, fulfill conditions as in* (5.93). *Then, the constants $M_n^s(\mathbf{p}, \widetilde{\mathbf{A}})$ and $M_{n-1}^s(\mathbf{p}, \widetilde{\mathbf{A}})$ are the best possible in* (6.58) *and* (6.59) *respectively.*

Proof. We follow the same procedure as in the proof of Theorem 6.9, that is, we suppose that the inequality

$$\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i) d\mathbf{x} \leq C_n^* \prod_{i=1}^{n} \| x_i^{m_i - \frac{1}{p_i} - \widetilde{A}_i} \mathscr{D}_\pm^{m_i} f_i \|_{p_i}, \tag{6.62}$$

holds with a positive constant C_n^*, smaller than $M_n^s(\mathbf{p},\widetilde{\mathbf{A}})$. Now, we consider this inequality with the functions

$$\widetilde{f}_i^*(x_i) = \begin{cases} 0, & 0 < x_i < 1 \\ \frac{\Gamma\left(-\widetilde{A}_i+\frac{\varepsilon}{p_i}\right)}{\Gamma\left(-\widetilde{A}_i+m_i+\frac{\varepsilon}{p_i}\right)} x_i^{\widetilde{A}_i-\frac{\varepsilon}{p_i}}, & x_i \geq 1 \end{cases}, i = 1,\dots,n,$$

where ε is sufficiently small number. Then, similarly as in the proof of Theorem 6.9, we have the following lower bound for the left-hand side of (6.62):

$$\begin{aligned} &\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^n \widetilde{f}_i^*(x_i) d\mathbf{x} \\ &\geq \left(\frac{1}{\varepsilon} k_1\left(\widetilde{\mathbf{A}} - \varepsilon \mathbf{1}/\mathbf{p}\right) - O(1)\right) \prod_{i=1}^n \frac{\Gamma\left(-\widetilde{A}_i + \frac{\varepsilon}{p_i}\right)}{\Gamma\left(-\widetilde{A}_i + m_i + \frac{\varepsilon}{p_i}\right)}. \end{aligned} \tag{6.63}$$

On the other hand, since the m_i-th derivative of the function $x_i^{\widetilde{A}_i-\frac{\varepsilon}{p_i}}$ is equal to $(-1)^{m_i} \frac{\Gamma\left(-\widetilde{A}_i+m_i+\frac{\varepsilon}{p_i}\right)}{\Gamma\left(-\widetilde{A}_i+\frac{\varepsilon}{p_i}\right)} x_i^{\widetilde{A}_i-\frac{\varepsilon}{p_i}-m_i}$, it follows that

$$\mathscr{D}_\pm^{m_i} \widetilde{f}_i^*(x_i) = \begin{cases} 0, & 0 < x_i < 1 \\ x_i^{\widetilde{A}_i-\frac{\varepsilon}{p_i}-m_i}, & x_i > 1 \end{cases}, i = 1,\dots,n,$$

so the right-hand side of (6.62) reduces to

$$C_n^* \prod_{i=1}^n \| x_i^{m_i-\frac{1}{p_i}-\widetilde{A}_i} \mathscr{D}_\pm^{m_i} \widetilde{f}_i^* \|_{p_i} = \frac{C_n^*}{\varepsilon}. \tag{6.64}$$

Consequently, comparing (6.62), (6.63), and (6.64), it follows that

$$C_n^* \geq \left(k_1\left(\widetilde{\mathbf{A}} - \varepsilon \mathbf{1}/\mathbf{p}\right) - \varepsilon O(1)\right) \prod_{i=1}^n \frac{\Gamma\left(-\widetilde{A}_i + \frac{\varepsilon}{p_i}\right)}{\Gamma\left(-\widetilde{A}_i + m_i + \frac{\varepsilon}{p_i}\right)}.$$

Therefore, as $\varepsilon \to 0$, it follows that $M_n^s(\mathbf{p},\widetilde{\mathbf{A}}) \leq C_n^*$, which contradicts with our assumption. This means that the constant $M_n^s(\mathbf{p},\widetilde{\mathbf{A}})$ is the best possible in (6.58).

To conclude the proof, we suppose that, contrary to our claim, there exists a constant $0 < C_{n-1}^* < M_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}})$ such that the inequality

$$\begin{aligned} &\left[\int_{\mathbb{R}_+} x_n^{(p_n'-1)(1+p_n\widetilde{A}_n)} \left(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x}) \prod_{i=1}^{n-1} f_i(x_i) d^{\hat{n}}\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'} \\ &\leq C_{n-1}^* \prod_{i=1}^{n-1} \| x_i^{m_i-\frac{1}{p_i}-\widetilde{A}_i} \mathscr{D}_\pm^{m_i} f_i \|_{p_i} \end{aligned}$$

holds. Then, utilizing the Hölder inequality and the inequality (6.51), we have

$$\begin{aligned}
&\int_{\mathbb{R}_+^n} K(\mathbf{x})\prod_{i=1}^{n} f_i(x_i)d\mathbf{x}\\
&\quad= \int_{\mathbb{R}_+}\left[x_n^{\frac{1+p_n\widetilde{A}_n}{p_n}}\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x})\prod_{i=1}^{n-1} f_i(x_i)d\hat{}^n\mathbf{x}\right]\cdot[x_n^{-\frac{1+p_n\widetilde{A}_n}{p_n}} f_n(x_n)]dx_n\\
&\quad\le \left[\int_{\mathbb{R}_+} x_n^{(p_n'-1)(1+p_n\widetilde{A}_n)}\left(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x})\prod_{i=1}^{n-1} f_i(x_i)d\hat{}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}\\
&\qquad\times\left[\int_{\mathbb{R}_+} x_n^{-1-p_n\widetilde{A}_n} f_n^{p_n}(x_n)dx_n\right]^{1/p_n}\\
&\quad\le C_{n-1}^{*}\frac{\Gamma(-\widetilde{A}_n)}{\Gamma(-\widetilde{A}_n+m_n)}\prod_{i=1}^{n}\|x_i^{m_i-\frac{1}{p_i}-\widetilde{A}_i}\mathscr{D}_{\pm}^{m_i} f_i\|_{p_i}.
\end{aligned}$$

Now, according to our assumption, it follows that

$$0 < C_{n-1}^{*}\frac{\Gamma(-\widetilde{A}_n)}{\Gamma(-\widetilde{A}_n+m_n)} < M_{n-1}^{s}(\mathbf{p},\widetilde{\mathbf{A}})\frac{\Gamma(-\widetilde{A}_n)}{\Gamma(-\widetilde{A}_n+m_n)} = M_n^{s}(\mathbf{p},\widetilde{\mathbf{A}}),$$

which means that $M_n^s(\mathbf{p},\widetilde{\mathbf{A}})$ is not the best constant in (6.58). This is a clear contradiction of our assumption and the proof is completed. □

6.3.2 Applications and Concluding Remarks

In order to conclude this chapter, we consider the inequalities (6.56), (6.57), (6.58), and (6.59) in some particular settings. The resulting inequalities will include the best possible constants on their right-hand sides.

Taking the standard examples of homogeneous kernels from Subsection 5.3.2, we get the following particular inequalities. With the kernel $K_1(\mathbf{x}) = \frac{1}{(\sum_{i=1}^n x_i)^s}$, $s > 0$, inequalities (6.56), (6.57), (6.58), and (6.59) reduce respectively to

$$\int_{\mathbb{R}_+^n}\frac{1}{(\sum_{i=1}^n x_i)^s}\prod_{i=1}^{n} f_i(x_i)d\mathbf{x} \le \frac{1}{\Gamma(s)}\prod_{i=1}^{n}\Gamma(\widetilde{A}_i-m_i+1)\prod_{i=1}^{n}\|x_i^{m_i-\frac{1}{p_i}-\widetilde{A}_i}\mathscr{D}_{+}^{m_i} f_i\|_{p_i},$$

$$\begin{aligned}
&\left[\int_{\mathbb{R}_+} x_n^{(p_n'-1)(1+p_n\widetilde{A}_n)}\left(\int_{\mathbb{R}_+^{n-1}}\frac{1}{(\sum_{i=1}^n x_i)^s}\prod_{i=1}^{n-1} f_i(x_i)d\hat{}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}\\
&\quad\le \frac{\Gamma(1+\widetilde{A}_n)}{\Gamma(s)}\prod_{i=1}^{n-1}\Gamma(\widetilde{A}_i-m_i+1)\prod_{i=1}^{n-1}\|x_i^{m_i-\frac{1}{p_i}-\widetilde{A}_i}\mathscr{D}_{+}^{m_i} f_i\|_{p_i},
\end{aligned}$$

$$\int_{\mathbb{R}_+^n}\frac{1}{(\sum_{i=1}^n x_i)^s}\prod_{i=1}^{n} f_i(x_i)d\mathbf{x} \le \frac{1}{\Gamma(s)}\prod_{i=1}^{n}\frac{B(1+\widetilde{A}_i,-\widetilde{A}_i)}{\Gamma(-\widetilde{A}_i+m_i)}\prod_{i=1}^{n}\|x_i^{m_i-\frac{1}{p_i}-\widetilde{A}_i}\mathscr{D}_{\pm}^{m_i} f_i\|_{p_i},$$

and

$$\left[\int_{\mathbb{R}_+} x_n{}^{(p_n'-1)(1+p_n\widetilde{A}_n)}\left(\int_{\mathbb{R}_+^{n-1}}\frac{1}{(\sum_{i=1}^n x_i)^s}\prod_{i=1}^{n-1} f_i(x_i)\hat{d}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}$$
$$\leq \frac{\Gamma(1+\widetilde{A}_n)}{\Gamma(s)}\prod_{i=1}^{n-1}\frac{B(1+\widetilde{A}_i,-\widetilde{A}_i)}{\Gamma(-\widetilde{A}_i+m_i)}\prod_{i=1}^{n-1}\|x_i^{m_i-\frac{1}{p_i}-\widetilde{A}_i}\mathscr{D}_{\pm}^{m_i}f_i\|_{p_i}.$$

Another interesting example of a homogeneous kernel with degree $-s$, is the function

$$K_2(\mathbf{x})=\frac{1}{\max\{x_1^s,\ldots,x_n^s\}},\quad s>0.$$

Then the inequalities (6.56), (6.57), (6.58), and (6.59) reduce to

$$\int_{\mathbb{R}_+^n}\frac{1}{\max\{x_1^s,\ldots,x_n^s\}}\prod_{i=1}^n f_i(x_i)d\mathbf{x}\leq s\prod_{i=1}^n\frac{\Gamma(\widetilde{A}_i-m_i+1)}{\Gamma(\widetilde{A}_i+2)}\prod_{i=1}^n\|x_i^{m_i-\frac{1}{p_i}-\widetilde{A}_i}\mathscr{D}_{+}^{m_i}f_i\|_{p_i},$$

$$\left[\int_{\mathbb{R}_+} x_n{}^{(p_n'-1)(1+p_n\widetilde{A}_n)}\left(\int_{\mathbb{R}_+^{n-1}}\frac{1}{\max\{x_1^s,\ldots,x_n^s\}}\prod_{i=1}^{n-1} f_i(x_i)\hat{d}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}$$
$$\leq \frac{s}{(1+\widetilde{A}_n)}\prod_{i=1}^{n-1}\frac{\Gamma(\widetilde{A}_i-m_i+1)}{\Gamma(\widetilde{A}_i+2)}\prod_{i=1}^{n-1}\|x_i^{m_i-\frac{1}{p_i}-\widetilde{A}_i}\mathscr{D}_{+}^{m_i}f_i\|_{p_i},$$

$$\int_{\mathbb{R}_+^n}\frac{1}{\max\{x_1^s,\ldots,x_n^s\}}\prod_{i=1}^n f_i(x_i)d\mathbf{x}\leq s\prod_{i=1}^n\frac{\Gamma(-\widetilde{A}_i)}{(1+\widetilde{A}_i)\Gamma(-\widetilde{A}_i+m_i)}\prod_{i=1}^n\|x_i^{m_i-\frac{1}{p_i}-\widetilde{A}_i}\mathscr{D}_{\pm}^{m_i}f_i\|_{p_i},$$

and

$$\left[\int_{\mathbb{R}_+} x_n{}^{(p_n'-1)(1+p_n\widetilde{A}_n)}\left(\int_{\mathbb{R}_+^{n-1}}\frac{1}{\max\{x_1^s,\ldots,x_n^s\}}\prod_{i=1}^{n-1} f_i(x_i)\hat{d}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'}$$
$$\leq \frac{s}{(1+\widetilde{A}_n)}\prod_{i=1}^{n-1}\frac{\Gamma(-\widetilde{A}_i)}{(1+\widetilde{A}_i)\Gamma(-\widetilde{A}_i+m_i)}\prod_{i=1}^{n-1}\|x_i^{m_i-\frac{1}{p_i}-\widetilde{A}_i}\mathscr{D}_{\pm}^{m_i}f_i\|_{p_i},$$

where the constants appearing on their right-hand sides are the best possible.

Remark 6.5 The Hilbert-type inequalities involving differential operators in this chapter, as well as their consequences, are taken from [3], [4], [7] and [9].

Chapter 7

Hilbert-type Inequalities for Hilbert Space Operators

Discrete version of the Hilbert inequality (1.1) asserts that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{a_i b_j}{i+j} \leq \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left[\sum_{i=1}^{\infty} a_i^p\right]^{\frac{1}{p}}\left[\sum_{j=1}^{\infty} b_j^q\right]^{\frac{1}{q}}, \tag{7.1}$$

where $(a_i)_{i\in\mathbb{N}} \in l^p$, $(b_j)_{j\in\mathbb{N}} \in l^q$, and p, q are conjugate exponents, $p > 1$. We know that the constant $\pi/\sin\left(\frac{\pi}{p}\right)$ is the best possible in the sense that it can not be replaced with a smaller constant so that (7.1) still holds for all $(a_i)_{i\in\mathbb{N}} \in l^p$ and $(b_j)_{j\in\mathbb{N}} \in l^q$.

In the recent time a considerable attention is dedicated to inequalities for bounded self-adjoint operators on a Hilbert space (see e.g. [44]). Let $\mathscr{H}$ be a Hilbert space and let $\mathfrak{B}_h(\mathscr{H})$ be the semi-space of all bounded linear self-adjoint operators on $\mathscr{H}$. Further, let $\mathfrak{B}^+(\mathscr{H})$ and $\mathfrak{B}^{++}(\mathscr{H})$, respectively, denote the sets of all positive and positive invertible operators in $\mathfrak{B}_h(\mathscr{H})$. The weighted operator geometric mean $\sharp_\nu$, for $\nu \in [0,1]$ and $A, B \in \mathfrak{B}^{++}(\mathscr{H})$ is defined by

$$A \sharp_\nu B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\nu A^{\frac{1}{2}}. \tag{7.2}$$

Clearly, if A and B commute then $A \sharp_\nu B = A^{1-\nu}B^\nu$, that is, formula (7.2) reduces to the classical definition of the geometric mean.

Mond *et al.* [76], derived an operator version of the Hölder inequality

$$\sum_{i=1}^{n} A_i^p \sharp_{1/q} B_i^q \leq \left[\sum_{i=1}^{n} A_i^p\right] \sharp_{1/q} \left[\sum_{i=1}^{n} B_i^q\right], \tag{7.3}$$

where p,q are conjugate exponents, $A_i, B_i \in \mathfrak{B}^{++}(\mathscr{H})$, $i,j = 1,2,\ldots,n$, and the sign of inequality is taken with respect to an operator order. Obviously, in commuting case, relation (7.3) reduces to the classical Hölder inequality.

The geometric mean is the special case of a more general concept, these are operator means. The theory of operator means for positive linear operators on a Hilbert space, in connection with Löwner's theory for operator monotone functions, was established by Kubo and Ando [67].

A binary operation $(A,B) \in \mathfrak{B}^{+}(\mathscr{H}) \times \mathfrak{B}^{+}(\mathscr{H}) \to A\sigma B \in \mathfrak{B}^{+}(\mathscr{H})$ in the cone of positive operators on a Hilbert space $\mathscr{H}$ is called a connection if the following conditions are satisfied:

(C1) monotonicity: $A \le C$ and $B \le D$ imply $A\sigma B \le C\sigma D$,

(C2) upper continuity: $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n\sigma B_n \downarrow A\sigma B$,

(C3) transformer inequality: $T^*(A\sigma B)T \le (T^*AT)\sigma(T^*BT)$ for every T.

An operator mean is a connection with normalized condition

(C4) normalized condition: $I_{\mathscr{H}}\sigma I_{\mathscr{H}} = I_{\mathscr{H}}$.

In condition (C2) symbol $\downarrow$ denotes the convergence in the strong operator topology, while $I_{\mathscr{H}}$ in (C4) denotes the identity operator on a Hilbert space.

Connections posses numerous significant properties, one of them is the so called joint concavity. More precisely, if $A_1, A_2, B_1, B_2 \in \mathfrak{B}^{+}(\mathscr{H})$ and $0 \le \lambda \le 1$, then

$$(\lambda A_1 + (1-\lambda)B_1)\,\sigma\,(\lambda A_2 + (1-\lambda)B_2) \ge \lambda(A_1\sigma A_2) + (1-\lambda)(B_1\sigma B_2).$$

The Hölder operator inequality (7.3), derived in [76], is established with a help of the above joint concavity property (see also paper [62]). In fact, inequality (7.3) holds for every connection, but in our further discussion we shall also use some additional characteristics of geometric mean.

The main tool in obtaining the Hilbert-type inequalities is the Hölder inequality. Hence, the main objective of this chapter is to establish the Hilbert inequality for Hilbert space operators, with the help of the Hölder operator inequality (7.3).

7.1 The Hilbert Operator Inequality

By virtue of the Hölder operator inequality, in this section we establish the operator form of the Hilbert inequality for Hilbert space operators. Our results will be given in a more general form. More precisely, we estimate double sum $\sum_{i=1}^{m}\sum_{j=1}^{n} K(i,j)\, A_i^p \sharp_{1/q} B_j^q$ involving operators in $\mathfrak{B}^{++}(\mathscr{H})$ and a non-negative measurable kernel K that satisfies some additional properties.

Theorem 7.1 *Let $1/p+1/q=1$, $p>1$, and let $m,n\in\mathbb{N}$. If $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ is a non-negative measurable function strictly decreasing in each argument, and $\varphi,\psi:\mathbb{R}_+\to\mathbb{R}$ are non-negative measurable strictly increasing functions, then the inequality*

$$\sum_{i=1}^{m}\sum_{j=1}^{n}K(i,j)A_i^p\sharp_{1/q}B_j^q$$
$$\leq\left[\sum_{i=1}^{m}\left(\int_0^n\frac{K(i,t)}{\psi^p(t)}dt\right)\varphi^p(i)A_i^p\right]\sharp_{1/q}\left[\sum_{j=1}^{n}\left(\int_0^m\frac{K(t,j)}{\varphi^q(t)}dt\right)\psi^q(j)B_j^q\right]\tag{7.4}$$

holds for all positive invertible operators $A_1,\dots,A_m,B_1,\dots,B_n\in\mathfrak{B}^{++}(\mathscr{H})$.

Proof. Considering definition (7.2) of geometric mean, we immediately obtain the following property

$$(sX)\sharp_{1/q}(tY)=s^{\frac{1}{p}}t^{\frac{1}{q}}X\sharp_{1/q}Y,\qquad s,t>0,\tag{7.5}$$

where $X,Y\in\mathfrak{B}^{++}(\mathscr{H})$.

Therefore the left-hand side of inequality (7.4) can be rewritten in the form

$$\sum_{i=1}^{m}\sum_{j=1}^{n}K(i,j)A_i^p\sharp_{1/q}B_j^q=\sum_{i=1}^{m}\sum_{j=1}^{n}K(i,j)\frac{\varphi(i)}{\psi(j)}\cdot\frac{\psi(j)}{\varphi(i)}A_i^p\sharp_{1/q}B_j^q$$
$$=\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{K(i,j)\varphi^p(i)}{\psi^p(j)}A_i^p\right]\sharp_{1/q}\left[\frac{K(i,j)\psi^q(j)}{\varphi^q(i)}B_j^q\right],$$

that is, the Hölder operator inequality (7.3) yields inequality

$$\sum_{i=1}^{m}\sum_{j=1}^{n}K(i,j)A_i^p\sharp_{1/q}B_j^q$$
$$\leq\left[\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{K(i,j)\varphi^p(i)}{\psi^p(j)}A_i^p\right]\sharp_{1/q}\left[\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{K(i,j)\psi^q(j)}{\varphi^q(i)}B_j^q\right]$$
$$=\left[\sum_{i=1}^{m}\left(\sum_{j=1}^{n}\frac{K(i,j)}{\psi^p(j)}\right)\varphi^p(i)A_i^p\right]\sharp_{1/q}\left[\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\frac{K(i,j)}{\varphi^q(i)}\right)\psi^q(j)B_j^q\right].\tag{7.6}$$

Since the kernel $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ is strictly decreasing in each argument, and $\varphi,\psi:\mathbb{R}_+\to\mathbb{R}$ are increasing, the functions $K(i,t)\psi^{-p}(t)$ and $K(t,j)\varphi^{-q}(t)$, $i=1,2,\dots,m$, $j=1,2,\dots,n$ are strictly decreasing on $\mathbb{R}_+$. Hence, $\sum_{j=1}^{n}K(i,j)\psi^{-p}(j)$ and $\sum_{i=1}^{m}K(i,j)\varphi^{-q}(i)$ are the lower Darboux sums for the corresponding integrals, that is,

$$\sum_{j=1}^{n}\frac{K(i,j)}{\psi^p(j)}\leq\int_0^n\frac{K(i,t)}{\psi^p(t)}dt,\quad\sum_{i=1}^{m}\frac{K(i,j)}{\varphi^q(i)}\leq\int_0^m\frac{K(t,j)}{\varphi^q(t)}dt,$$

$i=1,2,\dots,m$, $j=1,2,\dots,n$. Therefore, due to monotonicity property (C1) of geometric mean we see that the right-hand side of inequality (7.6) is not greater than

$$\left[\sum_{i=1}^{m}\left(\int_0^n\frac{K(i,t)}{\psi^p(t)}dt\right)\varphi^p(i)A_i^p\right]\sharp_{1/q}\left[\sum_{j=1}^{n}\left(\int_0^m\frac{K(t,j)}{\varphi^q(t)}dt\right)\psi^q(j)B_j^q\right],$$

and the proof is completed. □

Clearly, inequality (7.4) provides a unified treatment to the Hilbert-type inequalities for operators in $\mathfrak{B}^{++}(\mathscr{H})$. Recall that the unified approach to Hilbert-type inequalities in the real case was developed in the paper [66].

In the sequel we are concerned with homogeneous kernels K with negative degree of homogeneity, and the power weight functions φ and ψ. Now, in order to present our result referring to homogeneous kernels, we define the integral

$$k(a;r_1,r_2) = \int_{r_1}^{r_2} K(1,t)t^{-a}dt, \quad 0 \le r_1 < r_2 \le \infty, \tag{7.7}$$

where the arguments a, r_1 and r_2 are assumed to be such that the integral converge (see also Section 2.1). In addition, if $r_1 = 0$ and $r_2 = \infty$, then the integral $k(a;0,\infty)$ will simply be denoted by $k(a)$, $k(a) = \int_0^\infty K(1,t)t^{-a}dt$.

Theorem 7.2 *Let $1/p+1/q=1$, $p>1$, $\alpha,\beta>0$, and let $m,n\in\mathbb{N}$. If $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ is a non-negative measurable homogeneous function of degree $-s$, $s>0$, strictly decreasing in each argument, then inequality*

$$\begin{aligned}&\sum_{i=1}^{m}\sum_{j=1}^{n} K(i,j)A_i^p \sharp_{1/q} B_j^q \\ &\le \left[\sum_{i=1}^{m} k(\beta p;0,\tfrac{n}{i})\, i^{1-s+(\alpha-\beta)p} A_i^p\right] \sharp_{1/q} \left[\sum_{j=1}^{n} k(2-\alpha q-s;\tfrac{j}{m},\infty)\, j^{1-s+(\beta-\alpha)q} B_j^q\right]\end{aligned} \tag{7.8}$$

holds for all positive invertible operators $A_1,\dots,A_m,B_1,\dots,B_n\in\mathfrak{B}^{++}(\mathscr{H})$.

Proof. The proof follows immediately from Theorem 7.1, i.e. from inequality (7.4) equipped with homogeneous kernel K of degree $-s$, $s>0$, and the power weight functions $\varphi(t)=t^\alpha$, $\psi(t)=t^\beta$, $\alpha,\beta>0$. Namely, using the homogeneity of kernel K and regarding definition (7.7) we have

$$\int_0^n \frac{K(i,t)}{t^{\beta p}}dt = i^{1-s-\beta p}k(\beta p;0,\tfrac{n}{i}), \quad i=1,2,\dots,n,$$

and

$$\int_0^m \frac{K(t,j)}{t^{\alpha q}}dt = j^{1-s-\alpha q}k(2-\alpha q-s;\tfrac{j}{m},\infty), \quad j=1,2,\dots,m,$$

which yields (7.8). □

Remark 7.1 Suppose $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ is homogeneous function of degree $-s$, $s>0$ and $\alpha,\beta>0$ are such that $k(\beta p)<\infty$ and $k(2-\alpha q-s)<\infty$. Then, taking into account definition (7.7) and definition of function $k(\cdot)$, we have

$$k(\beta p;0,\tfrac{n}{i}) \le k(\beta p), \quad \text{and} \quad k(2-\alpha q-s;\tfrac{j}{m},\infty) \le k(2-\alpha q-s).$$

Therefore, taking into account monotonicity property (C1) for connections and property (7.5), we conclude that the right-hand side of inequality (7.8) is not greater than

$$\left[k(\beta p)\sum_{i=1}^{m} i^{1-s+(\alpha-\beta)p}A_i^p\right] \sharp_{1/q} \left[k(2-\alpha q-s)\sum_{j=1}^{n} j^{1-s+(\beta-\alpha)q}B_j^q\right]$$
$$= k^{\frac{1}{p}}(\beta p)k^{\frac{1}{q}}(2-\alpha q-s)\left[\sum_{i=1}^{m} i^{1-s+(\alpha-\beta)p}A_i^p\right] \sharp_{1/q} \left[\sum_{j=1}^{n} j^{1-s+(\beta-\alpha)q}B_j^q\right].$$

In other words, inequality (7.8) implies inequality

$$\sum_{i=1}^{m}\sum_{j=1}^{n} K(i,j)A_i^p \sharp_{1/q} B_j^q$$
$$\leq k^{\frac{1}{p}}(\beta p)k^{\frac{1}{q}}(2-\alpha q-s)\left[\sum_{i=1}^{m} i^{1-s+(\alpha-\beta)p}A_i^p\right] \sharp_{1/q} \left[\sum_{j=1}^{n} j^{1-s+(\beta-\alpha)q}B_j^q\right], \quad (7.9)$$

provided that $k(\beta p) < \infty$ and $k(2-\alpha q-s) < \infty$.

As in the real case, it is possible to extend relation (7.9) for the infinite series, under certain assumptions on convergence. The following infinite variant of (7.9) is established with respect to the strong operator topology. Recall, the sequence $(A_i)_{i\in\mathbb{N}}$ in $\mathfrak{B}_h(\mathscr{H})$ converges strongly to $A \in \mathfrak{B}_h(\mathscr{H})$ if $A_n x$ converges to Ax for all $x \in \mathscr{H}$. Moreover, a double sum $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} A_{ij}$, where $A_{ij} \in \mathfrak{B}^{++}(\mathscr{H})$, means the limit of the sequence

$$S_k = \sum_{\substack{i,j\in\mathbb{N}\\ i+j\leq k+1}} A_{ij}, \quad k \in \mathbb{N},$$

provided that $(S_k)_{k\in\mathbb{N}}$ converges with respect to strong operator topology.

Theorem 7.3 *Let $1/p+1/q=1$, $p>1$, let $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ be a non-negative measurable homogeneous function of degree $-s$, $s>0$, strictly decreasing in each argument, and let $\alpha,\beta>0$ be such that $k(\beta p)<\infty$ and $k(2-\alpha q-s)<\infty$. Further, suppose $A_i, B_j \in \mathfrak{B}^{++}(\mathscr{H})$, $i,j\in\mathbb{N}$, are such that the series $\sum_{i=1}^{\infty} i^{1-s+(\alpha-\beta)p}A_i^p$ and $\sum_{j=1}^{\infty} j^{1-s+(\beta-\alpha)q}B_j^q$ converge strongly. Then, series $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} K(i,j)A_i^p\sharp_{1/q}B_j^q$ also converges strongly, yielding the inequality*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} K(i,j)A_i^p \sharp_{1/q} B_j^q$$
$$\leq k^{\frac{1}{p}}(\beta p)k^{\frac{1}{q}}(2-\alpha q-s)\left[\sum_{i=1}^{\infty} i^{1-s+(\alpha-\beta)p}A_i^p\right] \sharp_{1/q} \left[\sum_{j=1}^{\infty} j^{1-s+(\beta-\alpha)q}B_j^q\right]. \quad (7.10)$$

Proof. Since $A_i, B_j \in \mathfrak{B}^{++}(\mathscr{H})$, $i,j\in\mathbb{N}$, the strong operator convergence implies that $\sum_{i=1}^{\infty} i^{1-s+(\alpha-\beta)p}A_i^p \in \mathfrak{B}^{++}(\mathscr{H})$ and $\sum_{j=1}^{\infty} j^{1-s+(\beta-\alpha)q}B_j^q \in \mathfrak{B}^{++}(\mathscr{H})$, so that the geometric mean on the right-hand side of inequality (7.10) is well-defined.

Moreover, due to positivity of operators A_i, B_j, $i,j \in \mathbb{N}$, the following two inequalities are obviously valid for each $m,n \in \mathbb{N}$:

$$\sum_{i=1}^{m} i^{1-s+(\alpha-\beta)p} A_i^p \le \sum_{i=1}^{\infty} i^{1-s+(\alpha-\beta)p} A_i^p$$
$$\sum_{j=1}^{n} j^{1-s+(\beta-\alpha)q} B_j^q \le \sum_{j=1}^{\infty} j^{1-s+(\beta-\alpha)q} B_j^q.$$

Further, taking into account the monotonicity principle (C1) for operator geometric mean and the above two inequalities, relation (7.9) yields inequality

$$\sum_{i=1}^{m}\sum_{j=1}^{n} K(i,j) A_i^p \sharp_{1/q} B_j^q$$
$$\le k^{\frac{1}{p}}(\beta p) k^{\frac{1}{q}}(2-\alpha q - s)\left[\sum_{i=1}^{\infty} i^{1-s+(\alpha-\beta)p} A_i^p\right] \sharp_{1/q} \left[\sum_{j=1}^{\infty} j^{1-s+(\beta-\alpha)q} B_j^q\right], \tag{7.11}$$

which hold for every $m,n \in \mathbb{N}$.

Now, consider the monotone increasing sequence of positive operators

$$S_k = \sum_{\substack{i,j\in\mathbb{N} \\ i+j\le k+1}} K(i,j) A_i^p \sharp_{1/q} B_j^q, \quad k \in \mathbb{N}.$$

Since the right-hand side of inequality (7.11) is a bounded operator, there exist a constant $d > 0$ such that $|S_k| \le d$ for all $k \in \mathbb{N}$. This means that the sequence $(S_k)_{k\in\mathbb{N}}$ is norm bounded, which yields its convergence with respect to strong operator topology (see e.g. [83]). Hence, regarding the limit of the sequence $(S_k)_{k\in\mathbb{N}}$ as the sum of the corresponding double series, we have

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} K(i,j) A_i^p \sharp_{1/q} B_j^q = \lim_k S_k,$$

so (7.11) yields inequality (7.10) and the proof is completed. □

Taking into account considerations as in the proof of Theorem 7.3, the inequality (7.4) is also meaningful for infinite series. More precisely, assuming the convergence of the integrals and strong convergence of the series on the right-hand side of (7.4), the inequality (7.4) also holds for $m = \infty$ and $n = \infty$.

A typical example of a homogeneous kernel $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ fulfilling conditions as in Theorem 7.3 is $K(x,y) = (x+y)^{-s}$, $s > 0$. In that case the constant on the right-hand side of inequality (7.10) is expressed in terms of the Beta function.

Corollary 7.1 *Let p and q be conjugate exponents with $p > 1$, let $s > 0$, and let α, β be real parameters such that $\alpha q, \beta p \in (\max\{1-s,0\},1)$. If $A_i, B_j \in \mathfrak{B}^{++}(\mathscr{H})$, $i,j \in \mathbb{N}$, are such that the series $\sum_{i=1}^{\infty} i^{1-s+(\alpha-\beta)p} A_i^p$ and $\sum_{j=1}^{\infty} j^{1-s+(\beta-\alpha)q} B_j^q$ converge strongly, then the series*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{A_i^p \sharp_{1/q} B_j^q}{(i+j)^s}$$

also converges strongly, and

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{A_i^p\sharp_{1/q}B_j^q}{(i+j)^s}\leq l\left[\sum_{i=1}^{\infty}i^{1-s+(\alpha-\beta)p}A_i^p\right]\sharp_{1/q}\left[\sum_{j=1}^{\infty}j^{1-s+(\beta-\alpha)q}B_j^q\right],\tag{7.12}$$

where $l=B^{1/p}(s+\beta p-1,1-\beta p)B^{1/q}(s+\alpha q-1,1-\alpha q)$.

Inequality (7.12) and its consequences will be dealt with in the sequel. In such a way we are going to derive operator form of the Hilbert double series theorem (7.1).

7.2 The Best Possible Constants

In this section our attention will be focused on determining the conditions under which the constant factor $k^{1/p}(\beta p)k^{1/q}(2-\alpha q-s)$ is the best possible in inequality (7.10). Similarly to previous chapters we consider α and β such that

$$\alpha q+\beta p=2-s,\tag{7.13}$$

so that inequality (7.10) takes form

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}K(i,j)A_i^p\sharp_{1/q}B_j^q\leq k(\beta p)\left[\sum_{i=1}^{\infty}i^{\alpha pq-1}A_i^p\right]\sharp_{1/q}\left[\sum_{j=1}^{\infty}j^{\beta pq-1}B_j^q\right].\tag{7.14}$$

In the sequel, we are going to show that constant $k(\beta p)$ is the best possible in inequality (7.14).

Remark 7.2 Let K be the kernel satisfying conditions as in the statement of Theorem 7.3. Observe that the homogeneity of degree $-s$ implies the following sequence of identities:

$$k(a)=\int_0^{\infty}K\left(\frac{1}{u},1\right)u^{-s-a}du=\int_0^{\infty}K(u,1)u^{s+a-2}du,$$

while from the strict decrease of the kernel in each argument we obtain that K is strictly positive on $\mathbb{R}_+\times\mathbb{R}_+$. In particular, for $a\geq 1$, monotonicity of K in the second argument and the fact that $K(1,1)>0$ yield

$$k(a)=\int_0^{\infty}K(1,u)u^{-a}du\geq\int_0^{1}K(1,u)u^{-a}du\geq K(1,1)\int_0^1u^{-a}du=\infty.$$

Analogous result holds also for $a\leq 1-s$, since

$$k(a)=\int_0^{\infty}K(u,1)u^{s+a-2}du\geq\int_0^{1}K(u,1)u^{s+a-2}du\geq K(1,1)\int_0^1u^{s+a-2}du=\infty.$$

Therefore, the interval $(1-s,1)$ covers all arguments a for which $k(a)$ may converge. In order to establish the best possible constant factor in (7.14), the integral $k(a)$ will assumed to converge on this interval.

Theorem 7.4 *Let $1/p+1/q=1$, $p>1$, and let $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ be a non-negative measurable homogeneous function of degree $-s$, $s>0$, strictly decreasing in each argument such that $k(a)<\infty$ for $a\in(1-s,1)$. If $K(1,t)$ is bounded on $(0,1)$ and α,β are such that $\alpha q,\beta p\in(\max\{1-s,0\},1)$ and $\alpha q+\beta p=2-s$, then the constant $k(\beta p)$ is the best possible in inequality* (7.14).

Proof. Suppose the constant $k(\beta p)$ is not the best possible in inequality (7.14). This means that there exist a constant k', $0<k'<k(\beta p)$, so that the inequality

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}K(i,j)A_i^p\sharp_{1/q}B_j^q\le k'\left[\sum_{i=1}^{\infty}i^{\alpha pq-1}A_i^p\right]\sharp_{1/q}\left[\sum_{j=1}^{\infty}j^{\beta pq-1}B_j^q\right]\tag{7.15}$$

holds for all operators $A_i,B_j\in\mathfrak{B}^{++}(\mathscr{H})$, $i,j\in\mathbb{N}$, such that the series $\sum_{i=1}^{\infty}i^{\alpha pq-1}A_i^p$ and $\sum_{j=1}^{\infty}j^{\beta pq-1}B_j^q$ converge strongly.

Consider the operators $\widetilde{A}_i=i^{-\alpha q-\varepsilon/p}I_{\mathscr{H}}$ and $B_j=j^{-\beta p-\varepsilon/q}I_{\mathscr{H}}$, $i,j\in\mathbb{N}$, where $0<\varepsilon<q-\beta pq$. In this setting the series on the right-hand side of inequality (7.15) converge strongly, i.e.

$$\sum_{i=1}^{\infty}i^{\alpha pq-1}\widetilde{A}_i^p=\sum_{j=1}^{\infty}j^{\beta pq-1}\widetilde{B}_j^q=\left(\sum_{i=1}^{\infty}i^{-1-\varepsilon}\right)I_{\mathscr{H}},$$

so that the right-hand side of inequality (7.15) becomes $k'\left(\sum_{i=1}^{\infty}i^{-1-\varepsilon}\right)I_{\mathscr{H}}$. Moreover, since $\sum_{i=1}^{\infty}i^{-1-\varepsilon}\le 1+\int_1^{\infty}t^{-1-\varepsilon}dt=1+1/\varepsilon$, we have that the right-hand side of inequality (7.15) is not greater than

$$\left[k'+\frac{k'}{\varepsilon}\right]I_{\mathscr{H}},\tag{7.16}$$

of course, with respect to operator order.

On the other hand, since the right-hand side of inequality (7.15) is a bounded operator for operators $\widetilde{A}_i$, $\widetilde{B}_j$, $i,j\in\mathbb{N}$, as above, the series of operators

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}K(i,j)\widetilde{A}_i^p\sharp_{1/q}\widetilde{B}_j^q$$

converges strongly, as well. Moreover, since $\widetilde{A}_i^p\sharp_{1/q}\widetilde{B}_j^q=i^{-\alpha q-\varepsilon/p}j^{-\beta p-\varepsilon/q}I_{\mathscr{H}}$, $i,j\in\mathbb{N}$, we have

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}K(i,j)\widetilde{A}_i^p\sharp_{1/q}\widetilde{B}_j^q=\left[\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}K(i,j)i^{-\alpha q-\frac{\varepsilon}{p}}j^{-\beta p-\frac{\varepsilon}{q}}\right]I_{\mathscr{H}}.$$

Now, since the function $K(x,y)x^{-\alpha q-\frac{\varepsilon}{p}}y^{-\beta p-\frac{\varepsilon}{q}}$ is strictly decreasing in both arguments x and y, we have

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}K(i,j)i^{-\alpha q-\frac{\varepsilon}{p}}j^{-\beta p-\frac{\varepsilon}{q}}&\ge\int_1^{\infty}\int_1^{\infty}K(x,y)x^{-\alpha q-\frac{\varepsilon}{p}}y^{-\beta p-\frac{\varepsilon}{q}}dxdy\\&=\int_1^{\infty}x^{-1-\varepsilon}\left(\int_{1/x}^{\infty}K(1,t)t^{-\beta p-\frac{\varepsilon}{q}}dt\right)dx.\end{aligned}$$

Further, since $K(1,t)$ is bounded on $(0,1)$, there exist a constant $c > 0$ such that $K(1,t) \le c$, $t \in (0,1)$, hence

$$\int_{1/x}^{\infty} K(1,t)t^{-\beta p-\frac{\varepsilon}{q}}dt \ge k\Big(\beta p+\frac{\varepsilon}{q}\Big) - c\int_0^{1/x} t^{-\beta p-\frac{\varepsilon}{q}}dt = k\Big(\beta p+\frac{\varepsilon}{q}\Big) - \frac{cx^{\beta p+\frac{\varepsilon}{q}-1}}{1-\beta p-\frac{\varepsilon}{q}},$$

wherefrom we obtain inequality

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} K(i,j)i^{-\alpha q-\frac{\varepsilon}{p}}j^{-\beta p-\frac{\varepsilon}{q}} \ge \frac{1}{\varepsilon}k\Big(\beta p+\frac{\varepsilon}{q}\Big) - \frac{c}{\left(1-\beta p-\frac{\varepsilon}{q}\right)\left(1-\beta p+\frac{\varepsilon}{p}\right)}.$$

Therefore, the left-hand side of inequality (7.15), equipped with the above operators $\widetilde{A}_i$, $\widetilde{B}_j$, $i,j \in \mathbb{N}$, is not less than

$$\left[\frac{1}{\varepsilon}k\Big(\beta p+\frac{\varepsilon}{q}\Big) - \frac{c}{\left(1-\beta p-\frac{\varepsilon}{q}\right)\left(1-\beta p+\frac{\varepsilon}{p}\right)}\right] I_{\mathscr{H}}. \tag{7.17}$$

Finally, considering (7.15), (7.16) and (7.17) we conclude that

$$\frac{1}{\varepsilon}k\Big(\beta p+\frac{\varepsilon}{q}\Big) - \frac{c}{\left(1-\beta p-\frac{\varepsilon}{q}\right)\left(1-\beta p+\frac{\varepsilon}{p}\right)} \le k' + \frac{k'}{\varepsilon},$$

that is,

$$k\Big(\beta p+\frac{\varepsilon}{q}\Big) \le k' + \varepsilon O(1), \quad 0 < \varepsilon < q - \beta pq.$$

Now, by letting $\varepsilon \to 0$ we have $k(\beta p) \le k'$, which contradicts with our assumption $0 < k' < k(\beta p)$. The proof is now completed. □

Considering inequality (7.14) equipped with the kernel $K(x,y) = (x+y)^{-s}$, $s > 0$, the corresponding constant is expressed in terms of the Beta function, i.e. $k(\beta p) = B(s+\beta p - 1, 1-\beta p)$.

As an example of parameters α and β fulfilling condition (7.13), we consider $\alpha = \beta = \frac{2-s}{pq}$, where $2 - \min\{p,q\} < s < 2$. In this setting, inequality (7.14) yields the following consequence.

Corollary 7.2 *Let $1/p + 1/q = 1$, $p > 1$, and let $2 - \min\{p,q\} < s < 2$. If $A_i, B_j \in \mathfrak{B}^{++}(\mathscr{H})$, $i,j \in \mathbb{N}$, are such that the series $\sum_{i=1}^{\infty} i^{1-s}A_i^p$ and $\sum_{j=1}^{\infty} j^{1-s}B_j^q$ converge strongly, then the series $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} K(i,j)A_i^p \sharp_{1/q} B_j^q$ also converges strongly and*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{A_i^p \sharp_{1/q} B_j^q}{(i+j)^s} \le B\left(\tfrac{s+p-2}{p}, \tfrac{s+q-2}{q}\right)\left[\sum_{i=1}^{\infty} i^{1-s}A_i^p\right] \sharp_{1/q} \left[\sum_{j=1}^{\infty} j^{1-s}B_j^q\right]. \tag{7.18}$$

Moreover, constant $B\left(\frac{s+p-2}{p}, \frac{s+q-2}{q}\right)$ is the best possible in inequality (7.18).

Remark 7.3 If $s=1$ then the constant in inequality (7.18) reduces to the form $B(1/q,1/p) = \pi/\sin\left(\frac{\pi}{p}\right)$, providing the operator version of the Hilbert double series theorem (7.1):

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{A_i^p\sharp_{1/q}B_j^q}{i+j} \le \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left[\sum_{i=1}^{\infty}A_i^p\right]\sharp_{1/q}\left[\sum_{j=1}^{\infty}B_j^q\right].$$

7.3 An Improvement of the Hilbert Operator Inequality via the Hermite-Hadamard Inequality

As in the real case, we can also investigate some improvements of the Hilbert operator inequality. In this section, we are going to derive a general improvement of the Hilbert operator inequality, based on the Hermite-Hadamard inequality (2.7) (see Section 2.1). In the following theorem we are going to adjust the Hermite-Hadamard inequality in order to derive an improvement of Theorem 7.1. Of course, this requires some additional assumptions concerning convexity, but as a consequence, we shall obtain a better result than in Theorem 7.1.

Theorem 7.5 *Let $1/p+1/q=1$, $p>1$, and let $m,n\in\mathbb{N}$. Suppose that $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$, $\varphi,\psi:\mathbb{R}_+\to\mathbb{R}$ are non-negative measurable functions fulfilling the following conditions:*

(i) functions $K(i,t)\psi^{-p}(t)$, $i=1,2,\ldots,m$, are convex on interval $[\frac{1}{2},n+\frac{1}{2}]$;

(ii) functions $K(t,j)\varphi^{-q}(t)$, $j=1,2,\ldots,n$, are convex on interval $[\frac{1}{2},m+\frac{1}{2}]$.

Then the inequality

$$\sum_{i=1}^{m}\sum_{j=1}^{n}K(i,j)A_i^p\sharp_{1/q}B_j^q \tag{7.19}$$
$$\le\left[\sum_{i=1}^{m}\left(\int_{\frac{1}{2}}^{n+\frac{1}{2}}\frac{K(i,t)}{\psi^p(t)}dt\right)\varphi^p(i)A_i^p\right]\sharp_{1/q}\left[\sum_{j=1}^{n}\left(\int_{\frac{1}{2}}^{m+\frac{1}{2}}\frac{K(t,j)}{\varphi^q(t)}dt\right)\psi^q(j)B_j^q\right]$$

holds for all positive invertible operators $A_1,\ldots,A_m,B_1,\ldots,B_n\in\mathfrak{B}^{++}(\mathscr{H})$.

Proof. We use the same procedure as in proof of Theorem 7.1, except that we use more accurate estimates for sums $\sum_{j=1}^{n}K(i,j)\psi^{-p}(j)$ and $\sum_{i=1}^{m}K(i,j)\varphi^{-q}(i)$. Namely, since the functions $K(i,t)\psi^{-p}(t)$, $i=1,2,\ldots,m$, are convex on interval $[\frac{1}{2},n+\frac{1}{2}]$, application of Hermite-Hadamard inequality on intervals $\left[j-\frac{1}{2},j+\frac{1}{2}\right]$ yields the series of inequalities

$$\frac{K(i,j)}{\psi^p(j)}\le\int_{j-\frac{1}{2}}^{j+\frac{1}{2}}\frac{K(i,t)}{\psi^p(t)}dt,\quad j=1,2,\ldots,n,$$

that is,

$$\sum_{j=1}^{n} \frac{K(i,j)}{\psi^p(j)} \leq \int_{\frac{1}{2}}^{n+\frac{1}{2}} \frac{K(i,t)}{\psi^p(t)} dt, \quad i = 1,2,\dots,m.$$

In the same way we have

$$\sum_{i=1}^{m} \frac{K(i,j)}{\varphi^q(i)} \leq \int_{\frac{1}{2}}^{m+\frac{1}{2}} \frac{K(t,j)}{\varphi^q(t)} dt, \quad j = 1,2,\dots,n,$$

so the result follows from (7.6) and monotonicity property (C1) of geometric mean. □

Remark 7.4 Suppose functions K, φ, ψ simultaneously satisfy conditions as in Theorems 7.1 and 7.5, so that both inequalities (7.4) and (7.19) hold. Since the functions $K(i,t)\psi^{-p}(t)$ and $K(t,j)\varphi^{-q}(t)$, $i = 1,2,\dots,m$, $j = 1,2,\dots,n$ are strictly decreasing we have

$$\int_{\frac{1}{2}}^{n+\frac{1}{2}} \frac{K(i,t)}{\psi^p(t)} dt \leq \int_0^n \frac{K(i,t)}{\psi^p(t)} dt \quad \text{and} \quad \int_{\frac{1}{2}}^{m+\frac{1}{2}} \frac{K(t,j)}{\varphi^q(t)} dt \leq \int_0^m \frac{K(t,j)}{\varphi^q(t)} dt,$$

hence, regarding the monotonicity property (C1) of geometric mean, we conclude that the right-hand side of inequality (7.19) is not greater than (7.4). In other words, inequality (7.19) is an improvement of (7.4).

Of course, assuming the convergence of integrals and strong convergence of the series on the right-hand side of (7.19), the inequality (7.19) is also meaningful for $m = \infty$ and $n = \infty$.

Now we turn back to the case of homogeneous kernels and power weight functions. In such a way we are going to establish improvements of corresponding results from Sections 7.1 and 7.2.

Corollary 7.3 *Let $1/p + 1/q = 1$, $p > 1$, let $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be a non-negative measurable homogeneous function of degree $-s$, $s > 0$, such that the functions $K(1,t)t^{-\beta p}$ and $K(t,1)t^{-\alpha q}$ are convex on $\mathbb{R}_+$, and let $\alpha, \beta > 0$ be real parameters such that $k(\beta p) < \infty$ and $k(2 - \alpha q - s) < \infty$. Then the inequality*

$$\begin{aligned} &\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} K(i,j) A_i^p \sharp_{1/q} B_j^q \\ &\leq \left[\sum_{i=1}^{\infty} i^{1-s+(\alpha-\beta)p} k\big(\beta p; \tfrac{1}{2i}, \infty\big) A_i^p \right] \sharp_{1/q} \left[\sum_{j=1}^{\infty} j^{1-s+(\beta-\alpha)q} k(2-\alpha q - s; 0, 2j) B_j^q \right] \end{aligned} \tag{7.20}$$

holds for all $A_i, B_j \in \mathfrak{B}^{++}(\mathscr{H})$, $i, j \in \mathbb{N}$, such that the series $\sum_{i=1}^{\infty} i^{1-s+(\alpha-\beta)p} A_i^p$ and $\sum_{j=1}^{\infty} j^{1-s+(\beta-\alpha)q} B_j^q$ converge strongly.

Proof. Since $k\big(\beta p;\frac{1}{2i},\infty\big)\le k(\beta p)$ and $k(2-\alpha q-s;0,2j)\le k(2-\alpha q-s)$, we have that the series on the right-hand side of inequality (7.20) converge strongly. Now, due to the homogeneity of kernel K we have

$$\int_{\frac{1}{2}}^{\infty} K(i,t)t^{-\beta p}dt \;=\; i^{1-s-\beta p}k\big(\beta p;\tfrac{1}{2i},\infty\big),$$
$$\int_{\frac{1}{2}}^{\infty} K(t,j)t^{-\alpha q}dt \;=\; j^{1-s-\alpha q}k(2-\alpha q-s;0,2j),$$

that is, result follows from (7.19). □

In the previous two sections we have considered the homogeneous kernel $K(x,y)=(x+y)^{-s}$, $s>0$. This kernel is also suitable for application of Corollary 7.3. Namely, considering the second derivative of function $f(t)=(1+t)^{-s}t^{-a}$, where $a>0$, we have

$$f''(t)=\frac{(s+a)(s+a+1)t^2+2a(s+a+1)t+a(a+1)}{t^{a+2}(1+t)^{s+2}},$$

that is, $f''(t)>0$ for $t\in\mathbb{R}_+$ since $a>0$ and $s>0$. Thus, due to the symmetry, the above kernel $K(x,y)=(x+y)^{-s}$ fulfills convexity conditions as in Corollary 7.3. Moreover, when applying Corollary 7.3 to this kernel, the weight functions will be expressed in terms of the incomplete Beta function (see Section 2.1)

$$B_r(a,b)=\int_0^r t^{a-1}(1-t)^{b-1}\,dt,\quad a,b>0.$$

Recall that for $r=1$ the incomplete Beta function coincides with the usual Beta function and obviously, $B_r(a,b)\le B(a,b)$, $a,b>0$, $0\le r\le 1$.

Corollary 7.4 *Let p and q be conjugate exponents with $p>1$, let $s>0$, and let α,β be real parameters such that $\alpha q,\beta p\in(\max\{1-s,0\},1)$. Then the inequality*

$$\begin{aligned}&\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{A_i^p\sharp_{1/q}B_j^q}{(i+j)^s}\\ &\le\left[\sum_{i=1}^{\infty} i^{1-s+(\alpha-\beta)p}B_{\frac{2i}{2i+1}}(s+\beta p-1,1-\beta p)A_i^p\right]\\ &\qquad\sharp_{1/q}\left[\sum_{j=1}^{\infty} j^{1-s+(\beta-\alpha)q}B_{\frac{2j}{2j+1}}(s+\alpha q-1,1-\alpha q)B_j^q\right]\end{aligned}\tag{7.21}$$

holds for all $A_i,B_j\in\mathfrak{B}^{++}(\mathscr{H})$, $i,j\in\mathbb{N}$, such that the series $\sum_{i=1}^{\infty} i^{1-s+(\alpha-\beta)p}A_i^p$ and $\sum_{j=1}^{\infty} j^{1-s+(\beta-\alpha)q}B_j^q$ converge strongly.

We conclude this chapter with a consequence of the previous corollary, regarding the same parameters α and β as in Corollary 7.2. Namely, considering $\alpha=\beta=\frac{2-s}{pq}$, where

$2-\min\{p,q\}<s<2$, Corollaries 7.2 and 7.4 yield the following interpolating series of inequalities:

$$
\begin{aligned}
&\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{A_i^p\sharp_{1/q}B_j^q}{(i+j)^s}\\
&\leq\left[\sum_{i=1}^{\infty}i^{1-s}B_{\frac{2i}{2i+1}}\left(\tfrac{s+p-2}{p},\tfrac{s+q-2}{q}\right)A_i^p\right]\sharp_{1/q}\left[\sum_{j=1}^{\infty}j^{1-s}B_{\frac{2j}{2j+1}}\left(\tfrac{s+q-2}{q'},\tfrac{s+p-2}{p}\right)B_j^q\right]\\
&\leq B\left(\tfrac{s+p-2}{p},\tfrac{s+q-2}{q}\right)\left[\sum_{i=1}^{\infty}i^{1-s}A_i^p\right]\sharp_{1/q}\left[\sum_{j=1}^{\infty}j^{1-s}B_j^q\right].
\end{aligned}
$$

In other words, inequality (7.21) refines previously deduced inequality (7.18).

Remark 7.5 The method and the results presented in this chapter were developed in paper [56].

Chapter 8

A Relation Between Hilbert-type and Carlson-type Inequalities

First, let us recall some *Carlson-type inequalities*. In 1935, Carlson [30], proved the following curious inequality: If $a_1, a_2, \ldots$ are real numbers, not all zero, then

$$\left(\sum_{n=1}^{\infty} a_n\right)^2 < \pi \left(\sum_{n=1}^{\infty} a_n^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} n^2 a_n^2\right)^{\frac{1}{2}}, \tag{8.1}$$

where π is the best possible constant. In 1937, Gabriel [45], proved a more general version of the Carlson inequality. In his work, Gabriel used a method similar to Carlson's original proof. However, he mentioned that Hardy's method could also be used. If $p > 1$, $a_n \geq 0$ and $0 < \delta \leq p-1$, then

$$\begin{aligned}\left(\sum_{n=1}^{\infty} a_n\right)^p &< \frac{2}{(2\delta)^{p-1}} \left(B\left(\frac{1}{2p-2}, \frac{1}{2p-2}\right)\right)^{p-1} \\ &\quad \times \left(\sum_{n=1}^{\infty} n^{p-1-\delta} a_n^p\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} n^{p-1+\delta} a_n^p\right)^{\frac{1}{2}},\end{aligned} \tag{8.2}$$

and the constant $\frac{2}{(2\delta)^{p-1}} \left(B\left(\frac{1}{2p-2}, \frac{1}{2p-2}\right)\right)^{p-1}$ is the best possible. For more details about the Carlson-type inequalities the reader is referred to [70].

In 2012, Azar [21], gave a new discrete inequality with conjugate parameters p and q, $p>1$, which is a relation between the Hilbert inequality and the Carlson inequality, as

$$\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sigma_{m,n}\right)^2 < \widehat{C}\left\{\sum_{n=1}^{\infty} m^{-1+pqA_1}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{-1+pqA_2}b_n^q\right\}^{\frac{1}{q}} \times\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m\sigma_{m,n}^2}{a_m b_n}\right\}^{pA_2}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{n\sigma_{m,n}^2}{a_m b_n}\right\}^{qA_1}, \tag{8.3}$$

where $a_m, b_n, \sigma_{m,n} > 0$, $A_1 \in \left(0,\frac{1}{q}\right), A_2 \in \left(0,\frac{1}{p}\right), pA_2+qA_1 = 1$, and the constant $\widehat{C} = \frac{B(pA_2,1-pA_2)}{(pA_2)^{pA_2}(qA_1)^{qA_1}}$ is the best possible.

The main objective of this chapter is to generalize the inequality (8.3) related to the inequality (8.2) with the best constant factor. First we derive general discrete and integral forms of inequality (8.3) with conjugate exponents in two-dimensional, and later on, in multidimensional integral case. It should be noticed here that we assume the convergence of series and integrals appearing in this chapter.

8.1 Generalizations on $\mathbb{R}_+^2$

Our result will be based on general Hilbert-type inequalities (1.33) and (1.14). We consider here the set of functions $H(r)$ (see Section 1.1, before Theorem 1.5) satisfying an extra condition $u((m_0-1)+) = 0$. We denote by $H_{m_0}(r)$ a subset of $H(r)$ fulfilling this condition.

If we let $u(x) \to \alpha u(x), v(x) \to \beta v(x)$, $K(u(m),v(n)) = (u(m)+v(n))^{-s/r}$ and $u_1(x) \to \alpha\varphi(x), u_2(y) \to \beta\psi(y)$, $s \to \frac{s}{r}$ $(\alpha,\beta>0)$, $K(u(x),v(y)) = (u(x)+v(y))^{-s/r}$, $n=2$ in (1.33) and (1.14), we have

$$\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{a_m b_n}{(\alpha u(m)+\beta v(n))^{\frac{s}{r}}} < L\left\{\sum_{m=m_0}^{\infty}[u(m)]^{-1+pqA_1}[u'(m)]^{1-p}a_m^p\right\}^{\frac{1}{p}} \times\left\{\sum_{n=n_0}^{\infty}[v(n)]^{-1+pqA_2}[v'(n)]^{1-q}b_n^q\right\}^{\frac{1}{q}}, \tag{8.4}$$

and

$$\int_a^b\int_c^d \frac{f(x)g(y)}{(\alpha\varphi(m)+\beta\psi(n))^{\frac{s}{r}}}dxdy < L\left\{\int_a^b[\varphi(x)]^{-1+pqA_1}[\varphi'(x)]^{1-p}f^p(x)dx\right\}^{\frac{1}{p}} \times\left\{\int_c^d[\psi(y)]^{-1+pqA_2}[\psi'(y)]^{1-q}g^q(y)dy\right\}^{\frac{1}{q}}, \tag{8.5}$$

where $L = \frac{B(1-pA_2,1-qA_1)}{\alpha^{1-qA_1}\beta^{1-pA_2}}$, $A_1 \in (\max\{\frac{1-\frac{s}{r}}{q},0\},\frac{1}{q})$, $A_2 \in (\max\{\frac{1-\frac{s}{r}}{p},0\},\frac{1}{p})$ and $pA_2+qA_1 = 2-\frac{s}{r}$.

8.1.1 A Discrete Inequality

The first result is a generalization of the inequality (8.3) involving some additional parameters and functions.

Theorem 8.1 *Let $p>1, \frac{1}{p}+\frac{1}{q}=1, r>1, \frac{1}{r}+\frac{1}{s}=1$ and $m_0,n_0 \in \mathbb{N}$. Suppose that $A_1 \in (\max\{\frac{1-\frac{s}{r}}{q},0\},\frac{1}{q})$, $A_2 \in (\max\{\frac{1-\frac{s}{r}}{p},0\},\frac{1}{p})$, $pA_2+qA_1 = 2-\frac{s}{r}>0$, $u \in H_{m_0}(qA_1)$ and $v \in H_{n_0}(pA_2)$. If $(a_m),(b_n)$ and $(\sigma_{m,n})$ are positive sequences, then*

$$\left(\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\sigma_{m,n}\right)^r < C\left\{\sum_{m=m_0}^{\infty} w_1(m)a_m^p\right\}^{\frac{r}{ps}}\left\{\sum_{n=n_0}^{\infty} w_2(n)b_n^q\right\}^{\frac{r}{qs}}$$
$$\times\left\{\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{u(m)\sigma_{m,n}^r}{(a_mb_n)^{\frac{r}{s}}}\right\}^{\frac{r(1-qA_1)}{s}}\left\{\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{v(n)\sigma_{m,n}^r}{(a_mb_n)^{\frac{r}{s}}}\right\}^{\frac{r(1-pA_2)}{s}}, \tag{8.6}$$

where $w_1(x) = [u(x)]^{-1+pqA_1}[u'(x)]^{1-p}, w_2(x) = [v(x)]^{-1+pqA_2}[v'(x)]^{1-q}$. In addition, the constant $C = \frac{s[B(1-pA_2,1-qA_1)]^{\frac{r}{s}}}{r(1-qA_1)^{\frac{r(1-qA_1)}{s}}(1-pA_2)^{\frac{r(1-pA_2)}{s}}}$ is the best possible.

Proof. Let $\alpha,\beta>0$. Utilizing the Hölder inequality and then, applying (8.4), we have

$$\begin{aligned}
&\left\{\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\sigma_{m,n}\right\}^r \\
&= \left\{\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\left(\frac{(a_mb_n)^{\frac{1}{s}}}{(\alpha u(m)+\beta v(n))^{\frac{1}{r}}}\right)\left(\frac{(\alpha u(m)+\beta v(n))^{\frac{1}{r}}}{(a_mb_n)^{\frac{1}{s}}}\sigma_{m,n}\right)\right\}^r \\
&\le \left\{\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{a_mb_n}{(\alpha u(m)+\beta v(n))^{\frac{s}{r}}}\right\}^{\frac{r}{s}}\left\{\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{\alpha u(m)+\beta v(n)}{(a_mb_n)^{\frac{r}{s}}}\sigma_{m,n}^r\right\} \\
&< \frac{[B(1-pA_2,1-qA_1)]^{\frac{r}{s}}}{\alpha^{\frac{r(1-qA_1)}{s}}\beta^{\frac{r(1-pA_2)}{s}}}\left\{\sum_{m=m_0}^{\infty} w_1(m)a_m^p\right\}^{\frac{r}{ps}}\left\{\sum_{n=n_0}^{\infty} w_2(n)b_n^q\right\}^{\frac{r}{qs}} \\
&\quad\times\left\{\alpha\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{u(m)\sigma_{m,n}^r}{(a_mb_n)^{\frac{r}{s}}}+\beta\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{v(n)\sigma_{m,n}^r}{(a_mb_n)^{\frac{r}{s}}}\right\} \\
&= [B(1-pA_2,1-qA_1)]^{\frac{r}{s}}\left\{\sum_{m=m_0}^{\infty} w_1(m)a_m^p\right\}^{\frac{r}{ps}}\left\{\sum_{n=n_0}^{\infty} w_2(n)b_n^q\right\}^{\frac{r}{qs}} \\
&\quad\times\left\{\left(\frac{\alpha}{\beta}\right)^{\frac{r(1-pA_2)}{s}}\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{u(m)\sigma_{m,n}^r}{(a_mb_n)^{\frac{r}{s}}}+\left(\frac{\beta}{\alpha}\right)^{\frac{r(1-qA_1)}{s}}\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{v(n)\sigma_{m,n}^r}{(a_mb_n)^{\frac{r}{s}}}\right\}.
\end{aligned}$$

Now, set $S=\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{u(m)\sigma_{m,n}^r}{(a_mb_n)^{\frac{r}{s}}}, T=\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{v(n)\sigma_{m,n}^r}{(a_mb_n)^{\frac{r}{s}}}, t=\frac{\alpha}{\beta}$ and consider the function $h(t)=t^{\frac{r(1-pA_2)}{s}}S+t^{\frac{r(qA_1-1)}{s}}T$. Since

$$h'(t)=\frac{r(1-pA_2)S}{s}t^{\frac{r(1-pA_2)}{s}-2}\left(t-\frac{(1-qA_1)T}{(1-pA_2)S}\right),$$

it follows that h attains its minimum at $t=\frac{(1-qA_1)T}{(1-pA_2)S}$. Thus, letting $\alpha=(1-qA_1)T$ and $\beta=(1-pA_2)S$, we obtain (8.6).

Now, in order to prove that C is the best constant, suppose that $\varepsilon>0$ is sufficiently small, $\widetilde{a}_m=[u(m)]^{-qA_1-\frac{\varepsilon}{p}}u'(m), \widetilde{b}_n=[v(n)]^{-pA_2-\frac{\varepsilon}{q}}v'(n)(m\geq m_0, n\geq n_0)$, and $\widetilde{\sigma}_{m,n}=\frac{\widetilde{a}_m\widetilde{b}_n}{(u(m)+v(n))^{\frac{s}{r}}}$. Then, considering the integral sums, we have

$$\begin{aligned}\frac{1}{\varepsilon[u(m_0)]^{\varepsilon}} &= \int_{m_0}^{\infty}[u(x)]^{-1-\varepsilon}d[u(x)]\\ &< \sum_{m=m_0}^{\infty}[u(m)]^{-1-\varepsilon}u'(m)\\ &= \sum_{m=m_0}^{\infty}[u(m)]^{-1+pqA_1}[u'(m)]^{1-p}\widetilde{a}_m^p\\ &< [u(m_0)]^{-1-\varepsilon}u'(m_0)+\int_{m_0}^{\infty}[u(x)]^{-1-\varepsilon}d[u(x)]\\ &= [u(m_0)]^{-1-\varepsilon}u'(m_0)+\frac{1}{\varepsilon[u(m_0)]^{\varepsilon}},\end{aligned}$$

and so $\sum_{m=m_0}^{\infty}[u(m)]^{-1+pqA_1}[u'(m)]^{1-p}\widetilde{a}_m^p=\frac{1}{\varepsilon[u(m_0)]^{\varepsilon}}+O(1)$. Similarly,

$$\sum_{n=n_0}^{\infty}[v(n)]^{-1+pqA_2}[v'(n)]^{1-q}\widetilde{b}_n^q=\frac{1}{\varepsilon[v(n_0)]^{\varepsilon}}+O(1).$$

In addition, substituting the above defined sequences $\widetilde{a}_m, \widetilde{b}_n$, and $\widetilde{\sigma}_{m,n}$ in the left-hand side of (8.6), we obtain the inequality

$$\begin{aligned}&\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{\widetilde{a}_m\widetilde{b}_n}{(u(m)+v(n))^{\frac{s}{r}}}\\ &>\int_{m_0}^{\infty}[u(x)]^{-qA_1-\frac{\varepsilon}{p}}\left(\int_{n_0}^{\infty}\frac{[v(y)]^{-pA_2-\frac{\varepsilon}{q}}}{(u(x)+v(y))^{\frac{s}{r}}}v'(y)dy\right)u'(x)dx\\ &=\int_{m_0}^{\infty}[u(x)]^{-1-\varepsilon}\left(\int_{\frac{v(n_0)}{u(x)}}^{\infty}\frac{t^{-pA_2-\frac{\varepsilon}{q}}}{(1+t)^{\frac{s}{r}}}dt\right)u'(x)dx\\ &=\int_{m_0}^{\infty}[u(x)]^{-1-\varepsilon}\left(\int_0^{\infty}\frac{t^{-pA_2-\frac{\varepsilon}{q}}}{(1+t)^{\frac{s}{r}}}dt-\int_0^{\frac{v(n_0)}{u(x)}}\frac{t^{-pA_2-\frac{\varepsilon}{q}}}{(1+t)^{\frac{s}{r}}}dt\right)u'(x)dx\end{aligned}$$

$$
\begin{aligned}
&> \frac{1}{\varepsilon[u(m_0)]^{\varepsilon}} B\left(1-qA_1-\frac{\varepsilon}{q}, 1-pA_2-\frac{\varepsilon}{q}\right) - \int_{m_0}^{\infty} [u(x)]^{-1-\varepsilon} u'(x) \int_0^{\frac{v(n_0)}{u(x)}} t^{-pA_2-\frac{\varepsilon}{q}} dt dx \\
&= \frac{1}{\varepsilon[u(m_0)]^{\varepsilon}} B\left(1-qA_1-\frac{\varepsilon}{q}, 1-pA_2-\frac{\varepsilon}{q}\right) \\
&\quad - \frac{1}{(1-pA_2-\frac{\varepsilon}{q})(1-pA_2+\frac{\varepsilon}{p})} \cdot \frac{[v(n_0)]^{1-pA_2-\frac{\varepsilon}{q}}}{[u(m_0)]^{1-pA_2+\frac{\varepsilon}{p}}} \\
&= \frac{1}{\varepsilon[u(m_0)]^{\varepsilon}} B\left(1-qA_1-\frac{\varepsilon}{q}, 1-pA_2-\frac{\varepsilon}{q}\right) - O(1).
\end{aligned}
$$

In the same way, we have

$$
\begin{aligned}
\sum_{m=m_0}^{\infty} \sum_{n=n_0}^{\infty} \frac{u(m)\widetilde{\sigma}_{m,n}^r}{(a_m b_n)^{\frac{r}{s}}} &= \sum_{m=m_0}^{\infty} [u(m)]^{1-qA_1-\frac{\varepsilon}{p}} u'(m) \sum_{n=n_0}^{\infty} \frac{[v(n)]^{-pA_2-\frac{\varepsilon}{q}} v'(n)}{(u(m)+v(n))^s} \\
&< \sum_{m=m_0}^{\infty} [u(m)]^{1-qA_1-\frac{\varepsilon}{p}} u'(m) \int_0^{\infty} \frac{[v(x)]^{-pA_2-\frac{\varepsilon}{q}} v'(x)}{(u(m)+v(x))^s} dx \\
&= \sum_{m=m_0}^{\infty} [u(m)]^{-1-\varepsilon} u'(m) \int_0^{\infty} \frac{t^{-pA_2-\frac{\varepsilon}{q}}}{(1+t)^s} dt \\
&= \frac{1+\varepsilon[u(m_0)]^{\varepsilon} O(1)}{\varepsilon[u(m_0)]^{\varepsilon}} B\left(s+pA_2+\frac{\varepsilon}{q}-1, 1-pA_2-\frac{\varepsilon}{q}\right) \\
&= \frac{1+\varepsilon[u(m_0)]^{\varepsilon} O(1)}{\varepsilon[u(m_0)]^{\varepsilon}} B\left(2-qA_1+\frac{\varepsilon}{q}, 1-pA_2-\frac{\varepsilon}{q}\right) \\
&= \frac{1+\varepsilon[u(m_0)]^{\varepsilon} O(1)}{\varepsilon[u(m_0)]^{\varepsilon}} \cdot \frac{r(1-qA_1+\frac{\varepsilon}{q})}{s} \\
&\quad \times B\left(1-qA_1+\frac{\varepsilon}{q}, 1-pA_2-\frac{\varepsilon}{q}\right),
\end{aligned}
$$

and similarly,

$$
\sum_{m=m_0}^{\infty} \sum_{n=n_0}^{\infty} \frac{v(n)\widetilde{\sigma}_{m,n}^r}{(a_m b_n)^{\frac{r}{s}}} < \frac{1+\varepsilon[v(n_0)]^{\varepsilon} O(1)}{\varepsilon[v(n_0)]^{\varepsilon}} \cdot \frac{r(1-pA_2+\frac{\varepsilon}{p})}{s} B\left(1-pA_2+\frac{\varepsilon}{p}, 1-qA_1-\frac{\varepsilon}{p}\right).
$$

If the constant C in (8.6) is not the best possible, then there exists a positive constant $\widetilde{C}$(with $\widetilde{C} < C$), such that (8.6) is still valid when we replace C by $\widetilde{C}$. In particular, utilizing the derived inequalities, we have

$$
\begin{aligned}
&\left(\frac{1}{\varepsilon[u(m_0)]^{\varepsilon}} B\left(1-qA_1-\frac{\varepsilon}{q}, 1-pA_2-\frac{\varepsilon}{q}\right) - O(1)\right)^r \\
&< \widetilde{C} \left\{\frac{1}{\varepsilon[u(m_0)]^{\varepsilon}} + O(1)\right\}^{\frac{r}{ps}} \left\{\frac{1}{\varepsilon[v(n_0)]^{\varepsilon}} + O(1)\right\}^{\frac{r}{qs}} \\
&\quad \times \left\{\frac{1+\varepsilon[u(m_0)]^{\varepsilon} O(1)}{\varepsilon[u(m_0)]^{\varepsilon}} \cdot \frac{r(1-qA_1+\frac{\varepsilon}{q})}{s} B\left(1-qA_1+\frac{\varepsilon}{q}, 1-pA_2-\frac{\varepsilon}{q}\right)\right\}^{\frac{r(1-qA_1)}{s}}
\end{aligned}
$$

$$\times \left\{ \frac{1+\varepsilon[v(n_0)]^{\varepsilon} O(1)}{\varepsilon[v(n_0)]^{\varepsilon}} \cdot \frac{r(1-pA_2+\frac{\varepsilon}{p})}{s} B\left(1-pA_2+\frac{\varepsilon}{p}, 1-qA_1-\frac{\varepsilon}{p}\right) \right\}^{\frac{r(1-pA_2)}{s}}. \tag{8.7}$$

Multiplying inequality (8.7) by ε^r and then, letting $\varepsilon \to 0^+$, it follows that

$$C = \frac{s[B(1-pA_2, 1-qA_1)]^{\frac{r}{s}}}{r(1-qA_1)^{\frac{r(1-qA_1)}{s}}(1-pA_2)^{\frac{r(1-pA_2)}{s}}} \le \widetilde{C},$$

which contradicts with the fact that $\widetilde{C} < C$. Hence, the constant C in (8.6) is the best possible. This completes the proof. □

Considering Theorem 8.1 with $\sigma_{m,n} = \frac{a_m b_n}{(u(m)+v(n))^{\frac{s}{r}}}$, $S = \sum_{m=m_0}^{\infty} \sum_{n=n_0}^{\infty} \frac{u(m)a_m b_n}{(u(m)+v(n))^s}$, $T = \sum_{m=m_0}^{\infty} \sum_{n=n_0}^{\infty} \frac{v(n)a_m b_n}{(u(m)+v(n))^s}$ and $S+T = \sum_{m=m_0}^{\infty} \sum_{n=n_0}^{\infty} \frac{a_m b_n}{(u(m)+v(n))^{\frac{s}{r}}}$, we obtain the following consequence:

Corollary 8.1 *Suppose the parameters p,q,r,s,A_1,A_2, and the functions $u,v : \mathbb{R}_+ \to \mathbb{R}$ are defined as in the statement of Theorem* 8.1. *If (a_m) and (b_n) are positive sequences, then,*

$$\sum_{m=m_0}^{\infty} \sum_{n=n_0}^{\infty} \frac{a_m b_n}{(u(m)+v(n))^{\frac{s}{r}}} < C_1 \left\{ \sum_{m=m_0}^{\infty} w_1(m) a_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=n_0}^{\infty} w_2(n) b_n^q \right\}^{\frac{1}{q}} \cdot R^{\frac{s}{r}}, \tag{8.8}$$

where

$$R = \frac{\left(\frac{S}{1-qA_1}\right)^{\frac{r(1-qA_1)}{s}} \left(\frac{T}{1-pA_2}\right)^{\frac{r(1-pA_2)}{s}}}{S+T},$$

$w_1(x) = [u(x)]^{-1+pqA_1}[u'(x)]^{1-p}$, $w_2(x) = [v(x)]^{-1+pqA_2}[v'(x)]^{1-q}$.
In addition, the constant $C_1 = \left(\frac{s}{r}\right)^{\frac{s}{r}} \cdot B(1-pA_2, 1-qA_1)$ is the best possible.

In particular, (**I**) for $A,B,\alpha,\beta > 0$, setting $u(x) = Ax^{\alpha}, v(x) = Bx^{\beta}, m_0 = n_0 = 1$, we have the inequality

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(Am^{\alpha}+Bn^{\beta})^{\frac{s}{r}}} < C_1 \left\{ \sum_{m=1}^{\infty} w_1(m) a_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} w_2(n) b_n^q \right\}^{\frac{1}{q}} \cdot R^{\frac{s}{r}},$$

where the constant

$$C_1 = \left(\frac{s}{r}\right)^{\frac{s}{r}} \cdot \frac{B(1-pA_2, 1-qA_1)}{A^{1-qA_1} B^{1-pA_2} \alpha^{\frac{1}{q}} \beta^{\frac{1}{q}}},$$

is the best possible and $w_1(m) = m^{p(\alpha qA_1 - \alpha + 1) - 1}, w_2(n) = n^{q(\beta pA_2 - \beta + 1) - 1}$.

(**II**) If $\alpha, \beta > 0$, putting $u(x) = \alpha \log x, v(x) = \beta \log x, m_0 = n_0 = 2$, we have

$$\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{a_m b_n}{(\alpha \log m + \beta \log n)^{\frac{s}{r}}} < C_1 \left\{ \sum_{m=2}^{\infty} w_1(m) a_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=2}^{\infty} w_2(n) b_n^q \right\}^{\frac{1}{q}} \cdot R^{\frac{s}{r}},$$

where

$$C_1 = \left(\frac{s}{r}\right)^{\frac{s}{r}} \cdot \frac{B(1-pA_2, 1-qA_1)}{\alpha^{1-qA_1}\beta^{1-pA_2}},$$

is the best constant and $w_1(m) = (\log m)^{-1+pqA_1}m^{p-1}, w_2(n) = (\log n)^{-1+pqA_2}n^{q-1}$.

(**III**) For $\alpha, \beta > 0$, set $u(x) = \alpha \log x, v(x) = \beta x, m_0 = 2, n_0 = 1$. Then,

$$\sum_{m=2}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{(\alpha \log m + \beta n)^{\frac{s}{r}}} < C_1 \left\{\sum_{m=2}^{\infty} w_1(m) a_m^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} w_2(n) b_n^q\right\}^{\frac{1}{q}} \cdot R^{\frac{s}{r}},$$

where

$$C_1 = \left(\frac{s}{r}\right)^{\frac{s}{r}} \cdot \frac{B(1-pA_2, 1-qA_1)}{\alpha^{1-qA_1}\beta^{1-pA_2}}$$

is the best constant and $w_1(m) = (\log m)^{-1+pqA_1}m^{p-1}, w_2(n) = n^{-1+pqA_2}$.

Theorem 8.2 *Inequality* (8.8) *refines inequality* (8.4).

Proof. Utilizing the Young inequality, we have

$$\begin{aligned} R &= \frac{\left(\frac{S}{1-qA_1}\right)^{\frac{r(1-qA_1)}{s}} \left(\frac{T}{1-pA_2}\right)^{\frac{r(1-pA_2)}{s}}}{S+T} \\ &\leq \frac{\frac{r(1-qA_1)}{s} \cdot \frac{S}{1-qA_1} + \frac{r(1-pA_2)}{s} \cdot \frac{T}{1-pA_2}}{S+T} = \frac{r}{s}. \end{aligned}$$

Now, the inequality (8.4) follows from (8.8), which completes the proof. □

Setting $u(x) = v(x) = x^\alpha, \alpha = \frac{p-q}{pq(qA_1-pA_2)} > 0, a_m = m^{\frac{k}{p}}, k = \alpha p(1-qA_1) - 1 - p, b_n = n^{\frac{l}{q}}, l = \alpha q(1-pA_2) - 1 - q$ and $\sigma_{m,n} = c_m c_n$ in Theorem 8.1, we obtain the following Gabriel-type inequality:

Corollary 8.2 *Suppose the parameters p, q, r, s, A_1, and A_2, are defined as in the statement of Theorem* 8.1. *If (c_m) is a positive sequence, then*

$$\left(\sum_{m=1}^{\infty} c_m\right)^r < C^* \left\{\sum_{m=1}^{\infty} m^{\alpha-\frac{rk}{sp}} c_m^r\right\}^{\frac{1}{2}} \left\{\sum_{m=1}^{\infty} m^{-\frac{rl}{sq}} c_m^r\right\}^{\frac{1}{2}},$$

where the constant $C^* = \sqrt{C} \cdot \left(\frac{\pi^2}{6\alpha}\right)^{\frac{r}{2s}}$ *is the best possible.*

8.1.2 An Associated Integral Form

Theorem 8.3 *Let* $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, *and* $r > 1, \frac{1}{r} + \frac{1}{s} = 1$. *Suppose that* $A_1 \in (\max\{\frac{1-\frac{s}{r}}{q}, 0\}, \frac{1}{q}), A_2 \in (\max\{\frac{1-\frac{s}{r}}{p}, 0\}, \frac{1}{p}), pA_2 + qA_1 = 2 - \frac{s}{r} > 0$, *$\varphi(x)$ and $\psi(y)$ are differentiable strictly increasing functions on (a,b) $(-\infty \leq a < b \leq \infty)$ and (c,d) $(-\infty \leq c <$

$d \le \infty$) respectively, such that $\varphi(a+) = \psi(c+) = 0$ and $\varphi(b-) = \psi(d-) = \infty$. If $f(x), g(y)$ and $G(x,y)$ are positive functions on $(a,b), (c,d)$ and $(a,b)\times(c,d)$ respectively, then the following inequality holds:

$$\begin{aligned}
&\left(\int_a^b \int_c^d G(x,y)dxdy\right)^r \\
&< C\left\{\int_a^b w_1(x) f^p(x)dx\right\}^{\frac{r}{ps}} \left\{\int_c^d w_2(y) g^q(y)dy\right\}^{\frac{r}{qs}} \\
&\times \left\{\int_a^b \int_c^d \frac{\varphi(x)G^r(x,y)}{(f(x)g(y))^{\frac{r}{s}}}dxdy\right\}^{\frac{r(1-qA_1)}{s}} \left\{\int_a^b \int_c^d \frac{\psi(y)G^r(x,y)}{(f(x)g(y))^{\frac{r}{s}}}dxdy\right\}^{\frac{r(1-pA_2)}{s}}.
\end{aligned} \tag{8.9}$$

Here, $w_1(x) = [\varphi(x)]^{-1+pqA_1}[\varphi'(x)]^{1-p}, w_2(y) = [\psi(y)]^{-1+pqA_2}[\psi'(y)]^{1-q}$ and the constant

$$C = \frac{s[B(1-pA_2, 1-qA_1)]^{\frac{r}{s}}}{r(1-qA_1)^{\frac{r(1-qA_1)}{s}}(1-pA_2)^{\frac{r(1-pA_2)}{s}}}$$

is the best possible.

Proof. Using the Hölder inequality, the Hilbert-type inequality (8.5) and following the lines as in the proof of Theorem 8.1, we have that (8.9) holds. Now, to prove the part with the best constant, suppose that $\varepsilon > 0$ is sufficiently small, and let

$$\widetilde{f}(x) = \begin{cases} 0, & \text{if } x \in (a, a_1) \ (a_1 = \varphi^{-1}(1)) \\ [\varphi(x)]^{-qA_1 - \frac{\varepsilon}{p}}\varphi'(x), & \text{if } x \in [a_1, b) \end{cases},$$

$$\widetilde{g}(y) = \begin{cases} 0, & \text{if } y \in (c, c_1) \ (c_1 = \psi^{-1}(1)) \\ [\psi(y)]^{-pA_2 - \frac{\varepsilon}{q}}\psi'(x), & \text{if } y \in [c_1, d) \end{cases},$$

and $\widetilde{G}(x,y) = \frac{\widetilde{f}(x)\widetilde{g}(y)}{(\varphi(x)+\psi(y))^{\frac{s}{r}}}$. Then we have

$$\left\{\int_a^b w_1(x)\widetilde{f}^p(x)dx\right\}^{\frac{r}{ps}} \left\{\int_c^d w_2(y)\widetilde{g}^q(y)dy\right\}^{\frac{r}{qs}} = \left(\frac{1}{\varepsilon}\right)^{\frac{r}{s}},$$

and

$$\begin{aligned}
&\int_a^b \int_c^d \widetilde{G}(x,y)dxdy \\
&= \int_a^b \int_c^d \frac{\widetilde{f}(x)\widetilde{g}(y)}{(\varphi(x)+\psi(y))^{\frac{s}{r}}}dxdy \\
&= \int_{a_1}^b [\varphi(x)]^{-1-\varepsilon}\varphi'(x)\int_{1/\varphi(x)}^\infty \frac{u^{-pA_2-\frac{\varepsilon}{q}}}{(1+u)^{\frac{s}{r}}}dudx \\
&= \int_{a_1}^b [\varphi(x)]^{-1-\varepsilon}\varphi'(x)\int_0^\infty \frac{u^{-pA_2-\frac{\varepsilon}{q}}}{(1+u)^{\frac{s}{r}}}dudx - \int_{a_1}^b [\varphi(x)]^{-1-\varepsilon}\varphi'(x)\int_0^{1/\varphi(x)} \frac{u^{-pA_2-\frac{\varepsilon}{q}}}{(1+u)^{\frac{s}{r}}}dudx
\end{aligned}$$

$$
\begin{aligned}
&> \frac{1}{\varepsilon}B\left(1-qA_1+\frac{\varepsilon}{q},1-pA_2-\frac{\varepsilon}{q}\right)-\int_{a_1}^{b}[\varphi(x)]^{-1-\varepsilon}\varphi'(x)\int_0^{1/\varphi(x)}u^{-pA_2-\frac{\varepsilon}{q}}dudx\\
&= \frac{1}{\varepsilon}B\left(1-qA_1+\frac{\varepsilon}{q},1-pA_2-\frac{\varepsilon}{q}\right)-\frac{1}{(1-pA_2-\frac{\varepsilon}{q})(1-pA_2+\frac{\varepsilon}{p})}\\
&= \frac{1}{\varepsilon}B\left(1-qA_1+\frac{\varepsilon}{q},1-pA_2-\frac{\varepsilon}{q}\right)-O(1).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\int_a^b\int_c^d\frac{\varphi(x)\widetilde{G}^r(x,y)}{(\widetilde{f}(x)\widetilde{g}(y))^{\frac{r}{s}}}dxdy &= \int_{a_1}^{b}[\varphi(x)]^{-1-\varepsilon}\varphi'(x)\int_{1/\varphi(x)}^{\infty}\frac{u^{-pA_2-\frac{\varepsilon}{q}}}{(1+u)^s}dudx\\
&< \int_{a_1}^{b}[\varphi(x)]^{-1-\varepsilon}\varphi'(x)\int_0^{\infty}\frac{u^{-pA_2-\frac{\varepsilon}{q}}}{(1+u)^s}dudx\\
&= \frac{1}{\varepsilon}B(2-qA_1+\frac{\varepsilon}{q},1-pA_2-\frac{\varepsilon}{q})\\
&= \frac{1}{\varepsilon}\frac{r(1-qA_1+\frac{\varepsilon}{q})}{s}B\left(1-qA_1+\frac{\varepsilon}{q},1-pA_2-\frac{\varepsilon}{q}\right),
\end{aligned}
$$

and similarly,

$$
\int_a^b\int_c^d\frac{\psi(y)\widetilde{G}^r(x,y)}{(\widetilde{f}(x)\widetilde{g}(y))^{\frac{r}{s}}}dxdy<\frac{1}{\varepsilon}\frac{r(1-pA_2+\frac{\varepsilon}{p})}{s}B\left(1-pA_2+\frac{\varepsilon}{p},1-qA_1-\frac{\varepsilon}{p}\right).
$$

Assuming that the constant C in (8.9) is not the best possible, then there exists a positive constant $\widetilde{C}<C$, such that (8.9) is still valid when we replace C by $\widetilde{C}$. In particular, utilizing the above inequalities, we have

$$
\begin{aligned}
&\left(\frac{1}{\varepsilon}B\left(1-qA_1-\frac{\varepsilon}{q},1-pA_2-\frac{\varepsilon}{q}\right)-O(1)\right)^r<\widetilde{C}\left(\frac{1}{\varepsilon}\right)^{\frac{r}{s}}\\
&\quad\times\left\{\frac{1+\varepsilon O(1)}{\varepsilon}\cdot\frac{r(1-qA_1+\frac{\varepsilon}{q})}{s}B\left(1-qA_1+\frac{\varepsilon}{q},1-pA_2-\frac{\varepsilon}{q}\right)\right\}^{\frac{r(1-qA_1)}{s}}\\
&\quad\times\left\{\frac{1+\varepsilon O(1)}{\varepsilon}\cdot\frac{r(1-pA_2+\frac{\varepsilon}{p})}{s}B\left(1-pA_2+\frac{\varepsilon}{p},1-qA_1-\frac{\varepsilon}{p}\right)\right\}^{\frac{r(1-pA_2)}{s}}.
\end{aligned}\tag{8.10}
$$

Now, multiplying inequality (8.10) by ε^r and then, letting $\varepsilon\to 0^+$, it follows that

$$
C=\frac{s[B(1-pA_2,1-qA_1)]^{\frac{r}{s}}}{r(1-qA_1)^{\frac{r(1-qA_1)}{s}}(1-pA_2)^{\frac{r(1-pA_2)}{s}}}\le\widetilde{C},
$$

which is in contrast to $\widetilde{C}<C$. The proof is now complete. □

Similarly to the discrete case, if $G(x,y) = \frac{f(x)g(y)}{(\varphi(x)+\psi(y))^{\frac{s}{r}}}$, then, setting

$$S = \int_a^b \int_c^d \frac{\varphi(x)f(x)g(y)}{(\varphi(x)+\psi(y))^s} dxdy, \; T = \int_a^b \int_c^d \frac{\psi(y)f(x)g(y)}{(\varphi(x)+\psi(y))^s} dxdy,$$

we easily obtain that $S+T = \int_a^b \int_c^d \frac{f(x)g(y)}{(\varphi(x)+\psi(y))^{\frac{s}{r}}} dxdy$, and the Theorem 8.3 yields the following consequence:

Corollary 8.3 *Suppose the parameters p,q,r,s,A_1,A_2, and the functions $\varphi,\psi : \mathbb{R}_+ \to \mathbb{R}$ are defined as in the statement of Theorem 8.3. If $f(x)$ and $g(x)$ are positive functions on $(0,\infty)$, then*

$$\int_a^b \int_c^d \frac{f(x)g(y)}{(\varphi(x)+\psi(y))^{\frac{s}{r}}} dxdy < C_1 \left\{ \int_a^b w_1(x) f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_c^d w_2(y) g^q(y) dy \right\}^{\frac{1}{q}} \cdot R^{\frac{s}{r}}, \tag{8.11}$$

where

$$R = \frac{\left(\frac{S}{1-qA_1}\right)^{\frac{r(1-qA_1)}{s}} \left(\frac{T}{1-pA_2}\right)^{\frac{r(1-pA_2)}{s}}}{S+T},$$

$w_1(x) = [\varphi(x)]^{-1+pqA_1}[\varphi'(x)]^{1-p}, w_2(y) = [\psi(y)]^{-1+pqA_2}[\psi'(y)]^{1-q}$.
In addition, the constant $C_1 = \left(\frac{s}{r}\right)^{\frac{s}{r}} \cdot B(1-pA_2, 1-qA_1)$ is the best possible.

It should be noticed here that the inequality (8.11) is more accurate than the inequality (8.5).

Theorem 8.4 *Inequality* (8.11) *refines inequality* (8.5).

Proof. The proof follows the lines of the proof of Theorem 8.2. □

If $\varphi(x) = \psi(x) = x^\alpha, 0 < \alpha < \min\left\{\frac{1}{1-qA_1}, \frac{1}{1-pA_2}\right\}, f(x) = g(x) = e^{-x}$ and $G(x,y) = \omega(x)\omega(y)$, the Theorem 8.3 yields the following integral Gabriel-type inequality:

Corollary 8.4 *Suppose the parameters p,q,r,s,A_1, and A_2, are defined as in the statement of Theorem 8.3. If $\omega(x)$ is a positive function on $(0,\infty)$, then*

$$\left(\int_0^\infty \omega(x)dx\right)^r < C^* \left\{ \int_0^\infty x^\alpha e^{\frac{rx}{s}} [\omega(x)]^r dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty e^{\frac{rx}{s}} [\omega(x)]^r dx \right\}^{\frac{1}{2}},$$

where $\mu = p + \alpha p(qA_1 - 1)$, $\nu = q + \alpha q(pA_2 - 1)$, and the constant $C^* = \sqrt{C}\left(\frac{1}{\alpha}\right)^{\frac{r}{2s}} \times \left(\frac{\Gamma(\mu)}{p^\mu}\right)^{\frac{r}{2ps}} \left(\frac{\Gamma(\nu)}{p^\nu}\right)^{\frac{r}{2qs}}$ is the best possible.

8.2 Multidimensional Integral Inequality

The starting point in this section is a multidimensional inequality (1.14) rewritten in a more suitable form. Namely, replacing u_i by $\alpha_i u_i$, $\alpha_i > 0$, and putting $\widetilde{A}_i = \lambda_i - 1$, $\lambda_i > 0$, $i = 1,2,\ldots,n$, $s = \lambda_1 + \lambda_2 + \ldots + \lambda_n := \lambda$, $K(u_1(x_1),\ldots,u_n(x_n)) = (u_1(x_1) + \cdots + u_n(x_n))^{-\lambda}$, inequality (1.14) reads

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \frac{\prod_{i=1}^{n} f_i(x_i)}{\left(\sum_{j=1}^{n} \alpha_j u_j(x_j)\right)^{\lambda}} dx_1 \cdots dx_n$$
$$< \frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \frac{\Gamma(\lambda_i)}{\alpha_i^{\lambda_i}} \prod_{i=1}^{n} \left\{ \int_{a_i}^{b_i} [u_i(x_i)]^{p_i(1-\lambda_i)-1} [u_i'(x_i)]^{1-p_i} f_i^{p_i}(x_i) dx_i \right\}^{\frac{1}{p_i}}. \tag{8.12}$$

Theorem 8.5 *Let $n \in \mathbb{N} \setminus \{1\}$, $\lambda > 0$, $\lambda_i > 0$, $p_i > 1$, $i = 1,2,\ldots,n$. Assume in addition*

$$\sum_{i=1}^{n} \lambda_i = \lambda = \frac{s}{r}, \quad \sum_{i=1}^{n} \frac{1}{p_i} = 1, \quad \frac{1}{s} + \frac{1}{r} = 1, \quad r > 1.$$

Further, suppose $u_i : (a_i,b_i) \to (0,\infty)$ $(i = 1,\ldots,n)$ are strictly increasing differentiable functions such that $u_i(a_i+) = 0$, and $u_i(b_i-) = \infty$. If the non-negative measurable functions $f_i : (a_i,b_i) \to (0,\infty), i = 1,\ldots,n$ and $F : \prod_{i=1}^{n}(a_i,b_i) \to \mathbb{R}$, satisfy

$$0 < \int_{a_i}^{b_i} [u_i(x_i)]^{p_i(1-\lambda_i)-1} [u_i'(x_i)]^{1-p_i} f_i^{p_i}(x_i) dx_i < \infty, \quad i = 1,\ldots,n,$$

and

$$0 < \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \frac{u_j(x_j)}{\left(\prod_{i=1}^{n} f_i(x_i)\right)^{\frac{1}{\lambda}}} F^r(x_1,\ldots,x_n) dx_1 \cdots dx_n < \infty, \quad j = 1,\ldots,n,$$

then

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} F(x_1,\ldots,x_n) dx_1 \cdots dx_n$$
$$< \lambda^{\frac{1}{r}} \left[\frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \frac{\Gamma(\lambda_i)}{\lambda_i^{\lambda_i}} \right]^{\frac{1}{s}} \prod_{i=1}^{n} \left\{ \int_{a_i}^{b_i} [u_i(x_i)]^{p_i(1-\lambda_i)-1} [u_i'(x_i)]^{1-p_i} f_i^{p_i}(x_i) dx_i \right\}^{\frac{1}{p_i s}}$$
$$\times \prod_{j=1}^{n} \left\{ \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \frac{u_j(x_j)}{\left(\prod_{i=1}^{n} f_i(x_i)\right)^{\frac{1}{\lambda}}} F^r(x_1,\ldots,x_n) dx_1 \cdots dx_n \right\}^{\frac{\lambda_j}{s}}, \tag{8.13}$$

where the constant $\lambda^{\frac{1}{r}} \left[\frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \frac{\Gamma(\lambda_i)}{\lambda_i^{\lambda_i}} \right]^{\frac{1}{s}}$ is the best possible.

In order to prove Theorem 8.5, we need the following two fundamental lemmas.

Lemma 8.1 *If $k \in \mathbb{N}, l \in \mathbb{N} \cup \{0\}, \lambda_i > 0$ $(i = 1, \ldots, k+1)$ and $\sum_{i=1}^{k+1} \lambda_i = \lambda$, then*

$$\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^k t_i^{\lambda_i - 1}}{(1 + \sum_{i=1}^k t_i)^{\lambda + l}} dt_1 \cdots dt_k = \frac{1}{\Gamma(\lambda)} \prod_{j=0}^{l-1} \frac{\lambda_{k+1} + j}{\lambda + j} \prod_{i=1}^{k+1} \Gamma(\lambda_i). \tag{8.14}$$

Proof. Setting $u = \frac{t_1}{1 + \sum_{i=2}^k t_i}$, we have

$$\begin{aligned}
&\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^k t_i^{\lambda_i - 1}}{(1 + \sum_{i=1}^k t_i)^{\lambda + l}} dt_1 \cdots dt_k \\
&= \int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=2}^k t_i^{\lambda_i - 1}}{(1 + \sum_{i=2}^k t_i)^{\lambda - \lambda_1 + l}} \left(\int_0^\infty \frac{u^{\lambda_1 - 1}}{(1+u)^{\lambda + l}} du \right) dt_2 \cdots dt_k \\
&= \frac{\Gamma(\lambda_1)\Gamma(\lambda - \lambda_1 + l)}{\Gamma(\lambda + l)} \int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=2}^k t_i^{\lambda_i - 1}}{(1 + \sum_{i=2}^k t_i)^{\lambda - \lambda_1 + l}} dt_2 \cdots dt_k.
\end{aligned}$$

Hence repeating the above process, we get

$$\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^k t_i^{\lambda_i - 1}}{(1 + \sum_{i=1}^k t_i)^{\lambda + l}} dt_1 \cdots dt_k = \frac{\Gamma(\lambda_{k+1} + l)}{\Gamma(\lambda + l)} \prod_{i=1}^k \Gamma(\lambda_i). \tag{8.15}$$

Moreover, by virtue of a well-known property of Gamma function $\Gamma(\lambda + 1) = \lambda\Gamma(\lambda)$ for $\lambda > 0$, one has

$$\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^k t_i^{\lambda_i - 1}}{(1 + \sum_{i=1}^k t_i)^{\lambda + l}} dt_1 \cdots dt_k = \frac{1}{\Gamma(\lambda)} \prod_{j=0}^{l-1} \frac{\lambda_{k+1} + j}{\lambda + j} \prod_{i=1}^{k+1} \Gamma(\lambda_i).$$

□

Lemma 8.2 *If $\lambda, \lambda_1, \ldots, \lambda_n, T_1, \ldots, T_n > 0$, and $\sum_{i=1}^n \lambda_i = \lambda$, then*

$$\min_{\alpha_1, \ldots, \alpha_n > 0} \left\{ \frac{1}{\prod_{i=1}^n \alpha_i^{\lambda_i / \lambda}} \sum_{j=1}^n \alpha_j T_j \right\} = \lambda \prod_{j=1}^n \left(\frac{T_j}{\lambda_j} \right)^{\frac{\lambda_j}{\lambda}}.$$

Proof. Applying the weighted arithmetic-geometric mean inequality, we obtain

$$\begin{aligned}
\frac{1}{\prod_{i=1}^n \alpha_i^{\lambda_i r/s}} \sum_{j=1}^n \alpha_j T_j &= \frac{1}{\prod_{i=1}^n \alpha_i^{\lambda_i r/s}} \sum_{j=1}^n \frac{\lambda_j r}{s} \cdot \frac{\alpha_j}{\frac{\lambda_j r}{s}} T_j \\
&\geq \frac{1}{\prod_{i=1}^n \alpha_i^{\lambda_i r/s}} \prod_{j=1}^n \frac{\alpha_j^{\frac{\lambda_j r}{s}}}{\left(\frac{\lambda_j r}{s}\right)^{\frac{\lambda_j r}{s}}} T_j^{\frac{\lambda_j r}{s}}
\end{aligned}$$

$$= \frac{s}{r}\prod_{j=1}^{n}\left(\frac{T_j}{\lambda_j}\right)^{\frac{\lambda_j r}{s}}.$$

□

Proof of Theorem 8.5. Set

$$T_j = \int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\frac{u_j(x_j)}{(\prod_{i=1}^{n} f_i(x_i))^{1/\lambda}}F^r(x_1,\ldots,x_n)dx_1\cdots dx_n.$$

Let $\alpha_i > 0,\ i = 1,\ldots,n$. Applying the Hölder inequality to the product

$$F(x_1,\ldots,x_n) = \left[\frac{(\prod_{i=1}^{n} f_i(x_i))^{\frac{1}{s}}}{\left(\sum_{j=1}^{n}\alpha_j u_j(x_j)\right)^{\frac{1}{r}}}\right]\left[\frac{\left(\sum_{j=1}^{n}\alpha_j u_j(x_j)\right)^{\frac{1}{r}}}{(\prod_{i=1}^{n} f_i(x_i))^{\frac{1}{s}}}F(x_1,\ldots,x_n)\right],$$

we have

$$\begin{aligned}
&\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}F(x_1,\ldots,x_n)dx_1\cdots dx_n\\
&\le\left\{\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\frac{\prod_{i=1}^{n} f_i(x_i)}{\left(\sum_{j=1}^{n}\alpha_j u_j(x_j)\right)^{\lambda}}dx_1\cdots dx_n\right\}^{\frac{1}{s}}\\
&\quad\times\left\{\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\frac{\sum_{j=1}^{n}\alpha_j u_j(x_j)}{(\prod_{i=1}^{n} f_i(x_i))^{\frac{1}{\lambda}}}F^r(x_1,\ldots,x_n)dx_1\cdots dx_n\right\}^{\frac{1}{r}}
\end{aligned}\tag{8.16}$$

Moreover, by virtue of (8.12), we have

$$\begin{aligned}
&\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}F(x_1,\ldots,x_n)dx_1\cdots dx_n\\
&<\left[\frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\frac{\Gamma(\lambda_i)}{\alpha_i^{\lambda_i}}\right]^{\frac{1}{s}}\prod_{i=1}^{n}\left\{\int_{a_i}^{b_i}[u_i(x_i)]^{p_i(1-\lambda_i)-1}[u_i'(x_i)]^{1-p_i}f_i^{p_i}(x_i)dx_i\right\}^{\frac{1}{p_i s}}\\
&\quad\times\left\{\sum_{j=1}^{n}\alpha_j\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\frac{u_j(x_j)}{(\prod_{i=1}^{n} f_i(x_i))^{\frac{r}{s}}}F^r(x_1,\ldots,x_n)dx_1\cdots dx_n\right\}^{\frac{1}{r}}\\
&=\left[\frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\Gamma(\lambda_i)\right]^{\frac{1}{s}}\prod_{i=1}^{n}\left\{\int_{a_i}^{b_i}[u_i(x_i)]^{p_i(1-\lambda_i)-1}[u_i'(x_i)]^{1-p_i}f_i^{p_i}(x_i)dx_i\right\}^{\frac{1}{p_i s}}\\
&\quad\times\left\{\frac{1}{\prod_{i=1}^{n}\alpha_i^{\lambda_i/\lambda}}\sum_{j=1}^{n}\alpha_j T_j\right\}^{\frac{1}{r}}.
\end{aligned}\tag{8.17}$$

Now, we optimize $\left\{\frac{1}{\prod_{i=1}^{n}\alpha_i^{\lambda_i/\lambda}}\sum_{j=1}^{n}\alpha_j T_j\right\}^{\frac{1}{r}}$ by adjusting $\alpha_1,\ldots,\alpha_n$. From Lemma 8.2 we conclude that the minimum value of expression $\left\{\lambda\prod_{j=1}^{n}\left(\frac{T_j}{\lambda_j}\right)^{\frac{\lambda_j}{\lambda}}\right\}^{\frac{1}{r}} = \lambda^{\frac{1}{r}}\prod_{j=1}^{n}\left(\frac{T_j}{\lambda_j}\right)^{\frac{\lambda_j}{s}}$ is attained when $\frac{\alpha_1 T_1}{\lambda_1}=\cdots=\frac{\alpha_n T_n}{\lambda_n}$. Therefore, if we put $\alpha_j=\frac{\lambda_j}{T_j}$ for each $i=1,\ldots,n$ in (8.17), we get (8.13).

The next step is to prove that the constant $\lambda^{\frac{1}{r}}\left[\frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\frac{\Gamma(\lambda_i)}{\lambda_i^{\lambda_i}}\right]^{\frac{1}{s}}$, appearing on the right-hand side of the inequality (8.13) is the best possible. Define $\widetilde{a}_i=u_i^{-1}(1)$. For $0<\varepsilon\ll 1$, we set

$$\widetilde{f}_i(x)=\begin{cases}0, & \text{if } x\in(a_i,\widetilde{a}_i)\\ [u_i(x)]^{\lambda_i-\frac{\varepsilon}{p_i}-1}u_i'(x), & \text{if } x\in[\widetilde{a}_i,b_i)\end{cases},$$

and

$$\widetilde{F}(x_1,\ldots,x_n)=\frac{\prod_{i=1}^{n}\widetilde{f}_i(x_i)}{\left(\sum_{i=1}^{n}u_i(x_i)\right)^{\lambda}}.$$

Define

$$\mathbb{D}_j=\{(t_1,\ldots,t_{n-1})\in(0,\infty)^{n-1};t_jt_n\le 1\}.$$

Then, we have

$$\prod_{i=1}^{n}\left\{\int_{a_i}^{b_i}[u_i(x_i)]^{p_i(1-\lambda_i)-1}[u_i'(x_i)]^{1-p_i}\widetilde{f}_i^{p_i}(x_i)dx_i\right\}^{\frac{1}{p_is}}$$
$$=\prod_{i=1}^{n}\left\{\int_{\widetilde{a}_i}^{b_i}[u_i(x_i)]^{-1-\varepsilon}u_i'(x_i)dx_i\right\}^{\frac{1}{p_is}}=\left(\frac{1}{\varepsilon}\right)^{\frac{1}{s}}.$$

Via the transforms

$$(x_1,\ldots,x_n)\mapsto(v_1,\ldots,v_n)=(u_1(x_1),\ldots,u_n(x_n))$$

and

$$(v_1,\ldots,v_n)\mapsto(t_1,\ldots,t_n)=(v_1v_n,\ldots,v_{n-1}v_n,v_n,),$$

together with

$$\sum_{i=1}^{n}\frac{1}{p_i}=1,\quad \sum_{i=1}^{n}\lambda_i=\lambda,$$

we have

$$\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\widetilde{F}(x_1,\ldots,x_n)dx_1\cdots dx_n$$
$$=\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\frac{\prod_{i=1}^{n}\widetilde{f}_i(x_i)}{\left(\sum_{i=1}^{n}u_i(x_i)\right)^{\lambda}}dx_1\cdots dx_n$$

$$= \int_{\widetilde{a}_1}^{b_1} \cdots \int_{\widetilde{a}_n}^{b_n} \frac{\prod_{i=1}^n [u_i(x_i)]^{\lambda_i - \frac{\varepsilon}{p_i} - 1} u_i'(x_i)}{(\sum_{i=1}^n u_i(x_i))^\lambda} dx_1 \cdots dx_n$$

$$= \int_1^\infty \cdots \int_1^\infty \frac{\prod_{i=1}^n v_i^{\lambda_i - \frac{\varepsilon}{p_i} - 1}}{(\sum_{i=1}^n v_i)^\lambda} dv_1 \cdots dv_n$$

$$= \int_1^\infty t_n^{-1-\varepsilon} \left(\int_{1/t_n}^\infty \cdots \int_{1/t_n}^\infty \frac{\prod_{i=1}^{n-1} t_i^{\lambda_i - \frac{\varepsilon}{p_i} - 1}}{\left(1 + \sum_{i=1}^{n-1} t_i\right)^\lambda} dt_1 \cdots dt_{n-1} \right) dt_n.$$

Taking into account overlapping of integration domains, we obtain

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \widetilde{F}(x_1, \ldots, x_n) dx_1 \cdots dx_n$$
$$\geq \int_1^\infty t_n^{-1-\varepsilon} \left(\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^{n-1} t_i^{\lambda_i - \frac{\varepsilon}{p_i} - 1}}{\left(1 + \sum_{i=1}^{n-1} t_i\right)^\lambda} dt_1 \cdots dt_{n-1} \right) dt_n$$
$$- \int_1^\infty t_n^{-1-\varepsilon} \left(\sum_{j=1}^{n-1} \int_{\mathbb{D}_j} \frac{\prod_{i=1}^{n-1} t_i^{\lambda_i - \frac{\varepsilon}{p_i} - 1}}{\left(1 + \sum_{i=1}^{n-1} t_i\right)^\lambda} dt_1 \cdots dt_{n-1} \right) dt_n$$
$$\geq \frac{1}{\varepsilon} \int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^{n-1} t_i^{\lambda_i - \frac{\varepsilon}{p_i} - 1}}{\left(1 + \sum_{i=1}^{n-1} t_i\right)^\lambda} dt_1 \cdots dt_{n-1}$$
$$- \int_1^\infty t_n^{-1} \left(\sum_{j=1}^{n-1} \int_{\mathbb{D}_j} \frac{\prod_{i=1}^{n-1} t_i^{\lambda_i - \frac{\varepsilon}{p_i} - 1}}{\left(1 + \sum_{i=1}^{n-1} t_i\right)^\lambda} dt_1 \cdots dt_{n-1} \right) dt_n. \tag{8.18}$$

Without loss of generality, it suffices to find the appropriate estimate for the integral

$$\int_{\mathbb{D}_1} \frac{\prod_{i=1}^{n-1} t_i^{\lambda_i - \frac{\varepsilon}{p_i} - 1}}{\left(1 + \sum_{i=1}^{n-1} t_i\right)^\lambda} dt_1 \cdots dt_{n-1}.$$

We choose $\varepsilon > 0$ so that

$$\lambda_1 > \varepsilon \left(\frac{1}{p_1} - 1 \right).$$

By the relation (8.14) with $l = 0$, we have

$$\int_{\mathbb{D}_1} \frac{\prod_{i=1}^{n-1} t_i^{\lambda_i - \frac{\varepsilon}{p_i} - 1}}{\left(1 + \sum_{i=1}^{n-1} t_i\right)^\lambda} dt_1 \cdots dt_{n-1}$$
$$\leq \int_0^{\frac{1}{t_n}} t_1^{\lambda_1 - \frac{\varepsilon}{p_1} - 1} \left(\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=2}^{n-1} t_i^{\lambda_i - \frac{\varepsilon}{p_i} - 1}}{\left(1 + \sum_{i=2}^{n-1} t_i\right)^\lambda} dt_2 \cdots dt_{n-1} \right) dt_1$$

$$\leq \int_0^{\frac{1}{t_n}} t_1^{\lambda_1-\frac{\varepsilon}{p_1}-1}\left(\int_0^\infty\cdots\int_0^\infty \frac{\prod_{i=2}^{n-1} t_i^{\lambda_i-\frac{\varepsilon}{p_i}-1}}{\left(1+\sum_{i=2}^{n-1} t_i\right)^{\lambda-\lambda_1+\varepsilon(\frac{1}{p_1}-1)}}dt_2\cdots dt_{n-1}\right)dt_1$$
$$= \frac{t_n^{\frac{\varepsilon}{p_1}-\lambda_1}}{\left(\lambda_1-\frac{\varepsilon}{p_1}\right)\Gamma\left(\lambda-\lambda_1+\varepsilon(\frac{1}{p_1}-1)\right)}\prod_{i=2}^{n}\Gamma\left(\lambda_i-\frac{\varepsilon}{p_i}\right)$$

due to Lemma 8.1. Hence, we have

$$\int_1^\infty t_n^{-1}\left(\int_{\mathbb{D}_1} \frac{\prod_{i=1}^{n-1} t_i^{\lambda_i-\frac{\varepsilon}{p_i}-1}}{\left(1+\sum_{i=1}^{n-1} t_i\right)^{\lambda}}dt_1\cdots dt_{n-1}\right)dt_n$$
$$\leq \frac{1}{\left(\lambda_1-\frac{\varepsilon}{p_1}\right)^2\Gamma\left(\frac{s}{r}-\lambda_1+\varepsilon(\frac{1}{p_1}-1)\right)}\prod_{i=2}^{n}\Gamma\left(\lambda_i-\frac{\varepsilon}{p_i}\right)<\infty.$$

As a consequence, from (8.18) we obtain

$$\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\widetilde{F}(x_1,\ldots,x_n)dx_1\cdots dx_n$$
$$\geq \frac{1}{\varepsilon}\int_0^\infty\cdots\int_0^\infty \frac{\prod_{i=1}^{n-1} t_i^{\lambda_i-\frac{\varepsilon}{p_i}-1}}{\left(1+\sum_{i=1}^{n-1} t_i\right)^{\lambda}}dt_1\cdots dt_{n-1}-O(1).$$

Now, from Lemma 8.1 we have

$$\begin{aligned}
&\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\widetilde{F}(x_1,\ldots,x_n)dx_1\cdots dx_n\\
&\geq \frac{1}{\varepsilon}\int_0^\infty\cdots\int_0^\infty \frac{\prod_{i=1}^{n-1} t_i^{\lambda_i-\frac{\varepsilon}{p_i}-1}}{\left(1+\sum_{i=1}^{n-1} t_i\right)^{\lambda}}dt_1\cdots dt_{n-1}-O(1) \qquad (8.19)\\
&= \frac{1}{\varepsilon}\cdot\frac{1}{\Gamma(\lambda)}\Gamma\left(\lambda_n+\varepsilon-\frac{\varepsilon}{p_n}\right)\prod_{i=1}^{n-1}\Gamma\left(\lambda_i-\frac{\varepsilon}{p_i}\right)-O(1)\\
&> \frac{1}{\varepsilon}\cdot\frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\Gamma\left(\lambda_i-\frac{\varepsilon}{p_i}\right)-O(1),
\end{aligned}$$

since $p_n>1$. On the other hand, taking into account the related definitions, we obtain

$$\prod_{j=1}^{n}\left\{\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\frac{u_j(x_j)}{\left(\prod_{i=1}^{n}\widetilde{f}_i(x_i)\right)^{\frac{r}{s}}}\widetilde{F}^r(x_1,\ldots,x_n)dx_1\cdots dx_n\right\}^{\frac{\lambda_j}{s}}$$
$$=\prod_{j=1}^{n}\left\{\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\frac{u_j(x_j)\prod_{i=1}^{n}[u_i(x_i)]^{\lambda_i-\frac{\varepsilon}{p_i}-1}u_i'(x_i)}{\left(\sum_{i=1}^{n}u_i(x_i)\right)^{s}}dx_1\cdots dx_n\right\}^{\frac{\lambda_j}{s}},$$

and employing (8.14) with $\lambda = \frac{s}{r}, l = 1$, it follows that

$$\prod_{j=1}^{n}\left\{\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\frac{u_j(x_j)}{\left(\prod_{i=1}^{n}\widetilde{f}_i(x_i)\right)^{\frac{r}{s}}}\widetilde{F}^r(x_1,\ldots,x_n)dx_1\cdots dx_n\right\}^{\frac{\lambda_j}{s}}$$

$$=\prod_{j=1}^{n}\left\{\int_0^{\infty}\cdots\int_0^{\infty}\frac{\chi_{[1,\infty)}(\min(t_1,t_2,\ldots,t_n)t_j)}{t_j^{1+\varepsilon}}\frac{\prod_{\substack{i=1\\ i\neq j}}^{n}t_i^{\lambda_i-\frac{\varepsilon}{p_i}-1}}{\left(1+\sum_{\substack{i=1\\ i\neq j}}^{n}t_i\right)^{s}}dt_1\cdots dt_n\right\}^{\frac{\lambda_j}{s}}$$

$$<\left(\frac{1}{\varepsilon}\right)^{\frac{1}{r}}\prod_{j=1}^{n}\left\{\int_0^{\infty}\cdots\int_0^{\infty}\frac{\prod_{\substack{i=1\\ i\neq j}}^{n}t_i^{\lambda_i-\frac{\varepsilon}{p_i}-1}}{\left(1+\sum_{\substack{i=1\\ i\neq j}}^{n}t_i\right)^{s}}dt_1\cdots dt_{j-1}dt_{j+1}\ldots dt_n\right\}^{\frac{\lambda_j}{s}}$$

$$=\left(\frac{1}{\varepsilon\lambda}\right)^{\frac{1}{r}}\prod_{j=1}^{n}\left\{\left(\lambda_j+\varepsilon-\frac{\varepsilon}{p_j}\right)\frac{\Gamma(\lambda_j+\varepsilon-\frac{\varepsilon}{p_j})}{\Gamma(\lambda)}\prod_{\substack{i=1\\ i\neq j}}^{n}\Gamma\left(\lambda_i-\frac{\varepsilon}{p_i}\right)\right\}^{\frac{\lambda_j}{s}}$$

$$\leq\left(\frac{1}{\varepsilon\lambda}\right)^{\frac{1}{r}}\prod_{j=1}^{n}\left\{\left(\lambda_j+\varepsilon-\frac{\varepsilon}{p_j}\right)\frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\Gamma\left(\lambda_i+\varepsilon-\frac{\varepsilon}{p_i}\right)\right\}^{\frac{\lambda_j}{s}}$$

$$=\left(\frac{1}{\varepsilon\lambda}\right)^{\frac{1}{r}}\left\{\prod_{j=1}^{n}\left(\lambda_j+\varepsilon-\frac{\varepsilon}{p_j}\right)^{\lambda_j}\right\}^{\frac{1}{s}}\left(\frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\Gamma\left(\lambda_i+\varepsilon-\frac{\varepsilon}{p_i}\right)\right)^{\frac{1}{r}}.$$

Now, assuming that the constant $\lambda^{\frac{1}{r}}\left[\frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\frac{\Gamma(\lambda_i)}{\lambda_i^{\lambda_i}}\right]^{\frac{1}{s}}$ in (8.13) is not the best possible, there exists a positive constant $\widetilde{C}<\lambda^{\frac{1}{r}}\left[\frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\frac{\Gamma(\lambda_i)}{\lambda_i^{\lambda_i}}\right]^{\frac{1}{s}}$, such that (8.13) is still valid when we replace $\lambda^{\frac{1}{r}}\left[\frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\frac{\Gamma(\lambda_i)}{\lambda_i^{\lambda_i}}\right]^{\frac{1}{s}}$ by $\widetilde{C}$. In particular, utilizing the derived inequalities, we have

$$\frac{1}{\varepsilon}\cdot\frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\Gamma\left(\lambda_i-\frac{\varepsilon}{p_i}\right)-O(1)$$

$$<\widetilde{C}\left(\frac{1}{\varepsilon\lambda}\right)^{\frac{1}{r}}\left(\frac{1}{\varepsilon}\right)^{\frac{1}{s}}\left\{\prod_{j=1}^{n}\left(\lambda_j+\varepsilon-\frac{\varepsilon}{p_j}\right)^{\lambda_j}\right\}^{\frac{1}{s}}\left(\frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\Gamma\left(\lambda_i+\varepsilon-\frac{\varepsilon}{p_i}\right)\right)^{\frac{1}{r}}.$$

Multiplying the above inequality by ε and then, letting $\varepsilon \to 0^+$, we obtain

$$\lambda^{\frac{1}{r}}\left[\frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\frac{\Gamma(\lambda_i)}{\lambda_i^{\lambda_i}}\right]^{\frac{1}{s}} \leq \widetilde{C},$$

which contradicts to the fact that $\widetilde{C} < \lambda^{\frac{1}{r}}\left[\frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\frac{\Gamma(\lambda_i)}{\lambda_i^{\lambda_i}}\right]^{\frac{1}{s}}$. Hence, the constant $\lambda^{\frac{1}{r}}\left[\frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\frac{\Gamma(\lambda_i)}{\lambda_i^{\lambda_i}}\right]^{\frac{1}{s}}$ in (8.13) is the best possible. This completes the proof of Theorem 8.5.

If $F(x_1,\ldots,x_n) = \frac{\prod_{i=1}^{n} f_i(x_i)}{\left(\sum_{i=1}^{n} u_i(x_i)\right)^{\lambda}}$, then

$$T_j = \int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\frac{u_j(x_j)\prod_{i=1}^{n} f_i(x_i)}{\left(\sum_{i=1}^{n} u_i(x_i)\right)^{s}}dx_1\cdots dx_n.$$

Therefore Theorem 8.5 yields the following consequence:

Corollary 8.5 *Under the same assumptions as in Theorem* 8.5, *inequality*

$$\begin{aligned}
&\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\frac{\prod_{i=1}^{n} f_i(x_i)}{\left(\sum_{i=1}^{n} u_i(x_i)\right)^{\lambda}}dx_1\cdots dx_n \\
&< \left[\frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\Gamma(\lambda_i)\right]^{\frac{1}{s}}\prod_{i=1}^{n}\left\{\int_{a_i}^{b_i}[u_i(x_i)]^{p_i(1-\lambda_i)-1}[u_i'(x_i)]^{1-p_i} f_i^{p_i}(x_i)dx_i\right\}^{\frac{1}{p_i s}} \\
&\quad\times\prod_{j=1}^{n}\left(\frac{s}{r\lambda_j}T_j\right)^{\frac{\lambda_j}{s}},
\end{aligned} \tag{8.20}$$

holds and the constant appearing on its right-hand side is the best possible.

It should be noticed here that the inequality (8.20) is more accurate than the inequality (8.12).

Theorem 8.6 *Inequality* (8.20) *refines inequality* (8.12).

Proof. It is not hard to see that

$$\sum_{j=1}^{n} T_j = \int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\frac{\prod_{i=1}^{n} f_i(x_i)}{\left(\sum_{i=1}^{n} u_i(x_i)\right)^{\lambda}}dx_1\cdots dx_n.$$

Thus, using the weighted arithmetic-geometric mean inequality, we obtain

$$\prod_{j=1}^{n}\left(\frac{s}{r\lambda_j}T_j\right)^{\frac{\lambda_j r}{s}} \leq \sum_{j=1}^{n}\frac{\lambda_j r}{s}\cdot\left(\frac{s}{r\lambda_j}T_j\right) = \int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\frac{\prod_{i=1}^{n} f_i(x_i)}{\left(\sum_{i=1}^{n} u_i(x_i)\right)^{\lambda}}dx_1\cdots dx_n.$$

The inequality (8.12) then follows from the preceding inequality (8.20). This completes the proof. □

Let

$$0 < \alpha < \min\left\{\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}\right\}.$$

Setting $u_i(x) = x^\alpha$, $f_i(x) = e^{-x}$, $i = 1, \dots, n$, and $F(x_1, \dots, x_n) = h(x_1) \cdots h(x_n)$, Theorem 8.5 yields the following Carlson-type inequality:

Corollary 8.6 *Suppose that the assumptions as in Theorem* 8.5 *are fulfilled and let h be a positive function on* $(0, \infty)$. *Then the inequality*

$$\left(\int_0^\infty h(x)dx\right)^r < C\left\{\int_0^\infty x^\alpha e^{\frac{rx}{s}} h^r(x)dx\right\}^{\frac{1}{n}} \left\{\int_0^\infty e^{\frac{rx}{s}} h^r(x)dx\right\}^{\frac{n-1}{n}}$$

holds, where the constant

$$C = \left[\frac{s}{r}\left(\frac{1}{\alpha^{\frac{(n-1)r}{s}}\Gamma(\lambda)} \prod_{i=1}^n \frac{p_i^{\alpha\lambda_i - 1}\Gamma(\lambda_i)(\Gamma(p_i - \alpha p_i \lambda_i))^{\frac{1}{p_i}}}{\lambda_i^{\lambda_i}}\right)^{\frac{r}{s}}\right]^{\frac{1}{n}}$$

is the best possible.

Finally, we propose the following open problem.

Open problem 5 *Find conditions so that the discrete versions of multidimensional inequalities from this section (with the best constants) hold.*

Remark 8.1 The inequalities presented in this chapter, as well as their consequences, are taken from [2] and [6]. For related results, the reader is referred to [19] and [21].

Chapter 9

On Some Hilbert-Pachpatte-type Inequalities

In this chapter we deal with a particular class of Hilbert-Pachpatte-type inequalities closely connected to Hilbert-type inequalities.

For example, some ten years ago, Pečarić *et al.* [80], established the following pair of Hilbert-Pachpatte-type inequalities: Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$, $\varphi,\psi:\mathbb{R}_+\to\mathbb{R}$ be non-negative functions. If $f,g:\mathbb{R}_+\to\mathbb{R}$ are absolutely continuous functions such that $f(0)=g(0)=0$, and $F(x)=\int_0^\infty K(x,y)\psi^{-p}(y)dy$, $G(y)=\int_0^\infty K(x,y)\varphi^{-q}(x)dx$, then the following inequalities hold:

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{K(x,y)|f(x)||g(y)|}{qx^{p-1}+py^{q-1}}dxdy\\
\leq\ &\int_0^\infty\int_0^\infty K(x,y)|f(x)||g(y)|d(x^{\frac{1}{p}})d(y^{\frac{1}{q}})\\
\leq\ &\frac{1}{pq}\left[\int_0^\infty\int_0^x \varphi^p(x)F(x)|f'(\tau)|^p d\tau dx\right]^{\frac{1}{p}}\left[\int_0^\infty\int_0^y \psi^q(y)G(y)|g'(\delta)|^q d\delta dy\right]^{\frac{1}{q}} \qquad (9.1)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\infty G^{1-p}(y)\psi^{-p}(y)\left[\int_0^\infty K(x,y)|f(x)|d(x^{\frac{1}{p}})\right]^p dy\\
\leq\ &\frac{1}{p^p}\int_0^\infty\int_0^x \varphi^p(x)F(x)|f'(\tau)|^p d\tau dx. \qquad (9.2)
\end{aligned}$$

For some applications of relations (9.1) and (9.2), as well as for some related results, the reader is referred to [63], [80], and references therein.

In this chapter we study a class of Hilbert-Pachpatte-type inequalities related to (9.1) and (9.2). More precisely, we give Hilbert-Pachpatte-type inequalities in more accurate forms, established by virtue of some recent refinements of arithmetic-geometric mean inequality. In addition, we also present weighted versions of such inequalities including fractional derivatives.

9.1 More Accurate Hilbert-Pachpatte-type Inequalities

In order to state and prove the corresponding inequalities we need some lemmas.

Lemma 9.1 *For $f \in C^n[a,b]$, $n \in \mathbb{N}$, the Taylor series of function f is given by*

$$f(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f^{(n)}(t)dt + \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k. \tag{9.3}$$

Define the subspace $C_a^n[a,b]$ of $C^n[a,b]$ as

$$C_a^n[a,b] = \{f \in C^n[a,b] : f^{(k)}(a) = 0, k = 0,1,\ldots,n-1\}.$$

Obviously, if $f \in C_a^n[a,b]$, then the right-hand side of (9.3) can be rewritten as

$$f(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f^{(n)}(t)dt. \tag{9.4}$$

Krnić *et al.* in [61] proved the following refinements and converses of the Young inequality in quotient and difference form. In order to state the corresponding results, denote $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, $P_n = \sum_{i=1}^n p_i$,

$$A_n(\mathbf{x}) = \frac{\sum_{i=1}^n x_i}{n}, \quad G_n(\mathbf{x}) = \left(\prod_{i=1}^n x_i\right)^{\frac{1}{n}},$$

and

$$M_r(\mathbf{x},\mathbf{p}) = \begin{cases} \left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i^r\right)^{\frac{1}{r}}, & r \neq 0 \\ \left(\prod_{i=1}^n x_i^{p_i}\right)^{\frac{1}{P_n}}, & r = 0 \end{cases}.$$

Lemma 9.2 (SEE [61]) *Let* $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ *and* $\boldsymbol{p} = (p_1, p_2, \dots, p_n)$ *be positive* $n-$*tuples such that* $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, *and*

$$\mathbf{x}^{\mathbf{p}} = (x_1^{p_1}, x_2^{p_2}, \dots, x_n^{p_n}), \quad \boldsymbol{p}^{-1} = \left(\frac{1}{p_1}, \frac{1}{p_2}, \dots, \frac{1}{p_n}\right).$$

Then

(i)

$$\left[\frac{A_n(\mathbf{x}^{\mathbf{p}})}{G_n(\mathbf{x}^{\mathbf{p}})}\right]^{n \min_{1\le i\le n}\{\frac{1}{p_i}\}} \le \frac{M_1(\mathbf{x}^{\mathbf{p}}, \boldsymbol{p}^{-1})}{M_0(\mathbf{x}^{\mathbf{p}}, \boldsymbol{p}^{-1})} \le \left[\frac{A_n(\mathbf{x}^{\mathbf{p}})}{G_n(\mathbf{x}^{\mathbf{p}})}\right]^{n \max_{1\le i\le n}\{\frac{1}{p_i}\}},$$

and

(ii)

$$\begin{aligned} n \min_{1\le i\le n}\left\{\frac{1}{p_i}\right\}[A_n(\mathbf{x}^{\mathbf{p}}) - G_n(\mathbf{x}^{\mathbf{p}})] &\le M_1(\mathbf{x}^{\mathbf{p}}, \boldsymbol{p}^{-1}) - M_0(\mathbf{x}^{\mathbf{p}}, \boldsymbol{p}^{-1}) \\ &\le n \max_{1\le i\le n}\left\{\frac{1}{p_i}\right\}[A_n(\mathbf{x}^{\mathbf{p}}) - G_n(\mathbf{x}^{\mathbf{p}})]. \end{aligned}$$

We first give improved form of the Hilbert-Pachpatte type inequality with a general kernel.

Theorem 9.1 *Let* $\frac{1}{p} + \frac{1}{q} = 1$ *with* $p, q > 1$, *and* $0 \le a < b \le \infty$. *If* $K : [a,b] \times [a,b] \to \mathbb{R}$ *is non-negative function,* $\varphi(x)$, $\psi(y)$ *are non-negative functions on* $[a,b]$ *and* $f, g \in C_a^n[a,b]$, *then the following inequalities hold*

$$\begin{aligned} &\int_a^b \int_a^b \frac{K(x,y)|f(x)|\,|g(y)|}{\left((x-a)^{\frac{1}{q(M-m)}} + (y-a)^{\frac{1}{p(M-m)}}\right)^{2(M-m)}} dxdy \\ &\le \frac{1}{4^{M-m}} \int_a^b \int_a^b \frac{K(x,y)|f(x)|\,|g(y)|}{(x-a)^{\frac{1}{q}}(y-a)^{\frac{1}{p}}} dxdy \qquad (9.5) \\ &\le \frac{1}{4^{M-m}[(n-1)!]^2} \left(\int_a^b \int_a^x (x-t)^{p(n-1)} \varphi^p(x) F(x) |f^{(n)}(t)|^p dtdx\right)^{\frac{1}{p}} \\ &\qquad \times \left(\int_a^b \int_a^y (y-t)^{q(n-1)} \psi^q(y) G(y) |g^{(n)}(t)|^q dtdy\right)^{\frac{1}{q}}, \end{aligned}$$

and

$$\begin{aligned} &\int_a^b G^{1-p}(y)\psi^{-p}(y) \left(\int_a^b K(x,y) \left(\int_a^x (x-t)^{p(n-1)} |f^{(n)}(t)|^p dt\right)^{\frac{1}{p}} dx\right)^p dy \\ &\le \int_a^b \int_a^x (x-t)^{p(n-1)} \varphi^p(x) F(x) |f^{(n)}(t)|^p dtdx, \qquad (9.6) \end{aligned}$$

where $m = \min\{\frac{1}{p}, \frac{1}{q}\}$, $M = \max\{\frac{1}{p}, \frac{1}{q}\}$, *and* $F(x)$ *and* $G(y)$ *are defined as in* (1.16).

Proof. By using (9.4) and Hölder's inequality, we have

$$\begin{aligned}
|f(x)| &= \frac{1}{(n-1)!}\left|\int_a^x (x-t)^{n-1} f^{(n)}(t)dt\right| \\
&\le \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}|f^{(n)}(t)|\cdot 1dt \\
&\le \frac{1}{(n-1)!}\left(\int_a^x (x-t)^{p(n-1)}|f^{(n)}(t)|^p dt\right)^{\frac{1}{p}}\left(\int_a^x 1^q dt\right)^{\frac{1}{q}} \\
&= \frac{(x-a)^{\frac{1}{q}}}{(n-1)!}\left(\int_a^x (x-t)^{p(n-1)}|f^{(n)}(t)|^p dt\right)^{\frac{1}{p}}, \qquad (9.7)
\end{aligned}$$

and similarly

$$|g(x)| \le \frac{(y-a)^{\frac{1}{p}}}{(n-1)!}\left(\int_a^y (y-t)^{q(n-1)}|g^{(n)}(t)|^q dt\right)^{\frac{1}{q}}. \qquad (9.8)$$

Now, from (9.7) and (9.8) we get

$$\begin{aligned}
|f(x)|\,|g(y)| \le{}& \frac{1}{[(n-1)!]^2}(x-a)^{\frac{1}{q}}(y-a)^{\frac{1}{p}} \\
&\times\left(\int_a^x (x-t)^{p(n-1)}|f^{(n)}(t)|^p dt\right)^{\frac{1}{p}} \qquad (9.9)\\
&\times\left(\int_a^y (y-t)^{q(n-1)}|g^{(n)}(t)|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Applying Lemma 9.2(i) (see also [61]), we have

$$4^{M-m}(x^p y^q)^{M-m} \le (x^p+y^q)^{2(M-m)}, \quad x\ge 0,\ y\ge 0, \qquad (9.10)$$

where $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$, and $m=\min\{\frac{1}{p},\frac{1}{q}\}$, $M=\max\{\frac{1}{p},\frac{1}{q}\}$. From (9.9) and (9.10) we observe that

$$\begin{aligned}
&\frac{4^{M-m}|f(x)|\,|g(y)|}{\left((x-a)^{\frac{1}{q(M-m)}}+(y-a)^{\frac{1}{p(M-m)}}\right)^{2(M-m)}} \le \frac{|f(x)|\,|g(y)|}{(x-a)^{\frac{1}{q}}(y-a)^{\frac{1}{p}}} \\
&\le \frac{1}{[(n-1)!]^2}\left(\int_a^x (x-t)^{p(n-1)}|f^{(n)}(t)|^p dt\right)^{\frac{1}{p}}\left(\int_a^y (y-t)^{q(n-1)}|g^{(n)}(t)|^q dt\right)^{\frac{1}{q}},
\end{aligned}$$

and therefore

$$\begin{aligned}
&4^{M-m}\int_a^b\int_a^b \frac{K(x,y)|f(x)|\,|g(y)|}{\left((x-a)^{\frac{1}{q(M-m)}}+(y-a)^{\frac{1}{p(M-m)}}\right)^{2(M-m)}}dxdy \\
&\le \int_a^b\int_a^b \frac{K(x,y)|f(x)|\,|g(y)|}{(x-a)^{\frac{1}{q}}(y-a)^{\frac{1}{p}}}dxdy \qquad (9.11)
\end{aligned}$$

$$\leq \frac{1}{[(n-1)!]^2} \int_a^b \int_a^b K(x,y) \left(\int_a^x (x-t)^{p(n-1)} |f^{(n)}(t)|^p dt \right)^{\frac{1}{p}}$$
$$\times \left(\int_a^y (y-t)^{q(n-1)} |g^{(n)}(t)|^q dt \right)^{\frac{1}{q}} dxdy.$$

Applying the substitutions

$$f_1(x) = \left(\int_a^x (x-t)^{p(n-1)} |f^{(n)}(t)|^p dt \right)^{\frac{1}{p}}, \quad g_1(y) = \left(\int_a^y (y-t)^{q(n-1)} |g^{(n)}(t)|^q dt \right)^{\frac{1}{q}}$$

and (1.17), we have

$$\int_a^b \int_a^b K(x,y) f_1(x) g_1(y) dxdy \tag{9.12}$$
$$\leq \left(\int_a^b \varphi^p(x) F(x) f_1^p(x) dx \right)^{\frac{1}{p}} \left(\int_a^b \psi^q(y) G(y) g_1^q(y) dy \right)^{\frac{1}{q}}$$
$$= \left(\int_a^b \int_a^x (x-t)^{p(n-1)} \varphi^p(x) F(x) |f^{(n)}(t)|^p dt dx \right)^{\frac{1}{p}}$$
$$\times \left(\int_a^b \int_a^y (y-t)^{q(n-1)} \psi^q(y) G(y) |g^{(n)}(t)|^q dt dy \right)^{\frac{1}{q}}.$$

By using (9.11) and (9.12) we obtain (9.5). The second inequality (9.6) follows by applying (1.18). □

Now we can apply our main result to non-negative homogeneous functions. To do this, we need the following lemma.

Lemma 9.3 *If $\lambda > 0$, $1-\lambda < \alpha < 1$ and $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a non-negative homogeneous function of degree $-\lambda$, then*

$$\int_0^\infty K(x,y) \left(\frac{x}{y} \right)^\alpha dy = x^{1-\lambda} k(\alpha), \tag{9.13}$$

and

$$\int_0^\infty K(x,y) \left(\frac{y}{x} \right)^\alpha dx = y^{1-\lambda} k(2-\lambda-\alpha). \tag{9.14}$$

Proof. We use the substitution $y = ux$. The proof follows easily from homogeneity of the function $K(x,y)$. □

Corollary 9.1 *Let $\frac{1}{p} + \frac{1}{q} = 1$, with $p,q > 1$. If $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a non-negative and homogeneous function of degree $-\lambda$, $\lambda > 0$, and $f,g \in C_0^n[0,\infty]$, then*

$$\int_0^\infty \int_0^\infty \frac{K(x,y)|f(x)|\,|g(y)|}{\left(x^{\frac{1}{q(M-m)}} + y^{\frac{1}{p(M-m)}} \right)^{2(M-m)}} dxdy \leq \frac{pq}{4^{M-m}} \int_0^\infty \int_0^\infty K(x,y)|f(x)|\,|g(y)| d(x^{\frac{1}{p}}) d(y^{\frac{1}{q}}) \tag{9.15}$$

$$\leq \frac{L}{4^{M-m}[(n-1)!]^2}\left(\int_0^\infty \int_0^x x^{p(A_1-A_2+n-1)+1-\lambda}|f^{(n)}(t)|^p dtdx\right)^{\frac{1}{p}}$$
$$\times \left(\int_0^\infty \int_0^y y^{q(A_2-A_1+n-1)+1-\lambda}|g^{(n)}(t)|^q dtdy\right)^{\frac{1}{q}},$$

and

$$\int_0^\infty y^{(p-1)(\lambda-1)+p(A_1-A_2)}\left(\int_0^\infty K(x,y)\left(\int_0^x (x-t)^{p(n-1)}|f^{(n)}(t)|^p dt\right)^{\frac{1}{p}} dx\right)^p dy$$
$$\leq L^p \int_0^\infty \int_0^x x^{p(A_1-A_2+n-1)+1-\lambda}|f^{(n)}(t)|^p dtdx, \tag{9.16}$$

where $A_1 \in (\frac{1-\lambda}{q}, \frac{1}{q})$, $A_2 \in (\frac{1-\lambda}{p}, \frac{1}{p})$, $L = k(pA_2)^{\frac{1}{p}} k(2-\lambda-qA_1)^{\frac{1}{q}}$, and M,m are defined as in Theorem 9.1.

Proof. Let $F(x)$, $G(y)$ be the functions defined by (1.16). Setting $\varphi(x) = x^{A_1}$ and $\psi(y) = y^{A_2}$ in (9.5), using the fact that $(x-t)^{p(n-1)} \leq x^{p(n-1)}$, for $x \geq 0$, $t \in [0,x]$, and applying Lemma 9.3, we get

$$\int_0^\infty \int_0^x (x-t)^{p(n-1)}\varphi^p(x)F(x)|f^{(n)}(t)|^p dtdx$$
$$\leq \int_0^\infty \int_0^x x^{p(A_1-A_2+n-1)}\left(\int_0^\infty K(x,y)\left(\frac{x}{y}\right)^{pA_2} dy\right)|f^{(n)}(t)|^p dtdx$$
$$= k(pA_2)\int_0^\infty \int_0^x x^{p(A_1-A_2+n-1)+1-\lambda}|f^{(n)}(t)|^p dtdx, \tag{9.17}$$

and similarly

$$\int_0^\infty \int_0^y (y-t)^{q(n-1)}\psi^q(y)G(y)|g^{(n)}(t)|^q dtdy \tag{9.18}$$
$$\leq k(2-\lambda-qA_1)\int_0^\infty \int_0^y y^{p(A_2-A_1+n-1)+1-\lambda}|g^{(n)}(t)|^q dtdy.$$

From (9.5), (9.17) and (9.18), we get (9.15). □

We proceed with some special homogeneous functions. First, by putting $K(x,y) = \frac{\log\frac{y}{x}}{y-x}$ in Corollary 9.1, we get the following result.

Corollary 9.2 *Let $\frac{1}{p}+\frac{1}{q} = 1$, with $p,q > 1$. Let M,m,f,g be defined as in Corollary* 9.1. *Then,*

$$\int_0^\infty \int_0^\infty \frac{\log\frac{y}{x}|f(x)|\,|g(y)|}{(y-x)\left(x^{\frac{1}{q(M-m)}}+y^{\frac{1}{p(M-m)}}\right)^{2(M-m)}}dxdy$$
$$\leq \frac{pq}{4^{M-m}}\int_0^\infty \int_0^\infty \frac{\log\frac{y}{x}|f(x)|\,|g(y)|}{y-x}d(x^{\frac{1}{p}})d(y^{\frac{1}{q}})$$

$$\leq \frac{L_1}{4^{M-m}[(n-1)!]^2}\left(\int_0^\infty\int_0^x x^{p(A_1-A_2+n-1)}|f^{(n)}(t)|^p dtdx\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^\infty\int_0^y y^{q(A_2-A_1+n-1)}|g^{(n)}(t)|^q dtdy\right)^{\frac{1}{q}},$$

and

$$\int_0^\infty y^{p(A_1-A_2)}\left(\int_0^\infty \frac{\log\frac{y}{x}}{y-x}\left(\int_0^x (x-t)^{p(n-1)}|f^{(n)}(t)|^p dt\right)^{\frac{1}{p}} dx\right)^p dy$$
$$\leq L_1^p\int_0^\infty\int_0^x x^{p(A_1-A_2+n-1)}|f^{(n)}(t)|^p dtdx,$$

where $A_1 \in (0,\frac{1}{q})$, $A_2 \in (0,\frac{1}{p})$, *and*

$$L_1 = \pi^2(\sin pA_2\pi)^{-\frac{2}{p}}(\sin qA_1\pi)^{-\frac{2}{q}}.$$

Similarly, for the homogeneous function of degree $-\lambda$, $\lambda > 0$, $K(x,y) = (\max\{x,y\})^{-\lambda}$, $A_1 = A_2 = \frac{2-\lambda}{pq}$, with $\lambda > 2-\min\{p,q\}$, we have:

Corollary 9.3 *Let* $\frac{1}{p}+\frac{1}{q}=1$, *with* $p,q>1$. *Let* M,m,f,g *be defined as in Corollary* 9.1. *Then,*

$$\int_0^\infty\int_0^\infty \frac{(\max\{x,y\})^{-\lambda}|f(x)|\,|g(y)|}{\left(x^{\frac{1}{q(M-m)}}+y^{\frac{1}{p(M-m)}}\right)^{2(M-m)}}dxdy$$
$$\leq \frac{pq}{4^{M-m}}\int_0^\infty\int_0^\infty \frac{|f(x)|\,|g(y)|}{(\max\{x,y\})^\lambda}d(x^{\frac{1}{p}})d(y^{\frac{1}{q}})$$
$$\leq \frac{L_2}{4^{M-m}[(n-1)!]^2}\left(\int_0^\infty\int_0^x x^{p(n-1)+1-\lambda}|f^{(n)}(t)|^p dtdx\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^\infty\int_0^y y^{q(n-1)+1-\lambda}|g^{(n)}(t)|^q dtdy\right)^{\frac{1}{q}},$$

and

$$\int_0^\infty y^{(p-1)(\lambda-1)}\left(\int_0^\infty (\max\{x,y\})^{-\lambda}\left(\int_0^x (x-t)^{p(n-1)}|f^{(n)}(t)|^p dt\right)^{\frac{1}{p}} dx\right)^p dy$$
$$\leq L_2^p\int_0^\infty\int_0^x x^{p(n-1)+1-\lambda}|f^{(n)}(t)|^p dtdx,$$

where $L_2 = k(\frac{2-\lambda}{q})$ *and* $k(\alpha) = \frac{\lambda}{(1-\alpha)(\lambda+\alpha-1)}$.

The following multidimensional inequality follows by virtue of the general Hilbert-type inequality (1.2) (see Section 1.1).

Theorem 9.2 *Let $n,l \in \mathbb{N}$, $l \geq 2$, $\sum_{i=1}^{l} \frac{1}{p_i} = 1$, $p_i > 1$, and let $\alpha_i = \prod_{j=1, j\neq i}^{l} p_j$, $i = 1,2,\ldots l$. If $K : [a,b]^l \to \mathbb{R}$ is a non-negative function, $\phi_{ij}(x_j)$, $i,j = 1,\ldots,l$, are non-negative functions on $[a,b]$, such that $\prod_{i,j=1}^{l} \phi_{ij}(x_j) = 1$, and $f_i \in C_a^n[a,b]$, $i = 1,\ldots,l$, then*

$$\int_{(a,b)^l} \frac{K(x_1,\ldots,x_l)\prod_{i=1}^{l}|f_i(x_i)|}{\left(\sum_{i=1}^{l}(x_i-a)^{\frac{1}{\alpha_i(M-m)}}\right)^{l(M-m)}} dx_1\ldots dx_l$$
$$\leq \frac{1}{l^{(M-m)l}} \int_{(a,b)^l} \frac{K(x_1,\ldots,x_l)\prod_{i=1}^{l}|f_i(x_i)|}{\prod_{i=1}^{l}(x_i-a)^{\frac{1}{\alpha_i}}} dx_1\ldots dx_l$$
$$\leq \frac{1}{l^{(M-m)l}[(n-1)!]^l} \prod_{i=1}^{l} \left(\int_a^b \int_a^{x_i} (x_i-t)^{p_i(n-1)} \phi_{ii}^{p_i}(x_i) F_i(x_i) |f_i^{(n)}(t)|^{p_i} dt dx_i\right)^{\frac{1}{p_i}},$$

where $m = \min_{1\leq i\leq l}\{\frac{1}{p_i}\}$, $M = \max_{1\leq i\leq l}\{\frac{1}{p_i}\}$, and $F_i(x_i)$, $i = 1,\ldots,l$ is defined by (1.4).

Obviously, Theorem 9.2 is a generalization of Theorem 9.1.

Remark 9.1 Applying Lemma 9.2 (ii) it follows that

$$x^p y^q \leq \left(\frac{x^p+y^q}{2} - \frac{1}{M-m}\right)^2, \quad x\geq 0,\ y\geq 0, \tag{9.19}$$

where $\frac{1}{p}+\frac{1}{q} = 1$ with $p > 1$, and $m = \min\{\frac{1}{p},\frac{1}{q}\}$, $M = \max\{\frac{1}{p},\frac{1}{q}\}$. Now, taking into account (9.19) and following the lines as in the proof of Theorem 9.1, we have

$$\int_a^b \int_a^b \frac{K(x,y)|f(x)|\,|g(y)|}{\left(\frac{1}{2}[(x-a)^{\frac{1}{q}} + (y-a)^{\frac{1}{p}}] - \frac{1}{M-m}\right)^2} dxdy$$
$$\leq \int_a^b \int_a^b \frac{K(x,y)|f(x)|\,|g(y)|}{(x-a)^{\frac{1}{q}}(y-a)^{\frac{1}{p}}} dxdy$$
$$\leq \frac{1}{[(n-1)!]^2} \left(\int_a^b \int_a^x (x-t)^{p(n-1)} \varphi^p(x) F(x) |f^{(n)}(t)|^p dt dx\right)^{\frac{1}{p}}$$
$$\times \left(\int_a^b \int_a^y (y-t)^{q(n-1)} \psi^q(y) G(y) |g^{(n)}(t)|^q dt dy\right)^{\frac{1}{q}},$$

where $F(x)$ and $G(y)$ are defined by (1.16) (see Section 1.1).

9.2 The Fractional Derivatives and Applications to Hilbert-Pachpatte Type Inequalities

First, we introduce some facts about fractional derivatives (for more details, see [29]). Let $[a,b]$, $-\infty < a < b < \infty$, be a finite interval on real axis $\mathbb{R}$. By $L_p[a,b]$, $1 \leq p < \infty$, we denote the space of all Lebesgue measurable functions f for which $|f^p|$ is Lebesgue integrable on $[a,b]$. For $f \in L_1[a,b]$ *the left-sided and the right-sided Riemann-Liouville integral of f of order α* are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt, \quad x > a,$$
$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad x < b.$$

For $f : [a,b] \to \mathbb{R}$ *the left-sided Riemann-Liouville derivative of f of order α* is defined by

$$D^{\alpha}_{a+}f(x) = \frac{d^n}{dx^n}J^{n-\alpha}_{a+}f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-t)^{n-\alpha-1}f(t)dt.$$

Our result with the Riemann-Liouville fractional derivative is based on the following result. By $AC^m[a,b]$ we denote the space of all functions $g \in C^{m-1}[a,b]$ with $g^{(m-1)} \in AC[a,b]$, where $AC[a,b]$ is the space of all absolutely continuous functions on $[a,b]$. For $\alpha > 0$, $[\alpha]$ denotes the integral part of α.

Lemma 9.4 (SEE [15]) *Let $\beta > \alpha \geq 0$, $m = [\beta]+1$, $n = [\alpha]+1$. The composition identity*

$$D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\beta-\alpha)}\int_0^x (x-t)^{\beta-\alpha-1}D^{\beta}_{a+}f(t)dt, \quad x \in [a,b],$$

is valid if one of the following conditions holds:

(i) $f \in J^{\beta}_{a+}(L_1[a,b]) = \{f : f = J^{\beta}_{a+}\varphi, \varphi \in L_1[a,b]\}$.

(ii) $J^{m-\beta}_{a+}f \in AC^m[a,b]$ *and* $D^{\beta-k}_{a+}f(a) = 0$ *for* $k = 1,\ldots,m$.

(iii) $D^{\beta-1}_{a+}f \in AC[a,b]$, $D^{\beta-k}_{a+}f \in C[a,b]$ *and* $D^{\beta-k}_{a+}f(a) = 0$ *for* $k = 1,\ldots,m$.

(iv) $f \in AC^m[a,b]$, $D^{\beta}_{a+}f$, $D^{\alpha}_{a+}f \in L_1[a,b]$, $\beta-\alpha \notin \mathbb{N}$, $D^{\beta-k}_{a+}f(a) = 0$ *for* $k = 1,\ldots,m$ *and* $D^{\alpha-k}_{a+}f(a) = 0$ *for* $k = 1,\ldots,n$.

(v) $f \in AC^m[a,b]$, $D^{\beta}_{a+}f$, $D^{\alpha}_{a+}f \in L_1[a,b]$, $\beta-\alpha = l \in \mathbb{N}$, $D^{\beta-k}_{a+}f(a) = 0$ *for* $k = 1,\ldots,l$.

(vi) $f \in AC^m[a,b]$, $D^{\beta}_{a+}f$, $D^{\alpha}_{a+}f \in L_1[a,b]$, *and* $f^{(k)}(a) = 0$ *for* $k = 0,\ldots,m-2$.

(vii) $f \in AC^m[a,b]$, $D^{\beta}_{a+}f$, $D^{\alpha}_{a+}f \in L_1[a,b]$, $\beta \notin \mathbb{N}$ *and* $D^{\beta-1}_{a+}f$ *is bounded in a neighborhood of* $m = a$.

By using Lemma 9.2 (see also Remark 9.1) and Lemma 9.4 we obtain the following result including the fractional derivative.

Theorem 9.3 *Let α, β, f, g be defined as in Lemma 9.4. If $K : [a,b]^2 \to \mathbb{R}$ is non-negative function, $\varphi(x)$, $\psi(y)$ are non-negative functions on $[a,b]$, then the following inequality holds*

$$\int_a^b \int_a^b \frac{K(x,y)|D_{a+}^{\alpha} f(x)|\,|D_{a+}^{\alpha} g(y)|}{\left(\frac{1}{2}[(x-a)^{\frac{1}{q}} + (y-a)^{\frac{1}{p}}] - \frac{1}{M-m}\right)^2} dxdy$$
$$\leq \int_a^b \int_a^b \frac{K(x,y)|D_{a+}^{\alpha} f(x)|\,|D_{a+}^{\alpha} g(y)|}{(x-a)^{\frac{1}{q}}(y-a)^{\frac{1}{p}}} dxdy$$
$$\leq \frac{1}{[\Gamma(\beta-\alpha)]^2} \left(\int_a^b \int_a^x (x-t)^{p(\beta-\alpha-1)} \varphi^p(x) F(x) |D_{a+}^{\alpha} f(t)|^p dt dx\right)^{\frac{1}{p}}$$
$$\times \left(\int_a^b \int_a^y (y-t)^{q(\beta-\alpha-1)} \psi^q(y) G(y) |D_{a+}^{\alpha} g(t)|^q dt dy\right)^{\frac{1}{q}},$$

where $m, M, F(x), G(y)$ are defined as in Theorem 9.1.

Proof. The proof is similar to the proof of Theorem 9.1. □

Let $\nu > 0$, $n = [\nu]$, and $\overline{\nu} = \nu - n$, $0 \leq \overline{\nu} < 1$. Let $[a,b] \subseteq \mathbb{R}$ and $x_0, x \in [a,b]$ such that $x \geq x_0$, where x_0 is fixed. For $f \in C[a,b]$ *the generalized Riemann-Liouville fractional integral of f of order ν* is given by

$$(J_\nu^{x_0} f)(x) = \frac{1}{\Gamma(\nu)} \int_{x_0}^x (x-t)^{\nu-1} f(t) dt, \quad x \in [x_0, b].$$

Further, define the subspace $C_{x_0}^\nu[a,b]$ of $C^n[a,b]$ as

$$C_{x_0}^\nu[a,b] = \{f \in C^n[a,b] : J_{1-\overline{\nu}}^{x_0} f^{(n)} \in C^1[x_0,b]\}.$$

For $f \in C_{x_0}^\nu[a,b]$ *the generalized Canavati $\nu-$fractional derivative of f over* $[x_0,b]$ is given by

$$D_{x_0}^\nu f = D J_{1-\overline{\nu}}^{x_0} f^{(n)},$$

where $D = d/dx$. Notice that

$$(J_{1-\overline{\nu}}^{x_0} f^{(n)})(x) = \frac{1}{\Gamma(1-\overline{\nu})} \int_{x_0}^x (x-t)^{-\overline{\nu}} f^{(n)}(t) dt$$

exists for $f \in C_{x_0}^\nu[a,b]$.
To obtain the result with generalized Canavati $\nu-$fractional derivative of f we need the following lemma.

Lemma 9.5 (see [29]) *Let $f \in C_{x_0}^\nu[a,b]$, $\nu > 0$ and $f^{(i)}(x_0) = 0$, $i = 0,1,\ldots,n-1$, $n = [\nu]$. Then*

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{x_0}^x (x-t)^{\nu-1} (D_{x_0}^\nu f)(t) dt,$$

for all $x \in [a,b]$ with $x \geq x_0$.

Theorem 9.4 *Let $\nu > 0$ and $x_0, y_0 \in [a,b]$. Let $K : [a,b]^2 \to \mathbb{R}$ be a non-negative function, and $\varphi(x)$, $\psi(y)$ be non-negative functions on $[a,b]$. If $f \in C^{\nu}_{x_0}[a,b]$ and $g \in C^{\nu}_{y_0}[a,b]$ are such that $f^{(i)}(x_0) = g^{(i)}(y_0) = 0$, $i = 0,1,\ldots,n-1$, $n = [\nu]$, then*

$$\begin{aligned}
&\int_a^b \int_a^b \frac{K(x,y)|f(x)|\,|g(y)|}{\left((x-x_0)^{\frac{1}{q(M-m)}} + (y-y_0)^{\frac{1}{p(M-m)}}\right)^{2(M-m)}} dxdy \\
&\le \frac{1}{4^{M-m}} \int_a^b \int_a^b \frac{K(x,y)|f(x)|\,|g(y)|}{(x-x_0)^{\frac{1}{q}}(y-y_0)^{\frac{1}{p}}} dxdy \qquad (9.20)\\
&\le \frac{1}{4^{M-m}[\Gamma(\nu)]^2} \left(\int_a^b \int_{x_0}^x (x-t)^{p(\nu-1)} \varphi^p(x) F(x) |(D^{\nu}_{x_0} f)(t)|^p dtdx \right)^{\frac{1}{p}} \\
&\qquad \times \left(\int_a^b \int_{y_0}^y (y-t)^{q(\nu-1)} \psi^q(y) G(y) |(D^{\nu}_{y_0} g)(t)|^q dtdy \right)^{\frac{1}{q}},
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b G^{1-p}(y)\ \psi^{-p}(y) \left(\int_a^b K(x,y) \left(\int_{x_0}^x (x-t)^{p(\nu-1)} |(D^{\nu}_{x_0} f)(t)|^p dt \right)^{\frac{1}{p}} dx \right)^p dy \\
&\qquad \le \int_a^b \int_{x_0}^x (x-t)^{p(\nu-1)} \varphi^p(x) F(x) |(D^{\nu}_{x_0} f)(t)|^p dtdx, \qquad (9.21)
\end{aligned}$$

where $m = \min\{\frac{1}{p}, \frac{1}{q}\}$, $M = \max\{\frac{1}{p}, \frac{1}{q}\}$, and $F(x)$ and $G(y)$ are defined by (1.16).

Proof. To prove the inequalities (9.20) and (9.21) we follow the same procedure as in the proof of Theorem 9.1, except that we use Lemma 9.5 instead of Lemma 9.1. □

In a similar manner as in the previous section, utilizing the inequality (1.2), we obtain a generalization of Theorem 9.4.

Theorem 9.5 *Let $\nu > 0$ and $\alpha_i = \prod_{j=1, j\neq i}^l p_j$, where $\sum_{i=1}^l \frac{1}{p_i} = 1$ with $p_i > 1$, $i = 1,\ldots,l$. Suppose that $K(x_1,\ldots,x_l)$, ϕ_{ij}, $i,j = 1,\ldots,l$, are defined as in Theorem 9.2. If $f_i \in C^{\nu}_{x_0^{(i)}}[a,b]$ ($x_0^{(i)} \in [a,b]$), $i = 1,\ldots,l$, are such that $f_i^{(j)}(x_0^{(i)}) = 0$, $j = 0,1,\ldots,n-1$, $n = [\nu]$, then*

$$\begin{aligned}
&\int_{(a,b)^l} \frac{K(x_1,\ldots,x_l)\prod_{i=1}^l |f_i(x_i)|}{\left(\sum_{i=1}^l (x_i - x_0^{(i)})^{\frac{1}{\alpha_i(M-m)}}\right)^{l(M-m)}} dx_1 \ldots dx_l \\
&\le \frac{1}{l^{(M-m)l}} \int_{(a,b)^l} \frac{K(x_1,\ldots,x_l)\prod_{i=1}^l |f_i(x_i)|}{\prod_{i=1}^l (x_i - x_0^{(i)})^{\frac{1}{\alpha_i}}} dx_1 \ldots dx_l \\
&\le \frac{1}{l^{(M-m)l}[\Gamma(\nu)]^l} \prod_{i=1}^l \left(\int_a^b \int_{x_0^{(i)}}^{x_i} (x_i - t)^{p_i(\nu-1)} \phi_{ii}^{p_i}(x_i) F_i(x_i) |(D^{\nu}_{x_0^{(i)}} f_i)(t)|^{p_i} dtdx_i \right)^{\frac{1}{p_i}},
\end{aligned}$$

where $m = \min_{1\le i\le l}\{\frac{1}{p_i}\}$, $M = \max_{1\le i\le l}\{\frac{1}{p_i}\}$, and $F_i(x_i)$, $i = 1,\ldots,l$ is defined by (1.4) *(see Section* 1.1*).*

For $\alpha > 0$, $f \in AC^n[a,b]$, where $n = [\alpha]+1$ if $\alpha \notin \mathbb{N}_0$ and $n = \alpha$ if $\alpha \in \mathbb{N}_0$, *the Caputo fractional derivative of f of order α ${}^cD^{\alpha}_{a+}f$ (left-sided) and ${}^cD^{\alpha}_{b-}f$ (right-sided)* are defined by

$$
\begin{aligned}
{}^cD^{\alpha}_{a+}f(x) &= D^{\alpha}_{a+}\left[f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(k+1)}(x-a)^k\right],\\
{}^cD^{\alpha}_{b-}f(x) &= D^{\alpha}_{b-}\left[f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{\Gamma(k+1)}(b-x)^k\right],
\end{aligned}
$$

where D^{α}_{a+}, D^{α}_{b-} denote the left-hand sided and the right-hand sided Riemann-Liouville derivatives.

Recently, Andrić *et al.* [16], proved the following result.

Theorem 9.6 *Let $\nu > \gamma \geq 0$, $n = [\nu]+1$, $m = [\gamma]+1$ and $f \in AC^k[a,b]$, $k = n$ if $\nu \notin \mathbb{N}_0$ and $k = n-1$ if $\nu \in \mathbb{N}_0$. Let ${}^cD^{\nu}_{a+}f$, ${}^cD^{\gamma}_{a+}f \in L^1[a,b]$. Suppose that one of the following conditions holds:*

(a) $\nu,\gamma \notin \mathbb{N}_0$ and $f^{(i)}(a) = 0$ for $i = m,\dots,n-1$.

(b) $\nu \in \mathbb{N}$, $\gamma \notin \mathbb{N}_0$ and $f^{(i)}(a) = 0$ for $i = m,\dots,n-2$.

(c) $\nu \notin \mathbb{N}$, $\gamma \in \mathbb{N}_0$ and $f^{(i)}(a) = 0$ for $i = m-1,\dots,n-1$.

(d) $\nu \in \mathbb{N}$, $\gamma \in \mathbb{N}_0$ and $f^{(i)}(a) = 0$ for $i = m-1,\dots,n-2$.

Then

$$
{}^cD^{\gamma}_{a+}f(x) = \frac{1}{\Gamma(\nu-\gamma)}\int_a^x (x-t)^{\nu-\gamma-1}D^{\nu}_{a+}f(t)dt.
$$

Applying Lemma 9.2 (i) and Theorem 9.6 (see also [16]), we obtain the following result.

Theorem 9.7 *Let ν,γ,f,g be defined as in Theorem 9.6. If $K:[a,b]^2 \to \mathbb{R}$ is non-negative function, $\varphi(x)$, $\psi(y)$ are non-negative functions on $[a,b]$, then*

$$
\begin{aligned}
&\int_a^b\int_a^b \frac{K(x,y)|{}^cD^{\gamma}_{a+}f(x)|\,|{}^cD^{\gamma}_{a+}g(y)|}{\left((x-a)^{\frac{1}{q(M-m)}}+(y-a)^{\frac{1}{p(M-m)}}\right)^{2(M-m)}}dxdy\\
&\leq \frac{1}{4^{M-m}}\int_a^b\int_a^b \frac{K(x,y)|{}^cD^{\gamma}_{a+}f(x)|\,|{}^cD^{\gamma}_{a+}g(y)|}{(x-x_0)^{\frac{1}{q}}(y-y_0)^{\frac{1}{p}}}dxdy\\
&\leq \frac{1}{4^{M-m}[\Gamma(\nu-\gamma)]^2}\left(\int_a^b\int_a^x (x-t)^{p(\nu-\gamma-1)}\varphi^p(x)F(x)|{}^cD^{\nu}_{a+}f(t)|^p dtdx\right)^{\frac{1}{p}}\\
&\qquad\times\left(\int_a^b\int_a^y (y-t)^{q(\nu-\gamma-1)}\psi^q(y)G(y)|{}^cD^{\nu}_{a+}g(t)|^q dtdy\right)^{\frac{1}{q}},
\end{aligned}
$$

where $m,M,F(x),G(y)$ are defined as in Theorem 9.1.

Remark 9.2 The general Hilbert-Pachpatte-type inequalities in this chapter are taken from [78]. For related results and some other forms of Hilbert-Pachpatte-type inequalities, the reader is reffered to [32], [102] and [103].

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Zagreb, December 2017